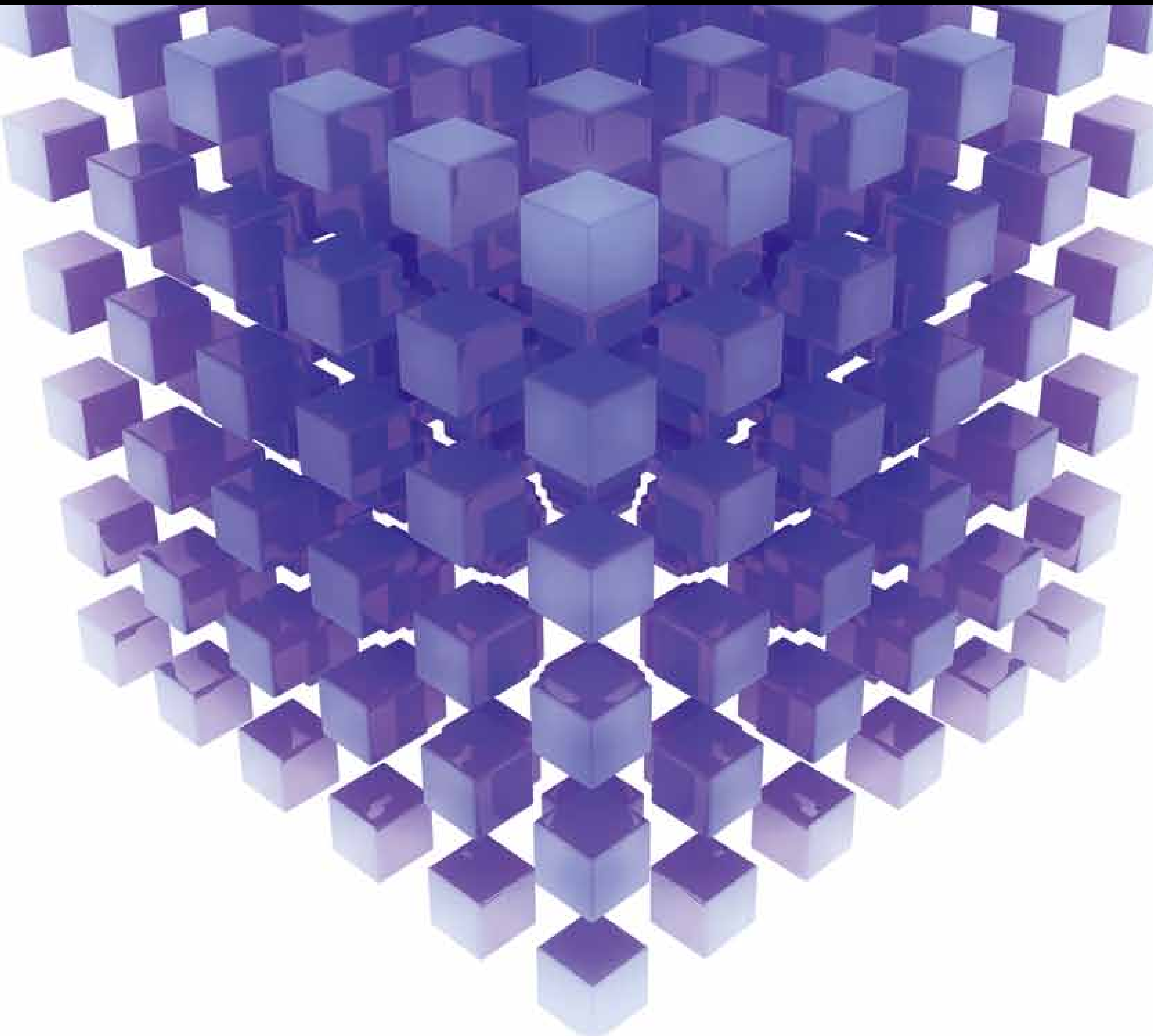
Mathematical Problems in Engineering

Stochastic Systems 2013

Guest Editors: Weihai Zhang, Xuejun Xie, Suiyang Khoo, Guangchen Wang, and Wuquan Li



Stochastic Systems 2013

Mathematical Problems in Engineering

Stochastic Systems 2013

Guest Editors: Weihai Zhang, Xuejun Xie, Suiyang Khoo, Guangchen Wang, and Wuquan Li

Editorial Board

Contents

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 924065, 2 pages
http://dx.doi.org/10.1155/2013/924065

Editorial

Stochastic Systems 2013

Weihai Zhang,[1] Xuejun Xie,[2] Suiyang Khoo,[3] Guangchen Wang,[4] and Wuquan Li[5]

[1] *College of Information and Electrical Engineering, Shandong University of Science and Technology, Qingdao 266590, China*
[2] *Institute of Automation, Qufu Normal University, Qufu 273165, China*
[3] *School of Engineering, Deakin University, Geelong, VIC 3216, Australia*
[4] *School of Control Science and Engineering, Shandong University, Jinan 250061, China*
[5] *School of Mathematics and Information, Ludong University, Yantai 264025, China*

Correspondence should be addressed to Weihai Zhang; w_hzhang@163.com

Received 6 August 2013; Accepted 6 August 2013

Randomness is inherent in reality, but it is often ignored due to the resulting difficulties. Stochastic control plays a central and significant role in modern control theory, which presents a valid tool for dealing with the randomness in the forms of Brownian motion and white noise. With the fast development of biology systems, mathematical finance, insurance, real estate, multiagent, and network control, a lot of new, challenging stochastic-control problems are springing up, which covers the fields of leader-follower game, stochastic filtering, stability, and so forth. These problems are desired to be deeply investigated by using more advanced theories and tools. To reflect the most recent advances in these fields, we are determined to organize this special issue.

This special issue focuses on stochastic-control systems governed by Ito-type stochastic differential equations and discrete-time stochastic difference equations together with their applications to control, filtering, communication, manufacturing, and fault detection. Topics covered in this issue include (i) stochastic modeling, stability, and stabilization analysis, (ii) stochastic robust/optimal/near-optimal/adaptive control, (iii) stochastic filtering and estimation, (iv) stochastic differential game, (v) small-gain approach to control of stochastic nonlinear systems, (vi) nonlinear risk measure including g-expectation, (vii) applications of stochastic-control theory to finance, economics, insurance, real estate, manufacturing systems, fault detection, and networked control systems. This special issue has received a total 51 of submitted papers with only 31 papers accepted.

There are 9 manuscripts on the subject "stochastic modeling, stability, and stabilization analysis." In the following, we give a brief summary. The paper entitled "*Risk-based predictive maintenance for safety-critical systems by using probabilistic inference*" by T. Xu et al. proposes a two-slice temporal Bayes net and risk-based maintenance model for safety critical systems. A new queuing model is proposed by V. Sağlam and M. Zobu in the paper "*A two-stage model queueing with no waiting line*," where the new Markovian model consists of two consecutive channels and no waiting line between channels. In "*Probabilistic value-centric optimization design for fractionated spacecrafts based on unscented transformation*," M. Xu et al. quantitatively assess the impacts of various fractionated spacecraft architecture strategies on the lifecycle cost, mass, propellant usage, and mission lifetime of pointing- intensive, remote sensing mission spacecraft. Two Markov chain models have been presented to simulate pitting corrosion in "*Markov chain models for the stochastic modeling of pitting corrosion*" by A. Valor et al. In "*Exponential stability of stochastic nonlinear dynamical price system with delay*," W. Zhu et al. give some sufficient conditions of exponential stability, and corollaries for such price systems are established by virtue of Lyapunov function. In "*Stochastic stability for time-delay Markovian jump systems with sector-bounded nonlinearities and more general transition probabilities*" by D. Ye et al., a more general transition probability matrix is considered, and the delay-dependent sufficient conditions for stability are

derived in terms of linear matrix inequalities. By constructing a novel piecewise Lyapunov-Krasovskii functional candidate, discrete-time stochastic exponential stability is studied in "*Exponential stability results of discrete-time stochastic neural networks with time-varying delays*" by Y. Li. A. Gu obtains the synchronization between two solutions and among different components of solutions under certain dissipative conditions in the manuscript "*Synchronization of coupled stochastic systems driven by α-stable Levy noises*." "*Robust passivity and feedback design for nonlinear stochastic systems with structural uncertainty*" by Z. Lin et al. discusses the robust passivity and global stabilization problems for a class of uncertain nonlinear stochastic systems with structural uncertainties.

Closely related to controlled stochastic differential equations are 3 contributions. J. Shi and Z. Yu's paper "*Relationship between maximum principle and dynamic programming for stochastic recursive optimal control problems and applications*" establishes a relationship between maximum principle and dynamic programming. A maximum principle and a verification theorem for optimal control are presented in "*Optimality conditions for optimal control of jump-diffusion SDEs with correlated observations noises*" by H. Xiao. The third contribution is for the relationship between functional forward-backward stochastic differential equations and path-dependent PDEs. Within the framework of functional Ito calculus, S. Ji and S. Yang introduce a path-dependent PDE and prove that its solution is uniquely determined by a functional forward-backward stochastic differential equation.

There are 5 papers concerned about stochastic filtering and estimation. J. Liu et al. propose an improved particle filter for multiple maneuvering target tracking in the paper "*Multiple maneuvering target tracking by improved particle filter based on multi-scan JPDA*." "*Maximum likelihood estimation of the VAR (1) model parameters with missing observations*" by H. Mouriño and M. I. Barão is about maximum likelihood estimation based on monotone missing data pattern. "*H_∞ estimates for discrete-time Markovian jump linear systems*" by M. H. Terra et al. deals with the H_∞ filtering problem for discrete-time Markovian jump linear systems based on the game theory. Based on the matrix inequality technique, the H_∞ filtering problem is studied by Z. Yan and Y. Huang in their paper "*Robust H_∞ filter design for Itô stochastic pantograph systems*." In the paper "*Optimal fusion filtering in multisensor stochastic systems with missing measurements and correlated noises*" by R. Caballero-Águila et al., the optimal least-square linear estimation problem is addressed for a class of discrete-time multisensor linear stochastic systems.

There are also 3 contributions on stochastic differential games. Using the theory of backward stochastic differential equations, L. Wei and Z. Wu study stochastic recursive zero-sum differential game and mixed zero-sum differential game problem. S. Ji et al. use a dynamic programming method to investigate stochastic differential games derived by functional forward-backward stochastic differential equations in "*The dynamic programming method of stochastic differential game for functional forward-backward stochastic system*." "*Governance mechanism for global greenhouse gas emissions: a stochastic differential game approach*" by W. Yu and B. Xin proposes cooperative and noncooperative stochastic differential game models to describe greenhouse gas emissions decision makings of developed and developing countries and calculates their feedback Nash equilibrium and the Pareto optimal solution and characterizes parameter spaces.

The subject on financial engineering and insurance has occupied 6 contributions. These contributions include, for example, convertible bond pricing, option pricing, optimal dividend strategy, and optimal portfolio selection. X. Guo and H. Wang study a pricing problem for convertible bond via reflected backward stochastic differential equations. X. Ruan et al. consider option pricing with risk-minimization criterion in an incomplete market with a finite difference method. Y. Fang and Z. Qu investigate a risk model including a constant force of interest and obtain an optimal strategy. W. Yu focuses on a discrete insurance risk model with stochastic premium income. Z. Huang and D. Zhang propose a model of corporate optimal investment with consideration of the influence of inflation and the difference between the market opening and market closure. J. A. Londoño considers a state-dependent utility problem in an incomplete market and develops a theory of markets when the processes are the generalization of Brownian flows on manifolds.

Other applications of stochastic-control theory can be found in the following: M. Zobu and V. Sağlam investigate the control of traffic intensity based on SPRT method. One paper entitled "*Fault detection for linear discrete time-varying systems with measurement packet dropping*" by Y. Li et al. introduces an adjoint operator-based optimization method; the analytical optimal solution is derived via solving a modified Riccati equation. In the paper "*A novel detection scheme for EBPSK system*," X. Chen and L. Wu introduce a novel solution for EBPSK communication systems, and a joint detection algorithm is given to eliminate the ISI coming from the SIF. Y. Fang et al. establish a model of the travel time of right-turning vehicles on secondary street at unsignalized intersections in congested urban traffic condition. X. Yin et al. discuss the discrete memoryless broadcast channels (DMBCs) with noiseless feedback; the entire capacity-equivocation regions of two models of the DMBCs with noiseless feedback are obtained.

Acknowledgments

This special issue represents an exciting and insightful snapshot of the current stochastic system research. As the Lead Guest Editor of this special issue, I would like to express my sincere gratitude to my four coeditors for helping me to undertake this project with a wonderful accomplishment. We are also deeply appreciative of interesting authors who have submitted their papers to the special issue.

Weihai Zhang
Xuejun Xie
Suiyang Khoo
Guangchen Wang
Wuquan Li

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 947104, 9 pages
http://dx.doi.org/10.1155/2013/947104

Research Article

Risk-Based Predictive Maintenance for Safety-Critical Systems by Using Probabilistic Inference

Tianhua Xu,[1] Tao Tang,[1] Haifeng Wang,[2] and Tangming Yuan[3]

[1] *State Key Laboratory of Rail Traffic Control and Safety, Beijing Jiaotong University, Beijing 100044, China*
[2] *National Engineering Research Centre of Rail Transportation Operation and Control Systems, Beijing Jiaotong University, Beijing 100044, China*
[3] *Computer Science Department, University of York, York YO10 5GH, UK*

Correspondence should be addressed to Tianhua Xu; thxubjtu@gmail.com

Received 27 January 2013; Revised 13 June 2013; Accepted 27 June 2013

Academic Editor: Suiyang Khoo

Risk-based maintenance (RBM) aims to improve maintenance planning and decision making by reducing the probability and consequences of failure of equipment. A new predictive maintenance strategy that integrates dynamic evolution model and risk assessment is proposed which can be used to calculate the optimal maintenance time with minimal cost and safety constraints. The dynamic evolution model provides qualified risks by using probabilistic inference with bucket elimination and gives the prospective degradation trend of a complex system. Based on the degradation trend, an optimal maintenance time can be determined by minimizing the expected maintenance cost per time unit. The effectiveness of the proposed method is validated and demonstrated by a collision accident of high-speed trains with obstacles in the presence of safety and cost constrains.

1. Introduction

Safety-critical systems, such as chemical factory, nuclear plant, and train control systems, are those where failures could result in loss of life, significant property damage, or damage to the environment. The loss caused by safety critical system failures is now becoming difficult to estimate. The efficient maintenance strategies are playing more important roles in preventing such system failures.

Over the last decade, reactive (fixing or replacing equipment after it fails) or blindly proactive strategies (also known as preventive strategies) have been used for system maintenance. The main disadvantage of both approaches is that they are extremely wasteful. As condition-based maintenance (CBM) systems have been implemented in a way to continuously output data that is calculated against the status and performance of the equipment, the decision making in CBM focuses on predictive maintenance (PdM) which promises to reduce downtime, spare inventory, maintenance cost, and safety hazards. Much work has been carried out in the area of predictive maintenance in order to improve safety. Generally speaking, current prognostic approaches can be classified into three categories,namely, model-based, data-driven, and hybrid prognostics [1]. For example, in [2], a DBN-HAZOP model was proposed to deduce the opportunistic predictive maintenance for complex multicomponent systems. The key idea behind the model is reliability-based maintenance. Krishnasamy et al. [3] proposed the risk-based maintenance (RBM) methodology, and a case study of a power-generating unit was used to illustrate the methodology. Arunraj and Maiti [4] identified the risk analysis and risk-based maintenance methodologies and classified them into suitable classes.

With the aforementioned research contributed to the efficient maintenance strategies of systems, to the best of our knowledge, however, the integration of dynamic system failure scenario into a risk-based maintenance model and the adoption of efficient inference methods for optimal maintenance strategy have received little attention. For the integration of dynamic system failure scenario with risk-based maintenance model, reference [2] proposed a component dynamic failure for reliability-based maintenance

model. In the model, the key focus is on reliability rather than risk. Regarding the inference approaches to manipulating the maintenance model, junction tree algorithms [5, 6] are quite common and widely used. However, junction algorithms are comparatively complex, which demands long digressions on graph theoretic concepts. Although there has been effort to explain junction tree algorithms without resorting to graphical concepts [7], the effort has not produced a variable elimination-like scheme for inference.

In order to tackle these problems, we propose a 2-TBN (two-slice temporal Bayes net) and risk-based maintenance model. By encoding the failure scenario into the conditional probability table (CPT) of risk based maintenance model, the risk of failure scenario is embedded. In order to facilitate efficient inference, an ad hoc bucket-elimination-based probabilistic inference is presented. Comparing with the complex junction-tree based inference, an attractive property of bucket elimination approaches is that it is relatively easy to understand and implement. Finally, by utilizing the optimal theory, the optimal maintenance time interval with minimal cost and risk constraints can be obtained.

The rest of the paper is organized as follows. In Section 2, the principle of RBM and the proposed RBM methodology are introduced. In Section 3, a maintenance model for degradation and risk prediction is presented. Section 4 gives optimal predictive maintenance strategies. A case study of a collision between a high speed train and an obstacle is discussed in Section 5. Section 6 draws the conclusion of the paper.

2. Risk-Based Maintenance (RBM) Methodology

Risk-based maintenance methodology provides a tool for maintenance planning and decision making to reduce the probability and consequences of failure of equipment. The resulting maintenance program minimizes the risk of the system and the maintenance cost. Figure 1 shows a general follow diagram of RBM. It consists of the following steps: (1) *identification of components, subsystems, system, and their relationships:* the system is divided into subsystems, and the components of each subsystem and their relationships are identified; in the following sections, we model the system structure by using a special case of dynamic Bayesian network, the 2-TBN; (2) *Collecting failure data, failure model and failure rate:* the information is encoded in the CPT in 2-TBNg based maintenance model.(3) *Risk assessment and evaluation*: by using probabilistic inference with bucket elimination, a consequence analysis is implemented to quantify the effect of the occurrence of each failure scenario and obtain quantitative measure for its associated risks. The risk is used to study maintenance costs including the costs incurred as a result of failure. (4) *Optimal maintenance strategy*: by defining different maintenance costs, the optimal maintenance scheme can be derived by applying the optimization theory to the risk quantitative measure computed in the aforementioned step.

3. Maintenance Model for Degradation and Risk Prediction

This section illustrates the first two steps of the RBM architecture discussed in Section 2 above. The main purpose is to encode the states, the dependency relations among components in each subsystem, subsystems, and the system. In order to facilitate the understanding of the optimization of predictive maintenance, we first introduce some basic notions including dynamic Bayesian network (DBN) and 2-TBN Model. We then prescribe the maintenance model based on 2-TBN and discuss the opportunistic predictive maintenance strategies.

3.1. Dynamic Bayesian Network and 2-TBN Model. A Bayesian network (BN) is a directed acyclic graph (DAG), which is a probability-based knowledge representation method and appropriate for the modeling of causal processes with uncertainty. The formal notion is defined as follows.

Definition 1 (see [8]). A Bayesian network (BN) is a triple (V, G, P), where V is a set of variables, G is a connected directed acyclic graph (DAG), and there is a one-to-one correspondence between nodes in G and variables in V. P is a set of probability distribution: $P = \{P(v \mid \pi(v)) \mid v \in V\}$, where $\pi(v)$ denotes the set of parents of v in G.

The statistic Bayesian network can be extended to a dynamic Bayesian network (DBN) by introducing relevant temporal dependences that capture the dynamic behaviors of the domain variables at different times of a static network. Definition 2 gives the formal definition of DBN.

Definition 2. A dynamic Bayesian network (DBN) is a quadruplet $G = (\bigcup_{t=0} V_t, \bigcup_{t=0} E_t, \bigcup_{t=0} E_t^{\rightarrow}, \bigcup_{t=0} P_t)$, and each V_t is a set of nodes labeled by variables, which represents the dynamic domain at time instant t $(0 \le t < k)$. Collectively, $\bigcup_{t=0}^{k} V_t$ represents the dynamic domain over k instants. Each E_t is a set of arcs among nodes in V_t, which represents dependencies among domain variables at time t. Each $E_t^{\rightarrow}$ is a set of temporal arcs each of which is directed from a node in V_{t-1} to a node in V_t $(0 < t < k)$. P_t is set of probability distributions, which can be referred to [8].

In this paper, we only consider a special class of DBNs, which is called 2-slice temporal Bayesian network (2-TBN) [9]. A 2-TBN is a DBN which satisfies the Markov property of order 1; that is, the future is independent of its past given its present.

3.2. 2-TBN Based Maintenance Model. 2-TBNs are general tools allowing the modeling of dynamic complex systems. Besides, it is important to note that using 2-TBNs to represent a variable depending on its own past is equivalent to the use of Markov chain to describe its local transition model. Consequently, we propose a 2-TBN based maintenance model capable of representing dynamic degradation and risk level of subsystems. We treat system state, system failure, and accidents as random variables and model dependencies

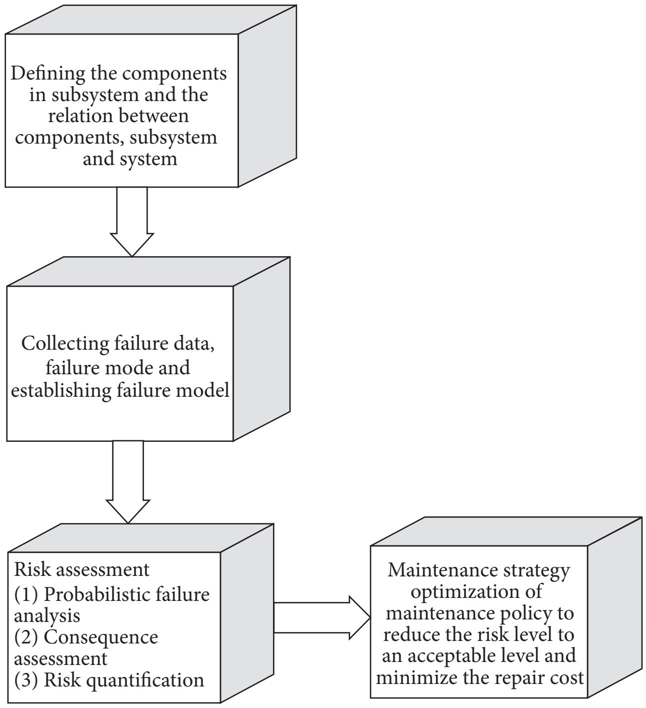
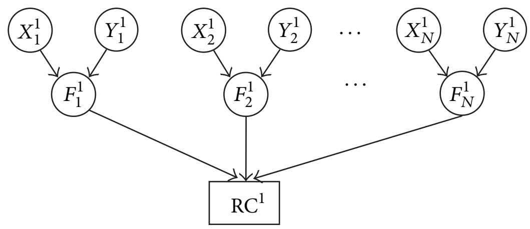
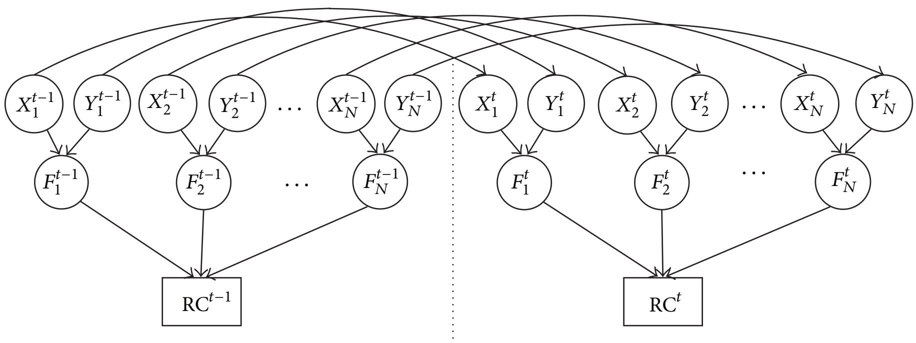


Figure 1: Architecture of RBM methodology (revised after [3]).



(a) Initial model



(b) Transition model

Figure 2: 2-TBN based maintenance model: (a) gives the initial state of the maintenance model while (b) depicts its transition model between time slice $t-1$ and t.

among them by exploiting the use of conditional probability tables (CPT). In order to simplify calculation, all of variables in our model are assumed to be discrete.

The 2-TBN based maintenance model is depicted in Figure 2. The model consists of the following variables: X_i^t $(1 \leq i \leq N,\ 1 \leq t)$ the state of component X (e.g., *failure* or *ok*) in the ith subsystem at time instant t (for the sake of simplicity, only two components X and Y are shown in the figure); F_i^t $(1 \leq i \leq N, 1 \leq t)$ denotes the states of ith subsystem (e.g., *failure* or *ok*) at time instant t; A^t $(1 \leq t)$ denotes the accident probability of the system due to the subsystem failure. RC^t $(1 \leq t)$ represents the corresponding

Table 1: Conditional probabilities for component state X_1, Y_1, and subsystem F_1.

X_1^t	Y_1^t	$\Pr(F_1^t \mid X_1^t, Y_1^t)$
Ok	Ok	Ok
Ok	Fail	Fail
Fail	Ok	Fail
Fail	Fail	Fail

Table 2: Temporal CPT for component X_1.

X_i^{t-1}	X_i^t	$\Pr(X_i^t \mid X_i^{t-1})$
Ok	Ok	$1-\lambda_p \Delta T$
Ok	Fail	$\lambda_p \Delta T$
Fail	Ok	0
Fail	Fail	1

maintenance cost till time instant t. From Figure 2(b), it can be seen that the current component state, for example, X_i^t depends on the previous component states, for example, X_i^{t-1}.

To complete the 2-TBN based maintenance model, the conditional probabilities must be specified for (1) the state transition of components between different time slice and (2) the dependency of components output on the subsystem, system, and accident state. For example, assume the state of component X_i^t $(1 \le i \le N,\ 1 \le t)$ has only two values: *ok* or *fail*, then its dependencies among F_i^t $(1 \le i \le N,\ 1 \le t)$ and X_i^t, Y_i^t $(1 \le i \le N,\ 1 \le t)$ can be illustrated by the CPT as shown in Table 1. In other words, any failure in component X_1 and/or Y_1 will lead to the failure of subsystem F_1.

Similarly, under the assumption that the failure rate of a component follows an exponential distribution where all these transition rates are constant, the transition relations between consecutive nodes for the different components maintenance model are obtained as follows (the failure rate is denoted by λ_i, the time interval between two successive trials is denoted by ΔT, and the components are assumed to be new on the initial trial $t = 0$):

$$\begin{pmatrix} \Pr\left(X_i^t = 0\right) \\ \Pr\left(X_i^t = 1\right) \end{pmatrix} = \begin{pmatrix} e^{-\lambda_k \Delta T} & 0 \\ 1 - e^{-\lambda_k \Delta T} & 1 \end{pmatrix} \begin{pmatrix} \Pr\left(X_i^{t-1} = 0\right) \\ \Pr\left(X_i^{t-1} = 1\right) \end{pmatrix}. \quad (1)$$

So the conditional probabilities for state transitions can be obtained directly from the above equation. For example, $\Pr(X_i^{t+1} = a \mid X_i^t = 0)$ can be obtained, where a denotes normal or failure state:

$$\begin{pmatrix} \Pr\left(X_i^{t+1} = 0 \mid X_i^t = 0\right) \\ \Pr\left(X_i^{t+1} = 1 \mid X_i^t = 0\right) \end{pmatrix} = \begin{pmatrix} e^{-\lambda_k \Delta T} \\ 1 - e^{-\lambda_k \Delta T} \end{pmatrix}. \quad (2)$$

The corresponding temporal CPT for component X_i^t $(1 \le i \le N,\ 1 \le t)$ is obtained as shown in Table 2.

Finally, the consequence resulting from different subsystem failures (i.e., F_i^t $(1 \le i \le N,\ 1 \le t)$) can be classified as shown in Table 3. The specific consequence can be determined by different failure remain so manually.

Table 3: Consequence resulted from different subsystem failure scenario.

F_1^t	F_2^t	$\cdots$	F_N^t	Consequence
Ok	Ok	$\cdots$	Ok	No
Fail	Ok	$\cdots$	Ok	Insignificant
Fail	Fail	$\cdots$	Ok	Marginal (minor injury)
Fail	Fail	$\cdots$	Ok	Critical (single severe injury)
Fail	Fail	$\cdots$	Fail	Catastrophic (fatalities)

The risk can be computed by integration of consequence and probability resulting from different failure scenarios. Please note that the probability of subsystem failure scenario can be calculated by using probability inference from the 2-TBN based maintenance model. The detailed procedures are discussed in the following sections.

4. Optimal Predictive Maintenance Strategies

This section discusses the calculation of optimal predictive maintenance strategies which consists of the calculation of the failure and accident probability of the maintenance model and optimal maintenance time under the repairing cost constraints.

4.1. Calculation of the Failure Probability of a Component in Maintenance Model. The purpose of this subsection is to evaluate the probability of any failure scenario for a time length of T. In other words, the underlying problem boils down to the calculation of the following probability:

$$\Psi^t = \Pr\left(F^1, \ldots, F^T\right). \quad (3)$$

The following theorem gives a recursive characterization of Ψ^t based on the derivation of the bucket elimination method presented in [10].

Theorem 3 (recursive characterization of Ψ^t). *Let* $(X_1^t, Y_1^t, F_1^t, X_2^t, Y_2^t, F_2^t, \ldots, X_N^t, Y_N^t, F_N^t, A^t)_{t\ge1}$ *be the sequence of random variables associated to maintenance model. Then for any* $t \ge 1$, Ψ^t *can be recursively expressed as follows*

$$\Psi^t = \begin{cases} \displaystyle\sum_{(X_i^1, Y_i^1)_{1\le i\le N}, A^1} \prod_{i=1}^{N} \left[\Pr\left(X_i^1\right) \Pr\left(Y_i^1\right)\right] \\ \displaystyle\cdot \prod_{j=1}^{N} \Pr\left(F_j^1 \mid X_j^1, Y_j^1\right) \cdot \Pr\left(A^1 \mid F_1^1, \ldots, F_N^1\right) & t = 1, \\ \displaystyle\sum_{(X_i^t, Y_i^t, A^t)_{1\le i\le N}} \prod_{i=1}^{N} \left[\Pr\left(X_i^t \mid X_i^{t-1}\right) \Pr\left(Y_i^t \mid Y_i^{t-1}\right)\right] \\ \displaystyle\cdot \prod_{j=1}^{N} \Pr\left(F_j^1 \mid X_j^t, Y_j^t\right) \cdot \Pr\left(A^t \middle| F_1^t, \ldots, F_N^t\right) \\ \cdot \Psi^{t-1} & t \ge 2. \end{cases} \quad (4)$$

Proof. Let's proceed by induction. For $t = 1$, the result is straight forward since

$$\Psi^1 = \sum_{(X_i^1, Y_i^1)_{1\le i\le N}, A^1} \Pr\left(X_1^1, Y_1^1, Z_1^1, X_2^1, Y_2^1, Z_2^1, \dots, X_N^1, Y_N^1, Z_N^1, A^1\right)$$
$$= \sum_{(X_i^1, Y_i^1)_{1\le i\le N}, A^1} \prod_{i=1}^{N} \left[\Pr\left(X_i^1\right) \Pr\left(Y_i^1\right)\right] \cdot \prod_{j=1}^{N} \Pr\left(F_j^1 \mid X_j^1, Y_j^1\right) \cdot \Pr\left(A^1 \mid F_1^1, \dots, F_N^1\right). \tag{5}$$

Assume $\Psi^{t-1} = \Pr(F^1, F^2, \dots, F^{t-1})$ for $t \ge 2$. Using simple probability manipulation rules, (5) can be rewritten as

$$\Psi^t = \Pr\left(F^1, F^2, \dots, F^t\right)$$
$$= \sum_{(X_i^\tau, Y_i^\tau, A^\tau)_{1\le i\le N, 1\le\tau\le t}} \Pr\left(X_1^\tau, Y_1^\tau, F_1^\tau, X_2^\tau, Y_2^\tau, F_2^\tau, \dots, X_N^\tau, Y_N^\tau, F_N^\tau, A^\tau\right)_{1\le\tau\le t}$$
$$= \sum_{(X_i^\tau, Y_i^\tau, A^\tau)_{1\le i\le N, 1\le\tau\le t}} \Pr\left(X_1^t, Y_1^t, F_1^t, X_2^t, Y_2^t, F_2^t, \dots, X_N^t, Y_N^t, F_N^t, A^t \mid \left(X_1^\tau, Y_1^\tau, F_1^\tau, X_2^\tau, Y_2^\tau, F_2^\tau, \dots, X_N^\tau, Y_N^\tau, F_N^\tau, , A^\tau\right)_{1\le\tau\le t-1}\right)$$
$$\cdot \Pr\left(\left(X_1^\tau, Y_1^\tau, F_1^\tau, X_2^\tau, Y_2^\tau, F_2^\tau, \dots, X_N^\tau, Y_N^t, F_N^\tau, A^\tau\right)_{1\le\tau\le t-1}\right). \tag{6}$$

According to the 2-TBN factorization property, the conditional probability distribution of $\Pr(X_1^t, Y_1^t, F_1^t, X_2^t, Y_2^t, F_2^t, \dots, X_N^t, Y_N^t, F_N^t, A^t \mid (X_1^\tau, Y_1^\tau, F_1^\tau, X_2^\tau, Y_2^\tau, F_2^\tau, \dots, X_N^\tau, Y_N^\tau, F_N^\tau, A^\tau)_{1\le\tau\le t-1})$ is given by

$$\Pr\left(X_1^t, Y_1^t, F_1^t, X_2^t, Y_2^t, F_2^t, \dots, X_N^t, Y_N^t, F_N^t, A^t \mid \left(X_1^\tau, Y_1^\tau, F_1^\tau, X_2^\tau, Y_2^\tau, F_2^\tau, \dots, X_N^\tau, Y_N^\tau, F_N^\tau, A^\tau\right)_{1\le\tau\le t-1}\right)$$
$$= \Pr\left(X_1^t, Y_1^t, F_1^t, X_2^t, Y_2^t, F_2^t, \dots, X_N^t, Y_N^t, F_N^t, A^t \mid X_1^{t-1}, Y_1^{t-1}, X_2^{t-1}, Y_2^{t-1}, \dots, X_N^{t-1}, Y_N^{t-1}\right)$$
$$= \prod_{i=1}^{N} \left[\Pr\left(X_i^t \mid X_i^{t-1}\right) \Pr\left(Y_i^t \mid Y_i^{t-1}\right)\right] \cdot \prod_{j=1}^{N} \Pr\left(F_j^t \mid X_j^t, Y_j^t\right) \cdot \Pr\left(A^t \mid F_1^t, \dots, F_N^t\right). \tag{7}$$

In addition, according to the 2-TBN interface property and (7), we can rewrite (6) into

$$\Psi^t = \Pr\left(F^1, F^2, \dots, F^t\right)$$
$$= \sum_{(X_i^\tau, Y_i^\tau, A^\tau)_{1\le i\le N, 1\le\tau\le t}} \Pr\left(X_1^\tau, Y_1^\tau, F_1^\tau, X_2^\tau, Y_2^\tau, F_2^\tau, \dots, X_N^\tau, Y_N^\tau, F_N^\tau, A^\tau\right)_{1\le\tau\le t}$$
$$= \sum_{(X_i^\tau, Y_i^\tau, A^\tau)_{1\le i\le N, 1\le\tau\le t}} \Pr\left(X_1^t, Y_1^t, F_1^t, X_2^t, Y_2^t, F_2^t, \dots, X_N^t, Y_N^t, F_N^t, A^t \mid \left(X_1^\tau, Y_1^\tau, F_1^\tau, X_2^\tau, Y_2^\tau, F_2^\tau, \dots, X_N^\tau, Y_N^\tau, F_N^\tau, A^\tau\right)_{1\le\tau\le t-1}\right)$$
$$\cdot \Pr\left(\left(X_1^\tau, Y_1^\tau, F_1^\tau, X_2^\tau, Y_2^\tau, F_2^\tau X_N^\tau, Y_N^\tau, F_N^\tau, A^\tau\right)_{1\le\tau\le t-1}\right)$$
$$= \sum_{(X_i^t, Y_i^t, A^t)_{1\le i\le N}} \prod_{i=1}^{N} \left[\Pr\left(X_i^t \mid X_i^{t-1}\right) \Pr\left(Y_i^t \mid Y_i^{t-1}\right)\right] \cdot \prod_{j=1}^{N} \Pr\left(F_j^t \mid X_j^t, Y_j^t\right) \cdot \Pr\left(A^t \middle| F_1^t, \dots, F_N^t\right)$$
$$\cdot \sum_{(X_i^\tau, Y_i^\tau, A^\tau)_{1\le i\le N, 1\le\tau\le t-1}} \Pr\left(\left(X_1^\tau, Y_1^\tau, F_1^\tau, X_2^\tau, Y_2^\tau, F_2^\tau, \dots, X_N^\tau, Y_N^\tau, F_N^\tau, A^\tau\right)_{1\le\tau\le t-1}\right). \tag{8}$$

Let remark the last term in (8) that

$$\sum_{(X_i^\tau, Y_i^\tau, A^\tau)_{1\le i\le N, 1\le \tau\le t-1}} \Pr\left(\left(X_1^\tau, Y_1^\tau, F_1^\tau, X_2^\tau, Y_2^\tau, F_2^\tau, \dots, X_N^\tau, Y_N^\tau, F_N^\tau, A^\tau\right)_{1\le\tau\le t-1}\right). \quad (9)$$
$$= \Pr\left(F^1, F^2, \dots, F^{t-1}\right).$$

Then, the computation of Ψ^t can be simplified as follows:

$$\Psi^t = \sum_{(X_i^t, Y_i^t, A^t)_{1\le i\le N}} \prod_{i=1}^{N} \left[\Pr\left(X_i^t \mid X_i^{t-1}\right)\Pr\left(Y_i^t \mid Y_i^{t-1}\right)\right] \cdot \prod_{j=1}^{N} \Pr\left(F_j^t \mid X_j^t, Y_j^t\right) \cdot \Pr\left(A^t \mid F_1^t, \dots, F_N^t\right) \cdot \Psi^{t-1}. \quad (10)$$

So the theorem can be proved now. □

Remark 4. The bucket-elimination based inference approach presented in the risk-based maintenance model aims at efficiently computing the failure probability densities of components in a maintenance model which is represented in a dynamic Bayesian network. The construction of bucket tree simplifies the presentation and produces an algorithm that is easy to grasp and implement. The algorithm relies only on independency relations and probability manipulation and does not use graphical concepts such as triangulations and cliques, and it focuses solely on the probability densities and avoids complex digressions on graph theoretic concepts.

4.2. Calculation of the Optimization Maintenance Time. This subsection concerns the optimization of predictive maintenance under the criterion of minimizing its life time operation and repair costs. Similar to [2], two types of costs need to be considered: (1) the cost of repairing component degradation of failure which is termed as "repairing cost" and (2) production losses caused by the shutdown of the system to undertake repairs which is related to the time lost in these tasks. There are two kinds of repairing costs: *corrective repairing cost* needs to be charged when component failure occurs before proactive schedule time, and *proactive repairing cost* is charged when component is under repair or replacement at certain proactive scheduled time without failure. For the ith component, the specific corrective and proactive repair costs are denoted as RC_i^c and RC_i^p, respectively. We consider the latter less than the former because the former contains production loss, personal injures, and environment contamination. For the ith component, the expected total cost per unit time of predictive maintenance is given by

$$\mathrm{RC}_{r\_i}(t) = \frac{\mathrm{RC}_i^c F_i(t) + \mathrm{RC}_i^p\left(1 - F_i(t)\right)}{t}, \quad (11)$$

where t is the time for a proactive repair of component i and $F_i(t)$ is its failure probability distribution. It represents the cumulative distribution function of random variable F_i "time to failure," which is the output of the 2-TBN based maintenance model. If the system contains N components, that is, $G = \{1, 2, \dots, N\}$, the expected group repair cost rates are given as follows:

$$\mathrm{RC}_r(t) = \frac{\sum_{i=1}^{N}\left[\mathrm{RC}_i^c F_i(t) + \mathrm{RC}_i^p\left(1 - F_i(t)\right)\right]}{t}. \quad (12)$$

Unlike [2], the associated production loss depends on different failure scenarios with difference severity. So the production loss rate is given as

$$\mathrm{PL}_r(t) = \left(\sum_{i=1}^{AccTypeNum}\left[L_i \prod_{j=1}^{N} \Pr\left(F_j(t) \mid A^i(t)\right) \cdot \prod_{l\in M\backslash j}\left(1 - \Pr\left(F_l(t) \mid A^i(t)\right)\right)\right]\right) \times (t)^{-1}, \quad (13)$$

where $AccTypeNum$ denotes all the kinds of failure types, N the component number, L_i the loss due to the accident with type i,and j and l are the failure and normal component indices in accident A^i.

The expected total cost per unit time of predictive maintenance for the system is given by

$$C_r(t) = \mathrm{RC}_{r\_i}(t) + \mathrm{PL}_r(t) = \frac{\sum_{i=1}^{k}\left[\mathrm{RC}_i^C F_i(t) + \mathrm{RC}_i^p\left(1 - F_i(t)\right)\right]}{t} + \left(\sum_{i=1}^{AccTypeNum}\left[L_i \prod_{j=1}^{M} \Pr\left(Fail_j \mid Acc_i\right) \cdot \prod_{l\in M''}^{M} \Pr\left(Succ_l \mid Acc_i\right)\right]\right) \times (t)^{-1}. \quad (14)$$

The optimal predictive maintenance time is boiled down the optimal problem and can be solved by many numerical optimal tools such as Matlab:

$$T_{\mathrm{opt}} = \arg\min_t C_r(t), \quad F_i(t) \in \mathrm{2TBNMM}, \quad (15)$$

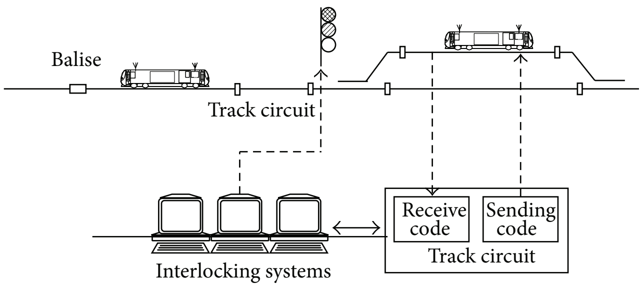


Figure 3: Configuration of collision of a high speed train with an obstacle.

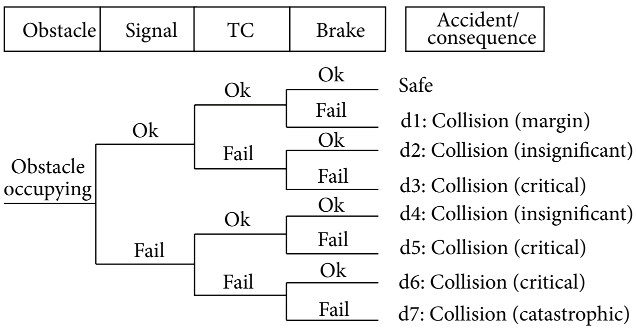


Figure 4: Event tree analysis of collision accident of high-speed trains.

where 2TBNMM denotes the 2-TBN based maintenance model.

5. Case Study

In this section, an accident for a high speed train with an obstacle located on the rail segment is considered to demonstrate the feasibility and effectiveness of the proposed approach. Figure 3 shows the configuration of the accident which consists of signal, track circuit, computer interlocking system, and train control system. Signals are placed between track segments and show different aspects. These aspects inform the train driver to go or stop safely; track circuit is monitored by electrical equipment to detect the presence of a train. It can also be used to send allowable train velocity code to assure the train moving safely. Computer interlocking system (CI) is used to give the right route for a train to enter the station. If a route is successfully established, CI will inform the signal to display green aspect. Otherwise, the red aspect will be displayed. Train control system receives the allowable train velocity code from the track circuit and the signal aspect and then determines whether the train accelerates or decelerates by applying a braking system.

The event tree analysis for train collision is shown in Figure 4. Three barriers,namely, *Signal*, *decelerate code* by track circuit, and *Brake systems*, have been established to decrease the risk caused by the train collision. Each of the barriers has two possible states: *ok* or *fail*. As a result of the analysis, eight collision accidents/consequences are distinguished.

Table 4: Failure rate of components.

Parameter	Meaning	Value
λ_{InfSend}	Failure rate of code sending module	$1e-7$/h
λ_{InfRev}	Failure rate of code receiving module	$1e-7$/h
λ_{Monitor}	Failure rate of monitor	$1e-5$/h
λ_{CI}	Failure rate of CI	$1e-6$/h
λ_{ATP}	Failure rate of ATP	$1e-06$/h
λ_{Brake}	Failure rate of brake	$7e-06$/h

Table 5: Corrective and proactive cost.

Parameter	TC	Signal	BrakSys
RC^C	1500	500	3000
RC^P	800	50	1000

Table 6: Product loss of different accident levels.

Parameter	L_1	L_2	L_3	L_4
LC	5000	10000	30000	100000

For example, when an external obstacle occupies the track, and the monitor system can successfully detect the presence of the obstacle and send the information to CI via track circuit (TC), the CI will then inform the signal to display red aspect (i.e., signal is *ok*). At the same time the TC sends deceleration code to train (i.e., TC is *ok*) and the braking system is normal (i.e., brake is *ok*); then the collision will be prevented and the consequence is "safe". On the other hand, when an external obstacle occupies the track and the monitor system, TC, CI, signal, and brake system all fail then the collision will be inevitable and the resulting consequence is catastrophic (d7). Given the failure rate of the different components, the "equivalent risk" for each accident is estimated by the numerical results derived from the probability inference discussed in Section 4 above.

Figure 5 illustrated the maintenance model of obstacle collision with high-speed train. The model consists of three subsystems: *track circuit* (TC), *Signal,* and brake system (BrakSys). The reliability of subsystem depends on its constituted components. For example, the reliability of TC subsystem depends on code sending module (InfSend) and code receiving module (InfRev), signal subsystem on monitor system (Monitor) and CI, and brake subsystem on automatic train protection (ATP), and brake equipment (Brake). The failure rate of components, corrective and proactive cost, and the product loss of different accident levels are given in Tables 4, 5, and 6, respectively.

The reliability probability distribution of component *Signal*, *track circuit* (TC), and *Brake system* (BrakSys) is shown in Figure 6. The total mission time is assumed to be 31 time units (i.e., month). Thirty-one months are sufficient for this purpose because, for predictive maintenance, it is inaccurate and meaningless to predict future deterioration for complex industrial system due to operational regulation, environmental changes, and human activity. The result of the mean values of expected repair cost rate of *Signal*, TC, and *Brake* component is shown in Figure 7. Figure 8 illustrates the

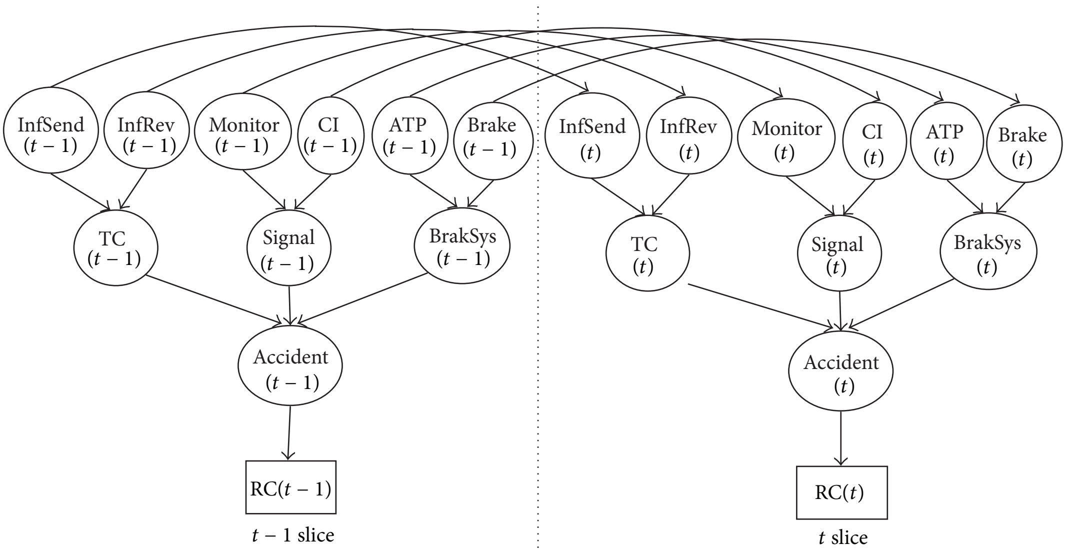


Figure 5: Maintenance model for high- speed train.

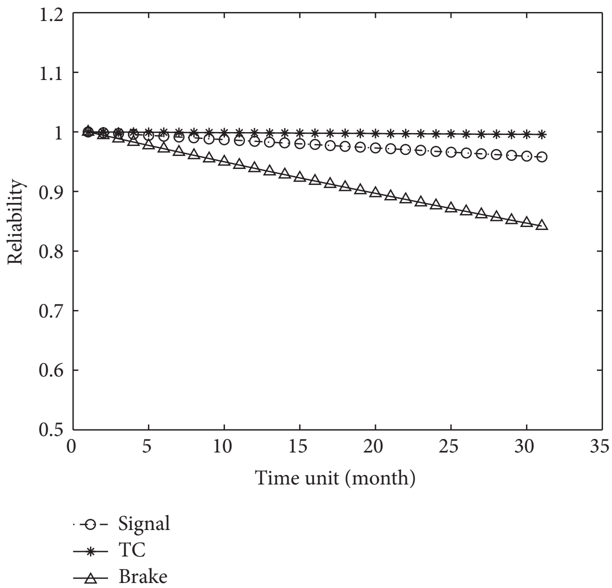


Figure 6: Reliability probability distribution of *Signal*, TC, and *Brake* component.

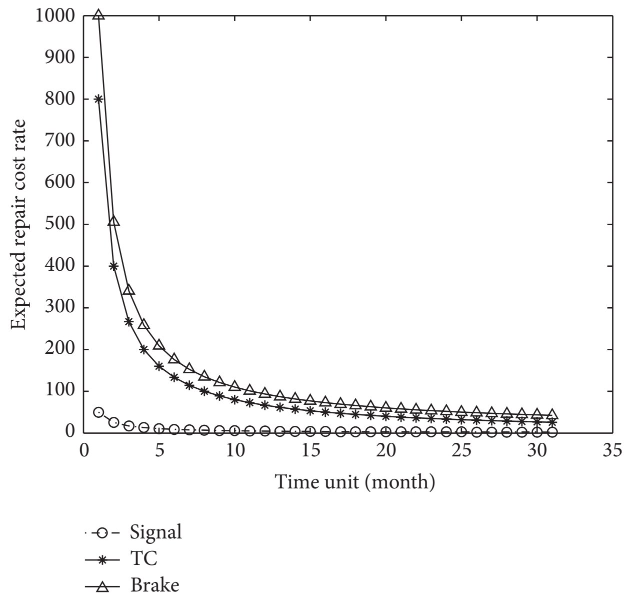


Figure 7: Mean values of expected repair cost rate of *Signal*, TC, and *Brake* component.

mean values of total repair cost rate, total production loss rate, and the total cost rate. The latter is the sum of total repair cost rate and total production loss rate. From Figure 8, it can be seen that the optimal maintenance time is 10 time units. From (11), the corresponding reliability of for *Signal*, TC, and *Brake* component is 0.98711, 0.99856, and 0.94971, respectively.

6. Conclusions

The paper presents a methodology for the optimization of maintenance strategies. This approach ensures that not only the safety of equipment is increased but also that the cost of maintenance including the cost of failure is reduced. The work reported will contribute to the "availability" of the safety critical systems. In order to calculate the failure probability and consequence of each failure scenario, a maintenance model based on 2-TBN has been created. An ad hoc inference procedure along with its proof of correctness is provided to efficiently compute the probability of component failure rates. The consequence of different failure scenarios is coded in conditional probability table (CPT) as part of the associated maintenance model. In the approach proposed in the paper, only the system's optimal maintenance time was considered.

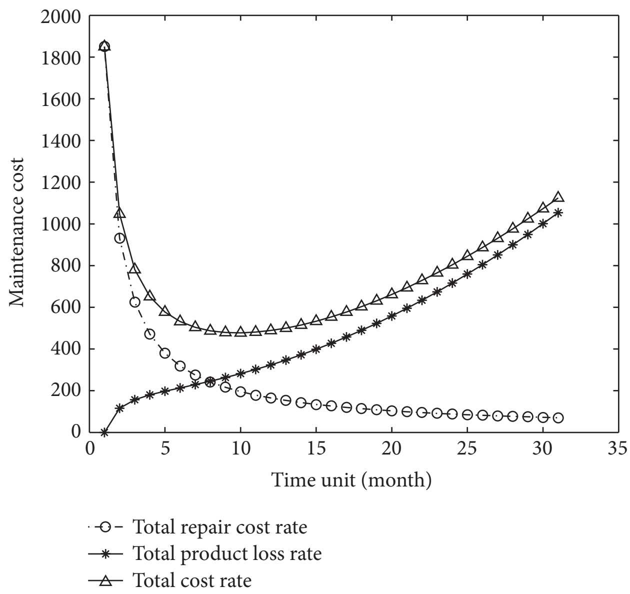


Figure 8: Mean values of total repair cost rate, total production loss rate, and the total cost rate.

However, the study can be extended so that each component's optimal maintenance time can be calculated in the same way.

Acknowledgments

The authors would like to thank the support of the International Science & Technology Cooperation Program of China under Grant no. 2012DFG81600, the Railway Ministry Science and Technology Research and Development Program (no. 2013X015-B), and the State Key laboratory of Rail Traffic Control and Safety of Beijing Jiaotong University within the frame of the project (no. RCS2012ZT005).

References

[1] J. Lee, J. Ni, D. Djurdjanovic, H. Qiu, and H. Liao, "Intelligent prognostics tools and e-maintenance," *Computers in Industry*, vol. 57, no. 6, pp. 476–489, 2006.

[2] J. Hu, L. Zhang, and W. Liang, "Opportunistic predictive maintenance for complex multi-component systems based on DBN-HAZOP model," *Process Safety and Environmental Protection*, vol. 90, pp. 376–388, 2012.

[3] L. Krishnasamy, F. Khan, and M. Haddara, "Development of a risk-based maintenance (RBM) strategy for a power-generating plant," *Journal of Loss Prevention in the Process Industries*, vol. 18, no. 2, pp. 69–81, 2005.

[4] N. S. Arunraj and J. Maiti, "Risk-based maintenance-techniques and applications," *Journal of Hazardous Materials*, vol. 142, no. 3, pp. 653–661, 2007.

[5] F. V. Jensen, S. L. Lauritzen, and K. G. Olesen, "Bayesian updating in causal probablisitic networks by local computations," *Computational Statistics Quarterly*, vol. 4, pp. 269–292, 1990.

[6] A. P. Dawid, "Applications of a general propagation algorithm for probabilistic expert systems," *Statistics and Computing*, vol. 2, no. 1, pp. 25–36, 1992.

[7] D. Draper, "Clustering without (thinking about) triangulation," in *Proceedings of the 11th Conference on Uncertainty in Artificial Intelligence*, 1995.

[8] X. An, Y. Xiang, and N. Cercone, "Dynamic multiagent probabilistic inference," *International Journal of Approximate Reasoning*, vol. 48, no. 1, pp. 185–213, 2008.

[9] R. Donat, P. Leray, L. Bouillaut, and P. Aknin, "A dynamic Bayesian network to represent discrete duration models," *Neurocomputing*, vol. 73, no. 4-6, pp. 570–577, 2010.

[10] R. Dechter, "Bucket elimination: a unifying framework for reasoning," *Artificial Intelligence*, vol. 113, no. 1, pp. 41–85, 1999.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 102069, 13 pages
http://dx.doi.org/10.1155/2013/102069

Research Article

Capacity-Equivocation Regions of the DMBCs with Noiseless Feedback

Xinxing Yin,[1] Zhi Xue,[1] and Bin Dai[2]

[1] *Department of Electrical Engineering, Shanghai Jiao Tong University, Shanghai 200240, China*
[2] *Department of Computer Science and Engineering, Shanghai Jiao Tong University, Shanghai 200240, China*

Correspondence should be addressed to Xinxing Yin; yinxinxing@sjtu.edu.cn

Received 30 January 2013; Accepted 15 July 2013

Academic Editor: Weihai Zhang

The discrete memoryless broadcast channels (DMBCs) with noiseless feedback are studied. The entire capacity-equivocation regions of two models of the DMBCs with noiseless feedback are obtained. One is the degraded DMBCs with rate-limited feedback; the other is the *less* and *reversely less noisy* DMBCs with causal feedback. In both models, two kinds of messages are transmitted. The common message is to be decoded by both the legitimate receiver and the eavesdropper, while the confidential message is only for the legitimate receiver. Our results generalize the secrecy capacity of the degraded wiretap channel with rate-limited feedback (Ardestanizadeh et al., 2009) and the restricted wiretap channel with noiseless feedback (Dai et al., 2012). Furthermore, we use a simpler and more intuitive deduction to get the single-letter characterization of the capacity-equivocation region, instead of relying on the recursive argument which is complex and not intuitive.

1. Introduction

Secure data transmission is an important requirement in wireless communication. Wyner first studied the degraded (the wiretap channel is said to be (physically) degraded if $X \rightarrow Y \rightarrow Z$ form a Markov chain, where X is the channel input and Y and Z are the channel outputs of the legitimate receiver and wiretapper, resp.) wiretap channel in [1], where the output Z^N of the channel to the wiretapper is degraded to the output Y^N of the channel to the legitimate receiver. In Wyner's model, the transmitter aimed to send a confidential message S to the legitimate receiver and keep the wiretapper as ignorant of the message as possible. Wyner obtained the secrecy capacity (the secrecy capacity is the best data transmission rate under perfect secrecy; i.e., the equivocation at the wiretapper $H(S \mid Z^N) = 0$. The formal definition of the secrecy capacity is given in Remark 3) and demonstrated that provable secure communication could be implemented by using information theoretic methods. This model was extended to a more general case by Csiszár and Körner [2], where broadcast channel with confidential messages was studied; see Figure 1. They considered transmitting not only the confidential messages S to the legitimate receiver, but also the common messages W to both the legitimate receiver and the eavesdropper. The capacity-equivocation region for the extended model was determined in [2]. This region contains all the achievable rate triples (R_0, R_1, R_e), where R_0 and R_1 are the rates of the common and confidential messages and R_e is the rate of the confidential message's equivocation. Nevertheless, neither Wyner's model nor Csiszár's model considered feedback.

To explore more ways in achieving secure data transmission, [3–5] studied the effects of the feedback on the capacities of several channel models. They all showed that feedback could help enhance the secrecy in wireless transmission. In [3], Ahlswede and Cai presented both the inner and outer bounds on the secrecy capacity of the wiretap channel with secure causal feedback from the decoder and showed that the outer bound was tight for the degraded case. It was proved that, by using feedback, the secrecy capacity of the (degraded) wiretap channel was increased. After Ahlswede's exploration, Ardestanizadeh et al. studied the wiretap channel with secure rate-limited feedback [4]. The main difference between Ardestanizadeh's model and Ahlswede's model is that the feedback in [4] is independent of the channel outputs, while the feedback in [3] is originated causally from the

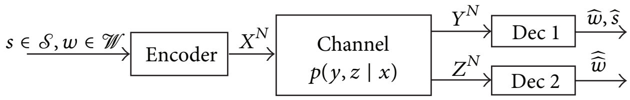


Figure 1: Broadcast channel with confidential messages.

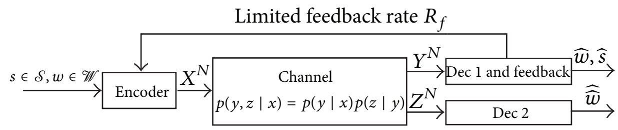


Figure 2: Degraded DMBCs with rate-limited feedback.

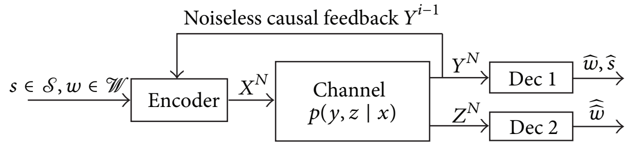


Figure 3: Less and reversely less noisy DMBCs with noiseless causal feedback.

outputs of the channel to the legitimate receiver. In [4], the authors got an outer bound on the wiretap channel with rate-limited feedback through a recursive argument which was effective but not intuitive. They also showed the outer bound was tight for the degraded case. In addition, Dai et al. investigated the secrecy capacity of the restricted wiretap channel with noiseless causal feedback under the assumption that the main channel is independent of the wiretap channel [5].

However, all of these explorations [3–5] focused on sending only the confidential messages. They did not consider sending both the common and confidential messages. In fact, transmitting the two kinds of messages can be seen in many systems with feedback. For example, in the satellite television service, some channels are available to all users for free, but some other channels are only for those who have paid for them. Recently, [6] studied the problem of transmitting both the common and confidential messages in the degraded broadcast channels with feedback. Note that, like [3], the feedback in [6] was originated causally from the legitimate receiver's channel outputs and not rate-limited. Besides, [7–9] studied the broadcast channel with feedback where no secure constraints were imposed.

To further investigate the secure data transmission with both common and confidential messages and noiseless feedback, this paper determines the capacity-equivocation regions of the following two DMBCs with both common and confidential messages. They are unsolved in the previous exploration.

(i) Degraded DMBCs with rate-limited feedback, where the feedback rate is limited by R_f and the feedback is independent of the channel outputs; see Figure 2.

(ii) *Less* and *reversely less noisy* (let X be the input of the DMBC, Y the legitimate receiver's channel output, and Z the eavesdropper's channel output. A DMBC $p(y, z \mid x)$ is said to be *less noisy* if $I(U; Y) \geq I(U; Z)$ for all $p(u, x)$; a DMBC $p(y, z \mid x)$ is said to be *reversely less noisy* if $I(U; Y) \leq I(U; Z)$ for all $p(u, x)$, where u is the value of the auxiliary random variable U) DMBCs with noiseless causal feedback, where the feedback is originated causally from the legitimate receiver's channel outputs; see Figure 3.

The two channel models are characterized in Section 2. The main results presented in Section 2 subsume some important previous findings about the secure data transmission with feedback. (1) By setting the auxiliary random variable U to be constant in the secrecy capacity of the first model (see (9) in Remark 3), the secrecy capacity of the degraded wiretap with rate-limited feedback [4] is obtained. (2) By eliminating the common message in the second model, the capacity-equivocation region of restricted wiretap channel with noiseless feedback [5] is obtained. (3) We utilize a simpler and more intuitive deduction to get the single-letter characterization of the capacity-equivocation region, instead of relying on the recursive argument (see [4]) which is complex and not intuitive. (4) We find that even if the eavesdropper is in a better position than the legitimate receiver, provable secure communication could also be implemented in the DMBCs with both common and confidential messages.

The remainder of the paper is organized as follows. Section 2 gives the notations and main results, that is, the capacity-equivocation regions of the two channel models. Section 3 proves Theorem 2. Section 4 proves Theorems 4 and 5. Section 5 concludes the whole work.

2. Channel Models and Main Results

2.1. Notations. Throughout this paper, we use calligraphic letters, for example, $\mathcal{X}$, $\mathcal{Y}$, to denote the finite sets and $\|\mathcal{X}\|$ to denote the cardinality of the set $\mathcal{X}$. Uppercase letters, for example, X, Y, are used to denote random variables taking values from finite sets, for example, $\mathcal{X}$, $\mathcal{Y}$. The value of the random variable X is denoted by the lowercase letter x. We use Z_i^j to denote the $(j-i+1)$-vectors $(Z_i, Z_{i+1}, \ldots, Z_j)$ of random variables for $1 \leq i \leq j$ and will always drop the subscript when $i = 1$. Moreover, we use $X \sim p(x)$ to denote the probability mass function of the random variable X. For $X \sim p(x)$ and $0 \leq \epsilon \leq 1$, the set of the typical N-sequences x^N is defined as $\mathcal{T}_X^N(\epsilon) = \{x^N : |\pi(x \mid x^N) - p(x)| \leq \epsilon p(x)$ for all $x \in \mathcal{X}\}$, where $\pi(x \mid x^N)$ denotes the frequency of occurrences of letter x in the sequence x^N (for more details about typical sequences, please refer to [10, Chapter 2]). The set of the conditional typical sequences, for example, $\mathcal{T}_{Y|X}^N(\epsilon)$, follows similarly.

2.2. Channel Models and Main Results. This paper studies the secure data transmission for two subclasses of DMBCs with noiseless feedback. One is the case where the feedback is rate-limited and independent of the channel outputs (see

Figure 2); the other is the case where the feedback is originated causally from the channel outputs (see Figure 3). Both models consist of a transmitter and two receivers, named receiver 1 (legitimate receiver) and receiver 2 (eavesdropper). The transmitter aims to convey a common message W to both receivers in addition to a confidential message S intended only for receiver 1. The confidential message S should be kept secret from receiver 2 as much as possible. We use equivocation at receiver 2 to characterize the secrecy of the confidential message. W and S are mutually independent and uniformly distributed over $\mathscr{W}$ and $\mathcal{S}$.

2.2.1. Degraded DMBCs with Rate-Limited Feedback. The degraded DMBCs with rate-limited feedback (see Figure 2) are under the condition that the channel to receiver 2 is physically degraded from the channel to receiver 1; that is, $p(y,z \mid x) = p(y \mid x)p(z \mid y)$ or $X \rightarrow Y \rightarrow Z$ form a Markov chain, where X is the channel input and Y, Z are observations of receiver 1 and 2. In this model, the encoder encodes the messages (W, S) and feedback into codewords X^N, where N is the length of the codeword. They are transmitted over a discrete memoryless channel (DMC) with transition probability $\prod_{i=1}^{N} p(y_i, z_i \mid x_i)$. Receiver 1 obtains Y^N and decodes the common and confidential messages $(\widehat{W}, \widehat{S})$. Receiver 2 obtains Z^N and decodes the common message $\widehat{W}$. More precisely, we define the encoder-decoder $(N, \Delta, P_{e1}, P_{e2})$ in Definition 1.

Definition 1. The encoder-decoder $(N, \Delta, P_{e1}, P_{e2})$ for the degraded DMBCs with rate-limited feedback (with rate limited by R_f) is defined as follows.

(i) The feedback alphabet $\mathscr{K}$ satisfies $\lim_{N\to\infty}(\log\|\mathscr{K}\|/N) \le R_f$. The feedback is generated independent of the channel output symbols.

(ii) The stochastic channel encoder φ is specified by a matrix of conditional probability distributions $\varphi(x^N \mid s, w, k)$ which denotes the probability that the message s, w and the feedback k are encoded as the channel input x^N, where $x^N \in \mathscr{X}^N, s \in \mathcal{S}, w \in \mathscr{W}, k \in \mathscr{K}$, and $\sum_{x^N} \varphi(x^N \mid s, w, k) = 1$. Note that $\mathcal{S}$ and $\mathscr{W}$ are the confidential and common message sets.

(iii) Decoder 1 is a mapping $h_1 : \mathscr{Y}^N \rightarrow \mathcal{S} \times \mathscr{W}$. The input of decoder 1 is Y^N, and the output is $\widehat{S}, \widehat{W}$. The decoding error probability of receiver 1 is defined as $P_{e1} = \Pr\{h_1(Y^N) \neq (S, W)\}$. Similarly, Decoder 2 is defined as a mapping $h_2 : \mathscr{Z}^N \rightarrow \mathscr{W}$. The input of decoder 2 is Z^N, and the output is $\widehat{\widehat{W}}$. The decoding error probability of receiver 2 is defined as $P_{e2} = \Pr\{h_2(Z^N) \neq W\}$.

(iv) The equivocation at receiver 2 is defined as

$$\Delta = \frac{1}{N} H\left(S \mid Z^N\right). \tag{1}$$

A rate triple (R_0, R_1, R_e) is said to be *achievable* for the model in Figure 2 if there exists a channel encoder-decoder $(N, \Delta, P_{e1}, P_{e2})$ defined in Definition 1, such that

$$\lim_{N\to\infty} \frac{\log\|\mathscr{W}\|}{N} = R_0, \tag{2}$$

$$\lim_{N\to\infty} \frac{\log\|\mathcal{S}\|}{N} = R_1, \tag{3}$$

$$\lim_{N\to\infty} \frac{\log\|\mathscr{K}\|}{N} = R_f' \le R_f, \tag{4}$$

$$\lim_{N\to\infty} \Delta \ge R_e, \tag{5}$$

$$P_{e1} \le \epsilon, \qquad P_{e2} \le \epsilon, \tag{6}$$

where ϵ is an arbitrary small positive real number, R_0, R_1, R_f' are the rates of the common messages, confidential messages, and feedback, and R_e is the equivocation rate of the confidential messages. Note that the feedback rate is limited by R_f. The capacity-equivocation region is defined as the convex closure of all achievable rate triples (R_0, R_1, R_e). The capacity-equivocation region of the degraded DMBCs with rate-limited feedback is shown in the following theorem.

Theorem 2. *For the degraded DMBCs with limited feedback rate R_f, the capacity-equivocation region is the set*

$$\begin{aligned} \mathscr{R}_d = \Big\{ (R_0, R_1, R_e) &: 0 \le R_e \le R_1, \\ R_0 &\le I(U; Z), \\ R_1 &\le I(X; Y \mid U), \\ R_e &\le I(X; Y \mid U) - I(X; Z \mid U) + R_f \Big\}, \end{aligned} \tag{7}$$

where U is an auxiliary random variable and $U \rightarrow X \rightarrow Y \rightarrow Z$ form a Markov chain.

The proof of Theorem 2 is given in Section 3. The remark of Theorem 2 is shown below.

Remark 3. (i) The secrecy capacity of the model in Figure 2 is defined as the maximum rate at which confidential messages can be sent to receiver 1 in perfect secrecy; that is,

$$C_s = \max_{(R_0=0, R_1, R_e=R_1)\in\mathscr{R}} R_1, \tag{8}$$

where $\mathscr{R}$ is the capacity-equivocation region. Therefore, by the definition in (8), the secrecy capacity of the degraded DMBCs with limited feedback rate R_f is

$$\begin{aligned} &C_{sd} \\ &= \max \min \left\{ I(X; Y \mid U), I(X; Y \mid U) - I(X; Z \mid U) + R_f \right\}. \end{aligned} \tag{9}$$

This result subsumes the secrecy capacity of the degraded wiretap channel with rate-limited feedback (see [4]) by setting the auxiliary random variable U to be constant in (9).

(ii) The capacity-equivocation region in (7) is bigger than that in [2] without feedback. This implies that feedback can be used to enhance the secrecy in the DMBCs. Note that this finding had already been verified in [3–6].

2.2.2. Less and Reversely Less Noisy DMBCs with Noiseless Causal Feedback. The model in Figure 3 is based on the assumption that the channel to receiver 1 is independent of the channel to receiver 2; that is, $p(y,z \mid x) = p(y \mid x)p(z \mid x)$. The definition of the encoder-decoder for this model is similar to Definition 1 except for the feedback and the encoder. Different from the model in Figure 2, the feedback in Figure 3 is originated causally from the channel outputs of receiver 1 to the transmitter. The stochastic encoder for this model at time i, $1 \le i \le N$, is defined as $f_i(x_i \mid w_i, s_i, y^{i-1})$, where $w_i \in \mathscr{W}$, $s_i \in \mathscr{S}$, $y^{i-1} \in \mathscr{Y}^{i-1}$ (the channel outputs of receiver 1 before time i) and $\sum_{x_i \in \mathscr{X}} f_i(x_i \mid w_i, s_i, y^{i-1}) = 1$.

A rate triple (R_0, R_1, R_e) is said to be *achievable* for the model in Figure 3 if there exists a channel encoder-decoder $(N, \Delta, P_{e1}, P_{e2})$ such that (2), (3), (5), and (6) hold. Note that the definition of "*achievable*" here does not include (4) since the feedback in the model of Figure 3 is not rate limited. The definition of secrecy capacity is the same as that in Remark 3. Then, we present the capacity-equivocation regions of the *less* and *reversely less noisy* DMBCs with noiseless causal feedback in Theorems 4 and 5, respectively.

Theorem 4. *For the less noisy DMBCs with noiseless causal feedback, the capacity-equivocation region is the set*

$$\begin{aligned} \mathscr{R}_l = \{(R_0, R_1, R_e) &: 0 \le R_e \le R_1, \\ R_0 &\le I(U;Z), \\ R_1 &\le I(X;Y \mid U), \\ R_e &\le H(Y \mid Z)\}, \end{aligned} \tag{10}$$

where $U \rightarrow X \rightarrow (Y,Z)$ *form a Markov chain.*

Theorem 5. *For the reversely less noisy DMBCs with noiseless causal feedback, the capacity-equivocation region is the set*

$$\begin{aligned} \mathscr{R}_{rl} = \{(R_0, R_1, R_e) &: 0 \le R_e \le R_1, \\ R_0 &\le I(U;Y), \\ R_1 &\le I(X;Y \mid U), \\ R_e &\le H(Y \mid X)\}, \end{aligned} \tag{11}$$

where $U \rightarrow X \rightarrow (Y,Z)$ *form a Markov chain.*

The proof of Theorems 4 and 5 is given in Section 4. The remark of Theorems 4 and 5 is given below.

Remark 6. (i) By the definition in (8), the secrecy capacity of the *less noisy* DMBCs with noiseless causal feedback is

$$C_{sl} = \max \min \{I(X;Y \mid U), H(Y \mid Z)\}. \tag{12}$$

The secrecy capacity of the *reversely less noisy* DMBCs with noiseless causal feedback is

$$C_{srl} = \max \min \{I(X;Y \mid U), H(Y \mid X)\}. \tag{13}$$

Setting the auxiliary random variable U to be constant in (12) and (13), the capacity-equivocation region of the model in [5] is obtained.

(ii) In the model of Figure 3, it is assumed that the channel to receiver 1 is independent of the channel to receiver 2; that is, $p(y,z \mid x) = p(y \mid x)p(z \mid x)$. This implies $Y \rightarrow X \rightarrow Z$. Therefore, it is easy to see $H(Y \mid X) = H(Y \mid XZ) \le H(Y \mid Z)$; that is, the upper bound on the equivocation rate R_e in (11) for the reversely less noisy case is smaller than that in (10) for the less noisy case. This tells that when the eavesdropper is in a better position than the legitimate receiver (see the *reversely less noisy* case), the uncertainty about the confidential messages at the eavesdropper is decreased. Besides, from (13), we see that even if the eavesdropper is in a better position, the secrecy capacity is a positive value, which means provable secure communication could also be implemented in such a bad condition.

3. Proof of Theorem 2

In this section, Theorem 2 is proved. The converse part of Theorem 2 gives the outer bound on the capacity-equivocation region of the degraded DMBCs with rate-limited feedback. The proof of the converse part is shown in Section 3.1. The key tools used in the proof include the identification of the random variables and Csiszár's sum equality [2]. In Section 3.2, to prove the direct part of Theorem 2, a coding scheme is provided to achieve the achievable rate triples in $\mathscr{R}_d$. The key ideas in the coding scheme are inspired by [4]. However, [4] only considers the transmission of the confidential messages. Our coding scheme considers both the confidential and common messages.

3.1. The Converse Part of Theorem 2. In order to find the identification of the auxiliary random variables that satisfy the capacity-equivocation region characterized by $\mathscr{R}_d$, we prove the converse part for the equivalent region (the fact that the two regions are equivalent follows similarly from [10, Chapter 5, problem 5.8]) containing all the rate triples (R_0, R_1, R_e) such that

$$0 \le R_e \le R_1, \tag{14}$$

$$R_0 \le I(U;Z), \tag{15}$$

$$R_0 + R_1 \le I(X;Y \mid U) + I(U;Z), \tag{16}$$

$$R_e \le I(X;Y \mid U) - I(X;Z \mid U) + R_f. \tag{17}$$

Now we show that all *achievable* triples (R_0, R_1, R_e) satisfy (14), (15), (16), and (17).

Condition (14) is proved as follows:

$$\begin{aligned} R_e &\le \lim_{N\to\infty} \Delta \\ &= \lim_{N\to\infty} \frac{H\left(S \mid Z^N\right)}{N} \\ &\le \lim_{N\to\infty} \frac{H(S)}{N} \\ &= R_1. \end{aligned} \tag{18}$$

To prove condition (15), we calculate

$$\begin{aligned} H(W) &= I\left(W; Z^N\right) + H\left(W \mid Z^N\right) \\ &\le^{(a3.1)} I\left(W; Z^N\right) + \epsilon_1 \\ &= \sum_{i=1}^{N} I\left(W; Z_i \mid Z^{i-1}\right) + \epsilon_1 \\ &= \sum_{i=1}^{N} I\left(W; Z_i \mid Z_{i+1}^N\right) + \epsilon_1 \\ &= \sum_{i=1}^{N} \left[I\left(WY^{i-1}; Z_i \mid Z_{i+1}^N\right) - I\left(Y^{i-1}; Z_i \mid Z_{i+1}^N W\right)\right] + \epsilon_1 \\ &\le \sum_{i=1}^{N} \left[I\left(WY^{i-1} Z_{i+1}^N; Z_i\right) - I\left(Y^{i-1}; Z_i \mid Z_{i+1}^N W\right)\right] + \epsilon_1 \\ &\le \sum_{i=1}^{N} I\left(WK^N Y^{i-1} Z_{i+1}^N; Z_i\right) + \epsilon_1, \end{aligned} \tag{19}$$

where $(a3.1)$ follows from Fano's inequality and ϵ_1 is a small positive number. Note that $K^N = (K_1, K_2, \ldots, K_N)$, where K_i is the feedback symbol at time i, $1 \le i \le N$.

To prove condition (16), we consider

$$\begin{aligned} H(S) + H(W) &= H\left(S \mid WK^N\right) + H(W) \\ &= I\left(S; Y^N \mid WK^N\right) + H\left(S \mid Y^N WK^N\right) \\ &\quad + I\left(W; Z^N\right) + H\left(W \mid Z^N\right) \\ &\le^{(a3.2)} I\left(S; Y^N \mid WK^N\right) + \epsilon_2 \\ &\quad + I\left(WK^N; Z^N\right) + \epsilon_1 \\ &= \sum_{i=1}^{N} I\left(S; Y_i \mid Y^{i-1} WK^N\right) \\ &\quad + \sum_{i=1}^{N} I\left(WK^N; Z_i \mid Z_{i+1}^N\right) + \epsilon_1 + \epsilon_2 \\ &= \sum_{i=1}^{N} \left[I\left(S Z_{i+1}^N; Y_i \mid Y^{i-1} WK^N\right) \right. \\ &\qquad \left. - I\left(Z_{i+1}^N; Y_i \mid Y^{i-1} WK^N S\right)\right] \\ &\quad + \sum_{i=1}^{N} \left[I\left(WK^N Y^{i-1}; Z_i \mid Z_{i+1}^N\right) \right. \\ &\qquad \left. - I\left(Y^{i-1}; Z_i \mid Z_{i+1}^N WK^N\right)\right] + \epsilon_1 + \epsilon_2 \\ &= \sum_{i=1}^{N} \left[I\left(Z_{i+1}^N; Y_i \mid Y^{i-1} WK^N\right) \right. \\ &\qquad + I\left(S; Y_i \mid Z_{i+1}^N Y^{i-1} WK^N\right) \\ &\qquad \left. - I\left(Z_{i+1}^N; Y_i \mid Y^{i-1} WK^N S\right)\right] \\ &\quad + \sum_{i=1}^{N} \left[I\left(WK^N Y^{i-1}; Z_i \mid Z_{i+1}^N\right) \right. \\ &\qquad \left. - I\left(Y^{i-1}; Z_i \mid Z_{i+1}^N WK^N\right)\right] + \epsilon_1 + \epsilon_2 \\ &\le \sum_{i=1}^{N} \left[I\left(Z_{i+1}^N; Y_i \mid Y^{i-1} WK^N\right) \right. \\ &\qquad + I\left(S; Y_i \mid Z_{i+1}^N Y^{i-1} WK^N\right) \\ &\qquad \left. - I\left(Z_{i+1}^N; Y_i \mid Y^{i-1} WK^N S\right)\right] \\ &\quad + \sum_{i=1}^{N} \left[I\left(WK^N Y^{i-1} Z_{i+1}^N; Z_i\right) \right. \\ &\qquad \left. - I\left(Y^{i-1}; Z_i \mid Z_{i+1}^N WK^N\right)\right] + \epsilon_1 + \epsilon_2 \\ &=^{(a3.3)} \sum_{i=1}^{N} \left[I\left(S; Y_i \mid Z_{i+1}^N Y^{i-1} WK^N\right) \right. \\ &\qquad \left. - I\left(Z_{i+1}^N; Y_i \mid Y^{i-1} WK^N S\right)\right] \\ &\quad + \sum_{i=1}^{N} I\left(WK^N Y^{i-1} Z_{i+1}^N; Z_i\right) + \epsilon_1 + \epsilon_2 \\ &\le \sum_{i=1}^{N} I\left(S; Y_i \mid Z_{i+1}^N Y^{i-1} WK^N\right) \\ &\quad + \sum_{i=1}^{N} I\left(WK^N Y^{i-1} Z_{i+1}^N; Z_i\right) + \epsilon_1 + \epsilon_2, \end{aligned} \tag{20}$$

where ϵ_2 is a small positive number and $(a3.2)$ and $(a3.3)$ follow from Fano's inequality and Csiszár's sum equality [2]; that is, $\sum_{i=1}^{N} I(Z_{i+1}^N; Y_i \mid Y^{i-1} WK^N) = \sum_{i=1}^{N} I(Y^{i-1}; Z_i \mid Z_{i+1}^N WK^N)$.

To prove condition (17), we calculate

$$\begin{aligned} H\left(S \mid Z^N\right) &= H\left(S \mid Z^N, W\right) + I\left(S; W \mid Z^N\right) \\ &\le H\left(S \mid Z^N, W\right) + H\left(W \mid Z^N\right) \\ &= I\left(S; K^N, Y^N \mid Z^N, W\right) \\ &\quad + H\left(S \mid Z^N, K^N, Y^N, W\right) + H\left(W \mid Z^N\right) \end{aligned}$$

$$\begin{aligned}
&\le I\left(S;K^N,Y^N \mid Z^N,W\right)\\
&\quad + H\left(S \mid K^N,Y^N\right) + H\left(W \mid Z^N\right)\\
&= I\left(S;K^N \mid Z^N,W\right) + I\left(S;Y^N \mid K^N,Z^N,W\right)\\
&\quad + H\left(S \mid K^N,Y^N\right) + H\left(W \mid Z^N\right)\\
&\le H\left(K^N\right) + I\left(S;Y^N \mid K^N,Z^N,W\right)\\
&\quad + H\left(S \mid K^N,Y^N\right) + H\left(W \mid Z^N\right)\\
&\le NR_f + I\left(S;Y^N \mid K^N,Z^N,W\right) + \epsilon_1 + \epsilon_2.
\end{aligned} \tag{21}$$

The last inequality in (21) follows from the Fano's inequality and the fact that the feedback rate is limited by R_f. Then, $I(S;Y^N \mid K^N,Z^N,W)$ will be calculated as follows:

$$\begin{aligned}
&I\left(S;Y^N \mid K^N,Z^N,W\right)\\
&= \sum_{i=1}^{N} I\left(S;Y_i \mid Y^{i-1}Z^N W K^N\right)\\
&= \sum_{i=1}^{N} I\left(S;Y_i \mid Y^{i-1},Z^{i-1},Z_i,Z_{i+1}^N,W,K^N\right)\\
&=^{(a3.4)} \sum_{i=1}^{N} \left[I\left(S;Y_i \mid Y^{i-1},Z^{i-1},Z_i,Z_{i+1}^N,W,K^N\right)\right.\\
&\qquad + I\left(Z^{i-1};Y_i \mid Y^{i-1},Z_i^N,W,K^N\right)\\
&\qquad \left.- I\left(Z^{i-1};Y_i \mid Y^{i-1},Z_i^N,W,S,K^N\right)\right]\\
&= \sum_{i=1}^{N} \left[I\left(S,Z^{i-1};Y_i \mid Y^{i-1},Z_i,Z_{i+1}^N,W,K^N\right)\right.\\
&\qquad \left.- I\left(Z^{i-1};Y_i \mid Y^{i-1},Z_i^N,W,S,K^N\right)\right]\\
&= \sum_{i=1}^{N} I\left(S;Y_i \mid Y^{i-1},Z_i,Z_{i+1}^N,W,K^N\right),
\end{aligned} \tag{22}$$

where $(a3.4)$ follows from the Markov chain $Y_i \to Y^{i-1}Z_i^N W K^N \to Z^{i-1}$ and $Y_i \to Y^{i-1}Z_i^N WSK^N \to Z^{i-1}$. Then, we introduce a random variable Q which is independent of $SWK^N X^N Y^N Z^N$ and uniformly distributed over $\{1,2,\ldots,N\}$. Set $U = Z_{Q+1}^N Y^{Q-1} W K^N Q$, $V = US$, $Y = Y_Q$, $X = X_Q$, $Z = Z_Q$. It is straightforward to see that $U \to V \to X \to Y \to Z$ form a Markov chain. After using the standard time sharing argument [10, Section 5.4], (19), (20), and (22) are simplified into

$$\begin{aligned}
H(W) &\le \sum_{i=1}^{N} I\left(WK^N Y^{i-1} Z_{i+1}^N;Z_i\right) + \epsilon_1\\
&= NI(U;Z) + \epsilon_1,
\end{aligned} \tag{23}$$

$$\begin{aligned}
H(S) + H(W) &\le \sum_{i=1}^{N} I\left(S;Y_i \mid Z_{i+1}^N Y^{i-1} W K^N\right)\\
&\quad + \sum_{i=1}^{N} I\left(WK^N Y^{i-1} Z_{i+1}^N;Z_i\right) + \epsilon_1 + \epsilon_2\\
&= NI(S;Y \mid U) + NI(U;Z) + \epsilon_1 + \epsilon_2\\
&= NI(V;Y \mid U) + NI(U;Z) + \epsilon_1 + \epsilon_2,
\end{aligned} \tag{24}$$

$$\begin{aligned}
I\left(S;Y^N \mid K^N,Z^N,W\right) &= \sum_{i=1}^{N} I\left(S;Y_i \mid Y^{i-1},Z_i,Z_{i+1}^N,W,K^N\right)\\
&= NI(S;Y \mid Z,U)\\
&= NI(US;Y \mid Z,U)\\
&= NI(V;Y \mid Z,U).
\end{aligned} \tag{25}$$

Substituting (25) into (21) and utilizing (5), we get

$$\begin{aligned}
R_e &\le \lim_{N\to\infty} \Delta\\
&= \lim_{N\to\infty} \frac{H\left(S \mid Z^N\right)}{N}\\
&\le \lim_{N\to\infty} \frac{NR_f + I\left(S;Y^N \mid K^N,Z^N,W\right) + \epsilon_1 + \epsilon_2}{N}\\
&= I(V;Y \mid Z,U) + R_f\\
&= I(V;Y \mid U) - I(V;Z \mid U) + R_f.
\end{aligned} \tag{26}$$

The last equality in (26) follows from the Markov chain $U \to V \to Y \to Z$.

To finish the proof of (16) and (17), we need to show that $I(V;Y \mid U) \le I(X;Y \mid U)$ and $I(V;Y \mid U) - I(V;Z \mid U) \le I(X;Y \mid U) - I(X;Z \mid U)$. We first prove $I(V;Y \mid U,X) = 0$ and $I(V;Z \mid U,X) = 0$:

$$\begin{aligned}
I(V;Y \mid U,X) &= H(Y \mid U,X) - H(Y \mid U,V,X)\\
&=^{(a3.5)} H(Y \mid X) - H(Y \mid X)\\
&= 0,\\
I(V;Z \mid U,X) &= H(Z \mid U,X) - H(Z \mid U,V,X)\\
&=^{(a3.6)} H(Z \mid X) - H(Z \mid X)\\
&= 0,
\end{aligned} \tag{27}$$

where $(a3.5)$ follows from the Markov chains $U \to X \to Y$ and $(UV) \to X \to Y$ and $(a3.6)$ follows from the Markov

chains $U \rightarrow X \rightarrow Z$ and $(UV) \rightarrow X \rightarrow Z$. Utilizing (27), we obtain

$$\begin{aligned} I(V;Y \mid U) &= I(V,X;Y \mid U) - I(X;Y \mid U,V) \\ &= I(X;Y \mid U) + I(V;Y \mid U,X) - I(X;Y \mid U,V) \\ &= I(X;Y \mid U) - I(X;Y \mid U,V), \end{aligned} \tag{28}$$

$$\begin{aligned} I(V;Z \mid U) &= I(V,X;Z \mid U) - I(X;Z \mid U,V) \\ &= I(X;Z \mid U) + I(V;Z \mid U,X) - I(X;Z \mid U,V) \\ &= I(X;Z \mid U) - I(X;Z \mid U,V). \end{aligned} \tag{29}$$

From (28), it is straightforward to see that $I(V;Y \mid U) \leq I(X;Y \mid U)$. This proves condition (16).

Then, we prove $I(V;Y \mid U) - I(V;Z \mid U) \leq I(X;Y \mid U) - I(X;Z \mid U)$. Since the channel model in Figure 1 is (physically) degraded, $I(X;Y \mid U = u, V = v) - I(X;Z \mid U = u, V = v) \geq 0$ holds for every (u, v), which implies

$$I(X;Y \mid U,V) - I(X;Z \mid U,V) \geq 0. \tag{30}$$

Therefore, utilizing (28), (29), and (30), we get

$$\begin{aligned} I(V;Y \mid U) - I(V;Z \mid U) &= I(X;Y \mid U) - I(X;Z \mid U) \\ &\quad - [I(X;Y \mid U,V) - I(X;Z \mid U,V)] \\ &\leq I(X;Y \mid U) - I(X;Z \mid U). \end{aligned} \tag{31}$$

This proves condition (17).

The converse part of Theorem 2 is proved.

3.2. A Coding Scheme Achieving $\mathscr{R}_d$. A coding scheme is provided to achieve the achievable triples $(R_0, R_1, R_e) \in \mathscr{R}_d$. The key methods used in the scheme include the superposition coding, rate splitting, and random binning. The confidential message is split into two parts. One part is reliably transmitted using superposition coding and random binning; the other part is securely transmitted with the help of the feedback. Note that Section 3.1 has already given the outer bound on the capacity-equivocation region. When $R_f \geq I(X;Z \mid U)$, it can be seen from (9) that the secrecy capacity for the degraded DMBCs with rate-limited feedback always equals to $I(X;Y \mid U)$. Therefore, in order to investigate the effects of the feedback, the feedback rate $R_f < I(X;Z \mid U)$ will only be considered in this subsection.

We need to prove that all the triples $(R_0, R_1, R_e) \in \mathscr{R}_d$ for the model of Figure 2 with any feedback rate R'_f limited by R_f are *achievable* (see Definition 1). This subsection is organized as follows. The codebook generation and encoding scheme is given in Section 3.2.1. The decoding scheme is given in Section 3.2.2. The analysis of error probability and equivocation are shown in Sections 3.2.3 and 3.2.4, respectively.

3.2.1. Codebook Generation and Encoding. Split the confidential message into two parts; that is, $\mathscr{S} = (\mathscr{M}_1, \mathscr{M}_2)$. The corresponding variables M_1, M_2 are uniformly distributed over $\{1, 2, 3, \ldots, 2^{NR'}\}$ and $\{1, 2, 3, \ldots, 2^{NR'_f}\}$, where (when $R_f \geq R_1$, the confidential message S can be totally protected by using part of the feedback (as the shared key between the transmitter and receiver 1). The remaining part of the feedback is redundant. Therefore, in order to study the effects of the feedback on the capacity region, only $R_f < R_1$ comes into our consideration)

$$\begin{aligned} 0 \leq R'_f &\leq R_f, \\ R' = R_1 - R'_f &> 0. \end{aligned} \tag{32}$$

It is important to notice that R_1 is the rate of the private message $\mathscr{S}$, which consists of $\mathscr{M}_1$ and $\mathscr{M}_2$. This means that

$$\begin{aligned} R_1 &= \lim_{N \to \infty} \frac{\log(\|\mathscr{M}_1\| \|\mathscr{M}_2\|)}{N} \\ &= \lim_{N \to \infty} \left(\frac{\log \|\mathscr{M}_1\|}{N} + \frac{\log \|\mathscr{M}_2\|}{N} \right). \end{aligned} \tag{33}$$

Define the index sets $\mathscr{J}_N$, $\mathscr{L}_N$, $\mathscr{F}_N$, and $\mathscr{M}_N$ satisfying

$$\begin{aligned} \lim_{N \to \infty} \frac{1}{N} \log \|\mathscr{J}_N\| &= I(X;Z \mid U) - R'_f, \\ \lim_{N \to \infty} \frac{1}{N} \log \|\mathscr{L}_N\| &= I(X;Y \mid U) - I(X;Z \mid U), \\ \lim_{N \to \infty} \frac{1}{N} \log \|\mathscr{F}_N\| &= R'_f, \\ \lim_{N \to \infty} \frac{1}{N} \log \|\mathscr{M}_N\| &= I(U;Z). \end{aligned} \tag{34}$$

We use $j \in \mathscr{J}_N, l \in \mathscr{L}_N, f \in \mathscr{F}_N, m \in \mathscr{M}_N$ to index the codeword x^N. Take $\mathscr{W} \subset \mathscr{M}_N$ such that (2) holds. Since $R_1 \leq I(X;Y \mid U)$, it is easy to see $\|\mathscr{J}_N \times \mathscr{L}_N \times \mathscr{F}_N\| \geq 2^{NR_1}$. Therefore, let $\mathscr{M}_1 = \mathscr{D}_N \times \mathscr{L}_N$, $\mathscr{M}_2 = \mathscr{F}_N$, where $\mathscr{D}_N$ is an arbitrary set such that (3) holds. Let g_j be a mapping of $\mathscr{J}_N$ into $\mathscr{D}_N$ partitioning $\mathscr{J}_N$ into subsets of size $\|\mathscr{J}_N\|/\|\mathscr{D}_N\|$; that is,

$$g_j : \mathscr{J}_N \longrightarrow \mathscr{D}_N, \tag{35}$$

where $g_j(j) = d, j \in \mathscr{J}_N, d \in \mathscr{D}_N$.

For each $w \in \mathscr{W}$, we generate a codeword $u^N(w)$ according to $\prod_{i=1}^{N} p(u_i)$. Then, for each $u^N(w)$, a codebook $\mathscr{CB}_w$ (see Figure 4) containing $\|\mathscr{J}_N\| \cdot \|\mathscr{L}_N\| \cdot \|\mathscr{F}_N\|$ codewords x^N_{jlfm} is constructed according to $\prod_{i=1}^{N} p(x_i \mid u_i)$, where $j \in \mathscr{J}_N, l \in \mathscr{L}_N, f \in \mathscr{F}_N, m = w \in \mathscr{M}$. Those x^N are put into $\|\mathscr{L}_N\| \cdot \|\mathscr{F}_N\|$ bins so that each bin contains $\|\mathscr{J}_N\|$ codewords. Each bin is indexed by (l, f), where $l \in \mathscr{L}_N$, $f \in \mathscr{F}_N$. Then, we divide each bin into $\|\mathscr{D}_N\|$ subbins such that each subbin contains $\|\mathscr{J}_N\|/\|\mathscr{D}_N\|$ codewords. The codebook structure is presented in Figure 4.

Let $\mathscr{K} = \{1, 2, 3, \ldots, 2^{NR'_f}\}$, where $k \in \mathscr{K}$ is the key sent to the transmitter from receiver 1 through the secure feedback

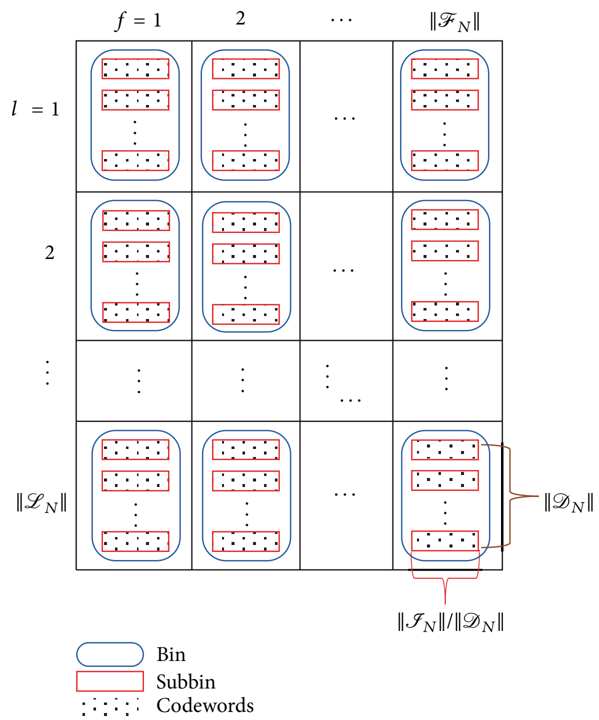


Figure 4: The codebook $\mathscr{CB}_w$ for each $u^N(w)$.

link. It is kept secret from receiver 2. The corresponding variable K is uniformly distributed over $\mathscr{K}$ and independent of S and W.

In order to send $s = (d, l, m_2) \in \mathscr{D}_N \times \mathscr{L}_N \times \mathscr{F}_N$ and $w \in \mathscr{W}$, a codeword x^N_{jlfm} is chosen as follows. According to the common message w, we first find the sequence $u^N(w)$. For the determined $u^N(w)$, there is a corresponding codebook $\mathscr{CB}_w$; see Figure 4. Then, the corresponding codeword x^N_{jlfm} is sent into the channel, where j is chosen randomly from the set $g_j^{-1}(d)$, $f = k \oplus m_2$, and $m = w$ (here $\oplus$ is modulo addition over $\mathscr{F}_N$). Figure 4 shows how to select x^N_{jlfm} in detail. According to $u^N(w)$, we can find the corresponding codebook $\mathscr{CB}_w$. In the codebook $\mathscr{CB}_w$, we choose the corresponding bin according to f and l. Then, in that bin, the subbin is found according to d. Finally, a codeword x^N (which is denoted by x^N_{jlfm}) is randomly chosen from that subbin.

3.2.2. Decoding. Receiver 2 tries to find a unique sequence $u^N(\widehat{\widehat{w}})$ such that $(u^N(\widehat{\widehat{w}}), z^N) \in T^N_{UZ}(\epsilon_1)$. If there exists such a unique sequence, decoder 2 outputs $\widehat{\widehat{w}}$; otherwise, an error is declared. Since the size of $\mathscr{W}$ is smaller than $2^{NI(U;Z)}$, the decoding error probability for receiver 2 approaches zero.

For receiver 1, he can also decode the common message $\widehat{w}$ since the output of channel 2 is a degraded version of the output of channel 1. Then, receiver 1 tries to find a unique codeword $x^N_{\widehat{j}\widehat{l}\widehat{f}\widehat{m}}$ indexed by $\widehat{j}, \widehat{l}, \widehat{f}, \widehat{m}$, such that $(x^N_{\widehat{j}\widehat{l}\widehat{f}\widehat{m}}, y^N) \in T^N_{XY|U}(\epsilon_2)$. If there exists such a unique codeword $x^N_{\widehat{j}\widehat{l}\widehat{f}\widehat{m}}$, receiver 1 calculates $\widehat{f} \ominus k$ as $\widehat{m}_2$ (here $\ominus$ is modulo subtraction over $\mathscr{F}_N$, and $\widehat{m} = \widehat{w}$) and finds $\widehat{d}$ according to $g_j(\widehat{j})$. Note that receiver 1 knows the secret key k. Decoder 1 outputs $\widehat{s} = (\widehat{d}, \widehat{l}, \widehat{m}_2)$ and $\widehat{w}$. If no such $x^N_{\widehat{j}\widehat{l}\widehat{f}\widehat{m}}$ or more than one such $x^N_{\widehat{j}\widehat{l}\widehat{f}\widehat{m}}$ exist, an error is declared.

3.2.3. Analysis of Error Probability. Since the number of $u^N(w)$ is upper bound by $2^{NI(U;Z)}$ and the DMBCs under discussion are degraded, both receivers can decode the common message w with error probability approaching zero by applying the standard channel coding theorem [11, Theorem 7.7.1]. Moreover, it can be calculated that given the codeword $u^N(w)$, the number of x^N is

$$\|\mathscr{F}_N\| \cdot \|\mathscr{J}_N\| \cdot \|\mathscr{L}_N\| = 2^{NI(X;Y|U)}. \tag{36}$$

So, after determining the codeword $u^N(w)$, receiver 1 can decode the codeword x^N with error probability approaching zero by applying the standard channel coding theorem [11, Theorem 7.7.1]. This proves (6).

3.2.4. Analysis of Equivocation. The proof of (5) is given below:

$$\begin{aligned} H(S \mid Z^N) &= H(M_1, M_2 \mid Z^N) \\ &= H(M_1 \mid Z^N) + H(M_2 \mid Z^N, M_1) \\ &\geq H(M_1 \mid Z^N) + H(M_2 \mid Z^N, M_1, K \oplus M_2) \\ &=^{(b3.1)} H(M_1 \mid Z^N) + H(M_2 \mid K \oplus M_2) \\ &=^{(b3.2)} H(M_1 \mid Z^N) + H(M_2) \\ &=^{(b3.3)} H(M_1 \mid Z^N) + NR'_f, \end{aligned} \tag{37}$$

where ($b3.1$) follows from the Markov chain $M_2 \to M_2 \oplus K \to (Z^N, M_1)$, ($b3.2$) follows from the fact that M_2 is independent of $M_2 \oplus K$, and ($b3.3$) follows from that M_2 is uniformly distributed over $\{1, 2, 3, \ldots, 2^{NR'_f}\}$. The proof of the fact that M_2 is independent of $M_2 \oplus K$ is shown as follows (the proof can also be seen in [6]):

$$\begin{aligned} &p(M_2 \oplus K = a) \\ &\quad = \sum_k p(M_2 \oplus K = a \mid K = k)\, p(K = k) \\ &\quad =^{(b3.4)} \sum_k p(M_2 \oplus K = a \mid K = k) \frac{1}{\|\mathscr{F}_N\|} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\|\mathcal{F}_N\|}\sum_k p\left(M_2 \oplus K = a \mid K = k\right) \\
&= \frac{1}{\|\mathcal{F}_N\|}\sum_k p\left(M_2 = a \ominus k \mid K = k\right) \\
&=^{(b3.5)} \frac{1}{\|\mathcal{F}_N\|}\sum_k p\left(M_2 = a \ominus k\right) \\
&= \frac{1}{\|\mathcal{F}_N\|}
\end{aligned}$$

$$\begin{aligned}
&p\left(M_2 \oplus K = a, M_2 = m_2\right) \\
&\quad = p\left(K = a \ominus m_2, M_2 = m_2\right) \\
&\quad =^{(b3.6)} p\left(K = a \ominus m_2\right) p\left(M_2 = m_2\right) \\
&\quad =^{(b3.7)} \frac{1}{\|\mathcal{F}_N\|} \cdot \frac{1}{\|\mathcal{F}_N\|},
\end{aligned} \tag{38}$$

where $(b3.5)$ and $(b3.6)$ follow from that M_2 is independent of K, and $(b3.4)$ and $(b3.7)$ follow from that M_2 and K are both uniformly distributed over $\mathcal{F}_N$. According to (38),

$$\begin{aligned}
&p\left(M_2 \oplus K = a, M_2 = m_2\right) \\
&\quad = p\left(M_2 \oplus K = a\right) p\left(M_2 = m_2\right).
\end{aligned} \tag{39}$$

Therefore, M_2 is independent of $M_2 \oplus K$.

Next, we focus on the first term in (37). The method of the equivocation analysis in [2] will be used:

$$\begin{aligned}
&H\left(M_1 \mid Z^N\right) \\
&\quad \ge H\left(M_1 \mid Z^N, W\right) \\
&\quad = H\left(M_1, Z^N \mid W\right) - H\left(Z^N \mid W\right) \\
&\quad = H\left(M_1, Z^N, X^N \mid W\right) - H\left(X^N \mid M_1, Z^N, W\right) \\
&\qquad - H\left(Z^N \mid W\right) \\
&\quad = H\left(M_1, X^N \mid W\right) + H\left(Z^N \mid M_1, X^N, W\right) \\
&\qquad - H\left(X^N \mid M_1, Z^N, W\right) - H\left(Z^N \mid W\right) \\
&\quad \ge H\left(X^N \mid W\right) + H\left(Z^N \mid M_1, X^N, W\right) \\
&\qquad - H\left(X^N \mid M_1, Z^N, W\right) - H\left(Z^N \mid W\right).
\end{aligned} \tag{40}$$

Note that W in inequality (40) is the random variable of the common message $\mathcal{W}$. The four terms $H(X^N \mid W)$, $H(Z^N \mid M_1, X^N, W)$, $H(X^N \mid M_1, Z^N, W)$, and $H(Z^N \mid W)$ will be bounded as follows.

Given $w \in \mathcal{W}$, the number of x^N is $\|\mathcal{J}_N\| \cdot \|\mathcal{L}_N\| \cdot \|\mathcal{F}_N\|$. By applying [12, Lemma 2.5], we obtain

$$\begin{aligned}
H\left(X^N \mid W\right) &\ge \log\left(\|\mathcal{J}_N\| \cdot \|\mathcal{L}_N\| \cdot \|\mathcal{F}_N\|\right) - 1 \\
&= NI\left(X; Y \mid U\right) - 1.
\end{aligned} \tag{41}$$

Since $(M_1, W) \rightarrow X^N \rightarrow Z^N$ and the channel to receiver 2 is discrete memoryless, it is easy to get

$$\begin{aligned}
H\left(Z^N \mid M_1, X^N, W\right) &= H\left(Z^N \mid X^N\right) \\
&= NH\left(Z \mid X\right).
\end{aligned} \tag{42}$$

With the knowledge of $(d, l) \in \mathcal{M}_1$ and $w \in \mathcal{W}$, the number of x^N is

$$\begin{aligned}
2^{NR'_f} \cdot \frac{\|\mathcal{J}_N\|}{\|\mathcal{D}_N\|} &< 2^{NR'_f} \cdot \|\mathcal{J}_N\| \\
&= 2^{NR'_f} \cdot 2^{N(I(X;Z|U) - R'_f)} \\
&= 2^{NI(X;Z|U)}.
\end{aligned} \tag{43}$$

So, receiver 2 can decode the codeword x^N with error probability approaching zero by using the standard channel coding theorem [11, Theorem 7.7.1]. Therefore, using Fano's inequality, we get

$$H\left(X^N \mid M_1, Z^N, W\right) \longrightarrow 0. \tag{44}$$

Moreover, using the similar deduction in [2, Section 4], we get

$$\begin{aligned}
H\left(Z^N \mid W\right) &\le \log\left\|T^N_{Z|U}\left(\epsilon_1\right)\right\| \\
&\le NH\left(Z \mid U\right).
\end{aligned} \tag{45}$$

Substituting (41), (42), (44), and (45) into (40), we get

$$\begin{aligned}
&H\left(M_1 \mid Z^N\right) \\
&\quad \ge NI\left(X; Y \mid U\right) + NH\left(Z \mid X\right) - NH\left(Z \mid U\right) \\
&\quad = NI\left(X; Y \mid U\right) - NI\left(X; Z \mid U\right),
\end{aligned} \tag{46}$$

where the equality in (46) follows from the Markov chain $U \rightarrow X \rightarrow Z$.

Finally, (5) is verified by substituting (46) into (37):

$$\begin{aligned}
\lim_{N \to \infty} \Delta &= \lim_{N \to \infty} \frac{H\left(S \mid Z^N\right)}{N} \\
&\ge \lim_{N \to \infty} \left(\frac{H\left(M_1 \mid Z^N\right)}{N} + R'_f\right) \\
&\ge \lim_{N \to \infty} \left(\frac{NI\left(X; Y \mid U\right) - NI\left(X; Z \mid U\right)}{N} + R'_f\right) \\
&= I\left(X; Y \mid U\right) - I\left(X; Z \mid U\right) + R'_f \\
&\ge R_e.
\end{aligned} \tag{47}$$

This completes the proof of Theorem 2.

4. Proof of Theorems 4 and 5

In this section, Theorems 4 and 5 are proved. In the model of Figure 3, it is assumed that the channel to receiver 1 is independent of the channel to receiver 2; that is, $p(y, z \mid x) = p(y \mid x)p(z \mid x)$. To prove Theorem 4, we first give the outer bound on the capacity-equivocation region of the *less noisy* DMBCs with noiseless causal feedback in Section 4.1. Then, a coding scheme is provided to achieve the outer bound. Similarly, to prove Theorem 5, the outer bound on the capacity-equivocation region of the *reversely less noisy* DMBCs with noiseless causal feedback is given in Section 4.2. Moreover, we also provide a coding scheme to achieve the outer bound. The methods used to prove the converse parts of the two theorems are from [5]. The coding schemes are inspired by [3, 5].

4.1. Less Noisy DMBCs with Noiseless Causal Feedback. We first show the converse part of Theorem 4, and then we prove the direct part of Theorem 4 by providing a coding scheme.

In order to find the identification of the auxiliary random variables that satisfy the capacity-equivocation region characterized by $\mathscr{R}_l$, we prove the converse part for the equivalent region containing all the rate triples (R_0, R_1, R_e) such that

$$0 \le R_e \le R_1, \tag{48}$$

$$R_0 \le I(U; Z), \tag{49}$$

$$R_0 + R_1 \le I(X; Y \mid U) + I(U; Z), \tag{50}$$

$$R_e \le H(Y \mid Z). \tag{51}$$

The proof of (48), (49), and (50) follows exactly the same line of proving (14), (15), and (16) in Section 3 except for the identification of the auxiliary random variable U, V (which will be given subsequently) and therefore is omitted. We focus on proving (51):

$$\begin{aligned} H\left(S \mid Z^N\right) &\le H\left(S \mid Z^N\right) + I\left(S; Z^N \mid Y^N\right) \\ &= H\left(S \mid Z^N\right) + H\left(S \mid Y^N\right) - H\left(S \mid Y^N, Z^N\right) \\ &= I\left(S; Y^N \mid Z^N\right) + H\left(S \mid Y^N\right) \\ &\le H\left(Y^N \mid Z^N\right) + H\left(S \mid Y^N\right) \\ &= \sum_{i=1}^{N} H\left(Y_i \mid Y^{i-1}, Z^N\right) + H\left(S \mid Y^N\right) \\ &\le \sum_{i=1}^{N} H\left(Y_i \mid Z_i\right) + \epsilon_3, \end{aligned} \tag{52}$$

where ϵ_3 is a small positive number. The last inequality in (52) follows from the fact that conditioning does not increase *entropy* and Fano's inequality. To complete the proof of (51), define a time-sharing random variable Q which is uniformly distributed over $1, 2, \ldots, N$ and independent of $SWX^NY^NZ^N$. Set $U = Z_{Q+1}^N Y^{Q-1} WQ$, $V = US$, $X = X_Q$, $Y = Y_Q$, $Z = Z_Q$. It is easy to see $U \to V \to X \to (Y, Z)$ form a Markov chain. After using the standard time-sharing argument [10, Section 5.4], (52) simplifies to

$$H\left(S \mid Z^N\right) \le NH(Y \mid Z) + \epsilon_3. \tag{53}$$

Finally, utilizing $\lim_{N\to\infty} \Delta \ge R_e$ in the definition of "*achievable*" and (53), we obtain (51). This completes the proof of the converse part of Theorem 4.

Next, a coding scheme is presented to achieve the rate triple $(R_0, R_1, R_e) \in \mathscr{R}_l$. We should prove that all triples $(R_0, R_1, R_e) \in \mathscr{R}_l$ are *achievable*. Note that the noiseless feedback for the less noisy DMBCs is causally transmitted from receiver 1 to the transmitter. The scheme includes codebook generation and encoding scheme in Section 4.1.1, decoding scheme in Section 4.1.2, analysis of error probability in Section 4.1.3, and equivocation analysis in Section 4.1.4. Techniques like block Markov coding, superposition coding, and random binning are used.

To serve the block Markov coding, let random vectors U^N, X^N, Y^N, and Z^N consist of n blocks of length N. Let $W^n \triangleq (W_1, \ldots, W_n)$ stand for the common messages of n blocks, where $W_1, \ldots, W_n$ are independent and identically distributed random variables over $\mathscr{W}$. Let $S^n \triangleq (S_2, \ldots, S_n)$ stand for the confidential messages of n blocks, where $S_2, \ldots, S_n$ are independent and identically distributed random variables over $\mathscr{S}$. Note that in the first block, there is no S_1. Let $\widetilde{Z}^n = (\widetilde{Z}_1, \widetilde{Z}_2, \ldots, \widetilde{Z}_n)$, $\widetilde{Z}^{\bar{b}} = (\widetilde{Z}_1, \widetilde{Z}_2, \ldots, \widetilde{Z}_{b-1}, \widetilde{Z}_{b+1}, \ldots, \widetilde{Z}_n)$, where $\widetilde{Z}_b$ is the output vector at receiver 2 at the end of the bth block, where $1 \le b \le n$. Similarly, $\widetilde{Y}_b$ denotes the output vector at receiver 1 at the end of the bth block, and $\widetilde{X}_b$ denotes the input vector of the channel in the bth block. These notations coincide with [6].

4.1.1. Codebook Generation and Encoding. Let the common message set $\mathscr{W}$ and the confidential message set $\mathscr{S}$ satisfy

$$\begin{aligned} \lim_{N\to\infty} \frac{\log \|\mathscr{W}\|}{N} &= R_0, \\ \lim_{N\to\infty} \frac{\log \|\mathscr{S}\|}{N} &= R_1, \end{aligned} \tag{54}$$

where R_0 and R_1 satisfy (10).

Fix $p(u)$ and $p(x \mid u)$. In the bth block, $1 \le b \le n$, we generate 2^{NR_0} independent and identically distributed (i.i.d) sequences $u^N(w_b)$ according to $\prod_{i=1}^{N} p(u_i)$, where $w_b \in$

$\mathcal{W}$ is the common message to be sent in the bth block. For each $u^N(w_b)$, generate $2^{NI(X;Y|U)}$ codewords $x^N(u^N(w_b))$ according to $\prod_{i=1}^{N} p(x_i \mid u_i)$. Put the $2^{NI(X;Y|U)}$ codewords into 2^{NR_1} bins, so each bin contains $2^{N(I(X;Y|U)-R_1)}$ codewords. The 2^{NR_1} bins are denoted by $Q_1, Q_2, \ldots, Q_{\|\mathcal{S}\|}$, where $\|\mathcal{S}\| = 2^{NR_1}$. The codebook structure is shown in Figure 5. Reveal all the codebooks to the transmitter, receiver 1, and receiver 2.

Let g be a mapping from $\mathcal{Y}^N$ into $\mathcal{S}$. Reveal the mapping g to the transmitter, receiver 1, and receiver 2. Define a random variable $S' = g(Y^N)$ uniformly distributed over $\mathcal{S}$ and independent of the confidential message S. It can be similarly proved from (39) that $S \oplus S'$ is independent of S. In the first block, that is, $b = 1$, to send the common message w_1 (note that there is no confidential message to be sent in the first block), the transmitter tries to find $u^N(w_1)$ and randomly choose a codeword $x^N(u^N(w_1))$ from the corresponding $2^{NI(X;Y|U)}$ codewords. In the bth block ($b = 2, 3, \ldots, n$), to send the common message w_b and confidential message s_b, the transmitter calculates $s'_b = g(\tilde{y}_{b-1})$ and randomly chooses a codeword $x^N(u^N(w_b), s_b)$ from the bin $Q_{s_b \oplus s'_b}$. Here, $\tilde{y}_{b-1}$ is the output vector of the $(b-1)$th block at receiver 1, and $\oplus$ is the modulo addition over $\mathcal{S}$.

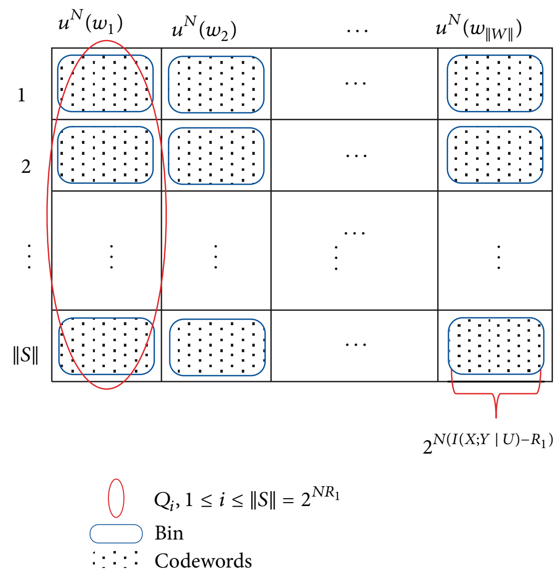


FIGURE 5: The codebook structure.

4.1.2. Decoding. In the first block, as there is no confidential message, only the common message needs to be decoded for both receivers. For receiver 2, he tries to find a unique sequence $u^N(\widehat{\widehat{w}}_1)$ such that $(u^N(\widehat{\widehat{w}}_1), \tilde{z}_1) \in T^N_{UZ}(\epsilon'_1)$, where ϵ'_1 is a small positive number. If there exists such a unique sequence, decoder 2 outputs $\widehat{\widehat{w}}_1$; otherwise, an error is declared. For receiver 1, he tries to find a unique sequence $u^N(\widehat{w}_1)$ such that $(u^N(\widehat{w}_1), \tilde{y}_1) \in T^N_{UY}(\epsilon''_1)$, where ϵ''_1 is a small positive number. If there exists such a unique sequence, output is $\widehat{w}_1$; otherwise, declare an error.

In the bth block, $2 \le b \le n$, receiver 2 aims to decode the common message, and receiver 1 aims to decode both confidential and common messages. The method of decoding the common message w_b for both receivers follows the same as that in the first block. Then, receiver 1 tries to find a unique sequence $x^N(u^N(\widehat{w}_b), \widehat{s}_b)$ such that $(x^N(u^N(\widehat{w}_b), \widehat{s}_b), \tilde{y}_b) \in T^N_{XY|U}(\epsilon'_2)$, where ϵ'_2 is a small positive number. If there exists such a unique sequence in one bin, denoting the corresponding index of that bin by s''_b, receiver 1 calculates $s''_b \ominus s'_b$ as $\widehat{s}_b$ (here $\ominus$ is modulo subtraction over $\mathcal{S}$, and receiver 1 knows $s'_b = g(\tilde{y}_{b-1})$); otherwise, declare an error.

4.1.3. Analysis of Error Probability. Since the number of $u^N(w_b)$ is upper bounded by $2^{NI(U;Z)}$, receiver 2 can decode the common message w_b with error probability approaching zero by applying the standard channel coding theorem [11, Theorem 7.7.1]. Moreover, since the DMBCs under discussion in Section 4.1 are *less noisy*, receiver 1 can also decode the common message with error probability approaching zero. It can be calculated that given the codeword $u^N(w_b)$, the number of x^N is $2^{NI(X;Y|U)}$. So, after determining the codeword $u^N(w_b)$, receiver 1 can decode the codeword x^N with error probability approaching zero by applying the standard channel coding theorem [11, Theorem 7.7.1] and obtain the confidential message with the help of the feedback.

4.1.4. Analysis of Equivocation. In this part, $\lim_{N\to\infty}\Delta \ge R_e$ is proved by utilizing the methods in [5, 6]:

$$
\begin{aligned}
\lim_{N\to\infty} \Delta &= \lim_{N,n\to\infty} \frac{H\left(S^n \mid \tilde{Z}^n\right)}{nN} \\
&= \lim_{N,n\to\infty} \sum_{i=2}^{n} \frac{H\left(S_i \mid S^{i-1}, \tilde{Z}^n\right)}{nN} \\
&=^{(a4.1)} \lim_{N,n\to\infty} \frac{\sum_{i=2}^{n} H\left(S_i \mid \tilde{Z}_i\right)}{nN} \\
&\ge \lim_{N,n\to\infty} \frac{\sum_{i=2}^{n} H\left(S_i \mid \tilde{Z}_i, \tilde{Z}_{i-1}, S_i \oplus S'_i\right)}{nN} \\
&=^{(a4.2)} \lim_{N,n\to\infty} \frac{\sum_{i=2}^{n} H\left(S_i \mid \tilde{Z}_{i-1}, S_i \oplus S'_i\right)}{nN} \\
&=^{(a4.3)} \lim_{N,n\to\infty} \frac{\sum_{i=2}^{n} \min\left\{NH\left(Y \mid Z\right), \log \|\mathcal{S}\|\right\}}{nN} \\
&= \lim_{n\to\infty} \frac{\sum_{i=2}^{n} \min\left\{H\left(Y \mid Z\right), R_1\right\}}{n} \\
&= \min\left\{H\left(Y \mid Z\right), R_1\right\} \\
&\ge^{(a4.4)} R_e.
\end{aligned}
\tag{55}
$$

In the above deduction, $(a4.1)$ follows from $S_i \to \tilde{Z}_i \to (S^{i-1}, \tilde{Z}^{\bar{i}})$. $(a4.2)$ follows from $S_i \to (S_i \oplus S'_i, \tilde{Z}_{i-1}) \to \tilde{Z}_i$. $(a4.3)$ follows from the fact that receiver 2 can choose a better way to intercept the secret key at will and $S_i \oplus S'_i$ is

independent of S_i and uniformly distributed over $\mathcal{S}$. $(a4.4)$ follows from (10).

This completes the proof of Theorem 4.

4.2. Reversely Less Noisy DMBCs with Noiseless Causal Feedback. In this subsection, Theorem 5 will be proved. The converse part will be shown first, and then a coding scheme is given for proving the direct part.

In order to find the identification of the auxiliary random variables that satisfy the capacity-equivocation region characterized by $\mathcal{R}_{rl}$, we prove the converse part for the equivalent region containing all the rate triples (R_0, R_1, R_e) such that

$$0 \le R_e \le R_1, \tag{56}$$

$$R_0 \le I(U;Y), \tag{57}$$

$$R_0 + R_1 \le I(X;Y \mid U) + I(U;Y), \tag{58}$$

$$R_e \le H(Y \mid X). \tag{59}$$

The inequalities (56), (57), and (58) can be proved using similar deduction of the converse part of Theorem 2 in Section 3 except for the identification of the auxiliary random variables. We focus on (59):

$$\begin{aligned} &H\left(S \mid Z^N\right) \\ &\quad = H\left(S \mid X^N, Z^N\right) + I\left(S; X^N \mid Z^N\right) \\ &\quad =^{(b4.1)} H\left(S \mid X^N\right) + I\left(X^N; S \mid Z^N\right) \\ &\quad = H\left(S, Y^N \mid X^N\right) - H\left(Y^N \mid X^N, S\right) \\ &\qquad + I\left(X^N; S \mid Z^N\right) \\ &\quad = H\left(Y^N \mid X^N\right) + H\left(S \mid Y^N, X^N\right) \\ &\qquad - H\left(Y^N \mid X^N, S\right) + I\left(X^N; S \mid Z^N\right) \\ &\quad \le H\left(Y^N \mid X^N\right) + H\left(S \mid Y^N, X^N\right) + I\left(X^N; S \mid Z^N\right) \\ &\quad \le H\left(Y^N \mid X^N\right) + H\left(S \mid Y^N, X^N\right) + H\left(X^N \mid Z^N\right) \\ &\quad =^{(b4.2)} H\left(Y^N \mid X^N\right) + H\left(S \mid Y^N, X^N\right) \\ &\qquad + H\left(X^N \mid Y^N\right) \\ &\quad = H\left(Y^N \mid X^N\right) + H\left(S, X^N \mid Y^N\right) \\ &\quad =^{(b4.3)} H\left(Y^N \mid X^N\right) + H\left(S \mid Y^N\right) \\ &\quad = \sum_{i=1}^{N} H\left(Y_i \mid Y^{i-1}, X^N\right) + H\left(S \mid Y^N\right) \\ &\quad \le^{(b4.4)} \sum_{i=1}^{N} H\left(Y_i \mid X_i\right) + \epsilon_3, \end{aligned} \tag{60}$$

where $(b4.1)$ from the Markov chain $S \rightarrow X^N \rightarrow Z^N$, $(b4.2)$ from the assumption that the channel is *reversely less noisy* (by setting $U = X$), $(b4.3)$ from that X^N is a function of S, Y^N, and $(b4.4)$ from the fact that conditioning does not increase *entropy* and Fano's inequality. To complete the proof of (59), define a time-sharing random variable Q which is uniformly distributed over $\{1, 2, \ldots, N\}$ and independent of $SWX^NY^NZ^N$. Set $U = Z_{Q+1}^N Y^{Q-1} WQ$, $V = US$, $X = X_Q$, $Y = Y_Q$, $Z = Z_Q$. It is easy to see $U \rightarrow V \rightarrow X \rightarrow (Y, Z)$ form a Markov chain. After using the standard time-sharing argument [10, Section 5.4], (60) simplifies to

$$H\left(S \mid Z^N\right) \le NH(Y \mid X) + \epsilon_3. \tag{61}$$

Finally, utilizing $\lim_{N\rightarrow\infty}\Delta \ge R_e$ in the definition of "*achievable*" and (61), we obtain (59). This completes the proof of the converse part of Theorem 5.

Next, a coding scheme will be provided for achieving the triple $(R_0, R_1, R_e) \in \mathcal{R}_{rl}$. We should prove that all triples $(R_0, R_1, R_e) \in \mathcal{R}_{rl}$ are *achievable*. The *codebook generation*, *encoding,* and *decoding* follow exactly the lines of the coding scheme for the *less noisy* case in Section 4.1. We present the *analysis of error probability* and *equivocation* as follows.

4.2.1. Analysis of Error Probability. Since the number of $u^N(w_b)$ is upper bounded by $2^{NI(U;Y)}$, receiver 1 can decode the common message w_b with error probability approaching zero by applying the standard channel coding theorem [11, Theorem 7.7.1]. Moreover, since the DMBCs under discussion in Section 4.2 are *reversely less noisy*, receiver 2 can also decode the common message with error probability approaching zero. It can be calculated that given the codeword $u^N(w_b)$, the number of x^N is $2^{NI(X;Y|U)}$. So, after determining the codeword $u^N(w_b)$, receiver 1 can decode the codeword x^N with error probability approaching zero by applying the standard channel coding theorem [11, Theorem 7.7.1] and obtain the confidential message with the help of the feedback.

4.2.2. Analysis of Equivocation. In this part, $\lim_{N\rightarrow\infty}\Delta \ge R_e$ will be proved. Special attention should be paid to receiver 2 since the DMBCs are *reversely less noisy*; that is, $I(U;Z) \ge I(U;Y)$ for all $p(u,x)$, which implies $2^{NI(X;Z|U)} \ge 2^{NI(X;Y|U)}$. Therefore, receiver 2 can also decode the codeword x^N. With the knowledge of x^N and z^N, receiver 2 can guess receiver 1's channel output y^N from the conditional typical set $\mathcal{T}_{Y|XZ}^N(\epsilon_3)$. Note that receiver 2 can intercept the confidential messages in two ways. One is guessing the secret key s_b' from $\mathcal{S}$ directly; the other is guessing the channel output $\widetilde{y}_{b-1}$ and finding s_b' through $g(\widetilde{y}_{b-1})$ indirectly. Intuitively, receiver 2 will always choose a better way to implement eavesdropping. More formally,

$$\begin{aligned} \lim_{N\rightarrow\infty} \Delta &= \lim_{N,n\rightarrow\infty} \frac{H\left(S^n \mid \widetilde{Z}^n\right)}{nN} \\ &= \lim_{N,n\rightarrow\infty} \sum_{i=2}^{n} \frac{H\left(S_i \mid S^{i-1}, \widetilde{Z}^n\right)}{nN} \end{aligned}$$

$$
\begin{aligned}
&\stackrel{(b4.5)}{=} \lim_{N,n\to\infty} \frac{\sum_{i=2}^{n} H\left(S_i \mid \widetilde{Z}_i, \widetilde{Z}_{i-1}\right)}{nN} \\
&\geq \lim_{N,n\to\infty} \frac{\sum_{i=2}^{n} H\left(S_i \mid \widetilde{Z}_i, \widetilde{Z}_{i-1}, \widetilde{X}_{i-1}, S_i \oplus S_i'\right)}{nN} \\
&\stackrel{(b4.6)}{=} \lim_{N,n\to\infty} \frac{\sum_{i=2}^{n} H\left(S_i \mid \widetilde{Z}_{i-1}, \widetilde{X}_{i-1}, S_i \oplus S_i'\right)}{nN} \\
&\stackrel{(b4.7)}{=} \lim_{N,n\to\infty} \frac{\sum_{i=2}^{n} \min\left\{NH\left(Y \mid XZ\right), \log \|\mathcal{S}\|\right\}}{nN} \\
&\stackrel{(b4.8)}{=} \lim_{N,n\to\infty} \frac{\sum_{i=2}^{n} \min\left\{NH\left(Y \mid X\right), \log \|\mathcal{S}\|\right\}}{nN} \\
&= \lim_{n\to\infty} \frac{\sum_{i=2}^{n} \min\left\{H\left(Y \mid X\right), R_1\right\}}{n} \\
&= \min\left\{H\left(Y \mid X\right), R_1\right\} \\
&\stackrel{(b4.9)}{\geq} R_e.
\end{aligned}
\tag{62}
$$

In the above deduction, $(b4.5)$ follows from $S_i \rightarrow (\widetilde{Z}_i, \widetilde{Z}_{i-1}) \rightarrow (S^{i-1}, \widetilde{Z}^{i-2}, \widetilde{Z}_{i+1}^{n})$. $(b4.6)$ follows from $S_i \rightarrow (S_i \oplus S_i', \widetilde{Z}_{i-1}, \widetilde{X}_{i-1}) \rightarrow \widetilde{Z}_i$. $(b4.7)$ follows from the fact that receiver 2 can choose a better way to intercept the secret key at will, and $S_i \oplus S_i'$ is independent of S_i and uniformly distributed over $\mathcal{S}$. Note that the number of $y^N \in T^N_{Y|XZ}(\epsilon_3)$ is about $2^{NH(Y|XZ)}$ based on the property of strong *typical* sequence [10]. $(b4.8)$ follows from the fact that Y is independent of Z conditioning on X, which is obtained from the assumption $p(y, z \mid x) = p(y \mid x)p(z \mid x)$. $(b4.9)$ follows from (11).

This completes the proof of Theorem 5.

5. Conclusion

This paper studies two models of the DMBCs with noiseless feedback. One is the degraded DMBCs with rate-limited feedback; the other is the *less* and *reversely less noisy* DMBCs with feedback. The difference between them is that the feedback in the first model is independent of the channel outputs and rate limited, while the feedback in the second model is originated causally from the channel outputs. The capacity-equivocation regions of the two models are obtained in this paper. We should point out that the second model studied in this paper is under the assumption that the channel to receiver 1 (the legitimate receiver) is independent of the channel to receiver 2 (the eavesdropper); that is, the channel output Y^N is independent of Z^N given the channel input X^N. However, without this assumption, the capacity-equivocation region remains unknown for the general DMBCs with noiseless feedback.

Acknowledgments

This work was supported in part by the National Science Foundation of China under Grant no. 60932003 and Grant no. 61271220. The authors also would like to thank the anonymous reviewers for helpful comments.

References

[1] A. D. Wyner, “The wire-tap channel,” *The Bell System Technical Journal*, vol. 54, no. 8, pp. 1355–1387, 1975.

[2] I. Csiszár and J. Körner, “Broadcast channels with confidential messages,” *IEEE Transactions on Information Theory*, vol. 24, no. 3, pp. 339–348, 1978.

[3] R. Ahlswede and N. Cai, “Transmission, identification and common randomness capacities for wire-tape channels with secure feedback from the decoder,” in *General Theory of Information Transfer and Combinatorics*, vol. 4123, pp. 258–275, Springer, Berlin, Germany, 2006.

[4] E. Ardestanizadeh, M. Franceschetti, T. Javidi, and Y.-H. Kim, “Wiretap channel with secure rate-limited feedback,” *IEEE Transactions on Information Theory*, vol. 55, no. 12, pp. 5353–5361, 2009.

[5] B. Dai, A. J. Han Vinck, Y. Luo, and Z. Zhuang, “Capacity region of non-degraded wiretap channel with noiseless feedback,” in *Proceedings of the IEEE International Symposium on Information Theory*, pp. 244–248, Cambridge, Mass, USA, July 2012.

[6] B. Dai, A. J. Han Vinck, and Z. Zhuang, “Degraded broadcast channel with side information,confidential messages and noiseless feedback,” in press, http://arxiv.org/abs/1201.2859.

[7] A. El Gamal, “The feedback capacity of degraded broadcast channels,” *IEEE Transactions on Information Theory*, vol. 24, no. 3, pp. 379–381, 1978.

[8] O. Shayevitz and M. Wigger, “On the capacity of the discrete memoryless broadcast channel with feedback,” *IEEE Transactions on Information Theory*, vol. 59, no. 3, pp. 1329–1345, 2013.

[9] R. Venkataramanan and S. Pradhan, “An achievable rate region for the broadcast channel with feedback,” *IEEE Transactions on Information Theory*. In press, http://arxiv.org/abs/1105.2311.

[10] A. El Gamal and Y.-H. Kim, *Network Information Theory*, Cambridge University Press, Cambridge, Mass, USA, 2011.

[11] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, John Wiley & Sons, New York, NY, USA, 1991.

[12] Y. Liang, H. V. Poor, and S. Shamai, “Information theoretic security,” in *Foundations and Trends in Communications and Information Theory*, vol. 5, pp. 355–580, Now Publishers, Hanover, Mass, USA, 2008.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 945342, 7 pages
http://dx.doi.org/10.1155/2013/945342

Research Article

$\mathscr{H}_\infty$ Estimates for Discrete-Time Markovian Jump Linear Systems

Marco H. Terra,[1] Gildson Jesus,[2] and João Y. Ishihara[3]

[1] *Department of Electrical Engineering, University of São Paulo at São Carlos, C.P. 359, 13566-590 São Carlos, SP, Brazil*
[2] *Department of Sciences and Technology, University of Santa Cruz, 45662-900 Ilhéus, BA, Brazil*
[3] *Department of Electrical Engineering, University of Brasília, C.P. 4386, 70.919-970 Brasília, DF, Brazil*

Correspondence should be addressed to Marco H. Terra; terra@sc.usp.br

Received 1 February 2013; Accepted 8 June 2013

Academic Editor: Weihai Zhang

This paper deals with the problem of $\mathscr{H}_\infty$ filtering for discrete-time Markovian jump linear systems. Predicted and filtered estimates are obtained based on the game theory. Both filters are solved through recursive algorithms. The Markovian system considered assumes that the jump parameters are not accessible. Necessary and sufficient conditions are provided to the existence of the filters. A numerical example is provided in order to show the effectiveness of the approach proposed.

1. Introduction

Filtering of Markovian jump linear systems (MJLSs) has been subject of intensive study in the last years. Different and creative approaches to deal with this class of filter have been considered in the literature. One of the main alternatives to solve this kind of problem is based on Kalman filter algorithms, see for instance [1, 2]. Filters proposed in these references are based on linear minimum mean square error estimates of discrete-time MJLS. They present an interesting feature related with recursiveness; however, they are not robust in nature.

When the robustness is relevant to the filtering process and demands an extra performance of the filtering approach, $\mathscr{H}_\infty$ techniques are always considered as one of the best solutions to be adopted. In this way, a closed-loop transfer function from the unknown disturbances to the estimation error is designed in order to satisfy a prescribed $\mathscr{H}_\infty$-norm constraint. In general, algorithms developed to deduce $\mathscr{H}_\infty$ Markovian filters are based on linear matrix inequalities (LMIs), see for instance [3–12]. In particular, [4, 6] assume that the jump parameter of the Markov chain is not available.

In this paper, we propose $\mathscr{H}_\infty$ filters for discrete-time MJLS which are calculated in terms of recursive algorithms based on Riccati equations. Following the approach considered in [13], we develop predicted and filtered estimates based on the two-players game theory. The idea of game theory was also adopted in sthochastic $\mathscr{H}_2/\mathscr{H}_\infty$ control, see for instance [14].

We define the status of the players in the filtering problems considered in this paper in order to reach an equilibrium between two contradictory objectives. The first player can be interpreted as the maximizer of the estimation cost, whereas the second player tries to find an estimate that brings the quadratic cost to a minimum. A solution exists for a specified γ-level if the resulting cost is positive. We assume in these Markovian filtering problems that the jump parameter is not accessible. The recursiveness of the approach we are proposing is the main advantage of these Markovian filters if compared with the filters aforementioned. As a by-product of this approach, we provide necessary conditions for the existence of them based on only known parameters of the Markovian system.

This paper is organized as follows: in Section 2, we present the problem statement; in Section 3, $\mathscr{H}_\infty$ filters for discrete-time MJLS are presented; and in Section 4, a comparative study, based on numerical example, between the approach we are proposing and the filter developed in [4] is performed.

2. Problem Definition

The $\mathscr{H}_\infty$ recursive filters developed in this paper are based on the following discrete-time MJLS:

$$\begin{aligned} x_{i+1} &= F_{i,\Theta_i} x_i + G_{i,\Theta_i} u_i, \quad i = 0, 1 \ldots, \\ y_i &= H_{i,\Theta_i} x_i + D_{i,\Theta_i} w_i, \\ s_i &= L_{i,\Theta_i} x_i + R_{i,\Theta_i} v_i, \end{aligned} \tag{1}$$

where $x_i \in \mathfrak{R}^n$ is the valued state, $y_i \in \mathfrak{R}^m$ is the valued output sequence, $s_i \in \mathfrak{R}^p$ is the valued signal to be estimated, $u_i \in \mathfrak{R}^p$, $w_i \in \mathfrak{R}^q$, and $v_i \in \mathfrak{R}^q$ are random disturbances; Θ_i is a discrete-time Markov chain with finite state space $\{1, \ldots, N\}$ and transition probability matrix $P = [p_{jk}]$. We set $\pi_{i,j} := P(\Theta_i = j)$, $F_{i,k} \in \mathfrak{R}^{n\times n}$, $G_{i,k} \in \mathfrak{R}^{n\times p}$, $H_{i,k} \in \mathfrak{R}^{m\times n}$, $D_{i,k} \in \mathfrak{R}^{m\times q}$, $L_{i,k} \in \mathfrak{R}^{p\times n}$, and $R_{i,k} \in \mathfrak{R}^{p\times q}$, $k \in \{1, \ldots, N\}$ for $i \geq 0$. The random disturbances $\{u_i\}$, $\{w_i\}$, and $\{v_i\}$ are assumed to be null mean with finite second order moments, independent wide sense stationary sequences mutually independent with covariance matrices equal to the U_i, W_i, and V_i, respectively. $x_0 1_{\{\Theta_0=k\}}$ are random vectors with $\mathbb{E}\{x_0 1_{\{\Theta_0=k\}}\} = \mu_k$ (where $1_{\{\cdot\}}$ denotes Dirac measure) and $\mathbb{E}\{x_0 x_0^T 1_{\{\Theta_0=k\}}\} = V_k$; x_0 and $\{\Theta_i\}$ are independent of $\{u_i\}$, $\{w_i\}$ and $\{v_i\}$. A scalar $\gamma > 0$ and a sequence of sets of observations

$$\{y_0\}, \{y_0, y_1\}, \ldots, \{y_0, \ldots, y_l\}, \ldots \tag{2}$$

are defined to find at each instant l a filtered estimate $s_{l|l}$ of s_l such that

$$\sup_{x_0} \frac{\left\| s_{0|0} - L_{0,\Theta_0} x_0 \right\|^2_{V_0^{-1}}}{\left\| x_0 - F_{-1}\overline{x}_0 \right\|^2_{P_0^{-1}} + \left\| y_0 - H_{0,\Theta_0} x_0 \right\|^2_{W_0^{-1}}} < \gamma^2, \tag{3}$$

is satisfied for $l = 0$, and

$$\sup_{\{x_i\}_{i=0}^{l}} \frac{\sum_{i=0}^{l} \left\| s_{i|l} - L_{i,\Theta_i} x_i \right\|^2_{V_i^{-1}}}{\left\| x_0 - F_{-1}\overline{x}_0 \right\|^2_{P_0^{-1}} + \sum_{i=0}^{l} \left\| y_i - H_{i,\Theta_i} x_i \right\|^2_{W_i^{-1}} + \sum_{i=0}^{l-1} \left\| x_{i+1} - F_{i,\Theta_i} x_i \right\|^2_{U_i^{-1}}} < \gamma^2 \tag{4}$$

is satisfied for $l > 0$. For the predicted estimate, given $\gamma > 0$ and a sequence of observations (2), the problem is to find at each instant l a prediction $s_{l+1|l}$ of s_{l+1} such that

$$\sup_{x_0} \frac{\left\| s_{0|-1} - L_{0,\Theta_0} x_0 \right\|^2_{V_0^{-1}}}{\left\| x_0 - F_{-1}\overline{x}_0 \right\|^2_{P_0^{-1}}} < \gamma^2 \tag{5}$$

is satisfied for $l = -1$ and

$$\sup_{\{x_i\}_{i=0}^{l+1}} \frac{\sum_{i=0}^{l+1} \left\| s_{i|l} - L_{i,\Theta_i} x_i \right\|^2_{V_i^{-1}}}{\left\| x_0 - F_{-1}\overline{x}_0 \right\|^2_{P_0^{-1}} + \sum_{i=0}^{l} \left\| y_i - H_{i,\Theta_i} x_i \right\|^2_{W_i^{-1}} + \sum_{i=0}^{l} \left\| x_{i+1} - F_{i,\Theta_i} x_i \right\|^2_{U_i^{-1}}} < \gamma^2 \tag{6}$$

is satisfied for $l \geq 0$. The sequence of solutions $s_{l|l}$ and $s_{l+1|l}$ is outputs of the respective $\mathscr{H}_\infty$ filters. Due to the hybrid nature of this class of system, at each instant of time a new model of a countable set of models is used to calculate the functionals (4) and (6). In general, to synthesize recursive filtering algorithms for this class of problems is not an easy task. Thanks to the augmented model of (1) proposed in [1] we can redefine these functionals in order to develop $\mathscr{H}_\infty$ recursive filters similar to those we find in the literature for standard state-space systems, without jumps. The augmented model of (1) can be written as follows:

$$\begin{aligned} z_{i+1} &= \mathscr{F}_i z_i + \psi_i, \quad i = 0, 1 \ldots \\ y_i &= \mathscr{H}_i z_i + \varphi_i, \\ s_i &= \mathscr{L}_i z_i + \sigma_i, \end{aligned} \tag{7}$$

where the parameter matrices are given by

$$\begin{aligned} \mathscr{F}_i &:= \begin{bmatrix} p_{11}F_{i,1} & \cdots & p_{N1}F_{i,N} \\ \vdots & \ddots & \vdots \\ p_{1N}F_{i,1} & \cdots & p_{NN}F_{i,N} \end{bmatrix}, \\ \mathscr{H}_i &:= \begin{bmatrix} H_{i,1} & \cdots & H_{i,N} \end{bmatrix}, \\ \mathscr{L}_i &:= \begin{bmatrix} L_{i,1} & \cdots & L_{i,N} \end{bmatrix}, \end{aligned} \tag{8}$$

whose dimensions are defined by $\mathscr{F}_i \in \mathfrak{R}^{Nn\times Nn}$, $\mathscr{H}_i \in \mathfrak{R}^{m\times Nn}$, and $\mathscr{L}_i \in \mathfrak{R}^{p\times Nn}$; and the state variable is defined as

$$\begin{aligned} z_i &:= \begin{bmatrix} z_{i,1}^T & \cdots & z_{i,N}^T \end{bmatrix}^T \in \mathfrak{R}^{Nn}, \\ z_{i,k} &:= x_i 1_{\{\Theta_i=k\}} \in \mathfrak{R}^n. \end{aligned} \tag{9}$$

We define for $i \geq 0$ and $k \in \{1, \ldots, N\}$,

$$\begin{aligned} Z_{i,k} &:= \mathbb{E}\left\{ z_{i,k} z_{i,k}^T \right\} \in \mathfrak{R}^{n\times n}, \\ Z_i &:= \mathbb{E}\left\{ z_i z_i^T \right\} = \operatorname{diag}\left[Z_{i,k} \right] \in \mathfrak{R}^{Nn\times Nn}, \end{aligned} \tag{10}$$

where $Z_{i,k}$ is given by the following recursive equation:

$$\begin{aligned} Z_{i+1,k} &= \sum_{j=1}^{N} p_{jk} F_{i,j} Z_{i,j} F_{i,j}^T + \sum_{j=1}^{N} p_{jk} \pi_{i,j} G_{i,j} U_i G_{i,j}^T, \\ Z_{0,k} &= V_k. \end{aligned} \tag{11}$$

We define also in the next lemma, weighting matrices Λ_i, Π_i, and Γ_i as variances of random disturbances given by ψ_i, φ_i, and σ_i, respectively.

Lemma 1. *Let ψ_i, φ_i, and σ_i defined by*

$$\begin{aligned} \psi_i &:= \mathscr{M}_{i+1} z_i + \vartheta_i, \\ \varphi_i &:= D_{i,\Theta_i} w_i, \\ \sigma_i &:= R_{i,\Theta_i} v_i, \end{aligned} \tag{12}$$

for $i \geq 0$ and $j, k \in \{1, \dots, N\}$, where

$$\mathscr{M}_{i+1}^T := \left[\mathscr{M}_{i+1,1}^T \cdots \mathscr{M}_{i+1,N}^T\right],$$

$$\mathscr{M}_{i+1,k} := \left[\left(1_{\{\Theta_{i+1}=k\}} - p_{1k}\right) F_{i,1} 1_{\{\Theta_i=1\}} \cdots \left(1_{\{\Theta_{i+1}=k\}} - p_{Nk}\right) F_{i,N} 1_{\{\Theta_i=N\}}\right], \tag{13}$$

$$\vartheta_i := \begin{bmatrix} 1_{\{\theta(i+1)=1\}} G_{i,\Theta_i} u_i \\ \vdots \\ 1_{\{\theta(i+1)=N\}} G_{i,\Theta_i} u_i \end{bmatrix}.$$

The variances of ψ_i, φ_i, and σ_i can be obtained by the following equations:

$$\Gamma_i := \operatorname{diag}\left[\sum_{j=1}^N p_{jk} F_{i,j} Z_{i,j} F_{i,j}^T\right] - \mathscr{F}_i Z_i \mathscr{F}_i^T + \operatorname{diag}\left[\sum_{j=1}^N p_{jk} \pi_{i,j} G_{i,j} U_i G_{i,j}^T\right], \tag{14}$$

$$\Pi_i := \mathscr{D}_i \mathscr{D}_i^T,$$

$$\Lambda_i := \mathscr{R}_i \mathscr{R}_i^T,$$

respectively, where

$$\mathscr{D}_i := \left[D_{i,1} \pi_{i,1}^{1/2} W_i^{1/2} \ \cdots \ D_{i,N} \pi_{i,N}^{1/2} W_i^{1/2}\right], \tag{15}$$

$$\mathscr{R}_i := \left[R_{i,1} \pi_{i,1}^{1/2} V_i^{1/2} \ \cdots \ R_{i,N} \pi_{i,N}^{1/2} V_i^{1/2}\right]. \tag{16}$$

Proof. The proof follows the arguments proposed in [15], Chapter 3, and in [1]. □

3. $\mathscr{H}_\infty$ Estimates for DMJLS

In this section, we propose recursive $\mathscr{H}_\infty$ estimates for discrete-time MJLS (DMJLS). It is known that for standard state-space systems, this kind of filtering approach is difficult to be implemented *online* due to the fact that we do not know the minimum γ for each step of the recursion. In general, it depends on the estimate error variance matrix which should be calculated at the same instant of time. In this sense, the existence condition of this filter is not known *a priori*. To ameliorate this limitation we provide, for this Markovian problem, necessary conditions to tune this parameter depending on the variances of the random disturbances ψ_i, φ_i, and σ_i and on the known parameter matrices of the augmented model of (1) given in (7).

3.1. $\mathscr{H}_\infty$ Filter. Consider (3) and (4) for the augmented model (7):

$$\sup_{z_0} \frac{\left\|s_{0|0} - \mathscr{L}_0 z_0\right\|^2_{\Lambda_0^{-1}}}{\left\|z_0 - \mathscr{F}_{-1}\bar{z}_0\right\|^2_{\bar{Z}_0^{-1}} + \left\|y_0 - \mathscr{H}_0 z_0\right\|^2_{\Pi_0^{-1}}} < \gamma_a^2, \tag{17}$$

for $l = 0$, and

$$\sup_{\{z_i\}_{i=0}^l} \frac{\sum_{i=0}^l \left\|s_{i|l} - \mathscr{L}_i z_i\right\|^2_{\Lambda_i^{-1}}}{\left\|z_0 - \mathscr{F}_{-1}\bar{z}_0\right\|^2_{\bar{Z}_0^{-1}} + \sum_{i=0}^l \left\|y_i - \mathscr{H}_i z_i\right\|^2_{\Pi_i^{-1}} + \sum_{i=0}^{l-1} \left\|z_{i+1} - \mathscr{F}_i z_i\right\|^2_{\Gamma_i^{-1}}} < \gamma_a^2, \tag{18}$$

for $l > 0$. We can rewrite this $\mathscr{H}_\infty$ filtering problem in terms of the following optimization problem:

$$\min_{\{z_{i|l}\}_{i=0}^l} \max_{\{s_{i|l}\}_{i=0}^l} J_l^f\left(\{y_i\}_{i=0}^l, \{s_{i|l}\}_{i=0}^l, \{z_{i|l}\}_{i=0}^l\right) > 0, \tag{19}$$

where

$$J_0^f := \left\|z_{0|0} - \mathscr{F}_{-1}\bar{z}_0\right\|^2_{\bar{Z}_0^{-1}} + \left\|y_0 - \mathscr{H}_0 z_{0|0}\right\|^2_{\Pi_0^{-1}} - \gamma_a^{-2}\left\|s_{0|0} - \mathscr{L}_0 z_{0|0}\right\|^2_{\Lambda_0^{-1}}, \quad l = 0, \tag{20}$$

$$J_l^f := \left\|z_{0|l} - \mathscr{F}_{-1}\bar{z}_0\right\|^2_{\bar{Z}_0^{-1}} + \sum_{i=0}^l \left\|y_i - \mathscr{H}_i z_{i|l}\right\|^2_{\Pi_i^{-1}} + \sum_{i=0}^{l-1} \left\|z_{i+1|l} - \mathscr{F}_i z_{i|l}\right\|^2_{\Gamma_i^{-1}} - \gamma_a^{-2} \sum_{i=0}^l \left\|s_{i|l} - \mathscr{L}_i z_{i|l}\right\|^2_{\Lambda_i^{-1}}, \quad l > 0. \tag{21}$$

Notice that it is not necessary to maximize J_l^f over $\{s_{i|l}\}_{i=0}^l$ in the optimization problem (19). The solution of the $\mathscr{H}_\infty$ filter is guaranteed if, and only if, there exists a $\{\hat{s}_{i|l}\}_{i=0}^l$ for which $J_l^f(\{y_i\}_{i=0}^l, \{\hat{s}_{i|l}\}_{i=0}^l, \{z_{i|l}\}_{i=0}^l) > 0$ has a minimum $\{\hat{z}_{i|l}\}_{i=0}^l$. Therefore, in order to deal with only the minimization problem (19), for each $l \geq 0$ it is easy to show that (21) can be rewritten as

$$J_l^f := \left(\mathfrak{U}_l \mathfrak{X}_{l|l} - \mathfrak{B}_l\right)^T \mathfrak{R}_l \left(\mathfrak{U}_l \mathfrak{X}_{l|l} - \mathfrak{B}_l\right), \tag{22}$$

where

$$\mathfrak{X}_{l|l} := \begin{bmatrix} z_{l|l} \\ \vdots \\ z_{1|l} \\ z_{0|l} \end{bmatrix}, \qquad \mathfrak{B}_l := \begin{bmatrix} \mathscr{Z}_l \\ \vdots \\ \mathscr{Z}_1 \\ \mathscr{Z}_0 \end{bmatrix},$$

$$\mathfrak{R}_l := \begin{bmatrix} \mathscr{R}_l & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathscr{R}_0 \end{bmatrix},$$

$$\mathfrak{U}_l := \begin{bmatrix} \mathscr{E}_l & \mathscr{A}_{l-1} & 0 & 0 & 0 \\ 0 & \mathscr{E}_{l-1} & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \mathscr{A}_1 & 0 \\ 0 & 0 & 0 & \mathscr{E}_1 & \mathscr{A}_0 \\ 0 & 0 & 0 & 0 & \mathscr{E}_0 \end{bmatrix},$$

$$\mathscr{E}_i := \begin{bmatrix} I \\ \mathscr{H}_i \\ \mathscr{L}_i \end{bmatrix}, \quad \mathscr{R}_i := \begin{bmatrix} \Gamma_{i-1}^{-1} & 0 & 0 \\ 0 & \Pi_i^{-1} & 0 \\ 0 & 0 & -\gamma_a^{-2}\Lambda_i^{-1} \end{bmatrix},$$

$$\mathscr{A}_{i-1} := \begin{bmatrix} -\mathscr{F}_{i-1} \\ 0 \\ 0 \end{bmatrix}, \quad \mathscr{Z}_i := \begin{bmatrix} 0 \\ y_i \\ s_{i|l} \end{bmatrix}, \quad 1 \le i \le l,$$

$$\mathbb{Z}_0 := \begin{bmatrix} \mathscr{F}_{-1}\overline{z}_0 \\ y_0 \\ s_{0|l} \end{bmatrix},$$

$$\Gamma_{-1} := \widetilde{Z}_0. \tag{23}$$

For all $l \ge 0$, we can conclude that the following recurrent relations are valid:

$$\mathfrak{R}_l := \begin{bmatrix} \mathscr{R}_l & 0 \\ 0 & \mathfrak{R}_{l-1} \end{bmatrix}, \qquad \mathfrak{B}_l := \begin{bmatrix} \mathscr{Z}_l \\ \mathfrak{B}_{l-1} \end{bmatrix},$$

$$\mathfrak{X}_{l|l} := \begin{bmatrix} z_{l|l} \\ \mathfrak{X}_{l-1|l-1} \end{bmatrix}, \qquad \mathfrak{U}_l := \begin{bmatrix} \mathscr{E}_l & \alpha_{l-1} \\ 0 & \mathfrak{U}_{l-1} \end{bmatrix}, \tag{24}$$

$$\alpha_{l-1} := \begin{bmatrix} \mathscr{A}_{l-1} & 0 & \cdots & 0 \end{bmatrix}.$$

According to [16], there exists a minimum for (22) if and only if the positiveness of $\mathfrak{U}_l^T \mathfrak{R}_l \mathfrak{U}_l$ is guaranteed.

Lemma 2. *Consider matrices $\mathfrak{U}_l$ and $\mathfrak{R}_l$ and column vectors $\mathfrak{B}_l$ and $\mathfrak{X}_{l|l}$ of appropriate dimensions with $\mathfrak{R}_l$ symmetric. For any $\mathfrak{B}_l$ we have*

$$\inf_x J_l^f > -\infty, \tag{25}$$

if and only if $\mathfrak{U}_l^T \mathfrak{R}_l \mathfrak{U}_l \ge 0$ and $\mathrm{Ker}(\mathfrak{U}_l^T \mathfrak{R}_l \mathfrak{U}_l) \subset \mathrm{Ker}(\mathfrak{R}_l \mathfrak{U}_l)$. *If the minimum is attained, it is unique if and only if $\mathfrak{U}_l^T \mathfrak{R}_l \mathfrak{U}_l > 0$ and the optimal solution is given by $\widehat{\mathfrak{X}}_{l|l} = (\mathfrak{U}_l^T \mathfrak{R}_l \mathfrak{U}_l)^{-1} \mathfrak{U}_l^T \mathfrak{R}_l \mathfrak{B}_l$.*

With this fundamental lemma in mind, we can obtain an existence condition for the filter we are proposing in the following, based on the known parameter matrices of the Markovian model. For $l > 0$, the term $\mathfrak{U}_l^T \mathfrak{R}_l \mathfrak{U}_l$ of $\widehat{\mathfrak{X}}_{l|l}$ can be written as

$$\begin{bmatrix} \mathscr{E}_l^T \mathscr{R}_l \mathscr{E}_l & \mathscr{E}_l^T \mathscr{R}_l \alpha_{l-1} \\ \alpha_{l-1}^T \mathscr{R}_l \mathscr{E}_l & \mathfrak{U}_{l-1}^T \mathfrak{R}_{l-1} \mathfrak{U}_{l-1} + \alpha_{l-1}^T \mathscr{R}_l \alpha_{l-1} \end{bmatrix}, \tag{26}$$

and a necessary condition for $\mathfrak{U}_l^T \mathfrak{R}_l \mathfrak{U}_l > 0$ is the positiveness of $\mathscr{E}_l^T \mathscr{R}_l \mathscr{E}_l$, which in terms of the augmented Markovian model of (1) can be established as follows:

$$\Gamma_{l-1}^{-1} + \mathscr{H}_l^T \Pi_l^{-1} \mathscr{H}_l - \gamma_a^{-2} \mathscr{L}_l^T \Lambda_l^{-1} \mathscr{L}_l > 0. \tag{27}$$

Remark 3. A lower bound for γ_a can be calculated through (27) based on the augmented model (7) and Lemma 1. Notice that (27) is a natural and interesting extension of the filtering of standard state-space systems [13], without jumps, to the DMJLS we are dealing with in this paper.

Now we are in a position to deduce the $\mathscr{H}_\infty$ recursive filtered estimate for DMJLS. The next theorem provides the solution for this problem based on the augmented model (7) and on the sequence of measurements (2) of the original Markovian system (1).

Theorem 4. *Consider the augmented Markovian model* (7). *There exists a recursive $\mathscr{H}_\infty$ filter for this system, defined by the following recursive equations:*

$$\widetilde{Z}_{0|0}^{-1} := \widetilde{Z}_0^{-1} + \mathscr{H}_0^T \Pi_0^{-1} \mathscr{H}_0 - \gamma_a^{-2} \mathscr{L}_0^T \Lambda_0^{-1} \mathscr{L}_0,$$

$$\widetilde{Z}_{l|l}^{-1} := \left(\Gamma_{l-1} + \mathscr{F}_{l-1} \widetilde{Z}_{l-1|l-1} \mathscr{F}_{l-1}^T\right)^{-1} + \mathscr{H}_l^T \Pi_l^{-1} \mathscr{H}_l - \gamma_a^{-2} \mathscr{L}_l^T \Lambda_l^{-1} \mathscr{L}_l, \tag{28}$$

$$\widehat{z}_{0|0} := \left(P_{0|0}^{-1} + \gamma_a^{-2} \mathscr{L}_0^T \Lambda_0^{-1} \mathscr{L}_0\right)^{-1} \times \left(\widetilde{Z}_0^{-1} \mathscr{F}_{-1}^T \overline{z}_0 + \mathscr{H}_0^T \Pi_0^{-1} y_0\right),$$

$$\widehat{z}_{l|l} := \left(\widetilde{Z}_{l|l}^{-1} + \gamma_a^{-2} \mathscr{L}_l^T \Lambda_l^{-1} \mathscr{L}_l\right)^{-1} \times \left(X_{l-1}^{-1} \mathscr{F}_{l-1} \widehat{z}_{l-1|l-1} + \mathscr{H}_l^T \Pi_l^{-1} y_l\right), \tag{29}$$

$$X_{l-1} := \Gamma_{l-1} + \mathscr{F}_{l-1} \widetilde{Z}_{l-1|l-1} \mathscr{F}_{l-1}^T,$$

$$\widehat{s}_{l|l} := \mathscr{L}_l \widehat{z}_{l|l}, \tag{30}$$

if and only if $\widetilde{Z}_{l|l} > 0$, $l = 0, 1, \ldots$, and the attenuation level γ_a is positive.

Proof. This proof follows standard arguments used to deduce Kalman filters and basically consists in checking the positiveness of (26) and in finding the equation of the filter through the Lemma 2. In order to have $\mathfrak{U}_l^T \mathfrak{R}_l \mathfrak{U}_l > 0$, we can rewrite this term as (26). The $(2,2)$ sub block of (26) must be positive definite. Assuming that the positiveness of $\mathfrak{U}_{l-1}^T \mathfrak{R}_{l-1} \mathfrak{U}_{l-1}$ is guaranteed and

$$\alpha_{l-1}^T \mathscr{R}_l \alpha_{l-1} = \begin{bmatrix} \mathscr{F}_{l-1}^T \Gamma_{l-1}^{-1} \mathscr{F}_{l-1} & 0 \\ 0 & 0 \end{bmatrix} \ge 0, \tag{31}$$

the $(2,2)$ term of (26) is positive definite and $\mathfrak{U}_l^T \mathfrak{R}_l \mathfrak{U}_l > 0$ if and only if the Schur complement of the $(2,2)$ block,

$$M_{l|l}^{-1} := \mathscr{E}_l^T \left(\mathscr{R}_l^{-1} + \alpha_{l-1} \left(\mathfrak{U}_{l-1}^T \mathfrak{R}_{l-1} \mathfrak{U}_{l-1}\right)^{-1} \alpha_{l-1}^T\right)^{-1} \mathscr{E}_l, \tag{32}$$

is positive definite. For $l-1$, $M_{l-1|l-1}$ is the $(1,1)$ block of $\left(\mathfrak{U}_{l-1}^T \mathfrak{R}_{l-1} \mathfrak{U}_{l-1}\right)^{-1}$. With α_{l-1} defined in (24), we obtain

$$M_{l|l}^{-1} := \mathscr{E}_l^T \left(\mathscr{R}_l^{-1} + \mathscr{A}_{l-1} M_{l-1|l-1} \mathscr{A}_{l-1}^T\right)^{-1} \mathscr{E}_l. \tag{33}$$

Considering $\widetilde{Z}_{l|l} := M_{l|l}$ in terms of the original data (23), we obtain (28). From Lemma 2 we have the solution for the minimization of (22). Based on the recurrent relations (23)

and (24), considering $\alpha_{l-1}\mathfrak{X}_{l-1|j} = \mathscr{A}_{l-1}x_{l-1|j}$ for $j \geq l-1$, and introducing

$$\begin{bmatrix} \widetilde{Z}_{l|l} & P_{12,l} \\ P_{21,l} & P_{22,l} \end{bmatrix} := \begin{bmatrix} \mathscr{E}_l^T \mathscr{R}_l \mathscr{E}_l & \mathscr{E}_l^T \mathscr{R}_l \alpha_{l-1} \\ \alpha_{l-1}^T \mathscr{R}_l \mathscr{E}_l & \mathfrak{U}_{l-1}^T \mathfrak{R}_{l-1} \mathfrak{U}_{l-1} + \alpha_{l-1}^T \mathscr{R}_l \alpha_{l-1} \end{bmatrix}^{-1}, \tag{34}$$

holds (29) taking into account that $s_{l|l} = \widehat{s}_{l|l} := \mathscr{L}_l \widehat{z}_{l|l}$. □

3.2. $\mathscr{H}_\infty$ Predictor. In this subsection, we present the recursive $\mathscr{H}_\infty$ predictor filter for DMJLS. The original $\mathscr{H}_\infty$ problem (5)-(6) can be rewritten in terms of the following optimization problem:

$$\min_{\{z_{i|l}\}_{i=0}^{l+1}} \max_{\{s_{i|l}\}_{i=0}^{l+1}} J_l^p \left(\{y_i\}_{i=0}^{l}, \{s_{i|l}\}_{i=0}^{l+1}, \{z_{i|l}\}_{i=0}^{l+1} \right) > 0, \tag{35}$$

where

$$\begin{aligned} J_{-1}^p &:= \left\| z_{0|-1} - \mathscr{F}_{-1}\overline{z}_0 \right\|_{\widetilde{Z}_0^{-1}}^2 \\ &\quad - \gamma_a^{-2} \left\| s_{0|-1} - \mathscr{L}_0 z_{0|-1} \right\|_{\Lambda_0^{-1}}^2, \quad l = -1, \\ J_l^p &:= \left\| z_{0|l} - \mathscr{F}_{-1}\overline{z}_0 \right\|_{\widetilde{Z}_0^{-1}}^2 + \sum_{i=0}^{l} \left\| y_i - \mathscr{H}_i z_{i|l} \right\|_{\Pi_i^{-1}}^2 \\ &\quad + \sum_{i=0}^{l} \left\| z_{i+1|l} - \mathscr{F}_i z_{i|l} \right\|_{\Gamma_i^{-1}}^2 \\ &\quad - \gamma_a^{-2} \sum_{i=0}^{l+1} \left\| s_{i|l} - \mathscr{L}_i z_{i|l} \right\|_{\Lambda_i^{-1}}^2, \quad l \geq 0. \end{aligned} \tag{36}$$

The $\mathscr{H}_\infty$ predictor filter exists at the instant l if and only if there exists a sequence $\{\widehat{s}_{i|l}\}_{i=0}^{l+1}$ such that $J_l^p(\{y_i\}_{i=0}^{l}, \{\widehat{s}_{i|l}\}_{i=0}^{l+1}, \{z_{i|l}\}_{i=0}^{l+1})$ has a minimum $\{\widehat{z}_{i|l}\}_{i=0}^{l+1}$ for which

$$J_l^p \left(\{y_i\}_{i=0}^{l}, \{\widehat{s}_{i|l}\}_{i=0}^{l+1}, \{\widehat{z}_{i|l}\}_{i=0}^{l+1} \right) > 0. \tag{37}$$

The prediction optimization problem is equivalent in nature to the filter problem, aforementioned. In the next theorem we present the $\mathscr{H}_\infty$ predictor filter whose proof follows the line of the filtered version.

Theorem 5. *The Markovian $\mathscr{H}_\infty$ prediction problem* (35) *is solvable if, and only if, $\widetilde{Z}_{l+1|l}^{-1} - \gamma_a^{-2}\mathscr{L}_{l+1}^T \Lambda_{l+1}^{-1} \mathscr{L}_{l+1} > 0$, where the sequence $\{\widetilde{Z}_{l+1|l}\}$ and the predictor filter are calculated by the recursions*

$$\begin{aligned} \widetilde{Z}_{0|-1} &:= \widetilde{Z}_0, \\ \widetilde{Z}_{l+1|l} &:= \Gamma_l + \mathscr{F}_l \widetilde{Z}_{l|l-1} \mathscr{F}_l^T - \mathscr{F}_l \widetilde{Z}_{l|l-1} \begin{bmatrix} \mathscr{H}_l \\ \mathscr{L}_l \end{bmatrix}^T \\ &\quad \times W_{e\infty,l}^{-1} \begin{bmatrix} \mathscr{H}_l \\ \mathscr{L}_l \end{bmatrix} \widetilde{Z}_{l|l-1} \mathscr{F}_l, \\ W_{e\infty,l} &:= \begin{bmatrix} \Pi_l & 0 \\ 0 & -\gamma_a^2 \Lambda_l \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathscr{H}_l \\ \mathscr{L}_l \end{bmatrix} \widetilde{Z}_{l|l-1} \begin{bmatrix} \mathscr{H}_l^T & \mathscr{L}_l^T \end{bmatrix}, \\ \widehat{z}_{l+1|l} &:= \mathscr{F}_l \widehat{z}_{l|l-1} \\ &\quad + \mathscr{F}_l M_{l|l-1} \mathscr{H}_l^T \left(\Pi_l + \mathscr{H}_l M_{l|l-1} \mathscr{H}_l^T \right)^{-1} \\ &\quad \times \left(y_l - \mathscr{H}_l \widehat{z}_{l|l-1} \right), \\ M_{l|l-1}^{-1} &:= \widetilde{Z}_{l|l-1}^{-1} - \gamma_a^{-2} \mathscr{L}_l^T \Lambda_l^{-1} \mathscr{L}_l, \\ \widehat{s}_{l+1|l} &:= \mathscr{L}_{l+1} \widehat{z}_{l+1|l}. \end{aligned} \tag{38}$$

In order to recover the original predicted and filtered state estimates of System (1), we calculate the following summations:

$$\widehat{x}_{i|i-1} = \sum_{j=1}^{N} \widehat{z}_{i,j|i-1}, \qquad \widehat{x}_{i|i} = \sum_{j=1}^{N} \widehat{z}_{i,j|i}, \tag{39}$$

where $\widehat{z}_{i|i-1} := \left[\widehat{z}_{i,1|i-1}^T \cdots \widehat{z}_{i,N|i-1}^T \right]^T \in \mathfrak{R}^{Nn}$ and $\widehat{z}_{i,k|i-1} := \widehat{x}_{i|i-1} 1_{\{\Theta_i = k\}} \in \mathfrak{R}^n$. $\widehat{x}_{i|i}$ is defined in the same way.

Remark 6. To establish the stability of the stationary $\mathscr{H}_\infty$ prediction filter, it is assumed that all matrices of (1) and the transition probabilities p_{jk} are time invariant; System (1) is mean square stable (MSS) and its Markov chain $\{\Theta_i\}$ is ergodic. There exists a unique positive-definite solution $\widetilde{Z}$ (with $i \to \infty$) to the algebraic Riccati equation

$$\begin{aligned} \widetilde{Z} &:= \Gamma + \mathscr{F}\widetilde{Z}\mathscr{F}^T - \mathscr{F}\widetilde{Z} \begin{bmatrix} \mathscr{H} \\ \mathscr{L} \end{bmatrix}^T W_{e\infty}^{-1} \begin{bmatrix} \mathscr{H} \\ \mathscr{L} \end{bmatrix} \widetilde{Z}\mathscr{F}, \\ W_{e\infty} &:= \begin{bmatrix} \Pi & 0 \\ 0 & -\gamma_a^2 \Lambda \end{bmatrix} + \begin{bmatrix} \mathscr{H} \\ \mathscr{L} \end{bmatrix} \widetilde{Z} \begin{bmatrix} \mathscr{H}^T & \mathscr{L}^T \end{bmatrix}, \end{aligned} \tag{40}$$

providing that $\widetilde{Z}^{-1} - \gamma_a^{-2}\mathscr{L}^T \Lambda^{-1} \mathscr{L} > 0$, for any γ_a fixed, which guarantees that $W_{e\infty} > 0$ and thus the above inverse is well defined. Following the guidelines defined in [1], the stability of the predictor filter is assured with

$$r_\sigma \left(\mathscr{F} - \mathscr{F} M \mathscr{H}^T \left(\Pi + \mathscr{H} M \mathscr{H}^T \right)^{-1} \mathscr{H} \right) < 1, \tag{41}$$

where $r_\sigma(\cdot)$ denotes spectral radius of the dynamic matrix of the filter with stationary gain. The asymptotic stability of the filtered version can also be assured following this line of argumentation.

Remark 7. Similar to the filtered case, we can also find a necessary existence condition to the Markovian $\mathscr{H}_\infty$ prediction filter given by

$$\begin{aligned} \widetilde{Z}_0^{-1} - \gamma_a^{-2} \mathscr{L}_0^T \Lambda_0^{-1} \mathscr{L}_0 > 0, &\quad l = -1, \\ \Gamma_l^{-1} - \gamma_a^{-2} \mathscr{L}_{l+1}^T \Lambda_{l+1}^{-1} \mathscr{L}_{l+1} > 0, &\quad l \geq 0. \end{aligned} \tag{42}$$

Remark 8. If we consider $\gamma_a \to \infty$, the $\mathscr{H}_\infty$ filter proposed in this paper reduces to the Kalman filter for MJLS proposed in [1]. For the case with no jumps ($N = 1$), the predicted and the filtered $\mathscr{H}_\infty$ estimates reduce to the $\mathscr{H}_\infty$ filters for state-space systems given in [17].

4. Numerical Example

In this section, we compare the filtering approach proposed with the filter developed in [4]. With two Markovian states, the model (1) is defined based on the following parameters:

$$\begin{gathered} P = \begin{bmatrix} 0.9 & 0.1 \\ 0.9 & 0.1 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} 0.7 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \\ F_2 = \begin{bmatrix} 0.6 & 0 \\ 0.1 & 0.2 \end{bmatrix}, \\ G_1 = G_2 = \begin{bmatrix} 0.8731 & 0 \\ 0 & 0.2089 \end{bmatrix}, \\ H_1 = H_2 = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, \\ D_1 = D_2 = \begin{bmatrix} 0.008 & 0 \end{bmatrix}, \\ L_1 = L_2 = \begin{bmatrix} 0.5 & 0 \end{bmatrix}, \\ R_1 = R_2 = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix}. \end{gathered} \tag{43}$$

We compare both filters in the predicted form, written in the following way:

$$\begin{aligned} \widehat{z}_{l+1|l} &= \mathscr{A}_l \widehat{z}_{l|l-1} + \mathscr{B}_l y_l, \\ \widehat{s}_{l+1|l} &= \mathscr{L}_{l+1} \widehat{z}_{l+1|l}, \end{aligned} \tag{44}$$

where

$$\begin{aligned} \mathscr{A}_l &= \mathscr{F}_l - \mathscr{F}_l M_{l|l-1} \mathscr{H}_l^T \left(\Pi_l + \mathscr{H}_l M_{l|l-1} \mathscr{H}_l^T \right)^{-1} \mathscr{H}_l, \\ \mathscr{B}_l &= \mathscr{F}_l M_{l|l-1} \mathscr{H}_l^T \left(\Pi_l + \mathscr{H}_l M_{l|l-1} \mathscr{H}_l^T \right)^{-1}. \end{aligned} \tag{45}$$

The filter of [4] is considered in the strictly proper form as

$$\begin{aligned} x_f(k+1) &= A_f x_f(k) + B_f y(k), \\ z_f(k) &= C_f x_f(k), \end{aligned} \tag{46}$$

whose solution is given in terms of linear matrix inequalities. We show in Figure 1 the root-mean-square errors (rms) of both filters. We performed 1000 Monte Carlo simulations from $i = 0, \ldots, 6$ with the values of Θ_i generated randomly. The initial condition x_0 is considered Gaussian with mean $[0.196 \;\; 0.295]^T$ and variance $\left[\begin{smallmatrix} 0.0384 & 0.0578 \\ 0.0578 & 0.870 \end{smallmatrix}\right]$; $\Theta_i \in \{1, 2\}$, u_i, w_i, and v_i are independent sequences of noises with U_i, W_i, and V_i identity matrices with appropriate dimensions, $\pi_1(0) = 0.05$, and $\pi_2(0) = 0.95$. We obtained $\gamma_a = 1.9523$ for the filter

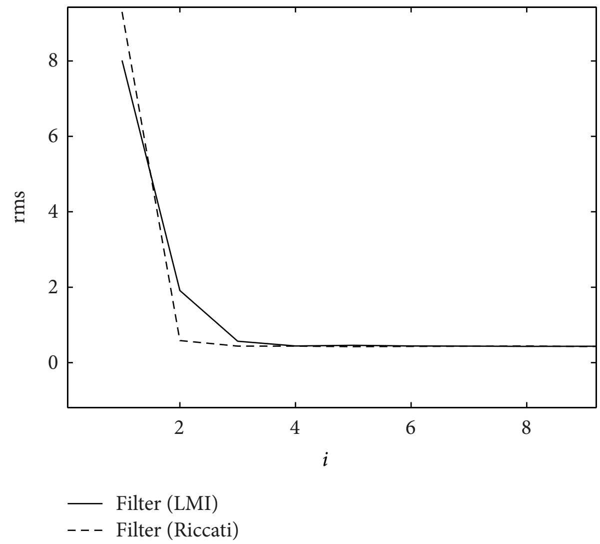


FIGURE 1: Root-mean-square errors for $\mathscr{H}_\infty$ filters based on Riccati recursive equation and linear matrix inequalities.

proposed in this paper and $\gamma = 0.2886$ for the filter proposed in [4]. The parameter matrices of our filter were computed as

$$\begin{gathered} \mathscr{A}_l = \begin{bmatrix} 0.0116 & 0 & -0.0784 & 0 \\ 0.0003 & 0.09 & 0.0003 & 0.18 \\ 0.0013 & 0 & -0.0087 & 0 \\ 0 & 0.01 & 0 & 0.02 \end{bmatrix}, \\ \mathscr{B}_l = \begin{bmatrix} 6.1843 \\ 0.8967 \\ 0.6871 \\ 0.0996 \end{bmatrix}, \qquad \mathscr{L}_{l+1} = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 \end{bmatrix}, \end{gathered} \tag{47}$$

and for the filter of [4] were computed as

$$\begin{gathered} A_f = \begin{bmatrix} -0.2154 & -0.0122 \\ -0.0342 & -0.3654 \end{bmatrix}, \\ B_f = \begin{bmatrix} 9.4196 \\ 2.4656 \end{bmatrix}, \qquad C_f = \begin{bmatrix} 0.4790 & 0.0962 \end{bmatrix}. \end{gathered} \tag{48}$$

In spite of the smaller γ provided by the filter of [4], the rms of both filters are equivalent. This difference in γ is due to the fact that the functionals of both approaches are different in nature. However, if we consider the same numerical value of all parameters but with matrices R_1 and R_2 multiplied by 10^6, we obtain for the proposed recursive filter $\gamma_a = 0.001$. On the other hand, we obtain an infeasible solution with the LMI proposed in Section IV of [4] (computed through Matlab 7.4.0). The mode-independent filter provided in [4] deals with only sufficient conditions. Considering *offline* computations, our approach provides necessary and sufficient conditions for the existence of the filter.

It is important to recall that the proposed approach enables all parameter matrices of (1) to be time varying. It also provides a practical estimate for the minimum admissible γ through the necessary condition (27), which depends on only known parameters of the Markovian system.

5. Conclusion

This paper developed $\mathscr{H}_\infty$ predicted and filtered estimates for DMJLS. They were deduced based on the assumption that jump parameter is not accessible. The numerical example showed the effectiveness of this approach. Through an augmented model of the standard Markovian system, these filters were deduced based on the game theory. It allows us to deal with recursive algorithms to solve this kind of filtering problem where the parameter matrices can be time varying for each operation mode of the Markovian system. As future works, we intend to solve nonlinear filtering and control problems for DMJLS based on the problems raised by [12, 18].

References

[1] O. L. V. Costa and S. Guerra, "Stationary filter for linear minimum mean square error estimator of discrete-time Markovian jump systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 8, pp. 1351–1356, 2002.

[2] M. H. Terra, J. Y. Ishihara, and G. Jesus, "Information filtering and array algorithms for discrete-time Markovian jump linear systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 1, pp. 158–162, 2009.

[3] Y. Dong, J. Sun, and Q. Wu, "$\mathscr{H}_\infty$ filtering for a class of stochastic Markovian jump systems with impulsive effects," *International Journal of Robust and Nonlinear Control*, vol. 18, no. 1, pp. 1–13, 2008.

[4] A. P. C. Gonçalves, A. R. Fioravanti, and J. C. Geromel, "$\mathscr{H}_\infty$ filtering of discrete-time Markov jump linear systems through linear matrix inequalities," *IEEE Transactions on Automatic Control*, vol. 54, no. 6, pp. 1347–1351, 2009.

[5] H. Liu, F. Sun, K. He, and Z. Sun, "Design of $\mathscr{H}_\infty$ reduced-order filter for Markovian jumping systems with time delay," *IEEE Transactions Circuits and Systems*, vol. 51, no. 11, pp. 607–612, 2004.

[6] C. E. Souza, "A mode-independent $\mathscr{H}_\infty$ filter design for discrete-time Markovian jump linear systems," in *Proceedings of the 42nd Conference on Decision and Control*, Maui, Hawaii, USA, December 2003.

[7] C. E. de Souza and M. D. Fragoso, "$\mathscr{H}_\infty$ filtering for Markovian jump linear systems," *International Journal of Systems Science*, vol. 33, no. 11, pp. 909–915, 2002.

[8] C. E. de Souza and M. D. Fragoso, "$\mathscr{H}_\infty$ filtering for discrete-time linear systems with Markovian jumping parameters," *International Journal of Robust and Nonlinear Control*, vol. 13, no. 14, pp. 1299–1316, 2003.

[9] C. E. de Souza, A. Trofino, and K. A. Barbosa, "Mode-independent $\mathscr{H}_\infty$ filters for Markovian jump linear systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 11, pp. 1837–1841, 2006.

[10] F. Sun, H. Liu, K. He, and Z. Sun, "Reduced-order $\mathscr{H}_\infty$ filtering for linear systems with Markovian jump parameters," *Systems & Control Letters*, vol. 54, no. 8, pp. 739–746, 2005.

[11] L. Wu, P. Shi, H. Gao, and C. Wang, "$\mathscr{H}_\infty$ filtering for 2D Markovian jump systems," *Automatica*, vol. 44, no. 7, pp. 1849–1858, 2008.

[12] Z. Lin, Y. Lin, and W. Zhang, "A unified design for state and output feedback $\mathscr{H}_\infty$ control of nonlinear stochastic Markovian jump systems with state and disturbance-dependent noise," *Automatica*, vol. 45, no. 12, pp. 2955–2962, 2009.

[13] J. Y. Ishihara, M. H. Terra, and B. M. Espinoza, "$\mathscr{H}_\infty$ filtering for rectangular discrete-time descriptor systems," *Automatica*, vol. 45, no. 7, pp. 1743–1748, 2009.

[14] B.-S. Chen and W. Zhang, "Stochastic $\mathscr{H}_2/\mathscr{H}_\infty$ control with state-dependent noise," *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp. 45–57, 2004.

[15] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems: Probability and Its Applications*, Springer, London, UK, 2005.

[16] S. Haykin, *Adaptive Filter Theory*, Prentice Hall, Upper Saddle River, NJ, USA, 2001.

[17] B. Hassibi, A. H. Sayed, and T. Kailath, *Indefinite-Quadratic Estimation and Control: A Unified Approach to $\mathscr{H}_2$ and $\mathscr{H}_\infty$ Theories*, SIAM Studies in Applied Mathematics, SIAM, Philadelphia, Pa, USA, 1999.

[18] W. Zhang, B.-S. Chen, and C.-S. Tseng, "Robust $\mathscr{H}_\infty$ filtering for nonlinear stochastic systems," *IEEE Transactions on Signal Processing*, vol. 53, no. 2, part 1, pp. 589–598, 2005.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 241241, 9 pages
http://dx.doi.org/10.1155/2013/241241

Research Article

Control of Traffic Intensity in Hyperexponential and Mixed Erlang Queueing Systems with a Method Based on SPRT

Müjgan Zobu[1] and Vedat Sağlam[2]

[1] *Department of Statistics, Amasya University, 05000 Amasya, Turkey*
[2] *Department of Statistics, Ondokuz Mayıs University, 55139 Samsun, Turkey*

Correspondence should be addressed to Müjgan Zobu; mujganzobu@hotmail.com

Received 6 December 2012; Revised 30 May 2013; Accepted 10 June 2013

Academic Editor: Wuquan Li

The control of traffic intensity (ρ) is one of the important problems in the study of queueing systems. Rao et al. (1984) developed a method to detect changes in the traffic intensity in queueing systems of the $M/G/1$ and $GI/M/c$ types based on the Sequential Probability Ratio Test (SPRT). In this paper, SPRT is theoretically investigated for two different phase-type queueing systems which consist of hyperexponential and mixed Erlang. Also, for testing $H_0 : \rho = \rho_0$ against $H_1 : \rho = \rho_1$, Operating Characteristic (OC) and Average Sample Number (ASN) functions are obtained with numerical methods using multipoint derivative equations according to different situations of α and β type errors. Afterward, numerical illustrations for each model are provided with Matlab programming.

1. Introduction

There are very few studies on the sequential detection of parameter changes of queueing systems in the literature. The theory of SPRT for a sequence of observations forming a finite Markov chain was given in [1]. After, based upon the theory of [1] was discussed statistical quality control and SPRT procedures for the control of traffic intensity in [2–6]. This method aims to detect changes in traffic intensity by observing only the number of customers in the system at successive departure epochs Q_n, which are embedded Markov points. Recently, a SPRT to regulate the traffic intensity based on the number of arrivals during the nth service periods for $M/E_k/1$ queue was proposed in [7] and an autoregressive process based on the number of customers at the departure point and its application to the queueing model were given in [8]. In this paper, the method based on the SPRT is first examined theoretically for the two different queueing phase types. Afterward, the observations obtained for different samples model are evaluated for each of the models by simulation through MATLAB 7.10.0 (R2010a) programming.

2. Queueing Process and SPRT

In many queueing applications, a performance characteristic of great importance and interest is the traffic intensity ρ (the ratio of mean arrival rate λ to mean service rate μ). The purpose of this test is to determine as quickly as possible the changes in traffic intensity and to take the appropriate corrective actions. With this objective, a procedure has been developed for testing the hypothesis $H_0 : \rho = \rho_0$ against $H_1 : \rho = \rho_1$ using Wald's SPRT for the systems $M/G/1$ and $GI/M/c$, in which queue length processes have imbedded Markov chains $\{Q_n\}$ in [6]. This procedure is explained as follows.

Consider the single server queue where arrivals occur according to a Poisson process with rate λ per unit time. The service times of customers are independent identically distributed (i.i.d.) random variables with the distribution $B(x)$. For this system, the queue lengths at service completion points form an imbedded Markov chain. Q_n will be the number of customers left behind by the nth departing customer. The capacity of the queueing system is restricted

to N. Then the transition probability matrix of the imbedded Markov chain is given by

$$P = \{P_{ij}(\rho)\}$$

$$= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \cdot \\ \vdots \\ \cdot \\ \cdot \\ N-1 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & & N-1 \end{array} \\ \begin{bmatrix} k_0 & k_1 & k_2 & \dots & 1-\sum_0^{k-2} k_j \\ k_0 & k_1 & k_2 & \dots & 1-\sum_0^{k-2} k_j \\ 0 & k_0 & k_1 & \dots & 1-\sum_0^{k-3} k_j \\ 0 & 0 & k_0 & \dots & . \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \dots & . \\ 0 & 0 & 0 & \dots & . \\ 0 & 0 & 0 & \dots & 1-k_0 \end{bmatrix} \end{array}, \quad (1)$$

where $k_n = P\{n \text{ arrivals during a service period}\} = \int_0^\infty e^{-\lambda t}((\lambda t)^j/j!)dB(t)$.

Let $t_0 < t_1 < t_2 < \cdots < t_n$ be the set of time points at which the $\mathcal{Q}(t)$ process exhibits the Markov property; namely,

$$\begin{aligned} P\{\mathcal{Q}(t_{n+1}) = j \setminus \mathcal{Q}(t_n) = i, \mathcal{Q}(t_{n-1}) = i_{n-1}, \dots, \mathcal{Q}(t_0) = i_0\} \\ = P\{\mathcal{Q}(t_{n+1}) = j \setminus \mathcal{Q}(t_n) = i\}. \end{aligned} \quad (2)$$

In the $M/G/1$ queue, t_n is the time of service completion of the nth customer and, in the case of the $GI/M/c$ queue, t_n is the arrival epoch of the nth customer. In the case of the $M/M/c/N$ queue, though the process is in Markovian continuous time, for purposes of control it is easier to examine the process at arrival points.

For simplicity, we will use the notation $\mathcal{Q}_n = \mathcal{Q}(t_n)$. Denoted by $P = \{P_{ij}(\rho)\}$ the transition probability matrix of the Markov chain $\{\mathcal{Q}_n\}$ defined over state space $E = \{0, 1, 2, \dots, N\}$, where

$$P_{ij}(\rho) = P\{\mathcal{Q}_{n+1} = j \setminus \mathcal{Q}_n = i; \rho\}. \quad (3)$$

It is assumed that P is known except for the value of the parameter ρ. The problem is to test the hypothesis $H_0 : \rho = \rho_0$ against $H_1 : \rho = \rho_1$.

Consider the sequence of observations $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_n$. The joint probability of observing this sequence under H_0 and H_1 is given by

$$\Pr\{\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_a; \rho_i\} = P(\mathcal{Q}_0; \rho_i) \prod_{j=1}^{n} P(\mathcal{Q}_j \setminus \mathcal{Q}_{j-1}; \rho_i), \quad i = 0, 1. \quad (4)$$

Then the likelihood ratio is

$$L = \frac{P(\mathcal{Q}_0; \rho_1) \prod_{j=1}^{n} P(\mathcal{Q}_j \setminus \mathcal{Q}_{j-1}; \rho_1)}{P(\mathcal{Q}_0; \rho_0) \prod_{j=1}^{n} P(\mathcal{Q}_j \setminus \mathcal{Q}_{j-1}; \rho_0)}. \quad (5)$$

$$Z_0 = \ln \frac{P(\mathcal{Q}_0; \rho_1)}{P(\mathcal{Q}_0; \rho_0)}, \quad (6)$$

$$Z_r = \ln \frac{P(\mathcal{Q}_r \setminus \mathcal{Q}_{r-1}; \rho_1)}{P(\mathcal{Q}_r \setminus \mathcal{Q}_{r-1}; \rho_0)}, \quad (r \ge 1). \quad (7)$$

Let $A = (1-\beta)/\alpha$ and $B = \beta/(1-\alpha)$, where α and β are the probabilities of the errors of the first and second types. Then, Wald's SPRT [9] to test $H_0 : \rho = \rho_0$ against $H_1 : \rho = \rho_1$ becomes as follows.

Observe $\{\mathcal{Q}_i\}$ $(i = 0, 1, 2, \dots)$ successively and at stage $n \ge 1$,

(1) accept H_0 if $\sum_0^n Z_r \le \ln B$;
(2) accept H_1 if $\sum_0^n Z_r \ge \ln A$;
(3) continue by observing $\mathcal{Q}_{n+1}$ if $\ln B < \sum_0^n Z_r < \ln A$.

If we assume $\mathcal{Q}_0 = i_0$ is specified and denote by n_{ij} the number of transitions $i \to j$ up to and including the nth transition, then the likelihood ratio given in (5) reduces to

$$\begin{aligned} L &= \frac{\prod_{i,j} P_{ij}^{n_{ij}}(\rho_1)}{\prod_{i,j} P_{ij}^{n_{ij}}(\rho_0)}, \\ \ln L &= \sum_{i,j} n_{ij} \ln \frac{P_{ij}(\rho_1)}{P_{ij}(\rho_0)}. \end{aligned} \quad (8)$$

Then the SPRT for testing the hypothesis $H_0 : \rho = \rho_0$ against $H_1 : \rho = \rho_1$ will have its continuation region

$$\ln B < \sum_{i,j} n_{ij} \ln \frac{P_{ij}(\rho_1)}{P_{ij}(\rho_0)} < \ln A. \quad (9)$$

In the case of $M/E_k/1$, $E_k/M/c$, and $M/M/c/N$ queues, we will show in the sequel that the logarithm of the likelihood ratio can be written in the form

$$\ln L = an + \sum_{i,j} n_{ij} c_{ij}, \quad (10)$$

where a, c_{ij}, and P_{ij} are constants depending upon the parameters ρ_1, ρ_0, and the transition probabilities P_{ij}. Thus (9) reduces to

$$\ln B - an < \sum_{i,j} n_{ij} c_{ij} < \ln A - an. \quad (11)$$

From this it follows that the continuation region of the test is bounded by straight lines $\ln A - an$ ve $\ln B - an$.

3. Operating Characteristic and Average Sample Number

Approximate formulas for the OC and ASN functions are given in [1]. In order to evaluate the OC and ASN functions,

$\lambda_0(t,\rho)$ is determined as the largest real positive latent root of the matrix

$$P(t) = \left\{ p_{ij}(\rho) \left[\frac{p_{ij}(\rho_1)}{p_{ij}(\rho_0)} \right]^t \right\}, \quad (12)$$

its derivate at $t = 0$ and the nonzero root $t(\rho)$ of the equation $\lambda_0(t,\rho) = 1$.

A numerical search technique was employed to find the root in [10]. For evaluating the derivate of $\lambda_0(t,\rho)$ at $t = 0$, the following five-point numerical differentiation formula has been used:

$$\lambda_0'(t,\rho) \cong \frac{1}{12h}\{\lambda_{-2} - 8\lambda_{-1} + 8\lambda_1 - \lambda_2\}, \quad (13)$$

$$\lambda_0''(t,\rho) \cong \frac{1}{12h^2}\{-\lambda_{-2} + 16\lambda_{-1} - 30\lambda_0 + 16h_1 - \lambda_2\}. \quad (14)$$

When the stated space of $\{\mathcal{Q}_n\}$ is finite, the OC function for the SPRT can be obtained as

$$\begin{aligned} L(\rho) &\cong \frac{A^{t_0(\rho)} - 1}{A^{t_0(\rho)} - B^{t_0(\rho)}} \quad \text{if } t_0(\rho) \neq 0 \\ &\cong \frac{\ln A}{\ln A - \ln B} \quad \text{if } t_0(\rho) = 0, \end{aligned} \quad (15)$$

where $t_0(\rho)$ is the nonzero real root of the equation $\lambda_0(t,\rho) = 1$. The ASN can then be obtained as

$$E(n;\rho) \cong \frac{L(\rho)\ln B + \{1 - L(\rho)\}\ln A}{\lambda_0'(0)} \quad \text{if } \lambda_0' \neq 0 \quad (16)$$

$$\cong \frac{L(\rho)\{\ln B\}^2 + \{1 - L(\rho)\}\{\ln A\}^2}{\lambda_0''} \quad \text{if } \lambda_0' = 0. \quad (17)$$

4. SPRT for Phase-Type Distribution

The exponential distribution is very widely used in performance modelling. The reason, of course, is that mathematical tractability flows from the memoryless property of this distribution. But sometimes mathematical tractability is not sufficient to overcome the need for a model process in which the exponential distribution is simply not adequate. This leads us to explore ways in which we can develop more general distributions while maintaining some of the tractability of the exponential. This is precisely what phase-type distributions permit us to do [11].

4.1. Hyperexponential-2 Service Model: The $M/H_2/1$ Queue. Consider the configuration presented in Figure 1, in which α_1 is the probability that the upper phase is taken and $\alpha_2 = 1 - \alpha_1$ the probability that the lower phase is taken. If such a distribution is used to model a service facility, then a customer entering a service will, with probability α_1, receive the service that is exponentially distributed with parameter μ_1 and then exit the server or else with probability α_2 receive the service that is exponentially distributed with parameter μ_2 and then exit the server. Once again, only one customer

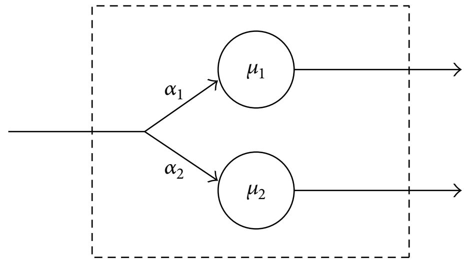


Figure 1: Two exponential phases in parallel.

can be in the process of receiving service at any one time; that is, both phases cannot be active at the same time [11].

Theorem 1. *The density function of the service time is given by*

$$\frac{dB(x)}{dx} = \left(\alpha_1\mu_1 e^{-\mu_1 x} + \alpha_2\mu_2 e^{-\mu_2 x}\right), \quad (x \geq 0). \quad (18)$$

In this case, one has

$$\begin{aligned} k_n = {} & \alpha_1 \left(\frac{\rho_1}{\rho_1 + 1}\right)^n \left(\frac{1}{\rho_1 + 1}\right) \\ & + \alpha_2 \left(\frac{\rho_2}{\rho_2 + 1}\right)^n \left(\frac{1}{\rho_2 + 1}\right), \end{aligned} \quad (19)$$

where $k_n = P$ {n arrivals during a service period}.

Proof. Consider

$$\begin{aligned} k_n &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} \left(\alpha_1\mu_1 e^{-\mu_1 t} + \alpha_2\mu_2 e^{-\mu_2 t}\right) d(t) \\ &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} \alpha_1\mu_1 e^{-\mu_1 t} d(t) \\ &\quad + \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} \alpha_2\mu_2 e^{-\mu_2 t} d(t) \\ &= \frac{\alpha_1\mu_1\lambda^n}{n!} \int_0^\infty e^{-\lambda t} t^n e^{-\mu_1 t} d(t) \\ &\quad + \frac{\alpha_2\mu_2\lambda^n}{n!} \int_0^\infty e^{-\lambda t} t^n e^{-\mu_2 t} d(t) \\ &= \frac{\alpha_1\mu_1\lambda^n}{n!} \int_0^\infty e^{-t(\lambda+\mu_1)} t^n d(t) \\ &\quad + \frac{\alpha_2\mu_2\lambda^n}{n!} \int_0^\infty e^{-t(\lambda+\mu_2)} t^n d(t) \\ &= \frac{\alpha_1\mu_1\lambda^n}{n!} \left(\frac{1}{\lambda+\mu_1}\right)^{n+1} \Gamma(n+1) \\ &\quad + \frac{\alpha_2\mu_2\lambda^n}{n!} \left(\frac{1}{\lambda+\mu_2}\right)^{n+1} \Gamma(n+1) \end{aligned}$$

$$= \frac{\alpha_1\mu_1\lambda^n n!}{n!}\left(\frac{1}{\lambda+\mu_1}\right)^{n+1} + \frac{\alpha_2\mu_2\lambda^n n!}{n!}\left(\frac{1}{\lambda+\mu_2}\right)^{n+1}$$
$$= \alpha_1\left(\frac{\lambda}{\lambda+\mu_1}\right)^n\left(\frac{\mu_1}{\lambda+\mu_1}\right) + \alpha_2\left(\frac{\lambda}{\lambda+\mu_2}\right)^n\left(\frac{\mu_2}{\lambda+\mu_2}\right)$$
$$= \alpha_1\left(\frac{\lambda/\mu_1}{\lambda/\mu_1+\mu_1/\mu_1}\right)^n\left(\frac{\mu_1/\mu_1}{\lambda/\mu_1+\mu_1/\mu_1}\right) + \alpha_2\left(\frac{\lambda/\mu_2}{\lambda/\mu_2+\mu_2/\mu_2}\right)^n\left(\frac{\mu_2/\mu_2}{\lambda/\mu_2+\mu_2/\mu_2}\right)$$
$$= \alpha_1\left(\frac{\rho_1}{\rho_1+1}\right)^n\left(\frac{1}{\rho_1+1}\right) + \alpha_2\left(\frac{\rho_2}{\rho_2+1}\right)^n\left(\frac{1}{\rho_2+1}\right). \tag{20}$$

Here

$$\begin{gathered} P_{ij}(\rho) = k_{j-i+1}, \quad i = 1,2,\ldots,N, \\ j = 0,1,\ldots,N-1, \quad j \ge i-1, \\ P_{0j}(\rho) = k_j, \quad j = 0,1,\ldots,N-1, \\ P_{0N}(\rho) = 1 - \sum_{0}^{N-1} k_n, \\ P_{iN}(\rho) = 1 - \sum_{0}^{N-i} k_n, \quad i = 1,2,\ldots,N, \\ P_{iN}(\rho) = 1 - \sum_{r=0}^{N-i} \alpha_1\left(\frac{\rho_1}{\rho_1+1}\right)^r\left(\frac{1}{\rho_1+1}\right) + \alpha_2\left(\frac{\rho_2}{\rho_2+1}\right)^r\left(\frac{1}{\rho_2+1}\right). \end{gathered} \tag{21}$$

With these values for $P_{ij}(\rho)$, the logarithm of the likelihood ratio will be

$$\ln L = \sum_{i,j} n_{ij} \ln \frac{P_{ij}(\rho_1)}{P_{ij}(\rho_0)},$$
$$\ln L = \sum_{i,j} n_{ij} \ln\left(\left(\alpha_1\left(\frac{\rho_{1(1)}}{\rho_{1(1)}+1}\right)^n\left(\frac{1}{\rho_{1(1)}+1}\right) + \alpha_2\left(\frac{\rho_{2(1)}}{\rho_{2(1)}+1}\right)^n\left(\frac{1}{\rho_{2(1)}+1}\right)\right) \times \left(\alpha_1\left(\frac{\rho_{1(0)}}{\rho_{1(0)}+1}\right)^n\left(\frac{1}{\rho_{1(0)}+1}\right) + \alpha_2\left(\frac{\rho_{2(0)}}{\rho_{2(0)}+1}\right)^n\left(\frac{1}{\rho_{2(0)}+1}\right)\right)^{-1}\right), \tag{22}$$

where

$$c_{ij} = \ln\left(\left(\alpha_1\left(\frac{\rho_{1(1)}}{\rho_{1(1)}+1}\right)^{j-i+1}\left(\frac{1}{\rho_{1(1)}+1}\right) + \alpha_2\left(\frac{\rho_{2(1)}}{\rho_{2(1)}+1}\right)^{j-i+1}\left(\frac{1}{\rho_{2(1)}+1}\right)\right) \times \left(\alpha_1\left(\frac{\rho_{1(0)}}{\rho_{1(0)}+1}\right)^{j-i+1}\left(\frac{1}{\rho_{1(0)}+1}\right) + \alpha_2\left(\frac{\rho_{2(0)}}{\rho_{2(0)}+1}\right)^{j-i+1}\left(\frac{1}{\rho_{2(0)}+1}\right)\right)^{-1}\right),$$
$$i = 1,2,\ldots,N, \; j = 0,1,\ldots,N-1, \; j \ge i-1,$$
$$c_{0j} = \ln\left(\left(\alpha_1\left(\frac{\rho_{1(1)}}{\rho_{1(1)}+1}\right)^{j}\left(\frac{1}{\rho_{1(1)}+1}\right) + \alpha_2\left(\frac{\rho_{2(1)}}{\rho_{2(1)}+1}\right)^{j}\left(\frac{1}{\rho_{2(1)}+1}\right)\right) \times \left(\alpha_1\left(\frac{\rho_{1(0)}}{\rho_{1(0)}+1}\right)^{j}\left(\frac{1}{\rho_{1(0)}+1}\right) + \alpha_2\left(\frac{\rho_{2(0)}}{\rho_{2(0)}+1}\right)^{j}\left(\frac{1}{\rho_{2(0)}+1}\right)\right)^{-1}\right),$$
$$j = 0,1,\ldots,N-1,$$
$$c_{iN} = \ln\left\{\left(1 - \sum_{r=0}^{N-i}\left(\alpha_1\left(\frac{\rho_{1(1)}}{\rho_{1(1)}+1}\right)^r\left(\frac{1}{\rho_{1(1)}+1}\right) + \alpha_2\left(\frac{\rho_{2(1)}}{\rho_{2(1)}+1}\right)^r\left(\frac{1}{\rho_{2(1)}+1}\right)\right)\right) \times \left(1 - \sum_{r=0}^{N-i}\left(\alpha_1\left(\frac{\rho_{0(1)}}{\rho_{0(1)}+1}\right)^r\left(\frac{1}{\rho_{0(1)}+1}\right) + \alpha_2\left(\frac{\rho_{0(1)}}{\rho_{0(1)}+1}\right)^r\left(\frac{1}{\rho_{0(1)}+1}\right)\right)\right)^{-1}\right\},$$
$$i = 1,2,\ldots,N,$$
$$c_{0N} = c_{1N}. \tag{23}$$

□

4.2. Mixed Erlang-k Service Model: The $M/mixE_k/1$ Queue. This is illustrated in Figure 2, where, with probability α, the top series of $k-1$ exponential phases is taken and, with probability $1-\alpha$, the bottom series of k phases is taken.

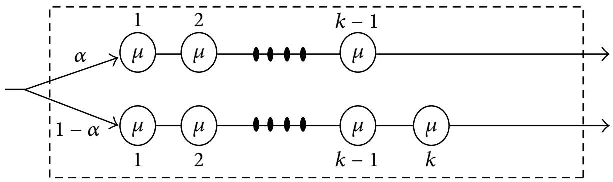


Figure 2: Mixed Erlang representation.

Theorem 2. *The density function of the service time is given by*

$$\frac{dB(x)}{dx} = \alpha\frac{\mu(\mu x)^{k-2}e^{-\mu x}}{(k-2)!} + (1-\alpha)\frac{\mu(\mu x)^{k-1}e^{-\mu x}}{(k-1)!}, \quad (x \geq 0), \tag{24}$$

where $k \geq 2$ is an integer.

In this case, one has

$$k_n = \left(\frac{\rho}{\rho+k}\right)^n\left(\frac{k}{\rho+k}\right)^{k-1}\frac{(n+k-2)!}{n!\,(k-2)!} \times\left(\alpha + (1-\alpha)\frac{(n+k-1)}{(k-1)}\left(\frac{k}{\rho+k}\right)\right), \tag{25}$$

where $k_n = P\{n$ arrivals during a service period$\} = \int_0^\infty e^{-\lambda t}((\lambda t)^j/j!)dB(t)$.

Proof. Consider

$$\begin{aligned}
k_n &= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} \times\left(\alpha\frac{(k\mu t)^{k-2}}{(k-2)!}e^{-ktx}k\mu+(1-\alpha)\frac{(k\mu t)^{k-1}}{(k-1)!}e^{-k\mu t}k\mu\right)d(t)\\
&= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!}\alpha\frac{(k\mu t)^{k-2}}{(k-2)!}e^{-ktx}k\mu\, d(t) + \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!}(1-\alpha)\frac{(k\mu t)^{k-1}}{(k-1)!}e^{-ktx}k\mu\, d(t)\\
&= \frac{\alpha k\mu\lambda^n(k\mu)^{k-2}}{n!\,(k-2)!}\int_0^\infty e^{-\lambda t}t^n e^{-k\mu t}t^{k-2}d(t) + \frac{(1-\alpha)k\mu\lambda^n(k\mu)^{k-1}}{n!\,(k-1)!}\int_0^\infty e^{-\lambda t}t^n e^{-k\mu t}t^{k-1}d(t)\\
&= \frac{\alpha k\mu\lambda^n k\mu^{(k-2)}}{n!\,(k-2)!}\int_0^\infty e^{-t(\lambda+k\mu)}t^{n+k-2}d(t) + \frac{\alpha k\mu\lambda^n k\mu^{(k-1)}}{n!\,(k-1)!}\int_0^\infty e^{-t(\lambda+k\mu)}t^{n+k-1}d(t)\\
&= \frac{\alpha\lambda^n(k\mu)^{k-1}}{n!\,(k-2)!}\int_0^\infty e^{-t(\lambda+k\mu)}t^{n+k-2}d(t) + \frac{\alpha\lambda^n(k\mu)^{k}}{n!\,(k-1)!}\int_0^\infty e^{-t(\lambda+k\mu)}t^{n+k-1}d(t)\\
&= \frac{k\mu\lambda^n k\mu^{(k-1)}}{n!\,(k-1)!}\int_0^\infty e^{-t(\lambda+k\mu)}t^{n+k-1}d(t)
\end{aligned}$$

$\longrightarrow$ Gama distribution $\Gamma(n+k)$

$$\begin{aligned}
&= \frac{\alpha\lambda^n(k\mu)^{k-1}}{n!\,(k-2)!}\left(\frac{1}{\lambda+k\mu}\right)^{n+k-1}\Gamma(n+k-1) + \frac{(1-\alpha)\lambda^n(k\mu)^{k}}{n!\,(k-1)!}\left(\frac{1}{\lambda+k\mu}\right)^{n+k}\Gamma(n+k)\\
&= \frac{(n+k-2)!\alpha\lambda^n(k\mu)^{k-1}}{n!\,(k-2)!}\left(\frac{1}{\lambda+k\mu}\right)^{n+k-1} + \frac{(n+k-1)!\,(1-\alpha)\lambda^n(k\mu)^{k}}{n!\,(k-1)!}\left(\frac{1}{\lambda+k\mu}\right)^{n+k}\\
&= \binom{n+k-2}{n}\left(\frac{1}{\lambda+k\mu}\right)^n\left(\frac{1}{\lambda+k\mu}\right)^{k-1}\alpha(k\mu)^{k-1}\lambda^n + \binom{n+k-1}{n}\left(\frac{1}{\lambda+k\mu}\right)^n\left(\frac{1}{\lambda+k\mu}\right)^{k}(1-\alpha)(k\mu)^{k}\lambda^n\\
&= \alpha\binom{n+k-2}{n}\left(\frac{\lambda}{\lambda+k\mu}\right)^n\left(\frac{k\mu}{\lambda+k\mu}\right)^{k-1} + (1-\alpha)\binom{n+k-1}{n}\left(\frac{\lambda}{\lambda+k\mu}\right)^n\left(\frac{k\mu}{\lambda+k\mu}\right)^{k}.
\end{aligned} \tag{26}$$

If the numerator and denominator are divided by μ,

$$\begin{aligned}
k_n &= \alpha\binom{n+k-2}{n}\left(\frac{\lambda/\mu}{(\lambda+k\mu)/\mu}\right)^n\left(\frac{(k\mu)/\mu}{(\lambda+k\mu)/\mu}\right)^{k-1} + (1-\alpha)\binom{n+k-1}{n}\left(\frac{\lambda/\mu}{(\lambda+k\mu)/\mu}\right)^n\left(\frac{(k\mu)/\mu}{(\lambda+k\mu)/\mu}\right)^{k}\\
&= \alpha\binom{n+k-2}{n}\left(\frac{\rho}{\rho+k}\right)^n\left(\frac{k}{\rho+k}\right)^{k-1} + (1-\alpha)\binom{n+k-1}{n}\left(\frac{\rho}{\rho+k}\right)^n\left(\frac{k}{\rho+k}\right)^{k}
\end{aligned} \tag{27}$$

and cobrackets are taken as follows:

$$k_n = \left(\frac{\rho}{\rho+k}\right)^n\left(\frac{k}{\rho+k}\right)^{k-1} \times\left(\alpha\frac{(n+k-2)!}{n!\,(k-2)!} + (1-\alpha)\frac{(n+k-1)(n+k-2)!}{n!\,(k-1)(k-2)!}\left(\frac{k}{\rho+k}\right)\right)$$

$$= \left(\frac{\rho}{\rho+k}\right)^n \left(\frac{k}{\rho+k}\right)^{k-1} \frac{(n+k-2)!}{n!\,(k-2)!} \times \left(\alpha + (1-\alpha)\frac{(n+k-1)}{(k-1)}\left(\frac{k}{\rho+k}\right)\right). \tag{28}$$

Here

$$\begin{aligned} P_{ij}(\rho) &= k_{j-i+1}, \quad i = 1,2,\ldots,N, \\ & j = 0,1,\ldots,N-1, \quad j \geq i-1, \\ P_{0j}(\rho) &= k_j, \quad j = 0,1,\ldots,N-1, \\ P_{0N}(\rho) &= 1 - \sum_{0}^{N-1} k_n, \\ P_{iN}(\rho) &= 1 - \sum_{0}^{N-i} k_n, \quad i = 1,2,\ldots,N, \\ &= 1 - \sum_{r=0}^{N-i}\left(\frac{\rho}{\rho+k}\right)^r \left(\frac{k}{\rho+k}\right)^{k-1} \binom{n+k-2}{n} \\ &\quad \times \left(\alpha + (1-\alpha)\frac{(r+k-1)}{(k-1)}\left(\frac{k}{\rho+k}\right)\right). \end{aligned} \tag{29}$$

With these values for $P_{ij}(\rho)$, the logarithm of the likelihood ratio will be

$$\ln L = \sum_{i,j} n_{ij} \ln \frac{P_{ij}(\rho_1)}{P_{ij}(\rho_0)} = an + \sum_{i,j} n_{ij} c_{ij},$$

$$\begin{aligned} \ln L &= \sum_{i,j} n_{ij} \ln \left(\left(\left(\frac{\rho_1}{\rho_1+k}\right)^n \left(\frac{k}{\rho_1+k}\right)^{k-1} \frac{(n+k-2)!}{n!\,(k-2)!} \right.\right. \\ &\quad \times \left(\alpha + (1-\alpha)\frac{(n+k-1)}{(k-1)}\left(\frac{k}{\rho_1+k}\right)\right)\Bigg) \\ &\quad \times \left(\left(\frac{\rho_0}{\rho_0+k}\right)^n \left(\frac{k}{\rho_0+k}\right)^{k-1} \frac{(n+k-2)!}{n!\,(k-2)!} \right. \\ &\quad \left.\left. \times \left(\alpha + (1-\alpha)\frac{(n+k-1)}{(k-1)}\left(\frac{k}{\rho_0+k}\right)\right)\right)^{-1}\right), \end{aligned}$$

$$\begin{aligned} \ln L &= \sum_{i,j} n_{ij} \ln \left(\left(\left(\frac{\rho_1}{\rho_1+k}\right)^n \left(\frac{k}{\rho_1+k}\right)^{k-1} \right.\right. \\ &\quad \times \left(\alpha + (1-\alpha)\frac{(n+k-1)}{(k-1)}\left(\frac{k}{\rho_1+k}\right)\right)\Bigg) \\ &\quad \times \left(\left(\frac{\rho_0}{\rho_0+k}\right)^n \left(\frac{k}{\rho_0+k}\right)^{k-1} \right. \\ &\quad \left.\left. \times \left(\alpha + (1-\alpha)\frac{(n+k-1)}{(k-1)}\left(\frac{k}{\rho_0+k}\right)\right)\right)^{-1}\right), \end{aligned} \tag{30}$$

where

$$\begin{aligned} k_n &= n(k-1)\ln\left(\frac{\rho_0+k}{\rho_1+k}\right) + \sum_{i,j} n_{ij} \ln\left(\frac{\rho_1}{\rho_0}\cdot\frac{\rho_0+k}{\rho_1+k}\right)^n + \sum_{i,j} n_{ij} \\ &\quad \times \ln\left(\frac{\alpha + (1-\alpha)((n+k-1)/(k-1))(k/(\rho_1+k))}{\alpha + (1-\alpha)((n+k-1)/(k-1))(k/(\rho_0+k))}\right). \end{aligned} \tag{31}$$

Here

$$a = (k-1)\ln\left(\frac{\rho_0+k}{\rho_1+k}\right),$$

$$\begin{aligned} c_{ij} + b_{ij} &= \ln\left(\frac{\rho_1}{\rho_0}\cdot\frac{\rho_0+k}{\rho_1+k}\right)^{j-i+1} \\ &\quad + \ln\left(\frac{\alpha + (1-\alpha)((j-i+k)/(k-1))(k/(\rho_1+k))}{\alpha + (1-\alpha)((j-i+k)/(k-1))(k/(\rho_0+k))}\right), \\ &\quad i = 1,2,\ldots,N, \; j = 0,1,\ldots,N-1, \; j \geq i-1, \end{aligned}$$

$$\begin{aligned} c_{0j} + b_{0j} &= \ln\left(\frac{\rho_1}{\rho_0}\cdot\frac{\rho_0+k}{\rho_1+k}\right)^{j} \\ &\quad + \ln\left(\frac{\alpha + (1-\alpha)((j+k-1)/(k-1))(k/(\rho_1+k))}{\alpha + (1-\alpha)((j+k-1)/(k-1))(k/(\rho_0+k))}\right), \\ &\quad j = 0,1,\ldots,N-1, \end{aligned}$$

$$\begin{aligned} c_{iN} &= \ln\left\{ \left(1 - \sum_{r=0}^{N-i}\left(\frac{\rho_1}{\rho_1+k}\right)^r \left(\frac{k}{\rho_1+k}\right)^{k-1} \binom{r+k-2}{r} \right.\right. \\ &\quad \times \left(\alpha + (1-\alpha)\frac{(r+k-1)}{(k-1)}\left(\frac{k}{\rho_1+k}\right)\right)\Bigg) \\ &\quad \times \left(1 - \sum_{r=0}^{N-i}\left(\frac{\rho_0}{\rho_0+k}\right)^r \left(\frac{k}{\rho_0+k}\right)^{k-1} \binom{r+k-2}{r} \right. \end{aligned}$$

$$\times\left(\alpha+(1-\alpha)\frac{(r+k-1)}{(k-1)}\left(\frac{k}{\rho_0+k}\right)\right)\Bigg)^{-1}\Bigg\}$$

$$i=1,2,\ldots,N,$$

$$c_{0N}=c_{1N}. \tag{32}$$

□

5. Numerical Results

Consider a $M/H_2/1$ queue (with poisson arrivals, two-phase-type exponential service, fixed N).

Let $\alpha_1 = 0.4$ and $\alpha_2 = 0.6$ be the probability for the upper phase and the probability for the lower phase and $\alpha = 0.005$ and $\beta = 0.05$ the first and second types of errors, respectively. Let $N = 4$ be the capacity of the queuing system.

The mean value of ρ is calculated using the following formula:

$$\rho = \frac{\lambda}{((\alpha_1/\mu_1)+(\alpha_2/\mu_2))}. \tag{33}$$

Suppose that we wish to maintain ρ at the level 0.5 and we wish to detect whether its value has increased. Then, the hypothesis test is $H_0 : \rho_0 = 0.5$ against alternative $H_1 : \rho_1 = 0.8$.

Let $t_0 = 0, t_1, t_2, t_3, \ldots$ be a discrete set of the number of customers remaining at points of departure in the $M/H_2/1$ queue. The number of customers remaining at the 48-point departure is given as follows:

$$\begin{gathered}0\ 0\ 0\ 3\ 2\ 1\ 0\ 1\ 0\ 3\ 3\ 2\ 2\ 1\ 0\ 0\ 1\ 3\ 2\ 1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 2\ 1\ 1\ 0\ 0\ 2\ 1\ 0\ 0.\end{gathered} \tag{34}$$

State space is $E = \{0, 1, 2, 3\}$. Table 1 shows n_{ij} that the number of transitions $i \to j$, and Table 2 shows the value of c_{ij} calculating for SPRT.

From $\ln B < \sum_{i,j} n_{ij}c_{ij} < \ln A$ decision region, $\sum_{i,j} n_{ij}c_{ij}$ is between $\ln A$ and $\ln B$ so that $-2.9907 < -1.1062 < 5.2470$. Therefore, the system is observed again and the number of customers remaining at the 49 points of departure is given as follows:

$$\begin{gathered}0\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 2\ 1\\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 2\ 1\ 0\ 0\ 2\ 2.\end{gathered} \tag{35}$$

Table 3 shows n_{ij} that the number of transitions $i \to j$.

From $\ln B < \sum_{i,j} n_{ij}c_{ij} < \ln A$, the decision region, $\sum_{i,j} n_{ij}c_{ij}$, is small $\ln B$ so that $-4.3517 < -2.9907$. Therefore, the hypothesis H_0 is accepted.

For the computation of the OC and ASN of the SPRT, the largest latent root $\lambda_0(t, \rho)$ of the $P(t)$ matrix has been computed by fixing the values for t and ρ. Figure 3 shows the graphs of $\lambda_0(t, \rho)$ which have been plotted against the values of t for different values of ρ.

