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Research Article

Barycentric Interpolation Collocation Method for Solving Fractional Linear Fredholm-Volterra Integro-Differential Equation

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In this article, barycentric interpolation collocation method (BICM) is presented to solve the fractional linear Fredholm-Volterra integro-differential equation (FVIDE). Firstly, the fractional order term of equation is transformed into the Riemann integral with Caputo definition, and this integral term is approximated by the Gauss quadrature formula. Secondly, the barycentric interpolation basis function is used to approximate the unknown function, and the matrix equation of BICM is obtained. Finally, several numerical examples are given to solve one-dimensional differential equation.

1. Introduction

The concept of the fractional calculus dates back to 1695. Fractional differential equations, as a generalization of integer differential equations, are suitable for describing materials and processes with genetic and memory properties. Compared with integer order model, fractional order model can simulate dynamic system and natural physical phenomena more accurately. Fractional models are widely used in many fields, such as biological engineering [1–3], mechanics [4, 5], physics [6], electromagnetism [7, 8], viscoelastic system [9, 10], and heat conduction engineering [11]. Moreover, many researchers have proposed some efficient methods to investigate the existence and uniqueness of the solutions of fractional differential equations [12–18].

Lately, many researchers insinuated some standards to classify fractional differential operators. The notion of offering a guideline in a field was satisfactory enough, although the list of items that were suggested presented a limitation along with the critics brought up that were not academically acceptable. As a result of these criticisms, numerous researchers investigated the list along with their outcomes rejected the index law; in [19], their outcomes invalidated that inclusion of index law in the field. In another research work, the authors did overall investigation of the diffusive function of some kernel [20] and the outcomes they presented suggested that only operators with nonindex law properties can have crossover diffusive behaviors. However, Caputo and Fabrizio proved that the suggested index law was not right or it was a restriction to the field, and in their turn, they offered a list of items to be followed [21]. Further, they also proved the necessity of nonsingular differential operators along with their applications to nature applications to nature. In [22], the authors presented an optimal control of diffusion using the Atangana–Baleanu fractional differential operator. They proved that the existence of the solution with Atangana–Baleanu derivatives was obtained when the fractional order $\alpha \in (0, 1)$, and they also mentioned that the existence of the solution with Riemann-Liouville and Caputo was achieved during $\alpha \in (0,0.5)$.

Furthermore, definitions of two well-known fractional derivatives, namely, Riemann-Liouville and Caputo [23], included a singular kernel. However, Caputo and Fabrizio introduced another definition having a nonsingular kernel and properties can be found in [24]. Another derivatives with nonsingular kernel were suggested in [25] which fundamentally generalized the Caputo and Fabrizio definition [26]. However, Riemann-Liouville fractional derivative be essential in the development of theory of fractional derivatives and integrals. But, this derivative barely able to generate physical interpretation of the initial conditions that are compulsory for the initial value issues containing fractional differential equations and also the boundary value issue both of the issues can be solved with the Caputo definition of fractional derivative for further details, refer [27]. Another difference is that the derivative of a constant is not zero for Riemann-Liouville, but it is equal to zero for Caputo. Additionally, the Riesz fractional derivatives have some shortcomings, such as it relies upon the values of whole interval also not sustaining the Leibniz rule for the product of two functions [28]. Besides, the Caputo fractional definition is easy to calculate and program. So the Caputo derivative is chosen in this manuscript.

In this paper, we mainly solve the FVIDE

$$\begin{aligned} &{}_0^C D_t^\alpha y(t) + \int_0^t K_v(t,x)y(x)dx + \int_a^b K_f(t,z)y(z)dz \\ &\quad = g(t), 0 \le t \le T, \end{aligned} \tag{1}$$

where $\int_0^t K_v(t,x)y(x)dx$ is the Volterra part, $\int_a^b K_f(t,z)y(z)dz$ is the Fredholm part, ${}_0^C D_t^\alpha y(t)$ is the fractional derivative part, and the fractional derivative is defined as the Caputo definition as follows:

$${}_0^C D_t^\alpha y(t) = \frac{1}{\Gamma(\xi-\alpha)} \int_0^t \frac{\partial^\xi y(\tau)}{\partial \tau^\xi} \frac{d\tau}{(t-\tau)^{\alpha+1-\xi}}, \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function.

The initial condition of one-dimensional differential equation is given as

$$y(0) = A. \tag{3}$$

In recent years, many methods are proposed to solve fractional differential equations. In [29], the Bell polynomials are introduced to solve fractional differential equations based on matrix and collocation points. In [30], the central difference and Crank-Nicolson method are used to obtain the full discrete scheme of spatial fractional convection-diffusion equation; then, the Richardson extrapolation method is used to further improve the calculation accuracy. In [31, 32], the finite element method is presented to solve fractional convection-diffusion equations. In [33–35], the element free Galerkin method is used to solve fractional differential equations. Compared with other algorithms, BICM has the advantages of high precision, easy programming, and simple formula. Therefore, this method has been applied to solve various equations, such as heat conduction equation [36], generalized Poisson equation [37], fractional differential equation [38], and fractional reaction-diffusion equation [39]. At the same time, the BICM is also utilized to solve some engineering problems, such as the plane elasticity problem [40], the bending problem of elliptic plate [41], and the numerical approximation of Darcy flow [42].

In this article, BICM is introduced to solve FVIDE. In Section 2, we provide relevant definitions of barycentric interpolation. In Sections 3–5, barycentric interpolation basis function is applied to approximate the unknown function, and matrix equations of the fractional derivative part, Volterra part, and Fredholm part are given. In Section 6, we obtain the matrix equation of FVIDE, and initial condition is dealt with by replacement method or additive method. In Section 7, some numerical examples are shown to prove feasibility of the algorithm.

2. Barycentric Interpolation

In this section, we will introduce barycentric interpolation for solving one-dimensional differential equation. First, $n+1$ equidistant nodes or Chebyshev's nodes are chosen as collocation points on the domain, i.e., (t_i), $i = 0, 1, \cdots, n$. The barycentric interpolation function is defined as

$$y_n(t) = \sum_{i=0}^{n} T_i(t) y_i, \tag{4}$$

where $y_i = y_n(t_i)$ and

$$T_i(t) = \frac{w_i/(t-t_i)}{\sum_{k=0}^{n} w_k/(t-t_k)}. \tag{5}$$

According to different definitions of weight functions w_i, barycentric interpolation can be divided into barycentric rational interpolation and barycentric Lagrange interpolation. The weight functions of barycentric Lagrange interpolation are defined as

$$w_i = \frac{1}{\prod_{j=0, j\ne i}^{n} t_i - t_j}, \tag{6}$$

the weight functions of barycentric rational interpolation are defined as

$$\begin{aligned} w_i &= \sum_{s\in D_i} (-1)^s \prod_{k=s, s\ne i}^{s+d} \frac{1}{t_i - t_k}, \\ D_i &= \{s : i-d \le s \le i\}, \end{aligned} \tag{7}$$

where $s \in \{0, 1, \cdots, n-d\}$, the parameter d is integer, and $0 \le d \le n$.

3. Matrix Equation of Fractional Derivative Part

Fractional terms are dealt with by integration by parts; then, we get

$$\begin{aligned} {}_0^C D_t^\alpha y(t) &= \frac{1}{\Gamma(\xi-\alpha)} \int_0^t \frac{\partial^\xi y(\tau)}{\partial \tau^\xi} \frac{d\tau}{(t-\tau)^{\alpha+1-\xi}} \\ &= \frac{1}{\Gamma(\xi+1-\alpha)} \frac{\partial^\xi y(0)}{\partial t^\xi} t^{\xi-\alpha} \\ &\quad + \frac{1}{\Gamma(\xi+1-\alpha)} \int_0^t \frac{\partial^{\xi+1} y(\tau)}{\partial \tau^{\xi+1}} \frac{d\tau}{(t-\tau)^{\alpha-\xi}} \\ &= \Gamma_\alpha^\xi \left[\frac{\partial^\xi y(0)}{\partial t^\xi} t^{\xi-\alpha} + \int_0^t \frac{\partial^{\xi+1} y(\tau)}{\partial \tau^{\xi+1}} \frac{d\tau}{(t-\tau)^{\alpha-\xi}} \right], \end{aligned} \tag{8}$$

where $\Gamma_\alpha^\xi = 1/(\Gamma(\xi+1-\alpha))$.

Substituting equation (4) into equation (8), we obtain

$${}_0^C D_t^\alpha y_n(t) = \Gamma_\alpha^\xi \sum_{i=0}^n \left[T_i^{(\xi)}(0) t^{\xi-\alpha} \right] y_i + \Gamma_\alpha^\xi \sum_{i=0}^n \left[\int_0^t \frac{T_i^{(\xi+1)}(\tau)}{(t-\tau)^{\alpha-\xi}} d\tau \right] y_i, \tag{9}$$

where

$$T_i(\tau) = \frac{w_i/(\tau-\tau_i)}{\sum_{k=0}^n w_k/(\tau-\tau_k)}. \tag{10}$$

Let $t = t_\theta$, formula (9) can be expressed as

$${}_0^C D_{t_\theta}^\alpha y_n(t_\theta) = \Gamma_\alpha^\xi \sum_{i=0}^n \left[T_i^{(\xi)}(0) t_\theta^{\xi-\alpha} \right] y_i + \Gamma_\alpha^\xi \sum_{i=0}^n \left[\int_0^{t_\theta} \frac{T_i^{(\xi+1)}(\tau)}{(t_\theta-\tau)^{\alpha-\xi}} d\tau \right] y_i, \tag{11}$$

where $\theta = 0, 1, \cdots, n$.

Let us write the integral term of the formula (11) as the following form:

$$P_{\theta i} = P_i(t_\theta) = \int_0^{t_\theta} T_i^{(\xi+1)}(\tau)(t_\theta-\tau)^{\xi-\alpha} d\tau, \quad i = 0, 1, \cdots, n. \tag{12}$$

Then, we have

$${}_0^C D_{t_\theta}^\alpha y_n(t_\theta) = \Gamma_\alpha^\xi \left\{ \sum_{i=0}^n \left[T_i^{(\xi)}(0) t_\theta^{\xi-\alpha} \right] + \sum_{i=0}^n [P_{\theta i}] \right\} y_i. \tag{13}$$

The integral term (12) is calculated using the Gauss quadrature formula with weights $\rho(\tau) = (t_\theta - \tau)^{\xi-\alpha}$; we get

$$P_{\theta i}^G = \sum_{j=1}^m T_i^{(\xi+1)}\left(\tau_j^{\theta,\alpha}\right) A_j^{\theta,\alpha}, \tag{14}$$

where $\tau_j^{\theta,\alpha}$ and $A_j^{\theta,\alpha}$ are the Gauss points and Gauss weights and m is the number of the Gauss points.

Using the Gauss-Legendre quadrature formula, equation (15) is given as

$$P_{\theta i}^{GL} = \frac{t_\theta}{2} \sum_{j=1}^m f\left(\tau_j^{\theta,l}\right) A_j^{\theta,l}, \tag{15}$$

where $\tau_j^{\theta,l}$ and $A_j^{\theta,l}$ are integral points and integral weights, m is the number of the integral points, $t_\theta/2$ is transformed coefficient, and $f(\tau_j^{\theta,l}) = \rho(\tau_j^{\theta,l}) T_i^{(\xi+1)}(\tau_j^{\theta,l})$.

Then, the formula (16) with the Gauss quadrature formula is obtained as

$$\begin{bmatrix} {}_0^C D_{t_0}^\alpha y_n(t_0) \\ \vdots \\ {}_0^C D_{t_n}^\alpha y_n(t_n) \end{bmatrix} = \Gamma_\alpha^\xi \left[T^{\xi,\alpha}\left(I_{n+1} \otimes M_1^{(\xi)}\right) + I_{n+1} \otimes P \right] \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix}, \tag{16}$$

where I_{n+1} is the identity matrix and $\otimes$ is the Kronecker product.

Briefly, the formula (16) can be written as

$$D = D^\alpha Y, \tag{17}$$

where

$$D^\alpha = \Gamma_\alpha^\xi \left[T^{\xi,\alpha}\left(I_{n+1} \otimes M_1^{(\xi)}\right) + I_{n+1} \otimes P \right],$$

$$Y = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix},$$

$$D = \begin{bmatrix} {}_0^C D_{t_0}^\alpha y_n(t_0) \\ \vdots \\ {}_0^C D_{t_n}^\alpha y_n(t_n) \end{bmatrix},$$

$$T^{\xi,\alpha} = \begin{bmatrix} t_\theta^{\xi-\alpha} & & & \\ & t_\theta^{\xi-\alpha} & & \\ & & \ddots & \\ & & & t_\theta^{\xi-\alpha} \end{bmatrix}_{N\times N},$$

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1N} \\ P_{21} & P_{22} & \cdots & P_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ P_{N1} & P_{N2} & \cdots & P_{NN} \end{bmatrix}_{N\times N},$$

$$N = n+1,$$

$$P_{11} = \sum_{j=1}^m T_0^{(\xi+1)}\left(\tau_j^{0,\alpha}\right) A_j^{0,\alpha},$$

$$P_{12} = \sum_{j=1}^{m} T_1^{(\xi+1)}\left(\tau_j^{0,\alpha}\right)A_j^{0,\alpha},$$

$$P_{1N} = \sum_{j=1}^{m} T_n^{(\xi+1)}\left(\tau_j^{0,\alpha}\right)A_j^{0,\alpha},$$

$$P_{21} = \sum_{j=1}^{m} T_0^{(\xi+1)}\left(\tau_j^{1,\alpha}\right)A_j^{1,\alpha},$$

$$P_{22} = \sum_{j=1}^{m} T_1^{(\xi+1)}\left(\tau_j^{1,\alpha}\right)A_j^{1,\alpha},$$

$$P_{2N} = \sum_{i=1}^{m} T_n^{(\xi+1)}\left(\tau_j^{1,\alpha}\right)A_j^{1,\alpha},$$

$$P_{N1} = \sum_{j=1}^{m} T_0^{(\xi+1)}\left(\tau_j^{n,\alpha}\right)A_j^{n,\alpha},$$

$$P_{N2} = \sum_{j=1}^{m} T_1^{(\xi+1)}(\tau_i^{n,\alpha})A_j^{n,\alpha},$$

$$P_{NN} = \sum_{j=1}^{m} T_n^{(\xi+1)}\left(\tau_j^{n,\alpha}\right)A_j^{n,\alpha}. \tag{18}$$

The relations between differential matrices and basis functions are defined as follows:

$$M^{(h)} = \left[M_{\theta i}^{(h)}\right]_{N\times N} = \left[T_i^{(h)}(t_\theta)\right]_{N\times N}, \tag{19}$$

where $N = n + 1$ and

$$M_{\theta i}^{(1)} = \begin{cases} \dfrac{w_i/w_\theta}{t_\theta - t_i}, & \theta \neq i, \\ -\sum_{i\neq\theta} M_{\theta i}^{(1)}, & \theta = i, \end{cases}$$

$$M_{\theta i}^{(\xi)} = \begin{cases} \xi\left(M_{\theta\theta}^{(\xi-1)}M_{\theta i}^{(1)} - \dfrac{M_{\theta i}^{(\xi-1)}}{t_\theta - t_i}\right), & \theta \neq i, \\ -\sum_{i\neq\theta} M_{\theta i}^{(\xi)}, & \theta = i. \end{cases} \tag{20}$$

Hence, we can get

$$M_1^{(\xi)} = \begin{bmatrix} -\sum_{i=1}^{n} M_{0i}^{(\xi)} & M_{01}^{(\xi)} & M_{02}^{(\xi)} & \cdots & M_{0n}^{(\xi)} \\ -\sum_{i=1}^{n} M_{0i}^{(\xi)} & M_{01}^{(\xi)} & M_{02}^{(\xi)} & \cdots & M_{0n}^{(\xi)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sum_{i=1}^{n} M_{0i}^{(\xi)} & M_{01}^{(\xi)} & M_{02}^{(\xi)} & \cdots & M_{0n}^{(\xi)} \end{bmatrix}_{N\times N}. \tag{21}$$

4. Matrix Equation of the Volterra Part

The Volterra part is expressed as $V(t)$; equation (22) is shown as follows:

$$V(t) = \int_0^t K_v(t,x)y(x)dx. \tag{22}$$

Substituting equation (4) into equation (22), we obtain

$$V_n(t) = \sum_{i=0}^{n}\left[\int_0^t K_v(t,x)T_i(x)dx\right]y_i, \tag{23}$$

where $T_i(x)$ is defined as shown in equation (10).

t is replaced by t_θ of formula (23), and we have

$$V_n(t_\theta) = \sum_{i=0}^{n}\left[\int_0^{t_\theta} K_v(t_\theta,x)T_i(x)dx\right]y_i, \tag{24}$$

where $\theta = 0, 1, \cdots, n$.

Formula (25) is expressed in the following form:

$$Q_{\theta i} = Q_i(t_\theta) = \int_0^{t_\theta} K_v(t_\theta,x)T_i(x)dx. \tag{25}$$

Using the Gauss quadrature formula with weights $\beta(x) = K_v(t_\theta, x)$, we get

$$Q_{\theta i}^{G} = \sum_{j=1}^{m} T_i\left(x_j^\theta\right)C_j^\theta, i = 0, 1, \cdots, n, \tag{26}$$

where x_j^θ and C_j^θ are the Gauss points and Gauss weights and m is the number of the Gauss points.

Formula (25) is calculated by the Gauss-Legendre quadrature formula, and we obtain

$$Q_{\theta i}^{GL} = \frac{t_\theta}{2}\sum_{j=1}^{m} q\left(x_j^{\theta,l}\right)C_j^{\theta,l}, \tag{27}$$

where $x_j^{\theta,l}$ and $C_j^{\theta,l}$ are integral points and integral weights, m is the number of the integral points, $t_\theta/2$ is transformed coefficient, and $q(x_j^{\theta,l}) = \beta(x_j^{\theta,l})T_i^{(\xi+1)}(x_j^{\theta,l})$.

Combining equation (24), equation (25), and equation (26), equation (28) is obtained

$$V_n(t_\theta) = \sum_{i=0}^{n}\left[Q_{\theta i}^{G}\right]y_i. \tag{28}$$

Let $V_i = V_n(t_i)$; formula (29) is obtained as follows:

$$V = QY, \tag{29}$$

where

$$V = \begin{bmatrix} V_0 \\ \vdots \\ V_n \end{bmatrix},$$

$$Y = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix},$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & \cdots & Q_{1N} \\ Q_{21} & Q_{22} & \cdots & Q_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{N1} & Q_{N2} & \cdots & Q_{NN} \end{bmatrix}_{N\times N},$$

$$N = n+1, Q_{11} = \sum_{j=1}^{m} T_0\left(x_j^0\right) C_j^0,$$

$$Q_{12} = \sum_{j=1}^{m} T_1\left(x_j^0\right) C_j^0, Q_{1N} = \sum_{j=1}^{m} T_n\left(x_j^0\right) C_j^0,$$

$$Q_{21} = \sum_{j=1}^{m} T_0\left(x_j^1\right) C_j^1, Q_{22} = \sum_{j=1}^{m} T_1\left(x_j^1\right) C_j^1,$$

$$Q_{2N} = \sum_{i=1}^{m} T_n\left(x_j^1\right) C_j^1, Q_{N1} = \sum_{j=1}^{m} T_0\left(x_j^n\right) C_j^n,$$

$$Q_{N2} = \sum_{j=1}^{m} T_1(x_i^n) C_j^n, Q_{NN} = \sum_{j=1}^{m} T_n\left(x_j^n\right) C_j^n. \tag{30}$$

5. Matrix Equation of the Fredholm Part

The Fredholm part is expressed as the following form:

$$I(t) = \int_a^b K_f(t,z) y(z) dx. \tag{31}$$

Substituting equation (4) into equation (31), we obtain

$$I_n(t) = \sum_{i=0}^{n} \left[\int_a^b K_f(t,z) T_i(z) dx \right] y_i, \tag{32}$$

where the definition of $T_i(z)$ is as shown in equation (10).

Let $t = t_\theta$, $\theta = i = 0, 1, \cdots, n$; we have

$$I_n(t_\theta) = \sum_{i=0}^{n} \left[\int_a^b K_f(t_\theta,z) T_i(z) dz \right] y_i. \tag{33}$$

Equation (34) is written as follows:

$$R_{\theta i} = R_i(t_\theta) = \int_a^b K_f(t_\theta,z) T_i(z) dz. \tag{34}$$

Formula (34) is calculated by the Gauss quadrature formula with weights $\eta(z) = K_f(t_\theta, z)$; we have

$$R_{\theta i}^{G} = \sum_{j=1}^{m} T_i\left(z_j^\theta\right) B_j^\theta, i = 0, 1, \cdots, n, \tag{35}$$

where z_j^θ and B_j^θ are the Gauss points and Gauss weights and m is the number of the Gauss points.

Using the Gauss-Legendre quadrature formula, we obtain

$$R_{\theta i}^{GL} = \frac{b-a}{2} \sum_{j=1}^{m} r\left(z_j^{\theta,l}\right) B_j^{\theta,l}, \tag{36}$$

where $z_j^{\theta,l}$ and $B_j^{\theta,l}$ are integral points and integral weights, m is the number of the integral points, $(b-a)/2$ is transformed coefficient, and $r(z_j^{\theta,l}) = \eta(z_j^{\theta,l}) T_i^{(\xi+1)}(z_j^{\theta,l})$.

Formula (33) is calculated by the Gauss quadrature formula; equation (37) is written as follows:

$$I_n(t_\theta) = \sum_{i=0}^{n} \left[R_{\theta i}^{G}\right] y_i. \tag{37}$$

Let $I_i = I_n(t_i)$; formula (38) is obtained

$$I = RY, \tag{38}$$

where

$$I = \begin{bmatrix} I_0 \\ \vdots \\ I_n \end{bmatrix},$$

$$Y = \begin{bmatrix} y_0 \\ \vdots \\ y_n \end{bmatrix},$$

$$R = \begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1N} \\ R_{21} & R_{22} & \cdots & R_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N1} & R_{N2} & \cdots & R_{NN} \end{bmatrix}_{N\times N},$$

$$N = n+1, R_{11} = \sum_{j=1}^{m} T_0\left(z_j^0\right) B_j^0,$$

$$R_{12} = \sum_{j=1}^{m} T_1\left(z_j^0\right) B_j^0, R_{1N} = \sum_{j=1}^{m} T_n\left(z_j^0\right) B_j^0,$$

$$R_{21} = \sum_{j=1}^{m} T_0\left(z_j^1\right) B_j^1, R_{22} = \sum_{j=1}^{m} T_1\left(z_j^1\right) B_j^1,$$

$$R_{2N} = \sum_{i=1}^{m} T_n\left(z_j^1\right) B_j^1, R_{N1} = \sum_{j=1}^{m} T_0\left(z_j^n\right) B_j^n,$$

$$R_{N2}=\sum_{j=1}^{m}T_1(z_i^n)B_j^n,\ R_{NN}=\sum_{j=1}^{m}T_n\left(z_j^n\right)B_j^n. \tag{39}$$

6. Matrix Equation for FVIDE

Equation (1) is treated by integration by parts; then, we get

$$\begin{aligned}&\Gamma_\alpha^\xi\left[\frac{\partial^\xi y(0)}{\partial t^\xi}t^{\xi-\alpha}+\int_0^t\frac{\partial^{\xi+1}y(\tau)}{\partial\tau^{\xi+1}}\frac{d\tau}{(t-\tau)^{\alpha-\xi}}\right]\\&+\int_0^t K_v(t,x)y(x)dx+\int_a^b K_f(t,z)y(z)dz=g(t).\end{aligned} \tag{40}$$

Substituting equation (4) into equation (40), equation (41) is obtained

$$\begin{aligned}&\Gamma_\alpha^\xi\left\{\sum_{i=0}^{n}\left[T_i^{(\xi)}(0)t^{\xi-\alpha}\right]+\sum_{i=0}^{n}\left[\int_0^t\frac{T_i^{(\xi+1)}(\tau)}{(t-\tau)^{\alpha-\xi}}d\tau\right]\right\}y_i\\&+\sum_{i=0}^{n}\left[\int_0^t K_v(t,x)T_i(x)dx\right]y_i\\&+\sum_{i=0}^{n}\left[\int_a^b K_f(t,z)T_i(z)dz\right]y_i=g(t).\end{aligned} \tag{41}$$

Taking $t=t_\theta$, $\theta=0,1,\cdots,n$, we get

$$\begin{aligned}&\Gamma_\alpha^\xi\left\{\sum_{i=0}^{n}\left[T_i^{(\xi)}(0)t_\theta^{\xi-\alpha}\right]+\sum_{i=0}^{n}\left[\int_0^{t_\theta}\frac{T_i^{(\xi+1)}(\tau)}{(t_\theta-\tau)^{\alpha-\xi}}d\tau\right]\right\}y_i\\&+\sum_{i=0}^{n}\left[\int_0^{t_\theta}K_v(t_\theta,x)T_i(x)dx\right]y_i\\&+\sum_{i=0}^{n}\left[\int_a^b K_f(t_\theta,z)T_i(z)dz\right]y_i=g(t_\theta).\end{aligned} \tag{42}$$

Let $g_i=g(t_i)$; combining (17), (29), and (38), we obtain the matrix equation as follows:

$$LY=G, \tag{43}$$

where

$$G=\begin{bmatrix}g_0\\ \vdots\\ g_n\end{bmatrix},\quad L=D^\alpha+Q+R. \tag{44}$$

The initial conditions are imposed by replacement method and additive method. When the replacement method is used to impose initial conditions, the 1st row element of matrix I_{n+1} is extracted to replace the corresponding row element of matrix L in the system (43). When the additive method is used to impose initial conditions, the 1st row element of matrix I_{n+1} is extracted and then added to the $n+2$ row of matrix L in the system (43).

Table 1: Errors of equidistant nodes for barycentric Lagrange interpolation with $m=6$ for Example 1.

t_i	$(n,\alpha)=(5,0.75)$	$(n,\alpha)=(10,0.75)$	$(n,\alpha)=(20,0.75)$
0	6.7117e-16	1.7418e-15	5.5914e-13
0.2	1.4468e-15	2.0154e-13	2.9617e-08
0.4	1.6376e-15	1.6798e-13	2.3200e-08
0.6	1.8041e-15	1.5907e-13	5.5773e-08
0.8	2.1094e-15	1.5510e-13	4.7605e-07
1	2.2204e-15	1.9784e-13	1.3536e-06

Table 2: Errors of equidistant nodes for barycentric rational interpolation with $m=6$ and $d=3$ for Example 1.

t_i	$(n,\alpha)=(5,0.75)$	$(n,\alpha)=(10,0.75)$	$(n,\alpha)=(20,0.75)$
0	2.6439e-16	4.0324e-16	1.0279e-15
0.2	1.3184e-16	1.0807e-15	2.5535e-15
0.4	6.1062e-16	1.4572e-15	2.8449e-15
0.6	8.3267e-16	1.8041e-15	3.2474e-15
0.8	9.9920e-16	2.9976e-15	1.9762e-14
1	8.8818e-16	8.8818e-16	2.4070e-13

Table 3: Errors of equidistant nodes for barycentric Lagrange interpolation using the Gauss-Legendre quadrature formula with $m=6$ for Example 1.

t_i	$(n,\alpha)=(5,0.75)$	$(n,\alpha)=(10,0.75)$	$(n,\alpha)=(20,0.75)$
0	7.6617e-17	1.7422e-15	7.1349e-14
0.2	2.0293e-05	2.0293e-05	2.0288e-05
0.4	1.6234e-04	1.6234e-04	1.6234e-04
0.6	5.4791e-04	5.4791e-04	5.4793e-04
0.8	1.2988e-03	1.2988e-03	1.2989e-03
1	2.5366e-03	2.5366e-03	2.5369e-03

7. Numerical Experiments

In this section, several numerical examples are given to illustrate the accuracy of BICM. All of numerical examples have been performed on MATLAB (version: R2020a). The error function is defined as

$$e_n(t)=\|y_n(t)-y(t)\|, \tag{45}$$

where $y_n(t)$ and $y(t)$ are approximate solution and exact solution of numerical examples.

Example 1. Consider the linear fractional Volterra integro-differential equation with the initial condition $y(0)=0$.

$$D^{0.75}y(t)+\frac{e^t t^2}{5}y(t)-\int_0^t e^t x y(x)dx=\frac{6t^{2.25}}{\Gamma(3.25)}, \tag{46}$$

where $0\le t\le 1$. The analytical solution is $y(t)=t^3$.

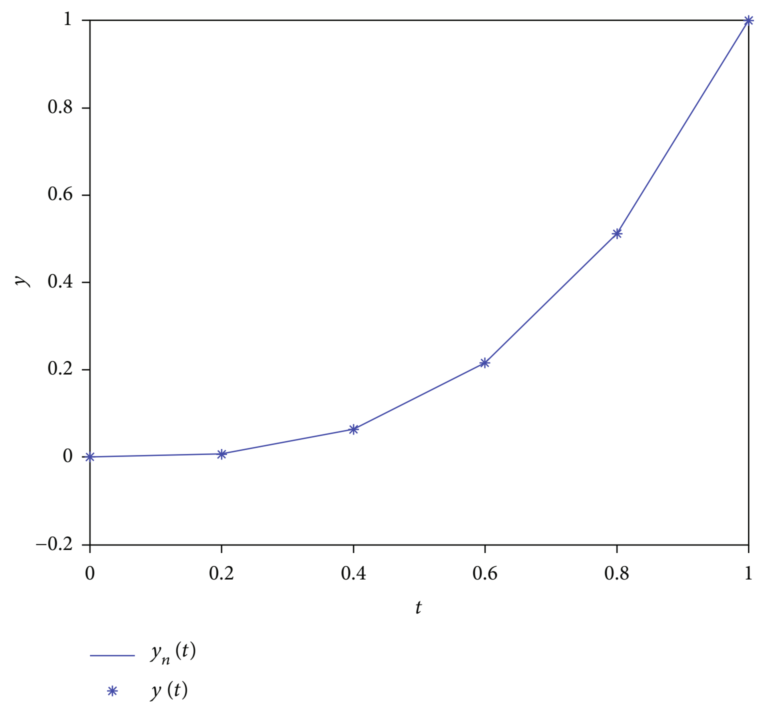

Figure 1: $y_n(t)$ and $y(t)$ of barycentric Lagrange interpolation using the Gauss quadrature formula with $m = 3$ at $n = 5$ for Example 1.

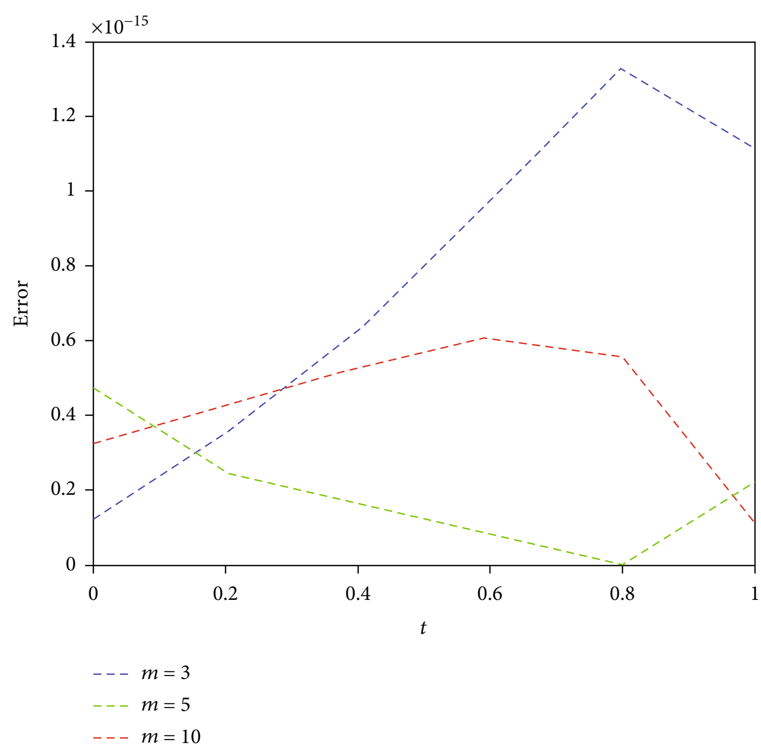

Figure 2: Errors of barycentric Lagrange interpolation using the Gauss quadrature formula with different Gauss points at $n = 5$ for Example 1.

In Tables 1 and 2, the errors of the Gauss quadrature formula are shown for $n = 5,10,20$ at $m = 6$. From Tables 1 and 2, we know that barycentric Lagrange interpolation and barycentric rational interpolation both get high error accuracy when $t = 0,0.2,0.4,0.6,0.8,1$. In Table 3, the errors of barycentric Lagrange interpolation with the Gauss-Legendre quadrature formula are shown. In Tables 1–3, initial conditions are imposed by replacement method. From

Table 4: Errors of equidistant nodes for barycentric Lagrange interpolation with $n = 5$ for Example 2.

t_i	$(m, \alpha) = (3,0.75)$	$(m, \alpha) = (5,0.75)$	$(m, \alpha) = (10,0.75)$
0	5.5339e-16	3.5112e-16	1.9950e-16
0.2	3.1745e-16	3.1745e-16	2.9317e-16
0.4	7.0777e-16	9.7145e-17	5.9674e-16
0.6	8.8818e-16	2.7756e-17	8.3267e-16
0.8	1.2212e-15	0	7.7716e-16
1	1.2212e-15	2.2204e-16	3.3307e-16

Table 5: Errors of equidistant nodes for barycentric rational interpolation with $n = 5$ and $d = 3$ for Example 2.

t_i	$(m, \alpha) = (3,0.75)$	$(m, \alpha) = (5,0.75)$	$(m, \alpha) = (10,0.75)$
0	7.8913e-17	3.4635e-16	3.4501e-16
0.2	3.4348e-16	6.1409e-16	6.3144e-16
0.4	1.3878e-17	4.8572e-16	4.5797e-16
0.6	1.3878e-16	4.1633e-16	5.5511e-16
0.8	2.2204e-16	3.3307e-16	2.2204e-16
1	0	1.1102e-15	6.6613e-16

Table 6: Errors of equidistant nodes for barycentric Lagrange interpolation using the additive method with $n = 5$ for Example 2.

t_i	$(m, \alpha) = (3,0.75)$	$(m, \alpha) = (5,0.75)$	$(m, \alpha) = (10,0.75)$
0	2.8897e-16	7.1029e-16	7.1941e-16
0.2	6.5399e-16	2.9317e-16	1.1293e-15
0.4	9.0206e-16	4.9960e-16	1.3600e-15
0.6	1.0825e-15	5.8287e-16	1.4433e-15
0.8	1.7764e-15	8.8818e-16	1.6653e-15
1	1.7764e-15	5.5511e-16	1.4433e-15

Tables 1 and 3, we know that error precision of barycentric Lagrange interpolation with the Gauss quadrature formula is higher than the Gauss-Legendre quadrature formula.

In Figure 1, approximate solution $y_n(t)$ and exact solution $y(t)$ are given for barycentric Lagrange interpolation using the Gauss quadrature formula with $m = 3$ at $n = 5$. Figure 2 shows errors of equidistant nodes for barycentric Lagrange interpolation using the Gauss quadrature formula with different Gauss points m. From Figures 1 and 2, we can see that higher error precision is attained when the lesser equidistant nodes are used.

Example 2. Consider the linear fractional Fredholm-Volterra integro-differential equation with the initial condition $y(0) = 1$.

$$D^{0.75}y(t) + \frac{e^t t^2}{5}y(t) - \int_0^t e^t x y(x)dx - \int_0^1 (t-x)y(x)dx = \frac{6t^{2.25}}{\Gamma(3.25)} - \frac{t}{4} + \frac{1}{5}, 0 \le t \le 1. \quad (47)$$

The analytical solution is $y(t) = t^3$.

In Tables 4–6, the errors of the Gauss quadrature formula are up to machine accuracy. In Tables 4 and 5, the initial conditions are imposed by replacement method. From Tables 4 and 6, we can find that replacement method or additive method can get high error precision.

In Figure 3, we can see that approximate solution $y_n(t)$ and exact solution $y(t)$ basically coincide. In Figure 4, errors of equidistant nodes are shown for barycentric Lagrange interpolation using the Gauss quadrature formula with different Gauss points $m = 3, 5, 10$ at $n = 5$.

Example 3. Consider the linear fractional Volterra integro-differential equation with the initial condition $y(0) = 1$.

$$D^{0.75}y(t) + ty(t) - \int_0^t txy(x)dx = \frac{t^{0.25}}{\Gamma(1.25)} - \frac{t^4}{3} - \frac{t^3}{2} - t^2 - t, 0 \le t \le 1. \quad (48)$$

The analytical solution is $y(t) = t + 1$.

Tables 7 and 8 show the errors of the Gauss quadrature formula for different m with replacement method. From these tables, BICM can obtain higher error accuracy with fewer interpolation nodes.

In Figure 5, approximate solution $y_n(t)$ and exact solution $y(t)$ are given with equidistant nodes. In Figure 6, errors of barycentric Lagrange interpolation are shown with different Gauss points m. From Figures 5 and 6, we know that

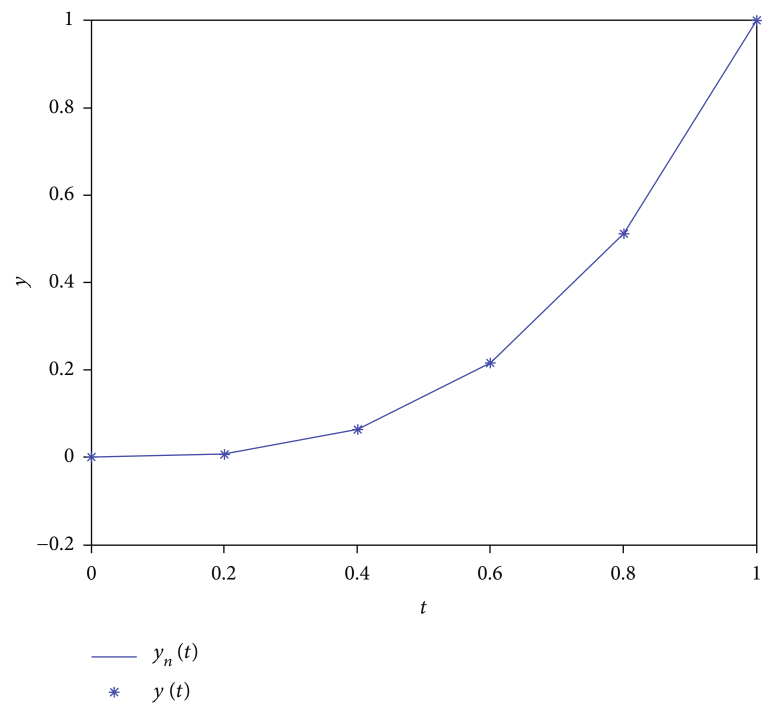

FIGURE 3: $y_n(t)$ and $y(t)$ of barycentric Lagrange interpolation using the Gauss quadrature formula with $m = 3$ at $n = 5$ for Example 2.

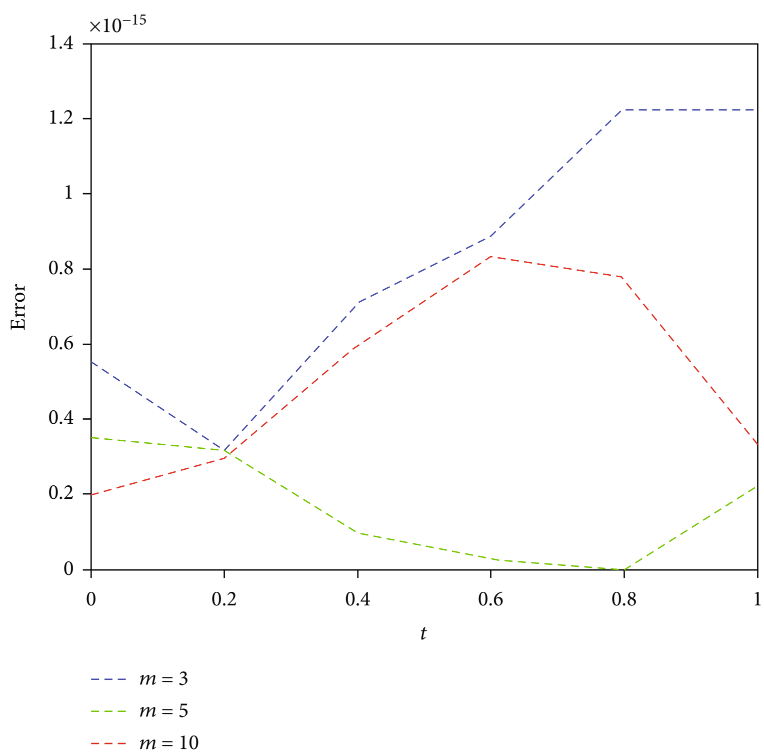

FIGURE 4: Errors of barycentric Lagrange interpolation using the Gauss quadrature formula with different Gauss points at $n = 5$ for Example 2.

Table 7: Errors of equidistant nodes for barycentric Lagrange interpolation with $n = 5$ for Example 3.

t_i	$(m, \alpha) = (3{,}0.75)$	$(m, \alpha) = (5{,}0.75)$	$(m, \alpha) = (10{,}0.75)$
0	8.8818e-16	1.3323e-15	1.1102e-16
0.2	5.5511e-15	4.8850e-15	5.1070e-15
0.4	5.9952e-15	0	2.6645e-15
0.6	8.8818e-15	8.8818e-16	1.1102e-15
0.8	1.1546e-14	2.2204e-16	3.1086e-15
1	1.4211e-14	3.9968e-15	1.1546e-14

Table 8: Errors of equidistant nodes for barycentric rational interpolation with $n = 5$ and $d = 3$ for Example 3.

t_i	$(m, \alpha) = (3{,}0.75)$	$(m, \alpha) = (5{,}0.75)$	$(m, \alpha) = (10{,}0.75)$
0	2.2204e-16	1.1102e-16	7.7716e-16
0.2	0	1.7764e-15	6.6613e-16
0.4	2.2204e-16	2.2204e-16	1.7764e-15
0.6	4.4409e-16	1.9984e-15	4.4409e-15
0.8	1.9984e-15	3.1086e-15	4.4409e-15
1	1.3323e-15	6.6613e-16	5.3291e-15

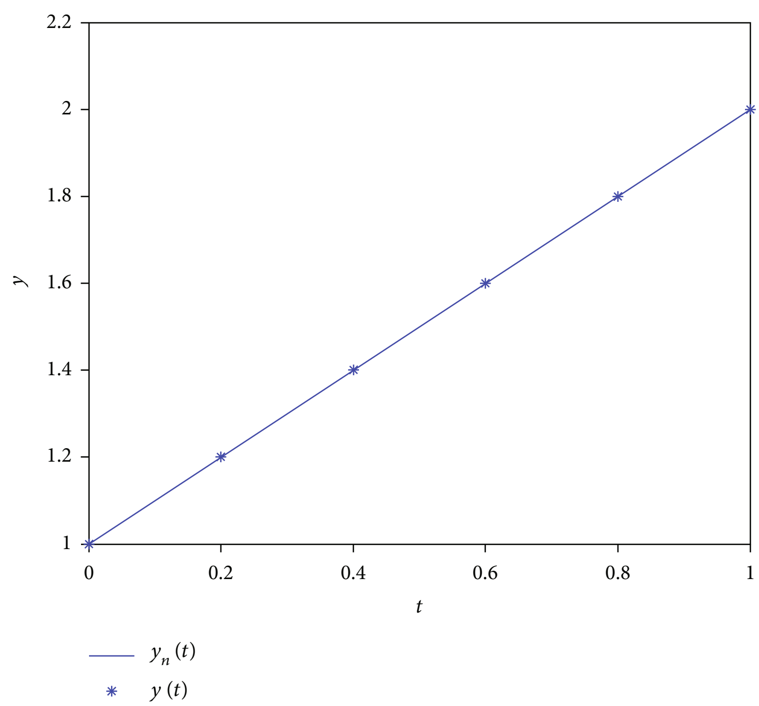

Figure 5: $y_n(t)$ and $y(t)$ of barycentric Lagrange interpolation using the Gauss quadrature formula with $m = 3$ at $n = 5$ for Example 3.

error accuracy of Barycentric Lagrange interpolation collocation method can achieve machine accuracy.

