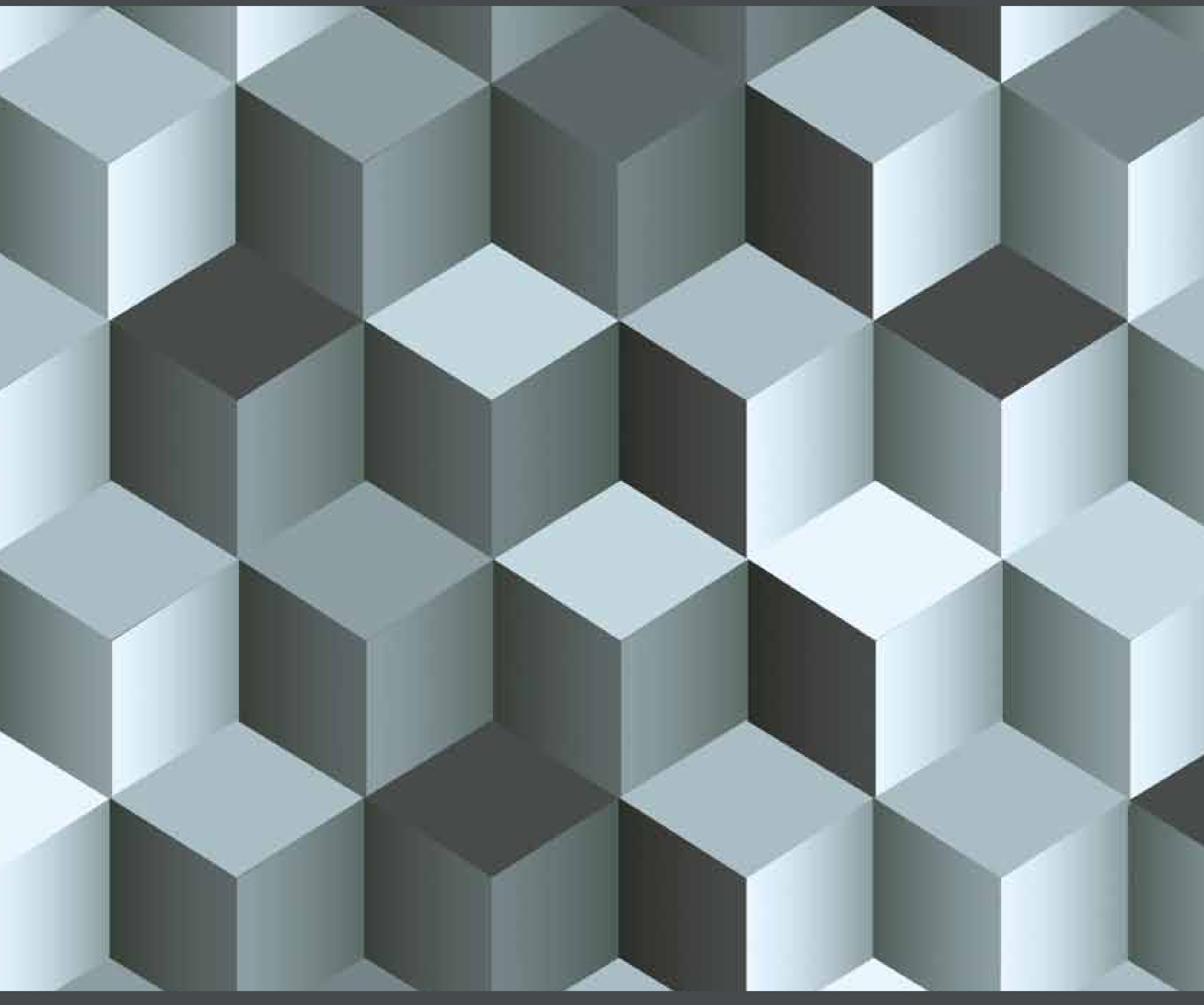
Journal of Function Spaces and Applications

Geometry of Banach Spaces, Operator Theory, and Their Applications

Guest Editors: Genqi Xu, Ji Gao, Pei Liu, and Satit Saejung



Geometry of Banach Spaces, Operator Theory, and Their Applications

Journal of Function Spaces

Geometry of Banach Spaces, Operator Theory, and Their Applications

Guest Editors: Genqi Xu, Ji Gao, Pei Liu, and Satit Saejung

Editorial Board

Contents

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2014, Article ID 635649, 1 page
http://dx.doi.org/10.1155/2014/635649

Editorial

Geometry of Banach Spaces, Operator Theory, and Their Applications

Genqi Xu,[1] Ji Gao,[2] Peide Liu,[3] and Satit Saejung[4]

[1] *Department of Mathematics, Tianjin University, Tianjin 300072, China*
[2] *Department of Mathematics, Community College of Philadelphia, Philadelphia, PA 19130-3991, USA*
[3] *College of Mathematics and Statistics, Wuhan University, Wuhan 430072, China*
[4] *Department of Mathematics, Khon Kaen University, Khon Kaen 40002, Thailand*

Correspondence should be addressed to Genqi Xu; gqxu@tju.edu.cn

Received 2 January 2014; Accepted 2 January 2014; Published 18 February 2014

In recent years, with developments of computer, high technique, and life science, more and more requirements were proposed to functional analysis. There were a large number of research articles published in the relative area and many major discoveries have been obtained. At the present time the ideas, terminology, and methods of functional analysis have penetrated deeply, not only into natural sciences, but also into applied disciplines. Many real world problems were analyzed and solved by using the ideas and results of function analysis. To fit the requirements, we edit this special issue.

The special issue represented the recent development in geometric structure of Banach spaces, Bergman spaces, and Minkowski spaces, random normed modules on function space, linear and nonlinear operator theory, and the applications of modern analysis in related areas of mathematics, as well as other disciplines, such as economics, finance and risk management, dynamic systems, the natural and life sciences, medicine, physics, and other real world problems.

The special issue consisted of twenty research articles contributed by participants of the 4th International Conference on Modern Analysis and Its Applications which was held at the Tianjin University, China, from August 1 to 4, 2013. With over 240 participants from different countries, the conference proved highly successful in bringing together experts and researchers and provided an opportunity to evaluate and disseminate new ideas and methods in modern analysis.

Each paper has been peer-reviewed and the editors thank the referees for their time and efforts to help ensure a high standard for this special issue.

We hope that readers interested in function spaces and operator theory will find in this special issue not only new information about development of functional analysis, but also important questions be resolved and directions for further study. At same time, we hope that in the near future, we can see new results to be published based on this special issue.

Acknowledgment

The auditors acknowledge Chinese Mathematical Society, National Nature Science Foundation of China, and the Department of Mathematics of Tianjin University, China. Without their support the conference would not have been possible. Thanks are also to editorial office of the Journal of Function Spaces and Applications.

Genqi Xu
Ji Gao
Peide Liu
Satit Saejung

Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2014, Article ID 501057, 6 pages
http://dx.doi.org/10.1155/2014/501057

Research Article

Composition Operators on Cesàro Function Spaces

Kuldip Raj, Suruchi Pandoh, and Seema Jamwal

School of Mathematics, Shri Mata Vaishno Devi University, Katra 182320, India

Correspondence should be addressed to Kuldip Raj; kuldeepraj68@rediffmail.com

Received 17 May 2013; Accepted 20 November 2013; Published 30 January 2014

Academic Editor: Satit Saejung

The compact, invertible, Fredholm, and closed range composition operators are characterized. We also make an effort to compute the essential norm of composition operators on the Cesàro function spaces.

1. Introduction and Preliminaries

Let (X, s, μ) be a σ-finite measure space and let $L^0 = L^0(X)$ denote the set of all equivalence classes of complex valued measurable functions defined on X, where $X = [0,1]$ or $X = [0,\infty)$. Then, for $1 \leq p < \infty$, the Cesàro function space is denoted by $\mathrm{Ces}_p(X)$ and is defined as

$$\mathrm{Ces}_p(X) = \left\{ f \in L^0(X) : \int_X \left(\frac{1}{x} \int_0^x |f(t)| \, d\mu(t) \right)^p d\mu(x) < \infty \right\}. \tag{1}$$

The Cesàro function space $\mathrm{Ces}_p(X)$ is a Banach space under the norm

$$\|f\| = \left(\int_X \left(\frac{1}{x} \int_0^x |f(t)| \, d\mu(t) \right)^p d\mu(x) \right)^{1/p}; \tag{2}$$

see [1].

The Cesàro functions spaces $\mathrm{Ces}_p[0,\infty)$ for $1 \leq p \leq \infty$ were considered by Shiue [2], Hassard and Hussein [3], and Sy et al. [4]. The space $\mathrm{Ces}_\infty[0,1]$ appeared already in 1948 and it is known as the Korenblyum, Krein, and Levin space K (see [5, 6]). Recently, in [7], it is proved that, in contrast to Cesàro sequence spaces, the Cesàro function spaces $\mathrm{Ces}_p(X)$ on both $X = [0,1]$ and $X = [0,\infty)$ for $1 < p < \infty$ are not reflexive and they do not have the fixed point property. In [8], Astashkin and Maligranda investigated Rademacher sums in $\mathrm{Ces}_p[0,1]$ for $1 \leq p \leq \infty$. The description is different for $1 \leq p < \infty$ and $p = \infty$.

Let $T : X \to X$ be a nonsingular measurable transformation; that is, $\mu T^{-1}(A) = \mu(T^{-1}(A)) = 0$, for each $A \in s$, whenever $\mu(A) = 0$. This condition means that the measure μT^{-1} is absolutely continuous with respect to μ. Let $f_0 = d\mu T^{-1}/d\mu$ be the Radon-Nikodym derivative. In addition, we assume that f_0 is almost everywhere finite valued or equivalently that $(X, T^{-1}(s), \mu)$ is σ-finite. An atom of the measure μ is an element $A \in s$ with $\mu(A) > 0$ such that, for each $F \in s$, if $F \subset A$, then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. Let A be an atom. Since μ is σ-finite, it follows that $\mu(A) < \infty$. Also every s-measurable function f on X is constant almost everywhere on A. It is a well-known fact that every sigma finite measure space (X, s, μ) can be decomposed into two disjoint sets X_1 and X_2 such that μ is atomic over X_1 and X_2 is a countable collection of disjoint atoms (see [9]).

Any nonsingular measurable transformation T induces a linear operator C_T from $\mathrm{Ces}_p(X)$ into the linear space of equivalence classes of s-measurable functions on X defined by $C_T f = f \circ T$, $f \in \mathrm{Ces}_p(X)$. Hence, the nonsingularity of T guarantees that the operator C_T is well defined. If C_T takes $\mathrm{Ces}_p(X)$ into itself, then we call that C_T is a composition operator on $\mathrm{Ces}_p(X)$. By $B(\mathrm{Ces}_p(X))$, we denote the set of all bounded linear operators from $\mathrm{Ces}_p(X)$ into itself.

So far as we know, the earliest appearance of a composition transformation was in 1871 in a paper of Schrljeder [10], where it is asked to find a function f and a number α such that

$$(f \circ T)(z) = \alpha f(z), \tag{3}$$

for every z, in an appropriate domain, whenever the function T is given. If z varies in the open unit disk and T is an analytic function, then a solution is obtained by Köenigs [11]. In 1925, these operators were employed in Littlewood's subordination theory [12]. In the early 1930s, the composition operators were used to study problems in mathematical physics and especially classical mechanics; see Koopman [13]. In those days, these operators were known as substitution operators. The systematic study of composition operators has relatively a very short history. It was started by Nordgren in 1968 in his paper [14]. After this, the study of composition operators has been extended in several directions by several mathematicians. For more details on composition operators, see [15–27] and references therein.

Associated with each σ-finite subalgebra $s_0 \subset s$, there exists an operator $E = E^{s_0}$, which is called conditional expectation operator; on the set of all nonnegative measurable functions f or for each $f \in L^0(X, s, \mu)$, the operator E is uniquely determined by the following conditions:

(i) $E(f)$ is s_0-measurable;

(ii) if A is any s_0-measurable set for which $\int_A f d\mu$ exists, we have $\int_A f d\mu = \int_A E(f) d\mu$.

The expectation operator E has the following properties:

(a) $E(f \cdot g \circ T) = E(f) \cdot (g \circ T)$;

(b) if $f \geq g$ almost everywhere, then $E(f) \geq E(g)$ almost everywhere;

(c) $E(1) = 1$;

(d) $E(f)$ has the form $E(f) = g \circ T$ for exactly one σ-measurable function g, in particular, $g = E(f) \circ T^{-1}$ is a well-defined measurable function;

(e) $|E(fg)|^2 \leq (E|f|^2)(E|g|^2)$; this is a Cauchy-Schwartz inequality for conditional expectation;

(f) for $f > 0$ almost everywhere, $E(f) > 0$ almost everywhere.

For deeper study of properties of E, see [28].

Let $\mathscr{B}$ be a Banach space and let $\mathscr{K}$ be the set of all compact operators on $\mathscr{B}$. For $U \in \mathscr{L}(\mathscr{B})$, the Banach algebra of all bounded linear operators on $\mathscr{B}$ into itself, the essential norm of U means the distance from U to $\mathscr{K}$ in the operator norm; namely,

$$\|U\|_e = \inf \{\|U - S\| : S \in \mathscr{K}\}. \tag{4}$$

Clearly, U is compact if and only if $\|U\|_e = 0$. As seen in [29], the essential norm plays an interesting role in the compact problems of concrete operators. Many people have computed the essential norm of various concrete operators. For the study of essential norm of composition operators, see [30–33] and reference therein.

The question of actually calculating the norm and essential norm of composition operators on Cesàro function spaces is not a trivial one. In spite of the difficulties associated with computing the essential norm exactly, it is often possible to find upper and lower bound for the essential norm of $C_T : \mathrm{Ces}_p(X) \to \mathrm{Ces}_p(X)$ under certain conditions on p and X.

The main purpose of this paper is to characterize the boundedness, compactness, closed range, and Fredholmness of composition operators on Cesàro function spaces. We also make an effort to compute the essential norm of composition operators in Section 3 of this paper.

2. Composition Operators

In this section of the paper, we will investigate the necessary and sufficient condition for a composition operator to be bounded.

Theorem 1. *Let (X, s, μ) be a σ-finite measure space and let $T : X \to X$ be nonsingular measurable transformation. Then, T induces a composition operator C_T on $\mathrm{Ces}_p(X)$ if and only if there exists $M > 0$ such that $\mu T^{-1}(E) \leq M\mu(E)$ for every $E \in s$. Moreover,*

$$\|C_T\| = \sup_{0<\mu(E)<\infty} \left(\left(\frac{\mu\left(T^{-1}(E)\right)}{\mu(E)} \right)^p \right)^{1/p}. \tag{5}$$

Proof. Suppose that C_T is a composition operator. If $E \in s$ such that $\mu(E) < \infty$, then $\chi_E \in \mathrm{Ces}_p(X)$ and

$$\mu T^{-1}(E) = \|C_T \chi_E\|^p \leq \|C_T\|^p \|\chi_E\|^p = \|C_T\|^p \mu(E). \tag{6}$$

Let $M = \|C_T\|^p$. Then,

$$\mu T^{-1}(E) \leq M\mu(E). \tag{7}$$

Conversely, suppose that the condition is true. Then, $\mu T^{-1} \ll \mu$ and hence the Radon-Nikodym derivative f_0 of μT^{-1} with respect to μ exists and $f_0 \leq M$ a.e.

Let $f \in \mathrm{Ces}_p(X)$. Then,

$$\begin{aligned} \|C_T f\|^p &= \int_X \left(\frac{1}{x} \int_0^x |(f \circ T)(t)| \, d\mu(t) \right)^p d\mu(x) \\ &\leq \int_X \left(\frac{1}{x} \int_0^x |f| \, d\mu T^{-1}(t) \right)^p d\mu(x) \\ &= \int_X \left(\frac{1}{x} \int_0^x |f| \, f_0 d\mu(t) \right)^p d\mu(x) \\ &\leq M^p \int_X \left(\frac{1}{x} \int_0^x |f(t)| \, d\mu(t) \right)^p d\mu(x). \end{aligned} \tag{8}$$

Therefore, $\|C_T f\| \leq M\|f\|$.

Now, Let $N = \sup_{0<\mu(E)<\infty}((\mu(T^{-1}(E))/\mu(E))^p)^{1/p}$. Then, $((\mu(T^{-1}(E))/\mu(E))^p)^{1/p} \leq N$ for all $E \in s$ and $\mu(E) \neq 0$. Thus $(\mu T^{-1}(E))^p \leq N^p(\mu(E))^p$ for all $E \in s$. By the first part of the theorem, we have

$$\|C_T f\| \leq N\|f\|, \quad \forall f \in \mathrm{Ces}_p(X). \tag{9}$$

Hence, $\|C_T\| = \sup_{f\neq 0}\|C_T f\|/\|f\| \leq N$. Thus,

$$\|C_T\| \leq \sup_{0<\mu(E)<\infty}\left(\left(\frac{\mu\left(T^{-1}(E)\right)}{\mu(E)}\right)^p\right)^{1/p}. \quad (10)$$

On the other hand, Let $M = \|C_T\| = \sup_{f\neq 0}\|C_T f\|/\|f\|$. Thus, $\|C_T f\|/\|f\| \leq M$ for all $f \in \mathrm{Ces}_p(X)$, $f \neq 0$. In particular, for $f = \chi_E$, such that $0 < \mu(E) < \infty$, $E \in s$, we have $f = \chi_E \in \mathrm{Ces}_p(X)$ and $\|C_T\chi_E\|/\|\chi_E\| = ((\mu(T^{-1}(E))/\mu(E))^p)^{1/p} \leq M$.

Therefore,

$$\sup_{0<\mu(E)<\infty}\left(\left(\frac{\mu\left(T^{-1}(E)\right)}{\mu(E)}\right)^p\right)^{1/p} \leq M = \|C_T\|. \quad (11)$$

From (10) and (11), we obtain

$$\|C_T\| = \sup_{0<\mu(E)<\infty}\left(\left(\frac{\mu\left(T^{-1}(E)\right)}{\mu(E)}\right)^p\right)^{1/p}. \quad (12)$$

□

Theorem 2. *If $C_T : Ces_p(X) \to Ces_p(X)$ is a linear transformation, then C_T is continuous.*

Proof. Let $\{f_n\}$ and $\{C_T f_n\}$ be sequences in $\mathrm{Ces}_p(X)$ such that

$$f_n \longrightarrow f, \quad C_T f_n \longrightarrow g \quad \text{for some } f, g \in \mathrm{Ces}_p(X). \quad (13)$$

Then, we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that

$$\| f_{n_k} - f \| (t) \longrightarrow 0 \quad \text{for } \mu\text{-almost all } t \in X. \quad (14)$$

From the nonsingularity of T,

$$\left(\left\|f_{n_k} - f\right\| \circ T\right)(t) \longrightarrow 0 \quad \text{for } \mu\text{-almost all } t \in X. \quad (15)$$

Then, from (13) and (15), we conclude that $C_T f = g$. This proves that graph of C_T is closed and hence, by closed graph theorem, C_T is continuous. □

3. Compactness and Essential Norm of Composition Operators

This section is devoted to the study of compact composition operators on Cesàro function spaces. A necessary and sufficient condition for a composition operator to be compact is reported in this section. The main aim of this section is to compute the essential norm of the composition operators.

Theorem 3. *Let $C_T \in B(Ces_p(X))$. Then, C_T is compact if and only if $Ces_p(X_\epsilon, \mu T^{-1})$ is finite dimensional, for each $\epsilon > 0$, where*

$$X_\epsilon = \left\{x \in X : \frac{d\mu T^{-1}}{d\mu}(x) \geq \epsilon\right\}. \quad (16)$$

Proof. For $f \in \mathrm{Ces}_p(X)$, we have

$$\begin{aligned}\|C_T f\|_\mu &= \left(\int_X \left(\frac{1}{x}\int_0^x \left|f\circ T\right|(t)\, d\mu(t)\right)^p d\mu(x)\right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x}\int_0^x \left|f\right| d\mu T^{-1}(t)\right)^p d\mu(x)\right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x}\int_0^x \left|f\right| d\mu T^{-1}(t)\right)^p d\mu T^{-1}(x)\right)^{1/p} \\ &= \|f\|_{\mu T^{-1}} \\ &= \|If\|_{\mu T^{-1}}.\end{aligned} \quad (17)$$

Then, C_T is compact if and only if $I : \mathrm{Ces}_p(X_\epsilon, \mu T^{-1}) \to \mathrm{Ces}_p(X_\epsilon, \mu T^{-1})$ is a compact operator if and only if $\mathrm{Ces}_p(X_\epsilon, \mu T^{-1})$ is a finite dimensional, where I is the identity operator. □

Corollary 4. *If (X, s, μ) is a nonatomic measure space. Then no nonzero composition operator on $Ces_p(X)$ is compact.*

Let $X = X_1 \cup X_2$ be the decomposition of X into nonatomic and atomic parts, respectively. If $X_2 = \phi$ or $\mu(X) = +\infty$ and X_2 consists of finitely many atoms, then, by Theorem 3, $\mathrm{Ces}_p(X)$ does not admit a nonzero compact composition operator. Thus, in this case, $\mathscr{K} = \{0\}$ and hence $\|C_T\|_e = \|C_T\|$.

Now, we present the main result of this section.

Theorem 5. *Let X_2 consists of finitely many atoms. Suppose that $Ces_p(\chi_\epsilon, \mu T^{-1})$ is a finite dimensional; that is, $X_\epsilon = \{x \in X : (d\mu T^{-1}/d\mu)(x) \geq \epsilon\}$ is a finite dimensional. Let $\alpha = \inf\{\epsilon > 0 : X_\epsilon$ is a finite dimensional$\}$. Then,*

(i) $\|C_T\|_e = 0$ *iff* $\alpha = 0$;

(ii) $\|C_T\|_e \geq \alpha$ *if* $0 < \alpha \leq 1$;

(iii) $\|C_T\|_e \leq \alpha$ *if* $\alpha > 1$.

Proof. (i) Theorem 3 implies that C_T is compact if and only if $\alpha = 0$. So (i) is the direct consequence of Theorem 3.

(ii) Suppose that $0 < \alpha \leq 1$. Take $0 < \epsilon < 2\alpha$ arbitrary. The definition of α implies that $F = \chi_{\alpha-(\epsilon/2)}$ either contains a nonatomic subset or has infinitely many atoms. If F contains a nonatomic subset, then there are measurable sets $E_n, n \in \mathbb{N}$ such that $E_{n+1} \subseteq E_n \subseteq F, 0 < \mu(E_n) < 1/n$. Define $f_n = 1/(\mu(E_n))^{1/p}\chi_{E_n}$. Then, $\|f_n\| = 1$, for all $n \in \mathbb{N}$. We claim that $f_n \to 0$ weakly. For this, we show that $\int_X f_n g\, d\mu \to 0$, for all $g \in (\mathrm{Ces}_q(X))$. Let $A \subseteq F$ with $\mu(A) < \infty$ and $g = \chi_A$. Then, we have

$$\begin{aligned}\left|\int_X f_n\chi_A d\mu\right| &= \frac{1}{(\mu(E_n))^{1/p}}\mu(A\cap E_n) \\ &\leq \left(\frac{1}{n}\right)^{1-(1/p)} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \quad (18)$$

Since simple functions are dense in $\mathrm{Ces}_q(X)$, thus f_n is proved to converge to zero weakly. Now, assume that F consists of infinitely many atoms. Let $\{E_n\}_{n=0}^{\infty}$ be disjoint atoms in F. Again, put f_n as above. It is easy to see that, for $A \subseteq F$ with $0 < \mu(A) < \infty$, we have $\mu(A \cap E_n) = 0$ for sufficiently large n. So, in both cases, $\int_X f_n g\, d\mu \to 0$. Now, we claim that $\|C_T f_n\| \geq \alpha - (\epsilon/2)$. Since $0 < \alpha - (\epsilon/2) < 1$, we see that

$$\begin{aligned}\|C_T f_n\| &= \left(\int_X \left(\frac{1}{x}\int_0^x |C_T f_n|\, d\mu(t)\right)^p d\mu(x)\right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x}\int_0^x |f_n|\, f_0(t)\, d\mu(t)\right)^p d\mu(x)\right)^{1/p} \\ &\geq \left(\int_X \left(\frac{1}{x}\int_0^x \left(\alpha - \frac{\epsilon}{2}\right)|f_n|\, d\mu(t)\right)^p d\mu(x)\right)^{1/p} \\ &= \left(\alpha - \frac{\epsilon}{2}\right)\left(\int_X \left(\frac{1}{x}\int_0^x |f_n|\, d\mu(t)\right)^p d\mu(x)\right)^{1/p} \\ &= \alpha - \frac{\epsilon}{2}.\end{aligned} \tag{19}$$

Finally, take a compact operator S on $\mathrm{Ces}_p(X)$ such that $\|C_T - S\| \leq \|C_T\|_e + (\epsilon/2)$. Then, we have

$$\begin{aligned}\|C_T\|_e &> \|C_T - S\| - \frac{\epsilon}{2} \geq \|C_T f_n - S f_n\| - \frac{\epsilon}{2} \\ &\geq \|C_T f_n\| - \|S f_n\| - \frac{\epsilon}{2} \\ &\geq \left(\alpha - \frac{\epsilon}{2}\right) - \|S f_n\| - \frac{\epsilon}{2},\end{aligned} \tag{20}$$

for all $n \in \mathbb{N}$. Since a compact operator maps weakly convergent sequences into norm convergent ones, it follows $\|S f_n\| \to 0$. Hence, $\|C_T\|_e \geq \alpha - \epsilon$. Since ϵ is arbitrary, we obtain $\|C_T\|_e \geq \alpha$.

(iii) Let $\alpha > 1$ and take $\epsilon > 0$ arbitrary. Put $K = X_{\alpha+\epsilon}$. The definition of α implies that K consists of finitely many atoms. So, we can write $K = (E_1, E_2, \ldots, E_m)$, where $E_1, E_2, \ldots, E_m$ are distinct. Since $(L_{\chi_K} C_T f)X = \sum_{i=1}^{m} \chi_K(E_i) f(T(E_i))$, for all $f \in \mathrm{Ces}_p(X)$, hence $L_{\chi_K} C_T$ has finite rank. Now, let $F \subseteq X \backslash K$ such that $0 < \mu(F) < \infty$. Then, we have

$$\mu T^{-1}(F) \leq (\alpha + \epsilon)\, \mu(F). \tag{21}$$

It follows that $\|\chi_F \circ T\| \leq (\alpha + \epsilon)\|\chi_F\|$. Since simple functions are dense in $\mathrm{Ces}_p(X)$. We obtain

$$\sup_{\|f\| \leq 1} \left\|\chi_{X\backslash K} f \circ T\right\| \leq \sup_{\|f\| \leq 1} \left\|\chi_{X\backslash K} f\right\| \leq \alpha + \epsilon. \tag{22}$$

Finally, Since $L_{\chi_K} C_T$ is a compact operator, we get

$$\begin{aligned}\left\|C_T - L_{\chi_K} C_T\right\| &= \sup_{\|f\| \leq 1} \left\|(1 - \chi_K)\, C_T f\right\| \\ &= \sup_{\|f\| \leq 1} \left\|\chi_{X\backslash K} C_T f\right\| \\ &\leq \alpha + \epsilon.\end{aligned} \tag{23}$$

It follows that $\|C_T\|_e \leq \alpha + \epsilon$ and, consequently, $\|C_T\|_e \leq \alpha$. □

Example 6. Let $X = (-\infty, 0] \cup \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Let μ be the Lebesgue measure on $(-\infty, 0]$ and $\mu(\{n\}) = 1/2^n$ if $n \in \mathbb{N}$. Define $T : \mathbb{N} \to \mathbb{N}$ as $T(1) = T(2) = T(3) = 1, T(4) = 2, T(5) = T(6) = 3$, and $T(2n+1) = 5$, for $n \geq 3$.

Consider $T(2n) = 2n - 2$, for $n \geq 4$, and $T(x) = 5x$, for all $x \in (-\infty, 0]$. Then, we can easily calculate $\|C_T\|_e = 3^{-1/p}$ on $\mathrm{Ces}_p(X)$ for $1 < p < \infty$.

4. Fredholm and Isometric Composition Operators

In this section, we first establish a condition for the composition operators to have closed range and then we make the use of it to characterize the Fredholm composition operators. We also make an attempt to compute the adjoint of the composition operators.

Holder's inequality for Cesàro measurable function spaces is that, if $f \in \mathrm{Ces}_p(X)$ and $g \in \mathrm{Ces}_q(X)$ such that $1/p + 1/q = 1$, then

$$\int fg\, d\mu \leq \|f\|_p \|g\|_q. \tag{24}$$

We find that every $g \in \mathrm{Ces}_q(X)$ gives rise to a bounded linear functional $Fg \in (\mathrm{Ces}_p(X))^*$ which is defined as

$$Fg(f) = \int fg\, d\mu, \quad \text{for every } f \in \mathrm{Ces}_p(X). \tag{25}$$

For each $f \in \mathrm{Ces}_p(X)$, there exists a unique $T^{-1}(s)$ measurable function $E(f)$ such that $\int gf\, d\mu = \int gE(f) d\mu$ for $T^{-1}(s)$ measurable function g for which the left integral exists. The function $E(f)$ is called conditional expectation of f with respect to the σ-algebra $T^{-1}(s)$. The operator $P_T : \mathrm{Ces}_p(X) \to \mathrm{Ces}_p(X)$ defined by $P_T f = f_0 E(f) \circ T^{-1}$ is called the Frobenius Perron operator where $E(f) \circ T^{-1} = g$ if and only if $E(f) = g \circ T$.

Theorem 7. *Let $C_T \in B(\mathrm{Ces}_p(X))$. Then C_T has closed range if and only if there exists $\delta > 0$ such that $f_0(x) \geq \delta$ for μ-almost all $x \in \operatorname{supp} f_0 = S$.*

Proof. If $f_0(x) \geq \delta$ for μ-almost all $x \in S$, then, for $\eta = \min(\delta, 1/\delta) \leq 1$,

$$\begin{aligned}1 &\geq \left(\int_X \left(\frac{1}{x}\int_0^x \frac{|C_T f|}{\|C_T f\|} d\mu(t)\right)^p d\mu(x)\right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x}\int_0^x f_0 \frac{|f|}{\|C_T f\|} d\mu(t)\right)^p d\mu(x)\right)^{1/p} \\ &\geq \left(\int_X \left(\frac{1}{x}\int_0^x \frac{\eta |f|}{\|C_T f\|} d\mu(t)\right)^p d\mu(x)\right)^{1/p}.\end{aligned} \tag{26}$$

Hence, $\|C_T f\| \geq \eta \|f\|$, for all $f \in \mathrm{Ces}_p(S)$, so that C_T has closed range.

Conversely, suppose that C_T has closed range. Then there exists $\delta \geq 0$ such that

$$\|C_T f\| \geq \delta \|f\|, \tag{27}$$

for every $f \in \mathrm{Ces}_p(S)$. Choose a positive integer n such that $1/n < \delta$. If the set $E = \{x \in X : f_0(x) < 1/n\}$ has positive measure, then, for a given measurable subset $F \subset \operatorname{supp} f_0$ such that $0 < \mu(F) < \infty$, we have

$$\mu T^{-1}(E) < \frac{1}{n}\mu(E) \tag{28}$$

or equivalently

$$\|C_T \chi_E\| \leq \frac{1}{n}\|\chi_E\|. \tag{29}$$

This contradicts inequality (27). Hence, f_0 is bounded away from zero on $\operatorname{supp} f_0$. □

Theorem 8. *Let $C_T \in B(Ces_p(X))$. Then,* ker C_T^* *is either zero-dimensional or infinite dimensional.*

Proof. Suppose $0 \neq g \in \ker C_T^*$. Then $E = \operatorname{supp} g$ is a set of nonzero measure. Now we can partition E into a sequence $\{E_n\}$ of measurable sets, $0 < \mu(E_n) < \infty$. We show that $g\chi_{E_n \circ T} \in \ker C_T^*$. Consider

$$\begin{aligned} C_T^*\left(g\chi_{E_n\circ T}\right)(f) &= \int_E \left(g\chi_{E_n\circ T}\right)(C_T f)\,d\mu \\ &= \int_E g C_T\left(\chi_{E_n} f\right) d\mu = 0. \end{aligned} \tag{30}$$

Hence, if ker C_T^* is not zero-dimensional, it is infinite dimensional. □

Corollary 9. *Let $C_T \in B(Ces_p(X))$. Then, C_T is injective if and only if T is surjective.*

Theorem 10. *Let $C_T \in B(Ces_p(X))$. Then, C_T has dense range if and only if $T^{-1}(s) = s$.*

Proof. Suppose that C_T has dense range. Let $E \in s$ such that $\chi_E \in \mathrm{Ces}_p(X)$. Then, there exists $\{f_n\} \subset \mathrm{Ces}_p(X)$ such that $C_T f_n \to \chi_E$. Now, we can find a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $C_T f_{n_k} \to \chi_E$ a.e. Now, each $C_T f_{n_k}$ is measurable with respect to $T^{-1}(s)$. Therefore, χ_E is measurable with respect to $T^{-1}(s)$ so that $\chi_E = \chi_{T^{-1}(F)}$. Hence, $T^{-1}(s) = s$ a.e.

Conversely, suppose $T^{-1}(s) = s$ a.e. If $E \in s, 0 < \mu(E) < \infty$, then there exists $F \in s$ such that $\mu(T^{-1}(F)\Delta E) = 0$. Since X is σ-finite, we can find an increasing sequence $\{F_n\}$ of sets of finite measure $F_n \uparrow F$ or $T^{-1}(F) \setminus T^{-1}(F_n) \downarrow 0$. Hence, for given $\epsilon > 0$, there exists a positive integer n_0 such that $\mu(T^{-1}(F \setminus F_n)) < \epsilon$ for every $n \geq n_0$. Hence,

$$\begin{aligned} \left\|C_T\chi_F - C_T\chi_{F_n}\right\| &= \left\|C_T\left(\chi_{F\setminus F_n}\right)\right\| \\ &= \left\|\chi_{T^{-1}(F\setminus F_n)}\right\| \\ &= \left(\int_X \left(\frac{1}{x}\int_0^x \chi_{T^{-1}(F\setminus F_n)} d\mu(t)\right)^p d\mu(x)\right)^p \\ &< \epsilon, \end{aligned} \tag{31}$$

for all $n \geq n_0$. Then $\chi_F = \chi_{T^{-1}(F)} \in \overline{\operatorname{ran} C_T}$. This proves that C_T has dense range. □

Theorem 11. *Let $C_T \in B(Ces_p(X))$. Then, C_T is Fredholm if and only if C_T is invertible.*

Proof. Assume that C_T is Fredholm. In view of Theorem 8, $\ker C_T$ and $\ker C_T^*$ are zero-dimensional so that C_T is injective and $T^{-1}(s) = s$ a.e. Therefore, by Theorem 10, C_T has dense range. Since $\operatorname{ran} C_T$ is closed, so C_T is surjective. This proves the invertibility of C_T. The proof of the converse part is obvious. □

Corollary 12. *Let $C_T \in B(Ces_p(X))$. Then, C_T is an isometry if and only if T is measure preserving.*

Proof. If T is measure preserving, then $f_0 = 1$ a.e. Therefore,

$$\begin{aligned} \|C_T f\| &= \left(\int_X \left(\frac{1}{x}\int_0^x \left|f \circ T\right|(t)\,d\mu(t)\right)^p d\mu(x)\right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x}\int_0^x f_0\left|f\right| d\mu(t)\right)^p d\mu(x)\right)^{1/p} \\ &= \left(\int_X \left(\frac{1}{x}\int_0^x \left|f\right| d\mu(t)\right)^p d\mu(x)\right)^{1/p} \\ &= \|f\|. \end{aligned} \tag{32}$$

Hence, C_T is an isometry. Conversely, if C_T is an isometry, then

$$\|C_T\chi_E\| = \|\chi_E\|. \tag{33}$$

This implies that $\mu(T^{-1}(E)) = \mu(E)$ for $E \in s$. Hence, $f_0 = 1$ a.e. □

Theorem 13. *Let $C_T \in B(Ces_p(X))$. Then, $C_T^* = P_T$.*

Proof. Let $A \in s$ such that $0 < \mu(A) < \infty$. For $g \in \mathrm{Ces}_q(X)$,

$$\begin{aligned}(C_T^* Fg)(\chi_A) &= Fg(C_T \chi_A) \\ &= \int C_T \chi_A g \, d\mu \\ &= \int \chi_A \circ T g \, d\mu \\ &= \int \chi_A E(g) \circ T^{-1} f_0 d\mu \\ &= F_{E(g)\circ T^{-1} f_0}(\chi_A).\end{aligned} \tag{34}$$

Hence, $C_T^* Fg = F_{E(g)\circ T^{-1} f_0}$. After identifying $g \in \mathrm{Ces}_q(X)$ with $Fg \in (\mathrm{Ces}_p(X))^*$, we can write $C_T^* g = E(g) \circ T^{-1} f_0 = P_T g$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] S. V. Astashkin and L. Maligranda, "Structure of Cesàro function spaces," *Indagationes Mathematicae*, vol. 20, no. 3, pp. 329–379, 2009.

[2] J.-S. Shiue, "A note on Cesàro function space," *Tamkang Journal of Mathematics*, vol. 1, no. 2, pp. 91–95, 1970.

[3] B. D. Hassard and D. A. Hussein, "On Cesàro function spaces," *Tamkang Journal of Mathematics*, vol. 4, pp. 19–25, 1973.

[4] P. W. Sy, W. Y. Zhang, and P. Y. Lee, "The dual of Cesàro function spaces," *Glasnik Matematički*, vol. 22, no. 1, pp. 103–112, 1987.

[5] B. I. Korenblyum, S. G. Kreĭn, and B. Y. Levin, "On certain nonlinear questions of the theory of singular integrals," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 62, pp. 17–20, 1948.

[6] E. Sawyer, "Boundedness of classical operators on classical Lorentz spaces," *Studia Mathematica*, vol. 96, no. 2, pp. 145–158, 1990.

[7] S. V. Astashkin and L. Maligranda, "Cesàro function spaces fail the fixed point property," *Proceedings of the American Mathematical Society*, vol. 136, no. 12, pp. 4289–4294, 2008.

[8] S. V. Astashkin and L. Maligranda, "Rademacher functions in Cesàro type spaces," *Studia Mathematica*, vol. 198, no. 3, pp. 235–247, 2010.

[9] A. C. Zaanen, *Integration*, North-Holland, 2nd edition, 1967.

[10] E. Schrljeder, "Uber iteratierte funcktionen," *Journal of Mathematical Analysis*, vol. 3, pp. 296–322, 1871.

[11] G. Köenigs, "Recherches sur le Integrable de certcuns equarions frontionalles," *Annales Scientifiques de l'École Normale Supérieure*, vol. 1, pp. 3–41, 1884.

[12] J. E. Littlewood, "On inequalities in the theory of functions," *Proceedings of the London Mathematical Society*, vol. 23, no. 1, pp. 481–519, 1925.

[13] B. O. Koopman, "Hamiltonian systems and transformations in Hilbert spaces," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 17, pp. 315–318, 1931.

[14] E. A. Nordgren, "Composition operators," *Canadian Journal of Mathematics*, vol. 20, pp. 442–449, 1968.

[15] Y. Cui, H. Hudzik, R. Kumar, and L. Maligranda, "Composition operators in Orlicz spaces," *Journal of the Australian Mathematical Society*, vol. 76, no. 2, pp. 189–206, 2004.

[16] R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Academic Press, New York, NY, USA, 1972.

[17] W. E. Hornor and J. E. Jamison, "Isometrically equivalent composition operators," *Contemporary Mathematics*, vol. 213, pp. 65–72, 1996.

[18] M. R. Jabbarzadeh, "A note on weighted composition operators on measurable function spaces," *Journal of the Korean Mathematical Society*, vol. 41, no. 1, pp. 95–105, 2004.

[19] B. S. Komal and R. S. Pathania, "Composition operators on a space of operators," *Indian Journal of Mathematics*, vol. 33, no. 1, pp. 11–17, 1991.

[20] A. Kumar, "Fredholm composition operators," *Proceedings of the American Mathematical Society*, vol. 79, no. 2, pp. 233–236, 1980.

[21] B. D. MacCluer, "Fredholm composition operators," *Proceedings of the American Mathematical Society*, vol. 125, no. 1, pp. 163–166, 1997.

[22] E. A. Nordgren, *Composition Operators on Hilbert Spaces*, vol. 693 of *Lectures Notes in Math*, Springer, New York, NY, USA, 1978.

[23] K. Raj, A. K. Sharma, and A. Kumar, "Weighted composition operators on Musielak-Orlicz spaces," *Ars Combinatoria*, vol. 107, pp. 431–439, 2012.

[24] K. Raj and S. K. Sharma, "Composition operators on Musielak-Orlicz spaces of Bochner type," *Mathematica Bohemica*, vol. 137, no. 4, pp. 449–457, 2012.

[25] W. C. Ridge, *Composition operators [Ph.D. thesis]*, Indiana University, 1969.

[26] R. K. Singh and J. S. Manhas, *Composition Operators on Function Spaces*, North-Holland, 1993.

[27] H. Takagi and K. Yokouchi, "Multiplication and composition operators between two L^p-spaces," *Contemporary Mathematics*, vol. 232, pp. 321–338, 1999.

[28] A. Lambert, "Localising sets for sigma-algebras and related point transformations," *Proceedings of the Royal Society of Edinburgh A*, vol. 118, no. 1-2, pp. 111–118, 1991.

[29] J. H. Shapiro, "The essential norm of a composition operator," *Annals of Mathematics*, vol. 125, no. 2, pp. 375–404, 1987.

[30] M. R. Jabbarzadeh, "The essential norm of a composition operator on Orlicz spaces," *Turkish Journal of Mathematics*, vol. 34, no. 4, pp. 537–542, 2010.

[31] A. Montes-Rodríguez, "The essential norm of a composition operator on Bloch spaces," *Pacific Journal of Mathematics*, vol. 188, no. 2, pp. 339–351, 1999.

[32] H. Takagi, T. Miura, and S.-E. Takahasi, "Essential norms and stability constants of weighted composition operators on $C(X)$," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 4, pp. 583–591, 2003.

[33] L. Zheng, "The essential norms and spectra of composition operators on H^∞," *Pacific Journal of Mathematics*, vol. 203, no. 2, pp. 503–510, 2002.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 856084, 10 pages
http://dx.doi.org/10.1155/2013/856084

Research Article

L^p Approximation Strategy by Positive Linear Operators

Wei Xiao

Science School, Nanjing University of Science and Technology, 200 Xiao Ling Wei Street, Nanjing 210094, China

Correspondence should be addressed to Wei Xiao; chinaxiao@sina.com

Received 31 March 2013; Accepted 28 June 2013

Academic Editor: Satit Saejung

Two important techniques to achieve the Jackson type estimation by Kantorovich type positive linear operators in L^p spaces are introduced in the present paper, and three typical applications are given.

1. Introduction

It is well known that Kantorovich type operators are usually used for approximation in L^p spaces. Let $f(x) \in L^p_{[0,1]}$, the class of all p power integrable functions on the interval $[0,1]$, and $\mathscr{K}_{n,j}(x)$ ($j = 0,1,2,\ldots,n$, or $j = 1,2,\ldots,n$) be given kernels satisfying

$$\sum_{j=1}^{n} \mathscr{K}_{n,j}(x) = 1 \qquad \left(\text{or } \sum_{j=0}^{n} \mathscr{K}_{n,j}(x) = 1\right),$$
$$\mathscr{K}_{n,j}(x) \geq 0, \quad x \in [0,1]. \tag{1}$$

This paper discusses how well the function $f(x) \in L^p_{[0,1]}$ can be approximated by the discrete Kantorovich-type operators such as

$$L_n(f,x) = (n+1)\sum_{j=0}^{n} \mathscr{K}_{n,j}(x) \int_{j/(n+1)}^{(j+1)/(n+1)} f(t)\,dt \tag{2}$$

or

$$L_n(f,x) = n\sum_{j=1}^{n} \mathscr{K}_{n,j}(x) \int_{(j-1)/n}^{j/n} f(t)\,dt, \tag{3}$$

and the L^p approximation is characterized by

$$\left\| f - L_n(f) \right\|_{L^p} \leq C\omega(f,\epsilon_n)_{L^p}, \tag{4}$$

where ϵ_n, a positive number depending on n only (such as $n^{-1/2}$, n^{-1}), is called *Jackson order* of L^p approximation; and $C > 0$, a constant depending sometimes upon p as well as the kernels $\{\mathscr{K}_{n,j}(x)\}$ (e.g., the kernels of the Shepard operators; it contains a parameter λ; see (10)), is called *Jackson constant*. Since the kernel $\{\mathscr{K}_{n,j}(x)\}$ can have two types, decided by the summation indices, respectively (see (1)), the Kantorovich type operators are therefore defined by (2) or (3). However, with similar arguments, we can only investigate the positive linear operators of the form of (3).

Sometimes, we need to write $Q_n(x,t) = \mathscr{K}_{n,j}(x)$ for $(j-1)/n \leq t < j/n$, $j = 1,2,\ldots,n$; then the operators (3) can be written as an integral form:

$$L_n(f,x) = n\int_0^1 Q_n(x,t)\,f(t)\,dt; \tag{5}$$

hence,

$$\int_0^1 Q_n(x,t)\,dt = n^{-1}. \tag{6}$$

This means, for Kantorovich type operators, there does not exist any difference whether (3) or (5) is taken. In particular, we give the following examples.

When the kernel function $Q_n(x,t)$ satisfies

$$\frac{\partial}{\partial x} Q_n(x,t) = \frac{n}{\phi(x)} Q_n(x,t)(t-x), \tag{7}$$

where $\phi(x) > 0$ satisfying $\phi(0) = \phi(1) = 0$ is a polynomial with a degree at most 2, then the operators $L_n(f,x)$ have exponential type, which have been studied deeply in [1–3], for example.

When the kernel function $\mathscr{K}_{n,j}(x)$ in (3) is taken as

$$R_j(x) = \frac{P_j(x)}{\sum_{k=1}^{n} P_k(x)},$$
$$P_k(x) = x^{\lambda_k} \prod_{l=1}^{k} \left(\frac{l}{n}\right)^{-\Delta\lambda_l}, \tag{8}$$

where $\{x^{\lambda_0}, x^{\lambda_1}, \ldots, x^{\lambda_n}, \ldots\}$ is a Müntz system satisfying that $\Delta\lambda_n = \lambda_n - \lambda_{n-1} \geq Mn$ (M is a given positive number), then the operator $L_n(f, x)$ becomes the rational Müntz operator:

$$M_n(f, x) = n \sum_{k=1}^{n} R_k(x) \int_{(k-1)/n}^{k/n} f(t)\, dt; \tag{9}$$

readers can refer to [4, 5].

When the kernel function $\mathscr{K}_{n,j}(x)$ of (2) is taken as

$$r_j(x) = \frac{|x - j/n|^{-\lambda}}{\sum_{k=0}^{n} |x - k/n|^{-\lambda}} \quad (\lambda > 1), \tag{10}$$

then the operator $L_n(f, x)$ is the well-known Shepard operator

$$S_{n,\lambda}(f, x) = (n+1) \sum_{k=0}^{n} r_k(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt; \tag{11}$$

one can check [6, 7], for example.

When the kernel $\mathscr{K}_{n,j}(x)$ in (2) is taken as

$$P_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}, \tag{12}$$

then we obtain the Kantorovich-Bernstein operator

$$B_n(f, x) = (n+1) \sum_{k=0}^{n} P_{n,k}(x) \int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\, dt, \tag{13}$$

which has been studied most widely among the positive linear operators of the form (2) (see [8–23]). Interested readers could also refer to the related papers for the other similar operators.

This paper will take the above three typical operators (9), (11), and (13) as examples to illustrate two quantitative methods on L^p approximation. Over discussion, we find out that the Jackson order in L^p spaces to approximate $f(x) \in L^p_{[0,1]}$ by the operators in (3) or (5) is decided completely by the kernels $\{\mathscr{K}_{n,j}(x)\}_{j=1}^{n}$, or by the kernel function $Q_n(x, t)$. Therefore, on applying this idea, we need only to investigate the properties of the kernels to obtain the magnitude of the Jackson order of the corresponding operators, which seems to be a different approach from the past L^p approximating methods.

2. Notations and Terminologies

In this section, we give all preliminary notations and terminologies. For $f(x) \in L^p_{[0,1]}$, the usual L^p norm is defined by

$$\|f\|_{L^p_{[a,b]}} = \left\{\int_a^b |f(x)|^p dx\right\}^{1/p}, \qquad \|f\|_{L^p} = \|f\|_{L^p_{[0,1]}}, \tag{14}$$

and the L^p modulus of continuity of $f \in L^p_{[0,1]}$ is defined by

$$\omega(f, \delta)_{L^p} = \omega(f, \delta)_{L^p_{[0,1-\delta]}} = \sup_{0<t\leq\delta} \|f(x+t) - f(x)\|_{L^p_{[0,1-\delta]}}. \tag{15}$$

To understand clearly L^p approximation by the positive linear operators, we need to make analysis of the kernels corresponding to the operators. Hence, some new terminologies on the kernels will be given and explained by some examples. For convenience, C always indicates an absolute positive constant and C_x indicates a positive constant depending upon at most x. C or C_x may have different values at different occurrences even at the same line. Sometimes we write $\mathscr{K}_n = \mathscr{K}_n(x) = \{\mathscr{K}_{n,j}(x)\}_{j=1}^{n}$.

Definition 1. For any $x \in [0, 1]$, if there exists a real sequence $\{\Phi(k)\}_{k=1}^{\infty}$ with $0 < \Phi(k) < 1$, $k = 1, 2, \ldots$, satisfying

$$\mathscr{K}_{n,r}(x) \leq C\Phi(|[nx] - r| + 1), \quad r = 1, 2, \ldots, n, \tag{16}$$

then $\{\Phi(k)\}_{k=1}^{\infty}$ is called *a global domination* of $\mathscr{K}_n(x)$, or we say $\mathscr{K}_n(x)$ is globally dominated by $\{\Phi(k)\}_{k=1}^{\infty}$, where $[x]$ indicates the greatest integer not exceeding x. In particular, if $\Phi(k) = q^k$, $0 < q < 1$, $k = 1, 2, \ldots$, then the kernel sequence $\mathscr{K}_n(x)$ is called *a globally dominated geometrical sequence*, or being dominated by a global geometrical sequence, or having (global) *geometrical order*. If $\Phi(k) = k^{-\rho}$, $\rho > 0$, $k = 1, 2, \ldots$, then the kernel sequence $\mathscr{K}_n(x)$ is called *a globally dominated arithmetic sequence with power* ρ, or being dominated by a global arithmetic sequence with power ρ, or having (global) *ρ-arithmetic order*.

Definition 2. For any $x \in [0, 1]$, if there exist a sequence $\{\Phi(k)\}_{k=1}^{\infty}$ with $0 < \Phi(k) < 1$, $k = 1, 2, \ldots$ and an integer subset $N_x \subset \{1, 2, \ldots, n\}$ satisfying that

$$\mathscr{K}_{n,r}(x) \leq C\Phi(|[nx] - r| + 1), \quad r \in N_x, \tag{17}$$

then $\{\Phi(k)\}_{k=1}^{\infty}$ is called *a local domination* of $\mathscr{K}_n(x)$. Like in Definition 1, *a locally dominated geometrical sequence* of $\mathscr{K}_n(x)$ and *a locally dominated ρ-arithmetic sequence* can be defined similarly.

The conceptions of Definitions 1 and 2 will be illustrated by the following examples.

Example 3. The kernel functions $R_k(x)$ of the rational Müntz operators defined by (9) satisfy

$$R_k(x) \leq C_M \exp(-C_M |[nx] - k|), \quad k = 0, 1, 2, \ldots, n \tag{18}$$

(see [4] or [5, Lemma 1]). The kernel functions $R_k(x)$ then have geometrical order.

Example 4. The kernels $r_k(x)$ of the Shepard operators $S_{n,\lambda}(f, x)$ $(\lambda > 1)$ defined by (11) satisfy

$$r_k(x) \leq \left(\frac{2}{|[(n+1)x] - k| + 1}\right)^{\lambda}, \quad k = 0, 1, 2, \ldots, n \tag{19}$$

(see [12, Lemma 1]). The kernels $r_k(x)$ have λ-arithmetic order.

Example 5. The kernels $P_{n,k}(x)$ of the Kantorovich-Bernstein operators (13) satisfy

$$P_{n,k}(x) \approx (2\pi n x(1-x))^{-1/2} \cdot \exp\left(-\frac{n}{2x(1-x)}\left(\frac{k}{n}-x\right)^2\right), \tag{20}$$

where $x \in (0,1)$, and k satisfies

$$\left|\frac{k}{n}-x\right| \leq n^{-\alpha} \tag{21}$$

for real number $\alpha > 1/3$ (see [15, Theorem 1.5.2]). That is to say, if the set of all k satisfying (21) is written as N_x, then, while $k \in N_x$, $P_{n,k}(x)$ has asymptotic expression (20). Hence, the kernels $P_{n,k}(x)$ of Bernstein operators, from Definition 2, have local geometrical order.

Definition 6. For $r = 0, 1, \ldots$, and any $x \in [0,1]$, denote

$$M^r(\mathscr{K}_n) := \sup_{n\geq 1}\sum_{j=1}^{n} |nx-j|^r \mathscr{K}_{n,j}(x), \tag{22}$$

particularly,

$$\begin{aligned} M(\mathscr{K}_n) &= M^1(\mathscr{K}_n), \\ D(\mathscr{K}_n) &= M^0(\mathscr{K}_n). \end{aligned} \tag{23}$$

Definition 7. For $r = 1, 2, \ldots$, write

$$DM^r(\mathscr{K}_n) := \sup_{n\geq 1}\sum_{k=1}^{n} k^r \Phi(k), \tag{24}$$

where $\{\Phi(k)\}_{k=1}^{\infty}$ is the global domination of $\mathscr{K}_n(x)$ in Definition 1. In particular,

$$DM(\mathscr{K}_n) = DM^1(\mathscr{K}_n). \tag{25}$$

Example 8. For any $\epsilon \in (0,1)$, from Example 3, we have

$$D(R_k^{\epsilon}(x)) \leq C_M \sum_{k=0}^{\infty} \exp(-C_M \epsilon k) < \infty, \tag{26}$$

where the kernels $R_k(x)$ (8), as well as $\Phi(k) = e^{-C_M k}$, from Definitions 6 and 7, satisfy $M^r(\mathscr{K}_n^{\epsilon}) < \infty$ and $DM^r(\mathscr{K}_n^{\epsilon}) < \infty$ for all $r = 1, 2, \ldots$ and any given $0 < \epsilon \leq 1$.