Table 1

i	j = 0	1	2	3	
0	17	5	1	2	$n_0 = 25$
1	8	2	1	1	$n_1 = 12$
2	0	5	1	0	$n_2 = 6$
3	0	0	3	1	$n_3 = 4$
					$n = 47$

$\ln L = \sum_{i,j} n_{ij}c_{ij} = -1.1062$.

Table 2: The value of c_{ij} calculating for SPRT.

i	j = 0	1	2	3
0	−0.15871	0.114953	0.348346	0.744993
1	−0.15871	0.114953	0.348346	0.744993
2	0	−0.15871	0.114953	0.523806
3	0	0	−0.15871	0.280086

$\ln L = \sum_{i,j} n_{ij}c_{ij} = -1.1062$.

Table 3

i	j = 0	1	2	3	
0	42	11	4	2	$n_0 = 59$
1	16	6	1	1	$n_1 = 24$
2	0	7	2	0	$n_2 = 9$
3	0	0	3	1	$n_3 = 4$
					$n = 96$

$\ln L = \sum_{i,j} n_{ij}c_{ij} = -4.35$.

As can be seen from Figure 3, there exists one and only one real $t_0 \neq 0$ such that $\lambda_0(t_0) = 1$. The derivative of $\lambda(t, \rho)$ at $t = 0$ has been computed. The OC and ASN functions are then evaluated using the expressions between (13) and (17). The results of the OC and ASN functions for testing $H_0 : \rho_0 = 0.5$ against $H_1 : \rho_1 = 0.8$ are given in Tables 4 and 5 for $\alpha = 0.005$, $\beta = 0.05$ and ve $\alpha = 0.05$, $\beta = 0.1$, respectively.

Figures 4 and 5 give the OC curve and the ASN curve for two different error probabilities in the same figure.

In this example, we wish to maintain ρ at the level 0.5 and we wish to detect whether its value has increased. Then, the hypothesis test is $H_0 : \rho_0 = 0.5$ against alternative $H_1 : \rho_1 = 0.8$. We accepted H_0 hypothesis. There is no need for any change in the system. So traffic intensity (ρ) is the desired level. If any change in ρ, the queueing system would need to be organized. Namely, if ρ has shifted to ρ_1 from ρ_0 ($\rho_1 > \rho_0$), the control action could be to increase the mean μ or add an extra server. If ρ has shifted to ρ_1 from ρ_0 ($\rho_1 < \rho_0$), the control action could be to decrease the mean μ or reduce the number of servers.

We can understand from Table 4 that the number of samples required for this hypothesis to be accepted is 82.1395

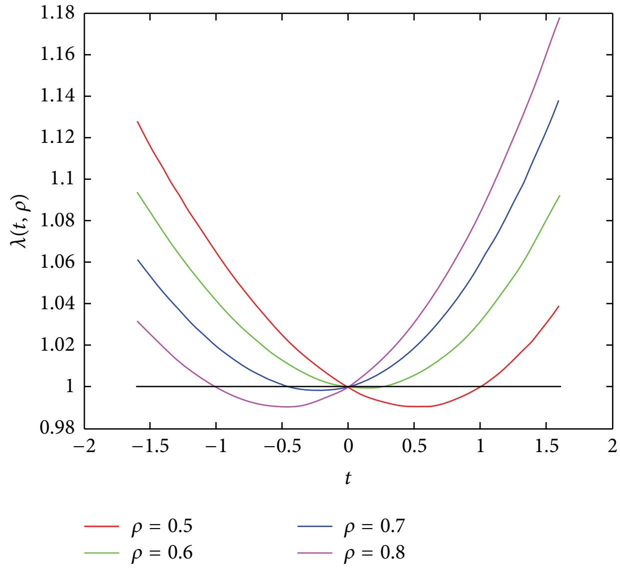


Figure 3: Graph of $\lambda(t, \rho)$ against t for testing $H_0 : \rho_0 = 0.5$, $H_1 : \rho_1 = 0.8$.

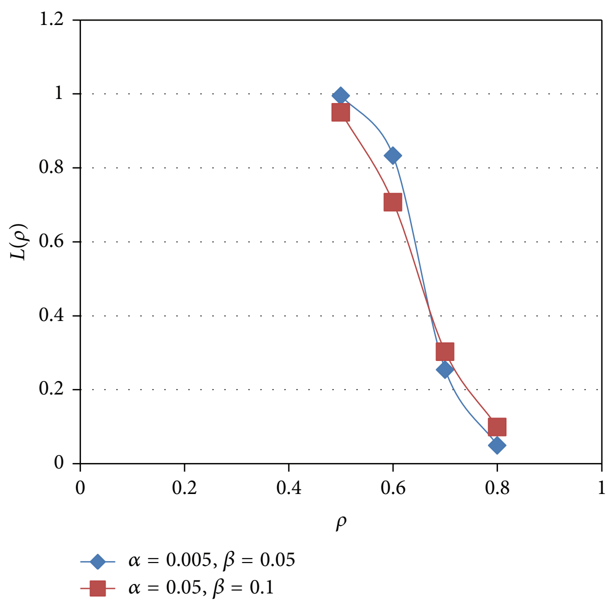


Figure 4: The OC curve for testing $H_0 : \rho_0 = 0.5$ against $H_1 : \rho_1 = 0.8$.

Table 4: The OC and ASN values for $\alpha = 0.005$ and $\beta = 0.05$.

ρ	$t(\rho)$	$\lambda_0'(0, \rho)$	$L(\rho)$	$E(n, \rho)$
0.5	1	−0.0359	0.9951	82.1395
0.6	0.2431	−0.0087	0.8332	184.7906
0.7	−0.4308	0.0165	0.2543	191.3422
0.8	−1	0.0395	0.0496	122.4006

and the number of samples required for it to be rejected is 122.4006.

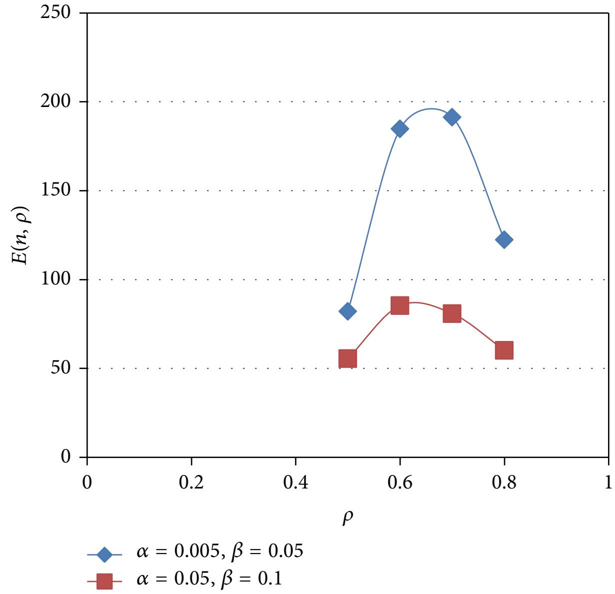


Figure 5: The ASN curve for testing $H_0 : \rho_0 = 0.5$ against $H_1 : \rho_1 = 0.8$.

Table 5: The OC and ASN values for $\alpha = 0.05$ and $\beta = 0.1$.

ρ	$t(\rho)$	$\lambda_0'(0, \rho)$	$L(\rho)$	$E(n, \rho)$
0.5	1	−0.0359	0.9503	55.5729
0.6	0.2431	−0.0087	0.7074	85.3826
0.7	−0.4308	0.0165	0.3031	80.8537
0.8	−1	0.0395	0.0994	60.1863

As can be understood from the tables and Figures 4 and 5, the smaller the error probability needed, the more the sampling number and power of a test are decreased.

6. Conclusions

The objective of the control technique is to detect changes in the traffic intensity ρ as quickly as possible, then take appropriate corrective action, and determine how much of a sample size is needed in the applications. Thus, the sequential probability ratio test provides a saving of up to fifty percent in the sample size according to traditional methods. At the same time, the use of SPRT is easy for observing only the number of customers in the system at successive departure epochs $\mathcal{Q}_n$, which are embedded Markov points.

In this paper, SPRT is theoretically investigated for two different phase-type queueing systems which consist of Hyperexponential and Mixed Erlang. Also the necessary coefficients have been obtained for SPRT.

In the application part, the queue lengths have been generated randomly for different arrival rates (λ), different service rates (μ), and selected system capacities (N) by using MATLAB 7.10.0 (R2010a) programming. Traffic intensities have also been calculated. With the obtained data and predetermined α and β, a simple hypothesis has been established. Accepting or rejecting the hypothesis has been examined by

SPRT. Afterward, the largest latent root $\lambda(t, \rho)$ of the $P(t)$ matrix has been computed by fixing the values for t and ρ. Their graphs have been drawn. It has been found that there exists one and only one real $t_0 \neq 0$ such that $\lambda_0(t_0) = 1$. The OC and ASN have been calculated with obtained values, and their graphs have been drawn by Microsoft Office Excel 2007 programming.

As a result, the length queue observed at the departure points can be maximum N, where $\rho > 1$, and the length queue observed at the departure points can be minimum N, where $\rho < 1$.

According to the given α and β, differences in sample size have been observed. It has been observed that when the error probability decreased the sample size grows and the power of test $(1 - \beta)$ increases. Here, the value of μ needed to calculate the traffic intensity was taken from the mean μ (33). If you recall, the purpose of a control technique is to detect changes in the traffic intensity ρ. When ρ has shifted to ρ_1 from the design level ρ_0 $(\rho_1 < \rho_0)$, an appropriate action is taken to bring ρ_1 back to the design level ρ_0. In the same way, when ρ has shifted to ρ_1 from the design level ρ_0 $(\rho_1 > \rho_0)$, an appropriate action is taken to bring ρ_1 back to the design level ρ_0. Consequently, the control action could be to increase mean μ or decrease mean μ for the phase-type queueing systems.

References

[1] R. M. Phatarfod, "Sequential analysis of dependent observations-I," *Biometrika*, vol. 52, pp. 157–165, 1965.

[2] U. N. Bhat and S. S. Rao, "A statistical technique for the control of traffic intensity in the queuing systems $M/G/1$ and $GI/M/1$," *Operations Research*, vol. 20, pp. 955–966, 1972.

[3] U. N. Bhat, "A statistical technique for the control of traffic intensity in Markovian queue," *Annals of Operations Research*, vol. 8, pp. 151–164, 1987.

[4] U. N. Bhat and S. S. Rao, "Statistical analysis of queueing systems," *Queueing Systems*, vol. 1, no. 3, pp. 217–247, 1987.

[5] K. Harishchandra and S. S. Rao, "Statistical inference about the traffic intensity parameter of $M/E_k/1$ and $E_k/M/1$ queues," Tech. Rep., Indian Institute of Management, Bangalore, India, 1984.

[6] S. S. Rao, U. N. Bhat, and K. Harishchandra, "Control of traffic intensity in a queue—a method based on SPRT," *Opsearch*, vol. 21, no. 2, pp. 63–80, 1984.

[7] S. Jain and J. G. C. Templeton, "Problem of statistical inference to control the traffic intensity," *Sequential Analysis*, vol. 8, no. 2, pp. 135–146, 1989.

[8] S. Jain, "An autoregressive process and its application to queueing mode," *Metron-International Journal of Statistics*, vol. 58, no. 1-2, pp. 131–138, 2000.

[9] A. Wald, *Sequential Analysis*, Dover, New York, NY, USA, 2004.

[10] S. S. Rao, U. N. Bhat, and K. Harishchanra, "OC and ASN of the SPRT for a finite markov chain and an application to the control of queues," Tech. Rep. 3, Indian Institute of Management, Bangalore, India, 1980.

[11] W. J. Stewart, *Probability, Markov Chains, Queues, and Simulation*, Princeton University Press, Princeton, NJ, USA, 2009.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 108386, 13 pages
http://dx.doi.org/10.1155/2013/108386

Research Article

Markov Chain Models for the Stochastic Modeling of Pitting Corrosion

A. Valor,[1] F. Caleyo,[1] L. Alfonso,[2] J. C. Velázquez,[3] and J. M. Hallen[1]

[1] *Departamento de Ingeniería Metalúrgica, IPN-ESIQIE, UPALM s/n, Edificio 7, Zacatenco, 07738 México, DF, Mexico*
[2] *Universidad Autónoma de la Ciudad de México, 09790 México, DF, Mexico*
[3] *Departamento de Ingeniería Química Industrial, ESIQIE-IPN, UPALM Edificio 7, Zacatenco, 07738 México, DF, Mexico*

Correspondence should be addressed to A. Valor; almavalor@gmail.com

Received 1 February 2013; Revised 19 April 2013; Accepted 3 May 2013

Academic Editor: Wuquan Li

The stochastic nature of pitting corrosion of metallic structures has been widely recognized. It is assumed that this kind of deterioration retains no memory of the past, so only the current state of the damage influences its future development. This characteristic allows pitting corrosion to be categorized as a Markov process. In this paper, two different models of pitting corrosion, developed using Markov chains, are presented. Firstly, a continuous-time, nonhomogeneous linear growth (pure birth) Markov process is used to model external pitting corrosion in underground pipelines. A closed-form solution of the system of Kolmogorov's forward equations is used to describe the transition probability function in a discrete pit depth space. The transition probability function is identified by correlating the stochastic pit depth mean with the empirical deterministic mean. In the second model, the distribution of maximum pit depths in a pitting experiment is successfully modeled after the combination of two stochastic processes: pit initiation and pit growth. Pit generation is modeled as a nonhomogeneous Poisson process, in which induction time is simulated as the realization of a Weibull process. Pit growth is simulated using a nonhomogeneous Markov process. An analytical solution of Kolmogorov's system of equations is also found for the transition probabilities from the first Markov state. Extreme value statistics is employed to find the distribution of maximum pit depths.

1. Introduction

Localized corrosion, specifically pitting corrosion of metals and alloys, constitutes one of the main failure mechanisms of corroding structures such as pressurized containers and pipes. Pits cause failure through perforation of the component wall. In other cases, pits become nucleation centers for cracks [1]. In the oil and gas industry, pitting corrosion is a major problem, especially for transporting pipelines [2].

Pitting corrosion comprises two main processes: pit initiation and stable pit growth (pit repassivation is not considered in this paper). It is accepted that pit initiation can be a consequence of the breakdown of the passive layer caused by random fluctuations in local conditions. This process takes some time, usually called induction (nucleation or initiation) period [3]. Passive layer breakdown, followed by localized metal dissolution, is the most common mechanism of pitting corrosion. However, pitting can also occur as the result of the active dissolution of certain regions of the material at its surface, such as nonmetallic inclusions, which are susceptible and dissolve faster than the rest of the surface [4]; in this case, the pitting induction time is typically shorter.

After a pit has nucleated, it can repassivate (stop growing) immediately or grow and then repassivate. This process is regarded as metastable pitting. If a pit is able to grow indefinitely, it becomes a stable growing pit [5]. Those pits that reach the stable growth regime become part of a pit population that shows a remarkable stochastic behavior [6, 7].

The time evolution of pit depth due to corrosion is often expressed as a power function of time [7–11]: $D(t) = \alpha(t - t_{\mathrm{ini}})^{\beta}$, where t is the exposure time, α and β are parameters related to the corrosion process, and t_{ini} stands for pit initiation time. This same function has been found to govern the pitting corrosion growth in stainless and mild steels and aluminum alloys [11, 12].

Pitting corrosion occurs in a wide range of metals and environments. This fact points out to the universality of this phenomenon and suggests that randomness is an inherent and unavoidable characteristic of this damage over time, so that stochastic models are better suited to describe pitting corrosion than deterministic ones. Localized corrosion cannot be explained without assuming stochastic points of view due to the large scatter in the measurable parameters such as corrosion rate, maximum pit depth, and time to perforation [13]. Many variables of the metal-environment system such as alloy composition and microstructure, and composition of the surrounding media and temperature, are all involved in the pitting process [4]. Such complexity imposes the development of theoretical models and simulation tools for a better understanding of the outcome of the pitting corrosion process. These tools help predict more accurately the time evolution of pit depth in corroding structures as the key factor in structural reliability assessment.

Another important characteristic of the pitting corrosion process that is worth noting is the time and pit-depth dependence of the corrosion rate [6, 7]. It has been established that, for a given pit, the growth rate decreases with time, while for pits with equal lifetimes, the corrosion rate is larger for deeper ones.

Provan and Rodriguez [14] are amongst the first authors to use a nonhomogenous Markov process to model pit depth growth. In their model, the authors divided the space of possible pit depths into discrete, non overlapping states and numerically solved the system of Kolmogorov's forward equations (1) for the transition probabilities $p_{i,j}(t)$ between damage states i and j. However, Provan and Rodriguez modeled pitting without taking into account the pit generation process and proposed an expression for the intensities $\lambda_i(t)$ of the process that conveyed no physical meaning. They did not discuss the method used to solve the system of equations either. This has made it impossible to reproduce their results (deeper discussion on this topic can be found in [15])

$$\frac{dp_{i,j}(t)}{dt} = \begin{cases} \lambda_{j-1}(t)\, p_{i,j-1}(t) - \lambda_j(t)\, p_{i,j}(t), & j \geq i+1 \\ \lambda_i(t)\, p_{i,i}(t). \end{cases} \tag{1}$$

Other authors who made use of Markov chains to model corrosion were Morrison and Worthingham [16]. Making use of a continuous time birth process with linear intensity λ, they determined the reliability of high-pressure corroding pipelines. For their purpose, these authors divided the space of the load-resistance ratio into discrete states and numerically solved the Kolmogorov's equations in order to find the intensities of transition between damage states. Afterwards, probability distribution function of the load-resistance ratio was estimated and compared to the distribution obtained from field measurements. Worthingham's model was further improved by Hong [17], who used an analytical solution to the system of Kolmogorov's equations for the same homogeneous continuous type of Markov process and derived the process probability transition matrix in order to evaluate the probability of failure. Hong also investigated the influence of pit depth on the load-resistance ratio. The merits and limitations of these two models are discussed in detail in [15, 18].

In recent years, modeling of pitting corrosion with Markov chains has shown new advances. For example, Bolzoni et al. [19] have modeled the first stages of localized corrosion using a continuous-time, three-state Markov process. The Markov states of the metal surface are passivity, meta stability, and stable pit growth. On the other hand, Timashev and coworkers [20] formulated a model based on the use of a continuous-time, discrete-state pure birth homogenous Markov process for stochastically describing the growth of corrosion-caused metal loss. The goal was to assess the conditional probability of pipeline failure and to optimize the maintenance of operating pipelines. In their model, the intensities of the process were calculated by iteratively solving the proposed system of Kolmogorov's forward equations. The drawbacks of Timashev's model are discussed in detail in [18].

In the present paper, a review of the Markov models developed by the authors in an attempt to describe pitting corrosion is presented. The first model intends to solve a problem that is crucial in reliability-based pipeline integrity management: the accurate estimation of future pit depth and growth rate distributions from a (single) measured or assumed pit depth distribution. It has been recognized [21] that such estimation can be carried out only if oversimplifications are made, or if additional information, besides the pit depth distribution, is available. In the developed model, it is postulated that in the case of external pitting corrosion in underground pipelines, this additional information can be attained from the available predictive models for pit growth as a function of the soil characteristics [8, 22]. A model for pit growth previously developed by the authors has been used to perform Monte Carlo simulations in order to predict the distribution of maximum pit depths as a function of the pipeline age and the physicochemical characteristics of the soil [9].

A nonhomogenous linear growth pure birth Markov process, with discrete states in continuous time, is used to model external pitting corrosion in underground pipelines. The system of forward Kolmogorov's equations (1) is analytically solved using the binomial closed-form solution for the transition probabilities between Markov states in a given time interval. This Markov framework is used to predict the time evolution of the pitting depth and rate distributions. The analytical solution becomes available under the assumption that Markov-derived stochastic mean of the pit depth distribution is equal to the deterministic mean of the distribution obtained through Monte Carlo simulations. This assumption is made for different exposure times and different soil classes defined according to soil physicochemical characteristics that are easy to measure in the field. In this way, the transition probabilities are obtained as a function of soil properties for a given time point, and the corrosion rate and future pit depth distributions are predicted. Real-life case studies are presented to illustrate the proposed Markov model. The main advantage of this model resides in its capability of correctly predicting the time evolution of pit depth and rate distributions over time.

One of the main goals of reliability analysis is to estimate the risk of perforation of in-service components produced by the deepest pit. Extreme value statistics is commonly used together with pit depth growth modeling to predict the risk of failure of in-service components and structures [23, 24].

It is recognized [6, 7] that the deepest pits grow at higher rates than the rest of the defects right from the beginning of the corrosion process, so that the maximum pit depths are commonly sampled from an exponential parent distribution that constitutes the right tail of the pit depth distribution of the whole pit population [7]. Besides that, in the corrosion literature [13, 14, 24, 25], it is a well-established fact that the Gumbel extreme value distribution [26, 27] is a good fit to the maximum pit depth distribution obtained by measuring the maximum pit depth on several areas. The cumulative function of the Gumbel distribution, with location parameter μ and scale parameter σ, is

$$G(x) = \exp\left[-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)\right], \quad -\infty < x < \infty. \quad (2)$$

It is important to underline that the previously mentioned Markov model has proved successful in modeling the time evolution of the entire pit depth population but failed to correctly describe the evolution of pit-depth extremes. The second model presented in this work focuses on the simulation of the time evolution of extreme pit depths. The model is based on the stochastic description of pitting corrosion, taking into account pit initiation and growth. A nonhomogeneous Poisson process is used to model pit initiation. The distribution of pit nucleation times is simulated using a Weibull process. Pit depth growth is also modeled as a nonhomogeneous Markov process. The system of forward Kolmogorov's equations (1) is solved analytically for the transition probability from the first state to any jth state during a given time interval [28]. From this solution, the cumulative distribution function of pit depth for the one-pit case can be found. This distribution function has an exponential character and corresponds to the parent distribution from which the extremes can be readily drawn. The extreme depths distribution for the m-pit case is obtained by the multiplication of the m parent cumulative distribution functions of the pits population. This stochastic model is able to predict the time evolution of the probability distribution of maximum pit depths.

2. Stochastic Models of Pitting Corrosion

In this section, two different Markov chain models are presented to describe pitting corrosion. The first one is focused on the description of time evolution of pit depth and rate distributions in operating underground pipelines. The second model deals with the description of maximum pit depths when multiple corrosion pits are taken into account in a laboratory (controlled) pitting corrosion experiment.

In the following, only the definitions relevant to the focus of the presented models are given. The reader is referred to [28, 29] for the formalism theory of Markov processes.

In both models, the possible pit depths constitute the Markov space. The material thickness (along which pits propagate) is divided into N discrete Markov states. The corrosion damage (pit) depth, at time t, is represented by a discrete random variable $D(t)$ such that $P\{D(t) = i\} = p_i(t)$, with $i = 1, 2, \ldots, N$. Furthermore, it is postulated that the probability that the damage that is at the ith state at the moment t increases by one state in a very small interval of time δt is expressed as $\lambda_i(t)\delta t + o(\delta t)$. For a continuous-time, nonhomogenous linear growth Markov process with intensities $\lambda_i(t) = i\lambda(t)$, the probability $p_{i,j}(t)$ that the process in state i will be in state j ($j \geq i$) at some later time obeys the system of Kolmogorov's forward equations presented in (1). In this infinitesimal transition scheme, $\lambda(t)$ can be interpreted as the jump frequency between the ith to the $(i+1)$th states during the time interval $[t, t+\delta t]$. This means that the number of states transited by the corrosion pit in a short time interval $[0, t]$ can be written as

$$\rho(t) = \int_0^t \lambda(\tau)\, d\tau. \quad (3)$$

It is not difficult to find out that the functions $\lambda(t)$ and $\rho(t)$ have direct physical meaning when Markov processes are used to model pitting corrosion. They are related to the pit growth rate and pit depth, respectively.

2.1. Markov Chain Modeling of Pitting Corrosion Depth and Rate in Underground Pipelines. In [28, page 304], it is shown that for a Markov process defined by the system of (1), the conditional probability $p_{m,n}(t_0, t) = P\{D(t) = n \mid D(t_0) = m\}$ of transition from the mth state to the nth state ($n \geq m$) in the interval (t_0, t) can be obtained analytically and has the form

$$p_{m,n}(t_0, t) = \binom{n-1}{n-m} e^{-\{\rho(t)-\rho(t_0)\}m}\left(1 - e^{-\{\rho(t)-\rho(t_0)\}}\right)^{n-m}, \quad (4)$$

where $\rho(t)$ is defined by (3).

Equation (4) shows that the increase in pit depth over $\Delta t = t - t_0$ follows a negative binomial distribution $\mathrm{NegBin}(r, p)$ with parameters $r = m$ and $p = p_s = e^{-\{\rho(t)-\rho(t_0)\}}$. From the transition probability $p_{m,n}(t_0, t)$ it is possible to estimate the probability distribution function $f(v)$ of the pitting corrosion rate v over the time interval Δt, when the pit depth is at the mth state as

$$f(v; m, t_0, t) = p_m(t_0)\, p_{m,m+v\Delta t}(t_0, t)\, \Delta t. \quad (5)$$

From the distribution function $f(v; m, t_0, t)$, it is straightforward to derive the pitting rate probability distribution associated with the entire pit population (all possible depths) as

$$f(v; t_0, t) = \sum_{m=1}^{N} f(v; m, t_0, t). \quad (6)$$

Until this point, we have the transition probabilities from any state m to the state n in the interval (t_0, t), given by (4). The corrosion rate distribution can also be derived through (6). In principle, if the probability distribution of the

corrosion depth at t_0 is known, that is, $P\{D(t_0) = m\} = p_m(t_0)$, the pit depth distribution at any future moment in time can be estimated using

$$p_n(t) = \sum_{m=1}^{n} p_m(t_0)\, p_{m,n}(t_0, t). \tag{7}$$

For the case of buried pipelines, the initial probability distribution of pit depths $p_m(t_0)$ can be obtained if the corrosion damage in the pipeline is monitored using in-line inspection (ILI), being t_0 the time of the inspection. The values of the probabilities p_m can be estimated from the ratio of the number of corrosion pits with depths in the mth state to the total number of observed pits.

It is evident from (4) that the transition probabilities $p_{m,n}(t_0, t)$ depend on the functions $\lambda(t)$ and $\rho(t)$. At this point, physically sound expression for $\rho(t)$ and $\lambda(t)$ should be proposed in order to estimate the evolution of the pit depth and corrosion rate distributions. For that, the knowledge about the pitting corrosion process in soils must be used. It is postulated that the stochastic mean pit depth $M(t)$ can be assumed to be equal to the deterministic mean $Đ(t)$ of the pitting process, which can be estimated from the observed evolution of the average pit depth over time as

$$M(t) = Đ(t). \tag{8}$$

This equality is true under certain simple assumptions that are explained in [29]. In summary, a sufficient condition for the equality of the stochastic and deterministic means is that for any positive integer q, the structure of the process starting from n individuals is identical to that of the sum of q separated systems each starting from n.

Cox and Miller [29] have shown that if the initial damage state is n_i at $t = t_i$, so that $D(t_i) = n_i$, then the time-dependent stochastic mean $M(t) = E[D(t)]$ of the linear growth Markov process can be expressed as

$$M(t) = n_i e^{\rho(t-t_i)}. \tag{9}$$

If we consider that a power function is an accurate deterministic representation of the pit growth process, one can postulate that the deterministic mean pit depth at time t is [8]

$$Đ(t) = \kappa\left(t - t_{\text{sd}}\right)^{\nu}, \tag{10}$$

where κ and ν are the pitting proportionality and exponent parameters, respectively, and t_{sd} is the starting time of the pitting corrosion process. In systems where passivity breakdown and/or inclusion dissolution are the prevalent mechanisms for pit initiation, t_{sd} would represent the initiation time of stable pit growth. In the case of underground pipelines, this parameter corresponds to the total elapsed time from pipeline commissioning to coating damage plus the time period when the cathodic protection is still effective preventing or attenuating external pitting corrosion after coating damage.

The increase of the deterministic mean pit depth in the time interval Δt is

$$\Delta Đ(t) = ƛ(t)\, Đ(t)\, \Delta t, \tag{11}$$

where $ƛ$ can be interpreted as the deterministic intensity (rate) of the process. The pitting rate, obtained by taking the time derivative of $Đ(t)$ (10), obeys the functional form of (11) with $ƛ = \nu/\tau$ and $\tau = t - t_{\text{sd}}$ as

$$\frac{dĐ(t)}{dt} = \frac{\nu}{\tau} Đ(t). \tag{12}$$

If δt represents the stay time of the corrosion pit in the first state of the chain, then $n_i = 1$ during the time interval between t_{sd} and $t_{\text{sd}} + \delta t$. If δt is significantly less than the simulation time span, it is easy to show from (8)–(10) that the value of the function $\rho(t)$ can be approximated as

$$\rho(t) = \ln\left[\kappa\left(t - t_{\text{sd}}\right)^{\nu}\right]. \tag{13}$$

From this, taking into account (3), it follows that

$$\lambda(t) = \frac{\nu}{t - t_{\text{sd}}}. \tag{14}$$

Note that the intensity of the Markov process $\lambda(t)$ is inversely proportional to the exposure time τ, as it is the case of the deterministic intensity $ƛ(t)$ according to (11) and (12).

One can substitute the expression for $\rho(t)$ from (13) into (4) and to show that the probability parameter $p_s = e^{-\{\rho(t)-\rho(t_0)\}}$ can be expressed as

$$p_s = \left(\frac{t_0 - t_{\text{sd}}}{t - t_{\text{sd}}}\right)^{\nu}, \quad t \geq t_0 \geq t_{\text{sd}}. \tag{15}$$

Suppose that a pit is in the state m at t_0. Let $E[n-m]/\Delta t$ be the average damage rate in the time interval $(t_0, t_0 + \Delta t)$. It can be shown [19] that the instantaneous pitting rate υ predicted by the stochastic model is

$$\upsilon(m, t_0) = \frac{E[n-m]}{\Delta t} \xrightarrow[\Delta t \to 0]{} \nu \frac{m}{t_0}. \tag{16}$$

The stochastically predicted instantaneous damage rate agrees with the deterministic rate given by (12). This coincidence is a sign of the adequacy of the proposed Markov pitting-corrosion model.

For the case of underground pipelines, the pitting corrosion damage evolution can be undertaken as follows. The measured or assumed pit depth probability distribution at t_0 is used as the initial corrosion damage distribution $p_m(t_0)$, where t_0 is the time of the pit depth measurement. The value of the probabilities p_m is estimated from the ratio of the number of corrosion pits with depths in the mth state to the total number of observed pits. The transition probability function $p_{m,n}(t_0, t)$ can be identified from (4) and (15) if the parameters t_{sd} and ν are known.

The estimation of the pitting parameters is possible thanks to the previously developed [8, 9] predictive model for localized corrosion in buried pipelines, which relates the physicochemical conditions of the pipe and soil to parameters t_{sd}, κ and ν. The model is based on a multivariate nonlinear regression analysis using (10), with the field-measured maximum pit depths as the dependent variable to be predicted and

Table 1: Estimated pit initiation times for underground pipelines (from [8]).

Soil class	t_{sd} (years)
Clay	3.0
Clay loam	3.1
Sandy clay loam	2.6
All	2.9

the pipe age and soil-pipe characteristics as the independent ones. The experimental details and corrosion data used to produce the model can be found elsewhere [8, 9]. From this model, the pitting initiation time was estimated to have the values displayed in Table 1 for different soil classes. In Table 1, class "All" corresponds to a general class that includes all the soils found in the field survey carried out in South Mexico [8]. The pitting parameters κ and ν in (10) were estimated as linear combinations of soil and pipe characteristics [8] for the different soil classes.

Later on, Caleyo et al. [9] used a Monte Carlo simulation based on (10) and on the obtained linear combinations of environmental variables for the pitting parameters to predict the time evolution of the maximum pit depth probability distribution in underground pipelines. The measured distributions of the considered soil-pipe independent variables were used as inputs in Monte Carlo simulations of the pitting process for each soil class to produce the simulated extreme pit depth distribution for each soil category and exposure time considered. The reader is referred to [9] for details about the used pipe-soil properties distributions and the Monte Carlo simulation framework. The Monte Carlo produced extreme pit depth values for each soil class at different time points were fitted to the generalized extreme value (GEV) distribution [30]:

$$\mathrm{GEV}(x) = \exp\left\{-\left[1+\zeta\left(\frac{x-\mu}{\sigma}\right)\right]^{-1/\zeta}\right\}, \qquad (17)$$

where ζ, σ, and μ are the shape, scale, and location parameters of the distribution, respectively.

Figure 1(a) shows the time evolution of the GEV distribution fitted to the Monte Carlo predicted extreme pit depths for a general soil class found in southern Mexico [8]. The inset of Figure 1(a) shows the evolution of the mean and variance of such distributions with time. These average values are used to estimate the pitting proportionality and exponent factors (κ and ν) associated with the typical (average) values of the predictor variables in this soil category. The time evolution of the mean of the simulated maximum pit depth distributions was fitted to (10) using the corresponding value of t_{sd} given in Table 1. Typical values κ_{Typ} and ν_{Typ}, derived following this method, are displayed in Table 2 for the analyzed soil textural classes.

From the estimated values of κ and ν for the general soil class and the value of t_{sd} shown in the last column of Table 1, it is possible to obtain the function $\rho(t)$ (13) and p_s (15) for typical conditions of the "All" soil category [8, 9].

Figure 1(b) shows the time evolution of $\rho(t)$ and p_s predicted from the parameters associated with the pitting

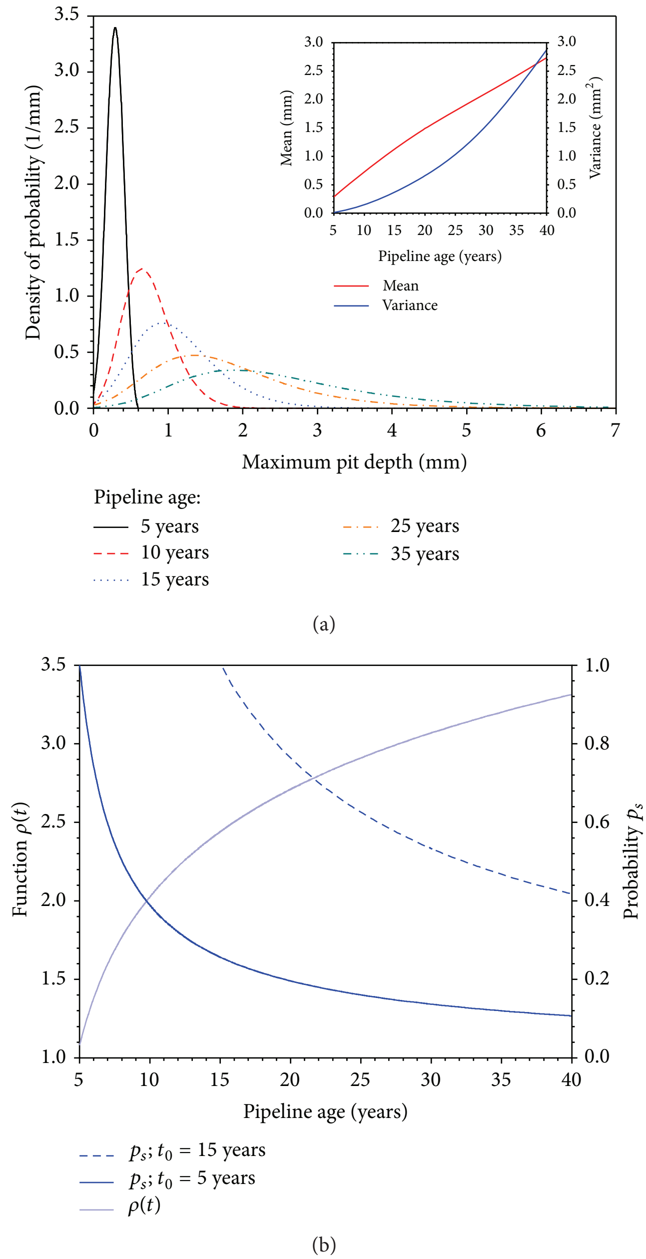


Figure 1: (a) Evolution of the Generalized Extreme Value distribution fitted to the extreme pit depths predicted in a Monte Carlo framework [7] for a general soil class. The time evolution of the mean value and variance of such distributions are shown in the inset. (b) Time evolution of the functions $\rho(t)$ (13) and p_s (15), calculated using the predicted pitting parameters κ, ν, and t_{sd}. The two curves of p_s correspond to two different values of t_0 in (15).

process in soils. Given that $\rho(t)$ is completely determined by the extent of the damage [15], its value is unique for each soil class. Also, the probability p_s is unique in each soil type. However, in Figure 1(b), two different curves of p_s are displayed to illustrate the dependence of p_s on time. They correspond to two distinct values of t_0 (see (15)), the time when the initial pit depth distribution is measured. The value of p_s at a given moment in time $t \geq t_0$ increases with increasing t_0 and

Table 2: Estimated pitting parameters (10) associated with the typical values of the physicochemical pipe-soil variables in each soil category [9].

Soil class	Typical pitting coefficient κ_{Typ} (mm/years$^{\nu_{\mathrm{Typ}}}$)[a]	Typical pitting exponent ν_{Typ} (years)[a]
Clay	0.178	0.829
Clay loam	0.163	0.793
Sandy clay loam	0.144	0.734
All	0.164	0.780

[a]Correspond to parameters in (10).

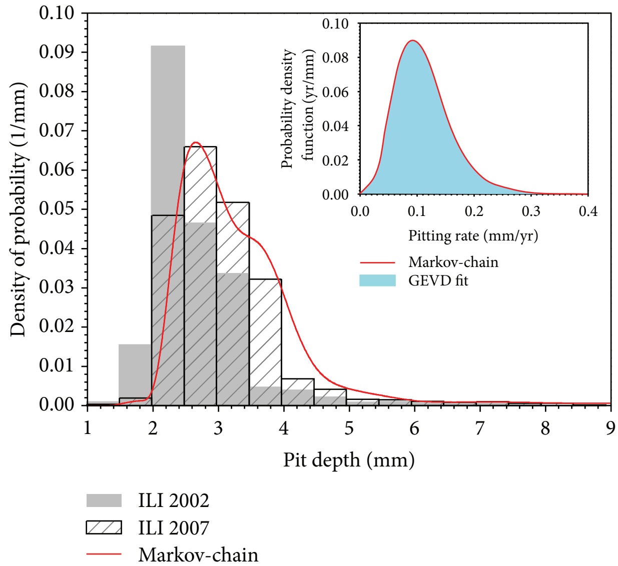


Figure 2: Histograms of the pipeline pit depth distributions measured by ILI in 2002 (grey) and in 2007 (hatched). The solid line is the Markov-estimated pit depth distribution for 2007. In the inset, the Markov-derived pitting rate distribution for the period 2002–2007 is presented.

decreases when the length of the interval (t_0, t) increases. The increase in p_s with t_0 indicates that the mean and variance of the pitting rate decrease as the lifetime of the pitting damage increases, which is in agreement with experimental observations. Also, the form of $\rho(t)$ in Figure 1(b) suggests that, for pits with equal lifetimes, the deeper the pit, the smaller the value of p_s and, therefore, the larger the mean and variance of the pitting rate. This characteristic of the developed model will permit accurately predicting the corrosion rate distribution in a pitting corrosion experiment.

To estimate the evolution of pit depth and pitting rate distributions using the proposed Markov model one can use (7) and (6), respectively. As has been already stated, from the results of an ILI of the pipe and the knowledge of the local soil characteristics, it would be possible to estimate the pitting corrosion damage evolution.

To illustrate the application of the proposed Markov chain model, it is employed in the analysis of an 82 km long operating pipeline, used to transport sweet gas since its commissioning in 1981. This pipeline is coal-tar coated and has a wall thickness of 9.52 mm (0.375″). It was inspected in 2002 and 2007 using magnetic flux leakage (MFL) ILI. The pit depth distributions present in the pipeline were obtained by calibrating the ILI tools using a methodology described by Caleyo et al. [31]. These distributions are shown in Figure 2 in the form of histograms. The first, grey-shadowed histogram represents the depth distribution of $N_{02} = 3577$ pits located and measured by ILI in 2002, while the hatched histogram represents the depth distribution of $N_{07} = 3851$ pits measured by ILI in mid-2007.

In order to apply the proposed model, the pipe wall thickness was divided in 0.1 mm thick non overlapping Markov states, so that the pitting damage is represented through Markov chains with states ranging from $m = 1$ to $m = N = 100$. Unless otherwise specified, this scheme of discretization of the pipe wall thickness is used hereafter to represent the pitting damage penetration.

The empirical depth distribution observed in 2002 was used as the initial distribution so that $t_0 = 21$ years, $\Delta t = 5.5$ years, and $p_m(t_0 = 21) = N_m/N_{02}$, N_m being the number of pits with measured depth in the mth state. The soil characteristics along the pipeline were taken as those of the "All" (generic) soil class and the value of t_{sd} was taken equal to 2.9 years, as indicated in Table 1. From Table 2, a value of 0.780 was assigned to ν in order to obtain the predicted pit depth distribution p_n ($t = 26$ years) for 2007. The values of $p_n(t)$ can be calculated using the following expression, which is deducted from (4), (7), and (15):

$$p_n(t) = \sum_{m=1}^{n} p_m(t_0)\binom{n-1}{n-m}\left(\frac{t_0 - t_{\mathrm{sd}}}{t_0 - t_{\mathrm{sd}}}\right)^{\nu m}\left[1-\left(\frac{t_0 - t_{\mathrm{sd}}}{t_0 - t_{\mathrm{sd}}}\right)^{\nu}\right]^{n-m}. \tag{18}$$

In Figure 2, the Markov chain-predicted pit depth distribution for 2007 is represented with a thick line.

In order to test the degree of coincidence between the empirical distribution (for 2007) and the Markov-model predicted distribution, the two-sample Kolmogorov Smirnov (K-S) and the two-sample Anderson-Darling (A-D) tests were used. This was done by means of a Monte Carlo simulation in which 1000 samples of 50 depth values were generated for each one of the distributions under comparison and then used in pairs for the tests. In the case of the Markov-predicted distribution the data samples were generated as (pseudo) random variates with probability density function (pdf) as displayed in Figure 2 and given by (18). The samples from the empirical data were produced by sampling the experimental 2007 dataset with replacement. The K-S test gave an average P value of 0.42, with only 7% of cases where the test failed at 5% significance. By its part, the A-D test produced an average P value of 0.39 with an 11% fraction of failed tests. This is an evidence of the validity of the null hypothesis about both samples, modeled and empirical, coming from the same distribution. The good agreement between the empirical pit depth distribution observed in 2007

and the Markov chain-modeled distribution also points to the accuracy of the proposed model.

The corrosion rate distribution $f(v; t_0, t)$ associated with the entire population of pits in the 2002–2007 period was also estimated using (19), which was derived from (4), (5), (6), and (15). Consider

$$f(v; t_0, t) = \sum_{m=1}^{N} p_m(t_0) \binom{m + v(t - t_0) - 1}{v(t - t_0)} \left(\frac{t_0 - t_{sd}}{t - t_{sd}}\right)^{\nu m} \times \left[1 - \left(\frac{t_0 - t_{sd}}{t - t_{sd}}\right)^{\nu}\right]^{v(t-t_0)}. \tag{19}$$

The estimated rate distribution is shown in the inset of Figure 2; it can be fitted to a GEV distribution (17) with negative shape parameter ζ, although it is very close to zero. This means that the Weibull and Gumbel subfamilies [27] of the GEV distribution are appropriate to describe the pitting rate in the pipeline over the selected estimation period.

It has to be noticed that the proposed Markov model can be used to predict the progression of other pitting processes. For this to be done, the values of the pitting exponent and starting time must be known for the process. These can be obtained, for example, from the analysis of repeated in-line inspections of the pipeline, from the study of corrosion coupons, or from the analysis of laboratory tests. To show this capability of the model, another example is given, using this time a different experimental data set gathered from two successive MFL-ILIs. The analysis involves a 28 km long pipeline used to transport natural gas since its commissioning in 1985. This pipeline is coal-tar coated, made of API 5L grade X52 steel [32], with a diameter of 457.2 mm (18″) and a wall thickness of 8.74 mm (0.344″). The first inspection was conducted in 1996 and the other in 2006. The ILIs were calibrated following procedures described elsewhere [31]. Before using the ILI data, the corrosion defects from both inspections were matched using the following criteria: (i) the matched defects should agree in location, and (ii) the depth of a defect in the second inspection must be equal to or greater than its depth in the first inspection. Only the matched defects were used in the analyses that followed to ensure that only the actual defect depth progression with time was considered (one-by-one if necessary) without including defects that might have nucleated in the interval between the two inspections. The depth distribution of the matched defects of the 1996 inspection was taken as the initial depth distribution fed into the proposed Markov model. The resulting depth distribution was subsequently compared to that of the defects observed in the 2006 inspection.

From the knowledge of the mean depth values from both inspections, an empirical value of the pitting exponent ν was estimated instead of using the value recommended in Table 2. It was calculated for the specific type of soil in which the pipe under analysis was buried by using the ratio between the empirical pit depth means $Đ_{96}$ and $Đ_{06}$, from the ILIs in 1996 and 2006, respectively. Assuming that the pit depth mean complies with (10) and that parameters κ, ν, and t_{sd} (which depend exclusively on the soil and pipe properties) are the same at the times of both inspections, it is possible to obtain the pitting exponential parameter ν from

$$\frac{Đ_{06}}{Đ_{96}} = \frac{\kappa(t_{06} - t_{sd})^{\nu}}{\kappa(t_{96} - t_{sd})^{\nu}}, \tag{20}$$

where $t_{96} = 11$ years and $t_{06} = 21$ years are the times of the inspections, and t_{sd} is the incubation time of the corrosion metal losses in the pipeline. The value of t_{sd} was taken equal to 2.9 years, the average pitting initiation time for a generic soil class (Table 1).

As in the previous example, the wall thickness was divided into N (0.1 mm thick) equally spaced Markov states. The observed defect depth was converted to Markov-state units, and the depth distribution was given in terms of the probability $p_m(t)$ for the depth in a state equal or less than m at time t. The proposed Markov model (18) was applied to estimate the pit depth distribution after a time interval $\delta t = t - t_0$ of 10 years, with $t_{sd} = 2.9$ years, $t_0 = 11$ years, and $t = 21$ years.

In Figure 3, the final (2006) pit depth distribution of the by-ILI measured and matched 179 defects in this pipeline is shown on a probability density scale in the form of a histogram, together with the pit depth distribution predicted by the Markov model for 2006. Again, the Monte Carlo framework described in the previous example was followed in order to apply the two-sample K-S test. The sample size was 50, and the sampling process was repeated 1000 times. The resulting average P value was 0.07, which is enough for not rejecting the null hypothesis that both the empirical and modeled distributions come from the same distribution. From the results of the test and from what it is shown in Figure 3, it can be concluded that the Markov model predicts a pit depth distribution that is close to the experimental one. More important, the model is also capable to correctly describe the form (shape) of the defect depth distribution, as seen also from this figure.

To further explore the capabilities of the model in estimating the time evolution of pit depths, the defects in the initial (1996) inspection were grouped into 6-state (0.6 mm) depth intervals in order to test the model efficacy in predicting the pit depth evolution by depth interval. An example of these grouping and modeling approach is shown in the inset of Figure 3. The chosen initial depth interval is that between 9 and 15 states (0.9 and 1.5 mm). In the inset, a histogram of the matched defect depths in the second ILI together with the distribution (solid line) predicted by the Markov model starting from the chosen initial depth subpopulation is depicted. Here as well, the model is capable of reproducing the experimental results. The same was proven true for the rest of depth intervals in the initial (1996) pit depth population. Therefore, it can be stated that the Markov model is not only accurate in estimating the time evolution of the entire defect depth distribution, but also in estimating the time evolution of defect subpopulations that differ in depths.

Additionally, to check the ability of the proposed Markov model to predict the corrosion rate (CR) distribution, an empirical CR distribution was determined using the data from both inspections based on the observed change in depth

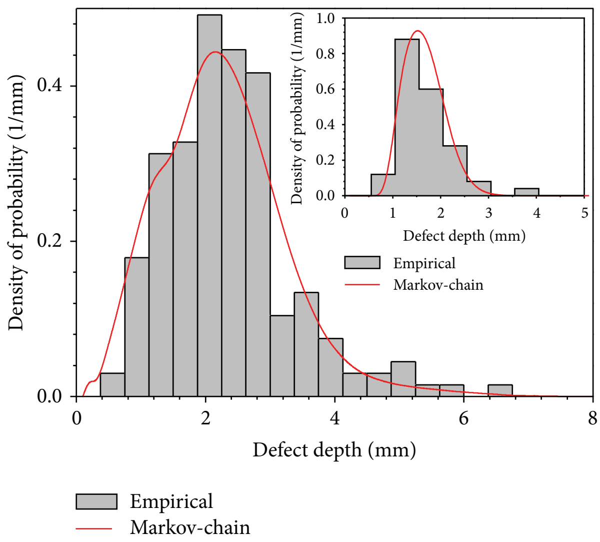


Figure 3: Histogram of the 179 matched defect depths, measured in 2006, together with the predicted pit depth distribution. The inset shows the histogram of the observed in the second inspection defect depths matched to 0.9–1.5 mm depths of first inspection, along with the Markov chain-predicted distribution (solid line).

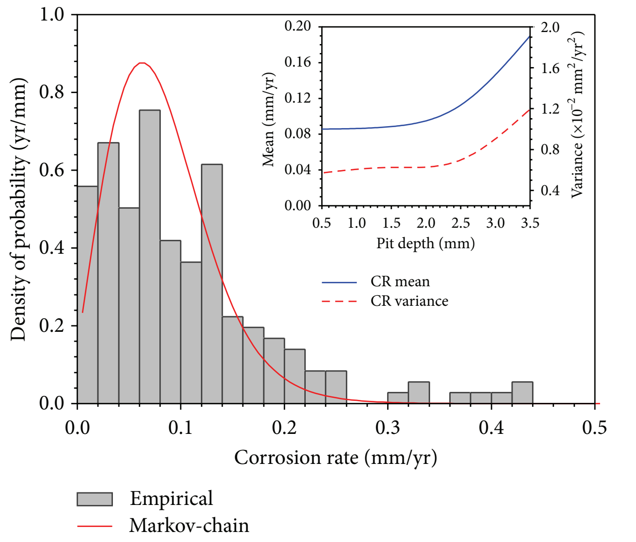


Figure 4: Comparison of the empirical corrosion rate (CR) distribution as determined from repeated ILIs (1996 and 2006) with the CR distribution predicted by the model. The inset is a plot of the experimental CR mean and variance against defect depth.

of the matched defects over the interval δt. The empirical rate distribution was then compared with the corrosion rate distribution predicted by the model (19). This comparison can be observed in Figure 4. The two-sample K-S test for 1000 pairs of 50-point CR samples yielded an average P value of 0.27 with only 12% of rejections at 5% significance level. Again, the Markov model satisfactorily reproduces the empirical CR distribution.

The reasons behind the foregoing results lie in the ability of the Markov chain model to capture the influence of both the depth and age of the corrosion defects on the deterministic pit depth growth together with its ability to reproduce the stochastic nature of the process, also as a function of pit depth and age. The inset of Figure 4 is a proof of this last statement. It shows the experimental corrosion rate mean and variance dependence on pit depth. To obtain these results, pit depths were grouped into depth intervals of ten states (1 mm) and the corrosion rate values, derived from the comparison of the two ILIs, were averaged within each group. From this inset, it can be concluded that both the corrosion rate mean and variance increase with defect depth, as was previously noted from different experimental evidence [7, 18]. This result is critical to support the criteria that a good corrosion rate model should consider both the age and depth of the to-be-evolved corrosion defects together with the actual shape of the function describing the observed dependence of the corrosion defect depth with time.

2.2. Markov Chain Modeling of Extreme Corrosion Pit Depths. The novelty in this second stochastic model is the consideration of multiple pits in a given corrosion area (called coupon hereafter). Pit initiation is modeled using a nonhomogeneous Poisson process, while the distribution of pit nucleation times is simulated using a Weibull process. The initiation time of each one of the m pits produced in a coupon is described as the time to the first failure of a part of the system [33]. So, if pit initiation time is understood as the time to first failure (passive layer breakdown or inclusion dissolution) of an individual part [33], one can assume that it follows a Weibull process with distribution

$$F(t) = 1 - \exp\left[-\left(\frac{t}{\varepsilon}\right)^{\xi}\right]. \tag{21}$$

The pit initiation times t_k are computed as the realizations of a Weibull probability distribution described by (21) with both the scale parameter ε and the time t expressed in days.

After a pit has been generated at time t_k, it is assumed that it starts to grow. The time evolution of pit depth $D(t)$ is then modeled using a nonhomogeneous Markov process. The transition probabilities $p_{i,j}$ from state i to state j satisfy the system of forward Kolmogorov's equations given in (1).

If it is assumed that the pitting damage (depth) is in state 1 at the initial time t_k (after initiation), then, for the Markov process defined by the system of (1), the transition probability from the first state to any jth state during the interval (t_k, t), that is, $p_{1,j}(t) = P\{D(t) = j \mid D(t_k) = 1\}$, can be found analytically [28] as

$$p_{1,j}(t) = \exp\left[-\rho(t) + \rho(t_k)\right]\left\{1 - \exp\left[-\rho(t) + \rho(t_k)\right]\right\}^{j-1}, \tag{22}$$

where $\rho(t)$ is defined by (3).

From (22) it follows that, for a single pit, the probability that its depth is equal or less than state i, after a time increment (t_k, t), is

$$F(i,t) = \sum_{j=1}^{i} p_{1,j}(t) = \frac{e^{-\rho(t)+\rho(t_k)}}{1-e^{-\rho(t)+\rho(t_k)}} \sum_{j=1}^{i} \left(1-e^{-\rho(t)+\rho(t_k)}\right)^j, \quad i = 1 \ldots N, \tag{23}$$

where N is the total number of states in the Markov chain.

It is easy to show [15] that expression (23) can be rewritten as

$$F(i,t) = 1 - \left\{1 - \exp\left[-\rho(t) + \rho(t_k)\right]\right\}^i. \tag{24}$$

Equation (24) represents the probability of finding a single pit in a state less than or equal to state i after a time interval (t_k, t). It is worth noting that in (22)–(24) the pit initiation and growth processes are combined by taking into account the pitting initiation time t_k.

As it was stated before, a direct physical meaning can be given to the functions $\lambda(t)$ and $\rho(t)$, which are related to the pit growth rate and pit depth, respectively. Taking into account that the dependence of pit depth on time is a power function (10), the functional dependence of ρ and λ on time was assumed to be

$$\rho(t) = \chi(t)^{\omega}, \tag{25}$$

$$\lambda(t) = \chi\omega(t-t_k)^{\omega-1}, \tag{26}$$

where χ has dimensions of distance over the ωth power of time and ω is less than 1.0.

In order to generalize from the single pit case to the m-pit case, the probability that the maximum damage state is less than or equal than a given value for a time interval is estimated under the assumption that all the pits are described by parent distributions $F(i; t_k, t)$ of the type given in (24). First, let us consider the simplest situation in which it is assumed that all the pits are generated at the same time $t_k = t_{\text{ini}}$. In this case, function (24) represents the cumulative distribution function of a parent population of m pits. To find the extreme depth value distribution, the distribution of (24) should be raised to the mth power [27]. It can be shown [15, 27] that for a large number of pits ($m \to \infty$) and under a suitable variable transformation, it follows that

$$\Phi(i,t) = [F(i,t)]^m \xrightarrow{m\to\infty} G(i,t). \tag{27}$$

Substituting (24) and (25) in expression (27), and after some transformations [15], $G(i,t)$ converges to a Gumbel function given by (2). The involved mathematical formalisms that led to this result can be found in Appendix A of [15].

Summarizing, when the pit initiation times are very short (smaller than the observation time) and/or when it can be considered that they are the same for all the pits, the distribution function for maximum pit depths is obtained by raising (24) to the power of the number of pits in the area of interest. The parameters of the model in this case are the number of pits m and the parameters of the $\rho(t)$ function: χ and ω (see expression (25)).

Consider now an alternative case, when m pits are generated at different times t_k, and expression (24) holds for each one of them. The m cumulative distribution functions, $F_k(i, t-t_k)$, $k = 1, \ldots, m$, must be combined in order to estimate the distribution of the deepest pit, under the assumption that pits nucleate and grow independently. In such a case, the probability that the deepest pit is in a state less than or equal to state i, at time t, can be estimated using the expression

$$\theta(i,t) = \prod_{k=1}^{m} \left\{1 - \left[1 - \exp\left(-\rho(t) + \rho(t_k)\right)\right]^i\right\}. \tag{28}$$

It can be shown that this cumulative function also follows a Gumbel distribution for large m [15].

In (28), expression (25) for $\rho(t)$ must be substituted together with the initiation times t_k, which are assumed to follow a Weibull distribution (21). Therefore, function $\theta(i,t)$ in (28) includes the model parameters ε, ξ, and m to simulate pit initiation, together with parameters χ and ω to model pit growth through the function $\rho(t)$.

At this point, owing to the fact that functions $\Phi(i,t)$ (from (27)) and $\theta(i,t)$ (from (28)) are extreme value distributions of the Gumbel type, it can be stated that function $F(i,t)$ (24) is the parent distribution that lies in the domain of attraction of the Gumbel distribution for maxima [27]. From (24), it can be observed that $F(i,t)$ is of exponential type; therefore, the parent distribution for pit depth extremes is actually an exponential function. This is in complete agreement with the findings in [7], where it was concluded, after measuring and analyzing all the pits in corroded low carbon steel coupons, that the exponential pit depth distribution fitted to the right tail of the pit depth distribution (of the Normal type, adjusted to the depths of the whole population of pits) is actually the extreme's parent distribution, which lies in the domain of attraction of the Gumbel extreme value distribution for maxima.

Following this idea, it is soundly to shift the initial Markov state to the depth value that constitutes the starting point of the exponential distribution tail. If we consider that this exponential distribution starts at some depth value u, then the Markov state i that appears in (24) should be changed to a new variable

$$i' = i + u. \tag{29}$$

Because the pits that contribute to the extreme pit depths values are only those whose depths exceed the threshold u, the number m of pits that should be taken into account when applying (27) or (28) can be equated to the average number of pits whose depths exceed the threshold value u. A more detailed justification of this procedure of variable change for the state i in (24) can be found elsewhere [34].

The empirical average number λ_e of exceedances over threshold u per coupon (to be compared to the model parameter m) can be estimated using (30), which relates λ_e with the

threshold value u and the parameters σ and μ of the Gumbel distribution fitted to the observed extreme pit depths. Consider

$$\lambda_e = \exp\left(\frac{\mu - u}{\sigma}\right). \quad (30)$$

This expression is readily obtained from the existing relation between the Gumbel and exponential parameters, as has been shown in [7].

To run the proposed Markov model for extremes, (27) (pits initiate all at the same time) or (28) (pits initiate at different times) is computed and fitted to a Gumbel distribution for several times t (2). The model parameters (three for the instantaneous pit initiation: χ, ω, and m, and five for the generation of pits at different times: ε, ξ, χ, ω, and m) are adjusted considering the experimental distributions for the deepest pits. A series of corrosion experiments is considered, each one of them consisting in the exposition of n_c coupons of a corrodible material to a corrosion environment for a given time. After exposure, the depth of the deepest pit in each coupon is measured [6, 7], and the distribution of maximum depths is fitted with a Gumbel distribution (2). If the experiment is carried out for N_t exposure times, and the observed behavior of depth extremes is to be described with the proposed model, the model parameters are adjusted by minimizing a total error function E_T defined as

$$E_T = \sum_{l=1}^{N_t}\left(\sqrt{\left(\mu_e^l - \mu_p^l\right)^2} + \sqrt{\left(\sigma_e^{2l} - \sigma_p^{2l}\right)^2}\right), \quad (31)$$

where (μ_e^l, σ_e^{2l}) and (μ_p^l, σ_p^{2l}) are the mean and variance values of the lth experimental and estimated extreme value distributions, respectively.

The model parameters are adjusted through minimization of the average value of the error function E_T computed during N_{MC} Monte Carlo simulations, with $N_{\mathrm{MC}} = 1000$.

2.2.1. Application of Markov Model for Extremes to Low Carbon Steel Corrosion Experiment. The described Markov model for extreme pit depths was applied to experimental data obtained in pitting corrosion experiments for low-carbon steel reported by Rivas et al. [7]. The details of the experiments can be found elsewhere [7].

In these experiments, groups of 20 coupons of API-5L X52 [32] pipeline steel, with $1 \times 1\,\mathrm{cm}^2$ of exposed area, were immersed in a corroding solution for 1, 3, 7, 15, 21, and 30 days. After the immersion, all the pits with depths greater than $5\,\mu$m were measured. The maximum pit depth in each coupon was also recorded. The Gumbel distribution (2) was fitted to the resulting maxima data sets using the Maximum Likelihood Estimator (MLE) [27].

The Gumbel probability density functions fitted to the experimental pit depth maxima for the six exposure times are displayed in Figure 5. They are represented with red solid lines. Table 3 shows the values of the Gumbel location (μ_E) and scale (σ_E) parameters for the fitted distributions.

In their work, Rivas et al. [7] concluded that, as has been previously recognized [4, 35], in low carbon steel pits initiate

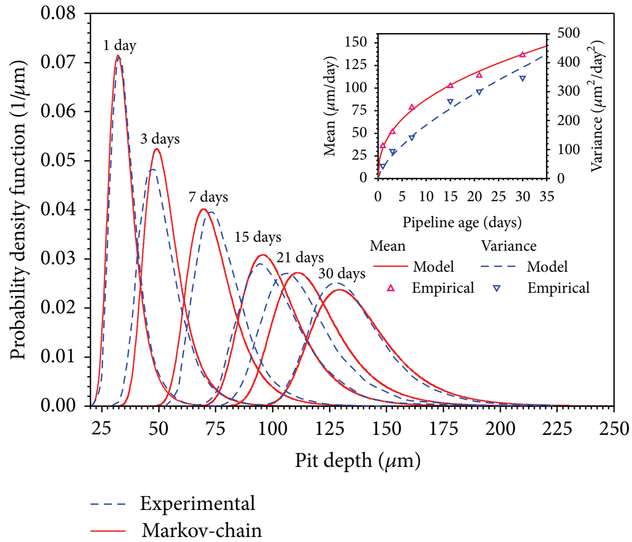


Figure 5: Comparison of the Gumbel probability density functions fitted to the experimental pit depth maxima with the model-predicted density functions for the six exposure times.

at the site of MnS inclusions. In the present experiment, the dissolution of these inclusions occurs in a matter of few hours after immersion, and then pits start propagating. The authors also established that the deepest pits of the entire pit population are the oldest ones, meaning that they are the defects that initiate first in time [7]. This conclusion leads to the use of (27) to model the extreme pits using Markov chains. Since the pit initiation time for this experiment has been considered to be very short, the nucleation time t_k in (27) was set to zero.

In order to apply the Markov model, the value of the threshold parameter u of (29) should be determined from the empirical corrosion data. Given that u represents the depth value from which the exponential tail of the whole pit depth distribution starts, it should coincide with the previously determined [7] threshold pit depth value u_P for the tail of the whole pit depth distribution.

When applying the so-called Peak over Threshold (POT) approach in pitting corrosion experiments, Rivas et al. proposed [7] a simplified method for determining the threshold depth value without the need of measuring the entire pit depth population. The method includes a proposition of the *a priori* determination of the threshold pit depth value u_P from the fitted Gumbel distribution to the pit depth extremes. It was suggested to take u_P as a depth value for which the Gumbel distribution shows a cumulative probability P in the range from 0.00005 to 0.005. This proposition was based on the empirical fact that in the analyzed pitting corrosion experiments [7], the Gumbel distribution for maximum depth starts to rise precisely at the beginning of the exponential tail of the normal distribution of the entire pit depth population. Therefore, as the starting point of the Gumbel distribution, it was proposed to use the 0.5%, 0.05%, or even the 0.005% quantile of the experimental pit depth distributions.

Following this suggestion, and taking into consideration the determined values of the location and scale from the

Table 3: Estimated location and scale parameters of the Gumbel distributions fitted to the experimental pit depth maxima and to the Markov model. The fourth and fifth columns correspond to the calculated pit depth threshold and average exceedances values, respectively. The last column displays the average P value of 1000 two-sample Kolmogorov-Smirnov tests.

Exp. time (days)	Estimated from the experimental data		$u_{0.005}$ [a]	λ_e [b]	Estimated from the modeled distribution		P value
	μ_E (μm)	σ_E (μm)			μ_M (μm)	σ_M (μm)	
1	32.8	5.15	21.2	10	32.0	5.15	0.50
3	47.0	7.58	34.49059	5	49.1	7.02	0.36
7	72.9	9.29	50.20122	12	69.7	9.17	0.31
15	94.9	12.75	70.36299	9	95.6	11.94	0.56
21	105.9	13.52	81.67309	6	111.0	13.52	0.29
30	127.9	14.54	95.65344	9	129.4	15.52	0.53

[a]Estimated from (32).
[b]Estimated from (30).

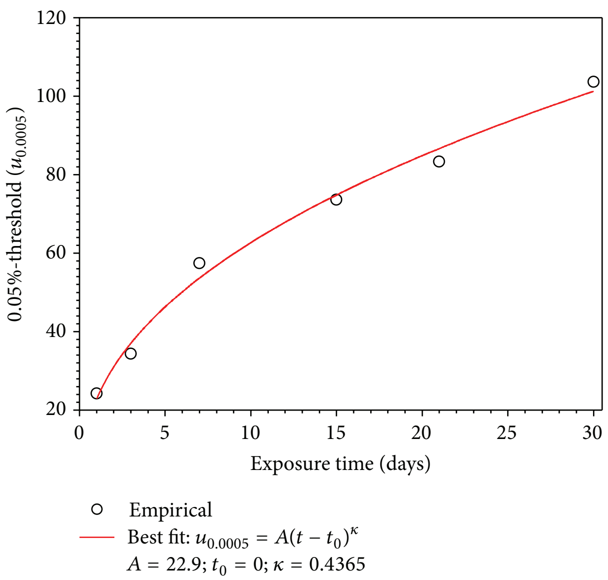


Figure 6: Plot of the calculated threshold depth values $u_{0.005}$ versus exposure time. The solid line is a power function fitted to the determined threshold values.

fitted Gumbel distribution, it is possible to determine the X_P quantile, being $P = 0.005, 0.0005$, or 0.00005. To achieve this, the following equation, obtained through the inversion of expression (1), is used:

$$X_P = \mu - \sigma \ln [- \ln (P)] . \tag{32}$$

Thus, the value of X_P can be used as the threshold u_P ($u_P = X_P$) [7]. For the analyzed experiment we are fixing the value of P to 0.0005. Substituting this value in (32) and using the estimated location (μ_E) and scale (σ_E) parameters for the empirical Gumbel distributions for each one of the exposure times (Table 3), six values of $u_{0.0005}$ are obtained. These values are plotted in Figure 6 against the exposure times. In this figure, a power function fitted to the empirical threshold values is also displayed. From this curve, the values of u to be substituted in (29) are taken. A more detailed proof of the correctness of this choice of u can be found in [34].

Once the value of the threshold is determined, the function $F(i, t)$ of (24) is raised to the mth power, and the resulting function $\Phi(i, t)$ (27) is fitted to a Gumbel distribution (2). The fitted function is compared with the empirical Gumbel distribution in order to adjust the model parameters. For this particular case, the parameters to take into account are the number of pits m, which should approximately match the average experimental number of exceedances over the threshold u per coupon, and the parameters χ and ω, which define the function $\Phi(i, t)$. The model parameters are refined by minimizing the error function of (31).

The minimization process gives an adjusted parameter $m = 7.6$. This value can be approximated to 8 pits. It represents the average number of pits per coupon whose depths exceed the threshold value u. This fact can be confirmed by comparing the adjusted value for m with the values of the empirical exceedances λ_e displayed in the fifth column of Table 3. The exceedances for each exposure period were calculated by substituting the corresponding estimated Gumbel parameters μ_E and σ_E and the threshold value $u_{0.0005}$ (all given in Table 3) into (30). The mean value of λ_e (from Table 3) equals 8, which is in good agreement with the model prediction.

The probability density functions corresponding to the distributions $\Phi(i, t)$ obtained after raising $F(i, t)$ to the power of the adjusted parameter $m = 7.6$ are shown in Figure 5 for the six exposure times. In this figure, the modeled distributions can be compared with the empirical Gumbel distributions. One can see a good agreement between the modeled and experimental Gumbel distributions. Even for this case, in which the number of pits m is small (see Section 2.1 and (27)), both the experimental and modeled distributions agree with the Gumbel model for maxima.

The Gumbel distribution (2) was also fitted to the functions predicted by the model using the Maximum Likelihood Estimator (MLE) [27]. In Table 3, the estimated parameters μ_M and σ_M for the fitted Gumbel distributions (to the Markov-predicted functions) are displayed together with the corresponding parameters of the empirical Gumbel distributions. The differences between the empirical and modeled parameters do not exceed 8%. The last column of Table 3 shows the average P values of the two-sample K-S test performed on 1000 pairs of 50-depth-point samples (one for

the empirical and one for the Markov-modeled distributions) for the six exposure periods. The smaller P value among them is 0.29, which leads to not reject the null hypothesis that the two samples belong to the same distribution.

The good agreement between the modeled and the empirical Gumbel distributions can also be established from the inset of Figure 5, where the time evolution of the Gumbel mean and variance for the observed data and for the result of the proposed model are compared. These results demonstrate the suitability of the proposed model to describe the initiation and evolution of the maximum pit depths in a pitting corrosion experiment, which is of great importance in many applications such as reliability assessment and risk management.

The advantages of the proposed model compared with previous Markov models (developed by other authors) can also be established. The details regarding this topic can be found in [15].

3. Concluding Remarks

Two Markov chain models have been presented to simulate pitting corrosion. They have been developed and validated using experimental pitting corrosion data. Both models are attractive due to the existence of analytical solutions of the system of Kolmogorov's forward equations.

The first model describes the time evolution of pit depths that correspond to the general population of defects in underground pipelines. It has been developed using a continuous-time, nonhomogenous linear growth (pure birth) Markov process, under the assumption that the Markov-chain-derived stochastic mean of the pitting damage equals the deterministic, empirical mean of the defect depths. Such an assumption allows the transition probability function to be identified only from the pitting starting time and exponent parameter. Moreover, this assumption requires that the functional form of the stochastic and deterministic instantaneous pitting rates also agree. This supports the idea that the intensities of the transitions in the Markov process are closely related to the pitting damage rate. This model permits predicting the time evolution of pit depth distribution, which is of paramount importance for reliability estimations. It is also able to capture the dependence of the pitting rate on the pit depth and lifetime. This is an advantage of the Markov chain approach over deterministic and other stochastic models for pitting corrosion. It could also be extended to investigate pitting corrosion in environments other than soils, for example, in laboratory experiments.

The second Markov chain pitting corrosion model gives account for the maximum pit depths. In it, pitting corrosion is modeled as the combination of two independent nonhomogeneous in time physical processes, one for pit initiation and one for pit growth. Both processes are combined using well-suited physical and statistical methodologies, such as extreme value statistics, to produce a unified stochastic model of pitting corrosion.

Pit initiation is described by means of a non-homogeneous Poisson process so that a set of multiple pit nucleation events can be modeled as a Weibull process. Pit growth is modeled as a nonhomogeneous Markov process. Given that the intensity of the process is related to the corrosion rate, its functional form can be proposed from the results of the experimental tests.

Taking into account the experimental evidence that the pit depth parent distribution that leads to the distribution for extremes is the exponential tail of the pit depth population distribution, the solution of the Markov chain is shifted to the threshold pit depth value, where the exponential tail of the pit depth distribution starts. The threshold value is determined from the pit depth value from which the empirical Gumbel distribution for maximum pit depth becomes significant.

Extreme value statistics has been used to show the accuracy of the model describing the experimental observations. The model is capable of predicting not only the correct Gumbel distributions for pitting corrosion maxima in low-carbon steel, but also of estimating the number of extreme pits that has physical sense and that matches the experimental findings.

In order to simulate the whole pitting process, five model parameters are necessary. Two parameters are required to simulate pit initiation as a Weibull process; another two parameters are required to simulate pit growth with the nonhomogeneous Markov process; finally, the number of pits is necessary to combine these two processes. If the assumption that all pits nucleate instantaneously holds, as is the case of the presented experimental example, only three parameters will be necessary to fit the model to the experimental data.

Given the fact that the model parameters and assumptions do not depend on the corroded material, nor on the corrosion environment, the model is suited for different corrosion systems. The model can be easily adapted to describe situations in which the distribution of pit initiation times and the functional form of the time dependence for pit growth differ from those considered in this work.

References

[1] G. S. Chen, K. C. Wan, G. Gao, R. P. Wei, and T. H. Flournoy, "Transition from pitting to fatigue crack growth—modeling of corrosion fatigue crack nucleation in a 2024-T3 aluminum alloy," *Materials Science and Engineering A*, vol. 219, no. 1-2, pp. 126–132, 1996.

[2] M. Nessin, "Estimating the risk of pipeline failure due to corrosion," in *Uhlig's Corrosion Handbook*, W. Revie, Ed., p. 85, John Wiley & Sons, 2nd edition, 2000.

[3] T. Shibata and T. Takeyama, "Pitting corrosion as a stochastic process," *Nature*, vol. 260, no. 5549, pp. 315–316, 1976.

[4] G. S. Frankel, "Pitting corrosion of metals: a review of the critical factors," *Journal of the Electrochemical Society*, vol. 145, no. 6, pp. 2186–2198, 1998.

[5] G. T. Burstein, C. Liu, R. M. Souto, and S. P. Vines, "Origins of pitting corrosion," *Corrosion Engineering Science and Technology*, vol. 39, no. 1, pp. 25–30, 2004.

[6] P. M. Aziz, "Application of the statistical theory of extreme values to the analysis of maximum pit depth data for aluminum," *Corrosion*, vol. 12, pp. 495t–506t, 1956.

[7] D. Rivas, F. Caleyo, A. Valor, and J. M. Hallen, "Extreme value analysis applied to pitting corrosion experiments in low carbon steel: comparison of block maxima and peak over threshold approaches," *Corrosion Science*, vol. 50, no. 11, pp. 3193–3204, 2008.

[8] J. C. Velázquez, F. Caleyo, A. Valor, and J. M. Hallen, "Predictive model for pitting corrosion in buried oil and gas pipelines," *Corrosion*, vol. 65, no. 5, pp. 332–342, 2009.

[9] F. Caleyo, J. C. Velázquez, A. Valor, and J. M. Hallen, "Probability distribution of pitting corrosion depth and rate in underground pipelines: A Monte Carlo Study," *Corrosion Science*, vol. 51, no. 9, pp. 1925–1934, 2009.

[10] Z. Szklarska-Smialowska, "Pitting corrosion of aluminum," *Corrosion Science*, vol. 41, no. 9, pp. 1743–1767, 1999.

[11] W. Zhang and G. S. Frankel, "Localized corrosion growth kinetics in AA2024 alloys," *Journal of the Electrochemical Society*, vol. 149, no. 11, pp. B510–B519, 2002.

[12] F. Hunkeler and H. Bohni, "Mechanism of pit growth on aluminum under open circuit conditions," *Corrosion*, vol. 40, no. 10, pp. 534–540, 1984.

[13] T. Shibata, "1996 W.R. Whitney Award Lecture: statistical and stochastic approaches to localized corrosion," *Corrosion*, vol. 52, no. 11, pp. 813–830, 1996.

[14] J. W. Provan and E. S. Rodriguez, "Part I: Development of a Markov description of pitting corrosion," *Corrosion*, vol. 45, no. 3, pp. 178–192, 1989.

[15] A. Valor, F. Caleyo, L. Alfonso, D. Rivas, and J. M. Hallen, "Stochastic modeling of pitting corrosion: a new model for initiation and growth of multiple corrosion pits," *Corrosion Science*, vol. 49, no. 2, pp. 559–579, 2007.

[16] T. B. Morrison and R. G. Worthingham, "Reliability of high pressure line pipe under external corrosion," in *Proceedings of the 11th ASME International Conference Offshore Mechanics and Arctic Engineering*, vol. 5, part B, pp. 401–408, Calgary, Canada, June 1992.

[17] H. P. Hong, "Inspection and maintenance planning of pipeline under external corrosion considering generation of new defects," *Structural Safety*, vol. 21, no. 3, pp. 203–222, 1999.

[18] F. Caleyo, J. C. Velázquez, A. Valor, and J. M. Hallen, "Markov chain modelling of pitting corrosion in underground pipelines," *Corrosion Science*, vol. 51, no. 9, pp. 2197–2207, 2009.

[19] F. Bolzoni, P. Fassina, G. Fumagalli, L. Lazzari, and E. Mazzola, "Application of probabilistic models to localised corrosion study," *Metallurgia Italiana*, vol. 98, no. 6, pp. 9–15, 2006.

[20] S. A. Timashev, M. G. Malyukova, L. V. Poluian, and A. V. Bushiskaya, "Markov description of corrosion defects growth and its application to reliability based inspection and maintenance of pipelines," in *Proceedings of the 7th ASME International Pipeline Conference (IPC '08)*, Calgary, Canada, September 2008, Paper IPC2008-64546.

[21] T. B. Morrison and G. Desjardin, "Determination of corrosion rates from single inline inspection of a pipeline," in *Proceedings of the NACE One Day Seminar*, Houston, Tex, USA, December 1998.

[22] Y. Katano, K. Miyata, H. Shimizu, and T. Isogai, "Predictive model for pit growth on underground pipes," *Corrosion*, vol. 59, no. 2, pp. 155–161, 2003.

[23] A. K. Sheikh, J. K. Boah, and D. A. Hansen, "Statistical modelling of pitting corrosion and pipeline reliability," *Corrosion*, vol. 46, no. 3, pp. 190–196, 1990.

[24] M. Kowaka, Ed., *Introduction to Life Prediction of Industrial Plant Materials: Application of the Extreme Value Statistical Method for Corrosion Analysis*, Allerton Press, New York, NY, USA, 1994.

[25] A. Valor, D. Rivas, F. Caleyo, and J. M. Hallen, "Discussion: statistical characterization of pitting corrosion—Part 1: Data analysis and part 2: Probabilistic modeling for maximum pit depth," *Corrosion*, vol. 63, no. 2, pp. 107–113, 2007.

[26] E. J. Gumbel, *Statistics of Extremes*, Columbia University Press, New York, NY, USA, 2004.

[27] E. Castillo, *Extreme Value Theory in Engineering*, Academic Press, San Diego, Calif, USA, 1988.

[28] E. Parzen, *Stochastic Processes*, vol. 24, SIAM, Philadelphia, Pa, USA, 1999.

[29] D. R. Cox and H. D. Miller, *The Theory of Stochastic Processes*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 1st edition, 1965.

[30] S. Coles, *An Introduction to Statistical Modeling of Extreme Values*, Springer Series in Statistics, Springer, London, UK, 2001.

[31] F. Caleyo, L. Alfonso, J. H. Espina-Hernández, and J. M. Hallen, "Criteria for performance assessment and calibration of in-line inspections of oil and gas pipelines," *Measurement Science and Technology*, vol. 18, no. 7, pp. 1787–1799, 2007.

[32] *Specification for Line Pipe, API Specification 5L*, American Petroleum Institute, Washington, DC, USA, 42nd edition, 2001.

[33] H. Ascher and H. Feingold, *Repairable Systems Reliability. Modeling, Inference, Misconceptions and Their Causes*, vol. 7, Marcel Dekker, New York, NY, USA, 1984.

[34] A. Valor, F. Caleyo, D. Rivas, and J. M. Hallen, "Stochastic approach to pitting-corrosion-extreme modelling in low-carbon steel," *Corrosion Science*, vol. 52, no. 3, pp. 910–915, 2010.

[35] G. Wranglen, "Pitting and sulphide inclusions in steel," *Corrosion Science*, vol. 14, no. 5, pp. 331–349, 1974.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 168169, 9 pages
http://dx.doi.org/10.1155/2013/168169

Research Article

Exponential Stability of Stochastic Nonlinear Dynamical Price System with Delay

Wenli Zhu,[1] Xinfeng Ruan,[1] Ye Qin,[2] and Jie Zhuang[3]

[1] *School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China*
[2] *Teaching Research Training Center in Xindu District of Chengdu, Chengdu 610500, China*
[3] *Sichuan University of Science and Engineering Library, Zigong 643000, China*

Correspondence should be addressed to Wenli Zhu; zhuwl@swufe.edu.cn

Received 30 January 2013; Revised 28 April 2013; Accepted 17 May 2013

Academic Editor: Wuquan Li

Based on Lyapunov stability theory, Itô formula, stochastic analysis, and matrix theory, we study the exponential stability of the stochastic nonlinear dynamical price system. Using Taylor's theorem, the stochastic nonlinear system with delay is reduced to an n-dimensional semilinear stochastic differential equation with delay. Some sufficient conditions of exponential stability and corollaries for such price system are established by virtue of Lyapunov function. The time delay upper limit is solved by using our theoretical results when the system is exponentially stable. Our theoretical results show that if the classical price Rayleigh equation is exponentially stable, so is its perturbed system with delay provided that both the time delay and the intensity of perturbations are small enough. Two examples are presented to illustrate our results.

1. Introduction

Let us make the following assumptions.

(H1.1) Demand for product is quadratic function with respect to price.

(H1.2) The price is not very sensitive to the change of inventory. That is, damping of nonlinear dynamical system λ is a ε-order infinitesimal ($\varepsilon > 0$ is small enough).

(H1.3) Stochastic noise is related to price. That is, it can be treated as Gaussian white noise, and the excitation coefficient is $\sqrt{\varepsilon}$-order infinitesimal.

The price system can be described by linear equations because of their convenience in mathematical treatment. Therefore linear equations play an important role in theory and their applications. However, they can not perfectly describe the process of the price fluctuation in nonlinear version. Then the nonlinear equations should be employed, for their virtues that can deeply reflect the rules of price fluctuation.

Suppose $S(t)$, $D(t)$, $P(t)$, and $Q(t)$ as supply, demand, price, and inventory at time t, respectively. S_0 is initial supply, a, b, c, λ, α and γ are constants, and alphabet with a bar is equilibrium value. Reference [1] gives a deterministic nonlinear price model as follows:

$$\frac{d^2P}{dt^2} = \lambda\left(aP^2 + bP + c\right)\frac{dP}{dt} - \lambda\alpha P - \lambda\left(S_0 - \overline{D}\right). \quad (1)$$

If there exists an equilibrium point in the above price system, denoted by $\overline{P}$, then $\overline{P} = (\overline{D} - S_0)/\alpha$ and $\overline{D} = S_0 + \alpha\overline{P}$. Let $x(t) = P(t) - \overline{P}$ and $t = \eta/\sqrt{\alpha\lambda}$, $\mu = \sqrt{\lambda/\alpha}$. Applying Lienard transformation, we obtain the classical price Rayleigh equation as follows:

$$\frac{dx}{d\eta} = -y + \varepsilon\tilde{\mu}\left(\frac{1}{3}ax^3 + \frac{1}{2}b_0x^2 + c_0x\right), \qquad \frac{dy}{d\eta} = x, \quad (2)$$

where $b_0 = 2a\overline{P} + b$, $c_0 = a\overline{P}^2 + b\overline{P} + c$, $\mu \triangleq \varepsilon\tilde{\mu}$.

For many real-world systems, there always exist random disturbances such as the measurement error and the control input of the system [2–5]. The basic source of random

disturbance is Gaussian white noise, which represents the joint effects of a large number of independent random forces acting on the systems, and the influence of individual is not significant. By (H1.3), the stochastic nonlinear dynamical price system can be described by stochastic differential equation (SDE for short) as follows [5]:

$$dx = \sqrt{\lambda\alpha}\left[-y + \varepsilon\widetilde{\mu}\left(\frac{1}{3}ax^3 + \frac{1}{2}b_0x^2 + c_0x\right)\right]dt + \varepsilon^{1/2}\gamma\, dW(t),$$
$$dy = \sqrt{\lambda\alpha}x\, dt, \tag{3}$$

where $\{W(t), t \geq 0\}$ is 1-dimensional Brownian motion. The above system also can be rewritten as the following matrix form:

$$dz(t) = f(t, z(t))\, dt + h(t, z(t))\, dW(t), \tag{4}$$

where $f : R^+ \times R^2 \rightarrow R^2$, $h : R^+ \times R^2 \rightarrow R^2$, $f(t,z(t)) = \sqrt{\lambda\alpha}(-y+\varepsilon\widetilde{\mu}(ax^3/3+b_0x^2/2+c_0x), x)^T$, $z(t) = (x(t), y(t))^T \in R^2$, and $h(t,z(t)) = (\sqrt{\varepsilon}\gamma, 0)^T$.

Supply is not only influenced by price and demand but also influenced by production management, information feedback, transportation, and so forth. Therefore, $S(t)$ not only depends on the situation at t but also on the certain period $t-\tau$ ($\tau > 0$ is a given time delay) in the past [6–9]. Furthermore, the parameter perturbation of the system's internal structure should also be taken into account in this paper.

Based on the abovementioned, the price system (4) can be extended to more general n-dimensional stochastic nonlinear price systems with delay as follows:

$$dx(t) = \left[f(t, x(t), x(t-\tau)) + \overline{f}(t, x(t), x(t-\tau))\right]dt + h(t, x(t), x(t-\tau))\, dW(t), \quad t \geq 0,$$
$$x(t) = \xi(t), \quad -\tau \leq t \leq 0, \tag{5}$$

where τ is a given time delay. The maps $f, \overline{f} \in C(R^+ \times R^n \times R^n, R^n)$, $h \in C(R^+ \times R^n \times R^n, R^{n\times m})$. $\overline{f}$ represents the uncertainty. $\{W(t), t \geq 0\}$ is an m-dimensional Brownian motion, and the term $h(t, x(t), x(t-\tau))\, dW(t)$ represents the stochastic disturbance. Furthermore, we always assume that $f(t,0,0) = \overline{f}(t,0,0) = h(t,0,0) \equiv 0$ for the stability purpose of this paper.

Stability is a very important dynamical feature for the stochastic price system with delay, and it is one of the main purposes of system designing [5, 6]. Keeping the price system steady within the cycle as long as possible to avoid inflation or deflation has the vital significance for the healthy development of the economy of the country. There is a rich literature on time delay system and stochastic system. Stability of stochastic system has been studied. See, for example, Liu and Feng [2], Liu and Deng [3], Yong and Zhou [4], Li and Xu [5], and Mao [10]. Stability of time delay system has been studied. See, for example, Kazmerchuk et al. [6], Lv and Liu [7], Lv and Zhou [8], Zhu et al. [11], Zhu and Yi [12], and Trinh and Aldeen [13]. Mao [14], Mao and Shah [15], Zhu and Hu [16], Zhu and Hu [17], and S. Xie and L. Xie [18] established some stability criteria of the stochastic system with delay by using an LMI approach. The Hopf bifurcation of price Rayleigh delayed equation on deterministic case has been studied extensively in recent years. See, for example, [6–8]. The stability and the optimal control of stochastic nonlinear dynamical price model has been studied in [5]. Unfortunately, there is a little previous literature on stochastic nonlinear dynamical price system with delay. Thus, we aim to fill this gap in this paper. We plug the time delay, the parameter perturbation, and the stochastic item into nonlinear dynamical price system (2). Such models may be identified as stochastic differential delayed equations (SDDEs for short). Our target in this paper is to derive some sufficient conditions of exponential stability for SDDEs.

Li and Xu only analyzed the stability for SDEs (4) in virtue of the marginal probability density about x in [5] but did not give the sufficient condition for stability. In this paper, using Taylor's theorem, the n-dimensional nonlinear SDDE (5) is reduced to an n-dimensional semilinear SDDE correspondingly. Some sufficient conditions of exponential stability and corollaries for such price system are established by using Lyapunov function. The time delay upper limit is solved by using our theoretical results when the system is exponentially stable. Thus, [5] is promoted and improved. Our theoretical results show that if the classical price Rayleigh equation (2) is exponentially stable, so is its perturbed system (5) with delay provided that both the time delay and the intensity of perturbations are small enough. Those results will help our government make a macrocontrol for price system and timely adjust their pricing strategies.

The rest of this paper is organized as follows. In Section 2, we introduce the definition of the exponential stability of SDDEs. Section 3 is devoted to the sufficient conditions for exponential stability and almost surely exponential stability of price system. Section 4 presents two simple examples to illustrate our results. Finally, Section 5 concludes the paper.

2. Preliminaries

Throughout this paper and unless specified, we let $W(t) = (W_1(t), \ldots, W_m(t))^T$ be an m-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ (i.e., $\mathcal{F}_t = \sigma\{W(s) : 0 \leq s \leq t\}$ and augmented by all the P-null sets in $\mathcal{F}$). Denote by $|\cdot|$ the Euclidean norm. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, denote by $\|A\|$ the operator norm of A, that is, $\|A\| = \sup\{|Ax| : |x| = 1\}$. $\xi(\cdot) \in C[-\tau, 0]$ is the initial path of x, where $\tau > 0$ is a given finite time delay and $C[-\tau, 0]$ is the set of continuous functions from $[-\tau, 0]$ into R^n. Moreover, denote by $L^p_{\mathcal{F}_0}(-\tau, 0; R^n)$ the family of R^n-valued adapted stochastic processes $\xi(s)$, $-\tau \leq s \leq 0$ such that $\xi(s)$ is $\mathcal{F}_0$-measurable and $\int_{-\tau}^{0} E|\xi(s)|^p ds < +\infty$ $(p > 1)$.

We define a norm mean square in S^n as follows:

$$|x|_{\mathrm{MS}} = \left(E|x|^2\right)^{1/2}, \quad \text{for any } x \in S^n, \tag{6}$$

where S^n is the set of random variable in probability space $(\Omega, \mathcal{F}, P)$. Similarly, we define $\|A\|_{\text{MS}} = \sup\{|Ax|_{\text{MS}} : |x|_{\text{MS}} = 1\}$.

Then, the Itô integral of $h(t, x(t))$ (from a to b) is defined by

$$(I) \int_a^b h(t, x(t))\, dW(t) = \lim_{\lambda \to 0} {}_{\text{MS}} \sum_{i=0}^{N} h(t_i, x(t_i)) [W(t_{i+1}) - W(t_i)] \quad \left(\text{limit in } L^2(P)\right), \tag{7}$$

where h is a stochastic process with value in $S^{n\times m}$ and $\int_a^b \|h(t,x(t))\|^2_{\text{MS}} dt < \infty$ $(0 \le a < b)$. Let $a = t_0 < t_1 < \cdots < t_N = b$, $\lambda = \max\{\Delta t_i, i = 0, 1, \ldots, N-1\}$, $\Delta t_i = t_{i+1} - t_i$, and $\lim_{\text{MS}}$ be a limit in mean square sense.

It is proved directly from the definition of Itô integral that

$$\int_a^b W(t)\, dW(t) = \frac{1}{2}\left[W^2(b) - W^2(a)\right] - \frac{1}{2}(b-a), \tag{8}$$

where $W(\cdot)$ is 1-dimensional Brownian motion. The extra term $-(b-a)/2$ shows that the Itô stochastic integral does not behave like ordinary integrals. It leads to Itô-type stochastic system being different from non-Itô-type. See [19] for the details.

Now, let us present an existence and uniqueness result for system (5). First, we make the following assumptions for the coefficients of (5).

(H2.1) The maps f, $\overline{f}$, and h are locally Lipschitz continuous.

(H2.2) The maps f, $\overline{f}$, and h satisfy the linear growth condition.

Theorem 1 (see [4]). *Let (H2.1) and (H2.2) hold. Then, for any $\xi(t) \in L^2_{\mathcal{F}_0}(-\tau, 0; R^n)$, (5) has a unique strong solution which is denoted by $x(t;\xi)$, and $x(t;\xi)$ is square integrable. So, (5) has a trivial solution $x(t;0) = 0$.*

For stochastic system, exponential stability in mean square and almost surely exponential stability are generally used [2].

Definition 2. The trivial solution of system (5) is said to be pth moment exponentially stable, if there exists a positive constant ε such that

$$\limsup_{t\to\infty} \frac{1}{t} \ln\left(E|x(t;\xi)|^p\right) \le -\varepsilon \tag{9}$$

for any $\xi \in L^p_{\mathcal{F}_0}(-\tau, 0; R^n)$, where $-\varepsilon$ is called pth moment Lyapunov exponent of the trivial solution.

In particular, $p = 2$; it is called mean square exponentially stable.

Definition 3. The trivial solution of system (5) is said to be almost surely exponentially stable, if there exists a positive constant λ such that

$$\limsup_{t\to\infty} \frac{1}{t} \ln|x(t;\xi)| \le -\lambda \quad \text{a.s.} \tag{10}$$

for any $\xi \in L^p_{\mathcal{F}_0}(-\tau, 0; R^n)$, where $-\lambda$ is called almost surely Lyapunov exponent of the trivial solution.

3. Exponential Stability for Stochastic Price System with Delay

To obtain sufficient condition of the exponential stability of system (5), we assume that the functions $f(t,x,y)$, $\overline{f}(t,x,y)$ are 1-order continuously differentiable in the neighbourhood of $(t,0,0)$ with respect to $(x,y) \in R^n\times R^n$. According to Taylor expansion, for $0 < \theta < 1$

$$\begin{aligned} f(t,x,y) &= f_x(t,\theta x,\theta y)x + f_y(t,\theta x,\theta y)y, \\ \overline{f}(t,x,y) &= \overline{f}_x(t,\theta x,\theta y)x + \overline{f}_y(t,\theta x,\theta y)y. \end{aligned} \tag{11}$$

Thus,

$$\begin{aligned} f(t,x,y) + \overline{f}(t,x,y) &= \left(f_x(t,\theta x,\theta y) + \overline{f}_x(t,\theta x,\theta y)\right)x \\ &\quad + \left(f_y(t,\theta x,\theta y) + \overline{f}_y(t,\theta x,\theta y)\right)y. \end{aligned} \tag{12}$$

Therefore, system (5) can be reduced to an n-dimensional semilinear stochastic differential delayed equation as follows:

$$\begin{aligned} dx(t) &= \left[f(t,x(t),x(t-\tau)) + \overline{f}(t,x(t),x(t-\tau))\right]dt \\ &\quad + h(t,x(t),x(t-\tau))\,dW(t), \quad t \ge 0, \\ x(t) &= \xi(t), \quad -\tau \le t \le 0, \end{aligned} \tag{13}$$

where A_1, B_1 are $n\times n$ matrices. $A_1(t)$, $A_2(t-\tau)$, $B_1(t)$, and $B_2(t-\tau)$ represent the uncertainties. They are bounded $n\times n$ matrix-valued functions. Here, τ, $W(t)$, $h(t,x(t),x(t-\tau))$, and $\xi(t)$ are the same as in the previous section.

We make the following assumption for the coefficients of system (13).

(H3.1) There exist nonnegative constants α_i, β_i $(i = 1,2,3)$, for any $t \ge 0$ such that

$$\begin{aligned} \|A_1(t)\| &\le \alpha_1, & \|A_2(t-\tau)\| &\le \alpha_2, \\ \|B_1(t)\| &\le \beta_1, & \|B_2(t-\tau)\| &\le \beta_2 \end{aligned} \tag{14}$$

and for any $(t,x(t),x(t-\tau)) \in R^+\times R^n\times R^n$ such that

$$\begin{aligned} &\operatorname{tr}\left[h^T(t,x(t),x(t-\tau))\,h(t,x(t),x(t-\tau))\right] \\ &\quad \le \alpha_3|x(t)|^2 + \beta_3|x(t-\tau)|^2. \end{aligned} \tag{15}$$

Theorem 4. *Let h satisfy (H2.1)-(H2.2), and condition* (14) *hold. Then, for any $\xi(t) \in L^2_{\mathcal{F}_0}(-\tau, 0; R^n)$, system* (13) *has a unique strong solution which is denoted by $x(t;\xi)$, and $x(t;\xi)$ is square integrable. So,* (13) *has a trivial solution $x(t;0) = 0$.*

See Mao [10] for the proof of Theorem 4.

In the study of mean square exponential stability, it is often to use a quadratic function as the Lyapunov function; that is, $V(t,x) = x^T(t)Gx(t)$, where G is a symmetric positive definite $n \times n$ matrix (see [11, 20]).

Theorem 5. *Let (H3.1) holds, and then the trivial solution of system* (13) *is exponentially stable in the mean square. Assume that there exists a pair of symmetric positive definite $n \times n$ matrices G and Q such that*

$$G(A+B) + (A+B)^T G = -Q, \tag{16}$$

$$\begin{aligned} &\lambda_{\min}(Q) \\ &\quad > \|G\| (2\alpha_1 + 2\alpha_2 + 2\beta_1 + 2\beta_2 + \alpha_3 + \beta_3) + 2\|GB\| \\ &\qquad \cdot \sqrt{2\tau \left[6\tau \left(\|A\|^2 + \|B\|^2 + \alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2\right) + \alpha_3 + \beta_3\right]}, \end{aligned} \tag{17}$$

where $\lambda_{\min}(Q) > 0$ is the smallest eigenvalue of Q.

Proof. Fix the initial data $\xi(t)$ arbitrarily, and write $x(t;\xi) = x(t)$ simply. Applying Itô's formula to $x^T(t)Gx(t)$, we have

$$\begin{aligned} &d\left[x^T(t) Gx(t)\right] \\ &\quad = 2x^T(t) G [Ax(t) + Bx(t-\tau)]\, dt \\ &\qquad + 2x^T(t) G \left[A_1(t) x(t) + B_1(t) x(t-\tau)\right] dt \\ &\qquad + 2x^T(t) G \left[A_2(t-\tau) x(t) + B_2(t-\tau) x(t-\tau)\right] dt \\ &\qquad + 2x^T G h(t, x(t), x(t-\tau))\, dW(t) \\ &\qquad + \operatorname{tr}\left[h^T(t, x(t), x(t-\tau)) G h(t, x(t), x(t-\tau))\right] dt. \end{aligned} \tag{18}$$

By (16), we can estimate the first item of (18) as follows:

$$\begin{aligned} &2x^T(t) G [Ax(t) + Bx(t-\tau)] \\ &\quad = 2x^T(t) G [Ax(t) + Bx(t-\tau)] \\ &\qquad + 2x^T(t) GBx(t) - 2x^T(t) GBx(t) \\ &\quad \le -\lambda_{\min}(Q) |x(t)|^2 + \beta |x(t)|^2 \\ &\qquad + \frac{1}{\beta} \|GB\|^2 \cdot |x(t) - x(t-\tau)|^2, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \beta = {} & \|GB\| \\ & \cdot \sqrt{2\tau \left[6\tau \left(\|A\|^2 + \|B\|^2 + \alpha_1^2 + \beta_1^2 + \alpha_2^2 + \beta_2^2\right) + \alpha_3 + \beta_3\right]}. \end{aligned} \tag{20}$$

By (14), we can estimate the second item and the third item of (18), respectively,

$$\begin{aligned} &2x^T(t) G \left[A_1(t) x(t) + B_1(t) x(t-\tau)\right] \\ &\quad \le \|G\| \left[(2\alpha_1 + \beta_1) |x(t)|^2 + \beta_1 |x(t-\tau)|^2\right], \\ &2x^T(t) G \left[A_2(t-\tau) x(t) + B_2(t) x(t-\tau)\right] \\ &\quad \le \|G\| \left[(2\alpha_2 + \beta_2) |x(t)|^2 + \beta_2 |x(t-\tau)|^2\right]. \end{aligned} \tag{21}$$

By (15), the last item of (18) yields

$$\begin{aligned} &\operatorname{tr}\left[h^T(t, x(t), x(t-\tau)) G h(t, x(t), x(t-\tau))\right] \\ &\quad \le \|G\| \left[\alpha_3 |x(t)|^2 + \beta_3 |x(t-\tau)|^2\right]. \end{aligned} \tag{22}$$

Substituting (21) and (22) into (18), we get

$$\begin{aligned} &d\left[x^T(t) Gx(t)\right] \\ &\quad \le -\left[\lambda_{\min}(Q) - \|G\| (2\alpha_1 + \beta_1 + 2\alpha_2 + \beta_2 + \alpha_3) - \beta\right] \\ &\qquad \cdot |x(t)|^2 dt + \|G\| (\beta_1 + \beta_2 + \beta_3) |x(t-\tau)|^2 dt \\ &\qquad + \frac{1}{\beta} \|GB\|^2 \cdot |x(t) - x(t-\tau)|^2 dt \\ &\qquad + 2x^T(t) G h(t, x(t), x(t-\tau))\, dW(t). \end{aligned} \tag{23}$$

If (17) holds, then we can choose $\varepsilon > 0$ small enough such that

$$\begin{aligned} &\lambda_{\min}(Q) \\ &\quad = \|G\| (2\alpha_1 + 2\alpha_2 + \beta_1 + \beta_2 + \alpha_3 + \varepsilon) \\ &\qquad + \|G\| (\beta_1 + \beta_2 + \beta_3) e^{\varepsilon\tau} + \beta \\ &\qquad + \frac{1}{\beta} \|GB\|^2 \left\{2\tau \left[6\tau \left(\|A\|^2 + \alpha_1^2 + \alpha_2^2\right) + \alpha_3\right] e^{\varepsilon\tau} \right. \\ &\qquad\qquad \left. + 2\tau \left[6\tau \left(\|B\|^2 + \beta_1^2 + \beta_2^2\right) + \beta_3\right] e^{2\varepsilon\tau}\right\}. \end{aligned} \tag{24}$$

Applying Itô's formula to $e^{\varepsilon t} x^T(t) Gx(t)$, we have

$$\begin{aligned} d\left[e^{\varepsilon t} x^T(t) Gx(t)\right] = {} & \varepsilon e^{\varepsilon t} x^T(t) Gx(t)\, dt \\ & + e^{\varepsilon t} d\left[x^T(t) Gx(t)\right], \quad \text{for any } t \ge 0. \end{aligned} \tag{25}$$

Substituting (23) into (25) yields

$$\begin{aligned}
&e^{\varepsilon t}x^T(t)Gx(t)\\
&\leq \xi^T(0)G\xi(0)+\varepsilon\int_0^t e^{\varepsilon s}x^T(s)Gx(s)\,ds\\
&\quad-\int_0^t e^{\varepsilon s}\left[\lambda_{\min}(Q)-\|G\|\left(2\alpha_1+2\alpha_2+\beta_1+\beta_2+\alpha_3\right)-\beta\right]\\
&\quad\cdot|x(s)|^2ds+\int_0^t e^{\varepsilon s}\cdot\|G\|\left(\beta_1+\beta_2+\beta_3\right)|x(s-\tau)|^2ds\\
&\quad+\int_0^t e^{\varepsilon s}\cdot\frac{1}{\beta}\|GB\|^2\cdot|x(s)-x(s-\tau)|^2ds\\
&\quad+\int_0^t e^{\varepsilon s}\cdot 2x^T(s)Gh(s,x(s),x(s-\tau))\,dW(s).
\end{aligned}\tag{26}$$

Taking the expectation in (26), we have

$$\begin{aligned}
&E\left(e^{\varepsilon t}x^T(t)Gx(t)\right)\\
&\quad\leq E\left(\xi^T(0)G\xi(0)\right)\\
&\qquad-\left[\lambda_{\min}(Q)-\|G\|\left(2\alpha_1+2\alpha_2+\beta_1+\beta_2+\alpha_3+\varepsilon\right)-\beta\right]\\
&\qquad\cdot\int_0^t e^{\varepsilon s}\cdot E|x(s)|^2ds\\
&\qquad+\|G\|\left(\beta_1+\beta_2+\beta_3\right)\int_0^t e^{\varepsilon s}\cdot E|x(s-\tau)|^2ds\\
&\qquad+\frac{1}{\beta}\|GB\|^2\int_0^t e^{\varepsilon s}\cdot E|x(s)-x(s-\tau)|^2ds.
\end{aligned}\tag{27}$$

In order to give an estimate of $E(e^{\varepsilon t}x^T(t)Gx(t))$, we now estimate the last two terms on the right-hand side of (27). First of all, for any $t\geq\tau$, we have

$$\begin{aligned}
&\int_0^t e^{\varepsilon s}\cdot E|x(s-\tau)|^2ds\\
&\quad=\int_0^\tau e^{\varepsilon s}\cdot E|x(s-\tau)|^2ds+\int_\tau^t e^{\varepsilon s}\cdot E|x(s-\tau)|^2ds\\
&\quad\leq e^{\varepsilon\tau}\cdot\int_0^\tau E|\xi(s-\tau)|^2ds+e^{\varepsilon\tau}\cdot\int_\tau^t e^{\varepsilon(s-\tau)}\cdot E|x(s-\tau)|^2ds\\
&\quad=e^{\varepsilon\tau}\cdot\int_{-\tau}^0 E|\xi(u)|^2du+e^{\varepsilon\tau}\\
&\qquad\cdot\int_0^{t-\tau}e^{\varepsilon u}\cdot E|x(u)|^2du\qquad(u=s-\tau)\\
&\quad=c_1e^{\varepsilon\tau}+e^{\varepsilon\tau}\cdot\int_0^{t-\tau}e^{\varepsilon s}\cdot E|x(s)|^2ds,
\end{aligned}\tag{28}$$

where $c_1=\int_{-\tau}^0 E|x(s)|^2ds$.

Next, recalling (14) and (15), for $s\geq\tau$, we derive

$$\begin{aligned}
&E|x(s)-x(s-\tau)|^2\\
&\quad\leq 2\tau\cdot E\int_{s-\tau}^s 6\left[\left(\|A\|^2+\alpha_1^2+\alpha_2^2\right)|x(t)|^2\right.\\
&\qquad\qquad\left.+\left(\|B\|^2+\beta_1^2+\beta_2^2\right)|x(t-\tau)|^2\right]dt\\
&\qquad+2E\int_{s-\tau}^s\left(\alpha_3|x(t)|^2+\beta_3x(t-\tau)^2\right)dt\\
&\quad\leq 2\left[6\tau\left(\|A\|^2+\alpha_1^2+\alpha_2^2\right)+\alpha_3\right]\int_{s-\tau}^s E|x(t)|^2dt\\
&\qquad+2\left[6\tau\left(\|B\|^2+\beta_1^2+\beta_2^2\right)+\beta_3\right]\int_{s-\tau}^s E|x(t-\tau)|^2dt.
\end{aligned}\tag{29}$$

Similar to (28), for $t\geq\tau$, we have

$$\begin{aligned}
&\int_0^t e^{\varepsilon s}\cdot E|x(s)-x(s-\tau)|^2ds\\
&\quad=\int_0^\tau e^{\varepsilon s}\cdot E|x(s)-x(s-\tau)|^2ds\\
&\qquad+\int_\tau^t e^{\varepsilon s}\cdot E|x(s)-x(s-\tau)|^2ds\\
&\quad\leq c_2+2\left[6\tau\left(\|A\|^2+\alpha_1^2+\alpha_2^2\right)+\alpha_3\right]\\
&\qquad\cdot\int_\tau^t e^{\varepsilon s}\int_{s-\tau}^s E|x(\nu)|^2d\nu\,ds\\
&\qquad+2\left[6\tau\left(\|B\|^2+\beta_1^2+\beta_2^2\right)+\beta_3\right]\\
&\qquad\cdot\int_\tau^t e^{\varepsilon s}\int_{s-\tau}^s E|x(\nu-\tau)|^2d\nu\,ds,
\end{aligned}\tag{30}$$

where $c_2=\int_0^\tau e^{\varepsilon s}\cdot E|x(s)-x(s-\tau)|^2ds$. Moreover,

$$\begin{aligned}
\int_\tau^t e^{\varepsilon s}\int_{s-\tau}^s E|x(\nu)|^2d\nu\,ds&=\int_0^t E|x(\nu)|^2\left(\int_{\nu\vee\tau}^{(\nu+\tau)\wedge t}e^{\varepsilon s}ds\right)d\nu\\
&\leq\tau e^{\varepsilon\tau}\int_0^t e^{\varepsilon\nu}\cdot E|x(\nu)|^2d\nu.
\end{aligned}\tag{31}$$

Similarly,

$$\begin{aligned}
&\int_\tau^t e^{\varepsilon s}\int_{s-\tau}^s E|x(\nu-\tau)|^2d\nu\,ds\\
&\quad=\int_0^t E|x(\nu-\tau)|^2\left(\int_{\nu\vee\tau}^{(\nu+\tau)\wedge t}e^{\varepsilon s}ds\right)d\nu\\
&\quad\leq\tau e^{\varepsilon\tau}\int_0^t e^{\varepsilon\nu}\cdot E|x(\nu-\tau)|^2d\nu.
\end{aligned}\tag{32}$$

Substituting (28) into (32) yields

$$\begin{aligned}&\int_{\tau}^{t} e^{\varepsilon s}\int_{s-\tau}^{s} E|x(\nu-\tau)|^2 d\nu\, ds\\&\quad\le \tau e^{\varepsilon\tau}\left(c_1 e^{\varepsilon\tau}+e^{\varepsilon\tau}\cdot\int_0^{t-\tau} e^{\varepsilon s}\cdot E|x(s)|^2 ds\right)\\&\quad< \tau c_1 e^{2\varepsilon\tau}+\tau e^{2\varepsilon\tau}\int_0^t e^{\varepsilon s}\cdot E|x(s)|^2 ds.\end{aligned}\tag{33}$$

Substituting (31) and (33) into (30), for $t\ge\tau$, we get

$$\begin{aligned}&\int_0^t e^{\varepsilon s}\cdot E|x(s)-x(s-\tau)|^2 ds\\&\quad\le c_2+2\left[6\tau\left(\|A\|^2+\alpha_1^2+\alpha_2^2\right)+\alpha_3\right]\\&\qquad\cdot\tau e^{\varepsilon\tau}\int_0^t e^{\varepsilon\nu}\cdot E|x(\nu)|^2 d\nu\\&\qquad+2\left[6\tau\left(\|B\|^2+\beta_1^2+\beta_2^2\right)+\beta_3\right]\\&\qquad\cdot\left(\tau c_1 e^{2\varepsilon\tau}+\tau e^{2\varepsilon\tau}\int_0^t e^{\varepsilon s}\cdot E|x(s)|^2 ds\right).\end{aligned}\tag{34}$$

Substituting (28) and (34) into (27) and recalling (24), we obtain that for $t\ge\tau$

$$\begin{aligned}&E\left(e^{\varepsilon t}x^T(t)Gx(t)\right)\\&\quad\le E\left(\xi^T(0)G\xi(0)\right)\\&\qquad-\left[\lambda_{\min}(Q)-\|G\|\left(2\alpha_1+2\alpha_2+\beta_1+\beta_2+\alpha_3+\varepsilon\right)-\beta\right]\\&\qquad\cdot\int_0^t e^{\varepsilon s}\cdot E|x(s)|^2 ds\\&\qquad+\|G\|\left(\beta_1+\beta_2+\beta_3\right)\left(c_1 e^{\varepsilon\tau}+e^{\varepsilon\tau}\cdot\int_0^{t-\tau} e^{\varepsilon s}\cdot E|x(s)|^2 ds\right)\\&\qquad+\frac{1}{\beta}\|GB\|^2\left\{c_2+2\left[6\tau\left(\|A\|^2+\alpha_1^2+\alpha_2^2\right)+\alpha_3\right]\right.\\&\qquad\qquad\cdot\tau e^{\varepsilon\tau}\int_0^t e^{\varepsilon s}\cdot E|x(s)|^2 ds\\&\qquad\qquad+2\left[6\tau\left(\|B\|^2+\beta_1^2+\beta_2^2\right)+\beta_3\right]\\&\qquad\qquad\left.\cdot\left(\tau c_1 e^{2\varepsilon\tau}+\tau e^{2\varepsilon\tau}\int_0^t e^{\varepsilon s}\cdot E|x(s)|^2 ds\right)\right\}\\&\quad= c_3,\end{aligned}\tag{35}$$

where

$$\begin{aligned}c_3&= E\left(\xi^T(0)G\xi(0)\right)+\|G\|\left(\beta_1+\beta_2+\beta_3\right)\cdot c_1 e^{\varepsilon\tau}\\&\quad+\frac{1}{\beta}\|GB\|^2\left\{c_2+2\left[6\tau\left(\|B\|^2+\beta_1^2+\beta_2^2\right)+\beta_3\right]\cdot\tau c_1 e^{2\varepsilon\tau}\right\}.\end{aligned}\tag{36}$$

Since G is positive definite,

$$x^T(t)Gx(t)\ge\lambda_{\min}(G)|x(t)|^2,\tag{37}$$

where $\lambda_{\min}(G)>0$ is the smallest eigenvalue of G.

Thus,

$$E\left(e^{\varepsilon t}x^T(t)Gx(t)\right)\ge E\left(e^{\varepsilon t}\lambda_{\min}(G)|x(t)|^2\right).\tag{38}$$

It then follows from (35) that

$$E|x(t)|^2\le\frac{c_3}{\lambda_{\min}(G)}\cdot e^{-\varepsilon t}\quad\text{for any } t\ge\tau.\tag{39}$$

Hence,

$$\begin{aligned}\frac{1}{t}\ln\left(E|x(t)|^2\right)&\le\frac{1}{t}\ln\left(\frac{c_3}{\lambda_{\min}(G)}\cdot e^{-\varepsilon t}\right)\\&=-\varepsilon+\frac{1}{t}\ln\left(\frac{c_3}{\lambda_{\min}(G)}\right).\end{aligned}\tag{40}$$

This easily yields

$$\limsup_{t\to\infty}\frac{1}{t}\ln\left(E|x(t)|^2\right)\le-\varepsilon.\tag{41}$$

Then, (13) is exponentially stable in the mean square. The proof is complete. □

Remark 6. In the proof we gave, the estimate for the second moment Lyapunov exponent should not be greater than $-\varepsilon$.

Theorem 7. *The trivial solution of system* (13) *is also almost surely exponentially stable under the same assumption as Theorem 5.*

Proof. For $t\in[k\tau,(k+1)\tau]$, $k=2,3,\ldots$, we have

$$\begin{aligned}&|x(t)|^2\\&\quad\le 3|x(k\tau)|^2\\&\qquad+3\left|\int_{k\tau}^{(k+1)\tau}\left[\left(A+A_1(t)+A_2(t-\tau)\right)x(t)\right.\right.\\&\qquad\qquad\left.\left.+\left(B+B_1(t)+B_2(t-\tau)\right)x(t-\tau)\right]dt\right|^2\\&\qquad+3\left|\int_{k\tau}^{(k+1)\tau}h(t,x(t),x(t-\tau))\,dW(t)\right|^2.\end{aligned}\tag{42}$$

Recalling (14) and (15), we derive

$$\begin{aligned}E&\left(\sup_{k\tau\le t\le(k+1)\tau}|x(t)|^2\right)\\&\le 3E|x(k\tau)|^2\\&\quad+18\tau\int_{k\tau}^{(k+1)\tau}\Big[\left(\|A\|^2+\alpha_1^2+\alpha_2^2\right)E|x(t)|^2\\&\qquad+\left(\|B\|^2+\beta_1^2+\beta_2^2\right)E|x(t-\tau)|^2\Big]dt\\&\quad+3\int_{k\tau}^{(k+1)\tau}\left[\alpha_3E|x(t)|^2+\beta_3E|x(t-\tau)|^2\right]dt.\end{aligned}\tag{43}$$

By (39), we easily get

$$\begin{aligned}E|x(k\tau)|^2&\le\frac{c_3}{\lambda_{\min}(G)}\cdot e^{-\varepsilon k\tau}\quad\text{for } k\tau\ge\tau,\\E|x(t-\tau)|^2&\le\frac{c_3}{\lambda_{\min}(G)}\cdot e^{-\varepsilon(t-\tau)}\quad\text{for } t-\tau\ge\tau.\end{aligned}\tag{44}$$

Substituting the above two into (43) yields

$$\begin{aligned}E&\left(\sup_{k\tau\le t\le(k+1)\tau}|x(t)|^2\right)\\&\le\frac{3c_3}{\lambda_{\min}(G)}\Big\{e^{-\varepsilon k\tau}+\left[6\tau\left(\|A\|^2+\alpha_1^2+\alpha_2^2\right)+\alpha_3\right]\\&\qquad\cdot\int_{k\tau}^{(k+1)\tau}e^{-\varepsilon t}dt+\left[6\tau\left(\|B\|^2+\beta_1^2+\beta_2^2\right)+\beta_3\right]\\&\qquad\cdot e^{\varepsilon\tau}\int_{k\tau}^{(k+1)\tau}e^{-\varepsilon t}dt\Big\}\\&\le\frac{3c_3}{\lambda_{\min}(G)}\Big\{e^{-\varepsilon k\tau}+\Big(\left[6\tau\left(\|A\|^2+\alpha_1^2+\alpha_2^2\right)+\alpha_3\right]\\&\qquad+\left[6\tau\left(\|B\|^2+\beta_1^2+\beta_2^2\right)+\beta_3\right]e^{\varepsilon\tau}\Big)\\&\qquad\cdot\frac{1}{\varepsilon}e^{-\varepsilon k\tau}\left(1-e^{-\varepsilon\tau}\right)\Big\}\\&\le\frac{3c_3}{\lambda_{\min}(G)}\Big\{e^{-\varepsilon k\tau}+\Big(\left[6\tau\left(\|A\|^2+\alpha_1^2+\alpha_2^2\right)+\alpha_3\right]\\&\qquad+\left[6\tau\left(\|B\|^2+\beta_1^2+\beta_2^2\right)+\beta_3\right]e^{\varepsilon\tau}\Big)\tau e^{-\varepsilon k\tau}\Big\}\\&=\frac{3\tau c_3}{\lambda_{\min}(G)}\Big\{\frac{1}{\tau}+\left[6\tau\left(\|A\|^2+\alpha_1^2+\alpha_2^2\right)+\alpha_3\right]\\&\qquad+\left[6\tau\left(\|B\|^2+\beta_1^2+\beta_2^2\right)+\beta_3\right]e^{\varepsilon\tau}\Big\}\cdot e^{-\varepsilon k\tau}\\&=c_4e^{-\varepsilon k\tau},\end{aligned}\tag{45}$$

where $c_4=(3\tau c_3/\lambda_{\min}(G))\{(1/\tau)+[6\tau(\|A\|^2+\alpha_1^2+\alpha_2^2)+\alpha_3]+[6\tau(\|B\|^2+\beta_1^2+\beta_2^2)+\beta_3]e^{\varepsilon\tau}\}$.

Let $\varepsilon_0\in(0,\varepsilon)$ be arbitrary. By Doob's martingale inequality, it follows from (45) that

$$P\left(\omega:\sup_{k\tau\le t\le(k+1)\tau}|x(t)|>e^{-(\varepsilon-\varepsilon_0)k\tau/2}\right)\le c_4e^{-\varepsilon_0k\tau}.\tag{46}$$

It then follows from the Borel-Cantelli lemma that for almost all $\omega\in\Omega$, there exists a $k_0(\omega)$, $k\ge k_0(\omega)$, and

$$\sup_{k\tau\le t\le(k+1)\tau}|x(t)|\le e^{-(\varepsilon-\varepsilon_0)k\tau/2}\tag{47}$$

holds in probability 1. Thus, for $k\tau\le t\le(k+1)\tau$, and $k\ge k_0(\omega)$, we have

$$\ln|x(t)|\le-\frac{(\varepsilon-\varepsilon_0)k\tau}{2}\quad\text{a.s.}\tag{48}$$

Hence,

$$\frac{1}{t}\ln|x(t)|\le-\frac{\varepsilon-\varepsilon_0}{2}\quad\text{a.s.}\tag{49}$$

This easily yields

$$\limsup_{t\to\infty}\frac{1}{t}\ln|x(t)|\le-\frac{\varepsilon-\varepsilon_0}{2}\quad\text{a.s.}\tag{50}$$

Since ε_0 is arbitrary, we must have

$$\limsup_{t\to\infty}\frac{1}{t}\ln|x(t)|\le-\frac{\varepsilon}{2}\quad\text{a.s.}\tag{51}$$

Then, (13) is almost surely exponentially stable. The proof is complete. □

Remark 8. Again in the proof we gave, the estimate for the almost surely Lyapunov exponent should not be greater than $-\varepsilon/2$.

Let us single out three important special cases.

Case 1. If $h(t,x(t),x(t-\tau))\equiv 0$, then (13) reduces to a semilinear differential delay equation

$$\begin{aligned}dx(t)=&\left[\left(A+A_1(t)+A_2(t-\tau)\right)x(t)\right.\\&\left.+\left(B+B_1(t)+B_2(t-\tau)\right)x(t-\tau)\right]dt.\end{aligned}\tag{52}$$

Corollary 9. *Let condition* (14) *holds. Assume that there exists a pair of symmetric positive definite n×n matrices G and Q such that*

$$G(A+B)+(A+B)^TG=-Q,\tag{53}$$

$$\begin{aligned}\lambda_{\min}&(Q)\\&>2\|G\|\left(\alpha_1+\alpha_2+\beta_1+\beta_2\right)\\&\quad+4\tau\|GB\|\cdot\sqrt{3\left(\|A\|^2+\|B\|^2+\alpha_1^2+\beta_1^2+\alpha_2^2+\beta_2^2\right)}.\end{aligned}\tag{54}$$

Then, (52) *is exponentially stable in the mean square.*

Case 2. Let us further assume that $A_1(t) = A_2(t-\tau) = B_1(t) = B_2(t-\tau) \equiv 0$, then (52) becomes an ordinary differential equation

$$dx(t) = (Ax(t) + Bx(t-\tau))\,dt. \tag{55}$$

Corollary 10. *Assume that there exists a pair of symmetric positive definite $n \times n$ matrices G and Q such that*

$$G(A+B) + (A+B)^T G = -Q, \tag{56}$$

$$\lambda_{\min}(Q) > 4\tau \|GB\| \cdot \sqrt{3\left(\|A\|^2 + \|B\|^2\right)}. \tag{57}$$

Then, (55) *is exponentially stable in the mean square.*

Remark 11. Corollary 10 clearly shows that if the price system $\dot{x}(t) = (A+B)x(t)$ (the general case of the classical price Rayleigh equation (2)) is exponentially stable (this is guaranteed by condition (16) and $\lambda_{\min}(Q) > 0$), then the corresponding delayed system (55) is also exponentially stable provided that the time delay τ is small enough (bounded by (57)).

Remark 12. Condition (17) controls the intensity of the uncertainties, that is, the parameters α_i, β_i ($i = 1, 2, 3$), and the time delay τ should be small enough to have the stability of (13) which is regarded as the perturbed system of (55). In other words, Theorem 5 shows that if (55) is exponentially stable, so is its perturbed system (13) provided that the intensity of perturbations is small enough.

Case 3. Let us assume that $B = A_2(t-\tau) = B_1(t) = B_2(t-\tau) \equiv 0$ and $\tau \equiv 0$, then (13) reduces to a stochastic differential equation

$$dx(t) = \left(A + A_1(t)\right)x(t)\,dt + h(t, x(t))\,dW(t). \tag{58}$$

Corollary 13. *Assume that there exists a pair of symmetric positive definite $n \times n$ matrices G and Q such that*

$$GA + A^T G = -Q, \tag{59}$$

$$\lambda_{\min}(Q) > \|G\|\left(2\alpha_1 + \alpha_3\right). \tag{60}$$

Then, (58) *is exponentially stable in the mean square and is also almost surely exponentially stable.*

4. Example

Let us now present two simple examples to illustrate our results which can help us find the time delay upper limit.

Example 14. Let us start with (13), where $\alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0.1$,

$$A = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}. \tag{61}$$

For convenience, let us choose Q = the 2-order identity matrix, and then $\lambda_{\min}(Q) = 1$. By plugging these into (16), it is easy to find that

$$G = \begin{bmatrix} \dfrac{1}{6} & 0 \\ 0 & \dfrac{1}{4} \end{bmatrix}. \tag{62}$$

We obtain via a simple calculation

$$\|A\|^2 = 2.618, \qquad \|B\|^2 = 5.2361, \qquad \|G\| = 0.25, \qquad \|GB\| = 0.4488. \tag{63}$$

Substituting (63) into (17), we derive that if $\tau < 0.08380$, then (13) is exponentially stable in the mean square and is also almost surely exponentially stable. If $h(t, x(t), x(t-\tau)) \equiv 0$, by Corollary 9, we would conclude that (52) is exponentially stable provided that $\tau < 0.09156$.

Example 15. Now, let us recall price system (4) with $a = -0.5$, $b_0 = 1.0$, $c_0 = 1.0$, $\gamma = 0.2$, $\varepsilon = 0.1$, $\sqrt{\lambda\alpha} = 1$, and $\tilde{\mu} = 0.06$. It is clear that $z(t) = (0, 0)^T$ is the trivial solution of system (4). Applying Taylor expansion to $f(t, z(t))$, we get

$$f(t, z(t)) = f(t, 0) + Az(t) + A_1(t)z(t), \tag{64}$$

where $A = \left[\begin{smallmatrix} 0.006 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, $\operatorname{tr}[h^T(t, x(t))h(t, x(t))] = 0.004$, and $\|A_1(t)\| \le 0.001$.

To compute conveniently, let us choose $\alpha_3 = 0.004$ and $Q = \left[\begin{smallmatrix} 0.01 & 0 \\ 0 & 0.01 \end{smallmatrix}\right]$, and then $\lambda_{\min}(Q) = 0.01$. By Corollary 13, it is easy to find

$$G = \begin{bmatrix} 1.66667 & -0.00500 \\ -0.00500 & 1.66670 \end{bmatrix}, \qquad \|G\| = 0.5285. \tag{65}$$

Substituting (65) into (60), it is easy to verify that (60) holds. So, the price system (4) with $a = -0.5$, $b_0 = 1.0$, $c_0 = 1.0$, $\gamma = 0.2$, $\varepsilon = 0.1$, $\sqrt{\lambda\alpha} = 1$, and $\tilde{\mu} = 0.06$ is exponentially stable in the mean square and is also almost surely exponentially stable. This result is the same as [5].

5. Concluding Remarks

In this paper, we study the exponential stability of the stochastic nonlinear dynamical price system. Using Taylor's theorem, the stochastic nonlinear system with delay is reduced to an n-dimensional semilinear stochastic differential equation with delay. Some sufficient conditions of exponential stability and corollaries for the price system are established by virtue of Lyapunov function. The time delay upper limit is solved by using our theoretical results when the system is exponentially stable, and [5] is promoted and improved. Remarks 11 and 12 show that if price Rayleigh equation (2) is exponentially stable, so is its perturbed system (13) provided that both the time delay and the intensity of perturbations are small enough. These results are very helpful for our government strengthening and improving macrocontrol and promoting

steady and rapid economic development. It is also an important guiding significance that our government can timely adjust their pricing strategies. Two examples are presented to illustrate our theoretical results, which are the same as [5]. Another challenging problem is to study a type of stochastic nonlinear dynamical price system with variable delay. We hope to study these problems in forthcoming papers.

Acknowledgments

This work is Supported by the Fundamental Research Funds for the Central Universities (JBK130213) and the Fundamental Research Funds for the Central Universities (JBK130401).

References

[1] S. Wang, *Differential Equation Model and Chaos*, China Science and Technology University Press, Hefei, China, 2002.

[2] Y. Liu and Z. Feng, "Stochastic stability & control," in *Large Power System Theory and Application*, South China University of Science and Technology Press, Guangzhou, China, 4th edition, 1992.

[3] Y. Liu and F. Deng, "Random system variable structure system," in *Large Power System Theory and Application*, South China University of Science and Technology Press, Guangzhou, China, 10th edition, 1992.

[4] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, vol. 43 of *Applications of Mathematics*, Springer, New York, NY, USA, 1999.

[5] J. R. Li and W. Xu, "Optimal control of a stochastic nonlinear dynamical price model," *Pure and Applied Mathematics*, vol. 24, no. 2, pp. 239–244, 2008.

[6] Y. Kazmerchuk, A. Swishchuk, and J. Wu, "The pricing of options for securities markets with delayed response," *Mathematics and Computers in Simulation*, vol. 75, no. 3-4, pp. 69–79, 2007.

[7] T. Lv and Z. W. Liu, "Hopf bifurcation of price Rayleigh equations with delay," *Journal of Jilin University*, vol. 47, no. 3, pp. 441–448, 2009.

[8] T. Lv and L. Zhou, "Hopf and codimension two bifurcation in price Rayleigh equation with two time delay," *Journal of Jilin University*, vol. 50, no. 3, pp. 441–448, 2012.

[9] X. Zhang, X. Chen, and Y. Chen, "A qualitative analysis of price model in differential equations of price," *Journal of Shenyang Institute of Aeronautical Engineering*, vol. 21, no. 1, pp. 83–86, 2004.

[10] X. Mao, *Exponential Stability of Stochastic Differential Equations*, vol. 182 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1994.

[11] W. Zhu, J. Hu, and J. Zhang, "Stability analysis of neural networks with time delays via energy functions approach," in *Proceedings of the 7th International Conference on Natural Computation (ICNC '11)*, vol. 1, pp. 232–236, 2011.

[12] W. Zhu and Z. Yi, "Integral input-to-state stability of nonlinear control systems with delays," *Chaos, Solitons & Fractals*, vol. 34, no. 2, pp. 420–427, 2007.

[13] H. Trinh and M. Aldeen, "On robustness and stabilization of linear systems with delayed nonlinear perturbations," *IEEE Transactions on Automatic Control*, vol. 42, no. 7, pp. 1005–1007, 1997.

[14] X. Mao, "Robustness of exponential stability of stochastic differential delay equations," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 442–447, 1996.

[15] X. Mao and A. Shah, "Exponential stability of stochastic differential delay equations," *Stochastics and Stochastics Reports*, vol. 60, no. 1-2, pp. 135–153, 1997.

[16] W. Zhu and J. Hu, "Stability analysis of stochastic delayed cellular neural networks by LMI approach," *Chaos, Solitons & Fractals*, vol. 29, no. 1, pp. 171–174, 2006.

[17] W. Zhu and J. Hu, "Exponential stability of stochastic recurrent neural network with time delays," *International Journal of Computational Intelligence Research*, vol. 2, no. 1, pp. 52–54, 2006.

[18] S. Xie and L. Xie, "Stabilization of a class of uncertain large-scale stochastic systems with time delays," *Automatica*, vol. 36, no. 1, pp. 161–167, 2000.

[19] B. Øksendal, *Stochastic Differential Equations: An Introduction with Applications*, Springer, Berlin, Germany, 5th edition, 1998.

[20] W. Zhu, J. Hu, P. Li, and C. Zhu, "On the energy functions of neural networks with delay," *Dynamics of Continuous, Discrete & Impulsive Systems B*, vol. 18, no. 5, pp. 589–599, 2011.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 679369, 5 pages
http://dx.doi.org/10.1155/2013/679369

Research Article

A Two-Stage Model Queueing with No Waiting Line between Channels

Vedat Sağlam[1] and Müjgan Zobu[2]

[1] Department of Statistics, Ondokuz Mayıs University, 55139 Samsun, Turkey
[2] Department of Statistics, Amasya University, 05100 Amasya, Turkey

Correspondence should be addressed to Müjgan Zobu; mujganzobu@hotmail.com

Received 5 December 2012; Revised 22 April 2013; Accepted 20 May 2013

Academic Editor: Suiyang Khoo

We consider a new queuing model with sequential two stations (stages), single server at each station, where no queue is allowed at station 2 and with no restriction at station 1. There is a FCFS service discipline in which the input stream is Poisson having rate λ. The service time of any customer at server i ($i = 1, 2$) is exponential with parameter μ_i. The state probabilities and loss probability of this model are given. The performance measures are obtained and optimized, and, additionally, the model is simulated. The simulation results, exact results, and optimal results of the performance measures are numerically computed for different parameters.

1. Introduction

New models of queuing theory have been needed lately concerning the developments in areas such as production line, communication, and computer systems. One of these necessary models is for a tandem queuing system. Many important studies have been done in this area. The mean waiting time and the mean customer number at two tandem channels (servers) which are Poisson arrival and exponential service time were given in [1]. The mean customer number, distribution of waiting time, and the probabilities of various numbers of the tandem queuing system at every stage of the Poisson arrival and the exponential service time of the tandem queuing system were found in [2]. In [3], it was proved that if arrivals to the system are Poisson process with the parameter, then the output of this system is also Poisson process with a parameter λ. A more complicated example with network analysis was studied in [4]. In queuing theory, it is usually assumed that service channels are homogeneous. However, it is seen that in real queuing systems, service channels sometimes are heterogeneous. Understanding such systems is important for finding solutions to both theoretical and technical problems. Under the condition that the sum of service rates is fixed, homogeneous systems have been compared with the heterogeneous systems for performance measures in [5–9]. The measures of effectiveness for tandem queues with blocking were calculated according to an approximation method and simulated in [10]. The tandem queues with one server in the first queue, and $n \geq 1$ servers in the second queue, where the arrivals to the system with Poisson process having parameter λ and there is no waiting room between the two stages, were analyzed and some probabilities for the number of customers were found in [11]. In the literature usually, for some similar models to ours, blocked tandem queueing systems have been studied and probabilities of number of customers have been obtained. Approximate and simulation results of performance measures have been obtained. In our proposed model, there is no restriction for the first stage, there is no waiting room between both stages and no blocking (i.e., customers leave the system after having service in the first stage if the second stage is busy). Probabilities of being-nonbeing of customers in the first and second stages, performance measures, and the optimal values of these measures are theoretically obtained by analyzing our proposed method. Also, we have compared some results by obtaining simulation results.

In real life, while there is waiting case in the service systems, because of obligations and urgency and unavailability of desired features, the loss may occur in the beginning of second stage as in our proposed model. Therefore, we have decided to construct such a model and analyze it. The following two examples can be proposed as potential applications of our model under some conditions.

The following examples can be proposed as the applications of our model on some topics.

(a) *Fruit-Vegetable Packing Line*. There is a company which exports fresh fruits and vegetables. This company has a product packing line which consists of two stages, for a special export fruit. The product comes to first stage for quality control. Later, if the product is not in the desirable size or quality, then it is taken away from the system (the loss occurs). The product is sent to the second stage to be packed with no waiting time, if it qualifies the desirable size and quality.

(b) *VoIP (Voice Over Internet Protocol)*. Internet telephony refers to communications services—voice, fax, SMS, and/or voice-messaging applications—that are transported via an IP network, rather than the public switched telephone network (PSTN). The steps involved in originating a VoIP telephone call are signalling and media channel setup, digitization of the analogy voice signal, encoding, packetization, and transmission as Internet Protocol (IP) packets over a packet-switched network. On the receiving side, similar steps (usually in the reverse order) such as reception of the IP packets, decoding of the packets, and digital-to-analogy conversion reproduce the original voice stream.

More generally, the server transfers voice messaging to the recipient. In this system, the server is first service, the recipient is second service. If the recipient is busy, and then the server destroys voice messaging at that moment.

2. The Model

Let n_1 and n_2 be the number of customers in the first and second stages, respectively, at any time of t, including those being served, where $n_1 = 0, 1, 2, \ldots; n_2 = 0, 1$.

For stage i, let $\xi_i(t)$ be defined as follows:

$$\xi_i(t) = \begin{cases} n_1, & i = 1, \\ n_2, & i = 2. \end{cases} \tag{1}$$

We can denote

$$P_{n_1,n_2}(t) = \text{Prob}\{\xi_1(t) = n_1,\ \xi_2(t) = n_2\}. \tag{2}$$

The random process

$$\{\xi_i(t) : i = 1, 2;\ t \geq 0\} \tag{3}$$

is a continuous-time two-dimensional Markov chain. For $n \geq 1$, the state space of this chain becomes

$$E = \{(0,0), (0,1), (n,0), (n,0)\}. \tag{4}$$

We wish to find the steady-state probability p_{n_1,n_2}:

$$p_{n_1,n_2} = \lim_{t\to\infty} P_{n_1,n_2}(t). \tag{5}$$

The usual procedure leads to the steady-state equations for this Markov chain:

$$0 = -\lambda p_{00} + \mu_2 p_{10}, \tag{6}$$

$$0 = -(\lambda + \mu_2) p_{01} + \mu_1 p_{10} + \mu_1 p_{11}, \tag{7}$$

$$0 = -(\lambda + \mu_1) p_{n_1,0} + \lambda p_{n_1-1,0} + \mu_2 p_{n_1,1}, \quad n_1 \geq 1, \tag{8}$$

$$\begin{aligned} 0 = & -(\lambda + \mu_1 + \mu_2) p_{n_1,1} + \lambda p_{n_1-1,1} \\ & + \mu_1 \left(p_{n_1+1,0} + p_{n_1+1,0}\right), \quad n_1 \geq 1. \end{aligned} \tag{9}$$

We define $\rho_i = \lambda/\mu_i$ for $i = 1, 2$ and $\rho = \lambda/(\mu_1 + \mu_2)$.

2.1. State Probabilities. The probability p_{n_1} denotes the probability of finding n_1 customers in the first stage at an arbitrary point in time (see [12]). We can write these as

$$p_{n_1} = p_{n_1,0} + p_{n_1,1} = \rho_1^{n_1}(1 - \rho_1), \quad n_1 \geq 0, \tag{10}$$

$$p_{00} + p_{01} = 1 - \rho_1. \tag{11}$$

Using this equation and (6), we obtain the following:

$$\begin{aligned} p_{01} &= \frac{\lambda(1-\rho_1)}{\lambda + \mu_2} = \frac{(1-\rho_1)\rho_2}{1+\rho_2}, \\ p_{00} &= \frac{\mu_2}{\lambda} p_{01} = \frac{1-\rho_1}{1+\rho_2}. \end{aligned} \tag{12}$$

By substituting the expression p_{n_1+1} in (9), we get

$$(\lambda + \mu_1 + \mu_2) p_{n_1,1} = \lambda p_{n_1-1,1} + \mu_1 p_{n_1+1}, \quad n_1 \geq 1. \tag{13}$$

Let us take $a = \rho/(1+\rho)$ and choose $p_{n_1,1}$ a place to put y_{n_1} in (13), for simplicity. In this case, the following equation can be obtained:

$$y_{n_1} = a y_{n_1-1} + a y_{n_1}, \quad n_1 \geq 1. \tag{14}$$

Both sides of (14) are divided into a^{n_1}, and later, the index n_1 is changed to k. Then, we sum this obtained value:

$$\frac{y_{n_1}}{a^{n_1}} - y_0 = \sum_{k=1}^{n_1} \frac{p_k}{a^{k-1}} = \frac{\rho_1(1-\rho_1)(a^{n_1} - \rho_1^{n_1})}{(1 - (\rho_1/a)) a^{n_1}}, \tag{15}$$

$$p_{n_1,1} - a^{n_1} p_{01} = a\frac{\rho_1(1-\rho_1)(a^{n_1} - \rho_1^{n_1})}{a - \rho_1}, \tag{16}$$

$$p_{n_1,1} = a\frac{\rho_1(1-\rho_1)(a^{n_1} - \rho_1^{n_1})}{a - \rho_1} + \frac{(1-\rho_1)\rho_2}{1+\rho_2} a^{n_1}. \tag{17}$$

When the value of a in (17) is substituted, the following equation is obtained:

$$p_{n_1,1} = \frac{(1-\rho_1)\rho_2}{1+\rho_2}\rho_1^{n_1}. \tag{18}$$

We get the probability $p_{n_1,0}$ from (10) and (18):

$$p_{n_1,0} = \frac{(1-\rho_1)}{1+\rho_2}\rho_1^{n_1}. \tag{19}$$

2.2. Loss Probability. The customer's loss probability is given as

$$p_L = \sum_{n_1=0}^{\infty} p_{n_1,1} = \frac{\rho_2}{1+\rho_2}. \tag{20}$$

In other way, the formula (20) can be obtained from "Erlang's loss formula" or "Erlang's B formula" for the $M/M/c/c$ queue. In [13], it is denoted as $B(c,\rho_2)$ and formulated as the follows:

$$B(c,\rho_2) = \frac{\rho_2/c!}{\sum_{n=0}^{c}\rho_2^n/n!}, \tag{21}$$

where ρ_2 is the utilization factor. Substituting $c = 1$ in (21), we have (20).

3. The Measures of Performance and Optimization of Performance Measures

3.1. The Mean Sojourn Time. Let T be a random variable that describes the sojourn time of customers in the system. Using the law of total expectation, we can write is as follows

$$E(T) = E(T \mid A)P(A) + E(T \mid \overline{A})P(\overline{A}), \tag{22}$$

where $P(A)$ is the probability of the loss of a customer. Now it is clear that

$$E(T \mid A) = \frac{1}{\mu_1-\lambda}, \qquad E(T \mid \overline{A}) = \frac{1}{\mu_1-\lambda} + \frac{1}{\mu_2}. \tag{23}$$

Thus,

$$E(T) = \frac{\mu_1+\mu_2}{(\mu_1-\lambda)(\mu_2+\lambda)}. \tag{24}$$

Our main results about the problem of minimizing the mean sojourn time can be explained by the following theorem.

Theorem 1. *If sum of two service rates $\mu_1 + \mu_2 = \mu$ is fixed, then the mean sojourn time of this tandem system attains its minimum value for $\mu_1 = \mu/2 + \lambda$ and $\mu_2 = \mu/2 - \lambda$.*

Proof. We will prove the theorem by using the following inequality:

$$\left(\prod_{i=1}^{m} a_i\right)^{1/m} \le \frac{1}{m}\sum_{i=1}^{m} a_i, \quad \forall a_i > 0,\ \forall m \in \mathbb{Z}^+. \tag{25}$$

From inequality (25), we have

$$\frac{1}{(\mu_1-\lambda)(\mu_2+\lambda)} \ge \frac{4}{\mu^2}. \tag{26}$$

If we replace the expressions $\mu_1+\mu_2 = \mu$ and $4/\mu^2$ in equality (24), we obtain the minimum value of $E(T)$ as follows:

$$\min E(T) = \frac{4}{\mu}, \tag{27}$$

where the equality (27) is provided with $\mu_1 = \mu/2 + \lambda$ and $\mu_2 = \mu/2 - \lambda$. □

3.2. The Mean Number of Customers. Let N be the random variable that describes the number of customers in the system:

$$\begin{aligned} E(N) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{1}(n_1+n_2)\,p_{n_1,n_2} \\ &= \frac{\lambda(\mu_1+\mu_2)}{(\mu_1-\lambda)(\mu_2+\lambda)}, \end{aligned} \tag{28a}$$

or

$$E(N) = \lambda E(T). \tag{28b}$$

The mean number of customers in the system is optimized from Theorem 1 and the equality (28b) as below:

$$\min E(N) = \lambda \min E(T) = \frac{4\lambda}{\mu}. \tag{29}$$

The independence of the number of customers can be expressed by the following theorem.

Theorem 2. *If the random variables N_1 and N_2 are taken as the number of customers in the first and second stages, respectively, then N_1 and N_2 are independent random variables.*

Proof. The joint probability mass functions of N_1 and N_2 random variables is

$$p_{n_1,n_2} = P(N_1 = n_1,\ N_2 = n_2), \quad n_1 = 0,1,2,\ldots;\ n_2 = 0,1. \tag{30}$$

If $n_2 = 0$ and ($n_2 = 1$) are substituted in (30), the equations (19) and (18) are obtained, respectively.

The marginal probability mass function of N_1 is given as (10).

TABLE 1: For $\lambda = 0.30$, $\mu_1 = 0.80$; $\mu_2 = 1.00$.

Iteration number	Simulation results		Exact results		Optimal results	
	$E(T)$	$E(N)$	$E(T)$	$E(N)$	$E(T)$	$E(N)$
100	2.8163	0.8452	2.7692	0.8308	2.2222	0.6667
1000	2.8111	0.8433	2.7692	0.8308	2.2222	0.6667
5000	2.8124	0.8436	2.7692	0.8308	2.2222	0.6667

TABLE 2: For $\lambda = 0.1$, $\mu_1 = 0.9$; $\mu_2 = 0.9$.

Iteration number	Simulation results		Exact results		Optimal results	
	$E(T)$	$E(N)$	$E(T)$	$E(N)$	$E(T)$	$E(N)$
100	2.2661	0.2263	2.2500	0.2250	2.2220	0.2222
1000	2.2599	0.2261	2.2500	0.2250	2.2220	0.2222
5000	2.2500	0.2260	2.2500	0.2250	2.2220	0.2222

TABLE 3: For $\lambda = 0.01$, $\mu_1 = 0.6$; $\mu_2 = 0.7$.

Iteration number	Simulation results		Exact results		Optimal results	
	$E(T)$	$E(N)$	$E(T)$	$E(N)$	$E(T)$	$E(N)$
100	3.1042	0.0311	3.1034	0.0310	3.0769	0.0308
1000	3.1034	0.0310	3.1034	0.0310	3.0769	0.0308
5000	3.1043	0.0310	3.1034	0.0310	3.0769	0.0308

The marginal probability mass function of N_2 is obtained from Erlang's B formula or (20):

$$p_0' = P(N_2 = 0) = \frac{1}{1+\rho_2}, \tag{31}$$

$$p_1' = P(N_2 = 1) = \frac{\rho_2}{1+\rho_2}, \tag{32}$$

$$p_{n_1,n_2} = p_{n_1} p_{n_2}', \quad n_1 = 0, 1, 2, \ldots;\ n_2 = 0, 1. \tag{33}$$

If (10) and (31) are substituted in (30), the equation (19) is obtained, and if (10) and (32) are substituted in (30), the equation (18) is obtained. Thus, the independence of N_1 and N_2 has been demonstrated. □

4. Numerical Results

The random arrivals and service times were generated from exponential distribution as seconds by using MATLAB 7.10.0 (R2010a) programming for this proposed model. The number of customers taken was 10000 and was performed in three iterations. Performance measures are calculated for different values of ρ_i $(i = 1, 2)$ and three different iterations steps, that is, 100, 1000, and 5000. These results were shown in Tables 1, 2, and 3.

5. Conclusions

A new queuing discipline is given for a Markov model which consists of two consecutive channels and no waiting line between channels. In this model, steady-state equations, the mean sojourn time, the mean number of customers, and loss probability are obtained. Additionally, two theorems are given which are about optimization of performance measures and the independent of the number of customers, respectively. Performance measures are calculated for different values of ρ_i $(i = 1, 2)$ and for three different iterations steps, that is, 100, 1000, and 5000. Moreover, results of these measures are compared in the tables above. It has been seen that the simulation results approximated the theoretical results. Although the iteration number is increased, the simulation results of performance measures have not changed. However, as ρ_i $(i = 1, 2)$ converges to zero, both the simulation results and exact results approximately are equal to optimal results. Thus, it is said that our proposed queueing model operates well.

For further research, a model, in which a customer who completed his service in channel 1, blocks channel 1 with probability π or leaves the system with probability $1-\pi$ while channel 2 is busy, can be studied.

References

[1] G. G. O.'Brien, "The solution of some queuing problems," *Journal of the Society For Industrial and Applied Mathematics*, vol. 2, pp. 133–142, 1954.

[2] R. R. P. Jackson, "Queuing system with phase-type service," *Operational Research Quarterly*, vol. 5, pp. 109–120, 1954.

[3] P. J. Burke, "The output of a queuing system," *Operations Research*, vol. 4, pp. 699–704, 1956.

[4] J. R. Jackson, "Network of waiting lines," *Operation Research*, vol. 5, pp. 518–524, 1957.

[5] R. B. Cooper, *Introduction to Queuing Theory*, The Macmillan Company, New York, NY, USA, 1972.

[6] D. Gross and C. M. Harris, *Fundamentals of Queueing Theory*, John Wiley & Sons, New York, NY, USA, 3rd edition, 1998.

[7] V. Sağlam and H. Torun, "On optimization of stochastic service system with two heterogeneous channels," *International Journal of Applied Mathematics*, vol. 17, no. 1, pp. 1–6, 2005.

[8] V. Sağlam and A. Shahbazov, "Minimizing loss probability in queuing systems with heterogeneous servers," *Iranian Journal of Science and Technology*, vol. 31, no. 2, pp. 199–206, 2007.

[9] A. H. Taha, *Operating Research*, Macmillian Publishing, New York, NY, USA, 1982.

[10] A. Brandwajn and Y.-L. L. Jow, "An approximation method for tandem queues with blocking," *Operations Research*, vol. 36, no. 1, pp. 73–83, 1988.

[11] A. A. Akinsete, "Blocked network of tandem queues with withdrawal," *Kragujevac Journal of Mathematics*, vol. 23, pp. 63–73, 2001.

[12] U. N. Bhat, *An Introduction to Queueing Theory*, Boston, Mass, USA, 2008.

[13] W. J. Stewart, *Probability, Markov Chains, Queues, and Simulation*, Princeton University Press, Princeton, NJ, USA, 2009.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 715869, 9 pages
http://dx.doi.org/10.1155/2013/715869

Research Article

Optimal Portfolio of Corporate Investment and Consumption Problem under Market Closure: Inflation Case

Zongyuan Huang[1] and Detao Zhang[2]

[1] *School of Mathematics, Shandong University, Jinan 250100, China*
[2] *School of Economics, Shandong University, Jinan 250100, China*

Correspondence should be addressed to Detao Zhang; zhangdetao@sdu.edu.cn

Received 21 January 2013; Revised 24 March 2013; Accepted 22 April 2013

Academic Editor: Guangchen Wang

We present the model of corporate optimal investment with consideration of the influence of inflation and the difference between the market opening and market closure. In our model, the investor has three market activities of his or her choice: investment in project A, investment in project B, and consumption. The optimal strategy for the investor is obtained using the Hamilton-Jacobi-Bellman equation which is derived using the dynamic programming principle. Further along, a specific case, the Hyperbolic Absolute Risk Aversion case, is discussed in detail, where the explicit optimal strategy can be obtained using a very simple and direct method. At the very end, we present some simulation results along with a brief analysis of the relationship between the optimal strategy and other factors.

1. Introduction

Inflation originally referred to the debasement of the currency. When the government increases the total number of the currency, the relative value of the currency decreases. As a result, consumers would need more money to exchange for the same goods and services. The term inflation usually refers to a measured rise in a broad price index that represents the overall level of prices in goods and services in the economy. The Consumer Price Index (CPI), the Personal Consumption Expenditures Price Index (PCEPI), and the GDP deflator are some examples of broad price indices. These factors play an important role in economics, because of which they are studied in the theory.

Portfolio diversification plays an important role in finance. In Merton [1] and Duffie [2], a stochastic model is first given to analyze the security market, while Karatzas [3], based on the stochastic analysis, presented the model considering the split of consumption and investment of the investors. Choi [4], on the other hand, paid attention to the international corporate investment where investments were made into real corporate projects. Afterwards, Bellalah and Wu [5] extended the theory of corporate international investment from Choi into an environment with the presence of incomplete information and taxation from governments. Their model was concerned with the international diversification problem in finance and gave reasonable analysis to the "home bias puzzle." In 2006, a model giving the optimal corporate portfolio and consumption choice was presented by Wu and Zhang [6], while Bellalah and Wu [7] modeled the market closure together with the international securities portfolio management with incomplete information.

In our paper, we present a model of corporate investment and consumption choice problem with market closure in inflation case, so that we can derive the optimal investment strategy. Our model is an extension of the usual corporate investment model from Bellalah and Wu [5, 7], Wu and Zhang [6], Huang and Wu [8], and D. T. Zhang and T. Zhang [9] since we take into consideration the influence of inflation and the investment on two projects.

In this model, we suppose that the investor has an option to invest his money into two different projects, A and B. In real life, the production of the factory may be different between the day and the night; therefore, in our model, we have equations describing the input price and the output price of the production in the daytime and night with specified

instantaneous expected rates and instantaneous volatilities. Also, we develop two pieces of production equations corresponding to the daytime and nighttime production which is a function of the output price. With the input and output price model and the production quantity model, we can easily derive the expression for the net profit of the corporate investment after considering taxation of the government.

Finally, instead of making investment, the investor has the option to make consumption for satisfaction as well.

During the nighttime, banking activities will not be allowed; therefore, the investment portfolio will be the same throughout the night. However, the investor is still allowed to consume to increase utility. Clearly, the consumption in the daytime and the night will be different.

In general, we consider corporate investment and consumption altogether to maximize the utility of wealth for the investor aiming to find out the optimal strategy for the investment. For the general model, it is difficult to solve for the optimal value function explicitly. Therefore, we considered a specific and important case of utility function, Hyperbolic Absolute Risk Aversion (HARA) case, to get the optimal solution. It turns out that we can write out the explicit optimal solution using a very simple and direct method which will be discussed in Section 3 of the paper.

In the next section, we will present the model and the assumptions of the optimal investment problem. In the HARA cases, besides giving the explicit optimal solution of the corporate investment model using dynamic programming principle, we will give analysis of the model in the economical point of view in Section 3.

At the end of this paper, some simulation results will be given to provide a more numerical interpretation of the model. Relationship between the optimal portfolio and important factors such as the volatility and the interest rate will be given together with computer-drawn graphs.

2. Model and Formulation of the Problem

Let $(\Omega, \mathscr{F}, P)$ be a complete probability space endowed with a filtration $\{\mathscr{F}_t : 0 \le t \le T\}$. $\{W_t\}_{0\le t\le T}$, $\{B_t^1\}_{0\le t\le T}$, and $\{B_t^2\}_{0\le t\le T}$ are three 1-dimensional Brownian motions defined in this space, which represent the external sources of uncertainty in the market, and the correlation coefficients between them are $\rho_1, \rho_2, \rho_{1,2}$ with $-1 \le \rho_1, \rho_2, \rho_{1,2} \le 1$.

Denote the rate of inflation by I_t, which is stochastic and depends on the situation of the economic. Like some pricing processes of the stochastic assets, we use a geometric Brownian motion to describe the inflation rate level at time t, that is,

$$dI_t = I_t\theta_t dt + I_t\eta_t dW_t. \tag{1}$$

The drift of the process θ is the expected rate of inflation per unit of time, and it is defined by

$$\theta_t = \lim_{\delta\downarrow 0} E\left(\frac{I_{t+\delta} - I_t}{\delta I_t}\right), \tag{2}$$

where $(\eta)^2$ is the variance of the process per unit of time defined by

$$(\eta_t)^2 = \lim_{\delta\downarrow 0} E\left[\frac{1}{\delta}\left(\frac{I_{t+\delta} - I_t}{I_t} - \delta\theta_t\right)^2\right]. \tag{3}$$

Without loss of generality, we assume that $I_0 = 1$; that is, there is no inflation at initial time $t = 0$.

In this model, we suppose that the investor can choose two real projects, A and B, with some production to get a higher return but with certain risk. The input price and output price are not the same during the day and the night. The duration of the production in the daytime is T, and the duration at the nighttime is N. Then the daytime is $[0,T]$, $[T+N, 2T+N], \ldots, [(n-1)(T+N), nT+(n-1)N]$, denoted by D, while the nighttime is $[T, T+N]$, $[2T+N, 2T+2N]$, $\ldots, [(nT+(n-1)N, n(T+N))]$, denoted by D^C.

For project A, we denote $P(t)$ and $\overline{P}(t)$ as the the input prices in the daytime and nighttime, respectively. $S(t)$ and $\overline{S}(t)$ are the output prices, while $Q(t)$ and $\overline{Q}(t)$ are the quantity of the production and $R(t)$, and $\overline{R}(t)$ are the cash flow.

In economic theory, the quantity of the production is the function of the input price, $Q(t) = [S(t)]^{\beta_1}$, where $\beta_1 < 0$ in general. The cash flow of project A in daytime and nighttime are as follows:

$$\begin{aligned} R(t) &= (1-\tau)[S(t) - P(t)]Q(t), \\ Q(t) &= [S(t)]^{\beta_1}, \quad t \in D, \\ \overline{R}(t) &= (1-\tau)\left[\overline{S}(t) - \overline{P}(t)\right]\overline{Q}(t), \\ \overline{Q}(t) &= \left[\overline{S}(t)\right]^{\beta_2}, \quad t \in D^C. \end{aligned} \tag{4}$$

Similarly, we can define the corresponding variables of project B using the letters which have the symbol $*$. And the cash flow of project B in daytime and nighttime are as follows:

$$\begin{aligned} R^*(t) &= (1-\tau)\left[S^*(t) - P^*(t)\right]Q^*(t), \\ Q^*(t) &= \left[S^*(t)\right]^{\beta_1^*}, \quad t \in D, \\ \overline{R}^*(t) &= (1-\tau)\left[\overline{S}^*(t) - \overline{P}^*(t)\right]\overline{Q}^*(t), \\ \overline{Q}^*(t) &= \left[\overline{S}^*(t)\right]^{\beta_2^*}, \quad t \in D^C. \end{aligned} \tag{5}$$

We suppose the input price and output price satisfy the following equation:

$$\begin{aligned} dP(t) &= \alpha_P(t)P(t)dt + \sigma(t)P(t)dB_t^1, \quad t \in D, \\ d\overline{P}(t) &= \overline{\alpha}_P(t)\overline{P}(t)dt + \overline{\sigma}(t)\overline{P}(t)dB_t^1, \quad t \in D^C, \\ dS(t) &= \alpha_S(t)S(t)dt + \sigma(t)S(t)dB_t^1, \quad t \in D, \\ d\overline{S}(t) &= \overline{\alpha}_S(t)\overline{S}(t)dt + \overline{\sigma}(t)\overline{S}(t)dB_t^1, \quad t \in D^C, \\ dP^*(t) &= \alpha_P^*(t)P^*(t)dt + \sigma^*(t)P^*(t)dB_t^2, \quad t \in D, \\ d\overline{P}^*(t) &= \overline{\alpha}_P^*(t)\overline{P}^*(t)dt + \overline{\sigma}^*(t)\overline{P}^*(t)dB_t^2, \quad t \in D^C, \\ dS^*(t) &= \alpha_S^*(t)S^*(t)dt + \sigma^*(t)S^*(t)dB_t^2, \quad t \in D, \\ d\overline{S}^*(t) &= \overline{\alpha}_S^*(t)\overline{S}^*(t)dt + \overline{\sigma}^*(t)\overline{S}^*(t)dB_t^2, \quad t \in D^C, \end{aligned} \tag{6}$$

where $P(t)$ and $S(t)$ have initial values P_0, S_0, respectively. Here α_P, $\overline{\alpha}_P$ represent the instantaneous expected input rates in the day and the night, respectively, while α_S, $\overline{\alpha}_S$ represent the instantaneous expected output rates. σ, $\overline{\sigma}$ are instantaneous volatilities in the day and the night, respectively.

According to Ito's formula, we know that the cash flow for project A and B satisfy the following equations:

$$\begin{aligned} dR(t) &= R(t)\left[f(t)\,dt + (1+\beta_1)\,\sigma(t)\,dB_t^1\right], \quad t \in D, \\ d\overline{R}(t) &= \overline{R}(t)\left[\overline{f}(t)\,dt + (1+\beta_2)\,\overline{\sigma}(t)\,dB_t^1\right], \quad t \in D^C, \end{aligned} \tag{7}$$

where

$$\begin{aligned} f(t) &= \frac{1}{2}(1+\beta_1)^2\sigma^2 + \frac{F'(t)}{F(t)}, \\ \overline{f}(t) &= \frac{1}{2}(1+\beta_2)^2\overline{\sigma}^2 + \frac{\overline{F}'(t)}{\overline{F}(t)}, \end{aligned}$$

$$\begin{aligned} dR^*(t) &= R^*(t)\left[f^*(t)\,dt + (1+\beta_1^*)\,\sigma^*(t)\,dB_t^2\right], \quad t \in D, \\ d\overline{R}^*(t) &= \overline{R}^*(t)\left[\overline{f}^*(t)\,dt + (1+\beta_2^*)\,\overline{\sigma}^*(t)\,dB_t^2\right], \quad t \in D^C, \end{aligned} \tag{8}$$

where

$$\begin{aligned} f(t) &= \frac{1}{2}(1+\beta_1)^2\sigma^2 + \frac{F'(t)}{F(t)}, \\ \overline{f}(t) &= \frac{1}{2}(1+\beta_2)^2\overline{\sigma}^2 + \frac{\overline{F}'(t)}{\overline{F}(t)}, \\ f^*(t) &= \frac{1}{2}(1+\beta_1^*)^2(\sigma^*)^2 + \frac{(F^*)'(t)}{F^*(t)}, \\ \overline{f}^*(t) &= \frac{1}{2}(1+\beta_2^*)^2(\overline{\sigma}^*)^2 + \frac{(\overline{F}^*)'(t)}{\overline{F}^*(t)}, \end{aligned} \tag{9}$$

with

$$\begin{aligned} F(t) = \Big\{ & S((k-1)(T+N)) \\ & \times \exp\left[\int_{(k-1)(T+N)}^{t}\left(\alpha_S(r) - \frac{1}{2}\sigma^2(r)\right)dr\right] \\ & - P((k-1)(T+N)) \\ & \times \exp\left[\int_{(k-1)(T+N)}^{t}\left(\alpha_P(r) - \frac{1}{2}\sigma^2(r)\right)dr\right]\Big\} \\ & \times \exp\left[\beta_1\int_{(k-1)(T+N)}^{t}\left(\alpha_S(r) - \frac{1}{2}\sigma^2(r)\right)dr\right], \\ \overline{F}(t) = \Big\{ & \overline{S}(kT+(k-1)N) \\ & \times \exp\left[\int_{kT+(k-1)N}^{t}\left(\overline{\alpha}_S(r) - \frac{1}{2}\overline{\sigma}^2(r)\right)dr\right] \\ & - P(kT+(k-1)N) \\ & \times \exp\left[\int_{kT+(k-1)N}^{t}\left(\overline{\alpha}_P(r) - \frac{1}{2}\sigma^2(r)\right)dr\right]\Big\} \\ & \times \exp\left[\beta_2\int_{kT+(k-1)N}^{t}\left(\overline{\alpha}_S(r) - \frac{1}{2}\sigma^2(r)\right)dr\right]. \\ F^*(t) = \cdots, & \qquad \overline{F}^*(t) = \cdots. \end{aligned} \tag{10}$$

Similarly, we can get $F^*(t)$ and $\overline{F}^*(t)$ just substitute α^* (σ^*) for α (σ) and substitute $\overline{\alpha}^*$ $(\overline{\sigma}^*)$ for $\overline{\alpha}$ $(\overline{\sigma})$.

Since the existence of the inflation, the actual value of the input price should be $U_t = P_t/I_t$. Notice that the proportional rate of change of the real return of input price and output price should be described by

$$\begin{aligned} dU(t) &= U(t)\left[\left(\alpha_P(t) - \theta_t + (\eta_t)^2 - \rho_1\sigma(t)\,\eta_t\right)dt \right. \\ &\qquad \left. +\sigma(t)\,dB_t^1 - \eta_t dW_t\right], \quad t \in D, \\ d\overline{U}(t) &= \overline{U}(t)\left[\left(\overline{\alpha}_P(t) - \theta_t + (\eta_t)^2 - \rho_1\overline{\sigma}(t)\,\eta_t\right)dt \right. \\ &\qquad \left. +\overline{\sigma}(t)\,dB_t^1 - \eta_t dW_t\right], \quad t \in D^C, \\ dV(t) &= V(t)\left[\left(\alpha_S(t) - \theta_t + (\eta_t)^2 - \rho_1\sigma(t)\,\eta_t\right)dt \right. \\ &\qquad \left. +\sigma(t)\,dB_t^1 - \eta_t dW_t\right], \quad t \in D, \\ d\overline{V}(t) &= \overline{V}(t)\left[\left(\overline{\alpha}_S(t) - \theta_t + (\eta_t)^2 - \rho_1\overline{\sigma}(t)\,\eta_t\right)dt \right. \\ &\qquad \left. +\overline{\sigma}(t)\,dB_t^1 - \eta_t dW_t\right], \quad t \in D^C, \\ dU^*(t) &= U^*(t)\left[\left(\alpha_P^*(t) - \theta_t + (\eta_t)^2 - \rho_2\sigma^*(t)\,\eta_t\right)dt \right. \\ &\qquad \left. +\sigma^*(t)\,dB_t^2 - \eta_t dW_t\right], \quad t \in D, \\ d\overline{U}^*(t) &= \overline{U}^*(t)\left[\left(\overline{\alpha}_P^*(t) - \theta_t + (\eta_t)^2 - \rho_2\overline{\sigma}^*(t)\,\eta_t\right)dt \right. \\ &\qquad \left. +\overline{\sigma}^*(t)\,dB_t^2 - \eta_t dW_t\right], \quad t \in D^C, \\ dV^*(t) &= V^*(t)\left[\left(\alpha_S^*(t) - \theta_t + (\eta_t)^2 - \rho_2\sigma^*(t)\,\eta_t\right)dt \right. \\ &\qquad \left. +\sigma^*(t)\,dB_t^2 - \eta_t dW_t\right], \quad t \in D, \\ d\overline{V}^*(t) &= \overline{V}^*(t)\left[\overline{\alpha}_S^*(t) - \theta_t + (\eta_t)^2 - \rho_2\overline{\sigma}^*(t)\,\eta_t dt \right. \\ &\qquad \left. +\overline{\sigma}^*(t)\,dB_t^2 - \eta_t dW_t\right], \quad t \in D^C. \end{aligned} \tag{11}$$

Finally the real cash flow for project A and B, $\Lambda(t) = R(t)/I(t)$, $\overline{\Lambda}(t), \ldots$, satisfy

$$\begin{aligned} d\Lambda(t) &= \Lambda(t)\left\{\left[f(t) - \theta_t + (\eta_t)^2 - (1+\beta_1)\,\rho_1\sigma(t)\,\eta_t\right]dt \right. \\ &\qquad \left. +(1+\beta_1)\,\sigma(t)\,dB_t^1 - \eta_t dW_t\right\}, \quad t \in D, \\ d\overline{\Lambda}(t) &= \overline{\Lambda}(t)\left\{\left[\overline{f}(t) - \theta_t + (\eta_t)^2 - (1+\beta_2)\,\rho_1\overline{\sigma}(t)\,\eta_t\right]dt \right. \\ &\qquad \left. +(1+\beta_2)\,\overline{\sigma}(t)\,dB_t^1 - \eta_t dW_t\right\}, \quad t \in D^C, \end{aligned} \tag{12}$$

$$\begin{aligned}
&d\Lambda^{*}(t)\\
&\quad=\Lambda^{*}(t)\left\{\left[f^{*}(t)-\theta_t+(\eta_t)^2-(1+\beta_1^{*})\rho_2\sigma^{*}(t)\eta_t\right]dt\right.\\
&\qquad\left.+(1+\beta_1^{*})\sigma^{*}(t)dB_t^2-\eta_t dW_t\right\},\quad t\in D,\\
&d\overline{\Lambda}^{*}(t)\\
&\quad=\overline{\Lambda}^{*}(t)\left\{\left[\overline{f}^{*}(t)-\theta_t+(\eta_t)^2-(1+\beta_2^{*})\rho_2\overline{\sigma}^{*}(t)\eta_t\right]dt\right.\\
&\qquad\left.+(1+\beta_2^{*})\overline{\sigma}^{*}(t)dB_t^1-\eta_t dW_t\right\},\quad t\in D^{C}.
\end{aligned}\tag{13}$$

Let $X^{\pi,x}(t)$ denote the total wealth at time t, and $\pi(t)$ represents the proportion of the wealth invested in project B; then $(1-\pi(t))X^{\pi,x}(t)$ is the amount invested in project A. Taking inflation into consideration, both of them are calculated by its actual value. So we have

$$\begin{aligned}
dX^{\pi,x}(t)&=\pi(t)X^{\pi,x}(t)\frac{d\Lambda^{*}(t)}{\Lambda^{*}(t)}\\
&\quad+(1-\pi(t))X^{\pi,x}(t)\frac{d\Lambda(t)}{\Lambda(t)},\quad t\in D,\\
dX^{\pi,x}(t)&=\pi(t)X^{\pi,x}(t)\frac{d\overline{\Lambda}^{*}(t)}{\overline{\Lambda}^{*}(t)}\\
&\quad+(1-\pi(t))X^{\pi,x}(t)\frac{d\overline{\Lambda}(t)}{\overline{\Lambda}(t)},\quad t\in D.
\end{aligned}\tag{14}$$

Substituting Λ_t, $\Lambda^{*}(t)$, $\overline{\Lambda}_t$, and $\overline{\Lambda}_t^{*}$ parts by (12) and (13), respectively, we get that, for an investor whose initial wealth is $x>0$, the total wealth at time t in inflation case satisfies

$$\begin{aligned}
&dX^{\pi,x}(t)\\
&\quad=X^{\pi,x}(t)\left[\pi(t)\left(f^{*}(t)-\theta_t+(\eta_t)^2\right.\right.\\
&\qquad\left.-(1+\beta_1^{*})\rho_2\sigma^{*}(t)\eta_t\right)+(1-\pi(t))\\
&\qquad\left.\times\left(f(t)-\theta_t+(\eta_t)^2-(1+\beta_1)\rho_1\sigma(t)\eta_t\right)\right]dt\\
&\qquad-c_t^1dt-\eta_tX^{\pi,x}(t)dW_t\\
&\qquad+\pi(t)X^{\pi,x}(t)(1+\beta_1^{*})\sigma^{*}(t)dB_t^2\\
&\qquad+(1-\pi(t))X^{\pi,x}(t)(1+\beta_1)\sigma(t)dB_t^1,\quad t\in D,\\
&dX^{\pi,x}(t)\\
&\quad=X^{\pi,x}(t)\left[\pi(t)\left(\overline{f}^{*}(t)-\theta_t+(\eta_t)^2\right.\right.\\
&\qquad\left.-(1+\beta_2^{*})\rho_2\overline{\sigma}^{*}(t)\eta_t\right)+(1-\pi(t))\\
&\qquad\left.\times\left(\overline{f}(t)-\theta_t+(\eta_t)^2-(1+\beta_2)\rho_1\overline{\sigma}(t)\eta_t\right)\right]dt\\
&\qquad-c_t^2dt-\eta_tX^{\pi,x}(t)dW_t\\
&\qquad+\pi(t)X^{\pi,x}(t)(1+\beta_2^{*})\overline{\sigma}^{*}(t)dB_t^2\\
&\qquad+(1-\pi(t))X^{\pi,x}(t)(1+\beta_2)\overline{\sigma}(t)dB_t^1,\quad t\in D^{C}.
\end{aligned}\tag{15}$$

In the daytime, the investor can choose investment with proportion π and consumption rate c^1 to maximize his wealth, but at the nighttime, he cannot change his portfolio and only can choose a different consumption rate c^2; the portfolio at the nighttime stays the same as the optimal portfolio in the daytime. So the investor wants to maximize the following utility of wealth by choosing his investment strategy π and consumption rate c^1 and c^2. Let $J^1(X)$ the value of wealth X starting the daytime and $J^2(X)$ be the value of the wealth X starting the nighttime. The whole investment duration can be divided into n days, that is, nT daytime duration and nN the nighttime duration. Consider

$$J^1(X_0)=\max_{(\pi,c^1)}\mathbb{E}\left[\int_0^T e^{-\gamma t}U\left(c^1(t)\right)dt+e^{-\gamma T}J^2(X_T)\right],\tag{16}$$

$$\begin{aligned}
&J^2(X_T)\\
&\quad=\max_{c^2}\mathbb{E}_T\left[\int_T^{T+N}e^{-\gamma(t-T)}U\left(c^2(t)\right)dt+e^{-\gamma N}J^1(X_{T+N})\right],\\
&\quad\vdots
\end{aligned}\tag{17}$$

$$\begin{aligned}
&J^1\left(X_{(n-1)(T+N)}\right)\\
&\quad=\max_{(\pi,c^1)}\mathbb{E}_{(n-1)(T+N)}\\
&\qquad\times\left[\int_{(n-1)(T+N)}^{nT+(n-1)N}e^{-\gamma(t-(n-1)(T+N))}U\left(c^1(t)\right)dt\right.\\
&\qquad\left.+e^{-\gamma T}J^2\left(X_{nT+(n-1)N}\right)\right],
\end{aligned}\tag{18}$$

$$\begin{aligned}
&J^2\left(X_{nT+(n-1)N}\right)\\
&\quad=\max_{c^2}\mathbb{E}_{nT+(n-1)N}\\
&\qquad\times\left[\int_{nT+(n-1)N}^{n(T+N)}e^{-\gamma(t-nT+(n-1)N)}U\left(c^2(t)\right)dt\right.\\
&\qquad\left.+e^{-\gamma N}h\left(X_{n(T+N)}\right)\right].
\end{aligned}\tag{19}$$

The optimal problem (15)–(19) can be solved by the classical methods, such as Pontryagin's maximum principle (cf., e.g., Yong and Zhou [10], Wang et al. [11], Wang and Wu [12], etc.) which gives the necessary conditions for the optimal strategies and dynamic programming principle (cf., e.g., Yong and Zhou [10]). In the following section, we will use a simple method, called construction method, to give an explicit solution to this problem in a special HARA case and give the economic analysis.

3. HARA Case: Optimal Solutions and Economic Analysis

In this section, we want to use the construction method to get the explicit optimal strategy in the special Hyperbolic

Absolute Risk Aversion (HARA) case. This method can also be found when solving the linear quadratic (LQ) optimal control problem. It is often along with the dynamic programming principle; see Yong and Zhou [10], for example. For simplicity, we only consider the case where the entire time interval is $T+N$, T is the duration of the daytime, and N is the duration of the nighttime. For general case, the whole interval is $n(T+N)$, and we can get the explicit result by repeating the procedure with the same method.

Let

$$J^1(X_0) = \max_{(\pi, c_t^1)} \mathbb{E}\left[\int_0^T e^{-\gamma t}\frac{\left(c_t^1\right)^{1-R}}{1-R}dt + e^{-\gamma T}J^2(X_T)\right], \tag{20}$$

$$J^2(X_T) = \max_{c_t^2} \mathbb{E}\left[\int_T^{T+N} e^{-\gamma t}\frac{\left(c_t^2\right)^{1-R}}{1-R}dt + e^{-\gamma N}K\frac{X_{T+N}^{1-R}}{1-R}\right]. \tag{21}$$

Here γ and R are constants, where $\gamma > 0$, $R \in (0,1)$. We try to get explicit optimal decision π, consumption rates c^1, c^2, and the optimal value function in this case.

Theorem 1. *Under all the above assumptions, the optimal strategies to the optimal portfolio choice problem* (18), (19), (20), *and* (21) *for the specific HARA case are given by*

$$\begin{gathered} J^1(X_0) = \frac{1}{1-R}\left(X_0^{\pi,x}\right)^{1-R}\Theta_0, \\ J^2(X_T) = \frac{1}{1-R}\left(X_T^{\pi,x}\right)^{1-R}\Theta_T, \\ \pi^*(t) = \frac{\Delta^u}{\Delta^d}, \quad t \in [0, T+N], \\ \left(c_t^1\right)^* = (1-R)^{-1/R}\left(\Theta_t\right)^{-1/R}X_t^{\pi,x}, \quad t \in [0,T], \\ \left(c_t^2\right)^* = (1-R)^{-1/R}\left(\Theta_t\right)^{-1/R}X_t^{\pi,x}, \quad t \in [T, T+N], \end{gathered} \tag{22}$$

where Θ_t satisfies ODEs (27), (41), *and Δ^u, Δ^d are denoted by* (37), (38), *respectively.*

Proof. We first consider the optimal problem (21) at nighttime. During the night, the investor cannot change his portfolio. His consumption c^2 is the only activity that varies. So, the wealth equation in the duration $[T, T+N]$ is,

$$\begin{aligned} dX^{\pi,x}(t) &= X^{\pi,x}(t)\Big[\pi^*(t)\left(\overline{f}^*(t) - \theta_t + (\eta_t)^2 - (1+\beta_2^*)\rho_2\overline{\sigma}^*(t)\eta_t\right) \\ &\quad + (1-\pi^*(t))\left(\overline{f}(t) - \theta_t + (\eta_t)^2 - (1+\beta_2)\rho_1\overline{\sigma}(t)\eta_t\right)\Big]dt \\ &\quad - c_t^2 dt - \eta_t X^{\pi,x}(t)dW_t \\ &\quad + \pi^*(t)X^{\pi,x}(t)(1+\beta_2^*)\overline{\sigma}^*(t)dB_t^2 \\ &\quad + (1-\pi^*(t))X^{\pi,x}(t)(1+\beta_2)\times\overline{\sigma}(t)dB_t^1, \quad t \in [T, T+N]. \end{aligned} \tag{23}$$

Here π^* is the optimal portfolio in the daytime, and we will solve it afterward.

We let Θ_t be one nonnegative deterministic continuous function whose dynamic will be given later. Applying Ito's formula to $(e^{-\gamma(t-T)}/(1-R))(X^{\pi,x}(t))^{1-R}\Theta_t$ from T to $T+N$, we have

$$\begin{aligned} &\mathbb{E}_T\left[\frac{e^{-\gamma N}}{1-R}\left(X_{T+N}^{\pi,x}\right)^{1-R}\Theta_{T+N}\right] \\ &= \frac{1}{1-R}\left(X_T^{\pi,x}\right)^{1-R}\Theta_T \\ &\quad + \mathbb{E}_T\int_T^{T+N}\Big\{\Theta_t\Big[-\gamma\frac{e^{-\gamma(t-T)}}{1-R}\left(X_t^{\pi,x}\right)^{1-R} \\ &\quad + \frac{e^{-\gamma(t-T)}}{1-R}\left(X_t^{\pi,x}\right)^{1-R}(1-R) \\ &\quad \times\Big[\pi^*(t)\left(\overline{f}^*(t) - \theta_t + (\eta_t)^2 - (1+\beta_2^*)\rho_2\overline{\sigma}^*(t)\eta_t\right) \\ &\quad + (1-\pi^*(t))\left(\overline{f}(t) - \theta_t + (\eta_t)^2 - (1+\beta_2)\rho_1\overline{\sigma}(t)\eta_t\right)\Big] \\ &\quad - \frac{1}{2}R(1-R)e^{-\gamma(t-T)}\left(X_t^{\pi,x}\right)^{1-R} \\ &\quad \times\left[\eta(t)^2 + (\pi^*(t))^2(1+\beta_2^*)^2(\overline{\sigma}^*(t))^2 + (1-\pi^*(t))^2(1+\beta_2)^2(\overline{\sigma}(t))^2\right] \\ &\quad - R(1-R)e^{-\gamma(t-T)}\left(X_t^{\pi,x}\right)^{1-R} \\ &\quad \times\big[-\eta(t)(\pi^*(t))(1+\beta_2^*)\overline{\sigma}^*(t)\rho_2 \\ &\quad - \eta(t)(1-\pi^*(t))(1+\beta_2)\overline{\sigma}(t)\rho_1 \\ &\quad + \pi^*(t)(1-\pi^*(t))(1+\beta_2^*) \\ &\quad \times(1+\beta_2)\overline{\sigma}(t)\overline{\sigma}^*(t)\rho_{1,2}\big] \\ &\quad - \frac{e^{-\gamma(t-T)}}{1-R}\left(X_t^{\pi,x}\right)^{-R}(1-R)c_t^2\Big] \\ &\quad + \frac{e^{-\gamma(t-T)}}{1-R}\left(X_t^{\pi,x}\right)^{1-R}\dot{\Theta}_t\Big\}dt. \end{aligned} \tag{24}$$

So we can write

$$J^2(X_T) = \frac{1}{1-R}\left(X_T^{\pi,x}\right)^{1-R}\Theta_T + I + II + III, \tag{25}$$

where

$$I = \max_{c_t^2} \mathbb{E}_T \int_T^{T+N} \frac{e^{-\gamma(t-T)}}{1-R} \left[\left(c_t^2\right)^{1-R} - (1-R)\left(X_t^{\pi,x}\right)^{-R} \Theta_t c_t^2 - R\left(X_t^{\pi,x}\right)^{1-R} \Theta_t^{1-(1/R)} \right] dt,$$

$$\begin{aligned} II = {} & \max_{c_t^2} \mathbb{E}_T \\ & \times \int_T^{T+N} \frac{e^{-\gamma(t-T)}}{1-R} \left(X_t^{\pi,x}\right)^{1-R} \\ & \times \Big\{ \dot{\Theta}_t + \Theta_t \\ & \times \Big[-\gamma + (1-R)\pi^*(t) \\ & \times \left[\overline{f}^*(t) - \theta_t + (\eta_t)^2 - (1+\beta_2^*)\rho_2 \overline{\sigma}^*(t)\eta_t \right] \\ & + (1-R)(1-\pi^*(t)) \\ & \times \left[\overline{f}(t) - \theta_t + (\eta_t)^2 - (1+\beta_2)\rho_1 \overline{\sigma}(t)\eta_t \right] \\ & - \frac{1}{2} R(1-R) \\ & \times \left[\eta(t)^2 + (\pi^*(t))^2 (1+\beta_2^*)^2 (\overline{\sigma}^*(t))^2 + (1-\pi^*(t))^2 (1+\beta_2)^2 (\overline{\sigma}(t))^2 \right] \\ & - R(1-R) \left[-\eta(t)(\pi^*(t))(1+\beta_2^*)\overline{\sigma}^*(t)\rho_2 - \eta(t)(1-\pi^*(t))(1+\beta_2)\overline{\sigma}(t) \times \rho_1 + \pi^*(t)(1-\pi^*(t))(1+\beta_2^*) \times (1+\beta_2)\overline{\sigma}(t)\overline{\sigma}^*(t)\rho_{1,2} \right] \Big] \\ & + R\Theta_t^{1-(1/R)} \Big\} dt, \end{aligned}$$

$$III = \max_{c^2} \mathbb{E}_T \left[\frac{e^{-\gamma N}}{1-R} \left(X_{T+N}^{\pi,x}\right)^{1-R} \left(K - \Theta_{T+N}\right) \right]. \tag{26}$$

Now we let Θ_t be nonnegative and the solution of the following ordinary differential equation:

$$\begin{gathered} -\dot{\Theta}_t = \overline{M}\Theta_t + R\Theta_t^{1-(1/R)}, \quad t \in [T, T+N], \\ \Theta_{T+N} = K. \end{gathered} \tag{27}$$

Here

$$\begin{aligned} \overline{M} = {} & -\gamma + (1-R)\pi^*(t) \\ & \times \left[\overline{f}^*(t) - \theta_t + (\eta_t)^2 - (1+\beta_2^*)\rho_2 \overline{\sigma}^*(t)\eta_t \right] \\ & + (1-R)(1-\pi^*(t)) \\ & \times \left[\overline{f}(t) - \theta_t + (\eta_t)^2 - (1+\beta_2)\rho_1 \overline{\sigma}(t)\eta_t \right] \\ & - \frac{1}{2} R(1-R) \left[\eta(t)^2 + (\pi^*(t))^2 (1+\beta_2^*)^2 (\overline{\sigma}^*(t))^2 + (1-\pi^*(t))^2 (1+\beta_2)^2 (\overline{\sigma}(t))^2 \right] \\ & - R(1-R) \left[-\eta(t)(\pi^*(t))(1+\beta_2^*)\overline{\sigma}^*(t)\rho_2 - \eta(t)(1-\pi^*(t))(1+\beta_2)\overline{\sigma}(t)\rho_1 + \pi^*(t)(1-\pi^*(t))(1+\beta_2^*) \times (1+\beta_2)\overline{\sigma}(t)\overline{\sigma}^*(t)\rho_{1,2} \right]. \end{aligned} \tag{28}$$

So

$$\Theta_t = e^{\overline{M}(T+N-t)} \left[K + \int_t^{T+N} e^{-(1/R)\overline{M}(T+N-s)} ds \right]^R, \quad t \in [T, T+N], \tag{29}$$

$$\Theta_T = e^{\overline{M}N} \left[K + \int_T^{T+N} e^{-(1/R)\overline{M}(T+N-s)} ds \right]^R.$$

We can easily get $II = 0$, $III = 0$.

If we take

$$\left(c_t^2\right)^* = (1-R)^{-1/R} \left(\Theta_t\right)^{-1/R} X_t^{\pi,x}, \quad t \in [T, T+N], \tag{30}$$

then we can check that I attains its maximum at point $(C^2)^*$ and $I = 0$. So

$$J^2(X_T) = \frac{1}{1-R} \left(X_T^{\pi,x}\right)^{1-R} \Theta_T. \tag{31}$$

And then applying Ito's formula to $(e^{-\gamma t}/(1-R))(X_t^{\pi,x})^{1-R}\Theta_t$ from 0 to T and taking expectation on both sides, we have

$$\begin{aligned} & \mathbb{E}\left[\frac{e^{-\gamma T}}{1-R} \left(X_T^{\pi,x}\right)^{1-R} \Theta_T \right] \\ & = \frac{1}{1-R} \left(X_0^{\pi,x}\right)^{1-R} \Theta_0 \\ & \quad + \mathbb{E} \int_0^T \Big\{ \Theta_t \Big[-\gamma \frac{e^{-\gamma t}}{1-R} \left(X_t^{\pi,x}\right)^{1-R} + \frac{e^{-\gamma t}}{1-R} \left(X_t^{\pi,x}\right)^{1-R} \\ & \quad \times (1-R) \Big[\pi(t) \left(f^*(t) - \theta_t + (\eta_t)^2 - (1+\beta_1^*)\rho_2 \sigma^*(t)\eta_t \right) \\ & \quad + (1-\pi(t)) \times \left(f(t) - \theta_t + (\eta_t)^2 - (1+\beta_1)\rho_1 \sigma(t)\eta_t \right) \Big] \\ & \quad - \frac{1}{2} R e^{-\gamma t} \left(X_t^{\pi,x}\right)^{1-R} \\ & \quad \times \left[(\eta_t)^2 + (\pi(t))^2 (1+\beta_1^*)^2 (\sigma^*(t))^2 + (1-\pi(t))^2 (1+\beta_1)^2 (\sigma(t))^2 \right] \\ & \quad - R e^{-\gamma t} \left(X_t^{\pi,x}\right)^{1-R} \end{aligned}$$

$$\times\left[-\eta_t\pi(t)\left(1+\beta_1^*\right)\sigma^*(t)\rho_2 - \eta_t(1-\pi(t))\left(1+\beta_1\right)\sigma(t)\rho_1 + \pi(t)(1-\pi(t))\left(1+\beta_1^*\right) \times\left(1+\beta_1\right)\sigma(t)\sigma^*(t)\rho_{1,2}\right] - \frac{e^{-\gamma t}}{1-R}\left(X_t^{\pi,x}\right)^{-R}(1-R)c_t^1\Big] + \frac{e^{-\gamma t}}{1-R}\left(X_t^{\pi,x}\right)^{1-R}\dot{\Theta}_t\Big\}\,dt. \quad (32)$$

So we can write

$$J^1(X_0) = \frac{1}{1-R}\left(X_0^{\pi,x}\right)^{1-R}\Theta_0 + IV + V + VI, \quad (33)$$

where

$$IV = \max_{(\pi,c_t^1)}\mathbb{E}\int_0^T \frac{e^{-\gamma(t-T)}}{1-R}\left[\left(c_t^1\right)^{1-R} - (1-R)\left(X_t^{\pi,x}\right)^{-R}\Theta_t c_t^1 - RX_t^{1-R}\Theta_t^{1-(1/R)}\right]dt,$$
$$V = \max_{(\pi,c_t^1)}\mathbb{E}\int_0^T e^{-\gamma t}\left(X_t^{\pi,x}\right)^{1-R}\Theta_t L(\pi(t))\,dt. \quad (34)$$

Here

$$L(x) = x\Big(f^*(t) - f(t) + (1-R)\eta_t \times\left[\left(1+\beta_1\right)\rho_1\sigma(t) - \left(1+\beta_1^*\right)\rho_2\sigma^*(t)\right] + R\left(1+\beta_1\right)^2(\sigma(t))^2 - R\left(1+\beta_1^*\right)\left(1+\beta_1\right)\sigma(t)\sigma^*(t)\rho_{1,2}\Big) - \frac{1}{2}x^2\Big(R\left[\left(1+\beta_1^*\right)^2(\sigma^*(t))^2 + \left(1+\beta_1\right)^2(\sigma(t))^2\right] - 2R\left(1+\beta_1\right)\left(1+\beta_1^*\right)\sigma(t)\sigma^*(t)\rho_{1,2}\Big) - \Delta, \quad (35)$$

$$\Delta = \frac{\left(\Delta^u\right)^2}{2\Delta^d}, \quad (36)$$

$$\Delta^u = f^*(t) - f(t) + (1-R)\eta_t \times\left[\left(1+\beta_1\right)\rho_1\sigma(t) - \left(1+\beta_1^*\right)\rho_2\sigma^*(t)\right] + R\left(1+\beta_1\right)^2(\sigma(t))^2 - R\left(1+\beta_1^*\right) \times\left(1+\beta_1\right)\sigma(t)\sigma^*(t)\rho_{1,2}, \quad (37)$$

$$\Delta^d = R\left[\left(1+\beta_1^*\right)^2(\sigma^*(t))^2 + \left(1+\beta_1\right)^2(\sigma(t))^2\right] - 2R\left(1+\beta_1\right)\left(1+\beta_1^*\right)\sigma(t)\sigma^*(t)\rho_{1,2}, \quad (38)$$

$$VI = \max_{(\pi,c_t^1)}\mathbb{E}\int_0^T \frac{e^{-\gamma t}}{1-R}X_t^{1-R} \times\Big\{\dot{\Theta}_t + \Theta_t\Big[-\gamma + (1-R) \times\left[f(t) - \theta_t + (\eta_t)^2 - \left(1+\beta_1\right)\rho_1\sigma(t)\eta_t\right] - \frac{1}{2}R(1-R) \times\left[\eta(t)^2 + \left(1+\beta_1\right)^2(\sigma(t))^2\right] + R(1-R)\eta(t)\left(1+\beta_1\right)\sigma(t)\rho_1 + (1-R)\Delta\Big] + R\Theta_t^{1-(1/R)}\Big\}. \quad (39)$$

If we take

$$\pi^*(t) = \frac{\Delta^u}{\Delta^d}, \quad (40)$$

it is easy to check that $L'(\pi^*(t)) = 0$ and $L''(\pi^*) < 0$. Thus the function $L(\pi)$ attains its maximum at point π^* and $L(\pi^*) = 0$, $V = 0$. Now we let P_t be the solution of the following ODE:

$$-\dot{\Theta}_t = M\Theta_t + R\Theta_t^{1-(1/R)}, \quad t\in[0,T],$$
$$\Theta_T = e^{\overline{M}N}\left[K + \int_T^{T+N} e^{-(1/R)\overline{M}(T+N-s)}ds\right]^R. \quad (41)$$

Here

$$M = -\gamma + (1-R)\left[f(t) - \theta_t + (\eta_t)^2 - \left(1+\beta_1\right)\rho_1\sigma(t)\eta_t\right] - \frac{1}{2}R(1-R)\left[\eta(t)^2 + \left(1+\beta_1\right)^2(\sigma(t))^2\right] + R(1-R)\eta(t)\left(1+\beta_1\right)\sigma(t)\rho_1 + (1-R)\Delta. \quad (42)$$

We have

$$P_t = e^{M(T-t)}\left[P_T^{1/R} + \int_t^T e^{-(1/R)M(T-s)}ds\right]^R, \quad t\in[0,T],$$
$$P_0 = e^{MT}\left[P_T^{1/R} + \int_0^T e^{-(1/R)M(T-s)}ds\right]^R. \quad (43)$$

Then $VI = 0$.

Let

$$\left(c_t^1\right)^* = (1-R)^{-1/R}\left(\Theta_t\right)^{-1/R}X_t^{\pi,x}, \quad t\in[0,T]. \quad (44)$$

One can check that IV attains its maximum at point $(c_t^1)^*$, $IV = 0$. Then the optimal value function is

$$J^1(X_0) = \frac{1}{1-R}\left(X_0^{\pi,x}\right)^{1-R}\Theta_0. \quad (45)$$

□

Remark 2. In HARA case, we can get from simple calculus that the Pratt-Arrow measure of relative aversion $A = R$, $R \in (0, 1)$.

So the constant R can indicate the investor's attitude to the risk in the investment.

Once knowing the amount of our wealth, we can make a decision on the strategy of the investment according to formula (22).

4. Simulation Results

From the history of price data in the market, we can use statistical method to estimate the parameters in the model. Now let us give a simulating example. In this example, let the coefficients be constants for simplicity, and we only consider the choice at initial time $t = 0$.

Example 3. We take the following parameters depending on the situation of the real market. Choose $S_0 = 30$, $S_0^* = 50$, $P_0 = 15$, $S_0^* = 26$, $\beta_1 = -0.28$, $\beta_2 = -0.1$, $\alpha_S = 1.5$, $\alpha_S^* = 1.2$, $\alpha_p = 1.4$, $\alpha_p^* = 1.1$, $\gamma = 0.6$, $\rho_1 = 0.25$, $\rho_2 = -0.1$, $\rho_{1,2} = -0.3$, $\eta = 0.1$, $T = 2$, and $R = 0.3$.

From the formula (22), we know that the optimal portfolio π is a 2-dimensional function with the volatility parameters σ and π^*, and we cannot get the explicitly monotone relationship. Next we will see the relationship of them by the 3D figure. For fixed $\sigma = 0.3$ (see Figure 1), we can get the relationship between the optimal proportion π and the volatility of the project B σ^*. Here π decreases down to 0. That means the high volatility leads to more capital investment in project A and less investment in project B. Taking $\sigma = 0.69$ for example, here $\pi^* = 0.5616$, that is, the invest's optimal choice is to invest almost half of his wealth to project B. When $\sigma = 0.84$, $\pi^* = 0.273$; that is, the investor would only devote about thirty percent of his wealth to the corporate project and invest the rest in project A.

In Figure 2, the curve goes just as we have expected. From the figure, we can easily get the relationship between the optimal proportion π^* and the volatility of the production σ and π^*. Taking $\sigma = 0.54$, $\sigma^* = 0.32$, for example, here $\pi^* = 0.5323$. In our model, $\pi(t)$ represents the proportion of the wealth invested in project B. The volatility of project B is bigger than project A. Then the investor will invest less wealth to project B.

Next for fixed $\sigma^* = 0.5$, we will discuss the relationship between the portfolio and the volatility σ and inflation ratio. In Figure 3, the curve goes just as we expect. Let us take the two points $(0.414, 0.097, 1.207)$ and $(0.414, 0.09, 0.4144)$ for example. From the figure, for fixed σ, we know that when the inflation ratio decreases, the investor will invest more money to project A.

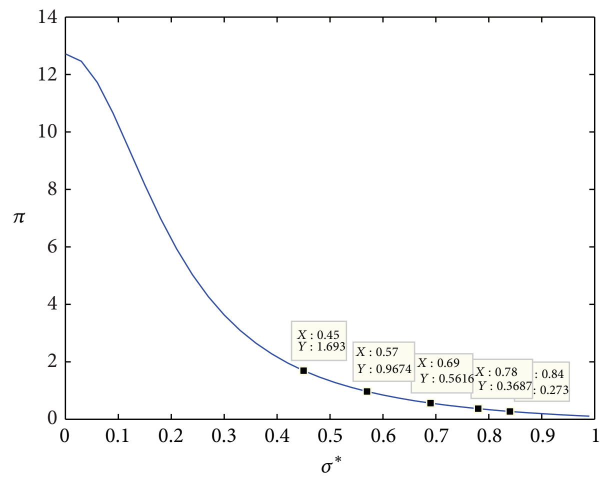


FIGURE 1: Relationship between the volatility of project B σ^* and the optimal proportion π^*.

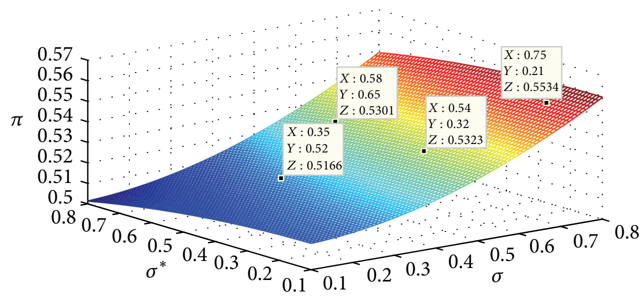


FIGURE 2: Relationship between the volatility of the project σ, σ^* and the optimal proportion π^*.

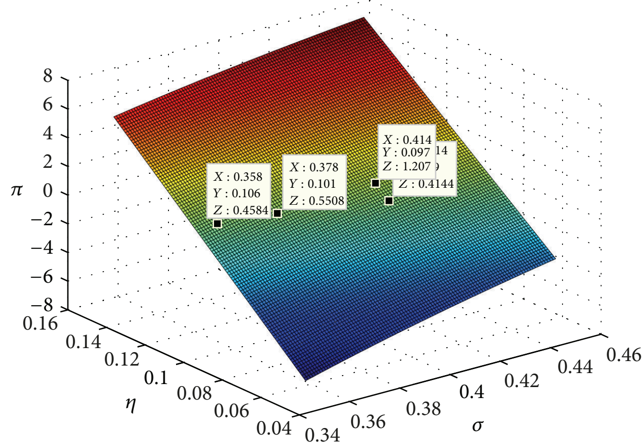


FIGURE 3: Relationship between the volatility of project A σ, the inflation ratio η, and the optimal proportion π^*.

5. Conclusions

In this paper, we present the model of corporate optimal investment with consideration of the influence of inflation and the difference between the daytime and nighttime. From Theorem 1, we know that the classical dynamical programming principle still holds for this optimal problem. The optimal investment and consumption choice have also been achieved by solving the corresponding H-J-B equation, and the economic analysis was then given for that. For a typical HARA utility function case, we got the explicit

optimal investment and consumption strategy and then some simulation results illustrating the influence of market's volatility parameters to the optimal choice. In addition, there are several interesting problems about the underlying topic, such as the corresponding convex restriction problem and the partial information problem which needs to use the Pontryagin's maximum principle under partial information (Wang and Wu [12, 13], etc.). Since these problems are more consistent with the practical situation in real market, we will furthermore investigate them in further work and then desire to solve some practical optimization investment problems in financial market.

Acknowledgments

Zongyuan Huang is supported in part by Chinese NSF Grant nos. 11126208 and 61174092. Detao Zhang is supported in part by Independent Innovation Foundation of Shandong University (2011GN018).

References

[1] R. Merton, "An equilibrium market model with incomplete information," *Journal of Finance*, vol. 42, no. 3, pp. 483–510, 1987.

[2] D. Duffie, *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton, NJ, USA, 1992.

[3] I. Karatzas, "Optimization problems in the theory of continuous trading," *SIAM Journal on Control and Optimization*, vol. 27, no. 6, pp. 1221–1259, 1989.

[4] J. J. Choi, "Diversification, exchange risks and corporate international investment," *Journal of International Business Studies*, vol. 20, no. 1, pp. 145–155, 1989.

[5] M. Bellalah and Z. Wu, "A simple model of corporate international investment under incomplete information and taxes," *Annals of Operations Research*, vol. 165, no. 1, pp. 123–143, 2009.

[6] Z. Wu and L. Y. Zhang, "The corporate optimal portfolio and consumption choice problem in the real project with borrowing rate higher than deposit rate," *Applied Mathematics and Computation*, vol. 175, no. 2, pp. 1596–1608, 2006.

[7] M. Bellalah and Z. Wu, "A model for market closure and international portfolio management within incomplete information," *International Journal of Theoretical and Applied Finance*, vol. 5, pp. 479–495, 2002.

[8] Z. Y. Huang and Z. Wu, "One kind of corporate international optimal investment and consumption choice problem," in *Proceedings of the 27th Chinese Control Conference (CCC '08)*, pp. 603–606, Kunming, China, July 2008.

[9] D. T. Zhang and T. Zhang, "Optimal portfolio of corporate investment and consumption under market closure," *International Journal of Business*, vol. 17, no. 1, pp. 25–38, 2012.

[10] J. M. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer, New York, NY, USA, 1999.

[11] G. Wang, Z. Wu, and J. Xiong, "Maximum principles for forward-backward stochastic control systems with correlated state and observation noises," *SIAM Journal on Control and Optimization*, vol. 51, no. 1, pp. 491–524, 2013.

[12] G. C. Wang and Z. Wu, "The maximum principles for stochastic recursive optimal control problems under partial information," *IEEE Transactions on Automatic Control*, vol. 54, no. 6, pp. 1230–1242, 2009.

[13] G. C. Wang and Z. Wu, "General maximum principles for partially observed risk-sensitive optimal control problems and applications to finance," *Journal of Optimization Theory and Applications*, vol. 141, no. 3, pp. 677–700, 2009.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 132920, 10 pages
http://dx.doi.org/10.1155/2013/132920

Research Article

Probabilistic Value-Centric Optimization Design for Fractionated Spacecrafts Based on Unscented Transformation

Ming Xu,[1] Jinlong Wang,[1] Ang Zhang,[1] and Shengli Liu[2]

[1] *Department of Aerospace Engineering, School of Astronautics, Beihang University, Beijing 100191, China*
[2] *DFH Satellite Co., Ltd., Beijing 100094, China*

Correspondence should be addressed to Ming Xu; xuming@buaa.edu.cn

Received 30 January 2013; Revised 19 March 2013; Accepted 8 April 2013

Academic Editor: Suiyang Khoo

Fractionated spacecrafts are of particular interest for pointing-intensive missions because of their ability to decouple physically the satellite bus and some imaging payloads, which possess a lesser lifecycle cost than a comparable monolithic spacecraft. Considering the probabilistic uncertainties during the mission lifecycle, the cost assessment or architecture optimization is essentially a stochastic problem. Thus, this research seeks to quantitatively assess different spacecraft architecture strategies for remote-sensing missions. A dynamical lifecycle simulation and parametric models are developed to evaluate the lifecycle costs, while the mass, propellant usage, and some other constraints on spacecraft are assessed using nonparametric, physics-based computer models. Compared with the traditional Monte Carlo simulation to produce uncertain distributions during the lifecycle, the unscented transformation is employed to reduce the computational overhead, just as it does in improving the extended Kalman filter. Furthermore, the genetic algorithm is applied to optimize the fractionated architecture based on the probabilistic value-centric assessments developed in this paper.

1. Introduction

The Defense Advanced Research Projects Agency (DARPA) implemented the Future, Fast, Flexible, Fractionated, Free-Flying (F6) program, aiming to demonstrate the concept of the fractionated architecture [1]. DARPA conducted extensive researches for the technological performances and economic advantages and invested Lockheed Martin Company (LM), Northrop Grumman Corporation (NG), Orbital Sciences Corporation (OSC), and Boeing Company (BC) for fractionated spacecrafts based on value-centric design methodologies (VCDM) in first phase of the F6 program [2]. This innovative design method for spacecraft has drawn great attentions from the astronautical community.

Considering the application of fractionated modules into responsive space, Richards et al. [3] analyzed the internal relations between the four aspects of technological, organizational, economic, and political supports. Brown and Eremenko [4, 5] summarized the achievements of VCDM in the F6 program from the viewpoint of the relationship between the innovative value-centered design standard and traditional monolithic design standard and then evaluated the concept of the fractionated spacecrafts compared with the traditional spacecraft architecture [5]. Mathieu and Weigel [6] evaluated the advantages and costs of the fractionated architecture in the fields of the attributes, strategies, and models. O'Neill [7] developed the semianalytic tool PIVOT for the model frame, risk, and the net present value and then optimized the PIVOT tool involved in the second phase of the F6 program. O'Neil and Mankins [8] achieved the conclusion that the fractionated spacecrafts were better on the lifecycle cost than a traditional spacecraft through the dynamical simulation and parameter models on the quality of the spacecraft [9]. Lafleur and Saleh [10, 11] developed another design tool GT-FAST, which instanced specific analysis on the F6 program for handling input, model, and attributes. Yao et al. [12, 13] proposed the multidisciplinary optimization about the uncertainty for the spacecraft conceptual design and verified the feasibility and effectiveness of this optimization. Daniels et al. [14] presented a heuristics-based decision model using

a Monte Carlo simulation to produce value distributions for satellite operator decision sets and a multi-stage decision process utilizing a dynamic programming algorithm to find value optimal decisions. Compared to the traditional measure metric on spacecraft cost, Collopy [15] derived a new metric rigorously from the view of probability, which places a focus on improving the probability of success rather than on reducing the cost.

One element necessary in enabling a probabilistic, value-centric analysis of such fractionated architecture is a systematic method for sizing and costing many candidate architectures that arise in aerospace engineering. In this paper, a dynamical lifecycle simulation and parametric models are developed to evaluate the lifecycle costs, while the mass, propellant usage, and some other constraints on spacecraft are assessed using nonparametric, physics-based computer models. The lifecycle is divided into three phases, that is, module development, launching, and on-orbit control. Furthermore, the genetic algorithm is applied to optimize the fractionated architecture based on the probabilistic value-centric assessments developed in this paper. To accelerate the optimization, one of the new techniques is to employ the graphic processing unit (GPU) accelerated genetic algorithm based on the Compute Unified Device Architecture (CUDA) to improve the searching efficiency because the assessments in any generation are parallel computerized. Another is to introduce the unscented transformation to reduce the amount of computations rather than the Monte Carlo simulation, just as it does in improving the extended Kalman filter (EKF) [16, 17].

2. Cost, Value, and Assessment Indicators

The cost assessment for a spacecraft design scheme includes module development costs, launch and operation costs, and risk costs from uncertainties during the lifecycle. All the types of the above costs are quantified by the monetary unit, for example, US dollar. The statistical sum of all the costs from the preparation phase to the end of the lifecycle constitutes the total cost of the spacecraft design scheme.

There are several valuation standards in the value-centric design methodologies to measure the practical spacecraft scheme by the quantitative assessments of the cost and value of fractionated spacecrafts. Quantitative assessment of the value gives a measurement indicator, as well as an evaluation methodology to provide information for the task of decision making. Considering the probabilistic uncertainties during the mission lifecycle, the value is acquired at the end of the lifecycle of the spacecraft from the quantitative criteria to quantify the value of spacecraft.

To study the cost and value of the trade-offs, the net present value (NPV) is usually used in the financial evaluation of the business assets [10, 11]. It refers to the investment scheme generating net cash flow to the capital cost for the discount rate discount and original investment present value variance. Therefore, the net present value NPV is used to judge the fractionated spacecraft design in this paper. The higher the value of NPV is, the better the spacecraft design scheme is, which means that the input cost is low but the output value is high. Hence, the input-output ratio could be weighted by the NPV value, which is formulized as follows [5]:

$$\mathrm{NPV} = \frac{N}{\left(1 + D_R\right)^{T_{\mathrm{yearloop}}-1}}, \tag{1}$$

where D_R is the discount rate, N is the free cash flow, and T_{yearloop} is the spacecraft lifecycle (unit: year).

Table 1: Parameter settings for fractionated components.

Components names	Weight (kg)	Power (W)	Cost (M$)	TRL	FIT
EO	40	15	15	9	5000
24/7 Comm	4	25	5	5	5000
HBD	10	25	2	9	5000
SSR	8	100	2	7	6000
MDP	8	18	1	6	5000
AIS	5	15	0.5	8	3500

3. Spacecraft Models

3.1. Fractionated Spacecraft Architecture. The spacecraft architecture modeling is a prerequisite to assess the cost and value of the spacecraft design scheme. But the fractionated spacecrafts are quite different from traditional spacecraft because of the different design principles on the two types of architectures. Common architecture level definitions on the fractionated spacecraft architecture modeling are listed from low to high order, as "component," "module," and "cluster" [12].

Component: it is the smallest unit in the fractionated spacecraft architecture modeling. The four companies modeled the components in the first phase of the DARPA mission, mainly including the following components: (1) payload component: the valuable payloads are the feasible assessment to increase the total value of the fractionated spacecraft for the purpose of using the least cost to achieve the more value; (2) measurement and control component: it realizes the continuous measurement and control through the relay module in the architecture constellation; (3) mission data processing component: usually the on-board computer is used for processing complex data; (4) digital communication component: it is responsible for communication between the space and ground, as well as responsible for the data download and upload; (5) data storage component: it is storing large amounts of data and preparing for data transmission, that is, solid-state drive (SSR).

Module: it consists of several components to achieve independent and free flight on orbit. In addition to payload and other functional components, it includes other associated components of the power supply and thermal control.

Cluster: it is composed of multiple fractionated modules in formation flight to complete certain missions.

Using six separable components in this paper, the parameters are shown in Table 1 [18], where FIT represents the number of failures in 1000 hours and the characterization

of component reliability parameters, the Earth observing payload (EO) is employed throughout this paper to meet the imaging missions, 24/7 Comm is the digital component, HBD is the high-bandwidth downloading communication component, SSR is the solid hard drive component, MDP is the mission data processing component, and AIS is the automatic identification component.

The spacecraft scheme includes the design of spacecraft architecture and the selection of launch vehicles [19]. To inject fractionated spacecrafts into space, six alternative categories for the launch vehicle are provided in this paper, that is, Minotaur I, Athena I, Taurus 2210, Taurus 3110, Minotaur IV, and Athena II [1], which have different launch capacities and reliabilities, and different launch vehicle costs. Therefore, the launch vehicle should be selected based upon the aerospace mission and the fractionated architecture.

The combination of separated components, the cluster segmentations, and the selection of launch vehicle impact the assessed cost and value of the fractionated spacecrafts. As a result, the optimization of the cluster segmentation and its separated components should be taken to get a higher input-output ratio.

3.2. Spacecraft Architecture Design Models

3.2.1. Cost Models. There are three parts of the fractionated spacecraft's cost: module development cost, launch and operation cost, and risk cost, where commercial insurance for spacecraft is involved in launch and operation cost, and the ground equipment cost and software development cost are not accounted in this paper. Risk cost includes the cost caused by time delay during the module development and the failure due to launch vehicle and on-orbit maintenance, which are considered to be governed by uncertainties.

According to the recyclability, the cost can be divided into nonrecyclable and recyclable types. Inherited from this attribution, the separated components in the cluster would be divided into the recyclable and nonrecyclable ones. This classification is primarily proposed for some reusable equipment and design schemes for the specified mission. In this case, the second cost is the result from the combination of the first cost and the learning curve rate.

(1) Module Development Costs. Different evaluations arise from the development costs with different components, or different development cycles and technology readiness levels (TRL) for the same components. For any spacecraft architecture scheme, the recyclable and nonrecyclable costs are calculated, respectively. For the specified type of modules, the cost on the module manufactured at the first time is higher than others because of its low TRL; however, the cost on the module built at the second or subsequent time is lower due to the cost of the recyclable components.

Considering the batch production, the cost of the unit module is estimated as follows:

$$C_{\mathrm{mod}_{i1}} = C_{\mathrm{mod}_{i1}}^{\mathrm{NRE}} + C_{\mathrm{mod}_{i1}}^{\mathrm{REC}},$$
$$C_{\mathrm{mod}_{iQ}} = C_{\mathrm{mod}_{i1}}^{\mathrm{REC}} \cdot Q^{(\ln L_R)/\ln 2}, \tag{2}$$

where $C_{\mathrm{mod}_{i1}}$ is the cost of the first module of the ith type module including nonrecyclable cost $C_{\mathrm{mod}_{i1}}^{\mathrm{NRE}}$ and recyclable cost $C_{\mathrm{mod}_{i1}}^{\mathrm{REC}}$, $C_{\mathrm{mod}_{iQ}}$ is the cost of the Qth module of the ith type module, and L_R is the learning curve rate of the module development and production. The nonrecyclable cost and recyclable cost are calculated by the components and satellite platform, the cost calculation formula with reference to the cost of small satellite model (abbr. SSCM07), which was developed by Mahr [20] for the Aerospace Cooperation and has potential applications in estimating the manufacturing cost empirically [10, 11, 13, 21].

SSCM07 is good at estimating the production cost; however, it is powerless in estimating other types of costs (such as operation, risk, and inflation cost) and their dynamic and uncertain evolutions during the spacecraft's lifecycle. Thus, we employed SSCM07 to estimate the production costs for fractionated modules and then developed a probabilistic method to measure all the uncertain costs during the lifecycle dynamically.

Taking inflation into account, the cost of the unit module is estimated as follows:

$$C_{\mathrm{mod}_{iQ}}^{\mathrm{inflated}} = C_{\mathrm{mod}_{iQ}} \cdot \left(1 + R_{\mathrm{inflated}}\right)^{T_{\mathrm{lateryear}}}, \tag{3}$$

where R_{inflated} is the year inflation rate and $T_{\mathrm{lateryear}}$ is expressed as module development time (unit: year). All module's development cost is as follows:

$$C_M = \sum_{i=1}^{N_{\mathrm{type}}} \left(\sum_{j=1}^{N_{\mathrm{mod}_i}^{\mathrm{type}}} C_{\mathrm{mod}_{ij}}^{\mathrm{inflated}} \right), \tag{4}$$

where N_{type} is the number of the spacecraft module's type and $N_{\mathrm{mod}_i}^{\mathrm{type}}$ is the number of certain type module.

(2) Launching and Operational Costs. The selection of a launch vehicle scheme is determined according to the module properties. Therefore, the cost of the launch vehicle is estimated by the design scheme of the spacecraft architecture. The requirement from the spacecraft and the capacity of the launch vehicle, respectively, restrict the feasible combinations between them, so the number of selecting launch vehicles is less than the number of modules. Thus, the launch cost is listed as follows:

$$C_{\mathrm{launch}} = \sum_{i=1}^{N_{\mathrm{launch}}} C_{\mathrm{launch}}^{i}, \tag{5}$$

where N_{launch} is the total expected number of launch vehicles and C_{launch}^{i} is the cost of the ith vehicle. Owning to the huge cost of launch vehicles, it is necessary to reduce the number of launch vehicles and to load as many modules as possible in every vehicle.

When the fractionated spacecrafts are on orbit, operating each module will generate the operational cost. In case of the annual operational cost C_i^{ops} for each module being equal to

2 M\$, the total operational cost C_{ops} during the spacecraft's lifecycle is as follows:

$$C_{\text{ops}} = \sum_{i=1}^{N_{\text{type}}} \left(\sum_{j=1}^{N_{\text{mod}_i}^{\text{type}}} C_i^{\text{ops}} \cdot T_{ij} \right), \tag{6}$$

where T_{ij} is the total orbiting time of the ith type module on the jth year. Then the launch and operational cost of fractionated spacecrafts during the lifecycle is as follows:

$$C_{\text{lo}} = C_{\text{launch}} + C_{\text{ops}}. \tag{7}$$

(3) Risk Costs. The uncertainties, such as task delay, launching failure, or on-orbit failure, exist in all the phases of spacecraft development, launching, and on-orbit operation, which are required to be maintained with extra costs. The risks are measured by the risk cost C_{risk} as the criterion of the robustness and reliability in spacecraft architecture design, which are allocated by launching failure and on-orbit failure in this paper.

Launching failure: this type of failure comes from the reliability of launch vehicles less than 100%. If a launch vehicle happens to fail, all the modules carried by this vehicle need to be redeveloped and then launched. Hence, the risk cost of launching failure can be formulized as follows:

$$C_{\text{launch}}^{\text{failure}} = \sum_{i \in I_{\text{launch}}^{\text{failure}}}^{N_{\text{launch}}} C_{\text{launch}}^{i} + \sum_{i \in I_{\text{launch}}^{\text{failure}}}^{N_{\text{launch}}} \left(\sum_{j \in I_{\text{mod}_i}^{\text{launchfail}}}^{N_{\text{mod}_{ij}}^{\text{launchfail}}} C_{\text{mod}_{jk}}^{\text{inflated}} \right), \tag{8}$$

where $I_{\text{launch}}^{\text{failure}}$ is the identification number of the failing launch vehicle, $N_{\text{mod}_{ij}}^{\text{launchfail}}$ and $I_{\text{mod}_i}^{\text{launchfail}}$ indicate the quantity and type of the modules carried by the failing vehicle "i," respectively.

On-orbit operation: if a module fails on-orbit, another module of the same type is expected to be launched to replace this faulty module. Thus, the cost of on-orbit operation is essentially the development and launching costs for the new module, which is listed as shown:

$$C_{\text{ops}}^{\text{failure}} = \sum_{i=1}^{N_{\text{ops}}^{\text{failure}}} C_{\text{launch}}^{i} + \sum_{i \in I_{\text{launch}}^{\text{failure}}}^{N_{\text{launch}}} \left(\sum_{j \in I_{\text{mod}_i}^{\text{opsfail}}}^{N_{\text{mod}_{ij}}^{\text{opsfail}}} C_{\text{mod}_{jk}}^{\text{inflated}} \right), \tag{9}$$

where $N_{\text{ops}}^{\text{failure}}$ is the identification number of the module failing on-orbit, $N_{\text{mod}_{ij}}^{\text{opsfail}}$ and $i_{\text{mod}_i}^{\text{opsfail}}$ indicate the quantity and type of the module failing on-orbit, respectively.

Therefore, the risk cost can be considered as the failure cost during the launching or operational phase, as

$$C_{\text{risk}} = C_{\text{launch}}^{\text{failure}} + C_{\text{ops}}^{\text{failure}}. \tag{10}$$

According to the above sections presenting the costs when developing, launching, and operating a module and the risk costs due to the failures in these phases, the total cost of the fractionated architecture can be expressed as following:

$$C = C_M + C_{\text{lo}} + C_{\text{risk}}. \tag{11}$$

3.2.2. Value Models. The revenue model gives a finical rule measuring the spacecraft gains, which are considered as the data products achieved from the payload modules and their communication links. Consequently, some empirical weight factors are introduced to monetize the imaging data to model the benefit value of a spacecraft. When the data have better resolution and positioning accuracy, a larger factor is weighted to price them; on the other hand, a smaller weight factor is expected.

The effectiveness of the communication link affects the value from the imaging data as well, depending upon the type of link hardware (low or high speed transmission), the frequency, and duration of the spacecraft passing through the data receiving stations.

In this paper, the payloads are defined to employ both the low-rate downlink (abbr. LR), high-rate downlink (abbr. HR), and space-ground interlink (abbr. SG) to implement the digital transmissions. Thus, the benefit value of the fractionated spacecrafts can be derived from the following equation:

$$R_{i+j} = \delta_i \left(N_i^{\text{LR}} + N_i^{\text{HR}} + N_i^{\text{SG}} \right) + \delta_j \left(N_j^{\text{LR}} + N_j^{\text{HR}} + N_j^{\text{SG}} \right), \tag{12}$$

where δ_i and δ_j are the weight factors of the payloads, respectively, and N_k^{LR}, N_k^{HR}, and N_k^{SG}, $k = i, j$, are, respectively, the amount of valid data obtained by the link between space and ground.