Example 4. Consider the linear fractional Volterra integro-differential equation with the initial condition $y(0) = 0$.

$$D^{0.75}y(t) + y(t) - 2\int_0^t \sin(x - t)y(x)dx = \frac{4}{\Gamma(0.25)}\left(t^{0.25} - \int_0^t \sin(\tau)(t-\tau)^{0.25}d\tau\right) - t\cos(t), \tag{49}$$

where $0 \le t \le 1$; the analytical solution is $y(t) = \sin(t)$.

Table 9 shows the errors of the Gauss quadrature formula for the Gauss points m with barycentric Lagrange interpolation. In Table 10, taking the parameter $d = 9$ of barycentric rational interpolation, we get the errors of BICM

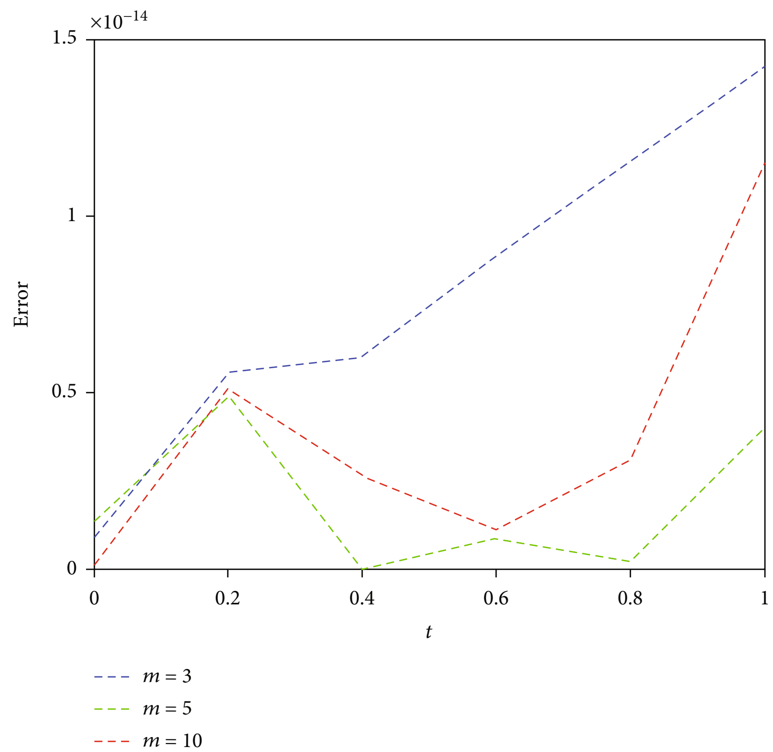

Figure 6: Errors of barycentric Lagrange interpolation using the Gauss quadrature formula with different Gauss points at $n = 5$ for Example 3.

Table 9: Errors of equidistant nodes for barycentric Lagrange interpolation with $n = 10$ for Example 4.

t_i	$(m, \alpha) = (3,0.75)$	$(m, \alpha) = (5,0.75)$	$(m, \alpha) = (10,0.75)$
0	2.7467*e*-15	1.0670*e*-14	2.3967*e*-14
0.2	5.3985*e*-13	9.6498*e*-13	1.1939*e*-12
0.4	7.5476*e*-12	1.0544*e*-12	1.3185*e*-12
0.6	2.3869*e*-10	1.1452*e*-12	1.4382*e*-12
0.8	2.7404*e*-09	1.1894*e*-12	1.4846*e*-12
1	1.7312*e*-08	1.0930*e*-12	1.4452*e*-12

Table 10: Errors of equidistant nodes for barycentric rational interpolation with $n = 10$ and $d = 9$ for Example 4.

t_i	$(m, \alpha) = (3,0.75)$	$(m, \alpha) = (5,0.75)$	$(m, \alpha) = (10,0.75)$
0	6.2870*e*-15	9.3112*e*-15	1.8457*e*-14
0.2	1.0277*e*-12	8.2812*e*-13	1.1412*e*-12
0.4	7.9721*e*-12	8.8335*e*-13	1.2341*e*-12
0.6	2.3779*e*-10	9.2648*e*-13	1.3328*e*-12
0.8	2.7337*e*-09	9.2382*e*-13	1.3728*e*-12
1	1.7293*e*-08	8.2778*e*-13	1.3872*e*-12

with equidistant nodes for different Gauss points m. Tables 9 and 10 also show the better error results.

In Figure 7, approximate solution $y_n(t)$ and exact solution $y(t)$ of the example are given at $n = 10$. In Figure 8, errors for barycentric Lagrange interpolation are shown with different Gauss points at $n = 10$. From Figures 7 and 8, barycentric Lagrange interpolation collocation method can get high error accuracy.

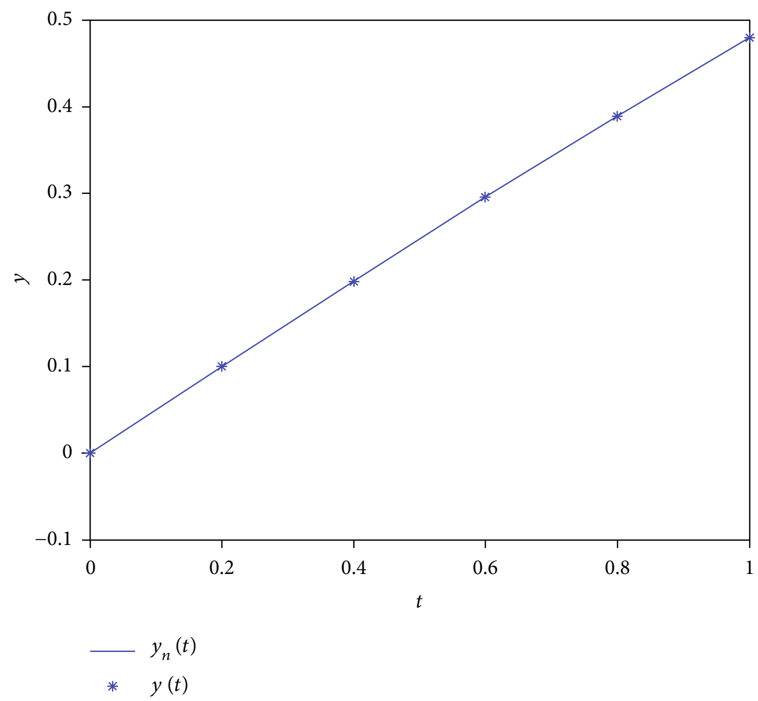

FIGURE 7: $y_n(t)$ and $y(t)$ of barycentric Lagrange interpolation using the Gauss quadrature formula with $m = 3$ at $n = 10$ for Example 4.

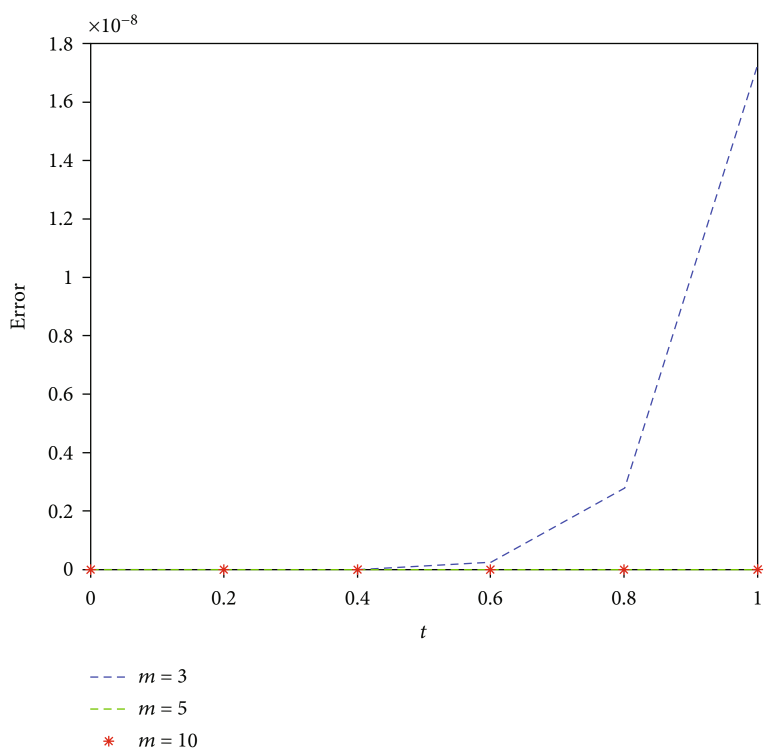

FIGURE 8: Errors of barycentric Lagrange interpolation using the Gauss quadrature formula with different Gauss points at $n = 10$ for Example 4.

8. Conclusion

BICM is proposed to solve the FVIDE. Integral terms of equation are dealt with by the Gauss quadrature formula or Gauss-Legendre quadrature formula. Compared with the Gauss-Legendre quadrature formula, barycentric Lagrange interpolation with the Gauss quadrature formula obtains higher error accuracy. The high-precise error results are gained when replacement method or additive method is chosen to deal with initial conditions. The errors of BICM are displayed by numerical examples, which illustrate that the method is available for solving one-dimensional FVIDE equation.

Data Availability

The table data and graph data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A Study on the New Class of Inequalities of Midpoint-Type and Trapezoidal-Type Based on Twice Differentiable Functions with Conformable Operators

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This paper derives some equalities via twice differentiable functions and conformable fractional integrals. With the help of the obtained identities, we present new trapezoid-type and midpoint-type inequalities via convex functions in the context of the conformable fractional integrals. New inequalities are obtained by taking advantage of the convexity property, power mean inequality, and Hölder's inequality. We show that this new family of inequalities generalizes some previous research studies by special choices. Furthermore, new other relevant results with trapezoid-type and midpoint-type inequalities are obtained.

1. Introduction

Fractional calculus and the theory of inequalities, which have recently received a lot of attention, have been the subject of many investigations in the mathematics. Mathematical modeling is one of the most important fields of this theory in which fractional operators are defined to design different fractional differential equations for describing the phenomena. For instance, one can mention to the third-order BVP with multistrip multipoint conditions [1], hybrid version and the Hilfer type of thermostat model [2, 3], fractional HIV model with the Mittag-Leffler-type kernel [4], mathematical fractional model of Q fever [5], fractional dynamics of mumps virus [6], fractional p-Laplacian equations [7], fractal-fractional version of AH1N1/09 virus along with the fractional Caputo-type version [8], etc.

In the last century, the Hermite–Hadamard inequality along with the midpoint and trapezoidal inequalities arising from this inequality has attracted many researchers. In addition, RL-fractional (Riemann-Liouville) integrals, conformable integrals, and many types of such integrals have been defined in these inequalities and have gained an important place in the literature.

More precisely, fractional calculus is a big part of mathematics in which the mathematicians develop and extend the existing classical ideas of integration and differentiation operators to noninteger orders. Recently, it has received the attention of many researchers from different areas like mathematicians, physicists, and engineers [9, 10]. For example, if we consider a fluid-dynamic traffic model, then we see that one can simulate the irregular oscillation of earthquakes via fractional derivatives. These operators are also utilized for modeling a main part of chemical and physical processes, biological processes, and engineering problems. For instance, biological population model [11], electrical circuits [12], viscous fluid and their semianalytical solutions [13], fractional gas dynamics [14], and fractal modeling of traffic flow [15] are applied examples of the application of fractional operators. Further, it is stated that fractional systems provide some numerical outcomes that are more appropriate than those given by integer-order systems [16, 17].

New investigations have developed a category of fractional integration operators and their application in various scientific fields. Using only the idea of the fundamental limit formulation for derivatives, a novel well-behaved fractional derivative was defined, entitled as the conformable derivative, by Khalil et al. in [18]. Some applied properties that cannot be derived by the Riemann-Liouville and Caputo operators are obtained by the conformable derivative. However, in [19], Abdelhakim stated that the conformable structure in [18] cannot yield acceptable data compared to the Caputo idea for special functions. This flaw in the conformable definition was overcome by giving several extensions of the conformable operators [20, 21]. Moreover, with the help of the well-known exponential and Mittag-Leffler functions and using them in the kernels, several researchers defined newly expanded fractional operators such as exponential discrete kernel-type operators [22], fractal-fractional operators [23], and some other derivatives [24, 25].

Inequalities are one of the important topics of mathematics, and in this field, convex functions and their generalizations play an important role. In [26–28], the authors focused on Hermite–Hadamard inequalities by using the majorization and some properties of convex functions. Later, some other researchers combined these notions with monocity and boundedness [29–31]. Over the years, many mathematicians have concentrated on acquired trapezoidal and midpoint-type inequalities that yield specific bounds via the R.H.S. and L.H.S. of the Hermite–Hadamard inequality, respectively. For instance, at first, Dragomir and Agarwal derived trapezoid inequalities in relation to the convex functions in [32], whereas Kirmac derived inequalities of midpoint type with the help of the convex functions in [33]. In addition, in [34], Qaisar and Hussain established a number of generalized inequalities of midpoint type. Moreover, Sarikaya et al. and Iqbal et al. derived some fractional trapezoid and midpoint-type inequalities for a family of the convex mappings in [35, 36], respectively. In [37, 38], studies obtained some extensions from midpoint inequalities involving the Riemann-Liouville operators. In [39], similar results are derived by Hyder et al. under the generalized Reimann-Liouville operators.

Researches on the differentiable functions of these inequalities also have an important place in the literature. Many researchers have focused on twice differentiable functions to obtain many important inequalities. For example, Barani et al. proved some inequalities under twice differentiable mappings having the convexity property which is connected to Hadamard-type inequalities in [40, 41]. In [42], several novel extensions of integral fractional inequalities of midpoint-trapezoid type for the abovementioned twice differentiable functions are established. In [43], authors obtained other class of novel inequalities in the sense of the Simpson and Hermite–Hadamard for some special functions whose absolute values of derivatives are convex.

The main goal of this paper is to acquire some new trapezoid-type and midpoint-type inequalities with the help of the twice differentiable function including conformable fractional integrals. We also establish that the newly obtained inequalities are a generalization of the existing trapezoid-type and midpoint type inequalities. The ideas and strategies for our results concerning trapezoid type and midpoint-type inequalities via conformable fractional integrals may open other directions for more research in this area.

2. Preliminaries

This section discusses the basics for building our main results. Here, definitions of the Riemann-Liouville integrals and conformable integrals, which are well known in the literature, are given. From the fact of fractional calculus theory, mathematical preliminaries will be given.

For $x, y > 0$ (real numbers), the famous gamma function and incomplete beta function are

$$\begin{aligned} \Gamma(x) &:= \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t, \\ \mathscr{B}(x, y, r) &:= \int_0^r t^{x-1} (1-t)^{y-1} \mathrm{d}t, \end{aligned} \tag{1}$$

respectively.

In 2006, Kilbas et al. [44] defined fractional integrals, also called the Riemann-Liouville integrals (RL-integral) as follows:

Definition 1 (see [44]). For $\hbar \in L^1[\nu, \omega]$, the Riemann-Liouville integrals $J_{\nu+}^{\varkappa}\hbar(x)$ and $J_{\omega-}^{\varkappa}\hbar(x)$ of order $\varkappa > 0$ are, respectively, given as

$$J_{\nu+}^{\varkappa}\hbar(x) = \frac{1}{\Gamma(\varkappa)} \int_\nu^x (x-t)^{\varkappa-1} \hbar(t) \mathrm{d}t, x > \nu, \tag{2}$$

$$J_{\omega-}^{\varkappa}\hbar(x) = \frac{1}{\Gamma(\varkappa)} \int_x^\omega (t-x)^{\varkappa-1} \hbar(t) \mathrm{d}t, x < \omega, \tag{3}$$

where $J_{\nu+}^{0}\hbar(x) = J_{\omega-}^{0}\hbar(x) = \hbar(x)$. By setting $\varkappa = 1$, the Riemann-Liouville integrals reduce to the classical integrals.

In 2017, Jarad et al. [25] formulated a novel fractional conformable integration operators. These researchers gave certain characteristics for these operators and some other fractional operators defined before. The fractional conformable integral operators are defined in the following definition:

Definition 2 (see [25]). For $\hbar \in L^1[\nu,\omega]$, the fractional conformable integral operator ${}^{\varkappa}\mathcal{J}^{\mu}_{\nu+}\hbar(x)$ and ${}^{\varkappa}\mathcal{J}^{\mu}_{\omega-}\hbar(x)$ of order $\varkappa \in C$, Re $(\varkappa) > 0$ and $\mu \in (0,1]$ are, respectively, given by

$$ {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+}\hbar(x) = \frac{1}{\Gamma(\varkappa)}\int_{\nu}^{x}\left(\frac{(x-\nu)^{\mu}-(t-\nu)^{\mu}}{\mu}\right)^{\varkappa-1} \cdot \frac{\hbar(t)}{(t-\nu)^{1-\mu}}\mathrm{d}t, t > \nu, \tag{4} $$

$$ {}^{\varkappa}\mathcal{J}^{\mu}_{\omega-}\hbar(x) = \frac{1}{\Gamma(\varkappa)}\int_{x}^{\omega}\left(\frac{(\omega-x)^{\mu}-(\omega-t)^{\mu}}{\mu}\right)^{\varkappa-1} \cdot \frac{\hbar(t)}{(\omega-t)^{1-\mu}}\mathrm{d}t, t < \omega. \tag{5} $$

It is notable that the fractional integral in (4) coincides with the fractional RL-integral in (2) when $\mu = 1$. Moreover, the fractional integral in (5) coincides with the fractional RL-integral in (3) when $\mu = 1$. For more studies about several recent results in relation to fractional integral inequalities, we can mention some versions in the context of the Caputo-Fabrizio operators [45, 46], proportional generalized operators [47, 48], some inequalities in the Maxwell fluid modeling with nonsingular operators [49], conformable integral inequalities [50], some inequalities based on the Caputo-type operators [51], the Katugampola-type inequalities [52, 53], and the references cited therein.

3. Trapezoid-Type Inequalities Based on Conformable Fractional Integrals

In this section, inequalities of trapezoid type are obtained for twice differentiable functions. We use the conformable fractional integral operators to obtain these inequalities.

To acquire conformable fractional integrals trapezoid-type inequalities, we consider the following lemma.

Lemma 3. *Let $\hbar : [\nu,\omega] \longrightarrow \mathbb{R}$ be a twice differentiable mapping on (ν,ω) such that $\hbar'' \in L_1([\nu,\omega])$. In this case, the equality*

$$ \frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu) + {}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right] = \frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\int_0^1\left(\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right) \cdot \hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)dt + \int_0^1\left(\int_0^t\left[\frac{1}{\mu^{\varkappa}} - \left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right)\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)dt\right], \tag{6} $$

holds.

Proof. Employing integration by parts, it yields

$$ \begin{aligned} I_1 &= \int_0^1\left(\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{ds}\right)\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\mathrm{d}t \\ &= \frac{2}{\omega-\nu}\left(\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{ds}\right)\hbar'\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\Bigg|_0^1 \\ &\quad - \frac{2}{\omega-\nu}\int_0^1\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-t)^{\mu}}{\mu}\right)^{\varkappa}\right]\hbar'\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\mathrm{d}t \\ &= \frac{2}{\omega-\nu}\left(\int_0^1\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{ds}\right)\hbar'\left(\frac{\nu+\omega}{2}\right) \\ &\quad - \frac{2}{\omega-\nu}\left\{\frac{2}{\omega-\nu}\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-t)^{\mu}}{\mu}\right)^{\varkappa}\right]\hbar\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\Bigg|_0^1 \right. \\ &\quad \left. + \frac{2\varkappa}{\omega-\nu}\int_0^1\left(\frac{1-(1-t)^{\mu}}{\mu}\right)^{\varkappa-1}(1-t)^{\mu-1}\hbar\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\mathrm{d}t\right\} \\ &= \frac{2}{\omega-\nu}\left(\int_0^1\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{ds}\right)f'\left(\frac{\nu+\omega}{2}\right) \\ &\quad + \left(\frac{2}{\omega-\nu}\right)^2\frac{\hbar(\nu)}{\mu^{\varkappa}} - \left(\frac{2}{\omega-\nu}\right)^2\frac{\Gamma(\varkappa+1)}{\Gamma(\varkappa)}\int_{\nu}^{\nu+\omega/2} \\ &\quad \cdot\left(\frac{1-(2/\omega-\nu(\nu+\omega/2-x))^{\mu}}{\mu}\right)^{\varkappa-1} \\ &\quad \cdot\left(\frac{2}{\omega-\nu}\left(\frac{\nu+\omega}{2}-x\right)\right)^{\mu-1}\frac{2}{\omega-\nu}\hbar(x)\mathrm{d}x \\ &= \frac{2}{\omega-\nu}\left(\int_0^1\left[\frac{1}{\mu^{\varkappa}}-\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}\mathrm{ds}\right)\hbar'\left(\frac{\nu+\omega}{2}\right) \\ &\quad + \left(\frac{2}{\omega-\nu}\right)^2\frac{\hbar(\nu)}{\mu^{\varkappa}} - \left(\frac{2}{\omega-\nu}\right)^{2+\mu\varkappa}\frac{\Gamma(\varkappa+1)}{\Gamma(\varkappa)} \\ &\quad \cdot\int_{\nu}^{\nu+\omega/2}\left(\frac{(\omega-\nu/2)^{\mu}-(\nu+\omega/2-x)^{\mu}}{\mu}\right)^{\varkappa-1} \\ &\quad \cdot\frac{\hbar(x)}{(\nu+\omega/2-x)^{1-\mu}}\hbar(x)\mathrm{d}x \\ &= \frac{2}{\omega-\nu}\left(\int_0^1\left[\frac{1}{\mu^{\varkappa}}-\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}\mathrm{ds}\right)\hbar'\left(\frac{\nu+\omega}{2}\right) \\ &\quad + \left(\frac{2}{\omega-\nu}\right)^2\frac{\hbar(\nu)}{\mu^{\varkappa}} - \left(\frac{2}{\omega-\nu}\right)^{2+\mu\varkappa}\Gamma(\varkappa+1)^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu). \end{aligned} \tag{7} $$

Likewise,

$$ \begin{aligned} I_2 &= \int_0^1\left(\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{ds}\right)\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)\mathrm{d}t \\ &= -\frac{2}{\omega-\nu}\left(\int_0^1\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]\mathrm{ds}\right)\hbar'\left(\frac{\nu+\omega}{2}\right) \\ &\quad + \left(\frac{2}{\omega-\nu}\right)^2\frac{\hbar(\omega)}{\mu^{\varkappa}} - \left(\frac{2}{\omega-\nu}\right)^{2+\mu\varkappa}\Gamma(\varkappa+1)^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega). \end{aligned} \tag{8} $$

Then, it follows that

$$\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}[I_1+I_2]=\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\cdot\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]. \tag{9}$$

So, the proof is accomplished. □

Theorem 4. *Consider $\hbar:[\nu,\omega]\longrightarrow\mathbb{R}$ as a twice differentiable mapping on (ν,ω) s.t. $\hbar''\in L_1([\nu,\omega])$. If $|\hbar''|$ is convex on $[\nu,\omega]$, then*

$$\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right|\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\Phi_1(\mu,\varkappa)\left(\left|\hbar''(\nu)\right|+\left|\hbar''(\omega)\right|\right), \tag{10}$$

where

$$\Phi_1(\mu,\varkappa)=\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|dt=\frac{1}{\mu^{\varkappa}}\int_0^1\left|t-\frac{1}{\mu}\mathscr{B}\left(\varkappa+1,\frac{1}{\mu},1-(1-t)^{\mu}\right)\right|dt. \tag{11}$$

Proof. Taking the absolute value of both sides of (6), we derive

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right|\\&\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|\cdot\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|dt+\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|\cdot\left|\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)\right|dt\right].\end{aligned} \tag{12}$$

By using the convexity property of the $|\hbar''|$, we establish

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right|\\&\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|\left[\frac{2-t}{2}\cdot\left|\hbar'(\nu)\right|+\frac{t}{2}\left|\hbar'(\omega)\right|\right]dt+\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|\cdot\left[\frac{t}{2}\left|\hbar'(\nu)\right|+\frac{2-t}{2}\left|\hbar'(\omega)\right|\right]dt\right]\\&=\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|dt\right)\cdot\left(\left|\hbar'(\nu)\right|+\left|\hbar'(\omega)\right|\right).\end{aligned} \tag{13}$$

The proof is ended. □

Remark 5. In Theorem 11, we have the inequalities as follows:

(i) If we set $\mu=1$ in (10), then Theorem 4 leads to [42], Corollary 7.

(ii) If we take $\mu=1$ and $\varkappa=1$ in (10), then Theorem 4 leads to [43], Proposition 2.

Theorem 6. *Assume that $\hbar:[\nu,\omega]\longrightarrow\mathbb{R}$ is a twice differentiable function on (ν,ω) s.t. $\hbar''\in L_p([\nu,\omega])$ with $\nu<\omega$. Let $|\hbar''|^q$ be convex on $[\nu,\omega]$ with $q>1$. Then, the inequality*

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right|\\&\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\Theta^{\varkappa}_{\mu}(p)\left[\left(\frac{3\left|\hbar''(\nu)\right|^q+\left|\hbar''(\omega)\right|^q}{4}\right)^{1/q}+\left(\frac{\left|\hbar''(\nu)\right|^q+3\left|\hbar''(\omega)\right|^q}{4}\right)^{1/q}\right]\\&\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{2^{3-2/p}}\Theta^{\varkappa}_{\mu}(p)\left[\left|\hbar''(\nu)\right|^q+\left|\hbar''(\omega)\right|^q\right],\end{aligned} \tag{14}$$

holds, where $1/q+1/p=1$ and

$$\Theta^{\varkappa}_{\mu}(p)=\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|^p dt\right)^{1/q}. \tag{15}$$

Proof. By employing the Hölder inequality on (12), we have

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right|\\&\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|^p dt\right)^{1/p}\cdot\left(\int_0^1\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q dt\right)^{1/q}\right.\\&\left.+\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|^p dt\right)^{1/p}\cdot\left(\int_0^1\left|\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)\right|^q dt\right)^{1/q}\right].\end{aligned} \tag{16}$$

For the sake of the convexity of $|\hbar''|^q$ on $[\nu,\omega]$, we get

$$\begin{aligned}&\int_0^1\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q dt\\&\leq\int_0^1\left[\frac{2-t}{2}\left|\hbar''(\nu)\right|^q+\frac{t}{2}\left|\hbar''(\omega)\right|^q\right]dt\\&=\frac{3\left|\hbar''(\nu)\right|^q+\left|\hbar''(\omega)\right|^q}{4},\end{aligned} \tag{17}$$

and similarly

$$\int_0^1 \left|\hbar''\left(\frac{t}{2}\nu + \frac{2-t}{2}\omega\right)\right|^q \mathrm{d}t \le \frac{|\hbar''(\nu)|^q + 3|\hbar''(\omega)|^q}{4}. \quad (18)$$

If we substitute the inequalities (17) and (18) in (16), the first inequality of (14) will be established.

The next inequality is derived directly if we let $\varpi_1 = 3|\hbar''(\nu)|^q, \rho_1 = |\hbar''(\omega)|^q, \varpi_2 = |\hbar''(\nu)|^q$, and $\rho_2 = 3|\hbar''(\omega)|^q$ and apply the inequality

$$\sum_{k=1}^n (\varpi_k + \rho_k)^s \le \sum_{k=1}^n \varpi_k^s + \sum_{k=1}^n \rho_k^s, 0 \le s < 1. \quad (19)$$

Thus, our deduction is ended. □

Corollary 7. *In Theorem 6, we have the inequalities as follows:*

(i) If we set $\mu = 1$ in Theorem 6, we derive

$$\begin{aligned}
&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\varkappa-1}\Gamma(\varkappa+1)}{(\omega-\nu)^\varkappa}\left[J^\varkappa_{\nu+\omega/2-}\hbar(\nu) + J^\varkappa_{\nu+\omega/2+}\hbar(\omega)\right]\right| \\
&\le \frac{(\omega-\nu)^2}{8}\left(\frac{1}{p+1} - \frac{1}{(\varkappa+1)^p(\varkappa p+p+1)}\right) \\
&\times \left[\left(\frac{3|\hbar''(\nu)|^q + |\hbar''(\omega)|^q}{4}\right)^{1/q} + \left(\frac{|\hbar''(\nu)|^q + 3|\hbar''(\omega)|^q}{4}\right)^{1/q}\right] \\
&\le \frac{(\omega-\nu)^2}{2^{3-2/p}}\left(\frac{1}{p+1} - \frac{1}{(\varkappa+1)^p(\varkappa p+p+1)}\right) \cdot \left[|\hbar''(\nu)|^q + |\hbar''(\omega)|^q\right].
\end{aligned} \quad (20)$$

Proof. For the proof, it will be sufficient to write down the solution of the integral below.

$$\begin{aligned}
\Theta^\varkappa_\mu(p) = \Theta^\varkappa_1(p) &= \left(\int_0^1 \left|\int_0^t (1-s^\varkappa)\mathrm{d}s\right|^p \mathrm{d}t\right)^{1/p} \\
&= \left(\int_0^1 \left|t - \frac{t^{\varkappa+1}}{\varkappa+1}\right|^p \mathrm{d}t\right)^{1/p}.
\end{aligned} \quad (21)$$

Under conditions $A > B > 0$ and $p > 1$, the inequality

$$|A-B|^p \le A^p - B^p \quad (22)$$

is satisfied.

From the inequality (22), $A = t$ and $B = t^{\varkappa+1}/\varkappa + 1$, we have

$$\begin{aligned}
\Theta^\varkappa_1(p) &\le \left(\int_0^1 t^p \mathrm{d}t - \int_0^1 \left(\frac{t^{\varkappa+1}}{\varkappa+1}\right)^p \mathrm{d}t\right)^{1/p} \\
&= \left(\frac{1}{p+1} - \frac{1}{(\varkappa+1)^p(\varkappa p+p+1)}\right)^{1/p}.
\end{aligned} \quad (23)$$

When the solution of $\Theta^\varkappa_\mu(p)$ is substituted for (14), the proof is clear.

(ii) If we take $\mu = 1$ and $\varkappa = 1$ in Theorem 6, then

$$\begin{aligned}
&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{1}{(\omega-\nu)}\int_\nu^\omega \hbar(x)dx\right| \\
&\le \frac{(\omega-\nu)^2}{8}\left(\frac{1}{p+1} - \frac{1}{2^p(2p+1)}\right) \\
&\cdot \left[\left(\frac{3|\hbar''(\nu)|^q + |\hbar''(\omega)|^q}{4}\right)^{1/q} + \left(\frac{|\hbar''(\nu)|^q + 3|\hbar''(\omega)|^q}{4}\right)^{1/q}\right] \\
&\le \frac{(\omega-\nu)^2}{2^{3-2/p}}\left(\frac{1}{p+1} - \frac{1}{2^p(2p+1)}\right) \cdot \left[|\hbar''(\nu)|^q + |\hbar''(\omega)|^q\right].
\end{aligned} \quad (24)$$

□

Theorem 8. *Consider $\hbar : [\nu, \omega] \longrightarrow \mathbb{R}$ as a twice differentiable mapping on (ν, ω) s.t. $\hbar'' \in L_q([\nu, \omega])$. Assume that $|\hbar''|^q$ admits the convexity property on $[\nu, \omega]$ with $q \ge 1$. Then,*

$$\begin{aligned}
&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2} - \frac{2^{\mu\varkappa-1}\mu^\varkappa\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2-}\hbar(\nu) + {}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2+}\hbar(\omega)\right]\right| \\
&\le \frac{(\omega-\nu)^2\mu^\varkappa}{8}(\Phi_1(\mu,\varkappa))^{1-1/q} \times \left[\left(\frac{(2\Phi_1(\mu,\varkappa) - \Phi_2(\mu,\varkappa))}{2} \cdot |\hbar''(\nu)|^q + \frac{\Phi_2(\mu,\varkappa)}{2}|\hbar''(\omega)|^q\right)^{1/q} + \left(\frac{\Phi_2(\mu,\varkappa)}{2} \cdot |\hbar''(\nu)|^q + \frac{(2\Phi_1(\mu,\varkappa) - \Phi_2(\mu,\varkappa))}{2}|\hbar''(\omega)|^q\right)^{1/q}\right],
\end{aligned} \quad (25)$$

holds, where

$$\begin{aligned}
\Phi_2(\mu,\varkappa) &= \int_0^1 t\left|\int_0^t \left[\frac{1}{\mu^\varkappa} - \left(\frac{1-(1-s)^\mu}{\mu}\right)^\varkappa\right] ds\right| dt \\
&= \frac{1}{\mu^\varkappa}\int_0^1 t\left|t - \frac{1}{\mu}\mathscr{B}\left(\varkappa+1, \frac{1}{\mu}, 1-(1-t)^\mu\right)\right| dt.
\end{aligned} \quad (26)$$

Proof. By employing the power-mean inequality in (12), we have

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)\right]\right|\\&\le\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|dt\right)^{1-1/q}\right.\\&\quad\times\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|\right.\\&\quad\left.\cdot\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q dt\right)^{1/q}\\&\quad+\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|dt\right)^{1-1/q}\\&\quad\times\left(\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|\right.\\&\quad\left.\left.\cdot\left|\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)\right|^q dt\right)^{1/q}\right].\end{aligned}\tag{27}$$

We know that $|\hbar'|^q$ is convex. Thus,

$$\begin{aligned}&\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q dt\\&\le\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|\\&\quad\cdot\left[\frac{2-t}{2}\left|\hbar''(\nu)\right|^q+\frac{t}{2}\left|\hbar''(\omega)\right|^q\right]dt\\&=\frac{(2\Phi_1(\mu,\varkappa)-\Phi_2(\mu,\varkappa))}{2}\left|\hbar''(\nu)\right|^q\\&\quad+\frac{\Phi_2(\mu,\varkappa)}{2}\left|\hbar''(\omega)\right|^q,\end{aligned}\tag{28}$$

and similarly

$$\begin{aligned}&\int_0^1\left|\int_0^t\left[\frac{1}{\mu^{\varkappa}}-\left(\frac{1-(1-s)^{\mu}}{\mu}\right)^{\varkappa}\right]ds\right|\left|\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)\right|^q dt\\&\le\frac{\Phi_2(\mu,\varkappa)}{2}\left|\hbar''(\nu)\right|^q+\frac{(2\Phi_1(\mu,\varkappa)-\Phi_2(\mu,\varkappa))}{2}\left|\hbar''(\omega)\right|^q.\end{aligned}\tag{29}$$

Substituting the inequalities (28) and (29) in (27), we derive the desired result. □

Corollary 9. *In Theorem 8, we have the inequalities as follows:*

(i) By choosing $\mu=1$ in Theorem 8, we derive

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{2^{\varkappa-1}\Gamma(\varkappa+1)}{(\omega-\nu)^{\varkappa}}\left[J^{\varkappa}_{\nu+\omega/2-}\hbar(\nu)+J^{\varkappa}_{\nu+\omega/2+}\hbar(\omega)\right]\right|\\&\le\frac{(\omega-\nu)^2}{8}\left(\frac{1}{2}-\frac{1}{(\varkappa+1)(\varkappa+2)}\right)^{1-1/q}\\&\quad\times\left[\left(\left(\frac{1}{3}-\frac{\varkappa+4}{2(\varkappa+1)(\varkappa+2)(\varkappa+3)}\right)\left|\hbar''(\nu)\right|^q\right.\right.\\&\quad\left.+\left(\frac{1}{6}-\frac{1}{2(\varkappa+1)(\varkappa+3)}\right)\left|\hbar''(\omega)\right|^q\right)^{1/q}\\&\quad+\left(\left(\frac{1}{6}-\frac{1}{2(\varkappa+1)(\varkappa+3)}\right)\left|\hbar''(\nu)\right|^q\right.\\&\quad\left.\left.+\left(\frac{1}{3}-\frac{\varkappa+4}{2(\varkappa+1)(\varkappa+2)(\varkappa+3)}\right)\left|\hbar''(\omega)\right|^q\right)^{1/q}\right].\end{aligned}\tag{30}$$

(ii) If we take $\mu=1$ and $\varkappa=1$ in Theorem 8, we derive

$$\begin{aligned}&\left|\frac{\hbar(\nu)+\hbar(\omega)}{2}-\frac{1}{(\omega-\nu)}\int_{\nu}^{\omega}\hbar(x)dx\right|\\&\le\frac{(\omega-\nu)^2}{24}\left[\left(\frac{11}{16}\left|\hbar''(\nu)\right|^q+\frac{5}{16}\left|\hbar''(\omega)\right|^q\right)^{1/q}\right.\\&\quad\left.+\left(\frac{5}{16}\left|\hbar''(\nu)\right|^q+\frac{11}{16}\left|\hbar''(\omega)\right|^q\right)^{1/q}\right].\end{aligned}\tag{31}$$

4. Midpoint-Type Inequalities Based on Conformable Fractional Integrals

In this section, midpoint-type inequalities are created for twice differentiable functions with the help of conformable fractional integrals. To formulate these inequalities, let us first set up the following identity.

Lemma 10. *Let $\hbar:[\nu,\omega]\longrightarrow\mathbb{R}$ be a twice differentiable map on (ν,ω) with $\hbar''\in L_1([\nu,\omega])$. Then, the equality*

$$\begin{aligned}&\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\\&=\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\int_0^1\left(\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right)\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)dt\right.\\&\quad\left.+\int_0^1\left(\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right)\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)dt\right]\end{aligned}\tag{32}$$

is valid.

Proof. With the help of the integration by parts

$$\begin{aligned}
I_3 &= \int_0^1 \left(\int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar'' \left(\frac{2-t}{2}\nu + \frac{t}{2}\omega \right) \mathrm{dt} \\
&= \frac{2}{\omega-\nu} \left(\int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar' \left(\frac{2-t}{2}\nu + \frac{t}{2}\omega \right) \Bigg|_0^1 \\
&\quad - \frac{2}{\omega-\nu} \int_0^1 \left[\frac{1-(1-t)^\mu}{\mu} \right]^\varkappa \hbar' \left(\frac{2-t}{2}\nu + \frac{t}{2}\omega \right) \mathrm{dt} \\
&= \frac{2}{\omega-\nu} \left(\int_0^1 \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar' \left(\frac{\nu+\omega}{2} \right) \\
&\quad - \frac{2}{\omega-\nu} \left\{ \frac{2}{\omega-\nu} \left(\frac{1-(1-t)^\mu}{\mu} \right)^\varkappa \hbar \left(\frac{2-t}{2}\nu + \frac{t}{2}\omega \right) \Bigg|_0^1 \right. \\
&\quad \left. - \frac{2\varkappa}{\omega-\nu} \int_0^1 \left(\frac{1-(1-t)^\mu}{\mu} \right)^{\varkappa-1} (1-t)^{\mu-1} \mathrm{dt}. \right.
\end{aligned} \tag{33}$$

By using variable change, equality is obtained as follows:

$$\begin{aligned}
I_3 &= \frac{2}{\omega-\nu} \left(\int_0^1 \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar' \left(\frac{\nu+\omega}{2} \right) \\
&\quad - \left(\frac{2}{\omega-\nu} \right)^2 \frac{1}{\mu^\varkappa} \hbar\left(\frac{\nu+\omega}{2} \right) + \left(\frac{2}{\omega-\nu} \right)^{2+\mu\varkappa} \frac{\Gamma(\varkappa+1)}{\Gamma(\varkappa)} \int_\nu^{\nu+\omega/2} \\
&\quad \cdot \left(\frac{(\omega-\nu/2)^\mu - (\nu+\omega/2-x)^\mu}{\mu} \right)^{\varkappa-1} \frac{\hbar(x)}{(\nu+\omega/2-x)^{1-\mu}} \hbar(x) \mathrm{dx} \\
&= \frac{2}{\omega-\nu} \left(\int_0^1 \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar' \left(\frac{\nu+\omega}{2} \right) - \left(\frac{2}{\omega-\nu} \right)^2 \\
&\quad \cdot \frac{1}{\mu^\varkappa} \hbar\left(\frac{\nu+\omega}{2} \right) + \left(\frac{2}{\omega-\nu} \right)^{2+\mu\varkappa} \Gamma(\varkappa+1)^\varkappa \mathcal{J}^\mu_{\nu+\omega/2-} \hbar(\nu).
\end{aligned} \tag{34}$$

In the same way,

$$\begin{aligned}
I_4 &= \int_0^1 \left(\int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar'' \left(\frac{t}{2}\nu + \frac{2-t}{2}\omega \right) \mathrm{dt} \\
&= -\frac{2}{\omega-\nu} \left(\int_0^1 \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right) \hbar' \left(\frac{\nu+\omega}{2} \right) \\
&\quad - \left(\frac{2}{\omega-\nu} \right)^2 \frac{1}{\mu^\varkappa} \hbar\left(\frac{\nu+\omega}{2} \right) \\
&\quad + \left(\frac{2}{\omega-\nu} \right)^{2+\mu\varkappa} \Gamma(\varkappa+1)^\varkappa \mathcal{J}^\mu_{\nu+\omega/2+} \hbar(\omega).
\end{aligned} \tag{35}$$

If (34) and (35) are added together and then multiplied by $(\omega-\nu)^2\mu^\varkappa/8$, the proof is completed. □

Theorem 11. *Assume $\hbar : [\nu,\omega] \longrightarrow \mathbb{R}$ as a twice differentiable function on (ν,ω) s.t. $\hbar'' \in L_1([\nu,\omega])$. By considering the convexity of $|\hbar''|$ on $[\nu,\omega]$, the inequality*

$$\begin{aligned}
&\left| \frac{2^{\mu\varkappa-1}\mu^\varkappa \Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}} \left[{}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2+}\hbar(\omega) + {}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2-}\hbar(\nu) \right] - \hbar\left(\frac{\nu+\omega}{2} \right) \right| \\
&\le \frac{(\omega-\nu)^2\mu^\varkappa}{8} Y_1(\mu,\varkappa) \left(|\hbar''(\nu)| + |\hbar''(\omega)| \right)
\end{aligned} \tag{36}$$

is satisfied, where $\mathcal{B}$ denotes the beta function and

$$\begin{aligned}
Y_1(\mu,\varkappa) &= \int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa ds \right| dt \\
&= \frac{1}{\mu^\varkappa} \int_0^1 \left| \frac{1}{\mu} \mathcal{B}\left(\varkappa+1, \frac{1}{\mu}, 1-(1-t)^\mu \right) \right| dt.
\end{aligned} \tag{37}$$

Proof. On both sides of (32), we take the absolute value and get

$$\begin{aligned}
&\left| \frac{2^{\mu\varkappa-1}\mu^\varkappa \Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}} \left[{}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2+}\hbar(\omega) + {}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2-}\hbar(\nu) \right] - \hbar\left(\frac{\nu+\omega}{2} \right) \right| \\
&\le \frac{(\omega-\nu)^2\mu^\varkappa}{8} \left[\int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right| \left| \hbar'' \left(\frac{2-t}{2}\nu + \frac{t}{2}\omega \right) \right| \mathrm{dt} \right. \\
&\quad \left. + \int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right| \left| \hbar'' \left(\frac{t}{2}\nu + \frac{2-t}{2}\omega \right) \right| \mathrm{dt} \right].
\end{aligned} \tag{38}$$

Since convexity of $|\hbar''|$, then we have

$$\begin{aligned}
&\left| \frac{2^{\mu\varkappa-1}\mu^\varkappa \Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}} \left[{}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2+}\hbar(\omega) + {}^\varkappa\mathcal{J}^\mu_{\nu+\omega/2-}\hbar(\nu) \right] - \hbar\left(\frac{\nu+\omega}{2} \right) \right| \\
&\le \frac{(\omega-\nu)^2\mu^\varkappa}{8} \left[\int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right| \right. \\
&\quad \cdot \left(\frac{2-t}{2} |\hbar''(\nu)| + \frac{t}{2} |\hbar''(\omega)| \right) dt \\
&\quad \left. + \int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right| \left(\frac{t}{2} |\hbar''(\nu)| + \frac{2-t}{2} |\hbar''(\omega)| \right) \mathrm{dt} \right] \\
&= \frac{(\omega-\nu)^2\mu^\varkappa}{8} \left(\int_0^1 \left| \int_0^t \left[\frac{1-(1-s)^\mu}{\mu} \right]^\varkappa \mathrm{ds} \right| \mathrm{dt} \right) \\
&\quad \cdot \left(|\hbar''(\nu)| + |\hbar''(\omega)| \right).
\end{aligned} \tag{39}$$

□

Remark 12. In Theorem 11:

(i) If we set $\mu = 1$, then we lead to [42], Theorem 1.5.