Example 9. Propose that $\lambda > 1$, write $\epsilon_\lambda = (\lambda+1)/2$. In view of Example 4, we get

$$D(r_k^{\epsilon_\lambda}(x)) \leq 2^{\lambda(\lambda+1)/2} \sum_{k=1}^{\infty} k^{-\lambda(\lambda+1)/2} < \infty. \tag{27}$$

Thus, from Definitions 6 and 7, the kernels $r_k(x)$ (10), as well as $\Phi(k) = k^{-\lambda}$, satisfy $M^r(\mathscr{K}_n^{\epsilon_\lambda}) < \infty$ and $DM^r(\mathscr{K}_n^{\epsilon_\lambda}) < \infty$ for some r.

Remark 10. We make a brief discussion on the above definitions.

(1) $M^r(\mathscr{K}_n^{\epsilon}) < \infty$ or $DM^r(\mathscr{K}_n^{\epsilon}) < \infty$ for some $0 < \epsilon < 1$ obviously yields that $M^r(\mathscr{K}_n) < \infty$ or $DM^r(\mathscr{K}_n) < \infty$; the converse may not be true.

(2) The kernels having geometrical order satisfy $M^r(\mathscr{K}_n^{\epsilon}) < \infty$ and $DM^r(\mathscr{K}_n^{\epsilon}) < \infty$ for all r and $0 < \epsilon \leq 1$.

(3) The kernels having ρ-arithmetic order satisfy $D(\mathscr{K}_n^{\epsilon}) < \infty$ for $\rho > 1$.

3. Elementary Approximation Technique (I)

This section gives one of the approximation techniques in L^p spaces by the Kantorovich type operators (3). We mainly apply K-functional and maximum principle to obtain the Jackson type estimation in L^p spaces.

Theorem 11. *Let $\{\delta_n\}$ be a positive null sequence. For any $f(x) \in L^p_{[0,1]}$, $1 < p < \infty$, the Kantorovich operators defined by* (3) *satisfy*

(i) *$\|L_n(f)\|_{L^p}$ which is uniformly bounded;*

(ii) *$L_n(|t-x|, x) = O(\delta_n)$.*

Then, the estimate

$$\|L_n(f) - f\|_{L^p} \leq C_p \omega(f, \delta_n)_{L^p} \tag{28}$$

holds, where C_p is a positive constant depending upon p only.

Proof. For any function $f(x) \in L^p_{[0,1]}$, from the definition of the K-functional

$$K(f,h)_{L^p} = \inf_{g\in AC_{[0,1]}, g'\in L^p} \left(\|f-g\|_{L^p} + h\|g'\|_{L^p}\right), \tag{29}$$

where $AC_{[0,1]}$ indicates the set of all absolute continuous functions on the interval $[0,1]$, we know that $K(f,h)_{L^p}$ and $\omega(f,h)_{L^p}$ are equivalent (see [24, Theorem 2.1]); that is,

$$\omega(f,h)_{L^p} \approx K(f,h)_{L^p}, \tag{30}$$

where $A_h \approx B_h$ we mean there is a constant $C > 0$ independent of h such that $C^{-1}A_h \leq B_h \leq CA_h$.

For given $f(x) \in L^p_{[0,1]}$, the Hardy-Littlewood maximum function is defined by

$$M(f,x) = \sup_{\delta>0}\frac{1}{\delta}\int_0^{\delta} |f(x+u)|\, du, \tag{31}$$

where if $x + u \notin [0,1]$, we simply set $f(x+u) = 0$. It is well known that (see [25])

$$\|M(f)\|_{L^p} \leq C_p \|f\|_{L^p}, \quad p > 1. \tag{32}$$

Since for any $g(x) \in AC_{[0,1]}$, $g'(x) \in L^p_{[0,1]}$, $p > 1$, we have

$$\begin{aligned} \|L_n(f) - f\|_{L^p} \leq{}& \|L_n(f-g) - (f-g)\|_{L^p} \\ &+ \|L_n(g) - g\|_{L^p}, \end{aligned} \tag{33}$$

then the condition (i) of Theorem 11 induces that

$$\|L_n(f) - f\|_{L^p} \le C\|f - g\|_{L^p} + \|L_n(g) - g\|_{L^p}. \tag{34}$$

It is easy to deduce from definition (3) that

$$\begin{aligned} &\left|L_n(g, x) - g(x)\right| \\ &\quad \le n\sum_{j=1}^{n} \mathscr{K}_{n,j}(x) \int_{(j-1)/n}^{j/n} \left|g(t) - g(x)\right| dt \\ &\quad = n\sum_{j=1}^{n} \mathscr{K}_{n,j}(x) \int_{(j-1)/n}^{j/n} \left|\int_x^t g'(u)\, du\right| dt \\ &\quad \le nM\left(g', x\right) \sum_{j=1}^{n} \mathscr{K}_{n,j}(x) \int_{(j-1)/n}^{j/n} |t - x|\, dt \\ &\quad = M\left(g', x\right) L_n\left(|t - x|, x\right). \end{aligned} \tag{35}$$

With the condition (ii) in Theorem 11, we obtain

$$\left|L_n(g, x) - g(x)\right| \le CM\left(g', x\right)\delta_n. \tag{36}$$

Combining (32) and (36) leads to

$$\|L_n(g) - g\|_{L^p} \le C_p\delta_n\|g'\|_{L^p}. \tag{37}$$

Together with (34) and (37), we get

$$\|L_n(f) - f\|_{L^p} \le C\|f - g\|_{L^p} + C_p\delta_n\|g'\|_{L^p}. \tag{38}$$

Finally, from the definition of the K-functional we achieve that

$$\|L_n(f) - f\|_{L^p} \le C_p K(f, \delta_n)_{L^p}. \tag{39}$$

Therefore, in view of (30) and (39), we have completed Theorem 11. □

4. Elementary Approximation Technique (II)

In Section 3, by applying the K-functional, we obtain the Jackson type estimation in L^p spaces for $p > 1$. However, the Jackson constant in that case must depend upon p, and thus we cannot establish corresponding result in L^1 space! In this section, we will exhibit another efficient technique in L^p spaces which will be used to obtain Jackson constant independent of p!

Theorem 12. *Let $f(x) \in L^p_{[0,1]}$, $1 \le p < \infty$, an ϵ be given with $0 < \epsilon < 1$ and the positive linear operators $L_n(f, x)$ defined by (3). If the kernels $\mathscr{K}_n(x)$ with (1) are dominated globally by $\{\Phi(k)\}_{k=1}^{\infty}$, and for some $0 < \epsilon < 1$ satisfy the following conditions:*

(i) $D(\mathscr{K}_n^{1-\epsilon}) < \infty$,

(ii) $DM(\mathscr{K}_n^{\epsilon}) < \infty$,

then, the estimate

$$\|L_n(f) - f\|_{L^p_{[0,1]}} \le C\omega(f, 1/n)_{L^p_{[0,1]}} \tag{40}$$

holds, where $C > 0$ is an absolute constant.

To prove Theorem 12, we first give two lemmas.

Lemma 13. *Given h with $0 < h < 1$ and $f(x) \in L^p_{[0,1]}$, write the Steklov function of $f(x)$ as*

$$f_h(x) = h^{-1}\int_0^h f(x + u)\, du, \tag{41}$$

where $x + u \in [0, 1]$. Then, the following results hold:

$$\text{(i)}\quad f_h'(x) = \frac{f(x + h) - f(x)}{h}; \tag{42}$$

$$\text{(ii)}\quad \|f - f_h\|_{L^p_{[0,1-h]}} \le \omega(f, h)_{L^p}. \tag{43}$$

Proof. Equations (42) and (43) can be directly verified from the definition of the Steklov function (41). □

Lemma 14. *Propose that $0 \le a, b \le 1$, $0 < \delta < 1$, $a + \delta \le 1$, and $b + \delta \le 1$, then for any $f(x) \in L^p_{[0,1]}$, one has*

$$\int_a^{a+\delta}\int_b^{b+\delta} \left|f(x) - f(y)\right|^p dx\, dy \le 2\delta\omega^p\left(f, |a - b| + \delta\right)_{L^p}. \tag{44}$$

Proof. This Lemma is proved in [7]; we give a sketch here for self-completeness. Due to the symmetries on a and b, as well as on x and y, we need only to prove the lemma under $b \ge a$. By calculations,

$$\begin{aligned} &\int_a^{a+\delta}\int_b^{b+\delta} \left|f(x) - f(y)\right|^p dx\, dy \\ &\quad = 2\int_a^{a+\delta}\left(\int_{b-a+x}^{b+\delta} \left|f(x) - f(y)\right|^p dy\right) dx \\ &\quad = 2\int_a^{a+\delta}\left(\int_{b-a}^{b+\delta-x} \left|f(x) - f(x + y)\right|^p dy\right) dx \\ &\quad = 2\int_{b-a}^{b-a+\delta}\left(\int_a^{b+\delta-y} \left|f(x) - f(x + y)\right|^p dx\right) dy \\ &\quad \le 2\delta\omega^p\left(f, b - a + \delta\right)_{L^p}. \end{aligned} \tag{45}$$

Lemma 14 is done. □

Proof of Theorem 12. Take $h = 1/n$ in the Steklov function (41); check

$$\begin{aligned} &\|L_n(f) - f\|_{L^p_{[1-h,1]}} \\ &\quad = \left\{\int_{1-h}^{1}\left(n\sum_{r=1}^{n} \mathscr{K}_{n,r}(x) \int_{(r-1)/n}^{r/n} \left(f(t) - f(x)\right) dt\right)^p dx\right\}^{1/p} \end{aligned} \tag{46}$$

by applying Minkowski inequality, we get

$$\begin{aligned} &\|L_n(f)-f\|_{L^p_{[1-h,1]}} \\ &\le n\sum_{r=1}^{n}\left\{\int_{1-1/n}^{1}\mathscr{K}^p_{n,r}(x)\left(\int_{(r-1)/n}^{r/n}|f(t)-f(x)|\,dt\right)^p dx\right\}^{1/p}. \end{aligned} \tag{47}$$

Then, apply Hölder inequality:

$$\begin{aligned} &\|L_n(f)-f\|_{L^p_{[1-h,1]}} \\ &\le n^{1/p}\sum_{r=1}^{n}\left\{\int_{1-1/n}^{1}\mathscr{K}^p_{n,r}(x)\int_{(r-1)/n}^{r/n}|f(t)-f(x)|^p\,dt\,dx\right\}^{1/p}. \end{aligned} \tag{48}$$

When $x\in[1-1/n,1]$, $[nx]=n-1$. From the definition of the domination, we know

$$\mathscr{K}_{n,r}(x)\le C\Phi(|[nx]-r|+1)=C\Phi(|n-r-1|+1). \tag{49}$$

Then,

$$\begin{aligned} &\|L_n(f)-f\|_{L^p_{[1-h,1]}} \\ &\le Cn^{1/p}\sum_{r=1}^{n}\Phi(|n-r-1|+1) \\ &\qquad\times\left\{\int_{1-1/n}^{1}\int_{(r-1)/n}^{r/n}|f(t)-f(x)|^p\,dt\,dx\right\}^{1/p}. \end{aligned} \tag{50}$$

Applying Lemma 14, we have

$$\begin{aligned} &\|L_n(f)-f\|_{L^p_{[1-h,1]}} \\ &\le C\sum_{r=1}^{n}\Phi(|n-r-1|+1)\,\omega\left(f,\frac{n-r+1}{n}\right)_{L^p} \\ &\le C\omega\left(f,\frac{1}{n}\right)_{L^p}\sum_{r=1}^{n}\Phi(|n-r-1|+1)(n-r+1) \\ &\le C\omega\left(f,\frac{1}{n}\right)_{L^p}\sum_{r=1}^{n}(|n-r-1|+1)\,\Phi(|n-r-1|+1). \end{aligned} \tag{51}$$

Applying (ii), we see that

$$\begin{aligned} &\sum_{r=1}^{n}(|n-r-1|+1)\,\Phi(|n-r-1|+1) \\ &\quad\le\sum_{r=1}^{n}(|n-r-1|+1)\,\Phi^{\epsilon}(|n-r-1|+1) \\ &\quad= DM(\mathscr{K}^{\epsilon}_n)<\infty, \end{aligned} \tag{52}$$

so that we obtain

$$\|L_n(f)-f\|_{L^p_{[1-1/n,1]}}\le C\omega\left(f,\frac{1}{n}\right)_{L^p}. \tag{53}$$

Now we verify the case when $x\in[0,1-1/n]$. It is not difficult to see from Lemma 13 that $n\int_{r/n}^{(r+1)/n}f(t)dt = f_h(r/n)$. By the definition of Kantorovich type operators, rewrite

$$\begin{aligned} &L_n(f,x)-f_h(x) \\ &\quad= n\sum_{r=1}^{n}\mathscr{K}_{n,r}(x)\int_{(r-1)/n}^{r/n}f(t)\,dt-f_h(x) \\ &\quad= \sum_{r=1}^{n}\mathscr{K}_{n,r}(x)\left(f_h\left(\frac{r-1}{n}\right)-f_h(x)\right) \\ &\quad= \sum_{r=1}^{n}\mathscr{K}_{n,r}(x)\int_{x}^{(r-1)/n}f_h'(u)\,du. \end{aligned} \tag{54}$$

This leads to

$$\begin{aligned} &\|L_n(f)-f_h\|^p_{L^p_{[0,1-n^{-1}]}} \\ &\quad= \int_0^{1-1/n}|L_n(f,x)-f_h(x)|^p\,dx \\ &\quad= \int_0^{1-1/n}\left|\sum_{r=1}^{n}\mathscr{K}_{n,r}(x)\int_x^{(r-1)/n}f_h'(u)\,du\right|^p dx \\ &\quad\le \int_0^{1-1/n}\left(\sum_{r=1}^{n}\mathscr{K}^{p(1-\epsilon)/(p-1)}_{n,r}(x)\right)^{p-1} \\ &\qquad\cdot\sum_{j=1}^{n}\mathscr{K}^{p\epsilon}_{n,j}(x)\left|\int_x^{(j-1)/n}f_h'(u)\,du\right|^p dx \\ &\quad\le \int_0^{1-1/n}\left(\sum_{r=1}^{n}\mathscr{K}^{1-\epsilon}_{n,r}(x)\right)^p \\ &\qquad\cdot\sum_{j=1}^{n}\mathscr{K}^{p\epsilon}_{n,j}(x)\left|\frac{j-1}{n}-x\right|^{p-1} \\ &\qquad\cdot\left|\int_x^{(j-1)/n}|f_h'(u)|^p\,du\right|dx. \end{aligned} \tag{55}$$

Since $D(\mathscr{K}^{1-\epsilon}_n)<\infty$,

$$\begin{aligned} &\|L_n(f)-f_h\|^p_{L^p_{[0,1-n^{-1}]}} \\ &\quad\le C^p\int_0^{1-1/n}\sum_{j=1}^{n}\mathscr{K}^{p\epsilon}_{n,j}(x)\left|\frac{j-1}{n}-x\right|^{p-1} \\ &\qquad\times\left|\int_x^{(j-1)/n}|f_h'(u)|^p\,du\right|dx \\ &\quad= C^p\sum_{l=1}^{n-1}\int_{(l-1)/n}^{l/n}\sum_{j=1}^{n}\mathscr{K}^{p\epsilon}_{n,j}(x)\left|\frac{j-1}{n}-x\right|^{p-1} \\ &\qquad\times\left|\int_x^{(j-1)/n}|f_h'(u)|^p\,du\right|dx \end{aligned}$$

$$= C^p \sum_{l=1}^{n-1}\sum_{j=1}^{n} \int_0^{1/n} \mathscr{K}_{n,j}^{p\epsilon}\left(t+\frac{l-1}{n}\right)\left|\frac{j-l}{n}-t\right|^{p-1} \times \left|\int_{t+((l-1)/n)}^{(j-1)/n} \left|f_h'(u)\right|^p du\right| dt$$
$$\le C^p n^{-p} \sum_{l=1}^{n-1}\sum_{j=1}^{n} \Phi^{p\epsilon}\left(\left|l-j-1\right|+1\right) \times \left(\left|l-j-1\right|+1\right)^{p-1} \times \left|\int_{\tau}^{(j-1)/n} \left|f_h'(u)\right|^p du\right|, \tag{56}$$

where

$$\tau = \begin{cases} \dfrac{l-1}{n}, & \text{for } j > l, \\ \dfrac{l}{n}, & \text{for } j \le l. \end{cases} \tag{57}$$

Note that

$$\sum_{l=1}^{n-1}\sum_{j=1}^{n} \Phi^{p\epsilon}\left(\left|l-j-1\right|+1\right)\left(\left|l-j-1\right|+1\right)^{p-1} \times \left|\int_{\tau}^{(j-1)/n} \left|f_h'(u)\right|^p du\right|$$
$$= \sum_{m=1}^{n} \sum_{\substack{|l-j-1|+1=m \\ 1\le l\le n-1, 1\le j\le n}} \Phi^{p\epsilon}(m)\, m^{p-1} \times \left|\int_{\tau}^{(j-1)/n} \left|f_h'(u)\right|^p du\right|$$
$$\le \sum_{m=1}^{n} \Phi^{p\epsilon}(m)\, m^{p-1} \cdot 2m \cdot \int_0^{1-h} \left|f_h'(u)\right|^p du$$
$$\le 2\int_0^{1-h} \left|f_h'(u)\right|^p du \cdot \left(\sum_{m=1}^{n} m\Phi^{\epsilon}(m)\right)^p. \tag{58}$$

Furthermore, condition (ii) implies that

$$\sum_{m=1}^{n} m\Phi^{\epsilon}(m) \le DM\left(\mathscr{K}_n^{\epsilon}\right) < \infty. \tag{59}$$

This means

$$\left\|L_n(f)-f_h\right\|_{L^p_{[0,1-1/n]}}^p \le C^p n^{-p} \int_0^{1-1/n} \left|f_h'(u)\right|^p du. \tag{60}$$

However, from Lemma 13,

$$\int_0^{1-1/n} \left|f_h'(u)\right|^p du = n^p \int_0^{1-1/n} \left|f\left(u+\frac{1}{n}\right)-f(u)\right|^p du \le n^p \omega^p\left(f,\frac{1}{n}\right)_{L^p}, \tag{61}$$

which leads to

$$\left\|L_n(f)-f_h\right\|_{L^p_{[0,1-n^{-1}]}} \le C\omega\left(f,\frac{1}{n}\right)_{L^p}. \tag{62}$$

Combining (62) with (43), we get

$$\left\|L_n(f)-f\right\|_{L^p_{[0,1-n^{-1}]}} \le C\omega\left(f,\frac{1}{n}\right)_{L^p}. \tag{63}$$

This, with (53), finishes Theorem 12. □

For L^1 space, we have the following result while conditions of Theorem 12 can be loosed.

Theorem 15. *Let $f(x) \in L^1_{[0,1]}$, $L_n(f,x)$ be defined by* (3). *If the kernels with* (1) *are dominated globally by $\{\Phi(k)\}$ and satisfy $DM(\mathscr{K}_n) < \infty$, then the estimate*

$$\left\|L_n(f)-f\right\|_{L^1} \le C\omega\left(f,\frac{1}{n}\right)_{L^1} \tag{64}$$

holds.

Proof. The argument of proof is similar, and we can just repeat the corresponding parts of the proof of Theorem 12. □

Remark 16. (1) If the kernels possess good properties, the conditions of Theorem 12 can be easily verified on the terminology of domination. For instance, if the kernels have geometric order, then the corresponding conditions of Theorem 12 are obviously satisfied (see the next section).

(2) There exists essential difference between Theorem 11 and Theorem 12. Theorem 11 requires weaker conditions than Theorem 12 does, but the latter obtains stronger result (the Jackson order is complete up to $1/n$, and the Jackson constant is independent of p!); we will make further illustrations in the coming section.

5. Applications

This section illustrates how to apply Theorems 11 and 12 to estimate L^p approximation. To check the efficiency of two techniques on L^p approximation by Kantorovich type positive linear operators, three examples will be exhibited. Those positive linear operators come from three different categories: rational Müntz operators from rational Müntz systems; the Shepard operators from general real rational function systems; and Bernstein polynomials from the polynomial system. Moreover, in our point of view, they represent three different types: positive linear operators with kernels of geometric order, positive linear operators of arithmetical order, and positive linear operators of local geometric order. It is because the kernels have different domination properties or different speeds of $\{\Phi(n)\}$ that the L^p approximations by the corresponding positive linear operators possess different Jackson orders.

To show the key role of the global (or local) domination on the kernels, the condition (ii) of Theorem 11 will be further explicated to the following lemma.

Lemma 17. *For any kernel $\mathscr{K}_n(x)$, one has*

$$L_n(|t-x|,x) = O(n^{-1})(M(\mathscr{K}_n)+1). \tag{65}$$

Especially, if the kernels $\mathscr{K}_n(x)$ are globally dominated, then

$$L_n(|t-x|,x) = O(n^{-1})DM(\mathscr{K}_n). \tag{66}$$

Proof. From definition (3),

$$L_n(|t-x|,x) = n\sum_{r=1}^{n}\mathscr{K}_{n,r}(x)\int_{(r-1)/n}^{r/n}|t-x|\,dt. \tag{67}$$

For any $x \in [0,1]$, there exists an integer m such that $(m-1)/n \le x < m/n$; then $\int_{(r-1)/n}^{r/n}|t-x|dt$ will be calculated according to the following three cases, respectively.

(a) When $1 \le r < m$,

$$\begin{aligned}\int_{(r-1)/n}^{r/n}|t-x|\,dt &= \int_{(r-1)/n}^{r/n}(x-t)\,dt \\ &= n^{-2}\left(nx-r+\frac{1}{2}\right).\end{aligned} \tag{68}$$

(b) When $m+1 \le r \le n$,

$$\begin{aligned}\int_{(r-1)/n}^{r/n}|t-x|\,dt &= \int_{(r-1)/n}^{r/n}(t-x)\,dt \\ &= n^{-2}\left(r-nx-\frac{1}{2}\right).\end{aligned} \tag{69}$$

(c) When $r = m$,

$$\int_{(m-1)/n}^{m/n}|t-x|\,dt \le \int_{(m-1)/n}^{m/n}n^{-1}\,dt = n^{-2}. \tag{70}$$

Combining (a), (b), and (c), we obtain that

$$\int_{(r-1)/n}^{r/n}|t-x|\,dt \le n^{-2}(|[nx]-r|+1). \tag{71}$$

From Definition 6, (67) and (71) yield (65). At the same time, since the kernels of $L_n(f,x)$ are globally dominated, we obtain by Definition 1 that

$$\mathscr{K}_{n,r}(x) \le C\Phi(|[nx]-r|+1), \quad r = 1,2,\ldots,n. \tag{72}$$

Thus,

$$L_n(|t-x|,x) \le Cn^{-1}\sum_{r=1}^{n}(|[nx]-r|+1)\Phi(|[nx]-r|+1); \tag{73}$$

that is, from Definition 7, (66) holds.

Lemma 17 is done. □

5.1. Rational Müntz Approximation. Rational Müntz Approximation has been researched in [5], shows the application of Theorem 12, and simplifies the proof of [5].

Write

$$\Lambda_n = \{x^{\lambda_1}, x^{\lambda_2}, \ldots, x^{\lambda_n}\}, \tag{74}$$

where span$\{x^{\lambda_k}\}$, $\lambda_k \in \Lambda_n$, is the class of all linear combinations of $\{x^{\lambda_k}\}_{k=1}^{n}$. For $f(x) \in L^p_{[0,1]}$, define

$$R_n(f,\Lambda) = \min_{r\in R(\Lambda_n)}\|f-r\|_{L^p}. \tag{75}$$

Corollary 18. *Given $f(x) \in L^p_{[0,1]}$, $1 \le p < \infty$, the rational Müntz operators are defined by (9). If $\Delta\lambda_n \ge Mn$, $n = 1,2,3,\ldots$, where $M > 0$ is an absolute constant, one has*

$$R_n(f,\Lambda) \le \|M_n(f)-f\|_{L^p} \le C_M\omega\left(f,\frac{1}{n}\right)_{L^p}, \tag{76}$$

where C_M is an absolute constant depending only upon M (independent of p!).

Proof. We verify that $M_n(f,x)$ satisfies all the requirements of Theorem 12. Take $\Phi(k) = e^{-C_M k}$, where $C_M > 0$ is the constant appeared in [4], or [5]; see the following inequality (77). Given a fixed ϵ with $0 < \epsilon < 1$, we verify that

(i) $D(R_k^{1-\epsilon}(x)) < \infty$.

From [4], or [5],

$$R_k(x) \le C_M\exp(-C_M|[nx]-k|). \tag{77}$$

Then,

$$\begin{aligned}\sum_{k=1}^{n}R_k^{1-\epsilon}(x) &\le C_M\sum_{k=1}^{n}\exp(-C_M|[nx]-k|(1-\epsilon)) \\ &\le C_M\sum_{k=0}^{\infty}\exp(-C_M k(1-\epsilon)) < \infty.\end{aligned} \tag{78}$$

(ii) $DM(R_k^{\epsilon}(x)) < \infty$.

Due to Example 3, we have $\Phi(k) = \exp(-C_M k)$. From Definition 7,

$$\begin{aligned}DM(R_k^{\epsilon}) &= \sup_{n>1}\sum_{k=1}^{n}k\exp(-C_M k\epsilon) \\ &\le \sum_{k=0}^{\infty}k\exp(-C_M k\epsilon) < \infty.\end{aligned} \tag{79}$$

Therefore, Corollary 18 can be deduced from Theorem 12. □

By the same argument of (ii), we obviously can obtain $DM(R_k) < \infty$. From (66), we know $M_n(|t-x|,x) = O(n^{-1})$. Moreover, it is simple to verify $\|M_n(f)\|_{L^p} < \infty$ (see, e.g., [7]). Hence, applying Theorem 11, we get

Corollary 19. *For rational Müntz operator $M_n(f,x)$ (9), one has*

$$R_n(f,\Lambda)_{L^p} \le C_{M,p}\omega\left(f,\frac{1}{n}\right)_{L^p}, \tag{80}$$

where $C_{M,p}$ is depending on M and p.

Remark 20. Since the kernels $R_k(x)$ (8) have geometrical order (see Example 3), they certainly satisfy the conditions of both Theorems 11 and 12. Hence, L^p approximation by these rational Müntz operators can always reach the Jackson order by applying both techniques of Theorems 12 and 11. However, for these operators, the conclusion of Corollary 18 surely contains Corollary 19 and makes the latter trivial.

5.2. The Shepard Operators. The approximation of the Shepard operators in the continuous function space $C_{[0,1]}$ has been studied very deeply (see [13, 26–32]). The L^p approximation by the Shepard operators is investigated in [6, 7].

Corollary 21. *Proposed that $f(x) \in L^p_{[0,1]}$, $p > 1$, the Shepard operators are defined by* (11). *Then,*

$$\left\|S_{n,\lambda}(f) - f\right\|_{L^p} \le C_{p,\lambda}\omega\left(f, \epsilon_n\right)_{L^p}, \tag{81}$$

where

$$\epsilon_n = \begin{cases} n^{-1}, & \text{if } \lambda > 2, \\ n^{-1}\log n, & \text{if } \lambda = 2, \\ n^{1-\lambda}, & \text{if } 1 < \lambda < 2. \end{cases} \tag{82}$$

Proof. By applying Theorem 11, we verify this result.

(i) $\|S_{n,\lambda}(f)\|_{L^p}$ are uniformly bounded (see [6]).

(ii) $S_{n,\lambda}(|t-x|, x) = \epsilon_n$ (Lemma 17, Inequality (66)).

Due to Example 4, we know that the global dominated sequence is $\Phi(k) = k^{-\lambda}$. Then, from Definition 7,

$$DM\left(r_k(x)\right) = \sup_{n>1}\sum_{k=1}^{n} k\Phi(k) = \sup_{n>1}\sum_{k=1}^{n} k^{1-\lambda}. \tag{83}$$

When $\lambda > 2$, $\sum_{k=1}^{\infty} k^{1-\lambda} < \infty$.
When $\lambda = 2$, $\sum_{k=1}^{n+1} k^{1-\lambda} \le 2\log n$.
When $1 < \lambda < 2$, $\sum_{k=1}^{n+1} k^{1-\lambda} \le C_\lambda n^{2-\lambda}$.
By Theorem 11, Corollary 21 holds. □

Corollary 22. *Propose that $f(x) \in L^p_{[0,1]}$, $1 \le p < \infty$, the Shepard operators are defined by* (11). *If $\lambda > 3$, then*

$$\left\|S_{n,\lambda}(f) - f\right\|_{L^p} \le C_\lambda\omega\left(f, \frac{1}{n+1}\right)_{L^p}, \tag{84}$$

where C_λ depends on λ only.

Proof. Theorem 12 will be applied to prove this result. The details can be referred in [7]. When $\lambda > 3$, let $\epsilon = (\lambda+1)/2\lambda$, then $0 < \epsilon < 1$, $\lambda\epsilon > 2$ and $\lambda(1-\epsilon) > 1$. Evidently, the Shepard kernels have λ-arithmetic order (see Example 4). We check the corresponding conditions of Theorem 12:

(i) $D(r_k^{1-\epsilon}(x)) < \infty$.

From Definition 6 and Example 4,

$$\begin{aligned} &\sum_{k=1}^{n} r_k^{1-\epsilon}(x) \\ &\quad\le C_\lambda \sum_{k=0}^{n}\left(\left|\left[(n+1)x\right] - k\right| + 1\right)^{-\lambda(1-\epsilon)} \\ &\quad\le C_\lambda \sum_{k=1}^{\infty} k^{-\lambda(1-\epsilon)} < \infty. \end{aligned} \tag{85}$$

Then,
(ii) $DM(r_k^{\epsilon}(x)) < \infty$.

Note that the present dominated sequence $\Phi(k) = k^{-\lambda}$; then from Definition 7

$$DM\left(r_k^{\epsilon}(x)\right) = \sup_{n>1}\sum_{k=1}^{n} k\Phi^{\epsilon}(k) \le \sup_{n>1}\sum_{k=1}^{\infty} k^{1-\lambda\epsilon} < \infty. \tag{86}$$

Corollary 22 is completed from Theorem 12 as (i) and (ii) hold. □

Remark 23. The difference between L^p approximation techniques (I) and (II) with respect to Theorems 11 and 12 is fully exhibited on the Shepard operators by Corollaries 21 and 22. Stronger requirements by applying technique (II) than by applying technique (I) are needed. However, if the conditions are satisfied, the former can obtain essentially better result (the Jackson constant is independent of p!). On this particular case, we can obtain the Jackson type estimation by applying technique (I) for $\lambda > 1$, while achieve the corresponding result by applying technique (II) only for $\lambda > 3$. We still do not know how to deal with the cases when $1 < \lambda \le 3$ by applying technique (II).

5.3. Bernstein Operators. There are many results on L^p approximation by Bernstein polynomials; interested readers may refer to [8–23]. These results can be classified into the following categories:

(1) uniform convergence; see [8, 12, 15, 18];

(2) quantitative estimations; see [10, 11, 14, 19];

(3) equivalence theorems; see [16, 17];

(4) saturation problems; see [9, 16, 20–23].

Here, we apply our technique (I) to test the corresponding Jackson type estimation as an example. The following proof is simpler than the past proof.

Corollary 24. *Given $f(x) \in L^p_{[0,1]}$, $p > 1$, one has the following estimation:*

$$\left\|B_n(f) - f\right\|_{L^p} \le C_p\omega\left(f, \frac{1}{\sqrt{n}}\right)_{L^p}. \tag{87}$$

Proof. $\|B_n(f)\|_{L^p}$ is uniformly bounded which is known from [33]. We only need to evaluate $B_n(|t-x|, x)$ from Theorem 11. It is known from [15, page 15] and Definition 6 that

$$M\left(P_{n,k}(x)\right) = \sup_{n\ge1}\sum_{k=0}^{n}|nx-k|\,P_{n,k}(x) \le C\sqrt{n}. \tag{88}$$

From (65) of Lemma 17 and (88),

$$B_n(|t-x|, x) = O\left(n^{-1}\right) M\left(P_{n,k}(x)\right) = O\left(n^{-1/2}\right) \qquad (89)$$

which, from Theorem 11, leads to Corollary 24. □

Remark 25. The approximation order of Corollary 24 is sharp which shows that (88) cannot be improved. In other words, $M(P_{n,k}(x))$ is unbounded. Hence, the approximation technique (II) or Theorem 12 cannot be applied in this case. That is to say, we can never obtain the Jackson constant independent of p for Bernstein polynomials in L^p spaces. Furthermore, Corollary 24 also exhibits that Kantorovich type operators (2) or (3) cannot reach Jackson type estimation with the Jackson constant independent of p unless their kernels possess good properties such as having globally geometric domination (rational Müntz approximation case) or having global ρ-arithmetic domination for sufficiently large ρ (the Shepard operators with $\lambda > 3$).

6. Conclusions

On the above discussions, the positive linear operators used in L^p approximation can be classified according to the properties of their kernels. We have three categories: kernels with geometrical order (such as the rational Müntz operators), kernels with arithmetic order (such as the Shepard operators), and kernels with local arithmetic order (such as the Bernstein operators). In another word, for characterizing the Jackson type estimate in L^p spaces by the Kantorovich type operators (3) or (5), it always plays an essential role how well the kernels of the operators under study behave.

References

[1] M. Felten, "Local and global approximation theorems for positive linear operators," *Journal of Approximation Theory*, vol. 94, no. 3, pp. 396–419, 1998.

[2] D. H. Mache and D. X. Zhou, "Characterization theorems for the approximation by a family of operators," *Journal of Approximation Theory*, vol. 84, no. 2, pp. 145–161, 1996.

[3] K. Satō, "Global approximation theorems for some exponential-type operators," *Journal of Approximation Theory*, vol. 32, no. 1, pp. 32–46, 1981.

[4] J. Bak, "On the efficiency of general rational approximation," *Journal of Approximation Theory*, vol. 20, no. 1, pp. 46–50, 1977.

[5] W. Xiao and S. Zhou, "On Müntz rational approximation rate in L^p spaces," *Journal of Approximation Theory*, vol. 111, no. 1, pp. 50–58, 2001.

[6] W. Xiao and S. P. Zhou, "A Jackson type estimate for Shepard operators in L^p spaces for $p \geq 1$," *Acta Mathematica Hungarica*, vol. 95, no. 3, pp. 217–224, 2002.

[7] W. Xiao, S. P. Zhou, and L. Y. Zhu, "L^p-approximation for the Shepard operators," *Journal of Zhejiang University*, vol. 27, no. 4, pp. 364–371, 2000 (Chinese).

[8] M. Becker and R. J. Nessel, "On the global approximation by Kantorovitch polynomials," in *Approximation Theory III*, pp. 207–212, Academic Press, New York, NY, USA, 1980.

[9] M. Becker and R. J. Nessel, "On global saturation for Kantorovitch polynomials," in *Approximation and Function Spaces*, pp. 89–101, North-Holland, Amsterdam, The Netherlands, 1981.

[10] H. Berens and R. A. Devore, "Quantitative Korovkin theorems for L^p-spaces," in *Approximation Theory II*, pp. 289–298, Academic Press, New York, NY, USA, 1976.

[11] R. Bojanic and O. Shisha, "Degree of L_1 approximation to integrable functions by modified Bernstein polynomials," *Journal of Approximation Theory*, vol. 13, pp. 66–72, 1975.

[12] P. L. Butzer, "Dominated convergence of Kantorovitch polynomials in the space L^p," *Transactions of the Royal Society of Canada*, vol. 46, pp. 23–27, 1952.

[13] Z. Ditzian, "A global inverse theorem for combinations of Bernšteǐn polynomials," *Journal of Approximation Theory*, vol. 26, no. 3, pp. 277–292, 1979.

[14] A. Grundmann, "Güteabschätzungen für den Kantorovič-operator in der L_1-norm," *Mathematische Zeitschrift*, vol. 150, no. 1, pp. 45–47, 1976.

[15] G. G. Lorentz, *Bernstein polynomials*, University of Toronto Press, Toronto, Canada, 1953.

[16] G. G. Lorentz and L. L. Schumaker, "Saturation of positive operators," *Journal of Approximation Theory*, vol. 5, pp. 413–424, 1972.

[17] D. H. Mache, "Equivalence theorem on weighted simultaneous L^p-approximation by the method of Kantorovič operators," *Journal of Approximation Theory*, vol. 78, no. 3, pp. 321–350, 1994.

[18] V. Maier, "L^p-approximation by Kantorovič operators," *Analysis Mathematica*, vol. 4, no. 4, pp. 289–295, 1978.

[19] M. W. Miller, "Die Güte der L^p-approximation durch Kantorovic-polynome," *Mathematische Zeitschrift*, vol. 152, pp. 243–247, 1976.

[20] S. D. Riemenschneider, "The L^p-saturation of the Bernšteǐn-Kantorovitch polynomials," *Journal of Approximation Theory*, vol. 23, no. 2, pp. 158–162, 1978.

[21] V. Totik, "$L^p(p > 1)$-approximation by Kantorovich polynomials," *Analysis*, vol. 3, no. 1–4, pp. 79–100, 1983.

[22] V. Totik, "Problems and solutions concerning Kantorovich operators," *Journal of Approximation Theory*, vol. 37, no. 1, pp. 51–68, 1983.

[23] V. Totik, "Approximation in L^1 by Kantorovich polynomials," *Acta Scientiarum Mathematicarum*, vol. 46, no. 1–4, pp. 211–222, 1983.

[24] R. A. Devore, "Degree of approximation," in *Approximation Theory II*, pp. 117–161, Academic Press, New York, NY, USA, 1976.

[25] E. M. Stein, *Singular Integrals*, Princeton, New Jersey, NJ, USA, 1970.

[26] G. Criscuolo and G. Mastroianni, "Estimates of the Shepard interpolatory procedure," *Acta Mathematica Hungarica*, vol. 61, no. 1-2, pp. 79–91, 1993.

[27] G. Somorjai, "On a saturation problem," *Acta Mathematica Academiae Scientiarum Hungaricae*, vol. 32, no. 3-4, pp. 377–381, 1978.

[28] J. Szabados, "Direct and converse approximation theorems for the Shepard operator," *Approximation Theory and Its Applications*, vol. 7, no. 3, pp. 63–76, 1991.

[29] B. Della Vecchia and G. Mastroianni, "Pointwise simultaneous approximation by rational operators," *Journal of Approximation Theory*, vol. 65, no. 2, pp. 140–150, 1991.

[30] B. Della Vecchia, "Direct and converse results by rational operators," *Constructive Approximation*, vol. 12, no. 2, pp. 271–285, 1996.

[31] P. Vértesi, "Saturation of the Shepard operator," *Acta Mathematica Hungarica*, vol. 72, no. 4, pp. 307–317, 1996.

[32] T. Xie, R. J. Zhang, and S. Zhou, "Three conjectures on Shepard interpolatory operators," *Journal of Approximation Theory*, vol. 93, no. 3, pp. 399–414, 1998.

[33] Z. X. Liu, "Approximation of the Kantorovič-Bézier operators in $L^p(0,1)$," *Northeastern Mathematical Journal*, vol. 7, no. 2, pp. 199–205, 1991.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 674191, 19 pages
http://dx.doi.org/10.1155/2013/674191

Research Article

Stability Analysis of a Repairable System with Warning Device and Repairman Vacation

Yangli Ren, Lina Guo, and Lingling Zhang

Department of Mathematics, Taiyuan University of Technology, Taiyuan 030024, China

Correspondence should be addressed to Lina Guo; guolina982@163.com

Received 19 April 2013; Accepted 7 August 2013

Academic Editor: Gen-Qi Xu

This paper considers a simple repairable system with a warning device and a repairman who can have delayed-multiple vacations. By Markov renewal process theory and the probability analysis method, the system is first transformed into a group of integrodifferential equations. Then, the existence and uniqueness as well as regularity of the system dynamic solution are discussed with the functional analysis method. Further, the asymptotic stability, especially the exponential stability of the system dynamic solution, is studied by using the strongly continuous semigroup theory or C_0 semigroup theory. The reliability indices and some applications (such as the comparisons of indices and profit of systems with and without warning device), as well as numerical examples, are presented at the end of the paper.

1. Introduction

A repairable system is a system which, after failing to perform one or more of its functions satisfactorily, can be restored to fully satisfactory performance by any method, rather than the replacement of the entire system. With different repair levels, repair can be broken down into three categories (see [1]): perfect repair, normal repair, and minimal repair. A perfect repair can restore a system to an "as good as new" state, a normal repair is assumed to bring the system to any condition, and a minimal repair, or imperfect repair, can restore the system to the exact state it was before failure.

Repairable system is not only a kind of important system discussed in reliability theory but also one of the main objects studied in reliability mathematics. Since the 1960s, various repairable system models have been established and researched.

However, in traditional repairable systems, it is assumed that the repairman or server remains idle until a failed component presents. But as Mobley [2] pointed out, one-third of all maintenance costs were wasted as the result of unnecessary or improper maintenance activities. Today, the role of maintenance tends to be a "profit contributor." Therefore, much more profit can be produced when the repairman in a system might take a sequence of vacations in the idle time. Repairman's vacation may literally mean a lack of work or repairman taking another assigned job. From the perspective of rational use of human resources, the introduction of repairman's vacation makes modeling of the repairable system more realistic and flexible. This is due to the fact that in practice, the vast majority of small-and medium-sized enterprises (SMEs) cannot afford to hire a full-time repairman. So, the repairman in SMEs usually plays two roles: one for looking after the equipment and one for other duties. Under normal circumstances, the repairman has to periodically check the status of the system. If he finds that the system failed, he repairs it immediately after the end of vacation; otherwise, he will leave the system for other duties or for a vacation.

Vacation model originally arised in queueing theory and has been well studied in the past three decades and successfully applied in many areas such as manufacturing/service and computer/communication network systems. Excellent surveys on the earlier works of vacation models have been reported by Doshi [3], Takagi [4], and Tian and Zhang [5]. A number of works (e.g., please see [6–10] and references therein) have recently appeared in the queueing literature in which concepts of different control operating politics along with vacations have been discussed. And Ke et al. [11] provided a summary of the most recent research works on vacation queueing systems in the past 10 years, in which a wide class of vacation policies for governing the vacation mechanism is presented.

In the past decade, inspired by the vacation queueing theory, some researchers introduced vacation model into repairable systems. The available references concerning repairman vacation in repairable systems can be classified into two categories: one is focused on the system indices and the other is the optimization problems.

For the first category, Jain and Rakhee [12] considered the bilevel control policy for a machining system having two repairmen. One turns on when queue size of failed units reaches a preassigned level. The other's provision in case of long queue of failed units may be helpful in reducing the backlog. The steady state queue size distribution is obtained by applying the recursive method. Hu et al. [13] studied the steady-state availability and the mean up-time of a series-parallel repairable system consisting of one master control unit, two slave units, and a single repairman who operates single vacation by using the supplementary variable method and the vector Markov process theory. Q. T. Wu and S. M. Wu [14] analyzed some reliability indices of a cold standby system consisting of two repairable units, a switch and a repairman who may not always be at the job site or take vacation. Yuan [15] and Yuan and Cui [16] studied a k-out-of-n:G system and a consecutive-k-out-of-n:F system, respectively, with R repairmen who can take multiple vacations and by using Markov model; the analytical solution of some reliability indices was discussed. Yuan and Xu [17] studied a deteriorating system with a repairman who can have multiple vacations. By means of the geometric process and the supplementary variable techniques, a group of partial differential equations of the system was presented, and some reliability indices were derived. Ke and Wu [18] studied a multiserver machine repair model with standbys and synchronous multiple vacations, and the stationary probability vectors were obtained by using the matrix-analytical approach and the technique of matrix recursive.

For the second category, Ke and Wang [19] studied a machine repair problem consisting of M operating machines with two types of spare machines and R servers (repairmen) who can take different vacation policies. The steady-state probabilities of the number of failed machines in the system as well as the performance measures were derived by using the matrix geometric theory, and a direct search algorithm was used to determine the optimal values of the number of two types of spares and the number of servers while maintaining a minimum specified level of system availability. Jia and Wu [20] considered a replacement policy for a repairable system that can not be repaired "as good as new" with a repairman who can have multiple vacations. By using geometric processes, the explicit expression of the expected cost rate was derived, and the corresponding optimal policy was determined analytically and numerically. Yuan and Xu [21, 22] considered, respectively, a deteriorating repairable system and a cold standby repairable system with two different components of different priority in use, both with one repairman who can take multiple vacations. The explicit expression of the expected cost rate was given, and an optimal replacement policy was discussed. Yu et al. [23] analyzed a phase-type geometric process repair model with spare device procurement lead time and repairman's multiple vacations. Employing the theory of renewal reward process, the explicit expression of the long-run average profit rate for the system was derived, and the optimal maintenance policy was also numerically determined.

However, to the best knowledge of the authors, whichever the catalogue, the references above only concentrated on the steady state (the steady-state indices or the steady-state optimization problems) of the systems. It is because that the transient behavior of a system is difficult to be studied. Therefore, in reliability study researchers usually substitute the steady-state solution for the instantaneous one of a system, for the steady-state solution can be easily obtained by Laplace transform and a limit theorem. Whereas, Laplace transform should be based on the two hypotheses: (1) the instantaneous solution of the interested system existed and (2) the instantaneous solution of the system is stable. Whether the hypotheses hold or not is still an open question and should be justified. Moreover, the substitution of the steady-state solution for the instantaneous one is not always rational. For detailed information or explanations, please see [24, 25].

Warning systems emerge in the background of repairable systems which are stepping into the times of requiring of both advanced warning and real-time fault detection. The so-called warning system is able to send emergency signals and report dangerous situations prior to disasters, catastrophes and/or other dangers need to watch out based on previous experiences and/or observed possible omens. Real-time warning systems play an important role in fault management in banking, telecommunications, securities, electric power, and other industries. If the warning prompts during system operation, operating staff can choose shut down the system, operate carefully, or repair the system. Warning systems can help users to achieve the 24-hour uninterrupted real-time monitoring and alerting during running of various types of network infrastructure sand application services. Therefore, there is a need to study the repairable systems with warning device.

This paper considers a simple repairable system with a warning device and a repairman who can have delayed-multiple vacations. The delayed-multiple vacations mean that the repairman will not leave for a vacation immediately if there is no component failed. However, there is a stochastic vacation-preparing period in which if a failed component appears he will stop the vacation preparing and serve it immediately; otherwise, he will take a rest on the end of the vacation-preparing period. When he returns from a vacation, he will either deal with the failed components waiting in the system or prepare for another vacation. In this paper, we are devoted to studying the asymptotic behavior of the system by strongly continuous semigroup theory and make comparisons of indices (such as reliability, availability, and the probability of the repairman's vacation) and profit of the two systems with and without warning device.

The paper is structured as follows. The coming section introduces the system model specifically and expresses it into a group of integrodifferential equations by Markov renewal process theory and the probability analysis method. Section 3 discusses the existence and uniqueness as well as the regularity of the system dynamic solution by the functional analysis method. Section 4 studies the asymptotic behavior of the system by strongly continuous semigroup theory or C_0 semigroup theory. Section 5 presents some reliability indices of the system, and the steady-state indices are discussed from the viewpoint of eigenfunction of the system operator. In Section 6, comparisons of indices and profit of systems with and without warning device are made. And a brief conclusion is offered in the last section.

2. System Formulation

The system model of interest is a simple repairable system (i.e., a repairable system with a unit and a repairman) with repairman vacation and a warning device. It is described specifically as follows: at the initial time $t = 0$, the unit is new, the system begins to work, and the repairman starts to prepare for the vacation. If the unit fails in the delayed-vacation period, the repairman deals with it immediately, and the delayed vacation is terminated. Otherwise, he leaves for a vacation after the delayed-vacation period ends. If the warning device sends alerts in the delayed-vacation period, the repairman will stay in the system until the unit fails. Whenever the repairman returns from a vacation, he either prepares for the next vacation if the unit is working or deals with the failed unit immediately or stays in the system if the warning device has sent alerts. The repair facility neither failed nor deteriorated. The unit is repaired as good as new. Further, we assume the following.

(1) The distribution function of the working time of the unit is $F(t) = 1 - e^{-\lambda t}$, $t \geq 0$, λ is a positive constant, and the distribution function of its repair time is $G(t) = \int_0^t g(x)\mathrm{d}x = 1 - e^{-\int_0^t \mu(x)\mathrm{d}x}$ and $\int_0^\infty t\mathrm{d}G(t) = 1/b$.

(2) The distribution function of the delayed-vacation time of the repairman is $D(t) = 1 - e^{-\varepsilon t}$, $t \geq 0$, ε is a positive constant, and the distribution function of his vacation time is $V(t) = 1 - e^{-\mu_0 t}$, μ_0 is a positive constant.

(3) The distribution function of the time of the warning device from its beginning to work to its first sending alerts is $U(t) = 1 - e^{-\alpha_0 t}$, $t \geq 0$; α_0 is a positive constant.

(4) The above stochastic variables are independent of each other.

Set $N(t)$ to be the state in which the system is at time t, and assume all the possible states as follows:

0: the system is working, and the repairman is preparing for the vacation;

1: the system is working, and the repairman is on vacation;

2: the system is warning, and the repairman is in the system;

3: the system is warning, and the repairman is on vacation;

4: the unit failed, and the repairman is on vacation;

5: the repairman is dealing with the failed unit.

Then, by using probability analysis method, the system model can be described as the following group of integrodifferential equations:

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}t} + \varepsilon + \alpha_0\right) P_0(t) &= \mu_0 P_1(t) + \int_0^\infty \mu(x) P_5(t,x)\,\mathrm{d}x, \\
\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mu_0 + \alpha_0\right) P_1(t) &= \varepsilon P_0(t), \\
\left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda\right) P_2(t) &= \alpha_0 P_0(t) + \mu_0 P_3(t), \\
\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mu_0 + \lambda\right) P_3(t) &= \alpha_0 P_1(t), \\
\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mu_0\right) P_4(t) &= \lambda P_3(t), \\
\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu(x)\right] P_5(t,x) &= 0.
\end{aligned} \tag{1}$$

The boundary condition is

$$P_5(t,0) = \lambda P_2(t) + \mu_0 P_4(t). \tag{2}$$

The initial conditions are

$$P_0(0) = 1, \text{ the others equal to } 0. \tag{3}$$

Here, $P_i(t)$ represents the probability that the system is in state i at time t, $i = 0, 1, \ldots, 4$, and $P_5(t,x)\mathrm{d}x$ represents the probability that the system is in state 5 with elapsed repair time lying in $[x, x + \mathrm{d}x)$ at time t.

Concerning the practical background, we can assume that

$$\mu(x) \geq 0, \quad \mu = \sup_{x\in[0,\infty)} \mu(x) < \infty. \tag{4}$$

3. Existence and Uniqueness of System Solution

In this section, we will study the existence and uniqueness as well as the regularity of the system solution. Firstly, we will transform the system (1)–(3) into an equivalent integral problem (P) by the method of characteristics. Secondly, the existence and uniqueness of the local solution of problem (P) are discussed by using the fixed point theory. Then, the existence and uniqueness of the global solution of problem (P) is further studied by a uniform priori estimate. Thus, the existence and uniqueness of the solution of system (1)–(3) are obtained. Moreover, the regularity or the C^1 continuity of the system solution is also discussed.

3.1. Unique Existence of System Local Solution. For convenience, we will give some notations. Let

$$L_1 = L^1[0,\infty), \qquad V_0 = C[0,T], \qquad V_1 = C([0,T], L_1) \tag{5}$$

with norm

$$\begin{aligned} \|\varphi\|_{L_1} &= \int_0^\infty |\varphi(x)|\,\mathrm{d}x, \\ \|f\|_{V_0} &= \max_{t\in[0,T]} |f(t)|, \\ \|g\|_{V_1} &= \max_{t\in[0,T]} \int_0^\infty |g(t,x)|\,\mathrm{d}x. \end{aligned} \tag{6}$$

Choose

$$M = \{q \in V_0 \mid q(0) = 1, q \geq 0, \|q\| \leq 2\}. \tag{7}$$

Clearly, M is a closed subspace of V_0.

By the method of characteristics [26], the following equivalent proposition can be easily obtained [27, 28].