3.2.3. Uncertainty Models. A systematic study on the investing cost and economic benefit assessments on the spacecraft architectures is implemented in this paper. Considering the probabilistic uncertainties during the mission lifecycle, the cost assessment or architecture optimization is essentially a stochastic problem. The uncertainties such as financial inflation, launching failure, and on-orbit operating failure are required to be maintained by extra costs, which are formulized by (3) in Section 3.2.1(1) and (8), (9), and (10) in Section 3.2.1(3).

In the numerical assessments, all the uncertainties are modeled by the stochastic noises. Generally, the upper and lower bounds of these uncertainties are determined according to the experience in previous work or technology readiness level (TRL), which are formulized as following:

$$C = F\left(C_s, \delta_{\max}, \delta_{\min}\right), \tag{13}$$

where C_s is the mean value and $\delta_{\max}$ and $\delta_{\min}$ are the upper and lower bounds, respectively. The annual inflation rate is assumed as the Gaussian noise approximately with its mean and variance assigned to 3.5% and 0.5%, respectively, according to the historical consumer price index (CPI) of China. The reliability of launch vehicles is set as $P_{\text{launch}} = 95.1\%$ according to the handbook of the vehicle type of Minotaur IV, which means the launching failure occurs when a random number from the uniform distribution $U(0,1)$ is larger than the reliability P_{launch}. The on-orbit operating failure is dependent on the mean time to failure θ of the components, which is modeled by the exponential

distribution as $f(t) = \lambda e^{-\lambda \cdot t}$, where λ is referred as aging rate equal to $1/\theta$ and is set as 5% in this paper.

For a specified design scheme, the traditional Monte Carlo simulation is employed by the cost and value assessment to cope with the uncertainties. More than 10000 scenarios are created by the Monte Carlo method to simulate all the combinations of the noises mentioned above. Generally, the Monte Carlo simulation costs a mass of computation to generate all the situations ergodicly. Therefore, this statistical method is feasible just for evaluating a specified design scheme within all kinds of scenarios; however, it is not available for refining the optimal one from many candidate schemes. For example, the best design scheme is expected to be selected from 5000 candidates, and then the optimization needs to be evaluated 5×10^7 (=5000×10000) times. The heavy computation makes the optimizing iteration very difficult or impossible.

Thus, it is necessary to introduce the unscented transformation (UT) firstly proposed by Julier and Uhlmann [16] to reduce the amount of computations rather than the Monte Carlo simulation. The most common use of UT is in the nonlinear projection of mean and covariance estimates in the context of nonlinear extensions of the Kalman filter [17]. The principal advantage of the approach is that the nonlinear function is fully exploited, as opposed to the extended Kalman filter which replaces it with a linear one. One immediate advantage is that the UT can be applied with any given function whereas linearization may not be possible for functions that are not differentiable. A practical advantage is that the UT can be easier to implement because it avoids the need to derive and implement a linearizing Jacobian matrix.

The mean and covariance of financial inflation, launching failure, and on-orbit operating failure will be exactly encoded by sigma points and then be propagated by the assessment procedure to each point. Hence, the mean and covariance of the transformed set of points then represent the desired transformed estimate.

The unscented transformation is defined as the application of a given function to any partial characterization of an otherwise unknown distribution, but its most common use is for the case in which only the mean and covariance are given. For the fractionated architecture with six modules, the random uncertainties (denoted by $\mathbf{x} \in \mathbf{R}^6$) from the annual inflation rate, launching failure, and on-orbit operating failure have the mean $\overline{\mathbf{x}}$ and variance $\mathbf{P}$. To yield the statistics of some combination of cost and value $\mathbf{y}$ propagated through the nonlinear mapping defined by the cost and value assessment procedure $\mathbf{y} = \mathbf{h}(\mathbf{x})$, a set of 13 sigma points $\{\tilde{\mathbf{x}}_i,\ i = 0, 1, \ldots, 12\}$ are calculated using the following general selection scheme [16]:

$$\mathbf{x}^{(0)} = \overline{\mathbf{x}},$$
$$\tilde{\mathbf{x}}^{(i)} = \overline{\mathbf{x}} + \left(\sqrt{6 \cdot \mathbf{P}}\right)^T, \quad i = 1, 2, \ldots, 6, \tag{14}$$
$$\tilde{\mathbf{x}}^{(i)} = \overline{\mathbf{x}} - \left(\sqrt{6 \cdot \mathbf{P}}\right)^T, \quad i = 7, 8, \ldots, 12.$$

Once the sigma points are calculated from the prior statistics as shown above, they are propagated through the nonlinear

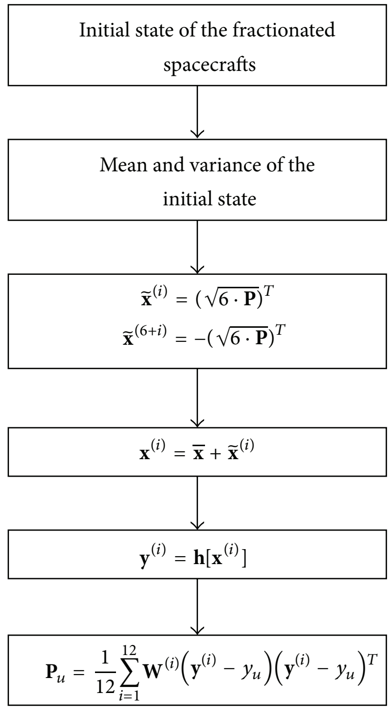


Figure 1: The process of applying UT to assess the cost and value for the fractionated architecture.

function $\mathbf{y}^{(i)} = \mathbf{h}[\mathbf{x}^{(i)}]$. Thus, the mean y_u and variance $\mathbf{P}_u$ of $\mathbf{y}$ approximated using a weighted sample mean and covariance of the posterior sigma-points are as follows:

$$y_u = \frac{1}{12}\sum_{i=1}^{12} \mathbf{W}^{(i)} \mathbf{y}^{(i)},$$
$$\mathbf{p}_u = \frac{1}{12}\sum_{i=1}^{12} \mathbf{W}^{(i)} \left(\mathbf{y}^{(i)} - y_u\right)\left(\mathbf{y}^{(i)} - y_u\right)^T, \tag{15}$$

where the weight coefficients $\mathbf{W}^{(i)}$ are restricted by the normalization condition $\sum_{i=0}^{12} \mathbf{W}^{(i)} = 1$. The process of applying the unscented transformation to calculate the statistical result of the cost and value assessment for the fractionated architecture is presented in Figure 1.

Compared with the traditional Monte Carlo simulation, this approach characterizes a probability distribution only in terms of few set of statistics. Furthermore, considering the ideal case without the uncertainties, the cost and benefit value (including the net present value NPV) will be the same as the single result achieved from the initial state of the fractionated spacecrafts.

4. Cost and Value Assessments

4.1. Cost and Value Assessment Procedure. Cost and value assessment for the fractionated architecture is quite dependent on all the stages divided in the whole lifecycle, that is, module development, launching, and on-orbit operation. Within the cost and value accumulated in each stage, the total

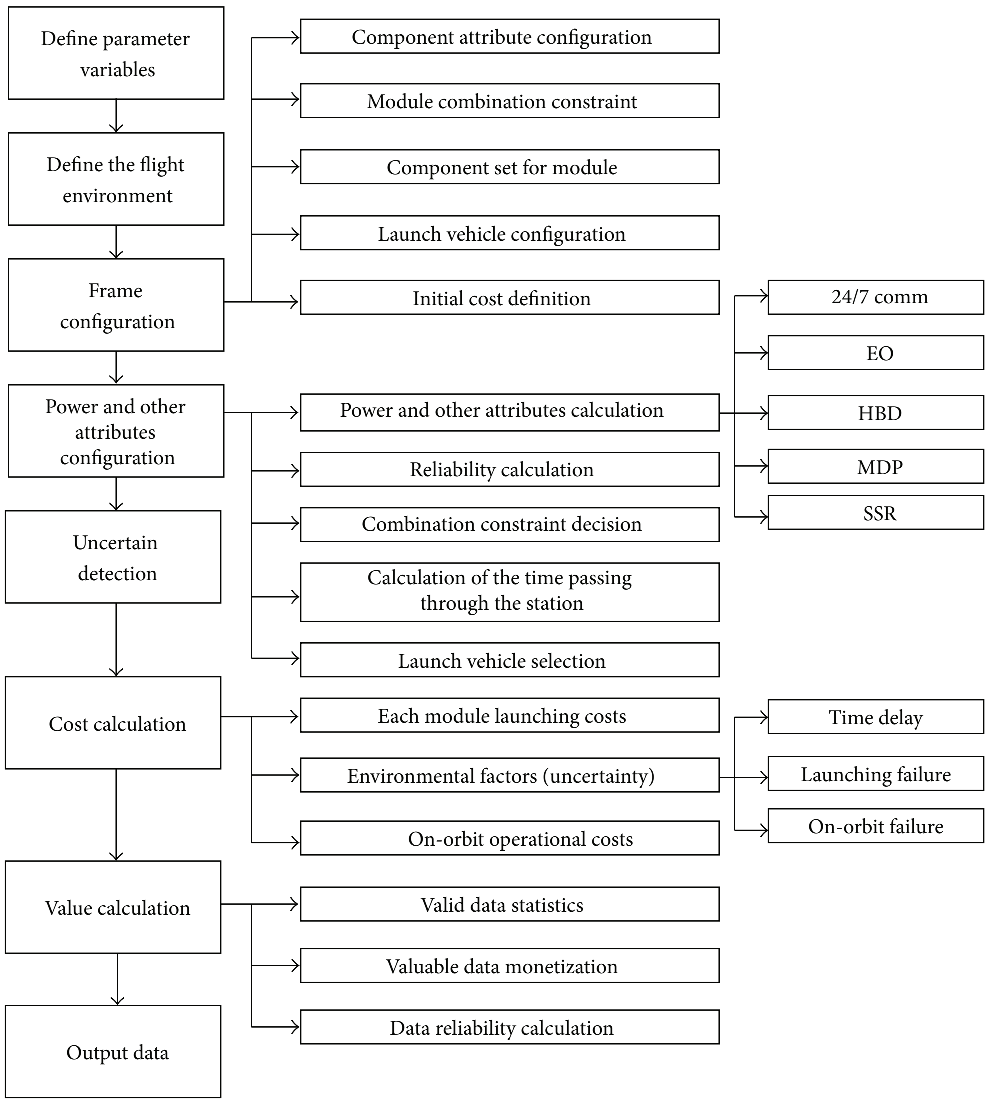


Figure 2: The flowchart of cost and value assessment for fractionated architecture.

cost and value will be yielded at the end of the lifecycle. The flowchart of the cost and value assessment is presented in Figure 2 for the fractionated architecture.

The main work is to define some relevant parameters for the fractionated spacecrafts in its early development stage, including setting the uncertainties due to the time delay and choosing the functional components assembled into the modules and their launch vehicles. The cost settlement at this stage should take the inflation rate and others into consideration. Moreover, no value is yielded in this stage because the spacecraft has not been injected into space and no imaging data are produced at the module development stage.

In the launch phase, the certain cost comes from a series of launch vehicles adapted by the developed modules, and the uncertain cost originates from the launching failure which requires more costs in redeveloping and relaunching modules. In addition, no value is yielded in this stage as well.

After injected into space successfully, the spacecraft is expected to be debugged on-orbit for about one month, and then some functional components (like on-board camera) will be put into use and able to produce valuable data, so that the spacecraft starts to benefit from this stage. Furthermore, the debugging stage costs the on-orbit operation a lot.

The uncertain cost in the operational stage results from a module failing on-orbit. In this case, another module is launched to replace this faulty module after redeveloped in accordance with the existing template. During the replacement, the faulty module and its successor have no contributions to the benefit value.

Therefore, the cost and value of the fractionated architecture can be obtained through the whole lifecycle according to the assessment flowchart listed in Figure 2.

4.2. Simulation of Cost and Value Assessment. In this paper, the nominal orbit is set as a sun-synchronous orbit with the orbital altitude of 500 km and the local time at descending node of 10:30 AM. The aerospace mission starts from 1st of January, and the development period of a module is assumed

Table 2: The net present value NPV in five-year lifecycle.

Time	NPV
The first year	−105.1077
The second year	−144.0231
The third year	118.7044
The fourth year	132.8142
The fifth year	14.1471

Table 3: The net present value NPV in ten-year lifecycle.

Time	NPV
The first year	−95.1603
The second year	−131.6379
The third year	55.4421
The fourth year	62.4822
The fifth year	51.2209
The sixth year	96.6323
The seventh year	62.1255
The eighth year	109.7646
The ninth year	119.1036
The tenth years	119.1036

Table 4: Average cost and benefit (unit: M$) in five-year lifecycle and their standard variances.

	Average cost	Cost standard variance	Average benefit	Benefit standard variance
Value	264.0081	32.1762	265.5474	59.9580

Table 5: Average cost and benefit (unit: M$) in ten-year lifecycle and their standard variances.

	Average cost	Cost standard variance	Average benefit	Benefit standard variance
Value	696.8454	37.8953	813.8701	88.1731

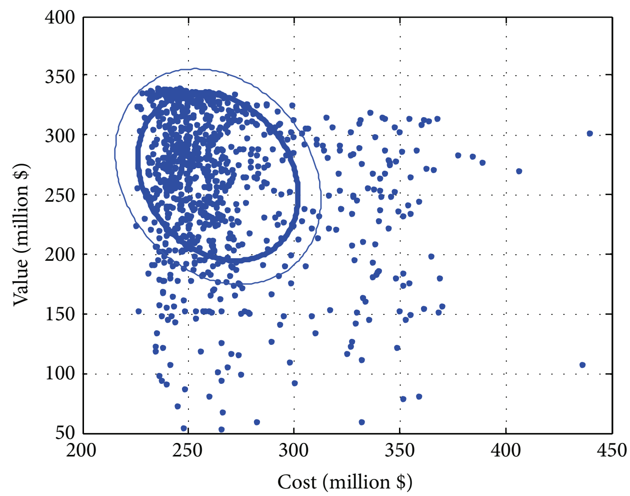


Figure 3: The assessments in five-year lifecycle labeled on the phase plane of cost and benefit values.

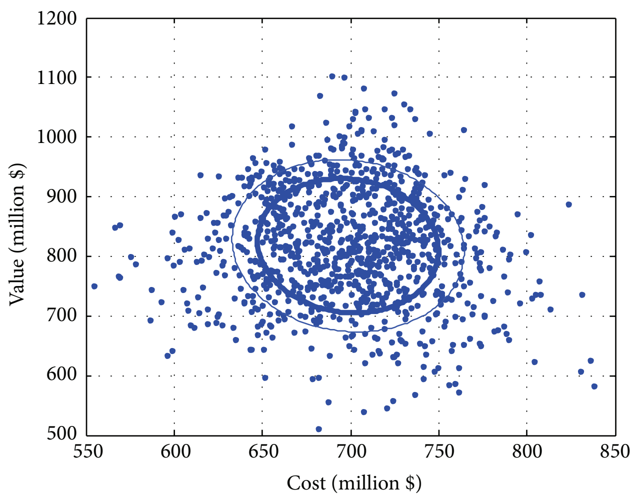


Figure 4: The assessments in ten-year lifecycle labeled on the phase plane of cost and benefit values.

as two years. All the modules are launched by the vehicle type of Minotaur IV with its reliability of 95.1%. The simulation cases are implemented in this paper with the lifecycles of five and ten years, respectively.

Considering the cost and value uncertainties from schedule change, launching failure, on-orbit failure, commercial insurance, module replacement, and other factors, the net present values counted numerically in the lifecycles of five and ten years are shown in Tables 2 and 3, and the average costs and benefits are shown in Tables 4 and 5. Then the assessments in the lifecycles of five and ten years are classified and labeled on the phase plane of cost and benefit values shown in Figures 3 and 4, where the two error ellipses are yielded by the unscented transformation with the confidence levels of 0.5 and 0.68, respectively, and the scattered points are plotted by the Monte Carlo method.

The numerical simulation indicates that the cost and benefit values quite depend on the mission lifecycle. The longer lifecycle increases the risk of on-orbit failure and the cost of module replacement and produces more valuable imaging data which raises the benefit values, which accounts for why the cost and value created in the lifecycle of ten years are more than five years. Moreover, most of the scattered points located inside the error ellipses validate the feasibility of the unscented transformation. Therefore, the assessments of many candidate schemes will be implemented in Section 5 by the unscented transformation to accelerate the optimization procedure.

However, there exists no relationship between the lifecycle of the fractionated spacecrafts and the net present value, which is accumulated by all the uncertainties during the lifecycle. Furthermore, the points far away from the confidence ellipses may happen with a very low probability, so that they are often ignored in aerospace engineering.

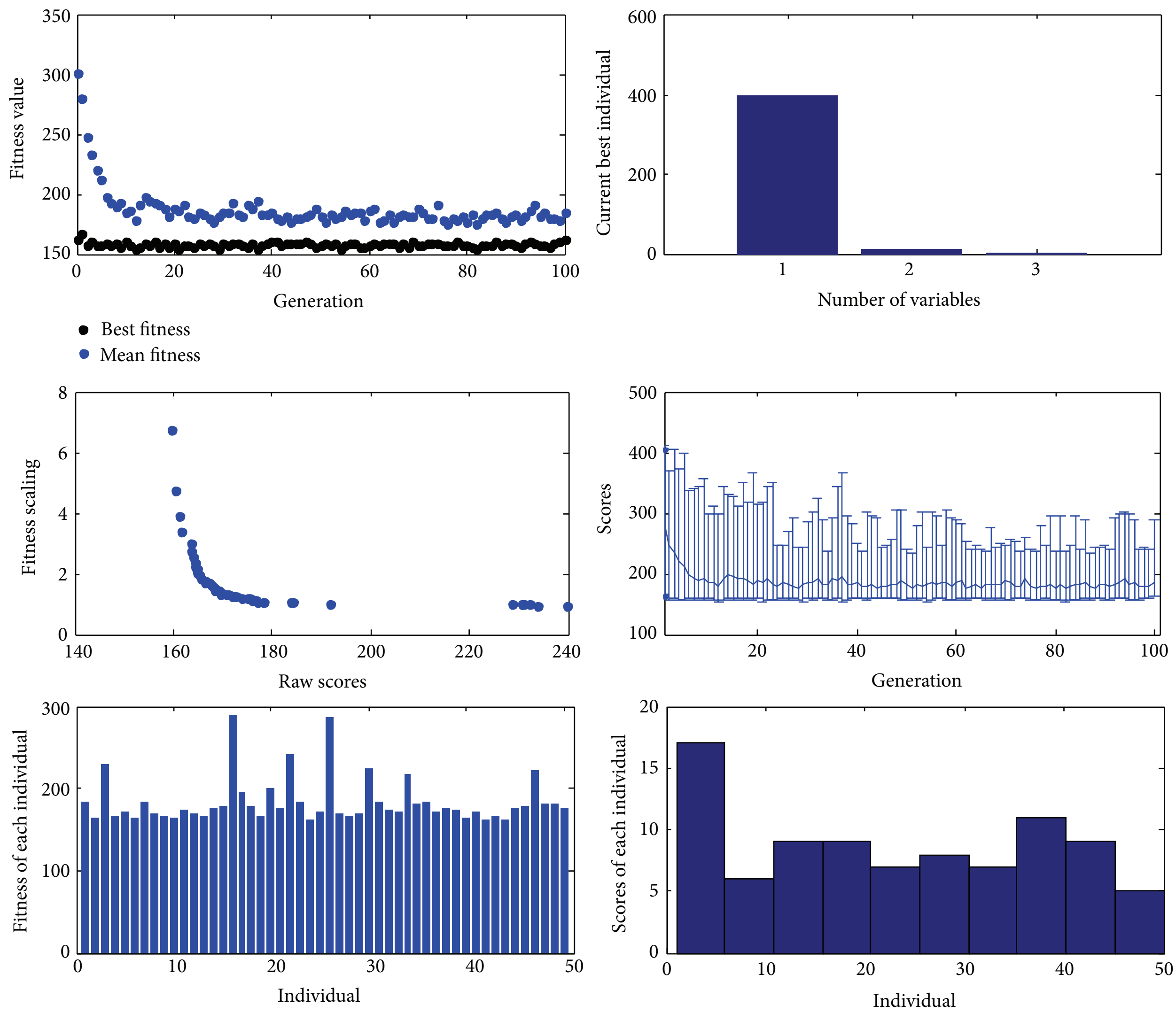


Figure 5: The optimization results by genetic algorithm.

5. Probabilistic Value-Centric Optimization for Fractionated Spacecraft Architecture

The value-centric assessment methodology can customize several criterions, such as the mean and variance of the cost, the benefit income, and the net present value NPV. Therefore, the optimization is essentially solving a multiobjective problem in Pareto's sense [13, 22].

For this multidisciplinary optimization problem, there exist analytic gradient functions to guide the refinement of iteration procedure of numerical optimization. Thus, the heuristic method, such as the genetic algorithm, is quite good at refining the optimal design scheme. Compared with the traditional algorithm, the graphic processing unit (GPU) accelerated genetic algorithm based on the Compute Unified Device Architecture (CUDA) is employed to improve the searching efficiency because the cost and value assessments in any generation are parallel computerized.

In this paper, the fractionated spacecrafts with five-years lifecycle are chosen to optimize their architecture. The objective function for this case is to maximize the net present value NPV with the optimized variables of orbital altitude, local time at descending node (LTDN), and the number of modules. The inequality constraints for the three variables are listed in Table 6.

For the genetic initialization, the population number within each generation is set as 100. The evolution after the fifty generations can refine the maximum net present value NPV, shown in Figure 5. In the optimal design scheme, the orbital altitude is 573.7150 km, the local time at descending node is 11:23 AM, and the number of modules employed by the fractionated architecture is six. Ideally, if the design models are accurate enough, the maximum NPV could be reached through the engineering practice.

According to the numerical simulations, the optimal design scheme of the fractionated architecture depends on

Table 6: Inequality constraints for the three variables.

Optimal conditions	Lower bound	Upper bound
Orbit altitude (km)	400	700
LTDN (hour)	9	13
Number of modules (—)	1	6

the assessment criterions. For the high-budget missions requiring more reliability or robustness, the variance of cost, income, or net present value is preferable to working as the objective function; however, the mean is selected by some low-budget missions which pay more attentions on the cost-performance ratio.

6. Conclusion

Considering the probabilistic uncertainties during the mission lifecycle, the cost assessment or architecture optimization is essentially a stochastic problem. One element necessary in enabling a probabilistic, value-centric analysis of such fractionated architecture is a systematic method for sizing and costing the many candidate architectures that arise. One of the contributions in this paper is to quantitatively assess the impacts of various fractionated spacecraft architecture strategies on the lifecycle cost, mass, propellant usage, and mission lifetime of pointing-intensive, remote-sensing mission spacecraft.

Based on the probabilistic value-centric assessments developed in this paper, the genetic algorithm is applied to optimize the fractionated spacecraft architecture from the viewpoint of probability. To accelerate the optimization, the second contribution is to employ the graphic processing unit (GPU) accelerated genetic algorithm based on the Compute Unified Device Architecture (CUDA) to improve the searching efficiency because the assessments in any generation are parallel computerized. Furthermore, another is to introduce the unscented transformation to reduce the amount of computations rather than the Monte Carlo method in stochastic simulation.

Finally, for future work, the models can ultimately be developed on a level of confidence such that the results of the surplus value model can be analyzed in detail, in particular to establish the design objective functions for components. This can then provide a platform for optimization to generate the best design scheme for different aerospace missions.

Acknowledgments

The research is supported by the National Natural Science Foundation of China (11172020), the National High Technology Research and Development Program of China (863 Program: 2012AA120601), Talent Foundation supported by the Fundamental Research Funds for the Central Universities, Aerospace Science and Technology Innovation Foundation of China Aerospace Science Corporation, and Innovation Fund of China Academy of Space Technology. The authors have no conflict of interests with the mentioned trademark(s).

References

[1] O. C. Brown, P. Eremenko, and P. D. Collopy, “Value-centric design methodologies for fractionated spacecraft: progress summary from phase 1 of the DARPA system F6 program,” in *Proceedings of the AIAA Space 2009 Conference and Exposition*, AIAA 2009-6540, Pasadena, Calif, USA, September 2009.

[2] M. G. O’Neill, H. Yue, S. Nag, P. Grogan, and O. L. deWeck, “Comparing and optimizing the DARPA system F6 program value-centric design methodologies,” in *Proceedings of the AIAA Space 2010 Conference and Exposition*, AIAA 2010-8828, Anaheim, Calif, USA, 2010.

[3] M. G. Richards, Z. Szajnfarber, M. G. O’Neill, and A. L. Weigel, “Implementation challenges for responsive space architectures,” in *Proceedings of the AIAA Responsive Space Conference*, AIAA-RS7-2009-2004, Los Angeles, Calif, USA, April 2009.

[4] O. Brown and P. Eremenko, “Application of value-centric design to space architectures: the case of fractionated spacecraft,” in *Proceedings of the AIAA Space 2008 Conference and Exposition*, AIAA 2009-6540, San Diego, Calif, USA, September 2008.

[5] O. Brown and P. Eremenko, “The value proposition for fractionated space architectures,” in *Proceedings of the Space 2006 Conference*, AIAA 2006-7506, pp. 2788–2809, Reston, Va, USA, September 2006.

[6] C. Mathieu and A. L. Weigel, “Assessing the flexibility provided by fractionated spacecraft,” in *Proceedings of the Space 2005 Conference*, AIAA 2005-6700, Long Beach, Calif, USA, August-September 2005.

[7] M. G. O’Neill, *Assessing the impacts of fractionation on pointing-intensive spacecraft [M.S. thesis]*, Institute of Technology, 2010.

[8] D. A. O’Neil and J. C. Mankins, “The advanced technology lifecycle analysis system (ATLAS),” in *Proceedings of the 55th International Astronautical Congress*, IAC-04-IAA.3.6.3.01, Vancouver, Canada, October 2004.

[9] O. Brown, A. Long, N. Shah, and P. Eremenko, “System lifecycle cost under uncertainty as a design metric encompassing the value of architectural flexibility,” in *Proceedings of the AIAA Space 2007 Conference*, AIAA 2007-6023, pp. 216–229, Long Beach, Calif, USA, September 2007.

[10] J. M. Lafleur and J. H. Saleh, “GT-FAST: a point design tool for rapid fractionated spacecraft sizing and synthesis,” in *Proceedings of the AIAA Space 2009 Conference and Exposition*, AIAA 2009-6563, Pasadena, Calif, USA, September 2009.

[11] J. M. Lafleur and J. H. Saleh, “Exploring the F6 fractionated spacecraft trade space with GT-FAST,” in *Proceedings of the AIAA Space 2009 Conference and Exposition*, AIAA 2009-6802, Pasadena, Calif, USA, September 2009.

[12] W. Yao, X. Q. Chen, and Y. Zhao, “Based on the uncertainty of MDO satellite overall optimization design,” *Chinese Journal of Aeronautics*, vol. 30, no. 5, pp. 68–76, 2009.

[13] W. Yao, X. Q. Chen, W. C. Luo, M. von Tooren, and J. Guo, “Review of uncertainty-based multidisciplinary design optimization methods for aerospace vehicles,” *Progress in Aerospace Sciences*, vol. 47, pp. 450–479, 2011.

[14] M. Daniels, J. Irvine, B. Tracey, W. Schram, and M. E. Paté-Cornell, “Probabilistic simulation of multi-stage decisions for operation of a fractionated satellite mission,” in *Proceedings of the IEEE Aerospace Conference (AERO ’11)*, pp. 1–16, March 2011.

[15] P. Collopy, “Value of the probability of success,” in *Proceedings of the AIAA Space 2008 Conference and Exposition*, AIAA 2008-7868, San Diego, Calif, USA, September 2008.

[16] S. J. Julier and J. K. Uhlmann, “Unscented filtering and nonlinear estimation,” *Proceedings of the IEEE*, vol. 92, no. 3, pp. 401–422, 2004.

[17] R. van der Merwe, E. A. Wan, and S. I. Julier, “Sigma-point kalman filters for nonlinear estimation and sensor-fusion: applications to integrated navigation,” in *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, AIAA 2004-5120, pp. 1735–1764, Providence, RI, USA, August 2004.

[18] P. D. Collopy, “Economic-based distributed optimal design,” in *Proceedings of the AIAA Space 2001 Conference and Exposition*, AIAA 2001-4675, Albuquerque, NM, USA, 2001.

[19] NASA, *NASA Systems Engineering Handbook*, SP-610S, NASA, 1995.

[20] E. M. Mahr, *Small Satellite Cost Model 2007 (SSCM07) User’s Manual*, ATR-2007(8617)-5, The Aerospace Corporation, 2007.

[21] A. Braukhane, “Lessons learned from one-week concurrent engineering study approach,” in *Proceedings of the 17th International Conference on Concurrent Enterprising (ICE ’01)*, pp. 1–10, Bremen, Germany, 2011.

[22] R. Hassan and W. Crossley, “Spacecraft reliability-based design optimization under uncertainty including discrete variables,” *Journal of Spacecraft and Rockets*, vol. 45, no. 2, pp. 394–405, 2008.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 418678, 14 pages
http://dx.doi.org/10.1155/2013/418678

Research Article

Optimal Fusion Filtering in Multisensor Stochastic Systems with Missing Measurements and Correlated Noises

R. Caballero-Águila,[1] I. García-Garrido,[2] and J. Linares-Pérez[2]

[1] *Departamento de Estadística, Universidad de Jaén, Paraje Las Lagunillas, 23071 Jaén, Spain*
[2] *Departamento de Estadística, Universidad de Granada, Avenida Fuentenueva, 18071 Granada, Spain*

Correspondence should be addressed to R. Caballero-Águila; raguila@ujaen.es

Received 30 January 2013; Accepted 28 April 2013

Academic Editor: Weihai Zhang

The optimal least-squares linear estimation problem is addressed for a class of discrete-time multisensor linear stochastic systems with missing measurements and autocorrelated and cross-correlated noises. The stochastic uncertainties in the measurements coming from each sensor (missing measurements) are described by scalar random variables with arbitrary discrete probability distribution over the interval $[0, 1]$; hence, at each single sensor the information might be partially missed and the different sensors may have different missing probabilities. The noise correlation assumptions considered are (i) the process noise and all the sensor noises are one-step autocorrelated; (ii) different sensor noises are one-step cross-correlated; and (iii) the process noise and each sensor noise are two-step cross-correlated. Under these assumptions and by an innovation approach, recursive algorithms for the optimal linear filter are derived by using the two basic estimation fusion structures; more specifically, both centralized and distributed fusion estimation algorithms are proposed. The accuracy of these estimators is measured by their error covariance matrices, which allow us to compare their performance in a numerical simulation example that illustrates the feasibility of the proposed filtering algorithms and shows a comparison with other existing filters.

1. Introduction

For a long time, the least-squares (LS) estimation problem in linear stochastic systems from measurements perturbed by additive noises has received considerable attention in the scientific community due to its wide applicability in many practical situations (e.g., video and laser tracking systems, satellite navigation, radar and meteorological applications, etc. [1]). As it is well known, one of the major contributions made to solve this problem is the Kalman filter, which provides a recursive algorithm for the optimal LS estimator when the additive white noises and the initial state are Gaussian and mutually independent (or, equivalently, uncorrelated due to the Gaussianity assumption) and, therefore, the optimal LS estimator is the optimal LS linear estimator. From the publication of the Kalman filter [2] in 1960, numerous results and several solution methods have been reported in the literature to address the state estimation problem from noisy observations, which depend on models representing possible relationships between the unknown state and the observable variables and also on the noise processes assumptions.

Specifically, during the past decades, there has been an increasing interest in the filtering problem in multisensor systems, where sensor networks are used to obtain the whole available information on the system state and its estimation must be carried out from the observations provided by all the sensors. A basic matter for this class of systems is how to fuse the measurement data from the different sensors to address the estimation problem. Commonly, two methods are used to process the measured data coming from multiple sensors: centralized and distributed fusion methods. In the centralized fusion method all the measured data from sensors are communicated to the fusion center for being processed; nevertheless, as is widely known, centralized estimators have many computational disadvantages, which motivate the research into other fusion methods. In the distributed fusion method, each sensor estimates the state based on its own single measurement data, and then it sends such

estimate to the fusion center for fusion according to a certain information fusion criterion. Although the use of sensor networks offers several advantages, the unreliable network characteristics usually cause problems during data transmission from sensors to the fusion center, such as missing measurements, random communication packet losses and/or delays. Taking into account these network uncertainties, the models representing the relationships between the state and measurements do not allow to apply the Kalman filter, and modifications of conventional estimation algorithms have been proposed (see e.g., [3–9] and references therein).

As in the Kalman filter, independent white noises are considered in all the mentioned papers; however, this assumption may not be realistic and can be a limitation in many real-world problems in which noise correlation may be present. This problem arises, for example, when a target is taking an electronic countermeasure, for example, noise jamming [10], or if the process noise and the sensor measurement noises are dependent on the system state, then there may be cross-correlation between different sensor noises and cross-correlation between process noise and sensor noises. Also, if all the sensors are observed in the same noisy environment, the measurement noises of different sensors are usually correlated.

For these reasons, the estimation problem in systems with correlated noises has received significant research interest in recent years. For example, the optimal Kalman filtering fusion problem in systems with cross-correlated sensor noises is addressed in [10], while [11, 12] study the same problem in systems with cross-correlated process noises and measurement noises; in these papers correlated noises at the same sampling time are considered. In general, the assumption of correlation and cross-correlation of the noise process and measurement noises in different sampling times makes difficult the identification of optimal estimators; this limitation has encouraged a wider research into suboptimal Kalman-type estimation problems. In [13], a Kalman-type recursive filter is presented for systems with finite-step correlated process noises, and the filtering problem with multistep correlated process and measurement noises is investigated in [14]. The optimal robust nonfragile Kalman-type recursive filtering problem is studied in [15] for a class of uncertain systems with finite-step autocorrelated measurement noises and multiple packet dropouts. The problem of distributed weighted robust Kalman filter fusion is studied in [16] for a class of uncertain systems with autocorrelated and cross-correlated noises. In [17], a stochastic singular system with correlated noises at the same sampling time is transformed into an equivalent nonsingular system with correlated noises at the same and neighboring sampling times. Also, in [18], an augmented parameterized system with correlated noises at the same and neighboring sampling times is used to describe the sensor delay, packet dropout, and uncertain observation phenomenons.

On the other hand, as noted above, the use of communication networks for transmitting measured data motivates the need of considering stochastic uncertainties. Missing measurements have been widely treated due to its applicability to model a large class of real-world problems, such as fading phenomena in propagation channels, target tracking or, in general, situations where there exist intermittent failures in the observation mechanism, accidental loss of some measurements, or inaccessibility of the data during certain times. The state estimation problem from missing measurement transmitted by multiple sensors has been studied based on the assumption that all the sensors are identical (see, e.g., [19–22]); however, this assumption can be unreasonable since some real systems usually involve multiple sensors with different characteristics. Recently, the filtering problem using missing measurements whose statistical properties are assumed not to be the same in all the sensors has been addressed by several authors under different approaches and hypotheses on the processes involved (see, e.g., [23–27]). In all the above papers, Bernoulli random variables are used to model the missing measurements phenomenon, and hence, it is assumed that the measurement signal is either completely lost (if the corresponding Bernoulli variable takes the value zero) or successfully transferred (when the Bernoulli variable is equal to one). Recently, this missing measurement model has been generalized considering any discrete distribution on the interval $[0, 1]$, which allows to cover some practical applications where only partial information is missing (see [28, 29] and references therein).

Motivated by the above considerations, our attention is focused on investigating the optimal LS linear centralized and distributed fusion estimation problems in multisensor systems with missing measurements and autocorrelated and cross-correlated noises. In each sensor, the missing measurement phenomenon is governed by a scalar random variable with arbitrary discrete probability distribution over the interval $[0, 1]$, and the different sensors may have different missing probabilities. Assume that the process noise and all the sensor noises are one-step autocorrelated; different sensor noises are one-step cross-correlated; and the process noise and each sensor noise are two-step cross-correlated. This paper makes a twofold substantial novel contribution: (1) unlike most previous results with correlated noises, in which suboptimal Kalman-type estimators are proposed, in this paper optimal LS linear estimators are obtained by using an innovation approach, which provides a simple derivation of the estimation algorithms due to the fact that the innovations constitute a white process; and (2) our missing measurement model considers at each sensor the possibility of observations containing only partial information about the state, or even only noise.

The paper is organized as follows. In Section 2 the system model with autocorrelated and cross-correlated noises and missing measurements coming from multiple sensors is described. Also, the suitable properties on the state and noise processes are specified and a brief description of the innovation approach to the optimal LS linear estimation problem is included. In Section 3 a recursive algorithm for the centralized optimal linear filter is presented for the considered model (the derivation has been deferred to Appendix 6). Next, in Section 4, the local LS linear filters and their corresponding error covariance matrices between any two local estimates are provided, and then

the distributed optimal weighted fusion estimators and their error covariance matrices are obtained by applying the optimal information fusion criterion weighted by matrices in the linear minimum variance sense. Finally, in Section 5, a numerical simulation example is presented to show the effectiveness of the estimation algorithms proposed in the current paper, and some conclusions are drawn in Section 6.

Notation. The notation used throughout the paper is standard. For any matrix A, the notation symbols A^T and A^{-1} represent its transpose and inverse, respectively; $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^{m\times n}$ is the set of all real matrices of dimension $m \times n$. The shorthand $\mathrm{Diag}(a_1, \ldots, a_r)$ denotes a diagonal matrix whose diagonal entries are $a_1, \ldots, a_r$. If the dimensions of matrices are not explicitly stated, they are assumed to be compatible for algebraic operations. δ_{k-s} is the Kronecker delta function, which is equal to one, if $k = s$, and zero otherwise. Moreover, for arbitrary random vectors α and β, we will denote $\mathrm{Cov}[\alpha, \beta] = E[(\alpha - E[\alpha])(\beta - E[\beta])^T]$ and $\mathrm{Cov}[\alpha] = \mathrm{Cov}[\alpha, \alpha]$, where $E[\cdot]$ stands for the mathematical expectation operator. Finally, $\widehat{\alpha}$ denotes the estimator of α and $\widetilde{\alpha} = \alpha - \widehat{\alpha}$ the estimation error.

2. Problem Formulation

Our aim is to obtain recursive algorithms for the optimal LS linear filtering problem in a class of discrete-time stochastic systems with missing measurements coming from multiple sensors, by using centralized and distributed fusion methods. In this section, firstly the system model and the assumptions about the state and noise processes are presented and, secondly, the optimal LS linear estimation problem is formulated using an innovation approach.

2.1. Stochastic System Model. Consider a discrete-time linear stochastic system with autocorrelated and cross-correlated noises and missing measurements coming from r sensors. The phenomenon of missing measurements occurs randomly and, for each sensor, a different sequence of scalar random variables with discrete distribution over the interval $[0, 1]$ is used to model this phenomenon. Specifically, the following system is considered:

$$x_k = F_{k-1}x_{k-1} + w_{k-1}, \quad k \geq 1, \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $\{w_k;\ k \geq 0\}$ is the process noise, and F_k, for $k \geq 0$, are known matrices with compatible dimensions.

Consider r sensors which, at any time k, provide scalar measurements of the system state, perturbed by additive and multiplicative noises according to the following model:

$$y_k^i = \theta_k^i H_k^i x_k + v_k^i, \quad k \geq 1, \quad i = 1, 2, \ldots, r, \tag{2}$$

where $\{y_k^i;\ k \geq 1\}$ are the measured data; $\{v_k^i;\ k \geq 1\}$ are measurement noises; $\{\theta_k^i;\ k \geq 1\}$ are scalar random variables sequences; H_k^i, for $k \geq 1$, are known time-varying matrices with compatible dimensions; superscript i denotes the ith sensor, and r is the number of sensors.

Next, the statistical properties assumed about the initial state and noise processes involved in (1) and (2) are specified.

(i) The initial state x_0 is a random vector with $E[x_0] = \overline{x}_0$ and $\mathrm{Cov}[x_0] = P_0$.

(ii) The process noise, $\{w_k;\ k \geq 0\}$, and the measurement noises, $\{v_k^i;\ k \geq 1\}$, $i = 1, 2, \ldots, r$, are zero-mean sequences with covariances and cross-covariances:

$$\begin{aligned}
\mathrm{Cov}\left[w_k, w_s\right] &= Q_{k,k}\delta_{k-s} + Q_{k,s}\delta_{k-s+1} + Q_{k,s}\delta_{k-s-1}, \\
\mathrm{Cov}\left[v_k^i, v_s^j\right] &= R_{k,k}^{ij}\delta_{k-s} + R_{k,s}^{ij}\delta_{k-s+1} + R_{k,s}^{ij}\delta_{k-s-1}, \\
\mathrm{Cov}\left[w_k, v_s^i\right] &= S_{k,k}^{i}\delta_{k-s} + S_{k,s}^{i}\delta_{k-s+1} + S_{k,s}^{i}\delta_{k-s+2}.
\end{aligned} \tag{3}$$

(iii) The multiplicative noises $\{\theta_k^i;\ k \geq 1\}$, $i = 1, 2, \ldots, r$, are white sequences of scalar variables with discrete distribution over the interval $[0, 1]$, with $E[\theta_k^i] = \overline{\theta}_k^i$ and $\mathrm{Var}[\theta_k^i] = V_k^{\theta^i}$.

(iv) The initial state x_0 and the multiplicative noises $\{\theta_k^i;\ k \geq 1\}$, for $i = 1, 2, \ldots, r$, are mutually independent, and they are independent of the additive noises $\{w_k;\ k \geq 0\}$ and $\{v_k^i;\ k \geq 1\}$, for $i = 1, 2, \ldots, r$.

Remark 1. From assumption (ii) the following correlation properties of the additive noises are easily deduced.

(1) The noise vectors w_k and w_s are correlated at consecutive sampling times, $|k - s| = 1$, and independent otherwise; the covariance matrices of w_k with w_{k-1}, and w_{k+1} are $Q_{k,k-1}$, and $Q_{k,k+1}$, respectively.

(2) For $i, j = 1, 2, \ldots, r$, the measurement noises v_k^i and v_s^j are cross-correlated at the same sampling time and at consecutive sampling times, $|k - s| = 0, 1$, and independent otherwise; the cross-covariances of v_k^i with v_k^j, v_{k-1}^j and v_{k+1}^j are $R_{k,k}^{ij}$, $R_{k,k-1}^{ij}$ and $R_{k,k+1}^{ij}$, respectively.

(3) For $i = 1, 2, \ldots, r$, the measurement noises v_k^i are correlated with the noise vectors w_s, for $s = k, k-1, k-2$, and independent otherwise; the cross-covariance matrices of v_k^i with w_k, w_{k-1} and w_{k-2} are $S_{k,k}^i$, $S_{k-1,k}^i$ and $S_{k-2,k}^i$, respectively.

The correlation conditions of the process noise and the measurement noises considered in this paper are the same as those in [16]. Systems with only finite-step correlated process noises or multistep correlated process and measurement noises are considered in [13–15], among others. The current study can be extended to more general systems involving finite-step autocorrelated and cross-correlated noises with no difficulty, except for a greater complexity in the mathematical derivations.

Remark 2. From the state equation (1) and assumptions (ii) and (iv), it is easy to deduce that $D_k = E[x_k x_k^T]$ is recursively calculated by

$$D_k = F_{k-1} D_{k-1} F_{k-1}^T + Q_{k-1,k-1} + F_{k-1} Q_{k-2,k-1} + Q_{k-1,k-2} F_{k-1}^T, \quad k \geq 2, \tag{4}$$
$$D_1 = F_0 D_0 F_0^T + Q_{0,0}, \qquad D_0 = P_0 + \overline{x}_0 \overline{x}_0^T.$$

Also, it is easy to see that the state x_k is correlated with the measurement noises v_k^i, for $i = 1, 2, \ldots, r$, and the expectations $E_k^i = E[x_k v_k^i]$ satisfy

$$E_k^i = F_{k-1} S_{k-2,k}^i + S_{k-1,k}^i, \quad k \geq 2; \quad E_1^i = S_{0,1}^i. \tag{5}$$

Remark 3. According to assumption (iii), the scalar random variables θ_k^i take values over the interval $[0, 1]$ and they can satisfy any arbitrary discrete probability distribution over such interval, for instance, a Bernoulli distribution. Usually, Bernoulli random variables have been used to model the phenomenon of missing measurements (see, e.g., [25] and references therein), with $\theta_k^i = 1$ meaning that the state x_k is present in the measurement y_k^i coming from the ith sensor at time k, while $\theta_k^i = 0$ means that the state is missing in the measured data at time k or, equivalently, that such observation only contains additive noise v_k^i. However, in practice, the information transmitted at a sampling time can usually be neither completely missing nor completely successful, but only part of the information can go through; in such situations, only partial information is missing and the proportion of missed data at one moment is a *fraction* other than 0 or 1 (see, e.g., [28, 29] and references therein).

2.2. Stacked Measurement Equation. As noted above, our aim is to solve the optimal LS linear estimation problem of the state x_k based on the measurements $\{y_1^i, y_2^i, \ldots, y_k^i\}$, for $i = 1, 2, \ldots, r$, by using centralized and distributed fusion methods to process the measured sensor data. The centralized fusion method considers that all the measurement data coming from r sensors are transmitted to a fusion center for being processed; for this purpose and to simplify the notation, the measurement equation (2) is rewritten in a stacked form as follows:

$$y_k = \Theta_k H_k x_k + v_k, \quad k \geq 1, \tag{6}$$

where $y_k = (y_k^1, \ldots, y_k^r)^T$, $v_k = (v_k^1, \ldots, v_k^r)^T$, $H_k = (H_k^{1T}, \ldots, H_k^{rT})^T$, and $\Theta_k = \mathrm{Diag}(\theta_k^1, \ldots, \theta_k^r)$.

The following properties of the noises in (6) are easily inferred from the model assumptions (ii)–(iv) previously stated.

(i) The additive noise $\{v_k;\ k \geq 1\}$ is a zero-mean process satisfying:

$$\mathrm{Cov}[v_k, v_s] = R_{k,k}\delta_{k-s} + R_{k,s}\delta_{k-s+1} + R_{k,s}\delta_{k-s-1},$$
$$\mathrm{Cov}[w_k, v_s] = S_{k,k}\delta_{k-s} + S_{k,s}\delta_{k-s+1} + S_{k,s}\delta_{k-s+2}, \tag{7}$$

where $R_{k,s} = (R_{k,s}^{ij})_{i,j=1,2,\ldots,r}$ and $S_{k,s} = (S_{k,s}^1, \ldots, S_{k,s}^r)$.

(ii) The state vector x_k and the measurement noise vector v_k are correlated with $E_k = E[x_k v_k^T]$ satisfying

$$E_k = F_{k-1} S_{k-2,k} + S_{k-1,k}, \quad k \geq 2, \quad E_1 = S_{0,1}. \tag{8}$$

(iii) The random matrices $\{\Theta_k;\ k \geq 1\}$ satisfy $E[\Theta_k] = \overline{\Theta}_k = \mathrm{Diag}(\overline{\theta}_k^1, \ldots, \overline{\theta}_k^r)$ and $E[(\Theta_k - \overline{\Theta}_k)^2] = \mathrm{Diag}(V_k^{\theta^1}, \ldots, V_k^{\theta^r})$; also, denoting $\theta_k = (\theta_k^1, \ldots, \theta_k^r)^T$, it is clear that $\mathrm{Cov}[\theta_k] = \mathrm{Diag}(V_k^{\theta^1}, \ldots, V_k^{\theta^r})$. Moreover, for any random matrix G independent of $\{\Theta_k;\ k \geq 1\}$, it is easily deduced that

$$E\left[\left(\Theta_k - \overline{\Theta}_k\right) G \left(\Theta_k - \overline{\Theta}_k\right)\right] = \mathrm{Cov}[\theta_k] \circ E[G], \tag{9}$$

where $\circ$ denotes the Hadamard product [23].

(iv) The initial state x_0 and $\{\Theta_k;\ k \geq 1\}$ are independent, and they are independent of $\{w_k;\ k \geq 0\}$ and $\{v_k;\ k \geq 1\}$.

2.3. Innovation Approach to the Optimal LS Linear Estimation Problem. To address the optimal LS linear estimation problem of the state x_k based on the measurements $\{y_1^i, y_2^i, \ldots, y_k^i\}$, $i = 1, 2, \ldots, r$, the centralized and distributed fusion methods will be used. In both cases, recursive algorithms for the LS linear estimators will be established using an innovation approach and the orthogonal projection Lemma (OPL); more specifically we have the following.

Centralized Fusion Estimation Problem. Our aim is to obtain the optimal LS linear filter, $\widehat{x}_{k/k}$, of the state x_k based on the measurements $\{y_1, y_2, \ldots, y_k\}$, given in (6), by recursive algorithms.

As known, the LS linear filter $\widehat{x}_{k/k}$ is the orthogonal projection of the state x_k over the linear space spanned by $\{y_1, y_2, \ldots, y_k\}$. These observations are generally nonorthogonal vectors, but the Gram-Schmidt orthogonalization procedure allows us to substitute them by a set of orthogonal vectors, called *innovations*, defined as the difference between each observation and its one-stage predictor. Due to the orthogonality property of the innovations and since the innovation process is uniquely determined by the observations, the LS linear filter, $\widehat{x}_{k/k}$, can be calculated as linear combination of the innovations; namely,

$$\widehat{x}_{k/k} = \sum_{s=1}^{k} \mathcal{X}_{k,s} \Pi_{s,s}^{-1} \mu_s, \quad k \geq 1, \tag{10}$$

where $\mu_s = y_s - \widehat{y}_{s/s-1}$ are the innovation vectors, with $\widehat{y}_{s/s-1}$ the one-stage observation predictor, $\Pi_{s,s} = E[\mu_s \mu_s^T]$, and $\mathcal{X}_{k,s} = E[x_k \mu_s^T]$.

Distributed Fusion Estimation Problem. To address the distributed fusion estimation problem, firstly, recursive algorithms to obtain local LS linear filters, $\widehat{x}_{k/k}^i$, for $i = 1, 2, \ldots, r$, and the error cross-covariance matrices between any two local estimates, are derived. Secondly, the distributed fusion filter, $\widehat{x}_{k/k}^D$, is established by applying the optimal information

fusion criterion weighted by matrices in the linear minimum variance sense [30].

Analogously to (10), denoting $\mu_s^i = y_s^i - \widehat{y}_{s/s-1}^i$, $\Pi_{s,s}^{ii} = E[\mu_s^i \mu_s^i]$, and $\mathcal{X}_{k,s}^i = E[x_k \mu_s^i]$, the local filter $\widehat{x}_{k/k}^i$ is expressed as

$$\widehat{x}_{k/k}^i = \sum_{s=1}^{k} \mathcal{X}_{k,s}^i \left(\Pi_{s,s}^{ii}\right)^{-1} \mu_s^i, \quad k \geq 1. \tag{11}$$

3. Optimal LS Linear Centralized Fusion Estimation

In this section a recursive algorithm for the centralized optimal (under the LS criterion) linear filter, $\widehat{x}_{k/k}$ is derived. Such algorithm is deduced using (10) and the OPL, and it is presented in Theorem 5. Firstly, in order to simplify the proof of Theorem 5, the following lemma is established.

Lemma 4. *Under assumptions (i)–(iv), the following results hold:*

$$\mathcal{W}_{k,k} = E\left[w_k \mu_k^T\right] = Q_{k,k-1} H_k^T \overline{\Theta}_k + S_{k,k}, \quad k \geq 1, \tag{12}$$

$$\mathcal{V}_{k,k-1} = E\left[v_k \mu_{k-1}^T\right] = S_{k-2,k}^T H_{k-1}^T \overline{\Theta}_{k-1} + R_{k,k-1}, \quad k \geq 2. \tag{13}$$

Proof. Since w_k is independent of $y_1, \ldots, y_{k-1}$, $E[w_k \widehat{y}_{k/k-1}^T] = 0$ and hence $\mathcal{W}_{k,k} = E[w_k y_k^T]$. Now, using (1) and (6), $\mathcal{W}_{k,k}$ can be calculated as follows:

$$\begin{aligned} \mathcal{W}_{k,k} &= E\left[w_k \left(\Theta_k H_k x_k + v_k\right)^T\right] \\ &= E\left[w_k x_k^T\right] H_k^T \overline{\Theta}_k + S_{k,k} \\ &= E\left[w_k \left(F_{k-1} x_{k-1} + w_{k-1}\right)^T\right] H_k^T \overline{\Theta}_k + S_{k,k} \\ &= Q_{k,k-1} H_k^T \overline{\Theta}_k + S_{k,k}. \end{aligned} \tag{14}$$

Taking into account that v_k is independent of $y_1, \ldots, y_{k-2}$, the calculation of $\mathcal{V}_{k,k-1}$ is similar to that of $\mathcal{W}_{k,k}$, and hence the proof is omitted. □

Theorem 5. *For the system model* (1) *and measurement model* (6), *under assumptions (i)–(iv), the optimal LS linear filter $\widehat{x}_{k/k}$ is obtained as*

$$\widehat{x}_{k/k} = \widehat{x}_{k/k-1} + \mathcal{X}_{k,k} \Pi_{k,k}^{-1} \mu_k, \quad k \geq 1, \quad \widehat{x}_{0/0} = \overline{x}_0, \tag{15}$$

where the state predictor, $\widehat{x}_{k/k-1}$, satisfies

$$\begin{aligned} \widehat{x}_{k/k-1} &= F_{k-1} \widehat{x}_{k-1/k-1} + \mathcal{W}_{k-1,k-1} \Pi_{k-1,k-1}^{-1} \mu_{k-1}, \quad k \geq 2, \\ \widehat{x}_{1/0} &= F_0 \widehat{x}_{0/0}. \end{aligned} \tag{16}$$

The innovation, μ_k, is given by

$$\begin{aligned} \mu_k &= y_k - \overline{\Theta}_k H_k \widehat{x}_{k/k-1} - \mathcal{V}_{k,k-1} \Pi_{k-1,k-1}^{-1} \mu_{k-1}, \quad k \geq 2, \\ \mu_1 &= y_1 - \overline{\Theta}_1 H_1 \widehat{x}_{1/0}. \end{aligned} \tag{17}$$

The matrix $\mathcal{X}_{k,k} = E[x_k \mu_k^T]$ is calculated by

$$\begin{aligned} \mathcal{X}_{k,k} &= P_{k/k-1} H_k^T \overline{\Theta}_k + E_k - \mathcal{X}_{k,k-1} \Pi_{k-1,k-1}^{-1} \mathcal{V}_{k,k-1}^T, \quad k \geq 2, \\ \mathcal{X}_{1,1} &= P_{1/0} H_1^T \overline{\Theta}_1 + E_1, \end{aligned} \tag{18}$$

where $\mathcal{X}_{k,k-1} = E[x_k \mu_{k-1}^T]$ satisfies

$$\mathcal{X}_{k,k-1} = F_{k-1} \mathcal{X}_{k-1,k-1} + \mathcal{W}_{k-1,k-1}, \quad k \geq 2. \tag{19}$$

The prediction error covariance matrix, $P_{k/k-1}$, is obtained by

$$\begin{aligned} P_{k/k-1} &= F_{k-1} P_{k-1/k-1} F_{k-1}^T + Q_{k-1,k-1} + F_{k-1} \mathcal{J}_{k-1} + \mathcal{J}_{k-1}^T F_{k-1}^T \\ &\quad - \mathcal{W}_{k-1,k-1} \Pi_{k-1,k-1}^{-1} \mathcal{W}_{k-1,k-1}^T, \quad k \geq 2, \\ P_{1/0} &= F_0 P_{0/0} F_0^T + Q_{0,0}, \end{aligned} \tag{20}$$

where $\mathcal{J}_k = E[\widetilde{x}_{k/k} w_k^T]$ is calculated by

$$\mathcal{J}_k = Q_{k-1,k} - \mathcal{X}_{k,k} \Pi_{k,k}^{-1} \mathcal{W}_{k,k}^T, \quad k \geq 1. \tag{21}$$

The filtering error covariance matrix, $P_{k/k}$, is given by

$$P_{k/k} = P_{k/k-1} - \mathcal{X}_{k,k} \Pi_{k,k}^{-1} \mathcal{X}_{k,k}^T, \quad k \geq 1, \quad P_{0/0} = P_0. \tag{22}$$

The innovation covariance matrix, $\Pi_{k,k}$, satisfies

$$\begin{aligned} \Pi_{k,k} = {} & \operatorname{Cov}\left(\theta_k\right) \circ \left(H_k D_k H_k^T\right) + R_{k,k} + \overline{\Theta}_k H_k \mathcal{X}_{k,k} \\ & + \mathcal{X}_{k,k}^T H_k^T \overline{\Theta}_k - \overline{\Theta}_k H_k P_{k/k-1} H_k^T \overline{\Theta}_k \\ & - \mathcal{V}_{k,k-1} \Pi_{k-1,k-1}^{-1} \mathcal{V}_{k,k-1}^T, \quad k \geq 2, \\ \Pi_{1,1} = {} & \operatorname{Cov}\left(\theta_1\right) \circ \left(H_1 D_1 H_1^T\right) + R_{1,1} + \overline{\Theta}_1 H_1 \mathcal{X}_{1,1} \\ & + \mathcal{X}_{1,1}^T H_1^T \overline{\Theta}_1 - \overline{\Theta}_1 H_1 P_{1/0} H_1^T \overline{\Theta}_1. \end{aligned} \tag{23}$$

The matrices D_k, E_k, $\mathcal{W}_{k,k}$, and $\mathcal{V}_{k,k-1}$ are given in (4), (8), (12), *and* (13), *respectively.*

Proof. See Appendix 6. □

Remark 6. In conventional estimation problems in systems with missing measurements and uncorrelated additive white noises, the one-stage state and observation predictors are calculated as $\widehat{x}_{k/k-1} = F_{k-1} \widehat{x}_{k-1/k-1}$ and $\widehat{y}_{k/k-1} = \overline{\Theta}_k H_k \widehat{x}_{k/k-1}$, respectively. However, this is not true for the problem at hand since, due to the correlation assumption (ii), the noise estimators $\widehat{w}_{k-1/k-1}$ and $\widehat{v}_{k/k-1}$ must be taken into account for the derivation of the predictors. Besides the fact of considering missing measurements, this is the main difference between the optimal estimators proposed in the current paper and the suboptimal Kalman-type ones proposed in [16], where the noise estimators are considered to be equal to zero.

4. Distributed Fusion Estimation

One of the main disadvantages of the centralized fusion estimators derived in Section 3 is that they may have a high computational cost due to augmentation. Moreover, as is widely known, the centralized approach has several other drawbacks, such as fault detection, isolation, poor reliability, and so forth. To overcome these disadvantages, our aim in this section is to address the optimal distributed fusion estimation problem, in which each single sensor provides its local LS linear estimator and their estimation error covariance matrices, and then these local estimators along with the covariances and cross-covariance matrices of the estimation errors between any two sensors are sent to the fusion center for fusion based on the matrices-weighted fusion estimation criterion in the linear minimum variance sense [30].

4.1. Local LS Linear Filtering Algorithms. For each single sensor subsystem of systems (1) and (2), the following theorem provides recursive formulas for the local LS linear filters, $\widehat{x}^i_{k/k}$, and their corresponding error covariance matrices, $P^{ii}_{k/k}$.

Theorem 7. *For the ith sensor subsystem of systems* (1) *and* (2) *under assumptions (i)–(iv), the local LS linear filter,* $\widehat{x}^i_{k/k}$, *is calculated by*

$$\widehat{x}^i_{k/k} = \widehat{x}^i_{k/k-1} + \mathcal{X}^i_{k,k}\left(\Pi^{ii}_{k,k}\right)^{-1}\mu^i_k, \quad k \ge 1, \quad \widehat{x}^i_{0/0} = \overline{x}_0, \tag{24}$$

where the local LS linear predictor, $\widehat{x}^i_{k/k-1}$, *satisfies*

$$\begin{gathered}\widehat{x}^i_{k/k-1} = F_{k-1}\widehat{x}^i_{k-1/k-1} + \mathcal{W}^i_{k-1,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\mu^i_{k-1}, \quad k \ge 2,\\ \widehat{x}^i_{1/0} = F_0\widehat{x}^i_{0/0},\end{gathered} \tag{25}$$

with $\mathcal{W}^i_{k,k} = \overline{\theta}^i_k Q_{k,k-1}H^{iT}_k + S^i_{k,k}, k \ge 1$.

The innovation, μ^i_k, *is given by*

$$\begin{gathered}\mu^i_k = y^i_k - \overline{\theta}^i_k H^i_k\widehat{x}^i_{k/k-1} - \mathcal{V}^{ii}_{k,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\mu^i_{k-1}, \quad k \ge 2,\\ \mu^i_1 = y^i_1 - \overline{\theta}^i_1 H^i_1\widehat{x}^i_{1/0},\end{gathered} \tag{26}$$

with $\mathcal{V}^{ii}_{k,k-1} = \overline{\theta}^i_{k-1}S^{iT}_{k-2,k}H^{iT}_{k-1} + R^{ii}_{k,k-1}, k \ge 2$.

The vector $\mathcal{X}^i_{k,k} = E[x_k\mu^i_k]$ *is calculated from the following expression*

$$\begin{gathered}\mathcal{X}^i_{k,k} = \overline{\theta}^i_k P^{ii}_{k/k-1}H^{iT}_k + E^i_k - \mathcal{X}^i_{k,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\mathcal{V}^{ii}_{k,k-1},\\ k \ge 2,\\ \mathcal{X}^i_{1,1} = \overline{\theta}^i_1 P^{ii}_{1/0}H^{iT}_1 + E^i_1,\end{gathered} \tag{27}$$

where $\mathcal{X}^i_{k,k-1} = F_{k-1}\mathcal{X}^i_{k-1,k-1} + \mathcal{W}^i_{k-1,k-1}, k \ge 2$.

The local prediction error covariance matrix, $P^{ii}_{k/k-1}$, *is obtained by*

$$\begin{gathered}P^{ii}_{k/k-1} = F_{k-1}P^{ii}_{k-1/k-1}F^T_{k-1} + Q_{k-1,k-1} + F_{k-1}\mathcal{J}^i_{k-1} + \mathcal{J}^{iT}_{k-1}F^T_{k-1}\\ - \mathcal{W}^i_{k-1,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\mathcal{W}^{iT}_{k-1,k-1}, \quad k \ge 2,\\ P^{ii}_{1/0} = F_0P^{ii}_{0/0}F^T_0 + Q_{0,0},\end{gathered} \tag{28}$$

where $\mathcal{J}^i_k = Q_{k-1,k} - \mathcal{X}^i_{k,k}(\Pi^{ii}_{k,k})^{-1}\mathcal{W}^{iT}_{k,k}, k \ge 1$, *and* $P^{ii}_{k/k}$, *the filtering error covariance matrix, is given by*

$$P^{ii}_{k/k} = P^{ii}_{k/k-1} - \mathcal{X}^i_{k,k}\left(\Pi^{ii}_{k,k}\right)^{-1}\mathcal{X}^{iT}_{k,k}, \quad k \ge 1, \quad P^{ii}_{0/0} = P_0. \tag{29}$$

The innovation variance, $\Pi^{ii}_{k,k}$, *satisfies*

$$\begin{gathered}\Pi^{ii}_{k,k} = V^{\theta^i}_k H^i_k D_k H^{iT}_k + R^{ii}_{k,k} + \overline{\theta}^i_k H^i_k\mathcal{X}^i_{k,k} + \overline{\theta}^i_k\mathcal{X}^{iT}_{k,k}H^{iT}_k\\ - \left(\overline{\theta}^i_k\right)^2 H^i_k P^{ii}_{k/k-1}H^{iT}_k - \left(\mathcal{V}^{ii}_{k,k-1}\right)^2\left(\Pi^{ii}_{k-1,k-1}\right)^{-1},\\ k \ge 2,\\ \Pi^{ii}_{1,1} = V^{\theta^i}_1 H^i_1 D_1 H^{iT}_1 + R^{ii}_{1,1} + \overline{\theta}^i_1 H^i_1\mathcal{X}^i_{1,1}\\ + \overline{\theta}^i_1\mathcal{X}^{iT}_{1,1}H^{iT}_1 - \left(\overline{\theta}^i_1\right)^2 H^i_1 P^{ii}_{1/0}H^{iT}_1.\end{gathered} \tag{30}$$

The matrix D_k *and the vector* E^i_k *are given in* (4) *and* (5), *respectively.*

Proof. The proof, based on the innovation approach and the OPL, is omitted for being analogous to that of Theorem 5. Nevertheless, it should be indicated that, in this proof, the Hadamard product is not used since, instead of the diagonal stochastic matrix Θ_k, the scalar variable θ^i_k is now involved in the derivation of the estimators. □

Remark 8. As indicated in Remark 6 for the centralized estimators, it must be noted that, due to the correlation assumption (ii) of the additive noises $\{w_k\}$ and $\{v^i_k\}$, the estimators $\widehat{w}^i_{k-1/k-1} = \mathcal{W}^i_{k-1,k-1}(\Pi^{ii}_{k-1,k-1})^{-1}\mu^i_{k-1}$ and $\widehat{v}^i_{k/k-1} = \mathcal{V}^{ii}_{k,k-1}(\Pi^{ii}_{k-1,k-1})^{-1}\mu^i_{k-1}$ are not equal to zero, and hence the optimal local state predictor, $\widehat{x}^i_{k/k-1} = F_{k-1}\widehat{x}^i_{k-1/k-1} + \widehat{w}^i_{k-1/k-1}$, and the observation predictor, $\widehat{y}^i_{k/k-1} = \overline{\theta}^i_k H^i_k\widehat{x}^i_{k/k-1} + \widehat{v}^i_{k/k-1}$, are quite different from conventional filtering algorithms with uncorrelated white noises. This issue, along with the consideration of missing measurements at each single sensor, constitutes the main difference between the current optimal local estimators and the suboptimal local estimators proposed in [16].

4.2. Cross-Covariance Matrices of Local Estimation Errors. To apply the optimal fusion criterion weighted by matrices in

the linear minimum variance sense, the filtering, $P^{ij}_{k/k}$, and prediction, $P^{ij}_{k/k-1}$, error cross-covariance matrices between local estimators of any two subsystems must be calculated.

For simplicity, besides the notation of Theorem 7, for $i \neq j$, $i, j = 1, 2 \ldots, r$, we introduce the following notation:

$$L^{ij}_k = E\left[\widehat{x}^i_{k/k-1}\mu^j_k\right], \qquad \Pi^{ij}_{k,s} = E\left[\mu^i_k\mu^j_s\right], \qquad \mathscr{V}^{ij}_{k,k-1} = E\left[v^i_k\mu^j_{k-1}\right]. \tag{31}$$

Also, in order to simplify the calculation of the error cross-covariance matrices, the following lemmas are given.

Lemma 9. *Under assumptions (i)–(iv), the following results hold.*

(a) *The expectation $E[\widehat{x}^i_{k/k-1}\mu^j_{k-1}]$ satisfies*

$$E\left[\widehat{x}^i_{k/k-1}\mu^j_{k-1}\right] = F_{k-1}L^{ij}_{k-1} + \mathscr{X}^i_{k,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k-1}, \quad k \geq 2. \tag{32}$$

(b) *The expectation $E[\widehat{x}^i_{k/k-1}v^j_k]$ satisfies*

$$E\left[\widehat{x}^i_{k/k-1}v^j_k\right] = \mathscr{X}^i_{k,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\mathscr{V}^{ji}_{k,k-1}, \quad k \geq 2, \tag{33}$$

where $\mathscr{V}^{ji}_{k,k-1} = \overline{\theta}^i_{k-1}H^i_{k-1}S^j_{k-2,k} + R^{ij}_{k-1,k}$.

(c) *The expectation $E[v^i_k\mu^j_k]$ satisfies*

$$E\left[v^i_k\mu^j_k\right] = \overline{\theta}^j_kE^{iT}_kH^{jT}_k + R^{ij}_{k,k} - \mathscr{V}^{ij}_{k,k-1}\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\left(\overline{\theta}^j_kH^j_k\mathscr{X}^j_{k,k-1} + \mathscr{V}^{jj}_{k,k-1}\right)^T, \quad k \geq 2. \tag{34}$$

Proof. (a) From (25) for $\widehat{x}^i_{k/k-1}$ and (24) for $\widehat{x}^i_{k-1/k-1}$, we have

$$\begin{aligned}
E\left[\widehat{x}^i_{k/k-1}\mu^j_{k-1}\right] &= F_{k-1}E\left[\widehat{x}^i_{k-1/k-1}\mu^j_{k-1}\right] + \mathscr{W}^i_{k-1,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k-1} \\
&= F_{k-1}E\left[\widehat{x}^i_{k-1/k-2}\mu^j_{k-1}\right] + F_{k-1}\mathscr{X}^i_{k-1,k-1} \times \left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k-1} + \mathscr{W}^i_{k-1,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k-1} \\
&= F_{k-1}L^{ij}_{k-1} + \left(F_{k-1}\mathscr{X}^i_{k-1,k-1} + \mathscr{W}^i_{k-1,k-1}\right) \times \left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k-1},
\end{aligned} \tag{35}$$

and since $\mathscr{X}^i_{k,k-1} = F_{k-1}\mathscr{X}^i_{k-1,k-1} + \mathscr{W}^i_{k-1,k-1}$, expression (32) is proved.

(b) Analogously, taking into account that $E[\widehat{x}^i_{k-1/k-2}v^j_k] = 0$, we have

$$\begin{aligned}
E\left[\widehat{x}^i_{k/k-1}v^j_k\right] &= F_{k-1}E\left[\widehat{x}^i_{k-1/k-1}v^j_k\right] + \mathscr{W}^i_{k-1,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}E\left[\mu^i_{k-1}v^j_k\right] \\
&= \left(F_{k-1}\mathscr{X}^i_{k-1,k-1} + \mathscr{W}^i_{k-1,k-1}\right)\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\mathscr{V}^{ji}_{k,k-1},
\end{aligned} \tag{36}$$

and expression (33) is immediately obtained. Finally, the derivation of expression $\mathscr{V}^{ji}_{k,k-1} = \overline{\theta}^i_{k-1}H^i_{k-1}S^j_{k-2,k} + R^{ij}_{k-1,k}$ is similar to that of (13) and hence it is omitted.

(c) Taking into account expression (26) for μ^j_k, with (2) for y^j_k, we have

$$\begin{aligned}
E\left[v^i_k\mu^j_k\right] = \overline{\theta}^j_kE^{iT}_kH^{jT}_k + R^{ij}_{k,k} - \overline{\theta}^j_kE\left[v^i_k\widehat{x}^{jT}_{k/k-1}\right]H^{jT}_k - \mathscr{V}^{ij}_{k,k-1}\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\mathscr{V}^{jj}_{k,k-1}, \quad k \geq 2,
\end{aligned} \tag{37}$$

and using (33) for $E[v^i_k\widehat{x}^{jT}_{k/k-1}]$, expression (34) is obtained. □

Lemma 10. *Under assumptions (i)–(iv), for $i \neq j$, $i, j = 1, 2 \ldots, r$, the expectations $L^{ij}_k = E[\widehat{x}^i_{k/k-1}\mu^j_k]$ are recursively obtained by*

$$\begin{aligned}
L^{ij}_k = {} & \overline{\theta}^j_k\left(P^{jj}_{k/k-1} - P^{ij}_{k/k-1}\right)H^{jT}_k - F_{k-1}L^{ij}_{k-1}\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\mathscr{V}^{jj}_{k,k-1} \\
& + \mathscr{X}^i_{k,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1} \times \left(\mathscr{V}^{ji}_{k,k-1} - \mathscr{V}^{jj}_{k,k-1}\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k-1}\right), \quad k \geq 2,
\end{aligned} \tag{38}$$

with initial condition $L^{ij}_1 = 0$.

Proof. Taking into account expression (26) for μ^j_k, with (2) for y^j_k, we have

$$\begin{aligned}
L^{ij}_k = {} & \overline{\theta}^j_kE\left[\widehat{x}^i_{k/k-1}x^T_k\right]H^{jT}_k + E\left[\widehat{x}^i_{k/k-1}v^j_k\right] - \overline{\theta}^j_kE\left[\widehat{x}^i_{k/k-1}\widehat{x}^{jT}_{k/k-1}\right]H^{jT}_k \\
& - E\left[\widehat{x}^i_{k/k-1}\mu^j_{k-1}\right]\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\mathscr{V}^{jj}_{k,k-1}, \quad k \geq 2.
\end{aligned} \tag{39}$$

From the OPL, $E[\widehat{x}^i_{k/k-1}x^T_k] = E[\widehat{x}^i_{k/k-1}\widehat{x}^{iT}_{k/k-1}]$; then, taking into account (32) for $E[\widehat{x}^i_{k/k-1}\mu^j_{k-1}]$, and (33) for $E[\widehat{x}^i_{k/k-1}v^j_k]$, it is enough to prove that

$$E\left[\widehat{x}^i_{k/k-1}\widehat{x}^{iT}_{k/k-1}\right] - E\left[\widehat{x}^i_{k/k-1}\widehat{x}^{jT}_{k/k-1}\right] = P^{jj}_{k/k-1} - P^{ij}_{k/k-1}, \tag{40}$$

which is easily deduced since

$$E\left[\widehat{x}^{i}_{k/k-1}\widehat{x}^{jT}_{k/k-1}\right] = P^{ij}_{k/k-1} - D_k + E\left[\widehat{x}^{i}_{k/k-1}\widehat{x}^{iT}_{k/k-1}\right] + E\left[\widehat{x}^{j}_{k/k-1}\widehat{x}^{jT}_{k/k-1}\right],$$
$$E\left[\widehat{x}^{j}_{k/k-1}\widehat{x}^{jT}_{k/k-1}\right] = D_k - P^{jj}_{k/k-1}. \tag{41}$$

□

Lemma 11. *Under assumptions (i)–(iv), for $i \neq j$, $i, j = 1, 2 \ldots, r$, the innovation cross-covariance $\Pi^{ij}_{k,k} = E[\mu^i_k \mu^j_k]$ satisfies*

$$\Pi^{ij}_{k,k} = \overline{\theta}^i_k H^i_k \left(\mathcal{X}^j_{k,k} - L^{ij}_k\right) + \overline{\theta}^j_k E^{iT}_k H^{jT}_k + R^{ij}_{k,k} - \mathcal{V}^{ii}_{k,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k} - \mathcal{V}^{ij}_{k,k-1}\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\left(\overline{\theta}^j_k H^j_k \mathcal{X}^j_{k,k-1} + \mathcal{V}^{jj}_{k,k-1}\right)^T, \quad k \geq 2,$$
$$\Pi^{ij}_{1,1} = \overline{\theta}^i_1 H^i_1 \mathcal{X}^j_{1,1} + \overline{\theta}^j_1 E^i_1 H^{jT}_1 + R^{ij}_{1,1}, \tag{42}$$

where $\Pi^{ij}_{k-1,k} = E[\mu^i_{k-1}\mu^j_k]$ is given by

$$\Pi^{ij}_{k-1,k} = \overline{\theta}^j_k \left(\mathcal{X}^i_{k,k-1} - F_{k-1}L^{ji}_{k-1} - \mathcal{X}^j_{k,k-1}\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\Pi^{ji}_{k-1,k-1}\right)^T H^{jT}_k + \mathcal{V}^{ji}_{k,k-1} - \Pi^{ij}_{k-1,k-1}\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\mathcal{V}^{jj}_{k,k-1}, \quad k \geq 2. \tag{43}$$

Proof. Taking into account expression (26) for μ^i_k, with (2) for y^i_k, we have

$$\Pi^{ij}_{k,k} = \overline{\theta}^i_k H^i_k E\left[x_k \mu^j_k\right] + E\left[v^i_k \mu^j_k\right] - \overline{\theta}^i_k H^i_k E\left[\widehat{x}^i_{k/k-1}\mu^j_k\right] - \mathcal{V}^{ii}_{k,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1} E\left[\mu^i_{k-1}\mu^j_k\right]$$
$$= \overline{\theta}^i_k H^i_k \left(\mathcal{X}^j_{k,k} - L^{ij}_k\right) + E\left[v^i_k \mu^j_k\right] - \mathcal{V}^{ii}_{k,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k}, \quad k \geq 2, \tag{44}$$

and, from (34) for $E[v^i_k \mu^j_k]$, expression for $\Pi^{ij}_{k,k}$ is clear.

Analogously, taking into account expression (26) for μ^j_k, with (2) for y^j_k, we have

$$\Pi^{ij}_{k-1,k} = \overline{\theta}^j_k E\left[\mu^i_{k-1} x^T_k\right] H^{jT}_k + E\left[\mu^i_{k-1} v^j_k\right] - \overline{\theta}^j_k E\left[\mu^i_{k-1}\widehat{x}^{jT}_{k/k-1}\right] H^{jT}_k - E\left[\mu^i_{k-1}\mu^j_{k-1}\right]\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\mathcal{V}^{jj}_{k,k-1},$$
$$= \overline{\theta}^j_k \left(\mathcal{X}^i_{k,k-1} - E\left[\mu^i_{k-1}\widehat{x}^{jT}_{k/k-1}\right]\right) H^{jT}_k + \mathcal{V}^{ji}_{k,k-1} - \Pi^{ij}_{k-1,k-1}\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\mathcal{V}^{jj}_{k,k-1}, \quad k \geq 2, \tag{45}$$

and, from (32) for $E[\mu^i_{k-1}\widehat{x}^{jT}_{k/k-1}]$, expression for $\Pi^{ij}_{k-1,k}$ is immediately derived. □

In the following theorem, recursive formulas to calculate the filtering and prediction error cross-covariance matrices, $P^{ij}_{k/k}$ and $P^{ij}_{k/k-1}$, respectively, are derived.

Theorem 12. *Under assumptions (i)–(iv), the cross-covariance matrices, $P^{ij}_{k/k}$, of the filtering errors between the ith and the jth sensor subsystems are recursively computed by*

$$P^{ij}_{k/k} = P^{ij}_{k/k-1} + \mathcal{X}^i_{k,k}\left(\Pi^{ii}_{k,k}\right)^{-1}\Pi^{ij}_{k,k}\left(\Pi^{jj}_{k,k}\right)^{-1}\mathcal{X}^{jT}_{k,k} - \left(\mathcal{X}^j_{k,k} - L^{ij}_k\right)\left(\Pi^{jj}_{k,k}\right)^{-1}\mathcal{X}^{jT}_{k,k} - \mathcal{X}^i_{k,k}\left(\Pi^{ii}_{k,k}\right)^{-1}\left(\mathcal{X}^i_{k,k} - L^{ji}_k\right)^T, \quad k \geq 1,$$
$$P^{ij}_{0/0} = P_0, \tag{46}$$

where $P^{ij}_{k/k-1}$, the cross-covariance matrix of the prediction error between the ith and the jth sensor subsystems, satisfies

$$P^{ij}_{k/k-1} = F_{k-1}P^{ij}_{k-1/k-1}F^T_{k-1} + Q_{k-1,k-1} + F_{k-1}\mathcal{J}^i_{k-1} + \mathcal{J}^{jT}_{k-1}F^T_{k-1} + \mathcal{W}^i_{k-1,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k-1} \times \left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\mathcal{W}^{jT}_{k-1,k-1} - \mathcal{G}^{ij}_{k-1}\left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\mathcal{W}^{jT}_{k-1,k-1} - \mathcal{W}^i_{k-1,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\mathcal{G}^{jiT}_{k-1}, \quad k \geq 2,$$
$$P^{ij}_{1/0} = F_0 P^{ij}_{0/0} F^T_0 + Q_{0,0}, \tag{47}$$

where $\mathcal{G}^{ij}_k = \mathcal{W}^j_{k,k} + F_k(\mathcal{X}^j_{k,k} - L^{ij}_k - \mathcal{X}^i_{k,k}(\Pi^{ii}_{k,k})^{-1}\Pi^{ij}_{k,k})$, $k \geq 1$. The vectors L^{ij}_k and the innovation cross-covariances $\Pi^{ij}_{k,k}$ are given in Lemmas 10 and 11, respectively.

Proof. By using (24) for $\widehat{x}^i_{k/k}$ and $\widehat{x}^j_{k/k}$, we have

$$P^{ij}_{k/k} = P^{ij}_{k/k-1} + \mathcal{X}^i_{k,k}\left(\Pi^{ii}_{k,k}\right)^{-1}\Pi^{ij}_{k,k}\left(\Pi^{jj}_{k,k}\right)^{-1}\mathcal{X}^{jT}_{k,k} - E\left[\left(x_k - \widehat{x}^i_{k/k-1}\right)\mu^{jT}_k\right]\left(\Pi^{jj}_{k,k}\right)^{-1}\mathcal{X}^{jT}_{k,k} - \mathcal{X}^i_{k,k}\left(\Pi^{ii}_{k,k}\right)^{-1}E\left[\mu^i_k\left(x_k - \widehat{x}^j_{k/k-1}\right)^T\right]. \tag{48}$$

Taking into account that $E[x_k\mu^j_k] = \mathcal{X}^j_{k,k}$ and $E[\widehat{x}^i_{k/k-1}\mu^j_k] = L^{ij}_k$, the recursive expression for the cross-covariance matrices of the local filtering errors is immediately deduced.

Following an analogous reasoning, using now (25) and taking into account that $E[(x_k - \widehat{x}^i_{k/k})w^T_k] = \mathcal{J}^i_k$ and $E[\mu^i_k w^T_k] = \mathcal{W}^{iT}_{k,k}$, it is easy to see that

$$\begin{aligned} P^{ij}_{k/k-1} = {} & F_{k-1}P^{ij}_{k-1/k-1}F^T_{k-1} + Q_{k-1,k-1} + F_{k-1}\mathcal{J}^i_{k-1} \\ & + \mathcal{J}^{jT}_{k-1}F^T_{k-1} + \mathcal{W}^i_{k-1,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k-1} \\ & \times \left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\mathcal{W}^{jT}_{k-1,k-1} \\ & - \left(\mathcal{W}^j_{k-1,k-1} + F_{k-1}E\left[\widetilde{x}^i_{k-1/k-1}\mu^j_{k-1}\right]\right) \\ & \times \left(\Pi^{jj}_{k-1,k-1}\right)^{-1}\mathcal{W}^{jT}_{k-1,k-1} \\ & - \mathcal{W}^i_{k-1,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1} \\ & \times \left(\mathcal{W}^i_{k-1,k-1} + F_{k-1}E[\widetilde{x}^j_{k/k}\mu^i_{k-1}]\right)^T. \end{aligned} \tag{49}$$

Finally, using again (24) for $\widehat{x}^i_{k-1/k-1}$, and since $E[\widehat{x}^i_{k-1/k-2}\mu^j_{k-1}] = L^{ij}_{k-1}$, we have

$$\begin{aligned} & E\left[\widetilde{x}^i_{k-1/k-1}\mu^j_{k-1}\right] \\ & \quad = \mathcal{X}^j_{k-1,k-1} - L^{ij}_{k-1} - \mathcal{X}^i_{k-1,k-1}\left(\Pi^{ii}_{k-1,k-1}\right)^{-1}\Pi^{ij}_{k-1,k-1} \end{aligned} \tag{50}$$

and the expression for the cross-covariance matrices of the local prediction errors is easily obtained. □

4.3. Distributed Fusion Filtering Estimators. Once the local LS linear filtering estimators $\widehat{x}^i_{k/k}$ and their error covariance matrices $P^{ii}_{k/k}$, given in Theorem 7, along with the error cross-covariance matrices, $P^{ij}_{k/k}$, given in Theorem 12, are available, the distributed optimal weighted fusion estimators and their error covariance matrices are obtained by applying the optimal information fusion criterion weighted by matrices in the linear minimum variance sense [30].

Theorem 13. *For the system model* (1) *and measurement model* (2)*, under assumptions (i)–(iv), the distributed optimal fusion filter, $\widehat{x}^D_{k/k}$, is given by*

$$\widehat{x}^D_{k/k} = A^1_k\widehat{x}^1_{k/k} + \cdots + A^r_k\widehat{x}^r_{k/k}, \quad k \geq 0, \tag{51}$$

where the local estimators $\widehat{x}^i_{k/k}, k \geq 0$ $(i = 1, 2 \ldots, r)$ are calculated by the recursive algorithm established in Theorem 7.

The optimal matrix weights A^i_k $(i = 1, 2, \ldots, r)$ are computed by

$$A_k = \Sigma^{-1}_{k/k}e\left(e^T\Sigma^{-1}_{k/k}e\right)^{-1}, \tag{52}$$

where the matrices $A_k = [A^1_k, \ldots, A^r_k]^T$ and $e = [I, \ldots, I]^T$ are both $nr \times n$ matrices, and

$$\begin{aligned} \Sigma_{k/k} & = E\left[\left(\widetilde{x}^1_{k/k}, \ldots, \widetilde{x}^r_{k/k}\right)\left(\widetilde{x}^1_{k/k}, \ldots, \widetilde{x}^r_{k/k}\right)^T\right] \\ & = \left(P^{ij}_{k/k}\right)_{i,j=1,2,\ldots,r} \end{aligned} \tag{53}$$

is an $nr \times nr$ positive definite symmetric block matrix, whose $n \times n$ matrix entries $P^{ij}_{k/k}$ are given in Theorems 7 and 12.

The error covariance matrices of the distributed weighted fusion filtering estimators are computed by

$$P^D_{k/k} = \left(e^T\Sigma^{-1}_{k/k}e\right)^{-1}, \quad k \geq 0, \tag{54}$$

and the following inequality holds: $P^D_{k/k} \leq P^{ii}_{k/k}, i = 1, 2, \ldots, r$.

Proof. The proof is omitted because it follows directly from the optimal information criterion weighted by matrices in the linear minimum variance sense [30]. □

Remark 14. The proposed distributed optimal LS linear fusion filter requires the computation of an $nr \times nr$ inverse matrix, with n the dimension of the system state and r the number of sensors. Consequently, the proposed distributed fusion method has a computational complexity of $O[(nr)^3]$, equal to that of the distributed Kalman-type filter in [16] and less than that of the distributed fusion filters based on the state augmentation approach. Hence, our distributed fusion method is superior to the filter proposed in [16] (since it has the same computation burden but better accuracy) and also to the distributed fusion filters based on state augmentation (since it has less computational complexity).

5. Numerical Simulation Example

In this section, a numerical simulation example is presented to illustrate the effectiveness of the centralized and distributed filtering algorithms proposed in this paper. Consider a scalar first-order autoregressive model with missing measurements coming from two sensors with autocorrelated and cross-correlated noises. According to the proposed observation model, two different independent sequences of random variables with a certain probability distribution over the interval $[0, 1]$ are used to model the missing phenomenon. Specifically, the following model is considered as follows:

$$\begin{aligned} x_k & = 0.95x_{k-1} + w_{k-1}, \quad k \geq 1 \\ y^i_k & = \theta^i_k x_k + v^i_k, \quad k \geq 1, \quad i = 1, 2, \end{aligned} \tag{55}$$

where the initial state x_0 is a zero-mean Gaussian variable with variance $P_0 = 1$. The noise processes $\{w_k;\ k \geq 0\}$ and $\{v^i_k;\ k \geq 1\}$, $i = 1, 2$, are defined by

$$\begin{aligned} w_k & = 0.6\left(\eta_{k+1} + \eta_{k+2}\right), \\ v^i_k & = c_i\left(\eta_k + \eta_{k+1}\right), \quad i = 1, 2, \end{aligned} \tag{56}$$

where the sequence of variables $\{\eta_k;\ k \geq 1\}$ is a zero-mean Gaussian white process with variance 0.5. Clearly, according

to assumption (ii), the additive noises $\{w_k\}$ and $\{v_k^i\}$ are one-step autocorrelated and two-step cross-correlated with

$$\begin{gathered} Q_{k,k} = 0.36, \qquad Q_{k,k+1} = 0.18, \\ S_{k,k}^i = 0.3c_i, \qquad S_{k-1,k}^i = 0.6c_i, \qquad S_{k-2,k}^i = 0.3c_i, \\ R_{k,k}^{ii} = c_i^2, \qquad R_{k,k+1}^{ii} = 0.5c_i^2, \\ R_{k,k}^{ij} = c_i c_j, \qquad R_{k,k+1}^{ij} = 0.5c_i c_j. \end{gathered} \tag{57}$$

The phenomenon of missing measurements for each sensor is described as follows.

(1) In the first sensor, a sequence of independent and identically distributed (i.i.d.) random variables, $\{\theta_k^1;\ k \geq 1\}$, is considered, with probability distribution given by

$$\begin{gathered} P\left[\theta_k^1 = 0\right] = 0.1, \qquad P\left[\theta_k^1 = 0.5\right] = 0.5, \\ P\left[\theta_k^1 = 1\right] = 0.4. \end{gathered} \tag{58}$$

If $\theta_k^1 = 0$, which occurs with probability 0.1, the state x_k is missing and the observation y_k^1 contains only noise v_k^1; if $\theta_k^1 = 0.5$, only partial information of the state x_k is missing in such observation, which happens with probability 0.5; and, finally, the state is present in the observation y_k^1 with probability 0.4 when $\theta_k^1 = 1$. The mean and variance of these variables are easily calculated, being $\overline{\theta}_k^1 = 0.65$ and $V_k^{\theta^1} = 0.1025$, for all k.

(2) In the second sensor, a sequence of i.i.d. Bernoulli random variables, $\{\theta_k^2;\ k \geq 1\}$, is considered, with $P[\theta_k^2 = 1] = p$; in this case, if $\theta_k^2 = 1$ the state x_k is present in the measurement y_k^2 with probability p, whereas if $\theta_k^2 = 0$ such observation only contains additive noise, v_k^2, with probability $1-p$. So, no partial missing information is considered in this sensor. Clearly, for all k, $\overline{\theta}_k^2 = p$ and $V_k^{\theta^2} = p(1-p)$.

To illustrate the feasibility and effectiveness of the proposed estimators we ran a program in MATLAB, in which fifty iterations of the proposed algorithms have been performed considering different values of c_i and p. Using simulated values of the state and the corresponding observations, both distributed and centralized filtering estimates of the state are calculated, as well as the corresponding error variances, which provide a measure of the estimation accuracy.

Firstly, for $p = 0.8$, the local, centralized, and distributed filtering error variances are displayed in Figure 1 considering the values $c_1 = 1$ and $c_2 = 0.5$. According to Theorem 13, this figure corroborates that the optimal fusion distributed filter performs quite better than each local filter, but lightly worse than the centralized filter. Nevertheless, although the distributed fusion filter has a bit lower accuracy than the centralized one, both filters perform similarly and provide good estimations. Moreover, this slight difference is compensated because the distributed fusion structure is in general more robust, reduces the computational cost, and improves the reliability due to its parallel structure. For these reasons, the distributed filter is generally preferred in practice.

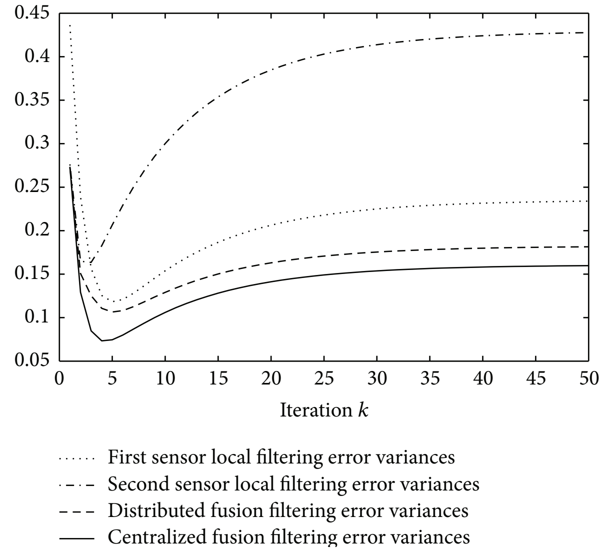


Figure 1: Local, centralized, and distributed fusion filtering error variances.

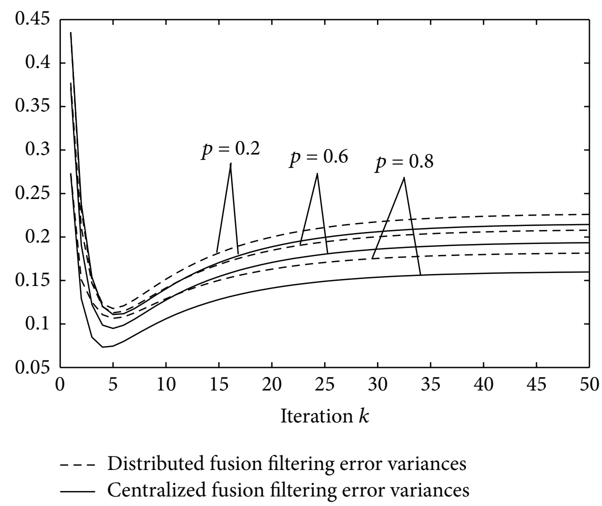


Figure 2: Centralized and distributed fusion filtering error variances for $p = 0.2, 0.6, 0.8$, when $c_1 = 1$, $c_2 = 0.5$.

Next, to analyze the performance of the proposed estimators versus the probability that the state x_k is present in the measurements of the second sensor, the centralized and distributed filtering error variances have been calculated for $c_1 = 1$, $c_2 = 0.5$ and different values of the probability $p = 0.2, 0.6$ and 0.8. The results are displayed in Figure 2; analysis of this figure reveals that as p increases (or, equivalently, the probability $1 - p$ that the state is missing in the observations from the second sensor decreases), the filtering error variances become smaller and, hence, better estimations are obtained. Also, this figure shows that, for all the considered probability values, the error variances corresponding to the centralized filter are always less than

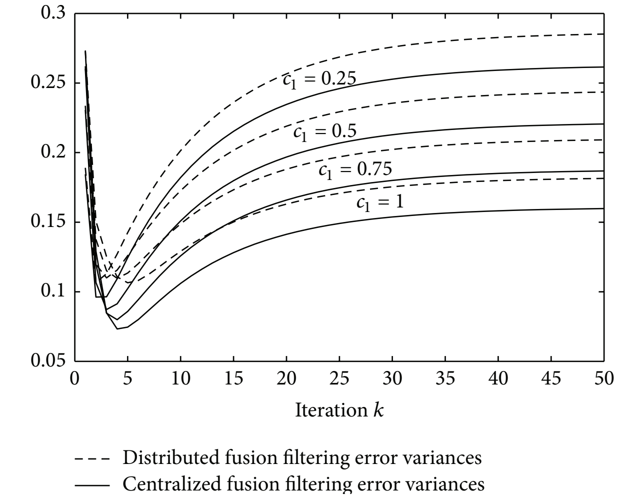


Figure 3: Centralized and distributed fusion filtering error variances for $c_1 = 0.25, 0.5, 0.75, 1$, when $c_2 = 0.5$ and $p = 0.8$.

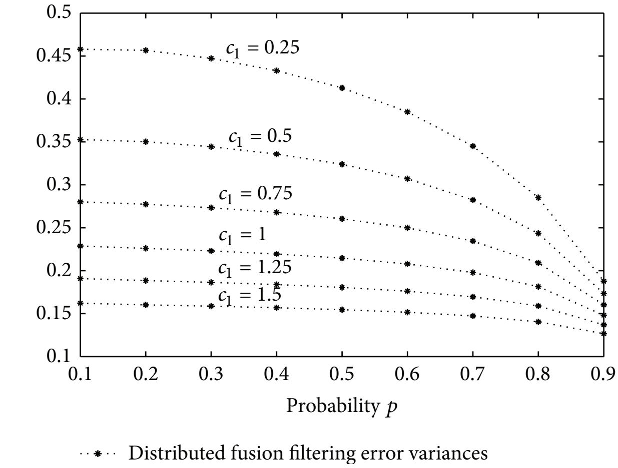


Figure 4: Distributed fusion filtering error variances versus p, with $c_1 = 0.25, 0.5, 0.75, 1, 1.25, 1.5$, when $c_2 = 0.5$.

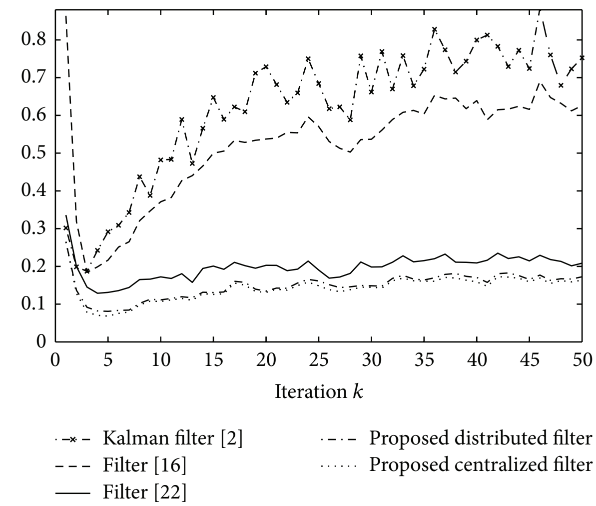


Figure 5: Comparison of MSE for different filters.

those of the distributed filter. Analogous results are obtained for other values of c_1, c_2 and the probability p.

On the other hand, to compare the performance of the estimators for different degrees of correlation between the state and the observation noises, the centralized and distributed filtering error variances have been calculated considering $c_2 = 0.5$, $p = 0.8$ and different values of c_1, specifically, $c_1 = 0.25, 0.5, 0.75$, and 1. These values provide different correlations between the noise process $\{w_k\}$ and the first sensor observation noise $\{v_k^1\}$ and, consequently, different correlations, E_k^1, between the state and the first sensor observation noise. The error variances are displayed in Figure 3, from which it is inferred that the error variances are smaller (and, consequently, the performance of the estimators is better) as the value c_1 is greater; these results were expected, since the correlation between the state and observations increases with c_1. Analogous results are obtained for different values of c_2 and other values of the probability p.

Now, completing the results of the two previous figures, the performance of the filters is analyzed when $c_2 = 0.5$, the probability p is varied from 0.1 to 0.9, and the values $c_1 = 0.25, 0.5, 0.75, 1, 1.25$, and 1.5 are considered. It must be noted that in all the cases examined, the error variances present insignificant variation from a certain iteration on and, consequently, only the values at a specific iteration (viz., $k = 50$) are shown. The results are presented in Figure 4 which, for the sake of clarity, only displays the distributed filtering error variances. Agreeing with the comments about Figures 2 and 3, this figure shows that, for a fixed value of c_1, the performance of the estimators improves as p becomes greater, and for a fixed value of p, also more accurate estimations are obtained as c_1 increases. Hence, from this figure it is gathered that, as c_1 or p decreases (which means that the correlation between the state and the first sensor observation noise decreases or the probability that the state is present in the second sensor measurements decreases, resp.), the filtering error variances become greater and, consequently, worse estimations are obtained.

Finally, a comparative analysis is presented between the classical Kalman filter [2], the Kalman-type filter with correlated and cross-correlated noises given in [16], the filter proposed in [23] for systems with different failure rates in multisensor networks, and the centralized and distributed filters proposed in this paper. For the comparison, the same parameter values as in Figure 1 are considered ($c_1 = 1$, $c_2 = 0.5$, and $p = 0.8$).

On the basis of one thousand independent simulations of the mentioned algorithms, a comparison between the different filtering estimates is performed using the mean square error (MSE) criteria. For $s = 1, \ldots, 1000$, let $\{x_k^{(s)}, k = 1, \ldots, 50\}$ denote the sth set of artificially simulated data (which is taken as the sth set of true values of the state), and $\widehat{x}_{k/k}^{(s)}$ the filtering estimate at the sampling time k in the sth simulation run. For each algorithm, the filtering MSE at time k is calculated by $\text{MSE}_k = (1/1000)\sum_{s=1}^{1000}(x_k^{(s)} - \widehat{x}_{k/k}^{(s)})^2$.

The values MSE_k, for $k = 1, \ldots, 50$, are displayed in Figure 5 which shows that, for all k, the proposed centralized

and distributed filters have approximately the same MSE_k values, which in turn are smaller than the MSE_k values of the filter in [23] and considerably less than those of the filters [2, 16]. Hence, we can conclude that, according to the MSE criterion, the proposed filtering estimates perform significantly better than other filters in the literature.

6. Conclusions

The LS linear estimation problem from missing measurements has been investigated for multisensor linear discrete-time systems with autocorrelated and cross-correlated noises. The main contributions are summarized as follows.

(1) Using both centralized and distributed fusion methods to process the measurement data from the different sensors, recursive optimal LS linear filtering algorithms are derived by an innovation approach.

(2) At each sensor, the possibility of missing measurements (i.e., observations containing only partial information about the state or even only noise) is modelled by a sequence of independent random variables taking discrete values over the interval $[0, 1]$.

(3) The multisensor system model considered in the current paper covers those situations where the sensor and process noises are one-step autocorrelated and two-step cross-correlated. Also, one-step cross-correlations between different sensor noises is considered. This correlation assumption is valid in a wide spectrum of applications, for example, in target tracking systems with process and measurement noises dependent on the system state, or situations where a target is observed by multiple sensors and all of them operate in the same noisy environment. Nevertheless, the current study can be extended to more general systems involving finite-step autocorrelated and cross-correlated noises with no difficulty, except for a greater complexity in the mathematical expressions.

(4) The applicability of the proposed centralized and distributed filtering algorithms is illustrated by a numerical simulation example, where a scalar state process generated by a first-order autoregressive model is estimated from missing measurements coming from two sensors with autocorrelated and cross-correlated noises. The results confirm that centralized and distributed fusion estimators have approximately the same error variances, with a slight inferiority of the distributed one which is compensated by a reduced computational burden and reduced communication demands for the sensor networks. Also, compared with some existing estimation methods, the proposed algorithms provide better estimations in the mean square error sense.

Appendix

Proof of Theorem 5

From (10), expression (15) for the state filter $\widehat{x}_{k/k}$ in terms of the one-stage predictor $\widehat{x}_{k/k-1}$ is immediately clear.

Expression (16) for the state predictor $\widehat{x}_{k/k-1}$ is obtained as follows:

$$\begin{aligned}
\widehat{x}_{k/k-1} &= \sum_{s=1}^{k-1} E\left[x_k \mu_s^T\right] \Pi_{s,s}^{-1} \mu_s \\
&= \sum_{s=1}^{k-1} E\left[\left(F_{k-1} x_{k-1} + w_{k-1}\right) \mu_s^T\right] \Pi_{s,s}^{-1} \mu_s \\
&= \sum_{s=1}^{k-1} F_{k-1} E\left[x_{k-1} \mu_s^T\right] \Pi_{s,s}^{-1} \mu_s \\
&\quad + E\left[w_{k-1} \mu_{k-1}^T\right] \Pi_{k-1,k-1}^{-1} \mu_{k-1} \\
&= F_{k-1} \widehat{x}_{k-1/k-1} + \mathscr{W}_{k-1,k-1} \Pi_{k-1,k-1}^{-1} \mu_{k-1}, \quad k \geq 2,
\end{aligned} \tag{A.1}$$

and clearly, $\widehat{x}_{1/0} = E[x_1] = F_0 E[x_0] = F_0 \widehat{x}_{0/0}$.

Now we show expression (17) for the innovation, $\mu_k = y_k - \widehat{y}_{k/k-1}$, for which it is enough to obtain an expression for $\widehat{y}_{k/k-1}$. A similar reasoning to that used to prove (16) leads to

$$\begin{aligned}
\widehat{y}_{k/k-1} &= \sum_{s=1}^{k-1} E\left[y_k \mu_s^T\right] \Pi_{s,s}^{-1} \mu_s \\
&= \sum_{s=1}^{k-1} E\left[\left(\Theta_k H_k x_k + v_k\right) \mu_s^T\right] \Pi_{s,s}^{-1} \mu_s \\
&= \overline{\Theta}_k H_k \sum_{s=1}^{k-1} E\left[x_k \mu_s^T\right] \Pi_{s,s}^{-1} \mu_s + E\left[v_k \mu_{k-1}^T\right] \Pi_{k-1,k-1}^{-1} \mu_{k-1} \\
&= \overline{\Theta}_k H_k \widehat{x}_{k/k-1} + \mathscr{V}_{k,k-1} \Pi_{k-1,k-1}^{-1} \mu_{k-1}, \quad k \geq 2,
\end{aligned} \tag{A.2}$$

with $\widehat{y}_{1/0} = E[y_1] = \overline{\Theta}_1 H_1 E[x_1] = \overline{\Theta}_1 H_1 \widehat{x}_{1/0}$. Hence,

$$\begin{aligned}
\widehat{y}_{k/k-1} &= \overline{\Theta}_k H_k \widehat{x}_{k/k-1} + \mathscr{V}_{k,k-1} \Pi_{k-1,k-1}^{-1} \mu_{k-1}, \quad k \geq 2, \\
\widehat{y}_{1/0} &= \overline{\Theta}_1 H_1 \widehat{x}_{1/0}
\end{aligned} \tag{A.3}$$

and expression (17) for the innovation is clear.

Next, expression (18) for the matrix $\mathscr{X}_{k,k} = E[x_k y_k^T] - E[x_k \widehat{y}_{k/k-1}^T]$ is derived. From (6) and the independence assumption, it is clear that

$$E\left[x_k y_k^T\right] = D_k H_k^T \overline{\Theta}_k + E_k, \quad k \geq 1. \tag{A.4}$$

From expression (A.3) for $\widehat{y}_{k/k-1}$ and the OPL, $E[x_k \widehat{y}_{k/k-1}^T]$ is calculated as follows:

$$\begin{aligned} E\left[x_k \widehat{y}_{k/k-1}^T\right] &= E\left[x_k \widehat{x}_{k/k-1}^T\right] H_k^T \overline{\Theta}_k + E\left[x_k \mu_{k-1}^T\right] \Pi_{k-1,k-1}^{-1} \mathscr{V}_{k,k-1}^T \\ &= E\left[\widehat{x}_{k/k-1} \widehat{x}_{k/k-1}^T\right] H_k^T \overline{\Theta}_k + \mathscr{X}_{k,k-1} \Pi_{k-1,k-1}^{-1} \mathscr{V}_{k,k-1}^T \\ &= \left(D_k - P_{k/k-1}\right) H_k^T \overline{\Theta}_k + \mathscr{X}_{k,k-1} \Pi_{k-1,k-1}^{-1} \mathscr{V}_{k,k-1}^T, \\ & \qquad k \geq 2, \\ E\left[x_1 \widehat{y}_{1/0}^T\right] &= \left(D_1 - P_{1/0}\right) H_1^T \overline{\Theta}_1. \end{aligned} \tag{A.5}$$

By substraction of the above expectations, expression (18) for $\mathscr{X}_{k,k} = E[x_k y_k^T] - E[x_k \widehat{y}_{k/k-1}^T]$ is obtained. From (1), expression (19) for $\mathscr{X}_{k,k-1} = E[x_k \mu_{k-1}^T]$ is immediately clear.

Expression (20) for the prediction error covariance matrix, $P_{k/k-1}$ is easily obtained by using (1) and (16); and, from (1) and (15), expression (21) for $\mathscr{J}_k = E[\widetilde{x}_{k/k} w_k^T] = E[x_k w_k^T] - E[\widehat{x}_{k/k} w_k^T]$ is also obvious. Expression (22) for the filtering error covariance matrix, $P_{k/k}$, is immediately derived by using (15).

Finally, we prove expression (23) for the innovation covariance matrix $\Pi_{k,k} = E[y_k y_k^T] - E[\widehat{y}_{k/k-1} \widehat{y}_{k/k-1}^T]$. From (6) and using (9), we have that

$$\begin{aligned} E\left[y_k y_k^T\right] &= E\left[\theta_k \theta_k^T\right] \circ \left(H_k D_k H_k^T\right) + R_{k,k} \\ &\quad + \overline{\Theta}_k H_k E_k + E_k^T H_k^T \overline{\Theta}_k. \end{aligned} \tag{A.6}$$

Using now (A.3) for $\widehat{y}_{k/k-1}$ and property (9), and taking into account that, from the OPL, $E[\widehat{x}_{k/k-1} \mu_{k-1}^T] = E[x_k \mu_{k-1}^T] = \mathscr{X}_{k,k-1}$, the following identity holds:

$$\begin{aligned} E\left[\widehat{y}_{k/k-1} \widehat{y}_{k/k-1}^T\right] &= \left(\overline{\theta}_k \overline{\theta}_k^T\right) \circ \left(H_k \left(D_k - P_{k/k-1}\right) H_k^T\right) \\ &\quad + \mathscr{V}_{k,k-1} \Pi_{k-1,k-1}^{-1} \mathscr{V}_{k,k-1}^T \\ &\quad + \overline{\Theta}_k H_k \mathscr{X}_{k,k-1} \Pi_{k-1,k-1}^{-1} \mathscr{V}_{k,k-1}^T \\ &\quad + \mathscr{V}_{k,k-1} \Pi_{k-1,k-1}^{-1} \mathscr{X}_{k,k-1}^T H_k^T \overline{\Theta}_k. \end{aligned} \tag{A.7}$$

From the above expectations and after some manipulations, expression (23) for the innovation covariance matrix $\Pi_{k,k}$ is obtained.

Acknowledgments

This research is supported by Ministerio de Ciencia e Innovación (Programa FPU and Grant no. MTM2011-24718) and Junta de Andalucía (Grant no. P07-FQM-02701).

References

[1] C. K. Chui and G. Chen, *Kalman Filtering with Real-Time Applications*, vol. 17 of *Springer Series in Information Sciences*, Springer, Berlin, Germany, 1999.

[2] R. Kalman, "A new approach to linear filtering and prediction problems," *Journal of Basic Engineering D*, vol. 82, pp. 35–45, 1960.

[3] J. D. Jiménez-López, J. Linares-Pérez, S. Nakamori, R. Caballero-Águila, and A. Hermoso-Carazo, "Signal estimation based on covariance information from observations featuring correlated uncertainty and coming from multiple sensors," *Signal Processing*, vol. 88, no. 12, pp. 2998–3006, 2008.

[4] R. Caballero-Águila, A. Hermoso-Carazo, and J. Linares-Pérez, "Least-squares linear estimators using measurements transmitted by different sensors with packet dropouts," *Digital Signal Processing*, vol. 22, no. 6, pp. 1118–1125, 2012.

[5] D. Y. Kim and M. Jeon, "Robust distributed Kalman filter for wireless sensor networks with uncertain communication channels," *Mathematical Problems in Engineering*, vol. 2012, Article ID 238597, 12 pages, 2012.

[6] J. Sun and C. Zhang, "Distributed optimal fusion estimators for multi-sensor systems with bounded random measurement delays and packet dropouts," *Journal of Computational Information Systems*, vol. 8, no. 10, pp. 4087–4094, 2012.

[7] M. J. García-Ligero, A. Hermoso-Carazo, and J. Linares-Pérez, "Distributed and centralized fusion estimation from multiple sensors with Markovian delays," *Applied Mathematics and Computation*, vol. 219, no. 6, pp. 2932–2948, 2012.

[8] W. A. Zhang, G. Feng, and L. Yu, "Multi-rate distributed fusion estimation for sensor networks with packet losses," *Automatica*, vol. 48, no. 9, pp. 2016–2028, 2012.

[9] J. Ma and S. Sun, "Information fusion estimators for systems with multiple sensors of different packet dropout rates," *Information Fusion*, vol. 12, no. 3, pp. 213–222, 2011.

[10] E. Song, Y. Zhu, J. Zhou, and Z. You, "Optimal Kalman filtering fusion with cross-correlated sensor noises," *Automatica*, vol. 43, no. 8, pp. 1450–1456, 2007.

[11] X. Wang, S. L. Sun, K. H. Ding, and J. Y. Xue, "Weighted measurement fusion white noise deconvolution filter with correlated noise for multisensor stochastic systems," *Mathematical Problems in Engineering*, vol. 2012, Article ID 257619, 16 pages, 2012.

[12] J. Feng and M. Zeng, "Optimal distributed Kalman filtering fusion for a linear dynamic system with cross-correlated noises," *International Journal of Systems Science*, vol. 43, no. 2, pp. 385–398, 2012.

[13] E. Song, Y. Zhu, and Z. You, "The Kalman type recursive state estimator with a finite-step correlated process noises," in *Proceedings of the IEEE International Conference on Automation and Logistics (ICAL '08)*, pp. 196–200, Qingdao, China, September 2008.

[14] A. Fu, Y. Zhu, and E. Song, "The optimal Kalman type state estimator with multi-step correlated process and measurement noises," in *Proceedings of the International Conference on Embedded Software and Systems (ICESS '08)*, pp. 215–220, Sichuan, China, July 2008.

[15] J. Feng, Z. Wang, and M. Zeng, "Optimal robust non-fragile Kalman-type recursive filtering with finite-step autocorrelated noises and multiple packet dropouts," *Aerospace Science and Technology*, vol. 15, no. 6, pp. 486–494, 2011.

[16] J. Feng, Z. Wang, and M. Zeng, "Distributed weighted robust Kalman filter fusion for uncertain systems with autocorrelated and cross-correlated noises," *Information Fusion*, vol. 14, pp. 78–86, 2013.

[17] S. L. Sun and J. Ma, "Optimal filtering and smoothing for discrete-time stochastic singular systems," *Signal Processing*, vol. 87, no. 1, pp. 189–201, 2007.

[18] J. Ma and S. Sun, "Optimal linear estimators for systems with random sensor delays, multiple packet dropouts and uncertain observations," *IEEE Transactions on Signal Processing*, vol. 59, no. 11, pp. 5181–5192, 2011.

[19] Z. Wang, F. Yang, D. W. C. Ho, and X. Liu, "Robust finite-horizon filtering for stochastic systems with missing measurements," *IEEE Signal Processing Letters*, vol. 12, no. 6, pp. 437–440, 2005.

[20] S. Nakamori, R. Caballero-Águila, A. Hermoso-Carazo, and J. Linares-Pérez, "Fixed-interval smoothing algorithm based on covariances with correlation in the uncertainty," *Digital Signal Processing*, vol. 15, no. 2, pp. 207–221, 2005.

[21] Z. Wang, F. Yang, D. W. C. Ho, and X. Liu, "Robust H_∞ filtering for stochastic time-delay systems with missing measurements," *IEEE Transactions on Signal Processing*, vol. 54, no. 7, pp. 2579–2587, 2006.

[22] S. Nakamori, R. Caballero-Águila, A. Hermoso-Carazo, and J. Linares-Pérez, "New recursive estimators from correlated interrupted observations using covariance information," *International Journal of Systems Science*, vol. 36, no. 10, pp. 617–629, 2005.

[23] F. O. Hounkpevi and E. E. Yaz, "Robust minimum variance linear state estimators for multiple sensors with different failure rates," *Automatica*, vol. 43, no. 7, pp. 1274–1280, 2007.

[24] X. Qu and J. Zhou, "The optimal robust finite-horizon Kalman filtering for multiple sensors with different stochastic failure rates," *Applied Mathematics Letters*, vol. 26, no. 1, pp. 80–86, 2013.

[25] R. Caballero-Águila, A. Hermoso-Carazo, and J. Linares-Pérez, "Linear and quadratic estimation using uncertain observations from multiple sensors with correlated uncertainty," *Signal Processing*, vol. 91, no. 2, pp. 330–337, 2011.

[26] B. Shen, Z. D. Wang, and Y. S. Hung, "Distributed H_∞-consensus filtering in sensor networks with multiple missing measurements: the finite-horizon case," *Automatica*, vol. 46, no. 10, pp. 1682–1688, 2010.

[27] M. Liu, D. Qi, S. Zhang, M. Qiu, and S. Zheng, "Optimal H_∞ fusion filters for a class of discrete-time intelligent systems with time delays and missing measurement," *Neurocomputing*, vol. 74, no. 18, pp. 3741–3752, 2011.

[28] G. Wei, Z. Wang, and H. Shu, "Robust filtering with stochastic nonlinearities and multiple missing measurements," *Automatica*, vol. 45, no. 3, pp. 836–841, 2009.

[29] J. Hu, Z. Wang, H. Gao, and L. K. Stergioulas, "Extended Kalman filtering with stochastic nonlinearities and multiple missing measurements," *Automatica*, vol. 48, no. 9, pp. 2007–2015, 2012.

[30] S. L. Sun and Z. L. Deng, "Multi-sensor optimal information fusion Kalman filter," *Automatica*, vol. 40, no. 6, pp. 1017–1023, 2004.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 848120, 13 pages
http://dx.doi.org/10.1155/2013/848120

Research Article

Maximum Likelihood Estimation of the VAR(1) Model Parameters with Missing Observations

Helena Mouriño and Maria Isabel Barão

Departamento de Estatística e Investigação Operacional, Faculdade de Ciências, Universidade de Lisboa, Edifício C6, Piso 4, Campo Grande, 1749-016 Lisboa, Portugal

Correspondence should be addressed to Helena Mouriño; mhnunes@fc.ul.pt

Received 4 January 2013; Revised 29 March 2013; Accepted 8 April 2013

Academic Editor: Xuejun Xie

Missing-data problems are extremely common in practice. To achieve reliable inferential results, we need to take into account this feature of the data. Suppose that the univariate data set under analysis has missing observations. This paper examines the impact of selecting an auxiliary complete data set—whose underlying stochastic process is to some extent interdependent with the former—to improve the efficiency of the estimators for the relevant parameters of the model. The Vector AutoRegressive (VAR) Model has revealed to be an extremely useful tool in capturing the dynamics of bivariate time series. We propose maximum likelihood estimators for the parameters of the VAR(1) Model based on monotone missing data pattern. Estimators' precision is also derived. Afterwards, we compare the bivariate modelling scheme with its univariate counterpart. More precisely, the univariate data set with missing observations will be modelled by an AutoRegressive Moving Average (ARMA(2,1)) Model. We will also analyse the behaviour of the AutoRegressive Model of order one, AR(1), due to its practical importance. We focus on the mean value of the main stochastic process. By simulation studies, we conclude that the estimator based on the VAR(1) Model is preferable to those derived from the univariate context.

1. Introduction

Statistical analyses of data sets with missing observations have long been addressed in the literature. For instance, Morrison [1] deduced the maximum likelihood estimators of the parameters of the multinormal mean vector and covariance matrix for the monotonic pattern with only a single incomplete variate. The exact expectations and variances of the estimators were also deduced. Dahiya and Korwar [2] obtained the maximum likelihood estimators for a bivariate normal distribution with missing data. They focused on estimating the correlation coefficient as well as the difference of the two means. Following this line of research and having in mind that the majority of the empirical studies are characterised by temporal dependence between observations, we will try to generalise the previous study by introducing a bivariate time series model to describe the relationship between the processes under consideration.

The literature on missing data has expanded in the last decades focusing mainly on univariate time series models [3–7], but there is still a lack of developments in the vectorial context.

This paper aims at analysing the main properties of the estimators from data generated by one of the most influential models in empirical studies, that is, the first-order Vector AutoRegressive (VAR(1)) Model, when the data set from the main stochastic process, designated by $\{Y_t\}_{t\in\mathbb{Z}}$, has missing observations. Therefore, we assume that there is also available a suitable auxiliary stochastic process, denoted by $\{X_t\}_{t\in\mathbb{Z}}$, which is to some extent interdependent with the main stochastic process. Additionally, the data set obtained from this process is complete. In this context, a natural question

arises: is it possible to exchange information between the two data sets to increase knowledge about the process whose data set has missing observations, or should we analyse the univariate stochastic process by itself? The goal of this paper is to answer this question.

Throughout this paper, we assume that the incomplete data set has a monotone missing data pattern. We follow a likelihood-based approach to estimate the parameters of the model. It is worth pointing out that, in the literature, likelihood-based estimation is largely used to manage the problem of missing data [3, 8, 9]. The precision of the maximum likelihood estimators is also derived.

In order to answer the question raised above, we must verify if the introduction of an auxiliary variable for estimating the parameters of the model increases the accuracy of the estimators. To accomplish this goal, we compare the precision of the estimators just cited with those obtained from modelling the dynamics of the univariate stochastic process $\{Y_t\}_{t\in\mathbb{Z}}$ by an AutoRegressive Moving Average (ARMA(2,1)) Model, which corresponds to the marginal model of the bivariate VAR(1) Model [10, 11]. The behaviour of the AutoRegressive Model of order one, AR(1), is also analysed due to its practical importance in time series modelling. Simulation studies allow us to assess the relative efficiency of the different approaches. Special attention is paid to the estimator for the mean value of the stochastic process about which information available is scarce. This is a reasonable choice given the importance of the mean function of a stochastic process in understanding the behaviour of the time series under consideration.

The paper is organised as follows. In Section 2, we review the VAR(1) Model and highlight a few statistical properties that will be used in the remaining sections. In Section 3, we establish the monotone pattern of missing data and factorise the likelihood function of the VAR(1) Model. The maximum likelihood estimators of the parameters are obtained in Section 4. Their precision is also deduced. Section 5 reports the simulation studies in evaluating different approaches to estimate the mean value of the stochastic process $\{Y_t\}_{t\in\mathbb{Z}}$. The main conclusions are summarised in Section 6.

2. Brief Description of the VAR(1) Model

In this section, a few properties of the Vectorial Autoregressive Model of order one are analysed. These features will play an important role in determining the estimators for the parameters when there are missing observations, as we will see in Section 4.

Hereafter, the stochastic process underlying the complete data set is denoted by $\{X_t\}_{t\in\mathbb{Z}}$, while the other one is represented by $\{Y_t\}_{t\in\mathbb{Z}}$. The VAR(1) Model under consideration takes the form

$$\begin{aligned} X_t &= \alpha_0 + \alpha_1 X_{t-1} + \epsilon_t, \\ &\qquad\qquad t = 0, \pm 1, \pm 2, \ldots, \\ Y_t &= \beta_0 + \beta_1 Y_{t-1} + \beta_2 X_{t-1} + \xi_t, \end{aligned} \tag{1}$$

where ϵ_t and ξ_t are Gaussian white noise processes with zero mean and variances σ_ϵ^2 and σ_ξ^2, respectively. The structure of correlation between the error terms is different from zero only at the same date t, that is, $\mathrm{Cov}(\epsilon_{t-i}, \xi_{t-j}) = \sigma_{\epsilon\,\xi}$, for $i = j$; $\mathrm{Cov}(\epsilon_{t-i}, \xi_{t-j}) = 0$, for $i \neq j$, $i, j \in \mathbb{Z}$. Exchanging information between both time series might introduce some noise in the overall process. Therefore, transfer of information from the smallest series to the largest one is not allowed here.

We have to introduce the restrictions $|\alpha_1| < 1$ and $|\beta_1| < 1$. They ensure not only that the underlying processes are ergodic for the respective means but also that the stochastic processes are covariance stationary (see Nunes [12, ch.3]). Hereafter, we assume that these restrictions are satisfied.

Next, we overview some relevant properties of the VAR(1) Model (1). Theoretical details can be found in Nunes [12, ch.3].

The mean values of X_t and Y_t are, respectively, given by

$$E(X_t) = \frac{\alpha_0}{1-\alpha_1}, \qquad E(Y_t) = \frac{\alpha_0\ \beta_2 + \beta_0(1-\alpha_1)}{(1-\alpha_1)(1-\beta_1)}. \tag{2}$$

Concerning the covariance structure of the process X_t,

$$\mathrm{Cov}(X_{t-i}, X_{t-j}) = \sigma_\epsilon^2 \frac{\alpha_1^{|i-j|}}{1-\alpha_1^2}, \quad \forall_{i,j\in\mathbb{Z}}. \tag{3}$$

For $\alpha_1 \neq \beta_1$, the covariance of the stochastic process Y_t is given by

$$\begin{aligned} \mathrm{Cov}(Y_{t-i}, Y_{t-j}) = {} & \frac{\sigma_\xi^2 \beta_1^{|i-j|}}{1-\beta_1^2} \\ & + \sigma_{\epsilon\xi}\ \beta_2 \left\{ \frac{1}{\beta_1-\alpha_1} \times \left(\frac{\beta_1^{|i-j|}}{1-\beta_1^2} - \frac{\alpha_1^{|i-j|}}{1-\alpha_1\beta_1} \right) + \frac{\beta_1\beta_1^{|i-j|}}{(1-\beta_1^2)(1-\alpha_1\beta_1)} \right\} \\ & + \sigma_\epsilon^2\ \frac{\beta_2^2}{(\beta_1-\alpha_1)(1-\alpha_1\beta_1)} \times \left(\frac{\beta_1\beta_1^{|i-j|}}{1-\beta_1^2} - \frac{\alpha_1\alpha_1^{|i-j|}}{1-\alpha_1^2} \right), \quad \forall_{i,j\in\mathbb{Z}}. \end{aligned} \tag{4}$$

Considering that $\alpha_1 = \beta_1$, we have

$$\begin{aligned} \mathrm{Cov}(Y_{t-i}, Y_{t-j}) = \frac{\beta_1^{|i-j|}}{1-\beta_1^2} \Bigg\{ & \sigma_\xi^2 + 2\ \sigma_{\epsilon\xi}\beta_2 \times \left(\frac{\beta_1}{1-\beta_1^2} + \frac{|i-j|}{\beta_1} \right) \\ & + \sigma_\epsilon^2\ \frac{\beta_2^2}{1-\beta_1^2} \left(|i-j| + \frac{1+\beta_1^2}{1-\beta_1^2} \right) \Bigg\}, \\ & \text{for } i, j \in \mathbb{Z}. \end{aligned} \tag{5}$$

In regard to the structure of covariance between the stochastic processes X_t and Y_t, for $\alpha_1 \neq \beta_1$, we have

$$\operatorname{Cov}\left(Y_{t-i}, X_{t-j}\right) = \sigma_{\epsilon\xi} \frac{\alpha_1^{|i-j|}}{1-\alpha_1\beta_1} + \sigma_\epsilon^2 \frac{\alpha_1 \beta_2 \alpha_1^{|i-j|}}{\left(1-\alpha_1\beta_1\right)\left(1-\alpha_1^2\right)}, \quad \forall_{i,j\in\mathbb{Z}}. \tag{6}$$

When $\alpha_1 = \beta_1$, the covariance function under study takes the form

$$\operatorname{Cov}\left(Y_{t-i}, X_{t-j}\right) = \sigma_\epsilon^2 \frac{\beta_2\alpha_1\alpha_1^{|i-j|}}{\left(1-\alpha_1^2\right)^2} + \sigma_{\epsilon\xi} \frac{\alpha_1^{|i-j|}}{1-\alpha_1^2}, \quad \forall_{i,j\in\mathbb{Z}}. \tag{7}$$

By writing out the stochastic system of (1) in matrix notation, the bivariate stochastic process $\mathbf{Z}_t = [X_t \quad Y_t]'$ can be expressed as

$$\mathbf{Z}_t = \begin{bmatrix}\alpha_0\\ \beta_0\end{bmatrix} + \begin{bmatrix}\alpha_1 & 0\\ \beta_2 & \beta_1\end{bmatrix}\begin{bmatrix}X_{t-1}\\ Y_{t-1}\end{bmatrix} + \begin{bmatrix}\epsilon_t\\ \xi_t\end{bmatrix} = \mathbf{c} + \mathbf{\Phi}_1\mathbf{Z}_{t-1} + \boldsymbol{\epsilon}_t, \quad t\in\mathbb{Z}, \tag{8}$$

where $\boldsymbol{\epsilon}_t = [\epsilon_t \quad \xi_t]'$ is the 2-dimensional Gaussian white noise random vector.

Hence, at each date $t, t \in \mathbb{Z}$, the conditional stochastic process $\mathbf{Z}_t|_{\mathbf{Z}_{t-1}=\mathbf{z}_{t-1}}$ follows a bivariate Gaussian distribution, $\mathbf{Z}_t|_{\mathbf{Z}_{t-1}=\mathbf{z}_{t-1}} \backsim \mathcal{N}_2(\boldsymbol{\mu}_{t|t-1}, \mathbf{\Omega}_{t|t-1})$, where the two-dimensional conditional mean value vector and the variance-covariance matrix are, respectively, given by

$$\boldsymbol{\mu}_{t|t-1} = \mathbf{c} + \mathbf{\Phi}_1\mathbf{Z}_{t-1}, \qquad \mathbf{\Omega}_{t|t-1} \equiv \mathbf{\Omega} = \begin{bmatrix}\sigma_\epsilon^2 & \sigma_{\epsilon\xi}\\ \sigma_{\epsilon\xi} & \sigma_\xi^2\end{bmatrix}. \tag{9}$$

Straightforward computations lead us to the following factoring of the probability density function of $\mathbf{Z}_t$ conditional to $\mathbf{Z}_{t-1} = \mathbf{z}_{t-1}$:

$$f_{\mathbf{Z}_t|\mathbf{Z}_{t-1}}\left(\mathbf{z}_t \mid \mathbf{z}_{t-1}\right) = f_{X_t|\mathbf{Z}_{t-1}}\left(x_t \mid \mathbf{z}_{t-1}\right) \times f_{Y_t|X_t,\mathbf{Z}_{t-1}}\left(y_t \mid x_t, \mathbf{z}_{t-1}\right). \tag{10}$$

Thus, the joint distribution of the pair X_t and Y_t conditional to the values of the process at the previous date $t-1$, $\mathbf{Z}_{t-1}$, can be decomposed into the product of the marginal distribution of $X_t|_{\mathbf{Z}_{t-1}}$ and the conditional distribution of $Y_t|_{X_t,\mathbf{Z}_{t-1}}$. Both densities follow univariate Gaussian probability laws:

$$X_t|_{\mathbf{Z}_{t-1}=\mathbf{z}_{t-1}} \backsim \mathcal{N}\left(\alpha_0 + \alpha_1 x_{t-1}, \sigma_\epsilon^2\right), \quad \text{for each date } t, \; t\in\mathbb{Z}. \tag{11}$$

Also, $Y_t|_{X_t=x_t,\mathbf{Z}_{t-1}=\mathbf{z}_{t-1}}$ follows a Gaussian distribution with

$$\begin{aligned} E\left(Y_t|_{X_t=x_t,\mathbf{Z}_{t-1}=\mathbf{z}_{t-1}}\right) &= \beta_0 + \beta_1 y_{t-1} + \beta_2 x_{t-1} + \frac{\sigma_{\epsilon\xi}}{\sigma_\epsilon^2}\left(x_t - \alpha_0 - \alpha_1 x_{t-1}\right) \\ &= \psi_0 + \psi_1 x_t + \psi_2 x_{t-1} + \beta_1 y_{t-1}, \end{aligned} \tag{12}$$

where $\psi_1 = (\sigma_{\epsilon\xi}/\sigma_\epsilon^2)$ or, for interpretive purposes, $\psi_1 = (\sigma_\xi/\sigma_\epsilon)\rho_{\epsilon\xi}$. The parameter ψ_1 describes, thus, a weighted correlation between the error terms ϵ_t and ξ_t. The weight corresponds to the ratio of their standard deviations. Moreover, $\psi_0 = \beta_0 - \psi_1, \psi_2 = \beta_2 - \psi_1\alpha_1$.

The variance has the following structure:

$$\begin{aligned} \operatorname{Var}\left(Y_t \mid X_t = x_t, \mathbf{Z}_{t-1} = \mathbf{z}_{t-1}\right) &= \sigma_\xi^2 - \frac{\sigma_{\epsilon\xi}^2}{\sigma_\epsilon^2} \\ &= \sigma_\xi^2\left(1-\rho_{\epsilon\xi}^2\right) \equiv \psi_3. \end{aligned} \tag{13}$$

The conditional distribution of $Y_t|_{X_t,\mathbf{Z}_{t-1}}$ can be interpreted as a straight-line relationship between Y_t and X_t, X_{t-1}, and Y_{t-1}. Additionally, it is worth mentioning that if $\rho_{\epsilon\xi} = \pm 1$ or $\sigma_\xi^2 = 0$, the above conditional distribution degenerates into its mean value. Henceforth, we will discard these particular cases, which means that $\psi_3 \neq 0$.

3. Factoring the Likelihood Based on Monotone Missing Data Pattern

We focus here on theoretical background for factoring the likelihood function from the VAR(1) Model when there are missing values in the data. Suppose that we have the following monotone pattern of missing data:

$$\begin{matrix} x_0 & x_2 & \cdots & x_{m-1} & x_m & \cdots & x_{n-1} \\ y_0 & y_2 & \cdots & y_{m-1} & \boxed{} & & \end{matrix}. \tag{14}$$

That is, there are n observations available from the stochastic process $\{X_t\}_{t\in\mathbb{Z}}$, whereas due to some uncontrolled factors it was only possible to record $m (m < n)$ observations from the stochastic process $\{Y_t\}_{t\in\mathbb{Z}}$. In other words, there are $n - m$ missing observations from Y_t.

Let the observed bivariate sample of size n with missing values:

$$\left\{\left(x_0, y_0\right), \left(x_1, y_1\right), \ldots, \left(x_{m-1}, y_{m-1}\right), x_m, \ldots, x_{n-1}\right\}; \tag{15}$$

denote a realisation of the random process $\mathbf{Z}_t = [X_t \quad Y_t]'$, $t \in \mathbb{Z}$, which follows a vectorial autoregressive model of order one. The likelihood function, $L(\boldsymbol{\theta})$, is given by

$$\begin{aligned} L(\boldsymbol{\theta}) &\equiv f_{\mathbf{Z}_0,\mathbf{Z}_1,\ldots,\mathbf{Z}_{m-1},X_m,\ldots,X_{n-1}} \times \left(\mathbf{z}_0, \mathbf{z}_1, \ldots, \mathbf{z}_{m-1}, x_m, \ldots, x_{n-1}\right) \\ &= f_{\mathbf{Z}_0}\left(\mathbf{z}_0\right) \prod_{t=1}^{m-1} f_{\mathbf{Z}_t|\mathbf{Z}_{t-1}}\left(\mathbf{z}_t \mid \mathbf{z}_{t-1}\right) f_{X_m|\mathbf{Z}_{m-1}}\left(x_m \mid \mathbf{z}_{m-1}\right) \\ &\quad \times \prod_{t=m+1}^{n-1} f_{X_t|X_{t-1}}\left(x_t \mid x_{t-1}; \boldsymbol{\theta}\right) \end{aligned}$$

$$= f_{\mathbf{Z}_0}(\mathbf{z}_0) \prod_{t=1}^{m-1} f_{\mathbf{Z}_t|\mathbf{Z}_{t-1}}(\mathbf{z}_t \mid \mathbf{z}_{t-1}) \times \prod_{t=m}^{n-1} f_{X_t|X_{t-1}}(x_t \mid x_{t-1}), \tag{16}$$

where $\boldsymbol{\theta} = [\alpha_0 \quad \alpha_1 \quad \sigma_\epsilon^2 \quad \beta_0 \quad \beta_1 \quad \beta_2 \quad \sigma_\xi^2 \quad \sigma_{\epsilon\xi}]'$ is the 8-dimensional vector of population parameters. To lighten notation, we assume that there is no need for conditioning the arguments of the above probability density functions on the values of the processes at date $t-1$. The likelihood function becomes

$$L(\boldsymbol{\theta}) = f_{\mathbf{Z}_0}(\mathbf{z}_0) \prod_{t=1}^{m-1} f_{\mathbf{Z}_t|\mathbf{Z}_{t-1}}(\mathbf{z}_t) \prod_{t=m}^{n-1} f_{X_t|X_{t-1}}(x_t). \tag{17}$$

Two points must be emphasised: first, we emphasise that the maximum likelihood estimators (m.l.e.) for the unknown vector of parameters will be obtained by maximising the natural logarithm of the above likelihood function. Second, a worthwhile improvement in reducing the complexity of the function to maximise is to determine the conditional maximum likelihood estimators regarding the first pair of random variables, $\mathbf{Z}_0 = [X_0 \ \ Y_0]'$, as deterministic and maximising the log-likelihood function conditioned on the values $X_0 = x_0$ and $Y_0 = y_0$. The loss of efficiency of the estimators obtained from such a procedure is negligible when compared with the exact maximum likelihood estimators computed by iterative techniques. Even for moderate sample sizes, the first pair of observations makes a negligible contribution to the total likelihood. Hence, the exact m.l.e. and the conditional m.l.e. turn out to have the same large sample properties, Hamilton [13]. Hereafter, we restrict the study to the conditional loglikelihood function.

Despite the above solutions for reducing the complexity of the problem, some difficulties still remain. The loglikelihood equations are intractable. To go over this problem we have to factorise the conditional likelihood function. From (17) we get

$$\begin{aligned} L(\boldsymbol{\theta}) &= \prod_{t=1}^{m-1} f_{(X_t,Y_t)|(X_{t-1},Y_{t-1})}(x_t, y_t) \prod_{t=m}^{n-1} f_{X_t|X_{t-1}}(x_t) \\ &= \prod_{t=1}^{m-1} \left(f_{X_t|(X_{t-1},Y_{t-1})}(x_t) \times f_{Y_t|X_t,(X_{t-1},Y_{t-1})}(y_t) \right) \\ &\quad \times \prod_{t=m}^{n-1} f_{X_t|X_{t-1}}(x_t) \\ &= \prod_{t=1}^{n-1} f_{X_t|X_{t-1}}(x_t) \prod_{t=1}^{m-1} f_{Y_t|X_t,(X_{t-1},Y_{t-1})}(y_t). \end{aligned} \tag{18}$$

So as to work out the analytical expressions for the unknown parameters under study, we have to decompose the entire likelihood function (18) into easily manipulated components.

For the Gaussian VAR processes, the conditional maximum likelihood estimators coincide with the least squares estimators [13]. Therefore, we may find a solution to the problem just raised in the geometrical context. The identification of such components relies on two of the most famous theorems in the Euclidean space: the Orthogonal Decomposition Theorem and the Approximation Theorem [14, Volume I, pages 572–575]. Based on these tools it is straightforward to establish that the estimation subspaces associated with the conditional distributions $X_t|_{X_{t-1}}$ and $Y_t|_{X_t,X_{t-1},Y_{t-1}}$ are, by construction, orthogonal to each other. This means that each element belonging to one of those subspaces is uncorrelated with each element that pertains to their orthogonal complement. Hence, events that happen on one subspace provide no information about events on the other subspace.

The aforementioned arguments guarantee that the decomposition of the joint likelihood in two components can be carried out with no loss of information for the whole estimation procedure. From (18) we can, thus, decompose the conditional loglikelihood function as follows:

$$\begin{aligned} l \equiv l(\boldsymbol{\theta}) = \log L(\boldsymbol{\theta}) &= \sum_{t=1}^{n-1} \log f_{X_t|X_{t-1}}(x_t) \\ &+ \sum_{t=1}^{m-1} \log f_{Y_t|X_t,(X_{t-1},Y_{t-1})}(y_t) = l_1 + l_2. \end{aligned} \tag{19}$$

Henceforth, l_1 denotes the loglikelihood from the marginal distribution of X_t, based on the whole sampled data with dimension n, that is, $x_0, x_1, \dots, x_{n-1}$. The function l_2 represents the loglikelihood from the conditional density of $Y_t|_{X_t,\mathbf{Z}_{t-1}}$ computed by the bivariate sample of size m:

$$(x_0, y_0), (x_1, y_1), \dots, (x_{m-1}, y_{m-1}). \tag{20}$$

The components l_1 and l_2 of (19) will be maximised separately in Section 4.1.

4. Maximum Likelihood Estimators for the Parameters

In Section 4.1 the m.l.e. of the parameters from the fragmentary VAR(1) Model are deduced. The precision of the estimators is examined in Section 4.2.

4.1. Analytical Expressions. Theoretical developments carried out in this section rely on solving the loglikelihood equations obtained from the factored loglikelihood given by (19). Before proceeding with theoretical matters, we introduce some relevant notation in the ensuing paragraphs.

Let $\overline{X}_k^{(l)} = (1/k)\sum_{t=1}^{k} X_{t-l}$ represent the sample mean lagged l time units, $l = 0, 1$. The subscript $k, k = 1, \dots, n-1$, allows us to identify the number of observations that takes part in the computation of the sample mean. A similar notation is used for denoting the sample mean of the random

sample $Y_0, \ldots, Y_k$, for $k = 1, \ldots, m-1$, $\overline{Y}_k^{(l)}$. According to this new definition, the sample variance of each univariate random variable based on k observations and lagged l time units is denoted by

$$\widehat{\gamma}_{X,k}^{(l)} = \frac{1}{k} \sum_{t=1}^{k} \left(X_{t-l} - \overline{X}_k^{(l)} \right)^2,$$
$$\widehat{\gamma}_{Y,k}^{(l)} = \frac{1}{k} \sum_{t=1}^{k} \left(Y_{t-l} - \overline{Y}_k^{(l)} \right)^2, \quad l = 0, 1. \tag{21}$$

Let $\widehat{\gamma}_{X,k}^{*}(1) = (1/k) \sum_{t=1}^{k} (X_t - \overline{X}_k^{(0)})(X_{t-1} - \overline{X}_k^{(1)})$ describe the sample autocovariance coefficient at lag one for the stochastic process X_t, based on k observations. Its counterpart for the stochastic process Y_t, $\widehat{\gamma}_{X,k}^{*}(1)$, is obtained by changing notation accordingly. The sample autocorrelation coefficient of the random process X_t at lag one is denoted by $\widehat{\rho}_{X,k}^{(1)} = \widehat{\gamma}_{X,k}^{*}(1) / \widehat{\gamma}_{X,k}^{(0)} \, \widehat{\gamma}_{X,k}^{(1)}$. The empirical covariance between the random processes X_t and Y_t lagged one time unit is represented by

$$\widehat{\gamma}_{XY}^{*}(1) = \frac{1}{m-1} \sum_{t=1}^{m-1} \left(X_t - \overline{X}_{m-1}^{(0)} \right) \left(Y_{t-1} - \overline{Y}_{m-1}^{(1)} \right), \quad \text{for lagged values on } Y,$$
$$\widehat{\gamma}_{YX}^{*}(1) = \frac{1}{m-1} \sum_{t=1}^{m-1} \left(X_{t-1} - \overline{X}_{m-1}^{(1)} \right) \left(Y_t - \overline{Y}_{m-1}^{(0)} \right), \quad \text{for lagged values on } X. \tag{22}$$

The sample covariance coefficient of X_t and Y_t computed from l time units lag for each series is given by

$$\widehat{\gamma}_{XY}^{(l)} = \frac{1}{m-1} \sum_{t=1}^{m-1} \left(X_{t-l} - \overline{X}_{m-1}^{(l)} \right) \left(Y_{t-l} - \overline{Y}_{m-1}^{(l)} \right), \quad \text{with } l = 0, 1. \tag{23}$$

(i) *Maximising the loglikelihood function* l_1: Using the results (11) and (19), we readily find the following m.l.e.

$$\widehat{\alpha}_0 = \overline{X}_{n-1}^{(0)} - \widehat{\alpha}_1 \overline{X}_{n-1}^{(1)}, \qquad \widehat{\alpha}_1 = \frac{\widehat{\gamma}_{X,n-1}^{*}(1)}{\widehat{\gamma}_{X,n-1}^{(1)}},$$
$$\widehat{\sigma}_{\epsilon}^2 = \frac{SS_R}{n-1}, \tag{24}$$

where SS_R is the respective residual sum of squares.

(ii) *Maximising the loglikelihood function* l_2: Based on (12) and (13) we get the loglikelihood function l_2

$$l_2 = \sum_{t=1}^{m-1} \log f_{Y_t | X_t, X_{t-1}, Y_{t-1}}(y_t) = -\frac{m-1}{2} \log(2\pi) - \frac{m-1}{2} \log \psi_3 - \frac{1}{2\,\psi_3} \times \sum_{t=1}^{m-1} \left(y_t - \psi_0 - \psi_1 x_t - \psi_2 x_{t-1} - \beta_1 y_{t-1} \right)^2. \tag{25}$$

We readily find out that the m.l.e. for the parameters under study are given by

$$\widehat{\psi}_0 = \overline{Y}_{m-1}^{(0)} - \widehat{\psi}_1 \overline{X}_{m-1}^{(0)} - \widehat{\psi}_2 \overline{X}_{m-1}^{(1)} - \widehat{\beta}_1 \overline{Y}_{m-1}^{(1)},$$
$$\widehat{\psi}_1 = \frac{1}{\widehat{\gamma}_{X,m-1}^{(0)}} \left\{ \widehat{\gamma}_{XY}^{(0)} - \widehat{\psi}_2 \widehat{\gamma}_{X,m-1}^{*}(1) - \widehat{\beta}_1 \widehat{\gamma}_{XY}^{*}(1) \right\},$$
$$\widehat{\psi}_2 = \frac{1}{\left(1 - \left(\widehat{\rho}_{X,m-1}^{(1)} \right)^2 \right) \widehat{\gamma}_{X,m-1}^{(1)}} \times \left\{ \widehat{\gamma}_{YX}^{*}(1) - \frac{\widehat{\gamma}_{XY}^{(0)} \widehat{\gamma}_{X,m-1}^{*}(1)}{\widehat{\gamma}_{X,m-1}^{(0)}} - \widehat{\beta}_1 \widehat{\gamma}_{XY}^{(1)} + \frac{\widehat{\beta}_1 \widehat{\gamma}_{XY}^{*}(1)\, \widehat{\gamma}_{X,m-1}^{*}(1)}{\widehat{\gamma}_{X,m-1}^{(0)}} \right\},$$
$$\widehat{\psi}_3 = \frac{SS_R^{*}}{m-1},$$
$$\widehat{\beta}_1 = \frac{1}{\left(\widehat{\gamma}_{Y,m-1}^{(1)} + \left(\overline{Y}_{m-1}^{(1)} \right)^2 \right)} \times \left\{ \widehat{\gamma}_{Y,m-1}^{*}(1) + \overline{Y}_{m-1}^{(0)} \overline{Y}_{m-1}^{(1)} - \widehat{\psi}_0 \, \overline{Y}_{m-1}^{(1)} - \widehat{\psi}_1 \, \widehat{\gamma}_{XY}^{*}(1) - \widehat{\psi}_1 \, \overline{X}_{m-1}^{(0)} \, \overline{Y}_{m-1}^{(1)} - \widehat{\psi}_2 \left(\widehat{\gamma}_{XY}^{(1)} - \overline{X}_{m-1}^{(1)} \, \overline{Y}_{m-1}^{(1)} \right) \right\}, \tag{26}$$

where SS_R^{*} denotes the corresponding residual sums of squares.

Using the results from Section 2 we get the following estimators for the original parameters:

$$\widehat{\beta}_0 = \widehat{\psi}_0 + \widehat{\psi}_1 \widehat{\alpha}_0, \qquad \widehat{\beta}_2 = \widehat{\psi}_2 + \widehat{\psi}_1 \widehat{\alpha}_1,$$
$$\widehat{\sigma}_{\epsilon\xi} = \widehat{\psi}_1 \widehat{\sigma}_{\epsilon}^2, \quad \widehat{\sigma}_{\xi}^2 = \widehat{\psi}_3 + \frac{\widehat{\sigma}_{\epsilon\xi}^2}{\widehat{\sigma}_{\epsilon}^2}. \tag{27}$$

Thus, the analytical expressions for the estimators of the mean values, variances, and covariances of the VAR(1) Model are given by

$$
\begin{aligned}
\widehat{\mu}_X &= \frac{\widehat{\alpha}_0}{1-\widehat{\alpha}_1}, \\
\widehat{\mu}_Y &= \frac{\widehat{\alpha}_0\widehat{\beta}_2 + \widehat{\beta}_0\left(1-\widehat{\alpha}_1\right)}{\left(1-\widehat{\alpha}_1\right)\left(1-\widehat{\beta}_1\right)}, \\
\widehat{\sigma}^2_X &= \frac{\widehat{\sigma}^2_\epsilon}{1-\widehat{\alpha}^2_1}, \\
\widehat{\sigma}^2_Y &= \frac{\widehat{\sigma}^2_\xi}{1-\widehat{\beta}^2_1} + \frac{2\widehat{\sigma}_{\epsilon\xi}\widehat{\beta}_1\widehat{\beta}_2}{\left(1-\widehat{\alpha}_1\widehat{\beta}_1\right)\left(1-\widehat{\beta}^2_1\right)} \\
&\quad + \frac{\widehat{\sigma}^2_\epsilon\widehat{\beta}^2_2\left(1+\widehat{\alpha}_1\widehat{\beta}_1\right)}{\left(1-\widehat{\alpha}^2_1\right)\left(1-\widehat{\beta}^2_1\right)\left(1-\widehat{\alpha}_1\widehat{\beta}_1\right)} \quad (\alpha_1 \neq \beta_1), \\
\widehat{\sigma}^2_Y &= \frac{\widehat{\sigma}^2_\xi}{1-\widehat{\alpha}^2_1} + 2\frac{\widehat{\sigma}_{\epsilon\xi}\widehat{\alpha}_1\widehat{\beta}_2}{\left(1-\widehat{\alpha}^2_1\right)^2} + \widehat{\sigma}^2_\epsilon\widehat{\beta}^2_2\frac{1+\widehat{\alpha}^2_1}{\left(1-\widehat{\alpha}^2_1\right)^3}, \\
\widehat{\sigma}_{XY} &= \frac{\widehat{\sigma}_{\epsilon\xi}}{1-\widehat{\alpha}_1\widehat{\beta}_1} + \frac{\widehat{\alpha}_1\widehat{\beta}_2\widehat{\sigma}^2_\epsilon}{\left(1-\widehat{\alpha}_1\widehat{\beta}_1\right)\left(1-\widehat{\alpha}^2_1\right)} \\
&\qquad \text{(at the same date } t, t \in \mathbb{Z}\text{)}.
\end{aligned} \tag{28}
$$

These estimators will play a central role in the following sections.

4.2. Precision of the Estimators. In the section, the precision of the maximum likelihood estimators underlying equations (28) is derived. The whole analysis will be separated in three stages. First, we study the statistical properties of the vector $\widehat{\mathbf{\Theta}}$, where $\widehat{\mathbf{\Theta}} = [\widehat{\mathbf{\Theta}}_1\ \ \widehat{\mathbf{\Theta}}_2]'$, with $\widehat{\mathbf{\Theta}}_1 = [\widehat{\alpha}_0\ \ \widehat{\alpha}_1\ \ \widehat{\sigma}^2_\epsilon]'$ and $\widehat{\mathbf{\Theta}}_2 = [\widehat{\psi}_0\ \ \widehat{\psi}_1\ \ \widehat{\psi}_2\ \ \widehat{\beta}_1\ \ \widehat{\psi}_3]'$. For notation consistency, the unknown parameter β_1 is either denoted by β_1 or ψ_4. That is, $\psi_4 \equiv \beta_1$. Secondly, we derive the precision of the m.l.e. of the original parameters of the VAR(1) Model (see (1)). Finally, we will focus our attention on the estimators for the mean vector and the variance-covariance matrix at lag zero of the VAR(1) model with a monotone pattern of missingness.

There are a few points worth mentioning. From Section 3 we know that there is no loss of information in maximising separately the loglikelihood functions l_1 and l_2 (19). As a consequence, the variance-covariance matrix associated with the whole set of estimated parameters is a block diagonal matrix. For sufficiently large sample size, the distribution of the maximum likelihood estimator is accurately approximated by the following multivariate Gaussian distribution:

$$
\widehat{\mathbf{\Theta}} \approx \mathcal{N}_8\left(\begin{bmatrix}\mathbf{\Theta}_1 \\ \mathbf{\Theta}_2\end{bmatrix}, \begin{bmatrix}\mathbf{I}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_2^{-1}\end{bmatrix}\right), \tag{29}
$$

where $\mathbf{I}_1$ and $\mathbf{I}_2$ denote the Fisher information matrices, respectively, from the components l_1 and l_2 of the loglikelihood function (see (19)). There is an asymptotic equivalence between the Fisher information matrix and the Hessian matrix (see [8, ch.2]). Moreover, as long as $\widehat{\mathbf{\Theta}} \to \mathbf{\Theta}$ there is also an asymptotic equivalence between the Hessian matrix computed at the points $\widehat{\mathbf{\Theta}}$ and $\mathbf{\Theta}$. Henceforth, the Fisher information matrices from (29) are estimated, respectively, by

$$
\widehat{\mathbf{I}}_1 = -\left(\frac{\partial^2 l_1}{\partial\mathbf{\Theta}_1\partial\mathbf{\Theta}_1'}\right)\Bigg|_{\mathbf{\Theta}_1=\widehat{\mathbf{\Theta}}_1}, \qquad \widehat{\mathbf{I}}_2 = -\left(\frac{\partial^2 l_1}{\partial\mathbf{\Theta}_2\partial\mathbf{\Theta}_2'}\right)\Bigg|_{\mathbf{\Theta}_2=\widehat{\mathbf{\Theta}}_2}. \tag{30}
$$

To lighten notation, from now on we suppress the "hat" from the consistent estimators of the information matrices.

The variance-covariance matrix for $\widehat{\mathbf{\Theta}}_1$ takes the following form:

$$
\mathbf{I}_1^{-1} = \frac{\widehat{\sigma}^2_\epsilon}{(n-1)\,\widehat{\gamma}^{(1)}_{X,n-1}(0)} \times \left[\begin{array}{cc|c}
\widehat{\gamma}^{(1)}_{X,n-1}(0) + \left(\overline{X}^{(1)}_{n-1}\right)^2 & -\overline{X}^{(1)}_{n-1} & 0 \\
-\overline{X}^{(1)}_{n-1} & 1 & 0 \\
\hline
0 & 0 & 2\widehat{\sigma}^2_\epsilon\ \widehat{\gamma}^{(1)}_{X,n-1}(0)
\end{array}\right]. \tag{31}
$$

We stress that there is orthogonality between the error and the estimation subspaces underlying the loglikelihood function l_1.

Calculating the second derivatives of the loglikelihood function l_2 results in the following approximate information matrix:

$$
\mathbf{I}_2 = \frac{1}{\widehat{\psi}_3}\left[\begin{array}{cccc|c}
m-1 & (m-1)\,\overline{X}_{m-1} & (m-1)\,\overline{X}^{(1)}_{m-1} & (m-1)\,\overline{Y}^{(1)}_{m-1} & 0 \\
(m-1)\,\overline{X}_{m-1} & \sum_{t=1}^{m-1} X_i^2 & \sum_{t=1}^{m-1} X_t X_{t-1} & \sum_{t=1}^{m-1} X_t Y_{t-1} & 0 \\
(m-1)\,\overline{X}^{(1)}_{m-1} & \sum_{t=1}^{m-1} X_i X_{i-1} & \sum_{t=1}^{m-1} X_{t-1}^2 & \sum_{t=1}^{m-1} X_{t-1} Y_{t-1} & 0 \\
(m-1)\,\overline{Y}^{(1)}_{m-1} & \sum_{t=1}^{m-1} X_i Y_{i-1} & \sum_{t=1}^{m-1} X_{t-1} Y_{t-1} & \sum_{t=1}^{m-1} Y_{t-1}^2 & 0 \\
\hline
0 & 0 & 0 & 0 & \dfrac{m-1}{2\,\widehat{\psi}_3}
\end{array}\right]. \tag{32}
$$

Once again, we mention that there is orthogonality between the error and the estimation subspaces underlying the loglikelihood function l_2. The matrix $\mathbf{I}_2$ can be written in a compact form:

$$\mathbf{I}_2 = \frac{1}{\widehat{\psi}_3}\left[\begin{array}{c|c} \mathbf{I}_{21} & \mathbf{0} \\ \hline \mathbf{0} & I_{22} \end{array}\right], \tag{33}$$

where the (4×4) submatrix $\mathbf{I}_{21}$ and the scalar I_{22} are, respectively, defined as

$$\mathbf{I}_{21} = \mathbf{U}'\mathbf{U}, \qquad I_{22} = \frac{m-1}{2(\widehat{\psi}_3)^2}, \tag{34}$$

with

$$\mathbf{U} = \begin{bmatrix} 1 & X_1 & X_0 & Y_0 \\ 1 & X_2 & X_1 & Y_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & X_{m-1} & X_{m-2} & Y_{m-2} \end{bmatrix}. \tag{35}$$

Using the above partition of $\mathbf{I}_2$ it is rather simple to compute the inverse matrix. In fact,

$$\mathbf{I}_2^{-1} = \widehat{\psi}_3\left[\begin{array}{c|c} \mathbf{I}_{21}^{-1} & \mathbf{0} \\ \hline \mathbf{0} & I_{22}^{-1} \end{array}\right], \tag{36}$$

with $\mathbf{I}_{21}^{-1} = (\mathbf{U}'\mathbf{U})^{-1}$ and $I_{22}^{-1} = (2/(m-1))\,(\widehat{\psi}_3)^2$.

Unfortunately, there is no explicit expression for the inverse matrix $\mathbf{I}_{21}^{-1}$. As a result, there are no explicit expressions for the approximate variance-covariance of the m.l.e. for the vector of unknown parameters $\widehat{\boldsymbol{\Theta}}_2$.

Now, we have to analyse the precision of the m.l.e. of the original parameters of the VAR(1) Model, that is, $\boldsymbol{\Upsilon} = [\alpha_0\ \ \alpha_1\ \ \sigma_\epsilon^2\ \ \beta_0\ \ \beta_1\ \ \beta_2\ \ \sigma_\xi^2\ \ \sigma_{\epsilon\xi}]'$.

Recalling from Section 2, the one-one monotone functions that relate the vector of parameters under consideration, that is,

$$\begin{gathered} \boldsymbol{\Theta}_2 = [\psi_0\ \ \psi_1\ \ \psi_2\ \ \psi_4\ \ \psi_3]', \\ \boldsymbol{\Upsilon}_2 = [\beta_0\ \ \beta_1\ \ \beta_2\ \ \sigma_\xi^2\ \ \sigma_{\epsilon\xi}]', \quad \text{are} \\ \psi_0 = \beta_0 - \alpha_0\psi_1, \qquad \psi_1 = \frac{\sigma_{\epsilon\xi}}{\sigma_\epsilon^2}, \qquad \psi_2 = \beta_2 - \psi_1\alpha_1 \\ \psi_3 = \sigma_\xi^2 - \sigma_\epsilon^2\,\psi_1^2, \qquad \psi_4 \equiv \beta_1\ . \end{gathered} \tag{37}$$

The parameters α_0, α_1, and σ_ϵ^2 remain unchanged. A key assumption in the following developments is that neither the estimates of the unknown parameters nor the true values fall on the boundary of the allowable parameter space.

The variance-covariance matrix of the m.l.e. for the vector of parameters $\boldsymbol{\Upsilon}$ is obtained by the first-order Taylor expansion at $\boldsymbol{\Upsilon}$. We also use the chain rule for derivatives of vector fields ([for details, see [14, Volume II, pages 269–275]).

Writing the vector of parameters $\boldsymbol{\Upsilon}$ as a function of the vector $\boldsymbol{\Theta}$, the respective first-order partial derivatives can be joined together in the following partitioned matrix:

$$\mathbf{D} = \left[\begin{array}{c|c} \mathbf{D}_1 & \mathbf{D}_2 \\ \hline \mathbf{D}_3 & \mathbf{D}_4 \end{array}\right], \tag{38}$$

where the (3×3) submatrix $\mathbf{D}_1$ corresponds to the first-order partial derivatives of the vector $\boldsymbol{\Upsilon}_1 \equiv \boldsymbol{\Theta}_1 = [\alpha_0\ \ \alpha_1\ \ \sigma_\epsilon^2]'$ with respect to itself, which means that $\mathbf{D}_1$ is nothing but the identity matrix of order 3, $\mathbf{D}_1 = \mathbf{I}_3$. On the other hand, this statement also means that the derivatives of the parameters under consideration with respect to either $\psi_0, \psi_1, \psi_2, \psi_3$, or ψ_4 are zero. In other words, the (3×5) submatrix $\mathbf{D}_2$ is equal to the null vector, that is, $\mathbf{D}_2 = \mathbf{0}$.

The (5×3) submatrix $\mathbf{D}_3$ and the (5×5) submatrix $\mathbf{D}_4$ are composed by the first-order partial derivatives of each component of the vector of parameters $\boldsymbol{\Upsilon}_2 = [\beta_0\ \ \beta_1\ \ \beta_2\ \ \sigma_\xi^2\ \ \sigma_{\epsilon\xi}]'$ with respect to, respectively, $\alpha_0, \alpha_1, \sigma_\epsilon^2$ and $\psi_0, \psi_1, \psi_2, \psi_4, \psi_3$. Their structures are, thus, given by

$$\begin{gathered} \mathbf{D}_3 = \begin{bmatrix} \psi_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \psi_1 & 0 \\ 0 & 0 & \psi_1^2 \\ 0 & 0 & \psi_1 \end{bmatrix}, \\ \mathbf{D}_4 = \left[\begin{array}{cccc|c} 1 & \alpha_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \alpha_1 & 1 & 0 & 0 \\ 0 & 2\psi_1\sigma_\epsilon^2 & 0 & 0 & 1 \\ \hline 0 & \sigma_\epsilon^2 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \mathbf{D}_{41} & \mathbf{D}_{42} \\ \hline \mathbf{D}_{43} & D_{44} \end{array}\right]. \end{gathered} \tag{39}$$

For finding out the approximate variance-covariance matrix of the maximum likelihood estimators for the unknown vector of parameters $\boldsymbol{\Upsilon}$, it is only necessary to pre- and postmultiply the variance-covariance matrix arising from expressions (29), (31), and (36) by, respectively, the matrix $\mathbf{D}$ and its transpose, $\mathbf{D}'$. More precisely,

$$\begin{aligned} \boldsymbol{\Sigma}_{\boldsymbol{\Upsilon}} &\approx \mathbf{D}\mathbf{I}^{-1}\mathbf{D}' \\ &= \left[\begin{array}{c|c} \mathbf{I}_3 & \mathbf{0}_{3\times5} \\ \hline \mathbf{D}_3 & \mathbf{D}_4 \end{array}\right]\left[\begin{array}{c|c} \mathbf{I}_1^{-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{I}_2^{-1} \end{array}\right]\left[\begin{array}{c|c} \mathbf{I}_3 & \mathbf{D}_3' \\ \hline \mathbf{0}_{5\times3} & \mathbf{D}_4' \end{array}\right]. \end{aligned} \tag{40}$$

Hence,

$$\boldsymbol{\Sigma}_{\boldsymbol{\Upsilon}} \approx \left[\begin{array}{c|c} \mathbf{I}_1^{-1} & \mathbf{I}_1^{-1}\mathbf{D}_3' \\ \hline \left(\mathbf{I}_1^{-1}\mathbf{D}_3'\right)' & \mathbf{D}_3\mathbf{I}_1^{-1}\mathbf{D}_3' + \mathbf{D}_4\mathbf{I}_2^{-1}\mathbf{D}_4' \end{array}\right], \tag{41}$$

with $\boldsymbol{\Sigma}_{\boldsymbol{\Upsilon}}$ denoting the variance-covariance matrix of the m.l.e. for the vector of unknown parameters $\boldsymbol{\Upsilon}$. A more detailed analysis of the variance-covariance matrix (41) can be found in Nunes [12, ch.3, p.91-92].

We can now deduce the approximate variance-covariance matrix of the maximum likelihood estimators for the mean vector and the variance-covariance matrix at lag zero of the VAR(1) Model with a monotone pattern of missingness, represented by $\boldsymbol{\Xi} = [\alpha_0\ \ \alpha_1\ \ \sigma_\epsilon^2\ \ \mu_X\ \ \mu_Y\ \ \sigma_X^2\ \ \sigma_Y^2\ \ \sigma_{XY}]'$. The first-order partial derivatives of the vector $\boldsymbol{\Xi}$ with respect to the vector $\boldsymbol{\Upsilon}$ are placed in a matrix that is denoted by $\mathbf{F}$. It takes the following form:

$$\mathbf{F} = \left[\begin{array}{c|c} \mathbf{F}_1 & \mathbf{F}_2 \\ \hline \mathbf{F}_3 & \mathbf{F}_4 \end{array}\right]. \tag{42}$$

According to the partition of the matrix $\mathbf{D}$ into four blocks—expression (38)—we partition the matrix $\mathbf{F}$ into the

following blocks: the (3×3) submatrix $\mathbf{F}_1$ corresponds to the partial derivatives of α_0,α_1, and σ_ϵ^2 with respect to themselves. As a consequence, $\mathbf{F}_1$ is the identity matrix of order 3, that is, $\mathbf{F}_1 = \mathbf{I}_3$. Regards to the (3×5) sub-matrix $\mathbf{F}_2$, its elements correspond to the partial derivatives of α_0,α_1, and σ_ϵ^2 with respect to $\beta_0,\beta_1,\beta_2,\sigma_\xi^2$, and $\sigma_{\epsilon\xi}$. Therefore, $\mathbf{F}_2 = \mathbf{0}$. The partial derivatives of $\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2$, and σ_{XY} with respect to α_0,α_1, and σ_ϵ^2 are gathered together in the (5×3) sub-matrix $\mathbf{F}_3$:

$$\mathbf{F}_3 = \begin{bmatrix} f_{11}^3 & f_{12}^3 & 0 \\ f_{21}^3 & f_{22}^3 & 0 \\ 0 & f_{32}^3 & f_{33}^3 \\ 0 & f_{42}^3 & f_{43}^3 \\ 0 & f_{52}^3 & f_{53}^3 \end{bmatrix}, \tag{43}$$

where

$$\begin{aligned}
f_{11}^3 &= \frac{1}{1-\alpha_1}, \qquad f_{12}^3 = \frac{\alpha_0}{(1-\alpha_1)^2},\\
f_{21}^3 &= \frac{\beta_2}{(1-\beta_1)(1-\alpha_1)},\\
f_{22}^3 &= \frac{\alpha_0\beta_2}{(1-\beta_1)(1-\alpha_1)^2}, \qquad f_{32}^3 = \frac{2\alpha_1\sigma_\epsilon^2}{(1-\alpha_1^2)^2},\\
f_{33}^3 &= \frac{1}{1-\alpha_1^2},\\
f_{42}^3 &= 2\beta_2\Big(\sigma_{\epsilon\xi}\beta_1^2(1-\alpha_1^2)^2\\
&\qquad +\sigma_\epsilon^2\beta_2\left(\beta_1\left(1-\alpha_1^2\right)+\alpha_1\left(1-\alpha_1^2\beta_1^2\right)\right)\Big)\\
&\qquad \times\left(\left((1-\alpha_1\ \beta_1)\left(1-\alpha_1^2\right)\right)^2\left(1-\beta_1^2\right)\right)^{-1},\\
f_{43}^3 &= \frac{\beta_2^2(1+\alpha_1\ \beta_1)}{(1-\alpha_1^2)(1-\beta_1^2)(1-\alpha_1\beta_1)},\\
f_{52}^3 &= \frac{\sigma_{\epsilon\xi}\beta_1}{(1-\alpha_1\beta_1)^2} + \sigma_\epsilon^2\beta_2\frac{1+\alpha_1^2(1-2\alpha_1\beta_1)}{(1-\alpha_1\beta_1)^2\ (1-\alpha_1^2)^2},\\
f_{53}^3 &= \frac{\alpha_1\beta_2}{(1-\alpha_1\beta_1)(1-\alpha_1^2)}.
\end{aligned} \tag{44}$$

The 5-dimensional square sub-matrix $\mathbf{F}_4$ corresponds to the partial derivatives of $\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2$, and σ_{XY} with respect to $\beta_0,\beta_1,\beta_2,\sigma_\xi^2,\sigma_{\epsilon\xi}$:

$$\mathbf{F}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ f_{21}^4 & f_{22}^4 & f_{23}^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & f_{42}^4 & f_{43}^4 & f_{44}^4 & f_{45}^4 \\ 0 & f_{52}^4 & f_{53}^4 & 0 & f_{55}^4 \end{bmatrix}, \tag{45}$$

with its nonnull elements taking the following analytical expressions:

$$\begin{aligned}
f_{21}^4 &= \frac{1}{1-\beta_1}, \qquad f_{22}^4 = \frac{\beta_0}{(1-\beta_1)^2} + \frac{\alpha_0\beta_2}{(1-\alpha_1)(1-\beta_1)^2},\\
f_{23}^4 &= \frac{\alpha_0}{(1-\alpha_1)(1-\beta_1)},\\
f_{42}^4 &= \frac{2\sigma_\xi^2\beta_1}{(1-\beta_1^2)^2} + \frac{2\sigma_{\epsilon\xi}\beta_2\left(1+\beta_1^2(1-2\alpha_1\beta_1)\right)}{(1-\alpha_1\ \beta_1)^2(1-\beta_1^2)^2}\\
&\quad + \frac{2\sigma_\epsilon^2\beta_2^2\ \left(\alpha_1\left(1-\beta_1^2\right)+\beta_1\left(1-\alpha_1^2\beta_1^2\right)\right)}{(1-\alpha_1^2)(1-\beta_1^2)^2\ (1-\alpha_1\beta_1)^2},\\
f_{43}^4 &= \frac{2}{(1-\alpha_1\beta_1)(1-\beta_1^2)}\left\{\beta_1\sigma_{\epsilon\xi} + \frac{\sigma_\epsilon^2\ \beta_2(1+\alpha_1\beta_1)}{1-\alpha_1^2}\right\},\\
f_{44}^4 &= \frac{1}{1-\beta_1^2}, \qquad f_{45}^4 = \frac{2\ \beta_1\beta_2}{(1-\alpha_1\beta_1)(1-\beta_1^2)},\\
f_{52}^4 &= \frac{\left(1-\alpha_1^2\right)\sigma_{\epsilon\xi}\alpha_1 + \sigma_\epsilon^2\alpha_1^2\beta_2}{(1-\alpha_1^2)(1-\alpha_1\beta_1)^2},\\
f_{53}^4 &= \frac{\sigma_\epsilon^2\alpha_1}{(1-\alpha_1\beta_1)(1-\alpha_1^2)}, \qquad f_{55}^4 = \frac{1}{1-\alpha_1\beta_1}.
\end{aligned} \tag{46}$$

Straightforward calculations have paved the way to the desired partitioned variance-covariance matrix, called here $\mathbf{\Sigma}_\Xi$,

$$\mathbf{\Sigma}_\Xi \approx \mathbf{F}\mathbf{\Sigma}_\mathbf{Y}\mathbf{F}' \approx \mathbf{F}\mathbf{D}\mathbf{I}^{-1}\mathbf{D}'\mathbf{F}' = \left[\begin{array}{c|c} \mathbf{\Sigma}_\Xi^{11} & \mathbf{\Sigma}_\Xi^{12} \\ \hline \mathbf{\Sigma}_\Xi^{21} & \mathbf{\Sigma}_\Xi^{22} \end{array}\right], \tag{47}$$

with its submatrices defined by

$$\begin{aligned}
\mathbf{\Sigma}_\Xi^{11} &= \mathbf{I}_1^{-1}, \quad \mathbf{\Sigma}_\Xi^{12} = \mathbf{I}_1^{-1}(\mathbf{F}_3+\mathbf{F}_4\mathbf{D}_3)' = \mathbf{I}_1^{-1}\left(\mathbf{F}_3'+\mathbf{D}_3'\mathbf{F}_4'\right),\\
\mathbf{\Sigma}_\Xi^{21} &= (\mathbf{F}_3+\mathbf{F}_4\mathbf{D}_3)\,\mathbf{I}_1^{-1} = \left(\mathbf{\Sigma}_F^{12}\right)',\\
\mathbf{\Sigma}_\Xi^{22} &= \mathbf{F}_3\mathbf{I}_1^{-1}\mathbf{F}_3' + \mathbf{F}_4\mathbf{D}_3\mathbf{I}_1^{-1}\mathbf{F}_3' + \mathbf{F}_3\mathbf{I}_1^{-1}\ \mathbf{D}_3'\mathbf{F}_4'\\
&\quad + \mathbf{F}_4\left(\mathbf{D}_3\ \mathbf{I}_1^{-1}\ \mathbf{D}_3' + \mathbf{D}_4\ \mathbf{I}_2^{-1}\ \mathbf{D}_4'\right)\mathbf{F}_4'\\
&= (\mathbf{F}_3+\mathbf{F}_4\mathbf{D}_3)\,\mathbf{I}_1^{-1}(\mathbf{F}_3+\mathbf{F}_4\mathbf{D}_3)' + \mathbf{F}_4\mathbf{D}_4\mathbf{I}_2^{-1}(\mathbf{F}_4\mathbf{D}_4)'\\
&= \mathbf{G}\ \mathbf{I}_1^{-1}\mathbf{G}' + \mathbf{H}\ \mathbf{I}_2^{-1}\mathbf{H}'.
\end{aligned} \tag{48}$$

The matrix $\mathbf{G}$ that has just been defined as $\mathbf{G} = \mathbf{F}_3 + \mathbf{F}_4\mathbf{D}_3$ corresponds to the first-order partial derivatives from the composite functions that relate $\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2$, and σ_{XY} with the vector of parameters $\mathbf{\Theta}_1$. The elements of the matrix $\mathbf{H} = \mathbf{F}_4\mathbf{D}_4$ are the first-order partial derivatives from the composite functions that relate $\mu_X,\mu_Y,\sigma_X^2,\sigma_Y^2$, and σ_{XY} with the vector of unknown parameters $\mathbf{\Theta}_2$.

The 3-dimensional square sub-matrix $\boldsymbol{\Sigma}_{\Xi}^{11}$ corresponds to the approximate covariance structure between the m.l.e. of the parameters α_0, α_1, and σ_ϵ^2. The (3×5) sub-matrix $\boldsymbol{\Sigma}_{\Xi}^{12}$ is composed of the approximate covariances between the m.l.e. that have just been cited and $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$, and σ_{XY}; its transpose is denoted by $\boldsymbol{\Sigma}_{\Xi}^{21}$. This is the reason why $\boldsymbol{\Sigma}_{\Xi}^{12}$, or $\boldsymbol{\Sigma}_{\Xi}^{21}$, results from the product of the variance-covariance matrix $\mathbf{I}_1^{-1}$ and $\mathbf{G}$. The 5-dimensional square sub-matrix $\boldsymbol{\Sigma}_{\Xi}^{22}$ is formed by the covariances between the m.l.e. for $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$, and σ_{XY}.

The main point of the section is to study the variances and covariances that take part of the sub-matrix $\boldsymbol{\Sigma}_{\Xi}^{22}$. Thus, it is of interest to further explore its analytical expression. The matrix $\mathbf{G}$ takes a cumbersome form. The most efficient way to deal with it is to consider its partition rather than the whole matrix at once.

Let

$$\mathbf{G} = \mathbf{F}_3 + \mathbf{F}_4\mathbf{D}_3 = \left[\begin{array}{c|c} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \hline \mathbf{G}_{21} & G_{22} \end{array}\right], \tag{49}$$

where the (4×2) sub-matrix $\mathbf{G}_{11}$ takes the form

$$\mathbf{G}_{11} = \begin{bmatrix} \dfrac{1}{1-\alpha_1} & \dfrac{\alpha_0}{(1-\alpha_1)^2} \\ \dfrac{\psi_1}{1-\beta_1} + \dfrac{\beta_2}{(1-\alpha_1)(1-\beta_1)} & \dfrac{\alpha_0}{(1-\alpha_1)(1-\beta_1)}\left(\psi_1 + \dfrac{\beta_2}{1-\alpha_1}\right) \\ 0 & \dfrac{2\alpha_1\sigma_\epsilon^2}{(1-\alpha_1^2)^2} \\ 0 & g_{42}^{11} \end{bmatrix}, \tag{50}$$

with

$$\begin{aligned} g_{42}^{11} = {} & \psi_1 f_{43}^4 + \frac{2\beta_2}{(1-\beta_1^2)(1-\alpha_1\beta_1)^2(1-\alpha_1^2)^2} \\ & \times \left(\sigma_{\epsilon\xi}\beta_1^2\left(1-\alpha_1^2\right)^2 + \sigma_\epsilon^2\,\beta_2\left(\beta_1\left(1-\alpha_1^2\right)\right.\right. \\ & \left.\left. + \alpha_1\left(1-\alpha_1^2\beta_1^2\right)\right)\right), \end{aligned} \tag{51}$$

where f_{43}^4 is defined by (45).

The 4-dimensional column vector $\mathbf{G}_{12}$, the 2-dimensional row vector $\mathbf{G}_{21}$ and the scalar G_{22} are, respectively, given by

$$\begin{aligned} \mathbf{G}_{12} = {} & \left[\begin{matrix} 0 & 0 & \dfrac{1}{1-\alpha_1^2} & \dfrac{\psi_1^2}{1-\beta_1^2} + \dfrac{\beta_2}{(1-\alpha_1\beta_1)(1-\beta_1^2)} \end{matrix}\right. \\ & \left. \times\left(2\psi_1\beta_1 + \frac{\beta_2(1+\alpha_1\beta_1)}{1-\alpha_1^2}\right)\right]', \\ \mathbf{G}_{21} = {} & \left[\begin{matrix} 0 & \dfrac{1}{(1-\alpha_1\beta_1)} \end{matrix}\right. \\ & \times\left\{\frac{\sigma_{\epsilon\xi}\beta_1}{1-\alpha_1\beta_1} + \frac{\sigma_\epsilon^2}{1-\alpha_1^2}\right. \\ & \left.\left. \times\left(\psi_1\alpha_1 + \frac{\beta_2\left(1+\alpha_1^2\left(1-2\alpha_1\beta_1\right)\right)}{(1-\alpha_1^2)(1-\alpha_1\beta_1)}\right)\right\}\right], \\ G_{22} = {} & \frac{1}{1-\alpha_1\beta_1}\left(\psi_1 + \frac{\alpha_1\beta_2}{1-\alpha_1^2}\right). \end{aligned} \tag{52}$$

On the other hand, we can also make the following partition of the matrix $\mathbf{H} = \mathbf{F}_4\mathbf{D}_4$:

$$\mathbf{H} = \left[\begin{array}{c|c} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \hline \mathbf{H}_{21} & H_{22} \end{array}\right], \tag{53}$$

where the sub-matrix $\mathbf{H}_{11}$ corresponds to the first order partial derivatives of the vector $[\mu_X \;\; \mu_Y \;\; \sigma_X^2 \;\; \sigma_Y^2]'$ with respect to the vector $[\psi_0 \;\; \psi_1 \;\; \psi_2 \;\; \psi_4]'$, whereas their derivatives with respect to the parameter ψ_3 constitute the sub-matrix $\mathbf{H}_{12}$. The sub-matrix $\mathbf{H}_{21}$ is composed of the first order partial derivatives of σ_{XY} with respect to each component of the vector $[\psi_0 \;\; \psi_1 \;\; \psi_2 \;\; \psi_4]'$. Finally, the scalar $H_{22} = \partial\sigma_{XY}/\partial\psi_3 = 0$.

The desired variance-covariance matrix can therefore be written in the following partitioned form:

$$\boldsymbol{\Sigma}_{\Xi}^{22} = \left[\begin{array}{c|c} \boldsymbol{\Sigma}_1^{22} & \boldsymbol{\Sigma}_2^{22} \\ \hline \boldsymbol{\Sigma}_3^{22} & \boldsymbol{\Sigma}_4^{22} \end{array}\right], \tag{54}$$

with

$$\begin{aligned} \boldsymbol{\Sigma}_1^{22} = {} & \widehat{\sigma}_\epsilon^2\mathbf{G}_{11}\left(\mathbf{U}_R'\mathbf{U}_R\right)^{-1}\mathbf{G}_{11}' + \frac{2\widehat{\sigma}_\epsilon^4}{n-1}\mathbf{G}_{12}\mathbf{G}_{12}' \\ & + \psi_1\mathbf{H}_{11}\left(\mathbf{U}'\mathbf{U}\right)^{-1}\mathbf{H}_{11}' \\ & + \frac{2\psi_3^2}{m-1}\,\mathbf{H}_{12}\mathbf{H}_{12}', \\ \boldsymbol{\Sigma}_2^{22} = {} & \widehat{\sigma}_\epsilon^2\mathbf{G}_{11}\left(\mathbf{U}_R'\mathbf{U}_R\right)^{-1}\mathbf{G}_{21}' + \frac{2\widehat{\sigma}_\epsilon^4}{n-1}\mathbf{G}_{12}\mathbf{G}_{22}' \\ & + \psi_3\mathbf{H}_{11}\left(\mathbf{U}'\mathbf{U}\right)^{-1}\mathbf{H}_{21}', \end{aligned}$$

$$\begin{aligned}\mathbf{\Sigma}_3^{22} &= \hat{\sigma}_\epsilon^2 \mathbf{G}_{21}\left(\mathbf{U}_R'\mathbf{U}_R\right)^{-1}\mathbf{G}_{11}' \\ &\quad + \frac{2\hat{\sigma}_\epsilon^4}{n-1}\,\mathbf{G}_{22}\mathbf{G}_{12}' + \psi_3\mathbf{H}_{21}\left(\mathbf{U}'\mathbf{U}\right)^{-1}\mathbf{H}_{11}', \\ \mathbf{\Sigma}_4^{22} &= \hat{\sigma}_\epsilon^2 \mathbf{G}_{21}\left(\mathbf{U}_R'\mathbf{U}_R\right)^{-1}\mathbf{G}_{21}' \\ &\quad + \frac{2\hat{\sigma}_\epsilon^4}{n-1}\,\mathbf{G}_{22}\mathbf{G}_{22}' + \psi_3\mathbf{H}_{21}\left(\mathbf{U}'\mathbf{U}\right)^{-1}\mathbf{H}_{21}',\end{aligned} \tag{55}$$

where the matrix $\mathbf{U}$ is defined by (35). The matrix $\mathbf{U}_R$ takes the form

$$\mathbf{U}_R = \begin{bmatrix} 1 & X_0 \\ 1 & X_1 \\ \vdots & \vdots \\ 1 & X_{m-2} \end{bmatrix}. \tag{56}$$

In short, the matrix defined by (54) corresponds to the approximate variance-covariance matrix of the m.l.e. for the mean vector and variance-covariance matrix at lag zero for the VAR(1) Model with missing data. We cannot write down explicit expressions for those variances and covariances. The limitation arises from the inability to invert the matrix product $\mathbf{U}'\mathbf{U}$ in analytical terms (see (36)). Hence, its inverse can only be accomplished by numerical techniques using the observed sampled data. This point will be pursued further in Section 5.

Despite the above restrictions, several investigations can be done regarding the amount of additional information obtained by making full use of the fragmentary data available. The strength of the correlation between the stochastic processes here plays a crucial role. These ideas will be developed in Section 5.

5. Simulation Studies

In this section, we analyse the effects of using different strategies to estimate the mean value of the stochastic process $\{Y_t, t \in \mathbb{Z}\}$, denoted by μ_Y. More precisely, the bivariate modelling scheme and its univariate counterparts are compared. Simulation studies are carried out to evaluate the relative efficiency of the estimators with interest.

The m.l.e. of the mean value of the stochastic process $\{Y_t, t \in \mathbb{Z}\}$ based on the VAR(1) Model is obtained by the second equation of the system of (28). We need to compare this estimator to those obtained by considering the univariate stochastic process $\{Y_t, t \in \mathbb{Z}\}$ itself. More precisely, having in mind that we are handling a bivariate VAR(1) Model, the corresponding marginal model is the ARMA(2,1) [10, 11]. On the other hand, the AR(1) Model is one of the most popular models due to its practical importance in time series modelling. Therefore, the behaviour of the AR(1) Model will be also evaluated. In short, we will compare the performance of the VAR(1) Model with both the ARMA(2,1) and the AR(1) Models.

To avoid any confusion between the parameters coming from the bivariate and the univariate modelling strategies, from now on we denote the parameter from the VAR(1) Model by μ_{VAR}, whereas those from the ARMA(2,1) and the AR(1) Models are represented by μ_{ARMA} and μ_{AR}, respectively.

The bivariate VAR(1) Model is described by the system of (1). Thus, the univariate stochastic process $\{Y_t, t \in \mathbb{Z}\}$ follows an ARMA(2,1) Model, and the m.l.e. of the mean value are given by

$$\hat{\mu}_{\text{ARMA}} = \frac{\hat{\beta}_0}{1 - \hat{\alpha}_1\left(1 - \hat{\beta}_1\right) - \hat{\beta}_1}. \tag{57}$$

On the other hand, if we assumed that $\{Y_t, t \in \mathbb{Z}\}$ followed an AR(1) Model, the m.l.e. of the mean value would be given by

$$\hat{\mu}_{\text{AR}} = \frac{\hat{\beta}_0}{1 - \hat{\beta}_1}. \tag{58}$$

Next, we will compare the performance of the estimators (57) and (58) with the m.l.e. based on the VAR(1) Model (second equation of the system (28)). It is important to stress that the strategy behind the AR(1) Model has not taken into account the relationship between the stochastic processes $\{X_t, t \in \mathbb{Z}\}$ and $\{Y_t, t \in \mathbb{Z}\}$. This feature will certainly introduce an additional noise in the overall estimation procedure.

Following the techniques used in Section 4.2 for determining the precision of the estimators under consideration, here we also have used the first-order Taylor expansion at the mean value μ_Y for computing the estimate of the variance of μ_Y.

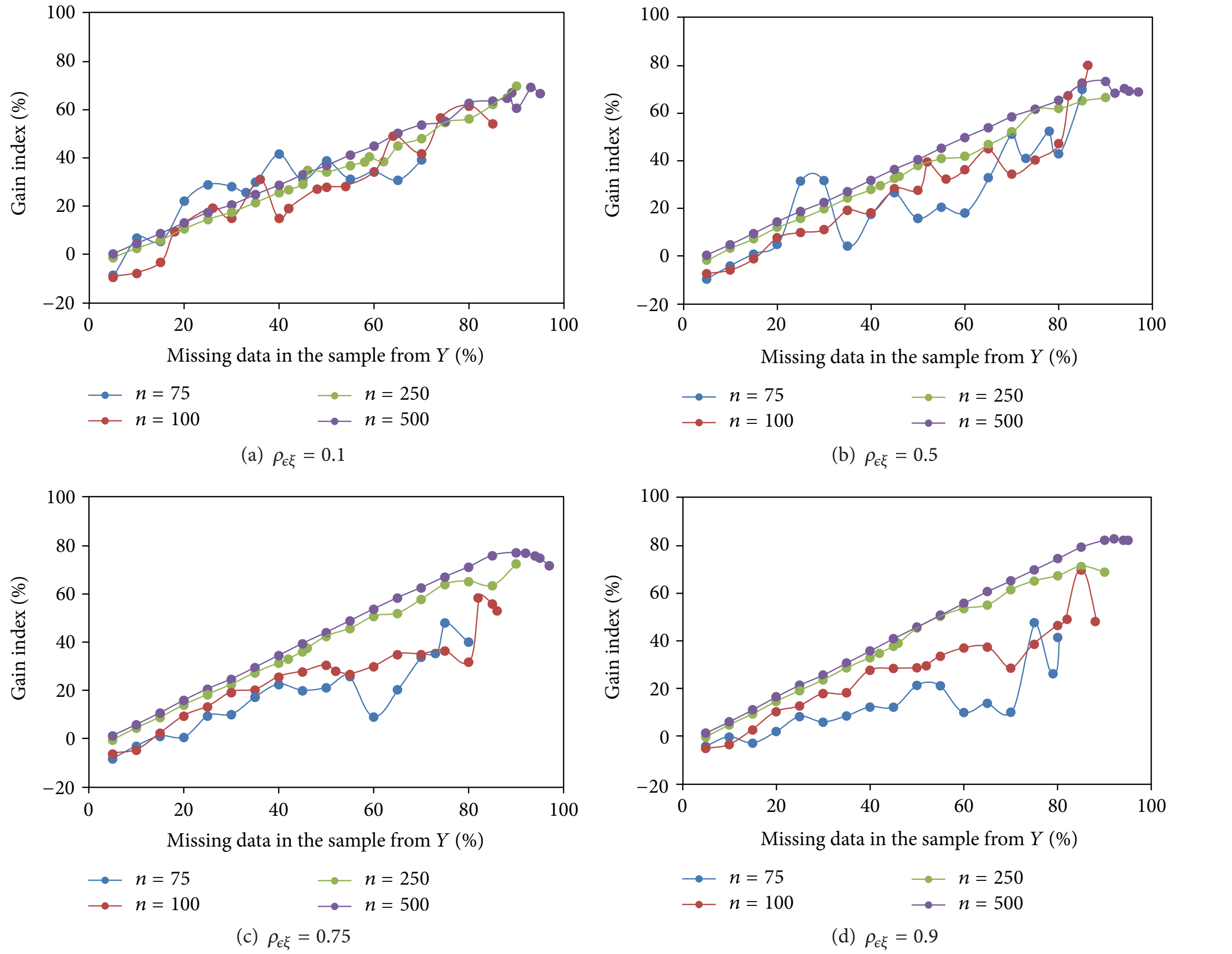
Considering the ARMA(2,1) Model, let $\boldsymbol{\theta} = [\beta_0\ \beta_1\ \alpha_1]'$ be the vector of the unknown parameters. Then,

$$\begin{aligned}\operatorname{Var}\left(\hat{\mu}_{\text{ARMA}}\right) &\approx \sum_{i=1}^{3}\left(\frac{\partial\hat{\mu}_{\text{ARMA}}}{\partial\theta_i}\Big|_{\theta_i=\hat{\theta}_i}\right)^2 \operatorname{Var}\left(\hat{\theta}_i\right) \\ &\quad + 2\sum_{i=1}^{3}\sum_{j=i+1}^{3}\frac{\partial\hat{\mu}_{\text{ARMA}}}{\partial\theta_i}\bigg|_{\theta_i=\hat{\theta}_i} \\ &\quad \times \frac{\partial\hat{\mu}_{\text{ARMA}}}{\partial\theta_j}\bigg|_{\theta_j=\hat{\theta}_j}\operatorname{Cov}\left(\hat{\theta}_i, \hat{\theta}_j\right).\end{aligned} \tag{59}$$

In regard to the AR(1) Model, $\hat{\mu}_{\text{AR}}$ is given by (58) and

$$\begin{aligned}\operatorname{Var}\left(\hat{\mu}_{\text{AR}}\right) &\approx \frac{2\hat{\beta}_0}{\left(1-\hat{\beta}_1\right)^3}\operatorname{Cov}\left(\hat{\beta}_0, \hat{\beta}_1\right) \\ &\quad + \frac{\operatorname{Var}\left(\hat{\beta}_0\right)}{\left(1-\hat{\beta}_1\right)^2} + \frac{\hat{\beta}_0^2\operatorname{Var}\left(\hat{\beta}_1\right)}{\left(1-\hat{\beta}_1\right)^4}.\end{aligned} \tag{60}$$

Improvements in choosing the sophisticated m.l.e. for μ_Y based on the VAR(1) Model rather than considering



(a) $\rho_{\epsilon\xi} = 0.1$

(b) $\rho_{\epsilon\xi} = 0.5$

(c) $\rho_{\epsilon\xi} = 0.75$

(d) $\rho_{\epsilon\xi} = 0.9$

Figure 1: Graphical representation of GI_1. The data were obtained from a VAR(1) Model, with $\alpha_0 = \beta_0 = 0$, $\alpha_1 = 0.6$, $\beta_1 = 0.7$, and $\beta_2 = 0.8$.

its univariate counterparts are next discussed. Simulation studies are carried out to evaluate the relative efficiency of the estimators under consideration.

The data were generated by the VAR(1) Model (system of (1)). In order to make comparisons on the same basis, a few assumptions to the parameters of the VAR(1) Model are made. We consider that $\mu_X = \mu_Y = 0$. These restrictions have no influence on the results because they are equivalent to $\alpha_0 = \beta_0 = 0$, that is, the constant terms of the VAR(1) Model are equal to zero (system of (1)). Additionally, we introduce the restriction $\sigma_\epsilon^2 = \sigma_\xi^2 = 1$.

Since the correlation coefficient regulates the supply of information between the stochastic processes $\{X_t\}_{t\in\mathbb{Z}}$ and $\{Y_t\}_{t\in\mathbb{Z}}$, particular emphasis is given to this parameter. Using the grid of points $\rho_{\epsilon\xi} = 0.1, 0.5, 0.75, 0.9$, the Gain index is computed. We stress that the value $\rho_{\epsilon\xi} = 1$ is not allowable in this context (see Section 2 for the details).

We analyse the performance of the estimators based on different sample sizes, $n = 50, 100, 250$, and 500. The simulations reported next are based on different percentages of missing observations referred to the dimension of the sampled data from the auxiliary random process $\{X_t\}_{t\in\mathbb{Z}}$. Simulation runs for each combination of the parameters are based on 1000 replicates.

It is worth emphasising that the estimates of the covariance terms that take part of the variances given by (59) and (60) were computed by the R package tseries [15].

The simulation goes as follows: after each simulation run, the relative efficiency of $\widehat{\mu}_{\mathrm{VAR}}$ with respect to each estimator $\widehat{\mu}_{\mathrm{ARMA}}$ and $\widehat{\mu}_{\mathrm{AR}}$ is quantified by the Gain index, GI_1 and GI_2, respectively, expressed as percentage:

$$\mathrm{GI}_1 = \frac{\mathrm{Var}(\widehat{\mu}_{\mathrm{ARMA}}) - \mathrm{Var}(\widehat{\mu}_{\mathrm{VAR}})}{\mathrm{Var}(\widehat{\mu}_{\mathrm{ARMA}})} \times 100\%,$$
$$\mathrm{GI}_2 = \frac{\mathrm{Var}(\widehat{\mu}_{\mathrm{AR}}) - \mathrm{Var}(\widehat{\mu}_{\mathrm{VAR}})}{\mathrm{Var}(\widehat{\mu}_{\mathrm{AR}})} \times 100\%. \quad (61)$$

A word of notation: the above quantities, that is, GI_1 and GI_2, were computed from the estimates of the corresponding variances. To lighten the notation, we skipped the conventional nomenclature used to represent the estimates.

If $\mathrm{GI}_1 > 0$, then $\widehat{\mu}_{\mathrm{VAR}}$ is more precise than $\widehat{\mu}_{\mathrm{ARMA}}$. Otherwise, $\widehat{\mu}_{\mathrm{VAR}}$ loses precision, and $\widehat{\mu}_{\mathrm{ARMA}}$ becomes a better

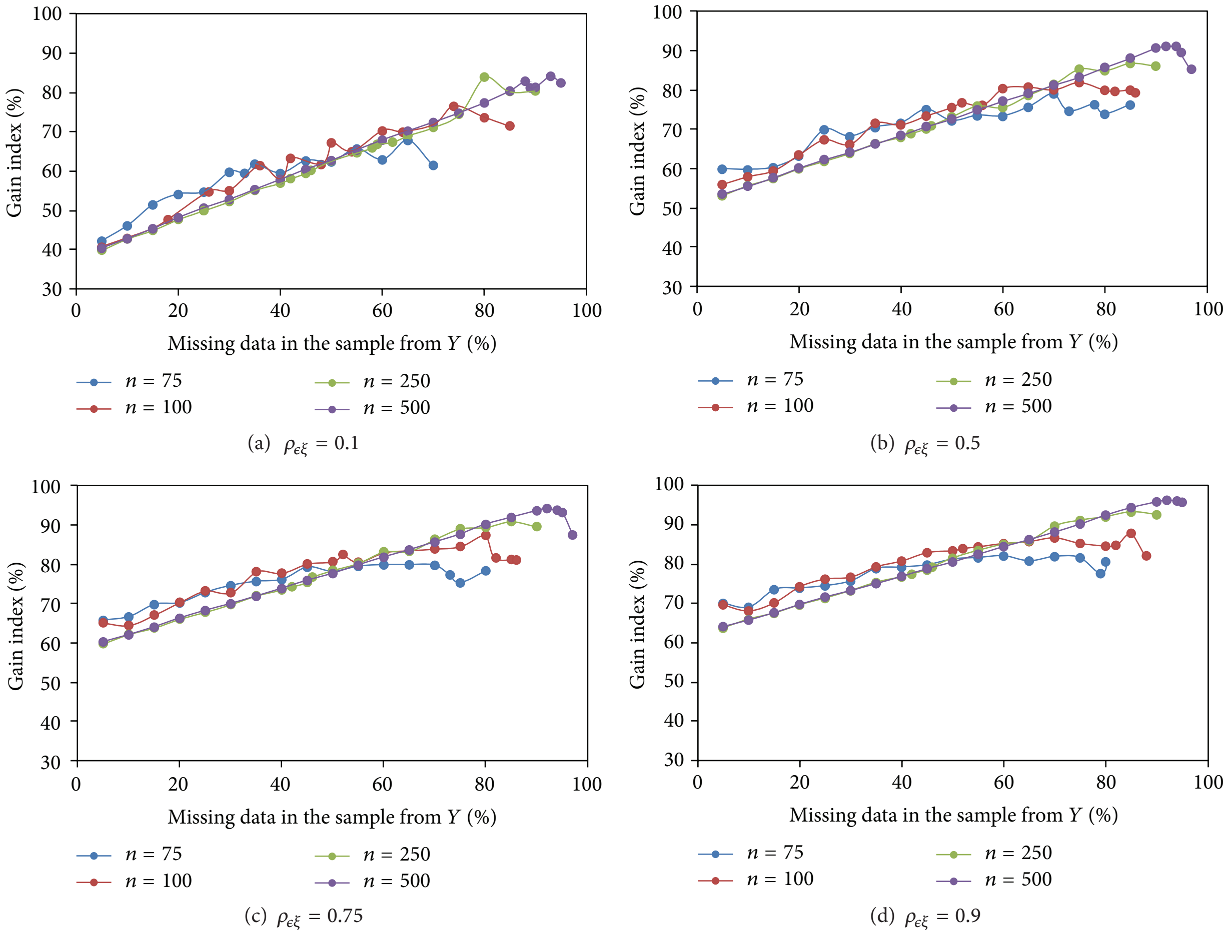


Figure 2: Graphical representation of GI_2. The data were obtained from a VAR(1) Model, with $\alpha_0 = \beta_0 = 0, \alpha_1 = 0.6, \beta_1 = 0.7$, and $\beta_2 = 0.8$.

estimator for the mean value of $\{Y_t\}_{t\in\mathbb{Z}}$. A similar reasoning applies to the comparison between $\widehat{\mu}_{\mathrm{VAR}}$ and $\widehat{\mu}_{\mathrm{AR}}$.

Figures 1 and 2 display the main results from the simulation studies. The estimators $\widehat{\mu}_{\mathrm{VAR}}$ and $\widehat{\mu}_{\mathrm{ARMA}}$ are compared in Figure 1, whereas Figure 2 exhibits the comparison between $\widehat{\mu}_{\mathrm{VAR}}$ and $\widehat{\mu}_{\mathrm{AR}}$. For each combination of the parameters of the model, we represent graphically the gain indexes as functions of the percentage of missing data in the sampled data from the stochastic process $\{Y_t\}_{t\in\mathbb{Z}}$.

Either Figures 1 or 2 shows that the plot of the gain index against the percentage of missing data in the sample from the stochastic process $\{Y_t\}_{t\in\mathbb{Z}}$ behaves roughly as a linear function, regardless of the combination of the parameters. In outline, the more the percentage of missing values in the sampled data is, the more precise is the estimator $\widehat{\mu}_{\mathrm{VAR}}$ when compared with the univariate context, that is, $\widehat{\mu}_{\mathrm{ARMA}}$ or $\widehat{\mu}_{\mathrm{AR}}$ (see Figures 1 and 2).

Further, the gain in precision by using the sophisticated estimator $\widehat{\mu}_{\mathrm{VAR}}$ rather than $\widehat{\mu}_{\mathrm{ARMA}}$ or $\widehat{\mu}_{\mathrm{AR}}$ increases as the strength of the linear relationship between the processes $\{X_t\}_{t\in\mathbb{Z}}$ and $\{Y_t\}_{t\in\mathbb{Z}}$ (described by the correlation coefficient) rises from $\rho = 0.1$ to $\rho = 0.9$. This statement is true for both the ARMA(2,1) and AR(1) modelling schemes (see Figures 1 and 2).

A final point to highlight from the comparison between Figures 1 and 2 is that the increase in precision obtained by using the estimator for the mean value of $\{Y_t\}_{t\in\mathbb{Z}}$ based on the VAR(1) Model is higher when we compare its performance with the results from the AR(1) Model than when we compare the VAR(1) Model with the ARMA(2,1) Model. This feature emphasises the idea that has already been raised that the ARMA(2,1) Model describes more accurately the dynamics of the stochastic process $\{Y_t\}_{t\in\mathbb{Z}}$ than the AR(1) Model does. In short, it seems that the AR(1) Model is not a good approach in this context because it incorporates a noise term related to the simulation scheme that we cannot control.

Summing up, the estimator $\widehat{\mu}_{\mathrm{VAR}}$ is preferable to those explored in the univariate context, that is, either $\widehat{\mu}_{\mathrm{ARMA}}$ or $\widehat{\mu}_{\mathrm{AR}}$.

6. Conclusions

This article deals with the problem of missing data in an univariate sample. We have considered an auxiliary complete data set, whose underlying stochastic process is serially correlated with the former by the VAR(1) Model structure. We have proposed maximum likelihood estimators for the relevant parameters of the model based on a monotone

missing data pattern. The precision of the estimators has also been derived. Special attention has been given to the estimator for the mean value of the stochastic process whose sampled data has missing values, μ_Y.

We have compared the performance of the estimator for μ_Y based on the VAR(1) Model with a monotone pattern of missing data with those obtained from both the ARMA(2,1) Model and the AR(1) Model. By simulation studies, we have showed that the estimator derived in this article based on the VAR(1) Model performs better than those derived from the univariate context. It is essential to emphasise that, even numerically, it was quite difficult to compute the precision of the later estimators as we have shown in Section 4.2.

A compelling question remains unresolved. From an applied point of view, it would be extremely useful to develop estimators for the dynamics of the stochastic processes. More precisely, we would like to get estimators for the correlation and cross-correlation matrices as well as their precision when there are missing observations in one of the data sets. It was not possible to achieve this goal based on maximum likelihood principles. As we have shown in Section 4.2, we have only developed estimators for the covariance matrix at lag zero. In future research, we will try to solve this problem in the framework of Kalman filter.

Acknowledgments

This work was financed by the Portuguese Foundation for Science and Technology (FCT), Projecto Estratégico PEst-OE/MAT/UI0209/2011. The authors are also thankful for the comments of the two anonymous referees.

References

[1] D. F. Morrison, "Expectations and variances of maximum likelihood estimates of the multivariate normal distribution parameters with missing data," *Journal of the American Statistical Association*, vol. 66, no. 335, pp. 602–604, 1971.

[2] R. C. Dahiya and R. M. Korwar, "Maximum likelihood estimates for a bivariate normal distribution with missing data," *The Annals of Statistics*, vol. 8, no. 3, pp. 687–692, 1980.

[3] V. Gómez and A. Maravall, "Estimation, prediction, and interpolation for nonstationary series with the Kalman filter," *Journal of the American Statistical Association*, vol. 89, no. 426, pp. 611–624, 1994.

[4] R. H. Jones, "Maximum likelihood fitting of ARMA models to time series with missing observations," *Technometrics*, vol. 22, no. 3, pp. 389–395, 1980.

[5] R. Kohn and C. F. Ansley, "Estimation, prediction, and interpolation for ARIMA models with missing data," *Journal of the American Statistical Association*, vol. 81, no. 395, pp. 751–761, 1986.

[6] M. Pourahmadi, "Estimation and interpolation of missing values of a stationary time series," *Journal of Time Series Analysis*, vol. 10, no. 2, pp. 149–169, 1989.

[7] V. Gómez, A. Maravall, and D. Peña, "Missing observations in ARIMA models: skipping approach versus additive outlier approach," *Journal of Econometrics*, vol. 88, no. 2, pp. 341–363, 1999.

[8] R. J. A. Little and D. B. Rubin, *Statistical Analysis with Missing Data*, John Wiley & Sons, New York, NY, USA, 1987.

[9] R. Sparks, "SUR models applied to an environmental situation with missing data and censored values," *Journal of Applied Mathematics and Decision Sciences*, vol. 8, no. 1, pp. 15–32, 2004.

[10] C. Heij, P. De Boer, P. H. Franses, T. Kloek, and H. K. van Dijk, *Econometric Methods with Applications in Business and Economics*, Oxford University Press, New York, NY, USA, 2004.

[11] R. S. Tsay, *Analysis of Financial Time Series*, John Wiley & Sons, Hoboken, NJ, USA, 3rd edition, 2010.

[12] M. H. Nunes, *Dynamics relating phytoplankton abundance with upwelling events. An approach to the problem of missing data in the gaussian context [Ph.D. thesis]*, University of Lisbon, Lisbon, Portugal, 2006.

[13] J. D. Hamilton, *Time Series Analysis*, Princeton University Press, Princeton, NJ, USA, 1994.

[14] T. M. Apostol, *Calculus*, John Wiley & Sons, Singapore, 2nd edition, 1969.

[15] A. Trapletti and K. Hornik, "tseries: Time series analysis and computational finance," R package version 0.10-25, 2011, http://CRAN.R-project.org/package=tseries.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 312585, 13 pages
http://dx.doi.org/10.1155/2013/312585

Research Article

Governance Mechanism for Global Greenhouse Gas Emissions: A Stochastic Differential Game Approach

Wei Yu[1,2] and Baogui Xin[1,3]

[1] *School of Management, Tianjin University, Tianjin 300072, China*
[2] *Department of Mathematics, Tianjin Polytechnic University, Tianjin 300387, China*
[3] *School of Economics and Management, Shandong University of Science and Technology, Qingdao 266590, China*

Correspondence should be addressed to Baogui Xin; xin@tju.edu.cn

Received 31 January 2013; Revised 2 May 2013; Accepted 3 May 2013

Academic Editor: Wuquan Li

Today developed and developing countries have to admit the fact that global warming is affecting the earth, but the fundamental problem of how to divide up necessary greenhouse gas reductions between developed and developing countries remains. In this paper, we propose cooperative and noncooperative stochastic differential game models to describe greenhouse gas emissions decision makings of developed and developing countries, calculate their feedback Nash equilibrium and the Pareto optimal solution, characterize parameter spaces that developed and developing countries can cooperate, design cooperative conditions under which participants buy the cooperative payoff, and distribute the cooperative payoff with Nash bargaining solution. Lastly, numerical simulations are employed to illustrate the above results.

1. Introduction

There seems to be rather compelling evidence that global warming is an issue that we seriously need to be concerned about today. One of the important facts on global warming is the result of greenhouse gases (GHGs) trapping heat and making the planet warmer. A GHG is one of several gases in an atmosphere that absorbs and emits infrared radiation in a planetary atmosphere. Many important GHGs are naturally occurring, such as water vapor, carbon dioxide, methane, nitrous oxide, and ozone, but others can also be added to the atmosphere by human activities, such as hydrofluorocarbons, perfluorocarbons, and sulfur hexafluoride. The United Nation's Intergovernmental Panel on Climate Change (IPCC) claims that the only way they can get their computerized climate models to produce the observed warming is with anthropogenic pollution. Undoubtedly, the problem of global warming has been arising for more than 150 years and will get worse as time goes on.

According to CDIAC, top 20 emitting countries by total fossil-fuel CO_2 emissions for 2009 were China, the United States, India, the Russian Federation, Japan, Germany, Iran, Canada, South Korea, South Africa, United Kingdom, Indonesia, Mexico, Saudi Arabia, Italy, Australia, Brazil, France, Poland, and Spain. Under the Kyoto Protocol of 2010, 37 developed countries and European Community commit themselves to a reduction of GHGs in the period of 2008–2012. The United States signed but did not ratify the protocol, and Canada withdrew from it in 2011. To respond to the extension of the Kyoto Protocol beyond 2012, several developed countries have communicated their intentions to set quantified economy-wide emission reduction targets up to 2020. From China's newest "12th Five-year Plan on Greenhouse Emission Control (Guofa [2011] No. 41)," China aims to reduce the carbon intensity by 17 percent by 2015 compared with 2010 levels, though it is still a developing country.

A lot of researchers have paid much attention to the problem of GHG emission reduction without game theory. Williams et al. [1] proposed a technology path to deep GHG emissions cuts by 2050. Bastianoni et al. [2] analyzed the problem of assigning responsibility for GHG emissions.

Parton et al. [3] studied agriculture's role in cutting GHG emissions. Tol [4] presented the marginal costs of GHG emissions. Ansuategi and Escapa [5] analyzed the relationship between economic growth and GHG emissions.

Other researchers have employed game theory to study the pollution management, including GHG emission reduction. Van der Ploeg and de Zeeuw studied noncooperative and cooperative pollution strategies and outcomes in a transboundary pollution in [6, 7]. Long [8] analyzed a two-player transboundary differential game. Rubio and Casino [9] show that it is necessary that the initial value of the stock of pollution to be higher than the cooperative solution stock. Yeung [10] presented a cooperative game of transboundary industrial pollution. Bahn and Haurie [11] designed equilibrium solutions with coupled constraints in dynamic games of greenhouse gas emissions abatement. Jørgensen et al. [12–14] designed a cooperative agreement which was time consistent, at no instant of time, no player or group of players wishes to defect from the agreement. Smala Fanokoa et al. [15] considered a two-player asymmetric differential game of pollution control. Breton et al. [16–18] proposed the differential game models to analyze joint implementation in environmental projects, one of the flexible mechanisms considered in the Kyoto Protocol. Petrosjan and Zaccour [19] studied time-consistent Shapley value allocation of pollution cost reduction. Dockner et al. [20] used differential games into economics and management science, including pollution management.

In fact, stock of GHGs not only depends on the amounts of GHG emission or natural absorption, but also is affected by the other stochastic factors, such as random climate, natural disaster, and man-made factor. So the stochastic differential game model should be more appropriate than the deterministic differential game model to describe the above problems.

In recent years, stochastic theory has been playing an important role in the study of in science, engineering, and sociology, such as multiobjective stochastic production-distribution planning problem [21], stochastic stabilization and control [22–28], market portfolio problem in a jump diffusion market [29], partial information problem [30], dynamic stochastic route choice model [31], stochastic heat dynamics systems [32], and stochastic Fokker-Planck-type problems [33]. In particular, the stochastic differential game has been successfully applied into the economics and management. For example, Leong and Huang [34] developed a stochastic differential game of capitalism to analyze the role of uncertainty. Yeung and Petrosyan [35] presented a cooperative stochastic differential game of transboundary industrial pollution with two novel features and analyzed the pollution management in the framework. Wang and Ewald [36] developed a stochastic differential fishery game for a two-species fish population with ecological interaction. Taksar and Zeng [37] studied stochastic differential games between two insurance companies who employ reinsurance to reduce risk exposure.

The remainder of this paper is organized as follows. In Section 2, we propose cooperative and noncooperative stochastic differential game models to describe greenhouse gas emissions decision makings of developed and developing countries. In Section 3, we calculate feedback Nash equilibrium and the Pareto optimal solution, characterize parameter spaces that developed and developing countries can cooperate, and design cooperative conditions under which participants buy the cooperative payoff. In Section 4, numerical simulations are employed to illustrate the above results. In Section 5, we study the assign the cooperative payoff with Nash bargaining solution. Finally, conclusions in Section 6 close the paper.

2. Stochastic Differential Game Models

For the sake of simplicity, global countries emitting GHGs can be divided into two interest groups: developing countries and developed countries, which are, respectively, denoted by player 1 and player 2. Inspired by [8, 15], we assume the two participants have asymmetric environmental behaviors, which mean player 1 is not affected by the pollution, while player 2 is affected by the pollution, or player 1 has no interest in sacrificing economic growth to reduce GHGs emission, whereas player 2 would like to take necessary measures to protect and improve the environment through GHG emission reduction. In order to clarify the above problem, we further assume the following.

(1) $q_i(t)$ is the output of player i ($i = 1, 2$) with $t \in [0, \infty)$. $e_i(t) = h_i(q_i(t))$ denotes the quantity of GHG emission of player i. $r_i(q_i(t))$ represents net revenue of player i with $q_i(t)$.

(2) $h_i(q_i(t))$ is a strictly increasing function, that is, $(d/dq)(h_i(q_i(t))) > 0$. So we can denote $r_i(q_i(t)) = r_i(h_i^{-1}(e_i(t)))$ and $R_i(e_i(t)) = r_i(h_i^{-1}(e_i(t)))$.

(3) $R_i(e_i(t))$ is a monotonically increasing concave function of $e_i(t)$, that is, $R_i'(e_i) \geq 0$, $R_i''(e_i) \geq 0$, $R_i(0) = 0$. $s(t)$ is the stock of GHGs. $D_i(s(t))$ is a monotone increasing convex function of $s(t)$, which is the cost of player i caused by the stock of GHGs $s(t)$.

(4) $\mu \geq 0$ is a constant that means the marginal impact on GHG emission. $\varepsilon > 0$ is a natural absorption rate. Without loss of generality, we assume that the marginal impact of GHG emission on both players is the same. $\sigma \geq 0$ is a constant affected by the weather, natural disasters, human factors, unpredictable factors, and so on. $B(t)$ is a Wiener process.

(5) Player i will, even if alone, maximize its expected payoff in case no agreement is reached between both players.

According to the above assumptions, we get cooperative and noncooperative stochastic differential game models for asymmetric GHG emission problems over an infinite horizon.

2.1. Noncooperative Stochastic Differential Game Model. Consider

$$ds(t) = (\mu(e_1(t) + e_2(t)) - \varepsilon s(t))\,dt + \sigma s(t)\,dB(t),\quad s(0) = s_0 \tag{1}$$

$$J_i = \max_{e_i} E\left\{\int_0^\infty e^{-rt}\left(R_i(e_i(t)) - D_i(s(t))\right)dt\right\},\quad i = 1, 2, \tag{2}$$

where s_0 is the initial GHGs stock with $t = 0$, r is the discount rate, and e^{-rt} is the discount factor.

Without consensus agreement between both players, they will, alone, maximize their payoffs as follows:

$$J_1 = \max_{e_1} E\left(\int_0^\infty e^{-rt} R_1(e_1(t))\,dt\right), \tag{3}$$

$$J_2 = \max_{e_2} E\left\{\int_0^\infty e^{-rt}\left(R_2(e_2(t)) - D_2(s(t))\right)dt\right\}, \tag{4}$$

which mean that player 1 maximizes its payoffs through its industrial activities without internalizing the damage of GHGs emission on the basis of (3), and player 2 maximizes its payoff based on (4).

We use $\Gamma(s_0, \infty - 0)$ to denote the noncooperative stochastic differential game Equations (1) and (3) and (1) and (4). In addition, we use $\Gamma(s_\tau, \infty - \tau)$ to represent the above game with the initial state $s_\tau \in S$ and the time $\tau \in [0, \infty)$.

The rational economic explanation for the noncooperative game is that (3) is the payoff of player 1 and implies its prior strategy of economic development, which means that developing countries will always maximize their economic benefit rather than give environmental protection a high priority.

2.2. Cooperative Stochastic Differential Game Model. In order to reach a cooperative agreement of GHG emission, we adopt a cooperative stochastic differential game to get its Nash equilibrium, that is, a cooperative agreement, which is a solution concept of a cooperative game involving two players, such that no player has an incentive to unilaterally change its GHG emission decision. In other words, players are in an equilibrium if a GHG emission decision change in strategies by any one of them would lead that player to earn less than if it remained with its current GHG emission strategy [38]. Two players' joint payoff is

$$J_1 + J_2 = \max_{e_1, e_2} E\left\{\sum_{i=1}^{2}\int_0^\infty e^{-rt}\left(R_i(e_i(t)) - D_i(s(t))\right)dt\right\}. \tag{5}$$

We use $\Psi(s_0, \infty - 0)$ to denote the cooperative stochastic differential game Equations (1) and (5). In addition, we use $\Psi(s_\tau, \infty - \tau)$ to represent the above game with the initial state $s_\tau \in S$ and the time $\tau \in [0, \infty)$.

3. Resolving Models

In order to reduce the difficulty of mathematical handling, we assume that $R_i(e_i(t))$ is a quadratic function of $e_i(t)$, $i = 1, 2$, and $D_i(s(t))$ is a quadratic function of $s(t)$, and cost parameters $\beta_2 \ge \beta_1 \ge 0$. Consider

$$R_i(e_i(t)) = \alpha_i e_i(t) - \frac{1}{2}e_i(t)^2,\quad i = 1, 2,$$
$$D_i(s(t)) = \frac{1}{2}\beta_i s^2(t),\quad i = 1, 2, \tag{6}$$

$\alpha_i > 0$, $i = 1, 2$ and $\beta_i \ge 0$, $i = 1, 2$ are constants.

3.1. Noncooperative Game

3.1.1. Noncooperative Game Solutions

Definition 1. When the state variable $s(t) = s_t$, the value function $V^{(\tau)i}(t, s_t)$, $i = 1, 2$ of a Nash equilibrium for the noncooperation stochastic differential game $\Gamma(s_\tau, \infty - \tau)$, that is, the current value of the expected payoff for player i, $i = 1, 2$ with the interval $[t, \infty)$ is given by

$$V^{(\tau)i}(t, s_t) = E\left\{\int_t^\infty e^{-r(y-\tau)}\left(R_i(e_i(y)) - D_i(s(y))\right)dy\right\}, \tag{7}$$

for $\tau \in [0, \infty)$, $t \in [\tau, \infty)$, $i = 1, 2$.

Theorem 2 (see [39]). *The strategy $\{\phi_1^*(t), \phi_2^*(t)\}$ is a feedback Nash equilibrium solution to the game $\Gamma(s_0, \infty - 0)$. If there exist continuous functions $V_i(s) : R \to R$, $i = 1, 2$ and continuous derivatives $V_i'(s)$, $V_i''(s)$, $i = 1, 2$ satisfy the following Hamilton-Jacobi-Bellman equation:*

$$rV_i(s) - \frac{\sigma^2 s^2}{2}V_i''(s) = \max_{e_i}\left\{\alpha_i e_i(t) - \frac{1}{2}e_i^2(t) - \frac{1}{2}\beta_i s^2(t) + V_i'(s)\left(\mu\left(e_i(t) + \phi_j^*(t)\right) - \varepsilon s(t)\right)\right\}, \tag{8}$$

where $\phi_j^*(t)$, $j = 1, 2$ is the optimal control of the player j, $j = 1, 2$.

