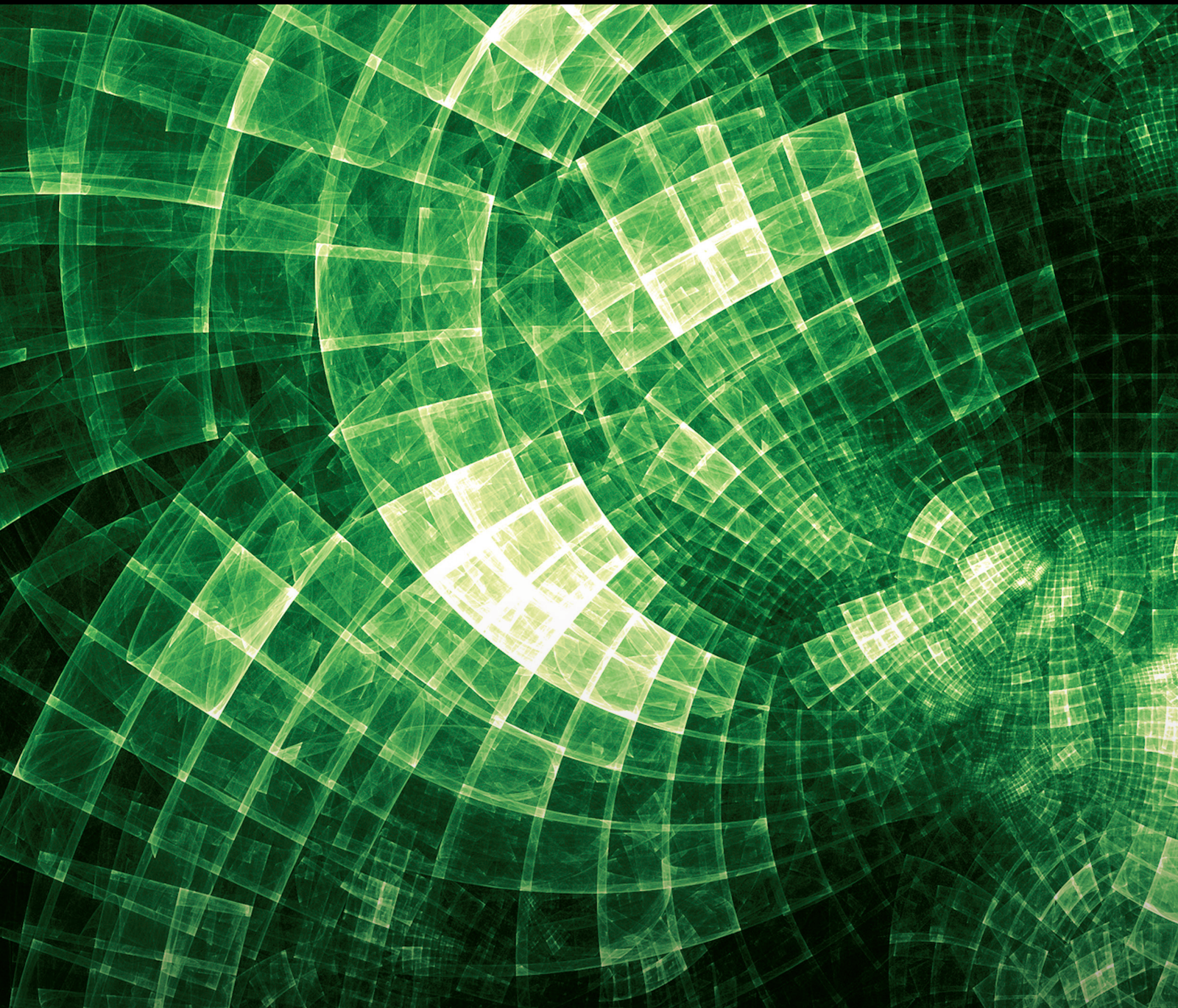


Recent Advances in Fixed Point Theory in Abstract Spaces

Lead Guest Editor: Hüseyin Işık

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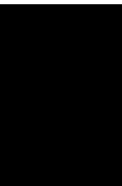


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Research Article

Thakur's Iterative Scheme for Approximating Common Fixed Points to a Pair of Relatively Nonexpansive Mappings

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Received 13 January 2021; Revised 6 February 2021; Accepted 9 February 2022; Published 22 April 2022

Academic Editor: A. Ghareeb

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In this work, we propose the three-step Thakur iterative process associated with two mappings in the setting of Banach space. Using this Thakur iteration, we approximate a common fixed point for a pair of noncyclic, relatively nonexpansive mappings. And we support our main result with a numerical example. Also, we give a stronger version of our main result by using von Neumann sequences. Finally, we provide some corollaries on the convergence of common best proximity points in uniformly convex Banach space.

1. Introduction and Preliminaries

In recent years, the convergence of iterative processes for fixed points and common fixed points has become an attractive problem in the theory of nonlinear analysis. In the literature, there are many articles that provide different kinds of iterative processes and their convergence results. At this point, Picard and Mann's iterative processes are well-known iterative procedures that often help to find fixed points of a mapping of the form $F: X \rightarrow X$, where X is Banach space. Here, we recall the following:

- (i) Picard iteration: let $w_0 \in X$. Then iteration is defined by

$$w_{n+1} = Fw_n. \quad (1)$$

- (ii) Mann iteration: let $w_0 \in X$. Then iteration is defined by

$$w_{n+1} = (1 - \eta_n)w_n + \eta_n Fw_n, \quad \eta_n \in [0, 1]. \quad (2)$$

The Picard iteration is a basic tool to find fixed points, and it was an important starting point for the improvement of other iterative processes. At the same time, the Picard iteration fails to converge a fixed point for the class of nonexpansive mappings (see [1]).

Later on, Ishikawa [2] iteration, a two-step iteration process helps to approximate fixed points of nonexpansive mappings. For a starting point $w_0 \in X$, this iterative scheme is defined by

$$\begin{cases} w_{n+1} = (1 - \eta_n)w_n + \eta_n Fw_n, \\ u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n, \end{cases} \quad (3)$$

where $\{\eta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.

Agarwal et al. [3] introduced two-step iteration process in 2007 for an arbitrary $w_0 \in X$, it is defined as

$$\begin{cases} w_{n+1} = (1 - \eta_n)Fw_n + \eta_n Fu_n, \\ u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n, \end{cases} \quad (4)$$

where $\{\eta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.

In 2000, Noor [4] introduced the following iteration scheme: starting with $w_0 \in X$, we define $\{w_n\}$ iteratively by

$$\begin{cases} w_{n+1} = (1 - \eta_n)w_n + \eta_n Fv_n, \\ v_n = (1 - \delta_n)w_n + \delta_n Fu_n, \\ u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n, \end{cases} \quad (5)$$

where $\{\eta_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$.

In the sequel, the following iterative process is defined by Thakur et al. in [5]: for an arbitrarily chosen element $w_0 \in X$, the sequence $\{w_n\}$ is generated by

$$\begin{cases} w_{n+1} = (1 - \eta_n)Fu_n + \eta_n Fv_n, \\ v_n = (1 - \delta_n)u_n + \delta_n Fu_n, \\ u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n, \end{cases} \quad (6)$$

where $\{\eta_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$.

Using the Mann iteration process, Eldred et al. [6] proved the convergence result of the fixed point for noncyclic, relatively nonexpansive mappings in the uniformly convex Banach space. One can note that the relatively nonexpansive mappings need not be continuous. Also, Gabeleh et al. [7] proved strong and weak convergence of the Ishikawa iterative scheme for noncyclic relatively nonexpansive mappings in uniformly convex Banach spaces. Gabriela et al. [8] proved the convergence of Thakur iteration for Suzuki-type nonexpansive mappings. This class of mappings properly contains the class of nonexpansive mappings. Recently, Abdeljawad et al. [9] approximated fixed points and best proximity points for relatively nonexpansive mappings through the Agarwal iterative process.

On the other hand, while solving systems of equations of the form $Fx = x, Gx = x$ (F and G are selfmappings), one needs to study common fixed points and their convergence theorems. So, the researchers are showing an interest in finding common fixed points for different kinds of mappings through the well-known iterative processes. In the literature, Rashwan [10] proved the convergence of Mann iteration to a common fixed point for a pair of mappings defined on Banach space. Later on, Ćirić et al. [11] proved the convergence of Ishikawa iteration to common fixed points of two self-mappings in complete convex metric space.

In the sequel, the researchers showed interest in approximating common fixed points for a pair of nonexpansive mappings in the setting of Banach space. For example, Maingé [12] approximated common fixed points for nonexpansive mappings in Hilbert space. Also, Song et al. [13] provided a strong convergence result of common fixed points for a family of nonexpansive mappings in the setting of reflexive Banach spaces. Later on, Gu et al. [14] proved the convergence of Ishikawa iterations associated with two mappings to the common fixed point in uniformly convex Banach space. Also, Gopi et al. [15] found a common fixed point for a pair of relatively nonexpansive mappings and found a common best proximity point for a pair of non-self relatively nonexpansive mappings via the Ishikawa iterative process. And, Praga-deeswarar et al. [16] proved the convergence of a common best proximity point for a pair of mean nonexpansive mappings.

In the light of the above literature survey, one can think of how the three-step Thakur iterative process will approach the common fixed point for a pair of noncyclic, relatively nonexpansive mappings. So, we want to approximate such a common fixed point using Thakur's iterative process.

The purpose of this paper is to present convergence results of the Thakur iterative process for common fixed points of a pair of noncyclic, relatively nonexpansive mappings in uniformly convex Banach space. Using the von Neumann sequence, we prove the strong convergence result of the Thakur iterative process. To support our main result, we provide a numerical example and we compare the Thakur iteration is how faster than other known iterative processes. Finally, we use projective operators to find the best common proximity point.

The following notations are used subsequently: let M and N be nonempty subsets of a Banach space X .

$$d(w, N) = \inf\{\|w - z\| : z \in N\},$$

$$d(M, N) = \inf\{\|w - z\| : w \in M, z \in N\},$$

$$P_M(w) = \{z \in M : \|w - z\| = d(w, M)\},$$

$$M_0 = \{w \in M : \|w - z'\| = d(M, N) \text{ for some } z' \in N\},$$

$$N_0 = \{z \in N : \|w' - z\| = d(M, N) \text{ for some } w' \in M\}.$$

(7)

If M is convex, a closed subset of a reflexive and strictly convex space, then $P_M(w)$ contains one element and if M and N are convex, closed subsets of a reflexive space, with either M or N is bounded, then $M_0 \neq \emptyset$.

First, we reconstruct the Thakur iteration associated with two noncyclic mappings $F, G: M \cup N \rightarrow M \cup N$, with M is convex, as follows: for an arbitrary chosen element $w_0 \in M$, the sequence $\{w_n\}$ is generated by

$$\begin{cases} w_{n+1} = (1 - \eta_n)Gu_n + \eta_n Fv_n, \\ v_n = (1 - \delta_n)u_n + \delta_n Gu_n, \\ u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n, \end{cases} \quad (8)$$

where $\{\eta_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following condition: (R) $0 < \varepsilon \leq \gamma_n(1 - \gamma_n)$.

Also, we provide the Abbas, Noor, Agarwal, and Ishikawa iterations associated with two noncyclic mappings $F, G: M \cup N \rightarrow M \cup N$, with M is convex, as follows:

(i) Abbas: let $w_0 \in M$. Then the iteration is defined by

$$\begin{cases} w_{n+1} = (1 - \eta_n)Fu_n + \eta_n Gv_n, \\ v_n = (1 - \delta_n)Fw_n + \delta_n Gu_n, \\ u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n, \quad \eta_n, \delta_n, \gamma_n \in [0, 1]. \end{cases} \quad (9)$$

(ii) Noor: let $w_0 \in M$. Then the iteration is defined by

$$\begin{cases} w_{n+1} = (1 - \eta_n)w_n + \eta_n Fv_n, \\ v_n = (1 - \delta_n)w_n + \delta_n Gu_n, \\ u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n, \quad \eta_n, \delta_n, \gamma_n \in [0, 1]. \end{cases} \quad (10)$$

(iii) Agarwal: let $w_0 \in M$. Then the iteration is defined by

$$\begin{cases} w_{n+1} = (1 - \eta_n)Fw_n + \eta_n Gu_n, \\ u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n, \quad \eta_n, \gamma_n \in [0, 1]. \end{cases} \quad (11)$$

(iv) Ishikawa: let $w_0 \in M$. Then the iteration is defined by

$$\begin{cases} w_{n+1} = (1 - \eta_n)w_n + \eta_n Gu_n, \\ u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n, \quad \eta_n, \gamma_n \in [0, 1]. \end{cases} \quad (12)$$

The following definitions and theorems are very useful to our results:

Definition 1. Let M and N be nonempty subsets of a metric space (X, d) . An element $w \in M$ is said to be the best proximity points of the nonself mapping $F: M \rightarrow N$ if it satisfies the condition that

$$d(w, Fw) = d(M, N). \quad (13)$$

Definition 2 (see [6]). Let M and N be nonempty subsets of a Banach space X . A mapping $F: M \cup N \rightarrow M \cup N$ is relatively nonexpansive, if

$$\|Fw - Fz\| \leq \|w - z\|, \text{ for all } w \in M, z \in N. \quad (14)$$

Definition 3 (see [17]). Let $(X, \|\cdot\|)$ be a Banach space. For every $\varepsilon \in [0, 2]$, we define the modulus of convexity of $\|\cdot\|$ by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : x, y \in B_X, \|x - y\| \geq \varepsilon \right\}, \quad (15)$$

where B_X is the unit ball of Banach space X .

The norm is called uniformly convex if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in [0, 2]$. The space $(X, \|\cdot\|)$ is then called uniformly convex space.

Definition 4 (see [18]). Let X be a Banach space. The pair of mappings $F, G: X \rightarrow X$ is said to be mean nonexpansive if

$$\begin{aligned} \|Fu - Gv\| &\leq a\|u - v\| + b\{\|u - Fu\| + \|v - Gv\|\} \\ &+ c\{\|u - Gv\| + \|v - Fu\|\}, \end{aligned} \quad (16)$$

for all $u, v \in X, a, b, c \in [0, 1]$ and $a + 2b + 2c \leq 1$.

Remark 1. In Definition 4, for $a = 1, b = c = 0$, then the pair of mappings $F, G: X \rightarrow X$ is said to be nonexpansive.

Lemma 1 (see [19]). Suppose X be a uniformly convex Banach space. Suppose $0 < a < b < 1$, and $\{t_n\}$ is a sequence in $[a, b]$. Suppose $\{w_n\}, \{v_n\}$ are sequences in X such that $\|w_n\| \leq 1, \|v_n\| \leq 1$ for all n . We define $\{z_n\}$ in X by

$$z_n = (1 - t_n)w_n + t_nv_n. \quad \text{If } \lim_{n \rightarrow \infty} \|z_n\| = 1, \quad \text{then } \lim_{n \rightarrow \infty} \|w_n - v_n\| = 0.$$

Lemma 2 (see [5]). Suppose E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $\limsup_{n \rightarrow \infty} \|x_n\| \leq r$, $\limsup_{n \rightarrow \infty} \|y_n\| \leq r$, and $\limsup_{n \rightarrow \infty} \|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Proposition 1 (see [20]). If X is a uniformly convex space and $\eta \in (0, 1)$ and $\varepsilon > 0$, then for any $d > 0$, if $w, z \in X$ are such that $\|w\| \leq d, \|z\| \leq d, \|w - z\| \geq \varepsilon$, then there exists $\delta = \delta(\varepsilon/d) > 0$, such that $\|\eta w + (1 - \eta)z\| \leq (1 - 2\delta(\varepsilon/d)) \min(\eta, 1 - \eta)d$.

Here, we prove a result that shows Thakur iteration converges to the common fixed point of a pair of nonexpansive self-mappings. This result helps to prove our main theorem.

Theorem 1. Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space X and suppose $G, F: K \rightarrow K$ is a pair of nonexpansive mappings with a nonempty common fixed point set. For an arbitrary chosen $w_0 \in K$, let the sequence $\{w_n\}$ be generated by (7) where $\eta_n, \delta_n, \gamma_n \in (\varepsilon, 1 - \varepsilon)$, and $\varepsilon \in (0, (1/2))$. Then $\lim_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$ and $\lim_{n \rightarrow \infty} \|Gu_n - Fv_n\| = 0$. Moreover, if $F(K)$ lies in a compact set, then $\{w_n\}$ converges to a common fixed point of G and F .

Proof. By assumption, there exists $z \in K$ such that $Gz = Fz = z$. Now, from (7), we have

$$\begin{aligned} \|u_n - z\| &= \|(1 - \gamma_n)w_n + \gamma_n Fw_n - z\| \\ &= \|(1 - \gamma_n)(w_n - z) + \gamma_n(Fw_n - z)\| \\ &\leq (1 - \gamma_n)\|w_n - z\| + \gamma_n\|Fw_n - z\| \\ &\leq (1 - \gamma_n)\|w_n - z\| + \gamma_n\|w_n - z\| \\ &= \|w_n - z\|. \end{aligned} \quad (17)$$

In the same way, we can obtain

$$\begin{aligned} \|v_n - z\| &= \|(1 - \delta_n)u_n + \delta_n Gu_n - z\| \\ &= \|(1 - \delta_n)(u_n - z) + \delta_n(Gu_n - z)\| \\ &\leq (1 - \delta_n)\|u_n - z\| + \delta_n\|Gu_n - z\| \\ &\leq (1 - \delta_n)\|u_n - z\| + \delta_n\|u_n - z\| \\ &= \|u_n - z\|. \end{aligned} \quad (18)$$

Now, using inequality equation (17), one gets

$$\|v_n - z\| \leq \|w_n - z\|. \quad (19)$$

Therefore, by equations (17) and (19), we obtain

$$\begin{aligned}
\|w_{n+1} - z\| &= \|(1 - \eta_n)Gu_n + \eta_n Fv_n - z\| \\
&= \|(1 - \eta_n)(Gu_n - z) + \eta_n(Fv_n - z)\| \\
&\leq (1 - \eta_n)\|Gu_n - Fz\| + \eta_n\|Fv_n - Gz\| \\
&\leq (1 - \eta_n)\|u_n - z\| + \eta_n\|v_n - z\| \\
&\leq (1 - \eta_n)\|w_n - z\| + \eta_n\|w_n - z\| \\
&= \|w_n - z\|.
\end{aligned} \tag{20}$$

This implies that the sequence $\{\|w_n - z\|\}$ is nonincreasing and bounded below by 0. Hence there exists $d \geq 0$, such that $\|w_n - z\| \rightarrow d$. \square

Case 1. Suppose $d = 0$. First

$$\begin{aligned}
\|w_n - Fw_n\| &\leq \|w_n - z\| + \|z - Fw_n\| \\
&\leq \|w_n - z\| + \|Gz - Fw_n\| \\
&\leq \|w_n - z\| + \|z - w_n\|.
\end{aligned} \tag{21}$$

As $n \rightarrow \infty$, we get $\|w_n - Fw_n\| \rightarrow 0$. From the Thakur iteration, we obtain

$$\begin{aligned}
\|w_{n+1} - w_n\| &= \|(1 - \eta_n)Gu_n + \eta_n Fv_n - w_n\| \\
&= \|Gu_n - w_n + \eta_n(Fv_n - Gu_n)\| \\
&\leq \|Gu_n - w_n\| + \eta_n\|Fv_n - Gu_n\|.
\end{aligned} \tag{22}$$

Now, by equation (17), we have.

$$\begin{aligned}
\|Gu_n - w_n\| &\leq \|Gu_n - z\| + \|z - w_n\| \\
&= \|Gu_n - Fz\| + \|z - w_n\| \\
&\leq \|u_n - z\| + \|z - w_n\| \\
&\leq 2\|w_n - z\|.
\end{aligned} \tag{23}$$

As $n \rightarrow \infty$, we obtain $\|Gu_n - w_n\| \rightarrow 0$. Also, by equations (17) and (19), we obtain

$$\begin{aligned}
\|Fv_n - Gu_n\| &\leq \|v_n - u_n\| \\
&= \|v_n - z\| + \|z - u_n\| \\
&\leq 2\|w_n - z\|.
\end{aligned} \tag{24}$$

As $n \rightarrow \infty$, we obtain $\|Fv_n - Gu_n\| \rightarrow 0$. So, we get $\|w_{n+1} - w_n\| \rightarrow 0$.

Since $F(K)$ is contained in a compact set, $\{Fw_n\}$ has a subsequence $\{Fw_{n_k}\}$ that converges to a point $z \in K$. Also $\{w_{n_k}\}$ and $\{w_{n_k+1}\}$ converge to z . This implies that $\{w_n\}$ converges to z . From the Thakur iteration, we can deduce that $\|u_n - w_n\| = \gamma_n\|Fw_n - w_n\|$, implies $\|u_n - w_n\| \rightarrow 0$. Then $u_n \rightarrow z$. And also $Gu_n \rightarrow z$, $Fw_n \rightarrow z$. Since F and G are continuous, it implies that $Gu_n \rightarrow Gz$, $Fw_n \rightarrow Fz$. Therefore $Fz = Gz = z$, which completes the proof.

Case 2. If $\|w_n - z\| \rightarrow d > 0$. Suppose there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and an $\epsilon > 0$ such that $\|w_{n_k} - Fw_{n_k}\| \geq \epsilon > 0$ for all k .

Since the modulus of convexity of δ of X is a continuous and increasing function, we choose $\xi > 0$ as small that $(1 - c\delta(\epsilon/(d + \xi)))(d + \xi) < d$, where $c > 0$.

Now we choose k , such that $\|w_{n_k} - z\| \leq d + \xi$. Now we have

$$\begin{aligned}
\|z - w_{n_k+1}\| &= \|z - ((1 - \eta_{n_k})Gu_{n_k} + \eta_{n_k}Fv_{n_k})\| \\
&= \|(1 - \eta_{n_k})z + \eta_{n_k}z - ((1 - \eta_{n_k})G((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k}) + \eta_{n_k}Fv_{n_k})\| \\
&\leq (1 - \eta_{n_k})\|z - G((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| + \eta_{n_k}\|z - Fv_{n_k}\| \\
&= (1 - \eta_{n_k})\|Fz - G((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| + \eta_{n_k}\|Gz - Fv_{n_k}\| \\
&\leq (1 - \eta_{n_k})\|z - ((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| + \eta_{n_k}\|z - v_{n_k}\|.
\end{aligned} \tag{25}$$

Now, by Proposition 1, we can obtain

$$\begin{aligned}
 & \|z - ((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| \\
 &= \|(1 - \gamma_{n_k})(z - w_{n_k}) + \gamma_{n_k}(z - Fw_{n_k})\| \\
 &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi).
 \end{aligned} \tag{26}$$

Also, using equation (26), we get

$$\begin{aligned}
 \|z - v_{n_k}\| &= \|z - ((1 - \delta_{n_k})u_{n_k} + \delta_{n_k}Gu_{n_k})\| \\
 &= \|(1 - \delta_{n_k})(z - u_{n_k}) + \delta_{n_k}(z - Gu_{n_k})\| \\
 &\leq (1 - \delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|z - Gu_{n_k}\| \\
 &= (1 - \delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|Fz - Gu_{n_k}\| \\
 &\leq (1 - \delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|z - u_{n_k}\| \\
 &= \|z - u_{n_k}\| \\
 &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi).
 \end{aligned} \tag{27}$$

Therefore, the equation (25) becomes

$$\|z - w_{n_k+1}\| \leq \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi). \tag{28}$$

Since there exists $l > 0$ such that $2\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\} \geq l$,

$$\begin{aligned}
 & \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi) \\
 &\leq \left(1 - l\delta\left(\frac{\varepsilon}{d + \xi}\right)\right)(d + \xi).
 \end{aligned} \tag{29}$$

Suppose that we choose very small $\xi > 0$, we have $(1 - l\delta(\varepsilon/d + \xi))(d + \xi) < d$, which is a contradiction. This implies that $\lim_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$.

Now we prove that $\|Gu_n - Fv_n\| \rightarrow 0$. For, we define $a_n = (w_{n+1} - z)/\|w_n - z\|$, $b_n = (Gu_n - z)/\|w_n - z\|$ and $c_n = (Fv_n - z)/\|w_n - z\|$. Now, using equation (17), we get.

$$\begin{aligned}
 \|Gu_n - z\| &= \|Gu_n - Fz\| \\
 &\leq \|u_n - z\| \\
 &\leq \|w_n - z\|,
 \end{aligned} \tag{30}$$

also, by (19), we obtain

$$\begin{aligned}
 \|Fv_n - z\| &= \|Fv_n - Gz\| \\
 &\leq \|v_n - z\| \\
 &\leq \|w_n - z\|.
 \end{aligned} \tag{31}$$

Therefore $\|b_n\| = (\|Gu_n - z\|/\|w_n - z\|) \leq (\|w_n - z\|/\|w_n - z\|) = 1$ and also $\|c_n\| = (\|Fv_n - z\|/\|w_n - z\|) \leq (\|w_n - z\|/\|w_n - z\|) = 1$. From Thakur's iteration, we obtain $w_{n+1} - z = (1 - \eta_n)(Gu_n - z) + \eta_n(Fv_n - z)$. Dividing by $\|w_n - z\|$, we get

$$\frac{w_{n+1} - z}{\|w_n - z\|} = (1 - \eta_n) \frac{(Gu_n - z)}{\|w_n - z\|} + \eta_n \frac{(Fv_n - z)}{\|w_n - z\|}. \tag{32}$$

Then $a_n = (1 - \eta_n)b_n + \eta_n c_n$. Now we prove that $\|a_n\| \rightarrow 1$. Now,

$$\lim_{n \rightarrow \infty} \|a_n\| = \lim_{n \rightarrow \infty} \frac{\|w_{n+1} - z\|}{\|w_n - z\|} = \frac{d}{d} = 1. \tag{33}$$

By Lemma 1, $\|b_n - c_n\| \rightarrow 0$. Therefore $\|Gu_n - Fv_n\| \rightarrow 0$.

Since $F(K)$ is contained in a compact set, $\{Fw_n\}$ has a subsequence $\{Fw_{n_k}\}$ that converges to a point $a \in K$. Also $\{w_{n_k}\}$ converges to a . Now, $\|Fw_{n_k} - Ga\| \leq \|w_{n_k} - a\|$. As $k \rightarrow \infty$, we obtain $a = Ga$. Since G is continuous, $Gw_{n_k} \rightarrow Ga$. So, we have $\|Gw_{n_k} - Fa\| \leq \|w_{n_k} - a\|$. As $k \rightarrow \infty$, we get $Fa = Ga$. Therefore, $Fa = Ga = a$. Since a is a common fixed point, implies $\lim_{n \rightarrow \infty} \|w_n - a\|$ exists. Therefore, $\lim_{n \rightarrow \infty} \|w_n - a\| = \lim_{k \rightarrow \infty} \|w_{n_k} - a\| = 0$. So, $w_n \rightarrow a$, which completes the proof.

Let M be a convex closed subset of a Hilbert Space X . Then for $w \in X$, we know that $P_M(w)$ is the nearest point to w and unique point of M . And also P_M is nonexpansive and distinguished by Kolmogorov's criterion as $\langle w - P_M w, P_M w - a \rangle \geq 0$, for all $w \in X$ and $a \in M$.

Let M and N be two convex closed subsets of X . We define.

$$P(w) = P_M(P_N(w)) \text{ for each } w \in X. \tag{34}$$

Then $\{P^n(w)\} \subset M$ and $\{P_N(P^n(w))\} \subset N$. When M and N are closed, the convergence of these sequences in norm was proved by von Neumann [21]. The sequences $\{P^n(w)\}$ and $\{P_N(P^n(w))\}$ are called von Neumann sequences or alternating projection algorithms for two sets.

Definition 5 (see [22]). Let M and N be nonempty closed convex subsets of a Hilbert space X . We say that (M, N) is boundedly regular if for each bounded subset S of X and for each $\varepsilon > 0$ there exists $\delta > 0$ such that.

$$\max\{d(w, M), d(w, N - v)\} \leq \delta \quad d(w, N) \leq \varepsilon, \quad \forall w \in X, \tag{35}$$

where $v = P_{N-M}(0)$ is the displacement vector from M to N . (v is the unique vector satisfying $\|v\| = d(M, N)$).

Theorem 2 (see [22]). If (M, N) is boundedly regular, then the von Neumann sequences converge in the norm.

Theorem 3 (see [22]). If M or N is boundedly compact, then (M, N) is boundedly regular.

Lemma 3 (see [19]). Let M be a nonempty closed and convex subset and N be a nonempty closed subset of a uniformly convex Banach space. Let $\{w_n\}$ and $\{a_n\}$ be sequences in M and $\{z_n\}$ be a sequence in N satisfying the following:

- (1) $\|w_n - z_n\| \rightarrow d(M, N)$,
- (2) $\|a_n - z_n\| \rightarrow d(M, N)$.

Then $\|w_n - a_n\|$ converges to zero.

Corollary 1 (see [19]). Let M be a nonempty closed convex subset and N be a nonempty closed subset of uniformly convex Banach space. Let $\{w_n\}$ be a sequence in M and $z_0 \in N$ such that $\|w_n - z_0\| \rightarrow d(M, N)$. Then $\{w_n\}$ converges to $P_M(z_0)$.

Proposition 2 (see [23]). Let M and N be two closed and convex subsets of a Hilbert space X . Then $P_N(M) \subseteq N$, $P_M(N) \subseteq M$, and $\|P_N w - P_M z\| \leq \|w - z\|$ for $w \in M$ and $z \in N$.

Lemma 4. Let M and N be two closed and convex subsets of a Hilbert space X . For each $w \in X$.

$$\|P^{n+1}(w) - a\| \leq \|P^n(w) - a\|, \text{ for each } a \in M_0 \cup N_0. \quad (36)$$

Lemma 5 (see [24]). Let (M, N) be a nonempty, bounded, closed, and convex pair in a reflexive and strictly convex Banach space X . We define $P: M_0 \cup N_0 \rightarrow M_0 \cup N_0$ as

$$P(x) = \begin{cases} P_{M_0}(x), & \text{if } x \in N_0, \\ P_{N_0}(x), & \text{if } x \in M_0, \end{cases} \quad (37)$$

Then the following statements hold:

- (1) $\|x - Px\| = d(M, N)$ for any $x \in M_0 \cup N_0$ and $P(M_0) \subseteq N_0$, $P(N_0) \subseteq M_0$.
- (2) P is an isometry, that is, $\|Px - Py\| = \|x - y\|$ for all $(x, y) \in M_0 \times N_0$.
- (3) P is affine.

Definition 6 (see [25]). If $M_0 \neq \emptyset$ then the pair (M, N) is said to have P -property if for any $u_1, u_2 \in M_0$ and $v_1, v_2 \in N_0$

$$\begin{cases} d(u_1, v_1) = d(M, N) \\ d(u_2, v_2) = d(M, N) \end{cases} \quad d(u_1, u_2) = d(v_1, v_2). \quad (38)$$

Lemma 6 (see [26]). Every, nonempty, bounded, closed and convex pair in a uniformly convex Banach space X has the P -property.

Lemma 7 (see [27]). Let (M, N) be a nonempty, closed, and convex pair in a uniformly convex Banach space X . Then for the projection mapping $P: M_0 \cup N_0 \rightarrow M_0 \cup N_0$ defined in equation (17) we have both $P|_{M_0}$ and $P|_{N_0}$ are continuous.

2. Main Results

Theorem 4. Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $F, G: M \cup N \rightarrow M \cup N$ satisfy

- (1) $G(M) \subseteq M$, $G(N) \subseteq N$, $F(M) \subseteq M$ and $F(N) \subseteq N$;
- (2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
- (3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$,

with a nonempty common fixed-point set. For an arbitrary chosen $w_0 \in M$, let the sequence $\{w_n\}$ be generated by (7) where $\eta_n, \delta_n, \gamma_n \in (\varepsilon, 1 - \varepsilon)$, where $\varepsilon \in (0, (1/2))$ and $n = 0, 1, 2, \dots$. Suppose $d(w_n, M_0) \rightarrow 0$, then $\lim_{n \rightarrow \infty} \|Gu_n - Fv_n\| = 0$, $\lim_{n \rightarrow \infty} \|u_n - Gu_n\| = 0$ and $\lim_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$. Moreover, if $F(M)$ lies in a compact set, then $\{w_n\}, \{v_n\}$ and $\{u_n\}$ converge to a common fixed point of G and F .

Proof. If $d(M, N) = 0$, then $M_0 = N_0 = M \cap N$ and by Theorem 1 we can prove the result from the truth that $F, G: M \cap N \rightarrow M \cap N$ is nonexpansive. Therefore, let us take that $d(M, N) > 0$. For a common fixed point $z \in N$ of F and G , we get

$$\begin{aligned} \|u_n - z\| &= \|(1 - \gamma_n)w_n + \gamma_n Fw_n - z\| \\ &= \|(1 - \gamma_n)(w_n - z) + \gamma_n(Fw_n - z)\| \\ &\leq (1 - \gamma_n)\|w_n - z\| + \gamma_n\|Fw_n - z\| \\ &= (1 - \gamma_n)\|w_n - z\| + \gamma_n\|Fw_n - Gz\| \\ &\leq (1 - \gamma_n)\|w_n - z\| + \gamma_n\|w_n - z\| \\ &= \|w_n - z\|. \end{aligned} \quad (39)$$

In the same way, we can obtain.

$$\begin{aligned} \|v_n - z\| &= \|(1 - \delta_n)u_n + \delta_n Gu_n - z\| \\ &= \|(1 - \delta_n)(u_n - z) + \delta_n(Gu_n - z)\| \\ &\leq (1 - \delta_n)\|u_n - z\| + \delta_n\|Gu_n - z\| \\ &= (1 - \delta_n)\|u_n - z\| + \delta_n\|Gu_n - Fz\| \\ &\leq (1 - \delta_n)\|u_n - z\| + \delta_n\|u_n - z\| \\ &= \|u_n - z\|. \end{aligned} \quad (40)$$

Now, using inequality equation (39), one gets

$$\|v_n - z\| \leq \|w_n - z\|. \quad (41)$$

Therefore, by equations (39) and (41), we obtain

$$\begin{aligned} \|w_{n+1} - z\| &= \|(1 - \eta_n)Gu_n + \eta_n Fv_n - z\| \\ &= \|(1 - \eta_n)(Gu_n - z) + \eta_n(Fv_n - z)\| \\ &\leq (1 - \eta_n)\|Gu_n - Fz\| + \eta_n\|Fv_n - Gz\| \\ &\leq (1 - \eta_n)\|u_n - z\| + \eta_n\|v_n - z\| \\ &\leq (1 - \eta_n)\|w_n - z\| + \eta_n\|w_n - z\| \\ &= \|w_n - z\|. \end{aligned} \quad (42)$$

This implies that the sequence $\{\|w_n - z\|\}$ is nonincreasing. Then we can find $d > 0$ such that $\lim_{n \rightarrow \infty} \|w_n - z\| = d$.

Suppose there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and an $\varepsilon > 0$ such that $\|w_{n_k} - Fw_{n_k}\| \geq \varepsilon > 0$ for all k . Since the modulus of convexity of δ of X is continuous and increasing function, we choose $\xi > 0$ as small that $(1 - c\delta(\varepsilon/(d + \xi)))(d + \xi) < d$, where $c > 0$. Now we choose k , such that $\|w_{n_k} - z\| \leq d + \xi$. Now we have

$$\begin{aligned} \|z - w_{n_k+1}\| &= \|z - ((1 - \eta_{n_k})Gu_{n_k} + \eta_{n_k}Fv_{n_k})\| \\ &= \|(1 - \eta_{n_k})z + \eta_{n_k}z - ((1 - \eta_{n_k})G((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k}) + \eta_{n_k}Fv_{n_k})\| \\ &\leq (1 - \eta_{n_k})\|z - G((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| + \eta_{n_k}\|z - Fv_{n_k}\| \\ &= (1 - \eta_{n_k})\|Fz - G((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| + \eta_{n_k}\|Gz - Fv_{n_k}\| \\ &\leq (1 - \eta_{n_k})\|z - ((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| + \eta_{n_k}\|z - v_{n_k}\|. \end{aligned} \quad (43)$$

Now, by Proposition 1, we can obtain

$$\begin{aligned} \|z - ((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k}Fw_{n_k})\| &= \|(1 - \gamma_{n_k})(z - w_{n_k}) + \gamma_{n_k}(z - Fw_{n_k})\| \\ &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi). \end{aligned} \quad (44)$$

Also, using equation (44), we get

$$\begin{aligned} \|z - v_{n_k}\| &= \|z - ((1 - \delta_{n_k})u_{n_k} + \delta_{n_k}Gu_{n_k})\| \\ &= \|(1 - \delta_{n_k})(z - u_{n_k}) + \delta_{n_k}(z - Gu_{n_k})\| \\ &\leq (1 - \delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|z - Gu_{n_k}\| \\ &= (1 - \delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|Fz - Gu_{n_k}\| \\ &\leq (1 - \delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|z - u_{n_k}\| \\ &= \|z - u_{n_k}\| \\ &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi). \end{aligned} \quad (45)$$

Therefore, the equation (43) becomes

$$\begin{aligned} \|z - w_{n_k+1}\| &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi). \\ &\quad (46) \\ \text{Since there exists } l > 0 \text{ such that } 2\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\} &\geq l, \\ &\left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi) \\ &\leq \left(1 - l\delta\left(\frac{\varepsilon}{d + \xi}\right)\right)(d + \xi). \end{aligned} \quad (47)$$

Suppose that we choose very small $\xi > 0$, we have $(1 - l\delta(\varepsilon/(d + \xi)))(d + \xi) < d$, which is a contradiction. This implies that $\lim_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$. From the Thakur iteration, we have $\|u_n - w_n\| = \gamma_n\|w_n - Fw_n\|$, which implies $\|u_n - w_n\| \rightarrow 0$.

Now we prove that $\|Gu_n - Fv_n\| \rightarrow 0$. For, we define $a_n = (w_{n+1} - z)/\|w_n - z\|$, $b_n = (Gu_n - z)/\|w_n - z\|$ and $c_n = (Fv_n - z)/\|w_n - z\|$. Now, using equation (17), we get

$$\begin{aligned}
\|Gu_n - z\| &= \|Gu_n - Fz\| \\
&\leq \|u_n - z\| \\
&\leq \|w_n - z\|,
\end{aligned} \tag{48}$$

also, by equation (19), we obtain

$$\begin{aligned}
\|Fv_n - z\| &= \|Fv_n - Gz\| \\
&\leq \|v_n - z\| \\
&\leq \|w_n - z\|.
\end{aligned} \tag{49}$$

Therefore $\|b_n\| = (\|Gu_n - z\|/\|w_n - z\|) \leq (\|w_n - z\|/\|w_n - z\|) = 1$ and also $\|c_n\| = (\|Fv_n - z\|/\|w_n - z\|) \leq (\|w_n - z\|/\|w_n - z\|) = 1$. From Thakur's iteration, we obtain $w_{n+1} - z = (1 - \eta_n)(Gu_n - z) + \eta_n(Fv_n - z)$. Dividing by $\|w_n - z\|$, we get

$$\frac{w_{n+1} - z}{\|w_n - z\|} = (1 - \eta_n) \frac{(Gu_n - z)}{\|w_n - z\|} + \eta_n \frac{(Fv_n - z)}{\|w_n - z\|}. \tag{50}$$

Then $a_n = (1 - \eta_n)b_n + \eta_n c_n$. Now we prove that $\|a_n\| \rightarrow 1$. Now,

$$\lim_{n \rightarrow \infty} \|a_n\| = \lim_{n \rightarrow \infty} \frac{\|w_{n+1} - z\|}{\|w_n - z\|} = \frac{d}{d} = 1. \tag{51}$$

By Lemma 1, $\|b_n - c_n\| \rightarrow 0$. Therefore $\|Gu_n - Fv_n\| \rightarrow 0$.

Since $\lim_{n \rightarrow \infty} \|w_n - z\| = d$, and from equations (39) and (41), we can obtain

$$\limsup_{n \rightarrow \infty} \|u_n - z\| \leq d, \tag{52}$$

and

$$\limsup_{n \rightarrow \infty} \|v_n - z\| \leq d. \tag{53}$$

Also, we have $\|Gu_n - z\| = \|Gu_n - Fz\| \leq \|u_n - z\|$.

Taking lim sup on both sides, we obtain

$$\limsup_{n \rightarrow \infty} \|Gu_n - z\| \leq d. \tag{54}$$

Now

$$\begin{aligned}
\|w_{n+1} - z\| &= \|(1 - \eta_n)Gu_n + \eta_n Fv_n - z\| \\
&= \|Gu_n - z\| + \eta_n \|Fv_n - Gu_n\|,
\end{aligned} \tag{55}$$

as $n \rightarrow \infty$, by $\|Fv_n - Gu_n\| \rightarrow 0$, we get

$$d \leq \liminf_{n \rightarrow \infty} \|Gu_n - z\|. \tag{56}$$

So, from equations (54) and (56), we obtain $\lim_{n \rightarrow \infty} \|Gu_n - z\| = d$. On the other hand, we have

$$\begin{aligned}
\|Gu_n - z\| &\leq \|Gu_n - Fv_n\| + \|Fv_n - z\| \\
&= \|Gu_n - Fv_n\| + \|Fv_n - Gz\| \\
&= \|Gu_n - Fv_n\| + \|v_n - z\|,
\end{aligned} \tag{57}$$

and this yields that

$$d \leq \liminf_{n \rightarrow \infty} \|v_n - z\|. \tag{58}$$

So, by equations (53) and (58), we deduce

$$\lim_{n \rightarrow \infty} \|v_n - z\| = d. \tag{59}$$

Using Lemma 2, we get $\|u_n - Gu_n\| \rightarrow 0$, as $n \rightarrow \infty$. From the Thakur iteration, we have $\|v_n - u_n\| = \delta_n \|u_n - Gu_n\|$, which implies $\|v_n - u_n\| \rightarrow 0$.

Since $F(M)$ is contained in a compact set, $\{Fw_n\}$ has a subsequence $\{Fw_{n_k}\}$ that converges to a point $a \in M$. Also $\{w_{n_k}\}$ and $\{u_{n_k}\}$ converge to a . Since $d(w_n, M_0) \rightarrow 0$, there exists $\{a_n\} \subseteq M_0$ such that $\|w_n - a_n\| \rightarrow 0$. Therefore, $a_{n_k} \rightarrow a$, which gives that $a \in M_0$. Let $D = d(M, N)$ and choose $b \in N_0$ such that $\|a - b\| = D$. So, we have $\|w_{n_k} - b\| \rightarrow \|a - b\| = D$ and $\|w_{n_k} - b\| \geq \|Fw_{n_k} - Gb\| \rightarrow \|a - Gb\|$. So $\|a - Gb\| = D$. By strict convexity of the norm, $Gb = b$. From $\|Fa - Gb\| \leq \|a - b\|$, follows $\|Fa - Gb\| = D$. Then $Fa = a$.

On the other hand, $\|u_{n_k} - b\| \rightarrow \|a - b\| = D$, and $\|u_{n_k} - b\| \geq \|Gu_{n_k} - Fb\| \rightarrow \|a - Fb\|$. So $\|a - Fb\| = D$. By strict convexity of the norm, $Fb = b$. From $\|Ga - Fb\| \leq \|a - b\|$, follows $\|Ga - Fb\| = D$. Then $Ga = a$. Therefore, $Fa = Ga = a$. Let $x \in M_0$. Then we have

$$\|Fx - GPx\| \leq \|x - Px\| = d(M, N). \tag{60}$$

Therefore, $\|Fx - GPx\| = d(M, N) = \|Fx - PFx\|$. By Lemma 6, we get $GPx = PFx$. In particular, $GPa = PFa$. In the same way, we can prove that $FPa = PGa$. So $F(Pa) = Pa$ and $GPa = Pa$. Since $Pa \in N_0$, we can obtain $\lim_{n \rightarrow \infty} \|w_n - Pa\|$ exists. Therefore,

$$\lim_{n \rightarrow \infty} \|w_n - Pa\| = \lim_{k \rightarrow \infty} \|w_{n_k} - Pa\| = \|a - Pa\| = d(M, N). \tag{61}$$

This implies $w_n \rightarrow a$. Also, $v_n \rightarrow a, u_n \rightarrow a$. \square

Corollary 2. Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $F, G: M \cup N \rightarrow M \cup N$ satisfy

(1) $G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ and $F(N) \subseteq N$;

(2) for $u \in M, v \in N$; and

(3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$,

with a nonempty common fixed-point set. For an arbitrary chosen $w_0 \in M_0$, let the sequence $\{w_n\}$ be generated by (7) where $\eta_n, \delta_n, \gamma_n \in (\varepsilon, 1 - \varepsilon)$, where $\varepsilon \in (0, (1/2))$ and $n = 0, 1, 2, \dots$, then $\lim_{n \rightarrow \infty} \|Gu_n - Fv_n\| = 0, \lim_{n \rightarrow \infty} \|u_n - Gu_n\| = 0$, and $\lim_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$. Moreover, if $F(M)$ lies in a compact set, then $\{w_n\}, \{v_n\}$, and $\{u_n\}$ converge to a common fixed point of G and F .

Corollary 3. Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $F, G: M \cup N \longrightarrow M \cup N$ satisfy

- (1) $G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ and $F(N) \subseteq N$;
- (2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
- (3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$,

with a nonempty common fixed-point set. Let $w_0 \in M_0$, and define $w_{n+1} = P^n((1 - \eta_n)Gu_n + \eta_n Fv_n)$ where $v_n = (1 - \delta_n)u_n + \delta_n Gu_n$, $u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n$, $\eta_n, \delta_n, \gamma_n \in (\varepsilon, 1 - \varepsilon)$, where $\varepsilon \in (0, (1/2))$ and $n = 0, 1, 2, \dots$ then $\lim_{n \rightarrow \infty} \|Gu_n - Fv_n\| = 0, \lim_{n \rightarrow \infty} \|u_n - Gu_n\| = 0$, and $\lim_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$. Moreover, if $F(M)$ lies in a compact set, then $\{w_n\}, \{v_n\}$, and $\{u_n\}$ converge to a common fixed point of G and F .

Proof. One can note that $P^n((1 - \eta_n)Gu_n + \eta_n Fv_n) = (1 - \eta_n)Gu_n + \eta_n Fv_n$. By Theorem 4, the result follows.

We illustrate the above theorem through the following example. \square

Example 1. Let $(\mathbb{R}^2, \|\cdot\|)$ with $\|(u_1, u_2) - (v_1, v_2)\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$. Let $M = \{(0, u) \in \mathbb{R}^2: u \in [0, 1]\}$ and $N = \{(1, u) \in \mathbb{R}^2: u \in [2, 3]\}$, then $d(M, N) = \sqrt{2}$. And we define a pair of mappings $F, G: M \cup N \longrightarrow M \cup N$ by $F(0, u) = (0, 1), F(1, u) = (1, u)$ and $G(0, u) = (0, u), G(1, u) = (1, 2)$. For $(0, u) \in M, (1, v) \in N$, we have

$$\|G(0, u) - F(1, v)\| = \|(0, u) - (1, v)\|. \quad (62)$$

For $(0, u) \in M, (1, v) \in N$, we have

$$\|F(0, u) - G(1, v)\| = \|(0, 1) - (1, 2)\| = \sqrt{2} \leq \|(0, u) - (1, v)\|. \quad (63)$$

Clearly, the set $\{(0, 1), (1, 2)\}$ is common fixed points of F and G . Fix $\eta_n = (3/4), \delta_n = (3/4), \gamma_n = (3/4) \forall n$. Let $(0, w_0) \in M$, then the Thakur iteration becomes

$$\begin{aligned} (0, u_n) &= \left(1 - \frac{3}{4}\right)(0, w_n) + \frac{3}{4}F(0, w_n) \\ &= \frac{1}{4}(0, w_n) + \frac{3}{4}(0, 1) \\ &= \left(0, \frac{w_n}{4}\right) + \left(0, \frac{3}{4}\right) \\ &= \left(0, \frac{w_n + 3}{4}\right) \\ (0, v_n) &= \left(1 - \frac{3}{4}\right)(0, u_n) + \frac{3}{4}G(0, u_n) \\ &= \frac{1}{4}\left(0, \frac{w_n + 3}{4}\right) + \frac{3}{4}G\left(0, \frac{w_n + 3}{4}\right) \\ &= \frac{1}{4}\left(0, \frac{w_n + 3}{4}\right) + \frac{3}{4}\left(0, \frac{w_n + 3}{4}\right) \\ &= \left(0, \frac{w_n + 3}{4}\right) \\ (0, w_{n+1}) &= \left(1 - \frac{3}{4}\right)G(0, u_n) + \frac{3}{4}F(0, v_n) \\ &= \frac{1}{4}G\left(0, \frac{w_n + 3}{4}\right) + \frac{3}{4}(0, 1) \\ &= \frac{1}{4}\left(0, \frac{w_n + 3}{4}\right) + \left(0, \frac{3}{4}\right) \\ &= \left(0, \frac{w_n + 3}{16}\right) + \left(0, \frac{3}{4}\right) \\ &= \left(0, \frac{w_n + 15}{16}\right). \end{aligned} \quad (64)$$

Using MATLAB coding, we give the following Table 1 to show that the iteration $\{(0, w_{n+1})\}, \{(0, u_n)\}$, and $\{(0, v_n)\}$, converge to a common fixed point of F, G for an initial point $(0, w_0) = (0, 0.1) \in M$.

In the same way, for the above example, the iterations (9), (10), (11), and (12) become

TABLE 1: Thakur iteration

n	$(0, v_n) = (0, v_n)$	$(0, w_{n+1})$
09	(0,0.999996566772461)	(0,0.999999999986903)
10	(0,0.999999141693115)	(0,0.999999999999181)
11	(0,0.999999785423279)	(0,0.99999999999949)
12	(0,0.99999946355820)	(0,0.99999999999997)
13	(0, 0.99999986588955)	(0,1.000000000000000)
\vdots	\vdots	\vdots
24	(0,0.99999999999997)	
25	(0,0.99999999999999)	
26	(0,1.000000000000000)	

TABLE 2: Comparative results.

n	Ishikawa	Noor	Agarwal	Abbas	Thakur
10	(0,0.999768781231887)	(0,0.999999141693115)	(0,0.999999951665723)	(0,0.999999997282962)	(0,0.999999999999181)
11	(0,0.999898841788951)	(0,0.999999785423279)	(0,0.99999990937323)	(0,0.99999999617984)	(0,0.99999999999949)
12	(0, 0.999955743282666)	(0,0.99999946355820)	(0,0.99999998300748)	(0,0.99999999946289)	(0,0.99999999999997)
13	(0,0.999980637686166)	(0,0.99999986588955)	(0,0.99999999681390)	(0,0.99999999992448)	(0,1.000000000000000)
14	(0,0.999991528987698)	(0,0.99999996647239)	(0,0.99999999940261)	(0,0.99999999998938)	(0,1.000000000000000)
15	(0, 0.99996293932118)	(0,0.99999999161810)	(0,0.99999999988799)	(0,0.99999999999851)	(0,1.000000000000000)
16	(0,0.99998378595301)	(0,0.99999999790452)	(0,0.99999999997900)	(0,0.99999999999979)	(0,1.000000000000000)
17	(0, 0.99999290635444)	(0,0.99999999947613)	(0,0.99999999999606)	(0,0.99999999999997)	(0,1.000000000000000)
18	(0,0.99999689653007)	(0,0.99999999986903)	(0,0.9999999999926)	(0,1.000000000000000)	(0,1.000000000000000)
19	(0,0.99999864223191)	(0,0.99999999996726)	(0,0.9999999999986)	(0,1.000000000000000)	(0,1.000000000000000)
20	(0,0.99999940597646)	(0,0.99999999999181)	(0,0.99999999999997)	(0,1.000000000000000)	(0,1.000000000000000)
21	(0, 0.99999974011470)	(0,0.99999999999795)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
22	(0,0.99999988630018)	(0,0.99999999999949)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
23	(0,0.99999995025633)	(0,0.99999999999987)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
24	(0,0.99999997823714)	(0,0.99999999999997)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
25	(0, 0.99999999047875)	(0,0.99999999999999)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
26	(0,0.99999999583445)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)	(0,1.000000000000000)
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
41	(0,0.999999999999998)				
42	(0,0.999999999999999)				
43	(0,1.000000000000000)				

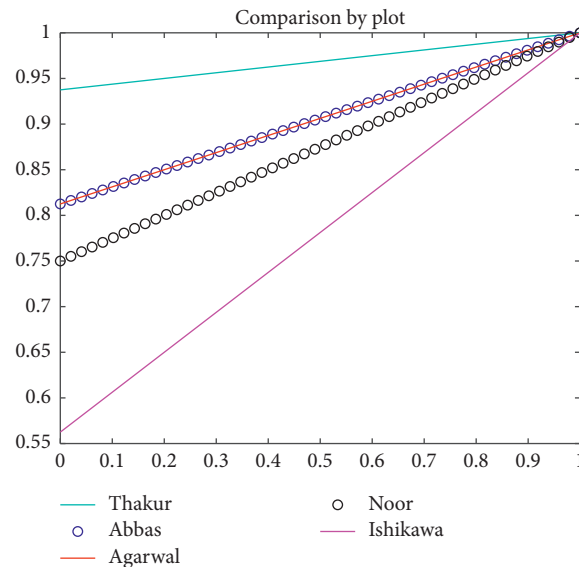


FIGURE 1: Convergence results.

(i) Abbas: for an initial point $(0, w_0) \in M$,

$$(0, w_{n+1}) = \left(0, \frac{9w_n + 55}{64}\right), \quad (65)$$

(ii) Agarwal: for an initial point $(0, w_0) \in M$,

$$(0, w_{n+1}) = \left(0, \frac{3w_n + 13}{16}\right), \quad (66)$$

(iii) Noor: for an initial point $(0, w_0) \in M$,

$$(0, w_{n+1}) = \left(0, \frac{w_n + 3}{4}\right), \quad (67)$$

(iv) Ishikawa: for an initial point $(0, w_0) \in M$,

$$(0, w_{n+1}) = \left(0, \frac{7w_n + 9}{16}\right). \quad (68)$$

Using MATLAB coding, we give the following Table 2, which compares Thakur iteration with Abbas, Agarwal, Noor, and Ishikawa iterations.

Using MATLAB coding, we give the following Figure 1, which compares convergence of Thakur iteration with Abbas, Agarwal, Noor, and Ishikawa iterations by the plot.

Now we omit the assumptions on constants $\{\eta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, and $d(w_n, M_0) \rightarrow 0$ in the above theorem and we provide the following theorem by using the condition (R) on constants $\{\eta_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$.

Lemma 8 (see [7]). *A Banach space X is uniformly convex if and only if for each fixed number $r > 0$, there exists a continuous strictly increasing function $\phi: [0, \infty) \rightarrow [0, \infty)$, $\phi(t) = 0$ if and only if $t = 0$, such that*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\phi(\|x - y\|), \quad (69)$$

for all $\lambda \in [0, 1]$ and all $x, y \in X$ such that $\|x\| \leq r$ and $\|y\| \leq r$.

Lemma 9 (see [7]). *We consider a strictly increasing function $\phi: [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$. If a sequence $\{r_n\}$ in $[0, \infty)$ satisfies $\lim_{n \rightarrow \infty} \phi(r_n) = 0$, then $\lim_{n \rightarrow \infty} r_n = 0$.*

Lemma 10 (see [7]). *Let (A, B) be a nonempty and closed pair in a uniformly convex Banach space X such that A is convex. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ be a sequence in B such that $\lim_{n \rightarrow \infty} \|x_n - y_n\| = d(A, B)$ and $\lim_{n \rightarrow \infty} \|z_n - y_n\| = d(A, B)$, then we have $\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0$.*

Theorem 5. *Let M and N be nonempty bounded closed convex subsets of a uniformly convex Banach space and suppose $F, G: M \cup N \rightarrow M \cup N$ satisfy*

- (1) $G(M) \subseteq M, G(N) \subseteq N, F(M) \subseteq M$ and $F(N) \subseteq N$;
- (2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
- (3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$,

with a nonempty common fixed-point set. For an arbitrary chosen $w_0 \in M_0$, let the sequence $\{w_n\}$ be generated by (7) where $\{\eta_n\}, \{\delta_n\}, \{\gamma_n\}$ satisfy (R) and $n = 0, 1, 2, \dots$. Then $\lim_{n \rightarrow \infty} \|Gu_n - Fv_n\| = 0, \lim_{n \rightarrow \infty} \|u_n - Gu_n\| = 0$ and $\lim_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$. Moreover, if $F(M)$ lies in a compact set, then $\{w_n\}, \{v_n\}$, and $\{u_n\}$ converge to a common fixed point of G and F .

Proof. Let $z \in N_0$ be a common fixed point of F and G . Then from Lemma 8, there exists continuous strictly increasing function $\phi: [0, \infty) \rightarrow [0, \infty)$ such that

$$\begin{aligned}
& \|w_{n+1} - z\|^2 \\
&= \|(1 - \eta_n)Gu_n + \eta_n Fv_n - z\|^2 \\
&= \|\eta_n(Fv_n - z) + (1 - \eta_n)(Gu_n - z)\|^2 \\
&\leq \eta_n \|Fv_n - z\|^2 + (1 - \eta_n) \|Gu_n - z\|^2 - \eta_n(1 - \eta_n) \phi(\|Fv_n - Gu_n\|) \\
&= \eta_n \|Fv_n - Gz\|^2 + (1 - \eta_n) \|Gu_n - Fz\|^2 - \eta_n(1 - \eta_n) \phi(\|Fv_n - Gu_n\|) \\
&\leq \eta_n \|v_n - z\|^2 + (1 - \eta_n) \|u_n - z\|^2 \\
&= \eta_n \|(1 - \delta_n)u_n + \delta_n Gu_n - z\|^2 + (1 - \eta_n) \|(1 - \gamma_n)w_n + \gamma_n Fw_n - z\|^2 \\
&= \eta_n \|\delta_n(Gu_n - z) + (1 - \delta_n)(u_n - z)\|^2 + (1 - \eta_n) \|\gamma_n(Fw_n - z) + (1 - \gamma_n)(w_n - z)\|^2 \\
&\leq \eta_n \delta_n \|Gu_n - z\|^2 + \eta_n(1 - \delta_n) \|u_n - z\|^2 - \eta_n \delta_n(1 - \delta_n) \phi(\|Gu_n - u_n\|) + (1 - \eta_n) \gamma_n \|Fw_n - z\|^2 \\
&\quad + (1 - \eta_n)(1 - \gamma_n) \|w_n - z\|^2 - (1 - \eta_n) \gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|) \\
&\leq \eta_n \delta_n \|u_n - z\|^2 + \eta_n(1 - \delta_n) \|u_n - z\|^2 + (1 - \eta_n) \gamma_n \|w_n - z\|^2 + (1 - \eta_n)(1 - \gamma_n) \|w_n - z\|^2 \\
&\quad - (1 - \eta_n) \gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|) \\
&\leq \eta_n \|u_n - z\|^2 + (1 - \eta_n) \|w_n - z\|^2 - (1 - \eta_n) \gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|) \\
&= \eta_n \|(1 - \gamma_n)w_n + \gamma_n Fw_n - z\|^2 + (1 - \eta_n) \|w_n - z\|^2 - (1 - \eta_n) \gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|) \\
&= \eta_n \|\gamma_n(Fw_n - z) + (1 - \gamma_n)(w_n - z)\|^2 + (1 - \eta_n) \|w_n - z\|^2 - (1 - \eta_n) \gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|) \\
&\leq \eta_n \gamma_n \|Fw_n - z\|^2 + \eta_n(1 - \gamma_n) \|w_n - z\|^2 - \eta_n \gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|) + (1 - \eta_n) \|w_n - z\|^2 \\
&\quad - (1 - \eta_n) \gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|) \\
&\leq \|w_n - z\|^2 - \gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|).
\end{aligned} \tag{70}$$

Therefore, we can deduce the following inequality:

$$\gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|) \leq \|w_n - z\|^2 - \|w_{n+1} - z\|^2. \tag{71}$$

Now, we proceed with the following:

Suppose $\{\eta_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ satisfy (R). From equation (73), we get

$$\sum_{n=1}^m \gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|) \leq \|w_1 - z\|^2 - \|w_{m+1} - z\|^2. \tag{72}$$

As $m \rightarrow \infty$, we get $\sum_{n=1}^{\infty} \gamma_n(1 - \gamma_n) \phi(\|Fw_n - w_n\|) < \infty$. In view of the fact that $\gamma_n(1 - \gamma_n) \geq \varepsilon$, implies $\phi(\|Fw_n - w_n\|) \rightarrow 0$, so $\|Fw_n - w_n\| \rightarrow 0$.

As in Theorem 4, we can prove $\|u_n - w_n\| \rightarrow 0$, $\|Gu_n - Fv_n\| \rightarrow 0$, $\|Gu_n - u_n\| \rightarrow 0$, and $\|v_n - u_n\| \rightarrow 0$. Now, since $F(M)$ lies in a compact subset then $\{Fw_n\}$ has a convergent subsequence $\{Fw_{n_k}\}$, converging to some point $u \in M_0$. Also, we have $w_{n_k} \rightarrow u$, $u_{n_k} \rightarrow u$, $Gu_{n_k} \rightarrow u$.

Now $\|w_{n_k} - Pu\| \geq \|Fw_{n_k} - GPu\| \rightarrow \|u - GPu\|$. So $\|u - GPu\| = D$. From $\|Fu - GPu\| \leq \|u - Pu\|$, follows $\|Fu - GPu\| = D$. Then, $Fu = u$.

On the other hand, $\|u_{n_k} - Pu\| \geq \|Gu_{n_k} - FPu\| \rightarrow \|u - FPu\|$. So $\|u - FPu\| = D$. From $\|Gu -$

$FPu\| \leq \|u - Pu\|$, follows $\|Gu - FPu\| = D$. Then $Gu = u$. Therefore, $Fu = Gu = u$.

Let $x \in M_0$. Then we have

$$\|Fx - GPx\| \leq \|x - Px\| = d(M, N). \tag{73}$$

Therefore $\|Fx - GPx\| = d(M, N) = \|Fx - PFx\|$. By Lemma 6, we get $GPx = PFx$. In particular $GPu = PFu$. In the same way, we can prove that $FPu = PGu$.

Since $F(Pu) = P(Gu) = Pu$ and $G(Pu) = P(Fu) = Pu$, we get that $\lim_{n \rightarrow \infty} \|w_n - Pu\|$ exists. So

$$\lim_{n \rightarrow \infty} \|w_n - Pu\| = \lim_{k \rightarrow \infty} \|w_{n_k} - Pu\| = \|u - Pu\| = d(M, N), \tag{74}$$

which gives $w_n \rightarrow u$.

In the next result, we provide a stronger version to approximate the common fixed point via von Neumann sequences. \square

Theorem 6. Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G: M \cup N \rightarrow M \cup N$ satisfy

- (1) $G(M) \subseteq M$, $G(N) \subseteq N$, $F(M) \subseteq M$ and $F(N) \subseteq N$;
- (2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
- (3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$,

with a nonempty common fixed-point set. Let $w_0 \in M$, and define $w_{n+1} = P^n((1 - \eta_n)Gu_n + \eta_n Fv_n)$ where $v_n = (1 - \delta_n)u_n + \delta_n Gu_n$, $u_n = (1 - \gamma_n)w_n + \gamma_n Fw_n$, $\eta_n, \delta_n, \gamma_n \in (\varepsilon, 1 - \varepsilon)$, where $\varepsilon \in (0, (1/2))$ and $n = 0, 1, 2, \dots$, then $\lim_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$. Moreover, if $F(M)$ lies in a compact set and $\|u_n - Gu_n\| \rightarrow 0$, $\|v_n - Fv_n\| \rightarrow 0$ then $\{w_n\}$ converges to a common fixed point of F, G .

Proof. If $d(M, N) = 0$, then $M_0 = N_0 = M \cap N$ and $F, G: M \cap N \rightarrow M \cap N$ is a pair of nonexpansive with $w_{n+1} = P^n((1 - \eta_n)Gu_n + \eta_n Fv_n) = (1 - \eta_n)Gu_n + \eta_n Fv_n$, the usual Thakur iteration. So, let us take that $d(M, N) > 0$. Let $z \in N_0$ be a common fixed point of F and G . Now, by equations (39) and (41), we obtain

$$\begin{aligned} \|w_{n+1} - z\| &= \|P^n((1 - \eta_n)Gu_n + \eta_n Fv_n) - z\| \\ &\leq \|(1 - \eta_n)Gu_n + \eta_n Fv_n - z\| \\ &= \|(1 - \eta_n)(Gu_n - z) + \eta_n(Fv_n - z)\| \\ &\leq (1 - \eta_n)\|u_n - z\| + \eta_n\|v_n - z\| \\ &\leq (1 - \eta_n)\|w_n - z\| + \eta_n\|w_n - z\| \\ &= \|w_n - z\|. \end{aligned} \quad (75)$$

This implies that the sequence $\{\|w_n - z\|\}$ is nonincreasing. Then we can find $d > 0$ such that $\lim_{n \rightarrow \infty} \|w_n - z\| = d$.

Suppose there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and an $\varepsilon > 0$ such that $\|w_{n_k} - Fw_{n_k}\| \geq \varepsilon > 0$ for all k .

Since the modulus of convexity of δ of X is continuous and increasing function we choose $\xi > 0$ as small that $(1 - c\delta(\varepsilon/(d + \xi)))(d + \xi) < d$, where $c > 0$.

Now we choose k , such that $\|w_{n_k} - z\| \leq d + \xi$. Now we have

$$\begin{aligned} \|z - w_{n_k+1}\| &= \|z - P^{n_k}((1 - \eta_{n_k})Gu_{n_k} + \eta_{n_k} Fv_{n_k})\| \\ &\leq \|z - ((1 - \eta_{n_k})Gu_{n_k} + \eta_{n_k} Fv_{n_k})\| \\ &= \|(1 - \eta_{n_k})z + \eta_{n_k}z - ((1 - \eta_{n_k})G((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k} Fw_{n_k}) + \eta_{n_k} Fv_{n_k})\| \\ &\leq (1 - \eta_{n_k})\|z - G((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k} Fw_{n_k})\| + \eta_{n_k}\|z - Fv_{n_k}\| \\ &\leq (1 - \eta_{n_k})\|z - ((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k} Fw_{n_k})\| + \eta_{n_k}\|z - v_{n_k}\|. \end{aligned} \quad (76)$$

Now

$$\begin{aligned} &\|z - ((1 - \gamma_{n_k})w_{n_k} + \gamma_{n_k} Fw_{n_k})\| \\ &= \|(1 - \gamma_{n_k})(z - w_{n_k}) + \gamma_{n_k}(z - Fw_{n_k})\| \\ &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi). \end{aligned} \quad (77)$$

Also, using equation (79), we get

$$\begin{aligned} \|z - v_{n_k}\| &= \|z - ((1 - \delta_{n_k})u_{n_k} + \delta_{n_k} Gu_{n_k})\| \\ &= \|(1 - \delta_{n_k})(z - u_{n_k}) + \delta_{n_k}(z - Gu_{n_k})\| \\ &\leq (1 - \delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|z - Gu_{n_k}\| \\ &\leq (1 - \delta_{n_k})\|z - u_{n_k}\| + \delta_{n_k}\|z - u_{n_k}\| \\ &= \|z - u_{n_k}\| \\ &\leq \left(1 - 2\delta\left(\frac{\varepsilon}{d + \xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi). \end{aligned} \quad (78)$$

Therefore, the equation (78) becomes

$$\|z - w_{n_k+1}\| \leq \left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi). \quad (79)$$

Since there exists $l > 0$ such that $2\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\} \geq l$,

$$\left(1 - 2\delta\left(\frac{\varepsilon}{d+\xi}\right)\min\{\gamma_{n_k}, 1 - \gamma_{n_k}\}\right)(d + \xi) \leq \left(1 - l\delta\left(\frac{\varepsilon}{d+\xi}\right)\right)(d + \xi). \quad (80)$$

Suppose that we choose very small $\xi > 0$, we have $(1 - l\delta(\varepsilon/(d + \xi)))(d + \xi) < d$, which is a contradiction. This implies that $\lim_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$. Now we prove that $\|w_{n+1} - w_n\| \rightarrow 0$. From the Thakur iteration, we get

$\|u_n - w_n\| = \gamma_n \|Fw_n - w_n\|$. Since $\lim_{n \rightarrow \infty} \|w_n - Fw_n\| = 0$ we obtain $\|u_n - w_n\| \rightarrow 0$.

Since $F(M)$ is contained in a compact set, $\{Fw_n\}$ has a subsequence $\{Fw_{n_k}\}$ that converges to a point $v_0 \in M$. Also $\{w_{n_k}\}$ converges to v_0 . From the given sequence, we obtain

$$\begin{aligned} \|w_{n_k+1} - w_{n_k}\| &= \|P^{n_k}((1 - \eta_{n_k})Gu_{n_k} + \eta_{n_k}Fv_{n_k}) - w_{n_k}\| \\ &\leq \|(1 - \eta_{n_k})Gu_{n_k} + \eta_{n_k}Fv_{n_k} - w_{n_k}\| \\ &= \|Gu_{n_k} - w_{n_k}\| + \eta_{n_k}\|Gu_{n_k} - Fv_{n_k}\| \\ &\leq \|Gu_{n_k} - u_{n_k}\| + \|u_{n_k} - w_{n_k}\| + \eta_{n_k}(\|Gu_{n_k} - u_{n_k}\| + \|u_{n_k} - v_{n_k}\| + \|v_{n_k} - Fv_{n_k}\|). \end{aligned} \quad (81)$$

Since $\|Gu_{n_k} - u_{n_k}\| \rightarrow 0$ implies $\|u_{n_k} - v_{n_k}\| \rightarrow 0$. Then $\|w_{n_k+1} - w_{n_k}\| \rightarrow 0$. Therefore, $w_{n_k+1} \rightarrow v_0$, which implies that $w_n \rightarrow v_0$. Also, we have $u_n \rightarrow v_0$, $Gu_n \rightarrow v_0$, $Fw_n \rightarrow v_0$ as $n \rightarrow \infty$.

Now, $\|Fw_n - G(P_N(v_0))\| \leq \|w_n - P_N(v_0)\|$, which implies that.

$$\|v_0 - G(P_N(v_0))\| \leq \|v_0 - P_N(v_0)\|. \quad \text{Hence,}$$

$$G(P_N(v_0)) = P_N(v_0).$$

Similarly, $\|Gu_n - F(P_N(v_0))\| \leq \|u_n - P_N(v_0)\|$, which implies that.

$$\|v_0 - F(P_N(v_0))\| \leq \|v_0 - P_N(v_0)\|. \quad \text{Hence,}$$

$$F(P_N(v_0)) = P_N(v_0).$$

Also, $\|G(P(v_0)) - P_N(v_0)\| = \|G(P(v_0)) - F(P_N(v_0))\| \leq \|P(v_0) - P_N(v_0)\|$.

$$\text{So } G(P(v_0)) = P(v_0).$$

And also $\|F(P(v_0)) - P_N(v_0)\| = \|F(P(v_0)) - G(P_N(v_0))\| \leq \|P(v_0) - P_N(v_0)\|$.

$$\text{So } F(P(v_0)) = P(v_0).$$

Now $\|GP_N(P(v_0)) - P(v_0)\| = \|GP_N(P(v_0)) - F(P(v_0))\| \leq \|P_N(P(v_0)) - P(v_0)\|$. Thus

$$GP_N(P(v_0)) = P_N(P(v_0)).$$

For any n , we have $F(P^n(v_0)) = P^n(v_0)$ and $GP_N(P^n(v_0)) = P_N(P^n(v_0))$.

Similarly, $\|FP_N(P(v_0)) - P(v_0)\| = \|FP_N(P(v_0)) - G(P(v_0))\| \leq \|P_N(P(v_0)) - P(v_0)\|$. Thus

$$FP_N(P(v_0)) = P_N(P(v_0)).$$

For any n , we have $G(P^n(v_0)) = P^n(v_0)$ and $FP_N(P^n(v_0)) = P_N(P^n(v_0))$. By Theorem 2, for each $u \in M$ the sequence $\{P^n(u)\}$ converges to some $r(u) \in M_0$. Now,

$$\begin{aligned} \|G(r(v_0)) - P_N(r(v_0))\| &\leq \lim_{n \rightarrow \infty} \|G(r(v_0)) - P_N(P^n(v_0))\| \\ &= \lim_{n \rightarrow \infty} \|G(r(v_0)) - F(P_N(P^n(v_0)))\| \\ &\leq \lim_{n \rightarrow \infty} \|r(v_0) - P_N(P^n(v_0))\| \\ &= \|r(v_0) - P_N(r(v_0))\|. \end{aligned} \quad (82)$$

So $\|G(r(v_0)) - P_N(r(v_0))\| \leq \|r(v_0) - P_N(r(v_0))\|$.

Therefore $G(r(v_0)) = r(v_0)$ and similarly, we get $GP_N(r(v_0)) = P_N(r(v_0))$.

In the same way, we prove that $F(r(v_0)) = r(v_0)$ and $FP_N(r(v_0)) = P_N(r(v_0))$.

Now we define $g_n: M \rightarrow \mathbb{R}$ by $g_n(u) = \|P^n(u) - r(u)\|$.

Since

$\|r(u) - r(v)\| = \lim_{n \rightarrow \infty} \|P^n(u) - P^n(v)\| \leq \|u - v\|$, then we conclude that r is continuous. Therefore $g_n(u)$ is continuous and converges pointwise to zero. Since $r(u) \in M_0$, by Lemma 4, we obtain $g_{n+1} \leq g_n$. Therefore g_n converges uniformly on the compact set.

$$F = \{(1 - \eta_{n_k})Gu_{n_k} + \eta_{n_k}Fv_{n_k}\} \cup \{v_0\}. \quad (83)$$

Therefore,

$$\begin{aligned} \lim_{k \rightarrow \infty} \|P^{n_k}((1 - \eta_{n_k})Gu_{n_k} + \eta_{n_k}Fv_{n_k}) \\ - r((1 - \eta_{n_k})Gu_{n_k} + \eta_{n_k}Fv_{n_k})\| = 0. \end{aligned} \quad (84)$$

Since $r((1 - \eta_{n_k})Gu_{n_k} + \eta_{n_k}Fv_{n_k}) \longrightarrow r(v_0)$, we get $w_{n_k+1} \longrightarrow r(v_0)$, which gives that $r(v_0) = v_0$. Therefore $Gv_0 = G(r(v_0)) = r(v_0) = v_0$ and $Fv_0 = F(r(v_0)) = r(v_0) = v_0$, which completes the proof.

Suppose X is a Hilbert space and let M and N be nonempty bounded closed convex subsets of X and suppose $F, G: M \cup N \longrightarrow M \cup N$ satisfy

- (1) $G(M) \subseteq N, G(N) \subseteq M, F(M) \subseteq N$ and $F(N) \subseteq M$;
- (2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
- (3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$.

We consider $P_M G: M \longrightarrow M, P_N F: N \longrightarrow N, P_N G: N \longrightarrow N$ and $P_M F: M \longrightarrow M$. From Proposition 2, $\|P_M F(u) - P_N G(v)\| \leq \|u - v\|$ for $u \in M$ and $v \in N$ and $\|P_N F(u) - P_M G(v)\| \leq \|u - v\|$ for $u \in N$ and $v \in M$, by Theorem 4 and Theorem 6, we give the following results on the convergence of best proximity points. \square

Corollary 4. Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G: M \cup N \longrightarrow M \cup N$ satisfy

- (1) $G(M) \subseteq N, G(N) \subseteq M, F(M) \subseteq N$ and $F(N) \subseteq M$;
- (2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
- (3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$.

If M is mapped into a compact subset of N , then for any $w_0 \in M_0$ the sequence is defined by $w_{n+1} = (1 - \eta_n)P_M Gu_n + \eta_n P_M Fv_n$, where $v_n = (1 - \delta_n)u_n + \delta_n P_M Gu_n, u_n = (1 - \gamma_n)w_n + \gamma_n P_M Fw_n$, converges to w in M_0 such that $\|w - Fw\| = \|w - Gw\| = d(M, N)$.

Corollary 5. Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G: M \cup N \longrightarrow M \cup N$ satisfy

- (1) $G(M) \subseteq N, G(N) \subseteq M, F(M) \subseteq N$ and $F(N) \subseteq M$;
- (2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
- (3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$.

If M is mapped into a compact subset of N , then for any $w_0 \in M$ the sequence defined by $w_{n+1} = (1 - \eta_n)P_M Gu_n + \eta_n P_M Fv_n$, where $v_n = (1 - \delta_n)u_n + \delta_n P_M Gu_n, u_n = (1 - \gamma_n)w_n + \gamma_n P_M Fw_n$ converges to w in M_0 such that $\|w - Fw\| = \|w - Gw\| = d(M, N)$, provided $d(w_n, M_0) \longrightarrow 0$.

Corollary 6. Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G: M \cup N \longrightarrow M \cup N$ satisfy

- (1) $G(M) \subseteq N, G(N) \subseteq M, F(M) \subseteq N$ and $F(N) \subseteq M$;
- (2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
- (3) for $u \in N, v \in M$.

If M is mapped into a compact subset of N , then for any $w_0 \in M_0$ the sequence defined by $w_{n+1} = P^n((1 - \eta_n)P_M Gu_n + \eta_n P_M Fv_n)$, where $v_n = (1 - \delta_n)u_n + \delta_n P_M$

$Gu_n, u_n = (1 - \gamma_n)w_n + \gamma_n P_M Fw_n$ converges to w in M_0 such that $\|w - Fw\| = \|w - Gw\| = d(M, N)$.

Proof. The result follows from Corollary 4. \square

Corollary 7. Let M and N be nonempty bounded closed convex subsets of a Hilbert space and suppose $F, G: M \cup N \longrightarrow M \cup N$ satisfy

- (1) $G(M) \subseteq N, G(N) \subseteq M, F(M) \subseteq N$ and $F(N) \subseteq M$;
- (2) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in M, v \in N$; and
- (3) $\|Fu - Gv\| \leq \|u - v\|$ for $u \in N, v \in M$.

Let $w_0 \in M$, and define $w_{n+1} = P^n((1 - \eta_n)P_M Gu_n + \eta_n P_M Fv_n)$, where $v_n = (1 - \delta_n)u_n + \delta_n P_M Gu_n, u_n = (1 - \delta_n)w_n + \delta_n P_M Fw_n, \eta_n, \delta_n \in (\varepsilon, 1 - \varepsilon)$, where $\varepsilon \in (0, (1/2))$ and $n = 0, 1, 2, \dots$. If M is mapped into a compact subset of N and $\|u_n - P_M Gu_n\| \longrightarrow 0, \|v_n - P_M Fv_n\| \longrightarrow 0$, then $\{w_n\}$ converges to w in M_0 such that $\|w - Fw\| = \|w - Gw\| = d(M, N)$.

Proof. The result follows from Theorem 6. \square

3. Conclusions

The fixed-point theorems provide sufficient conditions to ensure the existence of fixed points in different domains. Briefly, the fixed-point theorem possesses the solution of equations of the form $Fx = x$, where F is self-mapping. On the other hand, researchers want to find numerically such a fixed point by using different types of iterative processes for selfcontractive type operators in metric spaces, Hilbert spaces, or several classes of Banach spaces. One of the most famous iterative schemes is Picard's iterative process. Many research papers were presented for approaching the fixed point through Picard's iterative process. Later, for fast convergence, many iterative processes were found to approximate fixed points numerically. In this article, we consider the Thakur iterative process for fast convergence of common fixed points for relatively nonexpansive mappings in uniformly convex Banach spaces. Also, we approximate the common fixed point via the von Neumann iterative process in Hilbert space settings. We provide an example to illustrate our main result. As a consequence of our main results, we find common best proximity points for cyclic relatively nonexpansive mappings in Hilbert space.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest with this study.

Acknowledgments

This work has been partially funded by the Basque Government through Grant IT1207-19.

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Research Article

Lacunary \mathcal{F} -Invariant Convergence of Sequence of Sets in Intuitionistic Fuzzy Metric Spaces

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Received 28 July 2021; Revised 12 August 2021; Accepted 16 October 2021; Published 2 November 2021

Academic Editor: Huseyin Isik

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The concepts of invariant convergence, invariant statistical convergence, lacunary invariant convergence, and lacunary invariant statistical convergence for set sequences were introduced by Pancaroğlu and Nuray (2013). We know that ideal convergence is more general than statistical convergence for sequences. This has motivated us to study the lacunary \mathcal{F} -invariant convergence of sequence of sets in intuitionistic fuzzy metric spaces (briefly, IFMS). In this study, we examine the notions of lacunary \mathcal{F} -invariant convergence $(W\mathcal{F}_{\sigma\theta}^{(\eta,\nu)})$ (Wijsman sense), lacunary \mathcal{F}^* -invariant convergence $(W\mathcal{F}_{\sigma\theta}^{*(\eta,\nu)})$ (Wijsman sense), and q -strongly lacunary invariant convergence $([WN_{\sigma\theta}^{(\eta,\nu)}]_q)$ (Wijsman sense) of sequences of sets in IFMS. Also, we give the relationships among Wijsman lacunary invariant convergence, $[WN_{\sigma\theta}^{(\eta,\nu)}]_q$, $W\mathcal{F}_{\sigma\theta}^{(\eta,\nu)}$, and $W\mathcal{F}_{\sigma\theta}^{*(\eta,\nu)}$ in IFMS. Furthermore, we define the concepts of $W\mathcal{F}_{\sigma\theta}^{(\eta,\nu)}$ -Cauchy sequence and $W\mathcal{F}_{\sigma\theta}^{*(\eta,\nu)}$ -Cauchy sequence of sets in IFMS. Furthermore, we obtain some features of the new type of convergences in IFMS.

1. Introduction and Background

Fast [1] investigated the concept of statistical convergence. The publication of the study is affected deeply all the scientific fields. Nuray and Ruckle [2] redefined this concept which is known as generalized statistical convergence. A lot of development has been made in area about statistical convergence. Kostyrko et al. [3] defined ideal convergence, as a generalization of statistical convergence and worked some features of this convergence. Ideal convergence became a remarkable topic in summability theory after the studies of [4–8]. Fridy and Orhan [9] worked the notion of lacunary statistical convergence by using lacunary sequence.

Various authors involving Raimi [10], Schaefer [11], and Mursaleen [12] worked invariant convergent sequences. Nuray et al. [13] investigated \mathcal{F}_σ -convergence with the help of σ -uniform density. Mursaleen [14] put forward the idea of strongly σ -convergence. Savaş and Nuray [15] presented the opinion of σ -statistical convergence and lacunary σ -statistical convergence and proved some correlation theorems. Nuray and Uluslu [16] defined lacunary \mathcal{F} -invariant

convergence and lacunary \mathcal{F} -invariant Cauchy sequence of real numbers.

After the original study of Zadeh [17], a huge number of research works have appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Fuzzy sets (FSs) have been extensively applied in different disciplines and technologies. The theory of intuitionistic fuzzy sets (IFS) was presented by Atanassov [18]. The fuzzy sets and intuitionistic fuzzy sets have been widely used to solve many complex problems connected to different areas, especially in decision-making [19–22]. Kramosil and Michalek [23] worked fuzzy metric space (FMS) using the concepts fuzzy and probabilistic metric space. Park [24] rethought FMSs and investigated intuitionistic fuzzy metric space (IFMS). Park utilized George and Veeramani's [25] opinion of using t -norm and t -conorm to the FMS meantime describing IFMS and investigating its fundamental properties. In [26], motivated by Park's definition of an IF-metric, Lael and Nourouzi first defined an IF-normed space and then investigated, among other results, the fundamental theorems: open mapping, closed graph, and uniform

boundedness in IF-normed spaces. In order to have a different topology from the topology generated by the F -norm ψ , the condition $\psi + \phi \leq 1$ was omitted from Park's definition. Statistical convergence, ideal convergence, and different features of sequences in INFS were examined by several authors [27–31]. For the extraction of information by reflecting and modeling the hesitancy present in real-life situation, intuitionistic fuzzy set theory has been playing a significant role. The implementation of IF sets in place of fuzzy sets means the introduction of another degree of freedom into set description. IF fixed point theory has become a subject of great interest for expert in fixed point theory because this branch of mathematics has covered new possibilities for summability theory.

Convergence of sequences of sets has been examined by several authors. Nuray and Rhoades [32] presented a new convergence concept for sequences of sets called Wijsman statistical convergence. Ulusu and Nuray [33] examined the lacunary statistical convergence of sequence of sets. Kişi and Nuray [34] investigated ideal convergence for sequences of sets (Wijsman sense) and established some essential theorems. Convergence for sequences of sets became a notable topic in summability theory after the studies of [35–40].

Lacunary statistical convergence and lacunary strongly convergence for sequence of sets in IFMS were examined by Kişi [41]. Furthermore, Wijsman \mathcal{I} -convergence and Wijsman \mathcal{I}^* -convergence for sequence of sets in IFMS were investigated by Esi et al. [42].

The purpose of this study is to present some recent development in IFMS. The aim of the study is to examine some features of this new kind of convergence in IFMS. Also,

it is demonstrated that the new kind of convergence in IFMS is generally different from the known convergence in classical metric space. However; it is indicated that if certain conditions are met, every classical metric space can be a IFMS at the same time.

Throughout this work, we indicate \mathcal{I} to be the admissible ideal in \mathbb{N} , θ be a lacunary sequence, $(\mathcal{X}, \psi, \phi, *, \diamond)$ to be the IFMS, and $Y, \{Y_k\}$ to be nonempty closed subsets of \mathcal{X} .

2. Main Results

Definition 1. A sequence $\{Y_k\}$ of nonempty closed subsets of \mathcal{X} is called to be lacunary invariant convergent (Wijsman sense) to Y with regards to IFM (ψ, ϕ) , if for every $\xi \in (0, 1)$, for each $\alpha \in \mathcal{X}$ and for all $\tau > 0$, such that

$$\lim_{r \rightarrow \infty} \left| \frac{1}{h_r} \sum_{k \in I_r} \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right| = 1, \quad (1)$$

$$\lim_{r \rightarrow \infty} \left| \frac{1}{h_r} \sum_{k \in I_r} \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right| = 0, \quad (2)$$

uniformly in m .

Definition 2. A sequence $\{Y_k\}$ of nonempty closed subsets of \mathcal{X} is known as lacunary \mathcal{I} -invariant convergent or $W\mathcal{I}_{\sigma\theta}^{(\eta, \nu)}$ -convergent (Wijsman sense) to Y with regards to IFM (ψ, ϕ) , if for every $\xi \in (0, 1)$, for each $\alpha \in \mathcal{X}$ and for all $\tau > 0$, the set

$$P(\xi, \alpha, \tau) = \{k \in \mathbb{N} : |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| \geq \xi\} \in \mathcal{I}_{\sigma\theta}, \quad (3)$$

that is, $V_\theta(P(\xi, \alpha, \tau)) = 0$. We demonstrate this symbolically by $Y_k \xrightarrow{(\psi, \phi)} Y (W\mathcal{I}_{\sigma\theta}^{(\psi, \phi)})$.

Theorem 1. Let $\{Y_k\}$ be a bounded sequence. If $\{Y_k\}$ is $W\mathcal{I}_{\sigma\theta}^{(\psi, \phi)}$ -convergent to Y , then $\{Y_k\}$ is lacunary invariant convergent (Wijsman sense) to Y with regards to IFM (ψ, ϕ) .

Proof. Let $m \in \mathbb{N}$ be arbitrary and $\xi \in (0, 1)$. For each $\alpha \in \mathcal{X}$ and for all $\tau > 0$, we estimate

$$s(m, r, \alpha) := \left| \frac{1}{h_r} \sum_{k \in I_r} \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right|. \quad (4)$$

Then, for each $\alpha \in \mathcal{X}$ and for all $\tau > 0$, we get

$$s(m, r, \alpha) \leq s^1(m, r, \alpha) + s^2(m, r, \alpha), \quad (5)$$

where

$$s^1(m, r, \alpha) := \frac{1}{h_r} \sum_{k \in I_r, \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right| \geq 1 - \xi} \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right|, \quad (6)$$

$$s^2(m, r, \alpha) := \frac{1}{h_r} \sum_{k \in I_r, \left| \eta(\alpha, Y_{\sigma^k(m)}, \tau) - \eta(\alpha, Y, \tau) \right| < 1 - \xi} \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right|. \quad (7)$$

For every $m \geq 1$ and for every $\alpha \in \mathcal{X}$, it is obvious that $s^2(m, r, \alpha) < 1 - \xi$. Since $\{Y_k\}$ is bounded sequence, there is a $M > 0$, such that

$$|\psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau)| \leq M, \quad (k \in I_r, m \geq 1), \quad (8)$$

and so, we have

$$\begin{aligned} s^1(m, r, \alpha) &= \frac{1}{h_r} \sum_{k \in I_r, |\psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau)| \geq 1 - \xi} |\psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau)| \\ &\leq \frac{M}{h_r} \left| \left\{ k \in I_r : |\psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau)| \geq 1 - \xi \right\} \right| \\ &\leq M \frac{\max_m \left| \left\{ k \in I_r : |\psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau)| \geq 1 - \xi \right\} \right|}{h_r} = M \frac{S_r}{h_r}. \end{aligned} \quad (9)$$

Hence, we obtain

$$\lim_{r \rightarrow \infty} \left| \frac{1}{h_r} \sum_{k \in I_r} \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right| = 1. \quad (10)$$

Similarly, we have

$$\lim_{r \rightarrow \infty} \left| \frac{1}{h_r} \sum_{k \in I_r} \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right| = 0. \quad (11)$$

Hence, $\{Y_k\}$ is lacunary invariant convergent to Y (Wijsman sense) with regards to IFM (ψ, ϕ) . \square

Definition 3. Let $(\mathcal{X}, \psi, \phi, *, \diamond)$ be a separable IFMS, and \mathcal{I} be a proper ideal in \mathbb{N} . The sequence $\{Y_k\}$ is known as lacunary \mathcal{I}^* -invariant convergent or $W\mathcal{I}_{\sigma\theta}^{*(\psi, \phi)}$ -convergent (Wijsman sense) to Y with regards to IFM (ψ, ϕ) , if there is a set

$$M = \{m = (m_j) : m_j < m_{j+1}, j \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I}_{\sigma\theta}). \quad (12)$$

such that for each $\alpha \in \mathcal{X}$ and for all $\tau > 0$,

$$\lim_{r \rightarrow \infty} \psi(\alpha, Y_{m_k}, \tau) = \psi(\alpha, Y, \tau), \quad (13)$$

$$\lim_{r \rightarrow \infty} \phi(\alpha, Y_{m_k}, \tau) = \phi(\alpha, Y, \tau). \quad (14)$$

In that case, we write $Y_k \longrightarrow Y (W\mathcal{I}_{\sigma\theta}^{*(\psi, \phi)})$.

Theorem 2. If a sequence $\{Y_k\}$ is $W\mathcal{I}_{\sigma\theta}^{*(\psi, \phi)}$ -convergent to Y , then $\{Y_k\}$ is $W\mathcal{I}_{\sigma\theta}^{(\eta, \gamma)}$ -convergent to Y with regards to IFM (ψ, ϕ) .

Proof. Presume that $Y_k \longrightarrow Y (W\mathcal{I}_{\sigma\theta}^{*(\psi, \phi)})$. Then, $M = \{m = (m_j) : m_j < m_{j+1}, j \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I}_{\sigma\theta})$ (i.e., $\mathbb{N} \setminus M = H(\text{say}) \in \mathcal{I}_{\sigma\theta}$), such that

$$\lim_{r \rightarrow \infty} \psi(\alpha, Y_{m_k}, \tau) = \psi(\alpha, Y, \tau), \quad (15)$$

$$\lim_{r \rightarrow \infty} \phi(\alpha, Y_{m_k}, \tau) = \phi(\alpha, Y, \tau). \quad (16)$$

But then, for each $\xi \in (0, 1)$ and $\tau > 0$, there is $N > 0$, such that

$$|\psi(\alpha, Y_{m_k}, \tau) - \psi(\alpha, Y, \tau)| > 1 - \xi, \quad (17)$$

$$|\phi(\alpha, Y_{m_k}, \tau) - \phi(\alpha, Y, \tau)| < \xi. \quad (18)$$

for all $k > N$. Since

$$\left\{ (m_k) \in M : |\psi(\alpha, Y_{m_k}, \tau) - \psi(\alpha, Y, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_{m_k}, \tau) - \phi(\alpha, Y, \tau)| \geq \xi \right\} \quad (19)$$

is included in $\{m_1 < m_2 < \dots < m_{N-1}\}$ and the ideal $\mathcal{I}_{\sigma\theta}$ is admissible, we get

$$\left\{ (m_k) \in M : |\psi(\alpha, Y_{m_k}, \tau) - \psi(\alpha, Y, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_{m_k}, \tau) - \phi(\alpha, Y, \tau)| \geq \xi \right\} \in \mathcal{I}_{\sigma\theta}. \quad (20)$$

Hence,

$$\begin{aligned} \{k \in \mathbb{N}: |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| \geq \xi\} \\ \subseteq H \cup \{m_1 < m_2 < \dots < m_{N-1}\} \in \mathcal{F}_{\sigma\theta}, \end{aligned} \quad (21)$$

for all $\xi \in (0, 1)$ and $\tau > 0$. Therefore, we conclude that $Y_k \longrightarrow Y(W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)})$. \square

Theorem 3. Let the ideal $\mathcal{F}_{\sigma\theta}$ fulfill the property (AP). If $\{Y_k\}$ is a sequence in \mathcal{X} , such that $Y_k \longrightarrow Y(W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)})$, then $Y_k \longrightarrow Y(W\mathcal{F}_{\sigma\theta}^{*(\psi, \phi)})$.

Proof. Assume that $\mathcal{F}_{\sigma\theta}$ provides the feature (AP) and $Y_k \longrightarrow Y(W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)})$. Then, for every $\xi \in (0, 1)$, for each $\alpha \in \mathcal{X}$ and for all $\tau > 0$,

$$P(\xi, \alpha, \tau) = \{k \in \mathbb{N}: |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| \geq \xi\} \in \mathcal{F}_{\sigma\theta}. \quad (22)$$

We define the set Q_n for $n \in \mathbb{N}$ and $\tau > 0$ as

$$Q_n := \left\{ k \in \mathbb{N}: 1 - \frac{1}{n} \leq |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y, \tau)| < 1 - \frac{1}{n+1} \right. \\ \left. \text{or } \frac{1}{n+1} < |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| \leq \frac{1}{n} \right\}. \quad (23)$$

Clearly, $\{Q_1, Q_2, \dots\}$ is countable and belongs to $\mathcal{F}_{\sigma\theta}$ and $Q_i \cap Q_j = \emptyset$ for $i \neq j$. By the feature (AP), there is a sequence of $\{F_n\}_{n \in \mathbb{N}}$ such that the symmetric differences $Q_j \Delta F_j$ are finite sets for $j \in \mathbb{N}$ and $F = (\cup_{j=1}^{\infty} F_j) \in \mathcal{F}_{\sigma\theta}$. Now, to conclude the proof, it is enough to show that for $M = \mathbb{N} \setminus F$ and for each $\alpha \in \mathcal{X}$, we get

$$\lim_{r \rightarrow \infty} \psi(\alpha, Y_k, \tau) = \psi(\alpha, Y, \tau), \quad (24)$$

$$\lim_{r \rightarrow \infty} \phi(\alpha, Y_k, \tau) = \phi(\alpha, Y, \tau), \quad (25)$$

for $k \in M$. Let $\rho > 0$. Select $n \in \mathbb{N}$, such that $(1/n + 1) < \rho$. For each $\alpha \in \mathcal{X}$, we acquire

$$\begin{aligned} \{k \in \mathbb{N}: |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y, \tau)| \leq 1 - \rho \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| \geq \rho\} \\ \subset \left\{ k \in \mathbb{N}: |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y, \tau)| \leq 1 - \frac{1}{n+1} \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| \geq \frac{1}{n+1} \right\} \subset \bigcup_{j=1}^{n+1} Q_j. \end{aligned} \quad (26)$$

Since $Q_j \Delta F_j$ ($j = 1, 2, \dots, n+1$) are finite sets, there is a $k_0 \in \mathbb{N}$, such that

$$\left(\bigcup_{j=1}^{n+1} F_j \right) \cap \{k \in \mathbb{N}: k > k_0\} = \left(\bigcup_{j=1}^{n+1} Q_j \right) \cap \{k \in \mathbb{N}: k > k_0\}. \quad (27)$$

If $k > k_0$ and $k \notin F$, then

$$k \notin \bigcup_{j=1}^{n+1} F_j \text{ and by (27) } k \notin \bigcup_{j=1}^{n+1} Q_j. \quad (28)$$

Hence, for each $k > k_0$ and $k \in M$, we have

$$|\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y, \tau)| > 1 - \rho \text{ and } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| < \rho. \quad (29)$$

Since $\rho > 0$ is arbitrary, we obtain $T_k \longrightarrow T(W\mathcal{F}_{\sigma\theta}^{*(\psi, \phi)})$. \square

Definition 4. A sequence $\{Y_k\}$ is known as lacunary \mathcal{F} -invariant Cauchy sequence or $W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy sequence

(Wijsman sense) with regards to IFM (ψ, ϕ) if for each $\xi \in (0, 1)$, for each $\alpha \in \mathcal{X}$ and for all $\tau > 0$, there is $N = N(\varepsilon, x) \in \mathbb{N}$, such that

$$A(\xi, \alpha, \tau) = \{k \in \mathbb{N}: |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y_N, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y_N, \tau)| \geq \xi\} \in \mathcal{I}_{\sigma\theta}, \quad (30)$$

that is, $V_\theta(A(\xi, \alpha, \tau)) = 0$.

Definition 5. A sequence $\{Y_k\}$ is known as lacunary \mathcal{I}^* -invariant Cauchy sequence or $W\mathcal{I}_{\sigma\theta}^{*(\eta, \nu)}$ -Cauchy sequence (Wijsman sense) with regards to IFM (ψ, ϕ) provided that there is a set

$$M = \{m = (m_j): m_j < m_{j+1}, j \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I}_{\sigma\theta}), \quad (31)$$

such that

$$\lim_{k,l \rightarrow \infty} |\psi(\alpha, Y_{m_k}, \tau) - \psi(\alpha, Y_{m_l}, \tau)| = 1, \quad (32)$$

$$\lim_{k,l \rightarrow \infty} |\phi(\alpha, Y_{m_k}, \tau) - \phi(\alpha, Y_{m_l}, \tau)| = 0, \quad (33)$$

for each $\alpha \in \mathcal{X}$ and for all $\tau > 0$.

We give following theorems which indicate relationships among $W\mathcal{I}_{\sigma\theta}^{(\psi, \phi)}$ -convergence, $W\mathcal{I}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy sequence, and $W\mathcal{I}_{\sigma\theta}^{*(\psi, \phi)}$ -Cauchy sequence with regards to IFM (ψ, ϕ) .

Theorem 4. If a sequence $\{Y_k\}$ is $W\mathcal{I}_{\sigma\theta}^{(\psi, \phi)}$ -convergent, then $\{Y_k\}$ is $W\mathcal{I}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy sequence with regards to IFM (ψ, ϕ) .

Proof. Presume that $Y_k \longrightarrow Y(W\mathcal{I}_{\sigma\theta}^{(\psi, \phi)})$. Then, for every $\xi \in (0, 1)$, for each $\alpha \in \mathcal{X}$ and for all $\tau > 0$, the set

$$P(\xi, \alpha, \tau) = \{k \in \mathbb{N}: |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| \geq \xi\}, \quad (34)$$

belongs to $\mathcal{I}_{\sigma\theta}$. Since $\mathcal{I}_{\sigma\theta}$ is an admissible ideal, then there is $k_0 \in \mathbb{N}$ with the result that $k_0 \notin P(\xi, \alpha, \tau)$. Now, assume that

$$R(\xi, \alpha, \tau) = \{k \in \mathbb{N}: |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y_{k_0}, \tau)| \leq 1 - 2\xi \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y_{k_0}, \tau)| \geq 2\xi\}. \quad (35)$$

Thinking the inequality

$$|\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y_{k_0}, \tau)| \leq |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y, \tau)| + |\psi(\alpha, Y_{k_0}, \tau) - \psi(\alpha, Y, \tau)|, \quad (36)$$

$$|\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y_{k_0}, \tau)| \leq |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| + |\phi(\alpha, Y_{k_0}, \tau) - \phi(\alpha, Y, \tau)|. \quad (37)$$

Observe that if $k \in R(\xi, \alpha, \tau)$, therefore,

$$|\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y, \tau)| + |\psi(\alpha, Y_{k_0}, \tau) - \psi(\alpha, Y, \tau)| \leq 1 - 2\xi, \quad (38)$$

$$|\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| + |\phi(\alpha, Y_{k_0}, \tau) - \phi(\alpha, Y, \tau)| \geq 2\xi. \quad (39)$$

From another standpoint, since $k_0 \notin P(\xi, \alpha, \tau)$, we obtain

$$|\psi(\alpha, Y_{k_0}, \tau) - \psi(\alpha, Y, p)| > 1 - \xi \text{ and } |\phi(\alpha, Y_{k_0}, \tau) - \phi(\alpha, Y, \tau)| < \xi. \quad (40)$$

We reach that

$$|\psi(\alpha, Y_k, p) - \psi(\alpha, Y, p)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y, \tau)| \geq \xi. \quad (41)$$

Hence, $k \in P(\xi, \alpha, \tau)$. This gives that $R(\xi, \alpha, \tau) \subset P(\xi, \alpha, \tau) \in \mathcal{F}_{\sigma\theta}$ for every $\xi \in (0, 1)$ and $x \in \mathcal{X}$. Therefore, $R(\xi, \alpha, \tau) \in \mathcal{F}_{\sigma\theta}$, so $\{Y_k\}$ is $W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy sequence with regards to IFM (ψ, ϕ) . \square

Theorem 5. Let $(\mathcal{X}, \psi, \phi, *, \diamond)$ be a separable IFMS and \mathcal{F} be an admissible ideal. If a sequence $\{Y_k\}$ is $W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy sequence, then $\{Y_k\}$ is $W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy sequence with regards to IFM (ψ, ϕ) .

Proof. Assume that sequence $\{Y_k\}$ is $W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy with regards to IFM (ψ, ϕ) . Then, for each $\alpha \in \mathcal{X}$ and for each $\xi \in (0, 1)$, there is $M \in \mathcal{F}(\mathcal{F}_{\sigma\theta})$, where $M = \{m = (m_j) : m_j < m_{j+1}, j \in \mathbb{N}\}$, such that

$$\begin{aligned} Q(\xi, \alpha, \tau) &= \{k \in \mathbb{N} : |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y_N, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y_N, \tau)| \geq \xi\} \\ &\subset H \cup \{m_1, m_2, \dots, m_{k_0}\} \in \mathcal{F}_{\sigma\theta}. \end{aligned} \quad (46)$$

As a consequence, for all $\tau > 0$ and for each $\xi \in (0, 1)$, one can identify $N = N(\xi)$, such that $Q(\xi, \alpha, \tau) \in \mathcal{F}_{\sigma\theta}$, that is, sequence $\{Y_k\}$ is $W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy with regards to IFM (ψ, ϕ) . \square

Lemma 1 (see [4]). Assume \mathcal{F} be an admissible ideal with the feature (AP). Let there be a countable collection of subsets $\{Y_k\}_{k=1}^{\infty}$ of \mathbb{N} in such a way that $Y_k \in \mathcal{F}(\mathcal{F})$. As a result, there is a set $Y \subset \mathbb{N}$, such that $Y \setminus Y_k$ is finite for all $k \in \mathbb{N}$ and $Y \in \mathcal{F}(\mathcal{F})$.

$$A(\xi, \alpha, \tau) = \{k \in \mathbb{N} : |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y_N, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y_N, \tau)| \geq \xi\} \in \mathcal{F}_{\sigma\theta}. \quad (47)$$

Now, presume that

$$Q_j(\xi, \alpha, \tau) = \left\{k \in \mathbb{N} : \left| \psi(\alpha, Y_k, p) - \psi(\alpha, Y_{m_j}, \tau) \right| > 1 - \frac{1}{j} \text{ and } \left| \phi(\alpha, Y_k, \tau) - \phi(\alpha, Y_{m_j}, \tau) \right| < \frac{1}{j} \right\}, \quad (48)$$

$$|\psi(\alpha, Y_{m_k}, \tau) - \psi(\alpha, Y_{m_l}, \tau)| \leq 1 - \xi, \quad (42)$$

$$|\phi(\alpha, Y_{m_k}, \tau) - \phi(\alpha, Y_{m_l}, \tau)| \geq \xi, (\forall k, l > k_0 = k_0(\xi)). \quad (43)$$

Presume that $N = N(\xi) = m_{k_0+1}$. Therefore, for each $\xi \in (0, 1)$, one gets

$$|\psi(\alpha, Y_{m_k}, \tau) - \psi(\alpha, Y_N, \tau)| \leq 1 - \xi, \quad (44)$$

$$|\phi(\alpha, Y_{m_k}, \tau) - \phi(\alpha, Y_N, \tau)| \geq \xi. \quad (45)$$

for all $k > k_0$. Now, assume that $H = \mathbb{N} \setminus M$. Obviously, $H \in \mathcal{F}_{\sigma\theta}$ and

Theorem 6. Let $\mathcal{F}_{\sigma\theta}$ provides the feature (AP). Then, the notions $W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy sequence and $W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy sequence with regards to IFM (ψ, ϕ) coincide.

Proof. The direct part has been proved in Theorem 5.

Now, assume that the sequence is $W\mathcal{F}_{\sigma\theta}^{(\psi, \phi)}$ -Cauchy sequence with regards to IFM (ψ, ϕ) . Then, for each $\xi \in (0, 1)$, for each $x \in \mathcal{X}$, and for all $\tau > 0$, there is a $N = N(\xi, x) \in \mathbb{N}$, such that

where $m_j = m(1/j)$, $j = 1, 2, \dots$. Clearly, for $j = 1, 2, \dots$, $Q_j(\xi, \alpha, \tau) \in \mathcal{F}(\mathcal{I}_{\sigma\theta})$. Using Lemma 1, there is $Q \subset \mathbb{N}$, so that $Q \in \mathcal{F}(\mathcal{I}_{\sigma\theta})$ and $Q \setminus Q_j$ are finite for all j .

Now, we denote that

$$\lim_{k,l \rightarrow \infty} |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y_l, \tau)| = 1, \quad (49)$$

$$\lim_{k,l \rightarrow \infty} |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y_l, \tau)| = 0. \quad (50)$$

To demonstrate the above equations, let $\xi \in (0, 1)$ and $r \in \mathbb{N}$, such that $r > (2/\xi)$. If $k, l \in Q$, then $Q \setminus Q_j$ is a finite set; so, there is $\nu = \nu(r)$ in order that

$$|\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y_l, \tau)| > 1 - \frac{1}{r}, \quad |\psi(\alpha, Y_l, \tau) - \psi(\alpha, Y_l, \tau)| > 1 - \frac{1}{r}, \quad (51)$$

$$|\phi(\alpha, Y_k, p) - \phi(\alpha, Y_l, p)| < \frac{1}{r}, \quad |\phi(\alpha, Y_l, \tau) - \phi(\alpha, Y_l, \tau)| < \frac{1}{r}, \quad (52)$$

for all $k, l > \nu(r)$. From the above inequalities, we obtain

$$\begin{aligned} |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y_l, \tau)| &\leq |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y_l, \tau)| \\ &+ |\psi(\alpha, Y_l, \tau) - \psi(\alpha, Y_l, \tau)| > \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{r}\right) > 1 - \xi, \end{aligned} \quad (53)$$

$$\begin{aligned} |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y_l, \tau)| &\leq |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y_l, \tau)| \\ &+ |\phi(\alpha, Y_l, p) - \phi(\alpha, Y_l, p)| < \frac{1}{r} + \frac{1}{r} < \xi, \end{aligned} \quad (54)$$

for all $k, l > \nu(r)$.

Therefore, for every $\xi \in (0, 1)$, $\exists \nu = \nu(\xi)$, and $k, l \in Q \in \mathcal{F}(\mathcal{I}_{\sigma\theta})$, we acquire

$$\{k \in \mathbb{N}: |\psi(\alpha, Y_k, \tau) - \psi(\alpha, Y_l, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_k, \tau) - \phi(\alpha, Y_l, \tau)| \geq \xi\} \in \mathcal{I}_{\sigma\theta}. \quad (55)$$

This give that the sequence $\{Y_k\}$ is $W\mathcal{I}_{\sigma\theta}^{*(\psi, \phi)}$ -Cauchy. \square

uniformly in m , where $0 < q < \infty$. We indicate this symbolically by $Y_k \longrightarrow Y([WN_{\sigma\theta}^{(\psi, \phi)}]_q)$.

Definition 6. The sequence $\{Y_k\}$ is named to be q -strongly lacunary invariant convergent (Wijsman sense) to Y , provided that for each $\alpha \in \mathcal{X}$ and for all $\tau > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |\psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau)|^q = 1, \quad (56)$$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |\phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau)|^q = 0, \quad (57)$$

Theorem 7. Let $\mathcal{I}_{\sigma\theta} \subset 2^{\mathbb{N}}$ be an admissible ideal and $0 < q < \infty$.

- (i) If $Y_k \longrightarrow Y([WN_{\sigma\theta}^{(\psi, \phi)}]_q)$, then $Y_k \longrightarrow Y(W\mathcal{I}_{\sigma\theta}^{(\psi, \phi)})$
- (ii) If $\{Y_k\}$ is bounded and $Y_k \longrightarrow Y(W\mathcal{I}_{\sigma\theta}^{(\psi, \phi)})$, then $Y_k \longrightarrow Y([WN_{\sigma\theta}^{(\psi, \phi)}]_q)$
- (iii) If $\{Y_k\} \in L_{\infty}$, then $Y_k \longrightarrow Y(W\mathcal{I}_{\sigma\theta}^{(\psi, \phi)})$ iff $Y_k \longrightarrow Y([WN_{\sigma\theta}^{(\psi, \phi)}]_q)$

Proof.

(i) If $Y_k \longrightarrow Y([WN_{\sigma\theta}^{(\psi,\phi)}]_q)$, then for every $\xi \in (0, 1)$, for each $\alpha \in \mathcal{X}$ and for all $\tau > 0$, we obtain

$$\begin{aligned}
 & \sum_{k \in I_r} \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right|^q \text{ or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right|^q \\
 & \geq \sum_{\substack{k \in I_r, \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right| \leq 1-\xi \\ \text{or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right| \geq \xi}} \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right|^q \text{ or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right|^q \\
 & \geq \xi^q \left| \left\{ k \in I_r : \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right| \leq 1-\xi \text{ or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right| \geq \xi \right\} \right| \\
 & \geq \xi^q \max_m \left| \left\{ k \in I_r : \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right| \leq 1-\xi \text{ or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right| \geq \xi \right\} \right|.
 \end{aligned} \tag{58}$$

And so,

$$\begin{aligned}
 & \frac{1}{h_r} \sum_{k \in I_r} \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right|^q \text{ or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right|^q \\
 & \geq \xi^q \frac{\max_m \left| \left\{ k \in I_r : \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right| \leq 1-\xi \text{ or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right| \geq \xi \right\} \right|}{h_r} \\
 & = \xi^q \frac{S_r}{h_r}.
 \end{aligned} \tag{59}$$

For every $m = 1, 2, \dots$ This gives that $\lim_{r \rightarrow \infty} (S_r/h_r) = 0$, and hence, $T_k \longrightarrow T(W_{\mathcal{F}_{\sigma\theta}^{(\psi,\phi)}})$.

(ii) Presume that $\{Y_k\} \in L_\infty$, then $Y_k \longrightarrow Y(W_{\mathcal{F}_{\sigma\theta}^{(\psi,\phi)}})$. Let $\xi > 0$. By supposition, we get $V_\theta(P(\xi, \alpha, \tau)) = 0$. Since $\{Y_k\}$ is bounded, there is $M > 0$, such that for each $\alpha \in \mathcal{X}$ and for all k, m . Then, we acquire

$$\begin{aligned}
 & \frac{1}{h_r} \sum_{k \in I_r} \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right|^q \text{ or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right|^q \\
 & = \frac{1}{h_r} \sum_{\substack{k \in I_r, \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right| \leq 1-\xi \\ \text{or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right| \geq \xi}} \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right|^q \text{ or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right|^q \\
 & \quad + \frac{1}{h_r} \sum_{\substack{k \in I_r, \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right| > 1-\xi \\ \text{or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right| < \xi}} \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right|^q \text{ or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right|^q \\
 & \leq M \frac{\max_m \left| \left\{ k \in I_r : \left| \psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau) \right| \leq 1-\xi \text{ or } \left| \phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau) \right| \geq \xi \right\} \right|}{h_r} + \xi^q \\
 & \leq M \frac{S_r}{h_r} + \xi^q,
 \end{aligned} \tag{60}$$

For each $\alpha \in \mathcal{X}$. Hence, we obtain

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |\psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau)|^q = 1, \quad (61)$$

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \sum_{k \in I_r} |\phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau)|^q = 0, \quad (62)$$

Uniformly in m . As a result, we get $Y_k \longrightarrow Y([WN_{\sigma\theta}^{(\psi,\phi)}]_q)$.

(iii) It is clear from the consequence of (i) and (ii) \square

Definition 7. A sequence $\{Y_k\}$ is named to be lacunary invariant statistical convergent or $WS_{\sigma}^{\theta}(\psi, \phi)$ -convergent (Wijsman sense) to Y , if for each $\xi \in (0, 1)$, for each $\alpha \in \mathcal{X}$, and for all $\tau > 0$,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} \left| \left\{ k \in I_r : |\psi(\alpha, Y_{\sigma^k(m)}, \tau) - \psi(\alpha, Y, \tau)| \leq 1 - \xi \text{ or } |\phi(\alpha, Y_{\sigma^k(m)}, \tau) - \phi(\alpha, Y, \tau)| \geq \xi \right\} \right| = 0, \quad (63)$$

uniformly in m .

Theorem 8. A sequence $\{Y_k\}$ is $WS_{\sigma}^{\theta}(\psi, \phi)$ -convergent to Y iff it is $W\mathcal{F}_{\sigma\theta}^{(\psi,\phi)}$ -convergent to Y .

3. Conclusion

Fuzzy set theory is based on the assumption that reasoning is not crystal clear. This theory has a significant role in the areas of technology and science. Intuitionistic fuzzy set has many application areas; for example, sale analysis, new product marketing, and financial services. This study aims to find out the use of the notion of lacunary \mathcal{F} -invariant convergence of sequence of sets for demonstrating some results in the area of intuitionistic fuzzy metric space. With the help of its applications, we give the notions of Wijsman lacunary \mathcal{F} -invariant convergent, Wijsman lacunary \mathcal{F}^* -invariant convergent, and Wijsman q -strongly lacunary invariant convergent sequences in IFMS and acquired meaningful results for these notions. Also, we have examined the notions of Wijsman lacunary \mathcal{F} -invariant Cauchy and Wijsman \mathcal{F}^* -invariant Cauchy sequence in IFMS. The elements of IFMS have been studied. The results acquired here are more common than corresponding results for metric spaces. It is expected that new results will help to understand deeply the concept of this new type of convergence on IFMS. It could also be possible to work with the opinion of “Lacunary \mathcal{F} -invariant convergence of sequence of sets in probabilistic metric space” utilizing intuitionistic probability theory in the prospective studies.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Some $\alpha - \phi$ -Fuzzy Cone Contraction Results with Integral Type Application

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Received 1 July 2021; Revised 31 July 2021; Accepted 17 August 2021; Published 19 October 2021

Academic Editor: Naeem Saleem

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In this paper, we define α -admissible and α - ϕ -fuzzy cone contraction in fuzzy cone metric space to prove some fixed point theorems. Some related sequences with contraction mappings have been discussed. Ultimately, our theoretical results have been utilized to show the existence of the solution to a nonlinear integral equation. This application is also illustrative of how fuzzy metric spaces can be used in other integral type operators.

1. Introduction

The concept of fuzzy metric space (FM space) was first introduced by Kramosil and Michale [1] while George and Veeramani [2] illustrated some well-known FM space properties. In the sense of Kramosil and Michale [1], George and Sapena [3] and Grabice [4] introduced the idea of fuzzy contraction of complete FM spaces and developed some fixed point (FP) results. Some more related results can be found in [5–8]. Samet et al. [9] proposed the concept of α - ψ -contraction in complete metric spaces in 2012. Later, Gopal and Vetro [10] presented the concepts of $\alpha - \phi$ and $\beta - \psi$ fuzzy contractive mappings, as well as several novel FP theorems in FM spaces. More FP results in the context of FM spaces can be found in references (see, for example, [11–15]). Mohammadi et al. [16] proved some generalized contraction results in FM spaces with application in integral equations. Recently, the rational type fuzzy contraction concept in complete FM space is given by Rehman et al. [17], and they proved some FP results with an application.

Oner et al. [18] introduced the idea of fuzzy cone metric space (FCM space), proved some basic properties, and

developed the first version of “Banach contraction principle for fixed point” in FCM spaces which is stated as follows: “let $(\mathcal{U}, \tilde{M}_\alpha, *)$ be a complete FCM space in which fuzzy cone contractive sequences are Cauchy and let $h: \mathcal{U} \rightarrow \mathcal{U}$ be a fuzzy cone contractive mapping being $a \in (0, 1)$ the contractive constant. Then, h has a unique fixed point.” Ur Rehman et al. [19] presented some extended “fuzzy cone Banach contraction results” in FCM spaces for some weaker conditions. In topology and analysis, the definition of FCM spaces with different contractive conditions has been commonly used. For further reading, refer to [20–28].

The definition of α admissibility has been applied to certain directions by several writers. For a pair of functions, some authors expanded the concept of α admissibility. We may advise readers to look for more work in the field of α admissibility, as well as references (see [29–33]). Recently, Islam et al. [34] established some FP results in cone b_2 -metric space by using generalized a -admissible Hardy–Rogers’ contractions over Banach algebras with application.

In this paper, we define that a mapping h is α -admissible with respect to η and $\alpha - \phi$ fuzzy cone contraction in FCM

space. By using the concept of α -admissibility with respect to η under mapping h , we establish some FP theorems under the $\alpha - \phi$ fuzzy cone contraction conditions in FCM space with an example. In support of our work, we present an integral type application. By using this concept, one can prove different contractive type FP results for nonlinear mappings with different types of applications in the context of FM spaces. The paper is organized as follows. In Section 2, we introduce the preliminary concepts to support our work. In Section 3, we present some FP results by using different types of contraction conditions in FCM spaces with an illustrative example. In Section 4, we present an integral equation application to validate the concept defined in the paper. Finally, in Section 5, we discuss the conclusion.

2. Preliminaries

The continuous t -norm is defined by Schweizer and Sklar [35].

Definition 1 (see [35]). An operation $*$: $[0, 1]^2 \rightarrow [0, 1]$ is called continuous t' -norm if it satisfies the following conditions:

- (i) $*$ is commutative, associative, and continuous
- (ii) $1 * \varrho_1 = \varrho_1$ and $\varrho_1 * \varrho_2 \leq \varrho_3 * \varrho_4$, whenever, $\varrho_1 \leq \varrho_3$ and $\varrho_2 \leq \varrho_4$, for each $\varrho_1, \varrho_2, \varrho_3, \varrho_4 \in [0, 1]$

The basic t' -norms, i.e., product, minimum, and L Lukasiewicz continuous t' -norms are defined as follows (see [35]):

- (i) $\varrho_1 * \varrho_2 = \varrho_1 \varrho_2$
- (ii) $\varrho_1 * \varrho_2 = \min\{\varrho_1, \varrho_2\}$
- (iii) $\varrho_1 * \varrho_2 = \max\{\varrho_1 + \varrho_2 - 1, 0\}$

Definition 2 (see [36]). A subset \mathcal{C} of a real Banach space \mathbf{E} is called cone if

- (i) $\mathcal{C} \neq \emptyset$, closed, and $\mathcal{C} \neq \{\theta\}$
- (ii) $0 \leq \varrho_1, \varrho_2 < \infty$ and $\mu, \omega \in \mathcal{C}$, then $\varrho_1 \mu + \varrho_2 \omega \in \mathcal{C}$
- (iii) $-\mu, \mu \in \mathcal{C}$, then $\mu = \theta$

A cone $\mathcal{C} \subset \mathbf{E}$ and \preceq is a partial ordering on \mathbf{E} via \mathcal{C} which is defined by $\mu \preceq \omega$ iff $\omega - \mu \in \mathcal{C}$. $\mu < \omega$ stands for $\mu \preceq \omega$ and $\mu \neq \omega$, while $\mu \ll \omega$ stands for $\omega - \mu \in \text{int}(\mathcal{C})$. All the cones in this paper have a nonempty interior.

Definition 3 (see [18]). A 3-tuple $(\mathcal{U}, \ddot{M}_\alpha, *)$ is called FCM space if \mathcal{C} is a cone of \mathbf{E} , \mathcal{U} is an arbitrary set, $*$ is a continuous t -norm, and a mapping $\ddot{M}_\alpha: \mathcal{U} \times \mathcal{U} \times \text{int}(\mathcal{C}) \rightarrow [0, 1]$ satisfies the following axioms:

- (i) $\ddot{M}_\alpha(\mu, \omega, t') > 0$ and $\ddot{M}_\alpha(\mu, \omega, t') = 1$ iff $\mu = \omega$
- (ii) $\ddot{M}_\alpha(\mu, \omega, t') = \ddot{M}_\alpha(\omega, \mu, t')$
- (iii) $\ddot{M}_\alpha(\mu, \rho, s) * \ddot{M}_\alpha(\rho, \omega, t') \leq \ddot{M}_\alpha(\mu, \omega, s + t')$
- (iv) $\ddot{M}_\alpha(\mu, \omega, \cdot): \text{int}(\mathcal{C}) \rightarrow [0, 1]$ is continuous $\forall \mu, \omega, \rho \in \mathcal{U}$ and $s, t' \in \text{int}(\mathcal{C})$.

Definition 4 (see [18]). Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a FCM space, and $\mu \in \mathcal{U}$ and (μ_ℓ) be a sequence in \mathcal{U} . Then,

- (i) The sequence (μ_ℓ) converges to μ , if $t' \gg \theta$, $0 < r < 1$ and $\exists i_1 \in \mathbb{N}$ so that $\ddot{M}_\alpha(\mu_\ell, \mu, t') > 1 - r$, $\forall \ell \geq i_1$. We denote this by $\lim_{\ell \rightarrow \infty} \mu_\ell = \mu$ or $\mu_\ell \rightarrow \mu$ as $\ell \rightarrow \infty$.
- (ii) The sequence (μ_ℓ) is Cauchy sequence if $0 < r < 1$, $t' \gg \theta$, and $\exists \ell_1 \in \mathbb{N}$ so that $\ddot{M}_\alpha(\mu_\ell, \mu_j, t') > 1 - r$, $\forall \ell, j \geq \ell_1$.
- (iii) The sequence (μ_ℓ) is \mathcal{C} -C auchy sequence if $0 < r < 1$, $t' \gg \theta$ and $\exists \ell_1 \in \mathbb{N}$ so that $\ddot{M}_\alpha(\mu_\ell, \mu_j, t') = 1$, $\forall \ell, j \geq \ell_1$.
- (iv) $(\mathcal{U}, \ddot{M}_\alpha, *)$ is \mathcal{C} -complete if every \mathcal{C} -C auchy sequence is convergent in \mathcal{U} .
- (v) (μ_ℓ) is said to be fuzzy cone contractive if $\exists 0 < \varrho < 1$ so that

$$\frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t')} - 1 \leq \varrho \left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t')} - 1 \right), \quad \text{for } t' \gg \theta \ell \geq 1. \quad (1)$$

Definition 5 (see [19]). Let a FCM \ddot{M}_α be triangular in FCM space $(\mathcal{U}, \ddot{M}_\alpha, *)$ if

$$\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1 \leq \left(\frac{1}{\ddot{M}_\alpha(\mu, \rho, t')} - 1 \right) + \left(\frac{1}{\ddot{M}_\alpha(\rho, \omega, t')} - 1 \right), \quad (2)$$

$\forall \mu, \omega, \rho \in \mathcal{U}$ and $t' \gg \theta$.

Lemma 1 (see [18]). Let $\mu \in \mathcal{U}$ in a FCM space $(\mathcal{U}, \ddot{M}_\alpha, *)$ and (μ_ℓ) be a sequence in \mathcal{U} . Then, $\mu_\ell \rightarrow \mu \Leftrightarrow \ddot{M}_\alpha(\mu_\ell, \mu, t') \rightarrow 1$ as $\ell \rightarrow \infty$, for $t' \gg \theta$.

Definition 6 (see [18]). Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a FCM space and $A: \mathcal{U} \rightarrow \mathcal{U}$. Then, T is called fuzzy cone contractive if $\exists 0 < \varrho < 1$, so that

$$\frac{1}{\ddot{M}_\alpha(A\mu, A\omega, t')} - 1 \leq \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right), \quad \forall \mu, \omega \in \mathcal{U}, t' \gg \theta. \quad (3)$$

3. Main Result

Definition 7. Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a FCM space, and let $\alpha, \eta: \mathcal{U} \times \mathcal{U} \times \text{int}(\mathcal{E}) \rightarrow [0, \infty)$ be two functions. We say that $h: \mathcal{U} \rightarrow \mathcal{U}$ is α -admissible w.r.t η if

$$\begin{aligned} \alpha(\mu, \omega, t') &\geq \eta(\mu, \omega, t') \Rightarrow \alpha(h\mu, h\omega, t') \\ &\geq \eta(h\mu, h\omega, t'), \quad \forall \mu, \omega \in \mathcal{U}. \end{aligned} \quad (4)$$

Note that in a special case, if we take $\eta(\mu, \omega, t') = 1$, then Definition 7 is reduced to the α -admissible mapping (i.e., Definition 3.4 in the study by Gopal and Vetro [10]), and

also if we take $\alpha(\mu, \omega, t') = 1$, then we can say that h is a η -subadmissible mapping.

In the following, Φ denotes the family of all right continuous functions $\phi: [0, +\infty) \rightarrow [0, +\infty)$ with $\phi(\tau) < \tau, \forall \tau > 0$.

Definition 8. Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a FCM space, and the mapping $h: \mathcal{U} \rightarrow \mathcal{U}$ is called α - ϕ -fuzzy cone contractive if there exist three functions $\alpha, \eta: \mathcal{U} \times \mathcal{U} \times \text{int}(\mathcal{E}) \rightarrow [0, \infty)$ and $\phi \in \Phi$, so that

$$\alpha(\mu, \omega, t') \geq \eta(\mu, \omega, t') \Rightarrow \frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 \leq \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right), \quad (5)$$

where

$$\Omega(h, \mu, \omega, t') = \min \left\{ \begin{array}{l} \ddot{M}_\alpha(\mu, \omega, t'), \ddot{M}_\alpha(\mu, h\mu, t'), \\ \ddot{M}_\alpha(\omega, h\mu, t'), \ddot{M}_\alpha(\omega, h\omega, t') \end{array} \right\}, \quad \forall \mu, \omega \in \mathcal{U}, t' \gg \theta. \quad (6)$$

Theorem 1. Let a FM \ddot{M}_α be triangular in a \mathcal{G} -complete FCM space $(\mathcal{U}, \ddot{M}_\alpha, *)$ and let $h: \mathcal{U} \rightarrow \mathcal{U}$ be an α - ϕ -fuzzy cone contractive if the following axioms hold:

- (1) h is an α -admissible w.r.t η
- (2) $\exists \mu_0 \in \mathcal{U}$ such that $\alpha(\mu_0, h\mu_0, t') \geq \eta(\mu_0, h\mu_0, t')$, for $t' \gg \theta$
- (3) The sequence (μ_ℓ) in \mathcal{U} with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq \eta(\mu_\ell, \mu_{\ell+1}, t'), \forall \ell \in N, t' \gg \theta$ and $\mu_\ell \rightarrow z$ as $\ell \rightarrow +\infty$, then $\alpha(\mu_\ell, z, t') \geq \eta(\mu_\ell, z, t'), \forall \ell \in N$ and, $t' \gg \theta$

Then, h has a FP $z \in \mathcal{U}$ such that $hz = z$.

Proof. Let $\mu_0 \in \mathcal{U}$, such that

$$\alpha(\mu_0, h\mu_0, t') \geq \eta(\mu_0, h\mu_0, t'), \quad \text{for } t' \gg \theta. \quad (7)$$

We choose a sequence

$$(\mu_\ell) \in \mathcal{U}, \text{ i.e. } \mu_\ell = h\mu_{\ell-1} = h^\ell \mu_0, \quad \forall \ell \in N. \quad (8)$$

If $\mu_{\ell+1} = u_\ell$ for some $\ell \in N$, then $\mu_\ell = \mu$ is a FP of h in \mathcal{U} . Otherwise, we assume that $\mu_{\ell+1} \neq \mu_\ell, \forall \ell \in N$. Since the

mapping h is an α -admissible w.r.t η and $\alpha(\mu_0, h\mu_0, t') \geq \eta(\mu_0, h\mu_0, t')$, for $t' \gg \theta$, we have to find that

$$\begin{aligned} \alpha(\mu_1, \mu_2, t') &= \alpha(h\mu_0, h^2\mu_0, t') \\ &\geq \eta(h\mu_0, h^2\mu_0, t') = \eta(\mu_1, \mu_2, t'), \\ &\Rightarrow \alpha(\mu_1, \mu_2, t') \geq \eta(\mu_1, \mu_2, t'), \quad \text{for } t' \gg \theta. \end{aligned} \quad (9)$$

Continuing this process $\forall \ell \in N$, we get

$$\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq \eta(\mu_\ell, \mu_{\ell+1}, t'), \quad \text{for } t' \gg \theta. \quad (10)$$

From (5), we have

$$\begin{aligned} \frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t')} - 1 &= \frac{1}{\ddot{M}_\alpha(h\mu_{\ell-1}, h\mu_\ell, t')} - 1, \\ &\leq \phi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t')} - 1\right), \end{aligned} \quad (11)$$

where

$$\begin{aligned}
 \frac{1}{\Omega(h, \mu_{\ell-1}, \mu_{\ell}, t')} - 1 &= \frac{1}{\min \left\{ \begin{array}{l} \ddot{M}_{\alpha}(\mu_{\ell-1}, \mu_{\ell}, t'), \ddot{M}_{\alpha}(\mu_{\ell-1}, h\mu_{\ell-1}, t'), \\ \ddot{M}_{\alpha}(\mu_{\ell}, h\mu_{\ell-1}, t'), \ddot{M}_{\alpha}(\mu_{\ell}, h\mu_{\ell}, t') \end{array} \right\}} - 1, \\
 &= \frac{1}{\min \left\{ \begin{array}{l} \ddot{M}_{\alpha}(\mu_{\ell-1}, \mu_{\ell}, t'), \ddot{M}_{\alpha}(\mu_{\ell-1}, \mu_{\ell}, t'), \\ \ddot{M}_{\alpha}(\mu_{\ell}, \mu_{\ell+1}, t') \end{array} \right\}} - 1 \\
 &= \frac{1}{\min \{ \ddot{M}_{\alpha}(\mu_{\ell-1}, \mu_{\ell}, t'), \ddot{M}_{\alpha}(\mu_{\ell}, \mu_{\ell+1}, t') \}} - 1.
 \end{aligned} \tag{12}$$

Now, from (11), for $t' \gg \theta$, we have

$$\frac{1}{\ddot{M}_{\alpha}(\mu_{\ell}, \mu_{\ell+1}, t')} - 1 \leq \phi \left(\frac{1}{\min \{ \ddot{M}_{\alpha}(\mu_{\ell-1}, \mu_{\ell}, t'), \ddot{M}_{\alpha}(\mu_{\ell}, \mu_{\ell+1}, t') \}} - 1 \right). \tag{13}$$

If $\ddot{M}_{\alpha}(\mu_{\ell}, \mu_{\ell+1}, t')$ is minimum for some $\ell \in N$, then by (13), we obtain

$$\begin{aligned}
 \frac{1}{\ddot{M}_{\alpha}(\mu_{\ell}, \mu_{\ell+1}, t')} - 1 &\leq \phi \left(\frac{1}{\ddot{M}_{\alpha}(\mu_{\ell}, \mu_{\ell+1}, t')} - 1 \right), \\
 &< \frac{1}{\ddot{M}_{\alpha}(\mu_{\ell}, \mu_{\ell+1}, t')} - 1,
 \end{aligned} \tag{14}$$

which is not possible. Hence, $\forall \ell \in N$ and $t' \gg \theta$, we get

$$\begin{aligned}
 \frac{1}{\ddot{M}_{\alpha}(\mu_{\ell}, \mu_{\ell+1}, t')} - 1 &\leq \phi \left(\frac{1}{\ddot{M}_{\alpha}(\mu_{\ell-1}, \mu_{\ell}, t')} - 1 \right) \\
 &< \frac{1}{\ddot{M}_{\alpha}(\mu_{\ell-1}, \mu_{\ell}, t')} - 1.
 \end{aligned} \tag{15}$$

This implies that

$$\ddot{M}_{\alpha}(\mu_{\ell}, \mu_{\ell+1}, t') > \ddot{M}_{\alpha}(\mu_{\ell-1}, \mu_{\ell}, t'). \tag{16}$$

Thus, $(\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t'))$ is an increasing sequence in $[0, 1]$. Let $m_1(t') = \lim_{\ell \rightarrow \infty} \ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t')$, we have to show that $m_1(t') = 1$, for $t' \gg \theta$. Let $\exists t'^* \gg \theta \ni m(t'^*) < 1$.

$$\frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t'^*)} - 1 < \frac{1}{\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t'^*)} - 1. \quad (17)$$

Using the right side continuity of a function ϕ and let the limit $\ell \rightarrow +\infty$, we obtained the contradiction as follows:

$$\frac{1}{m(t'^*)} - 1 \leq \phi\left(\frac{1}{m(t'^*)} - 1\right) < \frac{1}{m(t'^*)} - 1. \quad (18)$$

This implies that $\lim_{\ell \rightarrow \infty} \ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t') = 1$, for $t' \gg \theta$. Let $j > \ell$, where $\ell, j \in N$ and $j = \ell + p$, for a fixed $p \in N$,

$$\begin{aligned} \frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_j, t')} - 1 &= \frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+p}, t')} - 1, \\ &\leq \frac{1}{\{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, (t'/p)) * \ddot{M}_\alpha(\mu_{\ell+1}, \mu_{\ell+2}, (t'/p)) * \dots * \ddot{M}_\alpha(\mu_{\ell+p-1}, \mu_{\ell+p}, (t'/p))\}} - 1 \rightarrow \theta, \quad \text{as } \ell \rightarrow \infty. \end{aligned} \quad (19)$$

This implies that $\lim_{\ell \rightarrow \infty} \ddot{M}_\alpha(\mu_\ell, \mu_j, t') = 1$. It is proved that the sequence (μ_ℓ) is \mathcal{G} -C auchy. Since $(\mathcal{U}, \ddot{M}_\alpha, *)$ is \mathcal{G} -complete, then $\exists z \in \mathcal{U}$ such that $\mu_\ell \rightarrow z$ as $\ell \rightarrow +\infty$, i.e.,

$$\lim_{\ell \rightarrow \infty} \ddot{M}_\alpha(\mu_\ell, z, t') = 1, \quad \text{for } t' \gg \theta. \quad (20)$$

Now, by the view of Definition 4 (iii),

$$\alpha(\mu_\ell, z, t') \geq \eta(\mu_\ell, z, t'), \quad \forall \ell \in N, t' \gg \theta. \quad (21)$$

If $hz \neq z$, i.e., $\ddot{M}_\alpha(z, hz, t') < 1$, then $t' \gg \theta$. Since \ddot{M}_α is triangular, we have that

$$\frac{1}{\ddot{M}_\alpha(z, hz, t')} - 1 \leq \left(\frac{1}{\ddot{M}_\alpha(z, \mu_{\ell+1}, t')} - 1 \right) + \left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell+1}, hz, t')} - 1 \right), \quad \text{for } t' \gg \theta. \quad (22)$$

Then, from (5) and (21), we have

$$\frac{1}{\ddot{M}_\alpha(\mu_{\ell+1}, hz, t')} - 1 = \frac{1}{\ddot{M}_\alpha(h\mu_\ell, hz, t')} - 1 \leq \phi\left(\frac{1}{\Omega(h, \mu_\ell, z, t')} - 1\right), \quad (23)$$

where

$$\begin{aligned}
& \frac{1}{\Omega(h, \mu_\ell, z, t')} - 1 \frac{1}{\min\{\ddot{M}_\alpha(\mu_\ell, z, t'), \ddot{M}_\alpha(\mu_\ell, h\mu_\ell, t'), \ddot{M}_\alpha(z, h\mu_\ell, t'), \ddot{M}_\alpha(z, hz, t')\}} - 1, \\
&= \frac{1}{\min\{\ddot{M}_\alpha(\mu_\ell, z, t'), \ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t'), \ddot{M}_\alpha(z, \mu_{\ell+1}, t'), \ddot{M}_\alpha(z, hz, t')\}} - 1 \\
&\longrightarrow \frac{1}{\min\{1, 1, 1, \ddot{M}_\alpha(z, hz, t')\}} - 1, \quad \text{as } \ell \longrightarrow \infty, \\
&= \frac{1}{\ddot{M}_\alpha(z, hz, t')}
\end{aligned} \tag{24}$$

Thus, to avoid the contradiction with $\phi(c) < c$, for $c > 0$,

$$\limsup_{\ell \longrightarrow \infty} \left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell+1}, hz, t')} - 1 \right) \leq \left(\frac{1}{\ddot{M}_\alpha(z, hz, t')} - 1 \right), \tag{25}$$

Thus, together with (20) and (22), we conclude that $\ddot{M}_\alpha(z, hz, t') = 1 \Rightarrow hz = z$. \square

Corollary 1. Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a \mathcal{G} -complete FCM space in which \ddot{M}_α is triangular, and let $h: \mathcal{U} \longrightarrow \mathcal{U}$ be an α -admissible. Assume that $\exists \phi \in \Phi$, so that

$$\alpha(\mu, \omega, t') \geq 1 \Rightarrow \frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 \leq \phi \left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1 \right), \tag{26}$$

where

$$\Omega(h, \mu, \omega, t') = \min\{\ddot{M}_\alpha(\mu, \omega, t'), \ddot{M}_\alpha(\mu, h\mu, t'), \ddot{M}_\alpha(\omega, h\mu, t'), \ddot{M}_\alpha(\omega, h\omega, t')\}, \quad \forall \mu, \omega \in \mathcal{U}, t' \gg \theta. \tag{27}$$

Assume that the following assertions hold:

- (i) $\exists \mu_0 \in \mathcal{U}$ such that $\alpha(\mu_0, h\mu_0, t') \geq 1$, for $t' \gg \theta$
- (ii) Any sequence (μ_ℓ) in \mathcal{U} with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq 1$, $\forall \ell \in \mathbb{N}$, $t' \gg \theta$ and $\mu_\ell \longrightarrow z$ as $\ell \longrightarrow +\infty$, then $\alpha(\mu_\ell, z, t') \geq 1$, $\forall \ell \in \mathbb{N}$ and, $t' \gg \theta$

Then, h has a FP in \mathcal{U} .

Corollary 2. Let a FM \ddot{M}_α be triangular in a \mathcal{G} -complete FCM space $(\mathcal{U}, \ddot{M}_\alpha, *)$, and let $h: \mathcal{U} \longrightarrow \mathcal{U}$ be an η -admissible. Assume that $\exists \phi \in \Phi$, so that

$$\eta(\mu, \omega, t') \leq 1 \Rightarrow \frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 \leq \phi \left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1 \right), \tag{28}$$

where

$$\Omega(h, \mu, \omega, t') = \min\{\ddot{M}_\alpha(\mu, \omega, t'), \ddot{M}_\alpha(\mu, h\mu, t'), \ddot{M}_\alpha(\omega, h\mu, t'), \ddot{M}_\alpha(\omega, h\omega, t')\}, \quad \forall \mu, \omega \in \mathcal{U}, t' \gg \theta. \tag{29}$$

Suppose that the following axioms hold:

- (i) $\exists \mu_0 \in \mathcal{U}$ such that $\eta(\mu_0, h\mu_0, t') \leq 1$, for $t' \gg \theta$

- (ii) Any sequence (μ_ℓ) in \mathcal{U} with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq 1$, $\forall \ell \in N$, $t' \gg \theta$ and $\mu_\ell \rightarrow z$ as $\ell \rightarrow +\infty$, then $\alpha(\mu_\ell, z, t') \geq 1$, $\forall \ell \in N$ and $t' \gg \theta$

Then, h has a FP in \mathcal{U} .

Now, to establish the unique FP of an α - ϕ -fuzzy cone contraction map, let the hypothesis (H) is given as follows:

(H) For all $\mu, \omega \in \mathcal{U}$, $t' \gg \theta$, $\exists \rho \in \mathcal{U}$ such that

$$\alpha(\mu, \rho, t') \geq \eta(\mu, \rho, t'), \alpha(\omega, \rho, t') \geq \eta(\omega, \rho, t'),$$

$$\lim_{\ell \rightarrow \infty} \ddot{M}_\alpha(h^{\ell-1}\rho, h^\ell\rho, t') = 1. \quad (30)$$

Theorem 2. Adding the hypothesis (H) in Theorem 1, we obtain the uniqueness of a FP of h provided the function $\phi \in \Phi$ is nondecreasing.

Proof. Assume that z and ρ be the two FPs of h in \mathcal{U} . If $\alpha(z, \rho, t') \geq \eta(z, \rho, t')$, then by (5), we get $z = \rho$. Suppose $\alpha(z, \rho, t') < \eta(z, \rho, t')$, then from (H), $\exists y \in \mathcal{U}$, so that

$$\alpha(z, y, t') \geq \eta(z, y, t'), \quad \text{and } \alpha(\rho, y, t') \geq \eta(\rho, y, t'), \quad \text{for } t' \gg \theta. \quad (31)$$

Since h is an α -admissible w.r.t η , then we get

$$\alpha(z, h^\ell y, t') \geq \eta(z, h^\ell y, t'), \quad \forall \ell \in N \text{ and } t' \gg \theta. \quad (32)$$

Now, we have to show that $\ddot{M}_\alpha(z, h^\ell y, t') \rightarrow 1$, as $\ell \rightarrow \infty$, for $t' \gg \theta$. Since \ddot{M}_α is triangular, then from (5) and (31),

$$\begin{aligned} \frac{1}{\ddot{M}_\alpha(z, h^\ell y, t')} - 1 &\leq \left(\frac{1}{\ddot{M}_\alpha(z, hz, t')} - 1 \right) + \left(\frac{1}{\ddot{M}_\alpha(hz, h^\ell y, t')} - 1 \right) \\ &= \left(\frac{1}{\ddot{M}_\alpha(z, hz, t')} - 1 \right) + \left(\frac{1}{\ddot{M}_\alpha(hz, h(h^{\ell-1}y), t')} - 1 \right) \leq \phi \left(\frac{1}{\Omega(h, z, h^{\ell-1}y, t')} - 1 \right), \end{aligned} \quad (33)$$

where z is the FP of h and for $t' \gg \theta$,

$$\begin{aligned} \frac{1}{\Omega(h, z, h^{\ell-1}y, t')} - 1 &= \left(\frac{1}{\min\{\ddot{M}_\alpha(z, h^{\ell-1}y, t'), \ddot{M}_\alpha(z, hz, t'), \ddot{M}_\alpha(h^{\ell-1}y, hz, t'), \ddot{M}_\alpha(h^{\ell-1}y, h^\ell y, t')\}} - 1 \right), \\ &= \left(\frac{1}{\min\{\ddot{M}_\alpha(z, h^{\ell-1}y, t'), 1, \ddot{M}_\alpha(h^{\ell-1}y, z, t'), \ddot{M}_\alpha(h^{\ell-1}y, h^\ell y, t')\}} - 1 \right) \\ &= \left(\frac{1}{\min\{\ddot{M}_\alpha(z, h^{\ell-1}y, t'), \ddot{M}_\alpha(h^{\ell-1}y, h^\ell y, t')\}} - 1 \right). \end{aligned} \quad (34)$$

Let $\ell_0 \in N$ such that Again by the triangular property of \ddot{M}_α and the view of (5) and (31),
 $\ddot{M}_\alpha(z, h^{\ell-1}y, t') \leq \ddot{M}_\alpha(h^{\ell-1}y, h^\ell y, t'), \forall \ell \geq \ell_0$ and by the hypothesis (H). Thus,

$$\frac{1}{\ddot{M}_\alpha(z, h^\ell y, t')} - 1 \leq \phi \left(\frac{1}{\ddot{M}_\alpha(z, h^{\ell-1}y, t')} - 1 \right), \quad \text{for } t' \gg \theta. \quad (35)$$

$$\begin{aligned} \frac{1}{\ddot{M}_\alpha(z, h^\ell y, t')} - 1 &\leq \phi \left(\frac{1}{\ddot{M}_\alpha(z, h^{\ell-1}y, t')} - 1 \right), \\ &\leq \phi \left(\left(\frac{1}{\ddot{M}_\alpha(z, hz, t')} - 1 \right) + \left(\frac{1}{\ddot{M}_\alpha(hz, h^{\ell-1}y, t')} - 1 \right) \right) \\ &= \phi \left(\left(\frac{1}{\ddot{M}_\alpha(z, hz, t')} - 1 \right) + \left(\frac{1}{\ddot{M}_\alpha(hz, h(h^{\ell-2}y), t')} - 1 \right) \right) \\ &\leq \phi^2 \left(\frac{1}{\Omega(h, z, h^{\ell-1}y, t')} - 1 \right), \end{aligned} \quad (36)$$

where z is the FP of h and for $t' \gg \theta$,

$$\begin{aligned} \frac{1}{\Omega(h, z, h^{\ell-2}y, t')} - 1 &= \left(\frac{1}{\min\{\ddot{M}_\alpha(z, h^{\ell-2}y, t'), \ddot{M}_\alpha(z, hz, t'), \ddot{M}_\alpha(h^{\ell-2}y, hz, t'), \ddot{M}_\alpha(h^{\ell-2}y, h^{\ell-1}y, t')\}} - 1 \right), \\ &= \left(\frac{1}{\min\{\ddot{M}_\alpha(z, h^{\ell-2}y, t'), 1, \ddot{M}_\alpha(h^{\ell-2}y, z, t'), \ddot{M}_\alpha(h^{\ell-2}y, h^{\ell-1}y, t')\}} - 1 \right) \\ &= \left(\frac{1}{\min\{\ddot{M}_\alpha(z, h^{\ell-2}y, t'), \ddot{M}_\alpha(h^{\ell-2}y, h^{\ell-1}y, t')\}} - 1 \right). \end{aligned} \quad (37)$$

Let $\ell_0 \in N$ such that $\ddot{M}_\alpha(z, h^{\ell-2}y, t') \leq \ddot{M}_\alpha(h^{\ell-2}y, h^{\ell-1}y, t'), \forall \ell \geq \ell_0$, and by the hypothesis (H). Thus,

$$\frac{1}{\ddot{M}_\alpha(z, h^\ell y, t')} - 1 \leq \phi^2 \left(\frac{1}{\ddot{M}_\alpha(z, h^{\ell-2}y, t')} - 1 \right), \quad \text{for } t' \gg \theta. \quad (38)$$

By continuing the same argument, we obtain

$$\frac{1}{\ddot{M}_\alpha(z, h^\ell y, t')} - 1 \leq \phi^{\ell-\ell_0} \left(\frac{1}{\ddot{M}_\alpha(z, h^{\ell_0}y, t')} - 1 \right), \quad \text{for } t' \gg \theta. \quad (39)$$

Then, by taking the limit $\ell \rightarrow \infty$, we get that

$$h^\ell y \rightarrow z. \quad (40)$$

Similarly, we can prove that

$$h^\ell y \rightarrow \rho, \quad \text{as } \ell \rightarrow \infty. \quad (41)$$

Now, from (40) and (41), we get the uniqueness, i.e., $z = \rho$. Subsequently, we use the following classes of functions in our results without \ddot{M}_α triangularity condition. Suppose that

$$\Psi = \{\psi: [0, +\infty) \rightarrow [0, +\infty)\}, \quad (42)$$

where ψ is nondecreasing and continuous, and

$$\Phi_0 = \{\phi: [0, +\infty) \rightarrow [0, +\infty)\}, \quad (43)$$

where ϕ is lower semicontinuous, where $\psi(c) = \phi(c) = 0 \Leftrightarrow c = 0$. \square

Theorem 3. Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a \mathcal{G} -complete FCM space, and let $h: \mathcal{U} \rightarrow \mathcal{U}$ be an α -admissible w.r.t η . Suppose that $\exists \psi \in \Psi$ and $\phi \in \Phi_0$, so that

$$\begin{aligned} \alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') &\geq \eta(\mu, h\mu, t')\eta(\omega, h\omega, t'), \\ \Rightarrow \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1\right) &\leq \psi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right), \end{aligned} \quad (44)$$

$\forall \mu, \omega \in \mathcal{U}$ and $t' \gg \theta$. Let the following axioms hold:

- (i) $\exists \mu_0 \in \mathcal{U}$ such that $\alpha(\mu_0, h\mu_0, t') \geq \eta(\mu_0, h\mu_0, t')$, for $t' \gg \theta$
- (ii) Any sequence (μ_ℓ) in \mathcal{U} with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq \eta(\mu_\ell, \mu_{\ell+1}, t'), \forall \ell \in N, t' \gg \theta$ and $\mu_\ell \rightarrow z \in \mathcal{U}$, as $\ell \rightarrow +\infty$, then $\alpha(\mu_\ell, z, t') \geq \eta(\mu_\ell, z, t')$ and $\alpha(z, hz, t') \geq \eta(z, hz, t'), \forall \ell \in N$ and $t' \gg \theta$

Then, h has an FP in \mathcal{U} .

Proof. Let $\mu_0 \in \mathcal{U}$ such that

$$\alpha(\mu_0, h\mu_0, t') \geq \eta(\mu_0, h\mu_0, t'), \quad \text{for } t' \gg \theta. \quad (45)$$

We define a sequence (μ_ℓ) in \mathcal{U} such that $\mu_\ell = h\mu_{\ell-1} = h^\ell \mu_0, \forall \ell \in N$. If $\mu_{\ell+1} = \mu_\ell$ for some $\ell \in N$, then $\mu_\ell = \mu$ is an FP of h in \mathcal{U} .

Otherwise, we assume that $\mu_{\ell+1} \neq \mu_\ell, \forall \ell \in N$. However, the mapping h is an α -admissible w.r.t η and $\alpha(\mu_0, h\mu_0, t') \geq \eta(\mu_0, h\mu_0, t')$. Now, we have to deduce that

$$\begin{aligned} \alpha(\mu_1, \mu_2, t') &= \alpha(h\mu_0, h^2\mu_0, t') \geq \eta(h\mu_0, h^2\mu_0, t') = \eta(\mu_1, \mu_2, t'), \\ \Rightarrow \alpha(\mu_1, \mu_2, t') &\geq \eta(\mu_1, \mu_2, t'), \quad \text{for } t' \gg \theta. \end{aligned} \quad (46)$$

Continuing this process $\forall \ell \in N$, we may obtain

$$\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq \eta(\mu_\ell, \mu_{\ell+1}, t'), \quad \text{for } t' \gg \theta. \quad (47)$$

Clearly,

$$\alpha(\mu_{\ell-1}, h\mu_{\ell-1}, t')\alpha(\mu_\ell, h\mu_\ell, t') \geq \eta(\mu_{\ell-1}, h\mu_{\ell-1}, t')\eta(\mu_\ell, h\mu_\ell, t'), \quad \text{for } t' \gg \theta. \quad (48)$$

Now, by (44), for $t' \gg \theta$,

$$\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t')} - 1\right) = \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu_{\ell-1}, h\mu_\ell, t')} - 1\right) \leq \psi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t')} - 1\right). \quad (49)$$

If $\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t') = 1$, then $\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t') = 1$. Otherwise, if $\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t') < 1$, then

$$\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t')} - 1\right) < \psi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t')} - 1\right), \quad \text{for } t' \gg \theta. \quad (50)$$

Since ψ is nondecreasing, we may obtain that $\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq \ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t')$, $\forall \ell \in N$ and $t' \gg \theta$. Thus, $(\ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t'))$ is nondecreasing sequence in $[0, 1]$. Let $m_1(t') = \lim_{\ell \rightarrow \infty} \ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t')$, $t' \gg \theta$. Now, we have to show that $m_1(t') = t'$, for $t' \gg \theta$. If not, then $\exists t' \gg \theta$ such

that $m_1(t') < 1$. Therefore, by taking the limit $\ell \rightarrow \infty$ on (50), we get

$$\psi\left(\frac{1}{m_1(t')} - 1\right) \leq \psi\left(\frac{1}{m_1(t')} - 1\right) - \phi\left(\frac{1}{m_1(t')} - 1\right), \quad (51)$$

which is a contradiction. Thus, we get that

$$\lim_{\ell \rightarrow \infty} \ddot{M}_\alpha(\mu_{\ell-1}, \mu_\ell, t') = 1, \quad \text{for } t' \gg \theta. \quad (52)$$

Let $\ell, j \in N$ such that $j > \ell$ and $j = \ell + p$, where $p \in N$. We have that

$$\ddot{M}_\alpha(\mu_\ell, \mu_{\ell+p}, t') \geq \ddot{M}_\alpha\left(\mu_\ell, \mu_{\ell+1}, \frac{t'}{p}\right) * \ddot{M}_\alpha\left(\mu_{\ell+1}, \mu_{\ell+2}, \frac{t'}{p}\right) * \cdots * \ddot{M}_\alpha\left(\mu_{\ell+p-1}, \mu_{\ell+p}, \frac{t'}{p}\right) \longrightarrow 1 * 1 * \cdots * 1 = 1, \quad (53)$$

as $\ell \rightarrow \infty$, for $t' \gg \theta$.

Thus, (μ_ℓ) is a \mathcal{G} -C auchy sequence, and since the space $(\mathcal{U}, \ddot{M}_\alpha, *)$ is \mathcal{G} -complete, therefore, $\mu_\ell \rightarrow z \in \mathcal{U}$. Now, for any sequence (μ_ℓ) in \mathcal{U} with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq \eta(\mu_\ell, \mu_{\ell+1}, t')$, $\forall \ell \in N$ and $t' \gg \theta$. By the completeness of \mathcal{U} , $\mu_\ell \rightarrow z$ as $\ell \rightarrow \infty$, $\alpha(\mu_\ell, z, t') \geq \eta(\mu_\ell, z, t')$ and $\alpha(z, hz, t') \geq \eta(z, hz, t')$, $\forall \ell \in N$ and $t' \gg \theta$. Then, easily we may obtain

$$\alpha(\mu_\ell, \mu_{\ell+1}, t') \alpha(z, hz, t') \geq \eta(\mu_\ell, \mu_{\ell+1}, t') \eta(z, hz, t'), \quad \text{for } t' \gg \theta. \quad (54)$$

Now, from (44),

$$\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell+1}, hz, t')} - 1\right) = \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu_\ell, hz, t')} - 1\right) \leq \psi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell, z, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell, z, t')} - 1\right), \quad \text{for } t' \gg \theta. \quad (55)$$

If $\ddot{M}_\alpha(\mu_\ell, z, t') = 1$, then $\ddot{M}_\alpha(\mu_{\ell+1}, hz, t') = 1$, for $t' \gg \theta$. If $\ddot{M}_\alpha(\mu_\ell, z, t') < 1$, for $t' \gg \theta$, then we have

$$\psi\left(\frac{1}{\ddot{M}_\alpha(\mu_{\ell+1}, hz, t')} - 1\right) < \psi\left(\frac{1}{\ddot{M}_\alpha(\mu_\ell, z, t')} - 1\right) \Rightarrow \ddot{M}_\alpha(\mu_{\ell+1}, hz, t') \geq \ddot{M}_\alpha(\mu_\ell, z, t') \longrightarrow 1, \quad \text{as } \ell \rightarrow \infty, \quad (56)$$

for $t' \gg \theta$. Thus, we get that $hz = z$. \square

Example 1. Let $\mathcal{U} = [0, \infty)$ and t' -norm is a continuous norm. Let a FM $\ddot{M}_\alpha: \mathcal{U} \times \mathcal{U} \times (0, \infty) \rightarrow [0, 1]$ be defined as

$$\ddot{M}_\alpha(\mu, \omega, t') = \frac{t'}{t' + |\mu - \omega|}, \quad \forall \mu, \omega \in \mathcal{U}, \text{ and } t' > 0. \quad (57)$$

Then, $(\mathcal{U}, \ddot{M}_\alpha, *)$ is a \mathcal{G} -complete FCM space and a FCM \ddot{M}_α is triangular. Now, we define a mapping $h: \mathcal{U} \rightarrow \mathcal{U}$ by

$$h(\mu) = \begin{cases} \frac{\mu}{3}, & \text{if } \mu \in [0, 1], \\ 3\mu, & \text{if } \mu \in (1, \infty). \end{cases} \quad (58)$$

Next, we define $\alpha, \eta: \mathcal{U} \times \mathcal{U} \times \text{int}(\mathcal{G}) \rightarrow [0, \infty)$ and $\phi, \psi: [0, +\infty) \rightarrow [0, +\infty)$, and we have

$$\alpha(\mu, \omega, t') = \begin{cases} 3, & \text{if } \mu, \omega \in [0, 1], \text{ and } t' \gg \theta, \\ \frac{3}{4}, & \text{otherwise,} \end{cases} \quad (59)$$

$$\eta(\mu, \omega, t') = \begin{cases} 2, & \text{if } \mu, \omega \in [0, 1], \text{ and } t' \gg \theta, \\ \frac{2}{5}, & \text{otherwise,} \end{cases} \quad (60)$$

$$\psi(\xi) = \frac{2\xi}{3}, \quad \text{and } \phi(\xi) = \frac{\xi}{3}, \quad \forall \xi \in [0, +\infty). \quad (61)$$

However, $(\mathcal{U}, \ddot{M}_\alpha, *)$ is a \mathcal{G} -complete FCM space. Now, first, we have to show that h is α -admissible w.r.t η . Since in (59) and (60), $\alpha(\mu, \omega, t') \geq 1$ and $\eta(\mu, \omega, t') \geq 1$ for all $\mu, \omega \in [0, 1]$ and $t' \gg \theta$ and $\alpha(\mu, \omega, t') \geq \eta(\mu, \omega, t')$, which shows that α is admissible w.r.t η and h is an α -admissible w.r.t η , that is, $\alpha(h\mu, h\omega, t') \geq \eta(h\mu, h\omega, t')$ for all $\mu, \omega \in [0, 1]$ and $t' \gg \theta$. On the other hand, $h(\mu) \leq 1$ for all $\mu \in [0, 1]$. If $\mu \notin [0, 1]$, then, again by using (59) and (60), we have that $\alpha(\mu, h\mu, t') = 3/4$ and $\eta(\mu, h\mu, t') = 2/5$ for $t' \gg \theta$ and so that $\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') = 9/16 < 1$ and $\eta(\mu, h\mu, t')\eta(\omega, h\omega, t') = 4/25 < 1$, for $t' \gg \theta$, which is contradiction. Similarly, if $\omega \notin [0, 1]$, then again we get that $\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') < 1$ and $\eta(\mu, h\mu, t')\eta(\omega, h\omega, t') < 1$, for $t' \gg \theta$, which is contradiction. Hence, $\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') \geq 1$ and $\eta(\mu, h\mu, t')\eta(\omega, h\omega, t') \geq 1$, for $t' \gg \theta$. It follows that the mapping h is both α -admissible and

η -admissible and $\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') \geq \eta(\mu, h\mu, t')\eta(\omega, h\omega, t')$, for $t' \gg \theta$.

Now, if (μ_ℓ) in a sequence in \mathcal{U} such that $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq 1$ and $\eta(\mu_\ell, \mu_{\ell+1}, t') \geq 1$ for all $\ell \in \mathbb{N} \cup \{0\}$ and $\mu_\ell \rightarrow z$ as $\ell \rightarrow +\infty$, then $(\mu_\ell) \subset [0, 1]$ and hence $z \in [0, 1]$. This implies that $\alpha(\mu_\ell, z, t') \geq 1$ and $\eta(\mu_\ell, z, t') \geq 1$ for all $\ell \in \mathbb{N}$ and $t' \gg \theta$. Next, we prove that inequality (44) of Theorem 3 is satisfied by using (59)–(61), for all $\mu, \omega \in [0, 1]$ and $t' \gg \theta$:

$$\begin{aligned} \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1\right) &= \psi\left(\frac{|h\mu - h\omega|}{t'}\right), \\ &= \frac{2|h\mu - h\omega|}{3t'} = \frac{2|\mu - \omega|}{9t'} \\ &\leq \frac{2|\mu - \omega|}{3t'} - \frac{|\mu - \omega|}{3t'} \\ &= \psi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right). \end{aligned} \quad (62)$$

That is,

$$\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') \geq \eta(\mu, h\mu, t')\eta(\omega, h\omega, t') \quad \text{for } t' \gg \theta,$$

$$\begin{aligned} &\Rightarrow \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1\right) \\ &\leq \psi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) \quad \text{for } t' \gg \theta. \end{aligned} \quad (63)$$

Hence, the conditions of Theorem 3 are satisfied, and the mapping h has a FP in \mathcal{U} , i.e., 0.

Note. In special case, by using metric space, the main result of Dutta and Choudhury [37] for the mapping h is not applicable; if we put $\mu = 3$ and $\omega = 4$, then we have

$$\psi(d(h\mu, h\omega)) = 2 > \frac{1}{3} = \psi(d(\mu, \omega)) - \phi(d(\mu, \omega)). \quad (64)$$

Note. if we take $\eta(\mu, \omega, t') = 1$ in Theorem 3, then we obtain the following two corollaries.

Corollary 3. Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a \mathcal{G} -complete FCM space, and let $h: \mathcal{U} \rightarrow \mathcal{U}$ be an α -admissible. Suppose that $\exists \psi \in \Psi$ and $\phi \in \Phi_0$, so that

$$\begin{aligned} \alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') &\geq 1 \\ &\Rightarrow \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1\right) \leq \psi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right), \end{aligned} \quad (65)$$

$\forall \mu, \omega \in \mathcal{U}$ and $t' \gg \theta$. Let the following axioms hold:

- (i) $\exists \mu_0 \in \mathcal{U}$ such that $\alpha(\mu_0, h\mu_0, t') \geq 1$, for $t' \gg \theta$
- (ii) Any sequence (μ_ℓ) in \mathcal{U} with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \geq 1$, $\forall \ell \in \mathbb{N}$, $t' \gg \theta$ and $\mu_\ell \rightarrow z \in \mathcal{U}$ as $\ell \rightarrow +\infty$, then $\alpha(\mu_\ell, z, t') \geq 1$ and $\alpha(z, hz, t') \geq 1$, $\forall \ell \in \mathbb{N}$, $t' \gg \theta$

Then, h has a FP in \mathcal{U} .

Corollary 4. Let $(\mathcal{U}, \ddot{M}_\alpha, *)$ be a \mathcal{G} -complete FCM space, and let $h: \mathcal{U} \rightarrow \mathcal{U}$ be an α -admissible. Suppose that $\exists \psi \in \Psi$ and $\phi \in \Phi_0$, so that

$$\alpha(\mu, h\mu, t')\alpha(\omega, h\omega, t') \leq 1 \Rightarrow \psi\left(\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1\right) \leq \psi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right) - \phi\left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1\right), \quad (66)$$

$\forall \mu, \omega \in \mathcal{U}$ and $t' \gg \theta$. Let the following conditions hold:

- (i) $\exists \mu_0 \in \mathcal{U}$ such that $\alpha(\mu_0, h\mu_0, t') \leq 1$, for $t' \gg \theta$
- (ii) Any sequence (μ_ℓ) in \mathcal{U} with $\alpha(\mu_\ell, \mu_{\ell+1}, t') \leq 1$, $\forall \ell \in \mathbb{N}$, $t' \gg \theta$ and $\mu_\ell \rightarrow z \in \mathcal{U}$ as $\ell \rightarrow +\infty$, then $\alpha(\mu_\ell, z, t') \leq 1$ and $\alpha(z, hz, t') \leq 1$, $\forall \ell \in \mathbb{N}$, $t' \gg \theta$

Then, h has a FP in \mathcal{U} .

4. Supportive Application

In this section, we present an integral type application for FP to support our result.

Let

$$\mu(r) = f(\tau) + \int_0^r \kappa(\tau, s, \mu(s)) ds, \quad \forall \tau \in [0, r] = I_0, \quad (67)$$

where $r > 0$. Let $\mathcal{U} = C(I_0, \mathcal{R})$ be a Banach space of all continuous functions defined on I_0 . The induced metric is defined by

$$\ddot{d}(\mu, \omega) = \sup_{\tau \in I_0} |\mu(\tau) - \omega(\tau)|, \quad \mu, \omega \in C(I_0, \mathcal{R}). \quad (68)$$

Let $\varrho_1 * \varrho_2 = \varrho_1 \varrho_2$, $\forall \varrho_1, \varrho_2 \in [0, 1]$ and consider the fuzzy metric be defined as follows:

$$\ddot{M}_\alpha(\mu, \omega, t') = \frac{t'}{t' + \ddot{d}(\mu, \omega)}, \quad (69)$$

for $t' > 0$ and $\mu, \omega \in C(I, \mathcal{R})$. The space $(C(I_0, \mathcal{R}), \ddot{M}_\alpha, *_3)$ is \mathcal{G} -complete FM space indeed by the Banach space $C(I_0, \mathcal{R})$. Now, here we discuss an integral type application

for FCM space and prove the existing solution for the integral equation (67).

Theorem 4. Let $h: C(I_0, \mathcal{R}) \rightarrow C(I_0, \mathcal{R})$ be an integral operator defined as follows:

$$h(\mu(\tau)) = f(\tau) + \int_0^\tau \kappa(\tau, s, \mu(s)) ds, \quad (70)$$

where $\tau \in I_0$, $f \in C(I_0, \mathcal{R})$, and $\kappa: I_0 \times I_0 \times \mathcal{R} \rightarrow \mathcal{R}$, that is, $\kappa \in C(I_0 \times I_0 \times \mathcal{R}, \mathcal{R})$ satisfies the following: $\exists g: I_0 * I_0 \rightarrow [0, +\infty)$ such that, $\forall r \in I_0$, $g(\tau, \cdot) \in L^1(I_0, \mathcal{R})$, $\forall \mu, \omega \in C(I_0, \mathcal{R})$, and $\forall \tau, s \in I_0$, and we have that

$$|\kappa(\tau, s, \mu(s)) - \kappa(\tau, s, \omega(s))| \leq g(\tau, s) \mathbf{B}(h, \mu, \omega), \quad (71)$$

where

$$\mathbf{B}(h, \mu, \omega) = \min \left\{ \begin{array}{l} \sup_{s \in I_0} |\mu(s) - \omega(s)|, \sup_{s \in I_0} |\mu(s) - h(\mu(s))|, \\ \sup_{s \in I_0} |\omega(s) - h(\mu(s))|, \sup_{s \in I_0} |\omega(s) - h(\omega(s))| \end{array} \right\}, \quad (72)$$

where $\int_0^\tau g(\tau, s) ds$ is bounded on I_0 and the $\sup_{\tau \in I_0} \int_0^\tau g(\tau, s) ds \leq \lambda < 1$. Then, the integral equation (67) has a solution $\mu^* \in C(I_0, \mathcal{R})$.

Proof. The integral operator h is defined in (70). Now, we have to apply Corollary 1, for all $\mu, \omega \in C(I_0, \mathcal{R})$, and by the view of (69)–(71), we have

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 &= \frac{d(h\mu, h\omega)}{t'}, \\
&= \frac{\sup_{\tau \in I_0} |h(\mu(\tau)) - h(\omega(\tau))|}{t'} \\
&= \frac{1}{t'} \sup_{\tau \in I_0} \int_0^\tau |\kappa(\tau, s, \mu(s)) - \kappa(\tau, s, \omega(s))| ds \\
&\leq \frac{1}{t'} \sup_{\tau \in I_0} \int_0^\tau g(\tau, s) \mathbf{B}(h, \mu, \omega) ds \\
&= \frac{1}{t'} \mathbf{B}(h, \mu, \omega) \sup_{\tau \in I_0} \int_0^\tau g(\tau, s) ds.
\end{aligned} \quad (73)$$

Then, we have the following four cases:

- (1) If $\sup_{s \in I_0} |\mu(s) - \omega(s)|$ is the minimum term in (72), then $\mathbf{B}(h, \mu, \omega) = \sup_{s \in I_0} |\mu(s) - \omega(s)|$. Now, by the view

of (69), (71), and (73), for $t' \gg \theta$, we have

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 &\leq \frac{1}{t'} \sup_{s \in I_0} |\mu(s) - \omega(s)| \sup_{\tau \in I_0} \int_0^\tau g(\tau, s) ds, \\
&\leq \frac{1}{t'} \ddot{d}(\mu, \omega) \sup_{\tau \in I_0} \int_0^\tau g(\tau, s) ds \\
&\leq \lambda \left(\frac{1}{\ddot{M}_\alpha(\mu, \omega, t')} - 1 \right) \\
&= \lambda \left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1 \right),
\end{aligned} \quad (74)$$

where $\Omega(h, \mu, \omega, t') = \ddot{M}_\alpha(\mu, \omega, t')$, for $t' \gg \theta$.

- (2) If $\sup_{s \in I_0} |\mu(s) - h(\mu(s))|$ is the minimum term in (72), then $\mathbf{B}(h, \mu, \omega) = \sup_{s \in I_0} |\mu(s) - h(\mu(s))|$. Now,

by the view of (69), (71), and (73), for $t' \gg \theta$, we have

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 &\leq \frac{1}{t'} \sup_{s \in I_0} |\mu(s) - h(\mu(s))| \sup_{\tau \in I_0} \int_0^\tau g(\tau, s) ds, \\
&\leq \frac{1}{t'} \ddot{d}(\mu, h\mu) \sup_{\tau \in I_0} \int_0^\tau g(\tau, s) ds \\
&\leq \lambda \left(\frac{1}{\ddot{M}_\alpha(\mu, h\mu, t')} - 1 \right) \\
&= \lambda \left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1 \right),
\end{aligned} \quad (75)$$

where $\Omega(h, \mu, \omega, t') = \ddot{M}_\alpha(\mu, h\mu, t')$, for $t' \gg \theta$.

- (3) If $\sup_{s \in I_0} |\omega(s) - h(\mu(s))|$ is the minimum term in (72), then $\mathbf{B}(h, \mu, \omega) = \sup_{s \in I_0} |\omega(s) - h(\mu(s))|$. Now, by

the view of (69), (71), and (73), for $t' \gg \theta$, we have

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 &\leq \frac{1}{t'} \sup_{s \in I_0} |\omega(s) - h(\mu(s))| \sup_{\tau \in I_0} \int_0^\tau g(\tau, s) ds, \\
&\leq \frac{1}{t'} \ddot{d}(\omega, h\mu) \sup_{\tau \in I_0} \int_0^\tau g(\tau, s) ds \\
&\leq \lambda \left(\frac{1}{\ddot{M}_\alpha(\omega, h\mu, t')} - 1 \right) \\
&= \lambda \left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1 \right),
\end{aligned} \quad (76)$$

where $\Omega(h, \mu, \omega, t') = \ddot{M}_\alpha(\omega, h\mu, t')$, for $t' \gg \theta$.

- (4) If $\sup_{s \in I_0} |\omega(s) - h(\omega(s))|$ is the minimum term in (72), then $\mathbf{B}(h, \mu, \omega) = \sup_{s \in I_0} |\omega(s) - h(\omega(s))|$. Now,

by the view of (69), (71), and (73), for $t' \gg \theta$, we have

$$\begin{aligned}
\frac{1}{\ddot{M}_\alpha(h\mu, h\omega, t')} - 1 &\leq \frac{1}{t'} \sup_{s \in I_0} |\omega(s) - h(\omega(s))| \sup_{\tau \in I_0} \int_0^\tau g(\tau, s) ds, \\
&\leq \frac{1}{t'} \ddot{d}(\omega, h\omega) \sup_{\tau \in I_0} \int_0^\tau g(\tau, s) ds \\
&\leq \lambda \left(\frac{1}{\ddot{M}_\alpha(\omega, h\omega, t')} - 1 \right) \\
&= \lambda \left(\frac{1}{\Omega(h, \mu, \omega, t')} - 1 \right),
\end{aligned} \quad (77)$$

where $\Omega(h, \mu, \omega, t') = \ddot{M}_\alpha(\omega, h\omega, t')$, for $t' \gg \theta$.

Hence, in all the cases, it is obvious that (26) holds with $\phi(c) = \lambda c, \forall c \geq 0$ and $\alpha(\mu, \omega, t') = 1, \forall \mu, \omega \in C(I_0, \mathcal{R})$ and $t' \gg \theta$. As we have mentioned above, $C(I_0, \mathcal{R})$ is complete and then the FCM space $(C(I_0, \mathcal{R}), \ddot{M}_\alpha, *_3)$ is \mathcal{G} -complete in which \ddot{M}_α is triangular. The other conditions of Corollary 1 are fulfilled immediately. We deduce that the operator h has a FP $\mu^* \in C(I_0, \mathcal{R})$ which is the required solution of the integral equation (67). \square

5. Conclusion

In this paper, we have presented the mappings, and h is α -admissible w.r.t η and α - ϕ -fuzzy cone contraction in FCM

spaces. Using this kind of contractions, we proved FP theorems for \mathcal{G} -complete FCM space in the sense of George and Veeramani with an illustrative example. Moreover, some extended results in the form of corollaries are discussed. An application is presented to support the concepts defined in the paper. This integral type application is also illustrative of how fuzzy metrics can be used in other integral type operators.

Data Availability

No data set were generated or analyzed during this current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors have equally contributed to the final manuscript. All the authors have read and approved the manuscript.

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Research Article

Some New Observations on Wijsman \mathcal{J}_2 -Lacunary Statistical Convergence of Double Set Sequences in Intuitionistic Fuzzy Metric Spaces

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Received 19 July 2021; Accepted 8 October 2021; Published 18 October 2021

Academic Editor: Huseyin Isik

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In this study, we investigate the notions of the Wijsman \mathcal{J}_2 -statistical convergence, Wijsman \mathcal{J}_2 -lacunary statistical convergence, Wijsman strongly \mathcal{J}_2 -lacunary convergence, and Wijsman strongly \mathcal{J}_2 -Cesàro convergence of double sequence of sets in the intuitionistic fuzzy metric spaces (briefly, IFMS). Also, we give the notions of Wijsman strongly \mathcal{J}_2^* -lacunary convergence, Wijsman strongly \mathcal{J}_2 -lacunary Cauchy, and Wijsman strongly \mathcal{J}_2^* -lacunary Cauchy set sequence in IFMS and establish noteworthy results.

1. Introduction and Background

Statistical convergence was firstly examined by Henry Fast [1]. This notion was redefined for double sequences by Mursaleen and Edely [2]. As a consequence of the study of ideal convergence defined by Kostyrko et al. [3], there has been valuable research to discover summability works of the classical theories. Das et al. [4] rethought \mathcal{J} -convergence of double sequences and worked some features of this convergence. Ideal convergence became a noteworthy topic in summability theory after the studies of [5–11].

Fridy and Orhan [12] examined the notion of lacunary statistical convergence by using lacunary sequence. The publication of the paper affected deeply all the scientific fields. Çakan and Altay [13] demonstrated multidimensional similarities of the conclusions given by Fridy and Orhan [12]. Some works in lacunary statistical convergence can be found in [13–17].

Theory of fuzzy sets (FSs) was firstly given by Zadeh [18]. This work affected deeply all the scientific fields. The theory of FSs has submitted to employ imprecise, vagueness, and inexact data [18]. FSs have been widely implemented in different disciplines and technologies. The theory of FSs cannot always cope with the lack of knowledge of

membership degrees. That is why Atanassov [19] investigated the theory of IFS which is the extension of the theory of FSs. Kramosil and Michalek [20] investigated fuzzy metric space (FMS) utilizing the concepts fuzzy and probabilistic metric space. The FMS as a distance between two points to be a nonnegative fuzzy number was examined by Kaleva and Seikkala [21]. George and Veeramani [22] gave some qualifications of FMS. Some basic features of FMS were given, and significant theorems were proved in [23]. Moreover, FMS has used in practical research studies, for example, decision-making, fixed point theory, and medical imaging. Park [24] generalized FMSs and defined IF metric space (IFMS). Park utilized George and Veeramani's [22] opinion of using t-norm and t-conorm to the FMS meantime describing IFMS and investigating its fundamental properties. The concept of IF-normed spaces (IFNS for shortly) was given by Lael and Nourouzi [25]. In order to have a different topology from the topology generated by the F -norm μ , the condition $\mu + \nu \leq 1$ was omitted from Park's definition. Statistical convergence, ideal convergence, and different features of sequences in INFS were examined by several authors [26–29].

Recently, convergence of sequences of sets was studied by several authors. Nuray and Rhoades [30] presented the

idea of statistical convergence of set sequences and established some essential theorems. Ulusu and Nuray [31] examined the lacunary statistical convergence of sequence of sets. Convergence for sequences of sets became a notable topic in summability theory after the studies of (see [32–38]).

Lacunary statistical convergence and lacunary strongly convergence for sequence of sets in IFMS were worked by Kisi [39]. Further, Wijsman \mathcal{I} -convergence and Wijsman \mathcal{I}^* -convergence for sequence of sets in IFMS were investigated by Esi et al. [40].

Throughout this work, we indicate \mathcal{I}_2 to be the admissible ideal in $\mathbb{N} \times \mathbb{N}$, $\theta_2 = \{(j_u, k_s)\}$ to be a double

lacunary sequence, $(\mathcal{X}, \eta, \nu, *, \diamond)$ to be the IFMS, and F_{wq} to be nonempty closed subsets of \mathcal{X} .

2. Main Results

Definition 1. A sequence $\{F_{wq}\}$ of nonempty closed subsets of \mathcal{X} is known as Wijsman \mathcal{I}_2 -statistical convergent to F or $S(\mathcal{I}_{W_2}^{(\eta, \nu)})$ -convergent to F with regard to IFM (η, ν) , if for every $\xi \in (0, 1)$, $\sigma > 0$, for each $x \in \mathcal{X}$, and for all $p > 0$,

$$\left\{ (s, t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{st} \left| \left\{ (w, q) : w \leq s, q \leq t, \begin{aligned} &|\eta(x, F_{wq}, p) - \eta(x, F, p)| \leq 1 - \xi \\ &\text{or } |\nu(x, F_{wq}, p) - \nu(x, F, p)| \geq \xi \end{aligned} \right\} \right| \geq \sigma \right\} \in \mathcal{I}_2. \quad (1)$$

We demonstrate this symbolically by $F_{wq} S(\mathcal{I}_{W_2}^{(\eta, \nu)}) F$ or $F_{wq} \longrightarrow F(S(\mathcal{I}_{W_2}^{(\eta, \nu)}))$. The set of all Wijsman \mathcal{I}_2 -statistical convergent sequences in IFMS is indicated by $S(\mathcal{I}_{W_2}^{(\eta, \nu)})$.

Example 1. Let $\mathcal{X} = \mathbb{R}^2$ and double sequence $\{F_{wq}\}$ be determined as follows:

$$F_{wq} := \begin{cases} (d, e) \in \mathbb{R}^2 : (d + w)^2 + (e + q)^2 = 1, & \text{if } w \text{ and } q \text{ are square integers,} \\ \{(1, 1)\}, & \text{otherwise.} \end{cases} \quad (2)$$

If $\mathcal{I}_2 = \mathcal{I}_2^\delta$ (\mathcal{I}_2^δ is the class of $K \subset \mathbb{N} \times \mathbb{N}$ with density of K equal to 0), then the sequence $\{F_{wq}\}$ is Wijsman \mathcal{I}_2 -statistical convergent to $F = \{(1, 1)\}$ with regard to IFM (η, ν) .

Definition 2. A sequence $\{F_{wq}\}$ is Wijsman strong \mathcal{I}_2 -Cesàro summable to F or $C_1[\mathcal{I}_{W_2}^{(\eta, \nu)}]$ -summable to F with regard to IFM (η, ν) , if for every $\xi \in (0, 1)$, for each $x \in \mathcal{X}$, and for all $p > 0$,

$$\left\{ (s, t) \in \mathbb{N} \times \mathbb{N} : \frac{1}{st} \sum_{w, q=1,1}^{s,t} \begin{aligned} &|\eta(x, F_{wq}, p) - \eta(x, F, p)| \leq 1 - \xi \\ &\text{or } \frac{1}{st} \sum_{w, q=1,1}^{s,t} |\nu(x, F_{wq}, p) - \nu(x, F, p)| \geq \xi \end{aligned} \right\} \in \mathcal{I}_2. \quad (3)$$

We write $F_{wq} \longrightarrow_{C_1} (\mathcal{I}_{W_2}^{(\eta, \nu)}) F$ or $F_{wq} \longrightarrow F(C_1[\mathcal{I}_{W_2}^{(\eta, \nu)}])$.

Example 2. Let $\mathcal{X} = \mathbb{R}^2$ and double sequence $\{F_{wq}\}$ be determined as follows:

$$F_{wq} := \begin{cases} (d, e) \in \mathbb{R}^2 : (d + 1)^2 + e^2 = \frac{1}{wq}; & \text{if } w \text{ and } q \text{ are square integers,} \\ \{(0, 1)\}; & \text{otherwise.} \end{cases} \quad (4)$$

If $\mathcal{J}_2 = \mathcal{J}_2^f$ (\mathcal{J}_2^f is the class of finite subsets of $\mathbb{N} \times \mathbb{N}$), then sequence $\{F_{wq}\}$ is Wijsman strong \mathcal{J}_2 -Cesàro summable to $F = \{(0, 1)\}$ with regard to IFM (η, ν) .

Definition 3. The sequence $\{F_{wq}\}$ is known as Wijsman \mathcal{J}_2 -lacunary statistically convergent to F or $S_{\theta_2}(\mathcal{J}_{W_2}^{(\eta, \nu)})$ -convergent to F with regard to IFM (η, ν) , if for every $\xi \in (0, 1)$, $\sigma > 0$, for each $x \in \mathcal{X}$, and for all $p > 0$,

$$\left\{ (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \left| \left\{ \begin{array}{l} (w, q) \in I_{us}: |\eta(x, F_{wq}, p) - \eta(x, F, p)| \leq 1 - \xi \\ \text{or } |\nu(x, F_{wq}, p) - \nu(x, F, p)| \geq \xi \end{array} \right\} \right| \geq \sigma \right\} \in \mathcal{J}_2. \quad (5)$$

In that case, we write $F_{wq} \xrightarrow{S_{\theta_2}(\mathcal{J}_{W_2}^{(\eta, \nu)})} F$ or $F_{wq} \longrightarrow F(S_{\theta_2}(\mathcal{J}_{W_2}^{(\eta, \nu)}))$.

Example 3. Let $\mathcal{X} = \mathbb{R}^2$ and double sequence $\{F_{wq}\}$ be determined as follows:

$$F_{wq} := \begin{cases} (d, e) \in \mathbb{R}^2: (d - w)^2 + (e + q)^2 = 1, & \text{if } (w, q) \in I_{us}; w \text{ and } q \text{ are square integers,} \\ \{(-1, 1)\}, & \text{otherwise.} \end{cases} \quad (6)$$

If we take $\mathcal{J}_2 = \mathcal{J}_2^\delta$, then the sequence $\{F_{wq}\}$ is Wijsman \mathcal{J}_2 -lacunary statistical convergent to $F = \{(-1, 1)\}$ with regard to IFM (η, ν) .

Definition 4. A double sequence $\{F_{wq}\}$ is Wijsman strong \mathcal{J}_2 -lacunary summable to F or $N_{\theta_2}[\mathcal{J}_{W_2}^{(\eta, \nu)}]$ -summable to F with regard to IFM (η, ν) , if for every $\xi \in (0, 1)$, for all $p > 0$, and for each $x \in \mathcal{X}$,

$$\left\{ (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w, q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| \leq 1 - \xi \right. \\ \left. \text{or } \frac{1}{h_{us}} \sum_{(w, q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| \geq \xi \right\} \in \mathcal{J}_2. \quad (7)$$

We shall write $F_{wq} \xrightarrow{N_{\theta_2}[\mathcal{J}_{W_2}^{(\eta, \nu)}]} F$ or $F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{J}_{W_2}^{(\eta, \nu)}])$.

Example 4. Let $\mathcal{X} = \mathbb{R}^2$ and double sequence $\{F_{wq}\}$ be determined as follows:

$$F_{wq} := \begin{cases} (d, e) \in \mathbb{R}^2: d^2 + (e - 1)^2 = \frac{1}{wq}; & \text{if } (w, q) \in I_{us}; w \text{ and } q \text{ are square integers,} \\ \{(1, 0)\}; & \text{otherwise.} \end{cases} \quad (8)$$

If we take $\mathcal{J}_2 = \mathcal{J}_2^f$, then the sequence $\{F_{wq}\}$ is Wijsman strong \mathcal{J}_2 -lacunary summable to $F = \{(1, 0)\}$ with regard to IFM (η, ν) .

$$F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{J}_{W_2}^{(\eta, \nu)}]) \implies F_{wq} \longrightarrow F(S_{\theta_2}(\mathcal{J}_{W_2}^{(\eta, \nu)})). \quad (9)$$

Theorem 1. Let $\theta_2 = \{(j_u, k_s)\}$ be a double lacunary sequence. Then,

Proof. Let $\xi \in (0, 1)$ and $F_{wq} \xrightarrow{N_{\theta_2}[\mathcal{J}_{W_2}^{(\eta, \nu)}]} F$. At that time, for every $x \in \mathcal{X}$, we acquire

$$\begin{aligned}
& \sum_{(w,q) \in I_{us}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \\
& \geq \sum_{\substack{(w,q) \in I_{us} \\ \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \text{ or} \\ \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \\
& \geq \xi \cdot \left| \left\{ (w, q) \in I_{us} : \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right|,
\end{aligned} \tag{10}$$

and so

$$\begin{aligned}
& \frac{1}{\xi h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \\
& \geq \frac{1}{h_{us}} \left| \left\{ (w, q) \in I_{us} : \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right|.
\end{aligned} \tag{11}$$

Then, for any $\sigma > 0$ and for every $x \in \mathcal{X}$,

$$\begin{aligned}
& \left\{ (u, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \left| \left\{ (w, q) \in I_{us} : \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \right. \right. \right. \\
& \quad \left. \left. \left. \text{or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \geq \sigma \right\} \\
& \subseteq \left\{ (u, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq (1 - \xi) \cdot \sigma \right. \\
& \quad \left. \text{or } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \cdot \sigma \right\}
\end{aligned} \tag{12}$$

This proof is concluded. \square

The set of all bounded double sequences of sets in IFMS is indicated by $L_{\infty}^2(\mathcal{X})$.

Theorem 2. Let $\theta_2 = \{(j_u, k_s)\}$ be a double lacunary sequence. Then, $\{F_{wq}\}$ is bounded ($\{F_{wq}\} \in L_{\infty}^2(\mathcal{X})$) and

$$F_{wq} \longrightarrow F(S_{\theta_2}(\mathcal{F}_{W_2}^{(\eta, \nu)})) \implies F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{F}_{W_2}^{(\eta, \nu)}]). \tag{13}$$

Proof. Presume that $F_{wq} \longrightarrow F(S_{\theta_2}(\mathcal{F}_{W_2}^{(\eta, \nu)}))$ and $\{F_{wq}\} \in L_{\infty}^2$. At this point, there is an $H > 0$ such that

$$\left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \geq 1 - H \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \leq H, \tag{14}$$

for every $x \in \mathcal{X}$ and all $w, q \in \mathbb{N}$. Given $\xi \in (0, 1)$, we obtain

$$\begin{aligned}
& \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \\
&= \frac{1}{h_{us}} \sum_{\substack{(w,q) \in I_{us} \\ \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - (\xi/2) \text{ or} \\ \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq (\xi/2)}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \\
&+ \frac{1}{h_{us}} \sum_{\substack{(w,q) \in I_{us} \\ \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| > 1 - (\xi/2) \text{ or} \\ \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| < (\xi/2)}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \\
&\leq \frac{H}{h_{us}} \left| \left\{ \begin{aligned} & (w, q) \in I_{us}: \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \frac{\xi}{2} \\ & \text{or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \frac{\xi}{2} \end{aligned} \right\} \right| + \frac{\xi}{2}.
\end{aligned} \tag{15}$$

As a consequence, for each $x \in \mathcal{X}$, we get

$$\begin{aligned}
& \left\{ \begin{aligned} & (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \\ & \text{or } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \end{aligned} \right\} \\
&\subseteq \left\{ (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \left| \left\{ \begin{aligned} & (w, q) \in I_{us}: \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \frac{\xi}{2} \\ & \text{or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \frac{\xi}{2} \end{aligned} \right\} \right| \geq \frac{\xi}{2H} \right\} \in \mathcal{F}_2.
\end{aligned} \tag{16}$$

This proof is concluded. \square

Proof. Presume that $\liminf_u q_u > 1$ and $\liminf_s q_s > 1$. Then, there are $\zeta > 0, \psi > 0$ such that

$$q_u \geq 1 + \zeta \text{ and } q_s \geq 1 + \psi, \tag{18}$$

Corollary 1. We have the following result:

$$\{S_{\theta_2}(\mathcal{F}_{W_2}^{(\eta, \nu)})\} \cap L_{\infty}^2(\mathcal{X}) = \{N_{\theta_2}[\mathcal{F}_{W_2}^{(\eta, \nu)}]\} \cap L_{\infty}^2(\mathcal{X}). \tag{17}$$

for sufficiently large u, s which gives that

$$\frac{h_{us}}{j_u k_s} \geq \frac{\zeta \psi}{(1 + \zeta)(1 + \psi)}. \tag{19}$$

Theorem 3. If $\liminf_u q_u > 1$ and $\liminf_s q_s > 1$, then $F_{wq} \longrightarrow F(S(\mathcal{F}_{W_2}^{(\eta, \nu)}))$ implies $F_{wq} \longrightarrow F(S_{\theta_2}^s(\mathcal{F}_{W_2}^{(\eta, \nu)}))$.

Assume that $F_{wq} \longrightarrow F(S(\mathcal{F}_{W_2}^{(\eta, \nu)}))$. For each $\xi \in (0, 1)$, for all $p > 0$, and for each $x \in \mathcal{X}$, we have

$$\begin{aligned}
& \frac{1}{j_u k_s} \left| \left\{ (w, q): w \leq j_u, q \leq k_s, \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \right. \right. \\
& \quad \left. \left. \text{or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \\
& \geq \frac{1}{j_u k_s} \left| \left\{ (w, q) \in I_{us}: \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \right. \right. \\
& \quad \left. \left. \text{or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \\
& = \frac{h_{us}}{j_u k_s} \frac{1}{h_{us}} \left| \left\{ (w, q) \in I_{us}: \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \right. \right. \\
& \quad \left. \left. \text{or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \\
& \geq \frac{\zeta \psi}{(1 + \zeta)(1 + \psi)} \frac{1}{h_{us}} \left| \left\{ (w, q) \in I_{us}: \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \right. \right. \\
& \quad \left. \left. \text{or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right|.
\end{aligned} \tag{20}$$

Thus, for any $\sigma > 0$,

$$\begin{aligned}
& \left\{ (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \left| \left\{ (w, q) \in I_{us}: \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \right. \right. \right. \\
& \quad \left. \left. \text{or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \geq \sigma \right\} \\
& \subseteq \left\{ (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{j_u k_s} \left| \left\{ (w, q): w \leq j_u, q \leq k_s, \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \right. \right. \right. \\
& \quad \left. \left. \text{or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \geq \frac{\zeta \psi \sigma}{(1 + \zeta)(1 + \psi)} \right\}.
\end{aligned} \tag{21}$$

Hence, by our supposition, the set on the right side belongs to \mathcal{J}_2 , and clearly the set on the left side belongs to \mathcal{J}_2 . As a result, we obtain $F_{wq} \longrightarrow F(S_{\theta_2}(\mathcal{J}_{W_2}^{(\eta, \nu)}))$. \square

Proof. Presume that $\limsup_u q_u < \infty$ and $\limsup_s q_s < \infty$. Then, there are $P, R > 0$ such that $q_u < P$ and $q_s < R$ for all u and s . Assume that $F_{wq} \longrightarrow F(S_{\theta_2}(\mathcal{J}_{W_2}^{(\eta, \nu)}))$ and let

Theorem 4. If $\limsup_u q_u < \infty$ and $\limsup_s q_s < \infty$, then $F_{wq} \longrightarrow F(S_{\theta_2}(\mathcal{J}_{W_2}^{(\eta, \nu)}))$ implies $F_{wq} \longrightarrow F(S(\mathcal{J}_{W_2}^{(\eta, \nu)}))$.

$$H_{us} := \left| \left\{ (w, q) \in I_{us}: \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right|. \tag{22}$$

Since $F_{wq} \longrightarrow F(S_{\theta_2}(\mathcal{J}_{W_2}^{(\eta, \nu)}))$, it holds for each $\xi \in (0, 1)$, $\sigma > 0$, for every $x \in \mathcal{X}$, and for all $p > 0$,

$$\begin{aligned}
& \left\{ (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \left| \left\{ (w, q) \in I_{us}: \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \right. \right. \right. \\
& \quad \left. \left. \text{or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \geq \sigma \right\} \\
& = \left\{ (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{H_{us}}{h_{us}} \geq \sigma \right\} \in \mathcal{J}_2.
\end{aligned} \tag{23}$$

So, we can select positive integers $u_0, s_0 \in \mathbb{N}$ such that $H_{us}/h_{us} < \sigma$ for all $u \geq u_0, s \geq s_0$. Now, take

$$T := \max\{H_{us} : 1 \leq u \leq u_0, 1 \leq s \leq s_0\}, \quad (24)$$

and let m and n be integers providing $j_{u-1} < m \leq j_u$ and $k_{s-1} < n \leq k_s$. Then, for every $\xi > 0$ and each $x \in \mathcal{X}$, we get

$$\begin{aligned} & \frac{1}{mn} \left| \left\{ (w, q) : w \leq m, q \leq n, \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \\ & \leq \frac{1}{j_{u-1}k_{s-1}} \left| \left\{ (w, q) : w \leq j_u, q \leq k_s, \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \\ & = \frac{1}{j_{u-1}k_{s-1}} \{H_{11} + H_{12} + H_{21} + H_{22} + \cdots + H_{u_0 s_0} + \cdots + H_{us}\} \\ & \leq \frac{u_0 s_0}{j_{u-1}k_{s-1}} \left(\max_{\substack{1 \leq w \leq u_0 \\ 1 \leq q \leq s_0}} \{H_{wq}\} \right) + \frac{1}{j_{u-1}k_{s-1}} \left\{ h_{u_0(s_0+1)} \frac{H_{u_0(s_0+1)}}{h_{u_0(s_0+1)}} + h_{(u_0+1)s_0} \frac{H_{(u_0+1)s_0}}{h_{(u_0+1)s_0}} + h_{(u_0+1)(s_0+1)} \frac{H_{(u_0+1)(s_0+1)}}{h_{(u_0+1)(s_0+1)}} + \cdots + h_{us} \frac{H_{us}}{h_{us}} \right\} \\ & \leq \frac{u_0 s_0 T}{j_{u-1}k_{s-1}} + \frac{1}{j_{u-1}k_{s-1}} \left(\sup_{\substack{u > u_0 \\ s > s_0}} \frac{H_{us}}{h_{us}} \right) \left(\sum_{\substack{w \geq u_0 \\ q \geq s_0}} h_{wq} \right) \\ & \leq \frac{u_0 s_0 T}{j_{u-1}k_{s-1}} + \sigma \frac{(j_u - j_{u_0})(k_s - k_{s_0})}{j_{u-1}k_{s-1}} \leq \frac{u_0 s_0 T}{j_{u-1}k_{s-1}} + \sigma q_u q_s \leq \frac{u_0 s_0 T}{j_{u-1}k_{s-1}} + \sigma PR. \end{aligned} \quad (25)$$

Since $j_{u-1}k_{s-1} \rightarrow \infty$ as $m, n \rightarrow \infty$, it concludes that for each $x \in \mathcal{X}$,

$$\frac{1}{mn} \left| \left\{ (w, q) : w \leq m, q \leq n, \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \rightarrow 0, \quad (26)$$

and as a result for any $\sigma_1 > 0$, the set

$$\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} \left| \left\{ (w, q) : w \leq m, q \leq n, \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| \leq 1 - \xi \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \geq \xi \right\} \right| \geq \sigma_1 \right\} \in \mathcal{J}_2. \quad (27)$$

It gives that $F_{wq} \rightarrow F(S(\mathcal{J}_{W_2}^{(\eta, \nu)}))$.

□

Theorem 5. Let θ_2 be a double lacunary sequence. If

$$1 < \liminf_u q_u < \limsup_u uq < \infty, \text{ and } 1 < \liminf_s q_s < \limsup_s sq < \infty, \quad (28)$$

then $F_{wq} \rightarrow F(S_{\theta_2}(\mathcal{J}_{W_2}^{(\eta, \nu)}))$ if $F_{wq} \rightarrow F(S(\mathcal{J}_{W_2}^{(\eta, \nu)}))$.

Proof. It obvious from Theorem 3 and Theorem 4. □

Theorem 6. Let \mathcal{F}_2 be a strongly admissible ideal providing feature (AP_2) , $\theta_2 \in \mathcal{F}(\mathcal{F}_2)$. If $\{F_{wq}\} \in S(\mathcal{F}_{W_2}^{(\eta, \nu)}) \cap S_{\theta_2}(\mathcal{F}_{W_2}^{(\eta, \nu)})$, then

$$S(\mathcal{F}_{W_2}^{(\eta, \nu)}) - \lim_{w, q \rightarrow \infty} F_{wq} = S_{\theta_2}(\mathcal{F}_{W_2}^{(\eta, \nu)}) - \lim_{w, q \rightarrow \infty} F_{wq}. \quad (29)$$

Proof. Presume that $S(\mathcal{F}_{W_2}^{(\eta, \nu)}) - \lim_{w, q \rightarrow \infty} F_{wq} = A$ and $S_{\theta_2}(\mathcal{F}_{W_2}^{(\eta, \nu)}) - \lim_{w, q \rightarrow \infty} F_{wq} = B$ and $A \neq B$. Let

$$0 < \xi < \frac{1}{2} |\eta(x, A, p) - \eta(x, B, p)|, \quad \text{and } 0 < \xi < \frac{1}{2} |\nu(x, A, p) - \nu(x, B, p)|, \quad (30)$$

for every $x \in \mathcal{X}$. Since \mathcal{F}_2 provides the feature (AP_2) , then there is $M \in \mathcal{F}(\mathcal{F}_2)$ (i.e., $\mathbb{N} \times \mathbb{N} \setminus M \in \mathcal{F}_2$) such that for every $x \in \mathcal{X}$ and for $(m, n) \in M$,

$$\lim_{m, n \rightarrow \infty} \frac{1}{mn} \left| \left\{ \begin{array}{l} w \leq m, q \leq n: |\eta(x, F_{wq}, p) - \eta(x, A, p)| \leq 1 - \xi \\ \text{or } |\nu(x, F_{wq}, p) - \nu(x, A, p)| \geq \xi \end{array} \right\} \right| = 0. \quad (31)$$

Let

$$\begin{aligned} T &= \left\{ w \leq m, q \leq n: |\eta(x, F_{wq}, p) - \eta(x, A, p)| \leq 1 - \xi, \text{ or } |\nu(x, F_{wq}, p) - \nu(x, A, p)| \geq \xi \right\}, \\ S &= \left\{ w \leq m, q \leq n: |\eta(x, F_{wq}, p) - \eta(x, B, p)| \leq 1 - \xi, \text{ or } |\nu(x, F_{wq}, p) - \nu(x, B, p)| \geq \xi \right\}. \end{aligned} \quad (32)$$

Then, $mn = |T \cup S| \leq |T| + |S|$. This gives that $1 \leq (|T|/mn) + (|S|/mn)$. Since $|S|/mn \leq 1$ and $\lim_{m, n \rightarrow \infty} |T|/mn = 0$, we have to get $\lim_{m, n \rightarrow \infty} |S|/mn = 1$.

Let $M^* = M \cap \theta_2 \in \mathcal{F}(\mathcal{F}_2)$. Then, for every $x \in \mathcal{X}$ and for $(w_k, q_j) \in M^*$, the $w_k q_j$ th term of the statistical limit expression

$$\frac{1}{mn} \left| \left\{ w \leq m, q \leq n: |\eta(x, F_{wq}, p) - \eta(x, B, p)| \leq 1 - \xi, \text{ or } |\nu(x, F_{wq}, p) - \nu(x, B, p)| \geq \xi \right\} \right|, \quad (33)$$

is

$$\begin{aligned} & \frac{1}{w_k q_j} \left| \left\{ (w, q) \in \bigcup_{u, s=1, 1}^{k, j} I_{us}: |\eta(x, F_{wq}, p) - \eta(x, B, p)| \leq 1 - \xi, \text{ or } |\nu(x, F_{wq}, p) - \nu(x, B, p)| \geq \xi \right\} \right| \\ &= \frac{1}{\bigcup_{u, s=1, 1}^{k, j} h_{us}} \bigcup_{u, s=1, 1}^{k, j} \nu_{us} h_{us}, \end{aligned} \quad (34)$$

where

$$\nu_{us} = \frac{1}{h_{us}} \left| \left\{ (w, q) \in I_{us}: |\eta(x, F_{wq}, p) - \eta(x, B, p)| \leq 1 - \xi, \text{ or } |\nu(x, F_{wq}, p) - \nu(x, B, p)| \geq \xi \right\} \right| \xrightarrow{\mathcal{F}_2} 0, \quad (35)$$

because $S_{\theta_2}(\mathcal{F}_{W_2}^{(\eta,\nu)}) - \lim_{w,q \rightarrow \infty} F_{wq} = B$. Since θ_2 is a lacunary sequence, (34) is a regular weighted mean transform of ν_{us} 's, and as a result, it is \mathcal{F}_2 -convergent to 0 as

$k, j \rightarrow \infty$, and also it has a subsequence which is convergent to 0 since \mathcal{F}_2 provides the feature (AP_2) . However, since this a subsequence of

$$\left\{ \frac{1}{mn} \left| \left\{ w \leq m, q \leq n: \left| \eta(x, F_{wq}, p) - \eta(x, B, p) \right| \leq 1 - \xi, \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, B, p) \right| \geq \xi \right\} \right| \right\}_{(m,n) \in M}, \quad (36)$$

we conclude that

$$\left\{ \frac{1}{mn} \left| \left\{ w \leq m, q \leq n: \left| \eta(x, F_{wq}, p) - \eta(x, B, p) \right| \leq 1 - \xi, \text{ or } \left| \nu(x, F_{wq}, p) - \nu(x, B, p) \right| \geq \xi \right\} \right| \right\}_{(m,n) \in M}, \quad (37)$$

which is not convergent to 1. This contradiction indicates that we cannot have $A \neq B$. \square

Theorem 7. If $\liminf q_u > 1$ and $\liminf_s q_s > 1$, then $C_1[\mathcal{F}_{W_2}^{(\eta,\nu)}] \subseteq N_{\theta_2}[\mathcal{F}_{W_2}^{(\eta,\nu)}]$.

Proof. Let $\liminf q_u > 1$ and $\liminf_s q_s > 1$. Then, there are $\zeta, \psi > 0$ such that $q_u \geq 1 + \zeta$ and $q_s \geq 1 + \psi$ for all u and s which gives that

$$\frac{j_u k_s}{h_{us}} \leq \frac{(1 + \zeta)(1 + \psi)}{\zeta \psi}, \quad \text{and} \quad \frac{j_{u-1} k_{s-1}}{h_{us}} \leq \frac{1}{\zeta \psi}. \quad (38)$$

Presume that $F_{wq} \rightarrow F(C_1[\mathcal{F}_{W_2}^{(\eta,\nu)}])$. For each $x \in \mathcal{X}$, we get

$$\begin{aligned} \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| - 1 &= \frac{1}{h_{us}} \sum_{w,q=1,1}^{j_u k_s} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| - \frac{1}{h_{us}} \sum_{w,q=1,1}^{j_{u-1} k_{s-1}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| - 1 \\ &= \frac{j_u k_s}{h_{us}} \left[\frac{1}{j_u k_s} \sum_{w,q=1,1}^{j_u k_s} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| - 1 \right] - \frac{j_{u-1} k_{s-1}}{h_{us}} \left[\frac{1}{j_{u-1} k_{s-1}} \sum_{w,q=1,1}^{j_{u-1} k_{s-1}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| - 1 \right]. \end{aligned} \quad (39)$$

Since $F_{wq} \rightarrow F(C_1[\mathcal{F}_{W_2}^{(\eta,\nu)}])$, then for each

$$\frac{1}{j_u k_s} \sum_{w,q=1,1}^{j_u k_s} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| - 1 \xrightarrow{\mathcal{F}_2} 0, \quad \text{and} \quad \frac{1}{j_{u-1} k_{s-1}} \sum_{w,q=1,1}^{j_{u-1} k_{s-1}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| - 1 \xrightarrow{\mathcal{F}_2} 0. \quad (40)$$

Hence, when the above equality is examined, for every $x \in \mathcal{X}$, we have

$$\frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta(x, F_{wq}, p) - \eta(x, F, p) \right| - 1 \xrightarrow{\mathcal{F}_2} 0. \quad (41)$$

Similarly, we obtain

$$\frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \nu(x, F_{wq}, p) - \nu(x, F, p) \right| \xrightarrow{\mathcal{F}_2} 0. \quad (42)$$

That is, $F_{wq} \rightarrow F(N_{\theta_2}[\mathcal{F}_{W_2}^{(\eta,\nu)}])$. As a result, we obtain $C_1[\mathcal{F}_{W_2}^{(\eta,\nu)}] \subseteq N_{\theta_2}[\mathcal{F}_{W_2}^{(\eta,\nu)}]$. \square

Theorem 8. If $\liminf q_u = 1$ and $\liminf_s q_s = 1$, then $N_{\theta_2}[\mathcal{F}_{W_2}^{(\eta,\nu)}] \subseteq C_1[\mathcal{F}_{W_2}^{(\eta,\nu)}]$.

Proof. Take $\liminf q_u = 1$, $\liminf_s q_s = 1$, and $\{F_{wq}\} \in N_{\theta_2}[\mathcal{F}_{W_2}^{(\eta,\nu)}]$. Then, for every $p > 0$, we acquire

$$H_{us} = \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| \xrightarrow{\mathcal{F}_2} 1, H'_{us} = \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| \xrightarrow{\mathcal{F}_2} 0, \quad (43)$$

as $u, s \rightarrow \infty$. Then, for $\xi > 0$, there are $u_0, s_0 \in \mathbb{N}$ such that $H_{us} < 1 + \xi$ for all $u > u_0, s > s_0$. Also, we can find $K > 0$ such

that $H_{us} < K$ and $H'_{us} < K, u, s = 1, 2, \dots$. Let m and n be an integer with $j_{u-1} < m \leq j_u$ and $k_{s-1} \leq n \leq k_s$. Then,

$$\begin{aligned} & \frac{1}{mn} \sum_{w,q=1,1}^{m,n} |\eta(x, F_{wq}, p) - \eta(x, F, p)| \leq \frac{1}{j_{u-1}k_{s-1}} \sum_{w,q=1,1}^{j_u, k_s} |\eta(x, F_{wq}, p) - \eta(x, F, p)|, \\ & = \frac{1}{j_{u-1}k_{s-1}} \left[\sum_{(w,q) \in I_{11}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| + \dots + \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| \right] \\ & = \sup_{1 \leq u \leq u_0, 1 \leq s \leq s_0} H_{us} \frac{j_{u_0}k_{s_0}}{j_{u-1}k_{s-1}} + \frac{h_{(u_0+1)(s_0+1)}}{j_{u-1}k_{s-1}} H_{(u_0+1)(s_0+1)} + \dots + \frac{h_{us}}{j_{u-1}k_{s-1}} H_{us} < K \frac{j_{u_0}k_{s_0}}{j_{u-1}k_{s-1}} + (1 + \xi) \frac{j_u k_s - j_{u_0} k_{s_0}}{j_{u-1}k_{s-1}}. \end{aligned} \quad (44)$$

Since $j_{u-1}k_{s-1} \rightarrow \infty$ as $m, n \rightarrow \infty$, it follows that $1/mn \sum_{w,q=1}^{m,n} |\eta(x, F_{wq}, p) - \eta(x, F, p)| \xrightarrow{\mathcal{F}_2} 1$. Similarly, we can show that $1/mn \sum_{w,q=1}^{m,n} |\nu(x, F_{wq}, p) - \nu(x, F, p)| \xrightarrow{\mathcal{F}_2} 0$. Hence, $\{F_{wq}\} \in C_1[\mathcal{F}_{W_2}^{(\eta, \nu)}]$. \square

Theorem 9. If $\{F_{wq}\} \in N_{\theta_2}[\mathcal{F}_{W_2}^{(\eta, \nu)}] \cap C_1[\mathcal{F}_{W_2}^{(\eta, \nu)}]$, then $N_{\theta_2}[\mathcal{F}_{W_2}^{(\eta, \nu)}] - \lim F_{wq} = C_1[\mathcal{F}_{W_2}^{(\eta, \nu)}] - \lim F_{wq}$.

Proof. Let $F_{wq} \rightarrow F_1(N_{\theta_2}[\mathcal{F}_{W_2}^{(\eta, \nu)}])$ and $F_{wq} \rightarrow F_2(C_1[\mathcal{F}_{W_2}^{(\eta, \nu)}])$. Assume $r \in \mathbb{N}$ and $\xi > 0$ in such way that $r > 2/\xi$. Then, for any $p > 0$, there are $u_0, s_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta\left(x, F_{wq}, \frac{p}{2}\right) - \eta\left(x, F_1, \frac{p}{2}\right) \right| > 1 - \frac{1}{r}, \quad \text{and} \quad \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \nu\left(x, F_{wq}, \frac{p}{2}\right) - \nu\left(x, F_1, \frac{p}{2}\right) \right| < \frac{1}{r}, \quad (45)$$

for all $u > u_0$ and $s > s_0$. Also, there are $m_0, n_0 \in \mathbb{N}$ such that

$$\frac{1}{mn} \sum_{w,q=1,1}^{m,n} \left| \eta\left(x, F_{wq}, \frac{p}{2}\right) - \eta\left(x, F_2, \frac{p}{2}\right) \right| > 1 - \frac{1}{r}, \quad \text{and} \quad \frac{1}{mn} \sum_{w,q=1,1}^{m,n} \left| \nu\left(x, F_{wq}, \frac{p}{2}\right) - \nu\left(x, F_2, \frac{p}{2}\right) \right| < \frac{1}{r}, \quad (46)$$

for all $m > m_0$ and $n > n_0$. Take $r_1 = \max\{u_0, m_0\}$ and $r_2 = \max\{s_0, n_0\}$. Then, we take $k, t \in \mathbb{N}$ such that

$$\begin{aligned} & \left| \eta\left(x, F_{kt}, \frac{p}{2}\right) - \eta\left(x, F_1, \frac{p}{2}\right) \right| \geq \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} \left| \eta\left(x, F_{wq}, \frac{p}{2}\right) - \eta\left(x, F_1, \frac{p}{2}\right) \right| > 1 - \frac{1}{r}, \\ & \left| \eta\left(x, F_{kt}, \frac{p}{2}\right) - \eta\left(x, F_2, \frac{p}{2}\right) \right| \geq \frac{1}{mn} \sum_{w,q=1,1}^{m,n} \left| \eta\left(x, F_{wq}, \frac{p}{2}\right) - \eta\left(x, F_2, \frac{p}{2}\right) \right| > 1 - \frac{1}{r}. \end{aligned} \quad (47)$$

Therefore, we get

$$|\eta(x, F_1, p) - \eta(x, F_2, p)| \leq \left| \eta\left(x, F_{kt}, \frac{p}{2}\right) - \eta\left(x, F_1, \frac{p}{2}\right) \right| + \left| \eta\left(x, F_{kt}, \frac{p}{2}\right) - \eta\left(x, F_2, \frac{p}{2}\right) \right| > \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{r}\right) > 1 - \xi. \quad (48)$$

Since $\xi > 0$ is arbitrary, we get $|\eta(x, F_1, p) - \eta(x, F_2, p)| = 1$ for all $p > 0$, which yields $F_1 = F_2$. \square

Throughout the following definitions and theorems, we consider $(\mathcal{X}, \eta, \nu, *, \diamond)$ to be a separable IFMS and \mathcal{I}_2 to be a strongly admissible ideal.

Definition 5. The sequence $\{F_{wq}\}$ is strongly \mathcal{I}_2 -lacunary Cauchy sequence (Wijsman sense) if for each $\xi \in (0, 1)$, for each $x \in \mathcal{X}$, and for all $p > 0$, there are $s = s(\xi, x), t = t(\xi, x) \in \mathbb{N}$ such that

$$A(\xi, x, p) = \left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{st}, p)| \leq 1 - \xi \\ \text{or, } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{st}, p)| \geq \xi \end{array} \right\} \in \mathcal{I}_2. \quad (49)$$

Theorem 10. Every Wijsman strongly \mathcal{I}_2 -lacunary convergent sequence of closed sets $\{F_{wq}\}$ is Wijsman strongly \mathcal{I}_2 -lacunary Cauchy with regard to IFM (η, ν) .

Proof. Let $F_{wq} \xrightarrow{N_{\theta_2}} (\mathcal{I}_{W_2}^{\eta, \nu}) F$. At that case, for each $\xi \in (0, 1)$, for every $x \in \mathcal{X}$, and for all $p > 0$,

$$A(\xi, x, p) = \left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| \leq 1 - \xi \\ \text{or, } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| \geq \xi \end{array} \right\} \in \mathcal{I}_2. \quad (50)$$

Since \mathcal{I}_2 is a strongly admissible ideal, the set

$$A^c(\xi, x, p) = \left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| > 1 - \xi \\ \text{and } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| < \xi \end{array} \right\}, \quad (51)$$

is nonempty and belongs to $\mathcal{F}(\mathcal{I}_2)$. So, we select positive integers u and s such that $(u, s) \notin A(\xi, x, p)$, and we get

$$\frac{1}{h_{us}} \sum_{(w_0, q_0) \in I_{us}} |\eta(x, F_{w_0 q_0}, p) - \eta(x, F, p)| > 1 - \xi, \text{ and } \frac{1}{h_{us}} \sum_{(w_0, q_0) \in I_{us}} |\nu(x, F_{w_0 q_0}, p) - \nu(x, F, p)| < \xi. \quad (52)$$

Now, presume that

$$B(\xi, x, p) = \left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q), (w_0,q_0) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{w_0q_0}, p)| \leq 1 - 2\xi \\ \text{or } \frac{1}{h_{us}} \sum_{(w,q), (w_0,q_0) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{w_0q_0}, p)| \geq 2\xi \end{array} \right\}. \quad (53)$$

Consider the inequality

$$\begin{aligned} \frac{1}{h_{us}} \sum_{(w,q), (w_0,q_0) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{w_0q_0}, p)| &\leq \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| + \frac{1}{h_{us}} \sum_{(w_0,q_0) \in I_{us}} |\eta(x, F_{w_0q_0}, p) - \eta(x, F, p)|, \\ \frac{1}{h_{us}} \sum_{(w,q), (w_0,q_0) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{w_0q_0}, p)| &\leq \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| + \frac{1}{h_{us}} \sum_{(w_0,q_0) \in I_{us}} |\nu(x, F_{w_0q_0}, p) - \nu(x, F, p)|. \end{aligned} \quad (54)$$

Notice that if $(u, s) \in B(\xi, x, p)$, therefore,

$$\begin{aligned} \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| + \frac{1}{h_{us}} \sum_{(w_0,q_0) \in I_{us}} |\eta(x, F_{w_0q_0}, p) - \eta(x, F, p)| &\leq 1 - 2\xi, \\ \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| + \frac{1}{h_{us}} \sum_{(w_0,q_0) \in I_{us}} |\nu(x, F_{w_0q_0}, p) - \nu(x, F, p)| &\geq 2\xi. \end{aligned} \quad (55)$$

From another point of view, since $(u, s) \notin A(\xi, x, p)$, we get

$$\frac{1}{h_{us}} \sum_{(w_0,q_0) \in I_{us}} |\eta(x, F_{w_0q_0}, p) - \eta(x, F, p)| > 1 - \xi, \text{ and } \frac{1}{h_{us}} \sum_{(w_0,q_0) \in I_{us}} |\nu(x, F_{w_0q_0}, p) - \nu(x, F, p)| < \xi. \quad (56)$$

We reach that

$$\frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| \leq 1 - \xi, \text{ or } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| \geq \xi. \quad (57)$$

Hence, $(u, s) \in A(\xi, x, p)$. This gives that $B(\xi, x, p) \subset A(\xi, x, p) \in \mathcal{S}_2$ for every $\xi \in (0, 1)$ and for each $x \in \mathcal{X}$. Therefore, $B(\xi, x, p) \in \mathcal{S}_2$, so the sequence is Wijsman strongly \mathcal{S}_2 -lacunary sequence. \square

Definition 6. The sequence $\{F_{wq}\}$ is Wijsman strongly \mathcal{S}_2^* -lacunary convergent to F iff there is a set $M = \{(w, q) \in \mathbb{N} \times \mathbb{N}\}$ such that $M' = \{(u, s) \in \mathbb{N} \times \mathbb{N}: (w, q) \in I_{us}\} \in \mathcal{F}(\mathcal{S}_2)$ for each $x \in \mathcal{X}$,

$$\lim_{u,s \rightarrow \infty} \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| = 1, \quad (58)$$

$$\lim_{u,s \rightarrow \infty} \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| = 0.$$

In this case, we write $F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{F}_{W_2}^{*(\eta, \nu)}])$.

Theorem 11. If the sequence $\{F_{wq}\}$ is Wijsman strongly \mathcal{F}_2^* -lacunary convergent to F , then $\{F_{wq}\}$ is Wijsman strongly \mathcal{F}_2 -lacunary convergent to F .

Proof. Presume that $F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{F}_{W_2}^{*(\eta, \nu)}])$. Then, there is a set $M = \{(w, q) \in \mathbb{N} \times \mathbb{N}\}$ such that

$$P(\xi, x, p) = \left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| \leq 1 - \xi \\ \text{or, } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| \geq \xi \end{array} \right\}, \quad (61)$$

$$\subset H \cup (M' \cap ((\{1, 2, \dots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (k_0 - 1)\}))),$$

for $\mathbb{N} \times \mathbb{N} \setminus M' = H \in \mathcal{F}_2$. Since \mathcal{F}_2 is an admissible ideal, we obtain

$$H \cup (M' \cap ((\{1, 2, \dots, (k_0 - 1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (k_0 - 1)\}))) \in \mathcal{F}_2, \quad (62)$$

and so $P(\xi, x, p) \in \mathcal{F}_2$. Hence, $F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{F}_{W_2}^{*(\eta, \nu)}])$. \square

Theorem 12. Let \mathcal{F}_2 be a strongly admissible ideal involving feature (AP_2) . Then, $F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{F}_{W_2}^{*(\eta, \nu)}])$ implies $F_{wq} \longrightarrow F(N_{\theta_2}[\mathcal{F}_{W_2}^{*(\eta, \nu)}])$.

$$\frac{1}{h_{us}} \sum_{(w,q),(s,t) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{st}, p)| > 1 - \xi, \quad \text{and} \quad \frac{1}{h_{us}} \sum_{(w,q),(s,t) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{st}, p)| < \xi, \quad (63)$$

for every $w, q, s, t \geq N$.

Theorem 13. Every Wijsman strongly \mathcal{F}_2^* -lacunary Cauchy sequence of closed sets is Wijsman strongly \mathcal{F}_2 -lacunary Cauchy in IFMS with regard to (η, ν) .

$$M' = \{(u, s) \in \mathbb{N} \times \mathbb{N}: (w, q) \in I_{us}\} \in \mathcal{F}(\mathcal{F}_2), \quad (59)$$

for each $x \in \mathcal{X}$,

$$\frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| > 1 - \xi, \quad (60)$$

$$\frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| < \xi.$$

for every $\xi > 0$ and for all $w, q \geq k_0 = k_0(\xi, x) \in \mathbb{N}$. Hereby, for each $\xi > 0$ and $x \in \mathcal{X}$, we get

Definition 7. The sequence $\{F_{wq}\}$ is known as Wijsman strongly \mathcal{F}_2^* -lacunary Cauchy sequence if for each $\xi \in (0, 1)$, for all $x \in \mathcal{X}$, and for all $p > 0$, and there is a set $M = \{(w, q) \in \mathbb{N} \times \mathbb{N}\}$ such that $M' = \{(u, s) \in \mathbb{N} \times \mathbb{N}: (w, q) \in I_{us}\} \in \mathcal{F}(\mathcal{F}_2)$ and $N = N(\varepsilon, x) \in \mathbb{N}$ such that

Proof. If the hypothesis is provided, then for each $\xi \in (0, 1)$, for each $x \in \mathcal{X}$, and for all $p > 0$, there is a set $M = \{(w, q) \in \mathbb{N} \times \mathbb{N}\}$ such that

$$M' = \{(u, s) \in \mathbb{N} \times \mathbb{N}: (w, q) \in I_{us}\} \in \mathcal{F}(\mathcal{F}_2), \quad (64)$$

and $N = N(\varepsilon, x) \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(w,q),(s,t) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{st}, p)| > 1 - \xi, \quad \text{and} \quad \frac{1}{h_{us}} \sum_{(w,q),(s,t) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{st}, p)| < \xi, \quad (65)$$

for each $w, q, s, t \geq N$. Let $H = \mathbb{N} \times \mathbb{N} \setminus M'$. It is clear that $H \in \mathcal{I}_2$ and

$$P(\xi, x, p) = \left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q), (s,t) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{st}, p)| > 1 - \xi, \\ \text{or } \frac{1}{h_{us}} \sum_{(w,q), (s,t) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{st}, p)| < \xi, \end{array} \right\}, \quad (66)$$

$$\subset H \cup (M' \cap ((\{1, 2, \dots, (N-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (N-1)\}))).$$

As \mathcal{I}_2 be a strongly admissible ideal, then

$$H \cup (M' \cap ((\{1, 2, \dots, (N-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \dots, (N-1)\}))) \in \mathcal{I}_2. \quad (67)$$

Therefore, we obtain $P(\xi, x, p) \in \mathcal{I}_2$; that is, $\{F_{wq}\}$ is strongly \mathcal{I}_2 -lacunary Cauchy sequence (Wijsman sense) with regard to (η, ν) . \square

Theorem 14. Let \mathcal{I}_2 be an admissible ideal involving property (AP_2) . Then, the concept of Wijsman strongly \mathcal{I}_2 -lacunary Cauchy sequence of sets coincides with Wijsman strongly \mathcal{I}_2^* -lacunary Cauchy sequence of sets.

Proof. If a set sequence is strongly \mathcal{I}_2^* -lacunary Cauchy sequence, then it is strongly \mathcal{I}_2 -lacunary Cauchy sequence according to Theorem 13, where \mathcal{I}_2 need not to have the feature (AP_2) .

Now, it is adequate to demonstrate that a sequence $\{F_{wq}\}$ in \mathcal{X} is a strongly \mathcal{I}_2^* -lacunary Cauchy sequence under assumption that it is a strongly \mathcal{I}_2 -lacunary Cauchy sequence. Let $\{F_{wq}\}$ be a Wijsman strongly lacunary Cauchy sequence. In this case, for each $\xi \in (0, 1)$, for all $x \in \mathcal{X}$, there is a number $s = s(\xi, x), t = t(\xi, x) \in \mathbb{N}$ such that

$$A(\xi, x, p) = \left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{st}, p)| \leq 1 - \xi \\ \text{or } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{st}, p)| \geq \xi \end{array} \right\} \in \mathcal{I}_2. \quad (68)$$

Let

$$P_j(\xi, x, p) = \left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{s_j t_j}, p)| > 1 - \frac{1}{j} \\ \text{or } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{s_j t_j}, p)| < \frac{1}{j} \end{array} \right\}, \quad (69)$$

where $s(j) = s(1/j)$ and $t(j) = t(1/j)$, $j = 1, 2, \dots$. Clearly, for $j = 1, 2, \dots$, $P_j(\xi, x, p) \in \mathcal{F}(\mathcal{I}_2)$. Since \mathcal{I}_2 has the property (AP_2) , then by Theorem 3.3 in [9], there is $P \subset \mathbb{N} \times \mathbb{N}$ so that $P \in \mathcal{F}(\mathcal{I}_2)$ and $P \setminus P_j$ is finite for all j . Now, we demonstrate that

$$\lim_{w,q,s,t \rightarrow \infty} \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}, (s,t) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{st}, p)| = 1, \quad (70)$$

\mathcal{I}_2 and

$$\lim_{w,q,s,t \rightarrow \infty} \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}, (s,t) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{st}, p)| = 0, \quad (71)$$

for all $w, q, s, t > u(r)$. So, it follows that

for each $x \in \mathcal{X}$ and $(w, q), (s, t) \in P$. To show these, let $\xi \in (0, 1)$ and $r \in \mathbb{N}$ such that $r > 2/\xi$. If $(w, q), (s, t) \in P$, then $P \setminus P_r$ is a finite set; therefore, there is $u = u(r)$ so that

$$\begin{aligned} \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{s_r t_r}, p)| &> 1 - \frac{1}{r}, \\ \frac{1}{h_{us}} \sum_{(s,t) \in I_{us}} |\eta(x, F_{st}, p) - \eta(x, F_{s_r t_r}, p)| &> 1 - \frac{1}{r}, \\ \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{s_r t_r}, p)| &< \frac{1}{r}, \\ \frac{1}{h_{us}} \sum_{(s,t) \in I_{us}} |\nu(x, F_{st}, p) - \nu(x, F_{s_r t_r}, p)| &< \frac{1}{r}, \end{aligned} \quad (72)$$

$$\begin{aligned} \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{st}, p)| &\leq \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{s_r t_r}, p)| \\ &+ \frac{1}{h_{us}} \sum_{(s,t) \in I_{us}} |\eta(x, F_{st}, p) - \eta(x, F_{s_r t_r}, p)| > \left(1 - \frac{1}{r}\right) + \left(1 - \frac{1}{r}\right) > 1 - \xi, \\ \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{st}, p)| &\leq \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{s_r t_r}, p)| \\ &+ \frac{1}{h_{us}} \sum_{(s,t) \in I_{us}} |\nu(x, F_{st}, p) - \nu(x, F_{s_r t_r}, p)| < \frac{1}{r} + \frac{1}{r} < \xi. \end{aligned} \quad (73)$$

Therefore, for each $\xi \in (0, 1)$, $\exists u = u(\xi)$, and $(w, q), (s, t) \in P \in \mathcal{F}(\mathcal{J}_2)$, we get

$$\left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F_{st}, p)| \leq 1 - \xi \\ \text{or } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F_{st}, p)| \geq \xi \end{array} \right\} \in \mathcal{J}_2. \quad (74)$$

This implies that $\{F_{wq}\}$ is Wijsman strongly \mathcal{J}_2^* -lacunary Cauchy. \square

Definition 8. A sequence $\{F_{wq}\}$ in IFMS is called to be Wijsman lacunary convergent to F with regard to IFM (η, ν) if, for every $p > 0$ and $\xi \in (0, 1)$, there is $m_0, n_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| > 1 - \xi, \quad \text{and} \quad \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| < \xi, \quad (75)$$

for all $u \geq m_0$ and $s \geq n_0$. We write $(\mu, \nu)^{\theta_2} - \lim F_{wq} = F$.

Definition 9. Take $(\mathcal{X}, \eta, \nu, *, \diamond)$ as a separable IFMS and take $\{F_{wq}\} \in \mathcal{X}$.

(a) $F \in \mathcal{X}$ is known as Wijsman lacunary \mathcal{J}_2 -limit point of $\{F_{wq}\}$ if there is set $M = \{(w_1, q_1) < (w_2, q_2) < \dots < (w_u, q_s) < \dots\} \subset \mathbb{N} \times \mathbb{N}$ such that the set

$$M' = \{(u, s) \in \mathbb{N} \times \mathbb{N}: (w_u, q_s) \in I_{us}\} \notin \mathcal{J}_2, \quad (76)$$

and $(\eta, \nu)^{\theta_2} - \lim F_{w_u q_s} = F$.

(b) $F \in \mathcal{X}$ is known as Wijsman lacunary \mathcal{J}_2 -cluster point of $\{F_{wq}\}$ if, for every $p > 0$ and $\varepsilon \in (0, 1)$, we get

$$\left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| > 1 - \xi \\ \text{and, } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| < \xi \end{array} \right\} \notin \mathcal{J}_2. \quad (77)$$

Here, $\Lambda_{(\eta, \nu)}^{\mathcal{J}_2, \theta_2}(F_{wq})$ denotes the set of all Wijsman lacunary \mathcal{J}_2 -limit points and $\Gamma_{(\eta, \nu)}^{\mathcal{J}_2, \theta_2}(F_{wq})$ indicates the set of all Wijsman lacunary \mathcal{J}_2 -cluster points in IFMS.

Theorem 15. For each sequence $\{F_{wq}\}$ in IFMS, we have $\Lambda_{(\eta, \nu)}^{\mathcal{J}_2, \theta_2}(F_{wq}) \subseteq \Gamma_{(\eta, \nu)}^{\mathcal{J}_2, \theta_2}(F_{wq})$.

Proof. Let $F \in \Lambda_{(\eta, \nu)}^{\mathcal{J}_2, \theta_2}(F_{wq})$. So, there is a set $M \subset \mathbb{N} \times \mathbb{N}$ such that $M' \notin \mathcal{J}_2$, where M and M' are as in Definition 9, satisfying $(\eta, \nu)^{\theta_2} - \lim F_{w_u q_s} = F$. Hence, for every $p > 0$ and $\xi \in (0, 1)$, there are $m_0, n_0 \in \mathbb{N}$ such that

$$\frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{w_u q_s}, p) - \eta(x, F, p)| > 1 - \xi, \quad \text{and} \quad \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{w_u q_s}, p) - \nu(x, F, p)| < \xi, \quad (78)$$

for all $u \geq m_0$ and $s \geq n_0$. Therefore,

$$B = \left\{ \begin{array}{l} (u, s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\eta(x, F_{wq}, p) - \eta(x, F, p)| > 1 - \xi \\ \text{and } \frac{1}{h_{us}} \sum_{(w,q) \in I_{us}} |\nu(x, F_{wq}, p) - \nu(x, F, p)| < \xi \end{array} \right\}, \quad (79)$$

$$\supseteq M' \setminus \{(w_1, q_1), (w_2, q_2), \dots, (w_{m_0}, q_{n_0})\}.$$

Now, with \mathcal{J}_2 being admissible, we must have $M' \setminus \{(w_1, q_1), (w_2, q_2), \dots, (w_{m_0}, q_{n_0})\} \notin \mathcal{J}_2$ and as such $B \notin \mathcal{J}_2$. Hence, $F \in \Gamma_{(\eta, \nu)}^{\mathcal{J}_2, \theta_2}(F_{wq})$. \square

3. Conclusion

In this study, we examined a version of ideal convergence, named Wijsman lacunary ideal convergence of double set sequences, in IFMS. We investigated new convergence concepts for double set sequences in IFMS and obtained some meaningful results. In addition, Wijsman lacunary \mathcal{J}_2 -limit points and Wijsman lacunary \mathcal{J}_2 -cluster points of

double set sequences in IFMS were defined. Some of the results presented in this article are analogous to the research studies in the relevant topic, but in most situations, the proofs follow a different approach.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Existence of Best Proximity Point with an Application to Nonlinear Integral Equations

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Received 10 July 2021; Accepted 17 August 2021; Published 8 October 2021

Academic Editor: Huseyin Isik

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Using the idea of modified ϱ -proximal admissible mappings, we derive some new best proximity point results for $\varrho - \vartheta$ -contraction mappings in metric spaces. We also provide some illustrations to back up our work. As a result of our findings, several fixed-point results for such mappings are also found. We obtain the existence of a solution for nonlinear integral equations as an application.

1. Introduction and Preliminaries

Problems originating in several disciplines of mathematical analysis, such as obtaining the existence of a solution for integral and differential equations, are solved using fixed-point theory. The study of fixed points for nonself-mapping, on the contrary, is also fascinating. A nonself-contraction $\mathfrak{F}: \zeta \longrightarrow E$ does not necessarily have a fixed point for two given nonempty closed subsets ζ and E of a complete metric space \mathfrak{N} . In this case, it is important to identify a point $\dot{u} \in \zeta$ such that $\wp(\dot{u}, \mathfrak{F}\dot{u})$ is minimum. Essentially, if

$$\wp(\dot{u}, \mathfrak{F}\dot{u}) = \text{dist}(\zeta, E) = \inf \left\{ \wp(\dot{u}, \tilde{v}) : \dot{u} \in \zeta, \tilde{v} \in E \right\}, \quad (1)$$

where $\wp(\dot{u}, \mathfrak{F}\dot{u})$ is the minimum value $\text{dist}(\zeta, E)$ and hence \dot{u} is an approximate solution of equation $\mathfrak{F}\dot{u} = \dot{u}$ with least possible error, such a solution is known as the best proximity point of mapping \mathfrak{F} .

Various academics have discovered a number of best proximity point theorems and associated fixed-point results in metric or normed linear spaces (see [1–7] and references cited therein). Inspired by the work of Geraghty [8] and Kutbi and Sintunavarat [9], we introduced a new class of nonself-contractive mappings known as $\varrho - \vartheta$ -contraction. The goal of this work is to use the modified ϱ -proximal admissible mappings concept to obtain some best proximity

point outcomes for $\varrho - \vartheta$ -contraction mappings in metric space. On a metric space enriched with an arbitrary binary relation, some optimal proximity point results are proved. For such a class of mappings, we also obtain certain fixed-point findings. As an application of our obtained results, we find the solution of a nonlinear integral equation.

The set of natural numbers and the set of real numbers are denoted by \mathbb{N} and \mathbb{R} , respectively, throughout this work.

Let ζ and E be two nonempty subsets of (\mathfrak{N}, \wp) , a metric space. Define

$$\begin{aligned} \wp(\zeta, E) &= \inf \left\{ \wp(\dot{u}, \tilde{v}) : \dot{u} \in \zeta, \tilde{v} \in E \right\}, \\ \zeta_0 &= \left\{ \dot{u} \in \zeta : \text{there exists some } \tilde{v} \right. \\ &\quad \left. \in E \text{ such that } \wp(\dot{u}, \tilde{v}) = \wp(\zeta, E) \right\}, \\ E_0 &= \left\{ \tilde{v} \in E : \text{there exists some } \dot{u} \right. \\ &\quad \left. \in \zeta \text{ such that } \wp(\dot{u}, \tilde{v}) = \wp(\zeta, E) \right\}. \end{aligned} \quad (2)$$

Definition 1. Let (ζ, E) be a pair of nonempty subsets of a metric space (\mathfrak{N}, \wp) with $\zeta_0 \neq \emptyset$. Then, the pair (ζ, E) is said to have \mathfrak{E} -property (see [10]) if and only if, for any $\dot{u}_1, \dot{u}_2 \in \zeta$ and $\tilde{v}_1, \tilde{v}_2 \in E$,

$$\begin{aligned}\wp(\dot{u}_1, \tilde{v}_1) &= \wp(\zeta, E), \\ \wp(\dot{u}_2, \tilde{v}_2) &= \wp(\zeta, E), \quad \text{then } \wp(\dot{u}_1, \dot{u}_2) = \wp(\tilde{v}_1, \tilde{v}_2).\end{aligned}\quad (3)$$

Many authors have refined the concept of \wp -admissible mappings developed by Samet [10] for proximal mappings (see [11, 12]).

Definition 2. Let ζ and E be two nonempty subsets of a metric space (\mathfrak{N}, \wp) . A mapping $\mathfrak{F}: \zeta \longrightarrow E$ is called modified \wp -proximal admissible if there exists a mapping $\varrho: E \times E \longrightarrow [0, \infty)$ such that $\varrho(\mathfrak{F}\dot{u}_0, \mathfrak{F}\dot{u}_1) \geq 1$:

$$\begin{aligned}\wp(\dot{x}_1, \mathfrak{F}\dot{u}_0) &= \wp(\zeta, E), \\ \wp(\dot{x}_2, \mathfrak{F}\dot{u}_1) &= \wp(\zeta, E), \quad \text{then } \varrho(\mathfrak{F}\dot{x}_1, \mathfrak{F}\dot{x}_2) \geq 1,\end{aligned}\quad (4)$$

for all $\dot{u}_0, \dot{u}_1, \dot{x}_1, \dot{x}_2 \in \zeta$.

Let (\mathfrak{N}, \wp) be a metric space and $^\circ$ be a binary relation over \mathfrak{N} . Denote

$$\kappa = ^\circ + ^\circ^{-1} \quad (5)$$

is the symmetric, transitive relation attached to $^\circ$. Clearly,

$$\begin{aligned}\dot{u}, \tilde{v} &\in \mathfrak{N}, \\ \dot{u}\kappa\tilde{v} &\Leftrightarrow \dot{u}^\circ\tilde{v} \text{ or } \tilde{v}^\circ\dot{u}.\end{aligned}\quad (6)$$

Definition 3. Let ζ and E be two nonempty subsets of metric space (\mathfrak{N}, \wp) . A mapping $\mathfrak{F}: \zeta \longrightarrow E$ is called modified proximal comparative mapping; if $\mathfrak{F}\dot{u}_0 \kappa \mathfrak{F}\dot{u}_1$,

$$\begin{aligned}\wp(\dot{x}_1, \mathfrak{F}\dot{u}_0) &= \wp(\zeta, E), \\ \wp(\dot{x}_2, \mathfrak{F}\dot{u}_1) &= \wp(\zeta, E), \quad \text{then } \mathfrak{F}\dot{x}_1 \kappa \mathfrak{F}\dot{x}_2,\end{aligned}\quad (7)$$

for all $\dot{u}_0, \dot{u}_1, \dot{x}_1, \dot{x}_2 \in \zeta$.

Definition 4. Let \mathfrak{N} be a metric space. A mapping $\varrho: \mathfrak{N} \times \mathfrak{N} \longrightarrow [0, \infty)$ is called transitive [13] if it satisfies the following condition:

$$\varrho(\dot{x}, \dot{y}) \geq 1, \varrho(\dot{y}, \dot{z}) \geq 1 \text{ implies } \varrho(\dot{x}, \dot{z}) \geq 1, \quad (8)$$

for all $\dot{x}, \dot{y}, \dot{z} \in \mathfrak{N}$.

2. Main Results

To start with, we have the following definition:

Definition 5. Let ζ and E be two nonempty subsets of a metric space (\mathfrak{N}, \wp) . A mapping $\mathfrak{F}: \zeta \longrightarrow E$ is an \wp - ϑ contraction mapping if there exist two functions $\varrho: E \times E \longrightarrow [0, \infty)$ and $\vartheta: \mathfrak{N} \longrightarrow [0, 1)$ for which $\vartheta(\mathfrak{F}(\dot{u})) \leq \vartheta(\dot{u})$ and $\limsup \vartheta(\dot{u}) < 1$ for all $\dot{u} \in \mathfrak{N}$ such that

$$\varrho(\mathfrak{F}(\dot{u}_1), \mathfrak{F}(\dot{u}_2)) \wp(\mathfrak{F}(\dot{u}_1), \mathfrak{F}(\dot{u}_2)) \leq \vartheta(\mathfrak{F}\dot{u}_1) \wp(\dot{u}_1, \dot{u}_2), \quad (9)$$

for all $\dot{u}_1, \dot{u}_2 \in \zeta$.

Theorem 1. Let ζ and E be two nonempty closed subsets of a complete metric space \mathfrak{N} such that ζ_0 is nonempty. Let $\varrho: E \times E \longrightarrow [0, \infty)$ be transitive and $\mathfrak{F}: \zeta \longrightarrow E$ be continuous mapping satisfying the following assertions:

- (i) \mathfrak{F} is \wp - ϑ contraction
- (ii) $\mathfrak{F}(\zeta_0) \subseteq E_0$ and (ζ, E) satisfy the \mathfrak{E} -property
- (iii) \mathfrak{F} is modified \wp -proximal admissible

Furthermore, suppose that there exists $\dot{u}_0, \dot{u}_1 \in \zeta_0$ and $\mathfrak{F}\dot{u}_0 \in E_0$ such that $\wp(\dot{u}_1, \mathfrak{F}\dot{u}_0) = \wp(\zeta, E)$ and $\varrho(\mathfrak{F}\dot{u}_0, \mathfrak{F}\dot{u}_1) \geq 1$. Then, the mapping \mathfrak{F} has a best proximity point.

Proof. By assumption, there exists $\dot{u}_0, \dot{u}_1 \in \zeta_0 \subseteq \zeta$ and $\mathfrak{F}\dot{u}_0 \in \mathfrak{F}(\zeta_0) \subseteq E_0 \subseteq E$ such that $\wp(\dot{u}_1, \mathfrak{F}\dot{u}_0) = \wp(\zeta, E)$ and $\varrho(\mathfrak{F}\dot{u}_0, \mathfrak{F}\dot{u}_1) \geq 1$.

Since $\dot{u}_1 \in \zeta_0$, then $\mathfrak{F}\dot{u}_1 \in \mathfrak{F}(\zeta_0) \subseteq E_0$. By definition of E_0 , there exists $\dot{u}_2 \in \zeta_0$ such that $\wp(\dot{u}_2, \mathfrak{F}\dot{u}_1) = \wp(\zeta, E)$. Since \mathfrak{F} is modified \wp -proximal admissible, and $\varrho(\mathfrak{F}\dot{u}_0, \mathfrak{F}\dot{u}_1) \geq 1$, we obtain

$$\begin{aligned}\wp(\dot{u}_1, \mathfrak{F}\dot{u}_0) &= \wp(\zeta, E), \\ \wp(\dot{u}_2, \mathfrak{F}\dot{u}_1) &= \wp(\zeta, E), \quad \text{then } \varrho(\mathfrak{F}\dot{u}_1, \mathfrak{F}\dot{u}_2) \geq 1.\end{aligned}\quad (10)$$

Again, $\dot{u}_2 \in \zeta_0$; then, $\mathfrak{F}\dot{u}_2 \in \mathfrak{F}(\zeta_0) \subseteq E_0$. By definition of E_0 , there exists $\dot{u}_3 \in \zeta_0$ such that $\wp(\dot{u}_3, \mathfrak{F}\dot{u}_2) = \wp(\zeta, E)$. Since \mathfrak{F} is modified \wp -proximal admissible, and $\varrho(\mathfrak{F}\dot{u}_1, \mathfrak{F}\dot{u}_2) \geq 1$, we have

$$\begin{aligned}\wp(\dot{u}_2, \mathfrak{F}\dot{u}_1) &= \wp(\zeta, E), \\ \wp(\dot{u}_3, \mathfrak{F}\dot{u}_2) &= \wp(\zeta, E), \quad \text{then } \varrho(\mathfrak{F}\dot{u}_2, \mathfrak{F}\dot{u}_3) \geq 1.\end{aligned}\quad (11)$$

Continuing in this fashion, we can construct a sequence $\{\dot{u}_n\}$ such that

$$\begin{cases} \wp(\dot{u}_{n+1}, \mathfrak{F}\dot{u}_n) = \wp(\zeta, E), \\ \wp(\dot{u}_n, \mathfrak{F}\dot{u}_{n-1}) = \wp(\zeta, E), \\ \varrho(\mathfrak{F}\dot{u}_n, \mathfrak{F}\dot{u}_{n+1}) \geq 1. \end{cases} \quad (12)$$

As a pair (ζ, E) satisfies \mathfrak{E} -property, we have

$$\wp(\dot{u}_n, \dot{u}_{n+1}) = \wp(\mathfrak{F}\dot{u}_{n-1}, \mathfrak{F}\dot{u}_n). \quad (13)$$

For $n \in \mathbb{N}$, we have

$$\begin{aligned} \wp(\dot{u}_n, \dot{u}_{n+1}) &= \wp(\mathfrak{T}(\dot{u}_{n-1}), \mathfrak{T}(\dot{u}_n)) \\ &\leq \wp(\mathfrak{T}\dot{u}_{n-1}, \mathfrak{T}\dot{u}_n) \wp(\mathfrak{T}\dot{u}_{n-1}, \mathfrak{T}\dot{u}_n) \\ &\leq \vartheta(\mathfrak{T}\dot{u}_{n-1}) \wp(\dot{u}_{n-1}, \dot{u}_n) \\ &\leq \vartheta(\dot{u}_{n-1}) \wp(\dot{u}_{n-1}, \dot{u}_n) \\ &< \wp(\dot{u}_{n-1}, \dot{u}_n), \end{aligned} \quad (14)$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\{\wp(\dot{u}_{n-1}, \dot{u}_n)\}$ is strictly decreasing and

$$\lim_{n \rightarrow +\infty} \wp(\dot{u}_{n-1}, \dot{u}_n) = \dot{d}, \quad (15)$$

for some $\dot{d} \geq 0$.

Next, we claim that $\dot{d} = 0$. Assume, on the contrary, that $\dot{d} > 0$. On taking limit as $n \rightarrow +\infty$ in equation (14), we obtain that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{\wp(\dot{u}_n, \dot{u}_{n+1})}{\wp(\dot{u}_{n-1}, \dot{u}_n)} &\leq \vartheta(\dot{u}_{n-1}), \\ 1 &\leq \vartheta(\dot{u}_{n-1}), \end{aligned} \quad (16)$$

which is a contradiction. Therefore, $\dot{d} = 0$.

Next, we show that $\{\dot{u}_n\}$ is a Cauchy sequence. Using triangle inequality, (12) and (13), we have, for all $m, n \in \mathbb{N}$ and $m > n$,

$$\begin{aligned} \wp(\dot{u}_m, \dot{u}_n) &\leq \wp(\dot{u}_m, \dot{u}_{m+1}) + \wp(\dot{u}_{m+1}, \dot{u}_{n+1}) + \wp(\dot{u}_{n+1}, \dot{u}_n) \\ &= \wp(\dot{u}_m, \dot{u}_{m+1}) + \wp(\mathfrak{T}\dot{u}_m, \mathfrak{T}\dot{u}_n) + \wp(\dot{u}_{n+1}, \dot{u}_n) \\ &\leq \wp(\mathfrak{T}\dot{u}_m, \mathfrak{T}\dot{u}_n) \wp(\dot{u}_m, \dot{u}_{m+1}) + \wp(\mathfrak{T}\dot{u}_m, \mathfrak{T}\dot{u}_n) \wp(\mathfrak{T}\dot{u}_m, \mathfrak{T}\dot{u}_n) + \wp(\mathfrak{T}\dot{u}_m, \mathfrak{T}\dot{u}_n) \wp(\dot{u}_{n+1}, \dot{u}_n) \\ &\leq \wp(\mathfrak{T}\dot{u}_m, \mathfrak{T}\dot{u}_n) [\wp(\dot{u}_m, \dot{u}_{m+1}) + \wp(\dot{u}_{n+1}, \dot{u}_n)] + \vartheta(\mathfrak{T}\dot{u}_m) \wp(\dot{u}_m, \dot{u}_n) \\ &\leq \wp(\mathfrak{T}\dot{u}_m, \mathfrak{T}\dot{u}_n) [\wp(\dot{u}_m, \dot{u}_{m+1}) + \wp(\dot{u}_{n+1}, \dot{u}_n)] + \vartheta(\dot{u}_m) \wp(\dot{u}_m, \dot{u}_n). \end{aligned} \quad (17)$$

It implies that

$$\begin{aligned} (1 - \vartheta(\dot{u}_m)) \wp(\dot{u}_m, \dot{u}_n) \\ \leq \wp(\mathfrak{T}\dot{u}_m, \mathfrak{T}\dot{u}_n) [\wp(\dot{u}_m, \dot{u}_{m+1}) + \wp(\dot{u}_{n+1}, \dot{u}_n)]. \end{aligned} \quad (18)$$

On taking limit as $m, n \rightarrow +\infty$ in (18), we obtain that

$$\lim_{m, n \rightarrow +\infty} \wp(\dot{u}_m, \dot{u}_n) = 0. \quad (19)$$

Therefore, $\{\dot{u}_n\}$ is a Cauchy sequence in ζ . Since ζ is a closed subset of a complete metric \mathbb{N} , we obtain

$$\lim_{n \rightarrow +\infty} \dot{u}_n = \dot{u}, \quad (20)$$

for some $\dot{u} \in \zeta$. Since \mathfrak{T} is continuous,

$$\lim_{n \rightarrow +\infty} \mathfrak{T}(\dot{u}_n) = \mathfrak{T}(\dot{u}), \quad (21)$$

for some $\mathfrak{T}(\dot{u}) \in E$. By (12), we obtain

$$\wp(\dot{u}, \mathfrak{T}\dot{u}) = \lim_{n \rightarrow +\infty} \wp(\dot{u}_{n+1}, \mathfrak{T}(\dot{u}_n)) = \wp(\zeta, E). \quad (22)$$

Hence, \dot{u} is a best proximity point of \mathfrak{T} .

The following hypotheses can be used to substitute \mathfrak{T} 's continuity hypothesis in Theorem 1.

(\mathfrak{R}) If $\{\dot{u}_n\}$ is a sequence in \mathbb{N} such that $\wp(\mathfrak{T}\dot{u}_n, \mathfrak{T}\dot{u}_{n+1}) \geq 1$, for all n and $\dot{u}_n \rightarrow \dot{u} \in \mathbb{N}$ as $n \rightarrow \infty$, then there exists a subsequence $\{\dot{u}_{n(k)}\}$ of $\{\dot{u}_n\}$ such that $\wp(\mathfrak{T}\dot{u}_{n(k)}, \mathfrak{T}\dot{u}) \geq 1$, for all k .

Theorem 2. Let ζ and E be two nonempty closed subsets of a complete metric space \mathbb{N} such that ζ_0 is nonempty. Let $\wp: E \times E \rightarrow [0, \infty)$ be transitive and $\mathfrak{T}: \zeta \rightarrow E$ be a mapping satisfying the following assertions:

- (i) \mathfrak{T} is $\wp - \vartheta$ contraction
- (ii) $\mathfrak{T}(\zeta_0) \subseteq E_0$ and (ζ, E) satisfy the \mathbb{E} -property
- (iii) \mathfrak{T} is modified \wp -proximal admissible

Furthermore, suppose that (\mathfrak{R}) holds and there exists $\dot{u}_0, \dot{u}_1 \in \zeta_0$ and $\mathfrak{T}\dot{u}_0 \in E_0$ such that $\wp(\dot{u}_1, \mathfrak{T}\dot{u}_0) = \wp(\zeta, E)$ and $\wp(\mathfrak{T}\dot{u}_0, \mathfrak{T}\dot{u}_1) \geq 1$. Then, the mapping \mathfrak{T} has a best proximity point.

Proof. Following the proof of Theorem 1, the sequence $\{\dot{u}_n\}$ is Cauchy and converges to some \dot{u}^* in (\mathbb{N}, \wp) .

From our assumption and using (12), there exists a subsequence $\{\dot{u}_{n(k)}\}$ of $\{\dot{u}_n\}$ such that $\wp(\mathfrak{T}\dot{u}_{n(k)}, \mathfrak{T}\dot{u}^*) \geq 1$, for all k .

Now, we shall show that \mathfrak{T} has a best proximity point. Using triangle inequality and (12), we obtain

$$\begin{aligned}
\wp(\dot{u}_{n(k)+1}, \mathfrak{S}\dot{u}^*) &\leq \wp(\dot{u}_{n(k)+1}, \mathfrak{S}\dot{u}_{n(k)}) + \wp(\mathfrak{S}\dot{u}_{n(k)}, \mathfrak{S}\dot{u}^*) \\
&\leq \wp(\zeta, E) + \wp(\mathfrak{S}\dot{u}_{n(k)}, \mathfrak{S}\dot{u}^*) \wp(\mathfrak{S}\dot{u}_{n(k)}, \mathfrak{S}\dot{u}^*) \\
&\leq \wp(\zeta, E) + \wp(\mathfrak{S}\dot{u}_{n(k)}) \wp(\dot{u}_{n(k)}, \dot{u}^*) \\
&\leq \wp(\zeta, E) + \wp(\dot{u}_{n(k)}) \wp(\dot{u}_{n(k)}, \dot{u}^*).
\end{aligned} \tag{23}$$

Again, by triangle inequality and equation (23), we have

$$\begin{aligned}
\wp(\zeta, E) &\leq \wp(\dot{u}^*, \mathfrak{S}\dot{u}^*) \leq \wp(\dot{u}^*, \dot{u}_{n(k)+1}) + \wp(\dot{u}_{n(k)+1}, \mathfrak{S}\dot{u}^*) \\
&\leq \wp(\dot{u}^*, \dot{u}_{n(k)+1}) + \wp(\zeta, E) + \wp(\dot{u}_{n(k)}) \wp(\dot{u}_{n(k)}, \dot{u}^*).
\end{aligned} \tag{24}$$

Taking $k \rightarrow +\infty$ in (24), we have

$$\wp(\zeta, E) = \wp(\dot{u}^*, \mathfrak{S}\dot{u}^*). \tag{25}$$

This shows that \mathfrak{S} has a best proximity point.

3. Best Proximity Points on a Metric Space Endowed with an Arbitrary Binary Relation

Using modified proximal comparative mapping, we establish some optimal proximity point findings on a metric space equipped with an arbitrary binary relation in this section.

Theorem 3. Let ζ and E be two nonempty closed subsets of a complete metric space (\mathfrak{N}, \wp) such that ζ_0 is nonempty and \circledast be a binary relation over \mathfrak{N} . Suppose that $\mathfrak{S}: \zeta \rightarrow E$ is a continuous mapping satisfying the following assertions:

(i) There exists $\wp: \mathfrak{N} \rightarrow [0, 1)$ such that, for all $\dot{u}_1, \dot{u}_2 \in \zeta$, $\dot{u}_1 \kappa \dot{u}_2$,

$$\wp(\mathfrak{S}\dot{u}_1, \mathfrak{S}\dot{u}_2) \leq \wp(\dot{u}_1) (\wp(\dot{u}_1, \dot{u}_2)). \tag{26}$$

(ii) $\mathfrak{S}(\zeta_0) \subseteq E_0$ and (ζ, E) satisfy the \mathfrak{E} -property.

(iii) \mathfrak{S} is modified proximal comparative mapping.

If there exists $\dot{u}_3, \dot{u}_4 \in \zeta_0$ and $\mathfrak{S}\dot{u}_3 \in E_0$ such that $\wp(\dot{u}_4, \mathfrak{S}\dot{u}_3) = \wp(\zeta, E)$, $\dot{u}_3 \kappa \dot{u}_4$, then the mapping \mathfrak{S} has a best proximity point.

Proof. Define a mapping $\wp: E \times E \rightarrow \mathbb{R}$ by

$$\wp(\mathfrak{S}\dot{u}_1, \mathfrak{S}\dot{u}_2) = \begin{cases} 2, & \text{if } \mathfrak{S}\dot{u}_1 \kappa \mathfrak{S}\dot{u}_2, \\ 1, & \text{otherwise.} \end{cases} \tag{27}$$

Suppose that $\wp(\mathfrak{S}\dot{u}_3, \mathfrak{S}\dot{u}_4) \geq 1$, such that

$$\begin{cases} \wp(\dot{u}_4, \mathfrak{S}\dot{u}_3) = \wp(\zeta, E), \\ \wp(\dot{u}_5, \mathfrak{S}\dot{u}_4) = \wp(\zeta, E). \end{cases} \tag{28}$$

Hold for some $\dot{u}_3, \dot{u}_4, \dot{u}_5 \in \zeta$. By definition of \wp , we obtain

$$\begin{aligned}
&\mathfrak{S}\dot{u}_3 \kappa \mathfrak{S}\dot{u}_4, \\
\wp(\dot{u}_4, \mathfrak{S}\dot{u}_3) &= \wp(\zeta, E), \\
\wp(\dot{u}_5, \mathfrak{S}\dot{u}_4) &= \wp(\zeta, E).
\end{aligned} \tag{29}$$

Condition (iii) implies that $\mathfrak{S}\dot{u}_3 \kappa \mathfrak{S}\dot{u}_5$, which gives us from the definition of \wp that $\wp(\mathfrak{S}\dot{u}_3, \mathfrak{S}\dot{u}_5) \geq 1$. This shows \mathfrak{S} is a modified \wp -proximal admissible mapping.

Furthermore, using assumption, we have

$$\begin{aligned}
\wp(\dot{u}_4, \mathfrak{S}\dot{u}_3) &= \wp(\zeta, E), \\
\wp(\mathfrak{S}\dot{u}_3, \mathfrak{S}\dot{u}_4) &\geq 1.
\end{aligned} \tag{30}$$

Condition (i) implies that

$$\wp(\mathfrak{S}\dot{u}_1, \mathfrak{S}\dot{u}_2) \wp(\mathfrak{S}\dot{u}_1, \mathfrak{S}\dot{u}_2) \leq \wp(\dot{u}_1) (\wp(\dot{u}_1, \dot{u}_2)), \quad \forall \dot{u}_1, \dot{u}_2 \in \zeta. \tag{31}$$

Thus, all the conditions of Theorem 1 are satisfied; then, \mathfrak{S} has a best proximity point.

We may replace the continuity hypothesis of \mathfrak{S} in Theorem 3 by the following hypothesis.

(\mathfrak{R}^*) If $\{\dot{u}_n\}$ is a sequence in \mathfrak{N} such that $\mathfrak{S}\dot{u}_n \kappa \mathfrak{S}\dot{u}_{n+1}$ for all n and $\dot{u}_n \rightarrow \dot{u} \in \mathfrak{N}$ as $n \rightarrow +\infty$, then there exists a subsequence $\{\dot{u}_{n(k)}\}$ of $\{\dot{u}_n\}$ such that $\mathfrak{S}\dot{u}_{n(k)} \kappa \mathfrak{S}\dot{u}$, for all k .

Theorem 4. Let ζ and E be two nonempty closed subsets of a complete metric space (\mathfrak{N}, \wp) such that ζ_0 is nonempty and \circledast be a binary relation over \mathfrak{N} . Suppose that $\mathfrak{S}: \zeta \rightarrow E$ is a mapping satisfying the following assertions:

(i) There exists $\wp: \mathfrak{N} \rightarrow [0, 1)$ such that, for all $\dot{u}_1, \dot{u}_2 \in \zeta$, $\dot{u}_1 \kappa \dot{u}_2$,

$$\wp(\mathfrak{S}\dot{u}_1, \mathfrak{S}\dot{u}_2) \leq \wp(\dot{u}_1) (\wp(\dot{u}_1, \dot{u}_2)). \tag{32}$$

(ii) $\mathfrak{S}(\zeta_0) \subseteq E_0$ and (ζ, E) satisfy the \mathfrak{E} -property.

(iii) \mathfrak{S} is modified proximal comparative mapping.

Furthermore, we assume that (\mathfrak{R}^*) holds and there exists $\dot{u}_3, \dot{u}_4 \in \zeta_0$ and $\mathfrak{S}\dot{u}_3 \in E_0$ such that $\wp(\dot{u}_4, \mathfrak{S}\dot{u}_3) = \wp(\zeta, E)$, $\dot{u}_3 \kappa \dot{u}_4$. Then, the mapping \mathfrak{S} has a best proximity point.

Proof. By evaluating the mapping \wp defined in Theorem 3 and noting that condition (\mathfrak{R}^*) implies condition (\mathfrak{R}), the result is derived from Theorem 2.

4. Consequences and Related Results

In this section, we obtain some results on the existence of fixed points as a result of our findings.

If we take $\zeta = E = \mathfrak{N}$ in Theorem 1, we obtain the following result:

Corollary 1 (see Theorem 10 in [9]). Let \aleph be a complete metric space. Let $\varrho: \aleph \times \aleph \rightarrow [0, \infty)$ be transitive and $\mathfrak{F}: \aleph \rightarrow \aleph$ be continuous mapping satisfying the following assertions:

- (i) \mathfrak{F} is ϱ - ϑ contraction
- (ii) \mathfrak{F} is ϱ admissible
- (iii) There exists $\dot{u}_0 \in \zeta$ such that $\varrho(\dot{u}_0, \mathfrak{F}\dot{u}_0) \geq 1$

Then, the mapping \mathfrak{F} has a fixed point.

By considering $\varrho(\dot{u}_1, \dot{u}_2) = 1$, for all $\dot{u}_1, \dot{u}_2 \in \zeta$ and $\vartheta(\dot{u}_1) = k$; $k \in [0, 1)$, for all $\dot{u}_1 \in \aleph$ in Corollary 1, we get the famous Banach contraction theorem.

Corollary 2 (see [14]). Let \aleph be a complete metric space and $\mathfrak{F}: \aleph \rightarrow \aleph$ be a mapping satisfying the following assertion:

$$\varrho(\mathfrak{F}\dot{u}_1, \mathfrak{F}\dot{u}_2) \leq k\varrho(\dot{u}_1, \dot{u}_2), \quad (33)$$

for all $\dot{u}_1, \dot{u}_2 \in \aleph$. Then, the mapping \mathfrak{F} has a fixed point.

If we take $\zeta = E = \aleph$ in Theorem 2, we obtain the following result.

Corollary 3 (see Theorem 12 in [9]). Let \aleph be a complete metric space. Let $\varrho: \aleph \times \aleph \rightarrow [0, \infty)$ be transitive and $\mathfrak{F}: \aleph \rightarrow \aleph$ be a mapping satisfying the following assertions:

- (i) \mathfrak{F} is ϱ - ϑ contraction
- (ii) \mathfrak{F} is ϱ admissible
- (iii) There exists $\dot{u}_0 \in \aleph$ such that $\varrho(\dot{u}_0, \mathfrak{F}\dot{u}_0) \geq 1$
- (iv) (\mathfrak{R}) holds

Then, the mapping \mathfrak{F} has a fixed point.

If we take $\zeta = E = \aleph$ in Theorems 3 and 4, we obtain the following fixed-point results:

Corollary 4. Suppose that $\mathfrak{F}: \aleph \rightarrow \aleph$ is a continuous mapping on a metric space (\aleph, ϱ) with a binary relation \circledast over \aleph , satisfying

$$\varrho(\mathfrak{F}\dot{u}_1, \mathfrak{F}\dot{u}_2) \leq \vartheta(\dot{u}_1)(\varrho(\dot{u}_1, \dot{u}_2)), \quad (34)$$

for $\dot{u}_1, \dot{u}_2 \in \aleph$, $\dot{u}_1 \kappa \dot{u}_2$, where $\vartheta: \aleph \rightarrow [0, 1)$. If there exists $\dot{u}_* \in \aleph$ such that $\dot{u}_* \kappa \mathfrak{F}\dot{u}_*$, then the mapping \mathfrak{F} has a fixed point.

Corollary 5. Suppose that $\mathfrak{F}: \aleph \rightarrow \aleph$ is a self-mapping on a metric space (\aleph, ϱ) with a binary relation \circledast over \aleph , satisfying

$$\varrho(\mathfrak{F}\dot{u}_1, \mathfrak{F}\dot{u}_2) \leq \vartheta(\dot{u}_1)(\varrho(\dot{u}_1, \dot{u}_2)), \quad (35)$$

for $\dot{u}_1, \dot{u}_2 \in \aleph$, $\dot{u}_1 \kappa \dot{u}_2$, where $\vartheta: \aleph \rightarrow [0, 1)$. If there exists $\dot{u}_* \in \aleph$ such that $\dot{u}_* \kappa \mathfrak{F}\dot{u}_*$ and (\mathfrak{R}^*) holds, then the mapping \mathfrak{F} has a fixed point.

5. Examples

We give several illustrations that support our findings in this section.

Example 1. Consider $\aleph = \mathbb{R}^2$ with metric

$$\varrho(\tilde{v}, \dot{u}_*) = \sqrt{(\tilde{v}_1 - \tilde{v}_2)^2 + (\dot{u}_1^* - \dot{u}_2^*)^2}, \quad (36)$$

for all $\tilde{v} = (\tilde{v}_1, \dot{u}_1^*)$, $\dot{u}^* = (\tilde{v}_2, \dot{u}_2^*) \in \mathbb{R}^2$.

Suppose $\zeta = \left\{((1/2), \dot{u}_1): 0 \leq \dot{u}_1 \leq 1\right\}$ and

$E = \left\{(0, \dot{u}_1): 0 \leq \dot{u}_1 \leq 1\right\}$, such that $\varrho(\zeta, E) = (1/2)$.

Define $\mathfrak{F}: \zeta \rightarrow E$ by

$$\mathfrak{F}(\dot{u}_1) = \begin{cases} (0, 1), & \dot{u}_1 = \left(\frac{1}{2}, 1\right), \\ \left(0, \frac{\dot{a}}{4}\right): 0 \leq \dot{a} \leq \dot{u}_1, & \text{otherwise,} \end{cases} \quad (37)$$

for all $\dot{u}_1 \in \zeta$.

Define $\varrho: E \times E \rightarrow [0, \infty)$ by

$$\varrho\left(\left(0, \dot{u}_1^*\right), \left(0, \dot{u}_2^*\right)\right) = \begin{cases} 1, & \text{if } \dot{u}_1^*, \dot{u}_2^* \in \left[0, \frac{1}{4}\right], \\ 2, & \text{otherwise,} \end{cases} \quad (38)$$

then ϱ is transitive. If $\dot{z}_1 = ((1/2), \dot{u}_1)$ and $\dot{z}_2 = ((1/2), \dot{u}_2)$ in ζ , for $\dot{u}_1, \dot{u}_2 \in [0, (1/2)]$, then

$$\begin{aligned} \mathfrak{F}\dot{z}_1 &= \left\{\left(0, \frac{\dot{a}}{4}\right): 0 \leq \dot{a} \leq \dot{u}_1\right\}, \\ \mathfrak{F}\dot{z}_2 &= \left\{\left(0, \frac{\dot{a}}{4}\right): 0 \leq \dot{a} \leq \dot{u}_2\right\}. \end{aligned} \quad (39)$$

Also, $\varrho(\mathfrak{F}\dot{z}_1, \mathfrak{F}\dot{z}_2) = 1$ and $\varrho(\dot{u}_1, \mathfrak{F}\dot{z}_1) = (1/2) = \varrho(\zeta, E)$ and $\varrho(\dot{u}_2, \mathfrak{F}\dot{z}_2) = (1/2) = \varrho(\zeta, E)$ if and only if $\dot{u}_1, \dot{u}_2 \in \left\{((1/2), (\dot{x}/4)): 0 \leq \dot{x} \leq (1/2)\right\}$. Then, $\varrho(\mathfrak{F}\dot{u}_1, \mathfrak{F}\dot{u}_2) = 1$. This shows \mathfrak{F} is a modified ϱ -proximal admissible and \mathfrak{F} is continuous. Since $\zeta_0 = \zeta$ and $E_0 = E$, then $\mathfrak{F}(\zeta_0) \subseteq E_0$, for each $\dot{u}_1 \in \zeta_0$.