Theorem 1. *For a given constant $T > 0$, $P_i(t) \in V_0$, $i = 0, 1, \ldots, 4$, $P_5(t,x) \in V_1$ are the solution to* (1)–(3) *if and only if they are the solution to the following integral problem* (*P*):

$$\begin{aligned} P_0(t) &= e^{-(\varepsilon+\alpha_0)t} + \int_0^t \left[\mu_0 P_1(s) + \int_0^\infty \mu(x) P_5(s,x)\,\mathrm{d}x\right] \\ &\quad \times e^{-(\varepsilon+\alpha_0)(t-s)}\,\mathrm{d}s, \\ P_1(t) &= \int_0^t \varepsilon P_0(s) e^{-(\mu_0+\alpha_0)(t-s)}\,\mathrm{d}s, \\ P_2(t) &= \int_0^t [\alpha_0 P_0(s) + \mu_0 P_3(s)] e^{-\lambda(t-s)}\,\mathrm{d}s, \\ P_3(t) &= \int_0^t \alpha_0 P_1(s) e^{-(\mu_0+\lambda)(t-s)}\,\mathrm{d}s, \\ P_4(t) &= \int_0^t \lambda P_3(s) e^{-\mu_0(t-s)}\,\mathrm{d}s, \\ P_5(t,x) &= \begin{cases} 0, & x \geq t \\ [\lambda P_2(t-x) + \mu_0 P_4(t-x)] e^{-\int_0^x \mu(\tau)\,\mathrm{d}\tau}, & x < t. \end{cases} \end{aligned} \tag{P}$$

Clearly, to get the existence and uniqueness of the solution of system (1)–(3), it is necessary to study the existence and uniqueness of the solution of the above integral problem (P). To this end, for any $q \in V_0$, we define six operators as follows:

$$\begin{aligned} K_0(q)(t) &= e^{-(\varepsilon+\alpha_0)t} \\ &\quad + \int_0^t \left[\mu_0 K_1(q)(s) \right. \\ &\qquad \left. + \int_0^\infty \mu(x) K_5(q)(s,x)\,\mathrm{d}x\right] \\ &\quad \times e^{-(\varepsilon+\alpha_0)(t-s)}\mathrm{d}s \end{aligned} \tag{8}$$

$$K_1(q)(t) = \int_0^t \varepsilon q(s) e^{-(\mu_0+\alpha_0)(t-s)}\mathrm{d}s, \tag{9}$$

$$K_2(q)(t) = \int_0^t [\alpha_0 q(s) + \mu_0 K_3(q)(s)] e^{-\lambda(t-s)}\mathrm{d}s, \tag{10}$$

$$K_3(q)(t) = \int_0^t \alpha_0 K_1(q)(s) e^{-(\mu_0+\lambda)(t-s)}\mathrm{d}s, \tag{11}$$

$$K_4(q)(t) = \int_0^t \lambda K_3(q)(s) e^{-\mu_0(t-s)}\mathrm{d}s, \tag{12}$$

$$\begin{aligned} K_5(q)(t,x) &= [\lambda K_2(q)(t-x) + \mu_0 K_4(q)(t-x)] \\ &\quad \times e^{-\int_0^x \mu(\tau)\mathrm{d}\tau}. \end{aligned} \tag{13}$$

It can be seen that for $q \in V_0$, if the operators K_i, $i = 1, 2, \ldots, 5$ are determined, it needs only to get the fixed point of the operator K_0 in order to get the existence and uniqueness of the solution of the integral problem (P).

From (9)–(13), the following two lemmas can be easily obtained.

Lemma 2. *For a given constant $T > 0$ such that $t \in [0,T]$, then for any $q \in M$, there exist unique and nonnegative $K_i(q) \in V_0$, $i = 1, \ldots, 4$ and $K_5(q) \in V_1$ satisfying* (9)–(13).

Lemma 3. *For a given constant $T > 0$ such that $t \in [0,T]$, then for any $q, \tilde{q} \in M$, the following estimations hold:*

$$\begin{aligned} &\|K_1(q)\| \leq 2\varepsilon T, \\ &\|K_1(q) - K_1(\tilde{q})\| \leq \varepsilon T \|q - \tilde{q}\|, \\ &\|K_3(q)\| \leq 2\alpha_0 \varepsilon T^2, \\ &\|K_3(q) - K_3(\tilde{q})\| \leq \alpha_0 \varepsilon T^2 \|q - \tilde{q}\|, \\ &\|K_2(q)\| \leq 2\alpha_0 T \left(1 + \mu_0 \varepsilon T^2\right), \\ &\|K_2(q) - K_2(\tilde{q})\| \leq \alpha_0 T \left(1 + \mu_0 \varepsilon T^2\right) \|q - \tilde{q}\|, \\ &\|K_4(q)\| \leq 2\lambda \alpha_0 \varepsilon T^3, \\ &\|K_4(q) - K_4(\tilde{q})\| \leq \lambda \alpha_0 \varepsilon T^3 \|q - \tilde{q}\|, \\ &\|K_5(q)\| \leq 2\lambda \alpha_0 T^2 \left(1 + 2\mu_0 \varepsilon T^2\right), \\ &\|K_5(q) - K_5(\tilde{q})\| \leq \lambda \alpha_0 T^2 \left(1 + 2\mu_0 \varepsilon T^2\right) \|q - \tilde{q}\|. \end{aligned} \tag{14}$$

Theorem 4. *There exists a $T = T_0 > 0$, such that K_0 has a unique fixed point on M.*

Proof. We prove the theorem in two steps. Firstly, we prove that the operator K_0 is a mapping from M to M. From the definition of K_0, we can know that if $q \in M$, then $K_0(q) \in V_0$ and $K_0(q)(0) = 1$. Choose $0 < T_0 < 1$ satisfying

$$T_0^2 [\mu_0 \varepsilon + \mu \lambda \alpha_0 (1 + 2\mu_0 \varepsilon)] < \frac{1}{2}. \tag{15}$$

Then, from (8) and Lemmas 2 and 3, it can be derived that

$$\begin{aligned}\|K_0(q)\| &= \max_{t\in[0,T]} |K_0(q)(t)| \le 1 + 2T^2 \\ &\quad \times \left[\mu_0\varepsilon + \mu\lambda\alpha_0 T\left(1 + 2\mu_0\varepsilon T^2\right)\right] \\ &< 1 + 2T^2 \left[\mu_0\varepsilon + \mu\lambda\alpha_0 \left(1 + 2\mu_0\varepsilon\right)\right] < 2.\end{aligned} \tag{16}$$

This implies that $K_0(q) \in M$.

Secondly, we prove that the operator K_0 is a strictly compressed mapping on M. For any $q, \tilde{q} \in M$, from (8) and Lemma 3, we have

$$\begin{aligned}&\|K_0(q) - K_0(\tilde{q})\| \\ &\quad \le T^2 \left[\mu_0\varepsilon + \mu\lambda\alpha_0 T\left(1 + 2\mu_0\varepsilon T^2\right)\right] \|q - \tilde{q}\| \\ &\quad < T^2 \left[\mu_0\varepsilon + \mu\lambda\alpha_0 \left(1 + 2\mu_0\varepsilon\right)\right] \|q - \tilde{q}\| \\ &\quad < \frac{1}{2} \|q - \tilde{q}\|.\end{aligned} \tag{17}$$

This means that K_0 is strictly compressed. According to the Banach contraction mapping principle combining the above two steps, it can be deduced readily that K_0 has a unique fixed point on M. The proof of Theorem 4 is completed. □

Theorems 1 and 4 combing Lemma 2 follows the existence and uniqueness of the local solution of system (1)–(3).

Theorem 5 (existence and uniqueness of local solution). *There exists a $T = T_0 > 0$ such that the system* (1)–(3) *has a unique nonnegative local solution* $(P_0, P_1, \ldots, P_4, P_5) \in V_0^5 \times V_1$.

3.2. Unique Existence of System Global Solution. In this section, we will prove the existence and uniqueness of the global solution of system (1)–(3) by a uniform priori estimate and extension theorem.

Lemma 6. *For a given constant $T > 0$, if $(P_0, P_1, \ldots, P_4, P_5) \in V_0^5 \times V_1$ is the nonnegative solution of system* (1)–(3)*, then one has:*

$$E(t) \le e^{QT}, \quad \forall t \in [0, T], \tag{18}$$

where $E(t) = \sum_{i=0}^{4} P_i(t) + \int_0^\infty P_5(t, x)\,\mathrm{d}x$, $Q = \max\{\varepsilon + \alpha_0, \mu_0 + \alpha_0, \mu_0 + \lambda, \mu\}$ and μ is defined in (4).

Proof. Because the solution of system (1)–(3) is the solution of problem (P), the estimation of the system solution can be obtained easily as follows:

$$\begin{aligned}P_0(t) &\le e^{-(\varepsilon+\alpha_0)t} + \int_0^t \mu_0 P_1(s)\,\mathrm{d}s \\ &\quad + \int_0^t \left[\int_0^\infty \mu(x) P_5(s, x)\,\mathrm{d}x\right] \mathrm{d}s \\ &\le 1 + \mu_0 \int_0^t P_1(s)\,\mathrm{d}s \\ &\quad + \mu \int_0^t \left[\int_0^\infty P_5(s, x)\,\mathrm{d}x\right] \mathrm{d}s, \\ P_1(t) &\le \varepsilon \int_0^t P_0(s)\,\mathrm{d}s, \\ P_2(t) &\le \alpha_0 \int_0^t P_0(s)\,\mathrm{d}s + \mu_0 \int_0^t P_3(s)\,\mathrm{d}s, \\ P_3(t) &\le \alpha_0 \int_0^t P_1(s)\,\mathrm{d}s, \qquad P_4(t) \le \lambda \int_0^t P_3(s)\,\mathrm{d}s, \\ \int_0^\infty P_5(t, x)\,\mathrm{d}x &\le \lambda \int_0^\infty P_2(t - x)\,\mathrm{d}x \\ &\quad + \mu_0 \int_0^\infty P_4(t - x)\,\mathrm{d}x \\ &= \lambda \int_0^t P_2(s)\,\mathrm{d}s + \mu_0 \int_0^t P_4(s)\,\mathrm{d}s.\end{aligned} \tag{19}$$

Thus,

$$\begin{aligned}E(t) &= \sum_{i=0}^{4} P_i(t) + \int_0^\infty P_5(t, x)\,\mathrm{d}x \\ &\le 1 + (\varepsilon + \alpha_0) \int_0^t P_0(s)\,\mathrm{d}s \\ &\quad + (\mu_0 + \alpha_0) \int_0^t P_1(s)\,\mathrm{d}s + \lambda \int_0^t P_2(s)\,\mathrm{d}s \\ &\quad + (\mu_0 + \lambda) \int_0^t P_3(s)\,\mathrm{d}s + \mu_0 \int_0^t P_4(s)\,\mathrm{d}s \\ &\quad + \mu \int_0^t \left[\int_0^\infty P_5(s, x)\,\mathrm{d}x\right] \mathrm{d}s.\end{aligned} \tag{20}$$

Let $Q = \max\{\varepsilon + \alpha_0, \mu_0 + \alpha_0, \mu_0 + \lambda, \mu\}$, then $E(t) \le 1 + Q\int_0^t E(s)\mathrm{d}s$. The Gronwall Inequality follows the estimation immediately: $E(t) \le e^{QT}$, for all $t \in [0, T]$. The proof of Lemma 6 is completed. □

From Theorem 5, Lemma 6, and extension theorem, the existence and uniqueness of the system solution can be derived readily as below.

Theorem 7 (existence and uniqueness of global solution). *For any $T > 0$, the system* (1)–(3) *has a unique nonnegative solution* $(P_0, P_1, \ldots, P_4, P_5) \in V_0^5 \times V_1$.

3.3. Regularity of System Solution. In this section, we discuss the regularity or the C^1 continuity of the solution of system (1)–(3).

From Theorem 1 and the expressions in problem (P) and noting the assumption (4), the following result is obvious.

Theorem 8. *For any* $T > 0$*, if* $(P_0, P_1, \ldots, P_4, P_5) \in V_0^5 \times V_1$ *is the nonnegative solution of system* (1)–(3)*, then* $P_i \in C[0,T]$*,* $i = 0, 1, \ldots, 4$ *and* $P_5 \in C([0,T] \times [0,\infty))$*.*

Theorem 9. *For any* $T > 0$*, assume* $\mu(x)$ *is continuous on* $[0,T]$*. If* $(P_0, P_1, \ldots, P_4, P_5) \in V_0^5 \times V_1$ *is the nonnegative solution of system* (1)–(3)*, then* $P_i \in C^1[0,T]$*,* $i = 0, 1, \ldots, 4$*,* $P_5 \in C^1(D)$*, where* $D = \{(t,x) \mid 0 < t \le T, 0 \le x < t\}$*.*

Proof. From Theorem 1 and the expressions in problem (P) combing the assumption (4), it is not difficult to know that for any $T > 0$, if $(P_0, P_1, \ldots, P_4, P_5) \in V_0^5 \times V_1$ is the nonnegative solution of system (1)–(3), then $P_i(t)$ is differentiable on $[0,T]$ and $P_i'(t) \in C[0,T]$ by Theorem 8, $i = 0, 1, \ldots, 4$. That is, $P_i(t) \in C^1[0,T]$, $i = 0, 1, \ldots, 4$. And with the expression of $P_5(t,x)$ in problem (P), we have

$$\begin{cases} \dfrac{\partial P_5(t,x)}{\partial t} = \left[\lambda P_2'(t-x) + \mu_0 P_4'(t-x)\right] e^{-\int_0^x \mu(\tau)\mathrm{d}\tau}, & x < t \\ 0, & x > t, \end{cases}$$

$$\begin{cases} \dfrac{\partial P_5(t,x)}{\partial x} = \left[-\lambda P_2'(t-x) - \mu_0 P_4'(t-x)\right] e^{-\int_0^x \mu(\tau)\mathrm{d}\tau} \\ \qquad -\mu(x)\left[\lambda P_2(t-x) + \mu_0 P_4(t-x)\right] e^{-\int_0^x \mu(\tau)\mathrm{d}\tau}, & x < t \\ 0, & x > t. \end{cases} \tag{21}$$

Then,

$$\begin{gathered} \left.\frac{\partial P_5(t,x)}{\partial t}\right|_{\substack{x<t \\ x\to t}} = \lambda\alpha_0 e^{-\int_0^x \mu(\tau)\mathrm{d}\tau}, \\ \left.\frac{\partial P_5(t,x)}{\partial t}\right|_{\substack{x>t \\ x\to t}} = 0, \\ \left.\frac{\partial P_5(t,x)}{\partial t}\right|_{\substack{x<t \\ x\to t}} = -\lambda\alpha_0 e^{-\int_0^x \mu(\tau)\mathrm{d}\tau}, \\ \left.\frac{\partial P_5(t,x)}{\partial t}\right|_{\substack{x>t \\ x\to t}} = 0. \end{gathered} \tag{22}$$

Therefore, by the continuity of $P_2'(t)$, $P_4'(t)$, and $\mu(x)$ on $[0,T]$, it can be yielded that $P_5(t,x) \in C^1(D)$, where $D = \{(t,x) \mid 0 < t \le T,\ 0 \le x < t\}$. The proof of Theorem 9 is completed. □

4. Stability of System Solution

In this section, we will study the asymptotic stability and exponential stability of the solution of system (1)–(3). For convenience, we will first translate the system equations into an abstract Cauchy problem in a Banach space. Then, the asymptotic stability of the system solution is discussed by analyzing the spectral distributions of the system operator and that of its adjoint operator. Further, the exponential stability of the system solution is studied by analyzing the essential spectrum bound of the system operator.

4.1. System Transformation. In this section, we will translate the system equations into an abstract Cauchy problem in a suitable Banach space.

First, choose the state space X to be

$$X = \Big\{ P = \left(P_0, P_1, \ldots, P_4, P_5(x)\right)^{\mathrm{T}} \mid P_i \in \mathbb{R},\ i = 0, 1, \ldots, 4, \quad P_5(x) \in L^1(\mathbb{R}_+),\ \|P\| = \sum_{i=0}^{4} |P_i| + \|P_5\|_{L^1(\mathbb{R}_+)} < \infty \Big\}. \tag{23}$$

Here, $\mathbb{R}_+$ denotes the set of nonnegative real numbers. Obviously, X is a Banach space.

Next, define operator A as follows:

$$AP = \begin{pmatrix} -(\varepsilon + \alpha_0) P_0 + \mu_0 P_1 + \int_0^\infty \mu(x) P_5(x)\,\mathrm{d}x \\ -(\alpha_0 + \mu_0) P_1 + \varepsilon P_0 \\ -\lambda P_2 + \alpha_0 P_0 + \mu_0 P_3 \\ -(\lambda + \mu_0) P_3 + \alpha_0 P_1 \\ -\mu_0 P_4 + \lambda P_3 \\ -P_5'(x) - \mu(x) P_5(x) \end{pmatrix},$$

$$\begin{aligned} D(A) = \Big\{ & P = \left(P_0, P_1, \ldots, P_4, P_5(x)\right)^{\mathrm{T}} \in X \mid P_5'(x) \in L^1(\mathbb{R}_+); \\ & P_5(x) \text{ is an absolutely continuous function satisfying } P_5(0) = \lambda P_2 + \mu_0 P_4 \Big\}. \end{aligned} \tag{24}$$

Then, the system (1)–(3) can be rewritten as an abstract Cauchy problem in the Banach space X:

$$\begin{gathered} \frac{\mathrm{d}P(t,\cdot)}{\mathrm{d}t} = AP(t,\cdot), \quad t \ge 0 \\ P(t,\cdot) = \left(P_0(t), \ldots, P_4(t), P_5(t,x)\right)^{\mathrm{T}} \\ P(0,\cdot) \triangleq P_0 = (1, 0, \ldots, 0)^{\mathrm{T}}. \end{gathered} \tag{25}$$

4.2. Properties of System Operator A. In this section, we will study some properties of the system operator A.

Lemma 10. *The system operator* A *is a densely closed dissipative operator.*

Proof. Firstly, we prove that A is a closed operator. Choose $P_n = (P_{n0}, P_{n1}, \ldots, P_{n4}, P_{n5}(x))^{\mathrm{T}} \in D(A)$, $P_n \to P = (P_0, P_1, \ldots, P_4, P_5(x))^{\mathrm{T}}$, $AP_n \to Q = (Q_0, Q_1, \ldots, Q_4, Q_5(x))^{\mathrm{T}}$, $n \to \infty$. By Proposition 1 ([29, II.2.10]), we know that the differential operator $\mathscr{D}$ is the infinitesimal generator of a left translation semigroup $\{T_l(t)\}_{t\geq 0}$ with domain

$$D(\mathscr{D}) = \left\{f \in L^1(\mathbb{R}_+) \mid f \text{ is absolutely continuous satisfying } f' \in L^1(\mathbb{R}_+)\right\}. \tag{26}$$

Because $D(\mathscr{D})$ is closed and $P_{n5} \in D(\mathscr{D})$, then $P_5 \in D(\mathscr{D})$, that is, $P_5'(x) \in L^1(\mathbb{R}_+)$, and $P_5(x)$ is absolutely continuous. Moreover, $P_{n5}(0) = \lambda P_{2n} + \mu_0 P_{4n} \to \lambda P_2 + \mu_0 P_4$, $n \to \infty$. Thus, $P \in D(A)$. Therefore, it is not difficult to get that $AP = Q$ by noting the bounded measure of $\mu(x)$. This implies that A is a closed operator.

Next, we prove that $D(A)$; the domain of A is dense in X. For any $F = (F_0, F_1, \ldots, F_4, F_5(x))^{\mathrm{T}} \in X$, let $P_i = F_i$, $i = 0, 1, \ldots, 4$. Because $F_5(x) \in L^1(\mathbb{R}_+)$, then for any $\varepsilon > 0$, there exist $\delta_1 > 0$ and $G > 0$ such that

$$\int_0^{\delta_1} |F_5(x)|\,\mathrm{d}x < \frac{\varepsilon}{6}, \qquad \int_G^{\infty} |F_5(x)|\,\mathrm{d}x < \frac{\varepsilon}{3}. \tag{27}$$

Set $\delta = \min\{\delta_1, 1/6(1 + |\lambda P_2 + \mu_0 P_4|)\}$, and define

$$P_5(x) = \begin{cases} \lambda P_2 + \mu_0 P_4, & 0 \leq x < \delta \\ g(x), & \delta \leq x \leq G \\ 0, & x > G. \end{cases} \tag{28}$$

Here, $g(x)$ is continuously differentiable function on $[\delta, G]$ satisfying $g(\delta) = \lambda P_5 + \mu P_4$, $g(G) = 0$ and

$$\int_\delta^G |P_5(x) - g(x)|\,\mathrm{d}x < \frac{\varepsilon}{3}. \tag{29}$$

Choose $P = (P_0, P_1, \ldots, P_4, P_5(x))^{\mathrm{T}}$, then $P \in D(A)$, and

$$\begin{aligned} \|P - F\| &= \int_0^\delta |P_5(x) - F_5(x)|\,\mathrm{d}x + \int_\delta^G |P_5(x) - F_5(x)|\,\mathrm{d}x + \int_G^\infty |P_5(x) - F_5(x)|\,\mathrm{d}x \\ &< \int_0^\delta |P_5(x)|\,\mathrm{d}x + \int_0^\delta |F_5(x)|\,\mathrm{d}x + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < (\lambda P_2 + \mu_0 P_4)\,\delta + \frac{\varepsilon}{6} + \frac{2\varepsilon}{3} \\ &< \varepsilon. \end{aligned} \tag{30}$$

This implies that $D(A)$ is dense in X.

Thirdly, we prove that A is a dissipative operator. In fact, for any $P = (P_0, P_1, \ldots, P_4, P_5(x))^{\mathrm{T}} \in D(A)$, choose $Q = (Q_0, Q_1, \ldots, Q_4, Q_5(x))^{\mathrm{T}}$, where $Q_i = \|P\| \operatorname{sgn}(P_i)$, $i = 0, 1, \ldots, 4$, $Q_5(x) = \|P\| \operatorname{sgn}(P_5(x))$. Clearly, $Q \in X^* = \mathbb{R}^5 \times L^\infty(\mathbb{R}_+)$, the dual space of X, and $\langle P, Q\rangle = \|P\|^2 = \|Q\|^2$. Moreover, it is not difficult to know that $\langle AP, Q\rangle \leq 0$. This manifests that A is a dissipative operator. The proof of Lemma 10 is completed. □

Lemma 11. *$\{\gamma \in \mathbb{C} \mid \operatorname{Re}\gamma > 0$ or $\gamma = ia$, $a \in \mathbb{R} \setminus \{0\}\} \subset \rho(A)$, the resolvent set of the system operator A.*

Proof. For any $G = (G_0, G_1, \ldots, G_4, G_5(x))^{\mathrm{T}} \in X$, consider the operator equation $(\gamma I - A)P = G$. That is,

$$(\gamma + \varepsilon + \alpha_0) P_0 = G_0 + \mu_0 P_1 + \int_0^\infty \mu(x) P_5(x)\,\mathrm{d}x, \tag{31}$$

$$(\gamma + \alpha_0 + \mu_0) P_1 = G_1 + \varepsilon P_0, \tag{32}$$

$$(\gamma + \lambda) P_2 = G_2 + \alpha_0 P_0 + \mu_0 P_3, \tag{33}$$

$$(\gamma + \lambda + \mu_0) P_3 = G_3 + \alpha_0 P_1, \tag{34}$$

$$(\gamma + \mu_0) P_4 = G_4 + \lambda P_3, \tag{35}$$

$$P_5'(x) + (\gamma + \mu(x)) P_5(x) = G_5(x), \tag{36}$$

$$P_5(0) = \lambda P_2 + \mu_0 P_4. \tag{37}$$

Solving (36) with the help of (37) yields

$$\begin{aligned} P_5(x) &= P_5(0)\, e^{-\int_0^x (\gamma + \mu(s))\mathrm{d}s} + \int_0^x G_5(\tau)\, e^{-\int_\tau^x (\gamma + \mu(s))\mathrm{d}s}\mathrm{d}\tau \\ &= (\lambda P_2 + \mu_0 P_4)\, e^{-\int_0^x (\gamma + \mu(s))\mathrm{d}s} + Y_5(x), \end{aligned} \tag{38}$$

where $Y_5(x) = \int_0^x G_5(\tau) e^{-\int_\tau^x [\gamma + \mu(s)]\mathrm{d}s}\mathrm{d}\tau$. By [30], there exists a constant N, such that

$$\int_t^\infty e^{-\int_0^x \mu(s)\mathrm{d}s}\mathrm{d}x \leq N, \quad \forall t \geq 0. \tag{39}$$

Thus, $P_5(x) \in L^1(\mathbb{R}_+)$.

Substituting (38) into (31) derives

$$\begin{aligned} (\gamma + \varepsilon + \alpha_0) P_0 &= G_0 + \mu_0 P_1 + \int_0^\infty \mu(x) \times \left[P_5(0)\, e^{-\int_0^x (\gamma + \mu(s))\mathrm{d}s} + Y_5(x)\right]\mathrm{d}x \\ &= \mu_0 P_1 + (\lambda P_2 + \mu_0 P_4) \times \int_0^\infty \mu(x)\, e^{-\int_0^x (\gamma + \mu(s))\mathrm{d}s}\mathrm{d}x + Y_0, \end{aligned} \tag{40}$$

where $Y_0 = G_0 + \int_0^\infty \mu(x) Y_5(x)\mathrm{d}x$.

Combing (40) and (32)–(35) follows the following matrix equation:

$$\begin{pmatrix} \gamma+\varepsilon+\alpha_0 & -\mu_0 & -\lambda\int_0^\infty \mu(x) e^{\int_0^x(\gamma+\mu(s))ds}dx & 0 & -\mu_0\int_0^\infty \mu(x) e^{-\int_0^x(\gamma+\mu(s))ds}dx \\ -\varepsilon & \gamma+\alpha_0+\mu_0 & 0 & 0 & 0 \\ -\alpha_0 & 0 & \gamma+\lambda & -\mu_0 & 0 \\ 0 & -\alpha_0 & 0 & \gamma+\lambda+\mu_0 & 0 \\ 0 & 0 & 0 & -\lambda & \gamma+\mu_0 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix}\begin{pmatrix} Y_0 \\ G_1 \\ G_2 \\ G_3 \\ G_4 \end{pmatrix}. \quad (41)$$

For $\operatorname{Re}\gamma > 0$ or $\gamma = ia$, $a \in \mathbb{R} \setminus \{0\}$, it is not difficult to get the following estimation from the definition of modulus of complex number:

$$\left|\int_0^\infty \mu(x) e^{-\int_0^x(\gamma+\mu(s))ds}dx\right| < 1. \quad (42)$$

Thus, the coefficient matrix of the matrix equation (41) is a strictly diagonally dominant matrix for column. So, it is inverse, and the matric equation (41) has a unique solution $(P_0, P_1, P_2, P_3, P_4)^T$. Combing (38), it can be seen that (31)–(37) have a unique solution $P = (P_0, P_1, P_2, P_3, P_4, P_5(x))^T \in D(A)$. This means that $\gamma I - A$ is surjective. Because $\gamma I - A$ is closed and $D(A)$ is dense in X, then $(\gamma I - A)^{-1}$ exists and is bounded by Inverse Operator Theorem, for any $\operatorname{Re}\ \gamma > 0$ or $\gamma = ia$, $a \in \mathbb{R} \setminus \{0\}$. The proof of Lemma 11 is completed. □

Lemma 12. *0 is an eigenvalue of the system operator A with algebraic multiplicity one.*

Proof. Consider the operator equation $(\gamma I - A)P = 0$. Let $D(\gamma)$ be the determinant of coefficient of the matrix equation (41), then we have

$$D(\gamma) = \begin{vmatrix} \gamma+\varepsilon+\alpha_0 & -\mu_0 & -\lambda(1-\gamma g(\gamma)) & 0 & -\mu_0(1-\gamma g(\gamma)) \\ -\varepsilon & \gamma+\alpha_0+\mu_0 & 0 & 0 & 0 \\ -\alpha_0 & 0 & \gamma+\lambda & -\mu_0 & 0 \\ 0 & -\alpha_0 & 0 & \gamma+\lambda+\mu_0 & 0 \\ 0 & 0 & 0 & -\lambda & \gamma+\mu_0 \end{vmatrix} = \gamma F(\gamma). \quad (43)$$

Here, $g(\gamma) = \int_0^\infty e^{-\int_0^x(\gamma+\mu(s))ds}$ and

$$\begin{aligned} F(\gamma) = {} & \alpha_0\lambda\mu_0 g(\gamma)(\lambda+\mu_0)(\varepsilon+\alpha_0+\mu_0) \\ & + \alpha_0\left(\varepsilon\mu_0^2 + \mu^3 + \mu_0^2\alpha_0\right) \\ & + \lambda(\alpha_0+\mu_0)[\lambda(\mu_0+\varepsilon) \\ & \qquad + \mu_0(\alpha_0+\varepsilon+\mu_0)] \\ & + \alpha_0\lambda\gamma g(\gamma)\left(2\varepsilon\mu_0 + 2\alpha_0\mu_0\right. \\ & \qquad \left. + 3\mu_0^2 + 2\lambda\mu_0 + \alpha_0\lambda\right) \\ & + \gamma\left[(\lambda\mu_0 + \lambda\alpha_0 + \alpha_0\mu_0)(4\mu_0 + 2\varepsilon + \lambda)\right. \\ & \qquad + \left(\lambda^2 + \mu_0^2\right)(\mu_0 + \varepsilon) \\ & \qquad \left. + 2\mu_0\alpha_0^2 + \varepsilon\lambda\mu_0 + \lambda\alpha_0^2\right] \\ & + \alpha_0\lambda\gamma^2 g(\gamma)(3\mu_0 + \alpha_0 + \lambda) \\ & + \gamma^2\left(5\lambda\mu_0 + 2\varepsilon\lambda + \lambda^2 + \alpha_0^2 + 3\mu_0^2\right. \\ & \qquad \left. + \varepsilon\alpha_0 + 5\mu_0\alpha_0 + 3\lambda\alpha_0 + 2\varepsilon\mu_0\right) \\ & + \gamma^3(2\lambda + \varepsilon + 2\alpha_0 + 3\mu_0 + \alpha_0\lambda g(\gamma)) \\ & + \gamma^4. \end{aligned} \quad (44)$$

Because

$$\begin{aligned} F(0) = {} & \alpha_0\lambda\mu_0 g(0)(\lambda+\mu_0)(\varepsilon+\alpha_0+\mu_0) \\ & + \alpha_0\left(\varepsilon\mu_0^2 + \mu^3 + \mu_0^2\alpha_0\right) + \lambda(\alpha_0+\mu_0) \\ & \times[\lambda(\mu_0+\varepsilon) + \mu_0(\alpha_0+\varepsilon+\mu_0)] > 0. \end{aligned} \quad (45)$$

This means $\gamma = 0$ is an eigenvalue of the system operator A with algebraic multiplicity one. The proof of Lemma 12 is completed. □

4.3. Properties of Adjoint Operator A^*. In this section, we will study some properties of A^*, the adjoint operator of system operator A.

The dual space of X is

$$X^* = \mathbb{R}^5 \times L^\infty(\mathbb{R}_+) \quad (46)$$

with norm $\|Q\| = \sup\{|Q_i|, \|Q_5\|_{L^\infty(\mathbb{R}_+)}, i = 0, 1, \ldots, 4\}$ for $Q = (Q_0, Q_1, \ldots, Q_4, Q_5(x))^T \in X^*$.

Lemma 13. *A^*, the adjoint operator of the system operator A is as follows:*

$$A^*Q = \begin{pmatrix} -(\varepsilon+\alpha_0)Q_0 + \varepsilon Q_1 + \alpha_0 Q_2 \\ -(\alpha_0+\mu_0)Q_1 + \mu_0 Q_0 + \alpha_0 Q_3 \\ -\lambda Q_2 + \lambda Q_5(0) \\ -(\lambda+\mu_0)Q_3 + \mu_0 Q_2 + \lambda Q_4 \\ -\mu_0 Q_4 + \mu_0 Q_5(0) \\ Q_5'(x) + \mu(x)[Q_0 - Q_5(x)] \end{pmatrix} \triangleq (C+D)Q \tag{47}$$

with domain

$$\begin{aligned} D(A^*) &= \{Q = (Q_0, Q_1, \ldots, Q_4, Q_5(x))^{\mathrm{T}} \\ &\quad \in X^* \mid Q_5'(x) \in L^\infty(\mathbb{R}_+), Q_5(x) \\ &\quad \text{is absolutely continuous satisfying} \\ &\quad Q_5(\infty) < \infty\}. \end{aligned} \tag{48}$$

Here,

$$\begin{aligned} CQ &= \operatorname{diag}\Big(-(\varepsilon+\alpha_0), -(\alpha_0+\mu_0), \\ &\quad -\lambda, -(\lambda+\mu_0), -\mu_0, \frac{\mathrm{d}}{\mathrm{d}x} - \mu(x)\Big)Q, \\ DQ &= \begin{pmatrix} 0 & \varepsilon & \alpha_0 & 0 & 0 & 0 \\ \mu_0 & 0 & 0 & \alpha_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda\theta(\cdot) \\ 0 & 0 & \mu_0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_0\theta(\cdot) \\ \mu(x) & 0 & 0 & 0 & 0 & 0 \end{pmatrix} Q \end{aligned} \tag{49}$$

and $\theta(\cdot): L^\infty(\mathbb{R}_+) \to \mathbb{C}$, satisfying $\theta(f) = f(0)$.

Proof. For any $P \in D(A)$ and $Q \in X^*$, A^* and its domain $D(A^*)$ can be readily derived by the equality $\langle AP, Q\rangle = \langle P, A^*Q\rangle$. The proof of Lemma 13 is completed. □

Lemma 14. *$S = \{\gamma \in \mathbb{C} \mid \sup\{(\varepsilon+\alpha_0)/|\gamma+\varepsilon+\alpha_0|, (\mu_0+\alpha_0)/|\gamma+\mu_0+\alpha_0|, \lambda/|\gamma+\lambda|, (\lambda+\mu_0)/|\gamma+\lambda+\mu_0|, \mu_0/|\gamma+\mu_0|, M/(\operatorname{Re}\gamma + M)\} < 1\} \subset \rho(A^*)$, the resolvent set of A^*, where $M = \sup_{x\geq 0}\mu(x)$.*

Proof. For any $W = (W_0, W_1, \ldots, W_4, W_5(x))^{\mathrm{T}} \in X^*$, consider the equation $(\gamma I - C)Q = DW$. That is,

$$(\gamma+\varepsilon+\alpha_0)Q_0 = \varepsilon W_1 + \alpha_0 W_2, \tag{50}$$

$$(\gamma+\alpha_0+\mu_0)Q_1 = \mu_0 W_0 + \alpha_0 W_3, \tag{51}$$

$$(\gamma+\lambda)Q_2 = \lambda W_5(0), \tag{52}$$

$$(\gamma+\lambda+\mu_0)Q_3 = \mu_0 W_2 + \lambda W_4, \tag{53}$$

$$(\gamma+\mu_0)Q_4 = \mu_0 W_5(0), \tag{54}$$

$$\frac{\mathrm{d}Q_5(x)}{\mathrm{d}x} = (\gamma+\mu(x))Q_5(x) - \mu(x)W_0. \tag{55}$$

Solving (50)–(54) yields

$$\begin{gathered} Q_0 = \frac{\varepsilon W_1 + \alpha_0 W_2}{\gamma+\varepsilon+\alpha_0}, \qquad Q_1 = \frac{\mu_0 W_0 + \alpha_0 W_3}{\gamma+\alpha_0+\mu_0}, \\ Q_2 = \frac{\lambda W_5(0)}{\gamma+\lambda}, \\ Q_3 = \frac{\mu_0 W_2 + \lambda W_4}{\gamma+\lambda+\mu_0}, \qquad Q_4 = \frac{\mu_0 W_5(0)}{\gamma+\mu_0}. \end{gathered} \tag{56}$$

Solving (55) derives

$$Q_5(x) = e^{\int_0^x(\gamma+\mu(s))\mathrm{d}s}\left[Q_5(0) - \int_0^x W_0\mu(\tau)e^{-\int_0^\tau(\gamma+\mu(s))\mathrm{d}s}\mathrm{d}\tau\right] \tag{57}$$

multiplying $e^{-\int_0^x(\gamma+\mu(s))\mathrm{d}s}$ to the two sides of (57) and letting $x \to \infty$ by noting $Q_5(\infty) < \infty$ follows

$$Q_5(0) = \int_0^\infty W_0\mu(\tau)e^{-\int_0^\tau(\gamma+\mu(s))\mathrm{d}s}\mathrm{d}\tau. \tag{58}$$

Substituting (58) into (57) yields

$$Q_5(x) = e^{\int_0^x(\gamma+\mu(s))\mathrm{d}s}\int_x^\infty W_0\mu(\tau)e^{-\int_0^\tau(\gamma+\mu(s))\mathrm{d}s}\mathrm{d}\tau. \tag{59}$$

Thus, we can get the following estimation:

$$\begin{aligned} \|Q_5\|_{L^\infty[0,\infty)} &= \sup_{x\in[0,\infty)}|Q_5(x)| \\ &= \sup_{x\in[0,\infty)}\Big|e^{\int_0^x(\gamma+\mu(s))\mathrm{d}s} \\ &\quad \times \int_x^\infty W_0\mu(\tau)e^{-\int_0^\tau(\gamma+\mu(s))\mathrm{d}s}\mathrm{d}\tau\Big| \\ &\leq \|W\| \sup_{x\in[0,\infty)} e^{\int_0^x(\operatorname{Re}\gamma+\mu(s))\mathrm{d}s} \\ &\quad \times \int_x^\infty -e^{-\operatorname{Re}\gamma\tau}\mathrm{d}e^{-\int_0^\tau\mu(s)\mathrm{d}s} \end{aligned}$$

$$
\begin{aligned}
&= \|W\| \sup_{x\in[0,\infty)} e^{\int_0^x (\operatorname{Re}\gamma+\mu(s))\mathrm{d}s} \\
&\quad \times \left[e^{-\int_0^x (\operatorname{Re}\gamma+\mu(s))\mathrm{d}s} - \operatorname{Re}\gamma \int_x^\infty e^{-\int_0^\tau (\operatorname{Re}\gamma+\mu(s))\mathrm{d}s} \mathrm{d}\tau \right] \\
&= \|W\| \sup_{x\in[0,\infty)} \left[1 - \operatorname{Re}\gamma e^{\operatorname{Re}\gamma x} \times \int_x^\infty e^{-\int_x^\tau (\operatorname{Re}\gamma+\mu(s))\mathrm{d}s} \mathrm{d}\tau \right] \\
&\le \|W\| \sup_{x\in[0,\infty)} \left[1 - \operatorname{Re}\gamma e^{\operatorname{Re}\gamma x} \times \int_x^\infty e^{-\int_x^\tau (\operatorname{Re}\gamma+M)\mathrm{d}s} \mathrm{d}\tau \right] \\
&= \|W\| \sup_{x\in[0,\infty)} \left[1 - \operatorname{Re}\gamma e^{(\operatorname{Re}\gamma+M)x} \times \int_x^\infty e^{-(\operatorname{Re}\gamma+M)\tau} \mathrm{d}\tau \right] \\
&= \|W\| \sup_{x\in[0,\infty)} \left[1 - \operatorname{Re}\gamma e^{(\operatorname{Re}\gamma+M)x} \times \frac{e^{-(\operatorname{Re}\gamma+M)x}}{\operatorname{Re}\gamma+M} \right] \\
&= \frac{M}{\operatorname{Re}\gamma+M} \|W\|,
\end{aligned}
\tag{60}
$$

where $M = \sup_{x\ge 0}\mu(x)$.

Equation (56) derive the following estimations:

$$
\begin{aligned}
|Q_0| &\le \frac{\varepsilon+\alpha_0}{|\gamma+\varepsilon+\alpha_0|}\|W\|, & |Q_1| &\le \frac{\mu_0+\alpha_0}{|\gamma+\alpha_0+\mu_0|}\|W\|, \\
|Q_2| &\le \frac{\lambda}{|\gamma+\lambda|}\|W\|, & |Q_3| &\le \frac{\lambda+\mu_0}{|\gamma+\lambda+\mu_0|}\|W\|, \\
|Q_4| &\le \frac{\mu_0}{|\gamma+\mu_0|}\|W\|, & |Q_5| &\le \frac{M}{\operatorname{Re}\gamma+M}\|W\|.
\end{aligned}
\tag{61}
$$

Then, for $\gamma \in S$, we have

$$
\begin{aligned}
\|Q\| &= \sum_{i=0}^{4} |Q_i| + \|Q_5\| \\
&\le \sup\left\{ \frac{\varepsilon+\alpha_0}{|\gamma+\varepsilon+\alpha_0|}, \frac{\mu_0+\alpha_0}{|\gamma+\alpha_0+\mu_0|}, \frac{\lambda}{|\gamma+\lambda|}, \frac{\lambda+\mu_0}{|\gamma+\lambda+\mu_0|}, \frac{\mu_0}{|\gamma+\mu_0|}, \frac{M}{\operatorname{Re}\gamma+M} \right\} \|W\| \\
&< \|W\|.
\end{aligned}
\tag{62}
$$

This implies that $\|(\gamma I - C)^{-1} D\| < 1$. Thus, $[I - (\gamma I - C)^{-1} D]$ is invertible. Therefore, $(\gamma I - A^*)$ is invertible and

$$
\begin{aligned}
(\gamma I - A^*)^{-1} &= [\gamma I - (C + D)]^{-1} \\
&= \left[I - (\gamma I - C)^{-1} D\right]^{-1} (\gamma I - C)^{-1}, \\
&\qquad \gamma \in S.
\end{aligned}
\tag{63}
$$

The proof of Lemma 14 is completed. □

Lemma 15. *0 is an eigenvalue of operator A^* with algebraic multiplicity one.*

Proof. We prove Lemma 15 in two steps. Firstly, we prove that 0 is an eigenvalue of operator A^*. For any $Q \in D(A^*)$, consider the operator equation $A^*Q = 0$, that is,

$$(\varepsilon+\alpha_0) Q_0 = \varepsilon Q_1 + \alpha_0 Q_2, \tag{64}$$

$$(\alpha_0+\mu_0) Q_1 = \mu_0 Q_0 + \alpha_0 Q_3, \tag{65}$$

$$\lambda Q_2 = \lambda Q_5(0), \tag{66}$$

$$(\lambda+\mu_0) Q_3 = \mu_0 Q_2 + \lambda Q_4, \tag{67}$$

$$\mu_0 Q_4 = \mu_0 Q_5(0), \tag{68}$$

$$Q_5'(x) = \mu(x) Q_5(x) - Q_0 \mu(x). \tag{69}$$

Solving (69) yields

$$Q_5(x) = e^{\int_0^x \mu(s)\mathrm{d}s} \left[Q_5(0) - Q_0 \int_0^x \mu(\tau) e^{-\int_0^\tau \mu(s)\mathrm{d}s} \mathrm{d}\tau \right] \tag{70}$$

multiplying $e^{-\int_0^x \mu(s)\mathrm{d}s}$ to the two sides of (70) and letting $x \to \infty$ by noting $Q_5(\infty) < \infty$ derive

$$Q_5(0) = Q_0 \int_0^\infty \mu(\tau) e^{-\int_0^\tau \mu(s)\mathrm{d}s} \mathrm{d}\tau = Q_0. \tag{71}$$

Combing (64)–(68) with (71) follows

$$Q_0 = Q_1 = Q_2 = Q_3 = Q_4 = Q_5(0). \tag{72}$$

Substituting (71) into (70) yields

$$Q_5(x) = Q_0 e^{\int_0^x \mu(s)\mathrm{d}s} \left[1 + \int_0^x \mathrm{d}e^{-\int_0^\tau \mu(s)\mathrm{d}s} \right] = Q_0. \tag{73}$$

This implies that ηQ^* $(\eta \neq 0)$ is the eigenfunction corresponding to eigenvalue 0 of operator A^*, where $Q^* = (1, 1, \ldots, 1)^{\mathrm{T}}$.

Next, we prove that the algebraic multiplicity of 0 in X^* is one. From the above step, we can see that the geometric multiplicity of 0 in X^* is one. Then, we only need to verify the algebraic index of 0 in X^* which is also one according to [31]. We use the reduction to absurdity. Suppose that the algebraic index of 0 in X^* is 2 without loss of generality, then there exists $Y \in X^*$, such that $A^*Y = Q^*$. It is obvious that

$$0 = \langle AP, Y \rangle = \langle P, A^*Y \rangle = \langle P, Q^* \rangle, \tag{74}$$

where P is the positive eigenfunction corresponding to eigenvalue 0 of A. However,

$$\langle P, Q^* \rangle = \sum_{i=0}^{4} P_i + \int_0^\infty P_5(x)\,\mathrm{d}x > 0 \tag{75}$$

which contradicts (74). Thus, the algebraic index of 0 in X^* is one. Therefore, the algebraic multiplicity of 0 in X^* is one. The proof of Lemma 15 is completed. □

4.4. Asymptotic Stability of System Solution. In this section, we will present the asymptotic stability of the system solution by using C_0 semigroup theory.

Recalling Phillips Theorem (see [32]) together with Lemmas 10, 11, and 3, we can obtain the following results.

Theorem 16. *The system operator A generates a positive C_0 semigroup of contraction $T(t)$.*

Theorem 17. *The system* (25) *has a unique nonnegative time-dependent solution $P(t,\cdot)$ which satisfies*

$$\|P(t,\cdot)\| = 1, \quad \forall t \in [0,\infty). \tag{76}$$

Proof. From Theorem 16 and [32], it can be derived that the system (25) has a unique nonnegative solution $P(t,\cdot)$ which can be expressed as

$$P(t,\cdot) = T(t)P_0, \quad \forall t \in [0,\infty). \tag{77}$$

Because $P(t,\cdot)$ satisfies (1)–(3), it is not difficult to know that

$$\frac{\mathrm{d}\|P(t,\cdot)\|}{\mathrm{d}t} = 0. \tag{78}$$

Therefore,

$$\|P(t,\cdot)\| = \|T(t)P_0\| = \|P_0\| = 1, \quad \forall t \in [0,\infty). \tag{79}$$

This just reflects the physical meaning of $P(t,\cdot)$. The proof of Theorem 17 is completed. □

Remark 18. Because the initial value P_0 of the system (25) belongs to the domain $D(A)$ of the system operator A, then the nonnegative time-dependent solution of the system expressed in (77) is the strong solution of the system (25).

Noting that the C_0 semigroup $T(t)$ generated by A is contractive according to Theorem 16, it is uniformly bounded certainly. Thus, recalling [33] combining Lemmas 11, 12, 14, and 15, we can know that the time-dependent solution of the system strongly converges to its steady-state solution. That is the following result.

Theorem 19. *Let $\widehat{P}$ be the eigenfunction corresponding to eigenvalue 0 of the system operator A satisfying $\|\widehat{P}\| = 1$, and let Q^* be defined in Lemma 15, then the time-dependent solution $P(t,\cdot)$ of the system* (25) *converges to the nonnegative steady-state solution $\widehat{P}$. That is,*

$$\lim_{t\to\infty} P(t,\cdot) = \langle P_0, Q^* \rangle \widehat{P} = \widehat{P}, \tag{80}$$

where P_0 is the initial value of the system.

4.5. Exponential Stability of System Solution. In Section 4.4, we have obtained the asymptotic stability of the system. In other words, the dynamic solution of the system asymptotically converges to its steady-state solution. However, there are still two problems: first, the convergence rate is unknown; second, the convergence is subject to some factors such as failure rate and repair rate. Both can be well settled if the system is exponentially stable. For this purpose, in this section, we will discuss the exponential stability of the system.

For simplicity, we will divide the system operator A into two operators. The one is a compact operator, and the other generates a quasicompact semigroup. Then, by the perturbation of compact operator, it is derived readily that the system operator also generates a quasicompact C_0 semigroup. Therefore, the system solution is exponentially stable.

For convenience, we will introduce three operators first:

$$\begin{gathered}
BP = \left(\int_0^\infty \mu(x) P_5(x)\,\mathrm{d}x + \mu_0 P_1, \varepsilon P_0, \alpha_0 P_0 \right. \\
\left. +\mu_0 P_3, \alpha_0 P_1, \lambda P_3, 0 \right)^{\mathrm{T}} \quad \text{with } D(B) = X, \\
\overline{A} = A - B \text{ with } D(\overline{A}) = D \\
= \left\{ \begin{array}{l} P = (P_0, P_1, \ldots, P_4, P_5(x))^{\mathrm{T}} \in X \mid P_5'(x) \in L^1(\mathbb{R}_+), \\ P_5(x) \text{ is an absolutely continuous function} \end{array} \right\} \\
A_0 = \overline{A} \quad \text{with } D(A_0) = \{ P \in D \mid P_5(0) = 0 \}.
\end{gathered} \tag{81}$$

It is easy to know that $\overline{A}$ and A_0 are both closed operators with dense domains in X. And with the perturbation of C_0 semigroup, it is clear that $\overline{A}$ also generates a C_0 semigroup $S(t)$.

Lemma 20. *Assume that the mean of the repair rate exists and greater than zero, that is,*

$$0 < \widehat{\mu} = \lim_{x\to\infty} \frac{1}{x} \int_0^x \mu(s)\,\mathrm{d}s. \tag{82}$$

Then, A_0 generates a quasicompact semigroup $T_0(t)$.

Proof. Firstly, we will prove that A_0 generates a C_0 semigroup $T_0(t)$. Consider the following abstract Cauchy problem:

$$\begin{gathered}
\frac{\mathrm{d}P(t,\cdot)}{\mathrm{d}t} = A_0 P(t,\cdot), \quad t \geq 0, \\
P(0,\cdot) = \Phi,
\end{gathered} \tag{83}$$

where $\Phi = (\varphi_0, \varphi_1, \dots, \varphi_4, \varphi_5(x))^{\mathrm{T}} \in X$. That is,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \varepsilon + \alpha_0\right) P_0(t) = 0, \quad (84)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \alpha_0 + \mu_0\right) P_1(t) = 0, \quad (85)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda\right) P_2(t) = 0, \quad (86)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda + \mu_0\right) P_3(t) = 0, \quad (87)$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mu_0\right) P_4(t) = 0, \quad (88)$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu(x)\right] P_5(t, x) = 0, \quad (89)$$

$$P_5(t, 0) = 0, \quad (90)$$

$$P_i(0) = \varphi_i, \quad i = 0, 1, \dots, 4, \quad (91)$$

$$P_5(0, x) = \varphi_5(x). \quad (92)$$

Solving (84)–(88) with the help of (91) yields

$$\begin{aligned} P_0(t) &= \varphi_0 e^{-(\varepsilon+\alpha_0)t}, \qquad P_1(t) = \varphi_1 e^{-(\alpha_0+\mu_0)t}, \\ P_2(t) &= \varphi_2 e^{-\lambda t}, \\ P_3(t) &= \varphi_3 e^{-(\lambda+\mu_0)t}, \qquad P_4(t) = \varphi_4 e^{-\mu_0 t}. \end{aligned} \quad (93)$$

Solving (89) with the help of (90) and (92) by the method of characteristics yields

$$P_5(t, x) = \begin{cases} P_5(t - x, 0) e^{-\int_0^x \mu(\tau)\mathrm{d}\tau} = 0, & t > x \\ P_5(0, x - t) e^{-\int_0^t \mu(x-t+\tau)\mathrm{d}\tau} \\ \quad = \varphi_5(x - t) e^{-\int_{x-t}^x \mu(\tau)\mathrm{d}\tau}, & t \le x. \end{cases} \quad (94)$$

Therefore, it is easy to prove that A_0 generates a C_0 semigroup $T_0(t)$ satisfying

$$(T_0(t)\Phi)(x) = \begin{cases} (\Psi_1, 0)^{\mathrm{T}}, & x < t \\ \left(\Psi_1, \varphi_5(x - t) e^{-\int_{x-t}^x \mu(\tau)\mathrm{d}\tau}\right)^{\mathrm{T}}, & x \ge t. \end{cases} \quad (95)$$

Here,

$$\Psi_1 = \left(\varphi_0 e^{-(\varepsilon+\alpha_0)t}, \varphi_1 e^{-(\alpha_0+\mu_0)t}, \varphi_2 e^{-\lambda t}, \varphi_3 e^{-(\lambda+\mu_0)t}, \varphi_4 e^{-\mu_0 t}\right). \quad (96)$$

Next, we will prove that $T_0(t)$ is quasicompact. We only need to prove that the essential growth bound $W_{\mathrm{ess}}(A_0)$ is less than zero.

The assumption condition (82) implies that for any $\varepsilon > 0$, there exists $t_0 > 0$ such that

$$\frac{1}{t}\int_{x-t}^x \mu(s)\,\mathrm{d}s > \hat{\mu} - \varepsilon, \quad x \ge t \ge t_0. \quad (97)$$

With the help of (97), it is not difficult to deduce that

$$\begin{aligned} \|T_0(t)\Phi\| &= |\varphi_0| e^{-(\varepsilon+\alpha_0)t} + |\varphi_1| e^{-(\alpha_0+\mu_0)t} \\ &\quad + |\varphi_2| e^{-\lambda t} + |\varphi_3| e^{-(\lambda+\mu_0)t} \\ &\quad + |\varphi_4| e^{-\mu_0 t} \\ &\quad + \int_t^\infty |\varphi_5(x - t)| e^{-\int_{x-t}^x \mu(\tau)\mathrm{d}\tau}\mathrm{d}x \\ &< |\varphi_0| e^{-(\varepsilon+\alpha_0)t} + |\varphi_1| e^{-(\alpha_0+\mu_0)t} \\ &\quad + |\varphi_2| e^{-\lambda t} + |\varphi_3| e^{-(\lambda+\mu_0)t} \\ &\quad + |\varphi_4| e^{-\mu_0 t} \\ &\quad + \int_t^\infty |\varphi_5(x - t)| e^{-(\hat{\mu}-\varepsilon)t}\mathrm{d}x \\ &= |\varphi_0| e^{-(\varepsilon+\alpha_0)t} + |\varphi_1| e^{-(\alpha_0+\mu_0)t} \\ &\quad + |\varphi_2| e^{-\lambda t} + |\varphi_3| e^{-(\lambda+\mu_0)t} \\ &\quad + |\varphi_4| e^{-\mu_0 t} \\ &\quad + e^{-(\hat{\mu}-\varepsilon)t}\int_0^\infty |\varphi_5(x)|\,\mathrm{d}x \\ &\le e^{-\min\{\varepsilon+\alpha_0, \alpha_0+\mu_0, \lambda, \lambda+\mu_0, \mu_0, \hat{\mu}-\varepsilon\}t}\,\|\Phi\|. \end{aligned} \quad (98)$$

This manifests that

$$\|T_0(t)\| \le e^{-\min\{\varepsilon+\alpha_0, \lambda, \mu_0, \hat{\mu}-\varepsilon\}t}. \quad (99)$$

Then,

$$\begin{aligned} W_{\mathrm{ess}}(A_0) \le W(A_0) &= \lim_{t\to\infty}\frac{\ln\|T_0(t)\|}{t} \\ &\le -\min\{\varepsilon + \alpha_0, \lambda, \mu_0, \hat{\mu} - \varepsilon\} < 0. \end{aligned} \quad (100)$$

Therefore, A_0 generates a quasicompact C_0 semigroup $T_0(t)$. The proof of Lemma 20 is completed. □

For $\gamma > 0$, $P \in X$, let

$$\Phi_\gamma(P)(x) = [\mathrm{diag}(0, 0, 0, 0, 0, \lambda P_2 + \mu_0 P_4)] \cdot E_\gamma(x), \quad (101)$$

where $E_\gamma(x) = (0, 0, 0, 0, 0, e^{-\int_0^x[\gamma+\mu(s)]\mathrm{d}s})^{\mathrm{T}} \in \mathrm{Ker}(\gamma I - \overline{A})$ and Φ_γ is a compact operator. Then, it is not difficult to obtain the following result.