(ii) If we allow $\mu = 1$ and $\varkappa = 1$, then Theorem 11 and [43], Proposition 1 are identical.

Theorem 13. *Let $\hbar : [\nu,\omega] \longrightarrow \mathbb{R}$ be a twice differentiable map on (ν,ω) s.t. $\hbar'' \in L_1([\nu,\omega])$. Let $|\hbar''|^q$ be convex on $[\nu,\omega]$ with $q > 1$. Then,*

$$\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2}{2}\left(Y^{\varkappa}_{\mu}(p)\right)^{1/p}\left[\left(\frac{3|\hbar''(\nu)|^q+|\hbar''(\omega)|^q}{4}\right)^{1/q}+\left(\frac{|\hbar''(\nu)|^q+3|\hbar''(\omega)|^q}{4}\right)^{1/q}\right]$$
$$\leq\frac{(\omega-\nu)^2}{2}\left(4Y^{\varkappa}_{\mu}(p)\right)^{1/p}\left[|\hbar''(\nu)|+|\hbar''(\omega)|\right],\quad(40)$$

where $1/p+1/q=1$, and

$$Y^{\varkappa}_{\mu}(p)=\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|^p dt.\quad(41)$$

Proof. Using the Hölder inequality in (38), we have

$$\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|^p dt\right)^{1/p}\cdot\left(\int_0^1\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q dt\right)^{1/q}+\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|^p dt\right)^{1/p}\cdot\left(\int_0^1\left|\hbar''\left(\frac{t}{2}\nu+\frac{2-t}{2}\omega\right)\right|^q dt\right)^{1/q}\right].\quad(42)$$

Since $|\hbar''|^q$ is convex, we obtain

$$\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|^p dt\right)^{1/p}\cdot\left(\int_0^1\left(\frac{2-t}{2}|\hbar''(\nu)|^q+\frac{t}{2}|\hbar''(\omega)|^q\right)dt\right)^{1/q}+\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|^p dt\right)^{1/p}\cdot\left(\int_0^1\left(\frac{2-t}{2}|\hbar''(\nu)|^q+\frac{t}{2}|\hbar''(\omega)|^q\right)dt\right)^{1/q}\right].\quad(43)$$

If we substitute the inequalities (17) and (18) in (43), we obtain the first inequality of (40).

The last inequality is established by letting $\varpi_1=3|\hbar''(\nu)|^q$, $\rho_1=|\hbar''(\omega)|^q$, $\varpi_2=|\hbar''(\nu)|^q$, and $\rho_2=3|\hbar''(\omega)|^q$ and with help of the inequality (19). □

Corollary 14. *In Theorem 13, we have the inequalities as follows:*

(i) If we set $\mu=1$ in Theorem 13, we derive

$$\left|\frac{2^{\varkappa-1}\Gamma(\varkappa+1)}{(\omega-\nu)^{\varkappa}}\left[J^{\varkappa}_{\nu+\omega/2+}\hbar(\omega)+J^{\varkappa}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2}{2(\varkappa+1)}\left(\frac{1}{\varkappa p+p+1}\right)^{1/p}\left[\left(\frac{3|\hbar''(\nu)|^q+|\hbar''(\omega)|^q}{4}\right)^{1/q}+\left(\frac{|\hbar''(\nu)|^q+3|\hbar''(\omega)|^q}{4}\right)^{1/q}\right]$$
$$\leq\frac{(\omega-\nu)^2}{2(\varkappa+1)}\left(\frac{4}{\varkappa p+p+1}\right)^{1/p}\left[|\hbar''(\nu)|+|\hbar''(\omega)|\right].\quad(44)$$

(ii) If we take $\mu=1$ and $\varkappa=1$ in Theorem 13, we have

$$\left|\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\hbar(x)dx-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2}{4}\left(\frac{1}{2p+1}\right)^{1/p}\left[\left(\frac{3|\hbar''(\nu)|^q+|\hbar''(\omega)|^q}{4}\right)^{1/q}+\left(\frac{|\hbar''(\nu)|^q+3|\hbar''(\omega)|^q}{4}\right)^{1/q}\right]$$
$$\leq\frac{(\omega-\nu)^2}{4}\left(\frac{4}{2p+1}\right)^{1/p}\left[|\hbar''(\nu)|+|\hbar''(\omega)|\right].\quad(45)$$

Theorem 15. *Let $\hbar:[\nu,\omega]\longrightarrow\mathbb{R}$ be a twice differentiable map on (ν,ω) s.t. $\hbar''\in L_1([\nu,\omega])$. Suppose that $|\hbar'|^q$ is convex on $[\nu,\omega]$ with $q\geq 1$. Then,*

$$\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathcal{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|$$
$$\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}(Y_1(\mu,\varkappa))^{1-1/q}\times\left[\left(\frac{2Y_1(\mu,\varkappa)-Y_2(\mu,\varkappa)}{2}\cdot|\hbar''(\nu)|^q+\frac{Y_2(\mu,\varkappa)}{2}|\hbar''(\omega)|^q dt\right)^{1/q}+\left(\frac{Y_2(\mu,\varkappa)}{2}\cdot|\hbar''(\nu)|^q+\frac{2Y_1(\mu,\varkappa)-Y_2(\mu,\varkappa)}{2}|\hbar''(\omega)|^q dt\right)^{1/q}\right],\quad(46)$$

in which $\mathscr{B}$ depicts the beta function, and $Y_1(\mu,\varkappa)$ is defined as in (37). Here,

$$Y_2(\mu,\varkappa)=\int_0^1\left|\int_0^t t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|dt=\frac{1}{\mu^{\varkappa}}\int_0^1 t\left|\frac{1}{\mu}\mathscr{B}\left(\varkappa+1,\frac{1}{\mu},1-(1-t)^{\mu}\right)\right|dt. \tag{47}$$

Proof. By utilizing the power-mean inequality in (38), it becomes

$$\begin{aligned}&\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{a+\omega}{2}\right)\right|\\&\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left[\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|dt\right)^{1-1/q}\right.\\&\times\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q dt\right)^{1/q}\\&+\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|dt\right)^{1-1/q}\\&\left.\times\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|\left|\hbar''\left(\frac{2-t}{2}\nu+\frac{t}{2}\omega\right)\right|^q dt\right)^{1/q}\right].\end{aligned} \tag{48}$$

Due to the convexity of $|\hbar''|^q$ on $[\nu,b]$, we may write

$$\begin{aligned}&\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|\\&\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|dt\right)^{1-1/q}\\&\times\left[\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|\frac{2-t}{2}\left|\hbar''(\nu)\right|^q+\frac{t}{2}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right.\\&\left.+\left(\int_0^1\left|\int_0^t\left[\frac{1-(1-s)^{\mu}}{\mu}\right]^{\varkappa}ds\right|\frac{t}{2}\left|\hbar''(\nu)\right|^q+\frac{2-t}{2}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right].\end{aligned} \tag{49}$$

It is clearly seen that

$$\begin{aligned}&\left|\frac{2^{\mu\varkappa-1}\mu^{\varkappa}\Gamma(\varkappa+1)}{(\omega-\nu)^{\mu\varkappa}}\left[{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2+}\hbar(\omega)+{}^{\varkappa}\mathscr{J}^{\mu}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|\\&\leq\frac{(\omega-\nu)^2\mu^{\varkappa}}{8}\left(Y_1(\mu,\varkappa)\right)^{1-1/q}\\&\times\left[\left(\frac{2Y_1(\mu,\varkappa)-Y_2(\mu,\varkappa)}{2}\left|\hbar''(\nu)\right|^q+\frac{Y_2(\mu,\varkappa)}{2}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right.\\&\left.+\left(\frac{Y_2(\mu,\varkappa)}{2}\left|\hbar''(\nu)\right|^q+\frac{2Y_1(\mu,\varkappa)-Y_2(\mu,\varkappa)}{2}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right].\end{aligned} \tag{50}$$

The proof is ended. □

Corollary 16. *In Theorem 15,*

(i) *if we set $\mu=1$, then we acquire*

$$\begin{aligned}&\left|\frac{2^{\varkappa-1}\Gamma(\varkappa+1)}{(\omega-\nu)^{\varkappa}}\left[J^{\varkappa}_{\nu+\omega/2+}\hbar(\omega)+J^{\varkappa}_{\nu+\omega/2-}\hbar(\nu)\right]-\hbar\left(\frac{\nu+\omega}{2}\right)\right|\\&\leq\frac{(\omega-\nu)^2}{8}\left(\frac{1}{(\varkappa+1)(\varkappa+2)}\right)^{1-1/q}\\&\times\left[\left(\frac{\varkappa+4}{2(\varkappa+1)(\varkappa+2)(\varkappa+3)}\left|\hbar''(\nu)\right|^q\right.\right.\\&\left.+\frac{1}{2(\varkappa+1)(\varkappa+3)}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\\&+\left(\frac{1}{2(\varkappa+1)(\varkappa+3)}\left|\hbar''(\nu)\right|^q\right.\\&\left.\left.+\left(\frac{\varkappa+4}{2(\varkappa+1)(\varkappa+2)(\varkappa+3)}\right)\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right],\end{aligned} \tag{51}$$

(ii) *if we take $\mu=1$ and $\varkappa=1$, we obtain*

$$\begin{aligned}&\left|\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\hbar(x)dx-\hbar\left(\frac{\nu+\omega}{2}\right)\right|\\&\leq\frac{(\omega-\nu)^2}{48}\left[\left(\frac{5}{8}\left|\hbar''(\nu)\right|^q+\frac{3}{8}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right.\\&\left.+\left(\frac{3}{8}\left|\hbar''(\nu)\right|^q+\frac{5}{8}\left|\hbar''(\omega)\right|^q dt\right)^{1/q}\right].\end{aligned} \tag{52}$$

5. Conclusion

In this research, we established new estimates of trapezoid type and midpoint-type inequalities via conformable fractional integrals under twice differentiable functions. These inequalities were proven to be generalizations of the Riemann-Liouville fractional integrals related to inequalities of trapezoid type and midpoint type. In future works, researchers can obtain likewise inequalities of midpoint type and trapezoid type via conformable fractional integrals for convex functions in the context of quantum calculus. Moreover, curious readers can investigate our obtained inequalities via different kinds of fractional integrals.

Data Availability

No data were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Conceptualization was performed by H.K. and H.B.; formal analysis was contributed by H.K., H.B., S.E., S.R., and

M.K.A.K.; methodology was performed by H.K., H.B., S.E., S.R., and M.K.A.K.; H.B. and S.E. were assigned for the software. All authors have read and agreed to the published version of the manuscript.

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Research Article

Discussions on Proinov-$\mathscr{C}_b$-Contraction Mapping on b-Metric Space

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In the present paper, we introduce the notion of Proinov-$\mathscr{C}_b$-contraction mapping and we discuss it within the most interesting abstract structure, namely, b-metric spaces. We further take into consideration the necessary conditions to guarantee the existence and uniqueness of fixed points for such mappings, as well as indicate the validity of the main results by providing illustrative examples.

1. Introduction and Preliminaries

The fixed point theory focuses on investigating the necessary and sufficient conditions on the operator as well as the abstract structure within which the operator is defined. Many research papers, on fixed point theory, aim to bring forth a new condition on the operator (contraction criteria) or suggest a new abstract structure, or both. The present paper highlights a new contraction condition, namely, a Proinov-$\mathscr{C}_b$-contraction, on the most interesting abstract structure of b-metric spaces.

The notion of b-metric has been approached by several researchers such as Bakhtin [1] and Czerwik [2, 3]. For instance, Berinde [4, 5] named this structure as "quasi-metric." To be more precise, by b-metric, we understand the natural successful extension of metric by weakening "the triangle inequality" with "the extended triangle inequality." In other words, the condition of metric $d(r, q) \leq d(r, p) + d(p, q)$ turns into the new condition $d(r, q) \leq s[d(r, p) + d(p, q)]$ for all p, q, r and for a real number $s \geq 1$. Evidently, in case of $s = 1$, these two notions coincide. Despite the high similarities of the definitions of the notion of metric and b-metric, there topological properties may differ. For instance, it is known that metric is a continuous map, but, as a mapping, b-metric is not necessarily continuous. Moreover, an open ball is not open and a closed ball is not a closed set. These differences make this structure very interesting to investigate. In particular, in [6], the authors characterized the weak ϕ-contractions in setting of b-metric spaces. In [7], the existence of the fixed point of certain set-valued mappings was discussed in the context of b-metric spaces. Additionally, Ulam Stability of the fixed point, in the framework of b-metric spaces, has been considered in [8]. On the other hand, in [9–12], the authors focused on the existence of distinct multivalued operators in the context of b-metric spaces. In [13], Pacurar dealt with a fixed point for ϕ-contractions in the same structures. Another fact worth mentioning is that Shukla [14] defined partial b-metric spaces while considering the fixed point theorem.

The notion of Proinov-$\mathscr{C}_b$-contraction mapping is based on two aspects: "Proinov-type mappings" [15] and "simulation functions" [16, 17]. Proinov [15] proved that several existing results are consequences of Skof's result [18] reported in 1977. On the other hand, the simulation function also helps to get a very general contraction condition whose consequences involve several existing fixed point theorems, including Banach's.

Throughout the paper, we presume that $\mathfrak{X}$ is a nonempty set.

The notion of simulation function, introduced by Joonaghany et al. [16], combine several existing results.

Definition 1 (see [16]). A function $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ is called a simulation function if

(ζ_1) $\zeta(0, 0) = 0$

(ζ_2) $\zeta(r, p) < p - r$ for all $r, p > 0$

(ζ_3) $\{r_n\}, \{p_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\longrightarrow\infty} r_n = \lim_{n\longrightarrow\infty} p_n > 0$, then

$$\limsup_{n\longrightarrow\infty} \zeta(r_n, p_n) < 0. \tag{1}$$

The set of all simulation functions will be denoted by $\mathcal{Z}$. On account of (ζ_2), we observe that

$$\zeta(t, t) < 0 \text{ for all } t > 0, \zeta \in \mathcal{Z}. \tag{2}$$

We also notice that in [17], it was shown that (ζ_1) is superfluous.

Definition 2 (see [16]). Let $(\mathfrak{X}, d)$ be a metric space and $\zeta \in \mathcal{Z}$. We say that a self-mapping T on $\mathfrak{X}$ is a $\mathcal{Z}$-contraction with respect to ζ, if

$$\zeta(d(T(x), T(y)), d(x, y)) \geq 0, \text{ for all } x, y \in \mathfrak{X}. \tag{3}$$

Considering $\zeta(r, p) = \kappa p - r$ with $\kappa \in [0, 1)$ and $r, p \in [0,\infty)$, it follows that the Banach contraction forms a $\mathcal{Z}$ -contraction with respect to ζ.

Theorem 3. *On a complete metric space, every $\mathcal{Z}$ -contraction has a unique fixed point.*

Definition 4. On a nonempty set X, let $b : \mathfrak{X} \times \mathfrak{X} \longrightarrow [0,\infty)$ be a function such that the following conditions hold:

(b_1) $b(x, y) = 0$ if and only if $x = y$

(b_2) $b(x, y) = b(y, x)$ for all $x, y \in X$

(b_3) $b(x, y) \leq s[b(x, u) + b(u, y)]$ for all $x, y, u \in \mathfrak{X}$, with $s \geq 1$

Then, we say that function b is a b -metric. In this case, the tripled $(\mathfrak{X}, b, s)$ forms a b-metric space.

Of course, for $s = 1$, the above function b defines a distance (or metric) on $\mathfrak{X}$.

An illustrative example of b-metric would be the following:

Example 1. Let the space

$$l_{1/2} = \left\{ x = (x_1, x_2, \cdots, x_m, \cdots) \colon \sum_{j=1}^{\infty} |x_j| < \infty \right\}. \tag{4}$$

Then, the function $b : l_{1/2} \times l_{1/2} \longrightarrow [0,\infty)$, where

$$b(x, y) = \left(\sum_{j=1}^{\infty} \sqrt{|x_j - y_j|} \right)^2 \tag{5}$$

is a b-metric, with $s = 2$.

The concepts of convergent and Cauchy sequences on b-metric spaces can be defined in a similar way to the case of ordinary metric spaces.

Definition 5. Let $\{x_m\}_{m\geq 0}$ be a sequence in the b-metric space $(\mathfrak{X}, b, s)$. We say that the sequence $\{x_m\}_{m\geq 0}$ is

(c) *convergent* $\Longleftrightarrow$ there exists $u \in \mathfrak{X}$ such that for any $e > 0$, there exists $N(e) \in \mathbb{N}$ such that $b(x_m, u) < e$, for all $m \geq N(e)$

This means, $\lim_{m\longrightarrow\text{infty}} b(x_m, u) = 0$; we write $x_m \longrightarrow u$, or $\lim_{m\longrightarrow\infty} x_m = u$.

(C) *Cauchy* $\Longleftrightarrow$ for any $e > 0$, there exists $N(e) \in \mathbb{N}$ such that $b(x_m, x_p) < e$, for all $m, p \geq N(e)$

In case every Cauchy sequence in $\mathfrak{X}$ is convergent, we say that the b-metric space $(\mathfrak{X}, \mathrm{b}, \mathrm{s})$ is *complete*.

Lemma 6 (see [19]). *Let $(\mathfrak{X}, b)$ be a b-metric space and $\{x_n\}$ be a sequence of elements in $\mathfrak{X}$ such that there exists $\kappa \in [0, 1)$ such that $b(x_{n+1}, x_{n+2}) \leq \kappa(x_n, x_{n+1})$ for every $n \in \mathbb{N}$. Then, $\{x_n\}$ is a Cauchy sequence.*

Definition 7. Let $(\mathfrak{X}, b)$, $s \geq 1$, be a b-metric space and a function $\zeta_b : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ satisfying the following:

(ζ_{b1}) $\zeta_b(r, t) < t - r$ for all $r, t \in \mathbb{R}^+$

(ζ_{b2}) If $\{r_n\}, \{t_n\}$ are two sequences in $[0, +\infty)$, such that for $p > 0$

$$\limsup_{n\longrightarrow\infty} t_n = s^p \lim_{n\longrightarrow\infty} r_n > 0, \tag{6}$$

then

$$\limsup_{n\longrightarrow\infty} \zeta_b(s^p r_n, t_n) < 0. \tag{7}$$

Thus, ζ_b is said to be a b-ψ-simulation function. We shall denote by $\mathcal{C}_b$ the family of all b-simulation functions.

(See, e.g., [16, 20, 21], for more details and examples.)

In [22], the authors considered several fixed point theorems, in the setting of b-metric spaces, for a family of contractions (called multiparametric contractions) depending on two functions (that are not defined in $t = 0$) and some parameters.

Definition 8 (see [22]). Let $(\mathfrak{X}, b)$ be a b-metric space and $T : \mathfrak{X} \longrightarrow \mathfrak{X}$ be a mapping. Let $\varkappa = \{\kappa_1, \kappa_2, \kappa_3, \kappa_4, \kappa_5\}$ be a set of five nonnegative real numbers, and we denote by

$$A_T : \mathfrak{X} \times \mathfrak{X} \longrightarrow [0,\infty) \tag{8}$$

the function defined, for all $x, y \in \mathfrak{X}$, by

$$A_T(x,y) = \kappa_1 b(x,y) + \kappa_2 b(x,Tx) + \kappa_3 b(y,Ty) + \kappa_4 b(x,Ty) + \kappa_5 b(y,Tx). \quad (9)$$

We say that T is a $(\psi, \phi, \varkappa, q)$-multiparametric contraction on $(\mathfrak{X}, b, s)$ if

$$\psi(s^q b(Tx,Ty)) \le \phi(A_T(x,y)) \quad \text{for all } x, y \in \mathfrak{X} \text{ such that } b(Tx,Ty) > 0, \quad (10)$$

where $\psi, \phi : (0,\infty) \longrightarrow \mathbb{R}$ are two auxiliary functions and $q \in [1,\infty)$.

Inspired by some results in [15], we will consider a pair of two functions $\psi, \phi : (0,\infty) \longrightarrow \mathbb{R}$ that satisfy the following:

(p_1) $\phi(u) < \psi(u)$ for any $u > 0$

(p_2) ψ is nondecreasing

Let $\mathcal{P}$ be the set of such pair of functions; that is,

$$\mathcal{P} = \{(\psi,\phi) | \psi, \phi \colon (0,\infty) \longrightarrow \mathbb{R}, \quad (p_1),(p_2) \text{ hold}\}. \quad (11)$$

2. Main Results

Definition 9. Let $(\mathfrak{X}, b, s)$ be a b-metric space. A mapping $T : \mathfrak{X} \longrightarrow \mathfrak{X}$ is a Proinov-$\mathcal{C}_b$-contraction mapping of type R_i if there exist $(\psi, \phi) \in \mathcal{P}$, $\zeta_b \in \mathcal{C}_b$, a number $\beta \ge 1$, and nonnegative real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, with $\alpha_1 + \alpha_2 + \alpha_3 > 0$, such that for all $x, y \in \mathfrak{X}$ with $b(Tx, Ty) > 0$, we have

$$\frac{1}{2s} \min\{b(x,Tx), b(y,Ty)\} \le b(x,y) \text{ implies } \zeta_b\left(\psi\left(s^\beta b(Tx,Ty)\right), \phi(R_i(x,y))\right) \ge 0, \quad (12)$$

where

$$\begin{aligned} R_1(x,y) &= \alpha_1 b(x,y) + \alpha_2 b(x,Tx) + \alpha_3 b(y,Ty) + \alpha_4 \frac{b(x,Tx)b(y,Ty)}{b(x,y)}, \quad \text{for any } x \ne y \\ R_2(x,y) &= \alpha_1 b(x,y) + \alpha_2 b(x,Tx) + \alpha_3 b(y,Ty) + \alpha_4 \frac{b(y,Ty))(1+b(x,Tx))}{1+b(x,y)}, \\ R_3(x,y) &= \alpha_1 b(x,y) + \alpha_2 b(x,Tx) + \alpha_3 b(y,Ty) + \alpha_4 \frac{b(x,Tx)b(x,Ty) + b(y,Ty)b(y,Tx)}{1+\max\{b(x,Ty), b(y,Tx)\}} \\ &\quad ++\alpha_5 \frac{b(x,Tx)b(x,Ty) + b(y,Ty)b(y,Tx)}{1+s \ \max\{b(x,Tx), b(y,Ty)\}}. \end{aligned} \quad (13)$$

Remark 10. We mention that following Corollary 11 in [22], we have that, for $\alpha_1 + \alpha_2 + \alpha_3 > 0$, either T admits at least one fixed point or $R_i(x,y) > 0$, $i = \overline{1,3}$, for all distinct $x, y \in \mathfrak{X}$.

Theorem 11. *On a complete b-metric space $(\mathfrak{X}, b, s)$, any continuous Proinov-$\mathcal{C}_b$-contraction mapping of type R_1T has a unique fixed point provided that $\sum_{k=1}^{4} \alpha_k < s^\beta$.*

Proof. Starting with a point $x_0 \in \mathfrak{X}$, we can consider the sequence $\{x_n\}$ in $\mathfrak{X}$, build as follows:

$$x_1 = Tx_0, \cdots x_{n+1} = Tx_n \quad \text{for all } n \in \mathbb{N}_0. \quad (14)$$

We observe that if there is some $m_0 \in \mathbb{N}$ such that $x_{m_0} = x_{m_0+1}$, it follows that $x_{m_0} = Tx_{m_0}$, so x_{m_0} is a fixed point of the mapping T. With this in mind, we will presume that $x_n \ne x_{n+1}$ for all n. Thus, since

$$\frac{1}{2s} \min\{b(x_n, Tx_n), b(x_{n+1}, Tx_{n+1})\} = \frac{1}{2s} \min\{b(x_n, x_{n+1}), b(x_{n+1}, x_{n+2})\} \le b(x_n, x_{n+1}), \quad (15)$$

by (12),

$$\zeta_b\left(\psi\left(s^\beta b(x_n, x_{n+1}), \phi(R_1(x_n, x_{n+1}))\right)\right) \ge 0, \quad (16)$$

which is equivalent, taking (ζ_{b1}) into account, with

$$\phi(R_1(x_n, x_{n+1})) - \psi\left(s^\beta b(Tx_n, Tx_{n+1})\right) > 0. \quad (17)$$

Moreover, since

$$\begin{aligned} R_1(x_n, x_{n+1}) &= \alpha_1 b(x_n, x_{n+1}) + \alpha_2 b(x_n, Tx_n) + \alpha_3 b(x_{n+1}, Tx_{n+1}) + \alpha_4 \frac{b(x_n, Tx_n) b(x_{n+1}, Tx_{n+1})}{b(x_n, x_{n+1})} \\ &= (\alpha_1 + \alpha_2) b(x_n, x_{n+1}) + (\alpha_3 + \alpha_4) b(x_{n+1}, x_{n+2}), \end{aligned} \quad (18)$$

the above inequality becomes

$$\psi\left(s^\beta \mathrm{b}(x_{n+1}, x_{n+2})\right) < \phi((\alpha_1 + \alpha_2)\mathrm{b}(x_n, x_{n+1}) + (\alpha_3 + \alpha_4) d(x_{n+1}, x_{n+2})). \quad (19)$$

Since the pair $(\psi, \phi) \in \mathcal{P}$, it follows

$$\begin{aligned} \psi\Big(s^\beta b(x_{n+1}, x_{n+2}) &< \phi((\alpha_1 + \alpha_2) b(x_n, x_{n+1}) + (\alpha_3 + \alpha_4) b(x_{n+1}, x_{n+2})) \\ &< \psi((\alpha_1 + \alpha_2) d(x_n, x_{n+1}) + (\alpha_3 + \alpha_4) d(x_{n+1}, x_{n+2})). \end{aligned} \quad (20)$$

Consequently,

$$s^{\beta}b(x_{n+1}, x_{n+2}) < (\alpha_1 + \alpha_2)b(x_n, x_{n+1}) + (\alpha_3 + \alpha_4)b(x_{n+1}, x_{n+2}),$$
$$b(x_{n+1}, x_{n+2}) < \frac{\alpha_1 + \alpha_2}{s^{\beta} - \alpha_3 - \alpha_4} b(x_n, x_{n+1}). \tag{21}$$

Let $\kappa = (\alpha_1 + \alpha_2)/(s^{\beta} - \alpha_3 - \alpha_4) < 1$. Consequently,

$$b(x_{n+1}, x_{n+2}) < \kappa b(x_n, x_{n+1}) < \kappa^{n+1} b(x_0, x_1) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{22}$$

Moreover, by Lemma 6, it follows that the sequence $\{x_n\}$ is Cauchy, and taking into account the completeness of the b-metric space $\mathfrak{X}$, we find that there exists $\omega \in \mathfrak{X}$ such that

$$\lim_{n \longrightarrow \infty} x_n = \omega. \tag{23}$$

But, the mapping T was supposed to be continuous, so that

$$T\omega = T\left(\lim_{n \longrightarrow \infty} x_n\right) = \lim_{n \longrightarrow \infty} T(x_n) = \lim_{n \longrightarrow \infty} x_{n+1} = \omega. \tag{24}$$

Thereupon, $T\omega = \omega$; that is, ω is a fixed point of the mapping T.

Supposing that there exists another point $v \in \mathfrak{X}$, such that $Tv = v \neq \omega = T\omega$, we have

$$\begin{aligned}&\frac{1}{2s} \min\{b(\omega, T\omega), b(v, T(v)\}\\ &\quad = 0 < b(\omega, v) \Longrightarrow \zeta_b\left(\psi\left(s^{\beta} b(T\omega, Tv)\right), \phi(R_1(\omega, v))\right) \geq 0.\end{aligned} \tag{25}$$

Thus,

$$\begin{aligned}&\phi(R_1(\omega, v)) - \psi\left(s^{\beta} b(T\omega, Tv)\right) > 0\\ &\quad \Longleftrightarrow \psi\left(s^{\beta} b(T\omega, Tv)\right) < \phi(R_1(\omega, v)),\end{aligned} \tag{26}$$

where

$$\begin{aligned}R_1(\omega, v) &= \alpha_1 b(\omega, v) + \alpha_2 b(\omega, T\omega) + \alpha_3 b(v, Tv)\\ &\quad + \alpha_4 \frac{b(\omega, T\omega) b(v, Tv)}{b(\omega, v)}\\ &= \alpha_1 b(\omega, v).\end{aligned} \tag{27}$$

We have in this case

$$\psi\left(s^{\beta} b(\omega, v)\right) = \psi\left(s^{\beta} b(T\omega, Tv)\right) < \phi(\alpha_1 b(\omega, v)) < \psi(\alpha_1 b(\omega, v)), \tag{28}$$

or, since ψ is nondecreasing,

$$0 < s^{\beta} b(\omega, v) < \alpha_1 b(\omega, v), \tag{29}$$

which is a contradiction. Therefore, the mapping T admits a unique fixed point. □

Example 2. Let $\mathfrak{X} = [-1, 1]$, the function $b : \mathfrak{X} \longrightarrow \mathfrak{X} \longrightarrow [0, \infty)$, and $b(x, y) = |x - y|^2$ be a b -metric with $s = 2$, and let $T : \mathfrak{X} \longrightarrow \mathfrak{X}$ be a continuous mapping, where

$$Tx = \begin{cases} -1, & \text{for } x \in [-1, 0),\\ \dfrac{x}{4} - 1, & \text{for } x \in [0, 1]. \end{cases} \tag{30}$$

Let the pair $(\psi, \phi) \in \mathscr{P}$, with $\psi(u) = u$, $\phi(u) = u/2$, for any $u > 0$, and $\zeta_b \in \mathscr{C}_b$, $\zeta_b(r, t) = (10/11)t - r$, for $r, t \geq 0$. Thus, choosing $\beta = 1$, $\alpha_1 = 1$, $\alpha_2 = \alpha_4 = 1/16$, and $\alpha_3 = 3/4$, we have

$$\begin{aligned}&\zeta_b\left(\psi\left(s^{\beta} b(Tx, Ty)\right), \phi(R_1(x, y))\right)\\ &\quad = \frac{10}{11}\phi(R_1(x, y)) - \psi(2b(Tx, Ty))\\ &\quad = \frac{5}{11}\left(b(x, y) + \frac{1}{16} b(x, Tx) + \frac{3}{4} b(y, Ty)\right.\\ &\qquad \left. + \frac{1}{16} \cdot \frac{b(x, Tx) b(y, Ty)}{b(x, y)}\right) - 2b(Tx, Ty).\end{aligned} \tag{31}$$

For $x, y \in [0, 1]$ such that $1/4 \min\{b(x, Tx), b(y, Ty)\} = 1/4 \min\{(3x/4 + 1)^2, (3y/4 + 1)^2\} \leq |x - y|^2 = b(x, y)$, we have $b(Tx, Ty) = |(x/4) - 1 - (y/4) + 1|^2 = (|x - y|^2)/16$ and

$$\begin{aligned}&\zeta_b\left(\psi\left(s^{\beta} b(Tx, Ty)\right), \phi(R_1(x, y))\right)\\ &\quad = \frac{5}{11}\left(|x - y|^2 + \frac{1}{16}\left(\frac{3x}{4} + 1\right)^2 + \frac{3}{4}\left(\frac{3y}{4} + 1\right)^2 + \frac{1}{16}\right.\\ &\qquad \left. \cdot \frac{((3x/4) + 1)^2 \cdot (3/4)((3y/4) + 1)^2}{b(x, y)}\right) - 2\frac{|x - y|^2}{16}\\ &\quad = \frac{5}{11}\left(\frac{29}{40}|x - y|^2 + \frac{1}{16}\left(\frac{3x}{4} + 1\right)^2 + \frac{3}{4}\left(\frac{3y}{4} + 1\right)^2 + \frac{1}{16}\right.\\ &\qquad \left. \cdot \frac{((3x/4) + 1)^2 \cdot (3/4)((3y/4) + 1)^2}{b(x, y)}\right) \geq 0.\end{aligned} \tag{32}$$

For $x \in [-1, 0)$, $y \in [0, 1]$ such that $1/4 \min\{b(x, Tx), b(y, Ty)\} = 1/4 \min\{(x + 1)^2, ((3y/4) + 1)^2\} \leq |x - y|^2 = b(x, y)$, we have $b(Tx, Ty) = |-1 - (y/4) + 1|^2 = y^2/16$ and

$$\begin{aligned}
&\zeta_b\left(\psi\left(s^\beta b(Tx,Ty)\right),\phi(R_1(x,y))\right)\\
&=\frac{5}{11}\left(|x-y|^2+\frac{1}{16}(x+1)^2+\frac{3}{4}\left(\frac{3y}{4}+1\right)^2+\frac{1}{16}\right.\\
&\quad\left.\cdot\frac{(x+1)^2((3y/4)+1)^2}{b(x,y)}\right)-2\frac{y^2}{16}\\
&=\frac{5}{11}\left(|x-y|^2+\frac{1}{16}(x+1)^2+\frac{3}{4}\left(\frac{9y^2}{16}+\frac{3y}{2}+1\right)+\frac{1}{16}\right.\\
&\quad\left.\cdot\frac{(x+1)^2((3y/4)+1)^2}{b(x,y)}\right)-\frac{y^2}{8}\\
&=\frac{5}{11}\left(|x-y|^2+\frac{1}{16}(x+1)^2+\frac{3y}{2}+1\right)+\frac{1}{16}\\
&\quad\cdot\frac{(x+1)^2((3y/4)+1)^2}{b(x,y)}\Bigg)+\left(\frac{5}{11}\cdot\frac{3}{4}\cdot\frac{9}{16}-\frac{1}{8}\right)y^2\geq 0.
\end{aligned}\tag{33}$$

Therefore, T is a continuous Proinov-$\mathscr{C}_b$-contraction mapping of type R_1, and from Theorem 11, it follows that T has a unique fixed point.

Corollary 12. *Let $(\mathfrak{X},b,s)$ be a complete b-metric space and $T:\mathfrak{X}\longrightarrow\mathfrak{X}$ be a continuous mapping such that there exist $(\psi,\phi)\in\mathscr{P}$, $\zeta\in\mathscr{C}_b$, a number $\beta\geq 1$, and nonnegative real numbers $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ such that for all $x,y\in\mathfrak{X}$ with $b(Tx,Ty)>0$, we have*

$$\zeta_b\left(\psi\left(s^\beta b(Tx,Ty)\right),\phi(R_1(x,y))\right)\geq 0,\tag{34}$$

where

$$\begin{aligned}
R_1(x,y)&=\alpha_1 b(x,y)+\alpha_2 b(x,Tx)+\alpha_3 b(y,Ty)\\
&\quad+\alpha_4\frac{b(x,Tx)b(y,Ty)}{b(x,y)},\text{for any } x\neq y.
\end{aligned}\tag{35}$$

Then, T has a unique fixed point provided that $\sum_{k=1}^4\alpha_k<s^\beta$.

Theorem 13. *On a complete b-metric space $(\mathfrak{X},b,s)$ any T Proinov-$\mathscr{C}_b$-contraction mapping of type R_2 has a unique fixed point provided that $\sum_{k=1}^4\alpha_k<s^\beta$.*

Proof. Let $\{x_n\}$ be the sequence in $\mathfrak{X}$ defined by (14), with $x_n\neq x_{n+1}$, for all $n\in\mathbb{N}$. Thus, by (12),

$$\begin{aligned}
&\frac{1}{2s}\min\{b(x_n,Tx_n),b(x_{n+1},Tx_{n+1}\}\\
&\quad=\frac{1}{2s}\min\{b(x_n,x_{n+1}),b(x_{n+1},x_{n+2}\}\\
&\quad\Longrightarrow\zeta_b\left(\psi\left(s^\beta b(x_n,x_{n+1})\right),\phi(R_2(x_n,x_{n+1}))\right)\geq 0.
\end{aligned}\tag{36}$$

Thus, using (ζ_{b1}), it follows

$$\phi(R_2(x_n,x_{n+1}))-\psi\left(s^\beta b(x_n,x_{n+1})\right)>0,\tag{37}$$

where

$$\begin{aligned}
R_2(x_n,x_{n+1})&=\alpha_1 b(x_n,x_{n+1})+\alpha_2 b(x_n,Tx_n)+\alpha_3 b(x_{n+1},Tx_{n+1})\\
&\quad+\alpha_4\frac{b(x_{n+1},Tx_{n+1})(1+b(x_n,Tx_n))}{1+b(x_n,x_{n+1})}\\
&=\alpha_1 b(x_n,x_{n+1})+\alpha_2 b(x_n,x_{n+1})+\alpha_3 b(x_{n+1},x_{n+2})\\
&\quad+\alpha_4\frac{b(x_{n+1},x_{n+2})(1+b(x_n,x_{n+1}))}{1+b(x_n,x_{n+1})}\\
&=(\alpha_1+\alpha_2)b(x_n,x_{n+1})+(\alpha_3+\alpha_4)b(x_{n+1},x_{n+2}).
\end{aligned}\tag{38}$$

Since $R_2(x_n,x_{n+1})=R_1(x_n,x_{n+1})$, proceeding in the previous proof, it follows that $\{x_n\}$ is a convergent sequence in $\mathfrak{X}$. Thus, there exists $\omega\in\mathfrak{X}$, such that $\lim_{n\longrightarrow\infty}x_n=\omega$.