Next, we prove, that \mathfrak{F} is ϱ - ϑ contraction. Let $\vartheta(t) = (1/3)$, for all $t \in \aleph$. Take $\dot{z}_1 = ((1/2), \dot{u}_1)$ and $\dot{z}_2 = ((1/2), \dot{u}_2)$ in ζ , where $0 \leq \dot{u}_1, \dot{u}_2 \leq (1/2)$. Consider

$$\begin{aligned} \varrho(\mathfrak{F}\dot{z}_1, \mathfrak{F}\dot{z}_2) &= \sqrt{(0-0)^2 + \left(\frac{\dot{u}_1}{4} - \frac{\dot{u}_2}{4}\right)^2} \\ &= \sqrt{\left(\frac{\dot{u}_1}{4} - \frac{\dot{u}_2}{4}\right)^2} = \left|\frac{\dot{u}_1}{4} - \frac{\dot{u}_2}{4}\right| \\ &= \frac{1}{4}(\dot{u}_1 - \dot{u}_2) \leq \frac{1}{3}(\dot{u}_1 - \dot{u}_2). \end{aligned} \quad (40)$$

It implies that

$$\varrho(\mathfrak{F}z_1, \mathfrak{F}z_2) \varrho(\mathfrak{F}z_1, \mathfrak{F}z_2) \leq \vartheta(z_1) \varrho(z_1, z_2), \quad (41)$$

for all $z_1, z_2 \in \zeta$. Hence, \mathfrak{F} is a $\varrho - \vartheta$ contraction. All conditions of Theorem 1 are satisfied and \mathfrak{F} has a best proximity point $((1/2), 1)$.

Example 2. Consider $\aleph = \mathbb{R}^2$ with metric

$$\varrho(\tilde{v}, \dot{u}_*) = \sqrt{(\tilde{v}_1 - \tilde{v}_2)^2 + (\dot{u}_1^* - \dot{u}_2^*)^2}, \quad (42)$$

for all $\tilde{v} = (\tilde{v}_1, \dot{u}_1^*)$, $\dot{u}^* = (\tilde{v}_2, \dot{u}_2^*) \in \mathbb{R}^2$. Suppose that

$$\begin{aligned} \zeta &= \left\{ \left(\dot{u}_1, \dot{u}_2 \right) : \dot{u}_1^2 + \dot{u}_2^2 = 3 \text{ and } \dot{u}_2 \geq 0 \right\}, \\ E &= \left\{ \left(\dot{u}_1, \dot{u}_2 \right) : \dot{u}_1^2 + \dot{u}_2^2 = 1 \text{ and } \dot{u}_2 \geq 0 \right\}. \end{aligned} \quad (43)$$

Then, $\varrho(\zeta, E) = 2$. Define $\mathfrak{F} : \zeta \longrightarrow E$ by

$$\mathfrak{F}(\dot{u}_1, \dot{u}_2) = \frac{(\dot{u}_1, \dot{u}_2)}{3}, \quad (44)$$

for all $(\dot{u}_1, \dot{u}_2) \in \zeta$.

Define $\varrho : \aleph \times \aleph \longrightarrow [0, \infty)$ by

$$\varrho((\dot{u}_1, \dot{u}_1^*), (\dot{u}_2, \dot{u}_2^*)) = 2, \quad (45)$$

for all $(\dot{u}_1, \dot{u}_1^*), (\dot{u}_2, \dot{u}_2^*) \in \aleph$. Hence, ϱ is transitive. If $z_1, z_2 \in \zeta$, then $\varrho(\mathfrak{F}z_1, \mathfrak{F}z_2) = 2 > 1$, $\varrho(\dot{u}_1, \mathfrak{F}z_1) = 2 = \varrho(\zeta, E)$ and $\varrho(\dot{u}_2, \mathfrak{F}z_2) = 2 = \varrho(\zeta, E)$ if and only if $\dot{u}_1, \dot{u}_2 \in \{(0, 3), (3, 0)\}$. Therefore, $\varrho(\mathfrak{F}\dot{u}_1, \mathfrak{F}\dot{u}_2) = 2 > 1$. This shows \mathfrak{F} is a modified ϱ -proximal admissible and \mathfrak{F} is continuous. Since $\zeta_0 = \zeta$ and $E_0 = E$, then $\mathfrak{F}(\zeta_0) \subseteq E_0$, for each $\dot{u}_1 \in \zeta_0$.

Next, we prove that \mathfrak{F} is $\varrho - \vartheta$ contraction.

Let $\vartheta(t) = (1/2)$. Take $z_1 = (\dot{x}_1, \dot{y}_1)$ and $z_2 = (\dot{x}_2, \dot{y}_2)$ in ζ ; then,

$$\begin{aligned} \varrho(\mathfrak{F}z_1, \mathfrak{F}z_2) &= \sqrt{\left(\frac{\dot{x}_1}{3} - \frac{\dot{x}_2}{3} \right)^2 + \left(\frac{\dot{y}_1}{3} - \frac{\dot{y}_2}{3} \right)^2} \\ &= \sqrt{\frac{1}{3^2} (\dot{x}_1 - \dot{x}_2)^2 + \frac{1}{3^2} (\dot{y}_1 - \dot{y}_2)^2} \\ &= \frac{1}{3} \sqrt{(\dot{x}_1 - \dot{x}_2)^2 + (\dot{y}_1 - \dot{y}_2)^2} \\ &= \frac{1}{3} \varrho(z_1, z_2). \end{aligned} \quad (46)$$

It implies that

$$\varrho(\mathfrak{F}z_1, \mathfrak{F}z_2) \varrho(\mathfrak{F}z_1, \mathfrak{F}z_2) \leq \vartheta(z_1) \varrho(z_1, z_2), \quad (47)$$

for all $z_1, z_2 \in \aleph$. Therefore, \mathfrak{F} is a $\varrho - \vartheta$ contraction. All conditions of Theorem 1 are satisfied and \mathfrak{F} has a best proximity point $(3, 0)$.

6. Application to Integral Equations

In this section, we obtain the solution of integral equation as an application of our obtained results.

If we take $\zeta = E = \aleph$ in Theorem 1, we obtain the solution of nonlinear integral equation.

Theorem 5. Let $\mathbb{C}[\dot{a}, \dot{b}]$ be the set of all continuous functions on closed interval $[\dot{a}, \dot{b}]$, with metric defined by

$$\varrho(\dot{u}, \dot{v}) = \sup_{t \in [\dot{a}, \dot{b}]} |\dot{u}(t) - \dot{v}(t)|, \quad (48)$$

for all $\dot{u}, \dot{v} \in \mathbb{C}[\dot{a}, \dot{b}]$. Consider the nonlinear integral equation:

$$\dot{u}(t) = v(t) + \int_{\dot{a}}^{\dot{b}} \omega(t, \dot{x}, \dot{u}(\dot{x})) d\dot{x}, \quad (49)$$

where $t \in [\dot{a}, \dot{b}]$, $v : [\dot{a}, \dot{b}] \longrightarrow \mathbb{R}$, and $\omega : [\dot{a}, \dot{b}] \times [\dot{a}, \dot{b}] \times \mathbb{R} \longrightarrow \mathbb{R}$ for each $\dot{u} \in \mathbb{C}[\dot{a}, \dot{b}]$. Suppose that the following statements hold:

(i) v is continuous on $[\dot{a}, \dot{b}]$ and $\omega(t, \dot{x}, \dot{u}(\dot{x}))$ is integrable with respect to \dot{x} on $[\dot{a}, \dot{b}]$

(ii) $\mathfrak{F}\dot{u} \in \mathbb{C}[\dot{a}, \dot{b}]$, for all $\dot{u} \in \mathbb{C}[\dot{a}, \dot{b}]$, where $\mathfrak{F}\dot{u}(t) = v(t) + \int_{\dot{a}}^{\dot{b}} \omega(t, \dot{x}, \dot{u}(\dot{x})) d\dot{x}$, for all $t \in [\dot{a}, \dot{b}]$

(iii) For all $\dot{u} \in \mathbb{C}[\dot{a}, \dot{b}]$, $\dot{u}(\dot{x}) \geq 0$, and $\mathfrak{F}\dot{u}(\dot{x}) \geq 0$, for all $\dot{x} \in [\dot{a}, \dot{b}]$

(iv) For all $\dot{x}, t \in [\dot{a}, \dot{b}]$ and $\dot{u}, \dot{v} \in \mathbb{C}[\dot{a}, \dot{b}]$, $|\omega(t, \dot{x}, \dot{u}(\dot{x})) - \omega(t, \dot{x}, \dot{v}(\dot{x}))| \leq (\dot{k}'\dot{b} - \dot{a})(\max|\dot{u}(\dot{x}) - \dot{v}(\dot{x})|)$; $\dot{k}' \in [0, 1)$

(v) There exists $\dot{u}_1 \in \mathbb{C}[\dot{a}, \dot{b}]$ such that $\dot{u}_1(t) \geq 0$ and $\mathfrak{F}\dot{u}_1(t) \geq 0$, for all $t \in [\dot{a}, \dot{b}]$

Then, nonlinear integral equation (49) has a solution in $\mathbb{C}[\dot{a}, \dot{b}]$.

Proof. Define a mapping $\mathfrak{F} : \mathbb{C}[\dot{a}, \dot{b}] \longrightarrow \mathbb{C}[\dot{a}, \dot{b}]$ by

$$\mathfrak{F}\dot{u}(t) = v(t) + \int_{\dot{a}}^{\dot{b}} \omega(t, \dot{x}, \dot{u}(\dot{x})) d\dot{x}, \quad (50)$$

for all $\dot{u} \in \mathbb{C}[\dot{a}, \dot{b}]$ and for all $t \in [\dot{a}, \dot{b}]$. Then, \mathfrak{F} is a continuous mapping.

Define a mapping $\varrho : \mathbb{C}[\dot{a}, \dot{b}] \times \mathbb{C}[\dot{a}, \dot{b}] \longrightarrow \mathbb{R}$ by

$$\varrho(\dot{u}, \dot{v}) = \begin{cases} 2, & \text{if } \dot{u}(\dot{x}), \dot{v}(\dot{x}) \in [0, \infty), \dot{x} \in [\dot{a}, \dot{b}], \\ 1, & \text{otherwise.} \end{cases} \quad (51)$$

We shall show that \mathfrak{F} is a modified ϱ -proximal admissible mapping. Indeed, for $\dot{u}, \dot{v} \in \mathbb{C}[\dot{a}, \dot{b}]$ such that $\varrho(\mathfrak{F}\dot{u}, \mathfrak{F}\dot{v}) \geq 1$, we have $\dot{u}(\dot{x}), \dot{v}(\dot{x}) \geq 0$, for all $\dot{x} \in [\dot{a}, \dot{b}]$. It follows from condition (iii) that $\mathfrak{F}\dot{u}(\dot{x}), \mathfrak{F}\dot{v}(\dot{x}) \geq 0$.

Therefore, $\varrho(\mathfrak{F}\dot{u}(\dot{x}), \mathfrak{F}\dot{v}(\dot{x})) > 1$, and hence, \mathfrak{F} is a modified ϱ -proximal admissible mapping and ϱ is transitive. Now, we claim that \mathfrak{F} is a $\varrho - \vartheta$ contraction. Let $\vartheta(\dot{x}) = \dot{k}$, for all $\dot{x} \in [\dot{a}, \dot{b}]$. Consider

$$\begin{aligned} |\mathfrak{F}\dot{u}(\dot{x}) - \mathfrak{F}\dot{v}(\dot{x})| &= \left| \int_{\dot{a}}^{\dot{b}} \omega(t, \dot{x}, \dot{u}(\dot{x})) d\dot{x} - \int_{\dot{a}}^{\dot{b}} \omega(t, \dot{x}, \dot{v}(\dot{x})) d\dot{x} \right| \\ &\leq \int_{\dot{a}}^{\dot{b}} |\omega(t, \dot{x}, \dot{u}(\dot{x})) - \omega(t, \dot{x}, \dot{v}(\dot{x}))| d\dot{x} \\ &\leq \frac{\dot{k}}{\dot{b} - \dot{a}} (\max |\dot{u}(\dot{x}) - \dot{v}(\dot{x})|) \int_{\dot{a}}^{\dot{b}} d(\dot{x}) \\ &\leq \frac{\dot{k}}{\dot{b} - \dot{a}} (\dot{b} - \dot{a}) (\max |\dot{u}(\dot{x}) - \dot{v}(\dot{x})|) \\ &\leq \dot{k}(\varrho(\dot{u}, \dot{v})) \\ &\leq \vartheta(\dot{u})(\varrho(\dot{u}, \dot{v})). \end{aligned} \quad (52)$$

It implies that

$$\varrho(\mathfrak{F}(\dot{u}), \mathfrak{F}(\dot{v}))\varrho(\mathfrak{F}(\dot{u}), \mathfrak{F}(\dot{v})) \leq \vartheta(\dot{u})(\varrho(\dot{u}, \dot{v})). \quad (53)$$

Then, \mathfrak{F} is a $\varrho - \vartheta$ contraction. Let $\{\mathfrak{F}\dot{u}_n\} \subseteq \mathbb{C}[\dot{a}, \dot{b}]$ such that $\varrho(\mathfrak{F}\dot{u}_n, \mathfrak{F}\dot{u}_{n+1}) \geq 1$ and $\lim_{n \rightarrow +\infty} \mathfrak{F}\dot{u}_n = \mathfrak{F}\dot{u} \in \mathbb{C}[\dot{a}, \dot{b}]$. Then, $\dot{u}(\dot{x}), \dot{u}_n(\dot{x}) \in [0, \infty)$, for all $\dot{x} \in [\dot{a}, \dot{b}]$ and $n \geq 0$. Therefore, $\varrho(\mathfrak{F}\dot{u}_n, \mathfrak{F}\dot{u}) > 1$, for all $n \geq 1$. Therefore, we conclude all the hypotheses of Theorem 1 are satisfied. Thus, equation (49) has a solution $\dot{u} \in \mathbb{C}[\dot{a}, \dot{b}]$.

7. Conclusion

Using the concept of modified ϱ -proximal admissible in the setting of metric space (\mathbb{N}, ϱ) , we find some novel best proximity point results for $\varrho - \vartheta$ -contraction mappings. Many known results in the literature are also generalized by our findings.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The first author is grateful for a Junior Research Fellowship from the UGC-CSIR. The second author is grateful for the research funding 02011/11/2020/NBHM (RP)/R&D-II/7830 from the NBHM, DAE, India.

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Research Article

Generalised Presic Type Operators in Modular Metric Space and an Application to Integral Equations of Caratheodory Type Functions

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Received 7 June 2021; Accepted 7 September 2021; Published 7 October 2021

Academic Editor: Naeem Saleem

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Extending the Presic type operators to modular spaces, we introduce generalised Presic type w -contractive mappings and strongly w -contractive mappings in a modular metric space and establish fixed-point theorems for such contractions in modular spaces. Ulam–Hyers stability of the fixed-point equation involving Presic type operators is also discussed. Our results extend and generalise some known results in the literature. The results are supported by appropriate example and an application to Caratheodory type integral equation.

1. Introduction

Maurice Ren Fréchet [1] introduced the general and axiomatic form of distance as “ L —space.” Felix Hausdorff [2] reexamined it as a metric space in the setting of points which has been refined, discussed, and generalised in numerous ways. Bakhtin [3], Branciari [4], George et al. [5], and Mitrović and Radenović [6] introduced the notions of a b -metric, a rectangular metric, a rectangular b -metric, and b_v^s -metric, respectively. The basic concepts and theory of modular space was formulated in [7]. Later, in [8], the authors established fixed-point theorems in a modular function space. In 2008, Chistyakov [9] introduced the notion of a modular metric and the corresponding modular space. Chistyakov, in [10], established the existence of fixed point for contractive maps and strongly contractive maps in modular metric spaces. Umit et al. [11] introduced Bogin type w -contractions and

proved fixed-point theorems for such contractions in a w -complete modular metric space and provided application to antiperiodic boundary value problems. Furthermore, we see that Chaipunya et al. [12] introduced Geraghty type theorems and Turkoglu and Kilinc [13] introduced Caristi type theorems in a modular metric space and gave its applications in integral equations. For more results of fixed-point theorems in modular metric space and some applications of fixed-point theorems, readers may refer to [14–20]. On the other hand, Presic [21, 22] extended the Banach contraction principle to the product of a finite number of metric spaces and later Ciric and Presic [23] generalised the result of Presic [21] and proved the following.

Theorem 1. Let (X, d) be a metric space, k be a positive integer, and $T: X^k \rightarrow X$ be a mapping satisfying the following condition:

$$d(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \leq \lambda \cdot \max\{d(x_1, x_2), d(x_2, x_3), \dots, d(x_k, x_{k+1}), \dots\} \quad (1)$$

where x_1, x_2, \dots, x_{k+1} are arbitrary elements in X and $\lambda \in (0, 1)$. Then, there exists some $x \in X$ such that $x = T(x, x, \dots, x)$. Moreover, if x_1, x_2, \dots, x_k are arbitrary points in X and for $n \in \mathbb{N}$, $x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$, then the sequence $\langle x_n \rangle$ is convergent and if $\lim_{n \rightarrow \infty} x_n = x$, then $x = T(x, x, \dots, x)$. If, in addition, T satisfies $D(T(u, u, \dots, u), T(v, v, \dots, v)) < d(u, v)$, for all $u, v \in X$. Then, x is the unique point satisfying $x = T(x, x, \dots, x)$.

Later, the above results were further extended and generalised to a b -metric space, cone metric space, and cone b -metric space by many authors, and fruitful applications were found for such results (see [24–29]). The aim of this work is to introduce generalised Presic type contractions (which includes Ciric–Presic type contraction and Presic type contractions) in modular metric spaces and establish fixed-point theorems for such contraction mappings in a w -complete modular metric space. Our results show symmetric transformation of the well-known Ciric–Presic theorem [23] and Presic theorem [21] from ordinary metric space to a modular metric space. We have introduced Ulam–Hyers stability of fixed-point equations involving Presic type operators in a modular metric space. We have also provided an application of our result to prove the existence of solution of an integral equation of Caratheodory type functions. Our results extend and generalise many known results in the literature.

2. Preliminaries

The following basic concepts of modular spaces are from [9, 10]. Let $X \neq \emptyset$ and $w: (0, \infty) \times X \times X \rightarrow [0, \infty]$ be a given function. For any $\lambda \in \mathbb{R}$, $x, y \in X$, we write $w(\lambda, x, y)$ as $w_\lambda(x, y)$.

Definition 1 (see [10]). We call w a modular metric on X if for all $\lambda, \mu > 0$ and $x, y, z \in X$, the following relations hold:

$$\begin{aligned} X_w &= X_w(x_0) = \{x \in X: w_\lambda(x, x_0) \rightarrow 0 \text{ as } \lambda \rightarrow \infty\}, \\ X_w^* &= X_w^*(x_0) = \{x \in X: \text{there exists } \lambda = \lambda(x) > 0 \text{ such that } w_\lambda(x, x_0) < \infty\}, \end{aligned} \quad (6)$$

are said to be modular spaces (around x_0).

Clearly, $X_w \subset X_w^*$. Note that if w is metric modular on X , then modular space X_w can be equipped with a metric generated by w given by

$$d_w(x, y) = \inf\{\lambda > 0: w_\lambda(x, y) \leq \lambda\}, \quad (7)$$

for any $x, y \in X_w$. Moreover, if w is a convex modular on X , then by Section 3.5 and Theorem 3.6 of [9], $X_w = X_w^*$; i.e., the two modular spaces coincide, and this common set can be endowed with a metric d_w^* .

$$\begin{aligned} (\text{mm1}) \quad w_\lambda(x, y) &= 0 \Leftrightarrow x = y \\ (\text{mm2}) \quad w_\lambda(x, y) &= w(\lambda, y, x) \\ (\text{mm3}) \quad w_{\lambda+\mu}(x, y) &\leq w_\lambda(x, z) + w_\mu(z, y) \end{aligned}$$

In this case, w is said to be a metric modular on X . Instead of (mm1), if w satisfies only

$$w_\lambda(x, x) = 0, \quad (2)$$

for all $\lambda > 0$ and $x \in X$, then w is said to be pseudomodular on X .

Also, w is strict modular on X if instead of (mm1), it satisfies (2) and for $x, y \in X$, there exists a number $\lambda > 0$ (possibly depending on x and y), such that

$$w_\lambda(x, y) = 0 \implies x = y. \quad (3)$$

Clearly, if w is strict modular, then it is modular, which in turn implies w is pseudomodular on X .

A metric modular w on X is convex if it satisfies the (stronger) inequality, in lieu of (mm3):

$$w_{\lambda+\mu}(x, y) \leq \frac{\lambda}{\lambda+\mu} w_\lambda(x, z) + \frac{\mu}{\lambda+\mu} w_\mu(z, y), \quad (4)$$

for all $\lambda, \mu > 0$.

The main property of a modular w on a set X is that (see Section 2.3 of [9]), given $x, y \in X$, the function $w_\lambda(x, y)$ is nonincreasing on $(0, \infty)$. If $0 < \mu < \lambda$, then (mm1) and (mm3) imply

$$w_\lambda(x, y) \leq w_{\lambda-\mu}(x, x) + w_\mu(x, y) = w_\mu(x, y). \quad (5)$$

Definition 2 (see [10]). Given a pseudomodular w on X , the two sets

$$d_w^*(x, y) = \inf\{\lambda > 0: w_\lambda(x, y) \leq 1\}. \quad (8)$$

Henceforth, (X_w, w) and (X_w^*, w) represent modular metric spaces induced by w .

Definition 3 (see [10]). Let w be a metric modular on X . We have the following:

- (i) A sequence $\{x_n\}$ in X_w is w -convergent (or modular convergent) to some $x \in X$, if there exists $\lambda > 0$, such that $\lim_{n \rightarrow \infty} w_\lambda(x_n, x) = 0$

- (ii) A sequence $\{x_n\}$ in X is w -Cauchy if $\exists \lambda > 0$, possibly dependent on $\{x_n\}$, such that $w_\lambda(x_m, x_n) = 0$ as $m, n \rightarrow \infty$
- (iii) X is w -complete if every w -Cauchy sequence is w -convergent

From [10], we have that X_w and X_w^* are closed with respect to w convergence, if w is pseudomodular on X . Also, if w is strict, then the modular limit, if exists, is unique. If w is (not necessarily convex) modular on X , then $v_\lambda(x, y) = (w_\lambda(x, y)/\lambda)$ is always convex modular on X .

3. Main Results

In this section, we introduce Presic type operators in modular metric space and prove existence of unique fixed points for such operators.

Let R_+ be the set of real numbers, and consider a function $\Phi: R_+^k \rightarrow R_+$ such that

- (a) Φ is increasing in all variables
- (b) $\Phi(u, u, u, \dots, u) \leq u$, for all $u \in R$

Definition 4. Let w be metric modular on X and $T: X_w^{k*} \rightarrow X_w^*$ (where X_w^{k*} denotes $X_w^* \times X_w^* \times \dots \times X_w^*$ k times) for some positive integer k . Then, we have the following:

- (i) T is a generalised Presic type modular contraction (or a generalised Presic w -contraction), if

$$w_{\xi\lambda}(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \leq \Phi(w_\lambda(x_1, x_2), w_\lambda(x_2, x_3), \dots, w_\lambda(x_k, x_{k+1})), \quad (9)$$

for some $0 < \xi < 1$ and $\lambda_0 > 0$ possibly depending on ξ and for all $0 < \lambda \leq \lambda_0$ and $x_1, x_2, \dots, x_{k+1} \in X_w^*$.

- (ii) T is a generalised Presic type strongly modular contraction (or a generalised Presic strong w -contraction), if

$$w_{\xi\lambda}(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \leq \xi \cdot \Phi(w_\lambda(x_1, x_2), w_\lambda(x_2, x_3), \dots, w_\lambda(x_k, x_{k+1})), \quad (10)$$

for some $0 < \xi < 1$ and $\lambda_0 > 0$ possibly depending on ξ and for all $0 < \lambda \leq \lambda_0$ and $x_1, x_2, \dots, x_{k+1} \in X_w^*$.

- (iii) T is a Ciric–Presic type modular contraction (or a Ciric–Presic w -contraction), if

$$w_{\xi\lambda}(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \leq \max\{w_\lambda(x_1, x_2), w_\lambda(x_2, x_3), \dots, w_\lambda(x_k, x_{k+1})\}, \quad (11)$$

for some $0 < \xi < 1$ and $\lambda_0 > 0$ possibly depending on ξ and for all $0 < \lambda \leq \lambda_0$ and $x_1, x_2, \dots, x_{k+1} \in X_w^*$.

- (iv) T is a Ciric–Presic type strongly modular contraction (or a Ciric–Presic strong w -contraction), if

$$w_{\xi\lambda}(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \leq \xi \cdot \max\{w_\lambda(x_1, x_2), w_\lambda(x_2, x_3), \dots, w_\lambda(x_k, x_{k+1})\}, \quad (12)$$

for some $0 < \xi < 1$ and $\lambda_0 > 0$ possibly depending on ξ and for all $0 < \lambda \leq \lambda_0$ and $x_1, x_2, \dots, x_{k+1} \in X_w^*$.

- (v) T is a Presic type modular contraction (or a Presic w -contraction), if there exists $\beta_i \geq 0, i = 1, 2, \dots, k$, $\sum_{i=1}^k \beta_i < 1$, such that

$$w_{\xi\lambda}(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \leq \beta_1 w_\lambda(x_1, x_2) + \beta_2 w_\lambda(x_2, x_3) + \dots + \beta_k w_\lambda(x_k, x_{k+1}), \quad (13)$$

for some $0 < \xi < 1$ and $\lambda_0 > 0$ possibly depending on ξ and for all $0 < \lambda \leq \lambda_0$ and $x_1, x_2, \dots, x_{k+1} \in X_w^*$.

Theorem 2. Let w be a strict metric modular on X and (X_w^*, w) be w -complete. For any positive integer k , let

$T: X_w^{k*} \longrightarrow X_w^*$ be a generalised Presic type strongly w -contractive mapping. If there exists x_1, x_2, \dots, x_k in X , such that $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, then T has a fixed point; that is, there exists an $x \in X$ such that $x = T(x, x, \dots, x)$. Moreover, for any x_1, x_2, \dots, x_k in X , with $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, the sequence $\langle x_n \rangle$ given by $x_{k+n} = T(x_1, x_2, \dots, x_k)$ converges to a fixed point of T . Furthermore, if T satisfies

$$w_{\xi\lambda}(T(u, u, \dots, u), T(v, v, \dots, v)) < w_\lambda(u, v), \quad (14)$$

for all $u, v \in X$ or if $\xi \in (0, (1/k))$, then the fixed point x is unique.

Proof. For arbitrary elements x_1, x_2, \dots, x_k in X with $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda$

$(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, define the sequence $\langle x_n \rangle$ in X given by $x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$, $n \in \mathbb{N}$. By the w -contractivity of T , there exist two numbers $0 < \xi < 1$ and $\lambda_0 = \lambda_0(\xi)$ such that condition (10) holds. Let $\alpha_{n_{\xi p\lambda}} = w_{\xi p\lambda}(x_n, x_{n+1})$, where $\lambda \leq \lambda_0$. Then, $\xi^p \lambda < \lambda < \lambda_0$ for any positive integer p . Using the method of induction, we will prove that for any nonnegative integer p ,

$$\alpha_{n_{\xi p\lambda}} \leq R_{\xi\lambda} \theta^n, \quad (15)$$

where $\theta = \xi^{(1/k)}$ and $R_{\xi\lambda} = \max((\alpha_{1\lambda}/\theta), (\alpha_{2\lambda}/\theta^2), \dots, (\alpha_{k\lambda}/\theta^k))$. Clearly, by the definition of $R_{\xi\lambda}$, (15) is true for $n = 1, 2, \dots, k$ and $p = 0$. Let the $k+1$ inequalities $\alpha_{n_{\xi p\lambda}} \leq R_{\xi\lambda} \theta^n, \alpha_{(n+1)_{\xi p\lambda}} \leq R_{\xi\lambda} \theta^{n+1}, \dots, \alpha_{(n+k)_{\xi p\lambda}} \leq R_{\xi\lambda} \theta^{n+k}$ hold true. Then,

$$\begin{aligned} \alpha_{(n+k)_{\xi^{p+1}\lambda}} &= w_{\xi^{p+1}\lambda}(x_{n+k}, x_{n+k+1}) \\ &= w_{\xi^{p+1}\lambda}(T(x_n, x_{n+1}, \dots, x_{n+k-1}), T(x_{n+1}, x_{n+2}, \dots, x_{n+k})) \\ &\leq \xi \Phi(w_{\xi^p\lambda}(x_n, x_{n+1}), w_{\xi^p\lambda}(x_{n+1}, x_{n+2}), \dots, w_{\xi^p\lambda}(x_{n+k-1}, x_{n+k})) \\ &= \xi \Phi(\alpha_{n_{\xi p\lambda}}, \alpha_{n+1_{\xi p\lambda}}, \dots, \alpha_{(n+k-1)_{\xi p\lambda}}) \\ &\leq \xi \Phi(R_{\xi\lambda} \theta^n, R_{\xi\lambda} \theta^{n+1}, \dots, R_{\xi\lambda} \theta^{n+k-1}) \\ &\leq \xi R_{\xi\lambda} \theta^n \\ &= R_{\xi\lambda} \theta^{n+k}, \end{aligned} \quad (16)$$

$$\begin{aligned} \alpha_{(n+k+1)_{\xi p\lambda}} &= w_{\xi p\lambda}(x_{n+k+1}, x_{n+k+2}) \\ &= w_{\xi p\lambda}(T(x_{n+1}, x_{n+2}, \dots, x_{n+k}), T(x_{n+2}, x_{n+3}, \dots, x_{n+k+1})) \\ &\leq \xi \Phi(w_{\xi^{p-1}\lambda}(x_{n+1}, x_{n+2}), w_{\xi^{p-1}\lambda}(x_{n+2}, x_{n+3}), \dots, w_{\xi^{p-1}\lambda}(x_{n+k}, x_{n+k+1})) \\ &= \xi \Phi(\alpha_{n+1_{\xi^{p-1}\lambda}}, \alpha_{n+2_{\xi^{p-1}\lambda}}, \dots, \alpha_{(n+k)_{\xi^{p-1}\lambda}}) \\ &= \xi \Phi(\alpha_{n+1_{\xi p\lambda}}, \alpha_{n+2_{\xi p\lambda}}, \dots, \alpha_{(n+k)_{\xi p\lambda}}) \\ &\leq \xi \Phi(R_{\xi\lambda} \theta^{n+1}, R_{\xi\lambda} \theta^{n+2}, \dots, R_{\xi\lambda} \theta^{n+k}) \\ &\leq \xi R_{\xi\lambda} \theta^{n+1} \\ &= R_{\xi\lambda} \theta^{n+k+1}. \end{aligned} \quad (17)$$

Thus, inductive proof of (15) is complete. Set $\lambda_1 = (1 - \xi)\lambda_0$; then, $\xi^i \lambda_1 < \lambda_1 < \lambda_0$. Let integers n and m be

such that $n > m$. We set $\lambda = \lambda(n, m) = \xi^m \lambda_1 + \xi^{m+1} \lambda_1 + \dots + \xi^{n-1} \lambda_1 = \xi^m ((1 - \xi^{n-m})/(1 - \xi)) \lambda_1$. Then, we have

$$\begin{aligned}
w_\lambda(x_m, x_n) &\leq w_{\xi^m \lambda_1}(x_m, x_{m+1}) + w_{\xi^{m+1} \lambda_1}(x_{m+1}, x_{m+2}) + \cdots + w_{\xi^{n-1} \lambda_1}(x_{n-1}, x_n) \\
&\leq [R_{\xi \lambda_1} \theta^m + R_{\xi \lambda_1} \theta^{m+1} + \cdots + R_{\xi \lambda_1} \theta^n] \\
&\leq R_{\xi \lambda_1} (\theta)^m (1 + \theta + \theta^2 + \cdots) \\
&= \frac{R_{\xi \lambda_1} \theta^m}{1 - \theta}.
\end{aligned} \tag{18}$$

Taking into account that $\lambda_0 = (\lambda_1 / (1 - \xi)) > \xi^m ((1 - \xi^{n-m}) / (1 - \xi)) \lambda_1 = \lambda$ for all $n > m$, we get

$$\begin{aligned}
w_{\lambda_0}(x_m, x_n) &\leq w_\lambda(x_m, x_n) \\
&\leq \frac{R_{\lambda_0} (\xi \theta)^m}{1 - \theta} \\
&\longrightarrow 0, \quad \text{as } m \longrightarrow \infty.
\end{aligned} \tag{19}$$

Therefore, $\{x_n\}$ is w -Cauchy in X_w^* . By w -completeness of X_w^* , there exists x in X_w^* such that $\lim_{n \rightarrow \infty} w_{\lambda_0}(x_n, x) = 0$, and by strictness of w , the limit is unique. Let $\epsilon > 0$ be arbitrary. Then, we can find a natural number N such that $w_{\lambda_0}(x_n, x_{n+1}) < \epsilon$ and $w_{\lambda_0}(x_n, x) < \epsilon$ for all $n \geq N$. Then, for any integer $n \geq N$, we have

$$\begin{aligned}
w_{(1+k\xi)\lambda_0}(x, T(x, x, \dots, x)) &\leq w_{\lambda_0}(x, x_{n+k}) + w_{k\xi\lambda_0}(x_{n+k}, T(x, x, \dots, x)) \\
&= w_{\lambda_0}(x, x_{n+k}) + w_{\xi\lambda_0 + \xi\lambda_0 + \cdots k \text{ times}}(T(x_n, x_{n+1}, \dots, x_{n+k-1}), T(x, x, \dots, x)) \\
&\leq w_{\lambda_0}(x, x_{n+k}) + w_{\xi\lambda_0}(T(x, x, \dots, x), T(x, x, \dots, x_n)) \\
&\quad + w_{\xi\lambda_0}(T(x, x, \dots, x_n), T(x, x, \dots, x_{n+1})) \\
&\quad + \cdots + w_{\xi\lambda_0}(T(x, x_n, \dots, x_{n+k-2}), T(x_n, x_{n+1}, \dots, x_{n+k-1})) \\
&\leq w_{\lambda_0}(x, x_{n+k}) + \xi \Phi(w_{\lambda_0}(x, x), w_{\lambda_0}(x, x_n)) \\
&\quad + \xi \Phi(w_{\lambda_0}(x, x), w_{\lambda_0}(x, x_n), w_{\lambda_0}(x_n, x_{n+1})) + \cdots \\
&\quad + \xi \Phi(w_{\lambda_0}(x, x_n), w_{\lambda_0}(x_n, x_{n+1}), \dots, w_{\lambda_0}(x_{n+k-2}, x_{n+k-1})) \\
&< \epsilon.
\end{aligned} \tag{20}$$

As $\epsilon \longrightarrow 0$, we get $w_{(1+k\xi)\lambda_0}(x, T(x, x, \dots, x)) = 0$. By the strictness of w , $T(x, x, \dots, x) = x$. For uniqueness, suppose there exists $y \in X_w^*$, where $y \neq x$, such that $T(y, \dots, y) = y$. Then, using (14), we get

$$w_{\xi\lambda_0}(x, y) = w_{\xi\lambda_0}(T(x, \dots, x), T(y, \dots, y)) < w_{\lambda_0}(x, y), \tag{21}$$

a contradiction (as $\xi\lambda_0 < \lambda_0$). Thus, the uniqueness of x is established.

Now, if $\xi \in (0, (1/k))$, then

$$\begin{aligned}
 w_{\xi\lambda_0}(x, y) &= w_{\xi\lambda_0}(T(x, \dots, x), T(y, \dots, y)) \\
 &\leq \frac{\xi\lambda_0}{k\xi\lambda_0} w_{(\xi\lambda_0/k)}(T(x, x, \dots, x), T(x, x, \dots, y)) \\
 &\quad + \frac{\xi\lambda_0}{k\xi\lambda_0} w_{(\xi\lambda_0/k)}(T(x, x, \dots, x, y), T(x, x, \dots, x, y, y)) \\
 &\quad + \dots + \frac{\xi\lambda_0}{k\xi\lambda_0} w_{(\xi\lambda_0/k)}(T(x, y, \dots, y), T(y, y, \dots, y)) \\
 &\leq \xi \frac{1}{k} \Phi\left(w_{(\lambda_0/k)}(x, x), w_{(\lambda_0/k)}(x, y)\right) + \xi \frac{1}{k} \Phi\left(w_{(\lambda_0/k)}(x, x), w_{(\lambda_0/k)}(x, y), w_{(\lambda_0/k)}(y, y)\right) \\
 &\quad + \dots + \xi \frac{1}{k} \Phi\left(w_{(\lambda_0/k)}(x, y), w_{(\lambda_0/k)}(y, y)\right) \\
 &\leq \xi \frac{k}{k} w_{(\lambda_0/k)}(x, y) \\
 &< w_{(\lambda_0/k)}(x, y),
 \end{aligned} \tag{22}$$

a contradiction as $\xi\lambda_0 < (\lambda_0/k)$. Hence, $T(x, x, \dots, x) = x$ is unique in X . \square

In the next result, using a convex modular metric, we prove the existence of a fixed point for a generalised Presic type modular contractive mapping.

Theorem 3. Let w be a strict convex modular on X and (X_w^*, w) be w -complete. For any positive integer k , let $T: X_w^{k*} \rightarrow X_w^*$ be a generalised Presic w -contractive mapping. If there exists x_1, x_2, \dots, x_k in X , such that $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, then T has a fixed point; that is, there exists an $x \in X$ such that $x = T(x, x, \dots, x)$. Moreover, for any x_1, x_2, \dots, x_k in X , with $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, the sequence $\langle x_n \rangle$ given by $x_{k+n} = T(x_1, x_2, \dots, x_k)$ converges to a fixed point of T . Furthermore, if T satisfies

$$w_{\xi\lambda}(T(u, u, \dots, u), T(v, v, \dots, v)) < w_\lambda(u, v), \tag{23}$$

for all $u, v \in X$ or if $\xi \in (0, (1/k))$, then the fixed point x is unique.

Proof. We take $u_\lambda(x, y) = \lambda \cdot w_\lambda(x, y)$ for all $x, y \in X$ and $\lambda > 0$. Then, for all $\lambda, \mu > 0$ and $x, y, z \in X$, we have

$$\begin{aligned}
 u_{\lambda+\mu}(x, y) &= (\lambda + \mu)w_{\lambda+\mu}(x, y) \\
 &\leq (\lambda + \mu) \left\{ \frac{\lambda}{\lambda + \mu} \left(w_\lambda(x, z) + \frac{\mu}{\lambda + \mu} (w_\mu(y, z)) \right) \right. \\
 &= \lambda \cdot w_\lambda(x, z) + \mu \cdot w_\mu(y, z) \\
 &= u_\lambda(x, z) + u_\mu(y, z).
 \end{aligned} \tag{24}$$

So, u is strict modular on X . Also, by (9), for all $0 < \lambda \leq \lambda_0$ and $x_1, x_2, \dots, x_{k+1} \in X_w^*$, we have

$$\begin{aligned}
u_{\xi\lambda}(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) &= \xi\lambda w_{\xi\lambda}(T(x_1, x_2, \dots, x_k), T(x_2, x_3, \dots, x_{k+1})) \\
&\leq \xi\lambda\Phi(w_\lambda(x_1, x_2), w_\lambda(x_2, x_3), \dots, w_\lambda(x_k, x_{k+1})) \\
&= \xi\Phi(\lambda w_\lambda(x_1, x_2), \lambda w_\lambda(x_2, x_3), \dots, \lambda w_\lambda(x_k, x_{k+1})) \\
&= \xi\Phi(u_\lambda(x_1, x_2), u_\lambda(x_2, x_3), \dots, u_\lambda(x_k, x_{k+1})).
\end{aligned} \tag{25}$$

Thus, T is a generalised Presic strong w -contraction mapping for the modular u . Clearly, $w_{\xi\lambda}(x_1, x_2) < \infty, w_{\xi\lambda}(x_2, x_3) < \infty, \dots, w_{\xi\lambda}(x_{k-1}, x_k) < \infty, w_{\xi\lambda}(x_k, T(x_1, x_2, \dots, x_k)) < \infty$ implies $u_{\xi\lambda}(x_1, x_2) < \infty, u_{\xi\lambda}(x_2, x_3) < \infty, \dots, u_{\xi\lambda}(x_{k-1}, x_k) < \infty, u_{\xi\lambda}(x_k, T(x_1, x_2, \dots, x_k)) < \infty$. By Theorem 2, there exists $x \in X$ such that $u_{(1+k\xi)\lambda_0}(x, T(x, x, \dots, x)) = 0$. The remaining part of the proof follows the same line as in Theorem 2. \square

Taking $\Phi(u_1, u_2, \dots, u_k) = \max\{u_1, u_2, \dots, u_k\}$ in Theorems 1 and 2, we get the following.

Theorem 4. Let w be strict metric modular on X and (X_w^*, w) be w -complete. For any positive integer k , let $T: X_w^{k*} \rightarrow X_w^*$ be a Ciric-Presic strongly w -contractive mapping. If there exists x_1, x_2, \dots, x_k in X , such that $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, then T has a fixed point; that is, there exists an $x \in X$ such that $x = T(x, x, \dots, x)$. Moreover, for any x_1, x_2, \dots, x_k in X , with $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, the sequence $\langle x_n \rangle$ given by $x_{k+n} = T(x_1, x_2, \dots, x_k)$ converges to a fixed point of T . Furthermore, if T satisfies

$$w_{\xi\lambda}(T(u, u, \dots, u), T(v, v, \dots, v)) < w_\lambda(u, v), \tag{26}$$

for all $u, v \in X$ or if $\xi \in (0, (1/k))$, then the fixed point x is unique.

Theorem 5. Let w be strict convex modular on X and (X_w^*, w) be w -complete. For any positive integer k , let $T: X_w^{k*} \rightarrow X_w^*$ be a Ciric-Presic w -contractive mapping. If there exists x_1, x_2, \dots, x_k in X , such that $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, then T has a fixed point; that is, there exists an $x \in X$ such that $x = T(x, x, \dots, x)$. Moreover, for any x_1, x_2, \dots, x_k in X , with $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, the sequence $\langle x_n \rangle$ given by $x_{k+n} = T(x_1, x_2, \dots, x_k)$ converges to a fixed point of T . Furthermore, if T satisfies

$$w_{\xi\lambda}(T(u, u, \dots, u), T(v, v, \dots, v)) < w_\lambda(u, v), \tag{27}$$

for all $u, v \in X$ or if $\xi \in (0, (1/k))$, then the fixed point x is unique.

Taking $\Phi(u_1, u_2, \dots, u_k) = \beta_1 u_1 + \beta_2 u_2, \dots, \beta_k u_k$, with $0 \leq \beta_i, \sum_{i=1}^k \beta_i < 1$ in Theorem 3, we get the following.

Theorem 6. Let w be strict convex modular on X and (X_w^*, w) be w -complete. For any positive integer k , let $T: X_w^{k*} \rightarrow X_w^*$ be a Presic type w -contractive mapping. If there exists x_1, x_2, \dots, x_k in X , with $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, then T has a fixed point; that is, there exists an $x \in X$ such that $x = T(x, x, \dots, x)$. Moreover, for any x_1, x_2, \dots, x_k in X , with $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, the sequence $\langle x_n \rangle$ given by $x_{k+n} = T(x_1, x_2, \dots, x_k)$ converges to a fixed point of T . Furthermore, if T satisfies

$$w_{\xi\lambda}(T(u, u, \dots, u), T(v, v, \dots, v)) < w_\lambda(u, v), \tag{28}$$

for all $u, v \in X$ or if $\xi \in (0, (1/k))$, then the fixed point x is unique.

Remark 1. For $k = 1$, Theorems 4 and 5 reduce to Theorem 10 and Theorem 5.4 of [10].

Remark 2. Theorems 4 and 5 extend the results of Ciric and Presic [23] to a modular space.

Remark 3. Theorem 6 extends the result of Presic [21] to a modular space.

Example 1. Let $X = [0, 1]$. Define the mapping $w: (0, \infty) \times X \times X \rightarrow [0, \infty]$ by $w_\lambda(a_1, a_2) = (4/15\lambda^2)|a_1 - a_2|$. Note that $w_\lambda(a_1, a_2) < \infty$ for all $a_1, a_2 \in X$, then $X = X_w^*$ and (X_w^*, w) is a complete modular metric space. Let $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by $\Phi(t_1, t_2) = \max\{t_1, t_2\}$. Let $T: X_w^* \times X_w^* \rightarrow X_w^*$ given by $T(a_1, a_2) = (32/1000)(a_1 + a_2)$. Then,

$$\begin{aligned}
w_{\xi\lambda}(T(a_1, a_2), T(a_2, a_3)) &= w_{\xi\lambda}\left(\frac{32}{1000}(a_1 + a_2), \frac{32}{1000}(a_2 + a_3)\right) \\
&= \frac{4}{15\xi^2\lambda^2} \left| \frac{32}{1000}(a_1 + a_2) - \frac{32}{1000}(a_2 + a_3) \right| \\
&= \frac{4}{15\xi^2\lambda^2} \cdot \frac{32}{1000} |(a_1 + a_2) - (a_2 + a_3)| \\
&= \frac{4}{15\xi^2\lambda^2} \cdot \frac{32}{1000} |(a_1 - a_2) + (a_2 - a_3)| \\
&= \frac{8}{15\xi^2\lambda^2} \cdot \frac{32}{1000} \frac{|(a_1 - a_2) + (a_2 - a_3)|}{2} \\
&= \frac{2}{\xi^2} \cdot \frac{32}{1000} \frac{|(4/15\lambda^2)(a_1 - a_2) + (4/15\lambda^2)(a_2 - a_3)|}{2} \\
&\leq \frac{2}{\xi^2} \cdot \frac{32}{1000} \max\{w_\lambda(a_1, a_2), w_\lambda(a_2, a_3)\}.
\end{aligned} \tag{29}$$

If $\xi = 0.4 \in (0, (1/2))$, then $w_{0.4\lambda}(T(a_1, a_2), T(a_2, a_3)) \leq 0.4\Phi(w_\lambda(a_1, a_2), w_\lambda(a_2, a_3))$ for all $a_1, a_2, a_3 \in X_w$. Thus, T is a generalised Presic type strongly modular contraction and Ciric–Presic type strongly modular contraction with constant $\xi = 0.4$. Theorems 2 and 4 are applicable, and $(0, 0) \in X_w$ is the unique fixed point of T .

Example 2. Let $X = [0, 1]$. Define the mapping $w: (0, \infty) \times X \times X \rightarrow [0, \infty]$ by $w_\lambda(a_1, a_2) = (|a_1 - a_2|/\lambda)$. Note that $w_\lambda(a_1, a_2) < \infty$ for all $a_1, a_2 \in X$; then, $X = X_w^*$ and (X_w^*, w) is a complete convex modular metric space. Let $\Phi: R^3 \rightarrow R$ be given by $\Phi(t_1, t_2, t_3) = t_1 + t_2 + t_3$, if $t_i \neq t_j$ for some i, j and $\Phi(t, t, t) = t$.

Let $T: X_w^* \times X_w^* \times X_w^* \rightarrow X_w^*$ be given by $T(a_1, a_2, a_3) = (1/1.01)(a_1 + a_2 + a_3)$. Then,

$$\begin{aligned}
w_{\xi\lambda}(T(a_1, a_2, a_3), T(a_2, a_3, a_4)) &= w_{\xi\lambda}\left(\frac{1}{1.01}(a_1 + a_2 + a_3), \frac{1}{1.01}(a_2 + a_3 + a_4)\right) \\
&= \frac{1}{\xi\lambda} \cdot \frac{1}{1.01} |(a_1 + a_2 + a_3) - (a_2 + a_3 + a_4)| \\
&= \frac{1}{\xi\lambda} \cdot \frac{1}{1.01} |(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4)|.
\end{aligned} \tag{30}$$

If $\xi = (1/1.01) \in (0, 1)$, then $w_{(1/1.01)\lambda}(T(a_1, a_2, a_3), T(a_2, a_3, a_4)) \leq \Phi(w_\lambda(a_1, a_2), w_\lambda(a_2, a_3), w_\lambda(a_3, a_4))$ for all $a_1, a_2, a_3, a_4 \in X_w$. Thus, T is a generalised Presic type modular contraction, with constant $\xi = (1/1.01)$. Thus, T satisfies the condition of Theorem 3 and $(0, 0, 0) \in X_w$ is the unique fixed point of T .

4. Ulam–Hyers Stability

In this section, we discuss the Ulam–Hyers stability of fixed-point equations involving Presic type operators in a modular metric space. We begin with the following concepts and definitions.

Definition 5. Let w be a metric modular on X and $T: X_w^{k*} \rightarrow X_w^*$ (where X_w^{k*} denotes $X_w^* \times X_w^* \times \dots \times X_w^*$ k times)

for some positive integer k . If for all $(x_1, x_2, \dots, x_k) \in X_w^{k*}$ with $w_\lambda(x_1, x_2) < \infty, w_\lambda(x_2, x_3) < \infty, \dots, w_\lambda(x_{k-1}, x_k) < \infty, w_\lambda(x_k, T(x_1, x_2, \dots, x_k)) < \infty$, the sequence x_{k+n} given by $x_{k+n} = T(x_n, x_{n+1}, \dots, x_{k+n-1})$ converges and the limit (which depends upon (x_1, x_2, \dots, x_k)) is a fixed point of T , then we say that T is a weakly Picard–Presic operator (in short, WPPO). If the fixed point of T is unique, then we say that T is a Picard–Presic operator.

Hereafter, we will make use of the following notations:

$$\begin{aligned}
T^n(x_1, x_2, \dots, x_k) &= x_{k+n}, \\
T^\infty(x_1, x_2, \dots, x_k) &= \lim_{n \rightarrow \infty} T^n(x_1, x_2, \dots, x_k) \\
&= \lim_{n \rightarrow \infty} x_{n+k}.
\end{aligned} \tag{31}$$

Definition 6. Let w be metric modular on X , $T: X_w^{k*} \rightarrow X_w^*$ be a weakly Picard–Presic operator, and $c > 0$ be a real number. We say that T is a c -weakly

Picard–Presic operator (in short, c -WPPO) if for all $x = (x_1, x_2, \dots, x_k) \in X_w^{k*}$, $1 \leq i \leq k$, and for some $0 < \xi < 1$ and $\lambda_0 > 0$ possibly depending on ξ and for all $0 < \lambda \leq \lambda_0$,

$$w_\lambda(x_i, T^\infty(x)) \leq c \max\{w_\lambda(x_1, x_2), w_\lambda(x_2, x_3), \dots, w_\lambda(x_{k-1}, x_k), w_\lambda(x_k, T(x))\}. \quad (32)$$

Let R_+ be the set of nonnegative real numbers, and consider a function $\Psi: R_+ \rightarrow R_+$ such that Ψ is continuous at 0 and $\Psi(0) = 0$.

Definition 7. Let w be a metric modular on X ; $T: X_w^{k*} \rightarrow X_w^*$ be a weakly Picard–Presic operator. We say

that T is a Ψ -weakly Picard–Presic operator (in short, Ψ -WPPO) if for all $x = (x_1, x_2, \dots, x_k) \in X_w^{k*}$, $1 \leq i \leq k$, and for some $0 < \xi < 1$ and $\lambda_0 > 0$ possibly depending on ξ and for all $0 < \lambda \leq \lambda_0$,

$$w_\lambda(x_i, T^\infty(x)) \leq \Psi(\max\{w_\lambda(x_1, x_2), w_\lambda(x_2, x_3), \dots, w_\lambda(x_{k-1}, x_k), w_\lambda(x_k, T(x))\}). \quad (33)$$

Proposition 1. Let $T: X_w^{k*} \rightarrow X_w^*$ be a Ciric–Presic strong w -contraction. Then, T is a c -WPPO.

Indeed, by Theorem 4, T is a WPPO. Now, for any $x = (x_1, x_2, \dots, x_k) \in X_w^{k*}$ and $0 < \xi < 1$, choose integers m, n , and r large enough so that $\xi^{m+1} + \xi^{m+2} + \dots + \xi^{m+n+1} \leq 1$ and $\xi^r + \xi^{r+1} + \dots + \xi^{r+k-i} \leq \xi^{m+1}$. Then, we have

$$\begin{aligned} w_\lambda(x_i, T^n(x)) &\leq w_{(\xi^{m+1} + \xi^{m+2} + \dots + \xi^{m+n+1})\lambda}(x_i, T^m(x)) \\ &\leq w_{\xi^{m+1}\lambda}(x_i, T(x)) + w_{\xi^{m+2}\lambda}(T(x), T^2(x)) + \dots + w_{\xi^{m+n+1}\lambda}(T^{n-1}(x), T^n(x)) \\ &\leq w_{(\xi^r + \xi^{r+1} + \dots + \xi^{r+k-i})\lambda}(x_i, T(x)) + w_{\xi^2\lambda}(T(x), T^2(x)) + \dots + w_{\xi^{n+1}\lambda}(T^{n-1}(x), T^n(x)) \\ &\leq w_{(\xi^r + \xi^{r+1} + \dots + \xi^{r+k-i})\lambda}(x_i, x_{k+1}) + w_{\xi^2\lambda}(x_{k+1}, x_{k+2}) + \dots + w_{\xi^{n+1}\lambda}(x_{n-1}, x^n) \\ &\leq w_{(\xi^r)\lambda}(x_i, x_{i+1}) + w_{(\xi^{r+1})\lambda}(x_{i+1}, x_{i+2}) + \dots + w_{(\xi^{r+k-i-1})\lambda}(x_{k-1}, x_k) + w_{(\xi^{r+k-i})\lambda}(x_k, x_{k+1}) \\ &\quad + w_{\xi^2\lambda}(x_{k+1}, x_{k+2}) + \dots + w_{\xi^{n+1}\lambda}(x_{n-1}, x^n). \end{aligned} \quad (34)$$

Using (15), we get

$$w_\lambda(x_i, T^n(x)) \leq R_{\xi\lambda} [\xi^{(i/k)} + \xi^{(i+1/k)} + \dots + \xi^{(k-1/k)} + \xi^{(k/k)} + \dots + \xi^{(n-1/k)}]. \quad (35)$$

As $n \rightarrow \infty$, we get

$$\begin{aligned}
 w_\lambda(x_i, T^\infty(x)) &\leq R_{\xi\lambda} \left[1 + \xi^{(i/k)} + \xi^{(i+1/k)} + \dots + \xi^{(k-1/k)} + \xi^{(k/k)} + \dots \right] \\
 &\leq \frac{1}{1 - \xi^{(1/k)}} \max \left(\frac{\alpha_{1_\lambda}}{\theta}, \frac{\alpha_{2_\lambda}}{\theta^2}, \dots, \frac{\alpha_{k_\lambda}}{\theta^k} \right) \\
 &\leq \frac{1}{1 - \xi^{(1/k)}} \max \left(\frac{w_\lambda(x_1, x_2)}{\xi^{(1/k)}}, \frac{w_\lambda(x_2, x_3)}{\xi^{(2/k)}}, \dots, \frac{w_\lambda(x_{k-1}, x_k)}{\xi^{(k-1/k)}}, \frac{w_\lambda(x_k, T(x))}{\xi^{(k/k)}} \right) \\
 &\leq \frac{1}{\xi(1 - \xi^{(1/k)})} \max(w_\lambda(x_1, x_2), w_\lambda(x_2, x_3), \dots, w_\lambda(x_{k-1}, x_k), w_\lambda(x_k, T(x))).
 \end{aligned} \tag{36}$$

Hence, T is a c -WPPO with $c = (1/\xi(1 - \xi^{(1/k)}))$.

For some $x^* \in \text{Fix}(X_w^*)$, the attraction basin of x^* with respect to T is given by

$$(AB)_T(x^*) = \{x = (x_1, x_2, \dots, x_k) \in X_w^{k*} : T^n(x) \text{ is defined for all } n \in \mathbb{N} \text{ and } T^n(x) \rightarrow x^*\}, \tag{37}$$

and the attraction basin of T is given by

$$(AB)_T = \bigcup_{x^* \in \text{Fix}(T)} (AB)_T(x^*). \tag{38}$$

For some $T: X_w^{k*} \rightarrow X_w^*$, we consider the fixed-point equation

$$x = T(x, x, \dots, x), \quad x \in X_w^*, \tag{39}$$

and the inequality

$$w_\lambda(y, T(y, y, \dots, y)) \leq \epsilon, \quad \text{for all } \lambda > 0. \tag{40}$$

Definition 8. Equation (39) is Ulam–Hyers stable if there exists $c > 0$ such that for each $\epsilon > 0$ and each solution y^* of (40) with $(y^*, y^*, \dots, y^*) \in (AB)_T$, there exists a solution x^* of fixed-point equation (39) such that $w_\lambda(y^*, x^*) \leq c\epsilon$ for all $\lambda > 0$.

Definition 9. Equation (39) is generalised Ulam–Hyers stable if there exists a function $\phi: R^+ \rightarrow R^+$ increasing, continuous at 0 and $\phi(0) = 0$, such that for each $\epsilon > 0$ and each solution y^* of (40) with $(y^*, y^*, \dots, y^*) \in (AB)_T$, there exists a solution x^* of fixed-point equation (39) such that $w_\lambda(y^*, x^*) \leq \phi(\epsilon)$ for all $\lambda > 0$.

Theorem 7. Let w be strict metric modular on X and (X_w^*, w) be w -complete. For any positive integer k , let $T: X_w^{k*} \rightarrow X_w^*$ be a c -WPPO. Then, fixed-point equation (39) is Ulam–Hyers stable.

Proof. Let $\epsilon > 0$, $\lambda > 0$, and y^* be a solution of (39) with $y = (y^*, y^*, \dots, y^*) \in (AB)_T$; that is, $w_\lambda(y^*, T(y)) \leq \epsilon$. Then, there exists $x^* \in \text{Fix}(T)$; that is, x^* is a solution of (39) such that $T^n(y) \rightarrow x^*$. Since $T: X_w^{k*} \rightarrow X_w^*$ is a c -WPPO, we have

$$\begin{aligned}
 w_\lambda(y^*, T^\infty(y)) &\leq c \cdot \max\{w_\lambda(y^*, y^*), w_\lambda(y^*, y^*), \dots, w_\lambda(y^*, y^*), w_\lambda(y^*, T(y))\} \\
 &\leq c\epsilon.
 \end{aligned} \tag{41}$$

Thus, fixed-point equation (39) is Ulam–Hyers stable. \square

Remark 4. For $k = 1$, Definitions 6–9 reduce to Definitions 1, 2, 7, and 8, respectively, of [30].

5. Application to Integral Equation

Recently, some interesting applications of fixed-point theorems in proving the existence of solutions of various

generalised integral equations and differential equations have been found (see [31–33]. Moreover, in [33], the Ulam–Hyers stability of nonlinear implicit fractional differential equations with Riemann–Liouville fractional derivative was discussed. In this section, we present an application of our results to prove the existence of solution of a generalised integral equation of Caratheodory type and give the conditions under which such integral equations are Ulam–Hyers stable. Let $\phi: R^+ \rightarrow R^+$ be a continuous, convex, increasing, and unbounded function such that

$\phi(u) = 0$ iff $u = 0$. Moreover, ϕ admits the inverse function $\phi^{-1}: R^+ \rightarrow R^+$, which is continuous and strictly increasing, and $\phi^{-1}(u) = 0$ iff $u = 0$. Let \mathfrak{M} be a set of real-valued function on $[a, b] \subset R$ with $a < b$; that is,

$$\mathfrak{M} = \{u: [a, b] \rightarrow \mathfrak{A}, \quad \mathfrak{A} \subset R \text{ is complete.} \quad (42)$$

Define $w: (0, \infty) \times \mathfrak{M} \times \mathfrak{M} \rightarrow [0, \infty]$ for all $\gamma > 0$ and $u, w \in \mathfrak{M}$ by

$$w_\gamma(u, w) = \sup_{\pi} \sum_{i=1}^m \phi \left(\frac{|[u(\tau_i) + w(\tau_{i-1})] - [u(\tau_{i-1}) + w(\tau_i)]|}{\gamma(\tau_i - \tau_{i-1})} \right) (\tau_i - \tau_{i-1}), \quad (43)$$

where the supremum is taken over all partitions $\pi = \{\tau_i\}_{i=0}^n$ of the interval $[a, b]$; that is, $a = \tau_0 < \tau_1 < \tau_2 \cdots < \tau_n = b$. Then, it is known that $w_\gamma(u, w)$ is convex pseudomodular on \mathfrak{M} (see [9, 10]).

For some $u_0 \in R$, consider the constant function $u_0 \in \mathfrak{M}$ given by $u_0(\tau) = u_0$ for all $\tau \in [a, b]$. Define the convex pseudomodular metric space \mathfrak{M}_w^* as

$$\mathfrak{M}_w^* = \mathfrak{M}_w^*(u_0) = \{u \in \mathfrak{M}: \exists \gamma = \gamma(u) > 0, \text{ such that } w_\gamma(u, u_0) < \infty\}. \quad (44)$$

The space \mathfrak{M}_w^* is denoted by $GV_\phi([a, b])$ and is called the space of mappings of bounded generalised ϕ -variations (see [34]). Then, $u \in \mathfrak{M}_w^* = GV_\phi([a, b])$ if and only if $u: [a, b] \rightarrow R$, and there exists a constant $\gamma = \gamma(u) > 0$ such that

$$w_\gamma(u, u_0) = \sup_{\pi} \sum_{i=1}^n \phi \left(\frac{|u(\tau_i) - u(\tau_{i-1})|}{\gamma(\tau_i - \tau_{i-1})} \right) (\tau_i - \tau_{i-1}) < \infty. \quad (45)$$

Clearly, $w(\gamma, u, u_0)$ is independent of u_0 .

Consider the set \mathfrak{M}^1 given by

$$\mathfrak{M}^1 = \{u: [a, b] \rightarrow \mathfrak{A}, u(a) = u_0\} \subset \mathfrak{M}_w^*. \quad (46)$$

Lemma 1. The function w given in (43) defines a strict convex metric modular on \mathfrak{M}^1 .

Proof. It is enough to show that $w_\gamma(u, w) = 0 \implies u(\tau) = w(\tau)$ for all $u, w \in \mathfrak{M}^1$ and $\tau \in [a, b]$. Clearly, for any $\psi, \tau \in [a, b]$, with $\psi \neq \tau$,

$$\phi \left(\frac{|[u(\tau) + w(\psi)] - [u(\psi) + w(\tau)]|}{\gamma|\tau - \psi|} \right) (\tau - \psi) \leq w_\gamma(u, w), \quad (47)$$

which implies

$$|u(\tau) - w(\tau) + w(\psi) - u(\psi)| \leq \gamma|\tau - \psi| \phi^{-1} \left(\frac{w_\gamma(u, w)}{\tau - \psi} \right). \quad (48)$$

Let $w_\gamma(u, w) = 0$. Then, since $\phi^{-1}(0) = 0$, we get

$$|u(\tau) - w(\tau) + w(\psi) - u(\psi)| \leq \gamma|\tau - \psi| \phi^{-1}(0) = 0. \quad (49)$$

Equivalently,

$$u(\tau) - w(\tau) = u(\psi) - w(\psi). \quad (50)$$

Solving the system by taking $\psi = a$ in (50) and using $u(a) = w(a) = u_0$, we obtain $u(\tau) = w(\tau)$ for any $\tau \in [a, b]$. \square

We define

$$\begin{aligned} \mathfrak{M}_w^{*1} &= M_w^* \cap \mathfrak{M}^1 = GV_\phi([a, b]) \cap \mathfrak{M}^1 = \{u \in GV_\phi([a, b]), u(a) = u_0\} \\ &= \{u \in \mathfrak{M}^1: \exists \gamma = \gamma(u) > 0, \text{ such that } w_\gamma(u, u_0) < \infty\}. \end{aligned} \quad (51)$$

Then, \mathfrak{M}_w^{*1} is a metric modular space.

Lemma 2. The metric modular space \mathfrak{M}_w^{*1} is w -complete.

Proof. Let $\{u_n\} \subset \mathfrak{M}_w^{*1}$ be w -Cauchy. Then,

$$w_\gamma(u_n, u_m) \rightarrow 0, \quad \text{as } n, m \rightarrow \infty, \quad (52)$$

for some $\gamma = \gamma(\{u_n\}) > 0$. Therefore, for $n, m \in N$ and $t \in [a, b]$, we have

$$\begin{aligned} |u_n(\tau) - u_m(\tau)| &= |[u_n(\tau) - u_m(\tau)] - u_n(0) + u_m(0)| \\ &\leq |[u_n(\tau) + u_m(0)] - [u_m(\tau) + u_n(0)]| \\ &\leq \gamma|t - 0| \phi^{-1} \frac{w_\gamma(u_n, u_m)}{(t - 0)} \\ &\rightarrow 0, \quad \text{as } n, m \rightarrow \infty, \end{aligned} \quad (53)$$

which implies that $\lim_{n,m \rightarrow \infty} |\mathbf{u}_n(\tau) - \mathbf{u}_m(\tau)| = 0$. Since \mathfrak{A} is complete, the sequence $\{\mathbf{u}_n\}$ converges to some $\mathbf{u}: [a, b] \rightarrow \mathfrak{A}$ with $\mathbf{u}(a) = \mathbf{u}_0$. That is, for all $t \in [a, b]$, $\lim_{n \rightarrow \infty} |\mathbf{u}_n(\tau) - \mathbf{u}_m(\tau)| = 0$ holds for $\mathbf{u} \in \mathfrak{M}^1$. It remains to show that $\lim_{n \rightarrow \infty} w_\gamma(\mathbf{u}_n(\tau), \mathbf{u}(\tau)) = 0$. From lower semicontinuity of w (see [9], p.27), we have

$$w_\gamma(\mathbf{u}_n, \mathbf{u}) \leq \liminf_{n \rightarrow \infty} w_\gamma(\mathbf{u}_n, \mathbf{u}_m), \quad (54)$$

for every $n \in N$. Since $\{\mathbf{u}_n\}$ is w -Cauchy, for all $\epsilon > 0$, there exists $N(\epsilon) \in N$ such that $\forall n, m \geq N(\epsilon), w_\gamma(\mathbf{u}_n, \mathbf{u}_m) < \epsilon$.

Hence, for all $n \geq N(\epsilon)$,

$$\lim_{m \rightarrow \infty} \sup w_\gamma(\mathbf{u}_n, \mathbf{u}_m) \leq \sup_{m \geq N(\epsilon)} w_\gamma(\mathbf{u}_n, \mathbf{u}_m) < \epsilon. \quad (55)$$

Thus, for every $\epsilon > 0$, there exists $N(\epsilon) \in N$ such that

$$w_\gamma(\mathbf{u}_n, \mathbf{u}) \leq \liminf_{m \rightarrow \infty} w(\gamma, \mathbf{u}_n, \mathbf{u}_m) \leq \lim_{m \rightarrow \infty} \sup w(\gamma, \mathbf{u}_n, \mathbf{u}_m) < \epsilon. \quad (56)$$

Then, $\{\mathbf{u}_n\}$ is w -convergent to \mathbf{u} . As \mathfrak{M}_w^{*1} is closed under the modular convergence, we have $\mathbf{u} \in \mathfrak{M}_w^{*1}$, and thus, \mathfrak{M}_w^{*1} is w -complete. \square

Furthermore, if ϕ satisfies the Orlicz condition at infinity, that is, $(\phi(\mathfrak{w})/\mathfrak{w}) \rightarrow \infty$ as $\mathfrak{w} \rightarrow \infty$, then $w_1(\mathbf{u}, 0)$ is called the ϕ -variation of the function $\mathbf{u}: [a, b] \rightarrow R$; the function \mathbf{u} with $w_1(\mathbf{u}, 0) < \infty$ is said to be of bounded ϕ -variation on $[a, b]$ and we have

$$w_\gamma(\mathbf{u}, \mathfrak{w}) = w_\gamma(\mathbf{u} - \mathfrak{w}, 0) = w_1\left(\frac{\mathbf{u} - \mathfrak{w}}{\gamma}, 0\right), \quad \gamma > 0, \mathbf{u}, \mathfrak{w} \in \mathfrak{M}. \quad (57)$$

For the functions $\mathbf{u}: [a, b] \rightarrow \mathfrak{A}$ in the space $\mathfrak{M}_w^* = GV_\phi([a, b])$, it is known that (see [10, 35])

$$\mathbf{u} \in GV_\phi([a, b]) \Leftrightarrow w_\gamma(\mathbf{u}, 0) < \infty, \quad \text{for some } \gamma = \gamma(\mathbf{u}) > 0, \quad (58)$$

$$\Leftrightarrow \mathbf{u} \in AC[a, b], \mathbf{u}' \in L^1[a, b] \text{ with } w_\gamma(\mathbf{u}, 0) = \int_a^b \phi\left(\frac{|\mathbf{u}'(\tau)|}{\gamma}\right) d\tau < \infty, \quad (59)$$

where $AC[a, b]$ is the space of all absolutely continuous real-valued functions on $[a, b]$ and $L^1[a, b]$ is the space of all Lebesgue integrable functions on $[a, b]$.

Now, we apply the result given in Theorem 2 to the following integral equation:

$$\mathbf{u}(\psi) = \int_a^{\mu(\psi)} K(\psi, v, \mathbf{u}(v)) dv + \int_a^{\sigma(\psi)} J(\psi, v, \mathbf{u}(v)) dv + \mathfrak{w}(\psi), \quad (60)$$

for $\psi \in [a, b]$, where $\mathbf{u}, \mathfrak{w} \in \mathfrak{M}_w^{*1}$, $K, J: [a, b] \times [a, b] \times \mathfrak{A} \rightarrow R$ are Caratheodory type functions, $\mu, \sigma: [a, b] \rightarrow [a, b]$, and $\mu(a) = \sigma(a) = a$. Define the function $T: \mathfrak{M}_w^{*1} \times \mathfrak{M}_w^{*1} \rightarrow \mathfrak{M}_w^{*1}$ by

$$T(\mathbf{u}_1, \mathbf{u}_2)(\psi) = \int_a^{\mu(\psi)} K(\psi, v, \mathbf{u}_1(v)) dv + \int_a^{\sigma(\psi)} J(\psi, v, \mathbf{u}_2(v)) dv + \mathfrak{w}(\psi), \quad (61)$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathfrak{w} \in \mathfrak{M}_w^{*1}$. Then, problem (60) is equal to the fixed-point problem:

$$T(u, u) = u. \quad (62)$$

Definition 10. Equation (60) is Ulam–Hyers stable, if there exists $c > 0$ such that for each $\epsilon > 0$ and $u^* \in \mathfrak{M}_w^{*1}$ for which

$$\sup_{\pi} \sum_{i=1}^m \phi \left(\frac{|[u^*(\tau_i) + T(u^*, u^*)(\tau_{i-1})] - [u^*(\tau_{i-1}) + T(u^*, u^*)(\tau_i)]|}{\gamma(\tau_i - \tau_{i-1})} \right) (\tau_i - \tau_{i-1}) \leq \epsilon, \quad (63)$$

there exists a solution u of (60) such that $\sup_{\pi} \sum_{i=1}^m \phi(|[u^*(\tau_i) + u(\tau_{i-1})] - [u^*(\tau_{i-1}) + u(\tau_i)]|/\gamma(\tau_i - \tau_{i-1}))(\tau_i - \tau_{i-1}) \leq c\epsilon$, for all $\tau \in [a, b]$, where the supremum is taken over all partitions $\pi = \{\tau_i\}_{i=0}^n$ of the interval $[a, b]$; that is, $a = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_n = b$ and $\phi: R^+ \rightarrow R^+$ is a continuous, convex, increasing, and unbounded function such that $\phi(u) = 0$ iff $u = 0$ admits the inverse function $\phi^{-1}: R^+ \rightarrow R^+$, which is continuous and strictly increasing, and $\phi^{-1}(u) = 0$ iff $u = 0$.

If ϕ also satisfies Orlicz condition at infinity, then the above definition is equivalent to the following.

Definition 11. Equation (60) is Ulam–Hyers stable if there exists $c > 0$ such that for each $\epsilon > 0$ and $u^* \in \mathfrak{M}_w^{*1}$ for which $\int_a^b \phi(|u^{*'}(\tau) - K(\tau, v, u^*(\mu(\tau)) - K(\tau, v, u^*(\sigma(\tau)) - u'(\tau)|/\gamma) d\tau \leq \epsilon$ there exists a solution u of (60) such that $\int_a^b \phi(|u^{*'}(\tau) - u'(\tau)|/\gamma) d\tau \leq c\epsilon$ for all $\tau \in [a, b]$.

We will analyse problem (60) under the following conditions:

(m1) For every $u_j \in R$, where $j \in N$, the functions $K(\psi, v, u_j)$ and $J(\psi, v, u_j)$ are Lebesgue measurable on $[a, b]$ and there exist points $u_0, v_0 \in R$ such that

$$\int_a^b \phi \left(\frac{|K(\psi, v, u_0)|}{\gamma_K} \right) dv < \infty, \quad (64)$$

$$\int_a^b \phi \left(\frac{|J(\psi, v, v_0)|}{\gamma_J} \right) dv < \infty,$$

for some $\gamma_K = \gamma(K(\psi, v, u_0)) > 0$ and $\gamma_J = \gamma(J(\psi, v, v_0)) > 0$, where ϕ is a function satisfying the Orlicz condition at infinity and $\gamma > 0$.

(m2) There exists a constant $L > 0$ such that

$$|K(\psi, v, u_1) - K(\psi, v, u_2)| + |J(\psi, v, u_2) - J(\psi, v, u_3)| \leq L^2(b-a) \max\{|u_1 - u_2|, |u_2 - u_3|\}, \quad (65)$$

for almost all $\psi, v \in [a, b]$ and $u_1, u_2 \in \mathfrak{A}$.

Theorem 8. Under the assumptions (m1) and (m2), the operator T is a function from $\mathfrak{M}_w^{*1} \times \mathfrak{M}_w^{*1}$ to \mathfrak{M}_w^{*1} and the following inequality holds:

$$w_{L(b-a)\gamma}(T(u_1, u_2), T(u_2, u_3)) \leq L(b-a) \max\{w_{\gamma}(u_1, u_2), w_{\gamma}(u_2, u_3)\}, \quad (66)$$

for all $\gamma > 0$ and $u_1, u_2, u_3 \in \mathfrak{M}_w^{*1}$.

Proof. Applying Jensen's integral inequality with the convex function ϕ ,

$$\phi \left(\frac{1}{b-a} \int_a^b |u(\tau)| d\tau \right) \leq \frac{1}{b-a} \int_a^b \phi(|u(\tau)|) d\tau, \quad u \in L^1[a, b], \quad (67)$$

and by the property of the convex function ϕ ,

$$\phi(au(\tau)) \leq a\phi(u(\tau)), \quad 0 < a < 1, u(\tau) \in R, \quad (68)$$

where the integral on the RHS takes values in $[0, \infty]$. \square

Step 1. Claim: T is well defined on $\mathfrak{M}_w^{*1} \times \mathfrak{M}_w^{*1}$.

Let $u_1, u_2 \in \mathfrak{M}_w^{*1}$, i.e., $u_1, u_2 \in GV_{\phi}[a, b]$ and $u_1(a) = u_2(a) = u_0$. Since $u_1, u_2 \in AC[a, b]$, in the light of (m1) and (m2), the functions K and J are measurable on $[a, b]$. Let us prove that $K, J \in L^1[a, b]$. By Lebesgue's theorem, $u(v) = u_0 + \int_a^v u'(\psi) d\psi$ for all $v \in [a, b]$ and so, by (m2), we have

$$\begin{aligned}
& |K(\psi, v, \mathbf{u}_1(v))| + |J(\psi, v, \mathbf{u}_2(v))| \\
& \leq |K(\psi, v, \mathbf{u}_1(v)) - K(\psi, v, \mathbf{u}_0)| + |K(\psi, v, \mathbf{u}_0)| + |J(\psi, v, \mathbf{u}_2(v)) - J(\psi, v, v_0)| + |J(\psi, v, v_0)| \\
& \leq L^2(b-a) \max\{|\mathbf{u}_1(v) - \mathbf{u}_0|, |\mathbf{u}_2(v) - v_0|\} + |K(\psi, v, \mathbf{u}_0)| + |J(\psi, v, v_0)| \\
& \leq L^2(b-a) \max\left\{\int_a^b \mathbf{u}'_1(v)dv + |\mathbf{u}_0 - \mathbf{u}_0|, \int_a^b \mathbf{u}'_2(v)dv + |\mathbf{u}_0 - v_0|\right\} + K(\psi, v, \mathbf{u}_0) + J(\psi, v, v_0).
\end{aligned} \tag{69}$$

Therefore,

$$\begin{aligned}
|K(\psi, v, \mathbf{u}_1(v))| + |J(\psi, v, \mathbf{u}_2(v))| & \leq L^2(b-a) \max\left\{\int_a^b \mathbf{u}'_1(v)dv + |\mathbf{u}_0 - \mathbf{u}_0|, \right. \\
& \left. \int_a^b \mathbf{u}'_2(v)dv + |\mathbf{u}_0 - v_0|\right\} + |K(\psi, v, \mathbf{u}_0)| + |J(\psi, v, v_0)|,
\end{aligned} \tag{70}$$

for almost all $v \in [a, b]$. Since $\mathbf{u}_1, \mathbf{u}_2 \in \mathfrak{M}_w^{*1}$, we have $\mathbf{u}_1, \mathbf{u}_2 \in GV_\phi[a, b]$ and so there exist constants $\gamma_1 = \gamma_1(\mathbf{u}_1) > 0$ and $\gamma_2 = \gamma_2(\mathbf{u}_2) > 0$ such that

$$C_1 \equiv w_{\gamma_1}(\mathbf{u}_1, \mathbf{u}_0) = \int_a^b \phi\left(\frac{|\mathbf{u}'_1(v)|}{\gamma_1}\right)dv < \infty, \tag{71}$$

$$C'_1 \equiv w_{\gamma_2}(\mathbf{u}_2, \mathbf{u}_0) = \int_a^b \phi\left(\frac{|\mathbf{u}'_2(v)|}{\gamma_2}\right)dv < \infty.$$

Also by (m1), there exist constants $\gamma_K = \gamma(K(\psi, v, v_0)) > 0$ and $\gamma_J = \gamma(J(\psi, v, v_0)) > 0$ such that

$$C_2 = \int_a^b \phi\left(\frac{|K(\psi, v, \mathbf{u}_0)|}{\gamma_K}\right)dv < \infty, \tag{72}$$

$$C_3 = \int_a^b \phi\left(\frac{|J(\psi, v, v_0)|}{\gamma_J}\right)dv < \infty.$$

Setting $\gamma_0 = L^2(b-a)\gamma_1 + 1 + \gamma_K + \gamma_J$ and $\gamma'_0 = L^2(b-a)\gamma_2 + 1 + \gamma_K + \gamma_J$, we get

$$\frac{L^2(b-a)\gamma_1}{\gamma_0} + \frac{1}{\gamma_0} + \frac{\gamma_K}{\gamma_0} + \frac{\gamma_J}{\gamma_0} = 1, \tag{73}$$

$$\frac{L^2(b-a)\gamma_2}{\gamma'_0} + \frac{1}{\gamma'_0} + \frac{\gamma_K}{\gamma'_0} + \frac{\gamma_J}{\gamma'_0} = 1.$$

If $\max\left\{\int_a^b \mathbf{u}'_1(v)dv + |\mathbf{u}_0 - \mathbf{u}_0|, \int_a^b \mathbf{u}'_2(v)dv + |\mathbf{u}_0 - v_0|\right\} = \int_a^b \mathbf{u}'_1(v)dv + |\mathbf{u}_0 - \mathbf{u}_0|$ by the convexity of ϕ , we find

$$\begin{aligned}
& \phi\left(\frac{|K(\psi, v, \mathbf{u}_1(v))| + |J(\psi, v, \mathbf{u}_2(v))|}{\gamma_0}\right) \\
& \leq \phi\left(\frac{1}{\gamma_0} \left[L^2(b-a) \int_a^b \mathbf{u}'_1(v)dv + L^2(b-a)|\mathbf{u}_0 - \mathbf{u}_0| + |K(\psi, v, \mathbf{u}_0)| + |J(\psi, v, v_0)| \right] \right) \\
& = \phi\left(\frac{L^2(b-a)\gamma_1}{\gamma_0} \int_a^b \frac{\mathbf{u}'_1(v)}{\gamma_1} d\psi + \frac{1}{\gamma_0} L^2(b-a)|\mathbf{u}_0 - \mathbf{u}_0| + \frac{\gamma_K}{\gamma_0} \frac{|K(\psi, v, \mathbf{u}_0)|}{\gamma_K} + \frac{\gamma_J}{\gamma_0} \frac{|J(\psi, v, v_0)|}{\gamma_J} \right) \\
& \leq \frac{L^2(b-a)\gamma_1}{\gamma_0} \phi\left(\int_a^b \frac{\mathbf{u}'_1(v)}{\gamma_1} d\psi\right) + \frac{1}{\gamma_0} \phi(L^2(b-a)|\mathbf{u}_0 - \mathbf{u}_0|) + \frac{\gamma_K}{\gamma_0} \phi\left(\frac{|K(\psi, v, \mathbf{u}_0)|}{\gamma_K}\right) + \frac{\gamma_J}{\gamma_0} \phi\left(\frac{|J(\psi, v, v_0)|}{\gamma_J}\right) \\
& = \frac{L^2(b-a)\gamma_1}{\gamma_0} C_1 + \frac{1}{\gamma_0} \phi(L^2(b-a)|\mathbf{u}_0 - \mathbf{u}_0|) + \frac{\gamma_K}{\gamma_0} \phi\left(\frac{|K(\psi, v, \mathbf{u}_0)|}{\gamma_K}\right) + \frac{\gamma_J}{\gamma_0} \phi\left(\frac{|J(\psi, v, v_0)|}{\gamma_J}\right),
\end{aligned} \tag{74}$$

and so

$$\begin{aligned} & \int_a^b \phi \left(\frac{|K(\psi, v, \mathbf{u}_1(v))| + |J(\psi, v, \mathbf{u}_2(v))|}{\gamma_0} \right) dv \\ & \leq \frac{L^2(b-a)^2 \gamma_1 C_1}{\gamma_0} + \frac{(b-a)}{\gamma_0} \phi(L^2(b-a)|\mathbf{u}_0 - u_0|) + \frac{\gamma_K C_2}{\gamma_0} + \frac{\gamma_J C_3}{\gamma_0} \equiv C_0 < \infty. \end{aligned} \quad (75)$$

Now, it follows from (67) that

$$\begin{aligned} & \phi \left(\frac{1}{(b-a)} \int_a^b \frac{|K(\psi, v, \mathbf{u}_1(v))| + |J(\psi, v, \mathbf{u}_2(v))|}{\gamma_0} dv \right) \\ & \leq \frac{1}{(b-a)} \int_a^b \phi \left(\frac{|K(\psi, v, \mathbf{u}_1(v))| + |J(\psi, v, \mathbf{u}_2(v))|}{\gamma_0} \right) dv \leq \frac{C_0}{b-a}, \end{aligned} \quad (76)$$

which implies

$$\int_a^b |K(\psi, v, \mathbf{u}_1(v))| + |J(\psi, v, \mathbf{u}_2(v))| d\tau \leq \gamma_0(b-a) \phi^{-1} \frac{C_0}{b-a} < \infty. \quad (77)$$

Similarly, if

$$\max \left\{ \int_a^b \mathbf{u}'_1(\psi) d\psi + |\mathbf{u}_0 - u_0|, \int_a^b \mathbf{u}'_2(\psi) d\psi + |\mathbf{u}_0 - v_0| \right\} = \int_a^b \mathbf{u}'_2(\psi) d\psi + |\mathbf{u}_0 - v_0|, \quad (78)$$

then

$$\begin{aligned} & \int_a^b \phi \left(\frac{|K(\psi, v, \mathbf{u}_1(v))| + |J(\psi, v, \mathbf{u}_2(v))|}{\gamma'_0} \right) dv \\ & \leq \frac{L^2(b-a)^2 \gamma'_2 C'_1}{\gamma'_0} + \frac{1}{\gamma'_0} \phi(L^2(b-a)|\mathbf{u}_0 - v_0|) + \frac{\gamma'_K C'_2}{\gamma'_0} + \frac{\gamma'_J C'_3}{\gamma'_0} \equiv \frac{C'_0}{b-a}, \end{aligned} \quad (79)$$

implying

$$\int_a^b |K(\psi, v, \mathbf{u}_1(v))| + |J(\psi, v, \mathbf{u}_2(v))| dv \leq \gamma'_0(b-a) \phi^{-1} \left(\frac{C'_0}{b-a} \right) < \infty. \quad (80)$$

Hence, in both the cases,

$$\int_a^b |K(\psi, v, \mathbf{u}_1(v))| dv + \int_a^b |J(\psi, v, \mathbf{u}_{21}(v))| dv < \infty. \quad (81)$$

Therefore, $K, J \in L^1[a, b]$. Thus, the operator T is well defined on $\mathfrak{M}_w^{*1} \times \mathfrak{M}_w^{*1}$ and by (61), $T(\mathbf{u}_1, \mathbf{u}_2) \in AC[a, b]$ for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathfrak{M}_w^{*1}$ which implies that $(T(\mathbf{u}_1, \mathbf{u}_2))' \in L^1[a, b]$ and

$$(T(\mathbf{u}_1, \mathbf{u}_2))'(\psi) = K(\psi, v, \mathbf{u}_1(v)) + J(\psi, v, \mathbf{u}_2(v)), \quad (82)$$

for almost all $v \in [a, b]$.

Step 2. It is clear from (61) and $\mu(a) = \sigma(a) = a$ that given $\mathbf{u}_1, \mathbf{u}_2, \mathbf{w} \in \mathfrak{M}_w^{*1}$,

$$T(\mathbf{u}_1, \mathbf{u}_2)(a) = \int_a^{\mu(a)} K(\psi, v, \mathbf{u}_1(v)) dv + \int_a^{\sigma(a)} J(\psi, v, \mathbf{u}_2(v)) dv + \mathfrak{w}(a) = u_0, \quad (83)$$

and so $T(\mathbf{u}_1, \mathbf{u}_2) \in \mathfrak{M}^1 = \{\mathbf{u}: [a, b] \rightarrow R: \mathbf{u}(a) = u_0\}$.
Now, we will show that $T(\mathbf{u}_1, \mathbf{u}_2) \in \mathfrak{M}_w^*$. By virtue of (59) and (73)–(82), we have

$$w_{\gamma_0}(T(\mathbf{u}_1, \mathbf{u}_2), \mathbf{u}_0) = \int_a^b \phi\left(\frac{|(T(\mathbf{u}_1, \mathbf{u}_2))'(v)|}{\gamma_0}\right) dv \leq C_0, \quad (84)$$

or

$$w_{\gamma_0'}(T(\mathbf{u}_1, \mathbf{u}_2), \mathbf{u}_0) = \int_a^b \phi\left(\frac{|(T(\mathbf{u}_1, \mathbf{u}_2))'(v)|}{\gamma_0}\right) dv \leq C_0', \quad (85)$$

and so T maps $\mathfrak{M}_w^{*1} \times \mathfrak{M}_w^{*1} \rightarrow \mathfrak{M}_w^{*1}$.

Step 3. To obtain (66), let $\gamma > 0$ and $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathfrak{M}_w^{*1}$. By (58), (59), and (82), we find

$$\begin{aligned} w_{L(b-a)\gamma}(T(\mathbf{u}_1, \mathbf{u}_2), T(\mathbf{u}_2, \mathbf{u}_3)) &= w_{L(b-a)\gamma}(T(\mathbf{u}_1, \mathbf{u}_2) - T(\mathbf{u}_2, \mathbf{u}_3), 0) \\ &= \int_a^b \phi\left(\frac{|(T(\mathbf{u}_1, \mathbf{u}_2) - T(\mathbf{u}_2, \mathbf{u}_3))'(v)|}{L(b-a)\gamma}\right) dv \\ &= \int_a^b \phi\left(\frac{|[K(\psi, v, \mathbf{u}_1(v)) + J(\psi, v, \mathbf{u}_2(v))] - [K(\psi, v, \mathbf{u}_2(v)) + J(\psi, v, \mathbf{u}_3(v))]|}{L(b-a)\gamma}\right) dv \\ &= \int_a^b \phi\left(\frac{|[K(\psi, v, \mathbf{u}_1(v)) - J(\psi, v, \mathbf{u}_2(v))] + [K(\psi, v, \mathbf{u}_2(v)) - J(\psi, v, \mathbf{u}_3(v))]|}{L(b-a)\gamma}\right) dv. \end{aligned} \quad (86)$$

By (m2) and Lebesgue's theorem, we have for almost all $v \in [a, b]$ (since $\mathbf{u}_1(a) = \mathbf{u}_2(a) = \mathbf{u}_3(a) = u_0$),

$$|[K(\psi, v, \mathbf{u}_1(v)) + J(\psi, v, \mathbf{u}_2(v))] - [K(\psi, v, \mathbf{u}_2(v)) + J(\psi, v, \mathbf{u}_3(v))]| \leq L^2(b-a) \max\{|\mathbf{u}_1 - \mathbf{u}_2|, |\mathbf{u}_2 - \mathbf{u}_3|\}. \quad (87)$$

Therefore,

$$\begin{aligned} &|[K(\psi, v, \mathbf{u}_1(v)) + J(\psi, v, \mathbf{u}_2(v))] - [K(\psi, v, \mathbf{u}_2(v)) + J(\psi, v, \mathbf{u}_3(v))]| \\ &\leq L^2(b-a) \max\left\{\int_a^b (\mathbf{u}_1 - \mathbf{u}_2)'(\psi), \int_a^b (\mathbf{u}_2 - \mathbf{u}_3)'(\psi)\right\}. \end{aligned} \quad (88)$$

Applying (67) and (68), we get

$$\begin{aligned}
 & \phi\left(\frac{[K(\psi, v, \mathbf{u}_1(v)) - J(\psi, v, \mathbf{u}_2(v))] + [K(\psi, v, \mathbf{u}_2(v)) - J(\psi, v, \mathbf{u}_3(v))]}{L(b-a)\gamma}\right) \\
 & \leq \phi\left(L^2(b-a)\max\left\{\int_a^b \frac{(\mathbf{u}_1 - \mathbf{u}_2)'(\psi)}{L(b-a)\gamma} d\psi, \int_a^b \frac{(\mathbf{u}_2 - \mathbf{u}_3)'(\psi)}{L(b-a)\gamma} d\psi\right\}\right) \\
 & = \phi\left(L(b-a)\max\left\{\int_a^b \frac{(\mathbf{u}_1 - \mathbf{u}_2)'(\psi)}{L(b-a)\gamma} d\psi, \int_a^b \frac{(\mathbf{u}_2 - \mathbf{u}_3)'(\psi)}{(b-a)\gamma} d\psi\right\}\right) \\
 & \leq L(b-a)\phi\left(\max\left\{\frac{1}{b-a} \int_a^b \frac{(\mathbf{u}_1 - \mathbf{u}_2)'(\psi)}{\gamma} d\psi, \frac{1}{b-a} \int_a^b \frac{(\mathbf{u}_2 - \mathbf{u}_3)'(\psi)}{\gamma} d\psi\right\}\right) \\
 & \leq L \max\left\{\int_a^b \phi \frac{(\mathbf{u}_1 - \mathbf{u}_2)'(\psi)}{\gamma} d\psi, \int_a^b \phi \frac{(\mathbf{u}_2 - \mathbf{u}_3)'(\psi)}{\gamma} d\psi\right\} \\
 & = L \max\{w_\gamma(\mathbf{u}_1, \mathbf{u}_2), w_\gamma(\mathbf{u}_2, \mathbf{u}_3)\}.
 \end{aligned} \tag{89}$$

Therefore,

$$\begin{aligned}
 & w_{L(b-a)\gamma}(T(\mathbf{u}_1, \mathbf{u}_2), T(\mathbf{u}_2, \mathbf{u}_3)) \\
 & = \int_a^b \phi\left(\frac{[K(\psi, v, \mathbf{u}_1(v)) - J(\psi, v, \mathbf{u}_2(v))] + [K(\psi, v, \mathbf{u}_2(v)) - J(\psi, v, \mathbf{u}_3(v))]}{L(b-a)\gamma}\right) d\psi \\
 & \leq L(b-a)\max\{w_\gamma(\mathbf{u}_1, \mathbf{u}_2), w_\gamma(\mathbf{u}_2, \mathbf{u}_3)\}.
 \end{aligned} \tag{90}$$

Theorem 9. Under the assumptions (m1), (m2), and $0 < L(b-a) < 1$, integral equation (60) admits a solution $\mathbf{u} \in GV_\phi[a, b]$.

Proof. (i) By Lemmas 1 and 2, w is strict modular on

$$\mathfrak{M}^1 = \{\mathbf{u}; \mathbf{u}: [a, b] \longrightarrow \mathfrak{A}, \mathbf{u}(a) = \mathbf{u}_0\}, \tag{91}$$

and the modular space

$$\mathfrak{M}_w^{*1} = GV_\phi([a, b]) \cap \mathfrak{M}^1 = \{\mathbf{u} \in GV_\phi([a, b]), \mathbf{u}(a) = \mathbf{u}_0\}, \tag{92}$$

is w -complete. By Theorem 8, the operator $T: \mathfrak{M}_w^{*1} \times \mathfrak{M}_w^{*1} \longrightarrow \mathfrak{M}_w^{*1}$ is Ćirić–Presic type strongly w -contractive. We are left to show that for some $\gamma > 0$, $\exists \mathbf{u} \in \mathfrak{M}_w^{*1}$, such that $w_\gamma(\mathbf{u}, T\mathbf{u}) < \infty$. Clearly, for constant function $\mathbf{u}_0 \in \mathfrak{M}_w^{*1}$, we have

$$\begin{aligned}
 w_{\gamma_0}(T(\mathbf{u}_0, \mathbf{u}_0), \mathbf{u}_0) & = w_\gamma(T(\mathbf{u}_0, \mathbf{u}_0) - \mathbf{u}_0, 0) \\
 & = \int_a^b \phi\left(\frac{|T(\mathbf{u}_0, \mathbf{u}_0) - \mathbf{u}_0|'}{\gamma_0}\right) d\tau \\
 & = \int_a^b \phi\left(\frac{|T(\mathbf{u}_0, \mathbf{u}_0)|'}{\gamma_0}\right) d\tau \\
 & = \int_a^b \phi\left(\frac{|K(\psi, v, \mathbf{u}_0)| + |J(\psi, v, \mathbf{u}_0)|}{\gamma_0}\right) d\tau \\
 & \leq \frac{L^2(b-a)^2\gamma_1}{\gamma_0} \int_a^b \phi\left(\frac{|\mathbf{u}_0'(\psi)|}{\gamma_1}\right) d\psi + \frac{(b-a)}{\gamma_0} \phi(L^2(b-a)|\mathbf{u}_0 - \mathbf{u}_0|) + \frac{\gamma_K C_2}{\gamma_0} + \frac{\gamma_L C_3}{\gamma_0}.
 \end{aligned} \tag{93}$$

Since $u_0 \in \mathfrak{M}_w^{*1}$ is a constant function, $(u_0)' = 0$. Therefore,

$$w_{\gamma_0}(T(u_0, u_0), u_0) \leq \frac{b-a}{\gamma_0} \phi(L^2(b-a)|u_0 - u_0|) + \frac{\gamma_K}{\gamma_0} C_2 + \frac{\gamma_J}{\gamma_0} C_3 < \infty, \quad (94)$$

where $L^2(b-a)\gamma_1 + 1 + \gamma_K + \gamma_J = \gamma_0$. By Theorem 4, the integral operator T admits a fixed point and thus problem (60) has a solution. \square

Theorem 10. Under the assumptions (m1), (m2), and $0 < L(b-a) < 1$, integral equation (60) is Ulam–Hyers stable.

Proof. Let $\epsilon > 0$ and $u^* \in \mathfrak{M}_w^{*1}$ for which $\int_a^b \phi(|u^{*'}(\tau) - K(\tau, v, u^*(\mu(\tau)) - K(\tau, v, u^*(\sigma(\tau)) - w'(\tau)|/\gamma) d\tau \leq \epsilon$. Then, we have

$$\begin{aligned} & \int_a^b \phi\left(\frac{|u^*(\tau) - T(u^*(\mu(\tau)), u^*(\sigma(\tau)))'|}{\gamma}\right) d\tau \leq \epsilon \\ & \implies w_\lambda(u^* - T(u^*, u^*), 0) \leq \epsilon \\ & \implies w_\lambda(u^*, T(u^*, u^*)) \leq \epsilon. \end{aligned} \quad (95)$$

By Theorem 8, the operator $T: \mathfrak{M}_w^{*1} \times \mathfrak{M}_w^{*1} \longrightarrow \mathfrak{M}_w^{*1}$ is Ciric–Presic type strongly w -contractive and so by Proposition 1, T is c -WPPO. By Theorem 7, the fixed-point equation (62) is Ulam–Hyers stable; that is, there exists $u \in \mathfrak{M}_w^{*1}$ such that $u = T(u, u)$ and $w_\lambda(u^*, u) \leq c\epsilon$. Thus, we have

$$\begin{aligned} & w_\lambda(u^*, u) \leq c\epsilon \\ & \implies w_\lambda(u^* - u, 0) \leq c\epsilon \\ & \implies \int_a^b \phi\left(\frac{|u^{*'}(\tau) - u'(\tau)|}{\gamma}\right) d\tau \leq c\epsilon. \end{aligned} \quad (96)$$

\square

We now furnish a numerical example to validate the hypothesis of Theorem 9.

Example 3. Consider the integral equation

$$\begin{aligned} x(s) = & -\frac{s}{8} + \frac{3s^2}{4} + \frac{1}{4} \arctan\left(\frac{s}{2}\right) - \frac{1}{2} \arctan\left(\frac{s^2}{2}\right) - \frac{s^2}{8} \ln\left(1 + \frac{s^4}{4}\right) \\ & + \int_0^{s^2/2} \frac{\ln(1+x(v))}{4} dv + \int_0^{s/2} \frac{x(v)}{4(1+x(v))} dv, \quad s \in [0, 1], \end{aligned} \quad (97)$$

for some $x \in \mathfrak{M}_w^{*1}$, $\mathfrak{A} = [0, 100]$, and $u_0 = 0$. We will apply Theorem 9, to prove the existence of a solution of (97).

Proof. Define the operator $T: \mathfrak{M}_w^{*1} \longrightarrow \mathfrak{M}_w^{*1}$ as

$$\begin{aligned} Tx(s) = & -\frac{s}{8} + \frac{3s^2}{4} + \frac{1}{4} \arctan\left(\frac{s}{2}\right) - \frac{1}{2} \arctan\left(\frac{s^2}{2}\right) - \frac{s^2}{8} \ln\left(1 + \frac{s^4}{4}\right) \\ & + \int_0^{s^2/2} \frac{\ln(1+x(v))}{2} dv + \int_0^{s/2} \frac{x(v)}{2(1+x(v))} dv, \quad s \in [0, 1]. \end{aligned} \quad (98)$$

Now, set $K(s, v, x(v)) = (\ln(1+x(v))/4)$, $J(s, v, x(v)) = (x(v)/4(1+x(v)))$, $\mu(s) = (s^2/2)$, $\sigma(s) = (s/2)$, and $w(s) =$

$-(s/8) + (3s^2/4) + (1/4)\arctan(s/2) - (1/2)\arctan(s^2/2) - (s^2/8)\ln(1 + (s^4/4))$. We observe the following:

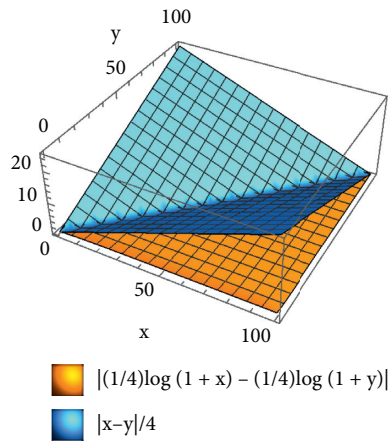


FIGURE 1: Inequality in (99).

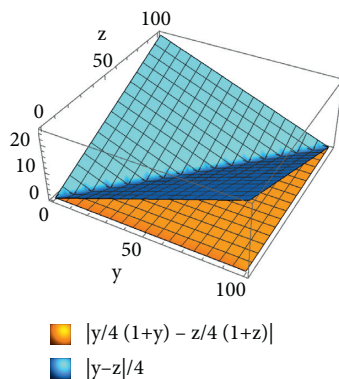


FIGURE 2: Inequality in (100).

- (i) Let $\phi(t) = t$ for all $t \in \mathbb{R}^+$. For $u_0 = v_0 = 9$ and $\gamma_K = 1, \gamma_I = 1$, we see that the functions K and J satisfy (m1).
- (ii) By using the Mathematica software, we found that the following inequalities are true for all $x, y, z \in [0, 100]$ (see Figures 1 and 2):

$$\left| \frac{\ln(1+x)}{4} - \frac{\ln(1+y)}{4} \right| \leq \frac{|x-y|}{4}, \quad (99)$$

$$\left| \frac{y}{4(1+y)} - \frac{z}{4(1+z)} \right| \leq \frac{|y-z|}{4}. \quad (100)$$

Thus, we get

$$\begin{aligned} & |K(s, v, x(v)) - K(s, v, y(v))| + |J(s, v, y(v)) - J(s, v, z(v))| \\ & \leq \left| \frac{\ln(1+x) - \ln(1+y)}{4} \right| + \left| \frac{y}{4(1+y)} - \frac{z}{4(1+z)} \right| \\ & \leq \frac{1}{2} \max\{|x-y|, |y-z|\}, \end{aligned} \quad (101)$$

and thus, K and J satisfies (m2), with $L^2 = (1/\sqrt{2})$.

Hence, all the conditions of Theorem 9 are satisfied. It is evident that integral equation (97) has a unique solution $x \in \mathfrak{M}_w^{*1}$ defined by $x(s) = s^2$. \square

6. Conclusion

The fixed-point technique is used to solve many mathematical problems as it gets involved with differential and integral equations, integro-differential equations, game theory, economics, and more disciplines. In this work, unique fixed points of generalised Presic type operators defined on modular metric space are obtained and the results are applied in proving the existence of solutions of integral equations involving Caratheodory type functions. Ulam–Hyers stability of fixed-point equations involving Presic type operators in a modular metric space is introduced. Our results pave way for further research in modular metric spaces. Following the techniques of our work, Presic–Bogin-type contractions, Presic–Chatterjee type contractions, and Presic–Hardy–Rogers type contractions can be extended to the setting of a modular metric space.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the Deanship of Scientific Research, Prince Sattam bin Abdulaziz University, Alkharj, Saudi Arabia, under Grant no. 2020/01/17320.

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Research Article

Convergence Classes of L -Filters in (L, M) -Fuzzy Topological Spaces

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Received 3 June 2021; Accepted 3 September 2021; Published 1 October 2021

Academic Editor: Kenan Yildirim

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An (L, M) -fuzzy topological convergence structure on a set X is a mapping which defines a degree in M for any L -filter (of crisp degree) on X to be convergent to a molecule in L^X . By means of (L, M) -fuzzy topological neighborhood operators, we show that the category of (L, M) -fuzzy topological convergence spaces is isomorphic to the category of (L, M) -fuzzy topological spaces. Moreover, two characterizations of L -topological spaces are presented and the relationship with other convergence spaces is concretely constructed.

1. Introduction

A convergence space of filters (as a generalization of a topological space based on the concept of convergence of filters as fundamental) is a pair (X, Conv) , where X is a set, Conv is a subset of $\mathbb{F}(X) \times X$, and $\mathbb{F}(X)$ is the set of filters on X . The pair $(\mathcal{F}, x) \in \text{Conv}$ means that \mathcal{F} is said to converge to x ($\forall \mathcal{F} \in \mathbb{F}(X)$). The convergence theory of filters is an important part in topology. It was proved that the category Conv of convergence spaces is a quasitopos which may be thought of as a nice category of spaces that includes Top (the category topological spaces) as a full subcategory (see [1–3] for details).

The corresponding convergence theory of filters in the fuzzy setting is also studied by many authors. E. Lowen and R. Lowen [4] gave a one-to-one correspondence between the set of limit functions on X and the set of stratified I -topologies on X based on I -filters (filters in the lattice $[0, 1]^X$ [5]). Jäger [6] introduced another fuzzy convergence structure (called stratified L -generalized convergence structure for L , a frame which was extended to the case of

complete residual lattices by Yao [7] later on). Here, the degree of convergence of a stratified L -fuzzy filter (called also stratified L -filter in [8]) is from L but it converges to a crisp point.

Jäger also gave some kinds of characterizations of stratified L -topological spaces (see [6, 9–11]). With the help of Jäger's work, Li [12] proved that there is a one-to-one correspondence between the set of all convergence functions of specific L -fuzzy filters on X and the set of all specific L -topologies on X for L , a complete residual lattice. Following papers [13, 14], Güloğlu and Çoker [15] proved that there exists a one-to-one correspondence between the set of I -fuzzy topological convergence structures on X and the set of I -fuzzy topologies on X . As a generalization, Pang and Fang [16] proved that there is a one-to-one correspondence between the set of topological L -fuzzy Q -convergence structures on X and the set of L -fuzzy topologies on X when L is a completely distributive complete lattice with an order-reversing involution $'$. Here, the L -fuzzy filter converges to a fuzzy point but the degree of convergence is from $\{0, 1\}$. Furthermore, Pang [17, 18] discussed categorical properties

in some fuzzy convergence spaces, such as L -fuzzifying convergence spaces and stratified L -convergence tower spaces.

In [19], the convergence theory of molecular nets in (L, M) -fuzzy topological spaces was discussed. As a generalization, Yao [20] presented a definition of (L, M) -fuzzy nets and established a Moore–Smith convergence in (L, M) -fuzzy topology. It is well known that every molecular net can induce an L -filter (a filter in lattice L^X) and an L -filter can induce a molecular net (see [21]). In addition, the degree of fuzzy topology, openness degree, and quotient degree [22] are interesting. Taking these in mind, we will use L -filters different from that of [6, 8, 16] to give some axioms of filter-theoretical convergence classes in (L, M) -fuzzy topological spaces. Such a convergence class on a set X will be called an (L, M) -fuzzy topological convergence structure, where the L -filter converges to a molecular in L^X and the degree of convergence is an element in M . This kind of convergence class is proved to have nice properties.

The present paper is arranged as follows. The rest of this section contains some basic definitions and notions which will be used in this paper. In Section 2, we prove that there exists a one-to-one correspondence between (L, M) -fuzzy topological convergence structures on a set X and (L, M) -fuzzy topologies on X . In Section 3, we give two approaches to $(L, 2)$ -fuzzy topological convergence structures. In Section 4, we discuss the relation between $(L, 2)$ -fuzzy topological convergence structures and Li's structures in [12].

2. Preliminaries

Now, we review some basic notions and results which will be used in this paper. Unless other explanations are given, L always stands for a complete lattice with a smallest element 0 and a largest element 1 in this paper. Obviously, for every set X , L^X (the set of all L -subsets of X) is also a complete lattice with the pointwise order. We will denote the L -set taking constant value a on X by a_X , and we write $\uparrow\mu = \{\nu \in L^X \mid \mu \leq \nu\}$ for each $\mu \in L^X$ ($\downarrow\mu$, $\uparrow a$, and $\downarrow a$ can be defined analogously for $a \in L$). For the sake of convenience, we denote the orders of L and L^X and their restrictions by the same symbol \leq . $a \in L - \{0\}$ is called a join irreducible element if, for any finite subset $J \subseteq L$ satisfying $a = \vee J$ (the supremum of J), there exists a $j \in J$ such that $a = j$. $a \in L - \{0\}$ is called a coprime element if, for any finite subset $J \subseteq L$ satisfying $a \leq \vee J$, there exists a $j \in J$ such that $a \leq j$. $p \in L - \{1\}$ is called a prime element of L iff it is a coprime element of L^{op} , where L^{op} denotes the opposite lattice of L . The set of all join irreducible elements (resp., all coprime elements and all prime elements) of L will be denoted by $J(L)$ (resp., $\text{Copr}(L)$ and $\text{Pr}(L)$). Clearly, $\text{Copr}(L^X) = \{x_a \mid x \in X, a \in \text{Copr}(L)\}$, where $x_a \in L^X$ is the L -subset taking value a at x and 0 elsewhere, and we call the members of $\text{Pt}(L^X) = \{x_a \mid x \in X, a \in L - \{0\}\}$ L -points. It is well known that $J(L) = \text{Copr}(L)$ if L is a distributive lattice. L is called a completely distributive complete lattice if it satisfies the following completely distributive laws:

- (i) (CD1) $\bigvee_{i \in I} \bigwedge_{j \in J_i} a_{i,j} = \bigwedge_{f \in \prod_{i \in I} J_i} \bigvee_{i \in I} a_{i,f(i)}, (\forall \{a_{i,j} \mid i \in I, j \in J_i\} \in 2^{L^L})$
- (ii) (CD2) $\bigwedge_{i \in I} \bigvee_{j \in J_i} a_{i,j} = \bigvee_{f \in \prod_{i \in I} J_i} \bigwedge_{i \in I} a_{i,f(i)} (\forall \{a_{i,j} \mid i \in I, j \in J_i\} \in 2^{L^L})$

It is also well known that $a = \vee(\downarrow a \cap \text{Copr}(L)) (\forall a \in L)$ when L is a completely distributive complete lattice, where $\downarrow a = \{b \in L \mid b \triangleleft a\}$ ($\uparrow a$, $\downarrow\mu$, and $\uparrow\mu$ can be defined analogously for $\mu \in L^X$), and $\triangleleft \subseteq L \times L$ is called the wedge-below relation on L which is defined as follows: $a \triangleleft b$ iff for any $S \subseteq L$ satisfying $b \leq \vee S$, there exists an $s \in S$ such that $a \leq s$ (\triangleright can be defined analogously).

An order-reversing involution $'$ on L is a self-map on L such that, for any $a, b \in L$, the following hold: (1) $a \leq b$ implies $b' \leq a'$; (2) $a'' = a$. The following hold for any subset $\{a_i \mid i \in I\} \subseteq L$:

- (1) $(\bigvee_{i \in I} a_i)' = \bigwedge_{i \in I} a_i'$
- (2) $(\bigwedge_{i \in I} a_i)' = \bigvee_{i \in I} a_i'$

For each mapping $f: X \longrightarrow Y$, we have a mapping (called L -forward powerset operator) $f_L^\rightarrow: L^X \longrightarrow L^Y$ which is defined by $f_L^\rightarrow(\mu)(y) = \vee\{\mu(x) \mid f(x) = y\}$ ($\forall \mu \in L^X, \forall y \in Y$). The right adjoint to f_L^\rightarrow (called L -backward powerset operator) is denoted by f_L^\leftarrow (therefore, the pair $(f_L^\rightarrow, f_L^\leftarrow)$ is a Galois connection on (L^X, \leq) and (L^Y, \leq)). It can be easily verified that f_L^\rightarrow preserves arbitrary supremal, f_L^\leftarrow preserves arbitrary supremal and arbitrary infimal (and also complements if L has an order-reversing involution $'$), and $f_L^\leftarrow(\nu) = \vee\{\mu \mid f_L^\rightarrow(\mu) \leq \nu\} = \nu \circ f$ ($\forall \nu \in L^Y$).

For other undefined notions, refer to [8, 23].

3. Relationship between (L, M) -Fuzzy Topological Spaces and (L, M) -Fuzzy Topological Convergence Spaces

In this section, we assume that L (resp., M) is a completely distributive complete lattice with an order-reversing involution $'$ (resp., $*$); in this case, $J(L) = \text{Copr}(L)$, $J(M) = \text{Copr}(M)$, and $J(L^X) = \text{Copr}(L^X)$. Our main task is to give a one-to-one correspondence between (L, M) -fuzzy topological convergence structures (resp., (L, M) -fuzzy principle convergence structures) on a set X and (L, M) -fuzzy topologies (resp., (L, M) -fuzzy preinterior operators) on X , which gives rise to an isomorphism between the category of (L, M) -fuzzy topological convergence spaces (resp., the category of (L, M) -fuzzy principle convergence spaces) and the category of (L, M) -fuzzy topological spaces (resp., the category of (L, M) -fuzzy preinterior spaces).

Definition 1 (See [13, 14])

- (1) (See [13, 14]). An L -fuzzy filter on X is a mapping $F: L^X \longrightarrow L$, which satisfies the following conditions:
 - (i) (FF1) $F(1_X) = 1, F(0_X) = 0$

- (ii) (FF2) If $A, B \in L^X$ and $A \leq B$, then $F(A) \leq F(B)$
- (iii) (FF3) $F(A) \wedge F(B) \leq F(A \wedge B)$ ($\forall A, B \in L^X$)
- (iv) An L -fuzzy filter is said to be stratified [8] if it satisfies the following condition: (Fs)
 $a \leq F(a_X) (\forall a \in L)$.

The set of all L -fuzzy filters (resp., stratified L -fuzzy filters) on X is denoted as $\mathbb{F}(X, L, L)$ (resp., $\mathbb{F}^S(X, L, L)$). Apparently, the mapping $[x]: L^X \longrightarrow L$, defined by $[x](A) = A(x)$, is a stratified L -fuzzy filter ($\forall x \in X$).

- (2) (See [5]). An L -filter on X is a family $F \subseteq L^X$ which satisfies the following conditions:

- (i) (F1) $1_X \in F$, $0_X \notin F$
- (ii) (F2) If $\mu \in F$ and $\mu \leq \nu$, then $\nu \in F$
- (iii) (F3) If $\mu, \nu \in F$, then $\mu \wedge \nu \in F$

The set of all L -filters on X is denoted as $\mathbb{F}(X, L, 2)$ (where $2 = \{0, 1\}$). Obviously, every L -filter may be looked upon as an L -fuzzy filter.

Remark 1. Apparently, $\uparrow x_a \in \mathbb{F}(X, L, 2)$, ($\forall x \in X, \forall a \in L$). Moreover, for every mapping $f: X \longrightarrow Y$ and every L -filter F on X , define

$$f_L \rightrightarrows (F) = \cap \{G \in \mathbb{F}(X, L, 2) \mid f_L \rightrightarrows (F) \subseteq G\}, \quad (1)$$

and then $f_L \rightrightarrows (F) \in \mathbb{F}(X, L, 2)$ and $f_L \rightrightarrows (F) = \{\mu \in L^Y \mid f_L^-(\mu) \in F\} = \{\mu \in L^Y \mid \mu \geq f_L \rightrightarrows (\nu) \text{ for some } \nu \in F\}$, where $f_L \rightrightarrows (F) = \{f_L \rightrightarrows (\nu) \mid \nu \in F\}$.

Definition 2 (see [24])

- (1) An (L, M) -fuzzy topology on X is a mapping $T: L^X \longrightarrow M$, which satisfies the following conditions:
 - (i) (LMFT1) $T(0_X) = T(1_X) = 1$
 - (ii) (LMFT2) $T(\mu) \wedge T(\nu) \leq T(\mu \wedge \nu)$ ($\forall \mu, \nu \in L^X$)
 - (iii) (LMFT3)
 $\bigwedge_{j \in J} T(\mu_j) \leq T(\bigvee_{j \in J} \mu_j)$, ($\{\mu_j\}_{j \in J} \subseteq L^X$)

In this case, $T(\mu)$ can be interpreted as the degree for μ to be an open set ($\forall \mu \in L^X$), and the pair (X, T) is called an (L, M) -fuzzy topological space. An $(L, 2)$ -fuzzy topology is also called L -topology.

- (2) A continuous mapping from one (L, M) -fuzzy topological space (X, T_X) to another (L, M) -fuzzy topological space (Y, T_Y) is a mapping $f: X \longrightarrow Y$, which satisfies $T_Y(\mu) \leq T_X(f_L^-(\mu))$ ($\forall \mu \in L^Y$). The category of (L, M) -fuzzy topological spaces and continuous mappings between them is denoted by (L, M) -FTop.

Definition 3 (see [25])

- (1) An (L, M) -fuzzy neighborhood operator on a set X is a mapping $N: J(L^X) \longrightarrow M^{L^X}$, which satisfies the following conditions:

- (i) (LMFN1) $N(x_a)(1_X) = 1$, $N(x_a)(0_X) = 0$,
 $(\forall x_a \in J(L^X))$
- (ii) (LMFN2) $N(x_a)(\mu) = 0$ ($x_a \leq \mu$)
- (iii) (LMFN3) $N(x_a)(\mu \wedge \nu) = N(x_a)(\mu) \wedge N(x_a)(\nu)$, ($\forall \mu, \nu \in L^X, \forall x_a \in J(L^X)$)
- (iv) (LMFN4) $N(x_a)(\mu) = \bigvee_{\nu \in \downarrow \mu \cap \uparrow x_a} \bigwedge_{y_b \in J(L^X) \cap \downarrow \nu} N(y_b)(\nu)$ ($\forall \mu \in L^X, \forall x_a \in J(L^X)$)

In this case, (X, N) is called an (L, M) -fuzzy neighborhood space.

- (2) A mapping $f: (X, N_X) \longrightarrow (Y, N_Y)$ (where (X, N_X) and (Y, N_Y) are both (L, M) -fuzzy neighborhood spaces) is said to be continuous if $N_Y(f(x)_a)(\mu) \leq N_X(x_a)(f_L^-(\mu))$ holds for any $\mu \in L^Y$ and any $x_a \in J(L^X)$. The category of (L, M) -fuzzy neighborhood spaces and continuous mappings between them will be denoted by (L, M) -FNS. In [25], Shi proved that (L, L) -FNS is isomorphic to (L, L) -FTop. Furthermore, it is easily proved that (L, M) -FNS is isomorphic to (L, M) -FTop.

Definition 4

- (1) An (L, M) -fuzzy preinterior operator on X is a mapping $I: L^X \longrightarrow M^{J(L^X)}$, which satisfies the following conditions:
 - (i) (LMPI1) $I(1_X)(x_a) = 1$, ($\forall x_a \in J(L^X)$)
 - (ii) (LMPI2) $I(\mu)(x_a) = 0$, ($x_a \leq \mu$)
 - (iii) (LMPI3) $I(\mu) \wedge I(\nu) = I(\mu \wedge \nu)$, ($\forall \mu, \nu \in L^X$)

In this case, (X, I) is called an (L, M) -fuzzy preinterior space.

- (2) A mapping $f: (X, I_X) \longrightarrow (Y, I_Y)$ (where (X, I_X) and (Y, I_Y) are both (L, M) -fuzzy preinterior spaces) is said to be continuous if $I_Y(\mu)(f(x)_a) \leq I_X(f_L^-(\mu))(x_a)$ holds for each $\mu \in L^Y$ and each $x_a \in J(L^X)$. The category of (L, M) -fuzzy preinterior spaces and continuous mappings between them will be denoted by (L, M) -FPIS.

Definition 5

- (1) An (L, M) -fuzzy convergence structure on X is a mapping $\text{Conv}: \mathbb{F}(X, L, 2) \longrightarrow M^{J(L^X)}$, which satisfies the following conditions:
 - (i) (LM1) $\text{Conv}(\uparrow x_a)(x_a) = 1 \forall x_a \in J(L^X)$
 - (ii) (LM2) If $F \subseteq G$, then $\text{Conv}(F) \leq \text{Conv}(G)$

In this case, (X, Conv) is called an (L, M) -fuzzy convergence space.

- (2) A mapping $f: (X, \text{Conv}_X) \longrightarrow (Y, \text{Conv}_Y)$ (where (X, Conv_X) and (Y, Conv_Y) are both (L, M) -fuzzy convergence spaces) is said to be continuous if $\text{Conv}_X(F)(x_a) \leq \text{Conv}_Y(f_L \rightrightarrows (F))(f(x)_a)$ for any $F \in \mathbb{F}(X, L, 2)$ and any $x_a \in J(L^X)$. The category of (L, M) -fuzzy convergence spaces and continuous

mappings between them will be denoted by (L, M) -FCS.

- (3) For an (L, M) -fuzzy convergence space (X, Conv) , we define a mapping $N_{\text{Conv}}: J(L^X) \longrightarrow M^{L^X}$ as follows:

$$N_{\text{Conv}}(x_a)(\mu) = \bigwedge_{\mu \notin F} (\text{Conv}(F)(x_a))^*, \quad (2)$$

$$(\forall x_a \in J(L^X), \forall \mu \in L^X).$$

Definition 6

- (1) An (L, M) -fuzzy convergence structure Conv on X is said to be an (L, M) -fuzzy principle convergence structure if it satisfies the following condition:

$$(\text{LM3}) \text{Conv}(F)(x_a) = \bigwedge_{\mu \notin F} (N_{\text{Conv}}(x_a)(\mu))^*, (\forall x_a \in J(L^X))$$

In this case, (X, Conv) is called an (L, M) -fuzzy pretopological convergence space.

- (2) The category of (L, M) -fuzzy pretopological convergence spaces and continuous mappings (see Definition 5) between them will be denoted by (L, M) -FPCS.

Definition 7

- (1) An (L, M) -fuzzy topological convergence structure on X is an (L, M) -fuzzy principle convergence

structure Conv , which satisfies the following condition:

$$(\text{LM4}) N_{\text{Conv}}(x_a)(\mu) = \bigvee_{\nu \in \downarrow \mu \cap \uparrow x_a} \bigwedge_{y_b \in \downarrow \nu \cap J(L^X)} N_{\text{Conv}}(y_b)(\nu), (\forall \mu \in L^X)$$

In this case, (X, Conv) is called an (L, M) -fuzzy topological convergence space.

- (2) The category of (L, M) -fuzzy topological convergence spaces and continuous mappings (see Definition 5) between them will be denoted by (L, M) -FTCS.

In the rest of this section, we will show that (L, M) -FNS is isomorphic to (L, M) -FTCS (thus, (L, M) -FTop is isomorphic to (L, M) -FTCS by [25]) and that (L, M) -FPIS is isomorphic to (L, M) -FPCS.

Proposition 1. For an (L, M) -fuzzy topological convergence structure Conv , the mapping $N_{\text{Conv}}: J(L^X) \longrightarrow M^{L^X}$ is an (L, M) -fuzzy neighborhood operator on X .

Proof. Since $\{F \in \mathbb{F}(X, L, 2) \mid 1_X \notin F\} = \emptyset$, we have $N_{\text{Conv}}(x_a)(1_X) = \bigwedge \emptyset = 1$. $\uparrow x_a$ is an L -filter on X and $0_X \notin \uparrow x_a$, $N_{\text{Conv}}(x_a)(0_X) \leq (\text{Conv}(\uparrow x_a)(x_a))^* = 0$ by (LM1). Thus, (LMFN1) is true. Again for each $\mu \in L^X$ and each $x_a \in J(L^X) - J(\mu)$, we have $N_{\text{Conv}}(x_a)(\mu) \leq (\text{Conv}(\uparrow x_a)(x_a))^* = 0$ since $\mu \notin \uparrow x_a$. (LMFN2) is also true.

Let $\mu, \nu \in L^X$ and $x_a \in J(L^X)$. Then, $\{F \in \mathbb{F}(X, L, 2) \mid \mu \wedge \nu \notin F\} = \{F \in \mathbb{F}(X, L, 2) \mid \mu \notin F\} \cup \{F \in \mathbb{F}(X, L, 2) \mid \nu \notin F\}$ holds. Thus, by definition of $N_{\text{Conv}}(x_a)$,

$$\begin{aligned} N_{\text{Conv}}(x_a)(\mu \wedge \nu) &= \bigwedge \{(\text{Conv}(F)(x_a))^* \mid F \in \mathbb{F}(X, L, 2), \mu \wedge \nu \notin F\} \\ &= \bigwedge \{(\text{Conv}(F)(x_a))^* \mid \mu \notin F\} \bigwedge \{(\text{Conv}(F)(x_a))^* \mid \nu \notin F\} \\ &= N_{\text{Conv}}(x_a)(\mu) \wedge N_{\text{Conv}}(x_a)(\nu), \end{aligned} \quad (3)$$

which means that (LMFN3) is true. (LMFN4) follows from (LM4). \square

Proposition 2. For each (L, M) -fuzzy neighborhood operator N on X , the mapping $\text{Conv}_N: \mathbb{F}(X, L, 2) \longrightarrow M^{J(L^X)}$, defined by

$$\begin{aligned} \text{Conv}_N(F)(x_a) &= \bigwedge \{N(x_a)(\mu)^* \mid \mu \in L^X, \mu \notin F\}, \\ &(\forall F \in \mathbb{F}(X, L, 2), \forall x_a \in J(L^X)), \end{aligned} \quad (4)$$

is an (L, M) -fuzzy topological convergence structure on X .

Proof. For any $\mu \in L^X$ and any $x_a \in J(L^X) - J(\mu)$, we have $N(x_a)(\mu) = 0$ by (LMFN2). Hence, $\text{Conv}_N(\uparrow x_a)(x_a) = 1$, which means that (LM1) is true.

Let $F, G \in \mathbb{F}(X, L, 2)$ and $F \subseteq G$. It can be easily checked that $\{\mu \in L^X \mid \mu \notin G\} \subseteq \{\mu \in L^X \mid \mu \notin F\}$. Thus, $\text{Conv}_N(F)$

$(x_a) = \bigwedge \{N(x_a)(\mu)^* \mid \mu \in L^X, \mu \notin F\} \leq \bigwedge \{N(x_a)(\mu)^* \mid \mu \in L^X, \mu \notin G\} = \text{Conv}_N(G)(x_a)$ $\mu \notin G$ for any $x_a \in J(L^X)$, which means that (LM2) is true.

Since Conv_N satisfies (LM1) and (LM2), the mapping $N_{\text{Conv}_N}: J(L^X) \longrightarrow M^{L^X}$ is well-defined (see Definition 5). In order to prove (LM3), it suffices to prove the equality $N_{\text{Conv}_N}(x_a)(\mu) = N(x_a)(\mu)$, $(\forall x_a \in J(L^X), \forall \mu \in L^X)$ by definition of Conv_N . By Definition 5 (3),

$$N_{\text{Conv}_N}(x_a)(\mu) = \bigwedge_{\mu \notin F} (\text{Conv}_N(F)(x_a))^* = \bigwedge_{\mu \notin F} \bigvee_{\nu \notin F} N(x_a)(\nu). \quad (5)$$

For each $F \in \mathbb{F}(X, L, 2)$ satisfying $\mu \notin F$, we have $\bigvee \{N(x_a)(\nu) \mid \nu \in L^X, \nu \notin F\} \geq N(x_a)(\nu)$. As F is arbitrary, the inequality $N_{\text{Conv}_N}(x_a)(\mu) \geq N(x_a)(\mu)$ holds. To prove the other inequality $N_{\text{Conv}_N}(x_a)(\mu) \leq N(x_a)(\mu)$, let $I = \{F \in \mathbb{F}(X, L, 2) \mid \mu \notin F\}$ and $J_F = \{\nu \in L^X \mid \nu \notin F\}$. As M is a completely distributive complete lattice,

$$N_{\text{Conv}_N}(x_a)(\mu) = \bigvee_{g \in \prod_{F \in I} J_F} \bigwedge_{F \in I} N(x_a)(g(F)). \quad (6)$$

First, we show that $F_b \in I$, where $F_b = \{\gamma \in L^X \mid N(x_a)(\gamma) \not\leq b\} (\forall b \in \uparrow(N(x_a)(\mu)) \cap \text{Pr}(M))$. As $\mu \notin F_b$ is obvious, we only need to show that F_b is an L -filter.

- (i) (F1) As $N(x_a)(1_X) = 1 \not\leq b$ and $N(x_a)(0_X) = 0 \leq b$, $1_X \in F_b$, and $0_X \notin F_b$.
- (ii) (F2) Assume that $\gamma \in F_b$ and $\gamma \leq \nu$; then $N(x_a)(\gamma) \leq N(x_a)(\nu)$. Further, $N(x_a)(\gamma) \not\leq b$, so we have $N(x_a)(\nu) \not\leq b$, and thus, $\nu \in F_b$.
- (iii) (F3) Let $\gamma, \nu \in F_b$. Then we declare that $N(x_a)(\gamma \wedge \nu) = N(x_a)(\gamma) \wedge N(x_a)(\nu) \not\leq b$ (i.e., $\gamma \wedge \nu \in F_b$). Otherwise, $N(x_a)(\gamma) \leq b$ or $N(x_a)(\nu) \leq b$ since $b \in \text{Pr}(M)$, which is a contradiction.

Next, it follows from $F_b \in I$ that $g(F_b) \notin F_b (\forall b \in \uparrow(N(x_a)(\mu)) \cap \text{Pr}(M), \forall g \in \prod_{F \in I} J_F)$, and thus, $N(x_a)(g(F_b)) \leq b$ (particularly, $\bigwedge \{N(x_a)(g(F)) \mid F \in I\} \leq b$). As b is arbitrary and M is a completely distributive complete lattice, $\bigwedge \{N(x_a)(g(F)) \mid F \in I\} \leq N(x_a)(\mu)$. As g is arbitrary, $N_{\text{Conv}_N}(x_a)(\mu) \leq N(x_a)(\mu)$.

Since $N_{\text{Conv}_N}(x_a)(\mu) = N(x_a)(\mu)$ and N is an (L, M) -fuzzy neighborhood operator, (LM4) holds. \square

Proposition 3. If $f: (X, N_X) \longrightarrow (Y, N_Y)$ is an (L, M) -FNS morphism, then $f: (X, \text{Conv}_{N_X}) \longrightarrow (Y, \text{Conv}_{N_Y})$ is an (L, M) -FTCS morphism.

Proof. For any $F \in \mathbb{F}(X, L, 2)$ and any $x_a \in J(L^X)$, as f is an (L, M) -FNS morphism, $N_Y(f(x_a)(\nu)) \leq N_X(x_a)(f_L^{\leftarrow}(\nu))$. Further, $*$ is an order-reversing mapping; $(N_X(x_a)(f_L^{\leftarrow}(\nu)))^* \leq (N_Y(f(x_a)(\nu)))^* (\forall \nu \in L^Y)$. It follows that

$$\begin{aligned} \text{Conv}_{N_X}(F)(x_a) &= \bigwedge_{\mu \notin F} (N_X(x_a)(\mu))^* \\ &\leq \bigwedge_{\nu \notin f_L^{\leftarrow}(F)} (N_X(x_a)(f_L^{\leftarrow}(\nu)))^* \\ &\leq \bigwedge_{\nu \notin f_L^{\leftarrow}(F)} (N_Y(f(x_a)(\nu)))^* \\ &= \text{Conv}_{N_Y}(f_L^{\leftarrow}(F))(f(x_a)). \end{aligned} \quad (7)$$

Therefore, $f: (X, \text{Conv}_{N_X}) \longrightarrow (Y, \text{Conv}_{N_Y})$ is an (L, M) -FTCS morphism. \square

Proposition 4. If $f: (X, \text{Conv}_X) \longrightarrow (Y, \text{Conv}_Y)$ is an (L, M) -FTCS morphism, then $f: (X, N_{\text{Conv}_X}) \longrightarrow (Y, N_{\text{Conv}_Y})$ is an (L, M) -FNS morphism.

Proof. For each $G \in \mathbb{F}(X, L, 2)$ and $x_a \in J(L^X)$, as f is an (L, M) -FTCS morphism, $\text{Conv}_X(G)(x_a) \leq \text{Conv}_Y(f_L^{\leftarrow}(G))(f(x_a))$. Further, $*$ is an order-reversing mapping; $(\text{Conv}_Y(f_L^{\leftarrow}(G))(f(x_a)))^* \leq (\text{Conv}_X(G)(x_a))^*$. For all $\mu \in L^Y$, it follows that

$$\begin{aligned} N_{\text{Conv}_Y}(f(x_a)(\mu)) &= \bigwedge_{\mu \notin F} (\text{Conv}_Y(F)(f(x_a)))^* \\ &\leq \bigwedge_{f_L^{\leftarrow}(\mu) \notin G} (\text{Conv}_Y(f_L^{\leftarrow}(G))(f(x_a)))^* \\ &\leq \bigwedge_{f_L^{\leftarrow}(\mu) \notin G} (\text{Conv}_X(G)(x_a))^* \\ &= N_{\text{Conv}_X}(x_a)(f_L^{\leftarrow}(\mu)). \end{aligned} \quad (8)$$

The first inequality holds since $\{F \in \mathbb{F}(Y, L, 2) \mid \mu \notin F\} \supseteq \{f_L^{\leftarrow}(G) \mid G \in \mathbb{F}(X, L, 2), f_L^{\leftarrow}(\mu) \notin G\}$. Therefore, $f: (X, N_{\text{Conv}_X}) \longrightarrow (Y, N_{\text{Conv}_Y})$ is an (L, M) -FNS morphism. \square

Proposition 5. For any (L, M) -fuzzy topological convergence structure Conv on X , one has $\text{Conv}_{N_{\text{Conv}}} = \text{Conv}$.

Proof. Let $F \in \mathbb{F}(X, L, 2)$ and $x_a \in J(L^X)$. We need to prove $\text{Conv}_{N_{\text{Conv}}}(F)(x_a) = \text{Conv}(F)(x_a)$. By Proposition 2 and Definition 5,

$$\text{Conv}_{N_{\text{Conv}}}(F)(x_a) = \bigwedge_{\mu \notin F} \bigvee_{\mu \notin G} \text{Conv}(G)(x_a). \quad (9)$$

For each $\mu \in L^X$ with $\mu \notin F$, we have

$$\bigvee_{\mu \notin G} \text{Conv}(G)(x_a) \geq \text{Conv}(F)(x_a), \quad (10)$$

and then the inequality $\text{Conv}_{N_{\text{Conv}}}(F)(x_a) \geq \text{Conv}(F)(x_a)$ holds since μ is arbitrary. To show the other inequality $\text{Conv}_{N_{\text{Conv}}}(F)(x_a) \leq \text{Conv}(F)(x_a)$, put $I = \{\mu \in L^X \mid \mu \notin F\}$ and $J_\mu = \{G \in \mathbb{F}(X, L, 2) \mid \mu \notin G\}$. As M is a completely distributive complete lattice,

$$\text{Conv}_{N_{\text{Conv}}}(F)(x_a) = \bigvee_{g \in \prod_{\mu \in I} J_\mu} \bigwedge_{\mu \in I} \text{Conv}(g(\mu))(x_a). \quad (11)$$

It suffices to prove

$$\bigwedge_{\mu \in I} \text{Conv}(g(\mu))(x_a) \leq \text{Conv}(F)(x_a), \quad \left(\forall g \in \prod_{\mu \in I} J_\mu \right). \quad (12)$$

For each $g \in \prod_{\mu \in I} J_\mu$, by (LM3), we have $\bigwedge_{\mu \in I} \text{Conv}(g(\mu))(x_a) = \bigwedge_{\mu \in I} \bigwedge_{\nu \notin g(\mu)} (N_{\text{Conv}}(x_a)(\nu))^* = \bigwedge_{\nu \notin \bigcap_{\mu \in I} g(\mu)} (N_{\text{Conv}}(x_a)(\nu))^* = \text{Conv}(\bigcap_{\mu \in I} g(\mu))(x_a)$. As $\bigcap_{\mu \in I} g(\mu) \in \mathcal{F}$, from (LM2), we have $\text{Conv}(\bigcap_{\mu \in I} g(\mu))(x_a) \leq \text{Conv}(F)(x_a)$, which means

$$\bigwedge_{\mu \in I} \text{Conv}(g(\mu))(x_a) \leq \text{Conv}(F)(x_a), \quad \left(\forall g \in \prod_{\mu \in I} J_\mu \right). \quad (13)$$

By Propositions 1–5, we have the following. \square

Theorem 1

- (1) (L, M) -FNS is isomorphic to (L, M) -FTCS.

Similar to [25], it is easy to check that (L, M) -FNS is isomorphic to (L, M) -FTop

- (2) Thus (L, M) -FTop is isomorphic to (L, M) -FTCS.

Remark 2

- (1) If Conv is an (L, M) -fuzzy topological convergence structure on X , then it satisfies the following: if $x_a \leq x_b$, then $\text{Conv}(F)(x_a) \leq \text{Conv}(F)(x_b)$ ($\forall x_a, x_b \in J(L^X), F \in \mathbb{F}(X, L, 2)$).
- (2) From Proposition 5, the following is true due to (LM3):

$$\bigwedge_{i \in I} \text{Conv}(F_i)(x_a) = \text{Conv}\left(\bigcap_{i \in I} F_i\right)(x_a). \quad (14)$$

- (3) Now we turn our attention to the Moore–Smith convergence theory; for basic notions, refer to [19, 21, 23].

For a molecule net $S = (S(m) \mid m \in D)$ ($S(m) \in J(L^X)$, and D is a directed set), we define an L -filter F_S associated with the net S as follows: $F_S = \{\mu \mid S \text{ is eventually in } \mu\}$. Conversely, for an L -filter F , define a molecule net $S_F = (S(m) \mid m \in D)$, where $D = \{(x_a, A) \mid x_a \in J(L^X), A \in F, x_a \leq A\}$ is a directed set, on which the order is equipped with the relation \leq : $(x_a, A) \leq (y_b, B)$ if and only if $A \geq B$, and the mapping is $S(x_a, A) = x_a$.

Now define convergence of S as

$$\text{Conv}(S)(x_a) = \bigwedge \{ (N(x_a)(\mu))^* \mid S \text{ is not eventually in } \mu \}. \quad (15)$$

In an (L, M) -fuzzy topological space (X, T) , the following hold. The proof is simple and is missing:

- (i) (1) $\text{Conv}(S)(x_a) = \text{Conv}(F_S)(x_a)$ for each molecule S and $x_a \in J(L^X)$.
- (ii) (2) $\text{Conv}(F)(x_a) = \text{Conv}(S_F)(x_a)$ for each $F \in \mathbb{F}(X, L, 2)$ and $x_a \in J(L^X)$.
- (iii) (4) In [16], Pang and Fang proposed the concept of topological L -fuzzy Q -convergence spaces (the corresponding category is denoted by L -QFTCS). In these convergence spaces, the value of an L -fuzzy filter converging to a fuzzy point is from $\{0, 1\}$. Thus, Pang and Fang's spaces are different from ours. But L -QFTCS is isomorphic to (L, L) -FTCS when L is a completely distributive lattice with an order-reversing involution.
- (iv) (5) For a given set X , let $\text{TopConv}(X, L, M)$ be a set of all (L, M) -fuzzy topological convergence structures on X , and the relation \leq on $\text{TopConv}(X, L, M)$ is defined by $\text{Conv}_1 \leq \text{Conv}_2$ if and only if $\text{Conv}_1(F)(x_a) \leq \text{Conv}_2(F)(x_a)$ ($\forall x_a \in J(L^X)$ and $F \in \mathbb{F}(X, L, 2)$).

The set of all (L, M) -fuzzy neighborhood operators on X is written as $\text{NO}(X, L, M)$, and the relation \leq on $\text{NO}(X, L, M)$ is defined by $N_1 \leq N_2$ if and only if $N_1(x_a)(\mu) \geq N_2(x_a)(\mu)$ ($\forall x_a \in J(L^X)$ and $\mu \in L^X$).

From Propositions 2 and 5, we have $N_{\text{Conv}_N} = N$ and for each $N \in \text{OP}(X, L, M)$, $\text{Conv}_{N_{\text{Conv}_N}} = \text{Conv}$ for each $\text{Conv} \in \text{TopConv}(X, L, M)$. This implies that there exists a bijection between $(\text{TopConv}(X, L, M), \leq)$ and $(\text{OP}(X, L, M), \leq)$. Similar to paper [26], it is easily checked that $(\text{TopConv}(X, L, M), \leq)$ and $(\text{OP}(X, L, M), \leq)$ are complete lattices and they are isomorphic.

By Theorem 1 (1), there exists an isomorphic functor between (L, M) -FNS and (L, M) -FTCS. Furthermore, the restriction of this functor to $(\text{OP}(X, L, M), \leq)$ is an order isomorphism.

Now we prove that (L, M) -FPIS is isomorphic to (L, M) -FPCS. Since its proof is similar to Theorem 1, we only give some propositions as follows.

Proposition 6. For each (L, M) -fuzzy preinterior operator I on X , the mapping $\text{Conv}_I: \mathbb{F}(X, L, 2) \longrightarrow M^{J(L^X)}$, defined by

$$\begin{aligned} \text{Conv}_I(F)(x_a) &= \bigwedge_{\mu \notin F} (I(\mu)(x_a))^*, \\ (\forall F \in \mathbb{F}(X, L, 2), x_a \in J(L^X)), \end{aligned} \quad (16)$$

is an (L, M) -fuzzy principal convergence structure on X .

Proposition 7. For each (L, M) -fuzzy principal convergence structure Conv on X , the mapping $I_{\text{Conv}}: L^X \longrightarrow M^{J(L^X)}$, defined by

$$\begin{aligned} I_{\text{Conv}}(\mu)(x_a) &= \bigwedge_{\mu \notin F} (\text{Conv}(F)(x_a))^*, \\ (\forall \mu \in L^X, x_a \in J(L^X)), \end{aligned} \quad (17)$$

is an (L, M) -fuzzy preinterior operator on X .

Proposition 8. For any (L, M) -fuzzy principal convergence structure Conv on X , one has $\text{Conv}_{I_{\text{Conv}}} = \text{Conv}$.

Proposition 9. For any (L, M) -fuzzy preinterior operator I on X , one has $I_{\text{Conv}_I} = I$.

Proposition 10. If $f: (X, I_X) \longrightarrow (Y, I_Y)$ is an (L, M) -FPIS morphism, then $f: (X, \text{Conv}_{I_X}) \longrightarrow (Y, \text{Conv}_{I_Y})$ is an (L, M) -FPCS morphism.

Proposition 11. If $f: (X, \text{Conv}_X) \longrightarrow (Y, \text{Conv}_Y)$ is an (L, M) -FPCS morphism, then $f: (X, I_{\text{Conv}_X}) \longrightarrow (Y, I_{\text{Conv}_Y})$ is an (L, M) -FPIS morphism.

By Propositions 6–11, we have the following.

Theorem 2. (L, M) -FPIS is isomorphic to (L, M) -FPCS.

Based on the above facts, we give the following example.

Example 1. Let $X = \{x\}$ be a single set, $L = \{1, 0.7, 0.5, 0.3, 0\}$ and $M = \{1, 0.5, 0\}$, where unary operation $*$ on M is defined as $1^* = 0$, $0.5^* = 0.5$, and $0^* = 1$. Define $T(0_X) = T(1_X) = 1$; otherwise, $T(A) = 0.5$, $A \in L^X$. Then

- (1) (X, T) is an (L, M) -fuzzy topological space.
- (2) Define $N(x_a)(A) = \vee \{T(B) \mid x_a \leq B \leq A\}$. It follows from Theorem 3.2 in [25] that (X, N) is an (L, M) -fuzzy neighborhood space. In this way, $N(x_{0.5})$, which is defined by $N(x_{0.5})(x_1) = 1$, $N(x_{0.5})(x_{0.7}) = N(x_{0.5})(x_{0.5}) = 0.5$, and $N(x_{0.5})(x_{0.3}) = N(x_{0.5})(x_0) = 0$, is the (L, M) -fuzzy neighborhood of $x_{0.5}$.
- (3) Let F be an L -filter, $\text{Conv}(F)(x_a) = \wedge \{N(x_a)(A)^* \mid A \notin F\}$. In this way, supposing $F = \{x_1, x_{0.7}\}$, we can see that $\text{Conv}(F)(x_{0.5}) = N(x_{0.5})(x_{0.5})^* \wedge N(x_{0.5})(x_{0.3})^* \wedge N(x_{0.5})(x_0)^* = 0.5$, which means that the degree of F convergence to $x_{0.5}$ is 0.5.

4. Two Approaches to L -Topological Convergence Structures

In this section, we restrict our attention to the case $M = 2$, and thus, we identify a set Y with its characteristic function χ_Y . We write $F \longrightarrow x_a$ whenever $\text{Conv}(F)(x_a) = 1$ and replace Conv by \longrightarrow in order to emphasize the convergence. In this manner, the notion of an $(L, 2)$ -fuzzy topological convergence space and the notion of an $(L, 2)$ -fuzzy neighborhood space can be easily given as follows.

Definition 8

- (1) A pair (X, \longrightarrow) is an L -convergence space (\longrightarrow is called an L -convergence structure on X) if it satisfies the following conditions:
 - (i) (L1) $\uparrow x_a \longrightarrow x_a$ ($\forall x_a \in J(L^X)$)
 - (ii) (L2) If $F \subseteq G$ and $F \longrightarrow x_a$, then $G \longrightarrow x_a$ ($\forall x_a \in J(L^X), \forall F, G \in \mathbb{F}(X, L, 2)$)
 - (Order) If $F \longrightarrow x_a$ and $x_a \leq x_b$, then $F \longrightarrow x_b$ ($\forall x_a, x_b \in J(L^X), \forall F \in \mathbb{F}(X, L, 2)$).
- (2) A pair (X, \longrightarrow) is an L -topological convergence space (\longrightarrow is called an L -topological convergence structure on X) if it satisfies the following conditions:
 - (i) (L1) $\uparrow x_a \longrightarrow x_a$ ($\forall x_a \in J(L^X)$)
 - (ii) (L2) If $F \subseteq G$ and $F \longrightarrow x_a$, then $G \longrightarrow x_a$ ($\forall x_a \in J(L^X), \forall F, G \in \mathbb{F}(X, L, 2)$)
 - (iii) (L3) $N(x_a) \longrightarrow x_a$, where $N(x_a) = \cap_{F \longrightarrow x_a} F$ ($\forall x_a \in J(L^X)$)
 - (iv) (L4) If $\mu \in N(x_a)$, then there exists $\nu \in L^X$ such that $\nu \in \uparrow x_a \cap \downarrow \mu$ and $\nu \in N(y_b)$ for each $y_b \in J(L^X) \cap \downarrow \nu$ ($\forall \mu \in L^X, \forall x_a \in J(L^X)$)

Obviously, an L -topological convergence space is an L -convergence space.

- (3) (See [25]) A pair (X, N) is an L -neighborhood space (N is called an L -neighborhood system on X) if it satisfies the following conditions:
 - (i) (LN0) $1_X \in N(x_a)$ and $0_X \notin N(x_a)$ ($\forall x_a \in J(L^X)$)

- (ii) (LN1) If $x_a \not\leq \mu$, then $\mu \notin N(x_a)$ ($\forall x_a \in J(L^X), \forall \mu \in L^X$)
- (iii) (LN2) If $\mu \leq \nu$ and $\mu \in N(x_a)$, then $\nu \in N(x_a)$ ($\forall x_a \in J(L^X), \forall \mu, \nu \in L^X$)
- (iv) (LN3) If $\mu, \nu \in N(x_a)$, then $\mu \wedge \nu \in N(x_a)$ ($\forall x_a \in J(L^X), \forall \mu, \nu \in L^X$)
- (v) (LN4) If $\mu \in N(x_a)$, then there exists $\nu \in L^X$ such that $\nu \in \uparrow x_a \cap \downarrow \mu$ and $\nu \in N(y_b)$ for each $y_b \in J(L^X) \cap \downarrow \nu$ ($\forall x_a \in J(L^X), \forall \mu \in L^X$)

Remark 3 (see [25]). For a given set X , let $\mathbb{T}(X, L)$ be the set of all L -topologies on X and $\mathbb{N}(X, L)$ the set of all L -topological neighborhood systems on X . Then, the mapping $\varphi_{12}: \mathbb{T}(X, L) \longrightarrow \mathbb{N}(X, L)$ defined by $N_T(x_a) = \{\nu \in L^X \mid \exists \mu \in T, \mu \in \uparrow x_a \cap \downarrow \nu\}$ ($\forall T \in \mathbb{T}(X, L), \forall x_a \in J(L^X)$) is a one-to-one correspondence, whose inverse $\varphi_{21}: \mathbb{N}(X, L) \longrightarrow \mathbb{T}(X, L)$ is given by $T_N = \{\nu \in L^X \mid \nu \in N(y_b) \text{ } (\forall y_b \in J(L^X) \cap \downarrow \nu)\}$, ($\forall N \in \mathbb{N}(X, L)$). Thus, by Theorem 1, there exists a one-to-one correspondence between the set of all L -topological convergence structures on X and the set of L -topologies on X .

In [9], Jäger gave a generalization of Kowalsky's diagonal condition. Based on different tools, we give another form of Kowalsky's diagonal condition [2] in fuzzy setting. In the presence of this form, we give a characterization of an L -topological convergence structure.

Theorem 3. (X, \longrightarrow) is an L -topological convergence space if and only if it satisfies (L1)–(L3), (order) in Definition 8, and the following:

- (i) (LK4) For any $G \in \mathbb{F}(X, L, 2)$ and any subfamily $\{F(y_b)\}_{y_b \in J(L^X)} \subseteq \mathbb{F}(X, L, 2)$ which satisfies (LFD) $F(y_a) \supseteq \mathcal{F}(y_b)$, whenever $y_a \leq y_b$, (18)

if $G \longrightarrow x_a$ and $F(y_b) \longrightarrow y_b$, then $G(F_-) \longrightarrow x_a$ ($x_a, y_b \in J(L^X)$), where the mapping $F_-: L^X \longrightarrow L^X$ and the subfamily $G(F_-) \subseteq L^X$ are defined as follows:

$$\begin{aligned} F_-(\mu) &= \vee \{x_a \mid \mu \in F(x_a)\}, \quad (\forall \mu \in L^X) \\ G(F_-) &= \{\mu \in L^X \mid F_-(\mu) \in G\}. \end{aligned} \quad (19)$$

To prove Theorem 3, we need several lemmas.

Lemma 1. Let $G \in \mathbb{F}(X, L, 2)$ and $\{G(x_a)\}_{x_a \in J(L^X)} \subseteq \mathbb{F}(X, L, 2)$ satisfy (LFD). Then $G(F_-) = \{\mu \in L^X \mid F_-(\mu) \in G\}$ is an L -filter on X .

Proof. As $1_X \in F(x_a)$ ($\forall x_a \in J(L^X)$), $F_-(1_X) = \vee J(L^X) = 1_X \in G$, which means $1_X \in G(F_-)$. Since $\{x_a \in J(L^X) \mid 0_X \in F(x_a)\} = \emptyset$, we have $F_-(0_X) = 0_X \notin G$; that is, $0_X \notin G(F_-)$. Thus, $G(F_-)$ satisfies (F1). Obviously, $G(F_-)$ satisfies (F2).

For any $\mu, \nu \in G(F_-)$, put $J_\mu = \{x_a \in J(L^X) \mid \mu \in F(x_a)\}$ and $J_\nu = \{x_a \in J(L^X) \mid \nu \in F(x_a)\}$. Then $F_-(\mu) = \vee J_\mu$, $F_-(\nu) = \vee J_\nu$ and $F_-(\mu) \wedge F_-(\nu) \in G$ by definition of $G(F_-)$. We first show the inequality $F_-(\mu) \wedge F_-(\nu) \leq \vee (J_\mu \cap J_\nu)$.

Suppose $x_a \in J(L^X)$ and $x_a \triangleleft F_-(\mu) \wedge F_-(\nu)$, and then $x_a \triangleleft F_-(\mu) = \vee J_\mu$. There exists an $x_b \in J_\mu$ (which implies $\mu \in F(x_b)$) such that $x_a \leq x_b$, and thus, $F(x_b) \subseteq F(x_a)$ since $\{F(x_a)\}_{x_a \in J(L^X)}$ satisfies (LFD). Therefore, $\mu \in F(x_a)$ (i.e., $x_a \in J_\mu$). Similarly, $x_a \in J_\nu$. Consequently $x_a \leq \vee(J_\mu \cap J_\nu)$, which implies $F_-(\mu) \wedge F_-(\nu) \leq \vee(J_\mu \cap J_\nu)$ since L^X is a completely distributive lattice. Notice that $(J_\mu \cap J_\nu) \subseteq \{x_a \in J(L^X) \mid \mu \wedge \nu \in F(x_a)\}$, we have $F_-(\mu) \wedge F_-(\nu) \leq F_-(\mu \wedge \nu)$, and thus, $F_-(\mu \wedge \nu) \in G$, which means $\mu \wedge \nu \in G(F_-)$. Hence, $G(F_-)$ also satisfies (F3). \square

Lemma 2. For a pair (X, N) satisfying conditions (LN0)-(LN3) in Definition 8 (3), the following are equivalent:

- (1) (LN4)
- (2) $\{N(x_a)\}_{x_a \in J(L^X)}$ satisfies (LFD) and $N(x_a) \subseteq N(x_a)(N_-)$ (N_-) ($\forall x_a \in J(L^X)$)

Proof

(1) \implies (2). Since (X, N) satisfies conditions (LN0), (LN2), and (LN3), $\{N(x_a)\}_{x_a \in J(L^X)} \subseteq \mathbb{F}(X, L, 2)$. For any $x_a, x_b \in J(L^X)$ satisfying $x_a \leq x_b$ and any $\mu \in N(x_b) = N_{T_N}(x_b)$, there exists a $\nu \in T_N$ such that $x_b \leq \nu \leq \mu$ since (X, N) satisfies conditions (LN0)-(LN4) (and thus, $N = N_{T_N}$; see [25] or Remark 3). It follows from $x_a \leq x_b$ that $\mu \in N_{T_N}(x_a) = N(x_a)$. Therefore, $\{N(x_a)\}_{x_a \in J(L^X)}$ satisfies (LFD), and thus, $N(x_a)(N_-)$ is well-defined from Lemma 1 ($\forall x_a \in J(L^X)$).

Suppose that $\mu \in N(x_a)$, then $x_a \leq \nu \leq \mu$ and $\nu \in N(y_c)$ ($\forall y_c \in J(L^X) \cap \downarrow \nu$) hold for some $\nu \in L^X$ by (LN4). It follows that $\mu \in N(y_c)$ ($\forall y_c \in J(L^X) \cap \downarrow \nu$) and $\nu \leq N_-(\mu)$. As $\nu \in T_N$ (see Remark 3), we have $\nu \in N_{T_N}(x_a) = N(x_a)$, and thus, $N_-(\mu) \in N(x_a)$ (i.e., $\mu \in N(x_a)(N_-)$). Therefore, $N(x_a) \subseteq N(x_a)(N_-)$ ($\forall x_a \in J(L^X)$).

(2) \implies (1). Assume $\mu \in N(x_a)$; then $\mu \in N(x_a)(N_-)$ by (2), and then $N_-(\mu) \in N(x_a)$ by definition of $N(x_a)(N_-)$. We will show that ν (put $\nu = N_-(\mu)$) is the required one in (LN4). Firstly, for all $y_b \in J(L^X)$ satisfying $\mu \in N(y_b)$, we have $y_b \leq \mu$ by (LN1) and thus, $N_-(\mu) \leq \mu$ by definition of $N_-(\mu)$. Further, it follows from $N_-(\mu) \in N(x_a)$ that $x_a \leq N_-(\mu)$. Therefore, $\nu \in \uparrow x_a \cap \downarrow \mu$. Secondly, we may show $N_-(\nu) = \nu$. On the one hand, as $\nu \leq \mu$, $N_-(\nu) \leq N_-(\mu) = \nu$. On the other hand, for each $y_b \in J(L^X)$ satisfying $\mu \in N(y_b)$, we have $\mu \in N(y_b)(N_-)$ by (2), and thus, $N_-(\mu) \in N(y_b)$ by definition of $N(y_b)(N_-)$, which implies $N_-(\mu) \leq N_-(N_-(\mu))$. Therefore, $\nu \leq N_-(\nu)$. Finally, for each $z_c \triangleleft \nu = N_-(\nu)$, there exists a $z_b \in J(L^X)$ such that $z_c \leq z_b$ and $\nu \in N(z_b)$. $\nu \in N(z_c)$ since $\{N(x_a)\}_{x_a \in J(L^X)}$ satisfies (LFD). That is, for each $z_c \triangleleft \nu$, $\nu \in N(z_c)$ holds. \square

Lemma 3. For a pair (X, \longrightarrow) satisfying conditions (L1)-(L3) and (order) in Definition 8, the following are equivalent:

- (1) (L4)
- (2) (LK4)

Proof

(i) (1) \implies (2). It suffices to prove $N(x_a) \subseteq G(F_-)$ by (L3). First, we show $G(N_-) \subseteq G(F_-)$. Suppose that $\mu \in G(N_-)$ (i.e., $N_-(\mu) \in G$). For each $y_b \in J(L^X)$ satisfying $\mu \in N(y_b)$, we have $N(y_b) \subseteq F(y_b)$ (and thus, $\mu \in F(y_b)$) for $F(y_b) \longrightarrow y_b$ and (L3), which implies $N_-(\mu) \leq F_-(\mu)$, and thus, $F_-(\mu) \in G$; that is $\mu \in G(F_-)$. Next, we show $N(x_a)(N_-) \subseteq G(N_-)$. Since $G \longrightarrow x_a$, we obtain $N(x_a) \subseteq G$. Suppose $\mu \in N(x_a)(N_-)$, then $N_-(\mu) \in N(x_a)$, and thus, $N_-(\mu) \in G$; that is, $\mu \in G(N_-)$. Finally, as $N(x_a) \subseteq N(x_a)(N_-)$ (see Lemma 2), we have $N(x_a) \subseteq G(F_-)$ by the preceding two conclusions.

(ii) (2) \implies (1). Take $G = N(x_a)$ and $F(y_b) = N(x_b)$ ($\forall y_b \in J(L^X)$). Since (order) is satisfied, $\{F(y_b)\}_{y_b \in J(L^X)}$ is a subfamily of $\mathbb{F}(X, L, 2)$ satisfying (LFD). $G \longrightarrow x_a$, and $F(y_b) \longrightarrow y_b$ ($\forall y_b \in J(L^X)$). Thus, $G(F_-) \longrightarrow x_a$ by (2); that is, $N(x_a) \subseteq G(F_-) = N(x_a)(N_-)$. It follows from Lemma 2 that (L4) holds. \square

Proof of Theorem 3. It follows from Lemma 3.

In paper [11], based on stratified L -fuzzy filter, Gähler's neighborhood condition is studied in stratified L -convergence space. Based on L -filter, Gähler's neighborhood condition is also studied in L -convergence space. \square

Theorem 4. (X, \longrightarrow) is an L -topological convergence space if and only if it satisfies (L1)-(L3), (order) in Definition 8, and the following:

- (i) (LG4) If $F \longrightarrow x_a$, then $F(\mathcal{N}_-) \longrightarrow x_a$ ($\forall x_a \in J(L^X)$, $F \in \mathbb{F}(X, L, 2)$)

Proof. Suppose (X, \longrightarrow) is an L -topological convergence space; we show that (X, \longrightarrow) satisfies (LG4). If $F \longrightarrow x_a$, then, by (L3), $N(x_a) \subseteq F$. It can be easily checked that $N(x_a)(N_-) \subseteq F(N_-)$. It follows from Lemma 2 that $N(x_a) \subseteq F(N_-)$; that is, $F(N_-) \longrightarrow x_a$. Conversely, it follows from (L3) that $N(x_a) \longrightarrow x_a$, and thus, $N(x_a)(N_-) \longrightarrow x_a$ by (LG4). This implies $N(x_a) \subseteq N(x_a)(N_-)$. Following Lemma 2, (L4) holds. \square

5. The Relation between (X, \longrightarrow) and Li's (X, \lim)

An L -topological stratified L -fuzzy convergence space [6] (where L is a complete Heyting algebra) is a pair (X, \lim) , where X is a set, and $\lim: \mathbb{F}^S(X, L, L) \longrightarrow L^X$ (called an

L -topological stratified L -fuzzy convergence structure on X is a mapping which satisfies the following conditions:

- (i) $(L1)^* \limx = 1 (\forall x \in X)$.
- (ii) $(L2)^* F \leq G$ implies $\lim F \leq \lim G (\forall F, G \in \mathbb{F}^S(X, L, L))$.
- (iii) $(Lp) \lim F(x) = \bigwedge_{\mu \in L^X} (N^x(\mu) \longrightarrow N(\mu))$,
- (iv) where

$$N^x(\mu) = \bigwedge_{G \in SF(X, L, L)} (\lim G(x) \longrightarrow G(\mu)), \quad (\forall x \in X). \quad (20)$$
- (v) $(Lt) N^x(\mu) \leq \bigvee \{N^x(\nu) \mid \nu(y) \leq N^y(\mu) (\forall y \in X)\},$
 $(\forall x \in X, \forall \mu \in L^X).$

From Proposition 6.4 in [6], there exists a one-to-one correspondence between the set of all L -topological stratified L -fuzzy convergence structures on X and the set of stratified L -topologies on X . Following Jäger's work, Li [12] proved that there exists a one-to-one correspondence between the set of all L -topological L -fuzzy convergence structures on X and the set of L -topologies on X , where an L -topological L -fuzzy convergence structure is a mapping $\lim: \mathbb{F}(X, L, L) \longrightarrow L^X$ which satisfies $(L1)^*$, $(L2)^*$, (Lp) , and (Lt) .

The spaces (X, \longrightarrow) and (X, \lim) are absolutely different in their forms. For the former, a filter is a family of subsets of L^X ; the point is fuzzy but the convergence is crisp. However, for the latter, a filter is a mapping from L^X to L ; the point is crisp but the convergence is fuzzy. From Remark 3, there exists a one-to-one correspondence between the set of all L -topological convergence structures on X and the set of L -topologies on X . Thus, these two notions (L -topological L -fuzzy convergence structures [12] and L -topological convergence structures) can be determined reciprocally for L being a completely distributive lattice. To say it more precisely, there exists a one-to-one correspondence φ_{13} from $\mathbb{A}(X, L)$ (the set of all L -topological convergence structures on X) to $\mathbb{B}(X, L)$ (the set of all L -topological L -fuzzy convergence structures on X), which is defined by

$$\varphi_{13}(\longrightarrow)F(x) = \bigwedge_{\mu \in L^X} (N^x(\mu) \longrightarrow F(\mu)), \quad (21)$$

$$(\forall F \in \mathbb{F}(X, L, L), \forall x \in X),$$

for each \longrightarrow in $\mathbb{A}(X, L)$, where $N^x(\mu) = \bigvee \{a \in J(L) \mid \mu \in N(x_a)\}, (\forall \mu \in L^X)$; the inverse mapping φ_{31} of φ_{13} is given by $\varphi_{31}(\lim) = \longrightarrow_{\lim}$ for each $\lim \in \mathbb{B}(X, L)$, which satisfies

$$F \longrightarrow_{\lim} x_a \iff N(x_a) \subseteq F, \quad (22)$$

where $N(x_a) = \{\mu \in L^X \mid N^x(\mu) \geq a\} (\forall x_a \in J(L^X), F \in \mathbb{F}(X, L, 2))$.

6. Conclusions

In the present paper, we propose the notion of (L, M) -fuzzy topological convergence structure (given by L -filters) and show that such a structure can be used to characterize an

(L, M) -fuzzy topology. With convergence theory of molecular nets [19], we give a Moore–Smith convergence theory in (L, M) -fuzzy topological space. For further issues, we are devoted to L -topological convergence structure and give two conditions, Kowalsky's diagonal condition and Gähler's neighborhood condition, in fuzzing setting. From these results, we consider that L -filter is an important and appropriate tool to (L, M) -fuzzy topological space and also to L -topological space.

It is well known that (L, M) -fuzzy topological space has bifuzziness: fuzziness of open sets and fuzziness of openness. So we are sure that there exist other kinds of convergence structures such that all these structures can be categorically isomorphic to (L, M) -fuzzy topologies. For example, with the help of L -filter and idea of Q -neighborhood operator (or L -fuzzy filter and idea of neighborhood operator), we could consider the corresponding convergence structures. This will be our future work. After these efforts, maybe, due to one kind of these convergence structures, we can study other properties of (L, M) -fuzzy topological spaces and (L, M) -fuzzy topological group [27] for more convenience.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundations of China (11501435, 11771263, and 61473181).

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Research Article

Generalized Reich–Ćirić–Rus-Type and Kannan-Type Contractions in Cone b -Metric Spaces over Banach Algebras

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Received 15 July 2021; Revised 10 August 2021; Accepted 30 August 2021; Published 27 September 2021

Academic Editor: Zoran Mitrovic

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In this paper, we firstly introduce the generalized Reich–Ćirić–Rus-type and Kannan-type contractions in cone b -metric spaces over Banach algebras and then obtain some fixed point theorems satisfying these generalized contractive conditions, without appealing to the compactness of X . Secondly, we prove the existence and uniqueness results for fixed points of asymptotically regular mappings with generalized Lipschitz constants. The continuity of the mappings is deleted or relaxed. At last, we prove that the completeness of cone b -metric spaces over Banach algebras is necessary if the generalized Kannan-type contraction has a fixed point in X . Our results greatly extend several important results in the literature. Moreover, we present some nontrivial examples to support the new concepts and our fixed point theorems.

1. Introduction and Preliminaries

It is well known that the fixed point theory is widely applied to almost all fields of quantitative sciences such as computer science, physics, and biology, especially since the famous Banach contraction principle was introduced in 1922 [1]. In 1968, Kannan [2] studied the following meaningful fixed point theorem, which is a generalization of Banach contraction principle.

Theorem 1. *Let (X, d) be a complete metric space and let $T: X \rightarrow X$ be a mapping such that there exists $K < (1/2)$ satisfying*

$$d(Tx, Ty) \leq K\{d(x, Tx) + d(y, Ty)\}, \quad (1)$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$ and for each $x \in X$, the iterated sequence $\{T^n x\}$ converges to z .

The mapping satisfying the contractive condition is known as Kannan-type contraction mapping, which is highly interesting since the contraction mapping does not

need to be continuous. In 1971, Reich [3] further extended the Banach and Kannan fixed point theorems as follows.

Theorem 2. *Let (X, d) be a complete metric space and let $T: X \rightarrow X$ be a mapping such that there exist $a_1, a_2, a_3 \geq 0, a_1 + a_2 + a_3 < 1$ satisfying*

$$d(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, y), \quad (2)$$

for all $x, y \in X$. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z .

The mapping satisfying (2) was originally called Reich-type contraction mapping. Since the importance of the Reich-type contraction is simultaneously proved by Ćirić [4] and Rus [5], we say that the mapping T is a Reich–Ćirić–Rus-type contraction mapping. Recently, Górnicki [6] proved the following theorems in compact metric spaces.

Theorem 3. *Let (X, d) be a compact metric space and let $T: X \rightarrow X$ be a continuous mapping satisfying*

$$d(Tx, Ty) < \frac{1}{2} [d(x, Tx) + d(y, Ty)], \quad (3)$$

for all $x, y \in X$ and $x \neq y$. Then T has a unique fixed point $z \in X$ and for each $x \in X$, the iterated sequence $\{T^n x\}$ converges to z .

Theorem 4. Let (X, d) be a compact metric space and let $T: X \rightarrow X$ be a continuous mapping such that there exist $a_1, a_2, a_3 \geq 0, a_1 + a_2 + a_3 = 1$ satisfying

$$d(Tx, Ty) < a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, y), \quad (4)$$

for all $x, y \in X$ and $x \neq y$. Then T has a unique fixed point $z \in X$ and for each $x \in X$, the iterated sequence $\{T^n x\}$ converges to z .

Note that the continuity of the mapping and the compactness of the metric space are essential conditions in Theorems 3 and 4. In order to improve these theorems, Garai et al. [7] investigated some meaningful fixed point theorems of Kannan-type contractive mappings in metric spaces by using the notions of bounded compactness, orbital continuity, and T -orbital compactness. Afterwards, Haokip and Goswami [8] extended some related results in b -metric spaces by using a subadditive altering distance function. In this paper, we further study the fixed point theorems about Kannan-type and Reich-Ćirić-Rus-type contractions in a much broader space.

The concept of b -metric space was derived from the work of Bakhtin [9] and Czerwik [10]. They gave a weaker condition than the triangular inequality, with the aim of extending Banach contraction principle. Moreover, in general, a b -metric is not a continuous function and thus is a generation of metric [11]. Subsequently, Hussian and Shah [12] introduced cone b -metric space which extended cone metric space [13] and b -metric space. In cone b -metric space, the distance between x and y is defined by a vector in an ordered Banach space, instead of the usual real line (see [14]). In 2013, Liu and Xu [15] introduced the concept of cone metric space over a Banach algebra by replacing Banach spaces with Banach algebras and considering the contractive constants to be vectors. Moreover, in their paper, it is significant to prove the nonequivalence of fixed point results between metric spaces and cone metric spaces over Banach algebras by some valid examples. In a similar way, the notion of cone b -metric space over a Banach algebra was defined by Huang and Radenović [16], which is also nonequivalent to b -metric space in terms of the existence of the fixed points of contractions with vector-valued coefficients. Since then, the fixed point theory in these abstract spaces is prompted to be investigated by lots of authors; for detail, see [17–19] and references therein.

In this paper, we prove some fixed point theorems about generalized Kannan-type and Reich-Ćirić-Rus-type contractions in cone b -metric spaces over Banach algebras by introducing the notions of bounded compactness, T -orbital compactness, orbital continuity, orbital completeness, and asymptotic regularity in cone b -metric spaces over Banach algebras. The main conclusion improves and extends some

important known results in the literature [1–3, 6, 7, 20, 21]. Moreover, we prove that the completeness of cone b -metric spaces over Banach algebras is necessary if the generalized Kannan-type contraction has a fixed point in X . Furthermore, there are some examples to present that our new notions and main conclusions are genuine improvements and extensions of the corresponding notions and works in the literature.

First, let us recall some preliminary concepts of Banach algebras and cone b -metric spaces.

Let \mathcal{A} be a real Banach algebra; i.e., \mathcal{A} is a real Banach space in which an operation of multiplication is defined, subject to the following properties: for all, $x, y, z \in \mathcal{A}, a \in \mathbb{R}$

- (1) $x(yz) = (xy)z$
- (2) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$
- (3) $a(xy) = (ax)y = x(ay)$
- (4) $\|xy\| \leq \|x\|\|y\|$

In this paper, we shall assume that the Banach algebra \mathcal{A} has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1} . For more details, we refer to [22].

A subset P of \mathcal{A} is called a cone if

- (i) P is nonempty and closed and $\{\theta, e\} \subset P$, where θ denotes the zero element of \mathcal{A}
- (ii) $\alpha P + \beta P \subset P$ for all nonnegative real numbers α, β
- (iii) $P^2 = PP \subset P$
- (iv) $P \cap (-P) = \{\theta\}$

For a given cone $P \subset \mathcal{A}$, we can define a partial ordering \preceq with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where $\text{int}P$ denotes the interior of P .

A cone P is called normal if there is a number $K > 0$ such that for all $x, y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \quad \text{implies} \quad \|x\| \leq K\|y\|. \quad (5)$$

The least positive number satisfying the above inequality is called the normal constant of P . Indeed, the number K cannot be less than 1; see [23]. A cone P is called regular if every increasing sequence which is bounded from above is convergent. In other words, if there is a $y \in \mathcal{A}$ such that

$$x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y, \quad (6)$$

then there exists $x \in \mathcal{A}$ such that $\lim_{n \rightarrow \infty} \|x_n - x\| = 0$. Equivalently, a cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that every regular cone is normal.

A cone P is called strongly minihedral if each subset of \mathcal{A} which is bounded from above has a supremum. If P is a strongly minihedral cone, then every subset of \mathcal{A} bounded below has an infimum (see [24, 25]).

Throughout this paper, we always assume that P is a cone over Banach algebra \mathcal{A} with $\text{int}P \neq \emptyset$ and \preceq is the partial ordering with respect to P .

Definition 1 (see [12, 16, 17, 18]). Let X be a nonempty set and $s \geq 1$ be a constant. Suppose that the mapping $d: X \times X \rightarrow \mathcal{A}$ satisfies the following:

- (d1) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$
- (d2) $d(x, y) = d(y, x)$ for all $x, y \in X$
- (d3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$

Then d is called a cone b -metric on X and (X, d) is called a cone b -metric space over Banach algebra \mathcal{A} .

Definition 2 (see [14, 16]). Let (X, d) be a cone b -metric space over a Banach algebra \mathcal{A} , $x \in X$ and $\{x_n\}$ a sequence in X . Then,

- (i) $\{x_n\}$ converges to x if, for every $c \in \mathcal{A}$ with $c \gg \theta$, there is a natural number N such that for all $n > N$, $d(x_n, x) \ll c$
- (ii) $\{x_n\}$ is a Cauchy sequence if, for every $c \in \mathcal{A}$ with $c \gg \theta$, there is a natural number N such that for all $n, m > N$, $d(x_n, x_m) \ll c$

- (iii) (X, d) is a complete cone b -metric space if every Cauchy sequence is convergent in X

It is worth mentioning that unlike the usual metric or cone metric with a normal cone, cone b -metric is not necessarily continuous in general even if the cone is normal, as the following example shows.

Example 1. Let $\mathcal{A} = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$. The multiplication is defined by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1), \quad (7)$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{A}$. It follows that \mathcal{A} is a Banach algebra with a unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2: x_1, x_2 \geq 0\}$. Then P is a normal cone with a normal constant $K = 1$. Let $X = \mathbb{N} \cup \{\infty\} \times \mathbb{N} \cup \{\infty\}$ for all $x = (x_1, x_2), y = (y_1, y_2)$. Define the cone b -metric $d: X \times X \rightarrow \mathcal{A}$ by

$$d((x_1, x_2), (y_1, y_2)) = \begin{cases} (0, 0), & x_1 = y_1, x_2 = y_2; \\ \left(\left| \frac{1}{x_1} - \frac{1}{y_1} \right|, \left| \frac{1}{x_2} - \frac{1}{y_2} \right| \right), & \text{one of } (x_1, x_2) \text{ and } (y_1, y_2) \text{ is odd and the other is odd or } \infty; \\ (6, 6), & \text{one of } (x_1, x_2) \text{ and } (y_1, y_2) \text{ is even and the other is even or } \infty; \\ (3, 3), & \text{otherwise.} \end{cases} \quad (8)$$

Note that for the definition of the cone b -metric d above, (x_1, x_2) is odd if both x_1 and x_2 are odd; (x_1, x_2) is ∞ if both x_1 and x_2 are ∞ . Then it is sufficient to check that (X, d) is a cone b -metric space over Banach algebra \mathcal{A} with the coefficient $s = 2$.

Let $z_n = (2n + 1, 2n + 3)$ for each $n \in \mathbb{N}$. Then

$$\begin{aligned} d(z_n, \infty) &= d((2n + 1, 2n + 3), (\infty, \infty)) \\ &= \left(\left| \frac{1}{2n + 1} - \frac{1}{\infty} \right|, \left| \frac{1}{2n + 3} - \frac{1}{\infty} \right| \right) \\ &= \left(\frac{1}{2n + 1}, \frac{1}{2n + 3} \right) \longrightarrow (0, 0), \quad (n \longrightarrow \infty). \end{aligned} \quad (9)$$

This gives $z_n \longrightarrow \infty$. However,

$$\begin{aligned} d(z_n, 2) &= d((2n + 1, 2n + 3), (2, 2)) = (3, 3) \nrightarrow (6, 6) \\ &= d((\infty, \infty), (2, 2)) = d(\infty, 2), \end{aligned} \quad (10)$$

as $n \longrightarrow \infty$. Therefore, the cone b -metric is not continuous even though the cone is normal.

Definition 3 (see [26]). Let P be a solid cone in a Banach algebra \mathcal{A} . A sequence $\{u_n\} \subset P$ is a c -sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.

Lemma 1 (see [16]). Let P be a solid cone in a Banach algebra \mathcal{A} and let $\{x_n\}$ and $\{y_n\}$ be sequences in P . If $\{x_n\}$ and $\{y_n\}$ are c -sequences and $\alpha, \beta \in P$, then $\{\alpha x_n + \beta y_n\}$ is a c -sequence.

Lemma 2 (see [22]). Let \mathcal{A} be a Banach algebra with a unit e and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,

$$r(x) = \lim_{n \longrightarrow \infty} \|x^n\|^{1/n} = \inf_{n \geq 1} \|x^n\|^{1/n} < 1, \quad (11)$$

then $e - x$ is invertible. Actually, $(e - x)^{-1} = \sum_{i=0}^{\infty} x^i$.

Inspired by Definition 3 in [24], we introduce a new notion of the distance between a set and a singleton in cone b -metric space over a Banach algebra \mathcal{A} .

Definition 4. Let (X, d) be a cone b -metric space over a Banach algebra \mathcal{A} and M be a nonempty subset of X . Let P

be a normal and strongly minihedral cone. The distance between the set M and the singleton $\{x\}$ is defined as follows:

$$d(x, M) = \inf\{d(x, y) : y \in M\}. \quad (12)$$

2. Bounded Compactness and T -Orbital Compactness

The concepts of bounded compactness and T -orbital compactness were discussed in usual metric spaces [7] and b -metric spaces [8], which were important to weaken the condition of compactness. In the following, we give the notions of generalized Kannan-type and Reich-Ćirić-Rus-type contractions, bounded compactness, and T -orbital compactness in the framework of cone b -metric spaces over Banach algebras, which are generalizations of metric spaces and b -metric spaces.

Definition 5. Let (X, d) be a cone b -metric space over a Banach algebra \mathcal{A} with a unit e . The mapping $T: X \rightarrow X$ is said to be a generalized Kannan-type contraction, if it satisfies

$$d(Tx, Ty) < \frac{e}{2} \{d(x, Tx) + d(y, Ty)\}, \quad (13)$$

for all $x, y \in X$ with $x \neq y$.

Definition 6. Let (X, d) be a cone b -metric space over a Banach algebra \mathcal{A} with a unit e . The mapping $T: X \rightarrow X$ is said to be a generalized Reich-Ćirić-Rus-type contraction, if it satisfies

$$d(Tx, Ty) < a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, y), \quad (14)$$

for all $x, y \in X$ with $x \neq y$, where $a_1, a_2, a_3 \in P$ with $a_1 + a_2 + a_3 = e$.

Definition 7. Let (X, d) be a cone b -metric space over a Banach algebra \mathcal{A} and T be a self-mapping on X . Let $x \in X$ and $O_T(x) = \{x, Tx, T^2x, T^3x, \dots\}$.

The space (X, d) is said to be boundedly compact, if every bounded sequence in X has a convergent subsequence.

The mapping T is said to be orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O_T(x)$ (for all $x \in X$), $x_n \rightarrow z$ as $n \rightarrow \infty$ implies $Tx_n \rightarrow Tz$ as $n \rightarrow \infty$. Clearly, every continuous mapping is orbitally continuous, but not the converse.

The set X is said to be T -orbitally compact set, if every sequence in $O_T(x)$ has a convergent subsequence for all $x \in X$.

Example 2. Let $\mathcal{A} = C_{\mathbb{R}}^1[0, 1] \times C_{\mathbb{R}}^1[0, 1]$ with the norm

$$\|(x_1, x_2)\| = \|x_1\|_{\infty} + \|x_2\|_{\infty} + \|x_1'\|_{\infty} + \|x_2'\|_{\infty}. \quad (15)$$

Define the multiplication by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1), \quad (16)$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{A}$. Then \mathcal{A} is a Banach algebra with a unit $e = (1, 0)$. Let $P = \{(x_1(t), x_2(t)) \in \mathcal{A} : x_1(t) \geq 0, x_2(t) \geq 0, t \in [0, 1]\}$.

- (1) Let $X = [0, \infty) \times [0, \infty)$ and define the cone b -metric $d: X \times X \rightarrow \mathcal{A}$ by $d((x_1, x_2), (y_1, y_2))(t) = (|x_1 - y_1|^2, |x_2 - y_2|^2) \cdot \alpha^t \in P, \forall x = (x_1, x_2), y = (y_1, y_2) \in X, \alpha > 0$. Then (X, d) is a complete cone b -metric space over Banach algebra \mathcal{A} with $s = 2$. Define mappings $T_1, T_2: X \rightarrow X$ as

$$T_1(x_1, x_2) = \left(\frac{x_1}{(n+1)^3}, \frac{x_2}{(m+1)^3} \right), \quad n \leq x_1 < n+1, m \leq x_2 < m+1, \quad (17)$$

$$T_2(x_1, x_2) = (3x_1 + 1, 3x_2 + 2),$$

for all $x \in X$ and $n \in \mathbb{N}$. This clearly gives that X is T_1 -orbitally compact and boundedly compact but not T_2 -orbitally compact.

- (2) Let $X = [0, 1) \times [0, 1)$. The cone b -metric is defined the same as above and $T: X \rightarrow X$ is defined by $T(x_1, x_2) = ((x_1/2), (x_2/3))$. We deduce that X is T -orbitally compact but not complete.
- (3) Let $X = [-2, 2] \times [-2, 2]$. The cone b -metric is defined the same as above and $T: X \rightarrow X$ is defined by

$$T(x_1, x_2) = \begin{cases} (0, 0), & (x_1, x_2) \in [-1, 1] \times [-1, 1]; \\ \left(\frac{x_1}{4}, \frac{x_2}{6} \right), & \text{otherwise.} \end{cases} \quad (18)$$

Then, for any $z_0 = (x_1, x_2) \in X, n \in \mathbb{N}, z_n = Tz_{n-1}, z_n \rightarrow \theta$ implies $Tz_n \rightarrow T\theta = \theta(\theta = (0, 0))$. So T is orbitally continuous but not continuous in X .

In the rest of this section, we always assume that (X, d) is a cone b -metric space over Banach algebra \mathcal{A} with regular cone P such that $d(x, y) \in \text{int}P$ for all $x, y \in X$ with $x \neq y$ and the cone b -metric d is continuous.

Theorem 5. Let (X, d) be a boundedly compact cone b -metric space over Banach algebra \mathcal{A} with a unit e and the coefficient $s \geq 1$. Let $T: X \rightarrow X$ be a generalized Reich-Ćirić-Rus-type contraction mapping and orbitally continuous. If $(e - a_2)^{-1}$ and $(e - s^2a_3)^{-1}$ exist, then T has a unique

fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z ; i.e., T is a Picard operator.

Proof. For an arbitrary $x_0 \in X$, let $x_n = Tx_{n-1} = T^n x_0$, $n \geq 1$. We assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Indeed, if for some $n \in \mathbb{N}$, $x_n = x_{n+1} = Tx_n$, then x_n is the fixed point of T . Denote $t_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. By (14), we have

$$\begin{aligned} t_n &= d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \\ &< a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) + a_3 d(x_{n-1}, x_n) \\ &= a_1 t_{n-1} + a_2 t_n + a_3 t_{n-1}, \end{aligned} \quad (19)$$

which gives that $t_n \leq (e - a_2)^{-1} (a_1 + a_3) t_{n-1}$. Let $h = (e - a_2)^{-1} (a_1 + a_3)$, then $h = e$ by $a_1 + a_2 + a_3 = e$. Therefore, we have

$$\theta < \dots < t_n < t_{n-1} < \dots < t_0 = d(x_0, x_1). \quad (20)$$

Because the cone is regular, there exists $b \geq \theta$ in \mathcal{A} such that $t_n \rightarrow b$ ($n \rightarrow \infty$). Thus, for all $n, m \in \mathbb{N}$, consider

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{m+1}) + s^2 d(x_{m+1}, x_m) \\ &< s d(x_n, x_{n+1}) + s^2 [a_1 d(x_n, x_{n+1}) + a_2 d(x_m, x_{m+1}) + a_3 d(x_n, x_m)] + s^2 d(x_{m+1}, x_m) \\ &= (se + s^2 a_1) d(x_n, x_{n+1}) + (s^2 a_2 + s^2 e) d(x_m, x_{m+1}) + s^2 a_3 d(x_n, x_m), \end{aligned} \quad (21)$$

which implies that

$$\begin{aligned} d(x_n, x_m) &\leq (e - s^2 a_3)^{-1} [(se + s^2 a_1) d(x_n, x_{n+1}) \\ &\quad + (s^2 e + s^2 a_2) d(x_m, x_{m+1})] \\ &< (e - s^2 a_3)^{-1} (se + s^2 a_1 + s^2 e + s^2 a_2) t_0. \end{aligned} \quad (22)$$

Therefore, $\{x_n\}$ is bounded. Since X is boundedly compact, there is a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $z \in X$ such that $x_{n_i} \rightarrow z$ as $i \rightarrow \infty$. So $Tx_{n_i} \rightarrow Tz$ by the orbital continuity of T . If $b > \theta$, then

$$\theta < b = \lim_{i \rightarrow \infty} d(x_{n_i}, Tx_{n_i}) = d(z, Tz). \quad (23)$$

Moreover,

$$\begin{aligned} \theta < b &= \lim_{i \rightarrow \infty} t_{n_i} = \lim_{i \rightarrow \infty} d(Tx_{n_i}, T^2 x_{n_i}) \\ &= d(Tz, T^2 z) < a_1 d(z, Tz) + a_2 d(Tz, T^2 z) + a_3 d(z, Tz), \end{aligned} \quad (24)$$

which means

$$b = d(Tz, T^2 z) < d(z, Tz) = b, \quad (25)$$

a contradiction. Thus, $b = \theta$ and $z = Tz$. That is, z is a fixed point of T . Then, the inequality (14) implies

$$\begin{aligned} d(x_{n+1}, z) &= d(Tx_n, Tz) \\ &< a_1 d(x_n, Tx_n) + a_2 d(z, Tz) + a_3 d(x_n, z) \\ &= a_1 d(x_n, x_{n+1}) + a_3 d(x_n, z) \\ &\leq a_1 d(x_n, x_{n+1}) + a_3 s^2 [d(x_n, x_{n+1}) + d(x_{n+1}, z)]. \end{aligned} \quad (26)$$

Hence, $(e - s^2 a_3) d(x_{n+1}, z) \leq (a_1 + s^2 a_3) d(x_n, x_{n+1}) \rightarrow \theta$. This gives $T^n x \rightarrow z$; i.e., T is a Picard operator.

Finally, the uniqueness of the fixed point can be obtained by (14). If y is another fixed point of T , then

$$\begin{aligned} d(y, z) &= d(Ty, Tz) < a_1 d(y, Ty) + a_2 d(z, Tz) + a_3 d(y, z) \\ &= a_3 d(y, z), \end{aligned} \quad (27)$$

leading to a contradiction. Therefore, z is the unique fixed point of T . \square

Theorem 6. Let (X, d) be a T -orbitally compact cone b -metric space over Banach algebra \mathcal{A} with a unit e and the coefficient $s \geq 1$, where $T: X \rightarrow X$ is a generalized Reich-Ćirić-Rus-type contraction mapping and orbitally continuous. If $(e - a_2)^{-1}$ and $(e - s^2 a_3)^{-1}$ exist, then T has a unique fixed point $z \in X$, and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z ; i.e., T is a Picard operator.

Proof. The analysis is similar to that in the proof of Theorem 5. We firstly get the sequence $x_n = Tx_{n-1} = T^n x_0$, $n \geq 1$. If there is an integer $n \in \mathbb{N}$ such that $x_n = x_{n+1} = Tx_n$, then x_n is the fixed point. Without loss of generality, we assume that $x_n \neq x_{n+1}$, $\forall n \in \mathbb{N}$. We can prove that $t_n \rightarrow b \geq \theta$ ($n \rightarrow \infty$), where t_n and b are the same as above. As X is T -orbitally compact, there is a convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $z \in X$ such that $x_{n_i} \rightarrow z$ as $n \rightarrow \infty$. By orbital continuity of T , we obtain $Tx_{n_i} \rightarrow Tz$. The rest proof is similar to Theorem 5. \square

Corollary 1. Let (X, d) be a boundedly compact cone b -metric space over Banach algebra \mathcal{A} with a unit e and the

coefficient $s \geq 1$. Let $T: X \longrightarrow X$ be a generalized Kannan-type contraction mapping which is orbitally continuous. Then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z ; i.e., T is a Picard operator.

Proof. Taking $a_1 = a_2 = (e/2)$ and $a_3 = \theta$, we obtain the conclusion by Theorem 5. \square

Corollary 2. Let (X, d) be a T -orbitally compact cone b -metric space over Banach algebra \mathcal{A} with a unit e and the coefficient $s \geq 1$, where $T: X \longrightarrow X$ is a generalized Kannan-type contraction mapping and orbitally continuous. Then T has a unique fixed point $z \in X$ and for each $x \in X$, the iterated sequence $\{T^n x\}$ converges to z ; i.e., T is a Picard operator.

Proof. The proof is analogous. \square

Remark 1. Theorems 5 and 6 greatly improve Theorem 2.3 in [6]. The assumptions of compactness and continuity considered in Theorem 2.3 of [6] are relaxed by bounded compactness or T -orbital compactness and T -orbital continuity, respectively. Corollaries 1 and 2 mainly improve and generalize Theorem 2.2 in [6] and Theorem 2.1 and Theorem 2.2 in [7].

Example 3. Let $\mathcal{A} = \mathbb{R}^2$ with the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$. The multiplication is defined by

$$xy = (x_1, x_2)(y_1, y_2) = (x_1 y_1, x_1 y_2 + x_2 y_1), \quad (28)$$

where $x = (x_1, x_2), y = (y_1, y_2) \in \mathcal{A}$. It follows that \mathcal{A} is a Banach algebra with a unit $e = (1, 0)$. Let $P = \{(x_1, x_2) \in \mathbb{R}^2: x_1, x_2 \geq 0\}$. Then P is a normal cone with a normal constant $K = 1$. Let $X = [-1, 2) \cup \{3\} \times [-1, 2) \cup \{3\}$ and define the cone b -metric $d: X \times X \longrightarrow \mathcal{A}$ by

$$d((x_1, x_2), (y_1, y_2)) = (k_1 |x_1 - y_1|^2, k_2 |x_2 - y_2|^2) \in P, \quad (29)$$

for all $x = (x_1, x_2), y = (y_1, y_2)$ in X , where $k_1, k_2 > 0$ are constants. Furthermore, define the mapping $T: X \longrightarrow X$ by

$$Tx = \begin{cases} \left(\frac{1}{7}x_1, \frac{1}{5}x_2\right), & x \neq 3; \\ (0, 0), & \text{otherwise,} \end{cases} \quad (30)$$

where $x \neq 3$ is equivalent to $x = (x_1, x_2) \neq (3, x_2)$ and $x = (x_1, x_2) \neq (x_1, 3)$. Obviously, T is not continuous but T -orbitally continuous. Moreover, (X, d) is an incomplete cone b -metric space over Banach algebra \mathcal{A} with $s = 2$ but T -orbitally compact. Let $a_1 = ((4/9), 0), a_2 = ((1/3), 0), a_3 = ((2/9), 0)$, then $a_1 + a_2 + a_3 = e$ and $(e - a_2)^{-1}, (e - s^2 a_3)^{-1}$ exist. In order to check the generalized Reich-Ćirić-Rus-type contraction, we have the following three cases:

(i) If $x \neq 3$ and $y = 3$, then

$$\begin{aligned} d(Tx, T3) &= \left(k_1 \left|\frac{1}{7}x_1 - 0\right|^2, k_2 \left|\frac{1}{5}x_2 - 0\right|^2\right) \\ &= \left(\frac{1}{49}k_1 x_1^2, \frac{1}{25}k_2 x_2^2\right) < \left(\frac{4}{9}, 0\right) \left(\frac{36}{49}k_1 x_1^2, \frac{16}{25}k_2 x_2^2\right) + \left(\frac{1}{3}, 0\right)(0, 0) + \left(\frac{2}{9}, 0\right)(k_1 |x_1 - 2|^2, k_2 |x_2 - 2|^2) \\ &= \left(\frac{4}{9}, 0\right) \left(k_1 \left|x_1 - \frac{1}{7}x_1\right|^2, k_2 \left|x_2 - \frac{1}{5}x_2\right|^2\right) + \left(\frac{1}{3}, 0\right)(0, 0) + \left(\frac{2}{9}, 0\right)(k_1 |x_1 - 2|^2, k_2 |x_2 - 2|^2) \\ &= a_1 d(x, Tx) + a_2 d(3, T3) + a_3 d(x, y). \end{aligned} \quad (31)$$

(ii) If $x, y \in X$ with $x \neq 3, y \neq 3$ and $x \neq y$, then

$$\begin{aligned} d(Tx, Ty) &= \left(k_1 \left|\frac{1}{7}x_1 - \frac{1}{7}y_1\right|^2, k_2 \left|\frac{1}{5}x_2 - \frac{1}{5}y_2\right|^2\right) \\ &< \left(\frac{4}{9}, 0\right) \left(\frac{36}{49}k_1 x_1^2, \frac{16}{25}k_2 x_2^2\right) + \left(\frac{1}{3}, 0\right) \left(\frac{36}{49}k_1 y_1^2, \frac{16}{25}k_2 y_2^2\right) + \left(\frac{2}{9}, 0\right)(k_1 |x_1 - y_1|^2, k_2 |x_2 - y_2|^2) \\ &= \left(\frac{4}{9}, 0\right) \left(k_1 \left|x_1 - \frac{1}{7}x_1\right|^2, k_2 \left|x_2 - \frac{1}{5}x_2\right|^2\right) + \left(\frac{1}{3}, 0\right) \left(k_1 \left|y_1 - \frac{1}{7}y_1\right|^2, k_2 \left|y_2 - \frac{1}{5}y_2\right|^2\right) \\ &\quad + \left(\frac{2}{9}, 0\right)(k_1 |x_1 - y_1|^2, k_2 |x_2 - y_2|^2) = a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, y). \end{aligned} \quad (32)$$

(iii) If $x = 3$ and $y = 3$, then

$$d(Tx, Ty) = (0, 0) < a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, y), \quad (33)$$

is clearly true. Therefore, the mapping T has a unique fixed point in X by Theorem 6.

3. Asymptotic Regularity and Orbital Completeness

In the following, we obtain some fixed point theorems of generalized contractive mappings in orbitally complete cone b -metric spaces over Banach algebras, under the condition of asymptotic regularity. The regularity or normality of the cone and the continuity of the cone b -metric are not necessary. Now, we give the definition of asymptotic regularity, which is a generalization of the counterpart in metric spaces.

Definition 8 (see [27]). Let (X, d) be a metric space. The mapping $T: X \rightarrow X$ is said to be asymptotically regular, if $\lim_{n \rightarrow \infty} d(T^{n+1}x, T^n x) = 0$ for all $x \in X$.

Definition 9. Let (X, d) be a cone b -metric space over a Banach algebra \mathcal{A} . The mapping $T: X \rightarrow X$ is said to be asymptotically regular, if for every $c \in \mathcal{A}$ with $c \gg \theta$, there is a natural number N such that for all $n \geq N, x \in X$, $d(T^{n+1}x, T^n x) \ll c$. That is, $\{d(T^{n+1}x, T^n x)\}$ is a c -sequence for all $x \in X$.

Compared with Definition 8, Definition 9 shows a great generalization. The condition that $\{d(T^{n+1}x, T^n x)\}$ is a c -sequence is a sharp improvement of that $\lim_{n \rightarrow \infty} d(T^{n+1}x, T^n x) = 0$. The latter is established only under normal cones (see Proposition 2.5 in [28]) or usual

metric spaces (see [6, 20, 21, 29]), while the following theorems are established in nonnormal cone b -metric space over Banach algebra \mathcal{A} . Inspired by the concept of T -orbitally complete in metric space [4], we give the similar concept in cone b -metric space over a Banach algebra \mathcal{A} as follows.

Definition 10. Let (X, d) be a cone b -metric space over a Banach algebra \mathcal{A} . The space (X, d) is said to be T -orbitally complete, if every Cauchy sequence which is contained in $O_T(x)$ for some $x \in X$ converges in X . Every complete cone b -metric space over Banach algebra \mathcal{A} is T -orbitally complete for any T , but a T -orbitally complete cone b -metric space over Banach algebra \mathcal{A} needs not be complete.

The continuity of the mapping and the cone b -metric is not necessary in the following theorems.

Theorem 7. Let $T: X \rightarrow X$ be an asymptotically regular mapping in the T -orbitally complete cone b -metric space over Banach algebra \mathcal{A} with a unit e and the coefficient $s \geq 1$. If there exist $a_1, a_2, a_3 \in P$ with $r(a_2) < (1/s)$ and $r(a_3) < (1/s^2)$ such that

$$d(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, y), \quad (34)$$

for all $x, y \in X$, then T has a unique fixed point $z \in X$ and, for each $x \in X$, the iterated sequence $\{T^n x\}$ converges to z ; i.e., T is a Picard operator.

Proof. For an arbitrary $x_0 \in X$, let $x_n = Tx_{n-1} = T^n x_0$, $n \geq 1$. Without loss of generality, we assume that $x_n \neq x_{n+1}$, $\forall n \in \mathbb{N}$. Indeed, if for some $n \in \mathbb{N}$, $x_n = x_{n+1} = Tx_n$, then x_n is the fixed point of T . By asymptotic regularity of T , $\{d(x_n, x_{n+1})\}$ is a c -sequence. For all $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &\leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{m+1}) + s^2 d(x_{m+1}, x_m) \\ &\leq s d(x_n, x_{n+1}) + s^2 [a_1 d(x_n, x_{n+1}) + a_2 d(x_m, x_{m+1}) + a_3 d(x_n, x_m)] + s^2 d(x_{m+1}, x_m) \\ &\leq s d(x_n, x_{n+1}) + s^2 [a_1 d(x_n, x_{n+1}) + a_2 d(x_m, x_{m+1}) + a_3 d(x_n, x_m)] + s^2 d(x_{m+1}, x_m). \end{aligned} \quad (35)$$

Now, $(e - s^2 a_3)$ is invertible, which is due to the fact that $r(s^2 a_3) = s^2 r(a_3) < 1$. It follows that

$$d(x_n, x_m) \leq (e - s^2 a_3)^{-1} [(se + s^2 a_1) d(x_n, x_{n+1}) + (s^2 e + s^2 a_2) d(x_m, x_{m+1})]. \quad (36)$$

By Lemma 1, $\{x_n\}$ is a Cauchy sequence in X . Since (X, d) is T -orbitally complete, there exists $z \in X$ such that

$x_n \rightarrow z$ as $n \rightarrow \infty$. We shall prove $Tz = z$; i.e., z is the fixed point of T . By the inequality (34), we have

$$\begin{aligned} d(z, Tz) &\leq s[d(z, Tx_n) + d(Tx_n, Tz)] \\ &\leq s[d(z, Tx_n) + a_1 d(x_n, Tx_n) + a_2 d(z, Tz) + a_3 d(x_n, z)], \end{aligned} \quad (37)$$

which implies that

$$d(z, Tz) \leq (e - sa_2)^{-1} s [d(z, x_{n+1}) + a_1 d(x_n, x_{n+1}) + a_3 d(x_n, z)]. \quad (38)$$

The right side of the above equality is a c -sequence by Lemma 1, so $z = Tz$. Using a similar analysis to Theorem 5, we can prove that z is unique. \square

Corollary 3. Let $T: X \longrightarrow X$ be an asymptotically regular mapping on the T -orbitally complete cone b -metric space over Banach algebra \mathcal{A} with a unit e and the coefficient $s \geq 1$. If there exists $a \in P$ with $r(a) < (1/s^2)$ such that

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty) + d(x, y)], \quad (39)$$

for all $x, y \in X$, then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z ; i.e., T is a Picard operator.

Now, if T is orbitally continuous, then the condition $r(a_2) < (1/s)$ can be deleted.

Theorem 8. Let $T: X \longrightarrow X$ be an asymptotically regular mapping on the T -orbitally complete cone b -metric space over Banach algebra \mathcal{A} with a unit e and the coefficient $s \geq 1$. There exist $a_1, a_2, a_3 \in P$ with $r(a_3) < (1/s^2)$ such that

$$d(Tx, Ty) \leq a_1 d(x, Tx) + a_2 d(y, Ty) + a_3 d(x, y), \quad (40)$$

for all $x, y \in X$. If T is orbitally continuous, then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z ; i.e., T is a Picard operator.

Proof. According to Theorem 7, we see that there exists $z \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$. Because T is orbitally continuous, we have $x_{n+1} = Tx_n \longrightarrow Tz$ as $n \longrightarrow \infty$. Then, $z = Tz$. Similar to Theorem 7, the conclusion is true. \square

Corollary 4. Let $T: X \longrightarrow X$ be an asymptotically regular mapping on the T -orbitally complete cone b -metric space over Banach algebra \mathcal{A} with a unit e and the coefficient $s \geq 1$. There exist $a, b \in P$ with $r(b) < (1/s^2)$ such that

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + b d(x, y), \quad (41)$$

for all $x, y \in X$. If T is orbitally continuous, then T has a unique fixed point $z \in X$ and for each $x \in X$ the iterated sequence $\{T^n x\}$ converges to z ; i.e., T is a Picard operator.

Remark 2. Corollary 3 is a generalization of Theorems 3.1 and 3.3 in [6]. Similarly, Corollary 4 is an extension of Theorem 2.6 in [20] and Theorem 2.1 in [21], since our cone is a nonnormal cone. In fact, we establish the contractive mappings with several generalized Lipschitz constants,

where the constants are all vectors but not usual real constants. These results are not equivalent to the theorems in cone b -metric spaces or b -metric spaces, which may offer us more applications since there are lots of nonnormal cones (see [23]). Moreover, we weaken the continuity of the mapping which is necessary in Theorem 2.6 of [20] by orbital continuity.

Example 4. Let $\mathcal{A} = C_{\mathbb{R}}^1[0, 1]$ and $X = [0, 1]$. For each $x \in \mathcal{A}$, $\|x\| = \|x\|_{\infty} + \|x'\|_{\infty}$. The multiplication is defined by its usual pointwise multiplication. Then \mathcal{A} is a Banach algebra with a unit $e = 1$. Define $d(x, y)(t) = |x - y|^2 \phi$ for all $x, y \in X$ and $\phi \in P = \{f(t) \in \mathcal{A} : f(t) \geq 0, t \in [0, 1]\}$. Then P is a nonnormal cone and (X, d) is a complete cone b -metric space over Banach algebra \mathcal{A} with coefficient $s = 2$. Choose $a_1(t) = 4t + 3, a_2(t) = 5t + 2$ and $a_3(t) = (1/9)t + (1/9)$. We deduce that

$$a_3^n(t) = \left(\frac{1}{9}t + \frac{1}{9}\right)^n, \quad (42)$$

$$(a_3^n(t))' = \frac{n}{9} \left(\frac{1}{9}t + \frac{1}{9}\right)^{n-1},$$

hence that ($t = 1$)

$$\begin{aligned} \|a_3^n\| &= \|a_3^n\|_{\infty} + \|(a_3^n)'\|_{\infty} = \left(\frac{2}{9}\right)^n + \frac{n}{9} \left(\frac{2}{9}\right)^{n-1} \\ &= \frac{n}{9} \left(\frac{2}{9}\right)^{n-1} \left(\frac{9}{n} \cdot \frac{2}{9} + 1\right) = \frac{n}{9} \left(\frac{2}{9}\right)^{n-1} \left(1 + \frac{2}{n}\right), \end{aligned} \quad (43)$$

and finally that

$$r(a_3) = \lim_{n \rightarrow \infty} \|a_3^n\|^{1/n} = \lim_{n \rightarrow \infty} \left(\frac{n}{9}\right)^{1/n} \left(\frac{2}{9}\right)^{(n-1)/n} \left(1 + \frac{2}{n}\right)^{1/n} = \frac{2}{9}. \quad (44)$$

This means

$$r(a_3) = \frac{2}{9} < \frac{1}{4} = \frac{1}{s^2}. \quad (45)$$

Define the mapping $T: X \longrightarrow X$ by

$$Tx = \begin{cases} \frac{x}{10} \sin \frac{x}{9}, & x \in \mathbb{Q} \cap X; \\ \log\left(1 + \frac{x}{4}\right), & x \in (\mathbb{R} \setminus \mathbb{Q}) \cap X. \end{cases} \quad (46)$$

Then T is asymptotically regular and orbitally continuous but not continuous. Now, we will show that inequality (40) is satisfied in the following three cases:

(1) For all $x, y \in \mathbb{Q} \cap X$,

$$\begin{aligned}
 d(Tx, Ty)(t) &= \left| \frac{x}{10} \sin \frac{x}{9} - \frac{y}{10} \sin \frac{y}{9} \right|^2 \phi \\
 &\leq \left(\frac{1}{90}x + \frac{1}{10} \right)^2 |x - y|^2 \phi \\
 &\leq (4t + 3) \left| x - \frac{x}{10} \sin \frac{x}{9} \right|^2 \phi + (5t + 2) \left| y - \log \left(1 + \frac{y}{4} \right) \right|^2 \phi + \left(\frac{1}{9}t + \frac{1}{9} \right) |x - y|^2 \phi \\
 &= a_1 d(x, Tx)(t) + a_2 d(y, Ty)(t) + a_3 d(x, y)(t).
 \end{aligned} \tag{47}$$

(2) For all $x, y \in (\mathbb{R} \setminus \mathbb{Q}) \cap X$,

$$\begin{aligned}
 d(Tx, Ty)(t) &= \left| \log \left(1 + \frac{x}{4} \right) - \log \left(1 + \frac{y}{4} \right) \right|^2 \phi \\
 &\leq \left(\frac{1}{4} \right)^2 \left| \frac{x}{4} - \frac{y}{4} \right|^2 \phi \\
 &\leq (4t + 3) \left| x - \log \left(1 + \frac{x}{4} \right) \right|^2 \phi + (5t + 2) \left| y - \log \left(1 + \frac{y}{4} \right) \right|^2 \phi + \left(\frac{1}{9}t + \frac{1}{9} \right) |x - y|^2 \phi \\
 &= a_1 d(x, Tx)(t) + a_2 d(y, Ty)(t) + a_3 d(x, y)(t).
 \end{aligned} \tag{48}$$

(3) For all $x \in \mathbb{Q} \cap X$, $y \in (\mathbb{R} \setminus \mathbb{Q}) \cap X$,

$$\begin{aligned}
 d(Tx, Ty)(t) &= \left| \frac{x}{10} \sin \frac{x}{9} - \log \left(1 + \frac{y}{4} \right) \right|^2 \phi \\
 &\leq \left| \frac{x}{10} \sin \frac{x}{9} \right|^2 \phi + \left| \log \left(1 + \frac{y}{4} \right) \right|^2 \phi \\
 &\leq \left| \frac{x}{10} \right|^2 \phi + \left| \frac{y}{4} \right|^2 \phi \\
 &\leq (4t + 3) \left| x - \frac{x}{10} \sin \frac{x}{9} \right|^2 \phi + (5t + 2) \left| y - \log \left(1 + \frac{y}{4} \right) \right|^2 \phi + \left(\frac{1}{9}t + \frac{1}{9} \right) |x - y|^2 \phi \\
 &= a_1 d(x, Tx)(t) + a_2 d(y, Ty)(t) + a_3 d(x, y)(t).
 \end{aligned} \tag{49}$$

Similarly, we can also prove that $d(Tx, Ty)(t) \leq a_1 d(x, Tx)(t) + a_2 d(y, Ty)(t) + a_3 d(x, y)(t)$ for all $x \in (\mathbb{R} \setminus \mathbb{Q}) \cap X$, $y \in \mathbb{Q} \cap X$. Therefore, T has a unique fixed point in X by Theorem 8.

4. Completeness and Fixed Point

By Corollaries 1 and 2, we know that if the generalized Kannan-type contraction mapping T is orbitally continuous

in boundedly compact or T -orbitally compact cone b -metric spaces over Banach algebras, then T has a unique fixed point. Conversely, if T has a unique fixed point in cone b -metric spaces over Banach algebras, then what conditions do (X, d) have to satisfy? Now, we prove an important theorem showing that the completeness of cone b -metric spaces over Banach algebras is necessary if the generalized Kannan-type contraction has a fixed point in X .

Theorem 9. Let (X, d) be a cone b -metric space over Banach algebra \mathcal{A} with a unit e and the coefficient $s \geq 1$. Let P be a normal and strongly minihedral cone. If every self-mapping T satisfying

$$d(Tx, Ty) < \frac{e}{2} \{d(x, Tx) + d(y, Ty)\}, \quad (50)$$

for all $x, y \in X$ with $x \neq y$ has a unique fixed point, then (X, d) must be a complete cone b -metric space over Banach algebra \mathcal{A} .

Proof. On the contrary, suppose that (X, d) is not complete; then, there exists a Cauchy sequence $\{x_n\}$ in X , which is not convergent. If it has a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow z \in X$ as $k \rightarrow \infty$, then

$$d(x_m, z) \leq s[d(x_m, x_{n_k}) + d(x_{n_k}, z)], \quad (51)$$

for all $m \geq n_k$. This is a c -sequence since $\{d(x_m, x_{n_k})\}$ and $\{d(x_{n_k}, z)\}$ are c -sequences. Thus, we can assume that all terms of the sequence $\{x_n\}$ are distinct. Let $M = \{x_n : n \in \mathbb{N}\}$; then, $d(x, M) > \theta$ for all $x \in X - M$ by the fact that the sequence $\{x_n\}$ does not converge in X . Let $x \in X$ be an arbitrary point. If $x \in X - M$, then there is an integer $n_x \in \mathbb{N}$ such that

$$d(x_m, x_{n_x}) < \frac{e}{2} d(x, M) \leq \frac{e}{2} d(x, x_n), \quad (52)$$

for all $m \geq n_x$ and arbitrary $n \in \mathbb{N}$. That is

$$d(x_m, x_{n_x}) < \frac{e}{2} d(x, x_n), \quad \forall m \geq n_x \text{ and } \forall n \in \mathbb{N}. \quad (53)$$

Suppose $x' \in M$; then $x' = x_{n_0}$ for some $n_0 \in \mathbb{N}$. Since $\{x_n\}$ is a Cauchy sequence, we can find some $n'_0 \in \mathbb{N}$ such that

$$d(x_m, x_{n'_0}) < \frac{e}{2} d(x_{n'_0}, x_{n_0}), \quad \forall m \geq n'_0 > n_0. \quad (54)$$

Now, define $T: X \rightarrow X$ by

$$Tx = \begin{cases} x_{n_x}, & \text{if } x \in X - M; \\ x_{n'_0}, & \text{if } x \in M \text{ and } x = x_{n_0} \end{cases} \quad (55)$$

For any $x, y \in X$ with $x \neq y$, we divide the following proof into three cases. \square

Case 1. If $x, y \in X - M$, then $Tx = x_{n_x}$ and $Ty = x_{n_y}$. Without loss of generality, we assume that $n_y \geq n_x$. By (53), we get

$$d(Tx, Ty) = d(x_{n_x}, x_{n_y}) < \frac{e}{2} d(x, x_{n_x}) = \frac{e}{2} d(x, Tx). \quad (56)$$

This gives $d(Tx, Ty) < (e/2)\{d(x, Tx) + d(y, Ty)\}$.

Case 2. If $x, y \in M$, then $x = x_{n_0}$ and $y = x_{m_0}$ for some $n_0, m_0 \in \mathbb{N}$. Then $Tx = x_{n'_0}$ and $Ty = x_{m'_0}$. Without loss of generality, we assume that $m'_0 \geq n'_0$. By (54), we deduce that

$$d(Tx, Ty) = d(x_{n'_0}, x_{m'_0}) < \frac{e}{2} d(x_{n'_0}, x_{n_0}) = \frac{e}{2} d(Tx, x), \quad (57)$$

which established the formula $d(Tx, Ty) < (e/2)\{d(x, Tx) + d(y, Ty)\}$.

Case 3. If $x \in X - M$, $y \in M$, then $y = x_{n_0}$ for some $n_0 \in \mathbb{N}$. Therefore $Tx = x_{n_x}$ and $Ty = x_{n'_0}$. If $n'_0 \geq n_x$, by (53), we have

$$d(Tx, Ty) = d(x_{n_x}, x_{n'_0}) = d(x_{n'_0}, x_{n_x}) < \frac{e}{2} d(x, x_{n_x}) = \frac{e}{2} d(x, Tx), \quad (58)$$

which yields $d(Tx, Ty) \leq (e/2)\{d(x, Tx) + d(y, Ty)\}$. If $n_x \geq n'_0$, by (54), we see that

$$d(Tx, Ty) = d(x_{n_x}, x_{n'_0}) < \frac{e}{2} d(x_{n'_0}, x_{n_0}) = \frac{e}{2} d(y, Ty), \quad (59)$$

which also gives $d(Tx, Ty) < (e/2)\{d(x, Tx) + d(y, Ty)\}$. Therefore, for all $x, y \in X$ with $x \neq y$, we always have $d(Tx, Ty) < (e/2)\{d(x, Tx) + d(y, Ty)\}$. That is, T is a generalized Kannan-type contraction mapping which has no fixed point in X , a contradiction. Hence, the assumption does not hold and the space (X, d) must be a complete cone b -metric space over Banach algebra \mathcal{A} . The conclusion is true.

Remark 3. According to the proof of Theorem 9, we see at once that inequality (50) can be replaced by

$$d(Tx, Ty) \leq k\{d(x, Tx) + d(y, Ty)\}, \quad (60)$$

for all $x, y \in X$ and a fixed point $k \in P$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was partially supported by the Undergraduate Teaching Quality Project of Guangdong Province (no. 520006) and the Basic Education Research Project of Hanshan Normal University in 2018 (no. ZD 201807).

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Research Article

Computing SS Index of Certain Dendrimers

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Received 25 June 2021; Accepted 28 August 2021; Published 25 September 2021

Academic Editor: Huseyin Isik

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The numerical descriptor gathers the data from the molecular graphs and helps to know the characteristics of the chemical structure known as topological index. The QSAR/QSPR/QSTR studies are benefited with the significant role played by topological indices in the drug design. Topological indices provide the information about the physical/chemical/biological properties of chemical compounds. The Zagreb indices are widely studied because of their extensive usage in chemical graph theory. Inspired by the earlier work on inverse sum indeg index (ISI index), novel topological index known as SS index is introduced and computed for four dendrimer structures. Also, the strong correlation coefficient between SS index and 5 physico-chemical characteristics such as boiling point (*bp*), molar volume (*mv*), molar refraction (*mr*), heats of vaporization (*hv*), and critical pressure (*cp*) of 67 alkane isomers have been determined. It is found that newly introduced index has shown good correlation in comparison with three most popular existing indices (ISI index and first and second Zagreb indices). In the last part, the mathematical properties of SS index are discussed.

1. Introduction and Terminologies

Every year, large number of new drugs are produced due to the rapid growth of medicine manufacturing. As a result, determining the pharmacological, chemical, and biological characteristics of a substance necessitates a significant amount of effort.

These new medications are becoming increasingly clumsy and clumped. In order to check the performance of new drugs and their side effects, sufficient reagents, equipment, and technicians are needed. However, in low-income countries, there is an insufficient funding to cover the costs of reagents and equipment needed to calculate biochemical properties. The existing studies have shown that the chemical and pharmacodynamic properties of drugs, as well as their molecular structures, are inextricably related. If we quantify measures of these drug molecular structures

with the aim of identifying topological indices, medical and pharmaceutical researchers will be able to understand their therapeutic properties, which can compensate for the shortcomings of medicine and chemical experiments. In this regard, the methods computing topological index are suitable and useful for developing countries, as they can produce accessible biological and medical knowledge about new drugs without the use of chemical experiment hardware. To calculate the characteristics of drug molecules, the PI index, Zagreb index, and eccentric index are used. The number of vertices and edges of a chemical compound counts to the computation of topological indices [1–8].

A topological index is a computational parameter derived from the graph structure mathematically [9–13]. To visualize the relationships between the data sets, graphs are crucial tools which make the concept better understandable. A descriptor that gives the data regarding arrangement of

atoms in a compound in numerical form of information regarding its shape, branching, and other data of a compound is a topological index.

The significant number of early drug studies suggests that the biomedical and pharmacology properties of drugs, as well as their molecular structures, have a clear inner relationship. Many scientists have developed various indices to quantify the characteristics of drug molecules over the last 40 years. The indices are of great use in the study of pharmacology, toxicology, and chemistry (QSAR/QSPR/QSTR) [14–16].

Dendrimers are also called “cascade molecules,” but this term is not in general use compared to the term dendrimers. In 1978, Fritz Vogtle was the first to bring these nanomolecules into light. Dendrites normally include a unique chemically addressable unit known as focus or core. The usage and popularity of dendrimers have been greatly increased. Since 2005, there have been over 5000 research papers and patents. A second group of the synthesized macromolecules is called arborols. We can say that the molecules of dendrimers are of architectural design. These thoroughly tailored architectural nanomolecules can be functionalized and modify their physico-chemical or biological characteristics.

The hyper-branched macromolecules have three phases in its structural constitution. An atom at the centre of the structure called the core of the dendrimer has some functional properties. Secondly, the branches are ejected out of the core and add on the branches repetitively. Finally, the terminal groups are situated on the surface of the dendritic structure. Dendrimer synthesis is divided into two methods: divergent synthesis and convergent synthesis. It is difficult to synthesize dendrimers using either approach, because the actual reactions require several steps to protect the active site. As a result, it is difficult to manufacture and prohibitively costly to buy. Dendrites have significant applications in biomedical field because of its characteristics, including hyper-branching, well-defined globular structures, outstanding structural uniformity, multivalency, varying chemical constitution, and higher biological compatibility.

In medical field, mathematical modelling is used to analyse the representation of emerging drugs, normally as an undirected graph, such that each vertex depicts an atom and an edge depicts a link between atoms. Every year new drugs are available and needs remarkable work to select the qualities of the emerging drugs. Dendrimers are a good option in the drug design because of its biological characteristics such as polyvalency, self-assembling, electrostatic interactions, chemical stability, low cytotoxicity, and solubility. The remarkable and emerging role of dendritic macromolecules is in therapies of anticancer and image diagnosis.

Various studies have revealed that there is a consistent correlation between the molecular structures of compounds, drugs, and their characteristics. Topological indices are numerical variants that assist researchers in understanding physical properties, chemical interactions, and biological activity [17–21]. Hence, the discussion on topological indices of chemical structures of drugs helps to know the theoretical

basis to prepare new drugs. In this study, SS index is defined and computed for porphyrin (D_nP_n), propyl ether imine (DPZ_n), zinc porphyrin (PETIM), and polyethylene amide amine (PETAA) dendrimers [22,23].

In this paper, the notations and terminologies pertaining to the graphs are found in [24].

Definition 1. The oldest and the most studied indices, the first and second Zagreb indices [25], proposed by Gutman and Trinajstić are defined as

$$\begin{aligned} M_1(G) &= \sum_{v\omega \in E(G)} (d_v + d_\omega), \\ M_2(G) &= \sum_{v\omega \in E(G)} (d_v \cdot d_\omega). \end{aligned} \quad (1)$$

Definition 2. Vukičević et al. introduced inverse sum indeg index [26] and stated as

$$ISI(G) = \sum_{v\omega \in E(G)} \frac{d_v \cdot d_\omega}{d_v + d_\omega}. \quad (2)$$

Definition 3. In this work, a novel invariant known as SS (Shilpa-Shanmukha) index is introduced and studied. This index is defined as follows:

$$SS(G) = \sum_{v\omega \in E(G)} \sqrt{\frac{d_v \cdot d_\omega}{d_v + d_\omega}}. \quad (3)$$

Throughout this article, d_v and d_ω represent the degrees of vertices v and ω , respectively.

2. Chemical Applicability of the SS Index through QSPR Analysis

Here, we discussed the proposed topological index known as SS index to study the physico-chemical properties, namely, bp , mv , mr , hv , and cp of 67 alkanes ranging from n -butanes to nonanes. The 5 physico-chemical properties of 67 alkane isomers can be found in [27] and Table 1 represents the computed values of four topological indices (SS, ISI, M_1 , and M_2) of 67 alkane isomers. The 5 characteristics of alkane isomers are correlated with SS index and it is found that SS index has shown good correlation with all the 5 properties compared to the existing three most popular indices M_1 , M_2 , and ISI considered in the study. The SS index is plotted against each of the 5 properties of alkane isomers which is depicted in Figure 1.

Regression model for properties of alkane isomers.

The linear regression model is given by

$$P = m(TI) + c, \quad (4)$$

where P is the physical property and TI is the topological index. Equation (4) results in the following linear regression models for various properties with SS index.

TABLE 1: The values of various topological indices of alkanes.

Sl. No.	Alkane	SS (G)	ISI (G)	M_1 (G)	M_2 (G)
1	Butane	2.8165	2.6667	10	8
2	2-Methyl propane	2.598	2.25	12	4
3	Pentane	2.9428	3.3333	14	12
4	2-Methyl butane	3.644	3.3667	16	14
5	2,2-Dimethyl propane	3.5777	3.2	20	16
6	Hexane	4.633	4.3333	18	16
7	2-Methyl pentane	4.644	4.3667	20	18
8	3-Methyl pentane	4.69	4.4833	20	19
9	2,2-Methyl butane	4.6545	4.4	24	22
10	2,3-Dimethyl butane	4.6888	4.5	22	21
11	Heptane	5.633	5.3333	22	20
12	2-Methyl hexane	5.644	5.3667	24	22
13	3-Methyl hexane	5.6899	5.4833	24	23
14	3-Ethyl pentane	5.7358	5.6	24	24
15	2,2-Dimethyl pentane	5.6544	5.4	28	26
16	2,3-Dimethyl pentane	5.7347	5.616	26	26
17	2,4-Dimethyl pentane	5.655	5.4	26	24
18	3,3-Dimethyl pentane	5.7312	5.6	28	28
19	Octane	6.633	6.333	26	24
20	2-Methyl heptane	6.644	6.3667	28	26
21	3-Methyl heptane	6.69	6.4833	28	27
22	4-Methyl heptane	6.68	6.49	28	27
23	3-Ethyl hexane	6.736	6.6	28	28
24	2,2-Dimethyl hexane	6.6545	6.4	32	30
25	2,3-Dimethyl hexane	6.7348	6.6166	30	30
26	2,4-Dimethyl hexane	6.7009	6.5166	30	29
27	2,5-Dimethyl hexane	6.655	6.4	32	30
28	3,3-Dimethyl hexane	6.7312	6.6	32	32
29	3,4-Dimethyl hexane	6.7807	6.7333	30	31
30	3-Ethyl-2-methyl pentane	6.7807	6.7333	30	22
31	3-Ethyl-3-methyl pentane	6.808	6.8	32	34
32	2,2,3-Trimethyl pentane	6.7706	6.731	34	35
33	2,2,4-Trimethyl pentane	6.6655	6.4333	34	32
34	2,3,3-Trimethyl pentane	6.8014	6.8143	34	36
35	2,3,4-Trimethyl pentane	6.7796	6.75	32	33
36	Nonane	7.6333	7.3333	30	28
37	2-Methyl octane	7.644	7.3667	32	30
38	3-Methyl octane	7.69	7.4833	32	31
39	4-Methyl octane	7.69	7.4833	32	31
40	3-Ethyl heptane	7.7358	7.6	32	32
41	4-Ethyl heptane	7.7358	7.6	32	32
42	2,2-Dimethyl heptane	7.6545	7.4	36	34
43	2,3-Dimethyl heptane	7.7348	7.6166	34	34
44	2,4-Dimethyl heptane	7.7009	7.5166	34	33
45	2,5-Dimethyl heptane	7.7963	7.7166	34	33
46	2,6-Dimethyl heptane	7.655	7.4	34	32
47	3,3-Dimethyl heptane	7.7312	7.6	36	36
48	3,4-Dimethyl heptane	7.7807	7.7333	34	35
49	3,5-Dimethyl heptane	7.7468	7.6333	34	34
50	4,4-Dimethyl heptane	7.7312	7.6	36	36
51	3-Ethyl-2-methyl hexane	7.7807	7.7333	34	35
52	4-Ethyl-2-methyl hexane	7.9246	8.0333	34	34
53	3-Ethyl-3-methyl hexane	7.808	7.8	36	36
54	2,2,4-Trimethyl hexane	7.7114	7.55	38	37
55	2,2,5-Trimethyl hexane	7.6655	7.4333	38	36
56	2,3,3-Trimethyl hexane	7.8014	7.8143	38	40
57	2,3,4-Trimethyl hexane	7.8255	7.8667	36	44
58	2,3,5-Trimethyl hexane	7.7458	7.65	36	44
59	3,3,4-Trimethyl hexane	7.8845	8	38	41
60	3,3-Diethyl pentane	7.866	7.931	44	40

TABLE 1: Continued.

Sl. No.	Alkane	SS (G)	ISI (G)	M_1 (G)	M_2 (G)
61	2,2-Dimethyl-3-ethyl pentane	7.878	8.0143	54	40
62	2,3-Dimethyl-3-ethyl pentane	7.8255	7.8666	46	42
63	2,4-Dimethyl-3-ethyl pentane	7.8575	8	44	38
64	2,2,3,3-Tetramethyl pentane	7.8575	8	42	46
65	2,2,3,4-Tetramethyl pentane	7.8154	7.643	40	42
66	2,2,4,4-Tetramethyl pentane	7.676	7.4667	42	40
67	2,3,3,4-Tetramethyl pentane	7.8715	8.0281	40	44

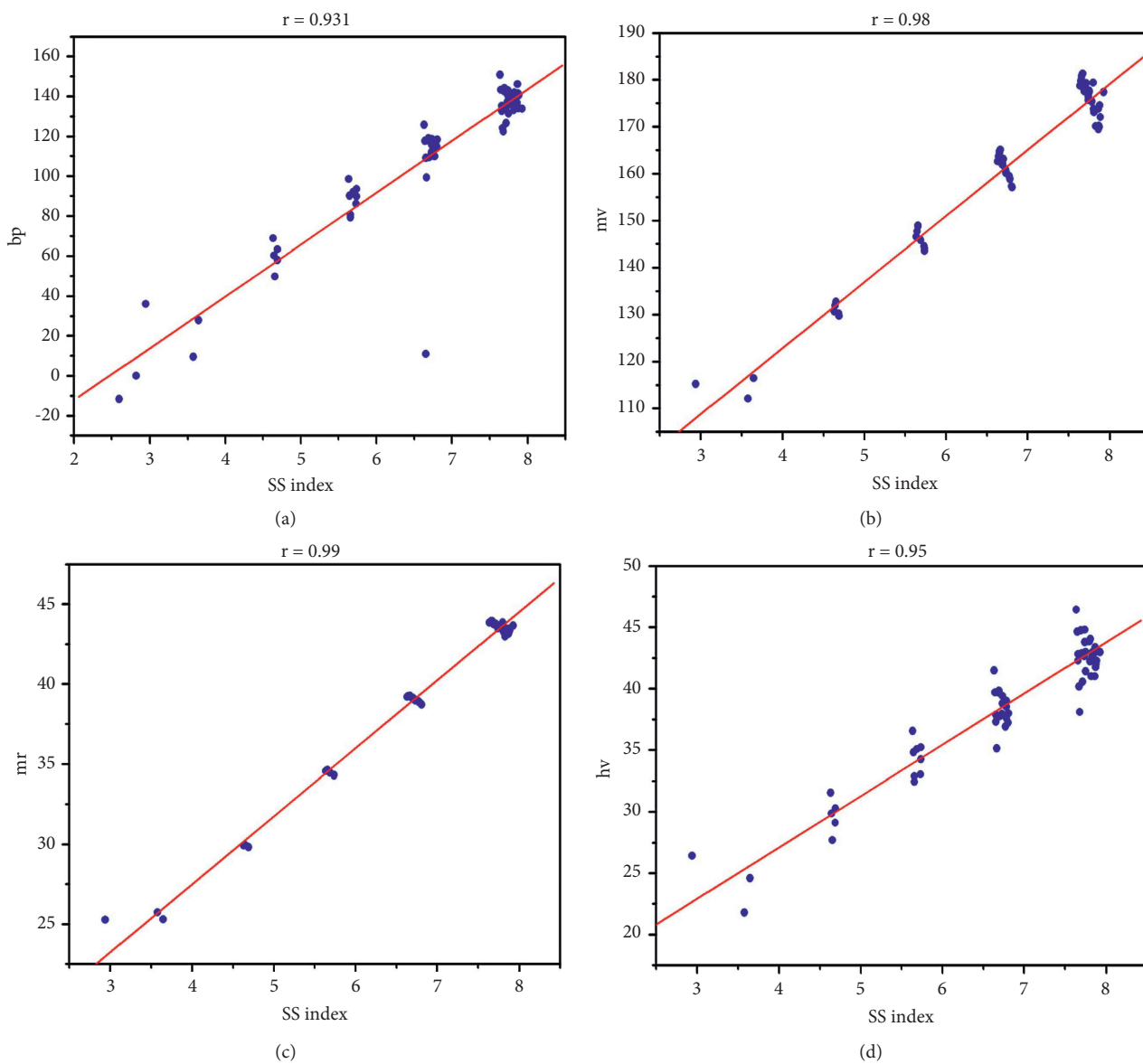


FIGURE 1: Continued.

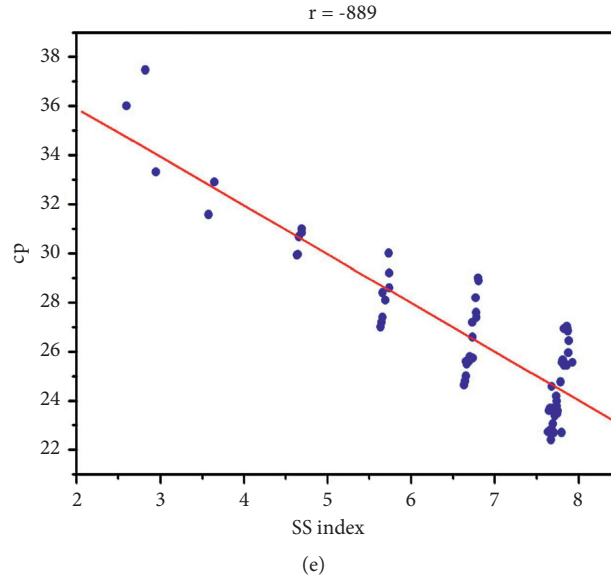


FIGURE 1: Correlation of SS index with properties of alkane isomers.

$$\begin{aligned}
 bp &= -64.28 + 25.99(SS), \\
 mv &= 66.57 + 14.07(SS), \\
 mr &= 10.46 + 4.26(SS), \\
 hv &= 10.4 + 4.173(SS), \\
 cp &= 39.87 - 1.98(SS).
 \end{aligned} \tag{5}$$

The SS index has correlation coefficients of 0.931, 0.98, 0.99, 0.951, and -0.889 with residual standard errors 14.46, 3.76, 0.57, 1.69, and 1.45 and all these models are statistically significant, since the level of significance value of all models is less than 0.05.

Some important observations from the data are presented in Table 2. The correlation coefficients of bp , mv , mr , and hv have shown high positive correlation for the introduced SS index. Also, it is interesting to know that the correlation coefficient for cp shows highly negative correlation for the SS index.

3. Dendrimers

Dendrimers, come from the Greek word which means “trees,” are branched at the core and they form a spherical three-dimensional structure. Dendrimers have attracted a lot of researchers globally in the study of topological indices

TABLE 2: The correlation coefficient between SS, ISI, M_1 , and M_2 indices with some physico-chemical properties of alkane isomers.

Properties	bp	mv	mr	hv	cp
SS (G)	0.931	0.98	0.99	0.951	-0.889
ISI (G)	0.931	0.96	0.984	0.942	-0.855
M_1 (G)	0.779	0.799	0.837	0.715	-0.684
M_2 (G)	0.816	0.8065	0.854	0.737	-0.688

[28–33]. The aim of this paper is to compute SS index of four dendrimer structures, namely, D_nP_n , DPZ_n , $PETIM$, and $PETAA$.

3.1. SS Index of Porphyrin Dendrimer (D_nP_n). Consider the porphyrin dendrimer family. This family of dendrimers is denoted by D_nP_n . The molecular graph of D_nP_n is shown in Figure 2.

Let G be the molecular graph of D_nP_n . By calculation, it is found that G consists of number of vertices and edges to be $96n - 10$ and $105n - 11$, respectively. Table 3 shows the six forms of edges in $D_nP_n(G)$ based on degrees of end vertices of each edge.

Theorem 1. Let D_nP_n be the family of porphyrin dendrimers. Then, the SS index of D_nP_n is given by

$$SS(D_nP_n) = \left[2\left(\frac{3}{4}\right)^{1/2} + 24\left(\frac{4}{5}\right)^{1/2} + 10 + 48\left(\frac{6}{5}\right)^{1/2} + 13\left(\frac{9}{6}\right)^{1/2} + 8\left(\frac{12}{7}\right)^{1/2} \right] n - \left[5 + 6\left(\frac{6}{5}\right)^{1/2} \right]. \tag{6}$$

Proof. From the definition of SS index and Table 3, we deduce

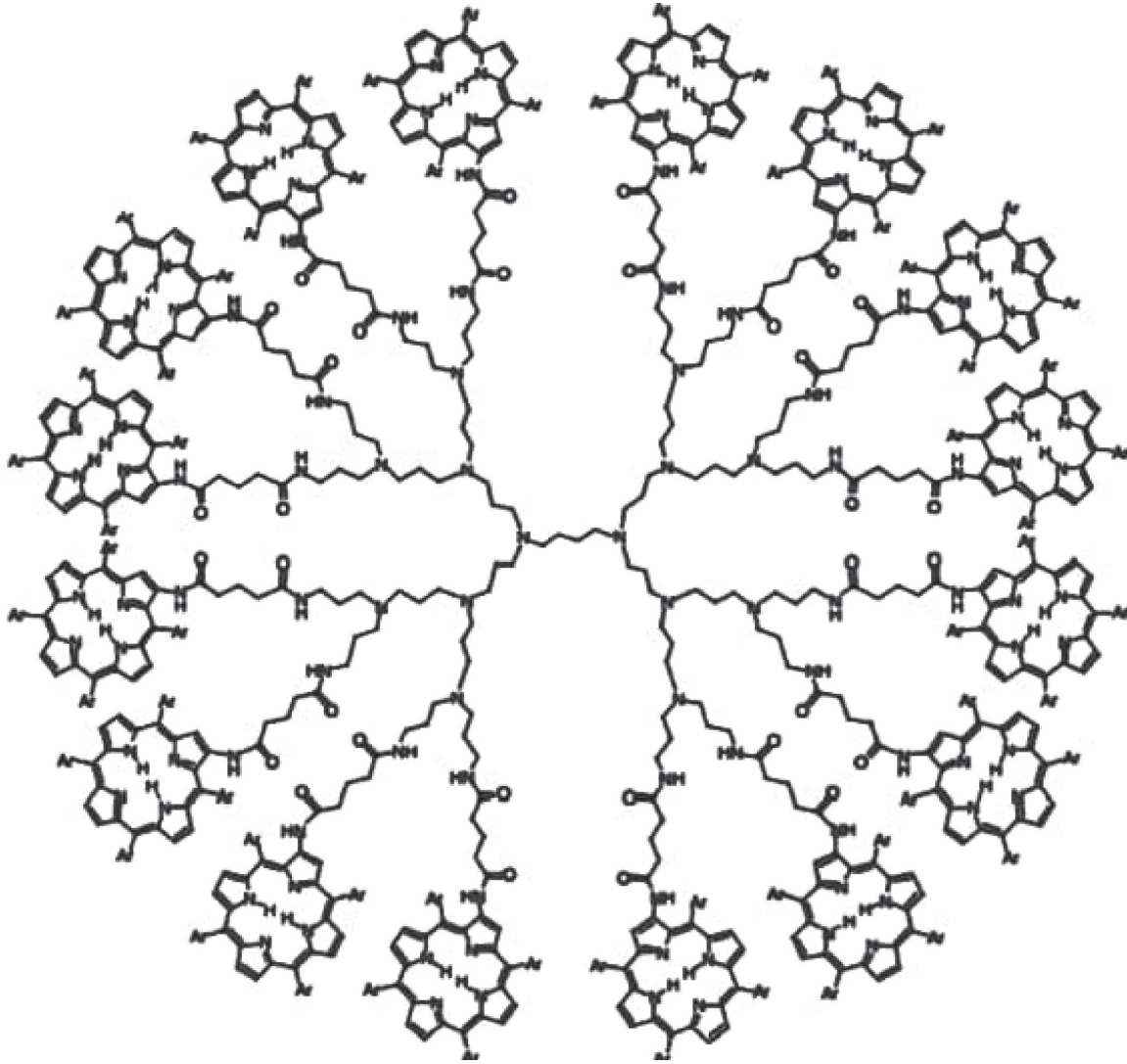


FIGURE 2: The molecular graph of porphyrin dendrimer.

TABLE 3: Edge partition of D_nP_n .

(d_v, d_w) , where $vw \in E(G)$	(1, 3)	(1, 4)	(2, 2)	(2, 3)	(3, 3)	(3, 4)
Number of edges	$2n$	$24n$	$10n - 5$	$48n - 6$	$13n$	$8n$

$$\begin{aligned}
 SS(D_nP_n) = \sum_{vw \in E(G)} \sqrt{\frac{d_v d_w}{d_v + d_w}} &= 2n \left(\frac{1 \times 3}{1 + 3} \right)^{1/2} + 24n \left(\frac{1 \times 4}{1 + 4} \right)^{1/2} + (10n - 5) \left(\frac{2 \times 2}{2 + 2} \right)^{1/2} + (48n - 6) \left(\frac{2 \times 3}{2 + 3} \right)^{1/2} \\
 &+ 13n \left(\frac{3 \times 3}{3 + 3} \right)^{1/2} + 8n \left(\frac{3 \times 4}{3 + 4} \right)^{1/2}, \quad (7)
 \end{aligned}$$

$$SS(D_nP_n) = \left[2 \left(\frac{3}{4} \right)^{1/2} + 24 \left(\frac{4}{5} \right)^{1/2} + 10 + 48 \left(\frac{6}{5} \right)^{1/2} + 13 \left(\frac{9}{6} \right)^{1/2} + 8 \left(\frac{12}{7} \right)^{1/2} \right] n - \left[5 + 6 \left(\frac{6}{5} \right)^{1/2} \right].$$

□

3.2. SS Index of Zinc Porphyrin Dendrimer (DPZ_n). Consider the zinc porphyrin dendrimer family. This family of dendrimers is represented by DPZ_n . The molecular graph of DPZ_n is depicted in Figure 3.

Let G be the molecular graph of DPZ_n . By calculation, it is found that G has $56 \times 2^n - 7$ vertices and $64 \times 2^n - 4$ edges. Table 4 shows the four forms of edges in DPZ_n based on degrees of end vertices of each edge.

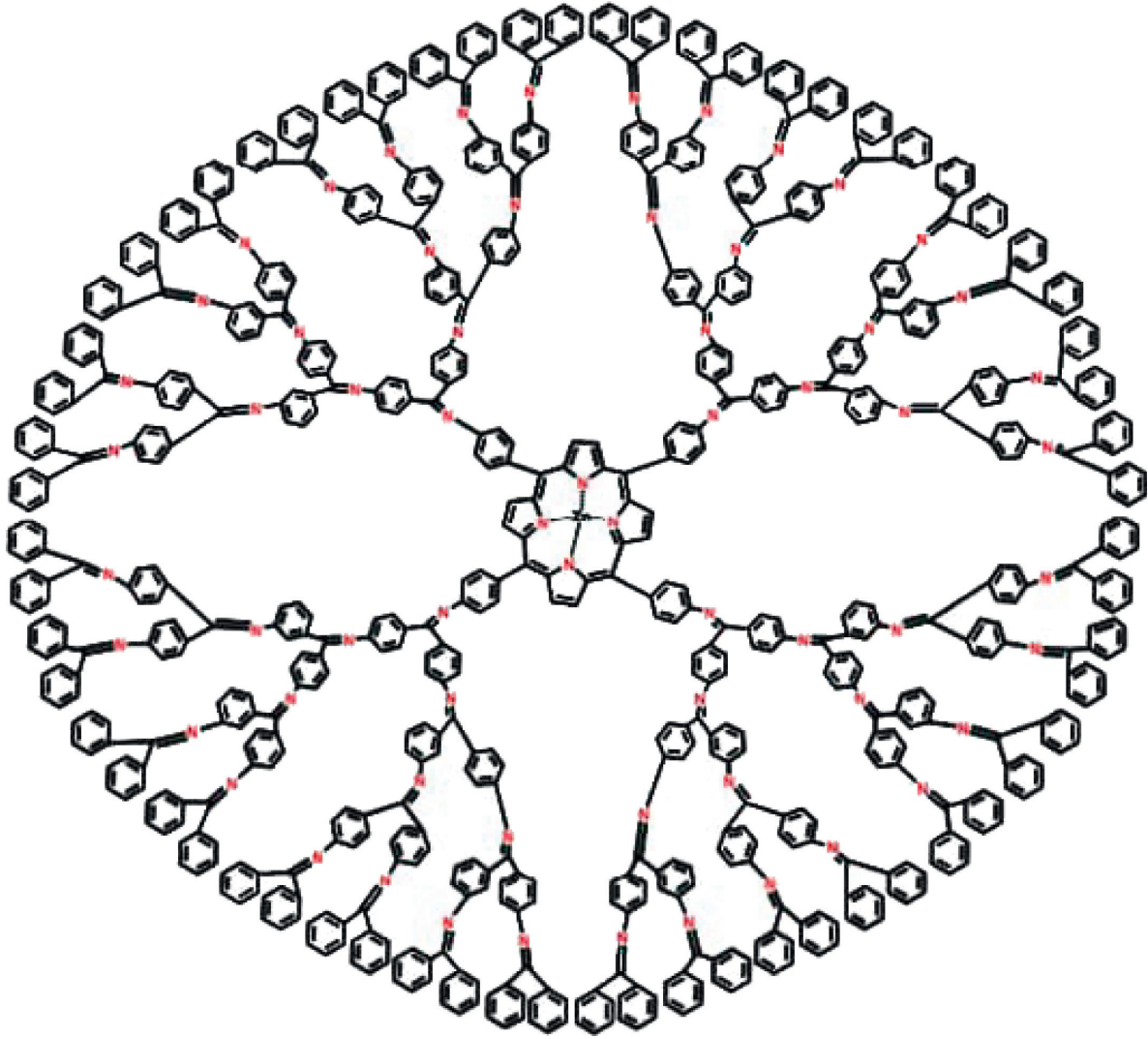


FIGURE 3: The molecular graph of zinc porphyrin dendrimer.

TABLE 4: Edge partition of DPZ_n .

(d_v, d_w) , where $v\omega \in E(G)$	(2, 2)	(2, 3)	(3, 3)	(3, 4)
Number of edges	$16 \times 2^n - 4$	$40 \times 2^n - 16$	$8 \times 2^n - 16$	4

Theorem 2. Let DPZ_n be the family of zinc porphyrin dendrimers. Then, the SS index of DPZ_n is given by

$$SS(DPZ_n) = \left[16 + 40\left(\frac{6}{5}\right)^{1/2} + 8\left(\frac{9}{6}\right)^{1/2} \right] 2^n - \left[4 + 16\left(\frac{16}{5}\right)^{1/2} + 16\left(\frac{9}{6}\right)^{1/2} - 4\left(\frac{12}{7}\right)^{1/2} \right]. \quad (8)$$

Proof. From the definition of SS index and Table 4, we deduce

$$\begin{aligned}
SS(DPZ_n) &= \sum_{v\omega \in E(G)} \sqrt{\frac{d_v d_\omega}{d_v + d_\omega}} = (16 \times 2^n - 4) \left(\frac{2 \times 2}{2 + 2} \right)^{1/2} + (40 \times 2^n - 16) \left(\frac{2 \times 3}{2 + 3} \right)^{1/2} \\
&\quad + (8 \times 2^n - 16) \left(\frac{3 \times 3}{3 + 3} \right)^{1/2} + 4 \left(\frac{3 \times 4}{3 + 4} \right)^{1/2},
\end{aligned} \tag{9}$$

$$SS(DPZ_n) = \left[16 + 40 \left(\frac{6}{5} \right)^{1/2} + 8 \left(\frac{9}{6} \right)^{1/2} \right] 2^n - \left[4 + 16 \left(\frac{16}{5} \right)^{1/2} + 16 \left(\frac{9}{6} \right)^{1/2} - 4 \left(\frac{12}{7} \right)^{1/2} \right].$$

3.3. SS Index of Propyl Ether Imine Dendrimer (PETIM). Consider the family of propyl ether imine dendrimers. This family of dendrimers is represented by PETIM. The molecular graph of PETIM is depicted in Figure 4.

Let G be the molecular graph of PETIM. By calculation, G has $24 \times 2^n - 23$ vertices and $24 \times 2^n - 24$ edges. Table 5 shows the three forms of edges in PETIM based on degrees of end vertices of each edge.

Theorem 3. Let PETIM be the family of propyl ether imine dendrimers. Then, the SS index of PETIM is given by

$$SS(PETIM) = \left[2 \left(\frac{2}{3} \right)^{1/2} + 16 + 6 \left(\frac{6}{5} \right)^{1/2} \right] 2^n - \left[18 + 6 \left(\frac{6}{5} \right)^{1/2} \right]. \tag{10}$$

Proof. From the definition of SS index and Table 5, we deduce

$$SS(PETIM) = \sum_{v\omega \in E(G)} \sqrt{\frac{d_v d_\omega}{d_v + d_\omega}} = (2 \times 2^n) \left(\frac{1 \times 2}{1 + 2} \right)^{1/2} + (16 \times 2^n - 18) \left(\frac{2 \times 2}{2 + 2} \right)^{1/2} + (6 \times 2^n - 6) \left(\frac{2 \times 3}{2 + 3} \right)^{1/2}, \tag{11}$$

$$SS(PETIM) = \left[2 \left(\frac{2}{3} \right)^{1/2} + 16 + 6 \left(\frac{6}{5} \right)^{1/2} \right] 2^n - \left[18 + 6 \left(\frac{6}{5} \right)^{1/2} \right].$$

3.4. SS Index of Polyethylene Amide Amine (PETAA) Dendrimer. Consider the family of polyethylene amide amine dendrimers. This family of dendrimers is represented by PETAA. The molecular graph of PETAA is depicted in Figure 5.

Let G be the molecular graph of PETAA. By calculation, G has $44 \times 2^n - 18$ vertices and $44 \times 2^n - 19$ edges. Table 6

shows the four forms of edges in PETAA based on degrees of end vertices of each edge.

Theorem 4. Let PETAA be the family of zinc porphyrin dendrimers. Then, the SS index of PETAA is given by

$$SS(PETAA) = \left[4 \left(\frac{2}{3} \right)^{1/2} + 4 \left(\frac{3}{4} \right)^{1/2} + 16 + 20 \left(\frac{6}{5} \right)^{1/2} \right] 2^n - \left[2 \left(\frac{3}{4} \right)^{1/2} + 8 + 9 \left(\frac{6}{5} \right)^{1/2} \right]. \tag{12}$$

Proof. From the definition of SS index and Table 6, we deduce

$$\begin{aligned}
SS(PETAA) &= \sum_{v\omega \in E(G)} \sqrt{\frac{d_v d_\omega}{d_v + d_\omega}} = (4 \times 2^n) \left(\frac{1 \times 2}{1 + 2} \right)^{1/2} + (4 \times 2^n - 2) \left(\frac{1 \times 3}{1 + 3} \right)^{1/2} + (16 \times 2^n - 8) \left(\frac{2 \times 2}{2 + 2} \right)^{1/2} \\
&\quad + (20 \times 2^n - 9) \left(\frac{2 \times 3}{2 + 3} \right)^{1/2},
\end{aligned} \tag{13}$$

$$SS(PETAA) = \left[4 \left(\frac{2}{3} \right)^{1/2} + 4 \left(\frac{3}{4} \right)^{1/2} + 16 + 20 \left(\frac{6}{5} \right)^{1/2} \right] 2^n - \left[2 \left(\frac{3}{4} \right)^{1/2} + 8 + 9 \left(\frac{6}{5} \right)^{1/2} \right].$$

□

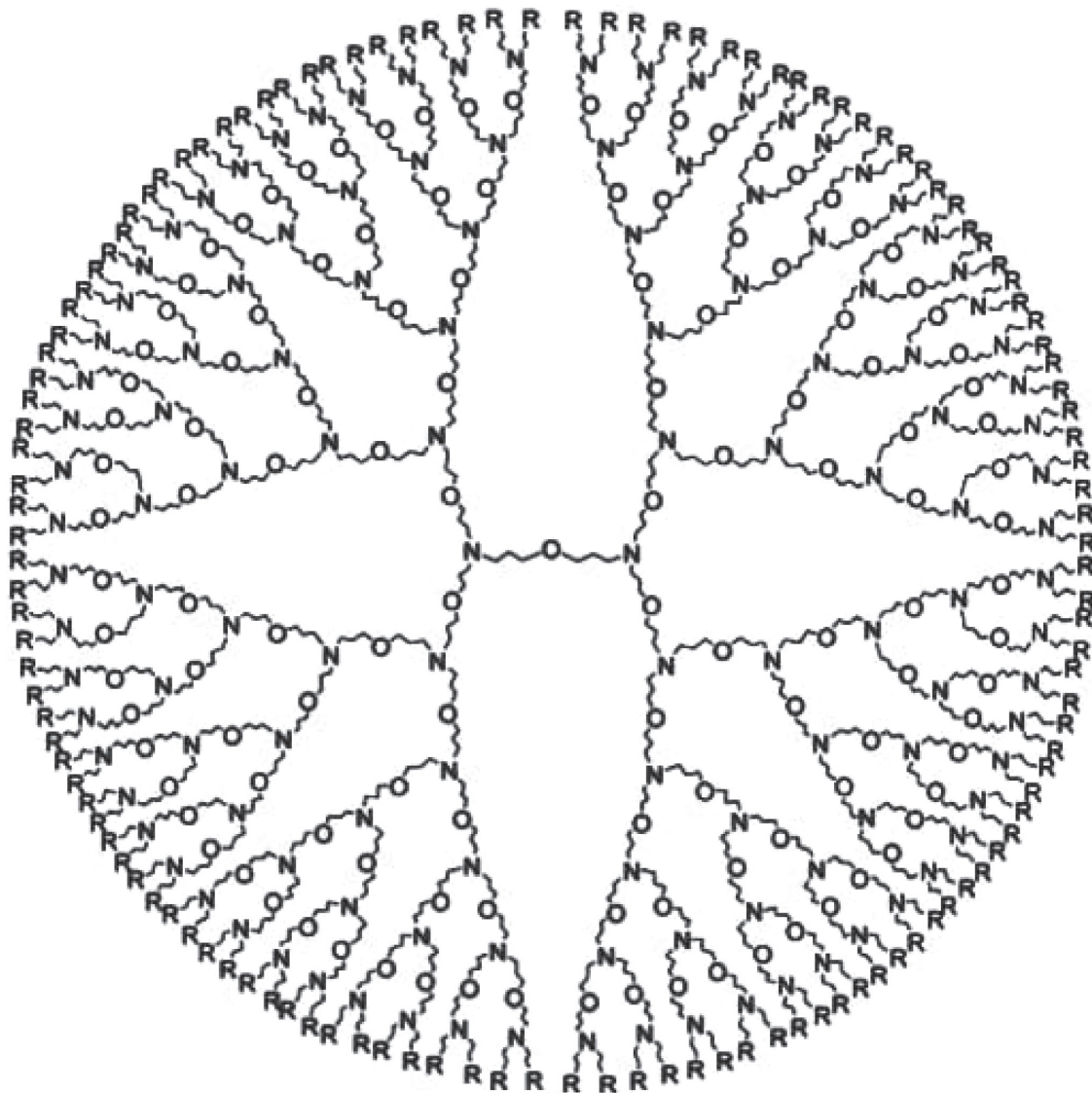


FIGURE 4: The molecular graph of propyl ether imine dendrimer.

TABLE 5: Edge partition of PETIM.

(d_v, d_w) , where $v\omega \in E(G)$	(1, 2)	(2, 2)	(2, 3)
Number of edges	2×2^n	$16 \times 2^n - 18$	$6 \times 2^n - 6$

4. Results and Discussion

In this work, novel topological index known as SS index is introduced and the proposed index is computed for 67 alkane isomers to study the physico-chemical properties, namely, bp , mv , mr , hv , and cp . A linear regression model of these physical properties with SS index is presented. From Table 2 and Figure 1, the SS index has highest correlation with molar refraction (mr) which is 0.99. Also, SS index is with boiling point (bp) 0.931, with molar volume (mv) 0.98, with heat of vaporization (hv) 0.951, and with critical pressure (cp) -0.889 . From Table 2 by inspection, it is clear that SS index has good correlation

with the physico-chemical properties compared to the existing indices, namely, inverse sum indeg index, first Zagreb index, and second Zagreb index. Also, the work focuses on computing the SS index for four dendrimer structures, namely, D_nP_n , DPZ_n , PETIM, and PETAA. The values of n are substituted for $n=1$ to 10. By inspection from Table 7, it is very clear that SS index increases as n increases. Also, it is observed that correlation coefficient of D_nP_n is $r=1$, which is more than the correlation coefficients of DPZ_n , PETIM, and PETAA which are 0.798837, 0.798841, and 0.798835, respectively. For each of the four structures, a graph as shown in Figure 6 is plotted against the values found in Table 7.

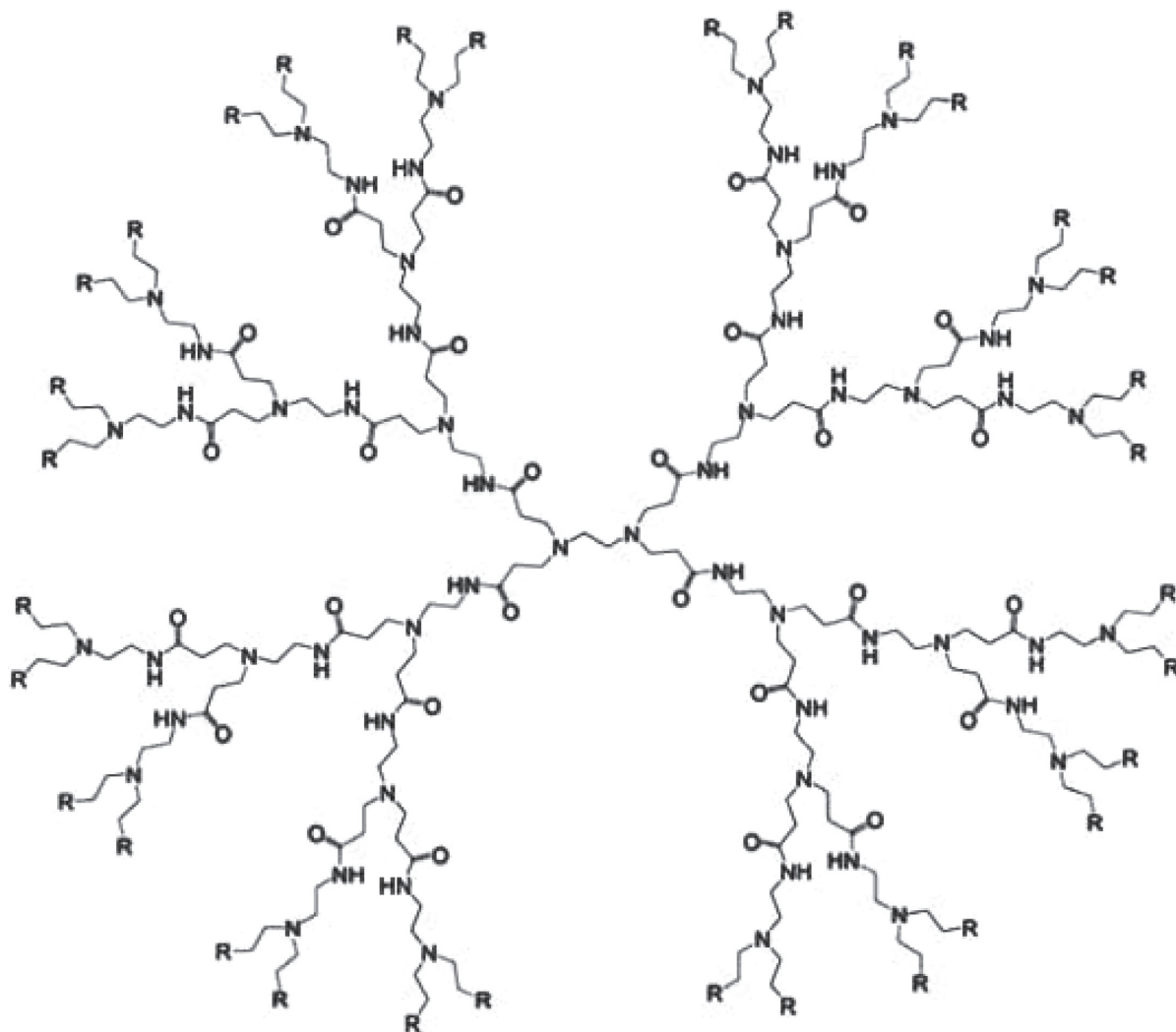


FIGURE 5: The molecular graph of polyethylene amide amine dendrimer.

TABLE 6: Edge partition of PETAA.

(d_v, d_w) , where $v\omega \in E(G)$	(1, 2)	(1, 3)	(2, 2)	(2, 3)
Number of edges	4×2^n	$4 \times 2^n - 2$	$16 \times 2^n - 8$	$20 \times 2^n - 9$

As the SS index is found to have very good correlation coefficient $r = 0.99 \cong 1$ with the above-discussed physico-chemical properties, the novel index is of great use in the QSPR/QSAR/QSTR analysis by the chemists.

5. Mathematical Properties of SS Index

In this section, the SS indexes of cycle, star, path, and simple graphs are computed [34–38].

Theorem 5. For a cycle C_n where n is the cardinality of vertices, then the SS index of C_n is given by $SS(C_n) = n$.

Proof. A cycle C_n has n vertices and n edges. The n edges of the cycle will be of type (2, 2). By considering all the n edges and using the definition of SS index, we get $SS(C_n) = n$. \square

Theorem 6. For a star S_n where n is the cardinality of vertices, then the SS index of S_n is given by

$$SS(S_n) = \frac{(n-1)^{3/2}}{\sqrt{n}}. \quad (14)$$

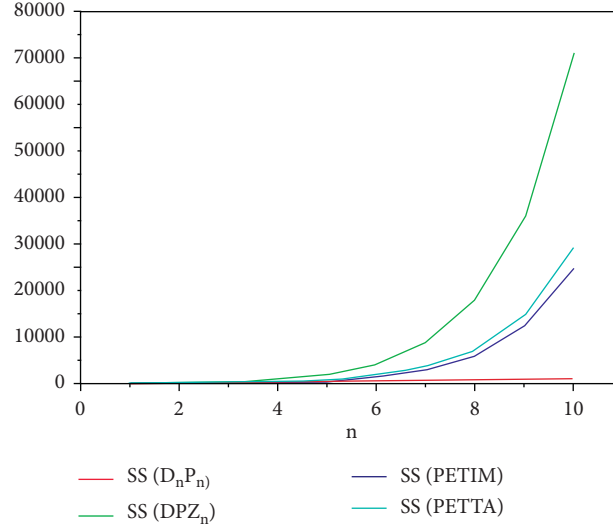
Proof. A star S_n has n vertices and $(n-1)$ edges. The $(n-1)$ edges of the star graph will be of type (1, $n-1$). By considering all the $(n-1)$ edges and using the definition of SS index, we get

$$SS(S_n) = \frac{(n-1)^{3/2}}{\sqrt{n}}. \quad (15)$$

\square

TABLE 7: Numerical comparison of $SS(D_nP_n)$, $SS(DPZ_n)$, $SS(PETIM)$, and $SS(PETAA)$ for $n = 1$ to 10.

n	1	2	3	4	5	6	7	8	9	10
D_nP_n	100.60	212.78	324.95	437.13	549.31	661.48	773.66	885.84	998.01	1110.2
DPZ_n	92.252	231.48	509.95	1066.9	2180.7	4408.4	8863.9	17775	35596	71240
PETIM	23.84	72.251	169.07	362.72	750.02	1524.6	3073.8	6172.2	12369	24762
PETAA	37.69	94.97	209.53	438.65	896.89	1813.4	3646.3	7312.2	14644	29308

FIGURE 6: Graphical comparison of $SS(D_nP_n)$, $SS(DPZ_n)$, $SS(PETIM)$, and $SS(PETAA)$ for $n = 1$ to 10.

Theorem 7. For a path P_n , where n is the cardinality of vertices, then the SS index of P_n is given by $SS(P_n) = (n - 3) + (2\sqrt{2}/\sqrt{3})$.

Proof. A path P_n has n vertices and $(n - 1)$ edges. The $(n - 1)$ edges of the path graph will be 2 edges of type (1, 2) and $(n - 3)$ edges of type (2, 2), respectively. By considering all the $(n - 1)$ edges and using the definition of SS index, we get $SS(P_n) = (n - 3) + (2\sqrt{2}/\sqrt{3})$.

The SS indexes of cycle, star, and path graphs are related as follows:

$$SS(S) = \frac{(n - 1)^{3/2}}{\sqrt{SS(C_n)}}, \quad (16)$$

$$SS(P_n) = SS(C_n) + \frac{2\sqrt{2}}{\sqrt{3}} - 3.$$

□

Theorem 8. Consider a simple graph G with m edges and cardinality n . Let p , Δ , and δ_1 be the pendent vertices and maximum and minimum vertex degrees of G , respectively. Then,

$$SS(G) \geq \frac{p\sqrt{\delta_1}}{\sqrt{1 + \delta_1}} + \frac{2\sqrt{n\delta_1\Delta}}{\Delta + \delta_1} \sqrt{ISI(G) - \frac{p\delta_1}{1 + \delta_1}}. \quad (17)$$

Proof. For $2 \leq \delta_1 \leq d_v, d_w \leq \Delta$,

$$\frac{d_v d_w}{d_v + d_w} \leq \frac{\Delta}{2}, \quad (18)$$

such that the equality holds iff $d_v = d_w = \Delta$. Also,

$$\frac{d_v d_w}{d_v + d_w} \geq \frac{\delta_1}{2}, \quad (19)$$

with equality holding if $d_v = d_w = \delta_1$. Using Polya–Szego inequality,

$$\left(\sum_{vw \in E(G): d_v, d_w \neq 1} \sqrt{\frac{d_v d_w}{d_v + d_w}} \right)^2 \geq \frac{4n\delta_1\Delta}{(\Delta + \delta_1)^2} \left(\sum_{vw \in E(G): d_v, d_w \neq 1} \frac{d_v d_w}{d_v + d_w} \right) \geq \frac{4n\delta_1\Delta}{(\Delta + \delta_1)^2} \left(ISI(G) - \sum_{vw \in E(G): d_v=1} \frac{d_w}{1 + d_w} \right), \quad (20)$$

$$\left(\sum_{vw \in E(G): d_v, d_w \neq 1} \sqrt{\frac{d_v d_w}{d_v + d_w}} \right) \geq \frac{2\sqrt{n\delta_1\Delta}}{(\Delta + \delta_1)} \sqrt{ISI(G) - \frac{p\delta_1}{1 + \delta_1}},$$

we have

$$SS(G) = \sum_{vw \in E(G)} \sqrt{\frac{d_w}{1+d_w}} + \left(\sum_{vw \in E(G): d_v, d_w \neq 1} \sqrt{\frac{d_v d_w}{d_v + d_w}} \right). \quad (21)$$

For $\Delta \geq d_v$, from (20) and (21), we get

$$SS(G) \geq \frac{p\sqrt{\delta_1}}{\sqrt{1+\delta_1}} + \frac{2\sqrt{n\delta_1\Delta}}{\Delta + \delta_1} \sqrt{ISI(G) - \frac{p\delta_1}{1+\delta_1}}. \quad (22)$$

□

Theorem 9. For a tree T with cardinality n and pendent vertices p , then the SS index is

$$SS(T) \geq \frac{p\sqrt{\delta_1}}{\sqrt{1+\delta_1}} + \frac{2\sqrt{(n-1)\delta_1\Delta}}{\Delta + \delta_1} \sqrt{ISI(G) - \frac{p\delta_1}{1+\delta_1}}. \quad (23)$$

$$\begin{aligned} \left(\sum_{vw \in E(G): d_v, d_w \neq 1} \sqrt{\frac{d_v d_w}{d_v + d_w}} \right)^2 &\leq (m-p) \left(\sum_{vw \in E(G): d_v, d_w \neq 1} \frac{d_v d_w}{d_v + d_w} \right), \\ &\leq (m-p) \left[ISI(G) - \sum_{vw \in E(G): d_v=1} \frac{d_w}{1+d_w} \right] \\ &\leq (m-p) \left[ISI(G) - \frac{p\Delta}{1+\Delta} \right] \end{aligned} \quad (25)$$

$$\left(\sum_{vw \in E(G): d_v, d_w \neq 1} \sqrt{\frac{d_v d_w}{d_v + d_w}} \right) \leq \sqrt{(m-p) \left(ISI(G) - \frac{p\Delta}{1+\Delta} \right)}$$

$$SS(G) \leq \frac{p\sqrt{\Delta}}{\sqrt{1+\Delta}} + \sqrt{(m-p) \left[ISI(G) - \frac{p\Delta}{1+\Delta} \right]}.$$

□

Theorem 11. The cardinality n of a tree T and pendent vertices p and then the SS index is

$$SS(T) \leq \frac{p\sqrt{\Delta}}{\sqrt{1+\Delta}} + \sqrt{(n-1-p) \left[ISI(G) - \frac{p\Delta}{1+\Delta} \right]}. \quad (26)$$

6. Conclusion

In this article, a novel index known as SS index is introduced and computed for four dendrimers such as D_nP_n , DPZ_n , PETIM, and PETAA. To validate the performance of this novel index, the chemical applicability of 67 alkane isomers is studied. It is observed that the proposed index has a very good correlation with alkane isomers considered in the study. The results obtained for the dendrimers have proved that they play a major role in drugs including anti-inflammatory, antimicrobial, and anticancer in administering the drug. Dendrimers play a vital role in the discovery of

Theorem 10. Consider simple graph G of order n with m edges and cardinality n . Here, p , Δ , and δ_1 represent pendent vertices and maximum vertex degree and minimum non-pendent vertex degree, respectively, and then

$$SS(G) \leq \frac{p\sqrt{\Delta}}{\sqrt{1+\Delta}} + \sqrt{(m-p) \left[ISI(G) - \frac{p\Delta}{1+\Delta} \right]}. \quad (24)$$

Proof. By Cauchy-Schwarz inequality,

drugs against the diseases such as Alzheimer's, HIV, and cancer. The article is concluded by mathematical properties of SS indexes for cycle, star, path, and simple graphs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally to this study. M. C. Shanmukha gave the idea and wrote the manuscript. A. Usha and K. C. Shilpa edited and verified the results. Weidong Zhao checked and corrected the initial manuscript and verified the results. M. Reza Farahani added some final

remarks and improved the overall paper. All authors read and approved the final draft.