Lemma 21. *$I + \Phi_\gamma$ is a bijection from $D(A_0)$ to $D(A)$ and*

$$[\gamma I - (A - B)](I + \Phi_\gamma) = \gamma I - A_0. \quad (102)$$

Lemma 22. *$S(t) - T_0(t)$ is a compact operator, for any $t \ge 0$. Here $S(t)$ is the C_0 semigroup generated by $\overline{A}$.*

Proof. From Lemma 21, we can see that $R(\gamma, A - B) \ge R(\gamma, A_0)$, for any $\gamma > 0$. Therefore $S(t) \ge T_0(t)$, for any $t \ge 0$.

For $P \in D(A_0)$, set

$$\Psi(s) P = S(t-s)\left(I+\Phi_\gamma\right) T_0(s) P, \tag{103}$$

where $0 \leq s \leq t$, $\gamma > 0$. Recalling the properties of C_0-semigroup and Lemma 21, we can obtain

$$\begin{aligned}\Psi'(s) P &= -S(t-s)(A-B)\left(I+\Phi_\gamma\right) T_0(s) P \\ &\quad + S(t-s)\left(I+\Phi_\gamma\right) A_0 T_0(s) P \\ &= S(t-s)\left[\gamma I-(A-B)\right]\left(I+\Phi_\gamma\right) T_0(s) P \\ &\quad + S(t-s)\left(I+\Phi_\gamma\right)\left[-\gamma I+A_0\right] T_0(s) P \\ &= S(t-s)\left[\gamma I-A_0\right] T_0(s) P \\ &\quad + S(t-s)\left(I+\Phi_\gamma\right)\left(-\gamma I+A_0\right) T_0(s) P \\ &= S(t-s) \Phi_\gamma\left(-\gamma I+A_0\right) T_0(s) P.\end{aligned} \tag{104}$$

Since $[\Psi(t)-\Psi(0)]P = \int_0^t \Psi'(s) P \mathrm{d}s$, then

$$[\Psi(t)-\Psi(0)] P = \int_0^t S(t-s) \Phi_\gamma\left(-\gamma I+A_0\right) T_0(s) P \mathrm{d}s. \tag{105}$$

That is,

$$\begin{aligned}S(t) P-T_0(t) P = &-\int_0^t S(t-s) \Phi_\gamma\left(-\gamma I+A_0\right) \\ &\times T_0(s) P \mathrm{d}s+\Phi_\gamma T_0(t) P-S(t) \Phi_\gamma P.\end{aligned} \tag{106}$$

Therefore, $S(t)-T_0(t)$ $(t \geq 0)$ is compact because the right-hand side of the above equation is the sum of three compact operators for the compactness of Φ_γ. The proof of Lemma 22 is completed. □

From the above preparations, we can present the main results of this section.

Theorem 23. *C_0 semigroup $T(t)$ generated by the system operator A is quasicompact.*

Proof. According to Proposition 9.20 (see [34]) combing Lemmas 22 and 20, we can deduce that

$$W_{\text{ess}}\left(\overline{A}\right) \leq W\left(A_0\right) < 0. \tag{107}$$

This shows that $S(t)$, the C_0 semigroup generated by $\overline{A}$ is quasicompact. Because B is a compact operator, then according to [35], it is evident that

$$W_{\text{ess}}(A) = W_{\text{ess}}(A-B) < 0. \tag{108}$$

This implies that $T(t)$ is quasicompact. The proof of Theorem 23 is completed. □

Theorem 24. *The time-dependent solution of the system* (1)–(3) *strongly converges to its steady-state solution. That is,*

$$\lim_{t \to \infty} P(t, \cdot) = \widehat{P}. \tag{109}$$

Moreover, there exist $C > 0$ and $\varepsilon > 0$ such that

$$\left\|P(t, \cdot)-\widehat{P}\right\| \leq C e^{-\varepsilon t}. \tag{110}$$

Here, $\widehat{P}$ is defined in Theorem 19.

Proof. Recalling Theorem 2.10 (see [35]) combing Theorem 23, we can derive that C_0 semigroup $T(t)$ generated by the system operator A can be decomposed as $T(t) = \overline{P}_0 + R(t)$, where $\overline{P}_0$ is the residue corresponding to eigenvalue 0 and $\|R(t)\| \leq Ce^{-\varepsilon t}$ for suitable constants $\varepsilon > 0$ and $C > 0$.

However, by Theorem 19, the nonnegative solution of the system (1)–(3) can be expressed as $P(t, \cdot) = T(t)P_0, t \in [0, \infty)$. Then, combing Theorem 12.3 in [36], we can derive that

$$\begin{aligned}P(t, \cdot) &= T(t) P_0 = \left(\overline{P}_0+R(t)\right) P_0 \\ &= \left\langle P_0, Q^*\right\rangle \widehat{P}+R(t) P_0 = \widehat{P}+R(t) P_0,\end{aligned} \tag{111}$$

where Q^* is defined in Lemma 15. Hence, we can get

$$\left\|P(t, \cdot)-\widehat{P}\right\| \leq C e^{-\varepsilon t}. \tag{112}$$

The proof of Theorem 24 is completed. □

5. Reliability Indices

In this section, we will discuss some reliability indices of the system. Noting that the eigenfunction corresponding to eigenvalue 0 of the system operator A is just the steady-state solution of system (1)–(3), we are dedicated to studying some primary steady-state indices of the system from the point of eigenfunction.

We first analyze the eigenfunction corresponding to eigenvalue 0 of the system operator A. In the proof process of Lemma 11, solving (31)–(36) with the help of (37) by letting $\gamma = 0$ and $G = 0$ derives

$$P_1 = \frac{\varepsilon}{\alpha_0+\mu_0} P_0, \tag{113}$$

$$P_2 = \frac{\alpha_0}{\lambda} P_0+\frac{\mu_0}{\lambda} P_3 = \left[\frac{\alpha_0}{\lambda}+\frac{\varepsilon \alpha_0 \mu_0}{\lambda\left(\lambda+\mu_0\right)\left(\alpha_0+\mu_0\right)}\right] P_0, \tag{114}$$

$$P_3 = \frac{\alpha_0}{\lambda+\mu_0} P_1 = \frac{\varepsilon \alpha_0}{\left(\lambda+\mu_0\right)\left(\alpha_0+\mu_0\right)} P_0, \tag{115}$$

$$P_4 = \frac{\lambda}{\mu_0} P_3 = \frac{\lambda \varepsilon \alpha_0}{\mu_0\left(\lambda+\mu_0\right)\left(\alpha_0+\mu_0\right)} P_0, \tag{116}$$

$$P_5(x) = \alpha_0\left(1+\frac{\varepsilon}{\alpha_0+\mu_0}\right) P_0 e^{-\int_0^x \mu(s) \mathrm{d}s}. \tag{117}$$

Let

$$P_5 = \int_0^{\infty} P_5(x) \mathrm{d}x = \alpha_0\left(1+\frac{\varepsilon}{\alpha_0+\mu_0}\right) P_0 g(0), \tag{118}$$

where $g(0)=\int_0^\infty e^{-\int_0^x \mu(s)\mathrm{d}s}\mathrm{d}x$ and

$$S=\sum_{i=0}^{5}P_i = \frac{\mu_0(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)(\lambda+\alpha_0+\lambda\alpha_0 g(0))+\lambda^2\varepsilon\alpha_0}{\lambda\mu_0(\alpha_0+\mu_0)(\lambda+\mu_0)}P_0. \tag{119}$$

Theorem 25. *The steady-state availability of the system is*

$$A_v=\frac{\mu_0(\lambda+\alpha_0)(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)}{\mu_0(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)(\lambda+\alpha_0+\lambda\alpha_0 g(0))+\lambda^2\varepsilon\alpha_0}. \tag{120}$$

Proof. The instantaneous availability of the system at time t is

$$A_v(t)=\sum_{i=0}^{3}P_i(t). \tag{121}$$

Let $t\to\infty$, then the steady-state availability of the system is obtained as follows:

$$A_v=\frac{\sum_{i=0}^{3}P_i}{S} = \frac{\mu_0(\lambda+\alpha_0)(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)}{\mu_0(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)(\lambda+\alpha_0+\lambda\alpha_0 g(0))+\lambda^2\varepsilon\alpha_0}. \tag{122}$$

The proof of Theorem 25 is completed. □

Theorem 26. *The steady-state probability of the repairman vacation is*

$$P_v=\frac{\lambda\varepsilon(\alpha_0+\mu_0)(\lambda+\mu_0)}{\mu_0(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)(\lambda+\alpha_0+\lambda\alpha_0 g(0))+\lambda^2\varepsilon\alpha_0}. \tag{123}$$

Proof. The instantaneous probability of the repairman vacation at time t is

$$P_v(t)=P_1(t)+P_3(t)+P_4(t). \tag{124}$$

Letting $t\to\infty$ derives the steady-state probability of the repairman vacation:

$$P_v=\frac{P_1+P_3+P_4}{S} = \frac{\lambda\varepsilon(\alpha_0+\mu_0)(\lambda+\mu_0)}{\mu_0(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)(\lambda+\alpha_0+\lambda\alpha_0 g(0))+\lambda^2\varepsilon\alpha_0}. \tag{125}$$

The proof of Theorem 26 is completed. □

Theorem 27. *The steady-state probability of the system in warning state is*

$$P_w=\frac{\alpha_0\mu_0(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)}{\mu_0(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)(\lambda+\alpha_0+\lambda\alpha_0 g(0))+\lambda^2\varepsilon\alpha_0}. \tag{126}$$

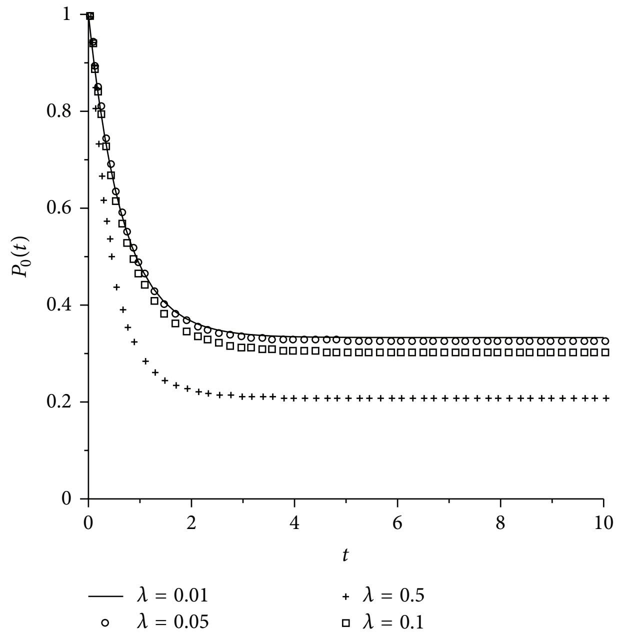


Figure 1: Instantaneous probabilities of the system without warning device in good state with different λ.

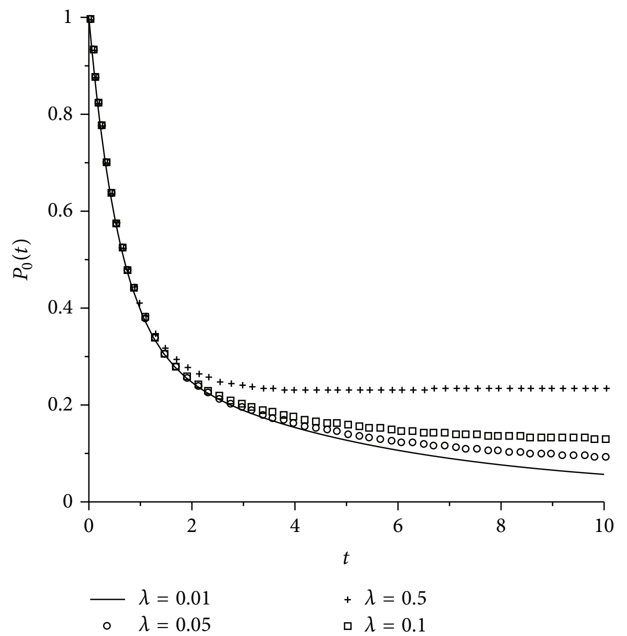


Figure 2: Instantaneous probabilities of the system with warning device in good state with different λ.

Proof. The instantaneous probability of the system in warning state at time t is

$$P_w(t)=P_2(t)+P_3(t). \tag{127}$$

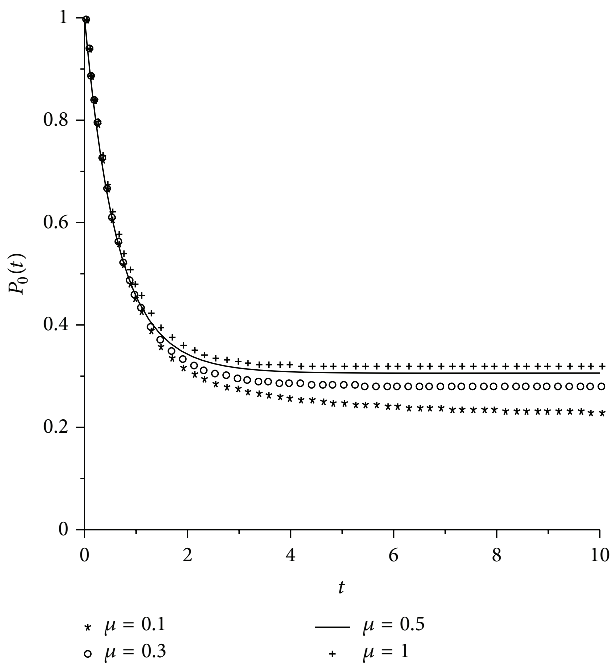


FIGURE 3: Instantaneous probabilities of the system without warning device in good state with different μ.

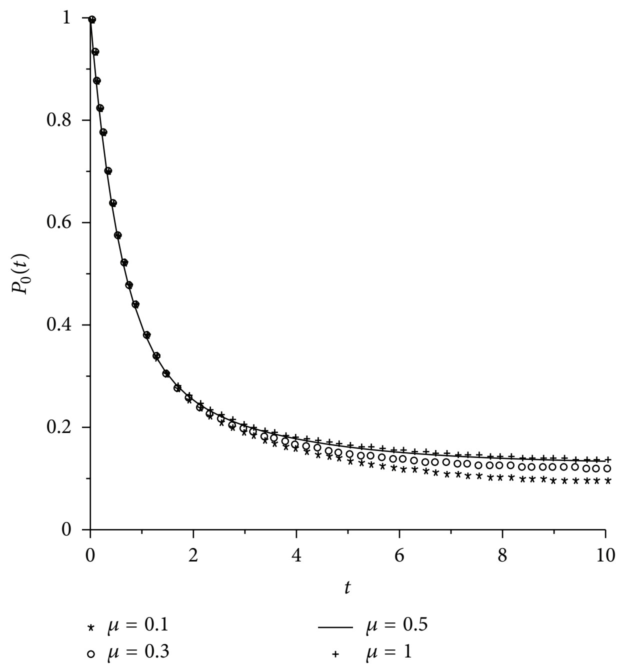


FIGURE 4: Instantaneous probabilities of the system with warning device in good state with different μ.

Letting $t \to \infty$ yields the steady-state probability of the system in warning state:

$$\begin{aligned} P_w &= \frac{P_2 + P_3}{S} \\ &= \frac{\alpha_0\mu_0(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)}{\mu_0(\lambda+\mu_0)(\alpha_0+\mu_0+\varepsilon)(\lambda+\alpha_0+\lambda\alpha_0 g(0)) + \lambda^2\varepsilon\alpha_0}. \end{aligned} \tag{128}$$

The proof of Theorem 27 is completed. □

Theorem 28. *The steady-state failure frequency of the system is*

$$W_f = \lambda P_w. \tag{129}$$

Proof. Let $P_5(t) = \int_0^\infty P_5(t,x)\mathrm{d}x$ and $\mu(t) = \int_0^\infty \mu(x)P_5(t,x)\mathrm{d}x/P_5(t)$. Then, the matrix of the transition probability of the system (1)–(3) can be obtained by (1)-(2) as follows:

$$T = \begin{pmatrix} -\varepsilon-\alpha_0 & \mu_0 & 0 & 0 & 0 & \mu(t) \\ \varepsilon & -\alpha_0-\mu_0 & 0 & 0 & 0 & 0 \\ \alpha_0 & 0 & -\lambda & \mu_0 & 0 & 0 \\ 0 & \alpha_0 & 0 & -\lambda-\mu_0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & -\mu_0 & 0 \\ 0 & 0 & \lambda & 0 & \mu_0 & -\mu(t) \end{pmatrix}. \tag{130}$$

Thus, by [37] the instantaneous failure frequency of the system at time t can be derived as

$$W_f(t) = \lambda\left[P_2(t) + P_3(t)\right]. \tag{131}$$

Let $t \to \infty$, then the steady-state failure frequency is immediate

$$W_f = \frac{\lambda(P_2+P_3)}{S} = \lambda P_w. \tag{132}$$

The proof of Theorem 28 is completed. □

6. Applications and Numerical Examples

Reference [38] discussed the effects of the delayed vacation and vacation policies on a system. That is, the shorter the delayed vacation time, the larger the reliability and failure frequency of a system; and the reliability of a system with multiple vacations is smaller than that of a system with single vacation, while the profit of a system with multiple vacations is larger than that of a system with single vacation. Therefore, in this section, we only concentrate on that how the warning device will affect the system. Specifically, we will compare the reliability, availability, and profit of the system with warning device and those of the system without warning device and present some numerical examples.

6.1. System without Warning Device. The simple repairable system without warning device and with a repairman following delayed-multiple vacations policy is as follows:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \varepsilon + \lambda\right)P_0(t) = \mu_0 P_1(t) + \int_0^\infty \mu(x)P_5(t,x)\mathrm{d}x, \tag{133}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mu_0 + \lambda\right)P_1(t) = \varepsilon P_0(t), \tag{134}$$

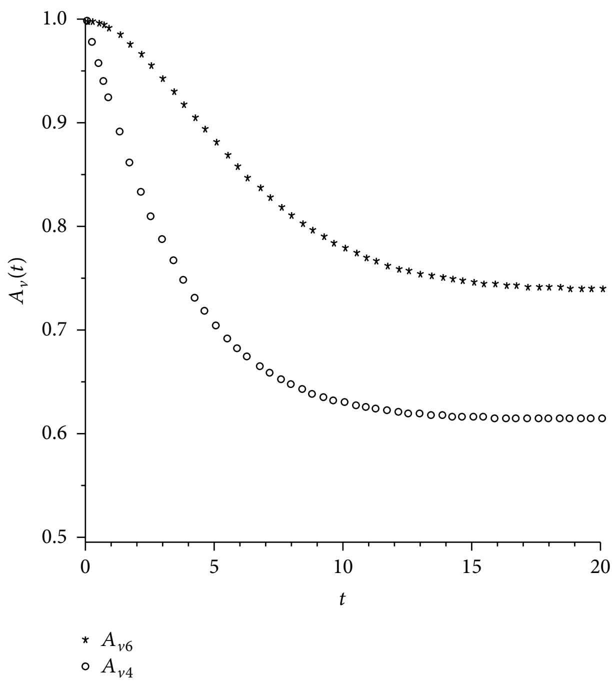


Figure 5: Instantaneous availabilities of the systems with and without warning device.

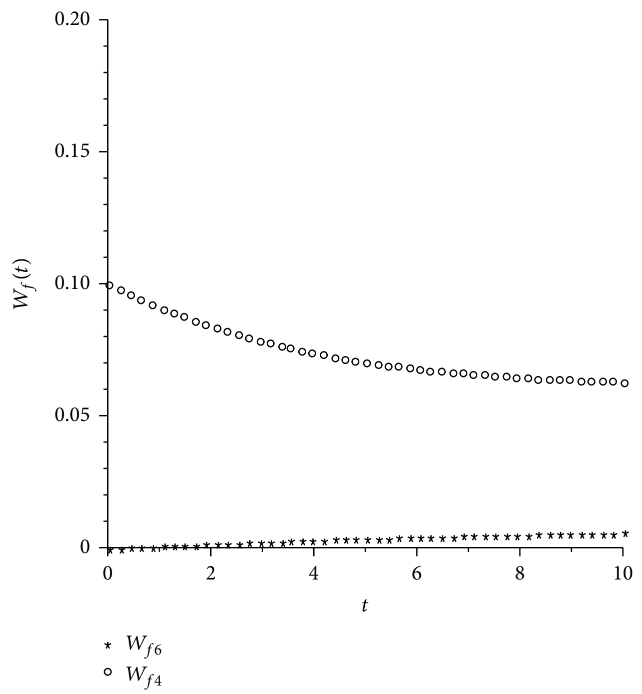


Figure 6: Instantaneous failure frequencies of the systems with and without warning device with $\alpha = 0.01$.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \mu_0\right) P_4(t) = \lambda P_1(t), \tag{135}$$

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \mu(x)\right] P_5(t,x) = 0 \tag{136}$$

with boundary condition

$$P_5(t,0) = \lambda P_0(t) + \mu_0 P_4(t) \tag{137}$$

and initial conditions

$$P_0(0) = 1, \quad \text{the others equal to } 0. \tag{138}$$

Then, by the same method of Theorems 25, 26, and 28, the corresponding reliability indices of the system (133)–(138) are as follows.

Theorem 29. *The steady-state availability of the system* (133)–(138) *is*

$$\widetilde{A}_v = \frac{\mu_0(\mu_0 + \lambda + \varepsilon)}{\mu_0(\mu_0 + \lambda + \varepsilon)(1 + \lambda g(0)) + \lambda\varepsilon}, \tag{139}$$

where $g(0)$ is defined in Section 5.

Theorem 30. *The steady-state probability of the repairman vacation of the system* (133)–(138) *is*

$$\widetilde{P}_v = \frac{\varepsilon(\mu_0 + \lambda)}{\mu_0(\mu_0 + \lambda + \varepsilon)(1 + \lambda g(0)) + \lambda\varepsilon}. \tag{140}$$

Theorem 31. *The steady-state failure frequency of the system* (133)–(138) *is*

$$\widetilde{W}_f = \lambda A_v. \tag{141}$$

6.2. Numerical Examples. By comparing the two groups of (120) and (139) and (129) and (141), it is not difficult to deduce the following results.

(i) The steady-state availability of the system with warning device (i.e., system (1)–(3)) is larger than that of the system without warning device (i.e., system (133)–(138)). That is, $A_v > \widetilde{A}_v$.

(ii) If $\alpha_0 \le \lambda$, then the steady-state failure frequency of the system with warning device is less than that of the system without warning device. That is, $W_f < \widetilde{W}_f$. While if $\alpha_0 > \lambda$, the magnitude of W_f and $\widetilde{W}_f$ cannot be determined.

Let I and $\widetilde{I}$ be the total profit of the system with and without warning device, respectively. That is,

$$\begin{aligned} I &= c_1 A_v - c_2 W_f + c_3 P_v, \\ \widetilde{I} &= c_1 \widetilde{A}_v - c_2 \widetilde{W}_f + c_3 \widetilde{P}_v. \end{aligned} \tag{142}$$

Here, c_1, c_2, and c_3 represent the income of the system for working unit per unit time, the loss of the system for failed unit per unit time, and the income of the system for the repairman vacation per unit time, respectively. Given $\varepsilon = 1$, $\mu_0 = 0.5$, $\alpha_0 = 0.2$, $c_1 = 50$, $c_2 = 15$, $c_3 = 30$, let $D = I - \widetilde{I}$. Then, D is a function of λ and μ. From Figure 10, we can know that the profit of the system with warning device is larger than that of the system without warning device.

From the above discussions, we can deduce that because both the availability and profit of a system with warning device are larger than those of a system without warning

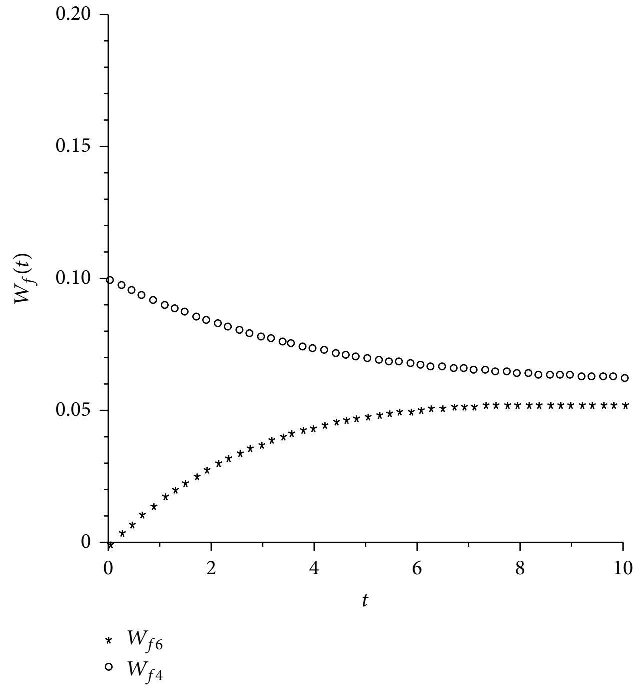


FIGURE 7: Instantaneous failure frequencies of the systems with and without warning device with $\alpha = 0.2$.

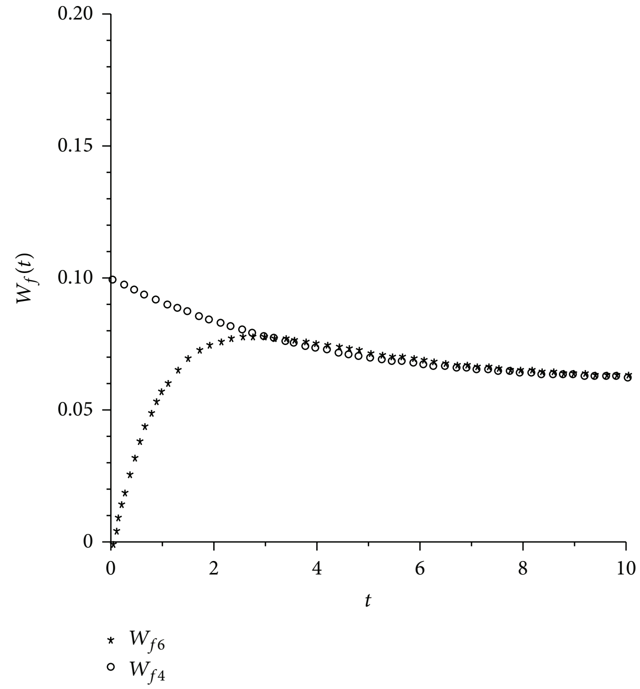


FIGURE 8: Instantaneous failure frequencies of the systems with and without warning device with $\alpha = 1$.

device. Then, a system with warning device is better than a system without warning device in practice.

In the following, we will present some numerical examples to illustrate the conclusions.

(1) Let $\varepsilon = 1$, $\mu_0 = \mu = 0.5$, $\alpha_0 = 0.2$, $\lambda = 0.01, 0.05, 0.1$, and 0.5, respectively. Figures 1 and 2 present the instantaneous probabilities of the systems with and without warning device in good state with different values of λ. Let $\varepsilon = 1$, $\lambda = 0.1$, $\mu_0 = 0.5$, $\alpha_0 = 0.2$, $\mu = 0.1, 0.3, 0.5$, and 1, respectively. Figures 3 and 4 present the instantaneous probabilities of the systems with and without warning device in good state with different values of μ. From the four figures, we can see that the instantaneous probabilities of the system without warning device in good state decrease with the increasing of λ, while the instantaneous probabilities of the system with warning device in good state increase with the increasing of λ. But both the instantaneous probabilities of the systems with and without warning device in good state increase with the increasing of μ in general.

(2) Let $\varepsilon = 1$, $\lambda = 0.1$, $\mu_0 = 0.5$, $\mu = 0.2$, $\alpha_0 = 0.2$. Figure 5 presents the instantaneous availabilities of the systems with and without warning device. From the figure, we can see that the availability of the system with warning device is larger than that of the system without warning device.

(3) Let $\varepsilon = 1$, $\lambda = 0.1$, $\mu_0 = 0.5$, $\mu = 0.2$ and $\alpha_0 = 0.01, 0.2, 1$, and 5, respectively. Figures 6, 7, 8, and 9 present the instantaneous failure frequencies of the systems with and without warning device. It can be derived from the four figures that when $\alpha_0 < \lambda$, the instantaneous failure frequency of the system with warning device is less than that of the system without warning device. But when $\alpha_0 \geq$

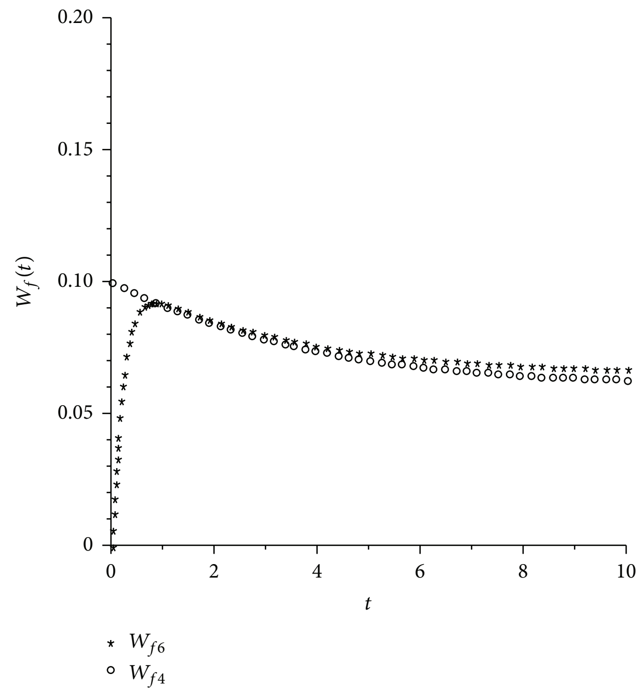


FIGURE 9: Instantaneous failure frequencies of the systems with and without warning device with $\alpha = 5$.

$1 \gg \lambda$, the instantaneous failure frequency of the system with warning device is larger than that of the system without warning device.

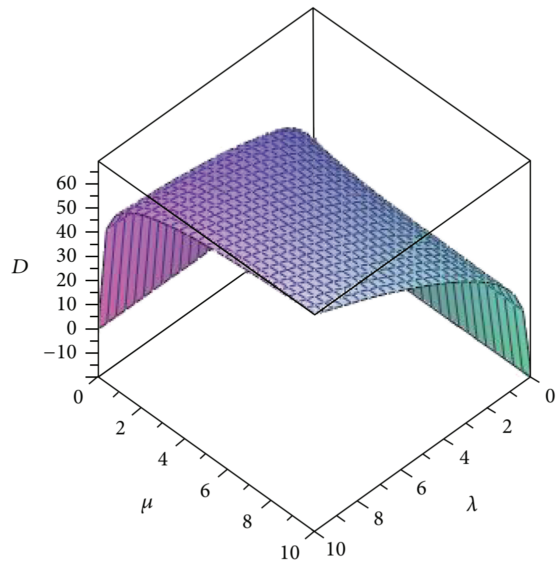


Figure 10: Profit difference between the systems with and without warning device.

(4) Let $\varepsilon = 1$, $\mu_0 = 0.5$, $\alpha_0 = 0.2$, $c_1 = 50$, $c_2 = 15$, $c_3 = 30$. Figure 10 presents the profit difference between the systems with and without warning device. From the figure, it can be derived easily that the profit the system with warning device is more than that of the system without warning device.

7. Conclusion

In this paper, we proposed a simple repairable system with a warning device and a repairman who can have delayed-multiple vacations. Because the two hypotheses used for Laplace transform in order to obtain the steady-state solution of a repairable system in traditional reliability research that needs to be verified, and the substitution of steady-state solution for the dynamic one that should be based on some conditions, the study of well-posedness of the time-dependent solution of a system is in demand in terms of theory and practice. In this paper, we first transformed the system model into a group of operator equations and obtained the existence and uniqueness as well as C^1 continuity of the system solution by functional analysis method. Then to study the stability of the system, we translated the system model into an abstract Cauchy problem in a suitable Banach space. The asymptotic stability and further the exponential stability of the system solution were derived by using C_0 semigroup theory and compact operator disturbance theorem. Because the stable solution of the system is just the eigenfunction corresponding to eigenvalue 0 of the system operator, we also presented some reliability indices, such as reliability, failure frequency, probabilities of repairman vacation, and system in warning state of the system in the viewpoint of eigenfunction. At the end of the paper, by the theoretical and numerical analyses, we give the conclusion that the system with warning device is better than the system without warning device in practice.

Acknowledgments

This work is supported by the TianYuan Special Funds of the National Natural Science Foundation of China under Grant no. 11226249, the Foundation Research Project of Shanxi Province (The Youth) under Grant no. 2012021015-5, the Youth Foundation of Taiyuan University of Technology under Grant no. 2012L031, the Special Funds of the National Natural Science Foundation of China under Grant no. 61250011, and the Foundation Research Project of Shanxi Province (The Natural Science) under Grant no. 2012011004-4.

References

[1] M. Yaez, F. Joglar, and M. Modarres, “Generalized renewal process for analysis of repairable systems with limited failure experience,” *Reliability Engineering and System Safety*, vol. 77, no. 2, pp. 167–180, 2002.

[2] R. K. Mobley, *An Introduction to Predictive Maintenance*, Elsevier Science, New York, NY, USA, 2nd edition, 2002.

[3] B. T. Doshi, “Queueing systems with vacations—a survey,” *Queueing Systems*, vol. 1, no. 1, pp. 29–66, 1986.

[4] H. Takagi, *Queueing Analysis: A Foundation of Performance Evaluation, Vacation and Priority Systems, Part I*, vol. 1, North-Holland, Amsterdam, The Netherlands, 1991.

[5] N. Tian and Z. G. Zhang, *Vacation Queueing Models: Theory and Applications*, International Series in Operations Research & Management Science, Springer, New York, NY, USA, 2006.

[6] J. Ke, “Batch arrival queues under vacation policies with server breakdowns and startup/closedown times,” *Applied Mathematical Modelling*, vol. 31, no. 7, pp. 1282–1292, 2007.

[7] B. K. Kumar, R. Rukmani, and V. Thangaraj, “An M/M/C retrial queueing system with Bernoulli vacations,” *Journal of Systems Science and Systems Engineering*, vol. 18, no. 2, pp. 222–242, 2009.

[8] Z. G. Zhang and N. Tian, “Discrete time *Geo*/*G*/1 queue with multiple adaptive vacations,” *Queueing Systems*, vol. 38, no. 4, pp. 419–429, 2001.

[9] Z. G. Zhang, “On the three threshold policy in the multi-server queueing system with vacations,” *Queueing Systems*, vol. 51, no. 1-2, pp. 173–186, 2005.

[10] J. Wu, Z. Liu, and Y. Peng, “On the BMAP/G/1 *G*-queues with second optional service and multiple vacations,” *Applied Mathematical Modelling. Simulation and Computation for Engineering and Environmental Systems*, vol. 33, no. 12, pp. 4314–4325, 2009.

[11] J. C. Ke, C. U. Wu, and Z. G. Zhang, “Recent developments in vacation models: a short survey,” *International Journal of Operations Research*, vol. 7, no. 4, pp. 3–8, 2010.

[12] M. Jain and M. S. Rakhee, “Bilevel control of degraded machining system with warm standbys, setup and vacation,” *Applied Mathematical Modelling*, vol. 28, no. 12, pp. 1015–1026, 2004.

[13] L. M. Hu, D. Q. Yue, and J. D. Li, “Probabilistic analysis of a series-parallel repairable system with three units and vacation,” *Applied Mathematical Modelling*, vol. 34, no. 10, pp. 2711–2721, 2010.

[14] Q. T. Wu and S. M. Wu, “Reliability analysis of two-unit cold standby repairable systems under Poisson shocks,” *Applied Mathematics and Computation*, vol. 218, no. 1, pp. 171–182, 2011.

[15] L. Yuan, “Reliability analysis for a k-out-of-n: G system with redundant dependency and repairmen having multiple vacations,” *Applied Mathematics and Computation*, vol. 218, no. 24, pp. 11959–11969, 2012.

[16] L. Yuan and Z. D. Cui, “Reliability analysis for the consecutive-k-out-of-n:F system with repairmen taking multiple vacations,” *Applied Mathematical Modelling*, vol. 37, no. 7, pp. 4685–4697, 2013.

[17] W. Z. Yuan and G. Q. Xu, “Modelling of a deteriorating system with repair satisfying general distribution,” *Applied Mathematics and Computation*, vol. 218, no. 11, pp. 6340–6350, 2012.

[18] J. C. Ke and C. H. Wu, “Multi-server machine repair model with standbys and synchronous multiple vacation,” *Computers and Industrial Engineering*, vol. 62, no. 1, pp. 296–305, 2012.

[19] J. C. Ke and K. H. Wang, “Vacation policies for machine repair problem with two type spares,” *Applied Mathematical Modelling*, vol. 31, no. 5, pp. 880–894, 2007.

[20] J. S. Jia and S. M. Wu, “A replacement policy for a repairable system with its repairman having multiple vacations,” *Computers and Industrial Engineering*, vol. 57, no. 1, pp. 156–160, 2009.

[21] L. Yuan and J. Xu, “A deteriorating system with its repairman having multiple vacations,” *Applied Mathematics and Computation*, vol. 217, no. 10, pp. 4980–4989, 2011.

[22] L. Yuan and J. Xu, “An optimal replacement policy for a repairable system based on its repairman having vacations,” *Reliability Engineering and System Safety*, vol. 96, no. 7, pp. 868–875, 2011.

[23] M. M. Yu, Y. H. Tang, L. P. Liu, and J. Cheng, “A phase-type geometric process repair model with spare device procurement and repairman’s multiple vacations,” *European Journal of Operational Research*, vol. 225, no. 2, pp. 310–323, 2013.

[24] L. N. Guo, H. B. Xu, C. Gao, and G. T. Zhu, “Stability analysis of a new kind series system,” *IMA Journal of Applied Mathematics*, vol. 75, no. 3, pp. 439–460, 2010.

[25] L. N. Guo, H. B. Xu, C. Gao, and G. T. Zhu, “Stability analysis of a new kind n-unit series repairable system,” *Applied Mathematical Modelling*, vol. 35, no. 1, pp. 202–217, 2011.

[26] M. E. Gurtin and R. C. MacCamy, “Non-linear age-dependent population dynamics,” *Archive for Rational Mechanics and Analysis*, vol. 54, pp. 281–300, 1974.

[27] Y. Yao, “Global existence and uniqueness of solutions to epidemic dynamic systems of equations,” *Chinese Annals of Mathematics A*, vol. 12, no. 2, pp. 218–230, 1991.

[28] P. Yan, “Well-posedness of a model for the dynamics of infectious diseases without immunity and with latent period,” *Journal of Biomathematics*, vol. 23, no. 2, pp. 245–256, 2008.

[29] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, vol. 194 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2000.

[30] H. B. Xu, W. H. Guo, J. Y. Yu, and G. T. Zhu, “The asymptotic stability of a series repairable system,” *Acta Mathematicae Applicatae Sinica*, vol. 29, no. 1, pp. 46–52, 2006.

[31] N. Dunford and J. T. Schwartz, *Linear Operators, Part I*, Wiley Classics Library, John Wiley & Sons, New York, NY, USA, 1958.

[32] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.

[33] G. Geni, X. Z. Li, and G. T. Zhu, *Functional Analysis Method in Queueing Theory*, Research Information Limited, Hertfordshire, UK, 2001.

[34] P. Clément, H. J. A. M. Heijmans, S. Angenent, C. J. van Duijn, and B. de Pagter, *One-Parameter Semigroup*, Elsevier Science, Amsterdam, The Netherlands, 1987.

[35] R. Nagel, *One-Parameter Semigroups of Positive Operators*, vol. 1184 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 1986.

[36] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, John Wiley & Sons, New York, NY, USA, 1980.

[37] J. H. Cao and K. Cheng, *Introduction to Reliability Mathematics*, Higher Education Press, Beijing, China, 1986.

[38] L. N. Guo, Z. X. Zhao, H. B. Xu, and G. T. Zhu, “Reliability and profit analysis of a series repairable system with repairman vacation,” in *Proceedings of the 30th Chinese Control Conference (CCC ’11)*, pp. 1009–1014, July 2011.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 194631, 7 pages
http://dx.doi.org/10.1155/2013/194631

Research Article

On the Distance between Three Arbitrary Points

Parin Chaipunya and Poom Kumam

Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi, Bangkok 10140, Thailand

Correspondence should be addressed to Poom Kumam; poom.kum@kmutt.ac.th

Received 25 March 2013; Accepted 1 August 2013

Academic Editor: Satit Saejung

We point out some equivalence between the results in (Sedghi et al., 2012) and (Khamsi, 2010). Then, we introduce the notion of a general distance between three arbitrary points and study some of its properties. In the final section, some fixed point results are proposed.

1. Introduction

The literature of a distance for any triple of points in a space was first considered during the sixties by Gähler [1, 2]. It is known as a 2-metric, the concept of which was later extended by Dhage [3] into a D-metric. Both notions are in no easy ways related to the classical concept of a metric. This led to the G-metric due to Mustafa and Sims [4].

On the other hands, Sedghi et al. [5] had introduced the notion of a D^*-metric, which has later been generalized by Sedghi et al. [6] into an S-metric.

These developments confirm that this kind of measurement is recently of many mathematicians' interests. One of the area that exploited these establishments largely is the fixed point theory, especially the ones involving some generalized contractions; see, for example, Sedghi et al. [5], Sedghi et al. [6], Mustafa et al. [7], Mustafa and Sims [8], Aydi et al. [9], Alghamdi and Karapinar [10], Chandok et al. [11], and Abbas et al. [12].

In this present paper, we divide our interests into three parts. Firstly, we give a remark on the existing fixed point result endowed in a S-metric space. Secondly, we propose and study a very general principle in measuring the distance between three arbitrary points called a $g-3ps$. Thirdly, we construct some fixed point theorems by utilizing the $g-3ps$ and its properties.

2. Preliminaries

This section is devoted to the recollection of important definitions and lemmas. We start with a sequence of definitions of G-, D^*-, and S-metric spaces.

Definition 1 (see [4]). Let X be a nonempty set. A function $G : X\times X\times X \to \mathbb{R}_+$ is said to be a G-metric if the following conditions are satisfied:

(1) for $x, y, z \in X$, $G(x, y, z) = 0$ if $x = y = z$;

(2) for $x, y \in X$ with $x \neq y$, $G(x, x, y) > 0$;

(3) for $x, y, z \in X$ with $y \neq z$, $G(x, x, y) \le G(x, y, z)$;

(4) for $x, y, z \in X$, $G(x, y, z) = G(\pi(x, y, z))$, where $\pi(x, y, z)$ is any permutation of $(x, y, z) \in X\times X\times X$;

(5) for $x, y, z, v \in X$, $G(x, y, z) \le G(x, v, v) + G(v, y, z)$.

The pair (X, G) is called a G-metric space. Moreover, if $G(x, x, y) = G(y, y, x)$ for all $x, y \in X$, then G is said to be *symmetric*.

Definition 2 (see [5]). Let X be a nonempty set. A function $D^* : X\times X\times X \to \mathbb{R}_+$ is said to be a D^*-metric if the following conditions are satisfied:

(1) for $x, y, z \in X$, $D^*(x, y, z) = 0$ if and only if $x = y = z$;

(2) for $x, y, z \in X$, $D^*(x, y, z) = D^*(\pi(x, y, z))$, where $\pi(x, y, z)$ is any permutation of $(x, y, z) \in X \times X \times X$;

(3) for $x, y, z, v \in X$, $D^*(x, y, z) \leq D^*(x, y, v) + D^*(v, z, z)$.

The pair (X, D^*) is called a D^*-metric space.

Definition 3 (see [6]). Let X be a nonempty set. A function $S : X \times X \times X \to \mathbb{R}_+$ is said to be a S-metric if the following conditions are satisfied:

(1) for $x, y, z \in X$, $S(x, y, z) = 0$ if and only if $x = y = z$;

(2) for $x, y, z, v \in X$, $S(x, y, z) \leq S(x, x, v) + S(y, y, v) + S(z, z, v)$.

The pair (X, S) is called a S-metric space.

Lemma 4 (see [6]). *Let (X, S) be a S-metric space, then $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.*

It can be seen that each symmetric G-metric is a D^*-metric and that each D^*-metric is a S-metric. In case of non-symmetric G-metric, the concepts of G-metric and S-metric are independent.

Definition 5 (see [6]). Let (X, S) be a S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x; r)$ as follows:

$$B_S(x; r) = \{y \in X; S(x, x, y) < r\}. \tag{1}$$

As in [6], one may consider the topology τ for X which is generated from the base containing all open balls in X. Some concepts are also introduced.

Definition 6 (see [6]). Let (X, S) be a S-metric space. A sequence (x_n) in X is called

(i) *Cauchy* if for any $\epsilon > 0$, we may find $N \in \mathbb{N}$ such that

$$n \in \mathbb{N},\ n \geq N \Longrightarrow S(x_m, x_m, x_n) < \epsilon; \tag{2}$$

(ii) *convergent* if there is a point $x \in X$ in which

$$\lim_{n \to \infty} S(x, x, x_n) = 0. \tag{3}$$

Moreover, if every Cauchy sequence in X is also convergent, X is said to be *complete*.

3. A S-Metric Space as a Metric Type Space

In this section, we shall be giving a small remark on a fixed point theorem due to [6]. According to [6], a self-operator f on a S-metric space (X, S) is called a contraction if it satisfies the following inequality:

$$S(fx, fx, fy) \leq \lambda S(x, x, y), \tag{4}$$

for all $x, y \in X$, where $0 \leq \lambda < 1$. The following result was introduced subsequently.

Theorem 7 (see [6]). *Let (X, S) be a complete S-metric space and let f be a contraction on X. Then, f has a unique fixed point.*

To expedite our remark, we shall first recall the notion of a metric type space, which was introduced by Khamsi in [13].

Definition 8 (see [13]). Let X be a nonempty set and let $D : X \times X \to \mathbb{R}_+$ be a function satisfying the following conditions:

(1) for $x, y \in X$, $D(x, y) = 0$ if and only if $x = y$;

(2) for $x, y \in X$, $D(x, y) = D(y, x)$;

(3) there exists a constant $K > 0$ such that for $x, y, z_1, z_2, \ldots, z_n \in X$,

$$D(x, y) \leq K[D(x, z_1) + D(z_1, z_2) + \cdots + D(z_n, y)]. \tag{5}$$

The triple (X, D, K) is called a metric type space.

In particular, if $K \leq 1$, then (X, D) is a metric space. A self-mapping operator f on a metric type space (X, D, K) is called *Lipschizian* if there exists $\lambda \geq 0$ such that

$$D(fx, fy) \leq \lambda D(x, y), \tag{6}$$

for all $x, y \in X$. The smallest $\lambda > 0$ satisfying such condition is denoted by $\mathrm{Lip}(f)$. Moreover, the following fixed point theorem was proposed.

Theorem 9 (see [13]). *Let (X, D, K) be a complete metric type space and let $f : X \to X$ be an operator such that the composition f^n is Lipschizian for each $n \in \mathcal{N}$ and $\sum_{n\in\mathbb{N}} Lip(f^n) < \infty$. Then, f has a unique fixed point.*

It follows that if f is a Lipschizian operator with $\mathrm{Lip}(f) < 1$, then f has a unique fixed point.

Now, let (X, S) be a S-metric space. Suppose that a function $D : X \times X \to \mathbb{R}_+$ is given by

$$D_S(x, y) := S(x, x, y), \quad \forall x, y \in X, \tag{7}$$

for $x, y \in X$, it is obvious that $D(x, y) = 0$ if and only if $x = y$ and $D(x, y) = D(y, x)$ for all $x, y \in X$. Now, observe for each $z_1, z_2, \ldots, z_n \in X$ that

$$\begin{aligned} D_S(x, y) &= S(x, x, y) \\ &\leq 2S(x, x, z_1) + S(y, y, z_1) \\ &= 2D_S(x, z_1) + D_S(z_1, y) \\ &\leq 2D_S(x, z_1) + 2D_S(z_1, z_2) + D_S(z_2, y) \\ &\vdots \\ &\leq 2D_S(x, z_1) + 2D_S(z_1, z_2) \\ &\quad + \cdots + 2D_S(z_{n-1}, z_n) + D_S(z_n, y) \\ &\leq 2[D_S(x, z_1) + D_S(z_1, z_2) \\ &\quad + \cdots + D_S(z_{n-1}, z_n) + D_S(z_n, y)]. \end{aligned} \tag{8}$$

Thus, $(X, D_S, 2)$ is a metric type space. Moreover, the balls $B_S(x; r)$ and $B_{D_S}(x; r)$ coincide.

Notice that we may rewrite the inequality (4) as follows:

$$D_S(fx, fy) \le \lambda D_S(x, y), \tag{9}$$

for all $x, y \in X$ with the same λ. Also notice that the definition of Cauchyness, convergence and completeness in a S-metric space (X, S) may be rewriten in terms of metric type spaces. So, these notions are transferred to the corresponding metric type space $(X, D_S, 2)$.

Now, if f is an operator satisfying (9), then each f^n, where $n \in \mathbb{N}$, is Lipschizian with $\mathrm{Lip}(f^n) = \lambda^n$. Therefore, Theorem 7 is obtained via Theorem 9.

Even though we set a new condition for an operator f, where $0 \le \lambda < 1$, to be

$$S(fx, fy, fz) \le \lambda S(x, y, z), \tag{10}$$

for each $x, y, z \in X$, the unique fixed point can still be obtained via Theorem 9, anyway. Note that not only the mentioned theorem, but also many theorems in the literature may be proved via this concept in metric type spaces. We shall give some results which seem more general than the setting of Theorem 7 but however equivalent.

Beforehand, we give the following useful lemma without proof since it is straight forward.

Lemma 10. *Let (X, S) be a S-metric space and let $x, y \in X$. Then, the following inequalities hold:*

(i) $S(x, y, y) \le S(x, x, y) = S(y, y, x)$;

(ii) $S(x, y, x) \le S(x, x, y) = S(y, y, x)$.

Theorem 11. *Let (X, S) be a complete S-metric space and let $f : X \to X$ be an operator such that there exists a sequence (λ_n) of nonnegative reals satisfying the condition:*

$$S(fx, fy, fz) \le \sum_{n\in\mathbb{N}} \lambda_n S(\pi_n(x, y, z)), \tag{11}$$

for all $x, y, z \in X$, where $\Lambda := \sum_{n\in\mathbb{N}} \lambda_n < 1$ and for each $n \in \mathbb{N}$, π_n is a fixed permutation in X^3. Then, f has a unique fixed point.

Proof. We shall show that (11) implies that f is Lipschizian with $\mathrm{Lip}(f) < 1$ in metric type space $(X, D_S, 2)$. For each $x, y \in X$, it is easy to verify that $S(\pi_n(x, x, y)) \le S(x, x, y)$ no matter which permutations are defined. Thus, we obtain that

$$\begin{aligned} D_S(fx, fy) &= S(fx, fx, fy) \\ &\le \sum_{n\in\mathbb{N}} \lambda_n S(\pi_n(x, x, y)) \\ &\le \Lambda S(x, x, y) \\ &= \Lambda D_S(x, y). \end{aligned} \tag{12}$$

Apply Theorem 9 (or Theorem 7) to obtain the desired result. □

4. A General Distance between Three Arbitrary Points

In this section, we shall be dealing with a new concept of a general distance between three arbitrary points (or $g - 3ps$).

To be able to define the $g - 3ps$, we first consider a nonempty set X together with a function $g : X\times X\times X \to \mathbb{R}_+$ for which $g(x, y, z) = 0$ if and only if $x = y = z$. Given $x \in X$ and $r > 0$, we define an open ball in the usual sense:

$$B_g(x; r) := \{y \in X; g(x, x, y) < r\}. \tag{13}$$

To be natural, we say that a subset $A \subset X$ is *bounded* if $\sup_{x,y,z\in A} g(x, y, z) < \infty$. Certainly, the assertion "$g(x, y, z) = 0$ *if and only if* $x = y = z$" is not enough to guarantee that open balls in X are bounded. We shall illustrate in the following.

Example 12. Let $X := [0, 1]$ and let $g : X \times X \times X \to \mathbb{R}_+$ be a function given by

$$g(x, y, z) := \begin{cases} 0, & \text{if } x = y = z, \\ \left|\left(\dfrac{x+y}{2}\right) - z\right|, & \text{if } x, y \in \mathbb{Q} \cap X \\ & \text{but } z \in \mathbb{Q}_\complement \cap X, \\ \left|\left(\dfrac{x+y}{2}\right) - z\right|, & \text{if } x, y \in \mathbb{Q}_\complement \cap X \\ & \text{but } z \in \mathbb{Q} \cap X, \\ \dfrac{1}{|((x+y)/2) - z|}, & \text{if } x, y, z \in \mathbb{Q} \cap X \\ & \text{and } [x \ne y \text{ or } y \ne z], \\ \dfrac{1}{|((x+y)/2) - z|}, & \text{if } x, y, z \in \mathbb{Q}_\complement \cap X \\ & \text{and } [x \ne y \text{ or } y \ne z], \\ 1, & \text{otherwise.} \end{cases} \tag{14}$$

It is clear that $g(x, y, z) = 0$ if and only if $x = y = z$.