We shall show that $T\omega=\omega$. First of all, we claim that

$$\frac{1}{2s}b(x_n,x_{n+1})\leq b(x_n,\omega)\tag{39}$$

or

$$\frac{1}{2s}b(x_{n+1},x_{n+2})\leq b(x_{n+1},\omega).\tag{40}$$

By contradiction, if we suppose that there exists $p_0\in\mathbb{N}$ such that neither (39) nor (40) hold, we have

$$\begin{aligned}
b\left(x_{p_0},x_{p_0+1}\right)&\leq s\cdot\left[b\left(x_{p_0},\omega\right)+b\left(\omega,x_{p_0+1}\right)\right]\\
&<s\cdot\left[\frac{1}{2s}b\left(x_{p_0},x_{p_0+1}\right)+\frac{1}{2s}b\left(x_{p_0+1},x_{p_0+2}\right)\right]\\
&=\frac{b\left(x_{p_0},x_{p_0+1}\right)+b\left(x_{p_0+1},x_{p_0+2}\right)}{2}\\
&<b\left(x_{p_0},x_{p_0+1}\right),
\end{aligned}\tag{41}$$

which is a contradiction. Consequently, at least one of (39) or (40) holds, so that we can find a subsequence $\{x_{n(i)}\}$ of $\{x_n\}$, such that

$$\begin{aligned}
&\frac{1}{2s}\min\left\{b\left(x_{n(i)},Tx_{n(i)}\right),b(\omega,T\omega)\right\}\\
&\quad=\frac{1}{2s}b\left(x_{n(i)},x_{n(i)+1}\right)\leq b\left(x_{n(i)},\omega\right).
\end{aligned}\tag{42}$$

Therefore, keeping (12) in mind,

$$\zeta_b\left(\psi\left(s^\beta b(Tx_n(i),T\omega)\right),\phi\left(R_2\left(x_{n(i)},\omega\right)\right)\right)\geq 0,\tag{43}$$

which is equivalent with

$$\psi\left(s^{\beta}b\left(Tx_{n(i)},T\omega\right)\right)<\phi\left(R_2\left(x_{n(i)},\omega\right)\right). \tag{44}$$

Moreover, since $(\psi,\phi)\in\mathcal{P}$,

$$\psi\left(s^{\beta}b\left(Tx_{n(i)},T\omega\right)\right)<\phi\left(R_2\left(x_{n(i)},\omega\right)\right)<\psi\left(R_2\left(x_{n(i)},\omega\right)\right), \tag{45}$$

and then,

$$s^{\beta}b\left(Tx_{n(i)},T\omega\right)<R_2\left(x_{n(i)},\omega\right). \tag{46}$$

But,

$$\begin{aligned}R_2\left(x_{n(i)},\omega\right)&=\alpha_1 b\left(x_{n(i)},\omega\right)+\alpha_2 b\left(x_{n(i)},Tx_{n(i)}\right)+\alpha_3 b(\omega,T\omega)\\&\quad+\alpha_4\frac{b(\omega,T\omega)\left(1+b\left(x_{n(i)},Tx_{n(i)}\right)\right)}{1+b\left(x_{n(i)},\omega\right)}\\&=\alpha_1 b\left(x_{n(i)},\omega\right)+\alpha_2 b\left(x_{n(i)},x_{n(i)+1}\right)+\alpha_3 b(\omega,T\omega)\\&\quad+\alpha_4\frac{b(\omega,T\omega)\left(1+b\left(x_{n(i)},x_{n(i)+1}\right)\right)}{1+b\left(x_{n(i)},\omega\right)}.\end{aligned} \tag{47}$$

Consequently, there exists $\lim_{n\longrightarrow\infty}R_2(x_{n(i)},\omega)$, and we have

$$\lim_{i\longrightarrow\infty}R_2\left(x_{n(i)},\omega\right)=(\alpha_3+\alpha_4)b(\omega,T\omega). \tag{48}$$

On the other hand,

$$\begin{aligned}0<b(\omega,T\omega)&\le s[b(\omega,Tx_n)+b(Tx_n,T\omega)]\\&\le sb(\omega,x_{n+1})+s^{\beta}b(Tx_n,T\omega)\\&<sb(\omega,x_{n+1})+R_2(x_n,\omega).\end{aligned} \tag{49}$$

Therefore,

$$\begin{aligned}0<b(\omega,T\omega)&<\limsup_{n\longrightarrow\infty}R_2(x_n,\omega)\\&=(\alpha_3+\alpha_4)b(\omega,T\omega)\\&\le b(\omega,T\omega),\end{aligned} \tag{50}$$

which is a contradiction. Thus, $T\omega=\omega$. Supposing that this point is not unique, we can find another point $\upsilon\in\mathfrak{X}$, such that $T\omega=\omega\neq\upsilon=T\upsilon$. In this case,

$$\begin{aligned}0=\frac{1}{2s}\min\{b(\omega,T\omega),b(\upsilon,T\upsilon)\}&<b(\omega,\upsilon)\\\Longrightarrow\zeta_b\left(\psi\left(s^{\beta}b(T\omega,T\upsilon)\right),\phi(R_2(\omega,\upsilon))\right)&\ge 0.\end{aligned} \tag{51}$$

We have,

$$\begin{aligned}\psi\left(s^{\beta}b(\omega,\upsilon)\right)&=\psi\left(s^{\beta}b(T\omega,T\upsilon)\right)\le\phi(R_2(\omega,\upsilon))\\&=\phi(\alpha_1 b(\omega,\upsilon))<\psi(\alpha_1 b(\omega,\upsilon)),\end{aligned} \tag{52}$$

and, taking (p_1) into account,

$$0<s^{\beta}b(\omega,\upsilon)<\alpha_1 b(\omega,\upsilon), \tag{53}$$

which is a contradiction, because $\alpha_1<s^{\beta}$. So, the mapping T possesses a unique fixed point. □

Corollary 14. *Let $(\mathfrak{X},b,s)$ be a complete b-metric space and $T:\mathfrak{X}\longrightarrow\mathfrak{X}$ be a continuous mapping such that there exist $(\psi,\phi)\in\mathcal{P}$, $\zeta\in\mathcal{C}_b$, a number $\beta\ge 1$, and nonnegative real numbers $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ such that for all $x,y\in\mathfrak{X}$ with $b(Tx,Ty)>0$, we have*

$$\zeta_b\left(\psi\left(s^{\beta}b(Tx,Ty)\right),\phi(R_2(x,y))\right)\ge 0, \tag{54}$$

where

$$\begin{aligned}R_2(x,y)&=\alpha_1 b(x,y)+\alpha_2 b(x,Tx)+\alpha_3 b(y,Ty)\\&\quad+\alpha_4\frac{b(x,Tx)b(y,Ty)}{b(x,y)},\text{for any } x\neq y.\end{aligned} \tag{55}$$

Then, T has a unique fixed point provided that $\sum_{k=1}^{4}\alpha_k<s^{\beta}$.

Theorem 15. *On a complete b-metric space $(\mathfrak{X},b,s)$, any Proinov- $\mathcal{C}_b$-contraction mapping of type R_3T has a unique fixed point provided that $\alpha_1+\alpha_2+\alpha_3+\alpha_4+2\alpha_5<s^{\beta}$ and $\alpha_3<1$.*

Proof. Let $\{x_n\}$ be the sequence in $\mathfrak{X}$ defined by (14), with $x_n\neq x_{n+1}$, for all $n\in\mathbb{N}$. Thus, by (12),

$$\begin{aligned}&\frac{1}{2s}\min\{b(x_n,Tx_n),b(x_{n+1},Tx_{n+1}\}\\&\quad=\frac{1}{2s}\min\{b(x_n,x_{n+1}),b(x_{n+1},x_{n+2}\}\\&\quad\Longrightarrow\zeta_b\left(\psi\left(s^{\beta}b(Tx_n,Tx_{n+1})\right),\phi(R_2(x_n,x_{n+1}))\right)\ge 0.\end{aligned} \tag{56}$$

Thus, using (ζ_{b1}), it follows

$$\phi(R_3(x_n,x_{n+1}))-\psi\left(s^{\beta}b(Tx_n,Tx_{n+1})\right)>0, \tag{57}$$

or, equivalent (keeping in mind (ζ_{b1}) and (p_1))

$$\psi\left(s^{\beta}b(Tx_n,Tx_{n+1})\right)<\phi(R_3(x_n,x_{n+1}))<\psi(R_3(x_n,x_{n+1})), \tag{58}$$

where

$$\begin{aligned} R_3(x_n,x_{n+1}) &= \alpha_1 b(x_n,x_{n+1}) + \alpha_2 b(x_n,Tx_n) + \alpha_3 b(x_{n+1},Tx_{n+1}) \\ &\quad ++\alpha_4 \frac{b(x_n,Tx_n)b(x_n,Tx_{n+1}) + d(x_{n+1},Tx_{n+1})b(x_{n+1},Tx_n)}{1+\max\{b(x_n,Tx_{n+1}),b(x_{n+1},Tx_n)\}} \\ &\quad ++\alpha_5 \frac{(b(x_n,Tx_n)b(x_n,Tx_{n+1}) + b(x_{n+1},Tx_{n+1})b(x_{n+1},Tx_n)}{1+s\max\{b(x_n,Tx_n),b(x_{n+1},Tx_{n+1})\}} \\ &= \alpha_1 b(x_n,x_{n+1}) + \alpha_2 b(x_n,x_{n+1}) + \alpha_3 b(x_{n+1},x_{n+2}) \\ &\quad ++\alpha_4 \frac{b(x_n,x_{n+1})b(x_n,x_{n+2}) + d(x_{n+1},x_{n+2})b(x_{n+1},x_{n+1})}{1+\max\{b(x_n,x_{n+2}),b(x_{n+1},x_{n+1})\}} \\ &\quad ++\alpha_5 \frac{(b(x_n,x_{n+1})b(x_n,x_{n+2}) + b(x_{n+1},x_{n+2})b(x_{n+1},x_{n+1})}{1+s\max\{b(x_n,x_{n+1}),b(x_{n+1},x_{n+2})\}} \\ &= \alpha_1 b(x_n,x_{n+1}) + \alpha_2 b(x_n,x_{n+1}) + \alpha_3 b(x_{n+1},x_{n+2}) \\ &\quad ++\alpha_4 \frac{b(x_n,x_{n+1})b(x_n,x_{n+2})}{1+b(x_n,x_{n+2})} \\ &\quad +\alpha_5 \frac{b(x_n,x_{n+1})b(x_n,x_{n+2})}{1+s\max\{b(x_n,x_{n+1}),b(x_{n+1},x_{n+2})\}} \\ &\le (\alpha_1+\alpha_2+\alpha_3)b(x_n,x_{n+1}) \\ &\quad +\alpha_5 \frac{s\cdot b(x_n,x_{n+1})(b(x_n,x_{n+1})+b(x_{n+1},x_{n+2}))}{1+s\max\{b(x_n,x_{n+1}),b(x_{n+1},x_{n+2})\}}. \end{aligned} \tag{59}$$

Assuming that there exists $p_0 \in \mathbb{N}$ such that $\max\{b(x_{p_0}, x_{p_0+1}), b(x_{p_0+1}, x_{p_0+2})\} = b(x_{p_0+1}, x_{p_0+2})$, we have

$$\begin{aligned} 0 &< R_3\left(x_{p_0},x_{p_0+1}\right) \\ &\le (\alpha_1+\alpha_2+\alpha_3)b\left(x_{p_0+1},x_{p_0+2}\right) + \alpha_5 \frac{2s\cdot\left(b\left(x_{p_0+1},x_{p_0+2}\right)\right)^2}{1+sb\left(x_{p_0+1},x_{p_0+2}\right)} \\ &\le (\alpha_1+\alpha_2+\alpha_3)b\left(x_{p_0+1},x_{p_0+2}\right) + 2\alpha_5 b\left(x_{p_0+1},x_{p_0+2}\right) = \phi. \end{aligned} \tag{60}$$

Therefore, by (58) and (59), together with (p_1), we get

$$\begin{aligned} \psi\left(s^\beta b\left(x_{p_0+1},x_{p_0+2}\right)\right) &= \psi\left(s^\beta b\left(Tx_{p_0},Tx_{p_0+1}\right)\right) \\ &< \phi\left(R_3\left(x_{p_0},x_{p_0+1}\right)\right) \\ &< \psi\Big((\alpha_1+\alpha_2+\alpha_3)b\left(x_{p_0+1},x_{p_0+2}\right) \\ &\quad + 2\alpha_5 b\left(x_{p_0+1},x_{p_0+2}\right)\Big), \end{aligned} \tag{61}$$

and taking (p_2) into account, it follows

$$s^\beta b\left(x_{p_0+1},x_{p_0+2}\right) < (\alpha_1+\alpha_2+\alpha_3+2\alpha_5)b\left(x_{p_0+1},x_{p_0+2}\right), \tag{62}$$

which is a contradiction.

Consequently, $b(x_n, x_{n+1}) > b(x_{n+1}, x_{n+2})$, for any $n \in \mathbb{N}$, and $\{b(x_n, x_{n+1})\}$ is a nonincreasing sequence; so, we can find $\rho \ge 0$ such that $\lim_{n\longrightarrow\infty} b(x_n, x_{n+1}) = \rho$. Moreover,

$$\begin{aligned} 0 &< R_3(x_n,x_{n+1}) \le \alpha_1 b(x_n,x_{n+1}) + \alpha_2 b(x_n,x_{n+1}) + \alpha_3 b(x_{n+1},x_{n+2}) \\ &\quad ++\alpha_4 b(x_n,x_{n+1}) + \alpha_5 \frac{b(x_n,x_{n+1})b(x_n,x_{n+2})}{1+sb(x_n,x_{n+1}),} \\ &\le (\alpha_1+\alpha_2+\alpha_4)b(x_n,x_{n+1}) + \alpha_3 b(x_{n+1},x_{n+2}) \\ &\quad ++\alpha_5 \frac{s\cdot b(x_n,x_{n+1})[b(x_n,x_{n+1})+b(x_{n+1},x_{n+2})]}{1+sb(x_n,x_{n+1})} \\ &\le (\alpha_1+\alpha_2+\alpha_4+\alpha_5)b(x_n,x_{n+1}) + (\alpha_3+\alpha_5)b(x_{n+1},x_{n+2}), \end{aligned} \tag{63}$$

and then, from (58) and (p_2),

$$\begin{aligned} s^\beta b(x_{n+1},x_{n+2}) &< R_3(x_n,x_{n+1}) \\ &< (\alpha_1+\alpha_2+\alpha_4+\alpha_5)b(x_n,x_{n+1}) \\ &\quad + (\alpha_3+\alpha_5)b(x_{n+1},x_{n+2}), \end{aligned} \tag{64}$$

which leads us to

$$b(x_{n+1},x_{n+2}) < \frac{\alpha_1+\alpha_2+\alpha_4+\alpha_5}{s^\beta-\alpha_3-\alpha_5} b(x_n,x_{n+1}). \tag{65}$$

Letting $\kappa_1 = (\alpha_1+\alpha_2+\alpha_4+\alpha_5)/(s^\beta-\alpha_3-\alpha_5) < 1$, we get $b(x_{n+1}, x_{n+2}) < \kappa_1 b(x_n, x_{n+1})$, for any $n \in \mathbb{N}$. Thus, Lemma 6 ensure that the sequence $\{x_n\}$ is Cauchy, that is,

$$\lim_{n,m\longrightarrow\infty} b(x_n,x_m) = 0. \tag{66}$$

Moreover, the b-metric space $(\mathfrak{X}, b, s)$ is supposed to be complete, so, we can find $\omega \in \mathfrak{X}$ such that

$$\lim_{m\longrightarrow\infty} x_m = \omega. \tag{67}$$

Further, from the proof of Theorem 13, we know that at least one of (39) or (40) holds, and for this reason, there exists a subsequence $\{x_k\}$ of $\{x_n\}$ such that

$$\frac{1}{2s}\min\{x_k,Tx_k),b(\omega,T\omega)\} \le \frac{1}{2s}b(x_k,x_{k+1}) \le b(x_k,\omega), \tag{68}$$

which implies

$$\zeta_b\left(\psi\left(s^\beta b(Tx_k,T\omega)\right),\phi(R_3(x_k,\omega))\right) \ge 0. \tag{69}$$

Therefore,

$$\psi\left(s^\beta b(Tx_k,T\omega)\right) < \phi(R_3(x_k,\omega)) < \psi(R_3(x_k,\omega)), \tag{70}$$

and, by (p_2),

$$s^\beta b(Tx_k,T\omega) < R_3(x_k,\omega). \tag{71}$$

Now, since

$$R_3(x_k,\omega) = \alpha_1 b(x_k,\omega) + \alpha_2 b(x_k,x_{k+1}) + \alpha_3 b(\omega,T\omega) + \alpha_4 \frac{b(x_k,x_{k+1})b(x_k,\omega)+b(\omega,T\omega)b(\omega,x_{k+1})}{1+\max\{b(x_k,T\omega),b(\omega,x_{k+1})\}} ++\alpha_4 \frac{b(x_k,x_{k+1})b(x_k,\omega)+b(\omega,T\omega)b(\omega,x_{k+1})}{1+s\max\{b(x_k,x_{k+1}),b(\omega,T\omega)\}}, \tag{72}$$

taking into account (66) and (67),

$$\limsup_{k\longrightarrow\infty} R_3(x_k,\omega) \le \alpha_3 b(\omega,T\omega) < b(\omega,T\omega). \tag{73}$$

But,

$$\begin{aligned} b(\omega,T\omega) &\le s[b(\omega,Tx_k)+b(Tx_k,T\omega)] \\ &\le sb(\omega,Tx_k)+s^\beta b(Tx_k,T\omega) \\ &< sb(\omega,Tx_k)+R_3(x_k,\omega), \end{aligned} \tag{74}$$

which combined with (73) showing that

$$b(\omega,T\omega) \le \limsup_{k\longrightarrow\infty} R_3(x_k,\omega) \le \alpha_3 b(\omega,T\omega). \tag{75}$$

But, this is a contradiction, so, $T\omega=\omega$.

We claim that ω is the only fixed point of T. Suppose that, on the contrary, there exists $\upsilon \in \mathfrak{X}$, such that $T\upsilon=\upsilon$ and $b(\upsilon,\omega)>0$. Thus,

$$0=\frac{1}{2s}\min\{b(\upsilon,T\upsilon),b(\omega,T\omega)\} < b(\upsilon,\omega) \Longrightarrow \zeta_b\left(\psi\left(s^\beta b(T\upsilon,T\omega)\right),\phi(R_3(\upsilon,\omega))\right) \ge 0, \tag{76}$$

and moreover,

$$\begin{aligned} \psi\left(s^\beta b(\upsilon,\omega)\right) &= \psi\left(s^\beta b(T\upsilon,T\omega)\right) \\ &< \phi(R_3(\upsilon,\omega)) \\ &= \phi(\alpha_1 b(\upsilon,\omega)) \\ &< \psi(\alpha_1 b(\upsilon,\omega)), \end{aligned} \tag{77}$$

which is a contradiction. □

Example 3. Let $\mathfrak{X}=\{q_1,q_2,q_3,q_4\}$ and a function $b:\mathfrak{X}\times\mathfrak{X}\longrightarrow[0,\infty)$, defined as follows:

$$\begin{array}{ccccc} b(x,y) & q_1 & q_2 & q_3 & q_4 \\ q_1 & 0 & \frac{1}{4} & \frac{5}{4} & 3 \\ q_2 & \frac{1}{4} & 0 & 2 & 3 \\ q_3 & \frac{5}{4} & 2 & 0 & 2 \\ q_4 & 3 & 3 & 2 & 0 \end{array}. \tag{78}$$

It is easy to check that b is a b-metric, with $s=2$. Let the mapping $T:\mathfrak{X}\longrightarrow\mathfrak{X}$, where

$$\begin{array}{ccccc} x & q_1 & q_2 & q_3 & q_4 \\ Tx & q_1 & q_1 & q_1 & q_2 \end{array}. \tag{79}$$

Let the pair $(\psi,\phi)\in\mathscr{P}$, where $\psi(u)=e^u$, $\phi(u)=1+\ln(1+u)$, for any $u>0$, and $\zeta_b\in\mathscr{C}_b$, $\zeta_b(r,t)=(11t/12)-r$. Choosing $\beta=1$ and $\alpha_1=\alpha_2=\alpha_4=\alpha_5=1/6$ and $\alpha_3=8/9$, we have

$$\begin{aligned} &\zeta_b\left(\psi\left(s^\beta b(Tx,Ty)\right),\phi(R_3(x,y))\right) \\ &= \frac{11}{12}\phi(R_3(x,y)) - \psi(2b(Tx,Ty)) \\ &= \frac{11}{12}(1+\ln(1+R_3(x,y))) - e^{2b(Tx,Ty)} \\ &= \frac{11}{12}\Bigg[1+\ln\Bigg(1+\frac{1}{6}(b(x,y)+b(x,Tx) + \frac{b(x,Tx)b(x,Ty)+b(y,Ty)b(y,Tx)}{1+\max\{b(x,Ty),b(y,Tx)\}} ++\frac{b(x,Tx)\mathrm{b}(x,Ty)+b(y,Ty)b(y,Tx)}{1+2\max\{b(x,Tx),b(y,Ty)\}}\Bigg) + \frac{8}{9}b(y,Ty))\Bigg] - e^{2b(Tx,Ty)}. \end{aligned} \tag{80}$$

We consider the following cases (which respect the condition $b(Tx,Ty)>0$):

(i) $x=q_j$, $y=q_4$, $j\in\{1,2\}$,

$$\frac{1}{4}\min\left\{b\left(q_j,Tq_j\right),b(q_4,Tq_4)\right\} < 3 = b\left(q_j,q_4\right) \Longrightarrow \zeta_b\left(\psi\left(s^\beta b\left(Tq_j,Tq_4\right)\right),\phi\left(R_3\left(q_j,q_4\right)\right)\right) \ge 0, \tag{81}$$

which means

$$\begin{aligned}e^{2b\left(Tq_j,Tq_4\right)} &= e^{2b(q_1,q_2)}\\ &= \sqrt{e} < \frac{11}{12}\left(1+\ln\frac{11}{3}\right)\\ &= \frac{11}{12}\left(1+\ln\left(1+\alpha_3 b(q_4,Tq_4)\right)\right)\\ &\le \frac{11}{12}\left(1+\ln\left(1+R_3\left(q_j,q_4\right)\right)\right).\end{aligned} \tag{82}$$

(ii) $x = q_3$, $y = q_4$,

$$\begin{aligned}&\frac{1}{4}\min\left\{b(q_3,Tq_3),b(q_4,Tq_4)\right\} < 2 = b(q_3,q_4)\\ &\quad\Longrightarrow \zeta_b\left(\psi\left(s^\beta b(Tq_3,Tq_4)\right),\phi(R_3(q_3,q_4))\right)\ge 0,\end{aligned} \tag{83}$$

which means

$$\begin{aligned}e^{2b(Tq_3,Tq_4)} &= e^{2b(q_1,q_2)}\\ &= \sqrt{e} < \frac{11}{12}\left(1+\ln\frac{11}{3}\right)\\ &= \frac{11}{12}\left(1+\ln\left(1+\alpha_3 b(q_4,Tq_4)\right)\right)\\ &\le \frac{11}{12}\left(1+\ln\left(1+R_3(q_3,q_4)\right)\right).\end{aligned} \tag{84}$$

Consequently, the mapping T is a Proinov-$\mathscr{C}_b$-contraction mapping of type R_3 and, by Theorem 15, it follows that T has a unique fixed point.

Corollary 16. *Let $(\mathfrak{X}, b, s)$ be a complete b-metric space and $T:\mathfrak{X}\longrightarrow\mathfrak{X}$ be a c mapping such that there exist $(\psi,\phi)\in\mathscr{P}$, $\zeta\in\mathscr{C}_b$, a number $\beta\ge 1$, and nonnegative real numbers $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ such that for all $x,y\in\mathfrak{X}$ with $b(Tx,Ty)>0$, we have*

$$\zeta_b\left(\psi\left(s^\beta b(Tx,Ty)\right),\phi(R_3(x,y))\right)\ge 0, \tag{85}$$

where

$$\begin{aligned}R_3(x,y) &= \alpha_1 b(x,y)+\alpha_2 b(x,Tx)+\alpha_3 b(y,Ty)\\ &\quad+\alpha_4\frac{b(x,Tx)b(x,Ty)+b(y,Ty)b(y,Tx)}{1+\max\left\{b(x,Ty),b(y,Tx)\right\}}\\ &\quad++\alpha_5\frac{(b(x,Tx))b(x,Ty)+b(y,Ty)b(y,Tx)}{1+s\max\left\{b(x,Tx),b(y,Ty)\right\}}.\end{aligned} \tag{86}$$

Then, T has a unique fixed point provided that $\alpha_1+\alpha_2+\alpha_3+\alpha_4+2\alpha_5<s^\beta$ and $\alpha_3<1$.

3. Conclusion

In this paper, we extend the renowned Proinov's result [15] in several directions: First of all, we investigate the contractions involving interesting rational forms. Secondly, the abstracted structure is chosen as a b-metric space that is one of the natural and novel generalizations of the concept of metric spaces. Thirdly, we use auxiliary simulation functions to improve Proinov's results [15].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Research Article

Solving Differential Equation via Orthogonal Branciari Metric Spaces

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In this paper, we investigate an orthogonal L^*-contraction map concept and prove the fixed-point theorem in an orthogonal complete Branciari metric space (OCBMS). We also provide illustrative examples to support our theorems. We demonstrated the existence of a uniqueness solution to the fourth-order differential equation using a more orthogonal L^* contraction operator in OCBMS as an application of the main results.

1. Introduction

The Branciari metric (BM) concept was introduced by Branciari [1] in the year 2000. The generalization is via the fact that the triangle inequality is replaced by the rectangular inequality $\mathfrak{b}(\lambda_1, \lambda_2) \leq \mathfrak{b}(\lambda_1, \lambda_3) + \mathfrak{b}(\lambda_3, \lambda_4) + \mathfrak{b}(\lambda_4, \lambda_2)$ for all pairwise distinct points $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of $\mathscr{P}$. Afterwards, many authors studied and elaborated the existence of old fixed-point theorems in the BMS (briefly Branciari metric spaces) [2–7]. The Θ-contraction concept was introduced by Jleli and Samet [8] in 2014. Later, some authors provided a variety of results based on Θ-contraction [9, 10]. Saleh et al. [11] introduced the concept of generalized L and L^*-contractions. And also proved fixed-point theorems in CBMS. Eshraghisamani et al. [12] initiated new contractive map and proved fixed-point theorem in BMS.

An orthogonality notion in metric spaces is presented by Gordji et al. in 2017 [13, 14]. Recently, many authors established a variety of fixed-point results in generalized orthogonal metric space (OMS). Nazam et al. [15] demonstrated the concept of (Ψ, Φ)-orthogonal interpolation contraction mappings. The notion of B metric-like space via a hybird pair of operators was introduced by Ali et al. [16] in 2022. In 2021, Hussain [17] presented another family of fractional symmetric α-η-contractions and builds up some new results for such contraction in the context of $\mathscr{F}$-metric space. Mukheimer et al. [18] initiated the concept of orthogonal L-contraction mapping and proved fixed-point results in OBMS.

From the above motivation, we prove some fixed-point results in the direction of OBMS. We also give some examples to argue that our results correctly generalize certain results in the literature.

In this article, we present basic definitions and examples in Section 2, prove some fixed-point theorems by orthogonal L^*-contractive mapping in an OCBMS in Section 3, and finally, obtain a unique solution of differential equation using orthogonal L^* contraction operator in Section 4.

2. Preliminaries

Throughout this article, we denote by $\mathscr{P}$, $\mathbb{N}$, and $\mathbb{R}_+$ the nonempty set, the set of positive integers, and the set of positive real numbers, respectively.

The Branciari metric space was introduced by Branciari [1] as follows.

Definition 1. Let $\mathscr{P} \neq \emptyset$ and a function $\mathfrak{b} : \mathscr{P} \times \mathscr{P} \longrightarrow \mathbb{R}_+$ s.t (briefly such that) $\forall \lambda_1, \lambda_2 \in \mathscr{P}$ and all $\lambda_3 \neq \lambda_4 \in \mathscr{P}/\{\lambda_1, \lambda_2\}$:

(BM1) $\mathfrak{b}(\lambda_1, \lambda_2) = 0$, iff $\lambda_1 = \lambda_2$;

(BM2) $\mathfrak{b}(\lambda_1, \lambda_2) = \mathfrak{b}(\lambda_2, \lambda_1)$;

(BM3) $\mathfrak{b}(\lambda_1, \lambda_2) \leq \mathfrak{b}(\lambda_1, \lambda_3) + \mathfrak{b}(\lambda_3, \lambda_4) + \mathfrak{b}(\lambda_4, \lambda_2)$.

The pair $(\mathscr{P}, \mathfrak{b})$ is called a BMS with Branciari metric $\mathfrak{b}$.

The following example is on the Branciari metric space (BMS).

Example 1. Let $\mathscr{P} = \{0, 2\} \cup \{(1/\imath) \colon \imath \in \mathbb{N}\}$, where $E = \{0, 2\}$ and $G = \{(1/\imath) \colon \imath \in \mathbb{N}\}$. Define $\mathfrak{b} : \mathscr{P} \times \mathscr{P} \longrightarrow \mathbb{R}_+$ as

$$\mathfrak{b}(\wp_1, \wp_2) = \begin{cases} 0, & \text{if } \wp_1 = \wp_2, \\ 1, & \text{if } \wp_1 \neq \wp_2 \text{ and } \{\wp_1, \wp_2\} \subset \mathrm{E} \text{ or } \{\wp_1, \wp_2\} \subset \mathrm{G}, \\ \wp_2, & \text{if } \wp_1 \in \mathrm{E} \text{ and } \wp_2 \in \mathrm{G}, \\ \wp_1, & \text{if } \wp_1 \in \mathrm{G} \text{ and } \wp_2 \in \mathrm{E}. \end{cases} \tag{1}$$

Then, $(\mathscr{P}, \mathfrak{b})$ is a CBMS (briefly complete Branciari metric space). However, we get

(1) $\lim_{\imath \longrightarrow \infty} \mathfrak{b}((1/\imath), (1/2)) \neq \mathfrak{b}(0, (1/2))$ although $\lim_{\imath \longrightarrow \infty} (1/\imath) = 0$, and hence, $\mathfrak{b}$ is discontinuous

(2) There is nonexistence $\ell > 0$ s.t $\mathrm{G}_\ell(0) \cap \mathrm{G}_\ell(2) = \phi$, and hence, the topology is not a Hausdorff

(3) $\mathrm{G}_{(2/3)} = \{0, 2, (1/3)\}$; however, there does not exist $\ell > 0$ s.t $\mathrm{G}_\ell(0) \subseteq \mathrm{G}_{(2/3)}(1/3)$, and thus, an open ball does not necessitate an open set

(4) $\{1/\imath\}_{\imath \in \mathbb{N}}$ is not a Cauchy sequence since it converges to both 0 and 2

Now, we give the following concepts, which are used in this paper.

Definition 2. Let $(\mathscr{P}, \mathfrak{b})$ be a BMS and $\{\alpha_\imath\}$ be a sequence in $\mathscr{P}$ and $\lambda_1 \in \mathscr{P}$.

(1) $\{\alpha_\imath\}$ is convergent to $\lambda_1 \Longleftrightarrow \mathfrak{b}(\alpha_\imath, \alpha_\ell) \longrightarrow 0$ as $\imath \longrightarrow \infty$. We denote this by $\alpha_\imath \longrightarrow \alpha$;

(2) $\{\alpha_\imath\}$ is Cauchy $\Longleftrightarrow \mathfrak{b}(\alpha_\imath, \alpha_\ell) \longrightarrow 0$ as $\imath, \ell \longrightarrow \infty$;

(3) $(\mathscr{P}, \mathfrak{b})$ is complete $\Longleftrightarrow$ every Cauchy sequence in $\mathscr{P}$ which converges to some element in $\mathscr{P}$.

Eshraghisamani et al. [12] introduced the concept of Θ-contraction as follows.

Definition 3. Let $(\mathscr{P}, \mathfrak{b})$ be a BMS. A map $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ is said to be Θ-contraction if there exist $\Theta \in \Gamma_{1,2,3}$ and $\nu \in (0, 1)$ s.t $(\forall \lambda_1, \lambda_2 \in \mathscr{P})$

$$\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) > 0 \Longrightarrow \Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)) \leq [\Theta(\mathfrak{b}(\lambda_1, \lambda_2))]^\nu, \tag{2}$$

where $\Gamma_{1,2,3}$ is the family of all functions $\Theta : (0, \infty) \longrightarrow (0,\infty)$ which satisfy the following axioms:

$(\Theta_1)\Theta$ is increasing

(Θ_2) For each sequence $\{\alpha_\imath\} \subset (0, \infty)$, $\lim_{\imath \longrightarrow \infty} \Theta(\alpha_i) = 1 \Longleftrightarrow \lim_{\imath \longrightarrow \infty} \alpha_\imath = 0^+$

$(\Theta_3)\Theta$ is continuous.

Using Definition 3, Eshraghisamani et al. [12] proved the following theorem.

Theorem 4. *Let $(\mathscr{P}, \mathfrak{b})$ be a CBMS and $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ a Θ-contraction function. Then, Φ has a **ufp** (briefly unique fixed point).*

The below example supports Theorem 4.

Example 2. Let $\varsigma_{\Phi,\mathfrak{J}} : [1, \infty) \times [1, \infty) \longrightarrow R$ be two functions defined as below:

$$\varsigma_{\Phi,\mathfrak{J}}(\sigma, \sigma_1) = \frac{\mathfrak{J}(\sigma_1)}{\Phi(\sigma_1)}, \forall \sigma, \sigma_1 \geq 1, \tag{3}$$

where $\Phi, \mathfrak{J} : [1, \infty) \longrightarrow [1, \infty)$ are upper semicontinuous from the right s.t $\mathfrak{J}(\sigma) < \sigma \leq \Phi(\sigma)$, for all $\sigma > 1$. Then, $\varsigma_{\Phi,\mathfrak{J}} \in \mathrm{L}$.

In Theorem 4, by replacing the condition (Θ_3), we get the following remark.

Remark 5. Let $\{\mathfrak{u}_\imath\}, \{\mathfrak{x}_\imath\}, \{\mathfrak{y}_\imath\}$ be the sequence of $\mathbb{R}_+$ s.t $\lim_{\imath \longrightarrow \infty} \mathfrak{u}_\imath = \mathfrak{u}$, $\lim_{\imath \longrightarrow \infty} \mathfrak{x}_\imath = \mathfrak{x}$ and $\lim_{\imath \longrightarrow \infty} \mathfrak{y}_\imath = \mathfrak{y}$. Then,

(1) $\lim_{\imath \longrightarrow \infty} \max\{\mathfrak{u}_\imath, \mathfrak{x}_\imath, \mathfrak{y}_\imath\} = \max\{\mathfrak{u}, \mathfrak{x}, \mathfrak{y}\}$,

(2) $\lim_{\imath \longrightarrow \infty} \min\{\mathfrak{u}_\imath, \mathfrak{x}_\imath, \mathfrak{y}_\imath\} = \min\{\mathfrak{u}, \mathfrak{x}, \mathfrak{y}\}$.

In 2017, Gordji et al. [13] introduced the concept of an orthogonal set as follows.

Definition 6. Let $\mathscr{P} \neq \emptyset$ and $\perp \subseteq \mathscr{P} \times \mathscr{P}$ be a binary relation. If $\perp$ holds

$$\exists \lambda_{10} \in \mathscr{P} : (\forall \lambda_1 \in \mathscr{P}, \lambda_1 \perp \lambda_{10}) \text{ or } (\forall \lambda_1 \in \mathscr{P}, \lambda_{10} \perp \lambda_1), \tag{4}$$

then $(\mathscr{P}, \perp)$ is called an orthogonal set.

The following example and Figure 1 are satisfied by Definition 6.

Example 3. Let $\mathscr{P}=Z$ and define $\lambda_2 \perp \lambda_1$ if $\exists \nu \in Z : \lambda_2 = \nu\lambda_1$. It is clear that $0 \perp \lambda_1, \forall \lambda_1 \in Z$. Hence, $(\mathscr{P}, \perp)$ is an orthogonal set.

Example 4. A wheel graph $\mathscr{W}_\imath$ with $\imath$ edge for every $\imath \geq 4$, a node connect to each node to every edge of $(\imath-1)$ -cycle. Let $\mathscr{P}$ be the set of all edge of $\mathscr{W}_\imath$ for every $\imath \geq 4$. Define $\lambda_1 \perp \lambda_2$ if there is a connection from λ_1 to λ_2. Then, $(\mathscr{P}, \perp)$ is an orthogonal set.

The following orthogonal sequence definition was introduced by Gordji et al. [13] which will be utilized in this paper to prove main results.

Definition 7. Let $(\mathscr{P}, \perp)$ be an orthogonal set. A sequence $\{\lambda_{1_\imath}\}$ is called an orthogonal sequence (shortly, O -sequence) if

$$(\forall \imath \in \mathbb{N}, \lambda_{1_\imath} \perp \lambda_{1_{\imath+1}}) \text{ or } (\forall \imath \in \mathbb{N}, \lambda_{1_{\imath+1}} \perp \lambda_{1_\imath}). \tag{5}$$

Again, the concepts of orthogonal continuous also introduced by Gordji et al. [13].

Definition 8. Let $(\mathscr{P}, \perp, \mathfrak{b})$ be a OMS. Then, a mapping $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ is called orthogonal continuous in $\lambda_1 \in \mathscr{P}$ if for every O -sequence $\{\lambda_{1_\imath}\}$ in $\mathscr{P}$ with $\lambda_{1_\imath} \longrightarrow \lambda_1$ as $\imath \longrightarrow \infty$, we have $\Phi(\lambda_{1_\imath}) \longrightarrow \Phi(\lambda_1)$ as $\imath \longrightarrow \infty$.

Definition 9. Let $(\mathscr{P}, \perp, \mathfrak{b})$ be a OBMS.

(1) $\{\lambda_{1_\imath}\}$, an orthogonal sequence in $\mathscr{P}$, converges at a point λ_1 if

$$\lim_{\imath \longrightarrow \infty} \Phi(\lambda_{1_\imath}, \lambda_1) = 0. \tag{6}$$

(2) $\{\lambda_{1_\imath}\}, \{\lambda_{1_m}\}$ are orthogonal sequences in $\mathscr{P}$ and are said to be orthogonal Cauchy sequence if

$$\lim_{\imath, m \longrightarrow \infty} \Phi(\lambda_{1_\imath}, \lambda_{1_m}) < \infty. \tag{7}$$

Gordji et al. [13] introduced the concept of an orthogonal complete as follows.

Definition 10. Let $(\mathscr{P}, \perp, \mathfrak{b})$ be a OMS. Then, $\mathscr{P}$ is called an orthogonal complete, if every orthogonal Cauchy sequence is convergent.

Finally, the following orthogonal-preserving concepts introduced by Gordji et al. [13] is of importance in this paper.

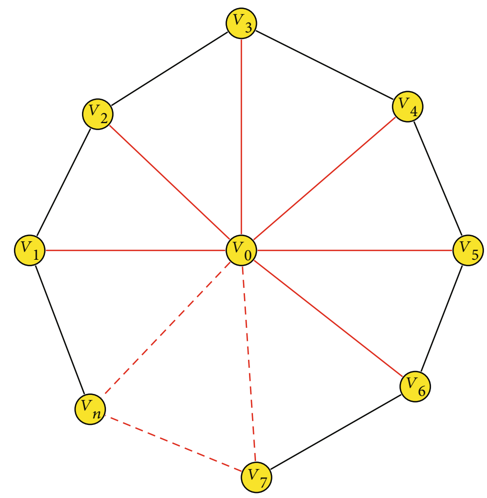

FIGURE 1: A wheel graph.

Definition 11. Let $(\mathscr{P}, \perp)$ be an orthogonal set. A function $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ is called a $\perp$ -preserving if $\Phi\lambda_1 \perp \Phi\lambda_2$ whenever $\lambda_1 \perp \lambda_2, \forall \lambda_1, \lambda_2 \in \mathscr{P}$.

Lemma 12. *Let $\{\lambda_{1_\imath}\}$ be an orthogonal Cauchy sequence in BMS $(\mathscr{P}, \mathfrak{b})$ s.t $\lim_{\imath \longrightarrow \infty} \mathfrak{b}(\lambda_{1_\imath}, \lambda_1) = 0$, for some $\lambda_1 \in \mathscr{P}$. Then, $\lim_{\imath \longrightarrow \infty} \mathfrak{b}(\lambda_{1_\imath}, \lambda_2) = \mathfrak{b}(\lambda_1, \lambda_2)$, for all $\lambda_1, \lambda_2 \in \mathscr{P}$, with $\lambda_1 \perp \lambda_2$.*

Eshraghisamani et al. [12] proved fixed-point result on Branciari metric space as follows.

Theorem 13. *Let $(\mathscr{P}, \mathfrak{b})$ be a complete generalized metric space and a map $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$. Suppose that there exist $\ell \in (0,1)$ and function $\pi : \mathscr{R}_+ \longrightarrow \mathscr{R}_+$, satisfying the following conditions:*

(i) For every $\{\beta_\imath\} \subset (0, \infty)$ and nonconstant

$$\lim_{\imath \longrightarrow \infty} \pi(\beta_\imath) = 0 \Longleftrightarrow \lim_{\imath \longrightarrow \infty} \beta_\imath = 0. \tag{8}$$

(ii) For every $\{\beta_\imath\} \subset (0,\infty)$ that $\beta_\imath \longrightarrow 0^+$, $\limsup_{\imath \longrightarrow \infty} \sqrt[\imath]{\pi(\beta_\imath)} < 1 \Longrightarrow \sum_1^\infty \beta_\imath < \infty$, such that

$$\pi(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)) \leq \ell\pi(\mathfrak{b}(\lambda_1, \lambda_2)), \tag{9}$$

*then ϕ has a **ufp**.*

3. Main Results

Before presenting our main result of this section, we are inspired by the concept of L^* contraction mapping defined by Saleh et al. [11]; we introduce a new concept of an orthogonal L^*-contraction mapping. Then, we prove a fixed-point results in OCBMS.

Definition 14. Let $(\mathcal{P},\perp,\mathfrak{b})$ be a OBMS and $\Phi:\mathcal{P}\longrightarrow\mathcal{P}$. Then, Φ is called an orthogonal L^* -contraction *w.r.t* $\zeta\in L$ if $\exists\Theta\in\Omega_{1,2,3}$ s.t.

$$\forall\lambda_1,\lambda_2\in\mathcal{P}\text{ with }\lambda_1\perp\lambda_2,\mathfrak{b}(\Phi\lambda_1,\Phi\lambda_2)>0\Longrightarrow\zeta[\Theta(\mathfrak{b}(\Phi\lambda_1,\Phi\lambda_2)),\Theta(\mathcal{M}(\lambda_1,\lambda_2))]\geq 1, \tag{10}$$

where $\mathcal{M}(\lambda_1,\lambda_2)=\max\{\mathfrak{b}(\lambda_1,\lambda_2),\mathfrak{b}(\lambda_1,\Phi\lambda_1),\mathfrak{b}(\lambda_2,\Phi\lambda_2)\}$.

Motivated by Theorem 13, we prove the below theorem.