Acknowledgments

This work was supported by the National Key Research and Development Program under Grant 2018YFB0904205. This research received no external funding.




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Research Article

On Some Fixed Point Results in E – Fuzzy Metric Spaces

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Received 12 June 2021; Accepted 6 September 2021; Published 20 September 2021

Academic Editor: Ali Jaballah

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In the existing literature, Banach contraction theorem as well as Meir-Keeler fixed point theorem were extended to fuzzy metric spaces. However, the existing extensions require strong additional assumptions. The purpose of this paper is to determine a class of fuzzy metric spaces in which both theorems remain true without the need of any additional condition. We demonstrate the wide validity of the new class.

1. Introduction

The well-known Banach's contraction principle (BCP) is a classic method in nonlinear analysis and is one of the most important and heavily researched fixed point theorems. It states that if $\mathcal{L}: \mathcal{O} \rightarrow \mathcal{O}$ is a contraction on the complete metric space \mathcal{O} , then

\mathcal{L} has a unique fixed point u in \mathcal{O} , and
 $\lim_n \mathcal{L}^n \kappa = u$ for all $\kappa \in \mathcal{O}$.

There exist many different concepts of a fuzzy metric space (cf. [1–5]). However, the definition of George and Veeramani [4] is reverently cited in most references in this area. The authors of [4] updated Kramosil and Michalek's definition of fuzzy metric space and obtained a Hausdorff topology for this kind of fuzzy metric space. The topology induced by a fuzzy metric space in the sense of George and Veeramani has recently been demonstrated to be metrizable in [6]. It would be very useful if the BCP remains true in fuzzy metric spaces. This has been discussed in [7, 8] and then in numerous other directions over the years (see, for

example, [9–11]). In [12], the generalized altering distance function has been defined and the Banach contraction principle in complete fuzzy metric spaces using altering distance has been extended.

In [7], in order to give the fuzzy version of the BCP, Grabiec introduced a new definition of a Cauchy sequence in fuzzy metric spaces as follows.

Definition 1. A sequence $\{\kappa_n\}_n$ in a fuzzy metric space $(\mathcal{O}, \mathcal{Q}, *)$ is Cauchy if $\lim_n \mathcal{Q}(\kappa_{n+p}, \kappa_n, t) = 1$ for each $t > 0$ and $p > 0$.

It can be seen easily that this definition of a Cauchy sequence is incorrect, for more details see [13–15], whereas, in [8], Gregori and Sapena extended the Banach fixed point theorem to fuzzy version stating the following theorem.

Theorem 1. Let $(\mathcal{O}, \mathcal{Q}, *)$ be a complete fuzzy metric space in which fuzzy contractive sequences are Cauchy. Let $\mathcal{L}: \mathcal{O} \rightarrow \mathcal{O}$ be a fuzzy contractive mapping with k serving as the contractive constant. Then, \mathcal{L} has a unique fixed point.

Here, authors added a fairly strong assumption, that is, “every contractive sequence is Cauchy.”

In this paper, we prove that the Banach contraction theorem as well as the Meir-Keeler fixed point theorem remain true in fuzzy metric spaces with only a slight modification in the definition of fuzzy spaces given by George and Veeramani. Last, in this paper, we give some results to illustrate the broad validity of our results.

Before stating the main results, we need the following definitions.

Definition 2 (Schweizer and Sklar [16]). A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a continuous t -norm if it satisfies the following assertions:

- (T1) $*$ is commutative and associative;
- (T2) $*$ is continuous;
- (T3) $a * 1 = a$ for all $a \in [0, 1]$;
- (T4) $a * b \leq c * d$ when $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0, 1]$.

Here is the definition of a fuzzy metric space given by George and Veeramani:

Definition 3 (George and Veeramani [4]). A fuzzy metric space is an ordered triple $(\mathcal{O}, \mathcal{Q}, *)$ in which \mathcal{O} is a non-empty set, $*$ is a continuous t -norm, and \mathcal{Q} is a fuzzy set on $\mathcal{O} \times \mathcal{O} \times (0, \infty) \rightarrow (0, 1]$ such that

- (T5) $\mathcal{Q}(\kappa, \omega, t) > 0$;
- (T6) $\mathcal{Q}(\kappa, \omega, t) = 1$ if and only if $\kappa = \omega$;
- (T7) $\mathcal{Q}(\kappa, \omega, t) = \mathcal{Q}(\omega, \kappa, t)$;
- (T8) $\mathcal{Q}(\kappa, \omega, t) * \mathcal{Q}(\omega, z, s) \leq \mathcal{Q}(\kappa, z, t + s)$;
- (T9) $\mathcal{Q}(\kappa, \omega, \cdot): (0, +\infty) \rightarrow (0, 1]$ is left continuous,

for all $\kappa, \omega, z \in \mathcal{O}$ and $t, s > 0$.

In this paper, we will consider the following class of fuzzy metric spaces.

Definition 4 (E -fuzzy metric space). Let \mathcal{O} denote a non-empty set, $*$ refers to a continuous t -norm, and \mathcal{Q} serves as a fuzzy set on $\mathcal{O} \times \mathcal{O} \times (0, \infty) \rightarrow (0, 1]$ such that

- (F1) $\mathcal{Q}(\kappa, \omega, t) > 0$;
- (F2) $\mathcal{Q}(\kappa, \omega, t) = 1$ if and only if $\kappa = \omega$;
- (F3) $\mathcal{Q}(\kappa, \omega, t) = \mathcal{Q}(\omega, \kappa, t)$;
- (F4) $\mathcal{Q}(\kappa, \omega, t) * \mathcal{Q}(\omega, z, s) \leq \mathcal{Q}(\kappa, z, t + s)$;
- (F5) $\mathcal{Q}(\kappa, \omega, \cdot): (0, +\infty) \rightarrow (0, 1]$ is left continuous.
- (F6) For some $r > 0$, the family $\{\mathcal{Q}(\kappa, \omega, \cdot): (0, r) \rightarrow (0, 1]; (\kappa, \omega) \in \mathcal{O}^2\}$ is uniformly equicontinuous,

for all $\kappa, \omega, z \in \mathcal{O}$, and $t, s > 0$. Then, the triple $(\mathcal{O}, \mathcal{Q}, *)$ is called an E -fuzzy metric space.

Remark 1. Obviously, all E -fuzzy metric space is a fuzzy metric space. So, all properties in fuzzy metric spaces remain true in E -fuzzy metric spaces.

Definition 5 (George and Veeramani [4]). Let $(\mathcal{O}, \mathcal{Q}, *)$ be a fuzzy metric space. Then,

- (i) A sequence $\{\kappa_n\}_n$ converges to $\kappa \in \mathcal{O}$ if and only if $\mathcal{Q}(\kappa_n, \kappa, t) \rightarrow 1$ as $n \rightarrow +\infty$ for all $t > 0$;
- (ii) A sequence $\{\kappa_n\}_n$ in \mathcal{O} is a Cauchy sequence if and only if for all $\varepsilon \in (0, 1)$ and $t > 0$, there exists n_0 such that $\mathcal{Q}(\kappa_n, \kappa_m, t) > 1 - \varepsilon$ for all $m, n \geq n_0$;
- (iii) The fuzzy metric space is complete if every Cauchy sequence converges to some $x \in \mathcal{O}$.

In the sequel, we use the following essential technical lemma.

Lemma 1. Let $(\mathcal{O}, \mathcal{Q}, *)$ be an E -fuzzy metric space, $\tilde{\mathcal{Q}}$ be the continuous extension of \mathcal{Q} up to $[0, \infty)$, and $\{\kappa_n\}_n$ be a sequence in \mathcal{O} such that $\lim_{n \rightarrow \infty} \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = 1$, for all $t > 0$. Then,

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) = 1. \quad (1)$$

Proof. For all $x, \omega \in \mathcal{O}$, function $t \mapsto \mathcal{Q}(x, \omega, t)$ is positive, continuous, and nondecreasing on $(0, +\infty)$, so $\tilde{\mathcal{Q}}$ is well defined. Let $\{t_n\}_n$ be a monotonically decreasing sequence of positive numbers, converging to 0, and $\{\kappa_n\}_n$ be a sequence in \mathcal{O} such that $\lim_{n \rightarrow \infty} \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = 1$, for all $t > 0$, i.e., for all $t > 0$ and for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$: for all $n \geq n_0$; $1 - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) < \varepsilon$.

From which it follows that for all $t > 0$ and for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$: for all $n \geq n_0$, and for all $k \in \mathbb{N}$,

$$1 - \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) + \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) + \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) < \frac{\varepsilon}{2}. \quad (2)$$

Therefore, for all $t > 0$ and for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$: for all $n \geq n_0$ and for all $k \in \mathbb{N}$,

$$1 - \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) < \frac{\varepsilon}{2} + |\tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k)| + |\mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t)|. \quad (3)$$

On the other hand, by the fact that $\lim_k \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) = \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0)$ and assumption (F6), we deduce that there exists $t_0 > 0$: for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, for all $n \geq n_0$, there exists $k_0 \in \mathbb{N}$, such that

$$|\tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k)| \leq \frac{\varepsilon}{4}, \quad (4)$$

$$|\mathcal{Q}(\kappa_n, \kappa_{n+1}, t_k) - \mathcal{Q}(\kappa_n, \kappa_{n+1}, t)| \leq \frac{\varepsilon}{4}, \quad (5)$$

for all $k > k_0$ and $t < t_0$. Hence, by relations (3)–(5), it yields for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$: for all $n \geq n_0$; $1 - \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) < \varepsilon$, and this means

$$\lim_n \tilde{\mathcal{Q}}(\kappa_n, \kappa_{n+1}, 0) = 1, \quad (6)$$

which achieves the proof of the lemma. \square

2. Main Results

Now, we will present our key finding.

Theorem 2. Let $(\mathcal{O}, \mathcal{Q}, *)$ be a complete E-fuzzy metric space. Let $\mathcal{L}: \mathcal{O} \rightarrow \mathcal{O}$ be a fuzzy contractive mapping with the contractive constant k , i.e., there exists $k \in]0, 1[$ such that

$$\frac{1}{\mathcal{Q}(\mathcal{L}\kappa, \mathcal{L}\omega, t)} - 1 \leq k \left(\frac{1}{\mathcal{Q}(\kappa, \omega, t)} - 1 \right), \quad (7)$$

for all κ, ω in \mathcal{O} and for all $t > 0$. Then, \mathcal{L} has a unique fixed point κ^* . Furthermore, for all $\kappa \in \mathcal{O}$, the sequence $\{\mathcal{L}^n \kappa\}$ converges to κ^* .

Proof. Let κ in \mathcal{O} and $\kappa_n = \mathcal{L}^n \kappa$ ($n \in \mathbb{N}$). Let $t > 0$ and $n \in \mathbb{N}$. By inequality (7), we obtain

$$\frac{1}{\mathcal{Q}(\kappa_{n+1}, \kappa_{n+2}, t)} - 1 \leq k \left(\frac{1}{\mathcal{Q}(\kappa_n, \kappa_{n+1}, t)} - 1 \right), \quad (8)$$

for all $t > 0$ and for all n in \mathbb{N} , which deduce that

$$\lim_{n \rightarrow \infty} \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = 1, \quad (9)$$

for all $t > 0$. Now, to prove that $\{\kappa_n\}_n$ is a Cauchy sequence, we assume to the contrary. Since $t \mapsto \mathcal{Q}(\kappa, \omega, t)$ is a non-decreasing function, there exists $\varepsilon \in (0, 1)$ and there exists $\xi > 0$ such that for all $p \in \mathbb{N}$, there exists $n_p (\geq p) < m_p \in \mathbb{N}$ so that

$$\mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t) \leq 1 - \varepsilon, \quad (10)$$

for all $t < \xi$. Let $t_0 < \min\{\xi, r\}$. By virtue of limit (9) and the last relation, we can write that there exists $\varepsilon \in (0, 1)$; for all $p \in \mathbb{N}$, there exists $n_p (\geq p) < m_p \in \mathbb{N}$:

$$\begin{aligned} \mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t_0) &\leq 1 - \varepsilon, \\ \mathcal{Q}(\kappa_{m_{p-1}}, \kappa_{n_p}, t_0) &> 1 - \varepsilon. \end{aligned} \quad (11)$$

Taking into account the continuity of the function $t \mapsto \mathcal{Q}(\kappa, \omega, t)$ and the fact that $\mathcal{Q}(\kappa_{m_{p-1}}, \kappa_{n_p}, t_0) > 1 - \varepsilon$, we can choose $q_0 \in \mathbb{N}$ such that

$$\mathcal{Q}\left(\kappa_{m_{p-1}}, \kappa_{n_p}, t_0 - \frac{1}{q_0}\right) > 1 - \varepsilon. \quad (12)$$

By virtue of assumptions (T4) and (F4) and relations (11) and (12), it follows that

$$\begin{aligned} 1 - \varepsilon &\geq \mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t_0) \\ &\geq \mathcal{Q}\left(\kappa_{m_p}, \kappa_{m_{p-1}}, \frac{1}{q_0}\right) * \mathcal{Q}\left(\kappa_{m_{p-1}}, \kappa_{n_p}, t_0 - \frac{1}{q_0}\right) \\ &\geq \tilde{\mathcal{Q}}(\kappa_{m_p}, \kappa_{m_{p-1}}, 0) * (1 - \varepsilon). \end{aligned} \quad (13)$$

So, according to assumptions (T2)-(T3), limit (9), and Lemma 1, one has

$$\lim_{p \rightarrow \infty} \mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t_0) = 1 - \varepsilon. \quad (14)$$

Suppose that for all $p_1 \geq 0$, there exists $p \geq p_1$ such that $\mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) \leq 1 - \varepsilon$ means, having in mind relations (7) and (14), that the sequence $\{\kappa_n\}_n$ has two subsequences $\{\kappa_{n_p}\}_p$ and $\{\kappa_{m_p}\}_p$ verifying

$$\lim_{p \rightarrow \infty} \mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t_0) = \lim_{p \rightarrow \infty} \mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) = 1 - \varepsilon, \quad (15)$$

(for the sake of simplicity, we have saved the same notation for the subsequence).

Now, we suppose that there exists $p_1 \geq 0$ such that $\mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) > 1 - \varepsilon$ for all $p \geq p_1$. We claim that $\lim_p \mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) = 1 - \varepsilon$. Suppose not, i.e., there exists $\alpha > 0$ and two subsequences $\{\kappa_{n_p}\}_p$ and $\{\kappa_{m_p}\}_p$ verifying

$$\mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) > \alpha + (1 - \varepsilon), \quad (16)$$

for all $p \in \mathbb{N}$.

Having $q \in \mathbb{N}$ satisfying $\mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0 - (1/q)) > \alpha + (1 - \varepsilon)$, we obtain

$$\begin{aligned} 1 - \varepsilon &\geq \mathcal{Q}(\kappa_{m_p}, \kappa_{n_p}, t_0) \\ &\geq \mathcal{Q}\left(\kappa_{m_p}, \kappa_{m_{p+1}}, \frac{1}{2q}\right) * \mathcal{Q}\left(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0 - \frac{1}{q}\right) \\ &\quad * \mathcal{Q}\left(\kappa_{n_{p+1}}, \kappa_{n_p}, \frac{1}{2q}\right) \\ &\geq \tilde{\mathcal{Q}}(\kappa_{m_p}, \kappa_{m_{p+1}}, 0) * [\alpha + (1 - \varepsilon)] * \tilde{\mathcal{Q}}(\kappa_{n_{p+1}}, \kappa_{n_p}, 0) \\ &\rightarrow \alpha + (1 - \varepsilon), \end{aligned} \quad (17)$$

as $p \rightarrow \infty$.

This is a contradiction. Then,

$$\lim_p \mathcal{Q}(\kappa_{m_{p+1}}, \kappa_{n_{p+1}}, t_0) = 1 - \varepsilon. \quad (18)$$

Relations (14), (15), and (18) drive to a clear contradiction with condition (7). So, $\{\kappa_n\}_n$ is a Cauchy sequence in the complete fuzzy metric space \mathcal{O} and we deduce that there exists $\kappa^* \in \mathcal{O}$ such that

$$\lim_n \mathcal{Q}(\kappa_n, \kappa^*, t) = 1, \quad (19)$$

for all $t > 0$, and by relation (7), we obtain

$$\frac{1}{\mathcal{Q}(\mathcal{L}\kappa_n, \mathcal{L}\kappa^*, t)} - 1 \leq k \left(\frac{1}{\mathcal{Q}(\kappa_n, \kappa^*, t)} - 1 \right), \quad (20)$$

for all $n \in \mathbb{N}$ and for all $t > 0$. Passing to the limit, having in mind the limit in (19), it follows that $\mathcal{Q}(\kappa^*, \mathcal{L}\kappa^*, t) = 1$, which, with assumption (F2) and relation (7), means that κ^* is the unique fixed point of mapping \mathcal{L} . This achieves the proof. \square

Theorem 3. (fuzzy Meir-Keeler fixed point theorem). Let $(\mathcal{O}, \mathcal{Q}, *)$ be a complete E -fuzzy metric space. Let $\mathcal{L}: \mathcal{O} \rightarrow \mathcal{O}$ be a fuzzy Meir-Keeler type mapping, i.e., for all $\varepsilon \in (0, 1)$, there exists $\delta > 0$ such that

$$\varepsilon - \delta < \mathcal{Q}(\kappa, \omega, t) \leq \varepsilon \implies \mathcal{Q}(\mathcal{L}\kappa, \mathcal{L}\omega, t) > \varepsilon, \quad (21)$$

for all κ, ω in \mathcal{O} and for all $t > 0$. Then, \mathcal{L} has a unique fixed point κ^* . Furthermore, for all $\kappa \in \mathcal{O}$, the sequence $\{\mathcal{L}^n \kappa\}$ converges to κ^* .

Proof. Let $\kappa \in \mathcal{O}$ and $\kappa_n = \mathcal{L}^n \kappa$ ($n \in \mathbb{N}$) and $t > 0$. Obviously, we have

$$\mathcal{Q}(\kappa, \mathcal{L}\kappa, t) - \delta < \mathcal{Q}(\kappa, \mathcal{L}\kappa, t) \leq \mathcal{Q}(\kappa, \mathcal{L}\kappa, t), \quad (22)$$

for all $\delta > 0$, and due to relation (21), we obtain $\mathcal{Q}(\mathcal{L}^2 \kappa, \mathcal{L}\kappa, t) > \mathcal{Q}(\kappa, \mathcal{L}\kappa, t)$. Recursively, we obtain a sequence $\{\mathcal{Q}(\kappa_n, \kappa_{n+1}, t)\}_n$ in $[0, 1]$ verifying

$$\mathcal{Q}(\kappa_n, \kappa_{n+1}, t) < \mathcal{Q}(\kappa_{n+1}, \kappa_{n+2}, t), \quad (23)$$

for all n in \mathbb{N} . It is a bounded increasing sequence. Then, there exists a function $u: (0, \infty) \rightarrow [0, 1]$ such that

$$\lim_{n \rightarrow +\infty} \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = \sup_{n \in \mathbb{N}} \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = u(t), \quad (24)$$

for all $t > 0$. We claim that $u(t) = 1$, for all $t > 0$. Suppose not, i.e., there exists $t_0 > 0$ such that $u(t_0) \in (0, 1)$. By the limit in (24), for all $\delta \in (0, u(t_0))$, there exists $n_0 \in \mathbb{N}$ such that

$$u(t_0) - \delta < \mathcal{Q}(\kappa_n, \kappa_{n+1}, t_0) \leq u(t_0), \quad (25)$$

for all $n \geq n_0$, which, with condition (21), implies that $\mathcal{Q}(\kappa_{n+1}, \kappa_{n+2}, t_0) > u(t_0)$. This is a clear contradiction with (24). Therefore,

$$\lim_n \mathcal{Q}(\kappa_n, \kappa_{n+1}, t) = 1, \quad (26)$$

for all $t > 0$. Now, we follow, exactly, the same lines as in the proof of Theorem 2 to deduce that $\{\kappa_n\}_n$ is a Cauchy sequence in the complete fuzzy metric space \mathcal{O} , which deduce that there exists $\kappa^* \in \mathcal{O}$ such that

$$\lim_n \mathcal{Q}(\kappa^*, \kappa_n, t) = 1. \quad (27)$$

On the other hand, for all $n \in \mathbb{N}$ and all $\delta \in (0, \mathcal{Q}(\kappa^*, \kappa_n, t))$, we have

$$\mathcal{Q}(\kappa^*, \kappa_n, t) - \delta < \mathcal{Q}(\kappa^*, \kappa_n, t) \leq \mathcal{Q}(\kappa^*, \kappa_n, t). \quad (28)$$

Condition (21) assures that

$$1 \geq \mathcal{Q}(\mathcal{L}\kappa^*, \mathcal{L}\kappa_n, t) > \mathcal{Q}(\kappa^*, \kappa_n, t), \quad (29)$$

which, with the limit in (27), gives $\lim_n \mathcal{Q}(\mathcal{L}\kappa^*, \kappa_n, t) = 1$, and finally

$$\kappa^* = \mathcal{L}\kappa^*. \quad (30)$$

For the uniqueness, we assume that there exists $\omega^* (\neq \kappa^*) \in \mathcal{O}$ such that $\omega^* = \mathcal{L}\omega^*$. It is clear that for all $\delta \in (0, \mathcal{Q}(\kappa^*, \omega^*, t))$, $\mathcal{Q}(\kappa^*, \omega^*, t) - \delta < \mathcal{Q}(\kappa^*, \omega^*, t) \leq \mathcal{Q}(\kappa^*, \omega^*, t)$.

Hence, by (21), $\mathcal{Q}(\mathcal{L}\kappa^*, \mathcal{L}\omega^*, t) > \mathcal{Q}(\kappa^*, \omega^*, t)$ or $\mathcal{Q}(\kappa^*, \omega^*, t) > \mathcal{Q}(\kappa^*, \omega^*, t)$, a contradiction, and this achieves the proof.

Now, we give the following corollary. \square

Corollary 1. Let (\mathcal{O}, d) be a complete metric space, and \mathcal{L} a Meir-Keeler mapping on \mathcal{O} , i.e., for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\kappa, \omega \in \mathcal{O}$,

$$\varepsilon \leq d(\kappa, \omega) < \varepsilon + \delta \implies d(\mathcal{L}(\kappa), \mathcal{L}(\omega)) < \varepsilon. \quad (31)$$

Let \mathcal{Q} be a function on $\mathcal{O} \times \mathcal{O} \times (0, +\infty)$ defined by

$$\mathcal{Q}(\kappa, \omega, t) = \frac{t + 1}{t + 1 + d(\kappa, \omega)}. \quad (32)$$

Then,

- (1) $(\mathcal{O}, \mathcal{Q}, \cdot)$ is an E -fuzzy metric space, where \cdot is the product t-norm.
- (2) For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon - \delta < \mathcal{Q}(\kappa, \omega, t) \leq \varepsilon \implies \mathcal{Q}(\mathcal{L}\kappa, \mathcal{L}\omega, t) > \varepsilon, \quad (33)$$

for all κ, ω in \mathcal{O} and for all $t > 0$.

Proof. $(\mathcal{O}, \mathcal{Q}, \cdot)$ is a fuzzy metric space (see [4]) and $\{t \mapsto \mathcal{Q}(\kappa, \omega, t); \kappa, \omega \in \mathcal{O}\}$ is a set of functions with common Lipschitz constant "1." So, it is uniformly equicontinuous. This means that $(\mathcal{O}, \mathcal{Q}, \cdot)$ is an E -fuzzy metric space. For the second assumption, it suffices to see that

For all $\varepsilon > 0$, $\delta \in (0, \varepsilon)$, $t > 0$ and all $\kappa, \omega \in \mathcal{O}$, we have

$$\begin{aligned} \varepsilon - \delta < \mathcal{Q}(\kappa, \omega, t) \leq \varepsilon &\iff \varepsilon - \delta < \frac{t + 1}{t + 1 + d(\kappa, \omega)} \leq \varepsilon \\ &\iff \varepsilon - \delta < \frac{1}{1 + (1/(t + 1))d(\kappa, \omega)} \leq \varepsilon \\ &\iff (t + 1) \left(\frac{1}{\varepsilon} - 1 \right) \leq d(\kappa, \omega) < (t + 1) \left(\frac{1}{\varepsilon - \delta} - 1 \right). \end{aligned} \quad (34)$$

Let $\varepsilon_0 = (t+1)((1/\varepsilon) - 1)$ and $\delta_{\varepsilon_0} > 0$ such that

$$\varepsilon_0 \leq d(\kappa, \omega) < \varepsilon_0 + \delta_{\varepsilon_0} \implies d(\mathcal{L}\kappa, \mathcal{L}\omega) < \varepsilon_0. \quad (35)$$

Now, we choose δ in (34) such that $(t+1)((1/(\varepsilon - \delta)) - 1) < (t+1)((1/\varepsilon) - 1) + \delta_{\varepsilon_0}$. Therefore, using relations (34) and (35), it follows that

$$\begin{aligned} \varepsilon - \delta < \mathcal{Q}(\kappa, \omega, t) \leq \varepsilon &\implies (t+1)\left(\frac{1}{\varepsilon} - 1\right) < d(\kappa, \omega) \leq (t+1)\left(\frac{1}{\varepsilon} - 1\right) + \delta_{\varepsilon_0} \\ &\implies d(\mathcal{L}\kappa, \mathcal{L}\omega) < (t+1)\left(\frac{1}{\varepsilon} - 1\right) \\ &\implies \frac{t+1}{t+1+d(\mathcal{L}\kappa, \mathcal{L}\omega)} > \varepsilon \\ &\implies \mathcal{Q}(\mathcal{L}\kappa, \mathcal{L}\omega, t) > \varepsilon, \end{aligned} \quad (36)$$

and this achieves the proof. \square

Theorem 4. Consider the integral operator \mathcal{L} on $C([0, I], \mathbb{R})$ as

$$\mathcal{L}\kappa(r) = g(r) + \int_0^r F(r, s, \kappa(s))ds. \quad (41)$$

3. Application

The purpose of this section is to give an example of the existence of a solution for an integral equation, where we can apply Theorem 2 to get its solution. For such integral equations, we refer the reader to [17] where the authors provide a common solution for a system of two integral equations.

Consider the integral equation,

$$\kappa(r) = g(r) + \int_0^r F(r, s, \kappa(s))ds, \quad \text{for all } r \in [0, I], I > 0, \quad (37)$$

and Banach space $C([0, I], \mathbb{R})$ of all continuous functions defined on $[0, I]$ equipped with supremum norm

$$\|\kappa\| = \sup_{r \in [0, I]} |\kappa(r)|, \quad \kappa \in C([0, I], \mathbb{R}), \quad (38)$$

with induced metric

$$d(\kappa, \omega) = \sup_{r \in [0, I]} |\kappa(r) - \omega(r)|. \quad (39)$$

Now, consider the fuzzy metric space with product t -norm as

$$\mathcal{Q}(\kappa, \omega, t) = \frac{t}{t + d(\kappa, \omega)}, \quad \text{for all } \kappa, \omega \in C([0, I], \mathbb{R}), t > 0. \quad (40)$$

According to George and Veeramani, standard fuzzy metric space and the corresponding metric space have same topologies. So, fuzzy metric space defined in (40) is complete.

Suppose that there exists $f: [0, I] \times [0, I] \rightarrow [0, \infty)$ such that $f \in L^1([0, I], \mathbb{R})$ and suppose that F satisfies the following condition:

$$|F(s, r, \kappa(r)) - F(s, r, \omega(r))| \leq f(r, s)|\kappa(s) - \omega(s)|, \quad (42)$$

for all $\kappa, \omega \in C([0, I], \mathbb{R})$ and for all $r, s \in [0, I]$ where

$$\sup_{r \in [0, I]} \int_0^r f(r, s)ds \leq k < 1. \quad (43)$$

Then, the integral equation (37) has a unique solution.

Proof. Let $\kappa, \omega \in C([0, I], \mathbb{R})$ and consider

$$\begin{aligned} &|\mathcal{L}\kappa(r) - \mathcal{L}\omega(r)| \\ &\leq \int_0^r |F(r, s, \kappa(s)) - F(r, s, \omega(s))|ds \\ &\leq \int_0^r f(r, s)|\kappa(s) - \omega(s)|ds \\ &\leq d(\kappa, \omega) \int_0^r f(r, s)ds \\ &\leq kd(\kappa, \omega). \end{aligned} \quad (44)$$

So,

$$d(\mathcal{L}\kappa, \mathcal{L}\omega) \leq kd(\kappa, \omega). \quad (45)$$

Using (40), we can write

$$\begin{aligned}
\frac{1}{Q(\kappa, \omega, t)} - 1 &= \frac{d(\kappa, \omega)}{t} \\
\frac{1}{Q(\mathcal{L}\kappa, \mathcal{L}\omega, t)} - 1 &= \frac{t + d(\mathcal{L}\kappa, \mathcal{L}\omega) - t}{t} \\
&= \frac{d(\mathcal{L}\kappa, \mathcal{L}\omega)}{t} \\
&\leq k \frac{d(\kappa, \omega)}{t} \\
&\leq k \left(\frac{1}{Q(\kappa, \omega, t)} - 1 \right).
\end{aligned} \tag{46}$$

Since all the conditions of Theorem 2 hold, (37) has a unique solution. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Exact Values of Zagreb Indices for Generalized T-Sum Networks with Lexicographic Product

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Received 16 April 2021; Accepted 17 August 2021; Published 31 August 2021

Academic Editor: Huseyin Isik

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The use of numerical numbers to represent molecular networks plays a crucial role in the study of physicochemical and structural properties of the chemical compounds. For some integer k and a network G , the networks $S_k(G)$ and $R_k(G)$ are its derived networks called as generalized subdivided and generalized semitotal point networks, where S_k and R_k are generalized subdivision and generalized semitotal point operations, respectively. Moreover, for two connected networks, G_1 and G_2 , $G_1[G_2]_{S_k}$ and $G_1[G_2]_{R_k}$ are T -sum networks which are obtained by the lexicographic product of $T(G_1)$ and G_2 , respectively, where $T \in \{S_k, R_k\}$. In this paper, for the integral value $k \geq 1$, we find exact values of the first and second Zagreb indices for generalized T -sum networks. Furthermore, the obtained findings are general extensions of some known results for only $k = 1$. At the end, a comparison among the different generalized T -sum networks with respect to first and second Zagreb indices is also included.

1. Introduction

Topological index (TI) being a molecular descriptor is a mathematical measure that associates a molecular network with a real number and predicts the underlying molecular network's biological, chemical, and structural properties. Molecular descriptors were used by Wiener [1] and Trinajstić and Gutman and Trinajstić [2] to determine the boiling point of paraffin and the total π -electron energy of the molecules, respectively. TIs are also used in the study of cheminformatics to classify molecules in terms of quantitative structure behavior and property relationships. Most notably, all TIs are invariants under the networks' isomorphism parameter [3, 4]. Many TIs exist in the literature for networks. Degree-based TIs, distance-based TIs, and polynomial-based TIs are the three major types of these. The degrees-based TIs are more familiar than the others [5].

In the field of chemical network theory, networks operations are frequently used to discover the new families of networks. Yan et al. [6] explained the subdivision (S_1) and semitotal point (R_1) operations on a molecular network G and attain the Wiener indices of the consequent networks

$S_1(G)$ and $R_1(G)$. After that, Eliasi and Taeri [7] explained the T -sum network $G_1 +_T G_2$ by using Cartesian product of G_1 and $S(G_2)$, where $T \in \{S_1, R_1\}$. Deng et al. [8] defined the T -sum network $(G_1[G_2])$ with the help of lexicographic product and also calculated the first and second Zagreb indices, and Akhter and Imran [9] also calculated the forgotten index of the T -sum networks. For more studies of the TIs under the operations of networks, see [10, 11].

Recently, for some integer $k \geq 1$, Liu et al. [12] defined the generalized subdivision (S_k) and generalized semitotal point (R_k) operations. They also computed the 1st and 2nd Zagreb indices for T -sum networks with the help of Cartesian product, i.e., $(G_1 +_{T_k} G_2)$ for $T_k \in \{S_k, R_k\}$ [13, 14]. In this study, we find the exact values of the Zagreb indices for the generalized T -sum networks which are constructed with the help of generalized subdivision, generalized semitotal point operations, and lexicographic product. In the remaining paper, Section 1 has some previous knowledge related to our work and Section 2 has some basic definition. The key findings are presented in Section 3, and the conclusion and applications of these indices are presented in Section 4.

2. Preliminaries

We consider undirected, connected, and simple networks, where $V(G)$ is a vertex set and $E(G) \subseteq V(G) \times V(G)$ is an edge set. In addition, $|V(G)| = n$ is order and $|E(G)| = m$ is size of G . Each vertices of a molecular network is referred to as an atom, and edges reflect the bonding between atoms.

The number of incident edges is called its degree. The networks $V(P_n) = \{b_j: 1 \leq j \leq n\}$ and $E(P_n) = \{b_j b_{j+1}: 1 \leq j \leq n-1\}$, $V(C_n) = \{b_j: 1 \leq j \leq n\}$ and $E(C_n) = \{b_i b_{i+1}: 1 \leq i \leq n-1\} \cup \{b_n b_1\}$ are called path (P_n), cycle (C_n), and complete (K_n), respectively [3].

Definition 1. Let G_1 and G_2 be two networks with their vertex sets as $V(G_1) = \{y_1, y_2, \dots, y_n\}$ and $V(G_2) = \{z_1, z_2, \dots, z_n\}$, respectively. The Cartesian product $G_1 \times G_2$ of these networks is defined as follows:

$$\begin{aligned} V(G_1 \times G_2) &= V(G_1) \times V(G_2), \\ E(G_1 \times G_2) &= \{(y_1, z_1)(y_2, z_2): (y_1, z_1), (y_2, z_2) \in V(G_1) \times V(G_2)\}, \end{aligned} \quad (1)$$

with conditions either $y_1 = y_2$ in $V(G_1)$ and $z_1 z_2 \in E(G_2)$ or $y_1 y_2 \in E(G_1)$ and $z_1 = z_2$ in $V(G_2)$.

Definition 2. Let G_1 and G_2 be two networks with their vertex sets as $V(G_1) = \{y_1, y_2, \dots, y_n\}$ and $V(G_2) = \{z_1, z_2, \dots, z_n\}$, respectively. Then, the lexicographic product $G_1[G_2]$ of these networks is defined as

$$\begin{aligned} V(G_1[G_2]) &= V(G_1) \times V(G_2), \\ E(G_1[G_2]) &= \{(y_1, z_1)(y_2, z_2): (y_1, z_1), (y_2, z_2) \in V(G_1) \times V(G_2)\}, \end{aligned} \quad (2)$$

with conditions either $y_1 = y_2$ in $V(G_1)$ and $z_1 z_2 \in E(G_2)$ or $y_1 y_2 \in E(G_1)$ and z_1, z_2 in $V(G_2)$.

For integer $k \geq 1$, Liu et al. [12] defined the following graphs using the generalized subdivision and semitotal point operations:

- (i) $S_k(G)$ network is found out by inserting k -vertices in each edge of G
- (ii) $R_k(G)$ is found out from $S_k(G)$ by connecting the old vertices which are adjacent in G

Eliasi and Taeri [7] proposed the generalized T -sum networks under the operation of two connected networks G_1

and G_2 based on the Cartesian product. Deng et al. [8] defined the generalized T -sum networks under the operation of two connected networks G_1 and G_2 based on the Lexicographic product as follows.

Definition 3. Let G_1 and G_2 be two networks and $T_k \in \{S_k, R_k\}$ and $T_k(G_1)$ be a network with vertex set $V(T_k(G_1))$ and edge set $E(T_k(G_1))$. Then, the generalized T_k -sum network $G_1[G_2]_{T_k}$ is a graph with the vertex set:

$$\begin{aligned} V(G_1[G_2]_{T_k}) &= V(T_k(G_1)) \times V(G_2), \\ E(G_1[G_2]_{T_k}) &= \{(y_1, z_1)(y_2, z_2): (y_1, z_1), (y_2, z_2) \in V(T_k(G_1)) \times V(G_2)\}, \end{aligned} \quad (3)$$

with conditions either $y_1 = y_2$ in $V(T_k(G_1))$ and $z_1 z_2 \in E(G_2)$ or $y_1 y_2 \in E(T_k(G_1))$ and z_1, z_2 in $V(G_2)$.

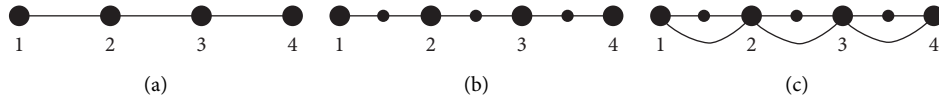
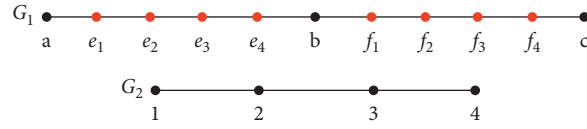
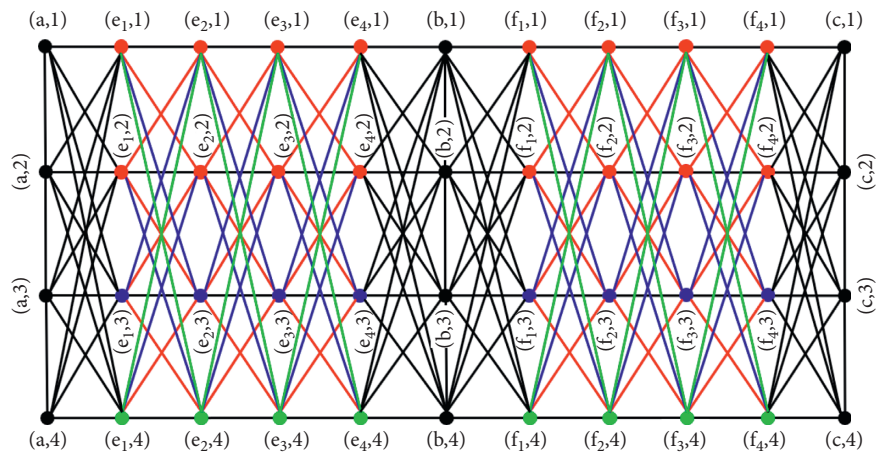
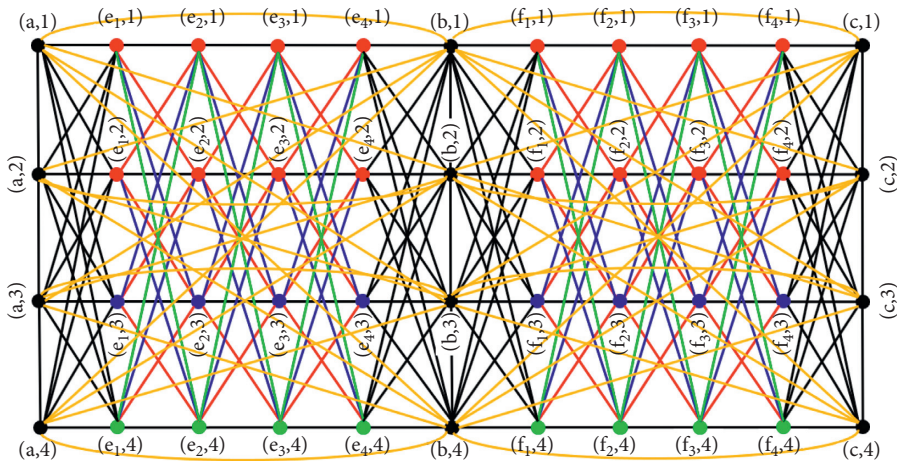
For more explanation, see Figures 1–4.

3. Main Results

In this part, we will learn about the Zagreb indices of T -sum networks obtained by the operations of generalized subdivision and generalized semitotal point and lexicographic product.

Theorem 1. Let G_1 and G_2 be two networks; then, for some integer k , the first Zagreb index of S_k -sum network is

$$\begin{aligned} M_1(G_1[G_2]_{S_k}) &= 8n_2 e_1 e_2 + n_1 M_1(G_2) + n_2^3 M_1(G_1) \\ &\quad + 4e_1 n_2^3 + 4e_1 n_2^2 (k-1) + 8n_2 e_1 (k-1) \\ &\quad + \sum_{i=1}^{n_2-1} (n_2 - i). \end{aligned} \quad (4)$$

FIGURE 1: (a) $G \cong P_4$, (b) $S_1(G) \cong S_1(P_4)$, and (c) $R_1(G) \cong R_1(P_4)$.FIGURE 2: (a) $S(G_1) \cong S(P_3)$ and (b) $G_2 \cong P_4$.FIGURE 3: $P_3[P_4]_{S_4}$.FIGURE 4: $P_3[P_4]_{R_4}$.

Proof. Let $d(u, v) = d_{G_1[G_2]_{S_k}}(u, v)$ be the degree of a vertex (u, v) in the network $G_1[G_2]_{S_k}$:

$$\begin{aligned}
 M_1(G_1[G_2]_{S_k}) &= \sum_{(u,v) \in V(G_1[G_2]_{S_k})} d^2(u, v) = \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1[G_2]_{S_k})} [d(u_1, v_1) + d(u_2, v_2)] \\
 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(S_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1=v_2=v \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(G_1) \\ u_1, u_2 \in V(S_k(G_1)) - V(G_1)}} [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(S_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)] \\
 &= \sum 1 + \sum 2 + \sum 3 + \sum 4.
 \end{aligned} \tag{5}$$

Now, first, we calculate

$$\begin{aligned}
 \sum 1 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2)] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [n_2 d_{G_1}(u) + d_{G_2}(v_1) + n_2 d_{G_1}(u) + d_{G_2}(v_2)] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [2n_2 d_{G_1}(u) + d_{G_2}(v_1) + d_{G_2}(v_2)] \\
 &= \sum_{u \in V(G_1)} e_2 \cdot 2n_2 d(u) + \sum_{u \in V(G_1)} M_1(G_2) = 2e_1 e_2 (2n_2) + n_1 M_1(G_2) = 4n_2 e_1 e_2 + n_1 M_1(G_2), \\
 \sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(S_k(G_1))} [d(u_1, v_1) + d(u_2, v_2)] \\
 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u \in (v(G_1)), a \in (E(G_1)) \\ u \text{ and } a \text{ are incident}}} [d(u, v_1) + d(a, v_2)] \\
 &= \sum_{v_1 \in V(G_2)} \sum_{\substack{v_2 \in V(G_2) \\ u \in (v(G_1)), a \in (E(G_1))}} [n_2 d_{G_1} u + d_{G_2} v_1 + 2n_2] \\
 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u \in (v(G_1))} [d_{G_1} u] [n_2 d_{G_1} u + d_{G_2} v_1 + 2n_2] \\
 &= n_2 n_2 n_2 \sum_{u \in (v(G_1))} [d_{G_1}^2 u] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} 2e_1 [d_{G_2} v_1 + 2n_2] \\
 &= n_2^3 M_1(G_1) + 2e_1 [2e_2 n_2 + 2n_2 n_2 n_2] = n_2^3 M_1(G_1) + 4n_2 e_1 e_2 + 4e_1 n_2^3,
 \end{aligned}$$

$$\begin{aligned}
\sum 3 &= \sum_{v_1=v_2=v \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} [d(u_1, v_1) + d(u_2, v_2)] \\
&\quad u_1 u_2 \in V(S_k(G_1)) - V(G_1) \\
&= n_2(k-1)e_1[2n_2 + 2n_2] = 4e_1 n_2^2(k-1), \\
\sum 4 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S_k(G_1))} [d(u_1, v_1) + d(u_2, v_2)] \\
&= [2n_2 + 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)(k-2)e_1 + [2n_2 + 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)e_1 \\
&= 8n_2 \sum_{i=1}^{n_2-1} (n_2 - i)(k-2)e_1 + 8n_2 \sum_{i=1}^{n_2-1} (n_2 - i)e_1 = 8n_2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i).
\end{aligned} \tag{6}$$

So, the result is

$$\begin{aligned}
&= 8n_2 e_1 e_2 + n_1 M_1(G_2) + n_2^3 M_1(G_1) + 4e_1 n_2^3 \\
&\quad + 4e_1 n_2^2(k-1) + 8n_2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i).
\end{aligned} \tag{7}$$

□

Theorem 2. Let G_1 and G_2 be two networks; then, for some integer k , the second Zagreb index of S_k -sum network is

$$\begin{aligned}
M_2[G_1[G_2]_{S_k}] &= n_2^2 e_2 M_1(G_1) + 2n_2 e_1 M_1(G_2) + n_1 M_2(G_2) + 2n_2^4 M_1(G_1) + 8n_2^2 e_1 e_2 + 4e_1 n_2^2(k-1) \\
&\quad + 8n_2^2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i).
\end{aligned} \tag{8}$$

Proof. Let $d(u, v) = d_{G_1[G_2]_{S_k}}(u, v)$ be the degree of a vertex (u, v) in the graph $G_1[G_2]_{S_k}$:

$$\begin{aligned}
M_1(G_1[G_2]_{S_k}) &= \sum_{(u,v) \in V(G_1[G_2]_{S_k})} d^2(u, v) = \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1[G_2]_{S_k})} [d(u_1, v_1)d(u_2, v_2)] \\
&= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d(u, v_1)d(u, v_2)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S_k(G_1))} [d(u_1, v_1)d(u_2, v_2)] \\
&= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)] \\
&= \sum 1 + \sum 2 + \sum 3 + \sum 4.
\end{aligned} \tag{9}$$

Now, first, we calculate

$$\begin{aligned}
\sum 1 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [d(u, v_1) d(u, v_2)] \\
&= \sum_{u \in V(G_1)} \sum_{v_1 v_2 \in E(G_2)} [n_2^2 d_{G_1}^2(u) + n_2 d_{G_1}(u) [d_{G_2}(v_1) + d_{G_2}(v_2)] + d_{G_2}(v_1) d_{G_2}(v_2)] \\
&= \sum_{u \in V(G_1)} [n_2^2 e_2 d_{G_1}^2(u) + n_2 d_{G_1}(u) M_1(G_2) + M_2(G_2)] = n_2^2 e_2 M_1(G_1) + 2n_2 e_1 M_1(G_2) + n_1 M_2(G_2), \\
\sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S_k(G_1))} [d(u_1, v_1) d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u \in (v(G_1)), a \in (E(G_1)) \\ u \text{ and } a \text{ are incident}}} [d(u, v_1) d(a, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u \in (v(G_1)), a \in (E(G_1)) \\ u \text{ and } a \text{ are incident}}} [n_2 d_{G_1}(u) + d_{G_2}(v_1)] [2n_2] \\
&= n_2 n_2 2n_2 \sum_{u \in (v(G_1))} [d_{G_1}^2(u)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} 2e_1 [2n_2 d_{G_2}(v_1)] \\
&= 2n_2^4 M_1(G_1) + 2n_2 n_2 2e_1 2e_2 = 2n_2^4 M_1(G_1) + 8n_2^2 e_1 e_2, \\
\sum 3 &= \sum_{v_1=v_2=v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 u_2 \in V(S_k(G_1)) - V(G_1)}} [d(u_1, v_1) d(u_2, v_2)] \\
&= n_2 (k-1) e_1 [2n_2 2n_2] = 4e_1 n_2^2 (k-1), \\
\sum 4 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(S_k(G_1))} [d(u_1, v_1) d(u_2, v_2)] \\
&= [2n_2 \times 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i) (k-2) e_1 + [2n_2 \times 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i) e_1 \\
&= 8n_2^2 \sum_{i=1}^{n_2-1} (n_2 - i) (k-2) e_1 + 8n_2^2 \sum_{i=1}^{n_2-1} (n_2 - i) e_1 = 8n_2^2 e_1 (k-1) \sum_{i=1}^{n_2-1} (n_2 - i).
\end{aligned} \tag{10}$$

So, the final result is

$$= n_2^2 e_2 M_1(G_1) + 2n_2 e_1 M_1(G_2) + n_1 M_2(G_2) + 2n_2^4 M_1(G_1) + 8n_2^2 e_1 e_2 + 4e_1 n_2^2 (k-1) + 8n_2^2 e_1 (k-1) \sum_{i=1}^{n_2-1} (n_2 - i). \tag{11}$$

Theorem 3. Let G_1 and G_2 be two networks; then, for some integer k , the first Zagreb index of R_k -sum network is

$$M_1(G_1[G_2]_{R_k}) = 16n_2 e_1 e_2 + n_1 M_1(G_2) + 4n_2^3 M_1(G_1) + 4e_1 n_2^3 + 4e_1 n_2^2 (k-1) + 8n_2 e_1 (k-1) \sum_{i=1}^{n_2-1} (n_2 - i). \tag{12}$$

Proof. Let $d(u, v) = d_{G_1[G_2]_{R_k}}(u, v)$ be the degree of a vertex (u, v) in the graph $G_1[G_2]_{R_k}$.

$$\begin{aligned}
 M_1(G_1[G_2]_{R_k}) &= \sum_{(u,v) \in V(G_1[G_2]_{R_k})} d^2(u, v) = \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1[G_2]_{R_k})} [d(u_1, v_1) + d(u_2, v_2)] \\
 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(R_k(G_2))} \\
 &\quad \cdot [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1=v_2=v \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(R_k(G_1)) \\ u_1, u_2 \in V[R_k(G_1)-V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(R_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)] \\
 &= \sum 1 + \sum 2 + \sum 3 + \sum 4.
 \end{aligned} \tag{13}$$

Now, first, we calculate

$$\begin{aligned}
 \sum 1 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1) + d(u, v_2)] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[n_2 d_{R_{kG_1}} u + d_{G_2} v_1 + n_2 d_{R_{kG_1}} u + d_{G_2} v_2 \right] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[2n_2 d_{R_{kG_1}} u + d_{G_2} v_1 + d_{G_2} v_2 \right] \\
 &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[4n_2 d_{G_1} u + d_{G_2} v_1 + d_{G_2} v_2 \right] \\
 &= 4n_2 \sum_{v_1, v_2 \in E(G_2)} \left[\sum_{u \in V(G_1)} [d_{G_1} u] + \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d_{G_2} v_1 + d_{G_2} v_2] \right] \\
 &= 4n_2 \sum_{v_1, v_2 \in E(G_2)} (e_1) + \sum_{u \in V(G_1)} M_1(G_2) \\
 &= 4n_2 (2e_2)(e_1) + n_1 M_1(G_2) = 8n_2 e_1 e_2 + n_1 M_1(G_2), \\
 \sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(G_1) \\ u_1, u_2 \in V[R_k(G_1)-V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(R_k(G_1)) \\ u_1, u_2 \in V[R_k(G_1)-V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
 &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1, u_2 \in E(R_k(G_1)) \\ u_1, u_2 \in V[R_k(G_1)-V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)],
 \end{aligned}$$

$$\begin{aligned}
\sum 2 &= \sum 2' + \sum 2'', \\
\sum 2' &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \left(n_2 d_{R_{kG_1}} u_1 + d_{G_2} v_1 \right) + \left(n_2 d_{R_{kG_1}} u_2 + d_{G_2} v_2 \right) \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} n_2 \left(d_{R_{kG_1}} u_1 + d_{R_{kG_1}} u_2 \right) + (d_{G_2} v_1 + d_{G_2} v_2) \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} 2n_2 (d_{G_1} u_1 + d_{G_1} u_2) + (d_{G_2} v_1 + d_{G_2} v_2) \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} 2n_2 M_1(G_1) + 2e_2 e_1 n_2 + 2e_1 e_2 n_2 \\
&= n_2 n_2 2n_2 M_1(G_1) + 4e_1 e_2 n_2 = 2n_2^3 M_1(G_1) + 4e_1 e_2 n_2, \\
\sum 2'' &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} \left[n_2 d_{R_{kG_1}} u_1 + d_{G_2} v_1 + n_2 d_{R_{kG_1}} u_2 \right] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} \left[2n_2 d_{G_1} u_1 + d_{G_2} v_1 + 2n_2 \right] \tag{14} \\
&= n_1 n_2 2n_2 \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} d_{G_1}(u_1) + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(R_{k2})} \sum_{\substack{u_1 u_2 \in E[R_k(G_1)] \\ u_1 \in V(G_1) u_2 \in V[R_k(G_1) - V(G_1)]}} [d_{G_2} v_1 + 2n_2] \\
&= 2n_2^3 \sum_{u_1 \in V(G_1)} d_{G_1}^2 u_1 \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 \in V(G_1)} d_{G_1} u_1 [d_{G_2} v_1 + 2n_2] \\
&= 2n_2^3 M_1(G_1) + 2e_1 \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} [d_{G_2} v_1 + 2n_2] \\
&= 2n_2^3 M_1(G_1) + 2e_1 [2e_2 n_2 + 2n_2 n_2 n_2] = 2n_2^3 M_1(G_1) + 4e_1 e_2 n_2 + 4e_1 n_2^3, \\
\sum 3 &= \sum_{v_1 = v_2 = v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R_k(G_1)) \\ u_1 u_2 \in V[R_k(G_1) - V(G_1)]}} [d(u_1, v_1) + d(u_2, v_2)] \\
&= n_2 (k-1) e_1 [2n_2 + 2n_2] = 4e_1 n_2^2 (k-1), \\
\sum 4 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R_k(G_1))} [d(u_1, v_1) + d(u_2, v_2)] \\
&= [2n_2 + 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)(k-2)e_1 + [2n_2 + 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)e_1 \\
&= 8n_2 \sum_{i=1}^{n_2-1} (n_2 - i)(k-2)e_1 + 8n_2 \sum_{i=1}^{n_2-1} (n_2 - i)e_1
\end{aligned}$$

So, the final result is

$$= 16n_2e_1e_2 + n_1M_1(G_2) + 4n_2^3M_1(G_1) + 4e_1n_2^3 + 4e_1n_2^2(k-1) + 8n_2e_1(k-1) \sum_{i=1}^{n_2-1}. \quad (15)$$

□

Theorem 4. Let G_1 and G_2 be two networks; then, for some integer k , the second Zagreb index of R_k -sum network is

$$M_2(G_1[G_2]_{R_k}) = 8n_2^2e_2M_1(G_1) + 4e_1n_2M_1(G_2) + n_1M_2(G_2) + 4n_2^4M_2(G_1) + 4n_2^4M_1(G_2) + 8n_2^2e_1e_2 + 4e_1e_2^2 + 4e_1n_2^2(k-1) + 8n_2^2e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i). \quad (16)$$

Proof. Let $d(u, v) = d_{G_1[G_2]_{R_k}}(u, v)$ be the degree of a vertex (u, v) in the network $G_1[G_2]_{R_k}$:

$$\begin{aligned} M_1(G_1[G_2]_{R_k}) &= \sum_{(u,v) \in V(G_1[G_2]_{R_k})} d^2(u, v) = \sum_{(u_1, v_1)(u_2, v_2) \in E(G_1[G_2]_{R_k})} [d(u_1, v_1)d(u_2, v_2)] \\ &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1)d(u, v_2)] + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(R_k(G_2))} [d(u_1, v_1)d(u_2, v_2)] \\ &\quad + \sum_{v_1=v_2=v \in V(G_2)} \sum_{u_1, u_2 \in E(R_k(G_1))} [d(u_1, v_1) + d(u_2, v_2)] \\ &\quad + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1, u_2 \in E(R_k(G_2))} [d(u_1, v_1) + d(u_2, v_2)] \\ &= \sum 1 + \sum 2 + \sum 3 + \sum 4. \end{aligned} \quad (17)$$

Now, first, we calculate

$$\begin{aligned} \sum 1 &= \sum_{u_1=u_2=u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} [d(u, v_1)d(u, v_2)] \\ &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[n_2 d_{R_{kG_1}} u + d_{G_2} v_1 \right] \left[n_2 d_{R_{kG_1}} u + d_{G_2} v_2 \right] \\ &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[2n_2 d_{G_1} u + d_{G_2} v_1 \right] \left[2n_2 d_{G_1} u + d_{G_2} v_2 \right] \\ &= \sum_{u \in V(G_1)} \sum_{v_1, v_2 \in E(G_2)} \left[4n_2^2 d_{G_1}^2 u + 2n_2 d_{G_1} u [d_{G_2} v_1 + d_{G_2} v_2] \right] \\ &= 4e_2 n_2^2 M_1(G_1) + 4e_1 n_2 M_1(G_2) + n_1 M_2(G_2), \end{aligned}$$

$$\begin{aligned}
\sum 2 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V(R_k(G_1) - V(G_1))}} [d(u_1, v_1) d(u_2, v_2)] \\
&+ \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R_k(G_1)) \\ u_1 \in V(G_1) u_2 \in V(R_k(G_1) - V(G_1))}} [d(u_1, v_1) d(u_2, v_2)], \\
\sum 2 &= \sum 2' + \sum 2'', \\
\sum 2' &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R_k(G_1))} [d(u_1, v_1) \cdot d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} [d(u_1, v_1) d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \left[n_2 d_{R_{kG_1}} u_1 + d_{G_2} v_1 \right] \left[n_2 d_{R_{kG_1}} u_2 + d_{G_2} v_2 \right] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \left[2n_2 d_{G_1} u_1 + d_{G_2} v_1 \right] \left[n_2 d_{G_1} u_2 + d_{G_2} v_2 \right] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(G_1)} \left[4n_2^2 d_{G_1} u_1 d_{G_1} u_2 + 2n_2 d_{G_1} u_1 d_{G_2} v_2 + 2n_2 d_{G_1} u_2 d_{G_2} v_1 + d_{G_2} v_1 d_{G_2} v_2 \right] \\
&= 4n_2^4 M_2(G_1) + 4e_2 n_2^2 \sum_{u_1 u_2 \in E(G_1)} [d_{G_1} u_1 + d_{G_1} u_2] + 4e_1 e_2^2 = 4n_2^4 M_2(G_1) + 4e_2 n_2^2 M_1(G_1) + 4e_1 e_2^2,
\end{aligned}$$

$$\begin{aligned}
\sum 2'' &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V(R_k(G_1) - V(G_1))}} [d(u_1, v_1) d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V(R_k(G_1) - V(G_1))}} [d(u_1, v_1) d(u_2, v_2)] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V(R_k(G_1) - V(G_1))}} \left[n_2 d_{R_{kG_1}} u_1 + d_{G_2} v_1 \right] \left[n_2 d_{R_{kG_1}} u_2 \right] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(G_1) \\ u_1 \in V(G_1) u_2 \in V(R_k(G_1) - V(G_1))}} \left[n_2 d_{R_{kG_1}} u_1 + d_{G_2} v_1 \right] [2n_2] \\
&= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R_k(G_1))} \left[4n_2^2 [d_{G_1} u_1 + 2n_2 d_{G_2} v_1] \right] \\
&= 4n_2^4 \sum_{u_1 u_2 \in E(N_2(G_1))} d_{G_1} u_1 + \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R_k(G_1))} [2n_2 d_{G_2} v_1] \\
&= 4n_2^4 \sum_{u_1 \in V(G_1)} d_{G_1}^2 u_1 + 4e_2 n_2^2 \sum_{u_1 \in V(G_1)} d_{G_1} u_1 = 4n_2^4 M_1(G_2) + 8e_1 e_2 n_2^2,
\end{aligned}$$

$$\sum 3 = \sum_{v_1=v_2=v \in V(G_2)} \sum_{\substack{u_1 u_2 \in E(R_k(G_1)) \\ u_1 u_2 \in V(R_k(G_1) - V(G_1))}} [d(u_1, v_1) d(u_2, v_2)],$$

TABLE 1: Numerical comparison for generalized network $G(P_n[P_m]_{T_k})$ for $k = 1$.

$[m, n]$	$M_1(G)(P_n[P_m]_{S_k})$	$M_2(G)(P_n[P_m]_{S_k})$	$M_1(G)(P_n[P_m]_{R_k})$	$M_2(G)(P_n[P_m]_{R_k})$
(5, 5)	4460	22920	10350	80636
(5, 6)	5634	29312	13184	104952
(5, 7)	6808	35704	16018	129268
(5, 8)	7982	42096	18852	153584
(5, 9)	9156	48488	21686	177900
(5, 10)	10330	54880	24520	202216
(6, 5)	7530	45752	17562	162672
(6, 6)	9516	58560	22380	211844
(6, 7)	11502	71368	27198	261016
(6, 8)	13488	841767	32016	310188
(6, 9)	15474	96984	36834	359360
(6, 10)	17460	109792	41652	408532
(7, 5)	11744	823647	27494	294900
(7, 6)	14846	105484	35048	384216
(7, 7)	17948	128604	42602	473532
(7, 8)	21050	151724	50156	562848
(7, 9)	24152	174844	57710	652164
(7, 10)	27254	197964	65264	741480
(8, 5)	17282	137400	40578	494408
(8, 6)	21852	176048	51740	644388
(8, 7)	26422	214696	62902	794368
(8, 8)	30992	253344	74064	944348
(8, 9)	35562	291992	85226	1094328
(8, 10)	40132	330640	96388	1244308
(9, 5)	24324	216176	57246	780780
(9, 6)	30762	277080	73008	1017944
(9, 7)	37200	337984	88770	1255108
(9, 8)	43638	398888	104532	1492272
(9, 9)	50076	459792	120294	1729436
(9, 10)	56514	520696	136056	1966600
(10, 5)	33050	324680	77930	1176096
(10, 6)	41804	416272	99404	1533732
(10, 7)	50558	507864	120878	1891368
(10, 8)	59312	599456	142352	2249004
(10, 9)	68066	691048	163826	2606640
(10, 10)	76820	782640	185300	2964276

$$\sum_{v_1=v_2=v \in V(G_2)} \sum_{u_1 u_2 \in E(G_2)} [d(u_1, v_1)d(u_2, v_2)] = n_2(k-1)e_1[2n_2 \cdot 2n_2] = 4e_1n_2^3(k-1),$$

$$\begin{aligned} \sum 4 &= \sum_{v_1 \in V(G_2)} \sum_{v_2 \in V(G_2)} \sum_{u_1 u_2 \in E(R_k(G_1))} [d(u_1, v_1)d(u_2, v_2)] \\ &= [2n_2 \times 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)(k-2)e_1 + [2n_2 \times 2n_2] \sum_{i=1}^{n_2-1} (2n_2 - 2i)e_1 \\ &= 8n_2^2 \sum_{i=1}^{n_2-1} (n_2 - i)(k-2)e_1 + 8n_2^2 \sum_{i=1}^{n_2-1} (n_2 - i)e_1 = 8n_2^2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i). \quad (18) \end{aligned}$$

So, the final result is

$$\begin{aligned} &= 8n_2^2 e_2 M_1(G_1) + 4e_1 n_2 M_1(G_2) + n_1 M_2(G_2) \\ &\quad + 4n_2^4 M_2(G_1) + 4n_2^4 M_1(G_2) + 8n_2^2 e_1 e_2 + 4e_1 e_2^2 \\ &\quad + 4e_1 n_2^3(k-1) \\ &\quad + 8n_2^2 e_1(k-1) \sum_{i=1}^{n_2-1} (n_2 - i). \end{aligned} \quad (19)$$

□

4. Conclusion

Let $G_1 = P_n$ and $G_2 = P_m$ be two networks; then, we have $n_1 = |V(G_1)| = n$, $n_2 = |V(G_2)| = m$, $e_1 = |E(G_1)| = n-1$, $e_2 = |E(G_2)| = m-1$, $M_1(G_1) = 4n-6$, $M_1(G_2) = 4m-6$, $M_2(G_1) = 4(n-2)$, and $M_2(G_2) = 4(m-2)$.

Applying these values on the above derived results in Theorems 1–4, we obtain Tables 1–4. The graphical

TABLE 2: Numerical comparison for generalized networks $G(P_n[P_m]_{N_k})$ for $k = 2$.

$[m, n]$	$M_1(G)(P_n[P_m]_{S_k})$	$M_2(G)(P_n[P_m]_{S_k})$	$M_1(G)(P_n[P_m]_{R_k})$	$M_2(G)(P_n[P_m]_{R_k})$
(5, 5)	6660	31120	12350	103136
(5, 6)	8374	39572	15684	129952
(5, 7)	10088	48024	19018	156768
(5, 8)	11802	56476	22352	183584
(5, 9)	13516	64928	25686	210400
(5, 10)	15230	73380	29020	237216
(6, 5)	11226	63368	21018	209328
(6, 6)	14124	80592	26700	263684
(6, 7)	17022	97816	32382	318040
(6, 8)	19920	115040	38064	372396
(6, 9)	22818	132264	43746	426752
(6, 10)	25716	149488	49428	481108
(7, 5)	17512	115796	32982	381336
(7, 6)	22042	147288	41908	480256
(7, 7)	26572	178780	50834	579176
(7, 8)	31102	210272	59760	678096
(7, 9)	35632	241764	68686	777016
(7, 10)	40162	273256	77612	875936
(8, 5)	25794	195448	48770	641864
(8, 6)	32476	248624	61980	808228
(8, 7)	39158	301800	75190	974592
(8, 8)	45840	354976	88400	1140956
(8, 9)	52522	408152	101610	1307320
(8, 10)	59204	461328	114820	1473684
(9, 5)	36348	310424	68910	1016976
(9, 6)	45774	394908	87588	1280384
(9, 7)	55200	479392	106266	1543792
(9, 8)	64626	563876	124944	1807200
(9, 9)	74052	648360	143622	2070608
(9, 10)	83478	732844	162300	2334016
(10, 5)	49450	469880	93930	1536096
(10, 6)	62284	597792	119404	1933732
(10, 7)	75118	725704	144878	2331368
(10, 8)	87952	853616	170352	2729004
(10, 9)	100786	981528	195826	3126640
(10, 10)	113620	1109440	221300	3524276

TABLE 3: Numerical comparison for generalized networks $G(P_n[P_m]_{N_k})$ for $k = 3$.

$[m, n]$	$M_1(G)(P_n[P_m]_{S_k})$	$M_2(G)(P_n[P_m]_{S_k})$	$M_1(G)(P_n[P_m]_{R_k})$	$M_2(G)(P_n[P_m]_{R_k})$
(5, 5)	8460	39520	14750	100636
(5, 6)	10634	50072	18684	129952
(5, 7)	12808	60624	22618	159268
(5, 8)	14982	71176	26552	188584
(5, 9)	17156	81728	30486	217900
(5, 10)	19330	92280	34420	247216
(6, 5)	14442	81224	25050	204144
(6, 6)	18156	102912	31740	263684
(6, 7)	21870	124600	38430	323224
(6, 8)	25584	146288	45120	382764
(6, 9)	29298	167976	51810	442304
(6, 10)	33012	189664	58500	501844
(7, 5)	22720	149508	39254	371732
(7, 6)	28566	189428	49748	480256
(7, 7)	34412	229348	60242	588780
(7, 8)	40258	269268	70736	697304
(7, 9)	46104	309188	81230	805828
(7, 10)	51950	349108	91724	914352

TABLE 3: Continued.

$[m, n]$	$M_1(G)(P_n[P_m]_{S_k})$	$M_2(G)(P_n[P_m]_{S_k})$	$M_1(G)(P_n[P_m]_{R_k})$	$M_2(G)(P_n[P_m]_{R_k})$
(8, 5)	33666	253816	57986	625480
(8, 6)	42332	321584	73500	808228
(8, 7)	50998	389352	89014	990976
(8, 8)	59664	457120	104528	1173724
(8, 9)	68330	524888	120042	1356472
(8, 10)	76996	592656	135556	1539220
(9, 5)	47652	405032	81870	990732
(9, 6)	59922	513168	103788	1280384
(9, 7)	72192	621304	125706	1570036
(9, 8)	84462	729440	147624	1859688
(9, 9)	96732	837576	169542	2149340
(9, 10)	109002	945712	191460	2438992
(10, 5)	65050	615480	111530	1496096
(10, 6)	81804	779792	141404	1933732
(10, 7)	98558	944104	171278	2371368
(10, 8)	115312	1108416	201152	2809004
(10, 9)	132066	1272728	231026	3246640
(10, 10)	148820	1437040	260900	3684276

TABLE 4: Numerical comparison for generalized networks $G(P_n[P_m]_{N_k})$ for $k = 4$.

$[m, n]$	$M_1(G)(P_n[P_m]_{S_k})$	$M_2(G)(P_n[P_m]_{S_k})$	$M_1(G)(P_n[P_m]_{R_k})$	$M_2(G)(P_n[P_m]_{R_k})$
(5, 5)	10460	47920	16750	110636
(5, 6)	13134	60572	21184	142452
(5, 7)	15808	73224	25618	174268
(5, 8)	18482	85876	30052	206084
(5, 9)	21156	98528	34486	237900
(5, 10)	23830	111180	38920	269716
(6, 5)	17898	99080	28506	224880
(6, 6)	22476	125232	36060	289604
(6, 7)	27054	151384	43614	354328
(6, 8)	31632	177536	51168	419052
(6, 9)	36210	203688	58722	483776
(6, 10)	40788	229840	66276	548500
(7, 5)	28208	183220	44742	410148
(7, 6)	35426	231568	56608	528276
(7, 7)	42644	279916	68474	646404
(7, 8)	49862	328264	80340	764532
(7, 9)	57080	376612	92206	882660
(7, 10)	64298	424960	104072	1000788
(8, 5)	41858	312184	66178	691016
(8, 6)	52572	394544	83740	890148
(8, 7)	63286	476904	101302	1089280
(8, 8)	74000	559264	118864	1288412
(8, 9)	84714	641624	136426	1487544
(8, 10)	95428	723984	153988	1686676
(9, 5)	59316	499640	93534	1095708
(9, 6)	74502	631428	118368	1411604
(9, 7)	89688	763216	143202	1727500
(9, 8)	104874	895004	168036	2043396
(9, 9)	120060	1026792	192870	2359292
(9, 10)	135246	1158580	217704	2675188
(10, 5)	81050	761080	127530	1656096
(10, 6)	101804	961792	161404	2133732
(10, 7)	122558	1162504	195278	2611368
(10, 8)	143312	1363216	229152	3089004
(10, 9)	164066	1563928	263026	3566640
(10, 10)	184820	1764640	296900	4044276

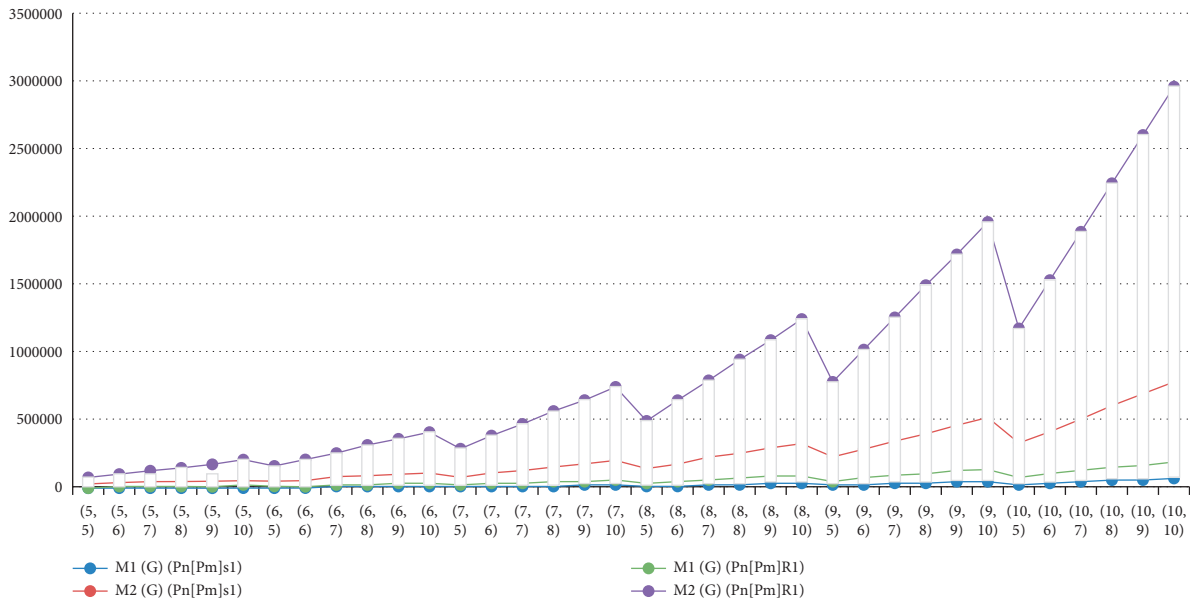


FIGURE 5: Numerical comparison of Zagreb indices of generalized subdivision and semitotal point operations with lexicographic product of the networks for $k = 1$.

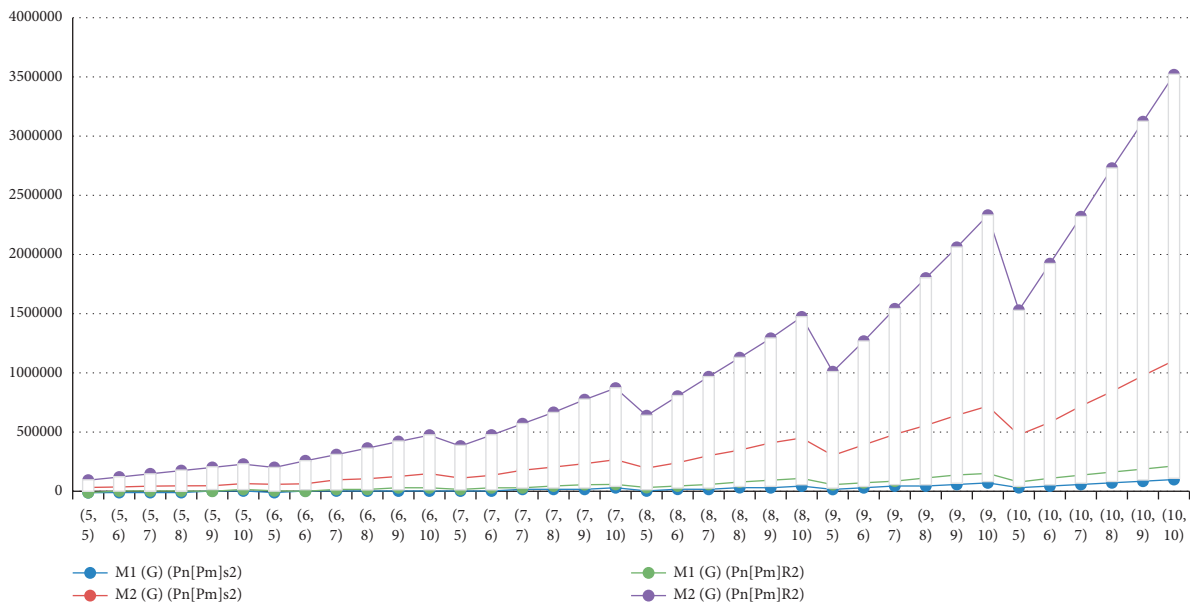


FIGURE 6: Numerical comparison of Zagreb indices of generalized subdivision and semitotal point operations with lexicographic product of the networks for $k = 2$.

representations of the abovementioned results in tables are shown in Figures 5–8. In Figure 9, we summarize the results of generalized T -sum networks from $k = 1$ to $k = 4$.

For $k = 1$,

- (a) $G_1(P_n[P_m]S) = 8m^3n - 10m^3 + 8m^2n - 8m^2 - 4mn + 8m - 6n$
- (b) $G_2(P_n[P_m]S) = 8m^4n - 12m^4 + 12m^3n - 14m^3 - 4m^2n + 6m^2 + 12m - 8n$
- (c) $G_1(P_n[P_m]R) = 20m^3n - 28m^3 + 16m^2n - 16m^2 - 12mn + 16m - 6n$

$$(d) G_2(P_n[P_m]R) = 32m^4n - 56m^4 + 40m^3n - 56m^3 - 20m^2n + 36m^2 - 36mn + 32m - 4n - 4$$

For $k = 2$,

- (a) $G_1(P_n[P_m]S) = 8m^3n - 10m^3 + 12m^2n - 12m^2 + 4mn + 8m - 6n + (8mn - 8m)\sum_{i=1}^{m-1}(m-i)$
- (b) $G_2(P_n[P_m]S) = 8m^4n - 12m^4 + 12m^3n - 14m^3 + 2m^2 - 8mn + 12m - 8n + (8m^2n - 8m^2)\sum_{i=1}^{m-1}(m-i)$
- (c) $G_1(P_n[P_m]R) = 20m^3n - 28m^3 + 20m^2n - 20m^2 - 12mn + 16m - 6n + (8mn - 8m)\sum_{i=1}^{m-1}(m-i)$

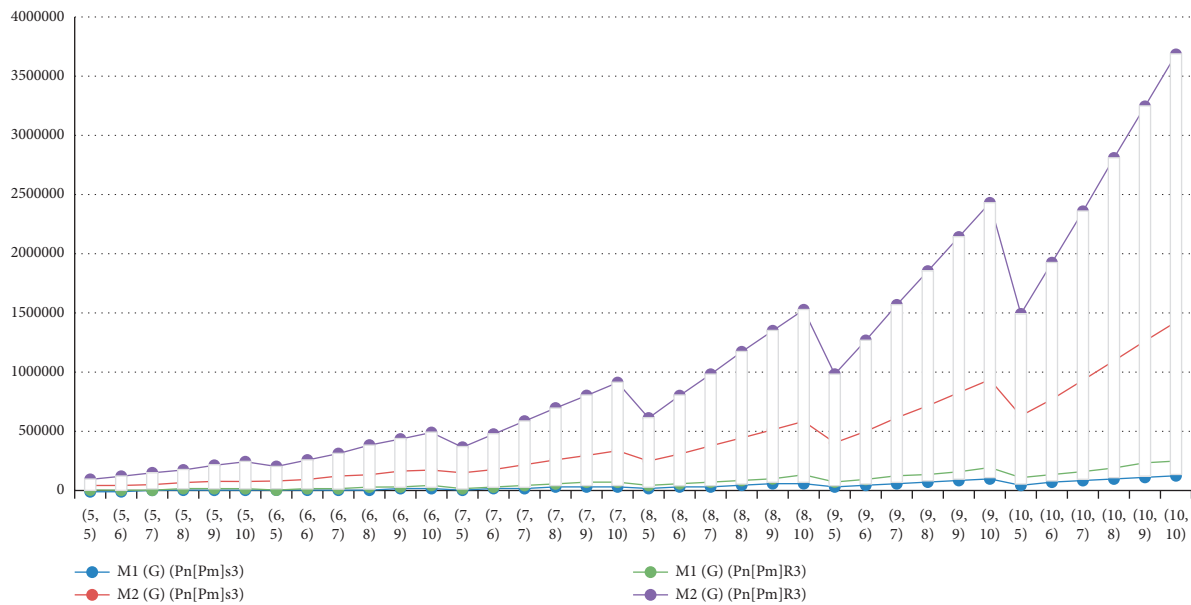


FIGURE 7: Numerical comparison of Zagreb indices of generalized subdivision and semitotal point operations with lexicographic product of the networks for $k = 3$.

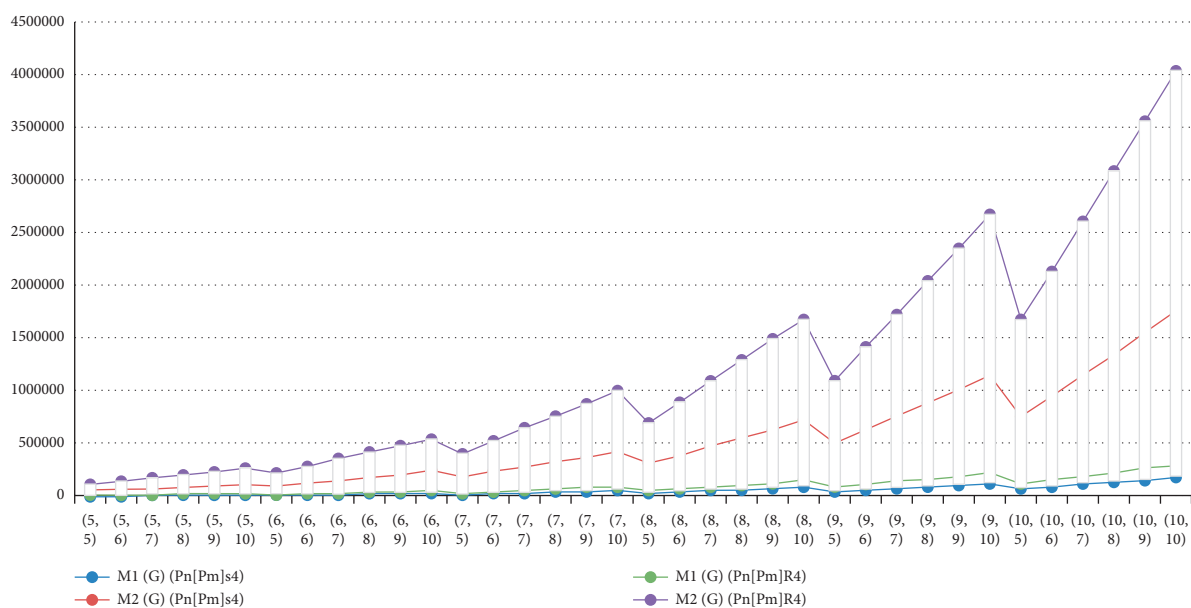


FIGURE 8: Numerical comparison of Zagreb indices of generalized subdivision and semitotal point operations with lexicographic product of the networks for $k = 4$.

$$(d) \quad G_2(P_n[P_m]R) = 32m^4n - 56m^4 + 44m^3n - 60m^3 - 20m^2n + 36m^2 - 36mn + 32m - 4n - 4 + (8m^2n - 8m^2)\sum_{i=1}^{m-1}(m-i)$$

For $k = 3$,

$$(a) \quad G_1(P_n[P_m]S) = 8m^3n - 10m^3 + 16m^2n - 16m^2 - 4mn + 8m - 6n + (16mn - 16m)\sum_{i=1}^{m-1}(m-i)$$

$$(b) \quad G_2(P_n[P_m]S) = 8m^4n - 12m^4 + 12m^3n - 14m^3 + 4m^2n - 2m^2 - 8mn + 12m - 8n + (16m^2n - 16m)\sum_{i=1}^{m-1}(m-i)$$

$$(c) \quad G_1(P_n[P_m]R) = 20m^3n - 28m^3 + 28m^2n - 28m^2 - 12mn + 16m - 6n + (16mn - 16m)\sum_{i=1}^{m-1}(m-i)$$

$$(d) \quad G_2(P_n[P_m]R) = 32m^4n - 56m^4 + 48m^3n - 64m^3 - 20m^2n + 36m^2 - 36mn + 32m - 4n - 4 + (16m^2n - 16m)\sum_{i=1}^{m-1}(m-i)$$

For $k = 4$,

$$(a) \quad G_1(P_n[P_m]S) = 8m^3n - 10m^3 + 20m^2n - 20m^2 - 4mn + 8m - 6n + (24mn - 24m)\sum_{i=1}^{m-1}(m-i)$$

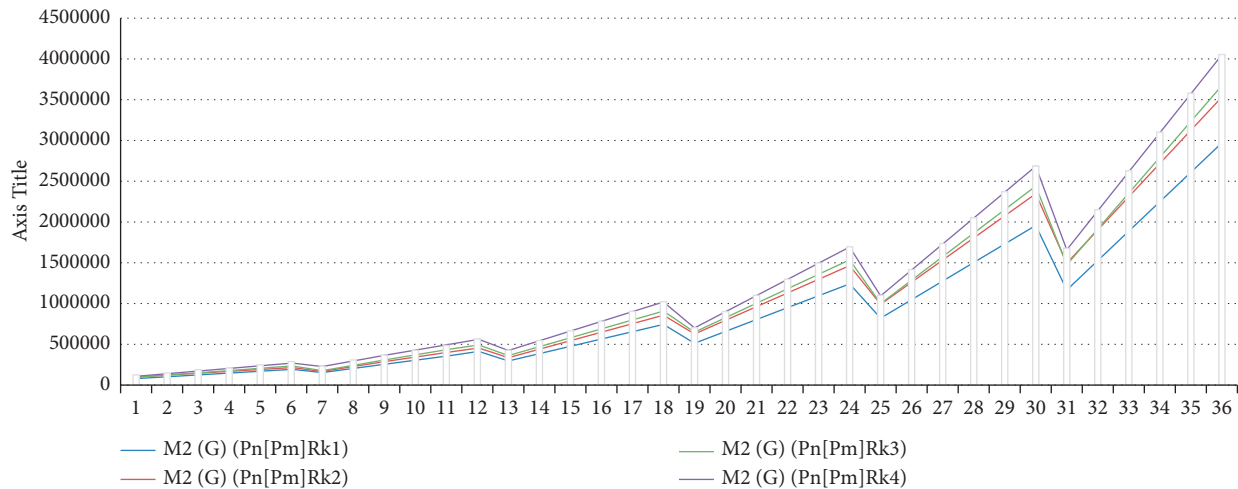


FIGURE 9: Numerical comparison of Zagreb indices of generalized subdivision and semitotal point operations with lexicographic product of the networks for $M_2(P_n[P_m]_{R_k})$, where $k \in \{1, 2, 3, 4\}$.

- (b) $G_2(P_n[P_m]S) = 8m^4n - 12m^4 + 12m^3n - 14m^3 + 8m^2n - 6m^2 - 8mn + 12m - 8n + (24m^2n - 24m^2)\text{sum}(m - i)$
- (c) $G_1(P_n[P_m]R) = 20m^3n - 28m^3 + 32m^2n - 32m^2 - 12mn + 16m - 6n + (24mn - 24m)\text{sum}(m - i)$
- (d) $G_2(P_n[P_m]R) = 32m^4n - 56m^4 + 52m^3n - 68m^3 - 20m^2n + 36m^2 - 36mn + 32m - 4n - 4 + (24m^2n - 24m^2)\text{sum}(m - i)$

In this paper, we proved the result of 1st and 2nd Zagreb indices of networks $G_1[G_2]_{T_k}$, by using generalized subdivision and generalized semitotal point operation and lexicographic product. The first and second Zagreb indices are in the form $M_i(G_1[G_2]_{T_k})$, where $(i = 1, 2)$. We plot the network of these derived exact values individually and after that plot a combine network for $k \in \{1, 2, 3, 4\}$ to compare these results. In this result, it is clearly shown that the 2nd Zagreb indices $M_2(G_1[G_2]_{T_k})$ shows better result as compared to others' exact values.

Data Availability

All the data are included within this paper. However, more details of the data can be obtained from the corresponding author upon request.

Conflicts of Interest

The authors have no conflicts of interest.

Acknowledgments

This work was supported by the Humanities and School science project of Anhui provincial Education Department: research on social crisis management in Anhui cities under the background of the data (subject no: SK2016A0233).

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Research Article

On the $(\alpha - \psi)$ -Contractive Mappings in C^* -Algebra Valued b -Metric Spaces and Fixed Point Theorems

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Received 5 June 2021; Revised 1 August 2021; Accepted 17 August 2021; Published 28 August 2021

Academic Editor: Naeem Saleem

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In this paper, for a unital C^* -algebra A , we introduce a version of α - ψ -contractive mappings in C^* -algebra valued b -metric spaces, and we prove some Banach fixed point theorems and give some examples to illustrate our results.

1. Introduction

Ma et al. [1] introduced the notion of C^* -algebra valued metric spaces, where the set of real number was replaced by the positive cone of a unital C^* -algebra. Later in [2], the class of C^* -algebra valued b -metric spaces is considered. Many results are introduced in this direction (see [3–10]). The notion of α - ψ -contractive mappings in metric spaces was introduced by Samet et al. [11]. Later in [12], Samet developed the notion of α - ψ -contractive mappings in b -metric spaces. Several results have been introduced in some related studies of α -admissible and $\alpha - \psi$ -contractive mappings and related fixed point theorems [13–22]. In this present work, we introduced a version of $\alpha - \psi$ -contractive mapping in a unital C^* -algebra valued b -metric spaces and proved some basic Banach fixed point theorems.

Some nontrivial examples are given to support our results. Suppose that A is a unital C^* -algebra with a unit I_A . Set $A_+ = \{x \in A : x = x^*\}$. An element $x \in A$ is a positive element, if $x = x^*$ and $\sigma(x) \subset \mathbb{R}^+$ is the spectrum of x . We define a partial ordering \leq on A as $x \leq y$ if $0_A \leq y - x$, where 0_A means the zero element in A , and we let A^+ denote the $\{x \in A : x \geq 0_A\}$ and $|x| = (x^*x)^{1/2}$.

Lemma 1. Suppose that A is a unital C^* -algebra with unit I_A . The following holds:

- (1) If $a \in A$, with $\|a\| < 1/2$, then $1 - a$ is invertible and $\|a(1 - a)^{-1}\| < 1$
- (2) For any $x \in A$ and $a, b \in A^+$, such that $a \leq b$, we have x^*ax and x^*bx which are positive element and $x^*ax \leq x^*bx$
- (3) If $0_A \leq a \leq b$, then $\|a\| \leq \|b\|$
- (4) If $a, b \in A^+$ and $ab = ba$, then $a.b \geq 0_A$
- (5) Let A' denote the set $\{a \in A : ab = ba \forall b \in A\}$ and let $a \in A'$, if $b, c \in A$ with $b \geq c \geq 0_A$ and $1 - a \in (A')^+$ is an invertible element, then $(I_A - a)^{-1}b \leq (I_A - a)^{-1}c$

We refer [23] for more C^* algebra details.

Definition 1. Let X be a nonempty set and $b \geq I_A$, $b \in A'$, suppose the mapping $d_A : X \times X \longrightarrow A$ satisfies the following:

- (1) $d_A(x, y) \geq 0_A$ for all $x, y \in X$ and $d_A(x, y) = 0_A \Leftrightarrow x = y$.
- (2) $d_A(x, y) = d_A(y, x)$ for all $x, y \in X$.
- (3) $d_A(x, z) \leq b[d_A(x, y) + d_A(y, z)]$ for all $x, y, z \in X$, where 0_A is zero element in A and I_A is the unit element in A . Then, d_A is called a C^* -algebra valued b -metric on X and (X, A, d_A) is called C^* -algebra valued b -metric space.

Definition 2. Let (X, A, d_A) be a C^* -algebra valued b-metric space, $x \in X$, and $\{x_n\}_{n=1}^\infty$ be a sequence in X , then

- (i) $\{x_n\}_{n=1}^\infty$ is convergent to x whenever, for every $c \in A$ with $c > 0_A$, there is a natural number $N \in \mathbb{N}$ such that

$$d_A(x_n, x) \leq c, \quad (1)$$

for all $n > N$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ as $n \rightarrow +\infty$.

- (ii) $\{x_n\}_{n=1}^\infty$ is said to be a Cauchy sequence whenever, for every $c \in A$ with $c > 0_A$, there is a natural number $N \in \mathbb{N}$ such that

$$d_A(x_n, x_m) \leq c, \quad (2)$$

for all $n, m > N$.

Lemma 2 (i) $\{x_n\}_{n=1}^\infty$ is a convergence sequence in X if for any element $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n > N$, $\|d(x_n, x)\| \leq \varepsilon$.
(ii) $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence in X , if for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\|d_A(x_n, x_m)\| \leq \varepsilon$, for all $n, m > N$. We say that (X, A, d_A) is a complete C^* -algebra valued b-metric space if every Cauchy sequence is convergent with respect to A .

Example 1. Let $X = \mathbb{R}$ and $A = M_2(\mathbb{C})$ be the set of all 2×2 matrices with entries in \mathbb{C} , and $M_2(\mathbb{C})$ is a C^* -algebra with the matrix norm. Define

$$d_A(a, b) = \begin{pmatrix} \lambda_1 |a_{11} - b_{11}|^p & 0 \\ 0 & \lambda_1 |a_{11} - b_{11}|^p \end{pmatrix}, \quad (3)$$

where $a = (a_{ij})_{i,j=1}^2$ and $b = (b_{ij})_{i,j=1}^2$ are two 2×2 -matrices, $a_{ij}, b_{ij} \in \mathbb{C}$, for all $i, j = 1, 2$, $\lambda_1, \lambda_2 > 0$.

One can define a partial ordering (\leq) on $M_2(\mathbb{C})$ as following $a \leq b$ if and only if $|a_{ij}| \leq |b_{ij}| \forall i, j = 1, 2$.

And an element $a \geq 0$ is positive in $M_2(\mathbb{C})$ if and only if $|a_{ij}| \geq 0$ for all $i, j = 1, 2$, we denote $M_2(\mathbb{C})^+$ the set of all positive element in $M_2(\mathbb{C})$. Then, $(X, M_2(\mathbb{C}), d_{M_2(\mathbb{C})})$ is C^* -algebra valued b-metric space.

Definition 4. If $\psi: A \rightarrow B$ is a linear mapping in C^* -algebra, it is said to be positive if $\psi(A^+) \subseteq B^+$. In this case, $\psi(A_h) \subseteq B_h$, and the restriction map: $\psi: A_h \rightarrow B_h$ is increasing.

Definition 5. Suppose that A and B are C^* -algebras. A mapping $\psi: A \rightarrow B$ is said to be C^* -homomorphism if

- (a) $\psi(ax + by) = a\psi(x) + b\psi(y)$ for all $a, b \in \mathbb{C}$ and $x, y \in A$
- (b) $\psi(xy) = \psi(x)\psi(y) \forall x, y \in A$
- (c) $\psi(x^*) = \psi(x)^* \forall x \in A$
- (d) ψ maps the unit in A to the unit in B

Definition 6. Let Ψ_A be the set of positive functions $\psi_A: A^+ \rightarrow A^+$ satisfying the following conditions:

- (a) $\psi_A(a)$ is continuous and nondecreasing
- (b) $\psi_A(a) = 0$ iff $a = 0$
- (c) $\sum_{n=1}^\infty \psi_A^n(a) < \infty$, $\lim_{n \rightarrow \infty} \psi_A^n(a) = 0$ for each $a > 0$, where ψ_A^n is n th iterate of ψ_A
- (d) The series $\sum_{k=0}^\infty b^k \psi_A^k(a) < \infty$ for $a > 0$ is increasing and continuous at 0

Corollary 1. Every C^* -homomorphism is contractive and hence bounded.

Lemma 3. Every $*$ -homomorphism is positive.

2. Main Results

In [11] Samet et al. and in [12] Samet introduced the concept of α - ψ -contractive mappings in metric space and α - ψ -contractive mappings in b-metric space, respectively. Here, we will develop the definitions in case of unital C^* -algebra and study some Banach fixed point theorems.

Definition 7 (see [11]). Let $T: X \rightarrow X$ be self map and $\alpha: X \times X \rightarrow [0, +\infty)$. Then, T is called α -admissible if for all $x, y \in X$ with $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Definition 8. Let X be a nonempty set and $\alpha_A: X \times X \rightarrow (A^+)^+$ be a function, we say that the self map T is α_A -admissible if $(x, y) \in X \times X$, $\alpha_A(x, y) \geq I_A \Rightarrow \alpha_A(Tx, Ty) \geq I_A$, where I_A the unit of A .

Definition 9. Let (X, A, d_A) be a C^* -algebra valued b-metric space and $T: X \rightarrow X$ is mapping, we say that T is an α_A - ψ_A -contractive mapping if there exist two functions $\alpha_A: X \times X \rightarrow A^+$ and $\psi_A \in \Psi_A$ such that

$$\alpha_A(x, y) d_A(Tx, Ty) \leq \psi_A(d_A(x, y)), \quad (4)$$

for all $x, y \in X$.

Theorem 1 (Banach version fixed point). Let (X, A, d_A) be a complete C^* -algebra valued b-metric space and $T: X \rightarrow X$ be an α_A - ψ_A -contractive mapping satisfying the following conditions:

- (i) T is α_A -admissible
- (ii) There exists $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \geq I_A$
- (iii) T is continuous

Then, T has a fixed point in X .

Proof. Let $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \geq I_A$ and define a sequence $\{x_n\}_{n=0}^\infty$ in X such that $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point for T .

Suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, since T is α_A -admissible, we get

$$\begin{aligned}\alpha_A(x_0, x_1) &= \alpha_A(x_0, Tx_0) \geq I_A \Rightarrow \\ \alpha_A(Tx_0, T^2x_0) &= \alpha_A(x_1, x_2) \geq I_A.\end{aligned}\quad (5)$$

By induction, we have

$$\alpha_A(x_n, x_{n+1}) \geq I_A, \quad \text{for all } n \in \mathbb{N}. \quad (6)$$

By inequalities (4) and (6), we get

$$\begin{aligned}d_A(x_n, x_{n+1}) &= d_A(Tx_{n-1}, Tx_n), \\ &\leq \alpha_A(x_{n-1}, x_n) d_A(Tx_{n-1}, Tx_n) \\ &\leq \psi_A(d_A(x_{n-1}, x_n)).\end{aligned}\quad (7)$$

By induction, we obtain

$$d_A(x_n, x_{n+1}) \leq \psi_A^n(d_A(x_0, x_1)), \quad \text{for all } n \in \mathbb{N}. \quad (8)$$

For $m \geq 1$ and $p \geq 1$, it follows that

$$\begin{aligned}d_A(x_m, x_{m+p}) &\leq b[d_A(x_m, x_{m+1}) + d_A(x_{m+1}, x_{m+p})] \\ &\leq b d_A(x_m, x_{m+1}) + b^2 d_A(x_{m+1}, x_{m+2}) + \cdots + b^{p-1} d_A(x_{m+p-2}, x_{m+p-1}) + b^p d_A(x_{m+p-1}, x_{m+p}) \\ &\leq b \psi_A^m(d_A(x_0, x_1)) + b^2 \psi_A^{m+1}(d_A(x_0, x_1)) + b^{p-1} \psi_A^{m+p-2}(d_A(x_0, x_1)) + b^p \psi_A^{m+p-1}(d_A(x_0, x_1)) \\ &= \sum_{k=1}^{p-1} b^k \psi_A^{m+k-1}(d_A(x_0, x_1)) + b^p \psi_A^{m+p-1}(d_A(x_0, x_1)).\end{aligned}\quad (9)$$

Since $b \geq I_A$, using Definition 6, we obtain

$$d_A(x_m, x_{m+p}) \leq \sum_{k=1}^{p-1} b^k \psi_A^{m+k-1}(d_A(x_0, x_1)) + b^p \psi_A^{m+p-1}(d_A(x_0, x_1)) \longrightarrow 0_A, \quad \text{as } n \longrightarrow +\infty. \quad (10)$$

Thus, $\{x_n\}_{n=0}^\infty$ is a Cauchy sequence in X . Since (X, A, d_A) is complete, there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow +\infty$, from continuity of T it follows that $x_{n+1} = Tx_n \longrightarrow Tx$ as $n \longrightarrow +\infty$. And by uniqueness of the limit, we get $Tx = x$, that is, x is a fixed point of T . To prove the uniqueness of the fixed point, we will consider the following condition. (H_A) : for all $x, y \in X$, there exists $z \in X$ such that $\alpha_A(x, z) \geq I_A$ and $\alpha_A(y, z) \geq I_A$.

Theorem 2. Adding condition (H_A) to the hypothesis of Theorem 1, we obtain the uniqueness of the fixed point of T .

Proof. Suppose that x and y are two fixed points of T . From (H_A) , there exists $z \in X$ such that

$$\begin{aligned}\alpha_A(x, z) &\geq I_A, \\ \alpha_A(y, z) &\geq I_A.\end{aligned}\quad (11)$$

Since T is α_A -admissible, we get

$$\begin{aligned}\alpha_A(x, T^n x) &\geq I_A, \\ \alpha_A(y, T^n y) &\geq I_A, \\ &\text{for all } n \in \mathbb{N}.\end{aligned}\quad (12)$$

Using (4) and (12), we obtain

$$\begin{aligned}d_A(x, T^n z) &= d_A(Tx, T(T^{n-1}z)), \\ &\leq \alpha_A(x, T^{n-1}z) d_A(Tx, T(T^{n-1}z)) \\ &\leq \psi_A^n(d_A(x, z)), \quad \text{for all } n \in \mathbb{N} \longrightarrow 0_A \text{ as } n \longrightarrow +\infty.\end{aligned}\quad (13)$$

Thus, $T^n z = x$. Similarly $T^n z = y$ as $n \longrightarrow +\infty$. So, the uniqueness of the limit gives $x = y$. This completes the proof.

Theorem 3 (Kannan version fixed point). Let (X, A, d_A) be a complete C^* -algebra valued b -metric space and $T: X \longrightarrow X$ be a mapping satisfying

$$\alpha_A(x, y) d_A(Tx, Ty) \leq \psi_A(d_A(Tx, x) + d_A(Ty, y)), \quad (14)$$

for $x, y \in X$, where

$$\alpha_A: X \times X \longrightarrow A^+, \quad \text{and } \psi_A \in \Psi_A, \quad (15)$$

and the following conditions holds:

- (i) T is α_A -admissible
- (ii) There exists $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \geq I_A$
- (iii) T is continuous

Then, T has a fixed point in X .

Proof. Following the proof of Theorem 1, we get

$$\alpha_A(x_n, x_{n+1}) \geq I_A, \quad (16)$$

for all $n \in \mathbb{N}$.

By inequalities (14) and (16), we obtain

$$\begin{aligned} d_A(x_n, x_{n+1}) &= d_A(Tx_{n-1}, Tx_n), \\ &\leq \alpha_A(x_{n-1}, x_n) d_A(Tx_{n-1}, Tx_n) \\ &\leq \psi_A(d_A(Tx_{n-1}, x_{n-1}) + d_A(Tx_n, x_n)) \\ &= \psi_A(d_A(x_n, x_{n-1}) + d_A(x_{n+1}, x_n)) \\ &= \psi_A(d_A(x_n, x_{n-1}) + \psi_A d_A(x_n, x_{n+1})) \\ (1 - \psi_A) d_A(x_n, x_{n+1}) &\leq \psi_A(d_A(x_n, x_{n-1})) \\ d_A(x_n, x_{n+1}) &\leq \psi_A(1 - \psi_A)^{-1} (d_A(x_n, x_{n-1})), \end{aligned} \quad (17)$$

from Lemma 1 and Definition 6, and let $\varphi = \psi_A$
 $(1 - \psi_A)^{-1} = \varphi_A \sum_{n=0}^{\infty} \psi_A^n = \sum_{n=0}^{\infty} \psi_A^n < \infty$.

So, we get $d_A(x_n, x_{n+1}) \leq \varphi_A(d_A(x_n, x_{n+1}))$.

By induction, we obtain

$$d_A(x_n, x_{n+1}) \leq \varphi_A^n(d_A(x_0, x_1)), \quad \text{for all } n \in \mathbb{N}. \quad (18)$$

For $m \geq 1$ and $p \geq 1$, it follows by similar calculation in Theorem 1 that

$$d_A(x_m, x_{m+p}) = \sum_{k=1}^{p-1} b^k \varphi_A^{m+k-1}(d_A(x_0, x_1)) + b^p \varphi_A^{m+p-1}(d_A(x_0, x_1)). \quad (19)$$

Since $b \geq I_A$, using Definition 6, we obtain

$$d_A(x_m, x_{m+p}) \leq \sum_{k=1}^{p-1} b^k \varphi_A^{m+k-1}(d_A(x_0, x_1)) + b^p \varphi_A^{m+p-1}(d_A(x_0, x_1)) \longrightarrow 0_A, \quad \text{as } n \longrightarrow +\infty. \quad (20)$$

Thus, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in X . Since (X, A, d_A) is complete, there exists $x \in X$ such that $x_n \longrightarrow x$ as $n \longrightarrow +\infty$, and from continuity of T , it follows that $x_{n+1} = Tx_n \longrightarrow Tx$ as $n \longrightarrow +\infty$.

And by uniqueness of the limit, we get $Tx = x$; that is, x is a fixed point of T .

Now, if $y (\neq x)$ is another fixed point of T , then

$$\begin{aligned} 0_A &\leq d_A(x, y) = d_A(Tx, Ty), \\ &\leq \alpha_A(x, y) d_A(Tx, Ty) \\ &\leq \psi_A(d_A(Tx, x) + d_A(Ty, y)) \\ &= \psi_A(d_A(x, x) + d_A(y, y)) \\ &= \psi_A(0) = 0_A. \end{aligned} \quad (21)$$

This implies that $d_A(x, y) = 0_A$. That is, $x = y$, and this complete the proof.

Theorem 4 (Banach–Kannan version fixed point). *Let (X, A, d_A) be a complete C^* -algebra valued b -metric space and $T: X \longrightarrow X$ be a mapping satisfying*

$$\alpha_A(x, y) d_A(Tx, Ty) \leq \psi_A(d_A(x, y) + d_A(Tx, x) + d_A(Ty, y)), \quad (22)$$

for $x, y \in X$, where

$$\alpha_A: X \times X \longrightarrow A^+, \quad \text{and } \psi_A \in \Psi_A, \quad (23)$$

such that $\psi_A(1 - \psi_A)^{-1} \leq 1/2I_A$, and the following conditions hold:

- (i) T is α_A -admissible
- (ii) There exists $x_0 \in X$ such that $\alpha_A(x_0, Tx_0) \geq I_A$
- (iii) T is continuous

Then, T has a fixed point in X .

Proof. Following the proof of Theorem 1, we get

$$\alpha_A(x_n, x_{n+1}) \geq I_A, \quad \text{for all } n \in \mathbb{N}. \quad (24)$$

By using inequalities (22) and (24), we have

$$\begin{aligned} d_A(x_n, x_{n+1}) &= d_A(Tx_{n-1}, Tx_n), \\ &\leq \alpha_A(x_{n-1}, x_n) d_A(Tx_{n-1}, Tx_n) \\ &\leq \psi_A(d_A(x_{n-1}, x_n) + d_A(Tx_{n-1}, x_{n-1}) \\ &\quad + d_A(Tx_n, x_n)) \\ &= \psi_A(d_A(x_n, x_{n-1}) 2I_A + d_A(x_n, x_{n+1})), \end{aligned} \quad (25)$$

Since ψ_A is additive, we get

$$\begin{aligned}
(1 - \psi_A)d_A(x_n, x_{n+1}) &\leq 2I_A\psi_A(d_A(x_n, x_{n-1})) \\
d_A(x_n, x_{n+1}) &\leq 2I_A(1 - \psi_A)^{-1}\psi_A(d_A(x_n, x_{n-1})),
\end{aligned} \tag{26}$$

and putting $(1 - \psi_A)^{-1}\psi_A = \varphi_A(I_A/2)$, we get

$$d_A(x_n, x_{n+1}) \leq \varphi_A^n d_A(x_0, x_1), \tag{27}$$

for all $n \in \mathbb{N}$; for $m \geq 1$ and $p \geq 1$ and by similar calculation as in the proof of Theorem 1, we get

$$\begin{aligned}
d_A(x_m, x_{m+p}) &\leq \sum_{k=1}^{p-1} b^n \varphi_A^{m+k+1}(d_A(x_0, x_1)) \\
&\quad + b^p \varphi_A^{m+p-1}(d_A(x_0, x_1)) \longrightarrow 0_A,
\end{aligned} \tag{28}$$

as $n \longrightarrow +\infty$. This x is a fixed point of T .

To prove the uniqueness part of the fixed point of T , if $y (\neq x)$ is another fixed point of T , we have

$$\begin{aligned}
0_A &\leq d_A(x, y) = d_A(Tx, Ty), \\
&\leq \alpha_A(x, y)d_A(Tx, Ty) \\
&\leq \psi_A(d_A(x, y) + d_A(Tx, x) + d_A(Ty, y)) \\
&= \psi_A(d_A(x, y) + d_A(x, x) + d_A(y, y)) \\
&\leq \psi_A(d_A(x, y)) \\
&\leq d_A(x, y).
\end{aligned} \tag{29}$$

This is a contraction, so $d_A(x, y) = 0_A$, and this gives $x = y$. This completes the proof.

Example 2. Let $X = \mathbb{R}$ and $A = M_2(\mathbb{C})$ as given in Example 1, define $T: X \longrightarrow X$, by $Tx = x/2$, and $\alpha_{M_2(\mathbb{C})}: X \times X \longrightarrow M_2(\mathbb{C})^+$ and $\alpha_{M_2(\mathbb{C})}(x, y) = I_{M_2(\mathbb{C})}$, so $\alpha_{M_2(\mathbb{C})}(Tx, Ty) = \alpha_{M_2(\mathbb{C})}(x/2, y/2) = I_{M_2(\mathbb{C})}$; thus, T is $\alpha_{M_2(\mathbb{C})}$ -admissible, where $M_2(\mathbb{C})^+$ is the set of all positive elements in $M_2(\mathbb{C})$. Define $\psi_{M_2(\mathbb{C})}: M_2(\mathbb{C})^+ \longrightarrow M_2(\mathbb{C})^+$, $\psi_{M_2(\mathbb{C})}(a) = a/2$. This is clear that $\alpha_{M_2(\mathbb{C})} - \psi_{M_2(\mathbb{C})}$ -contractive mapping and satisfies $\alpha_{M_2(\mathbb{C})}(x, y) \cdot (d_{M_2(\mathbb{C})}(Tx, Ty)) \leq \psi_{M_2(\mathbb{C})}(d_{M_2(\mathbb{C})}(x, y))$ for all $x, y \in X$.

3. Applications

In this section, we shall apply Theorem 1 to prove the existence and uniqueness of solution an integral equation in C^* -algebra.

Example 3. Let E be a compact Hausdorff space, we denote by $C(E)$ the algebra of all complex-valued continuous functions on E with pointwise addition and multiplication. The algebra $C(E)$ with the involution defined by $f^*(t) = \overline{f(t)}$ for each $f \in C(E)$, $t \in E$ and with the norm $\|f\|_\infty = \sup\{|f(t)|, t \in E\}$ is a commutative C^* -algebra, with unit $I_{C(E)}$ is the constant function. Let $C^+(E) = \{f \in C(E): f(t) = f(t), f(t) \geq 0\}$ denote the positive Cone of $C(E)$, with partial order relation $f \leq g$ if and only if $f(t) \leq g(t)$. Put $d_{C(E)}: C(E) \times C(E) \longrightarrow C(E)$ as $d_{C(E)}(f, g) = \sup_{t \in E}\{|f(t) - g(t)|^p\} \cdot I_{C(E)}$. It is clear that

$(C(E), C(E), d_{C(E)})$ is a complete C^* -algebra valued b-metric space.

Theorem 5 (Application). Consider the integral equation

$$x(t) = \int_E F(t, x(s))ds + h(t), \tag{30}$$

where E is the compact topological Hausdorff space. Suppose

(1) $F: E \times \mathbb{R} \longrightarrow \mathbb{R}$.

(2) There exists a continuous function $\phi: E \times E \longrightarrow \mathbb{R}$ and $k \in (0, 1)$ such that

$$|F(t, f(s)) - F(t, g(s))| \leq k|\phi(t, s)||f(s) - g(s)|, \tag{31}$$

for all $f, g \in C(E)$, $t, s \in E$.

(3) $\sup_{t \in E} \int_E |\phi(t, s)|ds \leq 1$, then the integral equation (30) has a unique solution $x^* = \bar{x} \in C(E)$.

Proof. Let $X = C(E)$ and $A = C(E)$, $d_{C(E)}$ as in the Example 3, $(C(E), C(E), d_{C(E)})$ is a complete C^* -algebra valued b-metric space, and let $T: C(E) \longrightarrow C(E)$ given by $Tx(t) = \int_E F(t, x(s))ds + h(t)$, $x, h \in C(E)$, $t, s \in E$. $\alpha_{C(E)}: C(E) \times C(E) \longrightarrow C^+(E)$ defined by $\alpha_{C(E)}(f, g) = (f, g) \cdot I_{C(E)}$. And $\psi_{C(E)}: C^+(E) \longrightarrow C^+(E)$ defined by $\psi_{C(E)}(f) = f$.

Now,

$$\begin{aligned}
d_{C(E)}(Tf, Tg) &= \sup_{t \in E} \{|Tf(t) - Tg(t)|^p\} \cdot I_{C(E)}, \\
&= \sup_{t \in E} \left| \int_E |F(t, f(s)) - F(t, g(s))|ds \right|^p \cdot I_{C(E)} dt \\
&\leq \sup_{t \in E} \int_E |\phi(t, s)|^p |f(s) - g(s)|^p \cdot I_{C(E)} ds \\
&\leq k^p d_{C(E)}(f, g).
\end{aligned} \tag{32}$$

Put $A = k^p$ since $k \in (0, 1)$, this gives $\|A\| \leq 1$, and we get

$$\begin{aligned}
\alpha_{C(E)}(d_{C(E)}(Tf, Tg)) &\leq \|A\| \psi_{C(E)}(d_{C(E)}(f, g)) \\
&\leq \psi_{C(E)}(d_{C(E)}(f, g)),
\end{aligned} \tag{33}$$

for all $f, g \in C(E)$.

Thus, T is an $\alpha_{C(E)} - \psi_{C(E)}$ -contractive mapping and satisfies Theorem 1. So, T has a unique fixed point, and the integral equation (30) has a unique solution $x^* = \bar{x} \in C(E)$.

4. Conclusions

In this paper, we define a new version of $\alpha_A - \psi_A$ -admissible in the case of self mappings $T: A \longrightarrow A$. We prove the principal Banach fixed point theorem, Kannan fixed point theorem, and Banach-Kannan fixed point theorem in the C^* -algebra valued b-metric space, which generalized the given results in [1, 2, 11, 12, 24].

Data Availability

No data were used to support the results.

Conflicts of Interest

The authors of this research declare that they have no conflicts of interest.

Acknowledgments

The first author thanks the South Valley University, Egypt, for partially supporting the study. The second author thanks the Jazan University, Saudi Arabia, for supporting the study.

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Research Article

Nonzero Solutions for Nonlinear Systems of Fourth-Order Boundary Value Problems

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Received 9 July 2021; Accepted 10 August 2021; Published 18 August 2021

Academic Editor: Huseyin Isik

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This study is devoted to the investigation of nonlinear systems of fourth-order boundary value problems. Namely, using some techniques from matrix analysis and ordinary differential equations, a Lyapunov-type inequality providing a necessary condition for the existence of nonzero solutions is obtained. Next, an estimate involving generalized eigenvalues is derived as an application of our main result.

1. Introduction

In this study, we investigate the system of differential equations

$$\begin{cases} y^{(4)}(t) = \rho(t)\mu(t, y, z), & 0 < t < 1, \\ z^{(4)}(t) = \sigma(t)\xi(t, y, z), & 0 < t < 1, \end{cases} \quad (1)$$

subjected to the boundary conditions

$$\begin{cases} y(0) = y'(0) = y''(1) = y^{(3)}(1) = 0, \\ z(0) = z'(0) = z''(1) = z^{(3)}(1) = 0, \end{cases} \quad (2)$$

where $\rho, \sigma: [0, 1] \rightarrow \mathbb{R}$ and $\mu, \xi: [0, 1] \times C([0, 1]) \times C([0, 1]) \rightarrow \mathbb{R}$ are the continuous functions with $\mu(\cdot, 0, 0) = \xi(\cdot, 0, 0) \equiv 0$. Clearly, $(y, z) \equiv (0, 0)$ is a trivial solution to (1) and (2). The aim of this study is to obtain necessary conditions for the existence of nonzero solutions to the considered problem. Namely, we establish new Lyapunov-type inequalities [1] for (1) and (2) under reasonable conditions on the nonlinearities $\mu(t, y, z)$ and $\xi(t, y, z)$. Our approach is based essentially on matrix analysis and some arguments from ordinary differential equations.

Fourth-order differential equations are useful in modeling many phenomena from physics ([2–7] and the references therein), which makes the study of such equations particularly interesting. In the literature, several contributions have been devoted to the investigation of sufficient conditions ensuring the existence of solutions to fourth-order boundary value problems ([2, 8–19] and the references therein). The study of necessary conditions for the existence of nontrivial solutions to fourth-order differential equations via Lyapunov-type inequalities has been investigated by some authors [20–22]. For instance, in [20], among other results, it was shown that, if y is a nontrivial solution to

$$\begin{cases} y^{(4)}(t) + \lambda(t)y(t) = 0, & a < t < b, \\ y(a) = y'(a) = y''(a) = y'''(b) = 0, \end{cases} \quad (3)$$

where $\lambda: [a, b] \rightarrow \mathbb{R}$ is continuous, then

$$\int_a^b \lambda^+(s) ds > \frac{512}{9(b-a)^3}, \quad (4)$$

where $\lambda^+(t) = \max\{\lambda(t), 0\}$.

For recent contributions related to Lyapunov-type inequalities, see e.g., [23–29] and the references therein.

This study is organized as follows. The next section is devoted to some preliminaries. In Section 3, the obtained results as well as their proofs are presented. Finally, some applications to generalized eigenvalue problems are given in Section 4.

2. Some Preliminaries

First, we fix some notations. We denote by $\leq_{\mathbb{R}^2}$ the partial order in the Euclidean space \mathbb{R}^2 defined as

$$\vec{u} \leq_{\mathbb{R}^2} \vec{v} \Leftrightarrow u_i \leq v_i, \quad i = 1, 2, \quad (5)$$

for every $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$.

We denote by M_2^+ the set of square matrices having nonnegative coefficients, i.e.,

$$M_2^+ = \left\{ (m_{ij})_{1 \leq i, j \leq 2} : m_{ij} \geq 0, \quad 1 \leq i, j \leq 2 \right\}. \quad (6)$$

For $C \in M_2^+$, the trace of C is denoted by $\text{Trace}(C)$, the determinant of C is denoted by $\det(C)$, and the spectral radius of C is denoted by ρ_C , i.e.,

$$\rho_C = \max\{|\lambda_i(C)| : i = 1, 2\}, \quad (7)$$

where $\lambda_i(C)$ are the complex eigenvalues of C .

We equip M_2^+ with the norm $\|\cdot\|$ defined as

$$\|C\| = \sup_{\vec{u} \in \mathbb{R}^2, \|\vec{u}\|_2 \neq 0} \frac{\|C\vec{u}\|_2}{\|\vec{u}\|_2}, \quad C \in M_2^+, \quad (8)$$

where $\|\cdot\|_2$ is the Euclidean norm in \mathbb{R}^2 .

The following lemmas will be useful later.

Lemma 1. Let $\vec{u}, \vec{v} \in \mathbb{R}^2$ with

$$\vec{0} \leq_{\mathbb{R}^2} \vec{u} \leq_{\mathbb{R}^2} \vec{v}, \quad (9)$$

where $\vec{0}$ is the zero vector in \mathbb{R}^2 . Then, $\|\vec{u}\|_2 \leq \|\vec{v}\|_2$.

Proof. The result follows from the fact that

$$P = \left\{ \vec{u} \in \mathbb{R}^2 : \vec{u} \geq_{\mathbb{R}^2} \vec{0} \right\} \quad (10)$$

is a normal cone in \mathbb{R}^2 with normal constant equal to 1 (e.g., [30]). \square

Lemma 2 (See e.g., [31]). Let $C \in M_2^+$. If $\rho_C < 1$, then

$$\lim_{n \rightarrow \infty} \|C\|^n = 0. \quad (11)$$

Lemma 3 (See [25]). Let $C \in M_2^+$. Then,

$$\rho_C = \frac{\text{Trace}(C) + \sqrt{[\text{Trace}(C)]^2 - 4\det(C)}}{2}. \quad (12)$$

Lemma 4 (See [12]). Let $x \in C^4((0, 1)) \cap C^1([0, 1]) \cap C^3((0, 1))$ be a solution to

$$\begin{cases} x^{(4)}(t) = f(t), & 0 < t < 1, \\ x(0) = x'(0) = x''(1) = x^{(3)}(1) = 0, \end{cases} \quad (13)$$

where $f \in C([0, 1])$. Then,

$$x(t) = \int_0^1 H(t, \tau) f(\tau) d\tau, \quad 0 \leq t \leq 1, \quad (14)$$

where

$$0 \leq H(t, \tau) = \begin{cases} \frac{(3t - \tau)\tau^2}{6}, & \text{if } 0 \leq \tau \leq t \leq 1, \\ \frac{(3\tau - t)t^2}{6}, & \text{if } 0 \leq t \leq \tau \leq 1. \end{cases} \quad (15)$$

Lemma 5. For all $0 < \tau < 1$,

$$\max_{0 \leq t \leq 1} H(t, \tau) = H(1, \tau) = \frac{(3 - \tau)\tau^2}{6}. \quad (16)$$

Proof. Fix $0 < \tau < 1$. Since $H(\cdot, \tau)$ is nondecreasing in $[\tau, 1]$, we deduce that

$$H(t, \tau) \leq H(1, \tau), \quad \text{for all } t \in [\tau, 1]. \quad (17)$$

Similarly, it can be easily shown that $H(\cdot, \tau)$ is nondecreasing in $[0, \tau]$, which yields

$$H(t, \tau) \leq H(\tau, \tau) = \frac{\tau^3}{3}, \quad \text{for all } t \in [0, \tau]. \quad (18)$$

Combining (17) with (18), for all $0 \leq t \leq 1$, we obtain

$$H(t, \tau) \leq \max\left\{\frac{(3 - \tau)\tau^2}{6}, \frac{\tau^3}{3}\right\} = \frac{(3 - \tau)\tau^2}{6}. \quad (19)$$

Hence, (16) is proved. \square

Throughout this study, we denote by $\|\cdot\|_\infty$ the norm in $C([0, 1])$ defined as

$$\|\vartheta\|_\infty = \max_{0 \leq t \leq 1} |\vartheta(t)|, \quad \vartheta \in C([0, 1]). \quad (20)$$

3. Results and Proofs

We investigate (1) and (2) under the following assumptions:

(A1) $\rho, \sigma: [0, 1] \rightarrow \mathbb{R}$ are continuous

(A2) $\mu, \xi: [0, 1] \times C([0, 1]) \times C([0, 1]) \rightarrow \mathbb{R}$ are continuous

(A3) $\mu(\cdot, 0, 0) = \xi(\cdot, 0, 0) \equiv 0$ (where 0 is the zero function)

(A4) For all $(t, y, z) \in [0, 1] \times C([0, 1]) \times C([0, 1])$,

$$\begin{aligned} |\mu(t, y, z)| &\leq \mu_{11}(t)\|y\|_\infty + \mu_{12}(t)\|z\|_\infty, \\ |\xi(t, y, z)| &\leq \xi_{21}(t)\|y\|_\infty + \xi_{22}(t)\|z\|_\infty, \end{aligned} \quad (21)$$

where $\mu_{11}, \mu_{12}, \xi_{21}, \xi_{22}: [0, 1] \rightarrow [0, \infty)$ are the continuous functions.

By solution to (1) and (2), we mean a pair of functions (y, z) , $y, z \in C^4((0, 1)) \cap C^1([0, 1]) \cap C^3((0, 1))$, satisfying (1) and the initial conditions (2). A solution (y, z) to (1) and (2) is said to be nontrivial, if $(y, z) \equiv (0, 0)$.

For $\delta, \eta \in C([0, 1])$, let

$$J_\delta(\eta) = \int_0^1 (3 - \tau)\tau^2 |\delta(\tau)| \eta(\tau) d\tau. \quad (22)$$

Theorem 1. *If (1) and (2) have a nontrivial solution, then*

$$J_\rho(\mu_{11}) + J_\sigma(\xi_{22}) + \sqrt{(J_\rho(\mu_{11}) - J_\sigma(\xi_{22}))^2 + 4J_\sigma(\xi_{21})J_\rho(\mu_{12})} \geq 12. \quad (23)$$

Proof. Let (y, z) be a nontrivial solution to (1) and (2), and suppose that

$$J_\rho(\mu_{11}) + J_\sigma(\xi_{22}) + \sqrt{(J_\rho(\mu_{11}) - J_\sigma(\xi_{22}))^2 + 4J_\sigma(\xi_{21})J_\rho(\mu_{12})} < 12. \quad (24)$$

By Lemma 4, $(y, z) \in C([0, 1]) \times C([0, 1])$ is a nontrivial solution to the system of integral equations:

$$\begin{cases} y(t) = \int_0^1 H(t, \tau) \rho(\tau) \mu(\tau, y, z) d\tau \\ z(t) = \int_0^1 H(t, \tau) \sigma(\tau) \xi(\tau, y, z) d\tau \end{cases}, \quad 0 \leq t \leq 1. \quad (25)$$

Using (A4) and (16), for all $0 \leq t \leq 1$, we obtain

$$\begin{aligned} |y(t)| &\leq \int_0^1 |H(t, \tau)| \rho(\tau) |\mu(\tau, y, z)| d\tau \\ &\leq \int_0^1 \frac{(3 - \tau)\tau^2}{6} |\rho(\tau)| (\mu_{11}(\tau) \|y\|_\infty + \mu_{12}(\tau) \|z\|_\infty) d\tau \\ &= \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\rho(\tau)| \mu_{11}(\tau) d\tau \right) \|y\|_\infty + \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\rho(\tau)| \mu_{12}(\tau) d\tau \right) \|z\|_\infty, \end{aligned} \quad (26)$$

which leads to

$$\begin{aligned} \|y\|_\infty &\leq \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\rho(\tau)| \mu_{11}(\tau) d\tau \right) \|y\|_\infty \\ &\quad + \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\rho(\tau)| \mu_{12}(\tau) d\tau \right) \|z\|_\infty. \end{aligned} \quad (27)$$

Similarly, by (A4) and (16), we obtain

$$\begin{aligned} \|z\|_\infty &\leq \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\sigma(\tau)| \xi_{21}(\tau) d\tau \right) \|y\|_\infty \\ &\quad + \left(\int_0^1 \frac{(3 - \tau)\tau^2}{6} |\sigma(\tau)| \xi_{22}(\tau) d\tau \right) \|z\|_\infty. \end{aligned} \quad (28)$$

Combining (27) with (28), we deduce that

$$\vec{0} \leq_{\mathbb{R}^2} \vec{\phi}_{y,z} \leq_{\mathbb{R}^2} C \vec{\phi}_{y,z}, \quad (29)$$

where $\vec{\phi}_{y,z} = \begin{pmatrix} \|y\|_\infty \\ \|z\|_\infty \end{pmatrix}$ and

$$C = \frac{1}{6} \begin{pmatrix} J_\rho(\mu_{11}) & J_\rho(\mu_{12}) \\ J_\sigma(\xi_{21}) & J_\sigma(\xi_{22}) \end{pmatrix}. \quad (30)$$

Next, using Lemma 3 and (24), we deduce that

$$\rho_C < 1. \quad (31)$$

On the other hand, using Lemma 1 and (29), we obtain

$$\|\vec{\phi}_{y,z}\|_2 \leq \|C \vec{\phi}_{y,z}\|_2 \leq \|C\| \|\vec{\phi}_{y,z}\|_2. \quad (32)$$

Since (y, z) is nontrivial, then $\vec{\phi}_{y,z} \neq \vec{0}$, and the above inequality leads to

$$\|C\| \geq 1. \quad (33)$$

But by Lemma 2 and (31), we know that

$$\lim_{n \rightarrow \infty} \|C\|^n = 0, \quad (34)$$

which contradicts (33). This proves (23). \square

Next, we discuss some particular cases of Theorem 1.

3.1. Nonlinearities Involving Trigonometric Functions. Consider the system of differential equations

$$\begin{cases} y^{(4)}(t) = \rho(t) \sin(y(t) + z(t)), & 0 < t < 1, \\ z^{(4)}(t) = \sigma(t) \arctan(y(t) + z(t)), & 0 < t < 1, \end{cases} \quad (35)$$

under the boundary conditions (2), where $\rho, \sigma \in C([0, 1])$. Observe that (35) is a particular case of (1) with

$$\begin{aligned} \mu(t, y, z) &= \sin(y(t) + z(t)), \\ (t, y, z) &\in [0, 1] \times C([0, 1]) \times C([0, 1]), \\ \xi(t, y, z) &= \arctan(y(t) + z(t)), \\ (t, y, z) &\in [0, 1] \times C([0, 1]) \times C([0, 1]). \end{aligned} \quad (36)$$

Obviously, μ and ξ satisfy (A3). Moreover, for all $(t, y, z) \in [0, 1] \times C([0, 1]) \times C([0, 1])$,

$$\begin{aligned} |\mu(t, y, z)| &= |\sin(y(t) + z(t))| \leq |y(t) + z(t)| \leq \|y\|_\infty + \|z\|_\infty, \\ |\xi(t, y, z)| &= |\arctan(y(t) + z(t))| \leq |y(t) + z(t)| \leq \|y\|_\infty + \|z\|_\infty. \end{aligned} \quad (37)$$

Then, (A4) is satisfied with

$$\mu_{11} = \mu_{12} = \xi_{21} = \xi_{22} = 1. \quad (38)$$

Hence, by Theorem 1, we deduce the following.

Corollary 1. *If (35) and (2) have a nontrivial solution, then*

$$J_\rho(1) + J_\sigma(1) + \sqrt{(J_\rho(1) - J_\sigma(1))^2 + 4J_\sigma(1)J_\rho(1)} \geq 12. \quad (39)$$

3.2. Nonlocal Source Terms. Consider the system of differential equations

$$\begin{cases} y^{(4)}(t) = \rho(t) \int_0^t (t-s)^{\alpha-1} (y(s) + z(s)) ds, & 0 < t < 1, \\ z^{(4)}(t) = \sigma(t) \int_0^t (t-s)^{\beta-1} (y(s) + z(s)) ds, & 0 < t < 1, \end{cases} \quad (40)$$

under the boundary condition (16), where $\rho, \sigma \in C([0, 1])$ and $\alpha, \beta > 0$. Problem (40) is a particular case of (1) with

$$\begin{aligned} \mu(t, y, z) &= \begin{cases} \int_0^t (t-s)^{\alpha-1} (y(s) + z(s)) ds, & \text{if } 0 < t \leq 1, \\ 0, & \text{if } t = 0, \end{cases} \\ \xi(t, y, z) &= \begin{cases} \int_0^t (t-s)^{\beta-1} (y(s) + z(s)) ds & \text{if } 0 < t \leq 1, \\ 0, & \text{if } t = 0, \end{cases} \end{aligned} \quad (41)$$

for all $y, z \in C([0, 1])$. Obviously, μ and ξ satisfy (A3). Moreover, for all $(t, y, z) \in [0, 1] \times C([0, 1]) \times C([0, 1])$,

$$\begin{aligned} |\mu(t, y, z)| &\leq \left(\int_0^t (t-s)^{\alpha-1} ds \right) (\|y\|_\infty + \|z\|_\infty) \\ &= \frac{t^\alpha}{\alpha} (\|y\|_\infty + \|z\|_\infty). \end{aligned} \quad (42)$$

and similarly

$$|\xi(t, y, z)| \leq \frac{t^\beta}{\beta} (\|y\|_\infty + \|z\|_\infty). \quad (43)$$

Then, (A4) is satisfied with

$$\begin{aligned} \mu_{11}(t) &= \mu_{12}(t) = \frac{t^\alpha}{\alpha}, & 0 \leq t \leq 1, \\ \xi_{21}(t) &= \xi_{22}(t) = \frac{t^\beta}{\beta}, & 0 \leq t \leq 1. \end{aligned} \quad (44)$$

Then, by Theorem 1, we deduce the following.

Corollary 2. *If (39) and (2) have a nontrivial solution, then*

$$\begin{aligned} &J_\rho\left(\frac{t^\alpha}{\alpha}\right) + J_\sigma\left(\frac{t^\beta}{\beta}\right) \\ &+ \sqrt{\left(J_\rho\left(\frac{t^\alpha}{\alpha}\right) - J_\sigma\left(\frac{t^\beta}{\beta}\right)\right)^2 + 4J_\sigma\left(\frac{t^\beta}{\beta}\right)J_\rho\left(\frac{t^\alpha}{\alpha}\right)} \geq 12. \end{aligned} \quad (45)$$

Example 1. Consider the system of differential equation (35) with $\rho \equiv 1$ and $\sigma(t) = 9/4(3-t)$ for all $t \in [0, 1]$. Elementary calculations show that

$$\begin{aligned} J_\rho(1) &= J_\sigma(1) = \frac{3}{4}, \\ J_\rho(1) + J_\sigma(1) + \sqrt{(J_\rho(1) - J_\sigma(1))^2 + 4J_\sigma(1)J_\rho(1)} &= 3 < 12. \end{aligned} \quad (46)$$

Hence, by Corollary 1, we deduce that (35) and (2) have no nontrivial solution.

4. Generalized Eigenvalues Problems

We say that $e = (e_{ij})_{1 \leq i, j \leq 2}$ is a generalized eigenvalue of the system of differential equations

$$\begin{cases} y^{(4)}(t) = e_{11}y(t) + e_{12}z(t), & 0 < t < 1, \\ z^{(4)}(t) = e_{21}y(t) + e_{22}z(t), & 0 < t < 1, \end{cases} \quad (47)$$

subjected to the boundary condition (2), if (2) and (44) admit a nonzero solution (y_e, z_e) . Notice that (44) is a particular case of (1) with

$$\begin{aligned} \rho(t) &= 1, \mu(t, y, z) = e_{11}y(t) + e_{12}z(t), \\ (t, y, z) &\in [0, 1] \times C([0, 1]) \times C([0, 1]), \\ \sigma(t) &= 1, \xi(t, y, z) = e_{21}y(t) + e_{22}z(t), \\ (t, y, z) &\in [0, 1] \times C([0, 1]) \times C([0, 1]). \end{aligned} \quad (48)$$

Moreover, (A2)–(A4) are satisfied with

$$\begin{aligned}
\mu_{11} &= e_{11}, \\
\mu_{12} &= e_{12}, \\
\xi_{21} &= e_{21}, \\
\xi_{22} &= e_{22}.
\end{aligned} \tag{49}$$

Hence, by Theorem 1, if e is a generalized eigenvalue of (2) and (44), then

$$J_1(e_{11}) + J_1(e_{22}) + \sqrt{(J_1(e_{11}) - J_1(e_{22}))^2 + 4J_1(e_{21})J_1(e_{12})} \geq 12. \tag{50}$$

On the other hand, for $\lambda \in \mathbb{R}$, we have

$$J_1(\lambda) = \lambda \int_0^1 (3 - \tau) \tau^2 d\tau = \frac{3\lambda}{4}. \tag{51}$$

Hence, (50) reduces to

$$e_{11} + e_{22} + \sqrt{(e_{11} - e_{22})^2 + 4e_{21}e_{12}} \geq 16. \tag{52}$$

Therefore, the following result follows.

Corollary 3. If $e = (e_{ij})_{1 \leq i, j \leq 2}$ is a generalized eigenvalue of (43) and (2), then

$$e_{11} + e_{22} + \sqrt{(e_{11} - e_{22})^2 + 4e_{21}e_{12}} \geq 16. \tag{53}$$

5. Conclusion

Using some techniques from matrix analysis and ordinary differential equations, a necessary condition for the existence of nonzero solutions to (1) and (2) is obtained (Theorem 1). As particular cases of (1), we discussed nonlinearities involving trigonometric functions (Corollary 1) and nonlocal source terms (Corollary 2). Finally, we applied our main result to obtain an estimate involving generalized eigenvalues (Corollary 3).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

The third author was supported by Researchers Supporting Project number, King Saud University, Riyadh, Saudi Arabia (RSP-2021/4).

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Research Article

Some Rational Coupled Fuzzy Cone Contraction Theorems in Fuzzy Cone Metric Spaces with an Application

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Received 13 June 2021; Accepted 3 August 2021; Published 18 August 2021

Academic Editor: Huseyin Isik

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In this paper, we establish the new concept of rational coupled fuzzy cone contraction mapping in fuzzy cone metric spaces and prove some unique rational-type coupled fixed-point theorems in the framework of fuzzy cone metric spaces by using “the triangular property of fuzzy cone metric.” To ensure the existence of our results, we present some illustrative unique coupled fixed-point examples. Furthermore, we present an application of a Lebesgue integral-type contraction mapping in fuzzy cone metric spaces and to prove a unique coupled fixed-point theorem.

1. Introduction

In 1965, the theory of fuzzy sets was introduced by Zadeh [1]. Kramosil and Michalek [2] introduced the notion of FMS by using continuous t -norm with fuzzy sets. Afterward, Grabiec [3] established the completeness property of the FMS and proved a “Fuzzy Banach Contraction Principle for a unique fixed point (FP) in complete FMS.” Since then, many contributed to this theory concerning FP results (e.g., see [4–6]). Later on, in 1994, George and Veeramani [7] modified the concept of FMS introduced by Kramosil and Michalek [2], and they presented the topological properties and proved Baire’s theorem on complete FMS. In 2002, some contractive-type FP theorems were proved by Gregory and Sapena [8] on complete FMS by using the concept of [2, 7]. Some related FP concepts in FMSs can be found in [9–12]. Recently, the rational-type fuzzy contraction concept in FMS is given by Rehman et al. [13], and they proved some FP results with an application.

Jaggi [14] proved the rational-type FP result for a contractive condition. However, Harjani et al. [15] modified the concept of Jaggi [14] and proved a generalized result in “partially ordered metric space.” In 2011, Luong and Thuan [16] proved generalized rational weak contraction results in “partially ordered metric space,” which is a generalization of the result of [14]. In [17], Guo and Lakshmikantham presented the concept of coupled FP with applications by using the nonlinear operator. Later on, Bhaskar [18] and Lakshmikantham [19] proved coupled FP results in “partially ordered metric space.” In [20], Sedghi et al. used commuting mappings and established some common coupled FP theorems in FMSs.

In 2007, the notion of cone metric space (CMS) was introduced by Huang and Zhang [21]. They proved some basic convergence properties and FP theorems on CMS. In 2008, Abbas et al. [22] proved some common FP theorems without continuity for noncommuting mappings on CMS. After that, many others contributed their ideas to the

problem of FP results in CMS. Some of their FP contributions can be found in [23–25].

Oner et al. [26] introduced the concept of fuzzy cone metric space (FCMS) and proved a “fuzzy cone Banach contraction theorem” for FP in complete FCMSs in which they assumed that the “fuzzy cone contractive (fc – contractive) sequences are Cauchy.” In [27], Rehman and Li proved some FP theorems in FCMSs without the assumption of “ fc – contractive sequences are Cauchy” by using the “triangular property of FCM.” Some more FP findings in the said space can be found in [28–31]. Recently, Chen et al. [32] and Rehman and Aydi [33] established some coupled FP and common FP results, respectively, in FCMs with integral types of applications. Waheed et al. [34] proved some coupled FP theorems in FCMSs depending on another function with an application to Volterra integral equations.

In this paper, we prove some rational-type unique coupled FP theorems in FCMSs under the rational type fc – contractive conditions with supportive examples. In addition, to verify the validity of our work, we present an application of a Lebesgue integral-type contraction condition theorem to support our work. The layout of this paper is as follows: Section 2 consists of some basic preliminary concepts. In Section 3, we define the rational coupled fc – contractive mapping in FCMS and prove some unique rational coupled FP results in complete FCMSs with suitable examples. Section 4 deals with the application of Lebesgue integral-type contraction mapping to get the existence result of unique coupled FP theorems in complete FCMSs.

2. Preliminaries

In this section, we recall some basic definitions and lemmas related to our main results. Throughout the complete paper, \mathbb{N} represents a set of natural numbers and τ -norm represents a continuous t -norm as defined in [35].

Definition 1. An operation $*$: $[0, 1]^2 \rightarrow [0, 1]$ would be a τ -norm if $*$ fulfils the following conditions:

- (1) $*$ is associative, commutative, and continuous
- (2) $1 * \kappa_1 = \kappa_1, \forall \kappa_1 \in [0, 1]$
- (3) $\kappa_1 * \kappa_2 \leq \kappa_3 * \kappa_4$ whenever $\kappa_1 \leq \kappa_3$ and $\kappa_2 \leq \kappa_4$, for $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in [0, 1]$

Definition 2. Let E be a real Banach space, $0 \in E$. Then, a subset $C \subset E$ is called a cone:

- (1) If $C \neq \emptyset$, closed, and $C \neq \{0\}$
- (2) If $\kappa_1, \kappa_2 \geq 0$ and $g, z \in C$, then $\kappa_1 g + \kappa_2 z \in C$
- (3) If $-z, z \in C$, then $z = 0$.

A partial ordering is defined on a given cone $C \subset E$ by $g \leq z \Leftrightarrow z - g \in C$. $g < z$ stands for $g \leq z$ and $g \neq z$, while $g \ll z$ stands for $z - g \in \text{int}(C)$. In this paper, all cones have a nonempty interior.

Definition 3. A 3-tuple $(G, M_r, *)$ is said to be a FMS if G is any set, $*$ is a τ -norm, and M_r is a fuzzy set on $G \times G \times (0, \infty)$ which satisfies the following:

- (1) $M_r(g_1, g_2, \tau) > 0$
- (2) $M_r(g_1, g_2, \tau) = 0 \Leftrightarrow g_1 = g_2$
- (3) $M_r(g_1, g_2, \tau) = M_r(g_2, g_1, \tau)$
- (4) $M_r(g_1, g_3, \tau) * M_r(g_3, g_2, s) \leq M_r(g_1, g_2, \tau + s)$
- (5) $M_r(g_1, g_2, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous;
 $\forall g_1, g_2, g_3 \in G$ and $s, \tau > 0$

Definition 4. A 3-tuple $(G, M_r, *)$ is said to be a FCMS if C is a cone of E , G is an arbitrary set, $*$ is a τ -norm, and M_r is a fuzzy set on $G \times G \times \text{int}(C)$ which satisfies the following:

- (1) $M_r(g_1, g_2, \tau) > 0$
- (2) $M_r(g_1, g_2, \tau) = 0 \Leftrightarrow g_1 = g_2$
- (3) $M_r(g_1, g_2, \tau) = M_r(g_2, g_1, \tau)$
- (4) $M_r(g_1, g_3, \tau) * M_r(g_3, g_2, s) \leq M_r(g_1, g_2, \tau + s)$
- (5) $M_r(g_1, g_2, \cdot): \text{int}(C) \rightarrow [0, 1]$ is continuous;
 $\forall g_1, g_2, g_3 \in G$, and $s, \tau \gg 0$

Definition 5. Let a 3-tuple $(G, M_r, *)$ be a FCMS and $q \in G$ and a sequence $\{g_J\}$ in G

- (1) Converges to q if $\gamma \in (0, 1)$ and $\tau \gg 0$ and there is $J_1 \in \mathbb{N} \ni M_r(g_J, q, \tau) > 1 - \gamma$, for $J \geq J_1$. We may write this $\lim_{J \rightarrow \infty} g_J = q$ or $g_J \rightarrow q$ as $J \rightarrow \infty$.
- (2) Is a Cauchy sequence if $\gamma \in (0, 1)$ and $\tau \gg 0$ and there is $J_1 \in \mathbb{N} \ni M_r(g_J, g_\ell, \tau) > 1 - \gamma$, for $J, \ell \geq J_1$.
- (3) $(G, M_r, *)$ is complete if every Cauchy sequence is convergent in G .
- (4) Is fc – contractive if $\exists \eta \in (0, 1)$, satisfying $(1/M_r(g_J, g_{J+1}, \tau)) - 1 \leq \eta(1/M_r(g_{J-1}, g_J, \tau) - 1)$, for $\tau \gg 0, J \geq 1$.

Lemma 1. Let $(G, M_r, *)$ be a FCMS, and let a sequence $\{g_J\}$ in G converge to a point $q \in G$ if $M_r(g_J, q, \tau) \rightarrow 1$ as $J \rightarrow \infty$, for $\tau \gg 0$.

Definition 6. Let $(G, M_r, *)$ be a FCMS. The FCM M_r is triangular, if $(1/M_r(g, h, \tau)) - 1 \leq ((1/M_r(g, z, \tau)) - 1) + ((1/M_r(z, h, \tau)) - 1), \forall g, h, z \in G, \tau \gg 0, \Gamma$.

Definition 7 (see [26]). Let $(G, M_r, *)$ be a FCMS and $\Gamma: G \rightarrow G$. Then, Γ is known as a fc – contractive if $\exists \eta \in (0, 1)$ such that

$$\frac{1}{M_r(\Gamma g, \Gamma h, \tau)} - 1 \leq \eta \left(\frac{1}{M_r(g, h, \tau)} - 1 \right), \quad \forall g, h \in G, \tau \gg 0. \quad (1)$$

Definition 8. An element $(g, h) \in G \times G$ is known as a coupled FP of a function $\tilde{F}: G \times G \rightarrow G$ if

$$\Gamma(g, h) = g, \quad \text{and } \Gamma(h, g) = h. \quad (2)$$

Furthermore, we shall study some unique coupled FP results in FCMSs under the rational coupled fc -contraction conditions with examples. Also, we present an application of the Lebesgue integral-type rational coupled fc -contraction mapping to get a unique rational coupled FP result in FCMSs.

3. Main Results

In this section, we shall present our main results with illustrative examples.

Definition 9. Let $(G, M_r, *)$ be a FCMS. A mapping $\Gamma: G \times G \longrightarrow G$ is called a rational coupled fc -contraction if $\exists \eta_1 \in (0, 1)$ and $\eta_2 \geq 0$ such tha

$$\begin{aligned} \frac{1}{M_r(\Gamma(g, h), \Gamma(\xi, \kappa), \tau)} - 1 &\leq \eta_1 \left(\frac{1}{M_r(g, \xi, \tau)} - 1 \right) \\ &+ \eta_2 \left(\frac{M_r(g, \xi, \tau)}{M_r(g, \Gamma(g, h), \tau) * M_r(\xi, \Gamma(g, h), 2\tau)} - 1 \right), \quad \forall g, h, \xi, \kappa \in G \text{ and } \tau \gg 0. \end{aligned} \quad (3)$$

Theorem 1. Assume that $(G, M_r, *)$ be a complete FCMS in which M_r is triangular and a mapping $\Gamma: G \times G \longrightarrow G$ is a rational coupled fc -contraction satisfying (3). Then, Γ has a unique coupled FP in G .

Proof. Let any $g_0, h_0 \in G$; we define sequences $\{g_J\}$ and $\{h_J\}$ in G such that

$$\Gamma(g_J, h_J) = g_{J+1}, \quad \text{and } \Gamma(h_J, g_J) = h_{J+1}, \quad \text{for } J \geq 0. \quad (4)$$

Now, from (3) and (4), for $\tau \gg 0$,

$$\begin{aligned} \frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 &= \frac{1}{M_r(\Gamma(g_{J-1}, h_{J-1}), \Gamma(g_J, h_J), \tau)} - 1, \\ &\leq \eta_1 \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g_{J-1}, g_J, \tau)}{M_r(g_{J-1}, \Gamma(g_{J-1}, h_{J-1}), \tau) * M_r(g_J, \Gamma(g_{J-1}, h_{J-1}), 2\tau)} - 1 \right) \\ &= \eta_1 \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g_{J-1}, g_J, \tau)}{M_r(g_{J-1}, g_J, \tau) * M_r(g_J, g_J, 2\tau)} - 1 \right). \end{aligned} \quad (5)$$

This implies that

$$\frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 \leq \eta_1 \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \right), \quad \text{for } \tau \gg 0. \quad (6)$$

Similarly,

$$\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \leq \eta_1 \left(\frac{1}{M_r(g_{J-2}, g_{J-1}, \tau)} - 1 \right), \quad \text{for } \tau \gg 0. \quad (7)$$

Now, from (6) and (7) and by induction, for $\tau \gg 0$, we have that

$$\begin{aligned} \frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 &\leq \eta_1 \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \right), \\ &\leq \eta_1^2 \left(\frac{1}{M_r(g_{J-2}, g_{J-1}, \tau)} - 1 \right) \\ &\leq \dots \leq \eta_1^J \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty. \end{aligned} \quad (8)$$

This shows $\{g_J\}$ is a fc -contractive sequence; therefore,

$$\lim_{J \rightarrow \infty} M_r(g_J, g_{J+1}, \tau) = 1, \quad \text{for } \tau \gg 0. \quad (9)$$

Now, for $\ell > J$ and for $\tau \gg 0$,

$$\begin{aligned}
\frac{1}{M_r(g_J, g_{\ell}, \tau)} - 1 &\leq \left(\frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 \right) + \left(\frac{1}{M_r(g_{J+1}, g_{J+2}, \tau)} - 1 \right) + \cdots + \left(\frac{1}{M_r(g_{\ell-1}, g_{\ell}, \tau)} - 1 \right) \\
&\leq \eta_1^J \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) + \eta_1^{J+1} \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) + \cdots + \eta_1^{\ell-1} \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) \\
&= (\eta_1^J + \eta_1^{J+1} + \cdots + \eta_1^{\ell-1}) \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) \\
&= \frac{\eta_1^J}{1 - \eta_1} \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty.
\end{aligned} \tag{10}$$

Hence, $\{g_J\}$ is a Cauchy sequence. Since, by the completeness of $(G, M_r, *)$, $\exists g \in G$, so that

$$\lim_{J \rightarrow \infty} M_r(g_J, g, \tau) = 1, \quad \text{for } \tau \gg 0. \tag{11}$$

Now, for sequence $\{h_J\}$ and from (3) and (4), for $\tau \gg 0$,

$$\begin{aligned}
\frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 &= \frac{1}{M_r(\Gamma(h_{J-1}, g_{J-1}), \Gamma(h_J, g_J), \tau)} - 1, \\
&\leq \eta_1 \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h_{J-1}, h_J, \tau)}{M_r(h_{J-1}, \Gamma(h_{J-1}, g_{J-1}), \tau) * M_r(h_J, \Gamma(h_{J-1}, g_{J-1}), 2\tau)} - 1 \right) \\
&= \eta_1 \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h_{J-1}, h_J, \tau)}{M_r(h_{J-1}, h_J, \tau) * M_r(h_J, h_J, 2\tau)} - 1 \right).
\end{aligned} \tag{12}$$

This implies that

$$\frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 \leq \eta_1 \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 \right), \quad \text{for } \tau \gg 0. \tag{13}$$

$$\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 \leq \eta_1 \left(\frac{1}{M_r(h_{J-2}, h_{J-1}, \tau)} - 1 \right), \quad \text{for } \tau \gg 0. \tag{14}$$

Now, from (13) and (14) and by induction, for $\tau \gg 0$, we have that

Similarly,

$$\begin{aligned}
\frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 &\leq \eta_1 \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 \right), \\
&\leq \eta_1^2 \left(\frac{1}{M_r(h_{J-2}, h_{J-1}, \tau)} - 1 \right) \\
&\leq \cdots \leq \eta_1^J \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty.
\end{aligned} \tag{15}$$

This shows $\{h_J\}$ is a fc -contractive sequence; therefore,

$$\lim_{J \rightarrow \infty} M_r(h_J, h_{J+1}, \tau) = 1, \quad \text{for } \tau \gg 0. \quad (16)$$

Now, for $\ell > J$ and for $\tau \gg 0$, we have

$$\begin{aligned} \frac{1}{M_r(h_J, h_\ell, \tau)} - 1 &\leq \left(\frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 \right) + \left(\frac{1}{M_r(h_{J+1}, h_{J+2}, \tau)} - 1 \right) + \cdots + \left(\frac{1}{M_r(h_{\ell-1}, h_\ell, \tau)} - 1 \right), \\ &\leq \eta_1^J \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) + \eta_1^{J+1} \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) + \cdots + \eta_1^{\ell-1} \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) \\ &= (\eta_1^J + \eta_1^{J+1} + \cdots + \eta_1^{\ell-1}) \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) \\ &= \frac{\eta_1^J}{1 - \eta_1} \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty. \end{aligned} \quad (17)$$

Hence, $\{h_J\}$ is a Cauchy sequence. Since, by the completeness of $(G, M_r, *)$, $\exists g \in G$ so that

$$\lim_{J \rightarrow \infty} M_r(h_J, h, \tau) = 1, \quad \text{for } \tau \gg 0. \quad (18)$$

Now, we shall prove that $\Gamma(g, h) = g$. Since M_r is triangular and by the view of (3), (9), and (11), for $\tau \gg 0$,

$$\begin{aligned} \frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 &\leq \left(\frac{1}{M_r(g, \Gamma(g_J, h_J), \tau)} - 1 \right) + \left(\frac{1}{M_r(\Gamma(g_J, h_J), \Gamma(g, h), \tau)} - 1 \right), \\ &\leq \left(\frac{1}{M_r(g, \Gamma(g_J, h_J), \tau)} - 1 \right) + \eta_1 \left(\frac{1}{M_r(g_\ell, g, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g_J, g, \tau)}{M_r(g_J, \Gamma(g_J, h_J), \tau) * M_r(g, \Gamma(g_J, h_J), 2\tau)} - 1 \right) \\ &= \left(\frac{1}{M_r(g, g_{J+1}, \tau)} - 1 \right) + \eta_1 \left(\frac{1}{M_r(g_J, g, \tau)} - 1 \right) \\ &\quad + \eta_2 \left(\frac{M_r(g_J, g, \tau)}{M_r(g_J, g_{J+1}, \tau) * M_r(g, g_{J+1}, 2\tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty. \end{aligned} \quad (19)$$

Hence, $M_r(g, \Gamma(g, h), \tau) = 1 \Rightarrow g = \Gamma(g, h)$ for $\tau \gg 0$. Next, we have to prove that $h = \Gamma(h, g)$; therefore, by the

triangular property of M_r and by the view of (3), (16), and (18), for $\tau \gg 0$,

$$\begin{aligned} \frac{1}{M_r(h, \Gamma(h, g), \tau)} - 1 &\leq \left(\frac{1}{M_r(h, \Gamma(h_J, g_J), \tau)} - 1 \right) + \left(\frac{1}{M_r(\Gamma(h_J, g_J), \Gamma(h, g), \tau)} - 1 \right) \\ &\leq \left(\frac{1}{M_r(h, \Gamma(h_J, g_J), \tau)} - 1 \right) + \eta_1 \left(\frac{1}{M_r(h_J, h, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h_J, h, \tau)}{M_r(h_J, \Gamma(h_J, g_J), \tau) * M_r(h, \Gamma(h_J, g_J), 2\tau)} - 1 \right) \\ &= \left(\frac{1}{M_r(h, h_{J+1}, \tau)} - 1 \right) + \eta_1 \left(\frac{1}{M_r(h_J, h, \tau)} - 1 \right) \\ &\quad + \eta_2 \left(\frac{M_r(h_J, h, \tau)}{M_r(h_J, h_{J+1}, \tau) * M_r(h, h_{J+1}, 2\tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty. \end{aligned} \quad (20)$$

Hence, $M_r(g, \Gamma(h, g), \tau) = 1 \Rightarrow \Gamma(h, g) = h$, for $\tau \gg 0$.

Uniqueness: suppose (g_1, h_1) and (h_1, g_1) are other coupled fixed-point pairs in $G \times G$ such that $\Gamma(g_1, h_1) = g_1$

and $\Gamma(h_1, g_1) = h_1$. Now, from (3) and by using Definition 4 (4), for $\tau \gg 0$,

$$\begin{aligned}
 \frac{1}{M_r(g, g_1, \tau)} - 1 &= \frac{1}{M_r(\Gamma(g, h), \Gamma(g_1, h_1), \tau)} - 1 \\
 &\leq \eta_1 \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g, g_1, \tau)}{M_r(g, \Gamma(g, h), \tau) * M_r(g_1, \Gamma(g, h), 2\tau)} - 1 \right) \\
 &= \eta_1 \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g, g_1, \tau)}{M_r(g, g, \tau) * M_r(g_1, g, 2\tau)} - 1 \right) \\
 &= \eta_1 \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) = \eta_1 \left(\frac{1}{M_r(\Gamma(g, h), \Gamma(g_1, h_1), \tau)} - 1 \right) \\
 &\leq \eta_1^2 \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) \leq \dots \leq \eta_1^J \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } J \rightarrow \infty.
 \end{aligned} \tag{21}$$

Hence, we get that $M_r(g, g_1, \tau) = 1 \Rightarrow g = g_1$ for $\tau \gg 0$. Similarly, again from (3) and by using Definition 4 (4), for $\tau \gg 0$,

$$\begin{aligned}
 \frac{1}{M_r(h, h_1, \tau)} - 1 &= \frac{1}{M_r(\Gamma(h, g), \Gamma(h_1, g_1), \tau)} - 1, \\
 &\leq \eta_1 \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h, h_1, \tau)}{M_r(h, \Gamma(h, g), \tau) * M_r(h_1, \Gamma(h, g), 2\tau)} - 1 \right) \\
 &= \eta_1 \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h, h_1, \tau)}{M_r(h, h_1, \tau) * M_r(h_1, h, 2\tau)} - 1 \right) \\
 &= \eta_1 \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) = \eta_1 \left(\frac{1}{M_r(\Gamma(h, g), \Gamma(h_1, g_1), \tau)} - 1 \right) \\
 &\leq \eta_1^2 \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) \leq \dots \leq \eta_1^J \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } J \rightarrow \infty.
 \end{aligned} \tag{22}$$

Hence, we get that $M_r(h, h_1, \tau) = 1 \Rightarrow h = h_1$ for $\tau \gg 0$. \square

Corollary 1. Let $\Gamma: G \times G \rightarrow G$ be a mapping on a complete FCMS $(G, M_r, *)$ in which M_r is triangular and Γ satisfies the inequality

$$\frac{1}{M_r(\Gamma(g, h), \Gamma(\xi, \kappa), \tau)} - 1 \leq \eta_1 \left(\frac{1}{M_r(g, \xi, \tau)} - 1 \right), \tag{23}$$

$\forall g, h, \xi, \kappa \in G, \tau \gg 0$, and $\eta_1 \in (0, 1)$. Then, Γ has a unique coupled FP in G .

Example 1. Let $G = [0, \infty)$; $*$ is a τ -norm, and $M_r: G \times G \times (0, \infty) \rightarrow [0, 1]$ is defined as

$$M_r(g, h, \tau) = \frac{\tau}{\tau + |g - h|}, \quad \forall g, h \in G \text{ and } \tau > 0. \tag{24}$$

Then, one can easily prove the triangular property of FCM from the above example and $(G, M_r, *)$ is a complete FCMS. We define a mapping $\Gamma: G \times G \rightarrow G$ by

$$\Gamma(g, h) = \begin{cases} \frac{3g}{8}, & g, h \in [0, 1], \\ \frac{g + 5h}{4} - \frac{7}{6}, & g, h \in [1, \infty). \end{cases} \tag{25}$$

Then, we have

$$\begin{aligned}
\frac{1}{M_r(\Gamma(g, h), \Gamma(\xi, \kappa), \tau)} - 1 &= \frac{1}{M_r((3g/8), (3\xi/8), \tau)} - 1, \\
&= \frac{3}{8\tau} |g - \xi| \\
&= \frac{3}{8} \left(\frac{1}{M_r(g, \xi, \tau)} - 1 \right).
\end{aligned}
\tag{26}$$

$\forall g, h, \xi, \kappa \in G$ and $\tau \gg 0$. Hence, we proved that $\Gamma: G \times G \longrightarrow G$ is a rational coupled fc -contractive. Now, by using Definition 4 (4), for $\tau \gg 0$,

$$\begin{aligned}
\frac{M_r(g, \xi, \tau)}{M_r(g, \Gamma(g, h), \tau) * M_r(\xi, \Gamma(g, h), 2\tau)} - 1 &\leq \frac{M_r(g, \xi, \tau)}{M_r(g, \Gamma(g, h), \tau) * M_r(\xi, g, \tau) * M_r(g, \Gamma(g, h), \tau)} - 1, \\
&= \frac{1}{M_r(g, \Gamma(g, h), \tau) * M_r(g, \Gamma(g, h), \tau)} - 1 \\
&= \frac{1}{(M_r(g, \Gamma(g, h), \tau))^2} - 1 = \frac{5g}{64\tau^2} (5g + 16\tau).
\end{aligned}
\tag{27}$$

$\forall g, h, \xi, \kappa \in G$. Hence, from the above, we conclude that all the conditions of Theorem 1 are satisfied with $\eta_1 = 3/8$, $\eta_2 \in [0, 8/13]$, and $\Gamma(g, h) = \Gamma(7/3, 7/3) = 7/3 \in [0, \infty)$.

Theorem 2. Let $\Gamma: G \times G \longrightarrow G$ be a mapping on a complete FCMS $(G, M_r, *)$ in which M_r is triangular and Γ satisfies the inequality

$$\begin{aligned}
\frac{1}{M_r(\Gamma(g, h), \Gamma(\xi, \kappa), \tau)} - 1 &\leq \eta_1 \left(\frac{1}{M_r(g, \xi, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g, \xi, \tau) * M_r(\xi, \Gamma(\xi, \kappa), \tau)}{M_r(g, \Gamma(g, h), \tau) * M_r(g, \Gamma(\xi, \kappa), 2\tau)} - 1 \right), \\
&+ \eta_3 \left(\frac{M_r(g, \Gamma(g, h), \tau)}{M_r(g, \Gamma(\xi, \kappa), 2\tau)} - 1 + \frac{M_r(\xi, \Gamma(\xi, \kappa), \tau)}{M_r(g, \Gamma(\xi, \kappa), 2\tau)} - 1 \right) \\
&+ \eta_4 \left(\frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 + \frac{1}{M_r(\xi, \Gamma(\xi, \kappa), \tau)} - 1 \right),
\end{aligned}
\tag{28}$$

$\forall g, h, \xi, \kappa \in G$, $\tau \gg 0$, $\eta_1 \in (0, 1)$, and $\eta_2, \eta_3, \eta_4 \geq 0$ with $(\eta_1 + \eta_2 + 2\eta_3 + 2\eta_4) < 1$. Then, Γ has a unique coupled FP in G .

$$\Gamma(g_J, h_J) = g_{J+1}, \quad \text{and } \Gamma(h_J, g_J) = h_{J+1}, \quad \text{for } J \geq 0. \tag{29}$$

Now, from (28) and (29), for $\tau \gg 0$,

Proof. Let any $g_0, h_0 \in G$; we define sequences $\{g_J\}$ and $\{h_J\}$ in G such that

$$\begin{aligned}
\frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 &= \frac{1}{M_r(\Gamma(g_{J-1}, h_{J-1}), \Gamma(g_J, h_J), \tau)} - 1, \\
&\leq \eta_1 \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g_{J-1}, g_J, \tau) * M_r(g_J, \Gamma(g_J, h_J), \tau)}{M_r(g_{J-1}, \Gamma(g_{J-1}, h_{J-1}), \tau) * M_r(g_{J-1}, \Gamma(g_J, h_J), 2\tau)} - 1 \right) \\
&+ \eta_3 \left(\frac{M_r(g_{J-1}, \Gamma(g_{J-1}, h_{J-1}), \tau)}{M_r(g_{J-1}, \Gamma(g_J, h_J), 2\tau)} - 1 + \frac{M_r(g_J, \Gamma(g_J, h_J), \tau)}{M_r(g_{J-1}, \Gamma(g_J, h_J), 2\tau)} - 1 \right)
\end{aligned}$$

$$\begin{aligned}
& + \eta_4 \left(\frac{1}{M_r(g_{J-1}, \Gamma(g_{J-1}, h_{J-1}), \tau)} - 1 + \frac{1}{M_r(g_J, \Gamma(g_J, h_J), \tau)} - 1 \right) \\
= & \eta_1 \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g_{J-1}, g_J, \tau) * M_r(g_J, g_{J+1}, \tau)}{M_r(g_{J-1}, g_J, \tau) * M_r(g_{J-1}, g_{J+1}, 2\tau)} - 1 \right) \\
& + \eta_3 \left(\frac{M_r(g_{J-1}, g_J, \tau)}{M_r(g_{J-1}, g_{J+1}, 2\tau)} - 1 + \frac{M_r(g_J, g_{J+1}, \tau)}{M_r(g_{J-1}, g_{J+1}, 2\tau)} - 1 \right) + \eta_4 \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 + \frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 \right).
\end{aligned} \tag{30}$$

Now, by Definition 4 (4), for $\tau \gg 0$,

$$\begin{aligned}
\frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 \leq & \eta_1 \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g_{J-1}, g_J, \tau) * M_r(g_J, g_{J+1}, \tau)}{M_r(g_{J-1}, g_J, \tau) * M_r(g_{J-1}, g_J, \tau) * M_r(g_J, g_{J+1}, \tau)} - 1 \right) \\
& + \eta_3 \left(\frac{M_r(g_{J-1}, g_J, \tau)}{M_r(g_{J-1}, g_J, \tau) * M_r(g_J, g_{J+1}, \tau)} - 1 + \frac{M_r(g_J, g_{J+1}, \tau)}{M_r(g_{J-1}, g_J, \tau) * M_r(g_J, g_{J+1}, \tau)} - 1 \right) \\
& + \eta_4 \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 + \frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 \right).
\end{aligned} \tag{31}$$

After simplification, we get that

$$\frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 \leq \rho \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \right), \quad \text{for } \tau \gg 0. \tag{32}$$

$$\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \leq \rho \left(\frac{1}{M_r(g_{J-2}, g_{J-1}, \tau)} - 1 \right), \quad \text{for } \tau \gg 0. \tag{33}$$

where $\rho = (\eta_1 + \eta_2 + \eta_3 + \eta_4) / (1 - \eta_3 - \eta_4) < 1$. Now, from (32) and (33) and by induction, for $\tau \gg 0$, we have that

where $\rho = (\eta_1 + \eta_2 + \eta_3 + \eta_4) / (1 - \eta_3 - \eta_4) < 1$. Similarly, again, by using (28) and Definition 4 (4), we get that

$$\begin{aligned}
\frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 & \leq \rho \left(\frac{1}{M_r(g_{J-1}, g_J, \tau)} - 1 \right), \\
& \leq \rho^2 \left(\frac{1}{M_r(g_{J-2}, g_{J-1}, \tau)} - 1 \right) \\
& \leq \dots \leq \rho^J \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty.
\end{aligned} \tag{34}$$

This shows $\{g_J\}$ is a fc -contractive sequence; therefore,

$$\lim_{\ell \rightarrow \infty} M_r(g_J, g_{J+1}, \tau) = 1, \quad \text{for } \tau \gg 0. \quad (35)$$

Now, for $\ell > J$ and for $\tau \gg 0$, we have

$$\begin{aligned} \frac{1}{M_r(g_J, g_\ell, \tau)} - 1 &\leq \left(\frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 \right) + \left(\frac{1}{M_r(g_{J+1}, g_{J+2}, \tau)} - 1 \right) + \cdots + \left(\frac{1}{M_r(g_{\ell-1}, g_\ell, \tau)} - 1 \right), \\ &\leq \rho^J \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) + \rho^{J+1} \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) + \cdots + \rho^{\ell-1} \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) \\ &= (\rho^J + \rho^{J+1} + \cdots + \rho^{\ell-1}) \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) \\ &= \frac{\rho^J}{1 - \rho} \left(\frac{1}{M_r(g_0, g_1, \tau)} - 1 \right) \rightarrow 0, \quad \text{as } J \rightarrow \infty. \end{aligned} \quad (36)$$

Hence, $\{g_J\}$ is a Cauchy sequence. Since, by the completeness of $(G, M_r, *)$, $\exists g \in G$ so that

$$\lim_{J \rightarrow \infty} M(g_J, g, \tau) = 1, \quad \text{for } \tau \gg 0. \quad (37)$$

Now, for sequence $\{h_J\}$, from (28) and (29), for $\tau \gg 0$,

$$\begin{aligned} \frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 &= \frac{1}{M_r(\Gamma(h_{J-1}, g_{J-1}), \Gamma(h_J, g_J), \tau)} - 1 \\ &\leq \eta_1 \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h_{J-1}, h_J, \tau) * M_r(h_J, \Gamma(h_J, g_J), \tau)}{M_r(h_{J-1}, \Gamma(h_{J-1}, g_{J-1}), \tau) * M_r(h_{J-1}, \Gamma(h_J, g_J), 2\tau)} - 1 \right) \\ &\quad + \eta_3 \left(\frac{M_r(h_{J-1}, \Gamma(h_{J-1}, g_{J-1}), \tau)}{M_r(h_{J-1}, \Gamma(h_J, g_J), 2\tau)} - 1 + \frac{M_r(h_J, \Gamma(h_J, g_J), \tau)}{M_r(h_{J-1}, \Gamma(h_J, g_J), 2\tau)} - 1 \right) \\ &\quad + \eta_4 \left(\frac{1}{M_r(h_{J-1}, \Gamma(h_{J-1}, g_{J-1}), \tau)} - 1 + \frac{1}{M_r(h_J, \Gamma(h_J, g_J), \tau)} - 1 \right) \\ &= \eta_1 \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h_{J-1}, h_J, \tau) * M_r(h_J, h_{J+1}, \tau)}{M_r(h_{J-1}, h_J, \tau) * M_r(h_{J-1}, h_{J+1}, 2\tau)} - 1 \right) \\ &\quad + \eta_3 \left(\frac{M_r(h_{J-1}, h_J, \tau)}{M_r(h_{J-1}, h_{J+1}, 2\tau)} - 1 + \frac{M_r(h_J, h_{J+1}, \tau)}{M_r(h_{J-1}, h_{J+1}, 2\tau)} - 1 \right) + \eta_4 \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 + \frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 \right). \end{aligned} \quad (38)$$

Now, again by Definition 4 (4), for $\tau \gg 0$,

$$\begin{aligned} \frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 &\leq \eta_1 \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h_{J-1}, h_J, \tau) * M_r(h_J, h_{J+1}, \tau)}{M_r(h_{J-1}, h_J, \tau) * M_r(h_{J-1}, h_J, \tau) * M_r(h_J, h_{J+1}, \tau)} - 1 \right) \\ &\quad + \eta_3 \left(\frac{M_r(h_{J-1}, h_J, \tau)}{M_r(h_{J-1}, h_J, \tau) * M_r(h_J, h_{J+1}, \tau)} - 1 + \frac{M_r(h_J, h_{J+1}, \tau)}{M_r(h_{J-1}, h_J, \tau) * M_r(h_J, h_{J+1}, \tau)} - 1 \right) \\ &\quad + \eta_4 \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 + \frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 \right). \end{aligned} \quad (39)$$

After simplification, we get that

$$\frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 \leq \rho \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 \right), \quad \text{for } \tau \gg 0, \quad (40)$$

where the value of ρ is same as in (32). Similarly, again by using (28) and Definition 4 (4), for $\tau \gg 0$, we get that

Now, from (40) and (41) and by induction, for $\tau \gg 0$, we have that

$$\begin{aligned} \frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 &\leq \rho \left(\frac{1}{M_r(h_{J-1}, h_J, \tau)} - 1 \right), \\ &\leq \rho^2 \left(\frac{1}{M_r(h_{J-2}, h_{J-1}, \tau)} - 1 \right) \\ &\leq \dots \leq \rho^J \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty. \end{aligned} \quad (42)$$

This shows $\{h_J\}$ is a fc -contractive sequence; therefore,

$$\lim_{J \rightarrow \infty} M_r(h_J, h_{J+1}, \tau) = 1, \quad \text{for } \tau \gg 0. \quad (43)$$

Now, for $\ell > J$ and for $\tau \gg 0$,

$$\begin{aligned} \frac{1}{M_r(h_J, h_J, \tau)} - 1 &\leq \left(\frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 \right) + \left(\frac{1}{M_r(h_{J+1}, h_{J+2}, \tau)} - 1 \right) + \dots + \left(\frac{1}{M_r(h_{\ell-1}, h_{\ell}, \tau)} - 1 \right), \\ &\leq \rho^J \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) + \rho^{J+1} \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) + \dots + \rho^{\ell-1} \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) \\ &= (\rho^J + \rho^{J+1} + \dots + \rho^{\ell-1}) \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) \\ &= \frac{\rho^J}{1 - \rho} \left(\frac{1}{M_r(h_0, h_1, \tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty. \end{aligned} \quad (44)$$

Hence, $\{h_J\}$ is a Cauchy sequence. Since, by the completeness of $(G, M_r, *)$, $\exists h \in G$ so that

$$\lim_{J \rightarrow \infty} M_r(h_J, h, \tau) = 1, \quad \text{for } \tau \gg 0. \quad (45)$$

Now, we shall prove that $\Gamma(g, h) = g$. Since M_r is triangular,

$$\begin{aligned} \frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 &= \left(\frac{1}{M_r(g, g_{J+1}, \tau)} - 1 \right) \\ &+ \left(\frac{1}{M_r(g_{J+1}, \Gamma(g, h), t)} - 1 \right), \quad \text{for } \tau \gg 0. \end{aligned} \quad (46)$$

Now, by the view of (28), (35), and (37), and by using Definition 4 (4), for $\tau \gg 0$,

$$\begin{aligned}
 \frac{1}{M_r(g_{J+1}, \Gamma(g, h), \tau)} - 1 &= \left(\frac{1}{M_r(\Gamma(g_J, h_J), \Gamma(g, h), \tau)} - 1 \right), \\
 &\leq \eta_1 \left(\frac{1}{M_r(g_J, g, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g_J, g, \tau) * M_r(g, \Gamma(g, h), \tau)}{M_r(g_J, \Gamma(g_J, h_J), \tau) * M_r(g_J, \Gamma(g, h), 2\tau)} - 1 \right) \\
 &\quad + \eta_3 \left(\frac{M_r(g_J, \Gamma(g_J, h_J), \tau)}{M_r(g_J, \Gamma(g, h), 2\tau)} - 1 + \frac{M_r(g, \Gamma(g, h), \tau)}{M_r(g_J, \Gamma(g, h), 2\tau)} - 1 \right) \\
 &\quad + \eta_4 \left(\frac{1}{M_r(g_J, \Gamma(g_J, h_J), \tau)} - 1 + \frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 \right) \\
 &\leq \eta_1 \left(\frac{1}{M_r(g_J, g, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g_J, g, \tau) * M_r(g, \Gamma(g, h), \tau)}{M_r(g_J, g_{J+1}, \tau) * M_r(g_J, g, \tau) * M_r(g, \Gamma(g, h), \tau)} - 1 \right) \\
 &\quad + \eta_3 \left(\frac{M_r(g_J, g_{J+1}, \tau)}{M_r(g_J, g, \tau) * M_r(g, \Gamma(g, h), \tau)} - 1 + \frac{M_r(g, \Gamma(g, h), \tau)}{M_r(g_J, g, \tau) * M_r(g, \Gamma(g, h), \tau)} - 1 \right) \\
 &\quad + \eta_4 \left(\frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 + \frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 \right) \longrightarrow (\eta_3 + \eta_4) \left(\frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 \right), \quad \text{as } J \longrightarrow \infty.
 \end{aligned} \tag{47}$$

Hence,

$$\begin{aligned}
 \limsup_{J \rightarrow \infty} \left(\frac{1}{M_r(g_{J+1}, \Gamma(g, h), \tau)} - 1 \right) \\
 \leq (\eta_3 + \eta_4) \left(\frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 \right), \quad \text{for } \tau \gg 0.
 \end{aligned} \tag{48}$$

Now, from (35), (46), and (48), for $\tau \gg 0$,

$$\begin{aligned}
 \frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 &\leq (\eta_3 + \eta_4) \left(\frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 \right), \\
 &\Rightarrow (1 - \eta_3 + \eta_4) \left(\frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 \right) \leq 0, \quad \text{for } \tau \gg 0,
 \end{aligned} \tag{49}$$

which is a contradiction. As $(1 - \eta_3 + \eta_4) \neq 0$, we get that $M_r(g, \Gamma(g, h), \tau) = 1 \Rightarrow \Gamma(g, h) = g$ for $\tau \gg 0$. Next, we prove that $\Gamma(h, g) = h$. Now, again from the triangularity of M ,

$$\frac{1}{M_r(h, \Gamma(h, g), \tau)} - 1 \leq \left(\frac{1}{M_r(h, h_{J+1}, \tau)} - 1 \right) + \left(\frac{1}{M_r(h_{J+1}, \Gamma(h, g), \tau)} - 1 \right), \quad \text{for } \tau \gg 0. \tag{50}$$

Now, by the view of (28), (43), and (45), and by using Definition 4 (4), for $\tau \gg 0$,

$$\begin{aligned}
 \frac{1}{M_r(h_{j+1}, \Gamma(h, g), \tau)} - 1 &= \left(\frac{1}{M_r(\Gamma(h_j, g_j), \Gamma(h, g), \tau)} - 1 \right), \\
 &\leq \eta_1 \left(\frac{1}{M_r(h_j, h, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h_j, h, \tau) * M_r(h, \Gamma(h, g), \tau)}{M_r(h_j, \Gamma(h_j, g_j), \tau) * M_r(h_j, \Gamma(h, g), 2\tau)} - 1 \right) \\
 &\quad + \eta_3 \left(\frac{M_r(h_j, \Gamma(h_j, g_j), \tau)}{M_r(h_j, \Gamma(h, g), 2\tau)} - 1 + \frac{M_r(h, \Gamma(h, g), \tau)}{M_r(h_j, \Gamma(h, g), 2\tau)} - 1 \right) \\
 &\quad + \eta_4 \left(\frac{1}{M_r(h_j, \Gamma(h_j, g_j), \tau)} - 1 + \frac{1}{M_r(h, \Gamma(h, g), \tau)} - 1 \right) \\
 &\leq \eta_1 \left(\frac{1}{M_r(h_j, h, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h_j, h, \tau) * M_r(h, \Gamma(h, g), \tau)}{M_r(h_j, h_{j+1}, \tau) * M_r(h_j, h, \tau) * M_r(h, \Gamma(h, g), \tau)} - 1 \right) \\
 &\quad + \eta_3 \left(\frac{M_r(h_j, h_{j+1}, \tau)}{M_r(h_j, h, \tau) * M_r(h, \Gamma(h, g), \tau)} - 1 + \frac{M_r(h, \Gamma(h, g), \tau)}{M_r(h_j, h, \tau) * M_r(h, \Gamma(h, g), \tau)} - 1 \right) \\
 &\quad + \eta_4 \left(\frac{1}{M_r(h_j, h_{j+1}, \tau)} - 1 + \frac{1}{M_r(h, \Gamma(h, g), \tau)} - 1 \right) \longrightarrow (\eta_3 + \eta_4) \left(\frac{1}{M_r(h, \Gamma(h, g), \tau)} - 1 \right), \quad \text{as } J \longrightarrow \infty.
 \end{aligned} \tag{51}$$

Hence,

$$\begin{aligned}
 \limsup_{j \rightarrow \infty} \left(\frac{1}{M_r(h_{j+1}, \Gamma(h, g), \tau)} - 1 \right) \\
 \leq (\eta_3 + \eta_4) \left(\frac{1}{M_r(h, \Gamma(h, g), \tau)} - 1 \right), \quad \text{for } \tau \gg 0.
 \end{aligned} \tag{52}$$

Now, from (43), (50), and (52), for $\tau \gg 0$,

$$\begin{aligned}
 \frac{1}{M_r(h, \Gamma(h, g), \tau)} - 1 &\leq (\eta_3 + \eta_4) \left(\frac{1}{M_r(h, \Gamma(h, g), \tau)} - 1 \right), \\
 &\Rightarrow (1 - \eta_3 + \eta_4) \left(\frac{1}{M_r(h, \Gamma(h, g), \tau)} - 1 \right) \leq 0, \quad \text{for } \tau \gg 0,
 \end{aligned} \tag{53}$$

which is a contradiction. As $(1 - \eta_3 + \eta_4) \neq 0$, we get that $M_r(h, \Gamma(h, g), \tau) = 1 \Rightarrow \Gamma(h, g) = h$ for $\tau \gg 0$.

Uniqueness: suppose (g_1, h_1) and (h_1, g_1) are other coupled FP pairs in $G \times G$ such that $\Gamma(g_1, h_1) = g_1$ and

$\Gamma(h_1, g_1) = h_1$. Now, from (28) and by using Definition 4 (4), for $\tau \gg 0$,

$$\begin{aligned}
 \frac{1}{M_r(g, g_1, \tau)} - 1 &= \frac{1}{M_r(\Gamma(g, h), \Gamma(g_1, h_1), \tau)} - 1, \\
 &\leq \eta_1 \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g, g_1, \tau) * M_r(g_1, \Gamma(g_1, h_1), \tau)}{M_r(g, \Gamma(g, h), \tau) * M_r(g, \Gamma(g_1, h_1), 2\tau)} - 1 \right) \\
 &\quad + \eta_3 \left(\frac{M_r(g, \Gamma(g, h), \tau)}{M_r(g, \Gamma(g_1, h_1), 2\tau)} - 1 + \frac{M_r(g_1, \Gamma(g_1, h_1), \tau)}{M_r(g, \Gamma(g_1, h_1), 2\tau)} - 1 \right) \\
 &\quad + \eta_4 \left(\frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 + \frac{1}{M_r(g_1, \Gamma(g_1, h_1), \tau)} - 1 \right) \\
 &= \eta_1 \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g, g_1, \tau) * M_r(g_1, g_1, \tau)}{M_r(g, g, \tau) * M_r(g, g_1, 2\tau)} - 1 \right) \\
 &\quad + \eta_3 \left(\frac{M_r(g, g, \tau)}{M_r(g, g_1, 2\tau)} - 1 + \frac{M_r(g_1, g_1, \tau)}{M_r(g, g_1, 2\tau)} - 1 \right) + \eta_4 \left(\frac{1}{M_r(g, g, \tau)} - 1 + \frac{1}{M_r(g_1, g_1, \tau)} - 1 \right) \quad (54) \\
 &= \eta_1 \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g, g_1, \tau)}{M_r(g, g_1, 2\tau)} - 1 \right) + \eta_3 \left(\frac{1}{M_r(g, g_1, 2\tau)} - 1 + \frac{1}{M_r(g, g_1, 2\tau)} - 1 \right) \\
 &\leq \eta_1 \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g, g_1, \tau)}{M_r(g, g_1, \tau) * M_r(g_1, g_1, \tau)} - 1 \right) \\
 &\quad + \eta_3 \left(\frac{1}{M_r(g, g_1, \tau) * M_r(g_1, g_1, \tau)} - 1 + \frac{1}{M_r(g, g_1, \tau) * M_r(g_1, g_1, \tau)} - 1 \right) \\
 &= (\eta_1 + 2\eta_3) \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) = (\eta_1 + 2\eta_3) \left(\frac{1}{M_r(\Gamma(g, h), \Gamma(g_1, h_1), \tau)} - 1 \right) \\
 &\leq (\eta_1 + 2\eta_3)^2 \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) \leq \dots \leq (\eta_1 + 2\eta_3)^J \left(\frac{1}{M_r(g, g_1, \tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty.
 \end{aligned}$$

Hence, $M_r(g, g_1, \tau) = 1 \Rightarrow g = g_1$ for $\tau \gg 0$. Next, we shall show that $h = h_1$, again from (28), and by using Definition 4 (4), for $\tau \gg 0$, we have

$$\begin{aligned}
 \frac{1}{M_r(h, h_1, \tau)} - 1 &= \frac{1}{M_r(\Gamma(h, g), \Gamma(h_1, g_1), \tau)} - 1 \\
 &\leq \eta_1 \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h, h_1, \tau) * M_r(h_1, \Gamma(h_1, g_1), \tau)}{M_r(h, \Gamma(h, g), \tau) * M_r(h, \Gamma(h_1, g_1), 2\tau)} - 1 \right) \\
 &\quad + \eta_3 \left(\frac{M_r(h, \Gamma(h, g), \tau)}{M_r(h, \Gamma(h_1, g_1), 2\tau)} - 1 + \frac{M_r(h_1, \Gamma(h_1, g_1), \tau)}{M_r(h, \Gamma(h_1, g_1), 2\tau)} - 1 \right) \\
 &\quad + \eta_4 \left(\frac{1}{M_r(h, \Gamma(h, g), \tau)} - 1 + \frac{1}{M_r(h_1, \Gamma(h_1, g_1), \tau)} - 1 \right)
 \end{aligned}$$

$$\begin{aligned}
&= \eta_1 \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h, h_1, \tau) * M_r(h_1, h_1, \tau)}{M_r(h, h_1, \tau) * M_r(h, h_1, 2\tau)} - 1 \right) \\
&\quad + \eta_3 \left(\frac{M_r(h, h_1, \tau)}{M_r(h, h_1, 2\tau)} - 1 + \frac{M_r(h_1, h_1, \tau)}{M_r(h, h_1, 2\tau)} - 1 \right) + \eta_4 \left(\frac{1}{M_r(h, h, \tau)} - 1 + \frac{1}{M_r(h_1, h_1, \tau)} - 1 \right) \\
&= \eta_1 \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h, h_1, \tau)}{M_r(h, h_1, 2\tau)} - 1 \right) + \eta_3 \left(\frac{1}{M_r(h, h_1, 2\tau)} - 1 + \frac{1}{M_r(h, h_1, 2\tau)} - 1 \right) \\
&\leq \eta_1 \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(h, h_1, \tau)}{M_r(h, h_1, \tau) * M_r(h_1, h_1, \tau)} - 1 \right) \\
&\quad + \eta_3 \left(\frac{1}{M_r(h, h_1, \tau) * M_r(h_1, h_1, \tau)} - 1 + \frac{1}{M_r(h, h_1, \tau) * M_r(h_1, h_1, \tau)} - 1 \right) \\
&= (\eta_1 + 2\eta_3) \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) = (\eta_1 + 2\eta_3) \left(\frac{1}{M_r(\Gamma(h, g), \Gamma(h_1, g_1), \tau)} - 1 \right) \\
&\leq (\eta_1 + 2\eta_3)^2 \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) \leq \dots \leq (\eta_1 + 2\eta_3)^J \left(\frac{1}{M_r(h, h_1, \tau)} - 1 \right) \longrightarrow 0, \quad \text{as } J \longrightarrow \infty.
\end{aligned} \tag{55}$$

Hence, $M_r(h, h_1, \tau) = 1 \Rightarrow h = h_1$ for $\tau \gg 0$.

□

Corollary 2. Let $\Gamma: G \times G \longrightarrow G$ be a mapping on a complete FCMS $(G, M_r, *)$ in which M_r is triangular and Γ satisfies the inequality

$$\begin{aligned}
\frac{1}{M_r(\Gamma(g, h), \Gamma(\xi, \kappa), \tau)} - 1 &\leq \eta_1 \left(\frac{1}{M_r(g, \xi, \tau)} - 1 \right) + \eta_2 \left(\frac{M_r(g, \xi, \tau) * M_r(\xi, \Gamma(\xi, \kappa), \tau)}{M_r(g, \Gamma(g, h), \tau) * M_r(g, \Gamma(\xi, \kappa), 2\tau)} - 1 \right) \\
&\quad + \eta_4 \left(\frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 + \frac{1}{M_r(\xi, \Gamma(\xi, \kappa), \tau)} - 1 \right),
\end{aligned} \tag{56}$$

$\forall g, h, \xi, \kappa \in G, \tau \gg 0, \eta_1 \in (0, 1),$ and $\eta_2, \eta_4 \geq 0$ with $(\eta_1 + \eta_2 + 2\eta_4) < 1$. Then, Γ has a unique coupled FP in G .

Corollary 3. Let $\Gamma: G \times G \longrightarrow G$ be a mapping on a complete FCMS $(G, M_r, *)$ in which M_r is triangular and Γ satisfies the inequality

$$\begin{aligned}
\frac{1}{M_r(\Gamma(g, h), \Gamma(\xi, \kappa), \tau)} - 1 &\leq \eta_1 \left(\frac{1}{M_r(g, \xi, \tau)} - 1 \right) \\
&\quad + \eta_3 \left(\frac{M_r(g, \Gamma(g, h), \tau)}{M_r(g, \Gamma(\xi, \kappa), 2\tau)} - 1 + \frac{M_r(\xi, \Gamma(\xi, \kappa), \tau)}{M_r(g, \Gamma(\xi, \kappa), 2\tau)} - 1 \right) \\
&\quad + \eta_4 \left(\frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 + \frac{1}{M_r(\xi, \Gamma(\xi, \kappa), \tau)} - 1 \right),
\end{aligned} \tag{57}$$

$\forall g, h, \xi, \kappa \in G, \tau \gg 0, \eta_1 \in (0, 1),$ and $\eta_3, \eta_4 \geq 0$ with $(\eta_1 + 2\eta_3 + 2\eta_4) < 1$. Then, Γ has a unique coupled FP in G .

Corollary 4. Let $\Gamma: G \times G \longrightarrow G$ be a mapping on a complete FCMS $(G, M_r, *)$ in which M_r is triangular and Γ satisfies the inequality

$$\frac{1}{M_r(\Gamma(g, h), \Gamma(\xi, \kappa), \tau)} - 1 \leq \eta_1 \left(\frac{1}{M_r(g, \xi, \tau)} - 1 \right) + \eta_4 \left(\frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 + \frac{1}{M_r(\xi, \Gamma(\xi, \kappa), \tau)} - 1 \right), \tag{58}$$

$\forall g, h, \xi, \kappa \in G, \quad \tau \gg 0, \quad \eta_1 \in (0, 1), \quad \text{and} \quad \eta_4 \geq 0 \quad \text{with}$
 $(\eta_1 + 2\eta_4) < 1$. Then, Γ has a unique coupled FP in G .

Example 2. From Example 1, we define a FM $M_r: G \times G \times (0, \infty) \longrightarrow [0, 1]$ by

$$M_r(g, h, \tau) = \frac{\tau}{\tau + |(g - h)/3|}, \quad \forall g, h \in G \text{ and } \tau > 0. \quad (59)$$

Then, one can easily prove the triangular property of FCM from the above example and $(G, M_r, *)$ is a complete FCMS. We define a mapping $\Gamma: G \times G \longrightarrow G$ by

$$\Gamma(g, h) = \begin{cases} \frac{2g}{5}, & g, h \in [0, 1], \\ \frac{2g+h}{h} + 1, & g, h \in [1, \infty). \end{cases} \quad (60)$$

Then,

$$\frac{1}{M_r(\Gamma(g, h), \Gamma(\xi, \kappa), \tau)} - 1 = \frac{2}{5} \left(\frac{1}{M_r(g, \xi, \tau)} - 1 \right), \quad (61)$$

$\forall g, h, \xi, \kappa \in G$ and $\tau \gg 0$. Hence, we proved that a mapping Γ is a coupled fc - contractive. Now, By using Definition 4 (4) to simplify the η_2 rational term of (28), for $\tau \gg 0$,

$$\begin{aligned} & \frac{M_r(g, \xi, \tau) * M_r(\xi, \Gamma(\xi, \kappa), \tau)}{M_r(g, \Gamma(g, h), \tau) * M_r(g, \Gamma(\xi, \kappa), 2\tau)} - 1 \\ & \leq \frac{M_r(g, \xi, \tau) * M_r(\xi, \Gamma(\xi, \kappa), \tau)}{M_r(g, \Gamma(g, h), \tau) * M_r(g, \xi, \tau) * M_r(\xi, \Gamma(\xi, \kappa), \tau)} - 1 = \frac{1}{M_r(g, \Gamma(g, h), \tau)} = \frac{g}{5\tau}, \end{aligned} \quad (62)$$

$\forall g, h, \xi, \kappa \in G$. Again, by using Definition 4 (4) and (61) to simplify the η_3 rational term of (28), for $\tau \gg 0$, we have

$$\begin{aligned} & \frac{M_r(g, \Gamma(g, h), \tau)}{M_r(g, \Gamma(\xi, \kappa), 2\tau)} - 1 + \frac{M_r(\xi, \Gamma(\xi, \kappa), \tau)}{M_r(g, \Gamma(\xi, \kappa), 2\tau)} - 1 \\ & \leq \frac{M_r(g, \Gamma(g, h), \tau)}{M_r(g, \Gamma(g, h), \tau) * M_r(\Gamma(g, h), \Gamma(\xi, \kappa), \tau)} - 1 + \frac{M_r(\xi, \Gamma(\xi, \kappa), \tau)}{M_r(g, \xi, \tau) * M_r(\xi, \Gamma(\xi, \kappa), \tau)} - 1 \\ & = \frac{1}{M_r(\Gamma(g, h), \Gamma(\xi, \kappa), \tau)} - 1 + \frac{1}{M_r(g, \xi, \tau)} - 1 = \frac{7}{5} \left(\frac{1}{M_r(g, \xi, \tau)} - 1 \right) = \frac{7}{15\tau} |g - \xi|, \end{aligned} \quad (63)$$

$\forall g, h, \xi, \kappa \in G$. After simple routine calculation, we can get the η_4 term result of (28) as follows:

$$\begin{aligned} & \frac{1}{M_r(g, \Gamma(g, h), \tau)} - 1 \\ & + \frac{1}{M_r(\xi, \Gamma(\xi, \kappa), \tau)} - 1 = \frac{1}{5\tau} |g + \xi|, \quad \text{for } \tau \gg 0. \end{aligned} \quad (64)$$

Hence, from the above, we conclude that all the conditions of the Theorem 2 are satisfied with $\eta_1 = 2/5, \eta_2 = 1/6, \eta_3 = \eta_4 = 1/15$, and $\Gamma(g, h) = \Gamma(4, 4) = 4 \in [0, \infty)$.

4. Application

In this section, we present an application on Lebesgue integral-type contraction mapping to support our main work.

In 2002, Branciari proved the following result on complete metric space for a unique FP (see [36]).

Theorem 3. Let (G, d) be a complete metric space, $\eta_1 \in (0, 1)$, and $\Gamma: G \longrightarrow G$ be a mapping such that for each $g, h \in G$,

$$\int_0^{d(\Gamma g, \Gamma h)} \varphi(\tau) d\tau \leq \eta_1 \int_0^{d(g, h)} \varphi(\tau) d\tau, \quad (65)$$

where $\varphi: [0, \infty) \longrightarrow [0, \infty)$ is a Lebesgue integrable mapping which is summable (i.e., with finite integral on each compact subset of $[0, \infty)$) and for each $\kappa > 0$,

$$\int_0^\kappa \varphi(s) ds > 0. \quad (66)$$

Then, Γ has a unique FP $u \in G$ such that for any $g \in G$, $\lim_{J \rightarrow \infty} \Gamma^J g = u$."

Now, we are in the position to use the above concept and to prove a unique coupled FP theorem in complete FCMSs.

Theorem 4. Let $\Gamma: G \times G \longrightarrow G$ be a mapping on a complete FCMS $(G, M_r, *)$ in which M_r is triangular and satisfies

$$\int_0^{((1/(M_r(\Gamma(g,h),\Gamma(\xi,\kappa),\tau)))^{-1})} \varphi(\tau) d\tau \leq \eta_1 \int_0^{((1/M_r(g,\xi,\tau))^{-1})} \varphi(\tau) d\tau + \eta_2 \int_0^{((M_r(g,\xi,\tau)/(M_r(g,\Gamma(g,h),\tau) * M_r(\xi,\Gamma(g,h),2\tau)))^{-1})} \varphi(\tau) d\tau, \quad (67)$$

for all $g, h, \xi, \kappa \in G$, $\tau \gg 0$, $\eta_1 \in (0, 1)$, and $\eta_2 \geq 0$, and $\varphi: [0, \infty) \longrightarrow [0, \infty)$ is a Lebesgue integrable mapping which is summable (i.e., with finite integral on each compact subset of $[0, \infty)$) and for each $\kappa > 0$,

$$\int_0^\kappa \varphi(\tau) d\tau > 0. \quad (68)$$

Then, Γ has a unique coupled FP in G .

Proof. Let any $g_o, h_o \in G$; we define sequences $\{g_J\}$ and $\{h_J\}$ in G such that

$$\Gamma(g_J, h_J) = g_{J+1}, \quad \text{and } \Gamma(h_J, g_J) = h_{J+1}, \quad \text{for } J \geq 0. \quad (69)$$

Now, from (67) and from the proof of Theorem 1, for $\tau \gg 0$,

$$\int_0^{((1/(M_r(g_J, g_{J+1}, \tau)))^{-1})} \varphi(\tau) d\tau = \int_0^{((1/M_r(\Gamma(g_{J-1}, h_{J-1}), \Gamma(g_J, h_J), \tau))^{-1})} \varphi(\tau) d\tau \leq \eta_1 \int_0^{((1/M_r(g_{J-1}, g_J, \tau))^{-1})} \varphi(\tau) d\tau. \quad (70)$$

Similarly, again by using the arguments, we have

$$\begin{aligned} & \int_0^{((1/(M_r(g_{J-1}, g_J, \tau)))^{-1})} \varphi(\tau) d\tau \\ & \leq \eta_1 \int_0^{((1/M_r(g_{J-2}, g_{J-1}, \tau))^{-1})} \varphi(\tau) d\tau, \quad \text{for } \tau \gg 0. \end{aligned} \quad (71)$$

Now, from (70) and (71) and by induction, for $\tau \gg 0$, we have

$$\begin{aligned} \int_0^{((1/(M_r(g_J, g_{J+1}, \tau)))^{-1})} \varphi(\tau) d\tau & \leq \eta_1 \int_0^{((1/M_r(g_{J-1}, g_J, \tau))^{-1})} \varphi(\tau) d\tau, \\ & \leq \eta_1^2 \int_0^{((1/M_r(g_{J-2}, g_{J-1}, \tau))^{-1})} \varphi(\tau) d\tau \\ & \leq \dots \leq \eta_1^J \int_0^{((1/M_r(g_0, g_1, \tau))^{-1})} \varphi(\tau) d\tau \longrightarrow 0, \quad \text{as } J \longrightarrow \infty. \end{aligned} \quad (72)$$

This shows that $\{g_J\}$ is a fc - contractive sequence, and therefore,

$$\lim_{J \rightarrow \infty} \int_0^{((1/(M_r(g_J, g_{J+1}, \tau)))^{-1})} \varphi(\tau) d\tau = 0 \Rightarrow \lim_{J \rightarrow \infty} \left(\frac{1}{M_r(g_J, g_{J+1}, \tau)} - 1 \right) = 0, \quad \text{for } \tau \gg 0. \quad (73)$$

Hence, we get that

$$\lim_{J \rightarrow \infty} M_r(g_J, g_{J+1}, \tau) = 1, \quad \text{for } \tau \gg 0. \quad (74)$$

Now, for $\ell > J$ and for $\tau \gg 0$,

$$\begin{aligned}
\int_0^{((1/M_r(g_J, g_\ell, \tau))^{-1})} \varphi(\tau) d\tau &\leq \int_0^{((1/M_r(g_J, g_{J+1}, \tau))^{-1})} \varphi(\tau) d\tau + \int_0^{((1/M_r(g_{J+1}, g_{J+2}, \tau))^{-1})} \varphi(\tau) d\tau + \cdots + \int_0^{((1/M_r(g_{\ell-1}, g_\ell, \tau))^{-1})} \varphi(\tau) d\tau, \\
&\leq \eta_1^J \int_0^{((1/M_r(g_0, g_1, \tau))^{-1})} \varphi(\tau) d\tau + \eta_1^{J+1} \int_0^{((1/M_r(g_0, g_1, \tau))^{-1})} \varphi(\tau) d\tau + \cdots + \eta_1^{\ell-1} \int_0^{((1/M_r(g_0, g_1, \tau))^{-1})} \varphi(\tau) d\tau \\
&= (\eta - 1^J + \eta_1^{J+1} + \cdots + \eta_1^{\ell-1}) \int_0^{((1/M_r(g_0, g_1, \tau))^{-1})} \varphi(\tau) d\tau \\
&= \frac{\eta_1^J}{1 - \eta_1} \int_0^{((1/M_r(g_0, g_1, \tau))^{-1})} \varphi(\tau) d\tau \longrightarrow 0, \quad \text{as } J \longrightarrow \infty.
\end{aligned} \tag{75}$$

We get that

$$\begin{aligned}
\lim_{J \longrightarrow \infty} \int_0^{((1/M_r(g_J, g_\ell, \tau))^{-1})} \varphi(\tau) d\tau &= 0 \\
\Rightarrow \lim_{J \longrightarrow \infty} \left(\frac{1}{M_r(g_J, g_\ell, \tau)} - 1 \right) &= 0, \quad \text{for } \tau \gg 0.
\end{aligned} \tag{76}$$

Hence, $\{g_J\}$ is a Cauchy sequence. Since, by the completeness of $(G, M_r, *)$, $\exists g \in G$ so that

$$\lim_{J \longrightarrow \infty} M_r(g_J, g, \tau) = 1, \quad \text{for } \tau \gg 0. \tag{77}$$

Now, for sequence $\{h_J\}$ from (67) and from the proof of Theorem 1, for $\tau \gg 0$, we have

$$\begin{aligned}
\int_0^{((1/M_r(h_J, h_{J+1}, \tau))^{-1})} \varphi(\tau) d\tau &= \int_0^{((1/M_r(\Gamma(h_{J-1}, g_{J-1}), \Gamma(h_J, g_J), t))^{-1})} \varphi(\tau) d\tau, \\
&\leq \eta_1 \int_0^{((1/M_r(h_{J-1}, h_J, \tau))^{-1})} \varphi(\tau) d\tau.
\end{aligned} \tag{78}$$

Similarly, again by using the same arguments, we have

$$\int_0^{((1/M_r(h_{J-1}, h_J, \tau))^{-1})} \varphi(\tau) d\tau \leq \eta_1 \int_0^{((1/M_r(h_{J-2}, g_{J-1}, \tau))^{-1})} \varphi(\tau) d\tau, \quad \text{for } \tau \gg 0. \tag{79}$$

Now, from (78) and (79) and by induction, for $\tau \gg 0$, we have

$$\begin{aligned}
\int_0^{((1/M_r(h_J, h_{J+1}, \tau))^{-1})} \varphi(\tau) d\tau &\leq \eta_1 \int_0^{((1/M_r(h_{J-1}, h_J, \tau))^{-1})} \varphi(\tau) d\tau, \\
&\leq \eta_1^2 \int_0^{((1/M_r(h_{J-2}, g_{J-1}, \tau))^{-1})} \varphi(\tau) d\tau \\
&\leq \cdots \leq \eta_1^J \int_0^{((1/M_r(h_0, h_1, \tau))^{-1})} \varphi(\tau) d\tau \longrightarrow 0, \quad \text{as } J \longrightarrow \infty.
\end{aligned} \tag{80}$$

This shows that $\{h_J\}$ is a fc -contractive sequence, and therefore,

$$\lim_{J \longrightarrow \infty} \int_0^{((1/M_r(h_J, h_{J+1}, \tau))^{-1})} \varphi(\tau) d\tau = 0 \Rightarrow \lim_{J \longrightarrow \infty} \left(\frac{1}{M_r(h_J, h_{J+1}, \tau)} - 1 \right) = 0, \quad \text{for } \tau \gg 0. \tag{81}$$

Hence, we get that

$$\lim_{J \rightarrow \infty} M_r(h_J, h_{J+1}, \tau) = 1, \quad \text{for } \tau \gg 0. \quad (82)$$

Now, for $\ell > J$ and for $\tau \gg 0$, we have

$$\begin{aligned} \int_0^{((1/M_r(h_J, h_\ell, \tau))^{-1})} \varphi(\tau) d\tau &\leq \int_0^{((1/M_r(h_J, h_{J+1}, \tau))^{-1})} \varphi(\tau) d\tau + \int_0^{((1/M_r(h_{J+1}, h_{J+2}, \tau))^{-1})} \varphi(\tau) d\tau + \cdots + \int_0^{((1/M_r(h_{\ell-1}, h_\ell, \tau))^{-1})} \varphi(\tau) d\tau, \\ &\leq \eta_2^J \int_0^{((1/M_r(h_o, h_1, \tau))^{-1})} \varphi(\tau) d\tau + \eta_1^{J+1} \int_0^{((1/M_r(h_o, h_1, \tau))^{-1})} \varphi(\tau) d\tau + \cdots + \eta_1^{\ell-1} \int_0^{((1/M_r(h_o, h_1, \tau))^{-1})} \varphi(\tau) d\tau \\ &= (\eta_1^J + \eta_1^{J+1} + \cdots + \eta_1^{\ell-1}) \int_0^{((1/M_r(h_o, h_1, \tau))^{-1})} \varphi(\tau) d\tau \\ &= \frac{\eta_1^J}{1 - \eta_1} \int_0^{((1/M_r(h_o, h_1, \tau))^{-1})} \varphi(\tau) d\tau \longrightarrow 0, \quad \text{as } J \longrightarrow \infty. \end{aligned} \quad (83)$$

We get that

$$\lim_{J \rightarrow \infty} \int_0^{((1/M_r(h_J, h_\ell, \tau))^{-1})} \varphi(\tau) d\tau = 0 \Rightarrow \lim_{J \rightarrow \infty} \left(\frac{1}{M_r(h_J, h_\ell, \tau)} - 1 \right) = 0, \quad \text{for } \tau \gg 0. \quad (84)$$

Hence, $\{h_J\}$ is a Cauchy sequence. Since, by the completeness of $(G, M_r, *)$, $\exists h \in G$ so that

$$\lim_{J \rightarrow \infty} M_r(h_J, h, \tau) = 1, \quad \text{for } \tau \gg 0. \quad (85)$$

Now, we prove $\Gamma(g, h) = g$. Then, from (67), (74), and (77), for $\tau \gg 0$,

$$\begin{aligned} \int_0^{(1/M_r(g, \Gamma(g, h), \tau))^{-1}} &\leq \int_0^{((1/M_r(g, g_{J+1}, \tau))^{-1})} + \int_0^{((1/M_r(\Gamma(g_J, h_J), \Gamma(g, h), \tau))^{-1})}, \\ &\leq \int_0^{((1/M_r(g, g_{J+1}, \tau))^{-1})} + \eta_1 \int_0^{((1/M_r(g_J, g, \tau))^{-1})} + \eta_2 \int_0^{((M_r(g_J, g, \tau) / (M_r(g_J, \Gamma(g_J, h_J), \tau) * M_r(g, \Gamma(g_J, h_J), 2\tau)))^{-1})} \\ &= \int_0^{((1/M_r(g, g_{J+1}, \tau))^{-1})} + \eta_1 \int_0^{((1/M_r(g_J, g, \tau))^{-1})} + \eta_2 \int_0^{(M_r(g_J, g, \tau) / M_r(g_J, g_{J+1}, \tau) * M_r(g, g_{J+1}, 2\tau))^{-1}} \longrightarrow 0, \quad \text{as } J \longrightarrow \infty. \end{aligned} \quad (86)$$

Hence, $M_r(g, \Gamma(g, h), \tau) = 1 \Rightarrow \Gamma(g, h) = g$ for $\tau \gg 0$. Next, we shall prove that $\Gamma(h, g) = h$; again from (67), (82), and (85), for $\tau \gg 0$,

$$\begin{aligned}
\int_0^{(1/M_r(h, \Gamma(h, g), \tau))^{-1}} &\leq \int_0^{((1/M_r(h, h_{j+1}, \tau))^{-1})} + \int_0^{((1/M_r(\Gamma(h_j, g_j), \Gamma(h, g), \tau))^{-1})}, \\
&\leq \int_0^{((1/M_r(h, h_{j+1}, \tau))^{-1})} + \eta_1 \int_0^{((1/M_r(h_j, h, \tau))^{-1})} + \eta_2 \int_0^{((M_r(h_j, h, \tau) / (M_r(h_j, \Gamma(h_j, g_j), \tau) * M_r(h, \Gamma(h_j, g_j), 2\tau)))^{-1})} \\
&= \int_0^{((1/M_r(h, h_{j+1}, \tau))^{-1})} + \eta_1 \int_0^{((1/M_r(h_j, h, \tau))^{-1})} + \eta_2 \int_0^{((M_r(h_j, h, \tau) / (M_r(h_j, h_{j+1}, \tau) * M_r(h, h_{j+1}, 2\tau)))^{-1})} \longrightarrow 0, \quad \text{as } J \longrightarrow \infty.
\end{aligned} \tag{87}$$

Hence, $M_r(h, \Gamma(h, g), \tau) = 1$ which implies $\Gamma(h, g) = h$ for $\tau \gg 0$.

Uniqueness: suppose (g_1, h_1) and (h_1, g_1) are other coupled fixed-point pairs in $G \times G$ such that $\Gamma(g_1, h_1) = g_1$

and $\Gamma(h_1, g_1) = h_1$. Now, from (67) and from the proof of Theorem 1, for $\tau \gg 0$,

$$\begin{aligned}
\int_0^{((1/M_r(g, g_1, \tau))^{-1})} \varphi(\tau) d\tau &= \int_0^{((1/M_r(\Gamma(g, h), \Gamma(g_1, h_1), \tau))^{-1})} \varphi(\tau) d\tau, \\
&\leq \eta_1 \int_0^{((1/M_r(g, g_1, \tau))^{-1})} \varphi(\tau) d\tau \\
&= \eta_1 \int_0^{((1/M_r(\Gamma(g, h), \Gamma(g_1, h_1), \tau))^{-1})} \varphi(\tau) d\tau \\
&\leq \eta_1^2 \int_0^{((1/M_r(g, g_1, \tau))^{-1})} \varphi(\tau) d\tau \\
&\leq \dots \leq \eta_1^J \int_0^{((1/M_r(g, g_1, \tau))^{-1})} \varphi(\tau) d\tau \longrightarrow 0, \quad \text{as } J \longrightarrow \infty.
\end{aligned} \tag{88}$$

Hence, we get that $M_r(g, g_1, \tau) = 1 \Rightarrow g = g_1$ for $\tau \gg 0$. Next, we have to prove $h = h_1$, and now, by using (67) and from the proof of Theorem 1, for $\tau \gg 0$, we have that

$$\begin{aligned}
\int_0^{(1/(M_r(h, h_1, \tau))^{-1})} \varphi(\tau) d\tau &= \int_0^{(1/M_r(\Gamma(h, g), \Gamma(h_1, g_1), \tau))^{-1})} \varphi(\tau) d\tau, \\
&\leq \eta_1 \int_0^{(1/(M_r(h, h_1, \tau))^{-1})} \varphi(\tau) d\tau \\
&= \eta_1 \int_0^{(1/M_r(\Gamma(h, g), \Gamma(h_1, g_1), \tau))^{-1})} \varphi(\tau) d\tau \\
&\leq \eta_1^2 \int_0^{(1/(M_r(h, h_1, \tau))^{-1})} \varphi(\tau) d\tau \\
&\leq \dots \leq \eta_1^J \int_0^{(1/(M_r(h, h_1, \tau))^{-1})} \varphi(\tau) d\tau \longrightarrow 0, \quad \text{as } J \longrightarrow \infty.
\end{aligned} \tag{89}$$

Hence, we get that $M_r(h, h_1, \tau) = 1 \Rightarrow h = h_1$ for $\tau \gg 0$. \square

5. Conclusion

We established the new concept of rational coupled fc-contraction mapping in FCMSs and proved some unique rational coupled FP theorems in FCMSs under the rational coupled

fc-contraction conditions by using the “triangular property of fuzzy cone metric” with the help of some suitable examples to unify our work. In the last section, we presented an application of the Lebesgue integral-type coupled contraction theorem for unique rational coupled FP in complete FCMSs. By using this concept, one can prove more rational coupled-type fc-contraction results in complete FCMSs with different integral types of application to prove unique coupled FP results.

Data Availability

Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The authors are grateful to the Deanship of Scientific Research, King Saud University for funding through Vice Deanship of Scientific Research Chairs.

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Research Article

The Existence of Fixed Points for a Different Type of Contractions on Partial b -Metric Spaces

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Received 10 April 2021; Accepted 9 July 2021; Published 14 August 2021

Academic Editor: Ching-Feng Wen

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The aim of this study is to present fixed point results in the setting of partial b -metric spaces. A different type of contractions is used to prove fixed point results in the given space, which are real generalization of many well-known results. The readers are also provided with some very interesting examples to illustrate the feasibility of the proposed work.

1. Introduction and Preliminaries

The notion of metric space was initiated in 1906 by French Mathematician Frechet [1]. In metric fixed point theory, the Banach contraction principle [2] is one of the most fundamental tools to investigate the existence and uniqueness of solutions for contraction maps in a complete metric space. Since the inception of this principle, many authors studied fixed point theory vividly and enriched this field with different ideas. This classical result was generalized in different spaces, and different structures were attained using this topic and one may recall the existing notions, partial b -metric spaces [3], \mathfrak{R} -partial b -metric spaces [4], fuzzy cone b -metric spaces [5], G_b -metric spaces [6], orthogonal partial b -metric spaces [7], orthogonal m -metric spaces [8], and several others. More details can be found in [9–13].

One important and extensively used generalization of metric space is the notion of b -metric spaces, which was introduced by Czerwik [14] and is defined as follows.

Definition 1 (see [14]). Let \mathfrak{F} be a nonempty set and $s \geq 1$ be a given real number. A function $b: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ is

referred as a b -metric if the following conditions hold for all $\sigma, v, \varsigma \in \mathfrak{F}$:

- (b_i) $b(\sigma, \varsigma) = 0 \Leftrightarrow \sigma = \varsigma$
- (b_{ii}) $b(\sigma, \varsigma) = b(\varsigma, \sigma)$
- (b_{iii}) $b(\sigma, \varsigma) \leq s[b(\sigma, v) + b(v, \varsigma)]$

The triplet (\mathfrak{F}, b, s) is called a b -metric space.

Example 1 (see [15]). Let $\mathfrak{F} = \{1, 2, 3, \dots\} \cup \infty$ and define $b: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ by

$$b(\sigma, \varsigma) = \begin{cases} \left| \frac{1}{\sigma} - \frac{1}{\varsigma} \right|, & \text{if } \sigma, \varsigma \text{ are even or } \sigma\varsigma = \infty, \\ 0, & \text{if } \sigma = \varsigma, \\ 5, & \text{if } \rho, v \text{ are odd or } \sigma \neq \varsigma, \\ 2, & \text{otherwise.} \end{cases} \quad (1)$$

Then, (\mathfrak{F}, b, s) is a b -metric space with $s = 3$.

Matthews [16], while working on networking, observed that a self-distance may not be zero, i.e., $d(\sigma, \sigma) \neq 0$. A loop is a good example of this case. He not only generalized the classical Banach fixed point theorem but also established some convergence criterion in this setting to ensure existence of a fixed point. His further investigations led him to the introduction of partial metric space, which is defined as follows.

Definition 2 (see [16]). Let \mathfrak{F} be a nonempty set. A mapping $\mathfrak{l}: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ is referred as a partial metric if, for all $\sigma, \varsigma, v \in \mathfrak{F}$,

$$\begin{aligned} (\mathfrak{l}_i) \quad & \sigma = \varsigma \Leftrightarrow \mathfrak{l}(\sigma, \sigma) = \mathfrak{l}(\sigma, \varsigma) = \mathfrak{l}(\varsigma, \varsigma) \\ (\mathfrak{l}_{ii}) \quad & \mathfrak{l}(\sigma, \sigma) \leq \mathfrak{l}(\sigma, \varsigma) \\ (\mathfrak{l}_{iii}) \quad & \mathfrak{l}(\sigma, \varsigma) = \mathfrak{l}(\varsigma, \sigma) \\ (\mathfrak{l}_{iv}) \quad & \mathfrak{l}(\sigma, \varsigma) \leq \mathfrak{l}(\sigma, v) + \mathfrak{l}(v, \varsigma) - \mathfrak{l}(v, v) \end{aligned}$$

Then, pair $(\mathfrak{F}, \mathfrak{l})$ is referred as a partial metric space.

Example 2 (see [16]). Let $\mathfrak{F} = [0, \infty)$ and define $\mathfrak{l}: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ by

$$\mathfrak{l}(\sigma, \varsigma) = \max\{\sigma, \varsigma\} \quad \text{for all } \sigma, \varsigma \in \mathfrak{F}. \quad (2)$$

Then, the pair $(\mathfrak{F}, \mathfrak{l})$ is a partial metric space.

Example 3 (see [16]). Let $\mathfrak{F} = \{[\sigma, \varsigma]: \sigma, \varsigma \in (0, \infty)\}$ and define $\mathfrak{l}: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ by

$$\mathfrak{l}([\sigma, \varsigma], [v, \rho]) = \max\{\varsigma, \rho\} - \min\{\sigma, v\}. \quad (3)$$

The pair $(\mathfrak{F}, \mathfrak{l})$ is a partial metric space.

Definition 3 (see [16]). Let $(\mathfrak{F}, \mathfrak{l})$ be a partial metric space and $\{\sigma_\kappa\}$ be a sequence in \mathfrak{F} . Then,

- (i) $\{\sigma_\kappa\}$ converges in $\mathfrak{F} \Leftrightarrow \mathfrak{l}(\varsigma, \varsigma) = \lim_{\kappa \rightarrow \infty} \mathfrak{l}(\varsigma, \sigma_\kappa)$.
- (ii) $\{\sigma_\kappa\}$ is referred as a Cauchy sequence if $\lim_{\kappa, j \rightarrow \infty} \mathfrak{l}(\sigma_\kappa, \sigma_j)$ exists and is finite.
- (iii) A partial metric space is complete if each Cauchy sequence converges in \mathfrak{F} satisfying

$$\mathfrak{l}(\varsigma, \varsigma) = \lim_{\kappa, j \rightarrow \infty} \mathfrak{l}(\sigma_\kappa, \sigma_j). \quad (4)$$

The concept of partial metric space was further extended to partial b -metric space by Shukla [3] in 2014 by combining the partial metric space and b -metric space.

Definition 4 (see [3]). Let \mathfrak{F} be a nonempty set and $s \geq 1$ be a given real number. A mapping $\mathfrak{l}_b: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ is referred as a partial b -metric if, for all $\sigma, \varsigma, v \in \mathfrak{F}$,

$$\begin{aligned} (\mathfrak{l}_{bi}) \quad & \sigma = v \Leftrightarrow \mathfrak{l}_b(v, v) = \mathfrak{l}_b(v, \sigma) = \mathfrak{l}_b(\sigma, \sigma) \\ (\mathfrak{l}_{bii}) \quad & \mathfrak{l}_b(v, v) \leq \mathfrak{l}_b(\sigma, v) \\ (\mathfrak{l}_{biii}) \quad & \mathfrak{l}_b(\varsigma, \sigma) = \mathfrak{l}_b(\sigma, \varsigma) \end{aligned}$$

$$(\mathfrak{l}_{biv}) \quad \mathfrak{l}_b(\sigma, \varsigma) \leq s[\mathfrak{l}_b(\sigma, v) + \mathfrak{l}_b(v, \varsigma)] - \mathfrak{l}_b(v, v)$$

The triplet $(\mathfrak{F}, \mathfrak{l}_b, s)$ is referred as a partial b -metric space with $s \in [1, \infty)$.

Definition 5 (see [3]). Let $(\mathfrak{F}, \mathfrak{l}_b, s)$ be a partial b -metric space and $\{\sigma_\kappa\}$ be a sequence in \mathfrak{F} . Then,

- (i) $\{\sigma_\kappa\}$ converges in $\mathfrak{F} \Leftrightarrow \mathfrak{l}_b(\varsigma, \varsigma) = \lim_{\kappa \rightarrow \infty} \mathfrak{l}_b(\varsigma, \sigma_\kappa)$.
- (ii) $\{\sigma_\kappa\}$ is referred as a Cauchy sequence if $\lim_{\kappa, j \rightarrow \infty} \mathfrak{l}_b(\sigma_\kappa, \sigma_j)$ exists and is finite.
- (iii) A partial b -metric space is complete if each Cauchy sequence converges in \mathfrak{F} such that

$$\mathfrak{l}_b(\varsigma, \varsigma) = \lim_{\kappa, j \rightarrow \infty} \mathfrak{l}_b(\sigma_\kappa, \sigma_j). \quad (5)$$

Remark 1

- (i) In the partial b -metric space, $\mathfrak{l}_b(\sigma, \varsigma) = 0 \Rightarrow \sigma = \varsigma$. The converse implication does not hold in general.
- (ii) A partial b -metric is a generalization of a partial metric.

Example 4 (see [3]). Let $\mathfrak{F} = [0, \infty)$ (with $\xi > 1$) and define $\mathfrak{l}_b: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ by

$$\mathfrak{l}_b(\kappa, \varsigma) = \{\max(\kappa, \varsigma)\}^\xi + (\kappa - \varsigma)^\xi \quad \text{for all } \kappa, \varsigma \in \mathfrak{F}. \quad (6)$$

\mathfrak{l}_b is referred as a partial b -metric with $s = 2^\xi$. For $\varsigma > 0$, we have $\mathfrak{l}_b(\varsigma, \varsigma) \neq 0$, so \mathfrak{l}_b is not a b -metric. On the contrary, for $\kappa, \sigma, \varsigma \in \mathfrak{F}$ with $\kappa > \sigma > \varsigma$, we have

$$(\kappa - \sigma)^\xi + (\sigma - \varsigma)^\xi < (\kappa - \varsigma)^\xi. \quad (7)$$

Observe that

$$\begin{aligned} \mathfrak{l}_b(\kappa, \varsigma) &= \kappa^\xi + (\sigma - \varsigma)^\xi, \\ \mathfrak{l}_b(\kappa, \sigma) + \mathfrak{l}_b(\sigma, \varsigma) - \mathfrak{l}_b(\sigma, \sigma) &= \kappa^\xi + (\kappa - \sigma)^\xi + (\kappa - \varsigma)^\xi. \end{aligned} \quad (8)$$

One can then write

$$\mathfrak{l}_b(\kappa, \sigma) + \mathfrak{l}_b(\sigma, \varsigma) - \mathfrak{l}_b(\sigma, \sigma) < \mathfrak{l}_b(\kappa, \varsigma). \quad (9)$$

Thus, \mathfrak{l}_b is not a partial metric.

Lemma 1 (see [3])

- (1) In a partial b -metric space $(\mathfrak{F}, \mathfrak{l}_b, s)$, $\{\sigma_\kappa\}$ is a \mathfrak{l}_b -Cauchy sequence \Leftrightarrow it is a b -Cauchy sequence in the b -metric space (\mathfrak{F}, b) .
- (2) A partial b -metric space $(\mathfrak{F}, \mathfrak{l}_b)$ is \mathfrak{l}_b -complete \Leftrightarrow the b -metric space (\mathfrak{F}, b) is b -complete. Moreover, $\lim_{\kappa \rightarrow \infty} b(\sigma, \sigma_\kappa) = 0 \Leftrightarrow$

$$\lim_{\kappa \rightarrow \infty} \mathfrak{I}_b(\sigma, \sigma_\kappa) = \lim_{\kappa, \tau \rightarrow \infty} \mathfrak{I}_b(\sigma_\tau, \sigma_\kappa) = \lim_{\kappa \rightarrow \infty} \mathfrak{I}_b(\sigma, \sigma). \quad (10)$$

2. Preliminary Results on Partial b -Metric Spaces

Following [17], we state the following.

Definition 6. A function $F: (0, \infty) \rightarrow (-\infty, \infty)$ belongs to Δ^s if

- (WF¹) F is strictly increasing
- (WF³) There exists $\xi \in (0, 1)$ such that $\lim_{\sigma \rightarrow \infty} (\sigma)^\xi F(\sigma) = 0$

In the above definition, we omitted Wardowski's second condition, that is, (WF²). For every sequence $\{\sigma_\kappa\} \subset (0, \infty)$,

$$\lim_{\kappa \rightarrow \infty} \sigma_\kappa = 0 \Leftrightarrow \lim_{\kappa \rightarrow \infty} F(\sigma_\kappa) = -\infty. \quad (11)$$

Definition 7. Let $(\mathfrak{F}, \mathfrak{I}_b, s)$ be a partial b -metric space. A mapping $\psi: \mathfrak{F} \rightarrow \mathfrak{F}$ is called an $F_{\mathfrak{I}_b}$ -contraction if there are $F \in \Delta_b^s$ and $\tau > 0$ so that

$$\begin{aligned} \sigma, \varsigma \in \mathfrak{F}, \mathfrak{I}_b(\psi(\sigma), \psi(\varsigma)) > 0 \Rightarrow \tau + F(s\mathfrak{I}_b(\psi(\sigma), \psi(\varsigma))) \\ \leq F(\mathfrak{I}_b(\sigma, \varsigma)). \end{aligned} \quad (12)$$

Remark 2. In Example 5, there is an $F_{\mathfrak{I}_b}$ -contraction mapping $\psi: \mathfrak{F} \rightarrow \mathfrak{F}$, but it does not satisfy

$$\mathfrak{I}_b(\psi(\sigma), \psi(\varsigma)) \leq p\mathfrak{I}_b(\sigma, \varsigma), \quad (13)$$

for all $p \in [0, 1)$.

Example 5. Let $\mathfrak{F} = \{\sigma_\kappa = 2^{\kappa/2}\kappa, \kappa \in \mathbb{N}\}$ and define $\mathfrak{I}_b: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ by $\mathfrak{I}_b(\sigma, \varsigma) = (\sigma \vee \varsigma)^2$. Then, the triplet $(\mathfrak{F}, \mathfrak{I}_b, s = 2)$ is a partial b -metric space (with $s = 2$). Consider $\psi: \mathfrak{F} \rightarrow \mathfrak{F}$ as

$$\psi(\sigma) = \begin{cases} \sigma_{k-1} = 2^{(k-1)/2}(\kappa - 1), & \text{if } \sigma = \sigma_\kappa \text{ (with } k = 1, 2, \dots), \\ 0, & \text{if } \sigma = \sigma_0 = 0. \end{cases} \quad (14)$$

We claim that ψ is an $F_{\mathfrak{I}_b}$ -contraction in the partial b -metric space \mathfrak{F} (with $\tau = 1$ and $F(\omega) = \omega$, which is in Δ_b^s : (WF¹) and (WF³) hold). For this, by this choice of F , we shall show that, for every $\sigma, \varsigma \in \mathfrak{F}$ so that $\psi(\sigma) \neq \psi(\varsigma)$, we have

$$2\mathfrak{I}_b(\psi(\sigma), \psi(\varsigma)) - \mathfrak{I}_b(\sigma, \varsigma) \leq -\tau. \quad (15)$$

For this, let $\sigma = \sigma_{\kappa+k}$ and $\varsigma = \sigma_\kappa$. Then,

$$\begin{aligned} & 2\mathfrak{I}_b(\psi(\sigma_{\kappa+k}), \psi(\sigma_\kappa)) - \mathfrak{I}_b(\sigma_{\kappa+k}, \sigma_\kappa) \\ &= 2\mathfrak{I}_b(2^{(\kappa+k-1)/2}(\kappa+k-1), 2^{(\kappa-1)/2}(\kappa-1)) - \mathfrak{I}_b(2^{(\kappa+k)/2}(\kappa+k), 2^{\kappa/2}\kappa) \\ &= (2^{(\kappa+k)/2}(\kappa+k-1) \vee 2^{\kappa/2}(\kappa-1))^2 - (2^{(\kappa+k)/2}(\kappa+k) \vee 2^{\kappa/2}\kappa)^2 \\ &= 2^\kappa \left((2^{\kappa/2}(\kappa+k-1) \vee (\kappa-1))^2 - (2^{\kappa/2}(\kappa+k) \vee \kappa)^2 \right) \\ &= 2^\kappa \left((2^{\kappa/2}(\kappa+k-1))^2 - (2^{\kappa/2}(\kappa+k))^2 \right) \\ &= 2^\kappa (2^k(\kappa+k-1)^2 - 2^k(\kappa+k)^2) \\ &= -2^{\kappa+k} \leq -1. \end{aligned} \quad (16)$$

On the contrary, ψ is not a Banach contraction. Indeed,

$$\begin{aligned}
\lim_{\kappa \rightarrow \infty} \frac{2\mathfrak{I}_b(\psi(\sigma_\kappa), \psi(\sigma_0))}{\mathfrak{I}_b(\sigma_\kappa, \sigma_0)} &= \lim_{\kappa \rightarrow \infty} \frac{2(\sigma_{\kappa-1} \vee \sigma_0)^2}{(\sigma_\kappa \vee \sigma_0)^2} \\
&= \lim_{\kappa \rightarrow \infty} \frac{2(2^{(\kappa-1)/2}(\kappa-1) \vee 0)^2}{(2^{\kappa/2}\kappa \vee 0)^2} = \lim_{\kappa \rightarrow \infty} \frac{(2^{\kappa/2}(\kappa-1) \vee 0)^2}{(2^{\kappa/2}\kappa \vee 0)^2} \\
&= \lim_{\kappa \rightarrow \infty} \frac{(2^{\kappa/2}(\kappa-1))^2}{(2^{\kappa/2}\kappa)^2} = \lim_{\kappa \rightarrow \infty} \frac{(\kappa-1)^2}{\kappa^2} \\
&= \lim_{\kappa \rightarrow \infty} \frac{\kappa^2 + 1 - 2\kappa}{\kappa^2} = \lim_{\kappa \rightarrow \infty} 1 + \frac{1}{\kappa^2} - \frac{2}{\kappa} = 1.
\end{aligned} \tag{17}$$

It shows that $\mathfrak{I}_b(\psi(\sigma), \psi(\varsigma)) \leq p\mathfrak{I}_b(\sigma, \varsigma)$ (for all $p \in [0, 1)$) does not hold.

In [11], the authors introduced the following assumption:

(CH⁴) for all $\kappa \in \mathbb{N}$, $\tau > 0$

$$\tau + F(s\sigma_\kappa) \leq F(\sigma_{\kappa-1}) \Rightarrow \tau + F(s^\kappa \sigma_\kappa) \leq F(s^{\kappa-1} \sigma_{\kappa-1})$$

Go back to Example 5. We proceed as follows. Let $\tau + F(s\sigma_\kappa) \leq F(\sigma_{\kappa-1})$, that is, $\tau + s\sigma_\kappa \leq \sigma_{\kappa-1}$. One writes

$$\begin{aligned}
F(s^\kappa \sigma_\kappa) + \tau &= s^\kappa \sigma_\kappa + \tau = s^{\kappa-1}(s\sigma_\kappa) + \tau \leq s^{\kappa-1}(\sigma_{\kappa-1} - \tau) + \tau \\
&= \tau + s^{\kappa-1}\sigma_{\kappa-1} - \tau s^{\kappa-1} = \tau(1 - s^{\kappa-1}) + s^{\kappa-1}\sigma_{\kappa-1} \\
&\leq s^{\kappa-1}\sigma_{\kappa-1} \\
&= F(s^{\kappa-1}\sigma_{\kappa-1}).
\end{aligned} \tag{18}$$

Thus, (CF⁴) holds.

Definition 8. Let $\{a_n\}$ be a sequence in $(0, \infty)$ and $\{b_n\}$ be a sequence in $[0, \infty)$. We call $\{b_n\} \in O(a_n)$ if there exists $C > 0$ satisfying $b_n \leq Ca_n$, for all $n \in \mathbb{N}$.

Lemma 2. Let $\{\sigma_d\}$ be a sequence in the partial b -metric space $(\mathfrak{S}, \mathfrak{I}_b, s)$. Assume that

$$\mathfrak{I}_b(\sigma_d, \sigma_{d+1}) \in \cup \{O(d^{-\delta})\}; \quad \delta > 1 + \log_2 s. \tag{19}$$

Then, $\{\sigma_d\}$ is a Cauchy sequence.

Proof. By (19), there exist $\delta > 1 + \log_2 s$ and $D > 0$ satisfying

$$\mathfrak{I}_b(\sigma_d, \sigma_{d+1}) \leq Dd^{-\delta} \quad \forall d \in \mathbb{N}. \tag{20}$$

Note that $2s < 2^\delta$. Choose $\mu \in \mathbb{N}$ satisfying $2s^{1+(1/\mu)} < 2^\delta$; then, we have

$$\begin{aligned}
2^\mu s^{1+\mu} &< 2^{\mu\delta} \\
&\Rightarrow 2^{\mu(1-\delta)} s^{1+\mu} < 1.
\end{aligned} \tag{21}$$

Define a mapping $\psi: \mathbb{N} \rightarrow \mathbb{N}$ by

$$\psi(d) = 2 + \frac{2^{\mu d} - 1}{2^\mu - 1}. \tag{22}$$

For $a, c, d \in \mathbb{N}$ satisfying $\psi(a) \leq c < d \leq \psi(a+1)$, we have

$$\begin{aligned}
\mathfrak{I}_b(\sigma_c, \sigma_d) &\leq s^{\psi(d)-c} \sum_{j=c}^{d-1} \mathfrak{I}_b(\sigma_j, \sigma_{j+1}) \\
&\leq s^{\psi(\psi(a+1)-\psi(a))} \sum_{j=\psi(a)}^{\psi(a+1)-1} \mathfrak{I}_b(\sigma_j, \sigma_{j+1}) \\
&= s^{\psi(2^{\mu a})} \sum_{j=\psi(a)}^{\psi(a+1)-1} \mathfrak{I}_b(\sigma_j, \sigma_{j+1}) \\
&= s^{\mu a} \sum_{j=\psi(a)}^{\psi(a+1)-1} \mathfrak{I}_b(\sigma_j, \sigma_{j+1}) \\
&\leq Ds^{\mu a} \sum_{j=\psi(a)}^{\psi(a+1)-1} j - \delta \\
&\leq Ds^{\mu a} \int_{\psi(a)-1}^{\psi(a+1)-1} t^{-\delta} dt \\
&\leq Ds^{\mu a} \int_{(\psi(a)-2+1)/(2^\mu-1)}^{(\psi(a+1)-2+1)/(2^\mu-1)} t^{-\delta} dt \\
&= \frac{Ds^{\mu a}}{1-\delta} \left[t^{1-\delta} \right]_{(\psi(a)-2+1)/(2^\mu-1)}^{(\psi(a+1)-2+1)/(2^\mu-1)} \\
&= Ds^{\mu a} \frac{1 - 2^{\mu(1-\delta)}}{\delta - 1} (2^\mu - 1)^{\delta-1} 2^{\mu a(1-\delta)} \\
&= M(s2^{1-\delta})^{\mu a},
\end{aligned} \tag{23}$$

where $M = ((D(1 - 2^{\mu(1-\delta)})(2^\mu - 1)^{\delta-1})/(\delta - 1)) > 0$. Since

$$(s2^{1-\delta})^\mu = \frac{s^{\mu+1} 2^{\mu(1-\delta)}}{s} < \frac{1}{s} \leq 1, \tag{24}$$

we have

$$\lim_{\mu \rightarrow \infty} M(s2^{1-\delta})^{\mu a} = 0. \tag{25}$$

For $a, l, c, d \in \mathbb{N}$ such that $a < l$, $\psi(a) \leq c < \psi(a+1)$ and $\psi(l) < d \leq \psi(l+1)$, then, from (21) and (23),

$$\begin{aligned}
\mathfrak{J}_b(\sigma_c, \sigma_d) &\leq s\mathfrak{J}_b(\sigma_c, \sigma_{\psi(a+1)}) + s\mathfrak{J}_b(\sigma_{\psi(a+1)}, \sigma_d) - \mathfrak{J}_b(\sigma_{\psi(a+1)}, \sigma_{\psi(a+1)}) \\
&\leq s\mathfrak{J}_b(\sigma_c, \sigma_{\psi(a+1)}) + s[s\mathfrak{J}_b(\sigma_{\psi(a+1)}, \sigma_{\psi(a+2)}) + s\mathfrak{J}_b(\sigma_{\psi(a+2)}, \sigma_d)] \\
&\quad - s\mathfrak{J}_b(\sigma_{\psi(a+2)}, \sigma_{\psi(a+2)}) - \mathfrak{J}_b(\sigma_{\psi(a+1)}, \sigma_{\psi(a+1)}) \\
&= s\mathfrak{J}_b(\sigma_c, \sigma_{\psi(a+1)}) + s^2\mathfrak{J}_b(\sigma_{\psi(a+1)}, \sigma_{\psi(a+2)}) + s^2\mathfrak{J}_b(\sigma_{\psi(a+2)}, \sigma_d) \\
&\quad - s^2\mathfrak{J}_b(\sigma_{\psi(a+2)}, \sigma_{\psi(a+2)}) - \mathfrak{J}_b(\sigma_{\psi(a+1)}, \sigma_{\psi(a+1)}) \\
&\leq sM(s2^{1-\delta})^{\mu a} + s^2M(s2^{1-\delta})^{\mu(a+1)} + s^3M(s2^{1-\delta})^{\mu(a+2)} \\
&\quad + s^3\mathfrak{J}_b(\sigma_{\psi(a+3)}, \sigma_d) - s^2\mathfrak{J}_b(\sigma_{\psi(a+3)}, \sigma_{\psi(a+3)}) - s^1M(s2^{1-\delta})^{\mu(a+1)} - s^0M(s2^{1-\delta})^{\mu(a)} \\
&\leq \sum_{\kappa=0}^2 s^{(\kappa+1)}M(s2^{1-\delta})^{\mu(a+\kappa)} + s^3\mathfrak{J}_b(\sigma_{\psi(a+3)}, \sigma_d) \\
&\quad - \sum_{\kappa=0}^1 s^\kappa M(s2^{1-\delta})^{\mu(a+\kappa)} - s^2\mathfrak{J}_b(\sigma_{\psi(a+3)}, \sigma_{\psi(a+3)}) \\
&\leq \dots \leq \sum_{\kappa=0}^{l-a-1} s^{(\kappa+1)}M(s2^{1-\delta})^{\mu(a+\kappa)} + s^{l-a}\mathfrak{J}_b(\sigma_{\psi(l)}, \sigma_d) \\
&\quad - \sum_{\nu}^{l-k-1} s^\kappa M(s2^{1-\delta})^{\mu(a+\kappa)} - s^{l-k}\mathfrak{J}_b(\sigma_{\psi(a+\kappa+1)}, \sigma_{\psi(a+\kappa+1)}) \\
&\leq \sum_{\kappa=0}^{l-a-1} s^{(\kappa+1)}M(s2^{1-\delta})^{\mu(a+\kappa)} + s^{l-a}M(s2^{1-\delta})^{\mu l} \\
&\quad - \sum_{\kappa=0}^{l-a-1} s^\kappa M(s2^{1-\delta})^{\mu(a+\kappa)} - s^{l-a}M(s2^{1-\delta})^{\mu(a+\kappa)} \leq \sum_{\kappa=0}^{l-a} s^{(\kappa+1)}M(s2^{1-\delta})^{\mu(a+\kappa)} \\
&\quad - \sum_{\kappa=0}^{l-a} s^\kappa M(s2^{1-\delta})^{\mu(a+\kappa)} \leq \sum_{\kappa=0}^{\infty} s^{(\kappa+1)}M(s2^{1-\delta})^{\mu(a+\kappa)} - \sum_{\kappa=0}^{\infty} s^\kappa M(s2^{1-\delta})^{\mu(a+\kappa)} \\
&= M(s2^{1-\delta})^{\mu(a+\kappa)} \left(\frac{s}{1 - s^{\mu+1}2^{\mu(1-\delta)}} - \frac{1}{1 - s^{\mu+1}2^{\mu(1-\delta)}} \right).
\end{aligned} \tag{26}$$

By (25), $\{\sigma_d\}$ is a Cauchy sequence. \square

Lemma 3 (see [9]). Let $\{\sigma_n\}$ be a decreasing sequence in $(0, \infty)$. Assume that there is a mapping $F: (0, \infty) \longrightarrow (-\infty, \infty)$, $\tau \in (0, \infty)$, and $k \in (0, 1)$ satisfying (WF^3) so that

$$n\tau + F(s^n\sigma_n) \leq F(\sigma_0). \tag{27}$$

Then, $\{\sigma_n\} \in O(n^{1/k})$.

Generally, a partial b -metric is discontinuous. Following Lemma 3, this statement is justified.

Lemma 4. Consider the convergent sequence $\{\sigma_\epsilon\}$ in the partial b -metric space $(\mathfrak{F}, \mathfrak{J}_b, s)$. Suppose that $\lim_{\epsilon \rightarrow \infty} \sigma_\epsilon = \sigma$ and $\sigma, \varsigma \in \mathfrak{F}$; then,

$$\frac{1}{s}\mathfrak{J}_b(\sigma, \varsigma) \leq \liminf_{\epsilon \rightarrow \infty} \mathfrak{J}_b(\sigma_\epsilon, \varsigma) \leq \limsup_{\epsilon \rightarrow \infty} \mathfrak{J}_b(\sigma_\epsilon, \varsigma) \leq s\mathfrak{J}_b(\sigma, \varsigma). \tag{28}$$

Proof. Since $(\mathfrak{F}, \mathfrak{J}_b, s)$ is a partial b -metric space, by $(\mathfrak{J}_{b_{iv}})$, we obtain

$$\begin{aligned}
\mathfrak{I}_b(\sigma, \varsigma) &\leq s[\mathfrak{I}_b(\sigma, \sigma_\varepsilon) + \mathfrak{I}_b(\sigma_\varepsilon, \varsigma)] - \mathfrak{I}_b(\sigma_\varepsilon, \sigma_\varepsilon) \\
&\leq s[\mathfrak{I}_b(\sigma, \sigma_\varepsilon) + \mathfrak{I}_b(\sigma_\varepsilon, \varsigma)] \\
&\Rightarrow \frac{1}{s}\mathfrak{I}_b(\sigma, \varsigma) - \mathfrak{I}_b(\sigma, \sigma_\varepsilon) \leq \mathfrak{I}_b(\sigma_\varepsilon, \varsigma).
\end{aligned} \tag{29}$$

Taking \liminf , we obtain

$$\frac{1}{s}\mathfrak{I}_b(\sigma, \varsigma) \leq \liminf_{\varepsilon \rightarrow \infty} \mathfrak{I}_b(\sigma_\varepsilon, \varsigma). \tag{30}$$

Again, by (\mathfrak{I}_{b_v}) , we obtain

$$\begin{aligned}
\mathfrak{I}_b(\sigma_\varepsilon, \varsigma) &\leq s[\mathfrak{I}_b(\sigma_\varepsilon, \sigma) + \mathfrak{I}_b(\sigma, \varsigma)] - \mathfrak{I}_b(\sigma, \sigma) \\
&\leq s[\mathfrak{I}_b(\sigma_\varepsilon, \sigma) + \mathfrak{I}_b(\sigma, \varsigma)] \\
&\Rightarrow \mathfrak{I}_b(\sigma_\varepsilon, \varsigma) \leq s\mathfrak{I}_b(\sigma_\varepsilon, \sigma) + s\mathfrak{I}_b(\sigma, \varsigma).
\end{aligned} \tag{31}$$

Taking \limsup ,

$$\limsup_{\varepsilon \rightarrow \infty} \mathfrak{I}_b(\sigma_\varepsilon, \varsigma) \leq s\mathfrak{I}_b(\sigma, \varsigma). \tag{32}$$

Also,

$$\liminf_{\varepsilon \rightarrow \infty} \mathfrak{I}_b(\sigma_\varepsilon, \varsigma) \leq \limsup_{\varepsilon \rightarrow \infty} \mathfrak{I}_b(\sigma_\varepsilon, \varsigma). \tag{33}$$

Combining (30), (32), and (33), we have

$$\begin{aligned}
\frac{1}{s}\mathfrak{I}_b(\sigma, \varsigma) &\leq \liminf_{\varepsilon \rightarrow \infty} \mathfrak{I}_b(\sigma_\varepsilon, \varsigma) \leq \limsup_{\varepsilon \rightarrow \infty} \mathfrak{I}_b(\sigma_\varepsilon, \varsigma) \leq s\mathfrak{I}_b(\sigma, \varsigma).
\end{aligned} \tag{34}$$

□

3. On (γ, F) -Weak Contractions

We shall investigate the existence of a fixed point for (γ, F) -weak contraction mappings in partial b -metric spaces. We shall also provide some examples in support of main results.

Definition 9. Let $\psi: \mathfrak{F} \rightarrow \mathfrak{F}$ and $\alpha_s: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ be two mappings and $(\mathfrak{F}, \mathfrak{I}_b, s \geq 1)$ be a partial b -metric space. Such ψ is α_s -admissible if

$$\alpha_s(\sigma, \varsigma) \geq s^2 \Rightarrow \alpha_s(\psi(\sigma), \psi(\varsigma)) \geq s^2 \quad \text{for all } \sigma, \varsigma \in \mathfrak{F}. \tag{35}$$

Example 6. Let $\mathfrak{F} = (0, \infty)$. Define $\psi: \mathfrak{F} \rightarrow \mathfrak{F}$ and $\alpha_s: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ by

$$\begin{aligned}
\psi(\sigma) &= \ln \sigma, \quad \text{for all } \sigma \in \mathfrak{F}, \\
\alpha_s(\sigma, \varsigma) &= \begin{cases} 2, & \text{if } \sigma \geq \varsigma, \\ 0, & \text{if } \sigma < \varsigma. \end{cases}
\end{aligned} \tag{36}$$

Then, ψ is α_s -admissible.

Definition 10. Let $(\mathfrak{F}, \mathfrak{I}_b, s)$ be a partial b -metric space and $\psi: \mathfrak{F} \rightarrow \mathfrak{F}$ and $\alpha_s: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ be two mappings. Then, ψ is triangular α_s -admissible if

$$(1) \alpha_s(\sigma, \varsigma) \geq s^2 \text{ implies } \alpha_s(\psi(\sigma), \psi(\varsigma)) \geq s^2$$

$$(2) \alpha_s(\sigma, \varsigma) \geq s^2, \alpha_s(\varsigma, v) \geq s^2 \text{ implies } \alpha_s(\sigma, v) \geq s^2, \text{ for all } \sigma, \varsigma, v \in \mathfrak{F}$$

Example 7. Let $\mathfrak{F} = (0, \infty)$, $\psi(\sigma) = \sqrt[3]{\sigma}$, and $\alpha_s(\sigma, \varsigma) = e^{\sigma-\varsigma}$. Here, ψ is triangular α_s -admissible. Indeed, if $\alpha_s(\sigma, \varsigma) = e^{\sigma-\varsigma} \geq s^2$, then $\sigma \geq \varsigma$, which implies $\psi(\sigma) \geq \psi(\varsigma)$. That is, $\alpha_s(\psi(\sigma), \psi(\varsigma)) = e^{\psi(\sigma)-\psi(\varsigma)} \geq s^2$. Also, if $\alpha_s(\sigma, \varsigma) \geq s^2$ and $\alpha_s(\varsigma, v) \geq s^2$, then $\sigma - \varsigma \geq 0, \varsigma - v \geq 0$. That is, $\sigma - v \geq 0$, so $\alpha_s(\sigma, v) = e^{\sigma-v} \geq s^2$.

Remark 3. For the given α_s , there is $\gamma: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ defined by $\gamma(\sigma, \varsigma) = (\alpha_s(\sigma, \varsigma)/s^2)$ having the following conditions:

(1) ψ is α_s -admissible iff ψ is γ -admissible, i.e.,

$$\gamma(\sigma, \varsigma) \geq 1 \Rightarrow \gamma(\psi(\sigma), \psi(\varsigma)) \geq 1, \quad \text{for all } \sigma, \varsigma \in \mathfrak{F}. \tag{37}$$

(2) The partial b -metric space is α_s -complete iff it is γ -complete.

Definition 11. Let $(\mathfrak{F}, \mathfrak{I}_b, s)$ be a partial b -metric space and $\psi, \phi: \mathfrak{F} \rightarrow \mathfrak{F}$ and $\gamma: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ be three mappings. Then, the pair (ψ, ϕ) is weakly γ -admissible if

$$\begin{aligned}
\gamma(\sigma, \varsigma) \geq 1 &\Rightarrow \gamma(\psi(\sigma), \phi\psi(\sigma)) \geq 1, \\
\gamma(\phi(\varsigma), \psi\phi(\varsigma)) &\geq 1, \quad \text{for all } \sigma, \varsigma \in \mathfrak{F}.
\end{aligned} \tag{38}$$

Definition 12. Let $(\mathfrak{F}, \mathfrak{I}_b, s)$ be a partial b -metric space and $\psi, \phi: \mathfrak{F} \rightarrow \mathfrak{F}$ and $\gamma: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ be three mappings. The pair (ψ, ϕ) is triangular weakly γ -admissible if

$$\begin{aligned}
(1) \gamma(\sigma, \varsigma) \geq 1 &\Rightarrow \gamma(\psi(\sigma), \phi\psi(\sigma)) \geq 1 \quad \text{and} \quad \gamma(\phi(\varsigma), \psi\phi(\varsigma)) \geq 1, \text{ for all } \sigma, \varsigma \in \mathfrak{F} \\
(2) \gamma(\sigma, \varsigma) \geq 1, \gamma(\varsigma, v) \geq 1 &\text{ implies } \gamma(\sigma, v) \geq 1, \text{ for all } \sigma, \varsigma, v \in \mathfrak{F}
\end{aligned}$$

Example 8. Let $\mathfrak{F} = [0, \infty)$ and define $\psi, \phi: \mathfrak{F} \rightarrow \mathfrak{F}$ by

$$\begin{aligned}
\psi(\sigma) &= \begin{cases} \sigma^{1/2} & \text{if } \sigma \in [0, 1), \\ 1, & \text{if } \sigma \in [1, \infty), \end{cases} \\
\phi(\sigma) &= \begin{cases} \sigma, & \text{if } \sigma \in [0, 1), \\ 1, & \text{if } \sigma \in [1, \infty). \end{cases}
\end{aligned} \tag{39}$$

Define $\gamma: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ by

$$\gamma(\sigma, \varsigma) = \begin{cases} e^{\sigma-\varsigma}, & \text{if } \sigma, \varsigma \in [0, 1), \\ 0, & \text{if } \sigma, \varsigma \in [1, \infty). \end{cases} \tag{40}$$

If $\gamma(\sigma, \varsigma) \geq 1$ and $\gamma(\varsigma, v) \geq 1$, then

$$\begin{aligned}
\sigma - \varsigma &\geq 0, \\
\varsigma - v &\geq 0 \\
\Rightarrow \sigma - v &\geq 0 \\
\Rightarrow \gamma(\sigma, v) &\geq 1.
\end{aligned} \tag{41}$$

Next, if $\gamma(\sigma, \varsigma) \geq 1$, then

$$\begin{aligned}\gamma(\psi(\sigma), \phi\psi(\sigma)) &= \gamma(\sigma^{1/2}, \phi(\sigma^{1/2})) = \gamma(\sigma^{1/2}, \sigma^{1/2}) \\ &= e^{\sigma^{1/2} - \sigma^{1/2}} \geq 1, \\ \gamma(\phi(\varsigma), \psi\phi(\varsigma)) &= \gamma(\varsigma, \psi(\varsigma)) = \gamma(\varsigma, \varsigma^{1/2}) = e^{\varsigma - \varsigma^{1/2}} \geq 1.\end{aligned}\quad (42)$$

Definition 13. Consider the partial b -metric space $(\mathfrak{F}, \mathfrak{I}_b, s)$ and two mappings $\gamma: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ and $\psi: \mathfrak{F} \rightarrow \mathfrak{F}$. ψ is γ -continuous if for given $v \in \mathfrak{F}$ and a sequence $\{\sigma_\kappa\}$ so that

$$\begin{aligned}\lim_{\kappa \rightarrow \infty} \mathfrak{I}_b(\sigma_\kappa, v) &= 0, \\ \gamma(\sigma_\kappa, \sigma_{\kappa+1}) &\geq 1 \\ \Rightarrow \lim_{\kappa \rightarrow \infty} \mathfrak{I}_b(\psi(\sigma_\kappa), \psi(v)) &= 0, \quad \text{for all } \kappa \in \mathbb{N}.\end{aligned}\quad (43)$$

Example 9. Let $\mathfrak{F} = [0, \infty)$ and define a partial b -metric $\mathfrak{I}_b: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ by

$$\mathfrak{I}_b(\sigma, \varsigma) = (\sigma \vee \varsigma)^2, \quad \text{for all } \sigma, \varsigma \in \mathfrak{F}. \quad (44)$$

Define $\gamma: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ and $\psi: \mathfrak{F} \rightarrow \mathfrak{F}$ by

$$\begin{aligned}\gamma(\sigma, \varsigma) &= \begin{cases} \sigma^2 + \varsigma^2 + 1, & \text{if } \sigma, \mu \in [0, 1], \\ 0, & \text{otherwise,} \end{cases} \\ \psi(\omega) &= \begin{cases} \sin(\pi\omega), & \text{if } \omega \in [0, 1], \\ \cos(\pi\omega) + 2, & \text{if } \omega \in (1, \infty). \end{cases}\end{aligned}\quad (45)$$

ψ is not continuous because

$$\lim_{\omega \rightarrow 1^-} \psi(\omega) \neq \lim_{\omega \rightarrow 1^+} \psi(\omega). \quad (46)$$

Next, we prove that ψ is γ -continuous. For given $v = 0 \in [0, \infty)$ and $\sigma_\kappa = 1/\kappa$, we obtain

$$\begin{aligned}\lim_{\kappa \rightarrow \infty} \mathfrak{I}_b(\sigma_\kappa, v) &= \lim_{\kappa \rightarrow \infty} \mathfrak{I}_b\left(\frac{1}{\kappa}, 0\right) = \lim_{\kappa \rightarrow \infty} \left(\frac{1}{\kappa} \vee 0\right) = 0, \\ \gamma\left(\frac{1}{\kappa}, \frac{1}{\kappa+1}\right) &= \left(1 + \left(\frac{1}{\kappa}\right)^2 + \left(\frac{1}{\kappa+1}\right)^2\right) \geq 1.\end{aligned}\quad (47)$$

Now,

$$\begin{aligned}\lim_{\kappa \rightarrow \infty} \mathfrak{I}_b(\psi(\sigma_\kappa), \psi(v)) &= \lim_{\kappa \rightarrow \infty} \mathfrak{I}_b\left(\psi\left(\frac{1}{\kappa}\right), \psi(0)\right) \\ &= \lim_{\kappa \rightarrow \infty} \left(\sin\left(\frac{\pi}{\kappa}\right) \vee \sin(0)\right) \\ &= \lim_{\kappa \rightarrow \infty} \left(\sin\left(\frac{\pi}{\kappa}\right)\right) = 0.\end{aligned}\quad (48)$$

Hence, ψ is γ -continuous.

Definition 14. Let $(\mathfrak{F}, \mathfrak{I}_b, s)$ be a partial b -metric space and γ be defined above. The partial b -metric space is γ -complete iff every Cauchy sequence $\{\sigma_\kappa\}$ in \mathfrak{F} satisfying $\gamma(\sigma_\kappa, \sigma_{\kappa+1}) \geq 1$ is convergent to $\sigma \in \mathfrak{F}$, for all $\kappa \in \mathbb{N}$.

Remark 4. If $(\mathfrak{F}, \mathfrak{I}_b, s)$ is a complete partial b -metric space, then $(\mathfrak{F}, \mathfrak{I}_b, s)$ is a γ -complete partial b -metric space, but not conversely.

Example 10. Let $\mathfrak{F} = (0, \infty)$ and $\mathfrak{I}_b: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ be the partial b -metric defined by

$$\mathfrak{I}_b(\sigma, \varsigma) = (\sigma \vee \varsigma)^2, \quad \text{for all } \sigma, \varsigma \in \mathfrak{F}. \quad (49)$$

Define $\gamma: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ by

$$\gamma(\sigma, \varsigma) = \begin{cases} e^{\mathfrak{I}_b(\sigma, \varsigma)}, & \text{if } \sigma, \varsigma \in [2, 5], \\ 0, & \text{otherwise.} \end{cases} \quad (50)$$

We can show that $(\mathfrak{F}, \mathfrak{I}_b, s)$ is not a complete partial b -metric space, but $(\mathfrak{F}, \mathfrak{I}_b, s)$ is a γ -complete partial b -metric space. Indeed, $\{\sigma_\kappa\}$ is a Cauchy sequence in \mathfrak{F} satisfying $\gamma(\sigma_\kappa, \sigma_{\kappa+1}) \geq 1$ for all $\kappa \in \mathbb{N}$; then, $\sigma_\kappa \in [2, 5]$. Since $[2, 5]$ is a closed subset of $(-\infty, \infty)$, we can check that $([2, 5], \mathfrak{I}_b, 2)$ is a complete partial b -metric space, and then, there is $v \in [2, 5]$ such that $\sigma_\kappa \rightarrow v$ as $\kappa \rightarrow \infty$.

Definition 15. Let $(\mathfrak{F}, \mathfrak{I}_b, s)$ be a partial b -metric space and $\gamma: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ be a mapping. $(\mathfrak{F}, \mathfrak{I}_b, s)$ is called γ -regular if, for every sequence $\{\sigma_\kappa\} \subset \mathfrak{F}$ satisfying $\gamma(\sigma_\kappa, \sigma_{\kappa+1}) \geq 1$ and $\sigma_\kappa \rightarrow v$ as $\kappa \rightarrow \infty$, we have $\gamma(\sigma_\kappa, v) \geq 1$, for all $\kappa \in \mathbb{N}$.

Definition 16. Consider the partial b -metric space $(\mathfrak{F}, \mathfrak{I}_b, s)$. Then, the pair of self-mappings (ψ, ϕ) forms a (γ, F) -weak contraction if there exist $F \in \Delta_b^s$ and $\tau > 0$ such that

$$\begin{aligned}\tau + F(s\gamma(\sigma, \varsigma)\mathfrak{I}_b(\psi(\sigma), \phi(\varsigma))) \\ \leq F(M_1(\sigma, \varsigma)), \quad \text{for all } \sigma, \varsigma \in \mathfrak{F},\end{aligned}\quad (51)$$

with $\gamma(\sigma, \varsigma) \geq 1$, whenever

$$\min\{\gamma(\sigma, \varsigma)\mathfrak{I}_b(\psi(\sigma), \phi(\varsigma)), M_1(\sigma, \varsigma)\} > 0, \quad (52)$$

where

$$M_1(\sigma, \varsigma) = \max\left\{\mathfrak{I}_b(\sigma, \varsigma), \mathfrak{I}_b(\sigma, \psi(\sigma)), \mathfrak{I}_b(\varsigma, \phi(\varsigma)), \frac{\mathfrak{I}_b(\sigma, \phi(\varsigma)) + \mathfrak{I}_b(\varsigma, \psi(\sigma))}{2s}\right\}. \quad (53)$$

Theorem 1. Let the pair of self-mappings (ψ, ϕ) be a (γ, F) -weak contraction defined on a γ -complete partial

b -metric space $(\mathfrak{F}, \mathfrak{I}_b, s)$. Let $k \in (0, (1/(1 + \log_2 s)))$. Suppose that

- (1) The pair (ψ, ϕ) is weakly γ -admissible.
- (2) There exists $\sigma_0 \in \mathfrak{F}$ such that $\gamma(\sigma_0, \psi(\sigma_0)) \geq 1$.
- (3) (a) Either \mathfrak{F} is γ -regular and F is continuous.
(b) One of ψ, ϕ is γ -continuous.

Then, we have a sequence $\{\sigma_\kappa\}$ in \mathfrak{F} such that $\sigma_\kappa \longrightarrow v \in \mathfrak{F}$. If $\gamma(v, v) \geq 1$, then v is a common fixed point of the pair (ψ, ϕ) . Moreover, if ω is another common fixed point of the pair (ψ, ϕ) satisfying $\gamma(v, \omega) \geq 1$, then $v = \omega$.

Proof. For $\sigma_1, \sigma_2 \in \mathfrak{F}$ such that $\sigma_1 \neq \sigma_2$, then $M_1(\sigma_1, \sigma_2) > 0$. Let $\sigma_0 \in \mathfrak{F}$ be as in (2). Consider the iterative sequence $\{\sigma_\kappa\}$ in \mathfrak{F} such that $\sigma_1 = \psi(\sigma_0)$ and $\sigma_2 = \phi(\sigma_1)$, and generally, $\sigma_{2\kappa+1} = \psi(\sigma_{2\kappa})$, $\sigma_{2\kappa} = \phi(\sigma_{2\kappa-1})$, for all $\kappa \in \mathbb{N} \cup \{0\}$.

Using the weakly γ -admissibility, we have

$$\begin{aligned} \gamma(\psi(\sigma_0), \phi\psi(\sigma_0)) &= \gamma(\sigma_1, \sigma_2) \geq 1, \\ \gamma(\psi(\sigma_1), \phi\psi(\sigma_1)) &= \gamma(\sigma_2, \sigma_3) \geq 1, \\ \gamma(\psi(\sigma_2), \phi\psi(\sigma_2)) &= \gamma(\sigma_3, \sigma_4) \geq 1, \\ \gamma(\psi(\sigma_3), \phi\psi(\sigma_3)) &= \gamma(\sigma_4, \sigma_5) \geq 1. \end{aligned} \quad (54)$$

Continuing in this way,

$$\begin{aligned} \gamma(\psi(\sigma_{2\kappa}), \phi\psi(\sigma_{2\kappa})) &= \gamma(\sigma_{2\kappa+1}, \sigma_{2\kappa+2}) \geq 1, \\ \gamma(\psi(\sigma_{2\kappa-1}), \phi\psi(\sigma_{2\kappa-1})) &= \gamma(\sigma_{2\kappa}, \sigma_{2\kappa+1}) \geq 1. \end{aligned} \quad (55)$$

Hence, $\gamma(\sigma_\kappa, \sigma_{\kappa+1}) \geq 1$, for all $\kappa \in \mathbb{N} \cup \{0\}$. If $\mathfrak{I}_b(\psi(\sigma_{2\kappa}), \phi(\sigma_{2\kappa+1})) = 0$, then $\sigma_{2\kappa}$ is a common fixed point of ψ, ϕ . Let $\mathfrak{I}_b(\psi(\sigma_{2\kappa}), \phi(\sigma_{2\kappa+1})) > 0$. From the contractive condition (51),

$$\begin{aligned} F(s\mathfrak{I}_b(\sigma_{2\kappa}, \sigma_{2\kappa+1})) &\leq F(s\gamma(\sigma_{2\kappa}, \sigma_{2\kappa+1})\mathfrak{I}_b(\psi(\sigma_{2\kappa}), \phi(\sigma_{2\kappa+1}))) \\ &\leq F(M_1(\sigma_{2\kappa}, \sigma_{2\kappa+1})) - \tau \quad \forall \kappa \in \mathbb{N} \cup \{0\}, \end{aligned} \quad (56)$$

where

$$\begin{aligned} M_1(\sigma_{2\kappa}, \sigma_{2\kappa+1}) &= \max \left\{ \mathfrak{I}_b(\sigma_{2\kappa}, \sigma_{2\kappa+1}), \mathfrak{I}_b(\sigma_{2\kappa+1}, \phi(\sigma_{2\kappa})), \mathfrak{I}_b(\sigma_{2\kappa+1}, \phi(\sigma_{2\kappa+1})), \frac{\mathfrak{I}_b(\sigma_{2\kappa}, \phi(\sigma_{2\kappa+1})) + \mathfrak{I}_b(\sigma_{2\kappa+1}, \psi(\sigma_{2\kappa}))}{2s} \right\} \\ &= \max \left\{ \mathfrak{I}_b(\sigma_{2\kappa}, \sigma_{2\kappa+1}), \mathfrak{I}_b(\sigma_{2\kappa}, \sigma_{2\kappa+1}), \mathfrak{I}_b(\sigma_{2\kappa+1}, \sigma_{2\kappa+2}), \frac{\mathfrak{I}_b(\sigma_{2\kappa}, \sigma_{2\kappa+2}) + \mathfrak{I}_b(\sigma_{2\kappa+1}, \sigma_{2\kappa+1})}{2s} \right\} \\ &\leq \max \{ \mathfrak{I}_b(\sigma_{2\kappa}, \sigma_{2\kappa+1}), \mathfrak{I}_b(\sigma_{2\kappa+1}, \sigma_{2\kappa+2}) \}. \end{aligned} \quad (57)$$

If $M_1(\sigma_{2\kappa}, \sigma_{2\kappa+1}) = \mathfrak{I}_b(\sigma_{2\kappa+1}, \sigma_{2\kappa+2})$, then

$$F(s\mathfrak{I}_b(\sigma_{2\kappa+1}, \sigma_{2\kappa+2})) \leq F(\mathfrak{I}_b(\sigma_{2\kappa+1}, \sigma_{2\kappa+2})) - \tau, \quad (58)$$

which contradicts (WF¹). Therefore,

$$F(s\mathfrak{I}_b(\sigma_{2\kappa+1}, \sigma_{2\kappa+2})) \leq F(\mathfrak{I}_b(\sigma_{2\kappa}, \sigma_{2\kappa+1})) - \tau, \quad \kappa \in \mathbb{N} \cup \{0\}. \quad (59)$$

Similarly, we have

$$F(s\mathfrak{I}_b(\sigma_{2\kappa+2}, \sigma_{2\kappa+3})) \leq F(\mathfrak{I}_b(\sigma_{2\kappa+1}, \sigma_{2\kappa+2})) - \tau, \quad \kappa \in \mathbb{N} \cup \{0\}. \quad (60)$$

From (59) and (60), we have

$$F(s\mathfrak{I}_b(\sigma_\kappa, \sigma_{\kappa+1})) \leq F(\mathfrak{I}_b(\sigma_{\kappa-1}, \sigma_\kappa)) - \tau, \quad \kappa \in \mathbb{N}. \quad (61)$$

Let $b_\kappa = \mathfrak{I}_b(\sigma_\kappa, \sigma_{\kappa+1})$, for each $\kappa \in \mathbb{N} \cup \{0\}$; then, from (61) and (CH⁴),

$$\tau + F(s^\kappa b_\kappa) \leq F(s^{\kappa-1} b_{\kappa-1}), \quad \kappa \in \mathbb{N}. \quad (62)$$

Continuing in this way, we obtain

$$F(s^\kappa b_\kappa) \leq F(b_0) - \kappa\tau, \quad \kappa \in \mathbb{N}. \quad (63)$$

So, $\{b_\kappa\} \in O(\kappa^{-1/k})$ by Lemma 3. Since $(1/k) \in (1 + \log_2 s, \infty)$, by Lemma 2, $\{\sigma_\kappa\}$ is a Cauchy sequence. We know that \mathfrak{F} is a γ -complete partial b -metric space; then, there is $v \in \mathfrak{F}$ satisfying $\sigma_{2\kappa+1} \longrightarrow v$ and $\sigma_{2\kappa+2} \longrightarrow v$ as $\kappa \longrightarrow \infty$. Using γ -continuity of ψ , one writes

$$\begin{aligned} v &= \lim_{\kappa \longrightarrow \infty} \sigma_\kappa = \lim_{\kappa \longrightarrow \infty} \sigma_{2\kappa+1} = \lim_{\kappa \longrightarrow \infty} \sigma_{2\kappa+2} = \lim_{\kappa \longrightarrow \infty} \psi(\sigma_{2\kappa+1}) \\ &= \psi \left(\lim_{\kappa \longrightarrow \infty} \sigma_{2\kappa+1} \right) = \psi(v). \end{aligned} \quad (64)$$

If $\mathfrak{I}_b(v, \phi(v)) > 0$ as $\gamma(v, v) \geq 1$, then by the contractive condition (51), we have

$$\begin{aligned} \tau + F(s\mathfrak{I}_b(v, \phi(v))) &\leq \tau + F(s\gamma(v, v)\mathfrak{I}_b(\psi(v), \phi(v))) \\ &\leq FM_1(v, v) = F(\mathfrak{I}_b(v, \phi(v))). \end{aligned} \quad (65)$$

It is a contradiction. Hence,

$$\begin{aligned} \mathfrak{I}_b(v, \phi(v)) &= 0 \\ \Rightarrow v &= \phi(v). \end{aligned} \quad (66)$$

Thus, we have $\psi(v) = \phi(v) = v$. Hence, (ψ, ϕ) has a common fixed point v .