Let $x \in X$ and let $r > 0$. Note that if $x \in \mathbb{Q} \cap X$, then

$$\mathbb{Q}_\complement \cap (x - r, x + r) \cap X \subset B_g(x; r). \tag{15}$$

Let (z_n) be a sequence in $\mathbb{Q}_\complement \cap (x-r, x+r) \cap X$ such that $|x - z_n| < r/(1+nr)$ for each $n \in \mathbb{N}$. Since $|z_n - z_{n+1}| < 2r/(1+nr)$, we have $g(z_n, z_n, z_{n+1}) > (1/2r)+(n/2)$ for all $n \in \mathbb{N}$. The same conclusion can be deduced also when $x \in \mathbb{Q}_\complement$. Therefore, $B_g(x; r)$ is not bounded at each $x \in X$ and $r > 0$.

This is not quite natural and does not meet the requirements we would like to have. So, we may add one more assumption at this stage and define the $g - 3ps$ space as follows.

Definition 13. Let X be a nonempty set. A function $g : X \times X \times X \to \mathbb{R}_+$ is said to be a $g - 3ps$ if the following conditions are satisfied:

(g1) for $x, y, z \in X$, $g(x, y, z) = 0$ if and only if $x = y = z$;

(g2) there exists some $r_0 > 0$ such that the balls $B_g(x; r_0)$ are bounded for all $x \in X$.

The pair (X, g) is called a $g - 3ps$ *space*.

Next, we shall give a characterization of a $g-3ps$ space.

Lemma 14. *Let X be a nonempty set and let $g : X\times X\times X \to \mathbb{R}_+$ be a function satisfying (g1). Then, the following are equivalent:*

(i) *g satisfies (g2);*

(ii) *there exist some constants $\delta,\eta > 0$ such that for any $x,u,v,w \in X$, one has*

$$g(x,x,u)+g(x,x,v)+g(x,x,w)<\delta \Longrightarrow g(u,v,w)<\eta. \tag{16}$$

Proof. [(i) ⇒ (ii)] Assume that (i) holds. Set $\delta := r/2$ and let $x,u,v,w \in X$ arbitrarily. If $g(x,x,u)+g(x,x,v)+g(x,x,w)<\delta$, then $u,v,w \in B_g(x;r)$. Thus, setting $\eta := 1+\sup_{x\in X}\sup_{a,b,c\in B_g(x;r)} g(a,b,c) < \infty$ and it follows that $g(u,v,w)<\eta$.

[(ii) ⇒ (i)] Assume that (ii) holds. Let $x\in X$ and suppose that $u,v,w \in B_g(x,\delta/3)$. Therefore, we have

$$g(x,x,u)+g(x,x,v)+g(x,x,w)<\delta. \tag{17}$$

From (ii), we obtain that $g(u,v,w) < \eta$. Thus, $\sup_{a,b,c\in B_g(X;\delta/3)} g(a,b,c) \le \eta$. Since $x \in X$ is arbitrary, the balls $B_g(x;\delta/3)$ are bounded for every $x\in X$. □

Remark 15. Suppose that (X,G) is a G-metric space and $x,u,v,w \in X$. Then, we have

$$\begin{aligned} G(u,v,w) &\le G(u,x,x)+G(x,v,w) \\ &= G(x,x,u)+G(v,x,w) \\ &\le G(x,x,u)+G(v,x,x)+G(x,x,w) \\ &= G(x,x,u)+G(x,x,v)+G(x,x,w). \end{aligned} \tag{18}$$

Thus, we can choose any $\delta > 0$ and let $\eta := \delta$ to fulfill the assumptions in Lemma 14. Hence, every G-metric space is in turn a $g-3ps$ space.

Remark 16. Suppose that (X,S) is a S-metric space and $x,u,v,w \in X$. Then, we have

$$\begin{aligned} S(u,v,w) &\le S(u,u,x)+S(v,v,x)+S(w,w,x) \\ &= S(x,x,u)+S(x,x,v)+S(x,x,w). \end{aligned} \tag{19}$$

The same argument is to be considered as in the previous remark. So, a S-metric space is a $g-3ps$ space. This immediately implies that a D^*-metric space is also a $g-3ps$ space.

Denoted by $\mathscr{U}$ the family of all open balls in X. Throughout this paper, we shall assume that $\mathscr{T} := \mathscr{T}(\mathscr{U})$ represents the topology having $\mathscr{U}$ as its subbase. Also, we write $\mathscr{U}^*$ to denote the base generated by $\mathscr{U}$.

Remark 17. The topology here is defined using different idea from those given in symmetric spaces or in semimetric spaces (see e.g., [14–17]).

Proposition 18. *The topology $\mathscr{T}$ for a $g-3ps$ space (X,g) is T_1-separable.*

Proof. Let $x,y \in X$ with $x\neq y$. So, we have $g(x,x,y) = r_1$ and $g(y,y,x) = r_2$, for some $r_1,r_2 > 0$. Observe that $y \notin B_g(x;r_1/2)$ and $x \notin B_g(y;r_2/2)$. The desired result is then followed. □

We shall now explicate an example of a $g-3ps$ space. In particular, this next example will even show that a $g-3ps$ space is no need to be T_2-separable.

Example 19. Let $X=[0,1]$ and define a function $g : X\times X\times X \to \mathbb{R}_+$ in the following:

$$g(x,y,z) := \begin{cases} 0, & \text{if } x=y=z, \\ z, & \text{if } x=y\neq z,\ z\neq 0, \\ 1, & \text{otherwise.} \end{cases} \tag{20}$$

It is clear that any subset in this space is always bounded. Hence, (X,g) is a $g-3ps$ space. Observe that for $x\in X$ and $r>0$, we have

$$B_g(x;r) = \begin{cases} X\cap[\{x\}\cup(0,r)], & \text{if } r\le 1, \\ X, & \text{if } r>1. \end{cases} \tag{21}$$

Therefore, any two balls intersect one another, implying that X is not T_2-separable.

We next introduce a new concept of convergence and compare it with the classical topological ones.

Definition 20. Let (X,g) be a $g-3ps$ space. A sequence (x_n) in X is said to be

(1) *Cauchy* if for any $\epsilon>0$, there exists $N\in\mathbb{N}$ such that

$$m,n\ge N \Longrightarrow g(x_m,x_m,x_n)<\epsilon; \tag{22}$$

(2) *g-convergent* if we can find a point $x\in X$ in which for any $\epsilon>0$, there exists $N\in\mathbb{N}$ satisfying

$$n\ge N \Longrightarrow g(x,x,x_n)<\epsilon. \tag{23}$$

In this case, we say that (x_n) g-converges to x and write $x_n \xrightarrow{g} x$.

Remark 21. Given a $g-3ps$ space (X,g), a sequence (x_n) in X g-converges to $x\in X$ if and only if for any set $U\in\mathscr{U}$ with $U\ni x$, there exists $N\in\mathbb{N}$ such that

$$n\ge N \Longrightarrow x_n\in U. \tag{24}$$

Lemma 22. *Let (X,g) be a $g-3ps$ space and let (x_n) be a sequence in X. Then, the following are equivalent:*

(i) *(x_n) converges to $x\in X$ in the topology $\mathscr{T}$;*

(ii) *for any neighborhood $V\in\mathscr{T}$ of x, on can find $N\in\mathbb{N}$ such that*

$$n\ge N \Longrightarrow x_n\in V; \tag{25}$$

(iii) *for any set $U \in \mathscr{U}^*$ containing x, we can find $N \in \mathbb{N}$ such that*

$$n \geq N \Longrightarrow x_n \in U. \quad (26)$$

Proof. (i) $\Leftrightarrow$ (ii) is by definition. So, we only need to show that (ii) $\Leftrightarrow$ (iii).

[(ii) $\Rightarrow$ (iii)] Since $\mathscr{U}^* \in \mathscr{T}$, we again apply (ii) to obtain our desired result.

[(iii) $\Rightarrow$ (ii)] By the definition of $\mathscr{T}$, for every $z \in F \in \mathscr{T}$, we can find $E \in \mathscr{U}^*$ in which $x \in E \subset F$. Suppose that $V \in \mathscr{T}$ is a neighborhood of x, then we can find some $U_0 \in \mathscr{U}^*$ such that $x \in U_0 \subset V$. Applying (ii), we obtain that $x_n \in V$ for every $n \geq N$ for some fixed $N \in \mathbb{N}$. □

We conclude the following lemma immediately from Remark 21 and Lemma 22.

Lemma 23. *Let (X, g) be a $g - 3ps$ space and let (x_n) be a sequence in X. If (x_n) converges to some point $x \in X$ in the topology $\mathscr{T}$, then it also g-converges to x.*

This lemma shows that the concept of g-convergence is weaker than convergence in topology. However, in case when (X, g) is a G-metric space or when (X, g) is a S-metric space, the two concepts coincide. Along this paper, we shall deal with this new kind of convergence, rather than those topological ones.

Definition 24. A $g - 3ps$ space (X, g) is called

(i) *g-Hausdorff* if every g-convergent sequence g-converges to at most one point;

(ii) *Σ-complete* if every sequence (x_n) satisfying $\sum_{n\in\mathbb{N}} g(x_n, x_n, x_{n+1}) < \infty$ g-converges;

(iii) *Cauchy-complete* if every Cauchy sequence g-converges.

Remark 25. A $g - 3ps$ space (X, g) is g-Hausdorff if and only if for any two distinct points $x, y \in X$, there exist two disjoint sets $U, V \in \mathscr{U}$ such that $U \ni x$ and $V \ni y$.

5. Some Fixed Point Theorems

Under this section, we propose some fixed point theorems in the framework of a $g - 3ps$ space.

We first introduce a lemma which will be used in our main theorems.

Lemma 26. *Let (X, g) be a $g - 3ps$ space. Suppose that $f : X \to X$ be an operator such that*

$$g(fx, fy, z) \leq \lambda g(\pi(x, y, z)), \quad (27)$$

for all $x, y, z \in X$, where $0 \leq \lambda < 1$ and π is a fixed permutation on X^3. Then, the following hold for every $x \in X$:

(i) $\sum_{n\in\mathbb{N}} g(f^n x, f^n x, f^{n+1} x) < \infty$;

(ii) $\sum_{n\in\mathbb{N}} g(f^{n+1} x, f^{n+1} x, f^n x) < \infty$.

Proof. Let $x \in X$ be arbitrary. We shall consider the permutation π case-by-case.

(i) *Case I*: $\pi(x, y, z) := (x, y, z)$ or $\pi(x, y, z) := (y, x, z)$.

Observe that

$$\begin{aligned} g(f^n x, f^n x, f^{n+1} x) &\leq \lambda g(f^{n-1} x, f^{n-1} x, f^n x) \\ &\leq \lambda^2 g(f^{n-2} x, f^{n-2} x, f^{n-1} x) \\ &\vdots \\ &\leq \lambda^n g(x, x, fx). \end{aligned} \quad (28)$$

(ii) *Case II*: $\pi(x, y, z) := (x, z, y)$.

Observe that

$$\begin{aligned} g(f^n x, f^n x, f^{n+1} x) &\leq \lambda g(f^{n-1} x, f^n x, f^{n-1} x) \\ &\leq \lambda^2 g(f^{n-2} x, f^{n-2} x, f^{n-1} x) \\ &\vdots \\ &\leq \lambda^n \Gamma_1, \end{aligned} \quad (29)$$

where $\infty > \Gamma_1 \geq \max\{g(x, x, fx), g(x, fx, x)\}$.

(iii) *Case III*: $\pi(x, y, z) := (y, z, x)$.

Observe that

$$\begin{aligned} g(f^n x, f^n x, f^{n+1} x) &\leq \lambda g(f^{n-1} x, f^n x, f^{n-1} x) \\ &\leq \lambda^2 g(f^{n-1} x, f^{n-2} x, f^{n-2} x) \\ &\leq \lambda^3 g(f^{n-3} x, f^{n-3} x, f^{n-2} x) \\ &\vdots \\ &\leq \lambda^n \Gamma_2, \end{aligned} \quad (30)$$

where $\infty > \Gamma_2 \geq \max\{g(x, x, fx), f(x, fx, x), f(fx, x, x)\}$.

(iv) *Case IV*: $\pi(x, y, z) := (z, x, y)$ or $\pi(x, y, z) := (z, y, x)$.

Observe that

$$\begin{aligned} g(f^n x, f^n x, f^{n+1} x) &\leq \lambda g(f^n x, f^{n-1} x, f^{n-1} x) \\ &\leq \lambda^2 g(f^{n-2} x, f^{n-2} x, f^{n-1} x) \\ &\vdots \\ &\leq \lambda^n \Gamma_1. \end{aligned} \quad (31)$$

In each case, we may conclude that

$$\sum_{n\in\mathbb{N}} g(f^n x, f^n x, f^{n+1} x) \leq \Gamma_2 \sum_{n\in\mathbb{N}} \lambda^n < \infty. \quad (32)$$

Similarly, we may prove that

$$\sum_{n\in\mathbb{N}} g(f^{n+1} x, f^{n+1} x, f^n x) \leq \Gamma_3 \sum_{n\in\mathbb{N}} \lambda^n < \infty, \quad (33)$$

where $\infty > \Gamma_3 \geq \max\{g(fx, fx, x), g(fx, x, fx), g(x, fx, fx)\}$. □

Now, we consider our fixed point results, which exploited the above lemma.

Theorem 27. *Let (X, g) be a g-Hausdorff Σ-complete $g-3ps$ space. Suppose that $f : X \to X$ is a g-sequentially continuous operator (i.e., $x_n \xrightarrow{g} x \Rightarrow fx_n \xrightarrow{g} fx$, for every sequence (x_n) in X) satisfying*

$$g(fx, fy, z) \leq \lambda g(\pi(x, y, z)), \tag{34}$$

for all $x, y, z \in X$, where $0 \leq \lambda < 1$ and π is a fixed permutation on X^3. Then, f has exactly one fixed point.

Proof. Let $x \in X$ be arbitrary. By (i) in Lemma 26, we have

$$\sum_{n\in\mathbb{N}} g\left(f^n x, f^n x, f^{n+1} x\right) < \infty. \tag{35}$$

Since X is Σ-complete, $(f^n x)$ g-converges to some point $x_* \in X$. Now, since f is g-sequentially continuous, $ff^n x = f^{n+1} x \xrightarrow{g} fx_*$. Since X is g-Hausdorff, we have $fx_* = x_*$.

Assume that $y_* \in X$ is also a fixed point of f. Observe that

$$g(x_*, x_*, y_*) = g(fx_*, fx_*, fy_*) \leq \lambda g(x_*, x_*, y_*). \tag{36}$$

The only possibility of the value of $g(x_*, x_*, y_*)$ allows us to conclude that $x_* = y_*$. So, the theorem is proved. □

Theorem 28. *Let (X, g) be a g-Hausdorff Cauchy-complete $g-3ps$ space. Suppose that $f : X \to X$ is a g-sequentially continuous operator satisfying*

$$g(fx, fy, fz) \leq \lambda g(\pi(x, y, z)), \tag{37}$$

for all $x, y, z \in X$, where $0 \leq \lambda \leq \delta/3\eta$ and $\delta, \eta > 0$ are given as in Lemma 14. Then, f has exactly one fixed point.

Proof. Let $x \in X$. From Lemma 26, we also have $g(f^n x, f^n x, f^{n+1} x) \to 0$ and $g(f^{n+1} x, f^{n+1} x, f^n x) \to 0$. So, we may choose $N \in \mathbb{N}$ such that

$$\begin{aligned} g\left(f^N x, f^N x, f^{N+1} x\right) &< \frac{\delta}{3}, \\ g\left(f^{N+1} x, f^{N+1} x, f^N x\right) &< \frac{\delta}{3}. \end{aligned} \tag{38}$$

Consequently, we obtain that

$$g\left(f^N x, f^N x, f^{N+1} x\right) < \eta. \tag{39}$$

We will show that $g(f^N x, f^N x, f^{N+n} x) < \eta$ for all $n \in \mathbb{N}$ via the mathematical induction. Now, we assume that

$$g\left(f^N x, f^N x, f^{N+n_0} x\right) < \eta, \tag{40}$$

for some $n_0 \in \mathbb{N}$. Then, observe that

$$\begin{aligned} &2g\left(f^{N+1} x, f^{N+1} x, f^N x\right) + g\left(f^{N+1} x, f^{N+1} x, f^{N+n_0+1} x\right) \\ &\quad \leq 2g\left(f^{N+1} x, f^{N+1} x, f^N x\right) + \lambda g\left(f^N x, f^N x, f^{N+n_0} x\right) \\ &\quad < 2g\left(f^{N+1} x, f^{N+1} x, f^N x\right) + \lambda \eta \\ &\quad < \delta. \end{aligned} \tag{41}$$

Hence, from Lemma 14, we have $g(f^N x, f^N x, f^{N+n_0+1} x) < \eta$. Therefore, we have $g(f^N x, f^N x, f^{N+n} x) < \eta$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$ be given and let $m, n \in \mathbb{N}$ with $N \leq m < n$. Thus, we may write

$$\begin{aligned} m &= N + \ell + p, \\ n &= N + \ell + q, \end{aligned} \tag{42}$$

for some $\ell, p, q \in \mathbb{N}$. Note that $p < q$. It follows that

$$\begin{aligned} g(f^m x, f^m x, f^n x) &= g\left(f^{N+\ell+p} x, f^{N+\ell+p} x, f^{N+\ell+q} x\right) \\ &\leq \lambda g\left(f^{N+\ell+p-1} x, f^{N+\ell+p-1} x, \right. \\ &\qquad \left. f^{N+\ell+q-1} x\right) \\ &\ \ \vdots \\ &\leq \lambda^\ell \lambda^p g\left(f^N x, f^N x, f^{N+q-p} x\right) \\ &< \lambda^\ell \lambda^p \eta. \end{aligned} \tag{43}$$

Since $\lambda \leq \delta/3\eta < 1$, if $M := N + \ell$ is chosen large enough so that $\lambda^\ell \eta < \epsilon$, then we ended up with $g(f^m x, f^m x, f^n x) < \epsilon$. Therefore, $(f^n x)$ is Cauchy. By mean of the Cauchy-completeness of X, it converges to some point $x_* \in X$. Since f is g-sequentially continuous, we have $ff^n x \xrightarrow{g} fx_*$. Moreover, since X is g-Hausdorff, $fx_* = x_*$.

Assume that $y_* \in X$ is also a fixed point of f. Observe that

$$g(x_*, x_*, y_*) = g(fx_*, fx_*, fy_*) \leq \lambda g(x_*, x_*, y_*). \tag{44}$$

This implies that $x_* = y_*$. Therefore, the uniqueness is proved. □

Example 29. Let $X = [0, 1]$ and define a function $g : X \times X \times X \to \mathbb{R}_+$ by

$$g(x, y, z) := \begin{cases} 0, & \text{if } x = y = z, \\ \dfrac{1}{8}, & \text{if } x, y, z \text{ are not pairwise equal} \\ & \text{and } x, y, z \in \{0, 1\}, \\ \dfrac{1}{2}, & \text{if } x, y, z \text{ are not pairwise equal} \\ & \text{and } \{0, 1\} \subset \{x, y, z\} \neq \{0, 1\}, \\ \dfrac{1}{8}, & \text{otherwise.} \end{cases} \tag{45}$$

It is easy to verify that (X, g) is a $g-3ps$ space which is g-Hausdorff and Σ-complete. Note also that g is neither a G-metric nor a S-metric.

Now, let us consider the map $f : X \to X$ given by

$$fx := \begin{cases} 1, & \text{if } x = 0 \text{ or } x = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (46)$$

Obviously, f is g-sequentially continuous.

Our results (Theorems 27 and 28) then guarantee the existence and uniqueness of the fixed point $f1 = 1$.

6. Conclusions

In this work, we pointed out that the results in [6] are obtainable through a metric type space. In addition, a S-metric generalizes a G-metric only in the case when G is symmetric. We then fill this gap by introducing a new space, namely, a $g-3ps$ space, which covers a S-metric space and also a G-metric space in which the symmetric is absent. We also study the underlying topology for this new space in the new direction, totally different from those studied in symmetric and semimetric spaces. We lastly give the sufficient conditions for a fixed point to exist and to be unique.

Acknowledgments

The authors were supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (NRU-CSEC no. 56000508).

References

[1] S. Gähler, “2-metrische Räume und ihre topologische Struktur,” *Mathematische Nachrichten*, vol. 26, pp. 115–148, 1963.

[2] S. Gähler, “Zur geomitric 2-metriche räume,” *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 11, pp. 664–669, 1996.

[3] B. C. Dhage, “Generalised metric spaces and mappings with fixed point,” *Bulletin of the Calcutta Mathematical Society*, vol. 84, no. 4, pp. 329–336, 1992.

[4] Z. Mustafa and B. Sims, “A new approach to generalized metric spaces,” *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[5] S. Sedghi, N. Shobe, and H. Zhou, “A common fixed point theorem in D^*-metric spaces,” *Fixed Point Theory and Applications*, vol. 2007, Article ID 27906, 13 pages, 2007.

[6] S. Sedghi, N. Shobe, and A. Aliouche, “A generalization of fixed point theorems in S-metric spaces,” *Matematički Vesnik*, vol. 64, no. 3, pp. 258–266, 2012.

[7] Z. Mustafa, H. Obiedat, and F. Awawdeh, “Some fixed point theorem for mapping on complete G-metric spaces,” *Fixed Point Theory and Applications*, vol. 2008, Article ID 189870, 12 pages, 2008.

[8] Z. Mustafa and B. Sims, “Fixed point theorems for contractive mappings in complete G-metric spaces,” *Fixed Point Theory and Applications*, vol. 2009, Article ID 917175, 10 pages, 2009.

[9] H. Aydi, M. Postolache, and W. Shatanawi, “Coupled fixed point results for (φ, ϕ)-weakly contractive mappings in ordered G-metric spaces,” *Computers & Mathematics with Applications*, vol. 63, no. 1, pp. 298–309, 2012.

[10] M. A. Alghamdi and E. Karapınar, “G-ψ-φ-contractive type mappings in G-metric spaces,” *Fixed Point Theory and Applications*, vol. 123, article 123, 17 pages, 2013.

[11] S. Chandok, W. Sintunavarat, and P. Kumam, “Some coupled common fixed points for a pair of mappings in partially ordered G-metric spaces,” *Mathematical Sciences*, vol. 2013, article 24, 7 pages, 2013.

[12] M. Abbas, W. Sintunavarat, and P. Kumam, “Coupled fixed point of generalized contractive mappings on partially ordered G-metric spaces,” *Fixed Point Theory and Applications*, vol. 2012, article 31, 14 pages, 2012.

[13] M. A. Khamsi, “Remarks on cone metric spaces and fixed point theorems of contractive mappings,” *Fixed Point Theory and Applications*, vol. 2010, Article ID 315398, 7 pages, 2010.

[14] I. D. Arandelović and D. J. Kečkić, “Symmetric spaces approach to some fixed point results,” *Nonlinear Analysis*, vol. 75, no. 13, pp. 5157–5168, 2012.

[15] K. Menger, “Untersuchungen über allgemeine,” *Mathematische Annalen*, vol. 100, no. 1, pp. 75–163, 1928.

[16] E. W. Chittenden, “On the equivalence of ecart and voisinage,” *Transactions of the American Mathematical Society*, vol. 18, no. 2, pp. 161–166, 1917.

[17] W. A. Wilson, “On semi-metric spaces,” *American Journal of Mathematics*, vol. 53, no. 2, pp. 361–373, 1931.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 989102, 13 pages
http://dx.doi.org/10.1155/2013/989102

Research Article

On Some Basic Theorems of Continuous Module Homomorphisms between Random Normed Modules

Guo Tiexin

School of Mathematics and Statistics, Central South University, Hunan Province, Changsha 410075, China

Correspondence should be addressed to Guo Tiexin; tiexinguo@csu.edu.cn

Received 24 April 2013; Accepted 22 July 2013

Academic Editor: Pei De Liu

We first prove the resonance theorem, closed graph theorem, inverse operator theorem, and open mapping theorem for module homomorphisms between random normed modules by simultaneously considering the two kinds of topologies—the (ϵ, λ)-topology and the locally L^0-convex topology for random normed modules. Then, for the future development of the theory of module homomorphisms on complete random inner product modules, we give a proof with better readability of the known orthogonal decomposition theorem and Riesz representation theorem in complete random inner product modules under two kinds of topologies. Finally, to connect module homomorphism between random normed modules with linear operators between ordinary normed spaces, we give a proof with better readability of the known result connecting random conjugate spaces with classical conjugate spaces, namely, $L^q(S^*) \cong (L^p(S))'$, where p and q are a pair of Hölder conjugate numbers with $1 \leq p < +\infty$, S a random normed module, S^* the random conjugate space of S, $L^p(S)(L^q(S^*))$ the corresponding L^p (resp., L^q) space derived from S (resp., S^*), and $(L^p(S))'$ the ordinary conjugate space of $L^p(S)$.

1. Introduction

The theory of probabilistic metric spaces initiated by K. Menger and subsequently developed by Schweizer and Sklar begins the study of randomizing the traditional space theory of functional analysis, where the randomness of "distance" or "norm" is expressed by probability distribution functions; compare [1]. The original notions of random metric spaces and random normed spaces occur in the course of the development of probabilistic metric and normed spaces, whereas the random distance between two points in a random metric space or the random norm of a vector in a random normed space is described by nonnegative random variables on a probability space; compare [1]. Probabilistic normed spaces are often endowed with the (ϵ, λ)-topology and not locally convex in general; a serious obstacle to the deep development of probabilistic normed spaces is that the taditional theory of conjugate spaces does not universally apply to probabilistic normed spaces. Although the traditional theory of conjugate spaces does not universally apply to random normed spaces either, the additional measure-theoretic structure and the stronger geometric structure peculiar to a random normed space enable us to introduce the notion of an almost surely bounded random linear functional and establish its Hahn-Banach extension theorem, which leads to the idea of the theory of random conjugate spaces for random normed spaces; compare [2–4].

The further development of the theory of random conjugate spaces motivates us to present the important notions of random normed modules, random inner product modules, and random locally convex modules; compare [3–5]. Independent of Schweizer, Sklar, and Guo's work, in [6] Haydon et al. also introduced random normed modules as a tool for the study of ultrapowers of Lebesgue-Bochner function spaces. All the work before 2009 was carried out under the (ϵ, λ)-topology.

In 2009, motivated by financial applications, in [7] Filipović et al. independently presented random normed modules and first applied them to the study of conditional risk measures. In particular, they introduced another kind of topology, namely, the locally L^0-convex topology, for random normed modules and random locally convex modules, and began the study of random convex analysis.

Relations between some basic results derived from the (ϵ, λ)-topology and the locally L^0-convex topology for a random locally convex module were further studied in [8]. Following [8], the advantage and disadvantage of the two kinds of topologies are gradually realized and the advantage of one can complement the disadvantage of the other, which also leads to a series of recent advances [9, 10] and in particular to a complete random convex analysis with applications to conditional risk measures [11, 12].

Up to now, the results obtained in random metric theory are of space-theoretical nature, whereas the study of module homomorphisms between random normed modules has not been fully carried out. With the development of random metric theory, we unavoidably need a deep theory of module homomorphisms; this paper gives some basic theorems of continuous module homomorphisms. These basic theorems are known under the (ϵ, λ)-topology, but their proofs were given before the definitive notions of random normed and inner product modules were presented in [3] so that these proofs do not have a good readability; in this paper we give better proofs and further give the versions of these basic theorems under the locally L^0-convex topology.

The remainder of this paper is organized as follows. In Section 2, we introduce some basic notions together with some simple facts subsequently used in this paper. In Section 3, we prove the resonance theorem, closed graph theorem, inverse operator theorem, and open mapping theorem for module homomorphisms between random normed modules endowed with the two kinds of topologies. In Section 4, we give a better proof of the known orthogonal decomposition theorem and Riesz representation theorem in complete random inner product modules under the two kinds of topologies for the future development of module homomorphisms between complete random inner product modules. Finally, Section 5 is devoted to a better proof of the known result connecting random conjugate spaces and ordinary conjugate spaces, namely, $(L^p(S))' \cong L^q(S^*)$.

2. Preliminaries

Throughout this paper, K denotes the scalar field, namely, the field R of real numbers or the field C of complex numbers, $(\Omega, \mathcal{F}, \mu)$ a σ-finite measure space, $L^0(\mathcal{F}, K)$ the algebra of equivalence classes of $\mathcal{F}$-measurable K-valued functions on $(\Omega, \mathcal{F}, \mu)$, $\overline{L}^0(\mathcal{F})$ the set of equivalence classes of extended real-valued $\mathcal{F}$-measurable functions on $(\Omega, \mathcal{F}, \mu)$ and $L^0(\mathcal{F}) := L^0(\mathcal{F}, R)$.

$\overline{L}^0(\mathcal{F})$ is partially ordered by $\xi \leq \eta$ if and only if $\xi^0(\omega) \leq \eta^0(\omega)$ *a.s.*, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η in $\overline{L}^0(\mathcal{F})$, respectively. It is well known from [13] that every subset H in $\overline{L}^0(\mathcal{F})$ has a supremum and infimum, denoted by $\vee H$ and $\wedge H$, respectively, and there are countable subsets $\{a_n, n \in N\}$ and $\{b_n, n \in N\}$ of H such that $\vee H = \vee_{n\geq 1} a_n$ and $\wedge H = \wedge_{n\geq 1} b_n$. Furthermore, if, in addition, H is upward directed or downward directed, then $\{a_n, n \in N\}$ and $\{b_n, n \in N\}$ can be chosen as nondecreasing and nonincreasing, respectively. In particular, $(L^0(\mathcal{F}), \leq)$ is conditionally complete, namely, every subset with an upper bound has a supremum.

Following are the notation and terminology frequently used in this paper:

$$\overline{L}^0_+(\mathcal{F}) = \{\xi \in \overline{L}^0(\mathcal{F}) \mid \xi \geq 0\},$$
$$L^0_+(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi \geq 0\},$$
$$\overline{L}^0_{++}(\mathcal{F}) = \{\xi \in L^0(\mathcal{F}) \mid \xi > 0 \text{ on } \Omega\},$$

where "$\xi > 0$ on Ω" means that $\xi^0(\omega) > 0$ *a.s.* for an arbitrarily chosen representative ξ^0 of ξ.

Definition 1 (see [3]). An ordered pair $(S, \|\cdot\|)$ is called a random normed space (briefly, an RN space) over K with base $(\Omega, \mathcal{F}, \mu)$ if S is a linear space over K and $\|\cdot\|$ is a mapping from S to $L^0_+(\mathcal{F})$ such that the following three conditions are satisfied:

(RN-1) $\|x\| = 0$ implies $x = \theta$ (the null in S),

(RN-2) $\|\alpha x\| = |\alpha|\|x\|$, for all $\alpha \in K$ and $x \in S$,

(RN-3) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in S$,

where $\|x\|$ is called the random norm of x. If $\|\cdot\|$ only satisfies (RN-2) and (RN-3), then it is called a random seminorm.

Furthermore, if, in addition, S is a left module over the algebra $L^0(\mathcal{F}, K)$ (briefly, an $L^0(\mathcal{F}, K)$-module) and the following additional condition is also satisfied:

(RNM-1) $\|\xi x\| = |\xi|\|x\|$, for all $\xi \in L^0(\mathcal{F}, K)$ and $x \in S$,

then $(S, \|\cdot\|)$ is called a random normed module (briefly, an RN module) over K with base $(\Omega, \mathcal{F}, \mu)$, at which time $\|\cdot\|$ is called an L^0 norm on S. Similarly, if $\|\cdot\|$ only satisfies (RN-3) and (RNM-1), then it is called an L^0-seminorm on S.

Remark 2. In [1], the original definition of an RN space was given by only requiring $(\Omega, \mathcal{F}, \mu)$ to be a probability space and defining $\|x\|$ to be a nonnegative random variable; the corresponding (RN-1) to (RN-3) are given in the following way:

(RN-1)′ $\|x\|(\omega) = 0$ *a.s.* implies $x = \theta$,

(RN-2)′ $\|\alpha x\|(\omega) = |\alpha|\|x\|(\omega)$ *a.s.*, for all $\alpha \in K$ and $x \in S$,

(RN-3)′ $\|x + y\|(\omega) \leq \|x\|(\omega) + \|y\|(\omega)$ *a.s.*, for all $x, y \in S$.

This definition is natural and intuitive from probability theory, but (RN-1)′ is difficult to satisfy when we construct examples. Thus we essentially have employed Definition 1 since our work [2] by saying that measurable functions or random variables that are equal *a.s.* are identified; in particular since 1999 we strictly distinguish between measurable functions and their equivalence classes in writings; compare [3].

Remark 3. At outset we consider both the real and complex cases in the study of RN spaces, whereas they only consider the real case in [6, 7] because of their special interests; an RN module over R is termed as a randomly normed L^0-module in [6] and an L^0-normed module in [7]. We still would like to continue to employ the terminology "an RN module over K with base $(\Omega, \mathcal{F}, \mu)$" in order to keep concordance with the earliest terminology used in [1].

Definition 4 (see [3, 5, 14]). Let $(S, \|\cdot\|)$ and $(S_1, \|\cdot\|_1)$ be two RN spaces over K with base $(\Omega, \mathcal{F}, \mu)$. A linear operator T from S to S_1 is said to be *a.s.* bounded if there exists $\xi \in L^0_+(\mathcal{F})$ such that $\|Tx\|_1 \le \xi\|x\|$, for all $x \in S$, in which case $\|T\|$ is defined to be $\wedge\{\xi \in L^0_+(\mathcal{F}) \mid \|Tx\|_1 \le \xi\|x\|, \forall x \in S\}$, called the random norm of T. Denote the linear space of *a.s.* bounded linear operators from S to S_1 by $B(S, S_1)$; then $(B(S, S_1), \|\cdot\|)$ still becomes an RN space over K with base $(\Omega, \mathcal{F}, \mu)$ when $\|T\|$ is defined as above for every $T \in B(S, S_1)$. In particular, when $S_1 = L^0(\mathcal{F}, K)$ and $\|\cdot\|_1 = |\cdot|$ (namely, the absolute value mapping), $S^* := B(S, S_1)$ is called the random conjugate space of S and an element in S^* is called an *a.s.* bounded random linear functional on S.

Remark 5. When $(S_1, \|\cdot\|_1)$ in Definition 4 is an RN module, $B(S, S_1)$ automatically becomes an RN module under the module operation $(\xi \cdot T)(x) = \xi \cdot (T(x))$, for all $\xi \in L^0(\mathcal{F}, K), T \in B(S, S_1)$, and $x \in S$. When S and S_1 are both RN modules, in [6] $B(S, S_1)$ is used to stand for the $L^0(\mathcal{F}, K)$-module of *a.s.* bounded module homomorphisms from S to S_1; we will show that in the special case an *a.s.* bounded linear operator must be a module homomorphism. Therefore, the two implications of $B(S, S_1)$ coincide in this case.

As in the classical functional analysis, we can similarly define a conjugate operator $T^* : S_2^* \to S_1^*$ for an *a.s.* bounded linear operator T from $(S_1, \|\cdot\|_1)$ to $(S_2, \|\cdot\|_2)$ as follows: $(T^* f)(x) = f(Tx)$, for all $f \in S_2^*, x \in S_1$. From the Hahn-Banach theorem for *a.s.* bounded random linear functional established in [2] (also see [8]), one has that $\|T^*\| = \|T\|$.

For the sake of convenience, let us recall some notation and terminology in the theory of probabilistic normed spaces.

Definition 6 (see [1]). A function $T : [0,1] \times [0,1] \to [0,1]$ is called a weak t-norm if the following are satisfied:

(t-1) $T(a, b) = T(b, a)$, for all $a, b \in [0, 1]$,

(t-2) $T(a, b) \le T(c, d)$, for all $a, b, c, d \in [0, 1]$ with $a \le c$, $b \le d$,

(t-3) $T(1, 0) = 0, T(1, 1) = 1$.

A weak t-norm $T : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if the following two additional conditions are satisfied:

(t-4) $T(a, T(b, c)) = T(T(a, b), c)$, for all $a, b, c \in [0, 1]$,

(t-5) $T(1, a) = a$, for all $a \in [0, 1]$.

Although t-norms are widely used in the theory of probabilistic metric spaces, weak t-norms have their own advantages, for example, for a family $\{T_\alpha, \alpha \in \wedge\}$ of weak t-norms, T defined by $T(a, b) = \sup\{T_\alpha(a, b) : \alpha \in \wedge\}$, for all $a, b \in [0, 1]$, is still a weak t-norm, whereas this is not true for t-norms.

Throughout this paper, $\Delta = \{F : [-\infty, +\infty] \to [0, 1] \mid F(-\infty) = 0, F(+\infty) = 1, F$ is nondecreasing and left continuous on $(-\infty, +\infty)\}$, $D = \{F \in \Delta \mid \lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to+\infty} F(x) = 1\}$, $\Delta^+ = \{F \in \Delta \mid F(0) = 0\}$, and $D^+ = \{F \in D \mid F(0) = 0\}$. For extended real random variable ξ on a probability space $(\Omega, \mathcal{F}, P)$, its (left continuous) distribution function N_ξ is defined by $N_\xi(t) = P\{\omega \in \Omega \mid \xi(\omega) < t\}$, for all $t \in [-\infty, +\infty)$ and $N_\xi(+\infty) = 1$.

In particular, ϵ_0 stands for the distribution function defined by $\epsilon_0(t) = 1$ when $t > 0$ and $\epsilon_0(t) = 0$ when $t \le 0$; namely, ϵ_0 is the distribution function of the constant 0.

Definition 7 (see [1]). A triple $(S, \mathcal{N}, T)$ is called a Menger probabilistic normed space (briefly, an M-PN space) over K if S is a linear space over K, $\mathcal{N}$ is a mapping from S to D^+, and T is a weak t-norm such that the following are satisfied:

(MPN-1) $N_x = \epsilon_0$ if and only if $x = \theta$ (the null element in S),

(MPN-2) $N_{\alpha x}(t) = N_x(t/|\alpha|)$, for all $t \ge 0$, $\alpha \in K \setminus \{0\}$ and $x \in S$,

(MPN-3) $N_{x+y}(u+v) \ge T(N_x(u), N_y(v))$, for all $u, v \ge 0$ and $x, y \in S$.

Here $N_x := \mathcal{N}(x)$ is called the probabilistic norm of x.

For an M-PN space $(S, \mathcal{N}, T)$, let $\mathcal{T} = \{\widetilde{T} \mid \widetilde{T}$ is a weak t-norm such that $(S, \mathcal{N}, \widetilde{T})$ is an M-PN space$\}$, and define $\widehat{T} : [0,1] \times [0,1] \to [0,1]$ by $\widehat{T}(a, b) = \sup\{\widetilde{T}(a, b) \mid \widetilde{T} \in \mathcal{T}\}$, for all $a, b \in [0, 1]$; then it is very easy to see that $\widehat{T} \in \mathcal{T}$. $\widehat{T}$ is called the largest weak t-norm of $(S, \mathcal{N})$ such that $(S, \mathcal{N})$ is an M-PN space under $\widehat{T}$. From now on, for an M-PN space $(S, \mathcal{N}, T)$, we always assume that T is the largest weak t-norm of $(S, \mathcal{N})$.

In [15], LaSalle introduced the notion of a pseudonormed linear space: let S be a linear space over K and $\{p_\alpha\}_{\alpha\in\wedge}$ a family of mappings from S to $R^+ := [0, +\infty)$ and indexed by a directed set $\wedge$; then $(S, \{p_\alpha\}_{\alpha\in\wedge})$ is called a pseudonormed linear space if the following are satisfied:

(PNS-1) $p_\alpha(\beta x) = |\beta| p_\alpha(x)$, for all $\beta \in K, x \in S$, and $\alpha \in \wedge$,

(PNS-2) $p_{\alpha_1}(x) \le p_{\alpha_2}(x)$, for all $\alpha_1, \alpha_2 \in \wedge$ such that $\alpha_1 \le \alpha_2$, $x \in S$,

(PNS-3) for any $\alpha \in \wedge$, there exists $\alpha' \in \wedge$ such that $p_\alpha(x + y) \le p_{\alpha'}(x) + p_{\alpha'}(y)$, for all $x, y \in S$.

Let $(S, \{p_\alpha\}_{\alpha\in\wedge})$ be a pseudonormed linear space. For any $\epsilon > 0$ and $\alpha \in \wedge$, let $U_\theta(\alpha, \epsilon) = \{x \in S \mid p_\alpha(x) < \epsilon\}$. Then $\mathcal{U}_\theta = \{U_\theta(\alpha, \epsilon) \mid \alpha \in \wedge, \epsilon > 0\}$ is a local base at the null element θ of some linear topology for S, called the linear topology induced by $\{p_\alpha\}_{\alpha\in\wedge}$. Conversely any linear topology for S can be induced by some $\{p_\alpha\}_{\alpha\in\wedge}$ such that $(S,\{p_\alpha\}_{\alpha\in\wedge})$ is a pseudonormed linear space.

To connect an M-PN space $(S,\mathcal{N},T)$ to a pseudonormed linear space, for each $r\in(0,1)$, define $p_r: S\to[0,+\infty)$ by $p_r(x)=\sup\{t\geq 0\mid N_x(t)<r\}$, for all $x\in S$. Then we have the following.

Theorem 8 (see [16]). *Let $(S,\mathcal{N},T)$ be an M-PN space. Then one has the following statements.*

(1) $\sup_{0<a<1}T(a,a)=1$ *if and only if $(S,\{p_r\}_{r\in(0,1)})$ is a pseudonormed linear space; namely, for each $r\in(0,1)$ there exists $s\in(0,1)$ such that $p_r(x+y)\leq p_s(x)+p_s(y)$, for all $x,y\in S$.*

(2) $T\geq \mathrm{Min}$, *namely, $T(a,b)\geq\min(a,b)$, for all $a,b\in[0,1]$, if and only if p_r is a seminorm on S for each $r\in(0,1)$; namely, $(S,\{p_r\}_{r\in(0,1)})$ is a B_0-type space.*

(3) $T(a,b)=1$ *for all $a,b\in[0,1]$ such that $a\cdot b>0$ if and only if there exists a norm $\|\cdot\|$ on S such that $p_r=\|\cdot\|$, for all $r\in(0,1)$.*

Theorem 8 was first studied in [17] in terms of isometric metrization and first given and strictly proved in its present form in [16].

Proposition 9 (see [1]). *Let $(S,\mathcal{N},T)$ be an M-PN space such that $\sup_{0<a<1}T(a,a)=1$. For any positive numbers ϵ and λ with $0<\lambda<1$, let $U_\theta(\epsilon,\lambda)=\{x\in S\mid N_x(\epsilon)>1-\lambda\}$; then $\mathcal{U}_\theta=\{U_\theta(\epsilon,\lambda)\mid\epsilon>0,0<\lambda<1\}$ forms a local base at θ of some metrizable linear topology for S, called the (ϵ,λ)-topology induced by $\mathcal{N}$.*

From Theorem 8, one can easily see that the (ϵ,λ)-topology for an M-PN space $(S,\mathcal{N},T)$ with $\sup_{0<a<1}T(a,a)=1$ is exactly the one induced by the family $\{p_r\}_{r\in(0,1)}$ of pseudonorms. Therefore as far as the study of linear homeomorphic invariants for a metrizable linear topological space is concerned, the theory of an M-PN space $(S,\mathcal{N},T)$ with $\sup_{0<a<1}T(a,a)=1$ and the theory of pseudonormed linear space $(S,\{p_r\}_{r\in(0,1)})$ are equivalent to each other, and hence either of them is also equivalent to the theory of a quasinormed space (see [18] for a quasinormed space) since a metrizable linear topology can be equivalently induced by a quasinorm as well as a family of pseudonorms such as $\{p_r\}_{r\in(0,1)}$. We find that the three kinds of frameworks have their own advantages and all will be used in this paper.

Definition 10 (see [1]). Let $(S,\mathcal{N},T)$ be an M-PN space with $\sup_{0<a<1}T(a,a)=1$ and A a subset of S. $D_A:[-\infty,+\infty]\to[0,1]$ is defined by $D_A(t)=\sup_{r<t}(\inf\{N_x(r): x\in A\})$, for all $t\in(-\infty,+\infty)$, $D_A(-\infty)=0$, and $D_A(+\infty)=1$, called the probabilistic diameter of A. If $D_A\in D^+$, then A is said to be probabilistically bounded.

Proposition 11 below is a straightforward verification by definition.

Proposition 11. *Let $(S,\mathcal{N},T)$ and A be the same as in Definition 10. Then A is probabilistically bounded if and only if A is bounded with respect to the (ϵ,λ)-topology (namely, A can be absorbed by every (ϵ,λ)-neighborhood of the null θ).*

Let $(\Omega,\mathcal{F},\mu)$ be a probability space and $(S,\|\cdot\|)$ an RN space over K with base $(\Omega,\mathcal{F},\mu)$. Define $\mathcal{N}: S\to D^+$ by $N_x(t)=\mu\{\omega\in\Omega\mid\|x\|(\omega)<t\}$, for all $t\geq 0$ and $x\in S$; namely, N_x is the distribution of $\|x\|$; then $(S,\mathcal{N},T)$ is an M-PN space with $T\geq W$, where $W(a,b)=\max(a+b-1,0)$, for all $a,b\in[0,1]$. $(S,\mathcal{N},T)$ is called the M-PN space determined by $(S,\|\cdot\|)$; the (ϵ,λ)-topology for $(S,\mathcal{N},T)$ is also called the (ϵ,λ)-topology for $(S,\|\cdot\|)$.

When $(\Omega,\mathcal{F},\mu)$ is a σ-finite measure space, let $\mathcal{F}_+=\{A\in\mathcal{F}\mid 0<\mu(A)<+\infty\}$; then the following definition is a slight generalization of the case when $(\Omega,\mathcal{F},\mu)$ is a probability space.

Definition 12 (see [3]). Let $(S,\|\cdot\|)$ be an RN space over K with base $(\Omega,\mathcal{F},\mu)$. For $A\in\mathcal{F}_+,\epsilon>0$ and $\lambda>0$, let $U_\theta(A,\epsilon,\lambda)=\{x\in S\mid\mu\{\omega\in A\mid\|x\|(\omega)<\epsilon\}>\mu(A)-\lambda\}$. Then $\mathcal{U}_\theta=\{U_\theta(A,\epsilon,\lambda)A\in\mathcal{F}_+(A),\epsilon>0,\lambda>0\}$ forms a local base at θ of some metrizable linear topology for S, called the (ϵ,λ)-topology for S induced by $\|\cdot\|$.

Proposition 13 below is a straightforward verification by definition.

Proposition 13. *Let $(S,\|\cdot\|)$ be an RN space over K with base $(\Omega,\mathcal{F},\mu)$ and $\{A_n\mid n\in N\}$ a countable partition of Ω to $\mathcal{F}$ such that $0<\mu(A_n)<+\infty$. Then one has the following.*

(1) *A sequence $\{x_n x\in N\}$ in S converges in the (ϵ,λ)-topology to x in S if and only if $\{\|x_n-x\|\mid n\in N\}$ converges to 0 locally in measure; namely, $\{\|x_n-x\|\mid n\in N\}$ converges to 0 in measure μ on each $A\in\mathcal{F}_+$.*

(2) *The (ϵ,λ)-topology for S is exactly the linear topology induced by the quasinorm $|||\cdot|||$ defined by $|||x|||=\sum_{n=1}^{\infty}(1/2^n)\int_{A_n}(\|x\|)/(1+\|x\|)d\mu$ for all $x\in S$.*

(3) *Let $P:\mathcal{F}\to[0,1]$ be defined by $P(A)=\sum_{n=1}^{\infty}(1/2^n)(\mu(A\cap A_n)/\mu(A_n))$; then P is a probability measure equivalent to μ and $(S,\|\cdot\|)$ has the same (ϵ,λ)-topology whether $(S,\|\cdot\|)$ is regarded as an RN space with base $(\Omega,\mathcal{F},\mu)$ or $(\Omega,\mathcal{F},P)$.*

Remark 14. When $(\Omega,\mathcal{F},\mu)$ is a σ-finite measure space, the (ϵ,λ)-topology for the special RN space $(L^0(\mathcal{F}),|\cdot|)$ is exactly the topology of convergence locally in measure. But the topology of convergence in measure is not a linear topology in general, so we choose the (ϵ,λ)-topology since not only is it a linear topology but also its convergence has almost all the nice properties of convergence in measure (see (1) of Proposition 13). (3) of Proposition 13 shows that we can always assume the base space of an RN space to be a probability space when only the linear homeomorphic invariants or those independent of the special choice of μ and P are studied. Finally, independently of B Schweizer and Sklar's work [1], the (ϵ,λ)-topology is also introduced in [6], called the L^0-topology.

Definition 15 (see [3, 5, 14]). Let $(S, \|\cdot\|)$ be an RN space over K with base $(\Omega, \mathcal{F}, \mu)$ and A a subset of S. A is said to be *a.s.* bounded if $\vee\{\|a\| : a \in A\} \in L^0_+(\mathcal{F})$.

In the sequel, the (ϵ, λ)-topology for every RN space is always denoted by $\mathcal{T}_{\epsilon,\lambda}$ and the quasinorm for every RN space is always denoted by $|||\cdot|||$ defined as in (2) of Proposition 13 when no confusion occurs.

Proposition 16 (see [3]). *Let $(S, \|\cdot\|)$ be an RN space with base $(\Omega, \mathcal{F}, \mu)$ and A a subset of S such that $\{\|a\| : a \in A\}$ is upward directed. Then A is a.s. bounded if and only if it is $\mathcal{T}_{\epsilon,\lambda}$-bounded, at which time and when $(\Omega, \mathcal{F}, \mu)$ is a probability space, $D_A = N_\xi$, where $\xi = \vee\{\|a\| : a \in A\}$ and N_ξ is the distribution function of ξ.*

Proof. We can, without loss of generality, assume that $(\Omega, \mathcal{F}, \mu)$ is a probability space. Necessity is clear. We prove the sufficiency as follows.

Since there exists a sequence $\{x_n, n \in N\}$ in A such that $\{\|x_n\|, n \in N\}$ converges *a.s.* to ξ in a nondecreasing manner. Let $(S, \mathcal{N}, T)$ be the M-PN space determined by $(S, \|\cdot\|)$; then $\{N_{x_n}, n \in N\}$ converges weakly to N_ξ; it is easy to check that $N_\xi = D_{\{x_n, n\in N\}}$ (namely, the probabilistic diameter of $\{x_n, n \in N\}$), and hence $N_\xi \geq D_A$. But $N_\xi \leq D_A$ is clear, then $N_\xi = D_A$. Since A is $\mathcal{T}_{\xi,\lambda}$-bounded, $D_A \in D^+$, which shows that $\xi \in L^0_+(\mathcal{F})$. □

Proposition 17 below gives a very general condition for $\{\|a\| : a \in A\}$ to be upward directed or downward directed.

Proposition 17. *Let $(S, \|\cdot\|)$ be an RN module with base $(\Omega, \mathcal{F}, \mu)$ and A a subset of S such that $\tilde{I}_D A + \tilde{I}_{D^C} A \subset A$ for any $D \in \mathcal{F}$, where $D^C = \Omega \setminus D$ and $\tilde{I}_D$ stands for the equivalence class of the characteristic function of D. Then $\{\|a\| : a \in A\}$ is both upward and downward directed.*

Proof. We only prove that $\{\|a\| : a \in A\}$ is upward directed; the case of being downward directed is similar. For any $a_1, a_2 \in A$, let $D = \{\omega \in \Omega \mid \|a_1\|^0(\omega) \leq \|a_2\|^0(\omega)\}$, where $\|a_1\|^0$ and $\|a_2\|^0$ are arbitrarily chosen representatives of $\|a_1\|$ and $\|a_2\|$, respectively. Then $a_3 := \tilde{I}_D a_2 + \tilde{I}_{D^C} a_1$ is such that $\|a_3\| = \tilde{I}_D \|a_2\| + \tilde{I}_{D^C}\|a_1\| = \|a_1\| \vee \|a_2\|$. Since $a_3 \in A$, the proof is complete. □

It is easy to see that $(L^0(\mathcal{F}, K), \mathcal{T}_{\epsilon,\lambda})$ is a topological algebra over K and $(S, \mathcal{T}_{\epsilon,\lambda})$ is a topological module over $(L^0(\mathcal{F}, K), \mathcal{T}_{\epsilon,\lambda})$ when $(S, \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, \mu)$. In 2009, another kind of topology for an RN module was introduced in [7].

Definition 18 (see [7]). Let $(S, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, \mu)$. A subset G of S is called a $\mathcal{T}_c$-open set if for each $x \in G$ there exists some $\epsilon \in L^0_{++}(\mathcal{F})$ such that $x + U_\theta(\epsilon) \subset G$, where $U_\theta = \{y \in S \mid \|y\| \leq \epsilon\}$. Denote by $\mathcal{T}_c$ the family of $\mathcal{T}_c$-open sets; then $\mathcal{T}_c$ forms a Hausdorff topology for S, called the locally L^0-convex topology induced by $\|\cdot\|$.

It is easy to check that the locally L^0-convex topology is much stronger than the (ϵ, λ)-topology for a given RN module; $(L^0(\mathcal{F}, K), \mathcal{T}_c)$ is, however, only a topological ring since it is unnecessarily a linear topological space (see [7]). Furthermore, for an RN module $(S, \|\cdot\|)$ over K with base $(\Omega, \mathcal{F}, \mu)$, $(S, \mathcal{T}_c)$ is a topological module over the topological ring $(L^0(\mathcal{F}, K), \mathcal{T}_c)$; compare [7]. From now on, the locally L^0-convex topology for every RN module is always denoted by $\mathcal{T}_c$ when no confusion occurs.

Definition 19. Let S be an $L^0(\mathcal{F}, K)$-module. A subset G of S is said to be L^0-convex if $\xi x + (1 - \xi) y \in G$, for all $x, y \in G$ and $\xi \in L^0_+(\mathcal{F})$ with $0 \leq \xi \leq 1$. A subset G of S is said to be L^0-balanced if $\xi x \in G$ for all $x \in G$ and $\xi \in L^0(\mathcal{F}, K)$ with $|\xi| \leq 1$. A subset G of S is said to be L^0-absorbed by a subset H of S if there exists some $\xi \in L^0_{++}(\mathcal{F})$ such that $\eta G \subset H$ for all $\eta \in L^0(\mathcal{F}, K)$ with $|\eta| \leq \xi$. Furthermore, if a subset G of S L^0-absorbs every element of S, then G is said to be L^0-absorbent.