Theorem 15. *Let $(\mathcal{P},\perp,\mathfrak{b})$ be a OCBMS and Φ is a self-map on $\mathcal{P}$. Suppose that $\exists\ell\in(0,1)$ and a function $\pi:\mathcal{R}_+\longrightarrow\mathcal{R}_+$ hold the axioms:*

(i) Φ is orthogonal-preserving

(ii) For every $\{\beta_\imath\}\subset(0,\infty)$ and nonconstant

$$\lim_{\imath\longrightarrow\infty}\pi(\beta_\imath)=0\Longleftrightarrow\lim_{\imath\longrightarrow\infty}\beta_\imath=0. \tag{11}$$

(iii) $\Phi_\perp$ with for every $\{\beta_\imath\}\subset(0,\infty)$ that $\beta_\imath\longrightarrow 0^+$, $\limsup_{\imath\longrightarrow\infty}\sqrt[\imath]{\pi(\beta_\imath)}<1\Longrightarrow\sum_1^\infty\beta_\imath<\infty$ such that

$$\forall\lambda_1,\lambda_2\in\mathcal{P}\text{ with }\lambda_1\perp\lambda_2\Longrightarrow\pi(\mathfrak{b}(\Phi\lambda_1,\Phi\lambda_2))\leq\ell\pi(\mathfrak{b}(\lambda_1,\lambda_2)), \tag{12}$$

*then Φ has a **ufp**.*

Proof. Since $(\mathcal{P},\perp)$ is orthogonal set,

$$\exists\lambda_2\in\mathcal{P}:(\forall\lambda_1\in\mathcal{P},\lambda_1\perp\lambda_2)\text{ or }(\forall\lambda_1\in\mathcal{P},\lambda_2\perp\lambda_1). \tag{13}$$

It follows that $\lambda_2\perp\Phi\lambda_2$ or $\Phi\lambda_2\perp\lambda_2$. Let

$$\lambda_{1_1}=\Phi\lambda_2,\lambda_{1_2}=\Phi\lambda_{1_1}=\Phi^2\lambda_2\cdots\cdots,\lambda_{1_{\imath+1}}=\Phi\lambda_{1_\imath}=\Phi^{\imath+1}\lambda_2,\forall\imath\in\mathbb{N}\cup\{0\}. \tag{14}$$

If $\lambda_{1_{\imath_0}}=\lambda_{1_{\imath_0+1}}$ for any $\imath\in\mathbb{N}\cup\{0\}$, then it is easy to see that λ_{10} is a fixed point of Φ. Consider that $\lambda_{1_{\imath_0}}\neq\lambda_{1_{\imath_0+1}}$ for all $\imath\in\mathbb{N}\cup\{0\}$. Since Φ is $\perp$-preserving, we have

$$\lambda_{1_{\imath_0}}\perp\lambda_{1_{\imath_0+1}}\text{or }\lambda_{1_{\imath_0+1}}\perp\lambda_{1_{\imath_0}}\forall\imath\in\mathbb{N}\cup\{0\}. \tag{15}$$

This implies that $\{\mathfrak{b}(\lambda_\imath,\lambda_{\imath+1})\}>0$ is an O-sequence.
First, we show that $\lim_{\imath\longrightarrow\infty}\mathfrak{b}(\lambda_\imath,\lambda_{\imath+1})=0$. Since Φ satisfies (12), for all $\imath\in\mathbb{N}$, we have

$$\pi(\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath+1}}))\leq\ell\pi(\mathfrak{b}(\lambda_{1_{\imath-1}},\lambda_{1_\imath})). \tag{16}$$

Since $\ell\in(0,1)$, we have

$$\pi(\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath+1}}))\leq\ell\pi(\mathfrak{b}(\lambda_{1_{\imath-1}},\lambda_{1_\imath}))\leq\pi(\mathfrak{b}(\lambda_{1_{\imath-1}},\lambda_{1_\imath})),\forall\imath\in\mathbb{N}. \tag{17}$$

Thus, $\{\pi(\mathfrak{b}(\lambda_{\imath+1},\lambda_\imath))\}$ is a decreasing sequence; hence, it is convergent and

$$\lim_{\imath\longrightarrow\infty}\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))=\mathfrak{u}\geq 0. \tag{18}$$

Now, we show that $\mathfrak{u}=0$. From (17), we have

$$\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))\leq\ell\pi(\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath-1}}))\leq\cdots\cdots\leq\ell^\imath\pi\left(\mathfrak{b}\left(\lambda_{1_1},\lambda_{10}\right)\right), \tag{19}$$

since $0<\ell<1$; therefore, $\lim_{\imath\longrightarrow\infty}\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))=0$. So, $\lim_{\imath\longrightarrow\infty}\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath})=0$ by (ii).
On the other hand from (19), we have

$$\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))\leq\ell^\imath\pi\left(\mathfrak{b}\left(\lambda_{1_1},\lambda_{10}\right)\right),\forall\imath\in\mathbb{N}. \tag{20}$$

Then,

$$\sqrt[\imath]{\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))}\leq\ell^\imath\sqrt[\imath]{\pi\left(\mathfrak{b}\left(\lambda_{1_1},\lambda_{10}\right)\right)},\forall\imath\in\mathbb{N}. \tag{21}$$

Thus,

$$\lim_{\imath\longrightarrow\infty}\sqrt[\imath]{\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath}))}\leq\ell<1. \tag{22}$$

Put $\beta_\imath=\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_\imath})$; using (22), and condition (iii) of π, we get

$$\sum_1^\infty\beta_\imath<\infty\text{ and also }\beta_\imath\longrightarrow 0. \tag{23}$$

Now, we will show that $\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath+2}})\longrightarrow 0$ as $\imath\longrightarrow\infty$.

$$0<\pi(\mathfrak{b}(\lambda_{1_{\imath+2}},\lambda_{1_\imath}))\leq\ell\pi(\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_{\imath-1}}))\leq\cdots\leq\ell^\imath\pi\left(\mathfrak{b}\left(\lambda_{1_2},\lambda_{10}\right)\right). \tag{24}$$

Therefore, $\mathfrak{b}(\lambda_{1_{\imath+2}},\lambda_{1_\imath})\longrightarrow 0$, as $\imath\longrightarrow\infty$. □

Now, to prove that the sequence $\{\lambda_{1_\imath}\}$ is Cauchy, we consider two cases.

Case 1. If $\mathfrak{m}=2\mathfrak{p}+1$, $\mathfrak{p}\geq 1$, then

$$\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath+\mathfrak{m}}})\leq\mathfrak{b}(\lambda_{1_\imath},\lambda_{1_{\imath+1}})+\mathfrak{b}(\lambda_{1_{\imath+1}},\lambda_{1_{\imath+2}})+..\cdots+\mathfrak{b}\left(\lambda_{1_{\imath+2\mathfrak{p}}},\lambda_{1_{\imath+2\mathfrak{p}+1}}\right)\leq\sum_\imath^{\imath+2\mathfrak{p}+1}\beta_\imath<\sum_\imath^\infty\beta_\imath. \tag{25}$$

Case 2. If $\mathfrak{m} = 2\mathfrak{p}$, $\mathfrak{p} \geq 2$, then

$$\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+\mathfrak{m}}}) \leq \mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+2}}) + \mathfrak{b}(\lambda_{1_{\imath+2}}, \lambda_{1_{\imath+3}}) + \dots \cdots + \mathfrak{b}\left(\lambda_{1_{\imath+2\mathfrak{p}-1}}, \lambda_{1_{\imath+2\mathfrak{p}}}\right) \leq \sum_{\imath}^{\imath+2\mathfrak{p}+1} \beta_\imath < \sum_{\imath}^{\infty} \beta_\imath. \tag{26}$$

Thus, combining these two cases and using (23), when $\imath \longrightarrow \infty$, we have

$$\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+\mathfrak{m}}}) \leq \sum_{\imath}^{\infty} \beta_\imath \longrightarrow 0, \text{as } \imath \longrightarrow \infty. \tag{27}$$

Thus, we deduce that $\{\Phi^\imath \lambda_1\}$ is an orthogonal Cauchy sequence.

Completeness of $(\mathscr{P}, \perp, \mathfrak{b})$ ensures $\lim_{\mathfrak{m} \longrightarrow \infty} \lambda_{1_\imath} = \mathfrak{z}$ for some $\mathfrak{z} \in \mathscr{P}$.

Now, we want to show that $\mathfrak{z}$ is a fixed point of $\mathscr{P}$. From (12), we have

$$\pi(\mathfrak{b}(\Phi\lambda_{1_\imath}, \Phi\mathfrak{z})) \leq \pi(\mathfrak{b}(\lambda_{1_\imath}, \mathfrak{z})). \tag{28}$$

Hence, $\mathfrak{b}(\lambda_{1_\imath}, \mathfrak{z}) \longrightarrow 0$, and $\pi(\mathfrak{b}(\lambda_{1_\imath}, \mathfrak{z})) \longrightarrow 0$, and therefore, $\lim_{\imath \longrightarrow \infty} \pi(\mathfrak{b}(\lambda_{1_{\imath+1}}, \Phi\mathfrak{z})) = 0$ as $\imath \longrightarrow \infty$. Again,

$$\lim_{\imath \longrightarrow \infty} \mathfrak{b}(\lambda_{1_{\imath+1}}, \Phi\mathfrak{z}) = 0, \tag{29}$$

by using (ii).

$$\mathfrak{b}(\mathfrak{z}, \Phi\mathfrak{z}) \leq \mathfrak{b}(\mathfrak{z}, \lambda_{1_\imath}) + \mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}}) + \mathfrak{b}(\lambda_{1_{\imath+1}}, \Phi\mathfrak{z}). \tag{30}$$

Thus, $\mathfrak{z} = \Phi\mathfrak{z}$, and hence, $\mathfrak{z}$ is a fixed point on Φ.

Now, we prove that Φ is unique. Conversely, assume that any two fixed points s.t $\mathfrak{b}(\lambda_1, \mathfrak{z}) = \mathfrak{b}(\Phi\lambda_1, \Phi\mathfrak{z}) > 0$. From (12), since Φ is preserving, $\forall \Phi\lambda_1 \perp \Phi\mathfrak{z}$, we have

$$\begin{aligned} &(\Phi^\imath\lambda_1 \perp \Phi^\imath\lambda_2 \text{ and } \Phi^\imath\lambda_1 \perp \Phi^\imath\mathfrak{z}) \text{ or} \\ &(\Phi^\imath\lambda_1 \perp \Phi^\imath\mathfrak{z} \text{ and } \Phi^\imath\lambda_1 \perp \Phi^\imath\lambda_2), \forall \imath \in \mathbb{N}. \end{aligned} \tag{31}$$

Now,

$$\mathfrak{b}(\lambda_2, \mathfrak{z}) = \mathfrak{b}(\Phi^\imath\lambda_2, \Phi^\imath\mathfrak{z}) \leq \mathfrak{b}(\Phi^\imath\lambda_2, \Phi^\imath\lambda_1) + \mathfrak{b}(\Phi^\imath\lambda_1, \Phi^\imath\mathfrak{z}). \tag{32}$$

This implies that

$$\pi(\mathfrak{b}(\lambda_2, \mathfrak{z})) < \pi(\mathfrak{b}(\lambda_2, \mathfrak{z})). \tag{33}$$

This is a contradiction. Then Φ has a **ufp**.

The below example validates the proof of Theorem 15.

Example 5. Let $\mathscr{P} = [-2,-1] \cup [1, 2]$ and $\mathfrak{b} : \mathscr{P} \times \mathscr{P} \longrightarrow [0,\infty)$ defined as follow $\mathfrak{b}(\lambda_1, \lambda_1) = 0$, for all $\lambda_1 \in \mathscr{P}$

$$\begin{aligned} \mathfrak{b}(1, 2) = \mathfrak{b}(2, 1) = 3, \mathfrak{b}(1,-1) = \mathfrak{b}(-1, 1) \\ = \mathfrak{b}(-1, 2) = \mathfrak{b}(2,-1) = 1, \end{aligned} \tag{34}$$

we define the relation $\lambda_1 \perp \lambda_2$ and $\mathfrak{b}(\lambda_1, \lambda_2) = |\lambda_1 - \lambda_2|$, otherwise.

We observe that

$$\mathfrak{b}(1, 2) > \mathfrak{b}(1,-1) + \mathfrak{b}(-1, 2). \tag{35}$$

Hence, $\Phi_\perp$-preserving, $\mathfrak{b}(\lambda_1, \lambda_2)$ is not a BMS. It is obvious that $\mathfrak{b}(\lambda_1, \lambda_2)$ is a OCBMS.

Let $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ be a map defined by

$$\Phi\lambda_1 = \begin{cases} \dfrac{3}{4}\lambda_1, & \lambda_1 \in \left[-2, -\dfrac{3}{2}\right) \cup \left(\dfrac{3}{2}, 2\right], \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

Now, we define $\pi : [0,\infty) \longrightarrow [0,\infty)$ by $\pi(\beta) = \sqrt{\beta}$.

Easily, we can show that π satisfies conditions (ii) and (iii) of Theorem 15, Φ satisfies (12), and $\lambda_1^* = 0$ is fixed point of Φ.

Saleh et al. [11] proved a new contractive maps and their fixed points on BMS as follows:

Theorem 16. *Let $(\mathscr{P}, \mathfrak{b})$ be a BMS and $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ be an L^* -contraction w.r.t (briefly with respect to) $\zeta \in L$. Then, Φ has a **ufp**.*

In the following theorem, we are going to prove fixed-point theorem on an orthogonal L^*-contraction mapping using continuity hypothesis of Φ.

Theorem 17. *Let $(\mathscr{P}, \perp, \mathfrak{b})$ be a OCBMS with an orthogonal element λ_2 and a function $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$, orthogonal L^* -contraction w.r.t $\zeta \in L$, the following axioms are satisfy:*

(i) Φ is orthogonal-preserving.

(ii) Φ is $\Phi_\perp$ with L^*-contraction mapping.

*Then, Φ has a **ufp**.*

Proof. Since $(\mathscr{P}, \perp)$ is orthogonal set,

$$\exists \lambda_2 \in \mathscr{P} : (\forall \lambda_1 \in \mathscr{P}, \lambda_1 \perp \lambda_2) \text{ or } (\forall \lambda_1 \in \mathscr{P}, \lambda_2 \perp \lambda_1). \tag{37}$$

It follows that $\lambda_2 \perp \Phi\lambda_2$ or $\Phi\lambda_2 \perp \lambda_2$. Let

$$\lambda_{1_1} = \Phi\lambda_2, \lambda_{1_2} = \Phi\lambda_{1_1} = \Phi^2\lambda_2 \cdots \cdots, \lambda_{1_{\imath+1}} = \Phi\lambda_{1_i} = \Phi^{\imath+1}\lambda_2, \tag{38}$$

for all $\imath \in \mathbb{N} \cup \{0\}$.

If $\lambda_{1_{\imath_0}} = \lambda_{1_{\imath_0+1}}$ for any $\imath \in \mathbb{N} \cup \{0\}$, then it is easy to see that λ_{10} is a fixed point of Φ. Consider $\lambda_{1_{\imath_0}} \neq \lambda_{1_{\imath_0+1}}, \forall \imath \in \mathbb{N} \cup \{0\}$. Since Φ is $\perp$-preserving, we have

$$\lambda_{1_{\imath_0}} \perp \lambda_{1_{\imath_0+1}} \text{ or } \lambda_{1_{\imath_0+1}} \perp \lambda_{1_{\imath_0}}, \tag{39}$$

for all $\imath \in \mathbb{N} \cup \{0\}$. Which implies that $\{\lambda_{1_\imath}\}$ is a O-sequence. □

Using equation (10) and (ζ_2^*), we have

$$\begin{aligned} 1 &\leq \zeta[\Theta(\mathfrak{b}(\Phi\lambda_{1_{\imath-1}}, \Phi\lambda_{1_\imath})), \Theta(\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}))] \\ &= \zeta[\Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})), \Theta(\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}))] \\ &< \frac{\Theta(\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}))}{\Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}}))}. \end{aligned} \tag{40}$$

Consequently, we obtain that

$$\Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})) < \Theta(\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_i})), \forall \imath \in \mathbb{N}, \tag{41}$$

where

$$\begin{aligned} &\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}) \\ &\quad = \max\{\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}), \mathfrak{b}(\lambda_{1_{\imath-1}}, \Phi\lambda_{1_{\imath-1}}), \mathfrak{b}(\lambda_{1_\imath}, \Phi\lambda_{1_\imath})\} \\ &\quad = \max\{\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}), \mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})\}. \end{aligned} \tag{42}$$

If $\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}) = \mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})$, then inequality (41) becomes

$$\Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})) < \Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})), \forall \imath \in \mathbb{N}. \tag{43}$$

This is a contradiction. Hence, we must have $\mathscr{M}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}) = \mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath})$, for all $\imath \in \mathbb{N}$. Therefore, inequality (41) becomes

$$\Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})) < \Theta(\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath})), \forall \imath \in \mathbb{N}, \tag{44}$$

which implies from (Θ_1) that

$$\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}}) < \mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}), \forall \imath \in \mathbb{N}. \tag{45}$$

Thus, $\{\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath})\}$ is decreasing sequence and boundary below by 0, so $\exists \mathfrak{r} \geq 0$ s.t $\lim_{\imath \longrightarrow \infty} \mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}) = \mathfrak{r}$. Suppose that $\mathfrak{r} \neq 0$, then from (Θ_2)

$$\lim_{\imath \longrightarrow \infty} \Theta(\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath})) > 1. \tag{46}$$

Taking $\alpha_\imath = \Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}}))$ and $\mathfrak{b}_\imath = \Theta(\mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}))$, $\forall \imath \in \mathbb{N}$, it is clear from (44), (46), and (Θ_3) that $\alpha_\imath < \mathfrak{b}_\imath, \forall \imath \in \mathbb{N}$, and $\lim_{\imath \longrightarrow \infty} \alpha_\imath = \lim_{\imath \longrightarrow \infty} \mathfrak{b}_\imath > 1$. Hence, using (ζ_3^*), we get

$$1 \leq \limsup_{\imath \longrightarrow \infty} \zeta(\alpha_\imath, \mathfrak{b}_\imath) < 1. \tag{47}$$

This is a contradiction. Therefore, $\mathfrak{r} = 0$, we have

$$\lim_{\imath \longrightarrow \infty} \mathfrak{b}(\lambda_{1_{\imath-1}}, \lambda_{1_\imath}) = 0, \forall \imath \in \mathbb{N}. \tag{48}$$

Now, let us assume that $\lambda_{1\mathfrak{m}} = \lambda_{1_\imath}$, for some $\mathfrak{m} > \imath$. Then, we have $\lambda_{1\mathfrak{m}+1} = \lambda_{1_{\imath+1}}$. Using (44), we get

$$\begin{aligned} &\Theta(\mathfrak{b}(\lambda_{1\mathfrak{m}}, \lambda_{1\mathfrak{m}+1})) \\ &\quad < \Theta(\mathfrak{b}(\lambda_{1\mathfrak{m}-1}, \lambda_{1\mathfrak{m}})) < \Theta(\mathfrak{b}(\lambda_{1\mathfrak{m}-2}, \lambda_{1\mathfrak{m}-1})) \\ &\quad < \cdots \cdots < \Theta(\mathfrak{b}(\lambda_{1_\imath}, \lambda_{1_{\imath+1}})) = \Theta(\mathfrak{b}(\lambda_{1\mathfrak{m}}, \lambda_{1\mathfrak{m}+1})). \end{aligned} \tag{49}$$

This is a contradiction. To summarize $\lambda_{1\mathfrak{m}} \neq \lambda_{1_\imath}$, for all $\mathfrak{m} \neq \imath$.

Next, to prove $\{\lambda_{1_\imath}\}$ is a orthogonal Cauchy sequence in $(\mathscr{P}, \perp, \mathfrak{b})$. Now, we consider it as not an orthogonal Cauchy; then, we can find two subsequences $\{\lambda_{1_{\imath_\ell}}\}$, and $\{\lambda_{1\mathfrak{m}_\ell}\}$ of $\{\lambda_{1_\imath}\}$ s.t $\imath_\ell$ is the smallest integer for which

$$\begin{gathered} \imath_\ell > \mathfrak{m}_\ell > \ell, \\ \mathfrak{b}\left(\lambda_{1\mathfrak{m}_\ell}, \lambda_{1_{\imath_\ell}}\right) \geq \varepsilon, \\ \mathfrak{b}\left(\lambda_{1\mathfrak{m}_\ell}, \lambda_{1_{\imath_\ell-2}}\right) < \varepsilon. \end{gathered} \tag{50}$$

By using a similar argument, we obtain

$$\lim_{\ell \longrightarrow \infty} \mathfrak{b}\left(\lambda_{1\mathfrak{m}_\ell}, \lambda_{1_{\imath_\ell}}\right) = \varepsilon = \lim_{\ell \longrightarrow \infty} \mathfrak{b}\left(\lambda_{1\mathfrak{m}_\ell-1}, \lambda_{1_{\imath_\ell-1}}\right). \tag{51}$$

Now, using (10) and (ζ_2^*), we have

$$\begin{aligned} 1 &\leq \zeta\left[\Theta\left(\mathfrak{b}\left(\Phi\lambda_{1\mathfrak{m}_\ell-1}, \Phi\lambda_{1_{\imath_\ell-1}}\right)\right), \Theta\left(\mathscr{M}\left(\lambda_{1\mathfrak{m}_\ell-1}, \lambda_{1_{\imath_\ell-1}}\right)\right)\right] \\ &= \zeta\left[\Theta\left(\mathfrak{b}\left(\lambda_{1\mathfrak{m}_\ell}, \lambda_{1_{\imath_\ell}}\right)\right), \Theta\left(\mathscr{M}\left(\lambda_{1\mathfrak{m}_\ell-1}, \lambda_{1_{\imath_\ell-1}}\right)\right)\right] \\ &< \frac{\Theta\left(\mathscr{M}\left(\lambda_{1\mathfrak{m}_\ell-1}, \lambda_{1_{\imath_\ell-1}}\right)\right)}{\Theta\left(\mathfrak{b}\left(\lambda_{1\mathfrak{m}_\ell}, \lambda_{1_{\imath_\ell}}\right)\right)}, \end{aligned} \tag{52}$$

which implies that

$$\Theta\left(\mathfrak{b}\left(\lambda_{1\mathfrak{m}_\ell}, \lambda_{1_{\imath_\ell}}\right)\right) < \Theta\left(\mathscr{M}\left(\lambda_{1\mathfrak{m}_\ell-1}, \lambda_{1_{\imath_\ell-1}}\right)\right), \forall \ell \in \mathbb{N}, \tag{53}$$

where

$$\begin{aligned} \mathscr{M}\left(\lambda_{1\mathfrak{m}_\ell-1}, \lambda_{1_{\imath_\ell-1}}\right) = \max\Big\{&\mathfrak{b}\left(\lambda_{1\mathfrak{m}_\ell-1}, \lambda_{1_{\imath_\ell-1}}\right), \mathfrak{b} \\ &\cdot \left(\lambda_{1\mathfrak{m}_\ell-1}, \lambda_{1\mathfrak{m}_\ell}\right), \mathfrak{b}\left(\lambda_{1_{\imath_\ell-1}}, \lambda_{1_{\imath_\ell}}\right)\Big\}. \end{aligned} \tag{54}$$

From (48), (51), and Remark 5, we get

$$\lim_{\ell \longrightarrow \infty} \mathscr{M}\left(\lambda_{1\mathfrak{m}_\ell-1}, \lambda_{1_{\imath_\ell-1}}\right) = \max\{\varepsilon, 0, 0\} = \varepsilon. \tag{55}$$

Now, let $\alpha_\ell = \Theta(\mathfrak{b}(\lambda_{1\mathrm{m}_\ell}, \lambda_{1_{\iota_\ell}}))$, and $\mathfrak{b}_\ell = \Theta(\mathcal{M}(\lambda_{1\mathrm{m}_\ell-1}, \lambda_{1_{\iota_\ell-1}}))$, for all $\ell \in \mathbb{N}$. In view of (51), (53), (55), and (Θ_3), we have $\alpha_\ell < \mathfrak{b}_\ell$, for all $\ell \in \mathbb{N}$ and $\lim_{\ell\longrightarrow\infty} \alpha_\ell = \lim_{\ell\longrightarrow\infty} \mathfrak{b}_\ell > 1$. Therefore, using (ζ_3^*), we obtain

$$1 \leq \limsup_{\ell\longrightarrow\infty} \zeta(\alpha_\ell, \mathfrak{b}_\ell) < 1, \tag{56}$$

which is contradiction. Hence, $\{\lambda_{1_\iota}\} \in (\mathscr{P},\perp,\mathfrak{b})$ is orthogonal Cauchy sequence. As $(\mathscr{P},\perp,\mathfrak{b})$ is complete, then there exists $\ell \in \mathscr{P}$ s.t

$$\lim_{\iota\longrightarrow\infty} (\mathfrak{b}(\lambda_{1_\iota}, \ell)) = 0. \tag{57}$$

Without loss of generality, we consider $\lambda_{1_\iota} \neq \ell$ and $\Phi\lambda_{1_\iota} \neq \Phi\ell$, for all $\iota \in \mathbb{N}$. Suppose that $\mathfrak{b}(\ell, \Phi\ell) > 0$, it follows from (10) and ζ_2^* that

$$\begin{aligned} 1 &\leq \zeta[\Theta(\mathfrak{b}(\Phi\lambda_{1_\iota}, \Phi\ell)), \Theta(\mathcal{M}(\lambda_{1_\iota}, \ell))] \\ &= \zeta[\Theta(\mathfrak{b}(\lambda_{1_{\iota+1}}, \Phi\ell)), \Theta(\mathcal{M}(\lambda_{1_\iota}, \ell))] \\ &< \frac{\Theta(\mathcal{M}(\lambda_{1_\iota}, \ell))}{\Theta(\mathfrak{b}(\lambda_{1_{\iota+1}}, \Phi\ell))}, \end{aligned} \tag{58}$$

where $\mathcal{M}(\lambda_{1_\iota}, \ell) = \max\{\mathfrak{b}(\lambda_{1_\iota}, \ell), \mathfrak{b}(\lambda_{1_\iota}, \lambda_{1_{\iota+1}}), \mathfrak{b}(\ell, \Phi\ell)\}$, which implies that

$$\Theta(\mathfrak{b}(\lambda_{1_{\iota+1}}, \Phi\ell)) < \Theta(\mathcal{M}(\lambda_{1_\iota}, \ell)). \tag{59}$$

From Remark 5 and Lemma 12, we have

$$\lim_{\iota\longrightarrow\infty} \mathfrak{b}(\lambda_{1_{\iota+1}}, \Phi\ell) = \lim_{\iota\longrightarrow\infty} \mathcal{M}(\lambda_{1_\iota}, \ell) = \mathfrak{b}(\ell, \Phi\ell) > 0. \tag{60}$$

Let $\alpha_\iota = \Theta(\mathfrak{b}(\lambda_{1_{\iota+1}}, \Phi\ell))$, and $\mathfrak{b}_\iota = \Theta(\mathcal{M}(\lambda_{1_\iota}, \ell))$, for all $\iota \in \mathbb{N}$; it follows from (10) and ζ_3^* that

$$1 \leq \limsup_{\iota\longrightarrow\infty} \zeta(\alpha_\iota, \mathfrak{b}_\iota) < 1. \tag{61}$$

This is a contradiction. Therefore, summarize $\ell = \Phi\ell$, that is, ℓ is a fixed point of Φ. Finally, prove that Φ is **ufp**.

Consider two different fixed points ℓ and $\mathfrak{z}$ in $\mathscr{P}$.

Then, $\mathfrak{b}(\ell, \mathfrak{z}) = \mathfrak{b}(\Phi\ell, \Phi\mathfrak{z}) > 0$, since Φ is an orthogonal-preserving, $\forall \Phi\ell \perp \Phi\mathfrak{z}$.

Using (10) and ζ_2^*, we deduce that

$$\begin{aligned} 1 &\leq \zeta[\Theta(\mathfrak{b}(\Phi\ell, \Phi\mathfrak{z})), \Theta(\mathcal{M}(\ell, \mathfrak{z}))] \\ &= \zeta[\Theta(\mathfrak{b}(\ell, \mathfrak{z})), \Theta(\mathcal{M}(\ell, \mathfrak{z}))] < \frac{\Theta(\mathcal{M}(\ell, \mathfrak{z}))}{\Theta(\mathfrak{b}(\ell, \mathfrak{z}))}, \end{aligned} \tag{62}$$

where $\mathcal{M}(\ell, \mathfrak{z}) = \max\{\mathfrak{b}(\ell, \mathfrak{z}), \mathfrak{b}(\ell, \Phi\ell), \mathfrak{b}(\mathfrak{z}, \Phi\mathfrak{z})\} = \mathfrak{b}(\ell, \mathfrak{z})$, which implies that

$$\Theta(\mathfrak{b}(\ell, \mathfrak{z})) < \Theta(\mathcal{M}(\ell, \mathfrak{z})) = \Theta(\mathfrak{b}(\ell, \mathfrak{z})). \tag{63}$$

This is a contradiction. Therefore, Φ has a **ufp**.

Corollary 18. *Let $(\mathscr{P},\perp,\mathfrak{b})$ be a OCBMS and $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$. Assume that (for all $\lambda_1, \lambda_2 \in \mathscr{P}$ with $\lambda_1 \perp \lambda_2$):*

(i) Φ is orthogonal-preserving

(ii) $\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) > 0 \Longrightarrow$

$$\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)) \leq \mathcal{M}(\lambda_1, \lambda_2) - \varphi(\mathcal{M}(\lambda_1, \lambda_2)), \forall \lambda_1, \lambda_2 \in \mathscr{P} \text{ with } \lambda_1 \perp \lambda_2, \tag{64}$$

*where $\mathcal{M}(\lambda_1, \lambda_2) = \max\{\mathfrak{b}(\lambda_1, \lambda_2, \mathfrak{b}(\lambda_1, \Phi\lambda_1), \mathfrak{b}(\lambda_2, \Phi\lambda_2))\}$, and $\varphi : [0, \infty) \longrightarrow [0, \infty)$ is nondecreasing and lower semicontinuous s.t $\varphi^{-1}(\{0\}) = 0$. Then, Φ has a **ufp**.*

Proof. Let $\Theta(\alpha) = e^\alpha$, for all $\alpha > 0$. From (64), we have

$$\begin{aligned} \Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)) &= e^{(\mathfrak{b}(\Phi\lambda_1,\Phi\lambda_2))} \leq e^{\mathcal{M}(\lambda_1,\lambda_2)-\varphi(\mathcal{M}(\lambda_1,\lambda_2))} \\ &= \frac{\Theta(\mathcal{M}(\lambda_1, \lambda_2)}{e^{\varphi(\mathcal{M}(\lambda_1,\lambda_2))}}, \end{aligned} \tag{65}$$

for all $\lambda_1, \lambda_2 \in \mathscr{P}$ with $\lambda_1 \perp \lambda_2$, and $\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) > 0$. Therefore, Φ is orthogonal-preserving.

Now, we define $\varphi(\alpha) = \mathrm{In}(\Phi(\Theta(\alpha)))$, for all $\alpha > 0$, where $\Phi : [1, \infty) \longrightarrow [1, \infty)$ is nondecreasing and lower semicontinuous s.t $\Phi^{-1}(\{1\}) = 1$.

From (65), we have

$$\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)) \leq \frac{\Theta(\mathcal{M}(\lambda_1, \lambda_2))}{\Phi(\Theta(\mathcal{M}(\lambda_1, \lambda_2)))}. \tag{66}$$

Taking $\zeta(\alpha, \mathfrak{b}) = ((\mathfrak{b}/\alpha)\Phi(\mathfrak{b}))$ and using (66), we have

$$\begin{aligned} 1 &\leq \frac{\Theta(\mathcal{M}(\lambda_1, \lambda_2))}{\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2))\Phi(\Theta(\mathcal{M}(\lambda_1, \lambda_2)))} \\ &= \zeta[\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)), \Theta(\mathcal{M}(\lambda_1, \lambda_2))]. \end{aligned} \tag{67}$$

Therefore, all conditions are satisfied in Theorem 17, and hence, Φ has a **ufp**. □

In the following example, validate the proof of Theorem 17.

Example 6. Let $\mathscr{P} = \Pi \cup \Psi$, where $\Pi = [1, 2]$ and $\Psi = \{(1/\iota): \iota = 2, 3, 4, 5\}$. Define a map $\mathfrak{b} : \mathscr{P} \times \mathscr{P} \longrightarrow [0, \infty)$ as follows:

(1) $\mathfrak{b}(1/2, 1/3) = \mathfrak{b}(1/4, 1/5) = 3/10$,

(2) $\mathfrak{b}(1/2, 1/5) = \mathfrak{b}(1/3, 1/4) = 2/10$,

(3) $\mathfrak{b}(1/2, 1/4) = \mathfrak{b}(1/5, 1/3) = 6/10$,

(4) $\mathfrak{b}(\lambda_1, \lambda_1) = 0$, $\mathfrak{b}(\lambda_1, \lambda_2) = \mathfrak{b}(\lambda_2, \lambda_1)$, $\forall \lambda_1, \lambda_2 \in \Psi$, and

(5) $\mathfrak{b}(\lambda_1, \lambda_2) = |\lambda_1 - \lambda_2|$ if $\lambda_1, \lambda_2 \in \Pi$ or $\lambda_1 \in \Pi, \lambda_2 \in \Psi$ or $\lambda_1 \in \Psi, \lambda_2 \in \Pi$.

Here, the triangle inequality is not satisfied, so $\mathfrak{b}$ is not a metric on $\mathscr{P}$; we have

$$\frac{6}{10} = \mathfrak{b}\left(\frac{1}{5}, \frac{1}{3}\right) > \mathfrak{b}\left(\frac{1}{5}, \frac{1}{4}\right) + \mathfrak{b}\left(\frac{1}{4}, \frac{1}{3}\right) = \frac{5}{10}. \tag{68}$$

It is easy to verify that $(\mathscr{P}, \mathfrak{b})$ is a OCBMS. Let $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ be defined as an orthogonality relation $\perp$ on $\mathscr{P}$ by

$$\Phi\lambda_1 = \begin{cases} \frac{1}{5}, & \text{if } \lambda_1 \in \left[1, \frac{3}{2}\right], \\ \frac{1}{4}, & \text{if } \lambda_1 \in \left(\frac{3}{2}, 2\right] \cup \Psi. \end{cases} \tag{69}$$

Since Φ is not continuous at $\lambda_1 = (3/2)$, and $\Phi - \perp$ is not continuous, then Φ is neither orthogonal Θ-contraction nor an orthogonal L^*-contraction.

Declare that Φ is an orthogonal L^*-contraction w.r.t $\zeta : [1, \infty) \times [1, \infty) \longrightarrow \mathbb{R}$, where

$$\zeta_\ell(\alpha, \mathfrak{b}) = \frac{\mathfrak{b}^\ell}{\alpha}, \forall \alpha, \mathfrak{b} \in [1,\infty), \ell \in \left[\frac{3}{8}, 1\right), \tag{70}$$

and $\Theta : (0, \infty) \longrightarrow (1, \infty)$, s.t $\Theta(\alpha) = \mathfrak{e}^\alpha, \forall \alpha \in (0, \infty)$.

Indeed, for $\lambda_1 \in [1, (3/2)]$, and $\lambda_2 \in [(3/2), 2] \cup \Psi$, we have

$$\begin{aligned} &\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) = \mathfrak{b}\left(\frac{1}{4}, \frac{1}{5}\right) = \frac{3}{10} > 0, \\ &\zeta[\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)), \Theta(\mathscr{M}(\lambda_1, \lambda_2))] \\ &\quad = \frac{[\Theta\mathscr{M}(\lambda_1, \lambda_2)]^\ell}{\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2))} \geq \frac{\mathfrak{e}^{4\ell/5}}{\mathfrak{e}^{3/10}} = \mathfrak{e}^{(1/5)(4\ell-(3/2))} \\ &\quad \geq 1, \text{for any } \ell \in \left[\frac{3}{8}, 1\right). \end{aligned} \tag{71}$$

Hence, all the hypotheses are satisfied in Theorem 17, and $\ell = 1/4$ is the **ufp** of Φ.

4. An Application

The following BVP of a fourth-order differential equation is taken from Saleh et al. [11].

In this section, as an application of Theorem 17, we present the following result which provides an existence and uniqueness solution to the BVP of a fourth-order differential equation through an orthogonal L^*-contraction.

$$\begin{cases} \lambda_1''''(\alpha) = \mathfrak{g}\left(\alpha, \lambda_1(\alpha), \lambda_1'(\alpha), \lambda_1''(\alpha), \lambda_1'''(\alpha)\right), \alpha \in [0, 1], \\ \lambda_1(0) = \lambda_1'(0) = \lambda_1''(1) = \lambda_1'''(1) = 0. \end{cases} \tag{72}$$

Let $\mathfrak{g} : [0, 1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ is a continuous function. Let $\mathscr{P} = \mathscr{C}[0, 1]$ represent the space of all continuous functions defined on the interval [0,1]. Define a metric $\Phi : \mathscr{P} \times \mathscr{P} \longrightarrow \mathbb{R}$ by

$$\Phi(\lambda_1, \lambda_2) = \max_{\alpha\in[0,1]} |\lambda_1(\alpha) - \lambda_2(\alpha)|, \text{for all } \lambda_1, \lambda_2 \in \mathscr{P}. \tag{73}$$

It is known that $(\mathscr{P}, \Phi)$ is a complete BMS. Define the green function associated with (72)

$$G(\mathfrak{b}, \alpha) = \begin{cases} \frac{1}{6}\alpha^2(3\mathfrak{b} - \alpha), & 0 \leq \alpha \leq \mathfrak{b} \leq 1, \\ \frac{1}{6}\mathfrak{b}^2(3\alpha - \mathfrak{b}), & 0 \leq \mathfrak{b} \leq \alpha \leq 1. \end{cases} \tag{74}$$

Now, we provide the following result regarding the BVP (72) solution.

Theorem 19. *Assume that the following axioms are satisfied:*

(P1) $\mathfrak{g} : [0, 1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ *is orthogonal continuous function*

(P2) there exist $\tau > 0$ *and s.t, for all* $\lambda_1, \lambda_2 \in \mathscr{P}, \lambda_1 \perp \lambda_2$, *and* $\mathfrak{b} \in [0, 1]$

$$\begin{aligned} &\left|\mathfrak{g}\left(\mathfrak{b}, \lambda_1, \lambda_1'\right) - \mathfrak{g}\left(\mathfrak{b}, \lambda_2, \lambda_2'\right)\right| \\ &\quad \leq 8e^{-\tau}[\max\{|\lambda_1(\mathfrak{b}) - \lambda_2(\mathfrak{b})|, |\lambda_1(\mathfrak{b}) \\ &\quad - \Phi\lambda_1(\mathfrak{b})|, |\lambda_2(\mathfrak{b}) - \Phi\lambda_2(\mathfrak{b})|\}], \end{aligned} \tag{75}$$

where $\Phi : \mathscr{P} \longrightarrow \mathscr{P}$ *is defined by*

$$\Phi\lambda_1(\alpha) = \int_0^1 G(\alpha, \mathfrak{b})\mathfrak{g}\left(\mathfrak{b}, \lambda_1(\mathfrak{b}), \lambda_1'(\mathfrak{b})\right) ds. \tag{76}$$

Then, (72) has a unique solution in $\mathscr{P}$.

Proof. Define the binary relation $\perp$ on $\mathscr{P}$ by

$$\begin{aligned} \lambda_1 \perp \lambda_2 \Leftarrow \lambda_1(\sigma)\lambda_2(\sigma) &\geq \lambda_1(\sigma) \text{ or } \lambda_1(\sigma)\lambda_2(\sigma) \\ &\geq \lambda_2(\sigma), \forall \sigma \in [0, 1]. \end{aligned} \tag{77}$$

Observe that $\lambda_1 \in \mathscr{P}$ is a solution of (72) iff $\lambda_1 \in \mathscr{P}$ is a solution of the differential equation

$$\lambda_1(\alpha) = \int_0^1 G(\alpha, \mathfrak{b})\mathfrak{g}\left(\mathfrak{b}, \lambda_1(\mathfrak{b}), \lambda_1'(\mathfrak{b})\right) ds, \forall \lambda_1 \in \mathscr{P}. \tag{78}$$

Then, Φ is an orthogonal-continuous.

Now, we show that Φ is orthogonal-preserving, in (P2), for all $\lambda_1, \lambda_2 \in \mathscr{P}$ with $\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) > 0$ and for all $\alpha \in [0, 1]$. Then, Φ is an orthogonal-preserving.

Next, we claim that Φ is orthogonal $\mathscr{L}^{\star}$-contraction. We have

$$\begin{aligned}
&|\Phi\lambda_1(\alpha) - \Phi\lambda_2(\alpha)| \\
&= \left| \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) \mathfrak{g}\left(\mathfrak{b}, \lambda_1(\mathfrak{b}), \lambda_1'(\mathfrak{b})\right) ds \right. \\
&\quad \left. - \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) \mathfrak{g}\left(\mathfrak{b}, \lambda_2(\mathfrak{b}), \lambda_2'(\mathfrak{b})\right) ds \right| \\
&\le \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) \left| \mathfrak{g}\left(\mathfrak{b}, \lambda_1(\mathfrak{b}), \lambda_1'(\mathfrak{b})\right) - \mathfrak{g}\left(\mathfrak{b}, \lambda_2(\mathfrak{b}), \lambda_2'(\mathfrak{b})\right) \right| ds \\
&\le 8e^{-\tau} \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) [\max\{|\lambda_1 - \lambda_2|, |\lambda_1 - \Phi\lambda_1|, |\lambda_2 - \Phi\lambda_2|\}] ds \\
&\le 8e^{-\tau} [\mathscr{M}(\lambda_1, \lambda_2)] \left(\sup_{\alpha \in [0,1]} \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) ds \right),
\end{aligned} \tag{79}$$

where $\mathscr{M}(\lambda_1, \lambda_2) = \max\{\mathfrak{b}(\lambda_1, \lambda_2), \mathfrak{b}(\lambda_1, \Phi\lambda_1), \mathfrak{b}(\lambda_2, \Phi\lambda_2)\}$. As $\int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) ds = (\alpha^4/24) - (\alpha^3/6) + (\alpha^2/4)$, for all $\alpha \in [0, 1]$, $\sup_{\alpha \in [0,1]} \int_0^1 \mathrm{G}(\alpha, \mathfrak{b}) ds = 1/8$, we obtain

$$\begin{aligned}
\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2) &\le 8e^{-\tau}[\mathscr{M}(\lambda_1, \lambda_2)], \\
e^{\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)} &\le 8e^{-\tau}\left(e^{\mathscr{M}(\lambda_1, \lambda_2)}\right)^{e^{\tau}}.
\end{aligned} \tag{80}$$

Observe that $e^{\tau} \in (0, 1)$ as $\tau > 0$. It follows that Φ is an orthogonal $\mathscr{L}^{\star}$-contraction. Therefore, for all $\lambda_1, \lambda_2 \in \mathscr{P}$, we obtain

$$\begin{aligned}
&\zeta[\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)), \Theta(\mathscr{M}(\lambda_1, \lambda_2))] \\
&\quad = \frac{[\Theta\mathscr{M}(\lambda_1, \lambda_2)]^{\ell}}{\Theta(\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2))} \ge \frac{\left(e^{\mathscr{M}(\lambda_1, \lambda_2)^{e^{-\tau}}}\right)}{e^{\mathfrak{b}(\Phi\lambda_1, \Phi\lambda_2)}} \ge 1,
\end{aligned} \tag{81}$$

where $\Theta(\alpha) = e^{\alpha}, \zeta(\alpha, \mathfrak{b}) = (\mathfrak{b}^{\ell}/\alpha)$, and $\ell = e^{\tau}$. Thus, all the axioms of Theorem 17 are fulfilled. Therefore, Φ has a **ufp** in $\mathscr{P}$ which is a solution of (72). □

5. Conclusion

In this paper, we proved the fixed-point results for orthogonal $\mathscr{L}^{\star}$-contraction map on OCBMS. Furthermore, we presented some examples to strengthen our main results. Also, we provided an application to the BVP of a fourth-order differential equation.