If \mathfrak{F} is γ -regular and F is continuous, then there are two different cases. Firstly, if there is a subsequence $\{\sigma_{n_k}\}_{k \in \mathbb{N}} \subset \{\sigma_n\}_{n \in \mathbb{N}}$ satisfying

$$\sigma_{n_k} = \begin{cases} \psi(v), & \text{if } k \in 2n, \\ \phi(v), & \text{if } k \in 2n+1. \end{cases} \quad (67)$$

Then,

$$\begin{aligned} v &= \lim_{k \rightarrow \infty} \sigma_{n_k} = \lim_{k \rightarrow \infty} \psi(v) = \psi(v), \\ v &= \lim_{k \rightarrow \infty} \sigma_{n_k} = \lim_{k \rightarrow \infty} \phi(v) = \phi(v). \end{aligned} \quad (68)$$

It is completed. Second, if there is no such subsequence of $\{\sigma_k\}_{k \in \mathbb{N}}$, then there is $\mu_0 \in \mathbb{N}$ such that, for each $k \geq \mu_0$, we obtain

$$\begin{aligned} \mathfrak{I}_b(\psi(\sigma_{2k}), \phi(v)) &> 0, \\ \mathfrak{I}_b(\phi(\sigma_{2k+1}), \psi(v)) &> 0. \end{aligned} \quad (69)$$

Since \mathfrak{F} is γ -regular, one writes

$$\begin{aligned} \gamma(\sigma_{2k+1}, v) &\geq 1, \\ \gamma(\sigma_{2k}, v) &\geq 1. \end{aligned} \quad (70)$$

By the contractive condition (51), we have

$$\begin{aligned} &\tau + F(s\gamma(\sigma_{2k}, v)\mathfrak{I}_b(\psi(\sigma_{2k}), \phi(v))) \\ &\leq F \max \left\{ \mathfrak{I}_b(\sigma_{2k}, v), \mathfrak{I}_b(\sigma_{2k}, \psi(\sigma_{2k})), \mathfrak{I}_b(v, \phi(v)), \frac{\mathfrak{I}_b(\sigma_{2k}, \phi(v)) + \mathfrak{I}_b(v, \psi(\sigma_{2k}))}{2s} \right\}. \end{aligned} \quad (71)$$

Now, we prove that $\mathfrak{I}_b(v, \phi(v)) = 0$. Suppose, on the contrary, that $\mathfrak{I}_b(v, \phi(v)) = q > 0$. Put $\lambda_k = \mathfrak{I}_b(\sigma_k, v) = 0 \forall k \in \mathbb{N}$. Since $\lim_{k \rightarrow \infty} \sigma_k = v$, there exists $\mu_1 \in \mathbb{N}$ satisfying for each $k \geq \mu_1$, both $\lambda_k < (q/2)$ and $b_k < (q/2)$ hold. Consequently, by (71), we obtain

$$\begin{aligned} &\tau + F(s\gamma(\sigma_{2k}, v)\mathfrak{I}_b(\psi(\sigma_{2k}), \phi(v))) \\ &\leq F \left(\max \left\{ \lambda_{2k}, b_{2k}, q, \frac{\mathfrak{I}_b(\sigma_{2k}, \phi(v)) + \lambda_{2k+1}}{2s} \right\} \right) \\ &\leq F \left(\max \left\{ \lambda_{2k}, b_{2k}, q, \frac{s\lambda_{2k} + sq + \lambda_{2k+1}}{2s} \right\} \right) \\ &\leq F \left(\max \left\{ \frac{q}{2}, \frac{q}{2}, q, \frac{(sq/2) + sq + (q/2)}{2s} \right\} \right) = F(q). \end{aligned} \quad (72)$$

Then, for every $k \geq \max\{\mu_0, \mu_1\}$, we obtain

$$\tau + F(s\gamma(\sigma_{2k}, v)\mathfrak{I}_b(\psi(\sigma_{2k}), \phi(v))) \leq F(\mathfrak{I}_b(v, \phi(v))). \quad (73)$$

Since F is increasing and continuous, by Lemma 4 and (73), we have

$$\begin{aligned} \tau + F(\mathfrak{I}_b(v, \phi(v))) &\leq \tau + F(s\gamma(\sigma_{2k}, v) \liminf_{k \rightarrow \infty} \mathfrak{I}_b \\ &\quad \cdot (\psi(\sigma_{2k}), \phi(v))) \\ &\leq \tau + \liminf_{k \rightarrow \infty} F(s\gamma(\sigma_{2k}, v)\mathfrak{I}_b(\psi(\sigma_{2k}), \phi(v))) \\ &\leq F(\mathfrak{I}_b(v, \phi(v))). \end{aligned} \quad (74)$$

The above result shows that $\tau \leq 0$, which is a contradiction. Then, $\mathfrak{I}_b(v, \phi(v)) = 0$, and hence, $v = \phi(v)$. Similarly, we can show that $v = \psi(v)$, and consequently, v is a common fixed point of ψ and ϕ .

Now, we prove that self-mappings ψ, ϕ have a unique common fixed point. Let v and ω be two common fixed points of ψ and ϕ . Then, $\psi(v) = v \neq \omega = \phi(\omega)$. It follows that

$$\mathfrak{I}_b(\psi(v), \phi(\omega)) = \mathfrak{I}_b(v, \omega) > 0. \quad (75)$$

Since $\gamma(v, \omega) \geq 1$, by contractive condition, one writes

$$\begin{aligned} &\tau + F(s\gamma(v, \omega)\mathfrak{I}_b(\psi(v), \phi(\omega))) \\ &\leq F \left(\max \left\{ \mathfrak{I}_b(v, \omega), \mathfrak{I}_b(\psi(v), v), \mathfrak{I}_b(\phi(\omega), \omega), \frac{\mathfrak{I}_b(\phi(\omega), v) + \mathfrak{I}_b(\psi(v), \omega)}{2s} \right\} \right) \\ &= F(\mathfrak{I}_b(v, \omega)) \leq F(s\gamma(v, \omega)\mathfrak{I}_b(v, \omega)). \end{aligned} \quad (76)$$

It shows that $\tau \leq 0$, which is a contradiction. Hence, ψ and ϕ have a unique common fixed point. \square

Example 11. Let $\mathfrak{F} = [0, \infty)$ and define $\mathfrak{I}_b, \gamma: \mathfrak{F} \times \mathfrak{F} \rightarrow [0, \infty)$ by

$$\begin{aligned}\mathfrak{L}_b(\sigma, \varsigma) &= (\sigma \vee \varsigma)^2, \\ \gamma(\sigma, \varsigma) &= \begin{cases} K \geq 1, & \text{if } \sigma \geq \varsigma, \\ 0, & \text{if } \sigma < \varsigma. \end{cases}\end{aligned}\quad (77)$$

So, \mathfrak{S} is a γ -complete partial b -metric space with $s = 2$. Take $\psi, \phi: \mathfrak{S} \longrightarrow \mathfrak{S}$ as

$$\begin{aligned}\psi(\sigma) &= \ln\left(1 + \frac{\sigma}{6}\right), \\ \phi(\sigma) &= \ln\left(1 + \frac{\sigma}{7}\right).\end{aligned}\quad (78)$$

Definitely, ψ and ϕ are γ -continuous. To prove that pair (ψ, ϕ) is weakly γ -admissible, let $\sigma, \varsigma \in \mathfrak{S}$ be such that $\varsigma = \psi(\sigma)$, so $\varsigma = \ln(1 + (\sigma/6))$. We have

$$\begin{aligned}\psi(\sigma) = \varsigma &= \ln\left(\frac{\sigma}{6} + 1\right) \geq \ln\left(1 + \frac{\varsigma = \ln(1 + (\sigma/6))}{7}\right) \\ &= \ln\left(1 + \frac{\varsigma}{7}\right) = \phi(\varsigma) = \phi\psi(\sigma).\end{aligned}\quad (79)$$

Thus, $\gamma(\psi(\sigma), \phi\psi(\sigma)) \geq 1$. Now, let $v, \varsigma \in \mathfrak{S}$ be such that $v = \phi(\varsigma)$, so $v = \ln(1 + (\varsigma/7))$. Since

$$\begin{aligned}\phi(\varsigma) &= \ln\left(1 + \frac{\varsigma}{7}\right) \geq \ln\left(1 + \frac{\varsigma = \ln(1 + (\varsigma/7))}{6}\right) \\ &= \ln\left(1 + \frac{v}{6}\right) = \psi(v) = \psi\phi(\varsigma),\end{aligned}\quad (80)$$

we get that $\gamma(\phi(\varsigma), \psi\phi(\varsigma)) \geq 1$. Hence, the pair (ψ, ϕ) is weakly γ -admissible.

Next, for all $\sigma, \varsigma \in \mathfrak{S}$ with $\sigma \geq \varsigma$, choose ξ such that $(\xi/2K) > 1 + \log_2 s$; then,

$$\begin{aligned}2\gamma(\sigma, \varsigma)\mathfrak{L}_b(\psi(\sigma), \phi(\varsigma)) &= 2K(\psi(\sigma) \vee \phi(\varsigma))^2 \\ &= 2K\left(\ln\left(\frac{\sigma}{6} + 1\right) \vee \ln\left(1 + \frac{\varsigma}{7}\right)\right)^2 = 2K\left(\ln\left(1 + \frac{\varsigma}{6}\right)\right)^2 \\ &\leq 2K\left(\ln\left(1 + \frac{\sigma}{6}\right)\right)^2 \leq \frac{2K}{\xi}\mathfrak{L}_b(\sigma, \varsigma) \leq \frac{2K}{\xi}M_1(\sigma, \varsigma).\end{aligned}\quad (81)$$

The above result may be written as

$$\ln \frac{\xi}{2k} + \ln(2\gamma(\sigma, \varsigma)\mathfrak{L}_b(\psi(\sigma), \phi(\varsigma)) + 1) \leq \ln(M_1(\sigma, \varsigma) + 1).\quad (82)$$

Define $F: (0, \infty) \longrightarrow (-\infty, \infty)$ by

$$F(v) = \ln(v + 1), \quad \text{for all } v \in (0, \infty). \quad (83)$$

Then, $F \in \Delta_b^s$. Hence, for all $\sigma, \varsigma \in \mathfrak{S}$ such that $\mathfrak{L}_b(\psi(\sigma), \phi(\varsigma)) > 0$ and $\tau = \ln(\xi/2K)$, we obtain

$$\tau + F(2\gamma(\sigma, \varsigma)\mathfrak{L}_b(\psi(\sigma), \phi(\varsigma))) \leq F(M_1(\sigma, \varsigma)). \quad (84)$$

Consequently, the contractive condition (51) holds for all $\sigma, \varsigma \in \mathfrak{S}$. Hence, all the assumptions of Theorem 1 are verified. Thus, ψ and ϕ have a unique common fixed point which is $v = 0$.

Definition 17. Let $(\mathfrak{S}, \mathfrak{L}_b, s)$ be a partial b -metric space. Then, the self-mappings $\psi, \phi: \mathfrak{S} \longrightarrow \mathfrak{S}$ form a Hardy–Rogers (γ, F) -contraction if there exist $F \in \Delta_b^s$ and $\tau > 0$ satisfying

$$\tau + F(s\gamma(\sigma, \varsigma)\mathfrak{L}_b(\psi(\sigma), \phi(\varsigma))) \leq F(R(\sigma, \varsigma)), \quad \forall \sigma, \varsigma \in \mathfrak{S}, \quad (85)$$

with $\gamma(\sigma, \varsigma) \geq 1$, whenever

$$\min\{\gamma(\sigma, \varsigma)\mathfrak{L}_b(\psi(\sigma), \phi(\varsigma)), R(\sigma, \varsigma)\} > 0, \quad (86)$$

where

$$\begin{aligned}R(\sigma, \varsigma) &= \xi_1\mathfrak{L}_b(\sigma, \varsigma) + \xi_2\mathfrak{L}_b(\sigma, \psi(\sigma)) + \xi_3\mathfrak{L}_b(\varsigma, \phi(\varsigma)) \\ &\quad + \xi_4[\mathfrak{L}_b(\sigma, \phi(\varsigma)) + \mathfrak{L}_b(\psi(\sigma), \varsigma)],\end{aligned}\quad (87)$$

such that $\xi_1 + \xi_2 + \xi_3 + 2s\xi_4 = 1$.

Theorem 2. Given the pair $(\psi, \phi): \mathfrak{S} \longrightarrow \mathfrak{S}$ of a Hardy–Rogers-type (γ, F) -contraction defined on a γ -complete partial b -metric space $(\mathfrak{S}, \mathfrak{L}_b, s)$. Consider $k \in (0, (1/(1 + \log_2 s)))$. Suppose that

- (1) The pair (ψ, ϕ) is weakly γ -admissible
- (2) There exists $\sigma_0 \in \mathfrak{S}$ such that $\gamma(\sigma_0, \psi(\sigma_0)) \geq 1$
- (3) (a) \mathfrak{S} is γ -regular and F is continuous
(b) Either one of ψ, ϕ is γ -continuous

Then, we have a sequence $\{\sigma_n\}$ in \mathfrak{S} such that $\sigma_n \longrightarrow v \in \mathfrak{S}$. If $\gamma(v, v) \geq 1$, then v is a common fixed point of the pair (ψ, ϕ) . Moreover, if w is another fixed point of the pair (ψ, ϕ) satisfying $\gamma(v, w) \geq 1$, then $v = w$.

Proof. We have

$$\begin{aligned}R(\sigma, \varsigma) &= \xi_1\mathfrak{L}_b(\sigma, \varsigma) + \xi_2\mathfrak{L}_b(\psi(\sigma), \sigma) + \xi_3\mathfrak{L}_b(\phi(\varsigma), \varsigma) \\ &\quad + \xi_4[\mathfrak{L}_b(\phi(\varsigma), \sigma) + \mathfrak{L}_b(\varsigma, \psi(\sigma))] = \xi_1\mathfrak{L}_b(\sigma, \varsigma) \\ &\quad + \xi_2\mathfrak{L}_b(\psi(\sigma), \sigma) \\ &\quad + \xi_3\mathfrak{L}_b(\phi(\varsigma), \varsigma) + 2s\xi_4\left[\frac{\mathfrak{L}_b(\sigma, \phi(\varsigma)) + \mathfrak{L}_b(\varsigma, \phi(\sigma))}{2s}\right] \\ &\leq \xi_1M_1(\sigma, \varsigma) + \xi_2M_1(\sigma, \varsigma) + \xi_3M_1(\sigma, \varsigma) + 2s\xi_4M_1(\sigma, \varsigma) \\ &= (\xi_1 + \xi_2 + \xi_3 + 2s\xi_4)M_1(\sigma, \varsigma) = M_1(\sigma, \varsigma).\end{aligned}\quad (88)$$

Inequality (85) implies (51), so the rest of proof follows from Theorem 1. \square

4. Conclusion

This paper contains a comparative study on partial b -metric spaces with examples. The obtained results generalize corresponding ones in the literature. We developed a methodology to obtain fixed point theorems in partial b -metric spaces. This paper may lead many researchers of this subject to investigate new fixed point theorems in partial b -metric spaces.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

An Implicit Algorithm for Finding a Fixed Point of a Q-Nonexpansive Mapping in Locally Convex Spaces

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Received 26 March 2021; Accepted 29 July 2021; Published 13 August 2021

Academic Editor: Zoran Mitrovic

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In this paper, the notion of the q -duality mappings in locally convex spaces is introduced. An implicit method for finding a fixed point of a Q -nonexpansive mapping is provided. Finally, the convergence of the proposed implicit scheme is investigated. Some examples in order to illustrate of the main results are presented.

1. Introduction

Let C be a nonempty closed and convex subset of a Banach space E and E^* be the dual space of E . Let $\langle \cdot, \cdot \rangle$ denote the pairing between E and E^* . The normalized duality mapping $J: E \longrightarrow E^*$ is defined by

$$J(x) = \{f \in E^*: \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad (1)$$

for all $x \in E$. It is well known that if J is a single-valued mapping, then it is norm-weak* continuous (see, for more details, Theorem 2.6.10 in [1]). In this paper, the duality mapping of locally convex space is studied and denoted by J_q for the seminorm q .

Suppose that Q is a family of seminorms on a locally convex space E which determines the topology of E that will be denoted by τ_Q . Let C be a nonempty closed and convex subset of E . A mapping $T: C \longrightarrow C$ is called Q -nonexpansive if $q(Tx - Ty) \leq q(x - y)$, for all $x, y \in C$ and $q \in Q$. Also, the function $f: E \longrightarrow E$ is Q -contraction if there exists $\beta \in [0, 1)$ such that $q(f(x) - f(y)) \leq \beta q(x - y)$, for all $x, y \in C$ and $q \in Q$.

The methods of iterations, which find a number of real applications, are popular for problems in sciences and engineering; see, e.g., [2–6]. Recently, various iterative

algorithms have been investigated for fixed points of the nonlinear mapping, especially, for the mappings of nonexpansive type; see, e.g., [7–11] and the references therein.

Also, the general implicit algorithm for finding a fixed point of a Q -nonexpansive mapping is given by

$$z_n = \varepsilon_n f z_n + (1 - \varepsilon_n) T z_n, \quad (n = 1, 2, \dots), \quad (2)$$

where f is a Q -contraction and T is a Q -nonexpansive mapping. On the contrary, it is proven that there exists a sunny Q -nonexpansive retraction P of C onto $\text{Fix}(T)$ and $x \in C$ such that the sequence $\{z_n\}$ converges to Px with respect to the topology τ_Q .

Moreover, some notions and concepts in locally convex spaces are recalled. The Hahn–Banach theorem and Banach contraction principle will be generalized to locally convex spaces. The results of this article can be viewed as implicit and locally convex space version of some results given in [1, 12–17].

2. Methods and Preliminaries

Suppose that Q is a family of seminorms on a locally convex space E which determines the topology of E will be denoted by τ_Q .

Let $D \subseteq B \subseteq E$ and P be a retraction of B onto D , that is, $Px = x$, for each $x \in D$. Then, P is said to be sunny if, for each $x \in B$ and $t \geq 0$ with $Px + t(x - Px) \in B$, we have

$$P(Px + t(x - Px)) = Px. \quad (3)$$

A subset D of B is said to be sunny Q -nonexpansive retract of B if there exists a sunny Q -nonexpansive retraction P of B onto D . We know that if E is a smooth Banach space and P a retraction of B onto D , then P is sunny nonexpansive if and only if

$$\langle x - Px, J(z - Px) \rangle \leq 0, \quad (4)$$

for each $x \in B$ and $z \in D$. For more details, see [18]. In the sequel, we will prove inequality (4) for a real locally convex space.

Now, we recall the following definitions:

- (1) The locally convex topology τ_Q is separated if and only if the family of seminorms Q possesses the following property: for each $x \in E \setminus \{0\}$, there exists $q \in Q$ such that $q(x) \neq 0$ (see [19]) or equivalently

$$\bigcap_{q \in Q} \{x \in E: q(x) = 0\} = \{0\}. \quad (5)$$

- (2) Let E be a locally convex topological vector space over \mathbb{R} or \mathbb{C} . If $U \subset E$, the polar of U is denoted by

$$U^\circ = \{f \in E^*: |f(x)| \leq 1, \forall x \in U\}, \quad (6)$$

see [20].

- (3) Suppose G is a Hausdorff topological group, and (x_α) is a net in G defined on a directed set D . (x_α) is left Cauchy when the following happens: for every neighborhood B of the identity e of G , there exists an $\alpha_0 \in D$ such that

$$\begin{aligned} \forall \beta, \gamma \in D: \beta > \alpha_0, \\ \gamma > \alpha_0 \implies x_\beta^{-1} x_\gamma \in B, \end{aligned} \quad (7)$$

see [20].

Suppose that Q is a family of seminorms on a locally convex space X which determines the topology of X and $q \in Q$. Let Y be a subset of X and

$$q_Y^*(f) = \sup\{|f(y)|: y \in Y, q(y) \leq 1\}. \quad (8)$$

Also,

$$q^*(f) = \sup\{|f(x)|: x \in X, q(x) \leq 1\}, \quad (9)$$

for every linear functional f on X . Notice that if, for each $x \in X$, $q(x) \neq 0$, and $f \in X^*$, then

$$|\langle x, f \rangle| \leq q(x)q^*(f), \quad (10)$$

for more details, see [12].

We will make use of the following theorems.

Theorem 1 (see [12]). Suppose that Q is a family of seminorms on a real locally convex space X which determines the topology of X and $q \in Q$ is a continuous seminorm and Y is a vector subspace of X such that

$$Y \cap \{x \in X: q(x) = 0\} = \{0\}. \quad (11)$$

Let f be a real linear functional on Y such that $q_Y^*(f) < \infty$. Then, there exists a continuous linear functional h on X that extends f such that

$$q_Y^*(f) = q^*(h). \quad (12)$$

Theorem 2 (see [12]). Suppose that Q is a family of seminorms on a real locally convex space X which determines the topology of X and $q \in Q$ a nonzero continuous seminorm. Let x_0 be a point in X . Then, there exists a continuous linear functional f on X such that $q^*(f) = 1$ and $f(x_0) = q(x_0)$.

Next, we bring the following known results for easy reference.

Theorem 3 (see Proposition 2.5.2 in [1]). Let X be a topological space and $f: X \rightarrow (-\infty, +\infty]$ a function. Then, the following statements are equivalent: (a) f is lower semicontinuous; (b) for each $\alpha \in \mathbb{R}$, the level set $\{x \in X: f(x) \leq \alpha\}$ is closed.

Theorem 4 (see Theorem 3.26 (Banach-Alaoglu) in [20]). Suppose that X is a locally convex space and U is a neighborhood of zero in X . Then, U° , the polar set of U , is weak* compact.

For general topological spaces, we know the useful property as follows.

Theorem 5 (see Theorem 3.3.18 in [21]). For a topological space (X, τ) , the following are equivalent:

- (i) X is compact
- (ii) Each net in X has a convergent subnet

3. Main Result

First, we define our notation of the q -duality mapping as follows.

Suppose that Q is a family of seminorms on a real locally convex space X which determines the topology of X , $q \in Q$ is a continuous seminorm, and X^* is the dual space of X . A multivalued mapping $J_q: X \rightarrow 2^{X^*}$, defined by

$$J_q x = \{j \in X^*: \langle x, j \rangle = q(x)^2 = q^*(j)^2\}, \quad (13)$$

is called q -duality mapping. Obviously, $J_q(-x) = -J_q(x)$ and $J_q x \neq \emptyset$. Indeed, let $x \in X$; if $q(x) = 0$, then $j = 0 \in J_q x$, and if $q(x) \neq 0$, by applying Theorem 2, there exists a linear functional $f \in X^*$ such that $q^*(f) = 1$ and $\langle x, f \rangle = q(x)$. Putting $j = q(x)f$, then

$$\begin{aligned}
\langle x, j \rangle &= \langle x, q(x)f \rangle = q(x)\langle x, f \rangle = q(x)^2, \\
q^*(j) &= \sup\{|j(y)|: y \in X, q(y) \leq 1\} \\
&= \sup\{|q(x)f(y)|: y \in X, q(y) \leq 1\} \\
&= q(x)\sup\{|f(y)|: y \in X, q(y) \leq 1\} \\
&= q(x)q^*(f) = q(x),
\end{aligned} \tag{14}$$

and then, $j = q(x)f \in J_q x$.

Now, we present the following result that extends Proposition 4.1.1 in [1].

Lemma 1. *Let Q be a family of seminorms on a Hausdorff and complete locally convex space E which determines the topology of E , $\phi_q: E \rightarrow (-\infty, \infty]$ a bounded below and lower semicontinuous function, for each $q \in Q$, and $\{x_n\}$ a sequence in E such that $q(x_n - x_{n+1}) \leq \phi_q(x_n) - \phi_q(x_{n+1})$, for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $q \in Q$. Then, $\{x_n\}$ converges to a point $v \in E$, and for each $q \in Q$,*

$$q(x_n - v) \leq \phi_q(x_n) - \phi_q(v), \tag{15}$$

for all $n \in \mathbb{N}_0$.

Proof. Since $q(x_n - x_{n+1}) \leq \phi_q(x_n) - \phi_q(x_{n+1})$, for each $n \in \mathbb{N}_0$ and $q \in Q$, then $\{\phi_q(x_n)\}$ is a decreasing sequence for each $q \in Q$. Moreover,

$$\begin{aligned}
\sum_{n=0}^m q(x_n - x_{n+1}) &= q(x_0 - x_1) + q(x_1 - x_2) + \cdots + q(x_m - x_{m+1}) \\
&\leq \phi_q(x_0) - \phi_q(x_{m+1}) \leq \phi_q(x_0) - \inf_{n \in \mathbb{N}_0} \phi_q(x_n),
\end{aligned} \tag{16}$$

for $m \in \mathbb{N}_0$. Letting $m \rightarrow \infty$, we have

$$\sum_{n=0}^{\infty} q(x_n - x_{n+1}) < \infty, \tag{17}$$

and then, $\lim_n q(x_n - x_{n+1}) = 0$, for each $q \in Q$. This implies that $\{x_n\}$ is a left Cauchy sequence in E . Since E is Hausdorff and complete, there exists a unique $v \in E$ such that $\lim_{n \rightarrow \infty} x_n = v$. Let $m, n \in \mathbb{N}_0$ with $m > n$. Then,

$$\begin{aligned}
q(x_n - x_m) &\leq \sum_{i=n}^{m-1} q(x_i - x_{i+1}) \\
&\leq \phi_q(x_n) - \phi_q(x_m),
\end{aligned} \tag{18}$$

for each $q \in Q$. Since ϕ_q is lower semicontinuous and $q \in Q$ is continuous (Theorem 1.4 in [19]), letting $m \rightarrow \infty$, we conclude that

$$q(x_n - v) \leq \phi_q(x_n) - \lim_{m \rightarrow \infty} \phi_q(x_m) \leq \phi_q(x_n) - \phi_q(v), \tag{19}$$

for all $n \in \mathbb{N}_0$ and $q \in Q$. \square

Now, we state an extension of Banach contraction principle to locally convex spaces, and we call it Banach Q-contraction principle.

Theorem 6 (Banach Q-contraction principle). *Let Q be a family of seminorms on a separated and complete locally convex space E which determines the topology of E and $T: E \rightarrow E$ a Q-contraction mapping with Lipschitz constant $k \in (0, 1)$. Then,*

- (a) T has a unique fixed point $v \in E$.
- (b) For arbitrary $x_0 \in E$, the Picard iteration process defined by

$$x_{n+1} = T(x_n), \quad n \in \mathbb{N}_0, \tag{20}$$

converges to v .

- (c) $q(x_n - v) \leq (k^n / (1 - k))q(x_0 - x_1)$, for all $n \in \mathbb{N}_0$ and $q \in Q$.

Proof

- (a) For each $q \in Q$, let $\phi_q: E \rightarrow \mathbb{R}^+$ be a function defined by

$$\phi_q(x) = \frac{q(x - Tx)}{1 - k}, \tag{21}$$

for each $x \in E$. Since q is continuous, ϕ_q is also a continuous function. Also, T is a Q-contraction mapping; then,

$$q(Tx - T^2x) \leq kq(x - Tx), \tag{22}$$

for each $x \in E$ and $q \in Q$, which conclude that

$$q(x - Tx) - kq(x - Tx) \leq q(x - Tx) - q(Tx - T^2x). \tag{23}$$

Hence,

$$\begin{aligned}
q(x - Tx) &\leq \frac{1}{1 - k} [q(x - Tx) - q(Tx - T^2x)] \\
&\leq \phi_q(x) - \phi_q(Tx),
\end{aligned} \tag{24}$$

for each $q \in Q$. Let x be an arbitrary and fixed element in E . We define the sequence x_n in E by $x_n = T^n x$ ($n \in \mathbb{N}$). By using (24), we have

$$q(x_n - x_{n+1}) \leq \phi_q(x_n) - \phi_q(x_{n+1}), \tag{25}$$

for each $n \in \mathbb{N}$ and $q \in Q$. Since E is Hausdorff, by applying Lemma 1, there exists an element $v \in E$ such that

$$\begin{aligned}
\lim_{n \rightarrow \infty} x_n &= v, \\
q(x_n - v) &\leq \phi_q(x_n),
\end{aligned} \tag{26}$$

for each $n \in \mathbb{N}_0$ and $q \in Q$. Notice that T is continuous (page 3 in [19]) and $x_{n+1} = Tx_n$; therefore, $v = Tv$. Let $z \in E$ be another fixed point for T and $q(v - z) > 0$ for some $q \in Q$. Then,

$$0 < q(v - z) = q(Tv - Tz) \leq kq(v - z) < q(v - z), \tag{27}$$

which is a contradiction. Therefore, $q(v - z) = 0$, for each $q \in Q$. Since E is separated, we have $v = z$. Hence, T has unique fixed point $v \in E$.

(b) This assertion follows from part (a).

(c) By applying (22), we have $\phi_q(x_n) \leq k^n \phi_q(x_0)$, for each $q \in Q$. This implies from (26) that $q(x_n - v) \leq k^n \phi_q(x_0)$, for each $q \in Q$. \square

Next, we prove a generalization of Proposition 2.4.7 in [1] as follows.

Lemma 2. *Let E be a locally convex space. Then, for $x, y \in E$ and for each $q \in Q$ with $q(x) \neq 0$, the following statements are equivalent:*

- (a) $q(x) \leq q(x + ty)$, for all $t > 0$ with $q(x + ty) \neq 0$
- (b) There exists $j_q \in J_q x$ such that $\langle y, j_q \rangle \geq 0$

Proof

(a) \implies (b). Let $t > 0$ and $f_t \in J_q(x + ty)$, and we define $g_t = f_t / q^*(f_t)$. It is clear that $q^*(g_t) = 1$ and $g_t \in (1/q^*(f_t))J_q(x + ty)$; then,

$$\begin{aligned} q(x) \leq q(x + ty) &= \frac{\langle x + ty, f_t \rangle}{q^*(f_t)} \\ &= \langle x + ty, g_t \rangle = \langle x, g_t \rangle + t \langle y, g_t \rangle \\ &\leq q(x) + t \langle y, g_t \rangle. \end{aligned} \quad (28)$$

Putting $U = \{x \in E : q(x) \leq 1\}$, it is clear that

$$\{g_t\}_t \subset U^\circ = \{f \in E^* : |f(x)| \leq 1, \forall x \in U\}, \quad (29)$$

from Theorem 4, U° is weak* compact; then, by applying Theorem 5, the net $\{g_t\}$ has a limit point $g \in E^*$ such that $q^*(g) \leq 1$. From the above equations $q(x) \leq \langle x + ty, g_t \rangle$, let $y = 0$; by limiting, we have $q(x) \leq \langle x, g \rangle$. Also, since $t > 0$, by using (28) and limiting, we have $\langle y, g \rangle \geq 0$. Therefore,

$$q(x) \leq \langle x, g \rangle \leq q(x)q^*(g) \leq q(x), \quad (30)$$

which conclude that $\langle x, g \rangle = q(x)$ and $q^*(g) = 1$. Set $j_q = gq(x)$; then, $j_q(x) = g(x)q(x) = q(x)^2$ and

$$\begin{aligned} q^*(j_q) &= \sup \left\{ |j_q(z)| : z \in X, q(z) \leq 1 \right\} \\ &= \sup \{ |q(x)g(z)| : z \in X, q(z) \leq 1 \} \\ &= q(x) \sup \{ |g(z)| : z \in X, q(z) \leq 1 \} \\ &= q(x)q^*(g) = q(x). \end{aligned} \quad (31)$$

Hence, $j_q \in J_q x$ and $\langle y, j_q \rangle = \langle y, gq(x) \rangle = q(x) \langle y, g \rangle \geq 0$.

(b) \implies (a). Suppose, for $x, y \in X$ and $q \in Q$ with $q(x) \neq 0$, there exists $j_q \in J_q x$ such that $\langle y, j_q \rangle \geq 0$. We know that

$$|\langle x + ty, j_q \rangle| \leq q(x + ty)q^*(j_q) = q(x + ty)q(x). \quad (32)$$

Hence, for $t > 0$ with $q(x + ty) \neq 0$, we have

$$\begin{aligned} q(x)^2 &= \langle x, j_q \rangle \leq \langle x, j_q \rangle + \langle ty, j_q \rangle \\ &= \langle x + ty, j_q \rangle \leq q(x + ty)q(x), \end{aligned} \quad (33)$$

which implies that $q(x) \leq q(x + ty)$. This completes the proof. \square

The next theorem is significant in the sequel, and it extends Proposition 2.10.21 in [1].

Theorem 7. *Let E be a separated locally convex space, C a nonempty convex subset of E , and D a nonempty subset of C . Also, let $J_q : E \longrightarrow E^*$ be a single valued, for every $q \in Q$ and P a retraction of C onto D . Then, for each $q \in Q$, the following statements are equivalent:*

- (a) $\langle x - Px, J_q(y - Px) \rangle \leq 0$, ($x \in C, y \in D$)
- (b) P is sunny Q -nonexpansive

Proof

(a) \implies (b). First, we show that P is sunny. For $x \in C$ and $t > 0$, set $x_t = Px + t(x - Px)$. Since C is convex, $x_t \in C$, for each $t \in (0, 1]$. Hence, from (a), we have

$$\begin{aligned} \langle x - Px, J_q(Px - Px_t) \rangle &\geq 0, \\ \langle x_t - Px_t, J_q(Px_t - Px) \rangle &\geq 0. \end{aligned} \quad (34)$$

Because $x_t - Px = t(x - Px)$ and $\langle t(x - Px), J_q(Px - Px_t) \rangle \geq 0$, we have

$$\langle x_t - Px, J_q(Px - Px_t) \rangle \geq 0. \quad (35)$$

Combining (34) and (35), we have

$$\begin{aligned} q(Px - Px_t)^2 &= \langle Px - x_t + x_t - Px_t, J_q(Px - Px_t) \rangle \\ &= -\langle x_t - Px, J_q(Px - Px_t) \rangle \\ &\quad + \langle x_t - Px_t, J_q(Px - Px_t) \rangle \leq 0. \end{aligned} \quad (36)$$

Then, $q(Px - Px_t) = 0$, for each $q \in Q$. Thus, from the fact that E is separated, $Px = Px_t$. This means that P is sunny. Now, we show that P is Q -nonexpansive. For $x, z \in C$ and $q \in Q$, we have from (a) that $\langle x - Px, J_q(Px - Px) \rangle \geq 0$ and $\langle z - Px, J_q(Pz - Px) \rangle \geq 0$. Hence,

$$\begin{aligned}
\langle x - z - (Px - Pz), J_q(Px - Pz) \rangle &\geq 0, \\
\langle x - z, J_q(Px - Pz) \rangle &= \langle x - Px + Px - Pz + Pz - z, J_q(Px - Pz) \rangle \\
&= \langle x - Px, J_q(Px - Pz) \rangle - \langle z - Pz, J_q(Px - Pz) \rangle \\
&\quad + \langle Px - Pz, J_q(Px - Pz) \rangle \geq \langle Px - Pz, J_q(Px - Pz) \rangle \\
&= q^2(Px - Pz).
\end{aligned} \tag{37}$$

(notice that $J_q(Px - Pz)$ is single valued). On the contrary,

$$\begin{aligned}
\langle x - z, J_q(Px - Pz) \rangle &\leq q(x - z)q^*(J_q(Px - Pz)) \\
&= q(x - z)q(Px - Pz).
\end{aligned} \tag{38}$$

Therefore, be applying (37) and (38), we have

$$q(Px - Pz) \leq q(x - z), \tag{39}$$

it follows that P is Q -nonexpansive.

(b) \implies (a). Conversely, suppose that P is the sunny Q -nonexpansive retraction from C onto D , $x \in C$ and $y \in D$. Then, $Px \in D$ and $Py = y$. Since C is convex, $Px + t(x - Px) \in C$, for $0 < t \leq 1$. Also, P is sunny; then,

$$Px = P(Px + t(x - Px)), \tag{40}$$

and therefore,

$$\begin{aligned}
q(y - Px) &= q(Py - Px) = q(Py - P(Px + t(x - Px))) \\
&\leq q(y - (Px + t(x - Px))) \\
&= q(y - Px + t(Px - x)).
\end{aligned} \tag{41}$$

Hence, from Lemma 2, there exists $j_q \in J_q(y - Px)$ such that

$$\langle Px - x, j_q(y - Px) \rangle \geq 0, \quad x \in C, y \in D. \tag{42}$$

Since $J_q: E \longrightarrow E^*$ is single valued, we can say that

$$\langle x - Px, J_q(y - Px) \rangle \leq 0, \quad x \in C, y \in D. \tag{43}$$

□

Next, as a direct consequence of Theorem 6.5.3 in [18], we have the following result. To apply Theorem 6.5.3 in [18] in Theorems 8, 9, 11, and 12 and Corollary 1, it is considered that $\inf q(x) > 0$, for each $q \in Q$, when x is in K (28) and (34), C_0 , $\text{Fix}(T)$, and C , respectively.

Theorem 8. Let Q be a family of seminorms on a locally convex space E which determines the topology of E and K be a convex subset of E , $x \in E$ and $x_0 \in K$. Then, for each $q \in Q$, the following are equivalent:

$$(1) \quad q(x_0 - x) = \inf\{q(x - y): y \in K\}.$$

(2) There exists an $L_q \in E^*$ such that $L_q(y) \leq q(y)$, for each $y \in E$, $q^*(L_q) = 1$, and

$$\inf\{L_q(y - x): y \in K\} = q(x_0 - x). \tag{44}$$

(3) There exists an $L_q \in E^*$ such that $L_q(y) \leq q(y)$, for each $y \in E$, $q^*(L_q) = 1$, and

$$\begin{aligned}
\inf\{L_q(y): y \in K\} &= L_q(x_0), \\
L_q(x_0 - x) &= q(x_0 - x).
\end{aligned} \tag{45}$$

The next theorem is similar to Corollary 6.5.5 in [18].

Theorem 9. Let Q be a family of seminorms on a real locally convex space E which determines the topology of E and K be a convex subset of E , $x \in E$ and $x_0 \in K$. Then, for each $q \in Q$, the following statements are equivalent:

- (1) $q(x_0 - x) = \inf\{q(x - y): y \in K\}$
- (2) There exists an $f_q \in J_q(x - x_0)$ such that $f_q(x_0 - y) \geq 0$, for every $y \in K$

Proof

- (1) $(1) \implies (2)$. Let $q \in Q$ and $q(x_0 - x) = \inf\{q(x - y): y \in K\}$. Then, from Theorem 8, there exists an $L_q \in E^*$ such that $L_q(y) \leq q(y)$, for each $y \in E$, $q^*(L_q) = 1$, and

$$\inf\{L_q(y - x): y \in K\} = q(x_0 - x). \tag{46}$$

Set $f_q = -q(x_0 - x)L_q$. It is easy to see that $L_q(x_0 - x) = q(x_0 - x)$; therefore,

$$\begin{aligned}
f_q(x - x_0) &= -q(x_0 - x)L_q(x - x_0) \\
&= q^2(x_0 - x) = (q^*(f_q))^2.
\end{aligned} \tag{47}$$

Then, $f_q \in J_q(x - x_0)$. Hence, from (46) and (47), we have

$$\begin{aligned}
f_q(x_0 - y) &= f_q(x_0 - x + x - y) = -q^2(x_0 - x) \\
&\quad - q(x_0 - x)L_q(x - y) \\
&= -q^2(x_0 - x) + q(x_0 - x)L_q(y - x) \\
&\geq -q^2(x_0 - x) + q^2(x_0 - x) = 0.
\end{aligned} \tag{48}$$

(2) \implies (1). Since $f_q \in J_q(x - x_0)$, then

$$f_q(x - x_0) = q^2(x_0 - x) = (q^*(f_q))^2. \quad (49)$$

Also, $f_q(x_0 - y) \geq 0$, for every $y \in K$; therefore,

$$\begin{aligned} q^2(x_0 - x) &= f_q(x - x_0) = f_q(x - y + y - x_0) \\ &\leq q(x - y)q^*(f_q) + f_q(y - x_0) \\ &\leq q(x - y)q^*(f_q) = q(x - y)q(x_0 - x), \end{aligned} \quad (50)$$

for every $y \in K$; hence, we have

$$q(x_0 - x) \leq q(x - y), \quad (51)$$

for all $y \in K$. This means that

$$q(x_0 - x) = \inf\{q(x - y) : y \in K\}. \quad (52)$$

Corollary 1. Let Q be a family of seminorms on a locally convex space E which determines the topology of E , C a nonempty closed convex subset of E , $C_0 \subseteq C$, and P be a sunny Q -nonexpansive retraction of C onto C_0 . Let $J_q : E \rightarrow E^*$ be single-valued duality mapping for each $q \in Q$; then,

$$\langle x - Px, J_q(y - Px) \rangle \leq 0, \quad (53)$$

for each $x \in C$ and $y \in C_0$.

Proof. Let $x \in C$ and $y \in C_0$. Set $x_t = Px + t(x - Px)$, for each $0 \leq t \leq 1$; then, $x_t \in C$ and

$$q(Px - y) = q(Px_t - Py) \leq q(x_t - y), \quad (54)$$

for each $q \in Q$. Now, from Theorem 9, there exists $f_q \in J_q(y - Px)$ such that $\langle Px - x_t, f_q \rangle = f_q(Px - x_t) \geq 0$. Since J_q is single valued, we can say that

$$\langle Px - x_t, J_q(y - Px) \rangle \geq 0, \quad (55)$$

and since $Px - x_t = t(Px - x)$, then

$$\langle x - Px, J_q(y - Px) \rangle \leq 0. \quad (56)$$

We now extend the result of Theorem 2.6.10 in [1] as follows.

Theorem 10. Let E be a locally convex space and $J_q : E \rightarrow E^*$ be a single-valued duality mapping. Then, J_q is continuous from τ_Q to weak* topology, for each $q \in Q$.

Proof. Let $q \in Q$ be fixed and arbitrary, $\{x_\alpha\}_{\alpha \in I} \subset E$, and $x \in E$ such that $x_\alpha \rightarrow x$ in τ_Q , and we prove that $J_q x_\alpha \rightarrow J_q x$ in the weak* topology. First, assume that $q(x) = 0$; then, $J_q x = 0$ (from (13)). It is clear that $q^*(J_q x_\alpha) = q(x_\alpha)$, also $q \in Q$ is continuous Theorem 1.4 in [19]; therefore,

$$\lim_\alpha q^*(J_q x_\alpha) = \lim_\alpha q(x_\alpha) = 0, \quad (57)$$

and hence, $\lim_\alpha \langle y, J_q x_\alpha \rangle = 0$, for every $y \in E$ that $q(y) \leq 1$ (from (9)). On the contrary, if $q(y) > 1$, then $q(y/q(y)) = 1$; therefore, $\lim_\alpha \langle y/q(y), J_q x_\alpha \rangle = 0$. So, we conclude that

$\lim_\alpha \langle y, J_q x_\alpha \rangle = 0$, for every $y \in E$. Hence, $J_q x_\alpha \rightarrow 0$ in the weak* topology. Second, assume that $q(x) \neq 0$. Since

$$J_q x_\alpha = \{j \in X^* : \langle x_\alpha, j \rangle = q(x_\alpha)^2 = q^*(j)^2\}, \quad (58)$$

and J_q is single-valued set $f_\alpha^q = J_q x_\alpha$. Then,

$$\langle x_\alpha, f_\alpha^q \rangle = q(x_\alpha)q^*(f_\alpha^q), \quad q(x_\alpha) = q^*(f_\alpha^q). \quad (59)$$

Since $\{x_\alpha\}_\alpha$ is bounded with respect to τ_Q , then there exists $N_q > 0$ such that $q(x_\alpha) \leq N_q$, for each $\alpha \in I$. Now, it is shown that $\{f_\alpha^q\}_{\alpha \in I}$ is bounded in E^* .

One can see that E^* is a locally convex space and the family of seminorms $\{p_y : y \in E\}$ defines the weak* topology on it, where $p_y(x^*) = |\langle y, x^* \rangle|$, for each $x^* \in E$ (Chapter 5, Definition 1.1 in [22]). Hence,

$$p_y(f_\alpha^q) = |\langle y, f_\alpha^q \rangle| \leq q(y)q^*(f_\alpha^q) = q(y)q(x_\alpha) \leq q(y)N_q, \quad (60)$$

when $q(y) \neq 0$. Also, if $q(y) = 0$,

$$\begin{aligned} p_y(f_\alpha^q) &= |\langle y, f_\alpha^q \rangle| \leq \sup\{|\langle y, f_\alpha^q \rangle| : y \in E, q(y) \leq 1\} \\ &= q^*(f_\alpha^q) = q(x_\alpha) \leq N_q, \end{aligned} \quad (61)$$

for each $\alpha \in I$. By applying (60) and (61), there exists $M_{q,y} > 0$ related to each $y \in E$ and $q \in Q$, such that $p_y(f_\alpha^q) \leq M_{q,y}$, for each $\alpha \in I$. This means that $\{f_\alpha^q\}_{\alpha \in I}$ is bounded in the weak* topology of E^* . Putting $U_q = \{z \in E : q(z) < 1\}$, we have $\{(1/M_{q,y})f_\alpha^q\}_{\alpha \in I} \subset U_q^\circ$, for each $y \in E$, where U_q° is the polar of U_q and it is weak* compact (Theorem 4). Then, from Theorem 5, there exists a subnet $\{(1/M_{q,y})f_{\alpha_\beta}^q\}_\beta$ of $\{(1/M_{q,y})f_\alpha^q\}_\alpha$ such that $(1/M_{q,y})f_{\alpha_\beta}^q \rightarrow f \in U_q^\circ$ in the weak* topology, equivalently

$$\langle x, M_{q,y}f - f_{\alpha_\beta}^q \rangle \rightarrow 0. \quad (62)$$

We know that the function q^* on E^* is lower semicontinuous in the weak* topology. Indeed, if $\{g_\beta\}_\beta \subset \{f \in E^* : q^*(f) \leq \alpha\}$ such that $g_\beta \rightarrow g$ in the weak* topology, then

$$|\langle y, g_\beta - g \rangle| = p_y(g_\beta - g) \rightarrow 0, \quad (63)$$

for each $y \in E$; therefore, $\langle y, g_\beta \rangle \rightarrow \langle y, g \rangle$ and $q^*(g) \leq \alpha$, i.e.,

$$g \in \{f \in E^* : q^*(f) \leq \alpha\}. \quad (64)$$

So, by applying Theorem 3, q^* is lower semicontinuous in the weak* topology on E^* . Hence, we have

$$q^*(f) \leq \liminf_\beta q^*\left(\frac{1}{M_{q,y}}f_{\alpha_\beta}^q\right) = \frac{1}{M_{q,y}} \liminf_\beta q(x_{\alpha_\beta}) = \frac{1}{M_{q,y}} q(x). \quad (65)$$

Since $x_{\alpha_\beta} \rightarrow x$ in τ_Q and $x_{\alpha_\beta} \rightarrow x$ in weak topology, then

$$\langle x - x_{\alpha_\beta}, f_{\alpha_\beta}^q \rangle \rightarrow 0. \quad (66)$$

Therefore, using (62) and (66) implies that

$$\begin{aligned} \left| \langle x, M_{q,y}f \rangle - q(x_{\alpha_\beta})^2 \right| &= \left| \langle x, M_{q,y}f \rangle - \langle x_{\alpha_\beta}, f_{\alpha_\beta}^q \rangle \right| \\ &\leq \left| \langle x, M_{q,y}f - f_{\alpha_\beta}^q \rangle \right| + \left| \langle x - x_{\alpha_\beta}, f_{\alpha_\beta}^q \rangle \right| \longrightarrow 0. \end{aligned} \quad (67)$$

On the contrary, $q(x)^2(x_{\alpha_\beta}) \longrightarrow q(x)^2(x)$. Thus,

$$\langle x, M_{q,y}f \rangle = q(x)^2. \quad (68)$$

Since $q(x) \neq 0$, using (65), we have

$$q(x)^2 = \langle x, M_{q,y}f \rangle \leq q^*(M_{q,y}f)q(x) \leq q(x)^2, \quad (69)$$

and therefore,

$$\begin{aligned} \langle x, M_{q,y}f \rangle &= q(x)^2, \\ q(x) &= q^*(M_{q,y}f). \end{aligned} \quad (70)$$

This means that $M_{q,y}f = J_q x$. Finally, since $f_\alpha^q = J_q x_\alpha$, from (62), we conclude that $J_q x_\alpha \longrightarrow J_q x$. This completes the proof. \square

In the next theorem, we prove an existence theorem of a sunny Q-nonexpansive retract.

Theorem 11. *Let Q be a family of seminorms on a real separated and complete locally convex space E which determines the topology of E , C a nonempty closed convex and bounded subset of E such that every sequence in C has a convergent subsequence, and $T: C \longrightarrow C$ be a Q -nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$, the fixed points set of T . Let $J_q: E \longrightarrow E^*$ be single valued for every $q \in Q$. Then, $\text{Fix}(T)$ is a sunny Q -nonexpansive retract of C and the sunny Q -nonexpansive retraction of C onto $\text{Fix}(T)$ is unique.*

Proof. Let $x \in C$ be fixed; then, there exists a sequence $\{z_n\}$ in C such that

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)Tz_n, \quad n \in \mathbb{N}. \quad (71)$$

For this means, we define the function $K_n: C \longrightarrow C$, with

$$K_n(t) = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)T(t), \quad (72)$$

for each $n \in \mathbb{N}$ and $t \in C$. It is clear that K_n is Q -nonexpansive because

$$q(K_n(t_1) - K_n(t_2)) \leq \left(1 - \frac{1}{n}\right)q(T(t_1) - T(t_2)) \leq q(t_1 - t_2), \quad (73)$$

for every $t_1, t_2 \in C$. Then, by applying Theorem 6, there exists a unique point z_n in C such that $K_n(z_n) = z_n$, for each $n \in \mathbb{N}$; therefore,

$$z_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right)Tz_n, \quad n \in \mathbb{N}. \quad (74)$$

Since C is bounded in τ_Q , there exists $M_q > 0$ such that $q(y) \leq M_q$, for each $y \in C$ and $q \in Q$. Also,

$$q(Tz_n - z_n) = \frac{1}{n}q(Tz_n - x) \leq \frac{2}{n}M_q, \quad (75)$$

and then,

$$\lim_n q(Tz_n - z_n) = 0, \quad (76)$$

for each $q \in Q$. Next, we show that the sequence $\{z_n\}$ converges to an element of $\text{Fix}(T)$. In the other words, we show that the limit set of $\{z_n\}$ (denoted by $\mathfrak{S}\{z_n\}$) is a subset of $\text{Fix}(T)$. First, for each $z \in \text{Fix}(T)$ and $n \in \mathbb{N}$, we have

$$\langle z_n - x, J_q(z_n - z) \rangle \leq 0 \quad (77)$$

because $Tz_n - x = n(Tz_n - z_n)$ and

$$\begin{aligned} \langle Tz_n - Tz, J_q(z_n - z) \rangle &\leq q(Tz_n - Tz)q^*(J_q(z_n - z)) \\ &\leq q(z_n - z)q(z_n - z) = q^2(z_n - z). \end{aligned} \quad (78)$$

Also, since J_q is single valued, from definition (13), we have

$$\langle z - z_n, J_q(z_n - z) \rangle = -\langle z_n - z, J_q(z_n - z) \rangle = -q^2(z_n - z). \quad (79)$$

Therefore,

$$\begin{aligned} \langle z_n - x, J_q(z_n - z) \rangle &= \langle z_n - Tz_n + Tz_n - x, J_q(z_n - z) \rangle \\ &= \langle z_n - Tz_n + n(Tz_n - z_n), J_q(z_n - z) \rangle \\ &= (n-1)\langle Tz_n - z_n, J_q(z_n - z) \rangle \\ &= (n-1)\langle Tz_n - Tz, J_q(z_n - z) \rangle \\ &\quad + (n-1)\langle z - z_n, J_q(z_n - z) \rangle \\ &\leq (n-1)(q(Tz_n - Tz)q(z_n - z) - q(z_n - z)^2) \\ &\leq (n-1)(q(z_n - z)^2 - q(z_n - z)^2) = 0. \end{aligned} \quad (80)$$

From our assumption, $\{z_n\}$ has a subsequence converges to a point in C . Let $\{z_{n_i}\}$ and $\{z_{n_j}\}$ be subsequences of $\{z_n\}$ such that $\{z_{n_i}\}$ and $\{z_{n_j}\}$ converge to y and z , respectively. Then, (76) implies that

$$\begin{aligned} q(y - Ty) &\leq q(y - z_{n_i}) + q(z_{n_i} - Tz_{n_i}) + q(Tz_{n_i} - Ty) \\ &\leq 2q(y - z_{n_i}) + q(z_{n_i} - Tz_{n_i}) \longrightarrow 0, \end{aligned} \quad (81)$$

for each $q \in Q$. Since E is separated, $Ty = y$; hence, $y \in \text{Fix}(T)$ and similarly $z \in \text{Fix}(T)$.

Now, $z_{n_i} \longrightarrow y$ and $z \in \text{Fix}(T)$, and we claim that

$$\langle y - x, J_q(y - z) \rangle = \lim_{i \rightarrow \infty} \langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle. \quad (82)$$

$$\begin{aligned} & \left| \langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle - \langle y - x, J_q(y - z) \rangle \right| \\ &= \left| \langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle - \langle y - x, J_q(z_{n_i} - z) \rangle + \langle y - x, J_q(z_{n_i} - z) \rangle - \langle y - x, J_q(y - z) \rangle \right| \\ &\leq \left| \langle z_{n_i} - y, J_q(z_{n_i} - z) \rangle \right| + \left| \langle y - x, J_q(z_{n_i} - z) - J_q(y - z) \rangle \right| \\ &\leq q(z_{n_i} - y)q(z_{n_i} - z) + \left| \langle y - x, J_q(z_{n_i} - z) - J_q(y - z) \rangle \right| \\ &\leq 2q(z_{n_i} - y)M_q + \left| \langle y - x, J_q(z_{n_i} - z) - J_q(y - z) \rangle \right| \longrightarrow 0, \end{aligned} \quad (83)$$

and hence, by using (77),

$$\langle y - x, J_q(y - z) \rangle = \lim_{i \rightarrow \infty} \langle z_{n_i} - x, J_q(z_{n_i} - z) \rangle \leq 0. \quad (84)$$

Similarly, $\langle z - x, J_q(z - y) \rangle \leq 0$ since $J_q(y - z) = -J_q(z - y)$; then,

$$\begin{aligned} q(y - z)^2 &= \langle y - z, J_q(y - z) \rangle \\ &= \langle y - x, J_q(y - z) \rangle + \langle x - z, J_q(y - z) \rangle \\ &= \langle y - x, J_q(y - z) \rangle + \langle z - x, J_q(z - y) \rangle \leq 0, \end{aligned} \quad (85)$$

that is, $q(y - z) = 0$, for each $q \in Q$, and since E is separated, $y = z$. Thus, $\{z_n\}$ converges to an element of $\text{Fix}(T)$. Therefore, a mapping $P: C \longrightarrow C$ can be defined by $Px = \lim_n z_n$. Then, since $z_n \longrightarrow Px$, by using (84) and for each $z \in \text{Fix}(T)$,

$$\begin{aligned} \langle x - Px, J_q(z - Px) \rangle &= \langle Px - x, J_q(Px - z) \rangle, \\ &= \lim_n \langle z_n - x, J_q(z_n - z) \rangle \leq 0. \end{aligned} \quad (86) \quad (87)$$

It follows from Theorem 7 that P is a sunny Q -nonexpansive retraction of C onto $\text{Fix}(T)$. Let R be another sunny Q -nonexpansive retraction of C onto $\text{Fix}(T)$. Then, from Corollary 1,

$$\langle x - Rx, J_q(z - Rx) \rangle \leq 0, \quad (88)$$

for each $x \in C$ and $z \in \text{Fix}(T)$. Putting $z = Rx$ in (86) and $z = Px$ in (88), we have $\langle x - Px, J_q(Rx - Px) \rangle \leq 0$ and $\langle x - Rx, J_q(Px - Rx) \rangle \leq 0$; then,

$$q^2(Rx - Px) = \langle Rx - Px, J_q(Rx - Px) \rangle \leq 0, \quad (89)$$

for each $q \in Q$. Since E is separated, $Rx = Px$. This completes the proof. \square

Proposition 1. Let Q be a family of seminorms on a separated locally convex space E which determines the topology of E . Then,

$$q(x)^2 - q(y)^2 \geq 2\langle x - y, j \rangle, \quad (90)$$

Indeed, from the fact that J_q is single valued and continuous from τ_Q to weak* topology, we have

for all $x, y \in E$ and $j \in J_q y$ such that $q(y) \neq 0$.

Proof. Let $j \in J_q x$, $x \in E$. Then,

$$\begin{aligned} q(y)^2 - q(x)^2 - 2\langle y - x, j \rangle &= q(x)^2 + q(y)^2 - 2\langle y, j \rangle \\ &\geq q(x)^2 + q(y)^2 - 2q(x)q(y) \\ &\geq q(q(x) - q(y))^2 \geq 0. \end{aligned} \quad (91)$$

\square

Theorem 12. Let Q be a family of seminorms on a real separated and complete locally convex space E which determines the topology of E , C a nonempty closed convex and bounded subset of E such that every sequence in C has a convergent subsequence, and $T: C \longrightarrow C$ a Q -nonexpansive mapping such that $\text{Fix}(T) \neq \emptyset$. Let $J_q: E \longrightarrow E^*$ be single valued for each $q \in Q$ and f be a Q -contraction on C ; also, ε_n is a sequence in $(0, 1)$ such that $\lim_n \varepsilon_n = 0$. Then, there exists a unique $x \in C$ and sunny Q -nonexpansive retraction P of C onto $\text{Fix}(T)$ such that the following sequence $\{z_n\}$, generated by

$$z_n = \varepsilon_n f z_n + (1 - \varepsilon_n) T z_n, \quad n \in \mathbb{N}, \quad (92)$$

converges to Px .

Proof. Since f is a Q -contraction, there exists $0 \leq \beta < 1$ such that

$$q(f(x) - f(y)) \leq \beta q(x - y), \quad (93)$$

for each $x, y \in E$ and $q \in Q$. We divide the proof into five steps.

Step 1: the existence of z_n which satisfies (92).

This follows immediately from the fact that, for every $n \in \mathbb{N}$, the mapping $N_n: C \longrightarrow C$, given by

$$N_n x := \varepsilon_n f x + (1 - \varepsilon_n) T x, \quad x \in C, \quad (94)$$

is a Q -contraction. To see this, put $\beta_n = (1 + \varepsilon_n(\beta - 1))$, $0 \leq \beta_n < 1$, $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}
q(N_n x - N_n y) &\leq \varepsilon_n q(fx - fy) + (1 - \varepsilon_n)q(Tx - Ty) \\
&\leq \varepsilon_n \beta q(x - y) + (1 - \varepsilon_n)q(x - y) \\
&= (1 + \varepsilon_n(\beta - 1))q(x - y) \\
&= \beta_n q(x - y) < q(x - y).
\end{aligned} \tag{95}$$

Therefore, by Theorem 6, there exists a unique point $z_n \in C$ such that $N_n z_n = z_n$, that is,

$$z_n = \varepsilon_n f z_n + (1 - \varepsilon_n) T z_n. \tag{96}$$

Step 2: $\lim_n q(T z_n - z_n) = 0$, for each $q \in Q$.

Since C is bounded in τ_Q , there exists $M_q > 0$ such that $q(y) \leq M_q$, for each $y \in C$ and $q \in Q$.

Then,

$$\begin{aligned}
q(T z_n - z_n) &= \varepsilon_n q(T z_n - f z_n) \leq 2M_q \varepsilon_n, \\
\lim_n q(T z_n - z_n) &= 0,
\end{aligned} \tag{97}$$

for each $q \in Q$.

Step 3: $\mathfrak{S}(\{z_n\}) \subset \text{Fix}(T)$, where $\mathfrak{S}(\{z_n\})$ denotes the set of τ_Q -limit points of subsequences of $\{z_n\}$.

Let $z \in \mathfrak{S}(\{z_n\})$ and $\{z_{n_k}\}$ be a subsequence of $\{z_n\}$ such that $z_{n_k} \rightarrow z$. By using (97), we have

$$\begin{aligned}
q(Tz - z) &\leq q(Tz - Tz_{n_k}) + q(Tz_{n_k} - z_{n_k}) + q(z_{n_k} - z) \\
&\leq 2q(z_{n_k} - z) + q(Tz_{n_k} - z_{n_k}) \rightarrow 0,
\end{aligned} \tag{98}$$

for each $q \in Q$. Hence, $q(Tz - z) = 0$, for each $q \in Q$, and since E is separated, we conclude that $z \in \text{Fix}(T)$.

Step 4: there exists a unique sunny Q -nonexpansive retraction P of C onto $\text{Fix}(T)$ and $x \in C$ such that

$$K := \limsup_n \langle x - Px, J_q(z_n - Px) \rangle \leq 0. \tag{99}$$

We know from Theorem 10 that J_q is continuous from τ_Q to weak* topology; then, by Theorem 11, there exists a unique sunny Q -nonexpansive retraction P of C onto $\text{Fix}(T)$. Theorem 6 guarantees that fP has a unique fixed point $x \in C$. Since

$$K = \limsup_n \langle x - Px, J_q(z_n - Px) \rangle, \tag{100}$$

by the definition of \limsup , there exists subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$K = \lim_i \langle x - Px, J_q(z_{n_i} - Px) \rangle. \tag{101}$$

On the contrary, every sequence in C has a convergent subsequence; then, there exists subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $z_{n_k} \rightarrow z$ and $z \in C$.

By Step 3, we have $z \in \text{Fix}(T)$. From Theorem 7, we have

$$\langle x - Px, J_q(z - Px) \rangle \leq 0, \tag{102}$$

and since J_q is continuous, then

$$\lim_k \langle x - Px, J_q(z_{n_k} - Px) \rangle = \langle x - Px, J_q(z - Px) \rangle. \tag{103}$$

Therefore, by applying (101)–(103), we have $K \leq 0$.

Step 5: $\{z_n\}$ converges to Px in τ_Q .

Since $fPx = x$, thus $(f - I)Px = x - Px$. Now, from Proposition 1 and our assumption, we have

$$\begin{aligned}
&\varepsilon_n(\beta - 1)q(z_n - Px)^2 \\
&\geq [\varepsilon_n \beta q(z_n - Px) + (1 - \varepsilon_n)q(z_n - Px)]^2 - q(z_n - Px)^2 \\
&\geq [\varepsilon_n q(fz_n - f(Px)) + (1 - \varepsilon_n)q(Tz_n - Px)]^2 - q(z_n - Px)^2 \\
&\geq 2\langle \varepsilon_n(fz_n - f(Px)) + (1 - \varepsilon_n)(Tz_n - Px) \\
&\quad - (z_n - Px), J_q(z_n - Px) \rangle \\
&= -2\varepsilon_n \langle (f - I)Px, J_q(z_n - Px) \rangle \\
&= -2\varepsilon_n \langle x - Px, J_q(z_n - Px) \rangle,
\end{aligned} \tag{104}$$

for each $n \in \mathbb{N}$. Hence,

$$q(z_n - Px)^2 \leq \frac{2}{1 - \beta} \langle x - Px, J_q(z_n - Px) \rangle, \tag{105}$$

for each $q \in Q$. Since $Px \in \text{Fix}(T)$, from (99) and (105), we conclude that

$$\limsup_n q(z_n - Px)^2 \leq \frac{2}{1 - \beta} \limsup_n \langle x - Px, J_q(z_n - Px) \rangle \leq 0, \tag{106}$$

for each $q \in Q$. That is, $z_n \rightarrow Px$ in τ_Q . \square

4. Numerical Example

The following examples illustrate Theorem 12. The first example is in the setting of a locally convex space that is not normable and the rest in finite dimensional spaces.

Example 1. Let ε be an arbitrary positive number, $E = C([0, 1])$, the set of all continuous complex valued functions, and $\varepsilon_n = (1/n)$. For each $x \in [0, 1]$, the seminorm q on E is defined by $q_x(f) = |f(x)|$, for each $f \in E$. Note that E is a locally convex space with the topology induced by Q . It can be proved by Urysohn's Lemma that it is not normable.

In Theorem 12, let $C = N(x_0, \varepsilon, 0) = \{f: q_{x_0}(f) = |f(x_0)| < \varepsilon\}$ be convex neighborhood of 0 with $x_0 \in [0, 1]$ and $|x_0| < (\varepsilon/2)$. Suppose that I is the identity mapping on the complex numbers, and let $T, F: E \rightarrow E$ be defined by $T(f)$. The following examples illustrate Theorem 12. The first example is in the setting of a locally convex space that is not normable and the rest in finite dimensional spaces.

Example 2. Let ε be an arbitrary positive number and $E = C([0, 1])$ be the set of all continuous complex valued functions and $\varepsilon_n = (1/n)$. For each $x \in [0, 1]$, we define the seminorm q_x on E by $q_x(f) = |f(x)|$, for each $f \in C([0, 1])$. Note that E is a locally convex space with the topology induced by Q . It can be proved by Urysohn's Lemma which is not normable.

In Theorem 12, let $C = N(x_0, \varepsilon, 0) = \{f: q_{x_0}(f) = |f(x_0)| < \varepsilon\}$ be convex neighborhood of 0 with $x_0 \in [0, 1]$ and $|x_0| < (\varepsilon/2)$. Suppose that I is the identity mapping on the complex numbers, and let $T, F: E \rightarrow E$ be defined by $T(f) = (1/2)(f + I)$ and $F(f) = (1/2)f$, respectively. It is obvious that T is Q -nonexpansive and F is Q -contraction mapping on C . It can be verified that $P(f) = I$ is a sunny Q -nonexpansive retraction from C onto $\text{Fix}(T) = \{I\}$; then, obviously, we have the sequence $\{f_n\}_{n \in \mathbb{N}} = \{(1 - (1/n))I\}_{n \in \mathbb{N}}$ generated by (92) which converges to $PI = I$ with respect to the topology induced by Q .

Example 3. In Theorem 12, let $E = \mathbb{R}$, $\varepsilon_n = (1/n)$, and $q(x) = |x|$ be the only seminorm on \mathbb{R} , i.e., $Q = \{|\cdot|\}$. Let $C = [0, 2]$ and $T(x) = (1/2)x + 1$ be a Q -nonexpansive mapping and $f(x) = (1/2)x$ be a Q -contraction mapping on $[0, 2]$. It is obvious $P(x) = 2$ is a sunny Q -nonexpansive retraction from $[0, 2]$ onto $\text{Fix}(T) = \{2\}$ and the sequence $\{z_n\}_{n \in \mathbb{N}} = \{2 - (2/n)\}_{n \in \mathbb{N}}$ generated by (92) converges to $PI = 2$ and $F(f) = (1/2)f$, respectively. It is obvious that T is Q -nonexpansive and F is Q -contraction mapping on C . It can be verified that $P(f) = I$ is a sunny Q -nonexpansive retraction from C onto $\text{Fix}(T) = \{I\}$; then, obviously, we have the sequence $\{f_n\}_{n \in \mathbb{N}} = \{(1 - (1/n))I\}_{n \in \mathbb{N}}$ generated by (92) converges to $PI = I$ with respect to the topology induced by Q .

Example 4. In Theorem 12, let $E = \mathbb{R}$, $\varepsilon_n = (1/n)$, and $q(x) = |x|$ be the only seminorm on \mathbb{R} , i.e., $Q = \{|\cdot|\}$. Let $C = [0, 2]$ and $T(x) = (1/2)x + 1$ be a Q -nonexpansive mapping and $f(x) = (1/2)x$ be a Q -contraction mapping on $[0, 2]$. It is obvious $P(x) = 2$ is a sunny Q -nonexpansive retraction from $[0, 2]$ onto $\text{Fix}(T) = \{2\}$ and the sequence $\{z_n\}_{n \in \mathbb{N}} = \{2 - (2/n)\}_{n \in \mathbb{N}}$, generated by (92), converges to $PI = 2$.

Example 5. In Theorem 12, assume $E = \mathbb{R}^2$ and $\varepsilon_n = (1/n)$, and $q_1(x, y) = |x|$ and $q_2(x, y) = |y|$ are two seminorms on \mathbb{R}^2 , i.e., $Q = \{q_1, q_2\}$. Let $C = [0, 2] \times [0, 2]$ and $T(x, y) = ((1/2)x + 1, (1/3)y + 1)$ be a Q -nonexpansive mapping and $f(x, y) = ((1/2)x, (1/3)y)$ be a Q -contraction mapping on $[0, 2] \times [0, 2]$ in Theorem 11. It is easy to check that $P(x, y) = (2, (3/2))$ is a sunny Q -nonexpansive retraction from $[0, 2] \times [0, 2]$ onto $\text{Fix}(T) = \{(2, (3/2))\}$, and the sequence $\{z_n\}_{n \in \mathbb{N}} = \{(x_n, y_n)\}_{n \in \mathbb{N}} = \{(2 - (2/n), (3/2) - (3/2n))\}_{n \in \mathbb{N}}$ generated by (92) converges to $P(1, 1) = (2, (3/2))$.

Data Availability

The data used to support the findings of the study are available within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (no. 11671365).

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Research Article

Homeomorphism and Quotient Mappings in Infrasoftware Topological Spaces

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Received 16 July 2021; Accepted 29 July 2021; Published 10 August 2021

Academic Editor: Huseyin Isik

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In this paper, we contribute to infrasoftware topology which is one of the recent generalizations of software topology. Firstly, we redefine the concept of software mappings to be convenient for studying the topological concepts and notions in different software structures. Then, we introduce the concepts of open, closed, and homeomorphism mappings in the content of infrasoftware topology. We establish main properties and investigate the transmission of these concepts between infrasoftware topology and its parametric infratopologies. Finally, we define a quotient infrasoftware topology and infrasoftware quotient mappings and study their main properties with the aid of illustrative examples.

1. Introduction

We face vagueness, ambiguity, and representation of imperfect knowledge in different areas such as economics, engineering, medical science, sociality, and environmental sciences. Mathematicians, engineers, and scientists, particularly those who focus on artificial intelligence, are seeking for approaches to solve the problems that contain vagueness. But they experienced a trouble: how they can formulate uncertain concepts that may not involve mathematically definite results. This means that there is a need for alternative mathematical concepts. Therefore, they have begun to look for different fields of research which leads to initiate several set theories as an alternative to George Cantor's set theory such as fuzzy set (and its generalizations such as intuitionistic fuzzy set and pythagorean fuzzy set), rough set, multiset, and recently software set.

Software set was proposed by Molodtsov [1] as one of the nonstatistical mathematical approaches targeting to deal with ambiguous, undefined, and imprecise meaning. It is characterized by flexibility and fruitful applications compared with other uncertainty theories. Since Molodtsov put forward the concept of software sets, many scholars have applied it in several research areas such as decision-making problems [2], systems of linear equations [3], computer science [4], engineering [5], and medical sciences [6].

The year 2011 was the beginning point of the interaction between software set theory and topology. Simultaneously, Shabir and Naz [7] and Çağman et al. [8] initiated the concept of software topology. However, they used different techniques to formulate software topology. On the one hand, Shabir and Naz formulated software topology on the collection of software sets over a universal crisp set and a fixed set of parameters. On the other hand, Çağman et al. formulated software topology on the collection of software sets over an absolute software set and different sets of parameters. We conduct this study in the frame of Shabir and Naz's definition which is more analogous to the classical topology.

Kharal and Ahmad [9] defined software mappings using two ordinary (crisp) mappings, one of them between the sets of parameters and the other between the universal sets. However, we reformulate this definition using the concept of software points to be convenient for studying in different software structures. We classify software mappings by software spaces into different families such as continuous, open, closed, and homeomorphism mappings. In addition, software mappings enable us to classify topological concepts in terms of preservation under specific classes of software mappings. In [10], the authors presented another point of view to study software mappings with a medical application. Zorlutuna and Çakir [11] investigated continuity between software topological spaces. The authors of [12] presented new relations between ordinary points and software sets to define new types of software

separation axioms. Quite recently, Al-shami and Kočinac [13] investigated the conditions under which some concepts are kept between soft topology and its parametric topologies. Kočinac et al. [14] discussed selection principles in the context of soft sets.

Some generalizations of soft topology were introduced and studied. For example, El-Sheikh and Abd El-Latif [15] established the concept of suprasoft topological spaces by neglecting a finite intersection condition of a soft topology. Thomas and John [16] formulated the concept of soft generalized topological spaces, and Zakari et al. [17] originated the concepts of soft weak structures. Lately, Al-shami et al. [18] have initiated and investigated the concept of infrasoft topology. Then, Al-shami [19] studied infrasoft compactness and its application to fixed point theorem. Also, Al-shami and Abo-Tabl [20] defined the concepts of infrasoft connected and infrasoft locally connected spaces. As a continuation of this work, we conduct this study. We aim to redefine soft mappings and explore some types of them through the infrasoft topology content.

We organize this article as follows. After the introduction section, Section 2 mentions some concepts and notions that clarify the investigations of this paper. In Section 3, we show shortcoming of soft mapping defined in [9] and reformulate it simulating to the definition of (crisp) mappings. In Section 4, we define new types of infrasoft mappings and investigate main properties. Among other things, we prove that these infrasoft mappings are preserved under product of infrasoft topological spaces and investigate the transmission of these concepts between infrasoft topology and its parametric infratopologies. We devote Section 5 to study quotient topology and mappings in the content of infrasoft topological spaces. Finally, we outline the fundamental obtained results and suggest some upcoming works in Section 6.

2. Preliminaries

In this section, we mention the main concepts that make this paper self-contained.

Definition 1 (see [1]). We called a pair (H, A) a soft set over the universal set $X \neq \emptyset$ with a set of parameters A provided that H is a mapping from A to the power set 2^X of X . We usually write a soft set (H, A) as follows $(H, A) = \{(a, H(a)) : a \in A \text{ and } H(a) \in 2^X\}$. The symbol $S(X_A)$ denotes the set of all soft sets over X with any subset of A .

We recall some special types of soft sets.

- (i) If $H(a) = \emptyset$ for each $a \in A$, then (H, A) is called a null soft set [21].
- (ii) If $H(a) = X$ for each $a \in A$, then (H, A) is called an absolute soft set [21].
- (iii) If $H(a) = \{x\}$ and $H(a') = \emptyset$ for each $a' \in A \setminus \{a\}$, then (H, A) is called a soft point. It is denoted by P_a^x . The set of all soft points over X with A is denoted by $P(X_A)$. We say that P_a^x belongs to a soft set (F, A) , denoted by $P_a^x \in (F, A)$ if $x \in F(a)$ [22]. More details concerning soft points were given in [23].

Definition 2 (see [24]). A soft set (H^c, A) is called a relative complement of a soft set (H, A) provided that a mapping $H^c: A \rightarrow 2^X$ is defined by $H^c(a) = X \setminus H(a)$ for each $a \in A$.

Since an infrasoft topological space is defined under a constant set of parameters, we will recall the previous definitions and results under a constant set of parameters.

Definition 3 (see [21, 24, 25]). Let (F, A) and (H, A) be two soft sets over X .

- (i) (F, A) is a soft subset of a soft set (H, A) , symbolized by $(F, A) \sqsubseteq (H, A)$, if $F(a) \subseteq H(a)$ for all $a \in A$.

(F, A) and (H, A) are called soft equal if $(F, A) \sqsubseteq (H, A)$ and $(H, A) \sqsubseteq (F, A)$.

- (ii) The soft intersection of (F, A) and (H, A) , symbolized by $(F, A) \sqcap (H, A)$, is a soft set (M, A) such that a mapping $M: A \rightarrow 2^X$ is given by $M(a) = F(a) \cap H(a)$ for each $a \in A$.

- (iii) The soft union of (F, A) and (H, A) , symbolized by $(F, A) \sqcup (H, A)$, is a soft set (M, A) such that a mapping $M: A \rightarrow 2^X$ is given by $M(a) = F(a) \cup H(a)$ for each $a \in A$.

More details concerning soft intersection and union were given in [26].

Definition 4 (see [27]). The Cartesian product of (H, A) and (F, B) , which are defined over X and Y , respectively, is a soft set, denoted by $(H \times F, A \times B)$, given by $(H \times F)(a, b) = H(a) \times F(b)$ for each $(a, b) \in A \times B$.

Definition 5 (see [9]). Let $g: X \rightarrow Y$ and $\varphi: A \rightarrow B$ be two crisp mappings. Then, a soft mapping $g_\varphi: S(X_A) \rightarrow S(Y_B)$ is defined as follows: the image of a soft set (H, M) in $S(X_A)$ is a soft set $g_\varphi(H, M) = (g(H), E)$ in $S(Y_B)$ such that $E = \varphi(M) \subseteq B$, and a mapping $g(H)$ is given by

$$g(H)(e) = g\left(\bigcup_{\lambda \in \varphi^{-1}(e) \cap M} H(\lambda)\right), \quad \text{for each } e \in E. \quad (1)$$

Definition 6 (see [9]). Let $g_\varphi: S(X_A) \rightarrow S(Y_B)$ be a soft mapping. Then, the preimage of a soft set (U, N) in $S(Y_B)$ is a soft set $g_\varphi^{-1}(U, N) = (g^{-1}(U), D)$ in $S(X_A)$ such that $D = \varphi^{-1}(N) \subseteq A$, and a mapping $g^{-1}(U)$ is given by

$$g^{-1}(U)(d) = g^{-1}(U(\varphi(d))), \quad \text{for each } d \in D. \quad (2)$$

Definition 7 (see [9]). We call $g_\varphi: S(X_A) \rightarrow S(Y_B)$ an injective (resp. a surjective, a bijective) soft mapping if g and φ are injective (resp. surjective, bijective) mappings.

Proposition 1 (see [9]). Consider $g_\varphi: S(X_A) \rightarrow S(Y_B)$ is a soft mapping. Let (H_1, A) and (H_2, A) be two soft sets in $S(X_A)$ and (U_1, B) and (U_2, B) be two soft sets in $S(Y_B)$. Then, we have the following results:

- (i) If $(H_1, A) \sqsubseteq (H_2, A)$, then $g_\varphi(H_1, A) \sqsubseteq g_\varphi(H_2, A)$
- (ii) If $(U_1, B) \sqsubseteq (U_2, B)$, then $g_\varphi^{-1}(U_1, B) \sqsubseteq g_\varphi^{-1}(U_2, B)$
- (iii) $(H_1, A) \sqsubseteq g_\varphi^{-1}g_\varphi(H_1, A)$ and the equality relation holds if g_φ is injective
- (vi) $g_\varphi g_\varphi^{-1}(U_1, B) \sqsubseteq (U_1, B)$ and the equality relation holds if g_φ is surjective
- (v) $g_\varphi[(H_1, A) \sqcup (H_2, A)] = g_\varphi(H_1, A) \sqcup g_\varphi(H_2, A)$
- (vi) $g_\varphi[(H_1, A) \cap (H_2, A)] \sqsubseteq g_\varphi(H_1, A) \cap g_\varphi(H_2, A)$
- (vii) $g_\varphi^{-1}[(U_1, B) \sqcup (U_2, B)] = g_\varphi^{-1}(U_1, B) \sqcup g_\varphi^{-1}(U_2, B)$
- (viii) $g_\varphi^{-1}[(U_1, B) \cap (U_2, B)] = g_\varphi^{-1}(U_1, B) \cap g_\varphi^{-1}(U_2, B)$

Definition 8 (see [18]). The collection τ of soft sets over X under a fixed set of parameters A is called an infrasoftware topology on X if it is closed under finite soft intersection as well as it contains Φ .

The triple (X, τ, A) is called an infrasoftware topological space. The term given to each member of τ is called an infrasoftware open set, and the relative complement each member of τ is called an infrasoftware closed set.

$$\tau(\Omega) = \{(a, H(a)): a \in A\} \in S(X_A) \text{ such that } H(a) \in \Omega_a \text{ for each } a \in A, \quad (3)$$

which defines an infrasoftware topology on X (we called this type of infrasoftware topology as an infrasoftware topology generated by (crisp) classical infratopologies).

Theorem 1 (see [18]). A soft mapping $g_\varphi: (X, \tau, A) \rightarrow (Y, \theta, B)$ is infrasoftware continuous if and only if the inverse image of each infrasoftware open (resp. infrasoftware closed) set is an infrasoftware open (resp. infrasoftware closed) set.

Definition 11 (see [28]). Let $g: X \rightarrow Y$ be a mapping. A subfamily θ of the power set $P(Y)$ of Y is said to be a quotient topology over Y (with respect to g) if θ is the largest topology that makes g continuous.

3. Note on Soft Mappings

In this section, we target to achieve two goals; first, we update Definition 5 of soft mappings to be convenient for studying the concepts of open, closed, and homeomorphism mappings in different soft structures such as soft topology, suprasoft topology, and infrasoftware topology. Second, we simplify the formulation of Definition 5 using a soft point as the starting point.

We begin by the following example which helps us to clarify the followed approach to achieve the coveted goals.

Example 1. Let $X = \{u, v, w\}$ and $Y = \{x, y, z\}$ be two universal sets, and let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ be two sets of parameters. Consider a soft mapping g_φ from $S(X_A)$ to $S(Y_B)$, where the mappings $g: X \rightarrow Y$ and $\varphi: A \rightarrow B$ are defined as follows:

Definition 9 (see [18]). Let (H, A) be a subset of (X, τ, A) .

- (i) The intersection of all infrasoftware closed subsets of (X, τ, A) which contains a soft set (H, A) is called the infrasoftware closure points of (H, A) . It is denoted by $\text{Cl}(H, A)$.
- (ii) The union of all infrasoftware open subsets of (X, τ, A) which are contained in a soft set (H, A) is called the infrasoftware interior points of (H, A) . It is denoted by $\text{Int}(H, A)$.

Definition 10 (see [18]). Let (X, τ, A) be an infrasoftware topological space and U be a nonempty subset of X . A class $\tau_U = \{\tilde{U} \cap (G, A): (G, A) \in \tau\}$ is called an infrasoftware relative topology on U , and (U, ϑ_U, A) is called an infrasoftware subspace of (X, τ, A) .

Proposition 2 (see [18]). Suppose that $\Omega = \{\Omega_a\}_{a \in A}$ is a family of (crisp) classical infratopologies on X . Then,

$$\begin{aligned} g(u) &= g(v) = x, \\ g(w) &= z, \\ \varphi(a_1) &= \varphi(a_2) = b_2, \\ \varphi(a_3) &= b_3. \end{aligned} \quad (4)$$

For $(H, A) = \{(a_1, \{u\}), (a_2, X), (a_3, \{w\})\}$, we have $g_\varphi(H, A) = \{(b_2, \{x, z\}), (b_3, \{z\})\}$. Note that, for any infrasoftware topology (or any soft structure) on X and Y with the sets of parameters A and B , a soft mapping g_φ will not be soft open (soft closed) because the image of any soft sets under g_φ is a soft subset of a soft set $\{(b_2, \{x, z\}), (b_3, \{x, z\})\}$. To remove this shortcoming, we make a slight modification for Definition 5 to be appropriate for defining soft open and closed mappings.

Definition 12. The image of $(H, M) \in S(X_A)$ under a soft mapping $g_\varphi: S(X_A) \rightarrow S(Y_B)$, where $g: X \rightarrow Y$ and $\varphi: A \rightarrow B$ are given by $g_\varphi(H, M) = (g(H), B)$ such that

$$g(H)(b) = \begin{cases} \bigcup_{a \in \varphi^{-1}(b) \cap M} g(H(a)), & \varphi^{-1}(b) \neq \emptyset, \\ \emptyset, & \varphi^{-1}(b) = \emptyset, \end{cases} \quad (5)$$

for each $b \in B$.

If there is no confusion, we simplify the above formulation as follows:

$$g(H)(b) = g\left(\bigcup_{a \in \varphi^{-1}(b) \cap M} H(a)\right), \quad \text{for each } b \in B. \quad (6)$$

The following result is easy, but it will be useful for the investigation.

Proposition 3

- (i) The image of each soft point is a soft point
- (ii) The product of two soft points is a soft point

Definition 13. A relation Ω on \tilde{X} is a subset of $P(X_A) \times P(X_A)$.

Note that Definition 5 does not give meaning of a soft mapping as a self-contained concept. It only gives the method of calculating the image and preimage of soft sets. So, it is nature to wonder what is the formulation of soft mappings that simulates its counterpart on the (crisp) set theory? It is well known that a soft point represents the soft version of an ordinary point so that we redefine a soft mapping between two classes of soft points as follows.

Definition 14. Let $g: X \rightarrow Y$ and $\varphi: A \rightarrow B$ be two crisp mappings. A soft mapping g_φ of $P(X_A)$ into $P(Y_B)$ is a relation such that each soft point in $P(X_A)$ is related to one and only one soft point in $P(Y_B)$ such that

$$g_\varphi(P_a^x) = P_{\varphi(a)}^{g(x)}, \quad \text{for each } P_a^x \in P(X_A). \quad (7)$$

In addition, $g_\varphi^{-1}(P_b^y) = \sqcup_{a \in \varphi^{-1}(b)} P_a^x$ for each $P_b^y \in P(Y_B)$.

From the above definition, we note two matters; first, reduce calculation burden and its difficulty that arises from Definition 5. Second, Definition 14 gives a logical explanation (justification) for some soft concepts, for example, it can be easily seen why we determine that g_φ is injective or surjective according to its two crisp mappings g and φ .

Now, we prove the following results.

Lemma 1. Let $g_\varphi: P(X_A) \rightarrow P(Y_B)$ be a soft mapping. Then, $g_\varphi(H, A) = \sqcup_{P_a^x \in (H, A)} P_{\varphi(a)}^{g(x)}$.

Proof. Since it can be written a soft set as a soft union of its soft points, we obtain $g_\varphi(H, A) = g_\varphi(\sqcup_{P_a^x \in (H, A)} P_a^x) = \sqcup_{P_a^x \in (H, A)} g_\varphi(P_a^x) = \sqcup_{P_a^x \in (H, A)} P_{\varphi(a)}^{g(x)}$, as required. \square

Theorem 2. The image of soft sets obtained from Definitions 12 and 14 is identical, i.e., Definitions 12 and 14 are identical.

Proof. Let (H, M) be a soft set in $S(X_A)$. Then, $g_\varphi(H, A) = g_\varphi(\sqcup_{P_a^x \in (H, A)} P_a^x) = \sqcup_{P_a^x \in (H, A)} g_\varphi(P_a^x)$. It follows from (i) of Proposition 3 that $g_\varphi(P_a^x)$ is a soft point in $S(Y_B)$. According to Definition 12, we obtain $g_\varphi(P_a^x) = P_{\varphi(a)}^{g(x)}$. Thus, $g_\varphi(H, M) = \sqcup_{P_a^x \in (H, A)} P_{\varphi(a)}^{g(x)}$ which represents the image of (H, M) according to Definition 14. Hence, we obtain the coveted result. \square

Corollary 1. The preimage of soft sets obtained from Definitions 6 and 14 is identical.

We complete this part by presenting some amendments of some results given in [18]. First, the following result is the correct form of Proposition 7 in [18].

Proposition 4. Let (H, A) and (F, A) be subsets of (X, τ, A) , where X and A are finite sets. If $\text{cl}(H, A) \cap \text{cl}(F, A) = \Phi$, then $\text{int}((H, A) \sqcup (F, A)) = \text{int}(H, A) \sqcup \text{int}(F, A)$.

Proof. It is clear that $\text{int}(H, A) \sqcup \text{int}(F, A) \subseteq \text{int}((H, A) \sqcup (F, A))$. Conversely, let $P_a^x \in \text{int}((H, A) \sqcup (F, A))$. Then, τ contains an infrasoftware open set (G, A) such that $P_a^x \in (G, A) \subseteq (H, A) \sqcup (F, A)$. Since X and A are finite sets, we consider (G, A) as a smallest infrasoftware open set containing P_a^x . Now, we have three cases:

Case 1: $(G, A) \subseteq (H, A)$. Then, $P_a^x \in \text{int}(H, A)$.

Case 2: $(G, A) \subseteq (F, A)$. Then, $P_a^x \in \text{int}(F, A)$.

Case 3: $(G, A) \subseteq (H, A)$ and $(G, A) \subseteq (F, A)$. Then, $(G, A) \cap (H, A) \neq \Phi$ and $(G, A) \cap (F, A) \neq \Phi$. Since (G, A) is the smallest infrasoftware open set containing P_a^x , $P_a^x \in \text{cl}(G, A)$ and $P_a^x \in \text{cl}(F, A)$. But this contradicts assumption $\text{cl}(H, A) \cap \text{cl}(F, A) = \Phi$. Therefore, the only valid cases are Case 1 and Case 2. Thus, $\text{int}((H, A) \sqcup (F, A)) \subseteq \text{int}(H, A) \sqcup \text{int}(F, A)$. Hence, the proof is complete.

Second, we replace Definition 24 by Theorem 8 (which are given in [18]) to keep the systematic line of defining infrasoftware continuity, infrasoftware openness, and infrasoftware closedness. That is, we reformulate Definition 24 of an infrasoftware continuous map as follows. \square

Definition 15. A soft mapping $g_\varphi: (X, \tau, A) \rightarrow (Y, \theta, B)$ is said to be infrasoftware continuous if the preimage of each infrasoftware open set is infrasoftware open.

4. Infrasoftware Homeomorphism Mappings

In this section, we initiate the concepts of infrasoftware open, infrasoftware closed, and infrasoftware homeomorphism mappings. We show the relationships among them and study some properties. We construct some counterexamples to explain some invalid results.

Definition 16. A soft mapping $g_\varphi: (X, \tau, A) \rightarrow (Y, \theta, B)$ is said to be infrasoftware open (resp. infrasoftware closed) if the image of each infrasoftware open (resp. infrasoftware closed) set is an infrasoftware open (resp. infrasoftware closed) set.

Proposition 5. If $g_\varphi: (X, \tau, A) \rightarrow (Y, \theta, B)$ is an infrasoftware open mapping, then the following statements hold.

- (i) $g_\varphi(\text{int}(H, A)) \subseteq \text{int}(g_\varphi(H, A))$ for each $(H, A) \in S(X)_A$
- (ii) $g_\varphi^{-1}(\text{cl}(F, B)) \subseteq \text{cl}(g_\varphi^{-1}(F, B))$ for each $(F, B) \in S(Y)_B$

Proof. To prove (i), let $P_b^y \in g_\varphi(\text{int}(H, A))$. Then, there exists a soft point $P_a^x \in \text{int}(H, A)$ such that $g_\varphi(P_a^x) = P_b^y$. Therefore, there exists an infrasoftware open set (U, A) such that $P_a^x \in (U, A) \subseteq (H, A)$. Obviously, $P_b^y = g_\varphi(P_a^x) \in g_\varphi(U, A) \subseteq g_\varphi(H, A)$. By hypothesis, $g_\varphi(U, A)$ is an infrasoftware open set; hence, $g_\varphi(U, A) \subseteq \text{int}(g_\varphi(H, A))$, as required.

One can prove (ii) using a similar technique. \square

Corollary 2. If $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ is an infrasoftware open mapping, then the image of an infrasoftware neighborhood of $P_a^x \in \tilde{X}$ is an infrasoftware neighborhood of $g_\varphi(P_a^x)$.

Proposition 6. If $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ is an infrasoftware closed mapping, then $\text{cl}(g_\varphi(H, A)) \subseteq g_\varphi(\text{cl}(H, A))$ for each $(H, A) \in S(X)_A$.

Proof. Suppose that $P_b^y \notin g_\varphi(\text{cl}(H, A))$. Then, $g_\varphi^{-1}(P_b^y) \not\subseteq \text{cl}(H, A)$. This means for each $P_a^x \in g_\varphi^{-1}(P_b^y)$, there exists an infrasoftware open set (U, A) containing P_a^x such that $(U, A) \cap (H, A) = \emptyset$. Therefore, $(U, A) \cap \text{cl}(H, A) = \emptyset$. Now, $g_\varphi(H, A) \subseteq g_\varphi(\text{cl}(H, A)) \subseteq g_\varphi(U^c, A)$. By assumption, $P_b^y \notin g_\varphi(\text{cl}(H, A))$ so that $g_\varphi(P_a^x) = P_b^y \notin g_\varphi(U^c, A)$. Since g_φ is an infrasoftware closed mapping, then $\text{cl}(g_\varphi(H, A)) \subseteq g_\varphi(U^c, A)$. Thus, $P_b^y \notin \text{cl}(g_\varphi(H, A))$, as required.

In fact, some characterizations of soft open and closed mappings, which are the counterparts of infrasoftware open and infrasoftware closed mappings, are losing via the structure of infrasoftware topology. The following example illustrates this observation. \square

Example 2. Consider the following soft sets over $X = \{w_1, w_2\}$ under a parameter set $A = \{a_1, a_2\}$ defined as follows:

$$\begin{aligned} (U_1, A) &= \{(a_1, \{w_1\}), (a_2, \{w_1\})\}, \\ (U_2, A) &= \{(a_1, \emptyset), (a_2, \{w_2\})\}. \end{aligned} \quad (8)$$

Then, $\tau = \{\emptyset, \tilde{X}, (U_i, A): i = 1, 2\}$ is an infrasoftware topology on X . Consider θ is the indiscrete soft topology (of course, it will be an infrasoftware topology) on X . Let a soft mapping $g_\varphi: (X, \theta, A) \longrightarrow (X, \tau, A)$ be defined as follows:

$$\begin{aligned} g_\varphi(P_{a_1}^{w_1}) &= g_\varphi(P_{a_1}^{w_2}) = P_{a_1}^{w_1}, \\ g_\varphi(P_{a_2}^{w_1}) &= P_{a_2}^{w_1}, \\ g_\varphi(P_{a_2}^{w_2}) &= P_{a_2}^{w_2}. \end{aligned} \quad (9)$$

One can check that the two conditions given in Proposition 5 hold. Also, the image of any infrasoftware neighborhood of P_a^w is an infrasoftware neighborhood of $g_\varphi(P_a^w)$. Moreover, $\text{cl}(g_\varphi(H, A)) \subseteq g_\varphi(\text{cl}(H, A))$ for each $(H, A) \in S(X)_A$. On the contrary, the image of an infrasoftware clopen set \tilde{X} is $\{(a_1, \{w_1\}), (a_2, X)\}$ which is neither infrasoftware open nor infrasoftware closed in τ ; hence, g_φ is neither an infrasoftware open mapping nor an infrasoftware closed mapping.

Proposition 7. Let $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ be a bijective soft map. Then, g_φ is infrasoftware open if and only if it is infrasoftware closed.

Proof. The proof follows from the fact that a bijective soft map g_φ implies that $g_\varphi(H^c, A) = (g_\varphi(H, A))^c$. \square

Proposition 8. Let $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ be an infrasoftware open mapping and \tilde{U} be an infrasoftware open set in τ . Then, $g_{\varphi|U}: (U, \tau_U, A) \longrightarrow (Y, \theta, B)$ is an infrasoftware open mapping.

Proof. Let (G, A) be an infrasoftware open set in τ_U . Then, there is an infrasoftware open set (V, A) in τ such that $(G, A) = (V, A) \cap \tilde{U}$. This means that (G, A) is also an infrasoftware open set in τ . By hypothesis, $g_{\varphi|U}(G, A) = g_\varphi(G, A)$ is an infrasoftware open set in θ ; hence, $g_{\varphi|U}$ is an infrasoftware open mapping. \square

Definition 17. The composition of two soft mappings $f_\psi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ and $g_\varphi: (Y, \theta, B) \longrightarrow (Z, \nu, C)$ is a soft mapping $g_\varphi \circ f_\psi: (X, \tau, A) \longrightarrow (Z, \nu, C)$ such that $(g_\varphi \circ f_\psi)(P_a^x) = g_\varphi(f_\psi(P_a^x))$.

Proposition 9. Let $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ and $f_\psi: (Y, \theta, B) \longrightarrow (Z, \mu, C)$ be two infrasoftware mappings. Then, the following statements hold.

- (i) If g_φ and f_ψ are infrasoftware open mappings, then $f_\psi \circ g_\varphi$ is an infrasoftware open mapping
- (ii) If $f_\psi \circ g_\varphi$ is an infrasoftware open mapping and g_φ is a surjective infrasoftware continuous mapping, then f_ψ is an infrasoftware open mapping
- (iii) If $f_\psi \circ g_\varphi$ is an infrasoftware open mapping and f_ψ is an injective infrasoftware continuous mapping, then g_φ is an infrasoftware open mapping

Proof

- (i) Straightforward.
- (ii) Let (G, B) be an infrasoftware open set in θ . By hypothesis, $g_\varphi^{-1}(G, B)$ is an infrasoftware open set in τ ; therefore, $(f_\psi \circ g_\varphi)(g_\varphi^{-1}(G, B))$ is an infrasoftware open set in μ . Since g_φ is surjective, then $(f_\psi \circ g_\varphi)(g_\varphi^{-1}(G, B)) = f_\psi(g_\varphi(g_\varphi^{-1}(G, B))) = f_\psi(G, B)$. Thus, f_ψ is an infrasoftware open mapping.
- (iii) Let (G, A) be an infrasoftware open set in τ . By hypothesis, $(f_\psi \circ g_\varphi)(G, A)$ is an infrasoftware open set in μ ; therefore, $f_\psi^{-1}(f_\psi \circ g_\varphi(G, A))$ is an infrasoftware open set in θ . Since f_ψ is injective, then $f_\psi^{-1}(f_\psi \circ g_\varphi(G, A)) = (f_\psi^{-1} f_\psi)(g_\varphi(G, A)) = g_\varphi(G, A)$. Thus, g_φ is an infrasoftware open mapping.

Following similar arguments given in the above proof, one can prove the next result. \square

Proposition 10. Let $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ and $f_\psi: (Y, \theta, B) \longrightarrow (Z, \mu, C)$ be two infrasoftware mappings. Then, the following statements hold.

- (i) If g_φ and f_ψ are infrasoftware closed mappings, then $f_\psi \circ g_\varphi$ is an infrasoftware closed mapping
- (ii) If $f_\psi \circ g_\varphi$ is an infrasoftware closed mapping and g_φ is a surjective infrasoftware continuous mapping, then f_ψ is an infrasoftware closed mapping
- (iii) If $f_\psi \circ g_\varphi$ is an infrasoftware closed mapping and f_ψ is an injective infrasoftware continuous mapping, then g_φ is an infrasoftware closed mapping

Proposition 11. Let $\{(X_i, \tau_i, A_i): i \in I\}$ be a family of infrasoft topological spaces. Then, $\tau = \{\prod_{i \in I} (U_i, A_i): (U_i, A_i) \in \tau_i\}$ is an infrasoft topology on $X = \prod_{i \in I} X_i$ under a set of parameters $A = \prod_{i \in I} A_i$.

Proof. It is clear that \tilde{X} and Φ are members in τ . Now, let $\prod_{i \in I} (U_i, A_i)$ and $\prod_{i \in I} (V_i, A_i)$ are two members in τ . Since $(U_i, A_i) \cap (V_i, A_i) \in \tau_i$ for each i , then it follows from the fact $(\prod_{i \in I} (U_i, A_i)) \cap (\prod_{i \in I} (V_i, A_i)) = \prod_{i \in I} ((U_i, A_i) \cap (V_i, A_i))$ that τ is closed under finite soft intersection. Hence, the proof is complete.

We call τ given in the proposition above a product of infra soft topologies, and (X, τ, A) a product of infrasoft spaces. \square

Definition 18. Let $\{g_{i\varphi}: (X_i, \tau_i, A_i) \longrightarrow (Y_i, \theta_i, B_i): i \in I\}$ be a family of soft mappings. The product of these soft mappings is given by $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ such that $g_\varphi(P_{(a_i)_{i \in I}}^{(x_i)_{i \in I}}) = (g_\varphi(P_{a_i}^{x_i}))_{i \in I}$.

If I is countable, then we write it as follows:

$$g_\varphi\left(P_{(a_{i0}, a_{i1}, \dots)}^{(x_{i0}, x_{i1}, \dots)}\right) = (g_\varphi(P_{a_{i0}}^{x_{i0}}), g_\varphi(P_{a_{i1}}^{x_{i1}}), \dots). \quad (10)$$

Theorem 3. The product of infrasoft open mappings is an infrasoft open mapping.

Proof. Let $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ be the product of infrasoft open mappings of the family $\{g_{i\varphi}: (X_i, \tau_i, A_i) \longrightarrow (Y_i, \theta_i, B_i): i \in I\}$. Let (U, A) be an infrasoft open set in τ . Since $(U, A) = \prod_{i \in I} (U_i, A_i)$ where $(U_i, A_i) \in \tau_i$ for each i , then $g_\varphi(U, A) = g_\varphi(\prod_{i \in I} (U_i, A_i)) = \prod_{i \in I} g_{i\varphi}(U_i, A_i)$. By hypothesis, $g_{i\varphi}(U_i, A_i)$ is an infrasoft open set in θ_i for each i . According to the definition of the product of infrasoft spaces, $g_\varphi(U, A)$ is an infrasoft open set in θ . Hence, we obtain the coveted result. \square

Corollary 3. The product of infrasoft closed mappings $\{g_{i\varphi}: i \in I\}$ is an infrasoft closed mapping provided that $g_{i\varphi}$ is bijective for each i .

Theorem 4. The product of infrasoft continuous mappings is an infrasoft continuous mapping.

Proof. Similar to the proof of Theorem 3, one can prove this result. \square

Theorem 5. If $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ is an infrasoft open (resp. infrasoft closed) mapping such that φ is injective, then $g: (X, \tau_a) \longrightarrow (Y, \theta_{\varphi(a)})$ is an infraopen (resp. infraclosed) mapping.

Proof. Let U be an infraopen set in τ_a . Then, there exists an infrasoft open set (H, A) in τ such that $H(a) = U$. Since g_φ is an infrasoft open mapping, then $g_\varphi(H, A) = (g(H), B)$ is an infrasoft open set in θ . Therefore, $g(H)(b = \varphi(a)) = g(\cup_{a \in \varphi^{-1}(b)} H(a))$ is infrasoft open in $\theta_{\varphi(a)}$. Since φ is injective, then $g(H)(b = \varphi(a)) = g(\cup_{a \in \varphi^{-1}(b)} H(a)) = g(U)$.

Thus, $g(U)$ is an infraopen set in $\theta_{\varphi(a)}$. Hence, we obtain the coveted result.

Following the same arguments, the case between parenthesis can be proved.

Example below explains that the converse of the above theorem fails. \square

Example 3. Let $A = \{a_1, a_2\}$, and consider the two infrasoft topologies $\tau = \{\Phi, \tilde{X}, (U, A)\}$ and $\theta = \{\Phi, \tilde{X}, (V, A)\}$ on $X = \{w_1, w_2\}$, where

$$\begin{aligned} (U, A) &= \{(a_1, \{w_1\}), (a_2, \emptyset)\}, \\ (V, A) &= \{(a_1, \emptyset), (a_2, \{w_1\})\}. \end{aligned} \quad (11)$$

Taking $g_\varphi: (X, \tau, A) \longrightarrow (X, \theta, A)$ as the soft identity mapping, it is clear that $g: (X, \tau_a) \longrightarrow (X, \tau_{\varphi(a)=a})$ is an infraopen mapping and an infraclosed mapping for each $a \in A$. But g_φ is neither an infrasoft open mapping nor an infrasoft closed mapping because $g_\varphi(U, A) = (U, A)$ is not an infrasoft open set in θ and $g_\varphi(U^c, A) = (U^c, A)$ is not an infrasoft closed set in θ .

We show under which condition the converse of Theorem 5 is true.

Theorem 6. Let θ be an infrasoft topology induced from the (crisp) classical infratopologies and $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ be a soft mapping such that φ is injective. Then, $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ is infrasoft open (resp. infrasoft closed) if and only if $g: (X, \tau_a) \longrightarrow (Y, \theta_{\varphi(a)})$ is infraopen (resp. infraclosed).

Proof. Necessity: it follows from Theorem 5.

Sufficiency: let (H, A) be an infrasoft open set in τ . Since $g: (X, \tau_a) \longrightarrow (Y, \theta_{\varphi(a)})$ is an infraopen mapping, then $g(H(a))$ is an infraopen set in $\theta_{\varphi(a)}$ for each $a \in A$. Now, $g_\varphi(H, A) = (g(H), B)$ such that either $g(H)(b = \varphi(a)) = g(\cup_{a \in \varphi^{-1}(b)} H(a)) = g(H(a))$ is an infraopen set in $\theta_{\varphi(a)}$ or $g(H)(b) = g(\cup_{\varphi^{-1}(b)=\emptyset} H(a)) = \emptyset$ is an infraopen set in θ_b . Since θ is generated from the crisp infratopologies, then $g_\varphi(H, A)$ is an infrasoft open set in θ . Hence, the proof is complete.

Following the same arguments, the case between parenthesis can be proved. \square

Definition 19. A bijective soft mapping $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ is said to be an infrasoft homeomorphism if it is infrasoft continuous and infrasoft open.

We cancel the proofs of the next two propositions because they are easy.

Proposition 12. Let $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ and $f_\psi: (Y, \theta, B) \longrightarrow (Z, \mu, C)$ be two infrasoft homeomorphism mappings. Then, $f_\psi \circ g_\varphi$ is an infrasoft homeomorphism mapping.

Proposition 13. If $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ is a bijective soft mapping, then the following statements are equivalent.

(i) g_φ is an infrasoft homeomorphism

- (ii) g_φ and g_φ^{-1} are infrasoft continuous
- (iii) g_φ is infrasoft closed and infrasoft continuous

The proofs of the following two theorems follow from Theorems 5 and 6, respectively.

Theorem 7. If a soft mapping $f_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ is infrasoft homeomorphism, then a mapping $f: (X, \tau_a) \longrightarrow (Y, \theta_{\varphi(a)})$ is infrahomeomorphism.

Theorem 8. Let θ be an infrasoft topology induced from classical (crisp) infratopologies. Then, a soft mapping $f_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ is infrasoft homeomorphism if and only if a mapping $f: (X, \tau_a) \longrightarrow (Y, \theta_{\varphi(a)})$ is infrahomeomorphism.

Definition 20. A soft subset (H, A) of an infrasoft topological space (X, τ, A) is called an isolated soft set if there exists a soft point $P_a^x \in (H, A)$ such that $P_a^x \notin (H, A)^{i'}$.

Proposition 14. If $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ is an infrasoft homeomorphism mapping, then the following statements hold for each $(H, A) \in S(X)_A$.

- (i) $g_\varphi(\text{int}(H, A)) = \text{int}(g_\varphi(H, A))$
- (ii) $g_\varphi(\text{cl}(H, A)) = \text{cl}(g_\varphi(H, A))$
- (iii) $g_\varphi((H, A)^{i'}) = (g_\varphi(H, A))^{i'}$

Proof. We prove (i), and one can prove the other two cases similarly.

It follows from (i) of Proposition 5 that $g_\varphi(\text{int}(H, A)) \subseteq \text{int}(g_\varphi(H, A))$. Conversely, let $P_b^y \in \text{int}(g_\varphi(H, A))$. Then, there exists an infrasoft open set (U, B) such that $P_b^y \in (U, B) \subseteq g_\varphi(H, A)$. By hypothesis, $P_a^x = g_\varphi^{-1}(P_b^y) \in g_\varphi^{-1}(U, B) \subseteq (H, A)$ such that $g_\varphi^{-1}(U, B)$ is an infrasoft open set in τ . Therefore, $P_a^x \in \text{int}(H, A)$. This means that $P_b^y \in g_\varphi(\text{int}(H, A))$, as required. \square

Definition 21. A property is said to be an infrasoft topological invariant if the property possessed by an infratopological space (X, τ, A) is also possessed by each an infrasoft homeomorphic to (X, τ, A) .

Theorem 9. The property of an infrasoft dense set (isolated soft set) is an infrasoft topological invariant.

Proof. Let $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ be an infrasoft homeomorphism mapping, and let (H, A) an infrasoft dense subset of (X, τ, A) , i.e., $\text{cl}(H, A) = \tilde{X}$. It follows from (ii) of Proposition 14 that $\text{cl}(g_\varphi(H, A)) = g_\varphi(\text{cl}(H, A)) = g_\varphi(\tilde{X}) = \text{cl}(\tilde{Y}) = \tilde{Y}$. Therefore, $g_\varphi(H, A)$ is an infrasoft dense subset of (Y, θ, B) . Hence, the proof is complete. \square

Theorem 10. The product of two infrasoft homeomorphism mappings is an infrasoft homeomorphism mapping.

Proof. It is clear that the product of bijective soft mappings is bijective. Then, Theorem 3 and Corollary 4.15 finish the coveted result. \square

Proposition 15. Let $\{(X_i, \tau_i, A): i \in I\}$ be a family of pairwise disjoint infrasoft topological spaces. Then, the family $\tau = \{(U, A): (U, A) \cap \tilde{X}_i \in \tau_i \text{ for each } i \in I\}$ produces an infrasoft topology on $X = \cup_{i \in I} X_i$ with a constant set of parameters A .

Proof. It is clear that \tilde{X} and Φ are members of τ . To prove that τ is closed under finite soft intersections, let (U_1, A) and (U_2, A) be two members of τ . Then, $(U_1, A) \cap \tilde{X}_i \in \tau_i$ and $(U_2, A) \cap \tilde{X}_i \in \tau_i$ for each $i \in I$. Therefore, $[(U_1, A) \cap (U_2, A)] \cap \tilde{X}_i \in \tau_i$ for each $i \in I$. Thus, $(U_1, A) \cap (U_2, A) \in \tau$. Hence, τ is an infrasoft topology on X .

We call the infrasoft topological space given in the above proposition a sum of infrasoft topological spaces and is denoted by $(\oplus X_i, \tau, A)$. \square

Proposition 16. A soft subset (H, A) of $(\oplus X_i, \tau, A)$ is infrasoft closed if and only if $(H, A) \cap \tilde{X}_i$ is an infrasoft closed set in τ_i for each $i \in I$.

Proof. (H, A) is an infra soft closed set in $\tau \Leftrightarrow (H^c, A) \cap \tilde{X}_i$ is an infra soft open set in τ_i for any $i \in I \Leftrightarrow (H, A) \cap \tilde{X}_i$ is an infra soft closed set in τ_i for any $i \in I$. \square

Definition 22. Let $\{g_{i\varphi}: (X_i, \tau_i, A) \longrightarrow (Y_i, \theta_i, B): i \in I\}$ be a family of soft mappings. Then, we define a soft mapping $g_\varphi: (\oplus X_i, \tau, A) \longrightarrow (\oplus Y_i, \theta, B)$ as follows: the image of each $(U, A) \in \oplus X_i$ and the image of each $(V, B) \in \oplus Y_i$ are given by

- (1) $g_\varphi(U, A) = \sqcup_{i \in I} g_{i\varphi}([(U, A) \cap \tilde{X}_i])$
- (2) $g_\varphi^{-1}(V, B) = \sqcup_{i \in I} g_{i\varphi}^{-1}([(V, B) \cap \tilde{X}_i])$

Theorem 11. A soft mapping $g_\varphi: (\oplus X_i, \tau, A) \longrightarrow (\oplus Y_i, \theta, B)$ is infrasoft open (resp. infrasoft closed) if and only if all soft mappings $g_{i\varphi}: (X_i, \tau_i, A) \longrightarrow (Y_i, \theta_i, B)$ are infrasoft open (resp. infrasoft closed).

Proof. We prove the theorem in case of the soft mapping is infra soft open.

Necessity: let $g_\varphi: (\oplus X_i, \tau, A) \longrightarrow (\oplus Y_i, \theta, B)$ be an infrasoft open mapping. Taking an arbitrary soft mapping $g_{j\varphi}: (X_j, \tau_j, A) \longrightarrow (Y_j, \theta_j, B)$, where $j \in I$, let (U, A) be an infrasoft open set in τ_j . Then, (U, A) is an infrasoft open set in τ . Therefore, $g_\varphi(U, A)$ is an infrasoft open set in θ . Since $(U, A) \cap \tilde{X}_i = \Phi$ for each $i \neq j$, then $g_\varphi(U, A) = g_{j\varphi}(U, A)$. Thus, $g_{j\varphi}(U, A)$ is an infrasoft open set in θ_j , as required.

Sufficiency: let $g_{i\varphi}: (X_i, \tau_i, A) \longrightarrow (Y_i, \theta_i, B)$ be an infrasoft open mapping for each $i \in I$, and let (U, A) be an infrasoft open set in τ . Since $(U, A) \cap \tilde{X}_i$ is an infrasoft open set in τ_i for each $i \in I$, then $g_{i\varphi}[(U, A) \cap \tilde{X}_i]$ is an infrasoft open set in θ_i for each $i \in I$. According to the definition of sum of infrasoft topologies, we obtain $\sqcup_{i \in I} g_{i\varphi}[(U, A) \cap \tilde{X}_i]$ is an infrasoft open set in θ . Now, $g_\varphi(U, A) = \sqcup_{i \in I} g_{i\varphi}[(U, A) \cap \tilde{X}_i]$ so that $g_\varphi(U, A)$ is an infrasoft open set in $(\oplus Y_i, \theta, A)$.

Following similar arguments, one can prove the case between the parenthesis. \square

Corollary 4. A soft mapping $g_\varphi: (\oplus X_i, \tau, A) \longrightarrow (\oplus Y_i, \theta, B)$ is infrasoft homeomorphism if and only if all soft mappings $g_{i\varphi}: (X_i, \tau_i, A) \longrightarrow (Y_i, \theta_i, B)$ are infrasoft homeomorphism.

5. Infrasoft Quotient Mappings

In this section, we define the concepts of quotient infrasoft topologies and infrasoft quotient mappings. We establish their main properties and investigate transmission of them to (crisp) mappings defined between parametric infrasoft topological spaces.

Definition 23. Let $g_\varphi: (X, \tau, A) \longrightarrow P(Y_B)$ be a soft mapping. A family $\theta \sqsubseteq S(Y_B)$ is said to be a quotient infrasoft topology over Y (with respect to g_φ) if θ is the largest infrasoft topology that makes g_φ infrasoft continuous.

Note that $\theta = \{(U, B) \sqsubseteq \tilde{Y}: g_\varphi^{-1}(U, B) \in \tau\}$.

The following example points out how a quotient infrasoft topology is constructed.

Example 4. Let $\tau = \{\Phi, \tilde{X}, (H_1, A), (H_2, A)\}$ be an infrasoft topology on $X = \{u, v, w\}$ with a set of parameters $A = \{a_1, a_2\}$ such that

$$\begin{aligned} (H_1, A) &= \{(a_1, \{u\}), (a_2, X)\}, \\ (H_2, A) &= \{(a_1, \{v\}), (a_2, \emptyset)\}. \end{aligned} \quad (12)$$

Let $Y = \{x, y, z\}$ be another universal set with a set of parameters $B = \{b_1, b_2\}$, and consider a soft mapping g_φ from (X, τ, A) to $P(Y_B)$, where $g: X \longrightarrow Y$ and $\varphi: A \longrightarrow B$ are defined as follows:

$$\begin{aligned} g(u) &= x, \\ g(v) &= g(w) = y, \\ \varphi(a_1) &= b_1, \\ \varphi(a_2) &= b_2. \end{aligned} \quad (13)$$

Then, $\theta = \{\Phi, \tilde{Y}, (U_i, B): i = 1, 2, \dots, 10\}$ is the quotient infrasoft topology on Y (with respect to g_φ), where

$$\begin{aligned} (U_1, B) &= \{(b_1, \{x\}), (b_2, \{x, y\})\}, \\ (U_2, B) &= \{(b_1, \{x\}), (b_2, Y)\}, \\ (U_3, B) &= \{(b_1, \{x, z\}), (b_2, \{x, y\})\}, \\ (U_4, B) &= \{(b_1, \{x, z\}), (b_2, Y)\}, \\ (U_5, B) &= \{(b_1, \{x, y\}), (b_2, \{x, y\})\}, \\ (U_6, B) &= \{(b_1, Y), (b_2, \{x, y\})\}, \\ (U_7, B) &= \{(b_1, \{x, y\}), (b_2, Y)\}, \\ (U_8, B) &= \{(b_1, \{z\}), (b_2, \emptyset)\}, \\ (U_9, B) &= \{(b_1, \emptyset), (b_2, \{z\})\}, \\ (U_{10}, B) &= \{(b_1, \{z\}), (b_2, \{z\})\}. \end{aligned} \quad (14)$$

Note that, for any infrasoft topology on Y is proper finer than θ , g_φ is not infrasoft continuous.

Theorem 12. Let $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ be an infrasoft continuous mapping. Then, the following statements are equivalent.

- (i) θ is a quotient infrasoft topology
- (ii) (U, B) is an infrasoft open set in θ if $g_\varphi^{-1}(U, B)$ is an infrasoft open set in τ
- (iii) (V, B) is an infrasoft closed set if $g_\varphi^{-1}(V, B)$ is an infrasoft closed set

Proof

- (i) \implies (ii): since g_φ is infrasoft continuous, then $g_\varphi^{-1}(U, B)$ is an infrasoft open set in τ for each infrasoft open set (U, B) in θ . Conversely, let $g_\varphi^{-1}(U, B)$ be an infrasoft open set in τ . Since θ is a quotient infrasoft topology, then $(U, B) \in \theta$, as required.
- (ii) \implies (i): suppose that there exists an infrasoft topology ν that makes g_φ infrasoft continuous such that $\theta \sqsubset \nu$. Now, let $(U, B) \in \nu$. Then, $g_\varphi^{-1}(U, B) \in \tau$. By hypothesis, $(U, B) \in \theta$. Thus, $\theta = \nu$, as required.
- (ii) \implies (iii): straightforward. \square

Corollary 5. If g_φ is an infrasoft continuous mapping from (X, τ, A) onto (Y, θ, B) such that g_φ is either infrasoft open or infrasoft closed, then θ is a quotient infrasoft topology.

Proof. It is clear that $g_\varphi^{-1}(U, B)$ is an infrasoft open set in τ for each infrasoft open set (U, B) in θ on one hand. On the other hand, g_φ is a surjective; then, $(U, B) = g_\varphi(g_\varphi^{-1}(U, B))$, and since g_φ is an infrasoft open mapping, then $g_\varphi(g_\varphi^{-1}(U, B))$ is an infrasoft open set so that (U, B) is infrasoft open. It follows from (ii) of the above theorem that θ is a quotient infrasoft topology, as required.

The case of g_φ is an infrasoft closed mapping is proved in a similar manner. \square

Proposition 17. Let f_ψ be an infrasoft continuous mapping from (X, τ, A) onto a quotient infrasoft topological space (Y, θ, B) . Then, $g_\varphi: (Y, \theta, B) \longrightarrow (Z, \mu, C)$ is infrasoft continuous if $g_\varphi \circ f_\psi$ is infrasoft continuous.

Proof. Necessity: it follows from the fact that the composition of two infrasoft continuous mappings is an infrasoft continuous mapping.

Sufficiency: suppose that (U, C) is an infrasoft open set in μ . Since $g_\varphi \circ f_\psi$ is infrasoft continuous, then $(g_\varphi \circ f_\psi)^{-1}(U, C) = f_\psi^{-1}(g_\varphi^{-1}(U, C))$ is an infrasoft open set in τ . Since θ is a quotient infrasoft topology, it follows from Theorem 12 that $g_\varphi^{-1}(U, C)$ is an infrasoft open set in θ . Hence, we obtain the coveted result.

Recall that a mapping $g: X \longrightarrow Y$ is called a quotient inframapping if it is surjective and Y is equipped with the quotient infratopology with respect to g . \square

Definition 24. A soft mapping $g_\varphi: P(X_A) \longrightarrow P(Y_B)$ is said to be a quotient infrasoftware mapping if g_φ is surjective and Y is equipped with the quotient infrasoftware topology with respect to g_φ .

In other words, $g_\varphi: P(X_A) \longrightarrow P(Y_B)$ is said to be a quotient infrasoftware mapping if g_φ is surjective and a subset (U, B) of \tilde{Y} is infrasoftware open if $g_\varphi^{-1}(U, B)$ is infrasoftware open.

Proposition 18. If $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$ be a quotient infrasoftware mapping such that τ generated from crisp infratopologies, then $g: (X, \tau_a) \longrightarrow (Y, \theta_{\varphi(a)})$ is a quotient inframapping.

Proof. Firstly, it is clear that g is a surjective mapping. To prove that $\theta_{\varphi(a)}$ is a quotient infratopology, let U be an infraopen set in $\theta_{\varphi(a)}$. Then, there is an infrasoftware open set (H, B) in θ such that $H(\varphi(a)) = U$. Therefore, $g_\varphi^{-1}(H, B) = (g^{-1}(H), A)$ is an infrasoftware open set in τ ; thus, $g^{-1}(H)(a) = g^{-1}(H(\varphi(a))) = g^{-1}(U)$ is an infrasoftware open set in τ_a . Conversely, let $g^{-1}(U)$ be an infrasoftware open set in τ_a . Then, there is an infrasoftware open set (F, A) in τ such that $F(a) = g^{-1}(U)$. Since τ generated from crisp infratopologies, we can write $F(a') = \emptyset$ for each $a' \neq a$. Now, $(F, A) = g_\varphi^{-1}(H, B)$, where $H(\varphi(a)) = U$ and $H(b) = \emptyset$ for each $b \neq \varphi(a)$. By hypothesis, (H, B) is infrasoftware open set in θ . Thus, U is infrasoftware open set in $\theta_{\varphi(a)}$. Hence, the proof is complete.

The following example shows that a condition imposed on the infrasoftware topology of the domain is indispensable. \square

Example 5. Let $\theta = \{\Phi, \tilde{Y}\}$ be an infrasoftware topology on $Y = \{x, y\}$ with a set of parameters $B = \{b_1, b_2\}$, and let $\tau = \{\Phi, \tilde{X}, (H_1, A), (H_2, A)\}$ be an infrasoftware topology on $X = \{u, v, w\}$ with a set of parameters $A = \{a_1, a_2\}$, where

$$\begin{aligned} (H_1, A) &= \{(a_1, \{u\}), (a_2, \{v\})\}, \\ (H_2, A) &= \{(a_1, \{v\}), (a_2, \emptyset)\}. \end{aligned} \quad (15)$$

Consider a soft mapping $g_\varphi: (X, \tau, A) \longrightarrow (Y, \theta, B)$, where $g: X \longrightarrow Y$ and $\varphi: A \longrightarrow B$ are defined as follows:

$$\begin{aligned} g(u) &= x, \\ g(v) &= g(w) = y, \\ \varphi(a_1) &= b_1, \\ \varphi(a_2) &= b_2. \end{aligned} \quad (16)$$

It is clear that g_φ is a quotient infrasoftware mapping. Now, we have $\tau_{a_1} = \{\emptyset, X, \{u\}, \{v\}\}$ and $\tau_{a_2} = \{\emptyset, X, \{v\}\}$ are two parametric infratopologies on X , and $\theta_{b_1} = \theta_{b_2} = \{\emptyset, Y\}$ is a parametric infratopology on Y . On the contrary, $g: (X, \tau_{a_1}) \longrightarrow (Y, \theta_{\varphi(a_1)=b_1})$ is not a quotient inframapping because $g^{-1}(\{x\}) = \{u\}$ is an infrasoftware open set in τ_{a_1} in spite of $\{x\}$ is not an infrasoftware open set in θ_{b_1} .

6. Conclusion

We study some extensions of soft topology, which are defined by reducing the stipulations of soft topology, for

various purposes such as obtaining appropriate models to handle some real-life issues, or building some paradigms that demonstrate the relations among some topological notions and ideas, or keeping certain properties under fewer conditions of those given on soft topology. To this end, we have recently defined a new generalization of soft topology, namely, infrasoftware topology.

The principal focus of the article was revising the definition of soft mappings and studying some types of soft mapping in the frame of infrasoftware topological structures. The main contributions of this paper are listed as follows.

- (1) Improve the definition of soft mapping given in [9] using soft points
- (2) Introduce the concepts of infrasoftware open, infrasoftware closed, and infrasoftware homeomorphism mappings, and study several properties with the help of examples
- (3) Present the concepts of quotient infrasoftware topology and quotient infrasoftware mappings and investigate main features

To complete building the infrasoftware topological structure, we plan to do the following studies in the frame of infrasoftware topological spaces.

- (1) Define some types of separation axioms and show the relationships among them
- (2) Explore the concepts of compactness and Lindelöfness and establish main characterizations
- (3) Initiate the concept of connectedness and research fundamental properties

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Review Article

Leray–Schauder Fixed Point Theorems for Block Operator Matrix with an Application

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Received 10 March 2021; Revised 16 June 2021; Accepted 10 July 2021; Published 27 July 2021

Academic Editor: Huseyin Isik

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In this paper, we establish some new variants of Leray–Schauder-type fixed point theorems for a 2×2 block operator matrix defined on nonempty, closed, and convex subsets Ω of Banach spaces. Note here that Ω need not be bounded. These results are formulated in terms of weak sequential continuity and the technique of De Blasi measure of weak noncompactness on countably subsets. We will also prove the existence of solutions for a coupled system of nonlinear equations with an example.

1. Introduction

With the development of new problems in diverse fields of sciences as well as in physical, biological, and social sciences, the theory of fixed point and its applications are very diverse and continuously growing. Also, the theory of block operator matrix is a subject of great interest thanks to the useful applications for studying some systems of integral equations as well as systems of partial or ordinary differential equations. Recent work has employed the fixed point technique for the operator matrix with nonlinear entries acting on Banach spaces or Banach algebras for studying the existence of solutions for several classes of systems of nonlinear integral equations, see, for example, [1–5]. These operators are defined by a 2×2 block operator matrix:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (1)$$

Based on new generalized Schauder and Krasnoselskii fixed point theorems for the block operator matrix (1), Ben Amar et al., in [6], have established some results for a coupled system of differential equations on $L_p \times L_p$ for $p \in (1, \infty)$, under abstract boundary conditions of Rotenberg's model type. These last equations were proposed by M. Rotenberg and model the evolution of a cell population

[7]. Due to the lack of compactness on L_1 spaces, the study in [6] did not cover the case $p = 1$. Later, Jeribi et al. [1] proposed to extend the results of Amar et al. [6] to the case $p = 1$ by establishing new variants of fixed point theorems for (1), and their analysis was carried out via arguments of weak topology and, particularly, the technique of measures of weak noncompactness. In the above quoted works, the assumptions that $I - A$ or $I - D$ are invertible play a fundamental role in the arguments. Jeribi et al., in [8], were interested in studying the case when $I - A$ is not injective and established some fixed point theorems for operator (1), involving multivalued maps acting on Banach spaces. This way, their results were formulated in terms of weak sequential continuity and the technique of De Blasi measure of weak noncompactness. The results obtained are then applied to the two-dimensional nonlinear functional integral equation:

$$\begin{cases} x(t) = k(t, x(t)) + \left[q(t) + \int_0^{\sigma(t)} g(t, s, y(s)) ds \right] \cdot u, \\ y(t) = \phi \left(t, \int_0^t \frac{t}{t+s} w(s, x(s)) ds \right) \cdot v + a(t) y(t), \end{cases} \quad (2)$$

for all $t \in J = [0, 1]$, where $u, v \in X$ with $u \neq 0$ and $v \neq 0$; here, X is a Banach space. And, $\sigma: J \rightarrow J$, $g: J \times J \times X \rightarrow X$, $k, w: J \times J \rightarrow \mathbb{R}$, and $b: J \rightarrow X$ are suitably defined functions.

The main purpose of this paper is to obtain some new variants of Leray–Schauder-type fixed point theorem for operator (1) on a Banach space. From application, we discuss the existence of solutions to problem (2) in a suitable Banach space and an example of a nonlinear integral equation in the Banach space $\mathcal{C}([0, 1], \mathbb{R})$.

Note that system (2) can be written as a fixed point problem:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} k(t, \cdot) & G(t, \cdot) \\ \varphi(t, \cdot) & a(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad (3)$$

where

$$\begin{cases} G(t, y(t)) = \left[q(t) + \int_0^{\sigma(t)} g(t, s, y(s)) ds \right] \cdot u, \\ \varphi(t, x(t)) = \phi \left(t, \int_0^t \frac{t}{t+s} w(s, x(s)) ds \right) \cdot v. \end{cases} \quad (4)$$

$$\omega(M) = \inf \{ r > 0, \text{ there exists a weakly compact set } K \text{ such that } M \subseteq K + B_r \}. \quad (5)$$

The De Blasi measure of weak noncompactness satisfies the following properties. For a proof, we refer the reader to [9, 10].

Lemma 1. Let M_1 and M_2 be in $P_{bd}(X)$, and we have

- (i) If $M_1 \subset M_2$, then $\omega(M_1) \leq \omega(M_2)$
- (ii) $\omega(M_1) = 0$ if and only if M_1 is relatively weakly compact
- (iii) $\omega(\overline{M_1^w}) = \omega(M_1)$, where $\overline{M_1^w}$ is the weak closure of M_1
- (iv) $\omega(\lambda M_1) = |\lambda| \omega(M_1)$, for all $\lambda \in \mathbb{R}$
- (v) $\omega(\overline{\text{co}}(M_1)) = \omega(M_1)$, where $\overline{\text{co}}(M_1)$ is the closed convex hull of M_1
- (vi) $\omega(M_1 + M_2) \leq \omega(M_1) + \omega(M_2)$
- (vii) $\omega(M_1 \cup \{x\}) = \omega(M_1)$, for all $x \in X$

In [10], Appell and De Pascale proved that, in L^1 -spaces, the maps $\omega(\cdot)$ has the following form:

$$\omega(M) = \limsup_{\varepsilon \rightarrow 0} \left\{ \sup_{\psi \in M} \left[\int_D \|\psi(t)\|_X dt : \text{meas}(D) \leq \varepsilon \right] \right\}. \quad (6)$$

For all bounded subsets M of $L^1(\Omega, X)$, where X is a finite dimensional Banach space and $\text{meas}(\cdot)$ denotes the Lebesgue measure, we recall the following definitions.

The present paper is built up as follows. In Section 2, we introduce the necessary definitions and preliminary concepts. Section 3 is devoted to present some new variants of Leray–Schauder-type fixed point theorems for a 2×2 block operator matrix maps acting on Banach spaces. Finally, in Section 4, we apply Corollary 1 in order to discuss the existence of solutions for problem (2).

2. Basic Definitions and Preliminary Concepts

In this section, we give some essential definitions, properties, and theorems of fixed point theory, which should be used in the present paper. Throughout this paper, unless otherwise mentioned, X denotes a Banach space endowed with the norm $\|\cdot\|$ and with the zero element θ , B_r denotes the closed ball in X centered at θ with radius $r > 0$, and $P_{bd}(X)$ denotes the collection of all nonempty bounded subsets of X . Moreover, we write $x_n \rightarrow x$ and $x_n \rightharpoonup x$ to denote, respectively, the strong convergence and the weak convergence of a sequence $\{x_n\}_n$ to x . We say that a map $T: X \rightarrow X$ is weakly sequentially continuous if, for every sequence $\{x_n\}_n \subset X$ with $x_n \rightharpoonup x$, we have $Tx_n \rightarrow Tx$.

The De Blasi measure of weak noncompactness [9] is the map $\omega: P_{bd}(X) \rightarrow [0, \infty)$ defined by

Definition 1. Let Ω be a subset of a Banach space X , and $k \in [0, 1)$. Let $T: \Omega \rightarrow X$ be a mapping, we say that

- 1. T is k -contractive if $\omega(T(M)) \leq k\omega(M)$ for any bounded set $M \subset \Omega$
- (2) T is condensing if $\omega(T(M)) < \omega(M)$ for any bounded set $M \subset \Omega$ with $\omega(M) > 0$
- (3) T is countably k -contractive if $\omega(T(M)) \leq k\omega(M)$ for any countable bounded set $M \subset \Omega$
- (4) T is countably condensing if $\omega(T(M)) < \omega(M)$ for any countable bounded set $M \subset \Omega$ with $\omega(M) > 0$

Clearly, every k -contractive is countably k -contractive. Now, T is said to be k -Lipschitzian if $\|Tx - Ty\| \leq k\|x - y\|$ with $k \in [0, \infty)$. If $k \in [0, 1)$, T is called a contraction.

Definition 2. A mapping $T: X \rightarrow X$ is said to be weakly compact if $T(M)$ is relatively weakly compact for every nonempty bounded subset $M \subset X$.

Following [11], we recall the next definition.

Definition 3. A mapping $T: X \rightarrow X$ is called to be a separate contraction if there exist two functions $\varphi, \psi: [0, \infty) \rightarrow [0, \infty)$, satisfying

- (1) ψ is strictly increasing and $\psi(0) = 0$
- (2) $\|Tx - Ty\| \leq \varphi(\|x - y\|)$ for all $x, y \in X$
- (3) $\psi(r) + \varphi(r) \leq r$ for any $r > 0$

Note that every contraction mapping is a separate contraction mapping.

Definition 4 (see [12]). A mapping $T: X \longrightarrow X$ is a nonlinear contraction mapping if there exists a continuous nondecreasing function $\varphi: [0, \infty) \longrightarrow [0, \infty)$ satisfying

- (1) $\varphi(r) < r$ for any $r > 0$
- (2) $\|Tx - Ty\| \leq \varphi(\|x - y\|)$ for all $x, y \in X$

Definition 5. An operator $T: X \longrightarrow X$ is said to be ψ -expansive if there exists a function $\psi: [0, \infty) \longrightarrow [0, \infty)$ such that

- (1) $\psi(0) = 0$
- (2) $\psi(r) > r$, for any $r > 0$
- (3) ψ is either continuous or nondecreasing
- (4) $\|Tx - Ty\| \geq \psi(\|x - y\|)$ for all $x, y \in X$

In particular, for $\psi(t) = ht$ with $h > 1$, then T is an expansive mapping (see [13]). The following results are the nonlinear alternatives of Leray–Schauder-type states in [14].

Theorem 1. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$ such that $\overline{U^w}$ is a weakly compact subset of Ω and $F: \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous mapping.

Then, either

- (A1) F has a fixed point or
- (A2) there is a point $x \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $x = \lambda Fx$, where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Remark 1. In Theorem 1, the condition “ $\overline{U^w}$ is a weakly compact” can be replaced by “ $F(\overline{U^w})$ is relatively weakly compact,” for the proof, see Remark 3.2 in [14].

Theorem 2. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Assume that $F: \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous and condensing map with $F(\overline{U^w})$ bounded. Then, either

- (A1) F has a fixed point or
- (A2) there is a point $x \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $x = \lambda Fx$, where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Amar et al., in [15], showed that the condition “condensing” in Theorem 2 can be relaxed by the assumption “countably condensing.”

Theorem 3. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Assume that $F: \overline{U^w} \longrightarrow \Omega$ is a weakly sequentially continuous and countably condensing map with $F(\overline{U^w})$ bounded. Then, either

- (A1) F has a fixed point or
- (A2) there is a point $x \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $x = \lambda Fx$, where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

The following results are crucial for our purposes.

Lemma 2 (see [16]). Let Ω be a subset of a Banach space X and let $T: \Omega \longrightarrow X$ be a k -Lipschitzian map. Assume that T is a sequentially weakly continuous map. Then, $\omega(T(M)) \leq k\omega(M)$ for each bounded subset M of Ω ; here, $\omega(\cdot)$ stands for the De Blasi measure of weak noncompactness.

Lemma 3 (see [17]). Let K be a Hausdorff compact space and X be a Banach space. A bounded sequence $\{f_n\}_n \subset \mathcal{C}(K, X)$ converges weakly to $f \in \mathcal{C}(K, X)$ if and only if, for every $t \in K$, the sequence $\{f_n(t)\}_n$ converges weakly (in X) to $f(t)$.

3. Main Result

In this section, we give some new variants of Leray–Schauder for the operator (1). The first result is formulated as follows.

Theorem 4. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \longrightarrow X$ and $B, D: X \longrightarrow X$ be four operators such that

- (i) A is linear and bounded, and there is $p \in \mathbb{N}^*$ such that A^p is a separate contraction
- (ii) B and C are weakly sequentially continuous and $C(\overline{U^w})$ is relatively weakly compact
- (iii) D is linear and bounded, and there is $p \in \mathbb{N}^*$ such that D^p is a separate contraction
- (iv) $(I - A)^{-1}B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$

Then, either

- (A1) the block operator matrix (1) has a fixed point or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (7)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. If we refer to Lemma 1.2 in [11] and to page 39 in [18], we can prove that $(I - A)^{-1}$ and $(I - D)^{-1}$ exist and are weakly continuous. Hence, we can define the mapping $F: \overline{U^w} \longrightarrow \Omega$ by

$$Fx = (I - A)^{-1}B(I - D)^{-1}Cx. \quad (8)$$

In view of Theorem 1, it suffices to establish that F is weakly sequentially continuous and $F(\overline{U^w})$ is relatively weakly compact.

We have $(I - A)^{-1}$ and $(I - D)^{-1}$, and B and C are weakly sequentially continuous; then, F is weakly

sequentially continuous, and because $C(\overline{U^w})$ is relatively weakly compact, we get $F(\overline{U^w})$ is relatively weakly compact. Hence, either

- (1) F has a fixed point or
- (2) there is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u = \lambda Fu$.

In the first case, the vector $y = (I - D)^{-1}Cx$ solves the problem, whereas in the second case we use the vector $v = (I - D)^{-1}Cu$ to achieve the proof. \square

Remark 2

- (1) Theorem 4 remains true if we suppose that A^p or D^p is a nonlinear contraction
- (2) Theorem 4 is a generalization of Theorem 4 and Theorem 3.2 in [19]

We also have the following result.

Theorem 5. *Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four operators such that*

- (i) $A(\overline{U^w})$ and $C(\overline{U^w})$ are relatively weakly compact
- (ii) A, B , and C are weakly sequentially continuous
- (iii) D is linear and bounded, and there is $p \in \mathbb{N}^*$ such that D^p is a separate contraction
- (iv) $Ax + B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$

Then, either

- (A1) the block operator matrix (1) has a fixed point or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (9)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. The reasoning in the proof of Theorem 4 yields that $(I - D)^{-1}$ exists and is weakly continuous. Hence, we can define the mapping $F: \overline{U^w} \rightarrow \Omega$ by

$$Fx = Ax + B(I - D)^{-1}Cx. \quad (10)$$

We can see that F is weakly sequentially continuous because A, B, C , and $(I - D)^{-1}$ are weakly sequentially continuous. The assumption (i) proves that $F(\overline{U^w})$ is relatively weakly compact. Hence, by Theorem 1, we have

- (1) F has a fixed point or
- (2) There is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u = \lambda Fu$

In the first case, the vector $y = (I - D)^{-1}x$ solves the problem, whereas in the second case we use the vector $v = (I - D)^{-1}Cu$ to achieve the proof. \square

Remark 3

- (1) In Theorem 5, we can replace the assumption “ $Ax + B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$,” by “ $x = Ax + B(I - D)^{-1}Cy \in \Omega$, for all $y \in \overline{U^w}$ implies $x \in \Omega$ ”

- (2) Theorem 5 is a generalization of Theorem 3.3 in [19]

Thus, we give the following result

Theorem 6. *Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four operators such that*

- (i) $A(\overline{U^w})$ and $C(\overline{U^w})$ are relatively weakly compact
- (ii) A, B , and C are weakly sequentially continuous
- (iii) D is linear and bounded, and there is $p \in \mathbb{N}^*$ such that D^p is a separate contraction
- (iv) $Ax + B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$

Then, either

- (A1) the block operator matrix (1) has a fixed point or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ \lambda C & \lambda D \end{pmatrix} \begin{pmatrix} u \\ \frac{v}{\lambda} \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (11)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. Reasoning as in the proof of Theorem 5, we obtain

- (1) F has a fixed point or
- (2) There is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u = \lambda Fu$

In the first case, the vector $y = (I - D)^{-1}x$ solves the problem, whereas in the second case, we use the vector $v = \lambda(I - D)^{-1}Cu$ to achieve the proof. \square

Remark 4. Theorem 6 is a generalization of Theorem 3.4 in [19].

Theorem 7. *Let Ω be a nonempty closed and convex subset of Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four weakly sequentially continuous operators such that*

- (i) $A(\overline{U^w})$ and $C(\overline{U^w})$ are relatively weakly compact
- (ii) D is a contraction with $C(\overline{U^w}) \subset (I - D)(\Omega)$
- (iii) $Ax + B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$

Then, either

- (A1) the block operator matrix (1) has a fixed point or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (12)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. By assumption (ii), we can see that $(I - D)^{-1}$ exists and continuous; hence, from Theorem 5, we deduce the desired result.

Inspired by [20], we deduce the following result. \square

Corollary 1. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four weakly sequentially continuous operators such that

- (i) $A(\overline{U^w})$ and $C(\overline{U^w})$ are relatively weakly compact
- (ii) D is a contraction with $C(\overline{U^w}) \subset (I - D)(\Omega)$
- (iii) $Ax + B(I - D)^{-1}Cx \in \Omega$ for all $x \in \overline{U^w}$

In addition, assume that

$$\lambda Au + \lambda Bv \neq u, \quad (13)$$

for all $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$. Then, the set of fixed points of the block operator matrix (1) in $\overline{U^w} \times X$ is nonempty.

In the next results, we explore the case where A and D are nonlinear.

Theorem 8. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four weakly sequentially continuous operators such that

- (i) $(I - D)^{-1}$ exists on $C(\Omega)$
 - (ii) $A(\overline{U^w})$ is relatively weakly compact and B is countably β -contractive
 - (iii) C is countably γ -contractive and D is countably δ -contractive with $\gamma + \delta < 1$
 - (iv) $(A + B(I - D)^{-1}C)(\overline{U^w})$ is a bounded subset of Ω
- Then, either
- (A1) the block operator matrix (1) has a fixed point or
 - (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad (14)$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. We defend the operator $F: \overline{U^w} \rightarrow \Omega$ by

$$Fx = Ax + B(I - D)^{-1}Cx. \quad (15)$$

We show that F is weakly sequentially continuous and countably condensing.

Let $\{x_n\}_n$ be a sequence in Ω such that $x_n \rightarrow x$; since C is weakly sequentially continuous, then the set $\{Cx_n; n \in \mathbb{N}\}$ is relatively weakly compact. Note that

$$(I - D)^{-1}C = C + D(I - D)^{-1}C. \quad (16)$$

We have D is countably δ -contractive; then,

$$\begin{aligned} \omega(\{(I - D)^{-1}Cx_n; n \in \mathbb{N}\}) &\leq \omega(\{Cx_n; n \in \mathbb{N}\}) + \omega(\{D(I - D)^{-1}Cx_n; n \in \mathbb{N}\}) \\ &\leq \omega(\{D(I - D)^{-1}Cx_n; n \in \mathbb{N}\}) \\ &\leq \delta\omega(\{(I - D)^{-1}Cx_n; n \in \mathbb{N}\}) \\ &< \omega(\{(I - D)^{-1}Cx_n; n \in \mathbb{N}\}), \end{aligned} \quad (17)$$

which implies that $\omega(\{(I - D)^{-1}Cx_n; n \in \mathbb{N}\}) = 0$; then, $\{(I - D)^{-1}Cx_n; n \in \mathbb{N}\}$ is relatively weakly compact. Consequently, there exists a subsequence $\{x_{\sigma(n)}\}_{n \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ such that $(I - D)^{-1}Cx_{\sigma(n)} \rightarrow y$. From (16) and because C and D are weakly sequentially continuous, we get $Cx + Dy = y$; then, $(I - D)^{-1}Cx = y$; consequently, $(I - D)^{-1}Cx_{\sigma(n)} \rightarrow (I - D)^{-1}Cx$. Now, we show that $(I - D)^{-1}Cx_n \rightarrow (I - D)^{-1}Cx$. Suppose the contrary; then, there exists a subsequence $\{x_{\phi(n)}\}_{n \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ and a weak neighborhood V of $(I - D)^{-1}Cx$ such that $(I - D)^{-1}Cx_{\phi(n)} \notin V$ for all $n \in \mathbb{N}$. Since $x_{\phi(n)} \rightarrow x$, then arguing as before, we may extract a subsequence $\{x_{\psi(n)}\}_{n \in \mathbb{N}}$ of $\{x_{\phi(n)}\}_{n \in \mathbb{N}}$ such that $(I - D)^{-1}Cx_{\psi(n)} \rightarrow (I - D)^{-1}Cx$, which is a contraction. As a result, $(I - D)^{-1}C$ is weakly sequentially continuous. Because A and B are weakly

sequentially continuous, we get $A + B(I - D)^{-1}C$ is weakly sequentially continuous.

Now, we show that $A + B(I - D)^{-1}C$ is countably condensing. Let M be a countably subset of $\overline{U^w}$ such that $\omega(M) > 0$; by (16), we have

$$\begin{aligned} \omega((I - D)^{-1}C(M)) &\leq \omega(C(M)) + \omega(D(I - D)^{-1}C(M)) \\ &\leq \gamma\omega(M) + \delta\omega((I - D)^{-1}C(M)). \end{aligned} \quad (18)$$

Then,

$$\omega((I - D)^{-1}C(M)) \leq \frac{\gamma}{1 - \delta}\omega(M) \leq \omega(M). \quad (19)$$

Using the subadditivity of the De Blasi measure of weak noncompactness, we obtain

$$\begin{aligned}
\omega(F(M)) &\leq \omega(A(M)) + \omega(B(I-D)^{-1}C(M)) \\
&\leq \beta\omega(M) \\
&< \omega(M).
\end{aligned} \tag{20}$$

Hence, F is countably condensing.
Then, by Theorem 3, we obtain

- (1) F has a fixed point or
- (2) There is a point $u \in \partial_\Omega U$ and $\lambda \in (0, 1)$ with $u = \lambda Fu$

In the first case, the vector $y = (I - D)^{-1}Cx$ solves the problem, whereas in the second case, we use the vector $v = (I - D)^{-1}Cu$ to achieve the proof. \square

Remark 5. In Theorem 8, we can replace the De Blasi measure of weak noncompactness ω by any subadditive measures of weak noncompactness on X .

In Theorem 8, condition (i) is difficult to verify; in the following, we will change it under weaker conditions.

Theorem 9. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four weakly sequentially continuous operators such that

- (i) $(I - D)$ is ψ -expansive and $C(\Omega) \subset (I - D)(X)$
- (ii) $A(\overline{U^w})$ is relatively weakly compact and B is countably β -contractive
- (iii) C is countably γ -contractive and D is countably δ -contractive with $\gamma + \delta < 1$
- (iv) $(A + B(I - D)^{-1}C)(\overline{U^w})$ is a bounded subset of Ω

Then, either

- (A1) the block operator matrix (1) has a fixed point or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{21}$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. Let $x, y \in X$ such that $x \neq y$; since $(I - D)$ is ψ -expansive, we obtain

$$\|(I - D)x - (I - D)y\| \geq \psi(\|x - y\|) > 0, \tag{22}$$

Thus, $(I - D): X \rightarrow (I - D)(X)$ is bijective, and because $C(\Omega) \subset (I - D)(X)$, we get $(I - D)^{-1}$ is well defined on $C(\Omega)$. Now, reasoning as in the proof of Theorem 8, we get the desired result. \square

Theorem 10. Let Ω be a nonempty closed and convex subset of a Banach space X and $U \subset \Omega$ be a weakly open subset of Ω with $\theta \in U$. Let $A, C: \overline{U^w} \rightarrow X$ and $B, D: X \rightarrow X$ be four weakly sequentially continuous operators such that

- (i) B is countably β -contractive and $A(\overline{U^w})$ is relatively weakly compact
- (ii) T_y^p is expansive for some $p \in \mathbb{N}$ and each $y \in X$, where $T_y x = Dx + y$ for $x \in X$
- (iii) D is a contraction with constant $0 < \delta < 1$ and C is countably γ -contractive with $\gamma + \delta < 1$
- (iv) $(A + B(I - D)^{-1}C)(\overline{U^w})$ is a bounded subset of Ω

Then, either

- (A1) the block operator matrix (1) has a fixed point, or
- (A2) there exists $(u, v) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$ such that

$$\begin{pmatrix} \lambda A & \lambda B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{23}$$

where $\partial_\Omega U$ denotes the weak boundary of U in Ω .

Proof. By assumptions (i) and (ii) with Lemma 2.3 in [21], we get $(I - D)^{-1}C$ is well defined. And, by assumption (i) and Lemma 2, we have that D is countably δ -contractive. Now, reasoning as in the proof of Theorem 8, we get the desired result. \square

Remark 6. Theorem 10 remains true if we suppose that T_y^p is contractive.

4. Application

Let X be a Banach space. As usual, we will denote by $E = \mathcal{C}(J, X)$ the Banach space of all X -valued continuous functions defined on $J = [0, 1]$. We equip the space E with its standard norm:

$$\|x\|_\infty = \sup_{t \in J} \|x(t)\|. \tag{24}$$

The goal of this section is to apply Corollary 1 to study the existence of continuous solutions to the nonlinear functional integral equations (2).

Let us now introduce the following assumptions.

(H0)

- (i) The function $\sigma: J \rightarrow J$ is continuous and nondecreasing
- (ii) The function $q: J \rightarrow \mathbb{R}$ is continuous
- (iii) The function $a: J \rightarrow X$ is continuous, and $\|a\|_\infty < 1$

(H1) The operator $k: J \times X \rightarrow X$ is such that

- (i) For all $t \in J$, the operator $k(t, \cdot): X \rightarrow X$ is weakly sequentially continuous and weakly compact.
- (ii) For each $x \in X$, the operator $k(\cdot, x): J \rightarrow X$ is continuous.
- (iii) There exists a continuous function $\delta: J \rightarrow [0, +\infty)$ with bound $\Delta = \|\delta\|_\infty$ such that

$$\|k(t, x) - k(t, y)\| \leq \delta(t)\|x - y\|, \quad \text{for all } x, y \in X \text{ and } t \in J. \quad (25)$$

(H2) The operator $g: J \times J \times X \rightarrow \mathbb{R}$ is continuous such that, for each $s \in J$ and $x \in X$, and the operator $g(\cdot, s, t): J \rightarrow \mathbb{R}$ is continuous uniformly.

(H3) The operator $\phi: J \times X \rightarrow \mathbb{R}$ is such that

(i) For each $t \in J$, the operator $\phi(t, \cdot): X \rightarrow \mathbb{R}$ is weakly sequentially continuous.

(ii) For each $x \in X$, the operator $\phi(\cdot, x): J \rightarrow \mathbb{R}$ is continuous.

(iii) There exists a constant L such that

$$|\phi(t, x) - \phi(t, y)| \leq L\|x - y\|, \quad \text{for all } x, y \in X \text{ and } t \in J. \quad (26)$$

(H4) The operators $w: J \times X \rightarrow \mathbb{R}$ is such that

(i) For each $x \in X$, the operator $w(\cdot, x): J \rightarrow \mathbb{R}$ is measurable.

(ii) For each $t \in J$, the operator $w(t, \cdot): X \rightarrow \mathbb{R}$ is weakly sequentially continuous.

(iii) There exists a constant b and a function $m \in L^1([0, 1])$ such that

$$|w(t, x)| \leq m(t) \text{ and } \int_0^1 \frac{1}{t+s} m(s) ds \leq b. \quad (27)$$

(H5) Assume that there exists $r_0 > 1$ such that $|g(\cdot, \cdot, \cdot)| \leq r_0 - \|q\|_\infty$, $\sup_{t \in J} |w(t, 0)| \leq ((1 - \|a\|_\infty)r_0 - L\|m\|_{L^1}\|v\|_\infty)/\|v\|_\infty$, and $\sup_{t \in J} |k(t, 0)| \leq r_0(1 - \Delta - \|u\|_\infty)$.

Let us define the subset Ω of $\mathcal{C}(J, X)$ by

$$\Omega = \{x \in \mathcal{C}(J, X): \|x\| \leq r_0\}. \quad (28)$$

We can see that Ω is a nonempty closed and convex subset of E . Let U be a weakly open subset of Ω such that $0 \in \bar{U}$. Notice that (2) is equivalent to the system:

$$\begin{cases} x(t) = Ax(t) + By(t), \\ y(t) = Cx(t) + Dy(t), \end{cases} \quad (29)$$

where the operators A, B, C , and D defined by

$$Ax(t) = k(t, x(t)), \quad \text{for all } t \in J,$$

$$Bx(t) = \left[q(t) + \int_0^{\sigma(t)} g(t, s, x(s)) ds \right] \cdot u, \quad \text{for all } t \in J,$$

$$Cx(t) = \phi\left(t, \int_0^t \frac{t}{t+s} w(s, x(s)) ds\right) \cdot v, \quad \text{for all } t \in J,$$

$$Dy(t) = a(t)y(t), \quad \text{for all } t \in J.$$

(30)

Now, we have come to a place where we give the main result of this section.

Theorem 11. Assume that (H_0) , (H_1) , (H_2) , (H_3) , (H_4) , and (H_5) hold. In addition, assume that

$$\lambda Au_1 + \lambda Bu_2 \neq u_1, \quad (31)$$

for all $(u_1, u_2) \in \partial_\Omega U \times X$ and $\lambda \in (0, 1)$.

Then, system (2) has, at least, one solution in $\mathcal{C}(J, X) \times \mathcal{C}(J, X)$.

Proof. In order to apply Corollary 1, we divided the proof into four steps.

Step 1: in this step, we prove in (a), (b), and (c), respectively, that the operators A , B , and C are well defined and weakly sequentially continuous.

(a) Let $x \in \mathcal{C}(J, X)$, and let $\{t_n\}_n$ be a sequence in J such that $t_n \rightarrow t \in J$. We have

$$\begin{aligned} \|Ax(t_n)(t_n) - Ax(t)\| &= \|k(t_n, x(t_n)) - k(t, x(t))\| \leq \|k(t_n, x(t_n)) - k(t_n, x(t))\| + \|k(t_n, x(t)) - k(t, x(t))\| \\ &\leq \delta(t_n)\|x(t_n) - x(t)\| + \|k(t_n, x(t)) - k(t, x(t))\| \leq \Delta\|x(t_n) - x(t)\| + \|k(t_n, x(t)) - k(t, x(t))\|. \end{aligned} \quad (32)$$

Since $t_n \rightarrow t$ and taking into account the assumptions (H_1) (ii), we obtain $\|Ax(t_n) - Ax(t)\| \rightarrow 0$. Accordingly, $Ax \in E$. Now, we show that A is weakly sequentially continuous. To

see this, let $\{x_n\}_n$ be a sequence in Ω such that $x_n \rightarrow x \in \Omega$. Hence, $\{x_n\}_n$ is bounded; then, by Lemma 3, we have $x_n(t) \rightarrow x(t)$, for all $t \in J$; consequently,

$$Ax_n(t) = k(t, x_n(t)) \rightarrow k(t, x(t)) = Ax(t). \quad (33)$$

Using again Lemma 3, we get $Ax_n \rightarrow Ax$ because $\{Ax_n\}_n$ is bounded by $\Delta r_0 + \sup_{t \in J} \|k(t, 0)\|$; hence, A is weakly sequentially continuous.

(b) Let $x \in \mathcal{C}(J, X)$, and let $\{t_n\}_n$ be a sequence in J such that $t_n \rightarrow t \in J$. We have

$$\begin{aligned} \|Bx(t_n) - Bx(t)\| &\leq \left| \int_0^{\sigma(t_n)} g(t_n, s, x(s)) ds - \int_0^{\sigma(t)} g(t, s, x(s)) ds \right| \|u\|_\infty + |q(t_n) - q(t)| \|u\|_\infty \\ &\leq \left[\int_0^{\sigma(t_n)} |g(t_n, s, x(s)) - g(t, s, x(s))| ds \right] \|u\|_\infty + \left[\left| \int_{\sigma(t_n)}^{\sigma(t)} g(t, s, x(s)) ds \right| + |q(t_n) - q(t)| \right] \|u\|_\infty \\ &\leq \left[\int_0^1 |g(t_n, s, x(s)) - g(t, s, x(s))| ds \right] \|u\|_\infty + [r_0 |\sigma(t_n) - \sigma(t)| + |q(t_n) - q(t)|] \|u\|_\infty. \end{aligned} \quad (34)$$

By using assumption (H_2) with the dominated convergence theorem, we obtain

$$\int_0^1 |g(t_n, s, x(s)) - g(t, s, x(s))| ds \rightarrow 0. \quad (35)$$

Hence,

$$\|Bx(t_n) - Bx(t)\| \rightarrow 0, \quad (36)$$

This implies that the function Bx is continuous. Now, let $\{x_n\}_n$ be a sequence in E such that $x_n \rightarrow x \in E$. For all $t \in J$, we have $Bx_n(t) = r_n(t) \cdot u$, where

$$r_n(t) = q(t) + \int_0^{\sigma(t)} g(t, s, x_n(s)) ds. \quad (37)$$

Because $\{r_n(t), n \in \mathbb{N}\}$ is a real bounded sequence, we deduce that there is a renamed subsequence such that $r_n(t) \rightarrow r(t)$. Since $\{Bx_n, n \in \mathbb{N}\}$ is bounded, we obtain $Bx_n \rightarrow Bx$; hence, B is weakly sequentially continuous.

(c) Let $x \in \mathcal{C}(J, X)$, and let $\{t_n\}_n$ be a sequence in J such that $t_n \rightarrow t \in J$. We have

$$\begin{aligned} \|Cx(t_n) - Cx(t)\| &\leq \left| \phi\left(t_n, \int_0^{t_n} \frac{t_n}{t_n + s} w(s, x(s)) ds\right) - \phi\left(t, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) \right| \|v\|_\infty \\ &\leq \left| \phi\left(t_n, \int_0^{t_n} \frac{t_n}{t_n + s} w(s, x(s)) ds\right) - \phi\left(t_n, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) \right| \|v\|_\infty \\ &\quad + \left| \phi\left(t_n, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) - \phi\left(t, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) \right| \|v\|_\infty \\ &\leq L \left| \int_0^{t_n} \frac{t_n}{t_n + s} w(s, x(s)) ds - \int_0^t \frac{t}{t + s} w(s, x(s)) ds \right| \|v\|_\infty \\ &\quad + \left| \phi\left(t_n, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) - \phi\left(t, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) \right| \|v\|_\infty \\ &\leq L \left| \int_0^{t_n} \frac{s(t_n - t)}{(t_n + s)(t + s)} w(s, x(s)) ds - \int_{t_n}^t \frac{t}{t + s} w(s, x(s)) ds \right| \|v\|_\infty \\ &\quad + \left| \phi\left(t_n, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) - \phi\left(t, \int_0^t \frac{t}{t + s} w(s, x(s)) ds\right) \right| \|v\|_\infty \\ &\leq 2Lb|t_n - t| \|v\|_\infty + \|\phi(t_n, z) - \phi(t, z)\| \|v\|_\infty, \end{aligned} \quad (38)$$

where

$$z(t) = \int_0^t \frac{t}{t+s} w(s, x(s)) ds, \quad \text{for all } t \in J. \quad (39)$$

If we refer to assumption $(H_3)(ii)$, we deduce Cx is continuous.

Now, let $\{x_n\}_n$ be a sequence in E such that $x_n \rightarrow x \in E$; using assumption $(H_4)(ii)$, we have

$$\frac{t}{t+s} w(s, x_n(s)) \rightarrow \frac{t}{t+s} w(s, x(s)), \quad \text{in } \mathbb{R}. \quad (40)$$

Using the dominated convergence theorem, we obtain

$$(Cx_n)(t) \rightarrow (Cx)(t), \quad \text{in } X. \quad (41)$$

Since $\{Cx_n, n \in \mathbb{N}\}$ is bounded, then we can apply Lemma 3, and we get $Cx_n \rightarrow Cx$; then, C is weakly sequentially continuous.

Step 2: in this step, in (a) and (b), respectively, we prove that $C(\overline{U^w})$ and $A(\overline{U^w})$ are relatively weakly compact.

(a) Let $\{x_n\}_n$ be any sequence in Ω . By $(H_3)(iii)$, we have

$$\begin{aligned} \|Cx_n(t)\| &= \left\| \left[\phi\left(t, \int_0^t \frac{t}{t+s} w(s, x_n(s)) ds\right) - \phi(t, 0) \right] + |\phi(t, 0)| \right\| \|v\|_\infty \\ &\leq \left[L \left| \int_0^t \frac{t}{t+s} w(s, x_n(s)) ds \right| + \sup_{t \in J} |w(t, 0)| \right] \|v\|_\infty \\ &\leq \left[L \|m\|_{L^1} + \sup_{t \in J} |w(t, 0)| \right] \|v\|_\infty. \end{aligned} \quad (42)$$

For all $t \in [0, 1]$, this proves that $\{Cx_n, n \in \mathbb{N}\}$ is a uniformly bounded sequence in $C(\Omega)$. As a result, $C(\Omega)(t)$ is sequentially relatively weakly compact. Now, we proceed to show that it is also weakly equicontinuous. If we take $\varepsilon > 0$, $x \in \Omega$, and $t, t' \in [0, 1]$ (without loss of generality assume that $t < t'$), then we have

$$|x^*((Cx_n)(t) - (Cx_n)(t'))| \leq [L|M(\varepsilon)| + |\Phi(\varepsilon)|] \|x^*(v)\|, \quad (43)$$

where

$$\begin{aligned} M(\varepsilon) &= \int_0^t \frac{t}{t+s} w(s, x_n(s)) ds - \int_0^{t'} \frac{t'}{t'+s} w(s, x_n(s)) ds, \quad |t - t'| < \varepsilon, \text{ and,} \\ \Phi(\varepsilon) &= \phi\left(t, \int_0^{t'} \frac{t'}{t'+s} w(s, x(s)) ds\right) - \phi\left(t', \int_0^{t'} \frac{t'}{t'+s} w(s, x(s)) ds\right), \quad |t - t'| < \varepsilon. \end{aligned} \quad (44)$$

Using assumption (H_4) , we have

$$\begin{aligned} |M(\varepsilon)| &\leq |t - t'| \int_0^1 \frac{1}{t+s} w(s, x_n(s)) ds + \int_{t'}^t \frac{t'}{t'+s} w(s, x_n(s)) ds, \\ &\leq a|t - t'| + \int_{t'}^t m(s) ds. \end{aligned} \quad (45)$$

Since $M(\varepsilon) \rightarrow 0$ and $\Phi(\varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$, we obtain

$$|x^*((Cx_n)(t) - (Cx_n)(t'))| \rightarrow 0, \quad \text{as } t \rightarrow t'. \quad (46)$$

Applying the Arzelà-Ascoli's theorem [22], we get $C(\Omega)$ is sequentially relatively weakly compact in X , and an application of Eberlein-Smulian's theorem [23] shows that $C(\Omega)$ is relatively weakly compact.

(b) We have $A(\Omega) = \{Ax: \|x\| \leq r_0\}$; this subset is nothing else than

$$k(\cdot, \Omega) = \{k(\cdot, x): \|x\| \leq r_0\}, \quad (47)$$

Hence, by hypothesis (H_1) (i), we can see that $A(\Omega)$ is relatively weakly compact.

Step 3: in this step in (a), we show that $C(\overline{U^w}) \subset (I - D)(\Omega)$, and in (b), we prove that $Ax + B(I - D)^{-1}Cx \in \Omega$, for all $x \in \overline{U^w}$.

(a) Let $x \in \overline{U^w}$, we defined the mapping $\phi_x: \mathcal{C}(J, X) \longrightarrow \mathcal{C}(J, X)$ by $\phi_x(y) = Cx + Dy$. Because D is a contraction, we can see that ϕ_x is a contraction; then, an application of Banach's fixed point theorem yields there is a unique point $y \in \mathcal{C}(J, X)$ such that $y = Cx + Dy$, and this implies that $C(\overline{U^w}) \subset (I - D)(\mathcal{C}(J, X))$. Since $y \in \mathcal{C}(J, X)$, then there is $t^* \in J$ such that

$$\begin{aligned} \|y\|_\infty &= \|y(t^*)\| \\ &= Cx(t^*) + Dy(t^*) \\ &\leq \left| \phi \left(t^*, \int_0^{t^*} \frac{t^*}{t^* + s} w(s, x_n(s)) ds \right) \right| \|v\|_\infty + \|a\|_\infty \|y(t^*)\|. \end{aligned} \quad (48)$$

This implies that

$$\|y\|_\infty \leq \frac{[L\|m\|_{L^1} + \sup_{t \in J} |w(t, 0)|] \|v\|_\infty}{1 - \|a\|_\infty} \leq r_0. \quad (49)$$

Then, $C(\overline{U^w}) \subset (I - D)(\Omega)$.

(b) Let $y \in \mathcal{C}(J, X)$ and $x \in \overline{U^w}$ such that

$$y(t) = Ax(t) + B(I - D)^{-1}Cx(t). \quad (50)$$

or, equivalently, for all $t \in J$,

$$y(t) = Ax(t) + B(I - D)^{-1}Cx(t). \quad (51)$$

We have

$$\begin{aligned} \|y(t)\| &= \|Ax(t) + B(I - D)^{-1}Cx(t)\| \\ &\leq \|Ax(t)\| + \|B(I - D)^{-1}Cx(t)\| \\ &\leq \Delta \|x(t)\| + \sup_{t \in J} |k(t, 0)| + r_0 \|u\|_\infty \\ &\leq \Delta r_0 + \sup_{t \in J} |k(t, 0)| + r_0 \|u\|_\infty \\ &\leq r_0. \end{aligned} \quad (52)$$

This implies that

$$Ax + B(I - D)^{-1}Cx \in \Omega, \quad \text{for all } x \in \overline{U^w}. \quad (53)$$

Thus, all the hypotheses of Corollary 1 are satisfied, and therefore, system (2) has, at least, one solution in $\mathcal{C}(J, X) \times \mathcal{C}(J, X)$. \square

5. Example

Consider the Banach space $E = \mathcal{C}([0, 1], \mathbb{R})$ of all continuous real-valued functions on $J = [0, 1]$, with norm $\|x\|_\infty = \sup_{t \in [0, 1]} |x(t)|$. In this case, $X = \mathbb{R}$, and E is a reflexive Banach space. We consider the following coupled nonlinear integral equation in E :

$$\begin{cases} x(t) = \frac{1}{4}t^3 \cos\left(\frac{x(t)}{2}\right) + \frac{1}{8} \left[\frac{1}{4} + \int_0^t \frac{|y(s)|}{1 + |y(s)|} ds \right], \\ y(t) = t \int_0^t \frac{ts|x(s)|}{(t+s)e^{|x(s)|}} ds + \frac{1}{3} \sin(t)y(t). \end{cases} \quad (54)$$

To show that (54) has a solution in E , we will verify that all conditions of Theorem 11 are satisfied.

Here, for all $t, s \in J$ and $x, y \in E$, we have

$$\begin{aligned} \sigma(t) &= t, \\ q(t) &= \frac{1}{4}, \\ a(t) &= \frac{1}{3} \sin(t), \\ u &= \frac{1}{8}, \\ v &= 1, \end{aligned} \quad (55)$$

$$\phi(t, x(t)) = tx(t),$$

$$k(t, x(t)) = \frac{1}{4}t^3 \cos\left(\frac{x(t)}{2}\right),$$

$$g(t, s, y(s)) = \frac{|y(s)|}{1 + |y(s)|},$$

$$w(s, x(s)) = \frac{s(|x(s)| + 1)}{2e^{|x(s)| + 1}}.$$

For each $t \in J$, the operator $k(t, \cdot): X \longrightarrow X$ is continuous (then weakly sequentially continuous), and for each $x \in X$, the operator $k(\cdot, x): J \longrightarrow X$ is continuous. Now, let $x, y \in E$ and $t \in J$, and we have

$$|k(t, x(t)) - k(t, y(t))| \leq \frac{1}{8}t^3 |x(t) - y(t)|, \quad (56)$$

where the function $\delta: t \longrightarrow 1/8t^3$ is continuous with bound $\Delta = 1/8$.

Next, we have g is continuous, and for each $s \in J$ and $x \in \mathbb{R}$, the operator $g(\cdot, s, t): J \longrightarrow \mathbb{R}$ is continuous uniformly.

Moreover, for all $x, y \in \mathbb{R}$ and $t \in J$, we have

$$|\phi(t, x(t)) - \phi(t, y(t))| \leq |x(t) - y(t)|. \quad (57)$$

Thus, $L = 1$.

Next, we have

$$|w(t, x(t))| \leq \frac{t}{2}, \quad (58)$$

$$\text{and } \int_0^1 \frac{1}{t+s} m(s) ds \leq 1.$$

where $m(t) = t$, for all $t \in J$.

Choose $b = 1$ and $r_0 = 2$, we obtain $r_0 - \|q\|_\infty = 7/4$, $((1 - \|a\|_\infty)r_0 - L\|m\|_{L^1}\|v\|_\infty)/\|v\|_\infty = 5/6$, and $r_0(1 - \Delta - \|u\|_\infty) = 3/2$. Then, the inequalities $|g(\cdot, \cdot, \cdot)| \leq 7/4$, $\sup_{t \in J} |w(t, 0)| \leq 5/6$, and $\sup_{t \in J} |k(t, 0)| \leq 3/2$ are verified.

Now, let $U = \{x \in \mathcal{C}([0, 1], \mathbb{R}) : \|x\| < 2\}$, we have $\partial U = \{x \in \mathcal{C}([0, 1], \mathbb{R}) : \|x\| = 2\}$. Suppose that there exists $u_1 \times u_2 \in \partial U \times X$ and $\lambda \in (0, 1)$ such that $\lambda Au_1 + \lambda Bu_2 = u_1$. We have $\|\lambda Au_1 + \lambda Bu_2\| < 13/32$. And, $\|u_1\| = 2$ which is a contraction. Hence, for all $u_1 \times u_2 \in \partial U \times X$ and $\lambda \in (0, 1)$, we have $\lambda Au_1 + \lambda Bu_2 \neq u_1$. Hence, all conditions of Theorem 11 are verified.

6. Conclusion

In recent years, some works were devoted to the investigation of fixed point theorems for operator matrices with entries acting on Banach spaces and Banach algebras. The aim of the present paper was to establish some new variants of Leray–Schauder-type fixed point theorems for a 2×2 block operator matrix. The second aim of this study was to use our results to prove the existence of solutions for a coupled system of nonlinear equations. An example to illustrate our theory is included.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A Strong Convergence to a Common Fixed Point of a Subfamily of a Nonexpansive Evolution Family of Bounded Linear Operators on a Hilbert Space

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Received 21 May 2021; Revised 25 June 2021; Accepted 13 July 2021; Published 23 July 2021

Academic Editor: Naeem Saleem

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In this article, we establish some results for convergence in a strong sense to a common fixed point of a subfamily of a nonexpansive evolution family of bounded linear operators on a Hilbert space. The obtained results generalize some existing ones in the literature for semigroups of operators. An example and an open problem are also given at the end.

1. Introduction

In the 19th century, the fixed point theory was started by Poincaré [1]. In the 20th century, many mathematicians, such as Brouwer [2], Schauder [3], Tarski [4], and others, developed the field. The fixed point theory has a wide range of applications. It is one of the most important tools of modern mathematical analysis and is useful in various fields such as mathematics, engineering, physics, economics, and many more. Fixed point theory can be used as a tool to discuss the uniqueness and existence of solutions of many problems such as integral equations [5], differential equations [6, 7], and numerical equations and algebraic systems [8–11]. We refer to [12–19] for a more detailed study on fixed point theory and its applications in metric spaces.

Let $\mathcal{X} \neq \emptyset$, and $\tau: \mathcal{X} \rightarrow \mathcal{X}$ be a self-mapping. The element $\alpha \in \mathcal{X}$ is called a fixed point of τ if $\tau(\alpha) = \alpha$. Consider the autonomous system

$$\begin{cases} \dot{\mathbf{y}}(r) = A\mathbf{y}(r), & r \geq 0, \\ \mathbf{y}(0) = \Omega, \end{cases} \quad (1)$$

where A is a linear operator on a Hilbert space \mathcal{H} . The solutions of such a system lead us to a class of linear and bounded mappings, called a semigroup. A family $\mathcal{G} = \{\mathcal{G}(\mathfrak{s}): \mathfrak{s} \geq 0\}$ of linear and bounded operators is called a semigroup if it satisfies the following two conditions:

- (1) $G(0) = I$, the identity map
- (2) $G(s+t) = G(s)G(t)$ for all $s, t \geq 0$

The system becomes more difficult if the operator A depends on time, i.e., when A is replaced by $A(t)$ in the above system. Such a system is called nonautonomous, and its solution leads to the concept of an evolution family. Again, a family $\mathbf{L} = \{L(\mathfrak{s}, \mathfrak{r})\}$ of linear and bounded operators is said to be an evolution family, if the following hold:

- (1) $L(\mathfrak{s}, \mathfrak{s}) = I$, for all $\mathfrak{s} \geq 0$
- (2) $L(\mathfrak{s}, \mathfrak{t})L(\mathfrak{t}, \mathfrak{r}) = L(\mathfrak{s}, \mathfrak{r})$, for all $\mathfrak{s} \geq \mathfrak{t} \geq \mathfrak{r} \geq 0$

Remark 1 (see [20]). Every semigroup is an evolution family, but the converse is not true in general. In fact, if an evolution family is periodic at every period, then it becomes a semigroup.

The study of fixed points for semigroups is studied by many mathematicians, such as Suzuki [21, 22] and Buthinah et al. [23]. They proved different results concerning a strong convergence to a fixed point of a semigroup and the representation of the set of all common fixed points of the semigroups in a form of intersection of the sets of all common fixed points of only two operators from the family. Such results are of too importance in the field. Recently, such results were generalized to a subfamily of an evolution family acting on different spaces, see [20, 24].

In this paper, we will present some new results for fixed points of an evolution family of operators. We also generalize some other results from semigroups [21], to a subfamily of an evolution family.

2. Preliminaries

In this article, we will frequently use the following notations:

- (1) By \mathbb{R} , \mathbb{R}_+ , \mathbb{N} , and \mathbb{Z}_+ , we will denote the set of all reals, nonnegative reals, natural numbers, and nonnegative integers, respectively.
- (2) The semigroup, evolution family, and its subfamily will be denoted by \mathfrak{S} , \mathbf{L} , and \mathbf{L}_s , respectively.
- (3) The set of all common fixed points of the semigroup, evolution family, and its subfamily will be denoted and defined as $\mathcal{F}\mathbf{ix}(G) = \cap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s))$, $\mathcal{F}\mathbf{ix}(\mathbf{L}) = \cap_{\mathfrak{s} \geq \mathfrak{r} \geq 0} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, \mathfrak{r}))$, and $\mathcal{F}\mathbf{ix}(\mathbf{L}_s) = \cap_{\mathfrak{s} \geq 0} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0))$, respectively.
- (4) By \mathcal{C} , we will denote a closed and convex subset of the Hilbert space \mathcal{H} .

A map $\tau: \mathcal{C} \longrightarrow \mathcal{C}$ is nonexpansive if $\|\tau_x - \tau_y\| \leq \|x - y\|$ for all $x, y \in \mathcal{C}$.

We denote by $\mathcal{F}\mathbf{ix}(\tau)$ the set of all fixed points of τ . If τ is a nonexpansive self-map on \mathcal{C} , then $\mathcal{F}\mathbf{ix}(\tau)$ is nonempty, see [25].

For a fixed ζ in \mathcal{C} and $\varepsilon \in (0, 1)$, there is a unique point $x_\varepsilon \in \mathcal{C}$ such that $x_\varepsilon = (1 - \varepsilon)\mathcal{T}_{x_\varepsilon} + \varepsilon\zeta$. We see that the map $x \longrightarrow (1 - \varepsilon)\mathcal{T}_x + \varepsilon\zeta$ is a contraction. Indeed,

$$\|((1 - \varepsilon)\mathcal{T}_x + \varepsilon\zeta) - ((1 - \varepsilon)\mathcal{T}_y + \varepsilon\zeta)\| = (1 - \varepsilon)\|\mathcal{T}_x - \mathcal{T}_y\|. \quad (2)$$

The nonexpansiveness of \mathcal{T} ensures that the map is a contraction.

In 1967, Browder [26] provided the following result for self-mappings.

Theorem 1. Let τ be a self-mapping on \mathcal{C} and $\{\gamma_n\} \in (0, 1)$ be a sequence such that $\lim_{n \rightarrow \infty} \gamma_n = 0$. Then, for a fixed ζ in \mathcal{C} , the sequence

$$\zeta_n = (1 - \gamma_n)\tau_{\zeta_n} + \gamma_n\zeta. \quad (3)$$

converges to a fixed point of τ nearest to ζ in a strong sense.

In this paper, we will prove a theorem of Suzuki [21] for a subfamily \mathbf{L}_s of nonexpansive evolution operators on Hilbert spaces. Such a family needs not be a semigroup. The following example will illustrate this fact.

Example 1. The family defined by $\mathbf{L} = \{L(\mathfrak{s}, \mathfrak{r}) = (\mathfrak{r} + 1/\mathfrak{s} + 1): \mathfrak{s} \geq \mathfrak{r} \geq 0\}$ is clearly an evolution family acting on \mathbb{R}_+ . Since $L(\mathfrak{s}, \mathfrak{s}) = 1$ (the identity on \mathbb{R}_+), and

$$L(\mathfrak{s}, \mathfrak{t})L(\mathfrak{t}, \mathfrak{r}) = \left(\frac{\mathfrak{t} + 1}{\mathfrak{s} + 1}\right)\left(\frac{\mathfrak{r} + 1}{\mathfrak{t} + 1}\right) = \frac{\mathfrak{r} + 1}{\mathfrak{s} + 1} = L(\mathfrak{s}, \mathfrak{r}), \quad (4)$$

by setting $\mathfrak{r} = 0$, we have $\mathbf{L}_s = \{L(\mathfrak{s}, 0) = \mathfrak{s} + 1\}$ which is a subfamily of \mathbf{L} , but not a semigroup.

However, if we put a condition as given in Remark 1, then such a family becomes a semigroup.

3. Main Results

In this section, we will present our main results. The following lemma states that the set of all common fixed points of a semigroup can be represented on the closed unit interval in place of \mathbb{R}_+ .

Lemma 1. Let $G = \{G(s): s \geq 0\}$ be a semigroup on a Hilbert space \mathcal{H} , then

$$\mathcal{F}\mathbf{ix}(G) = \bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s)) = \bigcap_{0 \leq s \leq 1} \mathcal{F}\mathbf{ix}(G(s)). \quad (5)$$

Proof. The following inclusion

$$\bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s)) \subseteq \bigcap_{0 \leq s \leq 1} \mathcal{F}\mathbf{ix}(G(s)), \quad (6)$$

is obvious, and we will show the reverse inclusion.

Let $\mathbf{u} \in \cap_{0 \leq s \leq 1} \mathcal{F}\mathbf{ix}(G(s))$, then $G(s)\mathbf{u} = \mathbf{u}$ for all $0 \leq s \leq 1$. Let $s \geq 0$, then it can be written as $s = \mathbf{n} + \varrho$, for some $0 \leq \varrho \leq 1$ and some $\mathbf{n} \in \mathbb{Z}_+$.

Now, consider

$$\begin{aligned} G(s)\mathbf{u} &= G(\mathbf{n} + \varrho)\mathbf{u} \\ &= G(\mathbf{n})G(\varrho)\mathbf{u} \\ &= G^\mathbf{n}(1)G(\varrho)\mathbf{u} = G^\mathbf{n}(1)\mathbf{u} = \mathbf{u}. \end{aligned} \quad (7)$$

That is, $\mathbf{u} \in \cap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s))$. Thus, we conclude that

$$\bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s)) = \bigcap_{0 \leq s \leq 1} \mathcal{F}\mathbf{ix}(G(s)). \quad (8)$$

□

In [22], it is proved that the set of all common fixed points of a semigroup can be represented by the intersection of only two operators from the family.

Theorem 2. Let $G = \{G(s) : s \geq 0\}$ be a semigroup on a Hilbert space \mathcal{H} , then

$\mathcal{F}\mathbf{ix}(G) = \bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s)) = \mathcal{F}\mathbf{ix}(G(\alpha)) \cap \mathcal{F}\mathbf{ix}(G(\beta))$, where α, β are positive such that α/β is irrational.

Now, using Lemma 1 and Theorem 2, we have the following corollary.

Corollary 1. Let $G = \{G(s) : s \geq 0\}$ be a semigroup on a Hilbert space \mathcal{H} , then

$\mathcal{F}\mathbf{ix}(G) = \bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(G(s)) = \mathcal{F}\mathbf{ix}(G(\alpha)) \cap \mathcal{F}\mathbf{ix}(G(\beta))$, where $\alpha, \beta \in [0, 1]$, such that α/β is irrational.

Lemma 1 can be extended to a subfamily $\mathbf{L}_s = \{L(\mathfrak{s}, 0) : \mathfrak{s} \geq 0\}$ of a periodic evolution family. See the following lemma.

Lemma 2. Let $\mathbf{L}_s = \{L(\mathfrak{s}, 0) : \mathfrak{s} \geq 0\}$ be a subfamily of a periodic evolution family with period $q \in \mathbb{R}_+$, then

$$\bigcap_{\mathfrak{s} \geq 0} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)) = \bigcap_{0 \leq \mathfrak{s} \leq q} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)). \quad (9)$$

Proof. Since it is obvious that

$$\bigcap_{\mathfrak{s} \geq 0} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)) \subseteq \bigcap_{0 \leq \mathfrak{s} \leq q} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)), \quad (10)$$

we will again prove the reverse inclusion.

Let

$$\mathbf{u} \in \bigcap_{0 \leq \mathfrak{s} \leq q} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)), \quad (11)$$

then $L(\mathfrak{s}, 0)\mathbf{u} = \mathbf{u}$ for all $0 \leq \mathfrak{s} \leq q$.

Now, since any $\mathfrak{s} \geq 0$ can be written as $\mathfrak{s} = \mathbf{n}q + \varrho$, for some $\mathbf{n} \in \mathbb{Z}_+$ and some $0 \leq \varrho \leq q$, we have

$$\begin{aligned} L(\mathfrak{s}, 0)\mathbf{u} &= L(\mathbf{n}q + \varrho, 0)\mathbf{u} \\ &= L(\mathbf{n}q + \varrho, \mathbf{n}q)L(\mathbf{n}q, 0)\mathbf{u} \\ &= L(\varrho, 0)L^n(q, 0)\mathbf{u} \\ &= L(\varrho, 0)\mathbf{u} \\ &= \mathbf{u}. \end{aligned} \quad (12)$$

Hence,

$$\bigcap_{s \geq 0} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)) = \bigcap_{0 \leq \mathfrak{s} \leq q} \mathcal{F}\mathbf{ix}(L(\mathfrak{s}, 0)). \quad (13)$$

□

This completes the proof.

The Opial condition holds on every Hilbert space, given as follows.

Proposition 1 (see [10]). If $\{\beta_n\}$ is sequence in \mathcal{H} , converging to a point $\mathbf{a} \in \mathcal{H}$ in a weak sense, then

$$\liminf_{n \rightarrow \infty} \|\beta_n \mathbf{a}\| \leq \liminf_{n \rightarrow \infty} \|\beta_n\| \longrightarrow \infty, \quad \text{for all } \mathfrak{h} \in \mathcal{H}. \quad (14)$$

The next theorem is about the strong convergence of a sequence to a point near to the fixed point of the subfamily of an evolution family.

Theorem 3. Let $\mathbf{L}_s = \{L(s, 0) : s \geq 0\}$ be a subfamily of strongly continuous evolution operators on \mathcal{E} such that

$\mathcal{F}\mathbf{ix}(\mathbf{L}_s) \neq \emptyset$. Let $\{\gamma_n\} \in (0, 1)$ and $\{s_n\} \geq 0$ be two sequences of real numbers with the property that $\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} (\gamma_n/s_n) = 0$, (e.g., $s_n = (1/n)$ and $\gamma_n = (1/n^2)$).

Then, for a fixed ζ in \mathcal{E} , the sequence

$$\zeta_n = \gamma_n \zeta + (1 - \gamma_n)L(s_n, 0)\zeta_n, \quad \text{where } n \in \mathbb{N}, \quad (15)$$

converges to an element of $\mathcal{F}\mathbf{ix}(L(s, 0))$ nearest to ζ in a strong sense.

Proof. Let \mathfrak{s} be a point in $\mathcal{F}\mathbf{ix}(\mathbf{L}_s)$ nearest to ζ . From

$$\begin{aligned} \|\zeta_n - \mathfrak{s}\| &= \|\gamma_n \zeta + (1 - \gamma_n)L(s_n, 0)\zeta_n - \mathfrak{s}\| \\ &\leq \gamma_n \|\zeta - \mathfrak{s}\| + (1 - \gamma_n) \|L(s_n, 0)\zeta_n - \mathfrak{s}\| \\ &\leq \gamma_n \|\zeta - \mathfrak{s}\| + (1 - \gamma_n) \|\zeta_n - \mathfrak{s}\|, \end{aligned} \quad (16)$$

we find that

$$\|L(s_n, 0)\zeta_n - \mathfrak{s}\| \leq \|\zeta_n - \mathfrak{s}\| \leq \|\zeta - \mathfrak{s}\|, \quad \text{for } n \in \mathbb{N}. \quad (17)$$

Therefore, $\{\zeta_n\}$ and $\{L(s_n, 0)\zeta_n\}$ both are bounded. Let $\{\zeta_{n_i}\}$ be any arbitrary subsequence of $\{\zeta_n\}$, then there exists a subsequence of $\{\zeta_{n_i}\}$ (say $\{\zeta_{n_{ij}}\}$) which converges to x in a weak sense. Our claim is that $x \in \mathcal{F}\mathbf{ix}(\mathbf{L}_s)$.

For this, put $\omega_j = \zeta_{n_{ij}}$, $z_j = \gamma_{n_{ij}}$, $q = [t/t_j]$, and $t_j = r_{n_{ij}}$, for $n \in \mathbb{N}$. Fix $r > 0$. One writes

$$\begin{aligned} \|\omega_j - L(s, 0)x\| &\leq \sum_{k=0}^{q-1} \|L(k+1)t_j, 0\omega_j - L(kt_j, 0)\omega_j\| \\ &\quad + \|L(qt_j, 0)\omega_j - L(qt_j, 0)x\| + \|L(qt_j, 0)x - L(s, 0)x\| \\ &\leq q \|L(t_j, 0)\omega_j - \omega_j\| + \|\omega_j - x\| + \|L(t - qt_j, 0)x - x\| \\ &= q z_j \|L(t_j, 0)\omega_j - \zeta\| + \|\omega_j - x\| + \|L(t - tq, 0)x - x\| \\ &\leq \frac{t z_j}{t_j} \|L(t_j, 0)\omega_j - \zeta\| \\ &\quad + \|\omega_j - x\| + \max\{\|L(t_j, 0)x - x\| : 0 \leq t \leq t_j\}, \end{aligned} \quad (18)$$

for $j \in \mathbb{N}$. In above inequality, the first and last terms tend to zero as $j \rightarrow \infty$, so

$$\liminf_{j \rightarrow \infty} \|\omega_j - L(s, 0)x\| \leq \liminf_{j \rightarrow \infty} \|\omega_j - x\|. \quad (19)$$

By the Opial condition and Proposition 1, we get $L(s, 0)x = x$, and therefore, $x \in \mathcal{F}\mathbf{ix}(\mathbf{L}_s)$.

Lastly, we will show that $\{\omega_j\}$ converges to \mathfrak{s} in a strong sense. From

$$\begin{aligned} &\langle (\omega_j - L(t_j, 0)\omega_j - (\mathfrak{s} - L(t_j, 0)\mathfrak{s})), \omega_j - \mathfrak{s} \rangle \\ &\geq \|\omega_j - \mathfrak{s}\|^2 - \|L(t_j, 0)\omega_j - L(t_j, 0)\mathfrak{s}\| \cdot \|\omega_j - \mathfrak{s}\|, \\ &z_j \|\omega_j - \mathfrak{s}\|^2 + (1 - z_j) \langle (\omega_j - L(t_j, 0)\omega_j \\ &\quad - (\mathfrak{s} - L(t_j, 0)\mathfrak{s})), \omega_j - \mathfrak{s} \rangle = z_j \langle \zeta - \mathfrak{s}, \omega_j - \mathfrak{s} \rangle, \end{aligned} \quad (20)$$

we conclude that

$$z_j \|\omega_j - \mathfrak{s}\|^2 \leq z_j \langle \zeta - \mathfrak{s}, \omega_j - \mathfrak{s} \rangle. \quad (21)$$

That is,

$$\|\omega_j - \mathfrak{s}\|^2 \leq \langle \zeta - \mathfrak{s}, \omega_j - \mathfrak{s} \rangle. \quad (22)$$

Since ζ is nearest to \mathfrak{s} , we can write $\langle \zeta - \mathfrak{s}, x - \mathfrak{s} \rangle \leq 0$,

$$\begin{aligned} \|\omega_j - \mathfrak{s}\|^2 &\leq \langle \zeta - \mathfrak{s}, \omega_j - \mathfrak{s} \rangle \\ &= \langle \zeta - \mathfrak{s}, \omega_j - x \rangle + \langle \zeta - \mathfrak{s}, x - \mathfrak{s} \rangle \\ &\leq \langle \zeta - \mathfrak{s}, \omega_j - x \rangle, \end{aligned} \quad (23)$$

for $j \in \mathbb{N}$. We see that $\{\omega_j\}$ converges to \mathfrak{s} in a strong sense. As $\{\omega_j\} = \{\zeta_{n_j}\}$ is arbitrary, we obtain that $\{\zeta_n\}$ converges to \mathfrak{s} in a strong sense. \square

Remark 2. Here, we mention that the above result is not applicable for a discontinuous family, see [21].

Remark 3. If we put the condition of periodicity of every positive real number on the evolution family, then it becomes a semigroup using Remark 1. So, the results in [21] become a special case of this paper.

4. Example and Open Problem

Example 2. Let $\mathcal{H} = L^2([0, \pi], \mathbb{C})$ be the Hilbert space and let $\mathfrak{S} = \{\mathfrak{S}(\mathfrak{s}): \mathfrak{s} \geq 0\}$ be a semigroup defined by

$$(\mathfrak{S}(\mathfrak{s})v)(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-\mathfrak{s}n^2} c_n(v) \sin nt, \quad t \in [0, \pi], \mathfrak{s} \geq 0, \quad (24)$$

where $c_n(v) = \int_0^\pi x(s) \sin(ns) ds$. Clearly, it is a strongly continuous and nonexpansive semigroup on \mathcal{H} , and it is generated by the linear operator A given by $Av = \ddot{v}$ and the maximal domain of A is the set $D(A)$ of all $x \in \mathcal{H}$ such that v and \dot{v} are absolutely continuous, $\ddot{v} \in \mathcal{H}$ and $v(0) = v(\pi) = 0$.

Now, consider the nonautonomous Cauchy problem

$$\begin{cases} \frac{\partial u(t, \xi)}{\partial t} = h(t) \frac{\partial^2 u(t, \xi)}{\partial^2 \xi}, & t > 0, \xi \in [0, \pi], \\ u(t, 0) = u(t, \pi) = 0, & t \geq 0, \\ u(0, \xi) = z(\xi), \end{cases} \quad (25)$$

where $z(\cdot) \in \mathcal{H}$, and the function $h: \mathbb{R}_+ \rightarrow [1, \infty)$ is nonexpansive on \mathbb{R}_+ and obeys the periodicity condition, i.e., $h(t+q) = h(t)$ for all $t \in \mathbb{R}_+$ for some $q \geq 1$.

Let $H(t) = \int_0^t h(s) ds$. It is obvious that the solution $x(\cdot)$ of the above Cauchy problem will satisfy the evolution property:

$$x(t) = L(t, s)x(s), \quad (26)$$

where $L(t, s) = \mathfrak{S}(H(t) - H(s))$. See [22, Example 2.9b].

We can find $\nu \geq 0$ such that the function $t \mapsto e^{\nu t} \|u(t)\|$ is bounded on \mathbb{R}_+ . In fact, we have

$$\begin{aligned} \int_0^\infty \|L(t, 0)v\|^2 dt &= \frac{2}{\pi} \int_0^\infty \sum_{\nu=1}^\infty c_\nu^2(v) e^{-2\nu^2 H(t)} dt \\ &= \frac{2}{\pi} \sum_{\nu=1}^\infty c_\nu^2(v) \int_0^\infty e^{-2\nu^2 H(t)} dt \\ &= \|v\|_2^2 \int_0^\infty e^{-2\nu^2 H(t)} dt \\ &\leq \|v\|_2^2 \int_0^\infty e^{-2H(t)} dt. \end{aligned} \quad (27)$$

On the other hand,

$$\begin{aligned} \int_0^\infty e^{-2H(t)} dt &= \sum_{j=0}^\infty \int_{jq}^{(j+1)q} e^{-2H(t)} dt \\ &= \sum_{j=0}^\infty \int_0^q e^{-2H(jq+a)} da \\ &= \sum_{j=0}^\infty e^{-2jH(q)} \int_0^q e^{-2H(a)} da \\ &\leq q \sum_{j=0}^\infty e^{-2jH(q)} = \frac{qe^{2H(q)}}{e^{2H(q)} - 1} := C. \end{aligned} \quad (28)$$

Hence,

$$\int_0^\infty \|L(t, 0)v\|^2 dt \leq C \|v\|_2^2. \quad (29)$$

Using Theorem 3.2 in [27], we have $\omega_0(L) \leq -1/2M$, where $M \geq 1$ and $\omega_0(L)$ is the growth bound of the family L , and see [27] for further details. This shows that the evolution family is nonexpansive on \mathcal{H} , so Theorem 3 can be applicable for such a family and can help for the uniqueness and existence of a solution for the above system.

Open problem: we leave open the question whether Lemma 2 and Theorem 3 can be generalized for the whole periodic and then for general evolution families?

5. Conclusion

The idea of an evolution family is more general than the semigroups. In [21], Suzuki proved a strong convergence to a fixed point of a nonexpansive semigroup of operators on a Hilbert space. In this paper, we generalized the results to a subfamily of an evolution family which is not a semigroup. These results can open the way for researchers to prove such convergence for the whole evolution family of operators on a Hilbert space.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

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Research Article

Global Optimal Solutions for Proximal Fuzzy Contractions Involving Control Functions

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Received 7 May 2021; Accepted 9 June 2021; Published 30 June 2021

Academic Editor: Naeem Saleem

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In this study, we introduce new concepts of $\alpha - \mathcal{FL}$ -contraction and $\alpha - \psi - \mathcal{FL}$ -contraction and we discuss existence results of the best proximity points of such types of non-self-mappings involving control functions in the structure of complete fuzzy metric spaces. Our results extend, generalize, enrich, and improve diverse existing results in the current literature.

1. Introduction

Recent advancements in fixed point theory are one of the central and active research areas of nonlinear functional analysis, which provides a variety of mathematical methods, principles, and techniques for solving a variety of problems arising from various branches of mathematics as well as various fields in science and engineering. The Banach fixed point theorem is considered as one of the most fruitful results in this theory. Due to its vast and significant applicability in pure and applied mathematics, this principle has been generalized and developed in various approaches (see, e.g., [1–22]). In particular, Khojasteh et al. [23] presented an impressive technique to the investigation of fixed point theory by developing the notion of simulation functions, which exhibit a significant unifying power. The idea of simulation functions has been generalized, improved, and extended in different metric spaces (see, e.g., [11, 14, 24, 25]).

The best proximity theory is another expanding and prominent aspect of fixed point theory which plays a fundamental role in the investigation of requirements that guarantee the existence of an optimal approximate fixed point when the functional equation $\mathcal{L}x = x$ has no solution. Indeed, a non-self-mapping $\mathcal{L}: \mathcal{U} \rightarrow \mathcal{V}$ does not possess

necessarily a fixed point, with \mathcal{U} and \mathcal{V} are two nonempty subsets of a classical metric space (Λ, d) . Best proximity theory is a remarkable generalization of fixed point theorems. In fact, the best proximity point turned out to be a fixed point in a natural way if the mapping in question is a self-mapping. For more recent developments in best proximity theory and related techniques, refer to [9–11, 19, 26–32].

In the present study, following this line of research interest, we present a simulation function approach to best proximity point problems in fuzzy metric spaces. We initiate new concepts of $\alpha - \psi - \mathcal{FL}$ -contraction, $\alpha - \mathcal{FL}$ -contraction, and generalized $\alpha - \mathcal{FL}$ -contraction, and we discuss existence results of best proximity point of such classes of non-self-mappings involving control functions in the structure of complete fuzzy metric spaces. The furnished results enrich, generalize, and extend various existing findings in the literature.

2. Preliminaries

Throughout this study, \mathbb{N} and \mathbb{R} will represent natural and real numbers, respectively. First, we start with some notions and main properties of fuzzy metric spaces.

Definition 1 (see [33]). A binary operation $*$: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if it fulfills the following conditions:

- (CT1) $*$ is continuous
- (CT2) $*$ is commutative and associative
- (CT3) $\wp * 1 = \wp$ for all $\wp \in [0, 1]$
- (CT4) $\wp * \ell \leq \mathcal{F} * 1$ whenever $\wp \leq \mathcal{F}$ and $\ell \leq 1$, for all $\wp, \ell, \mathcal{F}, 1 \in [0, 1]$

Example 1. Three standard instances are as follows:

- (a) $\wp * \ell = \wp \cdot \ell$
- (b) $\wp * \ell = \min(\wp, \ell)$
- (c) $\wp * \ell = \max[0, \wp + \ell - 1]$

Definition 2 (see George and Veeramani [34]). Let Λ be an arbitrary set, $*$ is a continuous t-norm, and \mathcal{D} is a fuzzy set on $\Lambda \times \Lambda \times (0, \infty)$. The ordered triple $(\Lambda, \mathcal{D}, *)$ is said to be a fuzzy metric space if

- (\mathcal{MS}_1) $\mathcal{D}(\vartheta, \theta, \varsigma) > 0$
- (\mathcal{MS}_2) $\mathcal{D}(\vartheta, \theta, \varsigma) = 1$ if and only if $\vartheta = \theta$
- (\mathcal{MS}_3) $\mathcal{D}(\vartheta, \theta, \varsigma) = \mathcal{D}(\theta, \vartheta, \varsigma)$
- (\mathcal{MS}_4) $\mathcal{D}(\vartheta, \theta, \varsigma) * \mathcal{D}(\theta, \omega, \sigma) \leq \mathcal{D}(\vartheta, \omega, \varsigma + \sigma)$
- (\mathcal{MS}_5) $\mathcal{D}(\vartheta, \theta, .): (0, \infty) \longrightarrow (0, 1]$ is continuous

for all $\vartheta, \theta, \omega \in \Lambda$ and $\varsigma, \sigma > 0$.

For $\varsigma > 0$, the open ball with centre $\vartheta \in \Lambda$ and radius ρ , where $0 < \rho < 1$, is defined by

$$\mathcal{B}(\vartheta, \rho, \varsigma) = \{\theta \in \Lambda: \mathcal{D}(\vartheta, \theta, \varsigma) > 1 - \rho\}. \quad (1)$$

A subset O of a fuzzy metric space $(\Lambda, \mathcal{D}, *)$ is said to be open if given any point $\theta \in O$, there exists $0 < \rho < 1$ and $\varsigma > 0$ such that $\mathcal{B}(\theta, \rho, \varsigma) \subseteq O$. Let τ denote the collection of all open subsets of Λ ; hence, τ is a topology on Λ . This topology is Hausdorff and first countable. For further topological results, refer to [2, 34].

Example 2 (see [34]). Let (Λ, d) be a metric space and $*$ be the product t-norm, and define the function $\mathcal{D}: \Lambda^2 \times (0, \infty) \longrightarrow [0, 1]$ by

$$\mathcal{D}(\vartheta, \theta, \varsigma) = e^{(-d(\vartheta, \theta)/\varsigma)}, \quad (2)$$

for all $\vartheta, \theta \in \Lambda, \varsigma > 0$. Then, $(\Lambda, \mathcal{D}, *)$ is a fuzzy metric space on Λ .

Lemma 1 (see [1]). $\mathcal{D}(\vartheta, \theta, .)$ is nondecreasing for all ϑ, θ in Λ .

Definition 3 (see [34]). Let $(\Lambda, \mathcal{D}, *)$ be a fuzzy metric space.

- (1) A sequence $\{\vartheta_n\} \subseteq \Lambda$ is said to be convergent to $\vartheta \in \Lambda$ if and only if $\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_n, \vartheta, \varsigma) = 1$ for all $\varsigma > 0$

- (2) A sequence $\{\vartheta_n\} \subseteq \Lambda$ is said to be a Cauchy sequence iff for each $\varepsilon \in (0, 1)$ and $\varsigma > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathcal{D}(\vartheta_n, \vartheta_m, \varsigma) > 1 - \varepsilon$ for all $n, m \geq n_0$
- (3) A fuzzy metric space is called complete if every Cauchy sequence in Λ has a limit in Λ

In [2], Gregori and Sapena initiated the notion of a fuzzy contractive mapping as follows.

Definition 4 (see [2]). Let $(\Lambda, \mathcal{D}, *)$ be a fuzzy metric space. A mapping $\mathcal{L}: \Lambda \longrightarrow \Lambda$ is called a fuzzy contractive mapping if there exists $a \in (0, 1)$ such that

$$\frac{1}{\mathcal{D}(\mathcal{L}\vartheta, \mathcal{L}\theta, \varsigma)} - 1 \leq a \left(\frac{1}{\mathcal{D}(\vartheta, \theta, \varsigma)} - 1 \right), \quad (3)$$

for each $\vartheta, \theta \in \Lambda$ and $\varsigma > 0$.

Definition 5 (see [4]). Let Ψ be the class of nondecreasing functions $\psi: (0, 1] \longrightarrow (0, 1]$ fulfilling the following two conditions:

- (ψ_1) ψ is continuous
- (ψ_2) $\psi(c) > c$ for all $c \in (0, 1)$

A self-mapping $\mathcal{L}: \Lambda \longrightarrow \Lambda$ on a fuzzy metric space $(\Lambda, \mathcal{D}, *)$ is called a fuzzy ψ -contractive mapping if

$$\mathcal{D}(\mathcal{L}\vartheta, \mathcal{L}\theta, \varsigma) \geq \psi(\mathcal{D}(\vartheta, \theta, \varsigma)), \quad \text{for all } \vartheta, \theta \in \Lambda, \varsigma > 0. \quad (4)$$

Afterwards, Wardowski [5] proposed the idea of a fuzzy \mathcal{H} -contractive mapping as follows.

Definition 6 (see [5]). Let \mathcal{H} be the set of functions $\eta: (0, 1] \longrightarrow (0, \infty]$ satisfying the two conditions (\mathcal{W}_1) and (\mathcal{W}_2) given by

- (\mathcal{W}_1) η transforms $(0, 1]$ onto $[0, \infty)$
- (\mathcal{W}_2) η is strictly decreasing

A self-mapping $\mathcal{L}: \Lambda \longrightarrow \Lambda$ on a fuzzy metric space $(\Lambda, \mathcal{D}, *)$ is called a fuzzy \mathcal{H} -contractive with respect to the function $\eta \in \mathcal{H}$ if there exists $a \in (0, 1)$ such that the following inequality holds:

$$\eta(\mathcal{D}(\mathcal{L}\vartheta, \mathcal{L}\theta, \varsigma)) \leq a\eta(\mathcal{D}(\vartheta, \theta, \varsigma)), \quad \text{for all } \vartheta, \theta \in \Lambda, \varsigma > 0. \quad (5)$$

The following class of control functions has been introduced in [8], where we used the term class \mathcal{FX} instead of the present \mathcal{FX} -simulation functions.

Definition 7 (see [8]). The function $\xi: (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$ is said to be a \mathcal{FX} -simulation function if the following properties hold:

- (ξ_1) $\xi(1, 1) = 0$
- (ξ_2) $\xi(\mu, \nu) < (1/\nu) - (1/\mu)$ for each $\mu, \nu \in (0, 1)$
- (ξ_3) if $\{\mu_n\}, \{\nu_n\}$ are sequences in $(0, 1]$ such that $\lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} \nu_n < 1$, then $\lim_{n \rightarrow \infty} \sup \xi(\mu_n, \nu_n) < 0$

By \mathcal{FL} , we denote the collection of all \mathcal{FL} -simulation functions.

Definition 8 (see [8]). Let $(\Lambda, \mathcal{D}, *)$ be a fuzzy metric space, $\mathcal{L}: \Lambda \rightarrow \Lambda$ a mapping, and $\xi \in \mathcal{FL}$. Then, \mathcal{L} is said to be a \mathcal{FL} -contraction with respect to ξ if the following condition is satisfied:

$$\xi(\mathcal{D}(\mathcal{L}\vartheta, \mathcal{L}\theta, \varsigma), \mathcal{D}(\vartheta, \theta, \varsigma)) \geq 0, \quad \text{for all } \vartheta, \theta \in \Lambda, \varsigma > 0. \quad (6)$$

Example 3 (see [8]). The type of fuzzy contractive mappings developed by Gregori and Sapena [2] is a perfect example of \mathcal{FL} -contraction. It can be expressed facily from the previous definition by taking the \mathcal{FL} -simulation function as

$$\xi(\mu, \nu) = a\left(\frac{1}{\nu} - 1\right) - \frac{1}{\mu} + 1, \quad \text{for all } \mu, \nu \in (0, 1], \quad (7)$$

where $a \in (0, 1)$.

Example 4 (see [8]). The corresponding \mathcal{FL} -simulation function for the fuzzy ψ -contractive mapping is defined by

$$\xi(\mu, \nu) = \frac{1}{\psi(\nu)} - \frac{1}{\mu}, \quad \text{for all } \mu, \nu \in (0, 1] \text{ with } \psi \in \Psi. \quad (8)$$

Definition 9 (see [6]). Let $(\Lambda, \mathcal{D}, *)$ be a fuzzy metric space. We say that a mapping $\mathcal{L}: \Lambda \rightarrow \Lambda$ is α -admissible if there exists a function $\alpha: \Lambda \times \Lambda \times (0, +\infty) \rightarrow [0, +\infty)$ such that for all $\vartheta, \theta \in \Lambda, \varsigma > 0$,

$$\alpha(\vartheta, \theta, \varsigma) \geq 1 \implies \alpha(\mathcal{L}\vartheta, \mathcal{L}\theta, \varsigma) \geq 1. \quad (9)$$

In line with [15] (see also [16]), we use the notion of triangular weak- α -admissible function in the form that is as follows.

Definition 10. Let $\mathcal{L}: \Lambda \rightarrow \Lambda$ be a mapping and $\alpha: \Lambda \times \Lambda \times (0, \infty) \rightarrow [0, \infty)$ be a function. We say that \mathcal{L} is a triangular weak- α -admissible if

$$\begin{aligned} \alpha(\vartheta, \theta, \varsigma) &\geq 1, \\ \alpha(\theta, \omega, \varsigma) &\geq 1 \implies \alpha(\vartheta, \omega, \varsigma) \geq 1, \end{aligned} \quad (10)$$

for all $\vartheta, \theta, \omega \in \Lambda, \varsigma > 0$.

Definition 11 (see [19]). Let \mathcal{U} and \mathcal{V} be nonempty subsets of a fuzzy metric space $(\Lambda, \mathcal{D}, *)$. Define $\mathcal{U}_0(\varsigma)$ and $\mathcal{V}_0(\varsigma)$ by the following sets:

$$\begin{aligned} \mathcal{U}_0(\varsigma) &= \{u \in \mathcal{U}: \mathcal{D}(u, v, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \text{ for some } v \in \mathcal{V}\}, \\ \mathcal{V}_0(\varsigma) &= \{v \in \mathcal{V}: \mathcal{D}(u, v, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \text{ for some } u \in \mathcal{U}\}, \end{aligned} \quad (11)$$

where

$$\mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) = \sup\{\mathcal{D}(u, v, \varsigma): u \in \mathcal{U}, v \in \mathcal{V}\}. \quad (12)$$

Note that, a point $\omega \in \mathcal{U}$ is said to be a fuzzy best proximity point of the mapping \mathcal{L} , where $\mathcal{L}: \mathcal{U} \rightarrow \mathcal{V}, \mathcal{U}$ and \mathcal{V} are nonempty subsets of an abstract nonempty set Λ if $\mathcal{D}(\omega, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ for all $\varsigma > 0$.

3. Main Results

Firstly, we define the following concepts.

Definition 12. Let \mathcal{U} and \mathcal{V} be two nonempty subsets of fuzzy metric space $(\Lambda, \mathcal{D}, *)$ and $\alpha: \Lambda \times \Lambda \times (0, \infty) \rightarrow [0, \infty)$. We say that $\mathcal{L}: \mathcal{U} \rightarrow \mathcal{V}$ is an α -proximal admissible if

$$\begin{cases} \alpha(\vartheta, \theta, \varsigma) \geq 1, \\ \mathcal{D}(u, \mathcal{L}\vartheta, \varsigma) = \mathcal{D}(v, \mathcal{L}\theta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) \implies \alpha(u, v, \varsigma) \geq 1, \end{cases} \quad (13)$$

for all $u, v, \vartheta, \theta \in \Lambda$ and $\varsigma > 0$.

Remark 1. Note that if $\mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) = 1$, then Definition 12 reduces to Definition 9 of α -admissibility.

Definition 13. Let \mathcal{U} and \mathcal{V} be nonempty subsets of fuzzy metric space $(\Lambda, \mathcal{D}, *)$ and $\alpha: \Lambda \times \Lambda \times (0, \infty) \rightarrow [0, \infty)$. We say that $\mathcal{L}: \mathcal{U} \rightarrow \mathcal{V}$ is an α - \mathcal{FL} -contraction with respect to $\xi \in \mathcal{FL}$ if \mathcal{L} is an α -proximal admissible such that

$$\begin{cases} \alpha(\vartheta, \theta, \varsigma) \geq 1, \\ \mathcal{D}(u, \mathcal{L}\vartheta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) \implies \xi(\mathcal{D}(u, v, \varsigma), \mathcal{D}(\vartheta, \theta, \varsigma)) \geq 0, \\ \mathcal{D}(v, \mathcal{L}\theta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) \end{cases} \quad (14)$$

for all $u, v, \vartheta, \theta \in \mathcal{U}$ and $\varsigma > 0$.

Definition 14. Let \mathcal{U} and \mathcal{V} be nonempty subsets of fuzzy metric space $(\Lambda, \mathcal{D}, *)$, $\alpha: \Lambda \times \Lambda \times (0, \infty) \rightarrow [0, \infty)$, and $\psi \in \Psi$. We say that $\mathcal{L}: \mathcal{U} \rightarrow \mathcal{V}$ is an α - ψ - \mathcal{FL} -contraction with respect to $\xi \in \mathcal{FL}$ if \mathcal{L} is an α -proximal admissible such that

$$\begin{cases} \alpha(\vartheta, \theta, \varsigma) \geq 1, \\ \mathcal{D}(u, \mathcal{L}\vartheta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) \implies \xi(\mathcal{D}(u, v, \varsigma), \psi(\mathcal{D}(\vartheta, \theta, \varsigma))) \geq 0, \\ \mathcal{D}(v, \mathcal{L}\theta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) \end{cases} \quad (15)$$

for all $u, v, \vartheta, \theta \in \mathcal{U}$ and $\varsigma > 0$.

Remark 2. Note that Definition 14 cannot be reduced to Definition 13 since $\psi(t) = t$ does not belong to Ψ .

Definition 15. Let \mathcal{U} and \mathcal{V} be nonempty subsets of fuzzy metric space $(\Lambda, \mathcal{D}, *)$ and $\alpha: \Lambda \times \Lambda \times (0, \infty) \rightarrow [0, \infty)$. We say that $\mathcal{L}: \mathcal{U} \rightarrow \mathcal{V}$ is a generalized α - \mathcal{FL} -contraction with respect to $\xi \in \mathcal{FL}$ if \mathcal{L} is an α -proximal admissible such that

$$\begin{cases} \alpha(\vartheta, \theta, \varsigma) \geq 1, \\ \mathcal{D}(u, \mathcal{L}\vartheta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \implies \xi(\mathcal{D}(u, v, \varsigma), \mathcal{R}(\vartheta, \theta, \varsigma)) \geq 0, \\ \mathcal{D}(v, \mathcal{L}\theta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) \end{cases} \quad (16)$$

for all $u, v, \vartheta, \theta \in \mathcal{U}$ and $\varsigma > 0$, where

$$\mathcal{R}(\vartheta, \theta, \varsigma) = \min \left\{ \mathcal{D}(\vartheta, \theta, \varsigma), \frac{\mathcal{D}(\vartheta, u, \varsigma) \mathcal{D}(\theta, v, \varsigma)}{\mathcal{D}(\vartheta, \theta, \varsigma)} \right\}. \quad (17)$$

Next, we give our first main result.

Theorem 1. *Let \mathcal{U} and \mathcal{V} be nonempty subsets of a complete fuzzy metric space $(\Lambda, \mathcal{D}, *)$, $\alpha: \Lambda \times \Lambda \times (0, \infty) \longrightarrow [0, \infty)$, $\psi \in \Psi$, and $\xi \in \mathcal{FL}$ is nonincreasing in its second argument. Assume that $\mathcal{L}: \mathcal{U} \longrightarrow \mathcal{V}$ is an $\alpha - \psi - \mathcal{FL}$ -contraction with respect to ξ and*

- (i) \mathcal{L} is triangular weak- α -admissible
- (ii) \mathcal{U} is closed
- (iii) $\mathcal{L}(\mathcal{U}_0) \subseteq \mathcal{V}_0$
- (iv) There exists $\vartheta_0, \vartheta_1 \in \mathcal{U}$ such that $\mathcal{D}(\vartheta_0, \mathcal{L}\vartheta_0, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ and $\alpha(\vartheta_0, \vartheta_1, \varsigma) \geq 1$ for all $\varsigma > 0$
- (v) \mathcal{L} is continuous.

Then, there exists $\omega \in \mathcal{U}$ such that $\mathcal{D}(\omega, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ for all $\varsigma > 0$; that is, \mathcal{L} has a best proximity point $\omega \in \mathcal{U}$.

Proof. Due to condition (iv), there exists $\vartheta_0, \vartheta_1 \in \mathcal{U}$ such that $\alpha(\vartheta_0, \vartheta_1, \varsigma) \geq 1$ and

$$\mathcal{D}(\vartheta_1, \mathcal{L}\vartheta_0, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma). \quad (18)$$

Regarding (iii), we deduce that $\mathcal{L}\vartheta_1 \in \mathcal{V}_0$; hence, there exists $\vartheta_2 \in \mathcal{U}$ such that

$$\mathcal{D}(\vartheta_2, \mathcal{L}\vartheta_1, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma). \quad (19)$$

Since $\alpha(\vartheta_0, \vartheta_1, \varsigma) \geq 1$ and \mathcal{L} is an α -proximal admissible, consequently, $\alpha(\vartheta_1, \vartheta_2, \varsigma) \geq 1$. Recursively, a sequence $\{\vartheta_n\} \subset \mathcal{U}_0$ can be defined as follows:

$$\alpha(\vartheta_n, \vartheta_{n+1}, \varsigma) \geq 1, \quad \text{for all } n \in \mathbb{N}, \quad (20)$$

$$\mathcal{D}(\vartheta_{n+1}, \mathcal{L}\vartheta_n, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \quad \text{for all } n \in \mathbb{N}. \quad (21)$$

If there exists $n_0 \in \mathbb{N}$ such that $\vartheta_{n_0+1} = \vartheta_{n_0}$, we obtain

$$\mathcal{D}(\vartheta_{n_0}, \mathcal{L}\vartheta_{n_0}, \varsigma) = \mathcal{D}(\vartheta_{n_0+1}, \mathcal{L}\vartheta_{n_0}, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \quad (22)$$

which means that ϑ_{n_0} is a best proximity point of \mathcal{L} . Thus, to continue our proof, we suppose that $\vartheta_n \neq \vartheta_{n+1}$ for all $n \in \mathbb{N}$. Making use of (20) and (21), we obtain

$$\begin{aligned} \mathcal{D}(\vartheta_n, \mathcal{L}\vartheta_{n-1}, \varsigma) &= \mathcal{D}(\vartheta_{n+1}, \mathcal{L}\vartheta_n, \varsigma) \\ &= \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (23)$$

Regarding that \mathcal{L} is an $\alpha - \psi - \mathcal{FL}$ -contraction with respect to $\xi \in \mathcal{FL}$, together with (20), (21), and (ξ_2) , we obtain

$$\begin{aligned} 0 &\leq \xi(\mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma), \psi(\mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma))) \\ &< \frac{1}{\psi(\mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma))} - \frac{1}{\mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma)}. \end{aligned} \quad (24)$$

Consequently, we have

$$\mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma) < \psi(\mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma)) < \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma), \quad (25)$$

which means that $\{\mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma)\}$ is a nondecreasing sequence of positive real numbers in $(0, 1]$. Then, there exists $c(\varsigma) \leq 1$ such that $\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma) = c(\varsigma) \geq 1$ for all $\varsigma > 0$. We shall prove that $c(\varsigma) = 1$. Reasoning by contradiction, suppose that $c(\varsigma_0) < 1$ for some $\varsigma_0 > 0$. Now, if we take the sequences $\{\tau_n = \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma_0)\}$ and $\{\delta_n = \mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma_0)\}$ and considering (ψ_2) and (ξ_3) and that ξ is nonincreasing with respect to its second argument, we obtain

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \sup \xi(\mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma_0), \psi(\mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma_0))) \\ &\leq \lim_{n \rightarrow \infty} \sup \xi(\mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma_0), \mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma_0)) \\ &< 0, \end{aligned} \quad (26)$$

which is a contradiction and yields

$$\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma) = 1, \quad \text{for all } \varsigma > 0. \quad (27)$$

Next, we show that the sequence $\{\vartheta_n\}$ is Cauchy. Reasoning by contradiction, suppose that $\{\vartheta_n\}$ is not a Cauchy sequence. Thus, there exists $\varepsilon \in (0, 1)$, $\varsigma_0 > 0$, and two subsequences $\{\vartheta_{n_k}\}$ and $\{\vartheta_{m_k}\}$ of $\{\vartheta_n\}$ with $n_k > m_k \geq k$ for all $k \in \mathbb{N}$ such that

$$\mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0) \leq 1 - \varepsilon. \quad (28)$$

Taking into account Lemma 1, we derive

$$\mathcal{D}\left(\vartheta_{m_k}, \vartheta_{n_k}, \frac{\varsigma_0}{2}\right) \leq 1 - \varepsilon. \quad (29)$$

By choosing m_k as the smallest index satisfying (29), we have

$$\mathcal{D}\left(\vartheta_{m_k}, \vartheta_{n_k-1}, \frac{\varsigma_0}{2}\right) > 1 - \varepsilon. \quad (30)$$

On account of (28), (30), and (\mathcal{MS}_4) , we have

$$\begin{aligned} 1 - \varepsilon &\geq \mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0) \\ &\geq \mathcal{D}\left(\vartheta_{m_k}, \vartheta_{n_k-1}, \frac{\varsigma_0}{2}\right) * \mathcal{D}\left(\vartheta_{n_k-1}, \vartheta_{n_k}, \frac{\varsigma_0}{2}\right) \\ &> (1 - \varepsilon) * \mathcal{D}\left(\vartheta_{n_k-1}, \vartheta_{n_k}, \frac{\varsigma_0}{2}\right). \end{aligned} \quad (31)$$

Taking limit as $k \rightarrow \infty$ and employing (27), we derive

$$\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0) = 1 - \varepsilon. \quad (32)$$

On the other hand, we have

$$\begin{aligned}\mathcal{D}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0) &\geq \mathcal{D}(\vartheta_{m_k-1}, \vartheta_{m_k}, \frac{\varsigma_0}{3}) * \mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \frac{\varsigma_0}{3}) * \mathcal{D}(\vartheta_{n_k}, \vartheta_{n_k-1}, \frac{\varsigma_0}{3}), \\ \mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0) &\geq \mathcal{D}(\vartheta_{m_k}, \vartheta_{m_k-1}, \frac{\varsigma_0}{3}) * \mathcal{D}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \frac{\varsigma_0}{3}) * \mathcal{D}(\vartheta_{n_k-1}, \vartheta_{n_k}, \frac{\varsigma_0}{3}),\end{aligned}\quad (33)$$

which imply that

$$\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0) = 1 - \varepsilon. \quad (34)$$

Furthermore, given that \mathcal{L} is triangular weak- α -admissible and taking into account (20), we deduce that

$$\alpha(\vartheta_n, \vartheta_m, \varsigma) \geq 1, \quad \text{for all } n, m \in \mathbb{N} \text{ with } n > m. \quad (35)$$

So that

$$\alpha(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0) \geq 1, \quad (36)$$

$$\begin{aligned}\mathcal{D}(\vartheta_{m_k}, \mathcal{L}\vartheta_{m_k-1}, \varsigma_0) &= \mathcal{D}(\vartheta_{n_k}, \mathcal{L}\vartheta_{n_k-1}, \varsigma_0) \\ &= \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma_0), \quad \text{for all } k \in \mathbb{N}.\end{aligned}\quad (37)$$

Regarding the fact that \mathcal{L} is an $\alpha - \psi - \mathcal{F}\mathcal{L}$ -contraction with respect to $\xi \in \mathcal{F}\mathcal{L}$ and making use of (35) and (36), we have

$$0 \leq \xi(\mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0), \psi(\mathcal{D}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0))), \quad \text{for all } k \in \mathbb{N}. \quad (38)$$

From (32) and (34), we see that the sequences $\{\mu_k = \mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0)\}$ and $\{\nu_k = \mathcal{D}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0)\}$ have the same limit $1 - \varepsilon < 1$, taking into consideration that ξ is nonincreasing with respect to its second argument; by the property ξ_3 , we conclude that

$$\begin{aligned}0 &\leq \lim_{n \rightarrow \infty} \sup \xi(\mu_k, \psi(\nu_k)) \\ &\leq \lim_{n \rightarrow \infty} \sup \xi(\mu_k, \nu_k) \\ &< 0,\end{aligned}\quad (39)$$

which is a contradiction. So that $\{\vartheta_n\}$ is a Cauchy sequence in \mathcal{U} . As \mathcal{U} is closed subset of a complete fuzzy metric space $(\Lambda, \mathcal{D}, *)$, there exists $\omega \in \mathcal{U}$ such that

$$\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_n, \omega, \varsigma) = 1. \quad (40)$$

As \mathcal{L} is continuous, we conclude that $\mathcal{L}\vartheta_n$ converges to $\mathcal{L}\omega$; thus,

$$\lim_{n \rightarrow \infty} \mathcal{D}(\mathcal{L}\vartheta_n, \mathcal{L}\omega, \varsigma) = 1. \quad (41)$$

Due to the continuity of \mathcal{D} , we have $\mathcal{D}(\vartheta_{n+1}, \mathcal{L}\vartheta_n, \varsigma) \rightarrow \mathcal{D}(\omega, \mathcal{L}\omega, \varsigma)$. From (21), we deduce

$$\mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) = \lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_{n+1}, \mathcal{L}\vartheta_n, \varsigma) = \mathcal{D}(\omega, \mathcal{L}\omega, \varsigma), \quad (42)$$

which means that $\omega \in \mathcal{U}$ is a best proximity point of \mathcal{L} .

In the next theorem, we substitute the continuity of \mathcal{L} in Theorem 1 with the following condition.

(C): if $\{\vartheta_n\}$ is a sequence in \mathcal{U} such that $\alpha(\vartheta_n, \vartheta_{n+1}, \varsigma) \geq 1$ for all $n \in \mathbb{N}$, $\varsigma > 0$, and $\vartheta_n \rightarrow \vartheta \in \mathcal{U}$ as $n \rightarrow \infty$, then there exists a subsequence $\{\vartheta_{n(k)}\}$ of $\{\vartheta_n\}$ such that $\alpha(\vartheta_{n(k)}, \vartheta, \varsigma) \geq 1$ for all $k \in \mathbb{N}$ and $\varsigma > 0$. \square

Theorem 2. Let \mathcal{U} and \mathcal{V} be nonempty subsets of a complete fuzzy metric space $(\Lambda, \mathcal{D}, *)$ and $\alpha: \Lambda \times \Lambda \times (0, \infty) \rightarrow [0, \infty)$, $\psi \in \Psi$, and $\xi \in \mathcal{F}\mathcal{L}$ is nonincreasing in its second argument. Assume that $\mathcal{L}: \mathcal{U} \rightarrow \mathcal{V}$ is an $\alpha - \psi - \mathcal{F}\mathcal{L}$ -contraction with respect to $\xi \in \mathcal{F}\mathcal{L}$ and

- (i) \mathcal{L} is triangular weak- α -admissible
- (ii) \mathcal{U} is closed
- (iii) $\mathcal{L}(\mathcal{U}_0) \subseteq \mathcal{V}_0$
- (iv) There exists $\vartheta_0, \vartheta_1 \in \mathcal{U}$ such that $\mathcal{D}(\vartheta_1, \mathcal{L}\vartheta_0, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ and $\alpha(\vartheta_0, \vartheta_1, \varsigma) \geq 1$ for all $\varsigma > 0$
- (v) If $\{\vartheta_n\}$ is a sequence in \mathcal{U} such that $\alpha(\vartheta_n, \vartheta_{n+1}, \varsigma) \geq 1$ for all $n \in \mathbb{N}$, $\varsigma > 0$, and $\vartheta_n \rightarrow \vartheta \in \mathcal{U}$ as $n \rightarrow \infty$, then there exists a subsequence $\{\vartheta_{n(k)}\}$ of $\{\vartheta_n\}$ such that $\alpha(\vartheta_{n(k)}, \vartheta, \varsigma) \geq 1$ for all $k \in \mathbb{N}$ and $\varsigma > 0$

Then, there exists $\omega \in \mathcal{U}$ such that $\mathcal{D}(\omega, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ for all $\varsigma > 0$.

Proof. Following the lines of the proof of Theorem 1, we deduce that there exists a Cauchy sequence $\{\vartheta_n\}$ in \mathcal{U}_0 which converges to $\omega \in \mathcal{U}_0$. Since $\mathcal{L}(\mathcal{U}_0) \subseteq \mathcal{V}_0$, we have $\mathcal{L}\omega \in \mathcal{V}_0$, and then

$$\mathcal{D}(a_1, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \quad \text{for some } a_1 \in \mathcal{U}_0. \quad (43)$$

By condition (v), there exists a subsequence $\{\vartheta_{n(k)}\}$ of $\{\vartheta_n\}$ such that

$$\alpha(\vartheta_{n_k}, \omega, \varsigma) \geq 1, \quad \text{for all } k \in \mathbb{N}, \varsigma > 0. \quad (44)$$

Regarding that \mathcal{L} is an α -proximal admissible and

$$\mathcal{D}(a_1, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\vartheta_{n_k+1}, \mathcal{L}\vartheta_{n_k}, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \quad (45)$$

we obtain that $\alpha(\vartheta_{n_k+1}, a_1, \varsigma) \geq 1$. Hence,

$$0 \leq \xi(\mathcal{D}(a_1, \vartheta_{n_k+1}, \varsigma), \psi(\mathcal{D}(\omega, \vartheta_{n_k}, \varsigma))), \quad \text{for all } k \in \mathbb{N}. \quad (46)$$

Applying the property (ξ_2) , it follows that

$$\mathcal{D}(\omega, \vartheta_{n_k}, \varsigma) < \psi(\mathcal{D}(\omega, \vartheta_{n_k}, \varsigma)) < \mathcal{D}(a_1, \vartheta_{n_k+1}, \varsigma), \quad (47)$$

which yields $\lim_{k \rightarrow \infty} \mathcal{D}(a_1, \vartheta_{n_k+1}, \varsigma) = 1$. Then, $a_1 = \omega$; from (45), we derive that $\mathcal{D}(\omega, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$. \square

Theorem 3. Let \mathcal{U} and \mathcal{V} be nonempty subsets of a complete fuzzy metric space $(\Lambda, \mathcal{D}, *)$ and

$\alpha: \Lambda \times \Lambda \times (0, \infty) \longrightarrow [0, \infty)$. Assume that $\mathcal{L}: \mathcal{U} \longrightarrow \mathcal{V}$ is a generalized $\alpha - \mathcal{FL}$ -contraction with respect to $\xi \in \mathcal{FL}$ and

- (i) \mathcal{L} is triangular weak- α -admissible
- (ii) \mathcal{U} is closed
- (iii) $\mathcal{L}(\mathcal{U}_0) \subseteq \mathcal{V}_0$
- (iv) There exists $\vartheta_0, \vartheta_1 \in \mathcal{U}$ such that $\mathcal{D}(\vartheta_1, \mathcal{L}\vartheta_0, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ and $\alpha(\vartheta_0, \mathcal{L}\vartheta_0, \varsigma) \geq 1$ for all $\varsigma > 0$
- (v) \mathcal{L} is continuous

Then, there exists $\omega \in \mathcal{U}$ such that $\mathcal{D}(\omega, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ for all $\varsigma > 0$.

Proof. Using condition (iv), there exists $\vartheta_0, \vartheta_1 \in \mathcal{U}$ such that $\alpha(\vartheta_0, \vartheta_1, \varsigma) \geq 1$ and $\mathcal{D}(\vartheta_1, \mathcal{L}\vartheta_0, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$. Regarding (iii), we have $\mathcal{L}\vartheta_1 \in \mathcal{V}_0$ which yields that there exists $\vartheta_2 \in \mathcal{U}$ such that

$$\mathcal{D}(\vartheta_2, \mathcal{L}\vartheta_1, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma). \quad (48)$$

Since $\alpha(\vartheta_0, \vartheta_1, \varsigma) \geq 1$ and \mathcal{L} is an α -proximal admissible, it therefore follows that $\alpha(\vartheta_1, \vartheta_2, \varsigma) \geq 1$. We recursively construct the sequence $\{\vartheta_n\} \subset \mathcal{U}_0$ as follows:

$$\begin{aligned} \mathcal{R}(\vartheta_{n-1}, \vartheta_n, \varsigma) &= \min \left\{ \mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma), \frac{\mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma) \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma)}{\mathcal{D}(\vartheta_{n-1}, \vartheta_{n+1}, \varsigma)} \right\} \\ &= \min \{ \mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma), \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma) \}. \end{aligned} \quad (53)$$

Now, if

$$\min \{ \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma), \mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma) \} = \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma). \quad (54)$$

$$\begin{aligned} \alpha(\vartheta_n, \vartheta_{n+1}, \varsigma) &\geq 1, \\ \mathcal{D}(\vartheta_{n+1}, \mathcal{L}\vartheta_n, \varsigma) &= \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (49)$$

Suppose that there exists certain $m_0 \in \mathbb{N}$ such that $\vartheta_{m_0+1} = \vartheta_{m_0}$. Hence,

$$\mathcal{D}(\vartheta_{m_0}, \mathcal{L}\vartheta_{m_0}, \varsigma) = \mathcal{D}(\vartheta_{m_0+1}, \mathcal{L}\vartheta_{m_0}, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \quad (50)$$

which means that ϑ_{m_0} is a best proximity point of \mathcal{L} ; thus, the proof is finished. For this reason, to continue our proof, we assume that $\vartheta_n \neq \vartheta_{n+1}$ for all $n \in \mathbb{N}$. Making use of (56) and (60), we obtain

$$\begin{aligned} \mathcal{D}(\vartheta_n, \mathcal{L}\vartheta_{n-1}, \varsigma) &= \mathcal{D}(\vartheta_{n+1}, \mathcal{L}\vartheta_n, \varsigma) \\ &= \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (51)$$

Since \mathcal{L} is a generalized $\alpha - \mathcal{FL}$ -contraction with respect to $\xi \in \mathcal{FL}$, we have

$$0 \leq \xi(\mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma), \mathcal{R}(\vartheta_{n-1}, \vartheta_n, \varsigma)), \quad (52)$$

where

applying (ξ_2) , we get that

$$0 \leq \xi(\mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma), \mathcal{R}(\vartheta_{n-1}, \vartheta_n, \varsigma)) < \frac{1}{\mathcal{R}(\vartheta_{n-1}, \vartheta_n, \varsigma)} - \frac{1}{\mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma)}. \quad (55)$$

Thus,

$$\mathcal{R}(\vartheta_{n-1}, \vartheta_n, \varsigma) = \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma) < \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma), \quad (56)$$

which is a contradiction. Consequently,

$$\mathcal{R}(\vartheta_{n-1}, \vartheta_n, \varsigma) = \min \{ \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma), \mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma) \} = \mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma). \quad (57)$$

By (55), we obtain that

$$\mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma) < \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma), \quad \text{for all } n \in \mathbb{N}. \quad (58)$$

Hence, we deduce that $\{\mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma)\}$ is a nondecreasing sequence in $(0, 1]$. Thus, there exists $s(\varsigma) \leq 1$ such that $\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma) = s(\varsigma)$ for all $\varsigma > 0$. We shall

prove that $s(\varsigma) = 1$. Reasoning by contradiction, suppose that $s(\varsigma_0) < 1$ for some $\varsigma_0 > 0$. Now, if we take the sequences $\mathcal{T}_n = \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma_0)$ and $\mathcal{S}_n = \mathcal{D}(\vartheta_{n-1}, \vartheta_n, \varsigma_0)$ and consider (ξ_3) , we obtain

$$0 \leq \lim_{n \rightarrow \infty} \sup \xi(\mathcal{T}_n, \mathcal{S}_n) < 0, \quad (59)$$

which is a contradiction. Therefore,

$$\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_n, \vartheta_{n+1}, \varsigma) = 1, \quad \text{for all } \varsigma > 0. \quad (60)$$

Next, we show that $\{\vartheta_n\}$ is Cauchy sequence. On the contrary, assume that $\{\vartheta_n\}$ is not a Cauchy. Hence, there exist $\varepsilon \in (0, 1)$, $\varsigma_0 > 0$, and two subsequences $\{\vartheta_{n_k}\}$ and $\{\vartheta_{m_k}\}$ of $\{\vartheta_n\}$ with $n_k > m_k \geq k$ for all $k \in \mathbb{N}$ such that

$$\mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0) \leq 1 - \varepsilon. \quad (61)$$

Taking into account Lemma 1, we derive that

$$\mathcal{D}\left(\vartheta_{m_k}, \vartheta_{n_k}, \frac{\varsigma_0}{2}\right) \leq 1 - \varepsilon. \quad (62)$$

By choosing m_k as the smallest index satisfying (29), we have

$$\mathcal{D}\left(\vartheta_{m_k}, \vartheta_{n_k-1}, \frac{\varsigma_0}{2}\right) > 1 - \varepsilon. \quad (63)$$

Making use of (61) and (63) and the triangular inequality, we get

$$\begin{aligned} 1 - \varepsilon &\geq \mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0) \\ &\geq \mathcal{D}\left(\vartheta_{m_k}, \vartheta_{n_k-1}, \frac{\varsigma_0}{2}\right) * \mathcal{D}\left(\vartheta_{n_k-1}, \vartheta_{n_k}, \frac{\varsigma_0}{2}\right) \\ &> (1 - \varepsilon) * \mathcal{D}\left(\vartheta_{n_k-1}, \vartheta_{n_k}, \frac{\varsigma_0}{2}\right). \end{aligned} \quad (64)$$

Passing to the limit $k \rightarrow \infty$ and using (60), we derive that

$$\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0) = 1 - \varepsilon. \quad (65)$$

On the other hand,

$$\begin{aligned} \mathcal{D}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0) &\geq \mathcal{D}\left(\vartheta_{m_k-1}, \vartheta_{m_k}, \frac{\varsigma_0}{3}\right) * \mathcal{D}\left(\vartheta_{m_k}, \vartheta_{n_k}, \frac{\varsigma_0}{3}\right) * \mathcal{D}\left(\vartheta_{n_k}, \vartheta_{n_k-1}, \frac{\varsigma_0}{3}\right), \\ \mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0) &\geq \mathcal{D}\left(\vartheta_{m_k}, \vartheta_{m_k-1}, \frac{\varsigma_0}{3}\right) * \mathcal{D}\left(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \frac{\varsigma_0}{3}\right) * \mathcal{D}\left(\vartheta_{n_k-1}, \vartheta_{n_k}, \frac{\varsigma_0}{3}\right), \end{aligned} \quad (66)$$

which imply that

$$\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0) = 1 - \varepsilon. \quad (67)$$

Furthermore, since \mathcal{L} is triangular weak- α -admissible, we deduce that

$$\alpha(\vartheta_n, \vartheta_m, \varsigma) \geq 1, \quad \text{for all } n, m \in \mathbb{N} \text{ with } n > m. \quad (68)$$

Thus,

$$\alpha(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0) \geq 1, \quad (69)$$

$$\mathcal{D}(\vartheta_{m_k}, \mathcal{L}\vartheta_{m_k-1}, \varsigma_0) = \mathcal{D}(\vartheta_{n_k}, \mathcal{L}\vartheta_{n_k-1}, \varsigma_0) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma_0), \quad (70)$$

for all $k \in \mathbb{N}$. By the fact that \mathcal{L} is a generalized $\alpha - \mathcal{F}\mathcal{L}$ -contraction with respect to $\xi \in \mathcal{F}\mathcal{L}$ and using (69) and (70), we obtain that

$$0 \leq \xi(\mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0), \mathcal{R}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0)), \quad \text{for all } k \in \mathbb{N}, \quad (71)$$

where

$$\mathcal{R}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0) = \min \left\{ \mathcal{D}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0), \frac{\mathcal{D}(\vartheta_{m_k-1}, \vartheta_{m_k}, \varsigma_0) \mathcal{D}(\vartheta_{n_k-1}, \vartheta_{n_k}, \varsigma_0)}{\mathcal{D}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0)} \right\}. \quad (72)$$

Letting $k \rightarrow \infty$ in equality (72) and using (60), we derive

$$\begin{aligned}
& \lim_{k \rightarrow \infty} \mathcal{R}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0) \\
&= \min \left\{ \frac{1}{1-\varepsilon}, 1-\varepsilon \right\} \\
&= 1-\varepsilon.
\end{aligned} \tag{73}$$

Take the sequences $\mu_k = \mathcal{D}(\vartheta_{m_k}, \vartheta_{n_k}, \varsigma_0)$ and $\nu_k = \mathcal{R}(\vartheta_{m_k-1}, \vartheta_{n_k-1}, \varsigma_0)$ for all $k \in \mathbb{N}$. Applying ξ_3 , we derive that

$$0 \leq \lim_{n \rightarrow \infty} \sup \xi(\mu_k, \nu_k) < 0, \tag{74}$$

which is a contradiction. Then, $\{\vartheta_n\}$ is a Cauchy sequence in \mathcal{U} . Given that \mathcal{U} is closed subset of a complete fuzzy metric space $(\Lambda, \mathcal{D}, *)$, there exists $\omega \in \mathcal{U}$ such that

$$\lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_n, \omega, \varsigma) = 1. \tag{75}$$

As \mathcal{L} is continuous, we obtain that $\mathcal{L}\vartheta_n$ converges to $\mathcal{L}\omega$; thus,

$$\lim_{n \rightarrow \infty} \mathcal{D}(\mathcal{L}\vartheta_n, \mathcal{L}\omega, \varsigma) = 1. \tag{76}$$

As the metric function \mathcal{D} is continuous, we have $\mathcal{D}(\vartheta_{n+1}, \mathcal{L}\vartheta_n, \varsigma) \rightarrow \mathcal{D}(\omega, \mathcal{L}\omega, \varsigma)$. In view of (51), we get

$$\mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) = \lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_{n+1}, \mathcal{L}\vartheta_n, \varsigma) = \mathcal{D}(\omega, \mathcal{L}\omega, \varsigma). \tag{77}$$

Thus, $\omega \in \mathcal{U}$ is a best proximity point of \mathcal{L} . \square

Theorem 4. Let \mathcal{U} and \mathcal{V} be nonempty subsets of a complete fuzzy metric space $(\Lambda, \mathcal{D}, *)$ and $\alpha: \Lambda \times \Lambda \times (0, \infty) \rightarrow [0, \infty)$. Assuming that $\mathcal{L}: \mathcal{U} \rightarrow \mathcal{V}$ is an α - \mathcal{FL} -contraction with respect to $\xi \in \mathcal{FL}$,

- (i) \mathcal{L} is triangular weak- α -admissible
- (ii) \mathcal{U} is closed
- (iii) $\mathcal{L}(\mathcal{U}_0) \subseteq \mathcal{V}_0$
- (iv) There exists $\vartheta_0, \vartheta_1 \in \mathcal{U}$ such that $\mathcal{D}(\vartheta_0, \mathcal{L}\vartheta_0, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ and $\alpha(\vartheta_0, \vartheta_1, \varsigma) \geq 1$ for all $\varsigma > 0$
- (v) \mathcal{L} is continuous or (C) holds

Then, there exists $\omega \in \mathcal{U}$ such that $\mathcal{D}(\omega, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ for all $\varsigma > 0$.

Proof. Pursuant to the same arguments as those given in the proof of Theorem 3, we know that there exists a Cauchy sequence $\{\vartheta_n\}$ in \mathcal{U} which converges to $\omega \in \mathcal{U}$. Further,

$$\lim_{k \rightarrow \infty} \mathcal{D}(\vartheta_n, \omega, \varsigma) = 1, \quad \text{for all } n \in \mathbb{N}, \varsigma > 0. \tag{78}$$

If \mathcal{L} is continuous, then

$$\lim_{k \rightarrow \infty} \mathcal{D}(\mathcal{L}\vartheta_n, \mathcal{L}\omega, \varsigma) = 1, \quad \text{for all } n \in \mathbb{N}, \varsigma > 0. \tag{79}$$

Taking into account (21), we deduce

$$\mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) = \lim_{n \rightarrow \infty} \mathcal{D}(\vartheta_{n+1}, \mathcal{L}\vartheta_n, \varsigma) = \mathcal{D}(\omega, \mathcal{L}\omega, \varsigma), \tag{80}$$

which means that $\omega \in \mathcal{U}$ is a best proximity point of \mathcal{L} .

Now, suppose that (C) holds. Since $\mathcal{L}(\mathcal{U}_0) \subseteq \mathcal{V}_0$, we have $\mathcal{L}\omega \in \mathcal{V}_0$ and then

$$\mathcal{D}(a_1, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \quad \text{for some } a_1 \in \mathcal{U}_0. \tag{81}$$

By condition (C), there exists a subsequence $\{\vartheta_{n(k)}\}$ of $\{\vartheta_n\}$ such that

$$\alpha(\vartheta_{n_k}, \omega, \varsigma) \geq 1, \quad \text{for all } k \in \mathbb{N}, \varsigma > 0. \tag{82}$$

Regarding that \mathcal{L} is an α -proximal admissible and

$$\mathcal{D}(a_1, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\vartheta_{n_k+1}, \mathcal{L}\vartheta_{n_k}, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \tag{83}$$

we obtain that $\alpha(\vartheta_{n_k+1}, a_1, \varsigma) \geq 1$. Hence,

$$0 \leq \xi(\mathcal{D}(a_1, \vartheta_{n_k+1}, \varsigma), \mathcal{D}(\omega, \vartheta_{n_k}, \varsigma)), \quad \text{for all } k \in \mathbb{N}. \tag{84}$$

Applying the property (ξ_2) , it follows that

$$\mathcal{D}(\omega, \vartheta_{n_k}, \varsigma) < \mathcal{D}(a_1, \vartheta_{n_k+1}, \varsigma), \tag{85}$$

which yields $\lim_{k \rightarrow \infty} \mathcal{D}(a_1, \vartheta_{n_k+1}, \varsigma) = 1$. Then, $a_1 = \omega$; from (45), we derive that $\mathcal{D}(\omega, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$. This completes the proof.

Note that Theorem 4 cannot be deduced by combining Theorems 1 and 2 since the function $\psi(c) = c$ does not belong to Ψ . Moreover, in Theorems 1 and 2, we have an added condition that ξ is nonincreasing in its second argument. \square

4. Consequences

Now, we shall clarify that diverse consequences of the existence results can be easily derived and developed from our main results.

Corollary 1. Let \mathcal{U} and \mathcal{V} be nonempty subsets of a complete fuzzy metric space $(\Lambda, \mathcal{D}, *)$, $\alpha: \Lambda \times \Lambda \times (0, \infty) \rightarrow [0, \infty)$, $\psi \in \Psi$. Assume that $\mathcal{L}: \mathcal{U} \rightarrow \mathcal{V}$ is an α -admissible proximal mapping such that

$$\begin{cases} \alpha(\vartheta, \theta, \varsigma) \geq 1, \\ \mathcal{D}(u, \mathcal{L}\vartheta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \implies \mathcal{D}(u, v, \varsigma) \geq \psi(\mathcal{D}(\vartheta, \theta, \varsigma)), \\ \mathcal{D}(v, \mathcal{L}\theta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) \end{cases} \tag{86}$$

for all $u, v, \vartheta, \theta \in \mathcal{U}$ and $\varsigma > 0$. Assume also that

- (i) \mathcal{L} is triangular weak- α -admissible
- (ii) \mathcal{U} is closed
- (iii) $\mathcal{L}(\mathcal{U}_0) \subseteq \mathcal{V}_0$
- (iv) There exists $\vartheta_0, \vartheta_1 \in \mathcal{U}$ such that $\mathcal{D}(\vartheta_0, \mathcal{L}\vartheta_0, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ and $\alpha(\vartheta_0, \vartheta_1, \varsigma) \geq 1$ for all $\varsigma > 0$
- (v) \mathcal{L} is continuous or (C) holds

Then, there exists $\omega \in \mathcal{U}$ such that $\mathcal{D}(\omega, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ for all $\varsigma > 0$.

Proof. Define $\xi: (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$ by

$$\xi(\mu, \nu) = \frac{1}{\psi(\nu)} - \frac{1}{\mu}, \quad \text{for all } \mu, \nu \in (0, 1]. \quad (87)$$

Since $\xi \in \mathcal{F}\mathcal{L}$, Theorem 4 leads to the desired results. \square

Corollary 2. Let \mathcal{U} and \mathcal{V} be nonempty subsets of a complete fuzzy metric space $(\Lambda, \mathcal{D}, *)$, $\alpha: \Lambda \times \Lambda \times (0, \infty) \longrightarrow [0, \infty)$, $\eta \in \mathcal{H}$. Assume that $\mathcal{L}: \mathcal{U} \longrightarrow \mathcal{V}$ is an α -admissible proximal mapping such that

$$\begin{cases} \alpha(\vartheta, \theta, \varsigma) \geq 1, \\ \mathcal{D}(u, \mathcal{L}\vartheta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \implies \eta(\mathcal{D}(u, v, \varsigma)) \leq a\eta(\mathcal{D}(\vartheta, \theta, \varsigma)), \\ \mathcal{D}(v, \mathcal{L}\theta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) \end{cases} \quad (88)$$

for all $u, v, \vartheta, \theta \in \mathcal{U}$ and $\varsigma > 0$. Assume also that

- (i) \mathcal{L} is triangular weak- α -admissible
- (ii) \mathcal{U} is closed
- (iii) $\mathcal{L}(\mathcal{U}_0) \subseteq \mathcal{V}_0$
- (iv) There exists $\vartheta_0, \vartheta_1 \in \mathcal{U}$ such that $\mathcal{D}(\vartheta_0, \mathcal{L}\vartheta_0, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ and $\alpha(\vartheta_0, \vartheta_1, \varsigma) \geq 1$ for all $\varsigma > 0$
- (v) \mathcal{L} is continuous or (C) holds

Then, there exists $\omega \in \mathcal{U}$ such that $\mathcal{D}(\omega, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ for all $\varsigma > 0$.

Proof. It follows from Theorem 4 using the $\mathcal{F}\mathcal{L}$ -simulation function $\xi(\mu, \nu) = (1/\eta^{-1}(a.\eta(\nu))) - (1/\mu)$, for all $\mu, \nu \in (0, 1]$. \square

Corollary 3. Let \mathcal{U} and \mathcal{V} be nonempty subsets of a complete fuzzy metric space $(\Lambda, \mathcal{D}, *)$, $\alpha: \Lambda \times \Lambda \times (0, \infty) \longrightarrow [0, \infty)$. Assume that $\mathcal{L}: \mathcal{U} \longrightarrow \mathcal{V}$ is an α -admissible proximal mapping such that

$$\begin{cases} \alpha(\vartheta, \theta, \varsigma) \geq 1, \\ \mathcal{D}(u, \mathcal{L}\vartheta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma), \implies \left(\frac{1}{\mathcal{D}(u, v, \varsigma)} - 1 \right) \leq \phi \left(\frac{1}{\mathcal{D}(\vartheta, \theta, \varsigma)} - 1 \right), \\ \mathcal{D}(v, \mathcal{L}\theta, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma) \end{cases} \quad (89)$$

for all $u, v, \vartheta, \theta \in \mathcal{U}$ and $\varsigma > 0$, where $\phi: [0, \infty) \longrightarrow [0, \infty)$ with $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$. Assume also

- (i) \mathcal{L} is triangular weak- α -admissible
- (ii) \mathcal{U} is closed
- (iii) $\mathcal{L}(\mathcal{U}_0) \subseteq \mathcal{V}_0$
- (iv) There exists $\vartheta_0, \vartheta_1 \in \mathcal{U}$ such that $\mathcal{D}(\vartheta_0, \mathcal{L}\vartheta_0, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ and $\alpha(\vartheta_0, \vartheta_1, \varsigma) \geq 1$ for all $\varsigma > 0$
- (v) \mathcal{L} is continuous or (C) holds.

Then, there exists $\omega \in \mathcal{U}$ such that $\mathcal{D}(\omega, \mathcal{L}\omega, \varsigma) = \mathcal{D}(\mathcal{U}, \mathcal{V}, \varsigma)$ for all $\varsigma > 0$, i.e., \mathcal{L} has a best proximity point $\omega \in \mathcal{U}$.

Proof. It follows from Theorem 4 by taking $\xi(\mu, \nu) = \phi((1/\nu) - 1) - (1/\mu) + 1$ for all $\mu, \nu \in (0, 1]$. \square

Example 5. Let $\Lambda = \mathcal{U} = \mathcal{V} = (0, \infty)$ and \mathcal{D} be a fuzzy set on $\Lambda^2 \times (0, \infty)$ given by $\mathcal{D}(\vartheta, \theta, \varsigma) = (\min\{\vartheta, \theta\}/\max\{\vartheta, \theta\})$, $\varsigma > 0$, and $*$ is a t-norm given by $\mathcal{F} * 1 = \mathcal{F}.1$ for all $\mathcal{F}.1 \in [0, 1]$. $(\Lambda, \mathcal{D}, *)$ is a complete fuzzy metric space. Consider the mappings $\mathcal{L}: \Lambda \longrightarrow \Lambda$ by $\mathcal{L}(\vartheta) = \sqrt{\vartheta}$. Define $\alpha(\vartheta, \theta, \varsigma) = 1$ for all $\vartheta, \theta \in \Lambda$, $\varsigma > 0$, and $\xi: (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$ by

$$\xi(\mu, \nu) = \frac{1}{\psi(\nu)} - \frac{1}{\mu}, \quad \text{for all } \mu, \nu \in (0, 1], \text{ where } \psi(c) = \sqrt{c}. \quad (90)$$

It is easy to see that $\xi \in \mathcal{F}\mathcal{L}$ and $\xi(\mathcal{D}(\mathcal{L}\vartheta, \mathcal{L}\theta, \varsigma), \mathcal{D}(\vartheta, \theta, \varsigma)) \geq 0$. Therefore, all conditions of Theorem 4 are satisfied, and $x = 1$ is a fixed point of \mathcal{L} .

We must point to the fact that, by defining the control function ξ and the admissible mapping $\alpha(\vartheta, \theta)$ in a proper way, it is possible to particularize and derive a number of varied consequences of our main results. We skip making such a number of corollaries since they seem clear.

5. Conclusion

This paper has dealt with a $\mathcal{F}\mathcal{L}$ -simulation function approach to best proximity point problems in fuzzy metric spaces. We have initiated some classes of non-self-mappings and discussed existence results of the best proximity points of such types of non-self-mappings. Our results can be further extended by replacing the fuzzy metric space by various settings (e.g., partially ordered fuzzy metric spaces and complex valued fuzzy metric spaces), and more generalization can be obtained by the study of optimal coincidence points, optimal best proximity coincidence points, and the setting of cyclic mappings.

Data Availability

The data used to support the findings of this study are included in the references within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On the Upper Bounds of Fractional Metric Dimension of Symmetric Networks

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Received 14 April 2021; Accepted 19 May 2021; Published 17 June 2021

Academic Editor: Zoran Mitrovic

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Distance-based numeric parameters play a pivotal role in studying the structural aspects of networks which include connectivity, accessibility, centrality, clustering modularity, complexity, vulnerability, and robustness. Several tools like these also help to resolve the issues faced by the different branches of computer science and chemistry, namely, navigation, image processing, biometry, drug discovery, and similarities in chemical compounds. For this purpose, in this article, we are considering a family of networks that exhibits rotationally symmetric behaviour known as circular ladders consisting of triangular, quadrangular, and pentagonal faced ladders. We evaluate their upper bounds of fractional metric dimensions of the aforementioned networks.

1. Introduction

A network \aleph comprises of two collections of distinct objects, which are set containing nodes/vertices $V(\aleph)$ and set of objects that forms connection among those nodes/vertices $E(\aleph)$, where $E(\aleph) \subseteq V(\aleph) \times V(\aleph)$. The cardinality of $V(\aleph)$ constitutes the order of \aleph denoted by v , and size of \aleph is the cardinality of $E(\aleph)$ denoted by e . For $a, b \in V(\aleph)$, $d(a, b)$ represents the distance of shortest path between a and b . In order to have an in depth study of graph theory's strata, please refer to [1–3].

Day by day technological advancements are restructuring our lives by replacing manpower with robots, devices, and machinery. In the same instance, the necessity of restructuring is also making it inevitable for us to utilize these machineries which are keeping in view cost effectiveness and serving the large amount of ques in production line, health services, and public dealing areas with minimum number of them. Here, distance-based parameters come to fulfill the aforementioned objectives. Consider $\aleph = \{m_1, m_2, \dots, m_k\} \subseteq V(\aleph)$, then \aleph becomes an ordered set of vertices whenever some ordering is being imposed on them. The k -tuple metric form of \aleph with respect to a is

represented by $r(a|\aleph) = (d(a, m_1), d(a, m_2), d(a, m_3), \dots, d(a, m_k))$, where $d(a, m_t)$ for $1 \leq t \leq k$ shows the distance of a from m_t . \aleph changes into resolving set if for any pair of distinct vertices $a, b \in V(\aleph) - \aleph$ such that $r(a|\aleph) \neq r(b|\aleph)$. The resolving set having minimum number of vertices in \aleph presents the metric dimension (MD) of \aleph denoted by $\lambda(\aleph)$ [4, 5].

For any connected network, Slater [6, 7] laid the foundation of resolving set by calling it as locating set. After studying these terminologies at their own, Harary and Melter named it as the MD of a network. Since then, a number of researchers considered different families of networks for the computation of MD. The results regarding the families of networks with constant and bounded MD can be found in [8]. Chartrand et al. proved the results regarding the metric dimensions of path and cycle in [4]. Similarly, $\mathbb{P}(n, 2)$, \mathbb{A}_m , and \mathbb{C}_m^2 as networks bearing constant, MD has been shown in [8]. Moreover, in [9], the MD of the generalized Petersen network is proved to be bounded. Also the MD of the Jahangir network and infinite regular networks has been studied in [10, 11], respectively.

Chartrand et al. [4] employed MD to acquire the solution of integer programming problem (IPP). Later on, Currie and

Oellermann [12] established the concept of fractional metric dimension (FMD) and acquired the higher accuracy solution. Fehr et al. [13] utilized FMD to obtain the optimal solution of a certain linear programming relaxation problem. Arumugam and Mathew [14] uncovered some peculiar features of FMD. Afterwards, a bunch of results appeared related to the FMD of several networks that are the resultants of graph products, namely, Cartesian, hierarchial, corona, and lexicographic comb products (see [14–18]). Furthermore, the FMD of the generalized Jahangir network and permutation network can be found in [19, 20], respectively. In a recent time, Raza et al. [21] evaluated the FMD of metal organic networks.

Aisyah et al. defined the notion of local fractional metric dimension (LFMD) and calculated for corona product of two networks [22]. Similarly, Liu et al. [23] calculated the LFMD of rotationally symmetric and planar networks. More recently, Javaid et al. evaluated the sharp extremal values of LFMD of connected networks [24]. Results regarding the LFMD of cycle related networks can be found in [25]. In this article, we have considered a family of network bearing rotational symmetry known by circular ladders consisting of different triangular, quadrangular, and pentagonal faces. This article propels in the following manners: Section 1 is introduction, Section 2 covers the role of MD in Robotics and Chemistry, and Section 3 is of preliminaries. In Section 4, we will discuss the idea behind the creation of the circular ladders under consideration. Section 5, contains the main results, and Section 6 closes this article with some conclusions and future directions.

2. Applications

2.1. Robotics. The ever increasing demand of networking enlarged the study of distance-based dimensions. All such tools are used to allocate an appropriate region to an interpolator for its effective usage [6, 26]. The placement of robots at different sites such as restaurants, production units, and public health facilities come under study by using one such tool in [27]. The effective rectification of example and picture handling and handling of informational structures using the aforementioned parameters has been given in [28]. The parameter like these has not only made it possible to serve restaurants, factories, and public service areas with a minimum number of robots but also truncated the delay in their serving. To find more applications on this topic, we refer to [29, 30].

2.2. Chemistry. A chemical compound in graph theoretic form is regarded as a molecular graph having nodes as atoms and links between them as bonds between them [4]. In this picturesque form of a compound along with distance-based parameters give chemists an opportunity to remove discrepancies in a chemical structure and to find the sites showing similar properties in them. All such strategies can be found in [5, 6, 28, 31].

3. Preliminaries

A vertex $c \in \aleph$ is said to resolve a pair $\{a, b\}$ in \aleph if $d(a, c) \neq d(b, c)$. For $a, b \in V(\aleph)$, then the resolving neighbourhood (RN) of the pair $\{a, b\}$ is given by $R\{a, b\} = \{c \in V(\aleph) | d(a, c) \neq d(b, c)\}$.

Assume a connected network $\aleph(V(\aleph), E(\aleph))$ bearing v as its order. A function $\kappa: V(\aleph) \rightarrow [0, 1]$ is called resolving function (RF) of \aleph if each RN set $R\{a, b\}$, $\kappa(R\{a, b\}) = \sum_{c \in R\{a, b\}} \kappa(c) \geq 1$. An RF of \aleph is known as minimal RF if any function $\phi: V(\aleph) \rightarrow [0, 1]$ such that $\phi \leq \kappa$ and $\phi(c) \neq \kappa(c)$ for at least one $c \in V(\aleph)$, which is not an RF of \aleph . The FMD of the network \aleph is given by $\dim_f(\aleph) = \min\{|\kappa|: \kappa \text{ is minimal RF of } \aleph\}$, where $|\kappa| = \sum_{c \in V(\aleph)} \kappa(c)$ [14].

4. Construction of Networks

Here, we give the construction of networks covered in this article. A cycle C_3 is a network bearing the regularity of 2, 3 vertices, and 2 faces known as triangle. On the other hand, C_4 is a network having the regularity of 2 with 4 vertices and 2 faces. A 3-faced quadrangle is obtained after joining any pair of nonadjacent vertices of it. Figure 1(a) shows the planar quadrangles having 2 and 3 faces. A network (C_5) bearing the regularity of 2, 5 nodes, and 2 faces is known as pentagon. The 3-faced pentagons are formed by joining any two nonadjacent nodes of 2-faced pentagon, and 4-faced pentagons are formed by forming links between any two pairs of nonadjacent nodes of 3-faced pentagon. Figure 1(b) illustrates some of the possible 2-faced, 3-faced, and 4-faced planar pentagons.

4.1. Triangular Circular Ladder. The triangular circular ladder (TCL) $\aleph \cong \mathbb{T}_m$ is created by joining each edge $a_1^1 a_2^1, a_2^1 a_3^1, \dots, a_{m-1}^1 a_m^1, a_m^1 a_1^1$ of a cycle C_m with the corresponding nodes $a_1^2, a_2^2, \dots, a_m^2$, where $m \equiv 0 \pmod{2}$. Thus, each step of TCL consists of 2-faced triangle and its vertex and edge sets are as follows: $V(\aleph) = \{a_r^1, a_r^2 | 1 \leq r \leq m\}$, $E(\aleph) = \{a_r^1 a_{r+1}^1 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1\} \cup \{a_r^1 a_r^2, a_r^2 a_{r-1}^2 | 1 \leq r \leq m \wedge a_0^2 = a_m^2\}$ with $v = 2m$ and $e = 3m$ respectively. Figure 2 shows TCL.

4.2. Quadrangular Circular Ladders. For $m \geq 3$, the quadrangular circular ladder (QCL) \mathbb{Q}_m^1 (prism) is a cubic network which is the resultant of the Cartesian product $P_2 \times C_m$, with $v = 2m$ and $e = 3m$. Its sets of vertices and edges are, respectively, given by the following: $V(\mathbb{Q}_m^1) = \{a_r^1, a_r^2 | 1 \leq r \leq m\}$, $E(\mathbb{Q}_m^1) = \{a_r^1 a_{r+1}^1 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1\} \cup \{a_r^1 a_r^2, a_r^2 a_{r+1}^2 | 1 \leq r \leq m \wedge a_{m+1}^2 = a_1^2\}$.

The two possible QCLs $\aleph \cong \mathbb{Q}_m^2$ and $\aleph \cong \mathbb{Q}_m^3$ having each step of 3-faced pentagons are formed by forming edges either $a_r^1 a_{r-1}^2$ or $a_{r-1}^1 a_r^2$ in quadrangular circular ladder \mathbb{Q}_m^1 . The sets of vertices and edges of these networks are, respectively, given as follows: $V(\mathbb{Q}_m^2) = \{a_r^1, a_r^2 | 1 \leq r \leq m\}$ and $E(\mathbb{Q}_m^2) = \{a_r^1 a_{r+1}^1, a_r^1 a_r^2, a_r^2 a_{r+1}^2 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1, a_{m+1}^2 = a_1^2\} \cup \{a_r^1 a_{r-1}^2 | 1 \leq r \leq m \wedge a_0^2 = a_m^2\}$ and $V(\mathbb{Q}_m^3) = \{a_r^1, a_r^2 | 1 \leq r \leq m\}$ and $E(\mathbb{Q}_m^3) = \{a_r^1 a_{r+1}^1, a_r^1 a_r^2, a_r^2 a_{r+1}^2 | 1 \leq r \leq m \wedge$

$a_{m+1}^1 = a_1^1, a_{m+1}^2 = a_1^2 \cup \{a_r^1 a_{r+1}^2 | 1 \leq r \leq m \wedge a_{m+1}^2 = a_1^2\}$ with $v = 2m$ and $e = 4m$.

Figure 3 illustrates $\mathbb{Q}_m^1, \mathbb{Q}_m^2$, and \mathbb{Q}_m^3 . It can be observed from Figure 3 that $\mathbb{Q}^3 \cong \mathbb{Q}_m^2$.

4.3. Pentagonal Circular Ladders. The each step of a pentagonal circular ladder (PCL) \mathbb{P}_m^1 consists of 2-faced pentagon, and it is created by inserting a new vertex a_r^3 between the vertices a_r^2 and a_{r+1}^2 of \mathbb{Q}_m^1 , where $V(\mathbb{P}_m^1)$ and $E(\mathbb{P}_m^1)$ are given by $V(\mathbb{P}_m^1) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$ and $E(\mathbb{P}_m^1) = \{a_r^1 a_{r+1}^1 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r+1}^2 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$ with $v = 3m$ and $e = 4m$, respectively.

Figure 4 illustrates \mathbb{P}_m^1 .

The three possible PCLs $\mathbb{N} \cong \mathbb{P}_m^2, \mathbb{P}_m^3, \mathbb{P}_m^4$, each step of which is having 3-faced pentagons are formed by creating edges either $a_r^2 a_{r+1}^2, a_r^1 a_{r+1}^2$, or $a_r^1 a_{r-1}^2$ (shown by magenta colour in Figure 5) in pentagonal circular ladder \mathbb{P}_m^1 . Figure 5 illustrates $\mathbb{P}_m^2, \mathbb{P}_m^3$, and \mathbb{P}_m^4 .

As seen from Figure 5, $\mathbb{P}_m^4 \cong \mathbb{P}_m^3$. The sets $V(\mathbb{P}_m^2)$ and $E(\mathbb{P}_m^2)$ are given as follows: $V(\mathbb{P}_m^2) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$ and $E(\mathbb{P}_m^2) = \{a_r^1 a_{r+1}^1, a_r^2 a_{r+1}^2 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1, a_{m+1}^2 = a_1^2\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r+1}^2 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$, $V(\mathbb{P}_m^3) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$ and $E(\mathbb{P}_m^3) = \{a_r^1 a_{r+1}^1, a_r^2 a_{r-1}^2 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1, a_0^2 = a_m^2\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r-1}^2 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$, $V(\mathbb{P}_m^4) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$ and $E(\mathbb{P}_m^4) = \{a_r^1 a_{r+1}^1, a_{r-1}^1 a_r^2 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1, a_0^1 = a_m^1\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r-1}^2 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$ with $v = 3m$ and $e = 5m$.

The three possible PCLs $\mathbb{N} \cong \mathbb{P}_m^5, \mathbb{P}_m^6$ and \mathbb{P}_m^7 whose each step is having 4-faced pentagons are created by forming edge $a_r^1 a_{r-1}^2$ (shown by cyan colour in Figure 6) in \mathbb{P}_m^2 , edge $a_r^1 a_{r-1}^3$ in \mathbb{P}_m^2 , edges $a_{r-1}^1 a_r^2$ in \mathbb{P}_m^2 , or edges $a_r^1 a_r^2$ and $a_r^1 a_{r-1}^2$ in the plane network \mathbb{P}_m^1 . Figure 6(a) illustrates \mathbb{P}_m^5 , 6(b) illustrates \mathbb{P}_m^6 , and 6(c) illustrates \mathbb{P}_m^7 , respectively.

It can be seen from Figure 6 that $\mathbb{P}_m^6 \cong \mathbb{P}_m^5$.

The sets of vertices and edges of these networks are $V(\mathbb{P}_m^5) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$ $E(\mathbb{P}_m^5) = \{a_r^1 a_{r+1}^1, a_r^2 a_{r+1}^2, a_r^1 a_{r-1}^2 | 1 \leq r \leq m \wedge a_{m+1}^1 = a_1^1, a_{m+1}^2 = a_1^2\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r+1}^2 | 1 \leq r \leq m\}$, $V(\mathbb{P}_m^6) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$, $E(\mathbb{P}_m^6) = \{a_r^1 a_{r+1}^1, a_r^2 a_{r+1}^2, a_r^1 a_{r-1}^2 | 1 \leq r \leq m\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r-1}^2 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$, $V(\mathbb{P}_m^7) = \{a_r^1, a_r^2, a_r^3 | 1 \leq r \leq m\}$, $E(\mathbb{P}_m^7) = \{a_r^1 a_{r+1}^1, a_r^1 a_r^2, a_r^1 a_{r-1}^2 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\} \cup \{a_r^1 a_r^2, a_r^2 a_r^3, a_r^2 a_{r-1}^2 | 1 \leq r \leq m \wedge a_0^3 = a_m^3\}$ with $v = 3m$ and $e = 6m$, respectively.