Definition 20 (see [12]). Let $(S, \|\cdot\|)$ be an RN module and A a subset of S. A is said to be $\mathcal{T}_c$-bounded if A is L^0-absorbed by every $\mathcal{T}_c$-neighborhood of the null element.

It is also very easy to see that a subset of an RN module is $\mathcal{T}_c$-bounded if and only if it is *a.s.* bounded.

For the sake of convenience, I_A always denotes the characteristic function of $A \in \mathcal{F}$ and $\tilde{I}_A$ the equivalence class of I_A. As usual, $\{B \in \mathcal{F} \mid \mu(A \Delta B) = 0\}$ is called the equivalence class of $A \in \mathcal{F}$, denoted by $\tilde{A}$; we sometimes also use $I_{\tilde{A}}$ for $\tilde{I}_A$.

Theorem 21 below is a formal generalization of the corresponding results given in [5, 19] for a random linear functional; it was already frequently employed in [12, 14] but does not have yet a better proof; here we give a better proof. From now on, for convenience we always denote by (S, N) the M-PN space determined by a given RN space $(S, \|\cdot\|)$.

Theorem 21. *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be two RN modules over K with base $(\Omega, \mathcal{F}, \mu)$ and $T : S_1 \to S_2$ a linear operator. Then one has the following:*

(1) *T is a.s. bounded if and only if T is a continuous module homomorphism from $(S_1, \mathcal{T}_{\epsilon,\lambda})$ to $(S_2, \mathcal{T}_{\epsilon,\lambda})$;*

(2) *T is a.s. bounded if and only if T is a continuous module homomorphism from $(S_1, \mathcal{T}_c)$ to $(S_2, \mathcal{T}_c)$.*

Proof.

(1) Necessity. Since T is *a.s.* bounded, T must be continuous from $(S_1, \mathcal{T}_{\epsilon,\lambda})$ to $(S_2, \mathcal{T}_{\epsilon,\lambda})$; it remains to prove that T is also a module homomorphism; it suffices to prove that $T(\tilde{I}_A x) = \tilde{I}_A T(x)$, for all $x \in S_1$ and $A \in \mathcal{F}$, since T is linear. Since $\|T(\tilde{I}_A x)\|_2 \leq \|T\| \cdot \|\tilde{I}_A x\|_1 = \tilde{I}_A \|T\| \cdot \|x\|_1$, $\tilde{I}_{A^c} \cdot T(\tilde{I}_A x) = \theta$, for all $A \in \mathcal{F}$. Then $T(\tilde{I}_A x) = (\tilde{I}_A + \tilde{I}_{A^c}) \cdot T(\tilde{I}_A x) = \tilde{I}_A \cdot T(\tilde{I}_A x)$, for all $A \in \mathcal{F}$. On the other hand, $\tilde{I}_A T(x) = \tilde{I}_A \cdot T(\tilde{I}_A x + \tilde{I}_{A^c} x) = \tilde{I}_A \cdot T(\tilde{I}_A x)$, for all $x \in S_1$ and for all $A \in \mathcal{F}$. So, $\tilde{I}_A \cdot T(x) = T(\tilde{I}_A x)$.

Sufficiency. $S_1(1) := \{x \in S_1 \mid \|x\|_1 \le 1\}$ is *a.s.* bounded, and hence also $\mathcal{T}_{\epsilon,\lambda}$-bounded; further $T(S_1(1))$ is $\mathcal{T}_{\epsilon,\lambda}$-bounded since T is a continuous linear operator. Besides, $\tilde{I}_A \cdot T(S_1(1)) + \tilde{I}_{A^c} \cdot T(S_1(1)) \subset T(S_1(1))$ for all $A \in \mathcal{F}$ since $S_1(1)$ has this property and T is a module homomorphism. Then $T(S_1(1))$ is *a.s.* bounded; namely, $\xi := \vee\{\|T(x)\|_2 : x \in S_1(1)\} \in L^0_+(\mathcal{F})$ by Propositions 16 and 17. Since $1/(\|x\|_1 + (1/n)) \cdot x \in S_1(1)$, for all $x \in S_1$ and $n \in N$, $\|T(1/(\|x\|_1 + (1/n)) \cdot x)\|_2 \le \xi$, which implies that $\|T(x)\|_2 \le \xi \cdot \|x\|_1$, for all $x \in S_1$; that is to say, T is *a.s.* bounded, at which time it is also clear that $\|T\| = \vee\{\|T(x)\|_2 : x \in S_1(1)\}$.

(2) Necessity. From the proof of necessity of (1), if T is *a.s.* bounded then T is a module homomorphism. The fact that T is *a.s.* bounded also obviously implies that T is continuous from $(S_1, \mathcal{T}_c)$ to $(S_2, \mathcal{T}_c)$.

Sufficiency. Since $S_2(1) := \{y \in S_2 \mid \|y\| \le 1\}$ is a $\mathcal{T}_c$-neighborhood of the null element of S_2 there exists some $\epsilon \in L^0_{++}(\mathcal{F})$ such that $T(U_\theta(\epsilon)) \subset S_2(1)$, where $U_\theta(\epsilon) = \{x \in S_1 \mid \|x\|_1 \le \epsilon\}$. Thus for any $x \in S_1$, $\|T(1/(\|x\|_1 + (1/n)) \cdot x)\|_2 \le 1$, for all $x \in S_1$, and $n \in N$; namely, $\|T(x)\|_2 \le (1/\epsilon)(\|x\|_1 + (1/n))$ by the fact that T is a module homomorphism, which shows that $\|T(x)\|_2 \le (1/\epsilon)\|x\|_1$, for all $x \in S_1$; namely, T is *a.s.* bounded. □

Remark 22. (1) of Theorem 21 was independently obtained by Guo in [5] and Haydon et al. in [6] although it is stated in a different way in [6, Proposition 5.6], one careful reader can see that Proposition 5.6 of [6] exactly amounts to (1) of Theorem 21. Our proof is different from Haydon et al.'s in that we make use of something from the theory of Menger-PN spaces (see the proof of Proposition 16) and our method may also be used in the proofs of some important results of Section 3.

Remark 23. Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be two RN modules over K with base $(\Omega, \mathcal{F}, \mu)$; when T is a continuous module homomorphism from $(S_1, \mathcal{T}_{\epsilon,\lambda})$ to $(S_2, \mathcal{T}_{\epsilon,\lambda})$ or from $(S_1, \mathcal{T}_c)$ to $(S_2, \mathcal{T}_c)$, the process of proof of Theorem 21 has implied that $\|T\| = \vee\{\|T(x)\|_2 \mid x \in S_1(1)\}$; further we have $N_T = D_{T(S_1(1))}$ by Proposition 16, where N_T is the probabilistic norm of T, namely, the distribution function of $\|T\|$, and $D_{T(S_1(1))}$ is the probabilistic diameter of $T(S_1(1))$.

The proof of Proposition 24 below is similar to that of (1) of Theorem 21, so is omitted, but this proposition is very useful in the proof of the resonance theorem in Section 3 of this paper; we state it as follows.

Proposition 24 (see [14]). *Let $(S, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, \mu)$ and $f : S \to L^0_+(\mathcal{F})$ such that the following two conditions are satisfied:*

(1) *$f(\alpha x) = \alpha \cdot f(x)$, for all $x \in S$ and all nonnegative numbers α;*

(2) *$f(x + y) \le f(x) + f(y)$, for all $x, y \in S$.*

Then f is a.s. bounded; namely, there is some $\xi \in L^0_+(\mathcal{F})$ such that $f(x) \le \xi\|x\|$, for all $x \in S$, if and only if f is continuous from $(S, \mathcal{T}_{\epsilon,\lambda})$ to $(L^0(\mathcal{F}), \mathcal{T}_{\epsilon,\lambda})$ and $f(\eta x) = \eta \cdot f(x)$, for all $x \in S$ and $\eta \in L^0_+(\mathcal{F})$, at which time $\|f\| = \vee\{f(x) \mid x \in S(1)\}$, where $\|f\| = \wedge\{\xi \in L^0_+(\mathcal{F}) \mid f(x) \le \xi\|x\|, \forall x \in S\}$; furthermore if, in addition, $(\Omega, \mathcal{F}, \mu)$ is a probability space, then N_f (the distribution function of $\|f\|) = D_{f(S(1))}$.

It is well known that $B(S_1, S_2)$ is a Banach space when S_1 and S_2 are normed spaces and S_2 is complete. Similarly, $B(S_1, S_2)$ is $\mathcal{T}_{\epsilon,\lambda}$-complete when $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ are RN modules and S_2 is $\mathcal{T}_{\epsilon,\lambda}$-complete, which is independently pointed out by Guo in [5, 14] and Haydon et al. in [6]; in particular S^* is $\mathcal{T}_{\epsilon,\lambda}$-complete for every RN module S. In fact, a more general result is proved in [14], namely, the following.

Proposition 25 (see [14]). *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be two RN spaces over K with base $(\Omega, \mathcal{F}, \mu)$ such that S_2 is $\mathcal{T}_{\epsilon,\lambda}$-complete; then $B(S_1, S_2)$ is $\mathcal{T}_{\epsilon,\lambda}$-complete.*

When S_1 and S_2 are both RN modules, since $\|T\| = \vee\{\|T(x)\|_2 \mid x \in S_1(1)\}$, for any $T \in B(S_1, B_2)$, the proof of Proposition 25 is similar to that of the classical case. But when S_1 and S_2 are only RN spaces, its proof needs Lemma 26 below. To state it, let us recall the canonical embedding mapping J from an RN space $(S, \|\cdot\|)$ to $(S^{**}, \|\cdot\|^{**})$, where $S^{**} = (S^*)^*$, $J(x)$ is defined by $(J(x))(f) = f(x)$, for all $f \in S^*$ and $x \in S$. It is easy to see that J is random-norm preserving. As usual, S is said to be random reflexive if J is surjective. Generally, the $\mathcal{T}_{\epsilon,\lambda}$-closed submodule generated by $J(S)$ in S^{**} is called the $\mathcal{T}_{\epsilon,\lambda}$-closed submodule generated by S, denoted by $M(S)$; it is, obviously, a $\mathcal{T}_{\epsilon,\lambda}$-complete RN module.

Lemma 26 below is given and proved in [14]; here we give it a better proof.

Lemma 26 (see [14]). *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be two RN spaces over K with base $(\Omega, \mathcal{F}, \mu)$ such that S_2 is $\mathcal{T}_{\epsilon,\lambda}$-complete. Then $B(S_1, B_2)$ is isomorphic to a $\mathcal{T}_{\epsilon,\lambda}$-closed subspace of $B(M(S_1, M(S_2))$ in a random-norm-preserving way.*

Proof. Define $L : B(S_1 \cdot S_2) \to B(M(S_1), M(S_2))$ by $L(T) = T^{**}|_{M(S_1)}$, where T^{**} is the random conjugate operator of T^*.

First, L is well defined, namely, $L(T)(M(S_1)) \subset M(S_2)$, and isometric. Let $J_1 : S_1 \to S_1^{**}$ and $J_2 : S_2 \to S_2^{**}$ be the corresponding canonical embedding mappings; it is easy to check that $T^{**} \circ J_1 = J_2 \circ T$, which not only shows that $T^{**}(J_1(S_1)) \subset J_2(S_2)$ but also shows that $\|T\| = \|T^{**}|_{J_1(S_1)}\| \le \|L(T)\| \le \|T^{**}\|$. Since $\|T\| = \|T^{**}\|, \|L(T)\| = \|T\|$. Further by (1) of Theorem 21 one can easily see that $L(T)(M(S_1)) \subset M(S_2)$.

Second, $L(B(S_1, S_2))$ is a $\mathcal{T}_{\epsilon,\lambda}$-closed subspace of $B(M(S_1), M(S_2))$. Let $\{T_n, n \in N\}$ be a sequence in $B(S_1, S_2)$ such that $\{L(T_n), n \in N\}$ converges in the (ϵ, λ)-topology to some $\overline{T} \in B(M(S_1), M(S_2))$. Then $\{T_n, n \in N\}$ is also $\mathcal{T}_{\epsilon,\lambda}$-Cauchy in $B(S_1, S_2)$. We can, without loss of generality, assume that $\{\|T_n\|, n \in N\}$ converges *a.s.* to some ξ. Define $T : S_1 \to S_2$ by $T(x) = \mathcal{T}_{\epsilon,\lambda} - \lim_{n\to\infty} T_n(x)$, for all $x \in S_1$; then T is well defined since S_2 is $\mathcal{T}_{\epsilon,\lambda}$-complete, and T is *a.s.* bounded since $\|T(x)\|_2 \le \xi\|x\|$, for all $x \in S_1$. Finally, it is easy to check that $L(T) = \overline{T}$. □

Remark 27. Since $B(M(S_1), M(S_2))$ is always $\mathcal{T}_{\epsilon,\lambda}$-complete, so is $B(S_1, S_2)$ when S_2 is $\mathcal{T}_{\epsilon,\lambda}$-complete by Lemma 26.

3. Some Basic Principles of Continuous Module Homomorphisms between Random Normed Modules

The main purpose of this section is to generalize some classical basic principles such as the resonance theorem, open mapping theorem, closed graph theorem, and inverse operator theorem to the context of random normed modules. It turns out that the counterparts under the (ϵ, λ)-topology are consequences of the corresponding classical theorems on ordinary operators between quasinormed spaces except for the proof of the resonance theorem which is somewhat complicated. However, the counterparts under the locally L^0-convex topology are another thing since the usual reasoning fails to be valid; for example, the Baire category argument is no longer valid. Owing to the relations established in [8], we can prove them by converting their proofs to the case for the (ϵ, λ)-topology.

The following surprisingly general uniform boundedness theorem is known (see [18]). But for the sake of reader's convenience, we state it as follows.

Proposition 28 (see [18]). *Let X be a linear topological space over K of second category and $(Y, |||\cdot|||)$ a quasinormed linear space. Let $\{T_\alpha, \alpha \in \wedge\}$ be a family of continuous mappings from X to Y such that the following three properties are satisfied:*

(1) $|||T_\alpha(x+y)||| \le |||T_\alpha(x)||| + |||T_\alpha(y)|||$, *for all $x, y \in X$ and $\alpha \in \wedge$;*

(2) $|||T_\alpha(ax)||| = |||aT_\alpha(x)|||$, *for all $x \in X$, $\alpha \in \wedge$, and $a \ge 0$;*

(3) $\{T_\alpha(x), \alpha \in \wedge\}$ *is bounded with respect to the linear topology induced by $|||\cdot|||$ for each $x \in X$.*

Then $\lim_{x\to\theta} T_\alpha(x) = \theta$ uniformly in $\alpha \in \wedge$.

Theorem 29. *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_1)$ be two RN modules over K with base $(\Omega, \mathcal{F}, \mu)$ such that S_1 is $\mathcal{T}_{\epsilon,\lambda}$-complete. Let $\{T_\alpha : \alpha \in \wedge\}$. be a family of continuous module homomorphisms from $(S_1, \mathcal{T}_{\epsilon,\lambda})$ to $(S_2, \mathcal{T}_{\epsilon,\lambda})$. Then, one has the following:*

(1) $\{T_\alpha : \alpha \in \wedge\}$ *is $\mathcal{T}_{\epsilon,\lambda}$-bounded in $B(S_1, S_2)$ if and only if $\{T_\alpha(x) : \alpha \in \wedge\}$ is $\mathcal{T}_{\epsilon,\lambda}$-bounded in S_2 for each $x \in S_1$;*

(2) $\{T_\alpha : \alpha \in \wedge\}$ *is a.s. bounded in $B(S_1, S_2)$ if and only if $\{T_\alpha(x) : \alpha \in \wedge\}$ is a.s. bounded in S_2 for each $x \in S_1$.*

Proof. We can, without loss of generality, assume that $(\Omega, \mathcal{F}, \mu)$ is a probability space.

(1) Necessity. Let $\{T_\alpha : \alpha \in \wedge\}$ be $\mathcal{T}_{\epsilon,\lambda}$-bounded in $B(S_1, S_2)$; namely, $D_{\{T_\alpha:\alpha\in\wedge\}} \in D^+$. For each $x \in S_1$, since $\|T_\alpha(x)\|_2 \le \|T_\alpha\|\|x\|_1$, for all $\alpha \in \wedge$, $\mu\{\omega \in \Omega \mid \|T_\alpha(x)\|_2(\omega) < t\} \ge \mu(\{\omega \in \Omega \mid \|T_\alpha\|_2(\omega) < 2\sqrt{t}\} \cap \{\omega \in \Omega \mid \|x\|_1(\omega) < (1/2)\sqrt{t}\}) \ge \mu\{\omega \in \Omega \mid \|T_\alpha\|_2(\omega) < 2\sqrt{t}\} + \mu\{\omega \in \Omega \mid \|x\|_1(\omega) < (1/2)\sqrt{t}\} - 1$; namely, $N_{T_\alpha(x)}(t) \ge N_{T_\alpha}(2\sqrt{t}) + N_x((1/2)\sqrt{t}) - 1$, for all $\alpha \in \wedge$ and $t > 0$. Then, $D_{\{T_\alpha(x):\alpha\in\wedge\}}(t) \ge W(D_{\{T_\alpha:\alpha\in\wedge\}}(2\sqrt{t}), N_x((1/2)\sqrt{t}))$, for all $x \in S_1$ and $t > 0$, where $W(a,b) = \max(a+b-1, 0)$, for all $a, b \in [0,1]$. Since $D_{\{T_\alpha:\alpha\in\wedge\}} \in D^+$ and $N_x \in D^+$ for any $x \in S_1$, then $D_{\{T_\alpha(x):\alpha\in\wedge\}} \in D^+$; namely, $\{T_\alpha(x) : \alpha \in \wedge\}$ is $\mathcal{T}_{\epsilon,\lambda}$-bounded in S_2 for each $x \in S_1$.

Sufficiency. Let $|||\cdot||| : S_2 \to [0, +\infty)$ be defined by $|||y||| = \int_\Omega (\|y\|/(1+\|y\|))d\mu$ for any $y \in S_2$; then $(S_2, |||\cdot|||)$ is a quasinormed linear space and $|||\cdot|||$ induces the (ϵ, λ)-topology for S_2. Since $(S_1, \mathcal{T}_{\epsilon,\lambda})$ is a linear topological space of the second category and it is also clear that $\{T_\alpha : \alpha \in \wedge\}$ satisfies all the three conditions of Proposition 28, $\lim_{x\to\theta} T_\alpha(x) = 0$ uniformly in $\alpha \in \wedge$ by Proposition 28, which certainly implies that $\bigcup_{\alpha\in\wedge} T_\alpha(A)$ is $\mathcal{T}_{\epsilon,\lambda}$-bounded in S_2 for each $\mathcal{T}_{\epsilon,\lambda}$-bounded set A in S_1, in particular $\bigcup_{\alpha\in\wedge} T_\alpha(S_1(1))$ is $\mathcal{T}_{\epsilon,\lambda}$-bounded. By Remark 23, $N_{T_\alpha} = D_{T_\alpha(S_1(1))}$, for all $\alpha \in \wedge$. Define $F : [-\infty, +\infty] \to [0,1]$ by $F(t) = \inf_{\alpha\in\wedge} D_{T_\alpha(S_1(1))}(t)$, for all $t \in [-\infty, +\infty]$, and $l^- F : [-\infty, +\infty]$ by $(l^- F)(-\infty) = 0$, $(l^- F)(+\infty) = 1$, and $(l^- F)(t) = \sup\{F(t') \mid t' < t\}$, for all $t \in (-\infty, +\infty)$; denote $\bigcup_{\alpha\in\wedge} T_\alpha(S_1(1))$ by A; then one can easily check that $D_A = l^- F = D_{\{T_\alpha:\alpha\in\wedge\}}$; then $\{T_\alpha : \alpha \in \wedge\}$ is $\mathcal{T}_{\epsilon,\lambda}$-bounded since $D_A \in D^+$.

(2) Necessity of (2) Is Clear. We prove sufficiency of (2) as follows.

Denote the family of finite subsets of $\wedge$ by $\mathcal{F}(\wedge)$. For any $F \in \mathcal{F}(\wedge)$, define $f_F : S_1 \to L^0_+(\mathcal{F})$ by $f_F(x) = \vee\{\|T_\alpha x\|_2 \mid \alpha \in F\}$, for all $x \in S_1$; then f_F is continuous from $(S_1, \mathcal{T}_{\epsilon,\lambda})$ to $(L^0(\mathcal{F}, K), \mathcal{T}_{\epsilon,\lambda})$ and $f_F(\xi x) = \xi \cdot f_F(x)$, for all $\xi \in L^0_+(\mathcal{F})$ and $x \in S_1$, and hence f_F is *a.s.* bounded and $\|f_F\| = \vee\{f_F(x) \mid x \in S_1(1)\} = \vee\{\|T_\alpha\| : \alpha \in F\}$. It is obvious that $\{\|f_F\| \mid F \in \mathcal{F}(\wedge)\} = \vee\{\|T_\alpha\| \mid \alpha \in \wedge\}$, so we only need to prove that $\{\|f_F\| \mid F \in \mathcal{F}(\wedge)\}$ is *a.s.* bounded, which is equivalent to the fact that $\{\|f_F\| \mid F \in \mathcal{F}(\wedge)\}$ is $\mathcal{T}_{\epsilon,\lambda}$-bounded in $L^0_+(\mathcal{F})$ by Proposition 16. Since $\vee\{f_F(x) \mid F \in \mathcal{F}(\wedge)\} = \vee\{\|T_\alpha(x)\| \mid \alpha \in \wedge\}$ for each $x \in S_1$, $\{f_F(x) \mid F \in \mathcal{F}(\wedge)\}$ is *a.s.* bounded and hence also $\mathcal{T}_{\epsilon,\lambda}$-bounded for each $x \in S_1$. In the process of proof of sufficiency of (1), by replacing S_2 with $L^0(\mathcal{F}, K)$ and the same reasoning we have that $\{\|f_F\| \mid F \in \mathcal{F}(\wedge)\}$ is $\mathcal{T}_{\epsilon,\lambda}$- bounded since $\{f_F \mid F \in \mathcal{F}(\wedge)\}$ still satisfies all the three conditions of Proposition 28. □

Theorems 30, 31, and 32 below are essentially known since they can be regarded as a special case of the classical closed graph theorem, open mapping theorem, and inverse operator theorem between Fréchét spaces only by noticing that a $\mathcal{T}_{\epsilon,\lambda}$-complete RN space is a Fréchét space, but we would like to state them for the convenience of subsequent applications.

Theorem 30. *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be $\mathcal{T}_{\epsilon,\lambda}$-complete RN modules over K with base $(\Omega, \mathcal{F}, \mu)$ and $T : S_1 \to S_2$ a module homomorphism. Then T is continuous from $(S_1, \mathcal{T}_{\epsilon,\lambda})$ to $(S_2, \mathcal{T}_{\epsilon,\lambda})$ if and only if T is $\mathcal{T}_{\epsilon,\lambda}$-closed (namely, the graph of T is $\mathcal{T}_{\epsilon,\lambda}$-closed in $S_1 \times S_2$).*

Theorem 31. *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be $\mathcal{T}_{\epsilon,\lambda}$-complete RN modules over K with base $(\Omega, \mathcal{F}, \mu)$ and T a surjective continuous module homomorphism from $(S_1, \mathcal{T}_{\epsilon,\lambda})$ to $(S_2, \mathcal{T}_{\epsilon,\lambda})$. Then T is $\mathcal{T}_{\epsilon,\lambda}$-open; namely, $T(G)$ is $\mathcal{T}_{\epsilon,\lambda}$-open for each $\mathcal{T}_{\epsilon,\lambda}$- open subset G of S_1.*

Theorem 32. *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be $\mathcal{T}_{\epsilon,\lambda}$-complete RN modules over K with base $(\Omega, \mathcal{F}, \mu)$ and T a bijective continuous module homomorphism from $(S_1, \mathcal{T}_{\epsilon,\lambda})$ to $(S_2, \mathcal{T}_{\epsilon,\lambda})$. Then T^{-1} is also a continuous module homomorphism from $(S_2, \mathcal{T}_{\epsilon,\lambda})$ to $(S_1, \mathcal{T}_{\epsilon,\lambda})$.*

To give the versions of Theorems 29 up to 32 under the locally L^0-convex topology, let us first recall the notion of countable concatenation property of a set or an $L^0(\mathcal{F}, K)$-module. The introducing of the notion utterly results from the study of the locally L^0-convex topology, the reader will see that this notion is ubiquitous in the theory of the locally L^0-convex topology; From now on, we always suppose that all the $L^0(\mathcal{F}, K)$-modules E involved in this paper have the property that for any $x, y \in E$, if there is a countable partition $\{A_n, n \in N\}$ of Ω to $\mathcal{F}$ such that $\tilde{I}_{A_n}x = \tilde{I}_{A_n}y$ for each $n \in N$, then $x = y$. Guo already pointed out in [8] that all random locally convex modules possess this property, so the assumption is not too restrictive.

Definition 33 (see [8]). Let S be an $L^0(\mathcal{F}, K)$-module. A subset G of S is said to have the countable concatenation property if for each sequence $\{g_n, n \in N\}$ in G and each countable partition $\{A_n, n \in N\}$ of Ω to $\mathcal{F}$, there is $g \in G$ such that $\tilde{I}_{A_n}g = \tilde{I}_{A_n}g_n$, for all $n \in N$.

Two propositions below are key in this paper.

Proposition 34 (see [8]). *Let $(S, \|\cdot\|)$ be an RN module and G a subset with the countable concatenation property. Then $\overline{G}_{\epsilon,\lambda} = \overline{G}_c$, where $\overline{G}_{\epsilon,\lambda}$ and $\overline{G}_c$ stand for the closures of G under the (ϵ, λ)-topology and the locally L^0-convex topology, respectively.*

Proposition 35 (see [8]). *An RN module $(S, \|\cdot\|)$ is $\mathcal{T}_{\epsilon,\lambda}$-complete if and only if it is $\mathcal{T}_c$-complete and S has the countable concatenation property.*

Theorem 36 below has been used to establish random convex analysis; compare [12].

Theorem 36. *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be two RN modules over K with base $(\Omega, \mathcal{F}, \mu)$ such that S_1 is $\mathcal{T}_c$-complete and has the countable concatenation property. Let $\{T_\alpha : \alpha \in \wedge\}$ be a family of continuous module homomorphism from $(S_1, \mathcal{T}_c)$ to $(S_2, \mathcal{T}_c)$; then $\{T_\alpha : \alpha \in \wedge\}$ is $\mathcal{T}_c$-bounded if and only if $\{T_\alpha(x) : \alpha \in \wedge\}$ is $\mathcal{T}_c$-bounded for each $x \in S_1$.*

Proof. By Proposition 35, it follows from (2) of Theorem 29. □

Theorem 37. *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be two $\mathcal{T}_c$-complete RN modules over K with base $(\Omega, \mathcal{F}, \mu)$ such that S_1 and S_2 have the countable concatenation property. Then, a module homomorphism $T : S_1 \rightarrow S_2$ is continuous from $(S_1, \mathcal{T}_c)$ to $(S_2, \mathcal{T}_c)$ if and only if T is $\mathcal{T}_c$-closed (namely, the graph of T is $\mathcal{T}_c$-closed in $S_1 \times S_2$).*

Proof. It is clear that the graph of T has the countable concatenation property. By Theorem 21, T is continuous from $(S_1, \mathcal{T}_{\epsilon,\lambda})$ to $(S_2, \mathcal{T}_{\epsilon,\lambda})$ if and only if it is continuous from $(S_1, \mathcal{T}_c)$ to $(S_2, \mathcal{T}_c)$. So, the proof follows from Propositions 34 and 35 and Theorem 30. □

Theorem 38. *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be two $\mathcal{T}_c$-complete RN modules over K with base $(\Omega, \mathcal{F}, \mu)$ such that S_1 and S_2 have the countable concatenation property. If $T : S_1 \rightarrow S_2$ is a bijective continuous module homomorphism from $(S_1, \mathcal{T}_c)$ to $(S_2, \mathcal{T}_c)$, then T^{-1} is also continuous module homomorphism from $(S_2, \mathcal{T}_c)$ to $(S_1, \mathcal{T}_c)$.*

Proof. It follows from Proposition 35 and Theorems 21 and 32. □

To give Theorem 40 below, we need Lemma 39 below.

Lemma 39. *Let $(S, \|\cdot\|)$ be a $\mathcal{T}_c$-complete RN module over K with base $(\Omega, \mathcal{F}, \mu)$ and M a $\mathcal{T}_c$-closed submodule of S such that both S and M have the countable concatenation property. Then, $(S/M, \|\cdot\|_M)$ is still a $\mathcal{T}_c$-complete RN module and S/M has the countable concatenation property, where S/M is the quotient module of S with respect to M and $\|\cdot\|_M : S/M \rightarrow L^0_+(\mathcal{F})$ is defined by $\|x + M\|_M = \wedge\{\|y\| \mid y \in x + M\}$.*

Proof. By Proposition 35 both S and M are $\mathcal{T}_{\epsilon,\lambda}$-complete; then $(S/M, \|\cdot\|_M)$ is a $\mathcal{T}_{\epsilon,\lambda}$-complete RN module by the theory of quotient spaces for Fréchét spaces. The proof again follows from Proposition 35. □

Theorem 40. *Let $(S_1, \|\cdot\|_1)$ and $(S_2, \|\cdot\|_2)$ be two $\mathcal{T}_c$-complete RN modules over K with base $(\Omega, \mathcal{F}, \mu)$ such that S_1 and S_2 have the countable concatenation property. If T is a surjective continuous module homomorphism from $(S_1, \mathcal{T}_c)$ to $(S_2, \mathcal{T}_c)$, then T is $\mathcal{T}_c$-open; namely, $T(G)$ is $\mathcal{T}_c$-open for each $\mathcal{T}_c$-open subset G of S_1.*

Proof. Let $M = \{x \in S_1 \mid T(x) = 0\}$; then M is $\mathcal{T}_c$-closed and has the countable concatenation property. Define $\widehat{T} : (S_1/M, \|\cdot\|_M) \rightarrow S_2$ by $\widehat{T}(x + M) = T(x)$, for all $x \in S_1$, where $(S_1/M, \|\cdot\|_M)$ is the quotient space of $(S_1, \|\cdot\|_1)$ with respect to M; it is clear that $\widehat{T}$ is a bijective continuous module homomorphism from $(S_1/M, \mathcal{T}_c)$ to $(S_2, \mathcal{T}_c)$. By Theorem 38, $\widehat{T}^{-1}$ is a continuous module homomorphism from $(S_2, \mathcal{T}_c)$ to $(S_1/M, \mathcal{T}_c)$. So, $\widehat{T}(\widehat{G})$ is a $\mathcal{T}_c$-open subset in S_2 for each $\mathcal{T}_c$-open subset $\widehat{G}$ of S_1/M. Observing that $T = \widehat{T} \circ J$, where $J : S_1 \rightarrow S_1/M$ is the canonical quotient mapping, then T is $\mathcal{T}_c$-open. □

Remark 41. Since a $\mathcal{T}_c$-complete RN module is not necessarily of second category, we can not obtain Theorem 40 by using the Baire category argument which is used in the proof of the classical open mapping theorem. In fact, the proof of Theorem 40 also gives a new proof of the classical open mapping theorem.

4. The Orthogonal Decomposition Theorem and Riesz Representation Theorem in Complete Random Inner Product Modules under the Two Kinds of Topologies

The orthogonal decomposition theorem in complete random inner product modules was already pointed out in [3, 20] without a detailed proof since it can be indirectly and similarly obtained from a best approximation result of [5, 21] in a special complete random inner product module. Here, we give it a detailed proof. The Riesz representation theorem in complete random inner product modules was proved in [20], but we did not strictly distinguish, by symbols, between measurable functions and their equivalence classes, so the readability of the proof given in [20] is not very good. Here, we also give a new proof for the sake of convenience for readers; the idea is, of course, due to [20].

Definition 42 (see [3]). An ordered pair $(S, \langle\cdot,\cdot\rangle)$ is called a random inner product space (briefly, an RIP space) over K with base $(\Omega, \mathcal{F}, \mu)$ if S is a linear space over K and $\langle\cdot,\cdot\rangle$ is a mapping from $S \times S \to L^0(\mathcal{F}, K)$ such that the following are satisfied:

(RIP-1) $\langle x, x\rangle \in L^0_+(\mathcal{F})$ and $\langle x, x\rangle = 0$ implies $x = 0$ (the null element of S);

(RIP-2) $\langle \alpha\cdot x, y\rangle = \alpha\cdot\langle x, y\rangle$, for all $\alpha \in K$ and $x, y \in S$;

(RIP-3) $\langle x, y\rangle = \overline{\langle y, x\rangle}$, for all $x, y \in S$, where $\overline{\langle y, x\rangle}$ stands for the complex conjugation of $\langle y, x\rangle$;

(RIP-4) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, for all $x, y, z \in S$,

where $\langle x, y\rangle$ is called the random inner product of x and y in S.

Furthermore, if, in addition, S is an $L^0(\mathcal{F}, K)$-module and the following is satisfied:

(RIPM-1) $\langle \xi \cdot x, y\rangle = \xi \cdot \langle x, y\rangle$, for all $\xi \in L^0(\mathcal{F}, K)$ and $x, y \in S$,

then $(S, \langle\cdot,\cdot\rangle)$ is called a random inner product module (briefly, an RIP module) over K with base $(\Omega, \mathcal{F}, \mu)$, at which time $\langle x, y\rangle$ is called the L^0-inner product of x and y in S; namely, an L^0-inner product is a random inner product with the property (RIPM-1).

In an RIP space $(S, \langle\cdot,\cdot\rangle)$, x is orthogonal to y, denoted by $x \perp y$, if $\langle x, y\rangle = 0$. For a subset M of S, $M^\perp := \{y \in S \mid \langle x, y\rangle = 0, \forall x \in M\}$ is the orthogonal complement of M in S. Define $\|\cdot\| : S \to L^0_+(\mathcal{F})$ by $\|x\| = \sqrt{\langle x, x\rangle}$, for all $x \in S$; then $(S, \|\cdot\|)$ is an RN space over K with base $(\Omega, \mathcal{F}, \mu)$ by the following random Schwartz inequality (namely, Lemma 43 below); $\|\cdot\|$ is the random norm derived from $\langle\cdot,\cdot\rangle$. It is also clear that $(S, \|\cdot\|)$ is an RN module if $(S, \langle\cdot,\cdot\rangle)$ is an RIP module.

Lemma 43. *Let $(S, \langle\cdot,\cdot\rangle)$ be an RIP space over K with base $(\Omega, \mathcal{F}, \mu)$. Then $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$, for all $x, y \in S$.*

Proof. Let x and y be fixed and then choose $\langle x, y\rangle^0$, $\|x\|^0$, and $\|y\|^0$ as given representatives of $\langle x, y\rangle$, $\|x\|$, and $\|y\|$, respectively. Since $\langle \alpha x + y, \alpha x + y\rangle = |\alpha|^2\|x\|^2 + 2\operatorname{Re}(\alpha \cdot \langle x, y\rangle) + \|y\|^2 \ge 0$, for all $\alpha \in K$. Let $K(1) = \{\beta \in K \mid |\beta| = 1\}$, then taking $\alpha = t\beta$ with $t \in R$ and $\beta \in K(1)$ yields that $t^2 \cdot \|x\|^2 + 2t \cdot \operatorname{Re}(\beta \cdot \langle x, y\rangle) + \|y\|^2 \ge 0$, for all $t \in R$ and $\beta \in K(1)$; namely, $t^2 \cdot (\|x\|^0(\omega))^2 + 2t \cdot \operatorname{Re}(\beta \cdot \langle x, y\rangle^0(\omega)) + (\|y\|^0(\omega))^2 \ge 0$ *a.s.*, for all $t \in R$ and $\beta \in K(1)$. Since R and $K(1)$ are separable, we can obtain an $\mathcal{F}$-measurable Ω_0 with $\mu(\Omega \setminus \Omega_0) = 0$ such that $t^2 \cdot (\|x\|^0(\omega))^2 + 2t \cdot \operatorname{Re}(\beta \cdot \langle x, y\rangle^0(\omega)) + (\|y\|^0(\omega))^2 \ge 0$ on Ω_0, for all $t \in R$ and $\beta \in K(1)$.

For each $\omega \in \Omega_0$, we can always take $\beta \in K(1)$ such that $\beta \cdot \langle x, y\rangle^0(\omega) = |\langle x, y\rangle^0(\omega)|$; then we have that $t^2 \cdot (\|x\|^0(\omega))^2 + 2t \cdot |\langle x, y\rangle^0(\omega)| + (\|y\|^0(\omega))^2 \ge 0$ on Ω_0, for all $t \in R$, so $|\langle x, y\rangle^0(\omega)| \le \|x\|^0(\omega) \cdot \|y\|^0(\omega)$, for all $\omega \in \Omega_0$; namely, $|\langle x, y\rangle| \le \|x\| \cdot \|y\|$. □

Remark 44. In the proof of Lemma 43, we use a technique, namely, making use of separability of the scalar field K, which was first used in the proof of extension theorem for complex random linear functionals; compare [2, 8].

Lemma 45. *Let $(S, \langle\cdot,\cdot\rangle)$ be an RIP space over K with base $(\Omega, \mathcal{F}, \mu)$, M a subspace of M, and $x_0 \in M$. Then $\|x - x_0\| = \wedge\{\|x - y\| : y \in M\}$ if and only if $x - x_0 \perp M$.*

Proof. Sufficiency is obvious. As for the necessity, since $\|x - x_0\|^2 \le \|x - x_0 - \alpha y\|^2$, for all $\alpha \in K$ and $y \in M$, namely, $2\operatorname{Re}(\alpha\langle y, x - x_0\rangle) \le |a|^2\|y\|^2$, taking $\alpha = (1/n)a$ yields that $2\operatorname{Re}(a\langle y, x - x_0\rangle) \le (1/n)|a|^2\|y\|^2$, which implies that $\operatorname{Re}(a\langle y, x - x_0\rangle) \le 0$, for all $a \in K$ and $y \in M$. Similar to the proof of Lemma 43, one can have that $\langle y, x - x_0\rangle = 0$, for all $y \in M$. □

Remark 46. x_0 in Lemma 45 is called a best approximation point of x in M; such a kind of idea was earlier used in [5, 21–23] for the study of best approximation problems in Lebesgue-Bochner function spaces.

Theorem 47. *Let $(S, \langle\cdot,\cdot\rangle)$ be a $\mathcal{T}_{\epsilon,\lambda}$-complete RIP module over K with base $(\Omega, \mathcal{F}, \mu)$ and M a $\mathcal{T}_{\epsilon,\lambda}$-closed subspace of S. Then $S = M \oplus M^\perp$ if and only if M is a submodule.*

Proof.

Sufficiency. For each $x \in S$, let $d(x, M) = \wedge\{\|x - y\| \mid y \in M\}$. By Proposition 17 there exists a sequence $\{y_n, n \in N\}$ in M such that $\{\|x - y_n\|, n \in N\}$ converges *a.s.* to $d(x, M)$ in a nonincreasing manner. Similar to the classical case, one can deduce that $\{y_n, x \in N\}$ is a $\mathcal{T}_{\epsilon,\lambda}$-Cauchy sequence and hence convergent to some $x_0 \in M$ such that $\|x - x_0\| = d(x, M)$. By Lemma 45, $x - x_0 \perp M$. Hence, each $x \in S$ can be written as $x - x_0 + x_0 \in M^\perp \oplus M$.

Necessity. We only need to prove that $\widetilde{I}_A x \in M$ for each $A \in \mathcal{F}$ and $x \in M$. Let $\widetilde{I}_A x = x_1 + x_2$ with $x_1 \in M$ and $x_2 \in M^\perp$; since $x_2 \perp \widetilde{I}_A x$ implies $x_2 = \theta$, $\widetilde{I}_A x = x_1 \in M$. □

Theorem 48. *Let $(S, \langle\cdot,\cdot\rangle)$ be a $\mathcal{T}_{\epsilon,\lambda}$-complete RIP module over K with base $(\Omega, \mathcal{F}, \mu)$. Then for each $f \in S^*$ there exists a unique $y_f \in S$ such that $f(x) = \langle x, y_f\rangle$, for all $x \in S$, and $\|f\| = \|y_f\|$.*

Before the proof of Theorem 48, let us first introduce some notation and terminology as follows. Let ξ be in $L^0(\mathcal{F}, K)$ with a chosen representative ξ^0. $(\xi^0)^{-1} : \Omega \to K$ is defined by $(\xi^0)^{-1}(\omega) = 1/\xi^0(\omega)$ if $\xi^0(\omega) \neq 0$ and 0 otherwise. Then the equivalence class determined by $(\xi^0)^{-1}$ is called the generalized inverse of ξ, denoted by ξ^{-1}, and $|\xi|^{-1}\xi$ is called the sign of ξ, denoted by $\operatorname{sgn}(\xi)$. It is obvious that $\xi \cdot \xi^{-1} = \tilde{I}_A$, where $A = \{\omega \in \Omega \mid \xi^0(\omega) \neq 0\}$, and $\overline{\operatorname{sgn}(\xi)} \cdot \xi = |\xi|$. Besides, for any ξ and η in $L^0(\mathcal{F})$, $[\xi \le \eta]$ denotes the equivalence class of the $\mathcal{F}$-measurable set $\{\omega \in \Omega \mid \xi^0(\omega) \le \eta^0(\omega)\}$, where ξ^0 and η^0 are any chosen representatives of ξ and η, respectively. Similarly, one can understand the meaning of $[\xi > \eta]$.

We can now prove Theorem 48.

Proof of Theorem 48. Let $\operatorname{Ran}(f) = \{f(x) \mid x \in S\}$ and $N(f) = \{x \in S \mid f(x) = 0\}$; then $\operatorname{Ran}(f)$ is a submodule of $L^0(\mathcal{F}, K)$ and $N(f)$ a $\mathcal{T}_{\epsilon,\lambda}$-closed submodule of S.

By Proposition 17 $\{|f(x)| \mid x \in S\}$ is upward directed and hence there exists a sequence $\{x_n, n \in N\}$ in S such that $\{|f(x_n)|, n \in N\}$ converges *a.s.* to $\xi := \vee\{|f(x)| \mid x \in S\}$ in a nondecreasing manner.

Denote $\xi_n = |f(x_n)|$, for all $n \in N$; let ξ^0 and ξ_n^0 be any chosen representatives of ξ and ξ_n, respectively, $B = \{\omega \in \Omega \mid \xi^0(\omega) > 0\}$, and $B_n = \{\omega \in \Omega \mid \xi_n^0(\omega) > 0\}$, for all $n \ge 1$. We can, without loss of generality, assume that $B_n \subset B_{n+1}$, for all $n \ge 1$, and $\bigcup_{n=1}^{\infty} B_n = B$. Further, let $B_0 = \varnothing$ and $A_n = B_n \setminus B_{n-1}$, for all $n \ge 1$; then $A_i \cap A_j = \varnothing$, where $i \neq j$, and $B = \bigcup_{n=1}^{\infty} A_n$. Since $\xi_n^0 > 0$ on A_n, we have $f(z_n) = \tilde{I}_{A_n}$, where $z_n = \tilde{I}_{A_n}(f(x_n))^{-1} \cdot x_n$, for all $n \ge 1$.

If $\mu(B) = 0$, then taking $y_f = \theta$ ends the proof. If $\mu(B) > 0$, then $f \neq 0$, in which case we can assume that $\mu(A_n) > 0$, for all $n \in N$. By Theorem 47 there exists a unique $\overline{z}_n \in N(f)^{\perp}$ such that $z_n - \overline{z}_n \in N(f)$, and hence $f(\overline{z}_n) = f(z_n) = \tilde{I}_{A_n}$, for all $n \in N$. Since $\tilde{I}_{A_n}x - \tilde{I}_{A_n}f(x)\overline{z}_n \in N(f)$, $\langle \tilde{I}_{A_n}x - \tilde{I}_{A_n}f(x)\overline{z}_n, \overline{z}_n\rangle = 0$, for all $n \in N$ and $x \in S$.

Since $\tilde{I}_{A_n} = f(\overline{z}_n) = |f(\overline{z}_n)| \le \|f\|\|\overline{z}_n\|$, $\|\overline{z}_n\| > 0$ on A_n and $\tilde{I}_{A_n}f(x) = \langle x, z_n^*\rangle$, for all $x \in S$, where $z_n^* = \tilde{I}_{A_n}(\|\overline{z}_n\|)^{-1}\overline{z}_n$, for all $n \in N$. Let $y_n = \sum_{i=1}^{n} z_i^*$, for all $n \in N$. By noticing that $I_B f(x) = f(x)$, for all $x \in S$, and $\tilde{I}_B = \lim_{n\to\infty}\tilde{I}_{B_n} = \lim_{n\to\infty}\sum_{i=1}^{n}\tilde{I}_{A_i}$, $f(x) = \lim_{n\to\infty}(\tilde{I}_{B_n}f(x)) = \lim_{n\to\infty}(\sum_{i=1}^{n}\tilde{I}_{A_i}f(x)) = \lim_{n\to\infty}\langle x, y_n\rangle$ (where convergence means the *a.s.* convergence). We can, without loss of generality, assume that $(\Omega, \mathcal{F}, \mu)$ is a probability space; then $\mu\{\omega \in \Omega \mid \|y_{n+m} - y_n\|(\omega) > \epsilon\} \le \sum_{i=n+1}^{n+m}\mu(A_i)$, for any $\epsilon > 0$ and $n, m \in N$, which implies that $\{y_n, n \in N\}$ is $\mathcal{T}_{\epsilon,\lambda}$-Cauchy and hence convergent to some $y_f \in S$, so $f(x) = \langle x, y_f\rangle$, for all $x \in S$.

$\|f\| \le \|y_f\|$ is obvious. Now, let $A = [\|y_f\| > 0]$ and $D = [|f(y_f)| \le \|f\|\|y_f\|]$; then $\mu(D) = 1$ and $I_A\|f\| \ge I_A\|y_f\|$, where $I_A = \tilde{I}_{A^0}$ with A^0 being any representative of A. On the other hand, $(1 - I_A)\|f\| \ge 0 = (1 - I_A)\|y_f\|$, so $\|f\| = I_A\|f\| + (1 - I_A)\|f\| \ge I_A\|y_f\| + (1 - I_A)\|y_f\| = \|y_f\|$. Finally, the uniqueness of y_f is obvious. □

The version of Theorem 48 under the locally L^0-convex topology, namely, Corollary 50 below, was given in [8]. Corollary 49 below is the version of Theorem 47 under the locally L^0-convex topology.

Corollary 49. *Let $(S, \|\cdot\|)$ be a $\mathcal{T}_c$-complete RIP module over K with base $(\Omega, \mathcal{F}, \mu)$ and M a $\mathcal{T}_c$-closed submodule of S such that both S and M have the countable concatenation property. Then $S = M \oplus M^{\perp}$.*

Proof. It follows from Propositions 34 and 35 and Theorem 47. □

Corollary 50. *Let $(S, \langle\cdot,\cdot\rangle)$ be a $\mathcal{T}_c$-complete RIP module over K with base $(\Omega, \mathcal{F}, \mu)$ such that S has the countable concatenation property. Then for each $f \in S^*$ there exists a unique $y_f \in S$ such that $f(x) = \langle x, y_f\rangle$, for all $x \in S$, and $\|f\| = \|y_f\|$.*

Proof. It follows from Proposition 35 and Theorem 48. □

Remark 51. Based on Theorem 48, we can establish the spectral representation theorem for random self-adjoint operators on complete complex random inner product modules, which has been used to establish the Stone's representation theorem for a group of random unitary operators in [24].

5. $(L^p(S))' \cong L^q(S^*)$

In this section, let $(S, \|\cdot\|)$ be a given RN module over K with base $(\Omega, \mathcal{F}, \mu)$ and S^* its random conjugate space. Further, let $1 \le p < +\infty$ and $1 < q \le +\infty$ be a pair of Hölder conjugate numbers.

Let r be an extended nonnegative number with $1 \le r \le +\infty$ and $(L^r(\mathcal{F}), |\cdot|_r)$ the Banach space of equivalence classes of r-integrable (when $r < +\infty$) or essentially bounded (when $r = +\infty$) real $\mathcal{F}$-measurable functions on $(\Omega, \mathcal{F}, \mu)$ with the usual L^r-norm $|\cdot|_r$. Further, let $L^r(S) = \{x \in S \mid \|x\| \in L^r(\mathcal{F})\}$ and let $\|\cdot\|_r : L^r(S) \to [0, +\infty)$ be defined by $\|x\|_r = |\|x\||_r$, for all $x \in L^r(S)$; then $(L^r(S), \|\cdot\|_r)$ is a normed space over K and $\mathcal{T}_{\epsilon,\lambda}$-dense in S; compare [22, 25]. Similarly, one can understand the implication of $L^r(S^*)$.

Theorem 52 below is proved in [25], a more general result is proved in [6] with $L^r(\mathcal{F})$ replaced by a Köthe function space, but the two proofs both only give the key idea of them. Here, we give a detailed proof of Theorem 52. Since our aim is to look for the tool for the development of the theory of RN modules together with their random conjugate spaces, Theorem 52 is enough for the aim.

Theorem 52. *$L^q(S^*) \cong (L^p(S))'$ under the canonical mapping T, where $(L^p(S))'$ denotes the classical conjugate space of $L^p(S)$*

and for each $f \in L^q(S^*)$, T_f *(denoting* $T(f)$*):* $L^p(S) \to K$ *is defined by* $T_f(x) = \int_\Omega f(x)d\mu$, *for all* $x \in L^p(S)$.

We will divide the proof of Theorem 52 into the following two Lemmas—Lemmas 53 and 54.

Lemma 53. *T is isometric.*

Proof. $|T_f(x)| = |\int_\Omega f(x)d\mu| \le \int_\Omega |f(x)|d\mu \le \int_\Omega \|f\|\,\|x\|\,d\mu \le \|f\|_q\|x\|_p$, for all $x \in L^p(S)$, so $\|T_f\| \le \|f\|_q$.

As for $\|f\|_q \le \|T_f\|$, we can, without loss of generality, assume that $(\Omega, \mathscr{F}, \mu)$ is a probability space. Let $\{x_n, n \in N\}$ be a sequence in $S(1) := \{x \in S \mid \|x\| \le 1\}$ such that $\{|f(x_n)|, n \in N\}$ converges *a.s.* to $\|f\|$ in a nondecreasing manner (such a sequence $\{x_n, n \in N\}$ does exist!).

When $p = 1$ and $q = +\infty$, for any positive number ϵ let $A(\epsilon) = [\|f\| > \|f\|_\infty - \epsilon]$; then $\mu(A(\epsilon)) > 0$, and hence there exists some $n_0 \in N$ such that $B(\epsilon) := [|f(x_{n_0})| > \|f\|_\infty - \epsilon]$ has a positive measure. Let $x_\epsilon = 1/\mu(B(\epsilon)) \cdot I_{B(\epsilon)} \cdot \overline{\operatorname{sgn}(f(x_{n_0}))} \cdot x_{n_0}$; then $\|x_\epsilon\|_1 \le 1$ and $|T_f(x_\epsilon)| > \|f\|_\infty - \epsilon$, which shows that $\|T_f\| \ge \|f\|_\infty$.

When $p > 1$,

$$\begin{aligned}\int_\Omega |f(x_n)|^q d\mu &= \int_\Omega |f(x_n)|^{q-1} |f(x_n)|\, d\mu \\ &= \int_\Omega |f(x_n)|^{q-1} \cdot \overline{\operatorname{sgn}(f(x_n))} \cdot f(x_n)\, d\mu \\ &= \int_\Omega f\left(|f(x_n)|^{q-1} \cdot \overline{\operatorname{sgn}(f(x_n))} \cdot x_n\right) d\mu \\ &= T_f\left(|f(x_n)|^{q-1} \cdot \overline{\operatorname{sgn}(f(x_n))} \cdot x_n\right) \\ &\le \|T_f\| \left(\int_\Omega |f(x_n)|^q d\mu\right)^{1/p},\end{aligned} \tag{1}$$

then $\|f(x_n)\|_q \le \|T_f\|$, for all $n \in N$. By the Levy theorem we have that $\|f\|_q \le \|T_f\|$. □

Lemma 54. *T is surjective.*

Proof. For any fixed $l \in (L^p(S))'$ and $x \in L^\infty(S)$, define the scalar measure $G_x : \mathscr{F} \to K$ and the vector measure $G : \mathscr{F} \to (L^\infty(S))'$ as follows:

$$G_x(E) = l(\widetilde{I}_E \cdot x), \text{ for all } E \in \mathscr{F},$$
$$G(E)(y) = l(\widetilde{I}_E \cdot y), \text{ for all } y \in L^\infty(S) \text{ and } E \in \mathscr{F}.$$

□

Since $|G(E)(y)| = |l(\widetilde{I}_E \cdot y)| \le \|l\| \cdot \|\widetilde{I}_E \cdot y\|_p \le \|l\| \cdot \|y\|_\infty \cdot \|\widetilde{I}_E\|_p$, for all $E \in \mathscr{F}$, $y \in L^\infty(S)$, both G and G_x are countably additive. Now, for any finite partition $\{E_1, E_2, \ldots, E_n\}$ of Ω to $\mathscr{F}$ and finitely many points $x_1, x_2, \ldots, x_n$, in the closed unit ball of $L^\infty(S)$, we have $|\sum_{i=1}^n G(E_i)(x_i)| = |l(\sum_{i=1}^n \widetilde{I}_{E_i} \cdot x_i)| \le \|l\| \cdot |\sum_{i=1}^n \widetilde{I}_{E_i}|_p = \|l\|$. Similarly, we have that $\sum_{i=1}^n |G_x(E_i)| = \sum_{i=1}^n \overline{\operatorname{sgn}(G_x(E_i))} \cdot G_x(E_i) = \sum_{i=1}^n G(E_i)(\overline{\operatorname{sgn}(G_x(E_i))} \cdot x) \le \|l\| \cdot \|x\|_\infty$. So $|G|(\Omega) \le \|l\|$ and $|G_x|(\Omega) \le \|l\| \cdot \|x\|_\infty$; namely, G and G_x are both of bounded variation and they are both absolutely continuous with respect to μ.