Theorem 3. *If above conditions are satisfied, its feedback Nash equilibrium is*

$$e_1^N(t) = \alpha_1,$$
$$e_2^N = \alpha_2 + \mu V_2'(s) = \alpha_2 + \mu\left(A_2 s + B_2\right), \tag{9}$$

where

$$A_2 = \frac{\left(r + 2\varepsilon - \sigma^2\right) - \sqrt{\left(r + 2\varepsilon - \sigma^2\right)^2 + 4\beta_2\mu^2}}{2\mu^2} < 0,$$
$$B_2 = \frac{(\alpha_1 + \alpha_2)\,\mu A_2}{r + \varepsilon - \mu^2 A_2} < 0. \tag{10}$$

Proof. For player 1,

$$\begin{aligned} rV_1(s) - \frac{\sigma^2 s^2}{2} V_1''(s) &= \max_{e_1} \left\{ \alpha_1 e_1(t) - \frac{1}{2} e_1^2(t) \right. \\ &\left. + V_1'(s)\left(\mu\left(e_1(t) + \phi_2^*(t)\right) - \varepsilon s(t)\right) \right\}. \end{aligned} \quad (11)$$

Differentiating (11) with respect to e_1 and using extreme conditions, we can get

$$e_1^*(t) = \alpha_1 + \mu V_1'(s). \quad (12)$$

For player 2,

$$\begin{aligned} rV_2(s) - \frac{\sigma^2 s^2}{2} V_2''(s) &= \max_{e_2} \left\{ \alpha_2 e_2(t) - \frac{1}{2} e_2^2(t) - \frac{1}{2}\beta_2 s^2(t) \right. \\ &\left. + V_2'(s)\left(\mu\left(e_2(t) + \phi_1^*(t)\right) - \varepsilon s(t)\right) \right\}. \end{aligned} \quad (13)$$

Differentiating (13) with respect to e_2 and using the extreme value theorem, we can get

$$e_2^*(t) = \alpha_2 + \mu V_2'(s). \quad (14)$$

Substituting $e_1^*(t), e_2^*(t)$ to (11),

$$\begin{aligned} rV_1(s) - \frac{\sigma^2 s^2}{2} V_1''(s) &= \alpha_1\left(\alpha_1 + \mu V_1'(s)\right) - \frac{1}{2}\left(\alpha_1 + \mu V_1'(s)\right)^2 \\ &+ V_1'(s)\left(\mu\left(\alpha_1 + \mu V_1'(s) + \alpha_2 + \mu V_2'(s)\right) - \varepsilon s(t)\right), \end{aligned} \quad (15)$$

hence, $V_1(s) = A_1$, $e_1^*(t) = \alpha_1$, $A_1 = \alpha_1^2/2r$.

Substituting $e_1^*(t)$ and $e_2^*(t)$ to (13),

$$\begin{aligned} rV_2(s) - \frac{\sigma^2 s^2}{2} V_2''(s) &= \alpha_2\left(\alpha_2 + \mu V_2'(s)\right) - \frac{1}{2}\left(\alpha_1 + \mu V_1'(s)\right)^2 - \frac{1}{2}\beta_2 s^2 \\ &+ V_2'(s)\left(\mu\left(\alpha_1 + \mu V_1'(s) + \alpha_2 + \mu V_2'(s)\right) - \varepsilon s(t)\right), \end{aligned} \quad (16)$$

hence, $V_2(s) = (1/2)A_2 s^2 + B_2 s + C_2$, $V_2'(s) = A_2 s + B_2$, $V_2''(s) = A_2$.

As [40], we have

$$\begin{aligned} A_2 &= \frac{\left(r + 2\varepsilon - \sigma^2\right) - \sqrt{\left(r + 2\varepsilon - \sigma^2\right)^2 + 4\beta_2\mu^2}}{2\mu^2} < 0, \\ B_2 &= \frac{\left(\alpha_1 + \alpha_2\right)\mu A_2}{r + \varepsilon - \mu^2 A_2} < 0, \\ C_2 &= \frac{\alpha_2^2 + 2\mu\alpha_1 B_2 + 2\mu\alpha_2 B_2 + \mu^2 B_2^2}{2r}, \end{aligned} \quad (17)$$

hence, $e_2^N = \alpha_2 + \mu V_2'(s) = \alpha_2 + \mu(A_2 s + B_2)$. □

Remark 4. (1) The first-order extreme value conditions show that player 2 determines its GHG emissions whose marginal profit $(\alpha_2 - e_2)$ is equal to marginal cost $-\mu V_2'(s) = -\mu(A_2 s + B_2)$. Since $V_2'(s) = A_2 s + B_2 < 0$, if $e_2^N = \alpha_2 + \mu V_2'(s) = \alpha_2 + \mu(A_2 s + B_2) \Leftrightarrow \alpha_2 > -\mu(A_2 s + B_2)$, the feedback Nash equilibrium is interior.

(2) From Theorem 3, one can know that the feedback Nash equilibrium strategy $\{e_1^N(t), e_2^N(t)\}$ possesses the Markov property, because it depends only upon the present state, not on the sequence of events that preceded it.

(3) Theorem 3 shows that optimal strategies of the subgame and the original game are the same at the same time and conditions.

The individual gross payoff in the noncooperative game can be written as

$$\begin{aligned} W_1^N &= V_1(s_0) = \frac{\alpha_1^2}{2r}, \\ W_2^N &= V_2(s_0) = \frac{1}{2}A_2 {s_0}^2 + B_2 s_0 + C_2. \end{aligned} \quad (18)$$

From Definition 1, one can obtain

$$\begin{aligned} V^{(\tau)i}(t, s_t) &= E\left\{\int_t^\infty e^{-r(y-\tau)}\left(R_i\left(e_i(y)\right) - D_i\left(s(y)\right)\right) dy\right\} \\ &= e^{-r(t-\tau)} E\left\{\int_t^\infty e^{-r(y-t)}\left(R_i\left(e_i(y)\right) - D_i\left(s(y)\right)\right) dy\right\} \\ &= e^{-r(t-\tau)} V^{(t)i}(t, s_t). \end{aligned} \quad (19)$$

Remark 5. For player i, $i = 1, 2$ in the noncooperative stochastic differential game $\Gamma(s_\tau, \infty - \tau)$ with the state variable $s(t) = s_t$; the discounted value of the subgame value function is equal to the current value of the expected payoff at the time $t \in [t, \infty)$.

3.1.2. The Expected Stock and Its Limitation

Theorem 6. *The expectation stock and its limit in noncooperative game feedback Nash equilibrium satisfy*

$$\begin{aligned} Es(t) &= s_0 e^{(\mu^2 A_2 - \varepsilon)t} + \frac{\mu\alpha_1 + \mu\alpha_2 + \mu^2 B_2}{\varepsilon - \mu^2 A_2}\left(1 - e^{(\mu^2 A_2 - \varepsilon)t}\right), \\ \lim_{t\to\infty} Es(t) &= \frac{\mu\alpha_1 + \mu\alpha_2 + \mu^2 B_1}{\varepsilon - \mu^2 A_2}. \end{aligned} \quad (20)$$

Proof. Substituting $e_1(t), e_2(t)$ to (1) leads to

$$ds(t) = \left(\mu\alpha_1 + \mu\alpha_2 + \mu^2 B_2 + \left(\mu^2 A_2 - \varepsilon\right)s\right)dt + \sigma s dB(t). \quad (21)$$

Solving the above stochastic differential equation leads to

$$s(t) = s_0 e^{(\mu^2 A_2 - \varepsilon - (1/2)\sigma^2)t + \sigma B(t)} + \int_0^t \left(\mu\alpha_1 + \mu\alpha_2 + \mu^2 B_2\right) \times e^{(\mu^2 A_2 - \varepsilon - (1/2)\sigma^2)(t-\tau) + \sigma(B(t) - B(\tau))} d\tau, \quad (22)$$

whose expectation is

$$\begin{aligned} Es(t) &= E\left\{s_0 e^{(\mu^2 A_2 - \varepsilon - (1/2)\sigma^2)t + \sigma B(t)}\right\} \\ &\quad + E\left\{\int_0^t \left(\mu\alpha_1 + \mu\alpha_2 + \mu^2 B_2\right) \times e^{(\mu^2 A_2 - \varepsilon - (1/2)\sigma^2)(t-\tau) + \sigma(B(t) - B(\tau))} d\tau\right\} \\ &= s_0 e^{(\mu^2 A_2 - \varepsilon)t} + \frac{\mu\alpha_1 + \mu\alpha_2 + \mu^2 B_2}{\varepsilon - \mu^2 A_2}\left(1 - e^{(\mu^2 A_2 - \varepsilon)t}\right). \end{aligned} \quad (23)$$

Since $A_2 < 0$ and $\mu^2 A_2 - \varepsilon < 0$ hold, one can obtain $\lim_{t\to\infty} Es(t) = (\mu\alpha_1 + \mu\alpha_2 + \mu^2 B_1)/(\varepsilon - \mu^2 A_2)$. □

3.2. Cooperative Game

3.2.1. Cooperative Game Solutions

Theorem 7 (see [39]). *The strategy $\{\varphi_1^*(t), \varphi_2^*(t)\}$ is a Pareto optimal solution to the game $\Psi(s_0, \infty - 0)$, if there exist continuous function $W_i(s) : R \to R$ and continuous derivatives $W_i'(s)$, $W_i''(s)$; $i = 1, 2$ satisfy the following Hamilton-Jacobi-Bellman equation:*

$$\begin{aligned} rW(s) - \frac{\sigma^2 s^2}{2} W''(s) &= \max_{e_1, e_2}\left\{\sum_{i=1}^{2}\left(\alpha_i e_i(t) - \frac{1}{2}e_i^2(t) - \frac{1}{2}\beta_i s^2(t)\right)\right. \\ &\quad \left. + W'(s)\left(\mu\left(e_i(t) + \varphi_j^*(t)\right) - \varepsilon s(t)\right)\right\}. \end{aligned} \quad (24)$$

Hence, the players will adopt the cooperative control $\{\varphi_1^*(t), \varphi_2^*(t)\}$ characterized in Theorem 7.

Theorem 8. *The Pareto optimal solution of the cooperative stochastic differential game $\Psi(s_0, \infty - 0)$ is given by*

$$e_i = \alpha_i + \mu W'(s) = \alpha_i + \mu(as + b), \quad i = 1, 2, \quad (25)$$

where

$$a = \frac{\left(r + 2\varepsilon - \sigma^2\right) - \sqrt{\left(r + 2\varepsilon - \sigma^2\right)^2 + 8\mu^2(\beta_1 + \beta_2)}}{4\mu^2} < 0,$$
$$b = \frac{\mu(\alpha_1 + \alpha_2)a}{r + \varepsilon - 2\mu^2 a} < 0. \quad (26)$$

Proof. Differentiating (24) with respect to e_i and using extreme value conditions, one can get

$$e_i = \alpha_i + \mu W'(s), \quad i = 1, 2. \quad (27)$$

Using $e_i = \alpha_i + \mu W'(s)$, $i = 1, 2$ into (24), one can obtain

$$\begin{aligned} rW(s) - \frac{\sigma^2 s^2}{2} W''(s) &= \frac{1}{2}\left(\alpha_1 + \mu W'(s)\right)^2 + \frac{1}{2}\left(\alpha_2 + \mu W'(s)\right)^2 \\ &\quad - \frac{1}{2}(\beta_1 + \beta_2)s^2 - \varepsilon s W'(s). \end{aligned} \quad (28)$$

From the above equation, it is rational to assume $W(s)$ is a quadratic polynomial.

Let $W(s) = (1/2)as^2 + bs + c$, one can get $2\mu^2 a^2 - (r + 2\varepsilon - \sigma^2)a - (\beta_1 + \beta_2) = 0$.

As [40], we have

$$a = \frac{\left(r + 2\varepsilon - \sigma^2\right) - \sqrt{\left(r + 2\varepsilon - \sigma^2\right)^2 + 8\mu^2(\beta_1 + \beta_2)}}{4\mu^2} < 0,$$
$$b = \frac{\mu(\alpha_1 + \alpha_2)a}{r + \varepsilon - 2\mu^2 a} < 0,$$
$$c = \frac{(\alpha_1 + \mu b)^2 + (\alpha_2 + \mu b)^2}{2r} > 0. \quad (29)$$

Hence, $e_i = \alpha_i + \mu W'(s) = \alpha_i + \mu(as + b)$, $i = 1, 2$. □

Remark 9. (1) The Pareto optimal solution $\{e_1, e_2\}$ possesses the Markov property.

(2) Theorem 8 shows that Pareto optimal solutions of the subgame and the original game are the same at the same time and conditions.

The cooperative value function is given by $W^C(s) = (1/2)as^2 + bs + c$, where

$$a = \frac{r + 2\varepsilon - \sigma^2 - \sqrt{\left(r + 2\varepsilon - \sigma^2\right)^2 + 8\mu^2(\beta_1 + \beta_2)}}{4\mu^2} < 0,$$
$$b = \frac{\mu(\alpha_1 + \alpha_2)a}{r + \varepsilon - 2\mu^2 a} < 0,$$
$$c = \frac{(\alpha_1 + \mu b)^2 + (\alpha_2 + \mu b)^2}{2r} > 0. \quad (30)$$

3.2.2. The Expected Stock and Its Limitation

Theorem 10. *The expectation stock and its limit in the Pareto optimal solution of the cooperative stochastic differential game $\Psi(s_0, \infty - 0)$ satisfy*

$$Es^C(t) = s_0 e^{(2\mu^2 a-\varepsilon)t} + \frac{\mu\alpha_1 + \mu\alpha_2 + 2\mu^2 a}{\varepsilon - \mu^2 a}\left(1 - e^{(2\mu^2 a-\varepsilon)t}\right),$$
$$\lim_{t\to\infty} Es^C(t) = \frac{\mu\alpha_1 + \mu\alpha_2 + 2\mu^2 b}{\varepsilon - 2\mu^2 a}. \tag{31}$$

Proof. Substituting $e_1(t), e_2(t)$ to (1) leads to

$$\begin{aligned} ds^C(t) &= \left(\mu\left(\alpha_1 + \mu\left(as^C(t) + b\right) + \alpha_2 + \mu\left(as^C(t) + b\right)\right)\right.\\ &\quad \left. -\varepsilon s^C(t)\right) dt + \sigma s^C(t)\, dB(t) \\ &= \left(\mu\alpha_1 + 2\mu^2 b + \mu\alpha_2 + \left(2\mu^2 a - \varepsilon\right) s^C(t)\right) dt \\ &\quad + \sigma s^C(t)\, dB(t). \end{aligned} \tag{32}$$

Solving the above stochastic differential equation leads to

$$\begin{aligned} s^C(t) &= s_0 e^{(2\mu^2 a-\varepsilon-(1/2)\sigma^2)t+\sigma B(t)} \\ &\quad + \int_0^t \left(\mu\alpha_1 + \mu\alpha_2 + 2\mu^2 b\right) \\ &\qquad \times e^{(2\mu^2 a-\varepsilon-(1/2)\sigma^2)(t-\tau)+\sigma(B(t)-B(\tau))} d\tau. \end{aligned} \tag{33}$$

Taking the expectation of the above equation leads to

$$\begin{aligned} Es^C(t) &= E\left\{s_0 e^{(2\mu^2 a-\varepsilon-(1/2)\sigma^2)t+\sigma B(t)}\right\} \\ &\quad + E\left\{\int_0^t \left(\mu\alpha_1 + \mu\alpha_2 + 2\mu^2 b\right)\right. \\ &\qquad \left. \times e^{(2\mu^2 a-\varepsilon-(1/2)\sigma^2)(t-\tau)+\sigma(B(t)-B(\tau))} d\tau\right\} \\ &= s_0 e^{(2\mu^2 a-\varepsilon)t} + \frac{\mu\alpha_1 + \mu\alpha_2 + 2\mu^2 a}{\varepsilon - \mu^2 a}\left(1 - e^{(2\mu^2 a-\varepsilon)t}\right). \end{aligned} \tag{34}$$

Since $a < 0$ and $2\mu^2 a - \varepsilon < 0$ hold, one can get $\lim_{t\to\infty} Es^C(t) = (\mu\alpha_1 + \mu\alpha_2 + 2\mu^2 b)/(\varepsilon - 2\mu^2 a)$. □

4. Numerical Simulations

4.1. Noncooperative Game. Let $r = 0.05$, $\sigma = 0.2$, $\varepsilon = 0.3$, $s_0 = 0.5$, $\beta_1 = 0.2$, $\beta_2 = 0.25$, $\alpha_1 = 2$, $\alpha_2 = 3$, and $\mu = 0.8$. For player 2, its noncooperative feedback Nash equilibrium e_2^N, as shown in Figures 1, 2, and 3, increases with ε but decreases with either s or σ or β_2. To be specific, the optimal quantity of GHG emission of player 2 is proportional to the natural absorption rate and inversely proportional to the

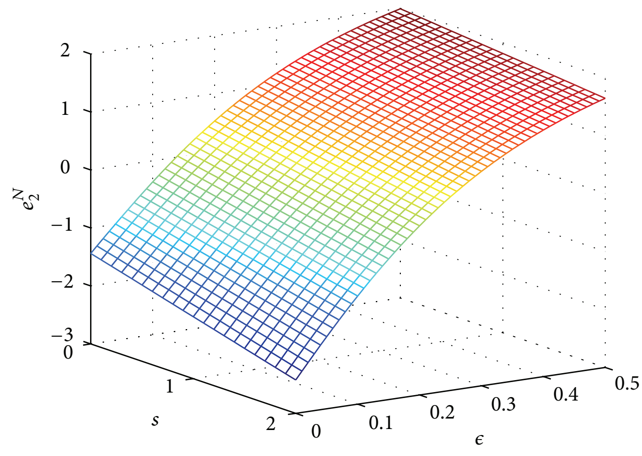


FIGURE 1: e_2^N versus s and ε.

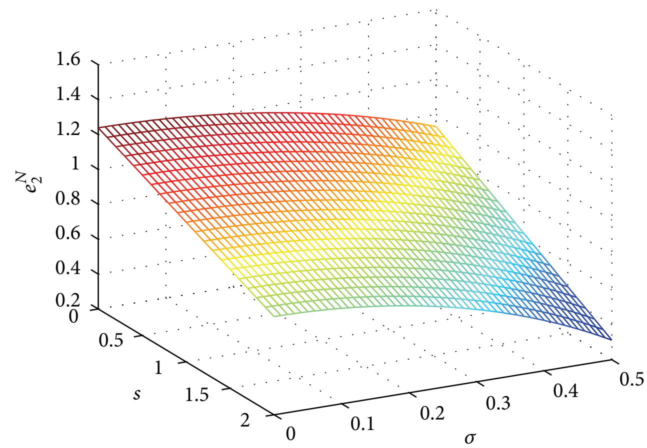


FIGURE 2: e_2^N versus s and σ.

stock of GHGs and constant affected by stochastic factors and cost parameter of player 2. Its limitation of GHG stock $\lim_{t\to\infty} Es^N(t) = 5.1297$ is shown in Figure 4. For the sake of simplicity, in this section we denote $EV_2(s)$ as $V_2(s)$. Its expected profit $EV_2(s)$ increases with ε but decreases with either σ or β_2 and changes little with s shown in Figures 5, 6, and 7. More specifically, the expected profit is proportional to the natural absorption rate, is inversely proportional to constant affected by stochastic factors and cost parameter of player 2, and has little to do with the stock of GHGs.

4.2. Cooperative Game Situation. Let $r = 0.05$, $\sigma = 0.2$, $\varepsilon = 0.3$, $s_0 = 0.5$, $\beta_1 = 0.2$, $\beta_2 = 0.25$, $\alpha_1 = 2$, $\alpha_2 = 3$, and $\mu = 0.8$. For player 1 and player 2, their Pareto optimal solutions e_1 and e_2, as shown in Figures 8, 9, 10, 11, 12, 13, 14, and 15, and their GHG emissions both increase with ε but decrease with s or σ or β_1 or β_2. Practically speaking, the optimal quantities of GHG emissions of player 1 and player 2 are all proportional

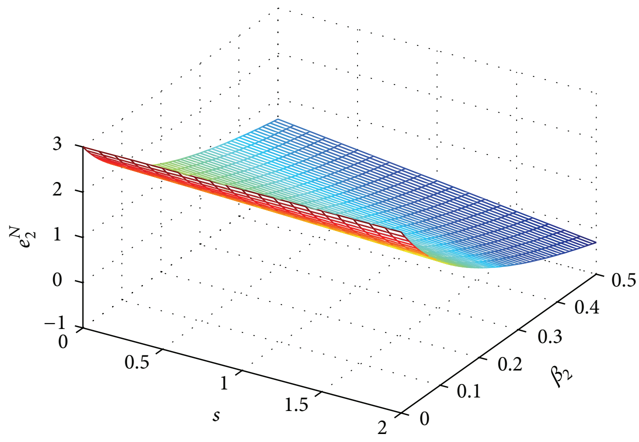


FIGURE 3: e_2^N versus s and β_2.

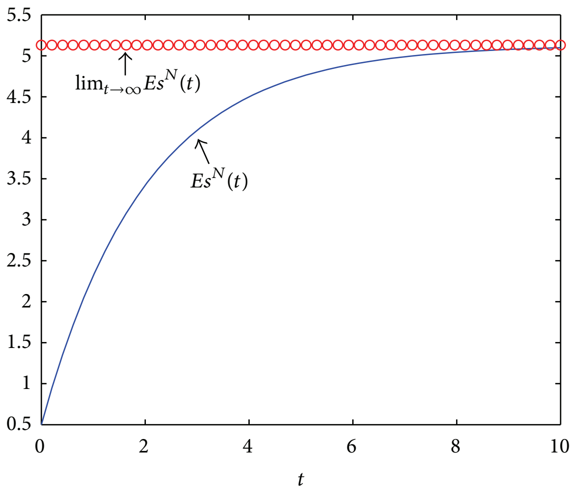


FIGURE 4: $Es^N(t)$ and $\lim_{t\to\infty} Es^N(t)$ versus t.

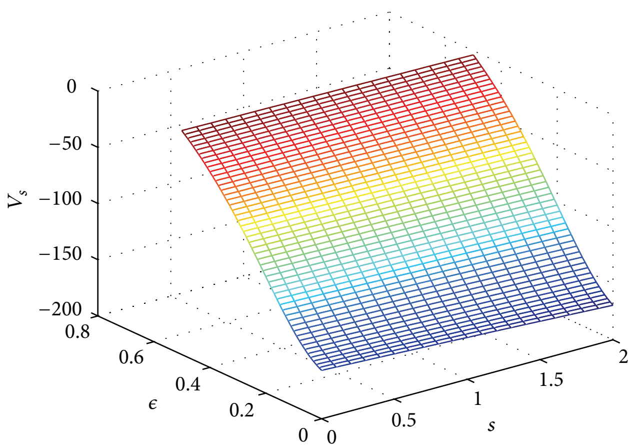


FIGURE 5: $V_2(s)$ versus s and ε.

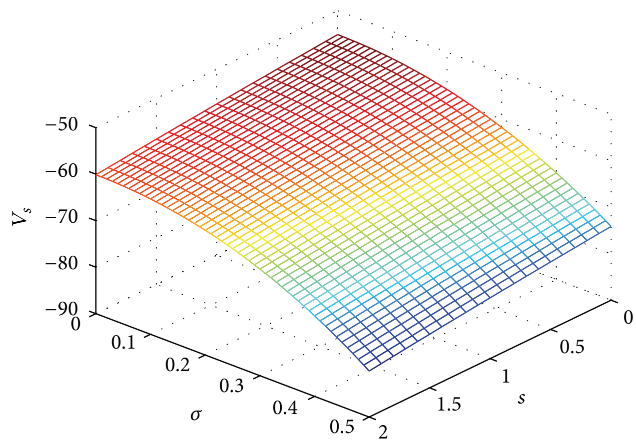


FIGURE 6: $V_2(s)$ versus s and σ.

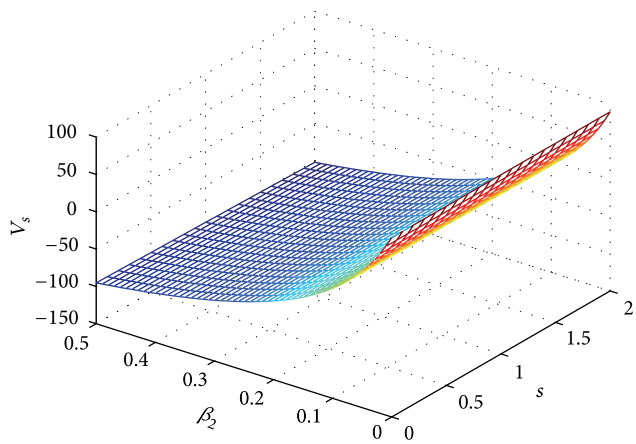


FIGURE 7: $V_2(s)$ versus s and β_2.

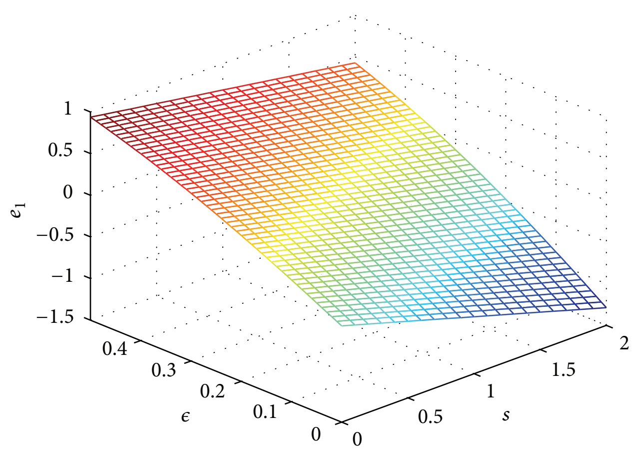


FIGURE 8: e_1 in terms of s and ε.

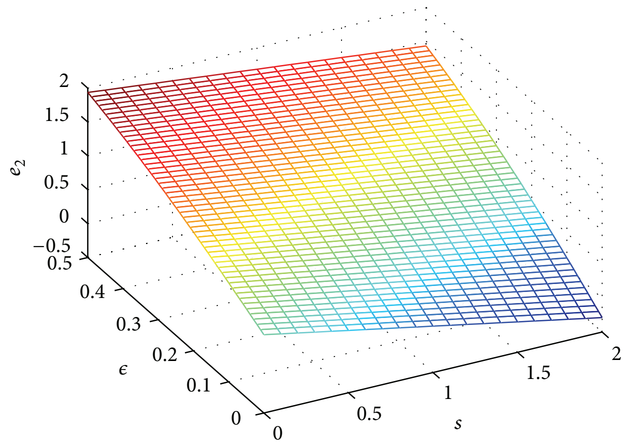


Figure 9: e_2 versus s and ε.

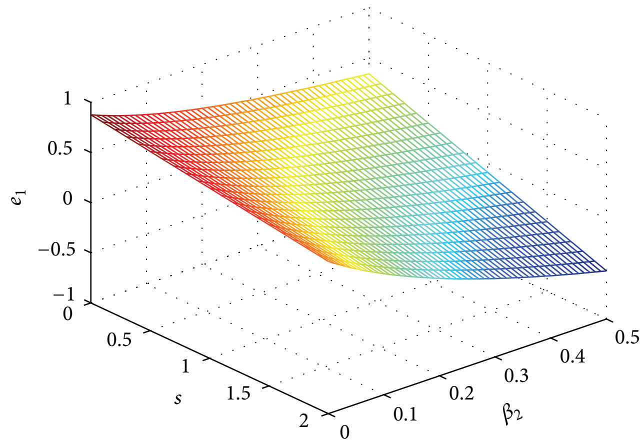


Figure 12: e_1 versus s and β_2.

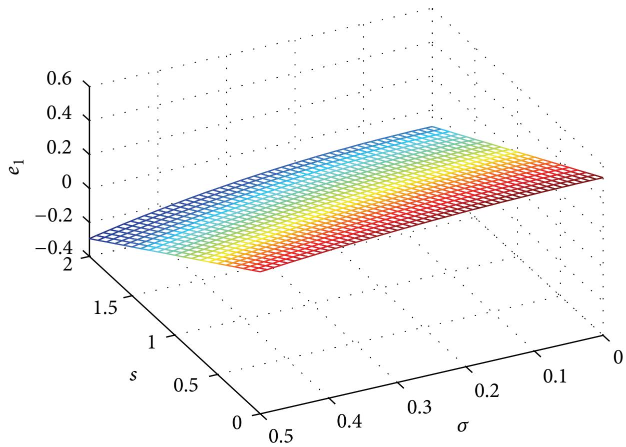


Figure 10: e_1 versus s and σ.

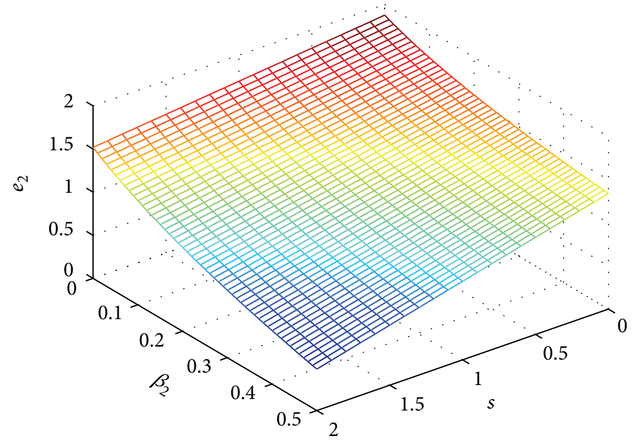


Figure 13: e_2 versus s and β_2.

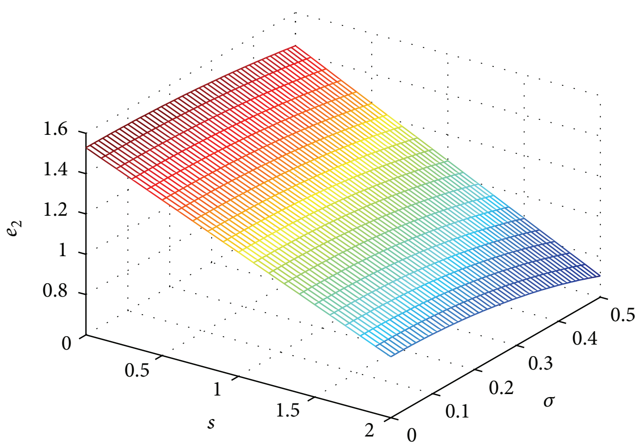


Figure 11: e_2 versus s and σ.

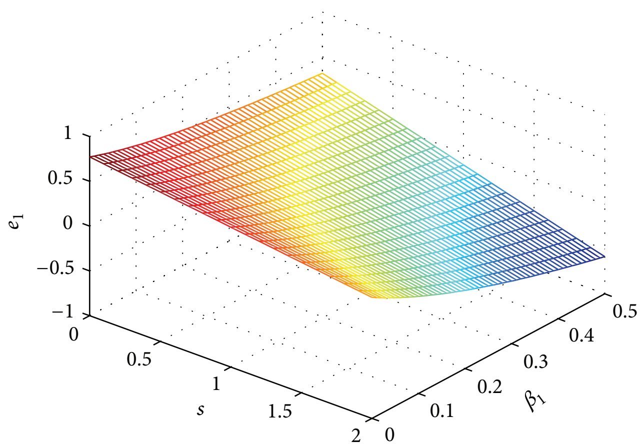


Figure 14: e_2 versus s and β_1.

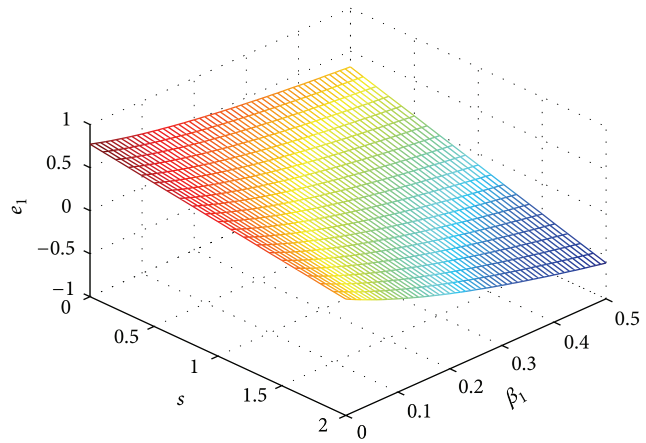


FIGURE 15: e_1 versus s and β_1.

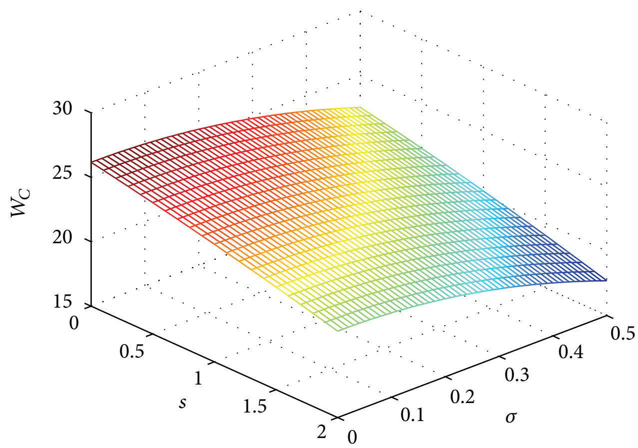


FIGURE 18: $W^C(s)$ versus s and σ.

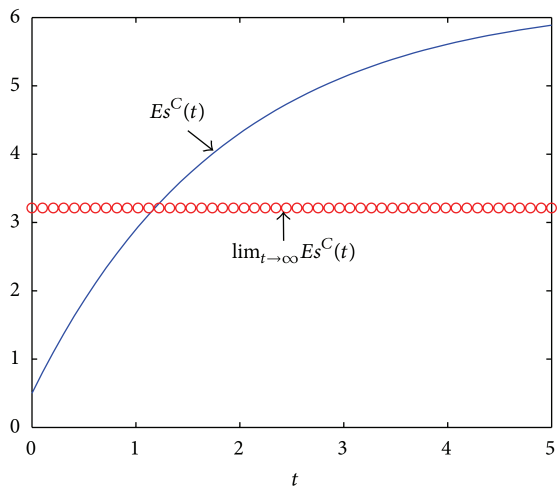


FIGURE 16: $Es^C(t), \lim_{t\to\infty} Es^C(t)$ versus t.

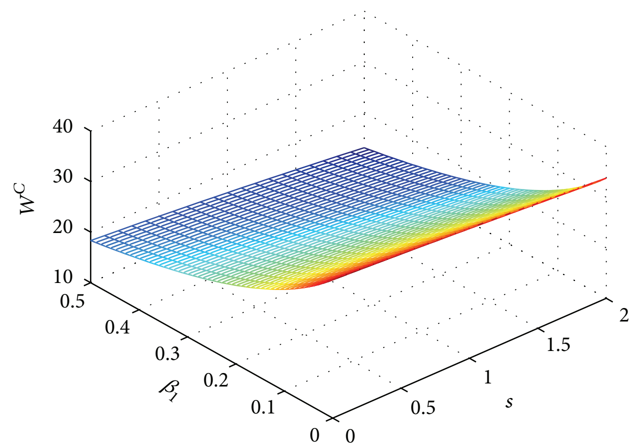


FIGURE 19: $W^C(s)$ versus s and β_1.

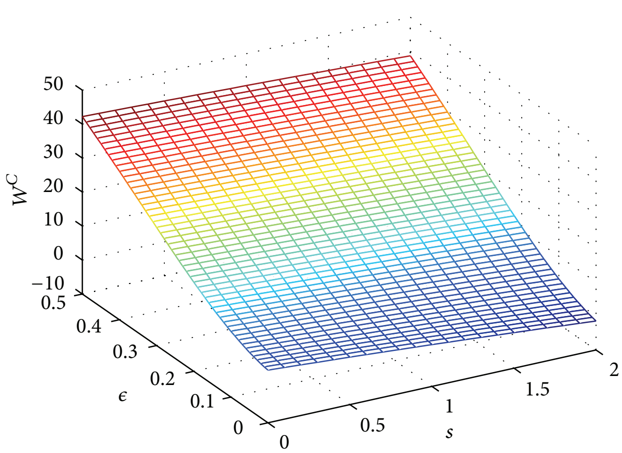


FIGURE 17: $W^C(s)$ versus s and ε.

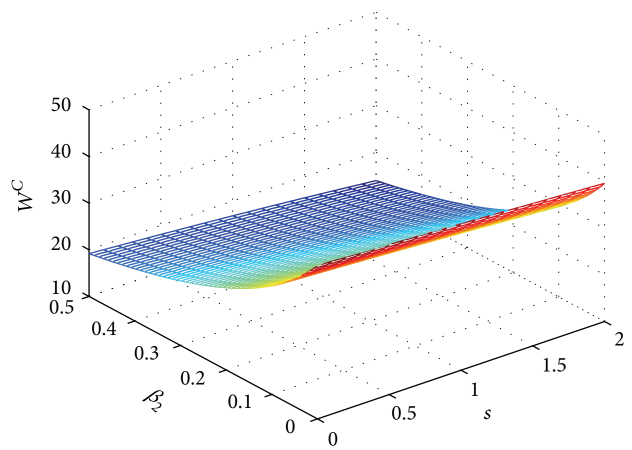


FIGURE 20: $W^C(s)$ versus s and β_2.

to the natural absorption rate and inversely proportional to the stock of GHGs and constant affected by stochastic factors and cost parameters of player 1 and player 2. The limitation of GHG stock $\lim_{t\to\infty} Es^C(t) = 3.2115$ is shown in Figure 16. For the sake of simplicity, we denote $EW^C(s)$ as $W^C(s)$ in this section. The whole expected cooperative profit $EW^C(s)$ decreases with either s or ε or σ or β_1 or β_2 shown in Figures 17, 18, 19, and 20. In fact, the whole expected cooperative profit is inversely proportional to the natural absorption rate and the stock of GHGs and constant affected by stochastic factors and cost parameters of player 1 and player 2.

5. The Distribution of Cooperative Dividend

We consider the long-term cooperation agreement related to the following crucial questions.

(1) When is a cooperative agreement globally feasible?

(2) In the parameter space where cooperation is achievable, how should the cooperative dividend be distributed among players?

(3) What can ensure that players abide by the cooperative agreement over time?

5.1. Global Feasible Conditions for the Cooperation Agreement. A cooperative agreement is globally feasible if the total cooperative payoff must be greater than the total individual noncooperative payoff. The difference between the two is called the cooperative dividend, which is denoted as DC:

$$\mathrm{DC}(s_0) = \frac{1}{2}(a - A_2)s_0^2 + (b - B_2)s_0 + C, \tag{35}$$

where $C = c - C_2 - (\alpha_1^2/2r) = (\mu/2r)(\mu(2b^2 - B_2^2) + 2(\alpha^1 + \alpha^2)(b - B_2))$.

Here, $\mathrm{DC}(s_0)$ is a quadratic polynomial of s_0. Let $\mathrm{DC}(s_0) = 0$, one can get

$$\begin{aligned} s_0^- &= \frac{-(b - B_2) - \sqrt{(b - B_2)^2 - 2(a - A_2)C}}{a - A_2}, \\ s_0^+ &= \frac{-(b - B_2) + \sqrt{(b - B_2)^2 - 2(a - A_2)C}}{a - A_2} \end{aligned} \tag{36}$$

for $(b - B_2)^2 - 2(a - A_2)C \geq 0$.

Remark 11. From Definition 1, the discount value of cooperative dividend DC along the optimal trajectory is equal to the cooperative dividend $\mathrm{DC}(s_0)$ in the initial state.

Theorem 12. *Let $\beta_2^+ = \beta_1 + ((r + 2\varepsilon - \sigma^2)/\mu)\sqrt{\beta_1}$.*

If $\beta_2 > \beta_2^+$, then

$$DC(s_0) \begin{cases} > 0, & \text{if } C \geq 0,\ \forall s_0 \geq 0, \\ < 0, & \text{if } C < 0,\ s_0 \in [0, s_0^+], \\ > 0, & \text{if } C < 0,\ s_0 > s_0^+. \end{cases} \tag{37}$$

If $\beta_2 \in [\beta_1, \beta_2^+]$ and $b - B_2 > 0$, then

$$DC(s_0) \begin{cases} > 0, & \text{if } C \geq 0,\ s_0 \in [0, s_0^-], \\ > 0, & \text{if } C < 0,\ s_0 \in [s_0^+, s_0^-]. \end{cases} \tag{38}$$

If $\beta_2 \in [\beta_1, \beta_2^+]$ and $b - B_2 < 0$, then $DC(s_0) < 0$ for all $s_0 > 0$.

Proof. It is easy to prove it by analyzing the quadratic function $\mathrm{DC}(s_0) = 0$ of s_0. □

Remark 13. The above theorem states that the cooperative agreement is achievable if $\mathrm{DC}(s_0) > 0$.

5.2. The Distribution Method. Assuming that the cooperation is globally feasible, then it is an important problem for us to tackle and how to allocate $\mathrm{DC}(s_0)$ between two players. From cooperative game or bargaining theory, one can find many solutions to the problems, such as core, Shapley value, Nucleolus, and Nash bargaining solution (NBS).

The NBS method will be applied into this paper. Let (W_1^N, W_2^N) denote the Nash equilibrium, and let $(W_1^{\mathrm{NBS}}, W_2^{\mathrm{NBS}})$ denote the Nash bargaining solution. One can get

$$\begin{aligned} W_1^{\mathrm{NBS}} &= W_1^N + \frac{\mathrm{DC}(s_0)}{2} \\ &= W_1^N + \frac{1}{2}\left(\left(W_1^C + W_2^C\right) - \left(W_1^N + W_2^N\right)\right), \\ W_2^{\mathrm{NBS}} &= W_2^N + \frac{\mathrm{DC}(s_0)}{2} \\ &= W_2^N + \frac{1}{2}\left(\left(W_1^C + W_2^C\right) - \left(W_1^N + W_2^N\right)\right), \end{aligned} \tag{39}$$

here

$$\begin{aligned} W_1^N &= \max_{e_1} E\left\{\int_0^\infty e^{-rt} R_1\left(e_1^N(t)\right) dt\right\}, \\ W_2^N &= \max_{e_2} E\left\{\int_0^\infty e^{-rt} \left(R_2\left(e_2^N(t)\right) - D_2\left(s^N(t)\right)\right) dt\right\}, \end{aligned} \tag{40}$$

$$W_i^C = \max_{e_i} E\left\{\int_0^\infty e^{-rt} \left(R_i\left(e_i^N(t)\right) - D_i\left(s^N(t)\right)\right) dt\right\}, \quad i = 1, 2. \tag{41}$$

Remark 14. (1) For non-cooperation stochastic differential game $\Gamma(s_\tau, \infty - \tau)$, the current value of expected payoff is equal to its subgame value function for the corresponding discount in the state variable $s(t) = s_t$ at the interval $[t, \infty)$.

(2) For cooperation stochastic differential game $\Psi(s_\tau, \infty - \tau)$, the current value of expected payoff is equal to its subgame value function for the corresponding discount in the state variable $s(t) = s_t$ at the interval $[t, \infty)$.

5.3. Abide by the Cooperative Agreement. It is crucial to build a credible basis for abiding strictly by the cooperative agreement in a long time. Inspired by [14, 35, 39, 41–43], we try to design a transfer payment mechanism for subgame consistent agreement based on the Nash bargaining solution.

Denote $W_i^{\mathrm{NBS}}(s^C(t)), W_i^N(s^C(t))$ as cooperation and noncooperative distribution, respectively.

Definition 15 (see [29]). $(W_1^{\mathrm{NBS}}(s^C(t)), W_2^{\mathrm{NBS}}(s^C(t)))$ is an effective distribution if it satisfies the following:

(1) $(W_1^{\mathrm{NBS}}(s^C(t)), W_2^{\mathrm{NBS}}(s^C(t)))$ is a Pareto optimal distribution,

(2) $W_i^{\mathrm{NBS}}(s^C(t)) \geq W_i^N(s^C(t)), i = 1, 2$,

where $s^C(t)$ is state trajectory of the cooperative state trajectory.

Definition 16 (see [29]). $(W_1^{\mathrm{NBS}}(s^C(t)), W_2^{\mathrm{NBS}}(s^C(t)))$ is a subgame consistent distribution if it satisfies

(1) $(W_1^{\mathrm{NBS}}(s^C(t)), W_2^{\mathrm{NBS}}(s^C(t)))$ is a Pareto optimal distribution,

(2) $W_i^{\mathrm{NBS}}(s^C(t)) \geq W_i^N(s^C(t)), i = 1, 2$ for $t \in [\tau, \infty)$,

where $s^C(t)$ is state trajectory of the cooperation state trajectory.

Denote $\xi_i(t)$, $i = 1, 2$ instantaneous payoff-distribution procedure of the subgame, which satisfies

$$\int_0^\infty e^{-rt}\xi_i(t)\,dt = W_i^{\mathrm{NBS}}(s_0), \quad i = 1, 2. \tag{42}$$

$$W_i^{\mathrm{NBS}}(s_0) = \int_0^t e^{-r\tau}\xi_i(\tau)\,d\tau + e^{-rt}W_i^{\mathrm{NBS}}(s^C(t)), \quad i = 1, 2, \tag{43}$$

where

$$W_i^{\mathrm{NBS}}(s^C(t)) = W_i^N(s^C(t)) + \frac{1}{2}\sum_{i=1}^{2}\left(W_i^C(s^C(t)) - W_i^N(s^C(t))\right), \quad i = 1, 2. \tag{44}$$

For the sake of simplicity, let $s^C(t)$ denote $Es^C(t)$, and let $\Gamma(s^C(t))$ denote a subgame in state $s_t^C = s^C(t)$. If a cooperative agreement is feasible in the subgame, player i, $i = 1, 2$ can get a Nash bargaining solution from (43).

Differential (43) with respect to t, one has

$$\xi_i(t) = rW_i^{\mathrm{NBS}}(s^C(t)) - \frac{d}{dt}\left(W_i^{\mathrm{NBS}}(s^C(t))\right), \quad i = 1, 2. \tag{45}$$

Equation (45) shows that the instantaneous payoff of player i in the time t is equal to its interest payoff minus the ratio of Nash bargaining solution.

Remark 17. If (45) is multiplied by discount factor e^{-rt}, by integrating both sides, one can get

$$\begin{aligned}
&\int_0^\infty e^{-rt}\xi_i(t)\,dt \\
&= \int_0^\infty \left(re^{-rt}W_i^{\mathrm{NBS}}(s^C(t)) - e^{-rt}\frac{d}{dt}\left(W_i^{\mathrm{NBS}}(s^C(t))\right)\right)dt \\
&= -e^{-rt}W_i^{\mathrm{NBS}}(s^C(t))\Big|_0^\infty = W_i^{\mathrm{NBS}}(s^C(0)) \\
&= W_i^{\mathrm{NBS}}(s_0), \quad i = 1, 2, \\
&W_i^{\mathrm{NBS}}(s^C(t)) - W_i^N(s^C(t)) \\
&= \frac{1}{2}\sum_{i=1}^{2}\left(W_i^C(s^C(t)) - W_i^N(s^C(t))\right) \\
&= \frac{1}{2}\mathrm{DC}(s^C(t)), \quad i = 1, 2,
\end{aligned} \tag{46}$$

where

$$\begin{aligned}
&\mathrm{DC}(s^C(t)) \\
&= \sum_{i=1}^{2}\left(W_i^C(s^C(t)) - W_i^N(s^C(t))\right) \\
&= \frac{1}{2}(a - A_2)(s^C(t))^2 + (b - B_2)s^C(t) + C \geq 0.
\end{aligned} \tag{47}$$

Hence, $W_i^{\mathrm{NBS}}(s^C(t)) - W_i^N(s^C(t))$, $i = 1, 2$ has the same sign as $(1/2)\mathrm{DC}(s^C(t))$.

In fact, if $\mathrm{DC}(s_0) < 0$, the cooperative game will not be played. So the next section will discuss the conditions of $\mathrm{DC}(s_0) \geq 0$.

5.4. Sufficient Conditions for $DC(s^C(t)) \geq 0$

Theorem 18. *If the cooperative solution is interior, and $DC(s^C(t))$ is increasing for $DC(s_0) > 0$, $DC(s^C(t)) \geq 0$ for all $t \in [0, \infty)$ holds.*

Proof. If for all $t \in [0, \infty)$, then $s^C(t) > s_0$, that is, the theorem is proved.

Next we prove $s^C(t) > s_0$.

From (34), one can have

$$s^C(t) - s_0 = \left(e^{(2\mu^2a^2-\varepsilon)t} - 1\right)\left(\frac{\mu\alpha_1 + \mu\alpha_2 + 2\mu^2 b}{2\mu^2a^2 - \varepsilon} - s_0\right). \tag{48}$$

Since the GHG cooperative emission is interior, and $e_i^C = \alpha_i + \mu(as + b)$, $i = 1, 2$, one can get $\alpha_i + \mu b > -\mu as > 0$, $i = 1, 2$, so $(\mu\alpha_1 + \mu\alpha_2 + 2\mu^2 b)/(2\mu^2a^2 - \varepsilon) < 0$.

According to $a < 0$ and $e^{(2\mu^2a^2-\varepsilon)t} - 1 < 0$, $s^C(t) > s_0$ holds. It is proved. □

From (45), one can get the instantaneous payoff $\xi_i(t)$, $i = 1, 2$ as follows:

$$\begin{aligned}
\xi_1(t) = {} & \frac{1}{2} r A_1 \\
& + \frac{1}{2}\left(r\left(\frac{1}{2}as^2 + bs + c\right)\right. \\
& \qquad \left. -(as+b)\left(\mu\left(e_1+e_2\right) - \varepsilon s\right) - \frac{1}{2}a\sigma^2\right) \\
& - \frac{1}{2}\left(r\left(\frac{1}{2}A_2 s^2 + B_2 s + C_2\right)\right. \\
& \qquad \left. -\left(A_2 s + B_2\right)\left(\mu\left(e_1+e_2\right) - \varepsilon s\right) - \frac{1}{2}A_2\sigma^2\right), \\
\xi_2(t) = {} & \frac{1}{2}\left(r\left(\frac{1}{2}A_2 s^2 + B_2 s + C_2\right)\right. \\
& \qquad \left. -\left(A_2 s + B_2\right)\left(\mu\left(e_1+e_2\right) - \varepsilon s\right) - \frac{1}{2}A_2\sigma^2\right) \\
& + \frac{1}{2}\left(r\left(\frac{1}{2}as^2 + bs + c\right) - (as+b)\right. \\
& \qquad \left. \times\left(\mu\left(e_1+e_2\right) - \varepsilon s\right) - \frac{1}{2}a\sigma^2\right) - \frac{1}{2}rA_1.
\end{aligned} \tag{49}$$

6. Conclusion

Taking the effect of uncertainty into consideration, we use stochastic differential game to build cooperative and non-cooperative game model for global GHG emission between developing and developed countries. Then we calculate the feedback Nash equilibrium and the Pareto optimal solution, give the globally feasible and sustainable conditions of the cooperative agreement, and propose the distribution method of payoff. At last, we illustrate the above results by numerical simulations. These results show that developing and developed countries would rationally adopt the Nash equilibrium as their GHGs emissions Pareto optimal solution in the long term.

Acknowledgments

This work is supported partly by Excellent Young Scientist Foundation of Shandong Province (Grant no. BS2011SF018), National Social Science Foundation of China (Grant no. 12BJY103), Humanities and Social Sciences Foundation of the Ministry of Education of China (Grant no. 11YJCZH200), and National Natural Science Foundation of China (Grant no. 71272148).

References

[1] J. H. Williams, A. DeBenedictis, R. Ghanadan et al., "The technology path to deep greenhouse gas emissions cuts by 2050: the pivotal role of electricity," *Science*, vol. 335, pp. 53–59, 2012.

[2] S. Bastianoni, F. M. Pulselli, and E. Tiezzi, "The problem of assigning responsibility for greenhouse gas emissions," *Ecological Economics*, vol. 49, no. 3, pp. 253–257, 2004.

[3] W. Parton, S. DelGrosso, E. Marx, and A. Swan, "Agriculture's role in cutting greenhouse gas emissions," *Issues in Science and Technology*, vol. 27, pp. 29–32, 2011.

[4] R. S. J. Tol, "The marginal costs of greenhouse gas emissions," *Energy Journal*, vol. 20, no. 1, pp. 61–81, 1999.

[5] A. Ansuategi and M. Escapa, "Economic growth and greenhouse gas emissions," *Ecological Economics*, vol. 40, no. 1, pp. 23–37, 2002.

[6] F. van der Ploeg and A. de Zeeuw, "A differential game of international pollution control," *Systems & Control Letters*, vol. 17, no. 6, pp. 409–414, 1991.

[7] F. van der Ploeg and A. J. de Zeeuw, "International aspects of pollution control," *Environmental & Resource Economics*, vol. 2, no. 2, pp. 117–139, 1992.

[8] N. V. Long, "Pollution control: a differential game approach," *Annals of Operations Research*, vol. 37, no. 1–4, pp. 283–296, 1992.

[9] S. J. Rubio and B. Casino, "A note on cooperative versus non-cooperative strategies in international pollution control," *Resource and Energy Economics*, vol. 24, no. 3, pp. 251–261, 2002.

[10] D. W. K. Yeung, "Dynamically consistent cooperative solution in a differential game of transboundary industrial pollution," *Journal of Optimization Theory and Applications*, vol. 134, no. 1, pp. 143–160, 2007.

[11] O. Bahn and A. Haurie, "A class of games with coupled constraints to model international GHG emission agreements," *International Game Theory Review*, vol. 10, no. 4, pp. 337–362, 2008.

[12] S. Jørgensen, G. Martín-Herrán, and G. Zaccour, "Agreeability and time consistency in linear-state differential games," *Journal of Optimization Theory and Applications*, vol. 119, no. 1, pp. 49–63, 2003.

[13] S. Jørgensen, G. Martín-Herrán, and G. Zaccour, "Sustainability of cooperation overtime in linear-quadratic differential games," *International Game Theory Review*, vol. 7, no. 4, pp. 395–406, 2005.

[14] S. Jørgensen and G. Zaccour, "Time consistent side payments in a dynamic game of downstream pollution," *Journal of Economic Dynamics & Control*, vol. 25, no. 12, pp. 1973–1987, 2001.

[15] P. Smala Fanokoa, I. Telahigue, and G. Zaccour, "Buying cooperation in an asymmetric environmental differential game," *Journal of Economic Dynamics & Control*, vol. 35, no. 6, pp. 935–946, 2011.

[16] M. Breton, G. Martín-Herrán, and G. Zaccour, "Equilibrium investment strategies in foreign environmental projects," *Journal of Optimization Theory and Applications*, vol. 130, no. 1, pp. 23–40, 2006.

[17] M. Breton, G. Zaccour, and M. Zahaf, "A differential game of joint implementation of environmental projects," *Automatica*, vol. 41, no. 10, pp. 1737–1749, 2005.

[18] M. Breton, G. Zaccour, and M. Zahaf, "A game-theoretic formulation of joint implementation of environmental projects," *European Journal of Operational Research*, vol. 168, no. 1, pp. 221–239, 2005.

[19] L. Petrosjan and G. Zaccour, "Time-consistent Shapley value allocation of pollution cost reduction," *Journal of Economic Dynamics & Control*, vol. 27, no. 3, pp. 381–398, 2003.

[20] E. J. Dockner, S. Jørgensen, N. V. Long, and G. Sorger, *Differential Games in Economics and Management Science*, Cambridge University Press, Cambridge, UK, 2000.

[21] S. M. J. Mirzapour Al-E-Hashem, A. Baboli, S. J. Sadjadi, and M. B. Aryanezhad, "A multiobjective stochastic production-distribution planning problem in an uncertain environment considering risk and workers productivity," *Mathematical Problems in Engineering*, vol. 2011, Article ID 406398, 14 pages, 2011.

[22] X.-J. Xie, N. Duan, and X. Yu, "State-feedback control of high-order stochastic nonlinear systems with SiISS inverse dynamics," *Institute of Electrical and Electronics Engineers*, vol. 56, no. 8, pp. 1921–1926, 2011.

[23] X.-J. Xie and L. Liu, "Further results on output feedback stabilization for stochastic high-order nonlinear systems with time-varying delay," *Automatica*, vol. 48, no. 10, pp. 2577–2586, 2012.

[24] J. Yin, S. Khoo, Z. Man, and X. Yu, "Finite-time stability and instability of stochastic nonlinear systems," *Automatica*, vol. 47, no. 12, pp. 2671–2677, 2011.

[25] W. Zhang, Y. Huang, and H. Zhang, "Stochastic H-2/H-infinity control for discrete-time systems with state and disturbance dependent noise," *Automatica*, vol. 43, no. 3, pp. 513–521, 2007.

[26] H. Ma, W. Zhang, and T. Hou, "Infinite horizon H-2/H-infinity control for discrete-time time-varying Markov jump systems with multiplicative noise," *Automatica*, vol. 48, no. 7, pp. 1447–1454, 2012.

[27] W. Li, Y. Jing, and S. Zhang, "State feedback control of a class of high-order stochastic nonlinear systems," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 14, pp. 1583–1601, 2012.

[28] W. Li, X. Liu, and S. Zhang, "Further results on adaptive state-feedback stabilization for stochastic high-order nonlinear systems," *Automatica*, vol. 48, no. 8, pp. 1667–1675, 2012.

[29] S. Mataramvura and B. Øksendal, "Risk minimizing portfolios and HJBI equations for stochastic differential games," *Stochastica*, vol. 80, no. 4, pp. 317–337, 2008.

[30] G. Wang and Z. Yu, "A partial information non-zero sum differential game of backward stochastic differential equations with applications," *Automatica*, vol. 48, no. 2, pp. 342–352, 2012.

[31] X. Zhao, G. Ren, C. Fan, and C. Ding, "A simulation-based dynamic stochastic route choice model for evacuation," *Mathematical Problems in Engineering*, vol. 2012, Article ID 396248, 18 pages, 2012.

[32] C. Lodi, P. Bacher, J. Cipriano, and Madsen, "Modelling the heat dynamics of a monitored test reference environment for building integrated photovoltaic systems using stochastic differential equations," *Energy and Buildings*, vol. 50, pp. 273–281, 2012.

[33] A. Kopp, I. Büsching, R. D. Strauss, and M. S. Potgieter, "A stochastic differential equation code for multidimensional Fokker-Planck type problems," *Computer Physics Communications*, vol. 183, no. 3, pp. 530–542, 2012.

[34] C. K. Leong and W. Huang, "A stochastic differential game of capitalism," *Journal of Mathematical Economics*, vol. 46, no. 4, pp. 552–561, 2010.

[35] D. W. K. Yeung and L. A. Petrosyan, "A cooperative stochastic differential game of transboundary industrial pollution," *Automatica*, vol. 44, no. 6, pp. 1532–1544, 2008.

[36] W.-K. Wang and C.-O. Ewald, "A stochastic differential fishery game for a two species fish population with ecological interaction," *Journal of Economic Dynamics & Control*, vol. 34, no. 5, pp. 844–857, 2010.

[37] M. Taksar and X. Zeng, "Optimal non-proportional reinsurance control and stochastic differential games," *Insurance*, vol. 48, no. 1, pp. 64–71, 2011.

[38] B. Xin and T. Chen, "On a master-slave Bertrand game model," *Economic Modelling*, vol. 28, no. 4, pp. 1864–1870, 2011.

[39] D. W. K. Yeung and L. A. Petrosyan, *Cooperative Stochastic Differential Games*, Springer, New York, NY, USA, 2006.

[40] C.-O. Ewald and W.-K. Wang, "Analytic solutions for infinite horizon stochastic optimal control problems via finite horizon approximation: a practical guide," *Mathematical Social Sciences*, vol. 61, no. 3, pp. 146–151, 2011.

[41] B. Tolwinski, A. Haurie, and G. Leitmann, "Cooperative equilibria in differential games," *Journal of Mathematical Analysis and Applications*, vol. 119, no. 1-2, pp. 182–202, 1986.

[42] M. Germain and V. Van Steenberghe, "Constraining equitable allocations of tradable CO_2 emission quotas by acceptability," *Environmental and Resource Economics*, vol. 26, no. 3, pp. 469–492, 2003.

[43] D. W. K. Yeung and L. A. Petrosyan, "Subgame consistent cooperative solutions in stochastic differential games," *Journal of Optimization Theory and Applications*, vol. 120, no. 3, pp. 651–666, 2004.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 932579, 10 pages
http://dx.doi.org/10.1155/2013/932579

Research Article

Dividends Sharing Convertible Bonds Pricing and Numerical Evaluation

Xu Guo[1] and Haiyang Wang[2]

[1] *Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong*
[2] *School of Mathematics, Shandong University, Jinan 250100, China*

Correspondence should be addressed to Haiyang Wang; health.sea@163.com

Received 23 January 2013; Accepted 23 April 2013

Academic Editor: Guangchen Wang

The convertible bond is becoming one of the most important financial instruments for the company to raise capital fund since it was first issued by American New York Erie Company in 1843. In this paper, it is the first time to study the pricing problem for convertible bond whose underlying stocks pay dividends via the reflected backward stochastic differential equations. Associating the solutions of reflected BSDEs with the obstacle problems for nonlinear parabolic PDEs, we establish the pricing formulas for convertible bonds with continuous and discrete dividends by means of the viscosity solutions for some PDEs. Besides, we also derive the price of convertible bonds with higher borrowing rate which is realistic in the financial market. Then the numerical evaluations are provided by the radial basis functions method. Moreover, we discuss the influence of dividends paying as well as higher borrowing rate on the convertible bond price at last.

1. Introduction

A convertible bond is a kind of corporate debt securities that gives the holder the right to forgo future coupon and principal payments and convert it to a specified number of shares of common stock of the issuing firm. In essence, a convertible bond is a hybrid security consisting of a straight bond and a call on the underlying equity, but various characteristics of realistic convertible bonds make it impossible to decouple the stock option from the bond part. Therefore, how to price the convertible bond fairly is more difficult than many other derivatives.

Theoretical pricing models for convertible bonds first appeared in the 1960s. Their general valuation procedure was to set the price of the convertible bond equal to the maximum of its value as an ordinary bond or its value in common stock (after conversion) at some point in the future and then discount this value to the present. This method or a slight modification thereof was employed by Poensgen [1, 2], Baumol et al. [3], Weil et al. [4], and no doubt others. Since the fundamental paper worked by Black and Scholes [5] for pricing the financial derivatives was published in 1973, the celebrated Black-Scholes formula is adopted to value the convertible bond as a contingent claim on the firm as a whole. There is also a rich literature along this line, for example, Ingersoll [6] and Brennan and Schwartz [7, 8], in all of which the authors took the firm value as the variable that determine the price of the convertible bond, while in McConnell and Schwartz [9], Ho and Pfeffer [10], and Tsiveriotis and Fernandes [11], the convertible bond is viewed and valued as a derivative of the underlying equity, which is commonly the stock of the issuing firm.

However, the backward stochastic differential equations (BSDEs) are a powerful alternative to price the contingent claims. Nonlinear BSDEs were introduced by Pardoux and Peng [12] and Duffie and Epstein [13] independently. El Karoui et al. [14] studied the property of BSDEs and their applications to optimal control and financial mathematics, such as European option pricing problem in the constraint case. Since the price of convertible bond is constrained to be greater than the payoff of the bond, it corresponds to the solution of a new type of backward equations called reflected BSDEs. El Karoui et al. [15] investigated the RBSDEs in detail and gave some important properties, among which is the connection between the solutions of RBSDEs and the related obstacle problems for parabolic PDEs. Our work is much inspired by them and attempts to price the convertible bonds

with continuous and discrete dividends as well as with higher borrowing rate.

Despite the difficulty of obtaining the analytic solutions of nonlinear parabolic PDEs, the radial basis functions (RBFs) play an important role in solving different types of PDEs numerically with many advantages such as mesh-free property, accuracy, stability, and efficiency. As one of notable RBFs, Hardy's Multiquadric (MQ) was firstly developed by Hardy [16] to approximate two-dimensional geographical surfaces. In Franke's [17] review paper, the MQ was rated one of the best methods among 29 scattered data interpolation schemes. Hon and Mao [18] applied this MQ as a spatial approximation to devise a new computation scheme for the numerical solution of the options value and its derivatives in the Black-Scholes equation. This transforms the Black-Scholes equation to a system of first-order equations in time, and the American options can then be approximated by using a high-order backward time integration scheme. In our paper, we adopt this kind of methods to evaluate the PDEs for pricing the convertible bonds, which makes our theoretical results more practical in the financial market.

The rest of this paper is organized as follows. We introduce some key characteristics of a convertible bond and recall the definition of reflected BSDEs and some important properties of it in Section 2. In Section 3, we establish the pricing formula for convertible bonds whose underlying stocks pay continuous and discrete dividends, respectively, by means of nonlinear parabolic PDEs. Furthermore, we investigate the situation with higher borrowing rate in Section 4. Then according to the computation scheme of RBF methods, some examples with all constant coefficients are presented in Section 5, and we can also see the influence of dividends paying as well as higher borrowing rate on the convertible bond price through the numerical results. The last section is devoted to conclude the novelty and distinctive feature of the paper and discuss the future research work in this field.

2. Preliminaries and Notations

In this section, at first, let us get familiar with some unique characteristics of the convertible bond and then recall some properties of reflected BSDEs and the associated obstacle problems for nonlinear parabolic PDEs.

2.1. A Convertible Bond Indenture Agreement. In finance, a convertible bond is a type of bond that the holder can convert into shares of common stock in the issuing company or cash of equal value, at an agreed-upon price. It is a hybrid security with both debt- and equity-like features. Although it typically has a coupon rate lower than that of similar, nonconvertible debt, the instrument carries additional value through the option to convert the bond to stock and thereby participates in further growth in the company's equity value. The investor receives the potential upside of conversion into equity while protecting downside with cash flow from the coupon payments and the return of principal upon maturity.

From the issuer's perspective, the key benefit of raising money by selling convertible bonds is a reduced cash interest payment. The advantage for companies of issuing convertible bonds is that, if the bonds are converted to stocks, companies' debt vanishes. However, in exchange for the benefit of reduced interest payments, the value of shareholder's equity is reduced due to the stock dilution expected when bondholders convert their bonds into new shares.

Generally speaking, there are several articles that determine the feature of a convertible bond in the indenture agreement. They are as follows.

(i) *Maturity Date T*: it is the date from now to which a holder can convert the bond to a specified number of common stocks at any time during this period, that is, the expiry date of a convertible bond.

(ii) *Conversion Ratio C*: it states the number of shares of common stock which can be obtained upon the surrender of one share of convertible bond.

(iii) *Face Price F*: it is the aggregate balloon payment that an investor can get at the maturity date T if he never exercises the convertible bond.

(iv) *Put Term*: it is an agreement that on each put date the investor can choose between holding the convertible bond and putting it to the issuer for the specified put value.

(v) *Call Term*: it is an agreement that when the issuer calls the convertible bond, the investor must elect to receive either the cash call price or the conversion value of the convertible bond.

2.2. Reflected Backward Stochastic Differential Equations. Let $(\Omega, \mathcal{F}, P)$ be a completed probability space endowed with a filtration $\{\mathcal{F}_t; 0 \le t \le T\}$. Also $\{W_t, 0 \le t \le T\}$ is a d-dimensional standard Brownian motion defined on this space. We also assume that $\{\mathcal{F}_t\}_{t\ge 0}$ is generated by the Brownian motion $\{W_t, t \ge 0\}$ and satisfies the usual conditions. If x belongs to $\mathbb{R}^n$, $|x|$ denotes its Euclidean norm.

There are some notations used throughout the paper:

$$
\begin{aligned}
&\mathcal{L}^2(\mathcal{F}_T, \mathbb{R}^m) \\
&\quad = \Big\{\xi \text{ is a } \mathbb{R}^m\text{-valued } \mathcal{F}_T\text{-measurable random variable} \\
&\qquad \text{s.t. } \mathbb{E}\left(|\xi|^2\right) < +\infty\Big\}, \\
&\mathcal{L}^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \\
&\quad = \left\{ \{\varphi_t, 0 \le t \le T\} \text{ is a } \mathbb{R}^m\text{-valued adapted process} \right. \\
&\qquad \left. \text{s.t. } \mathbb{E}\left(\int_0^T |\varphi_t|^2 dt\right) < +\infty \right\}, \\
&\mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R}^m) \\
&\quad = \left\{ \{\psi_t, 0 \le t \le T\} \text{ is a } \mathbb{R}^m\text{-valued adapted process} \right. \\
&\qquad \left. \text{s.t. } \mathbb{E}\left(\sup_{0\le t\le T} |\psi_t|^2\right) < +\infty \right\}.
\end{aligned}
\tag{1}
$$

Let us introduce the 1-dimensional reflected BSDEs. We are given three objects: the first is a terminal value ξ s.t.

(i) $\xi \in \mathscr{L}^2(\mathscr{F}_T, \mathbb{R})$.

The second is a "coefficient" f, which is a map

$$f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}, \tag{2}$$

such that

(ii) for all $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, $f(\cdot, y, z) \in \mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R})$,

(iii) for some $K > 0$ and all $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, a.s.,

$$\left|f(t,y,z) - f(t,y',z')\right| \le K\left(\left|y-y'\right| + \left|z-z'\right|\right). \tag{3}$$

The third is an "obstacle" $\{S_t; 0 \le t \le T\}$, which is a continuous progressively measurable real-valued process satisfying

(iv) $\mathbb{E}(\sup_{0\le t\le T}(S_t^+)^2) < \infty$.

We will always assume that $S_T \le \xi$ a.s.

Then we formulate the form of a reflected BSDE.

Definition 1. The solution of the reflected BSDE is a triple $\{(Y_t, Z_t, K_t), 0 \le t \le T\}$ of $\mathscr{F}_t$-progressively measurable processes taking values in $\mathbb{R}$, $\mathbb{R}^d$, and $\mathbb{R}_+$, respectively, and satisfying

(1) $Y \in \mathcal{S}^2_{\mathscr{F}}(0,T;\mathbb{R})$, $Z \in \mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$, and $K_T \in \mathscr{L}^2(\mathscr{F}_T, \mathbb{R})$,

(2) $Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dW_s$,

(3) $Y_t \ge S_t$, $0 \le t \le T$,

(4) $\{K_t\}$ is continuous and increasing, $K_0 = 0$, and $\int_0^T (Y_t - S_t)dK_t = 0$.

The existence and uniqueness of solutions of reflected BSDEs have been proved by El Karoui et al. [15].

Theorem 2. *Under the previous assumptions, in particular (i), (ii), (iii), and (iv), the reflected BSDE with* (1), (2), (3), *and* (4) *has a unique solution* (Y, Z, K).

Moreover, some properties of the reflected BSDEs are also summarized and presented there. An important fact is that the square-integrable solution $\{Y_t, 0 \le t \le T\}$ of reflected BSDE corresponds to the value of an optimal stopping time problem.

Proposition 3. *Let $\{(Y_t, Z_t, K_t), 0 \le t \le T\}$ be the solution of reflected BSDE defined by Definition 1. Then for each $t \in [0,T]$,*

$$Y_t = \operatorname*{ess\,sup}_{\nu\in\mathscr{T}_t} \mathbb{E}\left[\int_t^\nu f(s, Y_s, Z_s)\,ds + S_\nu \mathbf{1}_{\{\nu<T\}} + \xi \mathbf{1}_{\{\nu=T\}} \mid \mathscr{F}_t\right], \tag{4}$$

where $\mathscr{T}$ is the set of all stopping times dominated by T, and

$$\mathscr{T}_t = \{\nu \in \mathscr{T}; t \le \nu \le T\}. \tag{5}$$

2.3. Obstacle Problems for Nonlinear Parabolic PDEs. In this subsection, we will show that the reflected BSDEs allow us to give a probabilistic representation for solutions of some obstacle problems for PDEs. For that purpose, we will put the RBSDEs in a Markovian framework.

Let $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be continuous mappings, which are Lipschitz with respect to their second variable, uniformly with respect to $t \in [0,T]$. For each $(t,x) \in [0,T] \times \mathbb{R}^d$, let $\{X_s^{t,x}, t \le s \le T\}$ be the unique $\mathbb{R}^d$-valued solution of the SDE:

$$X_s^{t,x} = x + \int_t^s b\left(r, X_r^{t,x}\right)dr + \int_t^s \sigma\left(r, X_r^{t,x}\right)dW_r. \tag{6}$$

We suppose now that the data (ξ, f, S) of the RBSDE take the following form:

$$\begin{aligned} \xi &= g\left(X_T^{t,x}\right), \\ f(s,y,z) &= f\left(s, X_s^{t,x}, y, z\right), \\ S_s &= h\left(s, X_s^{t,x}\right), \end{aligned} \tag{7}$$

where g, f, and h are as follows. First, $g \in C(\mathbb{R}^d)$ has at most polynomial growth at infinity. Second,

$$f : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R} \tag{8}$$

is jointly continuous and for some $K > 0$, $p \in \mathbb{N}$, satisfies

$$\begin{gathered} \left|f(t,x,0,0)\right| \le K\left(1 + |x|^p\right), \\ \left|f(t,x,y,z) - f(t,x,y',z')\right| \le K\left(\left|y-y'\right| + \left|z-z'\right|\right), \end{gathered} \tag{9}$$

for $t \in [0,T]$, $x, z, z' \in \mathbb{R}^d$, $y, y' \in \mathbb{R}$. Finally,

$$h : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R} \tag{10}$$

is jointly continuous in t and x and satisfies

$$h(t,x) \le K\left(1 + |x|^p\right), \quad t \in [0,T], \; x \in \mathbb{R}^d. \tag{11}$$

We assume moreover that $h(T,x) \le g(x)$, $x \in \mathbb{R}^d$.

For each $t > 0$, we denote by $\{\mathscr{F}_s^t, t \le s \le T\}$ the natural filtration of Brownian motion $\{B_s - B_t, t \le s \le T\}$, argumented by the P-null sets of $\mathscr{F}$.

It follows from Theorem 2 that for each (t,x) there exists a unique triple $(Y^{t,x}, Z^{t,x}, K^{t,x})$ of $\mathscr{F}_s^t$-progressively measurable processes, which solves the following RBSDE:

(1) $\mathbb{E}\int_t^T (|Y_s^{t,x}|^2 + |Z_s^{t,x}|^2)ds < \infty$,

(2) $Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Y_r^{t,x})dr + K_T^{t,x} - K_s^{t,x} - \int_s^T Z_r^{t,x} dW_r$, $t \le s \le T$,

(3) $Y_s^{t,x} \ge h(s, X_s^{t,x})$, $0 \le t \le T$,

(4) $\{K_s^{t,x}\}$ is continuous and increasing, and $\int_t^T (Y_s^{t,x} - h(s, X_s^{t,x}))dK_s^{t,x} = 0$.

We now consider the related obstacle problem for a parabolic PDE. Roughly speaking, a solution of the obstacle problem is a function $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ which satisfie

$$\begin{gathered} \min\left(u(t,x) - h(t,x), -\frac{\partial u}{\partial t}(t,x) - L_t u(t,x) \right. \\ \left. - f(t,x,u(t,x),(\nabla u \sigma)(t,x)) \right) = 0, \\ (t,x) \in (0,T) \times \mathbb{R}^d, \\ u(T,x) = g(x), \quad x \in \mathbb{R}^d, \end{gathered} \tag{12}$$

where

$$L_t = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^*(t,x))_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x) \frac{\partial}{\partial x_i}. \tag{13}$$

We now define

$$u(t,x) \triangleq Y_t^{t,x}, \quad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{14}$$

which is a deterministic quantity; then we have the following result (see, e.g., El Karoui et al. [15]).

Theorem 4. *Defined by* (14), *$u(t,x)$ is the unique viscosity solution of the obstacle problem* (12).

3. Pricing Model with Dividends Paying

Suppose that there are two kinds of assets in a financial market: one is the bank account which obeys

$$dB_s = r_s B_s ds, \tag{15}$$

the other is the stock of the company with price process

$$dP_s = \mu_s P_s ds + \sigma_s P_s dW_s, \tag{16}$$

where r_t is the riskless interest rate and μ_t and σ_t are the expected interest rate and volatility rate of the stock, respectively.

Throughout the paper, we need the following assumptions for the market.

Assumption 5. Suppose that the financial market is like this.

(1) The capital markets are perfect with no transactions costs, no taxes, and equal access to information for all investor.

(2) The convertible bonds are not allowed to be called or putted, and the issuer will not default.

(3) The riskless interest rate r_t, the expected interest rate μ_t, and the volatility rate σ_t of the company's stock are all deterministic and bounded; σ_t is invertible, and the inverse σ_t^{-1} is also bounded.

(4) The convertible bond can be converted before any time of the maturity date T, and the conversion ratio C is a constant.

For any convertible bond in this financial market, its underlying stocks may pay dividends continuously or discretely. We will establish the pricing formulas for both cases in this section.

3.1. The Continuous Dividends Case. Generally speaking, the dividends can be influenced by many external factors, such as the underlying stock price, enterprise profits, and developing strategies. Nevertheless, under some circumstances, for example, when the stock price is relatively low or the stock is short after its going-to-market, the stock price is the elementary factor affecting the dividends paying. We study such a case in this subsection and need the following assumption.

Assumption 6. The dividends of the underlying stocks is paid continuously at rate $d(t,P_t)$, where $d(t,\cdot)$ is deterministic and bounded.

Thus, a convertible bond is similar to an American call option with the terminal value $\xi = \max(CP_T, F)$ and the "obstacle" process $\{CP_t, 0 \le t \le T\}$. So we adopt the methods for pricing American options employed by El Karoui et al. [19]. At each time t, pricing a convertible bond consists of the selection of a stopping time $\nu \in \mathcal{T}_t$ and a payoff CP_ν on exercise if $\nu < T$ and ξ if $\nu = T$. Denote

$$\widetilde{P}_s = CP_s \mathbf{1}_{\{s \le T\}} + \xi \mathbf{1}_{\{s=T\}}, \tag{17}$$

and fix $t \in [0,T]$. Suppose for a moment that the choice of $\nu \in \mathcal{T}_t$ has been made. Then there exists a unique hedging portfolio strategy $(X_s(\nu,\widetilde{P}_\nu), \pi_s(\nu,\widetilde{P}_\nu)) \in \mathcal{S}^2_{\mathcal{F}} \times \mathcal{L}^2_{\mathcal{F}}$, denoted also by (X_s^ν, π_s^ν), which replicates $\widetilde{P}_\nu$, that is, the solution of the classical BSDE associated with the terminal time ν, terminal value $\widetilde{P}_\nu$:

$$\begin{gathered} dX_s^\nu = \left[r_s X_s^\nu + (\mu_s - r_s + d(s,P_s)) \pi_s^\nu \right] ds + \sigma_s \pi_s^\nu dW_s, \\ X_\nu^\nu = \widetilde{P}_\nu. \end{gathered} \tag{18}$$

Set

$$Z_s^\nu = \sigma_s \pi_s^\nu, \qquad \theta_s = \sigma_s^{-1} (\mu_s - r_s + d(s,P_s)), \tag{19}$$

and thus (18) can be rewritten as

$$\begin{gathered} -dX_s^\nu = -(r_s X_s^\nu + \theta_s Z_s^\nu) ds - Z_s^\nu dW_s, \\ X_\nu^\nu = \widetilde{P}_\nu. \end{gathered} \tag{20}$$

Then, the price of the convertible bond at time t is given by the right-continuous adapted process $\{X_t^c, 0 \le t \le T\}$ satisfying

$$X_t^c = \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_t} X_t(\nu, \widetilde{P}_\nu). \tag{21}$$

By Proposition 3, it follows that the price process $\{X_t^c, 0 \le t \le T\}$ corresponds to the solution of a reflected BSDE.

Theorem 7. *Let Assumptions 5 and 6 hold. Then there exist* $(Z_t^c) \in \mathscr{L}_{\mathscr{F}}^2$ *and* (K_t^c) *an increasing adapted continuous process with* $K_0^c = 0$ *such that*

$$X_t^c = \xi + \int_t^T -\left(r_s X_s^c + \theta_s Z_s^c\right) ds + K_T^c - K_t^c - \int_t^T Z_s^c dW_s, \quad 0 \le t \le T, \tag{22}$$

where $X_t^c \ge CP_t$, $0 \le t \le T$, *and* $\int_0^T (X_t^c - CP_t) dK_t^c = 0$.

Furthermore, the stopping time $D_t = \inf\{t \le s \le T; X_s = CP_s\}$ *is the execution time of the convertible bond; that is*

$$X_t^c = \operatorname*{ess\,sup}_{\nu \in \mathscr{T}_t} X_t\left(\nu, \widetilde{P}_\nu\right) = X_t\left(D_t, \widetilde{P}_{D_t}\right). \tag{23}$$

The process $\{K_t^c, 0 \le t \le T\}$ may be interpreted as a cumulative consumption process. Such a triple $\{(Y_t^c, Z_t^c, K_t^c), 0 \le t \le T\}$ satisfying (22) with $X_t^c \ge CP_t$, $0 \le t \le T$, (but not necessarily $\int_0^T (X_t^c - CP_t) dK_t^c = 0$) is called a superhedging strategy for the convertible bond (CP_t, ξ). Consequently, the price X_t^c is equal to the so-called "upper price" defined as the smallest of the superhedging strategies for (CP_t, ξ).

Noticing the relationship between the solutions of reflected BSDEs and the obstacle problems for nonlinear parabolic PDEs, according to Theorem 4, the convertible bond price process $\{X_t^c, 0 \le t \le T\}$ is also the viscosity solution of a PDE.

Theorem 8. *Let Assumptions 5 and 6 hold. Then the following PDE*

$$\begin{gathered} \min\left(u(t,x) - Cx, \frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\sigma_t^2 x^2 \frac{\partial^2 u}{\partial x^2}(t,x) \right. \\ \left. + \left(r_t - d(t,x)\right) x \frac{\partial u}{\partial x}(t,x) - r_t u(t,x)\right) = 0, \\ (t,x) \in (0,T) \times \mathbb{R}, \\ u(T,x) = \max(Cx, F), \quad x \in \mathbb{R}, \end{gathered} \tag{24}$$

admits a unique viscosity solution $u(t,x)$. *The convertible bond price process* $\{X_t, 0 \le t \le T\}$ *can be given as*

$$X_t = u(t, P_t), \quad \forall t \in [0,T], \tag{25}$$

where P_t *is the price of underlying stock at time* t.

3.2. The Discrete Dividends Case. In a real financial market, the stock dividends are usually paid discretely at some future dates. Simultaneously, the stock price will fall down. More precisely, we suppose the following.

Assumption 9. The dividends is paid in the proportion δ of the stock price at some prefixed future dates $T_1, T_2, \ldots, T_n$ before the maturity date T.

Recalling the practical effects of dividends paying to the stock price, we suppose P_s admits

$$dP_s = \mu_s P_s ds + \sigma_s P_s dW_s + dL_s, \tag{26}$$

where L_s is a decreasing deterministic left continuous step function with jumps δP_{T_i} at T_i, $i = 1, 2, \ldots, n$, initialized by 0.

For simplicity and convenience, we first confine ourselves to the case of a single dividends paying before the maturity date temporarily. Correspondingly, L_s is a decreasing deterministic left continuous step function with jump δP_{T_1} at $T_1 < T$, starting at 0.

Thus, the price process (26) of the underlying stock consists of two stages:

$$\begin{gathered} dP_s = \mu_s P_s ds + \sigma_s P_s dW_s, \quad 0 \le s \le T_1, \\ P_0 = p_0, \end{gathered} \tag{27}$$

$$\begin{gathered} dP_s = \mu_s P_s ds + \sigma_s P_s dW_s, \quad T_1 < s \le T, \\ P_{T_{1+}} = (1-\delta) P_{T_1}. \end{gathered} \tag{28}$$

Firstly, we calculate the value of convertible bond (X_t^d) on the time interval $(T_1, T]$. Similar to the analysis in Section 3.1, the price process $\{X_t^d, T_1 < t \le T\}$ satisfies

$$\begin{aligned} X_t^d = \xi &+ \int_t^T -\left[\left(r_s X_s^d + \sigma_s^{-1}\left(\mu_s - r_s\right) Z_s^d\right)\right] ds \\ &+ K_T^d - K_t^d - \int_t^T Z_s^d dW_s, \quad T_1 < t \le T, \end{aligned} \tag{29}$$

where $X_t^d \ge CP_t$, $T_1 < t \le T$, and $\int_{T_1}^T (X_t^d - CP_t) dK_t^d = 0$.

Associating (29) with (28), by Theorem 4, we get the evaluation of PDE

$$\begin{gathered} \min\left(u^1(t,x) - Cx, \frac{\partial u^1}{\partial t}(t,x) + \frac{1}{2}\sigma_t^2 x^2 \frac{\partial^2 u^1}{\partial x^2}(t,x) \right. \\ \left. + r_t x \frac{\partial u^1}{\partial x}(t,x) - r_t u^1(t,x)\right) = 0, \\ (t,x) \in \left(T_1, T\right) \times \mathbb{R}, \\ u^1(T,x) = \max(Cx, F), \quad x \in \mathbb{R}. \end{gathered} \tag{30}$$

Thus, the convertible bond price $X_t^d = u^1(t, P_t)$ during the time $(T_1, T]$.

Secondly, we consider the situation on $[0, T_1]$. By (28), we know $X_{T_{1+}}^d = u^1(T_1, (1-\delta) P_{T_1})$. At the dividends paying date T_1, any reasonable investor will compare the value of converted stocks and the convertible bond and then decide whether to execute it or not. Again similarly to before, denote

$$\xi_1 = \max\left(CP_{T_1}, u^1\left(T_1, (1-\delta) P_{T_1}\right)\right). \tag{31}$$

Then, the convertible bond price process $\{X_t^d, 0 \le t \le T_1\}$ is given by

$$\begin{aligned} X_t^d = \xi_1 &+ \int_t^{T_1} -\left[\left(r_s X_s^d + \sigma_s^{-1}\left(\mu_s - r_s\right) Z_s^d\right)\right] ds \\ &+ K_{T_1}^d - K_t^d - \int_t^{T_1} Z_s^d dW_s, \quad 0 \le t \le T_1, \end{aligned} \tag{32}$$

where $X_t^d \ge CP_t$, $0 \le t \le T_1$, and $\int_0^{T_1} (X_t^d - CP_t) dK_t^d = 0$.

Also by Theorem 4, the evaluation of PDE is easily derived that

$$\min\left(u^0(t,x)-Cx,\frac{\partial u^0}{\partial t}(t,x)+\frac{1}{2}\sigma_t^2x^2\frac{\partial^2 u^0}{\partial x^2}(t,x)\right.$$
$$\left.+r_tx\frac{\partial u^0}{\partial x}(t,x)-r_tu^0(t,x)\right)=0, \tag{33}$$
$$(t,x)\in(0,T_1)\times\mathbb{R},$$
$$u^0(T_1,x)=\max\left(Cx,u^1(T_1,(1-\delta)x)\right),\quad x\in\mathbb{R},$$

and the fair price of convertible bond $X_t^d=u^0(t,P_t)$ during the time $[0,T_1]$, with P_t to be the stock price at time t.

Therefore, the convertible bond price can be calculated via two PDEs (30) and (33) which may be solved successively.

Analogously, for the general case where dividends paying are in the proportion δ of the stock price at dates $T_1,T_2,\ldots,T_n$ before the maturity date T, the convertible bond price is obtained via the following series of PDEs:

$$\min\left(u^n(t,x)-Cx,\frac{\partial u^n}{\partial t}(t,x)+\frac{1}{2}\sigma_t^2x^2\frac{\partial^2 u^n}{\partial x^2}(t,x)\right.$$
$$\left.+r_tx\frac{\partial u^n}{\partial x}(t,x)-r_tu^n(t,x)\right)=0, \tag{34}$$
$$(t,x)\in(T_n,T)\times\mathbb{R},$$
$$u^n(T,x)=\max(Cx,F),\quad x\in\mathbb{R},$$

$$\min\left(u^i(t,x)-Cx,\frac{\partial u^i}{\partial t}(t,x)+\frac{1}{2}\sigma_t^2x^2\frac{\partial^2 u^i}{\partial x^2}(t,x)\right.$$
$$\left.+r_tx\frac{\partial u^i}{\partial x}(t,x)-r_tu^i(t,x)\right)=0,$$
$$(t,x)\in(T_i,T_{i+1})\times\mathbb{R},$$
$$u^i(T_{i+1},x)=\max\left(Cx,u^{i+1}(T_{i+1},(1-\delta)x)\right),\quad x\in\mathbb{R}, \tag{35}$$

for $i=0,1,\ldots,n-1$ and $T_0=0$.

Theorem 10. *Let Assumptions 5 and 9 hold. Then, the convertible bond price is given by $u^i(t,P_t)$, where $u^i(t,x)$ is determined by (34) and (35) successively, $i=0,1,\ldots,n$. P_t is the stock price at time t.*

4. The Pricing Model with Higher Borrowing Rate

In this section, let us investigate a more realistic case in the financial market, that is as follows.

Assumption 11. The borrowing rate R_t is higher than the interest rate r_t, where R_t is also deterministic and bounded.

Then analogous to the discussion in Section 3.1, suppose that the dividends is still paid continuously at the rate $d(t,P_t)$. Then for any fixed $t\in[0,T]$ and $\nu\in\mathcal{T}_t$, there exists a unique hedging portfolio strategy $(\widetilde{X}_s(\nu,\widetilde{P}_\nu),\widetilde{\pi}_s(\nu,\widetilde{P}_\nu))\in\mathcal{S}_{\mathcal{F}}^2\times\mathcal{L}_{\mathcal{F}}^2$, denoted also by $(\widetilde{X}_s^\nu,\widetilde{\pi}_s^\nu)$ replicating $\widetilde{P}_\nu$; that is

$$-d\widetilde{X}_s^\nu=-\left[r_s\widetilde{X}_s^\nu+(\mu_s-r_s+d(s,P_s))\widetilde{\pi}_s^\nu\right.$$
$$\left.-(R_s-r_s)\left(\widetilde{X}_s^\nu-\widetilde{\pi}_s^\nu\right)^-\right]ds-\sigma_s\widetilde{\pi}_s^\nu dW_s,$$
$$\widetilde{X}_\nu^\nu=\widetilde{P}_\nu. \tag{36}$$

Thus, the price of convertible bond is given by the right-continuous adapted process $\{\widetilde{X}_t,0\le t\le T\}$ satisfying at each time t

$$\widetilde{X}_t=\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_t}\widetilde{X}_t\left(\nu,\widetilde{P}_\nu\right). \tag{37}$$

Again by Proposition 3, it follows that the price process $\{\widetilde{X}_t,0\le t\le T\}$ corresponds to the solution of a reflected BSDE.

Theorem 12. *Let Assumptions 5, 6, and 11 hold. Then, there exist $(\widetilde{\pi}_t)\in\mathcal{L}_{\mathcal{F}}^2$ and $(\widetilde{K}_t)$ an increasing adapted continuous process with $\widetilde{K}_0=0$ such that*

$$\widetilde{X}_t=\xi+\int_t^T-\left[r_s\widetilde{X}_s^\nu+(\mu_s-r_s+d(s,P_s))\widetilde{\pi}_s^\nu\right.$$
$$\left.-(R_s-r_s)\left(\widetilde{X}_s^\nu-\widetilde{\pi}_s^\nu\right)^-\right]ds \tag{38}$$
$$+\widetilde{K}_T-\widetilde{K}_t-\int_t^T\sigma_s\widetilde{\pi}_s^\nu dW_s,\quad 0\le t\le T,$$

where $\widetilde{X}_t\ge CP_t$, $0\le t\le T$, and $\int_0^T(\widetilde{X}_t-CP_t)d\widetilde{K}_t=0$.

Furthermore, the stopping time $\widetilde{D}_t=\inf\{t\le s\le T;\widetilde{X}_s=CP_s\}$ is the execution time of the convertible bond; that is

$$\widetilde{X}_t=\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_t}\widetilde{X}_t\left(\nu,\widetilde{P}_\nu\right)=\widetilde{X}_t\left(\widetilde{D}_t,\widetilde{P}_{\widetilde{D}_t}\right). \tag{39}$$

Combining Theorem 4, we easily obtain the following.

Theorem 13. *Let Assumptions 5, 6, and 11 hold. Then, the following PDE*

$$\min\left(\widetilde{u}(t,x)-Cx,\frac{\partial\widetilde{u}}{\partial t}(t,x)+\frac{1}{2}\sigma_t^2x^2\frac{\partial^2\widetilde{u}}{\partial x^2}(t,x)\right.$$
$$+(r_t-d(t,x))x\frac{\partial\widetilde{u}}{\partial x}(t,x)-r_t\widetilde{u}(t,x)$$
$$\left.+(R_t-r_t)\left(\widetilde{u}-x\frac{\partial\widetilde{u}}{\partial x}\right)^-(t,x)\right)=0, \tag{40}$$
$$(t,x)\in(0,T)\times\mathbb{R},$$
$$\widetilde{u}(T,x)=\max(Cx,F),\quad x\in\mathbb{R},$$

admits a unique viscosity solution $\widetilde{u}(t,x)$. The convertible bond price process $\{\widetilde{X}_t,0\le t\le T\}$ can be given as

$$\widetilde{X}_t=\widetilde{u}(t,P_t),\quad\forall t\in[0,T], \tag{41}$$

where P_t is the price of underlying stock at time t.

Remark 14. For the case that the dividends are paid discretely as in Assumption 9, one can easily derive the pricing formula for convertible bonds with higher borrowing rate by adopting the same methods stated previously. Thus, we just omit it here.

5. Numerical Computations and Analysis

From Theorems 8, 10, and 13, we can obtain the convertible bonds price in different situations by solving (24), (34), (35), and (40), respectively. However, there is no available analytical formula for the solutions of these PDEs. So we adopt the radial basis function (RBF) methods to solve PDEs and then present some examples to illustrate our theoretical results in above sections.

For simplicity, all functions R_t, r_t, μ_t, σ_t, and $d(t,\cdot)$ are assumed to be constants in this section. We take PDE (24), for example, to introduce the RBF methods as follows. Let us first consider a PDE:

$$\begin{gathered}\frac{\partial u}{\partial t}(t,x)+\frac{1}{2}\sigma^2x^2\frac{\partial^2 u}{\partial x^2}(t,x)\\ +(r-d)x\frac{\partial u}{\partial x}(t,x)-ru(t,x)=0, \qquad (42)\\ (t,x)\in(0,T)\times\mathbb{R},\\ u(T,x)=\max(Cx,F),\quad x\in\mathbb{R}.\end{gathered}$$

A simple transformation of $x=e^y$ changes (42) to

$$\begin{gathered}\frac{\partial u}{\partial t}(t,y)+\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2}(t,y)\\ +\left(r-d-\frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y}(t,y)-ru(t,y)=0, \qquad (43)\\ (t,y)\in(0,T)\times\mathbb{R},\\ u(T,y)=\max(Ce^y,F),\quad x\in\mathbb{R}.\end{gathered}$$

The idea of the proposed numerical scheme is to interpolate the unknown function u by the following radial function ϕ:

$$u(t,y)\simeq\sum_{j=1}^N\alpha_j(t)\phi\left(\left|y-y_j\right|\right), \qquad (44)$$

where α_j are unknown coefficients depending on time and $\phi_j(y)=\phi(|y-y_j|)$ are called RBFs because $|y-y_j|$ denotes the radial distance of each of the N scattered data points y_j, $j=1,2,\ldots,N$. As we mentioned before, in this paper, we take Hardy's Multiquadric, as the radial basis function for the computation scheme. Thus

$$\phi_j(y)=\sqrt{\left(y-y_j\right)^2+c^2}, \qquad (45)$$

where c is a positive constant called the shape parameter.

Collocating at the same N points y_j by substituting (44) into (43), we obtain the following system of linear equations, for $i=1,2,\ldots,N$,

$$\begin{gathered}\frac{\partial u}{\partial t}(t,y_i)+\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2}(t,y_i)\\ +\left(r-d-\frac{1}{2}\sigma^2\right)\frac{\partial u}{\partial y}(t,y_i)-ru(t,y_i)=0.\end{gathered} \qquad (46)$$

Since $\phi_j(y)$ does not depend on time, the time derivative of u is simply given in terms of the time derivatives of the coefficients:

$$\frac{\partial u}{\partial t}(t,y_i)=\sum_{j=1}^N\frac{d\alpha_j(t)}{dt}\phi\left(\left|y_i-y_j\right|\right). \qquad (47)$$

The first and second partial derivatives of u with respect to y are given, respectively, as follows:

$$\begin{aligned}\frac{\partial u}{\partial y}(t,y_i)&=\sum_{j=1}^N\alpha_j(t)\frac{\partial\phi}{\partial y}\left(\left|y_i-y_j\right|\right),\\ \frac{\partial^2 u}{\partial y^2}(t,y_i)&=\sum_{j=1}^N\alpha_j(t)\frac{\partial^2\phi}{\partial y^2}\left(\left|y_i-y_j\right|\right),\end{aligned} \qquad (48)$$

in which

$$\frac{\partial\phi}{\partial y}\left(\left|y_i-y_j\right|\right)=\frac{y_i-y_j}{\sqrt{\left(y_i-y_j\right)^2+c^2}}, \qquad (49)$$

$$\begin{aligned}\frac{\partial^2\phi}{\partial y^2}\left(\left|y_i-y_j\right|\right)=&\frac{1}{\sqrt{\left(y_i-y_j\right)^2+c^2}}\\ &-\frac{\left(y_i-y_j\right)^2}{\left(\sqrt{\left(y_i-y_j\right)^2+c^2}\right)^{3/2}}.\end{aligned} \qquad (50)$$

Thus, in matrix form, (46) can be expressed as

$$L\dot{\alpha}+\frac{1}{2}\sigma^2L_{yy}\alpha+\left(r-d-\frac{1}{2}\sigma^2\right)L_y\alpha-rL\alpha=0, \qquad (51)$$

where α denotes the vector containing the unknown coefficients $\alpha_j(t)$, $\dot{\alpha}$ the time derivatives, and L, L_y, and L_{yy} are $N\times N$ matrices with entries $\phi(|y_i-y_j|)$, $(\partial\phi/\partial y)(|y_i-y_j|)$, and $(\partial^2\phi/\partial y^2)(|y_i-y_j|)$ given by (45), (49), and (50), respectively. It has been proven by Powell [20] that the matrix L is invertible for distinct collocation points y_j, and hence L^{-1} exists. Thus, (51) can be rewritten as

$$\dot{\alpha}=-L^{-1}\left[\frac{1}{2}\sigma^2L_{yy}\alpha+\left(r-d-\frac{1}{2}\sigma^2\right)L_y\alpha-rL\alpha\right]\equiv G\alpha, \qquad (52)$$

where G is the $N\times N$ matrix

$$G=rI-\frac{1}{2}\sigma^2L^{-1}L_{yy}-\left(r-d-\frac{1}{2}\sigma^2\right)L^{-1}L_y, \qquad (53)$$

with I being the identity matrix of size N.

For fixed collocation points y_j, $j = 1, 2, \ldots, N$, (52) is a linear system of first-order homogeneous ordinary differential equations with constant coefficients, with the terminal condition

$$\alpha(T) = L^{-1}\left(u(T, y_1), u(T, y_2), \ldots, u(T, y_N)\right)^T \equiv L^{-1}U, \tag{54}$$

where $U = (u(T, y_1), u(T, y_2), \ldots, u(T, y_N))^T$, whose value is given by (43). Then, we can use explicit second-order backward time integration scheme to obtain the unknown coefficients α at each time step $T - \Delta t$. For notational convenience, let $U^n = L\alpha^n$ denote the approximation $U(T - n\Delta t, y_i)$ at each time step $T - n\Delta t$. We have

$$\begin{aligned} F_1 &= -\Delta t G \alpha^{n-1}, \\ F_2 &= -\Delta t G \left(\alpha^{n-1} + 0.5 F_1\right), \\ \alpha^n &= \alpha^{n-1} + 0.5\left(F_1 + F_2\right). \end{aligned} \tag{55}$$

Thus, the solution of (42) can be approximated by (44).

Now let us turn to (24). From the different forms of (24) and (42), it is easily seen that the evaluation of (24) can be performed by modifying the above procedure as follows.

Step 1: compute $U^n = L\alpha^n$.

Step 2: for $i = 1, 2, \ldots, N$, update the ith element $U^n(i) = \max(Ce^{y_i}, U^n(i))$.

Step 3: The updated α^n is simply $L^{-1}U^n$.

Then, the solution of (24) can also be approximated by (44).

Remark 15. For PDEs (34), (35), and (40), one can easily proceed the evaluation similarly as above. Here, we do not give the detailed derivation any more.

Remark 16. The accuracy of MQ is greatly affected by the choice of the shape parameter c, and an optimal recipe for the value of c is still under intensive research. We adopt the suggestion from Hardy [16] by choosing c to be $4d_{\min}$, where $d_{\min}$ is the minimum distance between any two collocation points y_j.

Now, let us present some examples of pricing the convertible bonds by above numerical calculation and discuss the influence of dividends as well as the higher borrowing rate.

Example 17. Suppose for four convertible bonds, $C = 4.36$, $F = 1000$ yuan, $r = 0.1121$, $\sigma = 0.30$, and $T = 15$ years are the same, while their underlying stocks pay dividends at different rates $d_1 = 0$, $d_2 = 0.005$, $d_3 = 0.010$, and $d_4 = 0.016$. Then according to Theorem 8, computing (24) by RBF method, we obtain their prices corresponding to the initial stock prices varying from 52.00 to 54.00 yuan as shown in Table 1.

From Table 1, we can see that the continuous dividends paying lower the price of convertible bond. The larger the dividends rate is, the lower the convertible bond price is. From the point of view of economics, the actual value of stocks will decrease for the dividends paying. Then by the comparison theorem of reflected BSDEs, the price of convertible bond will indeed lower correspondingly.

Table 1: The price of convertible bond with different dividends rates.

Stock price	Dividend rate			
	$d_1 = 0.000$	$d_2 = 0.005$	$d_3 = 0.010$	$d_4 = 0.016$
52.00	297.0862	288.5835	281.1576	273.7534
52.25	297.9239	289.3795	281.9206	274.4878
52.50	298.7629	290.1769	282.6853	275.2241
52.75	299.6033	290.9758	283.4515	275.9623
53.00	300.4450	291.7761	284.2194	276.7023
53.25	301.2880	292.5778	284.9889	277.4442
53.50	302.1323	293.3810	285.7600	278.1880
53.75	302.9779	294.1856	286.5326	278.9335
54.00	303.8248	294.9916	287.3069	279.6810

Table 2: The price of convertible bond with different dividends proportions.

Stock price	Dividend proportion			
	$\delta_1 = 0.000$	$\delta_2 = 0.005$	$\delta_3 = 0.010$	$\delta_4 = 0.016$
52.00	297.0862	289.1203	282.2663	275.4526
52.25	297.9239	289.9201	283.0373	276.2003
52.50	298.7629	290.7213	283.8099	276.9499
52.75	299.6033	291.5239	284.5842	277.7014
53.00	300.4450	292.3281	285.3601	278.4548
53.25	301.2880	293.1336	286.1376	279.2100
53.50	302.1323	293.9406	286.9167	279.9672
53.75	302.9779	294.7490	287.6975	280.7262
54.00	303.8248	295.5588	288.4798	281.4871

Example 18. Suppose for four convertible bonds, their underlying stocks pay dividends in different proportions $\delta_1 = 0$, $\delta_2 = 0.005$, $\delta_3 = 0.010$, and $\delta_4 = 0.016$ of the prices at the end of every year after issuing, while other features like C, F, r, σ, and T are the same as those in Example 17. Then according to Theorem 10, computing (34) and (35) successively by RBF method, we obtain their prices corresponding to the initial stock prices varying from 52.00 to 54.00 yuan as shown in Table 2.

This example shows that how the discrete dividends paying affects the prices of convertible bonds. The stock price will fall down explicitly after every time the dividends being paid. As the explanation of Example 17, it is reasonable for the convertible bond price to decline accordingly.

At last, let us see the influence of higher borrowing rate on the convertible bond price.

Example 19. Suppose all features about the convertible bonds keep the same with Example 17, while the borrowing rate $R = 0.150$, which is higher than the interest rate r in the financial market. Then according to Theorem 13, computing (40) by RBF method, we obtain their prices corresponding to the initial stock prices varying from 52.00 to 54.00 yuan as shown in Table 3.

Table 3: The price of convertible bond with higher borrowing rate.

Stock price	Dividend rate			
	$d_1 = 0.000$	$d_2 = 0.005$	$d_3 = 0.010$	$d_4 = 0.016$
52.00	297.0862	288.5835	281.1576	273.7534
52.25	297.9239	289.3795	281.9206	274.4878
52.50	298.7629	290.1769	282.6853	275.2241
52.75	299.6033	290.9758	283.4515	275.9623
53.00	300.4450	291.7761	284.2194	276.7023
53.25	301.2880	292.5778	284.9889	277.4442
53.50	302.1323	293.3810	285.7600	278.1880
53.75	302.9779	294.1856	286.5326	278.9335
54.00	303.8248	294.9916	287.3069	279.6810

Comparing Table 3 with Table 1, it is an interesting fact that the higher borrowing rate has no influence on the prices of convertible bonds. When the borrowing rate R is up to 0.200, one can obtain the same result. It means that, in order to hedge the contingent claim of convertible bond, one need not to borrow money from the bank; that is, the money invested into the stock market is always less than total wealth of the investor. Wang and Wu [21] have proved that this fact is always true when there are no dividends by means of Malliavin derivatives and comparison theorem for BSDEs. However, it needs more investigation for whether it still holds with dividends being paid.

6. Conclusion and Extension

It is the first attempt to formulate the pricing model for convertible bonds with dividends paying of their underlying stocks via reflected BSDEs, to the authors' knowledge. Compared with existing literatures devoting to this problem, there are three distinguishing features of our paper: (1) we consider the situation that underlying stocks pay dividends both continuously and discretely and the borrowing rate is higher than interest rate; (2) we establish the pricing formula for convertible bonds by the solution of obstacle problems for some PDEs; (3) we present a numerical evaluation method, the RBFs, to evaluate the convertible bond prices, illustrating the applications of our theoretical study.

In our numerical examples, we see that the higher borrowing rate has no effect on the price of convertible bond. However, it begs systematic study to prove whether this fact is always true no matter the dividends is paid continuously or discretely, which seems related to the field of Malliavin derivatives. We hope to develop some theories for this topic. Besides, during the numerical procedure, all coefficients are confined to constants, and the accuracy and stability of RBF methods need more investigation and improvement. We are also looking forward to a modified method, which can deal with time-dependent or even random coefficients.

Acknowledgments

The authors would like to thank Professor Zhen Wu for his enthusiastic discussions and enlightened suggestions related to this work. Also the authors are grateful to anonymous referees and editors for their valuable comments, which led to an improved version of this paper. This work is supported by the Natural Science Foundation of China (10921101 and 61174092).

References

[1] O. H. Poensgen, "The valuation of convertible bonds, part 1," *Industrial Management Review*, vol. 7, pp. 77–92, 1965.

[2] O. H. Poensgen, "The valuation of convertible bonds, part 2," *Industrial Management Review*, vol. 7, pp. 83–98, 1966.

[3] W. J. Baumol, B. G. Malkiel, and R. E. Quandt, "The valuation of convertible securities," *Quarterly Journal of Economics*, vol. 40, pp. 48–59, 1966.

[4] R. Weil, J. E. Segall, and D. Green, "Premiums on convertible bonds," *Journal of Finance*, vol. 23, pp. 445–463, 1968.

[5] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, pp. 637–654, 1973.

[6] J. E. Ingersoll, "A contingent-claims valuation of convertible securities," *Journal of Financial Economics*, vol. 4, pp. 289–382, 1977.

[7] M. J. Brennan and E. S. Schwartz, "Convertible bonds: valuation and optimal strategies for call and conversion," *Journal of Finance*, vol. 32, pp. 1699–1715, 1977.

[8] M. J. Brennan and E. S. Schwartz, "Analyzing convertible bonds," *Journal of Financal and Quantitative Analysis*, vol. 15, pp. 907–929, 1980.

[9] J. J. McConnell and E. S. Schwartz, "LYON taming," *Journal of Finance*, vol. 41, pp. 561–576, 1986.

[10] T. S. Y. Ho and D. M. Pfeffer, "Convertible bonds: model, value attribution, and analytics," *Financial Analysts Journal*, vol. 52, pp. 35–44, 1996.

[11] K. Tsiveriotis and C. Fernandes, "Valuing convertible bonds with credit risk," *Journal of Fixed Income*, vol. 8, pp. 95–102, 1998.

[12] E. Pardoux and S. G. Peng, "Adapted solution of a backward stochastic differential equation," *Systems and Control Letters*, vol. 14, no. 1, pp. 55–61, 1990.

[13] D. Duffie and L. G. Epstein, "Stochastic differential utility," *Econometrica*, vol. 60, no. 2, pp. 353–394, 1992.

[14] N. El Karoui, S. G. Peng, and M. C. Quenez, "Backward stochastic differential equations in finance," *Mathematical Finance*, vol. 7, no. 1, pp. 1–71, 1997.

[15] N. El Karoui, C. Kapoudjian, E. Pardoux, S. G. Peng, and M. C. Quenez, "Reflected solutions of backward SDE's, and related obstacle problems for PDE's," *The Annals of Probability*, vol. 25, no. 2, pp. 702–737, 1997.

[16] R. L. Hardy, "Multiquadric equations of topography and other irregular surfaces," *Journal of Geophysical Research*, vol. 76, pp. 1905–1915, 1971.

[17] R. Franke, "Scattered data interpolation: tests of some methods," *Mathematics of Computation*, vol. 38, no. 157, pp. 181–200, 1982.

[18] Y. C. Hon and X. Z. Mao, "A radial basis function method for solving options pricing models," *Financial Engineering*, vol. 8, pp. 31–50, 1999.

[19] N. El Karoui, E. Pardoux, and M. C. Quenez, "Reected backward SDEs and American options," *Numerical Methods in Finance*, vol. 6, pp. 215–231, 1997.

[20] M. J. D. Powell, "The theory of radial basis function approximation," in *Advances in Numerical Analysis*, vol. 3, pp. 105–210, Oxford University Press, New York, NY, USA, 1992.

[21] H. Y. Wang and Z. Wu, "Backward stochastic differential equations and application in pricing the convertible bond," *Journal of Mathematical Analysis and Applications*. In press.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 423101, 11 pages
http://dx.doi.org/10.1155/2013/423101

Research Article

Classical Solutions of Path-Dependent PDEs and Functional Forward-Backward Stochastic Systems

Shaolin Ji[1] and Shuzhen Yang[2]

[1] *Institute for Financial Studies and Institute of Mathematics, Shandong University, Jinan, Shandong 250100, China*
[2] *School of Mathematics, Shandong University, Jinan, Shandong 250100, China*

Correspondence should be addressed to Shuzhen Yang; yangsz@mail.sdu.edu.cn

Received 20 February 2013; Accepted 20 April 2013

Academic Editor: Guangchen Wang

In this paper we study the relationship between functional forward-backward stochastic systems and path-dependent PDEs. In the framework of functional Itô calculus, we introduce a path-dependent PDE and prove that its solution is uniquely determined by a functional forward-backward stochastic system.

1. Introduction

It is well known that quasilinear parabolic partial differential equations are related to Markovian forward-backward stochastic differential equations (see [1–3]), which generalizes the classical Feynman-Kac formula. Recently in the frame-work of functional Itô calculus, a path-dependent PDE was introduced by Dupire [4] and the so-called functional Feynman-Kac formula was also obtained. For a recent account and development of this theory we refer the reader to [5–11].

In this paper, we study a functional forward-backward system and its relation to a quasilinear parabolic path-dependent PDE. In more details, the functional forward-backward system is described by the following forward-backward SDE:

$$\begin{aligned} X^{\gamma_t}(s) &= \gamma_t(t) + \int_t^s b(X_r^{\gamma_t})\,dr + \int_t^s \sigma(X_r^{\gamma_t})\,dW(r), \\ X^{\gamma_t}(r) &= \gamma_t(r), \quad 0 \le r \le t, \\ Y^{\gamma_t}(s) &= g(X_T^{\gamma_t}) - \int_s^T h(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r))\,dr \\ &\quad - \int_s^T Z^{\gamma_t}(r)\,dW(r), \quad s \in [t,T]. \end{aligned} \tag{1}$$

Equation (1) is an uncoupled functional forward-backward system, its general the results of [8], and there are many applications of the uncoupled functional forward-backward system in optimal control problem. The main difference is that we give a weaker requirement of g and h about X, and we also establish some estimates and regularity results for the solution with respect to paths. Then, we prove that the solution of (1) is the unique classical solution of the following path-dependent PDE:

$$\begin{aligned} D_t u(\gamma_t) + \mathscr{L} u(\gamma_t) &= h(\gamma_t, u(\gamma_t), D_x u(\gamma_t)\sigma(\gamma_t)), \\ u(\gamma_T) &= g(\gamma_T), \quad \gamma_T \in \Lambda, \end{aligned} \tag{2}$$

where

$$\mathscr{L} u = \frac{1}{2}\operatorname{tr}\left[\sigma\sigma^T D_{xx} u\right] + \langle b, D_x u\rangle. \tag{3}$$

The paper is organized as follows: in Section 2, we give the notations and results on functional FBSDEs and functional Itô calculus. Some estimates and regularity results for the solution of FBSDEs are established in Section 3. Finally, we prove the relationship between functional FBSDEs and path-dependent PDEs in Section 4.

2. Preliminaries

2.1. Functional FBSDEs. Let $\Omega = C([0,T];\mathbb{R}^d)$ and let P be the Wiener measure on $(\Omega,\mathbb{B}(\Omega))$. We denote by $W = (W(t)_{t\in[0,T]})$ the cannonical Wiener process, with $W(t,\omega) = \omega(t)$, $t \in [0,T]$, $\omega \in \Omega$. For any $t \in [0,T]$ we denote by $\mathcal{F}_t$ the P-completion of $\sigma(W(s), s \in [0,t])$.

For any $t \in [0,T]$, we denote by $L^2(\Omega,\mathcal{F}_t;\mathbb{R}^n)$ the set of all square integrable $\mathcal{F}_t$-measurable random variables, $M^2(0,T;\mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\vartheta(\cdot)$ such that

$$E\int_0^T |\vartheta(s)|^2 ds < +\infty. \tag{4}$$

Let $t \in [0,T]$ and $\gamma_t \in \Lambda$. For every $s \in [t,T]$, we consider the following functional forward-backward SDEs:

$$X^{\gamma_t}(s) = \gamma_t(t) + \int_t^s b(X_r^{\gamma_t})\,dr + \int_t^s \sigma(X_r^{\gamma_t})\,dW(r), \tag{5}$$

$$\begin{aligned} Y^{\gamma_t}(s) &= g(X_T^{\gamma_t}) - \int_s^T h(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r))\,dr \\ &\quad - \int_s^T Z^{\gamma_t}(r)\,dW(r), \end{aligned} \tag{6}$$

where

$$X^{\gamma_t}(s) = \gamma_t(s), \quad s \in [0,t]. \tag{7}$$

The processes X, Y, Z take values in $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^{n\times d}$; b, h, σ, and g take values in $\mathbb{R}^n, \mathbb{R}^n, \mathbb{R}^{n\times d}$, and $\mathbb{R}^n$. Equations (5) and (6) can be rewritten as

$$\begin{aligned} dX^{\gamma_t}(s) &= b(X_s^{\gamma_t})\,ds + \sigma(X_s^{\gamma_t})\,dW(s), \\ dY^{\gamma_t}(s) &= h(X_s^{\gamma_t}, Y^{\gamma_t}(s), Z^{\gamma_t}(s))\,ds + Z^{\gamma_t}(s)\,dW(s), \\ X^{\gamma_t}(t) &= \gamma_t(t), \qquad Y^{\gamma_t}(T) = g(X_T^{\gamma_t}). \end{aligned} \tag{8}$$

For $z \in \mathbb{R}^{n\times d}$, we define $|z| = \{\mathrm{tr}(zz^T)\}^{1/2}$. For $z^1 \in \mathbb{R}^{n\times d}$, $z^2 \in \mathbb{R}^{n\times d}$,

$$((z^1, z^2)) = \mathrm{tr}(z^1(z^2)^T), \tag{9}$$

and for $u^1 = (y^1, z^1) \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$, $u^2 = (y^2, z^2) \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$

$$[u^1, u^2] = \langle y^1, y^2\rangle + ((z^1, z^2)). \tag{10}$$

We give the following assumption.

Assumption 1. For all x_T^1, $x_T^2 \in \Lambda$, $b(x_\cdot^1)$, $\sigma(x_\cdot^1) \in M^2$, and $t \in [0,T]$, there exists a constant $c_1 > 0$, such that

$$|b(x_t^1) - b(x_t^2)| + |\sigma(x_t^1) - \sigma(x_t^2)| \le c_1 \|x_t^1 - x_t^2\|, \quad \text{a.e.} \tag{11}$$

and for all $x_t \in \Lambda$,

$$|b(x_t)| + |\sigma(x_t)| \le c_1(1 + \|x_t\|), \quad \text{a.e.} \tag{12}$$

Definition 2. $X : [0,T] \times \Omega \to \mathbb{R}^n$ is called an adapted solution of (5), if $X \in M^2(0,T;\mathbb{R}^n)$, and it satisfies (5) P-a.s.

Then we have the following theorem (see [12]).

Theorem 3. *Let Assumption 1 hold, then there exists a unique adapted solution X for* (5).

2.2. Functional Itô Calculus. The following notations and tools are mainly from Dupire [4]. Let $T > 0$ be fixed. For each $t \in [0,T]$, we denote by Λ_t the set of càdlàg $\mathbb{R}^d$-valued functions on $[0,t]$. For each $\gamma \in \Lambda_T$ the value of γ at time $s \in [0,T]$ is denoted by $\gamma(s)$. Thus $\gamma = \gamma(s)_{0\le s\le T}$ is a càdlàg process on $[0,T]$ and its value at time s is $\gamma(s)$. The path of γ up to time t is denoted by γ_t, that is, $\gamma_t = \gamma(s)_{0\le s\le t} \in \Lambda_t$. We denote $\Lambda = \bigcup_{t\in[0,T]} \Lambda_t$. For each $\gamma_t \in \Lambda$ and $x \in \mathbb{R}^d$ we denote by $\gamma_t(s)$ the value of γ_t at $s \in [0,t]$ and $\gamma_t^x := (\gamma_t(s)_{0\le s<t}, \gamma_t(t) + x)$ which is also an element in Λ_t.

Let $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the inner product and norm in $\mathbb{R}^n$. We now define a distance on Λ. For each $0 \le t, \bar{t} \le T$ and $\gamma_t, \bar{\gamma}_{\bar{t}} \in \Lambda$, we denote

$$\begin{aligned} \|\gamma_t\| &:= \sup_{s\in[0,t]} |\gamma_t(s)|, \\ \|\gamma_t - \bar{\gamma}_{\bar{t}}\| &:= \sup_{s\in[0,t\vee\bar{t}]} |\gamma_t(s\wedge t) - \bar{\gamma}_{\bar{t}}(s\wedge\bar{t})|, \\ d_\infty(\gamma_t, \bar{\gamma}_{\bar{t}}) &:= \sup_{0\le s\le t\vee\bar{t}} |\gamma_t(s\wedge t) - \bar{\gamma}_{\bar{t}}(s\wedge\bar{t})| + |t - \bar{t}|. \end{aligned} \tag{13}$$

It is obvious that Λ_t is a Banach space with respect to $\|\cdot\|$ and d_∞ is not a norm.

Definition 4. A function $u : \Lambda \mapsto \mathbb{R}$ is said to be Λ-continuous at $\gamma_t \in \Lambda$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\bar{\gamma}_{\bar{t}} \in \Lambda$ with $d_\infty(\gamma_t, \bar{\gamma}_{\bar{t}}) < \delta$, we have $|u(\gamma_t) - u(\bar{\gamma}_{\bar{t}})| < \varepsilon$. u is said to be Λ-continuous if it is Λ-continuous at each $\gamma_t \in \Lambda$.

Definition 5. Let $u : \Lambda \mapsto \mathbb{R}$ and $\gamma_t \in \Lambda$ be given. If there exists $p \in \mathbb{R}^d$, such that

$$u(\gamma_t^x) = u(\gamma_t) + \langle p, x\rangle + o(|x|) \quad \text{as } x \longrightarrow 0,\ x \in \mathbb{R}^d, \tag{14}$$

then we say that u is vertically differentiable at γ_t and denote the gradient of $D_x u(\gamma_t) = p$. If $D_x u(\gamma_t)$ exists for each $\gamma_t \in \Lambda$, u is said to be vertically differentiable in Λ.

We can similarly define the Hessian $D_{xx} u(\gamma_t)$. It is an $\mathbb{S}(d)$-valued function defined on Λ, where $\mathbb{S}(d)$ is the space of all $d \times d$ symmetric matrices.

For each $\gamma_t \in \Lambda$ we denote

$$\gamma_{t,s}(r) = \gamma_t(r)\mathbf{1}_{[0,t)}(r) + \gamma_t(t)\mathbf{1}_{[t,s]}(r), \quad r \in [0,s]. \tag{15}$$

It is clear that $\gamma_{t,s} \in \Lambda_s$.

Definition 6. For a given $\gamma_t \in \Lambda$ if we have

$$u(\gamma_{t,s}) = u(\gamma_t) + a(s-t) + o(|s-t|) \quad \text{as } s \longrightarrow t,\ s \ge t, \tag{16}$$

then we say that $u(\gamma_t)$ is (horizontally) differentiable in t at γ_t and $D_t u(\gamma_t) = a$. u is said to be horizontally differentiable in Λ if $D_t u(\gamma_t)$ exists for each $\gamma_t \in \Lambda$.

Definition 7. Define $\mathbb{C}^{j,k}(\Lambda)$ as the set of function $u := (u(\gamma_t))_{\gamma_t \in \Lambda}$ defined on Λ which are j times horizontally and k times vertically differentiable in Λ such that all these derivatives are Λ-continuous.

The following Itô formula was firstly obtained by Dupire [4] and then generalized by Cont and Fournié [5–7].

Theorem 8 (functional Itô's formula). *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a probability space, if X is a continuous semi-martingale and u is in $\mathbb{C}^{1,2}(\Lambda)$, then for any $t \in [0,T)$,*

$$u(X_t) - u(X_0) = \int_0^t D_s u(X_s)\,ds + \int_0^t D_x u(X_s)\,dX(s) + \frac{1}{2}\int_0^t D_{xx} u(X_s)\,d\langle X\rangle(s), \quad P\text{-a.s.} \tag{17}$$

3. Regularity

We first recall some notions in Pardoux and Peng [2]. $C^n(\mathbb{R}^p;\mathbb{R}^q)$, $C_b^n(\mathbb{R}^p;\mathbb{R}^q)$, $C_p^n(\mathbb{R}^p;\mathbb{R}^q)$ will denote, respectively, the set of functions of class C^n from $\mathbb{R}^p$ into $\mathbb{R}^q$, the set of those functions of class C_b^n whose partial derivatives of order less than or equal to n are bounded, and the set of those functions of class C_p^n which, together with all their partial derivatives of order less than or equal to n, grow at most like a polynomial function of the variable x at infinity.

Now we give the definition of derivatives in our context. Under Assumption 1 we have that

$$\begin{gathered} dX^{\gamma_t}(s) = b(X_s^{\gamma_t})\,ds + \sigma(X_s^{\gamma_t})\,dW(s), \\ X^{\gamma_t}(t) = \gamma_t(t), \end{gathered} \tag{18}$$

has a unique solution. For $t \le s \le T$, set

$$\begin{gathered} \widetilde{\Lambda}_{\gamma_t,s} := \{\overline{\gamma}_s : \overline{\gamma}(h) = X^{\gamma_t}(h,\omega), 0 \le h \le s, \omega \in \Omega\}, \\ \widetilde{\Lambda}_{t,s} := \bigcup_{\gamma_t \in \Lambda_t} \widetilde{\Lambda}_{\gamma_t,s}, \qquad \widetilde{\Lambda}_t := \bigcup_{t\le s\le T} \widetilde{\Lambda}_{t,s}. \end{gathered} \tag{19}$$

Then the following definition of derivatives will be used frequently in the sequel.

Definition 9. An $\mathbb{R}^n$-valued function g is said to be in $C^2(\widetilde{\Lambda}_{\gamma_t,T})$, if for $\gamma_1 \in \widetilde{\Lambda}_{\gamma_t,T}$ and $\gamma_2 \in \widetilde{\Lambda}_{\gamma_t^y,T}$, there exist $p_1 \in \mathbb{R}^d$ and $p_2 \in \mathcal{S}^d$ ($\mathcal{S}^d$ is the set of all d order symmetric matrix) such that

$$g(\gamma_2) - g(\gamma_1) = \langle p_1, y\rangle + \frac{1}{2}\langle p_2 y, y\rangle + o(|y|^2), \quad x \in \mathbb{R}^d. \tag{20}$$

We denote $g'_{\gamma_t}(\gamma_1) := p_1$, and $g''_{\gamma_t}(\gamma_1) := p_2$. g is said to be in $C^2_{l,lip}(\widetilde{\Lambda}_{t,T})$ if $g'_{\gamma_t}(\gamma)$ and $g''_{\gamma_t}(\gamma)$ exist for each $\gamma_t \in \Lambda_t$, and there exist some constants $C \ge 0$ and $k \ge 0$ depending only on g such that for each $\gamma, \overline{\gamma} \in \Lambda_T$, $t, s \in [0,T]$,

$$|g(\gamma) - g(\overline{\gamma})| \le C\left(\|\gamma\|^k + \|\overline{\gamma}\|^k\right)\|\gamma - \overline{\gamma}\|, \tag{21}$$

and for each $\gamma \in \widetilde{\Lambda}_{t,T}$, $\overline{\gamma} \in \widetilde{\Lambda}_{s,T}$, $t, s \in [0,T]$,

$$\left|\Phi_{\gamma_t}(\gamma) - \Phi_{\gamma_s}(\overline{\gamma})\right| \le C\left(\|\gamma\|^k + \|\overline{\gamma}\|^k\right)(|t-s| + \|\gamma - \overline{\gamma}\|) \tag{22}$$

with $\Phi = g'_{\gamma_t}(\gamma), g''_{\gamma_t}(\gamma)$. We can also define $C^2(\widetilde{\Lambda}_{t,s})$, $C^2_{l,lip}(\widetilde{\Lambda}_{t,s})$, $C^1_{l,lip}(\widetilde{\Lambda}_{t,s})$, $C_{l,lip}(\widetilde{\Lambda}_{t,s})$ and $C^2(\widetilde{\Lambda}_t)$, $C^2_{l,lip}(\widetilde{\Lambda}_t)$, $C^1_{l,lip}(\widetilde{\Lambda}_t)$, $C_{l,lip}(\widetilde{\Lambda}_t)$.

Now we consider the solvability of (6).

Assumption 10. Let g be an $\mathbb{R}^n$-valued function on Λ_T. Moreover $g \in C^2_{l,lip}(\widetilde{\Lambda}_{t,T})$ with the Lipschitz constants C and k.

Assumption 11. Let $h(\gamma_t, y, z) = \overline{h}(t, \gamma(t), y, z)$, where $\overline{h} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \mapsto \mathbb{R}^n$ is such that $(t, r, y, z) \mapsto \overline{\Psi}(t, r, y, z)$ is of class $C_p^{0,3}([0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d};\mathbb{R}^n)$ and the first order partial derivatives in r, y, and z are bounded, as well as their derivatives of up to order two with respect to y, z.

It is obvious under Assumptions 1, 10, and 11 the FBSDE (5) and (6) has a unique solution (see [12–14]).

3.1. Regularity of the Solution of FBSDEs

3.1. Regularity of the Solution of FBSDEs. We assume the Lipschitz constants with respect to b, σ, h are C and k. Then we have the following estimates for the solutions of FBSDE (5) and (6).

Lemma 12. *Under Assumptions 1, 10, and 11, for all $p \ge 2$ there exist C_2 and q depending only on C, T, k, x such that*

$$\begin{gathered} E\left[\sup_{s\in[t,T]}\left|X^{\gamma_t}(s)\right|^p\right] \le C_2\left(1 + \|\gamma_t\|^p\right), \\ E\left[\sup_{s\in[t,T]}\left|Y^{\gamma_t}(s)\right|^p\right] \le C_2\left(1 + \|\gamma_t\|^q\right), \\ E\left[\left(\int_t^T \left|Z^{\gamma_t}(s)\right|^2 ds\right)^{p/2}\right] \le C_2\left(1 + \|\gamma_t\|^q\right). \end{gathered} \tag{23}$$

Proof. To simplify presentation, we only study the case $n = d = 1$, and $p = 2$.

The application of Itô's formula to $(Y_{\gamma_t,x}(s))^2 e^{\beta_1 s}$ yields that

$$\begin{aligned}(Y^{\gamma_t,}(s))^2 e^{\beta_1 s} &+ \int_s^T e^{\beta_1 r}\left[(Z^{\gamma_t}(r))^2 + \beta_1 (Y^{\gamma_t}(r))^2\right] dr \\ &= g^2(X_T^{\gamma_t}) e^{\beta_1 T} \\ &- \int_s^T 2e^{\beta_1 r} Y^{\gamma_t}(r) h(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) dr \\ &- \int_s^T 2e^{\beta_1 r} Y^{\gamma_t}(r) Z^{\gamma_t}(r) dW(r).\end{aligned} \tag{24}$$

So

$$\begin{aligned}(Y^{\gamma_t}(s))^2 &+ E\left[\int_s^T e^{\beta_1(r-s)}\left[(Z^{\gamma_t}(r))^2 \right.\right. \\ &\left.\left. +\beta_1 (Y^{\gamma_t}(r))^2\right] dr \mid \mathcal{F}_s\right] \\ &= E\left[g^2(X_T^{\gamma_t}) e^{\beta_1(T-s)} \mid \mathcal{F}_s\right] \\ &- E\left[\int_s^T 2e^{\beta_1(r-s)} Y^{\gamma_t}(r) \right. \\ &\left. \times h(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) dr \mid \mathcal{F}_s\right].\end{aligned} \tag{25}$$

Then we have

$$\begin{aligned}E\sup_{t\le s\le T} (Y^{\gamma_t}(s))^2 &+ E\left[\int_t^T e^{\beta_1(r-t)}\left[(Z^{\gamma_t}(r))^2 + \beta_1 (Y^{\gamma_t}(T))^2\right] dr\right] \\ &\le E\left[g^2(X_T^{\gamma_t}) e^{\beta_1(T-t)}\right] \\ &+ E\left[\int_t^T e^{\beta_1(r-t)} \frac{2}{\beta_1} h^2(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) dr\right] \\ &+ E\left[\int_t^T e^{\beta_1(r-t)} \frac{\beta_1}{2} (Y^{\gamma_t}(r))^2 d(r)\right], \\ E\sup_{t\le s\le T} (Y^{\gamma_t}(s))^2 &+ E\left[\int_t^T e^{\beta_1(r-t)}\left[(Z^{\gamma_t}(r))^2 + \frac{\beta_1}{2} (Y^{\gamma_t}(r))^2\right] dr\right] \\ &\le E\left[g^2(X_T^{\gamma_t}) e^{\beta_1(T-t)}\right] \\ &+ E\left[\int_t^T e^{\beta_1(r-t)} \frac{2}{\beta_1} h^2(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)) dr\right].\end{aligned} \tag{26, 27}$$

Applying Itô's formula to $(X^{\gamma_t,x}(s))^2$ yields that

$$\begin{aligned}(X^{\gamma_t}(s))^2 &= \gamma_t(t)^2 + \int_t^s 2X^{\gamma_t}(r) b(X_r^{\gamma_t}) dr \\ &+ \int_t^s 2X^{\gamma_t}(r) \sigma(X_r^{\gamma_t}) dW(r) + \int_t^s \sigma^2(X_r^{\gamma_t}) dr.\end{aligned} \tag{28}$$

By inequality $2ab \le a^2 + b^2$ and Burkholder-Davis-Gundy's inequality, there is a C_0 such that

$$\begin{aligned}E\sup_{t\le r\le s} (X^{\gamma_t}(r))^2 &\le C_0\left[\gamma_t(t)^2 + E\int_t^s b^2(X_r^{\gamma_t}) dr \right. \\ &\left. + E\int_t^s (X^{\gamma_t}(r))^2 dr + E\int_t^s \sigma^2(X_r^{\gamma_t}) dr\right].\end{aligned} \tag{29}$$

By Assumption 1 and Gronwall's inequality, from (29) we have (note that C_0 will change line by line)

$$E\sup_{t\le r\le T} (X^{\gamma_t}(r))^2 \le C_0\left(1 + \|\gamma_t\|^2\right). \tag{30}$$

By Assumptions 10 and 11 and taking $\beta_1 = 4C^2 + 1$, from (27) we have

$$\begin{aligned}E\sup_{t\le s\le T} (Y^{\gamma_t}(s))^2 &+ E\left[\int_t^T \left[(Z^{\gamma_t}(r))^2 + (Y^{\gamma_t}(r))^2\right] dr\right] \\ &\le C_0\left(1 + P\gamma_t P^q\right),\end{aligned} \tag{31}$$

where $q = 2(1 + k)$. Similarly we can get the same result for $p \ge 2$.

This completes the proof. □

Now we study the regularity properties of the solution of FBSDE (5), (6) with respect to the "parameter" γ_t. For $0 \le s < t \le T$, define $Y^{\gamma_t}(s) = Y^{\gamma_t}(s \vee t)$ and $Z^{\gamma_t}(s) = 0$.

Theorem 13. *Under Assumptions 1, 10 and 11, for all $p \ge 2$ there exist C_2 and q depending only on C, c_2, x such that for any $t, \bar{t} \in [0, T]$, $\gamma_t, \bar{\gamma}_{\bar{t}}$, and $h, \bar{h} \in \mathbb{R} \setminus \{0\}$*

(i)

$$\begin{aligned}E&\left[\sup_{u\in[t\vee\bar{t},T]} \left|Y^{\gamma_t}(u) - Y^{\bar{\gamma}_{\bar{t}}}(u)\right|^p\right] \\ &\le C_2\left(1 + \|\gamma_t\|^q + \|\bar{\gamma}_{\bar{t}}\|^q\right)\left(\|\gamma_t - \bar{\gamma}_{\bar{t}}\|^p + |t - \bar{t}|^{p/2}\right),\end{aligned} \tag{32}$$

(ii)

$$\begin{aligned}E&\left[\left|\int_{t\vee\bar{t}}^T \left|Z^{\gamma_t}(u) - Z^{\bar{\gamma}_{\bar{t}}}(u)\right|^2 du\right|^{p/2}\right] \\ &\le C_2\left(1 + \|\gamma_t\|^q + \|\bar{\gamma}_{\bar{t}}\|^q\right)\left(\|\gamma_t - \bar{\gamma}_{\bar{t}}\|^p + |t - \bar{t}|^{p/2}\right),\end{aligned} \tag{33}$$

(iii)

$$E\left[\sup_{u\in[t\vee\bar{t},T]}\left|\Delta_h^i Y^{\gamma_t}(u)-\Delta_h^i Y^{\bar{\gamma}_{\bar{t}}}(u)\right|^p\right]\le C_2\left(1+\|\gamma_t\|^q+\|\bar{\gamma}_{\bar{t}}\|^q+|h|^q+|\bar{h}|^q\right)\times\left(|h-\bar{h}|^p+\|\gamma_t-\bar{\gamma}_{\bar{t}}\|^p+|t-\bar{t}|^{p/2}\right), \tag{34}$$

(iv)

$$E\left[\left|\int_{t\vee\bar{t}}^T\left|\Delta_h^i Z^{\gamma_t}(u)-\Delta_h^i Z^{\bar{\gamma}_{\bar{t}}}(u)\right|^2du\right|^{p/2}\right]\le C_2\left(1+\|\gamma_t\|^q+\|\bar{\gamma}_{\bar{t}}\|^q+|h|^q+|\bar{h}|^q\right)\times\left(|h-\bar{h}|^p+\|\gamma_t-\bar{\gamma}_{\bar{t}}\|^p+|t-\bar{t}|^{p/2}\right), \tag{35}$$

where

$$\Delta_h^i Y^{\gamma_t,x}(s)=\frac{1}{h}\left(Y^{\gamma_t^{he_i}}(s)-Y^{\gamma_t}(s)\right),\quad \Delta_h^i Z^{\gamma_t}(s)=\frac{1}{h}\left(Z^{\gamma_t^{he_i}}(s)-Z^{\gamma_t}(s)\right) \tag{36}$$

and $(e_1,\dots,e_n)$ is an orthonormal basis of $\mathbb{R}^n$.

Proof. $(Y^{\gamma_t}-Y^{\bar{\gamma}_{\bar{t}}}, Z^{\gamma_t}-Z^{\bar{\gamma}_{\bar{t}}})$ can be formed as a linearized BSDE: for each $s\in[t\vee\bar{t},T]$,

$$\begin{aligned}
Y^{\gamma_t}(s)-Y^{\bar{\gamma}_{\bar{t}}}(s)
&=g(X_T^{\gamma_t})-g\left(X_T^{\bar{\gamma}_{\bar{t}}}\right)\\
&\quad+\int_s^T\left[h\left(X_r^{\gamma_t},Y^{\gamma_t}(r),Z^{\gamma_t}(r)\right)-h\left(X_r^{\bar{\gamma}_{\bar{t}}},Y^{\bar{\gamma}_{\bar{t}}}(r),Z^{\bar{\gamma}_{\bar{t}}}(r)\right)\right]dr\\
&\quad+\int_s^T\left(Z^{\gamma_t}(r)-Z^{\bar{\gamma}_{\bar{t}}}(r)\right)dW(r)\\
&=g(X_T^{\gamma_t})-g\left(X_T^{\bar{\gamma}_{\bar{t}}},\right)\\
&\quad-\int_s^T\left[\hat{\alpha}_{\gamma_t,\bar{\gamma}_{\bar{t}}}(r)+\hat{\beta}_{\gamma_t,\bar{\gamma}_{\bar{t}}}\left(Y^{\gamma_t}(r)-Y^{\bar{\gamma}_{\bar{t}}}(r)\right)+\hat{\delta}_{\gamma_t,\bar{\gamma}_{\bar{t}}}\left(Z^{\gamma_t}(r)-Z^{\bar{\gamma}_{\bar{t}}}(r)\right)\right]dr\\
&\quad+\int_s^T\left(Z^{\gamma_t}(r)-Z^{\bar{\gamma}_{\bar{t}}}(r)\right)dW(r),
\end{aligned} \tag{37}$$

where (with $U^{\gamma_t}=(Y^{\gamma_t},Z^{\gamma_t})$)

$$\begin{aligned}
\hat{\alpha}_{\gamma_t,\bar{\gamma}_{\bar{t}}}(r)&=h\left(X_r^{\gamma_t},Y^{\bar{\gamma}_{\bar{t}}}(r),Z^{\bar{\gamma}_{\bar{t}}}(r)\right)-h\left(X_r^{\bar{\gamma}_{\bar{t}}},Y^{\bar{\gamma}_{\bar{t}}}(r),Z^{\bar{\gamma}_{\bar{t}}}(r)\right),\\
\hat{\beta}_{\gamma_t,\bar{\gamma}_{\bar{t}}}(r)&=\int_0^1\frac{\partial h}{\partial y}\left(X_r^{\gamma_t},U^{\bar{\gamma}_{\bar{t}}}(r)+\theta\left(U^{\gamma_t}(r)-U^{\bar{\gamma}_{\bar{t}}}(r)\right)\right)d\theta,\\
\hat{\delta}_{\gamma_t,\bar{\gamma}_{\bar{t}}}(r)&=\int_0^1\frac{\partial h}{\partial z}\left(X_r^{\gamma_t},U^{\bar{\gamma}_{\bar{t}}}(r)+\theta\left(U^{\gamma_t}(r)-U^{\bar{\gamma}_{\bar{t}}}(r)\right)\right)d\theta.
\end{aligned} \tag{38}$$

Under Assumptions 10, 11, using the same method as in Lemma 12, we get the first three inequalities.

For the next three inequalities, we write $(\Delta_h^i Y^{\gamma_t},\Delta_h^i Z^{\gamma_t})$ as the solution of the following linearized BSDE:

$$\begin{aligned}
\Delta_h^i Y^{\gamma_t}(s)
&=\frac{1}{h}\left(g\left(X_T^{\gamma_t^{he_i}}\right)-g(X_T^{\gamma_t})\right)\\
&\quad-\int_s^T\left[\frac{1}{h}\hat{\alpha}_{\gamma_t,\gamma_t^{he_i}}(r)+\hat{\beta}_{\gamma_t,\gamma_t^{he_i}}(r)\Delta_h^i Y^{\gamma_t}(r)+\hat{\delta}_{\gamma_t,\gamma_t^{he_i}}\Delta_h^i Z^{\gamma_t}(r)\right]dr\\
&\quad-\int_t^T\Delta_h^i Z^{\gamma_t}(r)\,dW(r).
\end{aligned} \tag{39}$$

Then the same calculus implies that

$$E\left[\sup_{s\in[t,T]}\left|\Delta_h^i Y^{\gamma_t}(s)\right|^p+\left|\int_t^T\left|\Delta_h^i Z^{\gamma_t}\right|^2dr\right|^{p/2}\right]\le C_2\left(1+\|\gamma_t\|^q+|h|^q\right). \tag{40}$$

Consider

$$\begin{aligned}
\Delta_h^i Y^{\gamma_t}(s)-\Delta_{\bar{h}}^i Y^{\bar{\gamma}_{\bar{t}}}(s)
&=\frac{1}{h}\left(g\left(X_T^{\gamma_t^{he_i}}\right)-g(X_T^{\gamma_t})\right)-\frac{1}{\bar{h}}\left(g\left(X_T^{\bar{\gamma}_{\bar{t}}^{\bar{h}e_i}}\right)-g\left(X_T^{\bar{\gamma}_{\bar{t}}}\right)\right)\\
&\quad-\int_s^T\left(\Delta_h^i Z^{\gamma_t}(r)-\Delta_{\bar{h}}^i Z^{\bar{\gamma}_{\bar{t}}}(r)\right)dW(r)\\
&\quad-\left\{\int_s^T\left[\frac{1}{h}\hat{\alpha}_{\gamma_t,\gamma_t^{he_i}}(r)-\frac{1}{\bar{h}}\hat{\alpha}_{\bar{\gamma}_{\bar{t}},\bar{\gamma}_{\bar{t}}^{\bar{h}e_i}}(r)+\hat{\beta}_{\gamma_t,\gamma_t^{he_i}}(r)\Delta_h^i Y^{\gamma_t}(r)-\hat{\beta}_{\bar{\gamma}_{\bar{t}},\bar{\gamma}_{\bar{t}}^{\bar{h}e_i}}(r)\Delta_{\bar{h}}^i Y^{\bar{\gamma}_{\bar{t}}}(r)\right.\right.\\
&\quad\left.\left.+\hat{\delta}_{\gamma_t,\gamma_t}\Delta_h^i Z^{\gamma_t}(r)-\hat{\delta}_{\bar{\gamma}_{\bar{t}},\bar{\gamma}_{\bar{t}}^{\bar{h}e_i}}(r)\Delta_{\bar{h}}^i Z^{\bar{\gamma}_{\bar{t}}}(r)\right]dr\right\}.
\end{aligned} \tag{41}$$

Set

$$\left(\tilde{Y}(s),\tilde{Z}(s)\right):=\left(\Delta_h^i Y^{\gamma_t}(s)-\Delta_{\bar{h}}^i Y^{\bar{\gamma}_{\bar{t}}}(s),\Delta_h^i Z^{\gamma_t}(s)-\Delta_{\bar{h}}^i Z^{\bar{\gamma}_{\bar{t}}}(s)\right). \tag{42}$$

Then it solves the following BSDE:

$$\begin{aligned}
&\widetilde{Y}(s)\\
&= \frac{1}{h}\left(g\left(X_T^{\gamma_t^{h_{e_i}}}\right) - g\left(X_T^{\gamma_t}\right)\right) - \frac{1}{\bar{h}}\left(g\left(X_T^{\bar{\gamma}_{\bar{t}}^{\bar{h}_{e_i}}}\right) - g\left(X_T^{\bar{\gamma}_{\bar{t}}}\right)\right)\\
&\quad - \int_s^T \left[\widehat{\beta}_{\gamma_t,\gamma_t^{h_{e_i}}}(r)\,\widetilde{Y}(r) + \widehat{\delta}_{\gamma_t,\gamma_t^{h_{e_i}}}\widetilde{Z}(r) + \widetilde{h}(r)\right]dr\\
&\quad - \int_s^T \widetilde{Z}(r)\,dW(r),
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
&\widetilde{h}(r)\\
&:= \left[\widehat{\beta}_{\gamma_t,\gamma_t^{h_{e_i}}}(r) - \widehat{\beta}_{\bar{\gamma}_{\bar{t}},\bar{\gamma}_{\bar{t}}^{\bar{h}_{e_i}}}(r)\right]\Delta^i_{\bar{h}}Y^{\bar{\gamma}_{\bar{t}}}(r)\\
&\quad + \left[\widehat{\delta}_{\gamma_t,\gamma_t^{h_{e_i}}}(r) - \widehat{\delta}_{\bar{\gamma}_{\bar{t}},\bar{\gamma}_{\bar{t}}^{\bar{h}_{e_i}}}(r)\right]\Delta^i_{\bar{h}}Z^{\bar{\gamma}_{\bar{t}}}(r)\\
&\quad + \frac{1}{h}\widehat{\alpha}_{\gamma_t,\gamma_t^{h_{e_i}}}(r) - \frac{1}{\bar{h}}\widehat{\alpha}_{\bar{\gamma}_{\bar{t}},\bar{\gamma}_{\bar{t}}^{\bar{h}_{e_i}}}(r).
\end{aligned} \tag{44}$$

Thus, under Assumptions 10, 11, similarly as in Lemma 12, we can get the last three inequalities. □

Theorem 14. *For each $\gamma_t \in \Lambda$, $\{Y^{\gamma_t^z}(s), s \in [t,T], z \in \mathbb{R}^n\}$ has a version which is a.e. of class $C^{0,2}([0,T]\times\mathbb{R}^n)$.*

Proof. We only consider one dimensional case. Applying Lemma 12, for each $h, \bar{h} \in \mathbb{R}\setminus\{0\}$ and $k, \bar{k} \in \mathbb{R}$,

$$\begin{aligned}
&E\left[\sup_{u\in[t,T]}\left|Y^{\gamma_t^k}(u) - Y^{\gamma_t^{\bar{k}}}(u)\right|^p\right] \le C_p\left(1 + \|\gamma_t\|^q\right)\left|k-\bar{k}\right|^p,\\
&E\left[\left|\int_t^T \left|Z^{\gamma_t^k}(u) - Z^{\gamma_t^{\bar{k}}}\right|^p du\right|^{p/2}\right] \le C_2\left(1 + \|\gamma_t\|^q\right)\left|k-\bar{k}\right|^p,\\
&E\left[\sup_{u\in[t,T]}\left|\Delta^i_h Y^{\gamma_t^k}(u) - \Delta^i_{\bar{h}}Y^{\gamma_t^{\bar{k}}}\right|^p\right]\\
&\quad \le C_2\left(1 + \|\gamma_t\|^q + \|\bar{\gamma}_{\bar{t}}\|^q + |h|^q + \left|\bar{h}\right|^q\right)\\
&\quad \times\left(\left|k-\bar{k}\right|^p + \left|h-\bar{h}\right|^p\right),\\
&E\left[\left|\int_t^T \left|\Delta^i_h Z^{\gamma_t^k}(u) - \Delta^i_{\bar{h}}Z^{\gamma_t^{\bar{k}}}(u)\right|^2 du\right|^{p/2}\right]\\
&\quad \le C_2\left(1 + \|\gamma_t\|^q + \|\bar{\gamma}_{\bar{t}}\|^q + |h|^q + \left|\bar{h}\right|^q\right)\\
&\quad \times\left(\left|k-\bar{k}\right|^p + \left|h-\bar{h}\right|^p\right).
\end{aligned} \tag{45}$$

By Kolmogorov's criterion, there exists a continuous derivative of $Y^{\gamma_t^z}(s)$ with respect to z. There also exists a mean-square derivative of $Z^{\gamma_t^z}(s)$ with respect to z, which is mean square continuous in z. We denote them by

$$\left(D_z Y^{\gamma_t}, D_z Z^{\gamma_t}\right). \tag{46}$$

By Theorem 13 and Definition 9, $(D_z Y^{\gamma_t}, D_z Z^{\gamma_t})$ is the solution of the following BSDE:

$$\begin{aligned}
D_x Y^{\gamma_t}(s) &= g'_{\gamma_t}\left(X_T^{\gamma_t}\right) - \int_s^T D_x Z^{\gamma_t}(r)\,dW(r)\\
&\quad - \int_s^T \left[h'_{\gamma_t}\left(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)\right)\right.\\
&\quad + h'_y\left(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)\right)D_x Y^{\gamma_t}(r)\\
&\quad \left.+ h'_z\left(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)\right)D_x Z^{\gamma_t}(r)\right]dr.
\end{aligned} \tag{47}$$

It is easy to check that the above BSDE has a unique solution. Thus the existence of a continuous second order derivative of $Y^{\gamma_t^z}(s)$ with respect to z is proved in a similar way. □

Define

$$u(\gamma_t) := Y^{\gamma_t}(t), \quad \text{for } \gamma_t \in \Lambda. \tag{48}$$

We have the following results about $u(\gamma_t)$.

Lemma 15. *For all $t_1 \le t \le T$, one has $u(X_t^{\gamma_{t_1}}) = Y^{\gamma_{t_1}}(t)$.*

Proof. For given γ_{t_1}, $t_1 < t$, set $X_{t_1} = \gamma_{t_1}$. Consider the solutions of FBSDE (5) and (6) on $[t,T]$:

$$\begin{aligned}
X^{\gamma_{t_1}}(s) &= X^{\gamma_{t_1}}(t) + \int_t^s b\left(X_r^{\gamma_{t_1}}\right)dr + \int_t^s \sigma\left(X_r^{\gamma_{t_1}}\right)dW(r),\\
Y^{\gamma_{t_1}}(s)&\\
&= g\left(X_T^{\gamma_{t_1}}\right) - \int_s^T h\left(X_r^{\gamma_{t_1}}, Y^{\gamma_{t_1}}(r), Z^{\gamma_{t_1}}(r)\right)dr\\
&\quad - \int_s^T Z^{\gamma_{t_1}}(r)\,dW(r), \quad s \in [t,T].
\end{aligned} \tag{49}$$

We need to prove $u(X_t^{\gamma_{t_1}}) = Y^{\gamma_{t_1}}(t)$.

For each fixed $t \in [0,T]$ and positive integer n, we introduce a mapping $\gamma^n(\bar{\gamma}_s) : \Lambda_s \mapsto \Lambda_s$

$$\begin{aligned}
\gamma^n(\bar{\gamma}_s)(r) &= \bar{\gamma}_s(r)\,I_{[0,t_1)}\\
&\quad + \sum_{k=0}^{n-1}\bar{\gamma}_s\left(t_{k+1}^n \wedge s\right)I_{[t_k^n\wedge s, t_{k+1}^n\wedge s)}(r) + \bar{\gamma}_s(s)\,I_{\{s\}}(r),\\
&\qquad s \in [0,t],
\end{aligned} \tag{50}$$

where $t_k^n = t_1 + (k(t-t_1)/n)$, $k = 0, 1, \ldots, n$,

Denote

$$\gamma^n\left(X_t^{\gamma_{t_1}}\right)(r) = X^{n,\gamma_{t_1}}(r), \quad t_1 \le r \le t. \tag{51}$$

Set

$$X_t^{n,N;\gamma_{t_1}} := \sum_{i=1}^{N} I_{A_i} x_t^i, \tag{52}$$

where $\{A_i\}_{i=1}^N$ is a division of $\mathscr{F}_t$, $x_t^i \in \Lambda_t$, $i = 1, 2, \ldots, N$. For any i, $(Y^{x_t^i}(s), Z^{x_t^i}(s))$ is the solution of the following BSDE:

$$\begin{aligned} Y^{x_t^i}(s) &= g\left(X_T^{x_t^i}\right) \\ &\quad - \int_s^T h\left(X_r^{x_t^i}, Y^{x_t^i}(r), Z^{x_t^i}(r)\right) dr \\ &\quad - \int_s^T Z^{x_t^i}(r)\, dW(r), \quad s \in [t, T]. \end{aligned} \tag{53}$$

Multiplying by I_{A_i} and adding the corresponding terms, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} I_{A_i} Y^{x_t^i}(s) \\ &\quad = g\left(\sum_{i=1}^{N} I_{A_i} X_T^{x_t^i}\right) \\ &\quad - \int_s^T h\left(\sum_{i=1}^{N} I_{A_i} X_r^{x_t^i}, \sum_{i=1}^{N} I_{A_i} Y^{x_t^i}(r), \sum_{i=1}^{N} I_{A_i} Z^{x_t^i}(r)\right) dr \\ &\quad - \int_s^T \sum_{i=1}^{N} I_{A_i} Z^{x_t^i}(r)\, dW(r), \quad s \in [t, T]. \end{aligned} \tag{54}$$

By the uniqueness and existence theorem of BSDE, we have

$$Y^{X_t^{n,N;\gamma_{t_1}}}(s) = \sum_{i=1}^{N} I_{A_i} Y^{x_t^i}(s), \qquad Z^{X_t^{n,N;\gamma_{t_1}}}(s) = \sum_{i=1}^{N} I_{A_i} Z^{x_t^i}(s), \quad P\text{-a.s.} \tag{55}$$

Then, by the definition of u, we get

$$Y^{X_t^{n,N;\gamma_{t_1}}}(t) = \sum_{i=1}^{N} I_{A_i} Y^{x_t^i}(t) = \sum_{i=1}^{N} I_{A_i} u\left(x_t^i\right) = u\left(X_t^{n,N;\gamma_{t_1}}\right). \tag{56}$$

Note that

$$\lim_{n,N\to\infty} X_t^{n,N;\gamma_{t_1}} = X_t^{\gamma_{t_1}}, \quad P\text{-a.s.} \tag{57}$$

This completes the proof. □

By Theorem 13 and 14 and the definition of vertical derivative, we have the following corollary.

Corollary 16. *$u(\gamma_t)$ is Λ-continuous and $D_x u(\gamma_t)$, $D_{xx} u(\gamma_t)$ exist; moreover they are both Λ-continuous.*

Proof. By Theorem 14 we know that $D_x u(\gamma_t)$ and $D_{xx} u(\gamma_t)$ exist. In the following, we only prove $u(\gamma_t)$ is Λ-continuous. The proof for the continuous property of $D_x u(\gamma_t)$ and $D_{xx} u(\gamma_t)$ is similar. Taking expectation on both sides of (6),

$$u(\gamma_t) = Eg\left(X_T^{\gamma_t}\right) - E\int_t^T h\left(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)\right) dr. \tag{58}$$

For $\gamma_t, \bar{\gamma}_{\bar{t}} \in \Lambda$, $\bar{t} \ge t$, we have

$$\begin{aligned} &\left|u(\gamma_t) - u(\bar{\gamma}_{\bar{t}})\right| \\ &\quad \le E\left[\left|g\left(X_T^{\gamma_t}\right) - g\left(X_T^{\bar{\gamma}_{\bar{t}}}\right)\right|\right] \\ &\qquad + E\left[\int_t^{\bar{t}} \left|h\left(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)\right)\right| dr\right] \\ &\qquad + E\left[\int_{\bar{t}}^T \left|h\left(X_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)\right) \right.\right. \\ &\qquad\qquad \left.\left. - h\left(X_r^{\bar{\gamma}_{\bar{t}}}, Y^{\bar{\gamma}_{\bar{t}}}(r), Z^{\bar{\gamma}_{\bar{t}}}(r)\right)\right| dr\right] \\ &\quad \le E\left[C_1\left(1 + \left\|X_T^{\gamma_t}\right\|^k + \left\|X_T^{\bar{\gamma}_{\bar{t}}}\right\|^k\right)\left\|\gamma_t - \bar{\gamma}_{\bar{t}}\right\| \right. \\ &\qquad + 3(\bar{t} - t)^{1/2}\left(\int_t^{\bar{t}}\left(\left|h\left(X^{\bar{\gamma}_{\bar{t}}}(r), 0, 0\right)\right|^2 \right.\right. \\ &\qquad\qquad \left.\left. + \left|CY^{\bar{\gamma}_{\bar{t}}}(r)\right|^2 + \left|CY^{\bar{\gamma}_{\bar{t}}}(r)\right|^2\right) dr\right)^{1/2} \\ &\qquad \left. + C\int_{\bar{t}}^T \left(\left|Y^{\gamma_t}(r) - Y^{\bar{\gamma}_{\bar{t}}}(r)\right| + \left|Z^{\gamma_t}(r) - Z^{\bar{\gamma}_{\bar{t}}}(r)\right|\right) dr\right]. \end{aligned} \tag{59}$$

By Theorem 13, for some constant C_1 depending on C, k, and T,

$$\begin{aligned} &\left|u(\gamma_t) - u(\bar{\gamma}_{\bar{t}})\right| \\ &\quad \le C_1\left(1 + \left\|\gamma_t\right\|^k + \left\|\bar{\gamma}_{\bar{t}}\right\|^k\right)\left(\left\|\gamma_t - \bar{\gamma}_{\bar{t}}\right\| + \left|t - \bar{t}\right|^{1/2}\right). \end{aligned} \tag{60}$$

This completes the proof. □

3.2. Path Regularity of Process Z. In Pardoux and Peng [2], BSDE is only state-dependent, that is, $h = h(t, \gamma(t), y, z)$ and $g = g(\gamma(T))$. Under appropriate assumptions, Y and Z are related in the following sense:

$$Z^{\gamma_t}(s) = \nabla_x u\left(s, \gamma_t(t) + W(s) - W(t)\right), \quad P\text{-a.s.} \tag{61}$$

Under the conditions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ in [8], Peng and Wang extend this result to the path-dependent case. The corresponding BSDE is

$$\begin{aligned} &Y^{\gamma_t}(s) \\ &\quad = g\left(W_T^{\gamma_t}\right) - \int_s^T h\left(W_r^{\gamma_t}, Y^{\gamma_t}(r), Z^{\gamma_t}(r)\right) dr \\ &\qquad - \int_s^T Z^{\gamma_t}(r)\, dW(r), \quad s \in [t, T], \end{aligned} \tag{62}$$

where

$$W_T^{\gamma_t} = I_{s\leq t}\gamma_t(s) + I_{t<s\leq T}(\gamma_t(t) + W(s) - W(t)). \tag{63}$$

Then under some assumptions, they obtained

$$Z^{\gamma_t}(s) = D_x u(W_s^{\gamma_t}), \quad P\text{-a.s.} \tag{64}$$

In our context, we have the following theorem.

Theorem 17. *Under Assumptions 1, 10, and 11, for each $\gamma_t \in \Lambda$, the process $(Z^{\gamma_t}(s))_{s\in[t,T]}$ has a continuous version with the form*

$$Z^{\gamma_t}(s) = \sigma(X_s^{\gamma_t}) D_x u(X_s^{\gamma_t}), \quad \text{for } s \in [t,T] \ P\text{-a.s.} \tag{65}$$

To prove the above theorem, we need the following lemma essentially from Pardoux and Peng [2].

Lemma 18. *Let γ_t and some $\bar{t} \in [t,T]$ be given. Suppose that*

$$g(\gamma) = \varphi(\gamma(\bar{t}), \gamma(T) - \gamma(\bar{t})), \tag{66}$$

where φ is in $C_p^3(\mathbb{R}^{2d} \times \mathbb{R}^m; \mathbb{R}^m)$. For b, σ, h, suppose that

$$\begin{aligned} b(\gamma_s) &= b_1(s, \gamma_s(s)) I_{[0,\bar{t})}(s) \\ &\quad + b_2(s, \gamma_s(s) - \gamma_s(\bar{t})) I_{[\bar{t},T]}(s), \\ \sigma(\gamma_s) &= \sigma_1(s, \gamma_s(s)) I_{[0,\bar{t})}(s) \\ &\quad + \sigma_2(s, \gamma_s(s) - \gamma_s(\bar{t})) I_{[\bar{t},T]}(s), \\ h(\gamma_s, y, z) &= h_1(s, \gamma_s(s), y, z) I_{[0,\bar{t})}(s) \\ &\quad + h_2(s, \gamma_s(s) - \gamma_s(\bar{t}), y, z) I_{[\bar{t},T]}(s), \end{aligned} \tag{67}$$

where $b_i, \sigma_i, h_i \in C^{0,3}, i = 1, 2$. Then for each $s \in [t,T]$,

$$Z^{\gamma_t}(s) = \sigma(X_s^{\gamma_t}) D_x u(X_s^{\gamma_t}), \quad \text{for } s \in [t,T] \ P\text{-a.s.} \tag{68}$$

Proof. We only consider the one dimensional case.

For $s \in [\bar{t},T]$, the FBSDE (5), (6) can be rewritten as

$$\begin{aligned} X^{\gamma_s}(u) &= \gamma_s(s) + \int_s^u b_2(r, X^{\gamma_s}(s) - \gamma_s(\bar{t}))\,dr \\ &\quad + \int_s^u \sigma_2(r, X^{\gamma_s}(s) - \gamma_s(\bar{t}))\,dW(r), \\ Y^{\gamma_s}(u) &= \varphi(\gamma_s(\bar{t}), X^{\gamma_s}(T) - \gamma_s(\bar{t})) - \int_u^T Z^{\gamma_s}(r)\,dW(r) \\ &\quad - \int_u^T h_2(r, \gamma_s(\bar{t}), X^{\gamma_s}(r) - \gamma_s(\bar{t}), Y^{\gamma_s}(r), Z^{\gamma_s}(r))\,dr, \\ &\qquad u \in [s,T]. \end{aligned} \tag{69}$$

For $s \in [t,\bar{t}]$,

$$\begin{aligned} X^{\gamma_s}(u) &= \gamma_s(s) + \int_s^u b_1(r, X^{\gamma_s}(s))\,dr \\ &\quad + \int_s^u \sigma_1(r, X^{\gamma_s}(s))\,dW(r), \\ &\qquad u \in [s,\bar{t}], \\ X^{\gamma_s}(u) &= X^{\gamma_s}(\bar{t}) + \int_{\bar{t}}^u b_2(r, X^{\gamma_s}(s) - X^{\gamma_s}(\bar{t}))\,dr \\ &\quad + \int_s^u \sigma_2(r, X^{\gamma_s}(s) - X^{\gamma_s}(\bar{t}))\,dW(r), \\ &\qquad u \in [\bar{t},T], \\ Y^{\gamma_s}(u) &= \varphi(X^{\gamma_s}(\bar{t}), X^{\gamma_s}(T) - X^{\gamma_s}(\bar{t})) \\ &\quad - \int_u^T Z^{\gamma_s}(r)\,dW(r) \\ &\quad - \int_u^T h_2(r, X^{\gamma_s}(\bar{t}), X^{\gamma_s}(r) \\ &\qquad - X^{\gamma_s}(\bar{t}), Y^{\gamma_s}(r), Z^{\gamma_s}(r))\,dr, \\ &\qquad u \in [\bar{t},T], \\ Y^{\gamma_s}(u) & \\ &= Y^{\gamma_s}(\bar{t}) - \int_u^{\bar{t}} h_1(r, X^{\gamma_s}(\bar{t}), Y^{\gamma_s}(r), Z^{\gamma_s}(r))\,dr \\ &\quad - \int_u^{\bar{t}} Z^{\gamma_s}(r)\,dW(r), \quad u \in [s,\bar{t}]. \end{aligned} \tag{70}$$

Now consider the following system of quasilinear parabolic differential equations, which is defined on $[\bar{t},T] \times \mathbb{R}^2$ and parameterized by $x \in \mathbb{R}$:

$$\begin{aligned} &\partial_s u_2(s,x,y) + \mathscr{L}u_2(s,x,y) \\ &\quad = h_2(s,x,y,u_2(s,x,y), \partial_y u_2(s,x,y)\sigma(r_s)), \\ &u_2(T,x,y) = \varphi(x,y), \end{aligned} \tag{71}$$

where $\mathscr{L} = (1/2)\sigma^2(\partial^2/\partial_{xx}) + b(\partial/\partial_x)$. The other one is defined on $[t,\bar{t}] \times \mathbb{R}$:

$$\begin{aligned} &\partial_s u_1(s,x) + \mathscr{L}u_1(s,x) \\ &\quad = h_1(r,x,u_1(s,x), \partial_y u_1(s,x)\sigma(r)), \\ &u_1(\bar{t},x) = u_2(\bar{t},x,0), \end{aligned} \tag{72}$$

where $\mathscr{L} = (1/2)\sigma^2(\partial^2/\partial_{xx}) + b(\partial/\partial_x)$. By Corollary 16 and Theorems 3.1, 3.2 in Paroux-Peng [2], we have $u_2 \in C^{1,2}([\bar{t},T] \times \mathbb{R}^2; \mathbb{R})$, $u_1 \in C^{1,2}([t,\bar{t}] \times \mathbb{R}; \mathbb{R})$, and

$$\begin{aligned} u(\gamma_s) &= u_1(s, \gamma_s(s)) I_{[t,\bar{t})}(s) \\ &\quad + u_2(s, \gamma_s(\bar{t}), \gamma_s(s) - \gamma_s(\bar{t})) I_{[\bar{t},T]}(s). \end{aligned} \tag{73}$$

Then we obtain

$$Y^{\gamma_t}(s) = u_1(s, X^{\gamma_t}(s)), \quad t \le s < \bar{t},$$
$$Y^{\gamma_t}(s) = u_2(s, X^{\gamma_t}(\bar{t}), X^{\gamma_t}(s) - X^{\gamma_t}(\bar{t})), \quad \bar{t} \le s \le T,$$
$$Z^{\gamma_t}(s) = \partial_x u_1(s, X^{\gamma_t}(s))\,\sigma_1(X_s^{\gamma_t}), \quad t \le s < \bar{t}, \tag{74}$$
$$Z^{\gamma_t}(s) = \partial_x u_2(s, X^{\gamma_t}(\bar{t}), X^{\gamma_t}(s) - X^{\gamma_t}(\bar{t}))\,\sigma_2(X_s^{\gamma_t}), \quad \bar{t} \le s \le T.$$

Finally, for each $s \in [t, T]$,

$$Z^{\gamma_t}(s) = \sigma(X_s^{\gamma_t})\,D_x u(X_s^{\gamma_t}), \quad P\text{-a.s.} \tag{75}$$

In particular,

$$Z^{\gamma_t}(t) = \sigma(\gamma_t)\,D_x u(\gamma_t), \quad \gamma_t \in \Lambda. \tag{76}$$

This completes the proof. □

Now we give the proof of Theorem 17.

Proof. For each fixed $t \in [0, T]$ and positive integer n, we introduce a mapping $\gamma^n(\bar{\gamma}_s) : \Lambda_s \mapsto \Lambda_s$

$$\gamma^n(\bar{\gamma}_s)(r) = \bar{\gamma}_s(r) I_{[0,t)} + \sum_{k=0}^{n-1} \bar{\gamma}_s(t_{k+1}^n \wedge s) I_{[t_k^n \wedge s, t_{k+1}^n \wedge s)}(r) + \bar{\gamma}_s(s) I_{\{s\}}(r), \quad s \in [0, T], \tag{77}$$

where $t_k^n = t + (k(T-t)/n)$, $k = 0, 1, \ldots, n$

$$g^n(\bar{\gamma}) := g(\gamma^n(\bar{\gamma})), \qquad h^n(\bar{\gamma}_s, y, z) := h(\gamma^n(\bar{\gamma}_s), y, z). \tag{78}$$

For each n, there exist some functions φ_n defined on $\Lambda_t \times \mathbb{R}^{n\times d}$ and $\psi_n, \phi_n, \tilde{\phi}_n$ defined on $[t,T] \times \Lambda_t \times \mathbb{R}^{n\times d} \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$ such that

$$\begin{aligned}
g^n(\bar{\gamma}) &:= \varphi_n(\bar{\gamma}_t, \bar{\gamma}(t_1^n) - \bar{\gamma}(t), \ldots, \bar{\gamma}(t_n^n) - \bar{\gamma}(t_{n-1}^n)), \\
b^n(\bar{\gamma}_s) &:= \phi_n(s, \bar{\gamma}_t, \bar{\gamma}_s(t_1^n \wedge s) - \bar{\gamma}_s(t), \ldots, \bar{\gamma}_s(t_n^n \wedge s) - \bar{\gamma}_s(t_{n-1}^n \wedge s)), \\
\sigma^n(\bar{\gamma}_s) &:= \phi_n(s, \bar{\gamma}_t, \bar{\gamma}_s(t_1^n \wedge s) - \bar{\gamma}_s(t), \ldots, \bar{\gamma}_s(t_n^n \wedge s) - \bar{\gamma}_s(t_{n-1}^n \wedge s)), \\
h^n(\bar{\gamma}_s, y, z) &:= \psi_n(s, \bar{\gamma}_t, \bar{\gamma}_s(t_1^n \wedge s) - \bar{\gamma}_s(t), \ldots, \bar{\gamma}_s(t_n^n \wedge s) - \bar{\gamma}_s(t_{n-1}^n \wedge s), y, z).
\end{aligned} \tag{79}$$

Indeed, if we set

$$\begin{aligned}
\bar{\varphi}_n(\bar{\gamma}_t, x_1, \ldots, x_n) &:= g\left(\left(\bar{\gamma}_t(s) I_{[0,t)}(s) + \sum_{k=1}^{n} x_k I_{[t_{k-1}^n, t_k^n)}(s) + x_n I_{\{T\}}(s)\right)_{0\le s\le T}\right), \\
\varphi_n(\bar{\gamma}_t, x_1, \ldots, x_n) &:= \bar{\varphi}_n\left(\bar{\gamma}_t, \bar{\gamma}_t + x_1, \bar{\gamma}_t(t) + x_1 + x_2, \ldots, \bar{\gamma}_t(t) + \sum_{i=1}^{n} x_i\right),
\end{aligned} \tag{80}$$

then by Assumptions 1, 10, and 11, we obtain that, for each fixed $\bar{\gamma}_t$, $\varphi_n(\bar{\gamma}_t, x_1, \ldots, x_n)$ is a C_p^3-function of $x_1, \ldots, x_n$. In particular, for each $\bar{\gamma} \in \Lambda$,

$$\partial_{x_i} \varphi_n(\bar{\gamma}_t, \bar{\gamma}(t_1^n) - \bar{\gamma}(t), \ldots, \bar{\gamma}(t_n^n) - \bar{\gamma}(t_{n-1}^n)) = g'_{\gamma_{t_{i-1}^n}}(\gamma^n(\bar{\gamma})). \tag{81}$$

For any $\bar{t} \ge t$, $\bar{\gamma}_{\bar{t}} \in \Lambda_{\bar{t}}$, we consider the following FBSDE:

$$\begin{aligned}
X^{n,\bar{\gamma}_{\bar{t}}}(s) &= \bar{\gamma}_{\bar{t}}(\bar{t}) + \int_{\bar{t}}^{s} b^n(X_r^{n,\bar{\gamma}_{\bar{t}}})\,dr + \int_{\bar{t}}^{s} \sigma^n(X_r^{n,\bar{\gamma}_{\bar{t}}})\,dW(r), \\
Y^{n,\bar{\gamma}_{\bar{t}}}(s) &= g^n(X_T^{n,\bar{\gamma}_{\bar{t}}}) - \int_s^T h^n(X_r^{n,\bar{\gamma}_{\bar{t}}}, Y^{n,\bar{\gamma}_{\bar{t}}}(r), Z^{n,\bar{\gamma}_{\bar{t}}}(r))\,dr - \int_s^T Z^{n,\bar{\gamma}_{\bar{t}}}(r)\,dW(r), \quad \bar{t} \le s.
\end{aligned} \tag{82}$$

We denote

$$u^n(\bar{\gamma}_{\bar{t}}) := Y^{n,\bar{\gamma}_{\bar{t}}}(t), \qquad \bar{\gamma}_{\bar{t}} \in \Lambda. \tag{83}$$

Following the argument as in Lemma 18, for each $s \in [t, T]$, we have

$$Z^{n,\bar{\gamma}_{\bar{t}}}(s) = \sigma^n(X_s^{n,\bar{\gamma}_{\bar{t}}})\,D_x u^n(\bar{\gamma}_{\bar{t}}), \quad P\text{-a.s.} \tag{84}$$

By the definition of g^n, b^n, σ^n, h^n and (82) we have

$$\lim_n X_T^{n,\bar{\gamma}_{\bar{t}}} = X_T^{\bar{\gamma}_{\bar{t}}}, \quad P\text{-a.s},$$
$$\lim_n \left(Y^{n,\bar{\gamma}_{\bar{t}}}(s), Z^{n,\bar{\gamma}_{\bar{t}}}(s)\right) = \left(Y^{\bar{\gamma}_{\bar{t}}}(s), Z^{\bar{\gamma}_{\bar{t}}}(s)\right), \tag{85}$$
$$\text{a.e. } s \in [t, T], \ P\text{-a.s},$$

$$\begin{gathered}\lim_n u^n(\bar{\gamma}_{\bar{t}}) = u(\bar{\gamma}_{\bar{t}}), \qquad \lim_n D_x u^n(\bar{\gamma}_{\bar{t}}) = D_x u(\bar{\gamma}_{\bar{t}}), \\ \lim_n D_{xx} u^n(\bar{\gamma}_{\bar{t}}) = D_{xx} u(\bar{\gamma}_{\bar{t}}), \\ \lim_n \left(u^n\left(X_s^{n,\bar{\gamma}_t}\right), D_x u^n\left(X_s^{n,\bar{\gamma}_t}\right), D_{xx} u^n\left(X_s^{n,\bar{\gamma}_t}\right)\right) \\ = \left(u\left(X_s^{\bar{\gamma}_t}\right), D_x u\left(X_s^{\bar{\gamma}_t}\right), D_{xx} u\left(X_s^{\bar{\gamma}_t}\right)\right),\end{gathered} \tag{86}$$

a.e. $s \in [t, T]$, P-a.s.

Therefore

$$Z^{\gamma_t}(s) = \sigma\left(X_s^{\bar{\gamma}_t}\right) D_x u\left(X_s^{\bar{\gamma}_t}\right), \quad \text{a.e. } s \in [t, T], \ P\text{-a.s.} \tag{87}$$

This completes the proof. □

4. The Related Parabolic Path-Dependent PDEs

In this section, we relate FBSDE (5), (6) to the following path-dependent partial differential equation:

$$\begin{gathered}D_t u(\gamma_t) + \mathscr{L} u(\gamma_t) - h(\gamma_t, u(\gamma_t), \sigma(\gamma_t) D_x u(\gamma_t)) = 0, \\ u(\gamma_T) = g(\gamma_T), \quad \gamma_T \in \Lambda^n,\end{gathered} \tag{88}$$

where

$$\mathscr{L} u = \frac{1}{2} \operatorname{tr}\left[\left(\sigma\sigma^T\right) D_{xx} u\right] + \langle b, D_x u\rangle. \tag{89}$$

Theorem 19. *Suppose that Assumptions 1, 10, and 11 are fulfilled, and if $u \in \mathbb{C}^{1,2}(\Lambda)$, and that u is the solution of* (88)*, u is uniformly Lipschitz continuous, and bounded by $C(1 + \|\gamma_t\|)$, then the solution is unique, and for any $\gamma_t \in \Lambda$, $u(\gamma_t)$ is determined by* (5) *and* (6).

Proof. By the assumptions of this theorem, we know that $b(\gamma_t)$ and $\sigma(\gamma_t)$ are uniformly Lipschtiz continuous and the following SDE has a uniqueness solution:

$$\begin{gathered}dX^{\gamma_t}(s) = b\left(X_s^{\gamma_t}\right) ds + \sigma\left(X_s^{\gamma_t}\right) dW(s), \\ X_t = \gamma_t, \quad s \in [t, T].\end{gathered} \tag{90}$$

Set $Y(s) = u(X_s^{\gamma_t})$, $t \le s \le T$. Applying Itô's formula to $Y(s) = u(X_s^{\gamma_t})$, we have

$$\begin{aligned}&dY(s) \\ &\quad = -h\left(X_s^{\gamma_t}, Y(s), Z(s)\right) dr - \sigma\left(X_s^{\gamma_t}\right) D_x u\left(X_s^{\gamma_t}\right) dW(s). \\ &\qquad Y(T) = g\left(X_T^{\gamma_t}\right) \quad s \in [t, T].\end{aligned} \tag{91}$$

Then by the uniqueness and existence theorem of the functional FBSDE, we obtain the result. □

Now we prove the converse to the above result.

Theorem 20. *Under Assumptions 1, 10, and 11, the function $u(\gamma_t) = Y^{\gamma_t}(t)$ is the unique $\mathbb{C}^{1,2}(\Lambda)$-solution of the path-dependent PDE* (88).

Proof. We only study the one dimensional case. $u \in \mathbb{C}^{0,2}(\Lambda)$ follows from Corollary 16. Let $\delta > 0$ satisfying $t + \delta \le T$. By Lemma 15 we can get

$$u\left(X_{t+\delta}^{\gamma_t}\right) = Y^{\gamma_t}(t + \delta). \tag{92}$$

Hence

$$\begin{aligned}&u(\gamma_{t,t+\delta}) - u(\gamma_t) \\ &\quad = u(\gamma_{t,t+\delta}) - u\left(X_{t+\delta}^{\gamma_t}\right) + u\left(X_{t+\delta}^{\gamma_t}\right) - u(\gamma_t),\end{aligned} \tag{93}$$

By the proof of Theorem 17, we obtain

$$\begin{aligned}&u(\gamma_{t,t+\delta}) - u(\gamma_t) \\ &\quad = \lim_{n\to\infty}\left[u^n(\gamma_{t,t+\delta}) - u^n\left(X_{t+\delta}^{\gamma_t}\right)\right] \\ &\qquad + \int_t^{t+\delta} h\left(X_s^{\gamma_t}, Y^{\gamma_t}(s), Z^{\gamma_t}(s)\right) ds \\ &\qquad + \int_t^{t+\delta} Z^{\gamma_t}(s) \, dW(s).\end{aligned} \tag{94}$$

By Lemma 3.1 and Theorem 3.2 of Pardoux and Peng [2] and Theorem 4.4 of Peng and Wang [8], we deduce that

$$\begin{aligned}&u^n(\gamma_{t,t+\delta}) - u^n\left(X_{t+\delta}^{\gamma_t}\right) \\ &\quad = \int_t^{t+\delta} D_s u^n(\gamma_{t,s}) \, ds - \int_t^{t+\delta} D_s u^n\left(X_s^{\gamma_t}\right) ds \\ &\qquad - \int_t^{t+\delta} D_x u^n\left(X_s^{\gamma_t}\right) dX^{n,\gamma_t}(s) \\ &\qquad - \frac{1}{2}\int_t^{t+\delta} D_{xx} u^n\left(X_s^{\gamma_t}\right) d\left\langle X^{n,\gamma_t}\right\rangle(s).\end{aligned} \tag{95}$$

Thus by (86) and the dominated convergence theorem, we have

$$\begin{aligned}&u(\gamma_{t,t+\delta}) - u(\gamma_t) \\ &\quad = -\int_t^{t+\delta} D_x u\left(X_s^{\gamma_t}\right) dX^{\gamma_t}(s) \\ &\qquad - \frac{1}{2}\int_t^{t+\delta} D_{xx} u\left(X_s^{\gamma_t}\right) d\left\langle X^{\gamma_t}\right\rangle(s) \\ &\qquad + \int_t^{t+\delta} h\left(X_s^{\gamma_t}, Y^{\gamma_t}(s), Z^{\gamma_t}(s)\right) ds \\ &\qquad + \int_t^{t+\delta} Z^{\gamma_t}(s) \, dW(s) + \lim_{n\to\infty} C^n,\end{aligned} \tag{96}$$

where

$$C^n = \int_t^{t+\delta} D_s u^n(\gamma_{t,s})\, ds - \int_t^{t+\delta} D_s u^n(X_s^{\gamma_t})\, ds. \tag{97}$$

Note that $u^n(\gamma_t) \in \mathbb{C}^{0,2}_{l,lip}(\Lambda)$. By Lemma 12, we have

$$\left|D_s u^n(\gamma_{t,s}) - D_s u^n(X_s^{\gamma_t})\right| \le c \left\|\gamma_t - X_s^{\gamma_t}\right\|, \quad \text{a.e. } s \in [t,T],\ P\text{-a.s} \tag{98}$$

for some constant c depending on C, T, γ_t, and k. Hence

$$\left|C^n\right| \le c\delta \sup_{s\in[t,t+\delta]} \left|X^{\gamma_t}(s) - \gamma_t(t)\right|, \quad P\text{-a.s.} \tag{99}$$

Taking expectation on both sides of (96), we have

$$\lim_{\delta\to 0} \frac{u(\gamma_{t,t+\delta}) - u(\gamma_t)}{\delta} = -\mathscr{L}u(\gamma_t) + h(\gamma_t, u(\gamma_t), D_x u(\gamma_t)\sigma(\gamma_t)). \tag{100}$$

Thus $u(\gamma_t)$ belongs to $\mathbb{C}^{1,2}(\Lambda)$ and satisfies (88). □

Acknowledgments

This work was supported by National Natural Science Foundation of China (no. 11171187 and no. 11221061); by the Programme of Introducing Talents of Discipline to Universities of China (no. B12023); and by Program for New Century Excellent Talents in University of China.

References

[1] S. G. Peng, "Probabilistic interpretation for systems of quasilinear parabolic partial differential equations," *Stochastics and Stochastics Reports*, vol. 37, no. 1-2, pp. 61–74, 1991.

[2] É. Pardoux and S. Peng, "Backward stochastic differential equations and quasilinear parabolic partial differential equations," in *Stochastic Partial Diferential Equation and Their Applications*, B. L. Rozuvskii and R. B. Soeers, Eds., vol. 176 of *Lecture Notes in Control and Inform. Sci.*, pp. 200–217, Springer, Berlin, Germany, 1992.

[3] E. Pardoux and S. Tang, "Forward-backward stochastic differential equations and quasilinear parabolic PDEs," *Probability Theory and Related Fields*, vol. 114, no. 2, pp. 123–150, 1999.

[4] B. Dupire, "Functional Ito calculus," Portfolio Research Paper, Bloomberg, New York, NY, 2009.

[5] R. Cont and D. A. Fourni, "Functional Ito calculus and stochastic intergral representation of martingles," *Annals of Probability*, vol. 41, no. 1, pp. 109–133, 2010.

[6] R. Cont and D. Fournie, "A functional extension of the Ito formula," *Comptes Rendus Mathématique*, vol. 348, no. 1-2, pp. 57–61, 2010.

[7] R. Cont and D.-A. Fournié, "Change of variable formulas for non-anticipative functionals on path space," *Journal of Functional Analysis*, vol. 259, no. 4, pp. 1043–1072, 2010.

[8] S. Peng and F. Wang, "BSDE, path-dependent PDE and nonlinear Feynman-Kac formula ," http://arxiv.org/abs/1108.4317.

[9] S. Peng, "Note on viscosity solution of path-dependent PDE and G-martingales ," http://arxiv.org/abs/1106.1144 .

[10] I. Ekren, C. Keller, N. Touzi, and J. Zhang, "On viscosity solutions of path dependent PDEs ," http://arxiv.org/abs/1109.5971.

[11] A. Cosso, "Viscosity solutions of path-dependent PDEs and non-Markovian forward-backward stochastic systems," http://arxiv.org/abs/1202.2502.

[12] R. S. Lipster and A. N. Shiryaev, *Statistics of Random Processes I*, Springer, Berlin, Germany, 1978.

[13] E. Pardoux and S. Peng, "Adapted solution of a backward stochastic differential equation," *Systems and Control Letters*, vol. 14, no. 1, pp. 55–61, 1990.

[14] J. Ma and J. Yong, *Forward-Backward Stochastic Differential Equations and Their Applications*, vol. 1702 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1999.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 359701, 8 pages
http://dx.doi.org/10.1155/2013/359701

Research Article

State-Dependent Utilities and Incomplete Markets

Jaime A. Londoño

Departamento de Matemáticas, Universidad Nacional de Colombia, Bogotá, Colombia

Correspondence should be addressed to Jaime A. Londoño; jaime.a.londono@gmail.com

Received 23 January 2013; Accepted 1 May 2013

Academic Editor: Guangchen Wang

The problem of optimal consumption and investment for an agent that does not influence the market is solved. The optimization criteria are based on a state-dependent utility functional as proposed in Londoño (2009). The proposed solution is given in any market without state-tame arbitrage opportunities, includes several utilities structures, and includes *incomplete markets* where there are multiple state variables. The solutions obtained for optimal wealths consumptions, and portfolios are explicit and easily computable; the main condition for the result to hold is that the income process of each agent is hedgeable, requiring a natural condition on employer and employee to agree on a contract whose risk can be managed by both parties. In this paper we also developed a theory of markets when the processes are generalization of Brownian flows on manifolds, since this framework shows to be the natural one whenever the problem of intertemporal equilibrium is addressed.

1. Introduction

The problem of optimal consumption and investment for a "small investor" whose actions do not influence market prices in *complete markets* and where consumers have dependent utility structures has been studied in Londoño [1]. The modern treatment of this problem when the asset prices follow Itô processes started with the seminal works of Merton [2, 3]. Using a "martingale" approach, Cox and Huang [4] and Karatzas et al. [5] solved the problem in more general settings in the case of complete markets. Analytical and numerical problems with those earlier solutions and lack of agreement with empirical data motivated alternative treatments; see Londoño [1] for some literature related with problems associated with the standard models as well as some reviews of other approaches.

In incomplete markets, there are even more inconveniences associated with the theory of optimal consumption and investment. General results have been derived in Karatzas and Žitković [6], Hugonnier and Kramkov [7], and Cvitanić et al. [8]. The solutions obtained by them are very limited, and almost nontrivial cases have been solved explicitly. We notice that even though the utilities structures studied by some of them allow state dependence, they are not able to handle the case presented in this paper since they ask for *bounded* (with bounds that depend on time and value of consumption) utility random fields that do not include the ones considered here; see Karatzas and Žitković [6, Definition 3.1 and Proposition 3.5. (item 1)]. In the case presented in this paper the utility random fields are usually considered as unbounded (in the state variable) when the time and the consumption are fixed; see Remark 8. In case of hedgeable (insurable) random endowments and incomplete markets the solution is well known in the case of Markovian markets with non state variables. For instance Merton [3] stated that, in computing the optimal decision rules, the individual capitalizes the lifetime flow of wage income at the market (risk-free) rate of interest and then treats the capitalized value as an addition to the current stock of wealth. Similar results are obtained in more general Markovian markets, as is discussed in Karatzas and Shreve [9]. This result is even true for semimartingale complete markets as is pointed out in Karatzas and Žitković [6].

Whenever state variables are introduced, the solutions provided to complete markets are no longer available even in the presence of hedgeable income, or even with no income. Computable solutions are not known in general settings, since hedgeability of the income structure does not necessarily allow for the optimal portfolio to be hedgeable in the standard models (see Karatzas and shreve [9, Sections 3.6 and 4.4]). However, current models of equilibrium allow for

the existence of state variables that model the dynamics of the relevant variables of the economy as in Merton [10], Breeden [11], Cox et al. [12], or Londoño [13] to cite just a few.

The main result of this paper is the solution to the problem of optimal consumption and investment for an agent that does not influence the market. It is assumed that the utility maximization criteria used are based on a state-dependent utility functional as proposed in Londoño [1]. For the optimization problem of this paper, it is assumed that consumers are endowed with an initial capital and a hedgeable random endowment where the underlying market is not necessarily assumed to be complete.

Here, we extend further the approach presented in Londoño [1]. In this model utilities reflect the level of consumption satisfaction of flows of cash in future times as they are valued by the market when the economic agents are making their consumption and investment decisions. The utilities used in this paper were introduced in Londoño [1] and are equivalent to state-dependent utilities in standard settings, where dependence on the state is through the state price density process (see (7) and Remark 8). The main assumption of the theorem, besides the interpretation of the utility functionals, is that the income of the agent should be hedgeable; in plain English it is required that the employer and employee agree on a labor contract that allows each party to hedge any risk associated with the this contract, and in this way any economic agent would be able to cover the compromises of this contract in exchange for a fair price. In the context of this paper it is not needed to be able to cover all the financial liabilities in a given economy (for instance it is not needed that any "reasonable" derivative could be hedge).

The solutions of the optimal consumption and portfolio problem are obtained in a very general setting which includes several functional forms for utilities and considers general restrictions on allowable wealth that are used in the current literature. We obtained *simple* and *computable* solutions that are optimal consumption and investment strategies in all studied cases. In our model it is always true that when the endowment is hedgeable, the problem becomes equivalent to one where the entire endowment process is replaced by its present value, in the form of an augmented initial wealth; see Corollary 9.

The theoretical framework proposed in this paper is one of stochastic flows on manifolds; this is an extension of the framework proposed in Londoño [14]. In the spirit of Merton [10], Breeden [11], and Cox et al. [12] we assume a Markovian setting for the "state variables" that includes not only the price processes but also additional variables that describe the evolution of the economy. This setting proved to be the right framework to study equilibrium problems, since equilibrium defines restrictions on the variables that make them take values on manifolds; see Londoño [13].

This paper also depends on the valuation and arbitrage theory presented in Londoño [14]; this is an extension of the theory of state tameness (Londoño [15]) that provided a unified framework for valuation of financial instruments, of both European and American types, with an algebraic appealing character and economic justification. In Londoño [14, 15] the conditions presented for valuation of financial instruments of American type are the weakest possible. Additional characteristics of the framework developed in Londoño [14, 15] are weak conditions on the coefficients on the volatility matrix of the price process and a development of a theory of valuation of contingent claims with random expiration date.

To the best of our knowledge this theory of arbitrage and valuation is the most general existing setting in the case of (continuous) semimartingales driven by Brownian filtrations with continuous coefficients. For a review of the state of the art on valuation and arbitrage theory we suggest the reader to look at Londoño [15] and the references therein.

2. The Model

First we introduce some notation which will be frequently used in this paper. Let $\mathbb{D} \subset [s,T] \times \mathbb{R}^k$ be a measurable set for $0 \le s < T$, with section $\mathbb{D}(t) = \{x \in \mathbb{R}^k \mid (t,x) \in \mathbb{D}\}$ for $s \le t \le T$. We assume that for each t, $\mathbb{D}(t)$ has a differential structure, which at this point is not necessary to specify. Examples of this differential structure are sets whose sections $\mathbb{D}(t)$ are the solution sets $\{x \mid \varphi(t,x) = 0\}$ of a function $\varphi : [s,T] \times U \to \mathbb{R}^p$ for some $p < k$, where $U \subset \mathbb{R}^k$ is an open set, and φ is a continuous function, that is, differentiable in the spatial variable, and whose partial derivatives are continuous (also in the time variable). Moreover, in order to define a manifold structure it is customary to impose that the differential $D\varphi(t,\cdot)$ has maximal rank for each t. Other examples are sets whose sections $\mathbb{D}(t)$ are integral manifolds defined by $k-p$ $(1 \le p < k)$ continuous vector fields $X_1^s, \dots, X_{k-p}^s : [s,T] \times U \to \mathbb{R}^k$, where U is an open set as above, and $X_1^s(t,\cdot), \dots, X_{k-p}^s(t,\cdot)$ are continuously differentiable vector fields that are linearly independent at each space point. It is well known that these two conditions above imply a differential structure on each of the specified subsets; see Warner [16]. In this paper it is always assumed that the degree of "smoothness" of the differential structure is sufficient for every definition to make sense. By this we mean that the transition functions, the functions defining the solution sets or the vector fields, are sufficiently differentiable.

Let m be a nonnegative integer, and let $\mathbb{D}$ be a set defined as above. We say that $f \in C^{m,\delta}(\mathbb{D} : \mathbb{R}^n)$ if f is a continuous function $f \in C(\mathbb{D}, \mathbb{R}^n)$, and there exist an open set $\mathbb{U}$ (relative to the topology of $[0,T] \times \mathbb{R}^k$) with $\mathbb{D} \subset \mathbb{U} \subset [0,T] \times \mathbb{R}^k$ and a continuous extension $\tilde{f}$ of f such that $\tilde{f}(t,\cdot) \in C^{m,\delta}(\mathbb{U}(t) : \mathbb{R}^n)$, where this last space is the Fréchet space of m-times continuous differentiable functions whose mth-order derivatives are δ-Hölder continuous with seminorms $\|\cdot\|_{m,\delta:K}$ defined in Kunita [10, Section 3.1] with $\int_s^T \|\tilde{f}(t,\cdot)\|_{m,\delta:K}\, dt < \infty$, where $K \subset \mathbb{U}$ is a compact set, $0 \le \delta \le 1$. In case $m = 0$ (or $\delta = 0$), we denote $C^{m,\delta}(\mathbb{D} : \mathbb{R}^n)$ simply by $C^{\delta}(\mathbb{D} : \mathbb{R}^n)$ ($C^m(\mathbb{D} : \mathbb{R}^n)$), and whenever clear we denote the above spaces simply as $C^{m,\delta}$, C^{δ}, and C^m, respectively. We denote by $C^{m,0+}$ the set $\cup_{\delta>0} C^{m,\delta}$, and $C^{0+} = C^{0,0+}$. Let $\mathbb{D}' \subset [s,T] \times \mathbb{R}^k$ be a measurable set with sections $\mathbb{D}'(t)$; we say that a function f belongs to the class

$C^{m,\delta}(\mathbb{D} : \mathbb{D}')$, where $\delta > 0$ or $\delta = 0+$, if $f(t,\cdot)$ takes values in $\mathbb{D}'(t)$ and $f \in C^{m,\delta}(\mathbb{D} : \mathbb{R}^n)$. We assume in the following definitions that we have a differential structure sufficiently smooth according to the space defined. We notice that the definition of Hölder continuity is made with respect to the Euclidean distance derived from the fact that these sets are subsets of an Euclidean space.

Definitions of consistent processes that we review below are natural generalizations (of processes defined on embeddings) of the definitions made in Londoño [15].

We assume a d-dimensional Brownian motion $\{W(t), \mathcal{F}_t;\ 0 \le t \le T\}$ starting at 0 and defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathcal{F} = \mathcal{F}_T$ and $\{\mathcal{F}_t, 0 \le t \le T\}$ is the $\mathbf{P}$ augmentation by the null sets of the natural filtration $\mathcal{F}_t^W = \sigma(W(s), 0 \le s \le t)$. Let $(\mathcal{F}_{s,t}) = \{\mathcal{F}_{s,t}, 0 \le s \le t \le T\}$ be the two-parameter filtration, where $\mathcal{F}_{s,t}$ is the smallest sub-σ-field containing all null sets and $\sigma(W_s(u) \mid s \le u \le t)$, where $W_s(u) \equiv W(u) - W(s)$. For each $0 \le s \le T$ we also define the σ-field $\mathcal{P}_s$ of progressive measurable sets after time s as the σ-field of sets $P \in \mathcal{B}([s,T]) \otimes \mathcal{F}_{s,T}$, and the product σ-field, such that $\chi_P(t,\omega)$, $t \ge s$, is a $\mathcal{F}_{s,t}$ progressive measurable (in t) process, where χ is the indicator function. We denote by μ_s the measure on $\mathcal{P}_s$ defined by $\mu_s(P) = \mathbf{E}\int_s^T \chi_P(t,\omega)\, dt$.

Let $\varphi_{s,t}(x,\omega)$, $0 \le s \le t \le T$, $x \in \mathbb{D}(s)$, be a $\mathbb{R}^n$-valued random field on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, where $\mathbb{D}$ has a differential structure sufficiently smooth for the following definitions to make sense, and let $\mathbb{D}' \subset [0,T] \times \mathbb{R}^n$ be a measurable set. We say that $\varphi_{s,t}$ is a *continuous $C^{m,\delta}(\mathbb{D} : \mathbb{D}')$-semimartingale* if $\varphi_s : t \to \varphi_{s,t}(\cdot)$ is a continuous random field with values in $C^{m,\delta}(\mathbb{D} \cap ([s,T] \times \mathbb{R}^k) : \mathbb{D}' \cap ([s,T] \times \mathbb{R}^n))$, that is, a continuous $(\mathcal{F}_{s,\cdot})$ semimartingale process. In addition we assume that $\varphi_{s,t}(x)$ can be decomposed as $\varphi_{s,t}(x) = \varphi_{s,t}^{\mathrm{loc}}(x) + \varphi_{s,t}^{\mathrm{fv}}(x)$, where $\varphi_{s,\cdot}^{\mathrm{loc}}(\cdot)$ is a continuous $C^{m,\delta}(\mathbb{D} \cap ([s,T] \times \mathbb{R}^k) : \mathbb{D}' \cap ([s,T] \times \mathbb{R}^n))$-local-martingale, and $\varphi_{s,\cdot}^{\mathrm{fv}}(\cdot)$ is a continuous $C^{m,\delta}(\mathbb{D} \cap ([s,T] \times \mathbb{R}^k) : \mathbb{D}' \cap ([s,T] \times \mathbb{R}^n))$-process of bounded variation for each $0 \le s \le T$. A pair (a,b), where $a_{s,t}(x,y)$ and $b_{s,t}(x)$ are measurable random fields $\mathcal{F}_{s,t}$-progressive measurable in t for all $x, y \in \mathbb{D}(s)$, $0 \le s \le T$, is said to be the *local characteristics of* φ if $(a_{s,\cdot}(x,y), b_{s,\cdot}(x))$ is the local characteristic of $\varphi_s \equiv \varphi_{s,\cdot}(\cdot)$ for any $s \le T$, (see Kunita [17]). In addition, a pair (σ, b), where $\sigma_{s,t}(x)$ is a measurable random field, that is, $(\mathcal{F}_{s,t})$-progressive measurable in t with values in the set $L(\mathbb{R}^d : \mathbb{R}^n)$ of real-valued matrices with size $n \times d$, and b is as above, is said to be the *diffusion and drift processes of* φ if

$$\varphi_{s,t}^{\mathrm{loc}}(x)(\omega) = \int_s^t \sigma_{s,u}(x)\, dW_s(u) \tag{1}$$

for all x, s, t, and ω. If $b_{s,\cdot}(\cdot)$ and $\sigma_{s,\cdot}(\cdot)$ are processes of class $C^{m,\delta}$ for all $0 \le s \le T$, we will say that φ has *diffusion and drift of class $C^{m,\delta}$*.

Let $\varphi_{s,t}(x)$ and $\psi_{s,t}(x)$ be continuous $C(\mathbb{D} : \mathbb{R}^n)$ and $C(\mathbb{D} : \mathbb{D})$ semimartingales, respectively, where in addition, it is assumed that $\psi(s,s,x) = x$ for all $x \in \mathbb{D}(s)$. We say that the process φ is a *ψ-consistent semi-martingale process* if for each $0 \le s \le s' \le T$ there exists a set $N_{s,s'} \in \mathcal{P}_{s'}$ with $\mu_{s'}(N_{s,s'}) = 0$, such that $\varphi_{s,t}(x) = \varphi_{s',t}(\psi_{s,s'}(x))$ for all $(t,\omega) \notin N_{s,s'}$ and all $x \in \mathbb{D}(s)$. We say that the process φ is a *consistent semimartingale process* if φ is a φ-consistent process.

A few words should be said about the existence of solutions of stochastic differential equations on manifolds, and although our approach is surely not the more general, we believe that it is the simplest since it does not require any technicalities of differential geometry. The simplicity of our approach is due to our definitions of functions of type $C^{m,\delta}$ on a manifold. Assume that $b : \mathbb{D} \to \mathbb{R}^n$ and $\sigma : \mathbb{D} \to L(\mathbb{R}^n : \mathbb{R}^d)$ are functions in $C^{m,\delta}$, where $m = 0$ and $\delta = 1$ or $m \ge 1$ and $\delta > 0$. Let $\varphi_{s,t}(x)$ be the local (maximal) solution of

$$\varphi_{s,t}(x) = \tilde{b}(t, \varphi_{s,t}(x))\, dt + \tilde{\sigma}(t, \varphi_{s,t}(x))\, dW_s(t), \quad \varphi_{s,s}(x) = x \tag{2}$$

for $x \in \mathbb{U}(s)$, where $\tilde{b}$ and $\tilde{\sigma}$ are the extensions of b and σ to some open set $\mathbb{U}$. We assume that $\varphi_{s,t}(x)$ is the local solution of class $C^{m,\epsilon}$ for any $0 \le \epsilon < \delta$, see Kunita [17, Theorems 4.7.1 and 4.7.2]. If for any $x \in \mathbb{D}(s)$, $\varphi_{s,t}(x)$ has a nonexplosive solution with values in $\mathbb{D}$, we will say that $\varphi_{s,t}(x)$, $x \in \mathbb{D}(s)$ is the solution to the stochastic differential equation on $\mathbb{D}$. It is straightforward to see that this defines uniquely a process that does not depend on the extensions $\tilde{b}$ and $\tilde{\sigma}$ that are used.

Next, we describe a financial market. We assume an m-dimensional Itô process Θ of two parameters with values in $C^{2,0+}(\mathbb{K} : \mathbb{K})$ (for $m > 1$), where $\mathbb{K}(s) \subset \mathbb{R}_+ \times \mathbb{R}^{m-1}$ (where $\mathbb{R}_+ = (0,\infty)$) for each $0 \le s \le T$, and $\mathbb{K} \subset [0,T] \times \mathbb{R}_+ \times \mathbb{R}^{m-1}$ is some measurable set with a differential structure as discussed above. We assume an n+m-dimensional Itô process (P, Θ) of two parameters with values in $C^{0+}(\mathbb{D} : \mathbb{D})$, where $\mathbb{D}$ is a measurable set with sections $\mathbb{D}(s) \subset \mathbb{R}_+^n \times \mathbb{K}(s)$ for each s, where $\mathbb{R}_+^n = (\mathbb{R}_+)^n$ and with a differential structure as discussed above. We assume that for each initial condition $\vartheta = (\vartheta_0, \ldots, \vartheta_{m-1})^\top \in \mathbb{K}(s)$,

$$\frac{d\Theta_{s,t}^0}{\Theta_{s,t}^0} = \left(r(t, \Theta_{s,t})\, dt + \left\|\theta(t, \Theta_{s,t})\right\|^2\right) dt + \sum_{1 \le j \le d} \theta^j(t, \Theta_{s,t})\, dW_s^j(t), \quad \Theta_{s,s}^0(\vartheta) = \vartheta_0, \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm, and

$$\begin{gathered} \Theta_{s,t}^i = \rho^i(t, \Theta_{s,t})\, dt + \sum_{1 \le j \le d} \varrho^{i,j}(t, \Theta_{s,t})\, dW_s^j(t) \\ \Theta_{s,s}^i(\vartheta) = \vartheta_i, \quad i = 1, \ldots, m-1 \end{gathered} \tag{4}$$

for some continuous functions $r, \vartheta^j, \rho^i, \varrho^{i,j}$ for $j = 1, \ldots, d$ of class $C^{2,0+}$, and $i = 1, \ldots, m-1$ for which global solutions of these stochastic differential equations exit. We also assume that for each $(p_1, \ldots, p_n, \vartheta_0, \ldots, \vartheta_{m-1})^\top = (p^\top, \vartheta^\top)^\top \in \mathbb{D}(s)$,

$$\begin{gathered} \frac{dP_{s,t}^i}{P_{s,t}^i} = b^i(t, P_{s,t}, \Theta_{s,t})\, dt + \sum_{1 \le j \le d} \sigma^{i,j}(t, P_{s,t}, \Theta_{s,t})\, dW_s^j(t) \\ P_{s,s}^i(p, \vartheta) = p_i, \quad i = 0, \ldots, n, \end{gathered} \tag{5}$$

where, for $1 \le i \le n$, $P^i_{s,t} = \pi^i \circ P_{s,t}$ is *the price of shares outstanding for the i-stock* and where π^i denotes the projection on the ith-component for $1 \le i \le n$. We assume that $\sigma^{i,j}, b^i$ are continuous functions of class $C^{2,0+}$ for which global solutions of the set of differential equations above exist and are unique.

We point out that in a free of (state) arbitrage opportunities and (state) complete market, $\vartheta_0/\Theta^0_{s,t}$ is the process (with two parameters) that discounts the flow of money in every future state t, $s \le t \le T$ to bring the value of the flow to current time s; for instance see Londoño [15] and the definition of $H_{s,t}$ below.

The process of bounded variation $B_{s,t} \equiv B_{s,t}(\vartheta)$, whose evolution is given by the stochastic differential equation

$$dB_{s,t} = r(t, \Theta_{s,t}) B_{s,t} dt, \quad B_{s,s} = 1, \quad \text{for } s \le t \le T, \tag{6}$$

will be called the *bond price process*.

We define the *state price density process* $H_{s,t} \equiv H_{s,t}(\vartheta)$ to be the continuous $C^{0+}(\mathbb{K} : \mathbb{R}_+)$ process given by

$$H_{s,t} = B^{-1}_{s,t} Z_{s,t}, \quad s \le t \le T, \tag{7}$$

where $Z_{s,t} \equiv Z_{s,t}(\vartheta)$ is the process defined as

$$Z_{s,t} = \exp\left\{-\int_s^t \theta^\top(u, \Theta_{s,u})\, dW(u) - \frac{1}{2}\int_s^t \left\|\theta(u, \Theta_{s,u})\right\|^2 du\right\} \tag{8}$$

for $s \le t \le T$, and $B^{-1}_{s,t} = 1/B_{s,t}$. We point out that

$$H_{s,t} = \frac{\vartheta_0}{\Theta^0_{s,t}} \tag{9}$$

easily follows the Itô's lemma. Throughout this paper we will assume that $\theta(t,\vartheta) \in \ker^\perp(\sigma(t,p,\vartheta))$, where $\ker^\perp(\sigma(\cdot))$ denotes the orthogonal complement of the kernel of $\sigma(\cdot)$ and

$$b(t,p,\vartheta) + \delta(t,p,\vartheta) - r(t,\vartheta)\mathbf{1}_n = \sigma(t,p,\vartheta)\theta(t,\vartheta) \tag{10}$$

for all $(p^\top, \vartheta^\top)^\top \in \mathbb{D}(s)$ and t, where $\mathbf{1}^\top_n = (1,\ldots,1) \in \mathbb{R}^n$. This latter assumption implies that there are no state-tame arbitrage opportunities (see Londoño [15]). In addition to the above condition we assume in this paper that the market satisfies the following condition that we call smooth market condition. We notice that since $\theta(t,\vartheta) \in \ker^\perp(\sigma(t,p,\vartheta)) = \mathrm{Im}(\sigma^\top(t,p,\vartheta))$, the existence of a measurable function κ with the property expressed in (11) follows for any financial market, and therefore the condition below is indeed a weak condition on the smoothness of the mentioned property.

Condition 1 (smooth market condition). There exists a $C^{2,0+}$-matrix-valued function κ defined on $\mathbb{D}$ with the property that

$$\sigma^\top(t,p,\theta)\kappa(t,p,\vartheta) = \theta(t,\vartheta). \tag{11}$$

We point out that we require no dependence of prices on the evolution of the functions r and θ. We say that $b_{s,t} \equiv b(t, P_{s,t}, \Theta_{s,t})$ is the *return process*, $\sigma_{s,t} \equiv \sigma(t, P_{s,t}, \Theta_{s,t})$ is the *volatility process*, $\delta_{s,t} \equiv \delta(t, P_{s,t}, \Theta_{s,t})$ is the *process of dividends*, $r_{s,t} \equiv r(t, \Theta_{s,t})$ is the *interest rate process*, and $\theta_{s,t} \equiv \theta(t, \Theta_{s,t})$ is the *market price of risk process*.

For a structure as above, we say that $\mathscr{M} = (P, \Theta, \mathbb{D}, \mathbb{K}, b, \sigma, \delta, \theta, \rho, \varrho, r, p^0, \vartheta^0)$ is a *financial market with terminal time T and initial time 0, feasible set of values $\mathbb{D}$, feasible set of state values $\mathbb{K}$, vector of returns $b(t,\cdot) = (b_1(t,\cdot)\ldots,b_n(t,\cdot))^\top$, matrix of volatility coefficients $\sigma(t,\cdot) = (\sigma^{i,j}(t,\cdot))$, vector of dividends $\delta(t,\cdot) = (\delta_1(t,\cdot),\ldots,\delta_n(t,\cdot))^\top$, market price of risk $\theta(t,\cdot) = (\theta_1(t,\cdot),\ldots,\theta_d(t,\cdot))^\top$, interest rate $r(t,\cdot)$, drift of the state process $\rho(t,\cdot) = (\rho^1(t,\cdot),\ldots,\rho^m(t,\cdot))^\top$, diffusion matrix for the state process $\varrho(t,\cdot) = (\varrho^{i,j}(t,\cdot))$, vector of initial prices $p^0 \in \mathbb{R}^n_+$*, and initial state variables ϑ^0.

Next, we review and extend some definitions from Londoño [1] that are needed to describe equilibrium.

Definition 1. Assume a measurable set $\mathbb{X} \subset [0,T] \times \mathbb{R}^{n+m+1}$ with some differential structure as discussed in Section 2 with (nonempty) section $\mathbb{X}(s) \subset \mathbb{R} \times \mathbb{D}(s)$ for each $0 \le s \le T$. Assume a family of continuous Itô's processes

$$X = \left\{X_{s,t}(x,p,\vartheta); (x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s),\ 0 \le s \le t \le T\right\}, \tag{12}$$

such that (X, P, Θ) is a consistent process of class $C^{0+}(\mathbb{X} : \mathbb{X})$. We say that X is a *wealth evolution structure*; we will denote this by writing $X \in \mathscr{X}(\mathscr{M})$. For a detailed description of consistent processes and related ones see Londoño [1, 14]. One says that $\mathbb{X}$ is *a feasible set of values for* (X, P, Θ).

Let $(\pi^0, \pi) = \{(\pi^0_{s,t}(x,p,\vartheta),\ldots,\pi^n_{s,t}(x,p,\vartheta));\ (x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s), 0 \le s \le t \le T\}$ be a (X,P,Θ)-consistent progressive measurable process of class C^{0+} with $\pi_0 + (\pi)^\top \mathbf{1}_n = X$. Assume that $c = \{c_{s,t}(x,p,\vartheta) : (x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s), 0 \le s \le t \le T\}$ is a nonnegative consistent (X,P,Θ) progressive measurable process of class C^{0+} and that $Q = \{Q_{s,t}(p,\vartheta) : (p^\top, \vartheta^\top)^\top \in \mathbb{D}(s), 0 \le s \le t \le T\}$ is a nonnegative consistent (P,Θ) progressive measurable process of class C^{0+}. It is assumed that

$$\begin{aligned} B^{-1}_{s,t}(\vartheta) X_{s,t}(x,p,\vartheta) &= x + \int_s^t B^{-1}_{s,u}(\vartheta)\left(Q_{s,u}(p,\vartheta) - c_{s,u}(x,p,\vartheta)\right) du \\ &\quad + \int_s^t B^{-1}_{s,u}(\vartheta)(\pi)^\top_{s,u}(x,p,\vartheta)\sigma_{s,u}(p,\vartheta)\, dW_s(u) \\ &\quad \int_s^t B^{-1}_{s,u}(\vartheta)(\pi)^\top_{s,u}(x,p,\vartheta) \\ &\quad \times \left(b_{s,u}(p,\vartheta) + \delta_{s,u}(p,\vartheta) - r_{s,u}(\vartheta)\mathbf{1}_n\right) du \end{aligned} \tag{13}$$

for all $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s)$ and $0 \le s \le t \le T$. In addition it is assumed that

$$\mathbf{E}\left[\int_s^T H_{s,t}(\vartheta) c_{s,t}(x,p,\vartheta)\, dt\right] < \infty, \qquad \mathbf{E}\left[\int_s^T H_{s,t}(\vartheta) Q_{s,t}(p,\vartheta)\, dt\right] < \infty \tag{14}$$

for all $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s)$ and $0 \le s \le t \le T$. We say that $((\pi_0, \pi), c, Q)$ as above is a portfolio evolution structure with rate of consumption $c_{s,t}$ and rate of endowment Q. We say (X, c, Q) as above is a hedgeable rate of consumption and endowment evolution structure with feasible set of values $\mathbb{X}$. A (P, Θ)-consistent process Q (without any additional structure) that satisfies (14) is called a rate of endowment evolution process. If a wealth process $(X, 0, Q)$ is a hedgeable rate of consumption and endowment evolution structure with feasible set of values $\mathbb{X}$, we just say that (X, Q) is a hedgeable endowment evolution structure.

A subsistence random field L for the market $\mathscr{M}$ is a (P, Θ)-consistent process with drift and diffusion of class C^{0+}, where $L_{s,\cdot}(p, \vartheta) H_{s,\cdot}(\vartheta)$ is uniformly bounded below for all $(p^\top, \vartheta^\top)^\top \in \mathbb{D}(s)$ (where the bound might depend on p, ϑ, and s) such that

$$\mathbf{E}\left[H_{s,t}(\vartheta) L_{s,t}(p, \vartheta)\right] < \infty \tag{15}$$

for all t. We will say that the couple (π, c) of portfolio on stocks and rate of consumption is admissible for (L, Q) and write $(\pi, c) \in \mathscr{A}(L, Q)$ if for any $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s)$ with $x \ge L_{s,s}(p, \vartheta)$

$$X_{s,t}(x, p, \vartheta) \ge L_{s,t}(p, \vartheta) \quad \forall t. \tag{16}$$

If there does not exist a couple of portfolio on stocks and rate of consumption admissible for (L, Q), we say that the class cited above is empty, and we would denote this by $\mathscr{A}(L, Q) = \emptyset$.

3. Consumption and Portfolio Optimization

Throughout this paper we are mainly interested in portfolio evolution structures that are obtained as the result of the optimal behavior of consumers as it is explained below. Next, we review definitions and properties of the type of utilities that are used in this paper.

Definition 2. Consider a function $U : (0, \infty) \mapsto \mathbb{R}$ is continuous, strictly increasing, strictly concave, and continuously differentiable with $U'(\infty) = \lim_{x\to\infty} U'(x) = 0$ and $U'(0+) \triangleq \lim_{x\downarrow 0} U'(x) = \infty$. Such a function will be called a utility function.

Classic examples of utility functions are $U_\alpha(x) = x^\alpha/\alpha$ for some $\alpha \in (0, 1)$, $0 \le x < \infty$, and $U(x) = \log(x)$. For every utility function $U(\cdot)$, we will denote by $I(\cdot)$ the inverse of the derivative $U'(\cdot)$; both of these functions are continuous, strictly decreasing and map $(0, \infty)$ onto itself with $I(0+) = U'(0+) = \lim_{x\to 0^+} U'(x) = \infty$, $I(\infty) = \lim_{x\to\infty} I(x) = U'(\infty) = 0$. We extend U by $U(0) = U(0^+)$ (we keep the same notation to the extension to $[0, \infty)$ of U hoping that it will be clear to the reader). It is a well-known result that

$$\max_{0<x<\infty} \left(U(x) - xy\right) = U(I(y)) - yI(y), \quad 0 < y < \infty. \tag{17}$$

Definition 3. Consider a continuous function $U_1 : [0, T] \times (0, \infty) \mapsto \mathbb{R}$, such that $U_1(t, \cdot)$ is a utility function in the sense of Definition 2 for all $t \in [0, T]$. It follows that $I_1(t, x) \triangleq (\partial U_1(t, x)/\partial x)^{-1}$, the inverse of the derivative of U, is a continuous function. Similarly, if a utility function $U_2 : (0, \infty) \mapsto \mathbb{R}$ is given, then $I_2(x) \triangleq (\partial U_2(x)/\partial x)^{-1}$ is continuous. Let one denote

$$\mathscr{X}(t, y) \triangleq I_2(y) + \int_t^T I_1(t', y)\, dt'. \tag{18}$$

We call a couple of functions U_1 and U_2 a state preference structure.

Under the conditions outlined in the previous definition, it is easy to see that $\mathscr{X} : [0, T] \times (0, \infty) \to (0, \infty)$ is a continuous function with the property that for each t, $\mathscr{X}(t, \cdot)$ maps $(0, \infty)$ onto itself, strictly decreasing with $\mathscr{X}(t, 0+) = \lim_{y\downarrow 0} \mathscr{X}(t, y) = \infty$ and $\mathscr{X}(t, \infty) = \lim_{y\to\infty} \mathscr{X}(t, y) = 0$.

We extend U_1 and U_2 by defining $U_1(t, 0) = U(t, 0^+)$ for all $0 \le t \le T$ and $U_2(0) = U_2(0^+)$, and we keep the same notation to the extension of U_1 to $[0, T] \times [0, \infty)$ and the extension of U_2 to $[0, \infty)$.

We point out that $\mathscr{X}^{-1}$, defined for each t as $\mathscr{X}^{-1}(t, \cdot)$, the inverse of $\mathscr{X}(t, \cdot)$, shares the same properties stated above about $\mathscr{X}$. The discussion of those utility functions defined above, is given in Londoño [1].

For $s \le t$, define $\alpha(s, t) = \mathscr{X}(s, \mathscr{X}^{-1}(t, \cdot))$. Then $\alpha(s, t) = \alpha(s, t') \circ \alpha(t', t)$ for all s, t, and t' in $[0, T]$, where $\circ$ denotes standard composition of functions. We also observe that if $\alpha^I(s, t) \triangleq I_1(s, \mathscr{X}^{-1}(t, \cdot))$, then $\alpha^I(s, t) \circ \alpha(t, s) = \alpha^I(s, s)$. Some examples discussed in Londoño [1] include power utility structures $U_1(t, x) = x^\alpha h(t)$ and $U_2(x) = cx^\alpha$ with $\alpha \in (0, 1)$ and $c \ge 0$, where $h : [0, T] \to (0, \infty)$ is any continuous positive function. Logarithmic utility structures are also included, where h is as above, $U_1(t, x) = h(t)\log(x)$, and $U_2(x) = c\log(x)$ with $c \ge 0$. Finally in the cited paper "separable preference structures" are introduced, where h is as above, $U_1(t, x) = h(t)u(x/h(t))$, and $U_2(x) = cu(x/c)$, where $u(\cdot)$ is a utility function and $c > 0$.

Before we state the main result of this paper, we first require an additional definition.