5. Main Results

5.1. Triangular Circular Ladder

Lemma 1. Let $\mathbb{N} \cong \mathbb{T}_m$ be TCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq r, l \leq m$, $p \geq 1$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$, we have

- $|R_l| = |R\{a_l^1, a_l^2\}| = |R_r| = |R\{a_r^1, a_{r-1}^2\}| = m + 1$ and $|\cup_{l=1}^m R_l| \cup |\cup_{r=1}^m R_r| = 2m$, where $a_0^2 = a_m^2$.
- $|R_r| < |R_u| 2(m-1)$ for $1 \leq u \leq 8$, where $\bar{R}_1 = R\{a_r^1, a_{r+p}^2\}$, $\bar{R}_2 = R\{a_r^1, a_{r+s}^2\}$, $\bar{R}_3 = R\{a_r^2, a_{r+s}^2\}$, $\bar{R}_4 = R\{a_r^2, a_{r+p}^2\}$, $\bar{R}_5 = R\{a_r^1, a_{r+p}^2\}$, $\bar{R}_6 = R\{a_r^1, a_{r+s}^2\}$, $\bar{R}_7 =$

$R\{a_r^1, a_{r-p}^2\}$, $\bar{R}_8 = R\{a_r^1, a_{r-s}^2\}$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$ with $a_{m+1}^1 = a_1^1$.

Proof

- The RNs of $\{a_r^1, a_r^2\}$ and $\{a_r^1, a_{r-1}^2\}$ are $R_l = R\{a_l^1, a_l^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv l+1, l+2, \dots, l+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv l+1, l+2, \dots, l+(m/2)-1 \pmod{m}\}$ and $R_r = R\{a_r^1, a_{r-1}^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2)+1 \pmod{m}\} \cup \{a_h^2 | h \equiv r-2, r-3, \dots, r-(m/2) \pmod{m}\}$. We note that $\cup_{l=1}^m R_l = V(\mathbb{T}_m)$, $\cup_{r=1}^m R_r = V(\mathbb{T}_m)$ and $|\cup_{l=1}^m R_l| \cup |\cup_{r=1}^m R_r| = 2m$.
- The RNs of \bar{R}_u for $1 \leq u \leq 8$ are $\bar{R}_1 = R\{a_r^1, a_{r+p}^2\} = V(\mathbb{T}_m) - \{a_h^2 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2) \pmod{m}\} = \bar{R}_4 = R\{a_r^2, a_{r+p}^2\}$, $\bar{R}_2 = R\{a_r^1, a_{r+s}^2\} = V(\mathbb{T}_m) - \{a_h^2 | h \equiv r + (s/2), r + ((m+s)/2) \pmod{m}\} = \bar{R}_3 = R\{a_r^2, a_{r+s}^2\}$, $\bar{R}_5 = R\{a_r^1, a_{r+p}^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m+p-1)/2) \pmod{m}\}$, $\bar{R}_6 = R\{a_r^1, a_{r+s}^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + (s/2) \pmod{m}\}$, $\bar{R}_7 = R\{a_r^1, a_{r-p}^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv r - ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m-p-1)/2) \pmod{m}\}$ and $\bar{R}_8 = R\{a_r^1, a_{r-s}^2\} = V(\mathbb{T}_m) - \{a_h^1 | h \equiv r - (s/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m+s-2)/2) \pmod{m}\}$, respectively.

Clearly, $|\bar{R}_u| = 2(m-1)$. Since $|R_r| = m+1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2(m-1) \geq |R_r|$. \square

Theorem 1. If $\mathbb{N} \cong \mathbb{T}_m$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{T}_m) \leq (2m/(m+1))$.

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

We have seen from the above that Table 1 shows the RNs with maximum cardinality that is 10 and Table 2 shows the RNs with minimum cardinality that is 7, which is less than the cardinalities of all other RNs \bar{R}_r for $1 \leq t \leq 6$ that is 10. Moreover, $\cup_{r=1}^{12} R_r = V(\mathbb{T}_6)$ which implies $|\cup_{r=1}^{12} R_r| = 12$ and $|\bar{R}_u \cap \cup_{r=1}^{12} R_r| \geq |R_r|$, where $1 \leq u \leq 24$.

Now, we define a function $\kappa: V(\mathbb{T}_6) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = (1/7)$. As we can see, R_r for $1 \leq r \leq 12$ of \mathbb{T}_6 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{T}_6 such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{T}_6) \leq \sum_{r=1}^{12} (1/7) = (12/7)$.

Case 2. $m \geq 8$.

According to Lemma 1 the RNs with minimum cardinality are $R\{a_r^1, a_r^2\}$ and $R\{a_r^1, a_{r-1}^2\}$ having the cardinality of $m+1$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$, where $1 \leq u \leq 8$.

Let $\eta = |\cup_{r=1}^m R_r| = 2m$ and $\lambda = |R_r| = m+1$. Now, we define a function $\kappa: V(\mathbb{T}_m) \rightarrow [0, 1]$ such that

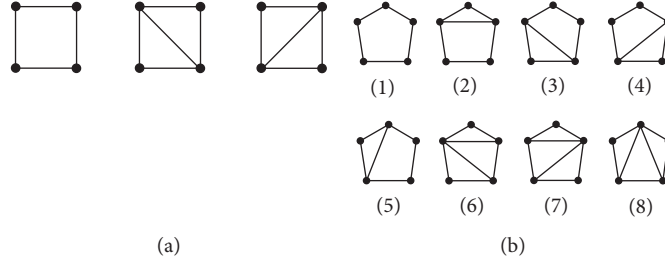
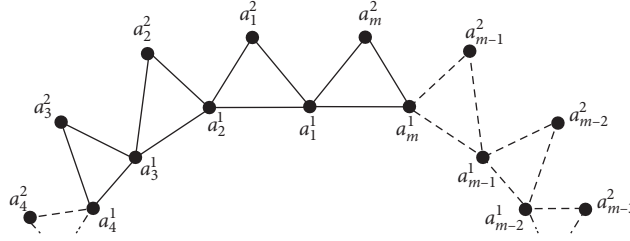
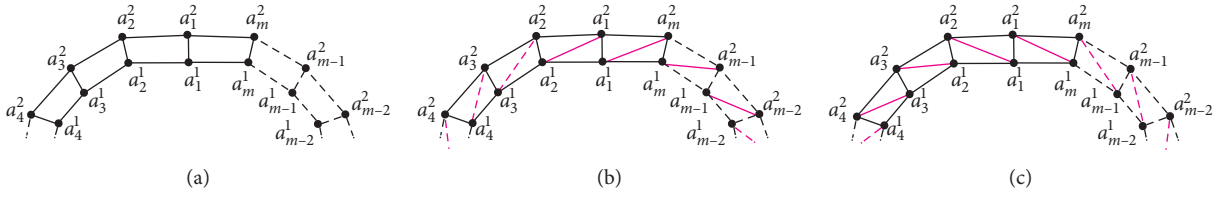
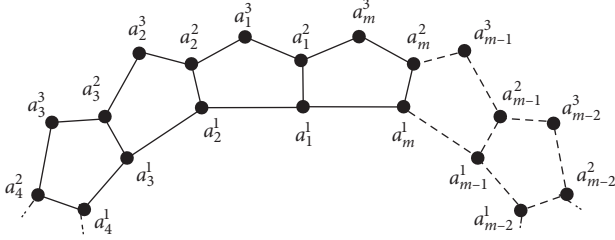


FIGURE 1: Quadrangles (a) and pentagons (b).

FIGURE 2: TCL \mathbb{T}_m .FIGURE 3: QCLs: (a) \mathbb{Q}_m^1 , (b) \mathbb{Q}_m^2 , and (c) \mathbb{Q}_m^3 .FIGURE 4: \mathbb{P}_m^1 .

$\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{T}_m) - \cup_{r=1}^m R_r. \end{cases}$ We can see that κ is a resolving function for \mathbb{T}_m because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(\mathbb{N})$. Suppose that there is another resolving function ρ , such that $\rho(x) \leq \kappa(x)$, for at least one $x \in V(\mathbb{T}_m)$ $\rho(x) \neq \kappa(x)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of \mathbb{T}_m having the minimum cardinality of λ . This shows that ρ is not a resolving function which is contradiction. Hence, κ is a minimal resolving function that achieves minimum $|\kappa|$ for \mathbb{T}_m . As a matter of fact R_r for $1 \leq r \leq m$ of \mathbb{T}_m are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{T}_m such that $|\bar{\kappa}| \leq |\kappa|$. As a result, we arrive at the following: $\dim_f(\mathbb{T}_m) \leq \sum_{r=1}^m (1/\lambda) = (2m/(m+1))$.

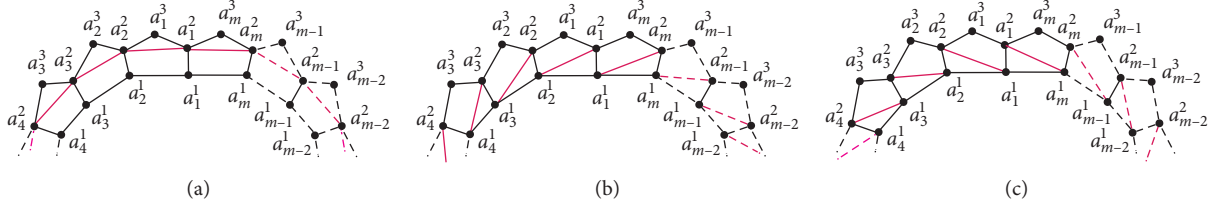
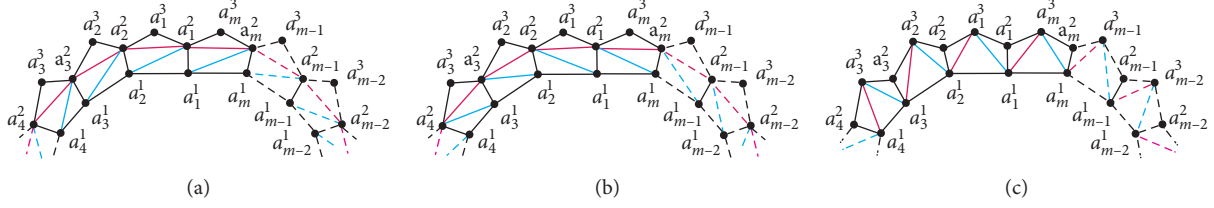
5.2. Quadrangular Circular Ladder

Lemma 2. Let $\mathbb{N} \cong \mathbb{Q}_m^1$ be the 2-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- (a) $|R_l| = |R\{a_l^1, a_{l+1}^2\}| = |R_r| = |R\{a_r^1, a_{r-1}^2\}| = m$ and $|\cup_{l=1}^m R_l| \cup |\cup_{r=1}^m R_r| = 2m$.
- (b) For $p \geq 3$, $s \geq 2$, $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_2| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_3| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_4| = |R\{a_r^1, a_{r-p}^2\}| = 2(m-2)$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2m$, where $1 \leq u \leq 4$.
- (c) For $p \geq 3$, $s \geq 0$, $|R_r| < |\bar{R}_5| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_6| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_7| = |R\{a_r^1, a_{r+s}^2\}| = |\bar{R}_8| = |R\{a_r^1, a_{r-s}^2\}| = 2m$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2m$, where $5 \leq u \leq 8$.

Proof

- (a) The RNs of a_r^1, a_{r+1}^2 and a_r^1, a_{r-1}^2 are given by $R_l = R\{a_l^1, a_{l+1}^2\} = V(\mathbb{Q}_m^1) - \{a_h^1 | h \equiv l+1, l+2, \dots, l+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv l, l-1, \dots, l-(m/2)+1 \pmod{m}\}$ and $R_r = R\{a_r^1, a_{r-1}^2\} = V(\mathbb{Q}_m^1) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r, r-1, \dots, r+(m/2)-1 \pmod{m}\}$, respectively. We note

FIGURE 5: PCLs: (a) \mathbb{P}_m^2 , (b) \mathbb{P}_m^3 , and (c) \mathbb{P}_m^4 .FIGURE 6: PCLs: (a) \mathbb{P}_m^5 , (b) \mathbb{P}_m^6 , and (c) \mathbb{P}_m^7 .TABLE 1: \bar{R}_u for $1 \leq u \leq 49$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_4^2\}$	$R\{a_4^1, a_5^1\}, R\{a_1^2, a_2^2\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{T}_6) - \{a_2^2, a_5^2\}$	$R\{a_5^1, a_6^1\}, R\{a_2^2, a_3^2\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{T}_6) - \{a_3^2, a_6^2\}$	$R\{a_1^1, a_6^1\}, R\{a_3^2, a_4^2\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_5^1\}$	$R\{a_1^2, a_3^2\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_6^1\}$	$R\{a_2^2, a_4^2\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_4^1\}$	$R\{a_3^2, a_5^2\}$
$R\{a_4^1, a_6^1\}$	$V(\mathbb{T}_6) - \{a_2^2, a_5^1\}$	$R\{a_4^2, a_6^2\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_6^1\}$	$R\{a_1^2, a_5^2\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{T}_6) - \{a_2^2, a_6^2\}$	$R\{a_1^2, a_6^2\}$
$R\{a_2^1, a_5^1\}$	$V(\mathbb{T}_6) - \{a_3^2, a_6^2\}$	$R\{a_2^2, a_5^2\}$
$R\{a_3^1, a_6^1\}$	$V(\mathbb{T}_6) - \{a_1^2, a_4^1\}$	$R\{a_3^2, a_6^2\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{T}_6) - \{a_4^2, a_6^2\}$	$R\{a_2^2, a_6^2\}$
$R\{a_1^1, a_2^2\}$	$V(\mathbb{T}_6) - \{a_1^1\} \cup \{a_4^2\}$	
$R\{a_2^1, a_3^2\}$	$V(\mathbb{T}_6) - \{a_3^1\} \cup \{a_5^2\}$	$R\{a_1^1, a_4^2\}$
$R\{a_3^1, a_4^2\}$	$V(\mathbb{T}_6) - \{a_4^1\} \cup \{a_6^2\}$	$R\{a_2^1, a_5^2\}$
$R\{a_4^1, a_5^2\}$	$V(\mathbb{T}_6) - \{a_5^1\} \cup \{a_2^2\}$	$R\{a_3^2, a_6^2\}$
$R\{a_5^1, a_6^2\}$	$V(\mathbb{T}_6) - \{a_6^1\} \cup \{a_3^2\}$	$R\{a_5^2, a_6^2\}$
$R\{a_6^1, a_1^2\}$	$V(\mathbb{T}_6) - \{a_1^1\} \cup \{a_4^2\}$	$R\{a_3^2, a_6^2\}$
$R\{a_1^1, a_2^2\}$	$V(\mathbb{T}_6) - \{a_5^1\} \cup \{a_2^2\}$	
$R\{a_2^1, a_3^2\}$	$V(\mathbb{T}_6) - \{a_6^1\} \cup \{a_3^2\}$	$R\{a_1^1, a_4^2\}$
$R\{a_3^1, a_4^2\}$	$V(\mathbb{T}_6) - \{a_2^1\} \cup \{a_5^2\}$	$R\{a_2^1, a_5^2\}$
$R\{a_4^1, a_5^2\}$	$V(\mathbb{T}_6) - \{a_3^1\} \cup \{a_6^2\}$	$R\{a_3^2, a_6^2\}$
$R\{a_1^1, a_2^2\}$	$V(\mathbb{T}_6) - \{a_4^1\} \cup \{a_2^2\}$	$R\{a_2^2, a_5^2\}$
$R\{a_2^1, a_3^2\}$	$V(\mathbb{T}_6) - \{a_5^1\} \cup \{a_3^2\}$	$R\{a_3^2, a_6^2\}$
$R\{a_3^1, a_4^2\}$	$V(\mathbb{T}_6) - \{a_6^1\} \cup \{a_4^2\}$	$R\{a_4^2, a_6^2\}$
$R\{a_4^1, a_5^2\}$	$V(\mathbb{T}_6) - \{a_1^1\} \cup \{a_5^2\}$	
$R\{a_5^1, a_6^2\}$	$V(\mathbb{T}_6) - \{a_2^1\} \cup \{a_6^2\}$	
$R\{a_6^1, a_1^2\}$	$V(\mathbb{T}_6) - \{a_3^1\} \cup \{a_2^2\}$	
$R\{a_1^1, a_2^2\}$	$V(\mathbb{T}_6) - \{a_4^1\} \cup \{a_3^2\}$	
$R\{a_2^1, a_3^2\}$	$V(\mathbb{T}_6) - \{a_5^1\} \cup \{a_4^2\}$	
$R\{a_3^1, a_4^2\}$	$V(\mathbb{T}_6) - \{a_6^1\} \cup \{a_5^2\}$	
$R\{a_4^1, a_5^2\}$	$V(\mathbb{T}_6) - \{a_1^1\} \cup \{a_6^2\}$	
$R\{a_5^1, a_6^2\}$	$V(\mathbb{T}_6) - \{a_2^1\} \cup \{a_1^2\}$	
$R\{a_6^1, a_1^2\}$	$V(\mathbb{T}_6) - \{a_3^1\} \cup \{a_2^2\}$	

that $\cup_{l=1}^m R_l = V(\mathbb{Q}_m^1)$, $\cup_{r=1}^m R_r = V(\mathbb{Q}_m^1)$ and $|(\cup_{l=1}^m R_l) \cup (\cup_{r=1}^m R_r)| = 2m$.

- (b) The RNs of \bar{R}_u for $1 \leq u \leq 4$ are as follows: $\bar{R}_1 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{Q}_m^1) - \{a_h^1 | h \equiv r - (s/2), r + (s/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r - (s/2), r + (s/2) \pmod{m}\} = \bar{R}_2 = R\{a_r^2, a_{r+s}^2\}$, $\bar{R}_3 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{Q}_m^1) - \{a_h^1 | h \equiv r + ((p+1)/2), r + ((p+m-1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((p-1)/2), r + ((m-p+1)/2) \pmod{m}\}$ and $\bar{R}_4 = R\{a_r^1, a_{r-p}^1\} = V(\mathbb{Q}_m^1) - \{a_h^1 | h \equiv$

$r+p, r - ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((p+1)/2), r - ((p-1)/2) \pmod{m}\}$ respectively. Clearly, $|\bar{R}_u| = 2(m-2)$. Since $|R_r| = 2(m-2) < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2(m-2) \geq |R_r|$, it can be easily seen that $\cup_{r=1}^m R_r = V(\mathbb{Q}_m^1)$ and $|\cup_{r=1}^m R_r| = 2m$.

- (c) The RNs of $a_r^1, a_{r+p}^1, a_r^2, a_{r+p}^2, a_r^1, a_{r+s}^1$ and a_r^1, a_{r-p}^1 are $\bar{R}_5 = R\{a_r^1, a_{r+p}^1\} = \bar{R}_6 = R\{a_r^2, a_{r+p}^2\} = \bar{R}_7 = R\{a_r^1, a_{r+s}^1\} = \bar{R}_8 = R\{a_r^1, a_{r-p}^1\} = V(\mathbb{G})$. We can see that $|\bar{R}_u| = 2m$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2n \geq |R_r|$. \square

TABLE 2: RNs having minimum cardinality.

RNs	Elements
$R_1 = R\{a_1^1, a_1^2\}$	$V(\mathbb{T}_6) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2, a_3^2\}$
$R_3 = R\{a_3^1, a_3^2\}$	$V(\mathbb{T}_6) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_4^2, a_5^2\}$
$R_5 = R\{a_5^1, a_5^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_1^2, a_2^2\}$
$R_7 = R\{a_7^1, a_7^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_5^2, a_6^2\}$
$R_9 = R\{a_9^1, a_9^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^2, a_2^2\}$
$R_{11} = R\{a_{11}^1, a_{11}^2\}$	$V(\mathbb{T}_6) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_3^2, a_4^2\}$
$R_2 = R\{a_2^1, a_2^2\}$	$V(\mathbb{T}_6) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_3^2, a_4^2\}$
$R_4 = R\{a_4^1, a_4^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_5^2, a_6^2\}$
$R_6 = R\{a_6^1, a_6^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^2, a_2^2\}$
$R_8 = R\{a_8^1, a_8^2\}$	$V(\mathbb{T}_6) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_1^2, a_2^2\}$
$R_{10} = R\{a_{10}^1, a_{10}^2\}$	$V(\mathbb{T}_6) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2, a_3^2\}$
$R_{12} = R\{a_{12}^1, a_{12}^2\}$	$V(\mathbb{T}_6) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_4^2, a_5^2\}$

Theorem 2. If $\mathbb{N} \cong \mathbb{Q}_m^1$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{Q}_m^1) \leq 2$.

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

We can see that Tables 3 and 4 show the RNs with maximum cardinalities, that is, 12 and 8, respectively, whereas, Table 5 shows the RNs with minimum cardinality, that is, 6, and $\bigcup_{r=1}^{12} R_r = V(\mathbb{Q}_6^1)$ which implies $|\bigcup_{r=1}^{12} R_r| = 12$ and $|\bar{R}_u \cap \bigcup_{r=1}^{12} R_r| \geq |R_r|$, where $1 \leq u \leq 36$.

Here, we define a function $\phi: V(\mathbb{Q}_6^1) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = (1/6)$. Also R_r for $1 \leq r \leq 12$ of \mathbb{Q}_6^1 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{Q}_6^1 such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{Q}_6^1) \leq \sum_{r=1}^{12} (1/6) = 2$.

Case 2. When $m \geq 6$.

We have seen from Lemma 2 that the RNs with minimum cardinality of m are $R\{a_r^1, a_r^2\}$ and $R\{a_r^1, a_{r-1}^2\}$, respectively.

Let $\eta = |\bigcup_{r=1}^m R_r| = 2m$ and $\lambda = |R_r| = 2m$. Then, we

define a function $\kappa: V(\mathbb{Q}_m^1) \rightarrow [0, 1]$ such that $\kappa(c) =$

$$\begin{cases} 1/\lambda & \text{for } c \in \bigcup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{Q}_m^1) - \bigcup_{r=1}^m R_r. \end{cases}$$

We see that κ is a resolving function for \mathbb{Q}_m^1 because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(\mathbb{N})$. Now, suppose on contrary that there is another resolving function ρ , such that $\rho(u) \leq \phi(u)$, for at least one $u \in V(\mathbb{Q}_m^1)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of \mathbb{Q}_m^1 having minimum cardinality λ . This shows that ρ is not a resolving function which is a contradiction to our supposition. Therefore, κ is a minimal resolving function that achieves minimum $|\kappa|$ for \mathbb{Q}_m^1 . Since all the RNs of \mathbb{Q}_m^1 are equal, hence \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{Q}_m^1 such that $|\bar{\kappa}| \leq |\kappa|$. Assigning $(1/\lambda)$ to all the vertices of \mathbb{Q}_m^1 and calculating their sum, we get: $\dim_f(\mathbb{Q}_m^1) \leq \sum_{r=1}^{\eta} (1/m) = 2$. \square

TABLE 3: \bar{R}_u for $1 \leq u \leq 21$.

Resolving neighbourhoods			Set/elements
$R\{a_1^2, a_2^2\}$	$R\{a_4^2, a_5^2\}$	$R\{a_1^2, a_4^2\}$	$V(\mathbb{Q}_6^1)$
$R\{a_2^2, a_3^2\}$	$R\{a_4^2, a_5^2\}$	$R\{a_1^2, a_5^2\}$	$V(\mathbb{Q}_6^1)$
$R\{a_3^2, a_4^2\}$	$R\{a_4^2, a_5^2\}$	$R\{a_2^2, a_5^2\}$	$V(\mathbb{Q}_6^1)$
$R\{a_3^2, a_6^2\}$	$R\{a_3^1, a_4^1\}$	$R\{a_2^1, a_3^1\}$	$V(\mathbb{Q}_6^1)$
$R\{a_1^1, a_3^1\}$	$R\{a_4^1, a_5^1\}$	$R\{a_5^1, a_6^1\}$	$V(\mathbb{Q}_6^1)$
$R\{a_1^1, a_6^1\}$	$R\{a_3^1, a_5^1\}$	$R\{a_2^1, a_4^1\}$	$V(\mathbb{Q}_6^1)$
$R\{a_2^1, a_6^1\}$	$R\{a_4^1, a_5^1\}$	$R\{a_1^1, a_5^1\}$	$V(\mathbb{Q}_6^1)$

Lemma 3. Let $\mathbb{N} \cong \mathbb{Q}_m^2$ be the 3-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then

- (a) $|R_l| = |R\{a_l^1, a_{l-1}^2\}| = |R_r| = |R\{a_r^1, a_r^2\}| = m$ with $a_0^2 = a_m^2$, $|\bigcup_{l=1}^m R_l| = 2m$, $|\bigcup_{r=1}^m R_r| = 2m$ and $|\bigcup_{l=1}^m R_l \cup \bigcup_{r=1}^m R_r| = 2m$.
- (b) $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_2| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_3| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_4| = |R\{a_r^1, a_{r+p}^2\}| = |\bar{R}_5| = |R\{a_r^1, a_{r+s}^2\}| = |\bar{R}_6| = |R\{a_r^1, a_{r-p}^1\}| = |\bar{R}_7| = |R\{a_r^1, a_{r-s}^1\}|$ and $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| \geq |R_r|$ with $a_{m+1}^1 = a_1^1$ and $1 \leq u \leq 8$.

Proof

- (a) The RNs of a_r^1, a_{r-1}^2 and a_r^1, a_r^2 are $R\{a_r^1, a_{r-1}^2\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r, r+1, \dots, r+(m/2)-1 \pmod{m}\}$ and $R\{a_r^1, a_r^2\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\}$ respectively.

We note that $\bigcup_{r=1}^m R_r = V(\mathbb{Q}_m^2)$ and $|\bigcup_{r=1}^m R_r| = 3m$.

- (b) The RNs of $\{a_r^1, a_{r+p}^1\}, \{a_r^1, a_{r+s}^1\}, \{a_r^2, a_{r+s}^2\}, \{a_r^1, a_{r+p}^2\}, \{a_r^1, a_{r+s}^2\}, \{a_r^1, a_{r-p}^1\}$ and $\{a_r^1, a_{r-s}^1\}$ are given by: $\bar{R}_1 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r + ((p-1)/2), r + ((m+p-1)/2) \pmod{m}\}$, $\bar{R}_2 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r + (s/2), r + ((m+s)/2) \pmod{m}\}$, $\bar{R}_3 = R\{a_r^2, a_{r+s}^2\} = V(\mathbb{Q}_m^2) - \{a_h^2 | h \equiv r + (s/2), r + ((m+s)/2) \pmod{m}\}$, $\bar{R}_4 = R\{a_r^1, a_{r+p}^2\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r - ((p-1)/2) \pmod{m}\}$, $\bar{R}_5 = R\{a_r^1, a_{r+s}^2\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m+s)/2) \pmod{m}\}$, $\bar{R}_6 = R\{a_r^1, a_{r-p}^2\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r - ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m+s)/2) \pmod{m}\}$, $\bar{R}_7 = R\{a_r^1, a_{r-s}^2\} = V(\mathbb{Q}_m^2) - \{a_h^1 | h \equiv r - (s/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r - (s/2) \pmod{m}\}$, $\bar{R}_8 = R\{a_r^2, a_{r+p}^2\} = V(\mathbb{Q}_m^2) - \{a_h^2 | h \equiv r + ((p+1)/2), r + ((m+p+1)/2) \pmod{m}\}$. Clearly, $|\bar{R}_u| = 2(m-1)$. Since $|R\{a_r^1, a_{r-1}^2\}| = |R\{a_r^1, a_r^2\}| = n < |R\{a_r^1, a_{r+1}^1\}|$ then $|R\{a_r^1, a_{r+1}^1\} \cap \bigcup_{r=1}^m R_r| \geq |R_r|$. \square

Theorem 3. Let $\mathbb{N} \cong \mathbb{Q}_m^2$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{Q}_m^2) \leq 2$.

TABLE 4: \bar{R}_u for $21 \leq u \leq 36$.

RNs	Elements	Equality
$R\{a_1^2, a_2^2\}$	$V(Q_6^1) - \{a_2^1, a_5^1\} \cup \{a_3^2, a_6^2\}$	$R\{a_1^1, a_3^1\}, R\{a_2^2, a_6^2\}, R\{a_4^1, a_6^1\}$
$R\{a_2^2, a_4^2\}$	$V(Q_6^1) - \{a_3^1, a_6^1\} \cup \{a_2^2, a_5^2\}$	$R\{a_2^1, a_5^1\}, R\{a_1^2, a_5^2\}, R\{a_1^1, a_5^1\}$
$R\{a_3^2, a_5^2\}$	$V(Q_6^1) - \{a_1^1, a_4^1\} \cup \{a_2^2, a_4^2\}$	$R\{a_3^1, a_5^1\}, R\{a_2^2, a_6^2\}, R\{a_2^1, a_6^1\}$
$R\{a_1^1, a_4^2\}$	$V(Q_6^1) - \{a_3^1, a_5^1\} \cup \{a_2^2, a_6^2\}$	
$R\{a_2^1, a_5^2\}$	$V(Q_6^1) - \{a_4^1, a_6^1\} \cup \{a_1^2, a_3^2\}$	
$R\{a_3^1, a_6^2\}$	$V(Q_6^1) - \{a_1^1, a_5^1\} \cup \{a_2^2, a_4^2\}$	

TABLE 5: RNs with minimum cardinality.

RNs	Elements
$R_1 = R\{a_1^1, a_2^2\}$	$V(Q_6^1) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2, a_5^2, a_6^2\}$
$R_3 = R\{a_3^1, a_4^2\}$	$V(Q_6^1) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_1^2, a_2^2, a_3^2\}$
$R_5 = R\{a_5^1, a_6^2\}$	$V(Q_6^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2, a_4^2, a_5^2\}$
$R_7 = R\{a_2^1, a_1^2\}$	$V(Q_6^1) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_9 = R\{a_4^1, a_3^2\}$	$V(Q_6^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2, a_3^2, a_5^2\}$
$R_{11} = R\{a_6^1, a_5^2\}$	$V(Q_6^1) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_1^2, a_2^2, a_6^2\}$
$R_2 = R\{a_2^1, a_3^2\}$	$V(Q_6^1) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_1^2, a_2^2, a_6^2\}$
$R_4 = R\{a_4^1, a_5^2\}$	$V(Q_6^1) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_6 = R\{a_6^1, a_5^2\}$	$V(Q_6^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2, a_4^2, a_5^2\}$
$R_8 = R\{a_3^1, a_2^2\}$	$V(Q_6^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2, a_4^2, a_5^2\}$
$R_{10} = R\{a_5^1, a_4^2\}$	$V(Q_6^1) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_1^2, a_5^2, a_6^2\}$
$R_{12} = R\{a_1^1, a_6^2\}$	$V(Q_6^1) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_1^2, a_2^2, a_3^2\}$

TABLE 6: \bar{R}_u for $1 \leq u \leq 52$.

RNs	Elements	Equality
$R\{a_1^2, a_2^2\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_2^2, a_6^2\}$
$R\{a_2^2, a_3^2\}$	$V(Q_6^2) - \{a_3^1, a_6^1\}$	$R\{a_4^2, a_5^2\}$
$R\{a_3^2, a_4^2\}$	$V(Q_6^2) - \{a_1^1, a_4^1\}$	$R\{a_2^2, a_6^2\}$
$R\{a_1^2, a_5^2\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_4^2, a_5^2\}$
$R\{a_2^2, a_6^2\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_2^2, a_6^2\}$
$R\{a_3^2, a_5^2\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_2^2, a_6^2\}$
$R\{a_1^2, a_4^2\}$	$V(Q_6^2) - \{a_3^1, a_6^1\}$	
$R\{a_2^2, a_5^2\}$	$V(Q_6^2) - \{a_1^1, a_4^1\}$	
$R\{a_3^2, a_6^2\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	
$R\{a_1^1, a_2^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_4^1, a_5^1\}$
$R\{a_2^1, a_3^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_5^1, a_6^1\}$
$R\{a_3^1, a_4^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	$R\{a_1^1, a_6^1\}$
$R\{a_1^1, a_3^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\} \cup \{a_2^2, a_5^2\}$	$R\{a_2^1, a_3^1\}$
$R\{a_2^1, a_4^1\}$	$V(Q_6^2) - \{a_3^1, a_6^1\} \cup \{a_2^2, a_6^2\}$	$R\{a_2^1, a_4^1\}$
$R\{a_3^1, a_5^1\}$	$V(Q_6^2) - \{a_1^1, a_4^1\} \cup \{a_2^2, a_4^2\}$	$R\{a_3^1, a_5^1\}$
$R\{a_4^1, a_6^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\} \cup \{a_2^2, a_6^2\}$	$R\{a_4^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(Q_6^2) - \{a_3^1, a_6^1\} \cup \{a_2^2, a_6^2\}$	$R\{a_1^1, a_5^1\}$
$R\{a_2^1, a_6^1\}$	$V(Q_6^2) - \{a_1^1, a_4^1\} \cup \{a_2^2, a_4^2\}$	$R\{a_2^1, a_6^1\}$
$R\{a_1^1, a_4^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	
$R\{a_2^1, a_5^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	
$R\{a_3^1, a_6^1\}$	$V(Q_6^2) - \{a_2^1, a_5^1\}$	
$R\{a_1^1, a_2^1\}$	$V(Q_6^2) - \{a_2^1\} \cup \{a_2^2\}$	
$R\{a_2^1, a_3^1\}$	$V(Q_6^2) - \{a_2^1\} \cup \{a_2^2\}$	
$R\{a_3^1, a_4^1\}$	$V(Q_6^2) - \{a_4^1\} \cup \{a_4^2\}$	
$R\{a_4^1, a_5^1\}$	$V(Q_6^2) - \{a_5^1\} \cup \{a_5^2\}$	
$R\{a_5^1, a_6^1\}$	$V(Q_6^2) - \{a_6^1\} \cup \{a_6^2\}$	
$R\{a_6^1, a_1^1\}$	$V(Q_6^2) - \{a_1^1\} \cup \{a_1^2\}$	
$R\{a_1^1, a_2^1\}$	$V(Q_6^2) - \{a_2^1\} \cup \{a_2^2\}$	
$R\{a_2^1, a_3^1\}$	$V(Q_6^2) - \{a_3^1\} \cup \{a_3^2\}$	
$R\{a_3^1, a_4^1\}$	$V(Q_6^2) - \{a_4^1\} \cup \{a_4^2\}$	
$R\{a_4^1, a_5^1\}$	$V(Q_6^2) - \{a_5^1\} \cup \{a_5^2\}$	
$R\{a_5^1, a_6^1\}$	$V(Q_6^2) - \{a_6^1\} \cup \{a_6^2\}$	
$R\{a_1^1, a_6^1\}$	$V(Q_6^2) - \{a_1^1\} \cup \{a_1^2\}$	
$R\{a_2^1, a_5^1\}$	$V(Q_6^2) - \{a_2^1\} \cup \{a_2^2\}$	
$R\{a_3^1, a_4^1\}$	$V(Q_6^2) - \{a_3^1\} \cup \{a_3^2\}$	
$R\{a_4^1, a_5^1\}$	$V(Q_6^2) - \{a_4^1\} \cup \{a_4^2\}$	
$R\{a_5^1, a_6^1\}$	$V(Q_6^2) - \{a_5^1\} \cup \{a_5^2\}$	
$R\{a_1^1, a_2^1\}$	$V(Q_6^2) - \{a_1^1\} \cup \{a_1^2\}$	
$R\{a_2^1, a_3^1\}$	$V(Q_6^2) - \{a_2^1\} \cup \{a_2^2\}$	
$R\{a_3^1, a_4^1\}$	$V(Q_6^2) - \{a_3^1\} \cup \{a_3^2\}$	
$R\{a_4^1, a_5^1\}$	$V(Q_6^2) - \{a_4^1\} \cup \{a_4^2\}$	
$R\{a_5^1, a_6^1\}$	$V(Q_6^2) - \{a_5^1\} \cup \{a_5^2\}$	
$R\{a_1^1, a_6^1\}$	$V(Q_6^2) - \{a_1^1\} \cup \{a_1^2\}$	

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

Tables 6 and 7 given above shows the RNs with cardinalities of 10 and 6 respectively. Clearly, the table shows the RNs bearing the minimum cardinality of 6 and $\cup_{r=1}^{12} R_r = V(Q_6^2)$ which implies $|\cup_{r=1}^{12} R_r| = 12$ and $|\bar{R}_u \cap \cup_{r=1}^{12} R_r| \geq |R_r|$, where $1 \leq u \leq 52$.

Now, we define a function $\kappa: V(Q_6^2) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = (1/6)$. Moreover, R_r for $1 \leq r \leq 12$ of Q_6^2 are pairwise overlapping; thus, \exists is another minimal resolving function $\bar{\kappa}$ of Q_6^2 such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(Q_6^2) \leq \sum_{r=1}^{12} (1/6) = 2$.

Case 2. $m \geq 8$

It is clear from Lemma 3 that the RNs bearing the minimum cardinality of m in Q_m^2 are $R_l = R\{a_l^1, a_l^2\}$ and $R_r = R\{a_r^1, a_{r-1}^2\}$ respectively, with $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 2n \geq |R_r|$.

Let $\eta = |\cup_{r=1}^m R_r| = 2m$ and $\lambda = |R_r| = m$. Then we define a mapping $\kappa: V(Q_m^2) \rightarrow [0, 1]$ such that $\kappa(c) =$

$$\begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(Q_m^2) - \cup_{r=1}^m R_r. \end{cases}$$

We see that κ is a resolving function for Q_m^1 because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(N)$. Now suppose on contrary that there is another resolving function ρ , such that $\rho(u) \leq \phi(u)$, for at least one $u \in V(Q_m^1)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is a RN of

Q_m^1 having minimum cardinality λ . This shows that ρ is not a resolving function which is a contradiction to our supposition. Therefore, κ is a minimal resolving function that achieves minimum $|\kappa|$ for Q_m^1 . Since all the RNs of Q_m^2 are equal, hence \exists another minimal resolving function $\bar{\kappa}$ of Q_m^2 such that $|\bar{\kappa}| \leq |\kappa|$. Assigning $(1/\lambda)$ to all the vertices of Q_m^2 and calculating their sum, we get $\dim_f(Q_m^2) \leq \sum_{r=1}^{\eta} (1/m) = 2$. \square

TABLE 7: R_r with minimum cardinalities for $1 \leq r \leq 12$.

RNs	Elements
$R_1 = R\{a_1^1, a_1^2\}$	$V(\mathbb{Q}_6^2) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_3 = R\{a_3^1, a_3^2\}$	$V(\mathbb{Q}_6^2) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_4^2, a_5^2, a_6^2\}$
$R_5 = R\{a_5^1, a_5^2\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_7 = R\{a_2^1, a_2^2\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_9 = R\{a_4^1, a_4^2\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_4^2, a_5^2, a_6^2\}$
$R_{11} = R\{a_6^1, a_6^2\}$	$V(\mathbb{Q}_6^2) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_2 = R\{a_1^1, a_1^2\}$	$V(\mathbb{Q}_6^2) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_4 = R\{a_1^1, a_1^2\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_6 = R\{a_6^1, a_6^2\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_8 = R\{a_3^1, a_3^2\}$	$V(\mathbb{Q}_6^2) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_{10} = R\{a_5^1, a_5^2\}$	$V(\mathbb{Q}_6^2) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2, a_3^2, a_4^2\}$
$R_{12} = R\{a_1^1, a_1^2\}$	$V(\mathbb{Q}_6^2) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2, a_4^2\}$

Corollary 1. Let $\aleph \cong \mathbb{Q}_m^3$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{Q}_m^3) \leq 2$.

Proof. The result is evident as $\mathbb{Q}_m^3 \cong \mathbb{Q}_m^2$. \square

5.3. Pentagonal Circular Ladders

Lemma 4. Let $\aleph \cong \mathbb{P}_m^1$ be the 2-faced QCL with $m \geq 10$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$, $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- $|R_l| = |R\{a_l^1, a_{l-2}^3\}| = |R_r| = |R\{a_r^1, a_{r+1}^3\}| = (5m/2) - 2$ with $a_{m+1}^3 = a_1^3$, $|\cup_{l=1}^m R_l| = |\cup_{r=1}^m R_r| = 3m$ and $|\cup_{l=1}^m R_l \cup \cup_{r=1}^m R_r| = 3m$.
- $|R_r| < |\bar{R}_1| = |R\{a_r^2, a_r^3\}| = |\bar{R}_2| = |R\{a_r^2, a_{r-1}^3\}| = (5m/2)$ with $a_0^3 = a_m^3$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_3| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_4| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_5| = |R\{a_r^2, a_{r+1}^2\}| = |\bar{R}_6| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_7| = |R\{a_r^3, a_{r+s}^3\}| = |\bar{R}_8| = |R\{a_r^1, a_{r+s}^2\}| = 3m - 2$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$ with $a_{m+1}^1 = a_1^1$.
- $|R_r| < |\bar{R}_9| = |R\{a_r^1, a_r^2\}| = |\bar{R}_{10}| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_{11}| = |R\{a_r^3, a_{r+1}^3\}| = |\bar{R}_{12}| = |R\{a_r^3, a_{r+p}^3\}| = |\bar{R}_{13}| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_{14}| = |R\{a_r^1, a_r^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{15}| = |R\{a_r^1, a_{r+1}^2\}| = |\bar{R}_{16}| = |R\{a_r^1, a_{r-1}^2\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.

Proof

- The RNs of $\{a_r^1, a_{r+1}^3\}$ and $\{a_1^1, a_{l-2}^3\}$ are $R_l = R\{a_r^1, a_{r+1}^3\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r+2, r+3, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r, r+4 \pmod{m}\}$ and $R_2 = R\{a_1^1, a_{l-2}^3\} = V(\mathbb{P}_m^1) - V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r+2, r+3, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r, r-5 \pmod{m}\}$, respectively. We can see that $|R_r| = (5m/2) - 2$ and $|\cup_{r=1}^m R_r| = |\cup_{l=1}^m R_l| = V(\mathbb{P}_m^1)$ and $|\cup_{l=1}^m R_l \cup \cup_{r=1}^m R_r| = 3m$.
- The RNs of $\{a_r^2, a_{r-1}^3\}$ and $\{a_r^2, a_r^3\}$ are $\bar{R}_1 = R\{a_r^2, a_{r-1}^3\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\}$ and $\bar{R}_1 = R\{a_r^2, a_r^3\} = V(\aleph) -$

$\{a_h^1 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \pmod{m}$, respectively.

Since for $1 \leq u \leq 2$, $|\bar{R}_u| > |R_r| = (5m/2) - 1$, thus $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m$.

- The RNs of $\{a_r^1, a_{r+1}^1\}$, $\{a_r^1, a_{r+p}^1\}$, $\{a_r^2, a_{r+1}^2\}$, $\{a_r^2, a_{r+p}^2\}$, $\{a_r^3, a_{r+s}^3\}$ and $\{a_r^1, a_{r+s}^2\}$ are $\bar{R}_3 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^1) - \{a_h^3 | h \equiv r, r+(m/2) \pmod{m}\} = \bar{R}_5 = R\{a_r^2, a_{r+1}^2\}$, $\bar{R}_4 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{P}_m^1) - \{a_h^3 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2) \pmod{m}\} = \bar{R}_6 = R\{a_r^2, a_{r+p}^2\}$, $\bar{R}_7 = R\{a_r^3, a_{r+s}^3\} = V(\mathbb{P}_m^1) - \{a_h^3 | h \equiv r + (s/2), r + ((m+s)/2) \pmod{m}\}$ and $\bar{R}_8 = R\{a_r^1, a_{r+s}^2\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r + (s/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r + ((m+s-2)/2) \pmod{m}\}$, respectively.

Clearly, for $3 \leq u \leq 8$, $|\bar{R}_u| = 3m - 2$. Since $|R\{a_r^2, a_r^3\}| = (5m/2) < |R\{a_r^1, a_{r+1}^1\}|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 2 \geq |R_r|$.

- The RN of $\{a_r^1, a_r^2\}$ is $\bar{R}_9 = R\{a_r^1, a_r^2\} = V(\mathbb{P}_m^1) - \{a_h^2 | h \equiv r-1, r+1 \pmod{m}\} \cup \{a_h^3 | h \equiv r-2, r+1 \pmod{m}\}$. $\bar{R}_{10} = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r + (s/2)r + ((m+s)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + (s/2), r + ((m+s)/2) \pmod{m}\}$, $\bar{R}_{11} = R\{a_r^3, a_{r+1}^3\} = \{a_r^1, a_{r+2}^1\}$, $\bar{R}_{12} = R\{a_r^3, a_{r+p}^3\} = \bar{R}_{13} = R\{a_r^2, a_{r+s}^2\} = R\{a_r^1, a_{r+s}^1\}$, $\bar{R}_{14} = R\{a_r^1, a_r^3\} = V(\mathbb{P}_m^1) - \{a_h^2 | h \equiv r, r+2 \pmod{m}\} \cup \{a_h^3 | h \equiv r-1, r+2 \pmod{m}\}$.

Clearly, for $9 \leq u \leq 14$, $|R_r| = (5m/2) < |\bar{R}_u|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.

- The RNs of $\{a_r^1, a_{r+1}^2\}$ and $\{a_r^1, a_{r-1}^2\}$ are $\bar{R}_{15} = R\{a_r^1, a_{r+1}^2\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{5}\} \cup \{a_h^2 | h \equiv r+(m/2) \pmod{m}\}$ and $\bar{R}_{16} = R\{a_r^1, a_{r-1}^2\} = V(\mathbb{P}_m^1) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r-(m/2) \pmod{m}\}$, respectively. We can see that $|R_r| = (5m/2)$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$. \square

Theorem 4. If $\aleph \cong \mathbb{P}_m^1$ with $m \geq 8$ and $m \equiv 0 \pmod{2}$. Then,

$$\dim_f(\mathbb{P}_m^1) \leq \begin{cases} 6m/(5m-2) & \text{for } 6 \leq m \leq 8, \\ 6m/(5m-4) & \text{for } m \geq 10 \end{cases}.$$

Proof

Case 1. (i) When $m = 6$.

The RNs are given as follows.

We can see that Tables 8 and 9 show the RNs with maximum cardinality of 16 and 15, respectively, whereas both Tables 10 and 11 show the RNs with minimum cardinality of 14. Moreover, $|\cup_{r=1}^{45} R_r| = V(\mathbb{P}_6^1)$ which implies $|\cup_{r=1}^{45} R_r| = 18$ and $|\bar{R}_r \cap \cup_{r=1}^{24} R_r| \geq |R_r|$.

Now, we define a mapping $\kappa: V(\mathbb{P}_6^1) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/14)$. It is observed that R_r for $1 \leq r \leq 45$ of \mathbb{P}_6^1 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_6^1

TABLE 8: \bar{R}_u for $1 \leq u \leq 23$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_1^3, a_4^3\}$	$R\{a_1^1, a_5^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_5^1\}, R\{a_1^2, a_6^2\}, R\{a_3^2, a_6^2\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_6^1) - \{a_2^3, a_3^3\}$	$R\{a_5^1, a_6^1\}, R\{a_2^2, a_3^2\}, R\{a_5^1, a_6^1\}, R\{a_1^1, a_4^1\}, R\{a_2^1, a_4^1\}, R\{a_1^1, a_4^1\}, R\{a_2^1, a_4^1\}, R\{a_1^3, a_3^3\}, R\{a_5^2, a_6^2\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_6^1) - \{a_3^3, a_6^3\}$	$R\{a_1^1, a_6^1\}, R\{a_3^2, a_4^2\}, R\{a_2^1, a_6^2\}, R\{a_1^1, a_5^1\}, R\{a_2^1, a_5^1\}, R\{a_1^1, a_6^1\}$

TABLE 9: \bar{R}_u for $24 \leq u \leq 35$.

RNs	Elements
$R\{a_1^2, a_3^3\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_3^1, a_4^1\}$
$R\{a_2^2, a_3^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_4^1, a_5^1\}$
$R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_6^1) - \{a_4^1, a_5^1, a_6^1\}$
$R\{a_4^2, a_4^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_5^1, a_6^1\}$
$R\{a_5^2, a_5^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1, a_6^1\}$
$R\{a_6^2, a_6^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1, a_3^1\}$
$R\{a_2^2, a_1^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_5^1, a_6^1\}$
$R\{a_3^2, a_2^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1, a_6^1\}$
$R\{a_4^2, a_3^3\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1, a_3^1\}$
$R\{a_5^2, a_4^3\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_3^1, a_4^1\}$
$R\{a_6^2, a_5^3\}$	$V(\mathbb{P}_6^1) - \{a_3^1, a_4^1, a_5^1\}$
$R\{a_2^2, a_6^3\}$	$V(\mathbb{P}_6^1) - \{a_4^1, a_5^1, a_6^1\}$

such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_6^1) \leq \sum_{r=1}^{16} (1/14) = (18/14) = (9/7)$.

(ii) When $m = 8$.

The RNs are given as follows.

We can see that Tables 12–14 shows the RNs with maximum cardinalities that is 22 and 20, respectively, whereas Table 15 shows the RNs with minimum cardinality of 19. Moreover, $\cup_{r=1}^{16} R_r = V(\mathbb{P}_8^1)$ which implies $|\cup_{r=1}^{16} R_r| = 19$ and $|\bar{R}_r \cap \cup_{r=1}^{24} R_r| \geq |R_r|$.

Now, we define a mapping $\kappa: V(\mathbb{P}_8^1) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/19)$. It is observed that R_r for $1 \leq r \leq 16$ of \mathbb{P}_8^1 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_8^1 such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_8^1) \leq \sum_{r=1}^{16} (1/19) = (24/19)$.

(iii) For $6 \leq m \leq 8$.

We have seen that $\dim_f(\mathbb{P}_6^1) \leq (9/7)$ whereas $\dim_f(\mathbb{P}_8^1) \leq (24/19)$. In generic form, we can write it as $\dim_f \leq (6m/(5m-2))$.

Case 2. When $m = 10$.

The RNs are given as follows.

We can see that Tables 16–18 show the RNs with maximum cardinalities, that is, 28 and 26 and 25, respectively, whereas Table 19 shows the RNs with minimum cardinality of 23. Moreover, $\cup_{r=1}^{16} R_r = V(\mathbb{P}_{10}^1)$ which implies $|\cup_{r=1}^{16} R_r| = 23$ and $|\bar{R}_r \cap \cup_{r=1}^{24} R_r| \geq |R_r|$.

Now, we define a mapping $\kappa: V(\mathbb{P}_{10}^1) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/23)$. It is observed that R_r for $1 \leq r \leq 16$ of \mathbb{P}_{10}^1 are pairwise

overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_{10}^1 such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_{10}^1) \leq \sum_{r=1}^{23} (1/19) \leq (30/23)$.

Case 3. $m \geq 12$.

From Lemma 4, we can see that the RNs with minimum cardinality of $(5m/2) - 2$ are $R\{a_r^1, a_{r+1}^3\}$ and $R\{a_r^1, a_{r-2}^3\}$, respectively. Moreover, for $1 \leq u \leq 15$, $|R_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.

Let $\eta = |\cup_{r=1}^m R_r| = 3m$ and $\lambda = |R_r| = (5m/2) - 2$. Then, we define a mapping $\kappa: V(\mathbb{P}_m^1) \rightarrow [0, 1]$

such that $\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{P}_m^1) - \cup_{r=1}^m R_r. \end{cases}$

We note that κ is a resolving function for \mathbb{P}_m^1 because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \in V(\mathbb{P}_m^1)$. Now, assume that there is another resolving function ρ , such that $\rho(u) \leq \kappa(u)$, for at least one $u \in V(\mathbb{P}_m^1)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is a RN of \mathbb{P}_m^1 bearing minimum cardinality λ . This shows that ρ is not a resolving function a contradiction. Therefore, κ is a minimal resolving function for \mathbb{P}_m^1 that achieves minimum $|\kappa|$. As a matter of fact R_r for $1 \leq t \leq 2m$ of \mathbb{P}_m^1 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_m^1 such that $|\bar{\kappa}| \leq |\kappa|$. Therefore, assigning $(1/\lambda)$ to all the vertices of \mathbb{P}_m^1 and calculating their sum, we get: $\dim_f(\mathbb{P}_m^1) \leq \sum_{r=1}^{\eta} 1/\lambda = (3m/((5m/2) - 2)) = (6m/(5m-4))$. \square

Lemma 5. Let $\aleph \cong \mathbb{P}_m^2$ be the 3-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- $|R_l| = |R\{a_l^2, a_l^3\}| = |R_r| = |R\{a_r^2, a_{r-1}^3\}| = (3m/2) + 1$, $|\cup_{l=1}^m R_l|$, $|\cup_{r=1}^m R_r| = 3m$ and $|\{(\cup_{l=1}^m R_l) \cup (\cup_{r=1}^m R_r)\}| = 3m$.
- $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_{r+1}^3\}| = |\bar{R}_2| = |R\{a_r^1, a_{r-1}^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_3| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_4| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_5| = |R\{a_r^2, a_{r+1}^2\}| = |\bar{R}_6| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_7| = |R\{a_r^3, a_{r+s}^3\}| = |\bar{R}_8| = |R\{a_r^1, a_{r+s}^2\}| = |\bar{R}_9| = |R\{a_r^1, a_{r+s}^3\}| = |\bar{R}_{10}| = |R\{a_r^1, a_{r+p}^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$ with $a_{m+1}^1 = a_1^1$ and $a_{m+1}^2 = a_1^2$.
- $|R_r| < |\bar{R}_{11}| = |R\{a_r^2, a_{r+p}^3\}| = |\bar{R}_{12}| = |R\{a_r^2, a_{r+s}^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{13}| = |R\{a_r^1, a_r^2\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.

TABLE 10: R_r for $1 \leq r \leq 27$.

RNs	Elements	Equality
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_6^1) - \{a_2^1, a_5^1\} \cup \{a_2^2, a_5^2\}$	$R\{a_1^1, a_6^1\}, R\{a_2^1, a_5^1\}, R\{a_2^2, a_5^2\}, R\{a_1^3, a_3^3\}, R\{a_4^3, a_5^3\}, R\{a_3^3, a_6^3\}$
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_6^1) - \{a_3^1, a_6^1\} \cup \{a_2^2, a_5^2\}$	$R\{a_1^1, a_5^1\}, R\{a_2^1, a_4^1\}, R\{a_2^2, a_5^2\}, R\{a_1^1, a_5^1\}, R\{a_2^1, a_5^1\}, R\{a_3^3, a_3^3\}, R\{a_5^3, a_6^3\}, R\{a_1^3, a_4^3\}, R\{a_3^3, a_6^3\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_4^1\} \cup \{a_2^2, a_5^2\}$	$R\{a_2^1, a_6^1\}, R\{a_2^2, a_5^2\}, R\{a_2^2, a_5^2\}, R\{a_1^1, a_6^1\}, R\{a_2^1, a_5^1\}, R\{a_3^3, a_3^3\}, R\{a_1^3, a_6^3\}, R\{a_2^3, a_5^3\}, R\{a_3^3, a_5^3\}$

TABLE 11: R_r for $28 \leq r \leq 45$.

RNs	Elements
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_6^2\} \cup \{a_2^3, a_5^3\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_6^2\} \cup \{a_3^3, a_4^3\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_6^2\} \cup \{a_3^3, a_6^3\}$
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_3^1, a_4^1\} \cup \{a_3^2, a_5^2\}$
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_6^1) - \{a_5^1, a_6^1\} \cup \{a_3^2, a_5^2\}$
$R\{a_5^1, a_6^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1\} \cup \{a_2^2, a_5^2\}$
$R\{a_2^1, a_6^1\}$	$V(\mathbb{P}_6^1) - \{a_5^1, a_6^1\} \cup \{a_3^3, a_4^3\}$
$R\{a_4^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_2^1\} \cup \{a_3^3, a_6^3\}$
$R\{a_6^1, a_3^1\}$	$V(\mathbb{P}_6^1) - \{a_3^1, a_4^1\} \cup \{a_2^2, a_5^2\}$
$R\{a_2^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_3^2\} \cup \{a_3^3, a_6^3\}$
$R\{a_4^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_5^2\} \cup \{a_3^3, a_5^3\}$
$R\{a_6^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_5^2\} \cup \{a_3^3, a_6^3\}$
$R\{a_2^1, a_2^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_5^1\} \cup \{a_2^2, a_5^2\}$
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_6^1\} \cup \{a_3^3, a_4^3\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_3^2\} \cup \{a_3^3, a_6^3\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_1^1, a_6^1\} \cup \{a_3^3, a_6^3\}$
$R\{a_5^1, a_3^1\}$	$V(\mathbb{P}_6^1) - \{a_2^2, a_3^2\} \cup \{a_3^3, a_4^3\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^1) - \{a_4^1, a_5^1\} \cup \{a_3^3, a_6^3\}$

Proof

- (a) The RNs of $a_r^2 a_r^3$ and $a_r^2 a_{r-1}^3$ are $R\{a_r^2 a_r^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r+1, r+2, \dots, r+(m/2)-1 \pmod{m}\}$ and $R\{a_r^2 a_{r-1}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r-2, r-3, \dots, r-(m/2)+1 \pmod{m}\}$. We note that $\cup_{r=1}^m R_r = V(\mathbb{P}_m^2)$ and $|\cup_{r=1}^m R_r| = 3m$.
- (b) The RNs of $\{a_r^1, a_{r-1}^3\}$ and $\{a_r^1, a_{r-1}^3\}$ are $\bar{R}_1 = R\{a_r^1, a_{r+1}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+2, r+3, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r, r+(m/2) \pmod{m}\}$ and $\bar{R}_2 = R\{a_r^1, a_{r-1}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r-2, r-3, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r \pmod{m}\}$ respectively. It is clear from the above that $|R_r| < |\bar{R}_u|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- (c) The RNs of $\{a_r^1, a_{r+1}^3\}, \{a_r^1, a_{r+p}^3\}, \{a_r^2, a_{r+1}^3\}, \{a_r^2, a_{r+p}^3\}, \{a_r^3, a_{r+s}^3\}, \{a_r^1, a_{r+s}^3\}, \{a_r^1, a_{r+s}^3\}$ and $\{a_r^1, a_{r+p}^3\}$ are $\bar{R}_3 = R\{a_r^1, a_{r+1}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r, r+(m/2) \pmod{m}\} = \bar{R}_5 = R\{a_r^2, a_{r+1}^3\}, \bar{R}_4 = R\{a_r^1, a_{r+p}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+((p-1)/2), r+((p+m-1)/2) \pmod{m}\} = \bar{R}_6 = R\{a_r^2, a_{r+p}^3\}, \bar{R}_7 = R\{a_r^3, a_{r+s}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+(s/2), r+(m+s)/2 \pmod{m}\}, \bar{R}_8 = R\{a_r^1, a_{r+s}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+(s/2), r+(s+m)/2 \pmod{m}\}, \bar{R}_9 = R\{a_r^1, a_{r+s}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+((s+2)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r+((s+m)/2) \pmod{m}\}, \bar{R}_{10} = R\{a_r^1, a_{r+p}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+((p+m+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r+((p-1)/2) \pmod{m}\}. Clearly, $|\bar{R}_u| = 3m-2$. Since $|R_r| = (3m/2)+1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m-2 \geq |R_r|$.$

- (d) The RNs of $\{a_r^2, a_{r+p}^3\}$ and $\{a_r^2, a_{r+s}^3\}$ are $\bar{R}_{11} = R\{a_r^2, a_{r+p}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r+((p+m+1)/2) \pmod{m}\}$ and $\bar{R}_{12} = R\{a_r^2, a_{r+s}^3\} = V(\mathbb{P}_m^2) - \{a_h^1 | h \equiv r+((m+s)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r+(s/2) \pmod{m}\}$. Clearly, $|\bar{R}_u| = 3(m-1)$. Since $|R_r| = (3m/2)+1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- (e) The RN of $\{a_r^1, a_r^2\}$ is $\bar{R}_{13} = R\{a_r^1, a_r^2\} = V(\mathbb{P}_m^2)$. Clearly, $|R\{a_r^2 a_r^3\}| = (3m/2)+1 < |R\{a_r^1 a_r^2\}|$ and $|R\{a_r^1 a_r^2\} \cap \cup_{r=1}^m R_r| = 3n \geq |R_r|$. \square

Theorem 5. If $\mathbb{N} \cong \mathbb{P}_m^2$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^2) \leq (6m/(3m+2))$.

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

Tables 20–22 show the RNs with maximum cardinality of 18, 16, and 14, respectively. On the other hand, Table 23 bears RNs with the minimum cardinality of 10. Also, it is observed that $\cup_{r=1}^{12} R_r = V(\mathbb{P}_6^2)$ which implies $|\cup_{r=1}^{12} R_r| = 18$ and $|\bar{R}_r \cap \cup_{r=1}^6 R_r| \geq |R_r|$.

Now, we define a function $\kappa: V(\mathbb{P}_6^2) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/10)$. As R_r for $1 \leq t \leq 12$ of \mathbb{P}_6^2 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_6^2 such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_6^2) \leq \sum_{r=1}^{18} (1/10) = (9/5)$.

Case 2. $m \geq 8$.

As Lemma 5 clears the fact that among the RNs of \mathbb{P}_m^2 , those that have the minimum cardinality of $(3m/2)+1$ are $R\{a_r^2, a_r^3\}$ and $R\{a_r^2, a_{r-1}^3\}$, respectively. Also, $|\bar{R}_u \cap \cup_{r=1}^m R_r| \leq |R_r|$, where $1 \leq u \leq 17$.

Let $\eta = |\cup_{r=1}^m R_r| = 3m$ and $\lambda = |R_r| = (3m/2)+1$. Then, we define a function $\kappa: V(\mathbb{P}_m^2) \rightarrow [0, 1]$ such that $\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{P}_m^2) - \cup_{r=1}^m R_r. \end{cases}$ It is seen that κ is a resolving function for \mathbb{P}_m^2 because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(\mathbb{P}_m^2)$. Now, suppose on contrary that there is another resolving function ρ , such that $\rho(u) \leq \kappa(u)$, for at least one $u \in V(\mathbb{P}_m^2)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of \mathbb{P}_m^2 having minimum cardinality λ . This shows that ρ is not a resolving function, a contradiction. Thus, κ is a minimal resolving function for \mathbb{P}_m^2 which attains minimum $|\kappa|$. Also, R_r for $1 \leq t \leq 2m$ of \mathbb{P}_m^2 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_m^2 such that

TABLE 12: \bar{R}_u for $1 \leq u \leq 30$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_8^1) - \{a_3^1, a_5^1\}$	$R\{a_1^1, a_6^1\}, R\{a_2^1, a_7^1\}, R\{a_3^1, a_8^1\}, R\{a_4^1, a_1^1\}, R\{a_5^1, a_2^1\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_6^1\}$	$R\{a_6^1, a_7^1\}, R\{a_2^1, a_3^1\}, R\{a_1^1, a_8^1\}, R\{a_5^1, a_7^1\}, R\{a_1^1, a_4^1\}, R\{a_2^1, a_1^1\}, R\{a_3^1, a_3^1\}, R\{a_5^1, a_2^1\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_3^1, a_7^1\}$	$R\{a_7^1, a_1^1\}, R\{a_3^1, a_4^1\}, R\{a_2^1, a_8^1\}, R\{a_2^1, a_5^1\}, R\{a_2^1, a_2^1\}, R\{a_1^1, a_6^1\}, R\{a_2^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_4^1, a_8^1\}$	$R\{a_1^1, a_8^1\}, R\{a_4^1, a_5^1\}, R\{a_2^1, a_8^1\}, R\{a_1^1, a_6^1\}, R\{a_2^1, a_6^1\}$

TABLE 13: \bar{R}_u for $28 \leq u \leq 38$.

RNs	Elements	Equality
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_5^1\} \cup \{a_2^2, a_5^2\}$	$R\{a_1^1, a_1^1\}, R\{a_2^1, a_2^1\}, R\{a_3^1, a_3^1\}, R\{a_4^1, a_4^1\}, R\{a_5^1, a_5^1\}, R\{a_6^1, a_6^1\}, R\{a_7^1, a_7^1\}, R\{a_8^1, a_8^1\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_6^1\} \cup \{a_3^2, a_6^2\}$	$R\{a_1^1, a_6^1\}, R\{a_2^1, a_4^1\}, R\{a_2^1, a_2^1\}, R\{a_1^1, a_1^1\}, R\{a_2^1, a_5^1\}, R\{a_3^1, a_3^1\}, R\{a_3^1, a_6^1\}, R\{a_1^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_4^1, a_7^1\} \cup \{a_4^2, a_7^2\}$	$R\{a_1^1, a_7^1\}, R\{a_3^1, a_5^1\}, R\{a_1^1, a_7^1\}, R\{a_1^1, a_6^1\}, R\{a_2^1, a_6^1\}, R\{a_3^1, a_4^1\}, R\{a_3^1, a_7^1\}, R\{a_3^1, a_3^1\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_5^1, a_8^1\} \cup \{a_5^2, a_8^2\}$	$R\{a_2^1, a_8^1\}, R\{a_3^1, a_5^1\}, R\{a_2^1, a_8^1\}, R\{a_3^1, a_7^1\}, R\{a_3^1, a_7^1\}, R\{a_4^1, a_5^1\}, R\{a_1^1, a_8^1\}, R\{a_3^1, a_7^1\}, R\{a_3^1, a_8^1\}$

TABLE 14: \bar{R}_u for $39 \leq u \leq 54$.

RNs	Elements
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_3^1, a_7^1\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_1^1, a_3^1\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_3^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_5^1, a_8^1\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_1^1, a_1^1\}$
$R\{a_1^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_1^1, a_1^1\}$
$R\{a_1^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_2^1, a_2^1\} \cup \{a_1^1, a_1^1\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_6^1, a_7^1, a_8^1\}$
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_2^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_6^1, a_7^1, a_8^1\}$
$R\{a_2^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_2^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_2^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_3^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_3^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_3^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_4^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_4^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_4^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_5^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_5^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_5^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_6^1, a_7^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_6^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$
$R\{a_7^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1, a_8^1\}$

TABLE 15: R_r with minimum cardinalities for $1 \leq r \leq 16$.

RNs	Elements
$R_1 = \{a_1^1, a_2^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_3^1, a_4^1\} \cup \{a_1^1, a_3^1\}$
$R_3 = \{a_1^1, a_4^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_3^1, a_6^1\} \cup \{a_1^1, a_3^1\}$
$R_5 = \{a_1^1, a_6^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_1^1, a_3^1\}$
$R_7 = \{a_1^1, a_8^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_3^1, a_4^1\} \cup \{a_1^1, a_3^1\}$
$R_9 = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_7^1, a_8^1\} \cup \{a_1^1, a_3^1\}$
$R_{11} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_8^1\} \cup \{a_1^1, a_3^1\}$
$R_{13} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_3^1, a_4^1\} \cup \{a_1^1, a_3^1\}$
$R_{15} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_1^1, a_3^1\}$
$R_2 = \{a_2^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_4^1, a_5^1\} \cup \{a_1^1, a_3^1\}$
$R_4 = \{a_4^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_1^1, a_3^1\}$
$R_6 = \{a_6^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^1, a_3^1\}$
$R_8 = \{a_8^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_4^1, a_5^1\} \cup \{a_1^1, a_3^1\}$
$R_{10} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_7^1, a_8^1\} \cup \{a_1^1, a_3^1\}$
$R_{12} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^1, a_3^1\}$
$R_{14} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_3^1, a_4^1\} \cup \{a_1^1, a_3^1\}$
$R_{16} = \{a_1^1, a_3^1\}$	$V(\mathbb{P}_8^1) - \{a_1^1, a_5^1, a_6^1, a_7^1\} \cup \{a_1^1, a_3^1\}$

$|\bar{\kappa}| \leq |\kappa|$. Therefore, assigning $(1/\lambda)$ to all the vertices of \mathbb{P}_m^2 and calculating their sum, we get $\dim_f(\mathbb{P}_m^2) \leq \sum_{r=1}^n (1/\lambda) = (3m / ((3m+2)/2)) = (6m / (3m+2))$. \square

Lemma 6. Let $\mathbb{N} \cong \mathbb{P}_m^3$ be the 3-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq r \leq m$, $p \geq 3$, $s \geq 2$, $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- $|R_r| = |R\{a_r^1, a_r^2\}| = |R_r| = 2m - 1$, $|\cup_{r=1}^m R_r| = 3m$ and $|\cup_{r=1}^m R_r| = 3m$.
- $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_2| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_3| = |R\{a_r^2, a_{r+1}^2\}| = |\bar{R}_4| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_5| = |R\{a_r^3, a_{r+1}^3\}| = |\bar{R}_6| = |R\{a_r^3, a_{r+p}^3\}| = |\bar{R}_7| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_8| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_9| = |R\{a_r^1, a_{r-1}^1\}| = |\bar{R}_{10}| = |R\{a_r^1, a_r^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$ with $a_{m+1}^1 = a_1^1$ and $a_{m+1}^2 = a_1^2$.

- $|R_r| < |\bar{R}_{11}| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_{12}| = |R\{a_r^3, a_{r+s}^3\}| = |\bar{R}_{13}| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_{14}| = |R\{a_r^1, a_{r+p}^1\}| |\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{15}| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_{16}| = |R\{a_r^3, a_{r+s}^3\}| = |\bar{R}_{17}| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_{18}| = |R\{a_r^1, a_{r+p}^1\}| |\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{19}| = |R\{a_r^1, a_{r-1}^1\}|$ and $|R\{a_r^1, a_{r-1}^1 \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{20}| = |R\{a_r^2, a_{r-1}^2\}|$ and $|R\{a_r^2, a_{r-1}^2 \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{21}| = |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3 \cap \cup_{r=1}^m R_r| \geq |R_r|$.

Proof

- The RN of $\{a_r^1, a_r^2\}$ is $R\{a_r^1, a_r^2\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r + 1, r + 2, \dots, r + (m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + 1, r + 2, \dots, r + (m/2) - 1 \pmod{m}\} \cup \{a_h^3 | h \equiv r + 1, r + 2 \pmod{m}\}$.

We note that $\cup_{r=1}^m R_r = V(\mathbb{P}_m^3)$ and $|\cup_{r=1}^m R_r| = 3m$.

- The RNs of $\{a_r^1, a_{r+1}^1\}, \{a_r^1, a_{r+p}^1\}, \{a_r^2, a_{r+1}^2\}, \{a_r^2, a_{r+p}^2\}, \{a_r^3, a_{r+1}^3\}, \{a_r^3, a_{r+p}^3\}, \{a_r^1, a_{r+s}^1\}, \{a_r^1, a_{r+1}^1\}, \{a_r^1, a_{r-1}^1\}$, and $\{a_r^1, a_r^3\}$ are $\bar{R}_1 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^3) -$

TABLE 16: \bar{R}_u for $1 \leq u \leq 57$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_{10}^1) - \{a_1^3, a_6^3\}$	$R\{a_6^1, a_7^1\}, R\{a_7^1, a_2^1\}, R\{a_6^2, a_7^2\}, R\{a_5^1, a_8^1\}, R\{a_5^2, a_8^2\}, R\{a_3^1, a_{10}^1\}, R\{a_3^2, a_{10}^2\}, R\{a_4^1, a_9^1\}, R\{a_4^2, a_9^2\}, R\{a_5^3, a_7^3\}$ $R\{a_7^1, a_8^1\}, R\{a_2^2, a_3^2\}, R\{a_7^2, a_8^2\}, R\{a_1^1, a_4^1\}, R\{a_6^2, a_7^2\}, R\{a_1^2, a_4^2\}, R\{a_6^1, a_9^1\}, R\{a_6^2, a_9^2\},$ $R\{a_1^3, a_3^3\}, R\{a_6^3, a_8^3\}, R\{a_6^1, a_9^1\}, R\{a_6^2, a_9^2\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^3, a_7^3\}$	$R\{a_8^1, a_9^1\}, R\{a_2^2, a_4^2\}, R\{a_8^2, a_9^2\}, R\{a_1^1, a_5^1\}, R\{a_2^2, a_5^2\}, R\{a_1^2, a_5^2\}, R\{a_1^1, a_{10}^1\}, R\{a_7^2, a_{10}^2\}, R\{a_2^3, a_4^3\}, R\{a_7^3, a_9^3\},$ $R\{a_1^2, a_6^2\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_{10}^1) - \{a_3^3, a_8^3\}$	$R\{a_9^1, a_{10}^1\}, R\{a_4^2, a_5^2\}, R\{a_9^2, a_{10}^2\}, R\{a_1^1, a_6^1\}, R\{a_3^2, a_6^2\}, R\{a_1^1, a_8^1\}, R\{a_2^2, a_8^2\}, R\{a_3^3, a_5^3\}, R\{a_8^3, a_{10}^3\}$ $R\{a_1^1, a_{10}^1\}, R\{a_4^2, a_5^2\}, R\{a_2^2, a_{10}^2\}, R\{a_1^2, a_7^2\}, R\{a_3^2, a_6^2\}, R\{a_2^2, a_9^2\}, R\{a_4^3, a_6^3\}, R\{a_1^3, a_9^3\}$

TABLE 17: \bar{R}_u for $58 \leq u \leq 122$.

RNs	Elements	Equality
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_{10}^1) - \{a_1^2, a_7^2\} \cup \{a_2^2, a_7^2\}$	$R\{a_6^1, a_8^1\}, R\{a_1^2, a_2^2\}, R\{a_2^2, a_2^2\}, R\{a_5^1, a_9^1\}, R\{a_5^2, a_9^2\}, R\{a_1^3, a_2^3\}, R\{a_6^3, a_7^3\}, R\{a_5^3, a_8^3\}, R\{a_3^3, a_{10}^3\}$ $R\{a_7^1, a_9^1\}, R\{a_2^2, a_4^2\}, R\{a_7^2, a_9^2\}, R\{a_1^1, a_5^1\}, R\{a_1^2, a_5^2\}, R\{a_6^1, a_{10}^1\}, R\{a_6^2, a_{10}^2\}, R\{a_3^3, a_3^3\}, R\{a_7^3, a_8^3\},$ $R\{a_1^1, a_4^1\}, R\{a_6^3, a_9^3\}$
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_{10}^1) - \{a_1^3, a_8^3\} \cup \{a_2^3, a_8^3\}$	$R\{a_8^1, a_{10}^1\}, R\{a_3^2, a_5^2\}, R\{a_2^2, a_{10}^2\}, R\{a_1^2, a_6^2\}, R\{a_2^2, a_6^2\}, R\{a_1^1, a_7^1\}, R\{a_1^2, a_7^2\}, R\{a_3^3, a_4^3\},$ $R\{a_3^3, a_9^3\}, R\{a_3^3, a_5^3\}, R\{a_3^3, a_{10}^3\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_{10}^1) - \{a_4^1, a_9^1\} \cup \{a_4^2, a_9^2\}$	$R\{a_1^1, a_9^1\}, R\{a_4^2, a_6^2\}, R\{a_1^2, a_9^2\}, R\{a_1^1, a_7^1\}, R\{a_3^2, a_7^2\}, R\{a_1^2, a_8^2\}, R\{a_4^3, a_5^3\}, R\{a_9^3, a_{10}^3\},$ $R\{a_1^2, a_8^2\}$
$R\{a_4^1, a_6^1\}$	$V(\mathbb{P}_{10}^1) - \{a_5^1, a_{10}^1\} \cup \{a_5^2, a_{10}^2\}$	$R\{a_2^1, a_{10}^1\}, R\{a_5^2, a_7^2\}, R\{a_2^2, a_{10}^2\}, R\{a_4^1, a_8^1\}, R\{a_4^2, a_8^2\}, R\{a_5^3, a_6^3\}, R\{a_1^3, a_{10}^3\}, R\{a_4^3, a_7^3\}, R\{a_2^3, a_9^3\}$ $R\{a_1^2, a_8^2\}$
$R\{a_5^1, a_7^1\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_6^1\} \cup \{a_2^1, a_6^1\}$	
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_{10}^2\} \cup \{a_3^2, a_9^2\}$	
$R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_{10}^1) - \{a_1^2, a_3^2\} \cup \{a_3^3, a_{10}^3\}$	
$R\{a_3^1, a_2^2\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_2^2\} \cup \{a_3^3, a_3^3\}$	
$R\{a_1^1, a_4^2\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_2^2\} \cup \{a_3^3, a_3^3\}$	
$R\{a_1^1, a_5^2\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_2^2\} \cup \{a_3^3, a_3^3\}$	
$R\{a_1^1, a_6^2\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_2^2\} \cup \{a_3^3, a_3^3\}$	
$R\{a_7^1, a_7^2\}$	$V(\mathbb{P}_{10}^1) - \{a_6^2, a_8^2\} \cup \{a_5^3, a_8^3\}$	
$R\{a_8^1, a_8^2\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_2^2\} \cup \{a_6^3, a_9^3\}$	
$R\{a_9^1, a_9^2\}$	$V(\mathbb{P}_{10}^1) - \{a_8^2, a_{10}^2\} \cup \{a_7^3, a_{10}^3\}$	
$R\{a_{10}^1, a_{10}^2\}$	$V(\mathbb{P}_{10}^1) - \{a_2^2, a_2^2\} \cup \{a_1^3, a_8^3\}$	

TABLE 18: \bar{R}_u for $123 \leq u \leq 142$.

RNs	Elements
$R\{a_1^2, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_1^1, a_6^1\}$
$R\{a_2^2, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_1^1, a_8^1\}$
$R\{a_5^2, a_5^3\}$	$V(\mathbb{P}_{10}^1) - \{a_6^1, a_7^1, a_8^1, a_9^1, a_{10}^1\}$
$R\{a_7^2, a_7^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_8^1, a_9^1, a_{10}^1\}$
$R\{a_9^2, a_9^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_2^1, a_4^1, a_{10}^1\}$
$R\{a_2^2, a_1^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_7^1, a_8^1, a_9^1, a_{10}^1\}$
$R\{a_4^2, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_9^1, a_{10}^1\}$
$R\{a_6^2, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_4^1, a_5^1\}$
$R\{a_8^2, a_7^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_4^1, a_5^1, a_6^1, a_7^1\}$
$R\{a_{10}^2, a_9^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_6^1, a_7^1, a_8^1, a_9^1\}$
$R\{a_2^2, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_6^1, a_7^1\}$
$R\{a_4^2, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_5^1, a_6^1, a_7^1, a_8^1, a_9^1\}$
$R\{a_6^2, a_6^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_7^1, a_8^1, a_9^1, a_{10}^1\}$
$R\{a_8^2, a_8^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_9^1, a_{10}^1\}$
$R\{a_{10}^2, a_{10}^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_4^1, a_5^1\}$
$R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_5^1, a_9^1, a_{10}^1\}$
$R\{a_5^2, a_4^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_4^1, a_{10}^1\}$
$R\{a_7^2, a_6^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_1^1, a_6^1\}$
$R\{a_9^2, a_8^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_7^1, a_8^1\}$
$R\{a_{10}^2, a_{10}^3\}$	$V(\mathbb{P}_{10}^1) - \{a_6^1, a_7^1, a_8^1, a_9^1, a_{10}^1\}$

TABLE 19: R_r with minimum cardinalities for $1 \leq r \leq 20$.

RNs	Elements
$R_1 = \{a_1^1, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_1^1, a_6^1\} \cup \{a_2^2\} \cup \{a_1^3, a_5^3\}$
$R_3 = \{a_3^1, a_4^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_6^1, a_7^1, a_8^1\} \cup \{a_7^2\} \cup \{a_3^3, a_7^3\}$
$R_5 = \{a_5^1, a_6^3\}$	$V(\mathbb{P}_{10}^1) - \{a_7^1, a_8^1, a_9^1, a_{10}^1\} \cup \{a_9^2\} \cup \{a_5^3, a_9^3\}$
$R_7 = \{a_7^1, a_8^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_9^1, a_{10}^1\} \cup \{a_7^2\} \cup \{a_7^3, a_7^3\}$
$R_9 = \{a_9^1, a_{10}^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_4^1\} \cup \{a_2^2\} \cup \{a_3^3, a_9^3\}$
$R_{11} = \{a_{10}^1, a_8^3\}$	$V(\mathbb{P}_{10}^1) - \{a_7^1, a_8^1, a_9^1, a_{10}^1\} \cup \{a_8^2\} \cup \{a_1^3, a_7^3\}$
$R_{13} = \{a_4^1, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_9^1, a_{10}^1\} \cup \{a_{10}^2\} \cup \{a_3^3, a_9^3\}$
$R_{15} = \{a_6^1, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_4^1\} \cup \{a_2^2\} \cup \{a_1^3, a_3^3\}$
$R_{17} = \{a_8^1, a_6^3\}$	$V(\mathbb{P}_{10}^1) - \{a_5^1, a_1^1, a_5^1, a_6^1\} \cup \{a_4^2\} \cup \{a_3^3, a_7^3\}$
$R_{19} = \{a_{10}^1, a_8^3\}$	$V(\mathbb{P}_{10}^1) - \{a_5^1, a_6^1, a_7^1, a_8^1\} \cup \{a_6^2\} \cup \{a_5^3, a_9^3\}$
$R_2 = \{a_2^1, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_4^1, a_5^1, a_6^1, a_7^1\} \cup \{a_6^2\} \cup \{a_2^3, a_6^3\}$
$R_4 = \{a_4^1, a_5^3\}$	$V(\mathbb{P}_{10}^1) - \{a_6^1, a_7^1, a_8^1, a_9^1\} \cup \{a_8^2\} \cup \{a_4^3, a_8^3\}$
$R_6 = \{a_6^1, a_7^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_1^1, a_9^1, a_{10}^1\} \cup \{a_{10}^2\} \cup \{a_6^3, a_{10}^3\}$
$R_8 = \{a_8^1, a_9^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_{10}^1\} \cup \{a_2^2\} \cup \{a_2^3, a_8^3\}$
$R_{10} = \{a_1^1, a_{10}^3\}$	$V(\mathbb{P}_{10}^1) - \{a_2^1, a_3^1, a_4^1, a_5^1\} \cup \{a_4^2\} \cup \{a_4^3, a_{10}^3\}$
$R_{12} = \{a_3^1, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_8^1, a_9^1, a_{10}^1\} \cup \{a_2^2\} \cup \{a_2^3, a_8^3\}$
$R_{14} = \{a_5^1, a_3^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_2^1, a_3^1, a_{10}^1\} \cup \{a_1^2\} \cup \{a_4^3, a_{10}^3\}$
$R_{16} = \{a_7^1, a_5^3\}$	$V(\mathbb{P}_{10}^1) - \{a_2^1, a_3^1, a_4^1, a_5^1\} \cup \{a_2^2\} \cup \{a_2^3, a_6^3\}$
$R_{18} = \{a_9^1, a_7^3\}$	$V(\mathbb{P}_{10}^1) - \{a_1^1, a_5^1, a_6^1, a_7^1\} \cup \{a_2^2\} \cup \{a_4^3, a_8^3\}$
$R_{20} = \{a_1^1, a_2^3\}$	$V(\mathbb{P}_{10}^1) - \{a_6^1, a_7^1, a_8^1, a_9^1\} \cup \{a_7^2\} \cup \{a_6^3, a_{10}^3\}$

$$\{a_h^2 | h \equiv r, r + (m/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r, r + (m/2)(\text{mod } m)\}, \bar{R}_2 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{P}_m^3) - \{a_h^2 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2)(\text{mod } m)\},$$

$$(\text{mod } m)), \bar{R}_3 = R\{a_r^2, a_{r+1}^2\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r + 1, r + ((m+2)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r, r + (m/2)(\text{mod } m)\}, \bar{R}_4 = R\{a_r^2, a_{r+p}^2\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r + ((p+1)/2), r + ((p+m+1)/2)(\text{mod } m)\} \cup \{a_h^3 | h$$

TABLE 20: \bar{R}_u for $1 \leq u \leq 6$.

Resolving neighbourhoods			Set/elements
$R\{a_1^1, a_2^1\}$	$R\{a_1^1, a_2^1\}$	$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^2)$
$R\{a_4^1, a_5^1\}$	$R\{a_5^1, a_6^1\}$	$R\{a_6^1, a_7^1\}$	$V(\mathbb{P}_6^2)$

TABLE 21: \bar{R}_u for $7 \leq u \leq 24$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$

TABLE 22: \bar{R}_u for $25 \leq u \leq 29$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\}$	$R\{a_1^1, a_2^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$

TABLE 23: R_r with minimum cardinalities for $1 \leq r \leq 12$.

RNs	Elements
$R_1 = R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_2 = R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_3 = R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_4 = R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_5 = R\{a_5^1, a_6^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_6 = R\{a_6^1, a_7^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_7 = R\{a_7^1, a_8^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_8 = R\{a_8^1, a_9^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_9 = R\{a_9^1, a_{10}^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_{10} = R\{a_{10}^1, a_{11}^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_{11} = R\{a_{11}^1, a_{12}^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$
$R_{12} = R\{a_{12}^1, a_{13}^1\}$	$V(\mathbb{P}_6^2) - \{a_3^1, a_5^1\} \cup \{a_2^1, a_5^1\} \cup \{a_3^1, a_5^1\}$

$\equiv r + ((p+1)/2), r + ((p+m+1)/2) \pmod{m}$,
 $\bar{R}_5 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r+1, r + ((m+2)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r+1, r + ((m+2)/2) \pmod{m}\}$,
 $\bar{R}_6 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r + ((p+1)/2), r + ((p+m+1)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r + ((p+1)/2), r + ((p+m+1)/2) \pmod{m}\}$,
 $\bar{R}_7 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r + (s/2), r + ((s+m)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r + ((s-2)/2), r + ((s+m-2)/2) \pmod{m}\} = \bar{R}_8 = R\{a_r^1, a_{r+1}^1\}$,
 $\bar{R}_9 = R\{a_r^1, a_{r-1}^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r - (s/2), r - ((s+m)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - ((s-2)/2), r - ((s+m-2)/2) \pmod{m}\}$,
 $\bar{R}_{10} = R\{a_r^1, a_r^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r, r + 2 \pmod{m}\} \cup \{a_h^1 | h \equiv r - 1, r + 2 \pmod{m}\}$,
 Clearly, $|\bar{R}_u| = 3m - 4$. Since $|R_r| = (3m/2) + 1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 2 \geq |R_r|$.

- (c) The RN of $\{a_r^1, a_{r+p}^1\}$, $\{a_r^1, a_{r+s}^1\}$, $\{a_r^1, a_{r-p}^1\}$, and $\{a_r^1, a_{r-s}^1\}$ are $\bar{R}_{11} = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r + ((p+m-1)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r + ((p-1)/2) \pmod{m}\}$,
 $\bar{R}_{12} = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r + (s/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r +$

$((m+s-2)/2) \pmod{m}\}$, $\bar{R}_{13} = R\{a_r^1, a_{r-p}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r - ((p+1)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - ((p+m-1)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - ((p-1)/2) \pmod{m}\}$,
 $\bar{R}_{14} = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r - ((m+s)/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - (s/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - ((m+s-2)/2) \pmod{m}\}$,
 Clearly, $|\bar{R}_u| = 3(m-1)$. Since $|R_r| = (3m/2) + 1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 2 \geq |R_r|$.

- (d) The RNs of $\{a_r^2, a_{r+s}^2\}$, $\{a_r^3, a_{r+s}^3\}$, $\{a_r^1, a_{r+p}^1\}$, and $\{a_r^1, a_{r+p}^1\}$ are $\bar{R}_{15} = \{a_r^2, a_{r+s}^2\} = V(\mathbb{G}) - \{a_h^2 | r \equiv r + (s/2), r + ((s+m-2)/2) \pmod{m}\}$,
 $\bar{R}_{16} = \{a_r^3, a_{r+s}^3\} = V(\mathbb{G}) - \{a_h^3 | r \equiv r + (s/2), r + ((s+m-2)/2) \pmod{m}\}$,
 $\bar{R}_{17} = \{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^1 | r \equiv r + ((s+2)/2), r + ((s+m+2)/2) \pmod{m}\}$,
 $\bar{R}_{18} = \{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^1 | r \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^1 | r \equiv r + ((p+m+1)/2) \pmod{m}\}$,
 Clearly, $|\bar{R}_u| = 3m - 2$. Since $|R_r| = (3m/2) + 1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 2 \geq |R_r|$.

- (e) The RNs of $\{a_r^1, a_{r+1}^1\}$ and $\{a_r^1, a_{r-1}^1\}$ are $\bar{R}_{19} = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^3) - \{a_h^1 | h \equiv r - 1, r - 2, \dots, r - (m/2) \pmod{m}\} \cup \{a_h^1 | h \equiv r - 2 \pmod{m}\} \cup \{a_h^1 | h \equiv r - 3 \pmod{m}\}$.
 Clearly, $|R\{a_r^1, a_{r+1}^1\}| = 2m - 1 < |R\{a_r^1, a_{r-1}^1\}|$ and $|R\{a_r^1, a_{r-1}^1\} \cap \cup_{r=1}^m R_r| = ((5m-4)/2) \geq |R_r|$.

- (f) The RN of $\{a_r^2, a_{r-1}^2\}$ is given by $\bar{R}_{20} = R\{a_r^2, a_{r-1}^2\} = V(\mathbb{P}_m^3) - \{a_h^2 | h \equiv r - 1, r - 2, \dots, r - (m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r - 1 \pmod{m}\}$.
 Clearly, $|R\{a_r^2, a_{r-1}^2\}| = 2m - 1 < |R\{a_r^2, a_{r-1}^2\}|$ and $|R\{a_r^2, a_{r-1}^2\} \cap \cup_{r=1}^m R_r| = ((5m-2)/2) \geq |R_r|$.

- (g) The RN of $\{a_r^2, a_r^3\}$ is given by $\bar{R}_{21} = R\{a_r^2, a_r^3\} = V(\mathbb{P}_m^3) - \{a_h^2 | h \equiv r - 1, r - 2, \dots, r - (m/2) + 1 \pmod{m}\}$.
 Clearly, $|R\{a_r^2, a_r^3\}| = 2m - 1 < |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3\} \cap \cup_{r=1}^m R_r| = ((5m+2)/2) \geq |R_r|$. \square

Theorem 6. If $\mathbb{N} \cong \mathbb{P}_m^3$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^3) \leq (3m/(2m-1))$.

Proof

Case 1. When $m = 6$

The RNs are given as follows.

We note that the cardinalities of RNs as shown in Tables 24–26 are 16, 14, and 13, respectively. Similarly, the cardinality of RNs given in Table 27 is 11 which is less than the least among all the RNs. Moreover, $\bigcup_{r=1}^6 R_r = V(\mathbb{P}_6^3)$ which implies $|\bigcup_{r=1}^6 R_r| = 18$ and $|\bar{R}_r \cap \bigcup_{r=1}^6 R_r| \geq |R_r|$.

Now, we define a function $\kappa: V(\mathbb{P}_6^3) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/11)$. Furthermore, R_r for $1 \leq t \leq 6$ of \mathbb{P}_6^3 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_6^3 such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_6^3) \leq \sum_{r=1}^{18} (1/11) = (18/11)$.

Case 2. $m \geq 8$

We have seen from Lemma 6 $|R\{a_r^1, a_r^2\}| = 2m - 1 \leq |R\{a, b\}|$ for all $\{a, b\} \subset V(\mathbb{P}_m^3)$ and $|R\{a, b\} \cap \bigcup_{r=1}^m R_r| \geq |R_r|$.

Let $\eta = |\bigcup_{r=1}^m R_r| = 3m$ and $\lambda = |R_r| = 2m - 1$. Then, we define a function $\kappa: V(\mathbb{P}_m^3) \rightarrow [0, 1]$ such that $\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \bigcup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{P}_m^3) - \bigcup_{r=1}^m R_r. \end{cases}$ We see that κ is a resolving function for \mathbb{P}_m^3 because $\kappa(R\{a, b\}) \geq 1 \forall uv \in E(\mathbb{P}_m^3)$. Now, suppose on contrary that there is another resolving function ρ , such that $\rho(u) \leq \kappa(u)$, for at least one $u \in V(\mathbb{P}_m^3)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of \mathbb{P}_m^3 having the minimum cardinality of λ . This shows that ρ is not a resolving function, a contradiction. Thus, κ is a minimal resolving function for \mathbb{P}_m^3 that achieves minimum $|\kappa|$. Moreover, R_r for $1 \leq r \leq m$ of \mathbb{P}_m^3 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_m^3 such that $|\bar{\kappa}| \leq |\kappa|$. Therefore, assigning $(1/\lambda)$ to all the vertices of \mathbb{P}_m^3 and calculating their sum, we get $\dim_f(\mathbb{P}_m^3) \leq \sum_{r=1}^{\eta} (1/\lambda) = (3m/(2m-1)) = (3m/(2m-1))$. \square

Corollary 2. If $\mathbb{N} \cong \mathbb{P}_m^4$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^4) \leq (3m/(2m-1))$.

Proof. The above assertion is valid as $\mathbb{P}_m^4 \cong \mathbb{P}_m^3$. \square

Lemma 7. Let $\mathbb{N} \cong \mathbb{P}_m^5$ be the 4-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- $|R_l| = |R\{a_l^1, a_l^2\}| = |R_r| = |R\{a_r^2, a_{r-1}^3\}| = (3m/2) + 1$ with $a_0^3 = a_m^3$, $a_0^2 = a_m^2$, $|\bigcup_{l=1}^m R_l| = 3m$, $|\bigcup_{r=1}^m R_r| = 3m$ and $|\bigcup_{l=1}^m R_l| \cup |\bigcup_{r=1}^m R_r| = 3m$,
- $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_r^3\}| = |\bar{R}_2| = |R\{a_r^2, a_{r+1}^3\}| = |\bar{R}_3| = |R\{a_r^2, a_{r-1}^3\}|$ and $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_4| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_5| = |R\{a_r^2, a_{r+1}^2\}| = |\bar{R}_6| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_7| = |R\{a_r^1, a_{r+s}^3\}|$ with $a_{m+1}^1 = a_1^1$ and $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| \geq |R_r|$.

TABLE 24: \bar{R}_u for $1 \leq u \leq 30$.

RNs	Elements	Equality
$R\{a_1^2, a_2^2\}$	$V(\mathbb{P}_6^3) - \{a_2^2, a_5^2\}$	$R\{a_4^2, a_6^2\}$
$R\{a_2^2, a_4^2\}$	$V(\mathbb{P}_6^3) - \{a_3^2, a_6^2\}$	$R\{a_5^1, a_6^1\}, R\{a_1^1, a_5^1\}$
$R\{a_3^2, a_5^2\}$	$V(\mathbb{P}_6^3) - \{a_1^2, a_4^2\}$	$R\{a_2^2, a_6^2\}$
$R\{a_1^3, a_3^3\}$	$V(\mathbb{P}_6^3) - \{a_2^3, a_5^3\}$	$R\{a_4^3, a_6^3\}$
$R\{a_2^3, a_5^3\}$	$V(\mathbb{P}_6^3) - \{a_3^3, a_6^3\}$	$R\{a_5^3, a_6^3\}, R\{a_1^3, a_5^3\}$
$R\{a_3^3, a_6^3\}$	$V(\mathbb{P}_6^3) - \{a_1^3, a_4^3\}$	$R\{a_2^3, a_6^3\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^3) - \{a_3^1, a_6^1\}$	$R\{a_2^3, a_6^3\}$
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_6^3) - \{a_3^1, a_6^1\}$	$R\{a_5^3, a_6^3\}$
$R\{a_3^1, a_6^1\}$	$V(\mathbb{P}_6^3) - \{a_4^1, a_6^1\}$	$R\{a_5^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_5^1\}$	
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^3) - \{a_5^1\} \cup \{a_3^2\}$	
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_6^3) - \{a_6^1\} \cup \{a_4^2\}$	
$R\{a_3^1, a_6^1\}$	$V(\mathbb{P}_6^3) - \{a_1^1\} \cup \{a_5^2\}$	
$R\{a_2^2, a_5^2\}$	$V(\mathbb{P}_6^3) - \{a_4^2, a_5^2\}$	
$R\{a_3^2, a_6^2\}$	$V(\mathbb{P}_6^3) - \{a_2^2, a_6^2\}$	
$R\{a_4^2, a_6^2\}$	$V(\mathbb{P}_6^3) - \{a_2^2, a_6^2\}$	
$R\{a_5^2, a_6^2\}$	$V(\mathbb{P}_6^3) - \{a_1^2, a_2^2\}$	
$R\{a_6^2, a_6^2\}$	$V(\mathbb{P}_6^3) - \{a_2^2, a_3^2\}$	
$R\{a_1^3, a_3^3\}$	$V(\mathbb{P}_6^3) - \{a_2^3, a_5^3\}$	

- $|R_r| < |\bar{R}_8| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_9| = |R\{a_r^1, a_{r+p}^1\}| = \bar{R}_{10} = |R\{a_r^2, a_{r+p}^2\}| = \bar{R}_{11} = |R\{a_r^2, a_{r-p}^2\}|$ and $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{12}| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_{13}| = |R\{a_r^1, a_{r+p}^2\}| = |\bar{R}_{14}| = |R\{a_r^1, a_{r+s}^2\}| = |\bar{R}_{15}| = |R\{a_r^1, a_{r+p}^3\}|$ and $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{16}| = |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3 \cap \bigcup_{r=1}^m R_r\}| \geq |R_r|$.

Proof

- The RNs of $\{a_l^1, a_l^2\}$ and $\{a_r^2, a_{r-1}^3\}$ are $R\{a_l^1, a_l^2\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv l+1, l+2, \dots, l+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv l-1, l-2, \dots, l-(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv l-1, l-2, \dots, l-(m/2)+1 \pmod{m}\}$ and $R\{a_r^2, a_{r-1}^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r-1, r-2, \dots, r-(m/2)+1 \pmod{m}\}$, respectively. We note that $\bigcup_{r=1}^m R_r = V(\mathbb{P}_m^5)$ and $|\bigcup_{r=1}^m R_r| = 3m$.
- The RNs of $\{a_r^1, a_r^3\}$, $\{a_r^2, a_{r+1}^3\}$ and $\{a_r^2, a_{r-1}^3\}$ are $\bar{R}_1 = R\{a_r^1, a_r^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r+2, r+3, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r \pmod{m}\} \cup \{a_h^3 | h \equiv r-1 \pmod{m}\}$, $\bar{R}_2 = R\{a_r^2, a_{r+1}^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r+2, r+3, \dots, r+(m+2)/2 \pmod{m}\} \cup \{a_h^2 | h \equiv r \pmod{m}\} \cup \{a_h^3 | h \equiv r+(m/2) \pmod{m}\}$ and $\bar{R}_3 = R\{a_r^2, a_{r-1}^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r-2, r-3, \dots, r-(m+2)/2 \pmod{m}\} \cup \{a_h^2 | h \equiv r \pmod{m}\} \cup \{a_h^3 | h \equiv r-(m/2) \pmod{m}\}$, respectively. We can see that $|\bar{R}_u| = (5m/2) + 2$. Since $|R\{a_r^1, a_r^2\}| = |R\{a_r^2, a_{r-1}^3\}| = (3m/2) + 1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \bigcup_{r=1}^m R_r| = 3m - 4 \geq |R_r|$, where $1 \leq u \leq 3$.
- The RNs of $\{a_r^1, a_{r+s}^1\}$, $\{a_r^2, a_{r+p}^2\}$ and $\{a_r^1, a_{r+s}^3\}$ are $\bar{R}_4 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r+(s/2), r+(m+s)/2 \pmod{m}\} \cup \{a_h^2 | h \equiv r+(s-2)/2, r+$

TABLE 25: \bar{R}_u for $31 \leq u \leq 78$.

RNs	Elements	Equality
$R\{a_2^2, a_1^3\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_5^2\}$	
$R\{a_3^2, a_2^3\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_4^2\}$	
$R\{a_4^2, a_3^3\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2\}$	
$R\{a_5^2, a_4^3\}$	$V(\mathbb{P}_6^3) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_3^2\}$	
$R\{a_6^2, a_5^3\}$	$V(\mathbb{P}_6^3) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_2^2\}$	
$R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_8^3) - \{a_2^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_2^1, a_3^2\}$	$V(\mathbb{P}_8^3) - \{a_2^2, a_5^2\} \cup \{a_3^3, a_4^3\}$	
$R\{a_3^1, a_4^2\}$	$V(\mathbb{P}_8^3) - \{a_3^2, a_6^2\} \cup \{a_3^3, a_4^3\}$	
$R\{a_4^1, a_5^2\}$	$V(\mathbb{P}_8^3) - \{a_4^2, a_5^2\} \cup \{a_3^3, a_4^3\}$	
$R\{a_5^1, a_6^2\}$	$V(\mathbb{P}_8^3) - \{a_5^2, a_6^2\} \cup \{a_3^3, a_4^3\}$	
$R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_2^1, a_3^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_3^1, a_4^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_4^1, a_5^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_5^1, a_6^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_2^1, a_3^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_3^1, a_4^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_4^1, a_5^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_5^1, a_6^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_2^1, a_3^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_3^1, a_4^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_4^1, a_5^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	
$R\{a_5^1, a_6^2\}$	$V(\mathbb{P}_8^3) - \{a_1^1, a_4^1\} \cup \{a_3^3, a_4^3\}$	

TABLE 26: \bar{R}_u for $79 \leq u \leq 84$.

RNs	Elements
$R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_5^2\} \cup \{a_3^3\}$
$R\{a_2^1, a_3^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_4^2\} \cup \{a_3^3\}$
$R\{a_3^1, a_4^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2\} \cup \{a_3^3\}$
$R\{a_4^1, a_5^2\}$	$V(\mathbb{P}_6^3) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_3^2\} \cup \{a_3^3\}$
$R\{a_5^1, a_6^2\}$	$V(\mathbb{P}_6^3) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_2^2\} \cup \{a_3^3\}$
$R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_5^2\} \cup \{a_3^3\}$
$R\{a_2^1, a_3^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_4^2\} \cup \{a_3^3\}$
$R\{a_3^1, a_4^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2\} \cup \{a_3^3\}$
$R\{a_4^1, a_5^2\}$	$V(\mathbb{P}_6^3) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_3^2\} \cup \{a_3^3\}$
$R\{a_5^1, a_6^2\}$	$V(\mathbb{P}_6^3) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_2^2\} \cup \{a_3^3\}$

TABLE 27: R_r with minimum cardinalities for $1 \leq r \leq 6$.

RNs	Elements
$R_1 = R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_6^3) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_2^2, a_3^2\} \cup \{a_3^3, a_4^3\}$
$R_2 = R\{a_2^1, a_3^2\}$	$V(\mathbb{P}_6^3) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_2^2, a_3^2\} \cup \{a_3^3, a_4^3\}$
$R_3 = R\{a_3^1, a_4^2\}$	$V(\mathbb{P}_6^3) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2\} \cup \{a_3^3, a_4^3\}$
$R_4 = R\{a_4^1, a_5^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_2^2, a_3^2\} \cup \{a_3^3, a_4^3\}$
$R_5 = R\{a_5^1, a_6^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_2^2, a_3^2\} \cup \{a_3^3, a_4^3\}$
$R_6 = R\{a_6^1, a_7^2\}$	$V(\mathbb{P}_6^3) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^2, a_3^2\} \cup \{a_3^3, a_4^3\}$

$((s+m-2)/2)(\text{mod } m)$, $\bar{R}_5 = R\{a_r^2, a_{r+1}^2\} = R\{a_r^1, a_{r+2}^1\}$, for $s \geq 4$, $\bar{R}_6 = R\{a_r^2, a_{r+p}^2\} = R\{a_r^1, a_{r+s}^1\}$, $\bar{R}_7 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r + ((s+2)/2), r + ((m+s)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r + (s/2), r + ((m+s)/2)(\text{mod } m)\}$. Clearly, $|\bar{R}_u| = 3m - 4$. Since $|R\{a_r^1, a_r^2\}| = |R\{a_r^2, a_{r-1}^3\}| = (3m/2) + 1 < |\bar{R}_u|$, then $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 4 \geq |R_r|$.

- (d) The RNs of $\{a_r^1, a_{r+p}^1\}$, $\{a_r^2, a_{r+p}^3\}$ and $\{a_r^2, a_{r-p}^3\}$ are $\bar{R}_8 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r, r + (m/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r(\text{mod } m)\}$, $\bar{R}_9 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r + ((p-1)/2)(\text{mod } m)\}$, $\bar{R}_{10} = R\{a_r^2, a_{r+p}^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r - 1(\text{mod } m)\} \cup \{a_h^3 | h \equiv r + ((p+1)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r + ((p+m-1)/2)(\text{mod } m)\}$ and $\bar{R}_{11} = R\{a_r^2, a_{r-p}^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r - 1(\text{mod } m)\} \cup \{a_h^3 | h \equiv r - ((p+1)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r - ((p+m+1)/2)(\text{mod } m)\}$.

$(\text{mod } m)$, respectively. We can see from these sets that $|\bar{R}_u| = 3(m-1) > |R_r|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 4 \geq |R_r|$.

- (e) The RNs of $\{a_r^2, a_{r+s}^2\}$, $\{a_r^1, a_{r+p}^2\}$, $\{a_r^1, a_{r+s}^2\}$, $\{a_r^1, a_{r+p}^3\}$ and $R\{a_r^1, a_{r+s}^1\}$ are $\bar{R}_{12} = R\{a_r^2, a_{r+s}^2\} = V(\mathbb{P}_m^5) - \{a_h^2 | h \equiv r + (s/2), r + ((s+m)/2)(\text{mod } m)\}$, $\bar{R}_{13} = R\{a_r^1, a_{r+p}^2\} = V(\mathbb{P}_m^5) - \{a_h^2 | h \equiv r + ((p-1)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r + ((p+m-1)/2)(\text{mod } m)\}$, $\bar{R}_{14} = R\{a_r^1, a_{r+s}^2\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r + ((m+s)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r + ((s-2)/2)(\text{mod } m)\}$, $\bar{R}_{15} = R\{a_r^1, a_{r+p}^3\} = V(\mathbb{P}_m^5) - \{a_h^1 | h \equiv r + ((p+1)/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r + ((p-1)/2)(\text{mod } m)\}$, respectively. We note that $|\bar{R}_u| = 3m - 2 > |R_r|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| = 3m - 2 \geq |R_r|$.
- (f) The RN of a_r^2, a_r^3 is $|\bar{R}_{16}| = R\{a_r^2, a_r^3\} = V(\mathbb{P}_m^5) - \{a_h^2 | h \equiv r + 1, r + 2, \dots, r + (m/2)(\text{mod } m)\} \cup \{a_h^3 | h \equiv r + 1, r + 2, \dots, r + (m/2) - 1(\text{mod } m)\}$.

Clearly, $|R\{a_r^1, a_r^2\}| = |R\{a_r^2, a_{r-1}^3\}| = (3m/2) + 1 < |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3\} \cap \cup_{r=1}^m R_r| = 2m + 1 \geq |R_r|$. \square

Theorem 7. If $\mathbb{N} \cong \mathbb{P}_m^5$ with $m \geq 6$ and $m \equiv 0(\text{mod } 2)$. Then, $\dim_f(\mathbb{P}_m^5) \leq (6m/(3m+2))$.

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

In Tables 28–31, the cardinalities of RNs are 16, 15, 14, and 13, respectively. In the same manner, Table 32 is having RNs withholding the cardinality of 10. Moreover, $\cup_{r=1}^{12} R_r = V(\mathbb{P}_6^5)$ which implies $|\cup_{r=1}^{12} R_r| = 18$ and $|\bar{R}_r \cap \cup_{r=1}^{12} R_r| \geq |R_r|$.

Now, we define a function $\kappa: V(\mathbb{P}_6^5) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/10)$. We have seen that R_r for $1 \leq r \leq 12$ of \mathbb{P}_6^5 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_6^5 such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_6^5) \leq \sum_{r=1}^{18} (1/10) = (9/5)$.

TABLE 28: \overline{R}_u for $1 \leq u \leq 31$.

RNs	Elements	Equality
$R\{a_1^2, a_2^2\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_3^2\}$	$R\{a_4^2, a_6^2\}$
$R\{a_2^2, a_4^2\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_6^2\}$	$R\{a_1^2, a_5^2\}$
$R\{a_3^2, a_6^2\}$	$V(\mathbb{P}_6^5) - \{a_4^2, a_6^2\}$	$R\{a_2^2, a_6^2\}$
$R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_6^5) - \{a_1^2\} \cup \{a_4^3\}$	$R\{a_2^2, a_6^2\}$
$R\{a_2^1, a_3^2\}$	$V(\mathbb{P}_6^5) - \{a_2^2\} \cup \{a_3^3\}$	$R\{a_1^2, a_4^3\}$
$R\{a_3^1, a_4^2\}$	$V(\mathbb{P}_6^5) - \{a_3^2\} \cup \{a_6^3\}$	$R\{a_2^2, a_5^3\}$
$R\{a_4^1, a_5^2\}$	$V(\mathbb{P}_6^5) - \{a_4^2\} \cup \{a_3^3\}$	$R\{a_3^2, a_6^3\}$
$R\{a_5^1, a_6^2\}$	$V(\mathbb{P}_6^5) - \{a_5^2\} \cup \{a_3^3\}$	
$R\{a_6^1, a_1^2\}$	$V(\mathbb{P}_6^5) - \{a_6^2\} \cup \{a_3^3\}$	
$R\{a_1^1, a_3^2\}$	$V(\mathbb{P}_6^5) - \{a_1^2\} \cup \{a_3^3\}$	
$R\{a_2^1, a_4^2\}$	$V(\mathbb{P}_6^5) - \{a_2^2\} \cup \{a_3^3\}$	
$R\{a_3^1, a_5^2\}$	$V(\mathbb{P}_6^5) - \{a_3^2\} \cup \{a_3^3\}$	
$R\{a_4^1, a_6^2\}$	$V(\mathbb{P}_6^5) - \{a_4^2\} \cup \{a_3^3\}$	
$R\{a_1^1, a_5^2\}$	$V(\mathbb{P}_6^5) - \{a_1^2\} \cup \{a_3^3\}$	
$R\{a_2^1, a_6^2\}$	$V(\mathbb{P}_6^5) - \{a_2^2\} \cup \{a_3^3\}$	
$R\{a_3^1, a_1^2\}$	$V(\mathbb{P}_6^5) - \{a_3^2\} \cup \{a_3^3\}$	
$R\{a_4^1, a_2^2\}$	$V(\mathbb{P}_6^5) - \{a_4^2\} \cup \{a_3^3\}$	
$R\{a_5^1, a_3^2\}$	$V(\mathbb{P}_6^5) - \{a_5^2\} \cup \{a_3^3\}$	
$R\{a_6^1, a_4^2\}$	$V(\mathbb{P}_6^5) - \{a_6^2\} \cup \{a_3^3\}$	
$R\{a_1^1, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2\} \cup \{a_3^3\}$	$R\{a_1^1, a_4^3\}$
$R\{a_2^1, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2\} \cup \{a_3^3\}$	$R\{a_2^1, a_5^3\}$
$R\{a_3^1, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_3^2\} \cup \{a_3^3\}$	$R\{a_3^1, a_6^3\}$
$R\{a_4^1, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_4^2\} \cup \{a_3^3\}$	
$R\{a_5^1, a_1^3\}$	$V(\mathbb{P}_6^5) - \{a_5^2\} \cup \{a_3^3\}$	
$R\{a_6^1, a_2^3\}$	$V(\mathbb{P}_6^5) - \{a_6^2\} \cup \{a_3^3\}$	

TABLE 29: \overline{R}_u for $32 \leq u \leq 47$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_4^2\} \cup \{a_3^3\}$	$R\{a_4^1, a_5^1\}, R\{a_3^1, a_6^1\}, R\{a_3^2, a_6^2\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_5^2\} \cup \{a_3^3\}$	$R\{a_1^1, a_4^1\}, R\{a_5^1, a_6^1\}, R\{a_2^2, a_4^2\}, R\{a_5^2, a_8^2\}, R\{a_3^3, a_3^3\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_6^2\} \cup \{a_3^3\}$	$R\{a_2^1, a_5^1\}, R\{a_1^1, a_6^1\}, R\{a_1^2, a_6^2\}$
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_6^5) - \{a_4^2\} \cup \{a_2^3\} \cup \{a_3^3\}$	
$R\{a_5^1, a_6^1\}$	$V(\mathbb{P}_6^5) - \{a_5^2\} \cup \{a_2^3\} \cup \{a_3^3\}$	
$R\{a_1^1, a_2^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2\} \cup \{a_2^3\} \cup \{a_3^3\}$	
$R\{a_2^1, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2\} \cup \{a_2^3\} \cup \{a_3^3\}$	
$R\{a_3^1, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_3^2\} \cup \{a_2^3\} \cup \{a_3^3\}$	
$R\{a_4^1, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_4^2\} \cup \{a_2^3\} \cup \{a_3^3\}$	
$R\{a_5^1, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_5^2\} \cup \{a_2^3\} \cup \{a_3^3\}$	

TABLE 30: \overline{R}_u for $48 \leq u \leq 62$.

RNs	Elements	Equality
$R\{a_1^1, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_5^2\} \cup \{a_2^3, a_2^3\}$	
$R\{a_2^1, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_6^2\} \cup \{a_2^3, a_2^3\}$	
$R\{a_3^1, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_1^2\} \cup \{a_2^3, a_2^3\}$	
$R\{a_4^1, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_4^2, a_6^2\} \cup \{a_2^3, a_2^3\}$	
$R\{a_1^1, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_5^2\} \cup \{a_2^3, a_2^3\}$	
$R\{a_2^1, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_6^2\} \cup \{a_2^3, a_2^3\}$	
$R\{a_3^1, a_1^3\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_1^2\} \cup \{a_2^3, a_2^3\}$	$R\{a_4^3, a_5^3\}, R\{a_3^3, a_4^3\}$
$R\{a_4^1, a_2^3\}$	$V(\mathbb{P}_6^5) - \{a_4^2, a_1^2\} \cup \{a_2^3, a_2^3\}$	$R\{a_5^3, a_6^3\}, R\{a_2^3, a_3^3\}$
$R\{a_5^1, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_5^2, a_1^2\} \cup \{a_2^3, a_2^3\}$	$R\{a_3^3, a_6^3\}, R\{a_3^3, a_6^3\}$
$R\{a_6^1, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_6^2, a_1^2\} \cup \{a_2^3, a_2^3\}$	$R\{a_4^1, a_6^1\}, R\{a_3^1, a_6^1\}, R\{a_2^2, a_2^2\}, R\{a_4^2, a_5^2\}, R\{a_7^2, a_7^2\}$
$R\{a_1^1, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_5^2\} \cup \{a_2^3, a_2^3\}$	$R\{a_1^1, a_5^1\}, R\{a_1^1, a_5^1\}, R\{a_2^2, a_2^2\}, R\{a_5^2, a_6^2\}, R\{a_2^2, a_5^2\}$
$R\{a_2^1, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_6^2\} \cup \{a_2^3, a_2^3\}$	$R\{a_2^1, a_6^1\}, R\{a_2^2, a_4^2\}, R\{a_2^2, a_6^2\}, R\{a_3^2, a_6^2\}$

Case 2. $m \geq 8$.

It can be seen from Lemma 7 that among all the RNs of \mathbb{P}_m^5 , $|R\{a_r^1 a_r^2\}| = |R\{a_r^1 a_{r-1}^3\}| = (3m/2) + 1 \leq |R\{a, b\}|$ for all $\{a, b\} \subset V(\mathbb{P}_m^5)$ and $|R\{a, b\} \cap \cup_{r=1}^m R_r| = (3m/2) + 1 \leq |R_r|$.

Let $\eta = |\cup_{r=1}^m R_r| = 3m$ and $\lambda = |R_r| = (3m/2) + 1$. Then, we define a function $\kappa: V(\mathbb{P}_m^5) \rightarrow [0, 1]$ such that $\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{P}_m^5) - \cup_{r=1}^m R_r. \end{cases}$ It is

found that κ is a resolving function for \mathbb{P}_m^5 because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(\mathbb{P}_m^5)$. On contrary, assume that there is another resolving function ρ , such that $\rho(u) \leq \kappa(u)$, for at least one $u \in V(\mathbb{P}_m^5)$ $\rho(u) \neq \kappa(u)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of \mathbb{P}_m^5 having the minimum cardinality of λ . This shows that ρ is not a resolving function, a contradiction. Thus, κ is a minimal resolving function for \mathbb{P}_m^5 that achieves minimum $|\kappa|$. Moreover, R_t for $1 \leq t \leq 2m$ of \mathbb{P}_m^5 are

TABLE 31: \bar{R}_u for $63 \leq u \leq 68$.

RNs	Elements
$R\{a_1^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_3^2, a_4^2\} \cup \{a_3^3, a_5^3\}$
$R\{a_2^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_4^2, a_5^2\} \cup \{a_3^3, a_4^3\}$
$R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_4^2, a_5^2, a_6^2\} \cup \{a_4^3, a_5^3\}$
$R\{a_4^2, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_4^3, a_6^3\}$
$R\{a_5^2, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_5^3, a_6^3\}$
$R\{a_6^2, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_3^2\} \cup \{a_6^3, a_3^3\}$

TABLE 32: R_r with minimum cardinalities for $1 \leq r \leq 12$.

RNs	Elements
$R_1 = R\{a_2^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_5^2, a_6^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_3^3, a_5^3\}$
$R_2 = R\{a_3^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_3^2, a_4^2, a_5^2\} \cup \{a_3^3, a_6^3\}$
$R_3 = R\{a_4^2, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_4^2, a_5^2, a_6^2\} \cup \{a_4^3, a_6^3\}$
$R_4 = R\{a_5^2, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_3^2, a_4^2\} \cup \{a_5^2, a_6^2, a_1^2\} \cup \{a_5^3, a_6^3\}$
$R_5 = R\{a_6^2, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_4^2, a_5^2\} \cup \{a_6^2, a_1^2, a_2^2\} \cup \{a_6^3, a_3^3\}$
$R_6 = R\{a_1^2, a_3^3\}$	$V(\mathbb{P}_6^5) - \{a_4^2, a_5^2, a_6^2\} \cup \{a_1^2, a_2^2, a_3^2\} \cup \{a_4^3, a_5^3\}$
$R_7 = R\{a_1^2, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_2^2, a_3^2, a_4^2\} \cup \{a_5^2, a_6^2, a_1^2\} \cup \{a_4^3, a_6^3\}$
$R_8 = R\{a_1^2, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_3^2, a_4^2, a_5^2\} \cup \{a_6^2, a_1^2, a_2^2\} \cup \{a_5^3, a_6^3\}$
$R_9 = R\{a_1^2, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_4^2, a_5^2, a_6^2\} \cup \{a_1^2, a_2^2, a_3^2\} \cup \{a_6^3, a_3^3\}$
$R_{10} = R\{a_4^2, a_4^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_5^2, a_6^2\} \cup \{a_2^2, a_3^2, a_4^2\} \cup \{a_4^3, a_5^3\}$
$R_{11} = R\{a_5^2, a_5^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_6^2\} \cup \{a_3^2, a_4^2, a_5^2\} \cup \{a_5^3, a_6^3\}$
$R_{12} = R\{a_6^2, a_6^3\}$	$V(\mathbb{P}_6^5) - \{a_1^2, a_2^2, a_3^2\} \cup \{a_4^2, a_5^2, a_6^2\} \cup \{a_6^3, a_3^3\}$

pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_m^5 such that $|\bar{\kappa}| \leq |\kappa|$. Therefore, assigning $(1/\lambda)$ to all the vertices of \mathbb{P}_m^5 and calculating their sum, we get $\dim_f(\mathbb{P}_m^5) \leq \sum_{r=1}^m (1/\lambda) = (3m/(3m+2)/2) = (6m/(3m+2))$. \square

Corollary 3. If $\aleph \cong \mathbb{P}_m^6$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^6) \leq (6m/(3m+2))$.

Proof. This can be derived easily as $\mathbb{P}_m^6 \cong \mathbb{P}_m^5$. \square

Lemma 8. Let $\aleph \cong \mathbb{P}_m^7$ be the 4-faced QCL with $m \geq 6$ and $m \equiv 0 \pmod{2}$. For $1 \leq l, r \leq m$, $p \geq 3$, $s \geq 2$ $p \equiv 1 \pmod{2}$ and $s \equiv 0 \pmod{2}$. Then,

- $|R_l| = |R\{a_l^1, a_l^3\}| = |R_r| = |R\{a_r^1, a_r^3\}|$, $|\cup_{l=1}^m R_l| = 3m$, $|\cup_{r=1}^m R_r| = 3m$ and $|\cup_{l=1}^m R_l| \cup |\cup_{r=1}^m R_r| = 3m$.
- $|R_r| < |\bar{R}_1| = |R\{a_r^1, a_{r+1}^1\}| = |\bar{R}_2| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_3| = |R\{a_r^2, a_{r+1}^2\}| = |\bar{R}_4| = |R\{a_r^2, a_{r+p}^2\}| = |\bar{R}_5| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_6| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_7| = |R\{a_r^2, a_{r+s}^2\}|$ with $a_{m+1}^1 = a_1^1$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_8| = |R\{a_r^1, a_r^2\}| = |\bar{R}_9| = |R\{a_r^1, a_{r+s}^1\}| = |\bar{R}_{10}| = |R\{a_r^2, a_{r+s}^2\}| = |\bar{R}_{11}| = |R\{a_r^3, a_{r+1}^3\}| = |\bar{R}_{12}| = |R\{a_r^3, a_{r+p}^3\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{13}| = |R\{a_r^1, a_{r+p}^1\}| = |\bar{R}_{14}| = |R\{a_r^1, a_{r+s}^1\}|$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{15}| = |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3 \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{16}| = |R\{a_r^2, a_{r-1}^2\}|$ and $|R\{a_r^2, a_{r-1}^2 \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- $|R_r| < |\bar{R}_{17}| = |R\{a_r^2, a_{r+p}^2\}|$ and $|\bar{R}_{17} \cap \cup_{r=1}^m R_r| \geq |R_r|$.

$$(h) |R_r| < |\bar{R}_{18}| = |R\{a_r^1, a_{r+1}^1\}| \text{ and } |\bar{R}_{18} \cap \cup_{r=1}^m R_r| \geq |R_r|.$$

Proof

- The RNs of $\{a_l^1, a_l^3\}$ and a_r^1, a_{r-1}^3 are $R_l = R\{a_l^1, a_l^3\} = V(\mathbb{P}_m^7) - \{a_h^1 | h \equiv l+1, l+2, \dots, l+(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv l, l+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^3 | t \equiv l+1, l+2, \dots, l+(m/2) \pmod{m}\}$ and $R_r = R\{a_r^1, a_{r-1}^3\} = V(\mathbb{P}_m^7) - \{a_h^1 | h \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r, r-2, \dots, r-(m/2) \pmod{m}\} \cup \{a_h^3 | t \equiv r-1, r-2, \dots, r-(m/2) \pmod{m}\}$, respectively. Clearly, $|R_r| = (3/2)m + 1$ for $1 \leq r \leq m$.

We note that $\cup_{r=1}^m R_r = V(\mathbb{P}_m^7)$ and $|\cup_{r=1}^m R_r| = 3m$.

- The RNs of $\{a_r^1, a_{r+1}^1\}$, $\{a_r^1, a_{r+p}^1\}$, $\{a_r^2, a_{r+p}^2\}$, $\{a_r^1, a_{r+s}^1\}$, $\{a_r^1, a_{r+p}^1\}$, and $\{a_r^2, a_{r+s}^2\}$ are $\bar{R}_1 = R\{a_r^1, a_{r+1}^1\} = V(\mathbb{G}) - \{a_h^3 | h \equiv r, r+(m/2) \pmod{m}\} = \bar{R}_3 = R\{a_r^2, a_{r+1}^2\}$, $\bar{R}_2 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^3 | h \equiv r + ((p-1)/2), r + ((p+m-1)/2) \pmod{m}\} = \bar{R}_4 = R\{a_r^2, a_{r+p}^2\} = \bar{R}_5 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{G}) - \{a_h^3 | h \equiv r + (s/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r + ((m+s-2)/2) \pmod{m}\}$, $\bar{R}_6 = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^3 | h \equiv r + ((p+1)/2), r + ((p+m+1)/2) \pmod{m}\}$ and $\bar{R}_7 = R\{a_r^2, a_{r+s}^2\} = V(\mathbb{G}) - \{a_h^3 | h \equiv r + (s/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r + ((m+s)/2) \pmod{m}\}$, respectively. We can see that $|R_r| < |\bar{R}_u| = 3m - 2$ and $|\bar{R}_u \cap \cup_{r=1}^m R_r| \geq |R_r|$.
- The RNs of $\{a_r^1, a_r^2\}$, $\{a_r^1, a_{r+s}^1\}$, $\{a_r^2, a_{r+s}^2\}$, $\{a_r^3, a_{r+1}^3\}$ and $\{a_r^3, a_{r+p}^3\}$ are $\bar{R}_8 = R\{a_r^1, a_r^2\} = V(\mathbb{P}_m^7) - \{a_h^2 | h \equiv r-1, r+1 \pmod{m}\} \cup \{a_h^3 | h \equiv r, r-1 \pmod{m}\}$, $\bar{R}_9 = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{P}_m^7) - \{a_h^1 | h \equiv r + (s/2), r + ((s+m)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + (s/2), r + ((s+m)/2) \pmod{m}\} = \bar{R}_{10} = \{a_r^2, a_{r+s}^2\}$, $\bar{R}_{11} = R\{a_r^3, a_{r+1}^3\} = R\{a_r^2, a_{r+2}^2\}$ and $\bar{R}_{12} = R\{a_r^3, a_{r+p}^3\} = R\{a_r^2, a_{r+s}^2\}$ where $s \geq 4$, respectively.

Clearly, $R_r = 3m - 4 < |R\{a_r^1, a_r^2\}|$ and $|R\{a_r^1, a_r^2\} \cap \cup_{r=1}^m R_r| = 3m - 4 \geq |R_r|$.

- The RNs of $\{a_r^1, a_{r+p}^1\}$ and $\{a_r^1, a_{r+s}^1\}$ are $\bar{R}_{13} = R\{a_r^1, a_{r+p}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((p+1)/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r + ((p+m-1)/2) \pmod{m}\}$ and $\bar{R}_{14} = R\{a_r^1, a_{r+s}^1\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r + ((m+s)/2) \pmod{m}\} \cup \{a_h^3 | h \equiv r + (s/2) \pmod{m}\}$, respectively. Clearly, $R_r = 3(m-1) < |R\{a_r^1, a_r^2\}|$ and $|R\{a_r^1, a_r^2\} \cap \cup_{r=1}^m R_r| = 3m - 4 \geq |R_r|$.
- The RN of a_r^2, a_r^3 is $\bar{R}_{15} = R\{a_r^2, a_r^3\} = V(\mathbb{P}_m^7) - \{a_h^1 | h \equiv r, r-1, \dots, r-(m/2)+1 \pmod{m}\}$. Clearly, $|R_r| = (5m/2) < |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3\} \cap \cup_{r=1}^m R_r| = ((5m+2)/2) \geq |R_r|$.
- The RN of a_r^2, a_{r-1}^2 is $\bar{R}_{16} = R\{a_r^2, a_r^3\} = V(\mathbb{P}_m^7) - \{a_h^1 | h \equiv r, r+1, \dots, r+(m/2)-1 \pmod{m}\}$. Clearly, $|R_r| = (5m/2) < |R\{a_r^2, a_r^3\}|$ and $|R\{a_r^2, a_r^3\} \cap \cup_{r=1}^m R_r| = ((5m+2)/2) \geq |R_r|$.

TABLE 33: \bar{R}_u for $1 \leq u \leq 39$.

RNs	Elements	Equality
$R\{a_1^1, a_2^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_5^1\}$	$R\{a_4^1, a_5^1\}, R\{a_2^1, a_3^1\}, R\{a_4^1, a_5^1\}, R\{a_4^1, a_7^1\}, R\{a_4^1, a_7^1\}$
$R\{a_1^1, a_3^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_5^1\}$	$R\{a_1^1, a_6^1\}, R\{a_2^1, a_3^1\}, R\{a_2^1, a_5^1\}, R\{a_1^1, a_4^1\}, R\{a_2^1, a_4^1\}, R\{a_5^1, a_8^1\}, R\{a_5^1, a_8^1\}, R\{a_5^1, a_8^1\}$
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_5^1\}$	$R\{a_7^1, a_8^1\}, R\{a_3^1, a_4^1\}, R\{a_7^1, a_8^1\}, R\{a_2^1, a_5^1\}, R\{a_2^1, a_5^1\}, R\{a_1^1, a_6^1\}, R\{a_1^1, a_6^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_5^1\}$	
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_4^1\}$	
$R\{a_1^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_1^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_2^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_3^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_3^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_3^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_4^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_4^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_4^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_5^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_5^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	
$R\{a_5^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_2^1\} \cup \{a_3^1\}$	

- (g) The RN of $\{a_r^2, a_{r+p}^3\}$ is $\bar{R}_{17} = R\{a_r^2, a_{r+p}^3\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r + ((p+m-1)/2) \pmod{m}\}$.
Clearly, $|R_r| < |\bar{R}_{17}| = 3m - 1$ and
 $|R\{a_r^2, a_{r+p}^3\} \cap \cup_{r=1}^m R_r| = ((5m+2)/2) \geq |R_r|$.
- (h) The RN of $\{a_r^1, a_{r+1}^2\}$ is $\bar{R}_{18} = R\{a_r^1, a_{r+1}^2\} = V(\mathbb{G}) - \{a_h^1 | h \equiv r+1, r+2, \dots, r+(m/2) \pmod{m}\} \cup \{a_h^2 | h \equiv r+3 \pmod{m}\} \cup \{a_h^3 | h \equiv r, r+2 \pmod{m}\}$.
Clearly, $|R_r| < |\bar{R}_{18}| = 3m - 1$ and
 $|R\{a_r^1, a_{r+1}^2\} \cap \cup_{r=1}^m R_r| = ((5m+2)/2) \geq |R_r|$. \square

Theorem 8. If $\mathbb{N} \cong \mathbb{P}_m^7$ with $m \geq 6$ and $m \equiv 0 \pmod{2}$. Then, $\dim_f(\mathbb{P}_m^7) \leq (6m/(3m+2))$.

Proof

Case 1. When $m = 6$.

The RNs are given as follows.

Tables 33–37 show the RNs of \mathbb{P}_6^7 with cardinalities 16, 15, 14, 13, and 10, respectively. Moreover, in Table 37, $\cup_{r=1}^{12} R_r = V(\mathbb{P}_6^7)$ which implies $|\cup_{r=1}^{12} R_r| = 18$ and $|\bar{R}_u \cap \cup_{r=1}^{12} R_r| \geq |R_r|$, where $1 \leq u \leq 81$.

Now, we define a function $\kappa: V(\mathbb{P}_6^7) \rightarrow [0, 1]$ such that $\kappa(a_r^1) = \kappa(a_r^2) = \kappa(a_r^3) = (1/10)$. Furthermore, R_r for $1 \leq r \leq 12$ are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_6^7 such that $|\bar{\kappa}| \leq |\kappa|$. As a result, $\dim_f(\mathbb{P}_6^7) \leq \sum_{r=1}^{18} (1/10) = (9/5)$.

Case 2. $m \geq 8$.

As Lemma 8 clears the fact that $|R\{a_r^2, a_r^3\}| = |R\{a_r^2, a_{r-1}^3\}| = (3/2)m + 1 \leq |R\{a, b\}|$ for all $\{a, b\} \subset V(\mathbb{P}_m^7)$ and $|R\{a, b\} \cap \cup_{r=1}^m R_r| \leq |R_r|$.

Let $\eta = |\cup_{r=1}^m R_r| = 3m$ and $\lambda = |R_r| = (3/2)m + 1$. Then, we define a function $\kappa: V(\mathbb{P}_m^7) \rightarrow [0, 1]$ such that $\kappa(c) = \begin{cases} 1/\lambda & \text{for } c \in \cup_{r=1}^m R_r, \\ 0 & \text{for } c \in V(\mathbb{P}_m^7) - \cup_{r=1}^m R_r. \end{cases}$ We find that κ is a resolving function for \mathbb{P}_m^7 because $\kappa(R\{a, b\}) \geq 1 \forall \{a, b\} \subset V(\mathbb{P}_m^7)$. Assume on contrary that

TABLE 34: \bar{R}_u for $40 \leq u \leq 51$.

RNs	Elements
$R\{a_1^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_1^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_1^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_1^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_1^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_3^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_2^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_3^1, a_4^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_3^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_3^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_3^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_3^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_4^1, a_5^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_4^1, a_6^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_4^1, a_7^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$
$R\{a_4^1, a_8^1\}$	$V(\mathbb{P}_6^7) - \{a_3^1\} \cup \{a_2^1\} \cup \{a_5^1\}$

there is another resolving function ρ , such that $\rho(u) \leq \kappa(u)$, for at least one $u \in V(\mathbb{P}_m^7)$. As a result, $\rho(R\{a, b\}) < 1$, where $R\{a, b\}$ is an RN of \mathbb{P}_m^7 having the minimum cardinality of λ . This shows that ρ is not a resolving function, a contradiction. Thus, κ is a minimal resolving function for \mathbb{P}_m^7 that achieves minimum $|\kappa|$. Also R_r for $1 \leq t \leq 2m$ of \mathbb{P}_m^7 are pairwise overlapping; hence, \exists another minimal resolving function $\bar{\kappa}$ of \mathbb{P}_m^7 such that $|\bar{\kappa}| \leq |\kappa|$. Therefore, assigning $(1/\lambda)$ to all the vertices of \mathbb{P}_m^7 and calculating their sum, we get $\dim_f(\mathbb{P}_m^7) \leq \sum_{r=1}^{\eta} (1/\lambda) = (3m/((3/2)m+1)) = (6m/(3m+2))$. \square

6. Conclusion

In this article,

- we have found the upper bounds of FMD of symmetric networks called by TCL (\mathbb{T}_m), QCLs (\mathbb{Q}_m^1 and \mathbb{Q}_m^2), and PCLs ($\mathbb{P}_m^1, \mathbb{P}_m^2, \mathbb{P}_m^3, \mathbb{P}_m^5$, and \mathbb{P}_m^7).
- Table 38 shows the summary of main results, and Table 39 gives the values of FMDs as they tend to ∞ .

TABLE 35: \bar{R}_u for $52 \leq u \leq 69$.

RNs	Elements	Equality
$R\{a_1^1, a_1^2\}$	$V(\mathbb{P}_6^7) - \{a_2^2, a_2^3\} \cup \{a_1^3, a_6^3\}$	
$R\{a_1^1, a_2^2\}$	$V(\mathbb{P}_6^7) - \{a_1^2, a_3^2\} \cup \{a_1^3, a_2^3\}$	
$R\{a_1^1, a_3^2\}$	$V(\mathbb{P}_6^7) - \{a_2^2, a_2^3\} \cup \{a_2^3, a_3^3\}$	
$R\{a_1^1, a_4^2\}$	$V(\mathbb{P}_6^7) - \{a_3^2, a_5^2\} \cup \{a_3^3, a_4^3\}$	
$R\{a_1^1, a_5^2\}$	$V(\mathbb{P}_6^7) - \{a_4^2, a_5^2\} \cup \{a_4^3, a_5^3\}$	
$R\{a_1^1, a_6^2\}$	$V(\mathbb{P}_6^7) - \{a_2^2, a_5^2\} \cup \{a_5^3, a_6^3\}$	
$R\{a_1^1, a_1^3\}$	$V(\mathbb{P}_6^7) - \{a_1^2, a_5^2\} \cup \{a_2^2, a_2^3\}$	$R\{a_1^1, a_6^1\}, R\{a_1^2, a_2^2\}, R\{a_4^2, a_6^2\}$
$R\{a_1^1, a_1^4\}$	$V(\mathbb{P}_6^7) - \{a_1^2, a_6^2\} \cup \{a_3^2, a_6^2\}$	$R\{a_1^1, a_5^1\}, R\{a_2^2, a_5^2\}, R\{a_1^2, a_5^2\}$
$R\{a_1^1, a_1^5\}$	$V(\mathbb{P}_6^7) - \{a_1^2, a_4^2\} \cup \{a_2^2, a_4^2\}$	$R\{a_2^1, a_6^1\}, R\{a_3^2, a_5^2\}, R\{a_2^2, a_6^2\}$

TABLE 36: \bar{R}_u for $70 \leq u \leq 81$.

RNs	Elements
$R\{a_1^2, a_1^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_4^3, a_5^3\}$
$R\{a_1^2, a_2^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_6^1\} \cup \{a_3^3, a_6^3\}$
$R\{a_1^2, a_3^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^3, a_5^3\}$
$R\{a_1^2, a_4^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_3^1, a_4^1\} \cup \{a_1^3, a_2^3\}$
$R\{a_1^2, a_5^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_4^1, a_5^1\} \cup \{a_2^3, a_3^3\}$
$R\{a_1^2, a_6^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_3^3, a_4^3\}$
$R\{a_2^2, a_1^3\}$	$V(\mathbb{P}_6^7) - \{a_2^1, a_3^1, a_4^1\} \cup \{a_3^3, a_5^3\}$
$R\{a_2^2, a_2^3\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_4^1, a_5^1\} \cup \{a_4^3, a_5^3\}$
$R\{a_2^2, a_3^3\}$	$V(\mathbb{P}_6^7) - \{a_4^1, a_5^1, a_6^1\} \cup \{a_5^3, a_6^3\}$
$R\{a_2^2, a_4^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_5^1, a_6^1\} \cup \{a_5^3, a_6^3\}$
$R\{a_2^2, a_5^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_1^3, a_2^3\}$
$R\{a_2^2, a_6^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} \cup \{a_2^3, a_3^3\}$

TABLE 37: R_r with minimum cardinalities for $1 \leq r \leq 12$.

RNs	Elements
$R_1 = R\{a_1^1, a_1^3\}$	$V(\mathbb{P}_6^7) - \{a_2^1, a_3^1, a_4^1\} - \{a_2^2, a_3^2, a_4^2\} - \{a_2^3, a_3^3\}$
$R_2 = R\{a_1^1, a_2^3\}$	$V(\mathbb{P}_6^7) - \{a_3^1, a_4^1, a_5^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_3^3, a_4^3\}$
$R_3 = R\{a_1^1, a_3^3\}$	$V(\mathbb{P}_6^7) - \{a_4^1, a_5^1, a_6^1\} - \{a_3^2, a_5^2, a_6^2\} - \{a_4^3, a_5^3\}$
$R_4 = R\{a_1^1, a_4^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_5^1, a_6^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_3^3, a_6^3\}$
$R_5 = R\{a_1^1, a_5^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_6^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_1^3, a_6^3\}$
$R_6 = R\{a_1^1, a_6^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_1^3, a_2^3\}$
$R_7 = R\{a_1^1, a_1^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_5^1, a_6^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_3^3, a_6^3\}$
$R_8 = R\{a_1^1, a_2^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_6^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_1^3, a_6^3\}$
$R_9 = R\{a_1^1, a_3^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_1^3, a_2^3\}$
$R_{10} = R\{a_1^1, a_4^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_2^1, a_3^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_1^3, a_2^3\}$
$R_{11} = R\{a_1^1, a_5^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_4^1, a_5^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_3^3, a_4^3\}$
$R_{12} = R\{a_1^1, a_6^3\}$	$V(\mathbb{P}_6^7) - \{a_1^1, a_4^1, a_5^1\} - \{a_2^2, a_2^3, a_2^4\} - \{a_3^3, a_4^3\}$

TABLE 38: FMD of rotationally symmetric networks having different FMD for $m \geq 8$.

Network	Upper bound of \dim_f	Comment
\mathbb{T}_m	$2m/(m+1)$	Bounded
\mathbb{Q}_m^1	2	Bounded and constant
\mathbb{Q}_m^2	2	Bounded and constant
\mathbb{P}_m^1	$6m/(5m-2)$	Bounded
\mathbb{P}_m^2	$6m/(3m+2)$	Bounded
\mathbb{P}_m^3	$3m/(2m-1)$	Bounded
\mathbb{P}_m^5	$6m/(3m+2)$	Bounded
\mathbb{P}_m^7	$6m/(3m+2)$	Bounded

TABLE 39: Values of FMD as they tends to ∞ , where $m \geq 8 \wedge m \equiv 0 \pmod{2}$.

Network	Values of FMD as they tends to ∞
\mathbb{T}_m	$\lim_{m \rightarrow \infty} (2m/(m+1))$
\mathbb{P}_m^2	$\lim_{m \rightarrow \infty} (6m/(5m-2))$
\mathbb{P}_m^2	$\lim_{m \rightarrow \infty} (6m/(3m+2))$
\mathbb{P}_m^3	$\lim_{m \rightarrow \infty} (3m/(2m-1))$
\mathbb{P}_m^5	$\lim_{m \rightarrow \infty} (6m/(3m+2))$
\mathbb{P}_m^7	$\lim_{m \rightarrow \infty} (6m/(3m+2))$

- (iii) the networks having the maximum FMD of 2 are \mathbb{Q}_m^1 and \mathbb{Q}_m^2 .
- (iv) in contrast, \mathbb{T}_m is the network bearing the minimum FMD of $(2m/(m+1))$.
- (v) the obtained results are the generalization of [23].
- (vi) to find the extremal values of FMD of asymmetric networks is still an open problem.
- (vii) moreover, the evaluation of the FMD and LFMD of connected networks such as convex polytopes is an open problem as well.

Data Availability

The data used to support the findings of this study are included within this paper and are available from the corresponding author upon request.

Conflicts of Interest

The authors have no conflicts of interest.

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Research Article

Computing the Narumi–Katayama Index and Modified Narumi–Katayama Index of Some Families of Dendrimers and Tetrathiafulvalene

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Received 6 May 2021; Revised 26 May 2021; Accepted 5 June 2021; Published 16 June 2021

Academic Editor: Naeem Saleem

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A dendrimer is an artificially manufactured or synthesized molecule built up from branched units called monomers. In mathematical chemistry, a particular attention is given to degree-based graph invariant. The Narumi–Katayama index and its modified Narumi–Katayama index of a graph G denoted by $NK(G)$ and $NK^*(G)$ are equal to the product of the degrees of the vertices of G . In this paper, we calculate the Narumi–Katayama Index and modified Narumi–Katayama index for some families of dendrimers.

1. Introduction

A molecular graph is a simple graph related to the structure of a chemical compound. Each vertex of a molecular graph represents an atom of the molecule and its edges to the bonds between atoms. Chemical Graph Theory has an important effect on the development of Chemical Sciences.

In Chemical Science, the multiplicative connectivity indices are used in the analysis of drug molecular structures which are helpful to find out the biological and chemical characteristics of drugs.

Dendrimers are a new class of polymeric materials. They are highly branched, monodisperse macromolecules. The structure of these materials has a great impact on their physical and chemical properties. In chemistry, biochemistry, and nanotechnology, different topological indices are used for modeling physicochemical, pharmacologic, toxicologic, biological, and other properties of chemical compounds. As a result of their unique behavior, dendrimers are suitable for a wide range of biomedical and industrial applications [1].

A molecular graph $G=(V, E)$ with the vertex set $V(G)$ and the edge set $E(G)$ is a graph whose vertices denote atoms and edges denote bonds between the atoms of any underlying chemical structure. The degree of a vertex v of G denoted by $d_G(v)$ is the number of edges that are incident to it (for simplicity, $d_G(v)=d_v$). A topological index $\text{Top}(G)$ of graph G is a number with the property, so $\text{Top}(H)=\text{Top}(G)$ means a graph H is isomorphic with a graph G . The idea of topological list originated from work done by Wiener [2]. In [3], Narumi and Katayama considered the product of d_v over all degrees of vertices in G as “simple topological index.” Then, the papers, mostly used the name “Narumi–Katayama index” for this index. So, we use from it in this paper, too. In [4–6], authors studied some properties of Narumi–Katayama indices as follows:

$$NK(G) = \prod_{u \in V(G)} d_u, \quad (1)$$

and the modified of Narumi–Katayama indices as follows:

$$NK^*(G) = \prod_{u \in V(G)} d_u^{d_u}. \quad (2)$$

Several articles contributed to determining the topological indices of some families of dendrimer structures and nanostar dendrimers (see [7–15]), porphyrin dendrimers (see [16]), and EThyleneAmidoAmine dendrimers (see [17, 18]).

In this paper, we compute the Narumi–Katayama index and modified Narumi–Katayama index for some families of dendrimers like $PD_1[n]$ be PAMAM dendrimers with n growth of stages and $n \in \mathbb{N}$. For example, the graph $PD_2[3]$ is shown in Figure 1. Another kind of dendrimers, namely, tetrathiafulvalene dendrimer (see Figures 2 and 3), is denoted by $TD_2[n]$, $n \in \mathbb{N} \cup \{0\}$. In Figure 3, we can see the graph $TD_2[0]$ and $TD_2[2]$.

2. Main Results

In this section, we shall compute the Narumi–Katayama indices and modified Narumi–Katayama index of some families of dendrimers, $PD_1[n]$, $PD_2[n]$, and $TD_2[n]$.

Theorem 1. Let $PD_1[n]$ be PAMAM dendrimers with n growth of stages where $n \in \mathbb{N} \cup \{0\}$. Then, the Narumi–Katayama index and modified Narumi–Katayama index of $PD_1[n]$ are given by

- (i) $NK(PD_1[n]) = 2^{30 \times 2^n - 15} \times 3^{9 \times 2^n - 5}$.
- (ii) $NK^*(PD_1[n]) = 2^{60 \times 2^n - 30} \times 3^{27 \times 2^n - 15}$.

Proof. Let $TD_1[n] = G_n$ where $n \in \mathbb{N} \cup \{0\}$. The number of vertices and edges in G_n is $48 \times 2^n - 23$ and $48 \times 2^n - 24$, respectively. The vertex set $V(G_n)$ can be divided into three vertex partitions based on degrees of vertices as V_1 , V_2 , and V_3 , where $V_i = \{u | u \in V(G_n), \deg(u) = i\}$; $1 \leq i \leq 3$. It is easy to see that $|V_1(G_n)| = 9 \times 2^n - 3$; moreover, we have

$$\begin{cases} V_1(G_n) + 2V_2(G_n) + 3V_3(G_n) = 2E(G_n) \\ V_1(G_n) + V_2(G_n) + V_3(G_n) = V(G_n) \end{cases} \quad (3)$$

Therefore, by solving the above system of equations, the number of vertices in $V_2(G_n)$ and $V_3(G_n)$ is $30 \times 2^n - 15$ and $9 \times 2^n - 5$. Now, by using (1) and (2), we have

$$\begin{aligned} NK(G_n) &= \prod_{u \in V(G_n)} du \\ &= \prod_{u_1 \in V_1(G_n)} du_1 \times \prod_{u_2 \in V_2(G_n)} du_2 \times \prod_{u_3 \in V_3(G_n)} du_3 \\ (i) \quad &= 1^{|V_1(G_n)|} \times 2^{|V_2(G_n)|} \times 3^{|V_3(G_n)|} \\ &= 1^1 \times 2^{30 \times 2^n - 15} \times 3^{9 \times 2^n - 5} \\ &= 2^{30 \times 2^n - 15} \times 3^{9 \times 2^n - 5}. \end{aligned}$$

$$\begin{aligned} NK^*(G_n) &= \prod_{u \in V(G_n)} du^{du} \\ &= \prod_{u_1 \in V_1(G_n)} du_1^{du_1} \times \prod_{u_2 \in V_2(G_n)} du_2^{du_2} \times \prod_{u_3 \in V_3(G_n)} du_3^{du_3} \\ (ii) \quad &= 1^{|V_1(G_n)|} \times 2^{2|V_2(G_n)|} \times 3^{3|V_3(G_n)|} \\ &= 2^{2 \times 30 \times 2^n - 15} \times 3^{3 \times 9 \times 2^n - 5} \\ &= 2^{60 \times 2^n - 30} \times 3^{27 \times 2^n - 15}. \end{aligned}$$

□

Theorem 2. Let $PD_2[n]$ be PAMAM dendrimers with n growth of stages and $n \in \mathbb{N}$. Then, the Narumi–Katayama index and its modified of $PD_2[n]$ are given by

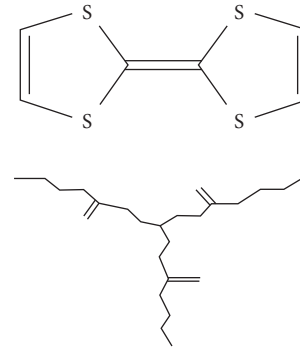


FIGURE 1: Tetrathiafulvalene ($H_2C_2S_2C$)₂ and the core of $PD_1[0]$.

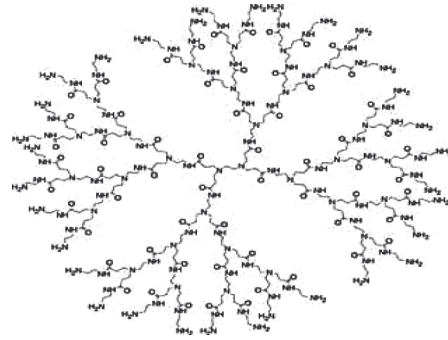


FIGURE 2: PAMAM dendrimers with 3 growth stages $PD_2[3]$ [18].

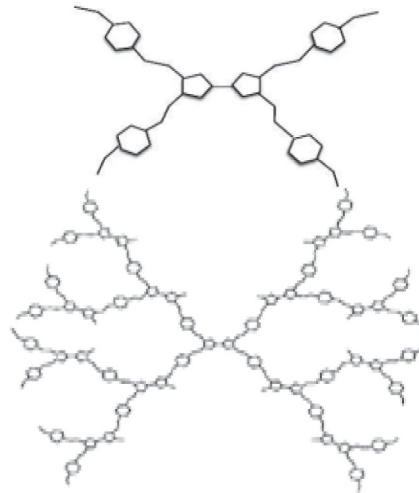


FIGURE 3: Tetrathiafulvalene dendrimer with 2 growth stages $TD_2[0]$ and $TD_2[2]$ [19–21].

- (i) $NK(PD_2[n]) = 2^{40 \times 2^n - 4} \times 3^{12 \times 2^n + 20}$.
- (ii) $NK^*(PD_2[n]) = 2^{80 \times 2^n - 8} \times 3^{36 \times 2^n + 60}$.

Proof. This is similar to the proofs of Theorem 1. □

Theorem 3. Let $TD_2[n]$ be tetrathiafulvalene dendrimer with n growth of stages and $n \in \mathbb{N} \cup \{0\}$. Then, the Narumi–Katayama index and its modified of $TD_2[n]$ are given by

- (i) $NK(TD_2[n]) = 2^{76 \times 2^n - 44} \times 3^{40 \times 2^n - 26}$.

$$(ii) NK^*(TD_2[n]) = 2^{152 \times 2^n - 88} \times 3^{120 \times 2^n - 78}.$$

Proof. This is similar to the proofs of Theorems 1 and 2. \square

Example 1. Consider tetrathiafulvalene dendrimer $TD_2[0] = G_0$ where $n \in \mathbb{N} \cup \{0\}$ is shown in Figure 3. Theorem 3, $|V(G_0)| = 50$ and $|E(G_0)| = 55$. The vertex partitions $V_1(G_0)$, $V_2(G_0)$, and $V_3(G_0)$ contain, respectively, 4, 32, and 14 vertices. Then,

$$\begin{aligned} NK(G_0) &= \prod_{u \in V(G_0)} du \\ (i) &= \prod_{u_1 \in V_1(G_0)} du_1 \times \prod_{u_2 \in V_2(G_0)} du_2 \times \prod_{u_3 \in V_3(G_0)} du_3 \\ &= 1^{|V_1(G_0)|} \times 2^{|V_2(G_0)|} \times 3^{|V_3(G_0)|} \\ &= 2^{32} \times 3^{14}. \\ NK^*(G_0) &= \prod_{u \in V(G_0)} du^{du} \\ (ii) &= \prod_{u_1 \in V_1(G_0)} du_1^{du_1} \times \prod_{u_2 \in V_2(G_0)} du_2^{du_2} \times \prod_{u_3 \in V_3(G_0)} du_3^{du_3} \\ &= 1^{|V_1(G_0)|} \times 2^{2^{|V_2(G_0)|}} \times 3^{3^{|V_3(G_0)|}} \\ &= 2^{64} \times 3^{42}. \end{aligned}$$

3. Conclusions

In this paper, we determined the Narumi–Katayama index and modified Narumi–Katayama index for some families of dendrimers, namely, PAMAM and tetrathiafulvalene dendrimer. In the future, we are interested to study and compute topological indices of various families of dendrimers or nanostructures, in general.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Common Fixed Point Results on Generalized Weak Compatible Mapping in Quasi-Partial b-Metric Space

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Received 21 February 2021; Revised 14 April 2021; Accepted 31 May 2021; Published 11 June 2021

Academic Editor: Huseyin Isik

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The focus of this paper is to acquaint with generalized condition (B) in a quasi-partial b-metric space and to establish coincidence and common fixed point theorems for weakly compatible pairs of mapping. Additionally, with the background of quasi-partial b-metric space, the outcomes obtained are exemplified to prove the existence and uniqueness of fixed point.

1. Introduction

In the early years of 20th century, the French mathematician Fréchet [1] commenced the concept of metric space, and due to its consequences and practicable implementations, the idea has been enlarged, upgraded, and generalized in different directions. In 1922, Banach [2] introduced the very important Banach contraction principle which holds a remarkable position in the field on nonlinear analysis. One such generalization was established by Künzi et al. [3] known as quasi-partial metric space by Karapinar et al. [4, 5]. In 1993, Czerwik [6] introduced the concept of b-metric space. Later, Gupta and Gautam [7, 8] generalized quasi-partial metric space to quasi-partial b-metric space and proved some fixed point results for such spaces. Several authors [9–18] have already proved the fixed point theorem in metric space, partial metric space [19], quasi-partial metric space, quasi-partial b-metric space [7], and many different spaces. After these classical results, some researchers [20–25] introduced the distinctive concepts and used fixed point theorems to demonstrate the uniqueness of a solution of the equations in different metric spaces such as multivalued contractive type mappings, Reich–Rus–Cirić

and Hardy–Rogers contraction mappings, and Chatterjea and cyclic Chatterjea contraction.

In this paper, we have introduced the generalized condition (B) in quasi-partial b-metric space to obtain coincidence and common fixed points. Moreover, some examples are given to exemplify the concept followed up with pictographic grid.

2. Preliminaries

Let us recall some definition.

Definition 1 (see [19]). A partial metric space on a nonempty set X is a function $M: X \times X \longrightarrow \mathbb{R}^+$ satisfying

- (1) $M(\tau, v) = M(v, \tau)$ (symmetry)
- (2) if $0 < M(\tau, \tau) = M(\tau, v) = M(v, v)$, then $\tau = v$ (indistancy implies equality)
- (3) $M(v, v) \leq M(\tau, v)$, then $\tau = v$ (small self-distances)
- (4) $M(\tau, \Upsilon) + M(v, v) \leq M(\tau, v) + M(v, \Upsilon)$ (triangularity)

for all $\tau, v, \Upsilon \in X$.

Definition 2 (see [4]). A quasi-partial metric on a nonempty set X is a function $q: X \times X \rightarrow \mathbb{R}^+$ satisfying

- (1) If $q(\tau, \tau) = q(\tau, v) = q(v, v)$, then $\tau = v$ (indistancy implies equality)
 - (2) $q(\tau, \tau) \leq q(\tau, v)$ (small self-distances)
 - (3) $q(\tau, \tau) \leq q(v, \tau)$ (small self-distances)
 - (4) $q(\tau, v) + q(Y, Y) \leq q(\tau, Y) + q(Y, v)$ (triangularity)
- for all $\tau, v, Y \in X$.

Definition 3 (see [20]). A quasi-partial b-metric on a nonempty set X is a function $qp_b: X \times X \rightarrow \mathbb{R}^+$ such that for some real number $\rho \geq 1$

- (1) If $qp_b(\tau, \tau) = qp_b(\tau, v) = qp_b(v, v)$, then $\tau = v$ (indistancy implies equality)
- (2) $qp_b(\tau, \tau) \leq qp_b(\tau, v)$ (small self-distances)
- (3) $qp_b(\tau, \tau) \leq qp_b(v, \tau)$ (small self-distances)
- (4) $qp_b(\tau, v) + qp_b(Y, Y) \leq \rho\{qp_b(\tau, Y) + qp_b(v, Y)\}$ (triangularity)

for all $\tau, v, Y \in X$. The infimum over all reals $\rho \geq 1$ satisfying condition (30) is called the coefficient of (X, qp_b) and represented by $R(X, qp_b)$.

Lemma 1 (see [6]). Let (X, qp_b) be a quasi-partial b-metric space. Then the following hold:

- (1) If $qp_b(\tau, v) = 0$, then $\tau = v$
- (2) If $\tau \neq v$, then $qp_b(\tau, v) > 0$ and $qp_b(v, \tau) > 0$

Definition 4 (see [6]). Let (X, qp_b) be a quasi-partial b-metric. Then

- (1) A sequence $\{\tau_n\} \subset X$ converges to $\tau \in X$ if and only if

$$qp_b(\tau, \tau) = \lim_{n, m \rightarrow \infty} qp_b(\tau, \tau_n) = \lim_{n, m \rightarrow \infty} qp_b(\tau_n, \tau). \quad (1)$$

- (2) A sequence $\{\tau_n\} \subset X$ is called a Cauchy sequence if and only if

$$\lim_{n, m \rightarrow \infty} qp_b(\tau_n, \tau_m), \quad \lim_{n, m \rightarrow \infty} qp_b(\tau_m, \tau_n) \text{ exist.} \quad (2)$$

- (3) The quasi-partial b-metric space (X, qp_b) is said to be complete if every Cauchy sequence $\{\tau_n\} \subset X$ converges with respect to x_{qp_b} to a point $\tau \in X$ such that

$$qp_b(\tau, \tau) = \lim_{n, m \rightarrow \infty} qp_b(\tau_n, \tau_m) = \lim_{n, m \rightarrow \infty} qp_b(\tau_m, \tau_n). \quad (3)$$

- (4) A mapping $f: X \rightarrow X$ is said to be continuous at $\tau_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(\tau_0, \delta)) \subset B(f(\tau_0), \varepsilon)$.

Lemma 2 (see [6]). Let (X, qp_b) be a quasi-partial b-metric space and (X, d_{qp_b}) be the corresponding b-metric space. Then (X, d_{qp_b}) is complete if (X, qp_b) is complete.

Definition 5 (see [26]). A self-mapping P on a metric space (X, d) satisfies condition (B), if there exist $\delta \in [0, 1]$ and $\omega > 0$ such that for all $\tau, v \in X$, we have,

$$d(P\tau, Pv) \leq \delta d(P\tau, Pv) + \omega \min\{d(P\tau, Pv), d(v, Pv), d(\tau, Pv), d(v, P\tau)\}. \quad (4)$$

Following Babu et al. [26], Abbas et al. [27] and Abbas and Illic [28] extended the concept of condition (B) to a pair of mappings. Abbas et al. [27] called it generalized condition (B), and Abbas and Illic [28] called it generalized almost A-contraction.

Definition 6 (see [27]). Let P and Q be two self-mappings on a metric space (X, d) . The mapping Q satisfies generalized condition (B) associated with P if there exist $\delta \in (0, 1)$ and $\omega \geq 0$ with $\rho \geq 1$ such that

$$d(Q\tau, Qv) \leq \delta \max\left\{d(P\tau, Pv), d(P\tau, Q\tau), d(Pv, Pv), \frac{1}{2\rho}\{d(P\tau, Qv) + d(Pv, Q\tau)\}\right\} \\ + \omega \min\{d(P\tau, Q\tau), d(Pv, Qv), d(P\tau, Qv), d(Pv, Q\tau)\}. \quad (5)$$

Clearly condition (B) implies generalized condition (B).

if $Px = Qx = w$, where w is called a point of coincidence of P and Q .

Definition 7 (see [29]). Let P and Q be self-mappings on a set X . A point $x \in X$ is called a coincidence point of P and Q

Definition 8 (see [30]). Let X be a nonempty set. Two mappings $P, Q: X \rightarrow X$ are said to be weakly compatible if

they commute at their coincidence point, that is, if $Pu = Qu$ for some $u \in X$, then $PQu = QPu$.

3. Main Results

Definition 9. Let P and R be two self-mappings on a quasi-partial b-metric space (X, qp_b) .

The mapping R satisfies generalized condition (B) associated with P (R is a generalized almost P contraction) if there exist $\delta \in (0, 1)$, $\rho \geq 1$, and $M \geq 0$ such that for all $\tau, v \in X$, we have

$$qp_b(R\tau, Rv) \leq \delta \max \left\{ qp_b(P\tau, Pv), qp_b(P\tau, R\tau), qp_b(Pv, Rv), \frac{1}{2\rho} (qp_b(R\tau, Pv) + qp_b(P\tau, Rv)) \right\} \\ + M \min \{ qp_b(P\tau, R\tau), qp_b(Pv, Rv), qp_b(P\tau, Rv), qp_b(Pv, R\tau) \}. \quad (6)$$

Definition 10. Let P, Q, R, S be four self-mappings on a quasi-partial b-metric space (X, qp_b) .

The pair of mapping (P, R) satisfies generalized condition (B) associated with (Q, S) ((P, R) is generalized almost

(Q, S) contraction) if there exist $\delta \in (0, 1)$, $\rho \geq 1$ and $M \geq 0$ such that for all $\tau, v \in X$, we have

$$qp_b(R\tau, Sv) \leq \delta \max \left\{ qp_b(P\tau, Qv), qp_b(P\tau, R\tau), qp_b(Qv, Sv), \frac{1}{2\rho} (qp_b(R\tau, Qv) + qp_b(P\tau, Sv)) \right\} \\ + M \min \{ qp_b(P\tau, R\tau), qp_b(Qv, Sv), qp_b(P\tau, Sv), qp_b(Qv, R\tau) \}. \quad (7)$$

Theorem 1. Let P, Q, R, S be four self-mappings on quasi-partial b-metric space (X, qp_b) and if we take the mappings in pair as (P, R) associated with (Q, S) for all $\tau, v \in X$, $\delta \in (0, 1)$, and $M \geq 0$, $\rho \geq 1$ and

- (1) $RX \subset QX$ and $SX \subset PX$
- (2) PX or QX is closed
- (3) $(1/\rho)(\delta + 2M) < 1$

then the pairs (P, R) and (Q, S) have a coincidence point. Also P, Q, R, S have a unique common fixed point, providing that pairs (P, R) and (Q, S) are weakly compatible.

Proof. Let $\tau^* \in X$. Since $RX \subset QX$ there exists $\tau_0 \in X$ such that $v_0 = Q\tau_0 = R\tau^*$. Suppose there exists a point $v_1 \in S\tau_0$ corresponding to the point v_0 . Also since $SX \subset PX$ there exist $\tau_1 \in X$ such that $v_1 = P\tau_1 = S\tau_0$. Going this way we get a sequence $\{v_n\} \in X$ as

$$v_{2m+1} = Q\tau_{2m+1} = R\tau_{2m}, \\ v_{2m+2} = P\tau_{2m+2} = R\tau_{2m+1}, \\ qp_b(v_{2m+1}, v_{2m+2}) = qp_b(R\tau_{2m}, S\tau_{2m+1}) \\ \leq \delta \max \{ qp_b(P\tau_{2m}, Q\tau_{2m+1}), qp_b(P\tau_{2m}, R\tau_{2m}), qp_b(Q\tau_{2m+1}, S\tau_{2m+1}), \\ \frac{1}{2\rho} (qp_b(R\tau_{2m}, Q\tau_{2m+1}) + qp_b(P\tau_{2m}, S\tau_{2m+1})) \} + \\ M \min \{ qp_b(P\tau_{2m}, R\tau_{2m}), qp_b(Q\tau_{2m+1}, S\tau_{2m+1}), \\ qp_b(P\tau_{2m}, R\tau_{2m+1}), qp_b(Q\tau_{2m+1}, R\tau_{2m}) \} \\ \leq \delta \max \{ qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2}), \\ \frac{1}{2\rho} (qp_b(v_{2m+1}, v_{2m+1}) + qp_b(v_{2m}, v_{2m+2})) \} + \\ M \min \{ qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2}) \\ = \delta \max \{ qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2}) \} + \\ M \min \{ qp_b(v_{2m}, v_{2m+2}), qp_b(v_{2m+1}, v_{2m+1}) \}. \quad (8)$$

This condition gives 4 cases.

Case 1.

$$\max\{qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2})\} = qp_b(v_{2m}, v_{2m+1}). \quad (9)$$

Also,

$$\min\{qp_b(v_{2m}, v_{2m+2}), qp_b(v_{2m+1}, v_{2m+1})\} = qp_b(v_{2m}, v_{2m+2}). \quad (10)$$

which implies

$$\begin{aligned} \rho qp_b(v_{2m+1}, v_{2m+2}) &\leq \delta qp_b(v_{2m}, v_{2m+1}) + M qp_b(v_{2m}, v_{2m+2}) \\ &\leq (\delta + M) qp_b(v_{2m}, v_{2m+1}) + \rho M qp_b(v_{2m+1}, v_{2m+2}) \\ &\leq \frac{\delta + M}{(1 - M)\rho} (qp_b(v_{2m}, v_{2m+1})). \end{aligned} \quad (11)$$

Let $\mu_1 = ((\delta + M)/(1 - M)\rho)$, $((\delta + 2M)/\rho) < 1$ and $M \geq 0$ then $\mu_1 < 1$.

Therefore, $qp_b(v_{2m+1}, v_{2m+2}) \leq \mu_1 qp_b(v_{2m}, v_{2m+1})$.

Let $\mu_3 = (M/(\rho(1 - \delta - M)))$, $((\delta + 2M)/\rho) < 1$ then $\mu_3 < 1$.

Therefore, $qp_b(v_{2m+1}, v_{2m+2}) \leq \mu_3 qp_b(v_{2m}, v_{2m+1})$.

Case 2.

$$\max\{qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2})\} = qp_b(v_{2m+1}, v_{2m+2}). \quad (12)$$

Also,

$$\min\{qp_b(v_{2m}, v_{2m+2}), qp_b(v_{2m+1}, v_{2m+1})\} = qp_b(v_{2m+1}, v_{2m+1}). \quad (13)$$

which implies,

$$\begin{aligned} qp_b(v_{2m+1}, v_{2m+2}) &\leq \frac{1}{\rho} \{\delta qp_b(v_{2m}, v_{2m+1})\} + M qp_b(v_{2m}, v_{2m+1}) \\ &\leq \frac{\delta + M}{\rho} qp_b(v_{2m}, v_{2m+1}). \end{aligned} \quad (14)$$

Let $\mu_2 = ((\delta + M)/\rho)$, $((\delta + 2M)/\rho) < 1$ then $\mu_2 < 1$.

Therefore, $qp_b(v_{2m+1}, v_{2m+2}) \leq \mu_2 qp_b(v_{2m}, v_{2m+1})$.

Case 3.

$$\max\{qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2})\} = qp_b(v_{2m+1}, v_{2m+2}). \quad (15)$$

Also,

$$\min\{qp_b(v_{2m}, v_{2m+2}), qp_b(v_{2m+1}, v_{2m+1})\} = qp_b(v_{2m}, v_{2m+2}). \quad (16)$$

which implies

$$\begin{aligned} qp_b(v_{2m+1}, v_{2m+2}) &\leq \delta qp_b(v_{2m+1}, v_{2m+2}) + M qp_b(v_{2m}, v_{2m+2}) \\ &\leq \frac{M}{\rho(1 - \delta - M)} qp_b(v_{2m}, v_{2m+1}). \end{aligned} \quad (17)$$

Case 4.

$$\max\{qp_b(v_{2m}, v_{2m+1}), qp_b(v_{2m+1}, v_{2m+2})\} = qp_b(v_{2m+1}, v_{2m+2}). \quad (18)$$

Also,

$$\min\{qp_b(v_{2m}, v_{2m+2}), qp_b(v_{2m+1}, v_{2m+1})\} = qp_b(v_{2m+1}, v_{2m+1}). \quad (19)$$

which implies

$$\begin{aligned} qp_b(v_{2m+1}, v_{2m+2}) &\leq \delta qp_b(v_{2m+1}, v_{2m+2}) + M qp_b(v_{2m+1}, v_{2m+1}) \\ &\leq \frac{M}{\rho(1 - \delta)} qp_b(v_{2m}, v_{2m+1}). \end{aligned} \quad (20)$$

Let $\mu_4 = (M/(\rho(1 - \delta)))$, $((\delta + 2M)/\rho) < 1$ then $\mu_4 < 1$.

Therefore, $qp_b(v_{2m+1}, v_{2m+2}) \leq \mu_4 qp_b(v_{2m+1}, v_{2m+2})$.

Choose $\mu = \max\{\mu_1, \mu_2, \mu_3, \mu_4\} \Rightarrow 0 < \mu < 1$.

$$\Rightarrow qp_b(v_{2m+1}, v_{2m+2}) \leq \mu qp_b(v_{2m}, v_{2m+1}). \quad (21)$$

Using mathematical induction,

$$qp_b(v_m, v_{m+1}) \leq \mu^m qp_b(v^*, v_0), \quad (22)$$

which tends to 0 as m tends to ∞

So, $\{v_m\}$ and its subsequence is convergent

Let PX be closed. Therefore, $\tau \in PX$, that is, there exists $Y \in X$ such that $\tau = PY$, and we need to show $\tau = RY$

By definition,

$$qp_b(RY, \tau) \leq \frac{\delta + M}{\rho} qp_b(RY, \tau), \quad (23)$$

which is a contradiction. Hence,

$$qp_b(RY, \tau) = 0 \Rightarrow RY = \tau. \quad (24)$$

So, $PY = RY$, that is, P and R have a coincidence point.

Similarly, Q and S have a coincidence point.

If we also assume QX is closed, then (P, R) and (Q, S) have a coincidence point.

Since (P, R) and (Q, S) are weakly compatible, we can prove there exists a common fixed point for P, Q, R, S by contradiction.

Example 1. Let $X = [0, 4]$ equipped with quasi-partial b-metric $qp_b(\tau, v) = |\tau - v| + |\tau|$. Let P, Q, R, S be self-mappings on quasi-partial b-metric defined by

$$\begin{aligned} P\tau &= \begin{cases} \frac{\tau}{2}, & \tau \in [0, 2], \\ \frac{5}{4}, & \tau \in (2, 4], \end{cases} \\ Q\tau &= \begin{cases} \frac{3\tau}{2}, & \tau \in [0, 2], \\ \frac{3}{2}, & \tau \in (2, 4] \end{cases} \\ R\tau &= \begin{cases} \frac{\tau}{6}, & \tau \in [0, 2], \\ \frac{1}{2}, & \tau \in (2, 4] \end{cases} \\ S\tau &= \begin{cases} \frac{\tau}{4}, & \tau \in [0, 2] \\ \frac{1}{4}, & \tau \in (2, 4]. \end{cases} \end{aligned} \quad (25)$$

Here,

$$\begin{aligned} PX &= \left[0, \frac{1}{2}\right] \cup \left\{\frac{5}{4}\right\} \\ QX &= \left[0, \frac{3}{2}\right], \\ SX &= \left[0, \frac{1}{4}\right] \subset PX \\ RX &= \left[0, \frac{1}{6}\right] \cup \left\{\frac{1}{2}\right\}. \end{aligned} \quad (26)$$

The point 0 is a coincidence point of these mapping. Furthermore, $PR0 = RP0 = 0$ and $SQ0 = QS0 = 0$, that is, the two pairs (P, R) and (Q, S) are weakly compatible.

Case 1. For $\tau, v \in [0, 2]$, we have

$$Q(R\tau, Sv) = \left\{ \left| \frac{\tau}{6} - \frac{v}{4} \right| + \left| \frac{\tau}{6} \right| \right\} \leq \frac{8}{5} \left\{ \left| \frac{\tau}{2} - \frac{3v}{2} \right| + \left| \frac{\tau}{2} \right| \right\}. \quad (27)$$

Dominance of right-hand side of equation (27) is easily visually checked in Figure 1. Thus the inequality required in Definition 10 holds for $\tau, v \in [0, 2]$.

Case 2. For $\tau \in [0, 2]$, $v \in [2, 4]$ we have

$$Q(R\tau, Sv) = \left\{ \left| \frac{\tau}{6} - \frac{1}{4} \right| + \left| \frac{\tau}{6} \right| \right\} \leq \frac{8}{5} \left\{ \left| \frac{3}{2} - \frac{1}{4} \right| + \left| \frac{3}{2} \right| \right\}. \quad (28)$$

Dominance of right-hand side of equation (28) is easily visually checked in Figure 2. Thus the inequality required in Definition 10 holds for $\tau \in [0, 2]$, $v \in [2, 4]$.

Case 3. For $\tau \in [2, 4]$, $v \in [0, 2]$ we have

$$Q(R\tau, Sv) = \left\{ \left| \frac{1}{2} - \frac{v}{4} \right| + \left| \frac{1}{2} \right| \right\} \leq \frac{8}{5} \left\{ \left| \frac{5}{4} - \frac{3v}{2} \right| + \left| \frac{5}{4} \right| \right\}. \quad (29)$$

Dominance of right-hand side of equation (29) is easily visually checked in Figure 3. Thus the inequality required in Definition 10 holds for $\tau \in [2, 4]$, $v \in [0, 2]$.

Case 4. For $\tau, v \in [2, 4]$, we have

$$Q(R\tau, Sv) = \left\{ \left| \frac{1}{2} - \frac{1}{4} \right| + \left| \frac{1}{2} \right| \right\} \leq \frac{8}{5} \left\{ \left| \frac{3}{2} - \frac{1}{4} \right| + \left| \frac{3}{2} \right| \right\}. \quad (30)$$

Dominance of right-hand side of equation (30) is easily visually checked in Figure 4. Thus the inequality required in Definition 10 holds for $\tau, v \in [2, 4]$.

As a result, all postulates of Theorem 1 are satisfied ($\delta = (4/5)$, $\rho = 2 \geq 1$, and $M = 0$) and 0 is a unique common fixed point of P, Q, R, S .

If $P = Q$ and $R = S$, we get a corollary.

Corollary 1. Let P and S be self-mappings on quasi-partial b-metric space (X, qp_b) . If for all $\tau, v \in X$, P satisfies the following conditions:

- (1) $SX \subset PX$
- (2) PX is closed
- (3) $((\delta + 2M)/\rho) < 1$

then P and S have a coincidence point. Also P and S have a common fixed point if (P, S) are weakly compatible.

Proof. Taking $P = Q$ and $R = S$ in Theorem 1, the above result can be obtained.

Theorem 2. Let P, Q, R, S be self-mappings on a quasi-partial b-metric space (X, qp_b) . If the pair (P, R) is associated with (Q, S) and satisfies

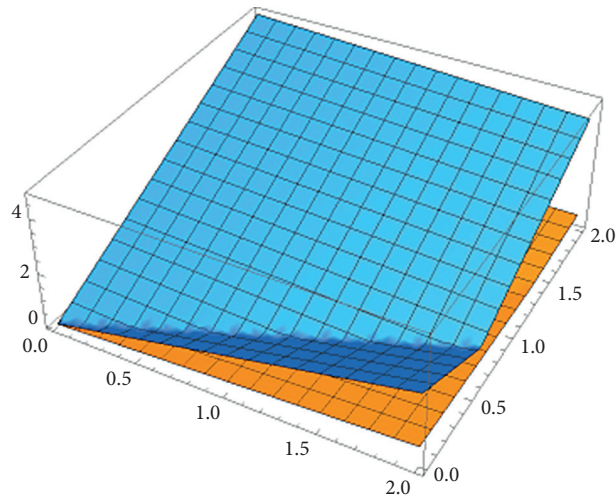


FIGURE 1: Dominance of right-hand side of equation (27) is visually checked for $\tau, v \in [0, 2]$.

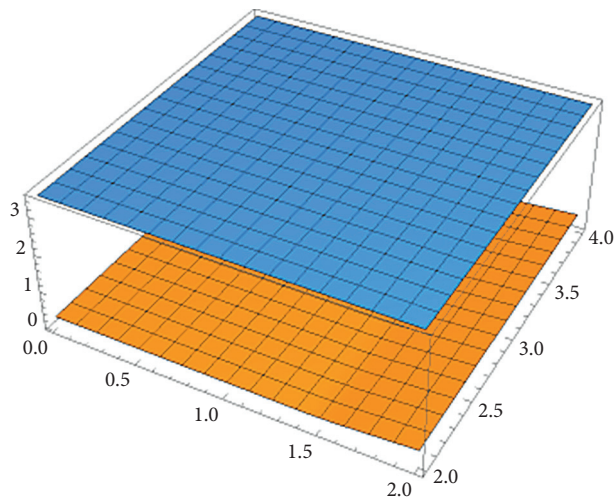


FIGURE 2: Dominance of right-hand side of equation (28) is visually checked for $\tau \in [0, 2], v \in [2, 4]$.

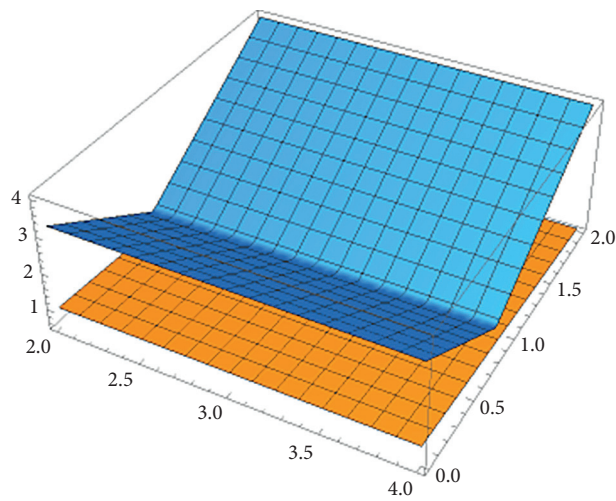


FIGURE 3: Dominance of right-hand side of equation (29) is visually checked for $\tau \in [2, 4], v \in [0, 2]$.

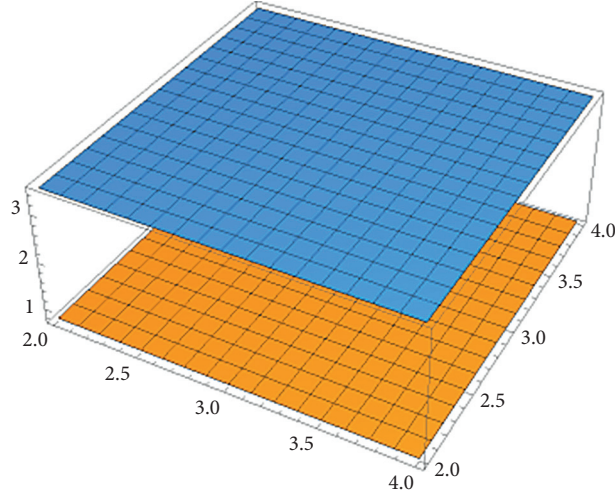


FIGURE 4: Dominance of right-hand side of equation (30) is visually checked for $\tau, v \in [2, 4]$.

$$qp_b(R\tau, Sv) \leq \delta \{ \max(qp_b(P\tau, R\tau), qp_b(P\tau, Sv), qp_b(Qv, Sv), qp_b(P\tau, Sv), qp_b(R\tau, Qv)) \} \\ + M \min\{qp_b(P\tau, R\tau), qp_b(Qv, Sv), qp_b(P\tau, Sv), qp_b(Qv, R\tau)\}, \quad (31)$$

with $\delta \in (0, 1)$, $M \geq 0$, and $p \geq 1$, for all $\tau, v \in X$, and

(1) $SX \subset PX$ and $RX \subset QX$

(2) $((\delta + 2M)/\rho) < 1$

then the pairs (P, R) and (Q, S) have a coincidence point. Also P, Q, R, S have a common fixed point.

Proof. This can be done following the same steps as the proof of Theorem 1.

Example 2. Let $X = [0, \infty)$ equipped with quasi-partial b-metric $qp_b(\tau, v) = |\tau - v| + |\tau|$. Let P, Q, R, S be self-mappings on quasi-partial b-metric defined by

$$\begin{aligned} P\tau &= \begin{cases} 2\tau, & \tau \in [0, 2], \\ 4, & \tau > 2, \end{cases} \\ Q\tau &= \begin{cases} \tau, & \tau \in [0, 2], \\ 2, & \tau > 2, \end{cases} \\ R\tau &= \begin{cases} \frac{\tau}{5}, & \tau \in [0, 2], \\ 2, & \tau > 2, \end{cases} \\ S\tau &= \begin{cases} \frac{2\tau}{5}, & \tau \in [0, 2], \\ 1, & \tau > 2. \end{cases} \end{aligned} \quad (32)$$

Here,

$$SX = \left[0, \frac{2}{5}\right] \cup \{1\} \cup [0, 2] \cup \{4\} = PX, \quad (33)$$

$$RX = \left[0, \frac{1}{5}\right] \cup \{2\} \subset [0, 1] \cup \{2\} = QX.$$

The point 0 is a coincidence point of these mapping. Furthermore, $PR0 = RP0 = 0$ and $SQ0 = QS0 = 0$, that is, the two pairs (P, R) and (Q, S) , are weakly compatible.

Case 1. For $\tau, v \in [0, 2]$, we have

$$\begin{aligned} Q(R\tau, Sv) &= \left\{ \left| \frac{\tau}{5} - \frac{2v}{5} \right| + \left| \frac{\tau}{5} \right| \right\} = \frac{1}{5} \{ |4\tau - 2v| + |2\tau| \} \\ &\leq \frac{10}{9} \{ |2\tau - v| + |2\tau| \}. \end{aligned} \quad (34)$$

Dominance of the right-hand side of equation (34) is easily visually checked in Figure 5. Thus the inequality required in theorem holds for $\tau, v \in [0, 2]$.

Case 2. For $\tau \in [0, 2]$, $v > 2$ we have

$$Q(R\tau, Sv) = \left\{ \left| \frac{\tau}{5} - 1 \right| + \left| \frac{\tau}{5} \right| \right\} \leq \frac{10}{9} \{ |2 - 1| + |2| \}. \quad (35)$$

Dominance of the right-hand side of equation (35) is easily visually checked in Figure 6. Thus the inequality required in theorem holds for $\tau \in [0, 2]$, $v > 2$.

Case 3. For $\tau > 2$, $v \in [0, 2]$, we have

$$Q(R\tau, Sv) = \left\{ \left| 2 - \frac{2v}{5} \right| + |2| \right\} \leq \frac{10}{9} \left\{ \left| 4 - \frac{2v}{5} \right| + |4| \right\}. \quad (36)$$

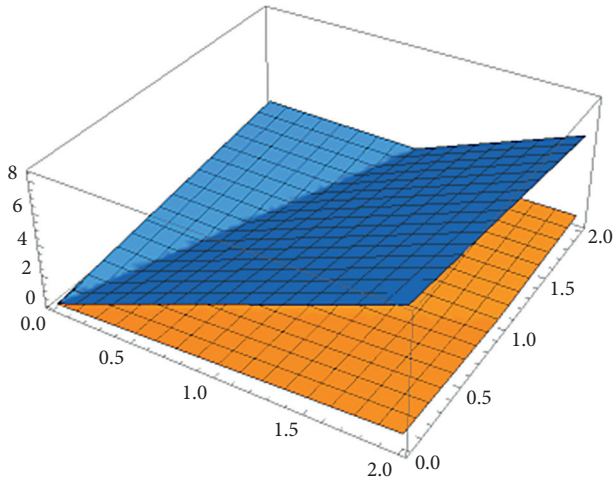


FIGURE 5: Dominance of the right-hand side of equation (34) is visually checked for $\tau, v \in [0, 2]$.

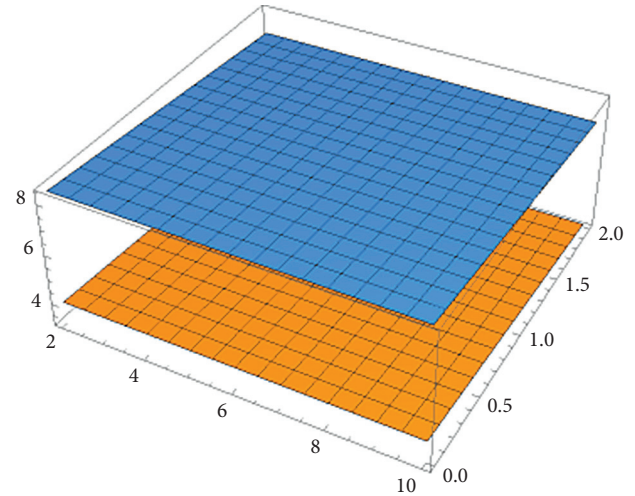


FIGURE 7: Dominance of the right-hand side of equation (36) is visually checked for $\tau > 2, v \in [0, 2]$.

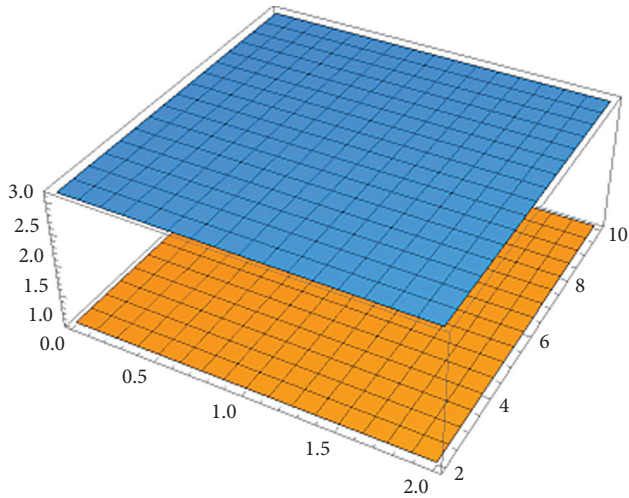


FIGURE 6: Dominance of the right-hand side of equation (35) is visually checked for $\tau \in [0, 2], v > 2$.

Dominance of the right-hand side of equation (36) is easily visually checked in Figure 7. Thus the inequality required in theorem holds for $\tau > 2, v \in [0, 2]$.

Case 4. For $\tau, v > 2$, we have

$$Q(R\tau, Sv) = \{|2 - 1| + |2|\} \leq \frac{30}{9}. \quad (37)$$

Dominance of the right-hand side of equation (37) is easily visually checked in Figure 8. Thus the inequality required in theorem holds for $\tau, v > 2$.

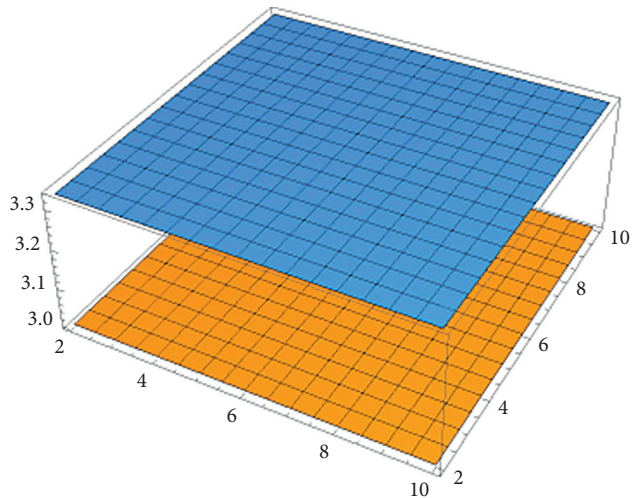


FIGURE 8: Dominance of the right-hand side of equation (37) is visually checked for $\tau, v > 2$.

As a result, all postulates of Theorem 2 are satisfied ($\delta = (5/9), \rho = 2 \geq 1$, and $M = 0$), and 0 is a unique common fixed point of P, Q, R, S .

If $P = Q$ and $R = S$, we get a corollary.

Corollary 2. Let P and S be self-mappings on quasi-partial b -metric space (X, qp_b) . If for all $\tau, v \in X$, the pair of mapping (P, S) satisfies

$$qp_b(R\tau, Sv) \leq \delta \{ \max \{ qp_b(P\tau, R\tau), qp_b(P\tau, Sv), qp_b(Qv, Sv), qp_b(P\tau, Sv), qp_b(R\tau, Qv) \} \} + M \min \{ qp_b(P\tau, R\tau), qp_b(Qv, Sv), qp_b(P\tau, Sv), qp_b(Qv, R\tau) \}, \quad (38)$$

and P satisfies the following conditions:

- (1) $SX \subset PX$
- (2) $((\delta + 2M)/\rho) < 1$

then the pair (P, S) has a coincidence point. Also P and S have a common fixed point if (P, S) are weakly compatible.

Proof. Taking $P = Q$ and $R = S$ in Theorem 2, the above result can be obtained.

4. Conclusion

This paper expounds a new notion in quasi-partial b-metric space which is generalized condition (B) that helped to demonstrate coincidence and common fixed point for two weakly compatible pairs of self-mappings. The incentive behind using quasi-partial b-metric space is the fact that the distance from point x to point y may be different to that from y to x , and the self-distance of a point need not always be zero; also the distance between two points x and z is not equal to the sum of the two distances having a point y in between x and z . Furthermore, the results acquired are validated by explanatory examples.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Some Results on Generalized Pata–Suzuki Type Contractive Mappings

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Received 14 April 2021; Accepted 27 May 2021; Published 9 June 2021

Academic Editor: Zoran Mitrovic

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In this work, we establish new fixed point theorems for generalized Pata–Suzuki type contraction via α -admissible mapping in metric spaces and to prove some fixed point results for such mappings. Moreover, we give an example to illustrate our main result. Consequently, the results presented in this paper generalize and improve the corresponding results of the literature.

1. Introduction and Preliminaries

The Banach contraction principle was introduced by Banach [1] which is one of the earlier and the most important result in fixed point theory. Because of its importance, over the years, many authors extended and generalized the Banach contraction principle in many directions.

The concept of almost contraction was given by Berinde and Pacurar [2]. Also this concept was compared with other contractions and proved some fixed point theorems by Berinde ([2–4]).

C-condition was introduced by Suzuki ([5–7]). Suzuki [8] proved generalized versions of Edelstein theorem [9] for compact metric spaces.

Firstly, Samet et al. [10] introduced an interesting contraction which is called α - ϱ -contractive and gave α -admissible mappings and investigated the existence and uniqueness of such mappings in the setting of complete metric spaces. Later, in addition to Samet et al. [10], some different fixed point theorems were introduced by Karapinar and Samet [11] and Babu et al. [12].

Recently, one of the famous generalizations of the Banach contraction principle for the existence and uniqueness of fixed points for self mappings on metric spaces is the theorem by Pata [13]. Since Pata's fixed point

theorem, some authors have studied this theorem in several ways (see [14–18]).

Using the ideas of Pata [13], we give a generalization of Pata-type contractions. The purpose of this paper is to introduce almost (α, ϱ) -Pata–Suzuki type contractive mapping and to prove some fixed point results for such mappings with admissibility condition. These results are generalizations of various results in the literature.

Now, we give some definitions and fundamental results.

Let (E, ρ) be a metric space; a point $u \in E$ is said to be a fixed point of $h: E \rightarrow E$ if $u = hu$. The set of all fixed points of f is denoted by $\text{Fix}(h)$.

Berinde and Pacurar [2] introduced the concept of almost contraction as a generalization of Banach contraction principle.

Definition 1 (see [2]). Let (E, ρ) be a metric space and $h: E \rightarrow E$ be a mapping which is called an almost contraction if there exists a constant $\vartheta \in (0, 1)$ and $L \geq 0$ such that

$$\rho(ht, hs) \leq \vartheta(t, s) + L\rho(s, ht), \quad (1)$$

for all $t, s \in E$.

In 2011, Pata [13] proved the following result.

Theorem 1 (see [13]). Let (E, ρ) be a complete metric space and Ψ denote the class of all increasing function $\psi: [0, 1] \rightarrow [0, \infty)$, which vanishes with continuity at zero. For an arbitrary $t_0 \in E$, we denote $\|t\| = \rho(t, t_0)$, $\forall t \in E$. Let $\Lambda \geq 0$, $\xi \geq 1$, and $\beta \in [0, \xi]$ be fixed constants and $\psi \in \Psi$ and $f: E \rightarrow E$ be a function. If for all $t, s \in E$, the following inequality is satisfied:

$$\rho(ht, hs) \leq (1 - \varepsilon)\rho(t, s) + \Lambda\varepsilon^\xi \psi(\varepsilon)[1 + \|t\| + \|s\|]^\beta, \quad (2)$$

for all $\varepsilon \in [0, 1]$, and then h has a unique fixed point $u = hu$.

Suzuki ([5–7]) introduced C-condition as follows.

Definition 2. Let (E, ρ) be a metric space and h be a given self mapping on E . h is said to satisfy C-condition if for all $t, s \in E$

$$\frac{1}{2}\rho(t, ht) \leq \rho(t, s) \text{ implies } \rho(ht, hs) \leq \rho(t, s). \quad (3)$$

Samet et al. [10] gave admissibility condition and the definition of α - ϱ -contractive mapping as follows.

Definition 3 (see [10]). Let (E, ρ) be a metric space, $h: E \rightarrow E$ be a map, and $\alpha: E \times E \rightarrow [0, +\infty)$ be a function.

- (i) If $\alpha(t, s) \geq 1$ implies $\alpha(ht, hs) \geq 1$ for all $t, s \in E$, then h is said to be α -admissible
- (ii) If h is α -admissible and $\alpha(t, z) \geq 1$ and $\alpha(t, s) \geq 1$ imply $\alpha(t, s) \geq 1$, then h is said to be triangular α -admissible.

Υ denotes the family of nondecreasing functions $\varrho: [0, +\infty) \rightarrow [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \varrho^n(t) < +\infty$ for each $t > 0$, where ϱ^n is the n -th iterate of ϱ .

Remark 1. Every function $\varrho \in \Upsilon$ satisfies $\lim_{n \rightarrow \infty} \varrho^n(t) = 0$ and $\varrho(t) < t$ for all $t > 0$.

Definition 4 (see [10]). Let (E, ρ) be a metric space and $h: E \rightarrow E$ be a mapping. We say that h is an α - ϱ -contractive mapping if there exist two functions $\alpha: E \times E \rightarrow [0, +\infty)$ and $\varrho \in \Upsilon$ such that $\alpha(t, s)\rho(ht, hs) \leq \varrho(\rho(t, s))$ for all $t, s \in E$.

2. Almost (α, ϱ) – Pata–Suzuki Type Contraction

In this section, we introduce concept of almost (α, ϱ) –Pata–Suzuki type contractions in metric spaces. We establish some fixed point results for such contractions on metric spaces.

The following lemma is necessary in our Theorem's proofs.

Lemma 1. Let (E, ρ) be a metric space and $\{t_n\}$ be a sequence in E such that $\rho(t_{n+1}, t_n) \rightarrow 0$ as $n \rightarrow \infty$. If $\{t_n\}$ is not a Cauchy sequence, then there exist a $\varsigma > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $\rho(t_{m_k}, t_{n_k}) \geq \varsigma$ and $\rho(t_{m_{k-1}}, t_{n_k}) \leq \varsigma$ and $\lim_{k \rightarrow \infty} \rho$

$(t_{m_{k-1}}, t_{n_{k+1}}) = \varsigma$, $\lim_{k \rightarrow \infty} \rho(t_{m_k}, t_{n_k}) = \varsigma$, and $\lim_{k \rightarrow \infty} \rho(t_{m_{k-1}}, t_{n_k}) = \varsigma$.

From Lemma 1, we obtain $\lim_{k \rightarrow \infty} \rho(t_{m_{k+1}}, t_{n_{k+1}}) = \varsigma$ and $\lim_{k \rightarrow \infty} \rho(t_{m_k}, t_{n_{k-1}}) = \varsigma$.

Let (E, ρ) be a complete metric space. Along this paper, Ψ denotes the class of all increasing function $\psi: [0, 1] \rightarrow [0, \infty)$, which vanishes with continuity at zero. For an arbitrary $t_0 \in E$, we denote $\|t\| = \rho(t, t_0)$, $\forall t \in E$.

First, we begin with the following definition.

Definition 5. Let $\Lambda \geq 0$, $\xi \geq 1$, and $\beta \in [0, \xi]$ be fixed constants, and $\psi \in \Psi$ and $h: E \rightarrow E$ be two functions. There exist two functions $\varrho \in \Upsilon$, $\alpha: E \times E \rightarrow [0, +\infty)$ and $L \geq 0$ such that, if for all $t, s \in E$, and $\varepsilon \in [0, 1]$, h satisfies the following inequality:

$$\frac{1}{2}\rho(t, ht) \leq \rho(t, s), \quad (4)$$

which implies

$$\begin{aligned} \alpha(t, s)\rho(ht, hs) &\leq (1 - \varepsilon)(\varrho(M(t, s)) + LN(t, s)) \\ &\quad + \Lambda\varepsilon^\xi \psi(\varepsilon)[1 + \|t\| + \|s\| + \|ht\| + \|hs\|]^\beta, \end{aligned} \quad (5)$$

where

$$M(t, s) = \max\left\{\rho(t, s), \frac{\rho(t, ht) + \rho(s, hs)}{2}, \frac{\rho(t, hs) + \rho(s, ht)}{2}\right\},$$

$$N(t, s) = \min\{\rho(t, ht), \rho(s, hs), \rho(t, hs), \rho(s, ht)\}. \quad (6)$$

Then, we say that h is an almost (α, ϱ) –Pata–Suzuki contractive mapping.

Now, we are ready to prove our first result.

Theorem 2. Let (E, ρ) be a complete metric space. $h: E \rightarrow E$ be an almost (α, ϱ) –Pata–Suzuki contractive mapping. Assume that

- (i) h is triangular α -admissible
- (ii) There exists $x_0 \in E$ such that $\alpha(t, ht_0) \geq 1$
- (iii) h is continuous
- (iv) For all $u, v \in \text{Fix}h$, $\alpha(u, v) \geq 1$

Then, h has a unique fixed point, that is, $u = hu$, $u \in E$.

Proof. From the hypothesis (ii) of Theorem 2, there exists $t_0 \in E$ such that $\alpha(t_0, ht_0) \geq 1$. Starting at the point $t_0 \in E$, the sequence $\{t_n\}$ is constructed by $t_n = ht_{n-1}$, $n \geq 1$. If $t_{n_0+1} = t_{n_0}$ for any $n_0 \in \mathbb{N}$, then x_{n_0} is a fixed point of h . Consequently, assume that $t_{n_0+1} \neq t_{n_0}$ for all $n_0 \in \mathbb{N}$. First of all, we show that $\alpha(t_n, t_{n+1}) \geq 1$ for all $n \in \mathbb{N}$. Since h is an α -admissible mapping, we have

$$\begin{aligned} \alpha(t_0, t_1) \geq 1 &= \alpha(t_0, ht_0) \geq 1 \text{ implies } \alpha(t_1, t_2) \geq 1, \\ \alpha(t_1, t_2) \geq 1 &\text{ implies } \alpha(t_2, t_3) \geq 1. \end{aligned} \quad (7)$$

By induction, we obtain

$$\alpha(t_n, t_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}. \quad (8)$$

Hence, by induction, we obtain

$$\alpha(t_n, t_m) \geq 1, \quad \text{for all } m > n \geq 0. \quad (10)$$

From the hypothesis (i) of Theorem 2, we have

$$\begin{aligned} \alpha(t_n, t_{n+1}) &\geq 1, \\ \alpha(t_{n+1}, t_{n+2}) &\geq 1 \text{ imply } \alpha(t_n, t_{n+2}) \geq 1. \end{aligned} \quad (9)$$

Now, we will show that $\{\rho(t_{n+1}, t_n)\}$ is a decreasing sequence. Using Remark 1, since $(1/2)\rho(t_{n-1}, t_n) \leq \rho(t_{n-1}, t_n)$, we obtain

$$\begin{aligned} \rho(t_{n+1}, t_n) &\leq (1 - \varepsilon) \left(\varrho \left(\max \left\{ \rho(t_n, t_{n-1}), \frac{\rho(t_{n+1}, t_n) + \rho(t_n, t_{n-1})}{2}, \frac{\rho(t_n, t) + \rho(t_{n-1}, t_{n+1})}{2} \right\} \right) \right. \\ &\quad \left. + L \min \{ \rho(t_{n+1}, t_n), \rho(t_n, t_{n-1}), \rho(t_n, t), \rho(t_{n-1}, t_{n+1}) \} \right) \\ &\quad + \Lambda \varepsilon^\xi \psi(\varepsilon) [1 + \|t_{n-1}\| + \|t_n\| + \|t_n\| + \|t_{n+1}\|]^\beta \\ &\leq (1 - \varepsilon) \left(\varrho \left(\max \left\{ \rho(t_n, t_{n-1}), \frac{\rho(t_{n+1}, t_n) + \rho(t_n, t_{n-1})}{2} \right\} \right) \right) + K \varepsilon^\xi \psi(\varepsilon) \\ &< (1 - \varepsilon) \left(\max \left\{ \rho(t_n, t_{n-1}), \frac{\rho(t_{n+1}, t_n) + \rho(t_n, t_{n-1})}{2} \right\} \right) + K \varepsilon^\xi \psi(\varepsilon), \end{aligned} \quad (11)$$

for some $K > 0$. If $\rho(t_n, t_{n-1}) < \rho(t_{n+1}, t_n)$, then we get $\rho(t_{n+1}, t_n) \leq (1 - \varepsilon)\rho(t_{n+1}, t_n) + K \varepsilon^\xi \psi(\varepsilon)$. In this way, we obtain $\rho(t_{n+1}, t_n) = 0$, a contraction. Thus, we have

$$\rho(t_{n+1}, t_n) < \rho(t_n, t_{n-1}) < \cdots < \rho(t_1, t) = \|t_1\|, \quad (12)$$

that is, $\{\rho(t_{n+1}, t_n)\}$ is a decreasing sequence. Since $\{\rho(t_{n+1}, t_n)\}$ is decreasing, it is convergent to a nonnegative real number. Let $\lim_{n \rightarrow \infty} (t_n, t_{n+1}) = \rho$. Now, we will show that the sequence $\{\|t_n\|\}$ is bounded. From triangle inequality, we have

$$\|t_n\| = \rho(t, t_0) \leq \rho(t_n, t_{n+1}) + \rho(t_{n+1}, t_1) + \rho(t_1, t_0). \quad (13)$$

We assert a claim that

$$\begin{aligned} \frac{1}{2} \rho(t_n, t_{n+1}) &\leq \rho(t_n, t_0) \\ \text{or } \frac{1}{2} \rho(t_{n-1}, t_n) &\leq \rho(t_{n-1}, t_0). \end{aligned} \quad (14)$$

On the contrary, suppose that

$$\begin{aligned} \frac{1}{2} \rho(t_n, t_{n+1}) &> \rho(t_n, t_0), \\ \frac{1}{2} \rho(t_{n-1}, t_n) &> \rho(t_{n-1}, t_0). \end{aligned} \quad (15)$$

From triangular inequality, we have

$$\begin{aligned} \rho(t_{n-1}, t_n) &\leq \rho(t_{n-1}, t_0) + \rho(t_0, t_n) < \frac{1}{2} [\rho(t_{n-1}, t_n) \\ &\quad + \rho(t_n, t_{n+1})] \leq \rho(t_{n-1}, t_n), \end{aligned} \quad (16)$$

which is a contradiction. Thus, (14) is satisfied. Since h is an almost (α, ϱ) -Pata-Suzuki contractive mapping and using (10) and Remark 1, we obtain

$$\begin{aligned}
\rho(t_1, t_{n+1}) &\leq \alpha(t_0, t_n) \rho(ht_0, ht_n) \\
&\leq (1 - \varepsilon) \left(\varrho \left(\max \left\{ \rho(t_n, t_0), \frac{\rho(t_n, t_{n+1}) + \rho(t_0, t_1)}{2}, \frac{\rho(t_n, t_1) + \rho(t_0, t_{n+1})}{2} \right\} \right) \right. \\
&\quad \left. + L \min \{ \rho(t_n, t_{n+1}), \rho(t_0, t_1), \rho(t_n, t_1), \rho(t_0, t_{n+1}) \} \right) \\
&\quad + \Lambda \varepsilon^\xi \psi(\varepsilon) [1 + \|t_n\| + 0 + \|t_{n+1}\| + \|t_1\|]^\beta \\
&\leq (1 - \varepsilon) \left(\varrho \left(\max \left\{ \rho(t_n, t_0), \frac{\rho(t_n, t_{n+1}) + \rho(t_0, t_1)}{2}, \frac{\rho(t_n, t_0) + \rho(t_1, t_0) + \rho(t_{n+1}, t_n) + \rho(t_n, t_0)}{2} \right\} \right) \right. \\
&\quad \left. + L \min \{ \rho(t_n, t_{n+1}), \rho(t_0, t_1), \rho(t_n, t_0) + \rho(t_0, t_1), \rho(t_0, t_{n+1}) \} \right) \\
&\quad + \Lambda \varepsilon^\xi \psi(\varepsilon) [1 + 2\|t_n\| + 2\|t_1\|]^\beta \\
&\leq (1 - \varepsilon) \left(\varrho(\max\{\|t_n\|, \|t_1\|, \|t_n\| + \|t_1\|\}) + L \min\{\|t_1\|, \|t_n\| + \|t_1\|, \|t_{n+1}\|\} \right) + \Lambda \varepsilon^\xi \psi(\varepsilon) [1 + 2\|t_n\| + 2\|t_1\|]^\beta \\
&\leq (1 - \varepsilon) \left(\varrho(\|t_n\| + \|t_1\|) + L\|t_{n+1}\| \right) + \Lambda \varepsilon^\xi \psi(\varepsilon) [1 + 2\|t_n\| + 2\|t_1\|]^\beta \\
&< (1 - \varepsilon) (\|t_n\| + \|t_1\| + L\|t_{n+1}\|) + \Lambda \varepsilon^\xi \psi(\varepsilon) [1 + 2\|t_n\| + 2\|t_1\|]^\beta.
\end{aligned} \tag{17}$$

Using $\beta \leq \xi$, we get

$$\begin{aligned}
\|t_n\| &< (1 - \varepsilon) (\|t_n\| + \|t_1\| + L\|t_1\|) + 2\|t_1\| + \Lambda \varepsilon^\xi \psi(\varepsilon) [1 + 2\|t_n\| + 2\|t_{n+1}\|]^\xi, \\
\varepsilon \|t_n\| &< k \varepsilon^\xi \psi(\varepsilon) \|t_n\|^\xi + l,
\end{aligned} \tag{18}$$

for some $k, l > 0$. By the same reason as in [13], the sequence $\{\|t_n\|\}$ is bounded. Let $\lim_{n \rightarrow \infty} \rho(t_n, t_{n+1}) = \rho$. Using Remark 1 and from (8), we have

$$\begin{aligned}
\rho(t_n, t_{n+1}) &\leq \alpha(t_{n-1}, t_n) \rho(t_n, t_{n+1}) \\
&\leq (1 - \varepsilon) \left(\varrho \left(\max \left\{ \rho(t_n, t_{n-1}), \frac{\rho(t_{n+1}, t) + \rho(t_n, t_{n-1})}{2}, \frac{\rho(t_n, t_n) + \rho(t_{n-1}, t_{n+1})}{2} \right\} \right) \right) \\
&\quad + L \min \{ \rho(t_{n+1}, t_n), \rho(t_n, t_{n-1}), \rho(t_n, t_n), \rho(t_{n-1}, t_{n+1}) \} \\
&\quad + \Lambda \varepsilon^\xi \psi(\varepsilon) [1 + \|t_n\| + \|t_{n-1}\| + \|t_{n+1}\| + \|t_n\|]^\beta \\
&\leq (1 - \varepsilon) \varrho \left(\max \left\{ \rho(t_n, t_{n-1}), \frac{\rho(t_{n+1}, t_n) + \rho(t_n, t_{n-1})}{2} \right\} \right) + K \varepsilon^\xi \psi(\varepsilon), \\
&< (1 - \varepsilon) \max \left\{ \rho(t_n, t_{n-1}), \frac{\rho(t_{n+1}, t_n) + \rho(t_n, t_{n-1})}{2} \right\} + K \varepsilon^\xi \psi(\varepsilon),
\end{aligned} \tag{19}$$

for some $K > 0$. Taking limit as $n \rightarrow \infty$, we obtain $\rho \leq K\varepsilon^\xi \psi(\varepsilon)$ and thus $\rho = 0$.

Next, we demonstrate that $\{t_n\}$ is a Cauchy sequence. Assume that the sequence $\{t_n\}$ is not a Cauchy sequence. From Lemma 1, there exist subsequence $\{t_{m_k}\}$ and $\{t_{n_k}\}$ with $n_k > m_k > k$ such that $\lim_{k \rightarrow \infty} \rho(t_{m_k-1}, t_{n_k+1}) = \varsigma$, $\lim_{k \rightarrow \infty} \rho(t_{m_k}, t_{n_k}) = \varsigma$, $\lim_{k \rightarrow \infty} \rho(t_{m_k-1}, t_{n_k}) = \varsigma$, $\lim_{k \rightarrow \infty} \rho(t_{m_k+1}, t_{n_k+1}) = \varsigma$, and $\lim_{k \rightarrow \infty} \rho(t_{m_k}, t_{n_k-1}) = \varsigma$. We assert that

$$\frac{1}{2}\rho(t_{m_k-1}, t_{m_k}) \leq \rho(t_{m_k-1}, t_{n_k-1}). \quad (20)$$

If we assume that

$$\frac{1}{2}\rho(t_{m_k-1}, t_{m_k}) > \rho(t_{m_k-1}, t_{n_k-1}), \quad (21)$$

then we obtain a contradiction. If we take limit $k \rightarrow \infty$ in inequality (21), we obtain contradiction. Since h is an almost (α, ϱ) -Pata-Suzuki contractive mapping, we have

$$\begin{aligned} \varsigma &\leq \rho(t_{m_k}, t_{n_k}) = \rho(ht_{m_k-1}, ht_{n_k-1}) \\ &\leq \alpha(t_{m_k}, t_{n_k})\rho(ht_{m_k-1}, ht_{n_k-1}) \\ &\leq (1-\varepsilon) \left(\varrho \left(\max \left\{ \rho(t_{m_k-1}, t_{n_k-1}), \frac{\rho(t_{m_k-1}, t_{m_k}) + \rho(t_{n_k-1}, t_{n_k})}{2}, \frac{\rho(t_{n_k-1}, t_{m_k}) + \rho(t_{m_k-1}, t_{n_k})}{2} \right\} \right) \right. \\ &\quad \left. + L \min \{ \rho(t_{m_k-1}, t_{m_k}), \rho(t_{n_k-1}, t_{n_k}), \rho(t_{n_k-1}, t_{m_k}), \rho(t_{m_k-1}, t_{n_k}) \} \right) \\ &\quad + \Lambda \varepsilon^\xi \psi(\varepsilon) \left[1 + \|t_{m_k}\| + \|t_{n_k}\| + \|t_{n_k+1}\| + \|t_{m_k+1}\| \right]^\beta \\ &< (1-\varepsilon) \left(\max \left\{ \rho(t_{m_k-1}, t_{n_k-1}), \frac{\rho(t_{m_k-1}, t_{m_k}) + \rho(t_{n_k-1}, t_{n_k})}{2}, \frac{\rho(t_{n_k-1}, t_{m_k}) + \rho(t_{m_k-1}, t_{n_k})}{2} \right\} \right. \\ &\quad \left. + L \min \{ \rho(t_{m_k-1}, t_{m_k}), \rho(t_{n_k-1}, t_{n_k}), \rho(t_{n_k-1}, t_{m_k}), \rho(t_{m_k-1}, t_{n_k}) \} \right) \\ &\quad + \Lambda \varepsilon^\xi \psi(\varepsilon) \left[1 + \|t_{m_k}\| + \|t_{n_k}\| + \|t_{n_k+1}\| + \|t_{m_k+1}\| \right]^\beta. \end{aligned} \quad (22)$$

Taking the limit as $k \rightarrow \infty$, we obtain

$$\varsigma \leq (1-\varepsilon)\varsigma + K\varepsilon\psi(\varepsilon), \quad (23)$$

and then

$$\varsigma \leq K\psi(\varepsilon), \quad (24)$$

that is, $\varsigma = 0$, which is a contradiction. Hence, the sequence $\{t_n\}$ is a Cauchy sequence in (E, ρ) . By the completeness of E , the sequence $\{t_n\}$ is convergent to some $u \in E$, that is,

$t_n \rightarrow u$ as $n \rightarrow +\infty$. Since h is continuous, $ht_n \rightarrow hu$ as $n \rightarrow +\infty$. By the uniqueness of the limit, we obtain $u = hu$; that is, u is a fixed point of h .

Now, we observe that fixed point of h is unique. Assume that u and v are fixed points of h . Since h satisfies the hypothesis (iv) of Theorem 2, h is an almost (α, φ) -Pata-Suzuki contractive mapping, Remark 1, and $(1/2)\rho(u, hu) \leq \rho(u, v)$, we have

$$\begin{aligned} \rho(hu, hv) &\leq \alpha(u, v)\rho(hu, hv) \\ &\leq (1-\varepsilon) \left(\varrho \left(\max \left\{ \rho(u, v), \frac{\rho(u, hu) + \rho(v, hv)}{2}, \frac{\rho(u, hv) + \rho(v, hu)}{2} \right\} \right) \right. \\ &\quad \left. + L \min \{ \rho(u, hu), \rho(v, hv), \rho(u, hv), \rho(v, hu) \} \right) + K\varepsilon\psi(\varepsilon) \\ &< (1-\varepsilon) \left(\max \left\{ \rho(u, v), \frac{\rho(u, hu) + \rho(v, hv)}{2}, \frac{\rho(u, hv) + \rho(v, hu)}{2} \right\} \right. \\ &\quad \left. + L \min \{ \rho(u, hu), \rho(v, hv), \rho(u, hv), \rho(v, hu) \} \right) + K\varepsilon\psi(\varepsilon). \end{aligned} \quad (25)$$

Hence, we obtain that $\rho(u, v) \leq K\psi(\varepsilon)$, and thus $u = v$. Hence, h has a unique fixed point in E .

Now, we prove the following theorem without the assumption of continuity of h . \square

Theorem 3. Let (E, ρ) be a complete metric space. $h: E \rightarrow E$ be an almost (α, φ) -Pata-Suzuki contractive mapping. Assume that

- (i) h is triangular α -admissible
- (ii) There exists $t_0 \in E$ such that $\alpha(t_0, ht_0) \geq 1$
- (iii) If $\{t_n\}$ is a sequence in E such that $\alpha(t_n, t_{n+1}) \geq 1$ for all n and $t_n \rightarrow u \in E$ as $n \rightarrow +\infty$, then $\alpha(t_n, u) \geq 1$ for all n
- (iv) For all $u, v \in \text{Fix}h$, $\alpha(u, v) \geq 1$

Then, h has a unique fixed point, that is, $u = hu$, $u \in E$.

Proof. Following the proof of Theorem 2, we have already known that $\{t_n\}$ is a Cauchy sequence in E . Since E is complete, we have $t_n \rightarrow u \in E$ as $n \rightarrow +\infty$. Now, we prove that $u = hu$. From (8) and hypothesis (iii) of Theorem 3, we have $\alpha(t_n, u) \geq 1$ for all n . For all $n \geq 1$, we assert that

$$\begin{aligned} \frac{1}{2}\rho(t_n, t_{n+1}) &\leq \rho(t_n, u) \\ \text{or } \frac{1}{2}\rho(t_{n-1}, t_n) &\leq \rho(t_{n-1}, u). \end{aligned} \quad (26)$$

We assume that

$$\begin{aligned} \frac{1}{2}\rho(t_n, t_{n+1}) &> \rho(t_n, u), \\ \frac{1}{2}\rho(t_{n-1}, t_n) &> \rho(t_{n-1}, u). \end{aligned} \quad (27)$$

Since $\{\rho(t_{n+1}, t_n)\}$ is a decreasing sequence,

$$\rho(t_{n-1}, t_n) \leq \rho(t_{n-1}, u) + \rho(u, t_n) < \frac{1}{2}[\rho(t_{n-1}, t_n) + \rho(t_n, t_{n+1})] \leq \rho(t_{n-1}, t_n), \quad (28)$$

which is a contradiction. Thus, (26) is held, and we have

$$\begin{aligned} \rho(hu, u) &= \rho(hu, t_{n+1}) + \rho(t_{n+1}, u) \\ &\leq \alpha(t_n, u)\rho(hu, t_{n+1}) + \rho(t_{n+1}, u) \\ &\leq (1 - \varepsilon) \left(\varrho \left(\max \left\{ \rho(u, t_n), \frac{\rho(u, hu) + \rho(t_n, t_{n+1})}{2}, \frac{\rho(u, t_{n+1}) + \rho(t_n, hu)}{2} \right\} \right) \right. \\ &\quad \left. + L \min \{ \rho(u, hu), \rho(t_n, t_{n+1}), \rho(u, t_{n+1}), \rho(t_n, hu) \} \right) \\ &\quad + \Lambda \varepsilon^\xi \psi(\varepsilon) [1 + \|t_n\| + \|u\| + \|hu\| + \|t_{n+1}\|]^\beta + d(t_{n+1}, u) \\ &< \max \left\{ \rho(u, t_n), \frac{\rho(u, tu) + \rho(t_n, t_{n+1})}{2}, \frac{\rho(u, t_{n+1}) + \rho(t_n, hu)}{2} \right\} \\ &\quad + L \min \{ \rho(u, hu), \rho(t_n, t_{n+1}), \rho(u, t_{n+1}), \rho(t_n, hu) \} + K \varepsilon^\xi \psi(\varepsilon) + \rho(t_{n+1}, u), \end{aligned} \quad (29)$$

for some $K > 0$. Taking the limit as $n \rightarrow \infty$, we obtain

$$\rho(hu, u) \leq \frac{1}{2}\rho(hu, u) + K \varepsilon^\xi \psi(\varepsilon). \quad (30)$$

Thus, we make an inference that $hu = u$ and that u is a fixed point of h . Similar to the proof of Theorem 2 the uniqueness of fixed point of h can be obtained.

We give an example related to our main theorem. \square

Example 1. Let $E = [0, 1]$ with the usual metric and define the mapping $h: E \rightarrow E$ by $h(t) = (t/3)$. Let $\varrho: [0, +\infty) \rightarrow [0, +\infty)$ be defined as $\varrho(t) = (t/2)$ and

$\alpha: E \times E \rightarrow [0, +\infty)$, $\alpha(t, s) = \begin{cases} 1 & t, s \in [0, 1] \\ 0 & t, s \notin [0, 1] \end{cases}$. It is easy to get h is triangular α -admissible. Our aim is to prove that h satisfies (5). For $x, y \in [0, 1]$,

$$\frac{1}{2}\rho(t, ht) = \frac{1}{2} \left| t - \frac{t}{3} \right| \leq \rho(t, s), \quad (31)$$

which implies

$$\rho(ht, hs) = \alpha(t, s)\rho(ht, hs) = \left| \frac{t}{3} - \frac{s}{3} \right| = \frac{1}{3}\rho(t, s). \quad (32)$$

Since $\rho(t, s) \leq M(t, s)$, we obtain

$$\rho(ht, hs) \leq \frac{1}{3}M(t, s) = \frac{2}{3}\varrho(M(t, s)), \quad (33)$$

and for $L = 0$, we can write

$$\rho(ht, hs) \leq \frac{2}{3}\varrho(M(t, s)) + LN(t, s). \quad (34)$$

For arbitrary $\varepsilon \in [0, 1]$, we can write inequality (34) as follows:

$$\begin{aligned} \rho(ht, hs) &\leq (1 - \varepsilon)(\varrho(M(t, s)) + LN(t, s)) + \left(\frac{2}{3} + \varepsilon - 1\right)M(t, s) \\ &\leq (1 - \varepsilon)(\varrho(M(t, s)) + LN(t, s)) + \left(\frac{2}{3} + \varepsilon - 1\right)(1 + \|t\| + \|s\| + \|ht\| + \|hs\|). \end{aligned} \quad (35)$$

Our aim is to prove that $\gamma \geq 0$ and $\Lambda \geq 0$ such that

$$\left(\frac{2}{3} + \varepsilon - 1\right)(1 + \|t\| + \|s\| + \|ht\| + \|hs\|) \leq \Lambda \varepsilon^{\gamma+1} (1 + \|t\| + \|s\| + \|ht\| + \|hs\|), \quad (36)$$

which satisfies for all $t, s \in [0, 1]$, and every $0 \leq \varepsilon \leq 1$. We can find $\Lambda \geq 0$ such that

$$\Lambda = \frac{(2/3) + \varepsilon - 1}{\varepsilon^{\gamma+1}}, \quad (37)$$

which satisfies for each $0 \leq \varepsilon \leq 1$ and some $\gamma \geq 0$. If we choose γ such that $(\gamma/\gamma + 1) > 1 - (2/3)$, then

$$\Lambda = \frac{\gamma^\gamma}{(\gamma + 1)^{\gamma+1} (1 - (2/3))^\gamma}. \quad (38)$$

Thus, we have that

$$\rho(t, ht) = \left|t - \frac{t}{3}\right| \leq \rho(t, s), \quad (39)$$

which implies

$$\alpha(t, s)\rho(ht, hs) \leq (1 - \varepsilon)(\varrho(M(t, s)) + LN(t, s)) + \Lambda \varepsilon^{\gamma+1} (1 + \|t\| + \|s\| + \|ht\| + \|hs\|), \quad (40)$$

which satisfies for all $t, s \in [0, 1]$ and each $\varepsilon > 0$. If $\varepsilon = 0$, it can be seen that (5) is satisfied. Hence, the conditions of Theorem 2 are satisfied with $\psi(\varepsilon) = \varepsilon^\gamma$, $\xi = \beta = 1$. By an application of Theorem 2, h has a unique fixed point in $E = [0, 1]$. It is seen that $u = 0$ is the unique fixed point of h in E .

The following result is directly obtained from Theorem 2 by taking $\varepsilon = 0$.

Corollary 1. Let $h: E \rightarrow E$ be a function. There exist two functions $\varphi \in Y$ and $\alpha: E \times E \rightarrow [0, +\infty)$ and $L \geq 0$ such that if for all $t, s \in E$, h satisfies the inequality

$$\frac{1}{2}\rho(t, ht) \leq \rho(t, s), \quad (41)$$

which implies

$$\alpha(t, s)\rho(ht, hs) \leq \varrho(M(t, s)) + LN(t, s), \quad (42)$$

where

$$M(t, s) = \max\left\{\rho(t, s), \frac{\rho(t, ht) + \rho(s, hs)}{2}, \frac{\rho(t, hs) + \rho(s, ht)}{2}\right\},$$

$$N(t, s) = \min\{\rho(t, ht), \rho(s, hs), \rho(t, ht), \rho(s, ht)\}, \quad (43)$$

and assume that

- (i) h is triangular α -admissible
- (ii) There exists $t_0 \in E$ such that $\alpha(t_0, ht_0) \geq 1$
- (iii) h is continuous
- (iv) For all $u, v \in \text{Fix}h$, $\alpha(u, v) \geq 1$

Then, h has a unique fixed point, that is, $u = hu$, $u \in E$.

Corollary 1 generalizes the results of Samet [10] and Karapinar [11].

Corollary 2. Let $\Lambda \geq 0$, $\xi \geq 1$ and $\beta \in [0, \xi]$ be fixed constants and $h: E \rightarrow E$ be a function. There exist two functions $\psi \in \Psi$ and $\alpha: E \times E \rightarrow [0, +\infty)$ and $L \geq 0$ such that if for all $t, s \in E$, and $\varepsilon \in [0, 1]$, h satisfies the inequality

$$\alpha(t, s)\rho(ht, hs) \leq (1 - \varepsilon)(M(t, s) + LN(t, s)) + \Lambda\varepsilon^\xi\psi(\varepsilon)[1 + \|t\| + \|s\| + \|ht\| + \|hs\|]^\beta, \quad (44)$$

where

$$M(t, s) = \max\left\{\rho(t, s), \frac{\rho(t, ht) + \rho(s, hs)}{2}, \frac{\rho(t, hs) + \rho(s, ht)}{2}\right\},$$

$$N(t, s) = \min\{\rho(t, ht), \rho(s, hs), \rho(t, hs), \rho(s, ht)\}, \quad (46)$$

and assume that

- (i) h is triangular α -admissible
- (ii) There exists $t_0 \in E$ such that $\alpha(t_0, ht_0) \geq 1$
- (iii) h is continuous
- (iv) For all $u, v \in \text{Fix}h$, $\alpha(u, v) \geq 1$

which implies

Then, h has a unique fixed point that is $u = hu$, $u \in E$.
If we take $L = 0$ in Theorem 2, then we say that h is (α, ϱ) -Pata-Suzuki contractive mapping and get the following corollary.

Corollary 3. Let $\Lambda \geq 0$, $\xi \geq 1$, and $\beta \in [0, \xi]$ be fixed constants; $\psi \in \Psi$ and $h: E \rightarrow E$ be two functions. There exist two functions $\varrho \in Y$ and $\alpha: E \times E \rightarrow [0, +\infty)$ such that if for all $t, s \in E$, and $\varepsilon \in [0, 1]$, h satisfies the inequality

$$\frac{1}{2}\rho(t, ht) \leq \rho(t, s), \quad (47)$$

which implies

$$\alpha(t, s)\rho(ht, hs) \leq (1 - \varepsilon)\varrho(M(t, s)) + \Lambda\varepsilon^\xi\psi(\varepsilon)[1 + \|t\| + \|s\| + \|ht\| + \|hs\|]^\beta, \quad (48)$$

where

$$M(t, s) = \max\left\{\rho(t, s), \frac{\rho(t, ht) + \rho(s, hs)}{2}, \frac{\rho(t, hs) + \rho(s, ht)}{2}\right\}, \quad (49)$$

and assume that

- (i) h is triangular α -admissible
- (ii) There exists $t_0 \in E$ such that $\alpha(t_0, ht_0) \geq 1$
- (iii) h is continuous
- (iv) For all $u, v \in \text{Fix}h$, $\alpha(u, v) \geq 1$

Then, h has a unique fixed point, that is, $u = hu$, $u \in E$.

If we take $\alpha(t, s) = 1$ for all $t, s \in E$, $M(t, s) = \rho(t, s)$ and $N(t, s) = \rho(s, ht)$, in Theorem 2, then we get the following corollary.

Corollary 4. Let (E, ρ) be a complete metric space, $\Lambda \geq 0$, $\xi \geq 1$, and $\beta \in [0, \xi]$ be fixed constants, and $\psi \in \Psi$, $\varrho \in Y$, and $h: E \rightarrow E$ be a function. There exist a function $\varphi \in Y$ and $L \geq 0$ such that if for all $t, s \in E$, h satisfies the inequality

$$\frac{1}{2}\rho(t, ht) \leq \rho(t, s), \quad (50)$$

which implies

$$\rho(ht, hs) \leq (1 - \varepsilon)(\varrho(\rho(t, s)) + L\rho(s, ht)) + \Lambda\varepsilon^\xi\psi(\varepsilon)[1 + \|t\| + \|s\| + \|ht\| + \|hs\|]^\beta, \quad (51)$$

is satisfied for all $\varepsilon \in [0, 1]$, and then h has a unique fixed point $u = hu$, $u \in E$.

If we take $\alpha(t, s) = 1$ for all $t, s \in E$, $M(t, s) = \rho(t, s)$ and $N(t, s) = \rho(s, ht)$, $\varrho(t) < t$, for all $t > 0$, in Theorem 2, then we get the following corollary.

Corollary 5. Let (E, ρ) be a complete metric space, $\Lambda \geq 0$, $\xi \geq 1$, and $\beta \in [0, \xi]$ be fixed constants, and $\psi \in \Psi$ and $h: E \rightarrow E$ be a function. There exists $L \geq 0$ such that if for all $t, s \in E$, and $\varepsilon \in [0, 1]$, h satisfies the inequality

$$\frac{1}{2}\rho(t, ht) \leq \rho(t, s), \quad (52)$$

which implies

$$\rho(ht, hs) \leq (1 - \varepsilon)(\rho(t, s) + L\rho(s, ht)) + \Lambda\varepsilon^\xi \psi(\varepsilon)[1 + \|t\| + \|s\| + \|ht\| + \|hs\|]^\beta, \quad (53)$$

is satisfied for all $\varepsilon \in [0, 1]$, and then h has a unique fixed point $u = hu$, $u \in E$.

Corollary 6. Let (E, ρ) be a complete metric space, $\Lambda \geq 0$, $\xi \geq 1$, and $\beta \in [0, \xi]$ be fixed constants, and $\psi \in \Psi$ and $h: E \longrightarrow E$ be a function. There exists $L \geq 0$ such that if for all $t, s \in E$, and $\varepsilon \in [0, 1]$, h satisfies the inequality

$$\rho(ht, hs) \leq (1 - \varepsilon)(\rho(t, s) + L\rho(s, ht)) + \Lambda\varepsilon^\xi \psi(\varepsilon)[1 + \|t\| + \|s\| + \|ht\| + \|hs\|]^\beta, \quad (54)$$

which is satisfied for all $\varepsilon \in [0, 1]$, and then h has a unique fixed point $u = hu$, $u \in E$.

Corollary 6 generalizes the results of Pata [13] and Banach [1].

Data Availability

The data used to support the findings of this study are included in the references within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The first author would like to thank TUBITAK (the Scientific and Technological Research Council of Turkey) for their financial supports during her PhD studies.

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Research Article

On Some New Fixed Point Results with Applications to Matrix Difference Equations

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Received 17 February 2021; Accepted 7 May 2021; Published 1 June 2021

Academic Editor: Huseyin Isik

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The aim of this article is to discuss the convergence of iterative sequences of the Prešić type involving new classes of operators satisfying Prešić type Θ -contractive condition in the context of metric spaces. Some examples are also provided to show the significance of the investigation of finding fixed points. Some convergence results for a class of matrix difference equations will be derived as application.

1. Introduction and Preliminaries

Banach's contraction principle [1] is one of the decisive results of fixed point theory. It states that if we have a self mapping \mathcal{L} on a complete metric space (\mathcal{S}, σ) and a constant $\lambda \in (0, 1)$ such that

$$\sigma(\mathcal{L}\mathbf{a}, \mathcal{L}\mathbf{b}) \leq \lambda \sigma(\mathbf{a}, \mathbf{b}), \quad (1)$$

holds, $\forall \mathbf{a}, \mathbf{b} \in \mathcal{S}$, then there exists a unique $\mathbf{a}^* \in \mathcal{S}$ such that $\mathbf{a}^* = \mathcal{L}\mathbf{a}^*$.

Because of its substance and accessibility, many authors have established various fascinating supplements and extensions of this principle (see [1–21] and references therein). From now to onward, we will consider k as positive integer and (\mathcal{S}, σ) as complete metric space.

In 1965, Prešić [2] generalized the famous Banach contraction principle and applied the obtained results to secure the convergence of a specific type of sequences. Prešić established the following theorem.

Theorem 1 (see [2]). Let $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ be a mapping satisfying the following contractive condition:

$$\begin{aligned} &\sigma(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_k, \mathbf{a}_{k+1})) \\ &\leq \lambda_1 \sigma(\mathbf{a}_1, \mathbf{a}_2) + \lambda_2 \sigma(\mathbf{a}_2, \mathbf{a}_3) + \dots + \lambda_k \sigma(\mathbf{a}_k, \mathbf{a}_{k+1}), \end{aligned} \quad (2)$$

for every $\mathbf{a}_1, \dots, \mathbf{a}_{k+1} \in \mathcal{S}$, where $\lambda_1, \lambda_2, \dots, \lambda_k$ are non-negative constants such that $\lambda_1 + \lambda_2 + \dots + \lambda_k < 1$. Then, $\forall \mathbf{a}^* \in \mathcal{S}$ such that $\mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*) = \mathbf{a}^*$ and it is unique. Moreover if $\mathbf{a}_1, \dots, \mathbf{a}_k$ are arbitrary points in \mathcal{S} and for $n \in \mathbb{N}$,

$$\mathbf{a}_{n+k} = \mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+k-1}), \quad (3)$$

then the sequence $\{\mathbf{a}_n\}$ is convergent and $\mathbf{a}^* = \lim \mathbf{a}_n = \mathcal{L}(\lim \mathbf{a}_n, \lim \mathbf{a}_n, \dots, \lim \mathbf{a}_n)$.

A mapping $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ satisfying inequality (2) is said to be a Prešić operator. A point $\mathbf{a}^* \in \mathcal{S}$ is called a fixed point of \mathcal{L} if $\mathbf{a}^* = \mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*)$. If $k = 1$ in Theorem 1, then we get the Banach contraction principle as a specific case.

Ćirić et al. [3] established the following theorem.

Theorem 2 (see [3]). Assume that $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ satisfies

$$\begin{aligned} &\sigma(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_k, \mathbf{a}_{k+1})) \\ &\leq \lambda \max\{\sigma(\mathbf{a}_1, \mathbf{a}_2), \sigma(\mathbf{a}_2, \mathbf{a}_3), \dots, \sigma(\mathbf{a}_k, \mathbf{a}_{k+1})\}, \end{aligned} \quad (4)$$

for any $\mathbf{a}_1, \dots, \mathbf{a}_{k+1} \in \mathcal{S}$, where $0 < \lambda < 1$. Then, $\exists \mathbf{a}^* \in \mathcal{S}$ such that $\mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*) = \mathbf{a}^*$. Moreover, for any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence given in (3) is convergent and

$$\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathcal{L}\left(\lim_{n \rightarrow \infty} \mathbf{a}_n, \dots, \lim_{n \rightarrow \infty} \mathbf{a}_n\right). \quad (5)$$

If in addition,

$$\sigma(\mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*), \mathcal{L}(\mathbf{a}', \dots, \mathbf{a}')) < \sigma(\mathbf{a}^*, \mathbf{a}'), \quad (6)$$

holds for all $\mathbf{a}^*, \mathbf{a}' \in \mathcal{S}$, with $\mathbf{a}^* \neq \mathbf{a}'$, then the fixed point in \mathcal{S} is unique.

Pwfâcurar [4] obtained a convergence theorem for Prešić–Kannan contraction in this way.

Definition 1 (see [4]). Assume that $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$. If $\exists \lambda \in \mathbb{R}$ with $0 < \lambda k(k+1) < 1$ such that

$$\begin{aligned} \sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) \\ \leq \lambda \sum_{i=1}^{k+1} \sigma(\mathbf{a}_i, \mathcal{L}(\mathbf{a}_i, \dots, \mathbf{a}_i)), \end{aligned} \quad (7)$$

holds for all $(\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$. Then,

- (i) \mathcal{L} has a unique fixed point $\mathbf{a}^* \in \mathcal{S}$.
- (ii) For any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ given by (3) converges to \mathbf{a}^* . Supplementary in this direction, we refer the readers to [5–8, 10].

Very recently, Jleli et al. [11] gave a new variety of contraction called Θ -contraction and established generalized results for these contractions.

Definition 2. Let $\Theta: (0, \infty) \rightarrow (1, \infty)$ be a mapping satisfying the following:

- (i) (Θ_1) Θ is nondecreasing.
- (ii) (Θ_2) $\forall \{t_n\} \subseteq \mathbb{R}^+$, $\lim_{n \rightarrow \infty} \Theta(t_n) = 1 \iff \lim_{n \rightarrow \infty} t_n = 0$.
- (iii) (Θ_3) $\exists 0 < h < 1$ and $l \in (0, \infty]$ such that $\lim_{t \rightarrow 0^+} ((\Theta(t) - 1)/t^h) = l$.

A mapping $\mathcal{L}: \mathcal{S} \rightarrow \mathcal{S}$ is called a Θ -contraction if there exist some $\lambda \in (0, 1)$ and a function Θ satisfying (Θ_1) – (Θ_3) such that

$$\mathcal{L}\mathbf{a} \neq \mathcal{L}\mathbf{b} \implies \Theta(\sigma(\mathcal{L}\mathbf{a}, \mathcal{L}\mathbf{b})) \leq [\Theta(\sigma(\mathbf{a}, \mathbf{b}))]^\lambda, \quad (8)$$

$\forall \mathbf{a}, \mathbf{b} \in \mathcal{S}$.

Theorem 3 (see [11]). Let $\mathcal{L}: \mathcal{S} \rightarrow \mathcal{S}$ be a Θ -contraction; then, \mathcal{L} has a unique fixed point.

We represent by the Ω set of all mappings $\Theta: (0, \infty) \rightarrow (1, \infty)$ satisfying (Θ_1) – (Θ_3) consistent with Samet et al. [11]. For more details, we refer the following [12, 14–16, 18–20] to the readers.

In this article, we discuss the convergence of $\{\mathbf{a}_n\}$ given by (3), where $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ is a Prešić type Θ -contraction. The given results unify and generalize various existing results of the literature.

2. Main Results

Motivated by the work of Samet et al. [11], we give the following definition.

Definition 3. A mapping $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ is called a Prešić type Θ -contraction if there exists some $\lambda \in (0, 1)$ such that

$$\begin{aligned} \Theta(\sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1}))) \\ \leq [\Theta(\max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\})]^\lambda, \end{aligned} \quad (9)$$

$\forall (\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$ with $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$.

Note that for $\Theta(t) = e^{\sqrt{t}}$, Prešić type Θ -contractive condition is reduced to

$$\begin{aligned} \sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) \\ \leq \lambda^2 \max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\}, \end{aligned} \quad (10)$$

$\forall (\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$, $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$.

Furthermore, for $(\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$ such that $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) = \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$, inequality (4) is also satisfied, i.e., \mathcal{L} is a Ćirić–Prešić contraction.

Remark 1. Every Prešić type Θ -contraction \mathcal{L} is a Prešić operator by (Θ_1) and (7), that is,

$$\begin{aligned} \sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) \\ < \max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\}, \end{aligned} \quad (11)$$

$\forall (\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$, $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$. Thus, each Prešić type Θ -contraction \mathcal{L} is a continuous function.

Now we present our main result in this way.

Theorem 4. Assume that $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ a Prešić type Θ -contraction. Then, for any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ given by (3) converges to \mathbf{a}^* in \mathcal{S} and \mathbf{a}^* is a fixed point of \mathcal{L} . Additionally, if $\sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}) \neq \mathcal{L}(\mathbf{b}, \dots, \mathbf{b}))$ implies that

$$\Theta(\sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}), \mathcal{L}(\mathbf{b}, \dots, \mathbf{b}))) \leq [\sigma(\mathbf{a}, \mathbf{b})]^\lambda, \quad (12)$$

$\forall \mathbf{a}, \mathbf{b} \in \mathcal{S}$ with $\mathbf{a} \neq \mathbf{b}$, then there exists unique $\mathbf{a}^* \in \mathcal{S}$ such that $\mathcal{L}\mathbf{a}^* = \mathbf{a}^*$.

Proof. Let $\mathbf{a}_1, \dots, \mathbf{a}_k$ be arbitrary k elements in \mathcal{S} . Define $\{\mathbf{a}_n\}$ in \mathcal{S} by

$$\mathbf{a}_{n+k} = \mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+k-1}), \quad (13)$$

for $n \in \mathbb{N}$. If for some $n_0 \in \{1, 2, 3, \dots, k\}$, we have $\mathbf{a}_{n_0} = \mathbf{a}_{n_0+1}$, then

$$\begin{aligned} \mathbf{a}_{n_0+k} &= \mathcal{L}(\mathbf{a}_{n_0}, \mathbf{a}_{n_0+1}, \dots, \mathbf{a}_{n_0+k-1}) \\ &= \mathcal{L}(\mathbf{a}_{n_0+k}, \mathbf{a}_{n_0+k}, \dots, \mathbf{a}_{n_0+k}), \end{aligned} \quad (14)$$

that is, \mathbf{a}_{n_0+k} is a fixed point of \mathcal{L} and we have nothing to prove. So, we assume that $\mathbf{a}_{n+k} \neq \mathbf{a}_{n+k+1}$ for all $n \in \mathbb{N}$. Represent $\beta_{n+k} = \sigma(\mathbf{a}_{n+k}, \mathbf{a}_{n+k+1})$ for $n \in \mathbb{N}$ and

$\mu = \max\{\sigma(\mathbf{a}_1, \mathbf{a}_2), \sigma(\mathbf{a}_2, \mathbf{a}_3), \dots, \sigma(\mathbf{a}_k, \mathbf{a}_{k+1})\}$; then, we obtain $\beta_{n+k} > 0$, $\forall n \in \mathbb{N}$, and $\mu > 0$. Thus, for $n \leq k$, we get

$$\begin{aligned}
 1 < \Theta(\beta_{k+1}) &= \Theta(\sigma(\mathbf{a}_{k+1}, \mathbf{a}_{k+2})) \\
 &= \Theta(\sigma(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_{k+1}))) \\
 &\leq [\Theta(\max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\})]^\lambda \\
 &= [\Theta(\mu)]^\lambda, \\
 1 < \Theta(\beta_{k+2}) &= \Theta(\sigma(\mathbf{a}_{k+2}, \mathbf{a}_{k+3})) \\
 &= \Theta(\sigma(\mathcal{L}(\mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_{k+1}), \mathcal{L}(\mathbf{a}_3, \mathbf{a}_4, \dots, \mathbf{a}_{k+2}))) \\
 &\leq [\Theta(\max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 2 \leq i \leq k+1\})]^\lambda \\
 &\leq [\Theta(\mu)]^{\lambda^2} \\
 &\vdots,
 \end{aligned} \tag{15}$$

and so on. Hence,

$$\begin{aligned}
 1 < \Theta(\beta_{k+n}) &= \Theta(\sigma(\mathbf{a}_{n+k}, \mathbf{a}_{n+k+1})) \\
 &= \Theta(\sigma(\mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+k-1}), \mathcal{L}(\mathbf{a}_{n+1}, \mathbf{a}_{n+2}, \dots, \mathbf{a}_{n+k}))) \\
 &\leq [\Theta(\mu)]^{\lambda^n},
 \end{aligned} \tag{16}$$

for $n \geq 1$. Letting $n \rightarrow \infty$, we get by (Θ_2) that

$$\lim_{n \rightarrow \infty} \Theta(\beta_{k+n}) = 1 \iff \lim_{n \rightarrow \infty} \beta_{k+n} = 0. \tag{17}$$

By (Θ_3) , $\exists 0 < h < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n \rightarrow \infty} \frac{\Theta(\beta_{k+n}) - 1}{\beta_{k+n}^h} = l. \tag{18}$$

Let $l < \infty$ and $\omega = (l/2) > 0$. By definition of the limit, $\exists n_1 \in \mathbb{N}$ such that

$$\left| \frac{\Theta(\beta_{k+n}) - 1}{\beta_{k+n}^h} - l \right| \leq \omega, \tag{19}$$

$\forall n > n_1$. It shows that

$$\frac{\Theta(\beta_{k+n}) - 1}{\beta_{k+n}^h} \geq l - \omega = \frac{l}{2} = \omega, \tag{20}$$

$\forall n > n_1$. Then,

$$n\beta_{k+n}^h \leq \omega n [\Theta(\beta_{k+n}) - 1], \tag{21}$$

$\forall n > n_1$, where $\omega = (1/\omega)$. Now we assume that $l = \infty$. Let $\omega > 0$. By definition of the limit, $\exists n_1 \in \mathbb{N}$ such that

$$\omega \leq \frac{\Theta(\beta_{k+n}) - 1}{\beta_{k+n}^h}, \tag{22}$$

$\forall n > n_1$. This implies that

$$n\beta_{k+n}^h \leq \omega n [\Theta(\beta_{k+n}) - 1], \tag{23}$$

$\forall n > n_1$, where $\omega = (1/\omega)$. Thus, in all cases, there exist $\omega > 0$ and $n_1 \in \mathbb{N}$ such that

$$n\beta_{k+n}^h \leq \omega n [\Theta(\beta_{k+n}) - 1], \tag{24}$$

for all $n > n_1$. Thus, by (10), we get

$$n\beta_{k+n}^h \leq \omega n ([\Theta(\mu)]^{\lambda^n} - 1). \tag{25}$$

Letting $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} n\beta_{k+n}^h = 0. \tag{26}$$

Thus, $\exists n_2 \in \mathbb{N}$ such that

$$n\beta_{k+n}^h \leq 1, \tag{27}$$

$\forall n > n_2$. Therefore, we get

$$\beta_{k+n} \leq \frac{1}{n^{1/h}}, \tag{28}$$

for all $n > n_2$. Now we show that $\{\mathbf{a}_n\}$ is a Cauchy sequence. For $m > n > n_2$, we have

$$\begin{aligned}
 \sigma(\mathbf{a}_{k+n}, \mathbf{a}_{k+m}) &= \sigma(\mathcal{L}(\mathbf{a}_n, \dots, \mathbf{a}_{k+n-1}), \mathcal{L}(\mathbf{a}_m, \dots, \mathbf{a}_{k+m-1})) \\
 &\leq \sigma(\mathcal{L}(\mathbf{a}_n, \dots, \mathbf{a}_{k+n-1}), \mathcal{L}(\mathbf{a}_{n+1}, \dots, \mathbf{a}_{k+n})) \\
 &\quad + \sigma(\mathcal{L}(\mathbf{a}_{n+1}, \dots, \mathbf{a}_{k+n}), \mathcal{L}(\mathbf{a}_{n+2}, \dots, \mathbf{a}_{k+n+1})) \\
 &\quad + \dots + \sigma(\mathcal{L}(\mathbf{a}_{m-1}, \dots, \mathbf{a}_{k+m-2}), \mathcal{L}(\mathbf{a}_m, \dots, \mathbf{a}_{k+m-1})) \\
 &= \sigma(\mathbf{a}_{n+k}, \mathbf{a}_{n+k+1}) + \sigma(\mathbf{a}_{n+k+1}, \mathbf{a}_{n+k+2}) + \dots + \sigma(\mathbf{a}_{m+k-1}, \mathbf{a}_{m+k}) \\
 &= \beta_{n+k} + \beta_{n+k+1} + \dots + \beta_{m+k-1} \\
 &= \sum_{i=n}^{m-1} \beta_{i+k} < \sum_{i=n}^{\infty} \beta_{i+k} \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/h}} < \infty.
 \end{aligned} \tag{29}$$

This proves that $\{\mathbf{a}_n\}$ is Cauchy in (\mathcal{S}, σ) . As (\mathcal{S}, σ) is complete, $\exists \mathbf{a}^*$ in \mathcal{S} such that

$$\lim_{n, m \rightarrow \infty} \sigma(\mathbf{a}_n, \mathbf{a}_m) = \lim_{n \rightarrow \infty} \sigma(\mathbf{a}_n, \mathbf{a}^*) = 0. \quad (30)$$

Now as \mathcal{L} is continuous, we get

$$\begin{aligned} \mathbf{a}^* &= \lim_{n \rightarrow \infty} \mathbf{a}_{n+k} = \lim_{n \rightarrow \infty} \mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+k-1}) \\ &= \mathcal{L}\left(\lim_{n \rightarrow \infty} \mathbf{a}_n, \lim_{n \rightarrow \infty} \mathbf{a}_{n+1}, \dots, \lim_{n \rightarrow \infty} \mathbf{a}_{n+k-1}\right) \\ &= \mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*). \end{aligned} \quad (31)$$

Now we show the uniqueness of fixed point of mapping \mathcal{L} . We suppose on the contrary that $\exists \mathbf{a}^*, \mathbf{a}' \in \mathcal{S}$ so that $\mathbf{a}^* = \mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*)$ and $\mathbf{a}' = \mathcal{L}(\mathbf{a}', \mathbf{a}', \dots, \mathbf{a}')$ with $\mathbf{a}^* \neq \mathbf{a}'$. Thus, $\mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*) \neq \mathcal{L}(\mathbf{a}', \mathbf{a}', \dots, \mathbf{a}')$. Hence, by given assumption, we have

$$\begin{aligned} \Theta(\sigma(\mathbf{a}^*, \mathbf{a}')) &= \Theta(\sigma(\mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*), \mathcal{L}(\mathbf{a}', \mathbf{a}', \dots, \mathbf{a}')))) \\ &\leq [\Theta(\sigma(\mathbf{a}^*, \mathbf{a}'))]^\lambda, \end{aligned} \quad (32)$$

a contradiction as $\lambda \in (0, 1)$. Therefore, $\mathbf{a}^* = \mathbf{a}'$. \square

Example 1. Consider the sequence $\{\mathbf{a}_n\}$ as follows:

$$\begin{aligned} \mathbf{a}_1 &= 1 \\ \mathbf{a}_2 &= 1 + 5 \\ &\vdots \\ \mathbf{a}_n &= 1 + 5 + 9 + \dots + (4n - 3) = n(2n - 1). \end{aligned} \quad (33)$$

Let $\mathcal{S} = \{\mathbf{a}_n : n \in \mathbb{N}\}$ and $\sigma(\mathbf{a}^*, \mathbf{a}') = |\mathbf{a}^* - \mathbf{a}'|$. Then, (\mathcal{S}, σ) becomes a complete metric space. Consider $\mathcal{L}: \mathcal{S}^2 \rightarrow \mathcal{S}$ by

$$\mathcal{L}(\mathbf{a}_n^*, \mathbf{a}_n') = \begin{cases} \frac{\mathbf{a}_{n-1}^* + \mathbf{a}_{n-1}'}{2}, & \text{for } n > 1, \\ \frac{\mathbf{a}_1^* + \mathbf{a}_1'}{2}, & \text{otherwise.} \end{cases} \quad (34)$$

For $n > 3$, we get

$$\begin{aligned} &\sigma(\mathcal{L}(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \mathcal{L}(\mathbf{a}_{n-1}, \mathbf{a}_n)) \\ &= \sigma\left(\frac{\mathbf{a}_{n-3} + \mathbf{a}_{n-2}}{2}, \frac{\mathbf{a}_{n-2} + \mathbf{a}_{n-1}}{2}\right) \\ &= \frac{1}{2} \left| ((n-3)(2n-7) + (n-2)(2n-5)) - ((n-2)(2n-5) + (n-1)(2n-3)) \right| \\ &= \frac{1}{2} \left| 2n^2 - 13n + 21 + 2n^2 - 9n + 10 - (2n^2 - 9n + 10 + 2n^2 - 5n + 3) \right| \\ &= \frac{1}{2} (8n - 18) = 4n - 9, \\ &\max\{\sigma(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \sigma(\mathbf{a}_{n-1}, \mathbf{a}_n)\} \\ &= \max\{|(n-2)(2n-5) - (n-1)(2n-3)|, |(n-1)(2n-3) - n(2n-1)|\} \\ &= \max\{4n-7, 4n-3\} = 4n-3. \end{aligned} \quad (35)$$

Now

$$\begin{aligned} &\lim_{n \rightarrow \infty} \frac{\sigma(\mathcal{L}(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \mathcal{L}(\mathbf{a}_{n-1}, \mathbf{a}_n))}{\max\{\sigma(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \sigma(\mathbf{a}_{n-1}, \mathbf{a}_n)\}} \\ &= \lim_{n \rightarrow \infty} \frac{4n-9}{4n-3} = 1. \end{aligned} \quad (36)$$

Thus,

$$\begin{aligned} &\sigma(\mathcal{L}(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \mathcal{L}(\mathbf{a}_{n-1}, \mathbf{a}_n)) \\ &\leq \lambda \max\{\sigma(\mathbf{a}_{n-2}, \mathbf{a}_{n-1}), \sigma(\mathbf{a}_{n-1}, \mathbf{a}_n)\} \end{aligned} \quad (37)$$

does not hold for $\lambda \in (0, 1)$. Thus, the main hypothesis of Theorem 2 in [3] is not satisfied. Now, by considering the mapping $\Theta: (0, \infty) \rightarrow (1, \infty)$ defined by

$$\Theta(t) = e^{te^t}, \quad (38)$$

we can easily show that $\Theta \in \Omega$ and \mathcal{L} is Prešić type Θ -contraction. Indeed, the following holds:

$$\begin{aligned} & e^{\sqrt{\sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})} e^{\sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})}} \\ & \leq e^{\lambda \sqrt{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), (\mathbf{a}_{i+1}, \mathbf{a}_{i+2})} e^{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), (\mathbf{a}_{i+1}, \mathbf{a}_{i+2})}}, \end{aligned} \quad (39)$$

for $\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1}) \neq \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})$, $i = 1, 2, \dots$, and for some $\lambda \in (0, 1)$. The above condition is equivalent to

$$\begin{aligned} & \sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2}) \\ & \cdot e^{\sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})} \\ & \leq \lambda^2 \max\{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), \sigma(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})\} \\ & \cdot e^{\max\{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), \sigma(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})\}}. \end{aligned} \quad (40)$$

So, we have to check that

$$\begin{aligned} & \frac{\sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2}) \cdot e^{\sigma(\mathcal{L}(\mathbf{a}_i, \mathbf{a}_{i+1})), \mathcal{L}(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})}}{\max\{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), \sigma(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})\} \cdot e^{\max\{\sigma((\mathbf{a}_i, \mathbf{a}_{i+1})), \sigma(\mathbf{a}_{i+1}, \mathbf{a}_{i+2})\}}} \\ & \leq \lambda^2, \end{aligned} \quad (41)$$

for some $\lambda \in (0, 1)$. We discuss these two cases.

Case 1. For $i = n = 1$, we have

$$\begin{aligned} & \frac{\sigma(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2)), \mathcal{L}(\mathbf{a}_2, \mathbf{a}_3) e^{\sigma(\mathcal{L}(\mathbf{a}_1, \mathbf{a}_2)), \mathcal{L}(\mathbf{a}_2, \mathbf{a}_3)}}{\max\{\sigma((\mathbf{a}_1, \mathbf{a}_2)), \sigma(\mathbf{a}_2, \mathbf{a}_3)\} e^{\max\{\sigma((\mathbf{a}_1, \mathbf{a}_1)), \sigma(\mathbf{a}_2, \mathbf{a}_3)\}}} \\ & = \frac{\sigma(((\mathbf{a}_1 + \mathbf{a}_1)/2), ((\mathbf{a}_1 + \mathbf{a}_2)/2)) e^{\sigma(((\mathbf{a}_1 + \mathbf{a}_1)/2), ((\mathbf{a}_1 + \mathbf{a}_2)/2))}}{\max\{\sigma((\mathbf{a}_1, \mathbf{a}_2)), \sigma(\mathbf{a}_2, \mathbf{a}_3)\} e^{\max\{\sigma((\mathbf{a}_1, \mathbf{a}_2)), \sigma(\mathbf{a}_2, \mathbf{a}_3)\}}} \\ & = \frac{\sigma(1, (7/2)) e^{\sigma(1, (7/2))}}{\max\{\sigma((1, 6)), \sigma(6, 15)\} e^{\max\{\sigma((1, 6)), \sigma(6, 15)\}}} \\ & = \frac{5}{18} e^{(-13/2)} \\ & < e^{-2}. \end{aligned} \quad (42)$$

Case 2. For $i = n > 1$, we have

$$\begin{aligned} & \frac{\sigma(\mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1})), \mathcal{L}(\mathbf{a}_{n+1}, \mathbf{a}_{n+2}) \cdot e^{\sigma(\mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1})), \mathcal{L}(\mathbf{a}_{n+1}, \mathbf{a}_{n+2})}}{\max\{\sigma((\mathbf{a}_n, \mathbf{a}_{n+1})), \sigma(\mathbf{a}_{n+1}, \mathbf{a}_{n+2})\} \cdot e^{\max\{\sigma((\mathbf{a}_n, \mathbf{a}_{n+1})), \sigma(\mathbf{a}_{n+1}, \mathbf{a}_{n+2})\}}} \\ & = \frac{\sigma(((\mathbf{a}_{n-1} + \mathbf{a}_n)/2), ((\mathbf{a}_n + \mathbf{a}_{n+1})/2)) \cdot e^{\sigma(((\mathbf{a}_{n-1} + \mathbf{a}_n)/2), ((\mathbf{a}_n + \mathbf{a}_{n+1})/2))}}{\max\{\sigma((\mathbf{a}_n, \mathbf{a}_{n+1})), \sigma(\mathbf{a}_{n+1}, \mathbf{a}_{n+2})\} \cdot e^{\max\{\sigma((\mathbf{a}_n, \mathbf{a}_{n+1})), \sigma(\mathbf{a}_{n+1}, \mathbf{a}_{n+2})\}}} \\ & = \frac{(1/2) \{ (4n^2 - 6n + 3) - (4n^2 + 2n + 1) \} \cdot e^{(1/2) \{ (4n^2 - 6n + 3) - (4n^2 + 2n + 1) \}}}{\max\{4n + 1, 2n + 2\} \cdot e^{\max\{4n + 1, 2n + 2\}}} \\ & = \frac{-4n + 1}{4n + 1} e^{-8n} \\ & < e^{-2}, \end{aligned} \quad (43)$$

with $\lambda = e^{-1}$.

Thus, all the conditions of Theorem 4 are satisfied and $(1, 1)$ is the unique fixed point of \mathcal{L} .

Example 2. Let $\mathcal{S} = [0, 1]$ and σ be such that

$$\sigma(\mathbf{a}^*, \mathbf{a}') = |\mathbf{a}^* - \mathbf{a}'|, \quad (44)$$

and $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ be defined by

$$\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) = \frac{\mathbf{a}_1 + \mathbf{a}_k}{4k} \quad \text{for all } \mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}. \quad (45)$$

Define the mapping $\Theta: (0, \infty) \rightarrow (1, \infty)$ given by

$$\Theta(t) = e^{\sqrt{t}}. \quad (46)$$

It is given in [11] that $\Theta \in \Omega$. Now for $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{k+1} \in \mathcal{S}$, we have

$$\sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) > 0. \quad (47)$$

We have

$$\begin{aligned}
& \Theta(\sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1}))) \\
&= e^{\sqrt{\sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1}))}} \\
&= e^{\sqrt{(1/4k)[(\mathbf{a}_1 - \mathbf{a}_2) + (\mathbf{a}_k - \mathbf{a}_{k+1})]}} \\
&= e^{(1/2\sqrt{k})\sqrt{[(\mathbf{a}_1 - \mathbf{a}_2) + (\mathbf{a}_k - \mathbf{a}_{k+1})]}} \\
&\leq e^{(1/2)\sqrt{\max\{\sigma(\mathbf{a}_1, \mathbf{a}_2), \sigma(\mathbf{a}_k, \mathbf{a}_{k+1})\}}} \\
&\leq e^{(1/\sqrt{2})\sqrt{\max\{\sigma(\mathbf{a}_j, \mathbf{a}_{j+1}): 1 \leq j \leq k\}}} \\
&= [\Theta(\max\{\sigma(\mathbf{a}_j, \mathbf{a}_{j+1}): 1 \leq j \leq k\})]^\lambda.
\end{aligned} \tag{48}$$

with $\lambda = 1/\sqrt{2}$. Moreover, for all $\mathbf{a}^*, \mathbf{a}' \in \mathcal{S}$ with $\mathbf{a}^* \neq \mathbf{a}'$, we have

$$\begin{aligned}
& \sigma(\mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*), \mathcal{L}(\mathbf{a}', \mathbf{a}', \dots, \mathbf{a}')) \\
&= \frac{|\mathbf{a}^* - \mathbf{a}'|}{2k} > 0, \\
& \Theta(\sigma(\mathcal{L}(\mathbf{a}^*, \mathbf{a}^*, \dots, \mathbf{a}^*), \mathcal{L}(\mathbf{a}', \mathbf{a}', \dots, \mathbf{a}')))) \\
&= \Theta\left(\frac{|\mathbf{a}^* - \mathbf{a}'|}{2k}\right) \\
&= e^{\sqrt{(|\mathbf{a}^* - \mathbf{a}'|/2k)}} = e^{(1/\sqrt{2k})\sqrt{|\mathbf{a}^* - \mathbf{a}'|}} \\
&\leq e^{(1/\sqrt{2})\sqrt{|\mathbf{a}^* - \mathbf{a}'|}} \\
&= [\Theta(\sigma(\mathbf{a}^*, \mathbf{a}'))]^\lambda,
\end{aligned} \tag{49}$$

with $\lambda = 1/\sqrt{2}$. Hence, all the hypotheses of Theorem 4 are satisfied. Furthermore, for some arbitrary $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ defined by (3) converges to $\mathbf{a}^* = 0$, which is the unique fixed point of mapping \mathcal{L} .

The upcoming result is an instant consequence of Theorem 4 by taking $\Theta(t) = e^{\sqrt{t}}$.

Corollary 1. Let $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ be a given mapping. Assume that there exists some $\lambda \in (0, 1)$ such that

$$\begin{aligned}
& \sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) \\
&\leq \lambda^2 \max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\},
\end{aligned} \tag{50}$$

for all $(\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$ with $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$. Then, for any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ given by (3) converges to \mathbf{a}^* , and \mathbf{a}^* is a fixed point of \mathcal{L} , that is, $\mathbf{a}^* = \mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*)$. Moreover, if

$$\sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}), \mathcal{L}(\mathbf{b}, \dots, \mathbf{b})) \leq \lambda^2 \sigma(\mathbf{a}, \mathbf{b}), \tag{51}$$

holds for all $\mathbf{a}, \mathbf{b} \in \mathcal{S}$ with $\mathbf{a} \neq \mathbf{b}$, then \mathbf{a}^* is the unique fixed point of \mathcal{L} .

Corollary 2. Let $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ be a given mapping. Suppose that there exist $\lambda_1, \lambda_2, \dots, \lambda_k$ non-negative constants with $\lambda_1 + \lambda_2 + \dots + \lambda_k < 1$ such that

$$\begin{aligned}
& \sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})) \\
&\leq \lambda_1 \sigma(\mathbf{a}_1, \mathbf{a}_2) + \lambda_2 \sigma(\mathbf{a}_2, \mathbf{a}_3) \\
&\quad + \dots + \lambda_k \sigma(\mathbf{a}_k, \mathbf{a}_{k+1}),
\end{aligned} \tag{52}$$

for all $(\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$ with $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$. Then, for any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ given by (3) converges to \mathbf{a}^* , where \mathbf{a}^* is the unique fixed point of \mathcal{L} .

Proof. Evidently, (16) \implies (15) with $\lambda^2 = \lambda_1 + \lambda_2 + \dots + \lambda_k$. Now, let $\mathbf{a}, \mathbf{b} \in \mathcal{S}$ with $\mathbf{a} \neq \mathbf{b}$. From (16), we have

$$\begin{aligned}
& \sigma(\mathcal{L}(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}), \mathcal{L}(\mathbf{b}, \mathbf{b}, \dots, \mathbf{b})) \\
&\leq \sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}), \mathcal{L}(\mathbf{a}, \dots, \mathbf{a}, \mathbf{b})) \\
&\quad + \sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}, \mathbf{b}), \mathcal{L}(\mathbf{a}, \dots, \mathbf{a}, \mathbf{b}, \mathbf{b})) \\
&\quad + \dots + \sigma(\mathcal{L}(\mathbf{a}, \mathbf{b}, \dots, \mathbf{b}), \mathcal{L}(\mathbf{b}, \mathbf{b}, \dots, \mathbf{b})) \\
&\leq (\lambda_k + \lambda_{k-1} + \dots + \lambda_1) \sigma(\mathbf{a}, \mathbf{b}) \\
&= \lambda^2 \sigma(\mathbf{a}, \mathbf{b}),
\end{aligned} \tag{53}$$

where $\lambda^2 = \lambda_k + \lambda_{k-1} + \dots + \lambda_1 \in (0, 1)$. Hence, all the hypotheses of Corollary 1 are satisfied.

Now consider the family Ω which contains large class of functions. For example, if

$$\Theta(t) = 2 - \frac{2}{\pi} \arctan\left(\frac{1}{t^\beta}\right), \tag{54}$$

where $0 < \beta < 1$ and $t > 0$, we can obtain the following theorem from our main theorem. \square

Theorem 5. Let $\mathcal{L}: \mathcal{S}^k \rightarrow \mathcal{S}$ be a given mapping. If there exist a mapping $\Theta \in \Omega$ and a constant $\beta, \lambda \in (0, 1)$ such that

$$\begin{aligned}
& 2 - \frac{2}{\pi} \arctan\left(\frac{1}{(\sigma(\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k), \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1}))^\beta)}\right) \\
&\leq \left[2 - \frac{2}{\pi} \arctan\left(\frac{1}{[\max\{\sigma(\mathbf{a}_i, \mathbf{a}_{i+1}): 1 \leq i \leq k\}]^\beta}\right)\right]^\lambda,
\end{aligned} \tag{55}$$

for all $(\mathbf{a}_1, \dots, \mathbf{a}_{k+1}) \in \mathcal{S}^{k+1}$ with $\mathcal{L}(\mathbf{a}_1, \dots, \mathbf{a}_k) \neq \mathcal{L}(\mathbf{a}_2, \dots, \mathbf{a}_{k+1})$, then for any arbitrary points $\mathbf{a}_1, \dots, \mathbf{a}_k \in \mathcal{S}$, the sequence $\{\mathbf{a}_n\}$ given by (3) converges to \mathbf{a}^* , and \mathbf{a}^* is a fixed point of \mathcal{L} , that is, $\mathbf{a}^* = \mathcal{L}(\mathbf{a}^*, \dots, \mathbf{a}^*)$. Moreover, if

$$2 - \frac{2}{\pi} \arctan \frac{1}{(\sigma(\mathcal{L}(\mathbf{a}, \dots, \mathbf{a}), \mathcal{L}(\mathbf{b}, \dots, \mathbf{b}))^\beta)} \leq \left[2 - \frac{2}{\pi} \arctan \left(\frac{1}{(\sigma(\mathbf{a}, \mathbf{b}))^\beta} \right) \right]^\lambda, \quad (56)$$

holds $\forall \mathbf{a}, \mathbf{b} \in \mathcal{S}$ with $\mathbf{a} \neq \mathbf{b}$, then \mathbf{a}^* is the unique fixed point of \mathcal{L} .

- (1) Theorem 1.3 in [3] and Theorem 1.2 in [2] are direct generalizations of Theorem 4.
- (2) Corollary 1 of Samet et al. in [11] can be deduced by putting $k = 1$ in Theorem 4.
- (3) Banach contraction principle [1] can be deduced from Corollaries 1 and 2 by taking $k = 1$.

3. Applications

We start this section with the definition of equilibrium point as follows.

Definition 6. Let $\mathcal{L}: \mathcal{S}^k \longrightarrow \mathcal{S}$. For given $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k \in \mathcal{S}$, consider the recursive sequence $\{\mathbf{a}_n\} \subset \mathcal{S}$ defined by

$$\mathbf{a}_{n+k} = \mathcal{L}(\mathbf{a}_n, \mathbf{a}_{n+1}, \dots, \mathbf{a}_{n+k-1}), \quad (57)$$

for all $n \in \mathbb{N}$. A point $\bar{\mathbf{a}}$ is said to be an equilibrium point of equation (17) if the following condition is satisfied:

$$\bar{\mathbf{a}} = \mathcal{L}(\bar{\mathbf{a}}, \dots, \bar{\mathbf{a}}). \quad (58)$$

Definition 7. An equilibrium point $\bar{\mathbf{a}}$ is said to be global attractor if for all $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_k \in \mathcal{S}$, we have $\sigma(\mathbf{a}_n, \bar{\mathbf{a}}) \longrightarrow 0$ as $n \longrightarrow \infty$.