Khalehoghli et al. [19, 20] presented a real generalization of the mentioned Banach's contraction principle by introducing R-metric spaces, where R is an arbitrary relation on L. We note that in a special case, R can be considered as $R = \preceq$ [partially ordered relation], $R = \perp$ [orthogonal relation], etc. If one can find a suitable replacement for a Banach theorem that may determine the values of fixed points, then many problems can be solved in this R-relation. This will provide a structural method for finding a value of a fixed point. It is an interesting open problem to study the fixed-point results on $\mathbb{R}$-complete R-metric spaces.

Data Availability

This clause is not applicable to this paper.

Additional Points

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

Fixed-Point Theorems for $\omega - \psi$-Interpolative Hardy-Rogers-Suzuki-Type Contraction in a Compact Quasipartial b-Metric Space

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This paper is aimed at proving the existence and uniqueness of a common fixed point for a pair of $\omega - \psi$-interpolative Hardy-Rogers-Suzuki-type contractions in the context of quasipartial b-metric space. Thus, several results in literature such as Hardy and Rogers, Suzuki, and others have been generalized in this work. We also offer a demonstrative example and an application of fractional differential equations to validate the findings.

1. Introduction and Preliminaries

Fixed-point theory is one of the fascinating research areas in pure mathematics, which has many applications in both pure and applied mathematics. Picard presented an iterative procedure for the solution of a functional equation first time in his research paper. This notion was later developed into an abstract framework by the Polish mathematician Stephan Banach [1] who presented a powerful tool known as the Banach contraction principle to determine the fixed point of mapping in complete metric space. It states as follows:

Theorem 1 (see [1]). *Let (M, d) be a complete metric space and let $f : M \longrightarrow M$ be a contraction; that is, there exists a number $k \in [0, 1)$ such that for all $u, v \in M$,*

$$d(fu, fv) \leq kd(u, v). \tag{1}$$

Then, f has a unique fixed point w in M.

By altering the contraction conditions, maps, and other conditions, several researchers have generalized the Banach contraction principle.

The Banach contraction principle needs continuity of the map involved in the contraction condition. In 1968, Kannan [2] relaxed the continuity condition and introduced a new fixed-point theorem with a new contraction condition as follows:

Theorem 2. *Let (M, d) be a complete metric space. A mapping $T : M \longrightarrow M$ is said to be a Kannan contraction if there exists $\lambda \in [0, 1/2)$ such that*

$$d(Tx, Ty) \leq \lambda[d(x, Tx) + d(y, Ty)], \tag{2}$$

for all $x, y \in X \setminus Fix(T)$. Then, T possesses a unique fixed point.

In 2018, Karapinar first established the interpolative Kannan-type contraction in his paper [3] as follows:

Definition 3. Let (M, d) be a metric space. We say that the self-mapping $T : M \longrightarrow M$ is an interpolative Kannan-type contraction, if there exists a constant $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$

such that

$$d(Tx, Ty) \leq \lambda [d(x, Tx)]^{\alpha} [\mathrm{d}(y, Ty)]^{1-\alpha}, \tag{3}$$

for all $x, y \in X$ with $x \neq Tx$.

Karapinar et al. [4] proved some results in the setting of $(\alpha, \beta, \psi, \phi)$ -interpolative contractions. Again in 2021, Khan et al. [5] proved some fixed-point results on the interpolative (ϕ, ψ)-type Z-contraction. For more results on interpolative-type contractions, one can see [6–8] and the references therein.

Following the results due to Karapinar et al. [9], Gaba and Karapinar [10] introduced a new approach to the interpolative contraction as follows:

Definition 4 (see [10]). Let (M, d) be a metric space and $f : M \longrightarrow M$ be a self-map. We shall call T a relaxed (λ, α, β) -interpolative Kannan contraction, if there exists $0 \leq \lambda, \alpha, \beta$ such that

$$d(fu, fv) \leq \lambda d(u, fu)^{\alpha} d(v, fv).^{\beta}. \tag{4}$$

Gaba and Karapinar [10] introduced the following definition of optimal interpolative triplet as follows:

Definition 5 (see [10]). Let (M, d) be a metric space and $f : M \longrightarrow M$ be a relaxed (λ, α, β)-interpolative Kannan contraction. The triplet (λ, α, β) will be called an "optimal interpolative triplet" if for any $\epsilon > 0$, the inequality (4) fails for at least one of the triplet $(\lambda - \epsilon, \alpha, \beta)$, $(\lambda, \alpha - \epsilon, \beta)$, and $(\lambda, \alpha, \beta - \epsilon)$.

In view of the above definitions, Gaba and Karapinar [10] proved the following theorem:

Theorem 6 (see [10]). *Let (M, d) be a complete metric space, and $f : M \longrightarrow M$ be a (λ, α, β)-interpolative Kannan contraction with $\lambda \in [0, 1)$, $\alpha, \beta \in (0, 1)$ so that $\alpha + \beta < 1$. Then, f has a fixed point in M.*

In 1973, Hardy and Rogers [11] introduced a natural modification of the Banach contraction principle.

Theorem 7. *Let (M, d) be a complete metric space. The mapping $f : M \longrightarrow M$ is called an interpolative Hardy-Rogers type of contraction if there exist positive real numbers $\alpha, \beta, \gamma, \delta > 0$, with $\beta + \alpha + \gamma + \delta < 1$ such that*

$$\begin{aligned} d(fu, fv) \leq {} & [\alpha d(u, v) + \beta d(u, fu) + \gamma d(v, fv)] \\ & + \delta \left[\frac{1}{2} (d(u, fv) + d(v, fu)) \right], \end{aligned} \tag{5}$$

for each $u, v \in M \setminus Fix(f)$. Then, a mapping f has a unique fixed point in M.

Later in 2018, Karapinar et al. [12] introduced the following notion of interpolative Hardy-Rogers-type contraction.

Theorem 8 (see [12]). *Let (M, d) be a complete metric space. The mapping $f : M \longrightarrow M$ is called an interpolative Hardy-Rogers type of contraction if there exist $\lambda \in (0, 1)$ and positive reals $\alpha, \beta, \gamma > 0$, with $\beta + \alpha + \gamma < 1$ such that*

$$d(fu, fv) \leq \lambda \left([d(u, v)]^{\beta} . [d(u, fu)]^{\alpha} . [d(v, fv)]^{\gamma} . \left[\frac{1}{2} (d(u, fv) + d(v, fu)) \right]^{1-\alpha-\beta-\gamma} \right), \tag{6}$$

for each $u, v \in M \setminus Fix(f)$. Then, a mapping f has a unique fixed point in M.

Several other versions of this type of results were proven by researchers. Some of them can be seen in [9, 13–15].

In 2008, Suzuki [16] introduced a generalization of the Banach contraction principle and characterizes the metric completeness of the underlying space. The generalized result is as follows:

Theorem 9 (see [16]). *Let (M, d) be a complete metric space and let $f : M \longrightarrow M$ be a mapping such that for all $u, v \in M$,*

$$\Phi(k) d(u, fu) \leq d(u, v) \Rightarrow d(fu, fv) \leqslant k d(u, v), \tag{7}$$

where $\Phi : [0, 1) \longrightarrow (1/2, 1)$ is a nonincreasing function defined by

$$\Phi(k) = \begin{cases} 1 & \text{if } 0 \leq k \leq \dfrac{\left(\sqrt{5} - 1\right)}{2}, \\ (1 - k) k^{-2} & \text{if } \dfrac{\left(\sqrt{5} - 1\right)}{2} \leq k \leq 2^{-1/2}, \\ (1 + k)^{-1} & \text{if } 2^{-1/2} \leq k < 1. \end{cases} \tag{8}$$

Then, there exists a unique fixed-point $w \in M$. A mapping f satisfying (7) is called as the Suzuki contraction.

Example 10 (see [16]). Let $M = \{(1, 1), (4, 1), (1, 4), (4, 5), (5, 4)\}$ with a metric d be defined by

$$d((u_1, u_2), (v_1, v_2)) = |u_1 - v_1| + |u_2 - v_2|. \tag{9}$$

Define a mapping

$$f(u_1, u_2) = \begin{cases} (u_1, 1) & \text{if } u_1 \geq u_2, \\ (1, u_2) & \text{if } u_1 > u_2. \end{cases} \quad (10)$$

Then, the map f satisfies all the hypotheses of Theorem 9, and $(1, 1)$ is the unique fixed point of f. However, for $u = (4, 5)$ and $v = (5, 4)$, $d(fu, fv) = 6 > 2 = d(u, v)$. Thus, f does not satisfy the assumptions in Theorem 9 for any $k \in [0, 1)$.

In 2021, Yeşilkaya [17] generalized the Banach contraction principle to (λ, α, β)-interpolative Kannan contraction as follows:

Definition 11 (see [17]). Let (M, d) be a metric space. The mapping $f : M \longrightarrow M$ is called an $\omega - \phi$ interpolative Hardy-Rogers contraction of the Suzuki type. If there exist $\psi \in \Psi$, $\omega : M \times M \longrightarrow [0,\infty)$, and positive reals $\alpha, \beta, \gamma > 0$, with $\alpha + \beta + \gamma < 1$, such that

$$\frac{1}{2} d(u, fu) \leq d(u, v) \Rightarrow \omega(u, v) d(fu, fv) \leq \psi\left\{ [d(u, v)]^{\beta}.[d(u, fu)]^{\alpha}.[d(v, fv)]^{\gamma}. \left[\frac{1}{2}(d(u, fv) + d(v, fu))\right]^{1-\alpha-\beta-\gamma} \right\}, \quad (11)$$

where Ψ is the set of all nondecreasing self-mappings ψ on $[0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$.

Similar results can be seen in [6, 7] and the references therein.

In 2012, Wardowski [18] generalized the Banach contraction principle into F-contraction mapping principle as follows:

Definition 12 (see [18]). Let (M, d) be a metric space. A mapping $f : M \longrightarrow M$ is called an F-contraction if there exist $\tau > 0$ and $F \in \mathscr{F}$ such that

$$\tau + F(d(fu, fv)) \leqslant F(d(u, v)), \quad (12)$$

holds for any $u, v \in M$ with $d(fu, fv) > 0$, where F is the set of all functions $F : R^+ \longrightarrow R$ satisfying the following conditions:

(F_1) F is strictly increasing: $u < v \Rightarrow F(u) < F(v)$,

(F_2) For each sequence $\{\alpha_n\}_{n \in N}$ in R^+, $\lim_{n \longrightarrow \infty} F(\alpha_n) = -\infty$,

(F_3) There exists $k \in (0, 1)$ such that $\lim_{\alpha \longrightarrow \infty} \alpha^k F(\alpha) = 0$.

We denote by $\mathscr{F}$ the set of all functions satisfying the conditions (F_1) and (F_2).

Example 13 (see [18]). The following $F : (0,+\infty)$ are the elements of $\mathscr{F}$

(1) $F\theta = \theta$,

(2) $F\theta = ln\theta + \theta$,

(3) $F\theta = -1/\sqrt{\theta}$,

(4) $F\theta = \ln(\theta^2 + \theta)$.

In 2013, Salimi et al. [19] and Hussain et al. [20] modified the notions of $\alpha - \phi$-contractive and α-admissible mappings and established certain fixed-point theorem as given below:

Definition 14 (see [19]). Let f be a self-mapping on M and $\alpha, \eta : M \times M \longrightarrow [0,+\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if $u, v \in M$,

$$\alpha(u, v) \geq \eta(u, v) \Rightarrow \alpha(fu, fv) \geq \eta(fu, fv). \quad (13)$$

Remark 15. It should be noted that Definition 14 reduces to α-admissible mapping definition due to Samet et al. [21] if we assume that $\alpha(u, v) = 1$. Furthermore, if we suppose that $\eta(u, v) = 1$, we may argue that f is an admissible η-sub admissible mapping.

Note that a self-map f can be ω-orbital admissible as stated in the definition below:

Definition 16 (see [11]). Let f be a self-map defined on M, and $\omega : M \times M \longrightarrow [0,\infty)$ be a function. f is said to be an ω-orbital admissible if for all $u \in M$, we have

$$\omega(u, fu) \geq 1 \Rightarrow \omega\left(u, f^2 u\right) \geq 1. \quad (14)$$

Gopal et al. [22] established the idea of α-type F-contractions and α-type F-weak contractions by combining the concepts of α-admissible mappings with F-contractions and F-weak contractions:

Definition 17 (see [22]). Let (M, d) be a metric space and $g : M \longrightarrow M$ be a mapping. Suppose $\alpha : M \times M \longrightarrow \{-\infty\} \cup (0,\infty)$ be a function. The function g is said to be an α-type F-contraction if there exists $\tau > 0$ such that for all $u, v \in M$,

$$d(fu, fv) > 0 \Rightarrow \tau + \alpha(u, v) F(d(gu, gv)) \leq F(d(u, v)). \quad (15)$$

In 2019, Dey et al. [23] introduced the notion of generalized α-F-contraction mapping as follows:

Theorem 18 (see [23]). *Let (M, d) be a metric space and $g : M \longrightarrow M$ be a mapping. Let $\alpha : M \times M \longrightarrow [0,\infty)$ be a function and $F \in \mathscr{F}$. The function g is said to be a modified generalized $\alpha - F$-contraction mapping if there exists $\tau > 0$ such that for all $u, v \in M$,*

$$d(gu, gv) > 0 \Rightarrow \tau + \alpha(u, v)F(d(gu, gv)) \leq F\left(N_{g(u,v)}\right), \tag{16}$$

where

$$N_{g(u,v)} = \max\left\{d(u, v), \frac{d(u, gv) + d(v, gu)}{2}, \frac{d(g^2u, u) + d(g^2u, gv)}{2}, d(g^2u, gu), d(g^2u, v), d(gu, v) + d(v, gv), d(g^2u, gv) + d(u, gu)\right\}. \tag{17}$$

Later, Wangwe and Kumar [24] proved results for $\alpha - F$-type contractions. One can see more results in [25–28] and the references therein.

$F -$contraction mapping of Hardy-Rogers type was introduced by Cosentino and Vetro [29] as follows:

Definition 19 (see [29]). Let (M, d) be a metric space. A self-mapping f on M is called an F-contraction of Hardy-Rogers type if there exists $F \in \mathscr{F}$ and $\tau \in S$ such that

$$\begin{aligned}\tau(d(u, v) + F(d(fu, fv))) \leq F[&\alpha d(u, v) + \beta d(u, fu)\\ &+ \gamma d(v, fv) + \delta d(u, fv) + Ld(v, fu)],\end{aligned} \tag{18}$$

for all $u, v \in M$ with $fu \neq fv$ where $\alpha, \beta, \gamma, \delta, L \in [0,+\infty)$,

$$\alpha + \beta + \gamma + 2\delta = 1. \tag{19}$$

Moreover, f is said to be a F-contraction of Suzuki-Hardy-Rogers type [30] if contraction Condition (18) holds for all $u, v \in M$ with $fu \neq fv$ and $d(u, fu)/2 < d(u, v)$.

Many researchers generalized the concept of metric space. The concept of b-metric space was first introduced by Bakhtin in 1989. By adding a variable $s \geq 1$ to the definition of metric space, the triangle inequality in this concept was relaxed as follows:

Definition 20 (see [31]). A b-metric on a nonempty set M is a function $d : M \times M \longrightarrow [0,\infty)$, such that for all $u, v, w \in M$ and for some real number $s \geq 1$, it satisfies the following:

(i) if $d(u, v) = 0$, then $u = v$,

(ii) $d(u, v) = d(v, u)$,

(iii) $d(u, v) \leqslant s[d(u, w) + d(w, v)]$,

then, a pair (M, d) is called b-metric space.

In 2021, Pauline and Kumar [32] presented an extension of the fixed-point theorem for T-Hardy-Rodgers contraction mappings in b-metric space. Czerwick [33] proved the existence of fixed point in b-metric space as follows:

Theorem 21 (see [33]). *Let ν be a topological space and let (M, d) be a complete b-metric space. Let $f : M \longrightarrow M$ be continuous and satisfy for each $w \in \nu$*

$$d[f(u, w), f(v, w)] \leq \alpha d(u, v), \tag{20}$$

for all $u, v \in M$, where $0 < \alpha < 1$. Then for each $w \in \nu$, there exists a unique fixed-point $u(w)$ of f, i.e., $f[u(w), w] = u(w)$ and the function $w \longrightarrow u(w)$ is continuous on ν.

In 1994, Matthews [34] introduced partial metric space as a result of the failure of metric functions in computer science as follows:

Definition 22 (see [34]). Let $M \neq \varnothing$. A partial metric is a function $p : M \times M \longrightarrow R^+$satisfying

(i) $p(u, v) = p(v, u)$,

(ii) If $0 \leqslant p(u, u) = p(u, v) = p(v, v)$, then $u = v$,

(iii) $p(u, v) + p(w, w) \leqslant p(u, w) + p(w, v)$ for all $u, v, w \in M$.

Then, a pair (M, p) is called partial metric space. It is clear that if $p(u, v) = 0$, then $u = v$; however, if $u = v$, then $p(u, v)$ may not be zero.

Remark 23 (see [34]). As partial metrics have a wider range of topological features and may easily support partial ordering, partial metrics are more versatile than metric spaces.

Künzi et al. [35] proposed the idea of partial quasimetric by eliminating symmetry condition from the notion of partial metric space.

Definition 24 (see [35]). A quasipartial metric on a nonempty set M is a function $qp : M \times M \longrightarrow [0,\infty)$ such that

(1) $qp(u,u) \leqslant qp(u,v)$ whenever $u, v \in M$,

(2) $qp(u,u) \leqslant qp(v,u)$ whenever $u, v \in M$,

(3) $qp(u,w) + qp(v,v) \leqslant (qp(u,v) + qp(v,w))$, whenever $u, v, w \in M$,

(4) $u = v$ if and only if $qp(u,u) = qp(u,v) = qp(v,v)$ whenever $u, v \in M$.

A pair (M, qp) is called a quasipartial metric space.

In 2015, Gupta and Gautam [36] introduced the notion of quasipartial b-metric space as follows:

Definition 25 (see [36]). A quasipartial b-metric on a nonempty set M is a function $qp_b : M \times M \longrightarrow [0,\infty)$ such that for some real number $s \geq 1$, it satisfies the following:

(i) if $qp_b(u,u) = qp_b(u,v) = qp_b(v,v)$, then $u = v$ (indistancy implies equality),

(ii) $qp_b(u,u) \leqslant qp_b(u,v)$ (small self-distances),

(iii) $qp_b(u,u) \leqslant qp_b(v,u)$ (small self-distances)

(iv) $qp_b(u,v) + qp_b(w,w) \leqslant s[qp_b(u,w) + qp_b(w,v)]$ (triangularity), for all $u, v \in M$.

Then, the pair (M, qp_b) is quasipartial b-metric on space M.

Example 26 (see [36]). Let $M = \mathbb{R}$ be the set of all real numbers. Define $qp_b : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$ by

$$qp_b(u,v) = |u - v| + |u|. \tag{21}$$

Then, it is a quasipartial b-metric on M.

Gautam et al. [37, 38] extended several results in quasipartial b-metric spaces.

In this article, we establish the existence and uniqueness of fixed-point theorems for $\omega - \psi$- interpolative Hardy-Rogers-Suzuki-type contraction in a compact quasipartial b-metric spaces with an application to fractional differential equations. An example is given to use the results that have been proven. The outcomes of this study will generalize several results obtained in [11, 12, 16–18, 25, 39, 40] and the references therein.

2. Main Results

To establish our first main results, we will begin by generalizing Definition 11 and extend it to a compact quasipartial b-metric space.

Definition 27. Let (M, qp_b) be a compact quasipartial b-metric space. A map $f : M \longrightarrow M$ is called $\omega - \psi$-interpolative Hardy-Rogers contraction of Suzuki type, if there exist $\psi \in \Psi$, where Ψ is the set of all nondecreasing self-mappings ψ on $[0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$ and $\alpha, \beta, \gamma > 0$, with $\alpha + \beta + \gamma < 1$,

$$\frac{1}{2} qp_b(u, fu) < qp_b(u,v) \Rightarrow \omega(u,v) qp_b(fu, fv) < \psi\left\{ [qp_b(u,v)]^{\beta} [qp_b(u,fu)]^{\alpha} [qp_b(v,fv)]^{\gamma} \left[\frac{1}{2}(qp_b(u,fv) + qp_b(v,fu))\right]^{1-\alpha-\beta-\gamma} \right\}, \quad \forall u, v \in M \setminus \mathrm{Fix}(f). \tag{22}$$

We now present our main theorem as follows:

Theorem 28. *Let (M, qp_b) be a compact quasipartial b-metric space and $f : M \longrightarrow M$ be $\omega - \psi$-interpolative Hardy-Rogers contraction of Suzuki type. If f is ω-orbital admissible mappings such that*

$$\omega(u_0, fu_0) \geq 1, \tag{23}$$

for some $u_0 \in M$. Then, a mapping f has a fixed point in M if at least one of the following properties holds

(i) (M, qp_b) is ω-regular

(ii) f is a continuous map

(iii) f^2 is continuous, $\omega(u, fu) \geq 1$ where $u \in Fix(f^2)$.

Proof. Let $u_0 \in M$ satisfies

$$\omega(u_0, fu_0) \geq 1. \tag{24}$$

We construct a sequence $\{u_n\}_{n=1}^{\infty}$ as shown below

$$u_1 = fu_0, u_2 = fu_1, \cdots, u_n = fu_{n-1}. \tag{25}$$

Assume that

$$u_{n_0} = u_{n_0+1} \tag{26}$$

for some $n_0 \in \mathbb{N}$, so that u_{n_0} is a fixed point of f. Thus on contrary, we can suppose that

$$u_n \neq u_{n+1}, \tag{27}$$

for each $n \in \mathbb{N} \cup \{0\}$. As f is ω-orbital admissible

$$\omega(u_0, fu_0) = \omega(u_0, u_1) \geq 1, \tag{28}$$

implies that

$$\omega(u_1, fu_1) = \omega(u_1, u_2) \geq 1. \tag{29}$$

Similarly, continuing this process, we get a sequence,

$$\omega(u_{n-1}, u_n) \geq 1. \tag{30}$$

By substituting $u = u_{n-1}$ and $v = fu_{n-1} = u_n$ in Definition 27, we obtain

$$\begin{aligned} \frac{1}{2} qp_b(u_{n-1}, fu_{n-1}) &= \frac{1}{2} qp_b(u_{n-1}, u_n) < qp_b(u_{n-1}, u_n) \\ &\Rightarrow \omega(u_{n-1}, u_n) qp_b(fu_{n-1}, fu_n) \\ &< \psi\Big([qp_b(u_{n-1}, u_n)]^{\beta} [qp_b(u_{n-1}, fu_{n-1})]^{\alpha} [qp_b(u_n, fu_n)]^{\gamma} \\ &\quad \times \left[\frac{1}{2}(qp_b(u_{n-1}, fu_n) + qp_b(u_n, fu_{n-1}))\right]^{1-\alpha-\beta-\gamma}\Big) \\ &= \psi\Big([qp_b(u_{n-1}, u_n)]^{\beta} [qp_b(u_{n-1}, u_n)]^{\alpha} [qp_b(u_n, u_{n+1})]^{\gamma} \\ &\quad \times \left[\frac{1}{2}(qp_b(u_{n-1}, u_{n+1}) + qp_b(u_n, u_n))\right]^{1-\alpha-\beta-\gamma}\Big). \end{aligned} \tag{31}$$

Thus, using $\psi(t) < t$ for $t > 0$, we have

$$\begin{aligned} qp_b(u_n, u_{n+1}) &< \psi\left([qp_b(u_{n-1}, u_n)]^{\beta} [qp_b(u_{n-1}, u_n)]^{\alpha} [qp_b(u_n, u_{n+1})]^{\gamma} \left[\frac{1}{2}(qp_b(u_{n-1}, u_{n+1}) + qp_b(u_n, u_n))\right]^{1-\alpha-\beta-\gamma}\right) \\ &< [qp_b(u_{n-1}, u_n)]^{\beta} [qp_b(u_{n-1}, u_n)]^{\alpha} [qp_b(u_n, u_{n+1})]^{\gamma} \left[\frac{1}{2}(qp_b(u_{n-1}, u_n) + qp_b(u_n, u_{n+1}))\right]^{1-\alpha-\beta-\gamma}. \end{aligned} \tag{32}$$

Assuming that,

$$qp_b(u_{n-1}, u_n) < qp_b(u_n, u_{n+1}), \tag{33}$$

for all $n \in \mathbb{N}$, then

$$\frac{1}{2}(qp_b(u_{n-1}, u_n) + qp_b(u_n, u_{n+1})) \leq qp_b(u_n, u_{n+1}), \tag{34}$$

Thus,

$$[qp_b(u_n, u_{n+1})]^{\alpha+\beta} < [qp_b(u_{n-1}, u_n)]^{\alpha+\beta}, \tag{35}$$

which is a contradiction. Hence, we get $\forall n \in \mathbb{N}$,

$$qp_b(u_n, u_{n+1}) \leq qp_b(u_{n-1}, u_n). \tag{36}$$

Then, the positive sequence $\{qp_b(u_{n-1}, u_n)\}$ is a nonincreasing sequence with positive terms, so we attain that there exists $a \geq 0$ such that

$$\lim_{n \longrightarrow \infty} qp_b(u_{n-1}, u_n) = a. \tag{37}$$

Accordingly, we get

$$\frac{1}{2}(qp_b(u_{n-1}, u_n) + qp_b(u_n, u_{n+1})) < qp_b(u_n, u_{n+1}). \tag{38}$$

Furthermore, using Equation (32),

$$[qp_b(u_n, u_{n+1})]^{1-\gamma} < \psi[qp_b(u_{n-1}, u_n)], \tag{39}$$

or equivalent

$$qp_b(u_n, u_{n-1}) < \psi(qp_b(u_{n-1}, u_n)). \tag{40}$$

Hence, by repeating this condition, we can write

$$\begin{aligned} qp_b(u_n, u_{n+1}) &< qp_b(u_{n-1}, u_n) < \psi^2 qp_b(qp_b(u_{n-2}, u_{n-1})) \\ &< \cdots < \psi^n qp_b(u_0, u_1). \end{aligned} \tag{41}$$

Now, we claim that $\{x_n\}$ is a Cauchy sequence in (X, qp_b). Then, we shall use the triangle inequality with Equation (41) for $s \geq 1$ and find that

$$\begin{aligned} qp_b(u_n, u_{n+l}) &\leq s(qp_b(u_n, u_{n+1}) + qp_b(u_{n+1}, u_{n+2}) \\ &\quad + \cdots + qp_b(u_{n+l-1}, u_{n+l}) - qp_b(u_{n+l-1}, u_{n+l-1})), \\ &< \psi^n\big(qp_b(u_0, u_1) + \psi^{n+1} qp_b(u_0, u_1) \\ &\quad + .. + \psi^{n+l-1} qp_b(u_0, u_1)\big), < \sum_{k=1}^{\infty} \psi^k(qp_b(u_0, u_1)). \end{aligned} \tag{42}$$

Letting $n \longrightarrow \infty$ in Equation (42), we find that $\{u_n\}$ is a Cauchy sequence in (M, qp_b). Regarding that (M, qp_b) is complete, there exists $t \in M$ such that

$$\lim_{n \longrightarrow \infty} qp_b(u_n, t) = 0. \quad (43)$$

We will show that the point t is a fixed point of f. If Equation (32) holds, that is, (M, qp_b) is ω-regular, then $\{u_n\}$ verify Equation (32), that

$$\omega(u_n, u_{n+1}) \geq 1. \quad (44)$$

and $\forall n \in \mathbb{N}$, we get

$$\omega(u_n, t) \geq 1. \quad (45)$$

We assert that

$$\frac{1}{2} qp_b(u_n, fu_n) \leq qp_b(u_n, t), \quad (46)$$

or

$$\frac{1}{2} qp_b(fu_n, f(fu_n)) \leq qp_b(fu_n, t), \quad (47)$$

$\forall n \in \mathbb{N}$. Assuming on the contrary that

$$\frac{1}{2} qp_b(u_n, fu_n) > qp_b(u_n, t), \quad (48)$$

and

$$\frac{1}{2} qp_b(fu_n, f(fu_n)) > qp_b(fu_n, t). \quad (49)$$

Using triangle inequality for $s \geq 1$, we obtain

$$\begin{aligned} qp_b(u_n, u_{n+1}) &= qp_b(u_n, fu_n) \leq s(qp_b(u_n, t) + qp_b(t, fu_n) - qp_b(t, t)) \\ &< \frac{1}{2} qp_b(u_n, u_{n+1}) + \frac{1}{2} qp_b(u_n, u_{n+2}) = qp_b(u_n, u_{n+1}), \end{aligned} \quad (50)$$

which is a contradiction. Therefore, $\forall n \in \mathbb{N}$, either

$$\frac{1}{2} qp_b(u_n, fu_n) \leq qp_b(u_n, t), \quad (51)$$

or

$$\frac{1}{2} qp_b(fu_n, f(fu_n)) \leq qp_b(fu_n, t), \quad (52)$$

holds. In case that inequality (46) holds, we get

$$\begin{aligned} qp_b(u_{n+1}, \mathrm{ft}) &< \omega(u_n, t).qp_b(fu_n, \mathrm{ft}) < \psi\left([(qp_b(u_n, t)]^{\beta} [qp_b(u_n, fu_n)]^{\alpha} [qp_b(t, \mathrm{ft})]^{\gamma} \left[\frac{1}{2}(qp_b(u_n, \mathrm{ft}) + qp_b(t, u_{n+1}))\right]^{1-\alpha-\beta-\gamma} \right). \\ &< [(qp_b(u_n, t)]^{\beta} [qp_b(u_n, u_{n+1})]^{\alpha} [qp_b(t, \mathrm{ft})]^{\gamma} \left[\frac{1}{2}(qp_b(u_n, \mathrm{ft}) + qp_b(t, u_{n+1}))\right]^{1-\alpha-\beta-\gamma}. \end{aligned} \quad (53)$$

If Equation (47) holds, we have

$$qp_b(u_{n+2}, \mathrm{ft}) < \omega(u_{n+1}, t) qp_b(f(fu_n), \mathrm{ft})) < \psi\left([qp_b(fu_n, t)]^{\beta} [qp_b(fu_n, f(fu_n))]^{\alpha} [qp_b(t, \mathrm{ft})]^{\gamma} \left[\frac{1}{2}(qp_b(fu_n, \mathrm{ft}) + qp_b(t, f(fu_n)))\right]^{1-\alpha-\beta-\gamma} \right), \quad (54)$$

$$= \psi\left([qp_b(u_{n+1}, t)]^{\beta} [qp_b(u_{n+1}, u_{n+2})]^{\alpha} [qp_b(t, \mathrm{ft})]^{\gamma} \left[\frac{1}{2}(qp_b(u_{n+1}, \mathrm{ft}) + qp_b(t, fu_{n+2}))\right]^{1-\alpha-\beta-\gamma} \right). \quad (55)$$

Therefore, letting $n \longrightarrow \infty$ in Equations (54) and (55), we get $qp_b(t, t) = 0$, that is,

$$\mathrm{ft} = t. \quad (56)$$

In case that assumption (47) is true, that is the mapping f is continuous,

$$t = \mathrm{ft} = \lim_{n \longrightarrow \infty} fu_n = \lim_{n \longrightarrow \infty} u_{n+1}, \quad (57)$$

and we want to show that also

$$\text{ft} = t. \tag{58}$$

Assuming on the contrary that

$$t \neq \text{ft}. \tag{59}$$

Since,

$$\frac{1}{2} qp_b\left(\text{ft}, f^2(t)\right) = \frac{1}{2} qp_b(\text{ft}, t) < qp_b(\text{ft}, t), \tag{60}$$

by Equation (47), we get

$$\begin{aligned} qp_b(t, \text{ft}) &< \omega(t, \text{ft}).qp_b\left(f^2 t, \text{ft}\right) < \psi\left([qp_b(\text{ft}, t)]^\beta \left[qp_b\left(\text{ft}, f^2 t\right)\right]^\alpha [qp_b(t, \text{ft})]^\gamma \left[\frac{1}{2}\left(qp_b(\text{ft}, \text{ft}) + qp_b\left(\text{ft}, f^2 t\right)\right)\right]^{1-\alpha-\beta-\gamma}\right) \\ &< [qp_b(\text{ft}, t)]^\beta [qp_b(\text{ft}, t)]^\alpha [qp_b(t, \text{ft})]^\gamma \left[\frac{1}{2} qp_b(\text{ft}, t)\right]^{1-\alpha-\beta-\gamma} < qp_b(t, \text{ft}), \end{aligned} \tag{61}$$

which is a contradiction. Consequently,

$$t = \text{ft}, \tag{62}$$

that is, t is a fixed point of f. □

The following corollary is obtained by substituting $\omega = 1$ in Theorem 28.

Corollary 29. *Let (M, qp_b) be a complete and compact metric space and f be self-mapping on M, such that*

$$\frac{1}{2} qp_b(u, fu) < qp_b(u, v), \tag{63}$$

implies

$$qp_b(fu, fv) < \psi\left([qp_b(u, v)]^\beta [qp_b(u, fu)]^\alpha [qp_b(v, fv)]^\gamma \left[\frac{1}{2}(qp_b(u, fv) + qp_b(v, fu))\right]^{1-\alpha-\beta-\gamma}\right), \tag{64}$$

for each $u, v \in M \setminus Fix(f)$, where $\psi \in \Psi$ and positive real β, $\alpha, \gamma > 0$, with $\alpha + \beta + \gamma < 1$. Then, f has a fixed point in M.

Proof. In Theorem 28, it is sufficient to get

$$\omega(u, v) = 1, \tag{65}$$

for proof. □

Further, taking $\psi(p) = p\lambda$, with $\lambda \in [0, 1)$ in Corollary 29, we obtain the following Corollary.

Corollary 30. *Let (M, qp_b) be a compact quasipartial b-metric space and f be a self-mapping on space M such that*

$$\frac{1}{2} qp_b(u, fu) < qp_b(u, v), \tag{66}$$

implies that

$$\begin{aligned} qp_b(fu, fv) &< \lambda [qp_b(u, v)]^\beta . [qp_b(u, fu)]^\alpha [qp_b(v, fv)]^\gamma \\ &\cdot \left[\frac{1}{2}(qp_b(u, fv) + qp_b(v, fu))\right]^{1-\alpha-\beta-\gamma}, \end{aligned} \tag{67}$$

for each $u, v \in M \setminus Fix(f)$, where positive reals $\alpha, \beta, \gamma > 0$, with $\alpha + \beta + \gamma < 1$. Then, f has a fixed point in M.

Remark 31. If we replace the quasipartial b-metric space by the metric space in Theorem 28, then we get the result due to Yeşilkaya [17] as a corollary.

Kumar [27] discussed the concept of orbital continuity. Using this concept, we formulate the following example which validates the result proved in Theorem 28.

Example 32. Let $M = [0, 2]$ and

$$qp_b = |u - v| + |u|. \tag{68}$$

Here, (M, qp_b) is a complete and compact quasipartial b-metric space defined by

$$f(u) = \begin{cases} \dfrac{1}{3} & \text{if}, 0 \leq u \leq 1, \\ \dfrac{u}{5} & \text{if}, 1 < u \leq 2, \end{cases} \tag{69}$$

and further, let

$$\omega(u,v)=\begin{cases}3, & \text{if}, 0\le u\le 1,\\ 1, & \text{if}, u=0, \text{and } v=2,\\ 0, & \text{otherwise.}\end{cases} \quad (70)$$

The mapping f is not continuous but since

$$f^2=\frac{1}{3}, \quad (71)$$

we have f^2 is continuous mapping. Let a function $\psi\in\Psi$ defined as $\psi=t/6$ and we choose $\beta=1/2, \alpha=1/3, \gamma=1/7$, and $t=1$. Then, we have to check if Theorem 28 holds. We have to consider the following cases:

(i) For $u, v\in[0,1]$, we have

$$\frac{1}{2}qp_b(u,fu)<qp_b(u,v), \quad (72)$$

implies

$$\omega(u,v)qp_b(fu,fv)<\psi\left([qp_b(u,v)]^\beta[qp_b(u,fu)]^\alpha[qp_b(v,fv)]^\gamma\left[\frac{1}{2}(qp_b(u,fv)+qp_b(v,fu))\right]^{1-\alpha-\beta-\gamma}\right), \quad \forall u,v\in M \quad (73)$$

(ii) For $u=0$ and $v=2$, we have

$$\frac{1}{2}qp_b(0,f0)=\frac{1}{3}<qp_b(0,2)=2, \quad (74)$$

implies

$$\omega(0,2)qp_b(f0,f2)=\frac{2}{5},<\frac{1}{6}\left([2]^{1/2}\right)\left(\left[\frac{2}{5}\right]^{1/3}\right)\left(\left[\frac{18}{5}\right]^{1/7}\right)\left(\frac{1}{2}\left[\frac{2}{5}+\frac{18}{5}\right]\right)^{1/42}. \quad (75)$$

For all other cases, Theorem 28 holds, since

$$\omega(u,v)=0. \quad (76)$$

As a result, the assumptions of Theorem 28 are satisfied, also the mappings f has a fixed point $u=1/3$.

3. An Application to Fractional Differential Equations

Several authors gave solutions of fractional differential equations using fixed-point theorems. Some of them are worth noting in this direction [41–45]. In this section, Theorem 28 is used to establish the existence and uniqueness of the solution of the fractional order differential equation. Here, we consider the following initial valued problem (IVP) of the form

$$D^\alpha u(t)=f(t,u_t),\forall t\in\gamma=[0,b], \alpha\in(0,1), \quad (77)$$

$$u(t)=\phi(t), t\in(-\infty,0), \quad (78)$$

where D^α is the standard Riemann-Liouville fractional derivative

$$f:\gamma\times A\longrightarrow\mathbb{R}, \phi\in A, \phi(0)=0, \quad (79)$$

and A is called a phase, space, or state space. Consider a quasipartial b-metric qp_b on X given by

$$qp_b(u,v)=|u-v|+|u|, \quad (80)$$

$\forall u,v\in M$ then, it is obvious that (M, qp_b) is a compact quasipartial b-metric space. If $u:(-\infty,b]\longrightarrow\mathbb{R}$, and $u_0\in\gamma$, then for every $t\in[0,b]u_t$ is a γ-valued continuous function on $[0,b]$. The space γ is complete by a solution of problems (77) and (78); we mean a space $\Omega=\{u:(-\infty, b]\longrightarrow\mathbb{R}: u|_{(-\infty,0)\in B} \text{ and } u|_{[0,b]}\}$. Therefore, a function $u\in\Omega$ is called a solution of Equations (77) and (78) if it satisfies the equation $D^s u(t)=f(t,u_t)$ on γ and condition $u(t)=\phi(t)$ on $(-\infty,0]$.

Lemma 33 (see [41]). *Let $0<\beta<1$ and $h:(0,b]\longrightarrow\mathbb{R}$ be continuous and*

$$\lim_{t\longrightarrow 0^+} v(t)=v(0^+)\in\mathbb{R}. \quad (81)$$

Then, u is a solution of the fractional integral equation

$$u(t)=\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}v(s)ds, \quad (82)$$

if and only if u is a solution of the initial value problem for the fractional differential equation

$$D^\beta u(t)=v(t), t\in(0,b], u(0)=0. \quad (83)$$

Theorem 34. *Let $f : \gamma \times A \longrightarrow \mathbb{R}$. Assume that there exists $q > 0$ such that*

$$|f(t,u)-f(t,v)|+|f(t,u)| \le q(|u-v|+|u|), \tag{84}$$

for $t \in \gamma$ and $\forall u, v \in A$. If $b^{\beta} k_b q / \Gamma(\beta+1) = k_1 \lambda < 1$ where $0 \le k_1 < 1/7$ and

$$k_b = \sup \{|k(t)|: t \in [0,b]\}, \tag{85}$$

then, there exists a unique solution for (IVP) (77) and (78) on the interval $(-\infty, b]$.

Proof. We first transform the given initial value problem into a fixed point problem. For this, we consider an operator $N : \Omega \longrightarrow \Omega$ defined by

$$N(u)(t) = \begin{cases} \phi(t) & \text{if, } t \in (-\infty,0], \\ \dfrac{1}{\Gamma(\beta)} \displaystyle\int_1^0 (t-s)^{\beta-1} f(s, y_s) & \text{if, } t \in [0,b]. \end{cases} \tag{86}$$

Let $\rho(\cdot)$: $(-\infty,b] \longrightarrow \mathbb{R}$ be a function defined by

$$\rho(t) = \begin{cases} \phi(t) & \text{if, } t \in (-\infty,0], \\ 0 & \text{if, } t \in (0,b). \end{cases} \tag{87}$$

Then, $\xi_0 = \phi$. For each $\eta \in C([0,b],\mathbb{R})$ with $\eta(0)=0$, we denote by $\bar{\eta}$ the function defined by

$$\bar{\eta}(t) = \begin{cases} 0 & \text{if, } t \in (-\infty,0], \\ \eta(t) & \text{if, } t \in (0,b). \end{cases} \tag{88}$$

If

$$u(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,u_s) ds, \tag{89}$$

for every $0 \le t \le b$ and the function $\eta(\cdot)$ satisfies

$$\eta(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, \bar{\eta}_s + \rho_s) ds. \tag{90}$$

Set

$$C_0 = \{\eta \in C([0,b],\mathbb{R}): \eta_0 = 0\}. \tag{91}$$

Now, let $f : C_0 \longrightarrow C_0$ be $\omega - \psi$ Hardy-Rogers-Suzuki operator be defined by

$$f\eta(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, \bar{\eta}_s + u_s). \tag{92}$$

The operator N has a fixed-point equivalent to f; hence, we have to prove that f has a fixed point. Indeed, if we consider that $\eta, \eta^* \in C_0$, then for all $t \in [0,b]$, we have

$$\begin{aligned} &|f\eta(t) - f\eta^*(t)| + |f\eta(t)| \\ &\quad = \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s, \bar{\eta}_s + u_s) ds \right. \\ &\qquad \left. - \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f\left(s, \bar{\eta^*}_s + u_s\right) ds \right| \\ &\qquad + \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{\eta}_s + u_s) ds \right| \\ &\quad < \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left| f(s, \bar{\eta}_s + u_s) - f\left(s, \bar{\eta^*} + \rho_s\right) \right| \\ &\qquad + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} |f(s, \bar{\eta}_s + \rho_s)| \\ &\quad < \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(\left| f(s, \bar{\rho}_s + \rho_s) - f\left(s, \bar{\eta^*} + \rho_s\right) \right| \right. \\ &\qquad + |f(s, \bar{\eta}_s + u_s)| \big) ds \\ &\quad < \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \left(q\left|\bar{\eta}_s - \bar{\eta^*}\right| + q|\bar{\eta}_s| \right) ds \\ &\quad < \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} q k_b \sup \left(|\eta(s) - \eta^*(s)| + |\eta(s)|\right) \\ &\quad < \frac{k_b}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} q ds |\eta - \eta^*| + |\eta|. \end{aligned} \tag{93}$$

Therefore,

$$\begin{aligned} |f(\eta) - f(\eta^*)| + |f(\eta)| &< \frac{q b^{\beta} k_b}{\Gamma(\beta+1)} |\eta - \eta^*|_b \\ &\quad + |\eta| q p_b(f(\eta), f(\eta^*)) < \lambda k_1 q p_b(\eta, \eta^*). \end{aligned} \tag{94}$$

Suppose $\psi \in \Psi$ and $\alpha, \beta, \gamma \ge 0$ with $\alpha + \beta + \gamma < 1$ such that

$$\frac{1}{2} q p_b(\eta, f\eta) < q p_b(\eta, \eta^*) \tag{95}$$

implies that

$$\begin{aligned} \omega(\eta, \eta^*) q p_b(f\eta, f\eta^*) &< \psi\left([q p_b(\eta, \eta^*)]^{\beta} . [q p_b(\eta, f\eta)]^{\beta} [q p_b(\eta^*, f\eta^*)]^{\gamma} \right) \\ &\quad \cdot \left(\frac{1}{2} (q p_b(\eta, f\eta)^*) + q p_b(\eta^*, f\eta) \right). \end{aligned} \tag{96}$$

Thus, we deduce that the operator f satisfy all the hypothesis of Theorem 28. Therefore, f has a unique fixed point. □

Data Availability

There is no data required in this research.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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Research Article

On Pata Convex-Type Contractive Mappings

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In this work, we introduce weak Pata convex contractions and weak E-Pata convex contractions via simulation functions in metric spaces to prove some fixed point results for such mappings. Also, we consider an example related to weak Pata convex contractions. Consequently, our results generalize and unify some results in the literature.