Definition 4. Assume a rate of endowment evolution process Q of class $C^{2,0+}$ with current value of future endowments L defined as

$$\begin{aligned}
&L_{s,t}(p, \vartheta) \\
&\quad = \frac{-1}{H_{s,t}(\vartheta)} \mathbf{E}\left[\int_t^T H_{s,u}(\vartheta) Q_{s,u}(p, \vartheta)\, du \mid \mathscr{F}_{s,t}\right] \\
&\quad = -\mathbf{E}\left[\int_t^T H_{t,u}(\Theta_{s,t}(\vartheta)) Q_{t,u}(P_{s,t}(p, \vartheta), \Theta_{s,t}(p, \vartheta))\, du\right] \\
&\quad = -\Pi(t, P_{s,t}(p, \vartheta), \Theta_{s,t}(\vartheta)),
\end{aligned} \tag{19}$$

where it is assumed that

$$\Pi(t, p, \vartheta) \triangleq -\mathbf{E}\left[\int_t^T H_{t,u}(\vartheta) Q_{t,u}(p, \vartheta)\, du\right] \tag{20}$$

is a function of class $C^{2,0+}$. If (L, Q) is a hedgeable endowment evolution structure with portfolio π_Q with feasible set of values $\mathbb{X}_Q$ defined as

$$\mathbb{X}_Q = \left\{(s, x, p, \vartheta) \mid x = -\Pi(s, p, \vartheta), (p^\top, \vartheta^\top)^\top \in \mathbb{D}(s)\right\}, \quad (21)$$

one will say that Q is a hedgeable income structure.

Remark 5. We point out that unless the market is state complete (see Londoño, [15, Theorem 3.1] and Londoño [14]), in which case the volatility matrix has maximal range (in the sense that its rank is equal to n), there might be an infinity number of portfolios that hedge the given income structure. If the market is not necessarily complete but has no state-tame arbitrage opportunities, it satisfies (10) therefore by definition of wealth associated to a portfolio

$$\begin{aligned} B_{s,t}^{-1}(\vartheta) L_{s,t}(p, \vartheta) &= x + \int_s^t B_{s,u}^{-1}(\vartheta) Q_{s,u}(p, \vartheta)\, du \\ &\quad + \int_s^t B_{s,u}^{-1}(\vartheta) \left(\sigma_{s,u}^\top(p, \vartheta) \pi_{s,u}(x, p, \vartheta)\right)^\top dW_s(u) \\ &\quad + \int_s^t B_{s,u}^{-1}(\vartheta) \left(\sigma_{s,u}^\top(p, \vartheta) \pi_{s,u}(x, p, \vartheta)\right)^\top \theta_{s,u}(\vartheta)\, du \end{aligned} \quad (22)$$

for all $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}_Q(s)$ and $0 \le s \le t \le T$, where $L_{s,t}$ is the process in Definition 4. It follows that the same wealth is hedged by any portfolio $\pi_{s,t} + \kappa_{s,t}$, where $\kappa_{s,t} \in \ker(\sigma_{s,t}^\top) \neq \emptyset$.

The following theorem extends the theory of optimal consumption and investment of Londoño [1] to incomplete markets which is the main result of this paper.

Theorem 6. *Assume that $\mathscr{M} = (P, \Theta, \mathbb{D}, \mathbb{K}, b, \sigma, \delta, \theta, \rho, \varrho, r, p^0, \vartheta^0)$ is a financial market that satisfies the smooth market condition (Condition 1) and* (10)*. Also, assume that (U_1, U_2) is a state preference structure for a consumer with hedgeable income structure Q (with hedging portfolio on the stocks $\pi_{s,t}^Q$).*

Denote by $\alpha_{s,t}$ and α_t^I the functions that correspond to the state preference structure (U_1, U_2) as defined after Definition 3. Define

$$\mathbb{X} = \left\{(s, x, p^\top, \vartheta^\top)^\top \mid x > \Pi(s, p, \vartheta), (p^\top, \vartheta^\top)^\top \in \mathbb{D}(s)\right\}. \quad (23)$$

Let ξ be defined as

$$\begin{aligned} \xi_{s,t}(x, p, \vartheta) &\triangleq \Pi(t, P_{s,t}(p, \vartheta), \Theta_{s,t}(\vartheta)) \\ &\quad + H_{s,t}^{-1}(\vartheta)\, \alpha_{t,s}(x - \Pi(s, p, \vartheta)), \end{aligned} \quad (24)$$

and let c be defined as

$$c_{s,t}(x, p, \vartheta) \triangleq H_{s,t}^{-1}(\vartheta) \left(\alpha_t^I \circ \alpha_{t,s}\right)(x - \Pi(s, p, \vartheta) \quad (25)$$

for any $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s)$. If ξ is a wealth evolution structure with feasible set of values $\mathbb{X}$, then it is a hedgeable cumulative consumption and endowment structure with portfolio $(\pi, c) \in \mathscr{A}(L, Q)$ which is an optimal solution for the problem of optimal consumption and investment. The optimality is in the sense that

$$\begin{aligned} &\mathbf{E}\left[\int_0^T U_1(t, H_{0,t}(\vartheta)\, c_{0,t}(x, p, \vartheta))\, dt + U_2(H_{0,T}(\vartheta)\, \xi_{0,T}(x, p, \vartheta))\right] \\ &\quad \ge \mathbf{E}\left[\int_0^T U_1(t, H_{0,t}(\vartheta)\, \tilde{c}_{0,t}(x, p, \vartheta))\, dt + U_2\left(H_{0,T}(\vartheta)\, \tilde{\xi}_{0,T}(x, p, \vartheta)\right)\right] \end{aligned} \quad (26)$$

for all $(x, p^\top, \vartheta^\top)^\top \in \mathbb{X}(s)$, where $(\tilde{\xi}, \tilde{c}, Q)$ is any hedgeable cumulative consumption and endowment structure with $(\tilde{\pi}, \tilde{c}) \in \mathscr{A}(L, Q)$, and

$$\mathbf{E}\left[\int_0^T U_1^-(t, H_{0,t}(\vartheta)\, \tilde{c}_{0,t}(x, p, \vartheta))\, dt + U_2^-\left(H_{0,T}(\vartheta)\, \tilde{\xi}_{0,T}(x, p, \vartheta)\right)\right] < \infty, \quad (27)$$

where $U_1^-(t, x) = -(U_1(t, x) \wedge 0)$ and $U_2^-(x) = -(U_2(x) \wedge 0)$. An optimal portfolio on stocks $\pi_{s,t}(x, p, \vartheta)$ is

$$\frac{\alpha_{t,s}(x - \Pi(s, p, \vartheta))}{H_{s,t}} \kappa(t, P_{s,t}, \Theta_{s,t}) - \pi_{s,t}^Q(-\Pi(s, p, \vartheta), p, \vartheta), \quad (28)$$

where κ is the function defined by (11)*. Moreover, $\pi_{s,t} + \eta_{s,t}$ is an optimal portfolio for any (X, P, Θ)-consistent process $\eta_{s,t}(x, p, \vartheta) \in \ker \sigma_{s,t}^\top(x, p, \vartheta)$ for $(x, p, \vartheta) \in \mathbb{X}(s)$.*

Proof. Define $\mathbb{X}_Q$ as the set given in Definition 4, and let π_Q be the portfolio that hedges the income process. It follows that

$$\begin{aligned} H_{s,t} L_{s,t} - \int_s^t H_{s,u} Q_{s,u}\, du &= -\mathbf{E}\left[\int_s^T H_{s,u} Q_{s,u}\, du \mid \mathscr{F}_{s,t}\right] \\ &= -\mathbf{E}\left[\int_s^T H_{s,u} Q_{s,u}\, du\right] \\ &\quad + \int_s^t H_{s,u}\left[\sigma_{s,u}^\top \pi_Q - L_{s,u} \theta_{s,u}\right]^\top dW_s(u), \end{aligned} \quad (29)$$

and on the other hand let us define $\pi_{s,t}^C(x - \Pi(s, p, \vartheta), p, \vartheta)$ by the process that satisfies

$$\sigma_{s,t}^\top \pi_{s,t}^C = \frac{\alpha_{t,s}(x - \Pi(s, p, \vartheta))}{H_{s,t}} \theta_{s,t}. \quad (30)$$

The existence of this process follows by Condition 1. Therefore

$$\begin{aligned}
&H_{s,t}H_{s,t}^{-1}\alpha_{t,s}\left(x-\Pi\left(s,p,\vartheta\right)\right)\\
&\quad+\int_s^t H_{s,u}H_{s,u}^{-1}\alpha_u^I\circ\alpha_{u,s}\left(x-\Pi\left(s,p,\vartheta\right)\right)du\\
&=\alpha_{t,s}\left(x-\Pi\left(s,p,\vartheta\right)\right)\\
&\quad+\int_s^t \alpha_u^I\circ\alpha_{u,s}\left(x-\Pi\left(s,p,\vartheta\right)\right)du\\
&=\left(x-\Pi\left(s,p,\vartheta\right)\right)\\
&=\left(x-\Pi\left(s,p,\vartheta\right)\right)\\
&\quad+\int_s^t H_{s,u}\left(\sigma_{s,u}^{\top}\pi_{s,u}^{C}\right.\\
&\qquad\left.-H_{s,u}^{-1}\alpha_{u,s}\left(x-\Pi\left(s,p,\vartheta\right)\right)\theta_{s,u}\right)^{\top}dW_s\left(u\right).
\end{aligned}\tag{31}$$

Then

$$\begin{aligned}
&H_{s,t}\xi_{s,t}+\int_s^t H_{s,u}\left(c_{s,u}-Q_{s,u}\right)du\\
&=x+\int_s^t H_{s,u}\left(\sigma_{s,u}^{\top}(\pi_{s,u}^{C}-\pi_{s,u}^{Q})-\xi_{s,u}\theta_{s.u}\right)^{\top}dW_s\left(u\right).
\end{aligned}\tag{32}$$

It follows as an application of Itô's rule that $\pi_{s,t}^{C}-\pi_{s,t}^{Q}$ is a portfolio that hedges $\xi_{s,t}$, and therefore (ξ,c,Q) is a hedgeable rate of consumption and endowment evolution structure. The proof that the consumption and endowment structure (ξ,c,Q) is optimal in the sense of the theorem is identical to the proof of [14, Theorem 2]. □

Remark 7. It is clear from the proof of Theorem 6 that even if a sufficient number of stocks are added in order to make a complete market, then the consumption, wealth, and portfolio processes obtained in the above mentioned theorem are still optimal solutions.

Remark 8. We notice that the optimization problem described above is equivalent to the standard optimization problem for (state-dependent) utility random fields when the random field is defined as follows:

$$\begin{aligned}
U_1\left(t,c,\omega\right)&=U_1\left(t,cH_{0,t}\left(\omega\right)\right),\quad \omega\in\Omega,\ c>0,\\
U_2\left(x,\omega\right)&=U_2\left(xH_{0,T}\left(\omega\right)\right),\quad \omega\in\Omega,\ x>0,
\end{aligned}\tag{33}$$

where the definition of utility random fields is the one presented by Karatzas and Žitković [6, Definition 3.1]. However we notice that Karatzas and Žitković [6] require the random fields to be bounded, and this is not the case for our random fields; therefore their results do not include ours.

As a result of the previous theorem, from a point of view of consumption of a consumer, he behaves as he sells the value of his income at the beginning and optimizes the consumption assuming no income at all.

Corollary 9. *Assume the conditions and notation of Theorem 6, and assume that the "augmented wealth"*

$$\begin{aligned}
&\xi'\left(x-\Pi\left(s,p,\vartheta\right),p,\vartheta\right)\\
&\quad=\xi_{s,t}\left(x,p,\vartheta\right)-\Pi\left(t,P_{s,t}\left(p,\vartheta\right),\Theta_{s,t}\left(\vartheta\right)\right),
\end{aligned}\tag{34}$$

where $\xi_{s,t}$ is the optimal wealth obtained in Theorem 6, is a wealth evolution structure with feasible set of values

$$\mathbb{Y}=\left\{\left(s,y,p^{\top},\vartheta^{\top}\right)^{\top}\mid y>0,\left(p^{\top},\vartheta^{\top}\right)^{\top}\in\mathbb{D}\left(s\right)\right\}.\tag{35}$$

Then, for any $x>\Pi(s,p,\vartheta)$ and $(p^{\top},\vartheta^{\top})^{\top}\in\mathbb{D}(s)$ and initial wealth $y=x-\Pi(s,p,\vartheta)$, the process of optimal hedgeable cumulative consumption and endowment structure (with no income) $(\xi',c,0)$ has the same optimal consumption as the one obtained in Theorem 6 (when it is assumed an income process $Q_{s,t}$).

Similar results to Theorem 6 can be obtained from the problems of maximizing the expected utility from discounted consumption (alone) or from the problem of maximizing the expected utility from final wealth (alone) as was discussed in Londoño [14, Theorems 3 and 4]. The proof follows on the lines of Theorem 6.

Acknowledgment

The authors would like to thank Professor Darrell Duffie for the detailed reading of this paper, positive and encouraging comments, and valuable suggestions that led to a substantial improvement of the paper. All remaining errors are the author's.

References

[1] J. A. Londoño, "State-dependent utility," *Journal of Applied Probability*, vol. 46, no. 1, pp. 55–70, 2009.

[2] R. C. Merton, "Lifetime portfolio selection under uncertainty: the continuos time case," *Review of Economics and Statistics*, vol. 51, no. 3, pp. 247–257, 1969.

[3] R. C. Merton, "Optimum consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, vol. 3, no. 4, pp. 373–413, 1971.

[4] J. C. Cox and C.-f. Huang, "Optimal consumption and portfolio policies when asset prices follow a diffusion process," *Journal of Economic Theory*, vol. 49, no. 1, pp. 33–83, 1989.

[5] I. Karatzas, J. P. Lehoczky, and S. E. Shreve, "Optimal portfolio and consumption decisions for a "small investor" on a finite horizon," *SIAM Journal on Control and Optimization*, vol. 25, no. 6, pp. 1557–1586, 1987.

[6] I. Karatzas and G. Žitković, "Optimal consumption from investment and random endowment in incomplete semimartingale markets," *The Annals of Probability*, vol. 31, no. 4, pp. 1821–1858, 2003.

[7] J. Hugonnier and D. Kramkov, "Optimal investment with random endowments in incomplete markets," *The Annals of Applied Probability*, vol. 14, no. 2, pp. 845–864, 2004.

[8] J. Cvitanić, W. Schachermayer, and H. Wang, "Utility maximization in incomplete markets with random endowment," *Finance and Stochastics*, vol. 5, no. 2, pp. 259–272, 2001.

[9] I. Karatzas and S. E. Shreve, *Methods of Mathematical Finance*, vol. 39 of *Applications of Mathematics*, Springer, New York, NY, USA, 1998.

[10] R. C. Merton, "An intertemporal capital asset pricing model," *Econometrica*, vol. 41, pp. 867–887, 1973.

[11] D. T. Breeden, "An intertemporal asset pricing model with stochastic consumption and investment opportunities," *Journal of Financial Economics*, vol. 7, no. 3, pp. 265–296, 1979.

[12] J. C. Cox, J. E. Ingersoll Jr., and S. A. Ross, "An intertemporal general equilibrium model of asset prices," *Econometrica*, vol. 53, no. 2, pp. 363–384, 1985.

[13] J. A. Londoño, *A Sensitive Intertemporal Equilibrium for Relative Well-Being*, Universidad Nacional de Colombia, Sede Bogota, Colombia, 2013.

[14] J. A. Londoño, "A more general valuation and arbitrage theory for Itô processes," *Stochastic Analysis and Applications*, vol. 26, no. 4, pp. 809–831, 2008.

[15] J. A. Londoño, "State tameness: a new approach for credit constrains," *Electronic Communications in Probability*, vol. 9, pp. 1–13, 2004.

[16] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, vol. 94 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1983.

[17] H. Kunita, *Stochastic Flows and Stochastic Differential Equations*, vol. 24 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1990.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 750547, 8 pages
http://dx.doi.org/10.1155/2013/750547

Research Article

Optimal Dividend and Capital Injection Strategies for a Risk Model under Force of Interest

Ying Fang[1] and Zhongfeng Qu[2]

[1] *School of Mathematical Sciences, Shandong Normal University, Jinan 250014, China*
[2] *School of Mathematical Sciences, University of Jinan, Jinan 250022, China*

Correspondence should be addressed to Zhongfeng Qu; quzhongfeng79@163.com

Received 30 January 2013; Accepted 14 April 2013

Academic Editor: Guangchen Wang

As a generalization of the classical Cramér-Lundberg risk model, we consider a risk model including a constant force of interest in the present paper. Most optimal dividend strategies which only consider the processes modeling the surplus of a risk business are absorbed at 0. However, in many cases, negative surplus does not necessarily mean that the business has to stop. Therefore, we assume that negative surplus is not allowed and the beneficiary of the dividends is required to inject capital into the insurance company to ensure that its risk process stays nonnegative. For this risk model, we show that the optimal dividend strategy which maximizes the discounted dividend payments minus the penalized discounted capital injections is a threshold strategy for the case of the dividend payout rate which is bounded by some positive constant and the optimal injection strategy is to inject capitals immediately to make the company's assets back to zero when the surplus of the company becomes negative.

1. Introduction

In the mathematical finance and the actuarial literature, the optimal dividend problem has attracted much attention. For example, there is a good deal of work on the problem of finding an optimal policy for paying out dividends. In the Brownian motion setting, it has been proved that the optimal dividend strategy is a threshold strategy for the case of the dividend payout rate is which bounded by some positive constant and a barrier strategy for the case with no restriction on the dividend payout rate by Asmussen and Taksar [1]. In the Cramér-Lundberg setting, it was first shown in [2] by a discrete approximation and then a limiting argument that the optimal dividend strategy is of the so-called band type. For exponentially distributed claim sizes this strategy simplifies to a barrier strategy. This result was rederived by stochastic control theory in [3, 4]. Recently, Albrecher and Thonhauser [5] studied the optimal dividend strategy by viscosity theory in the constant force of interest model. They pointed out that the optimal dividend strategy in the general case is again of band type and for exponential claim sizes collapses to a barrier strategy. In addition, Avram et al. [6] and Loeffen [7] considered the optimal dividend problem when the risk process is modeled by a spectrally negative Lévy process. They drew on the fluctuation theory of spectrally negative Lévy processes and gave sufficient conditions under which the optimal strategy is of barrier type.

Unfortunately, a drawback is that processes modeling the surplus of a risk business are absorbed at 0. In many cases, negative surplus does not necessarily mean that the business has to stop. Therefore, many authors suggested a model, where the above-mentioned fact should be taken into account.

One method is to make 0 as a reflecting barrier rather than absorbing barrier. For example, Shreve et al. [8] proposed that a diffusion process can be controlled by subtracting out a withdrawal (dividend) process and adding in a deposit (injection) process. The goal is then to maximize the expected discounted dividend payments minus the expected penalized discounted capital injections. If the surplus process is a Brownian motion with drift, they found that the optimal injection policy is to invest the minimum such that the controlled surplus remains positive. Furthermore, they also show that the optimal dividend policy is a barrier strategy.

Løkka and Zervos [9] added proportional costs to the deposits. In the more general framework of spectrally negative Lévy processes, Avram et al. [6] studied the optimality of barrier strategies with capital injections. In the Cramér-Lundberg model, Kulenko and Schmidli [10] showed that the optimal dividend strategy is a barrier strategy and the optimal injection policy is to inject the minimum such that the controlled surplus remains nonnegative. In this paper, we will solve the problem for a risk process under a constant force of interest.

Consider the following risk model for the reserve process $\{U(t)\}$ of an insurance portfolio:

$$U_t = xe^{rt} + c\int_0^t e^{r(t-s)}ds - \int_0^t e^{r(t-s)}d\sum_{i=1}^{N_s} X_i, \qquad (1)$$

where $x \geq 0$ is the initial capital, $c > 0$ is the premium rate, $r > 0$ is the constant force of interest, $\{N_t, t \geq 0\}$ is a homogeneous Poisson claim counting process with intensity $\lambda > 0$, and $\{X_i, i = 1, 2, \ldots\}$ are i.i.d. claim size random variables, which are independent of $\{N_t, t \geq 0\}$ and have common distribution function $P(\cdot)$ with $P(0) = 0$, density function $p(\cdot)$, and mean $E[X_i] = \mu$. Let π denote an admissible strategy consisting of an accumulated dividend process $\{D_t^\pi\}$ and an accumulated injection process $\{Z_t^\pi\}$. The accumulated dividend process $\{D_t^\pi\}$ is a nondecreasing and left continuous process with $D_0^\pi = 0$; the accumulated injection process $\{Z_t^\pi\}$ is a nondecreasing and right continuous process with $Z_{0-}^\pi = 0$. Then the surplus process modified by policy π becomes

$$U_t^\pi = U_t - \int_0^t e^{r(t-s)}dD_s^\pi + \int_0^t e^{r(t-s)}dZ_s^\pi. \qquad (2)$$

The set of admissible policies Π consists of those policies which satisfy

(i) U_t^π is nonnegative for $t > 0$,

(ii) $\int_0^\infty e^{-\delta t}dZ_t^\pi < \infty$ almost surely.

The value associated to the strategy π is defined as

$$V_\pi(x) = E\left[\int_0^\infty e^{-\delta t}dD_t^\pi - \kappa\int_0^\infty e^{-\delta t}dZ_t^\pi \mid U_0^\pi = x\right], \qquad (3)$$

where $\delta > 0$ is the force of interest for discounting the dividends and κ is the cost per unit injected capital or a penalizing factor. In this paper, we assume that $\delta > r$ and $\kappa > 1$. The maximal value function is denoted by

$$V(x) = \sup_{\pi\in\Pi} V_\pi(x). \qquad (4)$$

Let us illustrate the reason why we choose $\kappa > 1$. If we had chosen $\kappa < 1$, the maximal value function would be infinite because it is cheaper to pay dividends by injection capital than from the reserve. If we had chosen $\kappa = 1$, the cost of paying incoming claims from the reserve is the same as by an injection capital. However, the discount factor δ is greater than the force of interest r. Then it is better to pay dividends earlier. Therefore, it is optimal to pay out all positive surplus immediately as dividends and to pay all claims by injection capital. That is,

$$\begin{aligned} V(x) &= x + \int_0^\infty e^{-\delta t}c\,dt - E\left[\sum_{i=1}^\infty e^{-\delta T_i}X_i\right] \\ &= x + \frac{c-\lambda\mu}{\delta}, \end{aligned} \qquad (5)$$

where T_k is the time of the kth claim.

In addition, notice that from (3) it can not be optimal to inject capital before they are really necessary because of $\kappa > 1$ and $\delta > r$. Therefore, we only need to choose the dividend strategy $\{D_t^\pi\}$. The corresponding injection process $\{Z_t^{D^\pi}\}$ is defined as follows: when the surplus of the company with strategy $\{D_t^\pi\}$ becomes negative, the shareholders immediately inject the amount of the deficit such that the surplus can be restored to 0; when the surplus is nonnegative, no capitals are injected. If the initial capital is negative, then $Z_0 = |x|$, thus $V(x) = V(0) - \kappa|x|$.

In this paper, we impose restriction on the dividend stream to prevent ruin occurring almost surely. We assume that the dividends are paid at a dynamic rate d_t at time t. Stochastic process $\{d_t\}$ is called a control process and can only vary in $[0, \alpha]$ for some $0 < \alpha < c$. This optimal dividend problem with additional constraints is postulated by Jeanblanc-Picqué and Shiryaev [11] and Asmussen and Taksar [1].

The outline of the remainder of the paper is organized as follows. In Section 2, we prove some properties of the maximal dividend-value function and derive the Hamilton-Jacobi-Bellman equation for the problem. In Section 3, we give the optimal strategy by the HJB equation and the concavity of the value function. We show that the optimal dividend strategy is a threshold strategy. Some concluding remarks are given in Section 4.

2. The Maximal Value Function

We first prove some properties of the maximal dividend-value function.

Proposition 1. *The maximal value function $V(x)$ is increasing and converges to α/δ as x converges to ∞.*

Proof. Let π_ε^x be an ε-optimal strategy for initial capital x, that is, $V_{\pi_\varepsilon^x}(x) \geq V(x) - \varepsilon$. Let $\{d_t^{\pi_\varepsilon^x}\}$ be the corresponding dividend rate process. For initial capital y $(y > x)$, we define a policy π^y with dividend rate

$$d_t^{\pi^y} = d_t^{\pi_\varepsilon^x}. \qquad (6)$$

Note that this strategy is admissible for initial capital y and its injection process has the following relationship with $Z^{d^{\pi_\varepsilon^x}}$:

$$Z_t^{d^{\pi^y}} \leq Z_t^{d^{\pi_\varepsilon^x}}. \qquad (7)$$

Hence we have

$$\begin{aligned} V(y) &\geq E\left[\int_0^\infty e^{-\delta t} d_t^{\pi^y} dt - \kappa \int_0^\infty e^{-\delta t} dZ_t^{d^{\pi^y}}\right] \\ &\geq E\left[\int_0^\infty e^{-\delta t} d_t^{\pi_\varepsilon^x} dt - \kappa \int_0^\infty e^{-\delta t} dZ_t^{d^{\pi_\varepsilon^x}}\right] \\ &\geq V(x) - \varepsilon. \end{aligned} \tag{8}$$

Since this inequality holds for all $\varepsilon > 0$, then we have $V(y) \geq V(x)$.

Consider a strategy $\hat{\pi}$ with $d_t^{\hat{\pi}} = \alpha$. Then the time of the first injection under this strategy

$$\begin{aligned} T_{\hat{\pi}} = \inf \Big\{ t \geq 0 : x e^{rt} &+ \int_0^t e^{r(t-s)} (c - \alpha)\, ds \\ &- \int_0^t e^{r(t-s)} dS_s < 0 \Big\} \end{aligned} \tag{9}$$

converges to infinity as $x \to \infty$. Recall that capitals are injected only if the surplus process is below 0, then

$$\begin{aligned} &E \int_0^\infty e^{-\delta t} dZ_t^{\hat{\pi}} \\ &\leq E\left\{ e^{-\delta T_{\hat{\pi}}} \left| U_{T_{\hat{\pi}}}^{\hat{\pi}} \right| + \sum_{i=N_{T_{\hat{\pi}}}+1}^{\infty} e^{-\delta T_i} X_i \right\} \\ &= E\left\{ e^{-\delta T_{\hat{\pi}}} \left[\left| U_{T_{\hat{\pi}}}^{\hat{\pi}} \right| + \sum_{i=N_{T_{\hat{\pi}}}+1}^{\infty} e^{-\delta (T_i - T_{\hat{\pi}})} X_i \right] \right\} \\ &= E\left\{ e^{-\delta T_{\hat{\pi}}} \left| U_{T_{\hat{\pi}}}^{\hat{\pi}} \right| \right\} + E\left\{ e^{-\delta T_{\hat{\pi}}} \left[\sum_{i=N_{T_{\hat{\pi}}}+1}^{\infty} e^{-\delta (T_i - T_{\hat{\pi}})} X_i \right] \right\} \\ &= E\left\{ e^{-\delta T_{\hat{\pi}}} \left| U_{T_{\hat{\pi}}}^{\hat{\pi}} \right| \right\} + E\left[e^{-\delta T_{\hat{\pi}}} \right] E\left[\sum_{i=1}^{\infty} e^{-\delta T_i} X_i \right] \\ &\leq E\left\{ e^{-\delta T_{\hat{\pi}}} \left| U_{T_{\hat{\pi}}}^{\hat{\pi}} \right| \right\} + \frac{\lambda \mu}{\delta} E\left[e^{-\delta T_{\hat{\pi}}} \right]. \end{aligned} \tag{10}$$

Note that

$$\begin{aligned} &\lim_{x \to \infty} E\left[e^{-\delta T_{\hat{\pi}}} \mid U_0^{\hat{\pi}} = x \right] = 0, \\ &\lim_{x \to \infty} E\left\{ e^{-\delta T_{\hat{\pi}}} \left| U_{T_{\hat{\pi}}}^{\hat{\pi}} \right| \mid U_0^{\hat{\pi}} = x \right\} = 0. \end{aligned} \tag{11}$$

So,

$$\lim_{x \to \infty} E\left[\int_0^\infty e^{-\delta t} dZ_t^{\hat{\pi}} \mid U_0^{\hat{\pi}} = x \right] = 0. \tag{12}$$

Thus, as $x \to \infty$,

$$V_{\hat{\pi}}(x) = E\left[\int_0^\infty e^{-\delta t} \alpha\, dt - \kappa \int_0^\infty e^{-\delta t} dZ_t^{\hat{\pi}} \mid U_0^{\hat{\pi}} = x \right] \longrightarrow \frac{\alpha}{\delta}. \tag{13}$$

Naturally, it holds that

$$V_{\hat{\pi}}(x) \leq V(x) \leq \frac{\alpha}{\delta}. \tag{14}$$

Hence, by squeeze theorem, we obtain

$$\lim_{x \to \infty} V(x) = \frac{\alpha}{\delta}. \tag{15}$$

□

Proposition 2. *The maximal value function $V(x)$ is concave and Lipschitz continuous.*

Proof. Consider $x \geq 0$, $y \geq 0$, and $\theta \in (0,1)$. Let $\pi^x = (d^x, Z^{d^x})$ be an admissible strategy for the initial capital x and let $\pi^y = (d^y, Z^{d^y})$ be an admissible strategy for the initial capital y. We define strategy $(\theta d^x + (1-\theta) d^y, \theta Z^{d^x} + (1-\theta) Z^{d^y})$ and find that

$$\begin{aligned} &[\theta x + (1-\theta) y] + \int_0^t e^{r(t-s)} c\, ds - \int_0^t e^{r(t-s)} d \sum_{i=1}^{N_s} X_i \\ &\quad - \int_0^t e^{r(t-s)} \left[\theta d_s^x + (1-\theta) d_s^y \right] ds \\ &\quad + \int_0^t e^{r(t-s)} d \left[\theta Z_s^{d^x} + (1-\theta) Z_s^{d^y} \right] \\ &= \theta \left\{ x + \int_0^t e^{r(t-s)} (c - d_s^x)\, ds - \int_0^t e^{r(t-s)} d \sum_{i=1}^{N_s} X_i \right. \\ &\qquad \left. + \int_0^t e^{r(t-s)} dZ_s^{d^x} \right\} \\ &\quad + (1-\theta) \left\{ y + \int_0^t e^{r(t-s)} (c - d_s^y)\, ds \right. \\ &\qquad - \int_0^t e^{r(t-s)} d \sum_{i=1}^{N_s} X_i \\ &\qquad \left. + \int_0^t e^{r(t-s)} dZ_s^{d^y} \right\} \geq 0, \end{aligned} \tag{16}$$

which implies that policy $(\theta d^x + (1-\theta) d^y, \theta Z^{d^x} + (1-\theta) Z^{d^y})$ is an admissible dividend strategy for the initial capital $\theta x + (1-\theta) y$. Denote $\theta d^x + (1-\theta) d^y = d^{\theta x + (1-\theta) y}$ and let $Z^{d^{\theta x + (1-\theta) y}}$ be the corresponding injection process. Then we must have $Z_t^{d^{\theta x + (1-\theta) y}} \leq \theta Z_t^{d^x} + (1-\theta) Z_t^{d^y}$. Therefore, it follows that

$$\begin{aligned} V(\theta x + (1-\theta) y) \geq E\Big[&\int_0^\infty e^{-\delta t} d_t^{\theta x + (1-\theta) y} dt \\ &- \kappa \int_0^\infty e^{-\delta t} dZ_t^{d^{\theta x + (1-\theta) y}} \Big] \end{aligned}$$

$$\geq E\left[\int_0^{\infty} e^{-\delta t}\left[\theta d_t^x+(1-\theta) d_t^y\right] dt\right.$$
$$\left.-\kappa \int_0^{\infty} e^{-\delta t} d\left[\theta Z_t^{d^x}+(1-\theta) Z_t^{d^y}\right]\right]$$
$$=\theta V^{d^x}(x)+(1-\theta) V^{d^y}(y). \tag{17}$$

Taking the supremum over all admissible strategies in Π, we find that

$$V(\theta x+(1-\theta) y) \geq \theta V(x)+(1-\theta) V(y). \tag{18}$$

Then the concavity of the maximal value function $V(x)$ follows.

Let $\pi_y=\{d_t^y, Z_t^{d^y}\}$ be an arbitrary admissible strategy for initial capital y. Let h denote the time that the surplus reaches y from x if no claims occur, that is,

$$h=\frac{1}{r} \ln \frac{c+r y}{c+r x}. \tag{19}$$

For initial capital x, consider a dividend strategy

$$d_t^x= \begin{cases}0, & \text{for } t \leq h \text{ or } T_1 \leq h, \\ d_{t-h}^y, & \text{for } T_1 \wedge t>h.\end{cases} \tag{20}$$

Then the probability that there is dividend payment is the probability that the surplus reaches y without claims occurring first. This probability is $e^{-\lambda h}$. So, we have

$$V(x)$$
$$\geq E\left[\int_0^{\infty} e^{-\delta t} d_t^x dt-\kappa \int_0^{\infty} e^{-\delta t} d Z_t^{d^x}\right]$$
$$=P\left(T_1>h\right) E\left[\int_0^{\infty} e^{-\delta t} d_t^x dt-\kappa \int_0^{\infty} e^{-\delta t} d Z_t^{d^x} \mid T_1>h\right]$$
$$+P\left(T_1 \leq h\right) E\left[\int_0^{\infty} e^{-\delta t} d_t^x dt-\kappa \int_0^{\infty} e^{-\delta t} d Z_t^{d^x} \mid T_1 \leq h\right]$$
$$=e^{-\lambda h} e^{-\delta h} V_{\pi_y}(y)-\left(1-e^{-\lambda h}\right) E\left[\kappa \int_0^{\infty} e^{-\delta t} d Z_t^{d^x} \mid T_1 \leq h\right]$$
$$\geq e^{-(\lambda+\delta) h} V_{\pi_y}(y)-\kappa\left(1-e^{-\lambda h}\right) E\left[\sum_{i=1}^{\infty} e^{-\delta T_i} X_i\right]$$
$$=e^{-(\lambda+\delta) h} V_{\pi_y}(y)-\frac{\kappa \lambda \mu}{\delta}\left(1-e^{-\lambda h}\right). \tag{21}$$

Taking the supremum over all Π yields

$$V(x) \geq e^{-(\lambda+\delta) h} V(y)-\frac{\kappa \lambda \mu}{\delta}\left(1-e^{-\lambda h}\right). \tag{22}$$

Note that

$$1-e^{-(\lambda+\delta) h} \leq(\lambda+\delta) h \leq \frac{\lambda+\delta}{c}(y-x),$$
$$1-e^{-\lambda h} \leq \lambda h \leq \frac{\lambda}{c}(y-x). \tag{23}$$

Thus, by (22) and (23), we obtain

$$0 \leq V(y)-V(x) \leq\left(1-e^{-(\lambda+\delta) h}\right) V(y)+\frac{\kappa \lambda \mu}{\delta}\left(1-e^{-\lambda h}\right)$$
$$\leq \frac{(\lambda+\delta) \alpha+\kappa \lambda^2 \mu}{c \delta}(y-x), \tag{24}$$

which implies that Lipschitz continuity holds. Therefore, $V(x)$ is absolutely continuous. □

Lemma 3. *The maximal value function $V(x)$ is differentiable from the right, and the right derivative $V'(x+)$ satisfies HJB equation:*

$$\max_{0 \leq d \leq \alpha}\left\{(c-d+r x) V'(x+)-(\lambda+\delta) V(x)\right.$$
$$\left.+\lambda \int_0^{\infty} V(x-y) p(y) d y+d\right\}=0. \tag{25}$$

Proof. Take a constant $u \in[0, \alpha]$. Let $h>0$ be small enough and $\varphi(x, h)=x e^{r h}+\int_0^h e^{r(h-s)}(c-u) d s$. We can show analogously to the proof of Proposition 3.1 of [3, 12, 13] that the maximal dividend-value function fulfills dynamic programming principle for any stopping times τ:

$$V(x)=\sup_{\pi \in \Pi} E\left[\int_0^{\tau \wedge T_\pi} e^{-\delta t} d_t dt+e^{-\delta\left(\tau \wedge T_\pi\right)} V\left(U_{\tau \wedge T_\pi}^\pi\right)\right]. \tag{26}$$

Taking $\tau=T_1 \wedge h$ in (26), we derive

$$V(x)=\sup_{\pi \in \Pi} E\left[\int_0^{T_1 \wedge h} e^{-\delta t} d_t dt+e^{-\delta\left(T_1 \wedge h\right)} V\left(U_{T_1 \wedge h}^\pi\right)\right]$$
$$\geq e^{-\lambda h}\left[\int_0^h e^{-\delta t} u dt+e^{-\delta h} V(\varphi(x, h))\right]$$
$$+\int_0^h \lambda e^{-\lambda t}\left[\int_0^t e^{-\delta s} u ds\right.$$
$$\left.+e^{-\delta t} \int_0^{\infty} V(\varphi(x, t)-y) p(y) d y\right] dt. \tag{27}$$

Rearranging the terms and dividing by h yields

$$\frac{V(\varphi(x, h))-V(x)}{h}-\frac{1-e^{-(\lambda+\delta) h}}{h} V(\varphi(x, h))$$
$$+\frac{e^{-\lambda h}}{h} \int_0^h e^{-\delta t} u dt$$
$$+\frac{1}{h} \int_0^h \lambda e^{-\lambda t}\left[\int_0^t e^{-\delta s} u ds\right.$$
$$\left.+e^{-\delta t} \int_0^{\infty} V(\varphi(x, t)-y) p(y) d y\right] dt \leq 0. \tag{28}$$

On the other hand, from the dynamic programming principle, there exists a strategy $\pi^{h^2} = \{d_t^{h^2}\}$ such that

$$E\left[\int_0^{T_1\wedge h} e^{-\delta t} d_t^{h^2}\, dt + e^{-\delta(T_1\wedge h)} V\left(U_{T_1\wedge h}^{\pi^{h^2}}\right)\right] \geq V(x) - h^2. \tag{29}$$

Let u_t denote $d_t^{h^2}$ conditioned on $T_1 > t$ and $\phi(x,t) = xe^{rt} + \int_0^t e^{r(t-s)}(c-u_s)ds$. Using the same way as above, we get that

$$\begin{aligned} h &+ \frac{V(\phi(x,h)) - V(x)}{h} - \frac{1-e^{-(\lambda+\delta)h}}{h} V(\phi(x,h)) \\ &+ \frac{e^{-\lambda h}}{h}\int_0^h e^{-\delta t} u_t\, dt \\ &+ \frac{1}{h}\int_0^h \lambda e^{-\lambda t}\left[\int_0^t e^{-\delta s} u_s\, ds\right. \\ &\qquad \left. + e^{-\delta t}\int_0^\infty V(\phi(x,t) - y)\, p(y)\, dy\right] dt \geq 0. \end{aligned} \tag{30}$$

Notice that

$$\lim_{h\to 0+} \varphi(x,h) = \lim_{h\to 0+} \phi(x,h) = x. \tag{31}$$

Let

$$\begin{aligned} D^+V(x) &= \limsup_{s\to 0+} \frac{V(x+s) - V(x)}{s}, \\ D^-V(x) &= \liminf_{s\to 0+} \frac{V(x+s) - V(x)}{s}. \end{aligned} \tag{32}$$

We know that $D^+V(x)$ and $D^-V(x)$ are finite by Lipschitz continuity and monotonicity. We assume that

$$\lim_{h\to 0+} u_h = d_0. \tag{33}$$

Then we get

$$\lim_{h\to 0+} \frac{\phi(x,h) - x}{h} = c - d_0 + rx. \tag{34}$$

Choose a sequence $\{h_n^{(1)}\}$ such that

$$\lim_{n\to\infty} \frac{V\left(\phi\left(x,h_n^{(1)}\right)\right) - V(x)}{\phi\left(x,h_n^{(1)}\right) - x} = D^-V(x). \tag{35}$$

Then the limit is

$$\begin{aligned} &\lim_{n\to\infty} \frac{V\left(\phi\left(x,h_n^{(1)}\right)\right) - V(x)}{h_n^{(1)}} \\ &\quad = \lim_{n\to\infty} \frac{\phi\left(x,h_n^{(1)}\right) - x}{h_n^{(1)}} \cdot \frac{V\left(\phi\left(x,h_n^{(1)}\right)\right) - V(x)}{\phi\left(x,h_n^{(1)}\right) - x} \\ &\quad = (c - d_0 + rx) D^-V(x). \end{aligned} \tag{36}$$

Taking $h = h_n^{(1)}$ in (30) and letting $n \to \infty$, we get

$$\begin{aligned} &(c - d_0 + rx) D^-V(x) - (\lambda+\delta) V(x) \\ &\quad + d_0 + \lambda \int_0^\infty V(x-y)\, p(y)\, dy \geq 0. \end{aligned} \tag{37}$$

Using the same method as above, we choose a sequence $\{h_n^{(2)}\}$ such that

$$\lim_{n\to\infty} \frac{V\left(\varphi\left(x,h_n^{(2)}\right)\right) - V(x)}{\varphi\left(x,h_n^{(2)}\right) - x} = D^+V(x). \tag{38}$$

Then taking $u = d_0$ and $h = h_n^{(2)}$ in (28) and letting $n \to \infty$, we have

$$\begin{aligned} &(c - d_0 + rx) D^+V(x) - (\lambda+\delta) V(x) + d_0 \\ &\quad + \lambda \int_0^\infty V(x-y)\, p(y)\, dy \leq 0. \end{aligned} \tag{39}$$

From (37) and (39), we know that

$$D^+V(x) \leq D^-V(x). \tag{40}$$

On the other hand

$$D^+V(x) \geq D^-V(x). \tag{41}$$

So we have

$$D^+V(x) = D^-V(x) = V'(x+). \tag{42}$$

Therefore, the maximal dividend-value function is differentiable from the right.

For the sequence $\{h_n^{(1)}\}$ and $u = d_0$, inequality (28) holds and

$$\lim_{n\to\infty} \frac{V\left(\varphi\left(x,h_n^{(1)}\right)\right) - V(x)}{h_n^{(1)}} = (c - d_0 + rx) V'(x+). \tag{43}$$

Then taking limit on both sides of inequalities (28) and (30), we find that

$$\begin{aligned} &(c - d_0 + rx) V'(x+) - (\lambda+\delta) V(x) \\ &\quad + \lambda \int_0^\infty V(x-y)\, p(y)\, dy + d_0 = 0. \end{aligned} \tag{44}$$

From (28) we conclude that for any d

$$\begin{aligned} &(c - d + rx) V'(x+) - (\lambda+\delta) V(x) \\ &\quad + \lambda \int_0^\infty V(x-y)\, p(y)\, dy + d \leq 0. \end{aligned} \tag{45}$$

Thus, we get the HJB equation for $V'(x+)$

$$\begin{aligned} \max_{0\leq d\leq \alpha} \Big\{ &(c - d + rx) V'(x+) - (\lambda+\delta) V(x) \\ &+ \lambda \int_0^\infty V(x-y)\, p(y)\, dy + d \Big\} = 0. \end{aligned} \tag{46}$$

□

Lemma 4. *When $x > 0$, the maximal value function $V(x)$ is differentiable from the left and the left derivative $V'(x-)$ satisfies HJB equation:*

$$\max_{0\le d\le\alpha}\left\{(c-d+rx)V'(x-)-(\lambda+\delta)V(x)+\lambda\int_0^\infty V(x-y)p(y)\,dy+d\right\}=0. \tag{47}$$

Proof. Let h be small enough such that

$$\rho(x,h)=\left(x+\frac{c}{r}\right)e^{-rh}-\frac{c}{r} \tag{48}$$

is positive. Changing the initial capital x into $\rho(x,h)$ and using the dynamic programming principle, we obtain

$$\begin{aligned}
&\frac{V(\tilde\varphi(x,h))-V(\rho(x,h))}{h}-\frac{1-e^{-(\lambda+\delta)h}}{h}V(\tilde\varphi(x,h))\\
&\quad+\frac{e^{-\lambda h}}{h}\int_0^h e^{-\delta t}u\,dt\\
&\quad+\frac{1}{h}\int_0^h\lambda e^{-\lambda t}\left[\int_0^t e^{-\delta s}u\,ds\right.\\
&\quad\left.+e^{-\delta t}\int_0^\infty V(\tilde\varphi(x,h)-y)p(y)\,dy\right]dt\le 0,\\
&h+\frac{V(\tilde\phi(x,h))-V(\rho(x,h))}{h}-\frac{1-e^{-(\lambda+\delta)h}}{h}V(\tilde\phi(x,h))\\
&\quad+\frac{e^{-\lambda h}}{h}\int_0^h e^{-\delta t}u_t\,dt\\
&\quad+\frac{1}{h}\int_0^h\lambda e^{-\lambda t}\left[\int_0^t e^{-\delta s}u_s\,ds\right.\\
&\quad\left.+e^{-\delta t}\int_0^\infty V(\tilde\phi(x,t)-y)p(y)\,dy\right]dt\ge 0,
\end{aligned} \tag{49}$$

where

$$\begin{aligned}
\tilde\varphi(x,t)&=\left[\left(x+\frac{c}{r}\right)e^{-rh}-\frac{c}{r}\right]e^{rt}+\int_0^t e^{r(t-s)}(c-u)\,ds,\\
\tilde\phi(x,t)&=\left[\left(x+\frac{c}{r}\right)e^{-rh}-\frac{c}{r}\right]e^{rt}+\int_0^t e^{r(t-s)}(c-u_s)\,ds.
\end{aligned} \tag{50}$$

Let

$$\begin{aligned}
D_+V(x)&=\limsup_{s\to 0-}\frac{V(x+s)-V(x)}{s},\\
D_-V(x)&=\liminf_{s\to 0-}\frac{V(x+s)-V(x)}{s}.
\end{aligned} \tag{51}$$

Using the same method as Lemma 3, we obtain

$$D_+V(x)=D_-V(x)=V'(x-), \tag{52}$$

which implies the maximal dividend-value function which is differentiable from the left. Furthermore, the left derivative $V'(x-)$ fulfills HJB equation:

$$\max_{0\le d\le\alpha}\left\{(c-d+rx)V'(x-)-(\lambda+\delta)V(x)+\lambda\int_0^\infty V(x-y)p(y)\,dy+d\right\}=0. \tag{53}$$

□

Theorem 5. *The maximal value function $V(x)$ is continuously differentiable on $[0,\infty)$ and fulfills the following HJB equation:*

$$\max_{0\le d\le\alpha}\left\{(c-d+rx)V'(x)-(\lambda+\delta)V(x)+\lambda\int_0^\infty V(x-y)p(y)\,dy+d=0\right\}, \tag{54}$$

where $V'(0)$ means the right derivative $V'(0+)$.

Proof. From Lemmas 3 and 4, we know that function $V(x)$ is continuously differentiable and fulfills the HJB equation (54). □

3. Optimal Dividend Strategy

Since $V(x)$ is increasing, concave and converges to α/δ, there exists a finite constant:

$$b^*=\inf\left\{x:V'(x)\le 1\right\}, \tag{55}$$

such that

$$\begin{aligned}
V'(x)&>1,\quad \text{if } x<b^*,\\
V'(x)&\le 1,\quad \text{if } x\ge b^*.
\end{aligned} \tag{56}$$

From Theorem 5, we construct a strategy $\pi^*=\{d_t^*,Z_t^{d^*}\}$ as follows:

$$d_{t+}^*=\begin{cases}0, & \text{if } U_t^{\pi^*}<b^*,\\ \alpha, & \text{if } U_t^{\pi^*}\ge b^*.\end{cases} \tag{57}$$

$Z_t^{d^*}$ is the corresponding capital injection process. This policy is a double level strategy with a lower barrier at zero and an upper threshold at level b^*. Whenever the surplus is between 0 and b^*, no dividends are paid and no capitals are injected; whenever the surplus is at or above b^*, dividends will be paid at the maximal rate α, but no capitals are injected; whenever the surplus is below 0, a required amount of the capital which are equal to the amount of the deficit are injected immediately. Figure 1 gives an intuitional description for the strategy π^*.

Theorem 6. *The policy $\pi^*=\{d_t^*,Z_t^{d^*}\}$ which is given by* (57) *is optimal among all admissible policies.*

Proof. First of all, the policy $\pi^*=\{d_t^*,Z_t^{d^*}\}$ is an admissible strategy. Let $V_*(x)$ denote the corresponding value function.

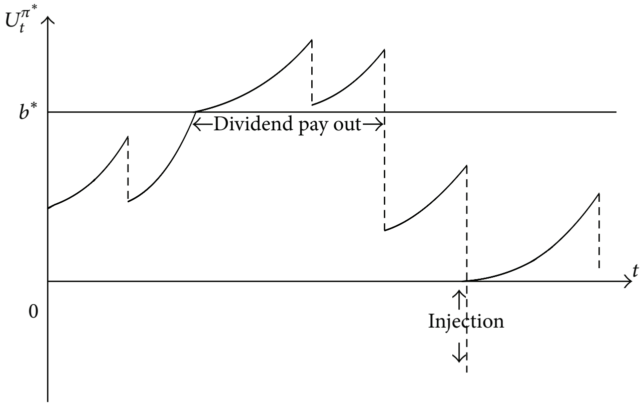


Figure 1: A typical sample path of the surplus process with strategy π^*.

By an application of Itô's formula to $e^{-\delta t}V(U_t^{\pi^*})$, it can be verified that

$$e^{-\delta t}V\left(U_t^{\pi^*}\right) = V(x) + \int_0^t e^{-\delta s}\left[\left(c - d_s^{\pi^*} + rU_{s-}^{\pi^*}\right) \times V'\left(U_{s-}^{\pi^*}\right) - \delta V\left(U_{s-}^{\pi^*}\right)\right]ds + \sum_{\substack{0<s\le t \\ U_{s-}^{\pi^*}\neq U_s^{\pi^*}}} e^{-\delta s}\left[V\left(U_s^{\pi^*}\right) - V\left(U_{s-}^{\pi^*}\right)\right]. \tag{58}$$

Note that the injections occur only at the claim times. Let $z_i^{\pi^*}$ denote the amount of injected capital at the occurrence time of the ith claim. If $U_{T_i-}^{\pi^*} - X_i \ge 0$, no capitals are injected, that is, $z_i^{\pi^*} = 0$. If $U_{T_i-}^{\pi^*} - X_i < 0$, $z_i^{\pi^*} = -(U_{T_i-}^{\pi^*} - X_i)$ and $U_{T_i}^{\pi^*} = U_{T_i-}^{\pi^*} - X_i + z_i^{\pi^*} = 0$. For both cases, the equality

$$V\left(U_{T_i}^{\pi^*}\right) = V\left(U_{T_i-}^{\pi^*} - X_i\right) + \kappa z_i^{\pi^*} \tag{59}$$

holds. Since $U_{s-}^{\pi^*} \neq U_s^{\pi^*}$ only at the occurrence time of a claim, then

$$\begin{aligned}\sum_{\substack{0<s\le t \\ U_{s-}^{\pi^*}\neq U_s^{\pi^*}}} & e^{-\delta s}\left[V\left(U_s^{\pi^*}\right) - V\left(U_{s-}^{\pi^*}\right)\right] \\ &= \sum_{i=1}^{N_t} e^{-\delta T_i}\left[V\left(U_{T_i-}^{\pi^*} - X_i + z_i^{\pi^*}\right) - V\left(U_{T_i-}^{\pi^*}\right)\right] \\ &= \sum_{i=1}^{N_t} e^{-\delta T_i}\left[V\left(U_{T_i-}^{\pi^*} - X_i\right) - V\left(U_{T_i-}^{\pi^*}\right)\right] + \kappa \sum_{i=1}^{N_t} e^{-\delta T_i} z_i^{\pi^*} \\ &= \sum_{i=1}^{N_t} e^{-\delta T_i}\left[V\left(U_{T_i-}^{\pi^*} - X_i\right) - V\left(U_{T_i-}^{\pi^*}\right)\right] + \kappa \int_0^t e^{-\delta s} dZ_s^{d^*}.\end{aligned} \tag{60}$$

Taking expectations in (58), we obtain

$$\begin{aligned}E&\left[e^{-\delta t}V\left(U_t^{\pi^*}\right)\right] \\ &= V(x) + E\int_0^t e^{-\delta s}\left[\left(c - d_s^{\pi^*} + rU_{s-}^{\pi^*}\right)V'\left(U_{s-}^{\pi^*}\right) - \delta V\left(U_{s-}^{\pi^*}\right)\right]ds \\ &\quad + \kappa E\int_0^t e^{-\delta s} dZ_s^{d^*} \\ &\quad + E\sum_{i=1}^{N_t} e^{-\delta T_i}\left[V\left(U_{T_i-}^{\pi^*} - X_i\right) - V\left(U_{T_i-}^{\pi^*}\right)\right].\end{aligned} \tag{61}$$

Note that

$$\sum_{i=1}^{N_t} e^{-\delta T_i}\left[V\left(U_{T_i-}^{\pi^*} - X_i\right) - V\left(U_{T_i-}^{\pi^*}\right)\right] - \lambda\int_0^t e^{-\delta s}\int_0^\infty \left[V\left(U_{s-}^{\pi^*} - y\right) - V\left(U_{s-}^{\pi^*}\right)\right] p(y)\,dy\,ds \tag{62}$$

is a martingale with zero-expectation. Then

$$\begin{aligned}E&\left[e^{-\delta t}V\left(U_t^{\pi^*}\right)\right] \\ &= V(x) + E\int_0^t e^{-\delta s}\left[\left(c - d_s^{\pi^*} + rU_{s-}^{\pi^*}\right)V'\left(U_{s-}^{\pi^*}\right) - (\lambda+\delta)V\left(U_{s-}^{\pi^*}\right) + \lambda\int_0^\infty V\left(U_{s-}^{\pi^*} - y\right)p(y)\,dy\right]ds \\ &\quad + \kappa E\int_0^t e^{-\delta s} dZ_s^{d^*}.\end{aligned} \tag{63}$$

By HJB equation (54), we find that

$$\left(c - d_s^{\pi^*} + rU_{s-}^{\pi^*}\right)V'\left(U_{s-}^{\pi^*}\right) - (\lambda+\delta)V\left(U_{s-}^{\pi^*}\right) + \lambda\int_0^{U_{s-}^{\pi^*}} V\left(U_{s-}^{\pi^*} - y\right)p(y)\,dy = -d_s^{\pi^*}. \tag{64}$$

Thus, we gather that

$$E\left[e^{-\delta t}V\left(U_t^{\pi^*}\right)\right] + E\int_0^t e^{-\delta s} d_s^{\pi^*} ds - \kappa E\int_0^t e^{-\delta s} dZ_s^{d^*} = V(x). \tag{65}$$

Taking limit in $t \to \infty$, by the boundedness of V, we obtain that

$$V_*(x) = E\left[\int_0^\infty e^{-\delta s} d_s^{\pi^*} ds - \kappa\int_0^\infty e^{-\delta s} dZ_s^{d^*}\right] = V(x). \tag{66}$$

This shows that the policy π^* is the optimal strategy. □

4. Concluding Remarks

In summary, this paper gives the optimal strategy which maximizes the discounted dividend payments minus the penalized discounted capital injections for a risk model including a constant force of interest. The optimal dividend strategy is a threshold strategy for the case of the dividend payout rate which is bounded by some positive constant, and the optimal injection strategy is to inject capitals immediately to make the company's assets back to zero when the surplus of the company becomes negative.

We also wish to point out that further research is needed. Further extensions of the analysis of the problems in the paper could remove the restriction on the dividend payout rate. We hope that this open problem can be addressed in future research.

Acknowledgments

The authors would like to thank the referees for their helpful comments. The research was supported by National Natural Science Foundation of China (Grant nos. 11201271, 11126093, and 71071088).

References

[1] S. Asmussen and M. Taksar, "Controlled diffusion models for optimal dividend pay-out," *Insurance: Mathematics and Economics*, vol. 20, no. 1, pp. 1–15, 1997.

[2] H. U. Gerber, "Entscheidungskriterien für den zusammengesetzten Poisson-Prozess," *Mitteilungen der Vereinigung Schweizerischer Versicherungsmathematiker*, vol. 69, pp. 185–228, 1969.

[3] P. Azcue and N. Muler, "Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model," *Mathematical Finance*, vol. 15, no. 2, pp. 261–308, 2005.

[4] H. Schmidli, *Stochastic Control in Insurance*, Springer, London, UK, 2008.

[5] H. Albrecher and S. Thonhauser, "Optimal dividend strategies for a risk process under force of interest," *Insurance: Mathematics and Economics*, vol. 43, no. 1, pp. 134–149, 2008.

[6] F. Avram, Z. Palmowski, and M. R. Pistorius, "On the optimal dividend problem for a spectrally negative levy process," *Annals of Applied Probability*, vol. 17, no. 1, pp. 156–180, 2007.

[7] R. L. Loeffen, "On optimality of the barrier strategy in de Finetti's dividend problem for spectrally negative Lévy processes," *Annals of Applied Probability*, vol. 18, no. 5, pp. 1669–1680, 2008.

[8] S. E. Shreve, J. P. Lehoczky, and D. P. Gaver, "Optimal consumption for general diffusions with absorbing and reflecting barriers," *SIAM Journal on Control and Optimization*, vol. 22, no. 1, pp. 55–75, 1984.

[9] A. Løkka and M. Zervos, "Optimal dividend and issuance of equity policies in the presence of proportional costs," *Insurance: Mathematics and Economics*, vol. 42, no. 3, pp. 954–961, 2008.

[10] N. Kulenko and H. Schmidli, "Optimal dividend strategies in a Cramér-Lundberg model with capital injections," *Insurance: Mathematics and Economics*, vol. 43, no. 2, pp. 270–278, 2008.

[11] M. Jeanblanc-Picqué and A. N. Shiryaev, "Optimization of the flow of dividends," *Russian Mathematical Surveys*, vol. 50, pp. 257–277, 1995.

[12] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, New York, NY, USA, 1993.

[13] J. Huang, G. Wang, and Z. Wu, "Optimal premium policy of an insurance firm: full and partial information," *Insurance: Mathematics and Economics*, vol. 47, no. 2, pp. 208–215, 2010.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 697345, 9 pages
http://dx.doi.org/10.1155/2013/697345

Research Article

Fault Detection for Linear Discrete Time-Varying Systems with Measurement Packet Dropping

Yueyang Li,[1] Shuai Liu,[2,3] and Zhonghua Wang[1]

[1] *School of Electrical Engineering, University of Jinan, Jinan 250022, China*
[2] *School of Control Science and Engineering, Shandong University, Jinan 250061, China*
[3] *School of Electrical and Electronic Engineering, Nanyang Technological University, Singapore 639798*

Correspondence should be addressed to Yueyang Li; cse_liyy@ujn.edu.cn

Received 24 January 2013; Accepted 8 April 2013

Academic Editor: Wuquan Li

The fault detection (FD) problem for linear discrete time-varying (LDTV) systems with measurement packet dropouts is considered. The objective is to design a new observer-based fault detection filter (FDF) as a residual generator through employing packet dropout information on the measurement sequence. Based on some new defined input-to-output operators, the FD problem is formulated in a framework of maximizing stochastic H_-/H_∞ or H_∞/H_∞ performance index. By introducing an adjoint-operator-based optimization method, the analytical optimal solution can be derived in terms of solving a modified Riccati equation. A numerical example is provided to demonstrate the effectiveness of the proposed approach.

1. Introduction

The problem of model-based fault detection and isolation has attracted much attention during the last two decades and bred numerous results. One of the most useful strategies is the observer-based fault detection (FD) approach which involves constructing an observer as a fault detection filter (FDF) for generating a residual signal that is sensitive to fault while insensitive to unknown input [1–7]. Among the developments of FD techniques, the H_∞ filtering scheme which formulates the FD problem into the framework of H_∞ filtering is widely used for systems with $\mathscr{L}_2$-norm bounded unknown inputs and faults [8–11]. Another efficient way is the so-called H_∞ optimization scheme which employs H_∞ norm as a robustness measure while the H_∞ or H_- norm is introduced as a fault sensitivity measure, and, then, the FD problem can be formulated as an H_∞/H_∞ or H_-/H_∞ optimization problem. In [12–14], the global optimal solution to H_∞/H_∞ and H_-/H_∞ optimization FD problem is given for linear time-invariant (LTI) systems by coprime factorization and linear matrix inequality (LMI) techniques, respectively. Recent years, some new results are dedicated to this problem for linear time-varying (LTV) systems, especially for linear discrete time-varying (LDTV) systems, under the background that real systems are intrinsically time varying and most industrial processes are operated by digital devices. In [15, 16], the unified approach in [12] is extended to H_∞/H_∞ optimization FD problem for linear discrete-time periodic systems. Finite horizon H_-/H_∞ and H_∞/H_∞ FD formulation for LDTV systems is proposed in [17–19] and the optimal solution is derived by solving a standard Kalman-like Riccati equation.

On another frontline, research on systems with intermittent measurements has gained growing interests due to that the communication networks are widely utilized in control systems [20–25]. In the literature of control and filtering, stochastic parameters such as the Bernoulli distributed variables and Markov chain are introduced to model the phenomenon of missing measurements, which can be divided into two fundamentally different categories. One is the UDP-like protocol-based model where there is no acknowledgement mechanism of successful delivery of data packets. The other model is based on the TCP-like protocol where successful transmissions of packets are acknowledged at the receiver [20]. With the increasing demands for system safety and reliability, FD for systems with measurement

packet dropping has become one of the most critical issues and many related results have emerged with the help of LMI technique. In [26], a robust H_∞ FDF is designed for LTI systems with missing measurements. In [27–29], the same idea is extended to Markovian jump singular systems, T-S fuzzy systems, and discrete-time switched systems under intermittent measurements condition, respectively. The H_∞ filtering FD scheme is also applied to networked control systems with both induced delays and incomplete measurements through augmenting retarded vectors into delay-free variables and numerical solutions are derived, we refer to [30–34] and references therein. However, it should be noted that most of the existing results are based on H_∞ filtering scheme under the assumption that the measurements are transmitted using UDP-like protocol-based model and only infinite-horizon cases are considered. There remain some problems to be figured out in the open area; for example, the existing LMI-based approach does not efficiently work for time-varying systems owing to large computational burden brought by optimization iterative algorithm at each time instant, and, consequently, how to detect a fault for LDTV systems with missing measurements in the TCP-like protocols scenario.

Motivated by the above discussions, the FD problem for LDTV systems with measurement packet dropout in TCP-like protocol frame will be investigated in this paper. The main contributions of the paper are twofold.

(1) A more generalized FD problem description in the framework of optimizing stochastic H_-/H_∞ or H_∞/H_∞ performance index is formulated by defining input-to-output operator that maps from fault or unknown input to residual signal. Then, a new observer-based FDF is proposed for generating residuals based on the acknowledgement mechanism of data packet transmission.

(2) An analytical solution to the aforementioned problem is explicitly expressed by solving a modified Riccati equation through a proposed adjoint-operator-based optimization method.

The remainder of this paper is organized as follows. In Section 2, the FD problem is formulated in the sense of maximizing a stochastic sensitivity/robustness ratio. In Section 3, a unified solution to optimizing stochastic H_-/H_∞ or H_∞/H_∞ performance index problem is derived by using the proposed adjoint-operator-based optimization scheme. Finally, an illustrative example is given in Section 4 to demonstrate the effectiveness of our proposed approach.

Notations. Throughout this paper, for a matrix X, X^{T} and X^{-1} stand for the transpose and inverse of X, respectively. R^n means the set of n-dimensional real vectors. I and 0 denote identity matrix and zero matrix with appropriate dimensions, respectively. $X > 0$ ($X < 0$) means that X is positive (negative) definite. $\mathrm{E}\{\vartheta(k)\}$ represents the mathematical expectation of $\vartheta(k)$. $\|\xi\|$ denotes the Euclidean norm of ξ. $\|\theta(k)\|_2$ stands for the deterministic l_2-norm of $\theta(k)$ with $\|\theta(k)\|_2^2 = \sum_{k=0}^{N}\theta^{\mathrm{T}}(k)\theta(k)$, while $\|\zeta(k)\|_{2,\mathrm{E}}$ for the stochastic case with $\|\zeta(k)\|_{2,\mathrm{E}}^2 = \mathrm{E}\{\sum_{k=0}^{N}\zeta^{\mathrm{T}}(k)\zeta(k)\}$, where N is a positive integer. $\langle(\mu_0,\mu(k)),(\varsigma_0,\varsigma(k))\rangle = \mathrm{E}\{\mu_0^{\mathrm{T}}\Pi\varsigma_0\} + \mathrm{E}\{\sum_{k=0}^{N}\mu^{\mathrm{T}}(k)\varsigma(k)\}$ gives the definition of the inner product on a Hilbert space for l_2-norm bounded vector $\mu(k)$ and $\varsigma(k)$ with appropriate dimensions, where Π is a positive-definite initial weighting matrix. $\mathrm{Prob}\{\cdot\}$ means the occurrence probability of the event "·".

2. Problem Formulation

Consider the following LDTV system with measurement packet dropping:

$$\begin{aligned} x(k+1) &= A(k)x(k) + B(k)u(k) \\ &\quad + B_d(k)d(k) + B_f(k)f(k), \\ y(k) &= \gamma(k)C(k)x(k) + D_d(k)d(k), \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $x(k) \in R^n$, $y(k) \in R^{n_y}$, $u(k) \in R^{n_u}$, $d(k) \in R^{n_d}$, and $f(k) \in R^{n_f}$ denote the state, measurement output, control input, unknown input, and fault, respectively. $A(k)$, $B(k)$, $B_d(k)$, $B_f(k)$, $C(k)$, and $D_d(k)$ are known time-varying matrices with appropriate dimensions. $f(k)$ and $d(k)$ are l_2-norm bounded signals. $\gamma(k)$ depicts the measurement packet dropouts and is assumed to be a scalar Bernoulli distributed binary stochastic variable; that is,

$$\begin{aligned} \mathrm{Prob}\{\gamma(k) = 1\} &= \mathrm{E}\{\gamma(k)\} = \rho, \\ \mathrm{Prob}\{\gamma(k) = 0\} &= 1 - \mathrm{E}\{\gamma(k)\} = 1 - \rho, \end{aligned} \tag{2}$$

where ρ is a known constant which can be obtained as prior knowledge through statistical test method [32].

Before describing the underlying problems, the following definition and assumptions are introduced.

Definition 1. System (1) is said exponentially stable in mean square sense (ESMS) with $u(k) = 0$, $d(k) = 0$, and $f(k) = 0$, if there exist $\beta \in (0, +\infty)$ and $q \in (0, 1)$ such that

$$\mathrm{E}\left\{\|x(k)\|^2\right\} \le \beta q^k \|x(0)\|^2, \tag{3}$$

for any initial condition x_0.

Assumption 2. $(A^{\mathrm{T}}(k), \rho C^{\mathrm{T}}(k), 0, C^{\mathrm{T}}(k))$ is stabilizable.

Assumption 3. $(A^{\mathrm{T}}(k), 0, B_d^{\mathrm{T}}(k))$ is exactly observable.

Assumption 4. $\gamma(k)$ is available at each time instant k.

The primary object of FD is to construct an FDF for generating a residual signal which is robust to unknown input $d(k)$ while sensitive to fault $f(k)$. Note that the indicator

$\gamma(k)$ is known due to Assumption 4, and hence $\gamma(k)$ can be involved in the following observer-based FDF:

$$\begin{aligned} \hat{x}(k+1) &= A(k)\hat{x}(k) + B(k)u(k) \\ &\quad + L(k)\left(y(k) - \gamma(k)C(k)\hat{x}(k)\right), \\ r(k) &= V(k)\left(y(k) - \gamma(k)C(k)\hat{x}(k)\right), \\ \hat{x}(0) &= \hat{x}_0, \end{aligned} \tag{4}$$

where $\hat{x}(k) \in R^n$ is an estimation of $x(k)$, $r(k) \in R^r$ is the generated residual, $\hat{x}_0$ is a guess of initial state, $L(k)$ is the observer gain matrices, and $V(k)$ is the (regular) post-filters to be determined.

Let $e(k) = x(k) - \hat{x}(k)$; it follows from (1) and (4) that

$$\begin{aligned} e(k+1) &= \left(A(k) - \gamma(k)L(k)C(k)\right)e(k) \\ &\quad + \left(B_d(k) - L(k)D_d(k)\right)d(k) + B_f(k)f(k), \\ r(k) &= V(k)\left(\gamma(k)C(k)e(k) + D_d(k)d(k)\right). \end{aligned} \tag{5}$$

Denote

$$\begin{aligned} r_f(k) &= r(k)|_{d_k=0}, \\ r_d(k) &= r(k)|_{f_k=0}, \end{aligned} \tag{6}$$

where $d_k = \left[e^{\mathrm{T}}(0)\ d^{\mathrm{T}}(0)\ \cdots\ d^{\mathrm{T}}(k)\right]^{\mathrm{T}}$ and $f_k = \left[f^{\mathrm{T}}(0)\ \cdots\ f^{\mathrm{T}}(k)\right]^{\mathrm{T}}$.

Similar to the perturbed operator defined by l_2-norm of input and output signals in [35], a linear operator that maps $f \mapsto r$ and a linear operator that maps $d \mapsto r$ can be defined as follows by the linearity of (1):

$$\begin{aligned} r_f(k) &= \mathcal{G}_{rf}f(k), \\ r_d(k) &= \mathcal{G}_{rd}d(k) \end{aligned} \tag{7}$$

with

$$\left\|\mathcal{G}_{rf}\right\|_\infty = \sup_{f\in l_2, \|f\|_2 \neq 0} \frac{\left\|r_f(k)\right\|_{2,\mathrm{E}}^2}{\left\|f(k)\right\|_2^2},$$

$$\left\|\mathcal{G}_{rd}\right\|_\infty = \sup_{d\in l_2, \|d\|_2 \neq 0} \frac{\left\|r_d(k)\right\|_{2,\mathrm{E}}^2}{\|d(k)\|_2^2 + e^{\mathrm{T}}(0)Se(0)}, \tag{8}$$

$$\left\|\mathcal{G}_{rf}\right\|_- = \inf_{f\in l_2, \|f\|_2 \neq 0} \frac{\left\|r_f(k)\right\|_{2,\mathrm{E}}^2}{\left\|f(k)\right\|_2^2},$$

where S is a positive-definite initial weighting matrix.

Based on the definitions above, we are now in the position to introduce an auxiliary FDF design problem for residual generation: find a suitable observer gain matrix $L(k)$ and a regular post-filter $V(k)$ such that system (5) is ESMS and satisfies the following performance:

$$\max_{L(k),V(k)} \frac{\left\|\mathcal{G}_{rf}\right\|_\infty}{\left\|\mathcal{G}_{rd}\right\|_\infty} \quad \text{or} \quad \max_{L(k),V(k)} \frac{\left\|\mathcal{G}_{rf}\right\|_-}{\left\|\mathcal{G}_{rd}\right\|_\infty}. \tag{9}$$

After designing the FDF, the remaining task is to evaluate the generated residual. In this paper, we adopt the residual evaluation function in the following forms:

$$J(k) = \sqrt{\frac{1}{k}\sum_{k=0}^{k=k_T} r^{\mathrm{T}}(k)r(k)}, \tag{10}$$

where k_T denotes the length of the evaluation time window [2]. The corresponding threshold is chosen as

$$J_{\mathrm{th}} = \sup_{f(k)=0} \mathrm{E}\{J(k)\}, \tag{11}$$

and hence the occurrence of faults can then be recognized based on the following rule:

$$\begin{aligned} J(k) &> J_{\mathrm{th}}, \quad \Longrightarrow \text{a fault is detected}, \\ J(k) &\le J_{\mathrm{th}}, \quad \Longrightarrow \text{no faults}. \end{aligned} \tag{12}$$

Remark 5. Assumptions 2 and 3 are given without loss of generality to guarantee the stability of the proposed FDF. For more details about the definitions on stabilizable and exactly observable, we refer to [24, 36] and references therein. With the aid of time-stamp technique, Assumption 4 is also reasonable due to the acknowledgements of successful transmissions of data packets for TCP-like protocols. Furthermore, under the condition that Assumption 4 holds, our proposed FDF is different from the ones given in [26–29] since the new FDF exploits additional information on the observation sequence.

Remark 6. It should be emphasized that since stochastic characteristic is introduced by $\gamma(k)$, the existing FD problem formulation and results in [17–19] for deterministic LTV systems would not be reasonably established. Our proposed performance index $\|\mathcal{G}_{rf}\|_\infty/\|\mathcal{G}_{rd}\|_\infty$ or $\|\mathcal{G}_{rd}\|_-/\|\mathcal{G}_{rd}\|_\infty$ is a generalized stochastic version comparing to that of the existing contributions. In what follows, a novel adjoint-operator-based optimization approach will be proposed with the well-defined linear operator $\mathcal{G}_{rd}$ and $\mathcal{G}_{rf}$, and we will show that detailed interpretation of $\mathcal{G}_{rd}$ and $\mathcal{G}_{rf}$ is not necessary owing to our proposed method.

3. Main Results

Before deriving the main results of this paper, the following definitions and lemmas should be given.

Definition 7 (see [37]). Let $\mathcal{G}_s$ denote an operator or a system mapping from l_2-norm bounded space S_1 to l_2-norm bounded space S_2. An operator $\mathcal{G}_s^{\sim}$ is called to be the adjoint operator of $\mathcal{G}_s$ from space S_2 to S_1 if $\langle \mathcal{G}_s(\mu_0, \mu), \varsigma \rangle = \langle (\mu_0, \mu), \mathcal{G}_s^{\sim}\varsigma \rangle$, for all $\mu \in S_1$ and $\varsigma \in S_2$, where μ_0 stands for the initial vector.

Definition 8 (see [38]). Let $\mathcal{G}_s$ denote an operator or a system mapping from l_2-norm bounded input space S_1 to l_2-norm bounded output space S_2; then, $\mathcal{G}_s$ is said to be coisometric if $\langle \mathcal{G}_s^{\sim}\varphi(k), \mathcal{G}_s^{\sim}\varphi(k) \rangle = \langle \varphi(k), \varphi(k) \rangle$ for all $\varphi(k) \in S_1$.

Lemma 9. *For two stochastic operators $A : y \mapsto z$ and $B : w \mapsto y$, where y, z, and w are l_2-norm bounded signals, one has*

$$\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty, \qquad \|AB\|_- \le \|A\|_\infty \|B\|_-. \tag{13}$$

Proof. Let $z = Ay$ and $y = Bw$; we can conclude that

$$\begin{aligned}
\|AB\|_\infty &= \sup_{w\in l_2,\|w\|_2\neq 0} \frac{\|z(k)\|_{2,\mathrm{E}}^2}{\|w(k)\|_2^2} = \sup_{w\in l_2,\|w\|_2\neq 0} \frac{\|ABw(k)\|_{2,\mathrm{E}}^2}{\|w(k)\|_2^2} \\
&= \sup_{w\in l_2,\|w\|_2\neq 0} \left\{ \frac{\|ABw(k)\|_{2,\mathrm{E}}^2}{\|Bw(k)\|_2^2} \times \frac{\|Bw(k)\|_{2,\mathrm{E}}^2}{\|w(k)\|_2^2} \right\} \\
&\le \|A\|_\infty \|B\|_\infty, \\
\|AB\|_- &= \inf_{w\in l_2,\|w\|_2\neq 0} \frac{\|z(k)\|_{2,\mathrm{E}}^2}{\|w(k)\|_2^2} = \inf_{w\in l_2,\|w\|_2\neq 0} \frac{\|ABw(k)\|_{2,\mathrm{E}}^2}{\|w(k)\|_2^2} \\
&= \inf_{w\in l_2,\|w\|_2\neq 0} \left\{ \frac{\|z(k)\|_{2,\mathrm{E}}^2}{\|y(k)\|_2^2} \times \frac{\|Bw(k)\|_{2,\mathrm{E}}^2}{\|w(k)\|_2^2} \right\} \\
&\le \sup_{w\in l_2,\|w\|_2\neq 0} \frac{\|z(k)\|_{2,\mathrm{E}}^2}{\|y(k)\|_2^2} \times \inf_{w\in l_2,\|w\|_2\neq 0} \frac{\|Bw(k)\|_{2,\mathrm{E}}^2}{\|w(k)\|_2^2} \\
&\le \|A\|_\infty \|B\|_-
\end{aligned} \tag{14}$$

which completes the proof. □

Lemma 10. *Consider the following residual generators:*

$$\begin{aligned}
\hat{x}^{(m)}(k+1) &= A(k)\hat{x}^{(m)}(k) + B(k)u(k) \\
&\quad + L^{(m)}(k)\left(y(k) - \gamma(k)C(k)\hat{x}^{(m)}(k)\right), \\
r^{(m)}(k) &= V^{(m)}(k)\left(y(k) - \gamma(k)C(k)\hat{x}^{(m)}(k)\right), \\
&\qquad m = 1, 2,
\end{aligned} \tag{15}$$

where $L^{(m)}(k)$ is the observer gain matrix such that system (15) *is ESMS and $V^{(m)}(k)$ is the post-filter. Then*

$$r^{(2)}(k) = \mathcal{Q}r^{(1)}(k), \tag{16}$$

where $\mathcal{Q}$ is an operator that maps $r^{(1)}(k) \mapsto r^{(2)}(k)$.

Proof. By applying Lemma 1 in [18] to (15), we have that for the following residual generators

$$\begin{aligned}
\hat{x}^{(m)}(k+1) &= A(k)\hat{x}^{(m)}(k) + B(k)u(k) \\
&\quad + L^{(m)}(k)\left(y(k) - \gamma(k)C(k)\hat{x}^{(m)}(k)\right), \\
\varepsilon^{(m)}(k) &= y(k) - \gamma(k)C(k)\hat{x}^{(m)}(k), \quad m = 1, 2,
\end{aligned} \tag{17}$$

where $L^{(m)}(k)$ is the observer gain matrix that ensures the stability of (15); an operator $\mathcal{Q}_\varepsilon$ that guarantees $\varepsilon^{(2)}(k) = \mathcal{Q}_\varepsilon \varepsilon^{(1)}(k)$ exists, which can be realized as

$$\begin{aligned}
\xi(k+1) &= \left(A(k) - L^{(2)}(k)\gamma(k)C(k)\right)\xi(k) \\
&\quad + \left(L^{(1)}(k) - L^{(2)}(k)\right)\nu(k), \\
\varepsilon^{\mathcal{Q}_\varepsilon}(k) &= \gamma(k)C(k)\xi(k) + \nu(k).
\end{aligned} \tag{18}$$

The regularity of $V(k)$ indicates there exists $V^+(k)$ such that

$$V^+(k)V(k) = I, \tag{19}$$

where $V^+(k)$ denotes the left inverse of $V(k)$, and then we have

$$\begin{aligned}
r^{(2)}(k) &= V^{(2)}(k)\varepsilon^{(2)}(k) = V^{(2)}(k)\mathcal{Q}_\varepsilon \varepsilon^{(1)}(k) \\
&= V^{(2)}(k)\mathcal{Q}_\varepsilon \left(V^{(1)}(k)\right)^+ r^{(1)}(k).
\end{aligned} \tag{20}$$

Thus, from (18), $\mathcal{Q}$ can be represented as

$$\begin{aligned}
\eta(k+1) &= \left(A(k) - L^{(2)}(k)\gamma(k)C(k)\right)\xi(k) \\
&\quad + \left(L^{(1)}(k) - L^{(2)}(k)\right)\nu(k), \\
r^{(2)}(k) &= V^{(2)}(k)\left(\gamma(k)C(k)\xi(k) + \nu(k)\right),
\end{aligned} \tag{21}$$

where $\nu(k) = V^{(2)}(k)\mathcal{Q}_\varepsilon(V^{(1)}(k))^+ r^{(1)}(k)$. This completes the proof. □

Lemma 11. *For system* (5)*, consider the operator $\mathcal{G}_{rd}$ that maps $d(k) \mapsto r_d(k)$ which is realized by the following discrete-time system:*

$$\begin{aligned}
e(k+1) &= A_e(k)e(k) + B_e(k)d(k), \\
r_d(k) &= C_e(k)e(k) + D_e(k)d(k), \\
e(0) &= e_0,
\end{aligned} \tag{22}$$

where $A_e(k) = A(k) - \gamma(k)L(k)C(k)$, $B_e(k) = B_d(k) - L(k)D_d(k)$, $C_e(k) = \gamma(k)V(k)C(k)$, and $D_e(k) = V(k)D_d(k)$. Let $\mathcal{G}_{\widetilde{rd}}$ be the adjoint operator of $\mathcal{G}_{rd}$; then $\mathcal{G}_{rd}$ is coisometric

if there exists a positive definite matrix $P(\cdot)$ satisfying the following equation:

$$\begin{aligned}
P(k+1) &= B_e(k) B_e^{\mathrm{T}}(k) + A(k) P(k) A^{\mathrm{T}}(k) \\
&\quad + \rho L(k) C(k) P(k) C^{\mathrm{T}}(k) L^{\mathrm{T}}(k) \\
&\quad - \rho L(k) C(k) P(k) A^{\mathrm{T}}(k) L^{\mathrm{T}}(k) \\
&\quad - \rho A(k) C(k) P(k) C^{\mathrm{T}}(k) L^{\mathrm{T}}(k), \\
0 &= \rho A(k) P(k) C^{\mathrm{T}}(k) V^{\mathrm{T}}(k) \\
&\quad - \rho L(k) C(k) P(k) C^{\mathrm{T}}(k) V^{\mathrm{T}}(k) \\
&\quad + B_e(k) D_e^{\mathrm{T}}(k), \\
I &= D_e(k) D_e^{\mathrm{T}}(k) + \rho V(k) C(k) P(k) C^{\mathrm{T}}(k) V^{\mathrm{T}}(k), \\
P(0) &= S^{-1}.
\end{aligned} \tag{23}$$

Proof. Denote $\mathscr{G}_{rd}^{\sim} r_d(k) = [\eta^{\mathrm{T}}(0) \ \ d_a^{\mathrm{T}}(k)]^{\mathrm{T}}$. From [39], the state space representation of $\mathscr{G}_{rd}^{\sim}$ can be obtained as

$$\begin{aligned}
\eta_a(k-1) &= A_e^{\mathrm{T}}(k) \eta_a(k) + C_e^{\mathrm{T}}(k) r_d(k), \\
d_a(k) &= B_e^{\mathrm{T}}(k) \eta_a(k) + D_e^{\mathrm{T}}(k) r_d(k), \\
\eta_a(N) &= 0.
\end{aligned} \tag{24}$$

For (24), define

$$\mathbb{V}(\eta_a(k), k) = \eta_a^{\mathrm{T}}(k) P(k+1) \eta_a(k), \quad P(\cdot) > 0. \tag{25}$$

Moreover, based on (22) and (24), we have

$$\begin{aligned}
&\mathrm{E}\left\{\sum_{k=0}^{N} d_a^{\mathrm{T}}(k) d_a(k)\right\} + \eta_a^{\mathrm{T}}(-1) S^{-1} \eta_a(-1) \\
&\quad = \mathrm{E}\left\{\sum_{k=0}^{N} d_a^{\mathrm{T}}(k) d_a(k) + \mathbb{V}(\eta_a(k-1), k-1) \right. \\
&\qquad \left. - \mathbb{V}(\eta_a(k), k) \right\} + \eta_a(-1)^{\mathrm{T}} S^{-1} \eta_a(-1) \\
&\qquad + \eta_a(N)^{\mathrm{T}} P(N+1) \eta_a(N) - \eta_a(-1)^{\mathrm{T}} P(0) \eta_a(-1) \\
&\quad = \mathrm{E}\left\{\sum_{k=0}^{N} \eta_a^{\mathrm{T}}(k) \left[B_e(k) B_e^{\mathrm{T}}(k) + A_e(k) P(k) A_e^{\mathrm{T}}(k) \right.\right. \\
&\qquad \left. - P(k+1) \right] \eta_a(k) + 2\eta_a^{\mathrm{T}}(k) \\
&\qquad \times \left[B_e(k) D_e^{\mathrm{T}}(k) + A_e(k) P(k) C_e^{\mathrm{T}}(k)\right] r_d(k) \\
&\qquad \left. + r_d^{\mathrm{T}} \left[D_e(k) D_e^{\mathrm{T}}(k) + C_e(k) P(k) C_e^{\mathrm{T}}(k)\right] r_d(k) \right\}.
\end{aligned} \tag{26}$$

Remembering that $\mathrm{E}\{\gamma(k)\} = \rho$ and $\mathrm{E}\{\gamma^2(k)\} = \rho$, then, from (26) and Definition 8, if

$$\langle (\eta(0), d_a(k)), (\eta(0), d_a(k)) \rangle = \langle r_d(k), r_d(k) \rangle = \|r_d\|_{2,\mathrm{E}}^2; \tag{27}$$

that is, (23) holds, $\mathscr{G}_{rd}$ is coisometric, which completes the proof. □

In view of the proposed lemmas above, we now present solutions to the optimization problem (9).

Theorem 12. *If Assumptions 2 and 3 hold, then the following matrix pair*

$$\begin{aligned}
L_o(k) &= \left(\rho A(k) P(k) C^{\mathrm{T}}(k) + B_d(k) D_d^{\mathrm{T}}(k)\right) R(k)^{-1}, \\
V_o(k) &= R(k)^{-1/2}
\end{aligned} \tag{28}$$

with

$$R(k) = \rho C(k) P(k) C^{\mathrm{T}}(k) + D_d(k) D_d^{\mathrm{T}}(k) > 0 \tag{29}$$

give a solution to the auxiliary FDF design problem and $P(k)$ is the solution to the following equation with $P(0) = S^{-1}$:

$$\begin{aligned}
P(k+1) &= A(k) P(k) A^{\mathrm{T}}(k) \\
&\quad - L(k) R(k) L^{\mathrm{T}}(k) + B_d(k) B_d^{\mathrm{T}}(k).
\end{aligned} \tag{30}$$

Proof. We first give the proof of the optimal solution to (9). Let $r_o(k)$ be the optimal generated residual in the sense of (9). Since system (1) is linear, by applying Lemma 10, we know that there exists an operator $\mathcal{Q}_r$ such that

$$\begin{aligned}
r(k) &= r_f(k) + r_d(k) = \mathscr{G}_{rf} f(k) + \mathscr{G}_{rd} d(k) \\
&= \mathcal{Q}_r r_o(k) = \mathcal{Q}_r \left(r_{o,f}(k) + r_{o,d}(k)\right),
\end{aligned} \tag{31}$$

where $r_{o,f}(k) = r_o(k)|_{d_k=0}$ and $r_{o,d}(k) = r_o(k)|_{f_k=0}$, which shows that

$$r_d(k) = \mathcal{Q}_r r_{o,d}(k). \tag{32}$$

On the other hand, consider the operator $\mathscr{G}_{rd}$ that maps $d \mapsto r$ in (5); we have

$$r_{o,d}(k) = \mathscr{G}_{rd,o} d(k), \tag{33}$$

where $\mathscr{G}_{rd,o} = \mathscr{G}_{rd}|_{L(k)=L_o(k), V(k)=V_o(k)}$, which concludes that

$$\mathscr{G}_{rd} = \mathcal{Q}_r \mathscr{G}_{rd,o}. \tag{34}$$

Similarly, we have

$$\mathscr{G}_{rf} = \mathcal{Q}_r \mathscr{G}_{rf,o}, \tag{35}$$

where $\mathscr{G}_{rf,o} = \mathscr{G}_{rf}|_{L(k)=L_o(k), V(k)=V_o(k)}$. Hence,

$$\frac{\|\mathscr{G}_{rf}\|_\infty}{\|\mathscr{G}_{rd}\|_\infty} = \frac{\|\mathcal{Q}_r \mathscr{G}_{rf,o}\|_\infty}{\|\mathcal{Q}_r \mathscr{G}_{rd,o}\|_\infty}. \tag{36}$$

Based on (24) and according to [40, 41] and Definition 8; that is, if $\mathcal{G}_{rd,o}$ is coisometric, we have that

$$\begin{aligned}\left\|\mathcal{G}_{rd}\right\|_{\infty} = \left\|\mathcal{G}_{rd}^{\sim}\right\|_{\infty} &= \sup_{r_d\in l_2, \|r_d\|_{2,\mathrm{E}}\neq 0} \frac{\left\langle d_a(k), d_a(k)\right\rangle}{\left\|r_d(k)\right\|_{2,\mathrm{E}}^2} \\ &= \sup_{r_d\in l_2, \|r_d\|_{2,\mathrm{E}}\neq 0} \frac{\left\langle \left(\mathcal{Q}_r\mathcal{G}_{rd,o}\right)^{\sim} r_d(k), \left(\mathcal{Q}_r\mathcal{G}_{rd,o}\right)^{\sim} r_d(k)\right\rangle}{\left\|r_d(k)\right\|_{2,\mathrm{E}}^2} \\ &= \sup_{r_d\in l_2, \|r_d\|_{2,\mathrm{E}}\neq 0} \frac{\left\langle r_d(k), \mathcal{Q}_r\mathcal{G}_{rd,o}\mathcal{G}_{rd,o}^{\sim}\mathcal{Q}_r^{\sim} r_d(k)\right\rangle}{\left\|r_d(k)\right\|_{2,\mathrm{E}}^2} \\ &= \left\|\mathcal{Q}_r\right\|_{\infty}\end{aligned} \tag{37}$$

and the following inequality is immediately established based on Lemma 9:

$$\frac{\left\|\mathcal{G}_{rf}\right\|_{\infty}}{\left\|\mathcal{G}_{rd}\right\|_{\infty}} = \frac{\left\|\mathcal{Q}_r\mathcal{G}_{rf,o}\right\|_{\infty}}{\left\|\mathcal{Q}_r\mathcal{G}_{rd,o}\right\|_{\infty}} \leq \frac{\left\|\mathcal{Q}_r\right\|_{\infty}\left\|\mathcal{G}_{rf,o}\right\|_{\infty}}{\left\|\mathcal{Q}_r\right\|_{\infty}} = \left\|\mathcal{G}_{rf,o}\right\|_{\infty} \tag{38}$$

which gives the optimal value of maximizing the performance index $\|\mathcal{G}_{rf}\|_{\infty}/\|\mathcal{G}_{rd}\|_{\infty}$.

Following the same idea, we can prove that

$$\frac{\left\|\mathcal{G}_{rf}\right\|_{-}}{\left\|\mathcal{G}_{rd}\right\|_{\infty}} = \frac{\left\|\mathcal{Q}_r\mathcal{G}_{rf,o}\right\|_{-}}{\left\|\mathcal{Q}_r\mathcal{G}_{rd,o}\right\|_{\infty}} \leq \frac{\left\|\mathcal{Q}_r\right\|_{\infty}\left\|\mathcal{G}_{rf,o}\right\|_{-}}{\left\|\mathcal{Q}_r\right\|_{\infty}} = \left\|\mathcal{G}_{rf,o}\right\|_{-}. \tag{39}$$

Furthermore, from Lemma 11, by solving the following equations in (23):

$$\begin{aligned}&\rho A(k)P(k)C^{\mathrm{T}}(k)V^{\mathrm{T}}(k) - \rho L(k)C(k)P(k)C^{\mathrm{T}}(k)V^{\mathrm{T}}(k) \\ &\quad + B_e(k)D_e^{\mathrm{T}}(k) = 0, \\ &\quad D_e(k)D_e^{\mathrm{T}}(k) + \rho V(k)C(k)P(k)C^{\mathrm{T}}V^{\mathrm{T}} = I\end{aligned} \tag{40}$$

we can obtain (28) and consequently

$$\begin{aligned}P(k+1) = {} & B_e(k)B_e^{\mathrm{T}}(k) + A(k)P(k)A^{\mathrm{T}}(k) \\ & + \rho L(k)C(k)P(k)C^{\mathrm{T}}(k)L^{\mathrm{T}}(k) \\ & - \rho L(k)C(k)P(k)A^{\mathrm{T}}(k)L^{\mathrm{T}}(k) \\ & - \rho A(k)C(k)P(k)C^{\mathrm{T}}(k)L^{\mathrm{T}}(k), \\ P(0) = {} & I\end{aligned} \tag{41}$$

turns to (30).

Next, we discuss the stability of the proposed FDF. In fact, by introducing $w(k) = \gamma(k) - \rho$ satisfying

$$\begin{aligned}\mathrm{E}\{w(k)\} &= 0, \\ \mathrm{E}\{w(i)w(j)\} &= \varepsilon\delta_{ij},\end{aligned} \tag{42}$$

where $\varepsilon = \rho - \rho^2$ and δ_{ij} means the Kronecker delta function, an artificial stochastic backward control system associated with (24) can be defined as follows:

$$\begin{aligned}x_a(k-1) = {} & \left(\overline{A}_e(k) + w(k)A_{ew}(k)\right)^{\mathrm{T}}(k)x_a(k) \\ & + \left(\overline{C}_e(k) + w(k)C_{ew}(k)\right)^{\mathrm{T}}(k)u_a(k), \\ x_a(N) = {} & 0,\end{aligned} \tag{43}$$

where $\overline{A}_e(k) = A(k) - \rho L(k)C(k)$, $A_{ew}(k) = L(k)C(k)$, $\overline{C}_e(k) = \rho V(k)C(k)$, and $C_{ew}(k) = V(k)C(k)$. From [36], if Assumptions 2 and 3 hold, there exists a unique positive solution $P(\cdot)$ to (30) in the sense of finding a linear quadratic (LQ) control law $u_a^*(k)$ to (43). Furthermore, following [42], the proposed FDF is ESMS. This completes the proof. □

Theorem 12 addresses a static solution to the two-objective optimization problem (9). A more generalized dynamic operator form of this theorem can be derived as follows.

Theorem 13. *Let $L(k)$ be any observer gain that guarantees the stability of the proposed FDF* (4)*; then $V_o(k)$ that maps $\nu(k)\mapsto r(k)$ in the following way presents a dynamic post-filter for the auxiliary FDF design problem:*

$$\begin{aligned}\xi_o(k+1) = {} & \left(A(k) - L_o(k)\gamma(k)C(k)\right)\xi_o(k) \\ & + \left(L(k) - L_o(k)\right)\nu(k), \\ r(k) = {} & V_o(k)\left(\gamma(k)C(k)\xi_o(k) + \nu(k)\right),\end{aligned} \tag{44}$$

where L_o and V_o are determined in (28) *and* (30).

Proof. The proof is straightforward as a result of Lemma 10. Hence, it is omitted here. □

Theorem 12 can be readily extended to the stationary FDF design over an infinite horizon. In this case, the system matrices $A(k)$, $B(k)$, $B_f(k)$, $B_d(k)$, $C(k)$, and $D_d(k)$ will be replaced by constant matrices A, B, B_f, B_d, C, and D_d, respectively. The following result, which is the stationary version of Theorem 12 for LTI systems with measurement packet dropping, can be obtained by redefining $\mathbb{V}(\eta_a(k)) = \eta_a^{\mathrm{T}}(k)P\eta_a(k)$ in Lemma 11 with a constant P and replacing the Riccati difference equation (30) by a corresponding algebraic Riccati equation. In addition, from [24, 42], the ESMS of the proposed FDF can be guaranteed when Assumptions 2 and 3 are satisfied.

Corollary 14. *Consider system* (1) *with constant system matrices and zero initial state. If Assumptions 2 and 3 hold, then the following matrix pair*

$$\begin{aligned}L_o &= \left(\rho APC^{\mathrm{T}} + B_dD_d^{\mathrm{T}}\right)R^{-1}, \\ V_o &= R^{-1/2}\end{aligned} \tag{45}$$

with

$$R = \rho CPC^{\mathrm{T}} + D_dD_d^{\mathrm{T}} > 0 \tag{46}$$

give a solution to (9) *in infinite horizon and P is the solution to the following equation:*

$$P = APA^{\mathrm{T}} - LRL^{\mathrm{T}} + B_d B_d^{\mathrm{T}}. \tag{47}$$

Remark 15. Theorem 12 gives unified analytical solutions to the associated FDF design problem of optimizing the defined stochastic H_-/H_∞ and H_∞/H_∞ performance indices. It can easily conclude the following.

(i) If $\rho \neq 1$; that is, there measurement packet dropping occurs, the solution to the FDF design problem is derived based on a modified Riccati equation (30).

(ii) If $\rho = 1$; that is, there is no measurement packet dropout, our results will coincide with the one given in [18, 19], where the modified Riccati equation (30) will reduce to a standard Kalman-like filtering Riccati equation.

Remark 16. Roughly speaking, the measurement packet dropping phenomenon depicted by the Bernoulli random variable can be divided into three categories, that is, measurement packet dropouts, stochastic finite step delay, and multiple packet dropouts [20]. It can be concluded that if the packet dropouts indicator (as $\gamma(k)$ in this paper) is on-line known, our proposed algorithm can be applied to these scenarios.

Remark 17. It should be pointed out that for system (1), the existing H_∞ filtering-based fault estimation approach in [39] cannot achieve the fault detection goal since the fault distribution matrix in the measurement $y(k)$ is zero. However, our solution is independent on the fault distribution matrices, which infers its more generalized availability. In addition, comparing to the LMI-based H_∞ fault estimation approaches given in [26–29], the proposed results are in analytical form and load less computational burden.

4. Numerical Example

To illustrate the effectiveness of the proposed method, we consider (1) with the following parameter matrices:

$$\begin{gathered} A(k) = \begin{bmatrix} 0.8 & 0.6 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 0.7 \end{bmatrix}, \qquad B_d(k) = \begin{bmatrix} 1.3 \\ 0.5 \\ 1 \end{bmatrix}, \\ B_f(k) = \begin{bmatrix} 0.7 \\ 1 \\ 0.8 \end{bmatrix}, \qquad C(k) = \begin{bmatrix} -0.5 & 1.5 & 0 \end{bmatrix}, \\ D_d(k) = 0.5. \end{gathered} \tag{48}$$

The missing measurement rate $\gamma(k)$ is displayed in Figure 1 with mean $\rho = 0.8$ and the unknown input $d(k)$ is simulated in Figure 2. A stepwise fault signal is assumed to be as follows:

$$f(k) = \begin{cases} 1, & k \in [31, 60] \\ 0, & \text{otherwise}. \end{cases} \tag{49}$$

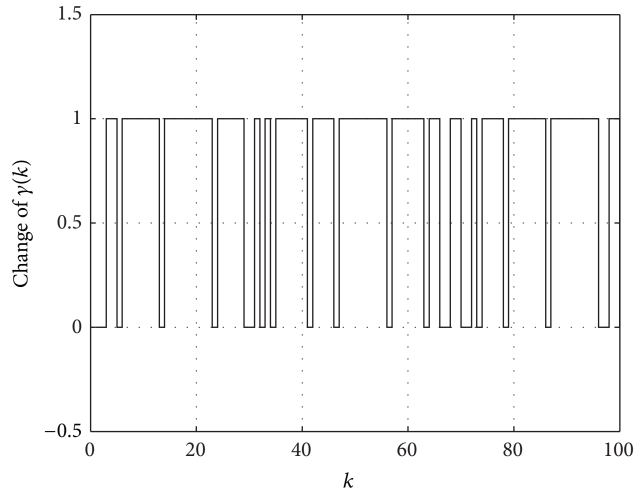


Figure 1: Change of $\gamma(k)$.

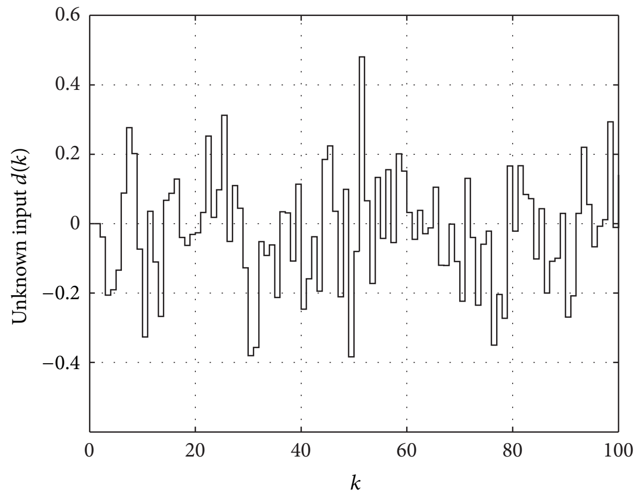


Figure 2: Unknown input $d(k)$.

Set $P(0) = I$. We design the FDF by directly applying Theorem 12. The generated residual $r(k)$ is shown in Figure 3 and its residual evaluation function $J(k)$ in (10) is displayed in Figure 4 with the corresponding threshold $J_{\mathrm{th}} = 0.7457$. It can be seen from the simulation results that $J(32) = 2.2504 > J_{\mathrm{th}}$ which implies that the residual can deliver fault alarm in 1 second after it occurs.

5. Conclusions

In this paper, the problem of FD for LDTV systems with measurement packet dropping has been dealt with. Under the condition that the packet dropout indicator is on-line available, an observer-based FDF has been employed as a residual generator through exploiting the information brought by the packet dropout indicator. The design of FDF has been formulated in the framework of maximizing stochastic H_-/H_∞ or H_∞/H_∞ performance index. An analytical optimal solution

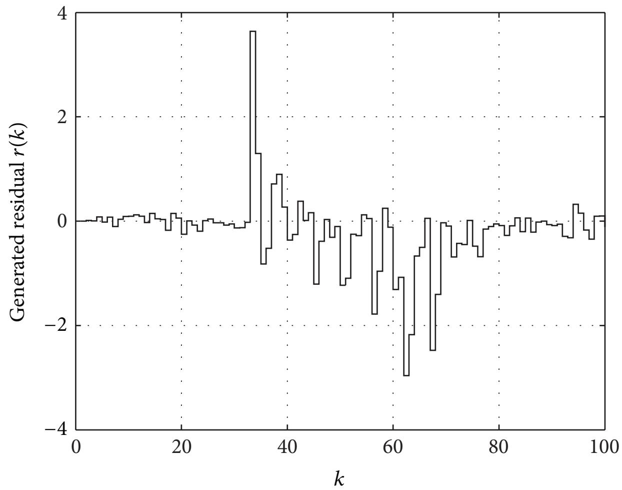


Figure 3: Residual $r(k)$.

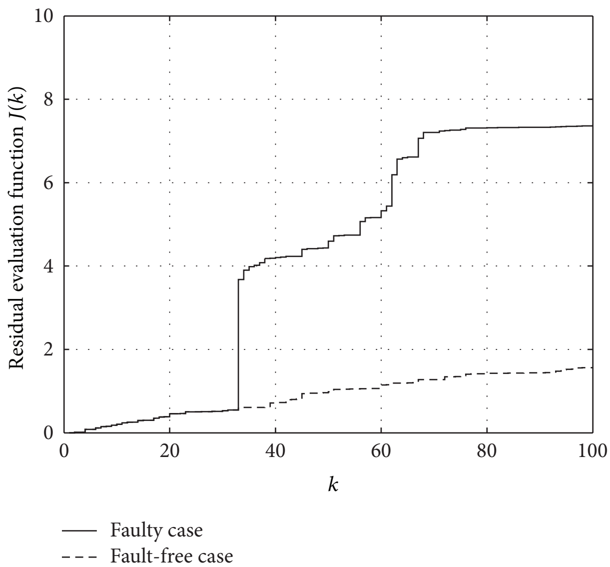


Figure 4: Residual evaluation function $J(k)$.

has been derived by solving a modified Riccati equation based on the proposed adjoint operator optimization method. The achieved result has been illustrated by a numerical example.

Acknowledgments

This work is partially supported by National Natural Science Foundation of China (no. 61203083 and no. 61074021), the Doctoral Foundation of University of Jinan (no. XBS1242).

References

[1] J. Chen and R. Patton, *Robust Model-Based Fault Diagnosis for Dynamic Systems*, Kluwer Academic Publishers, Boston, Mass, USA, 1999.

[2] S. X. Ding, *Model-Based Fault Diagnosis Techniques*, Springer, Berlin, Germany, 2008.

[3] B. Kulcsár and M. Verhaegen, "Robust inversion based fault estimation for discrete-time LPV systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1581–1586, 2012.

[4] Z. Wang, H. Guan, and C. Zheng, "Fault diagnosis observer design for discrete-time delayed complex interconnected networks with linear coupling," *Mathematical Problems in Engineering*, vol. 2012, Article ID 860489, 22 pages, 2012.

[5] J. Yu, G. Sun, and H. R. Karimi, "Fault-reconstruction-based cascaded sliding mode observers for descriptor linear systems," *Mathematical Problems in Engineering*, vol. 2012, Article ID 623426, 20 pages, 2012.

[6] Y. Yin, P. Shi, F. Liu, and J.-S. Pan, "Gain-scheduled fault detection on stochastic nonlinear systems with partially known transition jump rates," *Nonlinear Analysis. Real World Applications*, vol. 13, no. 1, pp. 359–369, 2012.

[7] L. Wu, X. Yao, and W. X. Zheng, "Generalized H_2 fault detection for two-dimensional Markovian jump systems," *Automatica*, vol. 48, no. 8, pp. 1741–1750, 2012.

[8] J. Chen and R. Patton, "Standard H_∞ filtering formulation of robust fault detection," in *Proceedings of SAFEPROCESS*, Budapest, Hungary, 2000.

[9] M.-Y. Zhong, S. Liu, and H.-H. Zhao, "Krein space-based H_∞ fault estimation for linear discrete time-varying systems," *Acta Automatica Sinica*, vol. 34, no. 12, pp. 1529–1533, 2008.

[10] Y. Li and M. Zhong, "Fault detection filter design for linear discrete time-varying systems with multiplicative noise," *Journal of Systems Engineering and Electronics*, vol. 22, no. 6, pp. 982–990, 2011.

[11] H. Dong, Z. Wang, and H. Gao, "Fault detection for Markovian jump systems with sensor saturations and randomly varying nonlinearities," *IEEE Transactions on Circuits and Systems I*, vol. 59, no. 10, pp. 2354–2362, 2012.

[12] S. X. Ding, T. Jeinsch, P. M. Frank, and E. L. Ding, "A unified approach to the optimization of fault detection systems," *International Journal of Adaptive Control and Signal Processing*, vol. 14, no. 7, pp. 725–745, 2000.

[13] J. Wang, G. Yang, and J. Liu, "An LMI approach to H_- index and mixed H_-/H_∞ fault detection observer design," *Automatica*, vol. 43, no. 9, pp. 1656–1665, 2008.

[14] A. Casavola, D. Famularo, and G. Franzè, "Robust fault detection of uncertain linear systems via quasi-LMIs," *Automatica*, vol. 44, no. 1, pp. 289–295, 2008.

[15] P. Zhang, S. X. Ding, G. Z. Wang, and D. H. Zhou, "Fault detection of linear discrete-time periodic systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 2, pp. 239–244, 2005.

[16] P. Zhang, S. X. Ding, and P. Liu, "A lifting based approach to observer based fault detection of linear periodic systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 2, pp. 457–462, 2012.

[17] X. Li, *Fault Detection Filter Design for Linear Systems [Ph.D. thesis]*, Louisiana State University, 2009.

[18] M. Zhong, S. X. Ding, and E. L. Ding, "Optimal fault detection for linear discrete time-varying systems," *Automatica*, vol. 46, no. 8, pp. 1395–1400, 2010.

[19] M. Zhong and S. X. Ding, "A projection-based method of fault detection for linear discrete time-varying system," *International Journal of Systems Science*, 2011.

[20] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[21] L. Bakule and M. de la Sen, "Decentralized resilient H_∞ observer-based control for a class of uncertain interconnected networked systems," in *Proceedings of the American Control Conference*, Baltimore, Md, USA, 2010.

[22] K. You, M. Fu, and L. Xie, "Mean square stability for Kalman filtering with Markovian packet losses," *Automatica*, vol. 47, no. 12, pp. 2647–2657, 2011.

[23] Y. Mo and B. Sinopoli, "Kalman filtering with intermittent observations: tail distribution and critical value," *IEEE Transactions on Automatic Control*, vol. 57, no. 3, pp. 677–689, 2012.

[24] H. Zhang, X. Song, and L. Shi, "Convergence and mean square stability of suboptimal estimator for systems with measurement packet dropping," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1248–1253, 2012.

[25] Y. Zhang, C. Liu, and X. Mu, "On stochastic finite-time control of discrete-time fuzzy systems with packet dropout," *Discrete Dynamics in Nature and Society*, vol. 2012, Article ID 752950, 18 pages, 2012.

[26] H. Gao, T. Chen, and L. Wang, "Robust fault detection with missing measurements," *International Journal of Control*, vol. 81, no. 5, pp. 804–819, 2008.

[27] Y. Zhao, J. Lam, and H. Gao, "Fault detection for fuzzy systems with intermittent measurements," *IEEE Transaction on Fuzzy Systems*, vol. 17, no. 2, pp. 398–410, 2009.

[28] X. Yao, L. Wu, and W. X. Zheng, "Fault detection filter design for Markovian jump singular systems with intermittent measurements," *IEEE Transactions on Signal Processing*, vol. 59, no. 7, pp. 3099–3109, 2011.

[29] D. Du, B. Jiang, and P. Shi, "Fault detection for discrete-time switched systems with intermittent measurements," *International Journal of Control*, vol. 85, no. 1, pp. 78–87, 2012.

[30] X. He, Z. Wang, Y. Ji, and D. Zhou, "Robust fault detection for networked systems with distributed sensors," *IEEE Transaction on Aerospace and Electronic Systems*, vol. 47, no. 1, pp. 166–177, 2008.

[31] X. He, Z. Wang, and D. H. Zhou, "Robust fault detection for networked systems with communication delay and data missing," *Automatica*, vol. 45, no. 11, pp. 2634–2639, 2009.

[32] D. Zhou, X. He, Z. Wang et al., "Leakage fault diagnosis for an internet-based three-tank system: an experimental study," *IEEE Transaction on Control Systems Technology*, vol. 20, no. 4, pp. 857–870, 2011.

[33] X. Wan and H. Fang, "Fault detection for networked nonlinear systems with time delays and packet dropouts," *Circuits, Systems, and Signal Processing*, vol. 31, no. 1, pp. 329–345, 2012.

[34] X. Wan, H. Fang, and F. Yang, "Fault detection for a class of networked nonlinear systems subject to imperfect measurements," *International Journal of Control, Automation, and Systems*, vol. 10, no. 2, pp. 265–274, 2012.

[35] W. Zhang, Y. Huang, and L. Xie, "Infinite horizon stochastic H_2/H_∞ control for discrete-time systems with state and disturbance dependent noise," *Automatica*, vol. 44, no. 9, pp. 2306–2316, 2008.

[36] Y. Huang, W. Zhang, and H. Zhang, "Infinite horizon LQ optimal control for discrete-time stochastic systems," in *Proceedings of the 6th World Congress on Intelligent Control and Automation*, Dalian, China, 2006.

[37] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Prentice Hall, Upper Saddle River, NJ, USA, 1995.

[38] X. Li and K. Zhou, "A time domain approach to robust fault detection of linear time-varying systems," *Automatica*, vol. 45, no. 1, pp. 94–102, 2009.

[39] Y. Li and M. Zhong, "On designing robust H_∞ fault detection filter for linear discrete time-varying systems with multiple packet dropouts," *Acta Automatica Sinica*, vol. 36, no. 12, pp. 1788–1796, 2010.

[40] E. Gershon, U. Shaked, and I. Yaesh, "H_∞ control and filtering of discrete-time stochastic systems with multiplicative noise," *Automatica*, vol. 37, no. 3, pp. 409–417, 2001.

[41] A. Bobrowski, *Functional Analysis for Probability and Stochastic Processes. An Introduction*, Cambridge University Press, Cambridge, UK, 2005.

[42] V. Dragan, T. Morozan, and A.-M. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems*, Springer, New York, NY, USA, 2010.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 165727, 9 pages
http://dx.doi.org/10.1155/2013/165727

Research Article

Option Pricing under Risk-Minimization Criterion in an Incomplete Market with the Finite Difference Method

Xinfeng Ruan,[1] Wenli Zhu,[1] Shuang Li,[2] and Jiexiang Huang[1]

[1] *School of Economic Mathematics, Southwestern University of Finance and Economics, Chengdu 611130, China*
[2] *Department of Mathematics and Statistics, Curtin University, Perth, WA 6102, Australia*

Correspondence should be addressed to Xinfeng Ruan; ruanxinf@gmail.com

Received 16 January 2013; Revised 22 March 2013; Accepted 13 April 2013

Academic Editor: Guangchen Wang

We study option pricing with risk-minimization criterion in an incomplete market where the dynamics of the risky underlying asset is governed by a jump diffusion equation with stochastic volatility. We obtain the Radon-Nikodym derivative for the minimal martingale measure and a partial integro-differential equation (PIDE) of European option. The finite difference method is employed to compute the European option valuation of PIDE.

1. Introduction

Option pricing problem is one of the predominant concerns in the financial market. Since the advent of the justly celebrated Black-Scholes option pricing formula in [1], there has been an increasing amount of the literature describing the theory and its practice. In the Black-Scholes model, the appreciation rate and the volatility rate are assumed to be constants. However, more and more empirical evidence has revealed that these assumptions are not consistent with reality. Hence, many option valuation models which relax some of the restrictive assumptions in the Black-Scholes framework have been proposed and tested, such as the stochastic volatility models in [2–8], the jump diffusion or the Lévy process models in [9–12]. The model to be studied in this paper takes into account not only stochastic volatility based on Heston's model in [8], but also the jump diffusion case. Therefore, several new problems and difficulties are exposed in our model.

Different from the Black-Scholes framework which uses jump diffusion to describe the price dynamics of underlying asset, the market of our model is incomplete; that is, it is not possible to replicate the payoff of every contingent claim by a portfolio, and there are several equivalent martingale measures. How to choose a consistent pricing measure from the set of equivalent martingale measures becomes an important problem. This means that we need to find some criterions to determine one from the set of equivalent martingale measures in some economically or mathematically motivated fashion. Föllmer and Leukert (2000), Kallsen (1999), Cvitanić et al. (2001), and Bielecki and Jeanblanc (2008) in [13–16] identified a unique equivalent martingale measure by utility maximization. Then, the option valuation under the minimal martingale measure was further developed by several researchers. Schweizer (1991) and Föllmer and Schweizer (1991) in [17, 18] found that under the minimal martingale measure, a unique risk-minimizing (or optimal) strategy hedging of contingent claims in incomplete market exists. In our paper, we name the criterion under the minimal martingale measure as a *risk-minimization criterion*. Thus, in the incomplete market, option pricing is approximately possible with risk-minimization criterion. As presented in this paper, our work is based on the task of Föllmer and Schweizer in [17, 18], and the purpose of this paper is to find the minimal martingale measure and the measure switch of asset prices processes with stochastic volatility and jump diffusion. By employing the minimal martingale measure, we obtain the Radon-Nikodym derivative and a partial integro-differential equation (PIDE) for the European option.

However, it is difficult to get the exact solution of the PIDE in our model. Several numerical methods have been proposed to solve the PIDE approximately. The methods

include numerical integration by Chiarella and Ziogas (2009) in [19], finite elements by Matache et al. (2005) in [20], the method of lines by Meyer (1998) in [21], and the finite difference methods (FDM) including those by Carr and Hirsa (2003) in [22], d'Halluin et al. (2004) in [23], and Briani et al. (2004) in [24]. This paper employs the FDM to compute the valuation of European option, and American option. Then further studies on using other methods to compute the solution of PIDE and derive the pricing formula of another option will be included in our future study.

The rest of the paper is organized as follows. In Section 2, we present the model for the underlying market and Doob-Meyer decomposition of the risky asset. In Section 3, we investigate an explicit representation of the density process of the minimal martingale measure. In Section 4, we derive a PIDE for the European option. The European option pricing with FDM is then studied in Section 5. Numerical results are shown in Section 6, and conclusions are given in Section 7.

2. The Model

In this paper, we consider the financial market with the following two basic assets:

(i) *a Bond* whose price B_t at time t is given by

$$dB_t = r_t B_t dt, \quad B_0 = 1; \tag{1}$$

(ii) *a Stock* whose price S_t at time t is given by

$$\frac{dS_t}{S_{t^-}} = \mu_t dt + \sqrt{V_t} dW_t^S + \int_{R_0} (y-1)\overline{N}(du,dy), \quad S_0 > 0,$$
$$dV_t = \kappa(\varphi - V_t)dt + \sigma\sqrt{V_t}dW_t^V, \quad V_0 > 0. \tag{2}$$

Here, $t \in [0,T]$, $y \in R_0 \subset \mathbb{R}/\{0\}$, and $T > 0$; on the filtered complete space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$, there are W_t^S, W_t^V which both are 1-dimensional Brownian motions with $dW_t^S dW_t^V = \rho dt$ and $\overline{N}(dt,dy) = N(dt,dy) - \nu(dy)dt$ which is the compensated jump measure of $\eta(\cdot)$. $N(dt,dy)$ is the jump measure, and $\nu(dy)$ is the Lévy measure of the Lévy process $\eta(\cdot)$. $\eta(t)$ is given by $\eta(t) = \int_0^t \int_{R_0}(y-1)\overline{N}(du,dy)$, $t \geq 0$. Additionally, $\int_{R_0}(y-1)^2\nu(dy) < \infty$.

From above settings, the risk is at least two dimensional including the Brownian motion and Lévy process but only one risky asset in the market. It is impossible to exactly replicate the payoff of a given option by a dynamic portfolio strategy of the basic assets which is self-financing. We all know that if the dynamic portfolio strategy exists, then the initial cost of the portfolio must equal the price of the option, and the market is called complete; otherwise, an arbitrage opportunity exists, and the market is called incomplete. Therefore, the market of our model is incomplete since the perfect replication is impossible.

Compared with the Black-Scholes model, our model remedies some serious drawbacks of the Black-Scholes model, such as the constant volatility assumption and market completeness assumption. Bakshi et al. in [10] systematically analyzed the performance of the stochastic volatility model, the jump diffusion model, and the stochastic interest rate model, and concluded that a model with stochastic volatility and the jump diffusion was a better alternative to the Black-Scholes model, because the former not only performed far better but also was practically implementable. An excellent innovation of our model is combining stochastic volatility model and the jump diffusion model. Consequently, our model is better than the Black-Scholes model to portray realistic financial markets in theory and practice.

Now, we want to find a unique optimal strategy hedging of contingent claims under *risk-minimization criterion*. Based on [18], it is equivalent to find the minimal martingale measure from the set of equivalent martingale measures and then obtain approximate prices of contingent claims.

With the Doob-Meyer decomposition, the discounted risky asset price process, $\widehat{S}_t = e^{-\int_0^t r_s ds} S_t$, is a special semimartingale and can be written as

$$\widehat{S}_t = \widehat{S}_0 + M_t + A_t, \tag{3}$$

with

$$M_t = \int_0^t \widehat{S}_{u^-}\sqrt{V_u}dW_u^S + \int_0^t\int_{R_0} \widehat{S}_{u^-} y\overline{N}(du,dy), \tag{4}$$

$$A_t = \int_0^t \widehat{S}_{u^-}(\mu_u - r_u)du, \tag{5}$$

where M_t is the martingale part of $\widehat{S}_t$ and A_t is the predictable process of finite variation.

3. Minimal Martingale Measure

We introduce the nations of minimal martingale measure in this section. Föllmer and Schweizer (1991) [18] noticed that the optimal hedging strategy can be computed in terms of the minimal martingale measure. Furthermore, it is uniquely determined. Hence, under the minimal martingale measure, the Radon-Nikodym derivative can be found and computed. Before that, we define the minimal martingale measure.

Definition 1 (see [18]). A local martingale measure $\widetilde{P}$, equivalent to the original measure P, is called minimal if $\widetilde{P} = P$ on $\mathscr{F}_t$ and if any square-integrable P-martingale L which is P orthogonal to M remains a local martingale under $\widetilde{P}$.

Theorem 2 (see [18]). *(i) The minimal martingale measure $\widetilde{P}$ is uniquely determined.*

(ii) $\widetilde{P}$ exists if and only if there exists a predictable process β_t that satisfies

$$Z_t = \frac{d\widetilde{P}}{dP} = 1 + \int_0^t \beta_t dM_t. \tag{6}$$

Using Theorem 2, we obtain the following theorem for computing the Radon-Nikodym derivative.

Theorem 3. *The Radon-Nikodym derivative under the minimal martingale measure $\widetilde{P}$ is*

$$Z_t = \exp\left\{-\int_0^t \theta_t \sqrt{V_u} dW_u^S - \frac{1}{2}\int_0^t \theta_u^2 V_u du + \int_0^t \int_{R_0} \ln\left(1-\theta_u (y-1)\right) N(du,dy) + \int_0^t \int_{R_0} \theta_u (y-1)\, v(dy)\, du\right\}. \quad (7)$$

Proof. The theory of the Girsanov transformation shows that the predictable process of bounded variation can also be computed in terms of Z_t:

$$-dA_t = \frac{1}{Z_{t-}} d\langle M, Z\rangle_t. \quad (8)$$

Throughout this paper, we make use of the notations that $\langle A, B\rangle_t$ defined the quadratic variation process between A and B and denote $\langle A\rangle_t \doteq \langle A, A\rangle_t$. Under $\widetilde{P}$, the predictable process of bounded variation in the Doob-Meyer decomposition of M is given by

$$\frac{1}{Z_{t-}} d\langle M, Z\rangle_t = \frac{1}{Z_{t-}} d\langle M\rangle_t = -dA_t. \quad (9)$$

Using (6) and (9), we have

$$Z_t = 1 - \int_0^t Z_{u-} \frac{dA_u}{d\langle M\rangle_u} dM_u. \quad (10)$$

Denote $dY_u = -(dA_u/d\langle M\rangle_u) dM_u$, then (10) can be written as

$$Z_t = 1 + \int_0^t Z_{u-} dY_u. \quad (11)$$

From (4), we get

$$\begin{aligned} \langle M\rangle_t &= \left\langle \int_0^t \widehat{S}_{u^-} \sqrt{V_u} dW_u^S + \int_0^t \int_{R_0} \widehat{S}_{u^-} (y-1)\, \overline{N}(du,dy)\right\rangle \\ &= \int_0^t \widehat{S}_{u^-}^2 \left(\sqrt{V_u}\right)^2 du + \int_0^t \int_{R_0} \widehat{S}_{u^-}^2 (y-1)^2 v(dy)\, du \\ &= \int_0^t \widehat{S}_{u^-}^2 \left(V_u + \int_{R_0} (y-1)^2 v(dy)\right) du. \end{aligned} \quad (12)$$

Hence,

$$\begin{aligned} dY_u &= -\frac{dA_u}{d\langle M\rangle_u} dM_u \\ &= -\frac{\widehat{S}_{u^-}(\mu_u - r_u)\, du}{\widehat{S}_{u^-}^2 \left(v_u + \int_{R_0} (y-1)^2 v(dy)\right) du} \times \left[\widehat{S}_{u^-}\left(\sqrt{V_u} dW_u^S + \int_{R_0} y\overline{N}(du,dy)\right)\right] \\ &= -\frac{(\mu_u - r_u)\left(\sqrt{V_u} dW_u^S + \int_{R_0} (y-1)\, \overline{N}(du,dy)\right)}{V_u + \int_{R_0} (y-1)^2 v(dy)}. \end{aligned} \quad (13)$$

From (11), we know that Z_t is the Doléans-Dade exponential. Thus, we obtain

$$\begin{aligned} Z_t &= 1 + \int_0^t Z_{u-} dY_u, \quad Z_0 = 1, \\ dY_u &= -\theta_u \left(\sqrt{V_u} dW_u^S + \int_{R_0} (y-1)\, \overline{N}(du,dy)\right), \end{aligned} \quad (14)$$

where

$$\theta_u = \frac{(\mu_u - r_u)}{V_u + \int_{R_0} (y-1)^2 v(dy)}. \quad (15)$$

Solving (14), we obtain Z_t in Theorem 3. □

Remark 4. The Brown motions under the minimal martingale measure $\widetilde{P}$ are

$$\begin{aligned} \widetilde{W}_t^S &= W_t^S + \int_0^t \theta_t \sqrt{V_u} du, \\ \widetilde{W}_t^V &= W_t^V + \rho \int_0^t \theta_t \sqrt{V_u} du, \end{aligned} \quad (16)$$

and the compensatory of $N(du, dy)$ is

$$\begin{aligned} \widetilde{v}(dy)\, du &= \left(1 - \theta_u (y-1)\right) v(dy)\, du, \\ \widetilde{N}(du,dy) &= N(du,dy) - \widetilde{v}(dy)\, du. \end{aligned} \quad (17)$$

Remark 5. Equation (2) under the minimal martingale measure $\widetilde{P}$ is written as

$$\begin{aligned} \frac{dS_t}{S_{t^-}} &= \mu_t dt + \sqrt{V_t} d\widetilde{W}_t^S + \int_{R_0} (y-1)\, \widetilde{N}(dt,dy) - \theta_t V_t dt - \int_{R_0} \theta_t (y-1)^2 v(dy)\, dt, \quad S_0 > 0, \\ dV_t &= \kappa(\varphi - V_t)\, dt + \sigma\sqrt{V_t} d\widetilde{W}_t^V - \rho\sigma\theta_t V_t dt. \end{aligned} \quad (18)$$

To guarantee that $\widehat{S}_t$ is a martingale under the minimal martingale measure $\widetilde{P}$, the following corollary is necessary.

Corollary 6. *Under the minimal martingale measure* $\widetilde{P}$, $\widehat{S}_t$ *is a martingale if and only if*

$$\mu_t - r_t - \theta_t V_t - \int_{R_0} \theta_t (y-1)^2 \nu(dy) = 0. \tag{19}$$

Proof. Substituting $S_t = e^{\int_0^t r_s ds}\widehat{S}_t$ into (18), since $\widehat{S}_t$ is a martingale, the drift term must be identical to zero. Then, we can get (19). □

4. Partial Integro-Differential Equation for European Call Option

Under the minimal martingale measure $\widetilde{P}$, the price of the European call option $C(t, S_t, V_t)$ at time t with strike price K and maturity date T is given by

$$C(t, S_t, V_t) = \mathbb{E}^{\widetilde{P}}\left[e^{-\int_t^T r_u du}(S_T - K)^+ \mid \mathscr{F}_t\right], \tag{20}$$

and $C(T, S_T, V_T) = (S_T - K)^+$.

By the fact that the discounted price of the European call option is a martingale under $\widetilde{P}$, we can obtain the following theorem.

Theorem 7. *The price of the European call option satisfies the following PIDE:*

$$\begin{aligned}0 = & -r_t C(t, S_t, V_t) + \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} r_t S_{t^-} \\ & + \frac{\partial C}{\partial V}(\kappa(\varphi - V_t) - \rho\sigma\theta_t V_t) \\ & + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} V_t S_{t^-}^2 + \frac{\partial^2 C}{\partial S \partial V} S_{t^-} V_t \sigma\rho \\ & + \frac{1}{2}\frac{\partial^2 C}{\partial V^2} V_t \sigma^2 + \int_{R_0}\Big(C(t, S_{t^-} y, V_{t^-}) - C(t, S_{t^-}, V_{t^-}) \\ & - (y-1)\frac{\partial C}{\partial S} S_{t^-}\Big)\tilde{\nu}(dy),\end{aligned} \tag{21}$$

and $C(T, S_T, V_T) = (S_T - K)^+$.

Proof. The total derivative of the discounted option price is

$$\begin{aligned} & d\left(e^{-\int_0^t r_u du} C(t, S_t, V_t)\right) \\ & = -r_t e^{-\int_0^t r_u du} C(t, S_t, V_t)\, dt \\ & + e^{-\int_0^t r_u du}\frac{\partial C}{\partial t} dt + e^{-\int_0^t r_u du}\frac{\partial C}{\partial S} dS^C \\ & + e^{-\int_0^t r_u du}\frac{\partial C}{\partial V} dV^C + \frac{1}{2} e^{-\int_0^t r_u du}\frac{\partial^2 C}{\partial S^2} dS^C dS^C \\ & + e^{-\int_0^t r_u du}\frac{\partial^2 C}{\partial S \partial V} dS^C dV^C + \frac{1}{2} e^{-\int_0^t r_u du}\frac{\partial^2 C}{\partial V^2} dV^C dV^C \\ & + e^{-\int_0^t r_u du}\int_{R_0}(C(t, S_{t^-} y, V_{t^-}) \\ & - C(t, S_{t^-}, V_{t^-}))\, N(dt, dy) \\ & = e^{-\int_0^t r_u du}\Big\{-r_t C(t, S_t, V_t)\, dt + \frac{\partial C}{\partial t} dt \\ & + \frac{\partial C}{\partial S} S_{t^-}\Big(\mu_t dt + \sqrt{V_t} d\widetilde{W}_t^S - \theta_t V_t dt \\ & - \int_{R_0}\theta_t (y-1)^2 \nu(dy)\, dt \\ & - \int_{R_0}(y-1)\tilde{\nu}(dy)\, dt\Big) \\ & + \frac{\partial C}{\partial V}\Big(\kappa(\varphi - V_t)\, dt + \sigma\sqrt{V_t} d\widetilde{W}_t^V \\ & - \rho\sigma\theta_t V_t dt\Big) \\ & + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} V_t S_{t^-}^2 dt + \frac{\partial^2 C}{\partial S \partial V} S_{t^-} V_t \sigma\rho dt \\ & + \frac{1}{2}\frac{\partial^2 C}{\partial V^2} V_t \sigma^2 dt \\ & + \int_{R_0}(C(t, S_{t^-} y, V_{t^-}) \\ & - C(t, S_{t^-}, V_{t^-}))\, \widetilde{N}(dt, dy) \\ & + \int_{R_0}(C(t, S_{t^-} y, V_{t^-}) \\ & - C(t, S_{t^-}, V_{t^-}))\, \tilde{\nu}(dy)\, dt\Big\} \\ & = e^{-\int_0^t r_u du}\Big\{-r_t C(t, S_t, V_t) + \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} r_t S_{t^-} \\ & + \frac{\partial C}{\partial V}(\kappa(\varphi - V_t) - \rho\sigma\theta_t V_t) \\ & + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} V_t S_{t^-}^2 + \frac{\partial^2 C}{\partial S \partial V} S_{t^-} V_t \sigma\rho \\ & + \frac{1}{2}\frac{\partial^2 C}{\partial V^2} V_t \sigma^2 \\ & + \int_{R_0}\Big(C(t, S_{t^-} y, V_{t^-}) \\ & - C(t, S_{t^-}, V_{t^-}) \\ & - (y-1)\frac{\partial C}{\partial S} S_{t^-}\Big)\tilde{\nu}(dy)\Big\}\, dt \\ & + e^{-\int_0^t r_u du}\Big\{\frac{\partial C}{\partial S} S_{t^-}\sqrt{v_t} d\widetilde{W}_t^S + \frac{\partial C}{\partial V}\sigma\sqrt{V_t} d\widetilde{W}_t^V \\ & + \int_{R_0}(C(t, S_{t^-} y, V_{t^-}) \\ & - C(t, S_{t^-}, V_{t^-}))\, \widetilde{N}(dt, dy)\Big\}.\end{aligned} \tag{22}$$

We make the drift term zero, since the discounted price of the European put option is a martingale. Then, we obtain in Theorem 7. □

5. Finite Difference Scheme of European Call Option

5.1. The Problem. In this section, we will derive the finite difference scheme for PIDE (21). Before that, we assume that $r_t = r$, $\mu_t = \mu$, $R_0 = [0, \infty]$ and $\nu(dy)dt = \lambda f(y)dydt$. In this paper, we take Merton's jump diffusion model with the density $f(y)$ which is from the log-normal distribution:

$$f(y) = \frac{1}{y\delta\sqrt{2\pi}} \exp\left(-\frac{(\ln y - \gamma)^2}{2\delta^2}\right). \tag{23}$$

Furthermore, using error function, we denote that

$$\begin{aligned}
K_0 &:= \int_{R_0} \nu(dy) = 1, \\
K_1 &:= \int_{R_0} (y-1)\nu(dy) = e^{(\delta^2/2)+\gamma} - 1, \\
K_2 &:= \int_{R_0} (y-1)^2 \nu(dy) \\
&= \frac{1}{\sqrt{1-2\delta^2}} e^{\gamma^2/(1-2\delta^2)} - 2e^{(\delta^2/2)+\gamma} + 1, \\
\theta_u &:= \frac{\mu - r}{V_u + \int_{R_0} (y-1)^2 \nu(dy)} = \frac{\mu - r}{V_u + K_2},
\end{aligned} \tag{24}$$

where r, μ, λ, δ, γ, K_0, K_1, and K_2 are constants, and $\delta^2 < 1/2$.

From (21) and (19), we get the PIDE with original parameters in the original model (2) as follows:

$$\begin{aligned}
0 = & -(r_t + K_0 - \theta_t K_1) C(t, S_t, V_t) \\
& + \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r_t - K_1 + \theta_t K_2) S_{t^-} \\
& + \frac{\partial C}{\partial V}(\kappa(\varphi - V_t) - \rho\sigma\theta_t V_t) \\
& + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} V_t S_{t^-}^2 + \frac{\partial^2 C}{\partial S \partial V} S_{t^-} V_t \sigma\rho + \frac{1}{2}\frac{\partial^2 C}{\partial V^2} V_t \sigma^2 \\
& + \int_{R_0} C(t, S_{t^-} y, V_{t^-}) \nu(dy) \\
& - \theta_t \int_{R_0} C(t, S_{t^-} y, V_{t^-})(y-1)\nu(dy).
\end{aligned} \tag{25}$$

To represent conveniently, we denote that

$$\begin{aligned}
\mathrm{I}(t, S, V) &:= \int_0^\infty C(t, S_{t^-}, V_{t^-}) f(y) dy, \\
\mathrm{II}(t, S, V) &:= \int_0^\infty C(t, S_{t^-}, V_{t^-})(y-1) f(y) dy.
\end{aligned} \tag{26}$$

Then, (25) can be written as

$$\begin{aligned}
0 = & -(r + K_0 - \theta K_1) C(t, S, V) + \frac{\partial C}{\partial t} \\
& + \frac{\partial C}{\partial S}(r - K_1 + \theta K_2) S \\
& + \frac{\partial C}{\partial V}(\kappa(\varphi - V) - \rho\sigma\theta V) \\
& + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} V S^2 + \frac{\partial^2 C}{\partial S \partial V} S V \sigma\rho \\
& + \frac{1}{2}\frac{\partial^2 C}{\partial V^2} V\sigma^2 + \lambda \mathrm{I}(t, S, V) - \lambda\theta \mathrm{II}(t, S, V), \\
& C(T, S, V) = (S - K)^+, \\
& C(t, 0, V) = 0, \\
& \lim_{S\to\infty} C(t, S, V) = (S_{\max} - K) e^{-r(T-t)}, \\
& \lim_{V\to\infty} C(t, S, V) = S, \\
0 = & -\left(r + K_0 - \frac{(\mu - r) K_1}{K_2}\right) C(t, S, 0) \\
& + \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r - K_1 + (\mu - r)) S \\
& + \frac{\partial C}{\partial V}\kappa\varphi + \lambda \mathrm{I}(t, S, 0) - \lambda\frac{(\mu - r)}{K_2} \mathrm{II}(t, S, 0).
\end{aligned} \tag{27}$$

Now, we denote $\tau := T - t$, $\widetilde{C}(\tau, S, V) := C(T - \tau, S, V) = C(t, S, V)$, $\widetilde{\mathrm{I}}(t, S, V) := \mathrm{I}(T - \tau, S, V) = \mathrm{I}(t, S, V)$, and $\widetilde{\mathrm{II}}(t, S, V) := \mathrm{II}(T - \tau, S, V) = \mathrm{II}(t, S, V)$. Then we have the following problem equivalent to (27).

Problem 1. Consider

$$\begin{aligned}
\frac{\partial \widetilde{C}}{\partial \tau} = & -(r + K_0 - \theta K_1) \widetilde{C}(\tau, S, V) \\
& + \frac{\partial \widetilde{C}}{\partial S}(r - K_1 + \theta K_2) S \\
& + \frac{\partial \widetilde{C}}{\partial V}(\kappa(\varphi - V) - \rho\sigma\theta V) + \frac{1}{2}\frac{\partial^2 \widetilde{C}}{\partial S^2} V S^2 \\
& + \frac{\partial^2 \widetilde{C}}{\partial S \partial V} S V \sigma\rho + \frac{1}{2}\frac{\partial^2 \widetilde{C}}{\partial V^2} V\sigma^2 \\
& + \lambda \widetilde{\mathrm{I}}(\tau, S, V) - \lambda\theta \widetilde{\mathrm{II}}(\tau, S, V), \\
& \widetilde{C}(0, S, V) = (S - K)^+, \\
& \widetilde{C}(\tau, 0, V) = 0, \\
& \lim_{S\to\infty} \widetilde{C}(\tau, S, V) = (S_{\max} - K) e^{-r\tau}, \\
& \lim_{V\to\infty} \widetilde{C}(\tau, S, V) = S,
\end{aligned}$$

$$0 = -\left(r + K_0 - \frac{(\mu - r) K_1}{K_2}\right)\widetilde{C}(\tau, S, 0) + \frac{\partial \widetilde{C}}{\partial \tau} + \frac{\partial \widetilde{C}}{\partial S}(r - K_1 + (\mu - r)) S + \frac{\partial \widetilde{C}}{\partial V}\kappa\varphi + \lambda \widetilde{\mathrm{I}}(\tau, S, 0) - \lambda \frac{(\mu - r)}{K_2}\widetilde{\mathrm{II}}(\tau, S, 0). \tag{28}$$

5.2. Discretization. To solve (38), we use a finite difference scheme with the following nation and approximations:

$$\begin{gathered}
\tau : 0 \leq \cdots n\Delta t \cdots \leq (N-1)\Delta t, \quad n = 0, 1, \ldots, N-1, \\
S : 0 \leq \cdots i\Delta X \cdots \leq (I-1)\Delta X = S_{\max}, \quad i = 0, 1, \ldots, I-1, \\
V : 0 \leq \cdots j\Delta V \cdots \leq J\Delta V = V_{\max}, \quad j = 0, 1, \ldots, J-1, \\
U_{i,j}^n = \widetilde{C}(n\Delta t, i\Delta X, j\Delta V), \\
\mathrm{I}_{i,j}^n = \widetilde{\mathrm{I}}(n\Delta t, i\Delta X, j\Delta V), \\
\mathrm{II}_{i,j}^n = \widetilde{\mathrm{II}}(n\Delta t, i\Delta X, j\Delta V), \\
\frac{\partial \widetilde{C}}{\partial \tau}(\tau, S, V) \approx \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t}, \\
\frac{\partial \widetilde{C}}{\partial S}(\tau, S, V) \approx \frac{U_{i+1,j}^n - U_{i-1,j}^n}{2\Delta X}, \\
\frac{\partial^2 \widetilde{C}}{\partial S^2}(\tau, S, V) \approx \frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{(\Delta X)^2}, \\
\frac{\partial^2 \widetilde{C}}{\partial S \partial V} = \frac{U_{i+1,j+1}^n + U_{i-1,j-1}^n - U_{i-1,j+1}^n - U_{i+1,j-1}^n}{4\Delta X \Delta V}, \\
\frac{\partial \widetilde{C}}{\partial V}(\tau, S, V) \approx \frac{U_{i,j+1}^n - U_{i,j-1}^n}{2\Delta V}, \\
\frac{\partial^2 \widetilde{C}}{\partial V^2}(\tau, S, V) \approx \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{(\Delta V)^2}.
\end{gathered} \tag{29}$$

Now, we evaluate the integral term. First, by performing a change of variable $z := S \cdot y$, $y = z/S$ and $dy = dz/S$, the $\widetilde{\mathrm{I}}(\tau, S_t, V_t)$ and $\widetilde{\mathrm{II}}(\tau, S_t, V_t)$ can be discretized by using the linear interpolation. Thus,

$$\begin{gathered}
z : 0 \leq \cdots k\Delta X \cdots \leq (I-2)\Delta X, \quad k = 0, 1, \ldots, I-2, \\
\widetilde{\mathrm{I}} = \int_{R_0} \frac{\widetilde{C}(\tau, z, V) f(z/S)}{S dz} \approx \int_0^\infty \frac{\widetilde{C}(n\Delta t, z, j\Delta V) f(z/i\Delta X)}{(i\Delta X) dz}, \\
\widetilde{\mathrm{II}} = \int_{R_0} \frac{\widetilde{C}(\tau, z, V)(z/S - 1) f(z/S)}{S dz} \approx \int_0^\infty \frac{\widetilde{C}(n\Delta t, z, j\Delta V)(z/(i\Delta X) - 1) f(z/i\Delta X)}{(i\Delta X) dz}.
\end{gathered} \tag{30}$$

Now, by using linear interpolation, we get approximation the following:

$$\mathrm{I}_{i,j}^n \approx A_{i,j}^n = \sum_{k=0}^{I-2} \widetilde{A}_{i,j}^{n,k}, \qquad \mathrm{II}_{i,j}^n \approx B_{i,j}^n = \sum_{k=0}^{I-2} \widetilde{B}_{i,j}^{n,k}, \tag{31}$$

where

$$\begin{aligned}
\widetilde{A}_{i,j}^{n,k} &= \int_{k\Delta X}^{(k+1)\Delta X} \frac{\widetilde{C}(n\Delta t, z, j\Delta V) f(z/i\Delta X)}{(i\Delta X) dz} \\
&= \int_{k\Delta X}^{(k+1)\Delta X} \frac{(k+1)\Delta X - z}{\Delta X} \times \frac{\widetilde{C}(n\Delta t, k\Delta X, j\Delta V) f(z/i\Delta X)}{(i\Delta X) dz} \\
&\quad + \int_{k\Delta X}^{(k+1)\Delta X} \frac{z - k\Delta X}{\Delta X} \times \frac{\widetilde{C}(n\Delta t, (k+1)\Delta X, j\Delta V) f(z/i\Delta X)}{(i\Delta X) dz} \\
&= \frac{1}{2}\left[k\widetilde{C}(n\Delta t, (k+1)\Delta X, j\Delta V) - (k+1)\widetilde{C}(n\Delta t, k\Delta X, j\Delta V)\right] \\
&\quad \cdot \left[\operatorname{erf}\left(\frac{(\gamma - \ln((k+1)/i))}{\sqrt{2}\delta}\right) - \operatorname{erf}\left(\frac{(\gamma - \ln(k/i))}{\sqrt{2}\delta}\right)\right] \\
&\quad + \frac{1}{2} i e^{\delta^2/2+\gamma}\left[\widetilde{C}(n\Delta t, k\Delta X, j\Delta V) - \widetilde{C}(n\Delta t, (k+1)\Delta X, j\Delta V)\right] \\
&\quad \cdot \left[\operatorname{erf}\left(\frac{(\gamma - \ln((k+1)/i) + \delta^2)}{\sqrt{2}\delta}\right) - \operatorname{erf}\left(\frac{(\gamma - \ln(k/i) + \delta^2)}{\sqrt{2}\delta}\right)\right].
\end{aligned} \tag{32}$$

Denote the following:

$$\begin{aligned}
E_{i,j}^{n,k} &= \frac{1}{2}\left[kU_{k+1,j}^{n} - (k+1)\,U_{k,j}^{n}\right],\\
F_{i,j}^{n,k} &= \frac{1}{2}\left[U_{k,j}^{n} - U_{k+1,j}^{n}\right],\\
G_{i}^{k} &= \left[\operatorname{erf}\left(\frac{\left(\gamma - \ln\left(\left(k+1\right)/i\right)\right)}{\sqrt{2}\delta}\right)\right.\\
&\qquad \left. - \operatorname{erf}\left(\frac{\left(\gamma - \ln\left(k/i\right)\right)}{\sqrt{2}\delta}\right)\right],\\
H_{i}^{k} &= \left[\operatorname{erf}\left(\frac{\left(\gamma - \ln\left(\left(k+1\right)/i\right) + \delta^{2}\right)}{\sqrt{2}\delta}\right)\right.\\
&\qquad \left. - \operatorname{erf}\left(\frac{\left(\gamma - \ln\left(k/i\right) + \delta^{2}\right)}{\sqrt{2}\delta}\right)\right].
\end{aligned} \tag{33}$$

Then, we have $A_{i,j}^{n,k} = E_{i,j}^{n,k} * G_{i}^{k} + ie^{(\delta^2/2)+\gamma}F_{i,j}^{n,k} * H_{i}^{k}$

$$\begin{aligned}
&\widetilde{B}_{i,j}^{n,k}\\
&= \int_{k\Delta X}^{(k+1)\Delta X} \frac{\widetilde{C}\left(n\Delta t, z, j\Delta V\right)\left(z/(i\Delta X)-1\right) f\left(z/(i\Delta X)-1\right)}{(i\Delta X)\, dz}\\
&= \int_{k\Delta X}^{(k+1)\Delta X} \frac{(k+1)\,\Delta X - z}{\Delta X}\\
&\qquad \times \widetilde{C}\left(n\Delta t, k\Delta X, j\Delta V\right)\\
&\qquad \times \left(\frac{z}{i\Delta X} - 1\right)\frac{1}{z\delta\sqrt{2\pi}}\\
&\qquad \times \exp\left(-\frac{\left(\ln\left(z/i\Delta X\right) - \gamma\right)^{2}}{2\delta^{2}}\right) dz\\
&\quad + \int_{k\Delta X}^{(k+1)\Delta X} \frac{z - k\Delta X}{\Delta X}\\
&\qquad \times \widetilde{C}\left(n\Delta t, (k+1)\,\Delta X, j\Delta V\right)\left(\frac{z}{i\Delta X} - 1\right)\\
&\qquad \times \frac{1}{z\delta\sqrt{2\pi}} \exp\left(-\frac{\left(\ln\left(z/i\Delta X\right) - \gamma\right)^{2}}{2\delta^{2}}\right) dz\\
&= \frac{1}{2}e^{(\delta^2/2)+\gamma}\left[k\widetilde{C}\left(n\Delta t, (k+1)\,\Delta X, j\Delta V\right)\right.\\
&\qquad \left. - (k+1)\,\widetilde{C}\left(n\Delta t, k\Delta X, j\Delta V\right)\right]\\
&\quad \cdot \left[\operatorname{erf}\left(\frac{\left(\gamma - \ln\left(\left(k+1\right)/i\right) + \delta^{2}\right)}{\sqrt{2}\delta}\right)\right.\\
&\qquad \left. - \operatorname{erf}\left(\frac{\left(\gamma - \ln\left(k/i\right) + \delta^{2}\right)}{\sqrt{2}\delta}\right)\right]\\
&\quad + \frac{1}{2i\sqrt{1-2\delta^{2}}} e^{\gamma^{2}/(1-2\delta^{2})}\\
&\qquad \times \left[\widetilde{C}\left(n\Delta t, k\Delta X, j\Delta V\right)\right.\\
&\qquad \left. -\widetilde{C}\left(n\Delta t, (k+1)\,\Delta X, j\Delta V\right)\right]\\
&\quad \cdot \left[\operatorname{erf}\left(\sqrt{1-2\delta^{2}}\frac{\left(\gamma - \ln\left(\left(k+1\right)/i\right)\right)}{\sqrt{2}\delta} - \frac{\sqrt{2}\delta\gamma}{\sqrt{1-2\delta^{2}}}\right)\right.\\
&\qquad \left. - \operatorname{erf}\left(\sqrt{1-2\delta^{2}}\frac{\left(\gamma - \ln\left(k/i\right)\right)}{\sqrt{2}\delta} - \frac{\sqrt{2}\delta\gamma}{\sqrt{1-2\delta^{2}}}\right)\right] - \widetilde{A}_{i,j}^{n,k}.
\end{aligned} \tag{34}$$

Denote the following:

$$\begin{aligned}
L_{i}^{k} = &\left[\operatorname{erf}\left(\sqrt{1-2\delta^{2}}\frac{\left(\gamma - \ln\left(\left(k+1\right)/i\right)\right)}{\sqrt{2}\delta} - \frac{\sqrt{2}\delta\gamma}{\sqrt{1-2\delta^{2}}}\right)\right.\\
&\left. - \operatorname{erf}\left(\sqrt{1-2\delta^{2}}\frac{\left(\gamma - \ln\left(k/i\right)\right)}{\sqrt{2}\delta} - \frac{\sqrt{2}\delta\gamma}{\sqrt{1-2\delta^{2}}}\right)\right].
\end{aligned} \tag{35}$$

Then, we have $B_{i,j}^{n,k} = e^{(\delta^2/2)+\gamma}E_{i,j}^{n,k} * H_{i}^{k} + (1/i\sqrt{1-2\delta^2})\, e^{\gamma^2/(1-2\delta^2)}F_{i,j}^{n,k} * L_{i}^{k} - A_{i,j}^{n,k}$, where $\operatorname{erf}(\cdot)$ is the error function defined by $\operatorname{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-y^2}dy$ with $\operatorname{erf}(\infty) = 1$ and $\operatorname{erf}(-x) = -\operatorname{erf}(x)$.

Then,

$$A_{i,j}^{n} = \sum_{k=0}^{I-2}\widetilde{A}_{i,j}^{n,k}, \qquad B_{i,j}^{n} = \sum_{k=0}^{I-2}\widetilde{B}_{i,j}^{n,k}. \tag{36}$$

Finally, discretization of (27) gives

$$\begin{aligned}
U_{i,j}^{n+1} = &-\left[-1 + \Delta t\left(r + K_0 - \frac{\mu - r}{j\Delta V + K_2}K_1\right)\right.\\
&\left. +\Delta t j\Delta V i^{2} + \frac{\Delta t j\sigma^{2}}{\Delta V}\right] U_{i,j}^{n}\\
&+ \left[\frac{\Delta t i}{2}\left(r - K_1 + \frac{\mu - r}{j\Delta V + K_2}K_2\right) + \frac{\Delta t j\Delta V i^{2}}{2}\right] U_{i+1,j}^{n}\\
&- \left[\frac{\Delta t i}{2}\left(r - K_1 + \frac{\mu - r}{j\Delta V + K_2}K_2\right) - \frac{\Delta t j\Delta V i^{2}}{2}\right] U_{i-1,j}^{n}\\
&+ \left[\frac{\Delta t}{2\Delta V}\left(\kappa\left(\varphi - j\Delta V\right) - \rho\sigma\frac{\mu - r}{j\Delta V + K_2}j\Delta V\right)\right.\\
&\left. +\frac{\Delta t j\sigma^{2}}{2\Delta V}\right] U_{i,j+1}^{n}
\end{aligned}$$

$$
\begin{aligned}
&- \left[\frac{\Delta t}{2\Delta V} \left(\kappa (\varphi - j\Delta V) - \rho\sigma \frac{\mu - r}{j\Delta V + K_2} j\Delta V \right) \right. \\
&\qquad \left. - \frac{\Delta t j \sigma^2}{2\Delta V} \right] U^n_{i,j-1} \\
&+ \left[\frac{\Delta t i j \sigma \rho}{4} \right] U^n_{i+1,j+1} + \left[\frac{\Delta t i j \sigma \rho}{4} \right] U^n_{i-1,j-1} \\
&- \left[\frac{\Delta t i j \sigma \rho}{4} \right] U^n_{i-1,j+1} - \left[\frac{\Delta t i j \sigma \rho}{4} \right] U^n_{i+1,j-1} \\
&+ \lambda \Delta t A^n_{i,j} - \frac{\lambda \Delta t (\mu - r)}{j\Delta V + K_2} B^n_{i,j}.
\end{aligned} \tag{37}
$$

It equals

$$
\begin{aligned}
U^{n+1}_{i,j} &= cU^n_{i,j} + dU^n_{i+1,j} + eU^n_{i-1,j} + gU^n_{i,j+1} + hU^n_{i,j-1} \\
&\quad + q\left(U^n_{i+1,j+1} + U^n_{i-1,j-1} - U^n_{i-1,j+1} - U^n_{i+1,j-1} \right) \\
&\quad + aA^n_{i,j} + bB^n_{i,j},
\end{aligned} \tag{38}
$$

where

$$
\begin{aligned}
c(i,j) &= 1 - \Delta t \left(r + K_0 - \frac{\mu - r}{j\Delta V + K_2} K_1 \right) \\
&\quad - \Delta t j \Delta V i^2 - \frac{\Delta t j \sigma^2}{\Delta V}, \\
d(i,j) &= \frac{\Delta t i}{2} \left(r - K_1 + \frac{\mu - r}{j\Delta V + K_2} K_2 \right) + \frac{\Delta t j \Delta V i^2}{2}, \\
e(i,j) &= -\frac{\Delta t i}{2} \left(r - K_1 + \frac{\mu - r}{j\Delta V + K_2} K_2 \right) + \frac{\Delta t j \Delta V i^2}{2}, \\
g(i,j) &= \frac{\Delta t}{2\Delta V} \left(\kappa (\varphi - j\Delta V) - \rho\sigma \frac{\mu - r}{j\Delta V + K_2} j\Delta V \right) + \frac{\Delta t j \sigma^2}{2\Delta V}, \\
h(i,j) &- \frac{\Delta t}{2\Delta V} \left(\kappa (\varphi - j\Delta V) - \rho\sigma \frac{\mu - r}{j\Delta V + K_2} j\Delta V \right) + \frac{\Delta t j \sigma^2}{2\Delta V}, \\
q(i,j) &= \frac{\Delta t i j \sigma \rho}{4}, \qquad a(i,j) = \lambda \Delta t, \\
b(i,j) &= -\frac{\lambda \Delta t (\mu - r)}{j\Delta V + K_2}.
\end{aligned} \tag{39}
$$

6. Numerical Results

In this section, we adopted the basic set of appropriate parameter values of the model shown in Table 1 as our basic set of parameter values to solve the finite difference equation (38).

Figure 1 shows the payoff with the price and the volatility. The solution of (38) is shown in Figure 2. All numerical experiments have been implemented by MATLAB R2011b software on a 2.0-GHz Intel Core PC.

Table 1: Basic set of appropriate parameter values of the model.

μ	0.23
κ	3.46
φ	$(0.0894)^2$
δ	0.0001
ρ	−0.82
T	1
$S_{\max}$	200
$V_{\max}$	1
r	0.039
σ	0.14
λ	0.47
γ	−0.086
K	100
N	2000
I	30
J	30

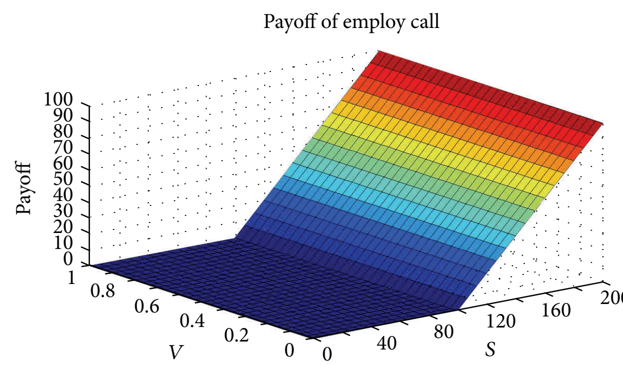


Figure 1: The payoff with the price and the volatility.

Table 2: European option prices.

Price	I
8.7858	10
9.3014	20
9.6929	30
9.7006	40

Now, we suppose that the underlying stock price process follows (2) and there is a European call option with $S_0 = 100$, $\sqrt{V_0} = 0.35$, and, other parameters in Table 1. Table 2 shows the European option prices solved by FDM with different I which stands for the different accuracy.

7. Conclusion

With risk-minimization criterion, we employ the minimal martingale measure to solve the pricing problem in an incomplete market. Then, we obtain the Radon-Nikodym derivative under the minimal martingale measure and a

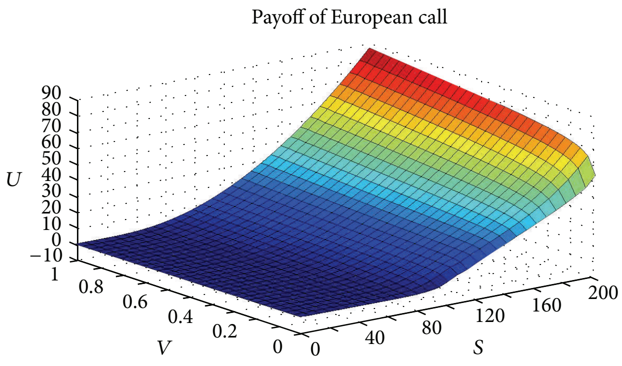


Figure 2: The solution of (38).

partial integro-differential equation (PIDE) for the European option. Since it is difficult to get the exact solution of PIDE in our model, a FDM scheme is proposed to compute the solution approximately. Finally, we complete a European call option price, and it is shown that our method is stable and locally accurate.

References

[1] F. Black and M. Scholes, "The pricing of options and corporate liabilities," *Journal of Political Economy*, vol. 81, no. 3, pp. 637–659, 1973.

[2] J. Hull and A. White, "The pricing of options on assets with stochastic volatilities," *The Journal of Finance*, vol. 42, no. 2, pp. 281–300, 1987.

[3] L. Scott, "Option pricing when the variance changes randomly: theory, estimation, and an application," *The Journal of Financial and Quantitative Analysis*, vol. 22, no. 4, pp. 419–438, 1987.

[4] J. Wiggins, "Option values under stochastic volatility: theory and empirical estimates," *Journal of Financial Economics*, vol. 19, no. 2, pp. 351–372, 1987.

[5] S. Melino and S. M. Turnbull, "Pricing foreign currency options with stochastic volatility," *Journal of Econometrics*, vol. 45, no. 1-2, pp. 239–265, 1990.

[6] S. Melino and S. M. Turnbull, "Misspecification and the pricing and hedging of long-term foreign currency options," *Journal of International Money and Finance*, vol. 14, no. 3, pp. 373–393, 1995.

[7] E. M. Stein and J. C. Stein, "Stock price distributions with stochastic volatility: an analytic approach," *The Review of Financial Studies*, vol. 4, no. 4, pp. 727–752, 1991.

[8] S. Heston, "A closed-form solution for options with stochastic volatility with applications to bond and currency options," *The Review of Financial Studies*, vol. 6, no. 2, pp. 327–343, 1993.

[9] R. Merton, "Option pricing when underlying stock returns are discontinuous," *Journal of Financial Economics*, vol. 3, no. 1-2, pp. 125–144, 1976.

[10] G. Bakshi, C. Cao, and Z. Chen, "Empirical performance of alternative option pricing models," *The Journal of Finance*, vol. 52, no. 5, pp. 2003–2049, 1997.

[11] F. E. Benth and T. Meyer-Brandis, "The density process of the minimal entropy martingale measure in a stochastic volatility model with jumps," *Finance and Stochastics*, vol. 9, no. 4, pp. 563–575, 2005.

[12] D. Bates, "The crash of '87: was it expected? The evidence from options markets," *The Journal of Finance*, vol. 46, no. 3, pp. 1009–1044, 1991.

[13] H. Föllmer and P. Leukert, "Efficient hedging: cost versus shortfall risk," *Finance and Stochastics*, vol. 4, no. 2, pp. 117–146, 2000.

[14] J. Kallsen, "A utility maximization approach to hedging in incomplete markets," *Mathematical Methods of Operations Research*, vol. 50, no. 2, pp. 321–338, 1999, Financial optimization.

[15] J. Cvitanić, W. Schachermayer, and H. Wang, "Utility maximization in incomplete markets with random endowment," *Finance and Stochastics*, vol. 5, no. 2, pp. 259–272, 2001.

[16] T. Bielecki and M. Jeanblanc, *Indifference Prices in Indifference Pricing, Theory and Applications, Financial Engineering*, Edited by R. Carmona, Princeton University Press, 2008.

[17] M. Schweizer, "Option hedging for semimartingales," *Stochastic Processes and their Applications*, vol. 37, no. 2, pp. 339–363, 1991.

[18] H. F. Föllmer and M. Schweizer, "Hedging of contingent claims under incomplete information," in *Applied Stochastic Analysis*, M. H. A. Davis and R. J. Elliott, Eds., vol. 5, pp. 389–414, Stochastics Monographs, London, UK, 1991.

[19] C. Chiarella and A. Ziogas, "American call options under jump-diffusion processes: a fourier transform approach," *Applied Mathematical Finance*, vol. 16, no. 1-2, pp. 37–79, 2009.

[20] A. M. Matache, C. Schwab, and T. P. Wihler, "Fast numerical solution of parabolic integrodifferential equations with applications in finance," *SIAM Journal on Scientific Computing*, vol. 27, no. 2, pp. 369–393, 2005.

[21] G. H. Meyer, "The numerical valuation of options with underlying jumps," *Acta Mathematica*, vol. 67, no. 1, pp. 69–82, 1998.

[22] P. Carr and A. Hirsa, "Why be backward? forward equations for American options," *Risk*, vol. 16, no. 1, pp. 103–107, 2003.

[23] Y. d'Halluin, P. A. Forsyth, and G. Labahn, "A penalty method for American options with jump diffusion processes," *Numerische Mathematik*, vol. 97, no. 2, pp. 321–352, 2004.

[24] M. Briani, C. La Chioma, and R. Natalini, "Convergence of numerical schemes for viscosity solutions to integro-differential degenerate parabolic problems arising in financial theory," *Numerische Mathematik*, vol. 98, no. 4, pp. 607–646, 2004.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 486257, 10 pages
http://dx.doi.org/10.1155/2013/486257

Research Article

Exponential Stability Results of Discrete-Time Stochastic Neural Networks with Time-Varying Delays

Yajun Li

Department of Electronics and Information Engineering, Shunde Polytechnic, Foshan 528300, China

Correspondence should be addressed to Yajun Li; lyjfirst@163.com

Received 31 January 2013; Accepted 7 April 2013

Academic Editor: Weihai Zhang

An innovative stability analysis approach for a class of discrete-time stochastic neural networks (DSNNs) with time-varying delays is developed. By constructing a novel piecewise Lyapunov-Krasovskii functional candidate, a new sum inequality is presented to deal with sum items without ignoring any useful items, the model transformation is no longer needed, and the free weighting matrices are added to reduce the conservatism in the derivation of our results, so the improvement of computational efficiency can be expected. Numerical examples and simulations are also given to show the effectiveness and less conservatism of the proposed criteria.

1. Introduction

In the past decades, neural networks (NNs) have attracted considerable attention due to their potential applications in associative memory, pattern recognition, optimization and signal processing, and so forth [1–3]. It is well known to us that stability is one of the preconditions in the design of neural networks. For example, if a neural network is employed to solve some optimization problems, it is highly desirable for the NNs to have a unique globally stable equilibrium. Therefore, stability analysis of NNs is a very important issue and has been studied extensively [4–11]. It is worth noting that most of the NNs have been analyzed by using a continuous-time model. However, when it comes to the implementation of continuous-time networks for the sake of computer-based simulation, experimentation or computation, it is necessary to discretize the continuous-time networks to formulate a discrete-time system. Under mild or no restriction on the discretization of step size, the dynamic characteristics of the continuous-time counterpart can be inherited by the discrete-time analogue to a certain extent, and the discrete-time model also remains similar to some other properties of the continuous-time system.

On the other hand, as a result of the finite switching speed of amplifiers and the inherent communication time of neurons, time delays are frequently encountered in neural networks in electronic implementations. Time delays can change the dynamic behaviors of neural networks evidently, which is very often the sources of instability, oscillation, and poor performance. Therefore, stability analysis of neural networks with time delays has been studied extensively during the past years; see [12–14] and the references therein. In practice, when modeling real neural systems, stochastic disturbances are probably part of the main sources leading to unwilling behaviors of neural networks. It has been proved that certain stochastic inputs could make the neural network unstable. Therefore, it is necessary to take into account both time delay and stochastic external flucation when modeling neural networks.

Recently, stochastic discrete-time system has been studied extensively; [15] presented a necessary and sufficient condition for the existence of $H_2/H\infty$ control to transform $H_2/H\infty$ controller design into solving coupled matrix-valued equation for discrete-time system with state and disturbance-dependent noise. In [16], the robust filtering analysis and synthesis of nonlinear stochastic systems with state and exogenous disturbance-dependent noise are presented. Instead of solving the Hamilton-Jacobi inequalities, a more convenient algorithm for practical applications is given by solving several linear matrix inequalities, and a few

examples have shown the effectiveness of the proposed methods. In [17], based on a nonlinear stochastic bounded real lemma and an exponential estimate formula, an exponential mean square $H\infty$ filtering design of nonlinear stochastic time-delay systems is presented via solving a Hamilton-Jacobi inequality.

But as pointed out in some previous articles, the discretization cannot preserve the dynamics of continuous-time system even for a small sampling period. Therefore, the study on the dynamics of discrete-time neural networks is crucially needed; see [18–22] and the references therein.

Based on the discussions previously mentioned, the problem of stability analysis for discrete stochastic neural networks (DSNNs) with time-varying delays has been investigated recently. In [23], the problem of exponentially stability analysis for uncertain discrete-time stochastic neural networks with time-varying delays is investigated by utilizing Lyapunov-Krasovskii method of converting the addressed stability analysis problem into a convex optimization problem. In [24], combining with the free weighting matrix method and a new Lyapunov-Krasovskii functional candidate, a delay-dependent stability condition has been obtained, which proves to be less conservative than [23]. In [25], a new Lyapunov-Krasovskii functional candidate and the delay partition idea are used to solve the problem of asymptotic stability analysis in the mean square for a class of DSNNs with time-varying delay; the stability analysis problem is converted into a feasibility problem of LMIs, and a numerical example is provided to show the usefulness of the proposed condition.

In [26], the global exponential stability problem for a class of discrete-time uncertain stochastic neural networks with time-varying delays is studied, and an improved result is obtained.

In [27], the midpoint of the time delay variation interval is introduced, and the variation interval is divided into two subintervals; a new Lyapunov-Krasovskii functional candidate is constructed, and the variation in the two subintervals are checked by LMIs; some novel delay-dependent stability criteria for the addressed neural networks are derived, and less conservation is obtained.

In [28], a new Lyapunov functional candidate with the idea of delay partitioning is introduced; the effects of both variation range and distribution probability of the time delay are taken into account at the same time; the time-varying delay is characterized by introducing a Bernoulli stochastic variable; the distribution probability of time delay is translated into parameter matrices of the transferred DSNNs model, so the conservation has been reduced further. However, one of the main issues in the stability criteria is how to reduce the possible conservatism induced by introduction of the Lyapunov-Krasovskii functional candidate when dealing with time delay, which leaves much room for further research by using the latest analysis techniques.

In this paper, we develop an innovative stability analysis approach for a class of discrete-time stochastic neural networks with time-varying delays. By constructing a novel piecewise Lyapunov-Krasovskii functional, a new sum inequality is presented to deal with sum items without ignoring any useful items, and the model transformation is no longer needed in the derivation of our results. All results are expressed in the form of LMIs, whose feasibility can be easily checked by using the numerically efficient Matlab LMI toolbox, and no tuning of parameters is required, so the improvement of computational efficiency can be expected. Numerical examples are also given to show the effectiveness and less conservatism of the proposed criteria.

Notation. Throughout this paper, if not explicit, matrices are assumed to have compatible dimensions. The notation $M > (\geq, <, \leq) 0$ means that the symmetric matrix M is positive definite (positive semidefinite, negative, negative semidefinite). The superscript T stands for the transpose of a matrix; the shorthand $\mathrm{diag}\{\cdots\}$ denotes the block diagonal matrix; $\|\cdot\|$ represents the Euclidean norm for vector or the spectral norm of matrices; and $\lambda_M(A), \lambda_m(A)$ denote the maximal and minimal eigenvalue of matrix A, respectively. I refers to an identity matrix of appropriate dimensions, $\mathbb{E}\{\cdot\}$ stands for the mathematical expectation, and $*$ means the symmetric terms. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise.

2. System Description

Consider the following n-neuron discrete stochastic neural networks with mixed time-varying delays:

$$\begin{aligned} x(k+1) = {} & Ax(k) + Bf(x(k)) + Wg(x(k-\tau(k))) \\ & + \sigma(k, x(k), x(k-\tau(k)))\,\omega(k), \end{aligned} \tag{1}$$

where $x(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^T \in \mathbb{R}^n$ is the neuron state vector; $f(x(k)) = [f_1(x_1(k)), f_2(x_2(k)), \ldots, f_n(x_n(k))]^T \in \mathbb{R}^n$ and $g(x(k)) = [g_1(x_1(k)), g_2(x_2(k)), \ldots, g_n(x_n(k))]^T \in \mathbb{R}^n$ denotes the neuron activation function. $A = \mathrm{diag}\{a_1, \ldots, a_n\}$ with $|a_i| < 1$ $(i = 1, 2, \ldots, n)$. B, W are the connection weight matrix and the delayed connection weight matrix, respectively, $\tau(k)$ represents the transmission time-varying delay, $\sigma : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function, and $\omega(k)$ is a Brown Motion defined on the complete probability space $(\Omega, \mathcal{F}, \{F_t\}_t)$ with $\mathbb{E}\{\omega(k)\} = 0$, $\mathbb{E}\{\omega^2(k)\} = 1$, $\mathbb{E}\{w(i)w(j)\} = 0$ $(i \neq j)$.

For further discussion, we introduce the following assumptions and lemmas.

Assumption 1. There exist two positive constants β_1 and β_1 such that

$$\begin{aligned} & \sigma(k, x(k), x(k-\tau(k)))^T \sigma(k, x(k), x(k-\tau(k))) \\ & \quad \leq \beta_1 x^T(k)\,x(k) + \beta_2 x^T(k-\tau(k))\,x(k-\tau(k)). \end{aligned} \tag{2}$$

Assumption 2. For $i \in 1, 2, \ldots, n$, the neuron activation functions in the DSNNS in (1) satisfy

$$l_i^- \leq \frac{f_i(x) - f_i(y)}{x - y} \leq l_i^+ \quad \forall x, y \in \mathbb{R},\ x \neq y,\ f_i(0) = 0,$$
$$\rho_i^- \leq \frac{g_i(x) - g_i(y)}{x - y} \leq \rho_i^+ \quad \forall x, y \in \mathbb{R},\ x \neq y,\ g_i(0) = 0, \tag{3}$$

where $l_i^-, l_i^+, \rho_i^+, \rho_i^-$ are some constants.

Remark 3. Assumption 2 previously mentioned on the activation function was widely used in many papers; see [16–21, 23–26], for example.

Lemma 4 (see [27]). *For any symmetric constant matrix $Q \in \mathbb{R}^{n\times n}$, $Q > 0$, integers $\tau_m < \tau_M$, and vector valued function $y(k) = x(k+1) - x(k)$, one has*

$$-(\tau_M - \tau_m) \sum_{i=k-\tau_M}^{k-\tau_m-1} y^T(i) Q y(i) \leq \begin{bmatrix} x(k-\tau_m) \\ x(k-\tau_M) \end{bmatrix}^T \begin{bmatrix} -Q & Q \\ Q & -Q \end{bmatrix} \begin{bmatrix} x(k-\tau_m) \\ x(k-\tau_M) \end{bmatrix}. \tag{4}$$

Remark 5. Lemma 4 is called discrete Jensen inequality, which is a very important tool for us to get the main results in this paper. The Lemma has been used in some literatures such as [28, 29].

Lemma 6 (see [30]). *For any constant matrix $R_i \in \mathbb{R}^{n\times n}$, $R_i = R_i^T \geq 0$, a scalar λ, a positive integer time-varying $\tau(k) \in [\tau_1, \tau_3]$, and vector function $\eta : [-\tau_3, -\tau_1] \to \mathbb{R}^n$, such that the following sum is well defined, the following inequalities hold:*

$$\Omega = -(\tau_3 - \tau_1) \sum_{\theta=k-\tau_3}^{k-\tau_1-1} \eta(\theta) R_i \eta(\theta) \leq \xi_1^T(k)[1+\pi_1]\mathscr{R}_i \xi_1(k) + \xi_2^T(k)[1+\pi_2]\mathscr{R}_i\xi_2(k). \tag{5}$$

Furthermore, if

$$\lambda + \xi_1^T(k)\, 3\mathscr{R}_i \xi_1(k) + \xi_2^T(k)\mathscr{R}_i\xi_2(k) \leq 0,$$
$$\lambda + \xi_1^T(k)\mathscr{R}_i \xi_1(k) + \xi_2^T(k)\, 3\mathscr{R}_i\xi_2(k) \leq 0, \tag{6}$$

then

$$\lambda + \xi_1^T(k)[1+\pi_1]\mathscr{R}_i \xi_1(k) + \xi_2^T(k)[1+\pi_2]\mathscr{R}_i\xi_2(k) \leq 0, \tag{7}$$

where

$$\pi_1 = -1, \quad \pi_2 = 0, \quad \text{when } \tau(k) = \tau_1,$$
$$\pi_1 = \frac{\tau_3 - \tau(k)}{\tau(k) - \tau_1}, \quad \pi_2 = \frac{1}{\pi_1}, \quad \text{when } \tau_1 < \tau(k) < \tau_3, \tag{8}$$
$$\pi_1 = 0, \quad \pi_2 = -1, \quad \text{when } \tau(k) = \tau_3$$

and $\xi_1(k) = [x^T(k-\tau_1)x^T(k-\tau(k))]^T$, $\xi_2(k) = [x^T(k-\tau(k))x^T(k-\tau_3)]^T$, $\eta(k) = x(k+1) - x(k)$,

$$\mathscr{R}_i = \begin{bmatrix} -R_i & R_i \\ R_i & -R_i \end{bmatrix}. \tag{9}$$

Remark 7. Lemma 6 is given and proved in [29], which is an effective method to reduce the conservation when studying the time delay stability problem for discrete system; see the literature mentioned previously [30]. Our main study is based on Lemma 6. It is worth mentioning that if $\xi_1^T(k)\pi_1\mathscr{R}_i\xi_1(k)$ and $\xi_2^T(k)\pi_2\mathscr{R}_i\xi_2(k)$ in the proof of (6) are directly ignored, then the following inequality can be derived:

$$\Omega \leq \xi_1^T(k)\mathscr{R}_i\xi_1(k) + \xi_2^T(k)\mathscr{R}_i\xi_2(k). \tag{10}$$

Compared with the literature [12–18], none of useful items is ignored in (5); what is more, (5) provides a tighter bound to deal with sum terms than those based on (10). However, since (6) is related to time-varying delay items π_1 and π_2, it cannot be directly solved based on MATLAB LMI toolbox. To tackle this problem, (6) is used to transfer the time-varying matrix inequality to a set of solvable LMIs. That is, the additional information $\xi_1^T(k)\pi_1\mathscr{R}_i\xi_1(k)$ and $\xi_2^T(k)\pi_2\mathscr{R}_i\xi_2(k)$ in (10) is effectively expressed by (7), and then the less conservative results can be expected.

3. Main Results

For convenience of presentation, we use the following notations:

$$\Sigma_1 = \operatorname{diag}\{l_1^- l_1^+, l_2^- l_2^+, \ldots, l_n^- l_n^+\},$$
$$\Sigma_2 = \operatorname{diag}\left\{\frac{l_1^- + l_1^+}{2}, \frac{l_2^- + l_2^+}{2}, \ldots, \frac{l_n^- + l_n^+}{2}\right\},$$
$$\Sigma_3 = \operatorname{diag}\{\varrho_1^- \varrho_1^+, \varrho_2^- \varrho_2^+, \ldots, \varrho_n^- \varrho_n^+\},$$
$$\Sigma_4 = \operatorname{diag}\left\{\frac{\varrho_1^- + \varrho_1^+}{2}, \frac{\varrho_2^- + \varrho_2^+}{2}, \ldots, \frac{\varrho_n^- + \varrho_n^+}{2}\right\},$$
$$\xi(k) = \left[x^T(k), x^T(k-\tau_m), x^T(k-\tau_0), x^T(k-\tau(k)), x^T(k-\tau_M), y^T(k), f^T(x(k)), g^T(k-\tau(k))\right]^T. \tag{11}$$

In this section, a new delay-dependent stability criteria is proposed for system (1) with time-varying delay satisfying (3); the sufficient conditions of stability are given as follows by Theorem 8.

Theorem 8. *For given τ_m, τ_0 and τ_M, diagonal matrices Σ_1, Σ_2, Σ_3 and Σ_4, the system (1) is said to be exponentially stable, if there exist scalar λ^*, diagonal matrices Λ_1, Λ_2 and symmetric*

matrices $P > 0$, $Q_i > 0$, $R_i > 0$ $(i = 1, 2, 3, 4)$ *such that the following LMIs* (12), (13) *and* (14) *hold for* $\alpha = 0, 1$:

$$P < \lambda^* I, \tag{12}$$

$$\Psi_{11}^1 (\alpha) < 0, \tag{13}$$

$$\Psi_{11}^2 (\alpha) < 0, \tag{14}$$

where

$$\Psi_{11}^1 (\alpha) = 2(1-\alpha)\Theta_1 + 2\alpha\Theta_2 + \Theta_1 + \Theta_2 + \Theta_3,$$

$$\Psi_{11}^2 (\alpha) = 2(1-\alpha)\overline{\Theta}_1 + 2\alpha\overline{\Theta}_2 + \overline{\Theta}_1 + \overline{\Theta}_2 + \overline{\Theta}_3,$$

$$\Theta_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_2 & 0 & 0 & 0 & 0 \\ 0 & R_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Theta_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_2 & -R_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{\Theta}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_3 & -R_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\overline{\Theta}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_3 & -R_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Theta_3 = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} & \Theta_{16} & \Theta_{17} & \Theta_{18} \\ * & \Theta_{22} & 0 & 0 & 0 & \Theta_{26} & 0 & \Theta_{28} \\ * & * & \Theta_{33} & \Theta_{34} & \Theta_{35} & \Theta_{36} & 0 & \Theta_{38} \\ * & * & * & \Theta_{44} & 0 & \Theta_{46} & \Theta_{47} & \Theta_{48} \\ * & * & * & * & \Theta_{55} & \Theta_{56} & 0 & \Theta_{58} \\ * & * & * & * & * & \Theta_{66} & \Theta_{67} & \Theta_{68} \\ * & * & * & * & * & * & \Theta_{77} & \Theta_{78} \\ * & * & * & * & * & * & * & \Theta_{88} \end{bmatrix},$$

$$\begin{aligned}
\Theta_{11} = {} & (\tau_M - \tau_m + 1) R_4 - R_1 \\
& + \sum_{i=1}^{4} Q_i - \frac{1}{\tau_m} R_1 - \Lambda_1 \Sigma_1 + \lambda^* \beta_1 I \\
& + U_1 (I - A) + (I - A)^T U_1^T,
\end{aligned}$$

$$\Theta_{12} = R_1 + U_2 (I - A), \qquad \Theta_{13} = U_3 (I - A),$$

$$\Theta_{14} = U_4 (I - A), \qquad \Theta_{15} = U_5 (I - A),$$

$$\Theta_{16} = U_6 (I - A) + U_1, \qquad \Theta_{17} = U_8 (I - A) - U_1 B + \Lambda_1 \Sigma_2,$$

$$\Theta_{18} = U_8 (I - A) - U_1 W, \qquad \Theta_{22} = -Q_1 - \tau_m R_1,$$

$$\Theta_{26} = U_2, \qquad \Theta_{27} = -U_2 B, \qquad \Theta_{28} = -U_2 W,$$

$$\Theta_{33} = Q_2 - \frac{1}{\tau_M - \tau_0} R_3, \qquad \Theta_{34} = R_2, \qquad \Theta_{35} = \frac{1}{\tau_M - \tau_0} R_3,$$

$$\Theta_{36} = U_3, \qquad \Theta_{37} = -U_3 B, \qquad \Theta_{38} = -U_3 W,$$

$$\Theta_{44} = -R_4 - Q_3 - \Lambda_2 \Sigma_1 + \lambda^* \beta_2 I,$$

$$\Theta_{46} = U_4, \qquad \Theta_{47} = -U_4 B, \qquad \Theta_{48} = -U_4 W + \Lambda_2 \Sigma_2,$$

$$\Theta_{55} = -Q_4 - \frac{1}{\tau_M - \tau_0} R_3, \qquad \Theta_{56} = U_5,$$

$$\Theta_{57} = -U_5 B, \qquad \Theta_{58} = -U_5 W,$$

$$\Theta_{66} = U_6 + U_6^T + P + \frac{1}{\tau_m} R_1 + \frac{1}{\tau_0 - \tau_m} R_2 + \frac{1}{\tau_M - \tau_0} R_3,$$

$$\Theta_{67} = -U_6 B, \qquad \Theta_{68} = -U_6 W,$$

$$\Theta_{77} = -V_1 - U_7 B - U_7^T B^T,$$

$$\Theta_{78} = -U_7 W - B^T U_8^T,$$

$$\Theta_{88} = -V_2 - U_8 W - U_8^T W^T,$$

$$\overline{\Theta}_3 = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} & \Theta_{16} & \Theta_{17} & \Theta_{18} \\ * & \overline{\Theta}_{22} & \overline{\Theta}_{23} & 0 & 0 & 0 & 0 & \Theta_{28} \\ * & * & \overline{\Theta}_{33} & \Theta_{34} & \overline{\Theta}_{35} & \Theta_{36} & 0 & \Theta_{38} \\ * & * & * & \Theta_{44} & 0 & \Theta_{46} & \Theta_{47} & \Theta_{48} \\ * & * & * & * & \overline{\Theta}_{55} & \Theta_{56} & 0 & \Theta_{58} \\ * & * & * & * & * & \Theta_{66} & \Theta_{67} & \Theta_{68} \\ * & * & * & * & * & * & \Theta_{77} & \Theta_{78} \\ * & * & * & * & * & * & * & \Theta_{88} \end{bmatrix}, \tag{15}$$

where

$$\overline{\Theta}_{22} = -Q_1 - \tau_m R_1 - \frac{1}{\tau_0 - \tau_m} R_2, \qquad \overline{\Theta}_{23} = -\frac{1}{\tau_0 - \tau_m} R_2,$$

$$\overline{\Theta}_{33} = Q_2 - \frac{1}{\tau_0 - \tau_m} R_2, \qquad \overline{\Theta}_{35} = 0, \qquad \Theta_{55} = -Q_4. \tag{16}$$

The other terms in $\overline{\Theta}_3$ *have the same expression as that in* Θ_3.

Proof. Define $y(k) = x(k+1) - x(k)$, $\eta(k) = [x^T(k) \;\; y^T(k)]^T$.

Construct the following Lyapunov-Krasovskii functional candidates:

$$V(k) = \sum_{i=1}^{4} V_i(k), \tag{17}$$

where

$$\begin{aligned}
V_1(k) &= x^T(k) P x(k), \\
V_2(k) &= \sum_{i=k-\tau(k)}^{k-1} x^T(k) R_4 x(k) + \sum_{j=k+1-\tau_M}^{k-\tau_m} \sum_{j=i}^{k-1} x^T(j) R_4 x(j), \\
V_3(k) &= \sum_{i=k-\tau_m}^{k-1} x^T(i) Q_1 x(i) + \sum_{i=k-\tau_0}^{k-1} x^T(i) Q_2 x(i) \\
&\quad + \sum_{i=k-\tau(k)}^{k-1} x^T(i) Q_3 x(i) + \sum_{i=k-\tau_M}^{k-1} x^T(i) Q_4 x(i), \\
V_4(k) &= \sum_{i=-\tau_m}^{-1} \sum_{\theta=k+i}^{k-1} y^T(\theta) R_1 y(\theta) \\
&\quad + \sum_{i=-\tau_0}^{-\tau_m-1} \sum_{\theta=k+i}^{k-1} y^T(\theta) R_2 y(\theta) \\
&\quad + \sum_{i=-\tau_M}^{-\tau_0-1} \sum_{\theta=k+i}^{k-1} y^T(\theta) R_3 y(\theta),
\end{aligned} \tag{18}$$

where if $(\tau_M - \tau_m)/2$ is an integer, then $\tau_0 = (\tau_M + \tau_m)/2$, else $\tau_0 = (\tau_M + \tau_m + 1)/2$.

By calculating the difference of $V(k)$ along the solution of the system (1), and taking the mathematical expectation, we have

$$\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\left\{\sum_{i=1}^{4} \Delta V_i(k)\right\}, \tag{19}$$

where

$$\begin{aligned}
\mathbb{E}\{\Delta V_1(k)\} &= \mathbb{E}\left\{x^T(k+1) P x^T(k+1) - x^T(k) P x(k)\right\} \\
&= \mathbb{E}\left\{2x^T(k) P y(k) + \sigma^T(k, x(k), x(k-\tau(k))) \right. \\
&\qquad \times P\sigma(k, x(k), x(k-\tau(k))) \\
&\qquad \left. + y^T(k) P y(k)\right\}.
\end{aligned} \tag{20}$$

From Assumption 2, we can obtain that

$$\begin{aligned}
&\sigma(k, x(k), x(k-\tau(k)))^T \sigma(k, x(k), x(k-\tau(k))) \\
&\quad \leq \lambda^* \left(\beta_1 x^T(k) x(k) + \beta_2 x^T(k-\tau(k)) x(k-\tau(k))\right).
\end{aligned} \tag{21}$$

So we can get that

$$\begin{aligned}
\mathbb{E}\{\Delta V_1(k)\} \leq \mathbb{E}\Big\{\Big(&2x^T(k) P y(k) + y^T(k) P y(k) \\
&+ \lambda^* \big(\beta_1 x^T(k) x(k) \\
&\quad + \beta_2 x^T(k-\tau(k)) x(k-\tau(k))\big)\Big)\Big\},
\end{aligned} \tag{22}$$

$$\begin{aligned}
\mathbb{E}\{\Delta V_2(k)\} &= \mathbb{E}\left\{x^T(k) R_4 x(k) - x^T(k-\tau(k)) \right. \\
&\quad \times R_4 x(k-\tau(k)) + \sum_{j=k+1-\tau_{k+1}}^{k-\tau_m} x^T(k) R_4 x(k) \\
&\quad + \sum_{j=k+1-\tau_m}^{k-1} x^T(k) R x(k) \\
&\quad - \sum_{j=k+1-\tau_k}^{k-1} x^T(k) R_4 x(k) \\
&\quad + \sum_{j=k+2-\tau_M}^{k+1-\tau_m} \sum_{j=i}^{k} x^T(j) R_4 x(j) \\
&\quad \left. - \sum_{j=k+1-\tau_M}^{k-\tau_m} \sum_{j=i}^{k-1} x^T(j) R_4 x(j)\right\} \\
&\leq \mathbb{E}\left\{x^T(k) R_4 x(k) - x^T(k-\tau(k)) R_4 x(k-\tau(k)) \right. \\
&\quad + \sum_{j=k+1-\tau_M}^{k-\tau_m} x^T(k) R_4 x(k) \\
&\quad + \sum_{j=k+1-\tau_M}^{k-\tau_m} \left(x^T(k) R_4 x(k) \right. \\
&\qquad \left.\left. - x^T(j) R_4 x(j)\right)\right\} \\
&= \mathbb{E}\left\{(\tau_M - \tau_m + 1) x^T(k) R_4 x(k) \right. \\
&\quad \left. - x^T(k-\tau(k)) R_4 x(k-\tau(k))\right\},
\end{aligned} \tag{23}$$

$$\begin{aligned}
\mathbb{E}\{\Delta V_3(k)\} &= \mathbb{E}\left\{x^T(k) \sum_{i=1}^{4} Q_i x(k) - x^T(k-\tau_m) Q_1 x(k-\tau_m) \right. \\
&\quad - x^T(k-\tau_0) Q_2 x(k-\tau_0) \\
&\quad - x^T(k-\tau(k)) Q_3 x(k-\tau(k)) \\
&\quad \left. - x^T(k-\tau_M) Q_4 x(k-\tau_M)\right\},
\end{aligned} \tag{24}$$

$$\begin{aligned}
\mathbb{E}\{\Delta V_4(k)\} &= \mathbb{E}\left\{\tau_m y^T(k) R_1 y(k) - \sum_{j=k-\tau_m}^{k-1} y^T(k) R_1 y(k) \right. \\
&\quad + (\tau_0 - \tau_m) y^T(k) R_2 y(k)
\end{aligned}$$

$$
\left.\begin{aligned}
&-\sum_{j=k-\tau_0}^{k-\tau_m-1} y^T(k) R_2 y(k) \\
&+ (\tau_M - \tau_0) y^T(k) R_3 y(k) \\
&-\sum_{j=k-\tau_M}^{k-\tau_0-1} y^T(k) R_3 y(k)
\end{aligned}\right\}. \tag{25}
$$

Now, we are in the position to prove that $\mathbb{E}\{\Delta V(k)\} < 0$ holds for both $\tau_M \geq \tau_k \geq \tau_0$ and $\tau_0 \geq \tau_k \geq \tau_m$.

Case I (when $\tau_0 \geq \tau_k \geq \tau_m$). Using Lemmas 4 and Lemma 6 to deal with the second sum items in the right side of (25), we have

$$
\begin{aligned}
&-\sum_{j=k-\tau_m}^{k-1} y^T(k) R_1 y(k) \\
&\quad \leq \frac{1}{\tau_m} \begin{bmatrix} x(k) \\ x(k-\tau_m) \end{bmatrix}^T \begin{bmatrix} -R_1 & R_1 \\ R_1 & -R_1 \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-\tau_m) \end{bmatrix}
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
&-\sum_{j=k-\tau_M}^{k-\tau_0-1} y^T(k) R_3 y(k) \\
&\quad \leq \frac{1}{(\tau_M - \tau_0)} \begin{bmatrix} x(k-\tau_0) \\ x(k-\tau_M) \end{bmatrix}^T \begin{bmatrix} -R_3 & R_3 \\ R_3 & -R_3 \end{bmatrix} \begin{bmatrix} x(k-\tau_0) \\ x(k-\tau_M) \end{bmatrix}
\end{aligned} \tag{27}
$$

$$
\begin{aligned}
&-\sum_{j=k-\tau_0}^{k-\tau_m-1} y^T(k) R_2 y(k) \\
&\quad \leq \frac{1}{(\tau_0 - \tau_m)} \xi_1^T(k) [1 + \overline{\pi}_1] \mathscr{R}_2 \xi_1(k) \\
&\quad + \xi_3^T(k) [1 + \overline{\pi}_2] \mathscr{R}_2 \xi_3(k),
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
\xi_1(k) &= \left[x^T(k-\tau_m)\, x^T(k-\tau(k))\right]^T, \\
\xi_3(k) &= \left[x^T(k-\tau_0)\, x^T(k-\tau(k))\right]^T.
\end{aligned} \tag{29}
$$

At the same time, for any matrices U of appropriate dimensions, we have

$$
\begin{aligned}
\mathbb{E}\Big\{\xi^T(k)\left(U+U^T\right)\big[&(A-I)x(k)+Bf(x(k)) \\
&+ Wg(x(k-\tau(k))) \\
&+ \sigma(k, x(k), x(k-\tau(k)))\omega(k) \\
&-y(k)\big]\Big\} = 0,
\end{aligned} \tag{30}
$$

where

$$
U = \left[U_1^T, U_2^T, U_3^T, U_4^T, U_5^T, U_6^T, U_7^T, U_8^T\right]^T. \tag{31}
$$

In addition, it can be deduced from Assumption 2 that there exist two positive diagonal matrices $\Lambda_i = \mathrm{diag}\{\varepsilon_{i,1}, \varepsilon_{i,2}, \ldots, \varepsilon_{i,n}\}$, $(i = 1, 2)$ such that

$$
\begin{aligned}
0 \leq\; & -\sum_{i=1}^{n} \varepsilon_{1,i} \left[f_i(x_i(k)) - l_i^- x_i(k)\right]\left[f_i(x_i(k)) - l_i^+ x_i(k)\right] \\
& -\sum_{i=1}^{n} \varepsilon_{2,i} \left[g_i(x_i(k-\tau(k))) - l_i^- x_i(k-\tau(k))\right] \\
& \qquad \times \left[g_i(x_i(k-\tau(k))) - l_i^+ x_i(k-\tau(k))\right] \\
=\; & -x^T(k)\Lambda_1\Sigma_1 x(k) + 2f^T(x(k))\Lambda_1\Sigma_2 x(k) \\
& - f^T(x(k))\Lambda_1 f(x(k)) \\
& - x^T(k-\tau(k))\Lambda_2\Sigma_3 x(k-\tau(k)) + 2g^T(x(k-\tau(k))) \\
& \times \Lambda_2\Sigma_4 x(k-\tau(k)) \\
& - g^T(x(k-\tau(k)))\Lambda_2 g(x(k-\tau(k))).
\end{aligned} \tag{32}
$$

By substituting (22)–(25) into (19), adding (30) and (32) into the right side of (19), and using (26) and (27), we can get that

$$
\begin{aligned}
\mathbb{E}\{\Delta V(k)\} \leq \mathbb{E}\Big\{&\lambda_1 + \frac{1}{(\tau_0 - \tau_m)} \xi_1^T(k)[1 + \overline{\pi}_1]\mathscr{R}_2\xi_1(k) \\
&+ \xi_3^T(k)[1 + \overline{\pi}_2]\mathscr{R}_2\xi_3(k)\Big\},
\end{aligned} \tag{33}
$$

where $\lambda_1 = \xi^T(k)\Theta_3\xi(k)$, $\overline{\pi}_1 = (\tau_0 - \tau(k))/(\tau(k) - \tau_m)$ and $\overline{\pi}_1 = 1/\overline{\pi}_2$,

$$
\begin{aligned}
\xi(k) &= \left[x^T(k)\;\; x^T(k-\tau_m)\;\; x^T(k-\tau_0)\;\; x^T(k-\tau(k))\;\; x^T(k-\tau_M)\;\; f^T(x(k))\;\; f^T(x(k-\tau(k)))\right]^T, \\
\xi(k) &= \left[x^T(k)\;\; x^T(k-\tau_m)\;\; x^T(k-\tau_0)\;\; x^T(k-\tau(k))\;\; x^T(k-\tau_M)\;\; y^T(x(k))\;\; f^T(x(k))\;\; f^T(x(k-\tau(k)))\right]^T.
\end{aligned} \tag{34}
$$

So (12) and (13) imply that

$$
\begin{aligned}
\lambda_1 + \xi_1^T(k)\, 3\mathscr{R}_2\xi_1(k) + \xi_2^T(k)\mathscr{R}_2\xi_2(k) \leq 0, \\
\lambda_1 + \xi_1^T(k)\mathscr{R}_2\xi_1(k) + \xi_2^T(k)\, 3\mathscr{R}_2\xi_2(k) \leq 0.
\end{aligned} \tag{35}
$$

So from Lemma 6, (35) guarantees that $\mathbb{E}\{\Delta V(k)\} \leq 0$, so there exists a positive scalar c_1 such that

$$
\mathbb{E}\{\Delta V(k)\} \leq c_1 \mathbb{E}\left\{\|x(k)\|^2\right\}. \tag{36}
$$

Case II (when $\tau_M \geq \tau_k \geq \tau_0$). Keeping (26) and by utilizing Lemmas 4 and 6 to deal with the accumulative items of (27) and (28), we can get

$$
\begin{aligned}
&-\sum_{j=k-\tau_m}^{k-1} y^T(k) R_1 y(k) \\
&\quad \leq \frac{1}{\tau_m}\begin{bmatrix} x(k) \\ x(k-\tau_m) \end{bmatrix}^T \begin{bmatrix} -R_1 & R_1 \\ R_1 & -R_1 \end{bmatrix} \begin{bmatrix} x(k) \\ x(k-\tau_m) \end{bmatrix} \\
&-\sum_{j=k-\tau_M}^{k-\tau_0-1} y^T(k) R_3 y(k) \leq \frac{1}{(\tau_M-\tau_0)} \bar{\xi}_2^T(k)\left[1+\bar{\pi}_1\right]\mathscr{R}_3\bar{\xi}_2(k) \\
&\qquad + \bar{\xi}_3^T(k)\left[1+\bar{\pi}_2\right]\mathscr{R}_3\bar{\xi}_3(k), \\
&-\sum_{j=k-\tau_0}^{k-\tau_m-1} y^T(k) R_2 y(k) \\
&\quad \leq \frac{1}{\tau_0-\tau_m}\begin{bmatrix} x(k-\tau_m) \\ x(k-\tau_0) \end{bmatrix}^T \begin{bmatrix} -R_2 & R_2 \\ R_2 & -R_2 \end{bmatrix} \begin{bmatrix} x(k-\tau_m) \\ x(k-\tau_0) \end{bmatrix},
\end{aligned} \tag{37}
$$

where

$$
\begin{aligned}
\bar{\xi}_2(k) &= \left[x^T(k-\tau_0) \;\; x^T(k-\tau(k))\right]^T, \\
\bar{\xi}_3(k) &= \left[x^T(k-\tau(k)) \;\; x^T(k-\tau_M)\right]^T.
\end{aligned} \tag{38}
$$

So the whole difference of Lyapunov-Krasovskii functional candidate is given as follows:

$$
\begin{aligned}
\mathbb{E}\{\Delta V(k)\} \leq \lambda_2 &+ \bar{\xi}_2^T(k)\left[1+\hat{\pi}_1\right]\mathscr{R}_3\bar{\xi}_2(k) \\
&+ \bar{\xi}_3^T(k)\left[1+\hat{\pi}_2\right]\mathscr{R}_3\bar{\xi}_3(k),
\end{aligned} \tag{39}
$$

where $\lambda_2 = \xi^T(k)[\overline{\Theta}_3\xi(k)$, $\hat{\pi}_1 = (\tau_M - \tau(k))/(\tau(k)-\tau_0)$ and $\hat{\pi}_2 = 1/\hat{\pi}_1$. For $\tau(k) \neq \tau_0$ and $\tau(k) \neq \tau_M$, we can also get that there exists a positive scalar c_2 such that

$$
\mathbb{E}\{\Delta V(k)\} \leq c_2 \mathbb{E}\left\{\|x(k)\|^2\right\}. \tag{40}
$$

Combining Cases I and II, we can conclude that (12), (13) and (14) guarantee that

$$
\mathbb{E}\{\Delta V(k)\} < -\min\{c_1, c_2\}\,\mathbb{E}\left\{\|x(k)\|^2\right\}. \tag{41}
$$

Defining a new function $\mathbb{V}(k, x(k)) = \mu^k V(k, x(k))$ and then using the similar analysis method of Theorem 1 in [18], we can easily get that the system (1) is globally exponentially stable in the mean square sense. This completes the proof. □

Remark 9. It can be seen from the proof previously mentioned that no model transformation has been employed to deal with the sum terms, and none of the useful items are ignored in the proof.

If we neglect the effect of the stochastic term $\omega(k)$ in (1), then $\beta_1 = \beta_2 = 0$ and (1) will reduce to

$$
x(k+1) = Ax(k) + Bf(x(k)) + Wf(x(k-\tau(k))). \tag{42}
$$

For system (42), we can obtain the following corollary based on Theorem 8.

Corollary 10. *For given τ_m, τ_0, and τ_M and diagonal matrices $\Sigma_1, \Sigma_2, \Sigma_3$, and Σ_4, the system (42) is said to be exponentially stable, if there exist diagonal matrices Λ_1, Λ_2, symmetric positive definite matrices $P > 0$, $Q_i > 0$ and $R_i > 0$ ($i = 1, 2, 3$) such that (12) and the following LMIs hold for $\alpha = 0, 1$:*

$$
\begin{aligned}
\Theta_4 &= \begin{bmatrix}
\Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} & \Theta_{16} & \Theta_{17} & \Theta_{18} \\
* & \Theta_{22} & 0 & 0 & 0 & \Theta_{26} & 0 & \Theta_{28} \\
* & * & \Theta_{33} & \Theta_{34} & \Theta_{35} & \Theta_{36} & 0 & \Theta_{38} \\
* & * & * & \Theta_{44} & 0 & \Theta_{46} & \Theta_{47} & \Theta_{48} \\
* & * & * & * & \Theta_{55} & \Theta_{56} & 0 & \Theta_{58} \\
* & * & * & * & * & \Theta_{66} & \Theta_{67} & \Theta_{68} \\
* & * & * & * & * & * & \Theta_{77} & \Theta_{78} \\
* & * & * & * & * & * & * & \Theta_{88}
\end{bmatrix}, \\
\overline{\Theta}_4 &= \begin{bmatrix}
\overline{\Theta}_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} & \Theta_{15} & \Theta_{16} & \Theta_{17} & \Theta_{18} \\
* & \overline{\Theta}_{22} & \overline{\Theta}_{23} & 0 & 0 & \Theta_{26} & 0 & \Theta_{28} \\
* & * & \overline{\Theta}_{33} & \Theta_{34} & \overline{\Theta}_{35} & \Theta_{36} & 0 & \Theta_{38} \\
* & * & * & \Theta_{44} & \overline{0} & \Theta_{46} & \Theta_{47} & \Theta_{48} \\
* & * & * & * & \overline{\Theta}_{55} & \Theta_{56} & 0 & \Theta_{58} \\
* & * & * & * & * & \Theta_{66} & \Theta_{67} & \Theta_{68} \\
* & * & * & * & * & * & \Theta_{77} & \Theta_{78} \\
* & * & * & * & * & * & * & \Theta_{88}
\end{bmatrix}, \\
\overline{\Theta}_{11} &= (\tau_M - \tau_m + 1) R_4 - R_1 \\
&\quad + \sum_{i=1}^{4} Q_i - \frac{1}{\tau_m} R_1 - \Lambda_1\Sigma_1 + U_1(I-A) + (I-A)^T U_1^T, \\
\Theta_{44} &= -R_4 - Q_3 - \Lambda_2\Sigma_1.
\end{aligned} \tag{43}
$$

Remark 11. The system (42) has been studied by many researchers, many stability criteria have been proposed, and many improved analysis results have been obtained; see [20–22].

4. Numerical Example

In this section, three examples are given to demonstrate the benefits of the proposed method.

Example 1. Consider the following stochastic discrete neural networks [25, 26]:

$$
\begin{aligned}
x(k+1) = {} & Ax(k) + Bf(x(k)) + Wg(x(k-\tau(k))) \\
& + \sigma(x(k), x(k-\tau(k)))\,\omega(k)
\end{aligned} \tag{44}
$$

with the following parameters:

$$
A = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.1 & 0.1 \\ -0.1 & 0.5 \end{bmatrix},
$$

$$
W = \begin{bmatrix} 0.05 & 0.1 \\ 0.5 & 0.5 \end{bmatrix},
$$

$$
\beta_1 = 0.2, \qquad \beta_2 = 0.2,
$$

TABLE 1: Allowable upper bound of τ_M for various τ_m.

Methods	$\tau_m = 2$	$\tau_m = 4$	$\tau_m = 6$	$\tau_m = 8$	$\tau_m = 10$	$\tau_m = 15$	$\tau_m = 20$
[26]	6	8	10	12	14	19	24
[25]	9	11	13	15	17	22	27
Theorem 8	10^6	10^6	10^6	10^6	10^6	10^6	10^6

$$f_1(s) = \sin(0.2s) - 0.6\cos(s), \qquad f_2(s) = \tanh(-0.4s),$$
$$g_1(s) = \tanh(0.83s) + 0.6\cos(s), \qquad g_2(s) = \tanh(0.2s). \tag{45}$$

So it can be verified that

$$\Sigma_1 = \operatorname{diag}\{-0.64, 0\}, \qquad \Sigma_2 = \operatorname{diag}\{0, -0.2\},$$
$$\Sigma_3 = \operatorname{diag}\{-0.6, 0\}, \qquad \Sigma_4 = \operatorname{diag}\{0.2, 0.1\}. \tag{46}$$

For $\tau_m = 2$, by using Matlab LMI toolbox, the maximum allowable value is $\tau_M = 11$. Setting $\tau_m = 2$ and $\tau_M = 5$ in Theorem 8, we can solve a set of feasible solutions for the LMIs (12), (13) and (14) which are listed as follows:

$$P = \begin{bmatrix} 2.8554 & -0.1283 \\ -0.1283 & 1.3300 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 0.0044 & 0.0021 \\ 0.0021 & 0.0022 \end{bmatrix},$$
$$R_2 = \begin{bmatrix} 0.0421 & 0.0037 \\ 0.0037 & 0.0128 \end{bmatrix}, \qquad R_3 = \begin{bmatrix} 0.0489 & 0.0024 \\ 0.0024 & 0.0136 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0.1793 & -0.0356 \\ -0.0356 & 0.1487 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 0.4279 & 0.0051 \\ 0.0051 & 0.0779 \end{bmatrix},$$
$$Q_3 = \begin{bmatrix} 0.3817 & -0.0218 \\ -0.0218 & 0.1063 \end{bmatrix}, \qquad \Lambda_1 = \begin{bmatrix} 0.0651 & 0 \\ 0 & 0.0904 \end{bmatrix},$$
$$\Lambda_2 = \begin{bmatrix} 0.0803 & 0 \\ 0 & 0.3411 \end{bmatrix}, \qquad \lambda^* = 4.7129. \tag{47}$$

At the same time, we set $\tau_M = 5$; the state curve can be obtained as Figure 1 by using Matlab simulation software. From Figure 1, we can see that the system (1) is asymptotically stable.

In our study, our purpose is to compare the maximum upper bound τ_M for different τ_m. Now assume the different lower bound τ_m; by solving the LMIs (12), (13) and (14), we can get the maximum delays of upper bound, which are listed in Table 1. From Table 1, we can see that Theorem 8 is less conservative than the criterion proposed in [25, 26].

Remark 12. According to Theorem 8, we can obtain that system (1) with the pervious parameters is mean square exponentially stable, and meantime it is very easy to check that our results have improved the conclusions in [26] by the Matlab LMI Toolbox. If we negelect the uncertainty effect and take $\rho_1 = 0.2, \rho_2 = 0.2$ then apply the criteria in [26], the value of τ_M for mean square exponentially stable of system (1) is 55, respectively. While by using Theorem 8 in this paper, taking τ_M 62 into LMIs (12), (13) and (14), we find that LMIs (12), (13) and (14) are feasible, which shows that our result is less conservative than [26] under the identical conditions.

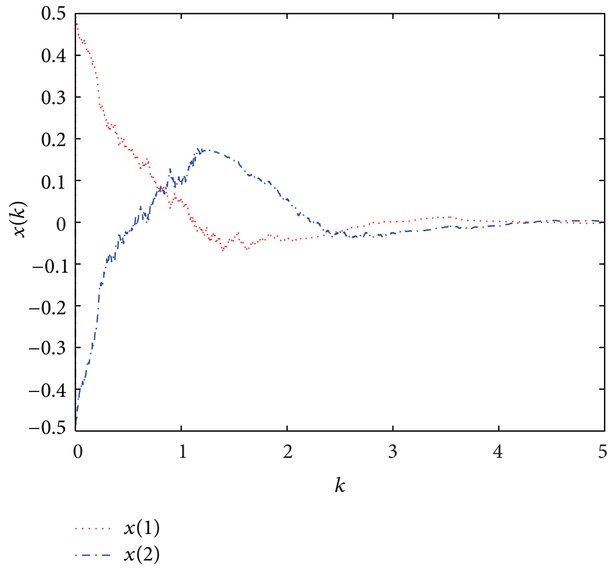


FIGURE 1: Trajectories of $x(k)$ of system in Example 1.

Example 2. Consider the neural networks (1) with the following parameters [23]:

$$A = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.3 & -0.1 & 0.2 \\ 0 & -0.3 & 0.2 \\ -0.1 & -0.1 & -0.2 \end{bmatrix},$$
$$W = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ -0.2 & 0.3 & 0.1 \\ 0.3 & -0.3 & 0.1 \end{bmatrix}. \tag{48}$$

Take the activation function as follows:

$$f_1(s) = \tanh(0.6s) - 0.2\sin(s), \qquad f_2(s) = \tanh(-0.4s),$$
$$f_3(s) = \tanh(-0.2s),$$
$$g_1(s) = \tanh(-0.4s) + 0.2\sin(s),$$
$$g_2(s) = \tanh(0.2s), \qquad g_3(s) = \tanh(0.4s). \tag{49}$$

From the previous parameters, it can be verified that

$$\Sigma_1 = \begin{bmatrix} -0.16 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Sigma_2 = \begin{bmatrix} -0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.1 \end{bmatrix},$$
$$\Sigma_3 = \begin{bmatrix} -0.12 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Sigma_4 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.2 \end{bmatrix}. \tag{50}$$

For the Corollary 1 in [23], using the previous parameters, we find that it is unsolvable, but it is solvable in our criteria in Theorem 8. By virtue of the Matlab Toolbox, we can obtain the feasible solutions as follows:

$$P = \begin{bmatrix} 41.9042 & -0.0335 & -0.2435 \\ -0.0335 & 43.3711 & 0.3170 \\ -0.2435 & 0.3170 & 43.0218 \end{bmatrix},$$
$$\Lambda_1 = \begin{bmatrix} 5.7657 & 0 & 0 \\ 0 & 11.3488 & 0 \\ 0 & 0 & 11.7506 \end{bmatrix},$$
$$\Lambda_2 = \begin{bmatrix} 19.8688 & 0 & 0 \\ 0 & 25.4560 & 0 \\ 0 & 0 & 22.0029 \end{bmatrix},$$
$$R_1 = \begin{bmatrix} 2.2528 & 0.0223 & -0.0231 \\ 0.0223 & 1.8145 & 0.0270 \\ -0.0231 & 0.0270 & 1.8773 \end{bmatrix},$$
$$R_2 = \begin{bmatrix} 0.5095 & 0.0818 & -0.0519 \\ 0.0818 & 25.4560 & -0.0313 \\ -0.0519 & -0.0313 & 0.6545 \end{bmatrix},$$
$$R_3 = \begin{bmatrix} 0.5494 & 0.0367 & -0.0231 \\ 0.0367 & 0.5644 & -0.0065 \\ -0.0231 & -0.0065 & 0.4522 \end{bmatrix},$$
$$\lambda^* = 46.4939. \tag{51}$$

Example 3. Consider the neural networks (1) with the following parameters [23, 27]:

$$A = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.3 & -0.1 & 0.2 \\ 0 & -0.3 & 0.2 \\ -0.1 & -0.1 & -0.2 \end{bmatrix}, \tag{52}$$
$$W = \begin{bmatrix} 0.2 & 0.1 & 0.1 \\ -0.2 & 0.3 & 0.1 \\ 0.3 & -0.3 & 0.1 \end{bmatrix}. \tag{53}$$

Take the activation function as follows:

$$\begin{aligned} &f_1(s) = \tanh(0.2s), && f_2(s) = \tanh(-0.4s), \\ &f_3(s) = \tanh(-0.2s), && g_1(s) = \tanh(-0.12s), \\ &g_2(s) = \tanh(0.2s), && g_3(s) = \tanh(0.4s). \end{aligned} \tag{54}$$

From the previous parameters, it can be verified that

$$\begin{aligned} \Sigma_1 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Sigma_2 = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}, \\ \Sigma_3 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Sigma_4 = \begin{bmatrix} -0.12 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}. \end{aligned} \tag{55}$$

For this example, these conditions in [23] cannot be satisfied. For $\tau_m = 2$, it has been verified that the maximum allowable time delay for system (42) is $\tau_M = 18$ in [27]. By letting $\tau_m = 2$ and $\tau_M = 26$ in Example 3, we find that systems (42) are feasible, which implies that the exponential stability result proposed in Corollary 10 in this paper provides less conservatism than in [23, 27].

5. Conclusion

An effective sum inequality has been introduced to derive the delay-dependent stability criteria for a class of discrete stochastic neural networks system with an interval time-varying delay. By choosing piecewise Lyapunov-Krasovskii functional candidate and employing the proposed sum inequalities, significant performance improvement has been achieved with noticeably reducing the number of LMIs scalar decision variables. All results are given by the form of LMIs. Numerical examples show that the achieved results are less conservative than some existing literatures.

References

[1] S. Arik, "An analysis of exponential stability of delayed neural networks with time varying delays," *Neural Networks*, vol. 17, no. 7, pp. 1027–1031, 2004.

[2] J. Cao and M. Xiao, "Stability and Hopf bifurcation in a simplified BAM neural network with two time delays," *IEEE Transactions on Neural Networks*, vol. 18, no. 2, pp. 416–430, 2007.

[3] J. Cao, K. Yuan, and H. X. Li, "Global asymptotical stability of recurrent neural networks with multiple discrete delays and distributed delays," *IEEE Transactions on Neural Networks*, vol. 17, no. 6, pp. 1646–1651, 2006.

[4] X. Lou, Q. Ye, and B. Cui, "Exponential stability of genetic regulatory networks with random delays," *Neurocomputing*, vol. 73, no. 4–6, pp. 759–769, 2010.

[5] H. Yang, T. Chu, and C. Zhang, "Exponential stability of neural networks with variable delays via LMI approach," *Chaos, Solitons and Fractals*, vol. 30, no. 1, pp. 133–139, 2006.

[6] H. Huang, Y. Qu, and H. X. Li, "Robust stability analysis of switched Hopfield neural networks with time-varying delay under uncertainty," *Physics Letters A*, vol. 345, no. 4–6, pp. 345–354, 2005.

[7] C. Song, H. Gao, and W. X. Zheng, "A new approach to stability analysis of discrete-time recurrent neural networks with time-varying delay," *Neurocomputing*, vol. 72, no. 10–12, pp. 2563–2568, 2009.

[8] J. Cao and J. Wang, "Global asymptotic and robust stability of recurrent neural networks with time delays," *IEEE Transactions on Circuits and Systems I*, vol. 52, no. 2, pp. 417–426, 2005.

[9] S. Xu, J. Lam, and D. W. C. Ho, "A new LMI condition for delay-dependent asymptotic stability of delayed Hopfield neural networks," *IEEE Transactions on Circuits and Systems II*, vol. 53, no. 3, pp. 230–234, 2006.

[10] Y. Zhang, D. Yue, and E. Tian, "New stability criteria of neural networks with interval time-varying delay: a piecewise delay method," *Applied Mathematics and Computation*, vol. 208, no. 1, pp. 249–259, 2009.

[11] Z. Wang, Y. Liu, and X. Liu, "On global asymptotic stability of neural networks with discrete and distributed delays," *Physics Letters A*, vol. 345, no. 4–6, pp. 299–308, 2005.

[12] H. Shao and Q. L. Han, "New stability criteria for linear discrete-time systems with interval-like time-varying delays," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 619–625, 2011.

[13] Z. Wu, H. Su, J. Chu, and W. Zhou, "Improved delay-dependent stability condition of discrete recurrent neural networks with time-varying delays," *IEEE Transactions on Neural Networks*, vol. 21, no. 4, pp. 692–697, 2010.

[14] Q. Song, J. Liang, and Z. Wang, "Passivity analysis of discrete-time stochastic neural networks with time-varying delays," *Neurocomputing*, vol. 72, no. 7–9, pp. 1782–1788, 2009.

[15] W. Zhang, Y. Huang, and H. Zhang, "Stochastic H_2/H_∞ control for discrete-time systems with state and disturbance dependent noise," *Automatica*, vol. 43, no. 3, pp. 513–521, 2007.

[16] W. Zhang, B. S. Chen, and C. S. Tseng, "Robust H_∞ filtering for nonlinear stochastic systems," *IEEE Transactions on Signal Processing*, vol. 53, no. 2, pp. 589–598, 2005.

[17] W. H. Zhang, G. Feng, and Q. H. Li, "Robust H_∞ filtering for general nonlinear stochastic state-delayed systems," *Mathematical Problems in Engineering*, vol. 2012, Article ID 231352, 15 pages, 2012.

[18] X. Wei, D. Zhou, and Q. Zhang, "On asymptotic stability of discrete-time non-autonomous delayed Hopfield neural networks," *Computers and Mathematics with Applications*, vol. 57, no. 11-12, pp. 1938–1942, 2009.

[19] S. Hu and J. Wang, "Global robust stability of a class of discrete-time interval neural networks," *IEEE Transactions on Circuits and Systems I*, vol. 53, no. 1, pp. 129–138, 2006.

[20] X. L. Zhu, Y. Wang, and G. H. Yang, "New delay-dependent stability results for discrete-time recurrent neural networks with time-varying delay," *Neurocomputing*, vol. 72, no. 13–15, pp. 3376–3383, 2009.

[21] W. Xiong and J. Cao, "Global exponential stability of discrete-time Cohen-Grossberg neural networks," *Neurocomputing*, vol. 64, no. 1–4, pp. 433–446, 2005.

[22] L. Wang and Z. Xu, "Sufficient and necessary conditions for global exponential stability of discrete-time recurrent neural networks," *IEEE Transactions on Circuits and Systems I*, vol. 53, no. 6, pp. 1373–1380, 2006.

[23] Y. Liu, Z. Wang, and X. Liu, "Robust stability of discrete-time stochastic neural networks with time-varying delays," *Neurocomputing*, vol. 71, no. 4–6, pp. 823–833, 2008.

[24] Q. Song and Z. Wang, "A delay-dependent LMI approach to dynamics analysis of discrete-time recurrent neural networks with time-varying delays," *Physics Letters A*, vol. 368, no. 1-2, pp. 134–145, 2007.

[25] Y. Ou, H. Liu, Y. Si, and Z. Feng, "Stability analysisof discrete-time stochastic neural networks with time-varying delays," *Neurocomputing*, vol. 73, no. 4–6, pp. 740–748, 2010.

[26] M. Luo, S. Zhong, R. Wang, and W. Kang, "Robust stability analysis for discrete-time stochastic neural networks systems with time-varying delays," *Applied Mathematics and Computation*, vol. 209, no. 2, pp. 305–313, 2009.

[27] Y. Zhang, S. Xu, and Z. Zeng, "Novel robust stability criteria of discrete-time stochastic recurrent neural networks with time delay," *Neurocomputing*, vol. 72, no. 13–15, pp. 3343–3351, 2009.

[28] Y. Zhang, D. Yue, and E. Tian, "Robust delay-distribution-dependent stability of discrete-time stochastic neural networks with time-varying delay," *Neurocomputing*, vol. 72, no. 4–6, pp. 1265–1273, 2009.

[29] X. M. Zhang and Q. L. Han, "A new finite sum inequality approach to delay-dependent H_∞ control of discrete-time systems with time-varying delay," *International Journal of Robust and Nonlinear Control*, vol. 18, no. 6, pp. 630–647, 2008.

[30] C. Peng, "Improved delay-dependent stabilisation criteria for discrete systems with a new finite sum inequality," *IET Control Theory & Applications*, vol. 6, no. 3, pp. 448–453, 2012.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 208263, 9 pages
http://dx.doi.org/10.1155/2013/208263

Research Article

Stochastic Stability for Time-Delay Markovian Jump Systems with Sector-Bounded Nonlinearities and More General Transition Probabilities

Dan Ye,[1,2] Quan-Yong Fan,[1] Xin-Gang Zhao,[2] and Guang-Hong Yang[1]

[1] *College of Information Science and Engineering, Northeastern University, Shenyang, Liaoning 110189, China*
[2] *State Key Laboratory of Robotics, Shenyang Institute of Automation, CAS, Shenyang, Liaoning 110016, China*

Correspondence should be addressed to Dan Ye; yedan@ise.neu.edu.cn

Received 29 January 2013; Accepted 20 March 2013

Academic Editor: Xuejun Xie

This paper is concerned with delay-dependent stochastic stability for time-delay Markovian jump systems (MJSs) with sector-bounded nonlinearities and more general transition probabilities. Different from the previous results where the transition probability matrix is completely known, a more general transition probability matrix is considered which includes completely known elements, boundary known elements, and completely unknown ones. In order to get less conservative criterion, the state and transition probability information is used as much as possible to construct the Lyapunov-Krasovskii functional and deal with stability analysis. The delay-dependent sufficient conditions are derived in terms of linear matrix inequalities to guarantee the stability of systems. Finally, numerical examples are exploited to demonstrate the effectiveness of the proposed method.