If we have the recursive sequence $\{\mathcal{S}_n\} \subset P(N)$ defined by

$$\mathcal{S}_{n+k} = Q + \frac{1}{k} \sum_{i=0}^{k-1} E^* \psi(\mathcal{S}_{n+i}) E, \quad (59)$$

for all $n \in \mathbb{N}$, then we explore the global attractivity of (59) in this application. In this application, we explore the global attractivity of (18). Here $P(N)$ (for $N \geq 2$) denotes family of $N \times N$ Hermitian positive definite matrices, Q is an $N \times N$ Hermitian positive semidefinite matrix, E is an $N \times N$ nonsingular matrix, E^* is the conjugate transpose of E , and ψ is a function from $P(N)$ to $P(N)$.

We first analyze the Thompson metric σ on $P(N)$, which is defined by

$$\sigma(E_1, E_2) = \max \left\{ \log M \left(\frac{E_1}{E_2} \right), \log M \left(\frac{E_2}{E_1} \right) \right\}, \quad (60)$$

for $E_1, E_2 \in P(N)$, where $M(E_1/E_2) = \inf \{ \delta > 0: E_1 \leq \delta E_2 \} = \delta^+ (E_2^{-1/2} E_1^{-1/2} E_2)$, the maximal eigenvalue of $E_2^{-1/2} E_1^{-1/2} E_2$. Here, $E_1 \leq E_2$ means that $E_2 - E_1$ is positive semidefinite and $E_1 < E_2$ means that $E_2 - E_1$ is positive definite. Nussbaum [21] proved that $P(N)$ is a complete metric space regarding σ and

$$\sigma(E_1, E_2) = \left\| \ln(E_1^{-1/2} E_2 E_1^{-1/2}) \right\|, \quad (61)$$

where $\|\cdot\|$ is a spectral norm [21, 22]. Now we directly initiate the graceful properties of σ , i.e.,

$$\sigma(E_1, E_2) = \sigma(E_1^{-1}, E_2^{-1}) = \sigma(M^* E_1 M, M^* E_2 M), \quad (62)$$

for any nonsingular matrix M . The second valuable result is the nonpositive curvature property of σ in this way.

$$\sigma(E_1^r, E_2^r) \leq c \sigma(E_1, E_2), \quad c \in [0, 1]. \quad (63)$$

In accordance with (62) and (63), we get

$$\sigma(M^* X^r M, M^* Y^r M) \leq |c| \sigma(X, Y), \quad c \in [-1, 1], \quad (64)$$

for all $E_1, E_2 \in P(N)$.

Lemma 1 (see [23]). For all $E_1, E_2, E_3, E_4 \in P(N)$, we have

$$\sigma(E_1 + E_2, E_3 + E_4) \leq \max \{ \sigma(E_1, E_3), \sigma(E_2, E_4) \}. \quad (65)$$

Moreover, for all positive semidefinite E_1 and $E_2, E_3 \in P(N)$, we have

$$\sigma(E_1 + E_2, E_1 + E_3) \leq \sigma(E_2, E_3). \quad (66)$$

Let $\psi: P(N) \longrightarrow P(N)$ be a Θ -contraction regarding σ . For $X_1, X_2, \dots, X_k \in P(N)$, consider $\{X_n\} \subset P(N)$ defined by (59).

Theorem 8. Equation (59) has a unique equilibrium point $\bar{X} \in P(N)$. Furthermore, \bar{X} is global attractor.

Proof. Define $\mathcal{L}: P(N)^k \longrightarrow P(N)$ by

$$\begin{aligned} \mathcal{L}(M_1, M_2, \dots, M_k) \\ = Q + \frac{1}{k} [E^* \psi(M_1) E + E^* \psi(M_2) E + \dots + E^* \psi(M_k) E], \end{aligned} \quad (67)$$

for all $M_1, M_2, \dots, M_k \in P(N)$.

Let $M_1, M_2, \dots, M_{k+1} \in P(N)$. By Lemma 1, we get

$$\begin{aligned}
& \sigma \mathcal{L}(M_1, M_2, \dots, M_k), \mathcal{L}(M_2, M_3, \dots, M_{k+1}) \\
&= \sigma \left(Q + \frac{1}{k} \sum_{i=1}^k E^* \psi(M_i) E, Q + \frac{1}{k} \sum_{j=2}^{k+1} E^* \psi(M_j) E \right) \\
&\leq \sigma \left(\frac{1}{k} \sum_{i=1}^k E^* \psi(M_i) E, \frac{1}{k} \sum_{j=2}^{k+1} E^* \psi(M_j) E \right) \\
&= \sigma \left(\sum_{i=1}^k ((1/\sqrt{k})E)^* \psi(M_i) ((1/\sqrt{k})E), \sum_{j=2}^{k+1} ((1/\sqrt{k})E)^* \psi(M_j) ((1/\sqrt{k})E) \right).
\end{aligned} \tag{68}$$

Denote $O = (1/\sqrt{k})E$. Then, using again Lemma 1, we have

$$\begin{aligned}
& \sigma(\mathcal{L}(M_1, M_2, \dots, M_k), \mathcal{L}(M_2, M_3, \dots, M_{k+1})) \\
&\leq \sigma \left(\sum_{i=1}^k O^* \psi(M_i) O, \sum_{j=2}^{k+1} O^* \psi(M_j) O \right) \\
&= \sigma \left(\begin{matrix} O^* \psi(M_1) O + O^* \psi(M_2) O + \dots + \\ O^* \psi(M_k) O, O^* \psi(M_2) O + \\ O^* \psi(M_3) O + \dots + O^* \psi(M_{k+1}) O \end{matrix} \right) \\
&\leq \max \left\{ \begin{matrix} \sigma(O^* \psi(M_1) O, O^* \psi(M_2) O), \\ \sigma(O^* \psi(M_2) O, O^* \psi(M_3) O), \dots, \\ \sigma(O^* \psi(M_k) O, O^* \psi(M_{k+1}) O) \end{matrix} \right\} \\
&= \max \{ \sigma(O^* \psi(M_j) O, O^* \psi(M_{j+1}) O) \},
\end{aligned} \tag{69}$$

for $j = 1, 2, \dots, k$. As A is nonsingular, O is also nonsingular. By (19), $\forall j = 1, 2, \dots, k$, we get

$$\sigma(O^* \psi(M_j) O, O^* \psi(M_{j+1}) O) = \sigma(\psi(M_j), \psi(M_{j+1})). \tag{70}$$

Now as ψ is a Θ -contraction, for all $j = 1, 2, \dots, k$, we have

$$\Theta(\sigma(O^* \psi(M_j) O, O^* \psi(M_{j+1}) O)) \leq [\Theta(\sigma(M_j, M_{j+1}))]^\lambda, \tag{71}$$

for some $\lambda \in (0, 1)$. Thus, we have

$$\begin{aligned}
& \Theta(\sigma(\mathcal{L}(M_1, M_2, \dots, M_k), \mathcal{L}(M_2, M_3, \dots, M_{k+1}))) \\
&\leq [\Theta(\max\{\sigma(M_i, M_{i+1}): i = 1, 2, \dots, k\})]^\lambda,
\end{aligned} \tag{72}$$

for all $M_1, M_2, \dots, M_{k+1} \in P(N)$. Thus, by Theorem 4, there exists a global attractor equilibrium point $\bar{X} \in P(N)$.

Now for $M_1, M_2 \in P(N)$ such that $\mathcal{L}(M_1, M_1, \dots, M_1) \neq \mathcal{L}(M_2, M_2, \dots, M_2)$, we have

$$\begin{aligned}
& \Theta(\sigma(\mathcal{L}(M_1, M_1, \dots, M_1), \mathcal{L}(M_2, M_2, \dots, M_2))) \\
&= \Theta(\sigma(Q + E^* \psi(M_1) E, Q + E^* \psi(M_2) E)) \\
&\leq \Theta(\sigma(E^* \psi(M_1) E, E^* \psi(M_2) E)) \\
&= \Theta(\sigma(\psi(M_1), \psi(M_2))) \\
&\leq [\Theta(\sigma(M_1, M_2))]^\lambda.
\end{aligned} \tag{73}$$

Again, by Theorem 4, we get the unique equilibrium point. \square

4. Conclusion

Jleli and Samet [11] very recently exploited the idea of Θ -contraction and established some generalized theorems for these contractions in generalized metric spaces. We continued their investigations and defined Prešić type Θ -contraction. In this paper, we studied the convergence of iterative sequences of the Prešić type involving new classes of operators satisfying Prešić type Θ -contractive condition in the context of metric spaces. Some examples are also provided to show the significance of the investigation of finding fixed points. As application, we derived some convergence results for a class of matrix difference equations.

Data Availability

The data supporting the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments


The first author expresses his thanks and gratitude to King Abdulaziz University for unlimited support.

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Research Article

Some Fixed Point Theorems of Contractive Mappings in \mathcal{P}_b^r -Cone Metric Spaces over Banach Algebras

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Received 10 April 2021; Revised 1 May 2021; Accepted 7 May 2021; Published 24 May 2021

Academic Editor: Naeem Saleem

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In this paper, we introduce the concept of a \mathcal{P}_b^r -cone metric space over Banach algebras and prove some fixed point results under various contractive mappings in such a space. Some examples are given to elucidate the results. Our results extend and generalize many existing results in the literature.

1. Introduction

In 2017, George et al. [1] introduced the concept of a rectangular cone b -metric space over Banach algebras as a generalization of metric space and many of its generalizations. They proved some fixed point results in such a space. Very recently, Fernandez et al. [2] introduced \mathcal{P}_b -cone metric space over Banach algebras as a generalization of partial metric space and many of its generalizations. Motivated and inspired by these papers [1, 2], we introduce the concept of a \mathcal{P}_b^r -cone metric space over Banach algebras which generalized both rectangular cone b -metric space over Banach algebras and \mathcal{P}_b -cone metric space over Banach algebras. Furthermore, we prove some fixed point results under various contractive mappings in such a space. Examples are also given to elucidate our results. Our results extend and generalize many results in [1, 3–10].

2. Preliminaries

We start with definitions and some basic facts needed in the sequel.

Definition 1 (see [11]). Let \mathcal{A} be a real Banach algebra, i.e., \mathcal{A} is a real Banach space in which an operation of

multiplication is defined, for all $y, z, x \in \mathcal{A}$ and $k \in \mathbb{R}$, and the following are satisfied:

- (i) $y(zx) = (yz)x$
- (ii) $y(z + x) = yz + yx$ and $(y + z)x = yx + zx$
- (iii) $k(yz) = (ky)z = y(kz)$
- (iv) $\|yz\| \leq \|y\|\|z\|$

A Banach algebra \mathcal{A} is called unital if there exists a unit $e \in \mathcal{A}$ such that $ey = ye = y$, for any $y \in \mathcal{A}$.

Definition 2 (see [5]). A subset \mathcal{K} of Banach algebra \mathcal{A} is called a cone if

- (i) \mathcal{K} is nonempty, closed, and $\{\theta, e\} \subset \mathcal{A}$, where θ is the zero of \mathcal{A}
- (ii) $\alpha\mathcal{K} + \beta\mathcal{K} \subset \mathcal{K}$ for all nonnegative real numbers α, β
- (iii) $\mathcal{K}^2 = \mathcal{K}\mathcal{K} \subset \mathcal{K}$
- (iv) $\mathcal{K} \cap (-\mathcal{K}) = \{\theta\}$

For a given cone $\mathcal{K} \subset \mathcal{A}$, we define a partial ordering \preceq with respect to \mathcal{K} by $y \preceq z$ if and only if $z - y \in \mathcal{K}$. The notation $y \ll z$ will stand for $z - y \in \mathcal{K}^\circ$, where \mathcal{K}° denotes the interior of \mathcal{K} . \mathcal{K} is called a solid cone if $\mathcal{K}^\circ \neq \emptyset$.

Definition 3 (see [6]). Let \mathcal{K} be a solid cone in a Banach algebra \mathcal{A} . A sequence $\{y_n\} \subset \mathcal{K}$ is said to be a c -sequence if, for every $c \in \mathcal{K}^\circ$, there exists $N \in \mathbb{N}$ such that $y_n \ll c$ for all $n > N$.

Lemma 1 (see [8]). Let \mathcal{K} be a solid cone in a Banach algebra \mathcal{A} :

- (1) If $\alpha, \beta \in \mathcal{A}$, $\gamma \in \mathcal{K}$, and $\alpha \leq \beta$, then $\gamma\alpha \leq \gamma\beta$
- (2) If $\alpha \leq \beta\alpha$, where $\alpha, \beta \in \mathcal{K}$ and $\rho(\beta) < 1$, then $\alpha = \theta$
- (3) If $\alpha \in \mathcal{K}$ and $\rho(\alpha) < 1$, then $\rho(\alpha^q) < 1$ for any fixed $q \in \mathbb{N}$

Lemma 2 (see [11, 12]). Let \mathcal{A} be a unital Banach algebra and $\alpha \in \mathcal{A}$; then, $\lim_{m \rightarrow \infty} \|\alpha^m\|^{(1/m)}$ exists and the spectral radius $\rho(\alpha)$ satisfies

$$\rho(\alpha) = \lim_{m \rightarrow \infty} \|\alpha^m\|^{(1/m)} = \inf_{m \geq 1} \|\alpha^m\|^{(1/m)}. \quad (1)$$

If $\rho(\alpha) < |\beta|$, then $(\beta e - \alpha)$ is invertible in \mathcal{A} . Moreover,

$$(\beta e - \alpha)^{-1} = \sum_{j=0}^{\infty} \frac{\alpha^j}{\beta^{j+1}}, \quad (2)$$

$$\rho[(\beta e - \alpha)^{-1}] \leq \frac{1}{|\beta| - \rho(\alpha)},$$

where β is a complex constant.

Lemma 3 (see [11]). Let \mathcal{A} be a unital Banach algebra and $\alpha, \beta \in \mathcal{A}$ such that α commutes with β . Then,

$$\begin{aligned} \rho(\alpha + \beta) &\leq \rho(\alpha) + \rho(\beta), \\ \rho(\alpha\beta) &\leq \rho(\alpha)\rho(\beta). \end{aligned} \quad (3)$$

Lemma 4 (see [13]). Let \mathcal{K} be a solid cone in a Banach algebra \mathcal{A} , $\{y_n\}$ and $\{z_n\}$ be two c -sequences in \mathcal{K} . If $\alpha, \beta \in \mathcal{K}$ are two given vectors, then $\{\alpha y_n + \beta z_n\}$ is also a c -sequence in \mathcal{K} .

Lemma 5 (see [13]). Let \mathcal{A} be a unital Banach algebra. Let $\alpha \in \mathcal{A}$ and $\rho(\alpha) < 1$. Then, $\{\alpha^n\}$ is a c -sequence in \mathcal{A} .

Lemma 6 (see [6]). Let \mathcal{K} be a solid cone in a Banach algebra \mathcal{A} :

- (1) If $\alpha, \beta, \gamma \in \mathcal{K}$ and $\alpha \leq \beta \ll \gamma$, then $\alpha \ll \gamma$
- (2) If $\alpha \in \mathcal{A}$ and $\theta \leq \alpha \ll \beta$ for each $\beta \in \mathcal{K}^\circ$, then $\alpha = \theta$
- (3) $\{y_n\} \subset \mathcal{K}$ is a c -sequence provided that $\{y_n\} \longrightarrow \theta$ as $n \longrightarrow \infty$

Definition 4 (see [2]). Let Y be a nonempty set and \mathcal{A} a Banach algebra. Suppose that, for all $y, z, x \in Y$, a mapping $\mathcal{P}_b: Y \times Y \longrightarrow \mathcal{A}$ satisfies

- (1) $y = z \Leftrightarrow \mathcal{P}_b(y, y) = \mathcal{P}_b(y, z) = \mathcal{P}_b(z, z)$
- (2) $\theta \leq \mathcal{P}_b(y, y) \leq \mathcal{P}_b(y, z)$

$$(3) \mathcal{P}_b(y, z) = \mathcal{P}_b(z, y)$$

$$(4) \mathcal{P}_b(y, z) \leq s[\mathcal{P}_b(y, x) + \mathcal{P}_b(x, z)] - \mathcal{P}_b(x, x)$$

Then, (Y, \mathcal{P}_b) is called a partial cone b -metric space over \mathcal{A} with coefficient $s \geq 1$.

Definition 5 (see [1]). Let Y be a nonempty set and \mathcal{K} a solid cone in a Banach algebra \mathcal{A} . Suppose that, for all $y, z \in Y$ and all distinct points $x_1, x_2 \in Y \setminus \{y, z\}$, a mapping $\mathcal{P}_{\text{rcb}}: Y \times Y \longrightarrow \mathcal{A}$ satisfies

$$(1) \theta \leq \mathcal{P}_{\text{rcb}}(y, z) \text{ and } \mathcal{P}_{\text{rcb}}(y, z) = \theta \Leftrightarrow y = z.$$

$$(2) \mathcal{P}_{\text{rcb}}(y, z) = \mathcal{P}_{\text{rcb}}(z, y).$$

$$(3) \text{ There exists } s \in \mathcal{K} \text{ with } e \leq s \text{ such that}$$

$$\mathcal{P}_{\text{rcb}}(y, z) \leq s[\mathcal{P}_{\text{rcb}}(y, x_1) + \mathcal{P}_{\text{rcb}}(x_1, x_2) + \mathcal{P}_{\text{rcb}}(x_2, z)]. \quad (4)$$

Then, \mathcal{P}_{rcb} is called a rectangular cone b -metric on Y , and $(Y, \mathcal{P}_{\text{rcb}})$ is called a rectangular cone b -metric space over \mathcal{A} with coefficient s .

3. Main Results

In this section, we introduce the concept of a partial rectangular cone b -metric space (\mathcal{P}_b^r -cone metric space) over Banach algebras and give some of its topological property. Furthermore, the notions of convergent sequence, θ -Cauchy sequence, and θ -completeness in the setting of this new space are defined. Moreover, some fixed point theorems under various contractive mappings are proved in such a space.

Definition 6. Let Y be a nonempty set and \mathcal{K} be a solid cone in a unital Banach algebra \mathcal{A} . Suppose that, for all $y, z \in Y$ and all distinct points $x_1, x_2 \in Y \setminus \{y, z\}$, a mapping $\mathcal{P}_b^r: Y \times Y \longrightarrow \mathcal{A}$ satisfies

$$(P1) y = z \Leftrightarrow \mathcal{P}_b^r(y, y) = \mathcal{P}_b^r(y, z) = \mathcal{P}_b^r(z, z).$$

$$(P2) \theta \leq \mathcal{P}_b^r(y, y) \leq \mathcal{P}_b^r(y, z).$$

$$(P3) \mathcal{P}_b^r(y, z) = \mathcal{P}_b^r(z, y).$$

$$(P4) \text{ There exists } s \in \mathcal{K} \text{ with } e \leq s \text{ such that}$$

$$\begin{aligned} \mathcal{P}_b^r(y, z) &\leq s[\mathcal{P}_b^r(y, x_1) + \mathcal{P}_b^r(x_1, x_2) + \mathcal{P}_b^r(x_2, z)] \\ &\quad - \mathcal{P}_b^r(x_1, x_1) - \mathcal{P}_b^r(x_2, x_2). \end{aligned} \quad (5)$$

Then, \mathcal{P}_b^r is called a partial rectangular cone b -metric on Y , and $(Y, \mathcal{P}_b^r, \mathcal{A})$ is called a partial rectangular cone b -metric space over \mathcal{A} with coefficient s (in short PRCbMS-BA).

Remark 1. In any PRCbMS-BA $(Y, \mathcal{P}_b^r, \mathcal{A})$, if $\mathcal{P}_b^r(y, z) = \theta$ for all $y, z \in Y$, then $y = z$, but the converse may not be true. Also, every rectangular cone b -metric space over \mathcal{A} is a \mathcal{P}_b^r -cone metric space over \mathcal{A} with zero (θ) self distance, but there are \mathcal{P}_b^r -cone metric spaces over \mathcal{A} which are not a rectangular cone b -metric space over \mathcal{A} .

Example 1. Let $\mathcal{A} = C_{\mathbb{R}}^1[0, 1]$ with the norm

$$\|y\| = \|y\|_\infty + \|y'\|_\infty, \quad \text{for all } y \in \mathcal{A}. \quad (6)$$

Define multiplication pointwisely on \mathcal{A} . Then, \mathcal{A} is a Banach algebra with unit $e(t) = 1, \forall t \in [0, 1]$. Let $\mathcal{K} = \{y \in \mathcal{A} : y = y(t) \geq 0, t \in [0, 1]\}$. Then, \mathcal{K} is a solid cone in \mathcal{A} . Let $Y = \{y_1, y_2, y_3, y_4\}$ and, for all $y, z \in Y$, define a mapping $\mathcal{P}_b^r : Y \times Y \longrightarrow \mathcal{K}$ by

$$\mathcal{P}_b^r(y, z)(t) = \begin{cases} \theta, & \text{if } y = z = y_1, \\ 2t, & \text{if } y, z \in \{y_1, y_2\}, y \neq z, \\ t, & \text{otherwise.} \end{cases} \quad (7)$$

Then, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is a PRCbMS-BA with coefficient $s = 4/3$ which is not a rectangular cone b -metric space over \mathcal{A} because $\mathcal{P}_b^r(y_2, y_2)(t) \neq \theta$ and $\mathcal{P}_b^r(y_1, y_2)(t) = 2t > t = \mathcal{P}_b^r(y_1, y_3)(t) + \mathcal{P}_b^r(y_3, y_4)(t) + \mathcal{P}_b^r(y_4, y_2)(t) - \mathcal{P}_b^r(y_3, y_3)(t) - \mathcal{P}_b^r(y_4, y_4)(t)$.

Definition 7. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PRCbMS-BA and \mathcal{K} be a solid cone in \mathcal{A} . For each $y \in Y$ and each $c \in \mathcal{K}^\circ$, let

$$B_{\mathcal{P}_b^r}(y, c) = \{z \in Y : \mathcal{P}_b^r(y, z) \ll c + \mathcal{P}_b^r(y, y)\}, \quad (8)$$

$$\mathcal{B} = \{B_{\mathcal{P}_b^r}(y, c) : y \in Y \text{ and } c \in \mathcal{K}^\circ\}.$$

Then,

$$\tau_{\mathcal{P}} = \left\{ \mathcal{U} \subset Y : \text{for all } y \in \mathcal{U} \text{ there exists } B_{\mathcal{P}_b^r} \in \mathcal{B} \text{ and } y \in B_{\mathcal{P}_b^r} \subset \mathcal{U} \right\} \cup \emptyset, \quad (9)$$

is a topology on Y , $B_{\mathcal{P}_b^r}(y, c)$ is a \mathcal{P}_b^r -ball in $(Y, \mathcal{P}_b^r, \mathcal{A})$, \mathcal{B} is a subbase for the topology $\tau_{\mathcal{P}}$ on Y , and \mathcal{U} is a base generated by the subbase \mathcal{B} .

Definition 8. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PRCbMS-BA, \mathcal{K} be a solid cone in \mathcal{A} , $y^* \in Y$, and $\{y_n\}$ be a sequence in Y . If, for every $c \in \mathcal{K}^\circ$, there exists $N \in \mathbb{N}$ such that $\mathcal{P}_b^r(y_n, y^*) \ll c + \mathcal{P}_b^r(y^*, y^*)$ for all $n > N$, then $\{y_n\}$ is said to be convergent in Y and converges to y^* . This fact is denoted by $y_n \longrightarrow y^*$ as $n \longrightarrow \infty$ or $\lim_{n \longrightarrow \infty} y_n = y^*$.

Definition 9. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PCbMS-BA, \mathcal{K} be a solid cone in \mathcal{A} , and $\{y_n\}$ be a sequence in Y . Then, $\{y_n\}$ is called a θ -Cauchy sequence if $\{\mathcal{P}_b^r(y_n, y_m)\}$ is a c -sequence in \mathcal{A} . That is, if, for every $c \in \mathcal{K}^\circ$, there exists $N \in \mathbb{N}$ such that $\mathcal{P}_b^r(y_n, y_m) \ll c$ for all $n, m > N$.

Definition 10. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PRCbMS-BA, \mathcal{K} be a solid cone in \mathcal{A} , $y^* \in Y$, and $\{y_n\}$ be a sequence in Y . Then, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is called θ -complete if every θ -Cauchy sequence $\{y_n\}$ in Y converges to a point $y^* \in Y$. That is,

$$\lim_{n, m \longrightarrow \infty} \mathcal{P}_b^r(y_n, y_m) = \lim_{n \longrightarrow \infty} \mathcal{P}_b^r(y_n, y^*) = \mathcal{P}_b^r(y^*, y^*) = \theta. \quad (10)$$

Lemma 7. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a PRCbMS-BA and $\{y_n\}$ be a sequence in Y . If $\{y_n\}$ converges to $y^* \in Y$, then

- (1) $\{\mathcal{P}_b^r(y_n, y^*)\}$ is a c -sequence.
- (2) For any $m \in \mathbb{N}$, $\{\mathcal{P}_b^r(y_n, y_{n+m})\}$ is a c -sequence.

Proof. It follows from Definitions 3, 6, and 8.

Firstly, we present a variant of the Banach contraction principle on \mathcal{P}_b^r -cone metric space over Banach algebra \mathcal{A} . \square

Theorem 1. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ -complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \leq s$. Suppose $F : Y \longrightarrow Y$ is a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \leq \alpha \mathcal{P}_b^r(y, z), \quad \text{for all } y, z \in Y, \quad (11)$$

where $\alpha \in \mathcal{K}$ such that α commutes with s and $\rho(\alpha) < 1$. Then, F has a unique fixed point.

Proof. Let y_0 be a point in Y . We define a sequence $\{y_n\}$ in Y by

$$y_n = Fy_{n-1} = F^n y_0, \quad \text{for all } n \geq 1. \quad (12)$$

If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then $y^* = y_n = Fy_n$ is a fixed point of F , and the result is proved. Hence, we assume that $y_n \neq y_{n+1}$ for all $n \geq 0$. We will show that $y_n \neq y_{n+q}$ for all $n \geq 0$ and $q \geq 1$. Suppose that $y_n = y_{n+q}$ for some $n \geq 0, q \geq 1$; then, $y_{n+1} = y_{n+q+1}$ and $Fy_n = Fy_{n+q}$. Then, (11) implies that

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_{n+1}) &= \mathcal{P}_b^r(y_{n+q}, y_{n+q+1}) \leq \alpha \mathcal{P}_b^r(y_{n+q-1}, y_{n+q}) \\ &\leq \cdots \leq \alpha^q \mathcal{P}_b^r(y_n, y_{n+1}). \end{aligned} \quad (13)$$

Using Lemma 1, we obtain that $\mathcal{P}_b^r(y_n, y_{n+1}) = \theta$, that is, $y_n = y_{n+1}$, which is a contradiction. Therefore, $y_n \neq y_m$ for all distinct $n, m \in \mathbb{N}$. Hence, from (11) and (12), we have that

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_{n+1}) &= \mathcal{P}_b^r(Fy_{n-1}, Fy_n) \leq \alpha \mathcal{P}_b^r(y_{n-1}, y_n) \\ &\leq \alpha^2 \mathcal{P}_b^r(y_{n-2}, y_{n-1}) \leq \cdots \\ \therefore \mathcal{P}_b^r(y_n, y_{n+1}) &\leq \alpha^n \mathcal{P}_b^r(y_0, y_1), \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (14)$$

Similarly, for all $n, m, q \in \mathbb{N}$, we obtain that

$$\begin{aligned} \mathcal{P}_b^r(y_{n+q}, y_{m+q}) &= \mathcal{P}_b^r(Fy_{n+q-1}, Fy_{m+q-1}) \\ &\leq \alpha \mathcal{P}_b^r(y_{n+q-1}, y_{m+q-1}) \\ &\leq \alpha^2 \mathcal{P}_b^r(y_{n+q-2}, y_{m+q-2}) \leq \cdots \\ \therefore \mathcal{P}_b^r(y_{n+q}, y_{m+q}) &\leq \alpha^q \mathcal{P}_b^r(y_n, y_m), \\ &\text{for all } n, m, q \in \mathbb{N}. \end{aligned} \quad (15)$$

Observe that $\rho(s)$ exists because of Lemma 2, and since $\rho(\alpha) < 1$, there exists $q_1 \in \mathbb{N}$ such that $\rho(s)\rho(\alpha)^{q_1} < 1$ holds. Since α commutes with s , by Lemmas 2 and 3, we have that

$$\rho(s\alpha^{q_1}) \leq \rho(s)\rho(\alpha)^{q_1} < 1 \text{ and } (e - s\alpha^{q_1}) \text{ is invertible in } \mathcal{A}. \quad (16)$$

Hence, by condition (P4), for all $y, z, x_1, x_2 \in Y$, we have

$$\begin{aligned} \mathcal{P}_b^r(y, z) &\leq s[\mathcal{P}_b^r(y, x_1) + \mathcal{P}_b^r(x_1, x_2) + \mathcal{P}_b^r(x_2, z)] - \mathcal{P}_b^r(x_1, x_1) - \mathcal{P}_b^r(x_2, x_2) \\ \therefore \mathcal{P}_b^r(y, z) &\leq s[\mathcal{P}_b^r(y, x_1) + \mathcal{P}_b^r(x_1, x_2) + \mathcal{P}_b^r(x_2, z)], \quad \text{for all } y, z, x_1, x_2 \in Y. \end{aligned} \quad (17)$$

This, using (15) and (16), implies that

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_m) &\leq s[\mathcal{P}_b^r(y_n, y_{n+q_1}) + \mathcal{P}_b^r(y_{n+q_1}, y_{m+q_1}) + \mathcal{P}_b^r(y_{m+q_1}, y_m)] \\ &\leq s[\alpha^n \mathcal{P}_b^r(y_0, y_{q_1}) + \alpha^{q_1} \mathcal{P}_b^r(y_n, y_m) + \alpha^m \mathcal{P}_b^r(y_{q_1}, y_0)] \\ &\quad \cdot (e - s\alpha^{q_1}) \mathcal{P}_b^r(y_n, y_m) \leq s[\alpha^n \mathcal{P}_b^r(y_0, y_{q_1}) + \alpha^m \mathcal{P}_b^r(y_{q_1}, y_0)] \\ &\quad \cdot (e - s\alpha^{q_1}) \mathcal{P}_b^r(y_n, y_m) \leq (e - s\alpha^{q_1})^{-1} s[\alpha^n \mathcal{P}_b^r(y_0, y_{q_1}) + \alpha^m \mathcal{P}_b^r(y_{q_1}, y_0)]. \end{aligned} \quad (18)$$

Using Lemmas 4 and 5, we deduce that $\{\mathcal{P}_b^r(y_n, y_m)\}$ is a c -sequence in \mathcal{A} . Therefore, $\{y_n\}$ is a θ -Cauchy sequence in Y . From the hypothesis, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is θ -complete; hence, there exists a point $y^* \in Y$ such that $\{y_n\}$ converges to y^* . That is,

$$\lim_{n \rightarrow \infty} \mathcal{P}_b^r(y_n, y^*) = \lim_{n, m \rightarrow \infty} \mathcal{P}_b^r(y_n, y_m) = \mathcal{P}_b^r(y^*, y^*) = \theta. \quad (19)$$

Next, we will show that y^* is the unique fixed point of F :

$$\begin{aligned} \mathcal{P}_b^r(y^*, Fy^*) &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(y_{n+1}, Fy^*)] \\ &\quad - \mathcal{P}_b^r(y_n, y_n) - \mathcal{P}_b^r(y_{n+1}, y_{n+1}) \\ &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(Fy_n, Fy^*)] \\ &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \alpha \mathcal{P}_b^r(y_n, y^*)] \\ &\leq s[(e + \alpha) \mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1})]. \end{aligned} \quad (20)$$

By Lemmas 6 and 7, we have $\mathcal{P}_b^r(y^*, y_n) \rightarrow \theta$ as $n \rightarrow \infty$ and $\mathcal{P}_b^r(y_n, y_{n+1}) \rightarrow \theta$ as $n \rightarrow \infty$. Hence, we deduce that $\mathcal{P}_b^r(y^*, Fy^*) = \theta$. That is, $y^* = Fy^*$. So, y^* is a fixed point of F . For uniqueness, we let z^* be another fixed point of F . Then, it follows from (11) that

$$\mathcal{P}_b^r(y^*, z^*) = \mathcal{P}_b^r(Fy^*, Fz^*) \leq \alpha \mathcal{P}_b^r(y^*, z^*). \quad (21)$$

By Lemma 1, we get that $\mathcal{P}_b^r(y^*, z^*) = \theta$, and hence, $y^* = z^*$.

Kindly, observe that Theorem 1 extends and generalizes Theorem 3.5 in [1], Theorem 3.1 in [3], Theorem 2.1 in [4], Theorem 2.1 in [5], and Theorem 3.1 in [6]. \square

Example 2. Let $\mathcal{A} = C_{\mathbb{R}}^1[0, 1]$ with the norm

$$\|y\| = \|y\|_{\infty} + \|y'\|_{\infty}, \quad \text{for all } y \in \mathcal{A}. \quad (22)$$

Define multiplication pointwisely on \mathcal{A} . Then, \mathcal{A} is a Banach algebra with unit $e(t) = 1, \forall t \in [0, 1]$. Let $\mathcal{K} = \{y \in \mathcal{A} : y = y(t) \geq 0, t \in [0, 1]\}$. Then, \mathcal{K} is a solid cone in \mathcal{A} . Let $Y = \{0, 1, 2, 3\}$ and, for all $y, z \in Y$, define a mapping $\mathcal{P}_b^r : Y \times Y \rightarrow \mathcal{K}$ by

$$\mathcal{P}_b^r(y, z)(t) = \begin{cases} y^2 t, & \text{if } y = z \neq 0, \\ 2(y^2 + z^2)t, & \text{if } y, z \notin \{2, 3\}, y \neq z, \\ (y^2 + z^2)t & \text{if } y, z \in \{2, 3\}, y \neq z, \\ \frac{1}{2}t, & \text{if } y = z = 0. \end{cases} \quad (23)$$

Then, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is a θ -complete PRCbMS-BA with coefficient $s = 2$. Define a mapping $F : Y \rightarrow Y$ as follows:

$$Fy = \begin{cases} 0, & \text{if } y \in \{0, 1\}, \\ 1, & \text{if } y \in \{2, 3\}. \end{cases} \quad (24)$$

Hence, the mapping F satisfies all the conditions of Theorem 1 and $y^* = 0 \in Y$ is the unique fixed point of F .

Secondly, we present a variant of the Reich contraction principle on \mathcal{P}_b^r -cone metric space over Banach algebra \mathcal{A} .

Theorem 2. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ -complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \leq s$. Suppose $F : Y \rightarrow Y$ is a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \leq \alpha \mathcal{P}_b^r(y, z) + \beta \mathcal{P}_b^r(y, Fy) + \gamma \mathcal{P}_b^r(z, Fz), \quad (25)$$

for all $y, z \in Y$, where $\alpha, \beta, \gamma \in \mathcal{K}$ commutes, $\rho(\alpha) + \rho(\beta + \gamma) < 1$, and $\min\{\rho(\beta), \rho(\gamma)\} < (1/\rho(s))$. Then, F has a unique fixed point.

Proof. Let y_0 be a point in Y . We define a sequence $\{y_n\}$ in Y by

$$y_{n+1} = Fy_n = F^{n+1}y_0, \quad \text{for all } n \geq 0. \quad (26)$$

From (25) and (26), we have

$$\begin{aligned} \mathcal{P}_b^r(y_{n+1}, y_n) &= \mathcal{P}_b^r(Fy_n, Fy_{n-1}) \\ &\leq \alpha \mathcal{P}_b^r(y_n, y_{n-1}) + \beta \mathcal{P}_b^r(y_n, Fy_n) \\ &\quad + \gamma \mathcal{P}_b^r(y_{n-1}, Fy_{n-1}) \\ \therefore (e - \beta) \mathcal{P}_b^r(y_{n+1}, y_n) &\leq (\alpha + \gamma) \mathcal{P}_b^r(y_n, y_{n-1}). \end{aligned} \quad (27)$$

On the contrary, we have

$$\begin{aligned} \mathcal{P}_b^r(y_{n+1}, y_n) &= \mathcal{P}_b^r(Fy_n, Fy_{n-1}) = \mathcal{P}_b^r(Fy_{n-1}, Fy_n) \\ &\leq \alpha \mathcal{P}_b^r(y_{n-1}, y_n) + \beta \mathcal{P}_b^r(y_{n-1}, Fy_{n-1}) + \gamma \mathcal{P}_b^r(y_n, Fy_n) \\ \therefore (e - \gamma) \mathcal{P}_b^r(y_{n+1}, y_n) &\leq (\alpha + \beta) \mathcal{P}_b^r(y_n, y_{n-1}). \end{aligned} \quad (28)$$

Adding up (27) and (28), we have

$$(2e - \lambda) \mathcal{P}_b^r(y_{n+1}, y_n) \leq (2\alpha + \lambda) \mathcal{P}_b^r(y_n, y_{n-1}), \quad (29)$$

where $\lambda = (\beta + \gamma) \in \mathcal{K}$. Now, we observe that

$$2\rho(\lambda) \leq 2\rho(\alpha) + 2\rho(\lambda) = 2[\rho(\alpha) + \rho(\beta + \gamma)] < 2. \quad (30)$$

This implies that $\rho(\lambda) < 1 < 2$; then, by Lemma 2, it follows that $(2e - \lambda)$ is invertible and $(2e - \lambda)^{-1} = \sum_{j=0}^{\infty} (\lambda^j / 2^{j+1})$. From (29), we obtain

$$\begin{aligned} \mathcal{P}_b^r(y_{n+1}, y_n) &\leq (2e - \lambda)^{-1} (2\alpha + \lambda) \mathcal{P}_b^r(y_n, y_{n-1}) \\ &\leq k \mathcal{P}_b^r(y_n, y_{n-1}), \end{aligned} \quad (31)$$

where $k = (2e - \lambda)^{-1} (2\alpha + \lambda) \in \mathcal{K}$. Hence,

$$\begin{aligned} \mathcal{P}_b^r(y_{n+1}, y_n) &\leq k \mathcal{P}_b^r(y_n, y_{n-1}) \leq k^2 \mathcal{P}_b^r(y_{n-1}, y_{n-2}) \leq \cdots \\ \therefore \mathcal{P}_b^r(y_{n+1}, y_n) &\leq k^n \mathcal{P}_b^r(y_1, y_0), \quad \text{for all } n \in \mathbb{N}. \end{aligned} \quad (32)$$

We claim that $\rho(k) < 1$. Indeed, since α commutes with $\lambda = \beta + \gamma$, it follows that

$$\begin{aligned} (2e - \lambda)^{-1} (2\alpha + \lambda) &= \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{2^{j+1}} \right) (2\alpha + \lambda) \\ &= 2 \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{2^{j+1}} \right) \alpha + \sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{2^{j+1}} \\ &= 2\alpha \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{2^{j+1}} \right) + \lambda \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{2^{j+1}} \right) \\ &= (2\alpha + \lambda) \left(\sum_{j=0}^{\infty} \frac{\lambda^j}{2^{j+1}} \right) \\ &= (2\alpha + \lambda) (2e - \lambda)^{-1}. \end{aligned} \quad (33)$$

Therefore, $(2\alpha + \lambda)$ commutes with $(2e - \lambda)^{-1}$. Then, by Lemmas 2 and 3, we obtain

$$\begin{aligned} \rho(k) &= \rho((2e - \lambda)^{-1} (2\alpha + \lambda)) \leq \rho((2e - \lambda)^{-1}) \rho(2\alpha + \lambda) \\ &\leq \frac{1}{2 - \rho(\lambda)} [2\rho(\alpha) + \rho(\lambda)] < 1 \quad (\text{since } \rho(\alpha) + \rho(\lambda) < 1). \end{aligned} \quad (34)$$

If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then $y^* = y_n = Fy_n$ is a fixed point of F , and the result is proved. Hence, we assume that $y_n \neq y_{n+1}$, for all $n \geq 0$. We will show that $y_n \neq y_{n+q}$, for all $n \geq 0$ and $q \geq 1$. Suppose that $y_n = y_{n+q}$, for some $n \geq 0, q \geq 1$; then, $y_{n+1} = y_{n+q+1}$ and $Fy_n = Fy_{n+q}$. Then, (31) implies that

$$\mathcal{P}_b^r(y_{n+1}, y_n) = \mathcal{P}_b^r(y_{n+q+1}, y_{n+q}) \leq k^q \mathcal{P}_b^r(y_{n+1}, y_n). \quad (35)$$

Using Lemma 1, we obtain that $\mathcal{P}_b^r(y_{n+1}, y_n) = \theta$, that is, $y_{n+1} = y_n$, which is a contradiction. Therefore, $y_n \neq y_m$ for all distinct $n, m \in \mathbb{N}$. Next, from (25), (26), and (32), we have

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_m) &= \mathcal{P}_b^r(Fy_{n-1}, Fy_{m-1}) \\ &\leq \alpha \mathcal{P}_b^r(y_{n-1}, y_{m-1}) + \beta \mathcal{P}_b^r(y_{n-1}, Fy_{n-1}) + \gamma \mathcal{P}_b^r(y_{m-1}, Fy_{m-1}) \\ &= \alpha \mathcal{P}_b^r(y_{n-1}, y_{m-1}) + \beta \mathcal{P}_b^r(y_{n-1}, y_n) + \gamma \mathcal{P}_b^r(y_{m-1}, y_m) \\ &\leq \alpha \mathcal{P}_b^r(y_{n-1}, y_{m-1}) + \beta k^{n-1} \mathcal{P}_b^r(y_0, y_1) + \gamma k^{m-1} \mathcal{P}_b^r(y_0, y_1) \\ \therefore \mathcal{P}_b^r(y_n, y_m) &\leq q \mathcal{P}_b^r(y_{n-1}, y_{m-1}) + (q^n + q^m) \mathcal{P}_b^r(y_0, y_1), \end{aligned} \quad (36)$$

where $q \in \{\alpha, \beta, \gamma, k\}$ such that $\rho(q) = \max\{\rho(\alpha), \rho(\beta), \rho(\gamma), \rho(k)\}$. Hence, from (36), we also obtain

$$\mathcal{P}_b^r(y_n, y_m) \leq q^p \mathcal{P}_b^r(y_{n-p}, y_{m-p}) + p(q^n + q^m) \mathcal{P}_b^r(y_0, y_1), \quad (37)$$

for all $p \in \{1, 2, \dots, \min\{n, m\}\}$. Observe that $\rho(s)$ exists because of Lemma 2, and since $\rho(q) < 1$, there exists $q_0 \in \mathbb{N}$

such that $\rho(s)\rho(q)^{q_0} < 1$ holds. Furthermore, since q commutes with s , by Lemmas 2 and 3, we have that

$$\rho(sq^{q_0}) \leq \rho(s)\rho(q)^{q_0} < 1 \text{ and } (e - sq^{q_0}) \text{ is invertible in } \mathcal{A}. \quad (38)$$

Therefore, from (37), we further obtain

$$\mathcal{P}_b^r(y_n, y_{n+q_0}) \leq q^n \mathcal{P}_b^r(y_0, y_{q_0}) + n(q^n + q^{n+q_0}) \mathcal{P}_b^r(y_0, y_1), \quad (39)$$

$$\mathcal{P}_b^r(y_{m+q_0}, y_m) \leq q^m \mathcal{P}_b^r(y_{q_0}, y_0) + m(q^{m+q_0} + q^m) \mathcal{P}_b^r(y_0, y_1), \quad (40)$$

$$\mathcal{P}_b^r(y_{n+q_0}, y_{m+q_0}) \leq q^{q_0} \mathcal{P}_b^r(y_n, y_m) + q_0(q^{n+q_0} + q^{m+q_0}) \mathcal{P}_b^r(y_0, y_1). \quad (41)$$

Hence, from (P4), (38), (39), (40) and (41), we have

$$\begin{aligned} \mathcal{P}_b^r(y_n, y_m) &\leq s[\mathcal{P}_b^r(y_n, y_{n+q_0}) + \mathcal{P}_b^r(y_{n+q_0}, y_{m+q_0}) + \mathcal{P}_b^r(y_{m+q_0}, y_m)] \\ &\quad - \mathcal{P}_b^r(y_{n+q_0}, y_{n+q_0}) - \mathcal{P}_b^r(y_{m+q_0}, y_{m+q_0}) \\ &\leq s \left[\begin{aligned} &q^n \mathcal{P}_b^r(y_0, y_{q_0}) + n(q^n + q^{n+q_0}) \mathcal{P}_b^r(y_0, y_1) + q^{q_0} \mathcal{P}_b^r(y_n, y_m) + q_0(q^{n+q_0} + q^{m+q_0}) \mathcal{P}_b^r(y_0, y_1) \\ &+ q^m \mathcal{P}_b^r(y_{q_0}, y_0) + m(q^{m+q_0} + q^m) \mathcal{P}_b^r(y_0, y_1) \end{aligned} \right] \\ (e - sq^{q_0}) \mathcal{P}_b^r(y_n, y_m) &\leq s[(q^n + q^m) \mathcal{P}_b^r(y_0, y_{q_0}) + [q^n(n + (n + q_0)q^{q_0}) \\ &\quad + q^m(m + (m + q_0)q^{q_0})] \mathcal{P}_b^r(y_0, y_1)] \\ \therefore \mathcal{P}_b^r(y_n, y_m) &\leq (e - sq^{q_0})^{-1} s[(q^n + q^m) \mathcal{P}_b^r(y_0, y_{q_0}) + [q^n(n + (n + q_0)q^{q_0}) \\ &\quad + q^m(m + (m + q_0)q^{q_0})] \mathcal{P}_b^r(y_0, y_1)]. \end{aligned} \quad (42)$$

Using Lemmas 4 and 5, we deduce that $\{\mathcal{P}_b^r(y_n, y_m)\}$ is a c -sequence in \mathcal{A} . Therefore, $\{y_n\}$ is a θ -Cauchy sequence in Y . From the hypothesis, $(Y, \mathcal{P}_b^r, \mathcal{A})$ is θ -complete; hence, there exists a point $y^* \in Y$ such that $\{y_n\}$ converges to y^* . That is,

$$\lim_{n \rightarrow \infty} \mathcal{P}_b^r(y_n, y^*) = \lim_{n, m \rightarrow \infty} \mathcal{P}_b^r(y_n, y_m) = \mathcal{P}_b^r(y^*, y^*) = \theta. \quad (43)$$

Next, we will show that y^* is the unique fixed point of F :

$$\begin{aligned} \mathcal{P}_b^r(y^*, Fy^*) &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(y_{n+1}, Fy^*)] \\ &\quad - \mathcal{P}_b^r(y_n, y_n) - \mathcal{P}_b^r(y_{n+1}, y_{n+1}) \\ &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \mathcal{P}_b^r(Fy_n, Fy^*)] \\ &\leq s[\mathcal{P}_b^r(y^*, y_n) + \mathcal{P}_b^r(y_n, y_{n+1}) + \alpha \mathcal{P}_b^r(y_n, y^*) + \beta \mathcal{P}_b^r(y_n, Fy_n) + \gamma \mathcal{P}_b^r(y^*, Fy^*)] \\ \therefore \mathcal{P}_b^r(y^*, Fy^*) &\leq s[(e + \alpha) \mathcal{P}_b^r(y^*, y_n) + (e + \beta) \mathcal{P}_b^r(y_n, y_{n+1}) + \gamma \mathcal{P}_b^r(y^*, Fy^*)]. \end{aligned} \quad (44)$$

On the contrary, we have

$$\begin{aligned}
 \mathcal{P}_b^r(Fy^*, y^*) &\leq s[\mathcal{P}_b^r(Fy^*, y_{n+1}) + \mathcal{P}_b^r(y_{n+1}, y_n) + \mathcal{P}_b^r(y_n, y^*)] \\
 &\quad - \mathcal{P}_b^r(y_{n+1}, y_{n+1}) - \mathcal{P}_b^r(y_n, y_n) \\
 &\leq s[\mathcal{P}_b^r(Fy^*, Fy_n) + \mathcal{P}_b^r(y_{n+1}, y_n) + \mathcal{P}_b^r(y_n, y^*)] \\
 &\leq s[\alpha\mathcal{P}_b^r(y^*, y_n) + \beta\mathcal{P}_b^r(y^*, Fy^*) + \gamma\mathcal{P}_b^r(y_n, Fy_n) + \mathcal{P}_b^r(y_{n+1}, y_n) + \mathcal{P}_b^r(y_n, y^*)] \\
 \therefore \mathcal{P}_b^r(y^*, Fy^*) &\leq s[(e + \alpha)\mathcal{P}_b^r(y^*, y_n) + (e + \gamma)\mathcal{P}_b^r(y_n, y_{n+1}) + \beta\mathcal{P}_b^r(y^*, Fy^*)].
 \end{aligned} \tag{45}$$

By Lemmas 6 and 7, we have $\mathcal{P}_b^r(y^*, y_n) \rightarrow \theta$ as $n \rightarrow \infty$ and $\mathcal{P}_b^r(y_n, y_{n+1}) \rightarrow \theta$ as $n \rightarrow \infty$. Hence, from (44) and (45), we deduce that $\mathcal{P}_b^r(y^*, Fy^*) \leq s\gamma\mathcal{P}_b^r(y^*, Fy^*)$ and $\mathcal{P}_b^r(y^*, Fy^*) \leq s\beta\mathcal{P}_b^r(y^*, Fy^*)$. Since $\min\{\rho(\beta), \rho(\gamma)\} < (1/\rho(s))$, by Lemma 1, we have $\mathcal{P}_b^r(y^*, Fy^*) = \theta$ so that $y^* = Fy^*$. That is, y^* is a fixed point of F . For uniqueness, we let z^* be another fixed point of F . Then, it follows from (25) that

$$\begin{aligned}
 \mathcal{P}_b^r(y^*, z^*) &= \mathcal{P}_b^r(Fy^*, Fz^*) \\
 &\leq \alpha\mathcal{P}_b^r(y^*, z^*) + \beta\mathcal{P}_b^r(y^*, Fy^*) + \gamma\mathcal{P}_b^r(z^*, Fz^*) \\
 &= \alpha\mathcal{P}_b^r(z^*, y^*) + \beta\mathcal{P}_b^r(y^*, y^*) + \gamma\mathcal{P}_b^r(z^*, z^*) \\
 &\leq \alpha\mathcal{P}_b^r(z^*, y^*) + \beta\mathcal{P}_b^r(y^*, z^*) + \gamma\mathcal{P}_b^r(z^*, y^*) \\
 \therefore \mathcal{P}_b^r(y^*, z^*) &\leq (\alpha + \beta + \gamma)\mathcal{P}_b^r(y^*, z^*).
 \end{aligned} \tag{46}$$

By Lemmas 1 and 3, we have that $\mathcal{P}_b^r(y^*, z^*) = \theta$, and hence, $y^* = z^*$, i.e., the fixed point of F is unique.

Note that Theorem 2 extends and generalizes Theorem 3.1 in [7].

Thirdly, we present a variant of the quasi-contraction principle on \mathcal{P}_b^r -cone metric space over Banach algebra \mathcal{A} . \square

Corollary 1. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ -complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \leq s$. Let $F: Y \rightarrow Y$ be a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \leq \alpha M(y, z), \quad \text{for all } y, z \in Y, \tag{47}$$

where $M(y, z) \in \{\mathcal{P}_b^r(y, z), \mathcal{P}_b^r(y, Fy), \mathcal{P}_b^r(z, Fz)\}$, $\alpha \in \mathcal{K}$ commutes with s , and $\rho(\alpha) < 1$. Then, F has a unique fixed point.

Proof. Since (47) implies (25), the proof follows from Theorem 2.

Note that Corollary 1 generalizes Theorem 3.1 in [8–10].

Finally, we present a variant of the Kannan contraction principle on \mathcal{P}_b^r -cone metric space over Banach algebra \mathcal{A} . \square

Corollary 2. Let $(Y, \mathcal{P}_b^r, \mathcal{A})$ be a θ -complete PRCbMS-BA with $s \in \mathcal{K}$ such that $e \leq s$. Suppose $F: Y \rightarrow Y$ is a function satisfying

$$\mathcal{P}_b^r(Fy, Fz) \leq \beta[\mathcal{P}_b^r(y, Fy) + \mathcal{P}_b^r(z, Fz)], \quad \text{for all } y, z \in Y, \tag{48}$$

where $\alpha \in \mathcal{K}$ such that $\rho(\beta) < 1/2$ and $\rho(s\beta) < 1$. Then, F has a unique fixed point.

Proof. Put $\alpha = \theta$ and $\beta = \gamma$ in Theorem 2, and the result follows.

Note that Corollary 2 generalizes Theorem 2.4 in [4], Theorem 2.3 in [5], and Theorem 3.3 in [6]. \square

4. Conclusion

The concept of a partial rectangular cone b -metric space over Banach algebras was introduced, and some fixed point results under various contractive mappings were proved in such a space. Some examples were also given to elucidate the results. Our results extend and generalized many existing results in [1, 3–10].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have contributed equally and significantly in writing this article.

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Research Article

Solution of Fractional Differential Equations Utilizing Symmetric Contraction

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Received 11 February 2021; Revised 8 March 2021; Accepted 4 April 2021; Published 22 May 2021

Academic Editor: Huseyin Isik

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The aim of this paper is to present another family of fractional symmetric α - η -contractions and build up some new results for such contraction in the context of \mathcal{F} -metric space. The author derives some results for Suzuki-type contractions and orbitally T -complete and orbitally continuous mappings in \mathcal{F} -metric spaces. The inspiration of this paper is to observe the solution of fractional-order differential equation with one of the boundary conditions using fixed-point technique in \mathcal{F} -metric space.

1. Preliminaries and Scope

Fixed-point theory has been promoted by a few particular works in the most recent decades [1–3]. One of the intriguing methodologies was presented in Karapinar et al.'s work [4] which starts a thought of interpolative kind of contractions and set up shiny new fixed-point results in partial metric space. Recently, Jleli and Samet [5] introduced a new generalization of metric space and named it as \mathcal{F} -metric space.

Definition 1 (see [5]). Let \mathcal{F} be the set function $f: (0, +\infty) \rightarrow (-\infty, +\infty)$ that meets the following conditions:

- (\mathcal{F}_1) f is nondecreasing; that is, for all $0 < c < d$, it implies $f(c) \leq f(d)$.
 - (\mathcal{F}_2) For each iteration $\{d_n\} \subset (0, +\infty)$, we have $\lim_{n \rightarrow +\infty} d_n = 0$, if and only if $\lim_{n \rightarrow +\infty} f(d_n) = -\infty$.
- (1)

The generalized notion of metric space is as follows.

Definition 2 (see [5]). Let $A \neq \emptyset$ with $D: A \times A \rightarrow [0, +\infty)$ be a given mapping. Suppose that there exists $(f, \mu) \in \mathcal{F} \times [0, +\infty)$ such that

$$(D_1) (w, v) \in A \times A, D(w, v) = 0 \Leftrightarrow w = v.$$

$$(D_2) D(w, v) = D(v, w) \text{ for all } (w, v) \in A \times A.$$

$$(D_3) \text{ Each } (w, v) \in A \times A, \forall N \in \mathbb{N}, N \geq 2, \text{ and for each } (u_i)_{i=1}^N \subset A \text{ with } (u_1, u_N) = (w, v), \text{ we have}$$

$$D(w, v) > 0 \text{ implies } f\left(D(w, v)\right) \leq f\left(\sum_{i=1}^{N-1} d(u_i, u_{i+1})\right) + \mu. \quad (2)$$

Then, it is said that D is an \mathcal{F} -metric on A .

Here, the pair (A, D) is called an \mathcal{F} -metric space and it is abbreviated as \mathcal{F} -MS. A sequence $\{w_n\}$ in (A, D) is \mathcal{F} -Cauchy, if $\lim_{n, m \rightarrow \infty} D(w_n, w_m) = 0$. Furthermore, (A, D) is \mathcal{F} -complete, if every \mathcal{F} -Cauchy sequence is \mathcal{F} -convergent in A .

The following example is stated in [5].

Example 1 (see [5]). The set of natural numbers $\mathbb{N} = A$ is an \mathcal{F} -MS if we define D by

$$D(w, v) = \begin{cases} (w - v)^2, & \text{if } (w, v) \in [0, 3] \times [0, 3], \\ |w - v|, & \text{if } (w, v) \notin [0, 3] \times [0, 3], \end{cases} \quad (3)$$

for all $(w, v) \in A \times A$, $f(t) = \ln(t)$, and $\mu = \ln(3)$. Moreover, D does not form a metric but it is an \mathcal{F} -MS.

Jleli and Samet proposed a simple Banach fixed-point theorem as follows.

Theorem 1 (see [5]). *Let (A, D) be an \mathcal{F} -MS. Let $g: A \rightarrow A$ be a self-mapping. Suppose that the following conditions are met:*

- (i) (A, D) is \mathcal{F} -complete.
- (ii) \exists a constant $k \in (0, 1)$ such that

$$D(g(w), g(v)) \leq k D(w, v), \quad (w, v) \in A \times A. \quad (4)$$

Then, g attains a unique fixed-point $w^ \in A$.*

In 2012, Samet et al. introduced a class of α -admissible mappings as follows.

Definition 3 (see [6]). Let $T: A \rightarrow A$ and $\alpha: A \times A \rightarrow [0, +\infty)$. T is said to be α -admissible if $w, v \in A$, and $\alpha(w, v) \geq 1$ implies that $\alpha(Tw, Tv) \geq 1$.

Next, Salimi et al. [7] modified the concept of α -admissible mapping as follows.

Definition 4 (see [7]). Let $T: A \rightarrow A$ and $\alpha, \eta: A \times A \rightarrow [0, +\infty)$ be two functions. T is called an α -admissible mapping with respect to η , if $w, v \in A$, and $\alpha(w, v) \geq \eta(w, v)$ implies that $\alpha(Tw, Tv) \geq \eta(Tw, Tv)$.

If $\eta(w, v) = 1$, then the above definition reduces to Definition 3. If $\alpha(w, v) = 1$, then T is called an η -subadmissible mapping.

Definition 5 (see [8]). Consider a metric space (A, d) and assume that $T: A \rightarrow A$ and $\alpha, \eta: A \times A \rightarrow [0, \infty)$ are two functions. A mapping T is considered as α - η -continuous mapping in (A, d) whenever $w \in A$ is given, and the sequence $\{w_n\}$ is as follows:

$$\begin{aligned} w_n &\longrightarrow w \text{ at } \infty, \\ \alpha(w_n, w_{n+1}) &\geq \eta(w_n, w_{n+1}), \\ \forall n \in \mathbb{N} &\text{ implies } Tw_n \longrightarrow Tw. \end{aligned} \quad (5)$$

For more details, see, for example, [9, 10].

A mapping $T: A \rightarrow A$ is called orbitally continuous in $v \in A$ if $\lim_{n \rightarrow \infty} T^n w = v$ implies that $\lim_{n \rightarrow \infty} T^n w = Tw$. T mapping is orbitally continuous on A if T is orbitally continuous $\forall v \in A$.

2. Fractional Symmetric α - η -Contraction of Type-I

In this segment, first we present a new fractional symmetric α - η -contraction of type-I.

Definition 6. Let $T: A \rightarrow A$ be an \mathcal{F} -metric space (A, D) and two functions $\alpha, \eta: A \times A \rightarrow [0, +\infty)$. We consider that T is a fractional symmetric α - η -contraction of type-I along with constants $\lambda \in [0, 1)$ and $\beta, \hat{w}, \gamma \in (0, 1)$ such that, whenever $\alpha(w, v) \geq \eta(w, v)$, we have

$$D(Tw, Tv)^p \leq \lambda (\tilde{S}_1(w, v)), \quad (6)$$

where

$$\begin{aligned} \tilde{S}_1(w, v) &= D(w, v)^p \cdot [D(w, Tw)]^{(p/(\beta-\hat{w})(\beta-\gamma))}, \\ &\quad [D(v, Tv)]^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p/(\hat{w}-\beta)(\hat{w}-\gamma))}, \\ &\quad [D(w, Tv) + D(v, Tw)]^{(p/(\gamma-\beta)(\gamma-\hat{w}))}, \end{aligned} \quad (7)$$

where $p \in [1, \infty)$, for all $w, v \in A \setminus \text{Fix}(T)$.

Example 2. Let $A = \{0, 1, 2, 3\}$ with grace of \mathcal{F} -metric D defined by

$$D(w, v) = \begin{cases} (w - v)^2, & \text{if } (w, v) \in A \times A, \\ |w - v|, & \text{if } (w, v) \notin A \times A, \end{cases} \quad (8)$$

and consider $f(t) = \ln(t)$ and $\mu = \ln(3)$. Define $T: A \rightarrow A$ by

$$T0 = 0, T1 = 1, T2 = T3 = 0, \quad (9)$$

and $\alpha, \eta: A \times A \rightarrow [0, +\infty)$ by

$$\begin{aligned} \alpha(w, v) &= \begin{cases} 1, & \text{if } w, v \in A, \\ 0, & \text{otherwise,} \end{cases} \\ \eta(w, v) &= \begin{cases} \frac{1}{2}, & \text{if } w, v \in A, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (10)$$

If $w, v \in A$, clearly $\alpha(w, v) \in A \geq \eta(w, v)$ such that

$$\begin{aligned}
D(T2, T3)^p &= 0 \leq \lambda \left[D(2, 3)^p \cdot D(2, T2)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(3, T3)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \right. \\
&\quad \left. (D(2, T2) + D(3, T3))^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma))} (D(2, T3) + D(3, T2))^{(p/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\
&= \lambda \left[1 \cdot D(2, 0)^{(p/(\beta-\widehat{w})(\beta-\gamma))} D(3, 0)^{(p/(\beta-\widehat{w})(\beta-\gamma))} (D(2, 0) + D(3, 0))^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \right. \\
&\quad \left. \cdot (D(2, 0) + D(3, 0))^{(p/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\
&= \lambda \left[(4)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot (9)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot (4+9)^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot (4+9)^{(p/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\
&\leq \lambda \left[(4)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot (9)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot (4 \cdot 9)^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot (4 \cdot 9)^{(p/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\
&= \lambda [(4) \cdot (9)]^{(p/(\beta-\widehat{w})(\beta-\gamma)) + (p/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p/(\gamma-\beta)(\gamma-\widehat{w}))} = \lambda.
\end{aligned} \tag{11}$$

By taking any value of constants $\lambda \in [0, 1)$ and $\beta, \widehat{w}, \gamma \in (0, 1)$, clearly, (6) holds for all $p \in [1, \infty)$, $w, v \in A \setminus \text{Fix}(T)$. Point out that T has two fixed points, which are 0 and 1.

Now, we initiate brand new fixed-point theorems for fractional symmetric α - η -contraction of type-I in the configuration of \mathcal{F} -complete \mathcal{F} -MS.

Theorem 2. Let (A, D) be a complete \mathcal{F} -metric space and T is a fractional symmetric α - η -contraction of type-I satisfying the following statements:

- (i) T is an α -admissible mapping concerning η
- (ii) There exists $w_0 \in A$ to such an extent that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$
- (iii) T is α - η -continuous

At that point, T possesses a fixed point at A .

Proof. Consider w_0 in A with the goal that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$. For $w_0 \in A$, we build a chain $\{w_n\}_{n=1}^\infty$ in such a way that $w_1 = Tw_0$ and $w_2 = Tw_1 = T^2w_0$. Proceeding with this exercise, $w_{n+1} = Tw_n = T^{n+1}w_0$, for every $n \in \mathbb{N}$. Presently, as long as mapping T is α -admissible with respect to η , at that time $\alpha(w_0, w_1) = \alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0) = \eta(w_0, w_1)$. Carrying on in this way, we get

$$\alpha(w_{n-1}, w_n) \geq \eta(w_{n-1}, w_n) = \eta(w_{n-1}, Tw_{n-1}), \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

Provided that $w_{n+1} = w_n$ for some $n \in \mathbb{N}$, then $w_n = w^*$ is a fixed point of T . So, we assume that $w_n \neq w_{n+1}$ accompanied by

$$D(Tw_{n-1}, Tw_n) = D(w_n, Tw_n) > 0, \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

As T is fractional symmetric α - η -contraction of type-I, a part of $n \in \mathbb{N}$, we have

$$\begin{aligned}
D(w_n, w_{n+1})^p &= D(Tw_{n-1}, Tw_n)^p \leq \lambda \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, Tw_{n-1})^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, Tw_n)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \right. \\
&\quad \left. [D(w_{n-1}, Tw_{n-1}) + D(w_n, Tw_n)]^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [D(w_{n-1}, Tw_n) + D(w_n, Tw_{n-1})]^{(p/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\
&= \lambda \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p/(\beta-\widehat{w})(\beta-\gamma))} \right. \\
&\quad \left. [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [D(w_{n-1}, w_{n+1}) + D(w_n, w_n)]^{(p/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\
&\leq \lambda \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p/(\beta-\widehat{w})(\beta-\gamma))} \right. \\
&\quad \left. [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\
&= \lambda \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p/(\beta-\widehat{w})(\beta-\gamma))} \right. \\
&\quad \left. [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\
&\leq \lambda \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p/(\beta-\widehat{w})(\beta-\gamma))} \right. \\
&\quad \left. [D(w_{n-1}, w_n) \cdot D(w_n, w_{n+1})]^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\
&= \lambda \left[D(w_{n-1}, w_n)^{p + (p/(\beta-\widehat{w})(\beta-\gamma)) + (p/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p/(\gamma-\beta)(\gamma-\widehat{w}))} \right. \\
&\quad \left. D(w_n, w_{n+1})^{(p/(\beta-\widehat{w})(\beta-\gamma)) + (p/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\
&= \lambda D(w_{n-1}, w_n)^p,
\end{aligned} \tag{14}$$

which implies that

$$D(w_n, w_{n+1})^p \leq \lambda D(w_{n-1}, w_n)^p, \quad (15)$$

and we deduce that

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n). \quad (16)$$

We conclude that $\{D(w_{n-1}, w_n)\}$ is a nonincreasing sequence with nonnegative terms. Thus, there is a non-negative constant ϱ such that $\lim_{n \rightarrow \infty} D(w_{n-1}, w_n) = \varrho$. Note that $\varrho \geq 0$. From (16), we have

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n) \leq \lambda^n D(w_0, w_1). \quad (17)$$

This provides

$$\sum_{i=n}^{m-1} D(w_i, w_{i+1}) \leq \frac{\lambda^n}{1-\lambda} D(w_0, w_1), \quad m > n. \quad (18)$$

Considering for as much as

$$\lim_{n \rightarrow +\infty} \frac{\lambda^n}{1-\lambda} D(w_0, w_1) = 0, \quad (19)$$

there subsist some $N \in \mathbb{N}$ corresponding to

$$0 < \frac{\lambda^n}{1-\lambda} D(w_0, w_1) < \delta, \quad n \geq N. \quad (20)$$

Let $\varepsilon > 0$ be fixed and let $(f, \mu) \in \mathcal{F} \times [0, \infty)$ be cognate and (D_3) is satisfied. By (\mathcal{F}_2) , there exists $\delta > 0$ which connotes that

$$0 < t < \delta \text{ implies } f(t) < f(\varepsilon) - \mu. \quad (21)$$

Hence, by (21) and (\mathcal{F}_1) , we get

$$f\left(\sum_{i=n}^{m-1} D(w_i, w_{i+1})\right) \leq f\left(\frac{\lambda^n}{1-\lambda} D(w_0, w_1)\right) < f(\varepsilon) - \mu, \quad (22)$$

where $m, n \in \mathbb{N}$ with the goal that $m > n \geq N$ together with $D(w_n, w_m) > 0$. Therefore, by using (D_3) and (22), we have

$$f(D(w_m, w_n)) \leq f\left(\sum_{i=n}^{m-1} (D(w_i, w_{i+1}))\right) + \mu < f(\varepsilon), \quad (23)$$

which implies that by (\mathcal{F}_1) , we have

$$D(w_m, w_n) < \varepsilon, \quad \text{for } m > n \geq N. \quad (24)$$

Consequently, $\{w_n\}$ is an \mathcal{F} -Cauchy sequence. Meanwhile (A, D) is an \mathcal{F} -complete metric space and there exists $w^* \in A$ such that w_n is \mathcal{F} -convergent to w^* ; that is,

$$\lim_{n \rightarrow \infty} D(w_n, w^*) = 0, \quad (25)$$

and T is α - η -continuous as well as $\alpha(w_{n-1}, w_n) \geq \eta(w_{n-1}, w_n)$, each one of $n \in \mathbb{N}$ at that point $w_{n+1} = Tw_n \rightarrow Tw^*$ as $n \rightarrow \infty$; in other words, $w^* = Tw^*$. Now we are going to prove that w^* is a fixed point of T . We argue by contradiction by supposing that $D(Tw^*, w^*) > 0$. By (D_3) , we have

$$f(D(Tw^*, w^*)^p) \leq f(D(Tw^*, Tw_n)^p + D(Tw_n, w^*)^p) + \mu, \quad n \in \mathbb{N}. \quad (26)$$

By using (\mathcal{F}_1) and the contractive condition, we have

$$\begin{aligned} f(D(Tw^*, w^*)) &\leq f\left(\lambda \left(D(w^*, w_n)^p \cdot D(Tw^*, w^*)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, Tw_n)^{(p/(\beta-\widehat{w})(\beta-\gamma))}\right.\right. \\ &\quad \cdot [D(Tw^*, w^*) + D(w_n, Tw_n)]^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \\ &\quad \left.\left.\cdot [D(Tw_n, w^*) + D(w_n, Tw^*)]^{(p/(\gamma-\beta)(\gamma-\widehat{w}))} + D(w_{n+1}, w^*)^p\right)\right) + \mu, \end{aligned} \quad (27)$$

for all $n \in \mathbb{N}$. In other words, by using (\mathcal{F}_2) and (25), we get

$$\lim_{n \rightarrow \infty} f(\lambda D(w^*, w_n)^p + D(w_{n+1}, w^*)^p) + \mu = -\infty, \quad (28)$$

which gives a contradiction. Therefore $D(Tw^*, w^*) = 0$; hence w^* possesses a fixed point of T . \square

Theorem 3. Let (A, D) be an \mathcal{F} -complete \mathcal{F} -metric space and let T be a fractional symmetric α - η -contraction of type-I fulfilling the accompanying affirmations:

- (i) T is an α -admissible mapping concerning η
- (ii) There exists a $w_0 \in A$ to such extent that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$

- (iii) An iteration $\{w_n\}$ in A is such that $\alpha(w_n, w_{n+1}) \geq \eta(w_n, w_{n+1})$ escorted by $w_n \rightarrow w^*$ at the same time $n \rightarrow \infty$; after that $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ holds for each $n \in \mathbb{N}$

Afterwards, T possesses a fixed point in A .

Proof. On closing lines of the proof of Theorem 2, we acquire $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ for each $n \in \mathbb{N}$. Using (D_3) , we have

$$f(D(Tw^*, w^*)^p) \leq f(D(Tw^*, Tw_n)^p + D(w_n, w^*)^p) + \mu. \quad (29)$$

From (6) connecting (\mathcal{F}_1) , we have

$$\begin{aligned}
f(D(Tw^*, w^*)^p) &\leq f((D(Tw^*, Tw_n)^p) + D(Tw_n, w^*)^p) + \mu \\
&\leq f\left(\lambda \left(D(w^*, w_n)^p \cdot D(Tw^*, w^*)^{(p/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, Tw_n)^{(p/(\beta-\widehat{w})(\beta-\gamma))}\right.\right. \\
&\quad \cdot [D(Tw^*, w^*) + D(w_n, Tw_n)]^{(p/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \\
&\quad \left.\left.\cdot [D(Tw_n, w^*) + D(w_n, Tw^*)]^{(p/(\gamma-\beta)(\gamma-\widehat{w}))} + D(w_{n+1}, w^*)^p\right)\right) + \mu.
\end{aligned} \tag{30}$$

Employ (25) with certitude that

$$\lim_{n \rightarrow \infty} D(w_n, w^*) = 0, \quad \text{together with} \quad \lim_{n \rightarrow \infty} D(w_{n+1}, w^*) = 0, \tag{31}$$

and we obtain

$$f(D(w^*, Tw^*)^p) \leq f(D(w^*, Tw^*)^p) + \mu, \tag{32}$$

and making use of (\mathcal{F}_2) , we have

$$\lim_{n \rightarrow \infty} f(D(w^*, Tw^*)^p) + \mu = -\infty, \tag{33}$$

which is a contradiction. Therefore, $D(w^*, Tw^*) = 0$; in other words, w^* possesses a fixed point of T . \square

Example 3. Let $A = (0, 1] \subset \mathbb{R}$ with an \mathcal{F} -metric $D: A \times A \rightarrow [0, \infty)$ by

$$D(w, v) = \begin{cases} (w - v)^2, & \text{if } (w, v) \in A \times A, \\ |w - v|, & \text{if } (w, v) \notin A \times A, \end{cases} \tag{34}$$

accompanied by $f(t) = \ln(t)$ together with $\mu = \ln(1)$, $p = 1$. Define $T: A \rightarrow A$ along with

$$Tw = \begin{cases} \frac{w^2}{e^6}, & \text{if } w \in A, \\ 0, & \text{if } w \notin A, \end{cases} \tag{35}$$

and $\alpha, \eta: A \times A \rightarrow [0, +\infty)$ by

$$\begin{aligned}
\alpha(w, v) &= \begin{cases} 2, & \text{if } w, v \in [0, \infty), \\ 0, & \text{otherwise,} \end{cases} \\
\eta(w, v) &= \begin{cases} 1, & \text{if } w, v \in [0, \infty), \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{36}$$

(i) Case I. If $w = v$, clearly $D(w, v) = 0$. Hence, every condition of Theorem 2 is satisfied.

(ii) Case II. If $w, v \in A$, clearly T is an α -admissible mapping with respect to η , whenever $\alpha(w, v) \geq \eta(w, v)$, such that

$$\begin{aligned}
D(Tw, Tv) &= \frac{1}{e^6} |w^2 - v^2| \leq \lambda \left[(v - w)^2 \cdot \left(w - \frac{w}{e}\right)^{(1/(\beta-\widehat{w})(\beta-\gamma))} \cdot \left(v - \frac{v}{e}\right)^{(1/(\beta-\widehat{w})(\beta-\gamma))} \right. \\
&\quad \left. \cdot \left(\left(w - \frac{w}{e}\right) + \left(v - \frac{v}{e}\right)\right)^{(1/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot \left(\left|w - \frac{v}{e}\right| + \left|v - \frac{w}{e}\right|\right)^{(1/(\gamma-\beta)(\gamma-\widehat{w}))} \right],
\end{aligned} \tag{37}$$

(iii) By taking constant $\lambda \in [0, 1)$, and $\beta, \widehat{w}, \gamma \in (0, 1)$, for all $w, v \in A \setminus \text{Fix}(T)$.

(iv) Case III. If any $w, v \notin A$, then we have

$$\begin{aligned}
D(Tw, Tv) &= 0 \leq \lambda \left[|v - w| \cdot |w|^{(1/(\beta-\widehat{w})(\beta-\gamma))} \cdot |v|^{(1/(\beta-\widehat{w})(\beta-\gamma))} \right. \\
&\quad \left. \cdot (|w| + |v|)^{(1/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot (|w| + |v|)^{(1/(\gamma-\beta)(\gamma-\widehat{w}))} \right].
\end{aligned} \tag{38}$$

Therefore, whole constraints of Theorem 2 are satisfied. Hence, T is fractional symmetric α - η -contraction of type-I.

Definition 7. Consider an \mathcal{F} -metric space (A, D) and two functions $\alpha, \eta: A \times A \rightarrow [0, +\infty)$. Then an \mathcal{F} -metric space

on A is said to be α - η -complete if and only if every \mathcal{F} -Cauchy sequence $\{w_n\}$, along with

$$\alpha(w_n, w_{n+1}) \geq \eta(w_n, w_{n+1}) \text{ each one of the } n \in \mathbb{N}. \tag{39}$$

\mathcal{F} -converges in A .

Remark 1. Theorems 2 and 3 also hold for α - η -complete \mathcal{F} -metric space instead of \mathcal{F} -complete \mathcal{F} -metric space (for details, see [10]).

3. Fractional Symmetric α - η -Contraction of Type-II

In this section, a fractional symmetric α - η -contraction of type-II is introduced and in the structure of \mathcal{F} -complete \mathcal{F} -metric space. Using this notion, we shall provide a fixed-point theorem.

Definition 8. Consider a self-map $T: A \longrightarrow A$ on an \mathcal{F} -metric space (A, D) and two functions $\alpha, \eta: A \times A \longrightarrow [0, +\infty)$. We presume that T is a fractional symmetric α - η -contraction of type-II provided that there are constants $\lambda \in [0, 1)$ and $\beta, \hat{w}, \gamma \in (0, 1)$ such that, whenever $\alpha(w, v) \geq \eta(w, v)$, we own

$$D(Tw, Tv)^p \leq \lambda (\check{S}_2(w, v)), \quad (40)$$

where

$$\check{S}_2(w, v) = \left\{ D(w, v) \cdot [D(w, Tw)]^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(v, Tv)]^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma))} \cdot [D(w, Tv) + D(v, Tw)]^{(p\gamma/(\gamma-\beta)(\gamma-\hat{w}))} \right\}, \quad (41)$$

where $p \in [1, \infty)$, for all $w, v \in A \setminus \text{Fix}(T)$.

Now we show and demonstrate our next theorem.

Theorem 4. Let (A, D) be an \mathcal{F} -complete \mathcal{F} -metric space and let T be a fractional symmetric α - η -contraction of type-II fulfilling the accompanying affirmations:

- (i) T is an α -admissible mapping concerning η
- (ii) There exists $w_0 \in A$ to such an extent that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$
- (iii) T is α - η -continuous

After that, T possesses a fixed point in A .

Proof. Consider w_0 in A correspondent to $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$. For $w_0 \in A$, we build an iteration $\{w_n\}_{n=1}^{\infty}$ in such a way that $w_1 = Tw_0$ and $w_2 = Tw_1 = T^2w_0$.

Proceeding with this exercise, $w_{n+1} = Tw_n = T^{n+1}w_0$, for all $n \in \mathbb{N}$. Now, as long as mapping T is α -admissible with respect to η , at that time $\alpha(w_0, w_1) = \alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0) = \eta(w_0, w_1)$. Carrying on in this way, we own

$$\alpha(w_{n-1}, w_n) \geq \eta(w_{n-1}, w_n) = \eta(w_{n-1}, Tw_{n-1}), \quad \text{for all } n \in \mathbb{N}. \quad (42)$$

Provided that $w_{n+1} = w_n$ for some $n \in \mathbb{N}$, then $w_n = w^*$ is a fixed point of T . So, we assume that $w_n \neq w_{n+1}$ accompanied by

$$D(Tw_{n-1}, Tw_n) = D(w_n, Tw_n) > 0, \quad \text{every } n \in \mathbb{N}. \quad (43)$$

As T is fractional symmetric α - η -contraction of type-II, a part of $n \in \mathbb{N}$, we own

$$\begin{aligned} D(w_n, w_{n+1})^p &= D(Tw_{n-1}, Tw_n)^p \leq \lambda \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, Tw_{n-1})^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot D(w_n, Tw_n)^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \right. \\ &\quad \left. [D(w_{n-1}, Tw_{n-1}) + D(w_n, Tw_n)]^{(p\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma))} \cdot [D(w_{n-1}, Tw_n) + D(w_n, Tw_{n-1})]^{(p\gamma/(\gamma-\beta)(\gamma-\hat{w}))} \right] \\ &= \lambda \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \right. \\ &\quad \left. [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma))} \cdot [D(w_{n-1}, w_{n+1}) + D(w_n, w_n)]^{(p\gamma/(\gamma-\beta)(\gamma-\hat{w}))} \right] \\ &\leq \lambda \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \right. \\ &\quad \left. [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma))} \cdot [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\gamma/(\gamma-\beta)(\gamma-\hat{w}))} \right] \\ &= \lambda \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \right. \\ &\quad \left. [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma)) + (p\gamma/(\gamma-\beta)(\gamma-\hat{w}))} \right] \\ &\leq \lambda \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \right. \\ &\quad \left. [D(w_{n-1}, w_n) \cdot D(w_n, w_{n+1})]^{(p\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma)) + (p\gamma/(\gamma-\beta)(\gamma-\hat{w}))} \right] \\ &= \lambda \left[D(w_{n-1}, w_n)^{p + (p\beta/(\beta-\hat{w})(\beta-\gamma)) + (p\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma)) + (p\gamma/(\gamma-\beta)(\gamma-\hat{w}))} \right. \\ &\quad \left. D(w_n, w_{n+1})^{(p\beta/(\beta-\hat{w})(\beta-\gamma)) + (p\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma)) + (p\gamma/(\gamma-\beta)(\gamma-\hat{w}))} \right] \\ &= \lambda D(w_{n-1}, w_n)^p, \end{aligned} \quad (44)$$

which implies that

$$D(w_n, w_{n+1})^p \leq \lambda D(w_{n-1}, w_n)^p, \quad (45)$$

and we deduce that

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n). \quad (46)$$

We conclude that $\{D(w_{n-1}, w_n)\}$ is a nonincreasing sequence with nonnegative terms. As a result, there is a nonnegative constant ρ such that $\lim_{n \rightarrow \infty} D(w_{n-1}, w_n) = \rho$. We shall indicate that $\rho > 0$. Indeed, from (46), we derive that

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w) \leq \lambda^n D(w_0, w_1). \quad (47)$$

The rest of the test follows the same lines of Theorem 2. \square

Theorem 5. Consider an \mathcal{F} -complete \mathcal{F} -metric space (A, D) and let T be a fractional symmetric α - η -contraction of type-II meeting the following assertions:

- (i) T is an α -admissible mapping with respect to η
- (ii) There exists $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$
- (iii) An iteration $\{w_n\}$ in A is such that $\alpha(w_n, w_{n+1}) \geq \eta(w_n, w_{n+1})$ escorted by $w_n \rightarrow w^*$ at the same time $n \rightarrow \infty$; after that $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ holds for each $n \in \mathbb{N}$

Afterwards, T possesses a fixed point in A .

Proof. Similar to the lines of Theorem 3, since, by (iii), $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ holds for every $n \in \mathbb{N}$. Using (D_3) , we meet

$$f(D(Tw^*, w^*)) \leq f(D(Tw^*, Tw_n) + D(w_n, w^*)) + \mu. \quad (48)$$

From (40) and (\mathcal{F}_1) , we have

$$\begin{aligned} f(D(Tw^*, w^*)) &\leq f((D(Tw^*, Tw_n)^p + D(Tw_n, w^*)^p) + \mu \\ &\leq f\left(\lambda \left(D(w^*, w_n)^p \cdot D(Tw^*, w^*)^{(p\beta/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, Tw_n)^{(p\beta/(\beta-\widehat{w})(\beta-\gamma))} \right. \right. \\ &\quad \cdot [D(Tw^*, w^*) + D(w_n, Tw_n)]^{(p\widehat{w}(\widehat{w}-\beta)(\widehat{w}-\gamma))} \\ &\quad \left. \left. \cdot [D(Tw_n, w^*) + D(w_n, Tw^*)]^{(p\gamma/(\gamma-\beta)(\gamma-\widehat{w}))} + D(w_{n+1}, w^*)^p\right) + \mu. \right. \end{aligned} \quad (49)$$

Making use of (25), we get

$$\lim_{n \rightarrow \infty} D(w_n, w^*) = 0 \text{ together } \lim_{n \rightarrow \infty} D(w_{n+1}, w^*) = 0, \quad (50)$$

and we procure

$$f(D(w^*, Tw^*)) \leq f(D(w^*, Tw^*)) + \mu. \quad (51)$$

Using (\mathcal{F}_2) , we have

$$\lim_{n \rightarrow \infty} f(D(w^*, Tw^*)) + \mu = -\infty, \quad (52)$$

which is a logical inconsistency. Along these lines $D(w^*, Tw^*) = 0$; that is, w^* possesses a fixed point of T . \square

4. Fractional Symmetric α - η -Contraction of Type-III

In this section, fractional symmetric α - η -contraction of type-III is considered in the environment of \mathcal{F} -complete \mathcal{F} -metric space. After stating a fixed-point theorem for such maps, we set up fractional symmetric α - η -contraction of type-III as follows.

Definition 9. Consider an \mathcal{F} -metric space (A, D) with a self-map $T: A \rightarrow A$ and two functions $\alpha, \eta: A \times A \rightarrow [0, +\infty)$. We say that T is fractional symmetric α - η -contraction of type-III along with constants $\lambda \in [0, 1)$ and $\beta, \widehat{w}, \gamma \in (0, 1)$ such that, whenever $\alpha(w, v) \geq \eta(w, v)$, we have

$$D(Tw, Tv)^p \leq \lambda (\check{S}_3(w, v)), \quad (53)$$

where

$$\begin{aligned} \check{S}_3(w, v) &= \lambda \max \left\{ D(w, v), [D(w, Tw)]^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \cdot [D(v, Tv)]^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p\widehat{w}^2/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \\ &\quad \left. \cdot [D(w, Tv) + D(v, Tw)]^{(p\gamma^2/(\gamma-\beta)(\gamma-\widehat{w}))} \right\}, \end{aligned} \quad (54)$$

where $p \in [1, \infty)$, for all $w, v \in A \setminus \text{Fix}(T)$.

Now we declare and demonstrate our next theorem.

Theorem 6. *Let an \mathcal{F} -complete (A, D) be an \mathcal{F} -metric space along with T being a fractional symmetric α - η -contraction of type-III which meets the following assertions:*

- (i) T is an α -admissible mapping concerning η
- (ii) There exists $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$
- (iii) T is α - η -continuous

After that, T possesses a fixed point in A .

Proof. Consider w_0 in A with the aim that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$. Take any $w_0 \in A$; we erect an recapitulate $\{w_n\}_{n=1}^{\infty}$ in such a way that $w_1 = Tw_0$ and

$w_2 = Tw_1 = T^2w_0$. Continuing with this practice, $w_{n+1} = Tw_n = T^{n+1}w_0$, every $n \in \mathbb{N}$. As long as mapping T is α -admissible with respect to η , at that time $\alpha(w_0, w_1) = \alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0) = \eta(w_0, w_1)$. Carrying on in this way, we find

$$\alpha(w_{n-1}, w_n) \geq \eta(w_{n-1}, w_n) = \eta(w_{n-1}, Tw_{n-1}), \quad \text{for all } n \in \mathbb{N}. \quad (55)$$

Provided that $w_{n+1} = w_n$ for some $n \in \mathbb{N}$, then $w_n = w^*$ is a fixed point of T . So, we assume that $w_n \neq w_{n+1}$ accompanied by

$$D(Tw_{n-1}, Tw_n) = D(w_n, Tw_n) > 0, \quad \text{each } n \in \mathbb{N}. \quad (56)$$

As T is fractional symmetric α - η -contraction of type-III, a part of $n \in \mathbb{N}$, we own

$$\begin{aligned} D(w_n, w_{n+1})^p &= D(Tw_{n-1}, Tw_n)^p \leq \lambda \max \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, Tw_{n-1})^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, Tw_n)^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \left. [D(w_{n-1}, Tw_{n-1}) + D(w_n, Tw_n)]^{(p\widehat{w}^2/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [D(w_{n-1}, Tw_n) + D(w_n, Tw_{n-1})]^{(p\gamma^2/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\ &= \lambda \max \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \left. [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\widehat{w}^2/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [D(w_{n-1}, w_{n+1}) + D(w_n, w_n)]^{(p\gamma^2/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\ &\leq \lambda \max \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \left. [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\widehat{w}^2/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\gamma^2/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\ &= \lambda \max \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \left. [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\widehat{w}^2/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p\gamma^2/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\ &\leq \lambda \max \left[D(w_{n-1}, w_n)^p \cdot D(w_{n-1}, w_n)^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \left. [D(w_{n-1}, w_n) \cdot D(w_n, w_{n+1})]^{(p\widehat{w}^2/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p\gamma^2/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\ &= \lambda \max \left[D(w_{n-1}, w_n)^p, D(w_{n-1}, w_n)^{p + (p\beta^2/(\beta-\widehat{w})(\beta-\gamma)) + (p\widehat{w}^2/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p\gamma^2/(\gamma-\beta)(\gamma-\widehat{w}))} \right. \\ &\quad \left. D(w_n, w_{n+1})^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma)) + (p\widehat{w}^2/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p\gamma^2/(\gamma-\beta)(\gamma-\widehat{w}))} \right] \\ &= \lambda \max \{ D(w_{n-1}, w_n)^p, D(w_n, w_{n+1})^p \}. \end{aligned} \quad (57)$$

Provided that $\max\{D(w_n, w_{n+1}), D(w_{n-1}, w_n)\} = D(w_n, w_{n+1})$, at that time,

$$D(w_n, w_{n+1}) \leq \lambda D(w_n, w_{n+1}), \quad (58)$$

which is a contradiction. We deduce that

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n). \quad (59)$$

We conclude that $\{D(w_{n-1}, w_n)\}$ is a nonincreasing sequence with nonnegative terms. As a result, there is a nonnegative constant ρ such that $\lim_{n \rightarrow \infty} D(w_{n-1}, w_n) = \rho$. We shall indicate that $\rho > 0$. Indeed, from (59), we derive that

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n) \leq \lambda^n D(w_0, w_1). \quad (60)$$

Pause the proof and go behind the closing lines of Theorem 2. \square

Theorem 7. Consider an \mathcal{F} -complete \mathcal{F} -metric space (A, D) and let T be a fractional symmetric α - η -contraction of type-III meeting the following assertions:

- (i) T is an α -admissible mapping with respect to η
- (ii) There exists $w_0 \in A$ such that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$
- (iii) An iteration $\{w_n\}$ in A is such that $\alpha(w_n, w_{n+1}) \geq \eta(w_n, w_{n+1})$ escorted by $w_n \rightarrow w^*$ at the same

time $n \rightarrow \infty$; after that $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ holds for each $n \in \mathbb{N}$

Afterwards, T possesses a fixed point in A .

Proof. Similar to the same lines of Theorem 3, considering (iii), $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ for all $n \in \mathbb{N}$. By (D_3) , we have

$$f(D(Tw^*, w^*)) \leq f(D(Tw^*, Tw_n) + D(w_n, w^*)) + \mu. \quad (61)$$

Using (40) along with (\mathcal{F}_1) , we have

$$\begin{aligned} f(D(Tw^*, w^*)) &\leq f((D(Tw^*, Tw_n)^p + D(Tw_n, w^*)^p) + \mu \\ &\leq f\left(\lambda \left(D(w^*, w_n)^p, D(Tw^*, w^*)^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, Tw_n)^{(p\beta^2/(\beta-\widehat{w})(\beta-\gamma))} \right. \right. \\ &\quad \cdot [D(Tw^*, w^*) + D(w_n, Tw_n)]^{(p\widehat{w}^2/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \\ &\quad \left. \left. \cdot [D(Tw_n, w^*) + D(w_n, Tw^*)]^{(p\gamma^2/(\gamma-\beta)(\gamma-\widehat{w}))} + D(w_{n+1}, w^*)^p \right) \right) + \mu. \end{aligned} \quad (62)$$

Using (25) the factuality is

$$\lim_{n \rightarrow \infty} D(w_n, w^*) = 0 \text{ as long as } \lim_{n \rightarrow \infty} D(w_{n+1}, w^*) = 0, \quad (63)$$

and we obtain

$$f(D(w^*, Tw^*)^p) \leq f(D(w^*, Tw^*)^p) + \mu. \quad (64)$$

Utilizing (\mathcal{F}_2) , we have

$$\lim_{n \rightarrow \infty} f(D(w^*, Tw^*)) + \mu = -\infty, \quad (65)$$

which is a logical inconsistency. Along these lines, $D(w^*, Tw^*) = 0$; that is, w^* possesses a fixed point of T . \square

5. Fractional Symmetric α - η -Contraction of Type-IV

In this part, we propose a new notion, fractional symmetric α - η -contraction of type-IV, in the framework of \mathcal{F} -complete \mathcal{F} -metric space.

(i) T is an α -admissible mapping concerning η

(ii) There exists $w_0 \in A$ which connotes that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$

(iii) T is α - η -continuous

Definition 10. Consider an \mathcal{F} -metric space (A, D) with a self-map $T: A \rightarrow A$ and two functions $\alpha, \eta: A \times A \rightarrow [0, +\infty)$. We say that T is a fractional symmetric α - η -contraction type-IV along with constants $\lambda \in [0, 1)$ and $\beta, \widehat{w}, \gamma \in (0, 1)$ with $\beta + \widehat{w} + \gamma < 1$ such that, whenever $\alpha(w, v) \geq \eta(w, v)$, we have

$$D(Tw, Tv)^p \leq \lambda (\check{S}_4(w, v)), \quad (66)$$

where

$$\begin{aligned} \check{S}_4(w, v) &= \lambda \left\{ D(w, v)^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w, Tw)^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p\widehat{w}^3/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \right. \\ &\quad \left. \cdot [D(w, Tv) + D(v, Tw)]^{(p\gamma^3/(\gamma-\beta)(\gamma-\widehat{w}))} \right\}, \end{aligned} \quad (67)$$

where $p \in [1, \infty)$, for all $w, v \in A \setminus \text{Fix}(T)$.

Now we declare and demonstrate our next theorem.

Theorem 8. Let an \mathcal{F} -complete (A, D) be an \mathcal{F} -metric space along with T being a fractional symmetric α - η -contraction of type-IV that meets the following assertions:

Proof. Consider w_0 in A with the aim that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$. Take any $w_0 \in A$; we build a chain $\{w_n\}_{n=1}^{\infty}$ in such a way that $w_1 = Tw_0$ and $w_2 = Tw_1 = T^2w_0$.

After that, T possesses a fixed point in A .

Proceeding with this exercise, $w_{n+1} = Tw_n = T^{n+1}w_0$, for every $n \in \mathbb{N}$. As long as mapping T is α -admissible with respect to η , at that time $\alpha(w_0, w_1) = \alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0) = \eta(w_0, w_1)$. Carrying on in this way, we get

$$\alpha(w_{n-1}, w_n) \geq \eta(w_{n-1}, w_n) = \eta(w_{n-1}, Tw_{n-1}), \quad \text{each } n \in \mathbb{N}. \quad (68)$$

Provided that $w_{n+1} = w_n$ for some $n \in \mathbb{N}$, then $w_n = w^*$ is a fixed point of T . So, we assume that $w_n \neq w_{n+1}$ accompanied by

$$D(Tw_{n-1}, Tw_n) = D(w_n, Tw_n) > 0, \quad \text{for all } n \in \mathbb{N}. \quad (69)$$

As T is fractional symmetric α - η -contraction of type-IV, a part of $n \in \mathbb{N}$, we have

$$\begin{aligned} D(w_n, w_{n+1})^p &= D(Tw_{n-1}, Tw_n)^p \leq \lambda \left[D(w_{n-1}, w_n)^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \cdot D(w_{n-1}, Tw_{n-1})^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \cdot [D(w_{n-1}, Tw_{n-1}) + D(w_n, Tw_n)]^{(p\widehat{w}^3/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \\ &\quad \cdot [D(w_{n-1}, Tw_n) + D(w_n, Tw_{n-1})]^{(p\gamma^3/(\gamma-\beta)(\gamma-\widehat{w}))} \Big] \\ &= \lambda \left[D(w_{n-1}, w_n)^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \cdot [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\widehat{w}^3/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [D(w_{n-1}, w_{n+1}) + D(w_n, w_n)]^{(p\gamma^3/(\gamma-\beta)(\gamma-\widehat{w}))} \Big] \\ &\leq \lambda \left[D(w_{n-1}, w_n)^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \cdot [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\widehat{w}^3/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\gamma^3/(\gamma-\beta)(\gamma-\widehat{w}))} \Big] \\ &= \lambda \left[D(w_{n-1}, w_n)^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \cdot [D(w_{n-1}, w_n) + D(w_n, w_{n+1})]^{(p\gamma^3/(\gamma-\beta)(\gamma-\widehat{w}))} \Big] \\ &\leq \lambda \left[D(w_{n-1}, w_n)^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \cdot D(w_n, w_{n+1})^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ &\quad \cdot [D(w_{n-1}, w_n) \cdot D(w_n, w_{n+1})]^{(p\widehat{w}^3/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p\gamma^3/(\gamma-\beta)(\gamma-\widehat{w}))} \Big] \\ &= \lambda \left[D(w_{n-1}, w_n)^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma)) + (p\widehat{w}^3/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p\gamma^3/(\gamma-\beta)(\gamma-\widehat{w}))} \right. \\ &\quad \cdot D(w_n, w_{n+1})^{(p\beta^3/(\beta-\widehat{w})(\beta-\gamma)) + (p\widehat{w}^3/(\widehat{w}-\beta)(\widehat{w}-\gamma)) + (p\gamma^3/(\gamma-\beta)(\gamma-\widehat{w}))} \Big] \\ &= \lambda \{ D(w_{n-1}, w_n) \cdot D(w_n, w_{n+1}) \}^{p(\beta+\widehat{w}+\gamma)} \\ &\leq \lambda \max \{ D(w_{n-1}, w_n)^p, D(w_n, w_{n+1})^p \}. \end{aligned} \quad (70)$$

On condition that $\max\{D(w_n, w_{n+1}), D(w_{n-1}, w_n)\} = D(w_n, w_{n+1})$, at that time,

$$D(w_n, w_{n+1}) \leq \lambda D(w_n, w_{n+1}), \quad (71)$$

which is a contradiction. We deduce that

$$D(w_n, w_{n+1}) \leq \lambda D(w_{n-1}, w_n). \quad (72)$$

Let up the closing lines of Theorem 2. \square

Theorem 9. Consider an \mathcal{F} -complete \mathcal{F} -metric space (A, D) and suppose that T is a fractional symmetric α - η -contraction of type-IV fulfilling the accompanying affirmations:

(i) T is an α -admissible mapping concerning η

(ii) There exists $w_0 \in A$ to such an extent that $\alpha(w_0, Tw_0) \geq \eta(w_0, Tw_0)$

(iii) An iteration $\{w_n\}$ in A is analogous to $\alpha(w_n, w_{n+1}) \geq \eta(w_n, w_{n+1})$ escorted by $w_n \longrightarrow w^*$ at the same time $n \longrightarrow \infty$; after that $\alpha(w_n, w^*) \geq \eta(w_n, w^*)$ holds for each $n \in \mathbb{N}$

Afterwards, T possesses a fixed point in A .

Whether $\eta(w, v) = 1$, in Theorems 2, 3, 4, and 5, we introduce the following corollaries.

Corollary 1. Consider an \mathcal{F} -complete \mathcal{F} -metric space (A, D) and suppose that T is a fractional symmetric α - η -contraction of type-I fulfilling the accompanying affirmations:

- (i) T is an α -admissible mapping
- (ii) There subsist $w_0 \in A$ parallel to $\alpha(w_0, Tw_0) \geq 1$
- (iii) T is α - η -continuous

Afterwards, T possesses a fixed point in A .

Corollary 2. Consider an \mathcal{F} -complete \mathcal{F} -metric space (A, D) and let T be fractional symmetric α - η -contraction of type-I fulfilling the accompanying affirmations:

- (i) T is an α -admissible mapping
- (ii) There subsist $w_0 \in A$ parallel to $\alpha(w_0, Tw_0) \geq 1$
- (iii) An iteration $\{w_n\}$ in A is analogous to $\alpha(w_n, w_{n+1}) \geq 1$ escorted by $w_n \longrightarrow w^*$ at the same time $n \longrightarrow \infty$; after that $\alpha(w_n, w^*) \geq 1$ holds for each $n \in \mathbb{N}$

Afterwards, T possesses a fixed point in A .

Corollary 3. Let an \mathcal{F} -complete (A, D) be \mathcal{F} -metric space and let T be a fractional symmetric α - η -contraction of type-II meeting the accompanying affirmations:

- (i) T is an α -admissible mapping
- (ii) There subsist $w_0 \in A$ parallel to $\alpha(w_0, Tw_0) \geq 1$
- (iii) T is α - η -continuous

Then T gets a fixed point in A .

Corollary 4. Let an \mathcal{F} -complete (A, D) be \mathcal{F} -metric space and let T be a fractional symmetric α - η -contraction of type-II meeting the accompanying affirmations:

- (i) T is an α -admissible mapping
- (ii) There subsist $w_0 \in A$ parallel to $\alpha(w_0, Tw_0) \geq 1$
- (iii) An iteration $\{w_n\}$ in A is analogous to $\alpha(w_n, w_{n+1}) \geq 1$ escorted by $w_n \longrightarrow w^*$ at the same time $n \longrightarrow \infty$; after that $\alpha(w_n, w^*) \geq 1$ holds for each $n \in \mathbb{N}$

Afterwards, T possesses a fixed point in A .

In similar fashion, we can deduce Corollaries 1, 2, 3, and 4 for fractional symmetric α - η -contraction of type-III and that of type-IV, respectively.

6. Consequences

As a consequence of our results, we derive some effect for Suzuki-type fractional symmetric contractions and orbitally T -complete and orbitally continuous mappings in \mathcal{F} -metric spaces.

Theorem 10. Consider an \mathcal{F} -metric space (A, D) and let T be a continuous self-mapping on A . Assume that there exists $r \in [0, 1)$ in addition to $\beta, \hat{w}, \gamma \in (0, 1)$ such that

$$D(w, Tw) \leq D(w, v) \text{ implies } D(Tw, Tv)^p \leq r(\check{S}_1(w, v)), \quad (73)$$

where

$$\begin{aligned} \check{S}_1(w, v) = & D(w, v)^p \cdot D(w, Tw)^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot D(v, Tv)^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p/(\hat{w}-\beta)(\hat{w}-\gamma))} \\ & \cdot [D(w, Tv) + D(v, Tw)]^{(p/(\gamma-\beta)(\gamma-\hat{w}))}, \end{aligned} \quad (74)$$

where $p \in [1, \infty)$, for all $w, v \in A \setminus \text{Fix}(T)$.

At that time, T possesses a fixed point in A .

Proof. Describe $\alpha, \eta: A \times A \longrightarrow [0, +\infty)$ by

$$\alpha(w, v) = D(w, v), \eta(w, v) = D(w, Tw), \quad \text{for all } w, v \in A, \quad (75)$$

and $\beta, \hat{w}, \gamma \in (0, 1)$, and $r \in [0, 1)$. It is clear that

$$\eta(w, v) \leq \alpha(w, v), \quad \text{for all } w, v \in A, \quad (76)$$

which means that conditions (i)-(iii) of our Theorem 2 hold true. Let

$$\eta(w, Tw) \leq \alpha(w, v) \text{ then } D(w, Tw) \leq D(w, v), \quad (77)$$

which implies contractive condition:

$$D(Tw, Tv)^p \leq r(\check{S}_1(w, v)). \quad (78)$$

Finally, every constraint of Theorem 2 holds true. Hence, T possesses a fixed point in A . \square

Theorem 11. Consider an \mathcal{F} -metric space (A, D) and let T be a self-mapping of A . Suppose that the following assertions hold:

- (i) (A, D) is an orbitally T -complete \mathcal{F} -metric space.
- (ii) There exists $r \in [0, 1)$ in addition to $\beta, \hat{w}, \gamma \in (0, 1)$ such that

$$D(Tw, Tv)^p \leq r(\check{S}_1(w, v)), \quad (79)$$

(i) where

$$\begin{aligned} \check{S}_1(w, v) = & D(w, v)^p \cdot D(w, Tw)^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot D(v, Tv)^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p/(\hat{w}-\beta)(\hat{w}-\gamma))} \\ & \cdot [D(w, Tv) + D(v, Tw)]^{(p/(\gamma-\beta)(\gamma-\hat{w}))}, \end{aligned} \quad (80)$$

- (ii) $p \in [1, \infty)$, for all $w, v \in O(\omega)$ for some $\omega \in A$, where $O(\omega)$ is an orbit of ω ,
 (iii) if $\{v_n\}$ is a sequence such that $\{v_n\} \subseteq O(\omega)$ with $v_n \rightarrow v^*$ as $n \rightarrow \infty$, then $v^* \in O(\omega)$.

Then, T possesses a fixed point.

Proof. Describe $\alpha, \eta: A \times A \rightarrow [0, +\infty)$, by $\alpha(w, v) = 3$ on $O(\omega) \times O(\omega)$ and $\alpha(w, v) = 0$; otherwise, $\eta(w, v) = 1$ for all

$w, v \in A$ (see Remark 6 [11]). Then (A, D) is an α - η -complete \mathcal{F} -metric and T is an α -admissible mapping with respect to η . Let $\alpha(w, v) \geq \eta(w, v)$; then $w, v \in O(\omega)$, and then, from (ii), we have

$$D(Tw, Tv)^p \leq r(\check{S}_1(w, v)), \quad (81)$$

where

$$\begin{aligned} \check{S}_1(w, v) = & D(w, v)^p \cdot D(w, Tw)^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot D(v, Tv)^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p/(\hat{w}-\beta)(\hat{w}-\gamma))} \\ & [D(w, Tv) + D(v, Tw)]^{(p/(\gamma-\beta)(\gamma-\hat{w}))}. \end{aligned} \quad (82)$$

That is, T is a fractional symmetric α - η -contraction of type-I. Let $\{v_n\}$ be a sequence commensurate with $\alpha(v_n, v_{n+1}) \geq \eta(v_n, v_{n+1})$ together with $v_n \rightarrow v^*$ for $n \rightarrow \infty$. So, $\{v_n\} \subseteq O(\omega)$. From (iii), $v^* \in O(\omega)$; that is, $\alpha(v_n, v^*) \geq \eta(v_n, v^*)$. Hence, every norm of Theorem 3 holds true. Thus, T possesses a fixed point. \square

Theorem 12. Consider an \mathcal{F} -metric space (A, D) and let T be a self-mapping of A . Suppose that the following assertions hold:

- (i) For all $w, v \in O(\omega)$, there exists $r \in [0, 1)$ along with $\beta, \hat{w}, \gamma \in (0, 1)$, such that

$$D(Tw, Tv)^p \leq r(\check{S}_1(w, v)), \quad (83)$$

- (ii) where

$$\begin{aligned} \check{S}_1(w, v) = & D(w, v)^p \cdot [D(w, Tw)]^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(v, Tv)]^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p/(\hat{w}-\beta)(\hat{w}-\gamma))} \\ & \cdot [D(w, Tv) + D(v, Tw)]^{(p/(\gamma-\beta)(\gamma-\hat{w}))}, \end{aligned} \quad (84)$$

- (iii) $p \in [1, \infty)$, for some $\omega \in A$,
 (iv) the operator T is orbitally continuous.

Afterwards, T possesses a fixed point.

Proof. Describe $\alpha, \eta: A \times A \rightarrow [0, +\infty)$, by $\alpha(w, v) = 3$ on $O(\omega) \times O(\omega)$ and $\alpha(w, v) = 0$; otherwise, $\eta(w, v) = 1$ (see Remark 1.1 [12]), and we know that T is an α - η -continuous

mapping. Let $\alpha(w, v) \geq \eta(w, v)$; then $w, v \in O(\omega)$. So $Tw, Tv \in O(\omega)$; that is, $\alpha(Tw, Tv) \geq \eta(Tw, Tv)$. Therefore, T is an α -admissible mapping with respect to η . From (i), we have

$$D(Tw, Tv)^p \leq r(\check{S}_1(w, v)), \quad (85)$$

where

$$\begin{aligned} \check{S}_1(w, v) = & D(w, v)^p \cdot D(w, Tw)^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot D(v, Tv)^{(p/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p/(\hat{w}-\beta)(\hat{w}-\gamma))} \\ & [D(w, Tv) + D(v, Tw)]^{(p/(\gamma-\beta)(\gamma-\hat{w}))}, \end{aligned} \quad (86)$$

and aforesaid T is a fractional symmetric α - η -contraction of type-I. Hence, each constraint of Theorem 2 holds true. Thus, T gets a fixed point. \square

Theorem 13. Consider an \mathcal{F} -metric space (A, D) and let T be a self-mapping of A . Suppose that the following assertions hold:

- (i) (A, D) is an orbitally T -complete \mathcal{F} -metric space;
 (ii) there subsist $r \in [0, 1)$ parallel to $\beta, \hat{w}, \gamma \in (0, 1)$ such that

$$D(Tw, Tv)^p \leq r(\check{S}_2(w, v)), \quad (87)$$

(iii) where

$$\check{S}_2(w, v) = D(w, v)^p \cdot D(w, Tw)^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot D(v, Tv)^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma))} \cdot [D(w, Tv) + D(v, Tw)]^{(p\gamma/(\gamma-\beta)(\gamma-\hat{w}))}, \quad (88)$$

- (iv) $p \in [1, \infty)$, for all $w, v \in O(\omega)$ for some $\omega \in A$, where $O(\omega)$ is an orbit of ω ;
 (v) if $\{v_n\}$ is a sequence such that $\{v_n\} \subseteq O(\omega)$ with $v_n \rightarrow v^*$ as $n \rightarrow \infty$, then $v^* \in O(\omega)$.

Then, T possesses a fixed point.

- (i) for all $w, v \in O(\omega)$, there subsist $r \in [0, 1)$ along with $\beta, \hat{w}, \gamma \in (0, 1)$, such that

$$D(Tw, Tv)^p \leq r(\check{S}_2(w, v)), \quad (89)$$

(ii) where

Theorem 14. Consider an \mathcal{F} -metric space (A, D) and let T be a self-mapping of A . Suppose that the following assertions hold:

$$\check{S}_2(w, v) = D(w, v)^p \cdot D(w, Tw)^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot D(v, Tv)^{(p\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot [D(w, Tw) + D(v, Tv)]^{(p\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma))} \cdot [D(w, Tv) + D(v, Tw)]^{(p\gamma/(\gamma-\beta)(\gamma-\hat{w}))}, \quad (90)$$

- (iii) $p \in [1, \infty)$, for some $\omega \in A$;
 (iv) the operator T is orbitally continuous.

Afterwards, T possesses a fixed point.

Theorems 10, 11, and 12 can be derived easily for fractional symmetric contraction of type-III and that of type-IV, respectively.

7. Application to Fractional-Order Differential Equations

The local and nonlocal fractional differential equations have been recently proved to be significant tools in the modeling of many phenomena in numerous fields of science and building. The fractional-order differential equations have numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetism, and so forth. For more details, see [2, 13–19]. Our aim is to give the existence and uniqueness of bounded solution of local fractional-order differential equation given in (93). Consider a function $f: (0, \infty) \rightarrow \mathbb{R}$. The conformable derivative of order α of f at $t > 0$ is defined by [20]

$$D^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}. \quad (91)$$

The conformable fractional integral associated with (91) is defined by [20, 21]

$$I_0^\alpha f(t) = \int_0^t s^{\alpha-1} f(s) ds. \quad (92)$$

We consider the following boundary value problem of a conformable fractional-order differential equation:

$$D^\alpha w(t) = \lambda f(t, w(t)), \quad t \in (0, 1), 1 < \alpha < 2, \quad (93)$$

with $w(0) = 0, w(1) = \int_0^1 w(s) ds$.

The integral representation of the solution to the boundary value problem (93) is

$$w(t) = \lambda \int_0^1 G(t, s) f(s, w(s)) ds, \quad (94)$$

where $G(t, s)$ is a Green's function defined by

$$G(t, s) = \begin{cases} -2ts^\alpha + s^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ -2ts^\alpha, & 0 \leq t \leq s \leq 1, \end{cases} \quad (95)$$

and $\int_0^1 w(s) ds$ denotes the Riemann integrable of w with respect to s and $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

$$\begin{aligned}
w(t) &= c_0 + c_1 t + \lambda \int_0^t s^{\alpha-1} f(s, w(s)) ds, \\
w(0) &= 0 \Rightarrow c_0 = 0, \\
w(1) &= c_1 + \lambda \int_0^1 s^{\alpha-1} f(s, w(s)) ds, \\
\int_0^1 w(s) ds &= \int_0^1 c_1 s ds + \lambda \int_0^1 \int_0^s z^{\alpha-1} f(z, w(z)) dz ds \\
&= \frac{1}{2} c_1 + \lambda \int_0^1 \int_z^1 z^{\alpha-1} f(z, w(z)) ds dz \\
&= \frac{1}{2} c_1 + \lambda \int_0^1 (1-z) z^{\alpha-1} f(z, w(z)) ds dz \\
&= \frac{1}{2} c_1 + \lambda \int_0^1 (s^{\alpha-1} - s^\alpha) f(s, w(s)) ds, \\
\frac{1}{2} c_1 &= -\lambda \int_0^1 s^{\alpha-1} f(s, w(s)) ds + \lambda \int_0^1 (s^{\alpha-1} - s^\alpha) f(s, w(s)) ds \\
&= -\lambda \int_0^1 s^\alpha f(s, w(s)) ds \\
c_1 &= -2\lambda \int_0^1 s^\alpha f(s, w(s)) ds.
\end{aligned} \tag{96}$$

So,

$$\begin{aligned}
w(t) &= -2t\lambda \int_0^1 s^\alpha f(s, w(s)) ds + \lambda \int_0^t s^{\alpha-1} f(s, w(s)) ds, \\
&= -2t\lambda \int_0^t s^\alpha f(s, w(s)) ds - 2t\lambda \int_t^1 s^\alpha f(s, w(s)) ds + \lambda \int_0^t s^{\alpha-1} f(s, w(s)) ds \\
&= \lambda \int_0^t (-2ts^\alpha + s^{\alpha-1}) f(s, w(s)) ds + \lambda \int_t^1 (-2ts^\alpha) f(s, w(s)) ds \\
&= \lambda \int_0^1 G(t, s) f(s, w(s)) ds.
\end{aligned} \tag{97}$$

Let $C(I)$ be the linear space of all continuous functions defined on $I = [0, 1]$, and let $D(w, v) = \|w - v\|_\infty = \max_{t \in I} |w(t) - v(t)|$ for all $w, v \in C(I)$. Then, $(C(I), D)$ is an \mathcal{F} -complete metric space.

We consider the following conditions:

- (a) There exists $r \in [0, 1)$, and $\zeta: \mathbb{R}^2 \rightarrow \mathbb{R}$ is a function for each $a, b \in \mathbb{R}$ with $\zeta(a, b) \geq \xi(a, b)$, such that

$$\begin{aligned}
|f(s, w(s)) ds - f(s, v(s)) ds| &\leq r |w(s) - v(s)| \cdot |w(s) - Tw(s)|^{(\beta/(\beta-\widehat{w}))(\beta-\gamma)} \cdot |v(s) - Tv(s)|^{(\beta/(\beta-\widehat{w}))(\beta-\gamma)} \\
&\cdot [|w(s) - Tw(s)| + |v(s) - Tv(s)|]^{(\widehat{w}/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [|w(s) - Tv(s)| + |v(s) - Tw(s)|]^{(\gamma/(\gamma-\beta)(\gamma-\widehat{w}))},
\end{aligned} \tag{98}$$

where $\beta, \hat{w}, \gamma \in (0, 1)$.

(b) There exists $w_1 \in C(I)$ such that

$$\begin{aligned} & \zeta\left(w_1(t), \int_0^1 G(t, s) f(s, w_1(s)) ds\right) \\ & \geq \xi\left(w_1(t), \int_0^1 G(t, s) f(s, w_1(s)) ds\right), \end{aligned} \quad (99)$$

for all $t \in I$;

(c) For each $w, v \in C(I)$, there exists $w_1, v_1 \in C(I)$ such that

$$\begin{aligned} \zeta(w(t), v(t)) \geq \xi(w(t), v(t)) & \text{ implies } \zeta\left(\int_0^1 G(t, s) f(s, w_1(s)) ds, \int_0^1 G(t, s) f(s, v_1(s)) ds\right) \\ & \geq \xi\left(\int_0^1 G(t, s) f(s, w_1(s)) ds, \int_0^1 G(t, s) f(s, v_1(s)) ds\right), \end{aligned} \quad (100)$$

for all $t \in I$;

(d) For any cluster point w of a sequence $\{w_n\}$ of points in $C(I)$ with

$$\begin{aligned} & \zeta(w_n, w_{n+1}) \geq \xi(w_n, w_{n+1}), \\ & \lim_{n \rightarrow \infty} \inf \zeta(w_n, w) \geq \lim_{n \rightarrow \infty} \inf \xi(w_n, w). \end{aligned} \quad (101)$$

Theorem 15. Suppose that conditions (a)-(d) are satisfied. Then, (93) has at least one solution $w^* \in C(I)$.

Proof. We know that $w \in C(I)$ is a solution of (93) if and only if $w \in C(I)$ is a solution of the fractional-order integral equation

$$w(t) = \lambda \int_0^1 G(t, s) f(s, w(s)) ds, \quad \text{for all } \lambda, t \in I. \quad (102)$$

We define a map $T: C(I) \rightarrow C(I)$ by

$$Tw(t) = \lambda \int_0^1 G(t, s) f(s, w(s)) ds, \quad \text{for all } t \in I. \quad (103)$$

Then, problem (93) is equivalent to finding $w^* \in C(I)$, that is, a fixed point of T . Let $w, v \in C(I)$, such that $\zeta(w(t), v(t)) \geq 0$, for all $t \in I$. For using (a), we get

$$\begin{aligned} |Tw(t) - Tv(t)| &= \left| \lambda \int_0^1 G(t, s) [f(s, w(s)) - f(s, v(s))] ds \right| \\ &\leq |\lambda| \int_0^1 G(t, s) |f(s, w(s)) - f(s, v(s))| ds \\ &\leq |\lambda| \int_0^1 G(t, s) r ds |w(s) - v(s)| \cdot |w(s) - Tw(s)|^{(\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot |v(s) - Tv(s)|^{(\beta/(\beta-\hat{w})(\beta-\gamma))} \\ &\quad \cdot [|w(s) - Tw(s)| + |v(s) - Tv(s)|]^{(\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma))} \cdot [|w(s) - Tv(s)| + |v(s) - Tw(s)|]^{(\gamma/(\gamma-\beta)(\gamma-\hat{w}))} \\ &\leq \max_{t \in I} \int_0^1 G(t, s) ds r \|w(s) - v(s)\|_\infty \cdot \|w(s) - Tw(s)\|_\infty^{(\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot \|v(s) - Tv(s)\|_\infty^{(\beta/(\beta-\hat{w})(\beta-\gamma))} \\ &\quad \cdot [\|w(s) - Tw(s)\|_\infty + \|v(s) - Tv(s)\|_\infty]^{(\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma))} \cdot [\|w(s) - Tv(s)\|_\infty + \|v(s) - Tw(s)\|_\infty]^{(\gamma/(\gamma-\beta)(\gamma-\hat{w}))} \\ &\leq r \|w(s) - v(s)\|_\infty \cdot \|w(s) - Tw(s)\|_\infty^{(\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot \|v(s) - Tv(s)\|_\infty^{(\beta/(\beta-\hat{w})(\beta-\gamma))} \cdot \|w(s) - Tv(s)\|_\infty^{(p\gamma-q\gamma)} \\ &\quad \cdot [\|w(s) - Tw(s)\|_\infty + \|v(s) - Tv(s)\|_\infty]^{(\hat{w}/(\hat{w}-\beta)(\hat{w}-\gamma))} \cdot [\|w(s) - Tv(s)\|_\infty + \|v(s) - Tw(s)\|_\infty]^{(\gamma/(\gamma-\beta)(\gamma-\hat{w}))}. \end{aligned} \quad (104)$$

Thus,

$$D(Tw, Tv) < \left\{ |w(s) - v(s)| \cdot |w(s) - Tw(s)|^{(\beta/(\beta-\widehat{w})(\beta-\gamma))} \cdot |v(s) - Tv(s)|^{(\beta/(\beta-\widehat{w})(\beta-\gamma))} \right. \\ \left. \cdot [|w(s) - Tw(s)| + |v(s) - Tv(s)|]^{(\widehat{w}/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [|w(s) - Tv(s)| + |v(s) - Tw(s)|]^{(\gamma/(\gamma-\beta)(\gamma-\widehat{w}))} \right\}, \quad (105)$$

for all $w, v \in C(I)$ such that $\zeta(w(t), v(t)) \geq \xi(w(t), v(t))$ for all $t \in I$. We define $\alpha: C(I) \times C(I) \longrightarrow [0, \infty)$ by

$$\alpha(w, v) = \begin{cases} 1, & \text{if } \zeta(w(t), v(t)) \geq 0, t \in I, \\ 0, & \text{otherwise} \end{cases}, \\ \eta(w, v) = \begin{cases} \frac{1}{2}, & \text{if } \xi(w(t), v(t)) \geq 0, t \in I, \\ 0, & \text{otherwise} \end{cases}. \quad (106)$$

$$D(Tw, Tv) \leq r |w(s) - v(s)| \cdot |w(s) - Tw(s)|^{(\beta/(\beta-\widehat{w})(\beta-\gamma))} \cdot |v(s) - Tv(s)|^{(\beta/(\beta-\widehat{w})(\beta-\gamma))} \\ \cdot [|w(s) - Tw(s)| + |v(s) - Tv(s)|]^{(\widehat{w}/(\widehat{w}-\beta)(\widehat{w}-\gamma))} \cdot [|w(s) - Tv(s)| + |v(s) - Tw(s)|]^{(\gamma/(\gamma-\beta)(\gamma-\widehat{w}))}. \quad (107)$$

Obviously, $\alpha(w, v) \geq \eta(w, v)$ for all $w, v \in C(I)$. If $\alpha(w, v) \geq \eta(w, v)$ for each $w, v \in C(I)$, then $\zeta(w(t), v(t)) \geq \xi(w(t), v(t))$. From (c), we have $\zeta(Tw(t), Tv(t)) \geq \xi(Tw(t), Tv(t))$ and so $\alpha(Tw, Tv) \geq \eta(Tw, Tv)$. Thus, T is an α -admissible map concerning η . From (b), there subsist $w_1 \in C(I)$ parallel to $\alpha(w_1, Tw_1) = \eta(w_1, Tw_1)$. By (d), we have that, for any cluster point w of a sequence $\{w_n\}$ of points in $C(I)$ with $\alpha(w_n, w_{n+1}) = \eta(w_n, w_{n+1})$, $\lim_{n \rightarrow \infty} \inf \alpha(w_n, w) = \lim_{n \rightarrow \infty} \inf \eta(w_n, w)$. By applying Theorem 2, if T has a fixed point in $C(I)$, there exists $w^* \in C(I)$ such that $Tw^* = w^*$, and w^* is a solution of (93). \square

7.1. Applications. The fractional-order differential equations emerge in various areas of engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, control theory, biology, economics, blood flow phenomena, signal and image processing, biophysics, aerodynamics, and fitting of experimental data.

7.2. Open Problem. What are the conditions for making a power of the contraction a nonnegative real number for fixed point and coincidence fixed point for two or more maps in various spaces?

8. Conclusion

The aim of this paper is to produce four new classes of type contractions. This research focuses on new idea of fractional symmetric α - η -contractions of type-I, type-II, type-III, and type-IV in the structure of \mathcal{F} -metric space, which is different

Then, for all $w, v \in C(I)$, $\alpha(w, v) \geq \eta(w, v)$, we have

from and more general than ordinary metric. This paper will open a new conspiracy of fractional fixed-point theory. We develop here Suzuki-type fixed point results in orbitally complete \mathcal{F} -metric space. These new investigations and applications would enhance the impact of new setup.

Data Availability

The data used to support the findings of this study are available upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Some New Results of Interpolative Hardy–Rogers and Ćirić–Reich–Rus Type Contraction

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Received 24 March 2021; Revised 18 April 2021; Accepted 30 April 2021; Published 20 May 2021

Academic Editor: Naeem Saleem

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In this paper, we present new concepts on completeness of Hardy–Rogers type contraction mappings in metric space to prove the existence of fixed points. Furthermore, we introduce the concept of g -interpolative Hardy–Rogers type contractions in b -metric spaces to prove the existence of the coincidence point. Lastly, we add a third concept, interpolative weakly contractive mapping type, Ćirić–Reich–Rus, to show the existence of fixed points. These results are a generalization of previous results, which we have reinforced with examples.

1. Introduction and Preliminaries

The theory of fixed points has known a lot of evolution. It has been given great merit and concern, thanks to its many uses in several fields of mathematics, such as differential equations, graph theory, and nonlinear analysis [1–8]. Besides, the emergence of the fixed theorem with Banach [4] in 1922 on complete normed space was followed by several improvements and generalizations of this theorem on two levels: the first level is related to the applications used, and the second to the spaces used in them. It first knew improvements with Kannan [9] in 1968, and later with other researchers such as Rus, Ćirić, Reich, Hardy, and Rogers. Afterwards, it took another turning with Karapinar [10] in 2018 in a new version, which has made several researchers pursue this field (see [11–19]). Thus, the concept has been applied in various spaces: metric space, b -metric space, rectangular b -metric spaces, and the Branciari distance. More recently, Errai et al. [14] have inserted g -interpolation over Ćirić–Reich–Rus type contraction. They have also introduced the concept of interpolative weakly contractive mapping, which makes us use these two concepts in this paper: the first concept on Hardy–Rogers type contraction and the second on Ćirić–Reich–Rus type contraction as a generalization of the previous findings, reinforced by various

examples. This leads us to come up with some remarks. Before starting, we will take some basic concepts that we will use in this article.

Definition 1 (see [20, 21]). Let $s \geq 1$ be a given real number and F be a nonempty set. A function $d: F \times F \rightarrow \mathbb{R}^+$ is a b -metric if the following conditions are met for all $v, y, z \in F$

- (b_1) $d(v, y) = 0$ if and only if $v = y$
- (b_2) $d(v, y) = d(y, v)$
- (b_3) $d(v, z) \leq s[d(v, y) + d(y, z)]$

The pair (F, d) is called a b -metric space.

It is worth mentioning that b -metric spaces are a broader category than metric spaces.

The definitions of b -convergent and b -Cauchy sequences, as well as b -complete b -metric spaces, are defined in the same way as usual metric spaces (see, e.g., [22]).

For the interesting examples and properties of b -metric, see the following papers [23–25] as examples.

Definition 2 (see [26, 27]). Let $\{v_n\}$ be a sequence in a b -metric space (F, d) . $S, h: F \rightarrow F$ and $v \in F$. v is said to be coincidence point of pair $\{S, h\}$ if $Sv = hv$.

Definition 3 (see [12, 13]). Let Ψ be the set of all nondecreasing functions $\psi: [0, \infty) \rightarrow [0, \infty)$, with $\sum_{k=0}^{\infty} \psi^k(t) < \infty$ for all $t > 0$. After that,

- (a) $\psi(t) < t$ for each $t > 0$
- (b) $\psi(0) = 0$

Remark 1 (see [22]). The following assertions apply in a b -metric space (F, d) :

- (1) Each b -convergent sequence is a b -Cauchy sequence.
- (2) A b -convergent sequence has a unique limit.
- (3) In general, a b -metric is not continuous.

To prove our results, the fact in the previous remark necessitates the following lemma regarding b -convergent sequences:

Lemma 1 (see [26]). Let (F, d) be a b -metric space with $s \geq 1$, and assume that $\{v_n\}$ and $\{y_n\}$ are b -convergent to v, y , respectively, so we have

$$\frac{1}{s}d(v, y) \leq \liminf_{n \rightarrow \infty} d(v_n, y_n) \leq \limsup_{n \rightarrow \infty} d(v_n, y_n) \leq s^2 d(v, y). \quad (1)$$

In particular, if $v = y$, then we have $\lim_{n \rightarrow \infty} d(v_n, y_n) = 0$. In addition, for each $z \in F$, we have

$$\frac{1}{s}d(v, z) \leq \liminf_{n \rightarrow \infty} d(v_n, z) \leq \limsup_{n \rightarrow \infty} d(v_n, z) \leq s d(v, z). \quad (2)$$

Lemma 2. Let $\{v_n\}$ be a sequence defined on a b -metric space (F, d, s) and meets the conditions:

- (i) $\{v_n\}$ is b -convergent sequence in (F, d, s)
- (ii) $d(v_{n+1}, v_n) \leq \psi(d(v_n, v_{n-1})) \leq \psi^2(d(v_{n-1}, v_{n-2})) \leq \dots \leq \psi^n(d(v_1, v_0))$, where $\psi \in \Psi$

Then, $\{v_n\}$ is a b -Cauchy sequence in (F, d, s) .

Proof. Let $p \in \mathbb{N} \setminus \{0\}$; using the triangle inequality of the b -metric space and condition (ii), we have

$$\begin{aligned} d(v_n, v_{n+p}) &\leq s[d(v_n, v_{n+1}) + d(v_{n+1}, v_{n+p})] \\ &\leq s[d(v_n, v_{n+1}) + s d(v_{n+1}, v_{n+2}) + s d(v_{n+2}, v_{n+p})] \\ &\vdots \\ &\leq s \psi^n(d(v_0, v_1)) + s^2 \psi^{n+1}(d(v_0, v_1)) + \dots + s^p \psi^{n+p-1}(d(v_0, v_1)) \\ &\leq s^p \sum_{k=n}^{n+p-1} \psi^k(d(v_0, v_1)) \\ &\leq s^p \sum_{k=n}^{\infty} \psi^k(d(v_0, v_1)). \end{aligned} \quad (3)$$

Since $\sum_{k=0}^{\infty} \psi^k(t) < \infty$ for each $t > 0$, then $\lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \psi^k(d(v_0, v_1)) = 0$, which implies for any finite integer $p \geq 1$:

$$\lim_{n \rightarrow \infty} d(v_n, v_{n+p}) = 0. \quad (4)$$

Then, $\{v_n\}$ is a b -Cauchy sequence in (F, d, s) . \square \square

2. Results

The set of functions $\xi: [0, \infty) \rightarrow [0, \infty)$ which satisfies $\xi(t) < t$ for all $t > 0$ is denoted by Ξ .

Definition 4. Consider the metric space (F, d) . If there exist $\alpha, \eta, \omega, \theta \in (0, 1)$ with $\alpha + \eta + \omega + \theta > 1$, the self mapping $T: F \rightarrow F$ is named a ξ -interpolative Hardy-Rogers type contraction, such that

$$d(Tv, Ty) \leq \xi \left([d(v, y)]^\alpha [d(v, Tv)]^\eta [d(y, Ty)]^\omega \left[\frac{d(v, Ty) + d(y, Tv)}{2} \right]^\theta \right), \quad (5)$$

for all $v, y \in F \setminus \text{Fix}(T)$, where $\text{Fix}(T) = \{a \in F \mid Ta = a\}$ and $\xi \in \Xi$.

The following is our key finding:

Theorem 1. In a complete metric space (F, d) , a ξ -interpolative Hardy-Rogers type contraction $T: F \rightarrow F$, we assume there exists $v \in F$ such that $d(v, Tv) < 1$. Then, T has a fixed point in F .

Proof. Let $\{v_n\}$ be the sequence defined by $v_0 = v$ and $v_{n+1} = Tv_n$ for all integer n . If there exists n_0 such that $v_{n_0} = v_{n_0+1}$, then v_{n_0} is a fixed point of T . The proof is complete. Suppose that $v_{n+1} \neq Tv_n$ for all $n \geq 0$.

By substituting the values $v = v_{n-1}$ and $y = v_n$ in (5), we have

$$\begin{aligned} d(v_n, v_{n+1}) &\leq \xi \left([d(v_{n-1}, v_n)]^\alpha [d(v_{n-1}, v_n)]^\eta [d(v_n, v_{n+1})]^\omega \left[\frac{d(v_{n-1}, v_{n+1}) + d(v_n, v_n)}{2} \right]^\theta \right) \\ &\leq \xi \left([d(v_{n-1}, v_n)]^{\alpha+\eta} [d(v_n, v_{n+1})]^\omega \left[\frac{d(v_{n-1}, v_n) + d(v_n, v_{n+1})}{2} \right]^\theta \right). \end{aligned} \quad (6)$$

Using the fact $\xi(t) < t$ for each $t > 0$, we obtain

$$\begin{aligned} d(v_n, v_{n+1}) &\leq \xi \left([d(v_{n-1}, v_n)]^{\alpha+\eta} [d(v_n, v_{n+1})]^\omega \left[\frac{d(v_{n-1}, v_n) + d(v_n, v_{n+1})}{2} \right]^\theta \right) \\ &< [d(v_{n-1}, v_n)]^{\alpha+\eta} [d(v_n, v_{n+1})]^\omega \left[\frac{d(v_{n-1}, v_n) + d(v_n, v_{n+1})}{2} \right]^\theta. \end{aligned} \quad (7)$$

If $d(v_{n-1}, v_n) < d(v_n, v_{n+1})$ for some $n \geq 1$, then

$$\frac{d(v_{n-1}, v_n) + d(v_n, v_{n+1})}{2} \leq d(v_n, v_{n+1}). \quad (8)$$

From (7), we obtain

$$d(v_n, v_{n+1}) < [d(v_{n-1}, v_n)]^{\alpha+\eta} [d(v_n, v_{n+1})]^{\omega+\theta}. \quad (9)$$

Thus,

$$[d(v_n, v_{n+1})]^{1-\omega-\theta} < [d(v_{n-1}, v_n)]^{\alpha+\eta}, \quad (10)$$

which implies

$$[d(v_{n-1}, v_n)]^{1-\omega-\theta} < [d(v_{n-1}, v_n)]^{\alpha+\eta}, \quad (11)$$

which contradicts with $1 - \omega - \theta < \alpha + \eta$ and $d(v_{n-1}, v_n) < 1$. Then, $d(v_{n-1}, v_n) \geq d(v_n, v_{n+1})$ for all $n \geq 1$, and the sequence $(d(v_{n-1}, v_n) + d(v_n, v_{n+1}))/2 \leq d(v_{n-1}, v_n)$. From (7), we deduce

$$[d(v_n, v_{n+1})]^{1-\omega} < [d(v_{n-1}, v_n)]^{\alpha+\eta+\theta} \quad \text{for all } n \geq 1. \quad (12)$$

Since $d(v_0, v_1) < 1$, so there exists a real $l \in (0, 1)$ such that $d(v_0, v_1) \leq l$ and $l = (d(v_0, v_1) + 1)/2$.

By (12), we obtain

$$d(v_1, v_2) \leq [d(v_0, v_1)]^{(\alpha+\eta+\theta)/(1-\omega)} \leq l^{(\alpha+\eta+\theta)/(1-\omega)}. \quad (13)$$

Assume that there exists a real $\rho(n)$ such that

$$d(v_n, v_{n+1}) \leq l^{\rho(n)} \quad \text{for all } n \geq 0. \quad (14)$$

From (12), we deduce

$$[d(v_{n+1}, v_{n+2})]^{1-\omega} < [d(v_n, v_{n+1})]^{\alpha+\eta+\theta} \leq l^{(\alpha+\eta+\theta)\rho(n)}, \quad (15)$$

which gives

$$d(v_{n+1}, v_{n+2}) < l^{\rho(n+1)}, \quad (16)$$

where $\rho(n+1) = ((\alpha + \eta + \theta)/(1 - \omega))\rho(n)$ for all $n \geq 1$ with $\rho(0) = 1$.

Since $(\alpha + \eta + \theta)/(1 - \omega) > 1$; we have $\lim_{n \rightarrow \infty} \rho(n) = +\infty$.

Consequently,

$$\sum_{n=0}^{+\infty} d(v_n, v_{n+1}) \leq \sum_{n=0}^{+\infty} l^{\rho(n)}, \quad (17)$$

which is convergent, so $\{v_n\}$ is Cauchy sequence in (F, d) , and then it converges to some $v \in F$. Suppose that $v \neq Tv$, we find by (5):

$$\begin{aligned}
 d(v_{n+1}, Tv) &\leq \xi \left([d(v_n, v)]^\alpha [d(v_n, v_{n+1})]^\eta [d(v, Tv)]^\omega \left[\frac{d(v_n, Tv) + d(v, v_{n+1})}{2} \right]^\theta \right) \\
 &< [d(v_n, v)]^\alpha [d(v_n, v_{n+1})]^\eta [d(v, Tv)]^\omega \left[\frac{d(v_n, Tv) + d(v, v_{n+1})}{2} \right]^\theta.
 \end{aligned} \tag{18}$$

Passing the limit as $n \rightarrow +\infty$, we get $d(v, Tv) \leq 0$. So $d(v, Tv) = 0$ which is a contradiction. Then, $Tv = v$. \square

Example 1. Let $E = [0, 3]$ be endowed with metric $\mu: E \times E \rightarrow [0, \infty)$, defined by

$$\mu(u, v) = \begin{cases} 0, & \text{if } u = v, \\ 2, & \text{if } u, v \in [0, 1] \text{ and } u \neq v, \\ 3, & \text{otherwise.} \end{cases} \tag{19}$$

Let $T: E \rightarrow E$ be defined as

$$Tu = \begin{cases} \frac{1}{3}, & \text{if } u \in [0, 1]; \\ \frac{u}{3}, & \text{if } u \in (1, 3]; \end{cases} \tag{20}$$

and the function $\xi(t) = 2/7t^2$ for all $t \in [0, \infty)$.

Choose $\alpha = 0.7$; $\eta = 0.6$; $\omega = 0.8$; and $\theta = 0.5$.

We have $\mu(Tu, Tv) \leq 2$ for all $u, v \in E$.

The following issues are discussed:

First case: if $u, v \in [0, 1]$ or $u = v$ for all $u, v \in [0, 3]$, we have $\xi(t) = 2/7t^2 \geq 0$ for all $t \in [0, \infty)$, and $\mu(Tu, Tv) = 0$ for all $u, v \in [0, 1]$ or $u = v$ for all $u, v \in [0, 3]$.

Consequently, in this case, inequality (5) is satisfied.

Second case: if $u, v \in (1, 3]$ and $u \neq v$, we have

$$\xi \left([\mu(u, v)]^\alpha [\mu(u, Tu)]^\eta [\mu(v, Tv)]^\omega \left[\frac{\mu(u, Tv) + \mu(v, Tu)}{2} \right]^\theta \right) = \xi(3^{\alpha+\eta+\omega+\theta}) = \frac{2}{7} \cdot 3^{5.2} \geq 2. \tag{21}$$

Third case: if $u \in [0, 1]$ and $v \in (1, 3]$ with $u \neq 1/3$, we have

$$\xi \left([\mu(u, v)]^\alpha [\mu(u, Tu)]^\eta [\mu(v, Tv)]^\omega \left[\frac{\mu(u, Tv) + \mu(v, Tu)}{2} \right]^\theta \right) = \xi(3^{\alpha+\omega} \cdot 2^{\eta-\theta} \cdot 5^\theta) = \frac{2}{7} \cdot 3^3 \cdot 2^{0.2} \cdot 5 \geq 2. \tag{22}$$

Fourth case: if $u \in (1, 3]$ and $v \in [0, 1]$ with $v \neq 1/3$, we have

$$\xi \left([\mu(u, v)]^\alpha [\mu(u, Tu)]^\eta [\mu(v, Tv)]^\omega \left[\frac{\mu(u, Tv) + d(v, Tu)}{2} \right]^\theta \right) = \xi(3^{\alpha+\eta} \cdot 2^{\omega-\theta} \cdot 5^\theta) = \frac{2}{7} \cdot 3^{2.6} \cdot 2^{0.6} \cdot 5 \geq 2. \tag{23}$$

Hence, in all cases, we have

$$\mu(Tu, Tv) \leq \xi \left([\mu(u, v)]^\alpha [\mu(u, Tu)]^\eta [\mu(v, Tv)]^\omega \left[\frac{\mu(u, Tv) + \mu(v, Tu)}{2} \right]^\theta \right). \tag{24}$$

for all $u, v \in [0, 3] \setminus \{1/3\}$.

As a result, all the conditions of Theorem 1 are fulfilled, and T has a fixed point, $u = 1/3$.

Example 2. Let $\mathbb{F} = \{a, q, m, r\}$ be endowed with the metric given in the following chart (Table 1).

Consider the self mapping T on \mathbb{F} as

$$T: \begin{pmatrix} a & q & m & r \\ q & q & a & r \end{pmatrix}. \quad (25)$$

Take $\xi(t) = (3^t - 1)/(3^t + 1)$, for all $t \in [0, \infty)$; $\alpha = 0.4$; $\eta = 0.8$; $\omega = 0.9$; and $\theta = 0.5$.

We have

$$d(Tu, Tv) \leq \xi \left([d(u, v)]^\alpha [d(u, Tu)]^\eta [d(v, Tv)]^\omega \left[\frac{d(u, Tv) + d(v, Tu)}{2} \right]^\theta \right), \quad (26)$$

for all $u, v \in \mathbb{F} \setminus \{q, r\}$.

Then, T has two fixed points, which are q and r .

If we replace $\xi(t) = \lambda t$ with $\lambda \in (0, 1)$ in Theorem 1, we get the following corollary.

Corollary 1. Let (\mathbb{F}, d) be a complete metric space and T is self mapping on \mathbb{F} such that

$$d(Tv, Ty) \leq \lambda [d(v, y)]^\alpha [d(v, Tv)]^\eta [d(y, Ty)]^\omega \left[\frac{d(v, Ty) + d(y, Tv)}{2} \right]^\theta \quad (27)$$

is satisfied for all $v, y \in \mathbb{F} \setminus \text{Fix}(T)$, where $\text{Fix}(T) = \{a \in \mathbb{F} | Ta = a\}$ and $\alpha, \eta, \omega, \theta, \lambda \in (0, 1)$ such that $\alpha + \eta + \omega + \theta > 1$.

If there exists $v \in \mathbb{F}$ such that $d(x, Tx) < 1$, then T has a fixed point in \mathbb{F} .

Definition 5. Let (\mathbb{F}, d, s) be a b -metric space and $T, g: \mathbb{F} \rightarrow \mathbb{F}$ be self mappings on \mathbb{F} . We say that T is a g -interpolative Hardy–Rogers type contraction if there exist $\psi \in \Psi$ and $\alpha, \eta, \omega \in (0, 1)$ such that

$$d(Tv, Ty) \leq \psi \left([d(gv, gy)]^\alpha [d(gv, Tv)]^\eta [d(gy, Ty)]^\omega \left[\frac{d(gv, Ty) + d(gy, Tv)}{2s} \right]^{1-\alpha-\eta-\omega} \right) \quad (28)$$

is satisfied for all $v, y \in \mathbb{F}$ such that $Tv \neq gv, Ty \neq gy$, and $gv \neq gy$.

Theorem 2. In a b -complete b -metric space (\mathbb{F}, d, s) , if T is a g -interpolative Hardy–Rogers type contraction such that

(1) $T\mathbb{F} \subseteq g\mathbb{F}$

(2) $g\mathbb{F}$ is closed

Then, T and g have a coincidence point in \mathbb{F} .

Proof. Let $v \in \mathbb{F}$, since $T\mathbb{F} \subseteq g\mathbb{F}$, we can inductively define a sequence $\{v_n\}$ such that

$$v_0 = v, \text{ and } gv_{n+1} = Tv_n, \text{ for all integer } n. \quad (29)$$

If there exists $n \in \{0, 1, 2, \dots\}$ such that $gv_n = Tv_n$, then v_n is a coincidence point of g and T . Assume that $gv_n \neq Tv_n$, for all n . By (28), we obtain

$$\begin{aligned}
d(Tv_{n+1}, Tv_n) &\leq \psi \left([d(gv_{n+1}, gv_n)]^\alpha [d(gv_{n+1}, Tv_{n+1})]^\eta [d(gv_n, Tv_n)]^\omega \left[\frac{d(gv_{n+1}, Tv_n) + d(gv_n, Tv_{n+1})}{2s} \right]^{1-\alpha-\eta-\omega} \right) \\
&= \psi \left([d(Tv_n, Tv_{n-1})]^\alpha [d(Tv_n, Tv_{n+1})]^\eta [d(Tv_{n-1}, Tv_n)]^\omega \left[\frac{d(Tv_n, Tv_n) + d(Tv_{n-1}, Tv_{n+1})}{2s} \right]^{1-\alpha-\eta-\omega} \right) \quad (30) \\
&\leq \psi \left([d(Tv_n, Tv_{n-1})]^{\alpha+\omega} [d(Tv_n, Tv_{n+1})]^\eta \left[\frac{d(Tv_{n-1}, Tv_n) + d(Tv_n, Tv_{n+1})}{2} \right]^{1-\alpha-\eta-\omega} \right).
\end{aligned}$$

Using the fact $\psi(t) < t$ for each $t > 0$, we obtain

$$d(Tv_{n+1}, Tv_n) \leq [d(Tv_n, Tv_{n-1})]^{\alpha+\omega} [d(Tv_n, Tv_{n+1})]^\eta \left[\frac{d(Tv_{n-1}, Tv_n) + d(Tv_n, Tv_{n+1})}{2} \right]^{1-\alpha-\eta-\omega}. \quad (31)$$

Suppose that $d(Tv_{n-1}, Tv_n) < d(Tv_n, Tv_{n+1})$ for some $n \geq 1$. Then,

$$\frac{d(Tv_{n-1}, Tv_n) + d(Tv_n, Tv_{n+1})}{2} \leq d(Tv_n, Tv_{n+1}). \quad (32)$$

Thus, from inequality (31), we have

$$d(Tv_{n+1}, Tv_n) \leq [d(Tv_n, Tv_{n-1})]^{\alpha+\omega} [d(Tv_n, Tv_{n+1})]^{1-\alpha-\omega}. \quad (33)$$

This implies

$$[d(Tv_{n+1}, Tv_n)]^{\alpha+\omega} \leq [d(Tv_n, Tv_{n-1})]^{\alpha+\omega}. \quad (34)$$

So, we get

$$d(Tv_{n+1}, Tv_n) \leq d(Tv_n, Tv_{n-1}), \quad (35)$$

which is a contradiction. Thus,

$$d(Tv_{n+1}, Tv_n) \leq d(Tv_n, Tv_{n-1}), \quad \text{for all } n \geq 1. \quad (36)$$

This means that the positive sequence $\{d(Tv_{n+1}, Tv_n)\}$ is monotone decreasing, and consequently, there exists $c \geq 0$ such that $\lim_{n \rightarrow +\infty} d(Tv_{n+1}, Tv_n) = c$.

From (36), we have

$$\begin{aligned}
&[d(Tv_n, Tv_{n-1})]^{\alpha+\omega} [d(Tv_n, Tv_{n+1})]^\eta \left[\frac{d(Tv_{n-1}, Tv_n) + d(Tv_n, Tv_{n+1})}{2} \right]^{1-\alpha-\eta-\omega} \\
&\leq [d(Tv_n, Tv_{n-1})]^{\alpha+\omega} [d(Tv_n, Tv_{n+1})]^\eta [d(Tv_{n-1}, Tv_n)]^{1-\alpha-\eta-\omega} = d(Tv_n, Tv_{n-1}).
\end{aligned} \quad (37)$$

So, (31) together with the nondecreasing character of ψ , we obtain

$$d(Tv_{n+1}, Tv_n) \leq \psi(d(Tv_n, Tv_{n-1})). \quad (38)$$

Similar to the previous method, we find

$$d(Tv_{n+1}, Tv_n) \leq \psi(d(Tv_n, Tv_{n-1})) \leq \psi^2(d(Tv_{n-1}, Tv_{n-2})) \leq \dots \leq \psi^n(d(Tv_1, Tv_0)). \quad (39)$$

Letting $n \rightarrow \infty$ in (39) and using the fact $\lim_{n \rightarrow \infty} \psi^n(t) = 0$ for each $t > 0$, we deduce that $c = 0$, that is,

$$\lim_{n \rightarrow \infty} d(Tv_{n+1}, Tv_n) = 0. \quad (40)$$

By Lemma 2, we deduce that $\{Tv_n\}$ is a b -Cauchy sequence, and consequently, $\{gv_n\}$ is also a b -Cauchy sequence. Let $z \in \mathbb{F}$ such that,

$$\lim_{n \rightarrow \infty} d(Tv_n, z) = \lim_{n \rightarrow \infty} d(gv_{n+1}, z) = 0. \quad (41)$$

And, since $z \in g\mathbb{F}$, there exists $u \in \mathbb{F}$ such that $z = gu$. We claim that u is a coincidence point of g and T . Thus, if we assume that $gu \neq Tu$, we obtain

$$\begin{aligned} d(Tv_n, Tu) &\leq \psi \left([d(gv_n, gu)]^\alpha [d(gv_n, Tv_n)]^\eta [d(gu, Tu)]^\omega \left[\frac{d(gv_n, Tu) + d(gu, Tv_n)}{2s} \right]^{1-\alpha-\eta-\omega} \right) \\ &< [d(gv_n, gu)]^\alpha [d(gv_n, Tv_n)]^\eta [d(gu, Tu)]^\omega \left[\frac{d(gv_n, Tu) + d(gu, Tv_n)}{2s} \right]^{1-\alpha-\eta-\omega}. \end{aligned} \quad (42)$$

Consequently,

$$\frac{1}{s} d(z, Tu) \leq [sd(z, gu)]^\alpha [s^2 d(z, z)]^\eta [d(gu, Tu)]^\omega \left[\frac{d(z, Tu) + d(gu, z)}{2} \right]^{1-\alpha-\eta-\omega} = 0, \quad (43)$$

which is a contradiction. This implies that

$$Tu = z = gu. \quad (44)$$

Then, u is a coincidence point in \mathbb{F} of T and g . \square

Example 3. Let the set $\mathbb{F} = \{a, r, i, m\}$ and a function $d: \mathbb{F} \times \mathbb{F} \rightarrow [0, \infty)$ defined as follows (Table 2)

One can check that the function d is a b -metric for $s=2$ by doing a simple calculation. Self-mappings g, T on \mathbb{F} are defined as

$$g: \begin{pmatrix} a & r & i & m \\ a & m & i & i \end{pmatrix}, T: \begin{pmatrix} a & r & i & m \\ i & m & m & i \end{pmatrix}. \quad (45)$$

Choose $\alpha = 0.4; \eta = 0.1; \omega = 0.7$; and $\psi(t) = (5^t - 1)/(5^t + 1)$ for all $t \in [0, \infty)$.

For all $u, v \in \mathbb{F} \setminus \{r, m\}$, it is obvious that g, T fulfills (28). Furthermore, r and m are two coincidence points of g and T .

Example 4. Let $\mathbb{F} = [0, +\infty)$ and $d: \mathbb{F} \times \mathbb{F} \rightarrow [0, \infty)$ be defined by

$$d(v, y) = \begin{cases} (v + y)^2, & \text{if } v \neq y; \\ 0, & \text{if } v = y. \end{cases} \quad (46)$$

Then, (\mathbb{F}, d) is a complete b -metric space.

Define two self-mappings T and g on \mathbb{F} by $g(v) = v^2$, for all $v \in \mathbb{F}$ and

$$Tx = \begin{cases} 1, & \text{if } v \in [0, 2]; \\ e^{-v}, & \text{if } v \in (2, +\infty). \end{cases} \quad (47)$$

T is a g -interpolative Hardy-Rogers type contraction for $\alpha = 0.3; \eta = 0.6$; and $\omega = 0.4$. Let $\psi(t) = 3/5t^2$, for all $t \in [0, +\infty)$.

Hence, the following issues are discussed.

First case: if $v, y \in [0, 2]$ or $v = y$ for all $v \in [0, +\infty)$, this is obvious.

Second case: if $v \in [0, 2] \setminus \{1\}$ and $y \in (2, +\infty)$.

Consider

$$M(v, y) = [d(gv, gy)]^\alpha [d(gv, Tv)]^\eta [d(gy, Ty)]^\omega \left[\frac{d(gv, Ty) + d(gy, Tv)}{2s} \right]^{1-\alpha-\eta-\omega}. \quad (48)$$

We have

$$d(Tv, Ty) = (1 + e^{-v})^2 \leq (1 + e^{-2})^2 \quad (49)$$

and

$$\begin{aligned}
\psi(M(v, y)) &= \psi\left([d(gv, gy)]^\alpha [d(gv, Tv)]^\eta [d(gy, Ty)]^\omega \left[\frac{d(gv, Ty) + d(gy, Tv)}{2s}\right]^{1-\alpha-\eta-\omega}\right) \\
&= \psi\left((v^2 + y^2)^{2\alpha} (v^2 + 1)^{2\eta} (y^2 + e^{-y})^{2\omega} \left[\frac{(v^2 + e^{-v})^2 + (y^2 + 1)^2}{4}\right]^{1-\alpha-\eta-\omega}\right) \\
&\geq \psi\left(4^{2\alpha} \cdot 1^{2\eta} \cdot 4^{2\omega} \left(\frac{5^2}{4}\right)^{1-\alpha-\eta-\omega}\right) \\
&= \psi(4^{1.7} \cdot 5^{-0.6}) \\
&= 3 \cdot 4^{3.4} \cdot 5^{-2.2} \geq (1 + e^{-2})^2.
\end{aligned} \tag{50}$$

Therefore,

$$d(Tv, Ty) \leq \psi\left([d(gv, gy)]^\alpha [d(gv, Tv)]^\eta [d(gy, Ty)]^\omega \left[\frac{d(gv, Ty) + d(gy, Tv)}{2s}\right]^{1-\alpha-\eta-\omega}\right). \tag{51}$$

Third case: if $v \in (2, +\infty)$ and $y \in [0, 2] \setminus \{1\}$.
We have

$$d(Tv, Ty) = (e^{-v} + 1)^2 \leq (1 + e^{-2})^2, \tag{52}$$

and

$$\begin{aligned}
\psi(M(v, y)) &= \psi\left([d(gv, gy)]^\alpha [d(gv, Tv)]^\eta [d(gy, Ty)]^\omega \left[\frac{d(gv, Ty) + d(gy, Tv)}{2s}\right]^{1-\alpha-\eta-\omega}\right) \\
&= \psi\left((v^2 + y^2)^{2\alpha} (v^2 + e^{-v})^{2\eta} (y^2 + 1)^{2\omega} \left[\frac{(v^2 + 1)^2 + (y^2 + e^{-v})^2}{4}\right]^{1-\alpha-\eta-\omega}\right) \\
&\geq \psi\left(4^{2\alpha} \cdot 4^{2\eta} \cdot 1^{2\omega} \left[\frac{5^2 + 0^2}{4}\right]^{1-\alpha-\eta-\omega}\right) \\
&= 3 \cdot 4^{4.2} \cdot 5^{-2.2} \\
&\geq (1 + e^{-2})^2.
\end{aligned} \tag{53}$$

Therefore,

$$d(Tv, Ty) \leq \psi\left([d(gv, gy)]^\alpha [d(gv, Tv)]^\eta [d(gy, Ty)]^\omega \left[\frac{d(gv, Ty) + d(gy, Tv)}{2s}\right]^{1-\alpha-\eta-\omega}\right). \tag{54}$$

Fourth case: if $v, y \in (2, +\infty)$ and $v \neq y$, we have

$$d(Tv, Ty) = (e^{-v} + e^{-y})^2 \leq 2e^{-4}. \tag{55}$$

Using the property of ψ , we get

$$\begin{aligned}
\psi(M(v, y)) &= \psi\left([d(gv, gy)]^\alpha [d(gv, Tv)]^\eta [d(gy, Ty)]^\omega \left[\frac{d(gv, Ty) + d(gy, Tv)}{2s}\right]^{1-\alpha-\eta-\omega}\right) \\
&= \psi\left((v^2 + y^2)^{2\alpha} (v^2 + e^{-v})^{2\eta} (y^2 + e^{-y})^{2\omega} \left[\frac{(v^2 + e^{-v})^2 + (y^2 + e^{-y})^2}{4}\right]^{1-\alpha-\eta-\omega}\right) \\
&\geq \psi\left(8^{2\alpha} \cdot 4^{2\eta} \cdot 4^{2\omega} \left[\frac{4^2 + 4^2}{4}\right]^{1-\alpha-\eta-\omega}\right) \\
&= \frac{3}{5} \cdot 2^{9.8} \geq 2e^{-4}.
\end{aligned} \tag{56}$$

Therefore,

$$d(Tv, Ty) \leq \psi\left([d(gv, gy)]^\alpha [d(gv, Tv)]^\eta [d(gy, Ty)]^\omega \left[\frac{d(gv, Ty) + d(gy, Tv)}{2s}\right]^{1-\alpha-\eta-\omega}\right). \tag{57}$$

For all $v, y \in \mathbb{F} \setminus \{1\}$, it is obvious that g, T fulfills (28). Furthermore, one is a coincidence point of g and T .

The two previous examples lead us to the following remark.

Remark 2. In the Theorem 2, T and g do not need a fixed point, just as T and g accept a coincidence point and are not necessarily unique.

Definition 6. In a metric space (\mathbb{F}, d) , we say that a self mapping $T: \mathbb{F} \longrightarrow \mathbb{F}$ is an interpolative weakly contractive mapping type Ćirić–Reich–Rus, if there exists a constant $\alpha, \eta \in (0, 1)$ such that

$$\zeta(d(Tv, Ty)) \leq \zeta(R(v, y)) - \varphi(R(v, y)), \tag{58}$$

for all $v, y \in \mathbb{F} \setminus \text{Fix}(T)$, where

$$\text{Fix}(T) = \{a \in X | Ta = a\}, \tag{59}$$

$$R(v, y) = [d(v, y)]^\alpha [d(v, Tv)]^\eta [d(y, Ty)]^{1-\alpha-\eta},$$

$\varphi: [0, \infty) \longrightarrow [0, \infty)$ is a lower semicontinuous function with $\varphi(t) = 0$ if and only if $t = 0$, $\zeta: [0, \infty) \longrightarrow [0, \infty)$ is a continuous monotone nondecreasing function with $\zeta(t) = 0$ if and only if $t = 0$.

Theorem 3. In a complete metric space (\mathbb{F}, d) , if $T: \mathbb{F} \longrightarrow \mathbb{F}$ is a interpolative weakly contractive mapping type Ćirić–Reich–Rus, then T has a fixed point.

Proof. For any $v_0 \in \mathbb{F}$, we consider a sequence $\{v_n\}$ by $v = v_0$ and $v_{n+1} = Tv_n$, $n = 0, 1, 2, \dots$

If there exists $n_0 \in \mathbb{N}$ such that $v_{n_0+1} = v_{n_0}$, then v_{n_0} is clearly a fixed point in \mathbb{F} . Otherwise, $v_{n+1} \neq v_n$ for each $n \geq 0$.

Substituting $v = v_n$ and $y = v_{n-1}$ in (58), we obtain that

$$\begin{aligned}
\zeta(d(v_{n+1}, v_n)) &\leq \zeta([d(v_n, v_{n-1})]^\alpha [d(v_n, v_{n+1})]^\eta [d(v_{n-1}, v_n)]^{1-\alpha-\eta}) \\
&\quad - \varphi([d(v_n, v_{n-1})]^\alpha [d(v_n, v_{n+1})]^\eta [d(v_{n-1}, v_n)]^{1-\alpha-\eta}) \\
&\leq \zeta([d(v_n, v_{n-1})]^{1-\eta} [d(v_n, v_{n+1})]^\eta).
\end{aligned} \tag{60}$$

Using property of function ζ , we get

$$d(v_{n+1}, v_n) \leq [d(v_n, v_{n-1})]^{1-\eta} [d(v_n, v_{n+1})]^\eta. \tag{61}$$

We derive

$$[d(v_{n+1}, v_n)]^{1-\eta} \leq [d(v_n, v_{n-1})]^{1-\eta}. \tag{62}$$

Therefore,

$$d(v_{n+1}, v_n) \leq d(v_n, v_{n-1}), \quad \text{for all } n \geq 1. \tag{63}$$

It follows that the positive sequence $\{d(v_{n+1}, v_n)\}$ is decreasing. Eventually, there exists $c \geq 0$ such that $\lim_{n \rightarrow \infty} d(v_{n+1}, v_n) = c$.

Taking $n \longrightarrow \infty$ in inequality (60), we obtain

$$\zeta(c) \leq \zeta(c) - \varphi(c). \tag{64}$$

We deduce that $c = 0$. Hence,

$$\lim_{n \rightarrow \infty} d(v_{n+1}, v_n) = 0. \tag{65}$$

Therefore, $\{v_n\}$ is a Cauchy sequence. Suppose not, then there exists a real number $\varepsilon > 0$, for any $k \in \mathbb{N}$, $\exists m_k \geq n_k \geq k$ such that

$$d(v_{m_k}, v_{n_k}) \geq \varepsilon \text{ and } d(v_{m_k-1}, v_{n_k}) < \varepsilon. \tag{66}$$

Putting $v = v_{m_k-1}$ and $y = v_{m_k-1}$ in (58) and using (66), we obtain

TABLE 1: Table of metric d for Example 2.

$d(x, y)$	a	q	m	r
a	0	0.2	3	1
q	0.2	0	2.8	0.8
m	3	2.8	0	2
r	1	0.8	2	0

TABLE 2: Table of metric d for Example 3.

$d(x, y)$	a	r	i	m
a	0	4	49/16	9/4
r	4	0	1/16	1/4
i	49/16	1/16	0	1/16
m	9/4	1/4	1/16	0

$$\zeta(\varepsilon) \leq \zeta(d(v_{m_k}, v_{n_k})) \leq \zeta(R(v_{m_k-1}, v_{n_k-1})) - \varphi(R(v_{m_k-1}, v_{n_k-1})), \quad \text{where} \quad (67)$$

$$R(v_{m_k-1}, v_{n_k-1}) = [d(v_{m_k-1}, v_{n_k-1})]^\alpha [d(v_{m_k-1}, v_{m_k})]^\eta [d(v_{n_k-1}, v_{n_k})]^{1-\alpha-\eta} \quad (68)$$

and

$$d(v_{m_k-1}, v_{n_k-1}) \leq d(v_{m_k-1}, v_{n_k}) + d(v_{n_k}, v_{n_k-1}) \leq \varepsilon + d(v_{n_k}, v_{n_k-1}). \quad (69)$$

Letting $k \rightarrow \infty$ and using (65), we conclude

$$\lim_{k \rightarrow \infty} R(v_{m_k-1}, v_{n_k-1}) = 0. \quad (70)$$

Then,

$$\zeta(\varepsilon) \leq \zeta(0) - \varphi(0) = 0, \quad (71)$$

which is contradiction with $\varepsilon > 0$. Thus, $\{v_n\}$ is a Cauchy sequence, and since (F, d) is complete, we obtain $\tau \in F$ such that $\lim_{n \rightarrow \infty} d(v_n, \tau) = 0$ and assume that $T\tau \neq \tau$, we have

$$\zeta(d(v_{n+1}, T\tau)) \leq \zeta(R(v_n, \tau)) - \varphi(R(v_n, \tau)) \quad \text{for all } n, \quad (72)$$

where

$$R(v_n, \tau) = [d(v_n, \tau)]^\alpha [d(v_n, v_{n+1})]^\eta [d(\tau, T\tau)]^{1-\alpha-\eta}. \quad (73)$$

Using (65), we get

$$\lim_{n \rightarrow \infty} R(v_n, \tau) = 0. \quad (74)$$

Letting $n \rightarrow \infty$ in (72), we get

$$\zeta(d(\tau, T\tau)) \leq \zeta(0) - \varphi(0) = 0, \quad (75)$$

which is a contradiction. Thus, $T\tau = \tau$. \square

Example 5. Consider the space $F = \{a, r, i, m\}$ equipped with the metric defined by the values of the following table (Table 3).

Consider the self mapping T on F as

$$T: \begin{pmatrix} a & r & i & m \\ i & m & i & m \end{pmatrix}. \quad (76)$$

For $\zeta(t) = e^t - 1$ and $\varphi(t) = (3/2)^t - 1$ for all $t \in [0, \infty)$, taking $\alpha = 0.2$ and $\eta = 0.7$, we have

$$\zeta(d(Tu, Tv)) \leq \zeta([d(u, v)]^\alpha [d(u, Tu)]^\eta [d(v, Tv)]^{1-\alpha-\eta}) - \varphi([d(u, v)]^\alpha [d(u, Tu)]^\eta [d(v, Tv)]^{1-\alpha-\eta}), \quad (77)$$

for all $u, v \in F \setminus \{i, m\}$.

Then, T possesses two fixed points: i and m .

$$\varrho(v, y) = \begin{cases} 0, & \text{if } v = y, \\ 5, & \text{if } v, y \in [0, 1) \text{ and } v \neq y, \\ 3, & \text{otherwise.} \end{cases} \quad (78)$$

Example 6. Let the set $\Lambda = [0, 5]$ and a function $\varrho: \Lambda \times \Lambda \rightarrow [0, \infty)$ be defined as follows:

Then, (Λ, ϱ) is a complete metric space.
Let $T: \Lambda \rightarrow \Lambda$ be defined as

TABLE 3: Table of metric d for Example 5.

$d(x, y)$	a	r	i	m
a	0	2	7	8
r	2	0	5	6
i	7	5	0	1
m	8	6	1	0

$$Tx = \begin{cases} 0, & \text{if } v \in [0, 1); \\ 3, & \text{if } v \in [1, 5]. \end{cases} \quad (79) \quad \zeta(\varrho(Tv, Ty)) = \zeta(\varrho(0, 3)) = \zeta(3) = 9, \quad (80)$$

Choose $\zeta(t) = t^2$ and $\varphi(t) = 1/3t$ for all $t \in [0, +\infty)$, taking $\alpha = 0.4$ and $\eta = 0.3$, the following issues are discussed:

First case: if $v = y$ or $v, y \in (0, 1)$, or $v, y \in [1, 5] \setminus \{3\}$ with $v \neq y$, this is obvious.

Second case: If $v \in (0, 1)$ and $y \in [1, 5] \setminus \{3\}$, we have

$$[\varrho(v, y)]^\alpha [\varrho(v, Tv)]^\eta [\varrho(y, Ty)]^{1-\alpha-\eta} = 3^\alpha \cdot 5^\eta \cdot 3^{1-\alpha-\eta} = 3 \cdot \left(\frac{5}{3}\right)^\eta. \quad (81)$$

Therefore,

$$\begin{aligned} & \zeta([\varrho(v, y)]^\alpha [\varrho(v, Tv)]^\eta [\varrho(y, Ty)]^{1-\alpha-\eta}) - \varphi([\varrho(v, y)]^\alpha [\varrho(v, Tv)]^\eta [\varrho(y, Ty)]^{1-\alpha-\eta}) \\ &= \left(\frac{5}{3}\right)^\eta \left[9 \cdot \left(\frac{5}{3}\right)^\eta - 1 \right] \geq 9 = \zeta(3) = \zeta(\varrho(Tv, Ty)). \end{aligned} \quad (82)$$

Third case: if $v \in [1, 5] \setminus \{3\}$ and $y \in (0, 1)$, we have

$$\zeta(\varrho(Tv, Ty)) = \zeta(\varrho(3, 0)) = 9 \quad (83)$$

and

Thus,

$$\begin{aligned} & \zeta([\varrho(v, y)]^\alpha [\varrho(v, Tv)]^\eta [\varrho(y, Ty)]^{1-\alpha-\eta}) - \varphi([\varrho(v, y)]^\alpha [\varrho(v, Tv)]^\eta [\varrho(y, Ty)]^{1-\alpha-\eta}) \\ &= \left(\frac{3}{5}\right)^{\alpha+\eta} \left[25 \cdot \left(\frac{3}{5}\right)^{\alpha+\eta} - \frac{5}{3} \right] \geq 9 = \zeta(3) = \zeta(\varrho(Tv, Ty)). \end{aligned} \quad (85)$$

Hence,

$$\zeta(d(Tu, Tv)) \leq \zeta([\varrho(v, y)]^\alpha [\varrho(v, Tv)]^\eta [\varrho(y, Ty)]^{1-\alpha-\eta}) - \varphi([\varrho(v, y)]^\alpha [\varrho(v, Tv)]^\eta [\varrho(y, Ty)]^{1-\alpha-\eta}), \quad (86)$$

for all $v, y \in \Lambda \setminus \{0, 3\}$.

Then, T possesses two fixed points: 0 and 3.

The two previous examples lead us to the next remark.

Remark 3. If T is an interpolative weakly contractive mapping type Ćirić–Reich–Rus, T accepts a fixed point that is not necessarily a single one.

We have the following corollary if $\zeta(t) = t$ in Theorem 3:

Corollary 2 Let (F, d) be complete metric space and $T: F \longrightarrow F$ self mapping on F . If there exists a constant $\alpha, \eta \in (0, 1)$ such that

$$d(Tv, Ty) \leq [d(v, y)]^\alpha [d(v, Tv)]^\eta [d(y, Ty)]^{1-\alpha-\eta} - \varphi([d(v, y)]^\alpha [d(v, Tv)]^\eta [d(y, Ty)]^{1-\alpha-\eta}), \quad (87)$$

for all $v, y \in \mathbb{F}$ and $v \neq Tv, y \neq Ty$, where $\varphi: [0, \infty) \rightarrow [0, \infty)$ is a lower semicontinuous function with $\varphi(t) = 0$ if and only if $t = 0$, then T has a fixed point.

If we use $\varphi(t) = (1 - \lambda)t$ for a constant $\lambda, \eta \in (0, 1)$ in Corollary 2, then we get the following corollary.

Corollary 3 Let (\mathbb{F}, d) be complete metric space and $T: \mathbb{F} \rightarrow \mathbb{F}$ self mapping on \mathbb{F} . If there exists a constant $\alpha, \eta \in (0, 1)$ such that

$$d(Tv, Ty) \leq \lambda [d(v, y)]^\alpha [d(v, Tv)]^\eta [d(y, Ty)]^{1-\alpha-\eta}, \quad (88)$$

for all $v, y \in \mathbb{F}$ and $v \neq Tv, y \neq Ty$, then T has a fixed point.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Set-Valued SU-Type Fixed Point Theorems via Gauge Function with Applications

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Received 26 December 2020; Revised 21 March 2021; Accepted 25 March 2021; Published 19 May 2021

Academic Editor: Naeem Saleem

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In this article, we have designed two existence of fixed point theorems which are regarding to set-valued SU-type θ_η -contraction and Γ_α -contraction via gauge function in the setting of metric spaces. An extensive set of nontrivial example will be given to justify our claim. At the end, we will give an application to prove the existence behavior for the system of functional equation in dynamical system and integral inclusion.

1. Introduction and Preliminaries

The most publicized famous result in nonlinear analysis is Banach contraction principle, which made clear a systematic rule to find the fixed point of a given mapping on a metric space. So far, numerous authors have studied this classical result to examine the existence and uniqueness of a solution for different forms of contractive structure.

In 2014, Jleli and Samet [1] introduced the concept of a new contraction known as the θ -contraction, which generalizes the Banach contraction principle in a beautiful way.

In 2015, Khojasteh et al. [2] introduced simulation function. Recently, many researchers have proved fixed point theorems for Suzuki-type (SU) mappings in metric space (see [3, 4]).

Let (X, δ) be a metric space. For $\mu \in X$ and $\beta_1 \subseteq X$, let $CL(X)$ and $CB(X)$ denote the family of all nonempty closed subsets and the family of all nonempty closed bounded subsets of X . Design the Pompeiu–Hausdorff metric H_d induced by δ on $CB(X)$ as

$$H_d(\beta_1, \beta_2) = \max\{\sup_{\mu_1 \in \beta_1} \delta(\mu_1, \beta_2), \sup_{\mu_2 \in \beta_2} \delta(\mu_2, \beta_1)\}, \quad (1)$$

for all $\beta_1, \beta_2 \in CL(X)$ and $\delta(\mu_1, \beta_1) = \inf\{\delta(\mu_1, \mu_2) : \mu_2 \in \beta_1\}$. A point $\mu \in X$ is said to be a fixed point of

$T: X \rightarrow CB(X)$, if $\mu \in T(\mu)$. If, for $\mu_0 \in X$, there exists a sequence $\{\mu_i\}$ in X such that $\mu_i \in T(\mu_{i-1})$, then $O(T, \mu_0) = \{\mu_0, \mu_1, \mu_2, \dots\}$ is said to be an orbit of $T: X \rightarrow CB(X)$. Mapping $f: X \rightarrow \mathbb{R}$ is said to be T -orbitally lower semi-continuous (o.l.s.c), if a sequence $\{\mu_i\}$ in $O(T, \mu_0)$ and $\mu_i \rightarrow \varrho \Rightarrow f(\varrho) \leq \liminf_i f(\mu_i)$.

A multivalued mapping $T: X \rightarrow CB(X)$ is called a Nadler-contraction, if there exists $\gamma \in (0, 1)$ such that

$$H_d(T(\mu_1), T(\mu_2)) \leq \gamma \delta(\mu_1, \mu_2) \quad \text{for all } \mu_1, \mu_2 \in X. \quad (2)$$

Nadler [5] obtained the multivalued version of the Banach contraction principle. Let (X, δ) be a complete metric space and $T: X \rightarrow CB(X)$ be a Nadler-contraction. Then, T has a fixed point. Recently, Vetro [6] proved the following result to μ^* .

Theorem 1. Let (X, δ) be a complete metric space and $T: X \rightarrow CB(X)$ be a multivalued mapping. Suppose that there exist $\theta \in \mathbb{E}$ and $k \in (0, 1)$ such that

$$\begin{aligned} \mu_1, \mu_2 \in X, H_d(T(\mu_1), T(\mu_2)) &> 0 \\ \Rightarrow \theta[H_d(T(\mu_1), T(\mu_2))] &\leq [\theta(\delta(\mu_1, \mu_2))]^k, \end{aligned} \quad (3)$$

where Ξ is the set of mapping $\theta: (0, \infty) \longrightarrow (1, \infty)$ satisfying $(\theta_i) - (\theta_{iii})$:

- (i) $(\theta_i)\theta$ is nondecreasing and right-continuous.
- (ii) (θ_{ii}) For each $\{s_i\}$ in $(0, \infty)$, $\lim_{i \rightarrow \infty} \theta(s_i) = 1 \Leftrightarrow \lim_{i \rightarrow \infty} (s_i) = 0$.
- (iii) (θ_{iii}) There exist $r \in (0, 1)$ and $\mu \in (0, +\infty]$ such that $\lim_{s \rightarrow 0^+} \theta(s) - 1/s^r = \mu$. Then, T has at least one fixed point.

Remark 1. Let (X, δ) be a metric space. If $T: X \longrightarrow CB(X)$ is a multivalued mapping satisfying the above theorem, then

$$\ln \theta(H_d(T(\mu_1), T(\mu_2))) \leq k \ln \theta(\delta(\mu_1, \mu_2)) < \ln \theta(\delta(\mu_1, \mu_2)). \quad (4)$$

Since θ is nondecreasing, we obtain

$$H_d(T\mu_1, T\mu_2) < \delta(\mu_1, \mu_2), \quad \text{for all } \mu_1, \mu_2 \in X, T\mu_1 \neq T\mu_2. \quad (5)$$

Example 1. The functions $\theta_1, \theta_2: (0, \infty) \longrightarrow (1, \infty)$, defined by $\theta_1(r) = e^{\sqrt{r}}$ and $\theta_2(r) = 1 + \sqrt{r}$, are in Ξ .

Lemma 1 (see [6]). Let (X, δ) be a metric space and $\beta_1, \beta_2 \in CB(X)$ with $H_d(\beta_1, \beta_2) > 0$. Then, for all $h > 1$ and $\mu \in \beta_1$, there exists $\nu = \nu(\mu) \in \beta_2$ such that

$$\delta(\mu, \nu) < hH_d(\beta_1, \beta_2). \quad (6)$$

Lemma 2 (see [6]). Let (X, δ) be a metric space, $\beta_2 \in CB(X)$, and $\mu \in X$. Then, for each $\epsilon > 0$, there exists $\nu \in \beta_2$ such that

$$\delta(\mu, \nu) \leq \delta(\mu, \beta_2) + \epsilon. \quad (7)$$

Definition 1 (see [2]). Mapping $\Gamma: \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ is called a simulation function such that

- (G1) $\Gamma(0, 0) = 0$.
- (G2) $\Gamma(\mu_1, \mu_2) < \mu_2 - \mu_1$ for all $\mu_1, \mu_2 > 0$.
- (G3) If $\{\mu_{1i}\}, \{\mu_{2i}\} \in (0, \infty)$ such that $\lim_{i \rightarrow \infty} \mu_{1i} = \lim_{i \rightarrow \infty} \mu_{2i} > 0$, then

$$\limsup_{i \rightarrow \infty} \Gamma(\mu_{1i}, \mu_{2i}) < 0. \quad (8)$$

Due to (G2), we have $\Gamma(\mu_1, \mu_1) < 0$ for all $\mu_1 > 0$. Here, the set ∇ satisfies the conditions (G1)–(G3).

Example 2 (see [2]). For $j = 1, 2$, let $\vartheta_j: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be continuous functions such that $\vartheta_j(\mu_1) = 0 \Leftrightarrow \mu_1 = 0$. Functions $\Gamma_j: \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ ($j = 1, 2$) are in ∇ :

- (i) $\Gamma_1(\mu_1, \mu_2) = \vartheta_1(\mu_2) - \vartheta_2(\mu_1)$ for all $\mu_1, \mu_2 \geq 0$, where $\vartheta_1(\mu_1) \leq \mu_1 \leq \vartheta_2(\mu_1)$.

- (ii) $\Gamma_2(\mu_1, \mu_2) = \mu_2 - \int_0^{\mu_1} \zeta(u) du$ for all $\mu_1, \mu_2 \geq 0$, where $\zeta: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a function such that

$$\int_0^\epsilon \zeta(u) du \text{ exists and } \int_0^\epsilon \zeta(u) du > \epsilon \text{ for all } \epsilon > 0, \quad (9)$$

Definition 2 (see [1]). Let (X, δ) be a metric space and Λ be a nonempty subset of X , and $T: \Lambda \longrightarrow CB(X)$ is known as α -admissible, if there exists a mapping $\alpha: \Lambda \times \Lambda \longrightarrow [0, \infty)$ such that

$$\alpha(\beta_1, \beta_2) \geq 1 \Rightarrow \alpha(\mu, \nu) \geq 1, \quad (10)$$

for all $\mu \in T(\beta_1) \cap \Lambda$ and $\nu \in T(\beta_2) \cap \Lambda$.

Lemma 3 (see [7]). Suppose there is a point $\mu_0 \in \Lambda$ (Λ is a closed subset of X) that satisfies

$$\delta(\mu_0, T(\mu_0)) \in \hat{E}, \quad (11)$$

and $\mu_i \in \Lambda$ for some $i \geq 0$. Then, $\delta(\mu_i, T(\mu_i)) \in \hat{E}$ where \hat{E} denotes an interval on \mathbb{R}^+ containing 0.

Definition 3 (see [7]). (inclusion ball) Suppose $\mu_0 \in \Lambda$ and $\delta(\mu_0, T(\mu_0)) \in \hat{E}$. Then, for all iterates μ_i ($i \geq 0$) which belongs to Λ , we define the closed-ball $\bar{b}(\mu_i, \rho_i)$ with center μ_i and radius $\rho_i = \sigma(\delta(\mu_i, T(\mu_i)))$, where $\sigma: \hat{E} \longrightarrow \mathbb{R}_+$ is defined by (13).

Lemma 4 (see [7]). Suppose there is a point $\mu_0 \in \Lambda$ that satisfies $\delta(\mu_0, T(\mu_0)) \in \hat{E}$ and $\bar{b}(\mu_i, \rho_i) \subset \Lambda$ for some $i \geq 0$; then, $\mu_{i+1} \in \Lambda$ and $\bar{b}(\mu_{i+1}, \rho_{i+1}) \subset \bar{b}(\mu_i, \rho_i)$.

Definition 4 (see [7]). Let $j \geq 1$, and $\eta: \hat{E} \longrightarrow \hat{E}$ is known as a gauge function of order j on \hat{E} , if it satisfies the following conditions:

- (i) $\eta(\lambda\mu) < \lambda^j \eta(\mu)$ for all $\lambda \in (0, 1)$ and $\mu \in \hat{E}$.
- (ii) $\eta(\mu) < \mu$ for all $\mu \in \hat{E} - \{0\}$.

Note that the first condition of Definition 4 is equivalent to $\eta(0) = 0$ and $\eta(\mu)/\mu^j$ is nondecreasing on $\hat{E} - \{0\}$.

Definition 5 (see [7]). A gauge function $\eta: \hat{E} \longrightarrow \hat{E}$ is said to be a Bianchini–Grandolfi gauge function on \hat{E} if

$$\sigma(\mu) = \sum_{i=0}^{\infty} \eta^i(\mu) < \infty, \quad \text{for all } \mu \in \hat{E}. \quad (12)$$

Note that a Bianchini–Grandolfi gauge function also satisfies the following functional equation:

$$\sigma(\mu) = \sigma(\eta(\mu)) + \mu. \quad (13)$$

2. Set-Valued SU_{η} -Contraction

The first main definition of this exposition is as follows.

Definition 6. Let (X, δ) be a metric space, Λ be a closed subset of X , and η be a Bianchini–Grandolfi gauge function on \widehat{E} . A mapping $T: \Lambda \longrightarrow \text{CB}(X)$ is known as set-valued SU θ_η -contraction, if there exists $\theta \in \Xi$ such that for $T(\mu) \cap \Lambda \neq \emptyset$,

$$\frac{1}{2} \min\{\delta(\mu, T(\mu) \cap \Lambda), \delta(\nu, T(\nu) \cap \Lambda)\} < \delta(\mu, \nu), \quad (14)$$

which implies that

$$\theta[H_d(T(\mu) \cap \Lambda, T(\nu) \cap \Lambda)] \leq [\theta(\eta(\Omega(\mu, \nu)))]^k, \quad (15)$$

where

$$\Omega(\mu, \nu) = \max\left\{\delta(\mu, \nu), \delta(\mu, T(\mu)), \delta(\nu, T(\nu)), \frac{\delta(\mu, T(\nu)) + \delta(\nu, T(\mu))}{2}\right\}, \quad (16)$$

for all $\mu \in \Lambda, \nu \in T(\mu) \cap \Lambda$ with $\delta(\mu, \nu) \in \widehat{E}$, where $k \in (0, 1)$.

Theorem 2. Let (X, δ) be a complete metric space and $T: \Lambda \longrightarrow \text{CB}(X)$ be a multivalued SU-contraction. Suppose $\mu_0 \in \Lambda$ such that $\delta(\mu_0, c^*) \in \widehat{E}$ for some $c^* \in T(\mu_0) \cap \Lambda$. Then, there exists an orbit $\{\mu_i\}$ of T in Λ and $\vartheta^* \in \Lambda$ such that $\lim_{i \rightarrow \infty} \mu_i = \vartheta^*$. In addition, ϑ^* is a fixed point of T if and only if the function $g(\mu) = \delta(\mu, T(\mu) \cap \Lambda)$ is T -o.l.s.c at ϑ^* .

Proof. Choose $\mu_1 = c^* \in T(\mu_0) \cap \Lambda$. In the case that $\delta(\mu_0, \mu_1) = 0$, μ_0 is a fixed point of T . Thus, we assume that $\delta(\mu_0, \mu_1) \neq 0$. On the other hand, we have

$$\frac{1}{2} \min\{\delta(\mu_0, T(\mu_0) \cap \Lambda), \delta(\mu_1, T(\mu_1) \cap \Lambda)\} < \delta(\mu_0, \mu_1). \quad (17)$$

Define $\rho = \sigma(\delta(\mu_0, \mu_1))$. From (13), we have $\sigma(r) \geq r$. Hence, $\delta(\mu_0, \mu_1) \leq \rho$, and so $\mu_1 \in \bar{b}(\mu_0, \rho)$. Since $\delta(\mu_0, \mu_1) \in \widehat{E}$, from (15) and (17), it follows that

$$\theta[H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda)] \leq [\theta(\eta(\delta(\mu_0, \mu_1)))]^k < [\theta(\Omega(\mu_0, \mu_1))]^k. \quad (18)$$

By right continuity of θ , there exists a real number $h_1 > 1$ such that

$$\theta[h_1 H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda)] \leq [\theta(\Omega(\mu_0, \mu_1))]^k. \quad (19)$$

From

$$\delta(\mu_1, T(\mu_1) \cap \Lambda) \leq H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda) < h_1 H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda), \quad (20)$$

by Lemma 1, there exists $\mu_2 \in T(\mu_1) \cap \Lambda$ such that $\delta(\mu_1, \mu_2) \leq h_1 H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda)$. Since θ is nondecreasing, by (19), this inequality gives that

$$\theta(\delta(\mu_1, \mu_2)) \leq \theta[h_1 H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda)] \leq [\theta(\Omega(\mu_0, \mu_1))]^k, \quad (21)$$

where

$$\begin{aligned} \Omega(\mu_0, \mu_1) &= \max\left\{\delta(\mu_0, \mu_1), \delta(\mu_0, T(\mu_0)), \delta(\mu_1, T(\mu_1)), \frac{\delta(\mu_0, T(\mu_1)) + \delta(\mu_1, T(\mu_0))}{2}\right\} \\ &\leq \max\left\{\delta(\mu_0, \mu_1), \delta(\mu_1, T(\mu_1)), \frac{\delta(\mu_0, T(\mu_1))}{2}\right\} \\ &\leq \max\{\delta(\mu_0, \mu_1), \delta(\mu_1, T(\mu_1))\}. \end{aligned} \quad (22)$$

We claim that

$$\begin{aligned} \theta(\delta(\mu_1, \mu_2)) &\leq \theta[h_1 H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda)] \\ &\leq [\theta(\delta(\mu_0, \mu_1))]^k. \end{aligned} \quad (23)$$

Let $\Phi = \max\{\delta(\mu_0, \mu_1), \delta(\mu_1, T(\mu_1))\}$. If $\Phi = \delta(\mu_1, T(\mu_1))$, we have $\mu_2 \in T(\mu_1) \cap \Lambda$, so we obtain

$$\theta(\delta(\mu_1, \mu_2)) \leq \theta[h_1 H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda)] \leq [\theta(\delta(\mu_1, \mu_2))]^k, \quad (24)$$

which is a contradiction. Thus, we conclude that $\Phi = \delta(\mu_0, \mu_1)$. We assume that $\delta(\mu_1, \mu_2) \neq 0$; otherwise, μ_1 is a fixed point of T . From Remark 1, we have $\delta(\mu_1, \mu_2) < \delta(\mu_0, \mu_1)$, and so $\delta(\mu_1, \mu_2) \in \widehat{E}$. Next, $\mu_2 \in \bar{b}(\mu_0, \rho)$ because

$$\begin{aligned} \delta(\mu_0, \mu_2) &\leq \delta(\mu_0, \mu_1) + \delta(\mu_1, \mu_2) \\ &\leq \delta(\mu_0, \mu_1) + \eta(\delta(\mu_0, \mu_1)) \\ &\leq \delta(\mu_0, \mu_1) + \sigma(\eta(\delta(\mu_0, \mu_1))) \\ &= \sigma(\delta(\mu_0, \mu_1)) = \rho. \end{aligned} \quad (25)$$

Also, since

$$\frac{1}{2} \min\{\check{\delta}(\mu_1, T(\mu_1) \cap \Lambda), \check{\delta}(\mu_2, T(\mu_2) \cap \Lambda)\} < \check{\delta}(\mu_1, \mu_2), \quad (26)$$

from (15), we get

$$\theta[H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)] \leq [\theta(\eta(\check{\delta}(\mu_1, \mu_2)))]^k < [\theta(\Omega(\mu_1, \mu_2))]^k. \quad (27)$$

Since θ is right-continuous, there exists a real number $h_2 > 1$ such that

$$\theta[h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)] \leq [\theta(\Omega(\mu_1, \mu_2))]^k. \quad (28)$$

Next, from

$$\check{\delta}(\mu_2, T(\mu_2) \cap \Lambda) \leq H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda) < h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda), \quad (29)$$

by Lemma 1, there exists $\mu_3 \in T(\mu_2) \cap \Lambda$ such that $\check{\delta}(\mu_2, \mu_3) \leq h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)$. By (28), this inequality gives that

$$\theta(\check{\delta}(\mu_2, \mu_3)) \leq \theta[h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)] \leq [\theta(\Omega(\mu_1, \mu_2))]^k \leq [\theta(\Omega(\mu_0, \mu_1))]^{k^2}. \quad (30)$$

where

$$\begin{aligned} \Omega(\mu_1, \mu_2) &= \max\left\{\check{\delta}(\mu_1, \mu_2), \check{\delta}(\mu_1, T(\mu_1)), \check{\delta}(\mu_2, T(\mu_2)), \frac{\check{\delta}(\mu_1, T(\mu_2)) + \check{\delta}(\mu_2, T(\mu_1))}{2}\right\} \\ &\leq \max\left\{\check{\delta}(\mu_1, \mu_2), \check{\delta}(\mu_2, T(\mu_2)), \frac{\check{\delta}(\mu_1, T(\mu_2))}{2}\right\} \\ &\leq \max\{\check{\delta}(\mu_1, \mu_2), \check{\delta}(\mu_2, T(\mu_2))\}. \end{aligned} \quad (31)$$

We claim that

$$\begin{aligned} \theta(\check{\delta}(\mu_2, \mu_3)) &\leq \theta[h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)] \\ &\leq [\theta(\check{\delta}(\mu_1, \mu_2))]^k \leq [\theta(\check{\delta}(\mu_0, \mu_1))]^{k^2} \end{aligned} \quad (32)$$

Let $\Phi = \max\{\check{\delta}(\mu_1, \mu_2), \check{\delta}(\mu_2, T(\mu_2))\}$. If $\Phi = \check{\delta}(\mu_2, T(\mu_2))$, we have $\mu_3 \in T(\mu_2) \cap \Lambda$, so we obtain

$$\theta(\check{\delta}(\mu_2, \mu_3)) \leq \theta[h_2 H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda)] \leq [\theta(\check{\delta}(\mu_2, \mu_3))]^k, \quad (33)$$

which is a contradiction. Thus, we conclude that $\Phi = \check{\delta}(\mu_1, \mu_2)$. We assume that $\check{\delta}(\mu_2, \mu_3) \neq 0$; otherwise, μ_2 is a fixed point of T . From Remark 1, we have $\check{\delta}(\mu_2, \mu_3) < \check{\delta}(\mu_1, \mu_2)$, and so $\check{\delta}(\mu_2, \mu_3) \in \hat{E}$. Also, we have $\mu_3 \in \bar{b}(\mu_0, \rho)$, since

$$\begin{aligned} \check{\delta}(\mu_0, \mu_3) &\leq \check{\delta}(\mu_0, \mu_1) + \check{\delta}(\mu_1, \mu_2) + \check{\delta}(\mu_2, \mu_3) \\ &\leq \check{\delta}(\mu_0, \mu_1) + \eta(\check{\delta}(\mu_0, \mu_1)) + \eta^2(\check{\delta}(\mu_0, \mu_1)) \\ &\leq \sum_{i=0}^{\infty} \eta^i(\check{\delta}(\mu_0, \mu_1)) \\ &= \sigma(\check{\delta}(\mu_0, \mu_1)) = \rho. \end{aligned} \quad (34)$$

Continuing this setup, we have two sequences $\{\mu_i\} \subset \bar{b}(\mu_0, \rho)$ and $\{h_i\} \subset (1, \infty)$ such that $\mu_{i+1} \in T(\mu_i) \cap \Lambda$, $\mu_i \neq \mu_{i+1}$ with $\check{\delta}(\mu_i, \mu_{i+1}) \in \hat{E}$ and

$$\begin{aligned} 1 < \theta(\check{\delta}(\mu_i, \mu_{i+1})) &\leq \theta(h_i H_d(T(\mu_{i-1}) \cap \Lambda, T(\mu_i) \cap \Lambda)) \\ &\leq [\theta(\check{\delta}(\mu_{i-1}, \mu_i))]^k, \end{aligned} \quad (35)$$

for all $i \in \mathbb{N}$. Then,

$$1 < \theta(\check{\delta}(\mu_i, \mu_{i+1})) \leq [\theta(\check{\delta}(\mu_0, \mu_1))]^{k^i}, \quad \text{for all } i \in \mathbb{N}, \quad (36)$$

which gives that

$$\lim_{i \rightarrow \infty} \theta(\check{\delta}(\mu_i, \mu_{i+1})) = 1, \quad (37)$$

and by (θ_{ii}) , we have

$$\lim_{i \rightarrow \infty} \check{\delta}(\mu_i, \mu_{i+1}) = 0. \quad (38)$$

Next, we prove that $\{\mu_i\}$ is a Cauchy sequence in X . Setting $\delta_i := \check{\delta}(\mu_i, \mu_{i+1})$, from (θ_{iii}) , there exist $r \in (0, 1)$ and $\mu \in (0, \infty]$ such that

$$\lim_{i \rightarrow \infty} \frac{\theta(\delta_i) - 1}{(\delta_i)^r} = \mu. \quad (39)$$

Take $\lambda \in (0, \mu)$. From the definition of limit, there exists $i_0 \in \mathbb{N}$ such that

$$[\delta_i]^r \leq \lambda^{-1} [\theta(\delta_i) - 1], \quad \text{for all } i > i_0. \quad (40)$$

Using (36) and the above inequality,

$$i[\delta_i]^r \leq \lambda^{-1} i([\theta(\delta_0)]^{k^i} - 1), \quad \text{for all } i > i_0. \quad (41)$$

This implies that

$$\lim_{i \rightarrow \infty} \iota[\delta_i]^r = \lim_{i \rightarrow \infty} \iota[d(\mu_i, \mu_{i+1})]^r = 0. \quad (42)$$

Hence, there exists $\iota_1 \in \mathbb{N}$ such that

$$d(\mu_i, \mu_{i+1}) \leq \frac{1}{i^{1/r}}, \quad \text{for all } i > \iota_1. \quad (43)$$

Let $p > \iota > \iota_1$. Then, using the triangular inequality and (43), we get

$$\delta(\mu_i, \mu_p) \leq \sum_{j=i}^{p-1} \delta(\mu_j, \mu_{j+1}) \leq \sum_{j=i}^{p-1} \frac{1}{j^{1/r}} < \sum_{j=i}^{\infty} \frac{1}{j^{1/r}}. \quad (44)$$

Owing to the convergence of the series $\sum_{j=i}^{\infty} 1/j^{1/r}$, $\{\mu_i\}$ is a Cauchy sequence in $\bar{b}(\mu_0, \rho)$. Since $\bar{b}(\mu_0, \rho)$ is closed in X , there exists $\vartheta^* \in \bar{b}(\mu_0, \rho)$ such that $\mu_i \rightarrow \vartheta^*$. Note that $\vartheta^* \in \Lambda$ because $\mu_{i+1} \in T(\mu_i) \cap \Lambda$. Now, we claim that

$$\frac{1}{2} \min\{\delta(\mu_i, T(\mu_i) \cap \Lambda), \delta(\vartheta^*, T(\vartheta^*) \cap \Lambda)\} < \delta(\mu_i, \vartheta^*), \quad (45)$$

or

$$\frac{1}{2} \min\{\delta(\vartheta^*, T(\vartheta^*) \cap \Lambda), \delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)\} < \delta(\mu_{i+1}, \vartheta^*), \quad (46)$$

for all $i \in \mathbb{N}$. Assume, on the contrary, that there exists $i' \in \mathbb{N}$ such that

$$\frac{1}{2} \min\{\delta(\mu_{i'}, T(\mu_{i'}) \cap \Lambda), \delta(\vartheta^*, T(\vartheta^*) \cap \Lambda)\} \geq \delta(\mu_{i'}, \vartheta^*), \quad (47)$$

and

$$\frac{1}{2} \min\{\delta(\vartheta^*, T(\vartheta^*) \cap \Lambda), \delta(\mu_{i'+1}, T(\mu_{i'+1}) \cap \Lambda)\} \geq \delta(\mu_{i'+1}, \vartheta^*). \quad (48)$$

By (47), we have

$$\begin{aligned} 2\delta(\mu_{i'}, \vartheta^*) &\leq \min\{\delta(\mu_{i'}, T(\mu_{i'}) \cap \Lambda), \delta(\vartheta^*, T(\vartheta^*) \cap \Lambda)\} \\ &\leq \min\{[\delta(\mu_{i'}, \vartheta^*) + \delta(\vartheta^*, T(\mu_{i'}) \cap \Lambda)], \\ &\quad \delta(\vartheta^*, T(\vartheta^*) \cap \Lambda)\} \\ &\leq [\delta(\mu_{i'}, \vartheta^*) + \delta(\vartheta^*, T(\mu_{i'}) \cap \Lambda)] \\ &< [\delta(\mu_{i'}, \vartheta^*) + \delta(\vartheta^*, T(\mu_{i'}) \cap \Lambda)] \\ &\leq [\delta(\mu_{i'}, \vartheta^*) + \delta(\vartheta^*, \mu_{i'+1})], \end{aligned} \quad (49)$$

which implies that

$$\delta(\mu_{i'}, \vartheta^*) \leq \delta(\vartheta^*, \mu_{i'+1}), \quad (50)$$

which together with (38) gives

$$\begin{aligned} \delta(\mu_{i'}, \vartheta^*) &\leq \delta(\vartheta^*, \mu_{i'+1}) \\ &\leq \frac{1}{2} \min\{\delta(\vartheta^*, T(\vartheta^*) \cap \Lambda), \delta(\mu_{i'+1}, T(\mu_{i'+1}) \cap \Lambda)\}. \end{aligned} \quad (51)$$

Since

$$\frac{1}{2} \min\{\delta(\mu_{i'}, T(\mu_{i'}) \cap \Lambda), \delta(\mu_{i'+1}, T(\mu_{i'+1}) \cap \Lambda)\} < \delta(\mu_{i'}, \mu_{i'+1}), \quad (52)$$

from contractive condition (15), we have

$$\begin{aligned} \theta(\delta(\mu_{i'+1}, \mu_{i'+2})) &\leq \theta[h_2 H_d(T(\mu_{i'}) \cap \Lambda, T(\mu_{i'+1}) \cap \Lambda)] \\ &\leq [\theta(\Omega(\mu_{i'}, \mu_{i'+1}))]^k, \end{aligned} \quad (53)$$

where

$$\begin{aligned} \Omega(\mu_{i'}, \mu_{i'+1}) &= \max \left\{ \begin{array}{l} \delta(\mu_{i'}, \mu_{i'+1}), \delta(\mu_{i'}, T(\mu_{i'})), \delta(\mu_{i'+1}, T(\mu_{i'+1})), \\ \frac{\delta(\mu_{i'}, T(\mu_{i'+1})) + \delta(\mu_{i'+1}, T(\mu_{i'}))}{2} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} \delta(\mu_{i'}, \mu_{i'+1}), \delta(\mu_{i'+1}, \mu_{i'+2}), \\ \frac{\delta(\mu_{i'}, \mu_{i'+2})}{2} \end{array} \right\} \\ &\leq \max\{\delta(\mu_{i'}, \mu_{i'+1}), \delta(\mu_{i'+1}, \mu_{i'+2})\}. \end{aligned} \quad (54)$$

We claim that

$$\begin{aligned} \theta(\delta(\mu_{i'+1}, \mu_{i'+2})) &\leq \theta[h_2 H_d(T(\mu_{i'}) \cap \Lambda, T(\mu_{i'+1}) \cap \Lambda)] \\ &\leq [\theta(\delta(\mu_{i'}, \mu_{i'+1}))]^k. \end{aligned} \quad (55)$$

Let $\Delta = \max\{\delta(\mu_{i'}, \mu_{i'+1}), \delta(\mu_{i'+1}, \mu_{i'+2})\}$. If $\Delta = \delta(\mu_{i'}, \mu_{i'+1})$, Since $\mu_{i'+2} \in T(\mu_{i'+1}) \cap \Lambda$, we have

$$\begin{aligned} \theta(\delta(\mu_{i'+1}, \mu_{i'+2})) &\leq \theta[h_2 H_d(T(\mu_{i'}) \cap \Lambda, T(\mu_{i'+1}) \cap \Lambda)] \\ &\leq [\theta(\delta(\mu_{i'+1}, \mu_{i'+2}))]^k, \end{aligned} \quad (56)$$

which is a contradiction. Thus, we conclude that $\Delta = \delta(\mu_{i'}, \mu_{i'+1})$. From Remark 1, we have

$$\delta(\mu_{i'+1}, \mu_{i'+2}) < \delta(\mu_{i'}, \mu_{i'+1}). \quad (57)$$

From (38), (43), and (47), we obtain

$$\begin{aligned} \delta(\mu_{i'+1}, \mu_{i'+2}) &< \delta(\mu_{i'}, \mu_{i'+1}) \\ &\leq [\delta(\mu_{i'}, \vartheta^*) + \delta(\vartheta^*, \mu_{i'+1})] \\ &\leq \left[\begin{array}{l} \frac{1}{2} \min\{\delta(\vartheta^*, T(\vartheta^*) \cap \Lambda), \delta(\mu_{i'+1}, T(\mu_{i'+1}) \cap \Lambda)\} \\ + \frac{1}{2} \min\{\delta(\vartheta^*, T(\vartheta^*) \cap \Lambda), \delta(\mu_{i'+1}, T(\mu_{i'+1}) \cap \Lambda)\} \end{array} \right] \\ &\leq \min\{\delta(\vartheta^*, T(\vartheta^*) \cap \Lambda), \delta(\mu_{i'+1}, T(\mu_{i'+1}) \cap \Lambda)\} \\ &\leq \delta(\mu_{i'+1}, T(\mu_{i'+1}) \cap \Lambda), \end{aligned} \quad (58)$$

which is a contradiction. Hence, (45) holds true:

$$\frac{1}{2} \min\{\check{\delta}(\mu_i, T(\mu_i) \cap \Lambda), \check{\delta}(\vartheta^*, T(\vartheta^*) \cap \Lambda)\} < \check{\delta}(\mu_i, \vartheta^*) \quad \text{for all } i \geq 2. \quad (59)$$

Also, we know that $\check{\delta}(\mu_i, \mu_{i+1}) \in \widehat{E}$ for all n . Thus, from (15), we have

$$\begin{aligned} \theta(\check{\delta}(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)) &\leq \theta[H_d(T(\mu_i) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda)] \\ &\leq [\theta(\eta(\Omega(\mu_i, \mu_{i+1})))^k] \\ &< \theta[(\Omega(\mu_i, \mu_{i+1}))^k], \end{aligned} \quad (60)$$

where

$$\begin{aligned} \Omega(\mu_i, \mu_{i+1}) &= \max \left\{ \begin{array}{l} \check{\delta}(\mu_i, \mu_{i+1}), \check{\delta}(\mu_i, T(\mu_i)), \check{\delta}(\mu_{i+1}, T(\mu_{i+1})), \\ \frac{\check{\delta}(\mu_i, T(\mu_{i+1})) + \check{\delta}(\mu_{i+1}, T(\mu_i))}{2} \end{array} \right\} \\ &\leq \max \left\{ \begin{array}{l} \check{\delta}(\mu_i, \mu_{i+1}), \check{\delta}(\mu_{i+1}, \mu_{i+2}), \\ \frac{\check{\delta}(\mu_i, \mu_{i+2})}{2} \end{array} \right\} \\ &\leq \max\{\check{\delta}(\mu_i, \mu_{i+1}), \check{\delta}(\mu_{i+1}, \mu_{i+2})\}. \end{aligned} \quad (61)$$

We claim that

$$\begin{aligned} \theta(\check{\delta}(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)) &\leq \theta[H_d(T(\mu_i) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda)] \\ &< \theta[(\check{\delta}(\mu_i, \mu_{i+1}))^k]. \end{aligned} \quad (62)$$

Let $\Phi = \max\{\check{\delta}(\mu_i, \mu_{i+1}), \check{\delta}(\mu_{i+1}, \mu_{i+2})\}$. If $\Phi = \check{\delta}(\mu_{i+1}, \mu_{i+2})$, we have $\mu_{i+2} \in T(\mu_{i+1}) \cap \Lambda$, so we obtain

$$\begin{aligned} \theta(\check{\delta}(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)) &\leq \theta[H_d(T(\mu_i) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda)] \\ &< \theta[(\check{\delta}(\mu_{i+1}, \mu_{i+2}))^k], \end{aligned} \quad (63)$$

which is a contradiction. Thus, we obtain $\Phi = \check{\delta}(\mu_i, \mu_{i+1})$. From Remark 1, we deduce

$$\check{\delta}(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) < \check{\delta}(\mu_i, \mu_{i+1}). \quad (64)$$

Taking limit $i \rightarrow \infty$ in (64),

$$\lim_{i \rightarrow \infty} \check{\delta}(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) = 0. \quad (65)$$

Since $g(\mu) = \check{\delta}(\mu, T(\mu) \cap \Lambda)$ is T -o.l.s.c at ϑ^* , then

$$\begin{aligned} \check{\delta}(\vartheta^*, T(\vartheta^*) \cap \Lambda) &= g(\vartheta^*) \leq \liminf_i g(\mu_{i+1}) \\ &= \liminf_i \check{\delta}(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) = 0. \end{aligned} \quad (66)$$

Since $T(\vartheta^*)$ is closed, we have $\vartheta^* \in T(\vartheta^*)$. Conversely, if ϑ^* is a fixed point of T , then $g(\vartheta^*) = 0 \leq \liminf_i g(\mu_i)$, since $\vartheta^* \in \Lambda$. \square

Corollary 1. Let $(X, \check{\delta})$ be a complete metric space, η be a Bianchini–Grandolfi gauge function on an interval \widehat{E} , and $T: \Lambda \rightarrow CB(X)$ be a given set-valued mapping. If $k \in (0, 1)$ and $T(\mu) \cap \Lambda \neq \emptyset$ exist,

$$\frac{1}{2} \min\{\check{\delta}(\mu, T(\mu) \cap \Lambda), \check{\delta}(\nu, T(\nu) \cap \Lambda)\} < \check{\delta}(\mu, \nu), \quad (67)$$

which implies that

$$\theta[H_d(T(\mu) \cap \Lambda, T(\nu) \cap \Lambda)] \leq [\theta(\eta(\check{\delta}(\mu, \nu)))^k], \quad (68)$$

for all $\mu \in \Lambda$, $\nu \in T(\mu) \cap \Lambda$ with $\check{\delta}(\mu, \nu) \in \widehat{E}$. Suppose $\mu_0 \in \Lambda$ such that $\check{\delta}(\mu_0, c^*) \in \widehat{E}$ for some $c^* \in T(\mu_0) \cap \Lambda$. Then, there exists an orbit $\{\mu_i\}$ of T in Λ and $\vartheta^* \in \Lambda$ such that $\lim_{i \rightarrow \infty} \mu_i = \vartheta^*$. In addition, ϑ^* is a fixed point of T if and only if the function $g(\mu) = \check{\delta}(\mu, T(\mu) \cap \Lambda)$ is T -o.l.s.c at point ϑ^* .

Corollary 2. Let $(X, \check{\delta})$ be a complete metric space, η be a Bianchini–Grandolfi gauge function on an interval \widehat{E} , and $T: \Lambda \rightarrow CB(X)$ be a given set-valued mapping. If $k \in (0, 1)$ and for $T\mu \cap \Lambda \neq \emptyset$ exist,

$$\frac{1}{2} \min\{\check{\delta}(\mu, T(\mu) \cap \Lambda), \check{\delta}(\nu, T(\nu) \cap \Lambda)\} < \check{\delta}(\mu, \nu), \quad (69)$$

implies that

$$\sqrt{H_d(T(\mu) \cap \Lambda, T(\nu) \cap \Lambda)} \leq k \sqrt{\eta(\check{\delta}(\mu, \nu))}, \quad (70)$$

for all $\mu \in \Lambda$, $\nu \in T(\mu) \cap \Lambda$, and $\check{\delta}(\mu, \nu) \in \hat{E}$. In addition, suppose $\mu_0 \in \Lambda$ such that $\check{\delta}(\mu_0, c^*) \in \hat{E}$ for some $c^* \in T(\mu_0) \cap \Lambda$. Then, there exists an orbit $\{\mu_i\}$ of T in Λ , $\vartheta^* \in \Lambda$ such that $\lim_{i \rightarrow \infty} \mu_i = \vartheta^*$ and ϑ^* is a fixed point of T if and only if the function $g(\mu) := \check{\delta}(\mu, T(\mu) \cap \Lambda)$ is T -o.l.s.c at ϑ^* .

Corollary 3. Let $(X, \check{\delta})$ be a complete metric space, η be a Bianchini–Grandolfi gauge function on \hat{E} , and $T: \Lambda \rightarrow \text{CB}(X)$ be a given set-valued mapping. If $\theta \in \Xi$ and $k \in (0, 1)$ exist, then

$$\frac{1}{2} \min\{\check{\delta}(\mu, T(\mu) \cap \Lambda), \check{\delta}(\nu, T(\nu) \cap \Lambda)\} < \check{\delta}(\mu, \nu), \quad (71)$$

$$\Rightarrow \theta[H_d(T(\mu), T(\nu))] \leq [\theta(\eta(\check{\delta}(\mu, \nu)))]^k,$$

for all $\mu \in X$, $\nu \in T(\mu)$, and $\check{\delta}(\mu, \nu) \in \hat{E}$. Suppose that $\mu_0 \in X$ such that $\check{\delta}(\mu_0, c^*) \in \hat{E}$ for some $c^* \in T\mu_0$. Then, there exists an orbit $\{\mu_i\}$ of T in X that converges to the fixed point $\vartheta^* \in \mathcal{F} = \{\mu \in X: \check{\delta}(\mu, \vartheta^*) \in \hat{E}\}$ of T .

Example 3. Let $X = [-10, \infty)$ be an usual metric $\check{\delta}$ and let $\hat{E} = [0, \infty)$. Mapping $T: \Lambda \rightarrow \text{CB}(X)$ is defined as

$$T(\mu) = \begin{cases} \left[0, \frac{\mu}{8}\right], & \mu \in [0, 4], \\ \{0, \mu\}, & \mu \in [-10, 0) \cup (4, \infty). \end{cases} \quad (72)$$

Clearly, $\frac{1}{2} \min\{\check{\delta}(\mu, T(\mu) \cap \Lambda), \check{\delta}(\nu, T(\nu) \cap \Lambda)\} < \check{\delta}(\mu, \nu)$ if and only if $\mu, \nu \in [0, 4]$. Let $\mu_0 = 4$; then, we have $c^* = 1/2 \in T(\mu_0)$ such that $\check{\delta}(\mu_0, c^*) \in \hat{E}$. Firstly, we claim that T satisfies inequality (68) with setting $\theta(r) = e^{\sqrt{r}e^r}$, $\eta(r) = r/2$, and $k = 1/2$. For $\mu \in [0, 4]$ and $\nu \in T(\mu)$, we obtain

$$\begin{aligned} \theta[H_d(T(\mu), T(\nu))] &= \theta\left(\frac{|\mu - \nu|}{8}\right) \\ &= e^{\sqrt{|\mu - \nu|/8}e^{|\mu - \nu|/8}} \\ &\leq e^{1/2\sqrt{|\mu - \nu|/2}e^{|\mu - \nu|/2}} \\ &= e^{1/2\sqrt{\eta(\hat{d}(\nu, \mu))}e^{\eta(\hat{d}(\nu, \mu))}} \\ &= [\theta(\eta(\check{\delta}(\mu, \nu)))]^k. \end{aligned} \quad (73)$$

Consequently, the requirements of Corollary 1 are fulfilled and 0 is a fixed point of T . For $\mu = 0$ and $\nu = 5$,

$$\begin{aligned} \theta[H_d(T(\mu), T(\nu))] &= \theta[H_d(T(0), T(5))] \\ &= \theta(5) > [\theta(5)]^k = [\theta(\check{\delta}(\mu, \nu))]^k, \end{aligned} \quad (74)$$

for all $\theta \in \Xi$ and $k \in (0, 1)$. Then, Corollary 1 cannot be satisfied.

3. Set-Valued SU-Type Γ_α -Contraction

In this section, we prove the existence of fixed point in the class of metric space with respect to a simulation function.

Definition 7. Let $(X, \check{\delta})$ be a metric space, Λ be a closed subset of X , and η be a Bianchini–Grandolfi gauge function on \hat{E} . Mapping $T: \Lambda \rightarrow \text{CB}(X)$ is known as set-valued SU-type Γ_α -contraction, if there exists $\Gamma \in \nabla$ such that for $T(\mu) \cap \Lambda \neq \emptyset$,

$$\frac{1}{2} \min\{\check{\delta}(\mu, T(\mu) \cap \Lambda), \check{\delta}(\nu, T(\nu) \cap \Lambda)\} < \check{\delta}(\mu, \nu), \quad (75)$$

which implies that

$$\Gamma[\alpha(\mu, \nu)H_d(T(\mu) \cap \Lambda, T(\nu) \cap \Lambda), \eta(\Omega(\mu, \nu))] \geq 0, \quad (76)$$

where

$$\Omega(\mu, \nu) = \max\left\{\check{\delta}(\mu, \nu), \check{\delta}(\mu, T(\mu)), \check{\delta}(\nu, T(\nu)), \frac{\check{\delta}(\mu, T(\nu)) + \check{\delta}(\nu, T(\mu))}{2}\right\}, \quad (77)$$

for all $\mu \in \Lambda$, $\nu \in T(\mu) \cap \Lambda$ with $\check{\delta}(\mu, \nu) \in \hat{E}$.

Theorem 3. Let $(X, \check{\delta})$ be a complete metric space and $T: \Lambda \rightarrow \text{CB}(X)$ be a set-valued SU-type Γ_α -contraction such that the following conditions are satisfied:

(a) T is α -admissible.

(b) There exists $\mu_0 \in \Lambda$ with $\check{\delta}(\mu_0, \mu_1) \in \hat{E}$ for some $\mu_1 \in T(\mu_0) \cap \Lambda$ such that $\alpha(\mu_0, \mu_1) \geq 1$. Then, there exists an orbit $\{\mu_i\}$ of T in Λ and $\vartheta^* \in \Lambda$ such that $\lim_{i \rightarrow \infty} \mu_i = \vartheta^*$. In addition, ϑ^* is a fixed point of T if

and only if the function $g(\mu) := \check{\delta}(\mu, T(\mu) \cap \Lambda)$ is T -o.l.s.c at ϑ^* .

Proof. By the hypothesis, there exists $\mu_0 \in \Lambda$ with $\check{\delta}(\mu_0, \mu_1) \in \hat{E}$ for some $\mu_1 \in T(\mu_0) \cap \Lambda$ such that $\alpha(\mu_0, \mu_1) \geq 1$. On the other hand, we have

$$\frac{1}{2} \min\{\check{\delta}(\mu_0, T(\mu_0) \cap \Lambda), \check{\delta}(\mu_1, T(\mu_1) \cap \Lambda)\} < \check{\delta}(\mu_0, \mu_1). \quad (78)$$

In the case that $\check{\delta}(\mu_0, \mu_1) = 0$, μ_0 is a fixed point of T . Thus, we assume that $\check{\delta}(\mu_0, \mu_1) \neq 0$. Define $\rho = \sigma(\check{\delta}(\mu_0, \mu_1))$. From (13), we have $\sigma(r) \geq r$. Hence, $\check{\delta}(\mu_0, \mu_1) \leq \rho$, and so $\mu_1 \in \bar{b}(\mu_0, \rho)$. Since $\alpha(\mu_0, \mu_1) \geq 1$ and $\check{\delta}(\mu_0, \mu_1) \in \hat{E}$, from (76) and (78), it follows that

$$\begin{aligned} 0 &\leq \Gamma[\alpha(\mu_0, \mu_1)H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda), \eta(\check{\delta}(\mu_0, \mu_1))] \\ &< \eta(\Omega(\mu_0, \mu_1)) - \alpha(\mu_0, \mu_1)H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda), \end{aligned} \quad (79)$$

which implies that

$$\alpha(\mu_0, \mu_1)H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda) < \eta(\Omega(\mu_0, \mu_1)). \quad (80)$$

We can choose $\varepsilon_1 > 0$ such that

$$\alpha(\mu_0, \mu_1)H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda) + \varepsilon_1 \leq \eta(\Omega(\mu_0, \mu_1)). \quad (81)$$

Thus,

$$\begin{aligned} \check{\delta}(\mu_1, T(\mu_1) \cap \Lambda) + \varepsilon_1 &\leq H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda) + \varepsilon_1 \\ &\leq \alpha(\mu_0, \mu_1)H_d(T(\mu_0) \cap \Lambda, T(\mu_1) \cap \Lambda) + \varepsilon_1 \\ &\leq \eta(\Omega(\mu_0, \mu_1)). \end{aligned} \quad (82)$$

It follows from Lemma 2 that there exists $\mu_2 \in T(\mu_1) \cap \Lambda$ such that

$$\check{\delta}(\mu_1, \mu_2) \leq \check{\delta}(\mu_1, T(\mu_1) \cap \Lambda) + \varepsilon_1. \quad (83)$$

From fd82(82) and (83), we infer that

$$\check{\delta}(\mu_1, \mu_2) \leq \eta(\Omega(\mu_0, \mu_1)), \quad (84)$$

where

$$\begin{aligned} \Omega(\mu_0, \mu_1) &= \max\left\{\check{\delta}(\mu_0, \mu_1), \check{\delta}(\mu_0, T(\mu_0)), \check{\delta}(\mu_1, T(\mu_1)), \frac{\check{\delta}(\mu_0, T(\mu_1)) + \check{\delta}(\mu_1, T(\mu_0))}{2}\right\} \\ &\leq \max\left\{\check{\delta}(\mu_0, \mu_1), \check{\delta}(\mu_1, T(\mu_1)), \frac{\check{\delta}(\mu_0, T(\mu_1))}{2}\right\} \\ &\leq \max\{\check{\delta}(\mu_0, \mu_1), \check{\delta}(\mu_1, T(\mu_1))\}. \end{aligned} \quad (85)$$

We claim that

$$\check{\delta}(\mu_1, \mu_2) \leq \eta(\check{\delta}(\mu_0, \mu_1)). \quad (86)$$

Let $\Phi = \max\{\check{\delta}(\mu_0, \mu_1), \check{\delta}(\mu_1, T(\mu_1))\}$. If $\Phi = \check{\delta}(\mu_1, T(\mu_1))$, we have $\mu_2 \in T(\mu_1) \cap \Lambda$, so we obtain

$$\check{\delta}(\mu_1, \mu_2) \leq \eta(\check{\delta}(\mu_1, T(\mu_1))), \quad (87)$$

which is a contradiction. Thus, we obtain $\Phi = \check{\delta}(\mu_0, \mu_1)$. We assume that $\check{\delta}(\mu_1, \mu_2) \neq 0$; otherwise, μ_1 is a fixed point of T . Since $\check{\delta}(\mu_1, \mu_2) \leq \eta(\check{\delta}(\mu_0, \mu_1)) < \check{\delta}(\mu_0, \mu_1)$, we deduce that $\check{\delta}(\mu_1, \mu_2) \in \hat{E}$. Next, $\mu_2 \in \bar{b}(\mu_0, \rho)$ because

$$\begin{aligned} \check{\delta}(\mu_0, \mu_2) &\leq \check{\delta}(\mu_0, \mu_1) + \check{\delta}(\mu_1, \mu_2) \\ &\leq \check{\delta}(\mu_0, \mu_1) + \eta(\check{\delta}(\mu_0, \mu_1)) \\ &\leq \check{\delta}(\mu_0, \mu_1) + \sigma(\eta(\check{\delta}(\mu_0, \mu_1))) \\ &= \sigma(\check{\delta}(\mu_0, \mu_1)) = \rho. \end{aligned} \quad (88)$$

Because T is α -admissible, $\alpha(\mu_1, \mu_2) \geq 1$. Also, since

$$\frac{1}{2} \min\{\check{\delta}(\mu_1, T(\mu_1) \cap \Lambda), \check{\delta}(\mu_2, T(\mu_2) \cap \Lambda)\} < \check{\delta}(\mu_1, \mu_2), \quad (89)$$

from (76), we get

$$\begin{aligned} 0 &\leq \Gamma[\alpha(\mu_1, \mu_2)H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda), \eta(\Omega(\mu_1, \mu_2))] \\ &< \eta(\Omega(\mu_1, \mu_2)) - \alpha(\mu_1, \mu_2)H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda). \end{aligned} \quad (90)$$

This implies that

$$\alpha(\mu_1, \mu_2)H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda) < \eta(\Omega(\mu_1, \mu_2)). \quad (91)$$

Now choose $\varepsilon_2 > 0$ such that

$$\alpha(\mu_1, \mu_2)H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda) + \varepsilon_2 \leq \eta(\Omega(\mu_1, \mu_2)). \quad (92)$$

Thus,

$$\begin{aligned} \check{\delta}(\mu_2, T(\mu_2) \cap \Lambda) + \varepsilon_2 &\leq H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda) + \varepsilon_2 \\ &\leq \alpha(\mu_1, \mu_2)H_d(T(\mu_1) \cap \Lambda, T(\mu_2) \cap \Lambda) \\ &\quad + \varepsilon_2 \\ &\leq \eta(\Omega(\mu_1, \mu_2)). \end{aligned} \quad (93)$$

It follows from Lemma 2 that there exists $\mu_3 \in T(\mu_2) \cap \Lambda$ such that

$$\check{\delta}(\mu_2, \mu_3) \leq \check{\delta}(\mu_2, T(\mu_2) \cap \Lambda) + \varepsilon_2. \quad (94)$$

From (93) and (94),

$$\check{\delta}(\mu_2, \mu_3) \leq \eta^2(\Omega(\mu_0, \mu_1)), \quad (95)$$

where

$$\begin{aligned}
\Omega(\mu_1, \mu_2) &= \max \left\{ \delta(\mu_1, \mu_2), \delta(\mu_1, T(\mu_1)), \delta(\mu_2, T(\mu_2)), \frac{\delta(\mu_1, T(\mu_2)) + \delta(\mu_2, T(\mu_1))}{2} \right\} \\
&\leq \max \left\{ \delta(\mu_1, \mu_2), \delta(\mu_2, T(\mu_2)), \frac{\delta(\mu_1, T(\mu_2))}{2} \right\} \\
&\leq \max \{ \delta(\mu_1, \mu_2), \delta(\mu_2, T(\mu_2)) \}.
\end{aligned} \tag{96}$$

We claim that

$$\delta(\mu_2, \mu_3) \leq \eta(\delta(\mu_1, \mu_2)). \tag{97}$$

Let $\Phi = \max\{\delta(\mu_1, \mu_2), \delta(\mu_2, T(\mu_2))\}$. If $\Phi = \delta(\mu_2, T(\mu_2))$. Since $\mu_3 \in T(\mu_2) \cap \Lambda$, we have

$$\delta(\mu_2, \mu_3) \leq \eta(\delta(\mu_2, \mu_3)), \tag{98}$$

which is a contradiction. Thus, we have $\Phi = \delta(\mu_1, \mu_2)$. We assume that $\delta(\mu_2, \mu_3) \neq 0$; otherwise, μ_2 is a fixed point of T . From (95), we have $\delta(\mu_2, \mu_3) < \delta(\mu_1, \mu_2)$, and so $\delta(\mu_2, \mu_3) \in \bar{E}$. Also, we have $\mu_3 \in \bar{b}(\mu_0, \rho)$, since

$$\begin{aligned}
\delta(\mu_0, \mu_3) &\leq \delta(\mu_0, \mu_1) + \delta(\mu_1, \mu_2) + \delta(\mu_2, \mu_3) \\
&\leq \delta(\mu_0, \mu_1) + \eta(\delta(\mu_0, \mu_1)) + \eta^2(\delta(\mu_0, \mu_1)) \\
&\leq \sum_{i=0}^{\infty} \eta^i(\delta(\mu_0, \mu_1)) \\
&= \sigma(\delta(\mu_0, \mu_1)) = \rho.
\end{aligned} \tag{99}$$

Continuing this setup, we obtain a sequence $\{\mu_i\} \subset \bar{b}(\mu_0, \rho)$ such that $\mu_{i+1} \in T(\mu_i) \cap \Lambda$, $\mu_i \neq \mu_{i+1}$ with $\alpha(\mu_i, \mu_{i+1}) \geq 1$ and $\delta(\mu_i, \mu_{i+1}) \in \bar{E}$ and

$$\delta(\mu_i, \mu_{i+1}) \leq \eta^i(\delta(\mu_0, \mu_1)), \quad \text{for all } i \in \mathbb{N}. \tag{100}$$

For $i, m \in \mathbb{N}$ with $m > i$, by using the triangular inequality and (100), we get

$$\begin{aligned}
\delta(\mu_i, \mu_m) &\leq \delta(\mu_i, \mu_{i+1}) + \delta(\mu_{i+1}, \mu_{i+2}) + \cdots + \delta(\mu_{m-1}, \mu_m) \\
&\leq \eta^i(\delta(\mu_0, \mu_1)) + \eta^{i+1}(\delta(\mu_0, \mu_1)) + \cdots \\
&\quad + \eta^{m-1}(\delta(\mu_0, \mu_1)) \\
&\leq \sum_{j=i}^{\infty} \eta^j(\delta(\mu_0, \mu_1)) < \infty.
\end{aligned} \tag{101}$$

To show that $\{\mu_i\}$ is a Cauchy sequence in $\bar{b}(\mu_0, \rho)$. Since $\bar{b}(\mu_0, \rho)$ is closed in X , there exists an $\vartheta^* \in \bar{b}(\mu_0, \rho)$ such that $\mu_i \rightarrow \vartheta^*$. Note that $\vartheta^* \in \Lambda$ because $\mu_{i+1} \in T(\mu_i) \cap \Lambda$. By same argument of Theorem 2, we have

$$\frac{1}{2s} \min\{\delta(\mu_i, T(\mu_i) \cap \Lambda), \delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda)\} < \delta(\mu_i, \mu_{i+1}). \tag{102}$$

Also, we know that $\alpha(\mu_i, \mu_{i+1}) \geq 1$ and $\delta(\mu_i, \mu_{i+1}) \in \bar{E}$ for all n . Thus, from (76), we have

$$\begin{aligned}
0 &\leq \Gamma[\alpha(\mu_i, \mu_{i+1})H_d(T\mu_i \cap \Lambda, T\mu_{i+1} \cap \Lambda), \eta(\Omega(\mu_i, \mu_{i+1}))] \\
&< \eta(\Omega(\mu_i, \mu_{i+1})) - \alpha(\mu_i, \mu_{i+1})H_d(T(\mu_i) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda),
\end{aligned} \tag{103}$$

which gives that

$$\alpha(\mu_i, \mu_{i+1})H_d(T\mu_i \cap \Lambda, T\mu_{i+1} \cap \Lambda) < \eta(\Omega(\mu_i, \mu_{i+1})). \tag{104}$$

Since $\mu_{i+1} \in T(\mu_i) \cap \Lambda$, from (100), we get

$$\begin{aligned}
\delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) &\leq \alpha(\mu_i, \mu_{i+1})H_d(T(\mu_i) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda) \\
&\leq \eta(\Omega(\mu_i, \mu_{i+1})) \\
&< (\Omega(\mu_0, \mu_1)),
\end{aligned} \tag{105}$$

where

$$\begin{aligned}
\Omega(\mu_i, \mu_{i+1}) &= \max \left\{ \delta(\mu_i, \mu_{i+1}), \delta(\mu_i, T\mu_i), \delta(\mu_{i+1}, T\mu_{i+1}), \right. \\
&\quad \left. \frac{\delta(\mu_i, T\mu_{i+1}) + \delta(\mu_{i+1}, T\mu_i)}{2} \right\} \\
&\leq \max \left\{ \delta(\mu_i, \mu_{i+1}), \delta(\mu_{i+1}, \mu_{i+2}), \right. \\
&\quad \left. \frac{\delta(\mu_i, \mu_{i+2})}{2} \right\} \\
&\leq \max\{\delta(\mu_i, \mu_{i+1}), \delta(\mu_{i+1}, \mu_{i+2})\}.
\end{aligned} \tag{106}$$

We claim that

$$\begin{aligned}
\delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) &\leq \alpha(\mu_i, \mu_{i+1})H_d(T(\mu_i) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda) \\
&< \eta(\delta(\mu_i, \mu_{i+1})).
\end{aligned} \tag{107}$$

Let $\Phi = \max\{\delta(\mu_i, \mu_{i+1}), \delta(\mu_{i+1}, \mu_{i+2})\}$. If $\Phi = \delta(\mu_{i+1}, \mu_{i+2})$. Since $\mu_{i+2} \in T(\mu_{i+1}) \cap \Lambda$, we have

$$\begin{aligned}
\delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) &\leq \alpha(\mu_i, \mu_{i+1})H_d(T(\mu_i) \cap \Lambda, T(\mu_{i+1}) \cap \Lambda) \\
&< \eta(\delta(\mu_{i+1}, \mu_{i+2})),
\end{aligned} \tag{108}$$

which is a contradiction. Thus, we have $\Phi = \delta(\mu_i, \mu_{i+1})$. Taking limit $i \rightarrow \infty$ in (105), we obtain

$$\lim_{i \rightarrow \infty} \delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) = 0. \quad (109)$$

Since $g(\mu) = \delta(\mu, T(\mu) \cap \Lambda)$ is T -o.l.s.c at ϑ^* , then

$$\begin{aligned} \delta(\vartheta^*, T(\vartheta^*) \cap \Lambda) &= g(\vartheta^*) \leq \liminf_i g(\mu_{i+1}) \\ &= \liminf_i \delta(\mu_{i+1}, T(\mu_{i+1}) \cap \Lambda) = 0. \end{aligned} \quad (110)$$

Since $T\vartheta^*$ is closed, $\vartheta^* \in T(\vartheta^*)$. Conversely, if ϑ^* is a fixed point of T , then $g(\vartheta^*) = 0 \leq \liminf_i g(\mu_i)$, since $\vartheta^* \in \Lambda$. \square

Taking $\Gamma(r, s) = s - \int_0^r \zeta(t)dt$ for all $r, s \geq 0$, in Theorem 3, we obtain the following theorem.

Corollary 4. Let (X, δ) be a complete metric space, η be a Bianchini–Grandolfi gauge function on an interval \widehat{E} , and $T: \Lambda \rightarrow CB(X)$ be a given set-valued mapping. If $T\mu \cap \Lambda \neq \emptyset$, then

$$\frac{1}{2} \min\{\delta(\mu, T(\mu) \cap \Lambda), \delta(\nu, T(\nu) \cap \Lambda)\} < \delta(\mu, \nu), \quad (111)$$

which implies that

$$\int_0^{\alpha(\mu, \nu)H_d(T(\mu) \cap \Lambda, T(\nu) \cap \Lambda)} \zeta(t)dt \leq \eta(\delta(\mu, \nu)), \quad (112)$$

for all $\mu \in \Lambda$, $\nu \in T(\mu) \cap \Lambda$, and $\delta(\mu, \nu) \in \widehat{E}$, where $\zeta: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a function such that $\int_0^\epsilon \zeta(t)dt$ exists and $\int_0^\epsilon \zeta(t)dt > \epsilon$ for all $\epsilon > 0$ such that the following holds:

(a) T is α -admissible.

(b) There exists $\mu_0 \in \Lambda$ with $\delta(\mu_0, \mu_1) \in \widehat{E}$ for some $\mu_1 \in T(\mu_0) \cap \Lambda$ such that $\alpha(\mu_0, \mu_1) \geq 1$. Then, there exists an orbit $\{\mu_i\}$ of T in Λ and $\vartheta^* \in \Lambda$ such that $\lim_{i \rightarrow \infty} \mu_i = \vartheta^*$. In addition, ϑ^* is a fixed point of T if and only if the function $g(\mu) = \delta(\mu, T(\mu) \cap \Lambda)$ is T -o.l.s.c at ϑ^* .

Corollary 5. Let (X, δ) be a complete metric space, η be a Bianchini–Grandolfi gauge function on an interval \widehat{E} , and $T: X \rightarrow CB(X)$ be a given set-valued mapping. If $\Gamma \in \nabla$ exists, then

$$\frac{1}{2} \min\{\delta(\mu, T(\mu) \cap \Lambda), \delta(\nu, T(\nu) \cap \Lambda)\} < \delta(\mu, \nu) \quad (113)$$

$$\Rightarrow \Gamma[\alpha(\mu, \nu)H_d(T(\mu), T(\nu)), \eta(\delta(\mu, \nu))] \geq 0,$$

for all $\mu \in X$, $\nu \in T(\mu)$, and $\delta(\mu, \nu) \in \widehat{E}$ such that the following holds:

(a) T is α -admissible.

(b) There exists $\mu_0 \in X$ with $\delta(\mu_0, \mu_1) \in \widehat{E}$ for some $\mu_1 \in T(\mu_0)$ such that $\alpha(\mu_0, \mu_1) \geq 1$. Then, there exists an orbit $\{\mu_i\}$ of T in X that converges to the fixed point $\vartheta^* \in \mathcal{F} = \{\mu \in X: \delta(\mu, \vartheta^*) \in \widehat{E}\}$ of T .

4. An Application to Dynamical System

Dynamical system is connected to a multistage operation reduced for solving the following functional equation:

$$T(\mu_1) = \sup_{\mu_2 \in H} \{h(\mu_1, \mu_2) + \delta(\mu_1, \mu_2, T(l(\mu_1 \mu_2)))\} \text{ for } \mu_1 \in \widetilde{\beta}, \quad (114)$$

where

$$\begin{aligned} l: \widetilde{\beta} \times H &\longrightarrow \widetilde{\beta}, \\ h, h': \widetilde{\beta} \times H &\longrightarrow (-\infty, \infty), \end{aligned} \quad (115)$$

$$D, D': \widetilde{\beta} \times H \times (-\infty, \infty) \longrightarrow (-\infty, \infty).$$

Assume that \widetilde{G}_1 and \widetilde{G}_2 are Banach spaces, $\widetilde{\beta} \subset \widetilde{G}_1$ is a state space, and $H \subset \widetilde{G}_2$ is a decision space. For more details, see [3]. Let $B(\widetilde{\beta})$ signify the set of all bounded real-valued functions on $\widetilde{\beta}$. Choose an arbitrary point $\sigma \in B(\widetilde{\beta})$ defined as $\|\sigma\| = \sup_{r \in \widetilde{\beta}} |\sigma(r)|$. $(B(\widetilde{\beta}), \|\cdot\|)$ endowed with the metric given by

$$\delta(\mu_1, \mu_2) = \sup_{r \in \widetilde{\beta}} |\mu_1(r) - \mu_2(r)|, \quad (116)$$

for all $\mu_1, \mu_2 \in B(\widetilde{\beta})$, are BS. Define $g: B(\widetilde{\beta}) \rightarrow B(\widetilde{\beta})$ by

$$g(\omega)(r) = \max_{t \in H} \{V(r, t, \omega(l(r, t))) + f(r, t)\}, \quad (117)$$

for all $\omega \in B(\widetilde{\beta})$ and $r \in \widetilde{\beta}$. Also,

$$\begin{aligned} H_d[g(\omega_1)(r), g(\omega_2)(r)] &= H_d[V^x(r, t, \omega_1(l(r, t))) + f^x(r, t), V^y(r, t, \omega_2(l(r, t))) + f^y(r, t)] \\ &\leq H_d[V^x(r, t, \omega_1(l(r, t))), V^y(r, t, \omega_2(l(r, t)))]. \end{aligned} \quad (118)$$

Consider $\phi: B(\widetilde{\beta}) \rightarrow B(\widetilde{\beta})$ such that

$$\frac{1}{2} \min\{\tilde{\mathfrak{d}}(\omega_1, \phi(\omega_1) \cap \Lambda), \tilde{\mathfrak{d}}(\omega_2, \phi(\omega_1) \cap \Lambda)\} < \tilde{\mathfrak{d}}(\omega_1, \omega_2), \omega_1, \omega_2 \in B(\tilde{\beta}), \quad (119)$$

and we have

$$\begin{aligned} & |V(r, t, \omega_1(r)) - V(r, t, \omega_2(r))| \\ & \leq \left[\left[1 + \sqrt{T(|\omega_1(r) - \omega_2(r)|)} \right]^\alpha - 1 \right]^2, \end{aligned} \quad (120)$$

for all $\omega_1, \omega_2 \in B(\tilde{\beta})$, where $r \in \tilde{\beta}$, $t \in V$ and $0 \leq \alpha < 1$.

Theorem 4. Let $\phi: B(\tilde{\beta}) \longrightarrow B(\tilde{\beta})$ be a l.s.c mapping as defined in (117) such that the following conditions are satisfied:

V and f are continuous and bounded.

There exists an orbit $\{\omega_i\} \in \Lambda$ of ϕ and $c^* \in \Lambda$ such that $\lim_{i \rightarrow \infty} \omega_i = c^*$.

c^* is a fixed point iff $\phi(\omega_1) = \tilde{\mathfrak{d}}(\omega_1, T(\omega_1) \cap \Lambda)$ is ϕ -o.l.s.c at c^* .

Then, functional (114) possesses a bounded solution.

Proof. Note that $(B(\tilde{\beta}), \tilde{\mathfrak{d}})$ is a complete metric, where $\tilde{\mathfrak{d}}(\mu_1, \mu_2)$ is the metric, as defined by (82). There exist $r \in \tilde{\beta}$, $t_1, t_2 \in V$ and $\psi: B(\tilde{\beta}) \times B(\tilde{\beta}) \longrightarrow (0, \infty)$ such that

$$\psi[\tilde{\mathfrak{d}}(\omega_1, \phi(\omega_2) \cap \Lambda), \tilde{\mathfrak{d}}(\omega_1, \omega_2)] < 0, \omega_1, \omega_2 \in B(\tilde{\beta}), \quad (121)$$

and we have

$$\begin{aligned} H_d[\phi_1(\omega_1)(r), \phi_1(\omega_2)(r)] & \leq H_d[V^x(r, t, \omega_1(l(r, t))), V^y(r, t, \omega_2(l(r, t)))] \\ & \leq \max_{r \in \tilde{\beta}} \left\{ \max_{t \in \beta} |V(r, t, \omega_1(l(r, t))) - V(r, t, \omega_2(l(r, t)))| \right\} \\ & \leq \max_{r \in \tilde{\beta}} \left\{ \max_{t \in \beta} \left\{ \left[\left[1 + \sqrt{\eta(|\omega_1(l(r, t)) - \omega_2(l(r, t))|)} \right]^\alpha - 1 \right]^2 \right\} \right\} \\ & \leq \max_{r \in \tilde{\beta}} \left\{ \left[\left[1 + \sqrt{\eta(\|\omega_1 - \omega_2\|)} \right]^\alpha - 1 \right]^2 \right\} \\ & \leq \left[\left[1 + \sqrt{\eta(\tilde{\mathfrak{d}}(\omega_1, \omega_2))} \right]^\alpha - 1 \right]^2. \end{aligned} \quad (122)$$

It implies that

$$H_d(\phi_1(\omega_1)(r), \phi_1(\omega_2)(r)) \leq \left[\left[1 + \sqrt{\eta(\tilde{\mathfrak{d}}(\omega_1, \omega_2))} \right]^\alpha - 1 \right]^2. \quad (123)$$

Owing to (123),

$$1 + \sqrt{\eta(\tilde{\mathfrak{d}}(\omega_1, \omega_2))} \leq \left[1 + \sqrt{\eta(\tilde{\mathfrak{d}}(\omega_1, \omega_2))} \right]^\alpha. \quad (124)$$

By $\theta \in \Xi$ and $\theta(z) = 1 + \sqrt{z}$ with (124), we obtain

$$\begin{aligned} & \theta[H_d \tilde{\mathfrak{d}}(\phi_1(\omega_1), \phi_1(\omega_2))] \\ & \leq [\theta(\eta(\tilde{\mathfrak{d}}(\omega_1, \omega_2)))]^\alpha \quad \text{for all } \omega_1, \omega_2 \in B(\tilde{\beta}). \end{aligned} \quad (125)$$

Furthermore, (1) – (3) are equivalent to (a) – (b) of Theorem 3. So, there exists a fixed point $c^* \in \Lambda$ in $B(\tilde{\beta})$, which is a bounded solution of functional (117). \square

4.1. An Application to Integral Inclusion. In this section, we consider the following set-valued integral inclusion:

$$\varsigma(r) \in \kappa + U \int_{r_0}^r V(t, \varsigma(t)) V t, \quad (126)$$

where $\kappa \in (-\infty, \infty)$, U is a bounded compact subset of $(-\infty, \infty)$, and $V(t, \varsigma(t))$ is l.s.c. Let $X = C(I)$ be the space of all continuous real-valued function and $C(I)$ is complete w.r.t the metric $\tilde{\mathfrak{d}}$, which defined by

$$\tilde{\mathfrak{d}}(\mu_1, \mu_2) = \sup_{r \in I} |\mu_1(r) - \mu_2(r)|. \quad (127)$$

Assume that there exists $\phi: B(\tilde{\beta}) \longrightarrow B(\tilde{\beta})$ and $V: (-\infty, \infty) \times (-\infty, \infty) \longrightarrow (-\infty, \infty)$ is continuous on

$$R = \left\{ (r, \varsigma): |r - r_0| \leq \left[\frac{1}{\alpha_1} \right]^{1/2} \text{ and } |\varsigma - \kappa| \leq \frac{1}{2\alpha_2} \right\}, \quad (128)$$

such that for

$$\frac{1}{2} \min\{\tilde{\mathfrak{d}}(\mu_1, \phi(\mu_1) \cap \Lambda), \tilde{\mathfrak{d}}(\mu_2, \phi(\mu_2) \cap \Lambda)\} < \tilde{\mathfrak{d}}(\omega_1, \omega_2),$$

$$\mu_1, \mu_2 \geq 0, \quad (129)$$

we have

$$|V(r, \varsigma_1(r)) - V(r, \varsigma_2(r))| \leq e^{\sqrt{\alpha_1/\alpha_2} |\varsigma_1(r) - \varsigma_2(r)|^\alpha}, \quad (130)$$

where $\alpha_2 = \max_{\bar{u} \in U} |\bar{u}|$, $0 < \alpha_1 \leq \alpha_2$, and $0 \leq \alpha < 1$.

$$|V(t, \varsigma)| < \frac{1}{2\alpha_1} \left[\frac{1}{\alpha_1} \right]^{1/2}. \quad (131)$$

Moreover, let $\check{C} = \{\varsigma \in C(I) : \check{\delta}(\varsigma, \kappa) \leq 1/2\alpha_2\}$ be a closed subspace of $C(I)$ and the operator ϕ be defined by

$$\phi(\varsigma(r)) \in \kappa + U \int_{r_0}^r V(t, \varsigma(t)) dt. \quad (132)$$

Set $V_x(r) = \int_{r_0}^r V(t, \varsigma(t)) dt$. Note that

$$\begin{aligned} H_d[\phi(\varsigma_1(r)), \phi(\varsigma_2(r))] &= H_d[\kappa + UV_x(r), \kappa + UV_y(r)] \\ &\leq H_d[UV_x(r), UV_y(r)] \\ &= \max \left\{ \max_{\bar{a} \in UV_x(r)} \check{\delta}(\bar{a}, UV_y(r)), \max_{\bar{b} \in UV_y(r)} \check{\delta}(\bar{b}, UV_x(r)) \right\}. \end{aligned} \quad (133)$$

Consider

$$\begin{aligned} \max_{\bar{a} \in UV_x(r)} \check{\delta}(\bar{a}, UV_y(r)) &= \max_{\bar{a} \in UV_x(r)} \min_{\bar{b} \in UV_y(r)} \check{\delta}(\bar{a}, \bar{b}) \\ &= \max_{\bar{u} \in U} \min_{\bar{v} \in U} \check{\delta}(\bar{u}V(r, \varsigma_1(r)), \bar{v}V(r, \varsigma_2(r))) \\ &= \max_{\bar{u} \in U} \min_{\bar{v} \in U} \sup_{r \in I} |\bar{u}V(r, \varsigma_1(r)) - \bar{v}V(r, \varsigma_2(r))| \\ &\leq \max_{\bar{u} \in U} \min_{\bar{v} \in U} \sup_{r \in I} [|\bar{u}V(r, \varsigma_2(r)) - \bar{v}V(r, \varsigma_2(r))| + |\bar{u}V(r, \varsigma_2(r)) - \bar{u}V(r, \varsigma_1(r))|] \\ &\leq \max_{\bar{u} \in U} \min_{\bar{v} \in U} \left[|\bar{u}| \sup_{r \in I} |V(r, \varsigma_2(r)) - V(r, \varsigma_1(r))| + |\bar{u} - \bar{v}| \sup_{r \in I} |V(r, \varsigma_2(r))| \right] \\ &= \max_{\bar{u} \in U} |\bar{u}| \sup_{r \in I} |V(r, \varsigma_2(r)) - V(r, \varsigma_1(r))| \\ &= \alpha_2 \sup_{r \in I} |V(r, \varsigma_2(r)) - V(r, \varsigma_1(r))|. \end{aligned} \quad (134)$$

This implies that

$$\max_{\bar{a} \in UV_x(r)} V(\bar{a}, UV_y(r)) \leq \alpha_2 \sup_{r \in I} |V(r, \varsigma_2(r)) - V(r, \varsigma_1(r))|. \quad (135)$$

Theorem 5. Let $X = C(I)$ and $\phi: (\check{C}, d) \longrightarrow (D(\check{C}), H_d)$ be a l.s.c mapping. Suppose that the following assumptions hold:

- (i) ϕ is defined for all $\varsigma \in \check{C}$.
- (ii) $\phi(\varsigma(r))$ is a CS of \check{C} for all $\varsigma \in \check{C}$.

Then, owing to (127)–(135), integral (126) has a solution on I .

Proof. Let $\kappa \in I$. Then, $|\kappa - r_0| \leq [1/\alpha_1]^{1/2}$. Hence, we have $|\varsigma(\kappa) - \kappa| \leq 1/2\alpha_1$. If $(\kappa, \varsigma(\kappa)) \in (-\infty, \infty)$, the integral equation in (132) exists. Since $\kappa \in (-\infty, \infty)$ is continuous, κ is defined for all $\kappa \in \check{C}$. Next, let $\vartheta(r) \in \phi(\varsigma(r))$. Then, $\vartheta(r) = \kappa + \bar{u}V_x(r)$ for $\bar{u} \in U$:

$$\begin{aligned} |\vartheta(r) - \kappa| &= |\bar{u}V_x(r)| = |\bar{u}| |V_x(r)| \\ &\leq \alpha_2 \int_{r_0}^r |V(t, \varsigma(t))| dt \\ &\leq \alpha_2 \int_{r_0}^r |V(t, \varsigma(t))| dt \\ &< \alpha_2 |r - r_0| \frac{1}{2\alpha_2} \left[\frac{1}{\alpha_1} \right]^{1/2} \\ &\leq \frac{1}{2\alpha_1}. \end{aligned} \quad (136)$$

Thus, $|\vartheta(r) - \kappa| \leq 1/2\alpha_1$ for each $\vartheta(r) \in \phi(\varsigma(r))$. So, $\phi(\varsigma(r))$ is a subset of \check{C} . Now, let $\{\varsigma_i\} \subset \phi(\varsigma(r))$; then, $\varsigma = \kappa + \bar{u}_i V_x(r)$ for $\bar{u}_i \in U$. Since U is compact, there exists subsequence $\bar{u}_{i^*} \in \bar{u}_i$ such that $\{\bar{u}_{i^*}\}$ is convergent to $\bar{u} \in U$. Let $\hat{u} = \kappa + \hat{u}V_x(r)$; then,

$$\begin{aligned}
d(\widehat{u_{i^*}}, \widehat{u}) &= \sup_{r \in I} (|\widehat{u_{i^*}} - \widehat{u}| |V_x(r)|) \\
&\leq \left| \widehat{u_{i^*}} - t\widehat{u} \right| \sup_{r \in I} |V_x(r)| \longrightarrow 0, \text{ as } i^* \longrightarrow \infty.
\end{aligned}
\tag{137}$$

Hence, $\phi(\zeta(r))$ is a CS of \check{C} for all $\zeta \in \check{C}$. Next,

$$\begin{aligned}
|V(r, \zeta_1(r)) - V(r, \zeta_2(r))| &\leq \int_{r_0}^r |V(t, \zeta_1(t)) - V(t, \zeta_2(t))| dt \\
&\leq e \int_{r_0}^r \sqrt{\alpha_1/\alpha_2} |\zeta_1(t) - \zeta_2(t)|^\alpha dt \\
&= e \sqrt{\alpha_1/\alpha_2} \int_{r_0}^r \sqrt{|\zeta_1(t) - \zeta_2(t)|^\alpha} dt \\
&\leq e \sqrt{\alpha_1/\alpha_2} (\sup_{r \in I} |\zeta_1(t) - \zeta_2(t)|^\alpha)^{1/2} \int_{r_0}^r dt \\
&= e \sqrt{\alpha_1/\alpha_2} |r - r_0| (\sup_{r \in I} |\zeta_1(t) - \zeta_2(t)|^\alpha)^{1/2} \\
&\leq e \sqrt{\alpha_1/\alpha_2} (1/\alpha_2)^{1/2} (\check{\delta}(\zeta_1(t) - \zeta_2(t))^\alpha)^{1/2} \\
&= e \sqrt{1/\alpha_2 \check{\delta}(\zeta_1(t) - \zeta_2(t))^\alpha}.
\end{aligned}
\tag{138}$$

It yields that

$$|V(r, \zeta_1(r)) - V(r, \zeta_2(r))| \leq e \sqrt{\alpha_1/\alpha_2} (1/\alpha_2)^{1/2} (\check{\delta}(\zeta_1(t) - \zeta_2(t))^\alpha)^{1/2}.
\tag{139}$$

By $\theta \in \Xi$, $\theta(z) = e^{\sqrt{z}}$ and $\eta(r) = 1/\alpha_2 r$ with (139), we get

$$\theta[H_d \check{\delta}(\phi(\omega_1), \phi(\omega_2))] \leq [\theta(\eta(\check{\delta}(\omega_1, \omega_2)))]^\alpha. \tag{140}$$

Furthermore, (i)-(ii) are equivalent to (a) - (b) of Theorem 3. So, there exists a fixed point $c^* (\in \Lambda)$ in \check{C} , which is a bounded solution of (126). \square

5. Conclusion

The paper deals with the set-valued fixed point theorems satisfying SU-type contraction via Bianchini–Grandolfi gauge function in the context of metric spaces. Within this framework, we introduced two related fixed point results in metric space. An extensive set of nontrivial example is given to justify our claim. Also, we have proven the existence theorem for the system of functional equation and integral inclusion.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

The authors are thankful to the Deanship, King Abdulaziz University, Jeddah, KSA, for funding this research.

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Research Article

Further Properties of 1- and 2-Dimensional U- and W-Convexity and Fixed Point of Nonexpansive Mappings in Banach Spaces X and X^*

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Received 7 February 2021; Accepted 26 April 2021; Published 4 May 2021

Academic Editor: Zoran Mitrovic

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In this paper, we further studied properties of the modulus of n -dimensional U -convexity and the modulus of n -dimensional U -flatness when $n = 1$ (2-dimensional character) and $n = 2$ (3-dimensional character). The new properties of these moduli are investigated, and the relationships between these moduli and other geometric parameters of Banach spaces are studied. Some results on fixed point theory for nonexpansive mappings and normal structure in Banach spaces are obtained.

1. Introduction

Let X be a real Banach space with the dual space X^* . Let $B(X) = \{x \in X: \|x\| \leq 1\}$ and $S(X) = \{x \in X: \|x\| = 1\}$ be the unit ball and the unit sphere of X , respectively. Let X^* be the dual space of X . Let $\nabla_x \subset S(X^*)$ denote the set of norm 1 supporting functionals of $x \in S(X)$.

Brodskiĭ and Mil'man [1] introduced the following geometric concepts in 1948.

Definition 1. Let X be a Banach space. A nonempty bounded and convex subset K of X is said to have normal structure if, for every convex subset C of K that contains more than one point, there is a point $x_0 \in C$ such that

$$\sup\{\|x_0 - y\|: y \in C\} < \text{diam } C. \quad (1)$$

A Banach space X is said to have

- (a) Normal structure if every bounded convex subset of X has normal structure.
- (b) Weak normal structure if every weakly compact convex set K of X has normal structure.

- (c) Uniform normal structure if there exists $0 < c < 1$ such that, for every bounded closed convex subset C of K that contains more than one point, there is a point $x_0 \in C$ such that

$$\sup\{\|x_0 - y\|: y \in C\} < c \cdot \text{diam } C. \quad (2)$$

Remark 1. The following facts are known:

- (a) Uniform normal structure \Rightarrow normal structure \Rightarrow weak normal structure
- (b) For a reflexive spaces, normal structure \Leftrightarrow weak normal structure

Kirk [2] proved that if a Banach space X has weak normal structure, then it has weak fixed point property, that is, every nonexpansive mapping from a weakly compact and convex subset of X to itself has a fixed point.

Let \mathbb{N} be the set of all natural numbers and $n \in \mathbb{N}$.

For two sets of vectors $\{x_i\}_{i=1}^{n+1} \subseteq X$ and $\{f_i\}_{i=2}^{n+1} \subseteq X^*$, the $(n+1) \times (n+1)$ matrix,

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \langle x_1, f_2 \rangle & \langle x_2, f_2 \rangle & \cdots & \langle x_{n+1}, f_2 \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_1, f_{n+1} \rangle & \langle x_2, f_{n+1} \rangle & \cdots & \langle x_{n+1}, f_{n+1} \rangle \end{bmatrix}, \quad (3)$$

is denoted by $m(x_1, x_2, \dots, x_{n+1}; f_2, f_3, \dots, f_{n+1})$ [3].

Gao and Saejung [3] introduced the concept of volume by the convex hull of x_1, x_2, \dots, x_{n+1} in X of

$$v(x_1, x_2, \dots, x_{n+1}) := \sup\{\det m(x_1, x_2, \dots, x_{n+1}; f_2, f_3, \dots, f_{n+1})\}, \quad (4)$$

where the supremum is taken over all $f_i \in \nabla_{x_i}$, where $i = 2, 3, \dots, n+1$.

Definition 2 (see [3]). Let $v_X^n = \sup\{v(x_1, x_2, \dots, x_{n+1}) : x_1, x_2, \dots, x_{n+1} \in S(X)\}$ be the upper bound of all n -dimensional volume in X .

The following result was proved [3].

Proposition 1. For a Banach space X with $\dim(X) > n$, we have $v_X^n \geq 2$.

Gao and Saejung introduced the concept of the modulus of n -dimensional U -convexity of X as follows.

Definition 3 (see [3]). Let X be a Banach space. Define

$$U_X^n(\varepsilon) = \inf \left\{ 1 - \frac{1}{n+1} \|x_1 + x_2 + \dots + x_{n+1}\| : \begin{array}{l} x_1, x_2, \dots, x_{n+1} \in S(X), \\ v(x_1, x_2, \dots, x_{n+1}) \geq \varepsilon \end{array} \right\}, \quad (5)$$

where $0 \leq \varepsilon \leq v_X^n$ is the modulus of n -dimensional U -convexity of X .

The following results were proved in [3] too.

Theorem 1

- (a) $U_X^n(\varepsilon)$ is an increasing and continuous function in $[0, v_X^n]$ for any $n \in \mathbb{N}$.
- (b) If X is a Banach space with $U_X^n(1) > 0$ for some $n \in \mathbb{N}$, then X is super-reflexive.
- (c) If X is a Banach space with $U_X^n(1) > 0$ for some $n \in \mathbb{N}$, then X has uniform normal structure

The following results were proved in [4].

Theorem 2. If X is a Banach space with $U_X^n(2n+1/2^n) > 1 - 1/n+1$ for some $n \in \mathbb{N}$ and n is even, then X is reflexive.

Let $n = 2$, and we have the following.

Corollary 1. If X is a Banach space with $U_X^2(5/4) > 2/3$, then X is reflexive.

The following results were also proved in [4].

Theorem 3. If X is a Banach space with $U_X^2(5/4) > 2/3$ and $U_X^2(5/2) > 5/6$, then X has uniform normal structure.

Gao and Saejung also introduced the concept of the modulus of n -dimensional U -flatness as follows.

Definition 4 (see [5]). Let X be a Banach space. Define

$$W_X^n(\varepsilon) = \sup \left\{ 1 - \frac{1}{n+1} \|x_1 + x_2 + \dots + x_{n+1}\| : \begin{array}{l} x_1, x_2, \dots, x_{n+1} \in S(X), \\ v(x_1, x_2, \dots, x_{n+1}) \leq \varepsilon \end{array} \right\}, \quad (6)$$

where $0 \leq \varepsilon \leq v_X^n$ is the modulus of n -dimensional U -flatness of X .

Remark 2. (a) The name of the modulus of U -flatness is defined by comparing with the name of the modulus of U -convexity of X . (b) $U_X^n(\varepsilon) \leq W_X^n(\varepsilon)$ in $[0, v_X^n]$.

The following results were proved in [5] too.

Theorem 4

- (a) $W_X^n(\varepsilon)$ is an increasing and continuous function in $[0, v_X^n]$ for any $n \in \mathbb{N}$.

- (b) If X is a Banach space with $W_X^1(5/3) < 2/3$, or $W_X^2(7/9) < 2/3$, then X is reflexive.

- (c) If X is a Banach space with $W_X^n(2n+1/2^n) < 1 - 1/n+1$, where $n \in \mathbb{N}$, then X is super-reflexive.

- (d) Suppose that X is a Banach space satisfying one of the following conditions:

- (i) $W_X^n(1) < 1 - 1/n+1$, for some $n \in \mathbb{N}$ with $n \geq 3$
- (ii) $W_X^n(2n+1/2^n) < 1 - 1/n+1$, for $n = 1$ or $n = 2$

Then, X has uniform normal structure.

Recently, Gabeleh introduced concepts of pointwise cyclic relatively nonexpansive mapping and weak proximal normal

structure for an extension of geometric property of normal structure. Some interesting results are obtained there [6].

In this paper, we studied further properties of the modulus of n -dimensional U -convexity and the modulus of n -dimensional U -flatness when $n=1$ (2-dimensional character) and $n=2$ (3-dimensional character). The new properties of these moduli are investigated and the relationships between these moduli and other geometric parameters of Banach spaces are studied. Some results on fixed point theory for nonexpansive mappings and normal structure in Banach spaces are obtained.

2. Main Results

Lemma 1 (Bishop-Phelps-Bollobás, see [7]). *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in B(X)$ and $h \in S(X^*)$ with $1 - \langle z, h \rangle < \varepsilon^2/4$, then there exist $y \in S(X)$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Since $\langle z, h \rangle$ is a continuous function in norm for $z \in X$ and norm for $h \in X^*$, Lemma 1 can be stated as follows.

Lemma 2. *Let X be a Banach space, and let $0 < \varepsilon < 1$. Given $z \in S(X)$ and $h \in B(X^*)$ with $1 - \langle z, h \rangle < \varepsilon^2/4$, then there exist $y \in S(X)$ and $g \in \nabla_y$ such that $\|y - z\| < \varepsilon$ and $\|g - h\| < \varepsilon$.*

Theorem 5 (see [8]). *Let X be a Banach space. Then, X is not reflexive if and only if, for any $0 < \delta < 1$, there are a sequence $\{x_n\} \subseteq S(X)$ and a sequence $\{f_n\} \subseteq S(X^*)$ such that*

- (a) $\langle x_m, f_n \rangle = \delta$ whenever $n \leq m$
- (b) $\langle x_m, f_n \rangle = 0$ whenever $n > m$

Theorem 6. *If X is a Banach space with $U_{X^*}^1(2) > 3/4$, or $W_{X^*}^1(2) < 1/2$, then X is reflexive.*

Proof. Suppose that X is not reflexive. Let $0 < \delta < 1$ be given.

Let $f_1 \in S(X^*)$, $-f_1 + f_2/2 \in B(X^*)$, and $x_1 \in S(X^{**})$, $-x_2 \in S(X^{**})$. Then, we have

$$m\left(f_1, -\frac{f_1 + f_2}{2}; x_1, -x_2\right) = \begin{bmatrix} 1 & 1 \\ \langle f_1, -x_2 \rangle & \left\langle -\frac{f_1 + f_2}{2}, -x_2 \right\rangle \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\delta & \delta \end{bmatrix} = 2\delta. \quad (7)$$

We also have

$$\frac{1}{4} \leq \left\langle x_1, \frac{1}{2} \left(f_1 - \frac{f_1 + f_2}{2} \right) \right\rangle \leq \frac{1}{2} \left\| f_1 - \frac{f_1 + f_2}{2} \right\| = \frac{1}{2} \left\| \frac{f_1 - f_2}{2} \right\| \leq \frac{1}{2}. \quad (8)$$

So,

$$\frac{1}{2} \leq 1 - \frac{\|f_1 - f_1 + f_2/2\|}{2} \leq \frac{3}{4}. \quad (9)$$

Since δ can be arbitrarily close to 1, by using Lemma 2 and the definition of $U_X^1(\varepsilon)$ and $W_X^1(\varepsilon)$, we have if $U_{X^*}^1(2) > 3/4$, or $W_{X^*}^1(2) < 1/2$, then X is reflexive. \square

Theorem 7. *If X is a Banach space with $U_{X^*}^2(2/3) > 2/3$, or $W_{X^*}^2(2/3) < 5/9$, then X is reflexive.*

Proof. Suppose that X is not reflexive. Let $0 < \delta < 1$ be given.

Let

$f_1 \in S(X^*)$, $-f_1 + f_2/2 \in B(X^*)$, $f_1 + f_2 + f_3/3 \in B(X^*)$, and $x_1 \in S(X^{**})$, $-x_2 \in S(X^{**})$, and $x_3 \in S(X^{**})$.

Then, we have

$$m\left(f_1, -\frac{f_1 + f_2}{2}, \frac{f_1 + f_2 + f_3}{3}; x_1, -x_2, x_3\right) = \begin{bmatrix} 1 & 1 & 1 \\ \langle f_1, -x_2 \rangle & \left\langle -\frac{f_1 + f_2}{2}, -x_2 \right\rangle & \left\langle \frac{f_1 + f_2 + f_3}{3}, -x_2 \right\rangle \\ \langle f_1, x_3 \rangle & \left\langle -\frac{f_1 + f_2}{2}, x_3 \right\rangle & \left\langle \frac{f_1 + f_2 + f_3}{3}, x_3 \right\rangle \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -\delta & \delta & -\frac{2}{3}\delta \\ \delta & -\delta & \delta \end{bmatrix} = \frac{2}{3}\delta^2. \quad (10)$$

We also have

$$\begin{aligned} \frac{1}{3} \left\| f_1 - \frac{f_1 + f_2}{2} + \frac{f_1 + f_2 + f_3}{3} \right\| &= \frac{\|5f_1 - f_2 + 2f_3\|}{18} \leq \frac{4}{9}, \\ \frac{1}{3} \left\| f_1 - \frac{f_1 + f_2}{2} + \frac{f_1 + f_2 + f_3}{3} \right\| &= \frac{\|5f_1 - f_2 + 2f_3\|}{18} \geq \left\langle \frac{5f_1 - f_2 + 2f_3}{18}, x_3 \right\rangle = \frac{1}{3}. \end{aligned} \quad (11)$$

We have

$$\frac{5}{9} \leq 1 - \frac{1}{3} \left\| f_1 - \frac{f_1 + f_2}{2} + \frac{f_1 + f_2 + f_3}{3} \right\| \leq \frac{2}{3}. \quad (12)$$

Since δ can be arbitrarily close to 1, by using Theorem 2.2 and the definition of $U_X^2(\varepsilon)$ and $W_X^2(\varepsilon)$, we have if $U_{X^*}^2(2/3) > 2/3$, or $W_{X^*}^2(2/3) < 5/9$, then X is reflexive.

The following result refers to a Banach space with weak* sequentially compact unit ball of the dual. Notice that this property is satisfied by reflexive or separable Banach spaces and by those that admit an equivalent smooth norm. \square

Lemma 3 (see [9]). *If X is a Banach space with $B(X^*)$ weak* sequentially compact and fails to have weak normal structure, then, for any $\varepsilon > 0$, there are sequence $\{x_n\} \subseteq S(X)$ and sequence $\{f_n\} \subseteq S(X^*)$ such that*

- (a) $\|x_i - x_j\| - 1 < \varepsilon$, whenever $i \neq j$
- (b) $\langle x_i, f_i \rangle = 1$, whenever $1 \leq i \leq \infty$
- (c) $|\langle x_j, f_i \rangle| < \varepsilon$, whenever $i \neq j$
- (d) $\|f_i - f_j\| > 2 - \varepsilon$, whenever $i \neq j$

Theorem 8. *If X is a Banach space with $U_{X^*}^1(2) > 1/2$, or $W_{X^*}^1(2) < 1/2$, then X has weak normal structure.*

Proof. Suppose that X does not have weak normal structure. Let $\varepsilon > 0$ be given. Since $2 \geq \|f_2 - f_1\| \geq \langle f_2 - f_1, x_2 - x_1 \rangle / 1 + \varepsilon \geq 2 - 2\varepsilon/1 + \varepsilon$, we have $1 - \varepsilon/1 + \varepsilon \leq \|f_2 - f_1\|/2 \leq 1$.

Let $f_1 \in S(X^*)$, $f_2 - f_1/2 \in B(X^*)$; and $x_1 \in S(X^{**})$ and $x_2 - x_1/\|x_2 - x_1\| \in S(X^{**})$. Then, we have

$$m\left(f_1, \frac{f_2 - f_1}{2}; x_1, \frac{x_2 - x_1}{\|x_2 - x_1\|}\right) = \left[\left\langle f_1, \frac{x_2 - x_1}{\|x_2 - x_1\|} \right\rangle \left\langle \frac{f_2 - f_1}{2}, \frac{x_2 - x_1}{\|x_2 - x_1\|} \right\rangle \right] = \left[\frac{1}{\|x_2 - x_1\|} \frac{1 + b\varepsilon}{\|x_2 - x_1\|} \right] = 2 + c\varepsilon, \quad (13)$$

where a , b , and c are constants and ε is arbitrarily small.

We also have

$$\begin{aligned} \frac{1}{2} - d\varepsilon &\leq \left\langle \frac{x_1 - x_2}{\|x_1 - x_2\|}, \frac{1}{2} \left(\frac{f_1 - f_2}{2} \right) \right\rangle \\ &\leq \frac{1}{2} \left\| f_1 + \frac{f_2 - f_1}{2} \right\| = \frac{1}{2} \left\| \frac{f_1 + f_2}{2} \right\| \leq \frac{1}{2}, \end{aligned} \quad (14)$$

where d is a constant.

So,

$$\frac{1}{2} \leq 1 - \frac{1}{2} \left\| f_1 + \frac{f_2 - f_1}{2} \right\| = \frac{1}{2} + d\varepsilon. \quad (15)$$

By using Theorem 2.2 and the definition of $U_X^1(\varepsilon)$ and $W_X^1(\varepsilon)$, we have if $U_{X^*}^1(2) > 1/2$, or $W_{X^*}^1(2) < 1/2$, then X has weak normal structure. \square

Theorem 9. *If X is a Banach space with $U_{X^*}^2(9/4) > 5/6$, or $W_{X^*}^2(9/4) < 2/3$, then X has weak normal structure.*

Proof. Suppose that X does not have weak normal structure. Let $\varepsilon > 0$ be given.