1. Introduction and Preliminaries

It is well known that Banach [1] pioneered in fixed point theory by introducing a novel notion, namely, Banach contraction principle in 1922. After this date, several authors generalized and extended this principle. A generalization was given by Pata [2] known as Pata contraction. Recently, Pata contraction has been studied by many authors. Some of the studies were for Pata contraction presented by [3–13].

Firstly, the concept of $\phi-$weak contraction was given by Alber et al. [14]. Zhang et al. and Rhoades's results [15, 16] extend previous results given by Alber et al., and they obtained fixed point results for single-valued mappings in Banach spaces, and Rhoades [15] got a unique common fixed point of such contractions, respectively.

In 2012, Samet et al. [17] suggested a novel notion, the so-called α-admissible. Later, Karapinar et al. [18] presented triangular α-admissible mappings, and then, Arshad et al. [19] introduced α-orbital admissible and triangular α-orbital admissible mappings. Due to the importance, many authors studied such mappings. For more knowledge and different examples related to admissible mappings, one can see [20–25].

Istratescu [26–28] gave the concept of contractions known as the convex contraction of order 2 and two-sided convex contraction mappings. Very recently on, Karapinar et al. [10] introduced the notion of α-almost Istratescu contraction of type E. Some notable generalizations related to Istratescu's results were obtained by [29–35].

In a recent work, Khojasteh et al. [36] introduced the notion of Z-contraction using simulation functions. Later, Karapinar [37] and Argoubi et al. [38] studied such contractions. After that, some new studies were obtained related to simulation functions in [39–44].

The aim of this paper is to establish some fixed point results for weak Pata convex contractive mapping and weak E-Pata convex contractive mapping via α-admissible mappings by using simulation functions in metric spaces. Our results are generalization of recent fixed point results derived by Karapinar et al. ([10, 32, 45]), Alber et al. [14], Zhang et al. [16], Istratescu [26], Pata [2], and Banach [1] and some other related results in the literature.

Firstly, we start this section by recalling some definitions related to our work.

In the course of this manuscript, $\mathbb{R}$, $\mathbb{N}$ denote the set of real numbers and the set of natural numbers, respectively. Let $FixS = \{w \in W : Sw = w.\}$

Alber et al. [14] gave the definition of $\phi-$weak contraction, stated below.

Definition 1. See [14]. Let (W, ρ) be a metric space. A mapping $S : W \longrightarrow W$ is called $\phi-$weak contraction, if there exists a map $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ with $\phi(0) = 0$ and $\phi(w) > 0$ for all $w > 0$ such that

$$\rho(Sw, Sv) \le \rho(w, v) - \phi(\rho(w, v)), \tag{1}$$

for all $w, v \in W$.

The concept of ϕ-weak contraction was generalized by Zhang et al. [16] as generalized ϕ-weak contraction.

Definition 2. See [16]. Let (W, ρ) be a metric space. A mapping $S : W \longrightarrow W$ is called generalized ϕ -weak contraction, if there exists a map $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ with $\phi(0) = 0$ and $\phi(w) > 0$ for all $w > 0$ such that

$$\rho(Sw, Sv) \leq M(w, v) - \phi(M(w, v)), \tag{2}$$

for all $w, v \in W$, where

$$M(w, v) = \max \left\{ \rho(w, v), \rho(w, Sw), \rho(v, Sv), \frac{\rho(w, Sv) + \rho(v, Sw)}{2} \right\}. \tag{3}$$

Samet et al. [17] and Karapinar et al. [18] introduced the following concepts, respectively.

Definition 3. Let (W, ρ) be a metric space, $S : W \longrightarrow W$ be a map, and $\alpha : W \times W \longrightarrow [0,+\infty)$ be a function.

(i) [17] If $\alpha(w, v) \geq 1$ implies $\alpha(Sw, Sv) \geq 1$ for all $w, v \in W$, then S is called $\alpha -$ admissible

(ii) [18] If S *is* $\alpha -$ admissible and $\alpha(w, z) \geq 1$ and $\alpha(z, v) \geq 1$ imply $\alpha(w, v) \geq 1$, then S is called triangular $\alpha -$ admissible

Example 4. Let $W = \mathbb{R}$, the mappings $S : W \longrightarrow W$ by

$$S(w) = \begin{cases} \dfrac{w^2 + 1}{3}, & w \in [0, 1), \\ \dfrac{1}{2}, & w \notin [0, 1), \end{cases} \tag{4}$$

and $\alpha : W \times W \longrightarrow [0,+\infty)$ by

$$\alpha(w, v) = \begin{cases} 1, & w, v \in [0, 1], \\ 0, & w, v \notin [0, 1]. \end{cases} \tag{5}$$

Thus, S is a triangular α-admissible mapping.

Khojasteh et al. [36] gave the simulation function and Z -contraction as follows.

Definition 5. See [36]. A mapping $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ is called a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0, 0) = 0$

(ζ_2) $\zeta(w, v) < w - v$

(ζ_3) if $\{w_n\}$ and $\{v_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\longrightarrow+\infty} w_n = \lim_{n\longrightarrow+\infty} v_n > 0$, then $\limsup_{n\longrightarrow+\infty} \zeta(w_n, v_n) < 0$.

Definition 6. See [36]. Let (W, ρ) be a metric space and $S : W \longrightarrow W$ be a mapping. If there exists $\zeta \in Z$ such that

$$\zeta(\rho(Sw, Sv), \rho(w, v)) \geq 0, \quad \text{for all } w, v \in W, \tag{6}$$

then, S is called $Z -$ contraction with respect to ζ.

(ζ_1) condition was removed in the above definition of simulation function by Argoubi et al. [38] in 2015. Also, Z' denotes the set of all simulation functions.

Example 7. See [36, 42, 44]. Let $\zeta : [0,\infty) \times [0,\infty) \longrightarrow \mathbb{R}$ and $\varphi_i : [0,\infty) \longrightarrow [0,\infty)$, $i = 1, 2, 3$ be continuous functions with $\varphi_i(w) = 0 \Leftrightarrow t = 0$.

$\zeta(w, v) = \varphi_1(w) - \varphi_2(v)$, for all $w, v \in [0,\infty)$, where $\varphi_1(w) < w \leq \varphi_2(v)$ for all $w > 0$.

$$\zeta(w, v) = \nu - \varphi_3(w) - w. \tag{7}$$

For the above examples and other examples related to simulation functions, one can see [36, 37, 42, 44] and references therein.

The following two concepts were defined by Istratescu [26] as follows.

Definition 8. See [26]. Let (W, ρ) be a metric space and $S : W \longrightarrow W$ be a self-mapping. For all $w, v \in W$, S is called convex contraction of order 2 if there exist $d_1, d_2 \in (0, 1)$ such that $d_1 + d_2 < 1$ and

$$\rho\left(S^2 w, S^2 v\right) \leq d_1 \rho(Sw, Sv) + d_2 \rho(w, v). \tag{8}$$

S is called two-sided convex contraction mappings if there exist $d_1, d_2, d_3, d_4 \in (0, 1)$ such that $d_1 + d_2 + d_3 + d_4 < 1$ and

$$\rho\left(S^2 w, S^2 v\right) \leq d_1 \rho(w, Sw) + d_2 \rho\left(Sw, S^2 w\right) + d_3 \rho(v, Sv) + d_4 \rho\left(Sv, S^2 v\right). \tag{9}$$

In the course of this work, Ψ denotes the set of all increasing function $\psi : [0, 1] \longrightarrow [0,\infty)$, which vanishes with continuity at zero. For a random $w_0 \in W$, we denote $\|w\| = \rho(w, w_0), \forall w \in W$.

Introducing a novel generalization of the Banach contraction principle, Pata [2] proved Theorem 9.

Theorem 9. See [2]. *Let (W, ρ) be a metric space and $\Lambda \geq 0$, $\xi \geq 1$ and $\vartheta \in [0, \xi]$ be fixed constants. $\psi \in \Psi$ and $S : W \longrightarrow W$ be functions. If for all $w, v \in W$, the inequality*

$$\rho(Sw, Sv) \leq (1 - \varepsilon)\rho(w, v) + \Lambda \varepsilon^{\xi} \psi(\varepsilon)[1 + \|w\| + \|v\|]^{\vartheta} \tag{10}$$

is satisfied for all $\varepsilon \in [0, 1]$; then S has a unique fixed point, $\omega = S\omega$, $\omega \in W$.

Pata-type contractions were studied by some authors. Karapinar et al. [11] introduced Pata-Ciric type contraction at a point. Alqahtani et al. [5] gave the α-Pata–Suzuki contraction and fixed point results for such contractions. After that, Karapinar and Himabindu [11] proved some common fixed point results for Pata–Suzuki Z-contraction.

We recall here the following important Lemma 10 that we will use to proof of our main results.

Lemma 10. See [46]. *Let (W, ρ) be a metric space and $\{w_n\}$ be a sequence in W such that $\rho(w_{n+1}, w_n) \longrightarrow 0$ as $n \longrightarrow \infty$. If $\{w_n\}$ is not a Cauchy sequence, then there exist a $\varsigma > 0$ and subsequences $\{w_{m_j}\}$ and $\{w_{n_j}\}$ of $\{w_n\}$ such that $\lim_{j\longrightarrow\infty} \rho(x_{m_j+1}, x_{n_j+1}) = \varsigma$, $\lim_{j\longrightarrow\infty}\rho(x_{m_j}, x_{n_j}) = \varsigma$, $\lim_{j\longrightarrow\infty}\rho(x_{m_j+1}, x_{n_j}) = \varsigma$ and $\lim_{j\longrightarrow\infty}\rho(x_{m_j}, x_{n_j+1}) = \varsigma$.*

2. Main Results

The main objective of this work is to give some new fixed point theorems via a combination of convex contraction, weak contraction and Pata type contractive mappings by introducing the concept of weak E-Pata convex contractions and weak Pata convex contractions in metric spaces. We will use simulation functions and admissible mappings when combining these concepts. Also, we will give an example that supports our conclusion.

In definitions and results in this paper, $\Lambda \geq 0$, $\xi \geq 1$, and $\vartheta \in [0, \xi]$ will be considered as fixed constants, and also, we will consider the following equations:

$$E_I(w, v) = \rho(Sw, Sv) + \left|\rho\left(Sw, S^2w\right) - \rho\left(Sv, S^2v\right)\right|,$$

$$M_I(w, v) = \max\left\{\rho(w, v), \rho(Sw, Sv), \rho(w, Sw), \rho(v, Sv), \rho\left(Sw, S^2w\right), \rho\left(Sv, S^2v\right)\right\},$$

$$P_I(w, v) = \left[1 + \|w\| + \|v\| + \|Sw\| + \|Sv\| + \left\|S^2w\right\| + \left\|S^2v\right\|\right]^{\vartheta}. \tag{11}$$

At first, we begin our work by giving the following definitions.

Definition 11. Let (W, ρ) be a metric space. We say that $S : W \longrightarrow W$ is weak Pata convex contractive mapping via simulation function if for all $w, v \in W$, and $\varepsilon \in [0, 1]$, there exist three functions $\zeta \in Z'$, $\psi \in \Psi$, and $\alpha : W \times W \longrightarrow [0,+\infty)$ such that S satisfies the inequality

$$\zeta\left(\alpha(w, v)\rho\left(S^2w, S^2v\right), (1 - \varepsilon)(M_I(w, v) - \phi(M_I(w, v))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w, v)\right) \geq 0, \tag{12}$$

where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(w) > 0$, for all $w > 0$.

Definition 12. Let (W, ρ) be a metric space. We say that $S : W \longrightarrow W$ is weak E -Pata convex contractive mapping via simulation function if for all $w, v \in W$, and $\varepsilon \in [0, 1]$, there exist three functions $\psi \in \Psi$, $\zeta \in Z'$, and $\alpha : W \times W \longrightarrow [0,+\infty)$ such that S satisfies the inequality

$$\zeta\left(\alpha(w, v)\rho\left(S^2w, S^2v\right), (1 - \varepsilon)(E_I(w, v) - \phi(E_I(w, v))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w, v)\right) \geq 0, \tag{13}$$

where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(w) > 0$, for all $w > 0$.

Now, we are in a position to present our main theorems.

Theorem 13. *Let (W, ρ) be a complete metric space, $\alpha : W \times W \longrightarrow [0,+\infty)$ and $S : W \longrightarrow W$ be a weak E -Pata convex mapping via simulation function. Suppose that*

(i) S is triangular α -admissible

(ii) there exists $w_0 \in W$ such that $\alpha(w_0, Sw_0) \geq 1$

(iii) S is continuous

(iv) for all $w, v \in FixS$, $\alpha(w, v) \geq 1$.

Then S has a unique fixed point in W.

Proof. From hypothesis (ii) of the Theorem 13, there exists $w_0 \in W$ such that $\alpha(w_0, Sw_0) \geq 1$. Firstly, we will show that $\alpha(S^n w_0, S^{n+1}w_0) \geq 1$ for all $n \in \mathbb{N}$. Since S is an α-admissible mapping, we have

$$\alpha(w_0, w_1) \geq 1 = \alpha(w_0, Sw_0) \geq 1 \Rightarrow \alpha\left(Sw_0, S^2w_0\right) \geq 1,$$
$$\alpha\left(Sw_0, S^2w_0\right) \geq 1 \Rightarrow \alpha\left(S^2w_0, S^3w_0\right) \geq 1. \tag{14}$$

By induction, we obtain that

$$\alpha\left(S^n w_0, S^{n+1}w_0\right) \geq 1, \quad \text{for all } n \in \mathbb{N}. \tag{15}$$

Taking into account hypothesis (i) of the Theorem 13, we have

$$\alpha\left(S^n w_0, S^{n+1}w_0\right) \geq 1 \text{ and } \alpha\left(S^{n+1}w_0, S^{n+2}w_0\right) \geq 1 \Rightarrow \alpha\left(S^n w_0, S^{n+2}w_0\right) \geq 1. \tag{16}$$

Again by induction, we obtain that

$$\alpha(S^n w_0, S^m w_0) \geq 1, \quad \text{for all } m > n \geq 0. \tag{17}$$

Now, we will show that $\{\rho(S^n w_0, S^{n+1}w_0)\}$ is a nonincreasing sequence. Since S is a weak E-Pata convex contractive mapping via simulation function, we have

$$\zeta\left(\alpha(w_0, Sw_0)\rho\left(S^2w_0, S^3w_0\right), (1-\varepsilon)\begin{pmatrix} E_I(w_0, Sw_0) \\ -\phi(E_I(w_0, Sw_0)) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_0, Sw_0)\right) \geq 0,$$
$$(1-\varepsilon)\begin{pmatrix} E_I(w_0, Sw_0) \\ -\phi(E_I(w_0, Sw_0)) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_0, Sw_0) - \alpha(w_0, Sw_0)\rho\left(S^2w_0, S^3w_0\right) \geq 0. \tag{18}$$

From hypothesis (ii) of the Theorem 13, we get

$$\begin{aligned}
\rho\left(S^2w_0, S^3w_0\right) &\leq \alpha(w_0, Sw_0)\rho\left(S^2w_0, S^3w_0\right) \\
&\leq (1-\varepsilon)(E_I(w_0, Sw_0) - \phi(E_I(w_0, Sw_0))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_0, Sw_0) \\
&= (1-\varepsilon)\begin{pmatrix} \rho\left(Sw_0, S^2w_0\right) + \left|\rho\left(Sw_0, S^2w_0\right) - \rho\left(S^2w_0, S^3w_0\right)\right| \\ -\phi\left(\rho\left(Sw_0, S^2w_0\right) + \left|\rho\left(Sw_0, S^2w_0\right) - \rho\left(S^2w_0, S^3w_0\right)\right|\right) \end{pmatrix} \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\begin{bmatrix} 1 + \|w_0\| + \|Sw_0\| + \|Sw_0\| \\ + \left\|S^2w_0\right\| + \left\|S^2w_0\right\| + \left\|S^3w_0\right\| \end{bmatrix}^{\vartheta} \\
&\leq (1-\varepsilon)\begin{pmatrix} \rho\left(Sw_0, S^2w_0\right) + \left|\rho\left(Sw_0, S^2w_0\right) - \rho\left(S^2w_0, S^3w_0\right)\right| \\ -\phi\left(\rho\left(Sw_0, S^2w_0\right) + \left|\rho\left(Sw_0, S^2w_0\right) - \rho\left(S^2w_0, S^3w_0\right)\right|\right) \end{pmatrix} \\
&\quad \cdot \left[1 + \|w_0\| + 2\|Sw_0\| + 2\left\|S^2w_0\right\| + \left\|S^3w_0\right\|\right]^{\vartheta} \\
&\leq (1-\varepsilon)\begin{pmatrix} \rho\left(Sw_0, S^2w_0\right) + \left|\rho\left(Sw_0, S^2w_0\right) - \rho\left(S^2w_0, S^3w_0\right)\right| \\ -\phi\left(\rho\left(Sw_0, S^2w_0\right) + \left|\rho\left(Sw_0, S^2w_0\right) - \rho\left(S^2w_0, S^3w_0\right)\right|\right) \end{pmatrix} \\
&\quad + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{19}$$

for some $K > 0$. If we assume that $\rho(Sw_0, S^2w_0) < \rho(S^2w_0, S^3w_0)$, then we have $\rho(Sw_0, S^2w_0) + |\rho(Sw_0, S^2w_0) - \rho(S^2w_0, S^3w_0)| = \rho(S^2w_0, S^3w_0)$. Hence, we have

$$\rho\left(S^2w_0, S^3w_0\right) \leq (1-\varepsilon)\left(\rho\left(S^2w_0, S^3w_0\right) - \phi\left(\rho\left(S^2w_0, S^3w_0\right)\right)\right) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{20}$$

The inequality (20) is true for all $\varepsilon \in [0, 1]$. For $\varepsilon = 0$, we obtain $\rho(S^2w_0, S^3w_0) < \rho(S^2w_0, S^3w_0)$ which is a contradiction. Therefore, we obtain

$$\rho\left(S^2w_0, S^3w_0\right) \leq \rho\left(Sw_0, S^2w_0\right). \tag{21}$$

Analogously, as S is a weak E-Pata convex *c*ontractive mapping via simulation function, we have

$$\zeta\left(\alpha\left(Sw_0, S^2w_0\right)\rho\left(S^3w_0, S^4w_0\right), (1-\varepsilon)\begin{pmatrix} E_I\left(Sw_0, S^2w_0\right) \\ -\phi\left(E_I\left(Sw_0, S^2w_0\right)\right) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(Sw_0, S^2w_0\right)\right) \geq 0,$$
$$\left((1-\varepsilon)\begin{pmatrix} E_I\left(Sw_0, S^2w_0\right) \\ -\phi\left(E_I\left(Sw_0, S^2w_0\right)\right) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(Sw_0, S^2w_0\right) - \alpha\left(Sw_0, S^2w_0\right)\rho\left(S^3w_0, S^4w_0\right)\right) \geq 0. \tag{22}$$

Now, we can write

$$\begin{aligned}
\rho\left(S^3w_0, S^4w_0\right) &\leq \alpha\left(Sw_0, S^2w_0\right)\rho\left(S^3w_0, S^4w_0\right) \\
&\leq (1-\varepsilon)\begin{pmatrix} \rho\left(S^2w_0, S^3w_0\right) + \left|\rho\left(S^2w_0, S^3w_0\right) - \rho\left(S^3w_0, S^4w_0\right)\right| \\ -\phi\left(\rho\left(S^2w_0, S^3w_0\right) + \left|\rho\left(S^2w_0, S^3w_0\right) - \rho\left(S^3w_0, S^4w_0\right)\right|\right) \end{pmatrix} \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|Sw_0\| + \left\|S^2w_0\right\| + \left\|S^2w_0\right\| + \left\|S^3w_0\right\| + \left\|S^3w_0\right\| + \left\|S^4w_0\right\|\right]^{\vartheta} \\
&\leq (1-\varepsilon)\begin{pmatrix} \rho\left(S^2w_0, S^3w_0\right) + \left|\rho\left(S^2w_0, S^3w_0\right) - \rho\left(S^3w_0, S^4w_0\right)\right| \\ -\phi\left(\rho\left(S^2w_0, S^3w_0\right) + \left|\rho\left(S^2w_0, S^3w_0\right) - \rho\left(S^3w_0, S^4w_0\right)\right|\right) \end{pmatrix} + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{23}$$

for some $K > 0$. In case that $\rho(S^2w_0, S^3w_0) < \rho(S^3w_0, S^4w_0)$; then we have $\rho(S^2w_0, S^3w_0) + |\rho(S^2w_0, S^3w_0) - \rho(S^3w_0, S^4w_0)| = \rho(S^3w_0, S^4w_0)$. So, we have

$$\rho\left(S^3w_0, S^4w_0\right) \leq (1-\varepsilon)\left(\rho\left(S^3w_0, S^4w_0\right) - \phi\left(\rho\left(S^3w_0, S^4w_0\right)\right)\right) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{24}$$

The inequality (24) is true for all $\varepsilon \in [0, 1]$. For $\varepsilon = 0$, we obtain $\rho(S^3w_0, S^4w_0) < \rho(S^3w_0, S^4w_0)$ which is again a contradiction. Therefore, we obtain

$$\rho\left(S^3w_0, S^4w_0\right) \leq \rho\left(S^2w_0, S^3w_0\right). \tag{25}$$

By induction, since S is a weak E-Pata convex *c*ontractive mapping via simulation function, we have

$$\begin{aligned}
&\zeta\left(\alpha\left(S^{n-2}w_0, S^{n-1}w_0\right)\rho\left(S^nw_0, S^{n+1}w_0\right), (1-\varepsilon)\right. \\
&\quad \cdot \left(E_I\left(S^{n-2}w_0, S^{n-1}w_0\right) - \phi\left(E_I\left(S^{n-2}w_0, S^{n-1}w_0\right)\right)\right) \\
&\quad \left. + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(S^{n-2}w_0, S^{n-1}w_0\right)\right) \geq 0, \\
&\left((1-\varepsilon)\begin{pmatrix} E_I\left(S^{n-2}w_0, S^{n-1}w_0\right) \\ -\phi\left(E_I\left(S^{n-2}w_0, S^{n-1}w_0\right)\right) \end{pmatrix}\right. \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(S^{n-2}w_0, S^{n-1}w_0\right) - \alpha\left(S^{n-2}w_0, S^{n-1}w_0\right) \\
&\quad \left. \cdot \rho\left(S^nw_0, S^{n+1}w_0\right)\right) \geq 0.
\end{aligned} \tag{26}$$

We have that

$$\begin{aligned}\rho\left(S^n w_0, S^{n+1} w_0\right) &\leq \alpha\left(S^{n-2} w_0, S^{n-1} w_0\right)\rho\left(S^n w_0, S^{n+1} w_0\right)\\ &\leq (1-\varepsilon)\begin{pmatrix}\rho\left(S^{n-1}w_0, S^n w_0\right) + \left|\rho\left(S^{n-1}w_0, S^n w_0\right) - \rho\left(S^n w_0, S^{n+1}w_0\right)\right| \\ -\phi\left(\rho\left(S^{n-1}w_0, S^n w_0\right) + \left|\rho\left(S^{n-1}w_0, S^n w_0\right) - \rho\left(S^n w_0, S^{n+1}w_0\right)\right|\right)\end{pmatrix}\\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \left\|S^{n-2}w_0\right\| + \left\|S^{n-1}w_0\right\| + \left\|S^{n-1}w_0\right\| + \left\|S^n w_0\right\| + \left\|S^n w_0\right\| + \left\|S^{n+1}w_0\right\|\right]^{\vartheta}\\ &\leq (1-\varepsilon)\begin{pmatrix}\rho\left(S^{n-1}w_0, S^n w_0\right) + \left|\rho\left(S^{n-1}w_0, S^n w_0\right) - \rho\left(S^n w_0, S^{n+1}w_0\right)\right| \\ -\phi\left(\rho\left(S^{n-1}w_0, S^n w_0\right) + \left|\rho\left(S^{n-1}w_0, S^n w_0\right) - \rho\left(S^n w_0, S^{n+1}w_0\right)\right|\right)\end{pmatrix} + K\varepsilon^{\xi}\psi(\varepsilon),\end{aligned} \tag{27}$$

for some $K > 0$. In case that $\rho(S^{n-1}w_0, S^n w_0) < \rho(S^n w_0, S^{n+1} w_0)$; then we have

$$\rho\left(S^n w_0, S^{n+1}w_0\right) < (1-\varepsilon)\left(\rho\left(S^n w_0, S^{n+1}w_0\right) - \phi\left(\rho\left(S^n w_0, S^{n+1}w_0\right)\right)\right) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{28}$$

Again, the inequality (28) is true for all $\varepsilon \in [0, 1]$ for $\varepsilon = 0$; we obtain $\rho(S^n w_0, S^{n+1} w_0) < \rho(S^n w_0, S^{n+1} w_0)$ is again a contradiction. Therefore, we obtain

$$\rho\left(S^n w_0, S^{n+1}w_0\right) \leq \rho\left(S^{n-1}w_0, S^n w_0\right). \tag{29}$$

Consequently, we find that

$$\begin{aligned}\rho\left(S^n w_0, S^{n+1}w_0\right) &\leq \rho\left(S^{n-1}w_0, S^n w_0\right) \leq \cdots \leq \rho\left(S^3 w_0, S^4 w_0\right)\\ &\leq \rho\left(S^2 w_0, S^3 w_0\right) \leq \rho\left(S w_0, S^2 w_0\right).\end{aligned} \tag{30}$$

If the point $w_0 \in W$ is taken as the starting point, the sequence $\{w_n\}$ is constructed by $w_n = Sw_{n-1} = S^n w_0$, $n \geq 1$. If $w_{n_0+1} = w_{n_0}$ for any $n_0 \in \mathbb{N}$, then w_{n_0} is a fixed point of S. As a result, supposing that $w_{n_0+1} \neq w_{n_0}$ for all $n_0 \in \mathbb{N}$ and let $\rho_n = \rho(w_{n-1}, w_n)$. So, we get that $\{\rho_n\}$ is a nonincreasing sequence. For this reason, there exists a $\delta \geq 0$ such that

$$\lim_{n\longrightarrow\infty}\rho(w_{n-1}, w_n) = \lim_{n\longrightarrow\infty}\rho_n = \delta. \tag{31}$$

We will demonstrate that $\delta = 0$. For this, we should demonstrate that the sequence $\{\|w_n\|\}$ is bounded. Since $\{\rho_n\}$ is a nonincreasing sequence, we have

$$\begin{aligned}\rho_{n+1} = \rho(w_n, w_{n+1}) &\leq \rho(w_{n-1}, w_n) \leq \cdots \leq \rho(w_3, w_4)\\ &\leq \rho(w_2, w_3) \leq \rho(w_1, w_2) = \rho_2 \leq \|w_1\| + \|w_2\|.\end{aligned} \tag{32}$$

By the triangle inequality, we have

$$\begin{aligned}\|w_n\| = \rho(w_n, w_0) &\leq \rho(w_n, w_{n+1}) + \rho(w_{n+1}, w_2) + \rho(w_2, w_0)\\ &= \rho_{n+1} + \rho(w_{n+1}, w_2) + \|w_2\| \leq \rho_2 + \rho(w_{n+1}, w_2)\\ &\quad + \|w_2\| \leq \|w_1\| + 2\|w_2\| + \rho(w_{n+1}, w_2).\end{aligned} \tag{33}$$

Since S is a weak E-Pata convex contractive mapping, we have

$$\zeta\left(\alpha(w_n, w_0)\rho(w_{n+1}, w_2), (1-\varepsilon)\begin{pmatrix}E_I(w_{n-1}, w_0)\\ -\phi(E_I(w_{n-1}, w_0))\end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_0)\right) \geq 0, \tag{34}$$

$$\begin{aligned}&\left((1-\varepsilon)(E_I(w_{n-1}, w_0) - \phi(E_I(w_{n-1}, w_0))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_0)\right)\\ &\quad - \alpha(w_n, w_0)\rho(w_{n+1}, w_2) \geq 0.\end{aligned} \tag{35}$$

Together with (35), we obtain

$$\begin{aligned}\rho(w_{n+1}, w_2) &\leq \alpha(w_n, w_0)\rho(w_{n+1}, w_2)\\ &\leq (1-\varepsilon)(E_I(w_{n-1}, w_0) - \phi(E_I(w_{n-1}, w_0)))\\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_0),\end{aligned} \tag{36}$$

where

$$\begin{aligned}E_I(w_{n-1}, w_0) &= \rho(w_n, w_1) + |\rho(w_n, w_{n+1}) - \rho(w_1, w_2)|\\ &\leq \rho(w_n, w_0) + \rho(w_1, w_0)\\ &\quad + |\rho(w_n, w_{n+1}) - \rho(w_1, w_2)| \leq \|w_n\| + \|w_1\|\\ &\quad + |\rho_{n+1} - \rho_2| = \|w_n\| + \|w_1\| + \rho_2 - \rho_{n+1}\\ &\leq \|w_n\| + 2\|w_1\| + \|w_2\| - \rho_{n+1} \leq \|w_n\|\\ &\quad + 2\|w_1\| + \|w_2\|,\\ P_I(w_{n-1}, w_0) &= [1 + \|w_{n-1}\| + \|w_0\| + \|w_n\| + \|w_1\| + \|w_{n+1}\|\\ &\quad + \|w_2\|]^{\vartheta} \leq [1 + \|w_1\| + \|w_2\| + \|w_n\| + \|w_n\|\\ &\quad + \|w_1\| + \|w_1\| + \|w_2\| + \|w_n\| + \|w_2\|]^{\vartheta}\\ &= [1 + 3\|w_1\| + 3\|w_2\| + 3\|w_n\|]^{\vartheta}.\end{aligned} \tag{37}$$

Now, we derive that

$$\begin{aligned}\|w_n\| &< \|w_1\| + 2\|w_2\| + (1-\varepsilon)(\|w_n\| + 2\|w_1\| + \|w_2\|\\ &\quad - \phi(\|w_n\| + 2\|w_1\| + \|w_2\|)) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|w_1\|\\ &\quad + 3\|w_2\| + 3\|w_n\|]^{\vartheta}.\end{aligned} \tag{38}$$

Using $\vartheta \leq \xi$, we get

$$\begin{aligned}\varepsilon\|w_n\| &< (3-2\varepsilon)\|w_1\| + (3-\varepsilon)\|w_2\| + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|w_1\|\\ &\quad + 3\|w_2\| + 3\|w_n\|]^{\vartheta} \leq (3-2\varepsilon)\|w_1\| + (3-\varepsilon)\|w_2\|\\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|w_1\| + 3\|w_2\| + 3\|w_n\|]^{\xi}\\ &= (3-2\varepsilon)\|w_1\| + (3-\varepsilon)\|w_2\| + \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1 + 3\|w_n\|)^{\xi}\\ &\quad \cdot \left(\frac{1 + 3\|w_1\| + 3\|w_2\|}{1 + 3\|w_n\|}\right)^{\xi} \leq 3\|w_1\| + 3\|w_2\| + \Lambda\varepsilon^{\xi}\psi\\ &\quad \cdot (\varepsilon)3^{\xi}\|w_n\|^{\xi}\left(\frac{1}{3\|w_n\|} + 1\right)^{\xi}(1 + 3\|w_1\| + 3\|w_2\|)^{\xi}.\end{aligned} \tag{39}$$

Conversely, we assume that $\{\|w_n\|\}$ is not bounded sequence. So, there exists a subsequence $\{\|w_{n_j}\|\}$ of $\{\|w_n\|\}$ such that $\lim_{j\longrightarrow\infty} w_{n_j} = \infty$. If we take $\varepsilon = \varepsilon_j = (1 + 3\|w_1\| + 3\|w_2\|)/\|w_{n_j}\|$ in (39) inequality; then we have

$$\begin{aligned}1 &\leq \Lambda 3^{\xi}\left(\varepsilon^{\xi}\|w_n\|^{\xi}\right)(1 + 3\|w_1\| + 3\|w_2\|)^{\xi}\left(\frac{1}{3\left\|w_{n_j}\right\|} + 1\right)^{\xi} \\ &\cdot \psi\left(\varepsilon_j\right) \leq \Lambda 3^{\xi}(1 + 3\|w_1\| + 3\|w_2\|)^{\xi}(1 + 3\|w_1\| + 3\|w_2\|)^{\xi} \\ &\cdot \left(\frac{1}{3\left\|w_{n_j}\right\|} + 1\right)^{\xi}\psi\left(\varepsilon_j\right) \leq \Lambda 3^{\xi}(1 + 3\|w_1\| + 3\|w_2\|)^{2\xi} \\ &\cdot \left(\frac{1}{3\left\|w_{n_j}\right\|} + 1\right)^{\xi}\psi\left(\varepsilon_j\right).\end{aligned} \tag{40}$$

If we take limit in (40) inequality as $j \longrightarrow \infty$, then we get

$$\Lambda 3^{\xi}(1 + 3\|w_1\| + 3\|w_2\|)^{2\xi}\left(\frac{1}{3\left\|w_{n_j}\right\|} + 1\right)^{\xi}\psi\left(\varepsilon_j\right) \longrightarrow 0, \tag{41}$$

which is a contradiction. Therefore, we demonstrate that the sequence$\{\|w_n\|\}$ is bounded. So, there exists $A > 0$ such that $\|w_n\| \leq A$ for all $n \in \mathbb{N}$. Following this line of work, we demonstrate that $\delta = 0$. Since S is a weak E-Pata convex contractive mapping, we have

$$\begin{aligned}&\zeta(\alpha(w_{n-1}, w_n)\rho(w_{n+1}, w_{n+2}), (1-\varepsilon)(E_I(w_{n-1}, w_n) \\ &\quad - \phi(E_I(w_{n-1}, w_n))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_n)) \geq 0, \\ &(1-\varepsilon)(E_I(w_{n-1}, w_n) - \phi(E_I(w_{n-1}, w_n))) \\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_n) - \alpha(w_{n-1}, w_n)\rho(w_{n+1}, w_{n+2}) \geq 0.\end{aligned} \tag{42}$$

Since $\rho_{n+1} \leq \rho_n$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned}E_I(w_{n-1}, w_n) &= \rho(w_n, w_{n+1}) + |\rho(w_n, w_{n+1}) - \rho(w_{n+1}, w_{n+2})| \\ &= 2\rho(w_n, w_{n+1}) - \rho(w_{n+1}, w_{n+2}) = 2\rho_{n+1} - \rho_{n+2}.\end{aligned} \tag{43}$$

Since the sequence$\{\|w_n\|\}$ is bounded, we have

$$\begin{aligned}P_I(w_{n-1}, w_n) &= \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + \|w_{n-1}\| + \|w_n\| + \|w_n\| + \|w_{n+1}\| \\ &\quad + \|w_{n+2}\| + \|w_{n+3}\|]^{\vartheta} \leq \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1 + 6A)^{\vartheta}.\end{aligned} \tag{44}$$

Now, we can write

$$\begin{aligned}\rho_{n+2} &= \rho(w_{n+1}, w_{n+2}) \leq \alpha(w_{n-1}, w_n)\rho(w_{n+1}, w_{n+2}) \\ &\leq (1-\varepsilon)(E_I(w_{n-1}, w_n) - \phi(E_I(w_{n-1}, w_n))) \\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_n) \leq (1-\varepsilon) \\ &\quad \cdot (2\rho_{n+1} - \rho_{n+2} - \phi(2\rho_{n+1} - \rho_{n+2})) \\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1 + 6A)^{\vartheta}.\end{aligned} \tag{45}$$

If we take the limit as $n \longrightarrow \infty$ in (45) inequality, then we obtain

$$\begin{aligned}\delta &\leq (1-\varepsilon)(\delta - \phi(\delta)) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1 + 6A)^{\vartheta}\delta \\ &\leq \Lambda\varepsilon^{\xi-1}\psi(\varepsilon)(1 + 6A)^{\vartheta}.\end{aligned} \tag{46}$$

$\delta \leq 0$ as $\varepsilon \longrightarrow 0$, that is $\lim_{n\longrightarrow\infty}\rho(w_{n+1}, w_{n+2}) = \delta = 0$. Now, we demonstrate that $\{w_n\}$ is a Cauchy sequence. On the contrary, assume that the sequence $\{w_n\}$ is not a Cauchy. From Lemma 10, there exist subsequence $\{w_{m_j}\}$and $\{w_{n_j}\}$ with $n_j > m_j > j$ such that $\lim_{k\longrightarrow\infty}\rho(x_{m_k-1}, x_{n_k+1}) = \varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k-1}, x_{n_k}) = \varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k}, x_{n_k}) = \varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k+1}, x_{n_k+1}) = \varsigma$, and $\lim_{k\longrightarrow\infty}\rho(x_{m_k}, x_{n_k-1}) = \varsigma$. Since S is a weak E – Pata convex contractive mapping, we have

$$\begin{aligned}&\zeta\left(\alpha\left(w_{n_j-1}, w_{m_j-1}\right)\rho\left(w_{n_j+1}, w_{m_j+1}\right), (1-\varepsilon)\right. \\ &\quad \left.\cdot\left(E_I\left(w_{n_j-1}, w_{m_j-1}\right) - \phi\left(E_I\left(w_{n_j-1}, w_{m_j-1}\right)\right)\right)\right) \geq 0, \\ &(1-\varepsilon)\left(E_I\left(w_{n_j-1}, w_{m_j-1}\right) - \phi\left(E_I\left(w_{n_j-1}, w_{m_j-1}\right)\right)\right) \\ &\quad - \alpha\left(w_{n_j-1}, w_{m_j-1}\right)\rho\left(w_{n_j+1}, w_{m_j+1}\right) \geq 0,\end{aligned} \tag{47}$$

where

$$\begin{aligned}E_I\left(w_{n_j-1}, w_{m_j-1}\right) &= \rho\left(w_{n_j}, w_{m_j}\right) \\ &\quad + \left|\rho\left(w_{n_j}, w_{n_j+1}\right) - \rho\left(w_{m_j}, w_{m_j+1}\right)\right|, \\ P_I\left(w_{n_j-1}, w_{m_j-1}\right) &= \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \left\|w_{n_j-1}\right\| + \left\|w_{m_j-1}\right\|\right. \\ &\quad \left. + \left\|w_{n_j}\right\| + \left\|w_{m_j}\right\| + \left\|w_{n_j+1}\right\| + \left\|w_{m_j+1}\right\|\right]^{\vartheta} \\ &= \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 6A]^{\vartheta}.\end{aligned} \tag{48}$$

Now, we have

$$\begin{aligned}\varsigma &\le \rho\left(w_{n_j+1}, w_{m_j+1}\right) \le \alpha\left(w_{n_j-1}, w_{m_j-1}\right)\rho\left(w_{n_j+1}, w_{m_j+1}\right)\\ &\le (1-\varepsilon)\left(E_I\left(w_{n_j-1}, w_{m_j-1}\right) - \phi\left(E_I\left(w_{n_j-1}, w_{m_j-1}\right)\right)\right)\\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(w_{n_j-1}, w_{m_j-1}\right)\\ &\le (1-\varepsilon)\begin{pmatrix}\rho\left(w_{n_j}, w_{m_j}\right) + \left|\rho\left(w_{n_j}, w_{n_j+1}\right) - \rho\left(w_{m_j}, w_{m_j+1}\right)\right|\\ -\phi\left(\rho\left(w_{n_j}, w_{m_j}\right) + \left|\rho\left(w_{n_j}, w_{n_j+1}\right) - \rho\left(w_{m_j}, w_{m_j+1}\right)\right|\right)\end{pmatrix}\\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1+6A]^{\vartheta}\\ &\le (1-\varepsilon)\left(\rho\left(w_{n_j}, w_{m_j}\right) + \left|\rho\left(w_{n_j}, w_{n_j+1}\right) - \rho\left(w_{m_j}, w_{m_j+1}\right)\right|\right)\\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1+6A]^{\vartheta}.\end{aligned} \tag{49}$$

If we take the limit as $j \longrightarrow \infty$, then we obtain

$$\varsigma \le (1-\varepsilon)\varsigma + K\varepsilon\psi(\varepsilon), \tag{50}$$

and so, we have

$$\varsigma \le K\psi(\varepsilon), \tag{51}$$

and thus, we get that $\varsigma = 0$, which is a contradiction. Therefore, we concluded that $\{w_n\}$ is a Cauchy sequence in (W, ρ). By the completeness of W, the sequence $\{w_n\}$ is convergent to some $\omega \in W$ that is $w_n \longrightarrow \omega$ as $n \longrightarrow +\infty$. Since S is continuous, $Sw_n \longrightarrow S\omega$ as $n \longrightarrow +\infty$. By the uniqueness of the limit, we obtain $\omega = S\omega$ that is ω is a fixed point of S.