1. Introduction

During the past decades, much attention has been devoted to the stochastic systems since stochastic modeling has an important science and engineering application [1, 2]. As an important class of stochastic systems, MJSs have attracted a lot of interest, since they can be used to model many practical dynamical systems, such as power systems, manufacturing systems, and economic systems in which they may experience failure or repairs, abrupt environmental disturbances, and abrupt changes in the operating point of a nonlinear plant (see [3, 4]). Theoretical results of this class of systems have been applied in many processes, such as target tracking, manufactory processes, solar thermal receivers, and fault-detection and fault-tolerant systems. Some examples can be found in [5–7].

For MJSs, the transition probabilities of the jumping process are important, but most of the previous issues on this kind of systems usually assumed that the elements of the transition probability matrix are completely known. Based on this condition, a lot of research results have been worked out in the literature, such as [8–13]. However, in many practical engineering applications, we cannot get all of the transition probabilities elements, and some elements might be time variable in some cases. Therefore, the study of Markov jump systems with partly known transition probabilities becomes necessary and some well-known results have been published, see [11–13]. In some cases, the exact value of some elements of the transition probability matrix can not be obtained, but their lower and upper bounds can be determined. The case has been addressed by [8], but its method cannot be used to deal with the case where there is no information available for transition probabilities. In [9], the more general partly known transition probabilities are investigated, which covers the cases that the transition probabilities are exactly known, unknown, and unknown but with known bounds. However, a conservative condition to relax inequality is used to deal with this kind of transition probabilities, and the time delay is not considered.

It is well known that time delay is very common in variously practical systems such as manufacturing systems, chemical processes, network control systems (NCSs), and telecommunication and economic systems. They are often the causes of oscillation, instability, and poor performance in

many control systems. In the past years, time-delay system has been widely studied, and many analysis results have been reported (see [14, 15]). However, many approaches for time-delay problems only employed partial information of the delay-related terms. The quadratic integral terms of the form $\int_{t-h(t)}^{t} f(\alpha)d(\alpha)$ have been the major choices to construct the Lyapunov-Krasovskii functional, ignoring the employment of the delay upper bound $\overline{h}$ (see [16]). Recently, Markovian systems with time delay have been widely studied (see [17–19]). But the time delays in [17, 19] are invariant, which is limited in the practical application. Although [18] considers the systems with time-varying delay, some useful terms are ignored to construct the Lyapunov-Krasovskii functional, which may lead to considerable conservativeness. Unfortunately, in the aforementioned papers about time delay, the elements of the considered transition probability matrix are completely known. To the best of the authors' knowledge, up to now, there are few papers concerning both time delay and more general transition probability to deal with stochastic stability for nonlinear Markov jump systems.

Motivated by the previous points, in this paper, we are concerned with the stochastic stability analysis for a class of nonlinear MJSs with time-varying delay and general transition probabilities. The considered nonlinearity is the so-called sector-bounded nonlinearity [20], which is quite more general than the usual Lipschitz conditions. By exploiting full information of the delay-related terms including $h(t)$ and $\overline{h}$ and inherent relationship between the elements in transition probabilities, less conservative stability criterions are obtained in the framework of LMIs. Numerical examples are given to demonstrate the superiorities and effectiveness of the proposed method.

Notations. The superscript T stands for matrix transposition, and R^n denotes the n dimensional Euclidean space. $E[\cdot]$ stands for the n mathematical expectation; $He(X)$ represents the sum of X and X^T. In addition, in symmetric block matrices or long matrix expressions, $*$ is used as an ellipsis for the terms that are introduced by symmetry. The notation $P > 0$ (≥ 0) means that P is real symmetric positive (semipositive) definite. $\varepsilon[\cdot]$ stands for the expectation operator with respect to the given probability measure P. $L^2_{\mathscr{F}_0}([-\overline{h}, 0]; \mathbb{R}^n)$ denotes the family of $\mathbb{R}^n$-valued stochastic process $x(t)$, $-\overline{h} \leq t \leq 0$, such that $x(t)$ is $\mathscr{F}_0$-measurable and $\int_{-\overline{h}}^{0} E\{\|x(t)\|^2\}dt < \infty$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Statement

Consider a class of stochastic nonlinear MJSs with sector-bounded nonlinearity and time-varying delay as follows:

$$\begin{aligned} \dot{x}(t) &= A(r(t))x(t) + A_d(r(t))x(t-h(t)) \\ &\quad + f(t, x(t), r(t)) + f_d(t, x(t-h(t)), r(t)), \\ x(t) &= \psi(t), \quad -\overline{h} \leq t \leq 0, \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state variable, and $A(r(t)) \in R^{n\times n}$ and $A_d(r(t)) \in R^{n\times n}$ are known matrix functions of the Markov jump process $r(t)$. $f(*)$ and $f_d(*)$ are the vector-valued nonlinear functions. Also $h(t) \in [0, \overline{h}]$, $\dot{h}(t) \leq \overline{\mu}$, and $\psi(t) \in L^2_{\mathscr{F}_0}([-\overline{h}, 0]; \mathbb{R}^n)$. $r(t)$ is a time-homogeneous Markov process with right continuous trajectories and takes values on the finite set $\mathbb{L} = \{1, 2, \ldots, \mathbb{N}\}$ with the following mode transition probabilities:

$$Pr\{r(t+dt) = j \mid r(t) = i\} = \begin{cases} \pi_{ij}dt + o(dt), & i \neq j, \\ 1 + \pi_{ii}dt + o(dt), & i = j, \end{cases} \tag{2}$$

where $dt > 0$, where $\lim_{dt\to 0}(o(dt)/dt) = 0$. $\pi_{ij} \geq 0$, for $i \neq j$, is the transition rate from mode i to mode j, and

$$\sum_{j=1, i\neq j}^{\mathbb{N}} \pi_{ij} = -\pi_{ii}, \quad i = (1, \ldots, \mathbb{N}). \tag{3}$$

In this paper, more general transition probabilities of the jumping process in [9] are considered. Namely, some elements in matrix have been exactly known, some ones have been merely known with lower and upper bounds, and others may have no information to use. For instance, the transition probability matrix might be described by matrix

$$\begin{bmatrix} \pi_{11} & ? & \pi_{13} & \cdots & \pi_{1\mathbb{N}} \\ ? & ? & \alpha & \cdots & \pi_{2\mathbb{N}} \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ \pi_{\mathbb{N}1} & \beta & \cdots & \cdots & \pi_{\mathbb{N}\mathbb{N}} \end{bmatrix}, \tag{4}$$

where "?" denotes the inaccessible elements, α and β have known lower and upper bounds ($\underline{\alpha} \leq \alpha \leq \overline{\alpha}$ and $\underline{\beta} \leq \beta \leq \overline{\beta}$) and π_{ij} is exactly known.

For notation clarity, for all $i, j \in \mathbb{L}$, we further rewrite $\mathbb{L} = \mathbb{L}_k^i \cup \mathbb{L}_{uk}^i$ with

$$\begin{aligned} \mathbb{L}_k^i &\triangleq \{j : \underline{\pi}_{ij} \leq \pi_{ij} \leq \overline{\pi}_{ij}\}, \\ \mathbb{L}_{uk}^i &\triangleq \{j : j \notin \mathbb{L}_k^i\}, \\ \mathscr{L}_k^i &\triangleq \{m \mid m \in \mathbb{L}_k^i, m \neq i\}, \\ \mathscr{L}_{uk}^i &\triangleq \{m \mid m \in \mathbb{L}_{uk}^i, m \neq i\}, \end{aligned} \tag{5}$$

where $\underline{\pi}_{ij}$ and $\overline{\pi}_{ij}$ are known lower and upper bounds of π_{ij}, respectively.

Remark 1. In order to get more general result, this paper considers that $\mathbb{L}_k^i$ includes two cases. One is that π_{ij} is exactly known, and the other is that it is unknown but with upper and lower known bounds, which has been considered in [8]. If π_{ij} is exactly known, it also can be seen as a rate with equal bounds.

The set $\mathbb{L}$ comprises the various operation modes of system (1), and for each possible value of $r(t) = i$ ($i \in \mathbb{L}$),

the matrices and the vector-valued nonlinear functions associated with the ith mode will be denoted by

$$A_i = A(r(t) = i), \qquad A_{di} = A_d(r(t) = i),$$
$$f_i(t) = f(t, x(t), r(t) = i), \tag{6}$$
$$f_{di}(t) = f_d(t, x(t - h(t)), r(t) = i).$$

In system (1), the vector-valued nonlinear functions $f(*)$ and $f_d(*)$ are assumed to satisfy the following sector-bounded conditions:

$$\begin{aligned} &[f(x) - f(y) - M_1(x-y)]^T \\ &\quad \times [f(x) - f(y) - M_2(x-y)] \le 0, \\ &[f_d(x) - f_d(y) - N_1(x-y)]^T \\ &\quad \times [f_d(x) - f_d(y) - N_2(x-y)] \le 0, \end{aligned} \tag{7}$$

where for all $x, y \in R^n$, $M_1, M_2, N_1, N_2 \in R_{n\times n}$ are known constant matrices. Without loss of generality, the following equations are always assumed:

$$f(0) = 0, \qquad f_d(0) = 0. \tag{8}$$

Remark 2. As in [20], the nonlinear functions $f(*)$, $f_d(*)$ are said to belong to sectors. In other words, the nonlinearities are bounded by sectors. Further, both the filter design problems and control analysis have been investigated; see [21, 22]. The nonlinear descriptions in (7) are more general than the usual sigmoid functions and the recently commonly used Lipschitz conditions [21], and M_1, M_2, N_1, N_2 are lower and upper slope bounds, respectively.

Lemma 3. *Assume that $f(*)$ is a vector-valued nonlinear function and M_1, M_2 are known constant matrices. If there exists*

$$[f(x) - M_1 x]^T [f(x) - M_2 x] \le 0, \tag{9}$$

it can be obtained that

$$\begin{bmatrix} x \\ f(x) \end{bmatrix}^T \begin{bmatrix} \overline{M}_1 & \overline{M}_2 \\ * & I \end{bmatrix} \begin{bmatrix} x \\ f(x) \end{bmatrix} \le 0, \tag{10}$$

with $\overline{M}_1 = (M_1^T M_2 + M_2^T M_1)/2$, $\overline{M}_2 = -(M_1^T + M_2^T)/2$.

Proof. Notice the fact that

$$\begin{aligned} &[f(x) - M_1 x]^T [f(x) - M_2 x] \le 0, \\ &[f(x) - M_2 x]^T [f(x) - M_1 x] \le 0; \end{aligned} \tag{11}$$

so,

$$\begin{aligned} 0.5 \{ &[f(x) - M_1 x]^T [f(x) - M_2 x] \\ &+ [f(x) - M_2 x]^T [f(x) - M_1 x] \} \le 0. \end{aligned} \tag{12}$$

Then, (10) can be obtained. Thus, the proof is complete. □

For the sake of simplicity, the solution $x(t, x_0, r_0)$ of system (1) with $r_0 \in \mathbb{L}$ is denoted by $x(t)$. It is known that $\{x(t), r(t)\}$ is a Markov process with an initial state (x_0, r_0), and its weak infinitesimal generator acting on function V is defined as follows:

$$\begin{aligned} &\xi V(x(t), r(t), t) \\ &\quad = \lim_{\Delta \to 0^+} \frac{1}{\Delta} [\varepsilon (V(x(t+\Delta), t+\Delta, r(t+\Delta)) \mid x(t), r(t) = i) \\ &\qquad\qquad - V(x(t), r(t), t)]. \end{aligned} \tag{13}$$

Throughout this paper, the following definition is used.

Definition 4. System (1) is said to be stochastically stable if for finite $\psi(t) \in R^n$ defined on $[-\overline{h}, 0]$ and $r(0) \in \mathbb{L}$, the following is satisfied:

$$\lim_{t\to\infty} \varepsilon \left\{ \int_0^t x^T(s, \psi, r(0))\, x(s, \psi, r(0))\, ds \right\} < \infty, \tag{14}$$

where $x(s, \psi, r(0))$ denotes the solution to system at time t under the initial conditions $\psi(t)$ and $r(0)$.

3. Stochastic Stability Analysis

In this section, new LMI-based sufficient conditions for system (1) will be presented based on a special Lyapunov-Krasovskii functional and Lemma 3.

For presentation convenience, it is necessary to define

$$\begin{aligned} \chi_i(t) &= \left[x^T(t), x^T(t - h(t)), x^T\left(t - \overline{h}\right), \right. \\ &\qquad \left. \dot{x}^T(t), f_i^T(t), f_{di}^T(t) \right]^T \\ &= e_1 x(t) + e_2 x(t - h(t)) + e_3 x\left(t - \overline{h}\right) \\ &\quad + e_4 \dot{x}(t) + e_5 f_i(t) + e_6 f_{di}(t), \end{aligned} \tag{15}$$

and the corresponding block entry matrices are e_j, $j \in \{1, \ldots, 6\}$. In addition, there are some other definitions. Consider

$$\begin{aligned} &\lambda_k^i = \sum_{m \in \mathcal{L}_k^i} \pi_{im}, \qquad \underline{\lambda}_k^i = \sum_{m \in \mathcal{L}_k^i} \underline{\pi}_{im}, \qquad \lambda_{uk}^i = \sum_{m \in \mathcal{L}_{uk}^i} \pi_{im}, \\ &\mathcal{P}_k^i = \sum_{m \in \mathcal{L}_k^i} \pi_{ij} P_m, \qquad \overline{\mathcal{P}}_k^i = \sum_{m \in \mathcal{L}_k^i} \overline{\pi}_{ij} P_m, \end{aligned} \tag{16}$$

Theorem 5. *For given scalars $\overline{h} > 0$ and $\overline{\mu}$, the delayed system (1) with $\dot{h}(t) \le \overline{\mu}$ is stochastically stable if there exist scalars $\eta_{1i} > 0$ and $\eta_{2i} > 0$ and matrices $P_i > 0$, $Q_{0i} > 0$, $Q_{1i} > 0$,*

$S_0 > 0$, $S_1 > 0$, $R > 0$, $Y_{11}^i, Y_{12}^i, Y_{22}^i, Z_{11}^i, Z_{12}^i, Z_{22}^i$, Φ_i, H_{1i}, and H_{2i} such that the following conditions hold:

$$\begin{aligned} &He\left(\Phi_i\left[A_i e_1^T + A_{di}e_2^T - e_4^T + e_5^T + e_6^T\right]\right. \\ &\qquad \left. + Z_{12}^i(e_2-e_3)^T + Y_{12}^i(e_1-e_2)^T\right) \\ &\quad + He\left(e_4 P_i e_1^T\right) - e_3 Q_{0i} e_3^T + \overline{h} e_4 \left(S_0 + S_1\right) e_4^T \\ &\quad - \left(1-\overline{\mu}\right) e_2 Q_{1i} e_2^T + \overline{h} Y_{11}^i + e_1\left(\overline{h}R + Q_{0i} + Q_{1i} + \Psi_i\right) e_1^T \\ &\quad - \eta_{1i}\left(e_1 \overline{M}_{1i} e_1^T + e_5 \overline{M}_{2i}^T e_1^T + e_1 \overline{M}_{2i} e_5^T + e_5 e_5^T\right) \\ &\quad - \eta_{2i}\left(e_2 \overline{N}_{1i} e_2^T + e_6 \overline{N}_{2i}^T e_2^T + e_2 \overline{N}_{2i} e_6^T + e_6 e_6^T\right) < 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &He\left(\Phi_i\left[A_i e_1^T + A_{di}e_2^T - e_4^T + e_5^T + e_6^T\right]\right. \\ &\qquad \left. + Z_{12}^i(e_2-e_3)^T + Y_{12}^i(e_1-e_2)^T\right) \\ &\quad + He\left(e_4 P_i e_1^T\right) - e_3 Q_{0i} e_3^T + \overline{h} e_4 S_0 e_4^T - \left(1-\overline{\mu}\right) e_2 Q_{1i} e_2^T \\ &\quad + \overline{h} Z_{11}^i + e_1\left(\overline{h}R + Q_{0i} + Q_{1i} + \Psi_i\right) e_1^T \\ &\quad - \eta_{1i}\left(e_1 \overline{M}_{1i} e_1^T + e_5 \overline{M}_{2i}^T e_1^T + e_1 \overline{M}_{2i} e_5^T + e_5 e_5^T\right) \\ &\quad - \eta_{2i}\left(e_2 \overline{N}_{1i} e_2^T + e_6 \overline{N}_{2i}^T e_2^T + e_2 \overline{N}_{2i} e_6^T + e_6 e_6^T\right) < 0, \end{aligned} \tag{18}$$

$$\begin{bmatrix} Y_{11}^i & Y_{12}^i \\ * & Y_{22}^i \end{bmatrix} \geq 0, \qquad \begin{bmatrix} Z_{11}^i & Z_{12}^i \\ * & Z_{22}^i \end{bmatrix} \geq 0, \tag{19}$$

$$S_0 \geq Z_{22}^i, \qquad S_0 + \left(1 - \overline{\mu} S_1\right) \geq Y_{22}^i, \tag{20}$$

$$i \in \mathbb{L}_k^i \begin{cases} \displaystyle\sum_{m\in\mathscr{L}_k^i} \overline{\pi}_{im}\left(Q_{0m}+Q_{1m}\right) + \overline{\pi}_{ii}\left(Q_{0i}+Q_{1i}\right) \\ \quad -\left(\underline{\lambda}_k^i + \underline{\pi}_{ii}\right)\left(Q_{0j}+Q_{1j}\right) - R < 0, \quad j \in \mathbb{L}_{uk}^i, \\ \displaystyle\sum_{m\in\mathscr{L}_k^i} \overline{\pi}_{im} Q_{0m} + \overline{\pi}_{ii} Q_{0i} - \left(\underline{\lambda}_k^i + \underline{\pi}_{ii}\right) Q_{0j} - R < 0, \\ \qquad\qquad\qquad\qquad j \in \mathbb{L}_{uk}^i, \end{cases} \tag{21}$$

$$i \in \mathbb{L}_{uk}^i \begin{cases} \displaystyle\sum_{m\in\mathbb{L}_k^i} \overline{\pi}_{im}\left(Q_{0m}+Q_{1m}\right) - \sum_{m\in\mathbb{L}_k^i} \underline{\pi}_{im} H_{2i} - R < 0, \\ \displaystyle\sum_{m\in\mathbb{L}_k^i} \overline{\pi}_{im} Q_{0m} - \sum_{m\in\mathbb{L}_k^i} \underline{\pi}_{im} H_{2i} - R < 0, \\ Q_{0j} + Q_{1j} - H_{2i} \leq 0, \quad j \in \mathbb{L}_{uk}^i,\ j \neq i, \\ Q_{0j} - H_{2i} \geq 0, \quad j \in \mathbb{L}_{uk}^i,\ j = i, \\ P_j - H_{1i} \leq 0, \quad j \in \mathbb{L}_{uk}^i,\ j \neq i, \\ P_j - H_{1i} \geq 0, \quad j \in \mathbb{L}_{uk}^i,\ j = i, \end{cases} \tag{22}$$

where

$$\Psi_i = \begin{cases} \overline{\mathscr{P}}_k^i + \overline{\pi}_{ii} P_i - \left(\underline{\pi}_{ii} + \underline{\lambda}_k^i\right) P_l, & i \in \mathbb{L}_k^i,\ l \in \mathbb{L}_{uk}^i, \\ \overline{\mathscr{P}}_k^i - \displaystyle\sum_{m\in\mathbb{L}_k^i} \underline{\pi}_{im} H_{1i}, & i \in \mathbb{L}_{uk}^i, \end{cases}$$

$$\begin{aligned} \overline{M}_1 &= \frac{\left(M_1^T M_2 + M_2^T M_1\right)}{2}, \qquad \overline{M}_2 = -\frac{\left(M_1^T + M_2^T\right)}{2}, \\ \overline{N}_1 &= \frac{\left(N_1^T N_2 + N_2^T N_1\right)}{2}, \qquad \overline{N}_2 = -\frac{\left(N_1^T + N_2^T\right)}{2}, \end{aligned} \tag{23}$$

Proof. First, in order to cast the model involved into the framework of the Markov processes, a new process is defined as $\{(x_t, r(t)), t \geq 0\}$ by

$$x_t(s) = x(t+s), \quad -2\overline{h} \leq s \leq 0; \tag{24}$$

so, $\{(x_t, r(t)), t \geq \overline{h}\}$ is a Markov process with initial state $(\psi(\cdot), r(0))$.

Then, the stochastic Lyapunov-Krasovskii functional is chosen as follows:

$$V\left(x_t, r(t), t\right) = \sum_{j=1}^{6} V_j\left(x_t, r(t), t\right), \tag{25}$$

where

$$\begin{aligned} V_1\left(x_t, r(t), t\right) &= x^T(t) P(r(t)) x(t), \\ V_2\left(x_t, r(t), t\right) &= \int_{t-\overline{h}}^{t} x^T(\alpha) Q_0(r(t)) x(\alpha)\, d\alpha, \\ V_3\left(x_t, r(t), t\right) &= \int_{t-h(t)}^{t} x^T(\alpha) Q_1(r(t)) x(\alpha)\, d\alpha, \\ V_4\left(x_t, r(t), t\right) &= \int_{-\overline{h}}^{0}\int_{t+\beta}^{t} \dot{x}^T(\alpha) S_0 \dot{x}(\alpha)\, d\alpha\, d\beta, \\ V_5\left(x_t, r(t), t\right) &= \int_{-h(t)}^{0}\int_{t+\beta}^{t} \dot{x}^T(\alpha) S_1 \dot{x}(\alpha)\, d\alpha\, d\beta, \\ V_6\left(x_t, r(t), t\right) &= \int_{-\overline{h}}^{0}\int_{t+\beta}^{t} x^T(\alpha) R x(\alpha)\, d\alpha\, d\beta, \\ P(r(t)) > 0, &\quad Q_0(r(t)) > 0, \quad Q_1(r(t)) > 0, \\ S_0 > 0, &\quad S_1 > 0, \quad R > 0. \end{aligned} \tag{26}$$

The infinitesimal generator of the Markov process acting on $V(\cdot)$ and emanating from the point $r(t) = i$ $(i \in \mathbb{L})$ is given as follows:

$$\begin{aligned} &\xi V\left(x_t, r(t), t\right) \\ &\quad = 2\chi_i^T(t) e_4 P_i e_1^T \chi_i(t) \\ &\qquad + \chi_i^T(t)\left[e_1 \sum_{j=1}^{\mathbb{N}} \pi_{ij} P_j e_1^T + e_1 Q_{0i} e_1^T - e_3 Q_{0i} e_3^T\right] \chi_i(t) \\ &\qquad + \int_{t-\overline{h}}^{t} x^T(\alpha) \sum_{j=1}^{\mathbb{N}} \pi_{ij} Q_{0j} x(\alpha)\, d\alpha \end{aligned}$$

$$
\begin{aligned}
&+\chi_i^T(t)\left[e_1Q_{1i}e_1^T-\left(1-\dot h(t)\right)e_2Q_{1i}e_2^T\right]\chi_i(t)\\
&+\int_{t-h(t)}^{t}x^T(\alpha)\sum_{j=1}^{\mathbb{N}}\pi_{ij}Q_{1j}x(\alpha)\,d\alpha\\
&+\bar h\chi_i^T(t)\,e_4S_0e_4^T\chi_i(t)\\
&-\int_{t-\bar h}^{t}\dot x^T(\alpha)\,S_0\dot x(\alpha)\,d\alpha+h(t)\,\chi_i^T(t)\,e_4S_1e_4^T\chi_i(t)\\
&-\left(1-\dot h(t)\right)\int_{t-h(t)}^{t}\dot x^T(\alpha)\,S_1\dot x(\alpha)\,d\alpha\\
&+\bar h\chi_i^T(t)\,e_1Re_1^T\chi_i(t)-\int_{t-\bar h}^{t}x^T(\alpha)\,Rx(\alpha)\,d\alpha.
\end{aligned}
\tag{27}
$$

Then, choose

$$
\begin{bmatrix} Y_{11}^i & Y_{12}^i\\ * & Y_{22}^i\end{bmatrix}\ge 0,\qquad \begin{bmatrix} Z_{11}^i & Z_{12}^i\\ * & Z_{22}^i\end{bmatrix}\ge 0; \tag{28}
$$

it follows that

$$
\begin{aligned}
&\int_{t-h(t)}^{t}\begin{bmatrix}\chi_i(t)\\ \dot x(\alpha)\end{bmatrix}^T\begin{bmatrix} Y_{11}^i & Y_{12}^i\\ * & Y_{22}^i\end{bmatrix}\begin{bmatrix}\chi_i(t)\\ \dot x(\alpha)\end{bmatrix}d\alpha\\
&\quad=\chi_i^T(t)\left[He\left(Y_{12}^i(e_1-e_2)^T\right)+h(t)\,Y_{11}^i\right]\chi_i(t)\\
&\qquad+\int_{t-h(t)}^{t}\dot x^T(\alpha)\,Y_{22}\dot x(\alpha)\,d\alpha\ge 0,\\
&\int_{t-\bar h}^{t-h(t)}\begin{bmatrix}\chi_i(t)\\ \dot x(\alpha)\end{bmatrix}^T\begin{bmatrix} Z_{11}^i & Z_{12}^i\\ * & Z_{22}^i\end{bmatrix}\begin{bmatrix}\chi_i(t)\\ \dot x(\alpha)\end{bmatrix}d\alpha\\
&\quad=\chi_i^T(t)\left[He\left(Z_{12}^i(e_2-e_3)^T\right)+\left(\bar h-h(t)\right)Z_{11}^i\right]\chi_i(t)\\
&\qquad+\int_{t-\bar h}^{t-h(t)}\dot x^T(\alpha)\,Z_{22}\dot x(\alpha)\,d\alpha\ge 0.
\end{aligned}
\tag{29}
$$

And the constraints of the model dynamics can be written as $\chi_i^T(t)\Phi_i[A_ie_1^T+A_{di}e_2^T-e_4^T+e_5^T+e_6^T]\chi_i(t)\equiv 0$ by introducing free variables Φ_i. In addition, according to Lemma 3 and choosing scalar $\eta_{1i}>0,\eta_{2i}>0$, the following inequalities hold:

$$
\begin{aligned}
&-\eta_{1i}\chi_i^T(t)\left(e_1\overline{M}_{1i}e_1^T+e_5\overline{M}_{2i}^Te_1^T+e_1\overline{M}_{2i}e_5^T+e_5e_5^T\right)\chi_i(t)\ge 0,\\
&-\eta_{2i}\chi_i^T(t)\left(e_2\overline{N}_{1i}e_2^T+e_6\overline{N}_{2i}^Te_2^T+e_2\overline{N}_{2i}e_6^T+e_6e_6^T\right)\chi_i(t)\ge 0.
\end{aligned}
\tag{30}
$$

So, $\xi V(x(t),r(t),t)$ can be upper bounded as

$$
\begin{aligned}
&\xi V\left(x_t,r(t),t\right)\\
&\quad\le\chi_i^T(t)\left\{\Omega_1(h(t))+e_1\sum_{j=1}^{\mathbb{N}}\pi_{ij}P_je_1^T\right\}\chi_i(t)+\Omega_2+\Omega_3,
\end{aligned}
\tag{31}
$$

where

$$
\begin{aligned}
\Omega_1(h(t))&=h(t)\left(Y_{11}^i+e_4S_1e_4^T\right)+\left(\bar h-h(t)\right)Z_{11}^i\\
&\quad+He\left[\Phi_i\left(A_ie_1^T+A_{di}e_2^T-e_4^T+e_5^T+e_6^T\right)\right.\\
&\qquad\left.+e_4P_ie_1^T+Y_{12}^i(e_1-e_2)^T+Z_{12}^i(e_2-e_3)^T\right]\\
&\quad+\bar he_4S_0e_4^T-e_3Q_{0i}e_3^T-(1-\bar\mu)\,e_2Q_{1i}e_2^T\\
&\quad+e_1\left(Q_{0i}+Q_{1i}+\bar hR\right)e_1^T\\
&\quad-\eta_{1i}\left(e_1\overline{M}_{1i}e_1^T+e_5\overline{M}_{2i}^Te_1^T+e_1\overline{M}_{2i}e_5^T+e_5e_5^T\right)\\
&\quad-\eta_{2i}\left(e_2\overline{N}_{1i}e_2^T+e_6\overline{N}_{2i}^Te_2^T+e_2\overline{N}_{2i}e_6^T+e_6e_6^T\right),\\
\Omega_2&=-\int_{t-\bar h}^{t-h(t)}\dot x^T(\alpha)\left(S_0-Z_{22}^i\right)\dot x(\alpha)\,d\alpha\\
&\quad-\int_{t-h(t)}^{t}\dot x^T(\alpha)\left(S_0-Y_{22}^i+(1-\bar\mu)\,S_1\right)\dot x(\alpha)\,d\alpha,\\
\Omega_3&=\int_{t-h(t)}^{t}x^T(\alpha)\left[\sum_{j=1}^{\mathbb{N}}\pi_{ij}\left(Q_{0j}+Q_{1j}\right)-R\right]x(\alpha)\,d\alpha\\
&\quad+\int_{t-\bar h}^{t-h(t)}x^T(\alpha)\left[\sum_{j=1}^{\mathbb{N}}\pi_{ij}Q_{0j}-R\right]x(\alpha)\,d\alpha.
\end{aligned}
\tag{32}
$$

Since $\Omega_1(h(t))$ is a convex combination of the matrices $Y_{11}^i+e_4S_1e_4^T$ and Z_{11}^i on $h(t)$, it can be handled nonconservatively by two corresponding boundary LMIs: one for $h(t)=\bar h$ and the other for $h(t)=0$.

So, $\Omega_1(h(t))<0$ is equivalent to

$$
\begin{aligned}
&\Omega_1(0)<0,\\
&\Omega_1\left(\bar h\right)<0.
\end{aligned}
\tag{33}
$$

Then, the proof is separated into two cases, $i\in\mathbb{L}_k^i$ and $i\in\mathbb{L}_{uk}^i$.

Case 1 ($i\in\mathbb{L}_k^i$). In this case, π_{ii} is bounded or exactly known. Based on $\sum_{j=1}^{\mathbb{N}}\pi_{ij}=\lambda_k^i+\pi_{ii}+\lambda_{uk}^i=0$ and $-\sum_{l\in\mathscr{L}_{uk}^i}\pi_{il}/(\pi_{ii}+\lambda_k^i)=1$, it is easy to obtain that

$$
\begin{aligned}
&\Omega_1(h(t))+e_1\sum_{j=1}^{\mathbb{N}}\pi_{ij}P_je_1^T\\
&\quad=\Omega_1(h(t))+e_1\left(\sum_{j\in\mathscr{L}_k^i}\pi_{ij}P_j+\pi_{ii}P_i+\sum_{l\in\mathscr{L}_{uk}^i}\pi_{il}P_l\right)e_1^T
\end{aligned}
$$

$$= -\frac{\sum_{l\in\mathscr{L}_{uk}^i}\pi_{il}}{\pi_{ii}+\lambda_k^i}\left\{\Omega_1(h(t)) + e_1\left[\sum_{j\in\mathscr{L}_k^i}\pi_{ij}P_j + \pi_{ii}P_i - \left(\pi_{ii}+\lambda_k^i\right)P_l\right]e_1^T\right\}. \tag{34}$$

Considering the fact that $0 \le -\pi_{il}/(\pi_{ii}+\lambda_k^i) \le 1$, it is easy to obtain that $\Omega_1(h(t)) + e_1[\sum_{j\in\mathscr{L}_k^i}\pi_{ij}P_j + \pi_{ii}P_i - (\pi_{ii}+\lambda_k^i)P_l]e_1^T < 0$, $l \in \mathbb{L}_{uk}^i$ can guarantee $\Omega_1(h(t)) + e_1\sum_{j=1}^{\mathbb{N}}\pi_{ij}P_je_1^T < 0$.

Since $\underline{\pi}_{ij} \le \pi_{ij} \le \overline{\pi}_{ij}$, the following inequality can be obtained:

$$\begin{aligned}\Omega_1(h(t)) &+ e_1\left[\sum_{j\in\mathscr{L}_k^i}\pi_{ij}P_j + \pi_{ii}P_i - \left(\pi_{ii}+\lambda_k^i\right)P_l\right]e_1^T \\ &\le \Omega_1(h(t)) + e_1\left[\overline{\mathscr{P}}_k^i + \overline{\pi}_{ii}P_i - \left(\underline{\pi}_{ii}+\underline{\lambda}_k^i\right)P_l\right]e_1^T.\end{aligned} \tag{35}$$

So, if (17) and (18) hold, $\Omega_1(h(t)) + e_1\sum_{j=1}^{\mathbb{N}}\pi_{ij}P_je_1^T < 0$ can be obtained.

Similarly, (21) $\Rightarrow \Omega_3 < 0$.

And (20) $\Rightarrow \Omega_2 < 0$.

Therefore, when (17)–(21) are satisfied, $\xi V(x_t, r(t), t) < 0$.

Case 2 ($i \in \mathbb{L}_{uk}^i$). Because $\sum_{j=1}^{\mathbb{N}}\pi_{ij} = 0$, there exist

$$\begin{aligned} -x^T(t)\sum_{j=1}^{\mathbb{N}}\pi_{ij}H_{1i}x(t) &= 0, \\ -\int_{t-h}^{t}x^T(\alpha)\sum_{j=1}^{\mathbb{N}}\pi_{ij}H_{2i}x(\alpha)\,d\alpha &= 0.\end{aligned} \tag{36}$$

Adding the left sides of (36) into (31), the inequality can be written as

$$\xi V(x_t, r(t), t) \le \chi^T(t)\Gamma_1(h(t))\chi(t) + \Gamma_2 + \Gamma_3, \tag{37}$$

where

$$\begin{aligned}\Gamma_1(h(t)) = {} & h(t)\left(Y_{11}^i + e_4S_1e_4^T\right) + \left(\overline{h} - h(t)\right)Z_{11}^i \\ & + He\left[\Phi_i\left(A_ie_1^T + A_{di}e_2^T - e_4^T + e_5^T + e_6^T\right)\right. \\ & \left. + e_4P_ie_1^T + Y_{12}^i(e_1-e_2)^T + Z_{12}^i(e_2-e_3)^T\right] \\ & + \overline{h}e_4S_0e_4^T - e_3Q_{0i}e_3^T - (1-\overline{\mu})e_2Q_{1i}e_2^T \\ & + e_1\left[Q_{0i} + Q_{1i} + \overline{h}R + \sum_{j\in\mathbb{L}_k^i}\pi_{ij}\left(P_j - H_{1i}\right)\right]e_1^T \\ & + e_1\sum_{j\in\mathbb{L}_{uk}^i}\pi_{ij}\left(P_j - H_{1i}\right)e_1^T \\ & - \eta_{1i}\left(e_1\overline{M}_{1i}e_1^T + e_5\overline{M}_{2i}^Te_1^T + e_1\overline{M}_{2i}e_5^T + e_5e_5^T\right) \\ & - \eta_{2i}\left(e_2\overline{N}_{1i}e_2^T + e_6\overline{N}_{2i}^Te_2^T + e_2\overline{N}_{2i}e_6^T + e_6e_6^T\right),\end{aligned}$$

$$\begin{aligned}\Gamma_2 = {} & \Omega_2, \\ \Gamma_3 = {} & \int_{t-h(t)}^{t}x^T(\alpha)\left[\sum_{j\in\mathbb{L}_k^i}\pi_{ij}\left(Q_{0j} + Q_{1j} - H_{2i}\right) - R + \sum_{j\in\mathbb{L}_{uk}^i}\pi_{ij}\left(Q_{0j} + Q_{1j} - H_{2i}\right)\right]x(\alpha)\,d\alpha \\ & + \int_{t-\overline{h}}^{t-h(t)}x^T(\alpha)\left[\sum_{j\in\mathbb{L}_k^i}^{\mathbb{N}}\pi_{ij}\left(Q_{0j} - H_{2i}\right) - R + \sum_{j\in\mathbb{L}_{uk}^i}^{\mathbb{N}}\pi_{ij}\left(Q_{0j} - H_{2i}\right)\right]x(\alpha)\,d\alpha.\end{aligned} \tag{38}$$

So, $\Gamma_1(h(t))$ is also a convex combination of the matrices $Y_{11}^i + e_4S_1e_4^T$ and Z_{11}^i on $h(t)$.

It is known that $\pi_{ii} < 0$. At the same time, the boundary information $\underline{\pi}_{ij} \le \pi_{ij} \le \overline{\pi}_{ij}$ of the transition probabilities need to be used. It is clear that if (17)–(20) and (22) hold, $\xi V(x_t, r(t), t) < 0$ can be achieved.

Taking into account (17)–(22), there exists a scalar $c > 0$ for each $i \in \mathbb{L}$ such that

$$\xi V(x(t), r(t), t) \le -c\|x(t)\|^2, \tag{39}$$

which implies that system (1) is stochastically stable.

Therefore, the proof is completed. □

If $f(t, x(t), r(t)) = \Delta A(t, r(t))x(t)$, $f_d(t, x(t-h(t)), r(t)) = \Delta A_d(t, r(t))x(t-h(t))$, system (1) becomes

$$\begin{aligned}\dot{x}(t) = {} & (A(r(t)) + \Delta A(t, r(t)))x(t) \\ & + \left(A_d(r(t)) + \Delta A_d(t, r(t))\right)x(t-h(t)), \\ x(t) = {} & \psi(t), \quad -\overline{h} \le t \le 0,\end{aligned} \tag{40}$$

where $\Delta A(t, r(t))$, $\Delta A_d(t, r(t))$ are unknown time-varying matrices with appropriate dimensions and are assumed to be of the following form:

$$\begin{aligned}&\left[\Delta A(t, r(t)) \quad \Delta A_d(t, r(t))\right] \\ &\quad = D(r(t))E(t, r(t))\left[F(r(t)) \quad F_d(r(t))\right],\end{aligned} \tag{41}$$

where $D(r(t))$, $F(r(t))$, $F_d(r(t))$ are known real constant matrices for all $r(t) \in \mathbb{L}$, and $E(t, r(t))$, for all $r(t) \in \mathbb{L}$, are the uncertain time-varying matrices satisfying

$$E(t, r(t))^TE(t, r(t)) \le I, \quad \forall r(t) \in \mathbb{L}. \tag{42}$$

It can be easily seen that system (40) is just the MJSs with norm bounded uncertainty (41).

Choosing $p(t,r(t)) = E(t,r(t))q(t,r(t))$ and $q(t,r(t)) = F(r(t))x(t) + F_d(r(t))x(t-h(t))$, system (40) is equivalent to the following system:

$$\dot{x}(t) = A(r(t))x(t) + A_d(r(t))x(t-h(t)) + D(r(t))p(t,r(t)), \quad x(t) = \psi(t), \quad -\overline{h} \le t \le 0. \tag{43}$$

For each possible value of $r(t) = i\,(i \in \mathbb{L})$, $p(t,r(t))$, $D(r(t))$, $F(r(t))$, and $F_d(r(t))$ can be denoted as $p_i(t)$, D_i, F_i, and F_{di}. Then, it is necessary to choose $\chi_i(t) = [x^T(t), x^T(t-h(t)), x^T(t-\overline{h}), \dot{x}^T(t), p_i^T(t)]^T$ and the corresponding block entry matrices as e_j, $j \in \{1,\ldots,5\}$.

Then, the following corollary is obtained.

Corollary 6. *If there exist matrices $P_i > 0$, $Q_{0i} > 0$, $Q_{1i} > 0$, $S_0 > 0$, $S_1 > 0$, $R > 0$, Y_{11}^i, Y_{12}^i, Y_{22}^i, Z_{11}^i, Z_{12}^i, Z_{22}^i, Φ_i, H_{1i}, and H_{2i} such that* (19)–(22) *and the following LMIs are feasible for $i = 1,2\ldots,\mathbb{N}$:*

$$\begin{aligned}
&He\left(\Phi_i\left[A_ie_1^T + A_{di}e_2^T - e_4^T + D_ie_5^T\right]\right.\\
&\quad\left.+Z_{12}(e_2-e_3)^T + Y_{12}(e_1-e_2)^T\right) + He\left(e_4P_ie_1^T\right)\\
&\quad - e_3Q_{0i}e_3^T + \overline{h}e_4\left(S_0+S_1\right)e_4^T - (1-\overline{\mu})\,e_2Q_{1i}e_2^T\\
&\quad + \overline{h}Y_{11} + e_1\left(\overline{h}R + Q_{0i} + Q_{1i} + \Psi_i\right)e_1^T\\
&\quad + \left(e_1F_i^T + e_2F_{di}^T\right)\left(F_ie_1^T + F_{di}e_2^T\right) - e_5e_5^T < 0,\\
&He\left(\Phi_i\left[A_ie_1^T + A_{di}e_2^T - e_4^T + D_ie_5^T\right]\right.\\
&\quad\left.+Z_{12}(e_2-e_3)^T + Y_{12}(e_1-e_2)^T\right)\\
&\quad + He\left(e_4P_ie_1^T\right) - e_3Q_{0i}e_3^T + \overline{h}e_4S_0e_4^T - (1-\overline{\mu})\,e_2Q_{1i}e_2^T\\
&\quad + \overline{h}Z_{11} + e_1\left(\overline{h}R + Q_{0i} + Q_{1i} + \Psi_i\right)e_1^T\\
&\quad + \left(e_1F_i^T + e_2F_{di}^T\right)\left(F_ie_1^T + F_{di}e_2^T\right) - e_5e_5^T < 0,
\end{aligned} \tag{44}$$

then the delayed uncertain system (40) *with partly known transition probabilities is stochastically stable.*

Proof. Here, the constraints of the model dynamics can be written as $\chi_i^T(t)\Phi_i[A_ie_1^T + A_{di}e_2^T - e_4^T + D_ie_5^T]\chi_i(t) \equiv 0$. And the additional uncertainty constraint (42) is given as follows:

$$\begin{aligned}
0 &\le q^T(t)q(t) - p^T(t)p(t)\\
&= \chi_i^T(t)\left\{\left(e_1F_i^T + e_2F_{di}^T\right)\left(F_ie_1^T + F_{di}e_2^T\right) - e_5e_5^T\right\}\chi_i(t).
\end{aligned} \tag{45}$$

Choose the Lyapunov functional as (25); so,

$$\xi V(x(t),r(t),t) \le \xi V(x(t),r(t),t) + q^T(t)q(t) - p^T(t)p(t). \tag{46}$$

Table 1: Comparison of maximum allowed $\overline{h}$.

π_{11}	-0.1	-0.5	-1
$\overline{h}$ by [16]	0.4261	0.4206	0.4161
$\overline{h}$ by [18]	0.5050	0.4887	0.4855
$\overline{h}$ by Theorem 5	0.5575	0.5499	0.5355

Table 2: Maximum allowed $\overline{h}$ by Corollary 6.

$\overline{\mu}$	0	0.2	0.6
$\overline{h}$ by Corollary 6	1.1739	1.0161	0.8618

Table 3: Maximum allowed $\overline{h}$ by Theorem 5.

$\overline{\mu}$	0	0.2	0.6
$\overline{h}$ by Theorem 5	1.2999	1.0500	0.9236

Use the same method as the proof of Theorem 5 to make $\xi V(x(t),r(t),t) + q^T(t)q(t) - p^T(t)p(t) < 0$, and then Corollary 6 can be obtained. □

4. Numerical Examples

In this section, numerical examples are given to demonstrate the benefits and effectiveness of the proposed methods.

First, Example 1 in [16] is given to show the superiority of Theorem 5 for MJS (1) with no nonlinear terms and completely known transition probability matrix.

Example 7. Consider MJS (1) with $f(t,x(t),r(t)) = 0$ and $f_d(t,x(t-h(t)),r(t)) = 0$. The two modes are given as

$$\begin{aligned}
A_1 &= \begin{bmatrix} -3.4888 & 0.8057 \\ -0.6451 & -3.2684 \end{bmatrix}, & A_2 &= \begin{bmatrix} -2.4898 & 0.2895 \\ 1.3396 & -0.0211 \end{bmatrix},\\
A_{d1} &= \begin{bmatrix} -0.8620 & -1.2919 \\ -0.6841 & -2.0729 \end{bmatrix}, & A_{d2} &= \begin{bmatrix} -2.8306 & 0.4978 \\ -0.8436 & -1.0115 \end{bmatrix}.
\end{aligned} \tag{47}$$

Supposing that $\mu = 0.8$ and $\pi_{22} = -0.8$, compare the stochastic stability result in Theorem 5 with the stability theorem in [16, 18]. For given π_{11}, the maximum $\overline{h}$, which satisfies the LMIs in (17)–(22), can be calculated.

Table 1 presents the comparison results, which shows that the stochastic stability result in Theorem 5 is less conservative than those in [16, 18] when no nonlinear terms exist and transition probabilities are completely known.

Next numerical example is for uncertain MJS (43) with more general transition probability.

Example 8. Consider the following MJS (43) with four modes, whose system matrices are

$$A_1 = \begin{bmatrix} -2.2460 & -1.4410 \\ -1.5937 & -2.9289 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1.8999 & 0.8156 \\ -0.6900 & -0.7881 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -0.7523 & 0.7500 \\ 1.5630 & -2.3540 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} -1.7840 & 0.3640 \\ 1.3670 & -1.5640 \end{bmatrix},$$
$$A_{d1} = \begin{bmatrix} -0.5 & -0.5 \\ 0.05 & 0.01 \end{bmatrix}, \qquad A_{d2} = \begin{bmatrix} 0.01 & 0 \\ 0.05 & -0.01 \end{bmatrix},$$
$$A_{d3} = \begin{bmatrix} -0.01 & -0.02 \\ 0.05 & 0.01 \end{bmatrix}, \qquad A_{d4} = \begin{bmatrix} 0.03 & 0.01 \\ -0.2 & -0.1 \end{bmatrix},$$
$$F_1 = F_2 = F_3 = F_4 = \begin{bmatrix} 1 & 0 \end{bmatrix},$$
$$F_{d1} = F_{d2} = F_{d3} = F_{d4} = \begin{bmatrix} -5 & 0.5 \end{bmatrix},$$
$$D_1 = D_2 = D_3 = D_4 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}. \tag{48}$$

The more general transition probability matrix is chosen as follows, which includes completely known elements, boundary known elements, and completely unknown elements:

$$\begin{bmatrix} -1.3 & 0.2 & ? & ? \\ ? & ? & 0.3 & 0.3 \\ 0.6 & ? & -1.5 & ? \\ ? & ? & ? & \alpha \end{bmatrix}. \tag{49}$$

Choose $-1.5 \leq \alpha \leq -0.9$. According to Corollary 6, Table 2 can be obtained for different values of $\overline{\mu}$. Also, the uncertain MJS (43) is stochastically stable. Moreover, as the upper bound of $\dot{h}(t)$, that is, $\overline{\mu}$, increases, the maximum allowed upper bound of $h(t)$ may be reduced.

Furthermore, a nonlinear MJS with sector-bounded nonlinearity is given.

Example 9. Consider MJS (1) with four models, whose A_i, A_{di} and the transition probability matrix are the same matrices as the ones of Example 8. The nonlinear functions are given as follows:

$$\begin{aligned} f_1(t) &= \begin{bmatrix} 0.1x_1(t)\cos^2(x_1(t)) - 0.05(x_1(t) - x_2(t)) \\ -0.05x_1(t) - 0.1x_2(t) \end{bmatrix}, f_2(t) \\ &= 2f_1(t), f_3(t) \\ &= \begin{bmatrix} 0.15x_2(t)\cos^2(x_1(t) + x_2(t)) + 0.1x_1(t) + 0.25x_2(t) \\ 0.15x_1(t) + 0.05x_2(t) \end{bmatrix}, f_4(t) \\ &= 2f_3(t), f_{d1}(t) \\ &= f_{d2}(t) = f_{d3}(t) = f_{d4}(t) \\ &= \begin{bmatrix} 0.12x_1(t-h(t))\sin^2(x_2(t-h(t))) + 0.04x_1(t-h(t)) + 0.08x_2(t-h(t)) \\ 0.04x_1(t-h(t)) + 0.04x_2(t-h(t)) \end{bmatrix}. \end{aligned} \tag{50}$$

According to (7), the corresponding matrices, which describe the sector bound, can be chosen as

$$\begin{aligned} M_{11} &= \begin{bmatrix} 0.05 & 0.05 \\ 0.05 & 0.05 \end{bmatrix}, \qquad M_{12} = 2M_{11}, \\ M_{13} &= \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & -0.05 \end{bmatrix}, \qquad M_{14} = 2M_{13}, \\ M_{21} &= \begin{bmatrix} -0.05 & 0.05 \\ -0.15 & -0.25 \end{bmatrix}, \qquad M_{22} = 2M_{21}, \\ M_{23} &= \begin{bmatrix} 0.1 & 0.25 \\ 0.2 & 0.15 \end{bmatrix}, \qquad M_{24} = 2M_{23}, \\ N_{11} &= N_{12} = N_{13} = N_{14} = \begin{bmatrix} 0.04 & 0.08 \\ -0.04 & 0.12 \end{bmatrix}, \\ N_{21} &= N_{22} = N_{23} = N_{24} = \begin{bmatrix} 0.16 & 0.08 \\ 0.12 & -0.04 \end{bmatrix}. \end{aligned} \tag{51}$$

Choose $-1.5 \leq \alpha \leq -1$. Using Matlab LMI toolbox and Theorem 5, the maximum allowed upper bound of $h(t)$ can be obtained for different values of $\overline{\mu}$.

From Table 3, it can be seen that the stochastic stability can be guaranteed for nonlinear MJS (1) when more general transition probabilities (4) are considered.

5. Conclusion

This paper has presented a new method for solving the stochastic stability analysis problem for MJS with time-varying delay and sector-bounded nonlinearity. A general transition probability matrix is adopted, which includes completely known elements, boundary known elements, and completely unknown elements. Based on a new mode-dependent stochastic Lyapunov-Krasovskii functional and some new techniques to deal with transition probabilities, sufficient conditions with less conservativeness are derived in terms of LMIs to obtain stochastic stability of systems.

Numerical examples have also demonstrated the effectiveness of the results.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (nos. 61273155, 61273355, 61273356, 61035005), New Century Excellent Talents in University (no. NCET-11-0083), Foundation for the Author of National Excellent Doctoral Dissertation of China (no. 201157), the Foundation of State Key Laboratory of Robotics (no. 2012-001).

References

[1] W. Zhang and L. Xie, "Interval stability and stabilization of linear stochastic systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 810–815, 2009.

[2] X. J. Xie and L. Liu, "A homogeneous domination approach to state feedback of stochastic high-Order nonlinear systems with time-varying delay," *IEEE Transactions on Automatic Control*, vol. 58, pp. 494–495, 2013.

[3] E. F. Costa and J. B. R. do Val, "On the observability and detectability of continuous-time Markov jump linear systems," *SIAM Journal on Control and Optimization*, vol. 41, no. 4, pp. 1295–1314, 2002.

[4] W. Li, H. Su, D. Wei, and K. Wang, "Global stability of coupled nonlinear systems with Markovian switching," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 6, pp. 2609–2616, 2012.

[5] Y. Long and G. H. Yang, "Fault detection for a class of non-homogeneous Markov jump systems based on delta operator approach," *Journal of Systems and Control Engineering*, 2012.

[6] M. Mariton, *Jump Linear Systems in Automatic Control*, Marcel Dekker, New York, NY, USA, 1990.

[7] B. C. Willsky and A. S. Levy, "Stochastic stability research for complex power systems," Tech. Rep., Massachesetts Institute of Technology, Cambridge, Mass, USA, 1979.

[8] E. K. Boukas, "H_∞ control of discrete-time Markov jump systems with bounded transition probabilities," *Optimal Control Applications & Methods*, vol. 30, no. 5, pp. 477–494, 2009.

[9] M. Shen and G.-H. Yang, "H_2 state feedback controller design for continuous Markov jump linear systems with partly known information," *International Journal of Systems Science*, vol. 43, no. 4, pp. 786–796, 2012.

[10] X.-M. Sun, G.-P. Liu, D. Rees, and W. Wang, "Stability of systems with controller failure and time-varying delay," *IEEE Transactions on Automatic Control*, vol. 53, no. 10, pp. 2391–2396, 2008.

[11] L. Zhang and E.-K. Boukas, "Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities," *Automatica*, vol. 45, no. 2, pp. 463–468, 2009.

[12] L. Zhang and J. Lam, "Necessary and sufficient conditions for analysis and synthesis of Markov jump linear systems with incomplete transition descriptions," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1695–1701, 2010.

[13] Y. Zhang, Y. He, M. Wu, and J. Zhang, "Stabilization for Markovian jump systems with partial information on transition probability based on free-connection weighting matrices," *Automatica*, vol. 47, no. 1, pp. 79–84, 2011.

[14] C. Lin, Q.-G. Wang, and T. H. Lee, "A less conservative robust stability test for linear uncertain time-delay systems," *IEEE Transactions on Automatic Control*, vol. 51, no. 1, pp. 87–91, 2006.

[15] P. Park and J. W. Ko, "Stability and robust stability for systems with a time-varying delay," *Automatica*, vol. 43, no. 10, pp. 1855–1858, 2007.

[16] S. Xu, J. Lam, and X. Mao, "Delay-dependent H_∞ control and filtering for uncertain Markovian jump systems with time-varying delays," *IEEE Transactions on Circuits and Systems*, vol. 54, no. 9, pp. 2070–2077, 2007.

[17] Y.-Y. Cao, J. Lam, and L. Hu, "Delay-dependent stochastic stability and H_∞ analysis for time-delay systems with Markovian jumping parameters," *Journal of the Franklin Institute*, vol. 340, no. 6-7, pp. 423–434, 2003.

[18] G. Wang, Q. Zhang, and V. Sreeram, "Robust delay-range-dependent stabilization for Markovian jump systems with mode-dependent time delays and nonlinearities," *Optimal Control Applications & Methods*, vol. 31, no. 3, pp. 249–264, 2010.

[19] Z. Wang, H. Qiao, and K. J. Burnham, "On stabilization of bilinear uncertain time-delay stochastic systems with Markovian jumping parameters," *IEEE Transactions on Automatic Control*, vol. 47, no. 4, pp. 640–646, 2002.

[20] H. K. Khalil, *Nonlinear Systems*, Prentice-Hall, Upper Saddle River, NJ, USA, 1996.

[21] X.-G. Guo and G.-H. Yang, "Delay-dependent reliable H_∞ filtering for sector-bounded nonlinear continuous-time systems with time-varying state delays and sensor failures," *International Journal of Systems Science*, vol. 43, no. 1, pp. 117–131, 2012.

[22] Q.-L. Han, "Absolute stability of time-delay systems with sector-bounded nonlinearity," *Automatica*, vol. 41, no. 12, pp. 2171–2176, 2005.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 613159, 7 pages
http://dx.doi.org/10.1155/2013/613159

Research Article

Optimality Conditions for Optimal Control of Jump-Diffusion SDEs with Correlated Observations Noises

Hua Xiao

School of Mathematics and Statistics, Shandong University, Weihai 264209, China

Correspondence should be addressed to Hua Xiao; xiao_hua@sdu.edu.cn

Received 25 January 2013; Accepted 5 March 2013

Academic Editor: Guangchen Wang

This paper is concerned with necessary and sufficient optimality conditions for optimal control of jump-diffusion stochastic differential equations. Compared with the existing literature, there are two distinguishing features: one is that the states are driven by Brownian motions and Poisson random measure; the other one is that the states and the observations are correlated. We derive a necessary and a sufficient conditions in the form of maximum principle when control domain is convex. A linear-quadratic example is worked out to illustrate the applications of the foregoing optimality conditions.

1. Introduction

The purpose of this paper is to establish maximum principle, also called necessary optimality conditions, for optimal control of jump-diffusion stochastic differential equations driven by Brownian motions and Poisson random measure where states and observations are correlated. There is rich literature on maximum principle for optimal control of stochastic differential equations (SDEs, for short). For example, Peng [1] proved a general stochastic maximum principle for optimal control of diffusion SDEs by introducing second-order variational equations; Tang and Li [2] derived a general necessary condition for optimal control of jump-diffusion SDEs; Pham [3] gave a survey on some recent aspects and developments in stochastic control of diffusion processes and discussed the main historical approaches and their modern exposition for studying stochastic control problems. In many situations, the states of the systems cannot be completely observed; however, some other processes related to the unobservable states can be observed. Then partially observable optimal control of SDEs also attracts much research attention. Such a subject has been discussed by many authors, such as Baras et al. [4], Bensoussan [5], Fleming [6], Li and Tang [7], Tang and Hou [8], Huang et al. [9], Wang and Wu [10, 11], Xiao [12], Zhou [13], and Hu and Øksendal [14].

In the real world, there usually exists certain correlated noises between the state and observation which is more general case than the case of independent noises. However, the literature mentioned above only deals with the case that states and observations are independent. To the author's best knowledge, there are only two papers about correlated observation noises. Tang [15] considered optimal control of diffusion SDEs and proved a general stochastic maximum principle; Xiao [16] obtained maximum principle for optimal control of Poisson point processes with correlated Gaussian white noisy observations. Up till now, there is no literature on optimal control of jump-diffusion SDEs with correlated observation noises. Here, we set out to study the optimal control problem of SDEs driven by Brownian motions and Poisson random measure in the case of correlated noisy observations. Due to the correlation between the state and the observation, there exists a weak solution rather than a strong solution to the state equation, which is different from the case of noncorrelated noisy observation. Meanwhile, the superfluous adjoint processes arise in the optimality condition, and we need some complicated matrices decomposition to eliminate them. We also make an effort to seek a suitable adjoint of the drift coefficient of the observation equation which plays an important role in defining the new Hamiltonian function, so that the improved necessary maximum principle is just the thing we want.

The rest of this paper is organized as follows. In Section 2, we formulate the optimal control problem of partially observable jump-diffusion SDEs with correlated observation noises.

In Section 3, we obtain the partially observed stochastic maximum principle when the control domain is convex. Section 4 presents a linear-quadratic example to illustrate the applications of the theoretical results derived in Section 3. Some conclusions are given in Section 5.

2. Formulation of Problem

Throughout this paper, we assume that $\mathscr{E}$ is a nonempty Borel subset of R^l, $\mathfrak{B}(\mathscr{E})$ is the Borel σ-algebra generated by $\mathscr{E}$, and $\pi(\cdot)$ is a σ-finite measure on $(\mathscr{E}, \mathfrak{B}(\mathscr{E}))$. Let $T > 0$ be a fixed real number. Let $\kappa(\cdot)$ be a stationary $\mathscr{F}_t$-Poisson point process on $\mathscr{E}$ with the characteristic measure $\pi(de)$. We denote by $N(de\,dt)$ the counting measure or Poisson measure induced by $\kappa(\cdot)$ and set $\widetilde{N}(de\,dt) = N(de\,dt) - \pi(de)dt$ satisfying $\int_{\mathscr{E}}(1 \wedge |e|^2)\pi(de) < \infty$ and $\pi(A) < \infty$ for every $A \in \mathfrak{B}(\mathscr{E})$. In general, we call $\pi(de)dt$ and $\widetilde{N}(de\,dt)$ the intensity and the compensated Poisson measure of $N(de\,dt)$, respectively (see Ikeda and Watanabe [17]). In addition, let $(\Omega, \mathscr{F}, P)$ be a complete probability space on which two mutually independent standard Brownian motions $B(\cdot)$ and $Y(\cdot)$ are defined, valued in R^d and R^r, respectively, independent of $\kappa(\cdot)$. Let $\mathscr{F}_t^B$, $\mathscr{F}_t^Y$, and $\mathscr{F}_t^N$ be the natural filtration generated by $B(\cdot)$, $Y(\cdot)$, and $N(\cdot)$, respectively. We assume that

$$\mathscr{F}_t := \mathscr{F}_t^B \vee \mathscr{F}_t^Y \vee \mathscr{F}_t^N \vee \mathscr{N}, \quad \mathscr{F} := \mathscr{F}_T, \tag{1}$$

where $\mathscr{N}$ denotes the totality of P-null sets. For a matrix, we use superscripts to indicate (when necessary) the number of its columns or its rows or the position of its components, and precise meaning can be specified from the context; the range of the superscripts will not be explicitly stated unless there is a danger of confusion.

Let $\mathcal{S}$ be a finite-dimensional space, and we denote by $\mathrm{L}^2(\Omega, \mathscr{F}_T; \mathcal{S})$ the space of $\mathcal{S}$-valued squared integrable $\mathscr{F}_T$-measurable random variables, by $\mathrm{L}^2_{\mathscr{F}}([0,T]; \mathcal{S})$ the space of $\mathcal{S}$-valued squared integrable $\mathscr{F}_t$-adapted processes, by $F^2_\pi(\mathscr{E}, \mathfrak{B}(\mathscr{E}), \pi; \mathcal{S})$ the space of square integrable functions $\delta : \mathscr{E} \to \mathcal{S}$, by $F^2_p([0,T]; \mathcal{S})$ the space of $\mathcal{S}$-valued $\mathscr{F}_t$-predictable processes $f(\cdot,\cdot,\cdot) : \Omega \times [0,T] \times \mathscr{E} \to \mathcal{S}$, such that $\mathbb{E}\int_0^T \int_{\mathscr{E}} |f(\omega, t, e)|^2 \pi(de)dt < \infty$. If $r(\cdot,\cdot,\cdot) \in F^2_p([0,T]; \mathcal{S})$, we write $r(t)$ for $\int_{\mathscr{E}} r(t,e)\pi(de)$. Unless otherwise stated, the notations $r(t)$ and $r(t,e)$ denote the above different meanings throughout the paper.

Let U be a nonempty convex subsets of some Euclidean space. A control is a stochastic process $u : \Omega \times [0,T] \to U$. For notational simplicity, hereinafter, we will omit ω in random functions. We define an admissible control set $\mathscr{U}_{\mathrm{ad}}$ by

$$\mathscr{U}_{\mathrm{ad}} = \Big\{ v(\cdot) \mid v(\cdot), \text{ valued in } U, \text{ is an } \mathscr{F}_t^Y\text{-adapted process and satisfies } \mathbb{E}\sup_{0\le t\le T} v(t)^2 < \infty \Big\}. \tag{2}$$

Every element of $\mathscr{U}_{\mathrm{ad}}$ is called an admissible control.

Introduce the mappings $f : [0,T] \times \mathrm{R}^n \times \mathrm{U} \to \mathrm{R}^n$, $\sigma : [0,T] \times \mathrm{R}^n \times \mathrm{U} \to \mathrm{R}^{n\times d}$, $\overline{\sigma} : [0,T] \times \mathrm{R}^n \times \mathrm{U} \to \mathrm{R}^{n\times r}$, $c : [0,T] \times \mathrm{R}^n \times \mathrm{U} \times \mathscr{E} \to \mathrm{R}^{n\times l}$, $h : [0,T] \times \mathrm{R}^n \times \mathrm{U} \to \mathrm{R}^r$, $l : [0,T] \times \mathrm{R}^n \times \mathrm{U} \to \mathrm{R}$, $m : \mathrm{R}^n \to \mathrm{R}$.

We make the following hypothesis.

(H1) f, σ, and c are continuously differentiable with respect to (x, v). They are bounded by $(1 + |x| + |v|)$ and their derivatives with respect to (x, v) are uniformly bounded. $\overline{\sigma}$ and h are uniformly bounded and continuously differentiable with respect to (x, v), whose derivatives with respect to (x, v) are also uniformly bounded. l and m are continuously differentiable with respect to (x, v). There exists a constant K_0 such that $|l_k(t, x, v)| \le K_0(1+|x|+|v|)$, $k = x$, v, $|l(t, x, v)| \le K_0(1+|x|^2+|v|^2)$, $(1+|x|^2)^{-1}|m(x)| + (1+|x|)^{-1}|m_x(x)| \le K_0$.

The partially observable optimal control problem is stated as follows.

Consider the state

$$\begin{aligned} dx^v(t) = {} & f(t, x^v(t), v(t))\,dt + \sigma(t, x^v(t), v(t))\,dB(t) \\ & + \overline{\sigma}(t, x^v(t), v(t))\,dW^v(t) \\ & + \int_{\mathscr{E}} c(t, x^v(t-), v(t), e)\,\widetilde{N}(de\,dt), \\ & x^v(0) = x_0, \quad 0 \le t \le T \end{aligned} \tag{3}$$

and the observation

$$\begin{aligned} dY(t) &= h(t, x^v(t), v(t))\,dt + dW^v(t), \\ Y(0) &= 0, \end{aligned} \tag{4}$$

where $W^v(\cdot)$ is a stochastic process depending on the control $v(\cdot)$. Note that if the diffusion term $\overline{\sigma} \neq 0$ in (3), then there exist the correlated noise W^v between the state and observation. Substituting (4) into (3), we have

$$\begin{aligned} dx^v(t) = {} & [f(t, x^v(t), v(t)) \\ & -\overline{\sigma}(t, x^v(t), v(t))\,h(t, x^v(t), v(t))]\,dt \\ & + \sigma(t, x^v(t), v(t))\,dB(t) \\ & + \overline{\sigma}(t, x^v(t), v(t))\,dY(t) \\ & + \int_{\mathscr{E}} c(t, x^v(t-), v(t), e)\,\widetilde{N}(de\,dt), \\ & x^v(0) = x_0, \quad 0 \le t \le T. \end{aligned} \tag{5}$$

For each $v(\cdot) \in \mathscr{U}_{\mathrm{ad}}$, there exists a unique strong solution to (5) (see Ikeda and Watanabe [17], which will be denoted by $x^v(\cdot) \in \mathrm{L}^2_{\mathscr{F}}([0,T]; \mathrm{R}^n)$. We define

$$\begin{aligned} \rho^v(t) := \exp\Big\{ & \int_0^t h^*(s, x^v(s), v(s))\,dY(s) \\ & -\frac{1}{2}\int_0^t |h(s, x^v(s), v(s))|^2 ds \Big\}, \end{aligned} \tag{6}$$

that is

$$d\rho^v(t) = \rho^v(t)\, h^*\left(t, x^v(t), v(t)\right) dY(t), \qquad \rho^v(0) = 1, \tag{7}$$

and $dP^v := \rho^v(T)dP$. Note that since h is uniformly bounded, we actually have that $\rho^v(t)$ is a martingale on $(\Omega, \mathscr{F}, P)$, then the expression $dP^v := \rho^v(T)dP$ is to say that $P^v \ll P$, that is, P^v is absolutely continuously w.r.t. P. Note that $\rho^v(T) > 0$ a.s., so we also have that $P \ll P^v$. Hence the two measures P^v and P are equivalent. Furthermore, $P^v(\Omega) = \mathbb{E}[\rho^v(T)] = 1$. Therefore P^v is also a probability measure. From Girsanov's theorem, it follows that $(P^v, x^v, B, W^v, \widetilde{N}, Y)$ is a weak solution on $(\Omega, \mathscr{F}, \mathscr{F}_t)$ to (3) and (4). Throughout this paper, the superscript symbol $*$ means the transpose of certain vector or matrix.

Consider the cost functional

$$J(v(\cdot)) = \mathbb{E}^v\left[\int_0^T l\left(t, x^v(t), v(t)\right) dt + m\left(x^v(T)\right)\right]. \tag{8}$$

Here, $\mathbb{E}^v$ denotes the expectation with respect to the probability space $(\Omega, \mathscr{F}, P^v)$. Our partially observed optimal control problem is to minimize the cost functional (8) over $v(\cdot) \in \mathscr{U}_{ad}$, subject to (3) and (4).

Obviously, the cost functional (8) can be rewritten as

$$\begin{aligned} J(v(\cdot)) &= \mathbb{E}\left[\int_0^T \rho^v(t)\, l\left(t, x^v(t), v(t)\right) dt + \rho^v(T)\, m\left(x^v(T)\right)\right]. \end{aligned} \tag{9}$$

So the original optimization problem is equivalent to minimizing the cost functional (9) over $v(\cdot) \in \mathscr{U}_{ad}$, subject to (5) and (7).

If an admissible control $u(\cdot)$ minimizes the cost functional (if it does exist), then it is called optimal and the corresponding weak solution $(P^u, x, B, W, \widetilde{N}, Y)$ to (3) and (4) is called the optimal trajectory.

3. Maximum Principle

In this section, we shall establish a necessary and a sufficient maximum principles.

We note that minimizing the cost functional (9) over $v(\cdot) \in \mathscr{U}_{ad}$, subject to (5) and (7), is similar to a completely observable optimal control problem except for the difference of admissible control set. It is a heuristic method to derive the maximum principle for partially observable optimal control from the maximum principle for completely observable optimal control. We now specify this point.

Let $u(\cdot)$ be an optimal control and $(P^u, x, B, W, \widetilde{N}, Y)$ the corresponding optimal trajectory. Set

$$\begin{gathered} X^v \doteq \begin{pmatrix} \rho^v \\ x^v \end{pmatrix}, \qquad X \doteq \begin{pmatrix} \rho \\ x \end{pmatrix} \doteq \begin{pmatrix} \rho^u \\ x^u \end{pmatrix}, \qquad X_0 \doteq \begin{pmatrix} 1 \\ x_0 \end{pmatrix}, \\ F(t, X^v, v) \doteq \begin{pmatrix} 0 \\ f(t, x^v, v) - \overline{\sigma}(t, x^v, v)\, h(t, x^v, v) \end{pmatrix}, \\ \Sigma(t, X^v, v) \doteq \begin{pmatrix} 0 \\ \sigma(t, x^v, v) \end{pmatrix}, \\ \overline{\Sigma}(t, X^v, v) \doteq \begin{pmatrix} \rho^v h^*(t, x^v, v) \\ \overline{\sigma}(t, x^v, v) \end{pmatrix}, \\ L(t, X^v, v) \doteq \rho^v l(t, x^v, v), \\ C(t, X^v, v, e) \doteq \begin{pmatrix} 0 \\ c(t, x^v, v, e) \end{pmatrix}, \qquad M(X^v) \doteq \rho^v m(x^v). \end{gathered} \tag{10}$$

Equations (5) and (7) can be compressed into the following form:

$$\begin{aligned} dX^v(t) = {} & F\left(t, X^v(t), v(t)\right) dt \\ & + \Sigma\left(t, X^v(t), v(t)\right) dB(t) \\ & + \overline{\Sigma}\left(t, X^v(t), v(t)\right) dY(t) \\ & + \int_{\mathscr{E}} C\left(t, X^v(t-), v(t), e\right) \widetilde{N}(de\, dt), \\ X^v(0) = {} & X_0, \quad 0 \le t \le T. \end{aligned} \tag{11}$$

The cost function (9) is rewritten as

$$J(v(\cdot)) = \mathbb{E}\left[\int_0^T L\left(t, X^v(t), v(t)\right) dt + M\left(X^v(T)\right)\right]. \tag{12}$$

Our partially observed optimal control problem becomes the following minimization problem: to minimize $J(v(\cdot))$ in (12) over $v(\cdot) \in \mathscr{U}_{ad}$, subject to (11). The present formulation of the partially observable optimal control problem is quite similar to a completely observed optimal control problem; the only difference lies in the admissible controls class $\mathscr{U}_{ad}$. We can follow the same arguments in the case of full information to derive the following desired maximum principle (see Tang and Li [2] in the case of complete information and convex control domain).

Define the Hamiltonian $H : [0, T] \times \mathrm{R}^{n+1} \times \mathbf{U} \times \mathrm{R}^{n+1} \times \mathrm{R}^{(n+1)\times d} \times \mathrm{R}^{(n+1)\times r} \times \mathrm{R}^{(n+1)\times l} \to \mathrm{R}$ as follows:

$$\begin{aligned} H\left(t, X^v, v, a, b, \overline{b}, \overline{c}\right) = {} & \langle a, F(t, X^v, v)\rangle + \langle b, \Sigma(t, X^v, v)\rangle \\ & + \left\langle \overline{b}, \overline{\Sigma}(t, X^v, v)\right\rangle + L(t, X^v, v) \\ & + \int_{\mathscr{E}} \langle \overline{c}, C(t, X^v, v, e)\rangle\, \pi(de). \end{aligned} \tag{13}$$

For simplicity, we introduce the notation

$$H(t,u(t)) \doteq H\left(t,X(t),u(t),a(t),b(t),\bar{b}(t),\bar{c}(t)\right). \tag{14}$$

Let $(a,b,\bar{b},\bar{c})$ be the unique $\mathscr{F}_t$-adapted square integrable solution of the first-order adjoint equation

$$\begin{aligned} -da(t) &= H_X^*(t,u(t))\,dt - b(t)\,dB(t) \\ &\quad - \bar{b}(t)\,dY(t) - \int_{\mathscr{E}} \bar{c}(t,e)\,\widetilde{N}(de\,dt), \\ a(T) &= M_X^*(X(T)), \quad 0 \le t \le \mathrm{T}. \end{aligned} \tag{15}$$

From the maximum principle in Tang and Li [2], we obtain the following necessary maximum principle for partially observable optimal control.

Lemma 1. *Assume that the hypothesis (H1) holds. Let $u(\cdot)$ be an optimal control and $(P^u, x, B, W, \widetilde{N}, Y)$ the optimal trajectory to* (3) *and* (4). *Then*

$$\left\langle \mathbb{E}\left(H_v^*(t,u(t)) \mid \mathscr{F}_t^Y\right), v(t)-u(t)\right\rangle \ge 0 \tag{16}$$

holds for $\forall\ v(\cdot) \in \mathscr{U}_{ad}$.

We note that since the variable $\rho(\cdot)$, which is regarded as the state variable for a moment, appears in the optimal control problem in a linear way, some adjoint processes are superfluous in the above maximum principle (16). Inspired by the method in Tang [15], we set about dispensing with these adjoint processes and reformulate the above maximum principle.

Decompose the matrices $a(t)$, $b(t)$, $\bar{b}(t)$, and $\bar{c}(t)$ into blocks in the following manner:

$$\begin{aligned} a(t) &= \overbrace{\begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix}}^{1} \begin{matrix} \}1 \\ \}n \end{matrix}, \qquad b(t) = \overbrace{\begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}}^{d} \begin{matrix} \}1 \\ \}n \end{matrix}, \\ \bar{b}(t) &= \overbrace{\begin{pmatrix} \bar{b}_1(t) \\ \bar{b}_2(t) \end{pmatrix}}^{r} \begin{matrix} \}1 \\ \}n \end{matrix}, \qquad \bar{c}(t,e) = \overbrace{\begin{pmatrix} \bar{c}_1(t,e) \\ \bar{c}_2(t,e) \end{pmatrix}}^{l} \begin{matrix} \}1 \\ \}n \end{matrix}. \end{aligned} \tag{17}$$

Then we can check the following:

$$\begin{aligned} F_X(t,X,u) &= \left(F_\rho(t,X,u), F_x(t,X,u)\right), \\ F_\rho(t,X,u) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ F_x(t,X,u) & \\ = \Big(& \begin{matrix} 0 \\ f_x(t,x,u) - \bar{\sigma}_x^j(t,x,u)\,h^j(t,x,u) - \bar{\sigma}(t,x,u)\,h_x(t,x,u) \end{matrix} \Big), \\ \Sigma_X^i(t,X,u) &= \left(\Sigma_\rho^i(t,X,u), \Sigma_x^i(t,X,u)\right) \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \sigma_x^i(t,x,u) \end{pmatrix}, \\ \bar{\Sigma}_X^j(t,X,u) &= \left(\bar{\Sigma}_\rho^j(t,X,u), \bar{\Sigma}_x^j(t,X,u)\right) \\ &= \begin{pmatrix} h^j(t,x,u) & \rho h_x^j(t,x,u) \\ 0 & \bar{\sigma}_x^j(t,x,u) \end{pmatrix}, \\ L_X(t,X,u) &= \left(L_\rho(t,X,u), L_x(t,X,u)\right) \\ &= \left(l(t,x,u), \rho(t)\,l_x(t,x,u)\right), \\ \mathrm{C}_X^k(t,X,u) &= \left(\mathrm{C}_\rho^k(t,X,u,e), \mathrm{C}_x^k(t,X,u,e)\right) \\ &= \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{C}_x^k(t,x,u,e) \end{pmatrix}, \\ M_X(X) &= \left(M_\rho(X), M_x(X)\right) = \left(m(x), \rho m_x(x)\right). \end{aligned} \tag{18}$$

We introduce a new Hamiltonian $\mathscr{H} : [0,T] \times \mathrm{R}^n \times \mathbf{U} \times \mathrm{R}^n \times \mathrm{R}^{n\times d} \times \mathrm{R}^{n\times r} \times \mathrm{R}^{n\times l} \times \mathrm{R}^r \to \mathrm{R}$ as follows:

$$\begin{aligned} &\mathscr{H}\left(t,x^v,v,q,k,\bar{k},\bar{r},\bar{K}\right) \\ &= \langle q, f(t,x^v,v)\rangle + \langle k, \sigma(t,x^v,v)\rangle + \left\langle \bar{k}, \bar{\sigma}(t,x^v,v)\right\rangle \\ &+ \int_{\mathscr{E}} \langle \bar{r}, c(t,x^v,v,e)\rangle\,\pi(de) \\ &+ \left\langle \bar{K}^*, h(t,x^v,v)\right\rangle + l(t,x^v,v). \end{aligned} \tag{19}$$

We set

$$\begin{aligned} Q(t) &\doteq a_1(t), \qquad K(t) \doteq b_1(t), \qquad \bar{K}(t) \doteq \bar{b}_1(t), \\ q(t) &\doteq \rho^{-1}(t)\,a_2(t), \qquad k(t) \doteq \rho^{-1}(t)\,b_2(t), \\ \bar{k}(t) &\doteq \rho^{-1}(t)\,\bar{b}_2(t) - \rho^{-1}(t)\,a_2(t)\,h^*(t,x(t),u(t)), \\ \bar{r}(t,e) &\doteq \rho^{-1}(t)\,\bar{c}_2(t,e), \qquad \bar{R}(t,e) \doteq \bar{c}_1(t,e). \end{aligned} \tag{20}$$

For simplicity, we introduce the notation: $h(t) \doteq h(t,x(t),u(t))$, and similar notation will be made for other functions f, σ, $\bar{\sigma}$, c, l. We denote

$$\mathscr{H}\left(t,x(t),u(t),q(t),k(t),\bar{k}(t),\bar{r}(t),\bar{K}(t)-q^*(t)\,\bar{\sigma}(t)\right) \tag{21}$$

by $\mathscr{H}(t,u(t))$. In view of the above calculations, (15) can be decomposed into the following equation, for $0 \le t \le T$:

$$\begin{aligned} -dQ(t) &= \left(\left\langle \overline{K}^*(t), h(t)\right\rangle + l(t)\right)dt - K(t)\,dB(t) \\ &\quad - \overline{K}(t)\,dY(t) - \int_{\mathscr{E}} \overline{R}(t,e)\,\widetilde{N}(de\,dt), \\ -dq(t) &= \left[\mathscr{H}_x^*(t,u(t)) + \overline{k}(t)\,h(t)\right]dt - k(t)\,dB(t) \\ &\quad - \overline{k}(t)\,dY(t) - \int_{\mathscr{E}} \overline{r}(t,e)\,\widetilde{N}(de\,dt), \\ Q(T) &= m(x(T)), \qquad q(T) = m_x^*(x(T)). \end{aligned} \tag{22}$$

It is the position to rewrite the maximum condition (16). We can verify the following:

$$H_v(t,u(t)) = \rho(t)\,\mathscr{H}_v(t,u(t)). \tag{23}$$

Then we obtain the following necessary maximum condition.

Theorem 2 (necessary maximum principle). *Assume that the hypothesis (H1) holds. Let $u(\cdot)$ be an optimal control and $\{(Q, K, \overline{K}, \overline{R}), (q, k, \overline{k}, \overline{r})\}$ the corresponding solution of (22). Then*

$$\left\langle \mathbb{E}^u\left[\mathscr{H}_v^*(t,u(t)) \mid \mathscr{F}_t^Y\right], v(t) - u(t)\right\rangle \ge 0, \quad a.s.,\ a.e., \tag{24}$$

is true for any $v(\cdot) \in \mathscr{U}_{ad}$.

Subsequently, we set out to derive the sufficient optimality conditions for the foregoing optimal control problem. We introduce the following assumption.

(H2) $m(\cdot)$ is convex in x. Function h is independent of variables x and v.

Inspired by Xiao [12] and Huang et al. [18], we have the following theorem.

Theorem 3 (sufficient maximum principle). *Let (H1) and (H2) hold, $\rho^v(\cdot)$ be $\mathscr{F}_t^Y$-adapted, and $u(\cdot) \in \mathscr{U}_{ad}$ be an admissible control with the corresponding trajectory $x(\cdot)$. Furthermore, one supposes that $\{(Q, K, \overline{K}, \overline{R}), (q, k, \overline{k}, \overline{r})\}$ satisfies (22), the Hamiltonian $\mathscr{H}(t,u(t))$ is convex in (x,v), and*

$$\mathbb{E}\left[\mathscr{H}(t,u(t)) \mid \mathscr{F}_t^Y\right] = \min_{v(t)\in U} \mathbb{E}\left[\mathscr{H}(t,v(t)) \mid \mathscr{F}_t^Y\right]. \tag{25}$$

Then $u(\cdot)$ is an optimal control.

Proof. By virtue of the convexity property of m and γ, we have for all $v(\cdot) \in \mathscr{U}_{\mathrm{ad}}$,

$$J(v(\cdot)) - J(u(\cdot)) \ge I + II + III \tag{26}$$

with

$$\begin{aligned} I &= \mathbb{E}\left[\left(\rho^v(T) - \rho(T)\right)\left(\int_0^T l(t)\,dt + m(x(T))\right)\right], \\ II &= \mathbb{E}^v\left[m_x(x(T))\left(x^v(T) - x(T)\right)\right], \\ III &= \mathbb{E}^v \int_0^T \left(l^v(t) - l(t)\right)dt. \end{aligned} \tag{27}$$

Applying Itô's formula to $\langle Q(t), \rho^v(t) - \rho(t)\rangle$ and $\langle \rho^v(t)q(t), x^v(t) - x(t)\rangle$, we deduce

$$\begin{aligned} I &= \mathbb{E}^v \int_0^T \left\langle \overline{K}^*(t), h^v(t) - h(t)\right\rangle dt, \\ II &= \mathbb{E}^v \int_0^T \Big[\big\langle q(t), f^v(t) - f(t) - \overline{\sigma}^v(t)\,h^v(t) \\ &\qquad + \overline{\sigma}(t)\,h(t)\big\rangle + \left\langle k(t), \sigma^v(t) - \sigma(t)\right\rangle \\ &\qquad + \left\langle \overline{k}(t), \overline{\sigma}^v(t) - \overline{\sigma}(t)\right\rangle + \left\langle \overline{r}(t), c^v(t) - c(t)\right\rangle \\ &\qquad - \big\langle \mathscr{H}_x^*(t,u(t)) - \overline{k}(t)\,h^v(t) \\ &\qquad + \overline{k}(t)\,h(t), x^v(t) - x(t)\big\rangle\Big]\,dt. \end{aligned} \tag{28}$$

Since the *Hamiltonian* $\mathscr{H}(t,u(t))$ is convex in (x,v), we have

$$\begin{aligned} III &= \mathbb{E}^v \int_0^T \Big[\mathscr{H}^v(t,v(t)) - \mathscr{H}(t,u(t)) \\ &\qquad - \left\langle \overline{K}^*(t), h^v(t) - h(t)\right\rangle \\ &\qquad - \left\langle k(t), \sigma^v(t) - \sigma(t)\right\rangle \\ &\qquad - \left\langle \overline{k}(t), \overline{\sigma}^v(t) - \overline{\sigma}(t)\right\rangle \\ &\qquad - \left\langle \overline{r}(t,e), c^v(t) - c(t)\right\rangle \\ &\qquad - \big\langle q(t), f^v(t) - f(t) + \overline{\sigma}(t)\,h(t) \\ &\qquad - \overline{\sigma}^v(t)\,h^v(t)\big\rangle\Big]\,dt \\ &\ge \mathbb{E}^v \int_0^T \Big[\left\langle \mathscr{H}_x^*(t,u(t)), x^v(t) - x(t)\right\rangle \\ &\qquad + \left\langle \mathscr{H}_v^*(t,u(t)), v(t) - u(t)\right\rangle \\ &\qquad - \big\langle q(t), f^v(t) - f(t) \\ &\qquad + \overline{\sigma}(t)\,h(t) - \overline{\sigma}^v(t)\,h^v(t)\big\rangle \\ &\qquad - \left\langle k(t), \sigma^v(t) - \sigma(t)\right\rangle \\ &\qquad - \left\langle \overline{k}(t), \overline{\sigma}^v(t) - \overline{\sigma}(t)\right\rangle \\ &\qquad - \left\langle \overline{r}(t,e), c^v(t) - c(t)\right\rangle \\ &\qquad - \left\langle \overline{K}^*(t), h^v(t) - h(t)\right\rangle\Big]\,dt. \end{aligned} \tag{29}$$

Since the function h is independent of variables x and v, we have $h^v(t) \equiv h(t)$. Substituting (28) and (29) into (26), it follows immediately that

$$\begin{aligned} &J(v(\cdot)) - J(u(\cdot)) \\ &\geq \mathbb{E}^v \int_0^T \left\langle \mathscr{H}_v^*(t, u(t)), v - u(t) \right\rangle dt \\ &= \mathbb{E} \int_0^T \rho^v(t) \, \mathrm{E}\left[\left\langle \mathscr{H}_v^*(t, u(t)), v - u(t) \right\rangle \mid \mathscr{F}^Y(t) \right] dt. \end{aligned} \tag{30}$$

In virtue of (25), we have

$$\mathbb{E}\left[\left\langle \mathscr{H}_v^*(t, u(t)), v(t) - u(t) \right\rangle \mid \mathscr{F}_t^Y \right] \geq 0. \tag{31}$$

Substituting (31) into (30), since $\rho^v(\cdot) \geq 0$, we have

$$J(v(\cdot)) - J(u(\cdot)) \geq 0. \tag{32}$$

We draw the desired conclusion. □

4. An LQ Example

To illustrate that the foregoing theories may find the interesting applications in practice, we work out an LQ example of partially observable optimal control of jump-diffusion system with correlated noisy observations. Firstly, by applying the necessary maximum principle, we find a candidate optimal control. Then by sufficient maximum principle, we verify that it is indeed an optimal control. Finally, by certain techniques of forward-backward stochastic differential equations filtering, we obtain an explicit expression of the optimal control.

Consider a partially observable 1-dimensional control system

$$\begin{aligned} dx^v(t) = {} & \left[a_1(t) x^v(t) + a_2(t) v(t)\right] dt + a_3(t) \, dB(t) \\ & + a_4(t) \, dW^v(t) + \int_{\mathscr{E}} a_5(t) \, \widetilde{N}(de\, dt), \\ x^v(0) = {} & x_0, \end{aligned} \tag{33}$$

with the observation

$$\begin{aligned} dY(t) &= D(t) \, dt + dW^v(t), \\ Y(0) &= 0, \end{aligned} \tag{34}$$

and the cost functional

$$J(v(\cdot)) = \mathbb{E}^v \left[\int_0^T \frac{1}{2} v(t)^2 dt + \frac{1}{2} x^v(T)^2 \right]. \tag{35}$$

Here, the coefficients a_i $(i = 1, \ldots, 5)$ and D are bounded and deterministic. The set of admissible controls is defined by

$$\begin{aligned} \mathscr{U}_{\mathrm{ad}} = \Big\{ & v(\cdot) \mid v(\cdot) \text{ is an } \mathrm{R}^1\text{-valued } \mathscr{F}_t^Y\text{-adapted} \\ & \text{and satisfies } \mathbb{E} \sup_{0 \leq t \leq T} v(t)^2 < \infty \Big\}. \end{aligned} \tag{36}$$

$\mathbb{E}^v$ denotes the expectation with respect to the probability space $(\Omega, \mathscr{F}, P^v)$, $dP^v = \rho^v(T) dP$ and

$$\begin{aligned} d\rho^v(t) &= \rho^v(t) D(t) \, dY(t). \\ \rho^v(0) &= 1. \end{aligned} \tag{37}$$

We aim to find an explicitly optimal control to minimize the cost functional $J(v(\cdot))$ over $v(\cdot) \in \mathscr{U}_{\mathrm{ad}}$, subject to (33) and (34).

Now we begin to seek the explicit optimal control by three steps.

First Step. Find candidate optimal controls.

We firstly write down the *Hamiltonian* function

$$\begin{aligned} &\mathscr{H}\left(t, x, v, q, k, \overline{k}, \overline{r}(\cdot), \overline{K} - a_4(t)\right) \\ &\doteq q\left(a_1(t) x + a_2(t) v\right) + k a_3(t) + \overline{k} a_4(t) \\ &\quad + D(t)\left(\overline{K} - a_4(t) q\right) + \frac{1}{2} v^2 + \int_{\mathscr{E}} a_5(t) \overline{r}(t, e) \pi(de), \end{aligned} \tag{38}$$

where x is the trajectory to (33) corresponding to the candidate optimal control $u(\cdot)$. By Theorem 2, we find a unique candidate optimal control $u(\cdot)$ which satisfies the following expression:

$$u(t) = -a_2(t) \mathbb{E}^u \left[q(t) \mid \mathscr{F}_t^Y \right], \tag{39}$$

where $(q(\cdot),\ k(\cdot),\ \overline{k}(\cdot),\ \overline{r}(\cdot,\cdot))$ is the solution of the following equations under measure P^u

$$\begin{aligned} -dq(t) &= a_1(t) q(t) \, dt - k(t) \, dB(t) \\ &\quad - \overline{k}(t) \, dW(t) - \int_{\mathscr{E}} \overline{r}(t-, e) \widetilde{N}(de\, dt), \\ -dQ(t) &= \frac{1}{2} u^2(t) \, dt - K(t) \, dB(t) \\ &\quad - \overline{K}(t) \, dW(t) - \int_{\mathscr{E}} \overline{R}(t-, e) \widetilde{N}(dedt), \\ q(T) &= x(T), \qquad Q(T) = \frac{1}{2} x(T)^2. \end{aligned} \tag{40}$$

Second Step. Verify $u(\cdot)$ in (39) is indeed optimal.

We can check that all conditions in Theorem 3 are satisfied, so $u(\cdot)$ in (39) is indeed optimal.

Third Step. Give explicit expression of optimal control.

Although we obtain its nominal form, the expression of optimal control $u(\cdot)$ in (39) is not quite explicit and satisfactory. From (33), we know that the state $x(\cdot)$ is dependent on the control $u(\cdot)$. Since $q(\cdot)$ and $x(\cdot)$ are coupled at time T and the control $u(\cdot)$ in (39) is dependent on the adjoint state $q(\cdot)$, (33) and (40) compose mutually coupled forward-backward stochastic systems which makes it difficult to find their explicit solution. Next, we try to get a more explicit and observable expression than the one in (39) by certain filtering techniques.

We introduce the notation $\hat{y}(t) = \mathbb{E}^u[y(t) \mid \mathcal{F}_t^Y]$, for $y = x, q, \bar{k}$. Then the optimal control $u(\cdot)$ in (39) can be rewritten as

$$u(t) = -a_2(t)\hat{q}(t). \tag{41}$$

Obviously, from (41), if we deduce the optimal filter $\hat{q}(\cdot)$ of $q(\cdot)$, then we obtain the optimal control $u(\cdot)$. Since $D(\cdot)$ is a deterministic function on t, we know from (34) that $\mathcal{F}_t^Y = \mathcal{F}_t^W = \mathcal{F}_t^{W^u}$. Then we have $\hat{y}(t) = \mathbb{E}^u[y(t) \mid \mathcal{F}_t^Y] = \mathbb{E}^u[y(t) \mid \mathcal{F}_t^W]$. Under measure P^u, $W(\cdot)$ is a Brownian motion, but $Y(\cdot)$ is not. In order to solve the explicit filter $\hat{q}(\cdot)$, from Xiong [19, Lemma 5.4], we can deduce the following group of filtering equations:

$$\begin{aligned} d\hat{x}(t) &= \left[a_1(t)\hat{x}(t) - a_2(t)^2\hat{q}(t)\right]dt + a_4(t)\,dW(t), \\ -d\hat{q}(t) &= a_1(t)\hat{q}(t)\,dt - \hat{\bar{k}}(t)\,dW(t), \\ \hat{x}(0) &= x_0, \qquad \hat{q}(T) = \hat{x}(T). \end{aligned} \tag{42}$$

In terms of the monotonicity conditions introduced by Peng and Wu [20], (42) has a unique solution $(\hat{x}(\cdot), \hat{q}(\cdot), \hat{\bar{k}}(\cdot))$. Substituting $\hat{x}(\cdot)$ and $u(\cdot)$ associated with $\hat{q}(\cdot)$ into (35), we derive the optimal cost functional.

5. Conclusion Remark

This paper has investigated the optimal control problem of partially observable jump-diffusion SDEs. The most distinguishing feature is, compared with the existing literature, that the states and observations are correlated. By transforming the partial observation problem to a related problem with full information, we established a necessary and a sufficient optimality conditions. Under the framework of convex control domain, our maximum principle can cover Tang and Hou [8], Tang [15] and Xiao [12]) as particular cases, but does not establish the relations among the adjoint processes. In the future work, we shall introduce some adjoint vector fields to characterize the adjoint processes and derive some other formulations of partially observable stochastic maximum principle.

Acknowledgments

This research project was funded by the National Nature Science Foundation of China (11201263, 11071144, and 11101242), the Nature Science Foundation of Shandong Province (ZR2012AQ004, BS2011SF010), and Independent Innovation Foundation of Shandong University (IIFSDU), China.

References

[1] S. Peng, "General stochastic maximum principle for optimal control problems," *SIAM Journal on Control and Optimization*, vol. 28, no. 4, pp. 966–979, 1990.

[2] S. Tang and X. Li, "Necessary conditions for optimal control of stochastic systems with random jumps," *SIAM Journal on Control and Optimization*, vol. 32, no. 5, pp. 1447–1475, 1994.

[3] H. Pham, "On some recent aspects of stochastic control and their applications," *Probability Surveys*, pp. 506–549, 2005.

[4] J. S. Baras, R. J. Elliott, and M. Kohlmann, "Partially observed stochastic minimum principle," *SIAM Journal on Control and Optimization*, vol. 27, no. 6, pp. 1279–1292, 1989.

[5] A. Bensoussan, "Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions," *Stochastics*, vol. 9, no. 3, pp. 169–222, 1983.

[6] W. Fleming, "Optimal control of partially observable diffusions," *SIAM Journal on Control and Optimization*, vol. 6, pp. 194–214, 1968.

[7] X. Li and S. Tang, "General necessary conditions for partially observed optimal stochastic controls," *Journal of Applied Probability*, vol. 32, pp. 1118–1137, 1995.

[8] S. Tang and S. H. Hou, "Optimal control of point processes with noisy observations: the maximum principle," *Applied Mathematics and Optimization*, vol. 45, no. 2, pp. 185–212, 2002.

[9] J. Huang, G. Wang, and Z. Wu, "Optimal premium policy of an insurance firm: full and partial information," *Insurance*, vol. 47, no. 2, pp. 208–215, 2010.

[10] G. Wang and Z. Wu, "Kalman-Bucy filtering equations of forward and backward stochastic systems and applications to recursive optimal control problems," *Journal of Mathematical Analysis and Applications*, vol. 342, no. 2, pp. 1280–1296, 2008.

[11] G. C. Wang and Z. Wu, "General maximum principles for partially observed risk-sensitive optimal control problems and applications to finance," *Journal of Optimization Theory and Applications*, vol. 141, no. 3, pp. 677–700, 2009.

[12] H. Xiao, "The maximum principle for partially observed optimal control of forward-backward stochastic systems with random jumps," *Journal of Systems Science and Complexity*, vol. 24, no. 6, pp. 1083–1099, 2011.

[13] X. Y. Zhou, "Sufficient conditions of optimality for stochastic systems with controllable diffusions," *IEEE Transactions on Automatic Control*, vol. 41, no. 8, pp. 1176–1179, 1996.

[14] Y. Hu and B. Øksendal, "Partial information linear quadratic control for jump diffusions," *SIAM Journal on Control and Optimization*, vol. 47, no. 4, pp. 1744–1761, 2008.

[15] S. Tang, "The maximum principle for partially observed optimal control of stochastic differential equations," *SIAM Journal on Control and Optimization*, vol. 36, no. 5, pp. 1596–1617, 1998.

[16] H. Xiao, "Maximum principle for optimal control of point processes with correlated noisy observations," in *Proceedinngs of the 30th Chinese Control Conference (CCC '11)*, pp. 1921–1924, Yantai, China, July 2011.

[17] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North- Holland, New York, NY, USA; Kodansha, Tokyo, Japan, 1989.

[18] J. Huang, X. Li, and G. Wang, "Maximum principles for a class of partial information risk-sensitive optimal controls," *IEEE Transactions on Automatic Control*, vol. 55, no. 6, pp. 1438–1443, 2010.

[19] J. Xiong, *An Introduction to Stochastic Filtering Theory*, Oxford University Press, London, UK, 2008.

[20] S. Peng and Z. Wu, "Fully coupled forward-backward stochastic differential equations and applications to optimal control," *SIAM Journal on Control and Optimization*, vol. 37, no. 3, pp. 825–843, 1999.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 747890, 8 pages
http://dx.doi.org/10.1155/2013/747890

Research Article

Robust H_∞ Filter Design for Itô Stochastic Pantograph Systems

Zhiguo Yan[1] and Yulin Huang[2]

[1] *School of Electrical Engineering and Automation, Shandong Polytechnic University, Jinan 250353, China*
[2] *School of Science, Shandong Polytechnic University, Jinan 250353, China*

Correspondence should be addressed to Zhiguo Yan; yanzg500@sina.com

Received 20 January 2013; Accepted 18 February 2013

Academic Editor: Weihai Zhang

The problem of robust H_∞ filter design is investigated for stochastic pantograph systems governed by linear Itô differential equation. First, a sufficient condition for asymptotic mean-square stability of stochastic pantograph systems is presented by means of Lyapunov approach. Then, based on matrix inequalities, the H_∞ filtering problem for this kind of systems is studied and a sufficient condition for the existence of the H_∞ filter is derived. Furthermore, the explicit expression of the desired filter parameters is characterized. Finally, an example is given to illustrate the results.

1. Introduction

Stochastic pantograph system which is treated as a special class of time-delay systems has also attracted more and more researchers [1–5]. Reference [1] gave the necessary analytical theory for the existence and uniqueness of a strong solution of the linear stochastic pantograph differential equation and presented the strong approximations to the solution obtained by a continuous extension of θ-Euler scheme. Reference [2] investigated the asymptotic growth and delay properties of solution of linear stochastic pantograph equation and gave the sufficient conditions on parameters when the solution grows at a polynomial rate in pth mean and almost sure sense. Reference [3] studied the αth moment stability for stochastic pantograph equation by using Razumikhin technique. Reference [4] investigated the convergence of the Euler method of stochastic pantograph equations and proved that the Euler approximation solution converges to the analytic solution in probability under weaker conditions. Reference [5] studied the almost surely asymptotic stability of the nonlinear stochastic pantograph differential equations with Markovian switching under the weakened linear growth condition. At present, most literatures on stochastic pantograph equation focus on the existence, uniqueness, and convergence of the numerical solution produced by kinds of approximate methods.

On the other hand, due to great many applications of robust H_∞ control and filtering in real world, the problems on these two have been studied extensively [6–14]. Compared with classical Kalman filter, one does not need to know the exact statistic information about the external disturbance in the H_∞ filtering design. H_∞ filtering requires one to design a filter such that the L_2 gain from the external disturbance to the estimation error is below a prescribed level $\gamma > 0$. Reference [10] studied the problem of H_∞ filtering for general continuous-time linear stochastic systems and gave a necessary and sufficient condition for the existence of H_∞ filter and furthermore designed H_2/H_∞ filter. Reference [11] gave a necessary and sufficient condition for reduced-order H_∞ filter of linear continuous and discrete-time stochastic systems. Reference [12] investigated the robust H_∞ filtering problem for nonlinear stochastic systems and gave a sufficient condition for the existence of H_∞ filter. Reference [13] studied the mixed H_2/H_∞ filtering for a class of nonlinear stochastic systems. Reference [14] considered the finite-time H_∞ filter design for a class of nonlinear stochastic systems. Nevertheless, to the best of our knowledge, the issue on the H_∞ filtering for stochastic linear pantograph systems with

state-dependent noise has not been investigated in previous literatures.

In this paper, we first consider the problem on the asymptotic mean-square stability and give a test criterion for stochastic pantograph systems by the Lyapunov approach. On this basis, a sufficient condition of the asymptotic mean square stability is obtained, which can be available for studying the H_∞ filtering of stochastic pantograph systems. Moreover, the H_∞ filter design is investigated and a sufficient condition for the existence of H_∞ filter is obtained in the form of linear matrix inequality. Finally, an example is given to illustrate our proposed methods.

This paper is organized as follows. Section 2 discusses the asymptotic mean-square stability of stochastic pantograph systems and presents a sufficient condition of stability by means of the Lyapunov approach. The H_∞ filtering problem of stochastic pantograph systems is investigated in Section 3. Section 4 provides a numerical example to demonstrate the effectiveness and applicability of the proposed methods. Section 5 concludes this paper.

2. Asymptotic Mean-Square Stability

Consider the following linear stochastic pantograph system:

$$\begin{gathered} dx(t) = \left(Ax(t) + A_1 x(qt)\right) dt + \left(Cx(t) + C_1 x(qt)\right) dw(t) \\ x(0) = x_0, \end{gathered} \tag{1}$$

where $x(t) \in R^n$ is system state; $0 < q < 1$; $w(t)$ is a one-dimension standard Wiener process defined on a complete probability space (Ω, F, F_t, P) with $F_t = \sigma(w(s) : 0 \le s \le t)$; A, A_1, C, C_1 are all constant matrices of $R^{n\times n}$. For initial value $x_0 \in R^n$ and $T > 0$, there exists a unique solution $x(t) \in L_F^2(R_T, R^n)$ [1].

Definition 1. The stochastic pantograph system (1) is said to be asymptotically mean square stable if for any initial value x_0, the corresponding state satisfies

$$\lim_{t\to\infty} \mathbb{E}\|x(t)\|^2 = 0. \tag{2}$$

Next, a test criterion for asymptotically mean-square stable of stochastic pantograph systems is given.

Lemma 2. *Stochastic pantograph system* (1) *is asymptotically mean-square stable if there exist some positive constant scalars $k_1 > 0$, $k_2 > 0$, and $k_3 > 0$ and a Lyapunov function $V(t,x)$ satisfying*

$$k_1\|x(t)\|^2 \le V(t,x) \le k_2\|x\|^2, \tag{3}$$

$$LV(t,x) \le -k_3\|x\|^2, \tag{4}$$

where

$$\begin{aligned} LV(t,x) &= \frac{\partial V(t,x)}{\partial t} + \frac{\partial V'(t,x)}{\partial x}\left(Ax(t) + A_1 x(qt)\right) \\ &\quad + \frac{1}{2}\left(Cx(t) + C_1 x(qt)\right)' \frac{\partial^2 V(t,x)}{\partial x^2}\left(Cx(t) + C_1 x(qt)\right). \end{aligned} \tag{5}$$

Proof. Expressing the difference $V(t,x(t)) - V(0,x_0)$ by means of Itô formula [15], calculating expectations, we get

$$\mathbb{E}V(t,x(t)) - V(0,x_0) = \int_0^t \mathbb{E}LV(s,x(s))\,ds. \tag{6}$$

Differentiating this equality with respect to t and using (3), (4), we see that

$$\frac{d}{dt}\mathbb{E}V(t,x(t)) \le -\frac{k_3}{k_2}\mathbb{E}V(t,x(t)). \tag{7}$$

This implies the estimate

$$\mathbb{E}V(t,x(t)) \le V(0,x_0)\exp\left\{-\frac{k_3}{k_2}t\right\}. \tag{8}$$

Together with (3), this estimate yields

$$\mathbb{E}\|x(t)\|^2 \le \frac{1}{k_1}V(0,x_0)\exp\left\{-\frac{k_3}{k_2}t\right\}. \tag{9}$$

Let $t \to \infty$; then (2) is obtained. This proof is complete. □

On the basis of Lemma 2, the following theorem gives a sufficient condition of the asymptotic mean-square stability is obtained, which can be available for studying the H_∞ filtering of stochastic pantograph systems.

Theorem 3. *If the following linear matrix inequality*

$$PA + A'P + P + 2C'PC + \frac{1}{q}\left(A_1'PA_1 + 2C_1'PC_1\right) < 0 \tag{10}$$

has a solution $P > 0$, then stochastic pantograph system (1) *is asymptotically mean-square stable.*

Proof. Take a Lyapunov function $V(t,x) = x'Px$, where $P > 0$ is the solution of (10). Applying Itô formula and by Cauchy inequality $X'PY + Y'PX \le X'PX + Y'PY$, we obtain

$$\begin{aligned} dV(x) &= \left[x'\left(PA + A'P + C'PC\right)x \right. \\ &\quad + x'(qt)A_1'Px + x'PA_1x(qt) + x'(qt)C_1'PCx \\ &\quad \left. + x'C'PC_1x(qt) + x'(qt)C_1'PC_1x(qt)\right] dt \end{aligned}$$

$$\begin{aligned}
&+\left[\left(Cx+C_1x\left(qt\right)\right)'Px+x'P\left(Cx+C_1x\left(qt\right)\right)\right]dw\left(t\right)\\
&\le\left[x'\left(PA+A'P+C'PC\right)x\right.\\
&\quad+x'\left(qt\right)A_1'PA_1x\left(qt\right)+x'Px+x'\left(qt\right)C_1'PC_1x\left(qt\right)\\
&\quad\left.+x'C'PCx+x'\left(qt\right)C_1'PC_1x\left(qt\right)\right]dt\\
&\quad+\left[\left(Cx+C_1x\left(qt\right)\right)'Px+x'P\left(Cx+C_1x\left(qt\right)\right)\right]dw\left(t\right)\\
&\le\left[x'\left(PA+A'P+P+2C'PC\right)x+x'\left(qt\right)Qx\left(qt\right)\right]dt\\
&\quad+\left[\left(Cx+C_1x\left(qt\right)\right)'Px+x'P\left(Cx+C_1x\left(qt\right)\right)\right]dw\left(t\right),
\end{aligned}\tag{11}$$

where $Q=(A_1'PA_1+2C_1'PC_1)\ge 0$ due to $P>0$. So for any $0\le s\le t$, taking integral from s to t, we have

$$\begin{aligned}
&\mathbb{E}V\left(x\left(t\right)\right)-\mathbb{E}V\left(x\left(s\right)\right)\\
&\quad\le\mathbb{E}\int_s^t x'\left(u\right)\left(PA+A'P+P+2C'PC\right)x\left(u\right)du\\
&\quad\quad+\mathbb{E}\int_s^t x'\left(qu\right)Qx\left(qu\right)du,
\end{aligned}\tag{12}$$

where

$$\begin{aligned}
&\mathbb{E}\int_s^t x'\left(qu\right)Qx\left(qu\right)du\\
&\quad=\frac{1}{q}\mathbb{E}\int_s^{qt}x'\left(u\right)Qx\left(u\right)du\le\frac{1}{q}\mathbb{E}\int_s^t x'\left(u\right)Qx\left(u\right)du.
\end{aligned}\tag{13}$$

The above last inequality is valid because of $Q\ge 0$ and $0<q<1$, so

$$\begin{aligned}
&\mathbb{E}V\left(x\left(t\right)\right)-\mathbb{E}V\left(x\left(s\right)\right)\\
&\quad\le\mathbb{E}\int_s^t x'\left(u\right)\left(PA+A'P+P+2C'PC+\frac{1}{q}Q\right)x\left(u\right)du.
\end{aligned}\tag{14}$$

Multiplying $1/(t-s)$ by both sides simultaneously and letting $t\to s$, we obtain

$$\mathbb{E}dV\left(x\left(t\right)\right)\le x'\left(t\right)\left(PA+A'P+P+2C'PC+\frac{1}{q}Q\right)x\left(t\right)dt.\tag{15}$$

Therefore, the infinitesimal generator of stochastic pantograph system (1) satisfies

$$\begin{aligned}
LV\left(x\left(t\right)\right)&\le x'\left(t\right)\left(PA+A'P+P+2C'PC+\frac{1}{q}Q\right)x\left(t\right)\\
&\le -k_3\|x\|^2,
\end{aligned}\tag{16}$$

where $PA+A'P+P+2C'PC+(1/q)Q\le -k_3<0$ for some $k_3>0$. By Lemma 2, the asymptotic mean-square stability of (1) is derived, which completes the proof. □

Remark 4. Inequality (10) is a linear matrix inequality, which provides more convenience to test the asymptotic mean-square stability of stochastic pantograph system (1).

Remark 5. When $A_1=C_1=0$, the pantograph system (1) becomes normal stochastic linear system

$$dx\left(t\right)=Ax\left(t\right)dt+Cx\left(t\right)dw\left(t\right),\tag{17}$$

and (10) is simplified by

$$PA+A'P+P+2C'PC<0,\tag{18}$$

which implies $PA+A'P+P+C'PC<0$, so (18) can also guarantee the asymptotic mean-square stability of (17) [15].

Remark 6. Let $\tau(t)=t-qt$; system (1) becomes

$$\begin{aligned}
dx\left(t\right)=&\left[Ax\left(t\right)+A_1x\left(t-\tau\left(t\right)\right)\right]dt\\
&+\left[Cx\left(t\right)+C_1x\left(t-\tau\left(t\right)\right)\right]dw\left(t\right).
\end{aligned}\tag{19}$$

System (19) is a time-vary delay system. Condition (10) guarantees that system (19) is asymptotically mean-square stable.

3. Robust H_∞ Filter Design

Based on the asymptotic mean-square stability of pantograph system discussed in the above section, we are in a position to deal with the H_∞ filtering problem for stochastic pantograph system.

Consider the following stochastic linear perturbed pantograph system with measurement output:

$$\begin{aligned}
dx\left(t\right)&=\left(Ax\left(t\right)+A_1x\left(qt\right)+Bv\left(t\right)\right)dt\\
&\quad+\left(Cx\left(t\right)+C_1x\left(qt\right)\right)dw\left(t\right)\\
dy\left(t\right)&=\left(A_2x\left(t\right)+B_1v\left(t\right)\right)dt+C_2x\left(t\right)dw\left(t\right)\\
z\left(t\right)&=Mx\left(t\right),
\end{aligned}\tag{20}$$

where $x(t)\in R^n$, $y(t)\in R^{n_y}$, $v(t)\in R^{n_v}$, and $z(t)\in R^{n_z}$ are the system state, the exogenous disturbance signal, the measurement output, and the state combination to be estimated, respectively. A, A_1, A_2, B, B_1, C, C_1, C_2, and M are constant matrices of suitable dimension. Here we suppose $v(t)\in L_F^2(R_+,R^{n_v})$, which guarantees that the system (20) has a unique solution $x(t)\in L_F^2(R_T,R^n)$ for any $T>0$.

The so-called H_∞ filtering problem is to design an estimator to estimate the unknown state combination $z(t)$ via output measurement $y(t)$, which guarantees the L_2 gain (from the external disturbance to estimation error) to be less than a prescribe level $\gamma>0$, and the extended system is internally stable. Here we construct the following linear pantograph filter via output measurement for the estimation of $z(t)$:

$$\begin{aligned}
d\hat{x}\left(t\right)&=\left(A_f\hat{x}\left(t\right)+B_f\hat{x}\left(qt\right)\right)dt+C_fdy\left(t\right)\\
\hat{x}\left(0\right)&=\hat{x}_0\\
\hat{z}\left(t\right)&=M_f\hat{x}\left(t\right),
\end{aligned}\tag{21}$$

where $\widehat{x}(t) \in R^n$, $A_f \in R^{n\times n}$, $B_f \in R^{n\times n}$, $C_f \in R^{n\times n_y}$, and $M_f \in R^{n_z\times n}$ are constant matrices to be determined subsequently. Let $\eta(t) = (x'(t)\ \ \widehat{x}'(t))'$, $\widetilde{z}(t) = z(t) - \widehat{z}(t)$; then the extended system is

$$\begin{aligned} d\eta(t) &= \left(\widetilde{A}\eta(t) + \widetilde{A}_1\eta(qt) + \widetilde{B}v(t)\right)dt \\ &\quad + \left(\widetilde{C}\eta(t) + \widetilde{C}_1\eta(qt)\right)dw(t), \\ \widetilde{z}(t) &= \widetilde{M}\eta(t), \end{aligned} \tag{22}$$

where

$$\begin{aligned} \widetilde{A} &= \begin{bmatrix} A & 0 \\ C_fA_2 & A_f \end{bmatrix}, \qquad \widetilde{A}_1 = \begin{bmatrix} A_1 & 0 \\ 0 & B_f \end{bmatrix}, \\ \widetilde{B} &= \begin{bmatrix} B \\ C_fB_1 \end{bmatrix}, \qquad \widetilde{C} = \begin{bmatrix} C & 0 \\ C_fC_2 & 0 \end{bmatrix}, \\ \widetilde{C}_1 &= \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widetilde{M} = \begin{bmatrix} 0 & -M_f \end{bmatrix}. \end{aligned} \tag{23}$$

For a given disturbance attenuation level $\gamma > 0$ and $v(t) \in L_F^2(R_+, R^{n_v})$, define the associated H_∞ filtering performance of (22) as

$$J_\infty = \mathbb{E}\int_0^\infty \|\widetilde{z}(t)\|^2 dt - \gamma^2\mathbb{E}\int_0^\infty \|v(t)\|^2 dt. \tag{24}$$

As in [10], the H_∞ filtering problem is formulated as follows.

Stochastic H_∞ Filtering Problem. Given $\gamma > 0$, find an estimator $\widehat{x}$ of the form (21) leading (22) to being internally stable. Moreover, for any $v(t) \neq 0$, $v(t) \in L_F^2(R_+, R^{n_v})$ with $\eta(0) = 0$, there always is $J_\infty < 0$.

In what follows, we will give the main result of H_∞ filtering problem and provide a technique to determine matrices A_f, B_f, C_f, M_f of filter (22).

Theorem 7. *If the following matrix inequality*

$$\begin{aligned} &P\widetilde{A} + \widetilde{A}'P + P + 2\widetilde{C}'P\widetilde{C} + \frac{1}{q}\left(\widetilde{A}_1'P\widetilde{A}_1 + 2\widetilde{C}'P\widetilde{C}\right) \\ &\quad + \widetilde{M}'\widetilde{M} + \gamma^{-2}P\widetilde{B}\widetilde{B}'P < 0 \end{aligned} \tag{25}$$

has a solution $P > 0$, then system (22) with $v(t) = 0$ is asymptotically mean-square stable, and $J_\infty < 0$ holds for any $v(t) \neq 0$, $v(t) \in L_F^2(R_+, R^{n_v})$ when $\eta(0) = 0$.

Proof. When $v(t) = 0$, from (25) we obtain

$$\begin{aligned} &P\widetilde{A} + \widetilde{A}'P + P + 2\widetilde{C}'P\widetilde{C} + \frac{1}{q}\left(\widetilde{A}_1'P\widetilde{A}_1 + 2\widetilde{C}'P\widetilde{C}\right) \\ &\quad < -\widetilde{M}'\widetilde{M} - \gamma^{-2}P\widetilde{B}\widetilde{B}'P < 0, \end{aligned} \tag{26}$$

so system (22) is asymptotical mean-square stable according to Theorem 3.

Next, we prove $J_\infty < 0$ for any nonzero $v(t) \in L_F^2(R_+, R^{n_v})$ with $\eta(0) = 0$, taking the Lyapunov function $V(\eta) = \eta'P\eta$, where $P > 0$ is a solution of (25), and following the outline of the proof in Theorem 3, we obtain that the infinitesimal generator of (22) satisfies

$$\begin{aligned} LV(\eta) \le \eta' &\left(P\widetilde{A} + \widetilde{A}'P + P + 2\widetilde{C}'P\widetilde{C}\right. \\ &\left. + \frac{1}{q}\left(\widetilde{A}_1'P\widetilde{A}_1 + 2\widetilde{C}'P\widetilde{C}\right)\right)\eta \\ &+ v'\widetilde{B}'P\eta + \eta'P\widetilde{B}v. \end{aligned} \tag{27}$$

Note that for $T > 0$,

$$\begin{aligned} &J_T(\eta, v) \\ &\quad = \mathbb{E}\int_0^T \|\widetilde{z}(t)\|^2 dt - \gamma^2\mathbb{E}\int_0^T \|v(t)\|^2 dt \\ &\quad = \mathbb{E}\int_0^T \|\widetilde{z}(t)\|^2 dt \\ &\qquad - \gamma^2\mathbb{E}\int_0^T \|v(t)\|^2 dt + d\left(\eta'P\eta\right) - d\left(\eta'P\eta\right) \\ &\quad = -\mathbb{E}\eta'(T)P\eta(T) + \mathbb{E}\int_0^T \left(\eta'\widetilde{M}'\widetilde{M} - \gamma^2 v'v + LV(\eta)\right)dt \\ &\quad \le \mathbb{E}\int_0^T \left(\eta\widetilde{M}'\widetilde{M} - \gamma^2 v'v + LV(\eta)\right)dt \\ &\quad \le \mathbb{E}\int_0^T \begin{bmatrix} \eta \\ v \end{bmatrix}' N \begin{bmatrix} \eta \\ v \end{bmatrix} dt, \end{aligned} \tag{28}$$

where

$$N = \begin{bmatrix} \begin{array}{c} P\widetilde{A} + \widetilde{A}'P + P + 2\widetilde{C}'P\widetilde{C} \\ +\frac{1}{q}\left(\widetilde{A}_1'P\widetilde{A}_1 + 2\widetilde{C}'P\widetilde{C}\right) + \widetilde{M}'\widetilde{M} \end{array} & P\widetilde{B} \\ \widetilde{B}'P & -\gamma^{-2}I \end{bmatrix}. \tag{29}$$

If $N < 0$, then there exists $\epsilon > 0$, such that

$$J_T(\eta, v) \le -\epsilon\mathbb{E}\int_0^T \left(\|\eta\|^2 + \|v\|^2\right)dt \le -\epsilon\mathbb{E}\int_0^T \|v\|^2 dt. \tag{30}$$

Let $T \to \infty$; then $J_\infty(\eta, v) \le -\epsilon\mathbb{E}\int_0^\infty \|v\|^2 dt < 0$. By Schur Complement, $N < 0$ is equivalent to (25), which ends the proof. □

It is difficult to solve the inequality (25) because of its nonlinearity, so Theorem 7 cannot be directly available for designing the filter. Next we will give a sufficient condition easy to be solved.

Theorem 8. *If the following LMI*

$$\begin{bmatrix} P_{11}A + A'P_{11} + P_{11} + \frac{2}{q}C_1'P_{11}C_1 & A_2'Z_1' & 2C'P_{11} & 2C_1'Z_1' & M' & P_{11}B & A_1'P_{11} & 0 \\ * & Z_3 + Z_3' + P_{22} & 0 & 0 & -M_f' & Z_1B_1 & 0 & Z_2' \\ * & * & -2P_{11} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -2P_{22} & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & * & * & -qP_{11} & 0 \\ * & * & * & * & * & * & * & -qP_{22} \end{bmatrix} < 0 \quad (31)$$

has solutions $P_{11} > 0$, $P_{22} > 0$, $Z_1 \in R^{n\times n_y}$, $Z_2 \in R^{n\times n}$, $Z_3 \in R^{n\times n}$, M_f, then system (22) is internally asymptotically mean-square stable, and filtering performance $J_\infty < 0$ holds for any $v(t) \neq 0$, $v(t) \in L_F^2(R_+, R^{n_v})$ with $\eta(0) = 0$. The corresponding H_∞ filter (21) can be formulated by

$$\begin{gathered} d\hat{x}(t) = \left(P_{22}^{-1}Z_3\hat{x}(t) + P_{22}^{-1}Z_2\hat{x}(qt)\right)dt + P_{22}^{-1}Z_1 dy(t), \\ \hat{x}(0) = \hat{x}_0, \\ \hat{z}(t) = M_f\hat{x}(t). \end{gathered} \quad (32)$$

Proof. By Schur Complement, (25) is equivalent to

$$\begin{bmatrix} P\widetilde{A} + \widetilde{A}'P + P + \frac{2}{q}\widetilde{C}_1'P\widetilde{C}_1 & 2\widetilde{C}'P & \widetilde{M}' & P\widetilde{B} & \widetilde{A}_1'P \\ * & -2P & 0 & 0 & 0 \\ * & * & -I & 0 & 0 \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & -qP \end{bmatrix} < 0. \quad (33)$$

Taking $P = \left[\begin{smallmatrix} P_{11} & 0 \\ 0 & P_{22} \end{smallmatrix}\right]$ and substituting (23) into (31), after a series computations, we have

$$\begin{bmatrix} P_{11}A + A'P_{11} + P_{11} + \frac{2}{q}C_1'P_{11}C_1 & A_2'C_f'P_{22} & 2C'P_{11} & 2C_1'C_f'P_{22} & M' & P_{11}B & A_1'P_{11} & 0 \\ * & P_{22}A_f + A_f'P_{22} + P_{22} & 0 & 0 & -M_f' & P_{22}C_fB_1 & 0 & B_f'P_{22} \\ * & * & -2P_{11} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -2P_{22} & 0 & 0 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & 0 & 0 \\ * & * & * & * & * & * & -qP_{11} & 0 \\ * & * & * & * & * & * & * & -qP_{22} \end{bmatrix} < 0. \quad (34)$$

Setting $P_{22}C_f = Z_1$, $P_{22}B_f = Z_2$, $P_{22}A_f = Z_3$, then (34) turns out to be (31). Therefore, $A_f = P_{22}^{-1}Z_3$, $B_f = P_{22}^{-1}Z_2$, $C_f = P_{22}^{-1}Z_1$; then the proof is complete. □

Remark 9. In the proof of Theorem 8, the matrix P is chosen as diag$\{P_{11}, P_{22}\}$ for simplicity. In order to reduce the conservatism of the conditions, the matrix P can also be chosen as $\left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{smallmatrix}\right]$. However, this case will increase the complexity of computation.

Remark 10. In many engineering applications, the performance constraint is often specified a priori. In Theorem 8, the filter is designed after H_∞ performance is prescribed. In fact, we can obtain an improved performance by optimization method. In addition, inequality (31) may be no feasible solution for very small q, that is, very large time delay. However, the smallest q can be found by numerical algorithm. The results in Theorem 8 suggest the following optimization problems.

(OP1): The optimal H_∞ filtering problem for stochastic pantograph systems is defined by

$$\begin{gathered} \min_{P_{11}>0, P_{22}>0, Z_1, Z_2, Z_3, M_f} \chi \\ \text{subject to (31) with } \chi = \gamma^2. \end{gathered} \quad (35)$$

Then the minimum value of optimal H_∞ performance γ^* is given by $\gamma^* = (\min \chi)^{1/2}$.

(OP2): The minimum value of γ corresponding to the different values of q in the interval $(0, 1)$ can be found.

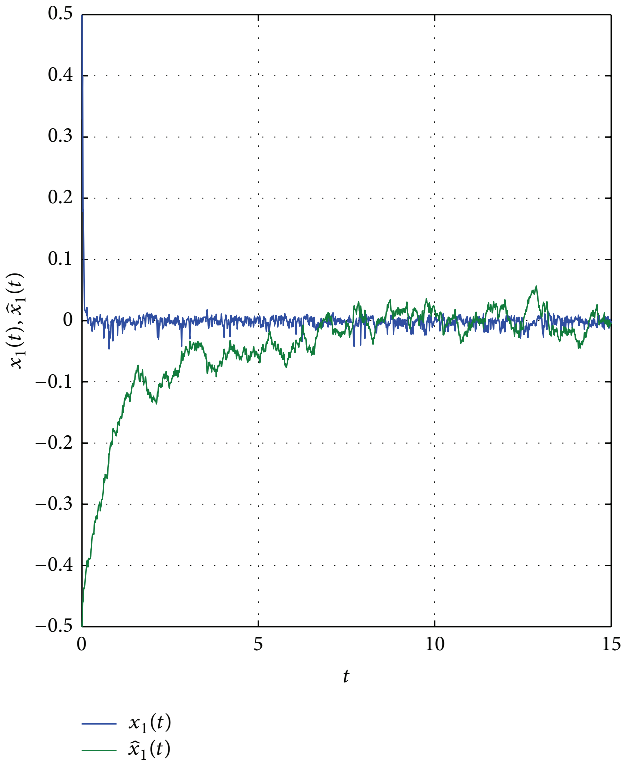


Figure 1: The trajectories of $x_1(t)$ and $\hat{x}_1(t)$.

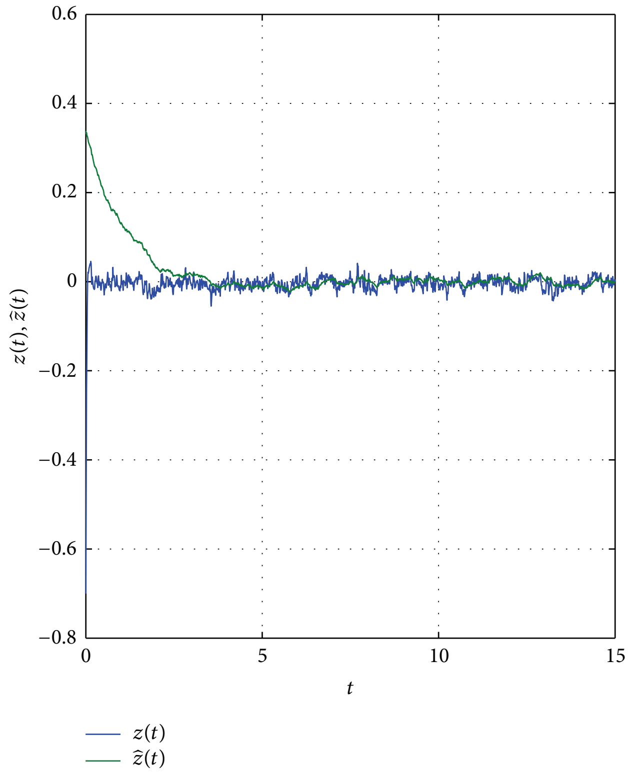


Figure 3: The trajectories of $z(t)$ and $\hat{z}(t)$.

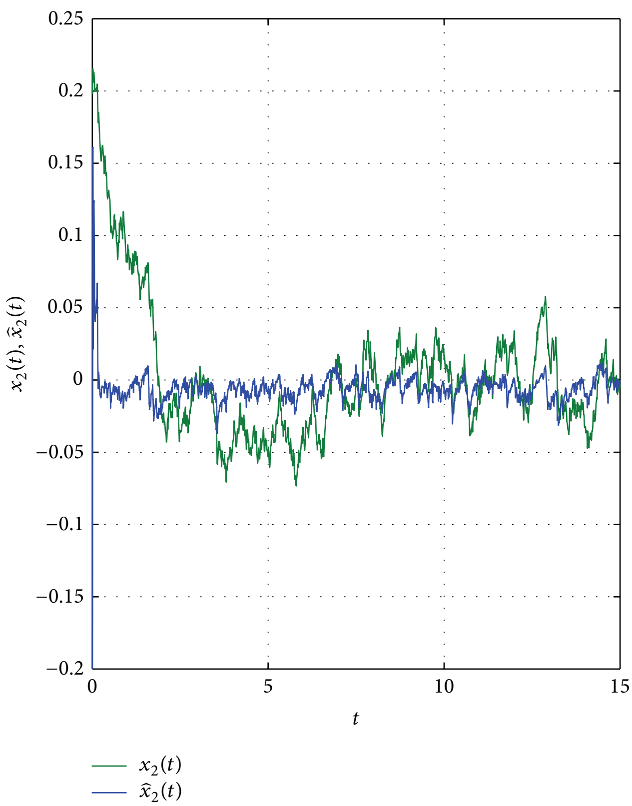


Figure 2: The trajectories of $x_2(t)$ and $\hat{x}_2(t)$.

Algorithm I. Consider the following steps.

Step 1. By simple search algorithms, if we find a series of q_i $(i = 1, \ldots, n)$ to make (31) have feasible solutions, then go to Step 2. Otherwise, go to Step 6.

Step 2. Set $i = 1$, take a q_i.

Step 3. Solving the following optimization problem OP1.

Step 4. Set $i = i + 1$, if $i + 1 > n$, then go to Step 5; otherwise $q_i = q_{i+1}$, go to Step 3.

Step 5. (31) has feasible solutions. Stop.

Step 6. (31) has no feasible solutions. Stop.

Remark 11. The smallest q may be obtained by Algorithm I.

4. Numerical Example

In this section, a numerical example is provided to demonstrate the effectiveness and applicability of the proposed methods. Consider the following Itô stochastic pantograph system:

$$dx(t) = \left(Ax(t) + A_1 x(qt) + Bv(t)\right) dt$$
$$+ \left(Cx(t) + C_1 x(qt)\right) dw(t)$$

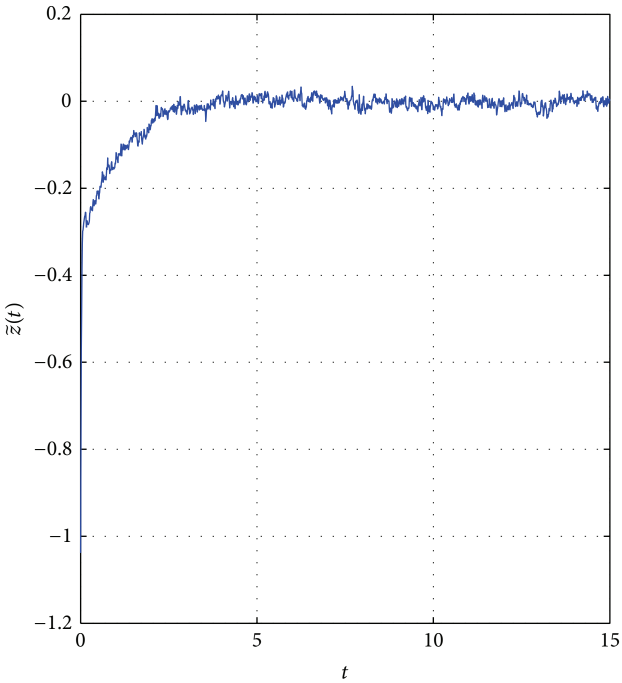


FIGURE 4: The trajectories of $\tilde{z}(t)$.

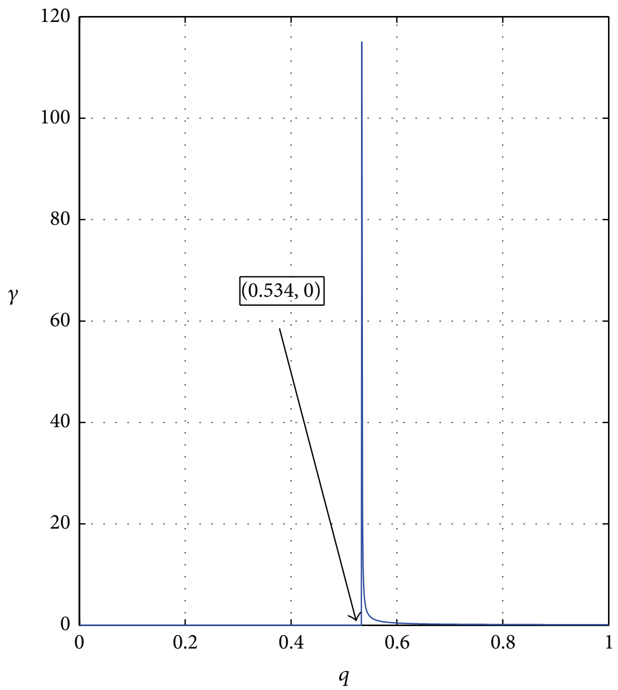


FIGURE 5: The minimum value of γ versus q in $(0, 1)$.

$$dy(t) = (A_2 x(t) + B_1 v(t))\,dt + C_2 x(t)\,dw(t)$$
$$z(t) = Mx(t), \tag{36}$$

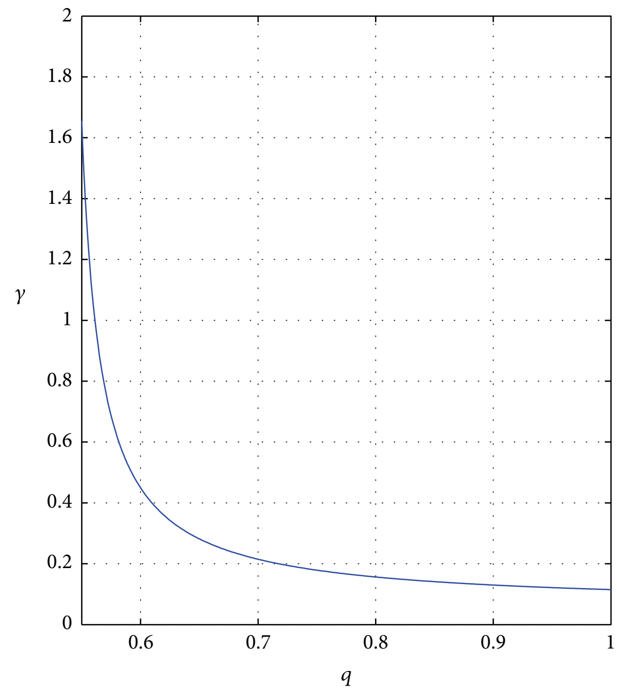


FIGURE 6: The minimum value of γ versus q in $(0.55, 1)$.

where

$$A = \begin{bmatrix} -50 & 0.8 \\ 0.4 & -10 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 2 & 1.5 \\ 0.2 & -2 \end{bmatrix},$$
$$B = \begin{bmatrix} 0.3 \\ -0.5 \end{bmatrix}, \qquad C = \begin{bmatrix} 4 & 0.5 \\ 3 & 1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} -0.8 & 1.2 \\ 0.8 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 15.6 & 1.1 \\ 0.6 & 8 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} 1.6 & 2 \\ 0 & 1 \end{bmatrix},$$
$$M = \begin{bmatrix} -1 & 1 \end{bmatrix}, \qquad q = 0.9. \tag{37}$$

Consider the following filter for estimation of $z(t)$:

$$d\hat{x}(t) = (A_f \hat{x}(t) + B_f \hat{x}(qt))\,dt + C_f\,dy(t),$$
$$\hat{x}(0) = \hat{x}_0, \tag{38}$$
$$\hat{z}(t) = M_f \hat{x}(t).$$

According to OP1, the minimum value of γ is 0.13 and the corresponding estimation gains of H_∞ filter are derived from theorem

$$A_f = \begin{bmatrix} -1.4414 & 0.0518 \\ 0.0512 & -1.5298 \end{bmatrix}, \qquad B_f = \begin{bmatrix} 0.5184 & -0.0095 \\ -0.0093 & 0.5418 \end{bmatrix},$$
$$C_f = \begin{bmatrix} 0.0610 & -0.0221 \\ -0.0219 & -0.0915 \end{bmatrix}, \qquad M_f = \begin{bmatrix} -0.3870 & 0.7240 \end{bmatrix}. \tag{39}$$

The initial condition in the simulation is assumed to be $\eta(0) = [0.5 \quad -0.2 \quad -0.5 \quad 0.2]$. Figures 1 and 2 show the trajectories of $x_1(t)$, $\hat{x}_1(t)$, $x_2(t)$, and $\hat{x}_2(t)$ by using the proposed H_∞ filter. Figure 3 shows the response of real state $z(t)$ and its estimation $\hat{z}(t)$. Figure 4 is the simulation result of the estimation error response of $\tilde{z}(t) = z(t) - \hat{z}(t)$, which demonstrates that the estimation error is asymptotically mean-square stable.

By the OP2, the minimum value of q can be given by $q = 0.534$. Figure 5 shows the minimum value of γ corresponding to different q in the interval $(0, 1)$. From Figure 5, we see that (31) has no feasible solution when q is in $(0, 0.534)$. In order to see the relationship between γ and q more clearly, Figure 6 gives the minimum value of γ corresponding to different q in the interval $(0.55, 1)$.

5. Conclusion

This paper has discussed infinite horizon H_∞ filtering for stochastic linear pantograph systems with state-dependent noise, which has not been studied for pantograph system in the previous literatures. A sufficient condition for asymptotic mean-square stability of stochastic linear pantograph systems is presented and a sufficient condition for the existence of the H_∞ filter is given in the form of linear matrix inequality. The results obtained in this paper may be significant in studying the other control/filtering problem such as H_2, H_2/H_∞ control/filtering for linear/nonlinear stochastic pantograph systems.

Notations

A':	The transpose of A
$A > 0$ ($A \geq 0$):	A is positive (nonnegative)
R^n:	The n-dimensional Euclidean space with $\|\cdot\|_2$
$R^{m\times n}$:	The set of all $m \times n$ matrices
R_+	$= [0, \infty)$
R_T	$= [0, T]$ for $T > 0$
I:	Identify matrix
$\mathbb{E}(\cdot)$:	The mathematical expectation operator
$L^2_F(R_r, R^k)$:	The space of nonanticipative square integrable stochastic processes $y(t) \in L^2(\Omega, R^k)$ with respect to an increasing σ-algebra satisfying F_t-measurable and $\mathbb{E}\int_0^T \|\cdot\|^2 dt < \infty$
$C^{1,2}(R_+ \times R^k; R_+)$:	The family of all nonnegative functions $V(t, x)$ on which are continuously once differentiable in t and twice differentiable in x.

Acknowledgments

This work is supported by the Starting Research Foundation of Shandong Polytechnic University under Grant 12045501, Outstanding Mid-Young Scientist Prize Foundation of Shandong Province (BS2011DX032), and a Project of Shandong Province Higher Educational Science and Technology Program (J10LG13).

References

[1] C. T. H. Baker and E. Buckwar, "Continuous θ-methods for the stochastic pantograph equation," *Electronic Transactions on Numerical Analysis*, vol. 11, pp. 131–151, 2000.

[2] J. A. Appleby and E. Buckwar, "Sufficient condition for polynomial asymptotic behavior of the stochastic pantograph equation," http://www4.dcu.ie/maths/research/preprint.shtml.

[3] Z. Fan, M. Song, and M. Liu, "The α-th moment stability for the stochastic pantograph equation," *Journal of Computational and Applied Mathematics*, vol. 233, no. 2, pp. 109–120, 2009.

[4] L. Ronghua, L. Min, and P. Wan-kai, "Convergence of numerical solutions to stochastic pantograph equations with Markovian switching," *Applied Mathematics and Computation*, vol. 215, no. 1, pp. 414–422, 2009.

[5] F. Jiang, H. Yang, and S. Wang, "Asymptotic stability of stochastic pantograph differential equations with Markovian switching," *Journal of Nonlinear Analysis and Optimization*, vol. 1, no. 1, pp. 9–16, 2010.

[6] B.-S. Chen and W. Zhang, "Stochastic H_2/H_∞ control with state-dependent noise," *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp. 45–57, 2004.

[7] W. Zhang and B.-S. Chen, "State feedback H_∞ control for a class of nonlinear stochastic systems," *SIAM Journal on Control and Optimization*, vol. 44, no. 6, pp. 1973–1991, 2006.

[8] Z. Lin, Y. Lin, and W. Zhang, "H_∞ filtering for non-linear stochastic Markovian jump systems," *IET Control Theory & Applications*, vol. 4, no. 12, pp. 2743–2756, 2010.

[9] X. Guo, G. Yang, and W. Che, "Insensitive dynamic output feedback control with mixed-H_∞ norm sensitivity minimization," *Journal of the Franklin Institute*, vol. 350, no. 1, pp. 72–91, 2013.

[10] E. Gershon, D. J. N. Limebeer, U. Shaked, and I. Yaesh, "Robust H_∞ filtering of stationary continuous-time linear systems with stochastic uncertainties," *IEEE Transactions on Automatic Control*, vol. 46, no. 11, pp. 1788–1793, 2001.

[11] S. Xu and T. Chen, "Reduced-order H_∞ filtering for stochastic systems," *IEEE Transactions on Signal Processing*, vol. 50, no. 12, pp. 2998–3007, 2002.

[12] W. Zhang, B.-S. Chen, and C.-S. Tseng, "Robust H_∞ filtering for nonlinear stochastic systems," *IEEE Transactions on Signal Processing*, vol. 53, no. 2, pp. 589–598, 2005.

[13] W. Zhang, B. S. Chen, L. Sheng, and M. Gao, "Robust H_2/H_∞ filter design for a class of nonlinear stochastic systems with state-dependent noise," *Mathematical Problems in Engineering*, vol. 2012, Article ID 750841, 16 pages, 2012.

[14] Z. G. Yan and G. S. Zhang, "Finite-time H_∞ filtering for a class of nonlinear stochastic uncertain systems," *Control and Decision*, vol. 27, no. 3, pp. 419–424, 2012.

[15] R. Z. Hasminskii, *Stochastic Stability of Differential Equations*, Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 104034, 6 pages
http://dx.doi.org/10.1155/2013/104034

Research Article

Travel Time Model for Right-Turning Vehicles of Secondary Street at Unsignalized Intersections

Feng Yu-Qin,[1] Leng Jun-Qiang,[2,3] Wang Peng,[4] He Yi,[3] and Zhang Gui-e[3]

[1] *School of Automobile and Traffic Engineering, Heilongjiang Institute of Technology, Harbin 150050, China*
[2] *School of Management, Harbin Institute of Technology, Harbin 150001, China*
[3] *School of Automobile Engineering, Harbin Institute of Technology, Weihai 264209, China*
[4] *Representative Office of PLA in Harbin Railway Administration, Harbin 150001, China*

Correspondence should be addressed to Leng Jun-Qiang; lengjunq@tom.com

Received 8 November 2012; Revised 9 February 2013; Accepted 17 February 2013

Academic Editor: Wuquan Li

The travel time of right-turning vehicles on secondary street at unsignalized intersection is discussed in this paper. Under the assumption that the major-street through vehicles' headway follows Erlang distribution and secondary-street right-turning vehicles' headway follows Poisson distribution. The right-turning vehicles travel time model is established on the basis of gap theory and M/G/1 queue theory. Comparison is done with the common model based on the assumption that the major-street vehicles' headway follows Poisson distribution. An intersection is selected to verify each model. The results show that the model established in this paper has stronger applicability, and its most relative error is less than 15%. In addition, the sensitivity analysis has been done. The results show that right-turning flow rate and major-street flow rate have a significant impact on the travel time. Hence, the methodology for travel time of right-turning vehicles at unsignalized intersection proposed in this paper is effective and applicable.

1. Introduction

As a bottleneck of urban road network, intersections are the emphasis in traffic management and control [1–3]. The travel time of vehicles at intersections is the basis to evaluate road traffic efficiency and the Intelligent Traffic System (ITS) applications. It is also one of the breakthrough points to calculate the vehicles delay at intersections [4, 5]. Based on the analysis of the conflict disciplines of the four-phase signalized intersection between right-turning vehicles and straight-going bicycles, Liang et al. established a theoretical model and a binary regression model of right-turning vehicles travel time. The former was based on gap theory and queue theory, and the latter was established on the basis of field observation data. In the binary regression model, it takes right-turning vehicles flow rate and bicycle through flow rate as independent variables. Meanwhile, their application conditions were studied, respectively [6]. Smith and Walsh established motor vehicles and nonmotor vehicles delay models under different mixed traffic conditions and traffic control methods [7]. Liu et al. took the basic link travel time and intersection delay as the basic unit of vehicle travel time in urban road network [8]. Ban et al. believed that vehicle travel time at signalized intersections contained discontinuities and nonsmoothness, and intersection delay pattern can be used to estimate real-time queue length at intersection [9]. Ahmed studied the travel time of indirect right-turning vehicles at intersection with a GPS device [10]. Lu et al. established the model of travel time at intersection with microscopic simulation data. The model considered the signal control effect at a variety of traffic volume combinations [11].

As mentioned above, there are more studies about vehicles delay at intersections, while few researches are about travel time of different turning movements at intersections [12–15]. Typically, Liang et al. established a right-turning motor vehicles' travel time model with the consideration of conflicts between motor vehicles and nonmotor vehicles at signalized intersections, and it provides a good reference for the following researches. The model was built on strict assumptions. For instance, nonmotor flow and right-turning motor flow were both subject to Poisson distributions. Non-motor flow only contained bicycles, the bicycle flow was only

in one line, and the service time followed a negative exponential distribution [6]. Poisson distribution is applicable to the random vehicle arriving, generally with flow rate less than 500 veh/h per lane. As urban road traffic is more and more congested while Poisson distribution is just applicable to free flow, it is hard to meet the actual traffic circumstances. As a consequence, Poisson distribution will be invalid when the flow rate is more than 500 veh/h per lane. Thus, it limits the application of the model to a large extent. Conversely, Erlang distribution, as a general distribution, has a high applicability to all traffic conditions.

Accordingly, this paper aims to establish the model of the travel time of right-turning vehicles on secondary-street at unsignalized intersections in congested urban traffic condition. This provides theoretical support for improving the traffic management, intersection control, designation, evaluation, and ITS application.

This paper is organized as follows. The first part gives a general introduction. The second part presents the model establishment methodology. Then, Section 3 takes an intersection as a numerical example to validate the models. In Section 4, the model sensitivity is analyzed. The final section then concludes the paper and gives suggestions for further study.

2. Model Establishment

2.1. Research Conditions Settings. Convenient for the study, the following assumptions were made in this paper.

(1) The following distribution models are usually used to describe traffic flow headway: negative exponential distribution, Weibull distribution, and Erlang distribution. The negative exponential distribution can be applied to the situation that vehicles arrive at random, generally in which the flow rate per hour per lane is less than 500 veh/h. Weibull distribution has a broader application range. Erlang distribution is also a general probability distribution of headway. Erlang distribution could be obtained by calculating the parameter l. Considering the applicability and conveniences of calibration, Erlang distribution is usually used to describe headway of major-street traffic flow. The negative exponential distribution is more suitable when the right-turning flow on the secondary street is not large.

(2) The vehicles on secondary street follow the right-in and right-out principle, and pedestrian is prohibited to cross, which is the management and control strategy of the unsignalized intersection.

(3) Drivers abide by the principle of the major-street priority strictly, and there is no grab-line phenomenon.

(4) In this research, the gap theory is adopted. The gap theory can be described as follows: at an unsignalized intersection, vehicles on the major street have the priority to cross, while the secondary-street vehicles have to wait for gaps long enough to cross. Therefore, the maximum flow rate of the secondary-street may be concluded by calculating the number of gaps provided by the major-street flow. The gap theory is used under the situation that secondary-street vehicles pass through the major-street flow perpendicularly at intersections. However, the actual right-turning movements at intersections do not cross the counter flow in perpendicularity. In this case, it is assumed that the right-turning movement crossed them perpendicularly.

(5) Queue theory: a system subjects to M/G/1, that is, vehicles arrival follows Poisson distribution, service time follows random distribution, and a single reception desk obeys the rule that the first comer should be served first under the condition that right-turning vehicles are waiting for services in the queue system. The service time is not definite, and there is only one right-turning lane.

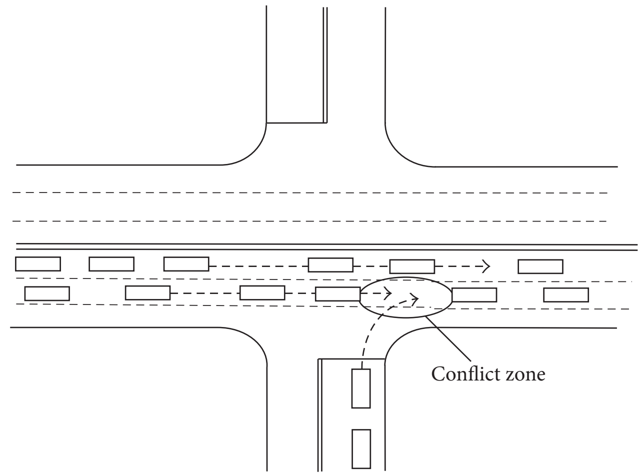


Figure 1: Service desk at unsignalized intersection.

2.2. Model Establishment. Based on the above research conditions settings, the conflict zone between right-turning vehicles on the secondary-street and opposing through vehicles on the major street can be seen as a single service desk as shown in Figure 1. The secondary-street right-turning vehicles receive service, and headway of opposing through flow provides service. As mentioned in the above assumptions, arrival of right-turning flow follows Poisson distribution, the service time follows Poisson distribution, and a single reception desk system obeys the rule of first comer to be served first, namely, the M/M/1 system.

In this research, the critical accepted headway of right-turning vehicles is τ_c, headway of the major-street through vehicles is h, and the following headway of right-turning vehicles is h_f. Considering that there is a vehicle queue on secondary-street, then we will have different conditions: when h is less than τ_c, no right-turning vehicles can merge into major street; when h is between τ_c and $\tau_c + h_f$, only one vehicle can cross; when h is between $\tau_c + h_f$ and $\tau_c + 2h_f$, two vehicles may merge into major street; when $\tau_c + (n-1)h_f \leq$

Table 1: Probability of headway.

Headway	Vehicles number/i	Probability/P_i
$\tau_c \le h < \tau_c + h_f$	1	$\sum_{i=0}^{l-1}(\lambda l\tau_c)^i \frac{e^{-\lambda\tau_c}}{i!} - \sum_{i=0}^{l-1}(\lambda l(\tau_c + h_f))^i \frac{e^{-\lambda(\tau_c+h_f)}}{i!}$
$\tau_c + h_f \le h < \tau_c + 2h_f$	2	$\sum_{i=0}^{l-1}(\lambda l(\tau_c + h_f))^i \frac{e^{-\lambda(\tau_c+h_f)}}{i!} - \sum_{i=0}^{l-1}(\lambda l(\tau_c + 2h_f))^i \frac{e^{-\lambda(\tau_c+2h_f)}}{i!}$
$\tau_c + 2h_f \le h < \tau_c + 3h_f$	3	$\sum_{i=0}^{l-1}(\lambda l(\tau_c + 2h_f))^i \frac{e^{-\lambda(\tau_c+2h_f)}}{i!} - \sum_{i=0}^{l-1}(\lambda l(\tau_c + 3h_f))^i \frac{e^{-\lambda(\tau_c+3h_f)}}{i!}$
$\vdots$	$\vdots$	$\vdots$
$\tau_c + (n-1)h_f \le h < \tau_c + nh_f$	n	$\sum_{i=0}^{l-1}(\lambda l(\tau_c + (n-1)h_f))^i \frac{e^{-\lambda(\tau_c+(n-1)h_f)}}{i!} - \sum_{i=0}^{l-1}(\lambda l(\tau_c + nh_f))^i \frac{e^{-\lambda(\tau_c+nh_f)}}{i!}$

$h < \tau_c + nh_f$, n vehicles may merge into major-street. The headway of major-street flow follows Erlang distribution, and it can be written as

$$P(h \ge t) = \sum_{i=0}^{l-1} (\lambda l t)^i \frac{e^{-\lambda t}}{i!}, \tag{1}$$

where λ is the arriving rate of major-street vehicles, veh/s; l is the distribution parameter, if $l = 1$, the formula above could be simplified as negative exponential distribution; while $l = \infty$, headway will follow uniform distribution. In practice, l could be determined by m and S^2 with rounding-off method. The formula may be expressed as

$$l = \text{int}\left(\frac{m^2}{S^2}\right), \tag{2}$$

where m is the observed average headway of major-street flow, s; S^2 is the variance of observed headway of major-street flow

$$P(h \ge t) = \sum_{i=0}^{l-1} (\lambda l t)^i \frac{e^{-\lambda t}}{i!}. \tag{3}$$

Consequently, the probability of different headway on major street can be expressed as

$$\begin{aligned} P_i &= P\left(\tau_c + (i-1)h_f \le h \le \tau_c + ih_f\right) \\ &= P\left(h \ge \tau_c + (i-1)h_f\right) - P\left(h \ge \tau_c + ih_f\right), \end{aligned} \tag{4}$$

where P_i is the probability of headway on the major street that allows i vehicles on the secondary street to merge into major-street flow. The probability of different headway is shown in Table 1.

The number of right-turning vehicles may merge into major-street flow per unit time and can be expressed as

$$\begin{aligned} N &= \sum_{i=1}^{n} i \cdot \lambda \cdot P_i \\ N &= \sum_{m=0}^{n-1}\sum_{i=0}^{l-1} \lambda\left(\lambda l\left(\tau_c + mh_f\right)\right)^i \frac{e^{-\lambda(\tau_c+mh_f)}}{i!} \\ &\quad - n\sum \lambda\left(\lambda l\left(\tau_c + nh_f\right)\right)^i \frac{e^{-\lambda(\tau_c+nh_f)}}{i!}, \end{aligned} \tag{5}$$

where N is the number of right-turning vehicles and may merge into major-street flow per unit time, veh/s; i is the number of right-turning vehicles and may merge into major-street flow in one headway; P_i is the probability of the headway and may provide i right-turning vehicles to merge into major-street flow; the other parameters have the same meanings as previously mentioned.

From the M/G/1 queue system, the average travel time of each vehicle in the queue is

$$T = \frac{\rho^2 + \lambda_1^2\delta^2}{2\lambda_1(1-\rho)} + \frac{1}{\mu}, \tag{6}$$

where λ_1 is the average arrival rate of right-turning flow, veh/s; μ is the service rate of right-turning flow, $\mu = N$, veh/s; δ^2 is the variance of vehicles service time; ρ is the service intensity or traffic intensity, $\rho = \lambda_1/\mu$, which reflects the traffic conditions. If $\rho < 1$, it means that the flow is stable, and each traffic condition will be repeated with a certain probability. If $\rho \ge 1$, the flow is unstable, and queue will become longer and longer.

When $\delta^2 = 0$, service time follows uniform distribution, which may be expressed as

$$T = \frac{\rho^2}{2\lambda_1(1-\rho)} + \frac{1}{\mu}. \tag{7}$$

When $\delta^2 = 1/\mu^2$, service time is subject to negative exponential distribution, which can be written as

$$T = \frac{\rho^2}{\lambda_1(1-\rho)} + \frac{1}{\mu}. \tag{8}$$

2.3. Existing Model. Most of the present researches assume that traffic flow arrival is subject to Poisson distribution, and the queue system is an M/M/1 one. The corresponding models are as follows.

The service rate of right-turning flow may be expressed as

$$\mu = \frac{\lambda e^{-\lambda\tau_c}}{1 - e^{-\lambda h_f}}. \tag{9}$$

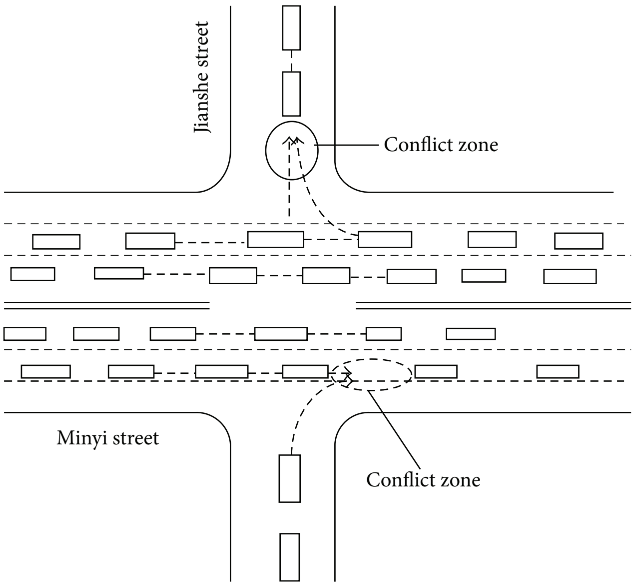


Figure 2: The sketch of study case.

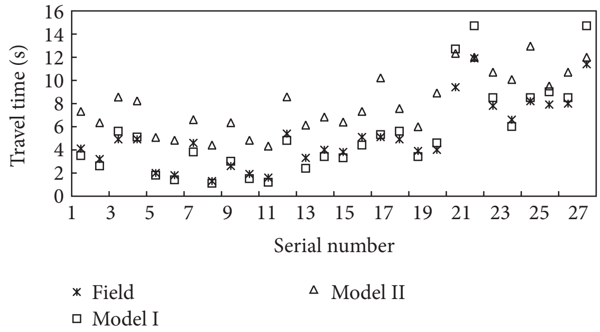


Figure 3: Model validation curve.

The travel time of right-turning vehicles merging into the main-street flow can be written as

$$T = \frac{1}{\mu - \lambda_1} = \frac{1}{\left(\lambda e^{-\lambda \tau_c} / \left(1 - e^{-\lambda h_f}\right)\right) - \lambda_1}. \tag{10}$$

The parameters have the same meanings as previously mentioned.

3. Models Validation

There are some differences between the application premise of gap theory and the research conditions in this paper. For example, the merging of right-turning vehicles into the major-street flow is not perpendicular, and the drivers do not strictly abide by the rule of major-street priority, especially when the right-turning flow rate is a bit large. Therefore, there still exists the phenomenon of grabbing line. It can be seen from the above that there exist some objective conditions that do not correspond with the practical situations. Thus, it is necessary to validate the models.

A real intersection of Harbin in China was selected to verify the accuracy of the above model based on Erlang distribution (Model-I) and the one based on Poisson distribution (Model-II). This unsignalized intersection lies in Nangang district of Harbin, and its geometric condition is shown in Figure 2. We conducted field observation in both rush and nonrush hours for one week. The following parameters were observed: flow rate and following headway of the major-street through vehicles, arrival rate and following headway of the secondary-street right-turning vehicles, and the time consumed to merge into the major-street flow and the critical headway accepted by right-turning vehicles. According to the field observation, critical headway τ_c evaluated by maximum likelihood estimation is 3.9 s, and h_f is 2.1 s. The above observation data and the results are applied to (8) and (10), and the results are shown in Table 2. The comparison curves of Model-I, Model-II, and the field observation data are shown in Figure 3.

It can be seen from the results that Model-I has better performance than Model-II, and most times the relative errors of Model-I are less than 15%, with the maximum one 35.11%. The higher accuracy is owing to the reasonable selected headway distribution. The main distinctions between Model-I and Model-II are the distribution models of the major-street traffic flow. Erlang distribution and Poisson distribution are used in Model-I and Model-II, respectively. It is well known that Poisson distributions are applicable to the random vehicle arriving, generally with flow rate less than 500 veh/h per lane. As a matter of fact, urban road traffic is more and more congested, while Poisson distribution is just applicable to free flow, and it is hard to meet the actual traffic circumstances. As a result, the Poisson distribution will be invalid when the flow rate is more than 500 veh/h per lane. Thus, it limits the application of Model-II. Conversely, Erlang distribution, as a general distribution, has a high applicability to all traffic conditions. Consequently, Model-I has better performance than Model-II.

The relative error of Model-I is relevant to the flow rate, indicating that the relative error shows an obvious rising trend with the increase of the flow rate. For instance, the most relative error is more than 15%, and the actual travel time is shorter than the theoretical value, while the flow rate of the major-street speeds up to 0.4 veh/s, and the flow rate of the right-turning vehicles reaches 0.04 veh/s. The reasons are as follows: with the increase of the right-turning vehicles flow rate, the number of headway long enough for right-turning vehicles to merge into major street decreases. When the flow rate of right-turning vehicles is increasing at the moment, there would be a formation of queues. Therefore, the right-turning vehicles may go after the front car to grab into the major street, and then the right-turning travel time will reduce, and the relative error will increase.

Table 2: Model validation data.

Serial number	τ_c (s)	h_f (s)	l	λ_1 (veh/s)	λ (veh/s)	Field (s)	Model-I		Model-II	
							Theoretical (s)	Relative error (%)	Theoretical (s)	Relative error (%)
1	3.9	2.1	2	0.03	0.36	4.1	3.5	14.63	7.3	78.44
2	3.9	2.1	2	0.03	0.32	3.2	2.6	18.75	6.3	98.11
3	3.9	2.1	3	0.04	0.38	4.9	5.6	14.29	8.5	74.39
4	3.9	2.1	3	0.04	0.37	4.9	5.1	4.08	8.2	67.57
5	3.9	2.1	2	0.02	0.27	2	1.8	10.00	5.1	152.95
6	3.9	2.1	2	0.03	0.24	1.8	1.4	22.22	4.8	167.40
7	3.9	2.1	3	0.02	0.35	4.6	3.8	17.39	6.6	43.28
8	3.9	2.1	2	0.04	0.2	1.3	1.1	15.38	4.4	238.38
9	3.9	2.1	3	0.03	0.32	2.6	3.0	15.38	6.3	143.82
10	3.9	2.1	3	0.03	0.24	1.9	1.5	21.05	4.8	153.33
11	3.9	2.1	3	0.02	0.22	1.6	1.2	25.00	4.3	169.25
12	3.9	2.1	3	0.05	0.36	5.4	4.8	11.11	8.6	58.70
13	3.9	2.1	2	0.03	0.31	3.3	2.4	27.27	6.1	85.46
14	3.9	2.1	2	0.02	0.36	4	3.4	15.00	6.8	70.43
15	3.9	2.1	2	0.01	0.36	3.8	3.3	13.16	6.4	67.95
16	3.9	2.1	3	0.03	0.36	5.1	4.4	13.73	7.3	43.45
17	3.9	2.1	2	0.05	0.40	5.1	5.3	3.92	10.2	100.27
18	3.9	2.1	3	0.02	0.39	4.9	5.6	14.29	7.6	54.14
19	3.9	2.1	3	0.01	0.34	3.9	3.4	12.82	6.0	53.62
20	3.9	2.1	2	0.04	0.39	4	4.6	15.00	8.9	122.42
21	3.9	2.1	3	0.05	0.44	9.4	12.7	35.11	12.3	31.06
22	3.9	2.1	3	0.04	0.46	11.9	14.7	23.53	12.0	0.64
23	3.9	2.1	3	0.05	0.41	7.8	8.5	8.97	10.7	37.06
24	3.9	2.1	2	0.04	0.42	6.6	6.0	9.09	10.1	52.59
25	3.9	2.1	2	0.05	0.45	8.2	8.5	3.66	12.9	57.81
26	3.9	2.1	3	0.03	0.43	7.9	9.0	13.92	9.5	20.35
27	3.9	2.1	3	0.05	0.41	8	8.5	6.25	10.7	33.63
28	3.9	2.1	3	0.04	0.46	11.4	14.7	28.95	12.0	5.05

4. Sensitivity Analyses

It is proved in the above section that the model established in this article has an overall better performance. The relationship between parameters and travel time was explored in this section. Specifically speaking, these variables include major-street through flow rate and secondary-street right-turning flow rate.

The influence of the variables is shown in Figure 4. To study the influence of major-street through flow rate and secondary-street right-turning flow rate, the travel times at different flow rates are shown in Figure 4. The trends in this figure indicate the following, (1) Right-turning vehicles travel times increase with the rising of the right-turning flow rate and the conflicting through flow rate. (2) When the conflicting through flow rate is low, the right-turning flow rate has a slight effect on right-turning vehicles travel times. With the increase of through flow rate, the right-turning flow rate would have a more obvious effect on travel time. Simultaneously, the threshold is about 0.30 veh/s. These findings indicate two facts: the larger the number of the opposing through vehicles is, the less acceptable the headway

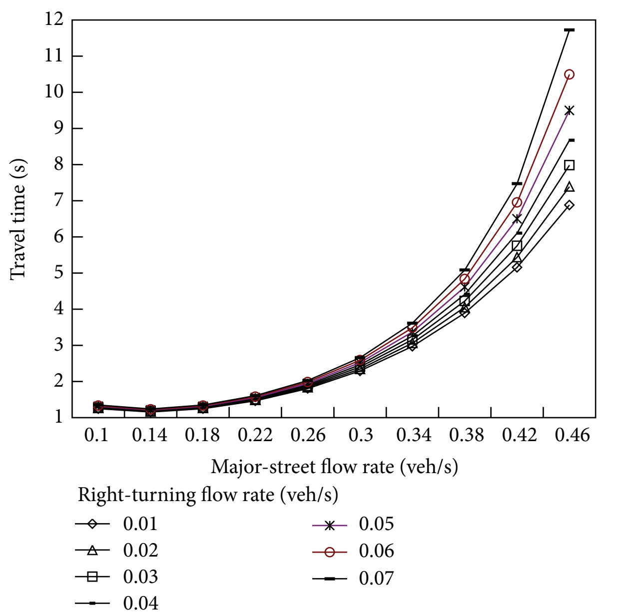


Figure 4: Influence of flow rate on right-turning vehicles travel time.

would be; the larger the right-turning flow rate is, the longer the queue and the longer the travel time would be.

5. Conclusions

(1) The travel time model based on Erlang distribution for major-street flow and M/G/1 queue system has a better applicability. The relative error has no obvious linear relationship with flow rate, and the error would still be stable, until the flow rate of the major street speeds up to 0.4 veh/s and the right-turning vehicles flow rate reaches 0.04 veh/s. Meanwhile, when the right-turning flow rate and major-street flow rate increase, the error also increases. This result shows the applicability of the model.

(2) The factors that have significant influence on the travel time are major-street flow rate and right-turning vehicles arrival rate. The travel time increases with the growth of them. In fact, there are other factors that have influence on the travel time, such as the intersection control pattern. Whether it is signalized or unsignalized, or it is interrupted by the pedestrian and the vehicles compositions or not, the intersection control pattern will always affect the travel time. Up to now, there is no such research on intersection control pattern, which we should spare no efforts to do within the coming future.

Acknowledgments

This research was supported by China Postdoctoral Science Foundation (2011M500676), National Education Ministry Humanities and Social Sciences Foundation (12YJCZH097), Heilongjiang Institute of Technology Doctoral Science Foundation (2011BJ05) and Technology Research and Development Program of Shandong (2012G0020129). The authors are also grateful to the anonymous referees for their helpful comments and constructive suggestions on an earlier version of the paper.

References

[1] S. M. Madanat, M. J. Cassidy, and M. H. Wang, "Probabilistic delay model at stop-controlled intersection," *Journal of Transportation Engineering*, vol. 120, no. 1, pp. 21–36, 1994.

[2] M. A. Qureshi and L. D. Han, "Delay model for right-turn lanes at signalized intersections with uniform arrivals and right turns on red," *Transportation Research Record*, vol. 1776, pp. 143–150, 2001.

[3] G. Abu-Lebdeh, R. F. Benekohal, and B. Al-Omari, "Models for right-turn-on-red and their effects on intersection delay," *Transportation Research Record*, vol. 1572, pp. 131–139, 1997.

[4] B. Hoeschen, D. Bullock, and M. Schlappi, "Estimating intersection control delay using large data sets of travel time from a global positioning system," *Transportation Research Record*, vol. 1917, pp. 18–27, 2005.

[5] T. Hagiwara, H. Hamaoka, T. Yaegashi, K. Miki, I. Ohshima, and M. Naito, "Estimation of time lag between right-turning vehicles and pedestrians approaching from the right side," *Transportation Research Record*, vol. 2069, pp. 65–76, 2008.

[6] C. Y. Liang, C. G. Wang, Z. Shen, and D. H. Wang, "Calculation method of travel time of right-turn vehicle at motor- and nonmotor-vehicle mixed traffic intersection," *Journal of Jilin University*, vol. 37, no. 5, pp. 1053–1057, 2007.

[7] R. L. Smith and T. Walsh, "Safety impacts of bicycle lanes," *Transportation Research Record*, vol. 1168, pp. 49–56, 1988.

[8] H. Liu, H. J. Van Zuylen, H. Van Lint, Y. Chen, and K. Zhang, "Prediction of urban travel times with intersection delays," in *Proceedings of the 8th International IEEE Conference on Intelligent Transportation Systems (ITSC '05)*, pp. 1062–1067, September 2005.

[9] X. G. Ban, P. Hao, and Z. B. Sun, "Real time queue length estimation for signalized intersections using travel times from mobile sensors," *Transportation Research C*, vol. 19, pp. 1133–1156, 2011.

[10] K. Ahmed, "Evaluation of low cost technique "indirect right turn" to reduce congestion at urbanized signalized intersection in developing countries," in *Proceedings of the 6th International Symposium on Highway Capacity and Quality of Service (ISHC '11)*, vol. 16, pp. 568–577, July 2011.

[11] C. Lu, F. Zhao, and M. Hadi, "A travel time estimation method for planning models considering signalized intersections," in *Proceedings of the 10th International Conference of Chinese Transportation Professionals—Integrated Transportation Systems: Green, Intelligent, Reliable (ICCTP '10)*, pp. 1993–2000, August 2010.

[12] J. A. Bonneson, "Delay to major-street through vehicles due to right-turn activity," *Transportation Research A*, vol. 32, no. 2, pp. 139–148, 1998.

[13] Y. Wang and N. L. Nihan, "Estimating the risk of collisions between bicycles and motor vehicles at signalized intersections," *Accident Analysis and Prevention*, vol. 36, no. 3, pp. 313–321, 2004.

[14] C. Quiroga, M. Perez, and S. Venglar, "Tool for measuring travel time and delay on arterial corridors," in *Proceedings of the 7th International Conference on: Applications of Advanced Technology in Transportation*, pp. 600–607, August 2002.

[15] J. Q. Leng, Y. Q. Feng, J. Zhai, L. Bao, and Y. He, "Travel time model of left-turning vehicles at signalized intersection," *Mathematical Problem in Engineering*, vol. 2010, Article ID 473847, 10 pages, 2012.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 460348, 9 pages
http://dx.doi.org/10.1155/2013/460348

Research Article

Robust Passivity and Feedback Design for Nonlinear Stochastic Systems with Structural Uncertainty

Zhongwei Lin, Jizhen Liu, and Yuguang Niu

State Key Laboratory of Alternate Electrical Power System with Renewable Energy Sources, School of Control and Computer Engineering, North China Electric Power University, Beijing 102206, China

Correspondence should be addressed to Zhongwei Lin; linzhongwei2003@tom.com

Received 30 January 2013; Accepted 18 February 2013

Academic Editor: Weihai Zhang

This paper discusses the robust passivity and global stabilization problems for a class of uncertain nonlinear stochastic systems with structural uncertainties. A robust version of stochastic Kalman-Yakubovitch-Popov (KYP) lemma is established, which sustains the robust passivity of the system. Moreover, a robust strongly minimum phase system is defined, based on which the uncertain nonlinear stochastic system can be feedback equivalent to a robust passive system. Following with the robust passivity theory, a global stabilizing control is designed, which guarantees that the closed-loop system is globally asymptotically stable in probability (GASP). A numerical example is presented to illustrate the effectiveness of our results.

1. Introduction

It is well known that passivity theory plays an important role in many engineering problems, which is a powerful technique in handling stability issue. Many problems on related topics have been investigated; see [1–4]. In 1990s, much more attention has been focused on the development of synthesis technique that combines the passivity theory with geometric nonlinear control theory; see [5–7] and the references therein. The work in [5] developed a framework for studying the stabilizability of minimum phase deterministic nonlinear systems. Especially, [8] has proposed a robust enhancement of the result in [5], which discussed the passivity and global stabilization for a class of uncertain minimum phase nonlinear systems, which considers the nonlinear systems with structural uncertainties, that is, gain-bounded uncertainties. The work in [9] extended the corresponding approach to solve the robust almost disturbance decoupling problem for such a class uncertain nonlinear systems.

On the other hand, due to the great many applications of stochastic Itô systems in real world [10], the study on feedback controller design for such a class of systems has received a great deal of attention; see [11–19] and the references therein. Especially, many researchers have extended the existing passivity theory from deterministic systems to stochastic systems. The work in [11] obtained necessary and sufficient conditions for the existence of diffeomorphisms that transform stochastic nonlinear systems to various canonical forms. The work in [12] studied the problem of feedback equivalent to a passive system for a particular class of interconnected stochastic systems. The work in [13] developed a systematic method for global asymptotic stabilization in probability of nonlinear control stochastic differential systems based on the passivity theory. However, compared with the deterministic case, up to date, there still requires much work of investigating the passivity theory of uncertain nonlinear stochastic systems with structural uncertainties.

This paper considers a class of uncertain nonlinear stochastic systems which are expressed by the Itô-type stochastic differential equations with structural uncertainty. We shall investigate the problem of feedback equivalent to a robust passive system and global stabilization for uncertain nonlinear stochastic system via robust passivity theory. A robust stochastic KYP lemma is proposed, which can be regarded as a robust stochastic extended results in deterministic case [2, 5]. Based on the above, we investigate the relationship between a robust passive system and the corresponding zero-output dynamics, where a robust strongly minimum phase property is proposed. Then, sufficient conditions for global asymptotic stabilization in probability are provided.

The remainder of the paper is organized as follows: Section 2 develops the robust passivity theory for a class of uncertain nonlinear stochastic systems, which presents a robust stochastic version of KYP lemma. In Section 3, through adopting the appropriate diffeomorphism, we discuss the relationship between the robust passivity of the system and the stability of the zero-output system, that is, the robust minimum phase system. Then, the global stabilizing control of the system can be determined through discussing the robust strongly minimum phase property. Based on the robust stochastic passivity theory, sufficient conditions are given for global stabilization of such a class of uncertain nonlinear stochastic systems. In Section 4, an example is given to illustrate the usefulness of our results. Section 5 concludes this paper.

For convenience, we adopt the following notations:

$\mathcal{S}_n$: the set of all real $n \times n$ symmetric matrices;

A': the transpose of a matrix or vector A;

$A \geq 0$ ($A > 0$): A is a positive semidefinite (positive definite) matrix;

$\mathscr{C}^2(U)$: the class of functions $V(x)$ twice continuously differentiable with respect to $x \in U$;

$L_g\lambda$: the Lie derivative of a smooth function λ along the vector field g, that is, $L_g\lambda := (\partial\lambda/\partial x)g$;

R^n: n-dimensional Euclidean space;

$\|x\|$: 2-norm of a vector $x \in R^n$.

2. Robust Stochastic Passivity Theory

First of all, let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, P)$ be a given filtered probability space where there lives a standard one-dimensional Brownian motion $w(t)$ on $[0, +\infty)$ with $w(0) = 0$ and $\mathscr{F}_t = \sigma\{w(s) \mid 0 \leq s \leq t\}$. The Brownian motion is assumed to be one dimensional only for simplicity, because there is no essential difference from the multidimensional case.

Consider the following uncertain nonlinear stochastic control system governed by Itô's differential equation:

$$\begin{aligned} dx = & \left(f(x) + \Delta f(x) + g(x)u\right)dt \\ & + l(x)\,dw, \quad f(0) = 0,\ l(0) = 0, \\ & y = h(x), \quad h(0) = 0, \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $x_0 \in R^n$ is the initial state, and $y(t) \in R$ is the controlled output. $w(t)$ is an one-dimensional Brownian motion. $u(t) \in R$ is the control input, which is an adaptive process with respect to $\{\mathscr{F}_t\}_{t\geq 0}$. f, g, l, and h are assumed to be smooth functions of appropriate dimensions. $\Delta f(x) : R^n \to R^n$ represents structural uncertainty or uncertain perturbation characterized by

$$\Delta f(x) = e(x)\,\delta(x), \quad \Delta f(0) = 0, \tag{2}$$

where $e(x) : R^n \to R^{n\times m}$, with $e(0) = 0$ being a known matrix whose entries are given smooth functions, and $\delta(x) : R^n \to R^m$ is an unknown vector-valued function. It is assumed that $\delta(x)$ is constrained to a given smooth function $n(x) : R^n \to R^m$, that is,

$$\Gamma = \{\delta(x) : \|\delta(x)\| \leq \|n(x)\|\}. \tag{3}$$

If $\delta(x) \in \Gamma$, then $\Delta f(x)$ is said to be admissible.

Definition 1. An uncertain system of the form (1) is said to be robust passive on $[s, \infty)$, $s \geq 0$ if there exists a nonnegative continuous function $V(x) : R^n \to R^+$, called the storage function, such that for all $t \geq s \geq 0$, $x(s) \in R^n$, and for all admissible $\Delta f(x)$,

$$EV(x(t)) - V(x(s)) \leq E\int_s^t y'(\tau)\,u(\tau)\,d\tau. \tag{4}$$

Here, (4) can be regarded as the robust version of the passive inequality for stochastic systems. For simplicity of our following discussion, we assume that the storage function of (4), if it exists, belongs to $\mathscr{C}^2(R^n)$.

In view of Definition 1 and [8], it is quite natural to introduce the following concept for uncertain nonlinear stochastic systems.

Definition 2. System (1) is said to have the robust KYP property if there exists a nonnegative function $V \in \mathscr{C}^2(R^n) : R^n \to R^+$, with $V(0) = 0$, such that

$$\begin{gathered} L_{f+\Delta f}V(x) + \frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l \leq 0, \\ L_g V(x) = h'. \end{gathered} \tag{5}$$

If the above inequality becomes a strict inequality that holds for a positive definite function $V(x)$, then system (1) is said to be robust strictly passive.

Now, we derive conditions under which an uncertain nonlinear stochastic system is robust passive, which can be viewed as a robust stochastic version of the nonlinear KYP lemma, which plays an important role in studying global robust stabilization for uncertain nonlinear stochastic systems.

Theorem 3. *System* (1) *is robust passive if and only if there exists a nonnegative function $V \in \mathscr{C}^2(R^n) : R^n \to R^+$, with $V(0) = 0$, such that*

$$\begin{gathered} L_f V(x) + \left\|L'_e V(x)\right\|\|n(x)\| + \frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l \leq 0, \\ L_g V(x) = h', \end{gathered} \tag{6}$$

where $L'_eV(x) = (L_eV(x))'$.

Proof. Sufficiency. According to the Cauchy inequality and considering the fact of $\|\delta(x)\| \leq \|n(x)\|$, $\forall\delta(x) \in \Gamma$, we have

$$\begin{aligned} L_e V(x)\,\delta(x) & \leq \left\|L'_e V(x)\right\|\|\delta(x)\| \\ & \leq \left\|L'_e V(x)\right\|\|n(x)\|, \quad \forall\delta(x) \in \Gamma, \end{aligned} \tag{7}$$

which reduces to

$$L_{f+\Delta f}V(x)+\frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l=L_fV(x)+L_eV(x)\delta(x)+\frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l\le 0. \tag{8}$$

The robust KYP property of system (1) is guaranteed through Definition 2. Then, by Itô's formula, we have

$$\mathscr{L}V(x)=L_fV(x)+L_{\Delta f}V(x)+L_gV(x)u+\frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l\le L_gV(x)u, \tag{9}$$

where $\mathscr{L}$ is the infinitesimal generator of system (1). From the second equation in (6), it follows that

$$\mathscr{L}V(x)\le y'u. \tag{10}$$

Integrating the above inequality from s to t for any $t\ge s\ge 0$, $x(s)\in R^n$, and for all admissible $\Delta f(x)$,

$$EV(x(t))-V(x(s))\le E\int_s^t y'(\tau)u(\tau)d\tau, \tag{11}$$

which implies that system (1) is robust passive by Definition 1.

Necessity. Assume that system (1) is robust passive which sustains a storage function $V(x)$; that is, (11) always holds for $\forall\delta(x)\in\Gamma$. By (11) with any $x(s)\in R^n$, we have

$$\frac{EV(x(t))-V(x(s))}{t-s}-\frac{E\int_s^t y'(\tau)u(\tau)d\tau}{t-s}\le 0,\quad \forall t>s. \tag{12}$$

By Itô's formula,

$$EV(x(t))-V(x(s))=E\int_s^t\left(L_fV(x)+L_eV(x)\delta(x)+L_gV(x)u+\frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l\right)d\tau. \tag{13}$$

Let $t\downarrow s$, it follows that

$$L_fV(x)+L_eV(x)\delta(x)+\frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l+\left(L_g'V(x)-y\right)'u\le 0 \tag{14}$$

always holds for any control input u, which implies that (5) holds; that is, the robust passive system (1) has the robust KYP property.

Denote

$$\alpha(x)=L_fV(x)+\frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l,\qquad \beta'(x)=L_eV(x). \tag{15}$$

So,

$$L_{f+\Delta f}V(x)+\frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l=\alpha(x)+\beta'(x)\delta(x)\le 0,\quad \forall\delta(x)\in\Gamma. \tag{16}$$

The rest is similar to the proof of Lemma 2 in [8], and we only note that through defining

$$\delta_0=\frac{\|n(x)\|}{\|\beta(x)\|}\beta(x),\quad \beta(x)\ne 0, \tag{17}$$

it can be found that $\delta_0\in\Gamma$, which also deduces that

$$\left|\beta'(x)\delta_0(x)\right|=\beta'(x)\beta(x)\frac{\|n(x)\|}{\|\beta(x)\|}=\|\beta(x)\|\,\|n(x)\|. \tag{18}$$

Obviously, the above equality also holds when $\beta(x)=0$. Considering the fact that

$$\sup_{\delta\in\Gamma}\left|\beta'(x)\delta(x)\right|\le\|\beta(x)\|\,\|n(x)\|, \tag{19}$$

we get to the result that

$$\sup_{\delta\in\Gamma}\left|\beta'(x)\delta(x)\right|=\|\beta(x)\|\,\|n(x)\|. \tag{20}$$

Hence, from (16) and (20), we have

$$\alpha(x)\le-\sup_{\delta\in\Gamma}\left|\beta'(x)\delta(x)\right|=-\|\beta(x)\|\,\|n(x)\|, \tag{21}$$

which also implies that

$$\alpha(x)+\|\beta(x)\|\,\|n(x)\|=L_fV(x)+\left\|L_e'V(x)\right\|\|n\|+\frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l\le 0. \tag{22}$$

The proof of Theorem 3 is completed. □

Remark 4. Indeed, Theorem 3 has established the equivalent relations between (4), (5), and (6). Obviously, the condition (6) is more convenient than (4) or (5), which has taken the bounding function $n(x)$ instead of the unknown function $\delta(x)$.

Remark 5. From the proof of Theorem 3, we can also discuss the robust strict passivity for system (1). There is no difference except that the inequality in (6) becomes a strict inequality which holds for a positive definite function $V(x)$. Moreover, it is easy to deduce from Theorem 3 that a sufficient condition for the robust passivity of system (1) is that there exists a real-valued function $\lambda(x)>0$, such that

$$L_fV(x)+\frac{\lambda}{2}\|L_eV(x)\|^2+\frac{1}{2\lambda}\|n(x)\|^2+\frac{1}{2}l'\frac{\partial^2 V(x)}{\partial x^2}l\le 0,\qquad L_gV(x)=h'. \tag{23}$$

Indeed, if (23) is also a strict inequality, it is equivalent to the strict inequalities (5) or (6). The proof can follow the line of Theorem 3 and [8] and is omitted.

In what follows, we recall some facts in the theory of stochastic stability, where only global stability is considered. Obviously, local stability results may also be achieved in a similar way.

Consider the uncertain stochastic unforced system with

$$\begin{gathered} dx = \left(f(x) + \Delta f(x)\right) dt + l(x)\, dw, \\ x(0) = x_0 \in R^n, \quad f(0) = 0, \\ \Delta f(0) = 0, \quad l(0) = 0. \end{gathered} \tag{24}$$

Definition 6 (see [20]). System (24) with the structural uncertainty is said to be stable in probability if for any $\varepsilon > 0$,

$$\lim_{x \to 0} P\left(\sup_{t \ge 0} \|x(t)\| > \varepsilon\right) = 0. \tag{25}$$

Additionally, if we also have $P(\lim_{t\to\infty} x(t) = 0) = 1$ for $\forall x_0 \in R^n$, system (24) is said to be GASP.

Definition 7. System (24) with the structural uncertainty is called zero-state detectable if for all $x(0) = x_0 \in R^n$,

$$\begin{aligned} & h(x(t)) = 0, \\ & \text{a.s. } \forall t \ge 0 \Longrightarrow P\left(\lim_{t\to\infty} x(t) = 0, x(0) = x_0\right) = 1. \end{aligned} \tag{26}$$

System (24) is globally zero-state observable if for all $x_0 \in R^n$, $h(x(t)) \equiv 0$ implies $x_0 \equiv 0$.

3. Feedback Equivalence and Global Stabilization

In this section, we discuss the feedback equivalence and global stabilization problems for the general nonlinear stochastic system (1) based on the above robust passivity theory. Similarly as the deterministic system case, we first present the following relative degree definition for uncertain nonlinear stochastic systems.

Definition 8 (relative degree). The nonlinear stochastic system (1) is said to have the relative degree ρ if

$$\begin{gathered} L_g L_f^{j-1} h(x) = 0, \\ L_g L_f^{\rho-1} h(x) \ne 0, \end{gathered} \tag{27}$$

where $j = 1, 2, \ldots, \rho - 1$.

In what follows, we only consider the simple case of $\rho = 1$. For system (1), by Itô's formula, we know that

$$\begin{aligned} dy &= \frac{\partial h'}{\partial x}(dx) + \frac{1}{2}(dx)' \frac{\partial^2 h}{\partial x^2}(dx) \\ &= \left(L_f h + L_{\Delta f} h + L_g h u + \frac{1}{2} l' \frac{\partial^2 h}{\partial x^2} l\right) dt + L_l h\, dw. \end{aligned} \tag{28}$$

From the above assumption, we have $L_g h(x) \ne 0$. If we take u as

$$u = \left(L_g h\right)^{-1} \left(v - L_f h - \frac{1}{2} l' \frac{\partial^2 h}{\partial x^2} l\right), \tag{29}$$

where v can be regarded as a new control input instead of u. Then, we obtain that

$$dy = v\, dt + L_{\Delta f} h\, dt + L_l h\, dw. \tag{30}$$

Motivated by the works of [5, 8], we assume that the vector fields $\tilde{g}_1(x), \tilde{g}_2(x), \ldots, \tilde{g}_p(x)$ are complete, where $[\tilde{g}_1(x), \ldots, \tilde{g}_p(x)] = g(x)[L_g h(x)]^{-1}$ and the distribution spanned by $\tilde{g}_1(x), \tilde{g}_2(x), \ldots, \tilde{g}_p(x)$ is involutive. Under these hypotheses, it is possible to find a global diffeomorphism $\phi(x)$, such that

$$\xi = \phi(x) = \begin{bmatrix} \theta(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} \eta \\ y \end{bmatrix}, \tag{31}$$

where $\eta = \theta(x) \in R^{n-1}$. Then, system (1) can be transformed into an interconnected system of the form

$$\begin{gathered} d\eta = \left(L_f \theta + L_{\Delta f} \theta + \frac{1}{2} l' \frac{\partial^2 \theta}{\partial x^2} l\right) dt + L_l \theta\, dw, \\ dy = v\, dt + L_{\Delta f} h\, dt + L_l h\, dw. \end{gathered} \tag{32}$$

For simplicity, we adopt that $\tilde{h}(\xi) = h(\phi^{-1}(\xi))$. Then, by taking the transformation $x = \phi^{-1}(\xi)$, system (1), in the new coordinate, is formulated as

$$\begin{gathered} d\eta = \tilde{f}(\xi)\, dt + \tilde{e}(\xi) \tilde{\delta}(\xi)\, dt + \tilde{l}(\xi)\, dw, \\ dy = v dt + \hat{e}(\xi) \tilde{\delta} dt + \hat{l}(\xi)\, dw, \end{gathered} \tag{33}$$

where

$$\begin{gathered} \tilde{f}(\xi) = L_f \theta + \frac{1}{2} l' \frac{\partial^2 \theta}{\partial x^2} l \Big|_{x=\phi^{-1}(\xi)}, \qquad \tilde{e}(\xi) = L_e \theta|_{x=\phi^{-1}(\xi)}, \\ \tilde{\delta}(\xi) = \delta\left(\phi^{-1}(\xi)\right), \qquad \tilde{l}(\xi) = L_l \theta|_{x=\phi^{-1}(\xi)}, \\ \hat{e}_i(\xi) = L_e h|_{x=\phi^{-1}(\xi)}, \qquad \hat{l}_i(\xi) = L_l h|_{x=\phi^{-1}(\xi)}. \end{gathered} \tag{34}$$

Here, we define

$$\begin{aligned}
\tilde{f}_1(\xi) &= \int_0^1 \frac{\partial \tilde{f}(\eta,\zeta)}{\partial \zeta}\bigg|_{\zeta=\tau y} d\tau, \\
\tilde{e}_1(\xi) &= \int_0^1 \frac{\partial \tilde{e}(\eta,\zeta)}{\partial \zeta}\bigg|_{\zeta=\tau y} d\tau, \\
\tilde{l}_1(\xi) &= \int_0^1 \frac{\partial \tilde{l}(\eta,\zeta)}{\partial \zeta}\bigg|_{\zeta=\tau y} d\tau, \\
\hat{e}_1(\xi) &= \int_0^1 \frac{\partial \hat{e}(\eta,\zeta)}{\partial \zeta}\bigg|_{\zeta=\tau y} d\tau, \\
\hat{l}_1(\xi) &= \int_0^1 \frac{\partial \hat{l}(\eta,\zeta)}{\partial \zeta}\bigg|_{\zeta=\tau y} d\tau.
\end{aligned} \tag{35}$$

Then, the following decomposition holds

$$\tilde{f}(\xi) = \tilde{f}_*(\eta) + \tilde{f}_1(\xi)\, y, \quad \text{where } \tilde{f}_*(\eta) = \tilde{f}(\eta, 0). \tag{36}$$

Similarly, we can also find $\tilde{l}$, $\hat{l}$, $\tilde{e}$, and $\tilde{n}$ with the following structures, respectively

$$\begin{aligned}
\tilde{l}(\xi) &= \tilde{l}_*(\eta) + \tilde{l}_1(\xi)\, y, \qquad \tilde{e}(\xi) = \tilde{e}_*(\eta) + \tilde{e}_1(\xi)\, y, \\
\hat{l}(\xi) &= \hat{l}_*(\eta) + \hat{l}_1(\xi)\, y, \qquad \tilde{n}(\xi) = n(\eta, 0) + \tilde{n}_1(\xi)\, y.
\end{aligned} \tag{37}$$

Moreover, we set the control input as

$$v = c(\xi) + \omega, \tag{38}$$

where c is a smooth function defined on R^n with $c(0) = 0$, and ω is the new control input. Equivalently, system (33) can be expressed with the form of

$$\begin{aligned}
d\xi &= F(\xi)\, dt + E(\xi)\, \tilde{\delta}\, dt + G(\xi)\, \omega\, dt + L(\xi)\, dw, \\
y &= \tilde{h}(\xi),
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
F(\xi) &= \begin{bmatrix} \tilde{f}(\xi) \\ c(\xi) \end{bmatrix}, \qquad E(\xi) = \begin{bmatrix} \tilde{e}(\xi) \\ \hat{e}(\xi) \end{bmatrix}, \\
G(\xi) &= \begin{bmatrix} 0 \\ I \end{bmatrix}, \qquad L(\xi) = \begin{bmatrix} \tilde{l}(\xi) \\ \hat{l}(\xi) \end{bmatrix}.
\end{aligned} \tag{40}$$

Let zero-output system describe the internal dynamic of a system which is consistent with the constraint $y \equiv 0$. Obviously, the zero-output dynamic of system (1) comes down to

$$d\eta = \tilde{f}_*(\eta)\, dt + \tilde{e}_*(\eta)\, \tilde{\delta}(\eta, 0) + \tilde{l}_*(\eta)\, dw. \tag{41}$$

As follows, we define the minimum phase system for nonlinear stochastic system (1).

Definition 9. System (1) is said to be

(i) Robust weakly minimum phase if the zero-output system (41) is stable in probability; that is, there exists a nonnegative function $W(\eta) \in \mathscr{C}^2 : R^{n-1} \rightarrow R^+$ with $W(0) = 0$, satisfying the following Lyapunov inequality:

$$L_{\tilde{f}_*} W(\eta) + \left\| L'_{\tilde{e}_*} W(\eta) \right\| \left\| n(\eta, 0) \right\| + \frac{1}{2} \tilde{l}'_* \frac{\partial^2 W(\eta)}{\partial \eta^2} \tilde{l}_* \le 0. \tag{42}$$

(ii) Robust minimum phase if for the zero-output system (41) is GASP; that is, there exists a positive definite function $W(\eta) \in \mathscr{C}^2 : R^{n-1} \rightarrow R^+$ with $W(0) = 0$, satisfying the following Lyapunov inequality:

$$L_{\tilde{f}_*} W(\eta) + \left\| L'_{\tilde{e}_*} W(\eta) \right\| \left\| n(\eta, 0) \right\| + \frac{1}{2} \tilde{l}'_* \frac{\partial^2 W(\eta)}{\partial \eta^2} \tilde{l}_* < 0. \tag{43}$$

(iii) Robust strongly minimum phase, if for the zero-output system (41) there exists positive definite functions $W(\eta) \in \mathscr{C}^2 : R^{n-1} \rightarrow R^+$ with $W(0) = 0$, and $\lambda(\xi) : R^n \rightarrow R^+$, satisfying the following Lyapunov-like inequality constraint:

$$\begin{aligned}
&L_{\tilde{f}_*} W(\eta) + \frac{\lambda}{2} \left\| L_{\tilde{e}_*} W(\eta) \right\|^2 + \frac{1}{2\lambda} \left\| n(\eta, 0) \right\|^2 \\
&\quad + \frac{1}{2} \tilde{l}'_* \frac{\partial^2 W(\eta)}{\partial \eta^2} \tilde{l}_* \le -\frac{1}{2} \hat{l}'_* \hat{l}_*.
\end{aligned} \tag{44}$$

The following theorem discusses the relationship between the passivity of system (39) and the stability of the zero-output system (41).

Theorem 10. *If system* (1) *is robust strongly minimum phase, then the system is feedback equivalent to a robust passive system* (39). *Conversely, the robust passivity of system* (39) *with a positive storage function implies that system* (1) *is robust weakly minimum phase.*

Proof. We construct the storage function $V(\xi) = W(\eta) + (1/2) y' y$ for system (39), where $W(\eta)$ satisfies the inequality (44). Obviously, we have

$$\begin{aligned}
&L_F V(\xi) + \left\| L'_E V(\xi) \right\| \left\| n(\xi) \right\| + \frac{1}{2} L' \frac{\partial^2 V(\xi)}{\partial \xi^2} L \\
&\le L_F V(\xi) + \frac{\lambda}{2} \left\| L_E V(\xi) \right\|^2 + \frac{1}{2\lambda} \left\| n(\xi) \right\|^2 \\
&\quad + \frac{1}{2} L' \frac{\partial^2 V(\xi)}{\partial \xi^2} L
\end{aligned}$$

$$\begin{aligned}
&= L_{\tilde f}W(\eta) + y'c + \frac{\lambda}{2}\left\|L_{\tilde e}W(\eta) + y'\hat e\right\|^2 \\
&\quad + \frac{1}{2\lambda}\|n(\xi)\|^2 + \frac{1}{2}\tilde l'\frac{\partial^2 W(\eta)}{\partial \eta^2}\tilde l + \frac{1}{2}\hat l'\hat l \\
&= L_{\tilde f_* + \tilde f_1 y}W(\eta) + y'c \\
&\quad + \frac{\lambda}{2}\left\|L_{\tilde e_*}W(\eta) + \tilde e_1 y + y'\hat e\right\|^2 \\
&\quad + \frac{1}{2\lambda}\|n(\eta,0) + n_1(\xi)y\|^2 \\
&\quad + \frac{1}{2}\left(\tilde l_* + \tilde l_1 y\right)'\frac{\partial^2 W(\eta)}{\partial \eta^2}\left(\tilde l_* + \tilde l_1 y\right) \\
&\quad + \frac{1}{2}\left(\hat l_* + \hat l_1 y\right)'\left(\hat l_* + \hat l_1 y\right) \\
&= L_{\tilde f_*}W(\eta) + L_{\tilde f_1}W(\eta)y + y'c \\
&\quad + \left\|L_{\tilde e_*}W(\eta)\right\|^2 + \frac{\lambda}{2}y'c_1 + \frac{1}{2\lambda}\|n(\eta,0)\|^2 \\
&\quad + \frac{1}{2\lambda}y'c_2 + \frac{1}{2}\tilde l_*'\frac{\partial^2 W(\eta)}{\partial \eta^2}\tilde l_* \\
&\quad + y'\left(\tilde l_1'\frac{\partial^2 W(\eta)}{\partial \eta^2}\tilde l_* + \frac{1}{2}\tilde l_1'\frac{\partial^2 W(\eta)}{\partial \eta^2}\tilde l_1 y\right) \\
&\quad + \frac{1}{2}\hat l_*'\hat l_* + y'\left(\hat l_1'\hat l_* + \frac{1}{2}\hat l_1'\hat l_1 y\right) \\
&= L_{\tilde f_*}W(\eta) + \left\|L_{\tilde e_*}W(\eta)\right\|^2 + \frac{1}{2\lambda}\|n(\eta,0)\|^2 \\
&\quad + \frac{1}{2}\tilde l_*'\frac{\partial^2 W(\eta)}{\partial \eta^2}\tilde l_* + \frac{1}{2}\hat l_*'\hat l_* \\
&\quad + y'\left(L_{\tilde f_1}'W(\eta) + c + \frac{\lambda}{2}c_1 + \frac{1}{2\lambda}c_2\right. \\
&\qquad + \tilde l_1'\frac{\partial^2 W(\eta)}{\partial \eta^2}\tilde l_* + \frac{1}{2}\tilde l_1'\frac{\partial^2 W(\eta)}{\partial \eta^2}\tilde l_1 y \\
&\qquad \left. + \hat l_1'\hat l_* + \frac{1}{2}\hat l_1'\hat l_1\right),
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
c_1 &= 2(L_{\tilde e_1}W + \hat e)'L_{\tilde e_*}'W + \left(L_{\tilde e_1}W + \hat e\right)(L_{\tilde e_1}W + \hat e)'y, \\
c_2 &= (n_1)'\left(2n(\eta,0) + n_1 y\right).
\end{aligned} \tag{46}$$

For (45), taking

$$\begin{aligned}
c = &-L_{\tilde f_1}'W - \frac{\lambda}{2}c_1 - \frac{1}{2\lambda}c_2 - \tilde l_1'\frac{\partial^2 W}{\partial \eta^2}\tilde l_* \\
&- \frac{1}{2}\tilde l_1'\frac{\partial^2 W}{\partial \eta^2}\tilde l_1 y - \hat l_1'\hat l_* - \frac{1}{2}\hat l_1'\hat l_1,
\end{aligned} \tag{47}$$

and considering the robust strongly minimum phase property, we have

$$L_F V(\xi) + \left\|L_E' V(\xi)\right\|\|n(\xi)\| + \frac{1}{2}L'\frac{\partial^2 V(\xi)}{\partial \xi^2}L \le 0. \tag{48}$$

Besides, it is obvious that

$$L_G V(\xi) = \left[\frac{\partial W'}{\partial \eta}, y'\right]\begin{bmatrix}0\\ I\end{bmatrix} = \tilde h'. \tag{49}$$

Then, by Theorem 3, we know that system (39) is robust passive.

On the other hand, suppose that there is a feedback control law v_* that renders the closed-loop system (39) robust passive with a storage function $V(\xi)$, which is positive definite and satisfies

$$\begin{gathered}
L_F V(\xi) + \left\|L_E' V(\xi)\right\|\|n(\xi)\| + \frac{1}{2}L'\frac{\partial^2 V(\xi)}{\partial \xi^2}L \le 0, \\
L_G V(\xi) = \tilde h'.
\end{gathered} \tag{50}$$

Note that

$$\begin{aligned}
\frac{\partial V'(\xi)}{\partial \xi} &= \left[\frac{\partial V'(\xi)}{\partial \eta}, \frac{\partial V'(\xi)}{\partial y}\right], \\
\frac{\partial^2 V(\xi)}{\partial \xi^2} &= \begin{bmatrix}\dfrac{\partial^2 V(\xi)}{\partial \eta^2} & \dfrac{\partial^2 V(\xi)}{\partial \eta y}\\[2mm] \dfrac{\partial^2 V(\xi)}{\partial y\eta} & \dfrac{\partial^2 V(\xi)}{\partial y^2}\end{bmatrix}.
\end{aligned} \tag{51}$$

Then, (50) deduces that

$$\begin{aligned}
&\frac{\partial V'(\xi)}{\partial \eta}\tilde f(\xi) + \frac{\partial V'(\xi)}{\partial y}c(\xi) \\
&\quad + \left\|\left(\frac{\partial V'(\xi)}{\partial \eta}\tilde e(\xi) + \frac{\partial V'(\xi)}{\partial y}\hat e(\xi)\right)'\right\|\|n(\xi)\| \\
&\quad + \frac{1}{2}\begin{bmatrix}\tilde l\\ \hat l\end{bmatrix}'\begin{bmatrix}\dfrac{\partial^2 V(\xi)}{\partial \eta^2} & \dfrac{\partial^2 V(\xi)}{\partial \eta y}\\[2mm] \dfrac{\partial^2 V(\xi)}{\partial y\eta} & \dfrac{\partial^2 V(\xi)}{\partial y^2}\end{bmatrix}\begin{bmatrix}\tilde l\\ \hat l\end{bmatrix} \le 0
\end{aligned} \tag{52}$$

holds for all $\xi \in R^n$. Obviously,

$$\frac{\partial V(\eta,0)}{\partial y} = 0, \qquad \frac{\partial^2 V(\eta,0)}{\partial y^2} = 0, \quad \forall \eta \in R^{n-1}. \tag{53}$$

Setting $y = 0$ in (52), it reduces to

$$\begin{aligned}
&L_{\tilde f_*}V(\eta,0) + \left\|L_{\tilde e_*}'V(\eta,0)\right\|\|n(\eta,0)\| \\
&\quad + \frac{1}{2}\tilde l_*'\frac{\partial^2 V(\eta,0)}{\partial \eta^2}\tilde l_* \le 0.
\end{aligned} \tag{54}$$

Then, by [20] and Definition 9(i), we know that the zero-output system (41) is stable in probability, and that system (1) is robust weakly minimum phase. The proof of Theorem 10 is completed. □

Corollary 11. *If the zero-output system* (41) *satisfies the following Lyapunov-like equality constraint, that is, there exists a positive definite and proper function $W(\eta) \in \mathscr{C}^2 : R^{n-1} \to R^+$ with $W(0) = 0$ and $\lambda(\xi) : R^n \to R^+$, such that*

$$L_{\widetilde{f}_*} W(\eta) + \frac{\lambda}{2} \left\| L_{\widetilde{e}_*} W(\eta) \right\|^2 + \frac{1}{2\lambda} \left\| n(\eta, 0) \right\|^2 + \frac{1}{2} \widetilde{l}_*' \frac{\partial^2 W(\eta)}{\partial \eta^2} \widetilde{l}_* < -\frac{1}{2} \widehat{l}_*' \widehat{l}_*, \tag{55}$$

then system (1) *is feedback equivalent to a robust strictly passive system* (39). *Conversely, the robust strict passivity of system* (39) *with a positive definite and proper storage function implies system* (1) *to be robust minimum phase.*

Proof. From Remark 5 and the proof of Theorem 10, the above conclusion is obvious. □

As follows, we use the above results to study the problem of global stabilization for system (39).

Theorem 12. *Suppose that system* (1) *is robust strongly minimum phase which sustains a positive definite and proper function $W(\eta) \in \mathscr{C}^2 : R^{n-1} \to R^+$ with $W(0) = 0$ and a positive definition function $\lambda(\xi) : R^n \to R^+$ satisfying* (44), *and that system* (39) *is zero-state detectable, then system* (39) *is globally asymptotically stabilizable in probability.*

Proof. We construct the storage function $V(\eta, y) = W(\eta) + (1/2)y'y$ for system (39), where $W(\eta)$ satisfies (44). Obviously, we have

$$\begin{aligned}
\mathscr{L}_\omega V &= L_F V(\xi) + L_E V(\xi) \widetilde{\delta} + L_G V(\xi) \omega + \frac{1}{2} L' \frac{\partial^2 V(\xi)}{\partial \xi^2} L \\
&\leq L_F V(\xi) + L_G V(\xi) \omega + \left\| L_E' V(\xi) \right\| \left\| n(\xi) \right\| \\
&\quad + \frac{1}{2} L' \frac{\partial^2 V(\xi)}{\partial \xi^2} L \\
&\leq L_F V(\xi) + L_G V(\xi) \omega + \frac{1}{2\lambda} \left\| L_E V(\xi) \right\|^2 \\
&\quad + \frac{1}{2\lambda} \left\| n(\xi) \right\|^2 + \frac{1}{2} L' \frac{\partial^2 V(\xi)}{\partial \xi^2} L,
\end{aligned} \tag{56}$$

where $\mathscr{L}_\omega$ is the infinitesimal generator of system (39). Similar as the proof of Theorem 10, we have

$$\begin{aligned}
\mathscr{L}_\omega V &\leq L_{\widetilde{f}_*} W(\eta) + \left\| L_{\widetilde{e}_*} W(\eta) \right\|^2 + \frac{1}{2\lambda} \left\| n(\eta, 0) \right\|^2 \\
&\quad + \frac{1}{2} \widetilde{l}_*' \frac{\partial^2 W(\eta)}{\partial \eta^2} \widetilde{l}_* + \frac{1}{2} \widehat{l}_*' \widehat{l}_* \\
&\quad + y' \left(\omega + c + \frac{\lambda}{2} c_1 + \frac{1}{2\lambda} c_2 + L_{\widetilde{f}_1}' W(\eta) \right. \\
&\quad + \widetilde{l}_1' \frac{\partial^2 W(\eta)}{\partial \eta^2} \widetilde{l}_* + \frac{1}{2} \widetilde{l}_1' \frac{\partial^2 W(\eta)}{\partial \eta^2} \widetilde{l}_1 y \\
&\quad \left. + \widehat{l}_1' \widehat{l}_* + \frac{1}{2} \widehat{l}_1' \widehat{l}_1 \right).
\end{aligned} \tag{57}$$

Taking the same c_1, c_2 and the corresponding c in (46)-(47) as in Theorem 10, we have

$$\mathscr{L}_\omega V \leq y' \omega. \tag{58}$$

Let $\psi(y) : R \to R$ be any smooth function satisfying $\psi(0) = 0$ and $y'\psi(y) > 0$ for each nonzero y. Then, we can construct the control law as $\omega = -\psi(y)$, which follows that

$$\mathscr{L}_{\omega = -\psi(y)} V \leq -y' \psi(y) \leq 0. \tag{59}$$

The rest is proved by using the version of LaSalle's invariance principle [21], which is similar to Theorem 4.6 of [12] and is omitted. The proof of Theorem 12 is completed. □

Corollary 13. *If the zero-output system* (41) *satisfies the Lyapunov-like equality constraint* (55) *with a positive definite and proper function $W(\eta) \in \mathscr{C}^2 : R^{n-1} \to R^+$ with $W(0) = 0$, then system* (39) *is globally asymptotically stabilizable in probability.*

Proof. The proof is similar to that of Theorem 12, and we only note that

$$\mathscr{L}_{\omega = -\psi(y)} V \leq -y' \psi(y) - S < 0. \tag{60}$$

□

Remark 14. Obviously, if the conditions of Theorem 12 or Corollary 13 are satisfied, then system (39) is GASP under the control law $\omega = -y$. Furthermore, system (1) is GASP under the corresponding control

$$u = \left(L_g h \right)^{-1} \left(-y + c(\xi) |_{\xi = \phi(x)} - L_f h - \frac{1}{2} l' \frac{\partial^2 h}{\partial x^2} l \right). \tag{61}$$

Remark 15. In this work, we only consider the relative degree $\rho = 1$ case. Further efforts should be concentrated on the robust passive control design for uncertain nonlinear stochastic system with any arbitrary relative degree case. In that case, we need to discuss the robust passivity through applying the backstepping technique for nonlinear stochastic systems.

4. Numerical Example

Consider the following nonlinear stochastic system:

$$\begin{aligned}
dx &= \begin{bmatrix} x_1 x_2 - 2x_1 \\ 8x_1^3 + x_1 x_2 \end{bmatrix} dt + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \delta(x) \, dt \\
&\quad + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \, dt + \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} dw,
\end{aligned} \tag{62}$$

$$y = x_1 + x_2.$$

The uncertainty $\delta(x)$ satisfies

$$\|\delta(x)\| \le \|n(x)\|, \tag{63}$$

where

$$n(x) = kx_1^2, \tag{64}$$

and k is a given constant satisfying $0 < k \le 1$.

Firstly, we construct the function $\phi(x)$ as

$$\xi = \phi(x) = \begin{bmatrix} \theta(x) \\ h(x) \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} \eta \\ y \end{bmatrix}, \tag{65}$$

then it is obvious that the following holds

$$\phi^{-1}(\eta, y) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\eta + y) \\ \frac{1}{2}(-\eta + y) \end{bmatrix}. \tag{66}$$

Also, it can be found that $(\partial h'/\partial x)g = 2$, $(\partial \theta'/\partial x)g = 0$ hold for the above system. Thus, all the assumptions are satisfied. According to the discussion in Section 3, we design the controller u as

$$u = v - 8x_1^3 - 2x_1x_2 + 2x_1. \tag{67}$$

Then, we change system (62) into the following form

$$\begin{aligned} d\eta &= \left(-(\eta + y)^3 - \eta - y + \eta\tilde{\delta}\right) dt + \eta\, dw, \\ dy &= \left(v + y\tilde{\delta}\right) dt + y\, dw, \end{aligned} \tag{68}$$

with

$$\begin{aligned} &\tilde{f}_* = -\eta^3 - \eta, \qquad \tilde{f}_1 = -1, \\ &\tilde{e}_* = \eta, \qquad \tilde{e}_1 = 0, \\ &\tilde{l}_* = \eta, \qquad \tilde{l}_1 = 0, \\ &\hat{e}_* = 0, \qquad \hat{e}_1 = \frac{1}{2}, \\ &\hat{l}_* = 0, \qquad \hat{l}_1 = \frac{1}{2}, \\ &\tilde{n}(\eta, 0) = \frac{k}{4}\eta^2, \qquad \tilde{n}_1 = \frac{k}{4}(2\eta + y). \end{aligned} \tag{69}$$

Taking $W = \eta^2$, then from (44), we obtain that

$$\begin{aligned} &L_{\tilde{f}_*} W(\eta) + \frac{\lambda}{2}\left\|L_{\tilde{e}_*} W(\eta)\right\|^2 \\ &\quad + \frac{1}{2\lambda}\|n(\eta, 0)\|^2 + \frac{1}{2}\tilde{l}_*' \frac{\partial^2 W(\eta)}{\partial \eta^2}\tilde{l}_* + \frac{1}{2}\hat{l}_*'\hat{l}_* \\ &= 2\eta\left(-\eta^3 - \eta\right) + 2\lambda\eta^4 + \frac{k^2}{32\lambda}\eta^4 + \eta^2. \end{aligned} \tag{70}$$

Leting $k = 1$ and $\lambda = 1/2$, we have

$$-\frac{15}{16}\eta^4 - \eta^2 \le 0. \tag{71}$$

Hence, system (62) is robust strongly minimum phase. By Theorem 10, we know that system (62) is feedback equivalent to the robust passive system (68), and we have

$$\begin{aligned} c_1 &= 2\eta^2 y + \frac{1}{4}y^3, \\ c_2 &= \frac{1}{4}\eta^3 + \frac{3}{8}\eta^2 y + \frac{1}{4}\eta y^2 + \frac{1}{16}y^3. \end{aligned} \tag{72}$$

Then, from (47), we have

$$c = -\frac{1}{4}\eta^3 - \frac{1}{8}y^3 - \frac{7}{8}\eta^2 y - \frac{1}{4}\eta y^2 + 2\eta - \frac{1}{8}. \tag{73}$$

Besides, according to Theorem 12 and Remark 14, we can construct the control law as $\omega = -y$, and it follows that

$$\begin{aligned} v = c(\eta, y) - y = &-\frac{1}{4}\eta^3 - \frac{1}{8}y^3 \\ &- \frac{7}{8}\eta^2 y - \frac{1}{4}\eta y^2 + 2\eta - y - \frac{1}{8}, \end{aligned} \tag{74}$$

which guarantees that the closed-loop system (68) is GASP.

5. Conclusion

In this paper, we have discussed the robust passivity, feedback equivalence, and global stabilization problems for a class of uncertain nonlinear stochastic systems, which contain the structural uncertainty. Through establishing the robust passivity theory, a robust stochastic version of KYP lemma has been presented for such a class of systems. Then, the feedback equivalence and global stabilization problems have been discussed through the robust strongly minimum phase property. However, more efforts should be concentrated on the robust passive control of uncertain nonlinear stochastic systems with any arbitrary relative degree.

Acknowledgments

This work is partially supported by the National Basic Research Program of China (973 Program) (Grant no. 2012CB215203), the National Natural Science Foundation of China (no. 61203043 and no. 51036002), and the Fundamental Research Funds for the Central Universities.

References

[1] J. C. Willems, "Dissipative dynamical systems. I. General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–351, 1972.

[2] D. J. Hill and P. J. Moylan, "The stability of nonlinear dissipative systems," *IEEE Transactions on Automatic Control*, vol. 21, no. 5, pp. 708–711, 1976.

[3] D. J. Hill and P. J. Moylan, "Connections between finite-gain and asymptotic stability," *IEEE Transactions on Automatic Control*, vol. 25, no. 5, pp. 931–936, 1980.

[4] I. G. Polushin, A. L. Fradkov, and D. D. Khill, "Passivity and passification of nonlinear systems," *Rossiĭskaya Akademiya Nauk. Avtomatika i Telemekhanika*, no. 3, pp. 3–37, 2000.

[5] C. I. Byrnes, A. Isidori, and J. C. Willems, "Passivity, feedback equivalence, and the global stabilization of minimum phase nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 36, no. 11, pp. 1218–1240, 1991.

[6] C. I. Byrnes and A. Isidori, "Asymptotic stabilization of minimum phase nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 36, no. 10, pp. 1122–1137, 1991.

[7] A. J. van der Schaft, *L_2-Gain and Passivity Techniques in Nonlinear Control*, Springer, New York, NY, USA, 2000.

[8] W. Lin and T. Shen, "Robust passivity and feedback design for minimum-phase nonlinear systems with structural uncertainty," *Automatica*, vol. 35, no. 1, pp. 35–47, 1999.

[9] T. Shen, L. Xie, and K. Tamura, "Robust almost disturbance decoupling for nonlinear systems with structural uncertainty," in *Proceedings of the 37th IEEE Conference on Decision and Control (CDC '98)*, pp. 4107–4108, Tampa, Fla, USA, December 1998.

[10] J. Yong and X. Y. Zhou, *Stochastic control: Hamiltonian Systems and HJB Equations*, Springer, New York, NY, USA, 1999.

[11] Z. G. Pan, "Canonical forms for stochastic nonlinear systems," *Automatica*, vol. 38, no. 8, pp. 1163–1170, 2002.

[12] P. Florchinger, "A passive system approach to feedback stabilization of nonlinear control stochastic systems," *SIAM Journal on Control and Optimization*, vol. 37, no. 6, pp. 1848–1864, 1999.

[13] P. Florchinger, "Stabilization of passive nonlinear stochastic differential systems by bounded feedback," *Stochastic Analysis and Applications*, vol. 21, no. 6, pp. 1255–1282, 2003.

[14] Z. G. Yan, G. S. Zhang, and J. K. Wang, "Non-fragile robust finite-time H_∞ control for nonlinear stochastic Ito systems using neural network," *International Journal of Control, Automation and Systems*, vol. 10, no. 5, pp. 873–882, 2012.

[15] W. H. Zhang and B. S. Chen, "State feedback H_∞ control for a class of nonlinear stochastic systems," *SIAM Journal on Control and Optimization*, vol. 44, no. 6, pp. 1973–1991, 2006.

[16] W. H. Zhang, H. S. Zhang, and B. S. Chen, "Generalized Lyapunov equation approach to state-dependent stochastic stabilization/detectability criterion," *IEEE Transactions on Automatic Control*, vol. 53, no. 7, pp. 1630–1642, 2008.

[17] W. H. Zhang and L. H. Xie, "Interval stability and stabilization of linear stochastic systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 810–815, 2009.

[18] Z. W. Lin, J. Z. Liu, W. H. Zhang, and Y. G. Niu, "Stabilization of interconnected nonlinear stochastic Markovian jump systems via dissipativity approach," *Automatica*, vol. 47, no. 12, pp. 2796–2800, 2011.

[19] Z. W. Lin, J. Z. Liu, and Y. G. Niu, "Dissipative control of nonlinear stochastic systems with Poisson jumps and Markovian switchings," *IET Control Theory and Applications*, vol. 6, no. 15, pp. 2367–2374, 2012.

[20] R. Z. Has'minskii, *Stochastic Stability of Differential Equation*, Sijthoff & Noordhoff, Alphen aan den Rijn, The Netherlands, 1980.

[21] H. J. Kushner, "Stochastic stability," in *Stability of Stochastic Dynamical Systems*, R. Curtain, Ed., vol. 294 of *Lecture Notes in Mathematics*, pp. 97–124, Springer, Berlin, Germany, 1972.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 685798, 10 pages
http://dx.doi.org/10.1155/2013/685798

Research Article

Synchronization of Coupled Stochastic Systems Driven by α-Stable Lévy Noises

Anhui Gu

College of Science, Guilin University of Technology, Guilin,Guangxi 541004, China

Correspondence should be addressed to Anhui Gu; gahui@glut.edu.cn

Received 9 October 2012; Revised 25 December 2012; Accepted 28 December 2012

Academic Editor: Weihai Zhang

The synchronization of the solutions to coupled stochastic systems of N-Marcus stochastic ordinary differential equations which are driven by α-stable Lévy noises is investigated ($N \in \mathbb{N}, 1 < \alpha < 2$). We obtain the synchronization between two solutions and among different components of solutions under certain dissipative conditions. The synchronous phenomena persist no matter how large the intensity of the environment noises. These results generalize the work of two Marcus canonical equations in X. M. Liu et al.' s (2010).