Let $f_1 \in S(X^*)$, $f_2 - f_1/2 \in B(X^*)$, $f_3 - f_2/2 \in B(X^*)$; and $x_1 \in S(X^{**})$, $x_2 - x_1/\|x_2 - x_1\| \in S(X^{**})$, and $x_3 - x_2/\|x_3 - x_2\| \in S(X^{**})$. Then, similar to the proof of Theorem 8, we have

$$\begin{aligned}
& m\left(f_1, \frac{f_2 - f_1}{2}, \frac{f_3 - f_2}{2}; x_1, \frac{x_2 - x_1}{\|x_2 - x_1\|}, \frac{x_3 - x_2}{\|x_3 - x_2\|}\right) \\
&= \begin{bmatrix} 1 & 1 & 1 \\ \left\langle f_1, \frac{x_2 - x_1}{\|x_2 - x_1\|} \right\rangle & \left\langle \frac{f_2 - f_1}{2}, \frac{x_2 - x_1}{\|x_2 - x_1\|} \right\rangle & \left\langle \frac{f_3 - f_2}{2}, \frac{x_2 - x_1}{\|x_2 - x_1\|} \right\rangle \\ \left\langle f_1, \frac{x_3 - x_2}{\|x_3 - x_2\|} \right\rangle & \left\langle \frac{f_2 - f_1}{2}, \frac{x_3 - x_2}{\|x_3 - x_2\|} \right\rangle & \left\langle \frac{f_3 - f_2}{2}, \frac{x_3 - x_2}{\|x_3 - x_2\|} \right\rangle \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 + a\varepsilon & 1 + b\varepsilon & -\frac{1}{2} + c\varepsilon \\ d\varepsilon & -\frac{1}{2} + e\varepsilon & 1 + f\varepsilon \end{bmatrix} = \frac{9}{4} + g\varepsilon, \quad (16)
\end{aligned}$$

where a, b, c, d, e, f , and g are constants and ε is arbitrarily small.

We also have

$$\begin{aligned}
\frac{1}{6} - \varepsilon &\leq \left\langle x_1, \frac{f_1 + f_3}{6} \right\rangle \leq \frac{1}{3} \left\| f_1 + \frac{f_2 - f_1}{2} + \frac{f_3 - f_2}{2} \right\| \\
&= \frac{\|f_1 + f_3\|}{6} \leq \frac{1}{3}, \quad (17)
\end{aligned}$$

and we have

$$\frac{2}{3} \leq 1 - \frac{\|f_1 + f_3\|}{6} \leq \frac{5}{6} + \varepsilon. \quad (18)$$

By using Theorem 2.2 and the definition of $U_X^2(\varepsilon)$ and $W_X^2(\varepsilon)$, we have if $U_{X^*}^2(9/4) > 5/6$, or $W_{X^*}^2(9/4) < 2/3$, then X has weak normal structure. \square

Definition 5 (see [10, 11]). Let X and Y be Banach spaces. We say that Y is finitely representable in X if, for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$, there is an isomorphism $T: N \rightarrow X$ such that, for any $y \in N$, $(1 - \varepsilon)\|y\| \leq \|Ty\| \leq (1 + \varepsilon)\|y\|$.

We say that X is super-reflexive if any space Y which is finitely representable in X is reflexive.

Theorem 10. Suppose that X is a Banach space satisfying one of the following conditions:

- (a) $U_{X^*}^1(2) > 3/4$
- (b) $U_{X^*}^2(2/3) > 2/3$
- (c) $W_{X^*}^1(2) < 1/2$
- (d) $W_{X^*}^2(2/3) < 5/9$

Then, X is super-reflexive.

Proof. The proof of this theorem follows from the fact that if a Banach space Y is finitely representable in X , then $U_Y^n(\varepsilon) = U_X^n(\varepsilon)$ and $W_Y^n(\varepsilon) = W_X^n(\varepsilon)$, for any $n \in \mathbb{N}$ and for any $\varepsilon \in \mathcal{V}_{X^*}^n$.

We consider the uniform normal structure. To discuss this result, let us recall the concept of the “ultra”-technique.

Let \mathcal{F} be a filter of an index set I , and let $\{x_i\}_{i \in I}$ be a subset in a Hausdorff topological space X , $\{x_i\}_{i \in I}$ is said to

converge to x with respect to \mathcal{F} , denoted by $\lim_{\mathcal{F}} x_i = x$, if, for each neighborhood U of x , $\{i \in I: x_i \in U\} \in \mathcal{F}$. A filter \mathcal{U} on I is called an ultrafilter if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called trivial if it is of the form $\{A: A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if \mathcal{U} is an ultrafilter, then

- (i) For any $A \subseteq I$, either $A \subseteq U$ or $I - A \subseteq U$
- (ii) If $\{x_i\}_{i \in I}$ has a cluster point x , then $\lim_{\mathcal{U}} x_i$ exists and equals to x

Let $\{X_i\}_{i \in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i \in I} \|x_i\| < \infty$. \square

Definition 6 (see [12, 13]). Let \mathcal{U} be an ultrafilter on I and let $N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i): \lim_{\mathcal{U}} \|x_i\| = 0\}$. The ultraproduct of $\{X_i\}_{i \in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote the element of the ultraproduct. It follows from remark (ii) above and the definition of quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|. \quad (19)$$

In the following, we will restrict our index set I to be \mathbb{N} , the set of natural numbers, and let $X_i = X, i \in \mathbb{N}$, for some Banach space X . For an ultrafilter \mathcal{U} on \mathbb{N} , we use $X_{\mathcal{U}}$ to denote the ultraproduct. Note that if \mathcal{U} is nontrivial, then X can be embedded into $X_{\mathcal{U}}$ isometrically.

Lemma 4 (see [13]). Suppose that \mathcal{U} is an ultrafilter on \mathbb{N} and X is a Banach space. Then, $(X^*)_{\mathcal{U}} \cong (X_{\mathcal{U}})^*$ if and only if X is super-reflexive; in this case, the mapping J defined by

$$\langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}}) \rangle = \lim_{\mathcal{U}} \langle x_i, f_i \rangle, \quad \text{for all } (x_i)_{\mathcal{U}} \in X_{\mathcal{U}}, \quad (20)$$

which is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

Theorem 11. Let X be a super-reflexive Banach space. Then, for any nontrivial ultrafilter \mathcal{U} on \mathbb{N} and for all $n \in \mathbb{N}$ and $\varepsilon > 0$, we have $U_{X_{\mathcal{U}}}^n(\varepsilon) = U_X^n(\varepsilon)$, and $W_{X_{\mathcal{U}}}^n(\varepsilon) = W_X^n(\varepsilon)$.

Proof. The proof is the same as the proof of Theorem 2.17 in [3]. \square

Lemma 5 (see [14]). *If X is a super-reflexive Banach space, then X has uniform normal structure if and only if $X_{\mathcal{U}}$ has normal structure.*

Theorem 12. *Suppose that X is a Banach space satisfying one of the following conditions:*

- (a) $U_{X^*}^1(2) > 3/4$
- (b) $U_{X^*}^2(2/3) > 2/3$ and $z_{X^*}^1(2) > 1/2$, or $U_{X^*}^2(2/3) > 2/3$ and $W_{X^*}^1(2) < 1/2$, or $U_{X^*}^2(2/3) > 2/3$ and $U_{X^*}^2(9/4) > 5/6$, or $U_{X^*}^2(2/3) > 2/3$ and $W_{X^*}^2(9/4) < 2/3$
- (c) $W_{X^*}^1(2) < 1/2$
- (d) $W_{X^*}^2(2/3) < 5/9$ and $U_{X^*}^1(2) > 1/2$, or $W_{X^*}^2(2/3) < 5/9$ and $W_{X^*}^1(2) < 1/2$, or $U_{X^*}^2(2/3) < 5/9$ and $U_{X^*}^2(9/4) > 5/6$, or $W_{X^*}^2(2/3) < 5/9$ and $W_{X^*}^2(9/4) < 2/3$

Then, X has uniform normal structure.

Proof. The results (a) and (c) follow directly from Theorem 8, Theorems 10 and 11, and Lemma 5. The results (b) and (d) follow directly from Theorems 9, 10, and 11 and Lemma 5. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Integral Criteria for Weighted Bloch Functions

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Received 10 March 2021; Accepted 5 April 2021; Published 23 April 2021

Academic Editor: Huseyin Isik

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The present manuscript gives analytic characterizations and interesting technique that involves the study of general ω -Besov classes of analytic functions by the help of analytic ω -Bloch functions. Certain special functions significant in both ω -Besov-norms and ω -Bloch norms framework and to introduce new important families of analytic classes. Interesting motivation of this concerned paper is to construct some new analytic function classes of general ω -Besov-type spaces via integrals on concerned functions view points. The introduced analytic ω -Bloch and ω -Besov type of functions with some interesting properties for these classes of function spaces are established within the constructions of their norms. Using the defined analytic function spaces, various important relations are also derived.

1. Introduction

Operator theory and special classes of holomorphic function spaces have interesting significant roles in different branches of pure and applied mathematics as well as in recent studies of theoretical physics. Some concerned problems arising in operator and measure theories are framed in terms of certain classes of function spaces. Most of these classes can be studied and treated by using numerous kinds of measures and some types of operators which provide and evolve new means of recent studies in mathematical analysis. Besov and Bloch-type classes are the focus of such studies. These types of spaces are extremely applied in computational mathematical analysis and theoretical physics problems. In addition, these special function classes allow the derivation of different useful identities in a fairly straightforward way and help in introducing new families of function classes. Throughout this concerned paper, the following notations and definitions will be used.

Let $H(\mathbb{D})$ denote the class of all concerned holomorphic functions on the concerned unit disk \mathbb{D} . For $a \in \mathbb{D}$, the known concerned Möbius transformation $\varphi_a(w)$ is symbolized by

$$\varphi_a(w) = \frac{a-w}{1-\bar{a}w}, \text{ with } w \in \mathbb{D}. \quad (1)$$

For a specific point $a \in \mathbb{D}$ and $0 < R < 1$, the supposed pseudo-hyperbolic disk $D(a, R)$ with the supposed center a and supposed radius R is symbolized also by $D(a, R) = \varphi_a(R\mathbb{D})$.

The supposed concerned pseudo-hyperbolic disk $D(a, R)$ can be considered an Euclidean disk: with a specific center and radius being $(1-R^2)a/1-R^2|a|^2$ and $(1-|a|^2)R/1-R^2|a|^2$, respectively ([1]). Let ζ denote the concerned normalized specific area of Lebesgue-type measure on \mathbb{D} . We will need the following interesting known identity:

$$1 - |\varphi_a(w)|^2 = \frac{(1-|a|^2)(1-|w|^2)}{|1-\bar{a}w|^2} = (1-|w|^2)|\varphi'_a(w)|. \quad (2)$$

For $a \in \mathbb{D}$, by concerned usual substitution $w = \varphi_a(w)$, we must consider the known Jacobian change in the concerned estimates given by $d\zeta(w) = |\varphi'_a(w)|^2 d\zeta(w)$. One should remark that $\varphi_a(\varphi_a(w)) = w$, and thus $\varphi_a^{-1}(w) = \varphi_a(w)$. For $a, w \in \mathbb{D}$ and $0 < r < 1$, the supposed pseudo-hyperbolic disc $D(a, R)$ is denoted by $D(a, R) = \{w \in \mathbb{D}: |\varphi_a(w)| < R\}$.

Let the concerned function of Green's type with concerned logarithmic singularity at the specific point $a \in \mathbb{D}$

$$g(w, a) = \log \left| \frac{1 - \bar{a}w}{w - a} \right| = \log \frac{1}{|\varphi_a(w)|}. \quad (3)$$

Assume that C_h and C_h^* are any two concerned quantities which depend on a holomorphic-type function h on \mathbb{D} . These concerned quantities are said to be equivalent and symbolized by $C_h \approx C_h^*$; when we find a concerned finite positive constant Υ not depending on the holomorphic function h , for every holomorphic-type function h on \mathbb{D} , the following inequalities hold:

$$\frac{1}{\Upsilon} C_h^* \leq C_h \leq \Upsilon C_h^*. \quad (4)$$

When the concerned quantities C_h and C_h^* are equivalent, we deduce that

$$C_h < \infty \Leftrightarrow C_h^* < \infty. \quad (5)$$

Note that we say $\Theta_1 \lesssim \Theta_2$ (for two concerned functions Θ_1 and Θ_2) when we find a specific concerned constant $\gamma > 0$ for which $\Theta_1 \leq \gamma \Theta_2$.

Next the following new general concerned function classes will be defined.

Definition 1. Let ϖ be a given concerned bounded nondecreasing and continuous function $\varpi: \mathbb{D} \rightarrow \mathbb{R}^+$. The

concerned function $h \in H(\mathbb{D})$ is said to belong to the weighted ϖ -Bloch-type space $\mathcal{B}_{(\varpi)}$, when

$$\|h\|_{\mathcal{B}_{(\varpi)}} = \sup_{w \in \mathbb{D}} \varpi(1 - |w|) |h'(w)| < \infty. \quad (6)$$

Definition 2. Let ϖ be a given concerned bounded nondecreasing and continuous function $\varpi: \mathbb{D} \rightarrow \mathbb{R}^+$. The concerned function $h \in H(\mathbb{D})$ is said to belong to the little weighted ϖ -Bloch-type space $\mathcal{B}_{(\varpi, 0)}$, when

$$\|h\|_{\mathcal{B}_{(\varpi, 0)}} = \lim_{|w| \rightarrow 1^-} \varpi(1 - |w|) |h'(w)| = 0. \quad (7)$$

Remark 1. One should obviously note that these new analytic-type classes are more general than the well-known Bloch and little Bloch analytic-type classes. If $\varpi(t) = t^\alpha$ with $\alpha \in (0, \infty)$, then the analytic α -Bloch classes which were introduced and studied in [2] will be followed. In addition, when $\alpha = 1$, we obtain the known analytic Bloch-type class that is given in [3].

The following new analytic weighted function classes will be introduced.

Definition 3. For a bounded nondecreasing continuous function $\varpi: \mathbb{D} \rightarrow \mathbb{R}^+$, let $0 < p < \infty$, and a holomorphic function $h \in H(\mathbb{D})$ is said to belong to the ϖ -Besov space $B_{p, \varpi}(\varphi)$, when

$$\|h\|_{\varpi, p; \varphi}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1 - |w|)^{2n-2} \varpi^p(1 - |w|^2) (1 - |\varphi_a(w)|)^p d\zeta(w) < \infty. \quad (8)$$

Definition 4. For a bounded nondecreasing continuous function $\varpi: \mathbb{D} \rightarrow (0, \infty)$, let $0 < p < \infty$, and an analytic

function h in \mathbb{D} is said to belong to the ϖ -Besov space $B_{p, \varpi}(g)$, when

$$\|h\|_{\varpi, p; g}^p = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1 - |w|)^{2n-2} \varpi^p(1 - |w|^2) g(w, a)^p d\zeta(w) < \infty. \quad (9)$$

Definition 5. (see [1]). Let $h \in H(\mathbb{D})$ and let $1 < p < \infty$. If

$$\|h\|_{B_p}^p = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1 - |w|^2)^{p-2} d\zeta(w) < \infty, \quad (10)$$

then h belongs to the Besov space B_p .

The aim of the current paper is to establish with concerned proofs various results on analytic-type function spaces with the help of holomorphic \mathcal{B}_{ϖ} -function type classes in some holomorphic concerned general Besov functions satisfying more extended general integral norm

conditions portrayed by some general weights in the known concerned complex disk. The new results generalize and evolve some results in [1, 2, 4]. To substantiate the authenticity of the obtained results and to clear the importance of the defined function classes, some related concerned corollaries are furnished obviously too.

Remark 2. It is an interesting considerable remark to mention the generalizations of some holomorphic-type function spaces in \mathbb{C}^n (see [1, 5–10]) and others. In the same way, there are some extensions by the use of hypercomplex functions (see [11–15]) and others.

2. Some Integral Criteria for \mathcal{B}_{ω} Functions

Some concerned interesting integral-type criteria for the analytic classes $\mathcal{B}_{(\omega)}$ and the analytic general Besov classes are established in this concerned present section. The obtained analytic results are generalizing and evolving the corresponding results in [1, 2, 16] and others.

Now, suppose that $\varphi_a(W) = a - w/1 - aw$ is a Möbius transformation of \mathbb{D} , let $D(a, r) = \{w \in \mathbb{D} : |\varphi_a(w)| < r\}$, and let $g(w, a) = \log 1/|\varphi_a(w)|$ be the known Green's function on \mathbb{D} with a specific concerned logarithmic singularity at the point $a \in \mathbb{D}$. Next, an interesting general result will be established.

Theorem 1. Let $r \in (0, 1)$, $p \in (0, \infty)$, $q \in [1, \infty)$. In addition, we let $h \in H(\mathbb{D})$. Therefore, the next concerned general quantities are equivalent:

$$(a) \|h\|_{\mathcal{B}_{(\omega)}}^p.$$

(b) For $0 < p < \infty$, we have

$$\sup_{a \in \mathbb{D}} \frac{1}{|D(a, r)|^{1-p/2}} \iint_{\mathbb{D}} |h'(w)|^p \omega^p(1-|w|) d\zeta(w) < \infty. \quad (11)$$

(c) For $0 < p < \infty$, we have

$$\sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (1-|w|)^{p-2} \omega^p(1-|w|) d\zeta(w) < \infty. \quad (12)$$

(d) For $0 < p < \infty$ and $1 < q < \infty$, we have

$$\sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (1-|w|)^{p-2} \omega^p(1-|w|) (1-|\varphi_a(w)|^2)^q d\zeta(w) < \infty. \quad (13)$$

(e) For $0 < p < \infty$, we have

$$\sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p \left(\log \frac{1}{|w|} \right) \omega^p(1-|w|) |\varphi'_a(w)|^2 d\zeta(w) < \infty. \quad (14)$$

(f)

$$\sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (g(w, a))^p (1-|w|)^{p-2} \omega^p(1-|w|) d\zeta(w) < \infty. \quad (15)$$

Proof. For any concerned analytic-type function g on \mathbb{D} , $|g|^p$ is a concerned subharmonic function, and thus we deduce that

$$|g(0)|^p \leq \frac{1}{\pi r^2} \iint_{D(0,r)} |g(w)|^p d\zeta(w). \quad (16)$$

Employing g by $h' \circ \varphi_a$, the next inequality can be obtained:

$$|h'(a)|^p \leq \frac{1}{\pi r^2} \iint_{D(0,r)} |h' \circ \varphi_a(w)|^p d\zeta(w) = \frac{1}{\pi r^2} \iint_{D(0,r)} |h'(w)|^p \frac{(1-|\varphi_a(w)|^2)^2}{(1-|w|^2)^2} d\zeta(w). \quad (17)$$

In view of the following concerned relations,

$$\begin{aligned} \frac{1-|\varphi_a(w)|^2}{1-|w|^2} &= |\varphi'_a(w)|, \text{ also } \frac{1-|\varphi_a(w)|^2}{1-|w|^2} \\ &\leq \frac{4}{1-|a|^2} a, \quad w \in \mathbb{D} \text{ (see [14])}. \end{aligned} \quad (18)$$

Thus, the next inequality can be deduced:

$$|h'(a)|^p \leq \frac{16}{\pi r^2 (1-|a|^2)^2} \iint_{D(a,r)} |h'(w)|^p d\zeta(w). \quad (19)$$

Hence, using the known specific estimations $(1-|a|^2)^2 \sim (1-|w|^2)^2 \sim |D(a, r)|$, with $w \in D(a, r)$, the following concerned inequalities can be deduced:

$$\begin{aligned} &|h'(a)|^p (1-|a|)^p \omega^p(1-|a|) \\ &\leq \frac{16(1-|a|)^p \omega^p(1-|a|)}{\pi r^2 (1-|a|^2)^2} \iint_{D(a,r)} |h'(w)|^p d\zeta(w). \end{aligned} \quad (20)$$

Because $(1 - |a|)^2 \sim (1 - |a|^2)^2$,

$$\begin{aligned} |h'(a)|^p (1 - |a|)^p \bar{\omega}^p (1 - |a|) &\leq \frac{16(1 - |a|)^{p\alpha} \bar{\omega}^p (1 - |a|)}{\pi r^2 (1 - |a|^2)^2} \iint_{D(a,r)} |h'(w)|^p d\zeta(w) \\ &\leq \frac{16\eta}{\pi r^2 |D(a,r)|^{1-p\alpha/2}} \iint_{\mathbb{D}} |h'(w)|^p \bar{\omega}^p (1 - |w|) d\zeta(w) = \frac{m(r)}{|D(a,r)|^{1-p/2}} \iint_{\mathbb{D}} |h'(w)|^p \bar{\omega}^p (1 - |w|) d\zeta(w), \end{aligned} \quad (21)$$

where η is a specific positive constant which is greater than zero and $m(r) = 16\eta/\pi r^2$ is a concerned constant which depends on r . Hence, for the two quantities (a) and (b), we have

$$(a) \leq k_1 (b), \quad (22)$$

where k_1 stands for a positive concerned constant.

Now, since $|D(a,r)| \sim (1 - |w|^2)^2$, $\forall w \in D(a,r)$, so it is clear that (b) \sim (c). From, the concerned inequality

$$1 - |\varphi_a(w)|^2 > 1 - r^2, \quad (23)$$

the following estimates can be simply obtained:

$$\begin{aligned} &\iint_{D(a,r)} |h'(w)|^p (1 - |w|)^{p-2} \bar{\omega}^p (1 - |w|) d\zeta(w) \\ &= \iint_{D(a,r)} |h'(w)|^p (1 - |w|)^{p-2} \bar{\omega}^p (1 - |w|) \frac{(1 - |\varphi_a(w)|^2)^q}{(1 - |\varphi_a(w)|^2)^q} d\zeta(w) \\ &\leq \frac{1}{(1 - r^2)^q} \iint_{D(a,r)} |h'(w)|^p (1 - |w|)^{p-2} \bar{\omega}^p (1 - |w|) (1 - |\varphi_a(w)|^2)^q d\zeta(w). \end{aligned} \quad (24)$$

Therefore, for the two quantities (c) and (d), we have

$$(c) \leq k_2 (d), \quad (25)$$

where k_2 stands for a positive concerned constant.

It is known that $1 - |\varphi_a(w)|^2 \leq 2g(w, a)$ for all $w, a \in \mathbb{D}$, and thus we can infer that the concerned quantity (d) is less than or equal to a concerned positive constant times the concerned quantity (f).

Using the next fundamental inequalities

$$\begin{aligned} &\iint_{D(a,r)} |h'(w)|^p (1 - |w|)^{p-2} \bar{\omega}^p (1 - |w|) (g(w, a))^p d\zeta(w) \\ &= \iint_{D(a,r)} |h'(\varphi_a(w))|^p (1 - |\varphi_a(w)|)^p \bar{\omega}^p (1 - |\varphi_a(w)|) \left(\log \frac{1}{|w|} \right)^q \frac{d\zeta(w)}{(1 - |w|^2)^2} \leq \|h\|_{\mathcal{B}_{\bar{\omega}}^\alpha}^p \iint_{D(a,r)} \left(\log \frac{1}{|w|} \right)^q \frac{d\zeta(w)}{(1 - |w|^2)^2}, \end{aligned} \quad (26)$$

where

$$\mathcal{T} = \iint_{D(a,r)} \left(\log \frac{1}{|w|} \right)^q (1 - |w|^2)^{-2} d\zeta(w) < \infty, \quad (27)$$

we can easily infer that the concerned quantity (e) is less than or equal to a specific positive concerned constant times the concerned quantity (a).

With the help of the known inequality $1 - |w|^2 \leq 2 \log 1/|w|$, where $w \in \mathbb{D}$, setting $q = 2$ in the concerned quantity

(d), we simply deduce that the concerned quantity (d) is also less than or equal to the concerned quantity (e). For the final step in the concerned proof, we consider the following estimates.

$$\begin{aligned}
 J &= \iint_{D(a,r)} |h'(w)|^p \left(\log \frac{1}{|w|} \right)^q \varpi^p (1-|w|) |\varphi'_a(w)|^2 d\zeta(w) \\
 &= \left(\iint_{D_{1/4}} + \iint_{\mathbb{D}/D_{(1/4)}} \right) |h'(w)|^p \left(\log \frac{1}{|w|} \right)^q \varpi^p (1-|w|) |\varphi'_a(w)|^2 d\zeta(w) \\
 &= J_1 + J_2,
 \end{aligned} \tag{28}$$

with $w \in D_{1/4} = \{w: |w| < 1/4\}$, $|\varphi'_a(w)|^2 = (1-|a|^2)/|1-\bar{a}w|^4 \leq 1/|1-|w||^4 \leq (4/3)^4$, and thus

$$\begin{aligned}
 J_1 &= \iint_{D_{1/4} \in J_1} |h'(w)|^p \left(\log \frac{1}{|w|} \right)^p \varpi^p (1-|w|) |\varphi'_a(w)|^2 d\zeta(w) \\
 &\leq \|h\|_{\mathcal{B}^a(\varpi)}^p \iint_{D_{1/4}} \left(\frac{(\log 1/|w|)}{(1-|w|)} \right)^p |\varphi'_a(w)|^2 d\zeta(w) \leq \|h\|_{\mathcal{B}^a(\varpi)}^p \left(\frac{4}{3} \right)^{p+4} \iint_{D_{1/4}} \left(\log \frac{1}{|w|} \right)^p d\zeta(w) = \left(\frac{4}{3} \right)^{p+4} T(p) \|h\|_{\mathcal{B}^a(\varpi)}^p,
 \end{aligned} \tag{29}$$

where the integral

$$T(p) = \iint_{D_{1/4}} \left(\log \frac{1}{|w|} \right)^p d\zeta(w) < \infty. \tag{30}$$

For each specific point $w \in \mathbb{D}/(1/4)$, the following useful estimates hold:

$$\log \frac{1}{|w|} \leq 4(1-|w|^2) \leq 8(1-|w|), \tag{31}$$

which implies that

$$\begin{aligned}
 J_2 &\leq 8 \\
 \iint_{\mathbb{D}/D_{(1/4)}} |h'(w)|^p \left(\log \frac{1}{|w|} \right)^q \varpi^p (1-|w|) |\varphi'_a(w)|^2 d\zeta(w) &\leq 8^p \|h\|_{\mathcal{B}^a(\varpi)}^p \iint_{\mathbb{D}/D_{(1/4)}} |\varphi'_a(w)|^2 d\zeta(w) \leq k_3 \|h\|_{\mathcal{B}^a(\varpi)}^p,
 \end{aligned} \tag{32}$$

where k_3 is a concerned positive constant. Thus, the concerned general quantity (e) is less than or equal to a concerned positive constant times the concerned general quantity (a). Thus, the interesting proof of Theorem 1 is completely established clearly. In Theorem 1, assuming that $\varpi(1-|w|) \equiv 1$, the following concerned corollary can be obtained clearly. \square

Corollary 1. Let $r \in (0, 1)$, $p \in (0, \infty)$, $q \in [1, \infty)$ and let $h \in H(\mathbb{D})$. Therefore, the next concerned general quantities are equivalent:

$$(a^*) \|h\|_{\mathcal{B}^a(\varpi)}^p.$$

(b*) For $0 < p < \infty$, we have

$$\sup_{a \in \mathbb{D}} \frac{1}{|D(a, r)|^{1-p/2}} \iint_{\mathbb{D}} |h'(w)|^p d\zeta(w) < \infty. \tag{33}$$

(c*) For $0 < p < \infty$, we have

$$\sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (1-|w|)^{p-2} d\zeta(w) < \infty. \tag{34}$$

(d*) For $0 < p < \infty$ and $1 < q < \infty$, we have

$$\sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (1-|w|)^{p-2} (1-|\varphi_a(w)|^2)^q d\zeta(w) < \infty. \tag{35}$$

(e*) For $0 < p < \infty$, we have

$$\sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p \left(\log \frac{1}{|w|} \right)^p |\varphi_a(w)|^2 d\zeta(w) < \infty. \quad (36)$$

(f*)

$$\sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (g(w, a))^p (1 - |w|)^{p-2} d\zeta(w) < \infty. \quad (37)$$

For the concerned holomorphic function classes $\mathcal{B}_{(\omega, 0)}$, we give the following corresponding result induced from Theorem 1 on the specific boundary of the concerned unit disc \mathbb{D} .

Theorem 2. Let $r \in (0, 1)$, $p \in (0, \infty)$, $q \in [1, \infty)$ and let $h \in H(\mathbb{D})$. Therefore, the next concerned general quantities are equivalent:

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p \omega^p (1 - |w|)^{p-2} \omega^p (1 - |w|) (1 - |\varphi_a(w)|^2) d\zeta(w) < \infty. \quad (40)$$

(a₅) For $0 < p < \infty$, we have

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p \left(\log \frac{1}{|w|} \right)^p \omega^p (1 - |w|) |\varphi'_a(w)|^2 d\zeta(w) < \infty. \quad (41)$$

(a₆)

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (g(w, a))^p (1 - |w|)^{p-2} \omega^p (1 - |w|) d\zeta(w) < \infty. \quad (42)$$

In Theorem 2, assuming that $\omega(1 - |w|) \equiv 1$, the next interesting corollary can be established directly.

Corollary 2. Let $r \in (0, 1)$, $p \in (0, \infty)$, $q \in [1, \infty)$ and let $h \in H(\mathbb{D})$. Therefore, the next concerned general statements are equivalent:

$$(b_1) \|h\|_{\mathcal{B}_{(\omega, 0)}}^p.$$

(b₂) For $0 < p < \infty$, we have

$$(a_1) \|h\|_{\mathcal{B}_{(\omega, 0)}}^p.$$

(a₂) For $0 < p < \infty$, we have

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \frac{1}{|D(a, r)|^{1-p/2}} \iint_{\mathbb{D}} |h'(w)|^p \omega^p (1 - |w|) d\zeta(w) < \infty. \quad (38)$$

(a₃) For $0 < p < \infty$, we have

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p \omega^p (1 - |w|)^{p-2} \omega^p (1 - |w|) d\zeta(w) < \infty. \quad (39)$$

(a₄) For $0 < p < \infty$ and $1 < q < \infty$, we have

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \frac{1}{|D(a, r)|^{1-p/2}} \iint_{\mathbb{D}} |h'(w)|^p d\zeta(w) < \infty. \quad (43)$$

(b₃) For $0 < p < \infty$, we have

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (1 - |w|)^{p-2} d\zeta(w) < \infty. \quad (44)$$

(b₄) For $0 < p < \infty$ and $1 < q < \infty$, we have

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (1-|w|)^{p-2} (1-|\varphi_a(w)|^2)^q d\zeta(w) < \infty. \quad (45)$$

(b₅) For $0 < p < \infty$, we have

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p \left(\log \frac{1}{|w|} \right)^p |\varphi'_a(w)|^2 d\zeta(w) < \infty. \quad (46)$$

(b₆)

$$\lim_{|a| \rightarrow 1^-} \sup_{a \in \mathbb{D}} \iint_{\mathbb{D}} |h'(w)|^p (g(w, a))^p (1-|w|)^{p-2} d\zeta(w) < \infty. \quad (47)$$

3. More Analytic Characterizations

Extension of Kwon et al.'s result (see [4]) is also established in this current section with the help of holomorphic $\mathcal{B}_{\bar{\omega}}$ -type functions. Holomorphic function significant in both weighted $\bar{\omega}$ -Besov norms and $\mathcal{B}_{\bar{\omega}}$ -Bloch norms framework and to introduce some new concerned families of holomorphic-type function classes.

Theorem 3. Let $0 < p < \infty$, $0 \leq s < \infty$, with $0 < p < s + 1$ and let $h \in H(\mathbb{D})$. Thus,

$$\|h\|_{\mathcal{B}_{\bar{\omega}}}^p \approx \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1-|w|^2)^{p-2} \bar{\omega}^p (1-|w|) (1-|\varphi_a(w)|^2)^p (g(w, a))^{-s} d\zeta(w). \quad (48)$$

Proof. In view of condition (b) in Theorem 1, we deduce that

$$\|h\|_{\mathcal{B}_{\bar{\omega}}}^p \approx \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1-|w|^2)^{p-2} \bar{\omega}^p (1-|w|) d\zeta(w). \quad (49)$$

Set a concerned finite specific positive constant δ_R by

$$\delta_R: (1-R^2)^p \left(\log \frac{1}{1-R^2} \right)^{-s}. \quad (50)$$

Thus, we deduce that

$$\|h\|_{\mathcal{B}_{\bar{\omega}}}^p \approx \sup_{a \in \mathbb{D}} \int_{D(a, R)} |h'(w)|^p (1-|w|^2)^{p-2} \bar{\omega}^p (1-|w|) \left(\log \frac{1}{1-R^2} \right)^{-s} (1-R^2)^p d\zeta(w). \quad (51)$$

Using the inequality

$$|\varphi_a(w)|^2 < R^2 \Leftrightarrow 1-|\varphi_a(w)|^2 > 1-R^2, \quad (52)$$

we infer that

$$\|h\|_{\mathcal{B}_{\bar{\omega}}}^p \approx \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1-|w|^2)^{p-2} \bar{\omega}^p (1-|w|) (1-|\varphi_a(w)|^2)^p (g(w, a))^{-s} d\zeta(w). \quad (53)$$

Thus,

$$\|h\|_{\mathcal{B}_\infty}^p \leq k^* \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1-|w|^2)^{p-2} \bar{\omega}^p (1-|w|)(1-|\varphi_a(w)|^2)^p (g(w,a))^{-s} d\zeta(w). \quad (54)$$

where k^* is a specific positive constant that does not depend on a .

Now for the other direction, in view of changing variable base, that is replacing w by $\varphi_a(w)$, the following estimates can be deduced:

$$\begin{aligned} & \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1-|w|^2)^{p-2} \bar{\omega}^p (1-|w|)(1-|\varphi_a(w)|^2)^p (g(w,a))^{-s} d\zeta(w) \\ & \leq \|h\|_{\mathcal{B}_\infty}^p \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (1-|w|^2)^{p-2} (1-|\varphi_a(w)|^2)^p (g(w,a))^{-s} d\zeta(w) \\ & \left(= \|h\|_{\mathcal{B}_\infty}^p \sup_{a \in \mathbb{D}} \right) \int_{\mathbb{D}} (1-|\varphi_a(w)|^2)^p (1-|w|^2)^p \left(\log \frac{1}{|w|} \right)^{-s} \left(\frac{1-|a|^2}{|1-\bar{a}w|^2} \right)^2 d\zeta(w). \end{aligned} \quad (55)$$

With the help of the known equality (1), the following estimates can be obtained:

$$\begin{aligned} & \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1-|w|^2)^{p-2} \bar{\omega}^p (1-|w|)(1-|\varphi_a(w)|^2)^p (g(w,a))^{-s} d\zeta(w) \\ & \leq \|h\|_{\mathcal{B}_\infty}^p \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (1-|w|^2)^{p-2} \left(\log \frac{1}{|w|} \right)^{-s} d\zeta(w) \\ & \leq (2)^{3p-2} \|h\|_{\mathcal{B}_\infty}^p \int_0^1 (1+R)^{-s} (1-R)^{p-s-2} dR \\ & \leq 4\pi (2)^{3p-s-2} \left| h_{\mathcal{B}_\infty}^p \int_0^1 (1-R)^{p-s-2} dR \right| < (2)^{3p-s} \pi \left| h_{\mathcal{B}_\infty}^p I(R, p, s), \right. \end{aligned} \quad (56)$$

where the specific integral $I(R, p, s) = \int_0^1 (1-R)^{p-s-2} dR < +\infty$ when $p < s+1$. Therefore,

$$\|h\|_{\mathcal{B}_\infty}^p \geq k_1^* \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |h'(w)|^p (1-|w|^2)^{p-2} \bar{\omega}^p (1-|w|)(1-|\varphi_a(w)|^2)^p (g(w,a))^{-s} d\zeta(w). \quad (57)$$

where the specific constant

$$k_1^* = \frac{1}{(2)^{s-3p} (\pi) I(p, s)}. \quad (58)$$

Thus, the concerned proof of our desired result is completely obtained. \square

Remark 3. Various research studies on advanced operator theory using numerous weighted classes of holomorphic functions have been actively appearing in some joyful important areas of mathematical sciences such as the theory of dynamical systems, the known probability theory, recent research in mathematical physics, and some branches of quantum mechanics. Such holomorphic

weighted classes of concerned function spaces are still under interest of considerations in the aforementioned applications. The defined holomorphic \mathfrak{Q} -Bloch spaces as well as the holomorphic \mathfrak{Q} -Besov spaces can be also applied in such interesting applications.

4. Conclusion

General analytic characterizations for some concerned extended classes of holomorphic Banach of function spaces of Bloch and Besov type are established and discussed in this article. The concerned proofs are obtained using two types of concerned holomorphic functions in \mathbb{D} . Both the considered functions and the proofs methods have concerned parameters which make deep help to the obtained results. As specific results, various new theorems and lemmas for the considered function classes are well derived. Further, the obtained concerned generalized results proved that the used function classes give better performance compared with existing results in the literature. The n -th partial derivative concerned quantities with general formulas including the extended type of functions are also well derived. The well-known analytic Bloch, the analytic Besov, and the Zygmund spaces are introduced as specific concerned special cases of the new defined classes. The concerned computations of the extended Bloch-type norm and the extended Besov-type norms related to these special cases can be established too.

The analytic characterizations of integral norms, the extended Besov space, and the specific analytic Bloch norms will give interesting general results and recent developments within this fascinating area of function spaces. Such concerned advanced aspects may be considered in some areas of further studies.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to thank Taif University Researchers Supporting Project (no. TURSP-2020/159), Taif University, Saudi Arabia. TURSP-2020/159

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Research Article

A Criterion for the Existence of the Unique Periodic Solution of One-Dimensional Periodic Differential Equation

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Received 10 February 2021; Revised 3 March 2021; Accepted 10 March 2021; Published 22 April 2021

Academic Editor: Huseyin Isik

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In this paper, we discuss one-dimensional differential equation with ω -period. By using the fixed point theory, the existence of a periodic solution is obtained; by using the second Lyapunov method, the uniqueness and stability of the periodic solution are obtained.

1. Introduction

Consider the first-order nonautonomous differential equation:

$$\frac{dx}{dt} = f(t, x), \quad (1)$$

where $t, x \in \mathbb{R}$ and $f(t, x)$ is continuous and is an ω -periodic function on t .

As we all know, the first-order differential equation (1) is widely used to establish mathematical models in many fields, such as physics, biology, economy, and medicine. Because the nonautonomous differential equations have important practical applications, it has been a research subject of many scientists for a long time. The main problems studied are the existence, stability, and number of periodic solutions (see [1–8]).

In [9], the authors gave several methods to study periodic solutions of one-dimensional ω -periodic differential equations, including Poincaré's map, the stability of zero solution and its multiplicity, normal form, and averaging method. They summarized and improved some existing results and obtained some new conclusions as well.

As we all know, the existence, uniqueness, and stability of periodic solutions of differential equations have always been an important research hotspot in the field of differential equations (see [10–20]). However, the above literatures are

basically about the study of periodic solutions of specific equations, rather than the study of general differential equations.

In [21], the existence of a periodic solution for Abel's differential equation is obtained first by using the fixed-point theorem. By constructing the Lyapunov function, the uniqueness, and stability of the periodic solution of the equation are obtained.

Stimulated by the works of [21], in this paper, we consider more general differential equation (1). By using the fixed point theorem and constructing Lyapunov function, we give a new criterion for the existence, uniqueness, and stability of the periodic solutions of the equation (1). These results generalize some related results in some literatures.

2. Some Lemmas and Abbreviations

Lemma 1 (see [22]). *Consider the equation:*

$$\frac{dx}{dt} = a(t)x + b(t), \quad (2)$$

where $a(t)$ and $b(t)$ are ω -periodic continuous functions on \mathbb{R} . If $\int_0^\omega a(t)dt \neq 0$, then (2) has a unique ω -periodic continuous solution $\eta(t)$, $\text{mod}(\eta(t)) \subseteq \text{mod}(a(t), b(t))$, and $\eta(t)$ can be written as

$$\eta(t) = \begin{cases} \int_{-\infty}^t e^{\int_s^t a(\mu) d\mu} b(s) ds, & \int_0^\omega a(t) dt < 0, \\ -\int_t^{+\infty} e^{\int_s^t a(\mu) d\mu} b(s) ds, & \int_0^\omega a(t) dt > 0. \end{cases} \quad (3)$$

Lemma 2 (see [23]). Assume that an ω -periodic sequence $\{f_n(t)\}$ is convergent uniformly on any compact set of \mathbb{R} , $f(t)$ is an ω -periodic function, and $\text{mod}(f_n) \subseteq \text{mod}(f)$ ($n = 1, 2, \dots$); then, $\{f_n(t)\}$ is convergent uniformly on \mathbb{R} .

Lemma 3 (see [24]). Assume \mathbb{V} is a metric space and \mathbb{C} is a convex closed set of \mathbb{V} ; its boundary is $\partial\mathbb{C}$ if $T: \mathbb{V} \rightarrow \mathbb{V}$ is a continuous compact mapping, such that $T(\partial\mathbb{C}) \subseteq \mathbb{C}$, then T has at least a fixed point on \mathbb{C} .

For convenience, assume that $f(t)$ is an ω -periodic continuous function on \mathbb{R} , we denote

$$\begin{aligned} f_M &= \sup_{t \in [0, \omega]} f(t), \\ f_L &= \inf_{t \in [0, \omega]} f(t). \end{aligned} \quad (4)$$

3. Main Results

In this section, firstly, we use the mean value theorem to transform equation (1) into an equivalent equation. Then, we use the fixed point theorem to get the existence of periodic solution of equation (1). Finally, we use Lyapunov function method to obtain the uniqueness and stability of the periodic solution.

Theorem 1. Consider equation (1), $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $f(t + \omega, x) = f(t, x)$, and f'_x exists and is a continuous function, and $r(t)$ is an ω -periodic continuous function on \mathbb{R} . Assume that the following conditions hold:

$$\begin{aligned} (H_1) f[t, r(t)] &= 0, \\ (H_2) f'_x(t, x) &\leq 0 \quad (\equiv 0), \end{aligned} \quad (5)$$

then (1) equation (1) has a unique ω -periodic continuous solution $\Phi(t)$ and (2) $\Phi(t)$ is globally attractive.

Proof. (1) By (H_1) and differential mean value theorem, equation (1) is transformed into the equation:

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) \\ &= f(t, x) - f[t, r(t)] \\ &= f'_x[t, r(t) + \theta_1(x)(x - r(t))](x - r(t)), \\ &\quad (0 < \theta_1(x) < 1). \end{aligned} \quad (6)$$

Noting that $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $f(t + \omega, x) = f(t, x)$, f'_x exists and is a continuous

function, and from (6), it follows that $f'_x[t, r(t) + \theta_1(x)(x - r(t))]$ is continuous and

$$\begin{aligned} &f'_x[t + \omega, r(t + \omega) + \theta_1(x)(x - r(t + \omega))] \\ &= f'_x[t, r(t) + \theta_1(x)(x - r(t))]. \end{aligned} \quad (7)$$

Defining

$$\mathbb{S} = \{\psi(t) \in C(\mathbb{R}, \mathbb{R}) | \psi(t + \omega) = \psi(t)\}, \quad (8)$$

$\forall \varphi(t), \psi(t) \in \mathbb{S}$, the distance is defined as follows:

$$\rho(\varphi, \psi) = \sup_{t \in [0, \omega]} |\varphi(t) - \psi(t)|, \quad (9)$$

and thus, (\mathbb{S}, ρ) is a complete metric space.

Take a convex closed set \mathbb{B} of \mathbb{S} as follows:

$$\mathbb{B} = \{\psi(t) \in \mathbb{S} | r_L \leq \psi(t) \leq r_M, \text{mod}(\psi) \subseteq \text{mod}(f)\}, \quad (10)$$

$\forall \psi(t) \in \mathbb{B}$, and consider the equation:

$$\begin{aligned} \frac{dx}{dt} &= f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))](x - r(t)) \\ &= f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))]x \\ &\quad - f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))]r(t) \\ &= f'_{x_\psi}(t)x - f'_{x_\psi}(t)r(t), \end{aligned} \quad (11)$$

where

$$f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))] \triangleq f'_{x_\psi}(t). \quad (12)$$

It follows from (7), (10), and (12) that $f'_{x_\psi}(t)$ is an ω -periodic continuous function on $t \in \mathbb{R}$; thus, there is a positive number δ such that

$$|f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))]| \leq \delta. \quad (13)$$

It follows from (H_2) that

$$\int_0^\omega f'_{x_\psi}(t) dt < 0. \quad (14)$$

Since $f'_{x_\psi}(t)$ and $r(t)$ are ω -periodic continuous functions, it follows

$$f'_{x_\psi}(t)r(t) \quad (15)$$

is an ω -periodic continuous function. It follows from (14) and Lemma 1 that equation (11) has a unique ω -periodic continuous solution as follows:

$$\eta(t) = - \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s)r(s) ds, \quad (16)$$

$$\text{mod}(\eta) \subseteq \text{mod}\left(f'_{x_\psi}(t), f'_{x_\psi}(t)r(t)\right). \quad (17)$$

It follows from (6), (10), and (11) that

$$\begin{aligned} \text{mod}\left(f'_{x_\psi}(t)\right) &\subseteq \text{mod}(f), \\ \text{mod}\left(f'_{x_\psi}(t)r(t)\right) &\subseteq \text{mod}(f), \end{aligned} \quad (18)$$

and hence, we have

$$\text{mod}(\eta) \subseteq \text{mod}(f). \quad (19)$$

It follows from (10), (14), and (16) that

$$\begin{aligned} \eta(t) &\geq -r_L \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) ds \\ &= r_L \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} d\left(\int_s^t f'_{x_\psi}(\mu) d\mu\right) \\ &= r_L \left[e^{\int_s^t f'_{x_\psi}(\mu) d\mu} \right]_{-\infty}^t \\ &= r_L \left[1 - e^{\int_{-\infty}^t f'_{x_\psi}(\mu) d\mu} \right] (-\infty < t < +\infty) \\ &= r_L, \\ \eta(t) &\leq -r_M \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) ds \\ &= r_M \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} d\left(\int_s^t f'_{x_\psi}(s) d\mu\right) \\ &= r_M \left[e^{\int_s^t f'_{x_\psi}(\mu) d\mu} \right]_{-\infty}^t \\ &= r_M \left[1 - e^{\int_{-\infty}^t f'_{x_\psi}(\mu) d\mu} \right] (-\infty < t < +\infty) \\ &= r_M, \end{aligned} \quad (20)$$

and therefore, $\eta(t) \in \mathbb{B}$.

Define a mapping as follows:

$$(T\psi)(t) = - \int_{-\infty}^t e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) r(s) ds, \quad (21)$$

and thus, if given any $\psi(t) \in \mathbb{B}$, then $(T\psi)(t) \in \mathbb{B}$; hence, $T: \mathbb{B} \rightarrow \mathbb{B}$.

Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\psi_n(t)\} \subseteq \mathbb{B} (n = 1, 2, \dots)$; then, it follows

$$r_L \leq \psi_n(t) \leq r_M, \text{mod}(\psi_n) \subseteq \text{mod}(f), \quad (n = 1, 2, \dots). \quad (22)$$

On the other hand, $(T\psi_n)(t) = x_{\psi_n}(t)$ satisfies

$$\begin{aligned} \frac{dx_{\psi_n}(t)}{dt} &= f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]x_{\psi_n}(t) \\ &\quad - f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]r(t) \\ s &= f'_{x_{\psi_n}}(t)x_{\psi_n}(t) - f'_{x_{\psi_n}}(t)r(t), \end{aligned} \quad (23)$$

where

$$f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))] \triangleq f'_{x_{\psi_n}}(t). \quad (24)$$

Since $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $f(t + \omega, x) = f(t, x)$, f'_x exists and is a continuous function, and from (23) and $\{\psi_n(t)\} \subseteq \mathbb{B} (n = 1, 2, \dots)$, it follows that $f'_{x_{\psi_n}}(t)$ is an ω -periodic continuous function on $t \in \mathbb{R}$.

It follows from (10), (13), (22), and (24) that

$$|f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]| \leq \delta. \quad (25)$$

Thus, we have

$$\left| \frac{dx_{\psi_n}(t)}{dt} \right| \leq 2|f'_{x_{\psi_n}}(t)|r|_M \leq 2\delta|r|_M, \quad (26)$$

$$\text{mod}(x_{\psi_n}(t)) \subseteq \text{mod}(f). \quad (27)$$

Hence, $\{dx_{\psi_n}(t)/dt\}$ is uniformly bounded; therefore, $\{x_{\psi_n}(t)\}$ is uniformly bounded and equi-continuous on \mathbb{R} . By the theorem of Ascoli-Arzelà, for any sequence $\{x_{\psi_n}(t)\} \subseteq \mathbb{B}$, there exists a subsequence (also denoted by $\{x_{\psi_n}(t)\}$) such that $\{x_{\psi_n}(t)\}$ is convergent uniformly on any compact set of \mathbb{R} . By (27), combined with Lemma 2, $\{x_{\psi_n}(t)\}$ is convergent uniformly on \mathbb{R} , that is to say, T is relatively compact on \mathbb{B} .

Next, we prove that T is a continuous mapping.

Assume $\{\psi_n(t)\} \subseteq \mathbb{B}$, $\psi(t) \in \mathbb{B}$, and

$$\psi_n(t) \rightarrow \psi(t), \quad (n \rightarrow \infty). \quad (28)$$

Since $f(t, x)$ has first-order continuous partial derivative on x , it follows from (28) that

$$\begin{aligned} &f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))] \\ &\rightarrow f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))], \end{aligned} \quad (29)$$

$(n \rightarrow \infty),$

that is,

$$f'_{x_{\psi_n}}(t) \rightarrow f'_{x_\psi}(t), \quad (n \rightarrow \infty). \quad (30)$$

It follows from (21) that

$$\begin{aligned}
 & |(T\psi_n)(t) - (T\psi)(t)| \\
 &= \left| \int_{-\infty}^t e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} f'_{x_{\psi_n}}(s) r(s) ds - \int_{-\infty}^t e^{\int_s^t f'_{x_{\psi}}(\mu) d\mu} f'_{x_{\psi}}(s) r(s) ds \right| \\
 &= \left| \int_{-\infty}^t e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(s) - f'_{x_{\psi}}(s)) r(s) ds \right. \\
 &\quad \left. + \int_{-\infty}^t \left(e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} - e^{\int_s^t f'_{x_{\psi}}(\mu) d\mu} \right) f'_{x_{\psi}}(s) r(s) ds \right| \\
 &= \left| \int_{-\infty}^0 e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(t+s) - f'_{x_{\psi}}(t+s)) r(t+s) ds \right. \\
 &\quad \left. + \int_{-\infty}^0 \left(e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} - e^{\int_{s+t}^t f'_{x_{\psi}}(\mu) d\mu} \right) f'_{x_{\psi}}(t+s) r(t+s) ds \right| \\
 &= \left| \int_{-\infty}^0 e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(t+s) - f'_{x_{\psi}}(t+s)) r(t+s) ds \right. \\
 &\quad \left. + \int_{-\infty}^0 e^{\xi} \left(\int_{s+t}^t (f'_{x_{\psi_n}}(\mu) - f'_{x_{\psi}}(\mu)) d\mu \right) f'_{x_{\psi}}(t+s) r(t+s) ds \right| \\
 &\leq \left(\int_{-\infty}^0 e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds + \int_{-\infty}^0 e^{\xi} \left(\int_{s+t}^t d\mu \right) |f'_{x_{\psi}}(s+t)| |r(s+t)| ds \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}) \\
 &= \left(\int_{-\infty}^{-T_0} e^{\int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds + \int_{-T_0}^0 e^{\int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds \right. \\
 &\quad \left. + \int_{-\infty}^{-T_0} e^{\xi} \left(\int_{t+s}^t d\mu \right) |f'_{x_{\psi}}(s+t)| |r(s+t)| ds \right. \\
 &\quad \left. + \int_{-T_0}^0 e^{\xi} \left(\int_{t+s}^t d\mu \right) |f'_{x_{\psi}}(s+t)| |r(s+t)| ds \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}),
 \end{aligned} \tag{31}$$

where ξ is between $\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu$ and $\int_{s+t}^t f'_{x_{\psi}}(\mu) d\mu$. Since $f'_{x_{\psi_n}}(t)$ and $f'_{x_{\psi}}(t)$ are ω -periodic continuous functions on $t \in \mathbb{R}$, it follows

$$\begin{aligned}
 \lim_{s \rightarrow -\infty} \frac{1}{-s} \int_s^0 f'_{x_{\psi_n}}(\mu) d\mu &= \frac{1}{\omega} \int_0^\omega f'_{x_{\psi_n}}(\mu) d\mu \triangleq -\lambda_1, \\
 \lim_{s \rightarrow -\infty} \frac{1}{-s} \int_s^0 f'_{x_{\psi}}(\mu) d\mu &= \frac{1}{\omega} \int_0^\omega f'_{x_{\psi}}(\mu) d\mu \triangleq -\lambda_2.
 \end{aligned} \tag{32}$$

Thus, there is a $T_0 > 0$ (also, assume it is T_0 in the above) when $s \leq -T_0$, such that

$$\begin{aligned}
 \left| \frac{-1}{s} \int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu + \lambda_1 \right| &< \frac{\lambda_1}{2}, \\
 \left| \frac{-1}{s} \int_{t+s}^t f'_{x_{\psi}}(\mu) d\mu + \lambda_2 \right| &< \frac{\lambda_2}{2},
 \end{aligned} \tag{33}$$

that is,

$$\begin{aligned}
 \int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu &< \frac{\lambda_1 s}{2}, \\
 \int_{t+s}^t f'_{x_{\psi}}(\mu) d\mu &< \frac{\lambda_2 s}{2}.
 \end{aligned} \tag{34}$$

Set

$$\frac{\lambda_0 s}{2} = \max\left(\frac{\lambda_1 s}{2}, \frac{\lambda_2 s}{2}\right). \tag{35}$$

Hence, we have

$$\begin{aligned}
 & |(T\psi_n)(t) - (T\psi)(t)| \\
 &\leq \left(\int_{-\infty}^{-T_0} e^{(\lambda_0 s/2)} |r|_M ds + T_0 e^{\delta T_0} |r|_M - \int_{-\infty}^{-T_0} e^{(\lambda_0 s/2)} s \delta |r|_M ds + T_0^2 e^{\delta T_0} |r|_M \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}) \\
 &= |r|_M \left(\frac{2}{\lambda_0} e^{-(\lambda_0 T_0/2)} + T_0 e^{\delta T_0} + \frac{4}{\lambda_0^2} e^{-(\lambda_0 T_0/2)} \delta + T_0^2 e^{\delta T_0} \delta \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}).
 \end{aligned} \tag{36}$$

It follows from (30) and above inequality that

$$(T\psi_n)(t) \longrightarrow (T\psi)(t), \quad (n \longrightarrow \infty). \quad (37)$$

Therefore, T is continuous. By (21), easy to see, $T(\partial\mathbb{B}) \subseteq \mathbb{B}$. According to Lemma 3, T has at least a fixed point on \mathbb{B} , the fixed point is the ω -periodic continuous solution $\Phi(t)$ of equation (1), and

$$r_L \leq \Phi(t) \leq r_M. \quad (38)$$

(2) Construct a Lyapunov function as follows:

$$V(t, x(t)) = (x(t) - \Phi(t))^2, \quad (39)$$

where $x(t)$ is the unique solution with initial value $x(t_0) = x_0$ of equation (1) and $\Phi(t)$ is the periodic solution of equation (1).

Differentiating both sides of (39) along the solution of equation (1), we have

$$\begin{aligned} \frac{dV(t, x(t))}{dt} &= 2(x(t) - \Phi(t)) \left[\frac{dx(t)}{dt} - \frac{d\Phi(t)}{dt} \right] \\ &= 2(x(t) - \Phi(t)) [f(t, x(t)) - f(t, \Phi(t))] \\ &= 2(x(t) - \Phi(t)) [f'_x(t, \zeta)(x(t) - \Phi(t))] \\ &= 2f'_x(t, \zeta)(x(t) - \Phi(t))^2, \end{aligned} \quad (40)$$

where ζ is between $x(t)$ and $\Phi(t)$.

By (H_2) , it follows

$$|x(t) - \Phi(t)| \longrightarrow 0, \quad (t \longrightarrow +\infty). \quad (41)$$

Thus, $x(t)$ cannot be a periodic solution of equation (1), and it is proved that equation (1) has a unique ω -periodic continuous solution $\Phi(t)$ which is globally attractive.

This is the end of the proof of Theorem 1. \square

Theorem 2. Consider equation (1), $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and $f(t + \omega, x) = f(t, x)$, and f'_x exists and is a continuous function; $r(t)$ is an ω -periodic continuous function on \mathbb{R} . Assume that the following conditions hold:

$$\begin{aligned} (H_1) f[t, r(t)] &= 0, \\ (H_2) f'_x(t, x) &\geq 0 \quad (\equiv 0). \end{aligned} \quad (42)$$

Then, (1) equation (1) has a unique ω -periodic continuous solution $\Phi(t)$ and (2) $\Phi(t)$ is unstable.

Proof.

(1) By (H_1) and differential mean value theorem, equation (1) is transformed into the equation

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) = f(t, x) - f[t, r(t)] \\ &= f'_x[t, r(t) + \theta_1(x)(x - r(t))](x - r(t)), \\ &\quad (0 < \theta_1(x) < 1). \end{aligned} \quad (43)$$

Noting that $f: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $f(t + \omega, x) = f(t, x)$, f'_x exists and is a continuous function, and from (43), it follows that $f'_x[t, r(t) + \theta_1(x)(x - r(t))]$ is continuous and

$$\begin{aligned} f'_x[t + \omega, r(t + \omega) + \theta_1(x)(x - r(t + \omega))] \\ = f'_x[t, r(t) + \theta_1(x)(x - r(t))]. \end{aligned} \quad (44)$$

Defining

$$\mathbb{S} = \{\psi(t) \in C(\mathbb{R}, \mathbb{R}) | \psi(t + \omega) = \psi(t)\}, \quad (45)$$

$\forall \varphi(t), \psi(t) \in \mathbb{S}$, the distance is defined as follows:

$$\rho(\varphi, \psi) = \sup_{t \in [0, \omega]} |\varphi(t) - \psi(t)|. \quad (46)$$

Thus, (\mathbb{S}, ρ) is a complete metric space.

Take a convex closed set \mathbb{B} of \mathbb{S} as follows:

$$\mathbb{B} = \{\psi(t) \in \mathbb{S} | r_L \leq \psi(t) \leq r_M, \text{mod}(\psi) \subseteq \text{mod}(f)\}, \quad (47)$$

$\forall \psi(t) \in \mathbb{B}$, and consider the equation

$$\begin{aligned} \frac{dx}{dt} &= f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))](x - r(t)) \\ &= f'_x[t, r(t) + \theta_1(\psi(t))(\psi - r(t))]x - f'_x[t, r(t) \\ &\quad + \theta_1(\psi(t))(\psi - r(t))]r(t) \\ &= f'_{x_\psi}(t)x - f'_{x_\psi}(t)r(t), \end{aligned} \quad (48)$$

where

$$f'_x[t, r(t) + \theta_1(\psi(t))(\psi - r(t))] \triangleq f'_{x_\psi}(t). \quad (49)$$

It follows from (44), (47), and (49) that $f'_{x_\psi}(t)$ is an ω -periodic continuous function on $t \in \mathbb{R}$; thus, there is a positive number δ such that

$$|f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))]| \leq \delta. \quad (50)$$

It follows from (H_2) that

$$\int_0^\omega f'_{x_\psi}(t) dt > 0. \quad (51)$$

Since $f'_{x_\psi}(t)$ and $r(t)$ are ω -periodic continuous functions, it follows

$$f'_{x_\psi}(t)r(t) \quad (52)$$

is an ω -periodic continuous function. It follows from (51) and Lemma 1 that equation (48) has a unique ω -periodic continuous solution as follows:

$$\eta(t) = \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) r(s) ds, \quad (53)$$

$$\text{mod}(\eta) \subseteq \text{mod}\left(f'_{x_\psi}(t), f'_{x_\psi}(t) r(t)\right). \quad (54)$$

It follows from (43), (47), and (48) that

$$\text{mod}\left(f'_{x_\psi}(t)\right) \subseteq \text{mod}(f), \quad (55)$$

$$\text{mod}\left(f'_{x_\psi}(t) r(t)\right) \subseteq \text{mod}(f).$$

Hence, we have

$$\text{mod}(\eta) \subseteq \text{mod}(f). \quad (56)$$

It follows from (47), (51), and (53) that

$$\begin{aligned} \eta(t) &\geq r_L \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) ds \\ &= -r_L \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} d\left(\int_s^t f'_{x_\psi}(\mu) d\mu\right) \\ &= -r_L \left[e^{\int_s^t f'_{x_\psi}(\mu) d\mu} \right]_t^{+\infty} \\ &= -r_L \left[e^{\int_{+\infty}^t f'_{x_\psi}(\mu) d\mu} - 1 \right], \quad (-\infty < t < +\infty) \\ &= r_L, \end{aligned}$$

$$\begin{aligned} \eta(t) &\leq r_M \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) ds \\ &= -r_M \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} d\left(\int_s^t f'_{x_\psi}(\mu) d\mu\right) \\ &= -r_M \left[e^{\int_s^t f'_{x_\psi}(\mu) d\mu} \right]_t^{+\infty} \\ &= -r_M \left[e^{\int_{+\infty}^t f'_{x_\psi}(\mu) d\mu} - 1 \right], \quad (-\infty < t < +\infty) \\ &= r_M. \end{aligned} \quad (57)$$

Therefore, $\eta(t) \in \mathbb{B}$.

Define a mapping as follows:

$$(T\psi)(t) = \int_t^{+\infty} e^{\int_s^t f'_{x_\psi}(\mu) d\mu} f'_{x_\psi}(s) r(s) ds. \quad (58)$$

Thus, if given any $\psi(t) \in \mathbb{B}$, then $(T\psi)(t) \in \mathbb{B}$, and hence, $T: \mathbb{B} \rightarrow \mathbb{B}$.

Now, we prove that the mapping T is a compact mapping.

Consider any sequence $\{\psi_n(t)\} \subseteq \mathbb{B}$ ($n = 1, 2, \dots$), and then, it follows

$$r_L \leq \psi_n(t) \leq r_M, \text{mod}(\psi_n) \subseteq \text{mod}(f), \quad (n = 1, 2, \dots). \quad (59)$$

On the other hand, $(T\psi_n)(t) = x_{\psi_n}(t)$ satisfies

$$\begin{aligned} \frac{dx_{\psi_n}(t)}{dt} &= f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]x_{\psi_n}(t) - f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]r(t) \\ &= f'_{x_{\psi_n}}(t)x_{\psi_n}(t) - f'_{x_{\psi_n}}(t)r(t), \end{aligned} \quad (60)$$

where

$$f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))] \triangleq f'_{x_{\psi_n}}(t). \quad (61)$$

Since $f: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $f(t + \omega, x) = f(t, x)$, f'_x exists and is a continuous function, and from (60) and

$\{\psi_n(t)\} \subseteq \mathbb{B}$ ($n = 1, 2, \dots$), it follows that $f'_{x_{\psi_n}}(t)$ is an ω -periodic continuous function on $t \in \mathbb{R}$.

It follows from (47), (50), (59), and (61) that

$$|f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))]| \leq \delta. \quad (62)$$

Thus, we have

$$\left| \frac{dx_{\psi_n}(t)}{dt} \right| \leq 2 \|f'_{x_{\psi_n}}(t)\| r(t) \leq 2\delta \|r\|_M, \quad (63)$$

$$\text{mod}(x_{\psi_n}(t)) \subseteq \text{mod}(f). \quad (64)$$

Hence, $\{dx_{\psi_n}(t)/dt\}$ is uniformly bounded, and therefore, $\{x_{\psi_n}(t)\}$ is uniformly bounded and equicontinuous on \mathbb{R} . By the theorem of Ascoli-Arzelà, for any sequence $\{x_{\psi_n}(t)\} \subseteq \mathbb{B}$, there exists a subsequence (also denoted by $\{x_{\psi_n}(t)\}$) such that $\{x_{\psi_n}(t)\}$ is convergent uniformly on any compact set of \mathbb{R} . By (64), combined with Lemma 2, $\{x_{\psi_n}(t)\}$ is convergent uniformly on \mathbb{R} , that is to say, T is relatively compact on \mathbb{B} .

Next, we prove that T is a continuous mapping.

Assume $\{\psi_n(t)\} \subseteq \mathbb{B}$, $\psi(t) \in \mathbb{B}$,

$$\psi_n(t) \longrightarrow \psi(t), \quad (n \longrightarrow \infty). \quad (65)$$

Since $f(t, x)$ has first-order continuous partial derivative on x , we have

$$\begin{aligned} & f'_x[t, r(t) + \theta_2(\psi_n(t))(\psi_n(t) - r(t))] \\ & \longrightarrow f'_x[t, r(t) + \theta_1(\psi(t))(\psi(t) - r(t))], \end{aligned} \quad (66)$$

($n \longrightarrow \infty$),

that is,

$$f'_{x_{\psi_n}}(t) \longrightarrow f'_{x_{\psi}}(t), \quad (n \longrightarrow \infty). \quad (67)$$

It follows from (58) that

$$\begin{aligned} & |(T\psi_n)(t) - (T\psi)(t)| \\ &= \left| \int_t^{+\infty} e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} f'_{x_{\psi_n}}(s) r(s) ds - \int_t^{+\infty} e^{\int_s^t f'_{x_{\psi}}(\mu) d\mu} f'_{x_{\psi}}(s) r(s) ds \right| \\ &= \left| \int_t^{+\infty} e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(s) - f'_{x_{\psi}}(s)) r(s) ds \right. \\ & \quad \left. + \int_t^{+\infty} \left(e^{\int_s^t f'_{x_{\psi_n}}(\mu) d\mu} - e^{\int_s^t f'_{x_{\psi}}(\mu) d\mu} \right) f'_{x_{\psi}}(s) r(s) ds \right| \\ &= \left| \int_0^{+\infty} e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(t+s) - f'_{x_{\psi}}(t+s)) r(t+s) ds \right. \\ & \quad \left. + \int_0^{+\infty} \left(e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} - e^{\int_{s+t}^t f'_{x_{\psi}}(\mu) d\mu} \right) f'_{x_{\psi}}(t+s) r(t+s) ds \right| \\ &= \left| \int_0^{+\infty} e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} (f'_{x_{\psi_n}}(t+s) - f'_{x_{\psi}}(t+s)) r(t+s) ds \right. \\ & \quad \left. + \int_0^{+\infty} e^{\xi} \left(\int_{s+t}^t (f'_{x_{\psi_n}}(\mu) - f'_{x_{\psi}}(\mu)) d\mu \right) f'_{x_{\psi}}(t+s) r(t+s) ds \right| \\ &\leq \left(\int_0^{+\infty} e^{\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds \right. \\ & \quad \left. + \int_0^{+\infty} e^{\xi} \left| \int_{s+t}^t d\mu \right| |f'_{x_{\psi}}(s+t)| \|r(s+t)\| ds \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}) \\ &= \left(\int_0^{T_0} e^{\int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds + \int_{T_0}^{+\infty} e^{\int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu} |r(t+s)| ds \right. \\ & \quad \left. + \int_0^{T_0} e^{\xi} \left| \int_{t+s}^t d\mu \right| |f'_{x_{\psi}}(s+t)| \|r(s+t)\| ds \right. \\ & \quad \left. + \int_{T_0}^{+\infty} e^{\xi} \left| \int_{t+s}^t d\mu \right| |f'_{x_{\psi}}(s+t)| \|r(s+t)\| ds \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}), \end{aligned} \quad (68)$$

where ξ is between $\int_{s+t}^t f'_{x_{\psi_n}}(\mu) d\mu$ and $\int_{s+t}^t f'_{x_{\psi}}(\mu) d\mu$. Since $f'_{x_{\psi_n}}(\mu)$ and $f'_{x_{\psi}}(\mu)$ are ω -periodic continuous functions on \mathbb{R} , it follows

$$\lim_{s \longrightarrow +\infty} \frac{1}{s} \int_0^s f'_{x_{\psi_n}}(\mu) dt = \frac{1}{\omega} \int_0^\omega f'_{x_{\psi_n}}(\mu) dt \triangleq \lambda_1, \quad (69)$$

$$\lim_{s \longrightarrow +\infty} \frac{1}{s} \int_0^s f'_{x_{\psi}}(\mu) dt = \frac{1}{\omega} \int_0^\omega f'_{x_{\psi}}(\mu) dt \triangleq \lambda_2,$$

and thus, there is a $T_0 > 0$ (also, assume it is T_0 in the above), when $s \geq T_0$, such that

$$\left| \frac{-1}{s} \int_{t+s}^t f'_{x_{\psi_n}}(\mu) dt - \lambda_1 \right| < \frac{\lambda_1}{2}, \quad (70)$$

$$\left| \frac{-1}{s} \int_{t+s}^t f'_{x_{\psi}}(\mu) dt - \lambda_2 \right| < \frac{\lambda_2}{2},$$

that is,

$$\begin{aligned} \int_{t+s}^t f'_{x_{\psi_n}}(\mu) d\mu &< -\frac{\lambda_1 s}{2}, \\ \int_{t+s}^t f'_{x_{\psi}}(\mu) d\mu &< -\frac{\lambda_2 s}{2}. \end{aligned} \quad (71)$$

Set

$$\frac{\lambda_0 s}{2} = \max\left(-\frac{\lambda_1 s}{2}, -\frac{\lambda_2 s}{2}\right), \quad (72)$$

and hence, we have

$$\begin{aligned} & |(T\psi_n)(t) - (T\psi)(t)| \\ & \leq \left(T_0 e^{\delta T_0} |r|_M + \int_{T_0}^{+\infty} e^{-(\lambda_0 s/2)} |r|_M ds + T_0^2 e^{\delta T_0} \delta |r|_M \right. \\ & \quad \left. + \int_{T_0}^{+\infty} e^{-(\lambda_0 s/2)} s \delta |r|_M ds \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}) \\ & = |r|_M \left(T_0 e^{\delta T_0} + \frac{2}{\lambda_0} e^{-(\lambda_0 T_0/2)} + T_0^2 e^{\delta T_0} \delta + \frac{4}{\lambda_0^2} e^{-(\lambda_0 T_0/2)} \delta \right) \rho(f'_{x_{\psi_n}}, f'_{x_{\psi}}). \end{aligned} \quad (73)$$

It follows from (67) and the above inequality that

$$(T\psi_n)(t) \longrightarrow (T\psi)(t), \quad (n \longrightarrow \infty). \quad (74)$$

Therefore, T is continuous. By (58), easy to see, $T(\partial\mathbb{B}) \subseteq \mathbb{B}$. According to Lemma 3, T has at least a fixed point on \mathbb{B} , the fixed point is the ω -periodic continuous solution $\Phi(t)$ of equation (1), and

$$r_L \leq \Phi(t) \leq r_M. \quad (75)$$

(2) Construct a Lyapunov function as follows:

$$V(t, x(t)) = (x(t) - \Phi(t))^2, \quad (76)$$

where $x(t)$ is the unique solution with initial value $x(t_0) = x_0$ of equation (1) and $\Phi(t)$ is the periodic solution of equation (1).

Differentiating both sides of (76) along the solution of equation (1), we have

$$\begin{aligned} \frac{dV(t, x(t))}{dt} &= 2(x(t) - \Phi(t)) \left[\frac{dx(t)}{dt} - \frac{d\Phi(t)}{dt} \right] \\ &= 2(x(t) - \Phi(t)) [f(t, x(t)) - f(t, \Phi(t))] \\ &= 2(x(t) - \Phi(t)) [f'_x(t, \zeta)(x(t) - \Phi(t))] \\ &= 2f'_x(t, \zeta)(x(t) - \Phi(t))^2, \end{aligned} \quad (77)$$

where ζ is between $x(t)$ and $\Phi(t)$.

By (H_2) , it follows

$$|x(t) - \Phi(t)| \longrightarrow +\infty, \quad (t \longrightarrow +\infty), \quad (78)$$

and thus, $x(t)$ cannot be a periodic solution of equation (1), and it is proved that equation (1) has a unique ω -periodic continuous solution $\Phi(t)$ which is unstable.

This is the end of the proof of Theorem 2. \square

4. Examples

The following examples show the feasibility of our main results.

Example 1. Consider the equation

$$\frac{dx}{dt} = -e^x + e^{\sin t}, \quad (79)$$

where

$$\begin{aligned} f(t, x) &= -e^x + e^{\sin t}, \\ f(t, \sin t) &= 0, \\ f'_x(t, x) &= -e^x \leq 0 \quad (\equiv 0), \end{aligned} \quad (80)$$

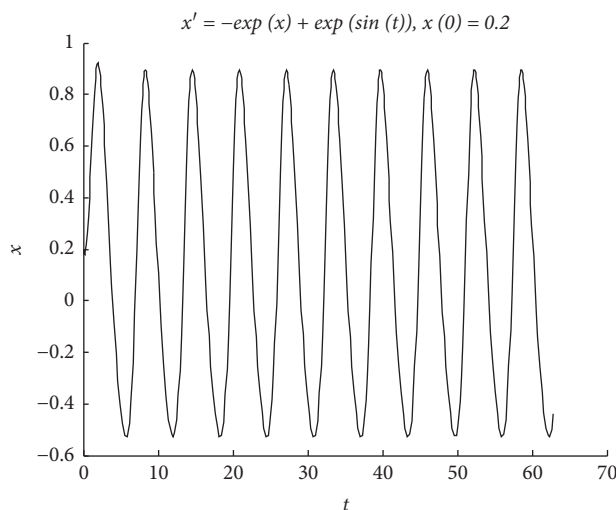
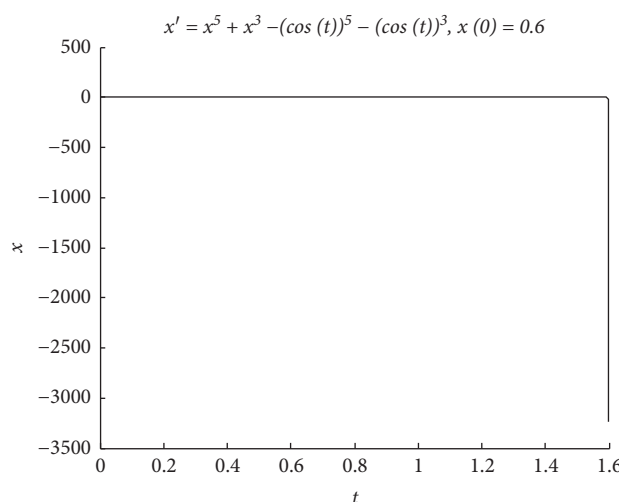
and hence, equation (79) satisfies all the conditions of Theorem 1. It follows from Theorem 1 that equation (79) has a unique 2π -periodic solution $r(t)$ which is globally attractive.

Clearly, according to the solution curve of equation (79), if given any initial value $x(0) = x_0$ (e.g., $x(0) = 0.2$), then the solution curve of equation (79) does tend to the curve of the periodic solution $r(t)$ (see Figure 1).

Example 2. Consider the equation

$$\frac{dx}{dt} = x^5 + x^3 - \cos^5 t - \cos^3 t, \quad (81)$$

where

FIGURE 1: The curve of the solution of equation (79) with initial value $x(0) = 0.2$.FIGURE 2: The curve of the solution of equation (81) with initial value $x(0) = 0.6$.

$$\begin{aligned} f(t, x) &= x^5 + x^3 - \cos^5 t - \cos^3 t, \\ f(t, \cos t) &= 0, \\ f'_x(t, x) &= 5x^4 + 3x^2 \geq 0 \quad (\equiv 0). \end{aligned} \quad (82)$$

Hence, equation (81) satisfies all the conditions of Theorem 2. It follows from Theorem 2 that equation (81) has a unique 2π -periodic solution $r(t)$ which is unstable.

Clearly, according to the solution curve of equation (81), if given any initial value $x(0) = x_0$ (e.g., $x(0) = 0.6$), then the solution curve of equation (81) arrives at enough large $(-\infty)$ at some time t^* (see Figure 2).

5. Conclusion

In this paper, a simple criterion for the existence, uniqueness, and stability of the periodic solution of one-dimensional periodic differential equations is given. This criterion can be applied to many one-dimensional periodic differential equations.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The research was supported by the Senior Talent Foundation of Jiangsu University (14JDG176).

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Research Article

Common Fixed Point Theorems via Inverse C_k – Class Functions in Metric Spaces

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Received 10 December 2020; Revised 7 January 2021; Accepted 31 March 2021; Published 21 April 2021

Academic Editor: Naeem Saleem

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In this paper, we firstly introduce a new notion of inverse C_k –class functions which extends the notion of inverse C –class functions introduced by Saleem et al., 2018. Secondly, some common fixed point theorems are stated under some compatible conditions such as weak semicompatible of type (A), weak semicompatibility, and conditional semicompatibility in metric spaces. Moreover, we introduce a new kind of compatibility called S_t –compatibility which is weaker than (E.A.) property and also present a common fixed point theorem in metric spaces via inverse C_k –class functions. Some examples are provided to support our results.

1. Introduction and Preliminaries

As a follow-up work of A.H. Ansari's research on fixed point (or common fixed point) theory via auxiliary C –class functions, very recently, Saleem et al. [1] introduced the new concept of inverse C –class functions and obtained some corresponding fixed point theorems under certain weak compatibility assumption via inverse C –class functions. In 1976, Jungck [2] defined the concept of commutative maps and initiated the study of the existence of a common fixed point of such maps in metric spaces. After which, Sessa [3] introduced the weak version of commuting maps called weak commuting maps. Next, Jungck [4, 5] provided some generalizations of weak commuting maps by providing the notions of compatible maps and compatible maps of type (A). Minor relaxations of compatible of type (A) are introduced by Pathak and Khan [6], which are well known as g –compatible and f –compatible (see [6], for more details).

Singh et al. [7] proposed the notion of compatibility of type (E) by making a minor modification of compatibility of type (A). By splitting the concept of compatibility of type (E), Singh et al. [7] also gave some relaxations of compatibility type of (E) which are known as g –compatibility of type (E) and f –compatibility of type (E).

Definition 1 (see [7]). Two self-maps f and g of a metric space (X, d) are said to be f –compatible of type (E), if $\lim_{n \rightarrow +\infty} f f x_n = \lim_{n \rightarrow +\infty} f g x_n = g t$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow +\infty} f x_n = \lim_{n \rightarrow +\infty} g x_n = t$, for some $t \in X$. Similarly, two self-maps f and g of a metric space (X, d) are said to be g –compatible of type (E), if $\lim_{n \rightarrow +\infty} g g x_n = \lim_{n \rightarrow +\infty} g f x_n = f t$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow +\infty} f x_n = \lim_{n \rightarrow +\infty} g x_n = t$, for some $t \in X$.

It is easy to see that compatibility of type (E) implies both g –compatibility of type (E) and f –compatibility of

type (E); however, g -compatibility or f -compatibility of type (E) does not imply compatibility of type (E).

In 1994, Pant [8] introduced the following definition.

Definition 2 (see [8]). Two self-maps f and g defined on a metric space (X, d) are said to be R -weakly commuting, if there exists a real number $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$, for all $x \in X$.

Note that $R = 1$; then, f and g are weakly commuting.

In 1997, Pathak et al. [9] introduced the notions of R -weak commuting of type (A_g) and R -weak commuting of type (A_f) as follows.

Definition 3 (see [9]). Two self-maps f and g of a metric space (X, d) are said to be R -weakly commuting of type (A_g) , if there exists a real number $R > 0$ such that $d(gfx, ffx) \leq Rd(fx, gx)$, for all $x \in X$.

Definition 4 (see [9]). Two self-maps f and g of a metric space (X, d) are said to be R -weakly commuting of type (A_f) , if there exists a real number $R > 0$ such that $d(fgx, ggx) \leq Rd(fx, gx)$, for all $x \in X$.

It is noted that compatible maps f and g are also R -weakly commuting of type (A_g) and R -weakly commuting of type (A_f) . Moreover, we can find suitable examples which show that R -weakly commuting mappings and R -weakly commuting of type (A_f) (or (A_g)) are independent concepts (see examples of [9, 10]).

In 2008, Gopal et al. [10] introduced the notions of g -absorbing and f -absorbing stated as follows.

Definition 5 (see [10]). Let f and g ($f \neq g$) be two self-maps of a metric space (X, d) ; then, f is said to be g -absorbing, if there exists a real number $R > 0$ such that $d(gx, gfx) \leq Rd(fx, gx)$ for all $x \in X$. Similarly, let f and g ($f \neq g$) be two self-maps of a metric space (X, d) ; then, g is said to be f -absorbing, if there exists a real number $R > 0$ such that $d(fx, fgx) \leq Rd(fx, gx)$ for all $x \in X$.

Jungck and Rhoades [11], in 1998, introduced the concept of weak compatibility which is weaker than the concept of compatibility.

Another generalization of compatible maps called semicompatible maps was firstly introduced by Cho et al. [12] under the setting of d -topological spaces in which a pair (S, T) of self-maps are called to be semicompatible if condition (i) $Sy = Ty$ implies that $STy = TSy$; (ii) for sequence $\{x_n\}$ in X and $x \in X$, whenever $\{Sx_n\} \rightarrow x$, $\{Tx_n\} \rightarrow x$, and then $STx_n \rightarrow Tx$, as $n \rightarrow +\infty$, hold. However, Singh and Jain [13] redefined this concept by using condition (ii) only stated as follows.

Definition 6 (see [13]). A pair (f, g) of self-maps of a metric space (X, d) is said to be semicompatible, if $\lim_{n \rightarrow +\infty} fgx_n = gt$ holds whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} gx_n = t$ for some $t \in X$.

It follows that if (f, g) is semicompatible and $fx = gx$, then $fgx = gfx$. It is also noted that if the pair (f, g) is semicompatible, then it is weak compatible; however, the

converse is not true. Further, the semicompatibility of the pair (f, g) does not imply the semicompatibility of the pair (g, f) (see Example 3.2 in [13]).

Now, we make a minor modification of semicompatibility to introduce the notion of semicompatible of type (A) as follows.

Definition 7. A pair (f, g) of self-maps of a metric space (X, d) is said to be semicompatible of type (A), if $\lim_{n \rightarrow +\infty} fgx_n = gt$ and $\lim_{n \rightarrow +\infty} gfx_n = ft$ hold whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} gx_n = t$ for some $t \in X$.

It is obvious that semicompatibility of type (A) implies semicompatibility; however, the converse is not true.

Recently, Saluja et al. [14, 15] introduced the weak semicompatible maps and conditional semicompatible maps and obtained corresponding fixed point theorems (see [14–16], for more details).

Definition 8 (see [14]). A pair (f, g) of self-maps of a metric space (X, d) is said to be weakly semicompatible, if $\lim_{n \rightarrow +\infty} fgx_n = gt$ or $\lim_{n \rightarrow +\infty} gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} gx_n = t$ for some $t \in X$.

Definition 9 (see [15]). A pair (f, g) of self-maps of a metric space (X, d) is said to be conditionally semicompatible; if whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} gx_n$ is nonempty, then there exists at least a sequence $\{y_n\}$ satisfying $\lim_{n \rightarrow +\infty} fy_n = \lim_{n \rightarrow +\infty} gy_n = t$ such that $\lim_{n \rightarrow +\infty} fgy_n = gt$ and $\lim_{n \rightarrow +\infty} gfx_n = ft$.

It is obvious that semicompatibility of type (A) implies weak semicompatibility. From the definition itself, it is clear that if a pair (f, g) of self-maps is semicompatible of type (A), then it is necessarily conditionally semicompatible; however, the conditionally semicompatible maps are not necessarily semicompatible of type (A).

Example 1. Let $X = [1, +\infty)$ and d be the usual metric on X . Define $f, g: X \rightarrow X$ as follows:

$$fx = \begin{cases} 1, & \text{if } x = 1, \\ \frac{x+7}{2}, & \text{if } 1 < x < 3, \\ 3, & \text{if } x \geq 3, \end{cases} \quad (1)$$

$$gx = \begin{cases} 1, & \text{if } x = 1, \\ x+3, & \text{if } 1 < x < 3, \\ x, & \text{if } x \geq 3. \end{cases}$$

Let us consider the sequence $x_n = 1 + (1/n)$; we have

$$\begin{aligned}
\lim_{n \rightarrow +\infty} f x_n &= \lim_{n \rightarrow +\infty} f\left(1 + \frac{1}{n}\right) = 4, \\
\lim_{n \rightarrow +\infty} g x_n &= \lim_{n \rightarrow +\infty} g\left(1 + \frac{1}{n}\right) = 4, \\
\lim_{n \rightarrow +\infty} f g x_n &= \lim_{n \rightarrow +\infty} f\left(4 + \frac{1}{n}\right) = 3 \neq g(4), \\
\lim_{n \rightarrow +\infty} g f x_n &= \lim_{n \rightarrow +\infty} g\left(4 + \frac{1}{2n}\right) = 4 \neq f(4).
\end{aligned} \tag{2}$$

However, if we take $y_n = 3 + (1/n)$, we have that

$$\begin{aligned}
\lim_{n \rightarrow +\infty} f y_n &= \lim_{n \rightarrow +\infty} f\left(3 + \frac{1}{n}\right) = 3, \\
\lim_{n \rightarrow +\infty} g y_n &= \lim_{n \rightarrow +\infty} g\left(3 + \frac{1}{n}\right) = 3, \\
\lim_{n \rightarrow +\infty} f g x_n &= \lim_{n \rightarrow +\infty} f\left(3 + \frac{1}{n}\right) = 3 = g(3), \\
\lim_{n \rightarrow +\infty} g f x_n &= \lim_{n \rightarrow +\infty} g(3) = 3 = f(3).
\end{aligned} \tag{3}$$

Thus, the pair (f, g) is conditional semicompatible.

Finally, we introduce a new kind of compatibility of a pair (f, g) of self-maps called S_τ -compatible firstly proposed by Jain et al. [17] as follows.

Definition 10. Let τ be a self-map defined on X satisfying $\lim_{n \rightarrow +\infty} \tau x_n = t$ for some sequence $\{x_n\} \in X$ and $t \in X$. Then, a pair (f, g) of self-maps defined on X is called S_τ -compatible if $\lim_{n \rightarrow +\infty} f \tau x_n = \lim_{n \rightarrow +\infty} g \tau x_n = \tau t$.

Example 2. Let $X = \mathbb{R}$, $fx = 2x$, $gx = 4 - 2x$, and $\tau x = 1 + x$. Take $\{x_n\} = 1/n$. Since $\lim_{n \rightarrow +\infty} \tau x_n = 1$ with $\lim_{n \rightarrow +\infty} f \tau x_n = 2 = \tau(1)$ and $\lim_{n \rightarrow +\infty} g \tau x_n = 2 = \tau(1)$, then pair (f, g) is S_τ -compatible. However, $\lim_{n \rightarrow +\infty} \tau x_n \neq \lim_{n \rightarrow +\infty} f x_n$ and $\lim_{n \rightarrow +\infty} \tau x_n \neq \lim_{n \rightarrow +\infty} g x_n$.

It is obvious that S_τ -compatibility of a pair (f, g) self-maps implies E.A. property of a pair (f, g) of self-maps by taking self-map τ as an identity map.

Let $X = \mathbb{R}$, $fx = x$, $gx = x^2$, and $\tau = I_x$ (identity function on X). Take $\{x_n\} = 1/n$. Here, $\lim_{n \rightarrow +\infty} \tau x_n = \lim_{n \rightarrow +\infty} g \tau x_n = \lim_{n \rightarrow +\infty} f \tau x_n = \lim_{n \rightarrow +\infty} x_n = 0$. Hence, pair self-maps (f, g) satisfy (E.A.) property.

Now, we introduce one more example of S_τ -compatibility including four maps as follows.

Example 3. If $X = [1, +\infty)$ with the usual metric. Define $f, g, B, T: X \rightarrow X$ by

$$\begin{aligned}
fx &= \begin{cases} x + 1, & \text{if } x \in [1, 2), \\ 3, & \text{if } x \in [2, +\infty), \end{cases} \\
gx &= \begin{cases} 2, & \text{if } x \in [1, 2), \\ \frac{x+4}{2}, & \text{if } x \in [2, +\infty), \end{cases} \\
Bx &= \begin{cases} 2x, & \text{if } x \in [1, 2), \\ x + 1, & \text{if } x \in [2, +\infty), \end{cases} \\
Tx &= \begin{cases} 3x - 1, & \text{if } x \in [1, 2), \\ 2x - 1, & \text{if } x \in [2, +\infty). \end{cases}
\end{aligned} \tag{4}$$

Choose $x_n = 1 + \varepsilon_n$, where $\varepsilon_n \rightarrow 0$ when $n \rightarrow +\infty$, then $\lim_{n \rightarrow +\infty} f x_n = 2$, $\lim_{n \rightarrow +\infty} g x_n = 2$, $\lim_{n \rightarrow +\infty} B x_n = 2$, and $\lim_{n \rightarrow +\infty} T x_n = 2$.

Since $\lim_{n \rightarrow +\infty} f B x_n = 3 = B(2)$ and $\lim_{n \rightarrow +\infty} g B x_n = 3 = B(2)$, then the pair (f, g) is S_B -compatible. Next, since $\lim_{n \rightarrow +\infty} f T x_n = 3 = T(2)$ and $\lim_{n \rightarrow +\infty} g T x_n = 3 = T(2)$, then the pair (f, g) is S_T -compatible. Further, since $\lim_{n \rightarrow +\infty} B g x_n = 3 = g(2)$ and $\lim_{n \rightarrow +\infty} T g x_n = 3 = g(2)$, then the pair (B, T) is S_g -compatible. Finally, since $\lim_{n \rightarrow +\infty} B f x_n = 3 = f(2)$ and $\lim_{n \rightarrow +\infty} T f x_n = 3 = f(2)$, then the pair (B, T) is S_f -compatible.

Ansari, in 2014, firstly [18], introduced the concept of C -class functions and proved some fixed point theorems via C -class functions (see [19, 20] for more details).

Definition 11 (see [18]). A mapping $F: [0, +\infty)^2 \rightarrow \mathbb{R}$ is called a C -class function if it is continuous and the following axioms hold:

- (1) $F(s, t) \leq s$ for all $s, t \in [0, +\infty)$
- (2) $F(s, t) = s$ implies that $s = 0$ or $t = 0$

Denote the family of C -class functions by \mathcal{C} .

Example 4 (see [18]). The following functions $F: [0, +\infty)^2 \rightarrow \mathbb{R}$ are elements of \mathcal{C} , for all $s, t \in [0, +\infty)$:

- (1) $F(s, t) = s - t$, $F(s, t) = s$ implies $t = 0$
- (2) $F(s, t) = ms$, for some $m \in (0, 1)$, $F(s, t) = s$ implies $s = 0$
- (3) $F(s, t) = (s/(1+t))^r$, for some $r \in (0, +\infty)$, $F(s, t) = s$ implies $s = 0$ or $t = 0$
- (4) $F(s, t) = \log_a[(t+a^s)/(1+t)]$, for some $a > 1$, $F(s, t) = s$ implies $s = 0$ or $t = 0$
- (5) $F(s, t) = \log_a[(1+a^s/2)]$, for $a > e$, $F(s, 1) = s$ implies $s = 0$

- (6) $F(s, t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1$, for $r \in (0, +\infty)$, $F(s, t) = s$ implies $t = 0$
- (7) $F(s, t) = s \log_{t+a} a$, for $a > 1$, $F(s, t) = s$ implies $s = 0$ or $t = 0$
- (8) $F(s, t) = s - (1 + s/2 + s)(t/1 + t)$, $F(s, t) = s$ implies $t = 0$
- (9) $F(s, t) = s\beta(s)$, where $\beta: [0, +\infty) \rightarrow [0, 1]$ is continuous, $F(s, t) = s$ implies $s = 0$
- (10) $F(s, t) = s - (t/k + t)$, $F(s, t) = s$ implies $t = 0$
- (11) $F(s, t) = s - \varphi(t)$, $F(s, t) = s$ implies $s = 0$, where $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$
- (12) $F(s, t) = sh(s, t)$, $F(s, t) = s$ implies $s = 0$, where $h: [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function such that $h(t, s) < 1$ for all $t, s > 0$
- (13) $F(s, t) = s - (2 + t/1 + t)t$, $F(s, t) = s$ implies $t = 0$
- (14) $F(s, t) = \sqrt[n]{\ln(1 + s^n)}$, $F(s, t) = s$ implies $s = 0$
- (15) $F(s, t) = \phi(s)$, $F(s, t) = s$ implies $s = 0$, where $\phi: [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function such that $\phi(0) = 0$ and $\phi(t) < t$, for $t > 0$

Afterward, by the motivation of C -class functions, Saleem et al. [1, 21] introduced a new notion of inverse C -class functions as follows.

Definition 12 (see [1]). A mapping $F: [0, +\infty)^2 \rightarrow \mathbb{R}$ is called an inverse C -class function if it is continuous and the following axioms hold:

- (1) $F(s, t) \geq s$ for all $s, t \in [0, +\infty)$
- (2) $F(s, t) = s$ implies that $s = 0$ or $t = 0$

Denote the family of inverse C -class functions by \mathcal{C}_{inv} .

Example 5 (see [1]). The following functions $F: [0, +\infty)^2 \rightarrow \mathbb{R}$ are elements of \mathcal{C}_{inv} , for all $s, t \in [0, +\infty)$:

- (1) $F(s, t) = s + t$, $F(s, t) = s$ implies $t = 0$
- (2) $F(s, t) = ms$, for some $m \in (1, +\infty)$, $F(s, t) = s$ implies $s = 0$
- (3) $F(s, t) = s(1 + t)^r$, for some $r \in (0, +\infty)$, $F(s, t) = s$ implies $s = 0$ or $t = 0$
- (4) $F(s, t) = \log_a[(t + a^s)(1 + t)]$, for some $a > 1$, $F(s, t) = s$ implies $t = 0$
- (5) $F(s, t) = \phi(s)$, $F(s, t) = s$ implies $s = 0$, where $\phi: [0, +\infty) \rightarrow [0, +\infty)$ is an upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) > t$, for $t > 0$

Motivated by the above definition, we now define inverse C_k -class functions as follows.

Definition 13. A mapping $F: [0, +\infty)^2 \rightarrow \mathbb{R}$ is called an inverse C_k -class function if it is continuous and the following axioms hold:

- (1) $F(s, t) \geq ks$ for all $s, t \in [0, +\infty)$ and some $k \geq 1$
- (2) $F(s, t) = ks$ implies that $s = 0$ or $t = 0$

Denote the family of inverse C_k -class functions by $\mathcal{C}_{\text{inv-}k}$. Every inverse C -class function and inverse C_k -class function are equivalent when $k = 1$; however, an inverse C_k -class function may not be an inverse C -class function.

Example 6. A mapping $F: [0, +\infty) \rightarrow \mathbb{R}$ is defined by $F(s, t) = 2s + t$ for all $s, t \in [0, +\infty)$. Then, clearly, F is an inverse C_k -class function for $k = 2$, but it is not an inverse C -class function.

Example 7. The following functions $F: [0, +\infty)^2 \rightarrow \mathbb{R}$ are elements of $\mathcal{C}_{\text{inv-}k}$, for all $s, t \in [0, +\infty)$:

- (1) $F(s, t) = ks + lt$, $F(s, t) = ks$ implies $t = 0$ for some $k \geq 1$ and $l > 0$
- (2) $F(s, t) = kms$, $F(s, t) = ks$ implies $s = 0$ for $m \in (1, +\infty)$ and some $k \geq 1$
- (3) $F(s, t) = ks(1 + kt)^r$, $F(s, t) = ks$ implies $s = 0$ or $t = 0$ for $r \in (0, +\infty)$ and some $k \geq 1$
- (4) $F(s, t) = \log_a[(kt + a^{ks})(1 + kt)]$, $F(s, t) = ks$ implies $t = 0$ for some $a > 1$ and some $k \geq 1$
- (5) $F(s, t) = k\phi(s)$, $F(s, t) = ks$ implies $s = 0$, where $\phi: [0, +\infty) \rightarrow [0, +\infty)$ is an upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) > t$, for $t > 0$ and some $k \geq 1$

Definition 14. Let Ψ denote the class of functions $\psi: [0, +\infty) \rightarrow [0, +\infty)$ which satisfy the following conditions:

- (a) ψ is continuous and increasing with $\psi(t) \geq t$
- (b) $\psi(t) = 0 \Leftrightarrow t = 0$

Definition 15 (see [22]). A function $\phi: [0, +\infty) \rightarrow [0, +\infty)$ is said to be ultra-altering distance function if (a) ϕ is non-decreasing and continuous; (b) $\phi(t) > 0$, for all $t > 0$ and $\phi(t) = 0 \Rightarrow t = 0$. Denote the class of ultra-altering distance functions by Φ .

Lemma 1. Every sequence $\{x_n\}$ in metric space (X, d) will be Cauchy if there exists $\lambda \in [0, 1)$ such that $d(x_{n+1}, x_n) \leq \lambda d(x_n, x_{n-1})$, for all $n \in \mathbb{N}$.

The aim of this presented paper is to provide some common fixed point theorems under several compatible conditions mentioned above via inverse C_k -class functions, which extend, generalize, and improve the existing results in the literature. Some examples are provided to illustrate the validity of our results.

2. Main Results

Theorem 1. Let (X, d) be a complete metric space and a let pair (g, f) of self-maps be semicompatible, satisfying the following assumptions:

- (A1) $f(X) \subseteq g(X)$
- (A2) $k\psi(d(fx, fy)) \geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y)$

Here,

$M(x, y) = a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)$ and $N(x, y) = \min\{d(fx, gx), d(fy, gx)\}$, for all $x, y \in X$. Moreover, $a_i > 0$, ($i = 1, 2, \dots, 5$), with $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, $a_3 + a_4 > a_1 + a_2$, $L \in \mathbb{R}$, $F \in \mathcal{C}_{inv-k}$, for some $k \geq 1$ and $\psi \in \Psi$, $\phi \in \Phi$. If the pair (g, f) is f -compatible of type (E), then f and g have a unique common fixed point t in X .

Proof. Let x_0 be any point in X . Since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that

$$fx_1 = gx_0 = y_0. \quad (5)$$

Continuing this way, we can construct a sequence $\{x_n\}$ in X satisfying

$$fx_{n+1} = gx_n = y_n. \quad (6)$$

By the assumption of (A2), we have

$$\begin{aligned} k\psi(d(fx_n, fx_{n+1})) &\geq F[\psi(M(x_n, x_{n+1})), \phi(M(x_n, x_{n+1}))] + LN(x_n, x_{n+1}) \\ &\geq k\psi(M(x_n, x_{n+1})) + LN(x_n, x_{n+1}) \\ &= k\psi(a_1 d(fx_n, gx_n) + a_2 d(fx_{n+1}, gx_{n+1}) + a_3 d(fx_n, gx_{n+1}) + a_4 d(fx_{n+1}, gx_n) + a_5 d(gx_n, gx_{n+1})) \\ &\quad + L \min\{d(fx_n, gx_n), d(fx_{n+1}, gx_n)\} \\ &= k\psi(a_1 d(y_{n-1}, y_n) + a_2 d(y_n, y_{n+1}) + a_3 d(y_{n-1}, y_{n+1}) + a_4 d(y_n, y_n) + a_5 d(y_n, y_{n+1})) \\ &\quad + L \min\{d(y_{n-1}, y_n), d(y_n, y_n)\} \\ &= k\psi(a_1 d(y_{n-1}, y_n) + a_2 d(y_n, y_{n+1}) + a_3 d(y_{n-1}, y_{n+1}) + a_5 d(y_n, y_{n+1})). \end{aligned} \quad (7)$$

By the monotonicity of ψ , we have

$$d(y_{n-1}, y_n) \geq a_1 d(y_{n-1}, y_n) + a_2 d(y_n, y_{n+1}) + a_3 d(y_{n-1}, y_{n+1}) + a_5 d(y_n, y_{n+1}). \quad (8)$$

Again, from the triangle inequality, that is, $d(y_n, y_{n+1}) \leq d(y_n, y_{n-1}) + d(y_{n-1}, y_{n+1})$, we have

$$\begin{aligned} d(y_{n-1}, y_n) &\geq a_1 d(y_{n-1}, y_n) + a_2 d(y_n, y_{n+1}) + a_3 (d(y_n, y_{n+1}) - d(y_n, y_{n-1})) + a_5 d(y_n, y_{n+1}) \\ &= (a_1 - a_3) d(y_{n-1}, y_n) + (a_2 + a_3 + a_5) d(y_n, y_{n+1}), \end{aligned} \quad (9)$$

which further yields that

$$d(y_n, y_{n+1}) \leq \frac{1 - a_1 + a_3}{a_2 + a_3 + a_5} d(y_{n-1}, y_n). \quad (10)$$

Since $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, then $0 \leq (1 - a_1 + a_3)/(a_2 + a_3 + a_5) < 1$; from Lemma 1, it follows that sequence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $t \in X$ such that

$$\lim_{n \rightarrow +\infty} y_n = \lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} gx_n = t. \quad (11)$$

Now, we will show that t is a common fixed point of f and g .

Since the pair (g, f) is semicompatible, we have

$$\lim_{n \rightarrow +\infty} gfx_n = ft. \quad (12)$$

Since f and g are f -compatible of type (E), it follows that

$$\lim_{n \rightarrow +\infty} ffx_n = \lim_{n \rightarrow +\infty} fgx_n = gt. \quad (13)$$

Now, by the definition of ψ and assumption (A2), we obtain

$$\begin{aligned}
k\psi(d(ffx_n, ft)) &\geq F[\psi(M(fx_n, t)), \phi(M(fx_n, t))] + LN(fx_n, t) \\
&= F[\psi(a_1d(ffx_n, gfx_n) + a_2d(ft, gt) + a_3d(ffx_n, gt) + a_4d(ft, gfx_n) + a_5d(gfx_n, gt)), \\
&\quad \phi(a_1d(ffx_n, gfx_n) + a_2d(ft, gt) + a_3d(ffx_n, gt) + a_4d(ft, gfx_n) + a_5d(gfx_n, gt))] \\
&\quad + L \min\{d(ffx_n, gfx_n), d(ft, gfx_n)\}.
\end{aligned} \tag{14}$$

Taking the limit as $n \longrightarrow +\infty$ in above inequality, it follows that

$$\begin{aligned}
k\psi(d(gt, ft)) &\geq F[\psi(a_1d(gt, ft) + a_2d(ft, gt) + a_3d(gt, gt) + a_4d(ft, ft) + a_5d(ft, gt)), \\
&\quad \phi(a_1d(gt, ft) + a_2d(ft, gt) + a_3d(gt, gt) + a_4d(ft, ft) + a_5d(ft, gt))] \\
&\quad + L \min\{d(gt, ft), d(ft, ft)\} \\
&= F[\psi(a_1d(ft, gt) + a_2d(ft, gt) + a_5d(ft, gt)), \phi(a_1d(ft, gt) + a_2d(ft, gt) + a_5d(ft, gt))] \\
&\geq k\psi((a_1 + a_2 + a_5)d(gt, ft)) \\
&\geq k\psi(d(gt, ft)),
\end{aligned} \tag{15}$$

which implies that

$$\begin{aligned}
d(ft, gt) &= 0 \\
\text{or } ft &= gt.
\end{aligned} \tag{16}$$

Again, it follows from the definition of ψ and assumption (A2) that

$$\begin{aligned}
k\psi(d(fx_n, ft)) &\geq F[\psi(a_1d(fx_n, gfx_n) + a_2d(ft, gt) + a_3d(fx_n, gt) + a_4d(ft, gfx_n) + a_5d(gfx_n, gt)), \\
&\quad \phi(a_1d(fx_n, gfx_n) + a_2d(ft, gt) + a_3d(fx_n, gt) + a_4d(ft, gfx_n) + a_5d(gfx_n, gt))] \\
&\quad + L \min\{d(fx_n, gfx_n), d(ft, gfx_n)\}.
\end{aligned} \tag{17}$$

Taking the limit as $n \longrightarrow +\infty$ in above inequality, we conclude that

$$\begin{aligned}
k\psi(d(t, ft)) &\geq F[\psi(a_3d(t, gt) + a_4d(ft, t) + a_5d(t, gt)), \phi(a_3d(t, gt) + a_4d(ft, t) + a_5d(t, gt))] \\
&= F[\psi(a_3d(t, ft) + a_4d(ft, t) + a_5d(t, ft)), \phi(a_3d(t, ft) + a_4d(ft, t) + a_5d(t, ft))] \\
&\geq k\psi((a_3 + a_4 + a_5)d(t, ft)).
\end{aligned} \tag{18}$$

Since $a_1 + a_2 < a_3 + a_4$ and $a_1 + a_2 + a_5 > 1$, it follows that

$$\begin{aligned}
k\psi(d(t, ft)) &\geq F[\psi((a_3 + a_4 + a_5)d(t, ft)), \phi((a_3 + a_4 + a_5)d(t, ft))] \\
&\geq k\psi((a_3 + a_4 + a_5)d(t, ft)) \\
&\geq k\psi((a_1 + a_2 + a_5)d(t, ft)) \\
&\geq k\psi(d(t, ft)),
\end{aligned} \tag{19}$$

which implies that

$$F[\psi((a_3 + a_4 + a_5)d(t, ft)), \phi((a_3 + a_4 + a_5)d(t, ft))] = k\psi(d(t, ft)). \quad (20)$$

From the definition of $\mathcal{C}_{\text{inv-}k}$ -class functions, we have

$$\begin{aligned} \psi((a_3 + a_4 + a_5)d(t, ft)) &= 0 \\ \text{or } \phi((a_3 + a_4 + a_5)d(t, ft)) &= 0. \end{aligned} \quad (21)$$

By the definitions of ψ and ϕ , we have that $d(ft, t) = 0$, which proves that $ft = gt = t$, that is, t is a common fixed point of f and g .

Next, we will prove the uniqueness of common fixed point of f and g in X .

Suppose that u is another common fixed point of f and g , that is $fu = gu = u$.

From above argument, it may be concluded that

$$\begin{aligned} k\psi(d(fu, ft)) &\geq F[\psi(M(u, t)), \phi(M(u, t))] + LN(u, t) \\ &= F[\psi(a_1d(fu, gu) + a_2d(ft, gt) + a_3d(fu, gt) + a_4d(ft, gu) + a_5d(gu, gt)), \\ &\quad \phi(a_1d(fu, gu) + a_2d(ft, gt) + a_3d(fu, gt) + a_4d(ft, gu) + a_5d(gu, gt))] \\ &\quad + L \min\{d(fu, gu), d(ft, gu)\}. \\ &= F[\psi(a_3d(u, t) + a_4d(t, u) + a_5d(u, t)), \phi(a_3d(u, t) + a_4d(t, u) + a_5d(u, t))] \\ &\geq k\psi((a_3 + a_4 + a_5)d(u, t)). \end{aligned} \quad (22)$$

Since $a_1 + a_2 < a_3 + a_4$ and $a_1 + a_2 + a_5 > 1$, it follows that

$$\begin{aligned} k\psi(d(u, t)) &\geq F[\psi((a_3 + a_4 + a_5)d(u, t)), \phi((a_3 + a_4 + a_5)d(u, t))] \\ &\geq k\psi((a_3 + a_4 + a_5)d(u, t)) \\ &\geq k\psi((a_1 + a_2 + a_5)d(u, t)) \\ &\geq k\psi(d(u, t)), \end{aligned} \quad (23)$$

which implies that

$$F[\psi((a_3 + a_4 + a_5)d(u, t)), \phi((a_3 + a_4 + a_5)d(u, t))] = k\psi(d(u, t)). \quad (24)$$

From the definition of $\mathcal{C}_{\text{inv-}k}$ -class functions, we have

$$\begin{aligned} \psi((a_3 + a_4 + a_5)d(u, t)) &= 0 \\ \text{or } \phi((a_3 + a_4 + a_5)d(u, t)) &= 0. \end{aligned} \quad (25)$$

By the definitions of ψ and ϕ , we have that $d(u, t) = 0$, which proves that $u = t$. \square

Remark 1. (i) The conclusion of Theorem 1 still holds under the assumptions of semi-compatibility of the pair (f, g) and g -compatibility of type (E) .

(ii) If the inequality in assumption $(A2)$ is replaced by

$$k\psi(d(fx, gx)) \geq F[\psi(M(x, y)), \phi(M(x, y))], \quad (26)$$

the conclusion still holds.

Here is an illustrated example to support the validity of Theorem 1 as follows.

Example 8. Let $X = [0, 2]$ be a usual metric space. We define f and g on X as follows:

$$fx = \frac{3}{2}, \quad \forall x \in X,$$

$$gx = \begin{cases} \frac{1}{2}, & x \in [0, 1], \\ \frac{3}{2}, & x \in (1, 2]. \end{cases} \quad (27)$$

To verify that the pair (g, f) is semicompatible as well as f -compatible of type (E) , we take any sequence

$\{x_n\} \in (1, 2]$; then, $\lim_{n \rightarrow +\infty} f x_n = \lim_{n \rightarrow +\infty} g x_n = 3/2$ and $\lim_{n \rightarrow +\infty} f g x_n = 3/2 = g(3/2)$, $\lim_{n \rightarrow +\infty} f f x_n = 3/2 = g(3/2)$, $\lim_{n \rightarrow +\infty} g f x_n = 3/2 = g(3/2)$. Define $F \in \mathcal{C}_{\text{inv-}k}$

by $F(s, t) = 2s$, with $k = 1$ and $\psi(t) = t = \phi(t)$, for all $t \in [0, +\infty)$. Then, for all $x, y \in [0, 1]$, we have

$$\begin{aligned}
 k\psi(d(fx, fy)) &= 0, \\
 F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y) &= 2\psi(M(x, y)) + LN(x, y) \\
 &= 2[a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)] \\
 &\quad + L \min\{d(fx, gx), d(fy, gx)\} \\
 &= 2\left[a_1 \left|\frac{3}{2} - \frac{1}{2}\right| + a_2 \left|\frac{3}{2} - \frac{1}{2}\right| + a_3 \left|\frac{3}{2} - \frac{1}{2}\right| + a_4 \left|\frac{3}{2} - \frac{1}{2}\right|\right] \\
 &\quad + L \min\left\{\left|\frac{3}{2} - \frac{1}{2}\right|, \left|\frac{3}{2} - \frac{1}{2}\right|\right\} \\
 &= 2[a_1 + a_2 + a_3 + a_4] + L.
 \end{aligned} \tag{28}$$

It is verified that assumption (A2) holds true provided by $a_1 = 2, a_2 = a_3 = 1, a_4 = 3$, and $a_5 > 0$ such that $1 \geq a_1 - a_3, a_1 + a_2 + a_5 > 1, a_3 + a_4 > a_1 + a_2$, and $L \leq -14$.

Also, it is obvious that assumption (A2) holds true, for all $x, y \in (1, 2]$.

Hence, functions f and g satisfy all conditions of Theorem 1 with $3/2$ are common fixed point.

Taking f as the identity map in Theorem 1, we have the following fixed point theorem.

Corollary 1. Let g be a self-map defined on a complete metric space (X, d) satisfying the following assumption:

$$k\psi(d(x, y)) \geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y), \tag{29}$$

where $M(x, y) = a_1 d(x, gx) + a_2 d(y, gy) + a_3 d(x, gy) + a_4 d(y, gx) + a_5 d(gx, gy)$ and $N(x, y) = \min\{d(x, gx), d(y, gy)\}$, for all $x, y \in X$. Moreover, $a_i > 0$, ($i = 1, 2, \dots, 5$) with $1 \geq a_1 - a_3, a_1 + a_2 + a_5 > 1, a_1 + a_2 < a_3 + a_4, L \in \mathbb{R}, F \in \mathcal{C}_{\text{inv-}k}$ for some $k \geq 1$ and $\psi \in \Psi, \phi \in \Phi$. Then, g has a unique fixed point in X .

Since semicompatibility of type (A) implies semicompatibility, we can obtain the following theorem.

Theorem 2. Let (X, d) be a complete metric space and let a pair (g, f) of self-maps be semicompatible of type (A) satisfying the following assumptions:

$$\begin{aligned}
 (A1) \quad & f(X) \subseteq g(X) \\
 (A2) \quad & k\psi(d(fx, fy)) \geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y)
 \end{aligned}$$

where $M(x, y) = a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)$ and $N(x, y) = \min\{d(fx, gx), d(fy, gx)\}$, for all $x, y \in X$. Moreover, $a_i > 0$, ($i =$

$1, 2, \dots, 5$) with $1 \geq a_1 - a_3, a_1 + a_2 + a_5 > 1, a_1 + a_2 < a_3 + a_4, L \in \mathbb{R}, F \in \mathcal{C}_{\text{inv-}k}$ for some $k \geq 1$ and $\psi \in \Psi, \phi \in \Phi$. If the pair (f, g) is either f -compatible of type (E) or g -compatible of type (E), then f and g have a unique common fixed point t in X .

Now, we will present a common fixed point theorem under weakly semicompatible condition via $\mathcal{C}_{\text{inv-}k}$ -class functions as follows.

Theorem 3. Let (X, d) be a complete metric space and let a pair (g, f) of self-maps be weakly semicompatible and R -weakly commuting type of A_f satisfying the following assumptions:

$$\begin{aligned}
 (A1) \quad & f(X) \subseteq g(X) \\
 (A2') \quad & k\psi(d(fx, fy)) \geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y)
 \end{aligned}$$

Here, $M(x, y) = a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)$ and $N(x, y) = \min\{d(fx, gx), d(fy, gx)\}$, for all $x, y \in X$. Moreover, $a_i > 0$, ($i = 1, 2, \dots, 5$), $L \in \mathbb{R}, F \in \mathcal{C}_{\text{inv-}k}$ for some $k \geq 1$ with $1 \geq a_1 - a_3, a_1 + a_2 + a_5 > 1, a_1 + a_2 < a_3 + a_4, k(a_1 + a_2 + a_3 + a_4) + L > 1$, and $\psi \in \Psi, \phi \in \Phi$. If the pair (f, g) is either f -compatible of type (E) or g -compatible of type (E), then f and g have a unique common fixed point t in X .

Proof. Let x_0 be any point in X . Since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that

$$fx_1 = gx_0 = y_0. \tag{30}$$

Continuing this way, we can have a sequence $\{x_n\}$ in X satisfying

$$fx_{n+1} = gx_n = y_n. \tag{31}$$

From the discussion of Theorem 1, we have that there exists a point $t \in X$ such that

$$\lim_{n \rightarrow +\infty} y_n = \lim_{n \rightarrow +\infty} f x_n = \lim_{n \rightarrow +\infty} g x_n = t. \quad (32)$$

Now consider the following possibilities:

Case 1. (f and g are f -compatible of type (E)).

Since f and g are weakly semicompatible, it follows that

$$\begin{aligned} \lim_{n \rightarrow +\infty} f g x_n &= g t \\ \text{or } \lim_{n \rightarrow +\infty} g f x_n &= f t. \end{aligned} \quad (33)$$

Firstly, we take

$$\lim_{n \rightarrow +\infty} g f x_n = f t. \quad (34)$$

Since f and g are f -compatible of type (E) , it yields that

$$\lim_{n \rightarrow +\infty} f f x_n = \lim_{n \rightarrow +\infty} f g x_n = g t. \quad (35)$$

Hence, the conclusion can directly follow from the proof of Theorem 1.

Secondly, we take

$$\lim_{n \rightarrow +\infty} f g x_n = g t. \quad (36)$$

Since f and g are f -compatible of type (E) , it yields that

$$\lim_{n \rightarrow +\infty} f f x_n = \lim_{n \rightarrow +\infty} f g x_n = g t. \quad (37)$$

Again since f and g are R -weakly commuting of type A_f , which implies

$$d(f g x_n, g g x_n) \leq R d(f x_n, g x_n). \quad (38)$$

Taking limit as $n \rightarrow +\infty$ in above inequality, we have

$$\lim_{n \rightarrow +\infty} g g x_n = g t. \quad (39)$$

From the monotonicity of ψ and assumption $(A2')$, we have

$$\begin{aligned} k\psi(d(f g x_n, f x_n)) &\geq F[\psi(M(g x_n, x_n)), \phi(M(g x_n, x_n))] + LN(g x_n, x_n) \\ &= F[\psi(a_1 d(f g x_n, g g x_n) + a_2 d(f x_n, g x_n) + a_3 d(f g x_n, g x_n) + a_4 d(f x_n, g g x_n) + a_5 d(g g x_n, g x_n)), \\ &\quad \phi(a_1 d(f g x_n, g g x_n) + a_2 d(f x_n, g x_n) + a_3 d(f g x_n, g x_n) + a_4 d(f x_n, g g x_n) + a_5 d(g g x_n, g x_n))] \\ &\quad + L \min\{d(f g x_n, g g x_n), d(f x_n, g g x_n)\}. \end{aligned} \quad (40)$$

Taking limit as $n \rightarrow +\infty$ in above inequality, we have

$$\begin{aligned} &k\psi(d(g t, t)) \\ &\geq F[\psi(a_1 d(g t, g t) + a_2 d(t, t) + a_3 d(g t, t) + a_4 d(t, g t) + a_5 d(g t, t)), \\ &\quad \phi(a_1 d(g t, g t) + a_2 d(t, t) + a_3 d(g t, t) + a_4 d(t, g t) + a_5 d(g t, t))] \\ &\quad + L \min\{d(g t, g t), d(t, g t)\} \\ &= F[\psi((a_3 + a_4 + a_5)d(t, g t)), \phi((a_3 + a_4 + a_5)d(t, g t))] \\ &\geq k\psi((a_3 + a_4 + a_5)d(g t, t)). \end{aligned} \quad (41)$$

Since $a_1 + a_2 < a_3 + a_4$ and $a_1 + a_2 + a_5 > 1$, it follows that

$$\begin{aligned}
& k\psi(d(gt, t)) \\
& \geq F[\psi((a_3 + a_4 + a_5)d(gt, t)), \phi((a_3 + a_4 + a_5)d(gt, t))] \\
& \geq k\psi((a_3 + a_4 + a_5)d(gt, t)) \\
& \geq k\psi((a_1 + a_2 + a_5)d(gt, t)) \\
& \geq k\psi(d(gt, t)),
\end{aligned} \tag{42}$$

which implies that

$$F[(a_3 + a_4 + a_5)d(gt, t), \phi((a_3 + a_4 + a_5)d(gt, t))] = k\psi(d(gt, t)). \tag{43}$$

From the definition of $\mathcal{C}_{\text{inv-}k}$ -class functions, we have

$$\begin{aligned}
& \psi((a_3 + a_4 + a_5)d(gt, t)) = 0 \\
& \text{or } \phi((a_3 + a_4 + a_5)d(gt, t)) = 0.
\end{aligned} \tag{44}$$

By the definitions of ψ and ϕ , we have that $d(gt, t) = 0$, which proves that $gt = t$.

Again, from the definition of ψ and assumption (A2'), we have

$$\begin{aligned}
0 &= k\psi(d(ft, ft)) \\
&\geq F[a_1d(ft, gt) + a_2d(ft, gt) + a_3d(ft, gt) + a_4d(ft, gt) + a_5d(gt, gt), \\
&\phi(a_1d(ft, gt) + a_2d(ft, gt) + a_3d(ft, gt) + a_4d(ft, gt) + a_5d(gt, gt))] \\
&\quad + L \min\{d(ft, gt), d(ft, gt)\} \\
&= F[\psi(a_1d(ft, gt) + a_2d(ft, gt) + a_3d(ft, gt) + a_4d(ft, gt)), \phi(a_1d(ft, gt) + a_2d(ft, gt) + a_3d(ft, gt) + a_4d(ft, gt))] \\
&\quad + Ld(ft, t) \geq k\psi((a_1 + a_2 + a_3 + a_4)d(ft, t)) + Ld(ft, t) \\
&\geq (k(a_1 + a_2 + a_3 + a_4) + L)d(ft, t).
\end{aligned} \tag{45}$$

Together with $a_i > 0, i = \{1, 2, \dots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from the above inequality, it follows that $d(ft, t) = 0$, which proves that $ft = t = gt$. Hence, t is a common fixed point of f and g .

Case 2 (f and g are g -compatible of type (E)).

Since f and g are weakly compatible, it follows that

$$\begin{aligned}
& \lim_{n \rightarrow +\infty} fgx_n = gt \\
& \text{or } \lim_{n \rightarrow +\infty} gfx_n = ft.
\end{aligned} \tag{46}$$

Firstly, we take

$$\lim_{n \rightarrow +\infty} fgx_n = ft. \tag{47}$$

Since f and g are g -compatible of type (E), it yields that

$$\lim_{n \rightarrow +\infty} gg x_n = \lim_{n \rightarrow +\infty} gfx_n = ft. \tag{48}$$

Therefore, the conclusion directly follows from Theorem 1 and (i) of Remark 1.

Secondly, we take

$$\lim_{n \rightarrow +\infty} gfx_n = ft. \tag{49}$$

Since f and g are g -compatible of type (E), it yields that

$$\lim_{n \rightarrow +\infty} gg x_n = \lim_{n \rightarrow +\infty} gfx_n = ft. \tag{50}$$

Also f and g are R -weakly commuting of type A_f , which implies

$$d(fgx_n, gg x_n) \leq R d(fx_n, gx_n). \tag{51}$$

Taking limit as $n \rightarrow +\infty$ in the above inequality, we have

$$\lim_{n \rightarrow +\infty} fgx_n = ft. \tag{52}$$

From the monotonicity of ψ and assumption (A2'), we have

$$\begin{aligned}
k\psi(d(fx_n, fgx_n)) &\geq F[\psi(M(x_n, gx_n)), \phi(M(x_n, gx_n))] + LN(x_n, gx_n) \\
&= F[\psi(a_1d(fx_n, gx_n) + a_2d(fgx_n, ggx_n) + a_3d(fx_n, ggx_n) + a_4d(fgx_n, gx_n) + a_5d(gx_n, ggx_n)) \\
&\quad \phi(a_1d(fx_n, gx_n) + a_2d(fgx_n, ggx_n) + a_3d(fx_n, ggx_n) + a_4d(fgx_n, gx_n) + a_5d(gx_n, ggx_n))] \\
&\quad + L \min\{d(fx_n, gx_n), d(fgx_n, ggx_n)\}.
\end{aligned} \tag{53}$$

Taking limit as $n \longrightarrow +\infty$ in the above inequality, we have

$$\begin{aligned}
k\psi(d(t, ft)) &\geq F[\psi(a_1d(t, t) + a_2d(ft, ft) + a_3d(t, ft) + a_4d(ft, t) + a_5d(t, ft)), \\
&\quad \phi(a_1d(t, t) + a_2d(ft, ft) + a_3d(t, ft) + a_4d(ft, t) + a_5d(t, ft))] \\
&\quad + L \min\{d(t, t), d(ft, t)\} \\
&= F[\psi((a_3 + a_4 + a_5)d(t, ft)), \phi((a_3 + a_4 + a_5)d(t, ft))] \\
&\geq k\psi((a_3 + a_4 + a_5)d(t, ft)).
\end{aligned} \tag{54}$$

Since $a_1 + a_2 < a_3 + a_4$ and $a_1 + a_2 + a_5 > 1$, it follows that

$$\begin{aligned}
k\psi(d(t, ft)) &\geq F[\psi((a_3 + a_4 + a_5)d(t, ft)), \phi((a_3 + a_4 + a_5)d(t, ft))] \\
&\geq k\psi((a_3 + a_4 + a_5)d(t, ft)) \\
&\geq k\psi((a_1 + a_2 + a_5)d(t, ft)) \\
&\geq k\psi(d(t, ft)),
\end{aligned} \tag{55}$$

which implies that

$$F[\psi((a_3 + a_4 + a_5)d(t, ft)), \phi((a_3 + a_4 + a_5)d(t, ft))] = k\psi(d(t, ft)). \tag{56}$$

From the definition of $\mathcal{C}_{\text{inv-}k}$ -class functions, we have

$$\begin{aligned}
&\psi((a_3 + a_4 + a_5)d(t, ft)) = 0 \\
&\text{or } \phi((a_3 + a_4 + a_5)d(t, ft)) = 0.
\end{aligned} \tag{57}$$

By the definitions of ψ and ϕ , we have that $d(t, ft) = 0$, which proves that $ft = t$.

Again, from the monotonicity of ψ and assumption $(A2')$, we have

$$\begin{aligned}
k\psi(d(fgx_n, ft)) &\geq F[\psi(M(gx_n, t)), \phi(M(gx_n, t))] + LN(gx_n, t) \\
&= F[\psi(a_1d(fgx_n, ggx_n) + a_2d(ft, gt) + a_3d(fgx_n, gt) + a_4d(ft, ggx_n) + a_5d(ggx_n, gt)), \\
&\quad \phi(a_1d(fgx_n, ggx_n) + a_2d(ft, gt) + a_3d(fgx_n, gt) + a_4d(ft, ggx_n) + a_5d(ggx_n, gt))] \\
&\quad + L \min\{d(fgx_n, ggx_n), d(ft, ggx_n)\}.
\end{aligned} \tag{58}$$

Taking limit as $n \rightarrow +\infty$ in above inequality and by the definition of ψ , we have

$$\begin{aligned}
 0 &= k\psi(d(ft, ft)) \\
 &\geq F[\psi(a_1d(ft, ft) + a_2d(ft, gt) + a_3d(ft, gt) + a_4d(ft, ft) + a_5d(ft, gt)), \\
 &\quad \phi(a_1d(ft, ft) + a_2d(ft, gt) + a_3d(ft, gt) + a_4d(ft, ft) + a_5d(ft, gt))] \\
 &\quad + L \min\{d(ft, ft), d(ft, gt)\} \\
 &= F[\psi((a_2 + a_3 + a_5)d(ft, gt)), \phi((a_2 + a_3 + a_5)d(ft, gt))] \\
 &\geq k\psi((a_2 + a_3 + a_5)d(ft, gt)).
 \end{aligned} \tag{59}$$

Since $a_i > 0, i = \{1, 2, \dots, 5\}$, it yields that $d(ft, gt) = 0$ or $ft = gt$, which further implies $ft = gt = t$; hence, t is a common fixed point of f and g .

Finally, the uniqueness of the common fixed point of f and g can be directly obtained from the second half proof of Theorem 1. \square

Remark 2. If f and g are assumed to be R -weakly commuting of type A_g , the conclusion of Theorem 3 still holds true.

Now, we provide an example to verify the validity of Theorem 3 as follows.

Example 9. Let $X = [1, 2]$ be a usual metric space and $x, y \in X$. We define f and g on X as follows: $f(x) = 1$ for all $x \in X$ and $g(x) = x$ for all $x \in X$. It is easy to check that $f(X) \subseteq g(X)$.

To show that the pair of mappings (f, g) is weak semicompatible and f -compatible of type (E) , we take any sequence $\{x_n\} \in (1, 2)$, then $\lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} gx_n = 1$. Hence, $\lim_{n \rightarrow +\infty} fgx_n = 1 = g(1)$ and $\lim_{n \rightarrow +\infty} ffx_n = 1 = g(1)$. It also can be observed that pair (f, g) satisfy R -weak commuting of type A_f for $R = 1$. To satisfy condition $(A2')$, we choose $F(s, t) = 2s$ with $k = 1$ and $\psi(t) = t/20$, for all $t \geq 0$. Then, for all $x, y \in [1, 2]$, we have

$$\begin{aligned}
 k\psi(d(fx, fy)) &= 0, \\
 F[\psi(M(x, y)), \phi(M(x, y))] &+ LN(x, y) \\
 &= \frac{2}{20} [a_1d(fx, gx) + a_2d(fy, gy) + a_3d(fx, gy) + a_4d(fy, gx) + a_5d(gx, gy)] \\
 &\quad + L \min\{d(fx, gx), d(fy, gx)\} \\
 &= \frac{1}{10} [a_1|1 - x| + a_2|1 - y| + a_3|1 - y| + a_4|1 - x| + a_5|x - y|] + L \min\{|1 - x|, |1 - x|\} \\
 &= \frac{1}{10} [a_1(x - 1) + a_2(y - 1) + a_3(y - 1) + a_4(x - 1) + a_5|x - y|] + L(x - 1).
 \end{aligned} \tag{60}$$

Here are three possible cases as follows:

Case (1): if $x > y$, taking $a_1 = 2, a_2 = 2, a_3 = 3, a_4 = 3, a_5 = 5$, and $-9 < L \leq -1$ satisfying $a_1 - a_3 \leq 1, a_1 +$

$a_2 < a_3 + a_4, a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$, then we have

$$\begin{aligned}
 0 &\geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y) \\
 &= \frac{1}{10} [a_1(x - 1) + a_2(y - 1) + a_3(y - 1) + a_4(x - 1) + a_5(x - y)] + L(x - 1) \\
 &= (x - 1) + L(x - 1),
 \end{aligned} \tag{61}$$

which proves that condition $(A2')$ holds true.

Case (2): if $x < y$, taking $a_1 = 2, a_2 = 2, a_3 = 3, a_4 = 3, a_5 = 5$, and $-9 < L \leq -1$ satisfying $a_1 - a_3 \leq 1$,

$a_1 + a_2 < a_3 + a_4, a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$, then we have

$$\begin{aligned} 0 &\geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y) \\ &= \frac{1}{10} [a_1(x-1) + a_2(y-1) + a_3(y-1) + a_4(x-1) + a_5(y-x)] + L(x-1) \\ &= (y-1) + L(x-1), \end{aligned} \quad (62)$$

which shows that condition $(A2')$ holds true.

Case (3): if $x = y$, taking $a_1 = 2, a_2 = 2, a_3 = 3, a_4 = 3, a_5 = 5$, and $-9 < L \leq -1$ satisfying $a_1 - a_3 \leq 1, a_1 + a_2 < a_3 + a_4, a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$, then we have

$$\begin{aligned} 0 &\geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y) \\ &= \frac{1}{10} (a_1 + a_2 + a_3 + a_4)(x-1) + L(x-1) \quad (63) \\ &= (x-1) + L(x-1), \end{aligned}$$

which shows that condition $(A2')$ holds true.

Hence, the pair (f, g) satisfies all conditions of Theorems 3, with 1 being the unique common fixed point.

Next, we will present some common fixed point theorems under conditional semicompatibility as follows.

Theorem 4. Let (X, d) be a complete metric space and a pair (g, f) of self-maps be conditional semicompatible satisfying the following assumptions:

$$(A1) \quad f(X) \subseteq g(X)$$

$$(A2') \quad k\psi(d(fx, fy)) \geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y)$$

Here, $M(x, y) = a_1d(fx, gx) + a_2d(fy, gy) + a_3d(fx, gy) + a_4d(fy, gx) + a_5d(gx, gy)$ and $N(x, y) = \min\{d(fx, gx), d(fy, gx)\}$, for all $x, y \in X$. Moreover, $a_i > 0$, $(i = 1, 2, \dots, 5)$, $L \in \mathbb{R}$, $F \in \mathcal{C}_{inv-k}$ for some $k \geq 1$ with $1 \geq a_1 - a_3$,

$a_1 + a_2 + a_5 > 1, a_1 + a_2 < a_3 + a_4, k(a_1 + a_2 + a_3 + a_4) + L > 1$, and $\psi \in \Psi, \phi \in \Phi$. If f is g -absorbing or g is f -absorbing, then f and g have a unique common fixed point t in X .

Proof. Similar to the first part of the proof in Theorems 1 or 3, we can construct a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} gx_n = t, \quad t \in X. \quad (64)$$

Again since f and g are conditional semicompatible and $\lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} gx_n = t$ (nonempty), then there exists a sequence $\{y_n\}$ satisfying $\lim_{n \rightarrow +\infty} fy_n = \lim_{n \rightarrow +\infty} gy_n = u$ such that $\lim_{n \rightarrow +\infty} fgy_n = gu$ and $\lim_{n \rightarrow +\infty} gfy_n = fu$.

Case 1: f is g -absorbing.

Since f is g -absorbing, this yields that

$$d(gy_n, gfy_n) \leq Rd(fy_n, gy_n). \quad (65)$$

Taking limit as $n \rightarrow +\infty$ in the above inequality, it follows that

$$\begin{aligned} \lim_{n \rightarrow +\infty} gy_n &= \lim_{n \rightarrow +\infty} gfy_n = u, \\ fu &= u. \end{aligned} \quad (66)$$

Since $f(X) \subseteq g(X)$, then there exists v in X such that $fu = gv$. Next, we will show that $fv = u$.

By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned} 0 &= k\psi(d(fv, fv)) \\ &\geq F[\psi(M(v, v)), \phi(M(v, v))] + LN(v, v) \\ &= F[a_1d(fv, gv) + a_2d(fv, gv) + a_3d(fv, gv) + a_4d(fv, gv) + a_5d(gv, gv)], \\ &\quad \phi(a_1d(fv, gv) + a_2d(fv, gv) + a_3d(fv, gv) + a_4d(fv, gv) + a_5d(gv, gv))] \\ &\quad + L \min\{d(fv, gv), d(fv, gv)\} \\ &= F[a_1d(fv, gv) + a_2d(fv, gv) + a_3d(fv, gv) + a_4d(fv, gv)], \\ &\quad \phi(a_1d(fv, gv) + a_2d(fv, gv) + a_3d(fv, gv) + a_4d(fv, gv))] + Ld(fv, gv) \\ &\geq k\psi((a_1 + a_2 + a_3 + a_4)d(fv, u)) + Ld(fv, u) \\ &\geq (k(a_1 + a_2 + a_3 + a_4) + L)d(fv, u). \end{aligned} \quad (67)$$

Together with $a_i > 0, i = \{1, 2, \dots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from above inequality, it follows that $d(fv, u) = 0$ or $fv = u = gv$.

Again, since f is g -absorbing, it yields

$$d(gv, gfv) \leq Rd(fv, gv), \quad (68)$$

which implies that $gv = gfv$ or $ggv = gv$. Now, we will show $fgv = gv$. By the definition of ψ , $s > 1$ and assumption $(A2')$, we have

$$\begin{aligned} 0 &= k\psi(d(fgv, fgv)) \\ &\geq F[\psi(M(gv, gv)), \phi(M(gv, gv))] + LN(gv, gv) \\ &= F[\psi(a_1d(fgv, ggv) + a_2d(fgv, ggv) + a_3d(fgv, ggv) + a_4d(fgv, ggv) + a_5d(ggv, ggv)), \\ &\quad \phi(a_1d(fgv, ggv) + a_2d(fgv, ggv) + a_3d(fgv, ggv) + a_4d(fgv, ggv) + a_5d(ggv, ggv))] \\ &\quad + L \min\{d(fgv, ggv), d(fgv, ggv)\} \\ &= F[\psi(a_1d(fgv, gv) + a_2d(fgv, gv) + a_3d(fgv, gv) + a_4d(fgv, gv)), \\ &\quad \phi(a_1d(fv, gv) + a_2d(fgv, gv) + a_3d(fgv, gv) + a_4d(fgv, gv))] + Ld(fgv, ggv) \\ &\geq k\psi((a_1 + a_2 + a_3 + a_4)d(fgv, gv)) + Ld(fgv, gv) \\ &\geq (k(a_1 + a_2 + a_3 + a_4) + L)d(fgv, gv). \end{aligned} \quad (69)$$

Together with $a_i > 0, i = \{1, 2, \dots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from the above inequality, it follows that $d(fgv, gv) = 0$ or $fgv = gv$.

Therefore, gv is a common fixed point of f and g .

Case 2: g is f -absorbing.

Since g is f -absorbing, this yields

$$d(fy_n, fgy_n) \leq Rd(fy_n, gy_n). \quad (70)$$

Taking limit as $n \rightarrow +\infty$ in the above inequality, it follows that

$$\begin{aligned} \lim_{n \rightarrow +\infty} gy_n &= \lim_{n \rightarrow +\infty} fgy_n = u, \\ gu &= u. \end{aligned} \quad (71)$$

Next, we will show $fu = gu$.

By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned} 0 &= k\psi(d(fu, fu)) \\ &\geq F[\psi(M(u, u)), \phi(M(u, u))] + LN(u, u) \\ &= F[\psi(a_1d(fu, gu) + a_2d(fu, gu) + a_3d(fu, gu) + a_4d(fu, gu) + a_5d(gu, gu)), \\ &\quad \phi(a_1d(fu, gu) + a_2d(fu, gu) + a_3d(fu, gu) + a_4d(fu, gu) + a_5d(gu, gu))] \\ &\quad + L \min\{d(fu, gu), d(fu, gu)\} \\ &= F[a_1d(fu, gu) + a_2d(fu, gu) + a_3d(fu, gu) + a_4d(fu, gu), \\ &\quad \phi(a_1d(fu, gu) + a_2d(fu, gu) + a_3d(fu, gu) + a_4d(fu, gu))] + Ld(fu, gu) \\ &\geq k\psi(a_1d(fu, gu) + a_2d(fu, gu) + a_3d(fu, gu) + a_4d(fu, gu)) + Ld(fu, gu) \\ &\geq (k(a_1 + a_2 + a_3 + a_4) + L)d(fu, gu). \end{aligned} \quad (72)$$

Together with $a_i > 0, i = \{1, 2, \dots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from the above inequality, it follows that $d(fu, gu) = 0$ or $fu = gu$. Therefore, u is a common fixed point of f and g .

The uniqueness of common fixed point of f and g can be directly obtained by following the same argument as Theorem 1. \square

Remark 3. The results of Theorem 4 still hold by replacing condition $(A2')$ with condition $(A2'')$ stated as follows.

$$(A2''): k\theta(d(fx, fy)) \geq F[\theta(M(x, y)), \phi(M(x, y))].$$

Here, $M(x, y) = a_1d(fx, gx) + a_2d(fy, gy) + a_3d(fx, gy) + a_4d(fy, gx) + a_5d(gx, gy)$, for all $x, y \in X$. Moreover, $a_i > 0, (i = 1, 2, \dots, 5)$ with $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, $F \in \mathcal{C}_{\text{inv-}k}$ for some $k \geq 1$ and $\theta \in \Theta$, $\phi \in \Phi$.

Moreover, θ is said to be an altering distance function (see [19]), which satisfies (i) $\theta: [0, +\infty) \rightarrow [0, +\infty)$ is continuous and increasing; (ii) $\theta(t) = 0 \Leftrightarrow t = 0$. Denote the class of altering distance functions by Θ .

Now, we provide an example to verify the validity of Theorem 4 as follows.

Example 10. Let $X = [2, +\infty)$, $x, y \in X$ with $x > y$ and let d be the usual metric on X . Define $f, g: X \rightarrow X$ as follows:

$$f x = \begin{cases} \frac{x+8}{2}, & 2 \leq x < 5, \\ 5, & x \geq 5, \end{cases} \quad (73)$$

$$g x = \begin{cases} x+3, & 2 \leq x < 5, \\ 5, & x \geq 5. \end{cases}$$

It is clear that $f(X) \subseteq g(X)$.

Taking $x_n = 2 + \varepsilon_n$, where $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, we have $\lim_{n \rightarrow +\infty} f x_n = 5$ and $\lim_{n \rightarrow +\infty} g x_n = 5$. Therefore, $\lim_{n \rightarrow +\infty} f x_n = \lim_{n \rightarrow +\infty} g x_n = 5$ (nonempty). Then, we have a sequence $y_n = 5 + \varepsilon_n$, where $\varepsilon_n \rightarrow 0$ as $n \rightarrow +\infty$, for which $\lim_{n \rightarrow +\infty} f y_n = \lim_{n \rightarrow +\infty} f(5 + \varepsilon_n) = 5$ and $\lim_{n \rightarrow +\infty} g y_n = \lim_{n \rightarrow +\infty} g(5 + \varepsilon_n) = 5$. Moreover, $\lim_{n \rightarrow +\infty} f g y_n = \lim_{n \rightarrow +\infty} f g(5 + \varepsilon_n) = \lim_{n \rightarrow +\infty} f(5) = 5 = g(5)$ and $\lim_{n \rightarrow +\infty} g f y_n = \lim_{n \rightarrow +\infty} g f(5 + \varepsilon_n) = \lim_{n \rightarrow +\infty} g(5) = 5 = f(5)$. Therefore, the pair (f, g) is conditional semicompatible.

For $2 \leq x < 5$, $g f x = g((x+8)/2) = (x+8)/2$, then $d(gx, g f x) = |(x+3) - (x+8)/2| = (x-2)/2$ and $d(gx, f x) = |(x+3) - (x+8)/2| = (x-2)/2$. Therefore, f and g satisfy $d(gx, g f x) \leq R d(fx, gx)$ with $R = 1$. Also, for $x \geq 5$, then $0 = d(gx, g f x) = R d(gx, f x)$ for all real number R . Therefore, f and g satisfy $d(gx, g f x) \leq R d(fx, gx)$, for all $x \in X$ and $R = 1$; that is, f is g -absorbing with $R = 1$.

To satisfy condition $(A2')$, we choose $F(s, t) = s + t$ with $k = 1$ and $\psi(t) = \phi(t) = t/40$, for all $t \geq 0$. Then, for all $x, y \in [2, +\infty)$ we have

$$k\psi(d(fx, fy)) = \begin{cases} \frac{|x-2|}{2}, & 2 \leq y < x < 5, \\ 0, & 5 \leq y < x < +\infty, \\ \frac{|y-2|}{2}, & 2 \leq y < 5 \leq x < +\infty. \end{cases} \quad (74)$$

Here are three possible cases:

Case (1): if $2 \leq y < x < 5$, we have

$$\begin{aligned} & F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y) \\ &= \frac{1}{20} [a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)] \\ &\quad + L \min\{d(fx, gx), d(fy, gx)\} \\ &= \frac{1}{20} \left[a_1 \frac{|x-2|}{2} + a_2 \frac{|y-2|}{2} + a_3 \frac{|x-2y+2|}{2} + a_4 \frac{|y-2x+2|}{2} + a_5 \frac{|y-2|}{2} \right] \\ &\quad L \min \left\{ \frac{|x-2|}{2}, \frac{|y-2x+2|}{2} \right\} \\ &\leq \frac{1}{20} \left[a_1 \frac{|x-2|}{2} + a_2 \frac{|y-2|}{2} + a_3 \left(\frac{|x-y|}{2} + \frac{|y-2|}{2} \right) + a_4 \left(\frac{|x-y|}{2} + \frac{|x-2|}{2} \right) + a_5 \frac{|y-2|}{2} \right] L \frac{|x-2|}{2} \\ &= \left(\frac{a_1 + a_4}{20} + L \right) \frac{|x-2|}{2} + \left(\frac{a_2 + a_3 + a_5}{20} \right) \frac{|y-2|}{2} + \left(\frac{a_3 + a_4}{20} \right) \frac{|x-y|}{2} \\ &\leq \left(\frac{a_1 + a_3 + 2a_4}{20} + L \right) \frac{|x-2|}{2} + \left(\frac{a_2 + 2a_3 + a_4 + a_5}{20} \right) \frac{|y-2|}{2} \\ &\leq \left(\frac{a_1 + a_2 + 3a_3 + 3a_4 + a_5}{20} + L \right) \frac{|x-2|}{2}. \end{aligned} \quad (75)$$

It is obvious that $k\psi(d(fx, fy)) \geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y)$ holds true by taking $a_1 = 1, a_2 = a_3 = a_4 = 2, a_5 = 5$, and $-6 < L \leq -1$ satisfying $a_1 - a_3 \leq 1$, $a_1 + a_2 < a_3 + a_4$, $a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$.

Case (2): if $5 \leq y < x < +\infty$, we have

$$F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y) = 0. \quad (76)$$

It is obvious that $k\psi(d(fx, fy)) \geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y)$ holds true by taking $a_i > 0$, ($i = 1, 2, \dots, 5$) and $L \in \mathbb{R}$.

Case (3): if $2 \leq y < 5 \leq x < +\infty$, we have

$$\begin{aligned}
& F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y) \\
&= \frac{1}{20} [a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)] \\
&\quad + L \min\{d(fx, gx), d(fy, gx)\} \\
&= \frac{1}{20} \left[a_2 \frac{|y-2|}{2} + a_3 |y-2| + a_4 \frac{|y-2|}{2} + a_5 \frac{|y-2|}{2} \right] \\
&= \frac{1}{20} \left(\frac{a_2}{2} + a_3 + \frac{a_4}{2} + \frac{a_5}{2} \right) |y-2|.
\end{aligned} \tag{77}$$

It is obvious that $k\psi(d(fx, fy)) \geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y)$ holds true by taking $a_1 = 1, a_2 = a_3 = a_4 = 2, a_5 = 5$, and $L \in \mathbb{R}$ satisfying $a_1 - a_3 \leq 1$, $a_1 + a_2 < a_3 + a_4$, $a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$.

To sum up, condition (A2') holds true by taking $a_1 = 1, a_2 = a_3 = a_4 = 2, a_5 = 5$, and $-6 < L \leq -1$ satisfying $a_1 - a_3 \leq 1$, $a_1 + a_2 < a_3 + a_4$, $a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$.

Therefore, f and g satisfy all the conditions of Theorems 4 with 5 being the unique common fixed point.

Theorem 5. Let (X, d) be a complete metric space and let a pair (g, f) of self-maps be conditional semicompatible satisfying the following assumptions:

$$(A1) \quad f(X) \subseteq g(X)$$

$$(A2') \quad k\psi(d(fx, fy)) \geq F[\psi(M(x, y)), \phi(M(x, y))] + LN(x, y)$$

Here, $M(x, y) = a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)$ and $N(x, y) = \min\{d(fx, gx), d(fy, gx)\}$, for all $x, y \in X$. Moreover, $a_i > 0$, ($i = 1, 2, \dots, 5$), $L \in \mathbb{R}$, $F \in \mathcal{C}_{inv-k}$ for some $k \geq 1$ with $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, and $\psi \in \Psi$, $\phi \in \Phi$. If pair (f, g) is R -weak

commuting either of type A_f or of type A_g , then f and g have a unique common fixed point t in X .

Proof. From the part of the proof in Theorems 1 or 3, we can construct a sequence $\{x_n\}$ in X such that $\lim_{n \rightarrow +\infty} f x_n = \lim_{n \rightarrow +\infty} g x_n = t$.

Again since f and g are conditional semicompatible and $\lim_{n \rightarrow +\infty} f x_n = \lim_{n \rightarrow +\infty} g x_n = t$ (nonempty), then there exists a sequence $\{y_n\}$ satisfying $\lim_{n \rightarrow +\infty} f y_n = \lim_{n \rightarrow +\infty} g y_n = u$ such that $\lim_{n \rightarrow +\infty} f g y_n = gu$ and $\lim_{n \rightarrow +\infty} g f y_n = fu$.

Case 1: pair (f, g) is R -weak commuting of type A_f . Since pair (f, g) is R -weak commuting of type A_f , it yields

$$d(f g y_n, g g y_n) \leq R d(f y_n, g y_n). \tag{78}$$

Taking limit as $n \rightarrow +\infty$ in the above inequality, it follows that

$$\lim_{n \rightarrow +\infty} f g y_n = \lim_{n \rightarrow +\infty} g g y_n = gu. \tag{79}$$

Now, we will show that $fu = gu$. By the definition of ψ and assumption (A2'), we have

$$\begin{aligned}
0 &= k\psi(d(fu, fu)) \\
&\geq F[\psi(M(u, u)), \phi(M(u, u))] + LN(u, u) \\
&= F[\psi(a_1 d(fu, gu) + a_2 d(fu, gu) + a_3 d(fu, gu) + a_4 d(fu, gu) + a_5 d(gu, gu)), \\
&\quad \phi(a_1 d(fu, gu) + a_2 d(fu, gu) + a_3 d(fu, gu) + a_4 d(fu, gu) + a_5 d(gu, gu))] \\
&\quad + L \min\{d(fu, gu), d(fu, du)\} \\
&= F[\psi(a_1 d(fu, gu) + a_2 d(fu, gu) + a_3 d(fu, gu) + a_4 d(fu, gu)), \\
&\quad \phi(a_1 d(fu, gu) + a_2 d(fu, gu) + a_3 d(fu, gu) + a_4 d(fu, gu))] + L d(fu, gu) \\
&\geq k\psi(a_1 d(fu, gu) + a_2 d(fu, gu) + a_3 d(fu, gu) + a_4 d(fu, gu)) + L d(fu, gu) \\
&\geq (k(a_1 + a_2 + a_3 + a_4) + L) d(fu, gu).
\end{aligned} \tag{80}$$

Together with $a_i > 0, i = \{1, 2, \dots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from the above inequality, it follows that $d(fu, gu) = 0$ or $fu = gu$.

Again, since pair (f, g) is R -weak commuting of type A_f , then we have

$$d(fgu, ggu) \leq R d(fu, gu), \quad (81)$$

which implies $fgu = ggu$.

Now, we will show $fgu = gu$.

By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned} & k\psi(d(fgu, fu)) \\ & \geq F[\psi(M(gu, u)), \phi(M(gu, u))] + LN(gu, u) \\ & = F[\psi(a_1 d(fgu, ggu) + a_2 d(fu, gu) + a_3 d(fgu, gu) + a_4 d(fu, ggu) + a_5 d(ggu, gu)), \\ & \quad \phi(a_1 d(fgu, ggu) + a_2 d(fu, gu) + a_3 d(fgu, gu) + a_4 d(fu, ggu) + a_5 d(ggu, gu))] \\ & \quad + L \min\{d(fgu, ggu), d(fu, ggu)\} \\ & = F[\phi(a_3 d(fgu, gu) + a_4 d(fgu, gu) + a_5 d(fgu, gu)), \\ & \quad \phi(a_3 d(fgu, gu) + a_4 d(fgu, gu) + a_5 d(fgu, gu))] \\ & \geq k\psi((a_3 + a_4 + a_5)d(fgu, gu)). \end{aligned} \quad (82)$$

Since $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, and $fu = gu$, it follows that

$$\begin{aligned} & k\psi(d(fgu, gu)) \\ & = F[\psi((a_3 + a_4 + a_5)d(fgu, gu)), \phi((a_3 + a_4 + a_5)d(fgu, gu))] \\ & \geq k\psi((a_3 + a_4 + a_5)d(fgu, gu)) \\ & \geq k\psi((a_1 + a_2 + a_5)d(fgu, gu)) \\ & \geq k\psi(d(fgu, gu)), \end{aligned} \quad (83)$$

which implies that

$$F[\psi((a_3 + a_4 + a_5)d(fgu, gu)), \phi((a_3 + a_4 + a_5)d(fgu, gu))] = k\psi(d(fgu, gu)). \quad (84)$$

From the definition of $\mathcal{C}_{\text{inv-}k}$ -class functions, we have

$$\begin{aligned} & \psi((a_3 + a_4 + a_5)d(fgu, gu)) = 0 \\ & \text{or } \phi((a_3 + a_4 + a_5)d(fgu, gu)) = 0. \end{aligned} \quad (85)$$

By the definitions of ψ and ϕ , we have that $d(fgu, gu) = 0$, which proves that $fgu = gu = ggu$.

Therefore, gu is a common fixed point of f and g .

Case 2: pair (f, g) is R -weak commuting of type A_g .

Since pair (f, g) is R -weak commuting of type A_f , it yields

$$d(gfy_n, ffy_n) \leq R d(fy_n, gfy_n). \quad (86)$$

Taking limit as $n \rightarrow +\infty$ in the above inequality, it follows that

$$\lim_{n \rightarrow +\infty} gfy_n = \lim_{n \rightarrow +\infty} ffy_n = fu. \quad (87)$$

Now, we will show that $fu = u$.

By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned}
& k\psi(d(ffy_n, f y_n)) \\
& \geq F[\psi(M(f y_n, y_n)), \phi(M(f y_n, y_n))] + LN(f y_n, y_n) \\
& = F[\psi(a_1 d(ffy_n, g f y_n) + a_2 d(f y_n, g y_n) + a_3 d(ffy_n, g y_n) + a_4 d(f y_n, g f y_n) + a_5 d(g f y_n, g y_n)), \\
& \phi(a_1 d(ffy_n, g f y_n) + a_2 d(f y_n, g y_n) + a_3 d(ffy_n, g y_n) + a_4 d(f y_n, g f y_n) + a_5 d(g f y_n, g y_n))] \\
& + L \min\{d(ffy_n, g f y_n), d(f y_n, g y_n)\}.
\end{aligned} \tag{88}$$

Taking limit as $n \longrightarrow +\infty$ in the above inequality, we have

$$\begin{aligned}
& k\psi(d(fu, u)) \geq F[a_1 d(fu, fu) + a_2 d(u, u) + a_3 d(fu, u) + a_4 d(u, fu) + a_5 d(fu, u), \\
& \phi(a_1 d(fu, fu) + a_2 d(u, u) + a_3 d(fu, u) + a_4 d(u, fu) + a_5 d(fu, u))] \\
& + L \min\{d(fu, fu), d(u, u)\} \\
& = F[\psi(a_3 d(fu, u) + a_4 d(u, fu) + a_5 d(fu, u)), \phi(a_3 d(fu, u) + a_4 d(u, fu) + a_5 d(fu, u))].
\end{aligned} \tag{89}$$

Since $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, it follows that

$$\begin{aligned}
& k\psi(d(fu, u)) \\
& \geq F[\psi(a_3 d(fu, u) + a_4 d(u, fu) + a_5 d(fu, u)), \phi(a_3 d(fu, u) + a_4 d(u, fu) + a_5 d(fu, u))] \\
& \geq k\psi((a_3 + a_4 + a_5)d(fu, u)) \\
& \geq k\psi((a_1 + a_2 + a_5)d(fu, u)) \\
& \geq k\psi(d(fu, u)),
\end{aligned} \tag{90}$$

which implies that

$$F[\psi((a_3 + a_4 + a_5)d(fu, u)), \phi((a_3 + a_4 + a_5)d(fu, u))] = k\psi(d(fu, u)). \tag{91}$$

From the definition of $\mathcal{E}_{\text{inv-}k}$ -class functions, we have

$$\begin{aligned}
& \psi((a_3 + a_4 + a_5)d(fu, u)) = 0 \\
& \text{or } \phi((a_3 + a_4 + a_5)d(fu, u)) = 0.
\end{aligned} \tag{92}$$

By the definitions of ψ and ϕ , we have that $d(fu, u) = 0$, which proves that $fu = u$.

Since $f(X) \subseteq g(X)$, there exists $v \in X$ such that $fu = gv$. Now, we will show that $fv = u$.

By the definition of ψ , $fu = gv = u$, and assumption $(A2')$, we have

$$\begin{aligned}
 0 &= k\psi(d(fv, fv)) \\
 &\geq F[\psi(M(v, v)), \phi(M(v, v))] + LN(v, v) \\
 &= F[\psi(a_1d(fv, gv) + a_2d(fv, gv) + a_3d(fv, gv) + a_4d(fv, gv) + a_5d(gv, gv)), \\
 &\quad \phi(a_1d(fv, gv) + a_2d(fv, gv) + a_3d(fv, gv) + a_4d(fv, gv) + a_5d(gv, gv))] \\
 &\quad + L \min\{d(fv, gv), d(fv, gv)\} \\
 &= F[\psi(a_1d(fv, gv) + a_2d(fv, gv) + a_3d(fv, gv) + a_4d(fv, gv), \\
 &\quad \phi(a_1d(fv, gv) + a_2d(fv, gv) + a_3d(fv, gv) + a_4d(fv, gv))] + Ld(fv, gv) \\
 &\geq k\psi(a_1d(fv, gv) + a_2d(fv, gv) + a_3d(fv, gv) + a_4d(fv, gv)) + Ld(fv, gv) \\
 &\geq (k(a_1 + a_2 + a_3 + a_4) + L)d(fv, gv).
 \end{aligned} \tag{93}$$

Together with $a_i > 0, i = \{1, 2, \dots, 5\}$, $k(a_1 + a_2 + a_3 + a_4) + L > 1$, from above inequality, it follows that $d(fv, gv) = d(fv, u) = 0$ or $fv = gv = u$.

Since pair (f, g) is R -weak commuting of type A_f , it yields

$$d(gfv, ffv) \leq R d(fv, gv), \tag{94}$$

which yields $ffv = gfv$.

Next, we will show $gfv = fv$.

By the definition of ψ and assumption $(A2')$, we have

$$\begin{aligned}
 0 &= k\psi(d(ffv, fv)) \\
 &\geq F[\psi(M(fv, v)), \phi(M(fv, v))] + LN(fv, v) \\
 &= F[\psi(a_1d(ffv, gfv) + a_2d(fv, gv) + a_3d(ffv, gv) + a_4d(fv, gfv) + a_5d(gfv, gv)), \\
 &\quad \phi(a_1d(ffv, gfv) + a_2d(fv, gv) + a_3d(ffv, gv) + a_4d(fv, gfv) + a_5d(gfv, gv))] \\
 &\quad + L \min\{d(ffv, gfv), d(fv, gv)\} \\
 &= F[\psi(a_3d(ffv, gv) + a_4d(fv, gfv) + a_5d(gfv, gv)), \\
 &\quad \phi(a_3d(ffv, gv) + a_4d(fv, gfv) + a_5d(gfv, gv))] \\
 &\geq k\psi((a_3 + a_4 + a_5)d(ffv, gv)).
 \end{aligned} \tag{95}$$

Since $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, $ffv = gfv$, and $fv = gv$, it follows that

$$\begin{aligned}
 &k\psi(d(gfv, fv)) \\
 &= F[\psi((a_3 + a_4 + a_5)d(gfv, fv)), \phi((a_3 + a_4 + a_5)d(gfv, fv))] \\
 &\geq k\psi((a_3 + a_4 + a_5)d(gfv, fv)) \\
 &\geq k\psi((a_1 + a_2 + a_5)d(gfv, fv)) \\
 &\geq k\psi(d(gfv, gv)),
 \end{aligned} \tag{96}$$

which implies that

$$F[\psi((a_3 + a_4 + a_5)d(gfv, fv)), \phi((a_3 + a_4 + a_5)d(gfv, fv))] = k\psi(d(gfv, fv)). \tag{97}$$

From the definition of $\mathcal{C}_{\text{inv-}k}$ -class functions, we have

$$\begin{aligned}\psi((a_3 + a_4 + a_5)d(gfv, fv)) &= 0 \\ \text{or } \phi((a_3 + a_4 + a_5)d(gfv, fv)) &= 0.\end{aligned}\quad (98)$$

By the definitions of ψ and ϕ , we have that $d(gfv, fv) = 0$, which proves that $gfv = fv = fgv$. Therefore, fv is a common fixed point of f and g .

The uniqueness of common fixed point of f and g is also obtained by following the same proof of Theorem 2. \square

Remark 4. The conclusions of Theorem 5 still hold true by replacing condition $(A2')$ with condition $(A2'')$ stated as before.

Now, we will provide an example to verify the validity of Theorem 5 as follows.

Example 11. Let $X = [0, 1]$ with the usual metric and $x, y \in X$. We define f and g on X as follows:

$$\begin{aligned}fx &= \begin{cases} \frac{4}{5}, & x \in \left[0, \frac{1}{2}\right), \\ \frac{3}{4}, & x \in \left(\frac{1}{2}, 1\right], \end{cases} \\ gx &= \begin{cases} \frac{4}{5}, & x \in \left[0, \frac{1}{2}\right), \\ \frac{1}{2}, & x = \frac{1}{2}, \\ \frac{3}{4}, & x \in \left(\frac{1}{2}, 1\right]. \end{cases}\end{aligned}\quad (99)$$

It is clear that $f(X) \subseteq g(X)$.

To show that the pair (f, g) of self-maps is conditional semicompatible, we take any sequence $\{x_n\} \in (0, 1/2)$, then $\lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} gx_n = 4/5$ (nonempty). Now, we choose another sequence $\{y_n\} \in (1/2, 1]$, then $\lim_{n \rightarrow +\infty} fy_n = 3/4 = g(3/2)$ and $\lim_{n \rightarrow +\infty} gfy_n = 3/4 = g(3/2)$. It

can be observed that the pair (f, g) satisfies R -weak commuting of type (A_f) in intervals $[0, 1/2)$ and $(1/2, 1]$ for all real numbers. Also, the pair (f, g) satisfies R -weak commuting of type (A_f) at $x = 1/2$ with $R = 1$. To verify assumption $(A2')$, we define $F(s, t) = 2s \in \mathcal{C}_{\text{inv-}k}$ with $k = 2$, $\psi(t) = t/10$, for all $t \geq 0$. For $x = y = 1/2$, we have

$$\begin{aligned}k\psi(d(fx, fy)) &= 0, \\ F[\psi(M(x, y)), \phi(M(x, y))] &= \frac{1}{5} [a_1 d(fx, gx) + a_2 d(fy, gy) + a_3 d(fx, gy) + a_4 d(fy, gx) + a_5 d(gx, gy)] \\ &\quad + L \min\{d(fx, gx), d(fy, gx)\} \\ &= \frac{1}{5} \left[a_1 \left| \frac{4}{5} - \frac{1}{2} \right| + a_2 \left| \frac{4}{5} - \frac{1}{2} \right| + a_3 \left| \frac{4}{5} - \frac{1}{2} \right| + a_4 \left| \frac{4}{5} - \frac{1}{2} \right| \right] \\ &\quad + L \min\left\{ \left| \frac{4}{5} - \frac{1}{2} \right|, \left| \frac{4}{5} - \frac{1}{2} \right| \right\} \\ &= \frac{1}{5} \left[\frac{3}{10} (a_1 + a_2 + a_3 + a_4) \right] + \frac{3}{10} L = \frac{3}{10} \left[\frac{1}{5} (a_1 + a_2 + a_3 + a_4) + L \right].\end{aligned}\quad (100)$$

It is obvious that $0 \geq (3/10)[(1/5)(a_1 + a_2 + a_3 + a_4) + L]$ holds true by taking $a_1 = a_2 = a_3 = 1$, $a_4 = 2$, $a_5 > 0$, and $-9 < L \leq -1$ such that $a_1 - a_3 \leq 1$, $a_1 + a_2 < a_3 + a_4$,

$a_1 + a_2 + a_5 > 1$, and $k(a_1 + a_2 + a_3 + a_4) + L > 1$. It is also clear that the above inequality holds true for all $x, y \in [0, 1/2)$ and $x, y \in (1/2, 1]$.

Therefore, f and g satisfy all the conditions of Theorem 5 with $3/4$ being the unique common fixed point.

Finally, we will present a common fixed point theorem under S_τ -compatibility as follows.

Theorem 6. Let (X, d) be a complete metric space and let f, g, B, T be self-maps satisfying the following assumptions:

$$(A1') \quad f(X) \subseteq T(X), \quad g(X) \subseteq B(X)$$

$$(A2'') \quad k\theta(d(fx, By)) \geq F[\theta(M'(x, y)), \phi(M'(x, y))]$$

Here, $M'(x, y) = a_1d(fx, gx) + a_2d(By, Ty) + a_3d(fx, Ty) + a_4d(gx, By) + a_5d(gx, Ty)$, for all $x, y \in X$. Moreover, $a_i > 0$, $(i = 1, 2, \dots, 5)$ with $1 \geq a_1 - a_3$, $a_1 +$

$a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, $F \in \mathcal{C}_{inv-k}$ for some $k \geq 1$ and $\theta \in \Theta$, $\phi \in \Phi$. If the pair (f, g) is S_B -compatible and (B, T) is S_g -compatible, then f, g, B, T have a unique common fixed point t in X .

Proof. Let $x_0 \in X$. Since $f(X) \subseteq T(X)$, then there exists $x_1 \in X$ such that $fx_1 = Tx_0 = y_0$. For $x_1 \in X$ and $g(X) \subseteq B(X)$, there exists $x_2 \in X$ such that $Bx_2 = gx_1 = y_1$. Continuing this way, we can construct the sequence $\{x_n\}$ in X such that $fx_n = Tx_{n-1} = y_{n-1}$ and $Bx_{n+1} = gx_n = y_n$.

By the assumption of $(A2'')$ and properties of θ and F , we have

$$\begin{aligned} & k\theta(d(fx_n, Bx_{n+1})) \\ & \geq F[\theta(M'(x_n, x_{n+1})), \phi(M'(x_n, x_{n+1}))] \\ & \geq k\theta(M'(x_n, x_{n+1})) \\ & = k\theta(a_1d(fx_n, gx_n) + a_2d(Bx_{n+1}, Tx_{n+1}) + a_3d(fx_n, Tx_{n+1}) + a_4d(gx_n, Bx_{n+1}) + a_5d(gx_n, Tx_{n+1})) \\ & = k\theta(a_1d(y_{n-1}, y_n) + a_2d(y_n, y_{n+1}) + a_3d(y_{n-1}, y_{n+1}) + a_4d(y_n, y_n) + a_5d(y_n, y_{n+1})) \\ & = k\theta(a_1d(y_{n-1}, y_n) + a_2d(y_n, y_{n+1}) + a_3d(y_{n-1}, y_{n+1}) + a_5d(y_n, y_{n+1})). \end{aligned} \quad (101)$$

By the similar procedure demonstrated in Theorem 1, we can get $d(y_n, y_{n+1}) \leq (1 - a_1 + a_3)/(a_2 + a_3 + a_5)d(y_{n-1}, y_n)$.

From Lemma 1, it follows that sequence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point $t \in X$ such that

$$\lim_{n \rightarrow +\infty} y_n = \lim_{n \rightarrow +\infty} fx_n = \lim_{n \rightarrow +\infty} gx_n = \lim_{n \rightarrow +\infty} Bx_n = \lim_{n \rightarrow +\infty} Tx_n = t. \quad (102)$$

Next, we will show that t is a common fixed point of f, g, B, T .

Since the pair (f, g) is S_B -compatible and the pair (B, T) is S_g -compatible and by equation (102), we have the following outcomes:

$$\lim_{n \rightarrow +\infty} fBx_n = \lim_{n \rightarrow +\infty} gBx_n = Bt, \quad (103)$$

$$\lim_{n \rightarrow +\infty} Bgx_n = \lim_{n \rightarrow +\infty} Tgx_n = gt. \quad (104)$$

By the assumption of $(A2'')$ and properties of θ and F , we have

$$\begin{aligned} & k\theta((d(fBx_n, Bx_n))) \\ & \geq F[\theta(M'(Bx_n, x_n)), \phi(M'(Bx_n, x_n))] \\ & \geq k\theta(M'(Bx_n, x_n)) \\ & = k\theta(a_1d(fBx_n, gBx_n) + a_2d(Bx_n, Tx_n) + a_3d(fBx_n, Tx_n) + a_4d(gBx_n, Bx_n) + a_5d(gBx_n, Tx_n)). \end{aligned} \quad (105)$$

Letting $n \longrightarrow +\infty$ in the above inequality, with equation (103), we have

$$\begin{aligned}
 & k\theta(d(Bt, t)) \\
 & \geq F[\theta(a_3d(Bt, t) + a_4d(Bt, t) + a_5d(Bt, t)), \phi(a_3d(Bt, t) + a_4d(Bt, t) + a_5d(Bt, t))] \\
 & \geq k\theta(a_3d(Bt, t) + a_4d(Bt, t) + a_5d(Bt, t)) \\
 & = k\theta((a_3 + a_4 + a_5)d(Bt, t)).
 \end{aligned} \tag{106}$$

Since $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, we have

$$\begin{aligned}
 & k\theta(d(Bt, t)) \\
 & \geq F[\theta(a_3d(Bt, t) + a_4d(Bt, t) + a_5d(Bt, t)), \phi(a_3d(Bt, t) + a_4d(Bt, t) + a_5d(Bt, t))] \\
 & \geq k\theta((a_3 + a_4 + a_5)d(Bt, t)) \\
 & \geq k\theta((a_1 + a_2 + a_5)d(Bt, t)) \\
 & \geq k\theta(d(Bt, t)),
 \end{aligned} \tag{107}$$

which implies that

$$F[\theta((a_3 + a_4 + a_5)d(Bt, t)), \phi((a_3 + a_4 + a_5)d(Bt, t))] = k\theta(d(Bt, t)). \tag{108}$$

From the definition of $\mathcal{C}_{\text{inv-}k}$ -class functions, we have

$$\begin{aligned}
 & \theta((a_3 + a_4 + a_5)d(Bt, t)) = 0 \\
 & \text{or } \phi((a_3 + a_4 + a_5)d(Bt, t)) = 0.
 \end{aligned} \tag{109}$$

By the definitions of θ and ϕ , we have that $d(Bt, t) = 0$, which proves that $Bt = t$.

We also have

$$\begin{aligned}
 & k\theta(d(fx_n, Bgx_n)) \\
 & \geq F[\theta(M'(x_n, gx_n)), \phi(M'(x_n, gx_n))] \\
 & \geq k\theta(M'(x_n, gx_n)) \\
 & = k\theta(a_1d(fx_n, gx_n) + a_2d(Bgx_n, Tgx_n) + a_3d(fx_n, Tgx_n) + a_4d(gx_n, Bgx_n) + a_5d(gx_n, Tgx_n)).
 \end{aligned} \tag{110}$$

Letting $n \longrightarrow +\infty$ in the above inequality, with equation (104), we have

$$\begin{aligned}
 & k\theta(d(t, gt)) \\
 & \geq F[\theta(a_3d(t, gt) + a_4d(t, gt) + a_5d(t, gt)), \phi(a_3d(t, gt) + a_4d(t, gt) + a_5d(t, gt))] \\
 & \geq k\theta(a_3d(t, gt) + a_4d(t, gt) + a_5d(t, gt)) \\
 & = k\theta((a_3 + a_4 + a_5)d(t, gt)).
 \end{aligned} \tag{111}$$

Since $a_1 + a_2 + a_5 > 1$, $a_1 + a_2 < a_3 + a_4$, we have

$$\begin{aligned}
 & k\theta(d(t, gt)) \\
 & \geq F[\theta(a_3d(t, gt) + a_4d(t, gt) + a_5d(t, gt)), \phi(a_3d(t, gt) + a_4d(t, gt) + a_5d(t, gt))] \\
 & \geq k\theta((a_3 + a_4 + a_5)d(t, gt)) \\
 & \geq k\theta((a_1 + a_2 + a_5)d(t, gt)) \\
 & \geq k\theta(d(t, gt)),
 \end{aligned} \tag{112}$$

which implies that

$$F[\theta((a_3 + a_4 + a_5)d(t, gt)), \phi((a_3 + a_4 + a_5)d(Bt, t))] = k\theta(d(t, gt)). \tag{113}$$

From the definition of $\mathcal{C}_{\text{inv-}k}$ -class functions, we have

$$\begin{aligned}
 & \theta((a_3 + a_4 + a_5)d(t, gt)) = 0 \\
 & \text{or } \phi((a_3 + a_4 + a_5)d(t, gt)) = 0.
 \end{aligned} \tag{114}$$

By the definitions of θ and ϕ , we have that $d(t, gt) = 0$, which proves that $t = gt$.

Now, we will prove $Tt = t$. If not, by the assumption of $(A2'')$ and properties of θ and F , we have

$$\begin{aligned}
 k\theta(d(fx_n, Bt)) & \geq F[\theta(M'(x_n, t)), \phi(M'(x_n, t))] \\
 & \geq k\theta(M'(x_n, t)) = k\theta(a_1d(fx_n, gx_n) + a_2d(Bt, Tt) + a_3d(fx_n, Tt) + a_4d(gx_n, Bt) + a_5d(gx_n, Tt)).
 \end{aligned} \tag{115}$$

Letting $n \rightarrow +\infty$ in the above inequality, we have

$$\begin{aligned}
 0 = k\theta(d(t, t)) & \geq F[\theta(a_1d(t, t) + a_2d(t, Tt)a_3d(t, Tt) + a_4d(t, t) + a_5d(t, Tt)), \\
 & \phi(a_1d(t, t) + a_2d(t, Tt)a_3d(t, Tt) + a_4d(t, t) + a_5d(t, Tt))] \\
 & \geq k\theta(a_1d(t, t) + a_2d(t, Tt)a_3d(t, Tt) + a_4d(t, t) + a_5d(t, Tt)) \\
 & = k\theta((a_2 + a_2 + a_5)d(t, Tt)).
 \end{aligned} \tag{116}$$

which implies $(a_2 + a_2 + a_5)d(t, Tt) = 0$; that is, $t = Tt$.

In the same manner, we can also get $t = ft$.

Hence, t is a common fixed point of f, g, B, T .

The uniqueness of t can also be obtained by the similar way stated in Theorem 1.

Now, we present the following example to support the validity of Theorem 6. \square

Example 12. Let $X = [0, 10]$ with the usual metric. We define f, g, B, T on X as follows:

$$f(x) = g(x) = \frac{x}{2}, \quad \text{for } \forall x \in X, \tag{117}$$

$$B(x) = T(x) = x, \quad \text{for } \forall x \in X.$$

It is clear that $f(X) \subseteq T(X)$ and $g(X) \subseteq B(X)$.

To show that pair (f, g) is S_B -compatible and pair (B, T) is S_g -compatible, we take $x_n = 1/n$; then, $\lim_{n \rightarrow +\infty} fx_n = 0$, $\lim_{n \rightarrow +\infty} gx_n = 0$, $\lim_{n \rightarrow +\infty} Bx_n = 0$, $\lim_{n \rightarrow +\infty} Tx_n = 0$ with $\lim_{n \rightarrow +\infty} fBx_n = \lim_{n \rightarrow +\infty} gBx_n = B(0)$, and $\lim_{n \rightarrow +\infty} Bgx_n = \lim_{n \rightarrow +\infty} Tgx_n = g(0)$.

Hence, the pair (f, g) is S_B -compatible and pair (B, T) is S_g -compatible.

To verify assumption $(A2'')$, we define $F(s, t) = ks \in \mathcal{C}_{\text{inv}-k}$ with $k \geq 1$, where $0 < \beta = a_3 + a_4 + a_5 < 1$ and $a_i > 0$, $(i = 1, 2, \dots, 5)$ satisfying $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$

and $a_1 + a_2 < a_3 + a_4$. Define $\theta(t) = t/(2\beta^2)$, for all $t \geq 0$. Then, for all $x, y \in X$, we have

$$\begin{aligned} k\theta(d(fx, By)) &= k \frac{d(fx, By)}{2\beta^2} \\ &= \frac{k}{2\beta^2} \left| \frac{x}{2} - y \right|, \\ F[\theta(M'(x, y)), \phi(M'(x, y))] & \\ &= \frac{k}{2\beta^2} [a_1 d(fx, gx) + a_2 d(By, Ty) + a_3 d(fx, Ty) + a_4 d(gx, By) + a_5 d(gx, Ty)] \\ &= \frac{k}{2\beta} \left| \frac{x}{2} - y \right|. \end{aligned} \tag{118}$$

It is easy to verify that $k\theta(d(fx, By)) \geq F[\theta(M'(x, y)), \phi(M'(x, y))]$ for all $x, y \in X$ satisfying $a_i > 0$, $(i = 1, 2, \dots, 5)$ with $0 < \beta = a_3 + a_4 + a_5 \leq 1$, $1 \geq a_1 - a_3$, $a_1 + a_2 + a_5 > 1$, and $a_1 + a_2 < a_3 + a_4$; that is, assumption $(A2'')$ holds true.

Then, f, g, B , and T satisfy all the conditions of Theorem 6; moreover, 0 is the unique common fixed point of f, g, B , and T .

3. Conclusion

In this paper, concepts of semicompatibility of type (A), S_r -compatibility are introduced, in which S_r -compatibility is weaker than (E.A) property. We also give a brief discussion on the relation between new notions and other existing types of compatibility. Motivated by the notion of inverse C -class functions, a distinct concept of inverse C_k class functions is introduced which extends the notion of inverse C -class functions introduced by Saleem et al. [1, 21]. Moreover, some common fixed point theorems are stated under some compatible conditions such as semi-compatibility, semicompatibility of type (A), weak semi-compatibility, conditional semicompatibility, and S_r -compatibility in metric spaces via inverse C_k class functions which are a valuable supplement to the common fixed point theory.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly to the writing of this article. All the authors read and approved the manuscript.

Acknowledgments

The authors thank the colleagues for their proofreading and other helpful suggestions. Xiao-lan Liu was partially supported by National Natural Science Foundation of China (no. 11872043), Sichuan Science and Technology Program (no. 2019YJ0541), Zigong Science and Technology Program (no. 2020YGJC03), Scientific Research Project of Sichuan University of Science and Engineering (nos. 2017RCL54, 2019RC42, and 2019RC08), the Opening Project of Key Laboratory of Higher Education of Sichuan Province for Enterprise Informationization and Internet of Things (no. 2020WYJ01), Opening Project of Sichuan Province University Key Laboratory of Bridge Non-Destruction Detecting and Engineering Computing (no. 2019QZJ03), the Open Fund Project of Artificial Intelligence Key Laboratory of Sichuan Province (no. 2018RYJ02), and 2020 Graduate Innovation Project of Sichuan University of Science and Engineering (no. y2020078).

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Research Article

New Results in Controlled Metric Type Spaces

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Received 3 February 2021; Revised 5 March 2021; Accepted 17 March 2021; Published 1 April 2021

Academic Editor: Naeem Saleem

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In this paper, a new class of functions denoted by Ψ_ν is introduced which we use to prove new interesting fixed point results in controlled metric type spaces. Also, we present examples to illustrate our work.

1. Introduction

Fixed point theory is one of the most interesting topics that was introduced by Banach [1] in 1922. Since that time and for the last decades, this area of research has become the inspiration for many researchers in the field of nonlinear analysis and its applications. We refer the reader to check these extensions of the Banach theorem in different metric spaces; for example, in [2], the authors proved the existence of a fixed point for a self-mapping on S_b -metric spaces that satisfies a more general contraction.

In 1989, Bakhtin [3] introduced an extension of metric spaces, called b -metric spaces, where many interesting fixed point results for some contractive mappings in b -metric spaces were studied. Also, in 1993, Czerwik [4] extended the results of b -metric spaces. In 2018, Shatanawi et al. [5] introduced the $\alpha - \psi$ -contraction on the extended b -metric spaces. Recently, many research studies were conducted on b -metric space under different contraction conditions. After that, many authors used $\alpha - \psi$ -contraction mapping on different metric spaces (see [6, 7]). In 2017, Kamran et al. [8] presented a very interesting generalization of the b -metric spaces, called extended b -metric spaces. An extension of the extended b -metric spaces,

called controlled metric type space, was introduced by Mlaiki et al. [9].

In this paper, we generalize the results of Mehmet [10] and Mukheimer [11] by introducing the $\alpha - \psi$ -contractive mapping on controlled metric type spaces.

2. Preliminaries

The concept of extended b -metric spaces was initiated by Kamran et al. [8] in 2017, and their work generalized many results in the literature (see, for example, [12–16]).

Definition 1 (see [8]). Let Y be a nonempty set and define the mappings $\nu: Y \times Y \rightarrow [1, \infty)$ and $\bar{\varphi}: Y \times Y \rightarrow [0, \infty)$, such that $\forall g, h, r \in Y$,

- (1) $\bar{\varphi}(g, h) = 0 \Leftrightarrow g = h$.
- (2) $\bar{\varphi}(g, h) = \bar{\varphi}(h, g)$.
- (3) $\bar{\varphi}(g, h) \leq \nu(g, h)[\bar{\varphi}(g, r) + \bar{\varphi}(r, h)]$.

Then, we say that the pair $(Y, \bar{\varphi})$ is an extended b -metric space.

A general extension of the b -metric spaces was considered by Mlaiki et al. in [9] where the concepts of controlled metric type spaces is defined as follows.

Definition 2 (see [9]). On a nonempty set Y , define the mappings $\nu: Y \times Y \longrightarrow [1, \infty)$ and $\bar{\rho}: Y \times Y \longrightarrow [0, \infty)$, such that $\forall g, h, r \in Y$, the following conditions hold:

- (d1) $\bar{\rho}(g, h) = 0$ if and only if $g = h$.
- (d2) $\bar{\rho}(g, h) = \bar{\rho}(h, g)$.
- (d3) $\bar{\rho}(g, h) \leq \nu(g, r)\bar{\rho}(g, r) + \nu(r, h)\bar{\rho}(r, h)$.

Then, the pair $(Y, \bar{\rho})$ is called a controlled metric type space.

To illustrate the above definition, we present the following examples.

Example 1 (see [9]). Choose $Y = \{1, 2, \dots\}$. Take $\bar{\rho}: Y \times Y \longrightarrow [0, \infty)$ such that

$$\bar{\rho}(g, h) = \begin{cases} 0, & \Leftrightarrow g = h, \\ \frac{1}{g}, & \text{if } g = 2n \text{ and } h = 2n + 1, \\ \frac{1}{h}, & \text{if } g = 2n + 1 \text{ and } h = 2n, \\ 1, & \text{otherwise.} \end{cases} \quad (1)$$

Consider $\nu: Y \times Y \longrightarrow [1, \infty)$ as

$$\nu(g, h) = \begin{cases} g, & \text{if } g = 2n \text{ and } h = 2n + 1, \\ h, & \text{if } g = 2n + 1 \text{ and } h = 2n, \\ 1, & \text{otherwise.} \end{cases} \quad (2)$$

It is clear that the conditions (d1) and (d2) are satisfied. Now, we investigate condition (d3).

Case 1. If $r = g$ or $r = h$, (d3) is satisfied.

Case 2. If $r \neq g$ and $r \neq h$, (d3) holds when $g = h$. Now, we may assume that $g \neq h$. Then, we have $g \neq h \neq r$. It is clear that (d3) holds in all following possible subcases:

- (1) g, r are even and $h = 2n + 1$.
- (2) $g = 2n$ and h, r are odd.
- (3) g, r are odd and $h = 2n$.
- (4) g, r are even and $h = 2n + 1$.
- (5) g, h, r are even.
- (6) g, h are even and $r = 2n + 1$.
- (7) g, h are odd and $r = 2n$.
- (8) g, h, r are odd.

Thus, $\bar{\rho}$ is a controlled metric type. Moreover, for $n = 2, 3, \dots$, we have

$$\begin{aligned} \bar{\rho}(2n + 1, 4n + 1) &= 1 > \frac{1}{n} = \nu(2n + 1, 4n + 1) \\ &\quad \cdot [\bar{\rho}(2n + 1, 2n) + \bar{\rho}(2n, 4n + 1)]. \end{aligned} \quad (3)$$

Therefore, $\bar{\rho}$ is not an extended b -metric.

Example 2 (see [9]). Take $Y = \{0, 1, 2\}$. Consider the function $\bar{\rho}$ given as

$$\begin{aligned} \bar{\rho}(0, 0) &= \bar{\rho}(1, 1) = \bar{\rho}(2, 2) = 0, \\ \bar{\rho}(0, 1) &= \bar{\rho}(1, 0) = 1, \\ \bar{\rho}(0, 2) &= \bar{\rho}(2, 0) = \frac{1}{2}, \\ \bar{\rho}(1, 2) &= \bar{\rho}(2, 1) = \frac{2}{5}. \end{aligned} \quad (4)$$

Define a symmetric function $\nu: Y \times Y \longrightarrow [1, \infty)$ such that

$$\begin{aligned} \nu(0, 0) &= \nu(1, 1) = \nu(2, 2) = \nu(0, 2) = 1, \\ \nu(1, 2) &= \frac{5}{4}, \\ \nu(0, 1) &= \frac{11}{10}. \end{aligned} \quad (5)$$

One can easily verify that $\bar{\rho}$ is a controlled metric type. Since

$$\bar{\rho}(0, 1) = 1 > \frac{99}{100} = \nu(0, 1)[\bar{\rho}(0, 2) + \bar{\rho}(2, 1)], \quad (6)$$

$\bar{\rho}$ is not an extended b -metric.

The concepts of Cauchy and convergent sequences in controlled metric type spaces are defined as follows.

Definition 3. Let $(Y, \bar{\rho})$ be a controlled metric type space and $\{g_n\}_{n \geq 0}$ be a sequence in Y .

- (1) The sequence $\{g_n\}$ converges to some $g \in Y$, if $\forall \epsilon > 0$, $\exists N = N(\epsilon) \in \mathbb{N}$ such that $\bar{\rho}(g_n, g) < \epsilon \forall n \geq N$. We write $\lim_{n \rightarrow \infty} g_n = g$.
- (2) We say that $\{g_n\}$ is Cauchy, if $\forall \epsilon > 0$, $\exists N = N(\epsilon) \in \mathbb{N}$ such that $\bar{\rho}(g_m, g_n) < \epsilon \forall m, n \geq N$.
- (3) If every Cauchy sequence is convergent, then the space $(Y, \bar{\rho})$ is called complete.

Definition 4 (see [9]). Let $(Y, \bar{\rho})$ be a controlled metric type space. Let $g \in Y$ and $\epsilon > 0$.

- (i) The open ball $B(g, \epsilon)$ is defined as

$$B(g, \epsilon) = \{h \in Y, \bar{\rho}(g, h) < \epsilon\}. \quad (7)$$

- (ii) A self-mapping ξ on Y is said to be continuous at $g \in Y$, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\xi(B(g, \delta)) \subseteq B(\xi g, \epsilon)$.

Remark 1. If for all g, y in Y , $\nu(g, h) = s \geq 1$, then $(Y, \bar{\rho})$ is a b -metric space. Therefore, we conclude that every b -metric space is a controlled metric type space. However, the converse is not always true.

Clearly, if a mapping ξ is continuous at g in the controlled metric type space $(Y, \bar{\rho})$, then $g_n \longrightarrow g$ implies that $\xi g_n \longrightarrow \xi g$ as $n \longrightarrow \infty$.

Let Ψ denote to the set of all functions $\psi: [0, \infty) \rightarrow [0, \infty)$ such that

- (1) ψ is nondecreasing.
- (2) $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$ for all $t > 0$, where ψ^n is the n -th iterate of ψ .

Now, we recall the following lemma.

Lemma 1 (see [5]). *If $\psi \in \Psi$, then $\psi(t) < t$, for all $t \in (0, +\infty)$.*

Next, we introduce the following class of functions.

Definition 5 (see [5]). Let Y be a nonempty set and $\nu: Y \times Y \rightarrow [1, +\infty)$ be a mapping. A function $\psi: [0, \infty) \rightarrow [0, \infty)$ is said to be controlled comparison function if ψ satisfies the following conditions:

- (1) ψ is nondecreasing.
- (2) $\sum_{n=1}^{\infty} \psi^n(t) \prod_{i=1}^n \nu(g_i, g_m) \nu(g_n, g_{n+1}) < +\infty$, and $\lim_{n \rightarrow \infty} \psi^n(t) \nu(g_n, g_{n+1}) < \infty$ for any sequence $\{g_n\}_{n=1}^{\infty}$ in Y , for all $t > 0$ and non-negative integer m where ψ^n is the n -th iterate of ψ .

The set of all controlled comparison functions is denoted by Ψ_ν which is an extension of b -comparison functions of Berinde.

Note that if $\psi \in \Psi_\nu$, then we have $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, since $\psi^n(t) \prod_{i=1}^n \nu(g_i, g_m) \geq \psi^n(t)$, for all $t > 0$. Hence, by Lemma 1, we have $\psi(t) < t$.

To show that the family Ψ_ν is a nonempty set, we present the following examples.

Example 3. Consider the controlled b -metric space $(Y, \bar{\rho})$ which was defined in Example 2. Define the mapping $\psi(t) = (kt/2)$, where $k < 1$. Note that $\nu(g, h) \leq 2$. Then, we have $\psi^n(t) \prod_{i=1}^n \nu(g_i, g_m) \nu(g_n, g_{n+1}) \leq (k^n t / 2^n) \cdot 2^{n+1} = 2k^n t$. Therefore, $\sum_{n=1}^{\infty} \psi^n(t) \prod_{i=1}^n \nu(g_i, g_m) \nu(g_n, g_{n+1}) \leq \sum_{n=1}^{\infty} 2k^n t < \infty$. Similarly, it is not difficult to see that $\lim_{n \rightarrow \infty} \psi^n(t) \nu(g_n, g_{n+1}) < \infty$.

3. Main Result

First, we define the α - ψ -contractive self-mapping in controlled metric type spaces.

Definition 6. Let ξ be a self-mapping on a complete controlled metric type space $(Y, \bar{\rho})$. We say that ξ is

α - ψ -contractive mapping if there exists a function $\alpha: Y^2 \rightarrow [0, \infty)$ and $\psi \in \Psi_\nu$ such that for all $g, y \in Y$, we have

$$\alpha(g, h) \bar{\rho}(\xi g, \xi h) \leq \psi(\bar{\rho}(g, h)). \quad (8)$$

Definition 7 (see [17]). Let $(Y, \bar{\rho})$ be a controlled metric type space. A mapping $\xi: Y \rightarrow Y$ is said to be an α -admissible if the following condition holds: if $\forall g, y \in Y$, with $\alpha(g, h) \geq 1$, then $\alpha(\xi g, \xi h) \geq 1$.

Now, we prove our first result.

Theorem 1. *Let $(Y, \bar{\rho})$ be a complete controlled metric type space and $\xi: Y \rightarrow Y$ be an α - ψ contractive mapping for some $\psi \in \Psi_\nu$. Assume that*

- (A) ξ is α -admissible.
- (B) There exists $g_0 \in Y$ such that $\alpha(g_0, \xi g_0) \geq 1$.
- (C) ξ is continuous.

Then, ξ has a fixed point. Moreover, if for any two fixed points of ξ in Y say a, b we have $\alpha(a, b) \geq 1$, then ξ has a unique fixed point in Y .

Proof. Take g_0 to be the point in Condition (2) in our theorem. Define the sequence $\{g_n\}_{n \geq 0}$ by $\xi g_n = g_{n+1}$.

First of all, note if there exists n such that $g_n = g_{n+1}$, then we are done and g_n is the fixed point of ξ .

So, we may assume that $g_n \neq g_{n+1}$ for all $n \geq 0$. Also, we know from the hypothesis of our theorem that $\alpha(g_0, \xi g_0) \geq 1$, $\alpha(g_0, g_1) \geq 1$, and using the fact that ξ is α -admissible, we can easily deduce that for all $n \geq 0$, $\alpha(g_n, g_{n+1}) \geq 1$.

Now, using the fact that ξ is an α - ψ contractive mapping, we deduce that

$$\begin{aligned} \bar{\rho}(g_n, g_{n+1}) &\leq \alpha(g_{n-1}, g_n) \bar{\rho}(g_n, g_{n+1}) \\ &= \alpha(g_{n-1}, g_n) \bar{\rho}(\xi g_{n-1}, \xi g_n) \\ &\leq \psi(\bar{\rho}(g_{n-1}, g_n)) \\ &\vdots \\ &\leq \psi^n(\bar{\rho}(g_0, g_1)). \end{aligned} \quad (9)$$

Hence, $\forall m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
\bar{\varphi}(g_n, g_m) &\leq \nu(g_n, g_{n+1})\bar{\varphi}(g_n, g_{n+1}) + \nu(g_{n+1}, g_m)\bar{\varphi}(g_{n+1}, g_m) \\
&\leq \nu(g_n, g_{n+1})\bar{\varphi}(g_n, g_{n+1}) + \nu(g_{n+1}, g_m)\nu(g_{n+1}, g_{n+2})\bar{\varphi}(g_{n+1}, g_{n+2}) \\
&\quad + \nu(g_{n+1}, g_m)\nu(g_{n+2}, g_m)\bar{\varphi}(g_{n+2}, g_m) \\
&\leq \nu(g_n, g_{n+1})\bar{\varphi}(g_n, g_{n+1}) + \nu(g_{n+1}, g_m)\nu(g_{n+1}, g_{n+2})\bar{\varphi}(g_{n+1}, g_{n+2}) \\
&\quad + \nu(g_{n+1}, g_m)\nu(g_{n+2}, g_m)\nu(g_{n+2}, g_{n+3})\bar{\varphi}(g_{n+2}, g_{n+3}) \\
&\quad + \nu(g_{n+1}, g_m)\nu(g_{n+2}, g_m)\nu(g_{n+3}, g_m)\bar{\varphi}(g_{n+3}, g_m) \\
&\leq \nu(g_n, g_{n+1})\psi^n(\bar{\varphi}(g_0, g_1)) \\
&\quad + \nu(g_{n+1}, g_{n+2})\nu(g_{n+1}, g_m)\psi^{n+1}(\bar{\varphi}(g_0, g_1)) \\
&\quad + \nu(g_{n+1}, g_m)\nu(g_{n+2}, g_m)\nu(g_{n+2}, g_{n+3})\psi^{n+2}(\bar{\varphi}(g_0, g_1)) \\
&\quad \vdots \\
&\quad + \nu(g_{n+1}, g_m)\nu(g_{n+2}, g_m)\nu(g_{n+3}, g_m) \cdots \nu(g_{m-2}, g_{m-1})\psi^{m-1}(\bar{\varphi}(g_0, g_1)) \\
&= \nu(g_n, g_{n+1})\psi^n(\bar{\varphi}(g_0, g_1)) + \sum_{j=n+1}^{m-2} \psi^j(\bar{\varphi}(g_0, g_1)) \prod_{i=n+1}^j \nu(g_i, g_m)\nu(g_j, g_{j+1}) \\
&= \nu(g_n, g_{n+1})\psi^n(\bar{\varphi}(g_0, g_1)) + \sum_{j=1}^{m-2} \psi^j(\bar{\varphi}(g_0, g_1)) \prod_{i=1}^j \nu(g_i, g_m)\nu(g_j, g_{j+1}) \\
&\quad - \sum_{j=1}^n \psi^j(\bar{\varphi}(g_0, g_1)) \prod_{i=1}^j \nu(g_i, g_m)\nu(g_j, g_{j+1}) \\
&= S_{m-2} - S_n,
\end{aligned} \tag{10}$$

where

$$S_m = \sum_{j=1}^m \psi^j(\bar{\varphi}(g_0, g_1)) \prod_{i=1}^j \nu(g_i, g_m)\nu(g_j, g_{j+1}), \tag{11}$$

and since $\psi \in \Psi$, we deduce that $\lim_{n,m \rightarrow \infty} [S_{m-2}, S_n] = 0$ and

$$\lim_{n \rightarrow \infty} \nu(g_n, g_{n+1})\psi^n(\bar{\varphi}(g_0, g_1)) < \infty. \tag{12}$$

Thus, the sequence $\{g_n\}_{n \geq 0}$ is a Cauchy sequence. The completeness of the controlled metric type space $(g, \bar{\varphi})$ implies that $\{g_n\}$ converges to some $g \in Y$.

Also, note that

$$\begin{aligned}
\bar{\varphi}(g, \xi g) &\leq \nu(g, g_{n+1})\bar{\varphi}(g, g_{n+1}) + \nu(g_{n+1}, \xi g)\bar{\varphi}(g_{n+1}, \xi g) \\
&= \nu(g, g_{n+1})\bar{\varphi}(g, g_{n+1}) + \nu(g_{n+1}, \xi g)\bar{\varphi}(\xi g_n, \xi g).
\end{aligned} \tag{13}$$

Taking the limit in above inequality and since $\lim_{n \rightarrow \infty} \bar{\varphi}(g, g_{n+1}) = 0$ and using the fact that ξ is continuous, we conclude that $\bar{\varphi}(g, \xi g) = 0$; that is, $\xi g = g$. Thus, ξ has a fixed point as desired. Now, assume that ξ has two fixed points a, b such that $\alpha(a, b) \geq 1$. Hence, using the fact that ξ is an $\alpha - \psi$ contractive mapping and ξ is α -admissible, we obtain

$$\begin{aligned}
\bar{\varphi}(a, b) &= \bar{\varphi}(\xi a, \xi b) \\
&\leq \alpha(a, b)\bar{\varphi}(\xi a, \xi b) \\
&\leq \psi(\bar{\varphi}(a, b)) \\
&\vdots \\
&\leq \psi^n(\bar{\varphi}(a, b)).
\end{aligned} \tag{14}$$

Since $\psi \in \Psi$, taking the limit in the above inequalities, we deduce that $\bar{\varphi}(a, b) = 0$ which implies that $a = b$. Thus, ξ has a unique fixed point.

Next, we present the following example as an application of Theorem 1. \square

Example 4. Let $(Y, \bar{\varphi})$ be the controlled metric type space that was defined in Example 2. Define the function $\alpha: Y \times Y \rightarrow (-\infty, \infty)$ such that

$$\alpha(g, h) = \begin{cases} 1, & \text{if } (g, h) = (1, 1), \\ \frac{1}{20}, & \text{if } (g, h) \neq (1, 1). \end{cases} \tag{15}$$

Define the self-mapping ξ on $Y = \{0, 1, 2\}$ by $\xi(1) = 1, \xi(0) = 2, \xi(2) = 1$ and the function $\psi(t) = (1/8)t$.

We want to verify that ξ satisfies the conditions of Theorem 1. It is clear that ξ is continuous for $g_0 = 1$. We have $\alpha(1, \xi(1)) = \alpha(1, 1) = 1 \geq 1$. So, ξ is α -admissible. Now,

we verify that ξ is α - ψ -contractive mapping. Note that $\alpha(g, h) = \alpha(h, g)$.

- (i) $\alpha(g, g)\overline{\varphi}(\xi g, \xi g) = 0 \leq \psi(\overline{\varphi}(g, g)) = \psi(0) = 0$, for all $g \in Y$.
- (ii) $\alpha(1, 2)\overline{\varphi}(\xi 1, \xi 2) = (1/20)\overline{\varphi}(1, 1) = 0 \leq \psi(\overline{\varphi}(1, 2)) = \psi(2/5) = (1/20)$.
- (iii) $\alpha(1, 0)\overline{\varphi}(\xi 1, \xi 0) = (1/20)\overline{\varphi}(1, 2) = (1/50) \leq \psi(\overline{\varphi}(1, 0)) = \psi(1) = (1/8)$.
- (iv) $\alpha(2, 0)\overline{\varphi}(\xi 2, \xi 0) = (1/20)\overline{\varphi}(1, 2) = (1/100) \leq \psi(\overline{\varphi}(2, 0)) = \psi(1/2) = (1/16)$.

Therefore, ξ satisfies the conditions in Theorem 1, and hence it has a unique fixed point $g = 1$.

Now, we present the following as an immediate consequence of Theorem 1.

Corollary 1. Let $(Y, \overline{\varphi})$ be a complete controlled metric type space and $\xi: Y \rightarrow Y$ be a mapping satisfying the following conditions:

- (1) ξ is continuous.
- (2) There exists $\psi \in \Psi_\psi$ such that $\overline{\varphi}(\xi g, \xi h) \leq \psi(\overline{\varphi}(g, h))$, for all $g, h \in Y$.

Then, ξ has a unique fixed point.

Proof. Define the function $\alpha: Y \times Y \rightarrow [0, +\infty)$ via $\alpha(g, h) = 1$. Note that ξ is α -admissible. Moreover, ξ satisfies all the conditions of Theorem 1. So, ξ has a unique fixed point.

In our next theorem, we replace the hypothesis of the continuity by a weaker condition. \square

Theorem 2. Let $(Y, \overline{\varphi})$ be a complete, controlled metric type space and $\xi: Y \rightarrow Y$ be an α - ψ -contractive mapping for some $\psi \in \Psi_\psi$. Suppose that the following conditions hold:

- (1) ξ is α -admissible.
- (2) There exists $g_0 \in Y$ such that $\alpha(g_0, \xi g_0) \geq 1$.
- (3) If $\{g_n\}_{n=1}^\infty$ is a sequence in Y such that $\alpha(g_n, g_{n+1}) \geq 1$ and $g_n \rightarrow x$ as $n \rightarrow \infty$, then $\alpha(g_n, g) \geq 1$ for all n .

Then, ξ has a fixed point.

Proof. In proving the result, we follow the same steps as in the proof of Theorem 1 to construct a sequence $\{g_n\}_{n=1}^\infty$ that converges to a point $g \in Y$. The constructed sequence has the property $\alpha(g_n, g_{n+1}) \geq 1$, for all natural numbers n . The last assumption of the result implies that $\alpha(g_n, g) \geq 1$. We finally prove that g is a fixed point for ξ . The triangle inequality implies that

$$\begin{aligned} \overline{\varphi}(g, \xi g) &\leq \nu(g, \xi g) [\overline{\varphi}(g, g_{n+1}) + \overline{\varphi}(g_{n+1}, \xi g)] \\ &= \nu(g, \xi g) \overline{\varphi}(g, g_{n+1}) + \nu(g, \xi g) \overline{\varphi}(g_{n+1}, \xi g). \end{aligned} \quad (16)$$

Note that the first term of the inequality, $\nu(g, \xi g) \overline{\varphi}(g, g_{n+1})$, converges to 0, since $\{g_n\}$ converges to 0. Also, the second term will be

$$\begin{aligned} \nu(g, \xi g) \overline{\varphi}(g_{n+1}, \xi g) &\leq \nu(g, \xi g) \overline{\varphi}(\xi g_n, \xi g) \alpha(g_n, g) \\ &\leq \nu(g, \xi g) \psi(\overline{\varphi}(g_n, g)) \\ &\leq \nu(g, \xi g) \overline{\varphi}(g_n, g), \end{aligned} \quad (17)$$

which converges to 0. Therefore, $\overline{\varphi}(g, \xi g) \leq 0$, and hence g is a fixed point for ξ . \square

4. Conclusion

Note that in Theorem 1, we used the continuity of the self-mapping on the controlled metric type space, which is a strong hypothesis. We want to bring to the reader's attention that we only used this hypothesis to prove that $\overline{\varphi}(a, Ta) = 0$, and thus a is a fixed point. So, the open question is can we omit the hypothesis of the continuity without replacing it with another weaker condition as in Theorem 2. In closing, we present the following open question:

Under what conditions we obtain the same results of Theorems 1 and 2 for self-mappings in double controlled metric type spaces [18].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors would like to thank Prince Sultan University for their support through NAMAM Research Group.

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Research Article

Data Dependence, Strict Fixed Point Results, and Well-Posedness of Multivalued Weakly Picard Operators

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Received 5 January 2021; Revised 31 January 2021; Accepted 1 March 2021; Published 16 March 2021

Academic Editor: Naeem Saleem

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In this paper, we introduce the notion of $(\mathfrak{g}, \mathfrak{r})$ -contractive multivalued weakly Picard operators via simulation functions, named as $\mathcal{Z}_{(\mathfrak{g}, \mathfrak{r})}$ -contractions. We present some related fixed point theorems. We investigate data dependence and strict fixed point results. The well-posedness for such operators is also considered. Moreover, we generalize the results of Moţ and Petruşel. To show the usability of our results, we give some examples and an application to resolve a functional equation arising in dynamical systems.

1. Introduction and Preliminaries

The fundamental theorem concerning the existence of a fixed point for a self-mapping on a metric space (\mathcal{N}, d) is due to Banach [1].

Definition 1. A self-map \mathcal{Q} on \mathcal{N} is called a contraction if there exists $\tilde{p} \in [0, 1)$ such that

$$d(\mathcal{Q}\mu, \mathcal{Q}\omega) \leq \tilde{p}d(\mu, \omega) \quad (1)$$

holds for all $\mu, \omega \in \mathcal{N}$.

On the basis of the above definition, Banach [1] in 1922 stated the following well-known contraction principle.

Theorem 1. Let $\mathcal{Q}: \mathcal{N} \longrightarrow \mathcal{N}$ be a contraction mapping defined on a complete metric space (\mathcal{N}, d) . Then, \mathcal{Q} possesses a unique fixed point μ^* in \mathcal{N} . Furthermore, for any $\mu \in \mathcal{N}$, we have

$$\lim_{n \rightarrow \infty} \mathcal{Q}^n \mu = \mu^*, \quad (2)$$

with

$$d(\mathcal{Q}^n \mu, \mu^*) \leq \frac{\tilde{p}^n}{1 - \tilde{p}} d(\mu, \mathcal{Q}\mu). \quad (3)$$

In 1969, investigating the case for multivalued mappings, Nadler [2] proved that the multivalued contraction mapping possesses at least one fixed point. Before moving towards this new generalization, we recall that $\mathfrak{P}(\mathcal{N})$, $\mathfrak{C}(\mathcal{N})$, $\mathfrak{CB}(\mathcal{N})$, and $\mathfrak{K}(\mathcal{N})$ the nonempty, closed, closed and bounded, and compact subsets of a metric space (\mathcal{N}, d) , respectively. For $\omega \in \mathcal{N}$ and $\mathcal{U}, \mathcal{V} \in \mathfrak{CB}(\mathcal{N})$, consider

$$\mathcal{D}(\mathcal{U}, \mathcal{V}) = \inf\{d(\sigma, \lambda) : \sigma \in \mathcal{U}, \lambda \in \mathcal{V}\}, \quad (4)$$

the Pompeiu–Hausdorff metric is defined by

$$\mathcal{H}(\mathcal{U}, \mathcal{V}) := \max \left\{ \sup_{u \in \mathcal{U}} \mathcal{D}(u, \mathcal{V}), \sup_{v \in \mathcal{V}} \mathcal{D}(v, \mathcal{U}) \right\}, \quad (5)$$

where $\mathcal{D}(\omega, \mathcal{V}) = \inf_{v \in \mathcal{V}} d(\omega, v)$.

Lemma 1 (see [2]). Suppose $\mathcal{U} \subseteq \mathcal{N}$ and $\xi > 1$. Then, for $\mu \in \mathcal{N}$, there is $v \in \mathcal{V}$ so that $d(\mu, v) \leq \xi \mathcal{D}(\mu, \mathcal{V})$.

Theorem 2 (see [2]). Let $\mathcal{Q}: \mathcal{N} \longrightarrow \mathcal{CB}(\mathcal{N})$ be a multivalued contraction mapping on a complete metric space (\mathcal{N}, d) . Then, \mathcal{Q} possesses a fixed point.

It is obvious that if (\mathcal{N}, d) is a complete metric space, then the pair $(\mathcal{CB}(\mathcal{N}), \mathcal{H})$ and $(\mathcal{C}(\mathcal{N}), \mathcal{H})$ is also complete (see e.g., [3–5]).

Definition 2 (see [6]). A mapping $\mathcal{Q}: \mathcal{N} \longrightarrow \mathcal{CB}(\mathcal{N})$ is called a multivalued weakly Picard (MWP) operator if, for all $\mu \in \mathcal{N}$ and $\omega \in \mathcal{Q}\mu$, there exists a sequence $\{\mu_n\}$ in \mathcal{N} such that

- (i) $\mu_0 = \mu, \mu_1 = \omega$
- (ii) $\mu_{n+1} \in \mathcal{Q}\mu_n$, for all $n \geq 0$
- (iii) $\{\mu_n\}$ is convergent and its limit is a fixed point of \mathcal{Q}

Popescu [7] defined the notion of $(\mathfrak{s}, \mathfrak{r})$ -contractive multivalued operators.

Definition 3 (see [7]). A multivalued operator $\mathcal{Q}: \mathcal{N} \longrightarrow \mathcal{CB}(\mathcal{N})$ on a complete metric space (\mathcal{N}, d) is called $(\mathfrak{s}, \mathfrak{r})$ -contractive if $\mathfrak{r} \in (0, 1)$, $\mathfrak{s} \geq \mathfrak{r}$, and $\mu, \omega \in \mathcal{N}$ such that

$$\mathcal{D}(\omega, \mathcal{Q}\mu) \leq \mathfrak{s}d(\omega, \mu), \quad \text{implies } \mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega) \leq rM_{\mathcal{T}}(\mu, \omega), \quad (6)$$

where

$$M_{\mathcal{T}}(\mu, \omega) = \max \left\{ d(\mu, \omega), \mathcal{D}(\mu, \mathcal{Q}\mu), \mathcal{D}(\omega, \mathcal{Q}\omega), \frac{\mathcal{D}(\mu, \mathcal{Q}\omega) + \mathcal{D}(\omega, \mathcal{Q}\mu)}{2} \right\}. \quad (7)$$

Popescu [7] showed that $(\mathfrak{s}, \mathfrak{r})$ -contractive multivalued operators are Picard, while in the single-valued case, it was shown that such operator possesses a unique fixed point. Later on, Kamran and Hussain [8] generalized the results of Popescu [7] to a weakly $(\mathfrak{s}, \mathfrak{r})$ -contractive multivalued operator. The set of fixed points of the mapping \mathcal{Q} is defined as $\text{Fix}(\mathcal{Q}) = \{\mu \in \mathcal{N}: \mu \in \mathcal{Q}\mu\}$, while the set of strict fixed points is defined as $\text{SFix}(\mathcal{Q}) = \{\mu \in \mathcal{N}: \{\mu\} = \mathcal{Q}\mu\}$. It is clear that $\text{SFix}(\mathcal{Q}) \subseteq \text{Fix}(\mathcal{Q})$.

Definition 4. (see [9, 10]). Let $Y \in \mathfrak{P}(\mathcal{N})$, where (\mathcal{N}, d) is a metric space, and $\mathcal{Q}: \mathcal{N} \longrightarrow \mathcal{C}(\mathcal{N})$ be a multivalued operator. Then, the fixed point problem is well-posed for \mathcal{Q} appropriate to \mathcal{D} if

- (i) $\text{Fix}(\mathcal{Q}) = \{\mu\}$
- (ii) For a sequence $\{\mu_n\}$ in Y , $\mathcal{D}(\mu_n, \mathcal{Q}\mu_n) \longrightarrow 0$ as $n \longrightarrow \infty$; then, $d(\mu_n, \mu) \longrightarrow 0$ as $n \longrightarrow \infty$

Observe that a fixed point problem, which is well-posed for \mathcal{Q} appropriate to \mathcal{D} , is also well-posed for \mathcal{Q} appropriate to \mathcal{H} . Moř and Petruřel in [11] proved the results of strict fixed point sets, well-posedness, and also data dependence of the fixed point sets.

An important class of functions was proposed by Khojasteh and Shukla [12] and was named as the set of simulation functions. Let $\xi: [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ verify the following conditions:

- $(\xi_0) \xi(0, 0) = 0$.
- $(\xi_1) \xi(\mu, \omega) < \omega - \mu$, for all $\mu, \omega > 0$.
- (ξ_2) If $\{\omega_n\}$ and $\{\mu_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \longrightarrow \infty} \omega_n = \lim_{n \longrightarrow \infty} \mu_n > 0$, then

$$\lim_{n \longrightarrow \infty} \sup \xi(\mu_n, \omega_n) < 0. \quad (8)$$

Such ξ is known as a simulation function proposed by Khojasteh et al. Utilizing such broad class of functions, they defined the notion of \mathcal{F} -contractions. We denote such class of functions by Δ .

Definition 5 (see [12]). A self-map \mathcal{Q} on \mathcal{N} is said to be \mathcal{F} -contraction appropriate to ξ if the inequality

$$\xi(d(\mathcal{Q}\mu, \mathcal{Q}\omega), d(\mu, \omega)) \geq 0, \quad \forall \mu, \omega \in \mathcal{N}, \quad (9)$$

is fulfilled.

On the basis of contraction mappings defined above, they gave a version of the contraction principle, which generalizes and unifies several existing fixed point results in the literature.

After this work, studies involving simulation functions have been performed by various researchers (see [13–15] and references therein). Later on, Argoubi et al. [16] reshaped the notion of a simulation function by withdrawing the assertion (ξ_0) . We denote such class of functions by Λ .

Example 1 (see [16]). Let $\xi_\lambda: [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ be a function defined by

$$\xi_\lambda(\omega, \mu) = \begin{cases} 1, & \text{if } (\omega, \mu) = (0, 0), \\ \lambda\mu - \omega, & \text{otherwise,} \end{cases} \quad (10)$$

where $\lambda \in (0, 1)$. Then, $\xi_\lambda \in \Lambda$.

Later on, the assertion (ζ_3) of a simulation function was replaced with (ζ'_3) by Roldán-López-de-Hierro et al. [17].

(ζ'_3) If $\{\mu_n\}$ and $\{\omega_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} \omega_n > 0$ and $\mu_n < \omega_n$, then

$$\lim_{n \rightarrow \infty} \sup \zeta(\mu_n, \omega_n) < 0. \quad (11)$$

The class of simulation functions ζ fulfilling (ζ_1) , (ζ_2) , and (ζ'_3) is known as the class of simulation functions in the manner of Roldan – Lopez-de-Hierro, and we denote it by Σ . We need the following lemma.

Lemma 2 (see [18]). *Let (\mathcal{N}, d) be a metric space and let $\{\mu_n\}$ be a sequence in X such that*

$$\lim_{n \rightarrow \infty} d(\mu_n, \mu_{n+1}) = 0. \quad (12)$$

If $\{\mu_n\}$ is not a Cauchy sequence in \mathcal{N} , then there exist $\varepsilon > 0$ and two sequences $\mu_{m(k)}$ and $\mu_{n(k)}$ of positive integers such that $\mu_{n(k)} > \mu_{m(k)} > k$ and the following sequences tend to ε^+ when $k \rightarrow \infty$:

$$\begin{aligned} & d(\mu_{m(k)}, \mu_{n(k)}), d(\mu_{m(k)}, \mu_{n(k)+1}), d(\mu_{m(k)-1}, \mu_{n(k)}), \\ & d(\mu_{m(k)-1}, \mu_{n(k)+1}), d(\mu_{m(k)+1}, \mu_{n(k)+1}). \end{aligned} \quad (13)$$

The purpose of this paper is to introduce the notion of weakly multivalued $\mathcal{F}_{(\mathfrak{s}, \mathfrak{r})}$ -contractions and to prove some fixed point results. We also discuss examples to illustrate and elaborate these new concepts. After that, we present data dependence, strict fixed point set, and well-posedness results. Following these ideas, we generalize the Moř and Petruřel result. Moreover, we present an application to functional equations arising in dynamical systems to show the usability of our results.

2. Main Results

We begin with the following definition.

Definition 6. Let (\mathcal{N}, d) be a metric space. A mapping $\mathcal{Q}: \mathcal{N} \rightarrow \mathcal{CB}(\mathcal{N})$ is called a weakly multivalued $\mathcal{F}_{(\mathfrak{s}, \mathfrak{r})}$ -contraction with respect to ζ if there are $\mathfrak{r} \in (0, 1)$, $\mathfrak{s} \geq \mathfrak{r}$, and $\mu, \omega \in \mathcal{N}$ so that

$$\mathcal{D}(\omega, \mathcal{Q}\mu) \leq \mathfrak{s}d(\omega, \mu) \quad (14)$$

implies

$$\zeta(\mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega), \mathfrak{r}M(\mu, \omega)) \geq 0, \quad (15)$$

where $\mathfrak{r} \in (0, 1)$, $\mathcal{L} = L \geq 0$, and

$$\begin{aligned} \mathcal{N}(\mu, \omega) &= \max \left\{ d(\mu, \omega), \mathcal{D}(\mu, \mathcal{Q}\mu), \mathcal{D}(\omega, \mathcal{Q}\omega), \right. \\ & \left. \frac{1}{2} (\mathcal{D}(\mu, \mathcal{Q}\omega) + \mathcal{D}(\omega, \mathcal{Q}\mu)) \right\} \\ &+ \mathcal{L} \min \{ d(\mu, \omega), d(\omega, \mathcal{Q}\mu) \}. \end{aligned} \quad (16)$$

Example 2. Let $\mathcal{N} = [0, 1]$ and $d(\mu, \omega) = |\mu - \omega|$. Define $\mathcal{Q}: \mathcal{N} \rightarrow \mathcal{CB}(\mathcal{N})$ by

$$\mathcal{Q}\mu = \left[0, \frac{\mu + 1}{\mu + 3} \right]. \quad (17)$$

Choose $\mathfrak{s} = 0.8$. For $\mu, \omega \in \mathcal{N}$, $\mu \neq \omega$, we have

$$\mathcal{D}(\mu, \mathcal{Q}\omega) < \mathfrak{s}d(\mu, \omega), \quad (18)$$

which implies

$$\begin{aligned} \zeta(\mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega), \mathfrak{r}M(\mu, \omega)) &= (\mathfrak{r}M(\mu, \omega)) - \mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega) \\ &\geq 0, \quad \because L \geq 0, \mathfrak{r} \in (0, 1). \end{aligned} \quad (19)$$

Hence, \mathcal{Q} is a weakly multivalued $\mathcal{F}_{(0.8, \mathfrak{r})}$ -contraction.

Theorem 3. Let (\mathcal{N}, d) be a complete metric space and $\mathcal{Q}: \mathcal{N} \rightarrow \mathcal{CB}(\mathcal{N})$ be a weakly multivalued $\mathcal{F}_{(\mathfrak{s}, \mathfrak{r})}$ -contraction. Then, \mathcal{Q} is a MWP operator.

Proof. Let $\mu_0 \in \mathcal{N}$, $\mu_1 \in \mathcal{Q}\mu_0$, and \mathfrak{t} be a real number such that $0 < \mathfrak{r} \leq \mathfrak{t} \leq \mathfrak{s} < 1$. Choose $\mu_2 \in \mathcal{Q}\mu_1$. Then,

$$\mathcal{D}(\mu_2, \mathcal{Q}\mu_1) = 0 \leq \mathfrak{s}d(\mu_2, \mu_1). \quad (20)$$

From (15) and $\zeta_1 \in \mathcal{F}$, we have

$$0 \leq \zeta(\mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2), \mathfrak{r}M(\mu_1, \mu_2)) < \mathfrak{r}M(\mu_1, \mu_2) - \mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2), \quad (21)$$

which implies

$$\begin{aligned} \mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2) &< \mathfrak{r}M(\mu_1, \mu_2), \\ \mathcal{D}(\mu_2, \mathcal{Q}\mu_2) &\leq \mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2) < \mathfrak{r}M(\mu_1, \mu_2). \end{aligned} \quad (22)$$

Hence,

$$\begin{aligned} & \mathcal{D}(\mu_2, \mathcal{Q}\mu_2) < \mathfrak{r}M(\mu_1, \mu_2) \\ &= \mathfrak{r} \max \left\{ d(\mu_1, \mu_2), \mathcal{D}(\mu_1, \mathcal{Q}\mu_1), \mathcal{D}(\mu_2, \mathcal{Q}\mu_2), \right. \\ & \left. \frac{1}{2} (\mathcal{D}(\mu_1, \mathcal{Q}\mu_2) + \mathcal{D}(\mu_2, \mathcal{Q}\mu_1)) \right\} \\ &+ L \min \{ d(\mu_1, \mu_2), d(\mu_2, \mathcal{Q}\mu_1) \} \end{aligned} \quad (23)$$

implies

$$d(\mu_2, \mu_3) < \mathfrak{r}d(\mu_1, \mu_2) + L \min \{ d(\mu_1, \mu_2), 0 \} = \mathfrak{r}d(\mu_1, \mu_2). \quad (24)$$

Similarly, we can get a sequence $\{\mu_n\}$ in \mathcal{N} such that $\{\mu_{n+1}\} \in \mathcal{Q}\mu_n$ and

$$d(\mu_{n+1}, \mu_{n+2}) < \mathfrak{r}d(\mu_n, \mu_{n+1}) < \mathfrak{r}^n d(\mu_1, \mu_2) \leq \mathfrak{t}^n d(\mu_1, \mu_2). \quad (25)$$

Since $n \rightarrow \infty$, $d(\mu_{n+1}, \mu_{n+2}) \rightarrow 0$, this shows that $\{\mu_n\}$ is a Cauchy sequence. \mathcal{N} is complete, so there exists $u \in \mathcal{N}$ such that

$$\lim_{n \rightarrow \infty} \mu_n = u. \quad (26)$$

We now show that there exists a subsequence $\{\mu_{n(m)}\}$ of $\{\mu_n\}$ such that

$$\mathcal{D}(u, \mathcal{Q}\mu_{n(m)}) \leq \mathfrak{s}d(u, \mu_{n(m)}), \quad \forall m \in \mathbb{N}. \quad (27)$$

On the contrary, we assume that there is a positive integer $N \in \mathbb{N}$ such that

$$\mathcal{D}(u, \mathcal{Q}\mu_n) > \mathfrak{s}d(u, \mu_n), \quad \forall n \geq N. \quad (28)$$

This implies

$$d(u, \mu_{n+1}) > \mathfrak{s}d(u, \mu_n), \quad \forall n \geq N. \quad (29)$$

By induction, we obtain

$$d(u, \mu_{n+p}) > \mathfrak{s}^p d(u, \mu_n), \quad \forall n \geq N, p \geq 1. \quad (30)$$

Recall that

$$\begin{aligned} d(\mu_{n+p}, \mu_n) &\leq d(\mu_n, \mu_{n+1}) + d(\mu_{n+1}, \mu_{n+2}) \\ &\quad + \cdots + d(\mu_{n+p-1}, \mu_{n+p}) \\ &< d(\mu_n, \mu_{n+1})(1 + \mathfrak{t} + \mathfrak{t}^2 + \mathfrak{t}^3 + \cdots + \mathfrak{t}^{p-1}) \\ &\leq \frac{1 - \mathfrak{t}^p}{1 - \mathfrak{t}} d(\mu_n, \mu_{n+1}) < \frac{1}{1 - \mathfrak{t}} d(\mu_n, \mu_{n+1}), \quad \forall n \geq N, p \geq 1. \end{aligned} \quad (31)$$

Taking $p \rightarrow \infty$, we get that

$$d(u, \mu_n) < \frac{1}{1 - \mathfrak{t}} d(\mu_n, \mu_{n+1}), \quad \forall n \geq 1. \quad (32)$$

Thus, we have

$$\begin{aligned} d(u, \mu_{n+p}) &< \frac{1}{1 - \mathfrak{t}} d(\mu_{n+p}, \mu_{n+p+1}) \\ &< \frac{\mathfrak{t}^p}{1 - \mathfrak{t}} d(\mu_n, \mu_{n+1}), \quad \forall n \geq N, p \geq 1. \end{aligned} \quad (33)$$

From (30) and (33), one writes

$$d(u, \mu_n) < \frac{(\mathfrak{t}/\mathfrak{s})^p}{1 - \mathfrak{t}} d(\mu_n, \mu_{n+1}), \quad \forall n \geq N. \quad (34)$$

By taking $p \rightarrow \infty$, we have $d(u, \mu_n) = 0$, for all $n \geq N$. It is a contradiction with respect to equation (30). Therefore, there is a subsequence $\{\mu_{n(m)}\}$ of $\{\mu_n\}$ such that

$$\mathcal{D}(u, \mathcal{Q}\mu_{n(m)}) \leq \mathfrak{s}d(u, \mu_{n(m)}), \quad \forall m \in \mathbb{N}, \quad (35)$$

which implies

$$\zeta(\mathcal{H}(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)}), \mathfrak{r}M(u, \mu_{n(m)})) \geq 0. \quad (36)$$

Now, from (36) and using (ζ_1) together with (ζ_2) , we have

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \sup [\zeta(\mathcal{H}(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)}), \mathfrak{r}M(u, \mu_{n(m)}))] \\ &< \lim_{n \rightarrow \infty} \sup [\mathfrak{r}M(u, \mu_{n(m)}) - \mathcal{H}(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)})] \\ &= \lim_{n \rightarrow \infty} \sup \left[\mathfrak{r} \max \left\{ d(u, \mu_{n(m)}), \mathcal{D}(u, \mathcal{Q}u), \mathcal{D}(\mu_{n(m)}, \mathcal{Q}\mu_{n(m)}), \frac{1}{2} (\mathcal{D}(u, \mathcal{Q}\mu_{n(m)}) + \mathcal{D}(\mu_{n(m)}, \mathcal{Q}u)) \right\} \right. \\ &\quad \left. + L \min \{ d(u, \mu_{n(m)}), \mathcal{D}(\mu_{n(m)}, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)}), \mathcal{D}(\mathcal{Q}\mu_{n(m)}, \mathcal{Q}u) \} \right] \\ &= \lim_{n \rightarrow \infty} \sup \left[\mathfrak{r} \max \left\{ d(u, \mu_{n(m)}), \mathcal{D}(u, \mathcal{Q}u), \mathcal{D}(\mu_{n(m)}, \mu_{n(m)+1}), \frac{1}{2} (\mathcal{D}(u, \mu_{n(m)+1}) + \mathcal{D}(\mu_{n(m)}, \mathcal{Q}u)) \right\} \right. \\ &\quad \left. + L \min \{ \mathcal{D}(u, \mu_{n(m)}), \mathcal{D}(\mu_{n(m)}, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, \mu_{n(m)+1}), \mathcal{D}(\mu_{n(m)+1}, \mathcal{Q}u) \} \right] \\ &= \sup \left[\mathfrak{r} \max \left\{ d(u, u), \mathcal{D}(u, \mathcal{Q}u), \mathcal{D}(u, u), \frac{1}{2} (\mathcal{D}(u, u) + \mathcal{D}(u, \mathcal{Q}u)) \right\} \right. \\ &\quad \left. + L \min \{ d(u, u), \mathcal{D}(u, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, u), \mathcal{D}(u, \mathcal{Q}u) \} \right] \\ &= \sup \left[\mathfrak{r} \max \left\{ \mathcal{D}(u, \mathcal{Q}u), \frac{\mathcal{D}(u, \mathcal{Q}u)}{2} \right\} + L \min \{ 0, \mathcal{D}(u, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, u), \mathcal{D}(u, \mathcal{Q}u) \} \right] \\ &= \sup [\mathfrak{r}d(u, \mathcal{Q}u) + 0 - \mathcal{D}(u, \mathcal{Q}u)] \\ &= (\mathfrak{r} - 1)\mathcal{D}(u, \mathcal{Q}u), \end{aligned} \quad (37)$$

where $\mathfrak{r} \in (0, 1)$. So,

$$0 \leq (\mathfrak{r} - 1)\mathcal{D}(u, \mathcal{Q}u) < 0. \quad (38)$$

From this contradiction, we get $\mathcal{D}(u, Qu) = 0$, that is, $u \in Qu$. Hence, Q is a MWP operator. \square

Example 3. Let $\mathcal{N} = \{1, 2, 3, 4, 5\}$ and $d(\mu, \omega) = |\mu - \omega|$. Define

$$Q\mu = \begin{cases} \{5\}, & \text{if } \mu \in \{5\}, \\ \{2, 3\}, & \text{if } \mu \in \{2, 3\}, \\ \{1, 4\}, & \text{otherwise.} \end{cases} \quad (39)$$

Now, for $\mu, \omega \in \mathcal{N}$, values of d , \mathcal{D} , and \mathcal{H} for all possible pair of points are given in Table 1.

By choosing $\mathfrak{s} = 0.5$, we have

$$\begin{aligned} \mathcal{D}(1, Q2) &> \mathfrak{s}d(1, 2), \mathcal{D}(3, Q5) > \mathfrak{s}d(3, 5), \\ \mathcal{D}(1, Q5) &> \mathfrak{s}d(1, 5), \mathcal{D}(4, Q3) > \mathfrak{s}d(4, 3), \\ \mathcal{D}(2, Q1) &> \mathfrak{s}d(2, 1), \mathcal{D}(4, Q5) > \mathfrak{s}d(4, 5), \\ \mathcal{D}(2, Q5) &> \mathfrak{s}d(2, 5), \mathcal{D}(5, Q3) > \mathfrak{s}d(5, 3), \\ \mathcal{D}(3, Q1) &> \mathfrak{s}d(3, 1), \mathcal{D}(5, Q4) > \mathfrak{s}d(5, 4), \\ \mathcal{D}(3, Q4) &> \mathfrak{s}d(3, 4). \end{aligned} \quad (40)$$

Furthermore, for $\mathfrak{s} = 0.5$, we also have

$$\begin{aligned} \mathcal{D}(1, Q1) &= \mathfrak{s}d(1, 1), \mathcal{D}(5, Q5) = \mathfrak{s}d(5, 5), \\ \mathcal{D}(1, Q3) &= \mathfrak{s}d(1, 3), \mathcal{D}(1, Q4) < \mathfrak{s}d(1, 4), \\ \mathcal{D}(2, Q2) &= \mathfrak{s}d(2, 2), \mathcal{D}(2, Q3) < \mathfrak{s}d(4, 5), \\ \mathcal{D}(3, Q3) &= \mathfrak{s}d(3, 3), \mathcal{D}(3, Q2) < \mathfrak{s}d(5, 3), \\ \mathcal{D}(4, Q2) &= \mathfrak{s}d(4, 2), \mathcal{D}(4, Q1) < \mathfrak{s}d(5, 4), \\ \mathcal{D}(4, Q4) &= \mathfrak{s}d(4, 4), \mathcal{D}(5, Q1) < \mathfrak{s}d(5, 1). \end{aligned} \quad (41)$$

Now, choosing $L = 1$ and $\mathfrak{r} = 0.4$, we have

$$\begin{aligned} \zeta(\mathcal{H}(Q1, Q1), \mathfrak{r}M(1, 1)) &= \mathfrak{r}M(1, 1) - \mathcal{H}(Q1, Q1) = (0.4)(0) + (1)(0) - 0 = 0, \\ \zeta(\mathcal{H}(Q1, Q3), \mathfrak{r}M(1, 3)) &= \mathfrak{r}M(1, 3) - \mathcal{H}(Q1, Q3) = (0.4)(2) + (1)(1) - 1 > 0, \\ \zeta(\mathcal{H}(Q2, Q2), \mathfrak{r}M(2, 2)) &= \mathfrak{r}M(2, 2) - \mathcal{H}(Q2, Q2) = (0.4)(0) + (1)(0) - 0 = 0, \\ \zeta(\mathcal{H}(Q3, Q3), \mathfrak{r}M(3, 3)) &= \mathfrak{r}M(3, 3) - \mathcal{H}(Q3, Q3) = (0.4)(0) + (1)(0) - 0 = 0, \\ \zeta(\mathcal{H}(Q4, Q2), \mathfrak{r}M(4, 2)) &= \mathfrak{r}M(4, 2) - \mathcal{H}(Q4, Q2) = (0.4)(2) + (1)(1) - 1 > 0, \\ \zeta(\mathcal{H}(Q4, Q4), \mathfrak{r}M(4, 4)) &= \mathfrak{r}M(4, 4) - \mathcal{H}(Q4, Q4) = (0.4)(0) + (1)(0) - 0 = 0, \\ \zeta(\mathcal{H}(Q5, Q5), \mathfrak{r}M(5, 5)) &= \mathfrak{r}M(5, 5) - \mathcal{H}(Q5, Q5) = (0.4)(0) + (1)(0) - 0 = 0, \\ \zeta(\mathcal{H}(Q1, Q4), \mathfrak{r}M(1, 4)) &= \mathfrak{r}M(1, 4) - \mathcal{H}(Q1, Q4) = (0.4)(3) + (1)(0) - 0 > 0, \\ \zeta(\mathcal{H}(Q2, Q3), \mathfrak{r}M(2, 3)) &= \mathfrak{r}M(2, 3) - \mathcal{H}(Q2, Q3) = (0.4)(1) + (1)(0) - 0 > 0, \\ \zeta(\mathcal{H}(Q3, Q2), \mathfrak{r}M(3, 2)) &= \mathfrak{r}M(3, 2) - \mathcal{H}(Q3, Q2) = (0.4)(1) + (1)(0) - 0 > 0, \\ \zeta(\mathcal{H}(Q4, Q1), \mathfrak{r}M(4, 1)) &= \mathfrak{r}M(4, 1) - \mathcal{H}(Q4, Q1) = (0.4)(3) + (1)(0) - 0 > 0, \\ \zeta(\mathcal{H}(Q5, Q1), \mathfrak{r}M(5, 1)) &= \mathfrak{r}M(5, 1) - \mathcal{H}(Q5, Q1) = (0.4)(1) + (1)(4) - 4 > 0. \end{aligned} \quad (42)$$

TABLE 1: Values of d , \mathcal{D} , and \mathcal{H} .

(μ, ω)	$d(\mu, \omega)$	$\mathcal{D}(\mu, \mathcal{Q}\mu)$	$\mathcal{D}(\omega, \mathcal{Q}\omega)$	$\mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega)$
(1, 1)	0	0	0	0
(1, 2)	1	0	1	1
(1, 3)	2	0	1	1
(1, 4)	3	0	0	0
(1, 5)	4	0	1	4
(2, 1)	1	0	1	1
(2, 2)	0	0	0	0
(2, 3)	1	0	0	0
(2, 4)	2	0	1	1
(2, 5)	3	0	2	3
(3, 1)	2	0	1	1
(3, 2)	1	0	0	0
(3, 3)	0	0	0	0
(3, 4)	1	0	1	1
(3, 5)	2	0	2	3
(4, 1)	3	0	0	0
(4, 2)	2	0	1	1
(4, 3)	1	0	1	1
(4, 4)	0	0	0	0
(4, 5)	1	0	1	4
(5, 1)	4	0	4	4
(5, 2)	3	0	3	3
(5, 3)	2	0	2	3
(5, 4)	1	0	1	4
(5, 5)	0	0	0	0

Hence, \mathcal{Q} is a weakly multivalued $Z_{(0.5,0.3)}$ -contraction. Thus, by Theorem 3, \mathcal{Q} is a MWP operator.

Remark 1. The weakly multivalued $\mathcal{L}_{(\mathfrak{s}, \mathfrak{r})}$ -contraction is the generalization of Popescu notion of $(\mathfrak{s}, \mathfrak{r})$ -contractions. By using Definition 3 for some $\mu, \omega \in \mathcal{N}$ and $\mathfrak{s} = 0.5$, we have

$$\mathcal{D}(1, \mathcal{Q}3) = \mathfrak{s}d(1, 3), \quad (43)$$

which implies

$$\mathfrak{r}M(1, 3) = 0.8 < \mathcal{H}(\mathcal{Q}1, \mathcal{Q}3) = 1, \quad (44)$$

$$\mathcal{D}(4, \mathcal{Q}2) = \mathfrak{s}d(4, 2), \quad (45)$$

which implies

$$\mathfrak{r}M(4, 2) = 0.8 < \mathcal{H}(\mathcal{Q}4, \mathcal{Q}2) = 1, \quad (46)$$

$$\mathcal{D}(5, \mathcal{Q}1) < \mathfrak{s}d(5, 1), \quad (47)$$

which implies

$$\mathfrak{r}M(5, 1) = 0.4 < \mathcal{H}(\mathcal{Q}5, \mathcal{Q}1) = 4. \quad (48)$$

Hence, from equations (44)–(48), it is clear that the notion of $(\mathfrak{s}, \mathfrak{r})$ -contraction defined by Popescu [7] is failed for Example 3.

Theorem 4. Let (\mathcal{N}, d) be a complete metric space and $\mathcal{Q}: \mathcal{N} \longrightarrow \mathcal{N}$ be a weakly single-valued $\mathcal{L}_{(\mathfrak{s}, \mathfrak{r})}$ -contraction operator. Then, \mathcal{Q} possesses a fixed point. Moreover, if $\mathfrak{s} \geq 1$ and $L + \mathfrak{r} < 1$, then \mathcal{Q} possesses a unique fixed point.

Proof. From Theorem 3, \mathcal{Q} possesses a fixed point. Suppose that $\mathfrak{s} \geq 1$ and $L + \mathfrak{r} < 1$. Assume that \mathcal{Q} possesses two distinct fixed points μ and ω . Then,

$$\begin{aligned} \mathcal{D}(\omega, \mathcal{Q}\mu) &= d(\mu, \omega) \leq \mathfrak{s}d(\mu, \omega) \\ \Rightarrow \zeta(\mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega), \mathfrak{r}M(\mu, \omega)) &\geq 0. \end{aligned} \quad (49)$$

From (49) and by using (ζ_1) , we have

$$\begin{aligned} \zeta(\mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega), \mathfrak{r}M(\mu, \omega)) &< \mathfrak{r}M(\mu, \omega) - \mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega) \\ &= (\mathfrak{r} + L)d(\mu, \omega) - d(\mu, \omega) = (\mathfrak{r} + L - 1)d(\mu, \omega). \end{aligned} \quad (50)$$

We know that $\mathfrak{r} + L - 1 < 0$. Therefore, we get $\zeta(\mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega), \mathfrak{r}M(\mu, \omega)) < 0$, which is a contradiction to our assumption. Thus, \mathcal{Q} possesses a unique fixed point. \square

Theorem 5. Let \mathcal{Q} be a weakly multivalued $\mathcal{L}_{(\mathfrak{s}, \mathfrak{r})}$ -contraction from \mathcal{N} into $\mathcal{CB}(\mathcal{N})$. Assume that there exist $\mathfrak{r}, \mathfrak{s} \in (0, 1)$ such that

$$\frac{1}{1 + \mathfrak{r}} \mathcal{D}(\mu, \mathcal{Q}\mu) \leq d(\mu, \omega) \leq \frac{1}{1 - \mathfrak{s}} \mathcal{D}(\mu, \mathcal{Q}\mu), \quad (51)$$

which implies

$$\zeta(\mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega), \mathfrak{r}M(\mu, \omega)) \geq 0. \quad (52)$$

Then, \mathcal{Q} is a Picard operator.

Proof. Without loss of generality choose \mathcal{Q} . Take a real number \mathfrak{t} such that $0 < \mathfrak{r} < \mathfrak{t} < \mathfrak{s}$. Let $\mu_1 \in \mathcal{N}$ and $\mu_2 \in \mathcal{Q}\mu_1$ such that $d(\mu_1, \mu_2) \leq ((1 - \mathfrak{t})/(1 - \mathfrak{s}))\mathcal{D}(\mu_1, \mathcal{Q}\mu_1)$.

Then,

$$\frac{1}{1 + \mathfrak{r}}\mathcal{D}(\mu_1, \mathcal{Q}\mu_1) \leq \mathcal{D}(\mu_1, \mathcal{Q}\mu_1) \leq d(\mu_1, \mu_2) \leq \frac{1}{1 - \mathfrak{s}}\mathcal{D}(\mu_1, \mathcal{Q}\mu_1). \quad (53)$$

By hypothesis, we have

$$0 \leq \zeta(\mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2), \mathfrak{r}M(\mu_1, \mu_2)) < \mathfrak{r}M(\mu_1, \mu_2) - \mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2). \quad (54)$$

So,

$$\mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2) < \mathfrak{r}M(\mu_1, \mu_2). \quad (55)$$

Following similar steps as in Theorem 3, we can easily obtain that $d(\mu_2, \mu_3) < \mathfrak{r}d(\mu_1, \mu_2)$. Therefore, a sequence $\{\mu_n\}$ can be constructed in \mathcal{N} such that $\mu_{n+1} \in \mathcal{Q}\mu_n$ and $d(\mu_{n+1}, \mathcal{Q}\mu_{n+2}) < \mathfrak{t}d(\mu_n, \mu_{n+2})$ for all $n \in \mathbb{N}$

$$\sum_{n=1}^{\infty} d(\mu_n, \mu_{n+1}) < \sum_{n=1}^{\infty} \mathfrak{t}^{n-1} d(\mu_1, \mu_2) < \infty. \quad (56)$$

That is, $\{\mu_n\}$ is a Cauchy sequence. Completeness of \mathcal{N} yields that there is $u \in \mathcal{N}$ so that $\{\mu_n\}$ converges to u .

Since

$$\begin{aligned} d(\mu_{n+p}, \mu_n) &\leq d(\mu_n, \mu_{n+1}) + d(\mu_{n+1}, \mu_{n+2}) + \cdots + d(\mu_{n+p-1}, \mu_{n+p}) \\ &< d(\mu_n, \mu_{n+1})(1 + \mathfrak{t} + \mathfrak{t}^2 + \mathfrak{t}^3 + \cdots + \mathfrak{t}^{p-1}) \\ &\leq \frac{1 - \mathfrak{t}^p}{1 - \mathfrak{t}} d(\mu_n, \mu_{n+1}), \quad \forall n \geq N, p \geq 1. \end{aligned} \quad (57)$$

By taking $p \rightarrow \infty$, we obtain

$$d(u, \mu_n) < \frac{1}{1 - \mathfrak{t}} d(\mu_n, \mu_{n+1}), \quad \forall n \geq 1. \quad (58)$$

We have

$$d(\mu_n, \mu_{n+1}) \leq \frac{1 - \mathfrak{t}}{1 - \mathfrak{s}} \mathcal{D}(\mu_n, \mathcal{Q}\mu_n), \quad \forall n \in \mathbb{N}, \quad (59)$$

$$d(u, \mu_{n+1}) < \frac{1}{1 - \mathfrak{s}} \mathcal{D}(\mu_n, \mathcal{Q}\mu_n), \quad \forall n \in \mathbb{N}.$$

Suppose now that there is $N > 0$, such that

$$d(u, \mu_n) < \frac{1}{1 + \mathfrak{r}} \mathcal{D}(\mu_n, \mathcal{Q}\mu_n), \quad \forall n \geq 0. \quad (60)$$

Therefore,

$$d(\mu_n, \mu_{n+1}) \leq d(u, \mu_n) + d(u, \mu_{n+1})$$

$$< \frac{1}{1 + \mathfrak{r}} [\mathcal{D}(\mu_n, \mathcal{Q}\mu_n) + \mathcal{D}(\mu_{n+1}, \mathcal{Q}\mu_{n+1})] \quad (61)$$

$$< \frac{1}{1 + \mathfrak{r}} [d(\mu_n, \mu_{n+1}) + \mathfrak{r}d(\mu_n, \mu_{n+1})]$$

implies

$$d(\mu_n, \mu_{n+1}) < \mathcal{D}(\mu_n, \mathcal{Q}\mu_n), \quad (62)$$

a contradiction. So, there exists a subsequence $\{\mu_{n(m)}\}$ of $\{\mu_n\}$ such that

$$d(u, \mu_{n(m)}) \geq \frac{1}{1 + \mathfrak{r}} \mathcal{D}(\mu_{n(m)}, \mathcal{Q}\mu_{n(m)}), \quad \forall m \geq N. \quad (63)$$

Since

$$d(u, \mu_{n(m)}) \leq \frac{1}{1 - \mathfrak{s}} \mathcal{D}(\mu_{n(m)}, \mathcal{Q}\mu_{n(m)}), \quad \forall n \geq 1, \quad (64)$$

thus, we have

$$\begin{aligned} \frac{1}{1 + \mathfrak{r}} \mathcal{D}(u, \mathcal{Q}\mu_{n(m)}) &\leq \mathcal{D}(u, \mathcal{Q}\mu_{n(m)}) \\ &\leq d(u, \mu_{n(m)}) \leq \frac{1}{1 - \mathfrak{s}} \mathcal{D}(u, \mathcal{Q}\mu_{n(m)}). \end{aligned} \quad (65)$$

This implies that

$$\zeta(\mathcal{H}(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)}), \mathfrak{r}M(u, \mu_{n(m)})) \geq 0. \quad (66)$$

From (66), (ζ_1) , and (ζ_2) , one obtains

$$\begin{aligned}
0 &\leq \lim_{n \rightarrow \infty} \sup \left[\zeta \left(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)} \right), \mathfrak{r}M(u, \mu_{n(m)}) \right] \\
&< \lim_{n \rightarrow \infty} \sup \left[\mathfrak{r}M(u, \mu_{n(m)}) - \mathcal{H}(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)}) \right] \\
&= \lim_{n \rightarrow \infty} \sup \left[\mathfrak{r} \max \left\{ d(u, \mu_{n(m)}), \mathcal{D}(u, \mathcal{Q}u), \mathcal{D}(\mu_{n(m)}, \mathcal{Q}\mu_{n(m)}), \frac{1}{2} (\mathcal{D}(u, \mathcal{Q}\mu_{n(m)}) + \mathcal{D}(\mu_{n(m)}, \mathcal{Q}u)) \right\} \right. \\
&\quad \left. + L \min \{ d(u, \mu_{n(m)}), \mathcal{D}(\mu_{n(m)}, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)}), \mathcal{D}(\mathcal{Q}\mu_{n(m)}, \mathcal{Q}u) \} \right] \\
&= \lim_{n \rightarrow \infty} \sup \left[\mathfrak{r} \max \left\{ d(u, \mu_{n(m)}), \mathcal{D}(u, \mathcal{Q}u), \mathcal{D}(\mu_{n(m)}, \mu_{n(m)+1}), \frac{1}{2} (\mathcal{D}(u, \mu_{n(m)+1}) + \mathcal{D}(\mu_{n(m)}, \mathcal{Q}u)) \right\} \right. \\
&\quad \left. + L \min \{ d(u, \mu_{n(m)}), \mathcal{D}(\mu_{n(m)}, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, \mu_{n(m)+1}), \mathcal{D}(\mu_{n(m)+1}, \mathcal{Q}u) \} \right] \quad (67) \\
&= \sup \left[\mathfrak{r} \max \left\{ d(u, u), \mathcal{D}(u, \mathcal{Q}u), \mathcal{D}(u, u), \frac{1}{2} (\mathcal{D}(u, u) + \mathcal{D}(u, \mathcal{Q}u)) \right\} \right. \\
&\quad \left. + L \min \{ d(u, u), \mathcal{D}(u, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, u), \mathcal{D}(u, \mathcal{Q}u) \} \right] \\
&= \sup \left[\mathfrak{r} \max \left\{ d(u, u), \frac{\mathcal{D}(u, \mathcal{Q}u)}{2} \right\} + L \min \{ 0, \mathcal{D}(u, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, u), \mathcal{D}(u, \mathcal{Q}u) \} \right] \\
&= \sup [\mathfrak{r}d(u, \mathcal{Q}u) + 0 - \mathcal{D}(u, \mathcal{Q}u)] \\
&= (\mathfrak{r} - 1)\mathcal{D}(u, \mathcal{Q}u),
\end{aligned}$$

where $\mathfrak{r} \in (0, 1)$. Hence,

$$0 \leq (\mathfrak{r} - 1)\mathcal{D}(u, \mathcal{Q}u) < 0. \quad (68)$$

This contradiction shows that $\mathcal{D}(u, \mathcal{Q}u) = 0$, that is, $u \in \mathcal{Q}u$. Hence, \mathcal{Q} is a Picard operator. \square

Corollary 1. Let \mathcal{Q} be a weakly single-valued $\mathcal{F}_{(\mathfrak{s}, \mathfrak{r})}$ -contraction mapping from \mathcal{N} into \mathcal{N} . Assume that there exist $\mathfrak{r}, \mathfrak{s} \in [0, 1]$ so that

$$\frac{1}{1 + \mathfrak{r}} \mathcal{D}(\mu, \mathcal{Q}\mu) \leq d(\mu, \omega) \leq \frac{1}{1 - \mathfrak{s}} \mathcal{D}(\mu, \mathcal{Q}\mu) \quad (69)$$

implies

$$\zeta(\mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega), \mathfrak{r}M(\mu, \omega)) \geq 0. \quad (70)$$

Then, there exists $u \in \mathcal{N}$ such that $u = \mathcal{Q}u$.

Proof. One can easily show that, for every $\mu_1 \in \mathcal{N}$, the sequence $\{\mu_n\}$ defined by $\mu_{n+1} = \mathcal{Q}\mu_n$ satisfies the relationship $d(\mu_{n+1}, \mu_{n+2}) < \mathfrak{r} d(\mu_n, \mu_{n+1})$ as done in Theorem 3. Thus, the sequence $\{\mu_n\}$ is Cauchy, so there is $u \in \mathcal{N}$ such that $\lim_{n \rightarrow \infty} \mu_n = u$. Following Theorem 5, we can show that $d(u, \mu_n) \leq (1/(1 - \mathfrak{r}))d(\mu_n, \mu_{n+1})$ for all $n \geq 1$ and there exists a subsequence $\{\mu_{n(m)}\}$ of $\{\mu_n\}$ such that $d(u, \mu_{n(m)}) \geq (1/(1 + \mathfrak{r}))d(\mu_{n(m)}, \mu_{n(m)+1})$ for all $m \geq N$. Therefore, we obtain that

$$\begin{aligned}
0 &\leq \lim_{n \rightarrow \infty} \sup \left[\zeta \left(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)} \right), \mathfrak{r}M(u, \mu_{n(m)}) \right] \\
&< \lim_{n \rightarrow \infty} \sup \left[\mathfrak{r}M(u, \mu_{n(m)}) - \mathcal{H}(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)}) \right] \\
&= \sup \left[\mathfrak{r} \max \left\{ d(u, u), \mathcal{D}(u, \mathcal{Q}u), d(u, u), \frac{1}{2} (d(u, u) \right. \right. \\
&\quad \left. \left. + \mathcal{D}(u, \mathcal{Q}u)) \right\} \right. \\
&\quad \left. + L \min \{ d(u, u), \mathcal{D}(u, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, u), \mathcal{D}(u, \mathcal{Q}u) \} \right] \\
&= (\mathfrak{r} - 1)\mathcal{D}(u, \mathcal{Q}u) < 0, \quad \because \mathfrak{r} \in (0, 1). \quad (71)
\end{aligned}$$

We deduce that $\mathcal{D}(u, \mathcal{Q}u) = 0$. Thus, we get $u = \mathcal{Q}u$. \square

2.1. Data Dependence of the Fixed Point Set. In this section, we study data dependence of the fixed point set for weakly multivalued $\mathcal{F}_{(\mathfrak{s}, \mathfrak{r})}$ -contractions.

Theorem 6. Let (\mathcal{N}, d) be a metric space and \mathcal{Q}_1 and \mathcal{Q}_2 be two multivalued operators. Assume that

- (1) \mathcal{Q}_i is a weakly $\mathcal{F}_{(\mathfrak{s}, \mathfrak{r}_i)}$ -contraction for each $i \in \{1, 2\}$
- (2) There exists a real number $\lambda > 0$ such that $\mathcal{H}(\mathcal{Q}_1\mu, \mathcal{Q}_2\mu) \leq \lambda, \forall \mu \in \mathcal{N}$

Then,

- (1) $\text{Fix}(\mathcal{Q}_i) \in \mathfrak{C}(\mathcal{N})$ for $i \in \{1, 2\}$.
 (2) \mathcal{Q}_1 and \mathcal{Q}_2 are weakly multivalued operators and

$$\mathcal{H}(\text{Fix}(\mathcal{Q}_1), \text{Fix}(\mathcal{Q}_2)) \leq \frac{\lambda}{1 - \max\{\mathbf{r}_1, \mathbf{r}_2\}}. \quad (72)$$

Proof. From Theorem 3, $\text{Fix}(\mathcal{Q}_i)$ is nonempty for $i \in \{1, 2\}$. First of all, we will show that the set of fixed point of a weakly

multivalued operator \mathcal{Q} is closed. Let $\{\mu_n\}$ be a sequence in $\text{Fix}(\mathcal{Q})$ such that $\mu_n \rightarrow u$ as $n \rightarrow \infty$. One obtains

$$\mathcal{D}(u, \mathcal{Q}\mu_n) \leq d(u, \mu_n). \quad (73)$$

This implies that

$$\zeta(\mathcal{H}(\mathcal{Q}\mu_n, \mathcal{Q}u), \mathbf{r}M(\mu_n, u)) \geq 0. \quad (74)$$

Now, from (74), (ζ_1) , and (ζ_2) , we have

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow \infty} \sup [\zeta(\mathcal{H}(\mathcal{Q}u, \mathcal{Q}\mu_n), \mathbf{r}M(u, \mu_n))] \\ &< \lim_{n \rightarrow \infty} \sup [\mathbf{r}M(u, \mu_n) - \mathcal{H}(\mathcal{Q}u, \mathcal{Q}\mu_n)] \\ &= \lim_{n \rightarrow \infty} \sup \left[\mathbf{r} \max \left\{ d(u, \mu_n), \mathcal{D}(u, \mathcal{Q}u), \mathcal{D}(\mu_n, \mathcal{Q}\mu_n), \frac{1}{2} (\mathcal{D}(u, \mathcal{Q}\mu_n) + \mathcal{D}(\mu_n, \mathcal{Q}u)) \right\} \right. \\ &\quad \left. + L \min \{ d(u, \mu_n), \mathcal{D}(\mu_n, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, \mathcal{Q}\mu_n), \mathcal{D}(\mathcal{Q}\mu_n, \mathcal{Q}u) \} \right] \\ &= \lim_{n \rightarrow \infty} \sup \left[\mathbf{r} \max \left\{ d(u, \mu_n), \mathcal{D}(u, \mathcal{Q}u), \mathcal{D}(\mu_n, \mu_{n+1}), \frac{1}{2} (\mathcal{D}(u, \mu_{n+1}) + \mathcal{D}(\mu_n, \mathcal{Q}u)) \right\} \right. \\ &\quad \left. + L \min \{ d(u, \mu_n), \mathcal{D}(\mu_n, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, \mu_{n+1}), \mathcal{D}(\mu_{n+1}, \mathcal{Q}u) \} \right] \\ &= \sup \left[\mathbf{r} \max \left\{ d(u, u), \mathcal{D}(u, \mathcal{Q}u), \mathcal{D}(u, u), \frac{1}{2} (\mathcal{D}(u, u) + \mathcal{D}(u, \mathcal{Q}u)) \right\} \right. \\ &\quad \left. + L \min \{ d(u, u), \mathcal{D}(u, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, u), \mathcal{D}(u, \mathcal{Q}u) \} \right] \\ &= (\mathbf{r} - 1) \mathcal{D}(u, \mathcal{Q}u) < 0, \quad \because \mathbf{r} \in [0, 1). \end{aligned} \quad (75)$$

Thus, from this contradiction, we deduce that $\mathcal{D}(u, \mathcal{Q}u) = 0$. Since $\mathcal{Q}u \in \mathfrak{C}(\mathcal{N})$, we have $u \in \mathcal{Q}u$. Hence, $u \in \text{Fix}(\mathcal{Q})$. Secondly, from Theorem 3, we get that a weakly multivalued $\mathcal{L}_{(1, \mathbf{r})}$ -contractive operator is a MWP operator. Let $\mathbf{q} > 1$ be a real number and $\mu_0 \in \text{Fix}(\mathcal{Q}_1)$ be arbitrary. Then, there exists $\mu_1 \in \mathcal{Q}_2\mu_0$ such that $d(\mu_0, \mu_1) \leq \mathcal{H}(\mathcal{Q}_1\mu_0, \mathcal{Q}_2\mu_0)$. Next, for $\mu_1 \in \mathcal{Q}_2\mu_0$, there is $\mu_2 \in \mathcal{Q}_2\mu_1$ such that $d(\mu_1, \mu_2) \leq \mathcal{H}(\mathcal{Q}_2\mu_0, \mathcal{Q}_2\mu_1)$. Since $\mu_1 \in \mathcal{Q}_2\mu_0$, $\mathcal{D}(\mu_1, \mathcal{Q}_2\mu_0) = 0 \leq d(\mu_0, \mu_1)$. So, we have

$$d(\mu_1, \mu_2) \leq \mathbf{q} \mathcal{H}(\mathcal{Q}_2\mu_0, \mathcal{Q}_2\mu_1) \leq \mathbf{q} \mathbf{r}_2 d(\mu_0, \mu_1). \quad (76)$$

Therefore, in a similar way, we obtain that the sequence of successive approximations for \mathcal{Q}_2 starting from μ_0 fulfills the following assertions:

$$\begin{aligned} \mu_{n+1} &\in \mathcal{Q}\mu_n, \\ d(\mu_n, \mu_{n+1}) &\leq (\mathbf{q} \mathbf{r}_2)^n d(\mu_0, \mu_1), \quad \forall n \geq 1. \end{aligned} \quad (77)$$

Hence, for all $n \geq N$ and $p \geq 1$,

$$\begin{aligned} d(\mu_{n+p}, \mu_n) &\leq d(\mu_n, \mu_{n+1}) + d(\mu_{n+1}, \mu_{n+2}) + \cdots + d(\mu_{n+p-1}, \mu_{n+p}) \\ &\leq (\mathbf{q} \mathbf{r}_2)^n d(\mu_0, \mu_1) + (\mathbf{q} \mathbf{r}_2)^{n+1} d(\mu_0, \mu_1) + \cdots + (\mathbf{q} \mathbf{r}_2)^{n+p-1} d(\mu_0, \mu_1) \\ &\leq \frac{(\mathbf{q} \mathbf{r}_2)^n}{1 - \mathbf{q} \mathbf{r}_2} d(\mu_0, \mu_1). \end{aligned} \quad (78)$$

Choosing now $1 < \mathbf{q} < \min\{(1/\mathbf{r}_1), (1/\mathbf{r}_2)\}$ and letting $n \rightarrow \infty$, we obtain that $\{\mu_n\}$ is a Cauchy sequence in (\mathcal{N}, d) . Then, there exists $u \in \mathcal{N}$ such that $\mu_n \rightarrow u$ as $n \rightarrow \infty$. We will prove that u is a fixed point for \mathcal{Q}_2 . Suppose that there exists a positive number N such that $\mathcal{D}(u, \mathcal{Q}\mu_n) > d(u, \mu_n) \forall n \geq N$. Then, $d(u, \mu_{n+1}) > d(u, \mu_n) \forall n \geq N$. Since $\mu_n \rightarrow u$ as $n \rightarrow \infty$, we get a contradiction. Hence, there exists a subsequence $\{\mu_{n(m)}\}$ such that $\mathcal{D}(u, \mathcal{Q}_2\mu_{n(m)}) \leq d(u, \mu_{n(m)}) \forall m \in \mathbb{N}$. Thus,

$$\zeta(\mathcal{H}(\mathcal{Q}_2u, \mathcal{Q}_2\mu_{n(m)}), \mathbf{r}_2M(u, \mu_{n(m)})) \geq 0. \quad (79)$$

Now, from (79), (ζ_1) , and (ζ_2) , we have

$$\begin{aligned}
0 &\leq \lim_{n \rightarrow \infty} \sup \left[\zeta \left(\mathcal{H}(\mathcal{Q}_2 u, \mathcal{Q}_2 \mu_n), \mathbf{r}_2 M(u, \mu_{n(m)}) \right) \right] \\
&< \lim_{n \rightarrow \infty} \sup \left[\mathbf{r}_2 M(u, \mu_{n(m)}) - \mathcal{H}(\mathcal{Q}_2 u, \mathcal{Q}_2 \mu_{n(m)}) \right] \\
&= \lim_{n \rightarrow \infty} \sup \left[\mathbf{r}_2 \max \left\{ d(u, \mu_{n(m)}), \mathcal{D}(u, \mathcal{Q}_2 u), \mathcal{D}(\mu_{n(m)}, \mathcal{Q}_2 \mu_{n(m)}), \frac{1}{2} (\mathcal{D}(u, \mathcal{Q}_2 \mu_{n(m)}) + \mathcal{D}(\mu_{n(m)}, \mathcal{Q}_2 u)) \right\} \right. \\
&\quad \left. + L \min \{ d(u, \mu_{n(m)}), \mathcal{D}(\mu_{n(m)}, \mathcal{Q}_2 u) \} - \max \{ \mathcal{D}(\mathcal{Q}_2 u, \mathcal{Q}_2 \mu_{n(m)}), \mathcal{D}(\mathcal{Q}_2 \mu_{n(m)}, \mathcal{Q}_2 u) \} \right] \\
&= \lim_{n \rightarrow \infty} \sup \left[\mathbf{r}_2 \max \left\{ d(u, \mu_{n(m)}), \mathcal{D}(u, \mathcal{Q}_2 u), \mathcal{D}(\mu_{n(m)}, \mu_{n(m)+1}), \frac{1}{2} (\mathcal{D}(u, \mu_{n(m)+1}) + \mathcal{D}(\mu_{n(m)}, \mathcal{Q}_2 u)) \right\} \right. \\
&\quad \left. + L \min \{ d(u, \mu_{n(m)}), \mathcal{D}(\mu_{n(m)}, \mathcal{Q}_2 u) \} - \max \{ \mathcal{D}(\mathcal{Q}_2 u, \mu_{n(m)+1}), \mathcal{D}(\mu_{n(m)+1}, \mathcal{Q}_2 u) \} \right] \\
&= \sup \left[\mathbf{r} \max \left\{ d(u, u), \mathcal{D}(u, \mathcal{Q}u), \mathcal{D}(u, u), \frac{1}{2} (\mathcal{D}(u, u) + \mathcal{D}(u, \mathcal{Q}u)) \right\} \right. \\
&\quad \left. + L \min \{ d(u, u), \mathcal{D}(u, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, u), \mathcal{D}(u, \mathcal{Q}u) \} \right] \\
&= \sup \left[\mathbf{r} \max \left\{ d(u, u), \frac{\mathcal{D}(u, \mathcal{Q}u)}{2} \right\} \right. \\
&\quad \left. + L \min \{ 0, \mathcal{D}(u, \mathcal{Q}u) \} - \max \{ \mathcal{D}(\mathcal{Q}u, u), \mathcal{D}(u, \mathcal{Q}u) \} \right] \\
&= \sup [\mathbf{r} \mathcal{D}(u, \mathcal{Q}u) + 0 - \mathcal{D}(u, \mathcal{Q}u)] \\
&= (\mathbf{r} - 1) \mathcal{D}(u, \mathcal{Q}u),
\end{aligned} \tag{80}$$

where $\mathbf{r}_2 \in (0, 1)$. So,

$$0 \leq (\mathbf{r}_2 - 1) \mathcal{D}(u, \mathcal{Q}_2 u) < 0. \tag{81}$$

It is a contradiction. So, we obtain that $\mathcal{D}(u, \mathcal{Q}_2 u) = 0$, that is, $u \in \mathcal{Q}_2 u$. Hence, $u \in \text{Fix}(\mathcal{Q}_2)$.

By taking $p \rightarrow \infty$ in (78), we have $d(u, \mu_n) \leq ((q\mathbf{r}_2)^n / (1 - q\mathbf{r}_2)) d(\mu_0, \mu_1)$ for each $n \in N$. Then, for $n = 0$, we get $d(\mu_0, u) \leq (1 / (1 - q\mathbf{r}_2)) d(\mu_0, \mu_1) \leq (q\lambda / (1 - q\mathbf{r}_2))$. In a similar way, we get that, for each $u_0 \in \text{Fix}(\mathcal{Q}_2)$, there exists $\mu \in \text{Fix}(\mathcal{Q}_1)$ such that $d(u_0, \mu) \leq (1 / (1 - q\mathbf{r}_1)) d(u_0, u_1) \leq (q\lambda / (1 - q\mathbf{r}_1))$. Hence,

$$\mathcal{H}(\text{Fix}(\mathcal{Q}_1), \text{Fix}(\mathcal{Q}_2)) \leq \frac{q\lambda}{1 - \max\{q\mathbf{r}_1, q\mathbf{r}_2\}}. \tag{82}$$

Letting $q \rightarrow 1$ completes the proof. Moreover, we get that \mathcal{Q}_i is a MWP operator for $i \in \{1, 2\}$. \square

2.2. Strict Fixed Point and Well-Posedness. Now, we prove the well-posedness for a weakly multivalued $\mathcal{F}_{(\mathfrak{s}, \mathbf{r})}$ -contractive operator with $\mathfrak{s} > 1$.

Theorem 7. Let (\mathcal{N}, d) be a complete metric space and $\mathcal{Q}: \mathcal{N} \rightarrow \mathfrak{C}(\mathcal{N})$ be a multivalued operator. Assume that

- (i) \mathcal{Q} is a weakly multivalued $\mathcal{F}_{(\mathfrak{s}, \mathbf{r})}$ -contractive operator with $\mathfrak{s} \geq 1$

(ii) $S\text{Fix}(\mathcal{Q}) \neq \emptyset$

Then,

(a) $\text{Fix}(\mathcal{Q}) = S\text{Fix}(\mathcal{Q}) = \{z\}$

(b) The fixed point problem is well-posed appropriate to \mathcal{H} if $\mathfrak{s} > 1$

Proof. (a) We will prove that $\text{Fix}(\mathcal{Q}) = z$. We suppose that $u, z \in \text{Fix}(\mathcal{Q})$ with $u \neq z$.

Since

$$\mathcal{D}(u, \mathcal{Q}z) = d(u, z) \leq \mathfrak{s} d(u, z), \tag{83}$$

one obtains

$$\zeta(\mathcal{H}(\mathcal{Q}u, \mathcal{Q}z), \mathbf{r}M(u, z)) \geq 0. \tag{84}$$

By using (ζ_1) , we have

$$0 \leq \zeta(\mathcal{H}(\mathcal{Q}u, \mathcal{Q}z), \mathbf{r}M(u, z)) < \mathbf{r}M(u, z) - \mathcal{H}(\mathcal{Q}u, \mathcal{Q}z). \tag{85}$$

Hence, we obtain $\mathcal{H}(\mathcal{Q}u, \mathcal{Q}z) < \mathbf{r}M(u, z)$. We know that $\mathcal{D}(\mathcal{Q}z, u) \leq \mathcal{H}(\mathcal{Q}u, \mathcal{Q}z)$. So,

$$\begin{aligned}
\mathcal{D}(\mathcal{Q}z, u) &= d(z, u) \leq \mathcal{H}(\mathcal{Q}u, \mathcal{Q}z) < \mathbf{r}M(u, z) \\
&= \mathbf{r}d(u, z) + L \min\{d(u, z), d(u, \mathcal{Q}z)\} = \mathbf{r}d(u, z).
\end{aligned} \tag{86}$$

Thus, $d(z, u) < r d(u, z)$, where $r \in [0, 1)$. It is a contradiction. Hence, we deduce that $u = z$ and $\text{Fix}(\mathcal{Q}) = \{z\}$.

- (b) Let $\{\mu_n\} \in \mathcal{N}$, $n \in \mathbb{N}$, be such that $\mathcal{D}(\mu_n, \mathcal{Q}\mu_n) \rightarrow 0$ as $n \rightarrow \infty$. We will prove that $d(\mu_n, u) \rightarrow 0$ as $n \rightarrow \infty$. Arguing by contradiction, we suppose that $d(\mu_n, u)$ does not converge to 0. Then, there exist $\varepsilon > 0$ and a subsequence $\{\mu_{n(m)}\}$ such that $d(\mu_{n(m)}, u) \geq \varepsilon \forall m \in \mathbb{N}$. If there exists a subsequence $\{\mu_{n(m)}\}$ of $\{\mu_{n(m)}\}$ such that

$$\mathcal{D}(u, \mathcal{Q}\mu_{n(m)}) \leq d(u, \mu_{n(m)}), \quad (87)$$

it implies

$$\zeta(\mathcal{H}(\mathcal{Q}u, \mathcal{Q}\mu_{n(m)}), rM(u, \mu_{n(m)})) \geq 0. \quad (88)$$

By following similar calculations as in Theorem 3, one gets $d(u, \mu_{n(m)}) \rightarrow 0$ as $n \rightarrow \infty$. Hence, we obtain $\varepsilon \leq d(\mu_{n(m)}, u) \rightarrow 0$ as $n \rightarrow \infty$, which is a contradiction to our supposition, that is, $\varepsilon > 0$.