By the classical Radon-Nikodým theorem there exists a unique $g(x) \in L^1(\mathscr{F}, K)$ for each $x \in L^\infty(S)$ such that $G_x(E) = \int_E g(x)d\mu$, for all $E \in \mathscr{F}$, and $|G_x|(E) = \int_E |g(x)|d\mu$, for all $E \in \mathscr{F}$, so we can obtain a mapping $g : L^\infty(S) \to L^1(\mu, K)$ such that

(1) $g(\alpha x + \beta y) = \alpha g(x) + \beta g(y)$, for all $\alpha, \beta \in K$, and $x, y \in L^\infty(S)$;

(2) $g(\xi x) = \xi \cdot g(x)$ for each simple element ξ in $L^0(\mathscr{F}, K)$, $x \in L^\infty(S)$.

We can now assert that $g(\xi x) = \xi g(x)$, for all $\xi \in L^\infty(\mathscr{F}, K)$ and $x \in L^\infty(S)$. In fact, for any $\xi \in L^\infty(\mathscr{F}, K)$ there are always a sequence $\{\xi_n, n \in N\}$ of simple elements in $L^0(\mathscr{F}, K)$ such that $\{\|\xi_n - \xi\|_\infty, n \in N\}$ converges to 0; then $\|g(\xi x) - g(\xi_n x)\|_1 = |G_{(\xi - \xi_n)x}|(\Omega) \le \|l\| \cdot \|\xi - \xi_n\|_\infty \cdot \|x\|_\infty \to 0$ $(n \to \infty)$. On the other hand, $\|\xi g(x) - \xi_n g(x)\|_1 \le \|\xi - \xi_n\|_\infty \|g(x)\|_1 \to 0$ $(n \to \infty)$, so $g(\xi x) = L^1 - \lim_{n\to\infty} g(\xi_n x) = L^1 - \lim_{n\to\infty}(\xi_n g(x)) = \xi g(x)$.

We prove that $\{|g(x)|\,|x \in S(1)|\}$ is upward directed as follows: for any x and $y \in S(1)$, let $E = [|g(x)| \le |g(y)|]$; then $|g(x)| \vee |g(y)| = I_E|g(y)| + (1 - I_E)|g(x)| = g(\overline{\operatorname{sgn}(g(y))} \cdot I_E \cdot y + \overline{\operatorname{sgn}(g(x))} \cdot (1 - I_E) \cdot x) = g(z) = |g(z)|$, where $z = \overline{\operatorname{sgn}(g(y))} \cdot I_E \cdot y + \overline{\operatorname{sgn}(g(x))} \cdot (1 - I_E) \cdot x \in S(1)$. Hence, there exists a sequence $\{x_n, n \in N\}$ in $S(1)$ such that $\{|g(x_n)|, n \in N\}$ converges *a.s.* to $\xi := \vee\{|g(x)| \mid x \in S(1)\}$ in a nondecreasing manner. By the Levy theorem $\|\xi\|_1 = \lim_{n\to\infty}\|g(x_n)\|_1 = \lim_{n\to\infty}|G_{x_n}|(\Omega) \le |G|(\Omega) \le \|l\| < +\infty$, so $\xi \in L^1(\mathscr{F}, K)$. Since for any positive number ϵ and $x \in L^\infty(S)$, it is clear that $|g((1/(\|x\| + \epsilon))x)| \le \xi$; namely, $|g(x)| \le \xi(\|x\| + \epsilon)$, which implies that $|g(x)| \le \xi\|x\|$, for all $x \in L^\infty(S)$. By the Hahn-Banach theorem for *a.s.* bounded random linear functionals (see [2, 8]) there is an *a.s.* bounded random linear functional $f \in S^*$ such that $f|_{L^\infty(S)} = g$; further f is unique since $L^\infty(S)$ is $\mathscr{T}_{\epsilon,\lambda}$-dense in S and $\|f\| \le \xi$; we also have that $\|f\| \in L^1(\mathscr{F}, K)$.

By the definition of G_x, $l(x) = G_x(\Omega) = \int_\Omega g(x)d\mu = \int_\Omega f(x)$, for all $x \in L^\infty(S)$. We prove that $f \in L^q(S^*)$ as follows: let $E_n = [\|f\| \le n]$ and $f_n = I_{E_n} \cdot f$; then $f_n \in L^q(S^*)$ and $\|f_n\| = I_{E_n}\|f\|$ (here, we can assume that μ is a probability measure). Since $\int_{E_n} f(x)d\mu = l(I_{E_n} \cdot x)$, for all $x \in L^\infty(S)$ and $n \in N$, then $T_{f_n}|_{L^\infty(S)} = l|_{I_E \cdot L^\infty(S)}$. From the fact that $L^\infty(S)$ is dense in $(L^p(S), \|\cdot\|_p)$, $T_{f_n}|_{L^p(S)} = l|_{I_{E_n} \cdot L^p(S)}$, so $\|T_{f_n}\| \le \|l\|$; letting $n \to +\infty$ yields that $\|\|f\|\|_q \le \lim_{n\to\infty}\|f_n\|_q = \lim_{n\to\infty}\|T_{f_n}\| \le \|l\| < +\infty$; namely, $f \in L^q(S^*)$. Further, we also have that $l = T_f$ since they coincide on the dense subspace $L^\infty(S)$ of $L^p(S)$.

Remark 55. Let $(B, \|\cdot\|)$ be a normed space over K and $L^0(\mathscr{F}, B)$ the $L^0(\mathscr{F}, K)$-module of equivalence classes of B-

valued $\mathscr{F}$-strongly measurable functions on $(\Omega, \mathscr{F}, \mu)$. Let B' be the classical conjugate space of B and $L^0(\mathscr{F}, B', w^*)$ the $L^0(\mathscr{F}, K)$-module of w^*-equivalence classes of B'-valued w^*-measurable functions on $(\Omega, \mathscr{F}, \mu)$. For any $x \in L^0(\mathscr{F}, B)$ with a representative x^0, the L^0-norm of x is defined to be the equivalence class of $\|x^0\|$, still denoted by $\|x\|$; then $(L^0(\mathscr{F}, B), \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathscr{F}, \mu)$. For any $y \in L^0(\mathscr{F}, B', w^*)$ with a representative y^0, the L^0-norm of y is defined to be the equivalence class of $\operatorname{esssup}\{|y^0(b)| \mid b \in B \text{ and } \|b\| \leq 1\}$ (namely, the essential supremum of $\{|y^0(b)| \mid b \in B \text{ and } \|b\| \leq 1\}$); then $L^0(\mathscr{F}, B', w^*)$ is also an RN module over K with base $(\Omega, \mathscr{F}, \mu)$. In [26] it is proved that $(L^0(\mathscr{F}, B))^* = L^0(\mathscr{F}, B', w^*)$, so if we take $S = L^0(\mathscr{F}, B)$ in Theorem 52 then $L^p(\mathscr{F}, B)' \cong L^q(\mathscr{F}, B', w^*)$. Generally speaking, σ-finite measure spaces are enough for various kinds of problems in analysis, but some more general measure spaces are sometimes necessary; for example, strictly localizable measure spaces are considered in [6]. Even in [27] we introduced the notion of an RN module with base being an arbitrary measure space $(\Omega, \mathscr{F}, \mu)$ by defining it to be a projective limit of a family of RN modules with base $(A, A \cap \mathscr{F}, \mu|_{A\cap\mathscr{F}})$, where $A \in \mathscr{F}$ satisfies $0 < \mu(A) < +\infty$, and further proved that Theorem 52 remains true for any measure space, so Theorem 52 unifies all the representation theorems of the dual spaces of Lebesgue-Bochner function spaces. As said in [6, 25], it is more interesting that Theorem 52 establishes the key connection between random conjugate spaces and classical conjugate spaces, which has played a crucial role in the subsequent development of random conjugate spaces; compare [22, 28, 29].

Remark 56. Since the Lebesgue-Bochner function space $L^p(\mathscr{F}, B)$ (or is written as $L^p(\mu, B)$) has the target space B, the simple functions in $L^p(\mathscr{F}, B)$ always play an active role in the study of the dual of $L^p(\mathscr{F}, B)$, whereas we do not have the counterparts of simple elements in $L^p(\mathscr{F}, B)$ for abstract spaces $L^p(S)$, so we are forced to replace simple elements in $L^p(\mathscr{F}, B)$ with elements in $L^\infty(S)$ in order to complete the proof of Theorem 52. In [26] we prove that a Banach space B is reflexive if and only if $L^0(\mathscr{F}, B)$ is random reflexive; the original motivation of Theorem 52 is to establish the following characterization.

Theorem 57. *Let $(S, \|\cdot\|)$ be a $\mathscr{T}_{\epsilon,\lambda}$-complete RN module over K with base $(\Omega, \mathscr{F}, \mu)$ and p any given positive number such that $1 < p < +\infty$. Then S is random reflexive if and only if $L^p(S)$ is reflexive.*

Proof. Let $J : S \to S^{**}$ and $j : L^p(S) \to (L^p(S))''$ be the corresponding canonical embedding mappings.

(1) Necessity. Since $1 < p < +\infty$, its Hölder conjugate number q satisfies $1 < q < +\infty$; then $(L^p(S))'' = (L^q(S^*))' = L^p(S^{**}) = L^p(S)$.

(2) Sufficiency. Let f^{**} be any given element in S^{**} and $E_n = [n-1 \leq \|f^{**}\| < n]$ for any $n \in N$. We can, without loss of generality, assume that μ is a probability measure; then $\sum_{n=1}^{\infty} \mu(E_n) = 1$. Since $I_{E_n} f^{**} \in L^p(S^{**}) = (L^p(S))''$, there exists $x_n \in L^p(S)$ such that $j(x_n) = I_{E_n} f^{**}$; namely, for each $f^* \in L^q(S^*)$, we have that $\int_\Omega f^*(x_n) d\mu = j(x_n)(f^*) = \int_\Omega I_{E_n} f^{**}(f^*) d\mu$. By replacing f^* with $I_E f^*$ we can obtain that $\int_E f^*(x_n) d\mu = \int_E I_{E_n} f^{**}(f^*) d\mu$, for all $E \in \mathscr{F}$ and $f^* \in L^q(S^*)$, which implies that $f^*(x_n) = I_{E_n} f^{**}(f^*)$, for all $f^* \in L^q(S^*)$. Since $L^q(S^*)$ is $\mathscr{T}_{\epsilon,\lambda}$-dense in S^*, $J(x_n) = I_{E_n} f^{**}$, for all $n \in N$.

Let $y_n = \sum_{k=1}^{n} I_{E_k} \cdot x_k$, for all $n \in N$; then $\{y_n, n \in N\}$ is $\mathscr{T}_{\epsilon,\lambda}$-Cauchy in S and hence convergent to some $y \in S$, which shows that $J(y) = \lim_{n\to\infty} J(y_n) = \lim_{n\to\infty} \sum_{k=1}^{n} I_{E_k} \cdot f^{**} = (\sum_{n=1}^{\infty} I_{E_n}) \cdot f^{**} = f^{**}$; namely, J is surjective. □

Remark 58. Concerning Theorem 57, a similar and more general result was given in [6] where $L^p(\mathscr{F})$ is replaced by a reflexive Köthe function space.

Acknowledgment

The author would like to thank Professor Quanhua Xu for kindly providing the reference [6] in February 2012, which makes us know, for the first time, the excellent work of Haydon, Levy, and Raynaud.

References

[1] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, Elsevier, New York, NY, USA, 1983, Reissued by Dover Publications, New York, NY, USA, 2005.

[2] T. X. Guo, *The theory of probabilistic metric spaces with applications to random functional analysis [M.S. thesis]*, Xi'an Jiaotong University, Xi'an, China, 1989.

[3] T. X. Guo, "Some basic theories of random normed linear spaces and random inner product spaces," *Acta Analysis Functionalis Applicata*, vol. 1, no. 2, pp. 160–184, 1999.

[4] T. X. Guo, "The relation of Banach-Alaoglu theorem and Banach-Bourbaki-Kakutani-Šmulian theorem in complete random normed modules to stratification structure," *Science in China. Series A*, vol. 51, no. 9, pp. 1651–1663, 2008.

[5] T. X. Guo, *Random metric theory and its applications [Ph.D. thesis]*, Xi'an Jiaotong University, Xi'an, China, 1992.

[6] R. Haydon, M. Levy, and Y. Raynaud, *Randomly Normed Spaces*, Hermann, Paris, France, 1991.

[7] D. Filipović, M. Kupper, and N. Vogelpoth, "Separation and duality in locally L^0-convex modules," *Journal of Functional Analysis*, vol. 256, no. 12, pp. 3996–4029, 2009.

[8] T. X. Guo, "Relations between some basic results derived from two kinds of topologies for a random locally convex module," *Journal of Functional Analysis*, vol. 258, no. 9, pp. 3024–3047, 2010.

[9] T. X. Guo and Y. J. Yang, "Ekeland's variational principle for an $\overline{L}^0$-valued function on a complete random metric space," *Journal of Mathematical Analysis and Applications*, vol. 389, no. 1, pp. 1–14, 2012.

[10] S. E. Zhao and T. X. Guo, "The random subreflexivity of complete random normed modules," *International Journal of Mathematics*, vol. 23, no. 3, Article ID 1250047, 14 pages, 2012.

[11] T. X. Guo, "Recent progress in random metric theory and its applications to conditional risk measures," *Science China. Series A*, vol. 54, no. 4, pp. 633–660, 2011.

[12] T. X. Guo, S. E. Zhao, and X. L. Zeng, "On random convex analysis—the analytic foundation of the module approach to conditional risk measures," http://arxiv.org/abs/1210.1848.

[13] N. Dunford and J. T. Schwartz, *Linear Operators(I)*, Interscience, New York, NY, USA, 1957.

[14] T. X. Guo, "Module homomorphisms on random normed modules," *Northeastern Mathematical Journal (China)*, vol. 12, no. 1, pp. 102–114, 1996.

[15] J. P. LaSalle, "Pseudo-normed linear spaces," *Duke Mathematical Journal*, vol. 8, pp. 131–135, 1941.

[16] T. X. Guo, "Metrics, probabilistic metrics and random metrics," *Systems Science and Mathematical Sciences (China)*, vol. 8, no. 2, pp. 182–186, 1995.

[17] Z. Y. You and L. H. Zhu, "The isometric metrization of probabilistic metric spaces," *Scientia Sinica, Series A*, vol. 1, pp. 19–24, 1989.

[18] K. Yosida, *Functional Analysis*, vol. 123, Beijing World Publishing, Beijing, China, 6th edition, 1999.

[19] T. X. Guo, "Extension theorems of continuous random linear operators on random domains," *Journal of Mathematical Analysis and Applications*, vol. 193, no. 1, pp. 15–27, 1995.

[20] T. X. Guo and Z. Y. You, "The Riesz representation theorem for random inner product modules and its applications," *Chinese Annals of Mathematics. Series A*, vol. 17, no. 3, pp. 361–364, 1996.

[21] Z. Y. You and T. X. Guo, "Pointwise best approximation in the space of strongly measurable functions with applications to best approximation in $L^p(\mu, X)$," *Journal of Approximation Theory*, vol. 78, no. 3, pp. 314–320, 1994.

[22] T. X. Guo and S. B. Li, "The James theorem in complete random normed modules," *Journal of Mathematical Analysis and Applications*, vol. 308, no. 1, pp. 257–265, 2005.

[23] T. X. Guo and Z. Y. You, "A note on pointwise best approximation," *Journal of Approximation Theory*, vol. 93, no. 2, pp. 344–347, 1998.

[24] T. X. Guo and X. Zhang, "Stone's representation theorem of a group of random unitary operators on complete complex random inner product modules," *Scientia Sinica, Series A*, vol. 42, pp. 181–202, 2012.

[25] T. X. Guo, "A characterization for a complete random normed module to be random reflexive," *Journal of Xiamen University. Natural Science*, vol. 36, no. 4, pp. 167–170, 1997.

[26] T. X. Guo, "Radon-Nikodým property of conjugate Banach spaces and w^*-equivalence theorems of w^*-measurable functions," *Science in China. Series A.*, vol. 39, no. 10, pp. 1034–1041, 1996.

[27] T. X. Guo, "Representation theorems of the dual of Lebesgue-Bochner function spaces," *Science in China. Series A*, vol. 43, no. 3, pp. 234–243, 2000.

[28] T. X. Guo, H. Xiao, and X. Chen, "A basic strict separation theorem in random locally convex modules," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 71, no. 9, pp. 3794–3804, 2009.

[29] M. Z. Wu, "The Bishop-Phelps theorem in complete random normed modules endowed with the (ϵ, λ)-topology," *Journal of Mathematical Analysis and Applications*, vol. 391, no. 2, pp. 648–652, 2012.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 284389, 13 pages
http://dx.doi.org/10.1155/2013/284389

Research Article

The Composition Operator and the Space of the Functions of Bounded Variation in Schramm-Korenblum's Sense

Z. Jesús,[1] O. Mejía,[1] N. Merentes,[1] and S. Rivas[2]

[1] *Departamento de Matemática, Universidad Central de Venezuela, Caracas 1220A, Venezuela*
[2] *Instituto Universitario Tecnológico Elías Calixto Pompa, Guatire, Venezuela*

Correspondence should be addressed to O. Mejía; odalism_18@yahoo.com

Received 20 March 2013; Accepted 20 July 2013

Academic Editor: Gen-Qi Xu

We show that the composition operator H, associated with $h : [a,b] \to \mathbb{R}$, maps the spaces $\mathrm{Lip}[a,b]$ on to the space $\kappa\mathrm{BV}_\phi[a,b]$ of functions of bounded variation in Schramm-Korenblum's sense if and only if h is locally Lipschitz. Also, verify that if the composition operator generated by $h : [a,b]\times\mathbb{R} \to \mathbb{R}$ maps this space into itself and is uniformly bounded, then regularization of h is affine in the second variable.

1. Introduction

The composition operator problem (or COP, for short) refers to determining the conditions on a function $h : \mathbb{R} \to \mathbb{R}$, such that the composition operator, associated with the function h, maps a space $\mathbb{X}$ of functions $u : [a,b] \to \mathbb{R}$ into itself [1, 2]. There are several spaces where the COP has been resolved. For example, in 1961, Babaev [3] showed that the composition operator H, associated with the function $h : \mathbb{R} \to \mathbb{R}$, maps the space $\mathrm{Lip}[a,b]$ of the Lipschitz functions into itself if and only if h is locally Lipschitz; in 1967, Mukhtarov [4] obtained the same result for the space $\mathrm{Lip}_\alpha[a,b]$ of the Hölder functions of order α $(0<\alpha<1)$.

The first work on the COP in the space of functions of bounded variation $\mathrm{BV}[a,b]$ was made by Josephy in 1981 [5]. In 1986, Ciemnoczołowski and Orlicz [6] got the same result for the space of the functions of bounded φ-variation in Wiener's sense. In 1974, Chaika and Waterman [7] reached a similar result for the space of functions of bounded harmonic variation $\mathrm{HBV}[a,b]$. In the years 1991 and 1995 Merentes showed a similar result for the spaces of absolutely continuous functions $\mathrm{AC}[a,b]$ and the space of function of bounded φ-variation in Riesz's sense $\mathrm{RV}_\varphi[a,b]$ (see [8, 9]), and in 1998 Merentes and Rivas achieved the same result when the composition operator maps the space $\mathrm{RV}_p[a,b]$ of the functions of bounded p-variation in Riesz's sense $(1<p<\infty)$ into the space $\mathrm{BV}[a,b]$ [10]. In 2003, Pierce and Waterman solved the COP for the spaces $\phi\mathrm{BV}[a,b]$ and $\Lambda\mathrm{BV}[a,b]$ [11]. More recently, in 2011, Appell et al. [1] conclude the same results verifying when the composition operator maps $\mathrm{Lip}[a,b]$ into $\mathrm{BV}[a,b]$. Finally, Appell and Merentes verify the same result for the space of functions of bounded κ-variation [12].

There exist spaces $\mathbb{X}$ of real functions defined on an interval $[a,b]$, such that H maps $\mathbb{X}$ into itself and h is not locally Lipschitz. For example, in the case the space of continuous functions $C[a,b]$ it follows from the Tietze-Urysohn theorem that the composition operator H acts from $C[a,b]$ into itself and the function h must be continuous; that is, h does not need to be Lipschitz. A similar result was obtained in the space of regulated functions [13].

A first objective of this work is to demonstrate that the composition operator, associated with the function h, maps the space $\mathrm{Lip}[a,b]$ of the Lipschitz functions into the space $\kappa\mathrm{BV}_\phi[a,b]$ of functions of bounded variation in Schramm-Korenblum's sense or into the space $\kappa\mathrm{BV}[a,b]$ of functions of bounded variation in Korenblum's sense if and only if h is locally Lipschitz. We also extend this result to function spaces $\mathbb{X},\mathbb{Y}$, such that $\mathrm{Lip}[a,b]\subset\mathbb{X}\subset\mathbb{Y}$, where $\mathbb{Y}\subset\kappa\mathrm{BV}[a,b]$ or $\mathbb{Y}\subset\mathrm{BV}_\phi[a,b]$.

In a seminal article of 1982, Matkowski [14] showed that if the composition operator H, associated with the function $h : [a,b]\times\mathbb{R}\to\mathbb{R}$, maps the space $\mathrm{Lip}[a,b]$ of the Lipschitzian functions into itself and is a globally Lipschitzian map, then the function h has the form

$$h(t,x)=\alpha(t)x+\beta(t),\quad t\in[a,b],\ x\in\mathbb{R},\tag{1}$$

for some $\alpha,\beta\in\operatorname{Lip}[a,b]$.

There are a variety of spaces besides $\operatorname{Lip}[a,b]$ that verify this result [15]. The spaces of Banach $(\mathbb{X},\|\cdot\|)$ that fulfill this property are said to satisfy the *Matkowski property* [1].

In 1984, Matkowski and Miś [16] considered the same hypotheses on the operator H for the space $\mathrm{BV}[a,b]$ of the function of bounded variation and concluded that (1) is true for the regularization h^- of the function h with respect of the first variable; that is,

$$h^-(t,x)=\alpha(t)x+\beta(t),\quad t\in[a,b],\ x\in\mathbb{R},\tag{2}$$

where $\alpha,\beta\in\mathrm{BV}^-[a,b]$. Spaces that satisfy this conditions said to be verified *Weak Matkowski Property* [1].

A second objective of this paper is to show that if function $h(t,\cdot)$ is continuous in the second variable, for each $t\in[a,b]$, and the composition operator H, associated with the function h, is uniformly bounded, then h satisfies (2).

2. Preliminaries

Let $[a,b]$ be a closed interval of the real line $\mathbb{R}$ ($a,b\in\mathbb{R}$, $a<b$). From now on, for a function $u:[a,b]\to\mathbb{R}$ denote by $L_a^b(u)$ the Lipschitz constant of u; that is,

$$L_a^b(u)=\sup\left\{\frac{|u(t)-u(s)|}{|s-t|}:t,s\in[a,b],\ s\neq t\right\}.\tag{3}$$

By $\operatorname{Lip}[a,b]=\{u:[a,b]\to\mathbb{R}:L_a^b(u)<\infty\}$ we will denote the space of the Lipschitz functions. It is well known that the space $\operatorname{Lip}[a,b]$ is a Banach space endowed with the norm

$$\|u\|_{\mathrm{Lip}}:=|u(a)|+L_a^b(u)\quad(u\in\operatorname{Lip}[a,b]).\tag{4}$$

Ever since the notion of a function of bounded variation appeared, it has led to an incredible number of generalizations. In 1881, Jordan [17] introduced the definition of function of bounded variation for a function $u:[a,b]\to\mathbb{R}$ and showed that these kinds of function can be decomposed as the difference of two monotone functions. As a consequence of this result we have that those functions satisfy the Dirichlet criterion, that is, the functions that have pointwise convergent Fourier series.

Jordan defined such functions in the following way.

Definition 1. Let $u:[a,b]\to\mathbb{R}$ and $\pi:a\le t_1<\cdots<t_n\le b$ be a partition of the interval $[a,b]$. Consider

$$\sigma(u,\pi):=\sum_{j=1}^{n-1}\left|u\left(t_{j+1}\right)-u\left(t_j\right)\right|,\qquad V(u;[a,b])=\sup_{\pi}\sigma(u,\pi),\tag{5}$$

where the supremum is taken over all partitions π of the interval $[a,b]$.

If $V(u;[a,b])<\infty$, then u has bounded variation on the interval $[a,b]$ and this number is called the variation in Jordan's sense on $[a,b]$. This space of function is denoted by $\mathrm{BV}[a,b]$.

The concept of bounded variation has been the subject of intensive research, and many applications, generalizations, and improvements of them can be found in the literature (see for instance [18–20]). Some generalizations have been introduced by De La Vallée Poussin, F. Riesz, N. Wiener, L. C. Young, Yu. T. Medvedev, D. Waterman, B. Korenblum and M. Schramm.

In 1975, Korenblum in [21] considered a new kind of variation, called κ-variation, introducing a function κ for distorting the expression $|t_j-t_{j-1}|$ in the partition itself rather than the expression $|f(t_j)-f(t_{j-1})|$ in the range. Subsequently, this class of functions has been studied in detail by Cyphert and Kelingos [22]. One advantage of this alternate approach is that a function of bounded κ-variation may be decomposed into the difference of two simpler functions called κ-decreasing functions (for the precise definition see the following).

Definition 2. A function $\kappa:[0,1]\to[0,1]$ is said to be a κ-function or distortion function if it satisfies the following properties:

(1) κ is continuous with $\kappa(0)=0$ and $\kappa(1)=1$,

(2) κ is concave, increasing, and

(3) $\lim_{t\to0^+}(\kappa(t)/t)=\infty$.

Simple examples of distortion functions are

$$\kappa(t)=t^{\alpha}\ (0<\alpha<1),\qquad\kappa(t)=t\left(1-\log t\right).\tag{6}$$

From Definition 2 we can see that κ is subadditive; that is,

$$\kappa(s+t)\le\kappa(s)+\kappa(t)\quad(0\le s,t\le1),\tag{7}$$

and since $\lim_{t\to0^+}(\kappa(t)/t)=\infty$, then without loss of generality we can assume that

$$\kappa(t)\ge t\quad(t\in[0,1]).\tag{8}$$

Furthermore Korenblum introduces the following concept of variation.

Definition 3. Let $\kappa:[0,1]\to[0,1]$ be a distortion function and $u:[a,b]\to\mathbb{R}$ and $\pi:a\le t_1<\cdots<t_n\le b$ a partition of the interval $[a,b]$. Consider

$$\kappa(u,\pi):=\frac{\sum_{j=1}^{n-1}\left|u\left(t_{j+1}\right)-u\left(t_j\right)\right|}{\sum_{j=1}^{n-1}\kappa\left(\left(t_{j+1}-t_j\right)/(b-a)\right)},\qquad\kappa V(u)=\kappa V(u;[a,b]):=\sup_{\pi}\kappa(u,\pi),\tag{9}$$

where the supremum is taken over all partitions π of the interval $[a,b]$, called the κ-variation of u on $[a,b]$. In the case,

$\kappa V(u;[a,b]) < \infty$ we say that u has κ-variation on $[a,b]$ and we will denote by $\kappa\mathrm{BV}[a,b]$ the space of functions of κ-variation on $[a,b]$.

In the case that $[a,b]=[0,1]$ the equality (9) becomes

$$\kappa(u,\pi) := \frac{\sum_{j=1}^{n-1}\left|u\left(t_{j+1}\right)-u\left(t_j\right)\right|}{\sum_{j=1}^{n-1}\kappa\left(t_{j+1}-t_j\right)}. \tag{10}$$

Some properties of the functions with bounded κ-variation are summarized in the following theorem.

Theorem 4 (see [22]). *Let $\kappa:[0,1]\to[0,1]$ be a distortion function. Then*

(1) $\kappa\mathrm{BV}[a,b]$ *is a Banach space endowed with the norm*

$$\|u\|_\kappa = |u(a)| + \kappa V(u;[a,b]) \quad (u\in\kappa\mathrm{BV}[a,b]). \tag{11}$$

(2) *If the function u is monotone, then $\kappa V(u;[a,b]) = |u(b)-u(a)|$.*

(3) *If $u\in\kappa\mathrm{BV}[a,b]$, then u is bounded and $\|u\|_\infty \le 2\|u\|_\kappa$.*

(4) $\mathrm{Lip}[a,b]\subset\mathrm{BV}[a,b]\subset\kappa\mathrm{BV}[a,b]\subset R[a,b]$, *where $R[a,b]$ denotes the space of regulated functions.*

(5) *If $u\in\kappa\mathrm{BV}[a,b]$, then u can be decomposed as difference of two κ-decreasing functions, that is, there exist functions $v:[a,b]\to\mathbb{R}$ such that $\sup_{s<t}((v(t)-v(s))/\kappa((t-s)/(b-a)))<\infty$.*

It is easy to show that if $\lim_{t\to 0}(k(t)/t)$ is finite, then $\kappa\mathrm{BV}[a,b]=\mathrm{BV}[a,b]$. In [1] it is shown that inclusions (4) of Theorem 4 are strict.

Throughout this paper a φ-function φ is a continuous increasing function $\varphi:[0,\infty)\to\mathbb{R}$, such that $\varphi(0)=0$ and $\lim_{t\to\infty}\varphi(t)=\infty$.

A ϕ-sequence is a sequence of decreasing $\phi=\{\varphi_n\}_{n\ge1}$ of convex φ-function that satisfies $\sum_{n\ge1}\varphi_n(t)$ diverge for $t>0$.

We denote by $F_{\mathbb{N}}[a,b]$ the collection of finite or numerable family of nonoverlapping interval $\{[a_n,b_n]\}_{n\ge1}$, such that $\bigcup_{n\ge1}[a_n,b_n]=[a,b]$.

In 1985, Schramm [23] introduced a new concept of variation as follows.

Definition 5. Let $\phi=\{\varphi_n\}_{n\ge1}$ be a ϕ-sequence, $\{I_n=[a_n,b_n]\}_{n\ge1}\in F_{\mathbb{N}}[a,b]$, and $u:[a,b]\to\mathbb{R}$. We define

$$\sigma_\phi(u,I_n) := \sum_{n=1}^{\infty}\varphi_n\left(\left|u(b_n)-u(a_n)\right|\right),$$
$$V_\phi(u) = V_\phi(u;[a,b]) := \sup_{I_n\in F_{\mathbb{N}}[a,b]}\sigma_\phi(u,I_n). \tag{12}$$

If $V_\phi(u;[a,b])<\infty$, we say that u has bounded ϕ-variation in the interval $[a,b]$ and this number denotes the ϕ-variation of u in Schramm's sense in $[a,b]$. The class of functions that have bounded ϕ-variation in the interval $[a,b]$ is denoted by $V_\phi[a,b]$. The vectorial space generated by this class is denoted by $\mathrm{BV}_\phi[a,b]$.

The next lemma is useful for building the space generated by several classes of functions.

Lemma 6. *Let $\mathbb{X}$ be a vector space and $A\subset\mathbb{X}$ a nonempty and symmetric set. Then*

(1) $0\in A$.

(2) *The vector space generated for A is equal to*

$$\langle A\rangle = \{x\in\mathbb{X} : \exists\lambda>0 \text{ such that } \lambda x\in A\} = \bigcup_{\lambda>0}\lambda A. \tag{13}$$

Some properties of functions of bounded ϕ-variation in Schramm's sense are given in the following theorem.

Theorem 7 (see [23]). *Let $\phi=\{\varphi_n\}_{n\ge1}$ be ϕ-sequence then*

(1) $\mathrm{BV}_\phi[a,b]$ *is a Banach space endowed with the norm*

$$\|u\|_\phi := |u(a)| + \mu(u) \quad \left(u\in\mathrm{BV}_\phi[a,b]\right), \tag{14}$$

where $\mu(u):=\inf_{\lambda>0}\{\lambda>0 : V_\phi(u/\lambda)\le1\}$.

(2) *If u is monotone, then $V_\phi(u)=\varphi_1(|u(b)-u(a)|)$.*

(3) *If $u\in V_\phi[a,b]$, then u is bounded and $\|u\|_\infty\le|u(a)|+2\varphi_1^{-1}(V_\phi(u))$.*

(4) *$V_\phi[a,b]$ is a symmetrical and convex set.*

(5) $\mathrm{BV}_\phi[a,b]=\{u:[a,b]\to\mathbb{R} : \exists\lambda>0$ *such that* $V_\phi(\lambda u)<\infty\}$.

(6) *$u\in V_\phi[a,b]$, then u has lateral limits at each point of $[a,b]$.*

In 1986, S. K. Kim and J. Kim [24] combined the concepts of κ-variation and ϕ-variation introduced by Korenblum and Schramm to create the concept of $\kappa\phi$-variation or variation in Schramm-Korenblum's sense.

Definition 8. Let $\kappa:[0,1]\to[0,1]$ be a distortion function, $\phi=\{\varphi_n\}_{n\ge1}$ a ϕ-sequence, $\{I_n=[a_n,b_n]\}_{n\ge1}\in F_{\mathbb{N}}[a,b]$, and $u:[a,b]\to\mathbb{R}$. We define

$$\kappa\sigma_\phi(u,I_n) := \frac{\sum_{n=1}^{\infty}\varphi_n\left(\left|u(b_n)-u(a_n)\right|\right)}{\sum_{n=1}^{\infty}\kappa\left((b_n-a_n)/(b-a)\right)},$$
$$\kappa V_\phi(u) = \kappa V_\phi(u;[a,b]) := \sup_{I_n\in F_{\mathbb{N}}[a,b]}\kappa\sigma_\phi(u,I_n). \tag{15}$$

If $\kappa V_\phi(u;[a,b])<\infty$, we say that u has bounded $\kappa\phi$-variation in the interval $[a,b]$ and this number denotes the $\kappa\phi$-variation of u in Schramm-Korenblum's sense in $[a,b]$. The class of functions that have bounded $\kappa\phi$-variation in the interval $[a,b]$ is denoted by $\kappa V_\phi[a,b]$. The vectorial space generated by this class is denoted by $\kappa\mathrm{BV}_\phi[a,b]$.

A particular case of ϕ-sequence is when all the functions φ_n, $n\in\mathbb{N}$ are equal to a fixed φ-function φ. In this situation the class $\kappa V_\varphi[a,b]$ is the class of the functions that have bounded $\kappa\varphi$-variation in Wiener-Korenblum's sense. This class of functions is denoted by $\kappa V_\varphi[a,b]$ and the vectorial space generated by this class of function is denoted by $\kappa\mathrm{BV}_{\varphi}[a,b]$.

Some properties of functions of bounded $\kappa\phi$-variation in Schramm-Korenblum's sense are given in the following theorem.

Theorem 9. *Let $\kappa : [0,1] \to [0,1]$ be a distortion function and $\phi = \{\varphi_n\}_{n\geq 1}$ a ϕ-sequence, then*

(1) $\kappa\mathrm{BV}_{\phi}[a,b]$ *is a Banach space endowed with the norm*

$$\|u\|_{\kappa\phi} := |u(a)| + \mu_{\phi}(u) \quad \left(u \in \kappa\mathrm{BV}_{\phi}[a,b]\right), \tag{16}$$

where $\mu_{\phi}(u) := \inf_{\lambda>0}\{\lambda > 0 : \kappa V_{\phi}(u/\lambda) \leq 1\}$.

(2) *If u is monotone, then* $\kappa V_{\phi}(u) = \varphi_1(|u(b) - u(a)|)$.

(3) *If* $u \in \kappa V_{\phi}[a,b]$, *then u is bounded and* $\|u\|_{\infty} \leq |u(a)| + 2\varphi_1^{-1}(\kappa V_{\phi}(u))$.

(4) $\kappa V_{\phi}[a,b]$ *is a symmetrical and convex set.*

(5) $\kappa\mathrm{BV}_{\phi}[a,b] = \{u : [a,b] \to \mathbb{R} : \exists\lambda > 0$ *such that* $\kappa V_{\phi}(\lambda u) < \infty\}$.

(6) $\mathrm{BV}[a,b] \subset V_{\phi}[a,b] \subset \kappa V_{\phi}[a,b]$, *and therefore* $\mathrm{BV}[a,b] \subset \mathrm{BV}_{\phi}[a,b] \subset \kappa\mathrm{BV}_{\phi}[a,b]$.

(7) $\kappa\mathrm{BV}[a,b] \subset \kappa\mathrm{BV}_{\phi}[a,b]$.

(8) $u \in \kappa V_{\phi}[a,b]$, *then u has lateral limits at each point of* $[a,b]$.

Proof.

Part (1). See [24].

Part (2). We take $\{[a_n, b_n]\}_{n\geq 1} \in F_{\mathbb{N}}[a,b]$, then since the functions $0 < t \to \varphi_n(t)/t$ are increasing, we obtain

$$\begin{aligned} &\frac{\sum_{n=1}^{\infty}\varphi_n(|u(b_n) - u(a_n)|)}{\sum_{n=1}^{\infty}\kappa((b_n - a_n)/(b-a))} \\ &\quad = \sum_{n=1}^{\infty}\frac{\varphi_n(|u(b_n) - u(a_n)|)}{|u(b_n) - u(a_n)|}|u(b_n) - u(a_n)| \\ &\qquad \times \left(\sum_{n=1}^{\infty}\kappa\left(\frac{b_n - a_n}{b-a}\right)\right)^{-1} \\ &\quad \leq \sum_{n=1}^{\infty}\frac{\varphi_n(|u(b) - u(a)|)}{|u(b) - u(a)|}|u(b_n) - u(a_n)| \\ &\qquad \times \left(\sum_{n=1}^{\infty}\kappa\left(\frac{b_n - a_n}{b-a}\right)\right)^{-1} \\ &\quad \leq \frac{\varphi_1(|u(b) - u(a)|)}{|u(b) - u(a)|}\sum_{n=1}^{\infty}|u(b_n) - u(a_n)| \\ &\quad = \varphi_1(|u(b) - u(a)|), \end{aligned} \tag{17}$$

from which is obtained

$$\kappa V_{\phi}(u) \leq \varphi_1(|u(b) - u(a)|). \tag{18}$$

From the Definition of $kV_{\phi}(u)$ we have the reciprocal inequality.

Part (3). We consider $a < t \leq b$, then

$$\frac{\varphi_1(|u(t) - u(a)|)}{\kappa((t-a)/(b-a)) + \kappa((b-t)/(b-a))} \leq \kappa V_{\phi}(u). \tag{19}$$

Then $\varphi_1(|u(t) - u(a)|) \leq 2\kappa V_{\phi}(u)$ and from this inequality we have the required relation.

Part (4). We get Part (4) by the convexity of functions φ_n, $n \in \mathbb{N}$ and the definition of $\kappa\phi$-variation.

Part (5). It follows from part (4) and Lemma 6.

Part (6). Let $u \in \mathrm{BV}[a,b]$ and consider $\{[a_n, b_n]\}_{n\geq 1} \in F_{\mathbb{N}}[a,b]$. Define

$$A := \{n \in \mathbb{N} : |u(b_n) - u(a_n)| \leq 1\}. \tag{20}$$

Then

$$\begin{aligned} &\sum_{n=1}^{\infty}\varphi_n(|u(b_n) - u(a_n)|) \\ &\quad \leq \varphi_1(1)\sum_{n\in A}|u(b_n) - u(a_n)| \\ &\qquad + \sum_{n\notin A}\varphi_n(|u(b_n) - u(a_n)|). \end{aligned} \tag{21}$$

The last sum has at most $[V(u)]$ terms, where $[x] = \max\{n : n \leq x\}$. Because otherwise it has at least $[V(u)] + 1$ summands.

Accordingly

$$V(u) \geq \sum_{n\notin A}|u(b_n) - u(a_n)| \geq [V(u)] + 1. \tag{22}$$

Which is a contradiction. Therefore

$$\begin{aligned} &\sum_{n=1}^{\infty}\varphi_n(|u(b_n) - u(a_n)|) \\ &\quad \leq \varphi_1(1)V(u) + \sum_{n\notin A}\varphi_n(|u(b_n) - u(a_n)|) \\ &\quad \leq \varphi_1(1)V(u) + \varphi_1(2\|u\|_{\infty})[V(u)]. \end{aligned} \tag{23}$$

This concludes that $\mathrm{BV}[a,b] \subset V_{\phi}[a,b]$.

Let us show that $V_{\phi}[a,b] \subset \kappa V_{\phi}[a,b]$. In Fact let $u \in V_{\phi}[a,b]$ and $\{[a_n, b_n]\}_{n\geq 1} \in F_{\mathbb{N}}[a,b]$, then

$$\begin{aligned} &\sum_{n=1}^{\infty}\varphi_n(|u(b_n) - u(a_n)|) \\ &\quad \leq V_{\phi}(u)\kappa(1) \leq V_{\phi}(u)\kappa\left(\sum_{n=1}^{\infty}\frac{b_n - a_n}{b-a}\right). \end{aligned} \tag{24}$$

Hence, we get that $\kappa V_{\phi}(u) \leq V_{\phi}(u)$.

Part (7). Let $u \in \kappa\mathrm{BV}[a,b]$, then by part (3) u is bounded in $[a,b]$. Let us fix $\lambda > 0$, such that $\|\lambda u\|_{\infty} \leq 1/2$. Let

$\{[a_n,b_n]\}_{n\geq 1} \in F_{\mathbb{N}}[a,b]$, then from the convexity of the functions φ_n, $n\in\mathbb{N}$, we have

$$\frac{\sum_{n=1}^{\infty}\varphi_n\left(\left|\lambda u\left(b_n\right)-\lambda u\left(a_n\right)\right|\right)}{\sum_{n=1}^{\infty}\kappa\left(\left(b_n-a_n\right)/\left(b-a\right)\right)} \leq \frac{\sum_{n=1}^{\infty}\varphi_n\left(1\right)\lambda\left|u\left(b_n\right)-u\left(a_n\right)\right|}{\sum_{n=1}^{\infty}\kappa\left(\left(b_n-a_n\right)/\left(b-a\right)\right)} \leq \varphi_1\left(1\right)\lambda\kappa V\left(u\right). \quad (25)$$

Thus Lemma 6 concludes that $u\in\kappa\mathrm{BV}_{\phi}[a,b]$.

Part (8). Suppose that there is $t^*\in(a,b]$ such that $\lim_{t\uparrow t^*}u(t)$ does not exist.

By part (3) u is bounded then

$$\alpha=\lim_{t\uparrow t^*}\inf u\left(t\right)<\lim_{t\uparrow t^*}\sup u\left(t\right)=\beta,\quad\left(\alpha,\beta\in\mathbb{R}\right). \quad (26)$$

For each integer n (large enough) we can choose t_n, t_n' such that

$$t^*-\frac{1}{n}<t_n<t_n'<t^*,\qquad \left|u\left(t_n\right)-u\left(t_n'\right)\right|\geq\beta-\alpha. \quad (27)$$

Using the definition of $\kappa\phi$-variation, we have

$$\varphi_1\left(\left|u\left(t_n\right)-u\left(t_n'\right)\right|\right)\times\left(\kappa\left(\frac{t_n-a}{b-a}\right)+\kappa\left(\frac{t_n'-t_n}{b-a}\right)+\kappa\left(\frac{b-t_n'}{b-a}\right)\right)^{-1}\leq\kappa V_{\phi}\left(u\right). \quad (28)$$

Therefore,

$$\varphi_1\left(\beta-\alpha\right)\leq\kappa V_{\phi}\left(u\right)\left(2+\kappa\left(\frac{1}{n\left(b-a\right)}\right)\right). \quad (29)$$

By taking limit when $n\to\infty$, we obtain $\varphi_1(\beta-\alpha)=0$, which is absurd. From each it follows that

$$\lim_{t\uparrow t^*}\inf u\left(t\right)=\lim_{t\uparrow t^*}\sup u\left(t\right). \quad (30)$$

By a similar argument it follows that there exist $\lim_{t\downarrow t^*}u(t)$, $t^*\in[a,b)$. □

Assuming that $\lim_{t\to 0}(k(t)/t)$ is finite, then $\mathrm{V}_{\phi}[a,b]=\kappa V[a,b]$. For the last part of Theorem 9, we can give the definition of left and right regularizations of the function $u\in\kappa V_{\phi}[a,b]$.

Definition 10. Let $u\in\kappa\mathrm{BV}_{\phi}[a,b]$, then

$$u^-\left(t\right):=\begin{cases}\lim_{s\uparrow t}u\left(s\right), & t\in(a,b]\\ u\left(a\right), & t=a,\end{cases}\qquad u^+\left(t\right):=\begin{cases}\lim_{s\downarrow t}u\left(s\right), & t\in[a,b)\\ u\left(b\right), & t=b.\end{cases} \quad (31)$$

The function u^- is called the left regularization of the function u and the function u^+ the right regularization of the function u.

Applying the previous definition and the last part of Theorem 9, we can define

$$\kappa\mathrm{BV}_{\phi}^-\left[a,b\right]:=\left\{u^-:u\in\kappa\mathrm{BV}_{\phi}\left[a,b\right]\right\}. \quad (32)$$

Similarly, we defined $\kappa\mathrm{BV}_{\phi}^+[a,b]$.

Recently Castillo et al. [25] introduced the concept of κp-variation in Riesz-Korenblum's sense in the following way.

Definition 11. Let $1<p<\infty$, $\kappa:[0,1]\to[0,1]$ be a distortion function and $u:[a,b]\to\mathbb{R}$ and $\pi:a\leq t_1<\cdots<t_n\leq b$ a partition of the interval $[a,b]$. We define

$$\kappa V_p^R\left(u\right)=\kappa V_p^R\left(u;[\mathrm{a},\mathrm{b}]\right):=\sup_{\pi}\kappa\sigma_p\left(u,\pi\right), \quad (33)$$

where

$$\kappa\sigma_p\left(u,\pi\right):=\left(\sum_{j=1}^{n-1}\frac{\left|u\left(t_{j+1}\right)-u\left(t_j\right)\right|^p}{\left|t_{j+1}-t_j\right|^{p-1}}\right)\times\left(\sum_{j=1}^{n-1}\kappa\left(\frac{t_{j+1}-t_j}{b-a}\right)\right)^{-1} \quad (34)$$

and the supremum is taken on the set of all partitions of $[a,b]$. If $\kappa V_p^R(u;[a,b])<\infty$, we say that u has bounded κp-variation in the interval $[a,b]$. The number $\kappa V_p^R(u;[a,b])$ denoted the κp-variation of u in Riesz-Korenblum's sense in $[a,b]$. The space of functions that have bounded κp-variation in the interval $[a,b]$ is denoted by $\kappa\mathrm{RV}_p[a,b]$.

Some properties of these functions are exposed in the following theorem.

Theorem 12 (see [25]). *Let $1<p<\infty$ and let $\kappa:[0,1]\to[0,1]$ be a distortion function, then*

(1) *$\kappa\mathrm{RV}[a,b]$ is a Banach space endowed with the norm*

$$\left\|u\right\|_{\kappa p}:=\left|u\left(a\right)\right|+\left(\kappa V_p^R\left(u\right)\right)^{1/p}\quad\left(u\in\kappa\mathrm{RV}_p\left[a,b\right]\right). \quad (35)$$

(2) $\mathrm{Lip}[a,b]\subset\mathrm{RV}_p[a,b]\subset\kappa\mathrm{RV}_p[a,b]\subset\kappa\mathrm{BV}[a,b]$, *where $\mathrm{RV}_p[a,b]$ denote the space of the functions u that have bounded p-variation in Riesz's sense [20].*

(3) *$\kappa\mathrm{RV}_p[a,b]$ is an algebra.*

This concept was generalized by Castillo et al. [26] as stated in the following definition.

Definition 13. Let φ be a φ-function, $\kappa:[0,1]\to[0,1]$ be a distortion function, and $u:[a,b]\to\mathbb{R}$ and $\pi:a\leq t_1<\cdots<t_n\leq b$ a partition of the interval $[a,b]$. We define

$$\kappa V_{\varphi}^R\left(u\right)=\kappa V_{\varphi}^R\left(u;[a,b]\right):=\sup_{\pi}\kappa\sigma_{\varphi}\left(u,\pi\right), \quad (36)$$

where

$$\kappa\sigma_{\varphi}(u,\pi) := \frac{\sum_{j=1}^{n-1}\varphi\left(\left|u\left(t_{j+1}\right)-u\left(t_j\right)\right|/\left|t_{j+1}-t_j\right|\right)\left|t_{j+1}-t_j\right|}{\sum_{j=1}^{n-1}\kappa\left(\left(t_{j+1}-t_j\right)/(b-a)\right)} \tag{37}$$

and the supremum is taken over all partitions of $[a,b]$. If $\kappa V_{\varphi}^{R}(u;[a,b]) < \infty$, we say that u has bounded $\kappa\varphi$-variation in the interval $[a,b]$ and this number denotes the $\kappa\varphi$-variation of u in Riesz-Korenblum's sense in $[a,b]$. The class of functions that have bounded $\kappa\varphi$-variation in the interval $[a,b]$ is denoted by $\kappa V_{\varphi}[a,b]$. The vectorial space generate by this class is denoted by $\kappa\mathrm{RV}_{\varphi}[a,b]$.

The space of all functions that have bounded $\kappa\varphi$-variation on $[a,b]$ is denoted by $\kappa\mathrm{RV}_{\varphi}[a,b]$. Some properties of these functions are exposed in the following theorem.

Theorem 14 (see [26]). *Let φ be a convex φ-function and $\kappa : [0,1] \to [0,1]$ a distortion function, then*

(1) $\mathrm{Lip}[a,b] \subset \kappa V_{\varphi}^{R}[a,b] \subset \kappa\mathrm{BV}[a,b]$.

(2) $\mathrm{RV}_{\varphi}[a,b] \subset \kappa\mathrm{RV}_{\varphi}[a,b] \subset \kappa\mathrm{BV}[a,b]$, *where* $\mathrm{RV}_{\varphi}[a,b]$ *denote the space generated by the class of functions of bounded φ-variation in Riesz's sense [20].*

(3) *If* $\lim_{t\to\infty}(\varphi(t)/t) < \infty$, *then* $\kappa V_{\varphi}^{R}[a,b] = \kappa\mathrm{BV}[a,b]$.

(4) *If* $u \in \kappa V_{\varphi}^{R}[a,b]$, *then u is bounded.*

(5) $\kappa V_{\varphi}^{R}[a,b]$ *is a convex and symmetric set.*

(6) $\kappa\mathrm{RV}_{\varphi}[a,b] = \{u : [a,b] \to \mathbb{R} : \exists\lambda > 0$ *such that* $\kappa V_{\phi}^{R}(\lambda u) < \infty\}$.

(7) $\kappa\mathrm{RV}_{\varphi}[a,b]$ *is a Banach space endowed with the norm*

$$\|u\|_{\kappa\varphi} := |u(a)| + \mu(u) \quad \left(u \in \kappa\mathrm{RV}_{\varphi}[a,b]\right), \tag{38}$$

where $\mu_{\varphi}(u) := \inf_{\lambda>0}\{\lambda > 0 : \kappa V_{\varphi}^{R}(u/\lambda) \le 1\}$.

3. Composition Operator between Lip$[a,b]$ and $\kappa\mathrm{BV}[a,b]$ or $\kappa\mathrm{BV}_{\phi}[a,b]$

Given a function $h : \mathbb{R} \to \mathbb{R}$, the composition operator H, associated to the function h (case autonomous), maps each function $u : [a,b] \to \mathbb{R}$ into the composition function $Hu : [a,b] \to \mathbb{R}$ defined by

$$Hu(t) := h(u(t)), \quad (t \in [a,b]). \tag{39}$$

More generally, given $h : [a,b] \times \mathbb{R} \to \mathbb{R}$, we consider operator H defined by

$$Hu(t) := h(t,u(t)), \quad (t \in [a,b]). \tag{40}$$

This operator is also called *superposition operator* or *substitution operator* or *Nemytskii operator*. In what follows, we will refer to (39) as the *autonomous case* and to (40) as the *nonautonomous case.*

A problem related with this operator is to establish necessary and sufficient conditions of function h so that the operators H map the space $\mathbb{X}$ of real functions defined on $[a,b]$ into itself, that is, $H(\mathbb{X}) \subset \mathbb{X}$, or in more general way that operator H maps the space $\mathbb{X}$ into space of functions $\mathbb{Y}(H(\mathbb{X}) \subset \mathbb{Y})$. This problem is sometimes referred to as the composition operator problem (or COP). The solution to this problem for given $\mathbb{X}$ is sometimes very easy and sometimes highly nontrivial. As we mentioned in the introduction of this paper in a variety of spaces the required condition is that function h is locally Lipschitz. Another interesting problem is to determine the smallest space of functions $\mathbb{X}$ and the bigger space $\mathbb{Y}$ such that $H(\mathbb{X}) \subset \mathbb{Y}$.

In order to obtain the main result of this section, we will use a function of the zig-zig type such as the employed by Appell et al. in [1, 15]. In this section we will show that the locally Lipschitz condition of the function h is a necessary and sufficient condition such that $H(\mathrm{Lip}[a,b]) \subset \kappa\mathrm{BV}[a,b]$ and that in this situation H is bounded.

The following lemma will be useful in the proof of our main theorem (Theorem 17).