1. Introduction

The synchronization of coupled systems is a ubiquitous phenomenon in the biological and physical science and is also known to occur in abundant of social science contexts; see for example [1–6] and references therein. In the recent book of Strogatz [4], a number of its diversity of occurrence and an extensive list of references can be found. Let $u(t), v(t) \in \mathbb{R}^d$ be two functions defined in $[t_0, \infty)$ ($t_0 \in \mathbb{R}$) and $u(t), v(t)$ are said to be synchronized if $\lim_{t\to\infty}\|u(t) - v(t)\| = 0$. Synchronization of deterministic coupled systems has been investigated both for autonomous systems and nonautonomous systems (see, e.g., [7–10]). For coupled systems of Itö stochastic differential equations with various Gaussian noises (in the terms of Brownian motion), the synchronization of solutions has been considered in the papers Caraballo and Kloeden [11], Caraballo et al. [12], Caraballo et al. [13] and Chueshov and Schmalfuß [14]. In [15], Shen et al. showed the synchronization of solutions for more general systems with multiplicative noise. Recently, Liu et al. [16, 17] studied the synchronization phenomenon for coupled systems driven by non-Gaussian noises (in terms of Lévy motion) and the analogous results also hold for the general systems with additive Lévy noises [18].

A Lévy motion L_t is a non-Gaussian process with independent and stationary increments; that is, increments $\Delta L_t = L_{t+\Delta t} - L_t$ are stationary and independent for any non overlapping time lags Δt. Moreover, its sample paths are only continuous in probability, namely, $\mathbb{P}(|L_t - L_{t_0}| \geq \epsilon) \to 0$ as $t \to t_0$ for any positive ϵ. With a suitable modification, these paths may be taken as càdlàg; that is, paths are continuous on the right and have limits on the left (see, e.g., [19, 20]). As a special case of Lévy processes, the symmetric α-stable Lévy motion plays an important role among stable processes just like Brownian motion among Gaussian processes. A stochastic process $\{L_t, t \geq 0\}$ is called the α-stable Lévy motion if (1) $L_0 = 0$ a.e., (2) L has independent increments, and (3) $L_t - L_s \sim \mathbf{S}_\alpha((t-s)^{1/\alpha}, \beta, 0)$ for $0 \leq s < t < \infty$ and for some $0 < \alpha \leq 2, -1 \leq \beta \leq 1$, where $\mathbf{S}_\alpha(\sigma, \beta, \nu)$ denotes the α-stable distribution with index of stability α, scale parameter σ, skewness parameter β, and shift parameter ν; in particular, $\mathbf{S}_2(\sigma, 0, \mu) = N(\mu, 2\sigma^2)$ denotes the Gaussian distribution. For more details on α-stable distributions, we can refer to [21, 22].

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, where $\Omega = D(\mathbb{R}, \mathbb{R}^d)$ of càdlàg functions with the Skorohod metric (see [23]) as the canonical sample space and denote by $\mathscr{F} := \mathfrak{B}(D(\mathbb{R}, \mathbb{R}^d))$ the Borel σ-algebra on Ω. Let μ_L be the (Lévy) probability measure on $\mathscr{F}$ which is given by the distribution of a two-sided Lévy process with paths in Ω, that is, $\omega(t) = L_t(\omega)$. Define $\theta = (\theta_t, t \in \mathbb{R})$ on Ω the shift by $(\theta_t\omega)(s) := \omega(t +$

$s) - \omega(t)$. Then, the mapping $(t,\omega) \rightarrow \theta_t\omega$ is continuous and measurable [24], and the (Lévy) probability measure is θ-invariant, that is, $\mu_L(\theta_t^{-1}(A)) = \mu_L(A)$, for all $A \in \mathscr{F}$; see [19] for more details.

Consider the following Marcus stochastic ordinary differential equations (MSODEs) system driven by α-stable Lévy noises in $\mathbb{R}^d$:

$$dX_t^{(j)} = f^{(j)}\left(X_t^{(j)}\right)dt + \sum_{i=1}^{m} c_i^{(j)} X_t^{(j)} \diamond dL_t^{(i)}, \quad j = 1, \ldots, N, \tag{1}$$

where $c_i^{(j)} \in \mathbb{R}$, $L_t^{(i)}$ are independent α-stable Lévy noises on $(\Omega, \mathscr{F}, \mathbb{P})$, $1 < \alpha < 2$, $\diamond$ denotes the Marcus integral (see, e.g., [25]), and $f^{(j)}$, $j = 1, \ldots, N$ are regular enough to ensure the existence and uniqueness of solutions and satisfy the one-sided dissipative Lipschitz condition

$$\left\langle x_1 - x_2, f^{(j)}(x_1) - f^{(j)}(x_2)\right\rangle \le -L\|x_1 - x_2\|^2, \quad j = 1, \ldots, N, \tag{2}$$

on $\mathbb{R}^d$ for some $L > 0$.

Set

$$x^{(j)}(t,\omega) = e^{-O_t^{(j)}} X_t^{(j)}(\omega), \quad t \in \mathbb{R}, \omega \in \Omega, \ j = 1, \ldots, N, \tag{3}$$

where

$$O_t^{(j)} := O_t^{(j)}(\omega) = \sum_{i=1}^{m} c_i^{(j)} e^{-t} \int_{-\infty}^{t} e^s dL_s^{(i)}, \quad j = 1, \ldots, N, \tag{4}$$

are the stationary solutions of the Ornstein-Uhlenbeck stochastic differential equations

$$dO_t^{(j)} = -O_t^{(j)}dt + \sum_{i=1}^{m} c_i^{(j)} \diamond dL_t^{(i)}, \quad j = 1, \ldots, N. \tag{5}$$

Then system (1) can be translated into the following random ordinary differential equations (RODEs):

$$\begin{aligned}\frac{dx^{(j)}}{dt_+} &= F^{(j)}\left(x^{(j)}, O_t^{(j)}\right) \\ &:= e^{-O_t^{(j)}} f^{(j)}\left(e^{O_t^{(j)}} x^{(j)}\right) + O_t^{(j)} x^{(j)}, \quad j = 1, \ldots, N,\end{aligned} \tag{6}$$

where $dx^{(j)}/dt_+$ is right-hand derivative of $x^{(j)}$ at t.

Now we consider the linear coupled RODEs of (6), as follows:

$$\frac{dx^{(j)}}{dt_+} = F^{(j)}\left(x^{(j)}, O_t^{(j)}\right) + \nu\left(x^{(j-1)} - 2x^{(j)} + x^{(j+1)}\right), \quad j = 1, \ldots, N, \tag{7}$$

with the coupled coefficient $\nu > 0$, where $x^{(0)} = x^{(N)}$ and $x^{(N+1)} = x^{(1)}$. Hence, (7) can be written as the following equivalent MSODEs:

$$\begin{aligned} dX_t^{(j)} &= f^{(j)}\left(X_t^{(j)}\right)dt \\ &\quad + \nu\left(e^{-\Delta\check{O}_t^{(j)}} X_t^{(j-1)} - 2X_t^{(j)} + e^{-\Delta\widehat{O}_t^{(j)}} X_t^{(j+1)}\right)dt \\ &\quad + \sum_{i=1}^{m} c_i^{(j)} X_t^{(j)} \diamond dL_t^{(i)}, \quad j = 1, \ldots, N, \end{aligned} \tag{8}$$

where $\Delta\check{O}_t^{(j)} = O_t^{(j)} - O_t^{(j-1)}$, $\Delta\widehat{O}_t^{(j)} = O_t^{(j)} - O_t^{(j+1)}$, $O_t^{(0)} = O_t^{(N)}$, and $O_t^{(N+1)} = O_t^{(1)}$.

For synchronization of solutions (in the sense of Carathéodory [26]) to RODEs system (7), there are two cases: one for any two solutions and the other for components of solutions. When $N = 2$, Liu et al. [17] consider both types of synchronization. Under the one-sided dissipative Lipschitz condition (2), they firstly proved that synchronization of any two solutions occurs and the random dynamical system generated by the solution of (7) $N = 2$ has a singleton sets random attractor, then they proved that the synchronization between any two components of solutions occurs as the coupled coefficient ν tends to infinity. The synchronization result implies that coupled dynamical systems share a dynamical feature in an asymptotic sense. Based on the work of [15, 17], we consider the synchronization of solutions of (7) in the case of $N \ge 3$ and obtain the similar results. We show that the random dynamical system (RDS) generated by the solution of the coupled RODEs system (7) has a singleton sets random attractor which implies the synchronization of any two solutions of (7). Moreover, the singleton set random attractor determines a stationary stochastic solution of the equivalently coupled RODEs system (8). We also show that any two solutions of RODEs system (7) converge to a solution $Z(t,\omega)$ of the averaged RODEs as follows:

$$\frac{dZ}{dt_+} = \frac{1}{N}\sum_{j=1}^{N} e^{-O_t^{(j)}} f^{(j)}\left(e^{O_t^{(j)}} Z\right) + \frac{1}{N}\sum_{j=1}^{N} O_t^{(j)} Z, \tag{9}$$

as the coupling coefficient $\nu \rightarrow \infty$.

When $\alpha = 2$, we have the standard Brownian motion, which the Marcus integral reduces to the Stratonovich stochastic integral, and both types of the synchronization of system (8) have been considered in [15]. It is worth mentioning that the generalization is not trivial because new techniques similar to [15] are needed. We restrict here that $\alpha \in (1,2)$, only in this case, the solutions of the Ornstein-Uhlenbech equations based on α-stable Lévy noises are stationary, which is crucial to our purpose. When $\alpha \in (0,1)$, dealing with such values of the parameter seems to be a new challenging for us.

The paper is organized as follows. In Section 2, we recall some basic facts on random dynamical systems, and then we give two lemmas which will be frequently used. In Section 3, we show the synchronization of two solutions to the coupled RODEs (7) and obtain the staionary stochastic solution to the equivalent MSODEs (8). In Section 4, we give the

synchronization of components of solutions to the coupled RODEs (7), which implies that the equivalent MSODEs (8) share the similarly synchronous phenomenon when driven by the same α-stable Lévy noises.

2. Random Dynamical Systems and Auxiliary Lemmas

We will frequently use the following results.

Lemma 1. *There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset $\overline{\Omega} \in \mathcal{F}$ of full measure for a.e. $\omega \in \overline{\Omega}$, and the sample paths $\omega(t)$ of L_t satisfy*

$$\lim_{t\to\pm\infty} \frac{\omega(t)}{t} = 0, \quad t \in \mathbb{R}. \tag{10}$$

In addition, for $j = 1, \ldots, N$, there exist random variables $\overline{O}^{(j)} = O_t^{(j)}$ and $T_\omega > 0$ such that

$$\overline{O}^{(j)}(\theta_t\omega) = O_t^{(j)}(\omega), \quad \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \overline{O}(\theta_s\omega)ds = 0, \quad \omega \in \overline{\Omega}, \tag{11}$$

$$e^{2\int_\tau^t O_s^{(j)}ds} \le e^{(L/2)(t-\tau)} \quad \text{for } -\tau, t > T_\omega. \tag{12}$$

Proof. The equalities (10) and (11) can be found in [17, Lemma 2]. By (11), we have $\lim_{t\to\infty}(1/t)\int_0^t O_s^{(j)}ds = 0$, then there exists $T_\omega(1) > 0$ such that $\int_0^t O_s^{(j)}ds \le (L/4)t$ for $t > T_\omega(1)$. Similarly, $\lim_{\tau\to-\infty}(1/\tau)\int_\tau^0 O_s^{(j)}ds = 0$, which implies that there exists $T_\omega(2) > 0$ such that $\int_\tau^0 O_s^{(j)}ds \le -(L/4)\tau$ for $\tau < -T_\omega(2)$. Denoting $T_\omega = \max\{T_\omega(1), T_\omega(2)\}$, we have $2\int_\tau^t O_s^{(j)}ds \le (L/2)(t-\tau)$ for $-\tau, t > T_\omega$, which completes the proof. □

Lemma 2 (Gronwall type inequality). *Suppose that $D(t)$ is an $n\times n$ matrix and $\Phi(t)$ and $\Psi(t)$ are n-dimensional vectors on $[T_0, T]$ $(T \ge T_0,\ T, T_0 \in \mathbb{R})$ which are sufficiently regular. If the following inequality holds in the componentwise sense:*

$$\frac{d}{dt_+}\Phi(t) \le D(t)\Phi(t) + \Psi(t), \quad t \ge T_0, \tag{13}$$

where $(d/dt_+)\Phi(t) := \lim_{h\downarrow 0^+}((\Phi(t+h) - \Phi(t))/h)$ is right-hand derivative of $\Phi(t)$, then

$$\Phi(t) \le \exp\left(\int_{T_0}^t D(s)ds\right)\Phi(T_0) + \int_{T_0}^t \exp\left(\int_\tau^t D(s)ds\right)\Psi(\tau)d\tau, \quad t \ge T_0. \tag{14}$$

Proof. See Lemma 2.8 in [27] and the proof of lemma 2.2 in [15]. □

Proposition 3 (Random attractor for càdlàg RDS (see [16])). *Let (θ, φ) be an RDS on $\Omega\times\mathbb{R}^d$ and let φ be continuous in space but càlàg in time. If there exists a family $\mathcal{B} = \{\mathcal{B}(\omega), \omega \in \Omega\}$ of nonempty measurable compact subsets $\mathcal{B}(\omega)$ of $\mathbb{R}^d$ and a $T_{B,\omega} \ge 0$ such that*

$$\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset \mathcal{B}(\omega), \quad \forall t \ge T_{B,\omega}, \tag{15}$$

for all families $B = \{B(\omega), \omega \in \Omega\}$ in a given attracting universe. then, the RDS (θ, φ) has a random attractor $\mathcal{A} = \{\mathcal{A}(\omega), \omega \in \Omega\}$ with the component subsets defined for each $\omega \in \Omega$ by

$$\mathcal{A}(\omega) = \bigcap_{s>0}\overline{\bigcup_{t\ge s}\varphi(t, \theta_{-t}\omega, B(\theta_{-t}\omega))}. \tag{16}$$

Furthermore, if the random attractor consist of singleton sets, that is, $\mathcal{A}(\omega) = \{X^*(\omega)\}$ for some random variable X^*, then $X_t^*(\omega) = X_t^*(\theta_t\omega)$ is a stationary stochastic process.

3. Synchronization of Two Solutions

Consider the coupled system (7) with the following initial data:

$$x^{(j)}(0, \omega) = x_0^{(j)}(\omega) \in \mathbb{R}^d, \quad \omega \in \Omega,\ j = 1, \ldots, N. \tag{17}$$

For asymptotic behavior of the difference between two solutions of RODEs system (7) with initial data (17) (omitting to RODEs system (7) for brevity), we get the following:

Lemma 4. *For any two solutions $(x_1^{(1)}(t), x_1^{(2)}(t), \ldots, x_1^{(N)}(t))^{\mathbf{T}}$ and $(x_2^{(1)}(t), x_2^{(2)}(t), \ldots, x_2^{(N)}(t))^{\mathbf{T}}$ of RODEs system (7),*

$$\lim_{t\to\infty}\left\|x_1^{(j)}(t) - x_2^{(j)}(t)\right\| = 0, \quad j = 1, \ldots, N, \tag{18}$$

that is, all solutions of the coupled RODEs system (7) converge pathwise to each other as time t tends to infinity.

Proof. By the dissipative Lipschitz condition (2), for $j = 1, \ldots, N$, we have

$$\begin{aligned}
&\frac{d}{dt_+}\left\|x_1^{(j)}(t) - x_2^{(j)}(t)\right\|^2 \\
&\quad = 2\left\langle x_1^{(j)}(t) - x_2^{(j)}(t), \frac{d}{dt}x_1^{(j)}(t) - \frac{d}{dt}x_2^{(j)}(t)\right\rangle \\
&\quad = 2e^{-O_t^{(j)}}\left\langle f^{(j)}\left(e^{O_t^{(j)}}x_1^{(j)}\right) - f^{(j)}\left(e^{O_t^{(j)}}x_2^{(j)}\right), x_1^{(j)}(t) - x_2^{(j)}(t)\right\rangle \\
&\qquad \times\left(2e^{O_t^{(j)}} - 4\nu\right)\left\|x_1^{(j)}(t) - x_2^{(j)}(t)\right\|^2 \\
&\qquad + 2\nu\left\langle x_1^{(j-1)}(t) - x_2^{(j-1)}(t), x_1^{(j)}(t) - x_2^{(j)}(t)\right\rangle \\
&\qquad + 2\nu\left\langle x_1^{(j+1)}(t) - x_2^{(j+1)}(t), x_1^{(j)}(t) - x_2^{(j)}(t)\right\rangle \\
&\quad \le \left(2e^{O_t^{(j)}} - 2L - 2\nu\right)\left\|x_1^{(j)}(t) - x_2^{(j)}(t)\right\|^2 \\
&\qquad + \nu\left\|x_1^{(j-1)}(t) - x_2^{(j-1)}(t)\right\|^2 + \nu\left\|x_1^{(j+1)}(t) - x_2^{(j+1)}(t)\right\|^2.
\end{aligned} \tag{19}$$

Define for $t \in \mathbb{R}$,

$$\mathbf{x}(t) = \left(\left\|x_1^{(1)}(t) - x_2^{(1)}(t)\right\|^2, \left\|x_1^{(2)}(t) - x_2^{(2)}(t)\right\|^2, \dots, \left\|x_1^{(N)}(t) - x_2^{(N)}(t)\right\|^2\right)^{\mathbf{T}},$$

$$D_\nu(t) = \begin{pmatrix} \lambda_\nu^{(1)}(t) & \nu & 0 & \cdots & \nu \\ \nu & \lambda_\nu^{(2)}(t) & \nu & 0 & \cdots \\ 0 & \nu & \lambda_\nu^{(3)}(t) & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \nu \\ \nu & \cdots & 0 & \nu & \lambda_\nu^{(N)}(t) \end{pmatrix}_{N\times N}, \tag{20}$$

where $\lambda_\nu^{(j)}(t) = 2e^{O_t^{(j)}} - 2L - 2\nu$, $j = 1, \dots, N$. Thus, the differential inequalities can be written as a simple form

$$\dot{\mathbf{x}}(t) \le D_\nu(t)\mathbf{x}(t), \quad -\text{componentwise.} \tag{21}$$

By Lemma 2, it yields from (21) that

$$\mathbf{x}(t) \le \exp\left(\int_0^t D_\nu(t)ds\right)\mathbf{x}(0), \quad -\text{componentwise.} \tag{22}$$

By [15, Lemma 3.2], we know that for $t \ge T_\omega$ defined in Lemma 1, and $\nu > 0$,

$$\left\|\exp\left(\int_0^t D_\nu(t)ds\right)\mathbf{x}(0)\right\| \le e^{-Lt}\|\mathbf{x}(0)\|, \tag{23}$$

which leads to

$$\lim_{t\to\infty}\left\|x_1^{(j)}(t) - x_2^{(j)}(t)\right\| = 0, \quad j = 1, \dots, N, \tag{24}$$

and completes the proof. □

Now, we use the theory of random dynamical systems which are generated by stochastic differential equations driven by α-stable Lévy noise to find what the solutions of (7) will converge to. Obviously by condition (2) and [16, Lemma 4], we know that the solution

$$\varphi(t,\omega) = \left(x^{(1)}(t,\omega), x^{(2)}(t,\omega), \dots, x^{(N)}(t,\omega)\right)^{\mathbf{T}}, \quad \omega \in \Omega, \tag{25}$$

of system (7) generates a càdlàg RDS over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ with state space $\Omega \times \mathbb{R}^{Nd}$.

Then, we have the result for this RDS φ.

Theorem 5. *Under the dissipative condition of* (2), *the RDS* $\varphi(t,\omega), t \in \mathbb{R}, \omega \in \Omega$, *has a singleton sets random attractor given by*

$$\mathcal{A}_\nu(\omega) = \left(\overline{x}_\nu^{(1)}(\omega), \overline{x}_\nu^{(2)}(\omega), \dots, \overline{x}_\nu^{(N)}(\omega)\right)^{\mathbf{T}}, \tag{26}$$

which implies the synchronization of any two solutions of system (7). *Furthermore,*

$$\left(\overline{x}_\nu^{(1)}(\theta_t\omega)e^{O_t^{(1)}(\omega)}, \overline{x}_\nu^{(2)}(\theta_t\omega)e^{O_t^{(2)}(\omega)}, \dots, \overline{x}_\nu^{(N)}(\theta_t\omega)e^{O_t^{(N)}(\omega)}\right)^{\mathbf{T}} \tag{27}$$

is the stationary stochastic solution of the equivalent coupled MSODEs (8).

Proof. For $j = 1, \dots, N$, we have

$$\begin{aligned}
\frac{d}{dt}\left\|x^{(j)}(t)\right\|^2 &= 2\left\langle x^{(j)}(t), \frac{d}{dt}x^{(j)}(t)\right\rangle \\
&= 2\left\langle e^{-O_t^{(j)}} f^{(j)}\left(e^{O_t^{(j)}}x^{(j)}(t)\right), x^{(j)}(t)\right\rangle \\
&\quad + 2\left\langle e^{O_t^{(j)}}x^{(j)}(t), x^{(j)}(t)\right\rangle - 4\nu\left\|x^{(j)}(t)\right\|^2 \\
&\quad + 2\nu\left\langle x^{(j)}(t), x^{(j-1)}(t)\right\rangle + 2\nu\left\langle x^{(j)}(t), x^{(j+1)}(t)\right\rangle \\
&\le \left(2e^{O_t^{(j)}} - 2L - 2\nu\right)\left\|x^{(j)}(t)\right\|^2 + \nu\left\|x^{(j-1)}(t)\right\|^2 \\
&\quad + \nu\left\|x^{(j+1)}(t)\right\|^2 + 2\left\|x^{(j)}(t)\right\|\left\|f^{(j)}(0)\right\|e^{-O_t^{(j)}} \\
&\le \left(2e^{O_t^{(j)}} - L - 2\nu\right)\left\|x^{(j)}(t)\right\|^2 + \nu\left\|x^{(j-1)}(t)\right\|^2 \\
&\quad + \nu\left\|x^{(j+1)}(t)\right\|^2 + \frac{e^{-2O_t^{(j)}}}{L}\left\|f^{(j)}(0)\right\|^2.
\end{aligned} \tag{28}$$

Analogous to (21), we get

$$\dot{\mathbf{y}}(t) \le \widetilde{D}_\nu(t)\mathbf{y}(t) + \mathbf{g}(t), \tag{29}$$

where $t \in \mathbb{R}$,

$$\mathbf{y}(t) = \left(\left\|x^{(1)}(t)\right\|^2, \left\|x^{(2)}(t)\right\|^2, \dots, \left\|x^{(N)}(t)\right\|^2\right)^{\mathbf{T}},$$

$$\mathbf{g}(t) = \left(\frac{e^{-2O_t^{(1)}}}{L}\left\|f^{(1)}(0)\right\|^2, \frac{e^{-2O_t^{(2)}}}{L}\left\|f^{(2)}(0)\right\|^2, \dots, \frac{e^{-2O_t^{(N)}}}{L}\left\|f^{(N)}(0)\right\|^2\right)^{\mathbf{T}},$$

$$\widetilde{D}_\nu(t) = \begin{pmatrix} \widetilde{\lambda}_\nu^{(1)}(t) & \nu & 0 & \cdots & \nu \\ \nu & \widetilde{\lambda}_\nu^{(2)}(t) & \nu & 0 & \cdots \\ 0 & \nu & \widetilde{\lambda}_\nu^{(3)}(t) & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots & \nu \\ \nu & \cdots & 0 & \nu & \widetilde{\lambda}_\nu^{(N)}(t) \end{pmatrix}_{N\times N}, \tag{30}$$

where $\widetilde{\lambda}_\nu^{(j)}(t) = 2e^{O_t^{(j)}} - L - 2\nu$, $j = 1, \dots, N$. Then by Lemma 2,

$$\mathbf{y}(t) \le \exp\left(\int_{t_0}^t \widetilde{D}_\nu(t)ds\right)\mathbf{y}(t_0) + \int_{t_0}^t \exp\left(\int_\tau^t \widetilde{D}_\nu(t)ds\right) \times \mathbf{g}(\tau)d\tau, \quad t \ge t_0. \tag{31}$$

By (23), we have

$$\left\| \exp\left(\int_{t_0}^{t} \widetilde{D}_\nu(t)ds\right)\mathbf{y}(t_0)\right\| \leq \exp\left(-\frac{L}{2}(t-t_0)\right) \times \|\mathbf{y}(t_0)\|, \quad t \geq t_0. \tag{32}$$

Define

$$\rho_\nu(\omega) := \int_{-\infty}^{0} \exp\left(\int_{\tau}^{0} \widetilde{D}_\nu(s)ds\right)\mathbf{g}(\tau)\,d\tau, \tag{33}$$

$$R_\nu^2(\omega) = 1 + \|\rho_\nu(\omega)\|^2, \tag{34}$$

and let $\mathbb{B}_\nu$ be a random ball in $\mathbb{R}^{Nd}$ centered at the origin with radius $R_\nu(\omega)$. Obviously, the infinite integrals on the right hand side of (33) and (34) are well defined by Lemma 1.

For a given attracting universe of tempered random bounded sets $\mathscr{D}$, that is, for any $\omega \in \Omega$, $B \in \mathscr{D}$, and all $\gamma > 0$, we have $\lim_{t\to\infty} e^{-\gamma t}\sup_{x\in B(\theta_{-t}\omega)}\|x\| = 0$. Note that for all $\gamma > 0$, if $\lim_{t\to\infty} e^{-\gamma t}\|\mathbf{y}(t_0)\| = 0$, then

$$\sum_{j=1}^{N}\left\|x^{(j)}(0)\right\|^2 < R_\nu^2(\omega) \quad \text{as} \quad t_0 \longrightarrow -\infty, \tag{35}$$

which implies that the closed random ball $\mathbb{B}_\nu(\omega)$ is a pullback absorbing set at $t = 0$ of the càdlàg RDS $\varphi(t,\omega)$; that is

$$\varphi(t,\theta_{-t}\omega)B(\theta_{-t}\omega) \subset \mathbb{B}_\nu(\omega), \quad \forall t \geq t_{\mathbb{B}_\nu}(\omega), \tag{36}$$

in the attracting universe $\mathscr{D}$. Hence by Proposition 3, the coupled system has a random attractor $\mathscr{A}_\nu = \{\mathscr{A}_\nu(\omega), \omega \in \Omega\}$ with $\mathscr{A}_\nu(\omega) \subset \mathbb{B}_\nu$ satisfying that $\mathscr{A}_\nu(\omega)$ is compact, φ-invariant, that is, $\varphi(t,\omega)\mathscr{A}_\nu(\omega) = \mathscr{A}_\nu(\theta_t\omega)$ for all $t \geq 0$, $\omega \in \Omega$, and attracting in $\mathscr{D}$, that is, for all $B \in \mathscr{D}$,

$$H_d^*(\varphi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), \mathscr{A}_\nu(\omega)) \longrightarrow 0 \qquad \text{as } t \longrightarrow \infty, \tag{37}$$

where H_d^* is the Hausdorff semidistance on $\mathbb{R}^{Nd}$. By Lemma 4, all solutions of (7) converge pathwise to each other; therefor, $\mathscr{A}_\nu(\omega)$ consists of singleton sets, that is,

$$\mathscr{A}_\nu(\omega) = \left(\overline{x}_\nu^{(1)}(\omega), \overline{x}_\nu^{(2)}(\omega), \ldots, \overline{x}_\nu^{(N)}(\omega)\right)^{\mathbf{T}}. \tag{38}$$

We transform the coupled RODEs (7) back to the coupled MSODEs (8), the corresponding pathwise singleton sets attractor is then equal to

$$\left(\overline{x}_\nu^{(1)}(\theta_t\omega)e^{O_t^{(1)}(\omega)}, \overline{x}_\nu^{(2)}(\theta_t\omega)\,e^{O_t^{(2)}(\omega)}, \ldots, \overline{x}_\nu^{(N)}\,(\theta_t\omega)e^{O_t^{(N)}(\omega)}\right)^{\mathbf{T}}, \tag{39}$$

which is exactly a stationary stochastic solution of the coupled MSODEs (8) because the Ornstein-Uhlenbeck process is stationary (see [17]). □

4. Synchronization of the Components of Solutions

It is known in Section 3 that all solutions of the coupled RODEs system (7) converge pathwise to each other in the future for a fixed positive coupling coefficient ν. Here, we would like to discuss what will happen to solutions of the coupled RODEs system (7) as $\nu \to \infty$. First, we will give some lemmas which play an important role in this section.

Similar to [15, Section 4], we can set up the following estimations. Suppose that $(x_\nu^{(1)}(t), x_\nu^{(2)}(t), \ldots, x_\nu^{(N)}(t))^{\mathbf{T}}$ is a solution of the coupled RODEs system (7). For any two different components $x_\nu^{(j)}(t), x_\nu^{(k)}(t)$ of the solution for all $j, k \in \{1, 2, \ldots, N\}$,

$$\begin{aligned}
d_\nu^{j,k}(t) &= 2\left\langle x_\nu^{(j)}(t) - x_\nu^{(k)}(t), F^{(j)}\left(x_\nu^{(j)}, O_t^{(j)}\right) - F^{(k)}\left(x_\nu^{(k)}, O_t^{(k)}\right)\right\rangle \\
&= 2\left\langle x_\nu^{(j)}(t) - x_\nu^{(k)}(t), e^{-O_t^{(j)}} f^{(j)}\left(e^{O_t^{(j)}} x_\nu^{(j)}\right) - e^{-O_t^{(k)}} f^{(k)}\left(e^{O_t^{(k)}} x_\nu^{(k)}\right)\right\rangle \\
&\quad + 2\left\langle x_\nu^{(j)}(t) - x_\nu^{(k)}(t), O_t^{(j)} x_\nu^{(j)}(t) - O_t^{(k)} x_\nu^{(k)}(t)\right\rangle \\
&\leq 2\left\|x_\nu^{(j)}(t) - x_\nu^{(k)}(t)\right\| \left(e^{-O_t^{(j)}} \left\|f^{(j)}\left(e^{O_t^{(j)}} x_\nu^{(j)}(t)\right)\right\| + \left|O_t^{(j)}\right| \left\|x_\nu^{(j)}(t)\right\|\right) \\
&\quad + 2\left\|x_\nu^{(j)}(t) - x_\nu^{(k)}(t)\right\| \left(e^{-O_t^{(k)}} \left\|f^{(k)}\left(e^{O_t^{(k)}} x_\nu^{(k)}(t)\right)\right\| + \left|O_t^{(k)}\right| \left\|x_\nu^{(k)}(t)\right\|\right),
\end{aligned} \tag{40}$$

thus, for fixed $\varrho > 0$, we have

$$\begin{aligned}
&-\varrho\nu\left\|x_\nu^{(j)}(t) - x_\nu^{(k)}(t)\right\|^2 + d_\nu^{k,j}(t) \\
&\leq \frac{1}{\nu}\left(\frac{4}{\varrho}e^{-2O_t^{(j)}}\left\|f^{(j)}\left(e^{O_t^{(j)}} x_\nu^{(j)}(t)\right)\right\|^2 + \frac{4\nu^2}{\varrho}\left|O_t^{(j)}\right|^2\left\|x_\nu^{(j)}(t)\right\|^2\right) \\
&\quad + \frac{1}{\nu}\left(\frac{4}{\varrho}e^{-2O_t^{(k)}}\left\|f^{(k)}\left(e^{O_t^{(k)}} x_\nu^{(k)}(t)\right)\right\|^2 + \frac{4\nu^2}{\varrho}\left|O_t^{(k)}\right|^2\left\|x_\nu^{(k)}(t)\right\|^2\right).
\end{aligned} \tag{41}$$

Let

$$C^{j,k,\varrho}_{T_1,T_2}(\nu,\omega) = \frac{4}{\varrho}\sup_{t\in[T_1,T_2]}\Bigg[\left(e^{-2O_t^{(j)}}\left\|f^{(j)}\left(e^{O_t^{(j)}}x_\nu^{(j)}(t)\right)\right\|^2 + \left|O_t^{(j)}\right|^2\left\|x_\nu^{(j)}(t)\right\|^2\right) + \left(e^{-2O_t^{(k)}}\left\|f^{(k)}\left(e^{O_t^{(k)}}x_\nu^{(k)}(t)\right)\right\|^2 + \left|O_t^{(k)}\right|^2\left\|x_\nu^{(k)}(t)\right\|^2\right)\Bigg] \tag{42}$$

in any bounded interval $[T_1,T_2]$. Note that $\rho_\nu(\omega)$ in (33) satisfies

$$\frac{d}{d\nu}\left\|\rho_\nu(\omega)\right\|^2 = 2\left\langle \rho_\nu(\omega), \frac{d}{d\nu}\rho_\nu(\omega)\right\rangle \le 0, \tag{43}$$

and, consequently, $\rho_\nu(\omega) \le \rho_1(\omega)$ for $\nu \ge 1$. Hence, $C^{j,k,\varrho}_{T_1,T_2}(\nu,\omega)$ is uniformly bounded in ν and

$$-\varrho\nu\left\|x_\nu^{(j)}(t)-x_\nu^{(k)}(t)\right\|^2 + d_\nu^{k,j}(t) \le \frac{1}{\nu}C^{j,k,\varrho}_{T_1,T_2}(\nu,\omega) \tag{44}$$

uniformly for $t\in[T_1,T_2]$ with

$$C^{j,k,\varrho}_{T_1,T_2}(\omega) = \sup_{\nu\ge 1} C^{j,k,\varrho}_{T_1,T_2}(\nu,\omega). \tag{45}$$

Now let us estimate the difference between any two components of a solution of the coupled RODEs system (7) as $\nu\to\infty$.

Lemma 6. *Provided condition* (2) *is satisfied, then any two components of a solution* $(x_\nu^{(1)}(t), x_\nu^{(2)}(t), \dots, x_\nu^{(N)}(t))^{\mathbf{T}}$ *of the coupled RODEs system* (7) *uniformly vanish in any bounded time interval when the coupling coefficient* $\nu\to\infty$*; that is, for any bounded interval* $[T_1,T_2]$ *and for all* $t\in[T_1,T_2]$*, it yields*

$$\lim_{\nu\to\infty}\left\|x_\nu^{(j)}(t)-x_\nu^{(k)}(t)\right\| = 0, \quad \forall j,k\in\{1,2,\dots,N\}. \tag{46}$$

Proof. The proof is quite similar to the proof of Lemma 4.2 in [15]. To prove the result, we can equivalently estimate the difference between any two adjacent components only because the first and the last components of the solution are considered to be adjacent. We will notice that only one new term appears in each step which continues the process, except the last step that ends the process.

For the difference of the first part of the solution $(x_\nu^{(1)}(t), x_\nu^{(2)}(t), \dots, x_\nu^{(N)}(t))^{\mathbf{T}}$,

$$\begin{aligned}
\frac{d}{dt_+}\left\|x_\nu^{(1)}(t)-x_\nu^{(2)}(t)\right\|^2 &= 2\Big\langle x_\nu^{(1)}(t)-x_\nu^{(2)}(t), F^{(1)}\left(x_\nu^{(1)},O_t^{(1)}\right) - F^{(2)}\left(x_\nu^{(2)},O_t^{(2)}\right)\Big\rangle \\
&\quad - 6\nu\left\|x_\nu^{(1)}(t)-x_\nu^{(2)}(t)\right\|^2 \\
&\quad + 2\nu\left\langle x_\nu^{(1)}(t)-x_\nu^{(2)}(t), x_\nu^{(N)}(t)-x_\nu^{(3)}(t)\right\rangle \\
&\le -5\left\|x_\nu^{(1)}(t)-x_\nu^{(2)}(t)\right\|^2 + \nu\left\|x_\nu^{(N)}(t)-x_\nu^{(3)}(t)\right\|^2 + d_\nu^{1,2}(t) \\
&\le -\delta\nu\left\|x_\nu^{(1)}(t)-x_\nu^{(2)}(t)\right\|^2 + \nu\left\|x_\nu^{(N)}(t)-x_\nu^{(3)}(t)\right\|^2 + \frac{1}{\nu}C^{1,2,5-\delta}_{T_1,T_2}(\omega)
\end{aligned} \tag{47}$$

uniformly for $t\in[T_1,T_2]$ by (44). Here, we can take

$$\delta = \begin{cases} 1-\cos\dfrac{N\pi}{N+2}, & N \text{ is even}, \\ 1-\cos\dfrac{(N-1)\pi}{N+1}, & N \text{ is odd}. \end{cases} \tag{48}$$

In fact, from [15, Lemma 4.1], we can take any $\delta\in(-2\cos(N\pi/(N+2)),2)$ when N is even and any $\delta\in(-2\cos((N-1)\pi/(N+1)),2)$ when N is odd.

We have seen that the estimations in (47) generate $x_\nu^{(3)}(t)-x_\nu^{(N)}(t)$. Now, we have

$$\begin{aligned}
\frac{d}{dt_+}\left\|x_\nu^{(3)}(t)-x_\nu^{(N)}(t)\right\|^2 &= 2\Big\langle x_\nu^{(3)}(t)-x_\nu^{(N)}(t), F^{(3)}\left(x_\nu^{(3)},O_t^{(3)}\right) - F^{(N)}\left(x_\nu^{(N)},O_t^{(N)}\right)\Big\rangle \\
&\quad - 4\nu\left\|x_\nu^{(3)}(t)-x_\nu^{(N)}(t)\right\|^2 \\
&\quad + 2\nu\left\langle x_\nu^{(3)}(t)-x_\nu^{(N)}(t), x_\nu^{(2)}(t)-x_\nu^{(1)}(t)\right\rangle \\
&\quad + 2\nu\left\langle x_\nu^{(3)}(t)-x_\nu^{(N)}(t), x_\nu^{(4)}(t)-x_\nu^{(N-1)}(t)\right\rangle \\
&\le -\delta\nu\left\|x_\nu^{(3)}(t)-x_\nu^{(N)}(t)\right\|^2 + \nu\left\|x_\nu^{(1)}(t)-x_\nu^{(2)}(t)\right\|^2 \\
&\quad + \nu\left\|x_\nu^{(4)}(t)-x_\nu^{(N-1)}(t)\right\|^2 + \frac{1}{\nu}C^{3,N,2-\delta}_{T_1,T_2}(\omega)
\end{aligned} \tag{49}$$

uniformly for $t\in[T_1,T_2]$.

Note that $x_\nu^{(1)}(t) - x_\nu^{(2)}(t)$ has been fixed and $x_\nu^{(4)}(t) - x_\nu^{(N-1)}(t)$ is generated. Similarly, it yields

$$\begin{aligned}\frac{d}{dt_+}&\left\|x_\nu^{(4)}(t) - x_\nu^{(N-1)}(t)\right\|^2\\ &\le -\delta\nu\left\|x_\nu^{(4)}(t) - x_\nu^{(N-1)}(t)\right\|^2 + \nu\left\|x_\nu^{(3)}(t) - x_\nu^{(N)}(t)\right\|^2\\ &\quad + \nu\left\|x_\nu^{(5)}(t) - x_\nu^{(N-2)}(t)\right\|^2 + \frac{1}{\nu}C_{T_1,T_2}^{4,N-1,2-\delta}(\omega)\end{aligned} \tag{50}$$

uniformly for $t \in [T_1, T_2]$.

Continuing such estimations, for $j = 2, 3, \ldots$, we get

$$\begin{aligned}\frac{d}{dt_+}&\left\|x_\nu^{(j+3)}(t) - x_\nu^{(N-j)}(t)\right\|^2\\ &\le -\delta\nu\left\|x_\nu^{(j+3)}(t) - x_\nu^{(N-j)}(t)\right\|^2\\ &\quad + \nu\left\|x_\nu^{(j+2)}(t) - x_\nu^{(N-j+1)}(t)\right\|^2\\ &\quad + \nu\left\|x_\nu^{(j+4)}(t) - x_\nu^{(N-j-1)}(t)\right\|^2 + \frac{1}{\nu}C_{T_1,T_2}^{j+3,N-j,2-\delta}(\omega)\end{aligned} \tag{51}$$

uniformly for $t \in [T_1, T_2]$.

We can divide the situation into two cases: N is even and N is odd, which is just as the same as [15] did. When N is even, we can rewrite the inequalities in the matrix form

$$\dot{\mathbf{u}}(t) \le \mathbf{H}_\nu \mathbf{u}(t) + \frac{1}{\nu}\mathbf{C} \tag{52}$$

uniformly for $t \in [T_1, T_2]$, where for $t \in \mathbb{R}$,

$$\begin{aligned}\mathbf{u}(t) &= \left(\left\|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\right\|^2, \left\|x_\nu^{(3)}(t) - x_\nu^{(N)}(t)\right\|^2, \ldots,\right.\\ &\qquad \left.\left\|x_\nu^{(N/2+1)}(t) - x_\nu^{(N/2+2)}(t)\right\|^2\right)^{\mathbf{T}},\\ \mathbf{C} &= \left(C_{T_1,T_2}^{1,2,5-\delta}(\omega), C_{T_1,T_2}^{3,N,2-\delta}(\omega), \ldots,\right.\\ &\qquad \left.C_{T_1,T_2}^{N/2,N/2+3,2-\delta}(\omega), C_{T_1,T_2}^{N/2+1,N/2+2,5-\delta}(\omega)\right)^{\mathbf{T}}\end{aligned} \tag{53}$$

are $(N/2)$-dimensional vectors, and

$$\mathbf{H}_\nu = \begin{pmatrix} -\delta\nu & \nu & 0 & \cdots & 0\\ \nu & -\delta\nu & \nu & \ddots & \vdots\\ 0 & \nu & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & -\delta\nu & \nu\\ 0 & \cdots & 0 & \nu & -\delta\nu \end{pmatrix}_{(N/2)\times(N/2)}. \tag{54}$$

By Lemma 2, it follows from (52) that

$$\mathbf{u}(t) \le e^{(t-t_0)\mathbf{H}_\nu}\mathbf{u}(t_0) + \frac{1}{\nu}\int_{t_0}^{t} e^{(t-s)\mathbf{H}_\nu}\mathbf{C}\,ds. \tag{55}$$

By [15, Lemma 4.1] again, $(1/\nu)\mathbf{H}_\nu$ is negative definite, then we have

$$\left\|e^{(t-t_0)\mathbf{H}_\nu}\mathbf{u}(t_0)\right\| \le e^{(t-t_0)\mu_{\max}}\left\|\mathbf{u}(t_0)\right\|, \tag{56}$$

where $\mu_{\max} = -\delta - 2\cos(N\pi/(N+2)) < 0$ is the maximal eigenvalue of $(1/\nu)\mathbf{H}_\nu$. Thus, (55) implies that

$$\begin{gathered}\mathbf{u}(t) \longrightarrow \mathbf{0} \quad \text{as } \nu \longrightarrow \infty,\\ \left\|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\right\|^2 \longrightarrow 0,\\ \left\|x_\nu^{(N/2+1)}(t) - x_\nu^{(N/2+2)}(t)\right\|^2 \longrightarrow 0\end{gathered} \tag{57}$$

uniformly for $t \in [T_1, T_2]$ as $\nu \to \infty$.

Similarly, when N is odd, we can rewrite the inequalities in the matrix form

$$\dot{\mathbf{v}}(t) \le \widetilde{\mathbf{H}}_\nu \mathbf{v}(t) + \frac{1}{\nu}\widetilde{\mathbf{C}} \tag{58}$$

uniformly for $t \in [T_1, T_2]$, where for $t \in \mathbb{R}$,

$$\begin{aligned}\mathbf{v}(t) &= \left(\left\|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\right\|^2, \left\|x_\nu^{(3)}(t) - x_\nu^{(N)}(t)\right\|^2, \ldots,\right.\\ &\qquad \left.\left\|x_\nu^{((N+1)/2)}(t) - x_\nu^{((N+1)/2+2)}(t)\right\|^2\right)^{\mathbf{T}},\\ \widetilde{\mathbf{C}} &= \left(C_{T_1,T_2}^{1,2,5-\delta}(\omega), C_{T_1,T_2}^{3,N,2-\delta}(\omega), \ldots,\right.\\ &\qquad \left.C_{T_1,T_2}^{(N-1)/2,(N+1)/2+3,2-\delta}(\omega), C_{T_1,T_2}^{(N+1)/2,(N+1)/2+2,5-\delta}(\omega)\right)^{\mathbf{T}}\end{aligned} \tag{59}$$

are $((N-1)/2)$-dimensional vectors, and

$$\widetilde{\mathbf{H}}_\nu = \begin{pmatrix} -\delta\nu & \nu & 0 & \cdots & 0\\ \nu & -\delta\nu & \nu & \ddots & \vdots\\ 0 & \nu & \ddots & \ddots & 0\\ \vdots & \ddots & \ddots & -\delta\nu & \nu\\ 0 & \cdots & 0 & \nu & -\delta\nu \end{pmatrix}_{((N-1)/2)\times((N-1)/2)}. \tag{60}$$

By Lemma 2, it follows from (58) that

$$\mathbf{v}(t) \le e^{(t-t_0)\widetilde{\mathbf{H}}_\nu}\mathbf{v}(t_0) + \frac{1}{\nu}\int_{t_0}^{t} e^{(t-s)\widetilde{\mathbf{H}}_\nu}\widetilde{\mathbf{C}}\,ds. \tag{61}$$

Just like the even case, for uniform $t \in [T_1, T_2]$, we have

$$\left\|x_\nu^{(1)}(t) - x_\nu^{(2)}(t)\right\|^2 \longrightarrow 0, \qquad \text{as } \nu \longrightarrow \infty. \tag{62}$$

For other adjacent components, the process above can be repeated. Hence, we can draw a conclusion that the difference between any adjacent components of a solution of the coupled RODEs system (7) tends to zero uniformly for $t \in [T_1, T_2]$ as the coupling coefficient goes to infinity which completes the proof. □

We know that all components of a solution of system (7) have the same limit uniformly for $t \in [T_1, T_2]$ as $\nu \to \infty$. Now, we are in the position to find what they converge to.

Lemma 7. *If assumptions* (2) *and* (6) *hold, the càdlàg random dynamical system $\varphi(t,\omega)$ generated by the solution of the averaged RODEs system*

$$\frac{dZ}{dt_+} = \frac{1}{N}\sum_{j=1}^{N} e^{O_t^{(j)}} f^{(j)}\left(e^{O_t^{(j)}} Z\right) + \frac{1}{N}\sum_{j=1}^{N} O_t^{(j)} Z \tag{63}$$

has a singleton sets random attractor denoted by $\{\overline{Z}(\omega)\}$. Furthermore,

$$\overline{Z}(\theta_t\omega)\exp\left(\frac{1}{N}\sum_{j=1}^{N} O_t^{(j)}(\omega)\right) \tag{64}$$

is the stationary stochastic solution of the equivalently averaged SODE system

$$dz = \frac{1}{N}\sum_{j=1}^{N} e^{-\zeta_t^{(j)}} f^{(j)}\left(e^{\zeta_t^{(j)}} z\right)dt + \frac{1}{N}\sum_{i=1}^{m}\sum_{j=1}^{N} c_i^{(j)} z \diamond dL_t^{(i)}, \tag{65}$$

where $\zeta_t^{(j)} = \sum_{k=1}^{N}(O_t^{(j)} - O_t^{(k)})$, $j = 1, \ldots, N$.

Proof. Assume that $Z_1(t)$ and $Z_2(t)$ are two solutions of (63), we have

$$\begin{aligned} &\frac{d}{dt_+}\left\|Z_1(t) - Z_2(t)\right\|^2 \\ &\quad \le \left(-2L + \frac{2}{N}\sum_{j=1}^{N} O_t^{(j)}\right)\left\|Z_1(t) - Z_2(t)\right\|^2. \end{aligned} \tag{66}$$

It follows from Gronwall's lemma (see [27, Lemma 2.8]) that

$$\begin{aligned} &\left\|Z_1(t) - Z_2(t)\right\|^2 \\ &\quad \le e^{-2t(L-(1/N)\sum_{j=1}^{N}(1/t)\int_0^t O_s^{(j)} ds)}\left\|Z_1(0) - Z_2(0)\right\|^2, \end{aligned} \tag{67}$$

which implies that

$$\lim_{t\to\infty}\left\|Z_1(t) - Z_2(t)\right\|^2 = 0. \tag{68}$$

Then, all solutions of (63) converge pathwise to each other.

Now, we have to give what they converge to based on the theory of càdlàg random dynamical systems. Let $Z(t)$ be a solution of (63), we get

$$\begin{aligned} \frac{d}{dt_+}\|Z(t)\|^2 \le &\left(-2L + \frac{2}{N}\sum_{j=1}^{N} O_t^{(j)}\right)\|Z(t)\|^2 \\ &+ \frac{1}{LN}\sum_{j=1}^{N} e^{-2O_t^{(j)}}\left\|f^{(j)}(0)\right\|^2. \end{aligned} \tag{69}$$

From Gronwall's lemma in [27], again, it yields for $-t_0, t > T_\omega$,

$$\begin{aligned} &\|Z(t)\|^2 \\ &\quad \le e^{-L(t-t_0)+(2/N)\sum_{j=1}^{N}\int_{t_0}^{t} O_s^{(j)} ds}\left\|Z(t_0)\right\|^2 + \frac{1}{LN} \\ &\quad\quad \times \sum_{j=1}^{N}\left\|f^{(j)}(0)\right\|^2 \int_{t_0}^{t} e^{-2O_\tau^{(j)}-L(t-\tau)+(2/N)\sum_{k=1}^{N}\int_\tau^t O_s^{(k)} ds} d\tau. \end{aligned} \tag{70}$$

By pathwise pullback convergence with $t_0 \to -\infty$, the random closed ball centered as the origin with random radius $\widetilde{R}(\omega)$ is a pullback absorbing set of $\varphi(t,\omega)$, where

$$\widetilde{R}^2(\omega) = 1 + \frac{1}{LN}\sum_{j=1}^{N}\left\|f^{(j)}(0)\right\|^2 \int_{-\infty}^{0} e^{L\tau-2O_\tau^{(j)}+(2/N)\sum_{k=1}^{N}\int_\tau^0 O_s^{(k)} ds} d\tau. \tag{71}$$

Obviously, by Lemma 1, the integral defined in the right hand side is well defined.

By Proposition 3, there exists a random attractor $\{\overline{Z}(\omega)\}$ for $\varphi(t,\omega)$. Since all solutions of (63) converge pathwise to each other, the random attractor $\{\overline{Z}(\omega)\}$ is composed of singleton sets.

Note that the averaged RODE (63) is transformed from the averaged SODE (65) by the following transformation:

$$Z(t,\omega) = z \exp\left(-\frac{1}{N}\sum_{j=1}^{N} O_t^{(j)}(\omega)\right), \tag{72}$$

so the pathwise singleton sets attractor $\overline{Z}(\theta_t\omega)\exp((1/N)\sum_{j=1}^{N} O_t^{(j)}(\omega))$ is a stationary solution of the averaged SODE (65) since the Ornstein-Uhlenbeck process is stationary. □

Now, we will present another main solution of this work.

Theorem 8. *Let*

$$\begin{aligned} &\left(\overline{x}_{\nu_n}^{(1)}(t,\omega), \overline{x}_{\nu_n}^{(2)}(t,\omega), \ldots, \overline{x}_{\nu_n}^{(N)}(t,\omega)\right)^{\mathbf{T}} \\ &\quad = \left(\overline{x}_{\nu_n}^{(1)}(\theta_t\omega), \overline{x}_{\nu_n}^{(2)}(\theta_t\omega), \ldots, \overline{x}_{\nu_n}^{(N)}(\theta_t\omega)\right)^{\mathbf{T}} \end{aligned} \tag{73}$$

be the singleton sets random attractor of the càdlàg random dynamical system $\varphi(t,\omega)$ generated by the solution of RODEs system (7), *then*

$$\begin{aligned} &\left(\left(\overline{x}_{\nu_n}^{(1)}(t,\omega), \overline{x}_{\nu_n}^{(2)}(t,\omega), \ldots, \overline{x}_{\nu_n}^{(N)}(t,\omega)\right)^{\mathbf{T}}\right) \\ &\quad \longrightarrow \left(\overline{Z}(t,\omega), \overline{Z}(t,\omega), \ldots, \overline{Z}(t,\omega)\right)^{\mathbf{T}} \end{aligned} \tag{74}$$

pathwise uniformly for t belongs to any bounded time interval $[T_1, T_2]$ for any sequence $\nu_n \to \infty$, where $\overline{Z}(t,\omega) = \overline{Z}(\theta_t\omega)$ is the solution of the averaged RODE (63) *and $\overline{Z}(\omega)$ is the singleton sets random attractor of the càdlàg random dynamical system $\varphi(t,\omega)$ which is generated by the solution of averaged RODE* (63).

Proof. Define

$$\overline{Z}_\nu(\omega) = \frac{1}{N}\sum_{j=1}^{N}\overline{x}_\nu^{(j)}(\omega), \tag{75}$$

where $\{\overline{x}_\nu^{(1)}(\omega), \overline{x}_\nu^{(2)}(\omega), \dots, \overline{x}_\nu^{(N)}(\omega)\}$ is the singleton sets random attractor of the càdlàg RDS generated by RODEs system (7). Thus, $\overline{Z}_\nu(t,\omega) = \overline{Z}_\nu(\theta_t\omega)$ satisfies

$$\frac{d\overline{Z}_\nu(t,\omega)}{dt_+} = \frac{1}{N}\sum_{j=1}^{N}\left(e^{-O_t^{(j)}} f^{(j)}\left(e^{O_t^{(j)}}\overline{x}_\nu^{(j)}(t,\omega)\right) + O_t^{(j)}\overline{x}_\nu^{(j)}(t,\omega)\right). \tag{76}$$

Then, we get

$$\left\|\frac{d\overline{Z}_\nu(t,\omega)}{dt_+}\right\|^2 \le \frac{2}{N}\sum_{j=1}^{N}\left(e^{-2O_t^{(j)}}\left\|f^{(j)}\left(e^{O_t^{(j)}}\overline{x}_\nu^{(j)}(t,\omega)\right)\right\|^2 + \left|O_t^{(j)}\right|^2\left\|\overline{x}_\nu^{(j)}(t,\omega)\right\|^2\right), \tag{77}$$

by the continuous property of the solutions and the fact that these solutions belong to the compact ball $\mathbb{B}_1(\omega)$, it follows that

$$\sup_{t\in[T_1,T_2]}\left\|\frac{d\overline{Z}_\nu(t,\omega)}{dt_+}\right\| \le \left(\frac{2}{N}\sum_{j=1}^{N}\frac{\varrho}{4}\mathbf{C}_{T_1,T_2}^{j,\bullet,\varrho}(\omega)\right)^{1/2} < \infty. \tag{78}$$

By the Ascoli-Arzelà theorem in a Skorohod space of bounded time intveral $D([T_1,T_2],\mathbb{R}^d)$ in [23], there exists a subsequence $\nu_{n_k} \to \infty$ such that $\overline{Z}_{\nu_{n_k}}(t,\omega)$ converges to $\overline{Z}(t,\omega)$ as $n_k \to \infty$.

Since difference between any two components of a solution of the coupled RODEs system (7) tends to zero uniformly for $t \in [T_1,T_2]$ as $\nu \to \infty$, from (75), we have

$$\begin{aligned}\overline{x}_{\nu_{n_k}}^{(j)}(t,\omega) &= \overline{Z}_{\nu_{n_k}}(t,\omega) + \frac{1}{N}\sum_{j'\ne j}\sum_{j''\ne j'}\left(\overline{x}_{\nu_{n_k}}^{(j'')}(t,\omega) - \overline{x}_{\nu_{n_k}}^{(j')}(t,\omega)\right)\\ &\longrightarrow \overline{Z}(t,\omega)\end{aligned} \tag{79}$$

uniformly for $t \in [T_1,T_2]$ as $\nu_{n_k} \to \infty$ for $j = 1,\dots,N$. Furthermore, it follows from (76) that, for $t \ge T_1$,

$$\overline{Z}_\nu(t,\omega) = \overline{Z}_\nu(T_1,\omega) + \frac{1}{N}\sum_{j=1}^{N}\int_{T_1}^{t}\left(e^{-O_s^{(j)}} f^{(j)}\left(e^{O_s^{(j)}}\overline{x}_\nu^{(j)}(s,\omega)\right) + O_s^{(j)}\overline{x}_\nu^{(j)}(s,\omega)\right)ds. \tag{80}$$

Thus,

$$\overline{Z}(t,\omega) = \overline{Z}(T_1,\omega) + \frac{1}{N}\sum_{j=1}^{N}\int_{T_1}^{t}\left(e^{-O_s^{(j)}} f^{(j)}\left(e^{O_s^{(j)}}\overline{Z}(s,\omega)\right) + O_s^{(j)}\overline{Z}(s,\omega)\right)ds, \tag{81}$$

uniformly for $t \in [T_1,T_2]$ as $\nu_{n_k} \to \infty$, which implies that $\overline{Z}_\nu(s,\omega)$ solves RODE (63). Then, we note that all possible sequences of $\overline{Z}_{\nu_{n_k}}(t,\omega)$ converge to the same limit $\overline{Z}(t,\omega)$ uniformly for $t \in [T_1,T_2]$ as $\nu_n \to \infty$. Since the RDS φ generated by the solutions of RODE (63) has a singleton sets random attractor $\{\overline{Z}(\omega)\}$, the stationary stochastic process $\overline{Z}(\theta_t\omega)$ must be equal to $\overline{Z}(t,\omega)$, that is, $\overline{Z}(t,\omega) = \overline{Z}(\theta_t\omega)$, which completes the proof. □

As an obvious result of Theorem 8, we get the following.

Corollary 9.

$$\begin{aligned}&\left(\left(\overline{x}_\nu^{(1)}(t,\omega), \overline{x}_\nu^{(2)}(t,\omega), \dots, \overline{x}_\nu^{(N)}(t,\omega)\right)^{\mathbf{T}}\right)\\ &\quad\longrightarrow \left(\overline{Z}(t,\omega), \overline{Z}(t,\omega), \dots, \overline{Z}(t,\omega)\right)^{\mathbf{T}}\end{aligned} \tag{82}$$

in Skorohod metric pathwise uniformly for $t \in [T_1,T_2]$ as $\nu \to \infty$.

Remark 10. The results in this paper hold just in almost everywhere because $\omega \in \overline{\Omega}$ in Lemma 1, and we still use Ω instead of $\overline{\Omega}$.

Remark 11. Although the same results hold when the systems perturbed by non-Gaussian noises (see, e.g., [17] and this paper for α-stable Lévy noises and [16, 18] for additive Lévy noises), there exists some difference between dealing with such stochastic systems when driven by Bronian motions and Lévy motions. Firstly, to some extent, the cases of Lévy noises have more general sense than the Bronian motions. For example, when $\alpha = 2$, the Lévy noise is the standard Brownian motion and the Marcus integral is reduced to the Stratonovich stochastic integral, that is, the case of multiplicative white noise (see [11, 15]). Here we only are restricted to $1 < \alpha < 2$. Secondly, We need to consider the càdlàg functions in the Skorohod metric, which are different from the continuous cases in the metric under the compact-open topology. Last but not least, we have to consider the solutions in the sense of Carthéodory and the right hand derivatives.

Acknowledgments

The author would like to thank the anonymous referees for their helpful comments and suggestions which largely improved the quality of the paper. This work is partially supported by NSF of China under Grant no. 11071165, Guangxi Provincial Department of Research Project (201010LX166),

and Program to Sponsor Teams for Innovation in the Construction of Talent Highlands in Guangxi Institutions of Higher Learning under Grant no. [2011]47.

References

[1] V. S. Afraimovich, S.-N. Chow, and J. K. Hale, "Synchronization in lattices of coupled oscillators," *Physica D*, vol. 103, no. 1–4, pp. 442–451, 1997.

[2] V. S. Afraimovich and W.-W. Lin, "Synchronization in lattices of coupled oscillators with Neumann/periodic boundary conditions," *Dynamics and Stability of Systems*, vol. 13, no. 3, pp. 237–264, 1998.

[3] V. S. Afraĭmovich, N. N. Verichev, and M. I. Rabinovich, "Stochastic synchronization of oscillations in dissipative systems," *Izvestiya Vysshikh Uchebnykh Zavedeniĭ*, vol. 29, no. 9, pp. 1050–1060, 1986.

[4] S. Strogatz, *Sync: The Emerging Science of Spontaneous Order*, Hyperion Press, 2003.

[5] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization, A Universal Concept in Nonlinear Sciences*, Cambridge University Press, 2001.

[6] L. Glass, "Synchronization and rhythmic processes in physiology," *Nature*, vol. 410, pp. 277–284, 2001.

[7] V. S. Afraimovich and H. M. Rodrigues, "Uniform dissipativeness and synchronization of nonau-tonomous equation," in *Proceedings of the International Conference on Differential Equations*, pp. 3–17, World Scientific, Lisbon, Portugal, 1995.

[8] P. E. Kloeden, "Synchronization of nonautonomous dynamical systems," *Electronic Journal of Differential Equations*, vol. 39, pp. 1–10, 2003.

[9] A. N. Carvalho, H. M. Rodrigues, and T. DThlotko, "Upper semicontinuity of attractors and synchronization," *Journal of Mathematical Analysis and Applications*, vol. 220, no. 1, pp. 13–41, 1998.

[10] H. M. Rodrigues, "Abstract methods for synchronization and applications," *Applicable Analysis*, vol. 62, no. 3-4, pp. 263–296, 1996.

[11] T. Caraballo and P. E. Kloeden, "The persistence of synchronization under environmental noise," *Proceedings of The Royal Society of London A*, vol. 461, no. 2059, pp. 2257–2267, 2005.

[12] T. Caraballo, P. E. Kloeden, and A. Neuenkirch, "Synchronization of systems with multiplicative noise," *Stochastics and Dynamics*, vol. 8, no. 1, pp. 139–154, 2008.

[13] T. Caraballo, I. D. Chueshov, and P. E. Kloeden, "Synchronization of a stochastic reaction-diffusion system on a thin two-layer domain," *SIAM Journal on Mathematical Analysis*, vol. 38, no. 5, pp. 1489–1507, 2007.

[14] I. Chueshov and B. Schmalfuß, "Master-slave synchronization and invariant manifolds for coupled stochastic systems," *Journal of Mathematical Physics*, vol. 51, no. 10, Article ID 102702, 23 pages, 2010.

[15] Z. W. Shen, S. F. Zhou, and X. Y. Han, "Synchronization of coupled stochastic systems with multiplicative noise," *Stochastics and Dynamics*, vol. 10, no. 3, pp. 407–428, 2010.

[16] X. M. Liu, J. Q. Duan, J. C. Liu, and P. E. Kloeden, "Synchronization of dissipative dynamical systems driven by non-Gaussian Lévy noises," *International Journal of Stochastic Analysis*, vol. 2010, Article ID 502803, 13 pages, 2010.

[17] X. M. Liu, J. Q. Duan, J. C. Liu, and P. E. Kloeden, "Synchronization of systems of Marcus canonical equations driven by α-stable noises," *Nonlinear Analysis*, vol. 11, no. 5, pp. 3437–3445, 2010.

[18] A. H. Gu, "Synchronization of coupled stochastic systems driven by non-Gaussian Lévy noises", *Stochastic and Dynamics*, submitted .

[19] D. Applebaum, *Lévy Processes and Stochastic Calculus*, Cambridge University Press, Cambridge, UK, 2004.

[20] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Léevy Processes*, Cambridge University Press, Cambridge, UK, 2007.

[21] K.-I. Sato, *Lévy Processes and Infinitely Divisible Distributions*, vol. 68 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 1999.

[22] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes*, Chapman & Hall, New York, NY, USA, 1994.

[23] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, New York, NY, USA, 1968.

[24] L. Arnold, *Random Dynamical Systems*, Springer Monographs in Mathematics, Springer, 1998.

[25] S. I. Marcus, "Modeling and approximation of stochastic differential equations driven by semimartingales," *Stochastics*, vol. 4, no. 3, pp. 223–245, 1981.

[26] M. Errami, F. Russo, and P. Vallois, "Itô's formula for $C^{1,\lambda}$-functions of a càdlàg process and related calculus," *Probability Theory and Related Fields*, vol. 122, no. 2, pp. 191–221, 2002.

[27] J. C. Robinson, *Infinite-Dimensional Dynamical Systems*, Cambridge University Press, Cambridge, UK, 2001.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 579534, 9 pages
http://dx.doi.org/10.1155/2013/579534

Research Article

Randomized Dividends in a Discrete Insurance Risk Model with Stochastic Premium Income

Wenguang Yu[1,2]

[1] *School of Mathematics, Shandong University, Jinan 250100, China*
[2] *School of Insurance, Shandong University of Finance and Economics, Jinan 250014, China*

Correspondence should be addressed to Wenguang Yu; yuwg@mail.sdu.edu.cn

Received 3 December 2012; Accepted 23 January 2013

Academic Editor: Guangchen Wang

The compound binomial insurance risk model is extended to the case where the premium income process, based on a binomial process, is no longer a constant premium rate of 1 per period and insurer pays a dividend of 1 with a probability q_0 when the surplus is greater than or equal to a nonnegative integer b. The recursion formulas for expected discounted penalty function are derived. As applications, we present the recursion formulas for the ruin probability, the probability function of the surplus prior to the ruin time, and the severity of ruin. Finally, numerical example is also given to illustrate the effect of the related parameters on the ruin probability.

1. Introduction

The classical compound binomial insurance risk model is a discrete time risk process with the following features. The premium received in each period is 1. The number of claims is regulated by a binomial process $N(t)$, $t = 0, 1, 2, \ldots$. In any time period, the probability of a claim is p $(0 < p < 1)$, and the probability of no claim is $q = 1 - p$. We assume that claims occur at the end of the period and denote by $\xi_t = 1$ the event where a claim occurs in the time period $(t-1, t]$ and denote by $\xi_t = 0$ the event where no claim occurs in the time period $(t-1, t]$. Then $N(t) = \sum_{k=1}^{t} \xi_k$ for $t \geq 1$ and $N(0) = 0$. The individual claim amounts $X_1, X_2, X_3, \ldots$ are independent, identically distributed (i.i.d.) positive integer-valued random variables with distribution $F(x) = 1 - \overline{F}(x) = \Pr(X \leq x)$ and probability function $f(x)$ with finite mean μ. They are independent of the binomial process $\{N(t)\}$. For $t = 1, 2, \ldots$, the surplus at time t is

$$U(t) = u + t - \sum_{k=1}^{t} X_k \xi_k, \tag{1}$$

where $U(0) = u > 0$ is the initial surplus. The risk model (1) was first proposed by Gerber [1] and has been further studied by many authors during the last few years. See, for example, Shiu [2], Willmot [3], and Dickson [4].

Recently, Gerber and Shiu [5] introduced a discounted penalty function with respect to the time of ruin, the surplus immediately before ruin, and the deficit at ruin, which has been proven to be a powerful analytical tool. Some related results can be found in Cheng et al. [6], Lin et al. [7], Wang and Wu [8], Wang et al. [9], and Willmot and Wood [10]. For the compound binomial model, Tan and Yang [11] consider the risk model (1) modified by the inclusion of dividends and derive recursion formulas and an asymptotic estimate for the ruin probability, the probability function of the surplus prior to the ruin time, and the severity of ruin, and so forth. Landriault [12] considers the compound binomial model with a multithreshold dividend structure and randomized dividend payments. Using the roots of a generalization of Lundberg's fundamental equation and the general theory of difference equations, he derives an explicit expression for the Gerber-Shiu discounted penalty function with any initial surplus. This result generalizes the main result of Tan and Yang [11]. Fang et al. [13] extend the risk model (1) to the case where the premium income process, based on a binomial process, is no longer a linear function and examine the expected discounted value of a penalty at ruin, which is

considered as a function of the initial surplus. Li [14] studies the moments of the present value of total dividend payments in the compound binomial risk model in the presence of a constant dividend barrier and stochastic interest rates. Some recent developments about compound binomial model can be found in Yuen and Guo [15], Liu and Zhao [16], Yu et al. [17], Hao and Yang [18], Tan and Yang [19], and Bao and Liu [20].

In this paper, motivated by the work of Tan and Yang [11] and Fang et al. [13], we consider the compound binomial model (1) modified by the inclusion of dividends and the stochastic premium income. We assume that the stochastic premiums are received at the beginning of each time period. It is no longer a constant premium rate of 1 per period but another binomial process with parameter p_1 $(0 \le p_1 < 1)$, independent of $\{N(t), t = 0, 1, 2, \ldots\}$ and $\{X_k, k = 1, 2, \ldots\}$. In addition, we suppose that the insurer will pay a dividend of 1 with a probability q_0 $(0 \le q_0 < 1)$ in each time period if the surplus is greater than or equal to a nonnegative integer b at the beginning of the period. It implies that the randomized dividend payments will only possibly occur at the beginning of each period, right after receiving the stochastic premium payment. We will derive the recursion formulas for the expected discounted penalty function, the ruin probability, the probability function of the surplus prior to the ruin time and the severity of ruin.

The rest of the paper proceeds as follows. In Section 2, the risk model is introduced. In Section 3, the recursion formulas for the expected discounted penalty function associated with the time of ruin when the discount factor $v = 1$ are derived. As applications, in Section 4, we give the recursion formulas for the ruin probability, the probability function of the surplus prior to the ruin time, and the severity of ruin. Finally, a numerical example is also given to illustrate the effect of parameters on the ruin probability.

2. The Risk Model

In this section, we consider a discrete time risk process based on the compound binomial model, which is the extension of the work of Tan and Yang [11]. The surplus process is given by

$$U(t) = u + M(t) - Z_t - S_t, \quad t = 0, 1, 2, \ldots, \tag{2}$$

where S_t is defined as $\sum_{k=1}^{t} X_k \xi_k$, for $t \ge 1$ and $S_0 = 0$. The binomial process $\{M(t), t = 0, 1, 2, \ldots\}$ with parameter p_1 is corresponding to the number of the customers up to time t. We denote by $\zeta_k = 1$ the event where a payment occurs in period $(t-1, t]$ denote by and $\zeta_k = 0$ the event where no payment occurs in period $(t-1, t]$. If the event occurs in period $(t-1, t]$, we suppose that the event happen at the beginning of the period. Let $\Pr(\zeta_k = 1) = p_1$, $\Pr(\zeta_k = 0) = q_1 = 1 - p_1$. Z_t is defined as

$$Z_t = \sum_{k=1}^{t} \eta_k I(U(k-1) \ge b), \tag{3}$$

for $t > 1$ and $Z_0 = 0$. b is a fixed nonnegative integer, $I(A)$ is the indicator function of a set A, and η_k $(k \ge 1)$ is a series of randomized decision functions that are mutually independent, identically distributed, and independent of $\sum_{k=1}^{t} X_k \xi_k$. In detail, we denote by $\eta_k = 1$ the event where a dividend of 1 is paid at the time k and denote by $\eta_k = 0$ the event where no dividend is paid at the time k. Assume $\Pr(\eta_k = 1) = q_0$, $\Pr(\eta_k = 0) = p_0 = 1 - q_0$. Then (2) is equivalent to

$$\begin{aligned} U(t) &= u + \sum_{k=1}^{t} \zeta_k - \sum_{k=1}^{t} \eta_k I(U(k-1) \ge b) \\ &\quad - \sum_{k=1}^{N(t)} X_k = u - V(t), \quad t = 0, 1, 2, \ldots. \end{aligned} \tag{4}$$

It is reasonable that the randomized decision functions η_k $(k \ge 1)$ are brought in to decide the periods with a dividend. This can be interpreted as follows: the return on the investment by an insurer in each time period can be regarded as being stochastic, and if the return on the investment in the present period is greater than a given level b, then the insurer will pay a benefit of 1 to the insured. But, in model (2), if the surplus is smaller than b at the beginning of the present period, then a decision for paying a benefit of 1 will be cancelled. Certainly, the randomized decisions can also be related to some other things that occur with probability q_0 and do not occur with probability p_0 in each time period, for example, natural disasters. The model (4) is a sort of generalization of the classic risk model and is exactly the risk model [13] if $q_0 = 0$. Hence our results in this paper include the corresponding results of the risk model [13].

In this paper, we always assume that the positive security loading condition holds; that is, if we denote by θ the relative security loading, then

$$\theta = \frac{p_1 - q_0}{p\mu} - 1. \tag{5}$$

Let $p(k) = \Pr(X = k)$, $k = 1, 2, \ldots$, be the common probability function of the claim amounts. (The value of $p(k)$ is zero if k is not a positive integer.)

Let

$$\begin{gathered} p(0) = 0, \qquad P(n) = \sum_{k=1}^{n} p(k), \quad \forall n \ge 1, \\ \overline{P}(n) = 1 - P(n), \quad \forall n \ge 0, \qquad P(0) = 0, \\ \mu = E[X] = \sum_{n=0}^{\infty} np(n) = \sum_{n=0}^{\infty} \overline{P}(n). \end{gathered} \tag{6}$$

Define

$$T = \inf\{t \ge 1 : U(t) < 0\} \quad (\inf \emptyset = \infty) \tag{7}$$

to be the time of ruin. Define the ultimate ruin probability

$$\psi(u) = \Pr(T < \infty \mid U(0) = u). \tag{8}$$

Except for the ruin probability, other important ruin quantities in ruin theory include the generating function of

the time of ruin, $E(r^T)$; the surplus before ruin, denoted by U_{T-1}; and the deficit at ruin, $|U_T|$. A unified method to study these ruin quantities is to consider the (expected discounted) penalty function associated with the time of ruin by defining

$$m_v(u) = E\left[v^T w\left(U_{T-1}, |U_T|\right) I(T < \infty) \mid U(0) = u\right], \quad (9)$$

where $w(x, y)$, $x \geq 0$, $y \geq 0$, is a nonnegative bounded function; $0 < v \leq 1$ is the discount factor.

In this paper, we only obtain the recursion formula for $m_1(u)$, that is, the discount factor $v = 1$. However, this is enough for us to study some important ruin quantities except for $E(r^T)$. Let $m_1(u) = m(u)$. In this paper, we adopt the convention that $\sum_a^d = 0$ when $d < a$.

3. Recursive Formulas for the Expected Discounted Penalty Function

In this section, we derive the renewal equations for the expected discounted penalty function $m(u)$.

Theorem 1. *(i) For all $u \geq b$, the penalty function $m(u)$ satisfies*

$$\begin{aligned} m(u+1) &= \frac{1 - qq_1p_0 - qp_1q_0}{qp_1p_0} m(u) - \frac{q_1q_0}{p_1p_0} m(u-1) \\ &\quad - \frac{pq_1}{qp_1p_0} \sum_{k=0}^{u} m(k)\left[p_0 p(u-k) + q_0 p(u-1-k)\right] \\ &\quad - \frac{pq_1}{qp_1p_0} \sum_{k=u+1}^{\infty} w(u, k-u)\left[p_0 p(k) + q_0 p(k-1)\right] \\ &\quad - \frac{p}{qp_0} \sum_{k=0}^{u} m(k)\left[p_0 p(u+1-k) + q_0 p(u-k)\right] \\ &\quad - \frac{p}{qp_0} \sum_{k=u+1}^{\infty} w(u, k-u)\left[p_0 p(k+1) + q_0 p(k)\right], \end{aligned} \quad (10)$$

$$\begin{aligned} &qp_1p_0 m(u+1) - qq_1q_0 m(u) \\ &\quad = pq_1 \sum_{k=0}^{u} m(k)\left[p_0 \overline{P}(u-k) + q_0 \overline{P}(u-1-k)\right] \\ &\quad + pp_1 \sum_{k=0}^{u} m(k)\left[p_0 \overline{P}(u+1-k) + q_0 \overline{P}(u-k)\right] \\ &\quad + pq_1 \sum_{k=u+1}^{\infty} \sum_{i=k+1}^{\infty} w(k, i-k)\left[p_0 p(i) + q_0 p(i-1)\right] \\ &\quad + pp_1 \sum_{k=u+1}^{\infty} \sum_{i=k+1}^{\infty} w(k, i-k)\left[p_0 p(i+1) + q_0 p(i)\right]. \end{aligned} \quad (11)$$

(ii) If $b \geq 1$, then $m(0), m(1), \ldots, m(b)$ satisfy the following linear equations:

$$p_1 q m(0) - q_0 m(b-1) = \delta, \quad (12)$$

$$\begin{aligned} &qp_1 m(u+1) + \left(qq_1 + pp_1 p(1) - 1\right) m(u) \\ &\quad + pq_1 \sum_{k=0}^{u-1} m(k)\, p(u-k) \\ &\quad + pp_1 \sum_{k=0}^{u-1} m(k)\, p(u+1-k) \\ &\quad = f(u+1), \quad (u = 0, 1, 2, \ldots, b-1), \end{aligned} \quad (13)$$

where

$$\begin{aligned} f(u+1) = &- pq_1 \sum_{k=u+1}^{\infty} w(u, k-u)\, p(k) \\ &- pp_1 \sum_{k=u+2}^{\infty} w(u, k-u-1)\, p(k), \end{aligned}$$

$$\begin{aligned} \delta = &\; pp_0 \sum_{k=0}^{b-1} \sum_{i=k+1}^{\infty} w(k, i-k)\left[p_1 p(i+1) + q_1 p(i)\right] \\ &+ pq_0 \sum_{k=0}^{b-2} \sum_{i=k+1}^{\infty} w(k, i-k)\left[p_1 p(i+1) + q_1 p(i)\right] \\ &+ pq_1 \sum_{k=b}^{\infty} \sum_{i=k+1}^{\infty} w(k, i-k)\left[p_0 p(i) + q_0 p(i-1)\right] \\ &+ pp_1 \sum_{k=b}^{\infty} \sum_{i=k+1}^{\infty} w(k, i-k)\left[p_0 p(i+1) + q_0 p(i)\right]. \end{aligned} \quad (14)$$

Proof. We consider $U(t)$ in the first time period $(0, 1]$ and separate the eight possible cases as follows.

(1) No claim occurs in $(0, 1]$, no premium arrives in $(0, 1]$ and paying no dividend in $(0, 1]$.

(2) No claim occurs in $(0, 1]$, no premium arrives in $(0, 1]$ and paying a dividend of 1 in $(0, 1]$ (if $u < b$, this case does not exist).

(3) No claim occurs in $(0, 1]$, a premium arrives in $(0, 1]$ and paying no dividend in $(0, 1]$.

(4) A claim occurs in $(0, 1]$, no premium arrives in $(0, 1]$ and paying no dividend in $(0, 1]$.

(5) A claim occurs in $(0, 1]$, a premium arrives in $(0, 1]$ and paying no dividend in $(0, 1]$.

(6) A claim occurs in $(0, 1]$, no premium arrives in $(0, 1]$ and paying a dividend of 1 in $(0, 1]$ (if $u < b$, this case does not exist).

(7) No claim occurs in $(0, 1]$, a premium arrives in $(0, 1]$ and paying a dividend of 1 in $(0, 1]$ (if $u < b$, this case does not exist).

(8) A claim occurs in $(0,1]$, a premium arrives in $(0,1]$ and paying a dividend of 1 in $(0,1]$ (if $u<b$, this case does not exist).

Considering whether $u<b$ or $u\geq b$ and using the laws of conditional probability, the penalty function is equal to

$$\begin{aligned} m(u) = {} & qq_1 m(u) + qp_1 m(u+1) \\ & + pq_1 \sum_{k=1}^{u} m(u-k)\,p(k) \\ & + pq_1 \sum_{k=u+1}^{\infty} w(u,k-u)\,p(k) \\ & + pp_1 \sum_{k=1}^{u+1} m(u+1-k)\,p(k) \\ & + pp_1 \sum_{k=u+2}^{\infty} w(u,k-u-1)\,p(k), \\ & \qquad \forall 0\leq u<b, \end{aligned} \tag{15}$$

$$\begin{aligned} m(u) = {} & qq_1 p_0 m(u) + qq_1 q_0 m(u-1) \\ & + qp_1 p_0 m(u+1) + qp_1 q_0 m(u) \\ & + pq_1 p_0 \sum_{k=1}^{u} m(u-k)\,p(k) \\ & + pq_1 p_0 \sum_{k=u+1}^{\infty} w(u,k-u)\,p(k) \\ & + pp_1 p_0 \sum_{k=1}^{u+1} m(u+1-k)\,p(k) \\ & + pp_1 p_0 \sum_{k=u+2}^{\infty} w(u,k-u-1)\,p(k) \\ & + pq_1 q_0 \sum_{k=1}^{u-1} m(u-1-k)\,p(k) \\ & + pq_1 q_0 \sum_{k=u}^{\infty} w(u,k-u+1)\,p(k) \\ & + pp_1 q_0 \sum_{k=1}^{u} m(u-k)\,p(k) \\ & + pp_1 q_0 \sum_{k=u+1}^{\infty} w(u,k-u)\,p(k), \\ & \qquad \forall u\geq b. \end{aligned} \tag{16}$$

Obviously, (16) leads to (10).

When $t\geq b$, summing (16) over u from b to t, we obtain

$$\begin{aligned} & qp_1 p_0 [m(t+1) - m(b)] - qq_1 q_0 [m(t) - m(b-1)] \\ & \quad = pq_1 \sum_{k=0}^{t} m(k) \left[p_0 \overline{P}(t-k) + q_0 \overline{P}(t-1-k)\right] \\ & \qquad - pq_1 \sum_{k=0}^{b-1} m(k) \left[p_0 \overline{P}(x-1-k) + q_0 \overline{P}(x-k-2)\right] \\ & \qquad - pq_1 \sum_{k=b}^{t} \sum_{i=k+1}^{\infty} w(k,i-k) \left[p_0 p(i) + q_0 p(i-1)\right] \\ & \qquad + pp_1 \sum_{k=0}^{t} m(k) \left[p_0 \overline{P}(t+1-k) + q_0 \overline{P}(t-k)\right] \\ & \qquad - pp_1 \sum_{k=0}^{b-1} m(k) \left[p_0 \overline{P}(x-k) + q_0 \overline{P}(x-1-k)\right] \\ & \qquad - pp_1 \sum_{k=b}^{t} \sum_{i=k+1}^{\infty} w(k,i-k) \left[p_0 p(i+1) + q_0 p(i)\right], \\ & \qquad \forall t\geq b. \end{aligned} \tag{17}$$

Note that $w(x,y)$ is defined as a bounded function. We define $\|w\| = \sup\{w(x,y)\}$ again; then $\|w\|<\infty$. According to the definition of $m(u)$, we know that

$$|m(u)| \leq \|w\| \Pr(T<\infty \mid U(0)=u) = \|w\|\,\psi(u). \tag{18}$$

Because the relative security loading $\theta > 0$ and $\lim_{u\to\infty}\psi(u)=0$, thus

$$\lim_{u\to\infty} m(u) = 0. \tag{19}$$

By the dominated convergence theorem and (19), and letting $t\to\infty$ in (17), we get

$$\begin{aligned} & -qp_1 p_0 m(b) + qq_1 q_0 m(b-1) \\ & \quad = -pq_1 \sum_{k=0}^{b-1} m(k) \left[p_0 \overline{P}(x-1-k) + q_0 \overline{P}(x-k-2)\right] \\ & \qquad - pq_1 \sum_{k=b}^{\infty} \sum_{i=k+1}^{\infty} w(k,i-k) \left[p_0 p(i) + q_0 p(i-1)\right] \\ & \qquad - pp_1 \sum_{k=0}^{b-1} m(k) \left[p_0 \overline{P}(x-k) + q_0 \overline{P}(x-1-k)\right] \\ & \qquad - pp_1 \sum_{k=b}^{\infty} \sum_{i=k+1}^{\infty} w(k,i-k) \left[p_0 p(i+1) + q_0 p(i)\right]. \end{aligned} \tag{20}$$

Plugging (20) into (17) and rearranging terms we obtain (11).

If $b \geq 1$ and $t < b$, summing (15) over u from 0 to t, we get

$$\begin{aligned} p_1 q\left[m(t+1)-m(0)\right] &= p\sum_{k=0}^{t} m(k)\left[p_1\overline{P}(t+1-k)+q_1\overline{P}(t-k)\right] \\ &\quad - p\sum_{k=0}^{t}\sum_{i=k+1}^{\infty} w(k,i-k)\left[p_1 p(i+1)+q_1 p(i)\right], \\ &\qquad \forall 0 \leq t < b. \end{aligned} \tag{21}$$

Replacing t by $t-1$ in (21) and considering $\sum_a^d = 0$ when $d < a$ yield

$$\begin{aligned} m(t)-q_1 q m(t)-p_1 q m(0) &= p\sum_{k=0}^{t} m(k)\left[p_1\overline{P}(t-k)+q_1\overline{P}(t-1-k)\right] \\ &\quad - p\sum_{k=0}^{t-1}\sum_{i=k+1}^{\infty} w(k,i-k)\left[p_1 p(i+1)+q_1 p(i)\right], \\ &\qquad \forall 0 \leq t < b. \end{aligned} \tag{22}$$

Multiplying (21) by p_0, multiplying (22) by q_0, and summing yield

$$\begin{aligned} p_0 p_1 q m(t+1)+q_0\left(1-q_1 q\right)m(t)-p_1 q m(0) &= p p_0\sum_{k=0}^{t} m(k)\left[p_1\overline{P}(t+1-k)+q_1\overline{P}(t-k)\right] \\ &\quad - p p_0\sum_{k=0}^{t}\sum_{i=k+1}^{\infty} w(k,i-k)\left[p_1 p(i+1)+q_1 p(i)\right] \\ &\quad + p q_0\sum_{k=0}^{t} m(k)\left[p_1\overline{P}(t-k)+q_1\overline{P}(t-1-k)\right] \\ &\quad - p q_0\sum_{k=0}^{t-1}\sum_{i=k+1}^{\infty} w(k,i-k)\left[p_1 p(i+1)+q_1 p(i)\right], \\ &\qquad \forall 0 \leq t < b. \end{aligned} \tag{23}$$

If $t = b-1$, from (23) and (20) we get (12). Equation (13) comes directly from (15). □

4. Application

4.1. Some Ruin Quantities. In this section, we give some examples of ruin quantities to illustrate applications of the recursive formulas derived in Section 3.

Example 2. Letting $w(x,y)=1$ and $v=1$, we have $m(u)=E[I(T<\infty)]=\psi(u)$. In this case, $f(u+1)=-p[q_1\overline{P}(u)+p_1\overline{P}(u+1)]$, $\delta = p(\mu - p_1)$. Thus, by Theorem 1, $\psi(u)$ satisfies

$$\begin{aligned} \psi(u+1) &= \frac{1-qq_1p_0-qp_1q_0}{qp_1p_0}\psi(u) \\ &\quad - \frac{q_1q_0}{p_1p_0}\psi(u-1) \\ &\quad - \frac{pq_1}{qp_1p_0}\sum_{k=0}^{u-1}\psi(k)\left[p_0p(u-k)+q_0p(u-1-k)\right] \\ &\quad - \frac{pq_1}{qp_1p_0}\left[p_0\overline{P}(u)+q_0\overline{P}(u-1)\right] \\ &\quad - \frac{p}{qp_0}\sum_{k=0}^{u}\psi(k)\left[p_0p(u+1-k)+q_0p(u-k)\right] \\ &\quad - \frac{p}{qp_0}\left[p_0\overline{P}(u+1)+q_0\overline{P}(u)\right] \quad \forall u \geq b, \end{aligned} \tag{24}$$

which can be rewritten as

$$\begin{aligned} \psi(u+1) &= \frac{q_0q_1}{p_0p_1}\psi(u) \\ &\quad + \frac{pq_1}{p_0p_1q}\sum_{k=0}^{u}\psi(k)\left[p_0\overline{P}(u-k)+q_0\overline{P}(u-1-k)\right] \\ &\quad + \frac{p}{p_0q}\sum_{k=0}^{u}\psi(k)\left[p_0\overline{P}(u+1-k)+q_0\overline{P}(u-k)\right] \\ &\quad + \frac{pq_1}{p_0p_1q}\sum_{k=u+1}^{\infty}\left[p_0\overline{P}(k)+q_0\overline{P}(k-1)\right] \\ &\quad + \frac{p}{p_0q}\sum_{k=u+1}^{\infty}\left[p_0\overline{P}(k+1)+q_0\overline{P}(k)\right], \quad \forall u \geq b, \end{aligned} \tag{25}$$

and the following linear equations: when $b=0$

$$\psi(0) = \frac{p}{p_0p_1q}\left(\mu - p_0p_1\right), \tag{26}$$

or, when $b \geq 1$,

$$p_1q\psi(0)-q_0\psi(b-1) = p\left(\mu - p_1\right), \tag{27}$$

$$\begin{aligned} qp_1\psi(u+1)+\left(qq_1+pp_1p(1)-1\right)\psi(u) &\\ + pq_1\sum_{k=0}^{u-1}\psi(k)\,p(u-k) &\\ + pp_1\sum_{k=0}^{u-1}\psi(k)\,p(u+1-k) &\\ = f(u+1), \quad (u=0,1,2,\ldots,b-1). & \end{aligned} \tag{28}$$

Example 3. Letting $w(x_1, x_2) = I(x_2 \le y)$ $(y = 1, 2, \ldots)$ and $v = 1$, then

$$m(u) = \Pr(|U_T| \le y, T < \infty) = H(u, y) \tag{29}$$

is the distribution function of the deficit at ruin. In this case,

$$\begin{aligned} f(u+1) = &-pq_1 [P(y+u) - P(u)] \\ &- pp_1 [P(y+u+1) - P(u+1)], \\ \delta = &\; p\sum_{k=1}^{y} \overline{P}(k) - pq_1 \overline{P}(y). \end{aligned} \tag{30}$$

Thus, $H(u, y)$ satisfies the following recursive formula ($\forall u \ge b$):

$$\begin{aligned} &H(u+1, y) \\ &= \frac{1 - qq_1p_0 - qp_1q_0}{qp_1p_0} H(u, y) \\ &- \frac{q_1q_0}{p_1p_0} H(u-1, y) \\ &- \frac{pq_1}{qp_1p_0} \sum_{k=0}^{u-1} H(k, y) [p_0 p(u-k) + q_0 p(u-1-k)] \\ &- \frac{pq_1}{qp_1p_0} \sum_{k=1}^{y} [p_0 p(u+k) + q_0 p(u+k-1)] \\ &- \frac{p}{qp_0} \sum_{k=0}^{u} H(k, y) [p_0 p(u+1-k) + q_0 p(u-k)] \\ &- \frac{p}{qp_0} \sum_{k=1}^{y} [p_0 p(u+k+1) + q_0 p(u+k)], \end{aligned} \tag{31}$$

when $b \ge 1$,

$$\begin{aligned} &p_1 q H(0, y) - q_0 H(b-1, y) = p\sum_{k=1}^{y} \overline{P}(k) - pq_1 \overline{P}(y), \\ &qp_1 H(u+1, y) + (qq_1 + pp_1 p(1) - 1) H(u, y) \\ &\quad + pq_1 \sum_{k=0}^{u-1} H(k, y) p(u-k) \\ &\quad + pp_1 \sum_{k=0}^{u-1} H(k, y) p(u+1-k) \\ &= f(u+1), \quad (u = 0, 1, 2, \ldots, b-1). \end{aligned} \tag{32}$$

Example 4. Letting $w(x_1, x_2) = r^{x_2}$ $(0 \le r < 1)$ and $v = 1$, then

$$m(u) = E\left[r^{|U_T|} I(T < \infty)\right] = \widehat{H}(u, r) \tag{33}$$

is the generating function of the deficit at ruin. In this case,

$$f(u+1) = -pq_1 \sum_{k=1}^{\infty} r^k p(u+k) - pp_1 \sum_{k=1}^{\infty} r^k p(u+k+1). \tag{34}$$

For all $u \ge b$, we have

$$\begin{aligned} &\widehat{H}(u+1, r) \\ &= \frac{1 - qq_1p_0 - qp_1q_0}{qp_1p_0} \widehat{H}(u, r) \\ &- \frac{q_1q_0}{p_1p_0} \widehat{H}(u-1, r) \\ &- \frac{pq_1}{qp_1p_0} \sum_{k=0}^{u-1} \widehat{H}(k, r) [p_0 p(u-k) + q_0 p(u-1-k)] \\ &- \frac{pq_1}{qp_1p_0} \sum_{k=1}^{\infty} [p_0 p(u+k) + q_0 p(u+k-1)] \\ &- \frac{p}{qp_0} \sum_{k=0}^{u} \widehat{H}(k, r) [p_0 p(u+1-k) + q_0 p(u-k)] \\ &- \frac{p}{qp_0} \sum_{k=1}^{\infty} [p_0 p(u+k+1) + q_0 p(u+k)]. \end{aligned} \tag{35}$$

When $b \ge 1$,

$$\begin{aligned} &p_1 q \widehat{H}(0, r) - q_0 \widehat{H}(b-1, r) = \frac{rp - G_X(r)(pp_1 + rpq_1)}{1 - r}, \\ &qp_1 \widehat{H}(u+1, r) + (qq_1 + pp_1 p(1) - 1) \widehat{H}(u, r) \\ &\quad + pq_1 \sum_{k=0}^{u-1} \widehat{H}(k, r) p(u-k) \\ &\quad + pp_1 \sum_{k=0}^{u-1} \widehat{H}(k, r) p(u+1-k) \\ &= f(u+1), \quad (u = 0, 1, 2, \ldots, b-1). \end{aligned} \tag{36}$$

4.2. Numerical Illustration. In the following example, we investigate the impact of the inclusion of the dividend payments, claim, and stochastic premium income on ruin probability. We suppose that the claim amounts X_i have a zero-truncated geometric distribution with mean $\mu = 6$ and the probability function $p(k) = 1/6 \times (1 - 1/6)^{k-1}, i = 1, 2, 3, \ldots$. Let $b = 5$. We aim to compute the ruin probabilities with different values of the q_0, p_1, and p. The results are given in Tables 1 and 2.

As expected, the tables show that ruin probability increases as q_0 and p increase while ruin probability decreases as p_1 increases. It is worth mentioning that when the initial surplus u is large enough (e.g., $u = 100$), the difference of ruin probability is very little. This illustrates the importance of enhancing the initial capital. Furthermore, in order to more vividly reflect the influence of the parameters on ruin probability we provide Figures 1, 2, and 3, which show the impact of p_1, q_0 and p on ruin probability, respectively.

Table 1: Exact values for $\psi(u)$.

u	$p = 0.08$, $q_0 = 0$, $p_1 = 0.9$	$q_0 = 0$, $p_1 = 0.95$	$q_0 = 0.1$, $p_1 = 0.9$	$q_0 = 0.1$, $p_1 = 0.95$	$q_0 = 0.15$, $p_1 = 0.9$	$q_0 = 0.15$, $p_1 = 0.95$
0	0.4928	0.4622	0.5423	0.5047	0.5746	0.5319
1	0.4511	0.4208	0.5047	0.4666	0.5396	0.4958
2	0.4130	0.3831	0.4703	0.4319	0.5076	0.4630
3	0.3781	0.3488	0.4388	0.4002	0.4784	0.4331
4	0.3461	0.3175	0.4099	0.3714	0.4515	0.4059
5	0.3168	0.2891	0.3835	0.3452	0.4270	0.3812
6	0.2900	0.2632	0.3558	0.3180	0.3993	0.3537
7	0.2655	0.2396	0.3300	0.2929	0.3733	0.3281
8	0.2430	0.2182	0.3062	0.2699	0.3490	0.3043
9	0.2225	0.1986	0.2840	0.2486	0.3263	0.2823
10	0.2037	0.1808	0.2634	0.2290	0.3051	0.2619
20	0.0842	0.0708	0.1243	0.1007	0.1557	0.1236
30	0.0347	0.0277	0.0596	0.0443	0.0795	0.0584
40	0.0143	0.0109	0.0276	0.0195	0.0406	0.0276
50	0.0059	0.0043	0.0130	0.0085	0.0207	0.0131
60	0.0024	0.0018	0.0061	0.0037	0.0106	0.0062
70	0.0009	0.0007	0.0029	0.0016	0.0054	0.0030
80	0.0003	0.0004	0.0023	0.0007	0.0028	0.0015
90	0.0001	0.0002	0.0006	0.0003	0.0014	0.0008
100	0.0000	0.0001	0.0003	0.0001	0.0008	0.0005

Table 2: Exact values for $\psi(u)$.

u	$p = 0.095$, $q_0 = 0$, $p_1 = 0.9$	$q_0 = 0$, $p_1 = 0.95$	$q_0 = 0.1$, $p_1 = 0.9$	$q_0 = 0.1$, $p_1 = 0.95$	$q_0 = 0.15$, $p_1 = 0.9$	$q_0 = 0.15$, $p_1 = 0.95$
0	0.5948	0.5580	0.6611	0.6146	0.7053	0.6514
1	0.5547	0.5169	0.6275	0.5787	0.6761	0.6190
2	0.5172	0.4788	0.5962	0.5455	0.6488	0.5890
3	0.4823	0.4436	0.5670	0.5148	0.6234	0.5612
4	0.4497	0.4109	0.5397	0.4863	0.5997	0.5354
5	0.4194	0.3806	0.5143	0.4599	0.5777	0.5115
6	0.3911	0.3526	0.4871	0.4320	0.5523	0.4845
7	0.3647	0.3266	0.4614	0.4059	0.5280	0.4589
8	0.3401	0.3025	0.4370	0.3831	0.5047	0.4346
9	0.3171	0.2802	0.4139	0.3581	0.4825	0.4116
10	0.2957	0.2596	0.3920	0.3364	0.4613	0.3898
20	0.1470	0.1208	0.2277	0.1801	0.2940	0.2264
30	0.0731	0.0562	0.1323	0.0964	0.1874	0.1315
40	0.0364	0.0261	0.0768	0.0516	0.1195	0.0763
50	0.0181	0.0121	0.0446	0.0276	0.0762	0.0443
60	0.0091	0.0056	0.0259	0.0148	0.0487	0.0257
70	0.0046	0.0026	0.0150	0.0080	0.0311	0.0149
80	0.0023	0.0012	0.0087	0.0043	0.0199	0.0086
90	0.0012	0.0005	0.0050	0.0023	0.0128	0.0050
100	0.0007	0.0003	0.0028	0.0012	0.0082	0.0028

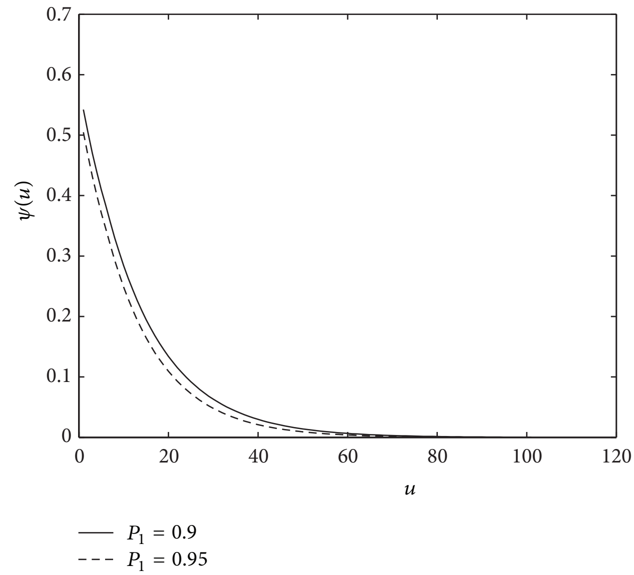


Figure 1: The impact of p_1 on ruin probability.

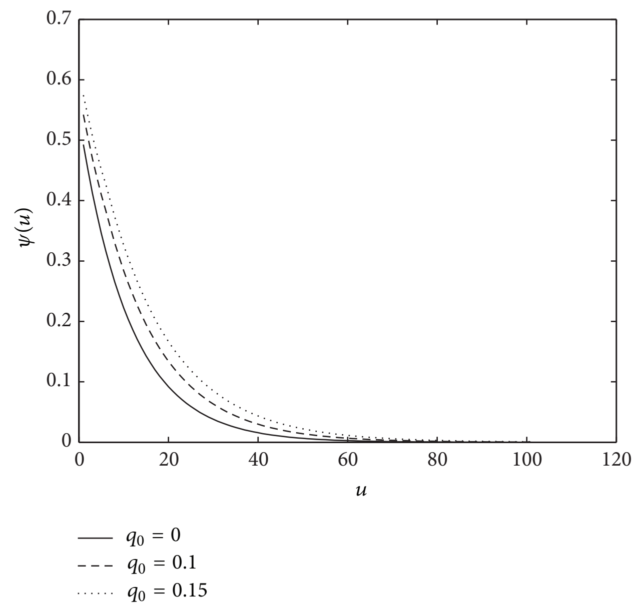


Figure 2: The impact of q_0 on ruin probability.

Acknowledgments

The author thanks the three anonymous referees for the thoughtful comments and suggestions that greatly improved the presentation of this paper. This work was supported by National Natural Science Foundation of China (Grant no. 11171187 and Grant no. 10921101), National Basic Research Program of China (973 Program, Grant no. 2007CB814906), Natural Science Foundation of Shandong Province (Grant no. ZR2012AQ013 and Grant no. ZR2010GL013), and Humanities and Social Sciences Project of the Ministry Education of China (Grant no. 10YJC630092 and Grant no. 09YJC910004).

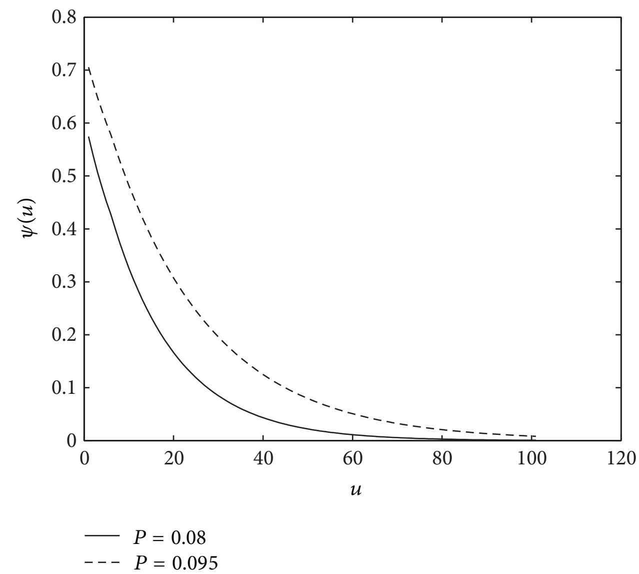


Figure 3: The impact of p on ruin probability.

References

[1] H. U. Gerber, "Mathematical fun with the compound binomial process," *Astin Bulletin*, vol. 18, no. 2, pp. 161–168, 1988.

[2] E. S. W. Shiu, "The probability of eventual ruin in the compound binomial model," *Astin Bulletin*, vol. 19, no. 2, pp. 179–190, 1989.

[3] G. E. Willmot, "Ruin probabilities in the compound binomial model," *Insurance: Mathematics & Economics*, vol. 12, no. 2, pp. 133–142, 1993.

[4] D. C. M. Dickson, "Some comments on the compound binomial model," *Astin Bulletin*, vol. 24, no. 1, pp. 33–45, 1994.

[5] H. U. Gerber and E. S. W. Shiu, "On the time value of ruin," *North American Actuarial Journal*, vol. 2, no. 1, pp. 48–78, 1998.

[6] S. Cheng, H. U. Gerber, and E. S. W. Shiu, "Discounted probabilities and ruin theory in the compound binomial model," *Insurance: Mathematics & Economics*, vol. 26, no. 2-3, pp. 239–250, 2000.

[7] X. S. Lin, G. E. Willmot, and S. Drekic, "The classical risk model with a constant dividend barrier: analysis of the Gerber-Shiu discounted penalty function," *Insurance: Mathematics & Economics*, vol. 33, no. 3, pp. 551–566, 2003.

[8] G. Wang and R. Wu, "The expected discounted penalty function for the perturbed compound Poisson risk process with constant interest," *Insurance: Mathematics & Economics*, vol. 42, no. 1, pp. 59–64, 2008.

[9] W. Wang, R. Ming, and Y. Hu, "On the expected discounted penalty function for risk process with tax," *Statistics & Probability Letters*, vol. 81, no. 4, pp. 489–501, 2011.

[10] G. E. Willmot and J.-K. Woo, "On the analysis of a general class of dependent risk processes," *Insurance: Mathematics & Economics*, vol. 51, no. 1, pp. 134–141, 2012.

[11] J. Y. Tan and X. Q. Yang, "The compound binomial model with randomized decisions on paying dividends," *Insurance: Mathematics & Economics*, vol. 39, no. 1, pp. 1–18, 2006.

[12] D. Landriault, "Randomized dividends in the compound binomial model with a general premium rate," *Scandinavian Actuarial Journal*, no. 1, pp. 1–15, 2008.

[13] S.-Z. Fang, P.-C. Zhao, and C.-M. Zhang, "The expected discounted penalty function at ruin of the discrete risk model with random income," *Mathematica Applicata*, vol. 21, no. 4, pp. 771–777, 2008.

[14] S. M. Li, "The moments of the present value of total dividends in the compound binomial model under a constant dividend barrier and stochastic interest rates," *Australian Actuarial Journal*, vol. 14, no. 2, pp. 175–192, 2008.

[15] K.-C. Yuen and J. Guo, "Some results on the compound Markov binomial model," *Scandinavian Actuarial Journal*, no. 3, pp. 129–140, 2006.

[16] G. Liu and J. Zhao, "Joint distributions of some actuarial random vectors in the compound binomial model," *Insurance: Mathematics & Economics*, vol. 40, no. 1, pp. 95–103, 2007.

[17] Y.-B. Yu, L.-X. Zhang, and Y. Zhang, "Joint and supremum distributions in the compound binomial model with Markovian environment," *Applied Mathematics*, vol. 26, no. 3, pp. 265–279, 2011.

[18] Y. Y. Hao and H. Yang, "On a compound Poisson risk model with delayed claims and random incomes," *Applied Mathematics and Computation*, vol. 217, no. 24, pp. 10195–10204, 2011.

[19] J. Y. Tan and X. Q. Yang, "The compound binomial model with a constant dividend barrier and periodically paid dividends," *Journal of Systems Science & Complexity*, vol. 25, no. 1, pp. 167–177, 2012.

[20] Z. H. Bao and H. Liu, "The compound binomial risk model with delayed claims and random income," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 1315–1323, 2012.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 958920, 14 pages
http://dx.doi.org/10.1155/2013/958920

Research Article

The Dynamic Programming Method of Stochastic Differential Game for Functional Forward-Backward Stochastic System

Shaolin Ji,[1] Chuanfeng Sun,[2] and Qingmeng Wei[2]

[1] *Institute for Financial Studies and Institute of Mathematics, Shandong University, Jinan 250100, China*
[2] *Institute of Mathematics, Shandong University, Jinan 250100, China*

Correspondence should be addressed to Shaolin Ji; jsl@sdu.edu.cn

Received 23 October 2012; Accepted 18 December 2012

Academic Editor: Guangchen Wang

This paper is devoted to a stochastic differential game (SDG) of decoupled functional forward-backward stochastic differential equation (FBSDE). For our SDG, the associated upper and lower value functions of the SDG are defined through the solution of controlled functional backward stochastic differential equations (BSDEs). Applying the Girsanov transformation method introduced by Buckdahn and Li (2008) , the upper and the lower value functions are shown to be deterministic. We also generalize the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equations to the path-dependent ones. By establishing the dynamic programming principal (DPP), we derive that the upper and the lower value functions are the viscosity solutions of the corresponding upper and the lower path-dependent HJBI equations, respectively.

1. Introduction

The theory of backward stochastic differential equations (BSDEs) has been studied widely since Pardoux and Peng [1] first introduced the nonlinear BSDEs in 1990. BSDEs have got applications in many fields, such as, stochastic control (see Peng [2]), stochastic differential games (SDGs) (see Hamadene and Lepeltier [3], Hamadene et al. [4]), mathematical finance (see El Karoui et al. [5]) and partial differential equation (PDE) theory (see Peng [6, 7]), and so forth.

In the aspect of finance, the BSDE theory presents a simple formulation of stochastic differential utilities introduced by Duffie and Epstein [8]. When the generator g of a BSDE does not depend on z, the solution Y is just the comparison theorem for BSDEs, we know

recursive utility presented in [8]. From the view of BSDE, by studying some important properties (such as, comparison theorem) of BSDEs, El Karoui et al. [5] gave the more general class of recursive utilities and their properties. And later the recursive optimal control problems whose cost functionals are described by the solution of BSDE are studied widely. Peng [7] obtained the Bellman's dynamic programming principle (DPP) for this kind of problem and proved the value function to be a viscosity solution of one kind of quasi-linear second order PDE, that is, Hamilton-Jacobi-Bellman (HJB) equation. Later, for the recursive optimal control problem introduced by a BSDE under Markovian framework, by introducing the notion of backward semigroup of BSDE, in Peng [9], the Bellman's DPP is derived and the value function is proved to be a viscosity solution of a generalized HJB equation.

By now, the DPP with related HJB equation has become a powerful approach to solving optimal control and game problems (see [10–13]). In [10], Buckdahn and Li studied a recursive SDG problem and interpreted the relationship between the controlled system and the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation. A point is worthy to mention: in order to derive the DPP, they introduced a Girsanov transformation method to prove the value functions are deterministic which is different from the method developed in Peng [9].

There really exist some systems which are modeled only by stochastic systems whose evolutions depend on the past history of the states. Based on this phenomenon, Ji and Yang

[14] investigated a controlled system governed by a functional forward-backward stochastic differential equation (FBSDE) and proved the value function to be the viscosity solution of the related path-dependent HJB equation.

In this paper, inspired by [10, 14], we will investigate the SDG problems of the functional FBSDEs. Precisely, the dynamics of our SDGs is described by the following functional SDE:

$$\begin{aligned} dX^{\gamma_t;u,v}(s) &= b\left(X_s^{\gamma_t;u,v}, u(s), v(s)\right) ds \\ &\quad + \sigma\left(X_s^{\gamma_t;u,v}, u(s), v(s)\right) dB(s), \quad s \in [t,T], \\ X_t^{\gamma_t,u,v} &= \gamma_t. \end{aligned} \tag{1}$$

And the cost functional $J(\gamma_t; u, v)$ is defined as $Y^{\gamma_t;u,v}(t)$ which is the solution of the following functional BSDE:

$$\begin{aligned} dY^{\gamma_t;u,v}(s) &= -f\left(X_s^{\gamma_t;u,v}, Y^{\gamma_t;u,v}(s), Z^{\gamma_t;u,v}(s),\right. \\ &\quad \left. u(s), v(s)\right) ds + Z^{\gamma_t;u,v}(s)\, dB(s), \\ Y^{\gamma_t,u,v}(T) &= \Phi\left(X_T^{\gamma_t;u,v}\right), \quad s \in [t,T], \end{aligned} \tag{2}$$

where γ_t is a path on $[0,t]$. The driver f and Φ can be interpreted as the running cost and the terminal cost, respectively. Also, they depend on the past history of the dynamics. Equations (1) and (2) compose a decoupled functional FBSDE. The concrete conditions on b, σ, f, Φ are shown in the later section.

In the context, we adopt the strategy against control form. The cost functional $J(\gamma_t; u, v)$ is explained as a payoff for player I and as a cost for player II. The aim of this paper is to show the following lower and upper value functions:

$$\begin{aligned} W(\gamma_t) &:= \operatorname*{essinf}_{\beta\in\mathscr{B}_{t,T}} \operatorname*{esssup}_{u\in\mathscr{U}_{t,T}} J(\gamma_t; u, \beta(u)), \\ U(\gamma_t) &:= \operatorname*{esssup}_{\alpha\in\mathscr{A}_{t,T}} \operatorname*{essinf}_{v\in\mathscr{V}_{t,T}} J(\gamma_t; \alpha(v), v) \end{aligned} \tag{3}$$

are the viscosity solutions of the following path-dependent HJBI equations, respectively,

$$\begin{aligned} & D_t W(\gamma_t) + \sup_{u\in U} \inf_{v\in V} H(\gamma_t, W, D_x W, D_{xx} W, u, v) = 0, \\ & W(\gamma_T) = \Phi(\gamma_T), \quad \gamma_T \in \Lambda, \\ & D_t U(\gamma_t) + \inf_{v\in V} \sup_{u\in U} H(\gamma_t, U, D_x U, D_{xx} U, u, v) = 0, \\ & U(\gamma_t) = \Phi(\gamma_T), \quad \gamma_T \in \Lambda, \end{aligned} \tag{4}$$

where

$$\begin{aligned} H(\gamma_t, y, p, X, u, v) &= \frac{1}{2} \operatorname{tr}\left(\sigma\sigma^T(\gamma_t, u, v) X\right) \\ &\quad + p \cdot b(\gamma_t, u, v) \\ &\quad + f(\gamma_t, y, p\cdot\sigma(\gamma_t, u, v), u, v), \end{aligned} \tag{5}$$

where $(\gamma_t, y, p, X) \in \Lambda \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ ($\mathbb{S}^d$ denotes the set of $d \times d$ symmetric matrices).

To solve the above SDG, we need the functional Itô's calculus and path-dependent PDEs which are recently introduced by Dupire [15] (for a recent account of this theory, the reader may consult [16–18]. And under the framework of functional Itô's calculus, for the non-Markovian BSDEs, Peng and Wang [19] derived a nonlinear Feynman-Kac formula for classical solutions of path-dependent PDEs. For the further development, the readers may refer to [20, 21]).

In this paper, we apply the Girsanov transformation method in Buckdahn and Li [10] to prove the determinacy of the value functions, which is different from the method introduced by Peng [7, 9]. Making use of this method and the functional Itô's calculus (introduced by Dupire [15] and developed by Cont and Fournié [16–18]), we complete the study of the zero-sum two-player SDGs in the non-Markovian case and present the lower and upper value functions of our SDG are the viscosity solutions of the corresponding path-dependent HJBI equations, respectively.

Different from the HJBI equations developed for stochastic delay systems, we establish the DPP and derive the HJBI equation in the new framework of functional Itô's calculus.

This paper is organized as follows. Section 2 recalls the functional Itô's calculus and the well-known results of BSDEs we will use later. In Section 3, we formulate our SDGs and get the corresponding DPP. Based on the obtained DPP, in Section 4 we derive the main result of the paper: the lower and upper value functions are the viscosity solutions of the associated path-dependent HJBI equations, respectively. And we add the proof for the DPP in the appendix.

2. Preliminaries

2.1. Functional Itô's Calculus. We present some preliminaries for functional Itô's calculus introduced firstly by Dupire [15]. Here we follow the notations in [15].

Let $T > 0$ be fixed. For each $t \in [0,T]$, we denote Λ_t the set of càdlàg functions from $[0,t]$ to $\mathbb{R}^d$.

For $\gamma \in \Lambda_T$, denote $\gamma(s)$ by the value of γ at time $s \in [0,T]$. Thus $\gamma = (\gamma(s))_{0\le s\le T}$ is a càdlàg process on $[0,T]$ and its value at time s is $\gamma(s)$. $\gamma_t = (\gamma(s))_{0\le s\le t} \in \Lambda_t$ is the path of γ up to time t. We denote $\Lambda = \bigcup_{t\in[0,T]} \Lambda_t$. For each $\gamma_t \in \Lambda$ and $x \in \mathbb{R}^d$, $\gamma_t(s)$ is denoted by the value of γ_t at $s \in [0,t]$ and $\gamma_t^x := (\gamma_t(s)_{0\le s<t}, \gamma_t(t) + x)$ which is also an element in Λ_t.

We now introduce a distance on Λ. Let $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the inner product and norm in $\mathbb{R}^d$. For each $0 \le t$, $\bar{t} \le T$ and $\gamma_t, \gamma_{\bar{t}} \in \Lambda$, we set

$$\begin{aligned} \|\gamma_t\| &:= \sup_{s\in[0,t]} |\gamma_t(s)|, \\ \|\gamma_t - \bar{\gamma}_{\bar{t}}\| &:= \sup_{s\in[0,t\vee\bar{t}]} |\gamma_t(s\wedge t) - \bar{\gamma}_{\bar{t}}(s\wedge\bar{t})|, \\ d_\infty(\gamma_t, \bar{\gamma}_{\bar{t}}) &:= \sup_{s\in[0,t\vee\bar{t}]} |\gamma_t(s\wedge t) - \bar{\gamma}_{\bar{t}}(s\wedge\bar{t})| + |t - \bar{t}|. \end{aligned} \tag{6}$$

It is obvious that Λ_t is a Banach space with respect to $\|\cdot\|$. Since Λ is not a linear space, d_∞ is not a norm.

Definition 1. A functional $u : \Lambda \mapsto \mathbb{R}$ is Λ-continuous at $\gamma_t \in \Lambda$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\overline{\gamma}_{\bar{t}} \in \Lambda$ with $d_\infty(\gamma_t, \overline{\gamma}_{\bar{t}}) < \delta$, one has $|u(\gamma_t) - u(\overline{\gamma}_{\bar{t}})| < \varepsilon$.

u is said to be Λ-continuous if it is Λ-continuous at each $\gamma_t \in \Lambda$.

Definition 2. Let $v : \Lambda \mapsto \mathbb{R}$ and $\gamma_t \in \Lambda$ be given. If there exists $p \in \mathbb{R}^d$, such that

$$v(\gamma_t^x) = v(\gamma_t) + \langle p, x\rangle + o(|x|), \quad \text{as} \quad x \longrightarrow 0, \ x \in \mathbb{R}^d. \tag{7}$$

Then we say that v is (vertically) differentiable at t and denote the gradient of $D_x v(\gamma_t) = p \cdot v$ is said to be vertically differentiable in Λ, if $D_x v(\gamma_t)$ exists for each $\gamma_t \in \Lambda$. The Hessian $D_{xx} v(\gamma_t)$ can be defined similarly. It is an $S(d)$-valued function defined on Λ, where $S(d)$ is the space of all $d \times d$ symmetric matrices.

For each $\gamma_t \in \Lambda$, we denote $\gamma_{t,s}(r) := \gamma_t(r)1_{[0,t)}(r) + \gamma_t(t)1_{[t,s]}(r),\ r \in [0, s]$. It is clear that $\gamma_{t,s} \in \Lambda_s$.

Definition 3. For a given $\gamma_t \in \Lambda$, if one has

$$v(\gamma_{t,s}) = v(\gamma_t) + a(s-t) + o(|s-t|), \quad \text{as } s \longrightarrow t, \ s \geq t, \tag{8}$$

then we say that $v(\gamma_t)$ is (horizontally) differentiable in t at γ_t and denote $D_t v(\gamma_t) = a$. v is said to be horizontally differentiable in Λ if $D_t v(\gamma_t)$ exists for each $\gamma_t \in \Lambda$.

Definition 4. Define $\mathbb{C}^{j,k}(\Lambda)$ as the set of function $v := (v(\gamma_t))_{\gamma_t \in \Lambda}$ defined on Λ which are j times horizontally and k times vertically differentiable in Λ such that all these derivatives are Λ-continuous.

The following is about the functional Itô's formula which was firstly obtained by Dupire [15] and then developed by Cont and Fournié [18] for a more general formulation.

Theorem 5 (functional Itô's formula). *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a probability space, if X is a continuous semimartingale and v is in $\mathbb{C}^{1,2}(\Lambda)$, then for any $t \in [0, T)$,*

$$\begin{aligned} v(X_t) - v(X_0) = &\int_0^t D_s v(X_s)\, ds + \int_0^t D_x v(X_s)\, dX(s) \\ &+ \frac{1}{2}\int_0^t D_{xx} v(X_s)\, d\langle X\rangle(s), \quad P\text{-a.s.} \end{aligned} \tag{9}$$

2.2. BSDEs. In this section, we recollect some important results which will be used in our SDG problems.

Let $(\Omega, \mathcal{F}, P)$ be the Wiener space, where Ω is the set of continuous functions from $[0, T]$ to $\mathbb{R}^d$ starting from 0 ($\Omega = C_0([0, T]; \mathbb{R}^d)$), T is an arbitrarily fixed real time horizon, $\mathcal{F}$ is the completed Borel σ-algebra over Ω, and P is the Wiener measure. Let B be the canonical process: $B(\omega, s) = \omega_s$, $s \in [0, T]$, $\omega \in \Omega$. We denote by $\mathbb{F} = \{\mathcal{F}_s, 0 \leq s \leq T\}$ the natural filtration generated by $\{B(t)\}_{t\geq 0}$ and augmented by all P-null sets, that is, $\mathcal{F}_s = \sigma\{B(r), r \leq s\} \vee \mathcal{N}_P$, $s \in [0, T]$; $\mathcal{F}_t^s = \sigma(B(r) - B(t), t \leq r \leq s) \vee \mathcal{N}_P$, where $\mathcal{N}_P$ is the set of all P-null subsets. First we present two spaces of processes as follows:

$$\begin{aligned} \mathcal{S}^2(0, T; \mathbb{R}^n) := \Big\{ &(\psi(t))_{0\leq t\leq T}\ \mathbb{R}^n\text{-valued} \\ &\mathbb{F}\text{-adapted continuous process}: \\ &E\Big[\sup_{0\leq t\leq T}|\psi(t)|^2\Big] < +\infty \Big\}; \\ \mathcal{H}^2(0, T; \mathbb{R}^n) := \Big\{ &(\psi(t))_{0\leq t\leq T}\ \mathbb{R}^n\text{-valued} \\ &\mathbb{F}\text{-progressively measurable process}: \\ &\|\psi\|^2 = E\Big[\int_0^T |\psi(t)|^2 dt\Big] < +\infty \Big\}. \end{aligned} \tag{10}$$

Consider $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, for every (y, z) in $\mathbb{R} \times \mathbb{R}^d$, $(g(t, y, z))_{t\in[0,T]}$ is progressively measurable and satisfies the following conditions:

(A1) there exists a constant $C \geq 0$ such that, P-a.s., for all $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|), \tag{11}$$

(A2) $g(\cdot, 0, 0) \in \mathcal{H}^2(0, T; \mathbb{R})$.

In the following, we suppose the driver g of a BSDE satisfies (A1) and (A2).

Lemma 6. *Under the assumptions (A1) and (A2), for any random variable $\xi \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, the BSDE*

$$\begin{aligned} y(t) = \xi + \int_t^T g(s, y(s), z(s))\, ds - \int_t^T z(s)\, dB(s), \\ 0 \leq t \leq T \end{aligned} \tag{12}$$

has a unique adapted solution

$$(y(t), z(t))_{t\in[0,T]} \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d). \tag{13}$$

The readers may refer to Pardoux and Peng [1] for the above well-known existence and uniqueness results on BSDEs.

Lemma 7 (comparison theorem). *Given two coefficients g_1, g_2 satisfying (A1), (A2), and two terminal values $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P; \mathbb{R})$, by (y^1, z^1) and (y^2, z^2) one denotes the solution of a BSDE with the data (ξ_1, g_1) and (ξ_2, g_2), respectively. Then one has the following.*

(i) *If $\xi_1 \geq \xi_2$ and $g_1 \geq g_2$, P-a.s., then $y^1(t) \geq y^2(t)$, P-a.s., for all $t \in [0, T]$.*

(ii) *(Strict monotonicity) If, in addition to* (i), *one also assumes that* $P(\xi_1 > \xi_2) > 0$, *then* $P(y^1(t) > y^2(t)) > 0$, $0 \le t \le T$, *and, in particular,* $y^1(0) > y^2(0)$.

With the notations in Lemma 7, we assume that, for some $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying (A1) and (A2), the drivers g_i have the following form:

$$g_i\left(s, y^i(s), z^i(s)\right) = g\left(s, y^i(s), z^i(s)\right) + \varphi_i(s), \quad dsdP\text{-a.e.}, \quad i = 1, 2, \tag{14}$$

where $\varphi_i \in \mathscr{H}^2(0,T;\mathbb{R})$. Then, the following results hold true for all terminal values $\xi_1, \xi_2 \in L^2(\Omega, \mathscr{F}_T, P; \mathbb{R})$.

Lemma 8. *The difference of the solutions* (y^1, z^1) *and* (y^2, z^2) *of BSDE* (12) *with the data* (ξ_1, g_1) *and* (ξ_2, g_2), *respectively, satisfies the following estimate:*

$$\begin{aligned} &\left|y^1(t) - y^2(t)\right|^2 + \frac{1}{2} E\left[\int_t^T e^{\beta(s-t)}\left(\left|y^1(s) - y^2(s)\right|^2 + \left|z^1(s) - z^2(s)\right|^2\right) ds \mid \mathscr{F}_t\right] \\ &\le E\left[e^{\beta(T-t)}\left|\xi_1 - \xi_2\right|^2 \mid \mathscr{F}_t\right] + E\left[\int_t^T e^{\beta(s-t)}\left|\varphi_1(s) - \varphi_2(s)\right|^2 ds \mid \mathscr{F}_t\right], \\ &P\text{-a.s.}, \ \forall 0 \le t \le T, \ \text{where } \beta = 16\left(1 + C^2\right). \end{aligned} \tag{15}$$

Proof. The reader may refer to Proposition 2.1 in El Karoui et al. [5] or Theorem 2.3 in Peng [9] for the details. □

3. A DPP for Stochastic Differential Games of Functional FBSDEs

In this section, we consider the SDGs of functional FBSDEs.

First we introduce the background of SDGs. Suppose that the control state spaces U, V are compact metric spaces. $\mathscr{U}$ (resp., $\mathscr{V}$) is the control set of all U (resp., V)-valued $\mathbb{F}$-progressively measurable processes for the first (resp., second) player. If $u \in \mathscr{U}$ (resp., $v \in \mathscr{V}$), we call u (resp., v) an admissible control.

Let us give the following mappings:

$$\begin{aligned} b &: \Lambda \times U \times V \longrightarrow \mathbb{R}^n, \\ \sigma &: \Lambda \times U \times V \longrightarrow \mathbb{R}^{n\times d}, \\ f &: \Lambda \times \mathbb{R} \times \mathbb{R}^d \times U \times V \longrightarrow \mathbb{R}. \end{aligned} \tag{16}$$

For given admissible controls $u(\cdot) \in \mathscr{U}$, $v(\cdot) \in \mathscr{V}$, and $t \in [0,T]$, $\gamma_t \in \Lambda$, we consider the following functional forward-backward stochastic system:

$$\begin{aligned} dX^{\gamma_t;u,v}(s) &= b\left(X_s^{\gamma_t;u,v}, u(s), v(s)\right) ds + \sigma\left(X_s^{\gamma_t;u,v}, u(s), v(s)\right) dB(s), \quad s \in [t,T], \\ dY^{\gamma_t;u,v}(s) &= -f\left(X_s^{\gamma_t;u,v}, Y^{\gamma_t;u,v}(s), Z^{\gamma_t;u,v}(s), u(s), v(s)\right) ds + Z^{\gamma_t;u,v}(s)\, dB(s), \\ X_t^{\gamma_t,u,v} &= \gamma_t, \\ Y^{\gamma_t,u,v}(T) &= \Phi\left(X_T^{\gamma_t;u,v}\right). \end{aligned} \tag{17}$$

(H) (i) For all $t \in [0,T]$, $u \in U$, $v \in V$, $x_t \in \Lambda$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $b(x_t, u, v)$, $\sigma(x_t, u, v)$, and $f(x_t, y, z, u, v)$ are $\mathscr{F}_t$-measurable.

(ii) There exists a constant $C > 0$, such that, for all $t \in [0,T]$, $u \in U$, $v \in V$, for any $x_t^1, x_t^2 \in \Lambda$,

$$\begin{aligned} &\left|b\left(x_t^1, u, v\right) - b\left(x_t^2, u, v\right)\right| + \left|\sigma\left(x_t^1, u, v\right) - \sigma\left(x_t^2, u, v\right)\right| \\ &\quad \le C\left\|x_t^1 - x_t^2\right\|, \end{aligned} \tag{18}$$

$$\left|b(x_t, u, v)\right| + \left|\sigma(x_t, u, v)\right| \le C\left(1 + \left\|x_t\right\|\right), \ \text{for any } x_t \in \Lambda. \tag{19}$$

(iii) There exists a constant $C > 0$, such that for all $t \in [0,T]$, $u \in U$, $v \in V$, for any $x_t^1, x_t^2 \in \Lambda$, $y^1, y^2 \in \mathbb{R}$, $z^1, z^2 \in \mathbb{R}^d$

$$\begin{aligned} &\left|f\left(x_t^1, y^1, z^1, u, v\right) - f\left(x_t^2, y^2, z^2, u, v\right)\right| \\ &\quad \le C\left(\left\|x_t^1 - x_t^2\right\| + \left|y^1 - y^2\right| + \left|z^1 - z^2\right|\right), \\ &\left|\Phi\left(x_T^1\right) - \Phi\left(x_T^2\right)\right| \le C\left\|x_T^1 - x_T^2\right\|, \\ &\left|f\left(x_t, 0, 0, u, v\right)\right| \le C\left(1 + \left\|x_t\right\|\right), \\ &\left|\Phi\left(x_T\right)\right| \le C\left(1 + \left\|x_T\right\|\right), \quad \text{for any } x_t \in \Lambda. \end{aligned} \tag{20}$$

Theorem 9. *Under the assumption* (H), *there exists a unique solution* $(X, Y, Z) \in \mathcal{S}^2(0,T;\mathbb{R}^n) \times \mathcal{S}^2(0,T;\mathbb{R}) \times \mathscr{H}^2(0,T;\mathbb{R}^d)$ *solving* (17).

We recall the subspaces of admissible controls and the definitions of admissible strategies as follows, which are similar to [10].

Definition 10. An admissible control process $u = (u_r)_{r\in[t,s]}$ (resp., $v = (v_r)_{r\in[t,s]}$) for Player I (resp., II) on $[t,s]$ is an $\mathscr{F}_r$-progressively measurable, U (resp., V)-valued process. The set of all admissible controls for Player I (resp., II) on $[t,s]$ is denoted by $\mathscr{U}_{t,s}$ (resp., $\mathscr{V}_{t,s}$). If $P\{u \equiv \bar{u}, a.e., \ in \ [t,s]\} = 1$, one will identify both processes u and $\bar{u}$ in $\mathscr{U}_{t,s}$. Similarly one interprets $v \equiv \bar{v}$ on $[t,s]$ in $\mathscr{V}_{t,s}$.

Definition 11. A nonanticipative strategy for Player I on $[t,s]$ $(t < s \le T)$ is a mapping $\alpha : \mathscr{V}_{t,s} \to \mathscr{U}_{t,s}$ such that, for any $\mathbb{F}$-stopping time $S : \Omega \to [t,s]$ and any $v_1, v_2 \in \mathscr{V}_{t,s}$, with $v_1 \equiv v_2$ on $[[t,S]]$, it holds that $\alpha(v_1) \equiv \alpha(v_2)$ on $[[t,S]]$. Nonanticipative strategies for Player II on $[t,s]$, $\beta : \mathscr{U}_{t,s} \to \mathscr{V}_{t,s}$, are defined similarly. The set of all nonanticipative strategies $\alpha : \mathscr{V}_{t,s} \to \mathscr{U}_{t,s}$ for Player I on $[t,s]$ is denoted by $\mathscr{A}_{t,s}$. The set of all nonanticipative strategies $\beta : \mathscr{U}_{t,s} \to \mathscr{V}_{t,s}$ for Player II on $[t,s]$ is denoted by $\mathscr{B}_{t,s}$. (Recall that $[[t,S]] = \{(r,\omega) \in [0,T] \times \Omega, t \le r \le S(\omega).\}$)

For given processes $u(\cdot) \in \mathscr{U}_{t,T}$, $v(\cdot) \in \mathscr{V}_{t,T}$, initial data $t \in [0,T]$, $\gamma_t \in \Lambda$, the cost functional is defined as follows:

$$J(\gamma_t; u, v) := Y^{\gamma_t;u,v}(t), \quad \gamma_t \in \Lambda, \tag{21}$$

where the process $Y^{\gamma_t;u,v}$ is defined by functional FBSDE (17).

For $\gamma_t \in \Lambda$, the lower and the upper value functions of our SDGs are defined as

$$W(\gamma_t) := \operatorname*{essinf}_{\beta \in \mathscr{B}_{t,T}} \operatorname*{esssup}_{u \in \mathscr{U}_{t,T}} J(\gamma_t; u, \beta(u)), \tag{22}$$

$$U(\gamma_t) := \operatorname*{esssup}_{\alpha \in \mathscr{A}_{t,T}} \operatorname*{essinf}_{v \in \mathscr{V}_{t,T}} J(\gamma_t; \alpha(v), v). \tag{23}$$

As we know, the essential infimum and essential supremum on a family of random variables are still random variables. But by applying the method introduced by Buckdahn and Li [10], we get $W(\gamma_t)$ and $U(\gamma_t)$ are deterministic.

Proposition 12. *For any $t \in [0,T]$, $\gamma_t \in \Lambda$, $W(\gamma_t)$ is a deterministic function in the sense that $W(\gamma_t) = E[W(\gamma_t)]$, P-a.s.*

Proof. Let H denote the Cameron-Martin space of all absolutely continuous elements $h \in \Omega$ whose derivative $\dot{h}$ belongs to $L^2([0,T];\mathbb{R}^d)$.

For any $h \in H$, we define the mapping $\tau_h\omega := \omega + h$, $\omega \in \Omega$. It is easy to check that $\tau_h : \Omega \to \Omega$ is a bijection, and its law is given by $P \circ [\tau_h]^{-1} = \exp\{\int_0^T \dot{h}(s)dB(s) - (1/2)\int_0^T |\dot{h}(s)|^2 ds\}P$. For any fixed $t \in [0,T]$, set $H_t = \{h \in H \mid h(\cdot) = h(\cdot \wedge t)\}$. The proof can be separated into the following four steps.

(1) For all $u \in \mathscr{U}_{t,T}$, $v \in \mathscr{V}_{t,T}$, $h \in H_t$, $J(\gamma_t; u, v)(\tau_h) = J(\gamma_t; u(\tau_h), v(\tau_h))$, P-a.s.

First, we make the transformation for the functional SDE as follows:

$$\begin{aligned}
X^{\gamma_t;u,v}(s) \circ (\tau_h) = {} & \left(\gamma_t(t) + \int_t^s b\left(X_r^{\gamma_t;u,v}, u(r), v(r)\right) dr\right. \\
& \left. + \int_t^s \sigma\left(X_r^{\gamma_t;u,v}, u(r), v(r)\right) dB(r)\right) \circ (\tau_h) \\
= {} & \gamma_t(t) + \int_t^s b\left(X_r^{\gamma_t;u,v}(\tau_h), u(\tau_h)(r),\right. \\
& \left. v(\tau_h)(r)\right) dr \\
& + \int_t^s \sigma\left(X_r^{\gamma_t;u,v}(\tau_h), u(\tau_h)(r),\right. \\
& \left. v(\tau_h)(r)\right) dB(r), \\
X^{\gamma_t;u(\tau_h),v(\tau_h)}(s) = {} & \gamma_t(t) + \int_t^s b\left(X_r^{\gamma_t;u(\tau_h),v(\tau_h)}, u(\tau_h)(r),\right. \\
& \left. v(\tau_h)(r)\right) dr \\
& + \int_t^s \sigma\left(X_r^{\gamma_t;u(\tau_h),v(\tau_h)}, u(\tau_h)(r),\right. \\
& \left. v(\tau_h)(r)\right) dB(r),
\end{aligned} \tag{24}$$

then, from the uniqueness of the solution of the functional SDE, we get

$$X^{\gamma_t;u,v}(s)(\tau_h) = X^{\gamma_t;u(\tau_h),v(\tau_h)}(s), \quad \text{for any } s \in [t,T], \ P\text{-a.s.} \tag{25}$$

Similarly, using the transformation to the BSDE in (17) and comparing the obtained equation with the BSDE obtained from (17) by replacing the transformed control process $u(\tau_h), v(\tau_h)$ for u, v, due to the uniqueness of the solution of functional BSDE, we obtain

$$\begin{gathered}
Y^{\gamma_t;u,v}(s)(\tau_h) = Y^{\gamma_t;u(\tau_h),v(\tau_h)}(s), \\
\text{for any } s \in [t,T], \ P\text{-a.s.}, \\
Z^{\gamma_t;u,v}(s)(\tau_h) = Z^{\gamma_t;u(\tau_h),v(\tau_h)}(s), \\
dsdP\text{-a.e. on } [0,T] \times \Omega.
\end{gathered} \tag{26}$$

Hence

$$J(\gamma_t; u, v)(\tau_h) = J(\gamma_t; u(\tau_h), v(\tau_h)), \quad P\text{-a.s.} \tag{27}$$

(2) For $\beta \in \mathscr{B}_{t,T}, h \in H_t$, let $\beta^h(u) := \beta(u(\tau_{-h}))(\tau_h), u \in \mathscr{U}_{t,T}$. Then, $\beta^h \in \mathscr{B}_{t,T}$.

Obviously, $\beta^h : \mathscr{U}_{t,T} \to \mathscr{V}_{t,T}$. And it is nonanticipating. In fact, given an $\mathbb{F}$-stopping time $S : \Omega \to [t,T]$ and $u_1, u_2 \in \mathscr{U}_{t,T}$, with $u_1 \equiv u_2$ on $[[t,S]]$. Accordingly, $u_1(\tau_{-h}) \equiv u_2(\tau_{-h})$ on $[[t, S(\tau_{-h})]]$. Thus,

$$\begin{aligned}
\beta^h(u_1) &= \beta(u_1(\tau_{-h}))(\tau_h) = \beta(u_2(\tau_{-h}))(\tau_h) \\
&= \beta^h(u_2) \ \text{ on } [[t,S]].
\end{aligned} \tag{28}$$

(3) For any $h \in H_t$, and $\beta \in \mathscr{B}_{t,T}$, we have

$$\begin{aligned}
&\left\{\operatorname*{esssup}_{u \in \mathscr{U}_{t,T}} J(\gamma_t; u, \beta(u))\right\}(\tau_h) \\
&\quad = \operatorname*{esssup}_{u \in \mathscr{U}_{t,T}} \{J(\gamma_t; u, \beta(u))(\tau_h)\}, \quad P\text{-a.s.}
\end{aligned} \tag{29}$$

In fact, for convenience, setting $I(\gamma_t; \beta) := \operatorname{esssup}_{u \in \mathscr{U}_{t,T}} J(\gamma_t; u, \beta(u))$, $\beta \in \mathscr{B}_{t,T}$, we know $I(\gamma_t; \beta) \ge J(\gamma_t; u, \beta(u))$. Then

$I(\gamma_t;\beta)(\tau_h) \geq J(\gamma_t;u,\beta(u))(\tau_h)$, P-a.s., for all $u \in \mathscr{U}_{t,T}$. Therefore,

$$\left\{\underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\, J(\gamma_t;u,\beta(u))\right\}(\tau_h) \geq \underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\{J(\gamma_t;u,\beta(u))(\tau_h)\}, \quad P\text{-a.s.} \tag{30}$$

From the definition of essential supremum, for any random variable ξ which satisfies $\xi \geq J(\gamma_t;u,\beta(u))(\tau_h)$, we have $\xi(\tau_{-h}) \geq J(\gamma_t;u,\beta(u))$, P-a.s. for all $u \in \mathscr{U}_{t,T}$. So $\xi(\tau_{-h}) \geq I(\gamma_t;\beta)$, P-a.s., that is, $\xi \geq I(\gamma_t;\beta)(\tau_h)$, P-a.s. Thus,

$$J(\gamma_t;u,\beta(u))(\tau_h) \geq \left\{\underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\, J(\gamma_t;u,\beta(u))\right\}(\tau_h), \quad P\text{-a.s., for any } u \in \mathscr{U}_{t,T}. \tag{31}$$

Therefore,

$$\underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\{J(\gamma_t;u,\beta(u))(\tau_h)\} \geq \left\{\underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\, J(\gamma_t;u,\beta(u))\right\}(\tau_h), \quad P\text{-a.s.} \tag{32}$$

From above we get

$$\left\{\underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\, J(\gamma_t;u,\beta(u))\right\}(\tau_h) = \underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\{J(\gamma_t;u,\beta(u))(\tau_h)\}, \quad P\text{-a.s.} \tag{33}$$

(4) Under the Girsanov transformation τ_h, $W(\gamma_t)$ is invariant, that is,

$$W(\gamma_t)(\tau_h) = W(\gamma_t), \quad P\text{-a.s., for any } h \in H. \tag{34}$$

In fact, we can prove $\{\text{essinf}_{\beta\in\mathscr{B}_{t,T}} I(\gamma_t;\beta)\}(\tau_h) = \text{essinf}_{\beta\in\mathscr{B}_{t,T}}\{I(\gamma_t;\beta)(\tau_h)\}$, P-a.s., for all $h \in H_t$, which is similar to the above step. From the above three steps, for all $h \in H_t$, we get

$$\begin{aligned} W(\gamma_t)(\tau_h) &= \left\{\underset{\beta\in\mathscr{B}_{t,T}}{\text{essinf}}\ \underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\, J(\gamma_t;u,\beta(u))\right\}(\tau_h), \\ &= \underset{\beta\in\mathscr{B}_{t,T}}{\text{essinf}}\ \underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\{J(\gamma_t;u,\beta(u))(\tau_h)\}, \\ &= \underset{\beta\in\mathscr{B}_{t,T}}{\text{essinf}}\ \underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\left\{J\left(\gamma_t;u(\tau_h),\beta^h(u(\tau_h))\right)\right\}, \\ &= \underset{\beta\in\mathscr{B}_{t,T}}{\text{essinf}}\ \underset{u\in\mathscr{U}_{t,T}}{\text{esssup}}\, J(\gamma_t;u,\beta(u)), \\ &= W(\gamma_t), \quad P\text{-a.s.}. \end{aligned} \tag{35}$$

Note that $\{u(\tau_h) \mid u(\cdot) \in \mathscr{U}_{t,T}\} = \mathscr{U}_{t,T}$ and $\{\beta^h \mid \beta \in \mathscr{B}_{t,T}\} = \mathscr{B}_{t,T}$ have been used in the above equalities. So, for any $h \in H_t, W(\gamma_t)(\tau_h) = W(\gamma_t)$, P-a.s. Thanks to $W(\gamma_t)$ is $\mathscr{F}_t$-measurable, this relation holds true for all $h \in H$.

To finish the proof, we also need the auxiliary lemma as follows. □

Lemma 13. *Let ζ be a random variable defined over the classical Wiener space $(\Omega, \mathscr{F}_T, P)$, such that $\zeta(\tau_h) = \zeta$, P-a.s., for any $h \in H$. Then $\zeta = E\zeta$, P-a.s.*

Proof. From Lemma 13 in Buckdahn and Li [10], we know for any $A \in \mathscr{B}(\mathbb{R}), \varphi \in L^2([0,T];\mathbb{R}^d)$,

$$E\left[1_{\{\zeta\in A\}} \exp\left\{\int_0^T \varphi(s)\,dB(s)\right\}\right] = E\left[1_{\{\zeta\in A\}}\right] E\left[\exp\left\{\int_0^T \varphi(s)\,dB(s)\right\}\right]. \tag{36}$$

For any $\varphi = \sum_{i=1}^N \varphi_i 1_{(t_{i-1},t_i]}$, where $\varphi_i \in \mathbb{R}^d$, for $0 \leq i \leq N$ and $\{t_i\}_{i=0}^N$ is a finite partition of $[0,T]$, from (36),

$$E\left[1_{\{\zeta\in A\}} \exp\left(\sum_{i=1}^N \varphi_i B(t_i) - B(t_{i-1})\right)\right] = E\left[1_{\{\zeta\in A\}}\right] E\left[\exp \sum_{i=1}^N \varphi_i \left(B(t_i) - B(t_{i-1})\right)\right]. \tag{37}$$

Therefore, for any nonnegative integer k_i

$$E\left[1_{\{\zeta\in A\}} \prod_{i=1}^N \left(B(t_i) - B(t_{i-1})\right)^{k_i}\right] = E\left[1_{\{\zeta\in A\}}\right] E\left[\prod_{i=1}^N \left(B(t_i) - B(t_{i-1})\right)^{k_i}\right]. \tag{38}$$

So, for any polynomial function Q, we have

$$\begin{aligned} &E\left[1_{\{\zeta\in A\}} Q\left(B(t_1) - B(t_0), \dots, B(t_n) - B(t_{n-1})\right)\right] \\ &\quad = E\left[1_{\{\zeta\in A\}}\right] E\left[Q\left(B(t_1) - B(t_0), \dots, B(t_n) - B(t_{n-1})\right)\right]. \end{aligned} \tag{39}$$

Furthermore, for any $Q \in C_b(\mathbb{R}^n)$, we still have (39). Combining the arbitrariness of $A \in \mathscr{B}(\mathbb{R})$, we obtain ζ is independent of $(B(t_1) - B(t_0), \dots, B(t_n) - B(t_{n-1}))$, for all partition of $[0,T]$. Therefore, ζ is independent of $\mathscr{F}_T$ which implies ζ is independent of itself, that is $E\zeta = \zeta$, P-a.s. □

In Ji and Yang [14], they proved the following estimates.

Lemma 14. *Under the assumption* (H)*, there exists some constant $C > 0$ such that, for any $t \in [0,T]$, $\gamma_t, \overline{\gamma}_t \in \Lambda$, $u(\cdot) \in \mathscr{U}$, $v(\cdot) \in \mathscr{V}$,*

$$E\left[\sup_{s\in[t,T]}\left|X^{\gamma_t,u,v}(s)\right|^2 \mid \mathscr{F}_t\right] \le C\left(1+\|\gamma_t\|^2\right),$$

$$E\left[\sup_{s\in[t,T]}\left|X^{\gamma_t,u,v}(s)-X^{\overline{\gamma}_t,u,v}(s)\right|^2 \mid \mathscr{F}_t\right] \le C\|\gamma_t-\overline{\gamma}_t\|^2,$$

$$E\left[\sup_{s\in[t,T]}\left|Y^{\gamma_t,u,v}(s)\right|^2+\int_t^T\left|Z^{\gamma_t,u,v}(s)\right|^2ds \mid \mathscr{F}_t\right] \le C\left(1+\|\gamma_t\|^2\right),$$

$$E\left[\sup_{s\in[t,T]}\left|Y^{\gamma_t,u,v}(s)-Y^{\overline{\gamma}_t,u,v}(s)\right|^2 + \int_t^T\left|Z^{\gamma_t,u,v}(s)-Z^{\overline{\gamma}_t,u,v}(s)\right|^2ds \mid \mathscr{F}_t\right] \le C\|\gamma_t-\overline{\gamma}_t\|^2. \quad (40)$$

From the definition of $W(\gamma_t)$ and Lemma 14, we have the following property.

Lemma 15. *There exists some constant $C > 0$ such that, for all $0 \le t \le T$, $\gamma_t, \overline{\gamma}_t \in \Lambda$,*

$$\begin{aligned}&(i)\quad \left|W(\gamma_t)-W(\overline{\gamma}_t)\right| \le C\|\gamma_t-\overline{\gamma}_t\|,\\ &(ii)\quad \left|W(\gamma_t)\right| \le C\left(1+\|\gamma_t\|\right).\end{aligned} \quad (41)$$

Now we adopt Peng's notion of stochastic backward semigroup (which was first introduced by Peng [9] to prove the DPP for stochastic control problems) to discuss a generalized DPP for our SDG (17), (22). First we define the family of backward semigroups associated with FBSDE (17).

For given $t \in [0,T]$, $\gamma_t \in \Lambda$, a number $\delta \in (0, T-t]$, admissible control processes $u(\cdot) \in \mathscr{U}_{t,t+\delta}$, $v(\cdot) \in \mathscr{V}_{t,t+\delta}$, we set

$$G^{\gamma_t;u,v}_{s,t+\delta}[\eta] := \widetilde{Y}^{\gamma_t;u,v}(s), \quad s\in[t,t+\delta], \quad (42)$$

where $\eta \in L^2(\Omega, \mathscr{F}_t^{t+\delta}, P; \mathbb{R})$, $(\widetilde{Y}^{\gamma_t;u,v}(s), \widetilde{Z}^{\gamma_t;u,v}(s))_{t\le s\le t+\delta}$ solves the following functional FBSDE on $[t,t+\delta]$:

$$\begin{aligned}d\widetilde{Y}^{\gamma_t;u,v}(s) &= -f\left(s, X_s^{\gamma_t;u,v}, \widetilde{Y}^{\gamma_t;u,v}(s), \widetilde{Z}^{\gamma_t;u,v}(s), u(s), v(s)\right)ds\\ &\quad + \widetilde{Z}^{\gamma_t;u,v}(s)\,dB(s),\\ \widetilde{Y}^{\gamma_t;u,v}&(t+\delta) = \eta, \qquad s\in[t,t+\delta].\end{aligned} \quad (43)$$

Also, we have

$$G^{\gamma_t;u,v}_{t,T}\left[\Phi\left(X_T^{\gamma_t;u,v}\right)\right] = G^{\gamma_t;u,v}_{t,t+\delta}\left[Y^{\gamma_t;u,v}(t+\delta)\right]. \quad (44)$$

Theorem 16. *Suppose* (H) *holds true, the lower value function $W(\gamma_t)$ satisfies the following DPP: for any $t\in[0,T]$, $\gamma_t\in\Lambda$, $\delta > 0$,*

$$W(\gamma_t) = \operatorname*{essinf}_{\beta\in\mathscr{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} G^{\gamma_t;u,\beta(u)}_{t,t+\delta}\left[W\left(X^{\gamma_t;u,\beta(u)}_{t+\delta}\right)\right]. \quad (45)$$

The proof is given in the appendix.

4. Viscosity Solutions of Path-Dependent HJBI Equation

Now we study the following path-dependent PDEs:

$$\begin{aligned}D_tW(\gamma_t) + H^-(\gamma_t, W, D_xW, D_{xx}W) &= 0,\\ W(\gamma_T) = \Phi(\gamma_T), \quad \gamma_T &\in \Lambda,\end{aligned} \quad (46)$$

$$\begin{aligned}D_tU(\gamma_t) + H^+(\gamma_t, U, D_xU, D_{xx}U) &= 0,\\ U(\gamma_t) = \Phi(\gamma_T), \quad \gamma_T &\in \Lambda,\end{aligned} \quad (47)$$

where

$$\begin{aligned}&H^-(\gamma_t, W, D_xW, D_{xx}W)\\ &\quad = \sup_{u\in U}\inf_{v\in V} H(\gamma_t, W, D_xW, D_{xx}W, u, v),\\ &H^+(\gamma_t, U, D_xU, D_{xx}U)\\ &\quad = \inf_{v\in V}\sup_{u\in U} H(\gamma_t, U, D_xU, D_{xx}U, u, v),\\ &H(\gamma_t, y, p, X, u, v)\\ &\quad = \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^T(\gamma_t,u,v)X\right) + p\cdot b(\gamma_t,u,v)\\ &\qquad + f(\gamma_t, y, p.\sigma(\gamma_t,u,v), u, v),\end{aligned} \quad (48)$$

where $(\gamma_t, y, p, X) \in \Lambda\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$ ($\mathbb{S}^d$ denotes the set of $d\times d$ symmetric matrices).

We will show that the value function $W(\gamma_t)$ (resp., $U(\gamma_t)$) defined in (22) (resp., (23)) is a viscosity solution of the corresponding equation (46) (resp., (47)). First we give the definition of viscosity solution for this kind of PDEs. For more information on viscosity solution, the reader is referred to Crandall et al. [22].

Definition 17. A real-valued Λ-continuous function $W \in \mathbb{C}(\Lambda)$ is called

(i) a viscosity subsolution of (46) if for any $\delta > 0$, $\Gamma \in \mathbb{C}^{1,2}_{l,b}(\Lambda)$, $\gamma_t \in \Lambda$ satisfying $\Gamma \ge W$ on $\bigcup_{0\le s\le\delta}\Lambda_{t+s}$ and $\Gamma(\gamma_t) = W(\gamma_t)$, one has

$$D_t\Gamma(\gamma_t) + H^-(\gamma_t, \Gamma, D_x\Gamma, D_{xx}\Gamma) \ge 0, \quad (49)$$

(ii) a viscosity supersolution of (46) if for any $\delta > 0$, $\Gamma \in \mathbb{C}^{1,2}_{l,b}(\Lambda)$, $\gamma_t \in \Lambda$ satisfying $\Gamma \le W$ on $\bigcup_{0\le s\le\delta}\Lambda_{t+s}$ and $\Gamma(\gamma_t) = W(\gamma_t)$, one has

$$D_t\Gamma(\gamma_t) + H^-(\gamma_t, \Gamma, D_x\Gamma, D_{xx}\Gamma) \le 0, \quad (50)$$

(iii) a viscosity solution of (46) if it is both a viscosity sub- and supersolution of (46).

Theorem 18. *Assume* (H) *holds, the lower value function W is a viscosity solution of path-dependent HJBI Equation* (46) *on Λ, the upper value function U is a viscosity solution of* (47).

First we prove some helpful lemmas. For some fixed $\Gamma \in \mathbb{C}^{1,2}_{l,b}(\Lambda)$, denote

$$\begin{aligned} &F(\gamma_s, y, z, u, v) \\ &\quad = D_s\Gamma(\gamma_s) + \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^T(\gamma_s, u, v) D_{xx}\Gamma(\gamma_s)\right) \\ &\qquad + D_x\Gamma(\gamma_s)\cdot b(\gamma_s, u, v) \\ &\qquad + f(\gamma_s, y + \Gamma(\gamma_s), z + D_x\Gamma(\gamma_s)\cdot\sigma(\gamma_s, u, v), u, v), \end{aligned} \tag{51}$$

where $(\gamma_s, y, z, u, v) \in \Lambda \times \mathbb{R} \times \mathbb{R}^d \times U \times V$.

Consider the following BSDE:

$$\begin{aligned} &-dY^{1,u,v}(s) \\ &\quad = F\left(X^{\gamma_t;u,v}_s, Y^{1,u,v}(s), Z^{1,u,v}(s), u(s), v(s)\right) ds \\ &\qquad - Z^{1,u,v}(s)\, dB(s), \\ &\qquad Y^{1,u,v}(t+\delta) = 0, \quad s \in [t, t+\delta]. \end{aligned} \tag{52}$$

Lemma 19. *For every $s \in [t, t+\delta]$, one has the following:*

$$Y^{1,u,v}(s) = G^{\gamma_t;u,v}_{s,t+\delta}\left[\Gamma\left(X^{\gamma_t;u,v}_{t+\delta}\right)\right] - \Gamma\left(X^{\gamma_t;u,v}_s\right), \quad P\text{-a.s.} \tag{53}$$

Proof. Note $G^{\gamma_t;u,v}_{s,t+\delta}[\Gamma(X^{\gamma_t;u,v}_{t+\delta})]$ is defined as $G^{\gamma_t;u,v}_{s,t+\delta}[\Gamma(X^{\gamma_t;u,v}_{t+\delta})] = Y^{u,v}(s)$ through the following BSDE:

$$\begin{aligned} -dY^{u,v}(s) &= f\left(X^{\gamma_t;u,v}_s, Y^{u,v}(s), Z^{u,v}(s), u(s), v(s)\right) ds \\ &\quad - Z^{u,v}(s)\, dB(s), \\ Y^{u,v}(t+\delta) &= \Gamma\left(X^{\gamma_t;u,v}_{t+\delta}\right), \quad s \in [t, t+\delta]. \end{aligned} \tag{54}$$

Using Itô's formula to $\Gamma(X^{\gamma_t;u,v}_s)$, we have

$$d\left(Y^{u,v}(s) - \Gamma\left(X^{\gamma_t;u,v}_s\right)\right) = dY^{1,u,v}(s). \tag{55}$$

Combined with $Y^{u,v}(t+\delta) - \Gamma(X^{\gamma_t;u,v}_{t+\delta}) = 0 = Y^{1,u,v}(t+\delta)$, we get the desired result. □

Now consider the following BSDE:

$$\begin{aligned} -dY^{2,u,v}(s) &= F\left(\gamma_t, Y^{2,u,v}(s), Z^{2,u,v}(s), u(s), v(s)\right) ds \\ &\quad - Z^{2,u,v}(s)\, dB(s), \end{aligned} \tag{56}$$

$$Y^{2,u,v}(t+\delta) = 0, \quad s \in [t, t+\delta]. \tag{57}$$

Then, we have the following lemma.

Lemma 20. *For every $u \in \mathscr{U}_{t,t+\delta}$, $v \in \mathscr{V}_{t,t+\delta}$, one has*

$$\left|Y^{1,u,v}(t) - Y^{2,u,v}(t)\right| \le C\delta^{3/2}, \quad P\text{-a.s.} \tag{58}$$

where C is independent of the control processes u, v.

Proof. From Lemma 14, we know there exists some constant $C > 0$ such that

$$E\left[\sup_{s\in[t,T]}\left|X^{\gamma_t;u,v}(s)\right|^2 \mid \mathscr{F}_t\right] \le C\left(1 + \|\gamma_t\|^2\right), \tag{59}$$

combined with

$$\begin{aligned} &E\left[\sup_{s\in[t,t+\delta]}\left|X^{\gamma_t;u,v}(s) - \gamma_t(t)\right|^2 \mid \mathscr{F}_t\right] \\ &\quad \le 2E\left[\sup_{s\in[t,t+\delta]}\left|\int_t^s b\left(X^{\gamma_t;u,v}_r, u(r), v(r)\right) dr\right|^2 \mid \mathscr{F}_t\right] \\ &\qquad + 2E\left[\sup_{s\in[t,t+\delta]}\left|\int_t^s \sigma\left(X^{\gamma_t;u,v}_r, u(r), v(r)\right) dB(r)\right|^2 \mid \mathscr{F}_t\right] \end{aligned} \tag{60}$$

we have

$$E\left[\sup_{s\in[t,t+\delta]}\left|X^{\gamma_t;u,v}(s) - \gamma_t(t)\right|^2 \mid \mathscr{F}_t\right] \le C\delta. \tag{61}$$

From (52) and (54), using Lemma 8, set

$$\begin{aligned} \xi_1 &= \xi_2 = 0, \\ g(s, y, z) &= F\left(X^{\gamma_t;u,v}_s, y, z, u_s, v_s\right), \\ \varphi_1(s) &= 0, \\ \varphi_2(s) &= F\left(\gamma_t, Y^{2,u,v}(s), Z^{2,u,v}(s), u_s, v_s\right) \\ &\quad - F\left(X^{\gamma_t;u,v}_s, Y^{2,u,v}(s), Z^{2,u,v}(s), u_s, v_s\right). \end{aligned} \tag{62}$$

Denote $\rho_0(r) = (1 + |x|^2)r$, due to b, σ, f are Lipishtiz and that they are of linear growth, $\Gamma \in \mathbb{C}^{1,2}_b$, $|\varphi_2(s)| \le \rho_0(|X^{\gamma_t;u,v}_s - \gamma_t(t)|)$, we have

$$\begin{aligned} &E\left[\int_t^{t+\delta}\left(\left|Y^{1,u,v}(s) - Y^{2,u,v}(s)\right|^2\right.\right. \\ &\qquad \left.\left. + \left|Z^{1,u,v}(s) - Z^{2,u,v}(s)\right|^2\right) ds \mid \mathscr{F}_t\right] \\ &\quad \le E\left[\int_t^{t+\delta}\rho_0^2\left(\left|X^{\gamma_t;u,v}_s - \gamma_t(t)\right|\right) ds \mid \mathscr{F}_t\right] \\ &\quad \le C\delta E\left[\sup_{s\in[t,t+\delta]}\rho_0^2\left(\left|X^{\gamma_t;u,v}_s - \gamma_t(t)\right|\right) \mid \mathscr{F}_t\right] \\ &\quad \le C\delta^2. \end{aligned} \tag{63}$$

So,

$$\begin{aligned}
&\left|Y^{1,u,v}(t)-Y^{2,u,v}(t)\right| \\
&\quad=\left|E\left[\left(Y^{1,u,v}(s)-Y^{2,u,v}(s)\right)\mid \mathscr{F}_t\right]\right| \\
&\quad=\left|E\left[\int_t^{t+\delta}\left(F\left(X_s^{\gamma_t;u,v},Y^{1,u,v}(s),Z^{1,u,v}(s),u(s),v(s)\right)\right.\right.\right. \\
&\qquad\left.\left.\left.-F\left(\gamma_t,Y^{2,u,v}(s),Z^{2,u,v}(s),u(s),v(s)\right)\right)ds\mid \mathscr{F}_t\right]\right| \\
&\quad\le CE\left[\int_t^{t+\delta}\left[\rho_0\left(\left|X_s^{\gamma_t;u,v}-\gamma_t\right|\right)+\left|Y^{1,u,v}(s)-Y^{2,u,v}(s)\right|\right.\right. \\
&\qquad\left.\left.+\left|Z^{1,u,v}(s)-Z^{2,u,v}(s)\right|\right]ds\mid \mathscr{F}_t\right] \\
&\quad\le CE\left[\int_t^{t+\delta}\rho_0\left(\left|X_s^{\gamma_t;u,v}-\gamma_t\right|\right)ds\mid \mathscr{F}_t\right] \\
&\qquad+C\delta^{1/2}E\left[\int_t^{t+\delta}\left|Y^{1,u,v}(s)-Y^{2,u,v}(s)\right|^2ds\mid \mathscr{F}_t\right]^{1/2} \\
&\qquad+C\delta^{1/2}E\left[\int_t^{t+\delta}\left|Z^{1,u,v}(s)-Z^{2,u,v}(s)\right|^2ds\mid \mathscr{F}_t\right]^{1/2} \\
&\quad\le C\delta^{3/2}.
\end{aligned} \tag{64}$$

□

Lemma 21. *Denote $Y_0(\cdot)$ by the solution of the following ordinary differential equation:*

$$\begin{aligned}
-dY_0(s)&=F_0\left(\gamma_t,Y_0(s),0\right)ds,\qquad s\in[t,t+\delta], \\
Y_0(t+\delta)&=0,
\end{aligned} \tag{65}$$

where $F_0(\gamma_t,y,z) := \sup_{u\in U}\inf_{v\in V}F(\gamma_t,y,z,u,v)$. Then, P-a.s.,

$$Y_0(t)=\operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}}\ \operatorname*{essinf}_{v\in\mathscr{V}_{t,t+\delta}}Y^{2,u,v}(t). \tag{66}$$

Proof. First we define a function as follows:

$$\begin{aligned}
F_1(\gamma_t,y,z,u)&:=\inf_{v\in V}F(\gamma_t,y,z,u,v), \\
&(\gamma_t,y,z,u)\in\Lambda\times\mathbb{R}\times\mathbb{R}^d\times U.
\end{aligned} \tag{67}$$

Consider the following equation:

$$\begin{aligned}
-dY^{3,u}(s)&=F_1\left(\gamma_t,Y^{3,u}(s),Z^{3,u}(s),u(s)\right)ds \\
&\quad-Z^{3,u}(s)\,dB(s),\quad s\in[t,t+\delta], \\
Y^{3,u}(t+\delta)&=0,
\end{aligned} \tag{68}$$

according to Lemma 6, for every $u\in\mathscr{U}_{t,t+\delta}$, there exists a unique $(Y^{3,u},Z^{3,u})$ solving (68). Also,

$$Y^{3,u}(t)=\operatorname*{essinf}_{v\in\mathscr{V}_{t,t+\delta}}Y^{2,u,v}(t),\quad P\text{-a.s. for every } u\in\mathscr{U}_{t,t+\delta}. \tag{69}$$

In fact, according to the definition of F_1 and Lemma 7, we have

$$\begin{aligned}
Y^{3,u}(t)\le Y^{2,u,v}(t),\quad &P\text{-a.s. for any } v\in\mathscr{V}_{t,t+\delta}, \\
&\text{for every } u\in\mathscr{U}_{t,t+\delta}.
\end{aligned} \tag{70}$$

On the other side, we have the existence of a measurable function $v^4:\mathbb{R}\times\mathbb{R}^d\times U\rightarrow V$ such that

$$\begin{aligned}
&F_1(\gamma_t,y,z,u) \\
&\quad=F\left(\gamma_t,y,z,u,v^4(y,z,u)\right),\quad\text{for any } y,z,u.
\end{aligned} \tag{71}$$

Set $\tilde{v}^4(s)=v^4(Y^{3,u}(s),Z^{3,u}(s),u(s))$, $s\in[t,t+\delta]$, we know $\tilde{v}^4\in\mathscr{V}_{t,t+\delta}$. Therefore, $(Y^{3,u},Z^{3,u})=(Y^{2,u,\tilde{v}^4},Z^{2,u,\tilde{v}^4})$ which is due to the uniqueness of the solution of (68). In particular, $Y^{3,u}(t)=Y^{2,u,\tilde{v}^4}(t)$, P-a.s. for every $u\in\mathscr{U}_{t,t+\delta}$. Hence, $Y^{3,u}(t)=\operatorname{essinf}_{v\in\mathscr{V}_{t,t+\delta}}Y^{2,u,v}(t)$, P-*a.s.* for every $u\in\mathscr{U}_{t,t+\delta}$.

Similarly, from $F_0(\gamma_t,y,z)=\sup_{u\in U}F_1(\gamma_t,y,z,u)$, we also derive

$$Y_0(t)=\operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}}Y^{3,u}(t)=\operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}}\ \operatorname*{essinf}_{v\in\mathscr{V}_{t,t+\delta}}Y^{2,u,v}(t),\quad P\text{-a.s.} \tag{72}$$

□

Lemma 22. *For every $u\in\mathscr{U}_{t,t+\delta}$, $v\in\mathscr{V}_{t,t+\delta}$, one has*

$$E\left[\int_t^{t+\delta}\left(\left|Y^{2,u,v}(s)\right|+\left|Z^{2,u,v}(s)\right|\right)ds\mid\mathscr{F}_t\right]\le C\delta^{3/2},\quad P\text{-a.s.}, \tag{73}$$

where the constant C is independent of the control processes u,v.

Proof. Due to $F(\gamma_t,\cdot,\cdot,u,v)$ is linear growth in (y,z), uniformly in (u,v), from Lemma 8, we know there exists a constant $C>0$ which does not depend on δ nor on the controls u and v, such that, P-a.s.

$$\begin{aligned}
\left|Y^{2,u,v}(s)\right|^2&\le C\delta, \\
E\left[\int_s^{t+\delta}\left|Z^{2,u,v}(r)\right|^2dr\mid\mathscr{F}_s\right]&\le C\delta.
\end{aligned} \tag{74}$$

Moreover, from (54), we have

$$\begin{aligned}
&\left|Y^{2,u,v}(s)\right| \\
&\quad\le E\left[\int_s^{t+\delta} F\left(\gamma_t, Y^{2,u,v}(r), Z^{2,u,v}(r), u(r), v(r)\right) dr \mid \mathscr{F}_s\right] \\
&\quad\le CE\left[\int_s^{t+\delta}\left(1+\|\gamma_t\|^2+\left|Y^{2,u,v}(r)\right|+\left|Z^{2,u,v}(r)\right|\right) dr \mid \mathscr{F}_s\right] \\
&\quad\le C\delta + C\delta^{1/2} E\left[\int_s^{t+\delta}\left|Z^{2,u,v}(r)\right|^2 dr \mid \mathscr{F}_s\right] \\
&\quad\le C\delta, \qquad s\in[t,t+\delta],
\end{aligned} \tag{75}$$

and applying Itô's formula, we get $E[\int_t^{t+\delta}|Z^{2,u,v}(s)|^2 ds \mid \mathscr{F}_t] \le C\delta^2$, P-a.s. Therefore,

$$\begin{aligned}
&E\left[\int_t^{t+\delta}\left(\left|Y^{2,u,v}(s)\right|+\left|Z^{2,u,v}(s)\right|\right) ds \mid \mathscr{F}_t\right] \\
&\quad\le C\delta^2 + C\delta^{1/2}\left(E\left[\int_t^{t+\delta}\left|Z^{2,u,v}(s)\right|^2 dr \mid \mathscr{F}_t\right]\right)^{1/2} \\
&\quad\le C\delta^{3/2}, \qquad P\text{-a.s.}
\end{aligned} \tag{76}$$

□

Now we continue the proof of Theorem 18.

Proof. (1) First we will prove $W(\gamma_t)$ is a viscosity supersolution. Given $\gamma_t\in\Lambda$, for any $\delta>0$, suppose $\Gamma(\gamma_t)=W(\gamma_t)$ and $W\ge\Gamma$ on $\bigcup_{0\le s\le\delta}\Lambda_{t+s}$.

According to the DPP (Theorem 16), we have

$$\Gamma(\gamma_t) = W(\gamma_t) = \operatorname*{essinf}_{\beta\in\mathscr{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} G^{\gamma_t;u,\beta(u)}_{t,t+\delta}\left[W\left(X^{\gamma_t;u,\beta(u)}_{t+\delta}\right)\right]. \tag{77}$$

From $W\ge\Gamma$ on $\bigcup_{0\le s\le\delta}\Lambda_{t+s}$ as well as the monotonicity property of $G^{\gamma_t;u,\beta(u)}_{t,t+\delta}[\cdot]$ (Lemma 7) we have

$$\operatorname*{essinf}_{\beta\in\mathscr{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} G^{\gamma_t;u,\beta(u)}_{t,t+\delta}\left[\Gamma\left(X^{\gamma_t;u,\beta(u)}_{t+\delta}\right)\right] - \Gamma(\gamma_t) \le 0. \tag{78}$$

By Lemma 19, we have

$$\operatorname*{essinf}_{\beta\in\mathscr{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} Y^{1,u,\beta(u)}(t) \le 0, \quad P\text{-a.s.} \tag{79}$$

Thus, from Lemma 20,

$$\operatorname*{essinf}_{\beta\in\mathscr{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} Y^{2,u,\beta(u)}(t) \le C\delta^{3/2}, \quad P\text{-a.s.} \tag{80}$$

Therefore, from $\operatorname{essinf}_{v\in\mathscr{V}_{t,t+\delta}} Y^{2,u,v}(t) \le Y^{2,u,\beta(u)}(t)$, $\beta\in\mathscr{B}_{t,t+\delta}$, we have

$$\begin{aligned}
&\operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} \operatorname*{essinf}_{v\in\mathscr{V}_{t,t+\delta}} Y^{2,u,v}(t) \\
&\quad\le \operatorname*{essinf}_{\beta\in\mathscr{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} Y^{2,u,\beta(u)}(t) \le C\delta^{3/2}, \quad P\text{-a.s.}
\end{aligned} \tag{81}$$

Thus, by Lemma 21, $Y_0(t)\le C\delta^{3/2}$, P-a.s., where $Y_0(\cdot)$ solves the ODE (65). Consequently,

$$C\delta^{1/2} \ge \frac{1}{\delta}Y_0(t) = \frac{1}{\delta}\int_t^{t+\delta} F_0(\gamma_t, Y_0(s), 0)\, ds, \quad \delta>0, \tag{82}$$

$$\sup_{u\in U}\inf_{v\in V} F(\gamma_t,0,0,u,v) = F_0(\gamma_t,0,0) \le 0. \tag{83}$$

Obviously, according to the definition of F, W is a viscosity supersolution of (46).

(2) Now we prove W is a viscosity subsolution.

Given $\gamma_t\in\Lambda$, for any $\delta>0$, suppose $\Gamma(\gamma_t)=W(\gamma_t)$ and $\Gamma\ge W$ on $\bigcup_{0\le s\le\delta}\Lambda_{t+s}$.

We need to prove

$$\sup_{u\in\mathscr{U}}\inf_{v\in V} F(\gamma_t,0,0,u,v) = F_0(\gamma_t,0,0) \ge 0. \tag{84}$$

Suppose it does not hold true. So we have some $\theta>0$ such that

$$F_0(\gamma_t,0,0) = \sup_{u\in\mathscr{U}}\inf_{v\in V} F(\gamma_t,0,0,u,v) \le -\theta < 0, \tag{85}$$

and there exists a measurable function $\psi: U\to V$ such that

$$F(\gamma_t,0,0,u,\psi(u)) \le -\frac{3}{4}\theta, \quad \forall u\in U. \tag{86}$$

Moreover, from the DPP (Theorem 16),

$$\Gamma(\gamma_t) = W(\gamma_t) = \operatorname*{essinf}_{\beta\in\mathscr{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} G^{\gamma_t;u,\beta(u)}_{t,t+\delta}\left[W\left(X^{\gamma_t,u,\beta(u)}_{t+\delta}\right)\right]. \tag{87}$$

Again from $W\le\Gamma$ on $\bigcup_{0\le s\le\delta}\Lambda_{t+s}$ as well as the monotonicity property of $G^{\gamma_t;u,\beta(u)}_{t,t+\delta}[\cdot]$ (Lemma 7) we get

$$\operatorname*{essinf}_{\beta\in\mathscr{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} G^{\gamma_t;u,\beta(u)}_{t,t+\delta}\left[\Gamma\left(X^{u,\beta(u)}_{t+\delta}\right)\right] - \Gamma(\gamma_t) \ge 0, \quad P\text{-a.s.} \tag{88}$$

Then, similar to (1), from the definition of backward semigroup, we have

$$\operatorname*{essinf}_{\beta\in\mathscr{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} Y^{1,u,\beta(u)}(t) \ge 0, \quad P\text{-a.s.}, \tag{89}$$

in particular,

$$\operatorname*{esssup}_{u\in\mathscr{U}_{t,t+\delta}} Y^{1,u,\psi(u)}(t) \ge 0, \quad P\text{-a.s.} \tag{90}$$

Setting $\psi_s(u)(\omega)=\psi(u(s)(\omega))$, $(s,\omega)\in[t,T]\times\Omega$, ψ can be regarded as an element of $\mathscr{B}_{t,t+\delta}$. Given any $\varepsilon>0$ we can select $u^\varepsilon\in\mathscr{U}_{t,t+\delta}$ satisfying $Y^{1,u^\varepsilon,\psi(u^\varepsilon)}(t)\ge-\varepsilon\delta$.

From Lemma 20 we have

$$Y^{2,u^\varepsilon,\psi(u^\varepsilon)}(t) \ge -C\delta^{3/2} - \varepsilon\delta, \quad P\text{-a.s.} \tag{91}$$

Moreover, from (54) we have

$$\begin{aligned}
&Y^{2,u^{\varepsilon},\psi(u^{\varepsilon})}(t)\\
&= E\left[\int_t^{t+\delta} F\left(\gamma_t, Y^{2,u^{\varepsilon},\psi(u^{\varepsilon})}(s), Z^{2,u^{\varepsilon},\psi(u^{\varepsilon})}(s),\right.\right.\\
&\qquad \left.\left. u^{\varepsilon}(s), \psi\left(u^{\varepsilon}(s)\right)\right) ds \mid \mathcal{F}_t\right]\\
&\leq E\left[\int_t^{t+\delta} \left(C\left|Y^{2,u^{\varepsilon},\psi(u^{\varepsilon})}(s)\right| + C\left|Z^{2,u^{\varepsilon},\psi(u^{\varepsilon})}(s)\right|\right.\right.\\
&\qquad \left.\left. + F\left(\gamma_t, 0, 0, u^{\varepsilon}(s), \psi\left(u^{\varepsilon}(s)\right)\right)\right) ds \mid \mathcal{F}_t\right]\\
&\leq C\delta^{3/2} - \frac{3}{4}\theta\delta, \quad P\text{-a.s.}
\end{aligned} \tag{92}$$

From (91) and (92), we have $-C\delta^{1/2} - \varepsilon \leq C\delta^{1/2} - (3/4)\theta$, P-a.s. Letting $\delta \downarrow 0$, and then $\varepsilon \downarrow 0$, we get $\theta \leq 0$, which produces a contradiction. Consequently,

$$\sup_{u\in U}\inf_{v\in V} F\left(\gamma_t, 0, 0, u, v\right) = F_0\left(\gamma_t, 0, 0\right) \leq 0. \tag{93}$$

According to the definition of F, W is a viscosity supersolution of (46). At last, from the above two steps, we derive that W is a viscosity solution of (46).

The similar argument for U, we also get that U is a viscosity solution of (47). □

Appendix

Proof of Theorem 16 (DPP)

Proof. For convenience, we set

$$W_\delta\left(\gamma_t\right) = \operatorname*{essinf}_{\beta\in\mathcal{B}_{t,t+\delta}} \operatorname*{esssup}_{u\in\mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{\gamma_t;u,\beta(u)}\left[W\left(X_{t+\delta}^{\gamma_t;u,\beta(u)}\right)\right]. \tag{A.1}$$

We want to prove $W_\delta(\gamma_t)$ and $W(\gamma_t)$ coincide. For it we only need to prove the following three lemmas. □

Lemma A.1. *$W_\delta(\gamma_t)$ is deterministic.*

The proof of this lemma is similar to the proof of Proposition 12, so we omit it here.

Lemma A.2. *$W_\delta(\gamma_t) \leq W(\gamma_t)$.*

Proof. For any fixed $\beta \in \mathcal{B}_{t,T}$, given a $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$, the restriction β_1 of β to $\mathcal{U}_{t,t+\delta}$ can be defined as follows:

$$\beta_1(u_1) := \beta\left(u_1 \oplus u_2\right)|_{[t,t+\delta]}, \quad u_1(\cdot) \in \mathcal{U}_{t,t+\delta}, \tag{A.2}$$

where $u_1 \oplus u_2 := u_1\mathbf{1}_{[t,t+\delta]} + u_2\mathbf{1}_{(t+\delta,T]}$, generalizes $u_1(\cdot)$ to an element of $\mathcal{U}_{t,T}$. Clearly, $\beta_1 \in \mathcal{B}_{t,t+\delta}$. Also, due to the nonanticipativity property of β we know β_1 is free of the choice of $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$. Therefore, due to the definition of $W_\delta(\gamma_t)$,

$$W_\delta\left(\gamma_t\right) \leq \operatorname*{esssup}_{u_1\in\mathcal{U}_{t,t+\delta}} G_{t,t+\delta}^{\gamma_t;u_1,\beta_1(u_1)}\left[W\left(X_{t+\delta}^{\gamma_t;u_1,\beta_1(u_1)}\right)\right], \quad P\text{-a.s.} \tag{A.3}$$

We put $I_\delta(\gamma_t, u, v) := G_{t,t+\delta}^{\gamma_t;u,v}[W(X_{t+\delta}^{\gamma_t;u,v})]$. Then there exists a sequence $\{u_i^1, i \geq \mathbf{1}\} \subset \mathcal{U}_{t,t+\delta}$, such that

$$\begin{aligned}
I_\delta\left(\gamma_t, \beta_1\right) &:= \operatorname*{esssup}_{u_1\in\mathcal{U}_{t,t+\delta}} I_\delta\left(\gamma_t; u_1, \beta_1\left(u_1\right)\right)\\
&= \sup_{i\geq 1} I_\delta\left(\gamma_t; u_i^1, \beta_1\left(u_i^1\right)\right), \quad P\text{-a.s.}
\end{aligned} \tag{A.4}$$

For any $\varepsilon > 0$, we set $\widetilde{\Gamma}_i := \{I_\delta(\gamma_t, \beta_1) \leq I_\delta(\gamma_t; u_i^1, \beta_1(u_i^1)) + \varepsilon\} \in \mathcal{F}_t$, $i \geq 1$. Then the sets $\Gamma_1 := \widetilde{\Gamma}_1$, $\Gamma_i := \widetilde{\Gamma}_i \setminus (\bigcup_{l=1}^{i-1}\widetilde{\Gamma}_l) \in \mathcal{F}_t$, $i \geq 2$, form an $(\Omega, \mathcal{F}_t)$-partition, Thus, $u_1^\varepsilon := \sum_{i\geq 1}\mathbf{1}_{\Gamma_i}u_i^1 \in \mathcal{U}_{t,t+\delta}$. Also, from the nonanticipativity of β_1 we have $\beta_1(u_1^\varepsilon) = \sum_{i\geq 1}\mathbf{1}_{\Gamma_i}\beta_1(u_i^1)$, and due to the uniqueness of the solution of the functional FBSDE, we have $I_\delta(\gamma_t; u_1^\varepsilon, \beta_1(u_1^\varepsilon)) = \sum_{i\geq 1}\mathbf{1}_{\Gamma_i}I_\delta(\gamma_t; u_i^1, \beta_1(u_i^1))$, P-a.s. So,

$$\begin{aligned}
W_\delta\left(\gamma_t\right) \leq I_\delta\left(\gamma_t; \beta_1\right) &\leq \sum_{i\geq 1}\mathbf{1}_{\Gamma_i}I_\delta\left(\gamma_t; u_i^1, \beta_1\left(u_i^1\right)\right) + \varepsilon\\
&= I_\delta\left(\gamma_t; u_1^\varepsilon, \beta_1\left(u_1^\varepsilon\right)\right) + \varepsilon\\
&= G_{t,t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}\left[W\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}\right)\right] + \varepsilon.
\end{aligned} \tag{A.5}$$

On the other hand, since $\beta_1(\cdot) := \beta(\cdot \oplus u_2) \in \mathcal{B}_{t,t+\delta}$ is independent of $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$, we define $\beta_2(u_2) := \beta(u_1^\varepsilon \oplus u_2)|_{[t+\delta,T]}$, for all $u_2(\cdot) \in \mathcal{U}_{t+\delta,T}$. According to the definition of $W(\gamma_t)$ we get, for any $y \in \mathbb{R}^n$,

$$W\left(\gamma_t\right) \leq \operatorname*{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J\left(\gamma_t; u_2, \beta_2\left(u_2\right)\right), \quad P\text{-a.s.} \tag{A.6}$$

Recalling that there exists a constant $C \in \mathbb{R}$ such that

$$\begin{aligned}
&\text{(i)}\ \left|W\left(\gamma_t\right) - W\left(\overline{\gamma}_t\right)\right| \leq C\left\|\gamma_t - \overline{\gamma}_t\right\| \quad \text{for any } \gamma_t, \overline{\gamma}_t \in \Lambda,\\
&\text{(ii)}\ \left|J\left(\gamma_t; u_2, \beta_2\left(u_2\right)\right) - J\left(\overline{\gamma}_t; u_2, \beta_2\left(u_2\right)\right)\right|\\
&\qquad \leq C\left\|\gamma_t - \overline{\gamma}_t\right\|, \quad P\text{-a.s. for any } u_2 \in \mathcal{U}_{t+\delta,T}.
\end{aligned} \tag{A.7}$$

We can show by approximating $X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}$ that

$$\begin{aligned}
&W\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}\right)\\
&\quad \leq \operatorname*{esssup}_{u_2\in\mathcal{U}_{t+\delta,T}} J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2\left(u_2\right)\right), \quad P\text{-a.s.}
\end{aligned} \tag{A.8}$$

Now we estimate the right side of the latter inequality, there exists some sequence $\{u_j^2, j \geq 1\} \subset \mathscr{U}_{t+\delta,T}$ such that

$$\begin{aligned} &\operatorname*{esssup}_{u_2 \in \mathscr{U}_{t+\delta,T}} J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)\right) \\ &\quad = \sup_{j\geq 1} J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2\left(u_j^2\right)\right), \quad P\text{-a.s.} \end{aligned} \tag{A.9}$$

Then, setting $\widetilde{\Delta}_j := \{\operatorname*{esssup}_{u_2 \in \mathscr{U}_{t+\delta,T}} J(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)) \leq J(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)) + \varepsilon\} \in \mathscr{F}_{t+\delta}$, $j \geq 1$; we have the sets $\Delta_1 := \widetilde{\Delta}_1$, $\Delta_j := \widetilde{\Delta}_j \setminus (\bigcup_{l=1}^{j-1} \widetilde{\Delta}_l) \in \mathscr{F}_{t+\delta}$, $j \geq 2$ forms an $(\Omega, \mathscr{F}_{t+\delta})$-partition and $u_2^\varepsilon := \sum_{j\geq 1} \mathbf{1}_{\Delta_j} u_j^2 \in \mathscr{U}_{t+\delta,T}$. Due to the nonanticipativity of β_2 we get $\beta_2(u_2^\varepsilon) = \sum_{j\geq 1} \mathbf{1}_{\Delta_j}\beta_2(u_j^2)$, also from the definitions of β_1, β_2, we have $\beta(u_1^\varepsilon \oplus u_2^\varepsilon) = \beta_1(u_1^\varepsilon) \oplus \beta_2(u_2^\varepsilon)$. Therefore, due to the uniqueness of the solution of functional FBSDE, we get

$$\begin{aligned} &J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2^\varepsilon, \beta_2\left(u_2^\varepsilon\right)\right) \\ &\quad = Y^{X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2^\varepsilon, \beta_2(u_2^\varepsilon)}(t+\delta) \\ &\quad = \sum_{j\geq 1} \mathbf{1}_{\Delta_j} Y^{X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2(u_j^2)}(t+\delta) \\ &\quad = \sum_{j\geq 1} \mathbf{1}_{\Delta_j} J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_j^2, \beta_2\left(u_j^2\right)\right), \quad P\text{-a.s.} \end{aligned} \tag{A.10}$$

So,

$$\begin{aligned} &W\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}\right) \\ &\quad \leq \operatorname*{esssup}_{u_2 \in \mathscr{U}_{t+\delta,T}} J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_2, \beta_2(u_2)\right) \\ &\quad \leq \sum_{j\geq 1} \mathbf{1}_{\Delta_j} J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_1^\varepsilon \oplus u_j^2, \beta\left(u_1^\varepsilon \oplus u_j^2\right)\right) + \varepsilon \\ &\quad = J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u_1^\varepsilon \oplus u_2^\varepsilon, \beta\left(u_1^\varepsilon \oplus u_2^\varepsilon\right)\right) + \varepsilon \\ &\quad = J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u^\varepsilon, \beta\left(u^\varepsilon\right)\right) + \varepsilon, \ P\text{-a.s.,} \end{aligned} \tag{A.11}$$

where $u^\varepsilon := u_1^\varepsilon \oplus u_2^\varepsilon \in \mathscr{U}_{t,T}$. From (A.5), (A.11) and the comparison theorem for BSDEs, we know

$$\begin{aligned} &W_\delta\left(\gamma_t\right) \\ &\quad \leq G_{t,t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}\left[J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u^\varepsilon, \beta\left(u^\varepsilon\right)\right) + \varepsilon\right] + \varepsilon \\ &\quad \leq G_{t,t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}\left[J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u^\varepsilon, \beta\left(u^\varepsilon\right)\right)\right] + (C+1)\,\varepsilon \\ &\quad = G_{t,t+\delta}^{\gamma_t;u^\varepsilon,\beta(u^\varepsilon)}\left[J\left(X_{t+\delta}^{\gamma_t;u_1^\varepsilon,\beta_1(u_1^\varepsilon)}; u^\varepsilon, \beta\left(u^\varepsilon\right)\right)\right] + (C+1)\,\varepsilon \\ &\quad = J\left(\gamma_t; u^\varepsilon, \beta\left(u^\varepsilon\right)\right) + (C+1)\,\varepsilon \\ &\quad = Y^{\gamma_t;u^\varepsilon,\beta(u^\varepsilon)}(t) + (C+1)\,\varepsilon, \\ &\quad \leq \operatorname*{esssup}_{u \in \mathscr{U}_{t,T}} Y^{\gamma_t;u,\beta(u)}(t) + (C+1)\,\varepsilon, \quad P\text{-a.s.} \end{aligned} \tag{A.12}$$

Because of the arbitrariness of $\beta \in \mathscr{B}_{t,T}$, (A.12) holds true for all $\beta \in \mathscr{B}_{t,T}$. Thus,

$$\begin{aligned} W_\delta\left(\gamma_t\right) &\leq \operatorname*{essinf}_{\beta \in \mathscr{B}_{t,T}} \operatorname*{esssup}_{u \in \mathscr{U}_{t,T}} Y^{\gamma_t;u,\beta(u)}(t) + (C+1)\,\varepsilon \\ &= W\left(\gamma_t\right) + (C+1)\,\varepsilon. \end{aligned} \tag{A.13}$$

Therefore, letting $\varepsilon \downarrow 0$, we have $W_\delta(\gamma_t) \leq W(\gamma_t)$. □

Lemma A.3. $W(\gamma_t) \leq W_\delta(\gamma_t)$.

Proof. We keep the notations in the above lemma. From the definition of $W_\delta(\gamma_t)$ we know

$$\begin{aligned} W_\delta\left(\gamma_t\right) &= \operatorname*{essinf}_{\beta_1 \in \mathscr{B}_{t,t+\delta}} \operatorname*{esssup}_{u_1 \in \mathscr{U}_{t,t+\delta}} G_{t,t+\delta}^{\gamma_t;u_1,\beta_1(u_1)}\left[W\left(X_{t+\delta}^{\gamma_t;u_1,\beta_1(u_1)}\right)\right] \\ &= \operatorname*{essinf}_{\beta_1 \in \mathscr{B}_{t,t+\delta}} I_\delta\left(\gamma_t, \beta_1\right), \end{aligned} \tag{A.14}$$

and the existence of some sequence $\{\beta_i^1, i \geq 1\} \subset \mathscr{B}_{t,t+\delta}$ such that

$$W_\delta\left(\gamma_t\right) = \inf_{i\geq 1} I_\delta\left(\gamma_t, \beta_i^1\right), \quad P\text{-a.s.} \tag{A.15}$$

For any $\varepsilon > 0$, we set $\widetilde{\Pi}_i := \{I_\delta(\gamma_t, \beta_i^1) - \varepsilon \leq W_\delta(\gamma_t)\} \in \mathscr{F}_t$, $i \geq 1$, $\Pi_1 := \widetilde{\Pi}_1$ and $\Pi_i := \widetilde{\Pi}_i \setminus (\bigcup_{l=1}^{i-1} \widetilde{\Pi}_l) \in \mathscr{F}_t$, $i \geq 2$. Then, $\{\Pi_i, i \geq 1\}$ forms an $(\Omega, \mathscr{F}_t)$-partition, $\beta_1^\varepsilon := \sum_{i\geq 1} \mathbf{1}_{\Lambda_i}\beta_i^1 \in \mathscr{B}_{t,t+\delta}$, combining the uniqueness of the solution of the functional FBSDE, we derive $I_\delta(\gamma_t, u_1, \beta_1^\varepsilon(u_1)) = \sum_{i\geq 1} \mathbf{1}_{\Lambda_i} I_\delta(\gamma_t, u_1, \beta_i^1(u_1))$, P-a.s., for all $u_1(\cdot) \in \mathscr{U}_{t,t+\delta}$.

$$\begin{aligned} W_\delta\left(\gamma_t\right) &\geq \sum_{i\geq 1} \mathbf{1}_{\Pi_i} I_\delta\left(\gamma_t, \beta_i^1\right) - \varepsilon \\ &\geq \sum_{i\geq 1} \mathbf{1}_{\Pi_i} I_\delta\left(\gamma_t, u_1, \beta_i^1\left(u_1\right)\right) - \varepsilon \\ &= I_\delta\left(\gamma_t, u_1, \beta_1^\varepsilon\left(u_1\right)\right) - \varepsilon \\ &= G_{t,t+\delta}^{\gamma_t,u_1,\beta_1^\varepsilon(u_1)}\left[W\left(X_{t+\delta}^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}\right)\right] - \varepsilon, \\ &\qquad P\text{-a.s.,} \ \forall u_1 \in \mathscr{U}_{t,t+\delta}. \end{aligned} \tag{A.16}$$

On the other hand, according to the definition of $W(\gamma_t)$, we deduce that, for any $x \in C([0,t+\delta];\mathbb{R}^n)$, there exists $\beta^{x,\varepsilon} \in \mathscr{B}_{t+\delta,T}$ such that

$$W(x_{t+\delta}) \geq \operatorname*{esssup}_{u_2\in\mathscr{U}_{t+\delta,T}} J(x_{t+\delta};u_2,\beta^{x,\varepsilon}(u_2)) - \varepsilon, \quad P\text{-a.s.} \tag{A.17}$$

Let $\{O_i\}_{i\geq1}$ be a Borel partition of $C([0,t+\delta];\mathbb{R}^n)$ and $\operatorname{diam}(O_i) \leq \varepsilon, i \geq 1$. Let γ^i be an arbitrarily fixed element of O_i, $i \geq 1$. Defining $[X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}] := \sum_{i\geq1}\gamma^i_{t+\delta}\mathbf{1}_{\{X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\in O_i\}}$, we have

$$\left|X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta} - \left[X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\right]\right| \leq \varepsilon, \quad \text{everywhere on } \Omega,\ \forall u_1 \in \mathscr{U}_{t,t+\delta}. \tag{A.18}$$

And, for every γ^i, there exists some $\beta^{\gamma^i,\varepsilon} \in \mathscr{B}_{t+\delta,T}$ satisfying (A.17), and, $\beta^\varepsilon_{u_1} := \sum_{i\geq1}\mathbf{1}_{\{X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\in O_i\}}\beta^{\gamma^i,\varepsilon} \in \mathscr{B}_{t+\delta,T}$.

Next we define the new $\beta^\varepsilon(u) := \beta_1^\varepsilon(u_1) \oplus \beta^\varepsilon_{u_1}(u_2), u \in \mathscr{U}_{t,T}$, where $u_1 = u|_{[t,t+\delta]}, u_2 = u|_{(t+\delta,T]}$ (which are the restrictions of u to $[t,t+\delta]\times\Omega$ and $(t+\delta,T]\times\Omega$, resp.). Clearly, $\beta^\varepsilon : \mathscr{U}_{t,T} \to \mathscr{V}_{t,T}$. And β^ε is nonanticipating. In fact, let $S:\Omega \to [t,T]$ be an $\mathbb{F}$-stopping time and $u,u' \in \mathscr{U}_{t,T}$ be such that $u \equiv u'$ on $[[t,S]]$. Decomposing u,u' into $u_1,u_1' \in \mathscr{U}_{t,t+\delta}, u_2,u_2' \in \mathscr{U}_{t+\delta,T}$ such that $u = u_1 \oplus u_1'$ and $u = u_2 \oplus u_2'$. From $u_1 \equiv u_1'$ on $[[t,S\wedge(t+\delta)]]$, we get $\beta_1^\varepsilon(u_1) = \beta_1^\varepsilon(u_1')$ on $[[t,S\wedge(t+\delta)]]$ (note that β_1^ε is nonanticipating). On the other side, from $u_2 \equiv u_2'$ on $[[t+\delta,S\vee(t+\delta)]](\subset(t+\delta,T]\times\{S > t+\delta\})$, and on $\{S > t+\delta\}$ we get $X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta} = X^{\gamma_t;u_1',\beta_1^\varepsilon(u_1')}_{t+\delta}$. Thus, from the definition, $\beta^\varepsilon_{u_1} = \beta^\varepsilon_{u_1'}$ on $\{S > t+\delta\}$ and $\beta^\varepsilon_{u_1}(u_2) = \beta^\varepsilon_{u_1'}(u_2')$ on$]]t+\delta,S\vee(t+\delta)]]$. This produces $\beta^\varepsilon(u) = \beta_1^\varepsilon(u_1)\oplus\beta^\varepsilon_{u_1}(u_2) = \beta_1^\varepsilon(u_1')\oplus\beta^\varepsilon_{u_1'}(u_2')$ on $[[t,S]]$, from which we know $\beta^\varepsilon \in \mathscr{B}_{t,T}$.

For an arbitrarily chosen $u \in \mathscr{U}_{t,T}$, decompose it into $u_1 = u|_{(t,t+\delta]} \in \mathscr{U}_{t,t+\delta}$, and $u_2 = u|_{[t+\delta,T]} \in \mathscr{U}_{t+\delta,T}$. From (A.16), (A.7)-(i), (A.18), and the comparison theorem, we get

$$\begin{aligned}
W_\delta(\gamma_t) &\geq G^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[W\left(X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\right)\right] - \varepsilon \\
&\geq G^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[W\left(\left[X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\right]\right) - C\varepsilon\right] - \varepsilon \\
&\geq G^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[W\left(\left[X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\right]\right)\right] - C\varepsilon \\
&= G^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[\sum_{i\geq1}1_{\{X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\in O_i\}}W\left(\gamma^i_{t+\delta}\right)\right] - C\varepsilon, \quad P\text{-a.s.}
\end{aligned} \tag{A.19}$$

Moreover, from (A.19), (A.7)-(ii), (A.17), and the comparison theorem, we have

$$\begin{aligned}
&W_\delta(\gamma_t) \\
&\geq G^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[\sum_{i\geq1}\mathbf{1}_{\{X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\in O_i\}}J\left(\gamma^i_{t+\delta};u_2,\beta^{\gamma^i,\varepsilon}(u_2)\right) - \varepsilon\right] - C\varepsilon \\
&\geq G^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[\sum_{i\geq1}\mathbf{1}_{\{X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\in O_i\}}J\left(\gamma^i_{t+\delta};u_2,\beta^{\gamma^i,\varepsilon}(u_2)\right)\right] - C\varepsilon \\
&= G^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[J\left(\left[X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta}\right];u_2,\beta^\varepsilon_{u_1}(u_2)\right)\right] - C\varepsilon \\
&\geq G^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[J\left(X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta};u_2,\beta^\varepsilon_{u_1}(u_2)\right) - C\varepsilon\right] - C\varepsilon \\
&\geq G^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t,t+\delta}\left[J\left(X^{\gamma_t;u_1,\beta_1^\varepsilon(u_1)}_{t+\delta};u_2,\beta^\varepsilon_{u_1}(u_2)\right)\right] - C\varepsilon \\
&= G^{\gamma_t;u,\beta^\varepsilon(u)}_{t,t+\delta}\left[Y^{\gamma_t;u,\beta^\varepsilon(u)}(t+\delta)\right] - C\varepsilon \\
&= Y^{\gamma_t;u,\beta^\varepsilon(u)}(t) - C\varepsilon, \quad P\text{-a.s., for any } u \in \mathscr{U}_{t,T}.
\end{aligned} \tag{A.20}$$

Therefore,

$$\begin{aligned}
W_\delta(\gamma_t) &\geq \operatorname*{esssup}_{u\in\mathscr{U}_{t,T}} J(\gamma_t;u,\beta^\varepsilon(u)) - C\varepsilon \\
&\geq \operatorname*{essinf}_{\beta\in\mathscr{B}_{t,T}}\operatorname*{esssup}_{u\in\mathscr{U}_{t,T}} J(\gamma_t;u,\beta(u)) - C\varepsilon \\
&= W(\gamma_t) - C\varepsilon, \quad P\text{-a.s.}
\end{aligned} \tag{A.21}$$

Letting $\varepsilon \downarrow 0$ we derive $W_\delta(\gamma_t) \geq W(\gamma_t)$. □

Remark A.4. (a) (i) For each $\beta \in \mathscr{B}_{t,t+\delta}$, there exists some $u^\varepsilon(\cdot) \in U_{t,t+\delta}$ such that

$$W(\gamma_t)\left(= W_\delta(\gamma_t)\right) \leq G^{\gamma_t;u^\varepsilon,\beta(u^\varepsilon)}_{t,t+\delta}\left[W\left(X^{\gamma_t;u^\varepsilon,\beta(u^\varepsilon)}_{t+\delta}\right)\right] + \varepsilon, \quad P\text{-a.s.} \tag{A.22}$$

(ii) There exists some $\beta^\varepsilon(\cdot) \in B_{t,t+\delta}$ such that, for all $u(\cdot) \in U_{t,t+\delta}$

$$W(\gamma_t)\left(= W_\delta(\gamma_t)\right) \geq G^{\gamma_t;u^\varepsilon,\beta^\varepsilon(u)}_{t,t+\delta}\left[W\left(X^{\gamma_t;u,\beta^\varepsilon(u)}_{t+\delta}\right)\right] - \varepsilon, \quad P\text{-a.s.} \tag{A.23}$$

(b) From Proposition 12, we know the lower value function W is deterministic. So, by choosing $\delta = T - t$ and taking expectation on both sides of (A.22), (A.23), we get $W(\gamma_t) = \inf_{\beta\in\mathscr{B}_{t,T}}\sup_{u\in\mathscr{U}_{t,T}} E[J(\gamma_t;u,\beta(u))]$.

Acknowledgments

This work was supported by National Natural Science Foundation of China (no. 11171187, no. 10871118, and no. 10921101);

the Programme of Introducing Talents of Discipline to Universities of China (no. B12023), and Program for New Century Excellent Talents in University of China.

References

[1] E. Pardoux and S. G. Peng, "Adapted solution of a backward stochastic differential equation," *Systems & Control Letters*, vol. 14, no. 1, pp. 55–61, 1990.

[2] S. G. Peng, "A general stochastic maximum principle for optimal control problems," *SIAM Journal on Control and Optimization*, vol. 28, no. 4, pp. 966–979, 1990.

[3] S. Hamadene and J.-P. Lepeltier, "Zero-sum stochastic differential games and backward equations," *Systems & Control Letters*, vol. 24, no. 4, pp. 259–263, 1995.

[4] S. Hamadene, J.-P. Lepeltier, and S. G. Peng, "BSDEs with continuous coefficients and stochastic differential games," in *Backward Stochastic Differential Equations*, vol. 364 of *Pitman Research Notes in Math Series*, pp. 115–128, El Karoui Mazliak, 1997.

[5] N. El Karoui, S. Peng, and M. C. Quenez, "Backward stochastic differential equations in finance," *Mathematical Finance*, vol. 7, no. 1, pp. 1–71, 1997.

[6] S. G. Peng, "Probabilistic interpretation for systems of quasilinear parabolic partial differential equations," *Stochastics and Stochastics Reports*, vol. 37, no. 1-2, pp. 61–74, 1991.

[7] S. G. Peng, "A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation," *Stochastics and Stochastics Reports*, vol. 38, no. 2, pp. 119–134, 1992.

[8] D. Duffie and L. G. Epstein, "Stochastic differential utility," *Econometrica*, vol. 60, no. 2, pp. 353–394, 1992.

[9] S. G. Peng, "Backward stochastic differential equations-stochastic optimization theory and viscosity solutions of HJB equations," in *Topics on Stochastic Analysis*, J. A. Yan, S. G. Peng, S. Z. Fang, and L. M. Wu, Eds., pp. 85–138, Science Press, Beijing, China, 1997.

[10] R. Buckdahn and J. Li, "Stochastic differential games and viscosity solutions of Hamilton-Jacobi-Bellman-Isaacs equations," *SIAM Journal on Control and Optimization*, vol. 47, no. 1, pp. 444–475, 2008.

[11] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, vol. 25, Springer, New York, NY, USA, Second edition, 2006.

[12] Z. Wu and Z. Y. Yu, "Dynamic programming principle for one kind of stochastic recursive optimal control problem and Hamilton-Jacobi-Bellman equation," *SIAM Journal on Control and Optimization*, vol. 47, no. 5, pp. 2616–2641, 2008.

[13] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, vol. 43, Springer, New York, NY, USA, 1999.

[14] S. L. Ji and S. Z. Yang, "An optimal control problem for functional forward-backward stochastic systems and related Path-dependent HJB equations," http://arxiv.org/abs/1204.6543 .

[15] B. Dupire, "Functional Itô's Calculus," Portfolio Research Paper, Bloomberg, 2009.

[16] R. Cont and D. A. Fournié, "A functional extension of the Itô's formula," *Comptes Rendus Mathématique*, vol. 348, no. 1-2, pp. 57–61, 2010.

[17] R. Cont and D. A. Fournié, "Change of variable formulas for non-anticipative functionals on path space," *Journal of Functional Analysis*, vol. 259, no. 4, pp. 1043–1072, 2010.

[18] R. Cont and D. A. Fournié, "Functional Itô's calculus and stochastic integral representation of martingales," *The Annals of Probability*, vol. 41, no. 1, pp. 109–133, 2013.

[19] S. G. Peng and F. L. Wang, "BSDE, path-dependent PDE and nonlinear Feynman-Kac formula," In press, http://arxiv.org/abs/1108.4317.

[20] I. Ekren, C. Keller, N. Touzi, and J. Zhang, "On viscosity solutions of path dependent PDEs," *Annals of Probability*. In press, http://arxiv.org/abs/1109.5971 .

[21] S. G. Peng, "Note on viscosity solution of path-dependent PDE and G-Martingales," *Probability*. In press, http://arxiv.org/abs/1106.1144.

[22] M. G. Crandall, H. Ishii, and P.-L. Lions, "User's guide to viscosity solutions of second order partial differential equations," *Bulletin of the American Mathematical Society*, vol. 27, no. 1, pp. 1–67, 1992.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 285241, 12 pages
http://dx.doi.org/10.1155/2013/285241

Research Article

Relationship between Maximum Principle and Dynamic Programming for Stochastic Recursive Optimal Control Problems and Applications

Jingtao Shi and Zhiyong Yu

School of Mathematics, Shandong University, Jinan 250100, China

Correspondence should be addressed to Jingtao Shi; shijingtao@sdu.edu.cn

Received 26 October 2012; Accepted 23 December 2012

Academic Editor: Guangchen Wang

This paper is concerned with the relationship between maximum principle and dynamic programming for stochastic recursive optimal control problems. Under certain differentiability conditions, relations among the adjoint processes, the generalized Hamiltonian function, and the value function are given. A linear quadratic recursive utility portfolio optimization problem in the financial engineering is discussed as an explicitly illustrated example of the main result.

1. Introduction

The nonlinear *backward stochastic differential equation* (BSDE) was introduced by Pardoux and Peng [1]. Independently, Duffie and Epstein [2] introduced BSDE from economic background. In [2], they presented a stochastic differential formulation of recursive utility. Recursive utility is an extension of the standard additive utility with the instantaneous utility depending not only on the instantaneous consumption rate but also on the future utility. As found by El Karoui et al. [3], the utility process can be regarded as a solution to a special BSDE. The optimal control problem that the cost functional is described by the solution to a BSDE is called a stochastic recursive optimal control problem. In this case, the control systems become *forward-backward stochastic differential equations* (FBSDEs). This kind of optimal control problems has found important applications in real-world problems such as mathematical economics, mathematical finance, and engineering (see Schroder and Skiadas [4], El Karoui et al. [3, 5], Ji and Zhou [6], Williams [7], and Wang and Wu [8]).

It is well known that Pontryagin's maximum principle and Bellman's dynamic programming are two of the most important tools in solving stochastic optimal control problems. See the famous reference book by Yong and Zhou [9] for systematic discussion. For stochastic recursive optimal control problems, Peng [10] first obtained a maximum principle when the control domain is convex. And then Xu [11] studied the nonconvex control domain case, but he needs to assume that the diffusion coefficient does not contain the control variable. Ji and Zhou [6] established a maximum principle when the forward state is constrained in a convex set at the terminal time. Wu [12] established a general maximum principle, where the control domain is nonconvex and the diffusion coefficient depends on the control variable. Maximum principle for stochastic recursive optimal control systems with Poisson jumps, and their applications in finance were studied in Shi and Wu [13], where the control domain is convex.

For another important approach—dynamic programming—to study stochastic recursive optimal control problems, Peng [14] (also see Peng [15]) first obtained the generalized dynamic programming principle and introduced a generalized *Hamilton-Jacobi-Bellman* (HJB) equation which is a second-order parabolic *partial differential equation* (PDE). Result that the value function is the viscosity solution to the generalized HJB equation is also proved in [14]. Wu and Yu [16] extended the results of [14, 15] with obstacle constraint for the cost functional described by the solution to a reflected backward stochastic differential equation and proved that

the value function is the unique viscosity solution to their generalized HJB equation. Li and Peng [17] generalized the results of [14, 15] by considering the cost functional defined by the controlled BSDE with jumps. They proved that the value function was the viscosity solution to the associated generalized HJB equation with integral-differential operators.

Hence, a natural question arises: are there any relations between these two methods? Such a topic was intuitively discussed by Bismut [18] and Bensoussan [19] and then studied by many researchers. Under certain differentiability conditions, the relationship between the maximum principle and dynamic programming is essentially the relationship between the derivatives of the value function and the solution to the adjoint equation along the optimal state. However, the smoothness conditions do not hold in general and are difficult to verify a priori, see Zhou [20] for the deterministic case and Yong and Zhou [9] for its stochastic counterpart. Zhou [21] first obtained the relationship between general maximum principle and dynamic programming using the viscosity solution theory (see also Zhou [22] or Yong and Zhou [9]), without the assumption that the value function is smooth. For diffusion with jumps, the relationship between maximum principle and dynamic programming was first given by Framstad et al. [23, 24] under certain differentiability conditions, and then Shi and Wu [25] eliminated these restrictions within the framework of viscosity solutions. For singular stochastic optimal control problem, the relationship between maximum principle and dynamic programming was given by Bahlali et al. [26], with the derivatives of the value function. For Markovian regime-switching jump diffusion model, the relationship between maximum principle and dynamic programming was given by Zhang et al. [27], also with the derivatives of the value function.

In this paper, we derive the relationship between maximum principle and dynamic programming for the stochastic recursive optimal control problem. For this problem, we connect the maximum principle of [10] with the dynamic programming of [14, 15], under certain differentiability conditions. Specifically, when the value function is smooth, we give relations among the adjoint processes, the generalized Hamiltonian function, and the value function. For this target, in Section 2, we first adopt some related results of [14, 15], which in this paper are stated as a stochastic verification theorem. Also we prove that under additional convexity conditions, the necessary conditions in the maximum principle of [10] are in fact sufficient. In Section 3, we show the relationship between maximum principle and dynamic programming under certain differentiability conditions for our stochastic recursive optimal control problem, by the martingale representation technique. In Section 4, we discuss a *linear quadratic* (LQ) recursive utility portfolio optimization problem in the financial engineering. In this problem, the state feedback optimal control is obtained by both the maximum principle and dynamic programming approaches, and the relations we obtained are illustrated explicitly. Finally, we end this paper with some concluding remarks in Section 5.

Notations. Throughout this paper, we denote by $\mathbf{R}^n$ the space of n-dimensional Euclidean space, by $\mathbf{R}^{n\times d}$ the space of $n\times d$ matrices, and by $\mathcal{S}^n$ the space of $n\times n$ symmetric matrices. $\langle\cdot,\cdot\rangle$ and $|\cdot|$ denote the scalar product and norm in the Euclidean space, respectively. $\top$ appearing in the superscripts denotes the transpose of a matrix.

2. Problem Statement and Preliminaries

Let $(\Omega,\mathcal{F},\mathbf{P})$ be a complete probability space equipped with a d-dimensional standard Brownian motion $\{W(t)\}_{t\geq 0}$. For fixed $t\geq 0$, the filtration $\{\mathcal{F}_s^t\}_{s\geq t}$ is generated as $\mathcal{F}_s^t=\sigma\{W(r)-W(t);\ t\leq r\leq s\}\bigvee\mathcal{N}$, where $\mathcal{N}$ contains all $\mathbf{P}$-null sets in $\mathcal{F}$ and $\sigma_1\bigvee\sigma_2$ denotes the σ-field generated by $\sigma_1\cup\sigma_2$. In particular, if $t=0$, we write $\mathcal{F}_s\equiv\mathcal{F}_s^t$.

Let $T>0$ be finite and let $\mathbf{U}\subset\mathbf{R}^k$ be nonempty, convex. For any initial time and state $(t,x)\in[0,T)\times\mathbf{R}^n$, consider the state $X^{t,x;u}(\cdot)\in\mathbf{R}^n$ given by the following controlled SDE:

$$\begin{aligned} dX^{t,x;u}(s) &= b\left(s,X^{t,x;u}(s),u(s)\right)ds \\ &\quad +\sigma\left(s,X^{t,x;u}(s),u(s)\right)dW(s),\quad s\in[t,T], \\ X^{t,x;u}(t) &= x. \end{aligned}\tag{1}$$

Here $b:[0,T]\times\mathbf{R}^n\times\mathbf{U}\to\mathbf{R}^n$, $\sigma:[0,T]\times\mathbf{R}^n\times\mathbf{U}\to\mathbf{R}^{n\times d}$ are given functions.

Given $t\in[0,T)$, we denote by $\mathcal{U}[t,T]$ the set of $\{\mathcal{F}_s^t\}_{s\geq t}$-adapted processes. For given $u(\cdot)\in\mathcal{U}[t,T]$ and $x\in\mathbf{R}^n$, an $\mathbf{R}^n$-valued process $X^{t,x;u}(\cdot)$ is called a solution to (1) if it is an $\mathcal{F}_s^t$-adapted process such that (1) holds. We refer to such $u(\cdot)\in\mathcal{U}[t,T]$ as an admissible control and $(X^{t,x;u}(\cdot),u(\cdot))$ as an admissible pair. We assume the following.

(H1) b, σ are uniformly continuous in (s,x,u), and there exists a constant $C>0$ such that for all $s\in[0,T]$, $x,\hat{x}\in\mathbf{R}^n$, $u,\hat{u}\in\mathbf{U}$,

$$\begin{aligned} &|b(s,x,u)-b(s,\hat{x},\hat{u})|+|\sigma(t,x,u)-\sigma(s,\hat{x},\hat{u})| \\ &\quad\leq C(|x-\hat{x}|+|u-\hat{u}|), \\ &|b(s,x,u)|+|\sigma(s,x,u)|\leq C(1+|x|). \end{aligned}\tag{2}$$

For any $u(\cdot)\in\mathcal{U}[t,T]$, under (H1), SDE (1) has a unique solution $X^{t,x;u}(\cdot)$ by the classical SDE theory (see, e.g., Yong and Zhou [9]).

Next, we introduce the following controlled BSDE coupled with controlled SDE (1):

$$\begin{aligned} -dY^{t,x;u}(s) &= f\left(s,X^{t,x;u}(s),Y^{t,x;u}(s),Z^{t,x;u}(s),u(s)\right)ds \\ &\quad -Z^{t,x;u}(s)\,dW(s),\quad s\in[t,T], \\ Y^{t,x;u}(T) &= \phi\left(X^{t,x;u}(T)\right). \end{aligned}\tag{3}$$

Here $f:[0,T]\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^d\times\mathbf{U}\to\mathbf{R}$, $\Phi:\mathbf{R}^n\to\mathbf{R}$ are given functions. We assume the following.

(H2) f, ϕ are uniformly continuous in (s,x,y,z,u) and there exists a constant $C>0$ such that for all $s\in[0,T]$, $x,\widehat{x}\in\mathbf{R}^n$, $y,\widehat{y}\in\mathbf{R}$, $z,\widehat{z}\in\mathbf{R}^d$, $u,\widehat{u}\in\mathbf{U}$,

$$\begin{aligned}&\left|f(s,x,y,z,u)-f(s,\widehat{x},\widehat{y},\widehat{z},\widehat{u})\right|\\&\qquad\le C\left(|x-\widehat{x}|+|y-\widehat{y}|+|z-\widehat{z}|+|u-\widehat{u}|\right),\\&\left|f(s,x,0,0,u)\right|+|\phi(x)|\le C(1+|x|),\\&\qquad|\phi(x)-\phi(\widehat{x})|\le C|x-\widehat{x}|.\end{aligned}\tag{4}$$

Then for any $u(\cdot)\in\mathcal{U}[t,T]$ and the given unique solution $X^{t,x;u}(\cdot)$ to (1), under (H2), BSDE (3) admits a unique solution $(Y^{t,x;u}(\cdot),Z^{t,x;u}(\cdot))$ by the classical BSDE theory (see Pardoux and Peng [1] or Yong and Zhou [9]).

Given $u(\cdot)\in\mathcal{U}[t,T]$, we introduce the cost functional

$$J(t,x;u(\cdot)):=-Y^{t,x;u}(s)\big|_{s=t},\quad (t,x)\in[0,T]\times\mathbf{R}^n.\tag{5}$$

Our recursive stochastic optimal control problem is the following.

Problem 1 (RSOCP). For given $(t,x)\in[0,T)\times\mathbf{R}^n$, to minimize (5) subject to (1)–(3) over $\mathcal{U}[t,T]$.

We define the value function

$$\begin{aligned}&V(t,x):=\inf_{u(\cdot)\in\mathcal{U}[t,T]}J(t,x;u(\cdot)),\quad(t,x)\in[0,T]\times\mathbf{R}^n,\\&V(T,x)=-\phi(x),\quad x\in\mathbf{R}^n.\end{aligned}\tag{6}$$

Any $\overline{u}(\cdot)\in\mathcal{U}[t,T]$ achieves the above infimum is called an optimal control, and the corresponding solutions $\overline{X}^{t,x;\overline{u}}(\cdot)$ to (1) and $(\overline{Y}^{t,x;\overline{u}}(\cdot),\overline{Z}^{t,x;\overline{u}}(\cdot))$ to (3) are called optimal state. For simplicity, we refer to $(\overline{X}^{t,x;\overline{u}}(\cdot),\overline{Y}^{t,x;\overline{u}}(\cdot),\overline{Z}^{t,x;\overline{u}}(\cdot),\overline{u}(\cdot))$ as the optimal quartet.

Remark 1. Because b,σ,f,g are all deterministic functions, then from [15, Proposition 5.1 of Peng], we know that under (H1) and (H2), the above value function is a deterministic function. So our definition (6) is meaningful.

We introduce the following generalized HJB equation:

$$\begin{aligned}&-v_s(t,x)+\sup_{u\in\mathbf{U}}G(t,x,-v(t,x),-v_x(t,x),\\&\qquad-v_{xx}(t,x),u)=0,\\&\qquad(t,x)\in[0,T)\times\mathbf{R}^n,\\&v(T,x)=-\phi(x),\quad\forall x\in\mathbf{R}^n,\end{aligned}\tag{7}$$

where the generalized Hamiltonian function $G:[0,T]\times\mathbf{R}^n\times\mathbf{R}\times\mathbf{R}^n\times\mathcal{S}^n\times\mathbf{U}\to\mathbf{R}$ is defined as

$$\begin{aligned}&G(t,x,r,p,A,u)\\&:=\frac{1}{2}\operatorname{tr}\{\sigma(t,x,u)^\top A\sigma(t,x,u)\}\\&\quad+\langle p,b(t,x,u)\rangle+f\left(t,x,r,\sigma(t,x,u)^\top p,u\right).\end{aligned}\tag{8}$$

We have the following result.

Lemma 2 (stochastic verification theorem). *Let (H1)-(H2) hold and let $(t,x)\in[0,T)\times\mathbf{R}^n$ be fixed. Suppose that $V\in C^{1,2}([0,T]\times\mathbf{R}^n)$ is a solution to (7), then*

$$\begin{aligned}V(t,x)\le J(t,x;u(\cdot)),\quad&\forall u(\cdot)\in\mathcal{U}[t,T],\\&(t,x)\in[0,T]\times\mathbf{R}.\end{aligned}\tag{9}$$

Furthermore, an admissible pair $(\overline{X}^{t,x;\overline{u}}(\cdot),\overline{u}(\cdot))$ is optimal for Problem (RSOCP) if and only if

$$\begin{aligned}&G\left(s,\overline{X}^{t,x;\overline{u}}(s),-V\left(s,\overline{X}^{t,x;\overline{u}}(s)\right),-V_x\left(s,\overline{X}^{t,x;\overline{u}}(s)\right),\right.\\&\quad\left.-V_{xx}\left(s,\overline{X}^{t,x;\overline{u}}(s)\right),\overline{u}(s)\right)\\&=\max_{u\in\mathbf{U}}G\left(s,\overline{X}^{t,x;\overline{u}}(s),-V\left(s,\overline{X}^{t,x;\overline{u}}(s)\right),\right.\\&\qquad\left.-V_x\left(s,\overline{X}^{t,x;\overline{u}}(s)\right),-V_{xx}\left(s,\overline{X}^{t,x;\overline{u}}(s)\right),u\right),\end{aligned}\tag{10}$$

a.e. $s\in[t,T]$, $\mathbf{P}$-a.s.