Thus, there exists $\mathcal{D}(u, \mu_{n(m)}) > \xi d(u, \mu_{n(m)})$ for all $m \geq m_1$. We know that $\mathcal{D}(\mu_n, \mathcal{Q}\mu_n) \rightarrow 0$ as $n \rightarrow \infty$, so there exists $\mathcal{D}(\mu_{n(m)}, \mathcal{Q}\mu_{n(m)}) < (\xi - 1)\varepsilon$ for all $m \geq m_2$. Thus,

$$\begin{aligned} (\xi - 1)\varepsilon &> \mathcal{D}(\mu_{n(m)}, \mathcal{Q}\mu_{n(m)}) \geq \mathcal{D}(z, \mathcal{Q}\mu_{n(m)}) \\ &- d(u, \mu_{n(m)}) > \xi d(u, \mu_{n(m)}) \leq (\xi - 1)\varepsilon, \end{aligned} \quad (89)$$

for all $m \geq m_2$. It is impossible. Therefore, $\mu_n \rightarrow u$ as $n \rightarrow \infty$. \square

2.3. An Extension of Moţ–Petruşel Theorem. Following Reich [19], Moţ and Petruşel initiated the following concepts.

Definition 7. Let (\mathcal{N}, d) be a metric space and $Y \in \mathfrak{P}(\mathcal{N})$. Then, $\mathcal{Q}: Y \rightarrow \mathcal{CB}(\mathcal{N})$ is called an (a, b, c) -KSR multivalued operator if there are $a, b, c \in R_+$ so that, for $\mu, \omega \in Y$, we have

$$\begin{aligned} \frac{1-b-c}{1+a} \mathcal{D}(\mu, \mathcal{Q}\mu) &\leq d(\mu, \omega) \\ \Rightarrow \mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega) &\leq ad(\mu, \omega) + b\mathcal{D}(\mu, \mathcal{Q}\mu) + c\mathcal{D}(\omega, \mathcal{Q}\omega). \end{aligned} \quad (90)$$

Theorem 8. Let (\mathcal{N}, d) be a complete metric space and $\mathcal{Q}: \mathcal{N} \rightarrow \mathcal{CB}(\mathcal{N})$ be an (a, b, c) -KSR multivalued operator. Then, $\text{Fix}(\mathcal{Q}) \neq \emptyset$. Moreover, \mathcal{Q} is a MWP operator.

Now, we prove the generalization of Theorem 8 in the case $b + c > 0$, by using a weakly multivalued $\mathcal{Z}_{(\xi, r)}$ -contractive operator.

Theorem 9. Let (\mathcal{N}, d) be a complete metric space and $\mathcal{Q}: \mathcal{N} \rightarrow \mathcal{CB}(\mathcal{N})$. Assume that there exist $a, b, c \in R_+$ with $a + b + c \in (0, 1)$, $b + c > 0$, so that, for all $\mu, \omega \in \mathcal{N}$, we have

$$\mathcal{D}(\omega, \mathcal{Q}\mu) \leq \frac{a+b+c}{1-b-c} d(\mu, \omega), \quad (91)$$

which implies

$$\zeta(\mathcal{H}(\mathcal{Q}\mu, \mathcal{Q}\omega), (a+b+c)M(\mu, \omega)) \geq 0. \quad (92)$$

Then, $\text{Fix}(\mathcal{Q}) \neq \emptyset$.

Proof. Let $\mu_0 \in \mathcal{N}$ and $\mu_1 \in \mathcal{Q}\mu_0$. Choose r_1 be a real number such that

$$0 < (a+b+c) < r_1 \leq \frac{a+b+c}{1-b-c} < 1. \quad (93)$$

One can choose $\mu_2 \in \mathcal{Q}\mu_1$, then

$$\mathcal{D}(\mu_2, \mathcal{Q}\mu_1) = 0 \leq \frac{a+b+c}{1-b-c} d(\mu_2, \mu_1), \quad (94)$$

which implies

$$\begin{aligned} 0 &\leq \zeta(\mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2), (a+b+c)M(\mu_1, \mu_2)) \\ &< (a+b+c)M(\mu_1, \mu_2) - \mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2). \end{aligned} \quad (95)$$

That is,

$$\mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2) < (a+b+c)M(\mu_1, \mu_2). \quad (96)$$

Therefore,

$$\begin{aligned} \mathcal{D}(\mu_2, \mathcal{Q}\mu_2) &\leq \mathcal{H}(\mathcal{Q}\mu_1, \mathcal{Q}\mu_2) < (a+b+c)M(\mu_1, \mu_2) \\ &= (a+b+c)d(\mu_1, \mu_2). \end{aligned} \quad (97)$$

Thus, we obtain

$$d(\mu_2, \mu_3) < (a+b+c)d(\mu_1, \mu_2). \quad (98)$$

From this way, we can easily construct the sequence $\{\mu_n\}$ in \mathcal{N} such that $\{\mu_{n+1}\} \in \mathcal{Q}\mu_n$ and

$$\begin{aligned} d(\mu_{n+1}, \mu_{n+2}) &< (a+b+c)d(\mu_n, \mu_{n+1}) \\ &< (a+b+c)^{n-1}d(\mu_1, \mu_2) \leq r_1^{n-1}d(\mu_1, \mu_2). \end{aligned} \quad (99)$$

As $n \rightarrow \infty$, one obtains

$$d(\mu_{n+1}, \mu_{n+2}) \rightarrow 0. \quad (100)$$

This shows that $\{\mu_n\}$ is a Cauchy sequence. Since \mathcal{N} is complete, there is $u \in \mathcal{N}$ so that $\lim_{n \rightarrow \infty} \mu_n = u$. Now, we suppose that there exists a subsequence $\{\mu_{n(m)}\}$ of $\{\mu_n\}$ such that

$$\mathcal{D}(u, \mathcal{Q}\mu_{n(m)}) \leq \frac{a+b+c}{1-b-c} d(u, \mu_{n(m)}), \quad (101)$$

for all $m \in \mathbb{N}$. Assume on the contrary that there is a positive integer $N \in \mathbb{N}$ such that

$$\mathcal{D}(u, \mathcal{Q}\mu_n) > \frac{a+b+c}{1-b-c} d(u, \mu_n), \quad \forall n \in \mathbb{N}. \quad (102)$$

Then, following similar procedure of Theorem 3, we can show that this contrary assumption leads to the contradiction. Therefore, there exists a subsequence $\{\mu_{n(m)}\}$ of $\{\mu_n\}$ such that

$$\mathcal{D}(u, \mathcal{Q}\mu_{\mathbf{n}(\mathbf{m})}) \leq \frac{a+b+c}{1-b-c} d(u, \mu_{\mathbf{n}(\mathbf{m})}), \quad (103)$$

for all $\mathbf{m} \in \mathbb{N}$. This implies that

$$\zeta(\mathcal{H}(\mathcal{Q}u, \mathcal{Q}\mu_{\mathbf{n}(\mathbf{m})}), (a+b+c)M(u, \mu_{\mathbf{n}(\mathbf{m})})) \geq 0. \quad (104)$$

By solving this inequality, using the definition of a simulation function, we will obtain $\mathcal{D}(u, \mathcal{Q}u) = 0$, that is, $u \in \mathcal{Q}u$. Hence, $\text{Fix}(\mathcal{Q}) \neq \emptyset$. \square

3. An Application

Many authors [20–22] have studied the existence and uniqueness of a solution of functional equations arising in dynamic programming. We make use of Theorem 4 to explore the existence and uniqueness of a solution for a class of functional equations. From now on, \mathcal{U} and \mathfrak{B} are Banach spaces and $\mathfrak{B} \subset \mathcal{U}$, $\mathfrak{D} \subset \mathfrak{B}$, and \mathfrak{R} is the set of real numbers. We denote by $\mathbb{B}(\mathfrak{B})$ the set of all bounded real-valued functions on \mathfrak{B} . The set $\mathbb{B}(\mathfrak{B})$ equipped with the metric,

$$d_{\mathbb{B}}(\mathfrak{h}, \mathfrak{m}) = \sup_{\mu \in \mathfrak{B}} |\mathfrak{h}(\mu) - \mathfrak{m}(\mu)|, \quad \mathfrak{h}, \mathfrak{m} \in \mathbb{B}(\mathfrak{B}), \quad (105)$$

is a complete metric space. Viewing \mathfrak{B} and \mathfrak{D} as the state and decision space, respectively, the problem of dynamic

programming reduces to the problem of solving the functional equation $\mathfrak{P}(\mu) = \sup_{\omega \in \mathfrak{D}} \mathcal{H}(\mu, \omega, p(\mathfrak{t}(\mu, \omega)))$, where $\mathfrak{t}: \mathfrak{B} \times \mathfrak{D} \rightarrow \mathfrak{B}$ represents the transformation of the process and $\mathfrak{P}(\mu)$ corresponds to the optimal return function with an initial functional equation:

$$\mathfrak{P}(\mu) = \sup_{\omega \in \mathfrak{D}} \{g(\mu, \omega) + G(\mu, \omega, p(\mathfrak{t}(\mu, \omega)))\}, \quad \mu \in \mathfrak{B}, \quad (106)$$

where $g: \mathfrak{B} \times \mathfrak{D} \rightarrow \mathfrak{R}$ and $G: \mathfrak{B} \times \mathfrak{D} \times \mathfrak{R} \rightarrow \mathfrak{R}$ are bounded functions. Let \mathcal{Q} be defined by $\mathcal{Q}(\mathfrak{h}(\mu)) = \sup_{\omega \in \mathfrak{D}} \{g(\mu, \omega) + G(\mu, \omega, p(\mathfrak{t}(\mu, \omega)))\}$, where $\mathfrak{h} \in \mathbb{B}(\mathfrak{B})$ and $\mu \in \mathfrak{B}$.

Theorem 10. Suppose that there exist $\mathfrak{r} \in (0, 1)$, $\mathfrak{s} > \mathfrak{r}$ such that, for every $(\mu, \omega) \in \mathfrak{B} \times \mathfrak{D}$, $\mathfrak{h}, \mathfrak{m} \in \mathbb{B}(\mathfrak{B})$ and $\mathfrak{t} \in \mathfrak{B}$, the inequality

$$|\mathfrak{m}(\mathfrak{t}) - \mathcal{Q}(\mathfrak{h}(\mathfrak{t}))| \leq \mathfrak{s}|\mathfrak{m}(\mathfrak{t}) - \mathfrak{h}(\mathfrak{t})| \quad (107)$$

implies

$$\zeta(|G(\mu, \omega, \mathfrak{h}(\mathfrak{t})) - G(\mu, \omega, \mathfrak{m}(\mathfrak{t}))|, \mathfrak{r}M(\mathfrak{h}(\mathfrak{t}), \mathfrak{m}(\mathfrak{t}))) \geq 0, \quad (108)$$

where

$$\begin{aligned} M(\mathfrak{h}(\mathfrak{t}), \mathfrak{m}(\mathfrak{t})) = \max \left\{ \left| \mathfrak{h}(\mathfrak{t}) - \mathfrak{m}(\mathfrak{t}) \right|, \left| \mathfrak{m}(\mathfrak{t}) - \mathcal{Q}(\mathfrak{m}(\mathfrak{t})) \right|, \left| \mathfrak{h}(\mathfrak{t}) - \mathcal{Q}(\mathfrak{h}(\mathfrak{t})) \right|, \right. \\ \left. \frac{|\mathfrak{h}(\mathfrak{t}) - \mathcal{Q}(\mathfrak{m}(\mathfrak{t}))| + |\mathfrak{m}(\mathfrak{t}) - \mathcal{Q}(\mathfrak{h}(\mathfrak{t}))|}{2} \right\} \\ + \mathcal{L} \min\{|\mathfrak{h}(\mathfrak{t}) - \mathfrak{m}(\mathfrak{t})|, |\mathfrak{m}(\mathfrak{t}) - \mathcal{Q}(\mathfrak{h}(\mathfrak{t}))|\}, \end{aligned} \quad (109)$$

with $\mathcal{L} = L \geq 0$. Then, the functional equation (106) possesses a bounded solution. Moreover, if $\mathfrak{s} \geq 1$ and $L + \mathfrak{r} < 1$, then such a solution is unique.

Proof. Let \mathcal{Q} be a self-map of $\mathbb{B}(\mathfrak{B})$. Let λ be an arbitrary real number and $\mathfrak{h}, \mathfrak{m} \in \mathbb{B}(\mathfrak{B})$. Pick $\mu \in \mathfrak{B}$. We can choose $\omega_1, \omega_2 \in \mathfrak{D}$ so that

$$\mathcal{Q}(\mathfrak{h}(\mu)) < g(\mu, \omega_1) + G(\mu, \omega_1, \mathfrak{h}(\mathfrak{t}_1)) + \lambda, \quad (110)$$

$$\mathcal{Q}(\mathfrak{m}(\mu)) < g(\mu, \omega_2) + G(\mu, \omega_2, \mathfrak{m}(\mathfrak{t}_2)) + \lambda, \quad (111)$$

where $\mathfrak{t}_i = \mathfrak{t}_i(\mu, \omega_i)$, $i \in \{1, 2\}$.

Using the definition of \mathcal{Q} , we obtain

$$\mathcal{Q}(\mathfrak{h}(\mu)) \geq g(\mu, \omega_2) + G(\mu, \omega_2, \mathfrak{h}(\mathfrak{t}_2)), \quad (112)$$

$$\mathcal{Q}(\mathfrak{m}(\mu)) \geq g(\mu, \omega_1) + G(\mu, \omega_1, \mathfrak{m}(\mathfrak{t}_1)). \quad (113)$$

If inequality (107) holds, then

$$\zeta(|G(\mu, \omega, \mathfrak{h}(\mathfrak{t})) - G(\mu, \omega, \mathfrak{m}(\mathfrak{t}))|, \mathfrak{r}M(\mathfrak{h}(\mu), \mathfrak{m}(\mu))) \geq 0. \quad (114)$$

From (114) and (ζ_1) , one writes

$$\begin{aligned} 0 &\leq \zeta(|G(\mu, \omega, \mathfrak{h}(\mathfrak{t})) - G(\mu, \omega, \mathfrak{m}(\mathfrak{t}))|, \mathfrak{r}M(\mathfrak{h}(\mu), \mathfrak{m}(\mu))) \\ &< \mathfrak{r}M(\mathfrak{h}(\mu), \mathfrak{m}(\mu)) - |G(\mu, \omega, \mathfrak{h}(\mathfrak{t})) - G(\mu, \omega, \mathfrak{m}(\mathfrak{t}))|. \end{aligned} \quad (115)$$

This yields that

$$|G(\mu, \omega, \mathfrak{h}(\mathfrak{t})) - G(\mu, \omega, \mathfrak{m}(\mathfrak{t}))| < \mathfrak{r}M(\mathfrak{h}(\mu), \mathfrak{m}(\mu)). \quad (116)$$

In view of (111), (114), and (117), we have

$$\begin{aligned} \mathcal{Q}(\mathfrak{h}(\mu)) - \mathcal{Q}(\mathfrak{m}(\mu)) &< G(\mu, \omega_1, \mathfrak{h}(\mathfrak{t}_1)) - G(\mu, \omega_1, \mathfrak{m}(\mathfrak{t}_1)) + \lambda \\ &< \mathfrak{r}M(\mathfrak{h}(\mu), \mathfrak{m}(\mu)) + \lambda. \end{aligned} \quad (117)$$

Similarly, from (112), (113), and (117), we have

$$\mathcal{Q}(\mathbf{m}(\mu)) - \mathcal{Q}(\mathbf{h}(\mu)) < \mathbf{r}M(\mathbf{h}(\mu), \mathbf{m}(\mu)) + \lambda. \quad (118)$$

Thus, from equations (117) and (118), the following,

$$|\mathcal{Q}(\mathbf{h}(\mu)) - \mathcal{Q}(\mathbf{m}(\mu))| < \mathbf{r}M(\mathbf{h}(\mu), \mathbf{m}(\mu)) + \lambda, \quad (119)$$

holds for all $\mu \in \mathfrak{B}$ and $\lambda > 0$. Hence, we get that

$$d_{\mathfrak{B}}(\mathbf{m}, \mathcal{Q}(\mathbf{h})) \leq \mathfrak{s}d_{\mathfrak{B}}(\mathbf{m}, \mathbf{h}) \quad (120)$$

implies

$$\zeta(d_{\mathfrak{B}}(\mathcal{Q}(\mathbf{h}), \mathcal{Q}(\mathbf{m})), \mathbf{r}M(\mathbf{h}, \mathbf{m})) \geq 0, \quad (121)$$

where

$$\begin{aligned} \mathbf{r}M(\mathbf{h}, \mathbf{m}) = \mathbf{r} \max \left\{ |\mathbf{h} - \mathbf{m}|, |\mathbf{m} - \mathcal{Q}(\mathbf{m})|, |\mathbf{h} - \mathcal{Q}(\mathbf{h})|, \frac{|\mathbf{h} - \mathcal{Q}(\mathbf{m})| + |\mathbf{m} - \mathcal{Q}(\mathbf{h})|}{2} \right\} \\ + L \min\{|\mathbf{h} - \mathbf{m}|, |\mathbf{m} - \mathcal{Q}(\mathbf{h})|\}, \quad \text{where } \mathbf{r}\mathcal{L} = L \geq 0. \end{aligned} \quad (122)$$

Hence, all assertions of Theorem 4 are fulfilled for the mapping \mathcal{Q} ; therefore, we get the required result. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this manuscript.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

On Generalized Rational α -Geraghty Contraction Mappings in G -Metric Spaces

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Received 13 December 2020; Revised 7 February 2021; Accepted 12 February 2021; Published 9 March 2021

Academic Editor: Efthymios G. Tsionas

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In this paper, we discuss about various generalizations of α -admissible mappings. Furthermore, we extend the concept of α -admissible to generalize rational α -Geraghty contraction in G -metric space. With this new contraction mapping, we establish some fixed-point theorems in G -metric space. The obtained result is verified with an example.

1. Introduction and Preliminaries

Samet et al. [1] make a remarkable contribution by introducing α -admissible and $\alpha - \psi$ -contractive mappings. They also showed that the most celebrated result, the Banach contraction principle and various other results, are consequences of their results. Geraghty [2] also made an improvement of the Banach contraction principle. Mustafa and Sims [3] introduced the concept of G -metric space and established the Banach contraction principle. In this paper, considering both the concepts of Samet et al. [1] and Geraghty [2], we introduce generalized rational α -Geraghty contraction in the framework of G -metric space and establish some theorems on fixed points.

Mustafa and Sims [3] give the following definition.

Definition 1 (see [3]). Let U be a nonempty set, and $G: U \times U \times U \longrightarrow \mathbb{R}^+$ satisfies the following:

- (i) $G(\xi, \theta, \phi) = 0$ if and only if $\xi = \theta = \phi$.
- (ii) $0 < G(\xi, \xi, \theta)$ for all $\xi, \theta \in U$ with $\xi \neq \theta$.
- (iii) $G(\xi, \xi, \theta) \leq G(\xi, \theta, \phi)$ for all $\xi, \theta, \phi \in U$ with $\theta \neq \phi$.
- (iv) $G(\xi, \theta, \phi) = G(\xi, \phi, \theta) = G(\theta, \phi, \xi) = \dots$ (symmetry in all three variables).
- (v) $G(\xi, \theta, \phi) \leq G(\xi, t, t) + G(t, \theta, \phi)$, for all $\xi, \theta, \phi, t \in U$.

Here, G is known as generalized metric or G -metric. The pair (U, G) is known as G -metric space.

The following example shows the relation between metric space and G -metric space given by Mustafa and Sims [3].

Example 1 (see [3])

- (i) Let (U, m) be an ordinary metric space, then

$$G(\xi, \theta, \phi) = \frac{1}{3} \{m(\xi, \theta) + m(\theta, \phi) + m(\xi, \phi)\}, \quad (1)$$

is a G -metric on U .

- (ii) Let (U, m) be an ordinary metric space, then

$$G(\xi, \theta, \phi) = \max\{m(\xi, \theta), m(\theta, \phi), m(\xi, \phi)\}, \quad (2)$$

is a G -metric on U .

- (iii) Let G be a G -metric on U , then

$$m(\xi, \theta) = G(\xi, \theta, \theta) + G(\xi, \xi, \theta), \quad (3)$$

is a metric on U .

Definition 2 (see [3]). Consider a G -metric space (U, G) and a sequence $\{\xi_n\}$ of points of U . $\{\xi_n\}$ is said to be

G -convergent to $\xi \in U$ provided $\lim_{n,m \rightarrow +\infty} G(\xi_n, \xi_m, \xi) = 0$; that is, there exists $K \in \mathbb{N}$ satisfying $G(\xi_n, \xi_m, \xi) < \varepsilon$ and $m, n \geq K$ where $\varepsilon > 0$. Here, ξ is known as the limit of the sequence $\{\xi_n\}$ and is denoted as $\xi_n \rightarrow \xi$ or $\lim_{n \rightarrow +\infty} \xi_n = \xi$.

Proposition 1 (see [3]). In a G -metric space (U, G) , we have the following equivalent statements:

- (i) $\{\xi_n\}$ is G convergent to ξ .
- (ii) $G(\xi_n, \xi_n, \xi) \rightarrow 0$ when $n \rightarrow +\infty$.
- (iii) $G(\xi_n, \xi, \xi) \rightarrow 0$ when $n \rightarrow +\infty$.
- (iv) $G(\xi_n, \xi_m, \xi) \rightarrow 0$ when $n, m \rightarrow +\infty$.

Definition 3 (see [3]). In a G -metric space (U, G) , the sequence $\{\xi_n\}$ is said to be G -Cauchy if for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ satisfying $G(\xi_n, \xi_m, \xi_l) < \varepsilon$ and $n, m, l \geq K$; that is, $G(\xi_n, \xi_m, \xi_l) \rightarrow 0$ when $n, m, l \rightarrow +\infty$.

Proposition 2 (see [3]). In a G -metric space (U, G) , we have the following equivalent statements:

- (i) The sequence $\{\xi_n\}$ is G -Cauchy.
- (ii) For any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that $G(\xi_n, \xi_m, \xi_m) < \varepsilon$ for all $n, m \geq K$.

Definition 4 (see [3]). A G -metric space (U, G) is said to be G -complete if every G -Cauchy sequence is G -convergent in (U, G) .

Lemma 1 (see [3]). In a G -metric space (U, G) , for $\xi, \theta, \phi, t \in U$, we have the following:

- (i) If $G(\xi, \theta, \phi) = 0$, then $\xi = \theta = \phi$.
- (ii) $G(\xi, \theta, \phi) \leq G(\xi, \xi, \theta) + G(\xi, \xi, \phi)$.
- (iii) $G(\xi, \theta, \theta) \leq 2G(\theta, \xi, \xi)$.
- (iv) $G(\xi, \theta, \phi) \leq G(\xi, t, \phi) + G(t, \theta, \phi)$.
- (v) $G(\xi, \theta, \phi) \leq (2/3)[G(\xi, \theta, t) + G(\xi, t, \phi) + G(t, \theta, \phi)]$.
- (vi) $G(\xi, \theta, \phi) \leq G(\xi, t, t) + G(\theta, t, t) + G(\phi, t, t)$.

Definition 5 (see [3]). In a G -metric space (U, G) , a mapping $R: U \rightarrow U$ is known as G -continuous if $\{R(\xi_n)\}$ is G -convergent to $R(\xi)$, where $\{\xi_n\}$ is any G -convergent sequence converging to ξ .

Here firstly, we recall the definition of α -admissible mappings and its generalizations in metric space and G -metric space.

Definition 6 (see [1]). Let R be a self-mapping on a metric space (U, d) , and let $\alpha: U \times U \rightarrow [0, +\infty)$ be a function. R is said to be an α -admissible if $\xi, \theta \in U$, and $\alpha(\xi, \theta) \geq 1$ makes $\alpha(R\xi, R\theta) \geq 1$.

Example 2 (see [1]). Consider $U = [0, +\infty)$ and define $R: U \rightarrow U$ and $\alpha: U \times U \rightarrow [0, +\infty)$ by $R\xi = 5\xi$ for all $\xi, \theta \in U$ and

$$\alpha(\xi, \theta) = \begin{cases} e^{(\theta/\xi)}, & \text{if } \xi \geq \theta, \xi \neq 0, \\ 0, & \text{if } \xi < \theta. \end{cases} \quad (4)$$

Then, R is α -admissible.

Definition 7 (see [4]). Let $R, S: U \rightarrow U$ and $\alpha: U \times U \rightarrow [0, +\infty)$. It is said that the pair (R, S) is α -admissible if $\xi, \theta \in U$ such that $\alpha(\xi, \theta) \geq 1$, then we have $\alpha(R\xi, S\theta) \geq 1$ and $\alpha(S\xi, R\theta) \geq 1$.

Definition 8 (see [5]). Let $R: U \rightarrow U$ and $\alpha: U \times U \rightarrow (-\infty, +\infty)$. It is said that R is a triangular α -admissible mapping if the following holds:

- (T1) $\alpha(\xi, \theta) \geq 1$ makes $\alpha(R\xi, R\theta) \geq 1, \xi, \theta \in U$.
- (T2) $\alpha(\xi, \phi) \geq 1, \alpha(\phi, \theta) \geq 1$, makes $\alpha(\xi, \theta) \geq 1, \xi, \theta, \phi \in U$.

Definition 9 (see [4]). Let $R, S: U \rightarrow U$ and $\alpha: U \times U \rightarrow [0, +\infty)$. It is said that a pair (R, S) is a triangular α -admissible mapping if the following holds:

- (T1) $\alpha(\xi, \theta) \geq 1$ makes $\alpha(R\xi, S\theta) \geq 1$ and $\alpha(S\xi, R\theta) \geq 1, \xi, \theta \in U$.
- (T2) $\alpha(\xi, \phi) \geq 1, \alpha(\phi, \theta) \geq 1$, makes $\alpha(\xi, \theta) \geq 1, \xi, \theta, \phi \in U$.

Definition 10 (see [6]). Let R be a self-mapping on a metric space (U, m) and let $\alpha, \eta: U \times U \rightarrow [0, +\infty)$ be two functions. It is said that R is α -admissible mapping with respect to η if $\xi, \theta \in U$, and $\alpha(\xi, \theta) \geq \eta(\xi, \theta)$ implies $\alpha(R\xi, R\theta) \geq \eta(R\xi, R\theta)$.

It can be noted that if $\eta(\xi, \theta) = 1$, then the above definition becomes Definition 6. If we take $\alpha(\xi, \theta) = 1$, then R is said to be an η -subadmissible mapping.

Lemma 2 (see [5]). Let $R: U \rightarrow U$ be a triangular α -admissible mapping. Let us take $\xi_0 \in U$ such that $\alpha(\xi_0, R\xi_0) \geq 1$. Form a sequence $\{\xi_n\}$ as $\xi_{n+1} = R\xi_n$. Then, $\alpha(\xi_n, \xi_m) \geq 1$, where $m, n \in \mathbb{N} \cup \{0\}, n < m$.

Lemma 3 (see [7]). Let $R, S: U \rightarrow U$ be triangular α -admissible mapping. Let us take $\xi_0 \in U$ such that $\alpha(\xi_0, R\xi_0) \geq 1$. Form sequences $\xi_{2i+1} = R\xi_{2i}$ and $\xi_{2i+2} = S\xi_{2i+1}$, where $i = 0, 1, 2, \dots$. Then, $\alpha(\xi_n, \xi_m) \geq 1$, where $m, n \in \mathbb{N} \cup \{0\}, n < m$.

Alghamdi and Karapinar [8] generalized the concept of α -admissible mappings in the context of G -metric space and called it β -admissible. The definition of β -admissible given by Alghamdi and Karapinar is defined as follows.

Definition 11 (see [8]). Let $R: U \rightarrow U$ and $\beta: U \times U \times U \rightarrow [0, +\infty)$, then R is said to be β -admissible if for all $\xi, \theta, \phi \in U$ then

$$\beta(\xi, \theta, \phi) \geq 1 \text{ implies } \beta(R\xi, R\theta, R\phi) \geq 1. \quad (5)$$

Alghamdi and Karapinar [8] introduced $G - \beta - \psi$ contractive mappings of type-I and type-II. They also introduced $G - \beta - \psi$ contractive mappings of type-A. They also gave the relation between these different types of $G - \beta - \psi$ contractions and equivalent Banach contractions.

Alghamdi and Karapinar [9] further generalized the results of Alghamdi and Karapinar [8] by introducing generalized $G - \beta - \psi$ contractive mappings of type-I and type-II.

Kutbi et al. [10] defined rectangular $G - \alpha$ -admissible mapping. They also defined weak $\alpha - \psi - \phi$ contractive mappings to establish some coincidence point theorems for coupled and tripled in G_b -metric space.

Definition 12 (see [10]). Let (U, G) be a G -metric space and let $R, S: U \longrightarrow U$ and $\alpha: U^3 \longrightarrow [0, +\infty)$. R is said to be a rectangular $G - \alpha$ -admissible mapping with respect to S if the following holds:

- (i) $\alpha(S\xi, S\theta, S\phi) \geq 1$ implies $\alpha(R\xi, R\theta, R\phi) \geq 1$, $\xi, \theta, \phi \in U$.
- (ii) $\alpha(S\xi, S\theta, S\theta) \geq 1$ and $\alpha(S\theta, S\phi, S\phi) \geq 1$ imply $\alpha(S\xi, S\theta, S\phi) \geq 1$, $\xi, \theta, \phi \in U$.

Hussain et al. [11] generalized the concept of rectangular $G - \alpha$ -admissible mappings used to obtain coupled and tripled fixed-point theorems.

Hussain et al. [12] established a generalized form of α -admissible mappings in order to prove coincidence points and common fixed points in the framework of G -metric spaces. Furthermore, several authors obtained different kinds of generalization of Banach contraction principle in different spaces (see for details [13–20]).

Definition 13 (see [12]). Let U be an arbitrary set, $\alpha: U \times U \times U \longrightarrow [0, +\infty)$, and $R: U \longrightarrow U$. The mapping R is called an α -dominating map on U if $\alpha(\xi, R\xi, R\xi) \geq 1$ or $\alpha(\xi, \xi, R\xi) \geq 1$ for each ξ in U .

Definition 14 (see [12]). In an arbitrary set U , let $R, S: U \longrightarrow U$ be given mappings and $\alpha: U \times U \times U \longrightarrow [0, +\infty)$ be a function. The pair (R, S) is said to be partially weakly $G - \alpha$ -admissible if and only if $\alpha(R\xi, SR\xi, SR\xi) \geq 1$ for all $\xi \in U$.

Definition 15 (see [12]). In an arbitrary set U , let $R, S: U \longrightarrow U$ be given mappings and $\alpha: U \times U \times U \longrightarrow [0, +\infty)$ be a function. The pair (R, S) is said to be partially weakly $G - \alpha$ -admissible with respect to T if and only if for all $\xi \in U$, $\alpha(R\xi, S\theta, S\theta) \geq 1$, where $\theta \in T^{-1}(R\xi)$.

In the above definition, if $R = S$, R is said to be partially weakly $G - \alpha$ -admissible (or α -admissible of rank 3) with respect to T .

If $T = I_U$ (the identity mapping on U), then the above definition becomes the definition of partially weakly $G - \alpha$ -admissible pair.

Ansari et al. [21] also studied α -admissible mappings in G -metric space by introducing $G - \eta$ -subadmissible mapping and α -dominating map. They also introduced η -subdominating map, α -regular in the framework of G -metric space, and partially weakly $G - \alpha$ -admissible and partially weakly $G - \eta$ -subadmissible mappings.

Definition 16 (see [21]). Let (U, G) be a G -metric space, and let R be a self-mapping on U and $\eta: U \times U \times U \longrightarrow [0, +\infty)$ be a function. R is said to be a $G - \eta$ -subadmissible (or η -subadmissible of rank 3) mapping if

$$\xi, \theta, \phi \in U, \eta(\xi, \theta, \phi) \leq 1, \text{ implies } \eta(R\xi, R\theta, R\phi) \leq 1. \quad (6)$$

Definition 17 (see [21]). Let U be an arbitrary set, $\eta: U \times U \times U \longrightarrow [0, +\infty)$, and $R: U \longrightarrow U$. A mapping R is called an η -subdominating map on U if $\eta(\xi, R\xi, R\xi) \leq 1$ or $\alpha(\xi, \xi, R\xi) \leq 1$ for each ξ in U .

Definition 18 (see [21]). In a G -metric space (U, G) , let $R, S: U \longrightarrow U$ be given mappings and $\eta: U \times U \times U \longrightarrow [0, +\infty)$ be a function. The pair (R, S) is said to be partially weakly $G - \eta$ -subadmissible (or η -subadmissible of rank 3) if and only if $\eta(R\xi, SR\xi, SR\xi) \leq 1$ for all $\xi \in U$.

Definition 19 (see [21]). In a G -metric space (U, G) , let $R, S: U \longrightarrow U$ be given mappings and $\eta: U \times U \times U \longrightarrow [0, +\infty)$ be a function. The pair (R, S) is said to be partially weakly $G - \eta$ -subadmissible (or η -subadmissible of rank 3) with respect to T if and only if for all $\xi \in U$, $\alpha(R\xi, S\theta, S\theta) \geq 1$, where $\theta \in T^{-1}(R\xi)$.

Hussain et al. [22] defined $G - (\alpha, \psi)$ -Meir-Keeler contractive mapping and used it in proving fixed-point theorems in the framework of G -metric spaces.

Definition 20 (see [22]). Let (U, G) be a G -metric space and $\psi \in \Psi$. Let $R: U \longrightarrow U$ be an α -admissible mapping satisfying the following: for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\varepsilon \leq \psi(G(\xi, \theta, \phi)) < \varepsilon + \delta$ implies $\alpha(\xi, \xi)\alpha(\theta, \theta)\alpha(\phi, \phi)\psi(G(R\xi, R\theta, R\phi)) < \varepsilon$ for all $\xi, \theta, \phi \in U$. Then, R is known as a $G - (\alpha, \psi)$ -Meir-Keeler contractive mapping.

In the above definition, Ψ is the collection of nondecreasing functions $\psi: [0, +\infty) \longrightarrow [0, +\infty)$ continuous in t such that $\psi(t) = 0$ if and only if $t = 0$ and $\psi(t + s) \leq \psi(t) + \psi(s)$.

The concept of α -admissible mappings is extended to S -metric space by Zhou et al. [23] and called it γ -admissible. They are defined as follows.

Definition 21 (see [23]). Let $R: U \longrightarrow U$ and $\gamma: U^3 \longrightarrow [0, +\infty)$, then R is said to be γ -admissible if for all $\xi, \theta, \phi \in U$:

$$\gamma(\xi, \theta, \phi) \geq 1 \text{ implies } \gamma(R\xi, R\theta, R\phi) \geq 1. \quad (7)$$

They also extended γ -admissibility for two mappings. Furthermore, they also introduced concepts of various contractive mappings viz. type A, type B, type C, type D, and type E.

Bulbul et al. [24] also derived the concept of generalized $S - \beta - \psi$ contractive-type mappings on the line of generalized $G - \beta - \gamma$ contractive-type mappings. Nabil et al. [25] also defined the concept of α -admissible mappings in S_b -metric space.

From these, what we observe is that β -admissible was for the first time used by Samet et al. [1] to represent α -admissible while dealing with coupled fixed point-related problems. Phiangsungnoen et al. [26] also used the name β -admissible mapping in order to represent α -admissible for fuzzy mappings. On the contrary, β -admissible of Alghamdi and Karapinar [9] and γ -admissible of Zhou et al. [23] are all extended versions of α -admissible mappings in G -metric space and S -metric space, respectively. Thus, we can remark that α -admissible and its various forms can be extended to G -metric as well as S -metric spaces and further to G_b -metric and S_b -metric spaces. With this idea, we introduce various forms of α -admissible mappings in the context of G -metric space and present following definitions. For notation, we use α_G for α -admissible mappings in G -metric space.

Definition 22. Let $R: U \longrightarrow U$ and $\alpha_G: U \times U \times U \longrightarrow [0, +\infty)$, then R is said to be α_G -admissible if $\xi, \theta, \phi \in U$, $\alpha_G(\xi, \theta, \phi) \geq 1$ implies $\alpha_G(R\xi, R\theta, R\phi) \geq 1$.

Definition 23. Let $R, S: U \longrightarrow U$ and $\alpha_G: U \times U \times U \longrightarrow [0, +\infty)$. We say that the pair (R, S) is α_G -admissible if $\xi, \theta, \phi \in U$ such that $\alpha_G(\xi, \theta, \phi) \geq 1$, then we have $\alpha_G(R\xi, S\theta, S\phi) \geq 1$ and $\alpha_G(S\xi, R\theta, R\phi) \geq 1$.

Definition 24. Let $R: U \longrightarrow U$ and $\alpha_G: U \times U \times U \longrightarrow [0, +\infty)$. We say that R is triangular α_G -admissible mapping if the following holds:

- (i) $\alpha_G(\xi, \theta, \phi) \geq 1$ implies $\alpha_G(R\xi, R\theta, R\phi) \geq 1, \xi, \theta, \phi \in U$.
- (ii) $\alpha_G(\xi, t, t) \geq 1$ and $\alpha_G(t, \theta, \phi) \geq 1$ implies $\alpha_G(\xi, \theta, \phi) \geq 1, \xi, \theta, \phi, t \in U$.

$$\nabla_1(\xi, \theta, \phi)$$

$$= \max \left\{ G(\xi, \theta, \phi), G(R\xi, S\theta, S\phi), \frac{G(\xi, R\xi, R\xi)G(\theta, S\theta, S\theta)}{1 + G(\xi, \theta, \phi) + G(R\xi, S\theta, S\phi)}, \frac{G(\theta, S\theta, S\theta)G(\phi, S\phi, S\phi)}{1 + G(\xi, \theta, \phi) + G(R\xi, S\theta, S\phi)}, \frac{G(\phi, S\phi, S\phi)G(\xi, R\xi, R\xi)}{1 + G(\xi, \theta, \phi) + G(R\xi, S\theta, S\phi)} \right\}. \quad (11)$$

Definition 28. In a G -metric space (U, G) , let $\alpha_G: U \times U \times U \longrightarrow [0, +\infty)$ be a function. We say that the mappings $R, S: U \longrightarrow U$ are a pair of generalized rational

Definition 25. Let $R: U \longrightarrow U$ and let $\alpha_G, \eta_G: U \times U \times U \longrightarrow [0, +\infty)$ be functions. We say that R is α_G -admissible mapping with respect to η_G if $\xi, \theta, \phi \in U$,

$$\alpha_G(\xi, \theta, \phi) \geq \eta_G(\xi, \theta, \phi) \text{ implies } \alpha_G(R\xi, R\theta, R\phi) \geq \eta_G(R\xi, R\theta, R\phi). \quad (8)$$

Note that if we take $\eta_G(\xi, \theta, \phi) = 1$, then this definition becomes Definition 22. Also, if we take $\alpha_G(\xi, \theta, \phi) = 1$, then it is said that R is an η_G -subadmissible mapping.

Definition 26. Let $R, S: U \longrightarrow U$ and $\alpha_G, \eta_G: U \times U \times U \longrightarrow [0, +\infty)$. We say that the pair (R, S) is α_G -admissible mapping with respect to η_G if $\xi, \theta, \phi \in U$ such that $\alpha_G(\xi, \theta, \phi) \geq \eta_G(\xi, \theta, \phi)$, then we have $\alpha_G(R\xi, S\theta, S\phi) \geq \eta_G(R\xi, S\theta, S\phi)$ and $\alpha_G(S\xi, R\theta, R\phi) \geq \eta_G(S\xi, R\theta, R\phi)$.

Lemma 4. Let $R, S: U \longrightarrow U$ are triangular α_G -admissible mappings. Suppose that there exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq 1$. Define sequences

$$\begin{aligned} \xi_{2i+1} &= R\xi_{2i}, \\ \xi_{2i+2} &= S\xi_{2i+1}, \quad \text{where } i = 0, 1, 2, \dots \end{aligned} \quad (9)$$

Then, we have $\alpha_G(\xi_n, \xi_m, \xi_m) \geq 1, m, n \in \mathbb{N} \cup \{0\}, n < m$.

2. Main Results

Let us take \mathcal{G} as the collection of functions $g: [0, +\infty) \longrightarrow [0, 1)$ such that $g(t_n) \longrightarrow 1$ gives $t_n \longrightarrow 0$, where $\{t_n\}$ is a bounded sequence of positive real numbers.

We start our results with the following definitions.

Definition 27. In a G -metric space (U, G) , let $\alpha_G: U \times U \times U \longrightarrow [0, +\infty)$ be a function. We say that mappings $R, S: U \longrightarrow U$ is a pair of generalized rational α_G -Geraghty contraction mappings of type-I if for all $\xi, \theta, \phi \in U$ and $g \in \mathcal{G}$,

$$\alpha_G(\xi, \theta, \phi)G(R\xi, S\theta, S\phi) \leq g(\nabla_1(\xi, \theta, \phi))\nabla_1(\xi, \theta, \phi), \quad (10)$$

where

α_G -Geraghty contraction mappings of type-II if for all $\xi, \theta \in U$ and $g \in \mathcal{G}$,

$$\alpha_G(\xi, \theta, \theta)G(R\xi, S\theta, S\theta) \leq g(\nabla_2(\xi, \theta, \theta))\nabla_2(\xi, \theta, \theta), \quad (12)$$

where

$$\nabla_2(\xi, \theta, \theta) = \max \left\{ G(\xi, \theta, \theta), G(R\xi, S\theta, S\theta), \frac{G(\xi, R\xi, R\xi)G(\theta, S\theta, S\theta)}{1 + G(\xi, \theta, \theta) + G(R\xi, S\theta, S\theta)}, \frac{G(\theta, S\theta, S\theta)G(\theta, S\theta, S\theta)}{1 + G(\xi, \theta, \theta) + G(R\xi, S\theta, S\theta)} \right\}. \quad (13)$$

If $R = S$, then we have the following.

Definition 29. In a G -metric space (U, G) , let $\alpha_G: U \times U \times U \rightarrow [0, +\infty)$ be a function. We say that mapping $R: U \rightarrow U$ is a generalized rational α_G -Geraghty contraction mappings of type-I if there exists $g \in \mathcal{G}$ such that for all $\xi, \theta, \phi \in U$,

$$\alpha_G(\xi, \theta, \phi)G(R\xi, R\theta, R\phi) \leq g(\nabla_1(\xi, \theta, \phi))\nabla_1(\xi, \theta, \phi), \quad (14)$$

where

$$\nabla_1(\xi, \theta, \phi) = \max \left\{ G(\xi, \theta, \phi), G(R\xi, R\theta, R\phi), \frac{G(\xi, R\xi, R\xi)G(\theta, R\theta, R\theta)}{1 + G(\xi, \theta, \phi) + G(R\xi, R\theta, R\phi)}, \frac{G(\theta, R\theta, R\theta)G(\phi, R\phi, R\phi)}{1 + G(\xi, \theta, \phi) + G(R\xi, R\theta, R\phi)}, \frac{G(\phi, R\phi, R\phi)G(\xi, R\xi, R\xi)}{1 + G(\xi, \theta, \phi) + G(R\xi, R\theta, R\phi)} \right\}. \quad (15)$$

Definition 30. In a G -metric space (U, G) , let $\alpha_G: U \times U \times U \rightarrow [0, +\infty)$ be a function. We say that the mapping $R: U \rightarrow U$ is a generalized rational α_G -Geraghty contraction mappings of type-II if there exists $g \in \mathcal{G}$ such that for all $\xi, \theta \in U$,

$$\alpha_G(\xi, \theta, \theta)G(R\xi, R\theta, R\theta) \leq g(\nabla_2(\xi, \theta, \theta))\nabla_2(\xi, \theta, \theta), \quad (16)$$

where

$$\nabla_2(\xi, \theta, \theta) = \max \left\{ G(\xi, \theta, \theta), G(R\xi, R\theta, R\theta), \frac{G(\xi, R\xi, R\xi)G(\theta, R\theta, R\theta)}{1 + G(\xi, \theta, \theta) + G(R\xi, R\theta, R\theta)}, \frac{G(\theta, R\theta, R\theta)G(\theta, R\theta, R\theta)}{1 + G(\xi, \theta, \theta) + G(R\xi, R\theta, R\theta)} \right\}. \quad (17)$$

Theorem 1. In a complete G -metric space (U, G) , let $\alpha_G: U \times U \times U \rightarrow [0, +\infty)$ be a function. Let $R, S: U \rightarrow U$ be two mappings satisfying the following:

- (i) R, S is pair of generalized rational α_G -Geraghty contraction mappings of type-I.
- (ii) (R, S) is a pair of triangular α_G -admissible mappings.
- (iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq 1$.
- (iv) R and S are continuous.

Then, a common fixed point exists for the pair (R, S) .

Proof. Let $\xi_1 \in U$ be such that $\xi_1 = R\xi_0$ and $\xi_2 = S\xi_1$. Inductively, we construct a sequence $\{\xi_n\}$ in U as follows:

$$\begin{aligned} \xi_{2i+1} &= R\xi_{2i}, \\ \xi_{2i+2} &= S\xi_{2i+1}, \end{aligned} \quad (18)$$

where $i = 0, 1, 2, 3, \dots$

By assumption $\alpha_G(\xi_0, \xi_1, \xi_1) \geq 1$ and the pair (R, S) is α_G -admissible, by Lemma 4, we have

$$\alpha_G(\xi_n, \xi_{n+1}, \xi_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \quad (19)$$

Then,

$$\begin{aligned} G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) &= G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1}) \\ &\leq \alpha_G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1}) \\ &\leq g(\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}))\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), \end{aligned} \quad (20)$$

for all $i \in \mathbb{N} \cup \{0\}$.

Now,

$$\begin{aligned}
 & \nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) \\
 &= \max \left\{ G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1}), \frac{G(\xi_{2i}, R\xi_{2i}, R\xi_{2i})G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})}, \frac{G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})}, \right. \\
 & \quad \left. \frac{G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})G(\xi_{2i}, R\xi_{2i}, R\xi_{2i})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})} \right\} \\
 &= \max \left\{ G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}), \frac{G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})}, \frac{G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})}, \right. \\
 & \quad \left. \frac{G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})} \right\} \\
 &= \max\{G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})\}.
 \end{aligned} \tag{21}$$

If $\max\{G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})\} = G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})$, then

$$\begin{aligned}
 G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) &\leq g(G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}))G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) \\
 &< G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}),
 \end{aligned} \tag{22}$$

which is a contradiction. Hence,

$$G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) < G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}). \tag{23}$$

This implies that

$$G(\xi_{n+1}, \xi_{n+2}, \xi_{n+2}) < G(\xi_n, \xi_{n+1}, \xi_{n+1}), \tag{24}$$

for all $n \in \mathbb{N} \cup \{0\}$.

So the sequence $\{G(\xi_n, \xi_{n+1}, \xi_{n+1})\}$ is nonnegative and nonincreasing. Now we prove that $G(\xi_n, \xi_{n+1}, \xi_{n+1}) \rightarrow 0$. It is clear that $\{G(\xi_n, \xi_{n+1}, \xi_{n+1})\}$ is a decreasing sequence. So, for some $r > 0$, we have $\lim_{n \rightarrow +\infty} G(\xi_n, \xi_{n+1}, \xi_{n+1}) = r$.

From (23),

$$\frac{G(\xi_{n+1}, \xi_{n+2}, \xi_{n+2})}{G(\xi_n, \xi_{n+1}, \xi_{n+1})} \leq g(G(\xi_n, \xi_{n+1}, \xi_{n+1})) \leq 1. \tag{25}$$

Now, by taking limit as $n \rightarrow +\infty$, we have

$$1 \leq \lim_{n \rightarrow +\infty} g(G(\xi_n, \xi_{n+1}, \xi_{n+1})) < 1, \tag{26}$$

that is,

$$\lim_{n \rightarrow +\infty} g(G(\xi_n, \xi_{n+1}, \xi_{n+1})) = 1. \tag{27}$$

By the property of g , we have

$$\lim_{n \rightarrow +\infty} G(\xi_n, \xi_{n+1}, \xi_{n+1}) = 0. \tag{28}$$

We have to show that $\{\xi_n\}$ is a Cauchy sequence. If possible, let $\{\xi_n\}$ is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and sequences $\{\xi_{m_k}\}$ and $\{\xi_{n_k}\}$ such that, for all positive integers k , we get $m_k > n_k > k$:

$$\begin{aligned}
 G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) &\geq \varepsilon, \\
 G(\xi_{n_k}, \xi_{m_k-1}, \xi_{m_k-1}) &< \varepsilon.
 \end{aligned} \tag{29}$$

Therefore,

$$\begin{aligned}
 \varepsilon &\leq G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) \\
 &\leq G(\xi_{n_k}, \xi_{m_k-1}, \xi_{m_k-1}) + G(\xi_{m_k-1}, \xi_{m_k}, \xi_{m_k}) \\
 &< \varepsilon + G(\xi_{m_k-1}, \xi_{m_k}, \xi_{m_k}).
 \end{aligned} \tag{30}$$

Taking $k \rightarrow +\infty$,

$$\varepsilon \leq \lim_{k \rightarrow +\infty} \sup G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) < \varepsilon. \tag{31}$$

Therefore,

$$\lim_{k \rightarrow +\infty} G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) = \varepsilon. \tag{32}$$

Also, from the triangular inequality, we have

$$\begin{aligned}
 G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) &\leq G(\xi_{n_k}, \xi_{n_k+1}, \xi_{n_k+1}) + G(\xi_{n_k+1}, \xi_{m_k}, \xi_{m_k}), \\
 G(\xi_{n_k+1}, \xi_{m_k}, \xi_{m_k}) &\leq G(\xi_{n_k+1}, \xi_{n_k}, \xi_{n_k}) + G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) \\
 &\leq G(\xi_{n_k+1}, \xi_{n_k+1}, \xi_{n_k}) + G(\xi_{n_k+1}, \xi_{n_k+1}, \xi_{n_k}) \\
 &\quad + G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}).
 \end{aligned} \tag{33}$$

Taking upper limit as $k \rightarrow +\infty$ above, we obtain

$$\begin{aligned}
 \varepsilon &\leq \lim_{k \rightarrow +\infty} \sup G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) \leq \lim_{k \rightarrow +\infty} \sup G(\xi_{n_k+1}, \xi_{m_k}, \xi_{m_k}), \\
 \lim_{k \rightarrow +\infty} \sup G(\xi_{n_k+1}, \xi_{m_k}, \xi_{m_k}) &\leq \lim_{k \rightarrow +\infty} G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) \leq \varepsilon.
 \end{aligned} \tag{34}$$

Thus,

$$\begin{aligned}
 \lim_{k \rightarrow +\infty} \sup G(\xi_{n_k+1}, \xi_{m_k}, \xi_{m_k}) &= \varepsilon, \\
 \lim_{k \rightarrow +\infty} \sup G(\xi_{n_k}, \xi_{m_k+1}, \xi_{m_k+1}) &= \varepsilon.
 \end{aligned} \tag{35}$$

By triangle inequality, we have

$$G(\xi_{n_k+1}, \xi_{m_k}, \xi_{m_k}) \leq G(\xi_{n_k+1}, \xi_{m_k+1}, \xi_{m_k+1}) + G(\xi_{m_k+1}, \xi_{m_k}, \xi_{m_k}). \quad (36)$$

Taking limit as $k \rightarrow +\infty$, we have

$$\varepsilon \leq \lim_{k \rightarrow +\infty} \sup G(\xi_{n_k+1}, \xi_{m_k+1}, \xi_{m_k+1}). \quad (37)$$

Following the above process, we have

$$\lim_{k \rightarrow +\infty} \sup G(\xi_{n_k+1}, \xi_{m_k+1}, \xi_{m_k+1}) \leq \varepsilon. \quad (38)$$

Combining, we have

$$\lim_{k \rightarrow +\infty} G(\xi_{n_k+1}, \xi_{m_k+1}, \xi_{m_k+1}) = \varepsilon. \quad (39)$$

By Lemma 4, $\alpha_G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) \geq 1$, we have

$$\begin{aligned} G(\xi_{n_k+1}, \xi_{m_k+1}, \xi_{m_k+1}) &= G(R\xi_{n_k}, S\xi_{m_k}, S\xi_{m_k}) \\ &\leq \alpha_G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) G(R\xi_{n_k}, S\xi_{m_k}, S\xi_{m_k}) \\ &\leq g(\nabla_1(\xi_{n_k}, \xi_{m_k}, \xi_{m_k})) \nabla_1(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}). \end{aligned} \quad (40)$$

Finally, we conclude that

$$\frac{G(\xi_{n_k+1}, \xi_{m_k+1}, \xi_{m_k+1})}{\nabla_1(\xi_{n_k}, \xi_{m_k}, \xi_{m_k})} \leq g(\nabla_1(\xi_{n_k}, \xi_{m_k}, \xi_{m_k})). \quad (41)$$

Applying $k \rightarrow +\infty$, we obtain

$$\lim_{k \rightarrow +\infty} g(G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k})) = 1. \quad (42)$$

So,

$$\lim_{k \rightarrow +\infty} G(\xi_{n_k}, \xi_{m_k}, \xi_{m_k}) = 0 < \varepsilon, \quad (43)$$

a contradiction. Thus, $\{\xi_n\}$ is a Cauchy sequence. By completeness of U , there exists $a \in U$ such that $\xi_n \rightarrow a$ implies

that $\xi_{2i+1} \rightarrow a$ and $\xi_{2i+2} \rightarrow a$. As R and S are continuous, we get $S\xi_{2i+1} \rightarrow Sa$ and $R\xi_{2i+2} \rightarrow Ra$. Thus, $a = Sa$. Similarly, $a = Ra$, and we have $Ra = Sa = a$. Then, (R, S) have common fixed point.

In the next theorem, we dropped the continuity condition. \square

Theorem 2. In a complete G -metric space (U, G) , let $\alpha_G: U \times U \times U \rightarrow \mathbb{R}$ be a function. Let $R, S: U \rightarrow U$ be two mappings satisfying the following:

- (i) (R, S) is a pair of generalized rational α_G -Geraghty contraction mappings of type-I.
- (ii) (R, S) is a pair of triangular α_G -admissible mappings.
- (iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq 1$.
- (iv) If $\{\xi_n\}$ is a sequence in U such that $\alpha_G(\xi_n, \xi_{n+1}, \xi_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\xi_n \rightarrow a \in U$ as $n \rightarrow +\infty$, then a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ exists satisfying $\alpha_G(\xi_{n_k}, a, a) \geq 1$ for all k .

Then, (R, S) have a common fixed point.

Proof. It follows the similar lines of Theorem 1. Define a sequence $\xi_{2i+1} = R\xi_{2i}$ and $\xi_{2i+2} = S\xi_{2i+1}$, where $i = 0, 1, 2, \dots$ converges to $a \in U$. By the hypothesis of (iv), a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ exists satisfying $\alpha_G(\xi_{2n_k}, a, a) \geq 1$ for all k . Now, we have

$$\begin{aligned} G(\xi_{2n_k+1}, Sa, Sa) &= G(R\xi_{2n_k}, Sa, Sa) \\ &\leq \alpha_G(\xi_{2n_k}, a, a) G(R\xi_{2n_k}, Sa, Sa) \\ &\leq g(\nabla_1(\xi_{2n_k}, a, a)) \nabla_1(\xi_{2n_k}, a, a), \end{aligned} \quad (44)$$

so that

$$G(\xi_{2n_k+1}, Sa, Sa) \leq g(\nabla_1(\xi_{2n_k}, a, a)) \nabla_1(\xi_{2n_k}, a, a). \quad (45)$$

On the contrary, we obtain

$$\begin{aligned} &\nabla_1(\xi_{2n_k}, a, a) \\ &= \max \left\{ G(\xi_{2n_k}, a, a), G(R\xi_{2n_k}, Sa, Sa), \frac{G(\xi_{2n_k}, R\xi_{2n_k}, R\xi_{2n_k}) G(a, Sa, Sa)}{1 + G(\xi_{2n_k}, a, a) + G(R\xi_{2n_k}, Sa, Sa)}, \frac{G(a, Sa, Sa) G(a, Sa, Sa)}{1 + G(\xi_{2n_k}, a, a) + G(R\xi_{2n_k}, Sa, Sa)}, \right. \\ &\quad \left. \frac{G(a, Sa, Sa) G(\xi_{2n_k}, R\xi_{2n_k}, R\xi_{2n_k})}{1 + G(\xi_{2n_k}, a, a) + G(R\xi_{2n_k}, Sa, Sa)} \right\} \\ &= \max \left\{ G(\xi_{2n_k}, a, a), G(\xi_{2n_k+1}, Sa, Sa), \frac{G(\xi_{2n_k}, \xi_{2n_k+1}, \xi_{2n_k+1}) G(a, Sa, Sa)}{1 + G(\xi_{2n_k}, a, a) + G(\xi_{2n_k+1}, Sa, Sa)}, \frac{G(a, Sa, Sa) G(a, Sa, Sa)}{1 + G(\xi_{2n_k}, a, a) + G(\xi_{2n_k+1}, Sa, Sa)}, \right. \\ &\quad \left. \frac{G(a, Sa, Sa) G(\xi_{2n_k}, \xi_{2n_k+1}, \xi_{2n_k+1})}{1 + G(\xi_{2n_k}, a, a) + G(\xi_{2n_k+1}, Sa, Sa)} \right\}. \end{aligned} \quad (46)$$

Letting $k \longrightarrow +\infty$, then we have

$$\lim_{k \longrightarrow +\infty} \nabla_1(\xi_{2n_k}, a, a) = G(a, Sa, Sa). \quad (47)$$

Suppose that $G(a, Sa, Sa) > 0$. From (47), for a large k , we have $\nabla_1(\xi_{2n_k}, a, a) > 0$, which implies that

$$g(\nabla_1(\xi_{2n_k}, a, a)) < 1. \quad (48)$$

Then, we have

$$G(\xi_{2n_k+1}, Sa, Sa) < \nabla_1(\xi_{2n_k}, a, a). \quad (49)$$

Letting $k \longrightarrow +\infty$ in (49), we claim that

$$G(a, Sa, Sa) < G(a, Sa, Sa), \quad (50)$$

which is a contradiction. Thus, we find that $G(a, Sa, Sa) = 0$ implies $a = Sa$.

Also $a = Ra$ showing that a in U is a common fixed point of R and S . \square

3. Consequences

If

$$\nabla_1(\xi, \theta, \phi)$$

$$= \max \left\{ G(\xi, \theta, \phi), G(R\xi, S\theta, S\phi), \frac{G(\xi, R\xi, R\xi)G(\theta, S\theta, S\theta)}{1 + G(\xi, \theta, \phi) + G(R\xi, S\theta, S\phi)}, \frac{G(\theta, S\theta, S\theta)G(\phi, S\phi, S\phi)}{1 + G(\xi, \theta, \phi) + G(R\xi, S\theta, S\phi)}, \frac{G(\phi, S\phi, S\phi)G(\xi, R\xi, R\xi)}{1 + G(\xi, \theta, \phi) + G(R\xi, S\theta, S\phi)} \right\}, \quad (51)$$

and $R = S$ in Theorems 1 and 2, we have the following corollaries.

Corollary 1. In a complete G -metric space (U, G) , let R be α_G -admissible mapping satisfying the following:

- (i) R is generalized rational α_G -Geraghty contraction mappings of type-I.
- (ii) R is triangular α_G -admissible.
- (iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq 1$.
- (iv) R is continuous.

Then, R has a fixed point $a \in U$, and R is a Picard operator; that is, $\{R^n \xi_0\}$ converges to a .

Corollary 2. In a complete G -metric space (U, G) , let R be α_G -admissible mapping satisfying the following:

- (i) R is a generalized rational α_G -Geraghty contraction mappings of type-I.
- (ii) R is triangular α_G -admissible.
- (iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq 1$.

(iv) If $\{\xi_n\}$ is a sequence in U such that $\alpha(\xi_n, \xi_{n+1}, \xi_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $\xi_n \longrightarrow a \in U$ as $n \longrightarrow +\infty$, then a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ exists satisfying $\alpha(\xi_{n_k}, a, a) \geq 1$ for all k .

Then, R has a fixed point $a \in U$ and R is a Picard operator; that is, $\{R^n \xi_0\}$ converges to a .

If

$$\nabla_1(\xi, \theta, \phi) = \max\{G(\xi, \theta, \phi), G(\xi, R\xi, R\xi), G(\theta, S\theta, S\theta), G(\phi, S\phi, S\phi)\} \quad (52)$$

in Theorems 1 and 2, we can have another result.

Let (U, G) be a G -metric space and let $\alpha_G, \eta_G: U \times U \times U \longrightarrow [0, +\infty)$ be functions. Mappings $R, S: U \longrightarrow U$ are called a pair of generalized rational α_G -Geraghty contraction-type mappings with respect to η_G if there exists $g \in \mathcal{G}$ such that for all $\xi, \theta, \phi \in U$:

$$\begin{aligned} \alpha_G(\xi, \theta, \phi) &\geq \eta_G(\xi, \theta, \phi) \\ \implies G(R\xi, S\theta, S\phi) &\leq g(\nabla_1(\xi, \theta, \phi))\nabla_1(\xi, \theta, \phi), \end{aligned} \quad (53)$$

where

$$\nabla_1(\xi, \theta, \phi)$$

$$= \max \left\{ G(\xi, \theta, \phi), G(R\xi, S\theta, S\phi), \frac{G(\xi, R\xi, R\xi)G(\theta, S\theta, S\theta)}{1 + G(\xi, \theta, \phi) + G(R\xi, S\theta, S\phi)}, \frac{G(\theta, S\theta, S\theta)G(\phi, S\phi, S\phi)}{1 + G(\xi, \theta, \phi) + G(R\xi, S\theta, S\phi)}, \frac{G(\phi, S\phi, S\phi)G(\xi, R\xi, R\xi)}{1 + G(\xi, \theta, \phi) + G(R\xi, S\theta, S\phi)} \right\}. \quad (54)$$

Theorem 3. In a complete G -metric space (U, G) , let R be α_G -admissible mapping with respect to η_G satisfying the following:

- (i) (R, S) is a pair of a generalized rational α_G -Geraghty contraction type mapping.

(ii) (R, S) is a pair of triangular α_G -admissible mappings.

(iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq \eta_G(\xi_0, R\xi_0, R\xi_0)$.

(iv) R and S are continuous.

Then, (R, S) have common fixed point.

Proof. Let $\xi_1 \in U$ be such that $\xi_1 = R\xi_0$ and $\xi_2 = S\xi_1$. Inductively, we form a sequence $\{\xi_n\}$ in U as follows:

$$\begin{aligned}\xi_{2i+1} &= R\xi_{2i}, \\ \xi_{2i+2} &= S\xi_{2i+1},\end{aligned}\quad (55)$$

where $i = 0, 1, 2, 3, \dots$

By assumption $\alpha_G(\xi_0, \xi_1, \xi_1) \geq \eta_G(\xi_0, \xi_1, \xi_1)$ and the pair (R, S) is α_G -admissible with respect to η_G , we have $\alpha_G(R\xi_0, S\xi_1, S\xi_1) \geq \eta_G(R\xi_0, S\xi_1, S\xi_1)$ from which we deduce that $\alpha_G(\xi_1, \xi_2, \xi_2) \geq \eta_G(\xi_1, \xi_2, \xi_2)$ which also implies that $\alpha_G(S\xi_1, R\xi_2, R\xi_2) \geq \eta_G(S\xi_1, R\xi_2, R\xi_2)$. Continuing in this way, we obtain $\alpha_G(\xi_n, \xi_{n+1}, \xi_{n+1}) \geq \eta_G(\xi_n, \xi_{n+1}, \xi_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$:

$$\begin{aligned}G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) &= G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1}) \\ &\leq g(\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}))\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}).\end{aligned}\quad (56)$$

Therefore,

$$G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) \leq \alpha_G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1}),\quad (57)$$

for all $i \in \mathbb{N} \cup \{0\}$.

Now,

$$\begin{aligned}\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) &= \max \left\{ G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1}), \frac{G(\xi_{2i}, R\xi_{2i}, R\xi_{2i})G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})}, \right. \\ &\quad \left. \frac{G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})}, \frac{G(\xi_{2i+1}, S\xi_{2i+1}, S\xi_{2i+1})G(\xi_{2i}, R\xi_{2i}, R\xi_{2i})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(R\xi_{2i}, S\xi_{2i+1}, S\xi_{2i+1})} \right\} \\ &= \max \left\{ G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}), \frac{G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})}, \right. \\ &\quad \left. \frac{G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})}, \frac{G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1})}{1 + G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) + G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})} \right\} \\ &= \max \{ G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}), G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) \}.\end{aligned}\quad (58)$$

From the definition of g , the case $\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) = G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2})$ is impossible.

$$\begin{aligned}G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) &\leq g(\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}))\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) \\ &\leq g(G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}))G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) \\ &< G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}),\end{aligned}\quad (59)$$

which is a contradiction. Otherwise, in other case,

$$\begin{aligned}G(\xi_{2i+1}, \xi_{2i+2}, \xi_{2i+2}) &\leq g(\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}))\nabla_1(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) \\ &\leq g(G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}))G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}) \\ &< G(\xi_{2i}, \xi_{2i+1}, \xi_{2i+1}).\end{aligned}\quad (60)$$

This implies that

$$G(\xi_{n+1}, \xi_{n+2}, \xi_{n+2}) < G(\xi_n, \xi_{n+1}, \xi_{n+1}),\quad (61)$$

for all $n \in \mathbb{N} \cup \{0\}$.

Following the similar lines of the Theorem 1, we can prove that R and S have a common fixed point. \square

Theorem 4. In a complete G -metric space (U, G) , let (R, S) be a pair of α_G -admissible mappings with respect to η_G satisfying the following:

- (i) The pair (R, S) is a generalized rational α_G -Geraghty contraction type mappings.
- (ii) The pair (R, S) is triangular α_G -admissible.
- (iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq \eta_G(\xi_0, R\xi_0, R\xi_0)$.
- (iv) If $\{\xi_n\}$ is a sequence in U such that $\alpha_G(\xi_n, \xi_{n+1}, \xi_{n+1}) \geq \eta_G(\xi_n, \xi_{n+1}, \xi_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $\xi_n \longrightarrow a \in U$ as $n \longrightarrow +\infty$, then a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ exists satisfying $\alpha_G(\xi_{n_k}, a, a) \geq \eta_G(\xi_{n_k}, a, a)$ for all k .

Then, R and S have common fixed point.

Proof. It follows the similar line of Theorem 2.

If

$$\nabla_1(\xi, \theta, \phi) = \max \left\{ G(\xi, \theta, \phi), G(R\xi, R\theta, R\phi), \frac{G(\xi, R\xi, R\xi)G(\theta, R\theta, R\theta)}{1 + G(\xi, \theta, \phi) + G(R\xi, R\theta, R\phi)}, \frac{G(\theta, R\theta, R\theta)G(\phi, R\phi, R\phi)}{1 + G(\xi, \theta, \phi) + G(R\xi, R\theta, R\phi)}, \frac{G(\phi, R\phi, R\phi)G(\xi, R\xi, R\xi)}{1 + G(\xi, \theta, \phi) + G(R\xi, R\theta, R\phi)} \right\}, \quad (62)$$

and $R = S$ in Theorems 3 and 4, we get the following corollaries. \square

Corollary 3. In a complete G -metric space (U, G) , let R be α_G -admissible mappings with respect to η_G satisfying the following:

- (i) R is a generalized rational α_G -Geraghty contraction type mapping.
- (ii) R is triangular α_G -admissible.
- (iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq \eta_G(\xi_0, R\xi_0, R\xi_0)$.
- (iv) R is continuous.

Then, R has a fixed point $a \in U$ and R be a Picard operator; that is, $\{R^n \xi_0\}$ converges to a .

Corollary 4. In a complete G -metric space (U, G) , let R be α_G -admissible mapping with respect to η_G satisfying the following:

- (i) R is a generalized rational α_G -Geraghty contraction type mapping.
- (ii) R is triangular α_G -admissible.
- (iii) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_0, R\xi_0, R\xi_0) \geq \eta_G(\xi_0, R\xi_0, R\xi_0)$.

- (iv) There exists $\xi_0 \in U$ such that $\alpha_G(\xi_n, R\xi_{n+1}, R\xi_{n+1}) \geq \eta_G(\xi_n, R\xi_{n+1}, R\xi_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $\xi_n \rightarrow a \in U$ as $n \rightarrow +\infty$, then a subsequence $\{\xi_{n_k}\}$ of $\{\xi_n\}$ exists satisfying $\alpha_G(\xi_{n_k}, a, a) \geq \eta_G(\xi_{n_k}, a, a)$ for all k .

Then, R has a fixed point $a \in U$, and R is a Picard operator; that is, $\{R^n \xi_0\}$ converges to a .

Example 3. Let $U = \{1, 2, 3\}$ with G -metric, then $G(1, 3, 3) = G(3, 1, 1) = (5/7)$, $G(1, 1, 1) = G(2, 2, 2) = G(3, 3, 3) = 0$, $G(1, 2, 2) = G(2, 1, 1) = 1$, $G(2, 3, 3) = G(3, 2, 2) = (4/7)$, and

$$\alpha_G(\xi, \theta, \phi) = \begin{cases} 1, & \text{if } \xi, \theta, \phi \in U, \\ 0, & \text{otherwise.} \end{cases} \quad (63)$$

Define the mappings $R, S: U \rightarrow U$ as follows $R\xi = 1$ for each $\xi \in U$, $S(1) = S(3) = 1$, $S(2) = 3$, and $g: [0, +\infty) \rightarrow [0, 1)$, then

$$\alpha_G(\xi, \theta, \phi)G(S\xi, S\theta, S\phi) \leq g(\nabla_1(\xi, \theta, \phi))\nabla_1(\xi, \theta, \phi). \quad (64)$$

Let $\xi = 2$, $\theta = 3$, $\phi = 3$, then condition (i) is not satisfied by the mapping S as $G(S2, S3, S3) = G(3, 1, 1) = (5/7)$, where

$$\begin{aligned} \nabla_1(\xi, \theta, \phi) &= \max \left\{ G(2, 3, 3), G(S2, S3, S3), \frac{G(2, S2, S2)G(3, S3, S3)}{1 + G(2, 3, 3) + G(S2, S3, S3)}, \frac{G(3, S3, S3)G(3, S3, S3)}{1 + G(2, 3, 3) + G(S2, S3, S3)} \right\} \\ &= \max \left\{ G(2, 3, 3), G(3, 1, 1), \frac{G(2, 3, 3)G(3, 1, 1)}{1 + G(2, 3, 3) + G(3, 1, 1)}, \frac{G(3, 1, 1)G(3, 1, 1)}{1 + G(2, 3, 3) + G(3, 1, 1)} \right\} \\ &= \max \left\{ \frac{4}{7}, \frac{5}{7}, \frac{5}{28}, \frac{25}{112} \right\} \\ &= \frac{5}{7}. \end{aligned} \quad (65)$$

Thus, $\alpha_G(\xi, \theta, \phi)G(S\xi, S\theta, S\phi) \leq g(\nabla_1(\xi, \theta, \phi))\nabla_1(\xi, \theta, \phi)$ is not true.

We prove that Theorem 1 can be applied to R and S . Let $\xi, \theta, \phi \in U$; clearly, (R, S) is α_G -admissible such that

$\alpha_G(\xi, \theta, \phi) \geq 1$. Let $\xi, \theta, \phi \in U$ so that $R\xi, S\theta, S\phi \in U$ and $\alpha_G(R\xi, S\theta, S\phi) = 1$. Hence, (R, S) is α_G -admissible. We know that condition (i) of Theorem 1 is satisfied.

If $\xi, \theta, \phi \in U$, then $\alpha_G(\xi, \theta, \phi) = 1$, and we have

$$\alpha_G(\xi, \theta, \phi)G(R\xi, S\theta, S\phi) \leq g(\nabla_1(\xi, \theta, \phi))\nabla_1(\xi, \theta, \phi), \quad (66)$$

where

$$\begin{aligned} \nabla_1(\xi, \theta, \phi) &= \max \left\{ G(2, 3, 3), G(R2, S3, S3), \frac{G(2, R2, R2)G(3, S3, S3)}{1 + G(2, 3, 3) + G(R2, S3, S3)}, \frac{G(3, S3, S3)G(3, S3, S3)}{1 + G(2, 3, 3) + G(R2, S3, S3)} \right\} \\ &= \max \left\{ G(2, 3, 3), G(1, 1, 1), \frac{G(2, 1, 1)G(3, 1, 1)}{1 + G(2, 3, 3) + G(1, 1, 1)}, \frac{G(3, 1, 1)G(3, 1, 1)}{1 + G(2, 3, 3) + G(1, 1, 1)} \right\} \\ &= \max \left\{ \frac{4}{7}, 0, \frac{5}{11}, \frac{25}{77} \right\} \\ &= \frac{4}{7}, \end{aligned} \quad (67)$$

and $G(R2, S3, S3) = G(1, 1, 1) = 0$.

$$\alpha_G(\xi, \theta, \phi)G(R\xi, S\theta, S\phi) \leq g(\nabla_1(\xi, \theta, \phi))\nabla_1(\xi, \theta, \phi). \quad (68)$$

Hence, the conditions of Theorem 1 are satisfied. So, R and S have a common fixed point.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Recent Fixed-Point Results for θ – Contraction Mappings in Rectangular M – Metric Spaces with Supportive Application

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Received 8 January 2021; Revised 29 January 2021; Accepted 6 February 2021; Published 23 February 2021

Academic Editor: Naeem Saleem

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The goal of this manuscript is to present a new fixed-point theorem on θ –contraction mappings in the setting of rectangular M -metric spaces (RMMSs). Also, a nontrivial example to illustrate our main result has been given. Moreover, some related sequences with θ –contraction mappings have been discussed. Ultimately, our theoretical result has been implicated to study the existence and uniqueness of the solution to a nonlinear integral equation (NIE).

1. Introduction

Fixed-point theory is one of the branches of functional analysis known as “Banach contraction principle (BCP)” [1] which plays a main role in several branches of mathematics and applied sciences. Its core subject is concerned with the stipulations for the existence of one or more fixed points of a mapping T from a topological space \aleph into itself, whereas we can obtain a fixed point such that $T1 = 1$ and $1 \in \aleph$. Later, this principle has been extended by using different forms of contractive conditions in various spaces (for example, see [2–7]).

The idea of a generalized metric space (GMS) (rectangular metric space or Branciari metric space) was introduced by Branciari [8] in 2000, where the triangle inequality is replaced by the inequality $d(1, \mathcal{J}) \leq d(1, \ell) + d(\ell, \hbar) + d(\hbar, \mathcal{J})$ for all pairwise distinct points $1, \mathcal{J}, \ell, \hbar \in \aleph$. Under this space, many fixed-point results were proposed and nice applications were discussed (see [9–13]).

Recently, GMSs extended to recent versions such as M –metric spaces (MMSs), rectangular metric spaces

(RMSs), extended rectangular b-metric spaces (ERbMSs), partial rectangular b-metric spaces (PRbMSs), and partial rectangular metric spaces (PRMSs) (see [8, 14–17]).

In 2018, a new version of a metric space was introduced by Özgür et al. [18], called a rectangular M –metric space (RMMS) which is also extended (MMS). Shukla [17] defined a PRMS which is the generalized RMS. In such spaces, the authors [17, 18] discussed and presented some fixed-point theorems for self-mappings. They presented some topological concepts about open balls and convergence and defined the notions of a circle and a fixed circle using these concepts with an application to fixed-circle problems. For more details of generalizing fixed-point results, see [8, 14, 17, 19–27].

Continuing in the same direction, in this manuscript, a new fixed-point theorem for θ –contraction mapping in RMMSs is proved and some related sequences are introduced. To support the theoretical result, a nontrivial example is obtained and the existence of a solution to an NIE has been found.

2. Preliminaries

In this part, we consider for all $1, \mathcal{J} \in \mathbb{N}$, $m_{1,\mathcal{J}} = \min\{m(1, 1), m(\mathcal{J}, \mathcal{J})\}$, $M_{1,\mathcal{J}} = \max\{m(1, 1), m(\mathcal{J}, \mathcal{J})\}$, $m_{r,1,\mathcal{J}} = \min\{m_r(1, 1), m_r(\mathcal{J}, \mathcal{J})\}$, and $M_{r,1,\mathcal{J}} = \max\{m_r(1, 1), m_r(\mathcal{J}, \mathcal{J})\}$.

Now, we give some previous results in the gradation of spaces.

Definition 1 (see [17], PRMS). Let \mathbb{N} be a nonempty set. A mapping $p: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}^+$ is said to be a PRM on \mathbb{N} if for any $1, \mathcal{J} \in \mathbb{N}$ and all distinct points $(\ell, h \in \mathbb{N}/\{1, \mathcal{J}\})$, the following stipulations hold:

- (i) (RP₁) $1 = \mathcal{J} \Leftrightarrow p(1, \mathcal{J}) = p(1, 1) = p(\mathcal{J}, \mathcal{J})$
- (ii) (RP₂) $p(1, 1) \leq p(1, \mathcal{J})$
- (iii) (RP₃) $p(1, \mathcal{J}) = p(\mathcal{J}, 1)$
- (iv) (RP₄) $p(1, \mathcal{J}) \leq p(1, \ell) + p(\ell, h) + p(h, \mathcal{J}) - p(\ell, h) - p(h, h)$

The pair (\mathbb{N}, p) is called a PRMS.

The above space was extended to an MMS by Asadi et al. [14] as follows.

Definition 2 (MMS). Assume that \mathbb{N} is a nonempty set. If the function $m: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}^+$ satisfies the following hypotheses, for all $1, \mathcal{J}, \ell \in \mathbb{N}$, then the pair (\mathbb{N}, m) is called an MMS:

- (i) (MM₁) $m(1, 1) = m(\mathcal{J}, \mathcal{J}) = m(1, \mathcal{J}) \Leftrightarrow 1 = \mathcal{J}$
- (ii) (MM₂) $m_{1,\mathcal{J}} \leq m(1, \mathcal{J})$
- (iii) (MM₃) $m(1, \mathcal{J}) = m(\mathcal{J}, 1)$
- (iv) (MM₄) $(m(1, \mathcal{J}) - m_{1,\mathcal{J}}) \leq (m(1, \ell) - m_{1,\ell}) + (m(\ell, \mathcal{J}) - m_{\ell,\mathcal{J}})$

Özgür et al. [18] generalized an MMS to a RMMS as follows.

Definition 3 (RMMS). Let \mathbb{N} be a nonempty set. If the function $m_r: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}^+$ verifies the following assumptions, for all $1, \mathcal{J} \in \mathbb{N}$, then the pair (\mathbb{N}, m_r) is called a RMMS:

- (i) (RM₁) $m_r(1, \mathcal{J}) = m_{r,1,\mathcal{J}} = M_{r,1,\mathcal{J}} \Leftrightarrow 1 = \mathcal{J}$
- (ii) (RM₂) $m_{r,1,\mathcal{J}} \leq m_r(1, \mathcal{J})$
- (iii) (RM₃) $m_r(1, \mathcal{J}) = m_r(\mathcal{J}, 1)$
- (iv) (RM₄) $m_r(1, \mathcal{J}) - m_{r,1,\mathcal{J}} \leq m_r(1, \ell) - m_{r,1,\ell} + m_r(\ell, h) - m_{r,\ell,h} + m_r(h, \mathcal{J}) - m_{r,h,\mathcal{J}}$, for all $\ell, h \in (\mathbb{N}/\{1, \mathcal{J}\})$

It is clear that every MMS is a RMMS.

The results given follow from Özgür et al. [18].

Remark 1. Let (\mathbb{N}, m_r) be a RMMS. Clearly, for every $1, \mathcal{J} \in \mathbb{N}$, we have

- (1) $0 \leq M_{r,1,\mathcal{J}} + m_{r,1,\mathcal{J}} = m_r(1, 1) + m_r(\mathcal{J}, \mathcal{J})$
- (2) $0 \leq M_{r,1,\mathcal{J}} - m_{r,1,\mathcal{J}} = |m_r(1, 1) - m_r(\mathcal{J}, \mathcal{J})|$

Also, it can be easily verified under special cases that

$$(3) \quad M_{r,1,\mathcal{J}} - m_{r,1,\mathcal{J}} \leq (M_{r,1,\ell} - m_{r,1,\ell}) + (M_{r,\ell,h} - m_{r,\ell,h}) + (M_{r,h,\mathcal{J}} - m_{r,h,\mathcal{J}})$$

Example 1. Assume that \mathbb{C} is the set of all complex numbers, and consider the set $\mathbb{N}_\varphi = \{z \in \mathbb{C} : \arg(z) = \varphi\} \cup \{0\}$ for a fixed $\varphi \in [0, 2\pi]$ and define the function $m_r: \mathbb{N} \times \mathbb{N} \rightarrow [0, \infty)$ by $m_r(1, \mathcal{J}) = (|1| + |\mathcal{J}|/2)$ for all $1, \mathcal{J} \in \mathbb{N}_\varphi$. Then, $(\mathbb{N}_\varphi, m_r)$ is a RMMS.

Proposition 1 (see [18]). Suppose that (\mathbb{N}, d) is a RMS and a function $\xi: [0, \infty) \rightarrow [\alpha, \infty)$ is a one-to-one and nondecreasing function with $\xi(0) = \alpha$ so that $\xi(1 + \mathcal{J} + h) = \xi(1) + \xi(\mathcal{J}) + \xi(h) - 2\alpha$, for all $1, \mathcal{J}, h \in \mathbb{N}$. Then, the function $m_r: \mathbb{N} \times \mathbb{N} \rightarrow [0, \infty)$ which is defined by $m_r(1, \mathcal{J}) = \xi(d(1, \mathcal{J}))$, for all $1, \mathcal{J} \in \mathbb{N}$ with the set \mathbb{N} , is a RMMS.

Example 2. Let (\mathbb{N}, d) be a RMS and a function $\xi: [0, \infty) \rightarrow [\alpha, \infty)$ be defined as $\xi(t) = mt + n$, with $\xi(0) = \alpha$ for all $t \in [0, \infty)$, and $m_r(1, \mathcal{J}) = m d(1, \mathcal{J}) + n$ is a RMMS with the set \mathbb{N} .

Definition 4. Assume that (\mathbb{N}, m_r) is a RMMS; a sequence $\{1_n\} \in \mathbb{N}$ is said to be

- (1) Converging to a point 1 iff

$$\lim_{n \rightarrow \infty} (m_r(1_n, 1) - m_{r,1_n,1}) = 0, \quad 1 = 0. \quad (1)$$

- (2) m_r -Cauchy sequence iff

$$\lim_{n,m \rightarrow \infty} (m_r(1_n, 1_m) - m_{r,1_n,1_m}) = 0, \quad (2)$$

$$\lim_{n,m \rightarrow \infty} (M_{r,1_n,1_m} - m_{r,1_n,1_m}) = 0,$$

exist and is finite.

- (3) A RMMS called m_r -complete if every m_r -Cauchy sequence $\{1_n\}$ converges to a point 1 so that

$$\lim_{n \rightarrow \infty} (m_r(1_n, 1) - m_{r,1_n,1}) = 0, \quad (3)$$

$$\lim_{n \rightarrow \infty} (M_{r,1_n,1} - m_{r,1_n,1}) = 0.$$

Lemma 1 (see [18]). Assume that $1_n \rightarrow 1$ and $\mathcal{J}_n \rightarrow \mathcal{J}$ as $n \rightarrow \infty$ in a RMMS (\mathbb{N}, m_r) . Then,

$$\lim_{n \rightarrow \infty} (m_r(1_n, \mathcal{J}_n) - m_{r,1_n,\mathcal{J}_n}) = m_r(1, \mathcal{J}) - m_{r,1,\mathcal{J}}. \quad (4)$$

Lemma 2 (see [18]). Assume that $1_n \rightarrow 1$ as $n \rightarrow \infty$ in a RMMS (\mathbb{N}, m_r) . Then,

$$\lim_{n \rightarrow \infty} (m_r(i_n, \mathcal{J}) - m_{r_{i_n, \mathcal{J}}}) = m_r(i, \mathcal{J}) - m_{r_{i, \mathcal{J}}} \text{ for all } \mathcal{J} \in \mathbb{N}. \quad (5)$$

Lemma 3 (see [18]). Suppose that $i_n \rightarrow i$ and $\mathcal{J}_n \rightarrow \mathcal{J}$ as $n \rightarrow \infty$ in a RMMS (\mathbb{N}, m_r) . Then, $m_r(i, \mathcal{J}) = m_{r_{i, \mathcal{J}}}$. Furthermore, if $m_r(i, i) = m_r(\mathcal{J}, \mathcal{J})$, then $i = \mathcal{J}$.

Lemma 4 (see [18]). Let $\{i_n\}$ be a sequence in a RMMS (\mathbb{N}, m_r) . If there is $r \in [0, 1)$ so that $m_r(i_{n+1}, i_n) \leq r m_r(i_n, i_{n-1})$ for all $n \in \mathbb{N}$, then

- (i) $\lim_{n \rightarrow \infty} m_r(i_n, i_{n-1}) = 0$
- (ii) $\lim_{n \rightarrow \infty} m_r(i_n, i_n) = 0$
- (iii) $\lim_{n, m \rightarrow \infty} m_{r_{i_n, i_m}} = 0$
- (iv) $\{i_n\}$ is an m_r -Cauchy sequence

Lemma 5 (see [18]). Let (\mathbb{N}, m_r) be a RMMS and T be a self-mapping on \mathbb{N} . If there is $k \in [0, 1)$ so that $m_r(Ti, T\mathcal{J}) \leq k m_r(i, \mathcal{J})$ for all $i, \mathcal{J} \in \mathbb{N}$, then $Ti_n \rightarrow Tu$ as $n \rightarrow \infty$, where $\{i_n\}_{n \geq 0}$ is a sequence defined by $i_{n+1} = Ti_n$ so that $i_n \rightarrow u$ as $n \rightarrow \infty$.

Theorem 1 (see [18]). Suppose that T is a self-mapping on a RMMS (\mathbb{N}, m_r) . If there is $k \in (0, 1)$ so that

$$m_r(Ti, T\mathcal{J}) \leq k m_r(i, \mathcal{J}) \text{ for all } i, \mathcal{J} \in \mathbb{N}, \quad (6)$$

then T has a unique fixed point u in \mathbb{N} , where $m_r(u, u) = 0$.

Recently, Jleli and Samet [10] established the fixed-point theorem as follows.

Theorem 2 (see [10]). Let (\mathbb{N}, d) be a complete GMS and $T: \mathbb{N} \rightarrow \mathbb{N}$ be a given map. Suppose that there is $\theta \in \Theta$ and $k \in (0, 1)$ such that

$$i, \mathcal{J} \in \mathbb{N}, \quad d(Ti, T\mathcal{J}) \neq 0 \Rightarrow \theta(d(Ti, T\mathcal{J})) \leq [\theta(d(i, \mathcal{J}))]^k, \quad (7)$$

then T has a unique fixed point.

Theorem 2 was extended by Jleli et al. [11] as follows.

Theorem 3 (see [11]). Let (\mathbb{N}, d) be a complete GMS and $T: \mathbb{N} \rightarrow \mathbb{N}$ be a given map. Suppose that there is $\theta \in \Theta$ and $k \in (0, 1)$ so that

$$i, \mathcal{J} \in \mathbb{N}, \quad d(Ti, T\mathcal{J}) \neq 0 \Rightarrow \theta(d(Ti, T\mathcal{J})) \leq [\theta(M(i, \mathcal{J}))]^k, \quad (8)$$

where

$$M(i, \mathcal{J}) = \max\{d(i, \mathcal{J}), d(i, Ti), d(\mathcal{J}, T\mathcal{J})\}, \quad (9)$$

then T has a unique fixed point.

In Theorems 2 and 3, the authors considered Θ is the set of all functions $\theta: (0, \infty) \rightarrow (1, \infty)$ so that the following conditions hold:

- (i) (Θ_1) θ is nondecreasing
- (ii) (Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n \rightarrow \infty} \theta(t_n) = 1$ iff $\lim_{n \rightarrow \infty} t_n = 0^+$
- (iii) (Θ_3) there is $r \in (0, 1)$ and $L \in (0, \infty]$ so that $\lim_{t \rightarrow 0^+} (\theta(t) - 1/t^r) = L$
- (iv) (Θ_4) θ is continuous

3. Main Result

We introduce in this part some new fixed-point results via θ -contraction mapping in RMMSs which generalizes the results of Jleli et al. [11].

Theorem 4. Let (\mathbb{N}, m_r) be a complete RMMS and $T: \mathbb{N} \rightarrow \mathbb{N}$ be a given mapping. Suppose that there is $\theta \in \Theta$ and $k \in (0, 1)$ so that

$$i, \mathcal{J} \in \mathbb{N}, \quad m_r(Ti, T\mathcal{J}) \neq 0 \Rightarrow \theta(m_r(Ti, T\mathcal{J})) \leq [\theta(N(i, \mathcal{J}))]^k, \quad (10)$$

where

$$N(i, \mathcal{J}) = \max\{m_r(i, \mathcal{J}), m_r(i, Ti), m_r(\mathcal{J}, T\mathcal{J})\}, \quad (11)$$

and θ is as defined in the above section. Then, T has a unique fixed point.

Proof. Let $i_0 \in \mathbb{N}$ be an arbitrary value and define a sequence $\{i_n\}_{n \in \mathbb{N}}$ by letting $i_1 = Ti_0$, $i_2 = Ti_1$, $i_3 = Ti_2, \dots, i_{n+1} = Ti_n = T^{n+1}i_0$. If for some $q \in \mathbb{N}$, we have $T^q i_1 = T^{q+1} i_1$, then $T^q i_1$ is a FP of T . So we suppose that $m_r(T^n i_1, T^{n+1} i_1) > 0 \forall n \in \mathbb{N}$. Now, from (10), we have

$$\theta(m_r(T^n i_1, T^{n+1} i_1)) \leq [\theta(N(T^{n-1} i_1, T^n i_1))]^k, \quad (12)$$

$$N(T^{n-1} i_1, T^n i_1) = \max \left\{ \begin{array}{l} m_r(T^{n-1} i_1, T^n i_1), m_r(T^{n-1} i_1, T^{n+1} i_1) \\ m_r(T^n i_1, T^{n+1} i_1) \end{array} \right\}, \quad (13)$$

$$\begin{aligned} N(T^{n-1} i_1, T^n i_1) &= \max \left\{ \begin{array}{l} m_r(T^{n-1} i_1, T^n i_1), m_r(T^{n-1} i_1, T^{n+1} i_1) \\ m_r(T^n i_1, T^{n+1} i_1) \end{array} \right\}, \\ &= \max\{m_r(T^{n-1} i_1, T^n i_1), m_r(T^n i_1, T^{n+1} i_1)\}. \end{aligned} \quad (14)$$

If $N(T^{n-1} i_1, T^n i_1) = m_r(T^n i_1, T^{n+1} i_1)$, then from (12), one can write

$$\theta(m_r(T^n i_1, T^{n+1} i_1)) \leq [\theta(m_r(T^n i_1, T^{n+1} i_1))]^k. \quad (15)$$

Taking \ln in both sides, we obtain

$$\ln[\theta(m_r(T^n i_1, T^{n+1} i_1))] \leq k \ln[\theta(m_r(T^n i_1, T^{n+1} i_1))], \quad (16)$$

which is a contradiction because $k \in (0, 1)$. Thus, we have

$$N(T^{n-1}1, T^n1) = m_r(T^{n-1}1, T^n1). \quad (17)$$

It follows from (12) that

$$\begin{aligned} \theta(m_r(T^n1, T^{n+1}1)) &\leq [\theta(m_r(T^{n-1}1, T^n1))]^k \\ &\leq [\theta(m_r(T^{n-2}1, T^{n-1}1))]^{k^2} \\ &\vdots \\ &\leq [\theta(m_r(1, T1))]^{k^n}. \end{aligned} \quad (18)$$

This leads to

$$1 \leq \theta(m_r(T^n1, T^{n+1}1)) \leq [\theta(m_r(1, T1))]^{k^n}, \quad \forall n \in \mathbb{N}. \quad (19)$$

As $n \rightarrow \infty$ in (19), we obtain that

$$\theta(m_r(T^n1, T^{n+1}1)) \rightarrow 1, \quad \text{as } n \rightarrow \infty, \quad (20)$$

which implies from (Θ_2) that $\lim_{n \rightarrow \infty} m_r(T^n1, T^{n+1}1) = 0$. Now, from condition (Θ_3) , there are $r \in (0, 1)$ and $L \in (0, \infty]$ so that

$$\lim_{n \rightarrow \infty} \frac{\theta(m_r(T^n1, T^{n+1}1)) - 1}{[m_r(T^n1, T^{n+1}1)]^r} = L. \quad (21)$$

Suppose that $L < \infty$. In this case, let $B = (L/2) > 0$. From the definition of the limit, there is $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta(m_r(T^n1, T^{n+1}1)) - 1}{[m_r(T^n1, T^{n+1}1)]^r} - L \right| \leq B, \quad \forall n \geq n_0. \quad (22)$$

This implies that

$$\frac{\theta(m_r(T^n1, T^{n+1}1)) - 1}{[m_r(T^n1, T^{n+1}1)]^r} \geq L - B = B, \quad \forall n \geq n_0. \quad (23)$$

Thus,

$$n[m_r(T^n1, T^{n+1}1)]^r \leq An[\theta(m_r(T^n1, T^{n+1}1)) - 1], \quad \forall n \geq n_0, \quad (24)$$

where $A = (1/B)$.

Now, consider $L = \infty$ and $B > 0$ as an arbitrary positive number. From the definition of the limit, there is $n_0 \in \mathbb{N}$ so that

$$\frac{\theta(m_r(T^n1, T^{n+1}1)) - 1}{[m_r(T^n1, T^{n+1}1)]^r} \geq B, \quad \forall n \geq n_0. \quad (25)$$

This leads to

$$n[m_r(T^n1, T^{n+1}1)]^r \leq An[\theta(m_r(T^n1, T^{n+1}1)) - 1], \quad \forall n \geq n_0, \quad (26)$$

where $A = (1/B)$.

Thus, in both cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ so that

$$n[m_r(T^n1, T^{n+1}1)]^r \leq An[\theta(m_r(T^n1, T^{n+1}1)) - 1], \quad \forall n \geq n_0. \quad (27)$$

Using (14), we obtain

$$n[m_r(T^n1, T^{n+1}1)]^r \leq An[\theta(m_r(1, T1))]^{k^n} - 1, \quad \forall n \geq n_0. \quad (28)$$

Taking limit as $n \rightarrow \infty$ in the above inequality, one can obtain

$$\lim_{n \rightarrow \infty} n[m_r(T^n1, T^{n+1}1)]^r = 0. \quad (29)$$

Thus, there is $n_1 \in \mathbb{N}$ so that

$$m_r(T^n1, T^{n+1}1) \leq \frac{1}{n^{(1/r)}}, \quad \forall n \geq n_1. \quad (30)$$

Now, we will discuss the following two steps.

St₁. When $T^n1 = T^m1$ for some integers $n \neq m$ with $m > n$, we have $T^{m-n}(T^n1) = T^m1$. Choose $\mathcal{J} = T^n1$ and $p = m - n$. Then, $T^p\mathcal{J} = \mathcal{J}$ and \mathcal{J} is a periodic point of T . By (10) and (14), we obtain

$$\theta(m_r(\mathcal{J}, T\mathcal{J})) = \theta(m_r(T^p\mathcal{J}, T^{p+1}\mathcal{J})) \leq [\theta(m_r(\mathcal{J}, T\mathcal{J}))]^{k^p}, \quad (31)$$

As $p \rightarrow \infty$, we have $m_r(\mathcal{J}, T\mathcal{J}) = 0$. On the other hand, we can write

$$\begin{aligned} \theta(m_r(\mathcal{J}, \mathcal{J})) &= \theta(m_r(T^p\mathcal{J}, T^p\mathcal{J})) \\ &\leq [\theta(m_r(T^{p-1}\mathcal{J}, T^{p-1}\mathcal{J}))]^k \\ &\vdots \\ &\leq [\theta(m_r(T\mathcal{J}, T\mathcal{J}))]^{k^{p-1}} \\ &\leq [\theta(m_r(\mathcal{J}, \mathcal{J}))]^{k^p}, \end{aligned} \quad (32)$$

which leads to

$$\begin{aligned} \ln \theta(m_r(\mathcal{J}, \mathcal{J})) &\leq k^{p-1} \ln [\theta(m_r(T\mathcal{J}, T\mathcal{J}))] \\ &\leq k^p \ln [\theta(m_r(\mathcal{J}, \mathcal{J}))], \end{aligned} \quad (33)$$

or equivalently

$$\theta(m_r(\mathcal{J}, \mathcal{J})) = \theta(m_r(T\mathcal{J}, T\mathcal{J})). \quad (34)$$

This yields $m_r(\mathcal{J}, \mathcal{J}) = m_r(T\mathcal{J}, T\mathcal{J}) = 0$. Hence, $T\mathcal{J} = \mathcal{J}$, that is, \mathcal{J} is a fixed point of T .

St₂. When $T^n1 \neq T^m1$ for every $n, m \in \mathbb{N}$ with $n \neq m$, we shall prove that T has a periodic point. Suppose on the contrary, utilizing (10) and (11), we obtain

$$\theta(m_r(T^n1, T^{n+2}1)) \leq [\theta(N(T^{n-1}1, T^{n+1}1))]^k, \quad (35)$$

where

$$N(T^{n-1}1, T^{n+1}1) = \max \left\{ \begin{array}{l} m_r(T^{n-1}1, T^{n+1}1), m_r(T^{n-1}1, T^n1) \\ m_r(T^{n+1}1, T^{n+2}1) \end{array} \right\}. \quad (36)$$

Since θ is continuous and nondecreasing, using (35) and (36), one can write

$$\theta(m_r(T^n_1, T^{n+2}_1)) \leq \left(\max \left\{ \begin{array}{c} \theta(m_r(T^{n-1}_1, T^{n+1}_1)) \\ \theta(m_r(T^{n-1}_1, T^n_1)) \\ \theta(m_r(T^{n+1}_1, T^{n+2}_1)) \end{array} \right\} \right)^k. \quad (37)$$

Let I be the set of $n \in \mathbb{N}$ such that

$$\begin{aligned} u_n &= \max\{\theta(m_r(T^{n-1}_1, T^{n+1}_1)), \theta(m_r(T^{n-1}_1, T^n_1)), \\ &\quad \theta(m_r(T^{n+1}_1, T^{n+2}_1))\} \\ &= \theta(m_r(T^{n-1}_1, T^{n+1}_1)). \end{aligned} \quad (38)$$

If $|I| < \infty$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$,

$$\begin{aligned} \max\{\theta(m_r(T^{n-1}_1, T^{n+1}_1)), \theta(m_r(T^{n-1}_1, T^n_1)), \\ \theta(m_r(T^{n+1}_1, T^{n+2}_1))\} \\ = \max\{\theta(m_r(T^{n-1}_1, T^n_1)), \theta(m_r(T^{n+1}_1, T^{n+2}_1))\}. \end{aligned} \quad (39)$$

In this case, we obtain from (37) that

$$\begin{aligned} 1 &\leq \theta(m_r(T^n_1, T^{n+2}_1)) \\ &\leq \left[\max\{\theta(m_r(T^{n-1}_1, T^n_1)), \theta(m_r(T^{n+1}_1, T^{n+2}_1))\} \right]^k, \end{aligned} \quad (40)$$

for all $n \geq N$. Letting $n \rightarrow \infty$ in the above inequality and using (20), we obtain

$$\theta(m_r(T^n_1, T^{n+2}_1)) \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (41)$$

If $|I| = \infty$, we can find a subsequence of $\{u_n\}$, which we denote also by $\{u_n\}$, such that

$$u_n = \theta(m_r(T^{n-1}_1, T^{n+1}_1)), \quad \text{for } n \text{ large enough.} \quad (42)$$

In this case, we obtain from (37) that

$$\begin{aligned} 1 &\leq \theta(m_r(T^n_1, T^{n+2}_1)) \leq [\theta(m_r(T^{n-1}_1, T^{n+1}_1))]^k \\ &\leq (\theta(m_r(T^{n-2}_1, T^n_1)))^{k^2} \leq \dots \leq [\theta(m_r(T^1_1, T^2_1))]^{k^n}, \end{aligned} \quad (43)$$

for n large enough. Letting $n \rightarrow \infty$ in the above inequality, we have

$$\theta(m_r(T^n_1, T^{n+2}_1)) \rightarrow 1, \quad \text{as } n \rightarrow \infty. \quad (44)$$

Then, in both cases, (44) holds. Using (44) and property (Θ_2) , we obtain

$$\lim_{n \rightarrow \infty} m_r(T^n_1, T^{n+2}_1) = 0. \quad (45)$$

Similarly, from condition (Θ_3) , there exists $n_2 \in \mathbb{N}$ such that

$$m_r(T^n_1, T^{n+2}_1) \leq \frac{1}{n^{(1/r)}}, \quad \text{for all } n \geq n_2. \quad (46)$$

Let $N' = \max\{n_0, n_1\}$. If $m > 2$ is odd, then we consider $m = 2q + 1$, where $q \geq 1$. Using (30) for all $n \geq N$, we can obtain

$$\begin{aligned} m_r(T^n_1, T^{n+m}_1) - m_{r_{T^n_1 T^{n+m}_1}} &\leq m_r(T^n_1, T^{n+1}_1) - m_{r_{T^n_1 T^{n+1}_1}} \\ &\quad + m_r(T^{n+1}_1, T^{n+2}_1) - m_{r_{T^{n+1}_1 T^{n+2}_1}} \\ &\quad + \dots + m_r(T^{n+2q}_1, T^{n+2q+1}_1) \\ &\quad - m_{r_{T^{n+2q}_1 T^{n+2q+1}_1}} \\ &\leq m_r(T^n_1, T^{n+1}_1) + m_r(T^{n+1}_1, T^{n+2}_1) \\ &\quad + \dots + m_r(T^{n+2q}_1, T^{n+2q+1}_1) \\ &\leq \frac{1}{n^{(1/r)}} + \frac{1}{(n+1)^{(1/r)}} + \dots \\ &\quad + \frac{1}{(n+2q)^{(1/r)}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{(1/r)}}. \end{aligned} \quad (47)$$

Now, if $m > 2$ is even, then writing $m = 2q$, where $q \geq 2$ and using (30) and (46), for all $n \geq N$, we obtain

$$\begin{aligned} m_r(T^n x, T^{n+m} x) - m_{r_{T^n x T^{n+m} x}} &\leq m_r(T^n x, T^{n+2} x) - m_{r_{T^n x T^{n+2} x}} \\ &\quad + m_r(T^{n+2} x, T^{n+3} x) \\ &\quad - m_{r_{T^{n+2} x T^{n+3} x}} \\ &\quad + \dots + m_r(T^{n+2q-1} x, T^{n+2q} x) \\ &\quad - m_{r_{T^{n+2q-1} x T^{n+2q} x}}, \\ &\leq m_r(T^n x, T^{n+2} x) \\ &\quad + m_r(T^{n+2} x, T^{n+3} x) \\ &\quad + \dots + m_r(T^{n+2q-1} x, T^{n+2q} x) \\ &\leq \frac{1}{n^{(1/r)}} + \frac{1}{(n+2)^{(1/r)}} + \dots \\ &\quad + \frac{1}{(n+2q-1)^{(1/r)}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{(1/r)}}. \end{aligned} \quad (48)$$

Combining (47) and (48), we have

$$m_r(T^n 1, T^{n+m} 1) - m_{r_{T^n 1, T^{n+m} 1}} \leq \sum_{i=n}^{\infty} \frac{1}{i^{(1/r)}}, \quad \text{for all } n \geq N', m \in \mathbb{N}. \quad (49)$$

Since the series $\sum_{i=n}^{\infty} (1/i^{(1/r)})$ is convergent, we find that $m_r(T^n 1, T^{n+m} 1) - m_{r_{T^n 1, T^{n+m} 1}}$ is also convergent as $n, m \rightarrow \infty$, which implies that

$$m_r(T^n 1, T^{n+m} 1) - m_{r_{T^n 1, T^{n+m} 1}} \leq m_r(T^n 1, T^{n+m} 1). \quad (50)$$

Thus, the sequence $\{T^n 1\}$ is m_r -Cauchy in the m_r -complete RMMS (\mathbb{N}, m_r) . Hence, there is some $u \in \mathbb{N}$ so that $T^n 1 \rightarrow u$ as $n \rightarrow \infty$. Without restriction of the generality, we can suppose that $T^n 1 \neq u$ for all n .

Assume that $m_r(u, Tu) > 0$. Using (10), we obtain

$$\theta(m_r(T^{n+1} 1, Tu)) \leq [\theta(N(T^n 1, u))]^k, \quad \text{for all } n \in \mathbb{N}, \quad (51)$$

where

$$N(T^n 1, u) = \max\{m_r(T^n 1, u), m_r(T^n 1, T^{n+1} 1), m_r(u, Tu)\}. \quad (52)$$

Taking limit as $n \rightarrow \infty$ in the above inequality and using (Θ_4) and Lemmas 1–3, we have

$$\theta(m_r(u, Tu)) \leq [\theta(m_r(u, Tu))]^k < \theta(m_r(u, Tu)), \quad (53)$$

which is a contradiction; therefore, $Tu = u$, which is also a contradiction with the supposition that T does not have a periodic point. Thus, T has a periodic point, say u , of period q . Assume that the set of all fixed points of T is empty. Then, we have

$$q > 1 \text{ and } m_r(u, Tu) > 0. \quad (54)$$

Using (10), we obtain that

$$\begin{aligned} \theta(m_r(u, Tu)) &= \theta(m_r(T^q u, T^{q+1} u)) \\ &\leq [\theta(m_r(u, Tu))]^{k^q} < \theta(m_r(u, Tu)), \end{aligned} \quad (55)$$

which is a contradiction. Thus, the set of fixed points of T is nonempty, that is, T has more than one fixed point.

Now, assume that $u, v \in \mathbb{N}$ are two fixed points of T so that $m_r(u, v) = m_r(Tu, Tv) > 0$. Utilizing (10), we obtain

$$\begin{aligned} \theta(m_r(u, v)) &= \theta(m_r(Tu, Tv)) \leq [\theta(m_r(u, v))]^k \\ &< \theta(m_r(u, v)), \end{aligned} \quad (56)$$

which is a contradiction. Then, T has a unique fixed point. This completes the proof. \square

Theorem 4 is still valid if we replace condition (10) with one of the following stipulations: for all $1, \mathcal{J}, u \in \mathbb{N}$ with $m_r(u, u) = 0$,

(i) There is $0 < k < 1$ so that

$$m_r(T1, T\mathcal{J}) \leq k \max\{m_r(1, \mathcal{J}), m_r(1, T1), m_r(\mathcal{J}, T\mathcal{J})\}. \quad (57)$$

(ii) There is $0 \leq k < (1/2)$ so that

$$m_r(T1, T\mathcal{J}) \leq k[m_r(1, T1) + m_r(\mathcal{J}, T\mathcal{J})]. \quad (58)$$

(iii) There is $0 < k < (\sqrt{3} - 1/2)$ so that

$$m_r(T1, T\mathcal{J}) \leq k[m_r(1, T\mathcal{J}) + m_r(\mathcal{J}, T1)]. \quad (59)$$

(iv) There are $\theta \in \Theta$ and $k \in (0, 1)$ so that

$$m_r(T1, T\mathcal{J}) \neq 0 \implies \theta(m_r(T1, T\mathcal{J})) \leq [\theta(m_r(1, \mathcal{J}))]^k. \quad (60)$$

We note that Θ contains a large class of functions. For instance,

$$\theta(t) = 2 - \frac{2}{\pi} \arctan\left(\frac{1}{t^\alpha}\right), \quad 0 < \alpha < 1, t > 0. \quad (61)$$

So we conclude the following corollary from Theorem 4.

Corollary 1. Let (\mathbb{N}, m_r) be a complete RMMS and $T: \mathbb{N} \rightarrow \mathbb{N}$ be a self-mapping. If there exists $\theta \in \Theta$ and $\alpha, k \in (0, 1)$ such that for all $1, \mathcal{J} \in \mathbb{N}$ and $T1 \neq T\mathcal{J}$,

$$\begin{aligned} &2 - \frac{2}{\pi} \arctan\left(\frac{1}{[m_r(T1, T\mathcal{J})]^\alpha}\right) \\ &\leq \left[2 - \frac{2}{\pi} \arctan\left(\frac{1}{[N(1, \mathcal{J})]^\alpha}\right)\right]^k, \end{aligned} \quad (62)$$

where

$$N(1, \mathcal{J}) = \max\{m_r(1, \mathcal{J}), m_r(1, T\mathcal{J}), m_r(\mathcal{J}, T\mathcal{J})\}. \quad (63)$$

Then, T has a unique fixed point.

The example below justifies all requirements of Theorem 4.

Example 3. Let $\mathbb{N}_\varphi = \{0\} \cup \{z \in \mathbb{C}: \arg(z) = \varphi\}$ for a fixed φ , where $0 \leq \varphi < 2\pi$. Define the function $m_r: \mathbb{N} \times \mathbb{N} \rightarrow [0, \infty)$ by $m_r(1, \mathcal{J}) = (|1| + |\mathcal{J}|/2)$ for all $1, \mathcal{J} \in \mathbb{N}_\varphi$. Then, $(\mathbb{N}_\varphi, m_r)$ is a RMMS. Take $\varphi = 0, (\pi/2)$ and the RMMS becomes $(\mathbb{N}_{0, (\pi/2)}, m_r)$ such that $\mathbb{N}_{0, (\pi/2)} = [0, \infty) \cup (A_i = \{\tilde{z} = ni: 1 \leq n < \infty\})$ and $m_r(\tilde{z}_1, \tilde{z}_2) = (|\tilde{z}_1| + |\tilde{z}_2|/2)$ for all $\tilde{z}_1, \tilde{z}_2 \in \mathbb{N}_{0, (\pi/2)}$. Let $\theta: (0, \infty) \rightarrow (1, \infty)$ be defined by $\theta(t) = e^{\sqrt{t}}$; it is clear that $\theta \in \Theta$. Define $T: \mathbb{N}_{0, (\pi/2)} \rightarrow \mathbb{N}_{0, (\pi/2)}$ by

$$T\tilde{z} = \begin{cases} \frac{\tilde{z}}{2+n}, & \tilde{z} \in A_i, 1 \leq n < \infty, \\ \frac{\tilde{z}}{2}, & \tilde{z} \in [0, 1], \\ \frac{\tilde{z}}{2^n}, & \tilde{z} \in (1, \infty), 2 \leq n < \infty. \end{cases} \quad (64)$$

Now, for $\tilde{z} \in A_i, 1 \leq n < \infty$, we have

$$\begin{aligned}
\theta(m_r(T\tilde{z}_1, T\tilde{z}_2)) &= e^{\sqrt{m_r(T\tilde{z}_1, T\tilde{z}_2)}} = \theta\left(\frac{|\tilde{z}_1/2 + n| + |\tilde{z}_2/2 + n|}{2}\right) = \theta\left(\frac{1}{2+n}\left(\frac{|\tilde{z}_1| + |\tilde{z}_2|}{2}\right)\right) \\
&= \theta\left(\frac{1}{2+n}m_r(\tilde{z}_1, \tilde{z}_2)\right) = e^{\sqrt{1/2+n}\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}} = \left(e^{\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}}\right)^{\sqrt{1/2+n}} \\
&\leq \left(e^{\sqrt{N(\tilde{z}_1, \tilde{z}_2)}}\right)^{\sqrt{1/2+n}} = [\theta(N(\tilde{z}_1, \tilde{z}_2))]^k.
\end{aligned} \tag{65}$$

Then, T satisfies the contraction with $0 < k = \sqrt{1/2+n} < 1$.

On the other hand, for $\tilde{z} \in [0, 1]$, we have

$$\begin{aligned}
\theta(m_r(T\tilde{z}_1, T\tilde{z}_2)) &= \theta\left(\frac{|\tilde{z}_1/2| + |\tilde{z}_2/2|}{2}\right) = \theta\left(\frac{1}{2}\frac{|\tilde{z}_1| + |\tilde{z}_2|}{2}\right) \\
&= \theta\left(\frac{1}{2}m_r(\tilde{z}_1, \tilde{z}_2)\right) \\
&= e^{(1/2)\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}} = \left(e^{\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}}\right)^{(1/2)} \\
&\leq \left(e^{\sqrt{N(\tilde{z}_1, \tilde{z}_2)}}\right)^{(1/2)} = [\theta(N(\tilde{z}_1, \tilde{z}_2))]^k.
\end{aligned} \tag{66}$$

Similarly, for $\tilde{z} \in (1, \infty)$, $2 \leq n < \infty$, we obtain

$$\begin{aligned}
\theta(m_r(T\tilde{z}_1, T\tilde{z}_2)) &= \theta\left(\frac{|\tilde{z}_1/2^n| + |\tilde{z}_2/2^n|}{2}\right) \\
&= \theta\left(\frac{1}{2^n}\frac{|\tilde{z}_1| + |\tilde{z}_2|}{2}\right) \\
&= \theta\left(\frac{1}{2^n}m_r(\tilde{z}_1, \tilde{z}_2)\right) \\
&= e^{(1/2^n)\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}} = \left(e^{\sqrt{m_r(\tilde{z}_1, \tilde{z}_2)}}\right)^{(1/2^n)} \\
&\leq \left(e^{\sqrt{N(\tilde{z}_1, \tilde{z}_2)}}\right)^{(1/2^n)} = [\theta(N(\tilde{z}_1, \tilde{z}_2))]^k,
\end{aligned} \tag{67}$$

where $k = (1/2^n) \in (0, 1)$. Hence, the conditions of Theorem 4 are fulfilled and $\tilde{z} = 0$ is a unique fixed point of T .

4. Supportive Application

In this section, we apply the theoretical results of Theorem 4 to discuss the existence of solution to a NIE:

$$1(t) = \int_0^t \tilde{E}(t, r)\tilde{\omega}(r, 1(r))dr, \quad t \in [0, D], D > 0, \tag{68}$$

where $\tilde{\omega}$ pounds: $[0, D] \times [0, D] \longrightarrow \mathbb{R}$ and $\tilde{\omega}: [0, D] \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

Define the operator $T: \mathfrak{N} \longrightarrow \mathfrak{N}$ by

$$T1(t) = \int_0^t \tilde{E}(t, r)\tilde{\omega}(r, 1(r))dr, \tag{69}$$

where $\mathfrak{N} = C([0, D])$ is the space of all continuous functions defined from $[0, D]$ to \mathbb{R} equipped with

$$\begin{aligned}
m_r(1, \mathcal{J}) &= \frac{\|1\| + \|\mathcal{J}\|}{2} = \max_{t \in [0, D]} \left(\frac{|1(t)| + |\mathcal{J}(t)|}{2} \right), \\
&\forall 1, \mathcal{J} \in \mathfrak{N}.
\end{aligned} \tag{70}$$

Obviously, (\mathfrak{N}, m_r) is a complete RMMS.

We will consider Problem (68) under the following assumptions:

(i) The function pounds: $[0, D] \times [0, D] \longrightarrow \mathbb{R}$ is continuous and $\max_{t, r \in [0, D]} |\text{pounds}(t, r)| \leq 1$

(ii) The function $\tilde{\omega}: [0, D] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous,

$$\begin{aligned}
|\tilde{\omega}(r, 1(r))| + |\tilde{\omega}(r, \mathcal{J}(r))| &< \lambda^2 e^{-\lambda} P(1, \mathcal{J}), \\
&\forall 1, \mathcal{J} \in \mathfrak{N}, r \in [0, D],
\end{aligned} \tag{71}$$

where $\lambda \geq 1$ and $P(1, \mathcal{J}) = \max\{(|1| + |\mathcal{J}|/2), (|1| + |T1|/2), (|\mathcal{J}| + |T\mathcal{J}|/2)\}$

(iii) For all $1, \mathcal{J} \in \mathfrak{N}$, $\|1\|_\lambda = \max_{t \in [0, D]} |1(t)|e^{-\lambda t}$ and

$$m_{r\lambda}(1, \mathcal{J}) = \frac{\|1\|_\lambda + \|\mathcal{J}\|_\lambda}{\lambda} = \max_{t \in [0, D]} \left(\frac{|1(t)| + |\mathcal{J}(t)|}{\lambda} \right) e^{-\lambda t}, \tag{72}$$

so that the pair $(\mathfrak{N}, m_{r\lambda})$ is a complete RMMS with $\lambda = 2$

(iv) For all $1, \mathcal{J} \in \mathfrak{N}$, define the function $\theta \in \Theta$ by $\theta(t) = e^{\sqrt{t}}$, where $t > 0$

Now, we can state and prove the important theorem in this part as follows.

Theorem 5. Via hypotheses (i)–(iv), Problem (68) has a unique solution in \mathfrak{N} .

Proof. Define the operator $T: \mathfrak{N} \longrightarrow \mathfrak{N}$ as in (69). If T has a unique fixed point, then the NIE (68) has a unique solution. From Hypotheses (i)–(iii), we can write

$$\begin{aligned}
|T_1(r)| + |T\mathcal{J}(r)| &= \left| \int_0^t E(t,r)\varpi(r,1(r))dr \right| + \left| \int_0^t E(t,r)\varpi(r,\mathcal{J}(r))dr \right| \\
&\leq \int_0^t |E(t,r)\varpi(r,1(r))|dr + \int_0^t |\tilde{E}(t,r)\varpi(r,\mathcal{J}(r))|dr \\
&\leq \int_0^t |E(t,r)|(|\varpi(r,1(r))| + |\varpi(r,\mathcal{J}(r))|)dr \\
&\leq \int_0^t (|\varpi(r,1(r))| + |\varpi(r,\mathcal{J}(r))|)dr \\
&< \int_0^t \lambda^2 e^{-\lambda} P(1(r), \mathcal{J}(r))dr \\
&= \lambda^2 e^{-\lambda} \int_0^t e^{r\lambda} \max \left\{ \frac{|1(r)| + |\mathcal{J}(r)|}{\lambda} e^{-r\lambda}, \frac{|1(r)| + |T_1(r)|}{\lambda} e^{-r\lambda}, \frac{|\mathcal{J}(r)| + |T\mathcal{J}(r)|}{\lambda} e^{-r\lambda} \right\} dr \\
&\leq \lambda^2 e^{-\lambda} \int_0^t e^{r\lambda} \max\{m_{r\lambda}(1, \mathcal{J}), m_{r\lambda}(1, T_1), m_{r\lambda}(\mathcal{J}, T\mathcal{J})\} dr \\
&= \lambda^2 e^{-\lambda} N(1, \mathcal{J}) \int_0^t e^{r\lambda} dr \\
&= \lambda^2 e^{-\lambda} N(1, \mathcal{J}) \frac{1}{\lambda} (e^{\lambda t} - 1) \\
&= \lambda e^{-\lambda} N(1, \mathcal{J}) e^{\lambda t} - \lambda e^{-\lambda} N(1, \mathcal{J}) \\
&\leq \lambda e^{-\lambda} N(1, \mathcal{J}) e^{\lambda t}.
\end{aligned} \tag{73}$$

Consequently, we have

$$\frac{|T_1(r)| + |T\mathcal{J}(r)|}{\lambda} e^{-\lambda t} \leq e^{-\lambda} N(1, \mathcal{J}). \tag{74}$$

This implies that

$$m_{r\lambda}(T_1, T\mathcal{J}) = \max_{t \in [0, D]} \frac{|T_1(r)| + |T\mathcal{J}(r)|}{\lambda} e^{-\lambda t} \leq e^{-\lambda} N(1, \mathcal{J}). \tag{75}$$

By Hypothesis (iv), we obtain

$$e^{\sqrt{m_{r\lambda}(T_1, T\mathcal{J})}} \leq e^{\sqrt{e^{-\lambda} N(1, \mathcal{J})}} = \left(e^{\sqrt{N(1, \mathcal{J})}} \right)^{e^{-(\lambda/2)}}, \tag{76}$$

yielding

$$\theta(m_{r\lambda}(T_1, T\mathcal{J})) \leq [\theta(N(1, \mathcal{J}))]^k, \quad \forall 1, \mathcal{J} \in \mathbb{N}, \tag{77}$$

where $k = e^{-(\lambda/2)} \in (0, 1)$ with $\lambda = 2$. Thus, all conditions of Theorem 4 are fulfilled. So T has a unique fixed point which is a unique solution to NIE (68). \square

5. Conclusions

In this paper, we established fixed points for θ -contraction self-mapping in RMMS. Our new results are extensions of recent fixed-point theorems of Jalili et al. [11]. Also, we gave an example to clarify the obtained results. Finally, we applied our main result to study the existence and uniqueness of a solution for a NIE. The new concepts lead to further investigations and applications.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interests regarding the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

Acknowledgments

The authors extend their appreciation to the Deanship of Post Graduate and Scientific Research at Dar Al Uloom University for funding this work.

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Research Article

Interpolative Kannan Contractions in T_0 -Quasi-Metric Spaces

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Received 24 December 2020; Revised 15 January 2021; Accepted 18 January 2021; Published 30 January 2021

Academic Editor: Zoran Mitrovic

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In this paper, we update the well-known fixed point theorem of Kannan using the interpolation notion in the realm of quasi-metric spaces. We consider some asymmetric versions. We also present some illustrative examples in support of the obtained results.

1. Introduction and Preliminaries

In 2018, Karapinar [1] published a new type of contractions obtained from the definition of the Kannan contraction by interpolation as follows.

Theorem 1 (see [11, Theorem 2.2]). *Let (X, d) be a complete metric space and $T: X \rightarrow X$ be an interpolative Kannan type contraction, i.e., a self-map such that there exist $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ so that*

$$d(Tx, Ty) \leq \lambda d(x, Tx)^\alpha d(y, Ty)^{1-\alpha}, \quad (1)$$

for all $x, y \in X$ with $x \neq Tx$. Then, T has a unique fixed point in X .

This theorem has been generalized in 2019 by Gaba et al. [2] where they introduced the concept of (λ, α, β) -interpolative Kannan contractions as follows.

Definition 1. Let (X, d) be a metric space and $T: X \rightarrow X$ be a self-map. We shall call T a relaxed (λ, α, β) -interpolative Kannan contraction, if there exist $0 \leq \lambda < 1$, $0 < \alpha, \beta \leq 1$ with $\alpha + \beta < 1$ such that

$$d(Tx, Ty) \leq \lambda d(x, Tx)^\alpha d(y, Ty)^\beta. \quad (2)$$

The interpolative method has been successfully applied to a diverse range of contractions (see [3–8]).

In 1982, Reilly et al. [9] obtained a quasi-metric version of the celebrated Banach contraction principle. Since then, and especially in the last decade, several authors have contributed to the development of the fixed point theory in the framework of quasi-metric spaces (see [10, 11]). It is in the same spirit that we provide here a quasi-metric version of the interpolative Kannan fixed point theorem.

Our basic reference for quasi-metric spaces is [12].

Definition 2 (see [13]). Let $q: X \times X \rightarrow [0, \infty)$ be a function where X is a nonempty set. This function is called a **quasi-pseudometric** (respectively, **T_0 -quasi-metric**) on X if (q_1) and (q_2) (respectively, $(q_1)^*$ and (q_2)) hold, where

$$(q_1) \quad q(\xi, \xi) = 0, \text{ for all } \xi \in X$$

$$(q_1)^* \quad q(\xi, \eta) = 0 = q(\eta, \xi) \text{ implies } \xi = \eta$$

$$(q_2) \quad q(\xi, \zeta) \leq q(\xi, \eta) + q(\eta, \zeta) \text{ for all } \xi, \eta, \zeta \in X$$

The condition $(q_1)^*$ is known as the T_0 -condition. Furthermore, given a quasi-pseudometric q on X , there is a natural function $q^{-1}: X \times X \rightarrow [0, \infty)$, defined by $q^{-1}(\xi, \eta) = q(\eta, \xi)$ for all $\xi, \eta \in X$. It is named as the

conjugate of q . For a T_0 -quasi-metric q on X , the distance function $q^s: X \times X \longrightarrow [0, \infty)$ defined by $q^s(\xi, \eta) = \max\{q(\xi, \eta), q(\eta, \xi)\}$ for all $(\xi, \eta) \in X \times X$ is a metric on X .

The classical example of a T_0 -quasi-metric is the **truncated difference**.

Example 1 (truncated difference). Set $\mathbb{R}_0^+ = [0, \infty)$. Given $\delta: \mathbb{R}_0^+ \times \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ as

$$\delta(\xi, \eta) = \max\{0, \xi - \eta\} \quad \xi, \eta \in X. \quad (3)$$

Under these conditions, δ forms a T_0 -quasi-metric. Further, the pair (\mathbb{R}_0^+, δ) becomes a T_0 -quasi-metric space.

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(\xi, \eta): x \in X, \varepsilon > 0\}$ where $B_d(\xi, \eta) = \{\xi \in X: d(\xi, \eta) < \varepsilon\}$.

If τ_d satisfies the separation axiom T_1 (resp. T_2) on X , we say that (X, d) is a T_1 (resp. a Hausdorff) quasi-metric space. Note that a T_0 -quasi-metric space (X, d) is T_1 if and only if for each $x, y \in X$, the condition $d(x, y) = 0$ implies $x = y$.

A quasi-metric space (X, d) is called bicomplete if the metric space (X, d^s) is complete.

Definition 3 (convergence in quasi-pseudometric spaces, see [12]).

Let (X, q) be a quasi-pseudometric space. We say that the sequence $\{\xi_n\}$ **q -converges** to ξ , or **left-converges** to ξ , if

$$q(\xi_n, \xi) \longrightarrow 0. \quad (4)$$

We denote this by $\xi_n \longrightarrow^d \xi$. More precisely, $\{\xi_n\}$ converges to ξ with respect to $\tau(q)$.

In a similar manner, a sequence $\{\xi_n\}$ **q^{-1} -converges** to ξ or **right-converges** to ξ with respect to $\tau(q^{-1})$, if

$$q(\xi, \xi_n) \longrightarrow 0. \quad (5)$$

We denote this by $\xi_n \longrightarrow^{q^{-1}} \xi$. A sequence $\{\xi_n\}$ in the setting of a quasi-pseudometric space (X, q) **q^s -converges** to ξ in the case that the sequence converges to ξ from left and right, that is,

$$\xi_n \xrightarrow{q} \xi, \xi_n \xrightarrow{q^{-1}} \xi. \quad (6)$$

Moreover, it is denoted as $\xi_n \xrightarrow{q^s} \xi$ (or, $\xi_n \longrightarrow \xi$, if there is no confusion).

Remark 1. From the definition of q^s -convergence, we have

$$q^s\text{-convergence implies } q\text{-convergence.} \quad (7)$$

As demonstrated in [7, Example 1.7.], the reverse implication does not hold in general.

Definition 4 (compare [13]).

(1) A sequence $\{\xi_n\}$ in a quasi-pseudometric (X, q) is called **left K -Cauchy** if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\text{for all } n, k: \quad n_0 \leq k \leq n, q(\xi_k, \xi_n) < \varepsilon. \quad (8)$$

(2) Similarly, we define **right K -Cauchy**. A sequence $\{\xi_n\}$ in a quasi-metric space (X, q) is called right K -Cauchy if it is a left K -Cauchy sequence in the quasi-metric space (X, q^{-1}) .

(3) The quasi-metric space (X, d) is called left (right) K -sequentially complete if every left (right) K -Cauchy sequence converges with respect to the topology τ_d .

(4) (X, q) is called q -sequentially complete if every Cauchy sequence in the metric space (X, q^s) converges with respect to the topology τ_d .

Remark 2. One can easily be convinced that both bicompleteness and left (right) K -sequential completeness imply q -sequential completeness. However, the rest of implications does not hold in general. For more details, the reader can consult [12].

2. Revisiting the Interpolative Kannan Mappings

Let us recall that an interpolative Kannan contraction on a metric space (X, m) is a self-mapping $T: X \longrightarrow X$ such that there exist $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ for which

$$m(Tx, Ty) \leq \lambda m(x, Tx)^\alpha m(y, Ty)^{1-\alpha}, \quad (9)$$

for all $x, y \in X$ with $x \neq Tx$.

Before proceeding to the main results of this paper, we would like to correct an inaccuracy that appears in the proof of Theorem 2.2 in the paper by Karapinar [1].

Theorem 2. (see [11, Theorem 2.2]).

Let (X, d) be a complete metric space and $T: X \longrightarrow X$ be an interpolative Kannan type contraction, i.e., a self-map such that there exist $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ so that

$$d(Tx, Ty) \leq \lambda d(x, Tx)^\alpha d(y, Ty)^{1-\alpha}, \quad (10)$$

for all $x, y \in X$ with $x \neq Tx$. Then, T has a unique fixed point in X .

Proof. Following the proof presented by Karapinar, let $x_0 \in X$, and construct the sequence $\{x_n\}$ by $x_{n+1} = T^n(x_0)$ for all positive integers n . Taking $x = x_n$ and $y = x_{n-1}$ in (10), we derive that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \leq \lambda [d(x_n, Tx_n)]^\alpha [d(x_{n-1}, Tx_{n-1})]^{1-\alpha} \\ &= \lambda [d(x_n, x_{n+1})]^\alpha [d(x_{n-1}, x_n)]^{1-\alpha}, \end{aligned} \quad (11)$$

which yields

$$[d(x_{n+1}, x_n)]^{1-\alpha} \leq \lambda [d(x_{n-1}, x_n)]^{1-\alpha}. \quad (12)$$

It is not difficult to see that inequality (10) is asymmetric, and to fill in this gap, we also consider the case of $x = x_{n-1}$ and $y = x_n$ in (10). So taking $x = x_{n-1}$ and $y = x_n$ in (10), we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq \lambda [d(x_{n-1}, Tx_{n-1})]^\alpha \\ &\quad [d(x_n, Tx_n)]^{1-\alpha} \\ &= \lambda [d(x_{n-1}, x_n)]^\alpha [d(x_n, x_{n+1})]^{1-\alpha}, \end{aligned} \quad (13)$$

which yields

$$[d(x_n, x_{n+1})]^\alpha \leq \lambda [d(x_{n-1}, x_n)]^\alpha. \quad (14)$$

Since $0 < \alpha < 1$, it is obvious that $0 < 1 - \alpha < 1$; hence, it is routine to check that (x_n) is a Cauchy sequence which converges to the unique fixed point x^* of T . \square

This gap was revealed in Example 2.3 of [1]. Indeed, according to [11, Example 2.3], let $X = \{x, y, z, w\}$ be a set endowed with a metric m such that $m(x, x) = m(y, y) = m(z, z) = m(w, w) = 0$, $m(x, y) = m(y, x) = 3$, $m(x, z) = m(z, x) = 4$, $m(y, z) = m(z, y) = 3/2$, $m(w, x) = m(x, w) = 5/2$, $m(w, y) = m(y, w) = 2$, $m(w, z) = m(z, w) = 3/2$. Define the self-mapping T on X by $Tx = x$, $Ty = w$, $Tz = x$ and $Tw = y$. The author claimed that for $\alpha = 1/8$ and $\lambda = 9/10$, the self-mapping T forms an interpolative Kannan type contraction. However, considering the pair (z, y) , note that $Tz \neq y$. We have

$$\begin{aligned} 2.5 &= m(Tz, Ty) = m(x, w) \\ &= \frac{5}{2} > \frac{9}{10} [m(z, Tz)]^{1/8} [m(y, Ty)]^{7/8} \\ &= (4)^{1/8} (2)^{7/8} \approx 1.962913. \end{aligned} \quad (15)$$

That is, inequality (10) fails for $m(Tz, Ty)$.

However, considering the pair (y, z) , note that $Ty \neq y$. We have

$$\begin{aligned} 2.5 &= m(Ty, Tz) = m(w, x) \\ &= \frac{5}{2} \leq \frac{9}{10} [m(y, Ty)]^{1/8} [m(z, Tz)]^{7/8} \\ &= \frac{9}{10} (2)^{1/8} (4)^{7/8} \approx 3.301214. \end{aligned} \quad (16)$$

This suggests a modification of the so-called **interpolative Kannan contraction** in the following way.

Definition 5. Let (X, m) be a metric space. A self-mapping $T: X \rightarrow X$ is called a generalized interpolative Kannan contraction if there exist $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ for which

$$\begin{aligned} m(Tx, Ty) &\leq \lambda \max\{m(x, Tx)^\alpha m(y, Ty)^{1-\alpha}, \\ &\quad m(x, Tx)^{1-\alpha} m(y, Ty)^\alpha\}, \end{aligned} \quad (17)$$

for all $x, y \in X$ with $x \neq Tx$ and $y \neq Ty$.

Remark 3. Observe then that for $\alpha = 1/2$ in (17), a generalized interpolative Kannan contraction is actually just an interpolative Kannan contraction in the sense and spirit of Karapinar.

Using this definition, we get the following.

Theorem 3 (compare [11, Theorem 2.2]). *Let (X, d) be a complete metric space and $T: X \rightarrow X$ be a generalized interpolative Kannan contraction. Then, T has a unique fixed point in X .*

Proof. The proof is merely a copy of that of [11, Theorem 2.2.] and needs not be repeated. One just has to observe that the right hand side of inequality (17) is symmetric in $x, y \in X$. This has actually been captured in the proof of Theorem 2. Moreover, [11, Example 2.3.] satisfies the hypothesis of Theorem 3. \square

3. Interpolative Kannan Mappings and Fixed Points in Quasi-Metric Spaces

The discussion we made in the previous section suggests, in a natural way, the following notions.

Definition 6. Let (X, d) be a quasi-metric space.

A d -interpolative Kannan contraction on (X, d) is a mapping $T: X \rightarrow X$ such that there exist $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ for which

$$d(Tx, Ty) \leq \lambda d(Tx, x)^\alpha d(y, Ty)^{1-\alpha}, \quad (18)$$

for all $x, y \in X$ with $x \neq Tx$ and $y \neq Ty$.

A d^{-1} -interpolative Kannan contraction on (X, d) is a mapping $T: X \rightarrow X$ such that there exist $\lambda \in [0, 1)$, $\alpha \in (0, 1)$ for which

$$d(Tx, Ty) \leq \lambda d(x, Tx)^\alpha d(Ty, y)^{1-\alpha}, \quad (19)$$

for all $x, y \in X$ with $x \neq Tx$, $y \neq Ty$.

Then, we have the following easy, but useful, consequence of the interpolative Kannan contraction for metric spaces.

Proposition 1. *Let (X, d) be a T_0 -quasi-metric space. If T is a d -interpolative Kannan contraction or a d^{-1} -interpolative Kannan contraction on (X, d) , then the following conditions hold:*

- (1) *T is a generalized interpolative Kannan contraction on the metric space (X, d^s) .*
- (2) *For any $x_0 \in X$, $(T_n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s) .*

Proof. For (1), suppose that T is a d -interpolative Kannan contraction on (X, d) . So there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ for which

$$\begin{aligned} d(Tx, Ty) &\leq \lambda d(Tx, x)^\alpha d(y, Ty)^{1-\alpha} \\ &\leq \lambda [d^s(y, Ty)]^\alpha [d^s(Tx, x)]^{1-\alpha}, \end{aligned} \quad (20)$$

for all $x, y \in X$ with $x \neq Tx$ and $y \neq Ty$. Similarly, given $x, y \in X$, we also have

$$\begin{aligned} d(Ty, Tx) &\leq \lambda d(Ty, y)^\alpha d(x, Tx)^{1-\alpha} \\ &\leq \lambda [d^s(x, Tx)]^\alpha [d^s(Ty, y)]^{1-\alpha}, \end{aligned} \quad (21)$$

for all $x, y \in X$ with $x \neq Tx$ and $y \neq Ty$.

Thus, given $x, y \in X$,

$$\begin{aligned} d^s(Tx, Ty) &\leq \lambda \max\{[d^s(y, Ty)]^\alpha [d^s(Tx, x)]^{1-\alpha}, \\ &\quad [d^s(y, Ty)]^\alpha [d^s(Tx, x)]^{1-\alpha}\}. \end{aligned} \quad (22)$$

Then, T is a generalized interpolative Kannan contraction on the metric space (X, d^s) .

(2) Since, by (1), T is a generalized interpolative Kannan contraction on the metric space (X, d^s) , it follows from the modified proof of classical interpolative Kannan contraction principle that $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s) . \square

By using the preceding proposition, three quasi-metric versions of the generalized interpolative Kannan principle are easily deduced.

Theorem 4. *Every d -(resp. every d^{-1} -) interpolative Kannan contraction on a bicomplete T_0 -quasi-metric space on (X, d) has a unique fixed point.*

Proof. Let T be a d or a d^{-1} interpolative Kannan contraction. Since (X, d^s) is a complete metric space and, by Proposition 1 (1), T is a generalized interpolative Kannan on (X, d^s) , we deduce a modified proof of classical interpolative Kannan contraction principle that T has a unique fixed point. \square

Theorem 5. *Every d -interpolative Kannan contraction on a Hausdorff d -sequentially complete T_0 -quasi-metric space (X, d) has a unique fixed point.*

Proof. Let T be a d -interpolative Kannan contraction on the Hausdorff d -sequentially complete T_0 -quasi-metric space (X, d) . Fix an $x_0 \in X$. By Proposition 1 (2), the sequence $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s) . Hence, there is $y \in X$ such that $(T^n x_0)_{n \in \mathbb{N}}$ converges to y with respect to τ_d , i.e., $d(y, T^n x_0) \rightarrow 0$ as $n \rightarrow \infty$. Since T is a d -interpolative Kannan contraction, there exist $\lambda \in [0, 1), \alpha \in (0, 1)$ for which

$$d(T^{n+1} x_0, Ty) \leq \lambda d(T^{n+1} x_0, T^n x_0)^\alpha d(y, Ty)^{1-\alpha}. \quad (23)$$

Consequently, $d(T^{n+1} x_0, Ty) \rightarrow 0$ as $n \rightarrow \infty$. From Hausdorffness of (X, d) , we deduce that $y = Ty$. Finally, suppose that $z \in X$ is a fixed point of T . From (10), we have

$$\begin{aligned} d(y, z) &= d(Ty, Tz) \leq \lambda [d(Ty, y)]^\alpha [d(z, Tz)]^{1-\alpha} \Rightarrow d(y, z) = 0, \\ d(z, y) &= d(Tz, Ty) \leq \lambda [d(Tz, z)]^\alpha [d(y, Ty)]^{1-\alpha} \Rightarrow d(z, y) = 0, \end{aligned} \quad (24)$$

and thus $y = z$ since the space (X, d) is T_0 . This concludes the proof. \square

Corollary 1. *Every d -interpolative Kannan contraction on a T_0 -quasi-metric space (X, d) such that (X, d^{-1}) is Hausdorff and d^{-1} -sequentially complete has a unique fixed point.*

Proof. Let T be a d -interpolative Kannan contraction on (X, d) . Put $q = d^{-1}$; then, T is a q -interpolative Kannan contraction on a Hausdorff q -sequentially complete T_0 -quasi-metric space (X, q) . From Theorem 5, we deduce that T has a unique fixed point. \square

Theorem 6. *Every d^{-1} -interpolative Kannan contraction on a T_1 -sequentially complete T_0 -quasi-metric space (X, d) has a unique fixed point.*

Proof. Let T be a d^{-1} -interpolative Kannan contraction on the T_1 -sequentially complete T_0 -quasi-metric space (X, d) . Fix $x_0 \in X$. As in the proof of Theorem 5 (see Proposition 1), $(T^n x_0)_{n \in \mathbb{N}}$ is a Cauchy sequence in the metric space (X, d^s) . Hence, there is $y \in X$ such that $(T^n x_0)_{n \in \mathbb{N}}$ converges to y with respect to τ_d , i.e., $d(y, T^n x_0) \rightarrow 0$ as $n \rightarrow \infty$. Since T is a d^{-1} -interpolative Kannan contraction, there exist $\lambda \in [0, 1), \alpha \in (0, 1)$ for which

$$d(T^{n+1} x_0, Ty) \leq \lambda d(T^n x_0, T^{n+1} x_0)^\alpha d(Ty, y)^{1-\alpha}, \quad (25)$$

for all $n \in \mathbb{N}$. Consequently, $d(T^{n+1} x_0, Ty) \rightarrow 0$ as $n \rightarrow \infty$. From the triangle inequality,

$$d(y, Ty) \leq d(y, T^n x_0) + d(T^n x_0, Ty). \quad (26)$$

We deduce $d(y, Ty) = 0$. Therefore, $y = Ty$ because (X, d) is a T_0 -quasi-metric space, which is T_1 . Finally, suppose that $z \in X$ is a fixed point of T . Then,

$$d(y, z) = d(Ty, Tz) \leq \lambda d(y, Ty)^\alpha d(Tz, z)^{1-\alpha}, \quad (27)$$

and thus $y = z$. This concludes the proof. \square

Corollary 2. *Every d -contraction on a T_0 -quasi-metric space (X, d) which is T_1 such that (X, d^{-1}) is d^{-1} -sequentially complete has a unique fixed point.*

Proof. Let T be a d -interpolative Kannan contraction on (X, d) . Put $q = d^{-1}$. Then, T is a q^{-1} -interpolative Kannan contraction on the T_1 -sequentially complete T_0 -quasi-metric space (X, q) . From Theorem 6, we deduce that T has a unique fixed point. \square

Remark 4. Note that Theorems 5 and 6 remain valid if “ d -sequentially complete” is replaced with “left K -sequentially complete” or “right K -sequentially complete.”

We conclude the paper with an example which shows that Theorem 5 cannot be generalized to T_1d -sequentially complete T_0 -quasi-metric spaces.

Example 2. Let $X = \mathbb{N}$ and let d be the T_1T_0 -quasi-metric on X given by $d(n, n) = 0$ for all $n \in X$, and $d(n, m) = 1/m$ otherwise. Clearly, (X, d) is both left and right K -sequentially complete, so it is d -sequentially complete. Now, define $T: X \rightarrow X$ as $Tn = 2n$ for all $n \in X$. Of course, T has no fixed point. However, it is a d -interpolative Kannan contraction with $\lambda = 3/4$ and $\alpha = 1/2$ since for each $n, m \in X$ with $n \neq m$, one has

$$\begin{aligned} d(Tn, Tm) &= d(2n, 2m) = \frac{1}{2m} \leq \frac{3}{4} \sqrt{n} \sqrt{2m} \\ &= \frac{3}{4} d(Tn, n)^{1/2} d(m, Tm)^{1/2}. \end{aligned} \quad (28)$$

In light of the above example, a question is in order.

Open Question 1. Can Theorem 6 be generalized to d -sequentially complete T_0 -quasi-metric spaces? It is our belief that the answer is “No”; however, so far, we have failed to provide a counter-example.

4. Conclusion

As mentioned in the introduction, the interesting concept of (λ, α, β) -interpolative Kannan contractions was introduced in [2]. So, the authors plan, in a different manuscript, to discuss asymmetric versions of such a concept by first outlining the asymmetric nature of this notion, even for a classical interpolative Kannan contraction in a metric space.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Disclosure

Our proofs are inspired by the recent work of Dağ et al. [14]. The presentation of the manuscript is as proceedings.

Conflicts of Interest

The authors declare that they have no conflicts of interest concerning the publication of this article.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

Acknowledgments

The first author (Y.U.G.) wishes to acknowledge that this work was carried out with the aid of a grant from the Carnegie Corporation provided through the African Institute for Mathematical Sciences.

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Research Article

Remarks on (α, β) -Admissible Mappings and Fixed Points under \mathcal{L} -Contraction Mappings

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Received 24 December 2020; Revised 13 January 2021; Accepted 15 January 2021; Published 28 January 2021

Academic Editor: Naeem Saleem

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In this paper, we discuss about different types of (α, β) -admissible mappings and introduce some new (α, β) -contraction-type mappings under simulation function. Furthermore, we present the definition of S-metric-like space and its topological properties. Some fixed point theorems in this space are established, proved, and verified with examples.

1. Introduction and Preliminaries

Banach contraction principle is considered as one of the most important tools for examining the existence and nonexistence of fixed point. Due to its simplicity and applicability, it is generalised in different directions.

Samet et al. [1] introduced the concept of α -admissible and $\alpha - \psi$ contraction to generalise the Banach contraction principle. These concepts were further generalised by many researchers to $\alpha - \beta$ admissible [2], β -admissible [3, 4], α -admissible in S-metric space [5], (α, β) -admissible [6, 7], and γ -admissible [8]. Some results on different types on contractive mappings can be seen in [9–11].

Different from the (α, β) -admissible mapping introduced by Alizadeh et al. [6], Chandok [7] introduced a new type of (α, β) -admissible mapping to obtain some fixed point results.

On the contrary, Khojasteh et al. [12] introduced simulation function and proved several fixed point theorems. Argoubi et al. [13] made some modifications to the definition of Khojasteh et al. [12]. It can be noted that the definition of Khojasteh et al. [12] implies the definition of Argoubi et al. [13], but the converse is not true.

Various generalizations of metric space are found in the literature. The concept of S-metric space [14] is one of them. More results on S-metric space can be found in [15, 16].

Hitzler [17] also introduced the concept of dislocated metric by generalizing metric space. Amini-Harandi [18] rediscovered dislocated metric space under the new name “metric-like.”

Mehravarani et al. [19] introduced the concept of dislocated S_b -metric space and dislocated S-metric space as a particular case of dislocated S_b -metric space when the parameter $b = 1$, but the topological properties of neither dislocated S_b -metric space nor dislocated S-metric space were given in [19].

In order to fill up the missing gap in [19], in our present study, first of all, we present the definition of dislocated S-metric space and its topological properties. For our convenience, it will be known as S-metric-like space in place of dislocated S-metric space.

Definition 1 (see [14]). In a nonempty set Ω , the function $S: \Omega^3 \rightarrow [0, +\infty)$ is said to be S-metric if it satisfies the following:

- (1) $S(\theta, \mu, \xi) \geq 0$,
- (2) $S(\theta, \mu, \xi) = 0$ if and only if $\theta = \mu = \xi$,
- (3) $S(\theta, \mu, \xi) \leq S(\theta, \sigma, \sigma) + S(\mu, \sigma, \sigma) + S(\xi, \sigma, \sigma)$,

for all $\theta, \mu, \xi, \sigma \in \Omega$. (Ω, S) is known as S-metric space.

Definition 2 (see [14]). In a S -metric space, we have $S(\theta, \theta, \mu) = S(\mu, \mu, \theta)$.

Definition 3 (see [14]). In a nonempty set Ω , the function $S: \Omega^3 \rightarrow [0, +\infty)$ is said to be S -metric-like if it satisfies the following:

- (1) $S(\theta, \mu, \xi) = 0$ implies $\theta = \mu = \xi$,
- (2) $S(\theta, \mu, \xi) \leq S(\theta, \sigma, \sigma) + S(\mu, \sigma, \sigma) + S(\xi, \sigma, \sigma)$,

for all $\theta, \mu, \xi, \sigma \in \Omega$. (Ω, S) is known as S -metric-like space.

Remark 1. It is true that every S -metric space is a S -metric-like space, but every S -metric-like space may not be a S -metric space.

Example 1. Let $\Omega = \{0, 2\}$ and

$$S(\theta, \mu, \xi) = \begin{cases} 3, & \theta = \mu = \xi, \\ 1, & \text{otherwise.} \end{cases} \quad (1)$$

Here, (Ω, S) is S -metric-like space. But $S(0, 0, 0) = 3 \neq 0$ and hence not S -metric space.

We discuss some topological properties of S -metric-like space. Let τ_s be a topology generated on S -metric-like space (Ω, S) with base as the family of open S -balls:

$$B_S(\theta, \varepsilon) = \{\mu \in \Omega : |S(\theta, \theta, \mu) - S(\theta, \theta, \theta)| < \varepsilon\}, \quad (2)$$

for all $\theta \in \Omega$ and $\varepsilon > 0$.

Mapping $R: \Omega \rightarrow \Omega$ in a S -metric-like (Ω, S) is said to be continuous at $\theta \in \Omega$ if there exists $\delta > 0$ for all $\varepsilon > 0$ satisfying $R(B_S(\theta, \delta)) \subseteq B_S(R\theta, \varepsilon)$.

If $R: \Omega \rightarrow \Omega$ is continuous, then $\lim_{n \rightarrow +\infty} \theta_n = \theta$ implies $\lim_{n \rightarrow +\infty} R\theta_n = R\theta$.

A sequence $\{\theta_n\}$ in Ω is said to be Cauchy if $\lim_{n, m \rightarrow +\infty} S(\theta_n, \theta_n, \theta_m)$ exists and is finite.

The S -metric-like (Ω, S) is said to be complete if every Cauchy sequence $\{\theta_n\}$ in Ω satisfies

$$\lim_{n \rightarrow +\infty} S(\theta_n, \theta_n, \theta) = S(\theta, \theta, \theta) = \lim_{n, m \rightarrow +\infty} S(\theta_n, \theta_n, \theta_m), \quad (3)$$

for some $\theta \in \Omega$.

A subset $P \subseteq \Omega$ is said to be bounded if there is a point $\xi \in \Omega$ satisfying

$$S(\theta, \mu, \xi) \leq L, \quad (4)$$

for all $\theta, \mu \in P$ and L is a positive constant.

Definition 4 (see [12]). A function $\zeta: [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$ is said to be a simulation function if ζ satisfies the following:

- (1) $\zeta(0, 0) = 0$.
- (2) $\zeta(v, \mu) < \mu - v$ for all $v, \mu > 0$.
- (3) If $\{v_n\}$ and $\{\mu_n\}$ are sequences in $(0, +\infty)$ such that $\lim_{n \rightarrow +\infty} v_n = \lim_{n \rightarrow +\infty} \mu_n = l \in (0, +\infty)$, then

$$\lim_{n \rightarrow +\infty} \sup \zeta(v_n, \mu_n) < 0. \quad (5)$$

Simulation function is modified by Argoubi et al. [13] as follows.

Definition 5 (see [13]). A mapping $\zeta: [0, +\infty) \times [0, +\infty) \rightarrow \mathbb{R}$ is said to be a simulation function if satisfying the conditions (2) and (3) of Definition 4.

Now, we present different forms of (α, β) -admissible mappings.

Alizadeh et al. [6] defined a type of (α, β) -admissible mappings by extending definition given by Salimi et al. [20] which they called as cyclic (α, β) -admissible mapping.

Definition 6 (see [6]). Let $R: \Omega \rightarrow \Omega$ be a mapping and $\alpha, \beta: \Omega \rightarrow \mathbb{R}^+$ be two functions. R is said to be a cyclic (α, β) -admissible mapping if the following holds:

- (i) For some $\theta \in \Omega$, $\alpha(\theta) \geq 1$ induces $\beta(R\theta) \geq 1$.
- (ii) For some $\theta \in \Omega$, $\beta(\theta) \geq 1$ induces $\alpha(R\theta) \geq 1$.

The following definition was given by Chandok [7].

Definition 7 (see [7]). Let $R: \Omega \rightarrow \Omega$ be a mapping in a nonempty set Ω and $\alpha, \beta: \Omega \times \Omega \rightarrow \mathbb{R}^+$. R is said to be (α, β) -admissible if $\alpha(\theta, \mu) \geq 1$ and $\beta(\theta, \mu) \geq 1$ implies $\alpha(R\theta, R\mu) \geq 1$ and $\beta(R\theta, R\mu) \geq 1$ for all $\theta, \mu \in \Omega$.

Shatanawi [2] also introduced a type of (α, β) -admissible mapping which he named as (α, β) -admissibility and defined as follows.

Definition 8 (see [2]). Let $S, T: \Omega \rightarrow \Omega$ be mappings in a nonempty set Ω and $\alpha, \beta: \Omega \times \Omega \rightarrow \mathbb{R}^+ \cup \{0\}$ be functions. Then, (R, S) is said to be a pair of (α, β) -admissibility if $\theta, \mu \in \Omega$ and $\alpha(\theta, \mu) \geq \beta(\theta, \mu)$ implies $\alpha(R\theta, S\mu) \geq \beta(R\theta, S\mu)$ and $\alpha(S\theta, R\mu) \geq \beta(S\theta, R\mu)$.

For our study, we will consider the definition given by Chandok [7]. This definition will be extended in the framework of S -metric-like space to define some new contractive types under simulation function defined by Khojasteh et al. [12].

Following is the extension of the property of metric-like space to S -metric-like space.

2. Main Results

We start our result with the following definitions.

Definition 9. Let R be a self-mapping on a nonempty set Ω and $\alpha, \beta: \Omega \times \Omega \times \Omega \rightarrow \mathbb{R}^+$. Then, R is said to be $(\alpha, \beta)_s$ -admissible if $\alpha(\theta, \mu, \xi) \geq 1$ and $\beta(\theta, \mu, \xi) \geq 1$ imply that $\alpha(R\theta, R\mu, R\xi) \geq 1$ and $\beta(R\theta, R\mu, R\xi) \geq 1$ for all $\theta, \mu, \xi \in \Omega$.

Definition 10. Let R be a self-mapping on a nonempty set Ω and $\alpha, \beta: \Omega \times \Omega \times \Omega \rightarrow \mathbb{R}^+$. Then, R is said to be triangular $(\alpha, \beta)_s$ -admissible mapping if it is $(\alpha, \beta)_s$ -admissible and $\alpha(\theta, \theta, t) \geq 1$, $\beta(\theta, \theta, t) \geq 1$, $\alpha(\mu, \mu, t) \geq 1$, $\beta(\mu, \mu, t) \geq 1$, and $\alpha(\xi, \xi, t) \geq 1$, $\beta(\xi, \xi, t) \geq 1$ imply $\alpha(\theta, \mu, \xi) \geq 1$, $\beta(\theta, \mu, \xi) \geq 1$.

Definition 11. Let R be a self-mapping on a S -metric-like space (Ω, S) and $\alpha, \beta: \Omega \times \Omega \times \Omega \rightarrow \mathbb{R}^+$. R is said to be an

$(\alpha, \beta)_s$ -admissible \mathcal{X} -contraction of type I with respect to ζ if

$$\zeta(\alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)S(R\theta, R\mu, R\xi), S(\theta, \mu, \xi)) \geq 0, \quad (6)$$

for all $\theta, \mu, \xi \in \Omega$.

Definition 12. Let R be a self-mapping on a S -metric-like space (Ω, S) and $\alpha, \beta: \Omega \times \Omega \times \Omega \longrightarrow \mathbb{R}^+$. R is said to be an $(\alpha, \beta)_s$ -admissible \mathcal{X} -contraction of type II with respect to ζ if

$$\zeta(\alpha(\theta, \theta, \mu)\beta(\theta, \theta, \mu)S(R\theta, R\theta, R\mu), S(\theta, \theta, \mu)) \geq 0, \quad (7)$$

for all $\theta, \mu \in \Omega$.

Lemma 1. Let (Ω, S) be a S -metric-like space and $\{\theta_n\}$ be a sequence in Ω such that $\theta_n \longrightarrow \theta$ with $S(\theta, \theta, \theta) = 0$. Then, for all $\mu \in \Omega$, we have $\lim_{n \rightarrow +\infty} S(\theta_n, \theta_n, \mu) = S(\theta, \theta, \mu)$.

Proof. From (2) of Definition 3, we have

$$|S(\theta_n, \theta_n, \mu) - S(\theta, \theta, \mu)| \leq 2S(\theta_n, \theta_n, \theta). \quad (8)$$

Taking limit as $n \longrightarrow +\infty$, we have

$$\lim_{n \rightarrow +\infty} S(\theta_n, \theta_n, \mu) = S(\theta, \theta, \mu). \quad (9)$$

Now, we prove the following theorem. \square

Theorem 1. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.
- (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.
- (c) R is $(\alpha, \beta)_s$ -admissible \mathcal{X} -contraction of type I on (Ω, S) .
- (d) R is S -continuous.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. By (b), let $\theta_0 \in \Omega$ with $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$. We construct the sequence $\{\theta_n\}$ by $\theta_{n+1} = R\theta_n$ for all $n \in \mathbb{N} \cup \{0\}$. For some n , if $\theta_n = \theta_{n+1}$, then we have $\theta_n = R\theta_n$. This gives that θ_n is a fixed point of R . In this case, proof is completed. Now, let $\theta_n \neq \theta_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By (a), we have

$$\begin{aligned} \alpha(\theta_0, \theta_0, R\theta_0) &= \alpha(\theta_0, \theta_0, \theta_1) \geq 1 \Rightarrow \alpha(R\theta_0, R\theta_0, R\theta_1) \\ &= \alpha(\theta_1, \theta_1, \theta_2) \geq 1. \end{aligned} \quad (10)$$

Deductively, we have

$$\alpha(\theta_n, \theta_n, \theta_{n+1}) \geq 1, \quad \text{for all } n \geq 0, \quad (11)$$

$$\beta(\theta_n, \theta_n, \theta_{n+1}) \geq 1, \quad \text{for all } n \geq 0. \quad (12)$$

By (6), (11), and (12), we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(R\theta_n, R\theta_n, R\theta_{n+1}), S(\theta_n, \theta_n, \theta_{n+1})) \\ &= \zeta(\alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}), S(\theta_n, \theta_n, \theta_{n+1})) \\ &< S(\theta_n, \theta_n, \theta_{n+1}) - \alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}). \end{aligned} \quad (13)$$

Hence,

$$\begin{aligned} S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) &\leq \alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) \\ &< S(\theta_n, \theta_n, \theta_{n+1}), \end{aligned} \quad (14)$$

for all $n \geq 0$.

This shows that $\{S(\theta_n, \theta_n, \theta_{n+1})\}$ is a decreasing sequence, then we have

$$\lim_{n \rightarrow +\infty} S(\theta_n, \theta_n, \theta_{n+1}) = r, \quad r \geq 0. \quad (15)$$

We prove that

$$\lim_{n \rightarrow +\infty} S(\theta_n, \theta_n, \theta_{n+1}) = 0. \quad (16)$$

Let $r > 0$. From (14), we have

$$\lim_{n \rightarrow +\infty} \alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) = r. \quad (17)$$

Taking $v_n = \alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})$ and $\mu_n = S(\theta_n, \theta_n, \theta_{n+1})$ and using (3) of Definition 4, we have

$$\begin{aligned} 0 &\leq \lim_{n \rightarrow +\infty} \sup \zeta(\alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}), \\ &S(\theta_n, \theta_n, \theta_{n+1})) < 0, \end{aligned} \quad (18)$$

a contradiction. Hence, $r = 0$.

Next, we have to prove that $\{\theta_n\}$ is Cauchy. If possible, let $\{\theta_n\}$ is not Cauchy. Then, there exists $\varepsilon > 0$ for which $\{\theta_n\}$ has subsequences $\{\theta_{n_k}\}$ and $\{\theta_{m_k}\}$ with $n_k > m_k > k$ such that for every k ,

$$\begin{aligned} S(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}) &\geq \varepsilon, \\ S(\theta_{n_k-1}, \theta_{n_k-1}, \theta_{m_k}) &< \varepsilon. \end{aligned} \quad (19)$$

We have

$$\begin{aligned} \varepsilon &\leq S(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}) \leq 2S(\theta_{n_k}, \theta_{n_k}, \theta_{n_k-1}) + S(\theta_{n_k-1}, \theta_{n_k-1}, \theta_{m_k}) \\ &< 2S(\theta_{n_k}, \theta_{n_k}, \theta_{n_k-1}) + \varepsilon. \end{aligned} \quad (20)$$

Taking $k \longrightarrow +\infty$ and using (16),

$$\lim_{n \rightarrow +\infty} S(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}) = \varepsilon. \quad (21)$$

Also,

$$|S(\theta_{n_k+1}, \theta_{n_k+1}, \theta_{m_k}) - S(\theta_{n_k}, \theta_{n_k}, \theta_{m_k})| \leq 2S(\theta_{n_k}, \theta_{n_k}, \theta_{n_k+1}). \quad (22)$$

Taking limit as $k \longrightarrow +\infty$ and by (16) and (21),

$$\lim_{k \rightarrow +\infty} S(\theta_{n_k+1}, \theta_{n_k+1}, \theta_{m_k}) = \varepsilon. \quad (23)$$

Similarly,

$$\begin{aligned}\lim_{k \rightarrow +\infty} S(\theta_{m_k+1}, \theta_{m_k+1}, \theta_{n_k}) &= \varepsilon, \\ \lim_{k \rightarrow +\infty} S(\theta_{n_k+1}, \theta_{n_k+1}, \theta_{m_k+1}) &= \varepsilon.\end{aligned}\quad (24)$$

is a contradiction, and hence, $\{\theta_n\}$ is a Cauchy sequence. By completeness of S -metric-like space (Ω, S) , we know that there exist some $\xi \in \Omega$ such that

$$\lim_{n \rightarrow +\infty} S(\theta_n, \theta_n, \xi) = S(\xi, \xi, \xi) = \lim_{n, m \rightarrow +\infty} S(\theta_n, \theta_n, \theta_m) = 0, \quad (27)$$

and thus, $S(\xi, \xi, \xi) = 0$. Since R is S -continuous,

$$\begin{aligned}\lim_{n \rightarrow +\infty} S(\theta_{n+1}, \theta_{n+1}, R\xi) &= \lim_{n \rightarrow +\infty} S(R\theta_n, R\theta_n, R\xi) \\ &= S(R\xi, R\xi, R\xi) = 0.\end{aligned}\quad (28)$$

Using Lemma 1 and (27), we have

$$\lim_{n \rightarrow +\infty} S(\theta_{n+1}, \theta_{n+1}, R\xi) = S(\xi, \xi, R\xi). \quad (29)$$

Thus, $S(R\xi, R\xi, R\xi) = S(\xi, \xi, R\xi)$, that is, $R\xi = \xi$. For proving uniqueness of the fixed point, let $\eta \in \Omega$ such that $R\eta = \eta$ and $\eta \neq \xi$. Then,

$$\begin{aligned}0 &\leq \zeta(\alpha(\xi, \xi, \eta)\beta(\xi, \xi, \eta)S(R\xi, R\xi, R\eta), S(\xi, \xi, \eta)) \\ &< S(\xi, \xi, \eta) - \alpha(\xi, \xi, \eta)\beta(\xi, \xi, \eta)S(R\xi, R\xi, \eta) \leq 0,\end{aligned}\quad (30)$$

is a contradiction, so $\xi = \eta$.

Next, we remove the continuity condition. \square

Theorem 2. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.
- (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.
- (c) R is $(\alpha, \beta)_s$ -admissible \mathcal{X} -contraction of type I on (Ω, S) .
- (d) If $\{\theta_n\}$ is a sequence in Ω such that $\alpha(\theta_n, \theta_n, \theta_{n+1}) \geq 1$ and $\beta(\theta_n, \theta_n, \theta_{n+1}) \geq 1$ for all n and $\theta_n \longrightarrow \xi \in \Omega$ as $n \longrightarrow +\infty$, then there exists a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\alpha(\theta_{n_k}, \theta_{n_k}, \xi) \geq 1$ and $\beta(\theta_{n_k}, \theta_{n_k}, \xi) \geq 1$ for all $k \in \mathbb{N}$.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. Proceeding as of Theorem 1, let $\{\theta_n\}$ be a sequence in Ω given by $\theta_{n+1} = R\theta_n$ converges to some $\xi \in \Omega$ with $\alpha(\theta_n, \theta_n, \theta_{n+1}) \geq 1$ and $\beta(\theta_n, \theta_n, \theta_{n+1}) \geq 1$ for all n and $S(\xi, \xi, \xi) = 0$. From (d), there exists a subsequence $\{\theta_{n_k}\}$ of

By triangular $(\alpha, \beta)_s$ -admissibility of R ,

$$\alpha(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}) \geq 1 \text{ and } \beta(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}) \geq 1. \quad (25)$$

Since R is $(\alpha, \beta)_s$ -admissible \mathcal{X} -contraction of type I and using (21), (25), and (3) of Definition 4, we have

$$0 \leq \lim_{k \rightarrow +\infty} \sup \zeta(\alpha(\theta_{n_k}, \theta_{n_k}, \theta_{m_k})\beta(\theta_{n_k}, \theta_{n_k}, \theta_{m_k})S(\theta_{n_k+1}, \theta_{n_k+1}, \theta_{m_k+1}), S(\theta_{n_k}, \theta_{n_k}, \theta_{m_k})) < 0, \quad (26)$$

$\{\theta_n\}$ such that $\alpha(\theta_{n_k}, \theta_{n_k}, \xi) \geq 1$ and $\beta(\theta_{n_k}, \theta_{n_k}, \xi) \geq 1$ for all $k \in \mathbb{N}$. Thus, applying (6) for all k , we have

$$\begin{aligned}0 &\leq \zeta(\alpha(\theta_{n_k}, \theta_{n_k}, \xi)\beta(\theta_{n_k}, \theta_{n_k}, \xi)S(R\theta_{n_k}, R\theta_{n_k}, R\xi), S(\theta_{n_k}, \theta_{n_k}, \xi)) \\ &= \zeta(\alpha(\theta_{n_k}, \theta_{n_k}, \xi)\beta(\theta_{n_k}, \theta_{n_k}, \xi)S(\theta_{n_k+1}, \theta_{n_k+1}, R\xi), S(\theta_{n_k}, \theta_{n_k}, \xi)) \\ &< S(\theta_{n_k}, \theta_{n_k}, \xi) - \alpha(\theta_{n_k}, \theta_{n_k}, \xi)\beta(\theta_{n_k}, \theta_{n_k}, \xi)S(\theta_{n_k+1}, \theta_{n_k+1}, R\xi),\end{aligned}\quad (31)$$

imply

$$\begin{aligned}S(\theta_{n_k+1}, \theta_{n_k+1}, R\xi) &\leq \alpha(\theta_{n_k}, \theta_{n_k}, \xi)\beta(\theta_{n_k}, \theta_{n_k}, \xi)S(\theta_{n_k+1}, \theta_{n_k+1}, R\xi) \\ &< S(\theta_{n_k}, \theta_{n_k}, \xi).\end{aligned}\quad (32)$$

Taking $k \longrightarrow +\infty$, we have $S(\xi, \xi, R\xi) = 0$, that is, $R\xi = \xi$. Proceeding as in Theorem 16, the uniqueness of fixed point of R can be proved. \square

Definition 13. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a S -metric-like space (Ω, S) and $\alpha, \beta: \Omega \times \Omega \times \Omega \longrightarrow \mathbb{R}^+$. R is said to be a generalised $(\alpha, \beta)_s$ -admissible \mathcal{X} -contraction of type I with respect to ζ if

$$\zeta(\alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)M(\theta, \mu, \xi), S(\theta, \mu, \xi)) \geq 0, \quad (33)$$

where

$$M(\theta, \mu, \xi) = \max\{S(\theta, \mu, \xi), S(\theta, \theta, R\theta), S(\mu, \mu, R\mu), S(\xi, \xi, R\xi)\}, \quad (34)$$

for all $\theta, \mu, \xi \in \Omega$.

Definition 14. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a S -metric-like space (Ω, S) and $\alpha, \beta: \Omega \times \Omega \times \Omega \longrightarrow \mathbb{R}^+$. R is said to be a generalised $(\alpha, \beta)_s$ -admissible \mathcal{X} -contraction of type II with respect to ζ if

$$\zeta(\alpha(\theta, \theta, \mu)\beta(\theta, \theta, \mu)M(\theta, \theta, \mu), S(\theta, \theta, \mu)) \geq 0, \quad (35)$$

where

$$M(\theta, \theta, \mu) = \{S(\theta, \theta, \mu), S(\theta, \theta, R\theta), S(\mu, \mu, R\mu)\}, \quad (36)$$

for all $\theta, \mu \in \Omega$.

Theorem 3. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.

- (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.
 (c) R is a generalised $(\alpha, \beta)_s$ -admissible \mathcal{L} -contraction of type I on (Ω, S) .
 (d) R is S -continuous.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. By (b), let $\theta_0 \in \Omega$ with $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$. We construct the sequence $\{\theta_n\}$ by $\theta_{n+1} = R\theta_n$ for all $n \in \mathbb{N} \cup \{0\}$. For some n , if $\theta_n = \theta_{n+1}$, then we have $\theta_n = R\theta_n$. This gives that θ_n is a fixed point of R . In this case, proof is completed. Now, let $\theta_n \neq \theta_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. By (a), we have

$$\begin{aligned}\alpha(\theta_0, \theta_0, R\theta_0) &= \alpha(\theta_0, \theta_0, \theta_1) \geq 1 \Rightarrow \alpha(R\theta_0, R\theta_0, R\theta_1) \\ &= \alpha(\theta_1, \theta_1, \theta_2) \geq 1.\end{aligned}\quad (37)$$

Deductively, we have

$$\alpha(\theta_n, \theta_n, \theta_{n+1}) \geq 1, \quad \text{for all } n \geq 0, \quad (38)$$

$$\beta(\theta_n, \theta_n, \theta_{n+1}) \geq 1, \quad \text{for all } n \geq 0. \quad (39)$$

By (33), (38), and (39),

$$0 \leq \zeta(\alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})M(\theta_n, \theta_n, \theta_{n+1}), S(\theta_n, \theta_n, \theta_{n+1})), \quad (40)$$

where

$$\begin{aligned}M(\theta_n, \theta_n, \theta_{n+1}) &= \max\{S(\theta_n, \theta_n, \theta_{n+1}), S(\theta_n, \theta_n, R\theta_n), \\ &\quad S(\theta_{n+1}, \theta_{n+1}, R\theta_{n+1})\} \\ &= \max\{S(\theta_n, \theta_n, \theta_{n+1}), S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})\}.\end{aligned}\quad (41)$$

Thus,

$$\begin{aligned}0 &\leq \zeta(\alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})\max\{S(\theta_n, \theta_n, \theta_{n+1}), S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})\}, S(\theta_n, \theta_n, \theta_{n+1})) \\ &< S(\theta_n, \theta_n, \theta_{n+1}) - \alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})\max\{S(\theta_n, \theta_n, \theta_{n+1}), S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})\}.\end{aligned}\quad (42)$$

Consequently, we have

$$S(\theta_n, \theta_n, \theta_{n+1}) > \alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})\max\{S(\theta_n, \theta_n, \theta_{n+1}), S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})\}. \quad (43)$$

If $\max\{S(\theta_n, \theta_n, \theta_{n+1}), S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})\} = S(\theta_n, \theta_n, \theta_{n+1})$ for all $n \geq 0$, then

$$S(\theta_n, \theta_n, \theta_{n+1}) > \alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_n, \theta_n, \theta_{n+1}) \geq S(\theta_n, \theta_n, \theta_{n+1}), \quad (44)$$

is a contradiction. Thus, $\max\{S(\theta_n, \theta_n, \theta_{n+1}), S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})\} = S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})$ for all $n \geq 0$. Hence,

$$S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) \leq \alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) < S(\theta_n, \theta_n, \theta_{n+1}), \quad (45)$$

for all $n \geq 0$.

This shows that $\{S(\theta_n, \theta_n, \theta_{n+1})\}$ is a decreasing sequence, and then, we have

$$\lim_{n \rightarrow +\infty} S(\theta_n, \theta_n, \theta_{n+1}) = r, \quad r \geq 0. \quad (46)$$

We prove that

$$\lim_{n \rightarrow +\infty} S(\theta_n, \theta_n, \theta_{n+1}) = 0. \quad (47)$$

Let $r > 0$. From (45), we have

$$\lim_{n \rightarrow +\infty} \alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}) = r. \quad (48)$$

Taking $v_n = \alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2})$ and $\mu_n = S(\theta_n, \theta_n, \theta_{n+1})$ and using (3) of Definition 4, we have

$$0 \leq \lim_{n \rightarrow +\infty} \sup \zeta(\alpha(\theta_n, \theta_n, \theta_{n+1})\beta(\theta_n, \theta_n, \theta_{n+1})S(\theta_{n+1}, \theta_{n+1}, \theta_{n+2}), S(\theta_n, \theta_n, \theta_{n+1})) < 0, \quad (49)$$

is a contradiction. Hence, $r = 0$.

Next, we have to prove that $\{\theta_n\}$ is Cauchy. If possible, let $\{\theta_n\}$ is not Cauchy. Then, there exists $\varepsilon > 0$ for which subsequences $\{\theta_{n_k}\}$ and $\{\theta_{m_k}\}$ of $\{\theta_n\}$ can be obtained with $n_k > m_k > k$ such that, for every k ,

$$\begin{aligned} S(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}) &\geq \varepsilon, \\ S(\theta_{n_k-1}, \theta_{n_k-1}, \theta_{m_k}) &< \varepsilon. \end{aligned} \quad (50)$$

We have

$$\begin{aligned} \varepsilon &\leq S(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}) \leq 2S(\theta_{n_k}, \theta_{n_k}, \theta_{n_k-1}) + S(\theta_{n_k-1}, \theta_{n_k-1}, \theta_{m_k}) \\ &< 2S(\theta_{n_k}, \theta_{n_k}, \theta_{n_k-1}) + \varepsilon. \end{aligned} \quad (51)$$

Taking $k \rightarrow +\infty$ and by (47),

$$\lim_{k \rightarrow +\infty} S(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}) = \varepsilon. \quad (52)$$

Also,

$$|S(\theta_{n_k+1}, \theta_{n_k+1}, \theta_{m_k}) - S(\theta_{n_k}, \theta_{n_k}, \theta_{m_k})| \leq 2S(\theta_{n_k}, \theta_{n_k}, \theta_{n_k+1}). \quad (53)$$

Taking limit as $k \rightarrow +\infty$ and by (47) and (52),

$$\lim_{k \rightarrow +\infty} S(\theta_{n_k+1}, \theta_{n_k+1}, \theta_{m_k}) = \varepsilon. \quad (54)$$

Similarly,

$$\begin{aligned} \lim_{k \rightarrow +\infty} S(\theta_{m_k+1}, \theta_{m_k+1}, \theta_{n_k}) &= \varepsilon, \\ \lim_{k \rightarrow +\infty} S(\theta_{n_k+1}, \theta_{n_k+1}, \theta_{m_k+1}) &= \varepsilon. \end{aligned} \quad (55)$$

By triangular $(\alpha, \beta)_s$ -admissibility of R ,

$$\alpha(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}) \geq 1 \text{ and } \beta(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}) \geq 1. \quad (56)$$

Since R is generalised $(\alpha, \beta)_s$ -admissible \mathcal{X} -contraction of type I and using (52), (56), and (3) of Definition 4, we have

$$0 \leq \lim_{k \rightarrow +\infty} \sup \zeta(\alpha(\theta_{n_k}, \theta_{n_k}, \theta_{m_k})\beta(\theta_{n_k}, \theta_{n_k}, \theta_{m_k})M(\theta_{n_k}, \theta_{n_k}, \theta_{m_k}), S(\theta_{n_k}, \theta_{n_k}, \theta_{m_k})) < 0, \quad (57)$$

is a contradiction, and hence, $\{\theta_n\}$ is Cauchy. By completeness of S -metric-like space (Ω, S) , we know that there exist some $\xi \in \Omega$ such that

$$\lim_{n \rightarrow +\infty} S(\theta_n, \theta_n, \xi) = S(\xi, \xi, \xi) = \lim_{n, m \rightarrow +\infty} S(\theta_n, \theta_n, \theta_m) = 0, \quad (58)$$

which implies that $S(\xi, \xi, \xi) = 0$. The continuity of R implies that

$$\begin{aligned} \lim_{n \rightarrow +\infty} S(\theta_{n+1}, \theta_{n+1}, R\xi) &= \lim_{n \rightarrow +\infty} S(R\theta_n, R\theta_n, R\xi) \\ &= S(R\xi, R\xi, R\xi) = 0. \end{aligned} \quad (59)$$

Using Lemma 1 and (58), we have

$$\lim_{n \rightarrow +\infty} S(\theta_{n+1}, \theta_{n+1}, R\xi) = S(\xi, \xi, R\xi). \quad (60)$$

Thus, $S(R\xi, R\xi, R\xi) = S(\xi, \xi, R\xi)$, that is, $R\xi = \xi$. To prove uniqueness of fixed point, let $\eta \in \Omega$ such that $R\eta = \eta$ and $\eta \neq \xi$. Then,

$$\begin{aligned} 0 &\leq \zeta(\alpha(\xi, \xi, \eta)\beta(\xi, \xi, \eta)S(R\xi, R\xi, R\eta), S(\xi, \xi, \eta)) \\ &< S(\xi, \xi, \eta) - \alpha(\xi, \xi, \eta)\beta(\xi, \xi, \eta)S(R\xi, R\xi, \eta) \leq 0, \end{aligned} \quad (61)$$

is a contradiction, so $\xi = \eta$.

By removing the continuity condition, we have the following result. \square

Theorem 4. Let $R: \Omega \rightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- R is triangular $(\alpha, \beta)_s$ -admissible.
- There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.
- R is generalised $(\alpha, \beta)_s$ -admissible \mathcal{X} -contraction of type I on (Ω, S) .
- If $\{\theta_n\}$ is a sequence in Ω such that $\alpha(\theta_n, \theta_n, \theta_{n+1}) \geq 1$ and $\beta(\theta_n, \theta_n, \theta_{n+1}) \geq 1$ for all n and $\theta_n \rightarrow \xi \in \Omega$ as $n \rightarrow +\infty$, then there exists a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\alpha(\theta_{n_k}, \theta_{n_k}, \xi) \geq 1$ and $\beta(\theta_{n_k}, \theta_{n_k}, \xi) \geq 1$ for all $k \in \mathbb{N}$.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. Proceeding as of Theorem 3, let $\{\theta_n\}$ be a sequence in Ω given by $\theta_{n+1} = R\theta_n$ converges to some $\xi \in \Omega$ with $\alpha(\theta_n, \theta_n, \theta_{n+1}) \geq 1$ and $\beta(\theta_n, \theta_n, \theta_{n+1}) \geq 1$ for all n and $S(\xi, \xi, \xi) = 0$. From (d), there exists a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\alpha(\theta_{n_k}, \theta_{n_k}, \xi) \geq 1$ and $\beta(\theta_{n_k}, \theta_{n_k}, \xi) \geq 1$ for all $k \in \mathbb{N}$. Thus, applying (33) for all k , we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(\theta_{n_k}, \theta_{n_k}, \xi)\beta(\theta_{n_k}, \theta_{n_k}, \xi)M(\theta_{n_k}, \theta_{n_k}, \xi), S(\theta_{n_k}, \theta_{n_k}, \xi)) \\ &< S(\theta_{n_k}, \theta_{n_k}, \xi) \\ &\quad - \alpha(\theta_{n_k}, \theta_{n_k}, \xi)\beta(\theta_{n_k}, \theta_{n_k}, \xi)M(\theta_{n_k}, \theta_{n_k}, \xi), \end{aligned} \quad (62)$$

where

$$\begin{aligned}
 M(\theta_{n_k}, \theta_{n_k}, \xi) &= \max\{S(\theta_{n_k}, \theta_{n_k}, \xi), S(\theta_{n_k}, \theta_{n_k}, R\theta_{n_k}), S(\xi, \xi, R\xi)\} \\
 &= \max\{S(\theta_{n_k}, \theta_{n_k}, \xi), S(\theta_{n_k}, \theta_{n_k}, \theta_{n_k+1}), S(\xi, \xi, R\xi)\}.
 \end{aligned} \tag{63}$$

From (62), we get

$$\begin{aligned}
 M(\theta_{n_k}, \theta_{n_k}, \xi) &\leq \alpha(\theta_{n_k}, \theta_{n_k}, \xi)\beta(\theta_{n_k}, \theta_{n_k}, \xi)M(\theta_{n_k}, \theta_{n_k}, \xi) \\
 &< S(\theta_{n_k}, \theta_{n_k}, \xi).
 \end{aligned} \tag{64}$$

Taking $k \longrightarrow +\infty$, we have $S(\xi, \xi, R\xi) = 0$, that is, $R\xi = \xi$. Similar to Theorem 3 uniqueness of fixed point of R can be proved. \square

Next, we give the following definition. \square

Definition 15. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a S -metric-like space (Ω, S) and $\alpha, \beta: \Omega \times \Omega \times \Omega \longrightarrow \mathbb{R}^+$. R is said to be a generalised rational $(\alpha, \beta)_s$ -admissible \mathcal{Z} -contraction of type I with respect to ζ if

$$\zeta(\alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)M_r(\theta, \mu, \xi), S(\theta, \mu, \xi)) \geq 0, \tag{65}$$

where

$$M_r(\theta, \mu, \xi) = \max \left\{ \begin{aligned} &S(\theta, \mu, \xi), S(R\theta, R\mu, R\xi), \frac{S(\theta, \theta, R\theta)S(\mu, \mu, R\mu)}{1 + S(\theta, \mu, \xi) + S(R\theta, R\mu, R\xi)}, \\ &\frac{S(\mu, \mu, R\mu)S(\xi, \xi, R\xi)}{1 + S(\theta, \mu, \xi) + S(R\theta, R\mu, R\xi)}, \frac{S(\xi, \xi, R\xi)S(\theta, \theta, R\theta)}{1 + S(\theta, \mu, \xi) + S(R\theta, R\mu, R\xi)} \end{aligned} \right\}, \tag{66}$$

for all $\theta, \mu, \xi \in \Omega$.

Definition 16. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a S -metric-like space (Ω, S) and $\alpha, \beta: \Omega \times \Omega \times \Omega \longrightarrow \mathbb{R}^+$. R is said to be a generalised rational $(\alpha, \beta)_s$ -admissible \mathcal{Z} -contraction of type II with respect to ζ if

$$\zeta(\alpha(\theta, \theta, \mu)\beta(\theta, \theta, \mu)M_r(\theta, \theta, \mu), S(\theta, \theta, \mu)) \geq 0, \tag{67}$$

where

$$M_r(\theta, \theta, \mu) = \max \left\{ S(\theta, \theta, \mu), S(R\theta, R\theta, R\mu), \frac{S(\theta, \theta, R\theta)S(\theta, \theta, R\theta)}{1 + S(\theta, \theta, \mu) + S(R\theta, R\theta, R\mu)}, \frac{S(\theta, \theta, R\theta)S(\mu, \mu, R\mu)}{1 + S(\theta, \theta, \mu) + S(R\theta, R\theta, R\mu)} \right\}, \tag{68}$$

for all $\theta, \mu, \xi \in \Omega$.

Theorem 5. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.
- (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.
- (c) R is a generalised rational $(\alpha, \beta)_s$ -admissible \mathcal{Z} -contraction of type I on (Ω, S) .
- (d) R is S -continuous.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. Similar to Theorem 3. \square

Theorem 6. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.
- (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.

(c) R is generalised rational $(\alpha, \beta)_s$ -admissible \mathcal{Z} -contraction of type I on (Ω, S) .

(d) If $\{\theta_n\}$ is a sequence in Ω such that $\alpha(\theta_n, \theta_n, \theta_{n+1}) \geq 1$ and $\beta(\theta_n, \theta_n, \theta_{n+1}) \geq 1$ for all n and $\theta_n \longrightarrow \xi \in \Omega$ as $n \longrightarrow +\infty$, then there exists a subsequence $\{\theta_{n_k}\}$ of $\{\theta_n\}$ such that $\alpha(\theta_{n_k}, \theta_{n_k}, \xi) \geq 1$ and $\beta(\theta_{n_k}, \theta_{n_k}, \xi) \geq 1$ for all $k \in \mathbb{N}$.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. Similar to Theorem 4. \square

3. Consequences

In this section, we give various results as consequences of the above results. First, we give a Banach type.

Corollary 1. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.
- (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.

- (c) $\alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)S(R\theta, R\mu, R\xi) \leq kS(\theta, \mu, \xi)$ for all $\theta, \mu, \xi \in \Omega$ and $k \in [0, 1)$.
 (d) R is S -continuous.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. Similar to Theorem 1, considering

$$\zeta(t, s) = ks - t. \quad (69)$$

Corollary 2. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.
 (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.
 (c) There exists a lower semicontinuous function $\phi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\phi^{-1}\{0\} = \{0\}$ such that

$$\alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)S(R\theta, R\mu, R\xi) \leq S(\theta, \mu, \xi) - \phi(S(\theta, \mu, \xi)), \quad (70)$$

for all $\theta, \mu, \xi \in \Omega$.

- (d) R is S -continuous.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. Consider

$$\zeta(t, s) = s - \phi(s) - t. \quad (71)$$

Taking $\alpha(\theta, \mu, \xi) = \beta(\theta, \mu, \xi) = 1$ for all $\theta, \mu, \xi \in \Omega$ in Theorem 1, we have the following. \square

Corollary 3. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) . Assume that there exists a simulation function ζ such that

$$\zeta(S(R\theta, R\mu, R\xi), S(\theta, \mu, \xi)) \geq 0, \quad (72)$$

for all $\theta, \mu, \xi \in \Omega$. Then, $\xi \in \Omega$ is a unique fixed point of R with $S(\xi, \xi, \xi) = 0$.

Corollary 4. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.
 (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.
 (c) $\alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)M(R\theta, R\mu, R\xi) \leq kS(\theta, \mu, \xi)$ for all $\theta, \mu, \xi \in \Omega$ and $k \in [0, 1)$.
 (d) R is S -continuous.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. Similar to Theorem 3, considering,

$$\zeta(t, s) = ks - t. \quad (73)$$

Corollary 5. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.
 (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.
 (c) There exists a lower semicontinuous function $\phi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\phi^{-1}\{0\} = \{0\}$ such that

$$\alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)M(R\theta, R\mu, R\xi) \leq S(\theta, \mu, \xi) - \phi(S(\theta, \mu, \xi)), \quad (74)$$

- (i) for all $\theta, \mu, \xi \in \Omega$.

- (d) R is S -continuous.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. Consider

$$\zeta(t, s) = s - \phi(s) - t. \quad (75)$$

If we consider in Theorem 20, $\alpha(\theta, \mu, \xi) = \beta(\theta, \mu, \xi) = 1$ for all $\theta, \mu, \xi \in \Omega$, we have \square

Corollary 6. Let $R: \Omega \longrightarrow \Omega$ be a S -continuous mapping on a complete S -metric-like space (Ω, S) . Suppose that there exists a simulation function ζ such that

$$\zeta(M(R\theta, R\mu, R\xi), S(\theta, \mu, \xi)) \geq 0, \quad (76)$$

for all $\theta, \mu, \xi \in \Omega$. Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Corollary 7. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.
 (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.
 (c) $\alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)M_r(R\theta, R\mu, R\xi) \leq kS(\theta, \mu, \xi)$ for all $\theta, \mu, \xi \in \Omega$ and $\theta \in [0, 1)$.
 (d) R is S -continuous.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. Similar to Theorem 5, considering,

$$\zeta(t, s) = ks - t. \quad (77)$$

Corollary 8. Let $R: \Omega \longrightarrow \Omega$ be a mapping on a complete S -metric-like space (Ω, S) satisfying the following:

- (a) R is triangular $(\alpha, \beta)_s$ -admissible.
 (b) There exists $\theta_0 \in \Omega$ such that $\alpha(\theta_0, \theta_0, R\theta_0) \geq 1$ and $\beta(\theta_0, \theta_0, R\theta_0) \geq 1$.
 (c) There exists a lower semicontinuous function $\phi: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ with $\phi^{-1}\{0\} = \{0\}$ such that

$$\begin{aligned} & \alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)M_r(R\theta, R\mu, R\xi) \\ & \leq S(\theta, \mu, \xi) - \phi(S(\theta, \mu, \xi)), \end{aligned} \quad (78)$$

for all $\theta, \mu, \xi \in \Omega$.

(d) R is S -continuous.

Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

Proof. It suffices to take

$$\zeta(t, s) = s - \phi(s) - t. \quad (79)$$

If we consider in Theorem 5, $\alpha(\theta, \mu, \xi) = \beta(\theta, \mu, \xi) = 1$ for all $\theta, \mu, \xi \in \Omega$, we have the following. \square

Corollary 9. Let $R: \Omega \longrightarrow \Omega$ be a S -continuous mapping on a complete S -metric-like space (Ω, S) . Suppose that there exists a simulation function ζ such that

$$\zeta(M_r(R\theta, R\mu, R\xi), S(\theta, \mu, \xi)) \geq 0, \quad (80)$$

for all $\theta, \mu, \xi \in \Omega$. Then, there is a unique fixed point $\xi \in \Omega$ of R with $S(\xi, \xi, \xi) = 0$.

We take the following example.

Example 2. Suppose $\Omega = [0, +\infty)$, $S(\theta, \mu, \xi) = (\theta + \mu) + (\mu + \xi)$ for all $\theta, \mu, \xi \in \Omega$ and $R: \Omega \longrightarrow \Omega$ as

$$R\theta = \begin{cases} \frac{\theta}{4}, & \text{if } 0 \leq \theta \leq 1, \\ 4\theta, & \text{otherwise.} \end{cases} \quad (81)$$

Consider

$$\zeta(t, s) = cs - t, \quad \text{where } 0 \leq \frac{1}{4} < c < 1. \quad (82)$$

Let $\alpha, \beta: \Omega \times \Omega \times \Omega \longrightarrow \mathbb{R}^+$ be defined as

$$\begin{aligned} \alpha(\theta, \mu, \xi) &= \begin{cases} \frac{4}{3}, & \text{if } 0 \leq \theta, \mu, \xi \leq 1, \\ 0, & \text{otherwise,} \end{cases} \\ \beta(\theta, \mu, \xi) &= \begin{cases} \frac{3}{2}, & \text{if } 0 \leq \theta, \mu, \xi \leq 1, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \quad (83)$$

We will verify Corollary 1. Here, (Ω, S) is a complete S -metric-like space. Let $\theta, \mu, \xi \in \Omega$ such that $\alpha(\theta, \mu, \xi) \geq 1$ and $\beta(\theta, \mu, \xi) \geq 1$. Then, $\theta, \mu, \xi \in [0, 1]$ and so $R\theta \in [0, 1]$, $R\mu \in [0, 1]$, $R\xi \in [0, 1]$, and $\alpha(R\theta, R\mu, R\xi) \geq 1$ and $\beta(R\theta, R\mu, R\xi) \geq 1$. Hence, R is triangular $(\alpha, \beta)_s$ -admissible. When $\theta_0 = 1$, condition (b) is true, $\theta_0 = R^n\theta_1 = (1/n)$ satisfies condition (d).

If $0 \leq \theta \leq 1$, then $\alpha(\theta, \mu, \xi) = (4/3)$ and $\beta(\theta, \mu, \xi) = (3/2)$. We have

$$\begin{aligned} & \zeta(\alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)S(R\theta, f\mu, f\xi), S(\theta, \mu, \xi)) = cS(\theta, \mu, \xi) - \alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)S(f\theta, f\mu, f\xi) \\ & = \frac{3}{4}\{(\theta + \mu) + (\mu + \xi)\} - 2 \cdot \frac{1}{4}\{(\theta + \mu) + (\mu + \xi)\} = \frac{1}{4}\{(\theta + \mu) + (\mu + \xi)\}. \end{aligned} \quad (84)$$

If $0 \leq \mu \leq 1$ and $\xi > 1$, then $\zeta(\alpha(\theta, \mu, \xi)\beta(\theta, \mu, \xi)S(f\theta, f\mu, f\xi), S(\theta, \mu, \xi)) \geq 0$ since $\alpha(\theta, \mu, \xi) = \beta(\theta, \mu, \xi) = 0$. Thus, $\xi = 0$ is the unique fixed point of f .

We also notice that (72) is not satisfied. In fact, for $\theta = 1$, $\mu = 2$, $\xi = 3$, we get

$$\begin{aligned} S(R1, R2, R3) &= S\left(\frac{1}{4}, 8, 12\right) = \left(\frac{1}{4} + 8\right) + (8 + 12) \\ &= \frac{33}{4} + 20 = \frac{113}{4} > 8 = S(1, 2, 3). \end{aligned} \quad (85)$$

4. Conclusion

In this paper, we present S -metric-like space and some of its topological properties. Also, we present some new (α, β) -contraction-type mappings under simulation function by extending the definition of (α, β) -admissible mappings and prove some fixed point results in the setting of S -metric-like space. We also open the scope for extending various contractions in S -metric-like space.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Common Fixed-Point Theorems of Generalized (ψ, φ) – Weakly Contractive Mappings in b – Metric-Like Spaces and Application

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Received 6 December 2020; Revised 26 December 2020; Accepted 30 December 2020; Published 20 January 2021

Academic Editor: Naeem Saleem

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In this paper, we prove some common fixed-point theorems of generalized (ψ, φ) –weakly contractive mappings in b –metric-like spaces. We also give two examples to support our results. Meanwhile, we present an application to the existence of solutions for a system of integral equations by means of our results.

1. Introduction

The Banach contraction mapping theorem [1] popularly known as Banach contraction mapping principle is a rewarding result in fixed-point theory. It has widespread applications in both pure and applied mathematics. This celebrated principle has been generalized by several authors. Recently, Saleem et al. [2] obtained fixed-point results of Suzuki-type generalized multivalued (f, θ, L) -almost contractions and coincidence and common fixed-point results of Suzuki-type generalized multivalued (f, θ, L) -almost contraction mapping in the setting of metric spaces. In 2019, Li et al. [3] defined a new contractive-type mapping called Z_θ -contraction and proved some fixed-point and Suzuki-type fixed-point results in the context of complete metric spaces. Another extension idea is the promotion of spaces. In 1993, Czerwik [4] introduced the concept of the b –metric space, and he also obtained some fixed-point theorems of contractive mappings in b –metric space. Since then, some papers have generalized the fixed-point of different contractive conditions in b –metric space. For example, Aydi et al. [5] proved common fixed-point results for weak ϕ –contractions on b –metric space, and Abbas et al. [6] also proved common fixed-point theorems of four mappings in b –metric space. In [7], Hussain et al. established some best

proximity point and coupled best proximity point results in the context of complete b –metric spaces based on the concepts of α –proximal admissible mappings and simulation function. Lately, Lael et al. [8] replied to an open problem related to a b –metric version of Banach’s fixed-point theorem and obtained some new fixed-point theorems for multi-valued mappings in b –metric spaces.

Inspired by Czerwik’s work, some researchers had studied the fixed-point theorems of various new types of contractive conditions in the generalized metric space and b –metric space. In 2011, Hussain and Shah in [9] introduced the notion of cone b –metric spaces, which means that it is a generalization of b –metric spaces and cone metric spaces. They also considered topological properties of cone b –metric spaces and results on KKM mappings in the setting of cone b –metric spaces. Fernandez et al. [10] introduced the concept of N_b –cone metric spaces over a Banach algebra as a generalization of N –cone metric spaces over a Banach algebra and b –metric spaces and studied some coupled common fixed-point theorems for generalized Lipschitz mappings in this framework. In 2019, Kanwal et al. [11] generalized Nadler’s theorem in weak partial b –metric space by using weak partial Hausdorff b –metric spaces. In 2020, Abbas et al. [12] introduced the concepts of ψ –contraction and monotone ψ –contraction correspondence in fuzzy

b -metric spaces and obtained fixed-point results for these contractive mappings. Ansari et al. [13] introduced the concept of inverse C -class function in G -metric setting and established some fixed-point theorems. Recently, Saleem et al. [14] proved some new fixed-point theorems, coincidence point theorems, and common fixed-point theorems for multivalued F -contractions involving a binary relation that is not necessarily a partial order, in the context of generalized metric spaces (in the sense of Jleli and Samet).

The concept of φ -contractive mappings was introduced by Rhoades [15]. Afterwards, some researchers introduced a few φ - and ψ, φ -weakly contractive conditions and discussed the existence of fixed and common fixed-point for these mappings [16–18]. In particular, Aghajani et al. [19]

presented several common fixed-point results of generalized weak contractive mappings in partially ordered b -metric spaces.

Our main concern is to study common fixed-point results involving generalized (ψ, φ) -weakly contractive conditions in b -metric-like space. Furthermore, we apply the given results to obtain existence of solutions of integral equations.

2. Preliminaries

Throughout this paper, let \mathbb{N} denote the set of all positive integers, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{R} = (-\infty, +\infty)$. Put

$$\begin{aligned}\Psi &= \{\psi: [0, +\infty) \longrightarrow [0, +\infty) \text{ is an increasing and continuous function}\}, \\ \Phi &= \{\varphi: [0, +\infty) \longrightarrow [0, +\infty) \text{ is an increasing and continuous function and, } \varphi(t) = 0 \text{ if and only if } t = 0\}.\end{aligned}\quad (1)$$

In order to get our main results, we introduce some definitions and lemmas as follows.

Definition 1 (see [4]). Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d: X \times X \longrightarrow [0, +\infty)$ is said to be a b -metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

- (1) $d(x, y) = 0$ if and only if $x = y$
- (2) $d(x, y) = d(y, x)$
- (3) $d(x, y) \leq s(d(x, z) + d(y, z))$

The pair (X, d) is called a b -metric space with parameter $s \geq 1$.

In general, the class of b -metric space is effectively larger than that of metric space, since a b -metric is a metric with $s = 1$. We can find several examples of b -metric spaces which are not metric spaces (see [20]).

Definition 2 (see [21]). Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d: X \times X \longrightarrow [0, +\infty)$ is said to be a b -metric-like if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

- (1) $d(x, y) = 0$ implies $x = y$
- (2) $d(x, y) = d(y, x)$
- (3) $d(x, y) \leq s(d(x, z) + d(y, z))$

The pair (X, d) is called a b -metric-like space with parameter $s \geq 1$.

Remark 1. We should note that in a b -metric-like space (X, d) , if $x, y \in X$ and $d(x, y) = 0$, then $x = y$. But the converse need not be true, and $d(x, x)$ may be positive for $x \in X$.

Example 1. Let $X = \mathbb{R}^+$, and let the mapping $d: X \times X \longrightarrow \mathbb{R}^+$ be defined by $d(x, y) = (x + y)^2$ for all

$x, y \in X$. Then, (X, d) is a b -metric-like space with parameter $s \geq 2$.

Proof. We can infer from the convexity of the function $f(x) = x^2$ ($x > 0$) that $(a + b)^2 \leq 2(a^2 + b^2)$ holds. Then, for $x, y, z \in X$, we have

$$\begin{aligned}d(x, y) &= (x + y)^2 \leq (x + z + y + z)^2 \\ &\leq 2[(x + z)^2 + (y + z)^2] = 2d(x, z) + 2d(y, z).\end{aligned}\quad (2)$$

In this case, (X, d) is a b -metric-like space with parameter $s \geq 2$. \square

Definition 3 (see [21]). Let (X, d) be a b -metric-like space with parameter $s \geq 1$ and $\{x_n\}$ be a sequence in X .

- (1) The sequence $\{x_n\}$ is said to be convergent to x if $\lim_{n \rightarrow +\infty} d(x_n, x) = d(x, x)$.
- (2) The sequence $\{x_n\}$ is said to be a Cauchy sequence if and only if $\lim_{n \rightarrow +\infty} d(x_n, x_m)$ exists and is finite.
- (3) (X, d) is said to be complete if for each Cauchy sequence $\{x_n\}$ in X , there exists an $x \in X$ such that $\lim_{n, m \rightarrow +\infty} d(x_n, x_m) = \lim_{n \rightarrow +\infty} d(x_n, x) = d(x, x)$.

Definition 4 (see [22]). Let f and g be two self-mappings on a nonempty set X . If $w = fx = gx$, for some $x \in X$, then x is said to be the coincidence point of f and g , where w is called the point of coincidence of f and g . Let $C(f, g)$ denote the set of all coincidence points of f and g .

Definition 5 (see [22]). Let f and g be two self-mappings defined on a nonempty set X . Then, f and g are said to be weakly compatible if they commute at every coincidence point, that is, $fx = gx \Rightarrow fgx = gfx$ for every $x \in C(f, g)$.

Lemma 1 (see [21]). Let (X, d) be a b -metric-like space with $s \geq 1$. We assume that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y , respectively. Then, we have

$$\begin{aligned} & \frac{1}{s^2}d(x, y) - \frac{1}{s}d(x, x) - d(y, y) \\ & \leq \liminf_{n \rightarrow +\infty} d(x_n, y_n) \leq \limsup_{n \rightarrow +\infty} d(x_n, y_n) \\ & \leq sd(x, x) + s^2d(y, y) + s^2d(x, y). \end{aligned} \quad (3)$$

In particular, if $d(x, y) = 0$, then we have $\lim_{n \rightarrow +\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have

$$\begin{aligned} & \frac{1}{s}d(x, z) - d(x, x) \leq \liminf_{n \rightarrow +\infty} d(x_n, z) \\ & \leq \limsup_{n \rightarrow +\infty} d(x_n, z), \quad (4) \\ & \leq sd(x, z) + sd(x, x). \end{aligned}$$

In particular, if $d(x, x) = 0$, then

$$\begin{aligned} & \frac{1}{s}d(x, z) \leq \liminf_{n \rightarrow +\infty} d(x_n, z) \leq \limsup_{n \rightarrow +\infty} d(x_n, z) \\ & \leq sd(x, z). \end{aligned} \quad (5)$$

Lemma 2 (see [23]). Let (X, d) be a b -metric-like space with $s \geq 1$. Then,

- (1) If $d(x, y) = 0$, then $d(x, x) = d(y, y) = 0$
- (2) If $\{x_n\}$ is a sequence such that $\lim_{n \rightarrow +\infty} d(x_n, x_{n+1}) = 0$, then we have $\lim_{n \rightarrow +\infty} d(x_n, x_n) = \lim_{n \rightarrow +\infty} d(x_{n+1}, x_{n+1}) = 0$
- (3) If $x \neq y$, then $d(x, y) > 0$

3. Main Results

In this section, we will show the existence and uniqueness of common fixed-point for generalized (ψ, φ) -weakly contractive mappings in complete b -metric-like space. Meanwhile, we give two examples to support our results.

Theorem 1. Let (X, d) be a complete b -metric-like space with parameter $s \geq 1$ and let $f, g: X \rightarrow X$ be given self-mappings satisfying $f(X) \subset g(X)$ where $g(X)$ is a closed subset of X . If there are functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(s^2[d(fx, fy)]^2) \leq \psi(N_1(x, y)) - \varphi(M_1(x, y)), \quad (6)$$

where

$$\begin{aligned} N_1(x, y) &= \max\{[d(fx, gx)]^2, [d(gx, gy)]^2, [d(fy, gy)]^2, d(fx, gx)d(fx, fy), d(fx, gx)d(gx, gy)\}, \\ M_1(x, y) &= \max\left\{[d(fy, gy)]^2, [d(fx, gy)]^2, [d(gx, gy)]^2, \frac{[d(fx, gx)]^2[1 + [d(gx, gy)]^2]}{1 + [d(fx, gy)]^2}\right\}, \end{aligned} \quad (7)$$

then f and g have a unique coincidence point in X . Moreover, f and g have a unique common fixed-point provided that f and g are weakly compatible.

Proof. Let $x_0 \in X$. As $f(X) \subset g(X)$, there exists $x_1 \in X$ such that $fx_0 = gx_1$. Now we define the sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N}$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then we have $y_n = y_{n+1} = fx_{n+1} = gx_{n+1}$ and f and g have a coincidence point. Without loss of generality, we assume that $y_n \neq y_{n+1}$ (by Lemma 2, we know that

$d(y_n, y_{n+1}) > 0$) for all $n \in \mathbb{N}$. Applying (6) with $x = x_n$ and $y = x_{n+1}$, we obtain

$$\begin{aligned} \psi(s^2[d(y_n, y_{n+1})]^2) &= \psi(s^2[d(fx_n, fx_{n+1})]^2) \\ &\leq \psi(N_1(x_n, x_{n+1})) - \varphi(M_1(x_n, x_{n+1})), \end{aligned} \quad (8)$$

where

$$N_1(x_n, x_{n+1}) = \max\{[d(y_n, y_{n-1})]^2, [d(y_{n-1}, y_n)]^2, [d(y_{n+1}, y_n)]^2, d(y_n, y_{n-1})d(y_n, y_{n+1}), [d(y_n, y_{n-1})]^2\}, \quad (9)$$

$$M_1(x_n, x_{n+1}) = \max\left\{[d(y_{n+1}, y_n)]^2, [d(y_n, y_n)]^2, [d(y_{n-1}, y_n)]^2, \frac{[d(y_n, y_{n-1})]^2[1 + [d(y_{n-1}, y_n)]^2]}{1 + [d(y_n, y_n)]^2}\right\}. \quad (10)$$

If $d(y_n, y_{n+1}) \geq d(y_n, y_{n-1}) > 0$, for some $n \in \mathbb{N}$, in view of (9) and (10), we have

$$\begin{aligned} N_1(x_n, x_{n+1}) &= [d(y_n, y_{n+1})]^2, \\ M_1(x_n, x_{n+1}) &\geq [d(y_n, y_{n+1})]^2. \end{aligned} \quad (11)$$

It follows from inequality (8) and the above inequalities that

$$\begin{aligned} \psi([d(y_n, y_{n+1})]^2) &\leq \psi(s^2[d(y_n, y_{n+1})]^2) \\ &\leq \psi(N_1(x_n, x_{n+1})) - \varphi(M_1(x_n, x_{n+1})) \\ &\leq \psi([d(y_n, y_{n+1})]^2) - \varphi([d(y_n, y_{n+1})]^2), \end{aligned} \quad (12)$$

which implies $\varphi([d(y_n, y_{n+1})]^2) = 0$, that is, $y_n = y_{n+1}$, a contradiction. Hence, $d(y_n, y_{n+1}) < d(y_n, y_{n-1})$ and $\{d(y_n, y_{n+1})\}$ is a nonincreasing sequence, and so there exists $r \geq 0$ such that $\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) = r$.

By virtue of (9) and (10), we have

$$\begin{aligned} N_1(x_n, x_{n+1}) &= [d(y_n, y_{n-1})]^2, \\ M_1(x_n, x_{n+1}) &= [d(y_n, y_{n-1})]^2. \end{aligned} \quad (13)$$

It follows that

$$\begin{aligned} \psi([d(y_n, y_{n+1})]^2) &\leq \psi(N_1(x_n, x_{n+1})) - \varphi(M_1(x_n, x_{n+1})) \\ &\leq \psi([d(y_n, y_{n-1})]^2) - \varphi([d(y_n, y_{n-1})]^2). \end{aligned} \quad (14)$$

Now suppose that $r > 0$. By taking the limit as $n \rightarrow +\infty$ in (12), we have $\psi(r^2) \leq \psi(r^2) - \varphi(r^2)$, a contradiction. This yields that

$$\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) = r = 0. \quad (15)$$

Now we shall prove that $\lim_{n, m \rightarrow +\infty} d(y_n, y_m) = 0$. Suppose on the contrary that $\lim_{n, m \rightarrow +\infty} d(y_n, y_m) \neq 0$. It follows that there exists $\varepsilon > 0$ for which one can find sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ where n_k is the smallest index for which $n_k > m_k > k$, $\varepsilon \leq d(y_{m_k}, y_{n_k})$, and $d(y_{m_k}, y_{n_{k-1}}) < \varepsilon$.

In view of the triangle inequality in b -metric-like space, we get

$$\begin{aligned} \varepsilon^2 &\leq [d(y_{m_k}, y_{n_k})]^2 \leq [sd(y_{m_k}, y_{n_{k-1}}) + sd(y_{n_{k-1}}, y_{n_k})]^2 \\ &= s^2[d(y_{m_k}, y_{n_{k-1}})]^2 + s^2[d(y_{n_{k-1}}, y_{n_k})]^2 + 2s^2d(y_{m_k}, y_{n_{k-1}})d(y_{n_{k-1}}, y_{n_k}) \\ &\leq s^2\varepsilon^2 + s^2[d(y_{n_{k-1}}, y_{n_k})]^2 + 2s^2d(y_{m_k}, y_{n_{k-1}})d(y_{n_{k-1}}, y_{n_k}). \end{aligned} \quad (16)$$

Using equality (15) and taking the upper limit as $k \rightarrow +\infty$ in the above inequality, we obtain

$$\varepsilon^2 \leq \limsup_{k \rightarrow +\infty} [d(y_{m_k}, y_{n_k})]^2 \leq s^2\varepsilon^2. \quad (17)$$

As the same arguments, we deduce the following results:

$$\begin{aligned} \varepsilon^2 &\leq [d(y_{m_k}, y_{n_k})]^2 \leq [sd(y_{m_k}, y_{n_{k-1}}) + sd(y_{n_{k-1}}, y_{n_k})]^2 \\ &= s^2[d(y_{m_k}, y_{n_{k-1}})]^2 + s^2[d(y_{n_{k-1}}, y_{n_k})]^2 + 2s^2d(y_{m_k}, y_{n_{k-1}})d(y_{n_{k-1}}, y_{n_k}), \end{aligned} \quad (18)$$

$$\begin{aligned} [d(y_{m_k}, y_{n_k})]^2 &\leq [sd(y_{m_k}, y_{m_{k-1}}) + sd(y_{m_{k-1}}, y_{n_k})]^2 \\ &= s^2[d(y_{m_k}, y_{m_{k-1}})]^2 + s^2[d(y_{m_{k-1}}, y_{n_k})]^2 + 2s^2d(y_{m_k}, y_{m_{k-1}})d(y_{m_{k-1}}, y_{n_k}), \end{aligned} \quad (19)$$

$$\begin{aligned} [d(y_{m_{k-1}}, y_{n_k})]^2 &\leq [sd(y_{m_{k-1}}, y_{m_k}) + sd(y_{m_k}, y_{n_k})]^2 \\ &= s^2[d(y_{m_{k-1}}, y_{m_k})]^2 + s^2[d(y_{m_k}, y_{n_k})]^2 + 2s^2d(y_{m_{k-1}}, y_{m_k})d(y_{m_k}, y_{n_k}). \end{aligned} \quad (20)$$

In view of (18), we have

$$\frac{\varepsilon^2}{s^2} \leq \limsup_{k \rightarrow +\infty} [d(y_{m_k}, y_{n_{k-1}})]^2 \leq \varepsilon^2. \quad (21)$$

Using (19) and (20), we obtain

$$\frac{\varepsilon^2}{s^2} \leq \limsup_{k \rightarrow +\infty} [d(y_{m_{k-1}}, y_{n_k})]^2 \leq s^4\varepsilon^2. \quad (22)$$

Similarly, we deduce that

$$\begin{aligned}
 [d(y_{m_k-1}, y_{n_k-1})]^2 &\leq [sd(y_{m_k-1}, y_{m_k}) + sd(y_{m_k}, y_{n_k-1})]^2 \\
 &= s^2 [d(y_{m_k-1}, y_{m_k})]^2 + s^2 [d(y_{m_k}, y_{n_k-1})]^2 + 2s^2 d(y_{m_k-1}, y_{m_k}) d(y_{m_k}, y_{n_k-1}), \\
 [d(y_{m_k}, y_{n_k})]^2 &\leq [sd(y_{m_k}, y_{m_k-1}) + sd(y_{m_k-1}, y_{n_k})]^2 = s^2 [d(y_{m_k}, y_{m_k-1})]^2 + s^2 [d(y_{m_k-1}, y_{n_k})]^2 \\
 &\quad + 2s^2 d(y_{m_k}, y_{m_k-1}) d(y_{m_k-1}, y_{n_k}) \\
 &\leq s^2 [d(y_{m_k}, y_{m_k-1})]^2 + s^2 [sd(y_{m_k-1}, y_{n_k-1}) + sd(y_{n_k-1}, y_{n_k})]^2 + 2s^2 d(y_{m_k}, y_{m_k-1}) [sd(y_{m_k-1}, y_{n_k-1}) \\
 &\quad + sd(y_{n_k-1}, y_{n_k})].
 \end{aligned} \tag{23}$$

It follows that

$$\frac{\varepsilon^2}{4} \leq \limsup_{k \rightarrow +\infty} [d(y_{m_k-1}, y_{n_k-1})]^2 \leq s^2 \varepsilon^2. \tag{24}$$

Through the definition of $N_1(x, y)$, we have

$$\begin{aligned}
 N_1(x_{m_k}, x_{n_k}) &= \max \left\{ [d(y_{m_k}, y_{m_k-1})]^2, [d(y_{m_k-1}, y_{n_k-1})]^2, [d(y_{n_k}, y_{n_k-1})]^2, d(y_{m_k}, y_{m_k-1}) d(y_{m_k}, y_{n_k}), \right. \\
 &\quad \left. d(y_{m_k}, y_{m_k-1}) d(y_{m_k-1}, y_{n_k-1}) \right\},
 \end{aligned} \tag{25}$$

which yields that

$$\limsup_{k \rightarrow +\infty} N_1(x_{m_k}, x_{n_k}) \leq \max \{0, s^2 \varepsilon^2, 0, 0, 0\} = \varepsilon^2 s^2. \tag{26}$$

Also,

$$M_1(x_{m_k}, x_{n_k}) = \max \left\{ [d(y_{n_k}, y_{n_k-1})]^2, [d(y_{m_k}, y_{n_k-1})]^2, [d(y_{m_k-1}, y_{n_k-1})]^2, \frac{[d(y_{m_k}, y_{m_k-1})]^2 [1 + [d(y_{m_k-1}, y_{n_k-1})]^2]}{1 + [d(y_{m_k}, y_{n_k-1})]^2} \right\}. \tag{27}$$

It is easy to show that

$$\liminf_{k \rightarrow +\infty} M_1(x_{m_k}, x_{n_k}) \geq \max \left\{ 0, \frac{\varepsilon^2}{s^2}, \frac{\varepsilon^2}{s^4}, 0 \right\} \geq \frac{\varepsilon^2}{s^4}. \tag{28}$$

Applying (6) with $x = x_{m_k}$ and $y = x_{n_k}$, we get

$$\begin{aligned}
 \psi([d(y_{m_k}, y_{n_k})]^2) &\leq \psi(s^2 [d(y_{m_k}, y_{n_k})]^2) \\
 &\leq \psi(N_1(x_{m_k}, x_{n_k})) - \varphi(M_1(x_{m_k}, x_{n_k})).
 \end{aligned} \tag{29}$$

In light of (26), one can obtain

$$\begin{aligned}
 \psi(s^2 \varepsilon^2) &\leq \psi(s^2 \limsup_{k \rightarrow +\infty} [d(fx_{m_k}, fx_{n_k})]^2) \\
 &\leq \psi(\limsup_{k \rightarrow +\infty} N_1(x_{m_k}, x_{n_k})) - \varphi(\liminf_{k \rightarrow +\infty} M_1(x_{m_k}, x_{n_k})) \\
 &\leq \psi(s^2 \varepsilon^2) - \varphi(\liminf_{k \rightarrow +\infty} M_1(x_{m_k}, x_{n_k})),
 \end{aligned} \tag{30}$$

which implies that

$$\liminf_{k \rightarrow +\infty} M_1(x_{n_k}, x_{n_k}) = 0, \quad (31)$$

a contradiction to (28). It follows that $\{y_n\}$ is a Cauchy sequence in X and $\lim_{n,m \rightarrow +\infty} d(y_n, y_m) = 0$. Since X is complete b -metric-like space, there exists $u \in X$ such that

$$\lim_{n \rightarrow +\infty} d(y_n, u) = \lim_{n \rightarrow +\infty} d(fx_n, u) = \lim_{n \rightarrow +\infty} d(gx_{n+1}, u) = \lim_{n,m \rightarrow +\infty} d(y_n, y_m) = d(u, u) = 0. \quad (32)$$

Furthermore, we have $u \in g(X)$ since $g(X)$ is closed. It follows that one can choose a $z \in X$ such that $u = gz$, and one can write (32) as

$$\begin{aligned} \lim_{n \rightarrow +\infty} d(y_n, gz) &= \lim_{n \rightarrow +\infty} d(fx_n, gz) \\ &= \lim_{n \rightarrow +\infty} d(gx_{n+1}, gz) = 0. \end{aligned} \quad (33)$$

where

$$\begin{aligned} \psi\left(s^2 [d(y_{n_k}, fz)]^2\right) &= \psi\left(s^2 [d(fx_{n_k}, fz)]^2\right) \\ &\leq \psi(N_1(x_{n_k}, z)) - \phi(M_1(x_{n_k}, z)), \end{aligned} \quad (34)$$

If $fz \neq gz$, taking $x = x_{n_k}$ and $y = z$ in contractive condition (6), we get

$$\begin{aligned} N_1(x_{n_k}, z) &= \max\left\{[d(y_{n_k}, y_{n_k-1})]^2, [d(y_{n_k-1}, gz)]^2, [d(fz, gz)]^2, d(y_{n_k}, y_{n_k-1})d(y_{n_k}, fz), d(y_{n_k}, y_{n_k-1})d(y_{n_k-1}, gz)\right\}, \\ M_1(x_{n_k}, z) &= \max\left\{[d(fz, gz)]^2, [d(y_{n_k}, gz)]^2, [d(y_{n_k-1}, gz)]^2, \frac{[d(y_{n_k}, y_{n_k-1})]^2 [1 + [d(y_{n_k-1}, gz)]^2]}{1 + [d(y_{n_k}, gz)]^2}\right\}, \end{aligned} \quad (35)$$

and then we obtain

$$\begin{aligned} \limsup_{k \rightarrow +\infty} N_1(x_{n_k}, z) &= \max\{0, 0, [d(gz, fz)]^2, 0, 0\} = [d(gz, fz)]^2, \\ \liminf_{k \rightarrow +\infty} M_1(x_{n_k}, z) &= \max\{[d(gz, fz)]^2, 0, 0, 0\} = [d(gz, fz)]^2. \end{aligned} \quad (36)$$

Taking the upper limit as $k \rightarrow +\infty$ in (34),

$$\begin{aligned} \psi([d(gz, fz)]^2) &= \psi\left(s^2 \cdot \frac{1}{s^2} [d(gz, fz)]^2\right) \leq \psi\left(s^2 \limsup_{k \rightarrow +\infty} [d(fx_{n_k}, fz)]^2\right) \\ &\leq \psi(\limsup_{k \rightarrow +\infty} N_1(x_{n_k}, z)) - \phi(\liminf_{k \rightarrow +\infty} M_1(x_{n_k}, z)) \\ &= \psi([d(gz, fz)]^2) - \phi([d(gz, fz)]^2), \end{aligned} \quad (37)$$

which implies that $\phi([d(fz, gz)]^2) = 0$.

It follows that $d(fz, gz) = 0$. That is, $fz = gz$. Therefore, $u = fz = gz$ is a point of coincidence for f and g . We also conclude that the point of coincidence is unique. Assume on the contrary that there exist $z, z' \in C(f, g)$ and $z \neq z'$; applying (6) with $x = z$ and $y = z'$, we obtain that

$$\begin{aligned} \psi([d(fz, fz')]^2) &\leq \psi(s^2 [d(fz, fz')]^2) \\ &\leq \psi(N_1(z, z')) - \phi(M_1(z, z')) \\ &\leq \psi([d(fz, fz')]^2) - \phi([d(fz, fz')]^2). \end{aligned} \quad (38)$$

Hence, $fz = fz'$. That is, the point of coincidence is unique. Considering the weak compatibility of f and g , it can be shown that z is a unique common fixed-point. This completes the proof. \square

Theorem 2. Let (X, d) be a complete b -metric-like space with parameter $s \geq 1$ and let $f, g: X \rightarrow X$ be given

self-mappings satisfying $f(X) \subset g(X)$ where $g(X)$ is a closed subset of X . If there are functions $\psi \in \Psi$ and $\varphi \in \Phi$ such that

$$\psi(s^2 [d(fx, fy)]^2) \leq \psi(N_2(x, y)) - \varphi(M_2(x, y)), \quad (39)$$

where

$$\begin{aligned} N_2(x, y) &= \max \left\{ d(fx, fy)d(gx, gy), [d(gx, gy)]^2, \frac{[d(fy, gy)]^2 + [d(fx, gy)]^2}{1 + 4s^2} \right\}, \\ M_2(x, y) &= \max \left\{ [d(fy, gy)]^2, [d(fx, gy)]^2, [d(gx, gy)]^2, \frac{[d(fx, gx)]^2 [1 + [d(gx, gy)]^2]}{1 + [d(fx, gy)]^2}, \right. \\ &\quad \left. \frac{[d(gx, gy)]^2 [1 + [d(gx, gy)]^2]}{1 + [d(fx, gx)]^2} \right\}, \end{aligned} \quad (40)$$

then f and g have a unique coincidence point in X . Moreover, f and g have a unique common fixed-point provided that f and g are weakly compatible.

Proof. It is the same as the proof of Theorem 1, and we also define the sequences $\{x_n\}$ and $\{y_n\}$ in X by $y_n = fx_n = gx_{n+1}$ for all $n \in \mathbb{N}$. We also suppose that $y_n \neq y_{n+1}$ for each $n \in \mathbb{N}$, and it follows from (39) that

$$\begin{aligned} \psi(s^2 [d(y_n, y_{n+1})]^2) &= \psi(s^2 [d(fx_n, fx_{n+1})]^2) \\ &\leq \psi(N_2(x_n, x_{n+1})) - \varphi(M_2(x_n, x_{n+1})), \end{aligned} \quad (41)$$

where

$$\begin{aligned} N_2(x_n, x_{n+1}) &= \max \left\{ d(y_n, y_{n+1})d(y_{n-1}, y_n), [d(y_{n-1}, y_n)]^2, \frac{[d(y_{n+1}, y_n)]^2 + [d(y_n, y_n)]^2}{1 + 4s^2} \right\}, \\ M_2(x_n, x_{n+1}) &= \max \left\{ [d(y_{n+1}, y_n)]^2, [d(y_n, y_n)]^2, [d(y_{n-1}, y_n)]^2, \frac{[d(y_n, y_{n-1})]^2 [1 + [d(y_{n-1}, y_n)]^2]}{1 + [d(y_n, y_n)]^2}, \right. \\ &\quad \left. \frac{[d(y_{n-1}, y_n)]^2 [1 + [d(y_{n-1}, y_n)]^2]}{1 + [d(y_n, y_{n-1})]^2} \right\}. \end{aligned} \quad (42)$$

$$\begin{aligned} M_2(x_n, x_{n+1}) &= \max \left\{ [d(y_{n+1}, y_n)]^2, [d(y_n, y_n)]^2, [d(y_{n-1}, y_n)]^2, \frac{[d(y_n, y_{n-1})]^2 [1 + [d(y_{n-1}, y_n)]^2]}{1 + [d(y_n, y_n)]^2}, \right. \\ &\quad \left. \frac{[d(y_{n-1}, y_n)]^2 [1 + [d(y_{n-1}, y_n)]^2]}{1 + [d(y_n, y_{n-1})]^2} \right\}. \end{aligned} \quad (43)$$

If we assume that for some $n \in \mathbb{N}$,

$$d(y_n, y_{n+1}) \geq d(y_{n-1}, y_n) > 0, \quad (44)$$

then from inequalities (42) and (43), we get that

$$\begin{aligned} N_2(x_n, x_{n+1}) &\leq [d(y_{n+1}, y_n)]^2, \\ M_2(x_n, x_{n+1}) &\geq [d(y_{n+1}, y_n)]^2. \end{aligned} \quad (45)$$

In view of (41), we have the following inequality:

$$\begin{aligned} \psi([d(y_n, y_{n+1})]^2) &\leq \psi(s^2 [d(y_n, y_{n+1})]^2) \\ &\leq \psi(N_2(x_n, x_{n+1})) - \varphi(M_2(x_n, x_{n+1})) \\ &\leq \psi([d(y_n, y_{n+1})]^2) - \varphi([d(y_n, y_{n+1})]^2), \end{aligned} \quad (46)$$

which gives that $d(y_n, y_{n+1}) = 0$, a contradiction to $d(y_n, y_{n+1}) > 0$. It follows that $d(y_n, y_{n+1}) < d(y_n, y_{n-1})$. Hence, $\{d(y_n, y_{n+1})\}$ is a nonincreasing sequence. Consequently, the limit of the sequence is a nonnegative number, say $r \geq 0$. That is, $\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) = r$.

According to (42) and (43), we have

$$\begin{aligned} N_2(x_n, x_{n+1}) &\leq [d(y_n, y_{n-1})]^2, \\ M_2(x_n, x_{n+1}) &\geq [d(y_n, y_{n-1})]^2. \end{aligned} \quad (47)$$

So,

$$\begin{aligned} \psi([d(y_n, y_{n+1})]^2) &\leq \psi(N_2(x_n, x_{n+1})) - \varphi(M_2(x_n, x_{n+1})) \\ &\leq \psi([d(y_n, y_{n-1})]^2) - \varphi([d(y_n, y_{n-1})]^2). \end{aligned} \quad (48)$$

If $r > 0$, then letting $n \rightarrow +\infty$ in above inequality, we obtain that $\psi(r^2) = \psi(r^2) - \varphi(r^2)$ which implies that $r = 0$, i.e., $\lim_{n \rightarrow +\infty} d(y_n, y_{n+1}) = 0$.

Now we prove that $\lim_{n,m \rightarrow +\infty} d(y_n, y_m) = 0$. If not, as the proof of Theorem 1, there exists $\varepsilon > 0$ for which one can find sequences $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ so that n_k is the smallest index for which $n_k > m_k > k$, and the following inequalities hold:

$$\begin{aligned} \varepsilon &\leq \limsup_{k \rightarrow +\infty} d(y_{m_k}, y_{n_k}) \leq s\varepsilon, \\ \frac{\varepsilon}{s} &\leq \limsup_{k \rightarrow +\infty} d(y_{m_k}, y_{n_k-1}) \leq \varepsilon, \\ \frac{\varepsilon}{s} &\leq \limsup_{k \rightarrow +\infty} d(y_{m_k-1}, y_{n_k}) \leq s^2\varepsilon, \\ \frac{\varepsilon}{s^2} &\leq \limsup_{k \rightarrow +\infty} d(y_{m_k-1}, y_{n_k-1}) \leq s\varepsilon. \end{aligned} \quad (49)$$

$$\begin{aligned} N_2(x_{m_k}, x_{n_k}) &= \max \left\{ d(y_{m_k}, y_{n_k})d(y_{m_k-1}, y_{n_k-1}), [d(y_{m_k-1}, y_{n_k-1})]^2, \frac{[d(y_{n_k}, y_{n_k-1})]^2 + [d(y_{m_k}, y_{n_k-1})]^2}{1 + 4s^2} \right\}, \\ M_2(x_{m_k}, x_{n_k}) &= \max \left\{ \begin{aligned} &[d(y_{n_k}, y_{n_k-1})]^2, [d(y_{m_k}, y_{n_k-1})]^2, [d(y_{m_k-1}, y_{n_k-1})]^2, \frac{[d(y_{m_k}, y_{m_k-1})]^2 [1 + [d(y_{m_k-1}, y_{n_k-1})]^2]}{1 + [d(y_{m_k}, y_{n_k-1})]^2}, \\ &\frac{[d(y_{m_k-1}, y_{n_k-1})]^2 [1 + [d(y_{m_k-1}, y_{n_k-1})]^2]}{1 + [d(y_{m_k}, y_{n_k-1})]^2} \end{aligned} \right\}, \end{aligned} \quad (50)$$

using (49), one can obtain that

$$\limsup_{k \rightarrow +\infty} N_2(x_{m_k}, x_{n_k}) \leq \max \left\{ s^2\varepsilon^2, s^2\varepsilon^2, \frac{\varepsilon^2}{1 + 4s^2} \right\} = s^2\varepsilon^2, \quad (51)$$

$$\liminf_{k \rightarrow +\infty} M_2(x_{m_k}, x_{n_k}) \geq \max \left\{ 0, \frac{\varepsilon^2}{s^2}, \frac{\varepsilon^2}{s^4}, 0, \frac{\varepsilon^2}{s^4} \left(1 + \frac{\varepsilon^2}{s^4} \right) \right\} \geq \frac{\varepsilon^2}{s^4}. \quad (52)$$

Taking $x = x_{m_k}$ and $y = x_{n_k}$ in (39), we get

$$\begin{aligned} \psi\left([d(y_{m_k}, y_{n_k})]^2\right) &\leq \psi\left(s^2[d(y_{m_k}, y_{n_k})]^2\right) \\ &\leq \psi(N_2(x_{m_k}, x_{n_k})) - \varphi(M_2(x_{m_k}, x_{n_k})). \end{aligned} \quad (53)$$

Therefore, we have

$$\begin{aligned} \psi(s^2\varepsilon^2) &\leq \psi\left(s^2\limsup_{k \rightarrow +\infty}[d(fx_{m_k}, fx_{n_k})]^2\right) \\ &\leq \psi(\limsup_{k \rightarrow +\infty}N_2(x_{m_k}, x_{n_k})) - \varphi(\liminf_{k \rightarrow +\infty}M_2(x_{m_k}, x_{n_k})) \\ &\leq \psi(s^2\varepsilon^2) - \varphi(\liminf_{k \rightarrow +\infty}M_2(x_{m_k}, x_{n_k})), \end{aligned} \quad (54)$$

and we conclude that $\liminf_{k \rightarrow +\infty}M_2(x_{m_k}, x_{n_k}) = 0$ which gives a contradiction to (52). Hence, $\lim_{n,m \rightarrow +\infty}d(y_n, y_m) = 0$.

The completeness of X ensures that there exists $u \in X$ such that

$$\lim_{n \rightarrow +\infty}d(y_n, u) = \lim_{n \rightarrow +\infty}d(fx_n, u) = \lim_{n \rightarrow +\infty}d(gx_{n+1}, u) = \lim_{n,m \rightarrow +\infty}d(y_n, y_m) = d(u, u) = 0. \quad (55)$$

In view of the hypothesis $g(X)$ is closed, we obtain that $u \in g(X)$. It follows that one can choose $z \in X$ such that $u = gz$, and we write the above equality as

$$\begin{aligned} \lim_{n \rightarrow +\infty}d(y_n, gz) &= \lim_{n \rightarrow +\infty}d(fx_n, gz) \\ &= \lim_{n \rightarrow +\infty}d(gx_{n+1}, gz) = 0. \end{aligned} \quad (56)$$

If $fx \neq gx$, putting $x = x_{n_k}$ and $y = z$ into contractive condition (39), we have

$$\psi\left(s^2[d(fx_{n_k}, fz)]^2\right) \leq \psi(N_2(x_{n_k}, z)) - \varphi(M_2(x_{n_k}, z)), \quad (57)$$

where

$$\begin{aligned} N_2(x_{n_k}, z) &= \max\left\{d(y_{n_k}, fz)d(y_{n_k-1}, gz), [d(y_{n_k-1}, gz)]^2, \frac{[d(fz, gz)]^2 + [d(y_{n_k}, gz)]^2}{1 + 4s^2}\right\}, \\ M_2(x_{n_k}, z) &= \max\left\{[d(fz, gz)]^2, [d(y_{n_k}, gz)]^2, [d(y_{n_k-1}, gz)]^2, \frac{[d(y_{n_k}, y_{n_k-1})]^2[1 + [d(y_{n_k-1}, gz)]^2]}{1 + [d(y_{n_k}, gz)]^2}, \right. \\ &\quad \left. \frac{[d(y_{n_k-1}, gz)]^2[1 + [d(y_{n_k-1}, gz)]^2]}{1 + d(y_{n_k}, y_{n_k-1})}\right\}. \end{aligned} \quad (58)$$

Consequently, we get

$$\begin{aligned} \limsup_{k \rightarrow +\infty}N_2(x_{n_k}, z) &= \max\left\{0, 0, \frac{[d(fz, gz)]^2}{1 + 4s^2}\right\} \leq [d(fz, gz)]^2, \\ \liminf_{k \rightarrow +\infty}M_2(x_{n_k}, z) &= \max\{[d(fz, gz)]^2, 0, 0, 0, 0\} = [d(fz, gz)]^2. \end{aligned} \quad (59)$$

Taking the upper limit as $k \rightarrow +\infty$ in (57), we have

$$\begin{aligned}
 \psi([d(gz, fz)]^2) &= \psi\left(s^2 \cdot \frac{1}{s^2} [d(gz, fz)]^2\right) \\
 &\leq \psi\left(s^2 \limsup_{k \rightarrow +\infty} [d(fx_{n_k}, fz)]^2\right) \\
 &\leq \psi\left(\limsup_{k \rightarrow +\infty} N_2(x_{n_k}, z)\right) \\
 &\quad - \varphi\left(\liminf_{k \rightarrow +\infty} M_2(x_{n_k}, z)\right) \\
 &\leq \psi([d(gz, fz)]^2) - \varphi([d(gz, fz)]^2), \tag{60}
 \end{aligned}$$

which implies that $d(fz, gz) = 0$. That is, $u = fz = gz$ is a point of coincidence for f and g . Using the same technique in the proof of Theorem 1, it can be proved that z is a unique common fixed-point. This completes the proof. \square

The following examples support Theorems 1 and 2.

Example 2. Let $X = [0, 1]$ be endowed with the b -metric-like $d(x, y) = (x + y)^2$ for all $x, y \in X$ and $s = 2$. Define mappings $f, g: X \rightarrow X$ by $fx = (x/64)$ and $gx = (x/2)$.

The control functions $\psi, \varphi: [0, +\infty) \rightarrow [0, +\infty)$ are defined as $\psi(t) = (5t/4)$, $\varphi(t) = (48545t/87846)$ for all $t \in [0, +\infty)$. It is clear that $f(X) \subset g(X)$ and $g(X)$ is closed. For all $x, y \in X$, we have

$$\begin{aligned}
 \psi(s^2 [d(fx, fy)]^2) &= \psi\left(4 \cdot \left(\frac{x}{64} + \frac{y}{64}\right)^4\right) = \frac{5}{4} \cdot 4 \left(\frac{x}{64} + \frac{y}{64}\right)^4 = \frac{5}{64^4} (x + y)^4, \\
 \psi(N_1(x, y)) &\geq \psi([d(gx, gy)]^2) = \frac{5}{4} \cdot \left(\frac{x}{2} + \frac{y}{2}\right)^4 = \frac{5}{64} (x + y)^4, \tag{61} \\
 \varphi(M_1(x, y)) &= \max \left\{ \left(\frac{y}{64} + \frac{y}{2}\right)^4, \left(\frac{x}{64} + \frac{y}{2}\right)^4, \left(\frac{x}{2} + \frac{y}{2}\right)^4, \frac{((x/64) + (x/2))^4 [1 + ((x/2) + (y/2))^4]}{1 + ((x/64) + (y/2))^4} \right\}.
 \end{aligned}$$

It follows that

$$\begin{aligned}
 \varphi(M_1(x, y)) &\leq \varphi\left(2 \cdot \left(\frac{33x}{64} + \frac{33y}{64}\right)^4\right) \\
 &= \frac{5 \cdot 64^3 - 5}{2 \cdot 33^4} \cdot 2 \cdot \left(\frac{33}{64}\right)^4 (x + y)^4 \tag{62} \\
 &= \frac{1310715}{64^4} (x + y)^4.
 \end{aligned}$$

It is easy to see that

$$\begin{aligned}
 \psi(s^2 [d(fx, fy)]^2) &= \psi([d(gx, gy)]^2) \\
 &\quad - \varphi\left(2 \cdot \left(\frac{33}{64}\right)^4 (x + y)^4\right) \tag{63} \\
 &\leq \psi(N_1(x, y)) - \varphi(M_1(x, y)).
 \end{aligned}$$

Therefore, the conditions of Theorem 1 are satisfied. It is obvious that 0 is the unique common fixed-point of f and g .

Example 3. Let $X = [-1, 1]$ and $d(x, y) = (x + y)^2$ for all $x, y \in X$, so (X, d) is a b -metric-like space with parameter $s = 2$. Let $f, g: X \rightarrow X$ be defined by the formulas

$$f(x) = \begin{cases} 0, & x \in [-1, 1), \\ \frac{1}{8}, & x = 1, \end{cases} \tag{64}$$

$$g(x) = \frac{x^2}{2}.$$

The control functions $\psi, \varphi: [0, +\infty) \rightarrow [0, +\infty)$ are defined as $\psi(t) = bt$ and $\varphi(t) = (b - 1)t$, for all $t \in [0, +\infty)$, where $1 < b \leq (2048/2045)$.

Now we consider four cases:

Case 1. $x \neq 1, y \neq 1$. It is clear that

$$\psi(s^2[d(fx, fy)]^2) = \psi(0) = 0,$$

$$\psi(N_2(x, y)) = bN_2(x, y) \geq b[d(gx, gy)]^2 = b\left(\frac{x^2}{2} + \frac{y^2}{2}\right)^4, \quad (65)$$

$$\varphi(M_2(x, y)) = (b-1)M_2(x, y) \leq (b-1) \cdot 2\left(\frac{x^2}{2} + \frac{y^2}{2}\right)^4,$$

so we have

$$\begin{aligned} \psi(s^2[d(fx, fy)]^2) &= 0 \leq b\left(\frac{x^2}{2} + \frac{y^2}{2}\right)^4 \\ &\quad - 2(b-1)\left(\frac{x^2}{2} + \frac{y^2}{2}\right)^4 \\ &\leq \psi(N_2(x, y)) - \varphi(M_2(x, y)). \end{aligned} \quad (66)$$

Case 2. $x = 1, y \neq 1$. It follows that

$$\psi(s^2[d(fx, fy)]^2) = 4b\left(\frac{1}{8}\right)^4 = \frac{b}{1024},$$

$$\begin{aligned} \psi(N_2(x, y)) &= bN_2(x, y) \geq bd(fx, fy)d(gx, gy) \\ &= \frac{b}{64}\left(\frac{1}{2} + \frac{y^2}{2}\right)^2 \\ &\geq \frac{1}{64} \cdot \frac{b}{4} = \frac{b}{256}, \end{aligned}$$

$$\begin{aligned} \varphi(M_2(x, y)) &= (b-1)M_2(x, y) \leq 2(b-1), \end{aligned} \quad (67)$$

and we obtain that

$$\begin{aligned} \psi(s^2[d(fx, fy)]^2) &= \frac{b}{1024} \leq \frac{b}{256} - 2(b-1) \\ &\leq \psi(N_2(x, y)) - \varphi(M_2(x, y)). \end{aligned} \quad (68)$$

Case 3. $x \neq 1, y = 1$. By concise calculation, we get

$$\psi(s^2[d(fx, fy)]^2) = 4b\left(\frac{1}{8}\right)^4 = \frac{b}{1024},$$

$$\begin{aligned} \psi(N_2(x, y)) &= bN_2(x, y) \\ &\geq bd(fx, fy)d(gx, gy) \\ &= \frac{b}{64}\left(\frac{1}{2} + \frac{x^2}{2}\right)^2 \\ &\geq \frac{1}{64} \cdot \frac{b}{4} = \frac{b}{256}, \end{aligned}$$

$$\varphi(M_2(x, y)) = (b-1)M_2(x, y) \leq 2(b-1). \quad (69)$$

It follows that

$$\begin{aligned} \psi(s^2[d(fx, fy)]^2) &= \frac{b}{1024} \leq \frac{b}{256} - 2(b-1) \\ &\leq \psi(N_2(x, y)) - \varphi(M_2(x, y)). \end{aligned} \quad (70)$$

Case 4. $x = 1, y = 1$. It is easy to see that

$$\psi(s^2[d(fx, fy)]^2) = 4b\left(\frac{1}{8} + \frac{1}{8}\right)^4 = \frac{b}{64},$$

$$\psi(N_2(x, y)) = bN_2(x, y)$$

$$\geq bd(fx, fy)d(gx, gy) = b\left(\frac{1}{8} + \frac{1}{8}\right)^2 \cdot \left(\frac{1}{2} + \frac{1}{2}\right)^2$$

$$= \frac{b}{16} \varphi(M_2(x, y)) = (b-1)M_2(x, y) \leq 2(b-1). \quad (71)$$

Obviously, for all $x, y \in [-1, 1]$, we obtain

$$\begin{aligned} \psi(s^2[d(fx, fy)]^2) &= \frac{b}{64} \leq \frac{b}{16} - 2(b-1) \\ &\leq \psi(N_2(x, y)) - \varphi(M_2(x, y)). \end{aligned} \quad (72)$$

It follows from Theorem 2 that f and g have a unique common fixed-point in X . It is easy to show that 0 is the unique common fixed-point of f and g .

According to Theorems 1 and 2, one can get the following results.

Corollary 1. Let (X, d) be a complete b -metric-like space with constant $s \geq 1$ and let $f, g: X \rightarrow X$ be given self-mappings satisfying $f(X) \subset g(X)$ where $g(X)$ is a closed subset of X . If the following condition is satisfied:

$$s^2[d(fx, fy)]^2 \leq N_1(x, y) - L[d(gx, gy)]^2, \quad (73)$$

where $L \in (0, 1)$ is a constant and then f and g have a unique coincidence point in X . Moreover, f and g have a unique common fixed-point provided that f and g are weakly compatible.

Corollary 2. Let (X, d) be a complete b -metric-like space with constant $s \geq 1$ and let $f, g: X \rightarrow X$ be given self-mappings satisfying $f(X) \subset g(X)$ where $g(X)$ is a closed subset of X . If the following condition is satisfied:

$$s^2[d(fx, fy)]^2 \leq N_2(x, y) - M_2(x, y), \quad (74)$$

then f and g have a unique coincidence point in X . Moreover, if f and g are weakly compatible, then f and g have a unique common fixed-point.

4. Application

In this section, we will use Corollary 1 to show that there is a solution to the following system of integral equations:

$$\begin{cases} x(t) = \int_0^t K(r, x(r))dr, \\ x(t) = \int_0^t x(r)dr. \end{cases} \quad (75)$$

Let $X = C([0, T])$ be the set of real continuous functions defined on $[0, T]$ for $T > 0$. Define the b -metric-like mapping $d: X \times X \rightarrow [0, +\infty)$ by

$$d(x, y) = \max_{t \in [0, T]} (|x(t)| + |y(t)|)^{(m/2)}, \quad (76)$$

for all $x, y \in X$, where $m > 2$. It is evident that (X, d) with $s = 2^{(m/2)} - 1$ is a complete b -metric-like space. Consider the mappings $f, g: X \rightarrow X$ by

$$\begin{aligned} fx(t) &= \int_0^t K(r, x(r))dr, \\ gx(t) &= \int_0^t x(r)dr. \end{aligned} \quad (77)$$

Theorem 3. Consider the system of integral equations (75) and suppose that the following conditions hold:

- (i) $K: [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^+$ is continuous
- (ii) If $K(r, x(r)) = x(r)$ for all $r \in [0, T]$, then we have $K(r, \int_0^t x(\omega)d\omega) = \int_0^t K(\omega, x(\omega))d\omega$
- (iii) There exists a continuous function $\gamma: [0, T] \rightarrow [0, 1]$ such that $|K(r, x(r))| \leq \gamma(r)|x(r)|$, for all $r \in [0, T]$, $x, y \in X$
- (iv) There exists constant L such that for all $r \in [0, T]$,

$$\sup_{t \in [0, T]} \gamma(t) \leq \left(\frac{1-L}{s^2} \right)^{1/m}. \quad (78)$$

Then, the system of integral equations (75) has a unique solution in $x \in X$.

Proof. Obviously, by condition (ii), f, g are weakly compatible. Let $x, y \in X$; from conditions (i), (iii), and (iv), for all $t \in [0, T]$, we have

$$\begin{aligned}
s^2(|fx(t)| + |fy(t)|)^m &= s^2\left(\left|\int_0^t K(r, x(r))dr\right| + \left|\int_0^t K(r, y(r))dr\right|\right)^m \\
&\leq s^2\left(\int_0^t |K(r, x(r))| + |K(r, y(r))|dr\right)^m \\
&\leq s^2\left(\int_0^t \gamma(r)(|x(r)| + |y(r)|)dr\right)^m \\
&\leq s^2\left(\sup_{t \in [0, T]} \gamma(t)\right)^m \cdot [d(gx(t), gy(t))]^2 \\
&\leq (1 - L)[d(gx(t), gy(t))]^2 \\
&\leq N_1(x(t), y(t)) - L[d(gx(t), gy(t))]^2,
\end{aligned} \tag{79}$$

which implies that

$$s^2[d(fx(t), fy(t))]^2 \leq N_1(x(t), y(t)) - L[d(gx(t), gy(t))]^2. \tag{80}$$

Consequently, letting $\psi(x) = x$ and $\varphi(x) = Lx$, all of the conditions of Corollary 1 are satisfied. As a result, the mappings f, g have a unique common fixed-point in X , which is the solution of the system of integral equations (75). \square

5. Conclusions

In this paper, we introduced new generalized (ψ, φ) -weakly contractive mappings and obtained common fixed-point theorems in the framework of b -metric-like space. Further we provided examples that elaborated the useability of our results. As an application of our result, we obtained a solution of the system of integral equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgments

This study was financially supported by the Science and Research Project Foundation of Liaoning Province Education Department (nos. LQN201902 and LJC202003).

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Research Article

Existence and Uniqueness Results for Two-Term Nonlinear Fractional Differential Equations via a Fixed Point Technique

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Received 3 December 2020; Revised 9 December 2020; Accepted 16 December 2020; Published 11 January 2021

Academic Editor: Naeem Saleem

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The work addressed in this paper is to ensure the existence and uniqueness of positive solutions for initial value problems for nonlinear fractional differential equations with two terms of fractional orders. By virtue of recent fixed point theorems on mixed monotone operators, we get some new straightforward results with a wide range of applications.

1. Introduction

In recent years, we see a great interest to consider fractional differential equations in various kinds of studies. Namely, these equations have striking applications in diverse science. Moreover, during the past decades, new areas for applications of fractional differential and integral equations with initial boundary conditions have been appeared. In the modeling of significant numbers of phenomena occurring in engineering and scientific subjects, we encountered a problem of such kinds. In multiple cases, modeling by fractional operators can also be done better than modeling by ordinary derivative operators. One can see fractional operators in applications of rheology, control, viscoelasticity, porous structures, chemical physics, electrochemistry, and significant number of other branches of science (see [1–3] for more details). For these reasons, there exist many articles about the existence and uniqueness of solutions of problems containing operators of fractional orders

and using fixed point techniques [4–27]. But, the number of research studies on the study of existence and uniqueness of solutions for FDEs with two or more terms of fractional operators is limited. Fujita [28] studied the following specific Cauchy problem:

$$\frac{\partial^{\alpha_1}}{\partial t} w(t, x) = \frac{\partial^{\alpha_2}}{\partial t} w(t, x), \quad 1 \leq \alpha_1, \alpha_2 \leq 2, \quad (1)$$

and established some existence and uniqueness results. In [29], the author considered an initial value problem of the following form:

$$D^\alpha y(t) = g(t, D^\beta y(t)), \quad 0 < t \leq 1, \quad (2)$$

$$y^{(k)}(0) = \mu_k, \quad 0 \leq k \leq n-1, k \in \mathbb{Z}, \quad (3)$$

where $n-1 < \beta < \alpha < n$, and studied existence and uniqueness of related solutions. Based on the results of the above paper, the following exacting conditions should be satisfied on g for the existence of solutions:

- (C1) g is a continuous and differentiable function on $[0, 1] \times \mathbf{R} \rightarrow \mathbf{R}$.
- (C2) For $t \in I$, $g(t, 0) \neq 0$ and $g(0, 0) = 0$, where I is any compact subinterval of $[0, 1]$.
- (C3) There exists a continuous function q which satisfies

$$C \sup_{t \in [0, 1]} \int_0^t (t-r)^{\alpha-\beta-1} q(r) dr < 1, \quad (n-1 < \beta < \alpha < n), \quad (4)$$

where $C = 1/\Gamma(\alpha - \beta)$.

To detect a unique solution other than (C1)–(C3), the following restriction should hold.

- (C4) Let w_1 and w_2 be any real numbers and $q \in C([0, 1], [0, \infty))$ be the continuous function appeared in (4), then

$$|g(t, w_1) - g(t, w_2)| \leq q(t)|w_1 - w_2|, \quad 0 \leq t \leq 1. \quad (5)$$

The authors in [16, 30] have studied the same problem. In [16], we observe that the necessary conditions to exist at least one solution for (2) and (3) are (C1), (C2), and the following condition.

- (C5) For a continuous function $y \in C[0, \infty)$, there exist positive numbers c and d such that

$$|g(t, y(t))| \leq c|y(t)| + d, \quad 0 \leq t \leq 1. \quad (6)$$

Also, the additional condition that $g(t, x(t))$ satisfies a Lipschitz condition is needed to prove the uniqueness of the solution:

$$|g(t, z_1(t)) - g(t, z_2(t))| \leq M|z_1(t) - z_2(t)|, \quad 0 \leq t \leq 1, \quad (7)$$

where M is the Lipschitz constant. In [31], the following general form is considered:

$$D^\alpha y(t) = g(t, y(t), D^\beta y(t)), \quad 0 < t \leq 1, \quad (8)$$

$$y^{(k)}(0) = \mu_k, \quad 0 \leq k \leq n-1, k \in \mathbf{Z}, \quad (9)$$

where $n-1 < \beta < \alpha < n$, ($n \in \mathbf{N}$) and D^α and D^β are the fractional derivative orders in the Caputo definition and $g \in C([0, 1] \times \mathbf{R})$. We have used an index fixed point theorem and obtained some new results about existence and also multiplicity of solutions. In this paper, we consider the same general form (8) and (9) and try to use new fixed point theorems proved for operators having mixed monotone property to establish new results with a wide range of applications for existence and uniqueness.

2. Preliminaries

For convenience of readers, we provide some useful definitions and previous results which we use throughout the paper.

Definition 1 (see [1, 3]). Let $\alpha > 0$. The operator defined by

$$I^\alpha y(t) = \int_0^t \frac{1}{(t-r)^{\alpha-1}\Gamma(\alpha)} y(r) dr, \quad (10)$$

where $y \in C[0, \infty) \cap L_{loc}^1[0, \infty)$, $\alpha \in (m-1, m)$ and $m \in \mathbf{N}$, is the fractional integral of y of order α in Riemann–Liouville definition.

After introducing the fractional integral operator, it is time to remind the fractional differentiation definitions. There are several ways to define fractional derivative. Here, we focus on the Caputo definition.

Definition 2 (see [1, 3]). Let $n = [\alpha] + 1$ where $\alpha \geq 0$. If $y(t) \in AC^n[0, 1]$, then the fractional derivative of y in Caputo sense exists on the interval $[0, 1]$, almost everywhere, and is obtained by

$${}^c D^\alpha y(t) = \int_0^t \frac{y^{(n)}(r)}{\Gamma(n-\alpha)(t-r)^{\alpha-n+1}} dr. \quad (11)$$

For the sake of convenience, here, we mention some properties of Caputo fractional operator. For the power function $f(t) = t^q$, $q \geq 0$, taking the Caputo derivative yields

$${}^c D^\alpha t^q = \begin{cases} \frac{\Gamma(q+1)}{\Gamma(q-\alpha+1)} t^{q-\alpha}, & (q \geq n-1), \\ 0, & (q \leq n-1). \end{cases} \quad (12)$$

By virtue of this formula, one can deduce ${}^c D^\alpha k = k {}^c D^\alpha t^0 = 0$. Therefore, unlike Riemann–Liouville derivative, the Caputo derivative of a constant function is zero. In the following equality:

$${}^c D^\alpha (a_0 t^r + a_1 t^{r-1} + \dots + a_1) = 0, \quad \alpha \in (m-1, m), \quad (13)$$

the polynomial's degree is less than or equal to $m-1$, i. e., $r \leq m-1$. From the following relation, it is clear that similar to the case of integer order, the α th fractional integral of the Caputo fractional derivative of order α requires to know the values of the function and its integer order derivatives:

$$I^\alpha {}^c D^\alpha g(t) = g(t) - \sum_{l=0}^{m-1} g^{(l)}(t_0^+) \frac{t^l}{l!}, \quad m-1 < \alpha \leq m. \quad (14)$$

The results of the paper about the existence and uniqueness are expressed by the famous beta function given as follows.

Definition 3. The following integral defines the beta function:

$$B(x, y) = \int_0^1 s^{x-1} (1-s)^{y-1} ds, \quad x, y > 0. \quad (15)$$

It is known that

$$B(r, s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}, \quad (16)$$

where Γ denotes the Gamma function.

The following lemma transforms problems (8) and (9) to an integral equation [29, 30], at which the nonlinear term g needs to satisfy the conditions (C1) and (C2).

Lemma 1 (see [29]). *Let $n \in \mathbf{N}$, $\beta < \alpha$ and $\alpha, \beta \in (n-1, n)$. If (C1) and (C2) hold, then $y \in C^n[0, 1]$ is a solution of the following problem:*

$$D^\alpha y(x) = g(x, y(x), D^\beta y(x)), \quad 0 < x \leq 1, \quad (17)$$

$$y^{(k)}(0) = \mu_k, \quad 0 \leq k \leq n-1, k \in \mathbf{Z}, \quad (18)$$

if and only if

$$y(x) = \sum_{k=0}^{n-1} \frac{x^k}{k!} \eta_k + \frac{1}{\Gamma(\beta)} \int_0^x (x-r)^{\beta-1} w(r) dr, \quad 0 \leq x \leq 1, \quad (19)$$

where w is a continuous function and satisfies the following integral equation:

$$w(x) = \int_0^1 K(x, r) g(r, w(r)) dr, \quad 0 \leq x \leq 1, \quad (20)$$

where

$$K(x, r) = \frac{1}{\Gamma(\alpha-\beta)} (x-r)_+^{\alpha-\beta-1}, \quad (21)$$

with

$$(x-r)_+^{\alpha-\beta-1} = \begin{cases} (x-r)^{\alpha-\beta-1}, & r \leq x, r, x \in [0, 1] \\ 0, & r \geq x. \end{cases} \quad (22)$$

Here and anywhere below, the function $K(x, r)$ is the Green function.

Lemma 2. *Let $n, m \in \mathbf{N}$ and $n-1 < \beta \leq m-1 < \alpha < m$. If (C1) and (C2) hold, then $y \in C^m[0, 1]$ is a solution of (17) and (18) if and only if*

$$y(x) = \sum_{l=0}^{n-1} \frac{x^l}{l!} \eta_l + \int_0^x \frac{(x-r)^{n-1} w(r) dr}{(n-1)!}, \quad 0 \leq x \leq 1, \quad (23)$$

where $w \in C[0, 1]$ satisfies the following integral equation:

$$w(x) = \sum_{l=0}^{m-n-1} \frac{x^l}{l!} \eta_{n+l} + \frac{1}{\Gamma(\alpha-n)} \int_0^x (x-r)^{\alpha-n-1} \times f(r, w(r), \frac{1}{\Gamma(n-\beta)}) \quad (24)$$

$$\int_0^r (r-h)^{n-\beta-1} w(h) dh dr, \quad 0 \leq x \leq 1.$$

Now, we present the following lemma which will be useful in proving an existence and uniqueness theorem of the solution of (8) and (9).

Lemma 3. *For the Green function $G(t, s)$, the following relations hold:*

$$\int_0^1 K(x, r) dr \leq B(\alpha-\beta, \alpha-\beta), \quad 0 \leq x \leq 1, \quad (25)$$

$$\int_0^1 K(x, r) dr \geq \frac{\Gamma(\alpha-\beta)}{\Gamma^2(\alpha-\beta)}, \quad 0 \leq x \leq 1.$$

Proof. By making use of (14), it is easy and straightforward to prove the properties for $K(x, r)$.

Now, we provide some definitions and a fixed point theorem involving mixed monotone operators. We suggest references [19, 20] for more details. \square

Definition 4. Let $(E, \|\cdot\|)$ be a Banach space and $P \subseteq E$ be a nonempty closed convex subset. P is a cone in E if

(I) For any $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$

(II) For any $x \in P$, if $-x \in P$ then $x = \theta$

where θ denotes the zero element of the Banach space E .

If there exists a constant $C > 0$ such that, for any $x, y \in P$ with $\theta \leq x \leq y$, we have $\|x\| \leq C\|y\|$, then P is called a normal cone. Then, we give the definition of a partially ordered Banach space. We say that E is partially ordered by P if $x \leq y$ iff $y-x \in P$, for any $x, y \in P$. For arbitrary $x_1, x_2 \in E$, the ordered interval is defined by $[x_1, x_2] = \{x \in E \mid x_1 \leq x \leq x_2\}$. Let $T: E \rightarrow E$ be an operator. If $u \leq v$ implies $Tu \leq Tv$, then T is called increasing.

Let us briefly recall that the operator $T: P \times P \rightarrow P$ is increasing in its first variable and decreasing in its second variable, if from $x_i, y_i \in P$ ($i = 1, 2$), $x_1 \leq x_2, y_1 \geq y_2$, one can deduce $T(x_1, y_1) \leq T(x_2, y_2)$. If $x \geq \theta$ implies that $T(x) \geq \theta$, then the operator T is called positive.

In the following definition, P is a cone on the Banach space $(E, \|\cdot\|)$.

Definition 5 (see [19, 20]). Suppose $T: P \times P \rightarrow P$ is increasing in its first variable and decreasing in its second variable, then T is called a mixed monotone operator.

If for an element $x \in P$, $T(x, x) = x$, then x is called a fixed point of T .

Theorem 1 (see [4]). Suppose $T: P \times P \rightarrow P$ is a mixed monotone operator and let the following conditions hold:

(i) If $c \in (0, 1), x, y \in P$, then there exists $\alpha(c, x, y) \in (1, \infty)$ such that

$$T(cx, y) \leq c^{\alpha(c, x, y)} T(x, y). \quad (26)$$

(ii) There exist two elements $u_0, v_0 \in P, r \in (0, 1)$ such that

$$\begin{aligned} u_0 &\leq rv_0, \\ T(u_0, v_0) &\geq u_0, \\ T(v_0, u_0) &\leq v_0. \end{aligned} \quad (27)$$

Then, T has a unique fixed point u^* in $[u_0, rv_0]$ such that $T(u^*, u^*) = u^*$. Moreover, one can construct successively the following iterates:

$$\begin{aligned} u_n &= T(u_{n-1}, v_{n-1}), \\ v_n &= T(v_{n-1}, u_{n-1}), \quad n = 1, 2, \dots, \end{aligned} \quad (28)$$

starting from $x_0, y_0 \in [u_0, rv_0]$ to get the fixed point of T . Also, $\|u_n - u^*\| \rightarrow 0, \|v_n - u^*\| \rightarrow 0$ as $n \rightarrow \infty$.

3. Main Results

In this section, we prove new existence and uniqueness results for nonlinear fractional differential equations with two terms of fractional derivatives. Compared with the results of the previous studies, our results are simple and straightforward. They are expressed with the values of beta and gamma functions. For these reasons, these results can be applied for a wide range of problems. To make effective our obtained result, some examples provided at the end of section cannot be considered by techniques of [16, 30]. In order to apply the fixed point results concerning the mixed monotone operators for the study of (8) and (9), we first consider $E = C([0, 1])$ as a suitable Banach space, and let the norm on E be defined by

$$\|y\| = \max\{|y(t)|, \quad 0 \leq t \leq 1\}. \quad (29)$$

Let us consider

$$P = \{y \in E | \min y(t) \geq 0, \quad 0 \leq t \leq 1, {}^C D^\alpha \text{ is positive}\}. \quad (30)$$

It is obvious that P is a normal cone in E .

In the following theorem, g is the function appeared in the right side of equation (8).

Theorem 2. Assume that

- (C6) $g(t, u, v)$ is a continuous function which is increasing in its first variable u and decreasing in its second variable v .
 (C7) For any $\eta \in (0, 1)$, let u and v be two arbitrary elements of P so that there exists $\lambda(\eta, u, v) \in (1, \infty)$ such that

$$g(t, \eta u, v) \leq \eta^{\lambda(\eta, u, v)} g(t, u, v). \quad (31)$$

- (C8) Let u_0 and v_0 be two elements of P and $\mu \in (0, 1)$ such that

$$\begin{aligned} u_0(t) &\leq \mu v_0(t), \\ g(t, u_0, D^\beta v_0) &\geq \frac{u_0 \Gamma[2(\alpha - \beta)]}{\Gamma(\alpha - \beta)}, \end{aligned} \quad (32)$$

$$g(t, u_0, D^\beta v_0) \leq \frac{v_0}{B(\alpha - \beta, \alpha - \beta)}.$$

Then, for equation (8) with initial conditions (9), we have a unique solution $u^* \in [u_0, \mu v_0]$. Furthermore, by computing the iterates,

$$\begin{aligned} u_{n+1}(t) &= \int_0^t K(t, r) g\left(r, u_n(r), \frac{\partial}{\partial r} v_n(r)\right) dr, \\ v_{n+1}(t) &= \int_0^t K(t, r) g\left(r, v_n(r), \frac{\partial}{\partial r} u_n(r)\right) dr. \end{aligned} \quad (33)$$

For $n = 0, 1, \dots$, one can find the unique solution. In other words, $\|u_n - u^*\| \rightarrow 0, \|v_n - u^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. First, we define the operator T as follows:

$$T(u, v)(t) = \int_0^t K(t, r) g(r, u(r), D^\beta v(r)) dr, \quad (34)$$

where $K(t, r)$ is the Green function (21). Taking the definition of the operator T into account and Lemma 3, it is clear that a certain function $u(t)$ is a solution of (2) and (3) iff $u(t)$ is a fixed point of T , that is, $T(u, u) = u$. Hence, it will be sufficient to show that T satisfies the conditions of Theorem 1. By virtue of (C6), it immediately follows that T is increasing in its first variable u and decreasing in its second variable v . On the other hand, under the hypothesis of (C7), if $\eta \in (0, 1)$ and u, v are two elements of P , then there exists $\lambda(\eta, u, v) \in (1, \infty)$ such that

$$\begin{aligned} T(\eta u(t), v(t)) &= \int_0^t K(t, r) g\left(r, \eta u(r), \frac{\partial}{\partial r} v(r)\right) dr \\ &\leq \eta^\alpha \int_0^t K(t, r) g\left(r, u(r), \frac{\partial}{\partial r} v(r)\right) dr \\ &= \eta^\alpha T(u(t), v(t)). \end{aligned} \quad (35)$$

Also, with the help of (C8), and using Lemma 3, we can choose u_0 and v_0 such that

$$\begin{aligned} T(u_0(t), v_0(t)) &= \int_0^t K(t, r) g\left(r, u_0(r), \frac{\partial}{\partial r} v_0(r)\right) dr \leq u_0, \\ T(v_0(t), u_0(t)) &= \int_0^t K(t, r) g\left(r, v_0(r), \frac{\partial}{\partial r} u_0(r)\right) dr \geq v_0. \end{aligned} \quad (36)$$

Consequently, it is obvious that all essentials of Theorem 1 are provided by (C6)–(C8), and therefore, there is a unique

positive solution (u^*, u^*) so that $T(u^*, u^*) = u^*$. It is the unique solution of (8) and (9). The proof is complete. \square

Example 1. Let us consider the following set of nonlinear differential equations of fractional order

$${}^c D^{m+(3/4)} y(t) = \psi(t) - \left[{}^c D^{m+(1/4)} y(t) \right]^2, \quad 0 < t \leq 1, \quad (37)$$

subject to

$$y^l(0) = 0, \quad l = 0, 1, 2, \dots, m, \quad (38)$$

where m is a positive integer, $\psi(t)$ is a continuous function such that (HTML translation failed), and $\eta_1 = \sqrt{\pi} \leq \psi(t) \leq m\pi = \eta_2$. In connection with Theorem 2, set $\alpha = m + (3/4)$ and $\beta = m + (1/4)$. By virtue of this theorem and Lemma 3, the existence of a unique solution for problems (37) and (38) is equivalent to a unique solution for the following integral equation:

$$y(t) = \int_0^t K(t, r) \left(\psi(r) - \left[{}^c D^{m+(1/4)} y(r) \right]^2 \right) dr, \quad (39)$$

where with the help of Lemma 3, one can deduce $L_1 = 1/\sqrt{\pi} \leq \int_0^1 K(t, r) dr \leq 2/\pi = L_2$. We note that if $u, v \in P$, defined by (30), then $g(t, u, v) = \psi(t) - [{}^c D^{m+(1/4)} v(t)]^2$ is increasing in u and decreasing in v . Now, let us define $A(u, v) = \int_0^t K(t, r) (\psi(r) - [{}^c D^{m+(1/4)} v(r)]^2) dr$ and choose $u_0 = 1, v_0 = 2m$. Then, considering that u_0, v_0 are constant, we can show that

$$u_0 \leq \frac{1}{2} v_0,$$

$$A(u_0, v_0) \geq L_1 \eta_1 \geq 1 = u_0, \quad (40)$$

$$A(v_0, u_0) \leq \frac{2}{\pi} \eta_2 \leq 2m = v_0.$$

Therefore, using Theorem 2, we conclude that problems (37) and (38) possess a unique positive solution such that $1 \leq y(t) \leq m$.

Example 2. In this example, we find a unique positive solution for the nonlinear fractional initial value problems of the following form:

$${}^c D^{n+(2/3)} y(t) = \chi(t) - \frac{1}{y(t)} - \sqrt{{}^c D^{n+(2/3)} y(t)}, \quad 0 < t \leq 1, \quad (41)$$

$$y^l(0) = 0, \quad l = 0, 1, 2, \dots, n, \quad (42)$$

where $n \in \mathbb{Z}^+$, $\chi(t)$ is a continuous function such that $\chi(0) \neq 0$, and $\tau_1 = 315/100 \leq \chi(t) \leq 945/100 = \tau_2$. In connection with Theorem 2, set $\alpha = n + (2/3)$ and $\beta = n + (1/3)$. By virtue of this theorem and Lemma 3, the existence of a unique solution for problems (41) and (42) is equivalent to a unique solution for the following integral equation:

$$y(t) = \int_0^t K(t, r) \left[\chi(r) - \frac{1}{y(r)} - \sqrt{{}^c D^{n+(1/3)} y(r)} \right] dr, \quad (43)$$

where with the help of Lemma 3, one deduces $2 \leq \int_0^1 K(t, r) dr \leq 53/10$. We note that if $u, v \in P$ are defined by (30), then $g(t, u, v) = \chi(t) - 1/u(t) - \sqrt{{}^c D^{n+(1/3)} v(t)}$ is increasing in u and decreasing in v .

For $c \in (0, 1)$, consider $1 < \alpha(c, u, v) < 2$; then, we have

$$g(t, cu(t), v(t)) = \chi(t) - \frac{1}{cu(t)} - \sqrt{{}^c D^{n+(1/3)} v(t)} \leq c^\alpha \left[\chi(t) - \frac{1}{u(t)} - \sqrt{{}^c D^{n+(1/3)} v(t)} \right] = c^\alpha g(t, u(t), v(t)). \quad (44)$$

Now, let us define $A(u, v) = \int_0^t K(t, r) [\chi(r) - 1/u(r) - \sqrt{{}^c D^{n+(1/3)} v(r)}] dr$ and choose $u_0 = 53/10, v_0 = 51$. Then, regarding that u_0, v_0 are constant functions, we get

$$\begin{aligned} A(u_0, v_0) &= \int_0^1 K(t, r) \left[\chi(r) - \frac{1}{u_0} - \sqrt{{}^c D^{n+(1/3)} v_0} \right] dr \geq 2\gamma_1 - 1 \geq \frac{53}{10} = u_0, \\ A(v_0, u_0) &= \int_0^1 K(t, r) \left[\chi(r) - \frac{1}{v_0} - \sqrt{{}^c D^{n+(1/3)} u_0} \right] dr \leq \frac{53}{10} \gamma_2 - \frac{2}{v_0} \leq 51 = v_0. \end{aligned} \quad (45)$$

Therefore, using Theorem 2, we conclude that for problems (38)–(41), there is a unique positive solution such that $y(t) \in [53/10, t51]$ for each m .

4. Conclusion

In this research, we considered nonlinear fractional differential equations of arbitrary order with two terms of fractional orders. By virtue of recent fixed point theorems on normal cones for mixed monotone operators, we obtained some new results for existence and uniqueness of solutions of these equations involving initial conditions, which are more applicable compared with previous existing results. The previous results can be proved based on very strict conditions on the function g . While, in our paper, the function $g(t, u, D^\alpha u) \equiv g(t, u, v)$ needs to be increasing in its first variable u and decreasing in its second variable v . Also, establishing the conditions (C7) and (C8) is not difficult. On the other hand, our results provided a constructive approach, based on an iterative relation, to find the solution. Indeed, the interval $(0, 1]$ can be generalized to $(0, T]$. The presented examples, at the end of the paper, cannot be considered by the previous works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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