Next, we will demonstrate the uniqueness of the fixed point. Suppose that T and ω are two fixed points of S. Since S satisfies the hypothesis (iv) of Theorem 13, S is an weak E-Pata convex contractive mapping; we have

$$\begin{aligned}\rho(\omega, T) &\le \alpha(\omega, T)\rho\left(S^2\omega, S^2T\right)\\ &\le (1-\varepsilon)(E_I(\omega, T) - \phi(E_I(\omega, T))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(\omega, T)\\ &\le (1-\varepsilon)\begin{pmatrix}\rho(S\omega, ST) + \left|\rho\left(S\omega, S^2\omega\right) - \rho\left(ST, S^2T\right)\right|\\ -\phi(\rho(S\omega, ST) + \left|\rho\left(S\omega, S^2\omega\right) - \rho\left(ST, S^2T\right)\right|\end{pmatrix}\Bigg)\\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|\omega\| + \|T\| + \|S\omega\| + \|ST\| + \left\|S^2\omega\right\|\right.\\ &\quad \left. + \left\|S^2T\right\|\right]^{\vartheta} \le (1-\varepsilon)\rho(\omega, T) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|\omega\| + 3\|T\|]^{\vartheta}.\end{aligned} \tag{52}$$

We obtain that $\rho(\omega, T) < K\psi(\varepsilon)$ for some $K \ge 0$, and so, we get $\omega = T$. Hence, S has a unique fixed point in W, that is $\omega = S\omega$, $\omega \in W$.

Following this line of work, Theorem 14 does not require the continuity of S. □

Theorem 14. *Let (W, ρ) be a complete metric space, $\alpha : W \times W \longrightarrow [0,+\infty)$ and $S : W \longrightarrow W$ be a weak Pata-convex mapping. Suppose that*

(i) S is triangular α – admissible

(ii) there exists $w_0 \in W$ such that $\alpha(w_0, Sw_0) \ge 1$

(iii) S^2 is continuous and for all $\omega \in FixS^2$, $\alpha(S\omega, \omega) \ge 1$

(iv) for all $w, \omega \in FixS^2$, $\alpha(w, \omega) \ge 1$

Then, S has a unique fixed point in W.

Proof. Following the proof of Theorem 13, we have already proved that $\{w_n\}$ is a Cauchy sequence in W. Since W is complete, we have $w_n \longrightarrow \omega \in W$ as $n \longrightarrow +\infty$. Taking into account hypothesis (iii) Theorem 14, we have $\lim_{n\longrightarrow\infty} \rho(w_n, S^2\omega) = \lim_{n\longrightarrow\infty}\rho(S^2 w_{n-2}, S^2\omega) = 0$. In the uniqueness of the limit, we obtain that $S^2\omega = \omega$. Next, we will prove that $\omega = S\omega$. On the contrary, we assume that ω is not fixed point of S. So, we have

$$\begin{aligned}0 &< \rho(S\omega, \omega) = \rho\left(S^2(S\omega), S^2\omega\right) \le \alpha(S\omega, \omega)\rho\left(S^2(S\omega), S^2\omega\right)\\ &\le (1-\varepsilon)(E_I(S\omega, \omega) - \phi(E_I(S\omega, \omega))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(S\omega, \omega)\\ &\le (1-\varepsilon)\begin{pmatrix}\rho\left(S\omega, S^2\omega\right) + \left|\rho\left(S\omega, S^2\omega\right) - \rho\left(S^2\omega, S^3\omega\right)\right|\\ -\phi\left(\rho\left(S\omega, S^2\omega\right) + \left|\rho\left(S\omega, S^2\omega\right) - \rho\left(S^2\omega, S^3\omega\right)\right|\right)\end{pmatrix}\\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|S\omega\| + \|\omega\| + \|S\omega\| + \left\|S^2\omega\right\| + \left\|S^3\omega\right\| + \left\|S^2\omega\right\|\right]^{\vartheta}\\ &\le (1-\varepsilon)\rho(S\omega, \omega) - \phi(\rho(S\omega, \omega)) + K\varepsilon^{\xi}\psi(\varepsilon),\end{aligned} \tag{53}$$

for some $K > 0$. We obtain

$$\rho(S\omega, \omega) < (1-\varepsilon)\rho(S\omega, \omega) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{54}$$

For $\varepsilon = 0$ in (54) which is a contradiction. Thus, we make an inference that $S\omega = \omega$, and so, ω is a fixed point of S. Following the proof of Theorem 13, the uniqueness of fixed point of S can be obtained.

Theorem 15 is other fundamental result of our work. □

Theorem 15. *Let (W, ρ) be a complete metric space, $\alpha : W \times W \longrightarrow [0,+\infty)$ and $S : W \longrightarrow W$ be a weak Pata convex contractive mapping via simulation function. On the assumption that all of the Theorem 13 hypotheses are satisfied, then h has a unique fixed point.*

Proof. In the proof of Theorem 13, we have got that

$$\alpha\left(S^n w_0, S^{n+1}w_0\right) \ge 1 \text{ for all } n \in \mathbb{N} \text{ and } \alpha(S^n w_0, S^m w_0) \ge 1 \text{ for all } m > n \ge 0. \tag{55}$$

Setting $\ell = \min\{\rho(w_0, Sw_0), \rho(Sw_0, S^2w_0)\}$ and now, we demonstrate that

$\{\rho(S^n w_0, S^{n+1}w_0)\}$ is a nonincreasing sequence. Since S is a weak Pata convex contractive mapping via simulation function, we have

$$\begin{aligned}&\zeta\left(\alpha(w_0, Sw_0)\rho\left(S^2w_0, S^3w_0\right), (1-\varepsilon)(M_I(w_0, Sw_0)\right.\\ &\quad \left. - \phi(M_I(w_0, Sw_0))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_0, Sw_0)\right) \ge 0.\end{aligned} \tag{56}$$

Using hypothesis (ii) of the Theorem 15, we get

$$\begin{aligned}
\rho(S^2w_0, S^3w_0) &\le \alpha(w_0, Sw_0)\rho(S^2w_0, S^3w_0) \le (1-\varepsilon) \\
&\quad \cdot (M_I(w_0, Sw_0) - \phi(M_I(w_0, Sw_0)) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_0, Sw_0) \\
&= (1-\varepsilon)\left(\begin{array}{c} \max\left\{\begin{array}{c} \rho(w_0, Sw_0), \rho(Sw_0, S^2w_0), \rho(w_0, Sw_0), \\ \rho(Sw_0, S^2w_0), \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0) \end{array}\right\} \\ -\phi\left(\max\left\{\begin{array}{c} \rho(w_0, Sw_0), \rho(Sw_0, S^2w_0), \\ \rho(w_0, Sw_0), \rho(Sw_0, S^2w_0), \\ \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0) \end{array}\right\}\right) \end{array}\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|w_0\| + 2\|Sw_0\| + 2\|S^2w_0\| + \|S^3w_0\|\right]^{\vartheta} \\
&\le (1-\varepsilon)\left(\begin{array}{c} \max\{\rho(w_0, Sw_0), \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0)\} \\ -\phi(\max\{\rho(w_0, Sw_0), \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0)\}) \end{array}\right) \\
&\quad + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{57}$$

for some $K > 0$. Assuming that $\max\{\ell, \rho(S^2w_0, S^3w_0)\} = \rho(S^2w_0, S^3w_0)$, then we have $\rho(Sw_0, S^2w_0) < \rho(S^2w_0, S^3w_0)$. Thus, we have

$$\begin{aligned}
\rho(S^2w_0, S^3w_0) &\le (1-\varepsilon)\left(\rho(S^2w_0, S^3w_0) - \phi(\rho(S^2w_0, S^3w_0))\right) \\
&\quad + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{58}$$

and since $\rho(S^2w_0, S^3w_0) \ge \rho(S^2w_0, S^3w_0) - \phi(\rho(S^2w_0, S^3w_0))$, we have

$$\rho(S^2w_0, S^3w_0) < (1-\varepsilon)\rho(S^2w_0, S^3w_0) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{59}$$

The inequality (59) is true for all $\varepsilon \in [0, 1]$. For $\varepsilon = 0$, we obtain $\rho(S^2w_0, S^3w_0) < \rho(S^2w_0, S^3w_0)$ which is a contradiction. Hence, we obtain

$$\rho(S^2w_0, S^3w_0) \le \ell. \tag{60}$$

Analogously, since S is a weak Pata convex *c*ontractive mapping via simulation function, we have

$$\begin{aligned}
&\zeta\left(\begin{array}{c} \alpha(Sw_0, S^2w_0)\rho(S^3w_0, S^4w_0), \\ (1-\varepsilon)(M_I(Sw_0, S^2w_0) - \phi(M_I(Sw_0, S^2w_0))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(Sw_0, S^2w_0) \end{array}\right) \ge 0, \\
&(1-\varepsilon)\left(\begin{array}{c} M_I(Sw_0, S^2w_0) \\ -\phi(M_I(Sw_0, S^2w_0)) \end{array}\right) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(Sw_0, S^2w_0) - \alpha(Sw_0, S^2w_0)\rho(S^3w_0, S^4w_0) \ge 0,
\end{aligned} \tag{61}$$

and we can write that

$$\begin{aligned}
\rho(S^3w_0, S^4w_0) &\le \alpha(Sw_0, S^2w_0)\rho(S^3w_0, S^4w_0) \\
&\le (1-\varepsilon)\left(\begin{array}{c} \max\left\{\begin{array}{c} \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0), \rho(Sw_0, S^2w_0), \\ \rho(S^2w_0, S^3w_0), \rho(S^2w_0, S^3w_0), \rho(S^3w_0, S^4w_0) \end{array}\right\} \\ -\phi\left(\max\left\{\begin{array}{c} \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0), \\ \rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0), \\ \rho(S^2w_0, S^3w_0), \rho(S^3w_0, S^4w_0) \end{array}\right\}\right) \end{array}\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|Sw_0\| + \|S^2w_0\| + \|S^2w_0\| + \|S^3w_0\| + \|S^3w_0\| + \|S^4w_0\|\right]^{\vartheta} \\
&\le (1-\varepsilon)\left(\max\{\rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0), \rho(S^3w_0, S^4w_0)\}\right) + K\varepsilon^{\xi}\psi(\varepsilon)
\end{aligned} \tag{62}$$

for some $K > 0$. In case that

$$\begin{aligned}
&\max\{\rho(Sw_0, S^2w_0), \rho(S^2w_0, S^3w_0), \rho(S^3w_0, S^4w_0)\} \\
&\quad = \rho(S^3w_0, S^4w_0),
\end{aligned} \tag{63}$$

then we have

$$\rho(S^3w_0, S^4w_0) < (1-\varepsilon)\rho(S^3w_0, S^4w_0) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{64}$$

The inequality (64) is true for all $\varepsilon \in [0, 1]$. For $\varepsilon = 0$, we obtain $\rho(S^3w_0, S^4w_0) < \rho(S^3w_0, S^4w_0)$ is again a contradiction. Therefore, we obtain

$$\rho(S^3w_0, S^4w_0) \le \rho(S^2w_0, S^3w_0) \le \ell. \tag{65}$$

Again, by induction, since S is a weak Pata convex *c*ontractive mapping via simulation function, we have

$$\begin{aligned}
&\zeta\left(\begin{array}{c} \alpha(S^{n-2}w_0, S^{n-1}w_0)\rho(S^nw_0, S^{n+1}w_0), (1-\varepsilon) \\ \cdot\left(\begin{array}{c} M_I(S^{n-2}w_0, S^{n-1}w_0) \\ -\phi(M_I(S^{n-2}w_0, S^{n-1}w_0)) \end{array}\right) \\ + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(S^{n-2}w_0, S^{n-1}w_0) \end{array}\right) \ge 0, \\
&\left(\begin{array}{c} (1-\varepsilon)\left(\begin{array}{c} M_I(S^{n-2}w_0, S^{n-1}w_0) \\ -\phi(M_I(S^{n-2}w_0, S^{n-1}w_0)) \end{array}\right) \\ + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(S^{n-2}w_0, S^{n-1}w_0) \\ - \alpha(S^{n-2}w_0, S^{n-1}w_0)\rho(S^nw_0, S^{n+1}w_0) \end{array}\right) \ge 0,
\end{aligned} \tag{66}$$

and we have that

$$\begin{aligned}
\rho\left(S^n w_0, S^{n+1} w_0\right) &\le \alpha\left(S^{n-2} w_0, S^{n-1} w_0\right)\rho\left(S^n w_0, S^{n+1} w_0\right) \\
&\le (1-\varepsilon)\left(\begin{array}{c} \max\left\{\begin{array}{c} \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^{n-1}w_0, S^n w_0\right), \rho\left(S^n w_0, S^{n+1} w_0\right) \end{array}\right\} \\ \phi\left(\max\left\{\begin{array}{c} \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^{n-1}w_0, S^n w_0\right), \rho\left(S^n w_0, S^{n+1} w_0\right) \end{array}\right\}\right) \end{array}\right) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \left\|S^{n-2}w_0\right\| + \left\|S^{n-1}w_0\right\| + \left\|S^{n-1}w_0\right\| + \left\|S^n w_0\right\| + \left\|S^n w_0\right\| + \left\|S^{n+1}w_0\right\|\right]^{\vartheta} \\
&< (1-\varepsilon)\left(\begin{array}{c} \max\left\{\begin{array}{c} \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^n w_0, S^{n+1} w_0\right) \end{array}\right\} \\ -\phi\left(\max\left\{\begin{array}{c} \rho\left(S^{n-2}w_0, S^{n-1}w_0\right), \rho\left(S^{n-1}w_0, S^n w_0\right), \\ \rho\left(S^n w_0, S^{n+1} w_0\right) \end{array}\right\}\right) \end{array}\right) + K\varepsilon^{\xi}\psi(\varepsilon),
\end{aligned} \tag{67}$$

for some $K > 0$. In case that $\max\{\rho(S^{n-2}w_0, S^{n-1}w_0), \rho(S^{n-1}w_0, S^n w_0), \rho(S^n w_0, S^{n+1}w_0)\} = \rho(S^n w_0, S^{n+1}w_0)$, then we have

$$\rho\left(S^n w_0, S^{n+1} w_0\right) < (1-\varepsilon)\rho\left(S^n w_0, S^{n+1} w_0\right) + K\varepsilon^{\xi}\psi(\varepsilon). \tag{68}$$

Again, the inequality (68) is true for all $\varepsilon \in [0, 1]$ and for $\varepsilon = 0$, we obtain $\rho(S^n w_0, S^{n+1}w_0) < \rho(S^n w_0, S^{n+1}w_0)$ is again a contradiction. Consequently, we can find that

$$\begin{aligned}
\rho\left(S^n w_0, S^{n+1} w_0\right) &\le \rho\left(S^{n-1} w_0, S^n w_0\right) \le \cdots \le \rho\left(S^3 w_0, S^4 w_0\right) \\
&\le \rho\left(S^2 w_0, S^3 w_0\right) \le \rho\left(S w_0, S^2 w_0\right).
\end{aligned} \tag{69}$$

Starting at the point $w_0 \in W$, the sequence $\{w_n\}$ is constructed by $w_n = Sw_{n-1} = S^n w_0$, $n \ge 1$. If $w_{n_0+1} = w_{n_0}$ for any $n_0 \in \mathbb{N}$, then w_{n_0} is a fixed point of S. Hereby, assume that $w_{n_0+1} \ne w_{n_0}$ for all $n_0 \in \mathbb{N}$ and let $\rho_n = \rho(w_{n-1}, w_n)$. Therefore, we get that $\{\rho_n\}$ is a nonincreasing sequence. Thereupon, there exists a $\delta \ge 0$ such that

$$\lim_{n\longrightarrow\infty} \rho(w_{n-1}, w_n) = \lim_{n\longrightarrow\infty} \rho_n = \delta. \tag{70}$$

We will demostrate that $\delta = 0$. For this, we should demostrate that the sequence $\{\|w_n\|\}$ is bounded. Since $\{\rho_n\}$ is a nonincreasing sequence, we have

$$\begin{aligned}
\rho_{n+1} = \rho(w_n, w_{n+1}) &\le \rho(w_{n-1}, w_n) \le \cdots \le \rho(w_3, w_4) \\
&\le \rho(w_2, w_3) \le \rho(w_1, w_2) = \rho_2 \le \|w_1\| + \|w_2\|.
\end{aligned} \tag{71}$$

From the triangle inequality, we can write

$$\begin{aligned}
\|w_n\| = \rho(w_n, w_0) &\le \rho(w_n, w_{n+1}) + \rho(w_{n+1}, w_2) + \rho(w_2, w_0) \\
&= \rho_{n+1} + \rho(w_{n+1}, w_2) + \|w_2\| \le \rho_2 + \rho(w_{n+1}, w_2) + \|w_2\| \\
&\le \|w_1\| + 2\|w_2\| + \rho(w_{n+1}, w_2).
\end{aligned} \tag{72}$$

Since S is a weak Pata convex contractive mapping via simulation function, we have

$$\begin{aligned}
&\zeta(\alpha(w_n, w_0)\rho(w_{n+1}, w_2), (1-\varepsilon)(M_I(w_{n-1}, w_0) \\
&\quad - \phi(M_I(w_{n-1}, w_0))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_0)) \ge 0, \\
&\left((1-\varepsilon)\left(\begin{array}{c} M_I(w_{n-1}, w_0) \\ -\phi(M_I(w_{n-1}, w_0)) \end{array}\right) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_0) \right. \\
&\quad \left. - \alpha(w_n, w_0)\rho(w_{n+1}, w_2)\right) \ge 0.
\end{aligned} \tag{73}$$

Together with (71), we obtain that

$$\begin{aligned}
\rho(w_{n+1}, w_2) &\le \alpha(w_n, w_0)\rho(w_{n+1}, w_2) \le (1-\varepsilon) \\
&\quad \cdot (M_I(w_{n-1}, w_0) - \phi(M_I(w_{n-1}, w_0))) \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1}, w_0).
\end{aligned} \tag{74}$$

From (71) and $\rho_2 \le \|w_1\| + \|w_2\|$, we have

$$\begin{aligned}
M_I(w_{n-1}, w_0) &= \max\left\{\begin{array}{c} \rho(w_{n-1}, w_0), \rho(w_n, w_1), \rho(w_{n-1}, w_n), \\ \rho(w_0, w_1), \rho(w_n, w_{n+1}), \rho(w_1, w_2) \end{array}\right\} \\
&= \max\{\rho(w_{n-1}, w_0), \rho(w_n, w_1), \rho_n, \rho_1, \rho_{n+1}, \rho_2\} \\
&\le \|w_1\| + \|w_2\| + \|w_n\|, \\
P_I(w_{n-1}, w_0) &= [1 + \|w_{n-1}\| + \|w_0\| + \|w_n\| + \|w_1\| + \|w_{n+1}\| + \|w_2\|]^{\vartheta} \\
&\le [1 + 3\|w_1\| + 3\|w_2\| + 3\|w_n\|]^{\vartheta}.
\end{aligned} \tag{75}$$

Now, we derive that

$$\begin{aligned}
\varepsilon\|w_n\| &< (2-\varepsilon)\|w_1\| + (3-\varepsilon)\|w_2\| + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|w_1\| \\
&\quad + 3\|w_2\| + 3\|w_n\|]^{\vartheta} \le (2-\varepsilon)\|w_1\| + (3-\varepsilon)\|w_2\| \\
&\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1 + 3\|w_n\|)^{\xi}\left(\frac{1 + 3\|w_1\| + 3\|w_2\|}{1 + 3\|w_n\|}\right)^{\xi} \\
&\le 2\|w_1\| + 3\|w_2\| + \Lambda\varepsilon^{\xi}\psi(\varepsilon)3^{\xi}\|w_n\|^{\xi}\left(\frac{1}{3\|w_n\|} + 1\right)^{\xi} \\
&\quad \cdot (1 + 3\|w_1\| + 3\|w_2\|)^{\xi}.
\end{aligned} \tag{76}$$

Contrarily, supposing that $\{\|w_n\|\}$ is not bounded sequence. Thence, there exists a subsequence $\{\|w_{n_j}\|\}$ of $\{\|w_n\|\}$ such that $\lim_{j\longrightarrow\infty} w_{n_j} = \infty$. If we take $\varepsilon = \varepsilon_j = (1 + 3\|w_1\| + 3\|w_2\|)/\|w_{n_j}\|$ in (76) inequality, then we have

$$1 \leq \Lambda 3^{\xi}\left(\varepsilon^{\xi}\|w_n\|^{\xi}\right)(1+3\|w_1\|+3\|w_2\|)^{\xi}\left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi} \cdot \psi(\varepsilon_j) \leq \Lambda 3^{\xi}(1+3\|w_1\|+3\|w_2\|)^{\xi}(1+3\|w_1\|+3\|w_2\|)^{\xi} \cdot \left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi}\psi(\varepsilon_j) \leq \Lambda 3^{\xi}(1+3\|w_1\|+3\|w_2\|)^{2\xi} \cdot \left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi}\psi(\varepsilon_j). \tag{77}$$

If we take limit in (77) inequality as $j \longrightarrow \infty$, then we get that

$$\Lambda 3^{\xi}(1+3\|w_1\|+3\|w_2\|)^{2\xi}\left(\frac{1}{3\left\|w_{n_j}\right\|}+1\right)^{\xi}\psi(\varepsilon_j) \longrightarrow 0 \tag{78}$$

is a contradiction. Next, we show that the sequence$\{\|w_n\|\}$ is bounded. So, there exists $A>0$ such that $\|w_n\| \leq A$ for all $n \in \mathbb{N}$. Following this line of work, we will demonstrate that $\delta = 0$. Since S is a weak Pata convex *c*ontractive mapping via simulation function, we have

$$\zeta\left(\alpha(w_{n-1},w_n)\rho(w_{n+1},w_{n+2}),(1-\varepsilon)\begin{pmatrix} M_I(w_{n-1},w_n) \\ -\phi(M_I(w_{n-1},w_n)) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1},w_n)\right) \geq 0,$$
$$\left((1-\varepsilon)\begin{pmatrix} M_I(w_{n-1},w_n) \\ -\phi(M_I(w_{n-1},w_n)) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1},w_n) - \alpha(w_{n-1},w_n)\rho(w_{n+1},w_{n+2})\right) \geq 0, \tag{79}$$

where

$$M_I(w_{n-1},w_n) = \max\left\{\begin{matrix} \rho(w_{n-1},w_n),\rho(w_n,w_{n+1}),\rho(w_{n-1},w_n), \\ \rho(w_n,w_{n+1}),\rho(w_n,w_{n+1}),\rho(w_{n+1},w_{n+2}) \end{matrix}\right\} = \max\{\rho(w_{n-1},w_n),\rho(w_n,w_{n+1}),\rho(w_{n+1},w_{n+2})\} \leq \|w_1\|+\|w_2\|+\|w_n\|. \tag{80}$$

Since the sequence $\{\|w_n\|\}$ is bounded, we have

$$P_I(w_{n-1},w_n) = \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1+\|w_{n-1}\|+\|w_n\|+\|w_n\| + \|w_{n+1}\|+\|w_{n+2}\|+\|w_{n+3}\|]^{\vartheta} \leq \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1+6A)^{\vartheta}. \tag{81}$$

Therefore, we have

$$\rho_{n+2} = \rho(w_{n+1},w_{n+2}) \leq \alpha(w_{n-1},w_n)\rho(w_{n+1},w_{n+2}) \leq (1-\varepsilon)(M_I(w_{n-1},w_n)-\phi(M_I(w_{n-1},w_n))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w_{n-1},w_n) \leq (1-\varepsilon) \cdot \begin{pmatrix} \max\{\rho(w_{n-1},w_n),\rho(w_n,w_{n+1}),\rho(w_{n+1},w_{n+2})\} \\ -\phi(\max\{\rho(w_{n-1},w_n),\rho(w_n,w_{n+1}),\rho(w_{n+1},w_{n+2})\}) \end{pmatrix} + \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1+6A)^{\vartheta}. \tag{82}$$

If the limit is taken as $n \longrightarrow \infty$ in (82) inequality, then we get

$$\delta \leq (1-\varepsilon)(\delta-\phi(\delta)) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)(1+6A)^{\vartheta}$$
$$\delta \leq \Lambda\varepsilon^{\xi-1}\psi(\varepsilon)(1+6A)^{\vartheta}. \tag{83}$$

$\delta \leq 0$ as $\varepsilon \longrightarrow 0$, that is $\lim_{n\longrightarrow\infty}\rho(w_{n+1},w_{n+2}) = \delta = 0$. Now, we demonstrate that $\{w_n\}$ is a Cauchy sequence. Contrarily, supposing that the sequence $\{w_n\}$ is not a Cauchy. From Lemma 10, we say that there exist subsequence $\{w_{m_j}\}$ and $\{w_{n_j}\}$ with $n_j > m_j > j$ such that $\lim_{k\longrightarrow\infty}\rho(x_{m_k-1},x_{n_k+1}) = \varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k},x_{n_k}) = \varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k-1},x_{n_k}) = \varsigma$, $\lim_{k\longrightarrow\infty}\rho(x_{m_k+1},x_{n_k+1}) = \varsigma$, and $\lim_{k\longrightarrow\infty}\rho(x_{m_k},x_{n_k-1}) = \varsigma$. Since S is a weak Pata convex contractive mapping, we have

$$\zeta\left(\alpha\left(w_{n_j-1},w_{m_j-1}\right)\rho\left(w_{n_j+1},w_{m_j+1}\right),(1-\varepsilon) \cdot \left(M_I\left(w_{n_j-1},w_{m_j-1}\right)-\phi\left(M_I\left(w_{n_j-1},w_{m_j-1}\right)\right)\right) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(w_{n_j-1},w_{m_j-1}\right)\right) \geq 0,$$
$$\left((1-\varepsilon)\left(M_I\left(w_{n_j-1},w_{m_j-1}\right)-\phi\left(M_I\left(w_{n_j-1},w_{m_j-1}\right)\right)\right) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(w_{n_j-1},w_{m_j-1}\right) - \alpha\left(w_{n_j-1},w_{m_j-1}\right) \cdot \rho\left(w_{n_j+1},w_{m_j+1}\right)\right) \geq 0 \tag{84}$$

where

$$M_I\left(w_{n_j-1},w_{m_j-1}\right) = \max\left\{\begin{matrix} \rho\left(w_{n_j-1},w_{m_j-1}\right),\rho\left(w_{n_j},w_{m_j}\right),\rho\left(w_{n_j-1},w_{n_j}\right), \\ \rho\left(w_{m_j-1},w_{m_j}\right),\rho\left(w_{n_j},w_{n_j+1}\right),\rho\left(w_{m_j},w_{m_j+1}\right) \end{matrix}\right\},$$

$$P_I\left(w_{n_j-1}, w_{m_j-1}\right) = \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \left\|w_{n_j-1}\right\| + \left\|w_{m_j-1}\right\| + \left\|w_{n_j}\right\| + \left\|w_{m_j}\right\| + \left\|w_{n_j+1}\right\| + \left\|w_{m_j+1}\right\|\right]^{\vartheta} = \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 6A]^{\vartheta}. \tag{85}$$

Now, we can write

$$\begin{aligned}\varsigma &\le \rho\left(w_{n_j+1}, w_{m_j+1}\right) \le \alpha\left(w_{n_j-1}, w_{m_j-1}\right)\rho\left(w_{n_j+1}, w_{m_j+1}\right)\\ &\le (1-\varepsilon)\left(M_I\left(w_{n_j-1}, w_{m_j-1}\right) - \phi\left(M_I\left(w_{n_j-1}, w_{m_j-1}\right)\right)\right) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I\left(w_{n_j-1}, w_{m_j-1}\right)\\ &\le (1-\varepsilon)\left(\max\left\{\begin{array}{c}\rho\left(w_{n_j-1}, w_{m_j-1}\right), \rho\left(w_{n_j}, w_{m_j}\right),\\ \rho\left(w_{n_j-1}, w_{n_j}\right), \rho\left(w_{m_j-1}, w_{m_j}\right),\\ \rho\left(w_{n_j}, w_{n_j+1}\right), \rho\left(w_{m_j}, w_{m_j+1}\right)\end{array}\right\} - \phi\left(\max\left\{\begin{array}{c}\rho\left(w_{n_j-1}, w_{m_j-1}\right), \rho\left(w_{n_j}, w_{m_j}\right),\\ \rho\left(w_{n_j-1}, w_{n_j}\right), \rho\left(w_{m_j-1}, w_{m_j}\right),\\ \rho\left(w_{n_j}, w_{n_j+1}\right), \rho\left(w_{m_j}, w_{m_j+1}\right)\end{array}\right\}\right)\right)\\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 6A]^{\vartheta}.\end{aligned} \tag{86}$$

If we take the limit as $j \longrightarrow \infty$, we get

$$\varsigma \le (1-\varepsilon)(\varsigma - \phi(\varsigma)) + K\varepsilon\psi(\varepsilon) \le (1-\varepsilon)\varsigma + K\varepsilon\psi(\varepsilon), \tag{87}$$

and so, we have

$$\varsigma \le K\psi(\varepsilon), \tag{88}$$

that is, we get $\varsigma = 0$ which is a contradiction. Therefore, we concluded that $\{w_n\}$ is a Cauchy sequence in (W, ρ). By the completeness of W, the sequence $\{w_n\}$ is convergent to some $\omega \in W$ that is $w_n \longrightarrow \omega$ as $n \longrightarrow +\infty$. Since S is continuous, $Sw_n \longrightarrow S\omega$ as $n \longrightarrow +\infty$. By the uniqueness of the limit, we obtain $\omega = S\omega$ that is ω is a fixed point of S.

Now, we will demonstrate that the fixed point is unique. Assuming that T and ω are two fixed points of S. From hypothesis (iv) of Theorem 15 and since S is an a weak Pata convex contractive mapping via simulation function, we have

$$\begin{aligned}&\zeta\left(\alpha(\omega, T)\rho\left(S^2\omega, S^2T\right), (1-\varepsilon)(M_I(\omega, T) - \phi(M_I(\omega, T))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(\omega, T)\right) \ge 0,\\ &(1-\varepsilon)(M_I(\omega, T) - \phi(M_I(\omega, T))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(\omega, T) - \alpha(\omega, T)\rho\left(S^2\omega, S^2T\right) \ge 0,\end{aligned} \tag{89}$$

and so, we have

$$\begin{aligned}\rho(\omega, T) &\le \alpha(\omega, T)\rho\left(S^2\omega, S^2T\right)\\ &\le (1-\varepsilon)(M_I(\omega, T) - \phi(M_I(\omega, T))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(\omega, T)\\ &\le (1-\varepsilon)\left(\max\left\{\begin{array}{c}\rho(\omega, T), \rho(S\omega, ST), \rho(\omega, S\omega),\\ \rho(T, ST), \rho\left(S\omega, S^2\omega\right), \rho\left(ST, S^2T\right)\end{array}\right\} - \phi\left(\max\left\{\rho(\omega, T), \rho(S\omega, ST), \rho(\omega, S\omega), \rho(T, ST), \rho\left(S\omega, S^2\omega\right), \rho\left(ST, S^2T\right)\right\}\right)\right)\\ &\quad + \Lambda\varepsilon^{\xi}\psi(\varepsilon)\left[1 + \|\omega\| + \|T\| + \|S\omega\| + \|ST\| + \left\|S^2\omega\right\| + \left\|S^2T\right\|\right]^{\vartheta}\\ &\le (1-\varepsilon)\rho(\omega, T) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)[1 + 3\|\omega\| + 3\|T\|]^{\vartheta}.\end{aligned} \tag{90}$$

We obtain that $\rho(\omega, T) < K\psi(\varepsilon)$ for some $K \ge 0$, and thus, we get $\omega = T$. Hence, S has a unique fixed point in W. □

Example 16. Let $(W, |.|)$ the usual metric space where $W = [0, (3/2)]$. Let define the mappings $S : W \longrightarrow W$ by

$$S(w) = \begin{cases}\dfrac{w^2 + 1}{3}, & w \in [0, 1),\\[2mm] \dfrac{1}{2}, & w \in \left[1, \dfrac{3}{2}\right],\end{cases} \tag{91}$$

$\phi : [0,+\infty) \longrightarrow 0, +\infty)$ by $\phi(w) = w/10$ and $\alpha : W \times W \longrightarrow [0,+\infty)$ by

$$\alpha(w, v) = \begin{cases}1, & w, v \in [0, 1],\\ 0, & w, v \notin [0, 1].\end{cases} \tag{92}$$

It is easily seen that S is a triangular α -admissible mapping, and also, $S^2w = (w^4 + 2w^2 + 10)/27, w \in [0, (3/2)]$. Though the mapping, S is discontinuous in $x = 1$ and S^2 is continuous on $W = [0, (3/2)]$. Now, we want to demonstrate that S satisfies (11). For $w, v \in [0, 1]$, we have

$$\begin{aligned}\rho\left(S^2w, S^2v\right) &= \left|\frac{w^4 + 2w^2}{27} - \frac{v^4 + 2v^2}{27}\right| \le \frac{2}{9}|w - v| + \frac{1}{2}|Sw - Sv|\\ &= \frac{3}{4}\left(\frac{8}{27}|w - v| + \frac{2}{3}|Sw - Sv|\right)\\ &\le \frac{3}{4}\max\{|w - v|, |Sw - hv|\} \le \frac{3}{4}M_I(w, v).\end{aligned} \tag{93}$$

Since $\phi(w) = w/10$ and $\alpha(w, v) = 1$, for $w, v \in [0, 1]$, we get that

$$\begin{aligned}\alpha(w, v)\rho\left(S^2w, S^2v\right) &\le \frac{5}{6}\frac{9}{10}(M_I(w, v))\\ &= \frac{5}{6}(M_I(w, v) - \phi(M_I(w, v))).\end{aligned} \tag{94}$$

For arbitrary $\varepsilon \in [0, 1]$, as one can see, the above inequality turns into the following inequality,

$$\begin{aligned}\alpha(w,v)\rho\left(S^2w,S^2v\right) &\le (1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v)))\\ &+\left(\frac{3}{4}+\varepsilon-1\right)M_I(w,v)\le(1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v)))\\ &+\left(\frac{3}{4}+\varepsilon-1\right)\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right].\end{aligned} \tag{95}$$

Now, our goal is to show that $\gamma \ge 0$ and $\Lambda \ge 0$ such that

$$\begin{aligned}&\left(\frac{3}{4}+\varepsilon-1\right)\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right]\\ &\quad\le \Lambda\varepsilon^{\gamma+1}\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right]\end{aligned} \tag{96}$$

holds for all $w, v \in [0,1]$, and every $0 \le \varepsilon \le 1$. We can find $\Lambda \ge 0$ such that

$$\Lambda = \frac{((3/4)+\varepsilon-1)}{\varepsilon^{\gamma+1}} \tag{97}$$

holds for each $0 \le \varepsilon \le 1$ and some $\gamma \ge 0$. If we choose γ such that $(\gamma/(\gamma+1)) > 1-(3/4)$, then

$$\Lambda = \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}(1-(3/4))^\gamma}. \tag{98}$$

Hence, we have that

$$\begin{aligned}\alpha(w,v)\rho\left(S^2w,S^2v\right) &\le (1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v)))+\Lambda\varepsilon^{\gamma+1}\\ &\cdot\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right].\end{aligned} \tag{99}$$

Now, we can write

$$\begin{aligned}&\Big((1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v)))+\Lambda\varepsilon^{\gamma+1}\\ &\quad\cdot\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right]\\ &\quad-\alpha(w,v)\rho\left(S^2w,S^2v\right)\Big)\ge 0,\end{aligned} \tag{100}$$

and for $\zeta \in Z'$, we have

$$\zeta\left(\begin{array}{c}\alpha(w,v)\rho\left(S^2w,S^2v\right),\\ (1-\varepsilon)(M_I(w,v)-\phi(M_I(w,v)))+\Lambda\varepsilon^{\gamma+1}\left[1+\|w\|+\|v\|+\|Sw\|+\|Sv\|+\left\|S^2w\right\|+\left\|S^2v\right\|\right]\end{array}\right)\ge 0, \tag{101}$$

which satisfies for each $\varepsilon > 0$ and all $w, v \in [0,1]$. If $\varepsilon = 0$, it can be seen that (11) is satisfied. Hence, all conditions of Theorem 15 are satisfied with $\xi = \vartheta = 1$ and $\psi(\varepsilon) = \varepsilon^\gamma$. By an application of Theorem 15, S has a unique fixed point in $W = [0,(3/2)]$.

Suppose that $\varepsilon = 0$ in Theorems 13 and 15; then we obtain the following corollaries.

Corollary 17. *Let (W,ρ) be a complete metric space and $\zeta \in Z'$ and $S: W \longrightarrow W$ be two functions. If for all $w, v \in W$, there exists a function, $\alpha: W\times W \longrightarrow [0,+\infty)$ such that S satisfies the inequality either*

$$\begin{aligned}&\zeta\left(\alpha(w,v)\rho\left(S^2w,S^2v\right),E_I(w,v)-\phi(E_I(w,v))\right)\ge 0\\ &\text{or}\ \zeta\left(\alpha(w,v)\rho\left(S^2w,S^2v\right),M_I(w,v)-\phi(M_I(w,v))\right)\ge 0,\end{aligned} \tag{102}$$

where $\phi: [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(w) > 0$, for all $w > 0$, and assuming that all of the hypotheses of Theorem 13 are satisfied, then S has a unique fixed point.

Karapinar's contractive conditions [10, 32, 45] are a special case of ours, and also, Corollary 17 generalizes the results of Samet [17] and Istratescu [26–28].

Corollary 18. *Let (W,ρ) be a complete metric space and $S: W \longrightarrow W$ be a function. If for all $w, v \in W$, there exist two functions, $\alpha: W\times W \longrightarrow [0,+\infty)$ such that S satisfies the inequality either*

$$\begin{aligned}&\alpha(w,v)\rho\left(S^2w,S^2v\right)\le E_I(w,v)-\phi(E_I(w,v))\\ &\text{or}\ \alpha(w,v)\rho\left(S^2w,S^2v\right)\le M_I(w,v)-\phi(M_I(w,v)),\end{aligned} \tag{103}$$

where $\phi: [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous and nondecreasing function with $\phi(0) = 0$ and $\phi(w) > 0$, for all $w > 0$, and assuming that all of the hypotheses of Theorem 13 are satisfied, then S has a unique fixed point.

In comparison with recent results such as Alber et al. [14] and Zhang [16], our results are a generalization of them.

Corollary 19. *Let (W,ρ) be a complete metric space and $S: W \longrightarrow W$ be a function. If for all $w, v \in W$, there exists a function $\alpha: W\times W \longrightarrow [0,+\infty)$ such that S satisfies the inequality either*

$$\begin{aligned}&\alpha(w,v)\rho\left(S^2w,S^2v\right)\le E_I(w,v)\\ &\text{or}\ \alpha(w,v)\rho\left(S^2w,S^2v\right)\le M_I(w,v),\end{aligned} \tag{104}$$

and assuming that all of the hypotheses of Theorem 13 are satisfied, then h has a unique fixed point.

Putting $\alpha(w,v) = 1$ in Theorems 13 and 15, we can see the following results.

Corollary 20. *Let (W,ρ) be a complete metric space and $\zeta \in Z'$ and $S: W \longrightarrow W$ be two functions. If for all $w, v \in W$, and $\varepsilon \in [0,1]$, there exists a function $\psi \in \Psi$, such that S satisfies the inequality either*

$$\zeta\left(\rho\left(S^2w, S^2v\right), (1-\varepsilon)(E_I(w,v) - \phi(E_I(w,v))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w,v)\right) \geq 0$$
$$\text{or}\ \zeta\left(\rho\left(S^2w, S^2v\right), (1-\varepsilon)(M_I(w,v) - \phi(M_I(w,v))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w,v)\right) \geq 0, \tag{105}$$

where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a continuous and nondecreasing function with* $\phi(0) = 0$ *and* $\phi(w) > 0$*, for all* $w > 0$*, and assuming that all of the hypotheses of Theorem 13 are satisfied, then S has a unique fixed point.*

Corollary 21. *Let* (W, ρ) *be a complete metric space and* $S : W \longrightarrow W$ *be a function. If for all* $w, v \in W$*, and* $\varepsilon \in [0, 1]$*, there exists a function* $\psi \in \Psi$*, such that S satisfies the inequality either*

$$\rho\left(S^2w, S^2v\right) \leq (1-\varepsilon)(E_I(w,v) - \phi(E_I(w,v))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w,v)$$
$$\text{or}\ \rho\left(S^2w, S^2v\right) \leq (1-\varepsilon)(M_I(w,v) - \phi(M_I(w,v))) + \Lambda\varepsilon^{\xi}\psi(\varepsilon)P_I(w,v), \tag{106}$$

where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a continuous and nondecreasing function with* $\phi(0) = 0$ *and* $\phi(w) > 0$*, for all* $w > 0$*, and assuming that all of the Theorem 13 hypotheses are satisfied, then S has a unique fixed point.*

Assume now that $\alpha(w, v) = 1$ and $\varepsilon = 0$ in Theorem 13 and Theorem 15; then we get the following corollaries.

Corollary 22. *Let* (W, ρ) *be a complete metric space and* $\zeta \in Z'$*, and* $S : W \longrightarrow W$ *be two functions. If for all* $w, v \in W$*, S satisfies the inequality either*

$$\zeta\left(\rho\left(S^2w, S^2v\right), E_I(w,v) - \phi(E_I(w,v))\right) \geq 0$$
$$\text{or}\ \zeta\left(\rho\left(S^2w, S^2v\right), M_I(w,v) - \phi(M_I(w,v))\right) \geq 0, \tag{107}$$

where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a continuous and nondecreasing function with* $\phi(0) = 0$ *and* $\phi(s) > 0$*, for all* $s > 0$ *and assume that S is continuous or* S^2 *is continuous. Then, S has a unique fixed point that is* $\omega = S\omega$*,* $\omega \in W$*.*

Corollary 23. *Let* (W, ρ) *be a complete metric space and* $S : W \longrightarrow W$ *be a function. If for all* $w, v \in W$*, S satisfies the inequality either*

$$\rho\left(S^2w, S^2v\right) \leq E_I(w,v) - \phi(E_I(w,v))$$
$$\text{or}\ \rho\left(S^2w, S^2v\right) \leq M_I(w,v) - \phi(M_I(w,v)), \tag{108}$$

where $\phi : [0,+\infty) \longrightarrow [0,+\infty)$ *is a continuous and nondecreasing function with* $\phi(0) = 0$ *and* $\phi(w) > 0$*, for all* $w > 0$ *and assume that S is continuous or* S^2 *is continuous. Then, S has a unique fixed point that is* $\omega = S\omega$*,* $\omega \in W$*.*

We derive that the main result of Pata [2] and Banach [1] can be expressed as a corollary of our main result.

3. Conclusion

We present the concept of weak E-Pata convex contractions and weak Pata convex contractions in metric spaces in this paper. After that, we investigate the existence of a fixed point for our novel type contraction and we state some consequences. Our results generalize and merge the results derived by Istratescu [26] and Pata [2] and some other related results in the literature. Besides the corollaries in this paper, to underline the novelty of our given results, we give an example that shows that Theorem 15 is a genuine generalization of Istratescu's results [26]. Our novel concept allows for further studies and applications.

Data Availability

The data used to support the findings of this study are included in the references within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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