Proof. For any $u(\cdot)\in\mathcal{U}[t,T]$ with the corresponding state $X^{t,x;u}(\cdot)$, applying Itô's formula to $V(s,X^{t,x;u}(s))$, we obtain the following:

$$\begin{aligned}&V(t,x)\\&=-\mathbb{E}\phi\left(X^{t,x;u}(T)\right)\\&\quad-\mathbb{E}\int_t^T\left\{V_s\left(s,\overline{X}^{t,x;\overline{u}}(s)\right)\right.\\&\qquad+\left\langle V_x\left(s,X^{t,x;u}(s)\right),b\left(s,X^{t,x;u}(s),u(s)\right)\right\rangle\\&\qquad+\frac{1}{2}\operatorname{tr}\left(\sigma\left(s,X^{t,x;u}(s),u(s)\right)^\top\right.\\&\qquad\times V_{xx}\left(s,X^{t,x;u}(s)\right)\\&\qquad\left.\left.\times\,\sigma\left(s,X^{t,x;u}(s),u(s)\right)\right)\right\}ds\\&=-\mathbb{E}\phi\left(X^{t,x;u}(T)\right)\\&\quad-\mathbb{E}\int_t^T\left\{V_s\left(s,X^{t,x;u}(s)\right)\right.\\&\qquad+G\left(s,X^{t,x;u}(s),-V\left(s,X^{t,x;u}(s)\right),\right.\\&\qquad-V_x\left(s,X^{t,x;u}(s)\right),\\&\qquad\left.-V_{xx}\left(s,X^{t,x;u}(s)\right),u(s)\right)\\&\qquad-f\left(s,X^{t,x;u}(s),-V\left(s,X^{t,x;u}(s)\right),\right.\\&\qquad-\sigma\left(s,X^{t,x;u}(s),u(s)\right)^\top\\&\qquad\left.\left.\times\,V_x\left(s,X^{t,x;u}(s)\right),u(s)\right)\right\}ds\end{aligned}$$

$$\begin{aligned}
&= J(t,x;u(\cdot)) \\
&\quad + \mathbb{E}\int_t^T \Big\{ -V_s\left(s, X_s^{t,x;u}\right) \\
&\qquad + G\left(s, X^{t,x;u}(s), -V\left(s, X^{t,x;u}(s)\right),\right. \\
&\qquad\qquad -V_x\left(s, X^{t,x;u}(s)\right), \\
&\qquad\qquad \left. -V_{xx}\left(s, X^{t,x;u}(s)\right), u(s)\right)\Big\} ds \\
&\le J(t,x;u(\cdot)) \\
&\quad + \mathbb{E}\int_t^T \Big\{ -V_s\left(s, X^{t,x;u}(s)\right) \\
&\qquad + \max_{u\in\mathbf{U}} G\left(s, X^{t,x;u}(s), -V\left(s, X^{t,x;u}(s)\right),\right. \\
&\qquad\qquad -V_x\left(s, X^{t,x;u}(s)\right), \\
&\qquad\qquad \left. -V_{xx}\left(s, X^{t,x;u}(s)\right), u\right)\Big\} ds \\
&= J(t,x;u(\cdot)).
\end{aligned} \tag{11}$$

Thus (9) holds. Next, applying the above inequality to $(\overline{X}^{t,x;\overline{u}}(\cdot), \overline{u}(\cdot))$, we have

$$\begin{aligned}
V(t,x) &= J(t,x;\overline{u}(\cdot)) \\
&\quad + \mathbb{E}\int_t^T \Big\{ -V_s\left(s, \overline{X}^{t,x;\overline{u}}(s)\right) \\
&\qquad + G\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right),\right. \\
&\qquad\qquad -V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \\
&\qquad\qquad \left. -V_{xx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \overline{u}(s)\right)\Big\} ds.
\end{aligned} \tag{12}$$

The desired result follows immediately from the fact that

$$\begin{aligned}
&-V_s\left(s, \overline{X}^{t,x;\overline{u}}(s)\right) + G\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right),\right. \\
&\qquad -V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \\
&\qquad \left. -V_{xx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \overline{u}(s)\right) \le 0,
\end{aligned} \tag{13}$$

which is due to the generalized HJB equation (7). The proof is complete. □

In convenient to state the maximum principle, we regard the above controlled SDE (1) and BSDE (3) as a controlled FBSDE:

$$\begin{aligned}
dX^{t,x;u}(s) &= b\left(s, X^{t,x;u}(s), u(s)\right) ds \\
&\quad + \sigma\left(s, X^{t,x;u}(s), u(s)\right) dW(s), \\
-dY^{t,x;u}(s) &= f\left(s, X^{t,x;u}(s), Y^{t,x;u}(s), Z^{t,x;u}(s), u(s)\right) ds \\
&\quad - Z^{t,x;u}(s)\, dW(s), \quad s \in [t,T], \\
X^{t,x;u}(t) &= x, \qquad Y^{t,x;u}(T) = \phi\left(X^{t,x;u}(T)\right).
\end{aligned} \tag{14}$$

This kind of control system was studied by Peng [10] and a maximum principle was obtained. In order to mention his result, we need the following assumption.

(H3) b, σ, ϕ are continuously differentiable in (x,u) and f is continuously differentiable in (x,y,z,u). Moreover, b_x, σ_x, f_x, f_y, f_z, b_u, σ_u, f_u are bounded and there exists a constant $C > 0$ such that

$$\left|\phi_x(x)\right| \le C(1+|x|), \quad \forall x \in \mathbf{R}^n. \tag{15}$$

Let $(\overline{X}^{t,x;\overline{u}}(\cdot), \overline{Y}^{t,x;\overline{u}}(\cdot), \overline{Z}^{t,x;\overline{u}}(\cdot), \overline{u}(\cdot))$ be an optimal quartet. For all $s \in [0,T]$, we denote

$$\begin{aligned}
\overline{b}(s) &:= b\left(s, \overline{X}^{t,x;\overline{u}}(s), \overline{u}(s)\right), \\
\overline{\sigma}(s) &:= b\left(s, \overline{X}^{t,x;\overline{u}}(s), \overline{u}(s)\right), \\
\overline{f}(s) &:= f\left(s, \overline{X}^{t,x;\overline{u}}(s), \overline{Y}^{t,x;\overline{u}}(s), \overline{Z}^{t,x;\overline{u}}(s), \overline{u}(s)\right),
\end{aligned} \tag{16}$$

and similar notations are used for all their derivatives.

We introduce the adjoint equation which is an FBSDE:

$$\begin{aligned}
-dp(s) &= \left[\overline{b}_x(s)^\top p(s) - \overline{f}_x(s)^\top q(s) + \overline{\sigma}_x(s) k(s)\right] ds \\
&\quad - k(s)\, dW(s), \\
dq(s) &= \overline{f}_y(s)^\top q(s)\, ds + \overline{f}_z(s)^\top q(s)\, dW(s), \quad s \in [t,T], \\
p(T) &= -\phi_x\left(\overline{X}^{t,x;\overline{u}}(T)\right)^\top q(T), \quad q(t) = 1,
\end{aligned} \tag{17}$$

and the Hamiltonian function $H : [0,T] \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^d \times \mathbf{U} \times \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^{n\times d} \to \mathbf{R}$ is defined as

$$\begin{aligned}
H(t,x,y,z,u,p,q,k) &:= \langle p, b(t,x,u)\rangle - \langle q, f(t,x,y,z,u)\rangle \\
&\quad + \operatorname{tr}\left[\sigma(t,x,u)^\top k\right].
\end{aligned} \tag{18}$$

Under (H1)–(H3), the forward equation in (17) admits an obvious unique solution $p(\cdot)$, and then the backward equation in (17) admits a unique solution $(q(\cdot), k(\cdot))$. We call p, q, k the adjoint processes. Next, the following result holds.

Lemma 3 (necessary maximum principle). *Let (H1)–(H3) hold and $(t,x) \in [0,T)\times\mathbf{R}^n$ be fixed. Suppose that $\overline{u}(\cdot)$ is an optimal control for Problem (RSOCP), and ($\overline{X}^{t,x;\overline{u}}(\cdot)$, $\overline{Y}^{t,x;\overline{u}}(\cdot)$, $\overline{Z}^{t,x;\overline{u}}(\cdot)$) is the corresponding optimal state. Let ($p(\cdot)$, $q(\cdot)$, $k(\cdot)$) be the adjoint processes. Then*

$$\Big\langle H_u\Big(s,\overline{X}^{t,x;\overline{u}}(s),\overline{Y}^{t,x;\overline{u}}(s),\overline{Z}^{t,x;\overline{u}}(s),\overline{u}(s),p(s),q(s),\\ k(s)\Big),u-\overline{u}(s)\Big\rangle\geq 0,\quad \forall u\in\mathbf{U}, \tag{19}$$

a.e. $s\in[t,T]$, $\mathbf{P}$-a.s.

Proof. It is an immediate consequence of [10, Theorem 4.4 of Peng]. □

As we mentioned in the introduction, we can also prove that, under some additional convexity conditions, the above necessary condition in Lemma 3 is also sufficient.

Lemma 4 (sufficient maximum principle). *Let (H1)–(H3) hold. Suppose that $\overline{u}(\cdot)$ is an admissible control, and ($\overline{X}^{t,x;\overline{u}}(\cdot)$, $\overline{Y}^{t,x;\overline{u}}(\cdot)$, $\overline{Z}^{t,x;\overline{u}}(\cdot)$) is the corresponding state, with $Y(T)=M_T^{\top}X(T)$, $M_T\in\mathbf{R}^n$. Let ($p(\cdot)$, $q(\cdot)$, $k(\cdot)$) be the adjoint processes. Suppose that the Hamiltonian function H is convex in (x,y,z,u). Then $\overline{u}(\cdot)$ is an optimal control for Problem (RSOCP) if it satisfies* (19).

Proof. For any $u(\cdot)\in\mathscr{U}[t,T]$ with the corresponding state ($X^{t,x;u}(\cdot)$, $Y^{t,x;u}(\cdot)$, $Z^{t,x;u}(\cdot)$). By Remark 1, we have for fixed $t\in[0,T]$,

$$J(t,x;\overline{u}(t))-J(t,x;u(t))=Y^{t,x;u}(t)-\overline{Y}^{t,x;\overline{u}}(t)\\ =\mathbb{E}\Big[Y^{t,x;u}(t)-\overline{Y}^{t,x;\overline{u}}(t)\Big]. \tag{20}$$

Applying Itô's formula to $\langle\overline{Y}^{t,x;\overline{u}}(s)-Y^{t,x;u}(s),q(s)\rangle+\langle\overline{X}^{t,x;\overline{u}}(s)-X^{t,x;u}(s),p(s)\rangle$, noting (14), (17), and $\overline{Y}^{t,x;\overline{u}}(T)-Y^{t,x;u}(T)=M_T^{\top}(\overline{X}^{t,x;\overline{u}}(T)-X^{t,x;u}(T))$, $p(T)=-M_Tq(T)$, we get

$$\begin{aligned}
&\mathbb{E}\Big(\overline{Y}^{t,x;\overline{u}}(T)-Y^{t,x;u}(T)\Big)q(T)\\
&\quad-\mathbb{E}\Big(\overline{Y}^{t,x;\overline{u}}(t)-Y^{t,x;u}(t)\Big)q(t)\\
&\quad+\mathbb{E}\Big\langle\overline{X}^{t,x;\overline{u}}(T)-X^{t,x;u}(T),p(T)\Big\rangle\\
&\quad-\mathbb{E}\Big\langle\overline{X}^{t,x;\overline{u}}(t)-X^{t,x;u}(t),p(t)\Big\rangle\\
&=-\mathbb{E}\Big[\overline{Y}^{t,x;\overline{u}}(t)-Y^{t,x;u}(t)\Big]\\
&=\mathbb{E}\int_t^T\Big\{\Big\langle\overline{Y}^{t,x;\overline{u}}(s)-Y^{t,x;u}(s),\overline{f}_y(s)^{\top}q(s)\Big\rangle\\
&\quad+\Big\langle\overline{Z}^{t,x;\overline{u}}(s)-Z^{t,x;u}(s),\overline{f}_z(s)^{\top}q(s)\Big\rangle\\
&\quad-\Big\langle\overline{f}(s)-f\Big(s,X^{t,x;u}(s),Y^{t,x;u}(s),\\
&\qquad Z^{t,x;u}(s),u(s),\Big),q(s)\Big\rangle\\
&\quad+\Big\langle\overline{X}^{t,x;\overline{u}}(s)-X^{t,x;u}(s),-\overline{b}_x(s)^{\top}p(s)\\
&\qquad+\overline{f}_x(s)^{\top}q(s)-\overline{\sigma}_x(s)k(s)\Big\rangle\\
&\quad+\Big\langle\overline{b}(s)-b\Big(s,X^{t,x;u}(s),u(s)\Big),p(s)\Big\rangle\\
&\quad+\operatorname{tr}\Big[\Big(\overline{\sigma}(s)-\sigma\Big(s,X^{t,x;u}(s),u(s)\Big)\Big)^{\top}k(s)\Big]ds\\
&=\mathbb{E}\int_t^T\Big\{H\Big(s,\overline{X}^{t,x;\overline{u}}(s),\overline{Y}^{t,x;\overline{u}}(s),\overline{Z}^{t,x;\overline{u}}(s),\overline{u}(s),\\
&\qquad p(s),q(s),k(s)\Big)\\
&\quad-H\Big(s,X^{t,x;u}(s),Y^{t,x;u}(s),Z^{t,x;u}(s),\\
&\qquad u(s),p(s),q(s),k(s)\Big)\\
&\quad-\Big\langle H_x\Big(s,\overline{X}^{t,x;\overline{u}}(s),\overline{Y}^{t,x;\overline{u}}(s),\overline{Z}^{t,x;\overline{u}}(s),\\
&\qquad \overline{u}(s),p(s),q(s),k(s)\Big),\\
&\qquad \overline{X}^{t,x;\overline{u}}(s)-X^{t,x;u}(s)\Big\rangle\\
&\quad-\Big\langle H_y\Big(s,\overline{X}^{t,x;\overline{u}}(s),\overline{Y}^{t,x;\overline{u}}(s),\overline{Z}^{t,x;\overline{u}}(s),\\
&\qquad \overline{u}(s),p(s),q(s),k(s)\Big),\\
&\qquad \overline{Y}^{t,x;\overline{u}}(s)-Y^{t,x;u}(s)\Big\rangle\\
&\quad-\Big\langle H_z\Big(s,\overline{X}^{t,x;\overline{u}}(s),\overline{Y}^{t,x;\overline{u}}(s),\overline{Z}^{t,x;\overline{u}}(s),\\
&\qquad \overline{u}(s),p(s),q(s),k(s)\Big),\\
&\qquad \overline{Z}^{t,x;\overline{u}}(s)-Z^{t,x;u}(s)\Big\rangle\Big\}ds.
\end{aligned} \tag{21}$$

Since H is convex in (x,y,z,u), then by the maximum condition (19),

$$\begin{aligned}
&J(t,x;\overline{u}(t))-J(t,x;u(t))\\
&\leq\mathbb{E}\int_t^T\Big\langle H_u\Big(s,\overline{X}^{t,x;\overline{u}}(s),\overline{Y}^{t,x;\overline{u}}(s),\overline{Z}^{t,x;\overline{u}}(s),\overline{u}(s),\\
&\qquad p(s),q(s),k(s)\Big),\overline{u}(s)-u(s)\Big\rangle\leq 0.
\end{aligned} \tag{22}$$

Thus $\bar{u}(\cdot)$ is really the optimal control for *Problem (RSOCP)*. The proof is complete. □

3. Relationship between Maximum Principle and Dynamic Programming

In this section, we investigate the relationship between the maximum principle and dynamic programming. That is, the connection between the value function V, the generalized Hamiltonian function G, and adjoint processes p, q, k. Our main result is the following.

Theorem 5. *Let (H1)–(H3) hold and let $(t,x) \in [0,T) \times \mathbf{R}^n$ be fixed. Suppose that $\bar{u}(\cdot)$ is an optimal control for Problem (RSOCP), and $(\overline{X}^{t,x;\bar{u}}(\cdot), \overline{Y}^{t,x;\bar{u}}(\cdot), \overline{Z}^{t,x;\bar{u}}(\cdot))$ is the corresponding optimal state. Let $(p(\cdot), q(\cdot), k(\cdot))$ be the adjoint processes. If $V \in C^{1,2}([0,T] \times \mathbf{R}^n)$, then*

$$\begin{aligned} &V_s\left(s, \overline{X}^{t,x;\bar{u}}(s)\right) \\ &\quad = G\left(s, \overline{X}^{t,x;\bar{u}}(s), -V\left(s, \overline{X}^{t,x;\bar{u}}(s)\right),\right. \\ &\qquad\quad \left. -V_x\left(s, \overline{X}^{t,x;\bar{u}}(s)\right), -V_{xx}\left(s, \overline{X}^{t,x;\bar{u}}(s)\right), \bar{u}(s)\right) \\ &\quad = \max_{u \in \mathbf{U}} G\left(s, \overline{X}^{t,x;\bar{u}}(s), -V\left(s, \overline{X}^{t,x;\bar{u}}(s)\right),\right. \\ &\qquad\quad \left. -V_x\left(s, \overline{X}^{t,x;\bar{u}}(s)\right), -V_{xx}\left(s, \overline{X}^{t,x;\bar{u}}(s)\right), u\right), \end{aligned} \tag{23}$$

a.e. $s \in [t,T]$, $\mathbf{P}$-a.s. Furthermore, if $V \in C^{1,3}([0,T] \times \mathbf{R}^n)$ and V_{sx} is also continuous, then

$$\begin{aligned} p(s) &= V_x\left(s, \overline{X}^{t,x;\bar{u}}(s)\right)^{\top} q(s), \quad \forall s \in [t,T], \ \mathbf{P}\text{-a.s.}, \\ k(s) &= \left\{ V_{xx}\left(s, \overline{X}^{t,x;\bar{u}}(s)\right)^{\top} \bar{\sigma}(s) + V_x\left(s, \overline{X}^{t,x;\bar{u}}(s)\right)^{\top} \right. \\ &\quad \times f_z\left(s, \overline{X}^{t,x;\bar{u}}(s), -V\left(s, \overline{X}^{t,x;\bar{u}}(s)\right),\right. \\ &\qquad \left.\left. -V_x\left(s, \overline{X}^{t,x;\bar{u}}(s)\right) \bar{\sigma}(s), \bar{u}(s)\right) \right\} q(s), \\ &\qquad\qquad a.e.\ s \in [t,T], \ \mathbf{P}\text{-a.s.}, \end{aligned} \tag{24}$$

where

$$\begin{aligned} q(s) = \exp &\left\{ \int_t^s \left[f_y\left(r, \overline{X}^{t,x;\bar{u}}(r), -V\left(r, \overline{X}^{t,x;\bar{u}}(r)\right),\right.\right.\right. \\ &\qquad \left. -V_x\left(r, \overline{X}^{t,x;\bar{u}}(r), \bar{u}(r)\right) \bar{\sigma}(r), \bar{u}(r)\right) \\ &\quad - \frac{1}{2} \left| f_z\left(r, \overline{X}^{t,x;\bar{u}}(r), -V\left(r, \overline{X}^{t,x;\bar{u}}(r)\right),\right.\right. \\ &\qquad - V_x\left(r, \overline{X}^{t,x;\bar{u}}(r)\right) \\ &\qquad \left.\left.\left. \times \bar{\sigma}(r), \bar{u}(r)\right) \right|^2 \right] dr \\ &+ \int_t^s f_z\left(r, \overline{X}^{t,x;\bar{u}}(r), -V\left(r, \overline{X}^{t,x;\bar{u}}(r)\right),\right. \\ &\qquad - V_x\left(r, \overline{X}^{t,x;\bar{u}}(r)\right) \\ &\qquad \left.\left. \times \bar{\sigma}(r), \bar{u}(r)\right) dW(r) \right\}. \end{aligned} \tag{25}$$

Proof. Obviously (25) can be obtained via solving the forward SDE in (17) directly. Now let us prove (24). By [15, Theorem 5.4 of Peng], for fixed $t \in [0,T)$, it is easy to obtain

$$\begin{aligned} V\left(s, \overline{X}^{t,x;\bar{u}}(s)\right) &= -\overline{Y}^{t,x;\bar{u}}(s) \\ &= -\mathbb{E}\left[\int_s^T \bar{f}(r)\, dr + \phi\left(\overline{X}^{t,x;\bar{u}}(T)\right) \mid \mathcal{F}_s^t \right], \\ &\qquad \forall s \in [t,T], \ \mathbf{P}\text{-a.s.} \end{aligned} \tag{26}$$

Define a square-integrable $\mathcal{F}_s^t$-martingale

$$m(s) := -\mathbb{E}\left[\int_t^T \bar{f}(r)\, dr + \phi\left(\overline{X}_T^{t,x;\bar{u}}\right) \mid \mathcal{F}_s^t \right], \quad s \in [t,T]. \tag{27}$$

Thus, by the martingale representation theorem (see Yong and Zhou [9]), there exists a unique $M(\cdot) \in L^2_{\mathcal{F}}([t,T]; \mathbf{R}^d)$ satisfying

$$m(s) = m(t) + \int_t^s M(r)\, dW(r), \tag{28}$$

where $t \in [0,T)$ is fixed. Then

$$\begin{aligned} V\left(s, \overline{X}^{t,x;\bar{u}}(s)\right) = &- \int_s^T \bar{f}(r)\, dr - \int_s^T M(r)\, dW(r) \\ &+ V\left(T, \overline{X}^{t,x;\bar{u}}(T)\right). \end{aligned} \tag{29}$$

On the other hand, applying Itô's formula to $V(s, \overline{X}^{t,x;\bar{u}}(s))$, we obtain

$$\begin{aligned} &dV\left(s, \overline{X}^{t,x;\bar{u}}(s)\right) \\ &\quad = \left\{ V_s\left(s, \overline{X}^{t,x;\bar{u}}(s)\right) + \left\langle V_x\left(s, \overline{X}^{t,x;\bar{u}}(s)\right), \bar{b}(s) \right\rangle \right. \\ &\qquad \left. + \frac{1}{2} \operatorname{tr}\left(\bar{\sigma}(s)^{\top} V_{xx}\left(s, \overline{X}^{t,x;\bar{u}}(s)\right) \bar{\sigma}(s) \right) \right\} ds \\ &\qquad + V_x\left(s, \overline{X}^{t,x;\bar{u}}(s)\right)^{\top} \bar{\sigma}(s)\, dW(s). \end{aligned} \tag{30}$$

Comparing the above two equalities, we conclude that

$$\begin{aligned} &V_s\left(s, \overline{X}^{t,x;\bar{u}}(s)\right) + \left\langle V_x\left(s, \overline{X}^{t,x;\bar{u}}(s)\right), \bar{b}(s) \right\rangle \\ &\qquad + \frac{1}{2} \operatorname{tr}\left(\bar{\sigma}(s)^{\top} V_{xx}\left(s, \overline{X}^{t,x;\bar{u}}(s)\right) \bar{\sigma}(s) \right) = \bar{f}(s), \\ &V_x\left(s, \overline{X}^{t,x;\bar{u}}(s)\right)^{\top} \bar{\sigma}(s) = M(s), \quad \text{a.e. } s \in [t,T], \ \mathbf{P}\text{-a.s.} \end{aligned} \tag{31}$$

However, by the uniqueness of solution to BSDE (3), we have

$$\overline{Y}^{t,x;\overline{u}}(s) = -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right),$$
$$\overline{Z}^{t,x;\overline{u}}(s) = -V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)^{\top} \overline{\sigma}(s), \quad \text{a.e. } s \in [t,T], \ \mathbf{P}\text{-a.s.} \tag{32}$$

Since $V \in C^{1,2}([0,T] \times \mathbf{R}^n)$, it satisfies the generalized HJB equation (7), which implies (23). Also, by (7), we have

$$\begin{aligned} 0 = & -V_s\left(s, \overline{X}^{t,x;\overline{u}}(s)\right) \\ & + G\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), -V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \right. \\ & \left. -V_{xx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \overline{u}(s)\right) \\ \geq & -V_s(s,x) + G(s,x,-V(s,x), -V_x(s,x), \\ & -V_{xx}(s,x), \overline{u}(s)), \quad \forall x \in \mathbf{R}^n. \end{aligned} \tag{33}$$

Consequently, if $V \in C^{1,3}([0,T] \times \mathbf{R}^n)$ and V_{sx} is also continuous, then

$$\begin{aligned} & \frac{\partial}{\partial x}\{-V_s(s,x) + G(s,x,-V(s,x), -V_x(s,x), \\ & -V_{xx}(s,x), \overline{u}(s))\}\big|_{x=\overline{X}^{t,x;\overline{u}}(s)} = 0, \\ & \forall s \in [t,T]. \end{aligned} \tag{34}$$

This is equivalent to (recall (8)), for all $s \in [t,T]$,

$$\begin{aligned} 0 = & -V_{sx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right) - V_{xx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{b}(s) \\ & - \overline{b}_x(s)^{\top} V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right) \\ & - \frac{1}{2}\operatorname{tr}\left(\overline{\sigma}(s)^{\top} V_{xxx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s)\right) \\ & - \overline{\sigma}_x(s)^{\top} V_{xx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s) \\ & + f_x\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \right. \\ & \left. -V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s), \overline{u}(s)\right) \\ & - f_y\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \right. \\ & \left. - V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s), \overline{u}(s)\right) \\ & \times V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right) \\ & - f_z\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \right. \\ & \left. - V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s), \overline{u}(s)\right) \\ & \times \left[V_{xx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s) + V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}_x(s)\right], \end{aligned} \tag{35}$$

where

$$\begin{aligned} & \operatorname{tr}\left(\overline{\sigma}^{\top} V_{xxx} \overline{\sigma}\right) \\ & := \left(\operatorname{tr}\left(\overline{\sigma}^{\top}\left(\left(V_x\right)^1\right)_{xx}\overline{\sigma}\right), \ldots, \operatorname{tr}\left(\overline{\sigma}^{\top}\left(\left(V_x\right)^n\right)_{xx}\overline{\sigma}\right)\right)^{\top}, \end{aligned} \tag{36}$$

with $((V_x)^1, \ldots, (V_x)^n)^{\top} = V_x$.

On the other hand, applying Itô's formula to $V_x(s, \overline{X}^{t,x;\overline{u}}(s))$, we get

$$\begin{aligned} & dV_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right) \\ & = -\left\{\overline{b}_x(s)^{\top} V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right) + \overline{\sigma}_x(s)^{\top}\right. \\ & \quad \times V_{xx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s) \\ & \quad - f_x\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \right. \\ & \quad \left. -V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s), \overline{u}(s)\right) \\ & \quad + f_y\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \right. \\ & \quad \left. -V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s), \overline{u}(s)\right) \\ & \quad \times V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right) \\ & \quad + f_z\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \right. \\ & \quad \left. -V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s), \overline{u}(s)\right) \\ & \quad \times \left[V_{xx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s) \right. \\ & \quad \left.\left. + V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}_x(s)\right]\right\} ds \\ & \quad + V_{xx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)^{\top}\overline{\sigma}(s)\, dW(s). \end{aligned} \tag{37}$$

Note that $V_x(T, \overline{X}^{t,x;\overline{u}}(T)) = -\phi_x(\overline{X}^{t,x;\overline{u}}(T))$. Applying again Itô's formula to $V_x(s, \overline{X}^{t,x;\overline{u}}(s))^{\top} q(s)$, we have

$$\begin{aligned} & dV_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)^{\top} q(s) \\ & = \left\{ -\overline{b}_x(s)^{\top} V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right) q(s) \right. \\ & \quad + f_x\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \right. \\ & \quad \left. -V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s), \overline{u}(s)\right)^{\top} q(s) \\ & \quad - \overline{\sigma}_x(s)^{\top}\left[V_{xx}\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)^{\top}\overline{\sigma}(s) \right. \\ & \quad + V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)^{\top} \\ & \quad \times f_z\left(s, \overline{X}^{t,x;\overline{u}}(s), -V\left(s, \overline{X}^{t,x;\overline{u}}(s)\right), \right. \\ & \quad \left.\left.\left. -V_x\left(s, \overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s), \overline{u}s\right)\right] q(s) \right\} ds \end{aligned}$$

$$+\left\{V_{xx}\left(s,\overline{X}^{t,x;\overline{u}}(s)\right)^{\top}\overline{\sigma}(s)+V_x\left(s,\overline{X}^{t,x;\overline{u}}(s)\right)^{\top}\right.$$
$$\times f_z\left(s,\overline{X}^{t,x;\overline{u}}(s),-V\left(s,\overline{X}^{t,x;\overline{u}}(s)\right),\right.$$
$$\left.-V_x\left(s,\overline{X}^{t,x;\overline{u}}(s)\right)\overline{\sigma}(s),\overline{u}(s)\right)\Big\}$$
$$\times q(s)\,dW(s). \tag{38}$$

Hence, by the uniqueness of the solutions to the adjoint equation (17), we obtain (24). The proof is complete. □

4. Applications to Financial Portfolio Optimization

In this section, we consider an LQ recursive utility portfolio optimization problem in the financial engineering. In this problem, the optimal portfolio in the state feedback form is obtained by both maximum principle and dynamic programming approaches, and the relations which we obtained in Theorem 5 are illustrated explicitly.

Suppose the investors have two kinds of securities in the market for possible investment choice.

(i) A risk-free security (e.g., a bond), where the price $S_0(t)$ at time t is given by

$$dS_0(t)=\rho_t S_0(t)\,dt,\quad S_0(0)>0, \tag{39}$$

here ρ_t is a bounded deterministic function.

(ii) A risky security (e.g., a stock), where the price $S_1(t)$ at time t is given by

$$dS_1(t)=\mu_t S_1(t)\,dt+\sigma_t S_1(t)\,dW(t),\quad S_1(0)>0, \tag{40}$$

here $W(\cdot)$ is a 1-dimensional Brownian motion and $\mu_t,\sigma_t\neq 0$ are bounded deterministic functions with $\mu_t>\rho_t$.

Let $u(t)$ denote the total market value of the investor's wealth invested in the risky security which we call portfolio. Given the initial wealth $X^u(0)=X_0\geq 0$, combining (39) and (40), we can get the following wealth dynamics:

$$\begin{aligned} dX^u(t)&=\left[\rho_t X^u(t)+(\mu_t-\rho_t)u(t)\right]dt\\ &\quad+\sigma_t u(t)\,dW(t),\quad t\geq 0, \\ X^u(0)&=X_0. \end{aligned} \tag{41}$$

We denote by $\mathbf{U}_{\mathrm{ad}}$ the set of admissible portfolios valued in $\mathbf{U}=\mathbf{R}$.

For any given initial wealth $X_0>0$, Kohlmann and Zhou [28] discussed a mean-variance portfolio optimization problem. That is, the investor's object is to find an admissible portfolio $u^*(t)$ which minimizes the variance $\mathrm{Var}[X^u(T)]:=\mathbb{E}[(X^u(T)-\mathbb{E}[X^u(T)])^2]$ at some future time $T>0$ under the condition that $\mathbb{E}[X^u(T)]=A$ for some given $A\in\mathbf{R}$. Using the Lagrange multiplier method, we know that it is equivalent to study the following problem:

$$\sup_{u(\cdot)\in\mathbf{U}_{\mathrm{ad}}}\mathbb{E}\left[-\frac{1}{2}(X^u(T)-a)^2\right], \tag{42}$$

where some $a\in\mathbf{R}$ is given. Using the completion of squares technique, the study of [28] obtained an optimal portfolio in the state feedback form by some stochastic Riccati equation and BSDE. The optimal value function was also obtained.

In this paper, we generalize the above mean-variance portfolio optimization problem to a recursive utility portfolio optimization problem. The recursive utility means that the utility at time t is a function of the future utility (in this section, we do not consider the consumption). In fact, in our framework, the recursive utility can be assumed to satisfy some controlled BSDE.

We consider a small investor, endowed with initial wealth $X_0>0$, who chooses at each time t his/her portfolio $u(t)$. The investor wants to choose an optimal portfolio $u^*(\cdot)\in\mathbf{U}_{\mathrm{ad}}$ to maximize the following recursive utility functional with generator:

$$f(t,u,x,y)=\rho_t x+(\mu_t-\rho_t)u-\beta y, \tag{43}$$

where constant $\beta\geq 0$.

Remark 6. In fact, the recursive utility functional (43) defined above stands for some standard additive utility of recursive type. It is a meaningful and nontrivial generalization of the classical standard additive utility and has many applications in mathematical economics and mathematical finance. For more details about utility functions, see Duffie and Epstein [2], Section 1.4 of El Karoui et al. [3] or Schroder and Skiadas [4].

More precisely, for any $u(\cdot)\in\mathbf{U}_{\mathrm{ad}}$, the investor's utility functional is defined by

$$\overline{J}(u(\cdot)):=Y^u(t)\big|_{t=0}, \tag{44}$$

where

$$\begin{aligned} Y^u(t) &:=\mathbb{E}\Big[-\frac{1}{2}(X^u(T)-a)^2\\ &\quad+\int_t^T\left[\rho_s X^u(s)+(\mu_s-\rho_s)u(s)-\beta Y^u(s)\right]ds\mid\mathcal{F}_t\Big],\\ &\quad\forall t\in[0,T]. \end{aligned} \tag{45}$$

In fact, in our framework, the wealth process $X^u(\cdot)$ and recursive utility process $Y^u(\cdot)$ can be regarded as the solution to the following controlled FBSDE:

$$\begin{aligned} dX^u(t)&=\left[\rho_t X^u(t)+(\mu_t-\rho_t)u(t)\right]dt+\sigma_t u(t)\,dW(t),\\ -dY^u(t)&=\left[\rho_t X^u(t)+(\mu_t-\rho_t)u(t)-\beta Y^u(t)\right]dt\\ &\quad-Z^u(t)\,dW(t),\quad t\in[0,T],\\ X^u(0)&=X_0,\qquad Y^u(T)=-\frac{1}{2}(X^u(T)-a)^2, \end{aligned} \tag{46}$$

and our portfolio optimization problem can be rewritten as (denoting $J = -\overline{J}$)

$$J\left(u^{*}(\cdot)\right) = \inf_{u(\cdot)\in\mathbf{U}_{\mathrm{ad}}} J\left(u(\cdot)\right). \tag{47}$$

Since we are going to involve the dynamic programming in treating the above problem, we will adopt the formulation as in Section 2. Let $T > 0$ be given. For any $(t, x) \in [0, T) \times \mathbf{R}$, consider the following controlled FBSDE:

$$\begin{aligned} dX^{t,x;u}(s) &= \left[\rho_s X^{t,x;u}(s) + (\mu_s - \rho_s) u(s)\right] ds \\ &\quad + \sigma_s u(s)\, dW(s), \\ -dY^{t,x;u}(s) &= \left[\rho_s X^{t,x;u}(s) + (\mu_s - \rho_s) u(s) - \beta Y^{t,x;u}(s)\right] ds \\ &\quad - Z^{t,x;u}(s)\, dW(s), \quad s \in [t, T], \\ X^{t,x;u}(t) &= x, \qquad Y^{t,x;u}(T) = -\frac{1}{2}\left(X^{t,x;u}(T) - a\right)^2. \end{aligned} \tag{48}$$

And our recursive utility portfolio optimization problem is to find an optimal portfolio $u^*(\cdot) \in \mathbf{U}_{\mathrm{ad}}$ to minimize the recursive utility functional $J(t, x; u(\cdot)) := -Y^{t,x;u}(s)|_{s=t}$. We define the value function as

$$V(t, x) := J\left(t, x; u^{*}(\cdot)\right) = \inf_{u(\cdot)\in\mathbf{U}_{\mathrm{ad}}} J\left(t, x; u(\cdot)\right). \tag{49}$$

We can check that all the assumptions in Section 2 are satisfied. Then we can use both the dynamic programming (Lemma 2) and maximum principle (Lemmas 3 and 4) approaches to solve the above problem (49).

4.1. Maximum Principle Approach. In this case, the Hamiltonian function (18) reduces to

$$\begin{aligned} H(s, x, y, z, u, p, q, k) &= p\left[\rho_s x + (\mu_s - \rho_s) u\right] \\ &\quad - q\left[\rho_s x + (\mu_s - \rho_s) u - \beta y\right] + k\sigma_s u, \end{aligned} \tag{50}$$

and the adjoint equation (17) writes

$$\begin{aligned} -dp(s) &= \rho_s\left[p(s) - q(s)\right] ds - k(s)\, dW(s), \\ dq(s) &= -\beta q(s)\, ds, \quad s \in [t, T], \\ p(T) &= \left[X^{t,x;u}(T) - a\right] q(T), \quad q(t) = 1. \end{aligned} \tag{51}$$

Noting that in this case the adjoint process $q(\cdot)$ reduced to a deterministic function because our generator f does not contain the process $z(\cdot)$. We immediately have

$$q(s) = e^{-\beta(s-t)}, \quad s \in [t, T]. \tag{52}$$

Due to the terminal condition of (51), we try a process $p(\cdot)$ of the form

$$p(s) = \left[\phi_s X^{t,x;u}(s) + \psi_s\right] e^{-\beta(s-t)}, \quad s \in [t, T], \tag{53}$$

where ϕ_s, ψ_s are deterministic differentiable functions. Applying Itô's formula to (53), we have

$$\begin{aligned} dp(s) &= e^{-\beta(s-t)}\left[\left(\dot{\phi}_s + (\rho_s - \beta)\phi_s\right) X^{t,x;u}(s)\right. \\ &\qquad \left. + \phi_s(\mu_s - \rho_s) u(s) + \dot{\psi}_s - \beta\psi_s\right] ds \\ &\quad + e^{-\beta(s-t)}\phi_s\sigma_s u(s)\, dW(s). \end{aligned} \tag{54}$$

Comparing (51) with (54), noting (52) and (53), we get

$$\begin{aligned} &\left(\dot{\phi}_s + (\rho_s - \beta)\phi_s\right) X^{t,x;u}(s) + \phi_s(\mu_s - \rho_s) u(s) + \dot{\psi}_s - \beta\psi_s \\ &\quad = -\rho_s\left(\phi_s X^{t,x;u}(s) + \psi_s - 1\right), \end{aligned} \tag{55}$$

$$k(s) = \phi_s\sigma_s u(s) e^{-\beta(s-t)}. \tag{56}$$

Let $u^*(\cdot)$ be a candidate optimal portfolio, $(X^{t,x;u^*}(\cdot), Y^{t,x;u^*}(\cdot), Z^{t,x;u^*}(\cdot))$ the corresponding solution to controlled FBSDE (48) with corresponding solution $(p^*(\cdot), q^*(\cdot), k^*(\cdot))$ to the adjoint equation (51). Now the Hamiltonian function (50) is

$$\begin{aligned} &H\left(s, X^{t,x;u^*}(s), Y^{t,x;u^*}(s), Z^{t,x;u^*}(s), u, p^*(s), q^*(s), k^*(s)\right) \\ &\quad = p^*(s)\left[\rho_s X^{t,x;u^*}(s) + (\mu_s - \rho_s) u\right] \\ &\qquad - q^*(s)\left[\rho_s X^{t,x;u^*}(s) + (\mu_s - \rho_s) u - \beta Y^{t,x;u^*}(s)\right] \\ &\qquad + k^*(s)\sigma_s u. \end{aligned} \tag{57}$$

Since this is a linear expression of u, by the maximum condition (19), we have

$$(\mu_s - \rho_s)\left(p^*(s) - q^*(s)\right) + \sigma_s k^*(s) = 0. \tag{58}$$

Substituting (52), (53) and (56) into (58), we can get

$$u^*(s) = \frac{(\rho_s - \mu_s)\left(\phi_s X^{t,x;u^*}(s) + \psi_s - 1\right)}{\phi_s\sigma_s^2}. \tag{59}$$

On the other hand, (55) gives

$$\begin{aligned} &u^*(s) \\ &\quad = \frac{\left(\dot{\phi}_s + 2\rho_s\phi_s - \beta\phi_s\right) X^{t,x;u^*}(s) + \dot{\psi}_s + (\rho_s - \beta)\psi_s - \rho_s}{\phi_s(\rho_s - \mu_s)}. \end{aligned} \tag{60}$$

Combining (59) and (60) (noting the terminal condition in (51)), we get

$$\dot{\phi}_s = \left[\frac{(\rho_s - \mu_s)^2}{\sigma_s^2} - 2\rho_s + \beta\right]\phi_s, \qquad \phi_T = 1, \tag{61}$$

$$\begin{aligned} \dot{\psi}_s &= \left[\frac{(\rho_s - \mu_s)^2}{\sigma_s^2} - \rho_s + \beta\right]\psi_s - \frac{(\rho_s - \mu_s)^2}{\sigma_s^2} + \rho_s, \\ \psi_T &= -a. \end{aligned} \tag{62}$$

The solutions to these equations are

$$\phi_s = e^{-\int_s^T [(\rho_r-\mu_r)^2/\sigma_s^2 - 2\rho_r+\beta]dr}, \tag{63}$$

$$\psi_s = e^{-\int_s^T [(\rho_r-\mu_r)^2/\sigma_r^2 - \rho_r+\beta]dr} \times \left\{ \int_s^T \left[\frac{(\rho_r-\mu_r)^2}{\sigma_r^2} - \rho_r \right] e^{\int_s^T [(\rho_\lambda-\mu_\lambda)^2/\sigma_\lambda^2 - \rho_\lambda+\beta]d\lambda} dr - a \right\}. \tag{64}$$

With this choice of ϕ_s and ψ_s, the processes

$$q^*(s) = e^{-\beta(s-t)}, \qquad p^*(s) = \left[\phi_s X^{t,x;u*}(s) + \psi_s\right] e^{-\beta(s-t)}, \qquad k^*(s) = \phi_s\sigma_s u^*(s)e^{-\beta(s-t)} \tag{65}$$

satisfying the adjoint equation (51) with $u^*(s)$ given by (59). With this choice of $u^*(s)$, the maximum condition (19) of Lemma 3 holds. Moreover, we can check that all conditions in Lemma 4 hold, then $u^*(s)$ given by (59) is really the optimal control.

Finally, let $t = 0$, then we can solve the initial problem (47) and give the explicit optimal portfolio in the state feedback form.

Theorem 7. *The optimal solution $u^*(\cdot)$ of our recursive utility portfolio optimization problem* (47)*, when the wealth dynamics obeys* (41)*, is given in the state feedback form by*

$$u^*(s, X^*) = \frac{(\rho_s-\mu_s)(\phi_s X^* + \psi_s - 1)}{\phi_s\sigma_s^2}, \tag{66}$$

for $s \in [0, T]$, where ϕ_s, ψ_s are given by (63) *and* (64)*, respectively.*

4.2. Dynamic Programming Approach. In this case, the value function V should satisfy the following generalized HJB equation:

$$\begin{gathered} -V_t(t,x) + \sup_{u\in\mathbf{U}} G(t, x, -V(t,x), -V_x(t,x), -V_{xx}(t,x), u) = 0, \quad (t,x)\in[0,T]\times\mathbf{R}^n, \\ V(T,x) = \frac{1}{2}(x-a)^2, \quad x\in\mathbf{R}^n, \end{gathered} \tag{67}$$

where the generalized Hamiltonian function (8) is

$$\begin{aligned} &G(t, x, -V(t,x), -V_x(t,x), -V_{xx}(t,x), u) \\ &= -\frac{1}{2}V_{xx}(t,x)\sigma_s^2 u^2 - V_x(t,x)\left[\rho_t x + (\mu_t-\rho_t)u\right] \\ &\quad + \rho_t x + (\mu_t-\rho_t)u + \beta V(t,x). \end{aligned} \tag{68}$$

We conjecture that $V(t,x)$ is quadratic in x, namely,

$$V(t,x) = \frac{1}{2}P_t x^2 + \varphi_t x + \lambda_t, \tag{69}$$

for some deterministic differentiable functions P_t, φ_t, and λ_t with

$$P_T = 1, \qquad \varphi_T = -a, \qquad \lambda_T = \frac{1}{2}a^2. \tag{70}$$

Substituting (69) into (68) and using completion of squares repeatedly, we get

$$\begin{aligned} &G(t, x, -V(t,x), -V_x(t,x), -V_{xx}(t,x), u) \\ &= -\frac{1}{2}P_t\sigma_t^2\left[u(t) - \frac{(\rho_t-\mu_t)(P_t x + \varphi_t - 1)}{P_t\sigma_t^2}\right]^2 \\ &\quad + \left[\frac{(\mu_t-\rho_t)^2 P_t}{2\sigma_t^2} + \frac{1}{2}\beta P_t - \rho_t P_t\right]x^2 \\ &\quad + \left[\frac{(\rho_t-\mu_t)^2}{\sigma_t^2}(\varphi_t - 1) + \beta\varphi_t - \rho_t\varphi_t + \rho_t\right]x \\ &\quad - \frac{(\rho_t-\mu_t)^2\varphi_t}{P_t\sigma_t^2} + \beta\lambda_t \\ &\le \left[\frac{(\mu_t-\rho_t)^2 P_t}{2\sigma_t^2} + \frac{1}{2}\beta P_t - \rho_t P_t\right]x^2 \\ &\quad + \left[\frac{(\rho_t-\mu_t)^2}{\sigma_t^2}(\varphi_t - 1) + \beta\varphi_t - \rho_t\varphi_t + \rho_t\right]x \\ &\quad - \frac{(\rho_t-\mu_t)^2\varphi_t}{P_t\sigma_t^2} + \beta\lambda_t, \end{aligned} \tag{71}$$

provided that $P_t > 0$ for all $t \in [0, T]$, which we will prove later. Then we see that the optimal state feedback portfolio is given by

$$u^*(t, X^*) = \frac{(\rho_t-\mu_t)(P_t X^* + \varphi_t - 1)}{P_t\sigma_t^2}. \tag{72}$$

In addition, the generalized HJB equation (67) now reads

$$\begin{aligned} &\frac{1}{2}\dot{P}_t x^2 + \dot{\varphi}_t x + \dot{\lambda}_t \\ &= \left[\frac{(\mu_t-\rho_t)^2 P_t}{2\sigma_t^2} + \frac{1}{2}\beta P_t - \rho_t P_t\right]x^2 \\ &\quad + \left[\frac{(\rho_t-\mu_t)^2}{\sigma_t^2}(\varphi_t - 1) + \beta\varphi_t - \rho_t\varphi_t + \rho_t\right]x \\ &\quad - \frac{(\rho_t-\mu_t)^2\varphi_t}{P_t\sigma_t^2} + \beta\lambda_t. \end{aligned} \tag{73}$$

Then noting (70), by comparing the quadratic terms and linear terms in x, we recover (61) and (62), respectively. That

is to say, we have that P_t coincides with ϕ_t and φ_t coincides with ψ_t. Then by (63), we have $\phi_t > 0$, for all $t \in [0,T]$ as expected before. And also we have

$$\dot{\lambda}_t = -\frac{(\rho_t - \mu_t)^2 \psi_t}{\phi_t \sigma_t^2} + \beta\lambda_t, \qquad \lambda_T = \frac{1}{2}a^2. \tag{74}$$

The solution to (74) is

$$\lambda_t = e^{-\beta(T-t)} \left\{ \int_t^T \left[\frac{(\rho_r - \mu_r)^2 \psi_r}{\phi_r \sigma_r^2} - \rho_r \right] e^{\beta(T-r)} dr + \frac{1}{2}a^2 \right\}. \tag{75}$$

Then the value function

$$V(t,x) = \frac{1}{2}\phi_t x^2 + \psi_t x + \lambda_t, \tag{76}$$

where ϕ_t, ψ_t, and λ_t are determined by (61), (62), and (75), respectively. By Lemma 2, we have proved the following

Theorem 8. *The optimal solution $u^*(\cdot)$ for our recursive utility portfolio optimization problem* (47)*, when the wealth dynamics obeys* (41)*, is given in the state feedback form by*

$$u^*(t, X^*) = \frac{(\rho_t - \mu_t)(\phi_t X^* + \psi_t - 1)}{\phi_t \sigma_t^2}, \tag{77}$$

for $t \in [0,T]$, and the value function is given by (76)*, where ϕ_t, ψ_t, and λ_t are determined by* (61), (62)*, and* (75)*, respectively.*

4.3. Relationship. We now can explicitly illustrate the relationships in Theorem 5. In fact, relationship (23) is obvious from (67). And (65) is exactly the relationship given in (25) and (24).

5. Concluding Remarks

In this paper, we have studied the relationship between maximum principle and dynamic programming for stochastic recursive optimal control problems. Under certain differentiability conditions, we give relations among the adjoint processes, the generalized Hamiltonian function, and the value function. A linear quadratic recursive utility portfolio optimization problem in the financial market is discussed as an explicitly illustrated example of our result.

An interesting and challenging problem remains open. For the stochastic recursive optimal control problem, what is the relationship between maximum principle and dynamic programming without the illusory differentiability conditions on the value function? This problem may be solved in the framework of nonsmooth analysis. Viscosity solution theory is certainly a nice tool (e.g., see Yong and Zhou [9]). A new result on stochastic verification theorem for forward-backward controlled system using viscosity solutions has been published very recently by Zhang [29]. However, at this moment, we do not have publishable results for the relationship within the framework of viscosity solutions. We hope to address this problem in the future work.

Acknowledgments

The authors would like to thank the anonymous referees for many constructive comments that led to an improved version of the paper. The authors also thank the Academic Editor for his efficient handling of this paper. Finally, many thanks are devoted to Dr. Qingxin Meng for his suggesting discussion during the revising progress. This work is supported by China Postdoctoral Science Foundation Funded Project (No. 20100481278), Postdoctoral Innovation Foundation Funded Project of Shandong Province (No. 201002026), the National Natural Sciences Foundations of China (No. 11201264 and 11101242) and Shandong Province (No. ZR2011AQ012, ZR2010AQ004), and the Scientific Research Foundation for the Returned Overseas Chinese Scholars, State Education Ministry, China.

References

[1] E. Pardoux and S. G. Peng, "Adapted solution of a backward stochastic differential equation," *Systems & Control Letters*, vol. 14, no. 1, pp. 55–61, 1990.

[2] D. Duffie and L. G. Epstein, "Stochastic differential utility," *Econometrica*, vol. 60, no. 2, pp. 353–394, 1992.

[3] N. El Karoui, S. G. Peng, and M. C. Quenez, "Backward stochastic differential equations in finance," *Mathematical Finance*, vol. 7, no. 1, pp. 1–71, 1997.

[4] M. Schroder and C. Skiadas, "Optimal consumption and portfolio selection with stochastic differential utility," *Journal of Economic Theory*, vol. 89, no. 1, pp. 68–126, 1999.

[5] N. El Karoui, S. G. Peng, and M. C. Quenez, "A dynamic maximum principle for the optimization of recursive utilities under constraints," *The Annals of Applied Probability*, vol. 11, no. 3, pp. 664–693, 2001.

[6] S. L. Ji and X. Y. Zhou, "A maximum principle for stochastic optimal control with terminal state constraints, and its applications," *Communications in Information and Systems*, vol. 6, no. 4, pp. 321–337, 2006.

[7] N. Williams, "On dynamic principal-agent problems in continuous time," Working paper, http://www.ssc.wisc.edu/~nwilliam/dynamic-pa1.pdf, 2008.

[8] G. C. Wang and Z. Wu, "The maximum principles for stochastic recursive optimal control problems under partial information," *IEEE Transactions on Automatic Control*, vol. 54, no. 6, pp. 1230–1242, 2009.

[9] J. M. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, NY, USA, 1999.

[10] S. G. Peng, "Backward stochastic differential equations and applications to optimal control," *Applied Mathematics and Optimization*, vol. 27, no. 2, pp. 125–144, 1993.

[11] W. S. Xu, "Stochastic maximum principle for optimal control problem of forward and backward system," *Journal of the Australian Mathematical Society B*, vol. 37, no. 2, pp. 172–185, 1995.

[12] Z. Wu, "A general maximum principle for optimal control problems of forward-backward stochastic control systems," In press.

[13] J. T. Shi and Z. Wu, "Maximum principle for forward-backward stochastic control system with random jumps and applications to finance," *Journal of Systems Science & Complexity*, vol. 23, no. 2, pp. 219–231, 2010.

[14] S. G. Peng, "A generalized dynamic programming principle and Hamilton-Jacobi-Bellman equation," *Stochastics & Stochastics Reports*, vol. 38, no. 2, pp. 119–134, 1992.

[15] S. G. Peng, "Backward stochastic differential equations-stochastic optimization theory and viscosity solutions of HJB equations," in *Topics on Stochastic Analysis*, J. Yan, S. Peng, S. Fang, and L. Wu, Eds., pp. 85–138, Science Press, Beijing, China, 1997.

[16] Z. Wu and Z. Y. Yu, "Dynamic programming principle for one kind of stochastic recursive optimal control problem and Hamilton-Jacobi-Bellman equation," *SIAM Journal on Control and Optimization*, vol. 47, no. 5, pp. 2616–2641, 2008.

[17] J. Li and S. G. Peng, "Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton-Jacobi-Bellman equations," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 4, pp. 1776–1796, 2009.

[18] J. M. Bismut, "An introductory approach to duality in optimal stochastic control," *SIAM Review*, vol. 20, no. 1, pp. 62–78, 1978.

[19] A. Bensoussan, *Lectures on Stochastic Control*, vol. 972 of *Lecture Notes in Mathematics*, Springer-Verlag, Berlin, Germany, 1982.

[20] X. Y. Zhou, "Maximum principle, dynamic programming, and their connection in deterministic control," *Journal of Optimization Theory and Applications*, vol. 65, no. 2, pp. 363–373, 1990.

[21] X. Y. Zhou, "The connection between the maximum principle and dynamic programming in stochastic control," *Stochastics & Stochastics Reports*, vol. 31, no. 1–4, pp. 1–13, 1990.

[22] X. Y. Zhou, "A unified treatment of maximum principle and dynamic programming in stochastic controls," *Stochastics & Stochastics Reports*, vol. 36, no. 3-4, pp. 137–161, 1991.

[23] N. C. Framstad, B. Øksendal, and A. Sulem, "Sufficient stochasticmaximum principle for the optimal control of jump diffusions and applications to finance," *Journal of Optimization Theory and Applications*, vol. 121, no. 1, pp. 77–98, 2004.

[24] N. C. Framstad, B. Øksendal, and A. Sulem, "Errata corrige. Sufficient stochastic maximum principle for the optimal control of jump diffusions and applications to finance," *Journal of Optimization Theory and Applications*, vol. 124, no. 2, pp. 511–512, 2005.

[25] J. T. Shi and Z. Wu, "Relationship between MP and DPP for the stochastic optimal control problem of jump diffusions," *Applied Mathematics and Optimization*, vol. 63, no. 2, pp. 151–189, 2011.

[26] K. Bahlali, F. Chighoub, and B. Mezerdi, "On the relationship between the stochastic maximum principle and dynamic programming in singular stochastic control," *Stochastics*, vol. 84, no. 2-3, pp. 233–249, 2012.

[27] X. Zhang, R. J. Elliott, and T. K. Siu, "A stochastic maximum principle for a Markov regime-switching jump-diffusion model and its application to finance," *SIAM Journal on Control and Optimization*, vol. 50, no. 2, pp. 964–990, 2012.

[28] M. Kohlmann and X. Y. Zhou, "Relationship between backward stochastic differential equations and stochastic controls: a linear-quadratic approach," *SIAM Journal on Control and Optimization*, vol. 38, no. 5, pp. 1392–1407, 2000.

[29] L. Q. Zhang, "Stochastic verification theorem of forward-backward controlled systems for viscosity solutions," *Systems & Control Letters*, vol. 61, no. 5, pp. 649–654, 2012.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 718714, 15 pages
doi:10.1155/2012/718714

Research Article

Stochastic Recursive Zero-Sum Differential Game and Mixed Zero-Sum Differential Game Problem

Lifeng Wei[1] and Zhen Wu[2]

[1] *School of Mathematical Sciences, Ocean University of China, Qingdao 266003, China*
[2] *School of Mathematics, Shandong University, Jinan 250100, China*

Correspondence should be addressed to Zhen Wu, wuzhensdu@gmail.com

Received 4 October 2012; Accepted 10 December 2012

Academic Editor: Guangchen Wang

Under the notable Issacs's condition on the Hamiltonian, the existence results of a saddle point are obtained for the stochastic recursive zero-sum differential game and mixed differential game problem, that is, the agents can also decide the optimal stopping time. The main tools are backward stochastic differential equations (BSDEs) and double-barrier reflected BSDEs. As the motivation and application background, when loan interest rate is higher than the deposit one, the American game option pricing problem can be formulated to stochastic recursive mixed zero-sum differential game problem. One example with explicit optimal solution of the saddle point is also given to illustrate the theoretical results.

1. Introduction

The nonlinear backward stochastic differential equations (BSDEs in short) had been introduced by Pardoux and Peng [1], who proved the existence and uniqueness of adapted solutions under suitable assumptions. Independently, Duffie and Epstein [2] introduced BSDE from economic background. In [2], they presented a stochastic differential recursive utility which is an extension of the standard additive utility with the instantaneous utility depending not only on the instantaneous consumption rate but also on the future utility. Actually, it corresponds to the solution of a particular BSDE whose generator does not depend on the variable Z. From mathematical point of view, the result in [1] is more general. Then, El Karoui et al. [3] and Cvitanic and Karatzas [4] generalized, respectively, the results to BSDEs with reflection at one barrier and two barriers (upper and lower).

BSDE plays an important role in the theory of stochastic differential game. Under the notable Isaacs's condition, Hamadène and Lepeltier [5] obtained the existence result of a saddle point for zero-sum stochastic differential game with payoff

$$J(u,v) = E^{(u,v)}\left[\int_t^T f(s,x_s,u_s,v_s)ds + g(x_T)\right]. \tag{1.1}$$

Using a maximum principle approach, Wang and Yu [6, 7] proved the existence and uniqueness of an equilibrium point. We note that the cost function in [5] is not recursive, and the game system in [6, 7] is a BSDE. In [8], El Karoui et al. gave the formulation of recursive utilities and their properties from the BSDE's pointview. The problem that the cost function (payoff) of the game system is described by the solution of BSDE becomes the recursive differential game problem. In the following Section 2, we proved the existence of a saddle point for the stochastic recursive zero-sum differential game problem and also got the optimal payoff function by the solution of one specific BSDE. Here, the generator of the BSDE contains the main variable solution y_t, and we extend the result in [5] to the recursive case which has much more significance in economics theory.

Then, in Section 3 we study the stochastic recursive mixed zero-sum differential game problem which is that the two agents have two actions, one is of control and the other is of stopping their strategies to maximize and minimize their payoffs. This kind of game problem without recursive variable and the American game option problem as this kind of mixed game problem can be seen in Hamadène [9]. Using the result of reflected BSDEs with two barriers, we got the saddle point and optimal stopping strategy for the recursive mixed game problem which has more general significance than that in [9].

In fact, the recursive (mixed) zero-sum game problem has wide application background in practice. When the loan interest rate is higher than the deposit one. The American game option pricing problem can be formulated to the stochastic recursive mixed game problem in our Section 3. To show the application of this kind of problem and our motivation to study our recursive (mixed) game problem, we analyze the American game option pricing problem and let it be an example in Section 4. We notice that in [5, 9], they did not give the explicit saddle point to the game, and it is very difficult for the general case. However, in Section 4, we also give another example of the recursive mixed zero-sum game problem, for which the explicit saddle point and optimal payoff function to illustrate the theoretical results.

2. Stochastic Recursive Zero-Sum Differential Game

In this section, we will study the existence of the stochastic recursive zero-sum differential game problem using the result of BSDEs.

Let $\{B_t, 0 \le t \le T\}$ be an m-dimensional standard Brownian motion defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t)_{t\ge 0}$ be the completed natural filtration of B_t. Moreover,

(i) $\mathcal{C}$ is the space of continuous functions from $[0,T]$ to R^m;

(ii) $\mathcal{P}$ is the σ-algebra on $[0,T]\times\Omega$ of $\mathcal{F}_t$-progressively sets;

(iii) for any stopping time $\nu, \mathcal{T}_\nu$ is the set of $\mathcal{F}_t$-measurable stopping time τ such that P-$a.s.$ $\nu \le \tau \le T$; $\mathcal{T}_0$ will simply be denoted by $\mathcal{T}$;

(iv) $\mathscr{H}^{2,k}$ is the set of $\mathcal{P}$-measurable processes $\omega = (\omega_t)_{t\leq T}$, R^k-valued, and square integrable with respect to $dt \otimes d\mathcal{P}$;

(v) $\mathcal{S}^2$ is the set of $\mathcal{P}$-measurable and continuous processes $\omega' = (\omega'_t)_{t\leq T}$, such that $E[\sup_{t\leq T}|\omega'_t|^2] < \infty$.

The $m \times m$ matrix $\sigma = (\sigma_{ij})$ satisfies the following:

(i) for any $1 \leq i, j \leq m, \sigma_{ij}$ is progressively measurable;

(ii) for any $(t,x) \in [0,T] \times \mathcal{C}$, the matrix $\sigma(t,x)$ is invertible;

(iii) there exists a constants K such that $|\sigma(t,x) - \sigma(t,x')| \leq K|x - x'|_t$ and $|\sigma(t,x)| \leq K(1 + |x|_t)$.

Then, the equation

$$x_t = x_0 + \int_0^t \sigma(s,x_s)dB_s, \quad t \leq T \tag{2.1}$$

has a unique solution (x_t).

Now, we consider a compact metric space A (resp. B), and $\mathcal{U}$ (resp. $\mathcal{V}$) is the space of $\mathcal{P}$-measurable processes $u := (u_t)_{t\leq T}$ (resp. $v := (v_t)_{t\leq T}$) with values in A (resp. B). Let $\Phi : [0,T] \times \mathcal{C} \times \mathcal{U} \times \mathcal{V} \longrightarrow R^m$ be such that

(i) for any $(t,x) \in [0,T] \times \mathcal{C}$, the mapping $(u,v) \longrightarrow \Phi(t,x,u,v)$ is continuous;

(ii) for any $(u,v) \in A \times B$, the function $\Phi(\cdot, x(\cdot), u, v)$ is $\mathcal{P}$-measurable;

(iii) there exists a constant K such that $|\Phi(t,x,u,v)| \leq K(1 + |x|_t)$ for any t, x, u, and v;

(iv) there exists a constant M such that $|\sigma^{-1}(t,x)\Phi(t,x,u,v)| \leq M$ for any t, x, u, and v.

For $(u,v) \in \mathcal{U} \times \mathcal{V}$, we define the measure $P^{u,v}$ as

$$\frac{dP^{u,v}}{dP} = \exp\left\{\int_0^T \sigma^{-1}(s,x_s)\Phi(s,x_s,u_s,v_s)dB_s - \frac{1}{2}\int_0^T \left|\sigma^{-1}(s,x_s)\Phi(s,x_s,u_s,v_s)\right|^2 ds\right\}. \tag{2.2}$$

Thanks to Girsanov's theorem, under the probability $P^{u,v}$, the process

$$B_t^{u,v} = B_t - \int_0^t \sigma^{-1}(s,x_s)\Phi(s,x_s,u_s,v_s)ds, \quad t \leq T, \tag{2.3}$$

is a Brownian motion, and for this stochastic differential equation

$$x_t = x_0 + \int_0^t \Phi(s,x_s,u_s,v_s)ds + \int_0^t \sigma(s,x_s)dB_s^{u,v}, \quad t \leq T, \tag{2.4}$$

$(x_t)_{t\leq T}$ is a weak solution.

Suppose that we have a system whose evolution is described by the process $(x_t)_{t\leq T}$. On that system, two agents c_1 and c_2 intervene. A control action for c_1 (resp. c_2) is a process

$u = (u_t)_{t\leq T}$ (resp. $v = (v_t)_{t\leq T}$) belonging to $\mathcal{U}$ (resp. $\mathcal{V}$). Thereby $\mathcal{U}$ (resp. $\mathcal{V}$) is called the set of admissible controls for c_1 (resp. c_2). When c_1 and c_2 act with, respectively, u and v, the law of the dynamics of the system is the same as the one of x under $P^{u,v}$. The two agents have no influence on the system, and they act to protect their advantages by means of $u \in \mathcal{U}$ and $v \in \mathcal{V}$ via the probability $P^{u,v}$.

In order to define the payoff, we introduce two functions $C(t,x,y,u,v)$ and $g(x)$ satisfying the following assumption: there exists $L > 0$, for all $x, x' \in \mathcal{H}^{2,m}$ and $Y, Y' \in \mathcal{S}^2$, such that

$$\begin{gathered}\left|C(t,x_t,Y_t,u,v) - C\left(t,x'_t,Y_t,u,v\right)\right| \leq L\left|x_t - x'_t\right|, \\ \left(Y_t - Y'_t\right)\left(C(t,x_t,Y_t,u,v) - C\left(t,x_t,Y'_t,u,v\right)\right) \leq L\left(Y_t - Y'_t\right)^2,\end{gathered} \tag{2.5}$$

and $g(x)$ is measurable, Lipschitz continuous function with respect to x. The payoff $J(x_0,u,v)$ is given by $J(x_0,u,v) = Y_0$, where Y satisfies the following BSDE:

$$\begin{gathered}-dY_s = C(s,x_s,Y_s,u_s,v_s)ds - Z_s dB_s^{u,v}, \\ Y_T = g(x_T).\end{gathered} \tag{2.6}$$

From the result in [10], there exists a unique solution (Y,Z) for u,v. The agent c_1 wishes to minimize this payoff, and the agent c_2 wishes to maximize the same payoff. We investigate the existence of a saddle point for the game, more precisely a pair (u^*,v^*) of strategies, such that $J(x_0,u^*,v) \leq J(x_0,u^*,v^*) \leq J(x_0,u,v^*)$ for each $(u,v) \in \mathcal{U}\times\mathcal{V}$.

For $(t,x,Y,Z,u,v) \in [0,T]\times\mathcal{C}\times R\times R^m\times\mathcal{U}\times\mathcal{V}$, we introduce the Hamiltonian by

$$H(t,x,Y,Z,u,v) = Z\sigma^{-1}(t,x)\Phi(t,x,u,v) + C(t,x,Y,u,v), \tag{2.7}$$

and we say that the Isaacs' condition holds if for $(t,x,Y,Z) \in [0,T]\times\mathcal{C}\times R\times R^m$,

$$\max_{v\in\mathcal{V}}\min_{u\in\mathcal{U}} H(t,x,Y,Z,u,v) = \min_{u\in\mathcal{U}}\max_{v\in\mathcal{V}} H(t,x,Y,Z,u,v). \tag{2.8}$$

We suppose now that the Isaacs' condition is satisfied. By a selection theorem (see Benes [11]), there exists $u^* : [0,T]\times\mathcal{C}\times R\times R^m \to \mathcal{U}$, $v^* : [0,T]\times\mathcal{C}\times R\times R^m \to \mathcal{V}$, such that

$$H(t,x,Y,Z,u^*,v) \leq H(t,x,Y,Z,u^*,v^*) \leq H(t,x,Y,Z,u,v^*). \tag{2.9}$$

Thanks to the assumption of σ, Φ, and C, the function $H(t,x,Y,Z,u^*(t,x,Y,Z), v^*(t,x,Y,Z))$ is Lipschitz in Z and monotone in Y like the function C.

Now we give the main result of this section.

Theorem 2.1. *(Y^*,Z^*) is the solution of the following BSDE:*

$$\begin{gathered}-dY_s^* = H(s,x_s,Y_s^*,Z_s^*,u^*(s,x,Y^*,Z^*),v^*(s,x,Y^*,Z^*))ds - Z_s^* dB_s, \\ Y_T^* = g(x_T).\end{gathered} \tag{2.10}$$

Then, Y_0^* is the optimal payoff $J(x_0, u^*, v^*)$, and the pair (u^*, v^*) is the saddle point for this recursive game.

Proof. We consider the following BSDE:

$$Y_t^* = g(x_T) + \int_t^T H(s, x_s, Y_s^*, Z_s^*, u^*(t, x, Y^*, Z^*), v^*(t, x, Y^*, Z^*))ds - \int_t^T Z_s^* dB_s. \tag{2.11}$$

Thanks to Theorem 2.1 in [10], the equation has a unique solution (Y^*, Z^*). Because Y_0^* is deterministic, so

$$\begin{aligned} Y_0^* &= E^{u^*,v^*}[Y_0^*] \\ &= E^{u^*,v^*}\left[g(x_T) + \int_0^T H(s, x_s, Y_s^*, Z_s^*, u^*(t, x, Y^*, Z^*), v^*(t, x, Y^*, Z^*))ds - \int_0^T Z_s^* dB_s\right] \\ &= E^{u^*,v^*}\left[g(x_T) + \int_0^T C(s, x_s, Y_s^*, u_s^*, v_s^*)ds - \int_0^T Z_s^* dB_s^{u^*,v^*}\right]. \end{aligned} \tag{2.12}$$

We can get $Y_0^* = J(x_0, u^*, v^*)$.

For any $u \in \mathcal{U}, v \in \mathcal{V}$, then we let

$$\begin{aligned} Y_t &= g(x_T) + \int_t^T C(s, x_s, Y_s, u_s^*, v_s)ds - \int_t^T Z_s dB_s^{u^*,v} \\ &= g(x_T) + \int_t^T H(s, x_s, Y_s, Z_s, u_s^*, v_s)ds - \int_t^T Z_s dB_s, \\ Y_t' &= g(x_T) + \int_t^T C(s, x_s, Y_s', u_s, v_s^*)ds - \int_t^T Z_s' dB_s^{u,v^*} \\ &= g(x_T) + \int_t^T H(s, x_s, Y_s', Z_s', u_s, v_s^*)ds - \int_t^T Z_s' dB_s. \end{aligned} \tag{2.13}$$

By the comparison theorem of the BSDEs and the inequality (2.9), we can compare the solutions of (2.11), and (2.13) and get $Y_t \le Y_t^* \le Y_t'$, $0 \le t \le T$, so $Y_0 = J(x_0, u^*, v) \le J(x_0, u^*, v^*) \le J(x_0, u, v^*) = Y_0'$ and (u^*, v^*) is the saddle point. □

3. Stochastic Recursive Mixed Zero-Sum Differential Game

Now, we study the stochastic recursive mixed zero-sum differential game problem. First, let us briefly describe the problem.

Suppose now that we have a system, whose evolution also is described by $(x_t)_{0\le t\le T}$, which has an effect on the wealth of two controllers C_1 and C_2. On the other hand, the controllers have no influence on the system, and they act so as to protect their advantages, which are antagonistic, by means of $u \in \mathcal{U}$ for C_1 and $v \in \mathcal{V}$ for C_2 via the probability $P^{u,v}$ in (2.2). The couple $(u, v) \in \mathcal{U} \times \mathcal{V}$ is called an admissible control for the game. Both controllers

also have the possibility to stop controlling at τ for C_1 and θ for C_2; τ and θ are elements of $\mathcal{T}$ which is the class of all $\mathcal{F}_t$-stopping time. In such a case, the game stops. The controlling action is not free, and it corresponds to the actions of C_1 and C_2. A payoff is described by the following BSDE:

$$\begin{aligned} Y_t^{u,\tau;v,\theta} &= U_\tau 1_{[\tau<\theta]} + L_\theta 1_{[\theta<\tau<T]} + Q_\tau 1_{[\tau=\theta<T]} + g(x_T) 1_{[\tau=\theta=T]} \\ &\quad + \int_t^{\tau\wedge\theta} C\left(s, x_s, Y_s^{u,\tau;v,\theta}, u_s, v_s\right) ds - \int_t^{\tau\wedge\theta} Z_s dB_s^{u,v}, \end{aligned} \tag{3.1}$$

and the payoff is given by

$$\begin{aligned} J(x_0; u, \tau; v, \theta) &= Y_0^{u,\tau;v,\theta} \\ &= E^{(u,v)}\left[\int_0^{\tau\wedge\theta} C\left(s, x_s, Y_s^{u,\tau;v,\theta}, u_s, v_s\right) ds + U_\tau 1_{[\tau<\theta]} + L_\theta 1_{[\theta<\tau<T]} \right. \\ &\quad \left. + Q_\tau 1_{[\tau=\theta<T]} + g(x_T) 1_{[\tau=\theta=T]}\right], \end{aligned} \tag{3.2}$$

where the $(U_t)_{t\le T}$, $(L_t)_{t\le T}$, and $(Q_t)_{t\le T}$ are processes of $\mathcal{S}^2$ such that $L_t \le Q_t \le U_t$. The action of C_1 is to minimize the payoff, and the action of C_2 is to maximize the payoff. Their terms can be understood as

(i) $C(s, x, Y, u, v)$ is the instantaneous reward for C_1 and cost for C_2;

(ii) U_τ is the cost for C_1 and for C_2 if C_1 decides to stop first the game;

(iii) L_θ is the reward for C_2 and cost for C_1 if C_2 decides stop first the game.

The problem is to find a saddle point strategy (one should say a fair strategy) for the controllers, that is, a strategy $(u^*, \tau^*; v^*, \theta^*)$ such that

$$J(x_0; u^*, \tau^*; v, \theta) \le J(x_0; u^*, \tau^*; v^*, \theta^*) \le J(x_0; u, \tau; v^*, \theta^*), \tag{3.3}$$

for any $(u, \tau; v, \theta) \in \mathcal{U} \times \mathcal{T} \times \mathcal{V} \times \mathcal{T}$.

Like in Section 2, we also define the Hamiltonian associated with this mixed stochastic game problem by $H(t, x, Y, Z, u, v)$, and thanks to the Benes's solution [11], there exist $u^*(t, x, Y, Z)$ and $v^*(t, x, Y, Z)$ satisfying

$$\begin{aligned} H(t, x, Y, Z, u^*(t, x, Y, Z), v^*(t, x, Y, Z)) &= \max_{v\in\mathcal{V}} \min_{u\in\mathcal{U}} \left[Z\sigma^{-1}(t, x)\Phi(t, x, u, v) + C(t, x, Y, u, v)\right] \\ &= \min_{u\in\mathcal{U}} \max_{v\in\mathcal{V}} \left[Z\sigma^{-1}(t, x)\Phi(t, x, u, v) + C(t, x, Y, u, v)\right]. \end{aligned} \tag{3.4}$$

It is easy to know that $H(t, x, Y, Z, u, v)$ is Lipschitz in Z and monotone in Y.

From the result in [12], the stochastic mixed zero-sum differential game problem is possibly connected with BSDEs with two reflecting barriers. Now, we give the main result of this section.

Theorem 3.1. $(Y^*, Z^*, K^{*+}, K^{*-})$ *is the solution of the following reflected BSDE:*

$$Y_t^* = g(x_T) + \int_t^T H(s, x_s, Y_s^*, Z_s^*, u_s^*, v_s^*)ds + \left(K_T^{*+} - K_t^{*+}\right) - \left(K_T^{*-} - K_t^{*-}\right) - \int_t^T Z_s^* dB_s, \quad (3.5)$$

satisfying for all $0 \le t \le T, L_t \le Y_t^* \le U_t$, *and* $\int_0^T (Y_s^* - L_s)dK_s^{*+} = \int_0^T (Y_s^* - U_s)dK_s^{*-} = 0$.

One defines $\tau^* = \inf\{s \in [0,T], Y_s^* = U_s\}$ *and* $\theta^* = \inf\{s \in [0,T], Y_s^* = L_s\}$.

Then $Y_0^* = J(x_0; u^*, \tau^*; v^*, \theta^*)$, $(u^*, \tau^*; v^*, \theta^*)$ *is the saddle point strategy.*

Proof. It is easy to know that the reflected BSDE (3.5) has a unique solution $(Y^*, Z^*, K^{*+}, K^{*-})$, then we have

$$\begin{aligned} Y_0^* &= g(x_T) + \int_0^T H(s, x_s, Y_s^*, Z_s^*, u_s^*, v_s^*)ds + K^{*+} - K_T^{*-} - \int_0^T Z_s^* dB_s \\ &= Y_{\tau^*\wedge\theta^*}^* + \int_0^{\tau^*\wedge\theta^*} C(s, x_s, Y_s^*, u_s^*, v_s^*)ds + K_{\tau^*\wedge\theta^*}^{*+} - K_{\tau^*\wedge\theta^*}^{*-} - \int_0^{\tau^*\wedge\theta^*} Z_s^* dB_s^{u^*,v^*}. \end{aligned} \quad (3.6)$$

Since K^{*+} and K^{*-} increase only when Y reaches L and U, we have $K_{\tau^*\wedge\theta^*}^{*+} = K_{\tau^*\wedge\theta^*}^{*-} = 0$. As $(\int_0^t Z_r dB_r^{u^*,v^*})_{t\le T}$ is an $(\mathcal{F}_t, P^{u^*,v^*})$-martingale, then we get

$$\begin{aligned} Y_0^* &= E^{u^*,v^*}\left[Y_{\tau^*\wedge\theta^*}^* + \int_0^{\tau^*\wedge\theta^*} C(s, x_s, Y_s^*, u_s^*, v_s^*)ds + K_{\tau^*\wedge\theta^*}^{*+} - K_{\tau^*\wedge\theta^*}^{*-} - \int_0^{\tau^*\wedge\theta^*} Z_s^* dB_s^{u^*,v^*}\right] \\ &= E^{u^*,v^*}\left[Y_{\tau^*\wedge\theta^*}^* + \int_0^{\tau^*\wedge\theta^*} C(s, x_s, Y_s^*, u_s^*, v_s^*)ds\right]. \end{aligned} \quad (3.7)$$

We know that $Y_{\tau^*\wedge\theta^*}^* = Y_{\tau^*}^* 1_{[\tau^*<\theta^*]} + Y_{\theta^*}^* 1_{[\theta^*<\tau^*]} + Y_{\theta^*}^* 1_{[\theta^*=\tau^*<T]} + g(x_T)1_{[\theta^*=\tau^*=T]}$ and $Y_{\tau^*}^* 1_{[\tau^*<\theta^*]} = U_{\tau^*} 1_{[\tau^*<\theta^*]}$, $Y_{\theta^*}^* 1_{[\theta^*<\tau^*]} = L_{\theta^*} 1_{[\theta^*<\tau^*]}$, $Y_{\theta^*}^* 1_{[\theta^*=\tau^*<T]} = Q_{\theta^*} 1_{[\theta^*=\tau^*<T]}$. So,

$$\begin{aligned} Y_0^* = E^{u^*,v^*}\Bigg[&U_{\tau^*} 1_{[\theta^*<\tau^*]} + L_{\theta^*} 1_{[\theta^*<\tau^*]} + Q_{\theta^*} 1_{[\theta^*=\tau^*<T]} + g(x_T) 1_{[\theta^*=\tau^*=T]} \\ &+ \int_0^{\tau^*\wedge\theta^*} C(s, x_s, Y_s^*, u_s^*, v_s^*)ds\Bigg] = J(x_0, u^*, \tau^*; v^*, \theta^*). \end{aligned} \quad (3.8)$$

Next, let v_t be an admissible control, and let $\theta \in \mathcal{T}$. We desire to show that $Y_0^* \geq J(x_0, u^*, \tau^*; v, \theta)$. We have

$$\begin{aligned}
Y_0^* &= Y_{\tau^*\wedge\theta}^* + \int_0^{\tau^*\wedge\theta} H(s, x_s, Y_s^*, Z_s^*, u_s^*, v_s^*)ds + K_{\tau^*\wedge\theta}^{*+} - \int_0^{\tau^*\wedge\theta} Z_s^* dB_s \\
&= U_{\tau^*} 1_{[\tau^*<\theta]} + Y_\theta^* 1_{[\theta<\tau^*]} + Q_\theta 1_{[\theta=\tau^*<T]} + g(x_T) 1_{[\theta=\tau^*=T]} \\
&\quad + \int_0^{\tau^*\wedge\theta} H(s, x_s, Y_s^*, Z_s^*, u_s^*, v_s^*)ds + K_{\tau^*\wedge\theta}^{*+} - \int_0^{\tau^*\wedge\theta} Z_s^* dB_s.
\end{aligned} \tag{3.9}$$

The payoff $J(x_0, u^*, \tau^*; v, \theta)$ can be described by the solution of following BSDE:

$$\begin{aligned}
Y_0 &= U_{\tau^*} 1_{[\tau^*<\theta]} + L_\theta 1_{[\theta<\tau^*<T]} + Q_\theta 1_{[\tau^*=\theta<T]} + g(x_T) 1_{[\tau^*=\theta=T]} \\
&\quad + \int_0^{\tau^*\wedge\theta} C(s, x_s, Y_s, u_s^*, v_s)ds - \int_0^{\tau^*\wedge\theta} Z_s dB_s^{u^*,v} \\
&= U_{\tau^*} 1_{[\tau^*<\theta]} + L_\theta 1_{[\theta<\tau^*<T]} + Q_\theta 1_{[\tau^*=\theta<T]} + g(x_T) 1_{[\tau^*=\theta=T]} \\
&\quad + \int_0^{\tau^*\wedge\theta} H(s, x_s, Y_s, Z_s, u_s^*, v_s)ds - \int_0^{\tau^*\wedge\theta} Z_s dB_s,
\end{aligned} \tag{3.10}$$

then

$$\begin{aligned}
Y_0 &= E^{u^*,v}\Bigg[U_{\tau^*} 1_{[\tau^*<\theta]} + L_\theta 1_{[\theta<\tau^*<T]} + Q_\theta 1_{[\tau^*=\theta<T]} + g(x_T) 1_{[\tau^*=\theta=T]} \\
&\qquad + \int_0^{\tau^*\wedge\theta} H(s, x_s, Y_s, Z_s, u_s^*, v_s)ds - \int_0^{\tau^*\wedge\theta} Z_s dB_s\Bigg] \\
&= E^{u^*,v}\left[U_{\tau^*} 1_{[\tau^*<\theta]} + L_\theta 1_{[\theta<\tau^*<T]} + Q_\theta 1_{[\tau^*=\theta<T]} + g(x_T) 1_{[\tau^*=\theta=T]} + \int_0^{\tau^*\wedge\theta} C(s, x_s, Y_s, u_s^*, v_s)ds\right],
\end{aligned} \tag{3.11}$$

and $J(x_0; u^*, \tau^*; v, \theta) = Y_0$. Thanks to $H(s, x_s, Y_s, Z_s, u_s^*, v_s^*) \geq H(s, x_s, Y_s, Z_s, u_s^*, v_s)$, $Y_\theta^* 1_{[\theta<\tau^*]} \geq L_\theta 1_{[\theta<\tau^*<T]}$, and $K_{\tau^*\wedge\theta}^{*+} \geq 0$ by the comparison theorem of BSDEs to compare (3.9) and (3.10) to get $Y_0^* \geq Y_0 = J(x_0; u^*, \tau^*; v, \theta)$.

In the same way, we can show that $Y_0^* = J(x_0; u^*, \tau^*; v^*, \theta^*) \leq J(x_0; u, \tau; v^*, \theta^*)$ for any $\tau \in \mathcal{T}$ and any admissible control u. It follows that $(u^*, \tau^*; v^*, \theta^*)$ is a saddle point for the recursive game.

Finally, let us show that the value of the game is Y_0^*. We have proved that

$$J(x_0; u^*, \tau^*; v, \theta) \leq Y_0^* = J(x_0; u^*, \tau^*; v^*, \theta^*) \leq J(x_0; u, \tau; v^*, \theta^*), \tag{3.12}$$

for any $(u,v) \in \mathcal{U} \times \mathcal{V}$ and $\tau, \theta \in \mathcal{T}$. Thereby,

$$Y_0^* \leq \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} J(x_0; u, \tau; v^*, \theta^*) \leq \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} J(x_0; u, \tau; v, \theta). \tag{3.13}$$

On the other hand,

$$Y_0^* \geq \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} J(x_0; u^*, \tau^*; v, \theta) \geq \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} J(x_0; u, \tau; v, \theta). \tag{3.14}$$

Now, due to the inequality

$$\inf_{u\in\mathcal{U},\tau\in\mathcal{T}} \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} J(x_0; u, \tau; v, \theta) \geq \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} J(x_0; u, \tau; v, \theta), \tag{3.15}$$

we have

$$Y_0^* = \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} J(x_0; u, \tau; v, \theta) = \sup_{v\in\mathcal{V},\theta\in\mathcal{T}} \inf_{u\in\mathcal{U},\tau\in\mathcal{T}} J(x_0; u, \tau; v, \theta). \tag{3.16}$$

The proof is now completed. □

4. Application

In this section, we present two examples to show the applications of Section 3.

The first example is about the American game option pricing problem. We formulate it to be one stochastic recursive mixed game problem. This can be regarded as the application background of our stochastic game problem.

Example 4.1. American game option when loan interest is higher than deposit interest is shown.

In El Karoui et al. [13], they proved that the price of an American option corresponds to the solution of a reflected BSDE. And Hamadène [9] proved that the price of American game option corresponds to the solution of a reflected BSDE with two barriers. Now, we will show that under some constraints in financial market such as when loan interest rate is higher than deposit interest rate, the price of an American game option corresponds to the value function of stochastic recursive mixed zero-sum differential game problem.

We suppose that the investor is allowed to borrow money at time t at an interest rate $R_t > r_t$, where r_t is the bond rate. Then, the wealth of the investor satisfies

$$\begin{gathered} -dX_t = b(t, X_t, Z_t)dt - dC_t - Z_t dW_t, \quad 0 \leq t \leq T, \\ b(t, X_t, Z_t) := -\left[r_t X_t + \theta_t Z_t - (R_t - r_t)\left(X_t - \frac{Z_t}{\sigma_t} \right)^- \right], \end{gathered} \tag{4.1}$$

where $Z_t := \sigma_t \pi_t$, $\theta_t := \sigma_t^{-1}(b_t - r_t)$. b_t represents the instantaneous expected return rate in stock, σ_t which is invertible represents the instantaneous volatility of the stock, and C_t

is interpreted as a cumulative consumption process. b_t, r_t, R_t, and σ_t are all deterministic bounded functions, and σ_t^{-1} is also bounded.

An American game is a contract between a broker c_1 and a trader c_2 who are, respectively, the seller and the buyer of the option. The trader pays an initial amount (the price of the option) which guarantees a payment of $(L_t)_{t\le T}$. The trader can exercise whenever he decides before the maturity T of the option. Thus, if the trader decides to exercise at θ, he gets the amount L_θ. On the other hand, the broker is allowed to cancel the contract. Therefore, if he chooses τ as the contract cancellation time, he pays the amount U_τ, and $U_\tau \ge L_\tau$. The difference $U_\tau - L_\tau$ is the premium that the broker pays for his decision to cancel the contract. If c_1 and c_2 decide together to stop the contract at the time τ, then c_2 gets a reward equal to $Q_\tau 1_{[\tau<T]} + \xi 1_{[\tau=T]}$. Naturally, $U_\tau \ge Q_\tau \ge L_\tau$. U_t, L_t, and Q_t are stochastic processes which are related to the stock price in the market.

We consider the problem of pricing an American game contingent claim at each time t which consists of the selection of a stopping time $\tau \in \mathcal{F}_\tau$ (or $\theta \in \mathcal{F}_\theta$) and a payoff U_τ (or L_θ) on exercise if $\tau < \theta < T$ (or $\theta < \tau < T$) and ξ if $\tau = T$. Set

$$\widetilde{S}_{\tau\wedge\theta} = \xi 1_{\{\tau=\theta=T\}} + Q_\tau 1_{\{\tau=\theta<T\}} + L_\theta 1_{\{\theta<\tau<T\}} + U_\tau 1_{\{\tau<\theta<T\}}, \quad 0 \le (\tau\wedge\theta) \le T, \tag{4.2}$$

then the price of American game contingent claim $(\widetilde{S}_{\tau\wedge\theta},\ 0 \le (\tau\wedge\theta) \le T)$ at time t is given by

$$X_t = \operatorname*{ess\,inf}_{\tau\in\mathcal{F}_\tau} \operatorname*{ess\,sup}_{\theta\in\mathcal{F}_\theta} X_t\left(\tau\wedge\theta, \widetilde{S}_{\tau\wedge\theta}\right), \tag{4.3}$$

where $X_t(\tau\wedge\theta, \widetilde{S}_{\tau\wedge\theta})$ noted by $X_t^{\tau\wedge\theta}$ satisfies BSDE

$$\begin{gathered} -dX_s^{\tau\wedge\theta} = b\left(s, X_s^{\tau\wedge\theta}, Z_s^{\tau\wedge\theta}\right)ds - dC_s - Z_s^{\tau\wedge\theta}dW_s, \\ X_{\tau\wedge\theta}^{\tau\wedge\theta} = \widetilde{S}_{\tau\wedge\theta}. \end{gathered} \tag{4.4}$$

For each (ω, t), $b(t, x, z)$ is a convex function of (x, z). It follows from [14] that we have $X_t^{\tau\wedge\theta} = \operatorname{ess\,sup}_{r_t\le\beta_t\le R_t} \operatorname{ess\,inf}_{C_t} X^{\beta,C,\tau\wedge\theta}$. Here, $X^{\beta,C,\tau\wedge\theta}$ satisfies

$$\begin{gathered} -dX_s^{\beta,C,\tau\wedge\theta} = b^\beta\left(s, X_s^{\beta,C,\tau\wedge\theta}, Z_s^{\beta,C,\tau\wedge\theta}\right)ds - dC_s - Z_s^{\tau\wedge\theta}dW_s, \\ X_{\tau\wedge\theta}^{\beta,C,\tau\wedge\theta} = \widetilde{S}_{\tau\wedge\theta}, \\ b^\beta(s, X_t, Z_t) := -\beta_t X_t - \left[\theta_t + \frac{r_t - \beta_t}{\sigma_t}\right] Z_t, \end{gathered} \tag{4.5}$$

where β_t is a bounded R-valued adapted process which can be regarded as an interest rate process in finance. So,

$$\begin{aligned} X_t &:= \operatorname*{ess\,inf}_{\tau\in\mathcal{F}_\tau} \operatorname*{ess\,sup}_{\theta\in\mathcal{F}_\theta} X_t\left(\tau\wedge\theta, \widetilde{S}_{\tau\wedge\theta}\right) \\ &= \operatorname*{ess\,inf}_{\tau\in\mathcal{F}_t, C_t} \operatorname*{ess\,sup}_{\theta\in\mathcal{F}_t, r_t\le\beta_t\le R_t} X_t^{\beta,C,\tau\wedge\theta} \\ &= \operatorname*{ess\,sup}_{\theta\in\mathcal{F}_t, r_t\le\beta_t\le R_t} \operatorname*{ess\,inf}_{\tau\in\mathcal{F}_t, C_t} X_t^{\beta,C,\tau\wedge\theta}. \end{aligned} \tag{4.6}$$

Here, $X_t^{\beta,C} := \operatorname{ess\,sup}_{\theta\in\mathcal{F}_t, r_t\le\beta_t\le R_t} \operatorname{ess\,inf}_{\tau\in\mathcal{F}_t, C_t} X_t^{\beta,C,\tau\wedge\theta}$. Then, from [13], there exist $Z_t^{\beta,C} \in H^2$ and $K_t^{\beta,C,+} K_t^{\beta,C,-}$, which are increasing adapted continuous processes with $K_0^{\beta,C,+} = 0$ and $K_0^{\beta,C,-} = 0$, such that $(X_t^{\beta,C}, Z_t^{\beta,C}, K_t^{\beta,C,+}, K_t^{\beta,C,-})$ satisfies the following reflected BSDE:

$$\begin{gathered} -dX_s^{\beta,C} = b^\beta\left(s, X_s^{\beta,C}, Z_s^{\beta,C}\right)ds - dC_s + dK_s^{\beta,C,+} - dK_s^{\beta,C,-} - Z_s^{\beta,C} dW_s, \\ X_T^{\beta,C} = \xi, \quad 0\le s\le T, \end{gathered} \tag{4.7}$$

with $U_t \ge X_t^{\beta,C} \ge L_t$, $0\le t\le T$, and $\int_0^T (X_t^{\beta,C} - L_t)^- dK_t^{\beta,C,+} = 0$, $\int_0^T (U_t - X_t^{\beta,C})^- dK_t^{\beta,C,-} = 0$. Then, the stopping time $\tau = \inf\{t\le s\le T;\ X_s^{\beta,C} = U_s\}$, and $\theta = \inf\{t\le s\le T;\ X_s^{\beta,C} = L_s\}$.

We formulate the pricing problem of American game option to the stochastic recursive mixed zero-sum differential game problem which was studied in Section 3, so the previous example provides the practical background for our problem. This is also one of our motivations to study the recursive mixed game problem in this paper.

In the following, we give another example, where we obtain the explicit saddle point strategy and optimal value of the stochastic recursive game. The purpose of this example is to illustrate the application of our theoretical results.

Example 4.2. We let the dynamics of the system $(x_t)_{t\le T}$ satisfy

$$dx_t = x_t dB_t, \quad t\le 1, \text{ where the initial value is } x_0. \tag{4.8}$$

The control action for c_1 (resp. c_2) is u (resp. v) which belongs to $\mathcal{U}$ (resp. $\mathcal{V}$). The $\mathcal{U}$ is $[0,1]$, and the $\mathcal{V}$ is $[0,1]$, while the function $\Phi = x_t(u_t + v_t)$. Then, by the Girsanov's theorem, we can define the probability $P^{u,v}$ by

$$\frac{dP^{u,v}}{dP} = \exp\left\{\int_0^T (u_s+v_s)dB_s - \frac{1}{2}\int_0^T (u_s+v_s)^2 ds\right\}. \tag{4.9}$$

Under the probability $P^{u,v}$, the process $B_t^{u,v} = B_t - \int_0^t (u_s+v_s)ds$ is a Brownian motion.

First, we consider the following stochastic recursive zero-sum differential game:

$$J(x_0,u,v) = Y_0 = E^{u,v}\left[x_T + \int_0^T \min\{|x_t|,2\} + Y_t(u_t+v_t)dt\right]. \tag{4.10}$$

$(Y_t)_{0\le t\le T}$ satisfies BSDE

$$\begin{gathered} -dY_s = \min\{|x_s|,2\} + Y_s(u_s+v_s)ds - Z_s dB_s^{u,v}, \\ Y_T = x_T. \end{gathered} \tag{4.11}$$

Therefore,

$$H(t,x,z,Y,u,v) = Z(u+v) + \min\{|x_t|,2\} + Y(u+v), \tag{4.12}$$

and obviously, the Isaacs condition is satisfied with $u^* = 1_{[Z+Y\le 0]}, v^* = 1_{[Z+Y\ge 0]}$. It follows that

$$\begin{aligned} \min_{u\in\mathcal{U}}\max_{v\in\mathcal{V}} H(t,x,Z,Y,u,v) &= \max_{v\in\mathcal{V}}\min_{u\in\mathcal{U}} H(t,x,Z,Y,u,v) = Z + \min\{|x_t|,2\} + Y, \\ J(x_0,u^*,v^*) &= Y_0 \\ &= x_T + \int_0^T (Z_t + \min\{|x_t|,2\} + Y_t)dtv - \int_0^T Z_t dB_t \\ &= E\left[x_0\exp(2B_T) + \int_0^T \exp\left(B_t + \frac{1}{2}t\right)\min\left\{\left|x_0\exp\left(B_t - \frac{1}{2}t\right)\right|,2\right\}dt\right]. \end{aligned} \tag{4.13}$$

We also can get the conclusion that the optimal game value $Y_0^* = J(x_0,u^*,v^*)$ is an increasing function with the initial value of the dynamics system x_0 from the previous representation. Now, we give the numerical simulation and draw Figure 1 to show this point. Let $T = 2$, when $x_0 = 1$, the optimal game value $Y_0 = 147.8$, $Z_0 = 147.8$ and the saddle point strategy $(u_0^*, v_0^*) = (0,1)$; when $x_0 = 2$, $Y_0 = 295.6$, $Z_0 = 295.6$, $(u_0^*, v_0^*) = (0,1)$; and $x_0 = 3$, $Y_0 = 443.4$, $Z_0 = 443.4$, and $(u_0^*, v_0^*) = (0,1)$. Y_0 is increasing function of x_0 which coincides with our conclusion.

Second, we consider the following stochastic recursive mixed zero-sum differential game:

$$\begin{aligned} J(x_0;u,\tau;v,\theta) = Y_0^{u,\tau;v,\theta} = E^{u,v}\Bigg[&\int_0^{\tau\wedge\theta} [\min\{|x_t|,2\} + Y_t(u_t+v_t)]dt \\ &+ (x_\tau + 1)I_{[\tau<\theta]} + (x_\theta - 1)I_{[\theta<\tau<T]} + x_T I_{[\theta=\tau]}\Bigg]. \end{aligned} \tag{4.14}$$

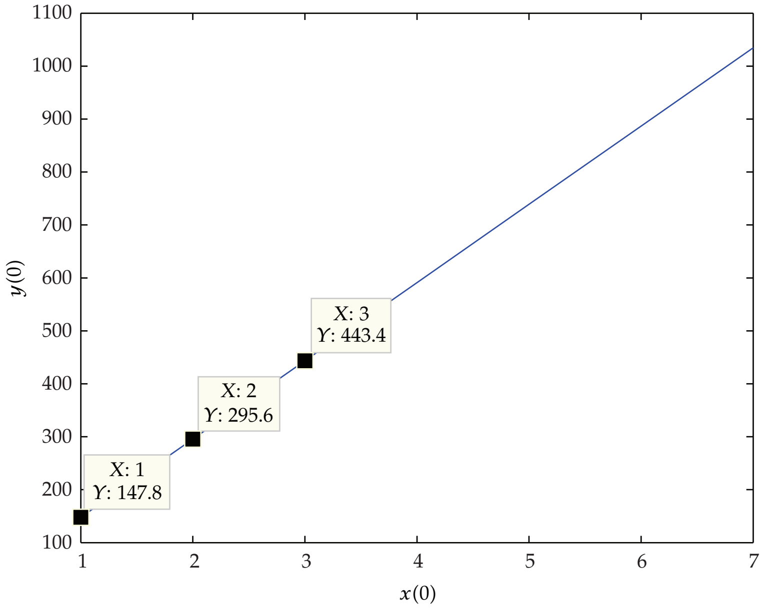


Figure 1: Y_0 stands for the optimal game value, and x_0 stand for the initial value of the dynamics system.

Then, $(Y_t)_{0\le t\le(\tau\wedge\theta)}$ satisfies the following BSDE:

$$\begin{aligned} Y_t &= (x_\tau+1)I_{[\tau<\theta]} + (x_\theta-1)I_{[\theta<\tau<T]} + x_T I_{[\theta=\tau]} \\ &\quad + \int_t^{\tau\wedge\theta}[\min\{|x_s|,2\} + Y_s(u_s+v_s)]ds - \int_t^{\tau\wedge\theta} Z_s dB_s^{u,v}. \end{aligned} \tag{4.15}$$

Therefore, $H(t,x,z,Y,u,v) = Z(u+v) + \min\{|x_t|,2\} + Y(u+v)$, and obviously, the Isaacs condition is satisfied with $u^* = 1_{[Z+Y\le 0]}, v^* = 1_{[Z+Y\ge 0]}$. It follows that

$$\begin{aligned} &\min_{u\in\mathcal{U}}\max_{v\in\mathcal{V}} H(t,x,Z,Y,u,v) = \max_{v\in\mathcal{V}}\min_{u\in\mathcal{U}} H(t,x,Z,Y,u,v) = Z + \min\{|x_t|,2\} + Y, \\ &J(x_0;u^*,\tau;v^*,\theta) = Y_0^{u^*,\tau;v^*,\theta} \\ &\qquad = Y_{\tau\wedge\theta} + \int_0^{\tau\wedge\theta}(Z_t + \min\{|x_t|,2\} + Y_t)dt - \int_0^{\tau\wedge\theta} Z_t dB_t \\ &\qquad = Y_{\tau\wedge\theta}\exp\left(\frac{1}{2}(\tau\wedge\theta) + B_{\tau\wedge\theta}\right) + \int_0^{\tau\wedge\theta}\min\{|x_t|,2\}\exp\left(\frac{1}{2}(t) + B_t\right)dt \\ &\qquad\quad - \int_0^{\tau\wedge\theta}\exp\left(\frac{1}{2}(t) + B_t\right)(Z_t + Y_t)dB_t, \end{aligned} \tag{4.16}$$

where $\tau^* = \inf\{t \in [0,T], Y_t \geq (x_t + 1)\}$, and $\theta^* = \inf\{t \in [0,T], Y_t \leq (x_t - 1)\}$, while $(u^*, \tau^*; v^*, \theta^*)$ is the saddle point. So, the optimal value is

$$\begin{aligned}
J(x_0; u^*, \tau^*; v^*, \theta^*) &= Y_0^{u^*,\tau^*;v^*,\theta^*} \\
&= Y_{\tau^*\wedge\theta^*} \exp\left(\frac{1}{2}(\tau^* \wedge \theta^*) + B_{\tau^*\wedge\theta^*}\right) + \int_0^{\tau^*\wedge\theta^*} \min\{|x_t|, 2\} \exp\left(\frac{1}{2}t + B_t\right) dt \\
&\quad - \int_0^{\tau^*\wedge\theta^*} \exp\left(\frac{1}{2}t + B_t\right)(Z_t + Y_t) dB_t \\
&= E\Bigg[x_0 \exp(2B_{\tau^*\wedge\theta^*}) + 1_{\tau^*<\theta^*} \exp\left(B_\tau^* + \frac{1}{2}\tau^*\right) - 1_{\theta^*<\tau^*} \exp\left(B_\theta^* + \frac{1}{2}\theta^*\right) \\
&\quad + \int_0^{\tau^*\wedge\theta^*} \min\left\{\left|x_0 \exp\left(B_t - \frac{1}{2}t\right)\right|, 2\right\} \exp\left(\frac{1}{2}t + B_t\right) dt\Bigg].
\end{aligned} \tag{4.17}$$

We also can get the conclusion that the optimal game value $Y_0^* = J(x_0, u^*, \tau^*; v^*, \theta^*)$ is an increasing function with the initial value of the dynamics system x_0 from the previous representation.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (no. 10921101, 61174092), the National Science Fund for Distinguished Young Scholars of China (no. 11125102), and the Special Research Foundation for Young teachers of Ocean University of China (no. 201313006).

References

[1] E. Pardoux and S. G. Peng, "Adapted solution of a backward stochastic differential equation," *Systems & Control Letters*, vol. 14, no. 1, pp. 55–61, 1990.
[2] D. Duffie and L. G. Epstein, "Stochastic differential utility," *Econometrica*, vol. 60, no. 2, pp. 353–394, 1992.
[3] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, and M. C. Quenez, "Reflected solutions of backward SDE's, and related obstacle problems for PDE's," *The Annals of Probability*, vol. 25, no. 2, pp. 702–737, 1997.
[4] J. Cvitanic and I. Karatzas, "Backward SDE's with reflection and Dynkin games," *The Annals of Probability*, vol. 24, no. 4, pp. 2024–2056, 1996.
[5] S. Hamadène and J.-P. Lepeltier, "Zero-sum stochastic differential games and backward equations," *Systems & Control Letters*, vol. 24, no. 4, pp. 259–263, 1995.
[6] G. Wang and Z. Yu, "A Pontryagin's maximum principle for non-zero sum differential games of BSDEs with applications," *IEEE Transactions on Automatic Control*, vol. 55, no. 7, pp. 1742–1747, 2010.
[7] G. Wang and Z. Yu, "A partial information non-zero sum differential game of backward stochastic differential equations with applications," *Automatica*, vol. 48, no. 2, pp. 342–352, 2012.
[8] N. El Karoui, S. Peng, and M. C. Quenez, "Backward stochastic differential equations in finance," *Mathematical Finance*, vol. 7, no. 1, pp. 1–71, 1997.
[9] S. Hamadène, "Mixed zero-sum stochastic differential game and American game options," *SIAM Journal on Control and Optimization*, vol. 45, no. 2, pp. 496–518, 2006.

[10] E. Pardoux, "BSDE's, weak convergence and homogenization of semilinear PDE's," in *Nonlinear Analysis, Differential Equations and Control*, F. H. Clarke and R. J. Stern, Eds., vol. 528, pp. 503–549, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.
[11] V. E. Benes, "Existence of optimal strategies based on specified information, for a class of stochastic decision problems," *SIAM Journal on Control and Optimization*, vol. 8, pp. 179–188, 1970.
[12] J. P. Lepeltier, A. Matoussi, and M. Xu, "Reflected BSDEs under monotonicity and general increasing growth conditions," *Advanced in Applied Probability*, vol. 37, pp. 134–159, 2005.
[13] N. El Karoui, E. Pardoux, and M. C. Quenez, "Reflected backward SDEs and American options," in *Numerical methods in Finance*, L. C. G. Rogers and D. Talay, Eds., vol. 13, pp. 215–231, Cambridge University Press, Cambridge, Mass, USA, 1997.
[14] S. Hamadène and I. Hdhiri, "Backward stochastic differential equations with two distinct reflecting barriers and quadratic growth generator," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 2006, Article ID 95818, 28 pages, 2006.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 372161, 25 pages
doi:10.1155/2012/372161

Research Article

Multiple Maneuvering Target Tracking by Improved Particle Filter Based on Multiscan JPDA

Jing Liu,[1] ChongZhao Han,[1] Feng Han,[1] and Yu Hu[2]

[1] *MOE Key Lab for Intelligent and Networked Systems, Institute of Integrated Automation, School of Electronics and Information Engineering, Xi'an Jiaotong University, Xi'an, Shaanxi Province 710049, China*
[2] *School of Aeronautics, Northwestern Polytechnical University, Xi'an, Shaanxi Province 710072, China*

Correspondence should be addressed to Jing Liu, elelj20080730@gmail.com

Received 5 September 2012; Revised 20 October 2012; Accepted 3 November 2012

Academic Editor: Suiyang Khoo

The multiple maneuvering target tracking algorithm based on a particle filter is addressed. The equivalent-noise approach is adopted, which uses a simple dynamic model consisting of target state and equivalent noise which accounts for the combined effects of the process noise and maneuvers. The equivalent-noise approach converts the problem of maneuvering target tracking to that of state estimation in the presence of nonstationary process noise with unknown statistics. A novel method for identifying the nonstationary process noise is proposed in the particle filter framework. Furthermore, a particle filter based multiscan Joint Probability Data Association (JPDA) filter is proposed to deal with the data association problem in a multiple maneuvering target tracking. In the proposed multiscan JPDA algorithm, the distributions of interest are the marginal filtering distributions for each of the targets, and these distributions are approximated with particles. The multiscan JPDA algorithm examines the joint association events in a multiscan sliding window and calculates the marginal posterior probability based on the multiscan joint association events. The proposed algorithm is illustrated via an example involving the tracking of two highly maneuvering, at times closely spaced and crossed, targets, based on resolved measurements.

1. Introduction

The problem of multiple maneuvering target tracking has received a considerable attention in recent years. Multiple model approaches are proposed to model highly maneuvering targets and the interacting multiple model (IMM) algorithm [1–3] is the most popular one among them. Moreover, in the presence of clutter, some of the sensor measurements may not relate to the target of interest. The problem of data association has to be solved. Effective approaches in a Bayesian framework include the probabilistic data association (PDA) approach [4, 5] and the integrated probabilistic data association (IPDA) approach [6–8] for tracking a single

target in clutter, the JPDA approach [4, 9, 10], and the joint integrated probabilistic data association (JIPDA) approach [11] for tracking multiple targets in clutter. The PDA and JPDA approaches address the track maintenance issue, while the IPDA and JIPDA approaches tackle the issues related to track initiation and track termination as well as track maintenance.

Different combinations of the IMM method and the PDA/JPDA/IPDA/JIPDA methods have been used to tackle multiple maneuvering target tracking problems. In [12], the IMM algorithm is combined with the PDA filter in a multiple sensor scenario proposing a combined IMMPDA algorithm. The IMMPDA algorithm has good tracking performance when the targets are sufficiently separated. In [13], an IMMJPDA-coupled filter is developed for situations where the measurements of two targets are unresolved during periods of close encounter. The filters corresponding to the individual targets are coupled through cross-target-covariance terms. In [14], JPDA is combined with a crude approximation of IMM, under the name IMMJPDA. In [15], an IMMJIPDA algorithm is used in situations with a small number of crossing targets and low clutter measurement density. In order to deal with more complex scenarios with a large number of targets, an IMM-LMIPDA algorithm has been proposed by integrating linear multitarget (LM) method with IMM-IPDA. The computational requirements of the proposed IMM-LMIPDA algorithm increase linearly in the number of targets and the number of measurements.

Recently, several methods which significantly outperform IMMJPDA have been proposed to deal with the multiple maneuvering target tracking problem. In [16], an IMMJPDA uncoupled fixed-lag smoothing algorithm is proposed, which performs far better than IMMJPDA filtering. In [17, 18], the descriptor system approach [19] is used to develop a track-coalescence-avoiding version of IMMJPDA. In [20], multiscan information is incorporated in JPDA to tackle the data association problem associated with the multiple maneuvering target tracking. In [21], a new track-coalescence-avoiding IMMCPDA method is proposed to improve the situation of tracking closely spaced targets by characterizing the exact Bayesian filter, and by developing novel target tracking combinations of IMM and PDA.

In the above methods, the data models are assumed Gaussian and weakly nonlinear, and the Kalman filter/extended Kalman filter (KF/EKF) is used to perform target state estimation. More recently, nonlinear filtering techniques have been gaining more attention. The most common one among them is the particle filter [22], a state estimate pdf sampling based algorithm.

There are not many reported works concerning the particle filter based multiple maneuvering target tracking methods. In [23], the problem of maintaining tracks of multiple maneuvering targets from unassociated measurements is formulated as a problem of estimating the hybrid state of a Markov jump linear system from measurements made by a descriptor system with independent, identically distributed (i.i.d.) stochastic coefficients. The possible multiple motion models and transition probability matrices are assumed to be known in the proposed method. In [24], the marginal filtering distributions for each of the targets are approximated with particles. The posterior marginal probability is calculated based on the joint association hypotheses, which are examined in a single scan. However, single scan information may not be enough for tracking maneuvering targets, especially in the condition of tracking two closely spaced targets.

In this work, a novel method is proposed for a multiple maneuvering target tracking, which is a combination of the particle filter based process noise identification algorithm and the particle filter based multiscan JPDA algorithm. The process noise identification process is effective in estimating both the maneuvering movement and the random acceleration of the target, using one general model for each of the maneuvering targets. The multiscan JPDA

is effective in maintaining the tracks of multiple targets using multiple scan information. Compared with the single scan JPDA method, the multiscan JPDA method uses richer information, which results in better estimated probabilities.

The rest of the paper is organized as follows. Section 2 provides a brief introduction to particle filter, and the multiple maneuvering target tracking problem is defined in Section 3. The particle filter based process noise identification method for tracking highly maneuvering target is introduced in Section 4. The data association method based on multiscan JPDA is discussed in Section 5. The proposed multiple maneuvering target tracking algorithm, which is a combination of the above two algorithms, is introduced in Section 6. The simulation results are shown in Section 7 and the paper is summarized in Section 8.

2. Basic Theory of Particle Filter

To define the problem of tracking, consider a dynamic system represented by the state equation:

$$x_k = f\left(x_{k-1}, v^*_{k-1}\right), \tag{2.1}$$

where x is the state, f is a possibly nonlinear function, and v^* is the known process noise with a zero mean Gaussian distribution. The objective of tracking is to recursively estimate x_k from a sequence of measurements up to time step k, $z_{1:k} = \{z_1, z_2, \ldots, z_k\}$. The observation model is described as follows:

$$z_k = h(x_k, n_k), \tag{2.2}$$

where h is a possibly nonlinear function. n is the observation noise with a zero mean Gaussian distribution. From the Bayesian perspective, the tracking problem is to recursively calculate the posterior distribution $p(x_k \mid z_{1:k})$.

In this paper, a particle filter is considered to solve the state estimation problem due to its ability to tackle the nonlinear and non-Gaussian systems. The posterior distribution $p(x_k \mid z_{1:k})$ is approximated by a set of particles with associated weights. The detailed introduction about particle filter algorithm can be found in [22]. The procedures associated with the standard particle filter are listed in the following.

Algorithm 2.1 (particle filter algorithm). (1) *Initialization*. Sample initial particles $\{x^i_0,\ i = 1, \ldots, H\}$ from the initial posterior distribution $p(x_0)$ and set the weights w^i_0 to $1/H$, $i = 1, \ldots, H$. H is the number of particles.

(2) *Prediction*. Particles at time step $k-1$, $\{x^i_{k-1},\ i = 1, \ldots, H\}$, are passed through the system model (2.3) to obtain the predicted particles at time step k, $\{\widehat{x}^i_k,\ i = 1, \ldots, H\}$:

$$\widehat{x}^i_k = f\left(x^i_{k-1}, v^{*,i}_{k-1}\right), \tag{2.3}$$

where $v^{*,i}_{k-1}$ is a sample drawn from the known zero mean Gaussian distribution.

(3) *Update*. Once the observation data, z_k, is measured, evaluate the importance weight of each predicted particle and obtain the normalized weight for each particle:

$$\widehat{w}_k^i = p\left(z_k \mid \widehat{x}_k^i\right), \qquad w_k^i = \frac{\widehat{w}_k^i}{\sum_{i=1}^{H} \widehat{w}_k^i}. \tag{2.4}$$

Thus at time step k we can obtain the estimate of the state, $\tilde{\tilde{x}}_k = \sum_{i=1}^{H} w_k^i \widehat{x}_k^i$.

(4) *Resample*. Resample the discrete distribution $\{w_k^i : i = 1, \ldots, H\}$, H times to generate particles $\{x_k^j : j = 1, \ldots, H\}$, so that for any j, $\Pr\{x_k^j = \widehat{x}_k^i\} = w_k^i$. Set the weights w_k^i to $1/H$, $i = 1, \ldots, H$, and move to Stage 2.

3. Multiple Maneuvering Target Tracking Problem Formulation

The number of maneuvering targets (M) to be tracked is assumed as fixed and known, where each target track has been initiated, and our objective is to maintain the tracks. Each target is parameterized by a state $x_{m,k}$, where m denotes the mth target and k denotes time step k. The combined state, $x_k = (x_{1,k}, \ldots, x_{M,k})$, is the concatenation of the individual target states. The individual targets are assumed to evolve independently according to a general motion model,

$$x_k = f\left(x_{k-1}, u_{k-1}, v_{k-1}^*\right), \tag{3.1}$$

where u is the maneuver acceleration and v^* is the process noise.

The observation vector z_k is composed of multiple sensor measurements $\{z_{j,k},\ j = 1, \ldots, N_k\}$, where N_k is the total number of measurements. It is assumed that there are no unresolved measurements (i.e., measurements associated with two or more targets simultaneously); any measurement is either associated with a single target or caused by clutter. Clutter is modeled as independently and identically distributed (IID) with an uniform spatial distribution over the surveillance area.

In this paper, the multiple maneuvering target tracking problem is divided into two parts: single maneuvering target tracking and multiple target tracking, which are introduced in Sections 4 and 5, respectively. The particle filter based process noise identification method for tracking a highly maneuvering target is introduced in Section 4. The data association method based on multiscan JPDA is discussed in Section 5. Finally, in Section 6, the combination of the two methods is combined to deal with the multiple maneuvering target tracking problem.

4. Maneuvering Target Tracking Based on Process Noise Identification

In this section, the particle filter based process noise identification method is proposed for tracking highly maneuvering target. The general motion model of a maneuvering target is

described by (3.1). In the equivalent-noise approach [25–27], it is assumed that the general motion model (3.1) that describes target motions can be simplified to

$$x_k = f(x_{k-1}, v_{k-1}), \tag{4.1}$$

with an adequate accuracy, where v is the equivalent noise that quantifies the error of this model in describing the target motions, in particular, maneuvers. The statistics of this noise v, nonstationary in general, are not known.

In this section, a novel method is proposed for process noise identification. The process noise is modeled as a dynamic system. The noise vector v_{k-1} is chosen as the state of the noise system. The observation vector is z_k, which is the same as in (2.2). The observation equation is defined in

$$z_k = h(x_k, n_k) = h\left[f(x_{k-1}, v_{k-1}), n_k\right]. \tag{4.2}$$

The aim of the proposed method is to estimate the posterior distribution of the process noise, $p(v_{k-1} \mid z_{1:k})$. According to the Bayesian inference theory,

$$p(v_{k-1} \mid z_{1:k}) = \frac{p(z_k \mid v_{k-1})p(v_{k-1} \mid z_{1:k-1})}{p(z_k \mid z_{1:k-1})}, \tag{4.3}$$

where $p(z_k \mid z_{1:k-1})$ is a normalizing constant and $1/p(z_k \mid z_{1:k-1})$ is defined as Υ. So we can obtain

$$p(v_{k-1} \mid z_{1:k}) = \Upsilon \cdot p(z_k \mid v_{k-1})p(v_{k-1} \mid z_{1:k-1}). \tag{4.4}$$

The noise v_{k-1} may be from random accelerations, sudden maneuvers, or both, and there is no information about the distribution of v_{k-1}. v_{k-1} is not dependent on the previous measurements $z_{1:k-1}$, which results in

$$p(v_{k-1} \mid z_{1:k}) = \Upsilon \cdot p(z_k \mid v_{k-1})p(v_{k-1}). \tag{4.5}$$

According to the Bayesian theory, the prior distribution of parameters, on which there is no information, could be considered as a uniform distribution. In this paper, the prior distribution of the process noise is considered as a uniform distribution, and the posterior distribution of the process noise is then obtained via a Monte Carlo deviation process. The general procedure of the particle filter based process noise identification algorithm is as follows.

Initially a number of process noise samples are generated from a uniform distribution since there is no information on the process noise distribution due to the uncertain dynamics. Each of the process noise samples is assigned with a weight based on the current measurement and the particles from the previous time step. According to Monte Carlo theory, the process noise samples and their associated weights approximate the true posterior distribution of the process noise. The process noise samples are then resampled according to their associate weights. The samples with large weights are duplicated, while

the samples with small weights are removed. The resampled process noise samples distribute approximately according to the true posterior distribution of the process noise.

Since there is no information about v_{k-1}, it is reasonable to assume that v_{k-1} is uniformly distributed in the range of $U(-d, d)$, where U denotes a uniform distribution and d is the known process noise bound accounting for the maximin uncertain dynamics. According to the Monte Carlo theory, $p(v_{k-1})$ can be represented by H samples, $\{\widehat{v}_{k-1}^{j},\ j = 1, \ldots, H\}$, from $U(-d, d)$. Consider

$$p(v_{k-1}) = \frac{1}{H}\sum_{j=1}^{H}\delta\left(v_{k-1} - \widehat{v}_{k-1}^{j}\right). \tag{4.6}$$

The number of process noise samples (H) is proportional to the magnitude of the maximum uncertain dynamics $(\|d\|)$. The posterior distribution of v_{k-1} can be represented as

$$\begin{aligned} p(v_{k-1} \mid z_{1:k}) &= \frac{\Upsilon}{H}\sum_{j=1}^{H}p\left(z_k \mid \widehat{v}_{k-1}^{j}\right)\delta\left(v_{k-1} - \widehat{v}_{k-1}^{j}\right) \\ &= \frac{\Upsilon}{H}\sum_{j=1}^{H}\xi_k^j \cdot \delta\left(v_{k-1} - \widehat{v}_{k-1}^{j}\right), \end{aligned} \tag{4.7}$$

where $\xi_k^j = p(z_k \mid \widehat{v}_{k-1}^{j})$ is defined as the weight assigned to the jth process noise sample $\widehat{v}_{k-1}^{j}$. The process noise samples $\{\widehat{v}_{k-1}^{j},\ j = 1, \ldots, H\}$ are then resampled according to $\{\xi_k^j,\ j = 1, \ldots, H\}$ based on the principle that $\Pr\{v_{k-1}^{i} = \widehat{v}_{k-1}^{j}\} = \xi_k^j$, where $\{v_{k-1}^{i},\ i = 1, \ldots, H\}$ are the process noise samples obtained from resampling. The obtained resampled process noise samples are approximately distributed according to the posterior distribution $p(v_{k-1} \mid z_{1:k})$.

In order to calculate ξ_k^j, the likelihood function $p(z_k \mid \widehat{v}_{k-1}^{j})$ is expanded based on the resampled state particles at time step $k-1$, $\{x_{k-1}^{i},\ i = 1, \cdots, H\}$. Consider

$$p\left(z_k \mid \widehat{v}_{k-1}^{j}\right) = \sum_{i=1}^{H}p\left(z_k \mid x_{k-1}^{i}, \widehat{v}_{k-1}^{j}\right)p\left(x_{k-1}^{i} \mid \widehat{v}_{k-1}^{j}\right). \tag{4.8}$$

Since x_{k-1}^{i} and $\widehat{v}_{k-1}^{j}$ are independent, $p(x_{k-1}^{i} \mid \widehat{v}_{k-1}^{j}) = p(x_{k-1}^{i})$.

The resmapled particles at time step $k-1$, $\{x_{k-1}^{i},\ i = 1, \ldots, H\}$, should be assigned with the same and equal weights, $1/H$. We can then obtain

$$p\left(x_{k-1}^{i}\right) = \frac{1}{H}. \tag{4.9}$$

To calculate $p(z_k \mid x_{k-1}^{i}, \widehat{v}_{k-1}^{j})$, define $\mu_k^{i,j}$ as the intermediate particle,

$$\mu_k^{i,j} = f\left(x_{k-1}^{i}, \widehat{v}_{k-1}^{j}\right), \quad i = 1, \ldots H,\ j = 1, \ldots, H \tag{4.10}$$

and expand $p(z_k \mid x_{k-1}^i, \hat{v}_{k-1}^j)$ based on $\mu_k^{i,j}$,

$$p\left(z_k \mid x_{k-1}^i, \hat{v}_{k-1}^j\right) = \sum_{p=1}^{H}\sum_{q=1}^{H}\left[p\left(z_k \mid \mu_k^{p,q}, x_{k-1}^i, \hat{v}_{k-1}^j\right)p\left(\mu_k^{p,q} \mid x_{k-1}^i, \hat{v}_{k-1}^j\right)\right]. \tag{4.11}$$

Since x_{k-1}^i and $\hat{v}_{k-1}^j$ are known and $\mu_k^{p,q}$ is obtained from a purely deterministic relationship in (4.10), we obtain

$$p\left(\mu_k^{p,q} \mid x_{k-1}^i, \hat{v}_{k-1}^j\right) = \begin{cases} 1, & \text{for } p = i \text{ and } q = j, \\ 0, & \text{for } p \neq i \text{ or } q \neq j, \end{cases} \tag{4.12}$$

$$p\left(z_k \mid x_{k-1}^i, \hat{v}_{k-1}^j\right) = p\left(z_k \mid \mu_k^{i,j}\right). \tag{4.13}$$

Combining (4.9) and (4.13) with (4.8) results in

$$p\left(z_k \mid \hat{v}_{k-1}^j\right) = \sum_{i=1}^{H} p\left(z_k \mid \mu_k^{i,j}\right) \cdot \frac{1}{H}. \tag{4.14}$$

At each time step, the process noise samples are drawn from a uniform distribution $U(-d, d)$. Each process noise sample $\hat{v}_{k-1}^j$ is evaluated and assigned its corresponding weight ξ_k^j. A resampling procedure is then used to redistribute the process noise samples, from which the process noise samples with large weights are replicated while the samples with small weights are eliminated.

The standard particle filter procedure for state estimation follows next. The predicted particles $\{\hat{x}_k^i : i = 1, \ldots, H\}$ are then obtained based on the resampled process noise samples $\{v_{k-1}^i : i = 1, \ldots, H\}$ through the dynamic model (4.1). The predicted particles are updated and resampled as in the conventional particle filter algorithm.

The complete algorithm including the process noise estimation and state estimation parts is summarized below.

Algorithm 4.1 (particle filter based process noise identification). (1) At time step $k-1$, draw process noise samples $\{\hat{v}_{k-1}^j : j = 1, \ldots, H\}$ from a uniform distribution $U(-d, d)$.

(2) Calculate the intermediate particles $\{\mu_k^{i,j} : i = 1, \ldots, H;\ j = 1, \ldots, H\}$ according to (4.10).

(3) Calculate the process noise sample weights $\{\xi_k^j : j = 1, \ldots, H\}$ via (4.14).

(4) Resample the discrete distribution $\{\xi_k^j : j = 1, \ldots, H\}$, H times to generate the resampled process noise samples $\{v_{k-1}^i : i = 1, \ldots, H\}$, so that for any i, $\Pr\{v_{k-1}^i = \hat{v}_{k-1}^j\} = \xi_k^j$. Set the weights ξ_k^j to $1/H$, $i = 1, \ldots, H$.

(5) Calculate the predicted particles at time step k, $\{\hat{x}_k^i,\ i = 1, \ldots, H\}$,

$$\hat{x}_k^i = f\left(x_{k-1}^i, v_{k-1}^i\right). \tag{4.15}$$

(6) and (7) are the same with the stages (3) and (4) in Algorithm 2.1.

Algorithm 4.2 (simplification of the proposed algorithm). In the proposed algorithm, at each iteration, $H * H$ intermediate particles are calculated through the permutation of particles and process noise samples in (4.10) and evaluated via (4.13). This increases the computation burden and the algorithm runs slowly compared to the conventional particle filter, which are based on H particles (Algorithm 2.1). It is observed that at each time step, after resampling the particles focus in some smaller area, a large portion of particles are with the same value (Algorithm 2.1, Step (4)). To simplify the algorithm, it is assumed that the particles (Algorithm 2.1, Step (4)) are less variable compared with the process noise samples (4.10). In (4.10), particles $\{x_{k-1}^{i},\ i = 1, \ldots, H\}$ are replaced by $\tilde{\tilde{x}}_{k-1}$, the estimate of the state at time step $k - 1$, which results in a simplified version of (4.10),

$$\mu_k^j = f\left(\tilde{\tilde{x}}_{k-1}, \hat{v}_{k-1}^j\right), \tag{4.16}$$

and $p(z_k \mid \hat{v}_{k-1}^j)$ is expanded directly on μ_k^j,

$$p\left(z_k \mid \hat{v}_{k-1}^j\right) = \sum_{\tau=1}^{H} p(z_k \mid \mu_k^\tau) p\left(\mu_k^\tau \mid \hat{v}_{k-1}^j\right). \tag{4.17}$$

Using the similar derivation process with (4.12), we can obtain

$$p\left(z_k \mid \hat{v}_{k-1}^j\right) = p\left(z_k \mid \mu_k^j\right). \tag{4.18}$$

In the simplified version of the proposed algorithm, the number of intermediate particles is reduced to H, which reduces the computation burden and increases the computing speed. More importantly, the performance of the algorithm with the simplification procedure is verified through simulation study. In the following sections, the particle filter based process noise identification algorithm refers to the simplified version.

Discussion. It is required to set the process noise bound d preliminarily, which could be estimated based on the maximin uncertain dynamics. For most of the manned air vehicles, the process noise bound d could be determined according to the flight envelop of the aircraft. For the unmanned air vehicles with unexpected maneuvers, d is set large enough to cover the unexpected maneuvers. If our knowledge about d is imprecise, tracking process may fail when coping with the maneuvers that are not covered by the process noise bound d.

5. Multiple Target Tracking Using Particle Filter Based Multiscan JPDAF

The number of targets (M) to be tracked is assumed as fixed and known, where each target track has been initiated, and our objective is to maintain the tracks. Each target is parameterized by a state $x_{m,k}$, where m denotes the mth target and k denotes time step k. The combined state, $x_k = (x_{1,k}, \ldots, x_{M,k})$, is the concatenation of the individual target states. The individual targets are assumed to evolve independently according to the Markovian dynamic models $p_m(x_{m,k} \mid x_{m,k-1})$. The observation vector z_k is composed of multiple sensor

measurements $\{z_{j,k},\ j = 1,\ldots,N_k\}$, where N_k is the total number of measurements. It is assumed that there are no unresolved measurements (i.e., measurements associated with two or more targets simultaneously); any measurement is either associated with a single target or caused by clutter. Clutter is modeled as independently and identically distributed (IID) with uniform spatial distribution over the surveillance area.

In the particle filter based JPDA algorithm, the distributions of interest are the marginal filtering distributions for each of the targets $p_m(x_{m,k} \mid z_{1:k}),\ m = 1\cdots M$, and these distributions are approximated with particles, $\{\hat{x}^i_{m,k},\ i = 1\cdots H,\ m = 1\cdots M\}$, and their associated weights $\{w^i_{m,k},\ i = 1\cdots H,\ m = 1\cdots M\}$, as in

$$p_m(x_{m,k} \mid z_{1:k}) = \sum_{i=1}^{H} w^i_{m,k}\delta\left(x_{m,k} - \hat{x}^i_{m,k}\right). \tag{5.1}$$

At each time step, when the new observation vector arrives, the marginal filtering distributions for each of the targets are updated through the Bayesian sequential estimation recursions [24].

In the standard single scan JPDA framework, a track is updated with a weighted sum of the measurements which could have reasonably originated from the target in track. The only information the standard JPDA algorithm uses is the measurements on the present scan and the state vectors. If more scans of measurements are used, additional information is available resulting in better computed probabilities [4]. Since the tracking systems are unable to store all of the measurements from all the scans, a Bayesian tracking system can at best rely on a sliding window of scans. In [28], the single scan JPDA filter is extended to the multiple scan JPDA filter for tracking multiple targets. And in this work the multiple scan JPDA filter is developed in a particle filter framework.

The multiple scan JPDA calculation examines multiple scan joint association events [28]. The measurement to target the association event of a multiple scan is defined as $\lambda_{k-L+1:k}$, where L denotes the length of the multiple scan sliding window. The multiple scan joint association events are mutually exclusive, and they form a complete set $\Lambda_{k-L+1:k}$ [11]. $\lambda_{k-L+1:k}$ is composed by the association vectors at each scan in the sliding window, $\lambda_{k-L+1:k} = (\theta_{k-L+1}, \theta_{k-L+2}, \ldots, \theta_k)$. The elements of the association vector at time step k, $\theta_k = (\zeta_{1,k}, \ldots, \zeta_{j,k}, \ldots, \zeta_{N_k,k})$ are given by

$$\zeta_{j,k} = \begin{cases} 0, & \text{if } z_{j,k} \text{ is due to clutter,} \\ m \in \{1\cdots M\}, & \text{if } z_{j,k} \text{ is from target } m. \end{cases} \tag{5.2}$$

The heart of the new algorithm is to find the posterior probability for the joint association event of multiple scans. That is to calculate $p(\lambda_{k-L+1:k} \mid z_{1:k})$ and it can be written as

$$\begin{aligned} p(\lambda_{k-L+1:k} \mid z_{1:k}) &\propto p(z_k \cdots z_{k-L+1} \mid \lambda_{k-L+1:k}, z_{1:k-L})p(\lambda_{k-L+1:k} \mid z_{1:k-L}) \\ &\propto p(z_k \cdots z_{k-L+1} \mid \lambda_{k-L+1:k}, z_{1:k-L})p(\lambda_{k-L+1:k}), \end{aligned} \tag{5.3}$$

where the conditioning of $\lambda_{k-L+1:k}$ on the history of measurements before the sliding window has been eliminated.

The distribution of the measurements in the sliding window based on a specific association event is given by

$$p(z_k \cdots z_{k-L+1} \mid \lambda_{k-L+1:k}, z_{1:k-L}) = \prod_{s=1}^{L} \left[\prod_{j=1}^{N_{k-L+s}} p\left(z_{j,k-L+s} \mid \lambda_{k-L+1:k}, z_{1:k-L}\right) \right]. \tag{5.4}$$

To reduce the notation, the index of the scan s in the sliding window is denoted by $k_s = k - L + s$. Then we can obtain

$$\begin{aligned} p(z_k \cdots z_{k-L+1} \mid \lambda_{k-L+1:k}, z_{1:k-L}) &= \prod_{s=1}^{L} \left[\prod_{j=1}^{N_{k_s}} p\left(z_{j,k_s} \mid \lambda_{k-L+1:k}, z_{1:k-L}\right) \right] \\ &= \prod_{s=1}^{L} \left[\prod_{j \in I_{0,k_s}} p_c\left(z_{j,k_s}\right) \cdot \prod_{j \in I_{k_s}} p\left(z_{j,k_s} \mid x_{\zeta_{j,k_s},k_s}\right) \right] \\ &= \prod_{s=1}^{L} \left[(V)^{-C_{k_s}} \cdot \prod_{j \in I_{k_s}} p\left(z_{j,k_s} \mid x_{\zeta_{j,k_s},k_s}\right) \right], \end{aligned} \tag{5.5}$$

where $I_{0,k_s} = \{j \in \{1, \ldots, N_{k_s}\} : \zeta_{j,k_s} = 0\}$ and $I_{k_s} = \{j \in \{1, \ldots, N_{k_s}\} : \zeta_{j,k_s} \neq 0\}$ are, respectively, the subsets of measurement indices corresponding to clutter measurements and measurements from the targets being tracked, on scan k_s. p_c denotes the clutter likelihood model, which is assumed to be uniform over the volume of the surveillance area V. The volume of the surveillance area could be calculated as per $V = 2\pi R_{\max}$, where $R_{\max}$ is the maximum range of the sensor. C_{k_s} is defined as the number of clutter measurements.

The joint association prior $p(\lambda_{k-L+1:k})$, can be calculated as in (5.6) according to [4, 16],

$$p(\lambda_{k-L+1:k}) = \prod_{s=1}^{L} \left[\frac{C_{k_s}! \varepsilon}{N_{k_s}!} \prod_{m=1}^{M} (P_D)^{\delta_m(\theta_{k_s})} (1 - P_D)^{1-\delta_m(\theta_{k_s})} \right], \tag{5.6}$$

where ε is a “diffuse” prior [16] and P_D is the detection probability. $\delta_m(\theta_{k_s})$ is a binary variable and set to one if the mth target is assigned with a measurement in the event θ_{k_s}.

The posterior probability for the joint association event of multiple scans is obtained as

$$p(\lambda_{k-L+1:k} \mid z_{1:k}) \propto p(\lambda_{k-L+1:k}) \prod_{s=1}^{L} \left[(V)^{-C_{k_s}} \cdot \prod_{j \in I_{k_s}} p\left(z_{j,k_s} \mid x_{\zeta_{j,k_s},k_s}\right) \right]. \tag{5.7}$$

The posterior probability that the jth measurement is associated with the mth target on scan k, β_{jm}, is calculated by summing over the probabilities of the corresponding joint association events via

$$\beta_{jm} = p\left(\zeta_{j,k} = m \mid z_{1:k}\right) = \sum_{\left\{\lambda_{k-L+1:k} \in \Lambda_{k-L+1:k} : \zeta_{j,k} = m\right\}} p(\lambda_{k-L+1:k} \mid z_{1:k}). \tag{5.8}$$

These approximations can, in turn, be used in (5.9) to approximate the target likelihood according to [24],

$$p_m(z_k \mid x_{m,k}) = \beta_{0m} + \sum_{j=1}^{N_k} \beta_{jm} p(z_{j,k} \mid x_{m,k}), \tag{5.9}$$

where β_{0m} is the posterior probability that the mth target is undetected. Finally, setting the new importance weights to

$$\begin{gathered} w_{m,k}^i \propto w_{m,k-1}^i \frac{p_m\left(z_k \mid \hat{x}_{m,k}^i\right) p_m\left(\hat{x}_{m,k}^i \mid x_{m,k-1}^i\right)}{q_m\left(\hat{x}_{m,k}^i \mid x_{m,k-1}^i\right)}, \\ \sum_{i=1}^{H} w_{m,k}^i = 1, \end{gathered} \tag{5.10}$$

where $q_m(\hat{x}_{m,k}^i \mid x_{m,k-1}^i)$ is the proposal distribution, which is used to generate the predicted particles. Equation (5.10) leads to the sample set $\{w_{m,k}^i, \hat{x}_{m,k}^i\}_{i=1}^H$ being approximately distributed according to the marginal filtering distribution $p_m(x_{m,k} \mid z_{1:k})$.

A summary of the particle filter based multiple scan JPDA filter algorithm is presented in what follows. Assuming that the sample sets $\{w_{m,k-1}^i, x_{m,k-1}^i\}_{i=1}^H$, $m = 1 \cdots M$, are approximately distributed according to the corresponding marginal filtering distributions at the previous time step $p_m(x_{m,k-1} \mid z_{1:k-1})$, $m = 1 \cdots M$, the algorithm proceeds as follows at the current time step.

Algorithm 5.1 (particle filter based multiscan JPDA filter). (1) For $m = 1 \cdots M$, $i = 1 \cdots H$, generate predicted particles for the target states $\hat{x}_{m,k}^i \sim q_m(\hat{x}_{m,k}^i \mid x_{m,k-1}^i)$.

(2) For $m = 1 \cdots M$, calculate $\tilde{x}_{m,k}$, the preapproximation of $x_{m,k}$, which is to be substituted into (5.7) to calculate the posterior probability of the joint association event in multiple scan, $p(\lambda_{k-L+1:k} \mid z_{1:k})$. Consider

$$\tilde{x}_{m,k} = \sum_{i=1}^{H} w_{m,k-1}^i \hat{x}_{m,k}^i. \tag{5.11}$$

(3) Enumerate all the valid joint measurement to target association events in the sliding window $k - L + 1 : k$ to form the set $\Lambda_{k-L+1:k}$.

(4) For each $\lambda_{k-L+1:k} \in \Lambda_{k-L+1:k}$, compute the posterior probability of the joint association event in multiple scan, $p(\lambda_{k-L+1:k} \mid z_{1:k})$, via (5.7).

(5) For $m = 1 \cdots M$, $j = 1 \cdots N_k$, compute the marginal association posterior probability, β_{jm}, via (5.8).

(6) For $m = 1 \cdots M$, $i = 1 \cdots H$, compute the target likelihood, $p_m(z_k \mid \hat{x}_{m,k}^i)$, via (5.9).

(7) For $m = 1 \cdots M$, $i = 1 \cdots H$, compute and normalize the particle weights, via (5.10).

(8) Resample the discrete distribution $\{w^i_{m,k} : i = 1, \ldots, H\}$, H times to generate particles $\{x^j_{m,k} : j = 1, \ldots, H\}$, so that for any j, $\Pr\{x^j_{m,k} = \hat{x}^i_{m,k}\} = w^i_{m,k}$. Set the weights $w^i_{m,k}$ to $1/H$, $i = 1, \ldots, H$, and move to Stage 1.

6. An Algorithm for Tracking Multiple Maneuvering Targets

The proposed multiple maneuvering target tracking algorithm is a combination of the particle filter based process noise identification algorithm (proposed in Section 4) and the particle filter based multiscan JPDA algorithm (proposed in Section 5). In the proposed algorithm, firstly the particle filter based process noise identification algorithm is used to estimate the maneuvering movement for each target. Then the particles of each target model are propagated to the next time step based on the new distributed process noise samples to obtain the predicted particles. In the update process, each predicated particle of one target model is assigned with a weight based on the multiscan JPDA process. The steps of the proposed multiple maneuvering target tracking algorithm are listed in the following.

Algorithm 6.1 (multiple maneuvering target tracking). (1) At time step $k-1$, for $m = 1 \cdots M$,

(a) draw process noise samples $\{\hat{v}^j_{m,k-1} : j = 1, \ldots, H\}$ from a uniform distribution $U(-d, d)$,

(b) calculate the intermediate particles $\{\mu^j_{m,k} : j = 1, \ldots, H\}$ according to

$$\mu^j_{m,k} = f\left(\tilde{\tilde{x}}_{m,k-1}, \hat{v}^j_{m,k-1}\right), \tag{6.1}$$

(c) calculate the process noise sample weights $\{\xi^j_{m,k} : j = 1, \ldots, H\}$ via (6.2) and normalize each weight:

$$\xi^j_{m,k} = p\left(z^m_k \mid \mu^j_{m,k}\right), \tag{6.2}$$

where z^m_k are the measurements that are close to $\mu^j_{m,k}$ obtained from the nearest neighbor method,

(d) resample the discrete distribution $\{\xi^j_{m,k} : j = 1, \ldots, H\}$, H times to generate the new process noise samples $\{v^i_{m,k-1} : i = 1, \ldots, H\}$, so that for any i, $\Pr\{v^i_{m,k-1} = \hat{v}^j_{m,k-1}\} = \xi^j_{m,k}$. Set the weights $\xi^j_{m,k}$ to $1/H$, $i = 1, \ldots, H$,

(e) obtain the predicted particles $\{\hat{x}^i_{m,k} : i = 1, \ldots, H\}$ at time step k from the new process noise samples via

$$\hat{x}^i_{m,k} = f\left(x^i_{m,k-1}, v^i_{m,k-1}\right). \tag{6.3}$$

(2) At time step k, for $m = 1 \cdots M$, calculate $\tilde{x}_{m,k}$, the postapproximation of $x_{m,k}$,

$$\tilde{x}_{m,k} = \sum_{i=1}^{H} \xi_{m,k}^{i} \hat{x}_{m,k}^{i}, \tag{6.4}$$

where $\{\xi_{m,k}^{i} : i = 1, \ldots, H\}$ are the process noise samples' weights and are calculated based on the measurements at the current time step (k). However, in (5.11) $\tilde{x}_{m,k}$ is calculated based on $\{w_{m,k-1}^{i} : i = 1, \ldots, H\}$, which rely on the measurements at the previous time step ($k-1$). As a result, in maneuvering target tracking, $\tilde{x}_{m,k}$ calculated via (6.4) is closer to the true state than that via (5.11) since the information from the current time step is considered in (6.4).

(3)–(8) are the same with the stages (3)–(8) in Algorithm 5.1.

(3) Thus we can obtain $\tilde{\tilde{x}}_{m,k} = \sum_{i=1}^{H} w_{m,k}^{i} \hat{x}_{m,k}^{i}$. $\tilde{\tilde{x}}_{m,k}$ is the estimate of the true state $x_{m,k}$.

7. Simulation Results and Analysis

This section consists of three parts. In the first and second part, the particle filter based process noise identification algorithm and the particle filter based multiscan JPDA algorithm are, respectively, used to track single maneuvering targets and two slow-maneuvering targets in clutter. In the third part, the combination of the two algorithms are used to track two highly maneuvering, at times closely spaced and crossed, targets.

7.1. Single Maneuvering Target Tracking

The simulation study using nearly coordinate turn model is performed. The maneuvering target tracking is done by setting up a 2D flight path in x-y plane, which is similar to the path considered in [16]. At time step 1, the target starts at location $[-310, 310]$ in Cartesian coordinates in meters with initial velocity (in m/s) $[10, -400]$. The following trajectory is considered: a straight line with constant velocity between 1 and 17 s, a coordinated turn (0.09 rad/s) between 17 and 34 s, a straight line with constant velocity between 34 and 51 s, a coordinated turn (0.09 rad/s) between 51 and 68 s, and a straight line with constant velocity between 68 and 100 s.

In the particle filter with process noise identification, a general model

$$X_k = \Phi X_{k-1} + \Gamma v_{k-1} \tag{7.1}$$

is adopted during the whole tracking process, where in (7.1),

$$\Phi = \begin{bmatrix} \Phi_b & 0 \\ 0 & \Phi_b \end{bmatrix}, \qquad \Phi_b = \begin{bmatrix} 1 & \Delta T & \dfrac{\Delta T^2}{2} \\ 0 & 1 & \Delta T \\ 0 & 0 & 1 \end{bmatrix}, \qquad \Gamma = I_{6\times 6}. \tag{7.2}$$

Matrix Φ is the transition matrix and ΔT is the sample interval. $X_k = [p_x, \gamma_x, a_x, p_y, \gamma_y, a_y]_k^T$ is the state vector; p_x, γ_x, and a_x denote, respectively, the position, velocity, and acceleration

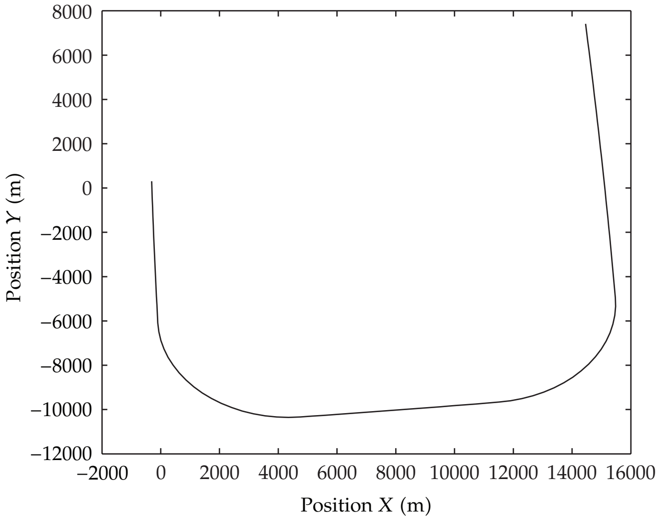


Figure 1: True trajectory of the single maneuvering target.

of the moving object along the x-axis of Cartesian frame and, p_y, γ_y, and a_y along the y-axis. The equivalent process noise, $v_{k-1} = [v_{p_x}, v_{\gamma_x}, v_{a_x}, v_{p_y}, v_{\gamma_y}, v_{a_y}]_{k-1}^T$, with unknown statistics is required to be identified. The bound of the process noise (d), which accounts for the uncertain dynamics, is chosen as $\{20\,\mathrm{m}, 20\,\mathrm{m/s}, 10\,\mathrm{m/s^2}, 20\,\mathrm{m}, 20\,\mathrm{m/s}, 10\,\mathrm{m/s^2}\}$. The number of the process noise samples is equal to the number of particles, which is set as 500. The algorithm is initialized with Gaussian around the initial state of the true target, and the standard deviation of the Gaussian distribution is chosen as $\{10\,\mathrm{m}, 10\,\mathrm{m/s}, 5\,\mathrm{m/s^2}, 10\,\mathrm{m}, 10\,\mathrm{m/s}, 5\,\mathrm{m/s^2}\}$.

A track-while-scan (TWS) radar is positioned at the origin of the plane. The measurement equation is as follows:

$$Z_k = h(X_k) + n_k, \tag{7.3}$$

where $Z_k = [z_1, z_2]_k$ is the observation vector. z_1 is the distance between the radar and the target, and z_2 is the bearing angle. The measurement noise $n_k = [n_{z_1}, n_{z_2}]_k$ is a zero mean Gaussian white noise process with standard deviations of $20\,\mathrm{m}$ (σ_{z_1}) and $0.01\,\mathrm{rad}$ (σ_{z_2}). Resolution of the sensor is selected after from [29] (twice of the standard deviations of the measurement noise). The sampling interval is $\Delta T = 1\,\mathrm{s}$.

The particle filter based process noise identification algorithm is compared to the IMM filter and the regularized particle filter. The IMM filter consists of three extended Kalman filter (EKF) with different motion models. The details regarding these models may be found in [16]. The initial model probabilities and the mode switching probability matrix are set the same values as in [16]. For the regularised particle filter, Epanechnikov kernel is chosen as the rescaled kernel density, which is the same as in [30]. Moreover, the proposed algorithm is compared to its complete version in the same simulation setup.

The simulation results are obtained from 1000 Monte Carlo runs except the results from the complete version of the particle filter based multiscan JPDA algorithm, which is run for 100 times. Figure 1 shows the true trajectory of the maneuvering target. The root mean-square errors (RMSEs) in position at each time step, respectively, using the four methods are shown

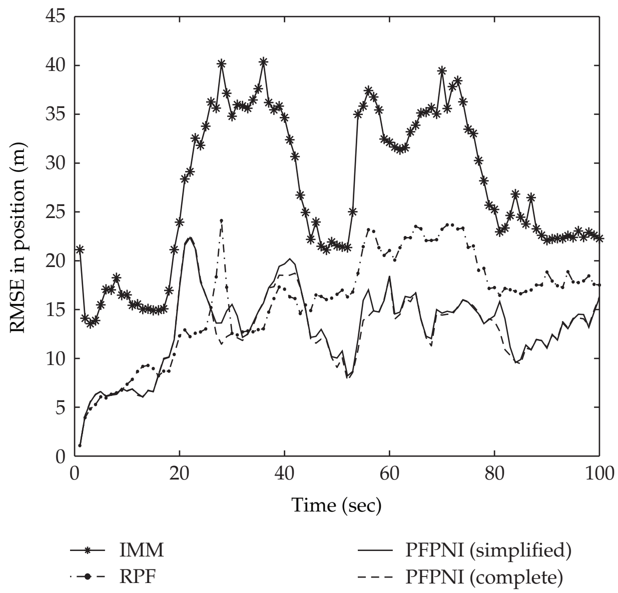


Figure 2: RMSE in position using IMM, RPF, PFPNI (simplified), and PFPNI (complete) algorithms.

Table 1: Performance comparison.

	RMSE (m)	ET (s)	TLR
IMM	28.34	0.0239	3.7%
RPF	16.66	0.8708	0
PFPNI (simplified)	13.67	0.4503	0
PFPNI (complete)	13.29	190.47	0

in Figure 2, where RPF and PFPNI represent the regularised particle filter algorithm and the particle filter based process noise identification algorithm. The performance of the methods is also compared via the global RMSE (in position), the tracking loss rate (TLR), and the executing time (ET), which are listed in Table 1. The tracking loss rate (TLR) is defined as the ratio of the number of simulations, in which the target is lost in track, to the total number of simulations carried out. The target is defined as lost in track when its global RMSE in position is larger than the ten times of the magnitude of the standard deviation in position. The executing time (ET) is the CPU time needed to execute one time step in MATLAB 7.1 on a 3 GHz (Mobile) Pentium IV operating under Windows 2000.

From the simulation results, it can be seen that the simplified version of the proposed particle filter based process noise identification algorithm outperforms the IMM filter and the regularised particle filter algorithm, with computing time within the limits of practically realizable systems. Moreover, the proposed algorithm needs neither the possible multiple motion models nor the transition probability matrices, which makes it a more general algorithm for maneuvering target tracking. From the simulation results, it can also be seen that the complete version of the particle filter based process noise identification algorithm is not suitable for practical application due to the long computing time, though it gains a 2.8% increase in accuracy (RMSE) compared with its simplified version.

Table 2: Performance comparison.

	RMSE (m)	ET (s)	TLR	SR
JPDA	T1: 20.24, T2: 22.87	0.07	24%	0
PFJPDA(1)	T1: 13.21, T2: 11.36	1.28	0	0
PFJPDA(2)	T1: 12.92, T2: 11.01	1.54	0	0
PFJPDA(3)	T1: 12.75, T2: 10.67	1.8	0	0

7.2. Multiple Target Tracking

The simulation is carried out for tracking two slow-maneuvering targets in clutter. The Wiener process acceleration model (7.1) is chosen as the motion model for the two targets. The process noise, $v_{k-1} = [v_{p_x}, v_{\gamma_x}, v_{a_x}, v_{p_y}, v_{\gamma_y}, v_{a_y}]_{k-1}^T$, is a zero mean Gaussian white noise process with standard deviations of $1\,\mathrm{m}(\sigma_{v_{p_x}})$, $1\,\mathrm{m/s}(\sigma_{v_{\gamma_x}})$, $20\,\mathrm{m/s}^2(\sigma_{v_{a_x}})$, $1\,\mathrm{m}(\sigma_{v_{p_y}})$, $1\,\mathrm{m/s}(\sigma_{v_{\gamma_y}})$, and $20\,\mathrm{m/s}^2(\sigma_{v_{a_y}})$.

At time step 1, target one starts at location $[-310, 310]$ in x-y Cartesian coordinates in meters with the initial velocity (in m/s) $[10, -400]$. Target two starts at location (in m) $[-310, -19000]$ with the initial velocity (in m/s) $[10, 400]$. The length of each simulation run is 50 seconds. A TWS radar is positioned at the origin of the plane, whose details are provided in Section 7.1.

The sampling interval is $\Delta T = 1\,\mathrm{s}$ and it is assumed that the probability of detection $P_D = 0.9$ for the radar. For generating measurements in simulations, the clutter is assumed uniformly distributed with density $1 \times 10^{-6}/\mathrm{m}^2$.

In the particle filter based multiscan JPDA methods, each target model is assigned with 500 particles. The algorithm is initialized with Gaussians around the initial states of the true targets, and the standard deviations of the two Gaussian distributions are chosen equally as $\{10\,\mathrm{m}, 10\,\mathrm{m/s}, 5\,\mathrm{m/s}^2, 10\,\mathrm{m}, 10\,\mathrm{m/s}, 5\,\mathrm{m/s}^2\}$.

In the simulations carried out, the length of the multiple scan sliding window (L) varies from 1 to 3. The corresponding particle filter based multiscan JPDA methods with $L = 1, 2, 3$ are utilized to track two slow-maneuvering targets individually. A comparison to the standard JPDA filter is also studied on the same simulation setup. The simulation results are obtained from 1000 Monte Carlo runs. Figure 3 shows the true trajectories of the two targets and Figure 4 shows the distance between the two targets along time, through which we can see that the two targets reach the smallest distance at time step 25. The RMSEs in position for the two targets are, respectively, shown in Figures 5 and 6, where PFJPDA represents the particle filter based JPDA filter with its following number in the bracket (e.g., (2.1)) denoting the scan number (L). The performance of the four methods is also compared in Table 2. The swap rate (SR) is defined as the ratio of the number of simulations, in which the two targets swap, to the total number of simulations.

Compared with the standard JPDA method (based on extended Kalman filter), the particle filter based JPDA methods (single scan and multiple scan) are much more accurate and robust, at the cost of longer computing time. This verifies that when dealing with nonlinear problem (nonlinear observation equation) and large random acceleration (large process noise), the performance of particle filter is better than extended Kalman filter using local linearization.

In the comparison between the three particle filter based JPDA methods ($L = 1, 2, 3$, resp.), from Figures 5 and 6 it can be seen that there is no significant deference between the three methods except around time step 25, when the two targets are very close to each

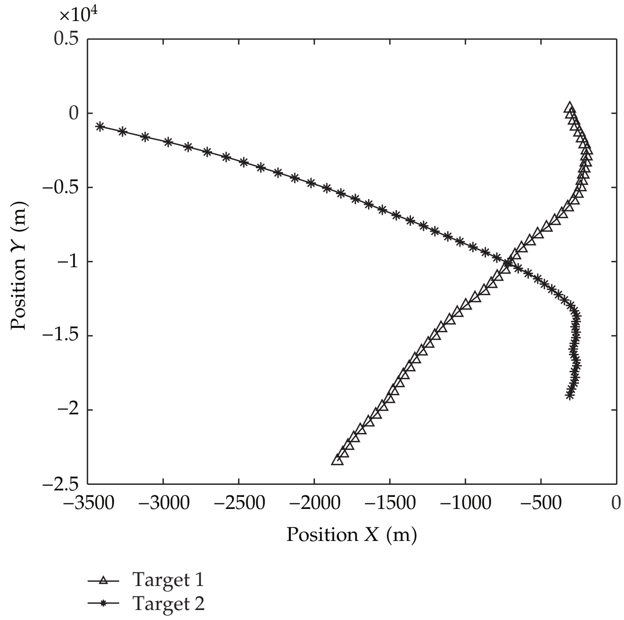


Figure 3: True trajectories of the two slow-maneuvering targets.

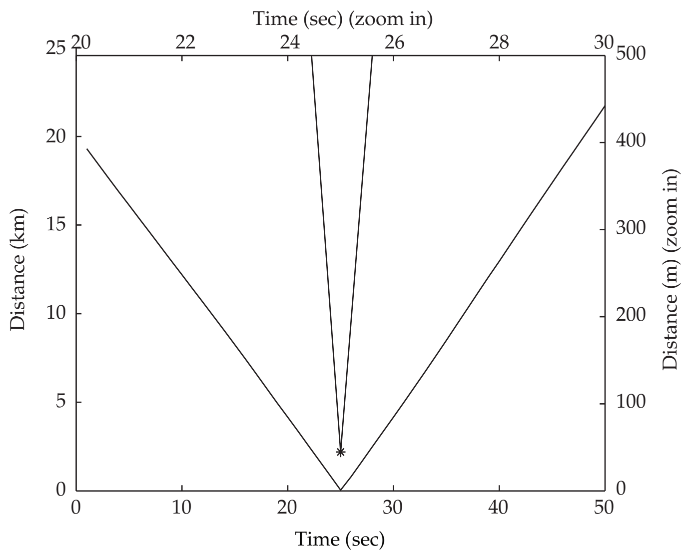


Figure 4: Distance between the two targets (- distance; -* zoom in distance).

other (Figure 4). As the scan number increases, the RMSE (in position) of the corresponding algorithm decreases significantly around time step 25 (Figures 5 and 6). This verified that the particle filter based multiscan JPDA method provides a better performance especially when the targets are very close. The additional information of more scans improve the association probabilities in such critical situation, resulting in lower estimation errors (RMSE) and larger robustness (tracking loss rate).

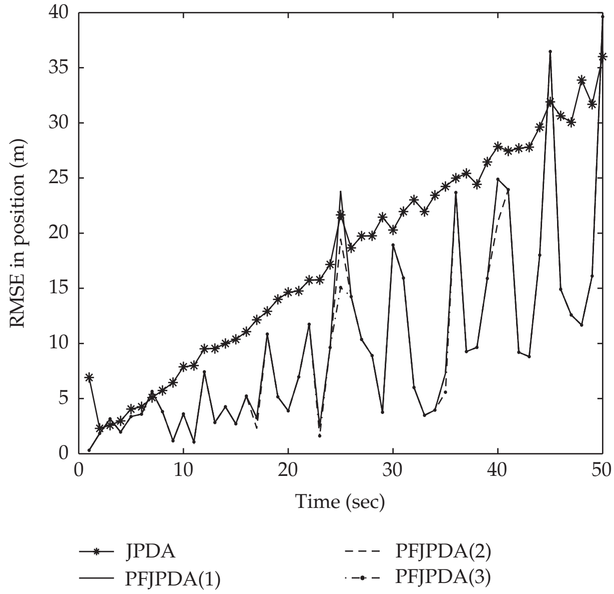


Figure 5: Target 1: RMSE in position using standard JPDA filter, particle filter based multiscan JPDA algorithms with a scan number equals to 1, 2, and 3, respectively.

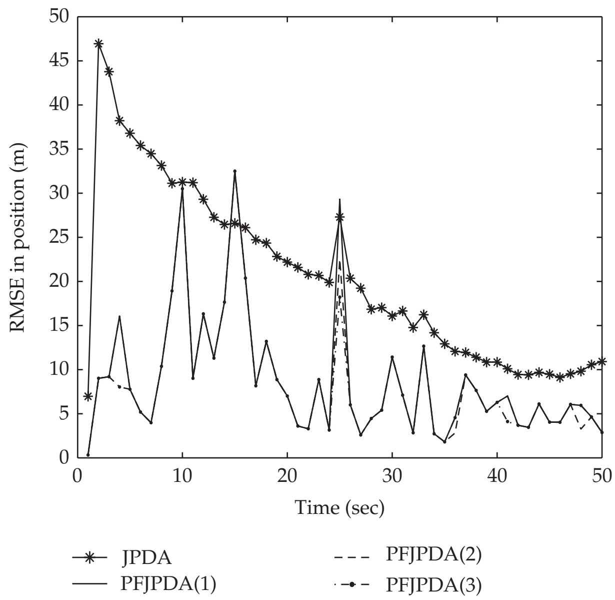


Figure 6: Target 2: RMSE in position using standard JPDA filter, particle filter based multiscan JPDA algorithms with a scan number equals to 1, 2, and 3, respectively.

7.3. Multiple Maneuvering Target Tracking

We now consider tracking two highly maneuvering targets in clutter. The true trajectories are shown in Figure 7 in x-y plane. The velocities of the targets are shown in Figure 8, where VX and VY represent the velocity, respectively, in x and y coordinate. The distance between

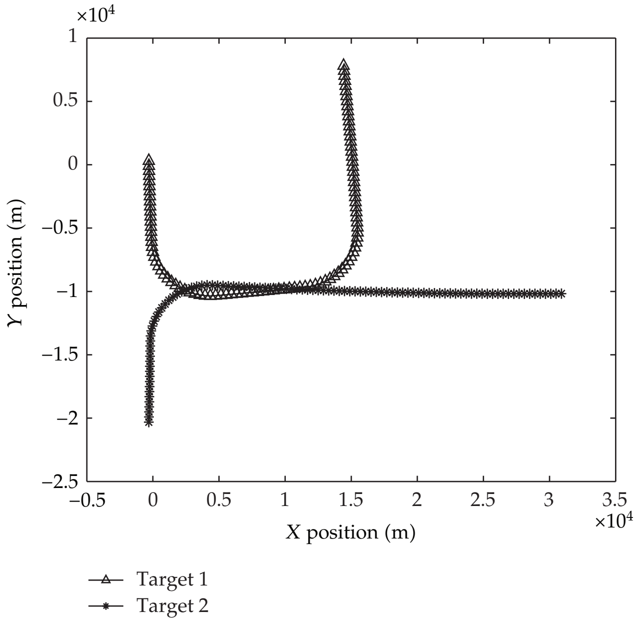


Figure 7: True trajectories of the two maneuvering targets.

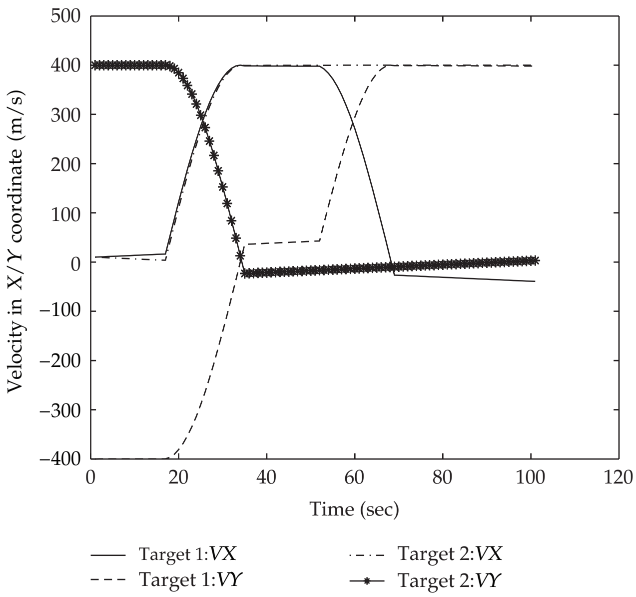


Figure 8: True velocities of the two maneuvering targets.

the two targets is shown in Figure 9. Target one is with the same setup as in the example proposed in Section 7.1. Target two starts at location $[-310, -20310]$ in Cartesian coordinates in meters with the initial velocity (in m/s) $[10, 400]$. Its trajectory is a straight line with constant velocity between 1 and 17 s, a coordinated turn (0.09 rad/s) between 17 and 34 s, and a straight line constant velocity between 34 and 100 s.

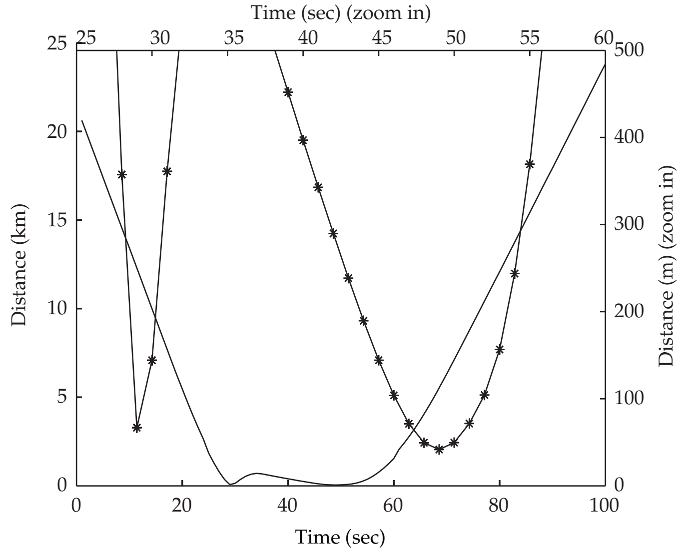


Figure 9: Distance between the two targets (- distance; -* zoom in distance).

A TWS radar is positioned at the origin of the plane (refer to Section 7.1). The sampling interval (ΔT), the probability of detection (P_D), and the clutter density are the same as in Section 7.2.

In the proposed method, each target model is assigned with 1000 particles. The length of the multiple scan sliding window (L) is chosen equal to 3. The bound of the process noise (d) is the same as in Section 7.1. The algorithm is initialized with Gaussians around the initial states of the true targets, and the standard deviations of the two Gaussian distributions are chosen equally as $\{10\,\text{m}, 10\,\text{m/s}, 5\,\text{m/s}^2, 10\,\text{m}, 10\,\text{m/s}, 5\,\text{m/s}^2\}$.

In this work, the proposed method is compared to an IMMJPDA method and a multiscan IMMJPDA method, whose details could be found in [16, 20], respectively.

The simulation results are obtained from 1000 Monte Carlo runs. The RMSEs (resp. in position and velocity) at each time step for the two targets are shown in Figures 10, 11, 12, and 13, where MS-IMMJPDA and PFPNI-PFMSJPDA represent, respectively, the multiscan IMMJPDA algorithm and the proposed algorithm. The performance comparison is listed in Table 3. Moreover, for the proposed algorithm, the influence of the particle number in its performance is studied and simulations are carried out based on different sample sizes. The results are listed in Table 4.

The simulation results show that the proposed algorithm is more accurate than the multiple scan IMMJPDA filter and IMMJPDA filter, though it takes longer computing time. However, the three algorithms are implemented using Matlab. The computing time is considerably reduced when coded in C++. For the proposed method, it takes about 0.8 second for one time step in C++.

In the scene of simulation, at the period of $46\,\text{s} \sim 52\,\text{s}$, the two targets are very near to each other, which easily results in track swap. In the IMMJPDA method, the judgement of which measurements belong to which target depends on the information from the current scan. If the measurements from two targets are very close, it is hard to distinguish the different targets based on the measurements from a single scan, which leads to a track swap. In the proposed method and the multiple scan JPDA method, the information from

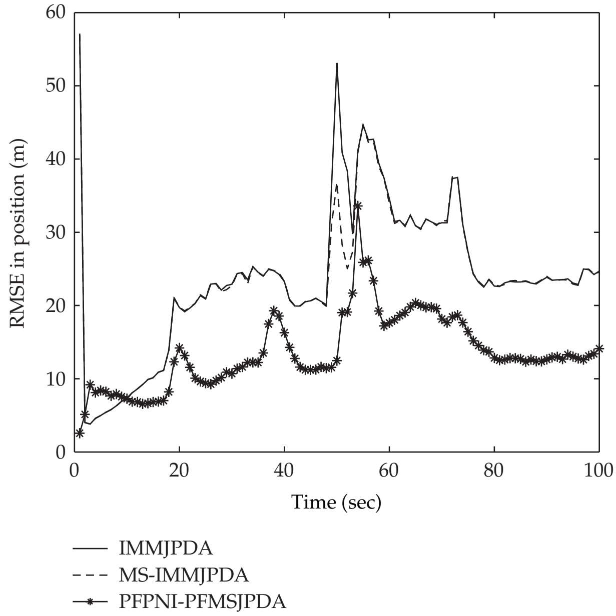


Figure 10: Target 1: RMSE in position using IMMJPDA filter, multiple scan IMMJPDA filter, and the proposed method.

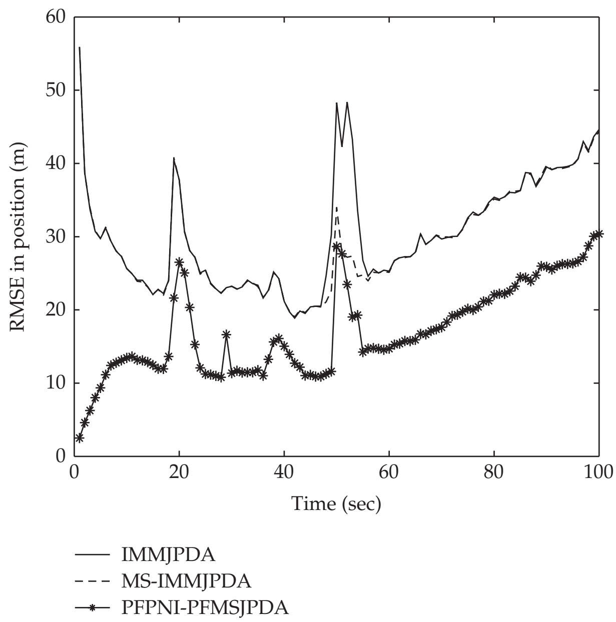


Figure 11: Target 2: RMSE in position using IMMJPDA filter, multiple scan IMMJPDA filter, and the proposed method.

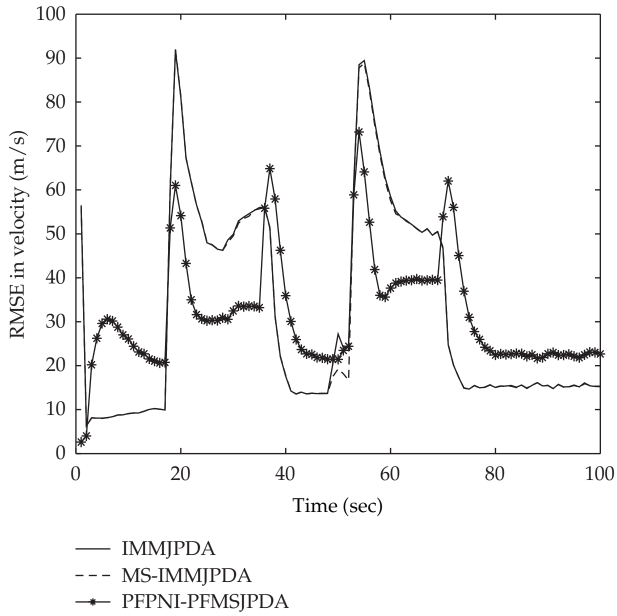


Figure 12: Target 1: RMSE in velocity using IMMJPDA filter, multiple scan IMMJPDA filter, and the proposed method.

Table 3: Performance comparison.

	RMSE in position (m)	RMSE in velocity (m/s)	ET (s)	TLR	SR
IMMJPDA	T1: 26.04, T2: 31.28	T1: 39.48, T2: 31.98	0.75	8.7%	13.9%
MS-IMMJPDA	T1: 25.30, T2: 30.26	T1: 39.22, T2: 31.76	3.78	0	4.5%
PFPNI-PFMSJPDA	T1: 14.42, T2: 18.14	T1: 34.44, T2: 30.58	4.0	0	3.7%

Table 4: Influence of particle number in the performance of the proposed algorithm for tracking multiple maneuvering target.

Particle number	RMSE in position (m)	ET (s)	TLR
200	NA	NA	100%
500	T1: 17.37, T2: 21.22	2.9	35.9%
1000	T1: 14.42, T2: 18.14	4.0	0
2000	T1: 13.75, T2: 17.93	20.2	0

several previous scans is combined with the information from current scan to calculate the association probabilities. The decision of which measurements belong to which target based on the information from multiple scans will be more accurate than single scan method, which reduces the swap rate effectively.

From Table 4, it can be seen that when the number of particles is increased, the performance of the proposed algorithm also increases. But when the number of particles exceeds some threshold (1000), the increase in the performance slows down.

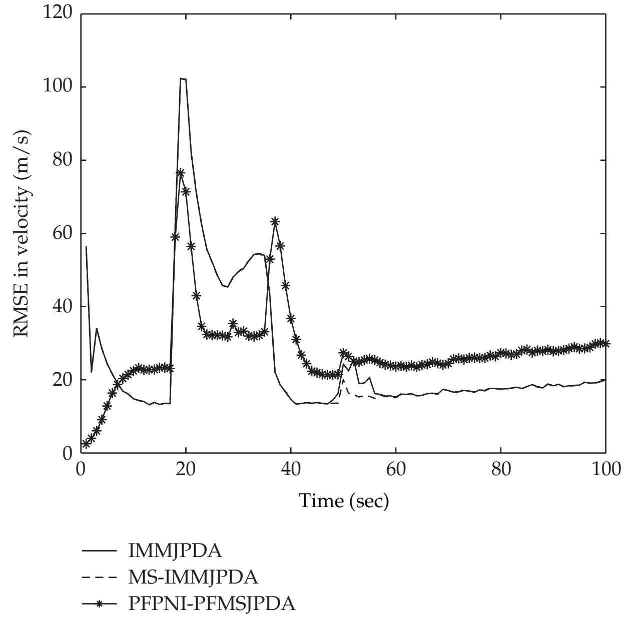


Figure 13: Target 2: RMSE in velocity using IMMJPDA filter, multiple scan IMMJPDA filter, and the proposed method.

8. Conclusions

A new algorithm is proposed for multiple maneuvering target tracking in the particle filter framework. In order to track the highly maneuvering target, the particle filter based process noise identification method is proposed to estimate the equivalent process noise induced by both maneuvering and random acceleration. Compared with the multiple model based methods for maneuvering target tracking, only one general model is adopted in the proposed method. In order to tackle the data association problem in multiple maneuvering target tracking, the particle filter based multiscan JPDA filter is adopted. Compared with the single scan JPDA method, the multiscan JPDA method uses richer information, which results in better estimated probabilities.

Acknowledgments

This work was supported by the Foundation for Innovative Research Groups of the National Natural Science Foundation of China (60921003), the Natural Science Foundations of China (nos. 61104051, 61074176, and 61004087), and the Program for New Century Excellent Talents in University.

References

[1] H. A. P. Blom and Y. Bar-Shalom, “The interacting multiple model algorithm for systems with a jump-linear smoothing application,” *IEEE Transaction on Automatic Control*, vol. 33, no. 8, pp. 780–783, 1988.

[2] Y. Bar-Shalom, K. Chang, and H. Blom, “Tracking a maneuvering target using input estimation versus the interacting multiple model algorithm,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 25, no. 2, pp. 296–300, 1989.

[3] A. Munir and P. Artherton, “Adaptive interacting multiple model algorithm for tracking a maneuvering target,” *IEE Proceedings Radar, Sonar and Navigation*, vol. 142, no. 1, pp. 11–17, 1995.

[4] Y. Bar-Shalom and T. E. Fortmann, *Tracking and Data Association*, vol. 179, Academic Press, Orlando, Fla, USA, 1988.

[5] Y. Bar-Shalom and X.-R. Li, *Estimation and Tracking*, Artech House, Norwood, NJ, USA, 1993.

[6] D. Musicki, R. Evans, and S. Stanković, “Integrated probabilistic data association,” *IEEE Transactions on Automatic Control*, vol. 39, no. 6, pp. 1237–1241, 1994.

[7] D. Mušicki and R. Evans, “Integrated probabilistic data association—finite resolution,” *Automatica*, vol. 31, no. 4, pp. 559–570, 1995.

[8] D. Mušicki and R. Evans, “Clutter map information for data association and track initialization,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 40, no. 2, pp. 387–398, 2004.

[9] Y. Bar-Shalom and X. Li, *Multitarget-Multisensor Tracking: Principlesand Techniques*, YBS Publishing, Storrs, Conn, USA, 1995.

[10] T. E. Fortmann, Y. Bar-Shalom, and M. Scheffe, “Sonar tracking of multiple targets using joint probabilistic data association,” *IEEE Journal of Oceanic Engineering*, vol. 8, pp. 173–184, 1983.

[11] D. Mušicki and R. Evans, “Joint integrated probabilistic data association: JIPDA,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 40, no. 3, pp. 1093–1099, 2004.

[12] A. Houles and Y. Bar-Shalom, “Multisensor tracking of a maneuvering target in clutter,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 25, pp. 176–188, 1989.

[13] Y. Bar-Shalom, K. Chang, and B. H. A. Blom, “Tracking splitting targets in clutter by using an interacting multiple model joint probabilistic data association filter,” in *Multitarget Multisensor Tracking: Applications and Advances*, Y. Bar-Shalom, Ed., vol. 2, pp. 93–110, Artech House, 1992.

[14] M. de Feo, A. Graziano, R. Miglioli, and A. Farina, “IMMJPDA versus MHT and Kalman filter with NN correlation: performance comparision,” *IEE Proceedings—Radar Sonar and Navigation*, vol. 144, pp. 49–56, 1997.

[15] D. Musicki and S. Suvorova, “Tracking in clutter using IMM-IPDA Based algorithms,” *IEEE Transaction on Aerospace and Electronic Systems*, vol. 44, pp. 111–126, 2008.

[16] B. Chen and J. K. Tugnait, “Tracking of multiple maneuvering targets in clutter using IMM/JPDA filtering and fixed-lag smoothing,” *Automatica*, vol. 37, no. 2, pp. 239–249, 2001.

[17] H. A. P. Blom and E. A. Bloem, “Combining IMM and JPDA for tracking multiple maneuvering targets in clutter,” in *Proceedings of the 5th International Conference on Information Fusion*, vol. 1, pp. 705–712, July 2002.

[18] H. A. P. Blom and E. A. Bloem, “Interacting multiple model joint probabilistic data association avoiding track coalescence,” in *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 3, pp. 3408–3415, December 2002.

[19] H. A. P. Blom and E. A. Bloem, “Probabilistic data association avoiding track coalescence,” *IEEE Transactions on Automatic Control*, vol. 45, no. 2, pp. 247–259, 2000.

[20] S. Puranik and J. K. Tugnait, “Tracking of multiple maneuvering targets using multiscan JPDA and IMM filtering,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 43, no. 1, pp. 23–35, 2007.

[21] H. A. P. Blom and E. A. Bloem, “Exact Bayesian filter and joint IMM coupled PDA tracking of maneuvering targets from possibly missing and false measurements,” *Automatica*, vol. 42, no. 1, pp. 127–135, 2006.

[22] M. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, “A tutorial on particle filters for online nonlinear/non-Gaussian Bayesian tracking,” *IEEE Transactions on Signal Processing*, vol. 50, pp. 174–188, 2002.

[23] H. A. P. Blom and E. A. Bloem, “Joint particle filtering of multiple maneuvering targets from unassociated measurements,” *Journal of Advances in Information Fusion*, vol. 1, no. 1, pp. 15–34, 2006.

[24] J. Vermaak, S. J. Godsill, and P. Pérez, “Monte Carlo filtering for multitarget tracking and data association,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 41, pp. 309–332, 2005.

[25] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic Press, New York, NY, USA, 1970.

[26] X. R. Li and Y. Bar-Shalom, “A recursive multiple model approach tonoise identification,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 30, no. 3, pp. 671–684, 1994.

[27] X. R. Li and Y. Bar-Shalom, “A recursive hybrid system approach to noise identification,” in *Proceedingsof the 1st IEEE Conference on Control Applications*, pp. 847–852, 1992.

[28] J. Roecker, “Multiple scan joint probabilistic data association,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 31, pp. 1204–1209, 1995.

[29] W. Koch and G. Keuk, "Multiple hypothesis track maintenance with possibly unresolved measurements," *IEEE Transaction on Aerospace and Electronic Systems*, vol. 33, pp. 883–892, 1997.

[30] C. Musso, N. Oudjane, and F. LeGland, "Improving regularized particle filters," in *Sequential Monte Carlo Methods in Practice*, A. Doucet, N. DeFreitas, and N. Gordon, Eds., pp. 247–271, Springer, New York, NY, USA, 2001.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 956191, 14 pages
doi:10.1155/2012/956191

Research Article

A Novel Detection Scheme for EBPSK System

Xianqing Chen and Lenan Wu

School of Information Science & Engineering, Southeast University, Sipailou 2, Nanjing 210096, China

Correspondence should be addressed to Xianqing Chen, xqchen213@126.com

Received 5 September 2012; Revised 21 October 2012; Accepted 23 October 2012

Academic Editor: Wuquan Li

We introduce the extended binary phase shift keying (EBPSK) communication system which is different from traditional communication systems by using a special impacting filter (SIF) for demodulation. The joint detection technique is applied at the demodulator side in order to improve the performance of the system under intersymbol interference (ISI). The main advantage of the joint detection technique, when compared to conventional threshold approaches, lies in its ability to use the amplitude and the correlation between neighboring bits, thus significantly improving performance, with low complexity. Moreover, we concentrate not only on increasing the bit rate of the system, but also on designing a bandwidth efficient communication system. Simulation results show that this new approach significantly outperforms the conventional method of using threshold decision by from 3.5 to 5 dB. The new system also occupies a narrower bandwidth. So joint detection is an effective method for EBPSK demodulation under ISI.

1. Introduction

In wireless communications systems, efficient use of the available spectrum is one of the most critical design issues. Therefore, modern communications systems must evolve to work as closely as possible to capacity, in order to achieve the required binary rates. In wireless sensor networks [1], sensor nodes are typically powered by batteries with a limited lifetime, and even though energy-scavenging mechanisms can be adopted to recharge batteries through solar panels and piezoelectric or acoustic transducers, energy is still a limited resource and must be used judiciously. Efficient use of the sensor node battery's energy is therefore an important aspect of sensor networks. For these reasons, many researchers have paid attention to this problem and proposed many energy management schemes [2–4]. In order to satisfy the ever increasing demand for such systems, an extended binary phase shift keying (EBPSK) system with very high spectra efficiency is introduced in [5]. A special impact filter (SIF) [6], which can produce high impact at the phase jumping point, with narrow bandwidth,

and great improvement in output signal noise ratio (SNR), was applied at the demodulator side. Therefore, following a simple amplitude detector would perform the demodulation of EBPSK signals. However, it is difficult to detect the signals of SIF output via a threshold decision under intersymbol interference (ISI), and the performance becomes worse [7]. ISI elimination in a communications system is a very important and difficult problem. Usually, the system which has been interfered with will have a very poor bit error rate (BER) performance. Therefore, many researchers are proposing some new complicated methods to deal with ISI [8–10].

Traditionally, channel equalization is used to eliminate ISI, which is a major issue in digital communications. Several detection procedures have been proposed to address this problem, each with varying degrees of success, including the optimal solution based on maximum likelihood sequence estimation (MLSE) [11] and the machine learning technique [12, 13], which can be used to approximate MLSE decisions at a lower computational cost. However, we need to send a training sequence, which will increase the computational cost with the length of sequence increased [14].

On the other hand, as opposed to traditional communication systems, an EBPSK system with the SIF at the demodulator side converts the phase changing to amplitude impacting. If the bit duration is short or a narrowband band-pass filter (NBPF) is added to achieve a bandwidth efficient transmission and suppress the interference to other channels, then the neighboring symbols will interfere with the others [15, 16].

In this paper, we concentrate not only on increasing the bit rate of the system, but also on designing a bandwidth efficient communications system. Given the characteristics of EBPSK modulation techniques, we introduce a joint detection algorithm to eliminate the ISI resulting from the SIF and to achieve a bandwidth efficient transmission.

The rest of the paper is organized as follows. Section 2 is devoted to introducing an EBPSK communications system. We also describe the generation of ISI and the threshold decision of amplitude. We present the receiver scheme with joint detection in Section 3. In Section 4, we include illustrative experiments to compare the performance of the proposed detectors. Section 5 includes our conclusions along with some final comments.

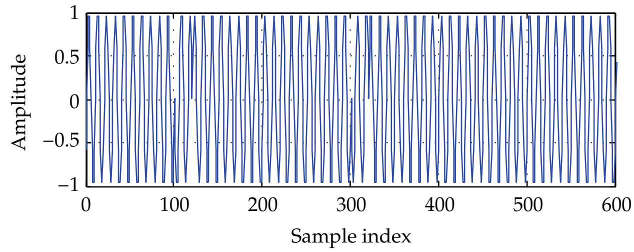
2. EBPSK Communication System

2.1. EBPSK Modulation

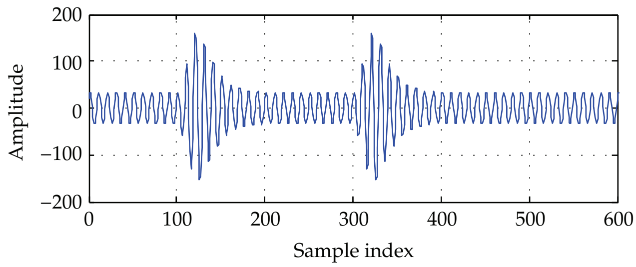
In this section, we will give a brief introduction of EBPSK modulation. EBPSK modulation is defined as follows:

$$
\begin{aligned}
f_0(t) &= A\sin\omega_c t, \quad 0 \le t < T, \\
f_1(t) &= \begin{cases} -B\sin\omega_c t, & 0 \le t < \tau, \\ A\sin(\omega_c t), & \tau \le t < T, \end{cases}
\end{aligned}
\tag{2.1}
$$

where f_0 and f_1 are modulation waveforms corresponding to bit "0" and bit "1," respectively, $T = 2\pi N/\omega_c$ is the bit duration, $\tau = 2\pi K/\omega_c$ is the phase modulation duration, and θ is the modulating angle. A and B are the amplitude of bit duration and phase modulation duration, respectively. Obviously, if $\tau = T$ and $\theta = \pi$, (2.1) degenerates to the classical binary



(a)



(b)

Figure 1: EBPSK modulation with $N = 10$, $\theta = \pi$, $K = 2$, $A = B = 1$ in (a) and SIF output in (b). The transmitted symbol sequence is $[0, 1, 0, 1, 0, 0]$.

phase shift keying (BPSK) modulation. As an example, Figure 1(a) is the waveform of EBPSK modulation and Figure 1(b) is the waveform of SIF output.

2.2. SIF and Demodulation

The waveforms of EBPSK modulation corresponding to “0” and “1” have very tiny differences. If we use the matched filter to demodulate, it has even high demand on input SNR. In order to improve the efficiency of the EBPSK signal as much as possible, the SIF method must be sought [17]. Figure 2 shows the amplitude-frequency response and phase-frequency response of the SIF. With one pair of conjugate zero poles, it reveals narrow notch-frequency-selecting performance near the center frequency.

The SIF with narrow bandwidth can produce high impact at the phase jumping point of EBPSK modulation waveform, with great improvement in output SNR. Obviously, following a simple amplitude detector would perform the demodulation of EBPSK signals, because of the existence of high impulse in coded 1s. Therefore, a direct threshold detector, which is the simple demodulation technique, can be used in the receiver.

Reference [18] gives the result which indicates we can get the optimal threshold if the symbols only interfere with additive white gaussian noise (AWGN). The threshold can be obtained as follows:

$$u_T = \frac{1}{2}\left(A_1 + A_0 + \frac{2\sigma^2}{A_0 - A_1}\ln\sqrt{\frac{A_0}{A_1}}\right), \tag{2.2}$$

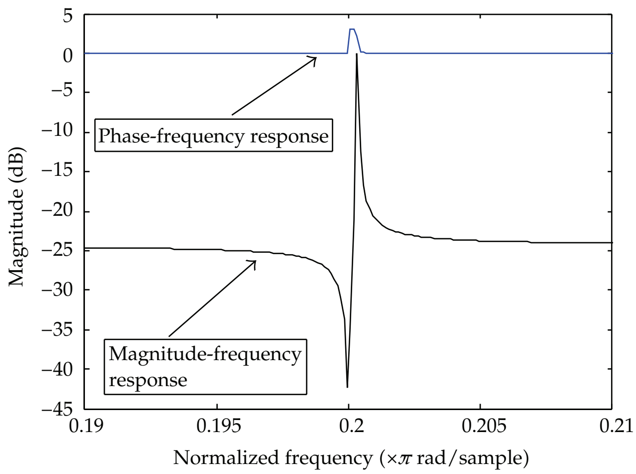


Figure 2: Amplitude-frequency response and phase-frequency response of the SIF with one pair of conjugate zero pole.

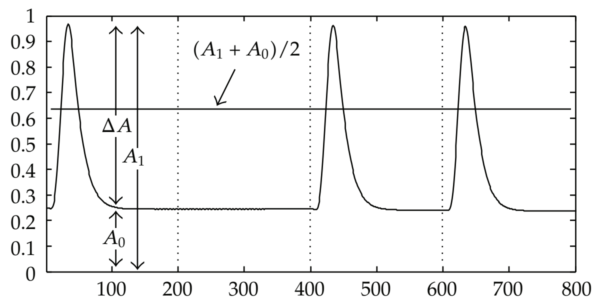


Figure 3: The threshold decision.

where σ^2 is the noise variance and A_0 and A_1 are the maximum amplitudes of the filter output corresponding to codes "0" and "1," respectively.

Figure 3 shows the envelope of the symbols where $\sigma^2 = 0$. Now, it is necessary to determine the value of A_0 and A_1. According to [6], the value of A_0 can be obtained through the following equation:

$$A_0 = A \cdot |H(\omega_c)| = \sqrt{\frac{2E_b}{T}} \cdot |H(\omega_c)|,$$
$$A_1 = A_0 + \Delta A. \tag{2.3}$$

On the premise of the existence of SIF, the value of ΔA has been deduced through the following results [18].

Assuming the transfer function of the impacting filter can be written as follows:

$$H(s) = \frac{A\prod (s - z_i)^{n_i}}{\prod_{i=1}^{m}(s - p_i)}, \tag{2.4}$$

then the transient response of the system can be calculated using the following [19]:

$$\begin{aligned} y(t) = {} & \sum_{i=1}^{m} \frac{-2A\omega\prod_{j=1}^{n}(p_i - z_j)^{n_j}}{(p_i^2 + \omega^2)\prod_{j=1, i\neq j}^{m}(p_i - p_j)} e^{p_i t}\varepsilon(t) \\ & + \sum_{i=1}^{m} \frac{2A\omega\prod_{j=1}^{n}(p_i - z_j)^{n_j}}{(p_i^2 + \omega^2)\prod_{j=1, i\neq j}^{m}(p_i - p_j)} e^{p_i(t-\tau)}\varepsilon(t-\tau) \\ & + \frac{-2A\prod_{i=1}^{n}(z_i^2 + \omega^2)^{n_i/2}}{\prod_{i=1}^{m}(p_i^2 + \omega^2)^{1/2}} \cdot \sin\left(\omega t - \sum_{i=1}^{m}\varphi_i + \sum_{i=1}^{n} n_i\phi_i\right)[\varepsilon(t) - \varepsilon(t-\tau)], \end{aligned} \tag{2.5}$$

where φ_i and ϕ_i are the phase angles of pole and zero points, respectively. Therefore, the value of ΔA is equal to the maximum of (2.5).

However, when the symbol interferes with the neighboring symbols, it becomes difficult to detect the received symbols that we will analyze in the next subsection.

2.3. ISI and Threshold Decision

Communication channels introduce linear and nonlinear distortions, and in most cases of interest, they cannot be considered to be devoid of memory. ISI, mainly a consequence of multipath in wireless channels, accounts for the linear distortion. Unlike traditional communication systems, in this paper we will discuss two cases of ISI generation in EBPSK communication systems. Because of the characteristics of EBPSK modulation, in order to increase the bit rate, the bit duration N becomes short, and so the signals of the SIF output will interfere with the others. On the other hand, most transmission systems have band limitations imposed by either the natural bandwidth of the transmission medium or by regulatory conditions. If a narrowband band-pass filter (NBPF) is added at the transmitting end of the communication system to achieve a bandwidth-efficient transmission, the symbols are also interfered with.

It is very difficult to detect the received symbols via a conventional threshold decision (CTD) when the symbols have interfered with the others, because the envelope of SIF output fluctuates considerably, as is shown in Figure 4. If we use a threshold decision, it is difficult to get the optimal threshold. However, the method which is referred to as an adaptive threshold may be considered. We could use dynamic threshold to adapt the envelope fluctuations; the threshold becomes high when the value of A_1 increases, and vice versa. However, even then we cannot ensure the correctness of the decision, because the value of ΔA changes considerably, as is shown in Part A of Figure 4. Therefore, this is not the proper methodology for EBPSK demodulation via threshold decision under ISI.

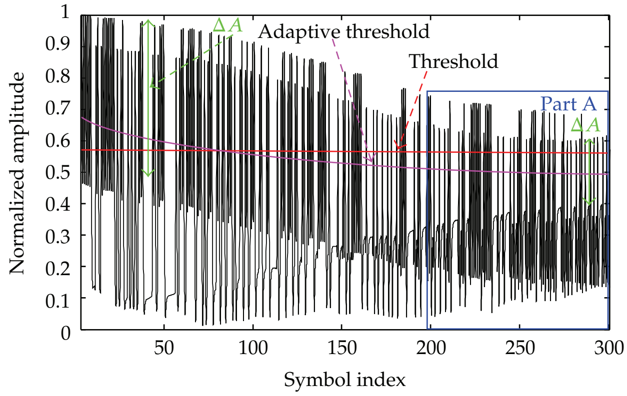


Figure 4: The signal envelope of SIF output under ISI.

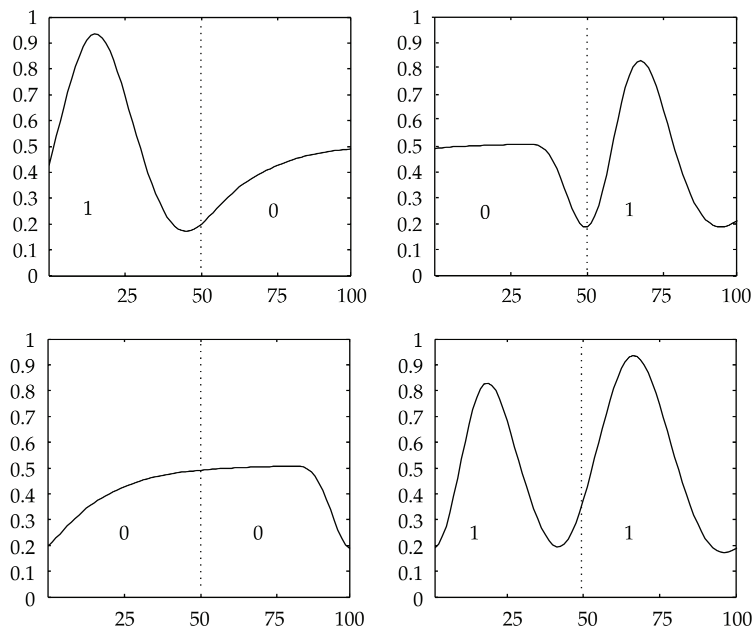


Figure 5: The signal template of $[0,0;0,1;1,0;1,1]$ under ISI.

3. Joint Detection

In the previous subsection, we have learned that it is difficult to detect the received symbols via threshold decision, and performance may worsen dramatically because of the ISI. In this section, we will introduce the detection based on the similarity of waveforms of SIF output and the correlation between symbols, which is referred to as joint detection (JD). The design approach is completely novel. We first divide two or more symbols into a group and then use the sampling points of the SIF output at intermediate frequency without downconversion to compute the correlation coefficient with the codebook, as shown in Figure 5.

Suppose $y(n)$ is the sequence of the SIF output which interfered with the AWGN and $x(n)$ is the noiseless template sequence. The similarity between them can be used to measure the error energy:

$$E^2 = \sum_{n=1}^{N} \left[x(n) - ky(n)\right]^2. \tag{3.1}$$

If $x(n)$ and $y(n)$ (or by multiplying factor k) are the same, then $E^2 = 0$. In general, the smaller the error energy E^2, the more similar the signals. In order to get the minimum value of k, let

$$\frac{\partial E^2}{\partial k} = 0. \tag{3.2}$$

Then

$$k = \frac{\sum_{n=1}^{N} x(n)y(n)}{\sum_{n=1}^{N} y^2(n)}. \tag{3.3}$$

Also, we can use the relative error to measure the similarity. Define relative error

$$\varepsilon^2 = \frac{E^2}{\sum_{n=1}^{N} x^2(n)}. \tag{3.4}$$

Then we can get

$$\varepsilon^2 = 1 - \frac{\left[\sum_{n=1}^{N} x(n)y(n)\right]^2}{\sum_{n=1}^{N} x^2(n) \cdot \sum_{n=1}^{N} y^2(n)} = 1 - \rho_{xy}^2, \tag{3.5}$$

where $E_{xy} = \sum_{n=1}^{N} x^2(n) \cdot \sum_{n=1}^{N} y^2(n)$ and ρ_{xy} is the correlation coefficient. According to the Schwartz inequality $[\sum_{n=1}^{N} x(n)y(n)]^2 \leq \sum_{n=1}^{N} x^2(n) \cdot \sum_{n=1}^{N} y^2(n)$, we can get $|\rho_{xy}| \leq 1$. So if $\rho_{xy} = 1$, $\varepsilon^2 = 0$, then $x(n)$ and $y(n)$ have a perfect correlation, and ρ_{xy} reflects the similarity between $x(n)$ and $y(n)$. Therefore, we can get the correlation coefficient ρ_{xy} between the detection signal and template signal and make a preliminary decision as to whether or not $\rho_{xy} \geq \rho_T$, where ρ_T is the correlation coefficient threshold, which will be discussed in detail later. The performance of detection will be affected due to noise levels. If we join the threshold ρ_T and the threshold u_T based on amplitude, which we deduced in the previous section as having the ability to detect the signal under ISI, we can make full use of the waveform and the amplitude of the signal. This detection is referred to as joint amplitude and waveforms detection (JAWD), which is described as follows.

Firstly, we divide two signal sequences of SIF output into one group and then compute the ρ_{xy} between the received sequence $y(n)$ and the template sequence $x(n)$. If $\rho_{xy} \geq \rho_T$ means that $x(n)$ and $y(n)$ are similar, so we can get the detected symbols. If on the other hand $\rho_{xy} < \rho_T$, this means we cannot determine the values of the symbols. Finally, we use

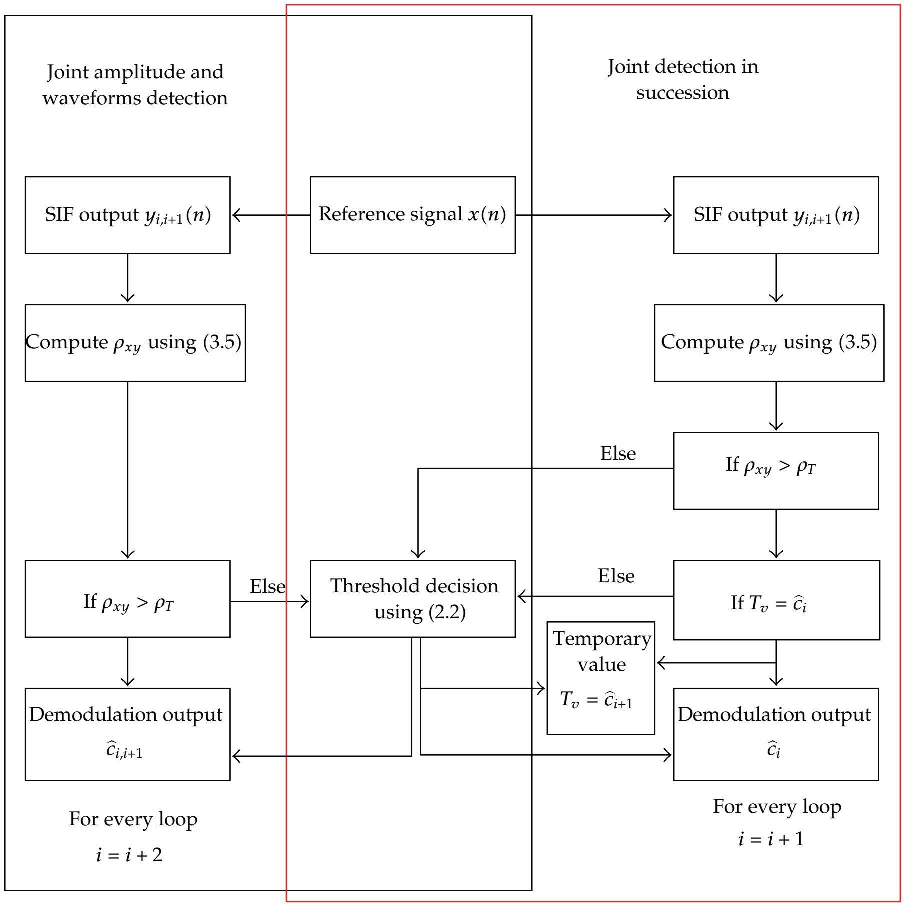


Figure 6: The procedure of JAWD and JDS.

the threshold u_T to make a final decision and acquire the symbols. For every decision loop, we can get two symbols $c_{i,i+1}$.

In order to make full use of the correlation between neighboring symbols, another joint detection, which is referred to as joint detection in succession (JDS), is introduced. All procedures are similar to the JAWD, but for every loop, we only get one symbol c_i. The other symbol c_{i+1} is stored in the temporary variable T_v for verification in the next loop. In the $i+1$th loop, if $T_v = c_i$, then we get the symbol c_i and update T_v with new c_{i+1}, or else we use the amplitude threshold to make a decision and update the $T_v = c_{i+1}$. However, compared to the JAWD detection, the JDS only gets one symbol for every loop, and the temporary variable T_v gives additional decisions. We can make full use of the correlation between every neighboring symbol. The procedures of JAWD and JDS are shown in Figure 6.

4. Simulation Results

In this section, we illustrate the performance of the proposed joint detection. Throughout our experiments, the reported BERs were computed using 10^5 symbols, and we averaged

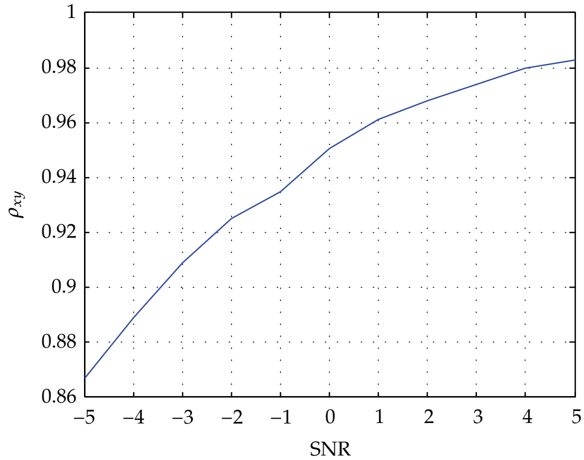


Figure 7: The value of ρ_{xy} for different SNR with $N = 5$.

the results over 100 independent trials. During simulation, we chose $K = 2$, $A = B = 1$, $\theta = \pi$ as the parameters of EBPSK modulation. The whole system was simulated under MATLAB.

In what follows, we label the performance of the different detection techniques by JD for joint detection, JDS for joint detection in succession, CTD for conventional threshold detection, and JAWD and JAWD-3 for two and three joint symbols, respectively. The BER performance of the narrowband system is labeled with NB-JAWD, NB-JDS, and NB-TD.

4.1. Experiment 1: High Bit Rate System

In this first experiment, we deal with the ISI generated by the short bit duration of the modulation parameter, which increases the bit rate of the system. We compare the JAWD with the JD, CTD, and artificial neural network (ANN) detection. Also, we will compare the performance of different joint detection methods with CTD and without ISI.

The JAWD we discussed in the previous section requires two thresholds; one is the conventional threshold u_T, and the other is the threshold of correlation coefficient ρ_T. The former can be obtained through (2.2), but the latter is a little hard to determine, because ρ_{xy} decreases with decreases in the SNR, as is shown in Figure 7. Therefore, the ρ_T would also change to adapt the ρ_{xy}. It is generally accepted that there is a strong correlation between $x(n)$ and $y(n)$ with $\rho_T > 0.85$. In Figure 7, we know that $\rho_T > 8.6$ with SNR > -5, so it meets our requirements. In order to enhance the decision of the correlation coefficient, we choose $\rho_T = 0.9$ with SNR > -3 and $\rho_T = 0.85$ with SNR ≤ -3.

In Figure 8, we can appreciate that the decisions provided by the conventional threshold, ANN classification, JAWD, and JD are quite different. The performance of JAWD outperforms the other detection technique and is superior by 1 to 2 dB when compared with the ANN and JD, respectively. The CTD only uses the amplitude and omits the waveforms, and JD is the opposite. Therefore, the JAWD significantly reduces the BER, because it not only concentrates on the correlation between neighboring symbols and the waveform of the symbol, but also uses the signal amplitude for decision when the demodulation by ρ_T failed.

We have shown that the JAWD technique is far superior to the CTD. In Figure 9, we compare the BER performance of different joint detections to the CTD method. We can

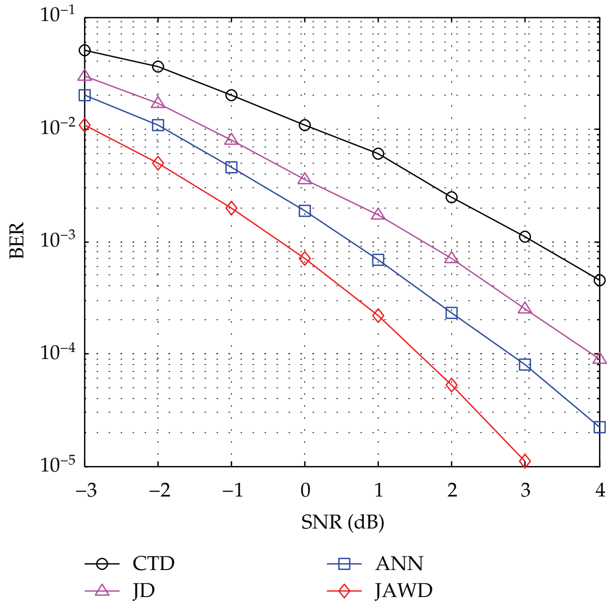


Figure 8: The performance comparison of different detection with $N = 5$.

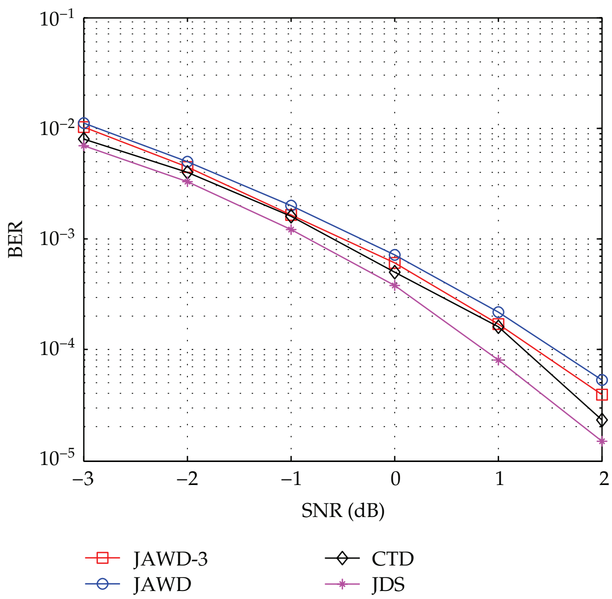


Figure 9: The performance of different joint detection techniques compared with CTD. We use $N = 5$ for JAWD, JAWD-3, and JDS; $N = 20$ for CTD.

appreciate that joint detection with two symbols performs similarly to three joint symbols. The performance of JDS is not only better than JD but also outperforms the threshold decision without ISI. This means that using CTD has difficulty allowing the system to perform to its fullest and that the JAWD or JDS techniques are a better choice.

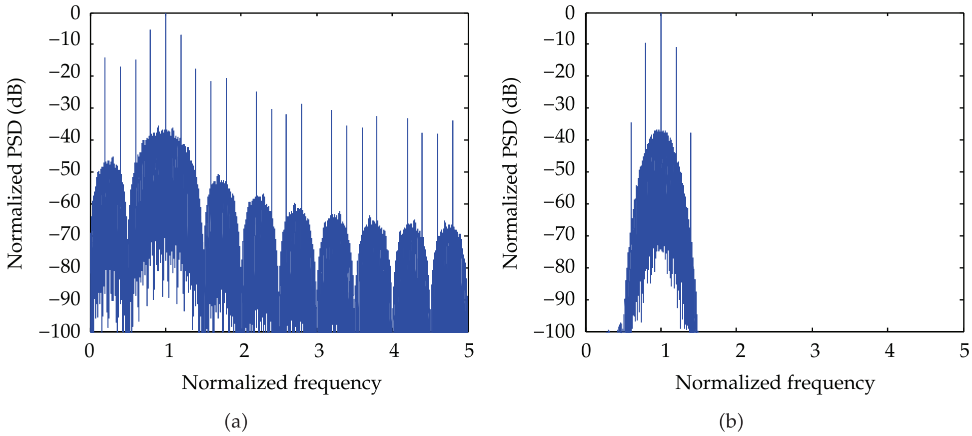


Figure 10: The power spectrum density of modulated signals and its filtered signals, respectively, in (a) and (b) for $K = 2$, $N = 5$.

From this first experiment, it is clear that we should use the JAWD or JDS for EBPSK demodulation while under ISI. Otherwise, we cannot get the desired BER performance. Also, we do not need to join three or even more symbols for detection, because the performances are almost identical, and the complexity is increased. Moreover, JDS is a little superior to JAWD and far superior to CTD, which even without ISI, because of the template variable, makes an additional decision for each symbol.

4.2. Experiment 2: Band-Limited System

In the next experiment, we face a bandwidth efficient communication model, which is proposed via a narrowband band-pass filter at the transmitting end of the system. In order to reduce the ISI caused by the filter, we use the detection technique as described in the previous section.

The bandwidth of the linear phase NBPF is designed to be $[0.98N/T, 1.02N/T]$. Power spectrum density (PSD) of the modulated signals is plotted in Figure 10(a). When this signal is filtered by the NBPF, its corresponding spectrum is illustrated in Figure 10(b).

First of all, we should determine the ρ_T by using JAWD. As described in the previous subsection, we have plotted the curve for ρ_{xy} in Figure 11. We know that, if $\rho_T > 0.85$, then we can assume the signals are similar, and we can make a final decision, so as is shown in Figure 11, the $\rho_T > 0.85$ when the SNR > -4 dB. In order to improve the detection accuracy, we use $\rho_T = 0.9$ with SNR > -1 dB and $\rho_T = 0.85$ with SNR ≤ -1.

The performance comparisons of different detection methods are presented in Figure 12. As is shown in Figure 12, both JAWD and JDS have the ability to significantly improve the quality of the receiver. For the JDS, the SNR gain over the JAWD is around 0.8 dB with BER $= 10^{-4}$. This demonstrates that significant performance gains can be obtained via the joint detection algorithm. The JDS outperforms the CTD by about 5 dB, when the BER $= 4 \times 10^{-3}$. This effect can be explained by noting that the joint detection algorithm makes full use of the relevance of the waveforms and amplitude of SIF output signals.

From these two experiments, a high bit rate communication system with narrow bandwidth can be obtained. In order to increase the bit rate, we can use short bit duration.

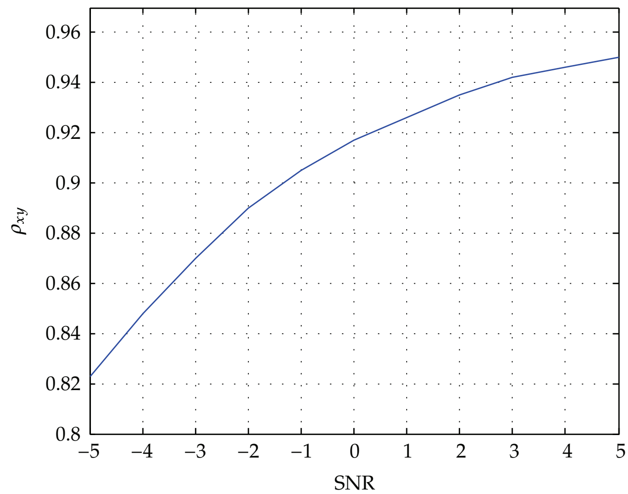


Figure 11: The value of ρ for different SNR with bandwidth efficient system.

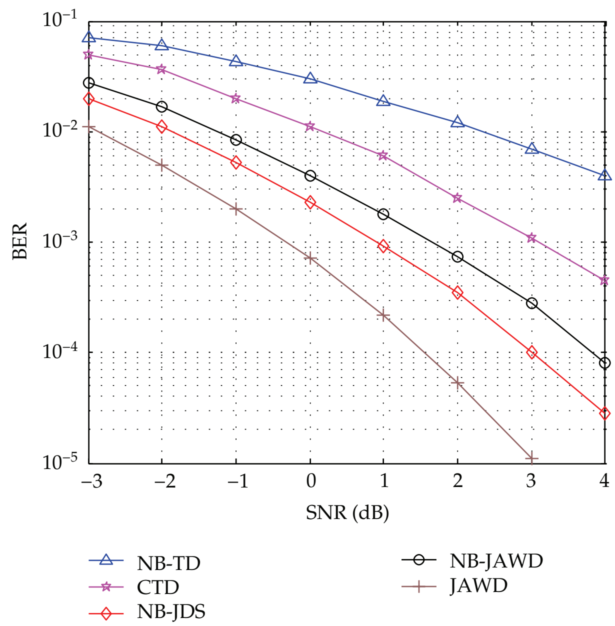


Figure 12: The performance comparisons of different detection techniques with bandwidth efficient systems and high rate systems.

The simulation shows that the BER performance is almost identical if we use JAWD or JDS, as compared to CTD with long bit duration. An NBPF added to the transmitting end can achieve a bandwidth efficient transmission by using JDS, which is only 1.5 dB inferior to the high bit rate, but a wide bandwidth system can be achieved by using JAWD.

5. Conclusions

In this paper, we introduced a novel solution for EBPSK communication systems based on joint detection technique. JDS and JAWD both use the amplitude and the correlation between two waveforms for detection, and the complexity of them is a little higher than CTD. We have shown that JDS and JAWD can significantly increase system performance under ISI and that the former outperforms the latter by more than 0.5 dB, with the cost of complexity.

A bandwidth efficient communication can work well without any channel equalizer, by using joint detection, which is different from traditional communication systems. Therefore, this system is always favorable, especially in a band-limited system, in which case the CTD may not work. Moreover, it is much simpler than an ANN demodulator and other equalizers, which need training before detection.

However, both the JDS and JAWD detectors are difficult to provide accurate posterior probability that can be exploited by a soft-input channel decoder to achieve capacity. The improvement of EBPSK performance by applying channel coding still has great potential. Therefore, future work will be focused on these problems.

Acknowledgments

The authors thank all of the reviewers for their valuable comments, which have considerably helped in improving the overall quality of the work presented in the revised paper. This work is supported by the State 863 Project (2008AA01Z227), the National Natural Science Foundation of China (NSFC), under the Grant 61271204.

References

[1] I. F. Akyildiz, S. u. Weilian, Y. Sankarasubramaniam, and E. Cayirci, "A survey on sensor networks," *IEEE Communications Magazine*, vol. 40, no. 8, pp. 102–114, 2002.

[2] S. Maheswararajah, S. Halgamuge, and M. Premaratne, "Energy efficient sensor scheduling with a mobile sink node for the target tracking application," *Sensors*, vol. 9, no. 2, pp. 696–716, 2009.

[3] C. Alippi, G. Anastasi, M. Di Francesco, and M. Roveri, "An adaptive sampling algorithm for effective energy management in wireless sensor networks with energy-hungry sensors," *IEEE Transactions on Instrumentation and Measurement*, vol. 59, pp. 335–344, 2010.

[4] G. Anastasi, M. Contib, M. Di Francescoa, A. Passarellab et al., "Energy conservation in wireless sensor networks: a survey," *Ad Hoc Networks*, vol. 7, no. 3, pp. 537–568, 2009.

[5] M. Feng and L. Wu, "Special non-linear filter and extension to Shannon's channel capacity," *Digital Signal Processing*, vol. 19, pp. 861–873, 2009.

[6] L. Wu and M. Feng, "On BER performance of EBPSK-MODEM in AWGN channel," *Sensors*, vol. 10, pp. 3824–3834, 2010.

[7] F. Man and W. Lenan, "Research on anti-fading performance of EBPSK system," in *Proceedings of 2nd International Symposium on Information Science and Engineering (ISISE '09)*, pp. 561–564, 2009.

[8] D. Shutin and B. H. Fleury, "Sparse variational Bayesian SAGE algorithm with application to the estimation of multipath wireless channels," *IEEE Transactions on Signal Processing*, vol. 59, no. 8, pp. 3609–3623, 2011.

[9] M. Miah, M. M. Rahman, T. K. Godder, and B. C Singh, "Performance analysis of an efficient wireless communication system in AWGN and slow fading channel," *Journal of Telecommunications*, vol. 5, no. 2, pp. 24–32, 2010.

[10] H. Zhao and J. Zhang, "Adaptively combined FIR and functional link artificial neural network equalizer for nonlinear communication channel," *IEEE Transactions on Neural Networks*, vol. 20, pp. 665–674, 2009.

[11] E. A. Lee and D. G. Messerschmitt, *Digital Communication*, Springer, 1994.

[12] F. Pérez-Cruz, A. Navia-Vázqueza, P. L. Alarcón-Dianab, A. Artés-Rodrígueza et al., "SVC-based equalizer for burst TDMA transmissions," *Signal Processing*, vol. 81, no. 8, pp. 1681–1693, 2001.

[13] F. Pérez-Cruz, J. J. Murillo-Fuentes, and S. Caro, "Nonlinear channel equalization with Gaussian processes for regression," *IEEE Transactions on Signal Processing*, vol. 56, no. 10, pp. 5283–5286, 2008.

[14] S. H. Kim, Y.-H. Sim, S.-W. Kim, C. Ahn, and D.-J. Kim, "Kalman-viterbi joint channel equalizer," Google Patents, 2009.

[15] Z. Mei, L. Wu, and S. Zhang, "Joint detection for a bandwidth efficient modulation method," in *Proceedings of the 9th international conference on Knowledge-Based Intelligent Information and Engineering Systems*, vol. 3683 of *Lecture Notes in Artificial Intelligence*, pp. 483–487, 2005.

[16] M. Feng and L. Wu, "Novel anti co-channel interference scheme for sensor networks," *Sensors*, vol. 10, pp. 3170–3179, 2010.

[17] M. Feng, L. Wu, and P. Gao, "From special analogous crystal filters to digital impacting filters," *Digital Signal Processing*, vol. 22, no. 4, pp. 690–696, 2012.

[18] F. Man, W.U. Lenan, J. Ding, and Q. I. Chenhao, "BER analysis and verification of EBPSK system in AWGN channel," *IEICE Transactions on Communications*, vol. 94, no. 3, pp. 806–809, 2011.

[19] N. Lu and L. Wu, "Transient response of filted PSK modulated signals," *Journal of Electrical & Electronic Education*, vol. 31, pp. 64–65, 2009.