Lemma 15. *Let* $u : [a,b] \to \mathbb{R}$, $a \le s < \eta < t \le b$, *then*

$$\frac{|u(t)-u(s)|}{t-s} \le \frac{|u(\eta)-u(s)|}{\eta-s} + \frac{|u(t)-u(\eta)|}{t-\eta}. \tag{41}$$

Proof. Let $a \le s < \eta < t \le b$. Then

$$\begin{aligned} &\frac{|u(t)-u(s)|}{t-s} \\ &\quad\le \frac{|u(\eta)-u(s)|}{t-s} + \frac{|u(t)-u(\eta)|}{t-s} \\ &\quad= \frac{|u(\eta)-u(s)|}{\eta-s}\left(\frac{\eta-s}{t-s}\right) + \frac{|u(t)-u(\eta)|}{t-\eta}\left(\frac{t-\eta}{t-s}\right) \\ &\quad\le \frac{|u(\eta)-u(s)|}{\eta-s} + \frac{|u(t)-u(\eta)|}{t-\eta}. \end{aligned} \tag{42}$$

□

Lemma 16. *Let* $\kappa : [0,1] \to [0,1]$ *be a distortion function,* $\phi = \{\varphi_n\}_{n\ge1}$ *a ϕ-sequence,* $u \in \kappa\mathrm{BV}_{\phi}[a,b]$, *and* $\lambda > 0$. *Then* $\mu_{\phi}(u) < \lambda$ *if and only if* $\kappa V_{\phi}(u/\lambda) < 1$.

Proof. Let $u \in \kappa\mathrm{BV}_{\phi}[a,b]$. Suppose that $\mu_{\phi}(u) < \lambda$; then, by definition of $\mu_{\phi}(u)$ there exists k such that $\lambda > k > \mu_{\phi}(u)$ and $\kappa V_{\phi}(u/k) \le 1$. Hence, by the convexity of the functions φ_n, we have

$$\kappa V_{\phi}\left(\frac{u}{\lambda}\right) = \kappa V_{\phi}\left(\frac{u}{k}\frac{k}{\lambda}\right) \le \frac{k}{\lambda}\kappa V_{\phi}\left(\frac{u}{k}\right) \le \frac{k}{\lambda} \le 1. \tag{43}$$

Conversely, assume $\kappa V_{\phi}(u/\lambda) < 1$, then $\lambda \in \{\lambda > 0 : \kappa V_{\phi}(u/\lambda) \le 1\}$; hence $\mu_{\phi}(u) < \lambda$. □

Theorem 17. *Let* $\kappa : [0,1] \to [0,1]$ *be a distortion function,* $\phi = \{\varphi_n\}_{n\ge1}$ *a ϕ-sequence,* $h : \mathbb{R} \to \mathbb{R}$, *and H the composition* *operator associated to h. H maps the space* Lip$[0,1]$ *into the space* $\kappa\mathrm{BV}_\phi[0,1]$ *or* $\kappa\mathrm{BV}[0,1]$ *if and only if h is locally Lipschitz. Furthermore operator H is bounded.*

Proof. Let $u \in \mathrm{Lip}[0,1]$, $r = \|u\|_\infty$ and suppose that h is locally Lipschitz, then there exist $k = k(r)$, such that

$$|h(t) - h(s)| \le k(r)|s - t|, \quad (s, t \in \mathbb{R},\ |s| \le r,\ |t| \le r). \tag{44}$$

Let $\{[a_n, b_n]\}_{n\ge1} \in F_{\mathbb{N}}[a,b]$ and $\lambda > 0$, such that $\lambda < 1/(2k(r)\|u\|_{\mathrm{Lip}}\|u\|_\infty + 1)$, then

$$\begin{aligned}&\frac{\sum_{n=1}^{\infty}\varphi_n(\lambda|h(u(b_n)) - h(u(a_n))|)}{\sum_{n=1}^{\infty}\kappa((b_n - a_n)/(b-a))}\\ &\quad\le \frac{\sum_{n=1}^{\infty}\varphi_n(\lambda k(r)|u(b_n) - u(a_n)|)}{\kappa(1)}\\ &\quad\le \lambda k(r)\varphi_1(1)\|u\|_{\mathrm{Lip}}\sum_{n=1}^{\infty}(b_n - a_n) < \infty.\end{aligned} \tag{45}$$

Then by Lemma 15 and Theorem 9, we have $H(u) \in \kappa\mathrm{BV}_\phi[a,b]$.

The proof of the only if direction will be by contradiction, that is, we assume $H(\mathrm{Lip}[0,1]) \subset \kappa\mathrm{BV}_\phi[0,1]$ and h is not locally Lipschitz. Since the identity function $I_d : [0,1] \to [0,1]$ belongs to Lip$[0,1]$, then $h \circ I_d \in \kappa\mathrm{BV}_\phi[0,1]$ and therefore h is bounded in the interval $[0,1]$. Without loss of generality we may assume that

$$\left\|h|_{[0,1]}\right\|_\infty \le \frac{1}{4}. \tag{46}$$

Since h is not locally Lipschitz in $\mathbb{R}$, there is a closed interval I such that h does not satisfy any Lipschitz condition. In order to simplify the proof we can assume that $I = [0,1]$. In this way for any increasing sequence of positive real numbers $\{k_n\}_{n\ge1}$ that converge to infinite that we will define later, we can choose sequences $\{a_n\}_{n\ge1}$, $\{b_n\}_{n\ge1}$, such that

$$|h(b_n) - h(a_n)| > k_n|b_n - a_n|, \quad (n \in \mathbb{N}). \tag{47}$$

In addition we can choose a_n, b_n such that

$$a_n < b_n, \quad (n \in \mathbb{N}). \tag{48}$$

Considering subsequences if necessary, we can assume that the sequence $\{a_n\}_{n\ge1}$ is monotone. We can assume without loss of generality that sequence $\{a_n\}_{n\ge1}$ is increasing.

Since $[0,1]$ is compact, from inequality (47) we have that there exist subsequences of $\{a_n\}_{n\ge1}$ and $\{b_n\}_{n\ge1}$ that we will denote in the same way, and that converge to $a_\infty \in [0,1]$.

Since the sequence $\{a_n\}_{n\ge1}$ is a Cauchy sequence, we can assume (taking subsequence if necessary) that

$$|a_m - a_n| < \frac{1}{k_n}, \quad (m > n). \tag{49}$$

Again considering subsequences if needed using the properties of the function κ we can assume that

$$\max\{\kappa(b_n - a_n), \kappa(a_m - a_n)\} < \frac{1}{k_n}, \quad (n \in \mathbb{N}, m \ge n). \tag{50}$$

Consider the new sequence $\{m_n\}_{n\ge1}$ defined by

$$m_n := \frac{1}{k_n(b_n - a_n)}, \quad (n \in \mathbb{N}). \tag{51}$$

From inequalities (46) and (47) it follows that $m_n > 2$; therefore

$$\frac{m_n}{2} < [m_n] \le m_n, \quad (n \in \mathbb{N}). \tag{52}$$

Consider the sequence defined recursively $\{t_n\}_{n\ge1}$ by

$$t_1 := 0, \qquad t_{n+1} := t_n + a_{n+1} - a_n + 2[m_n](b_n - a_n), \quad (n \in \mathbb{N}). \tag{53}$$

This sequence is strictly increasing and from the relations (49) and (50), we get

$$\begin{aligned}t_n \longrightarrow t_\infty &:= \sum(t_{n+1} - t_n)\\ &= \sum_{n=1}^{\infty}(a_{n+1} - a_n) + 2\sum_{n=1}^{\infty}[m_n](b_n - a_n)\\ &\le 3\sum_{n=1}^{\infty}\frac{1}{k_n}.\end{aligned} \tag{54}$$

Then to ensure that $t_\infty \in [0,1]$, it is sufficient to suppose that $\sum_{n=1}^{\infty}(1/k_n) \le 1/3$.

We define the continuous zig-zag function $u : [0,1] \to \mathbb{R}$, as shown in the following:

$$u(t) := \begin{cases} a_n, & t = t_n + 2i(b_n - a_n),\ i = 0, \dots, [m_n]\\ b_n, & t = t_n + (2i+1)(b_n - a_n),\ i = 0, \dots, [m_n] - 1\\ a_\infty, & t_\infty \le t \le 1\\ \text{affine}, & \text{other case}. \end{cases} \tag{55}$$

Put

$$t_{n,i} := t_n + i(b_n - a_n), \quad n \in \mathbb{N},\ i = 0, \dots, 2[m_n]. \tag{56}$$

We can write each interval $I_n = [t_n, t_{n+1}]$, $n \in \mathbb{N}$, as the union of the family of nonoverlapping intervals

$$\begin{aligned}I_{n,i} &:= [t_{n.i}, t_{n,i+1}], \quad i = 0, \dots, [m_n] - 1,\\ I_{n,2[m_n]} &:= [t_{n,2[m_n]}, t_{n+1}].\end{aligned} \tag{57}$$

And function u is defined on $I_{n,i}$, $i = 0, \dots, 2[m_n]$ as follows:

$$u(t) = t - (t_n + 2i(b_n - a_n)) + a_n, \quad (t \in I_{n,2i}), \tag{58}$$

$$u(t) = t - t_n + (2i+1)(b_n - a_n) + b_n, \quad (t \in I_{n,2i+1}), \tag{59}$$

$$u(t) = t - t_{n+1} + a_n, \quad (t \in I_{n,2[m]}). \tag{60}$$

In all these situations, the slopes of the segments of lines are equal to 1.

Hence, we have for $n \in \mathbb{N}$, the absolute value of the slope of the line segments in these ranges are bounded by 1, as shown below

$$2^{-n}\frac{|b_n - a_n|}{\kappa^{-1}(b_n - a_n)} \le 2^{-n}k_n(b_n - a_n) \le 1,$$
$$2^{-(n+1)}\frac{a_{n+1} - a_n}{t_n + a_{n+1} - a_n} \le 1. \tag{61}$$

We will show that $u \in \mathrm{Lip}[0,1]$.

Let $0 \le s < t \le 1$, and then there are the following possibilities for the location of s and t on $[0,1]$.

Case 1. If $s, t \in I_n$, $(n \in \mathbb{N})$ and are in the same interval $I_{n,i}$, $i = 0, \ldots, 2[m_n]$.

From relations (58), (59), and (60) it follows that $|u(t) - u(s)|/|t - s| = 1$.

Case 2. If $s, t \in I_n$, $(n \in \mathbb{N})$ and are in two different intervals $I_{n,i}$, $i = 0, \ldots, 2[m_n]$.

There are several possibilities.

(a) $s \in I_{n.i}$, $t \in I_{n,j}$, $i < j < 2[m_n]$.

(a_1) $j = i + 1$. By Lemma 15 and relations (58) and (59) we have

$$\frac{|u(t) - u(s)|}{|t - s|} \le \frac{|u(t_{n,i+1}) - u(s)|}{t_{n,i+1} - s} + \frac{|u(t) - u(t_{n,i+1})|}{t - t_{n,i+1}} \le 2. \tag{62}$$

(a_2) $j > i + 1$. Then

$$\frac{|u(t) - u(s)|}{|t - s|} \le \frac{b_n - a_n}{t_{n,i+2} - t_{n,,i+1}} = 1. \tag{63}$$

(b) $s \in I_{n.i}$, $t \in I_{n,j}$, $i < j = 2[m_n]$.

If $j = i + 1$, proceed as (a_1).

If $j > i + 1$, again using the Lemma 15 and relations (58), (59), and (60) we obtain

$$\frac{|u(t) - u(s)|}{|t - s|} \le \frac{\left|u\left(t_{n,2[m_n]}\right) - u(s)\right|}{t_{n,2[m_n]} - s} + \frac{\left|u(t) - u\left(t_{n,2[m_n]}\right)\right|}{t_{n,2[m_n]} - t} \le \frac{b_n - a_n}{t_{n,2[m_n]} - t_{n,2[m_n]-1}} + 1 \le 2. \tag{64}$$

Case 3. If $s \in I_n$, $t \in I_m$, $n, m \in \mathbb{N}$, $n < m$.

From Lemma 15 and Case 2, we conclude that

$$\frac{|u(t) - u(s)|}{t - s} \le \frac{|u(t_{n+1}) - u(s)|}{t_{n+1} - s} + \frac{|u(t) - u(t_m)|}{t - t_m} \le 4. \tag{65}$$

Case 4. If $s \in I_n$, $n \in \mathbb{N}$, $t = t_\infty$.

Then from Lemma 15

$$\frac{|u(t_\infty) - u(s)|}{t_\infty - s} \le \frac{|u(t_{n,i+1}) - u(s)|}{t_{n,i+1} - s} + \frac{|a_\infty - u(t_{n,i+1})|}{b_n - a_n} \le 1 + \frac{a_\infty - a_n}{b_n - a_n} \le 2. \tag{66}$$

Case 5. If $s < t_\infty < t \le 1$.

From Lemma 15 and Case 4,

$$\frac{|u(t) - u(s)|}{t - s} \le \frac{|u(t_\infty) - u(s)|}{t_\infty - s} \le 2. \tag{67}$$

Case 6. If $t_\infty \le s < t \le 1$.

In this circumstance $u(s) = u(t) = a_\infty$ and the situation is trivial. Therefore we have that

$$|u(t) - u(s)| \le |t - s|, \quad (s, t \in [0,1]). \tag{68}$$

So u is Lipschitz in $[0,1]$. Moreover, for each partition of the interval $[0,1]$ of the form

$$\begin{aligned}\pi : 0 = t_1 &< t_1 + (b_1 - a_1) < \cdots < t_1 + 2[m_1](b_1 - a_2) \\ &< t_2 < t_2 + (b_2 - a_2) < \cdots < t_k \\ &< \cdots < t_k + 2[m_k](b_k - a_k) < 1\end{aligned} \tag{69}$$

and $c > 0$, using the inequality (47), convexity of the function φ_n, $n \ge 1$, and definition of m_n, $n \in \mathbb{N}$, we have

$$\begin{aligned}\kappa V_\phi&(c(h \circ u); [0,1]) \\ &\ge \frac{\sum_{n=1}^{k} 2[m_n]\varphi_n(c)|h(b_n) - h(a_n)|}{\sum_{n=1}^{k}[2[m_n]\kappa(b_n - a_n) + \kappa(a_{n+1} - a_n)]} \\ &\ge \frac{\sum_{n=1}^{k} 2[m_n]\varphi_n(c)k_n(b_n - a_n)}{\sum_{n=1}^{k}[2[m_n]\kappa(b_n - a_n) + \kappa(a_{n+1} - a_n)]} \\ &\ge \frac{\sum_{n=1}^{k} 2[m_n]k_n(b_n - a_n)\varphi_n(c)}{\sum_{n=1}^{k}(1/k_n)} \ge \sum_{n=1}^{k}\varphi_n(c).\end{aligned} \tag{70}$$

As the series $\sum_{n=1}^{\infty}\varphi_n(c)$ diverge, $ch \circ u \notin \kappa BV_\phi[0,1]$, which is a contradiction.

Let us see that $h \circ u \notin \kappa BV[a,b]$. In fact, as in the case of κ-variation we have

$$V_\kappa(h \circ u; [0,1]) \ge \sum_{n=1}^{k} 2[m_k]k_n(b_n - a_n) \ge k. \tag{71}$$

Therefore, $h \circ u \notin \kappa BV[0,1]$.

To prove that the operator H is bounded, let $r > 0$ and $u \in \mathrm{Lip}[0,1]$, such that $\|u\|_{\mathrm{Lip}} \leq r$, then from the definition of $\|\cdot\|_{\mathrm{Lip}[0,1]}$, we have

$$\|u\|_{\infty} \leq |u(0)| + L_0^1(u) \leq r. \tag{72}$$

Without loss of generality assume that $r = 1$. As h is locally Lipschitz, there exists $k(1) > 0$, such that

$$|h(x) - h(y)| \leq k(1)\ |x - y|, \quad |x| \leq 1, |y| \leq 1. \tag{73}$$

As the identity function $I_d : [0,1] \to [0,1]$ belongs to $\mathrm{Lip}[0,1]$, then $h \circ I_d \in \kappa\mathrm{BV}_{\phi}[0,1]$. By Theorem 9 we have that $\|h \circ I_d\|_{\infty} < \infty$.

Let $\{[a_n, b_n]\}_{n\geq 1} \in F_{\mathbb{N}}[0,1]$ and choose λ such that $k(1)\ \kappa V_{\phi}(I_d) < \lambda$, then

$$\begin{aligned}
&\frac{\sum_{n=1}^{\infty} \varphi_n\left(|h(u(b_n)) - h(u(a_n))| / \lambda\right)}{\sum_{n=1}^{\infty} \kappa(b_n - a_n)} \\
&\quad \leq \sum_{n=1}^{\infty} \varphi_n\left(\frac{k(1)|u(b_n) - u(a_n)|}{\lambda}\right) \\
&\quad \leq \sum_{n=1}^{\infty} \varphi_n\left(\frac{k(1)(b_n - a_n)}{\lambda}\right) \\
&\quad \leq \sum_{n=1}^{\infty} \frac{k(1)}{\lambda}\ \varphi_n(b_n - a_n) \\
&\quad \leq \frac{k(1)}{\lambda}\kappa V_{\phi}(I_d) < 1.
\end{aligned} \tag{74}$$

From Lemma 6 we have $\mu(h \circ u) < \lambda(r)$. Thus we conclude that

$$\begin{aligned}
\|H(u)\|_{\kappa\phi} &= |h(u(a))| + \mu(h \circ u) \\
&\leq \|h \circ u\|_{\infty} + \lambda(r).
\end{aligned} \tag{75}$$

And so operator H is bounded. In the case that $H(\mathrm{Lip}[0,1]) \subset \kappa\mathrm{BV}[0,1]$, proceed similarly. □

In the following result we give a Lemma of invariance.

Lemma 18. *Let κ be a distortion function, $\phi = \{\varphi_n\}_{n\geq 1}$ a ϕ-sequence, and $\nu : [0,1] \to [a,b]$ affine function that maps $[0,1]$ on $[a,b]$ ($\nu(t) := (b-a)t + a,\ t \in [0,1]$).*

(1) *$u \in \kappa\mathrm{BV}_{\phi}[a,b]$ if and only if $u \circ \nu \in \kappa\mathrm{BV}_{\phi}[0,1]$.*

(2) *$u \in \kappa\mathrm{BV}[a,b]$ if and only if $u \circ \nu \in \kappa\mathrm{BV}_{\phi}[0,1]$.*

Proof. (1) Let $u \in \kappa\mathrm{BV}_{\phi}[a,b]$ and $\{[a_n, b_n]\}_{n\geq 1} \in F_{\mathbb{N}}[0,1]$, then $\{[\nu(a_n), \nu(b_n)]\}_{n\geq 1} \in F_{\mathbb{N}}[a,b]$ and

$$\begin{aligned}
&\frac{\sum_{n=1}^{\infty} |u(\nu(b_n)) - u(\nu(a_n))|}{\sum_{n=1}^{\infty} \kappa(b_n - a_n)} \\
&\quad = \frac{\sum_{n=1}^{\infty} |u(\nu(b_n)) - u(\nu(a_n))|}{\sum_{n=1}^{n-1} \kappa((\nu(b_n) - \nu(a_n)) / (b-a))} \\
&\quad \leq \kappa V_{\phi}(u; [a,b]).
\end{aligned} \tag{76}$$

Hence $u \circ \nu \in \kappa\mathrm{BV}_{\phi}[0,1]$.

Reciprocally, let us suppose that $u \circ \nu \in \kappa\mathrm{BV}_{\phi}[0,1]$ and let $\{[a_n, b_n]\}_{n\geq 1} \in F_{\mathbb{N}}[a,b]$ be a partition of the interval $[a,b]$. Define

$$I_n := \left[\nu^{-1}(b_n), \nu^{-1}(a_n)\right] \quad (n \in \mathbb{N}). \tag{77}$$

We have $\{I_n\}_{n\geq 1} \in F_{\mathbb{N}}[0,1]$ and

$$\begin{aligned}
&\frac{\sum_{j=1}^{n-1} |u(b_n) - u(a_n)|}{\sum_{n=1}^{\infty} \kappa((b_n - a_n)/(b-a))} \\
&\quad = \left(\sum_{n=1}^{\infty} |u(\nu((b_n - a)/(b-a))) \right. \\
&\qquad\qquad \left. - u(\nu((a_n - a)/(b-a)))| \right) \\
&\qquad \times \left(\sum_{n=1}^{\infty} \kappa((b_n - a)/(b-a) \right. \\
&\qquad\qquad \left. - (a_n - a)/(b-a)) \right)^{-1} \\
&\quad \leq \kappa V_{\phi}(u \circ \nu; [0,1]),
\end{aligned} \tag{78}$$

so $u \in \kappa\mathrm{BV}_{\phi}[a,b]$.

(2) The proof is similar from part (1). □

As consequence of Lemma 18 we have the following results

Lemma 19. *Let κ be a distortion function, $\phi = \{\varphi_n\}_{n\geq 1}$ a ϕ-sequence, $h : \mathbb{R} \to \mathbb{R}$, and H the composition operator associated to the function h.*

(1) *$H(\mathrm{Lip}[a,b]) \subset \kappa\mathrm{BV}_{\phi}[a,b]$ if and only if $H(\mathrm{Lip}[0,1]) \subset \kappa\mathrm{BV}_{\phi}[0,1]$.*

(2) *$H(\mathrm{Lip}[a,b]) \subset \kappa\mathrm{BV}[a,b]$ if and only if $H(\mathrm{Lip}[0,1]) \subset \kappa\mathrm{BV}[0,1]$.*

Corollary 20. *Let κ be a distortion function, $\phi = \{\varphi_n\}_{n\geq 1}$ a ϕ-sequence, and $h : \mathbb{R} \to \mathbb{R}$. Then the composition operator H, associated with function h, maps the space $\mathrm{Lip}[a,b]$ on $\kappa\mathrm{BV}_{\phi}[a,b]$ or on $\kappa\mathrm{BV}[a,b]$ if and only if h is locally Lipschitz.*

Corollary 21. *Let κ be a distortion function, $\phi = \{\varphi_n\}_{n\geq 1}$ a ϕ-sequence, $h : \mathbb{R} \to \mathbb{R}$, and $\mathbb{X}, \mathbb{Y}$ normed spaces such that $\mathrm{Lip}[a,b] \subset \mathbb{X} \subset \mathbb{Y}$, where $\mathbb{Y} \subset \kappa\mathrm{BV}_{\phi}[a,b]$ or $\mathbb{Y} \subset \kappa\mathrm{BV}[a,b]$. Then the composition operator H associated with the function h maps the space $\mathbb{X}$ in the space $\mathbb{Y}$ if and only if h is locally Lipschitz.*

Some particular cases of Corollary 21 are the following

(1) $\mathbb{X} = \mathbb{Y}$, where $\mathbb{X}$ is one of the following spaces $\mathrm{Lip}[a,b]$ [3], $H_{\alpha}[a,b]$ $(0 < \alpha < 1)$ [4], $\mathrm{BV}_{\varphi}[a,b]$ [6], $\mathrm{BV}[a,b]$ [5], HBV [7], AC $[a,b]$ [8], $\mathrm{RV}_{\varphi}[a,b]$ [9], ϕBV [23], ΛBV [11], and $\kappa\mathrm{BV}[a,b]$ [12].

(2) $\mathbb{X} =$ RV$_\varphi[a,b]$, $\mathbb{Y} =$ BV$[a,b]$. See [10].

(3) $\mathbb{X} =$ Lip$[a,b]$, $\mathbb{Y} =$ BV$[a,b]$. See [1].

From Corollary 21 we get the following new cases

(1) $\mathbb{X}$ is one of the following spaces: Lip$[a,b]$, H_α $(0 < \alpha < 1)$, RV$_\varphi[a,b]$, AC$[a,b]$, BV$[a,b]$, BV$_\phi[a,b]$, and κRV$_\varphi[a,b]$; $\mathbb{Y}$ is one of the following spaces: κBV$[a,b]$, κBV$_\phi[a,b]$.

(2) $\mathbb{X} = \kappa$RV$_p[a,b]$, $\mathbb{Y} = \kappa$RV$_q[a,b]$, $1 < p, q < \infty$.

(3) $\mathbb{X} = \kappa$BV$[a,b]$, $\mathbb{Y} = \kappa$BV$_\phi[a,b]$,

More generally

(1) $\mathbb{X} = \kappa_1$BV$[a,b]$, $\mathbb{Y} = \kappa_2$RV$_\phi[a,b]$;

(2) $\mathbb{X} = \kappa_1$RV$_\varphi[a,b]$, $\mathbb{Y} = \kappa_2$RV$_\psi[a,b]$;

(3) $\mathbb{X} = \kappa_1$RV$_\phi[a,b]$, $\mathbb{Y} = \kappa_2$RV$_\phi[a,b]$;

where κ_1, κ_2 are distortion functions.

4. Uniformly Continuous Composition Operator in the Space $\kappa\mathrm{BV}_\phi[a,b]$

In many problems solving equation where the composition operator appears to guarantee the existence of solution it is necessary to apply a Fixed Point Theorem. To ensure the application this type of results is necessary to request the condition of global Lipschitz operator H. In several works Matkowski and Mís have shown that this condition implies that the function h has the form (1) or (2) (see, e.g., [16, 27]). This means that we may apply the Banach contraction mapping principle only if the underlying problems are actually *linear* and therefore are not interesting.

More recently, Matkowski and other researchers have replaced the condition of global Lipschitz by uniform continuity conditions or uniform boundedness composition operator (see e.g., [14]).

In this section we present results in this direction for the space κBV$_\phi[a,b]$.

Theorem 22. *Let $\kappa : [0,1] \to [0,1]$ be a distortion function, $\phi = \{\varphi_n\}_{n\ge1}$ a ϕ-sequence, and $h : [a,b] \times \mathbb{R} \to \mathbb{R}$ such that the function $h(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous with respect to the second variable, for each $t \in (a,b]$. If the composition operator H, associated with h, maps $\kappa\mathrm{BV}_\phi[a,b]$ into itself and satisfies the inequality*

$$\|H(u) - H(v)\|_{\kappa\phi} \le \gamma\left(\|u - v\|_{\kappa\phi}\right), \quad \left(u, v \in \kappa\mathrm{BV}_\phi[a,b]\right), \tag{79}$$

then there exists $\alpha, \beta \in \kappa\mathrm{BV}_\phi^-[a,b]$, such that

$$h^-(t,x) = \alpha(t)x + \beta(t), \quad (t \in [a,b], x \in \mathbb{R}). \tag{80}$$

Proof. As for each $x \in \mathbb{R}$ fixed the function $u(t) = x$, $t \in [a,b]$ is in κBV$_\phi[a,b]$, then $H(u) = h(\cdot,x) \in \kappa$BV$_\phi[a,b]$. There is left regularization of $h^-(\cdot,x)$.

From inequality (79) and Lemma 16, we get

$$\kappa V_\phi\left(\frac{H(u) - H(v)}{\gamma\left(\|u - v\|_{\kappa\phi}\right)}\right) < 1, \quad \left(u, v \in \kappa\mathrm{BV}_\phi[a,b]\right). \tag{81}$$

Let $a < r < a_1 < b_1 < a_2 < \cdots < a_m < b_m = s < b$, $x_1, x_2 \in \mathbb{R}$, $x_1 \ne x_2$ and put the zig-zag continuous functions $u_k : [a,b] \to \mathbb{R}$, $k = 1, 2$, as

$$u_k(t) = \begin{cases} \dfrac{x_k + x_2}{2}, & a \le t \le a_1 \text{ or } t = a_i, i = 1,\dots,m \\ \dfrac{x_k + x_1}{2}, & b_m \le t \le b \text{ or } t = b_i, i = 1,\dots,m, k = 1,2 \\ \text{affine}, & \text{other case.} \end{cases} \tag{82}$$

The functions u_k, $k = 1, 2$ are Lipschitz and therefore belongs to κBV$_\phi[a,b]$. Furthermore

$$(u_1 - u_2)(t) = \frac{x_1 - x_2}{2}, \quad (t \in [a,b]). \tag{83}$$

From inequality (81) and the definition of $k\phi$-variation, we have

$$\begin{aligned} &\left(\sum_{n=1}^{m} \varphi_n\left(\left(\left|h(b_n, u_1(b_n)) - h(b_n, u_2(b_n))\right.\right.\right.\right. \\ &\qquad \left.-h(a_n, u_1(a_n)) + h(a_n, u_2(a_n))\right|\Big) \\ &\qquad \times\left(\gamma\left(\|u_1 - u_2\|_{\kappa\phi}\right)\right)^{-1}\Big)\Bigg) \\ &\times\left(\kappa\left(\frac{a_1 - a}{b - a}\right) + \sum_{n=1}^{m}\kappa\left(\frac{b_n - a_n}{b - a}\right)\right. \\ &\qquad \left.+\sum_{n=1}^{m-1}\kappa\left(\frac{a_{n+1} - b_n}{b - a}\right) + \kappa\left(\frac{b - b_m}{b - a}\right)\right)^{-1} \le 1. \end{aligned} \tag{84}$$

By the construction of the u_k, $k = 1, 2$, we get

$$\begin{aligned} &\left(\sum_{n=1}^{m} \varphi_n\left(\left(2h(b_n, x_1) - h\left(b_n, \frac{x_1 + x_2}{2}\right)\right.\right.\right. \\ &\qquad \left. -h\left(a_n, \frac{x_1 + x_2}{2}\right) + h(a_n, x_2)\right) \\ &\qquad \times\left(\gamma\left(2^{-1}|x_1 - x_2|\right)\right)^{-1}\Big)\Bigg) \\ &\times\left(\kappa\left(\frac{a_1 - a}{b - a}\right) + \sum_{n=1}^{m}\kappa\left(\frac{b_n - a_n}{b - a}\right)\right. \\ &\qquad \left.+\sum_{n=1}^{m-1}\kappa\left(\frac{a_{n+1} - b_n}{b - a}\right) + \kappa\left(\frac{b - b_m}{b - a}\right)\right)^{-1} \le 1. \end{aligned} \tag{85}$$

Let r tend to s in the above inequality; we obtain

$$\sum_{n=1}^{m}\varphi_n\left(\frac{\left|h^-(s,x_1)-2h^-(s,(x_1+x_2)/2)+h^-(s,x_2)\right|}{\gamma(2^{-1}|x_1-x_2|)}\right)\leq\kappa\left(\frac{s-a}{b-a}\right)+\kappa\left(\frac{b-s}{b-a}\right),\quad(s\in(a,b)).\tag{86}$$

Passing the limit as $m\to\infty$,

$$\sum_{n=1}^{\infty}\varphi_n\left(\frac{\left|h^-(s,x_1)-2h^-(s,(x_1+x_2)/2)+h^-(s,x_2)\right|}{\gamma(2^{-1}|x_1-x_2|)}\right)\leq\kappa\left(\frac{s-a}{b-a}\right)+\kappa\left(\frac{b-s}{b-a}\right),\quad(s\in(a,b)).\tag{87}$$

As the series $\sum_{n=1}^{\infty}\varphi_n(x)$ is divergent for each $x>0$, necessarily

$$\left|h^-(s,x_1)-2h^-\left(s,\frac{x_1+x_2}{2}\right)+h^-(s,x_2)\right|=0,\quad(s\in(a,b)).\tag{88}$$

So we conclude that $h^-(s,\cdot)$ satisfies the Jensen equation in $\mathbb{R}$ (see [28], page 315). The continuity of h with respect of the second variable implies that for every $t\in[a,b]$ there exist $\alpha,\beta:[a,b]\to\mathbb{R}$, such that

$$h^-(t,x)=\alpha(t)x+\beta(t),\quad(t\in[a,b],x\in\mathbb{R}).\tag{89}$$

Since $\beta(t)=h^-(t,0)$, $t\in[a,b]$, $\alpha(t)=h^-(t,1)-\beta(t)$ and $h^-(\cdot,x)\in\kappa\mathrm{BV}_\phi^-[a,b]$, for each $x\in\mathbb{R}$, we obtain that $\alpha,\beta\in\kappa\mathrm{BV}_\phi[a,b]$. □

Corollary 23. *Let $\kappa:[0,1]\to[0,1]$ be a distortion function, $\phi=\{\varphi_n\}_{n\geq1}$ a ϕ-sequence, and $h:[a,b]\times\mathbb{R}\to\mathbb{R}$. If the composition operator H, associated with h, maps $\kappa\mathrm{BV}_\phi[a,b]$ in itself and satisfies the inequality*

$$\|H(u)-H(v)\|_{\kappa\phi}\leq\gamma\left(\|u-v\|_{\kappa\phi}\right),\quad\left(u,v\in\kappa\mathrm{BV}_\phi[a,b]\right),\tag{90}$$

for any function $\gamma:[0,\infty)\to[0,\infty)$, verifying $\gamma(t)\to\gamma(0)=0$, when $t\downarrow0$, then there exists $\alpha,\beta\in\kappa\mathrm{BV}_\phi^-[a,b]$, such that

$$h^-(t,x)=\alpha(t)x+\beta(t),\quad(t\in[a,b],x\in\mathbb{R}).\tag{91}$$

Proof. Let us fix $x,y\in\mathbb{R}$, $x\neq y$, and define $u(t):=x$, $v(t):=y$, $t\in[a,b]$. Then $u,v\in\kappa\mathrm{BV}_\phi[a,b]$ and $\|u-v\|_{\kappa\phi}=|x-y|$. Then from inequality (90) and Lemma 16, we have

$$\varphi_1\left(\frac{|h(t,x)-h(t,y)-h(a,x)+h(a,y)|}{\gamma(|x-y|)}\right)\leq\kappa\left(\frac{t-a}{b-a}\right)+\kappa\left(\frac{b-t}{b-a}\right),\quad(t\in(a,b)).\tag{92}$$

And therefore

$$|h(t,x)-h(t,y)-h(a,x)+h(a,y)|\leq\varphi_1^{-1}(2)\ \gamma(|x-y|).\tag{93}$$

Proceeding as in the proof of Theorem 22 we have that there is left regularization of $h^-(\cdot,x)$, for each $x\in\mathbb{R}$ and if $a\leq r<a_1<b_1<a_2<\cdots<a_m<b_m=s\leq b$, $x,y\in\mathbb{R}$, $x\neq y$ we defined the zig-zag continuous functions $u,v:[a,b]\to\mathbb{R}$, as show below

$$u(t)=\begin{cases}x, & a\leq t\leq a_1\ \text{ or } t=a_i,\ i=1,\dots,m\\ y, & b_m\leq t\leq b\ \text{or } t=b_i,\ i=1,\dots,m\\ \text{affine}, & \text{other case},\end{cases}$$
$$u(t)=\begin{cases}y, & a\leq t\leq a_1\ \text{ or } t=a_i,\ i=1,\dots,m\\ x, & b_m\leq t\leq b\ \text{or } t=b_i,\ i=1,\dots,m\\ \text{affine}, & \text{other case},\end{cases}\tag{94}$$

then $u,v\in\kappa\mathrm{BV}_\phi[a,b]$, $(u-v)(t)=x-y$, $t\in[a,b]$ and for $s\in(a,b]$

$$\varphi_m\left(\frac{2|h^-(s,x)-h^-(s,y)|}{\gamma(|x-y|)}\right)\leq\sum_{n=1}^{m}\varphi_n\left(\frac{2|h^-(s,x)-h^-(s,y)|}{\gamma(|x-y|)}\right)\leq\kappa\left(\frac{s-a}{b-a}\right)+\kappa\left(\frac{b-s}{b-a}\right)\leq2.\tag{95}$$

Hence,

$$|h^-(s,x)-h^-(s,y)|\leq\frac{\varphi_m^{-1}(2)}{2}\gamma(|x-y|),\quad(s\in(a,b)).\tag{96}$$

And so $h^-(\cdot,x)$ is continuous in $(a,b]$. Now the result is a consequence of Theorem 22. □

Corollary 24. *Let $\kappa:[0,1]\to[0,1]$ be a distortion function, $\phi=\{\varphi_n\}_{n\geq1}$ a ϕ-sequence, $h:[a,b]\times\mathbb{R}\to\mathbb{R}$, and H the composition operator associated with h. Suppose that H maps $\kappa\mathrm{BV}_\phi[a,b]$ into itself and is uniformly continuous, then there exist $\alpha,\beta\in\kappa\mathrm{BV}_\phi^-[a,b]$, such that*

$$h^-(t,x)=\alpha(t)x+\beta(t),\quad(t\in[a,b],x\in\mathbb{R}),\tag{97}$$

where $h^-(\cdot,x)$ is the left regularization of $h(\cdot,x)$ for all $x\in\mathbb{R}$.

Proof. Consider the modulus of continuity associated with H; that is,

$$\gamma(t):=\sup\left\{\|H(u)-H(v)\|_{\kappa\phi}:\|u-v\|_{\kappa\phi}\leq t,\ u,v\in\kappa\mathrm{BV}_\phi[a,b]\right\},\quad(t\geq0).\tag{98}$$

Then $\gamma(t) \geq 0, \gamma(0) = 0$. Furthermore, if $t > 0$ and $\|u-v\|_{\kappa\phi} \leq t$, we obtain

$$\|H(u) - H(v)\|_{\kappa\phi} \leq \gamma(t). \quad (99)$$

Particularly, in the case where $t = \|u-v\|_{\kappa\phi}$, we have

$$\|H(u) - H(v)\|_{\kappa\phi} \leq \gamma(\|u-v\|_{\kappa\nu}), \quad (u, v \in \kappa\mathrm{BV}_{\phi}[a,b]). \quad (100)$$

From Corollary 23 we obtain the conclusion. □

Matkowski [27] introduced the notion of a uniformly bounded operator and proved that the generator of any uniformly bounded composition operator acting between general Lipschitz function normed spaces must be affine with respect to the function variable.

Definition 25. Let $\mathbb{Y}$ and $\mathbb{Z}$ be two metric (or normed) spaces. We say that a mapping $F : \mathbb{Y} \to \mathbb{Z}$ is *uniformly bounded*, if for any $t > 0$, there exists a nonnegative real number $\gamma(t)$ such that for any nonempty set $B \subset \mathbb{Y}$ we have

$$\operatorname{diam} B \leq t \Longrightarrow \operatorname{diam} F(B) \leq \gamma(t). \quad (101)$$

Remark 26. Every uniformly continuous operator or Lipschitzian operator is uniformly bounded.

Corollary 27. *Let $\kappa : [0,1] \to [0,1]$ be a distortion function, $\phi = \{\varphi_n\}_{n\geq 1}$ a ϕ-sequence, and $h : [a,b] \times \mathbb{R} \to \mathbb{R}$, such that function $h(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous in respect to the second variable, for each $t \in (a,b]$. If the composition operator H, associated with h, maps $\kappa\mathrm{BV}_{\phi}[a,b]$ in itself and is uniformly bounded, then exists $\alpha, \beta \in \kappa\mathrm{BV}_{\phi}^{-}[a,b]$, such that*

$$h^{-}(t,x) = \alpha(t)x + \beta(t), \quad (t \in [a,b], x \in \mathbb{R}). \quad (102)$$

Proof. Take any $t > 0$ and $u, v \in \kappa\mathrm{BV}_{\phi}[a,b]$ such that $\|u-v\|_{\kappa\phi} \leq t$. Since diam $\{u,v\} \leq t$, by uniform boundedness of H, we have diam $H(\{u,v\}) \leq \gamma(t)$; that is,

$$\|H(u) - H(v)\|_{\kappa\phi} = \operatorname{diam} H(\{u,v\}) \leq \gamma(\|u-v\|_{\kappa\phi}). \quad (103)$$

From Theorem 22 we have (102). □

Acknowledgment

This research has been party supported by the Central Bank of Venezuela. The authors want to thank the library staff of BCV for compiling the references.

References

[1] J. Appell, N. Guanda, and M. Väth, "Function spaces with the Matkowski property and degeneracy phenomena for composition operators," *Fixed Point Theory*, vol. 12, no. 2, pp. 265–284, 2011.

[2] J. Appell, Z. Jesús, and O. Mejía, "Some remarks on nonlinear composition operators in spaces of differentiable functions," *Bollettino Della Unione Matematica Italiana*, vol. 4, no. 3, pp. 321–336, 2011.

[3] A. A. Babaev, "On the structure of a certain nonlinear operator and itsapplication," *Uchen. Zap. Azerbajdzh. Gos. Univ*, vol. 4, pp. 13–16, 1961 (Russian).

[4] K. S. Mukhtarov, "On the properties of theoperator $F_u = f(u(x))$ in the space H_{φ}," *Sbornik Nauchm. Rabot Mat. Kaf. Dagestan Univ*, pp. 145–150, 1967 (Russian).

[5] M. Josephy, "Composing functions of bounded variation," *Proceedings of the American Mathematical Society*, vol. 83, no. 2, pp. 354–356, 1981.

[6] J. Ciemnoczołowski and W. Orlicz, "Composing functions of bounded ϕ-variation," *Proceedings of the American Mathematical Society*, vol. 96, no. 3, pp. 431–436, 1986.

[7] M. Chaika and D. Waterman, "On the invariance of certain classes of functions under composition," *Proceedings of the American Mathematical Society*, vol. 43, pp. 345–348, 1974.

[8] N. Merentes, "On the composition operator in $AC[a,b]$," *Collectanea Mathematica*, vol. 42, no. 3, pp. 231–238, 1991.

[9] N. Merentes, "On the composition operator in $RV_{\varphi}[a,b]$," *Collectanea Mathematica*, vol. 46, no. 3, pp. 231–238, 1995.

[10] N. Merentes and S. Rivas, "On the composition operator between $RV_p[a,b]$ and $BV[a,b]$," *Studia Mathematica*, vol. 1, no. 3, pp. 237–292, 1998.

[11] P. B. Pierce and D. Waterman, "On the invariance of classes ϕBV, ΛBV under composition," *Proceedings of the American Mathematical Society*, vol. 132, no. 3, pp. 755–760, 2004.

[12] J. Appell and N. Merentes, "Composing functions of bounded Koremblum variation," *Dynamic Systems and Applications*, vol. 22, pp. 197–206, 2013.

[13] W. Aziz, N. Merentes, and J. L. Sánchez, "A note on the composition of regularfunctions," *Zeitschrift für Analysis und ihre Anwendungen*. In press.

[14] J. Matkowski, "Functional equations and Nemytskiĭ operators," *Fako de l'Funkcialaj Ekvacioj Japana Matematika Societo*, vol. 25, no. 2, pp. 127–132, 1982.

[15] J. Appell, N. Guanda, N. Merentes, and J. L. Sánchez, "Some boundedness and continuity properties of nonlinear composition operators: a survey," *Communications on Pure and Applied Analysis*, vol. 15, no. 2–4, pp. 153–182, 2011.

[16] J. Matkowski and J. Miś, "On a characterization of Lipschitzian operators of substitution in the space $BV[a,b]$," *Mathematische Nachrichten*, vol. 117, pp. 155–159, 1984.

[17] C. Jordan, "Sur la série de Fourier," *Comptes Rendus De L'Académie Des Sciences*, pp. 228–230, 1881.

[18] J. Appell, J. Banas, and N. Merentes, Bounded variation and around. Gruyter. Series in nonlinear analysis and application.

[19] M. Avdispahić, "Concepts of generalized bounded variation and the theory of Fourier series," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 2, pp. 223–244, 1986.

[20] N. Merentes and S. Rivas, *El Operador De Composición En Espacios Con Algún Tipo De Variación Acotada*, IX Escuela Venezolana de Matemáticas, Mérida, Venezuela, 1996.

[21] B. Korenblum, "An extension of the Nevanlinna theory," *Acta Mathematica*, vol. 135, no. 3-4, pp. 187–219, 1975.

[22] D. S. Cyphert and J. A. Kelingos, "The decomposition of functions of bounded κ-variation into differences of κ-decreasing functions," *Polska Akademia Nauk*, vol. 81, no. 2, pp. 185–195, 1985.

[23] M. Schramm, "Functions of Φ-bounded variation and Riemann-Stieltjes integration," *Transactions of the American Mathematical Society*, vol. 287, no. 1, pp. 49–63, 1985.

[24] S. K. Kim and J. Kim, "Functions of $\kappa\phi$-bounded variation," *Bulletin of the Korean Mathematical Society*, vol. 23, no. 2, pp. 171–175, 1986.

[25] M. Castillo, M. Sanoja, and I. Zea, "The spaces of functions of bounded κ-variation in the sense ofRiesz-Koremblum," *Journal of Mathematical Control Science and Applications*. In press.

[26] M. Castillo, S. Rivas, M. Sanoja, and I. Zea, "functions of bounded K_φ-variation in the sense of Riesz-Korenblum," *Journal of Function Spaces and Applications*, vol. 2013, Article ID 718507, 12 pages, 2013.

[27] J. Matkowski, "Uniformly bounded composition operators between general Lipschitz function normed spaces," *Topological Methods in Nonlinear Analysis*, vol. 38, no. 2, pp. 395–406, 2011.

[28] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, Polish Scientific Editors and Silesian University, Warszawa, Poland, 1885.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 473876, 9 pages
http://dx.doi.org/10.1155/2013/473876

Research Article

Abstract Semilinear Evolution Equations with Convex-Power Condensing Operators

Lanping Zhu, Qianglian Huang, and Gang Li

School of Mathematics, Yangzhou University, Yangzhou 225002, China

Correspondence should be addressed to Lanping Zhu; zlpmath@aliyun.com

Received 14 May 2013; Revised 16 August 2013; Accepted 18 August 2013

Academic Editor: Ji Gao

By using the techniques of convex-power condensing operators and fixed point theorems, we investigate the existence of mild solutions to nonlocal impulsive semilinear differential equations. Two examples are also given to illustrate our main results.

1. Introduction

This paper is concerned with the existence of mild solutions for the following impulsive semilinear differential equations with nonlocal conditions

$$\begin{gathered} u'(t) = Au(t) + f(t, u(t)), \quad t \in [0, T], \ t \neq t_i, \\ u(0) = g(u), \\ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, p, \\ 0 < t_1 < t_2 < \cdots < t_p < T, \end{gathered} \tag{1}$$

where $A : D(A) \subseteq X \rightarrow X$ is the infinitesimal generator of strongly continuous semigroup $S(t)$ for $t > 0$ in a real Banach space X and $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ constitutes an impulsive condition. f and g are X-valued functions to be given later.

As far as we know, the first paper dealing with abstract nonlocal initial value problems for semilinear differential equations is due to [1]. Because nonlocal conditions have better effect in the applications than the classical initial ones, many authors have studied the following type of semilinear differential equations under various conditions on $S(t)$, f, and g:

$$\begin{gathered} u'(t) = Au(t) + f(t, u(t)), \quad t \in [0, T], \\ u(0) = g(u). \end{gathered} \tag{2}$$

For instance, Byszewski and Lakshmikantham [2] proved the existence and uniqueness of mild solutions for nonlocal semilinear differential equations when f and g satisfy Lipschitz type conditions. In [3], Ntouyas and Tsamatos studied the case with compactness conditions. Byszewski and Akca [4] established the existence of solution to functionaldifferential equation when the semigroup is compact and g is convex and compact on a given ball. Subsequently, Benchohra and Ntouyas [5] discussed the second-order differential equation under compact conditions. The fully nonlinear case was considered by Aizicovici and McKibben [6], Aizicovici and Lee [7], Aizicovici and Staicu [8], García-Falset [9], Paicu and Vrabie [10], Obukhovski and Zecca [11], and Xue [12, 13].

Recently, the theory of impulsive differential inclusions has become an important object of investigation because of its wide applicability in biology, medicine, mechanics, and control theory and in more and more fields. Cardinali and Rubbioni [14] studied the multivalued impulsive semilinear differential equation by means of the Hausdorff measure of noncompactness. Liang et al. [15] investigated the nonlocal impulsive problems under the assumptions that g is compact, Lipschitz, and g is not compact and not Lipschitz, respectively. All these studies are motivated by the practical interests of nonlocal impulsive Cauchy problems. For a more detailed bibliography and exposition on this subject, we refer to [14–18].

The present paper is motivated by the following facts. Firstly, the approach used in [9, 12, 13, 19, 20] relies on the assumption that the coefficient l of the function f about the measure of noncompactness satisfies a strong inequality, which is difficult to be verified in applications. Secondly, in

[21], it seems that authors have considered the inequality restriction on coefficient function $l(t)$ of f may be relaxed for impulsive nonlocal differential equations. However, in fact, they only solve the classical initial value problems $u(0) = u_0$ rather than the nonlocal initial problems $u(0) = u_0 + g(u)$. For more details, one can refer to the proof of Theorem 3.1 in [21] (see the inequalities (3.3) and (3.4) in page 5 and the estimations of the measure of noncompactness in page 6 and page 7 of [21]).

Therefore, we will continue to discuss the impulsive nonlocal differential equations under more general assumptions. Throughout this work, we mainly use the property of convex-power condensing operators and fixed point theorems to obtain the main result (Theorem 10). Indeed, the fixed point theorem about the convex-power condensing operators is an extension for Darbo-Sadovskii's fixed point theorem. But the former seems more effective than the latter at times for some problems. For example, in [22] we ever applied the former to study the nonlocal Cauchy problem and obtained more general and interesting existence results. Based on the results obtained, we discuss the impulsive nonlocal differential equations. Fortunately, applying the techniques of convex-power condensing operators and fixed point theorems solves the difficulty involved by coefficient restriction that is, the constraint condition for the coefficient function $l(t)$ of f is unnecessary (see Theorem 10). Therefore, our results generalize and improve many previous ones in this field, such as [9, 12, 13, 19, 20].

The outline of this paper is as follows. In Section 2, we recall some concepts and facts about the measure of noncompactness, fixed point theorems, and impulsive semilinear differential equations. In Section 3, we obtain the existence results of (1) when g is compact in $PC([0,T];X)$. In Section 4, we discuss the existence result of (1) when g is Lipschitz continuous, while Section 5 contains two illustrating examples.

2. Preliminaries

Let E be a real Banach space, we introduce the Hausdorff measure of noncompactness α defined on each bounded subset Ω of E by

$$\alpha(\Omega) = \inf \Big\{ r > 0;\ \text{there are finite points } x_1, x_2, \dots, x_n \in E \text{ with } \Omega \subset \bigcup_{i=1}^{n} B(x_i, r) \Big\}. \tag{3}$$

Now we recall some basic properties of the Hausdorff measure of noncompactness.

Lemma 1. *For all bounded subsets Ω, Ω_1, and Ω_2 of E, the following properties are satisfied:*

(1) Ω *is precompact if and only if* $\alpha(\Omega) = 0$;

(2) $\alpha(\Omega) = \alpha(\overline{\Omega}) = \alpha(\operatorname{co}\Omega)$, *where* $\overline{\Omega}$ *and* $\operatorname{co}\Omega$ *mean the closure and convex hull of* Ω, *respectively;*

(3) $\alpha(\Omega_1) \le \alpha(\Omega_2)$ *when* $\Omega_1 \subset \Omega_2$;

(4) $\alpha(\Omega_1 \bigcup \Omega_2) \le \max\{\alpha(\Omega_1), \alpha(\Omega_2)\}$;

(5) $\alpha(\lambda\Omega) = |\lambda|\alpha(\Omega)$, *for any* $\lambda \in R$;

(6) $\alpha(\Omega_1 + \Omega_2) \le \alpha(\Omega_1) + \alpha(\Omega_2)$, *where* $\Omega_1 + \Omega_2 = \{x + y; x \in \Omega_1, y \in \Omega_2\}$;

(7) *if* $\{W_n\}_{n=1}^{+\infty}$ *is a decreasing sequence of nonempty bounded closed subsets of* E *and* $\lim_{n\to\infty}\alpha(W_n) = 0$, *then* $\cap_{n=1}^{+\infty} W_n$ *is nonempty and compact in* E.

The map $Q : D \subset E \to E$ is said to be α-condensing if for every bounded and not relatively compact $B \subset D$, we have $\alpha(QB) < \alpha(B)$ (see [23]).

Lemma 2 (see [9]: Darbo-Sadovskii). *If $D \subset E$ is bounded closed and convex, the continuous map $Q : D \to D$ is α-condensing, then the map Q has at least one fixed point in D.*

In the sequel, we will continue to generalize the definition of condensing operator. First of all, we give some notations.

Let $D \subset E$ be bounded closed and convex, the map $Q : D \to D$, and $x_0 \in D$ for every $B \subset D$, set

$$\begin{gathered} Q^{(1,x_0)}(B) = Q(B), \\ Q^{(n,x_0)}B = Q\left(\overline{\operatorname{co}}\left\{Q^{(n-1,x_0)}B, x_0\right\}\right), \quad n = 2, 3, \dots, \end{gathered} \tag{4}$$

where $\overline{\operatorname{co}}$ means the closure of the convex hull.

Now we give the definition of a kind of new operator.

Definition 3. Let $D \subset E$ be bounded closed and convex, the map $Q : D \to D$ is said to be α-convex-power condensing if there exist $x_0 \in D$, $n_0 \in N$ and for every bounded and not relatively compact $B \subset D$, we have

$$\alpha\left(Q^{(n_0,x_0)}(B)\right) < \alpha(B). \tag{5}$$

From this definition, if $\alpha(Q^{(n_0,x_0)}(B)) = \alpha(B)$, one obtains $B \subset E$ as relatively compact.

Subsequently, we give the fixed point theorem about the convex-power condensing operator.

Lemma 4 (see [23]). *If $D \subset E$ is bounded closed and convex, the continuous map $Q : D \to D$ is α-convex-power condensing, then the map Q has at least one fixed point in D.*

Throughout this paper, let $(X, \|\cdot\|)$ be a real Banach space. We denote by $C([0,T];X)$ the Banach space of all continuous functions from $[0,T]$ to X with the norm $\|u\| = \sup\{\|u(t)\|, t \in [0,T]\}$ and by $L^1([0,T];X)$ the Banach space of all X-valued Bochner integrable functions defined on $[0,T]$ with the norm $\|u\|_1 = \int_0^T \|u(t)\|dt$. Let $PC([0,T];X) = \{u : u$ is a function from $[0,T]$ into X such that $u(t)$ is continuous at $t \ne t_i$ and the left continuous at $t = t_i$ and the right limit $u(t_i^+)$ exists for $i = 1, 2, \dots, p\}$. It is easy to check that $PC([0,T];X)$ is a Banach space with the norm $\|u\|_{PC} = \sup\{\|u(t)\|, t \in [0,T]\}$ and $C([0,T];X) \subseteq PC([0,T];X) \subseteq L^1([0,T];X)$. Moreover, we denote β by the

Hausdorff measure of noncompactness of X, denote β_c by the Hausdorff measure of noncompactness of $C([0,T];X)$ and denote β_{pc} by the Hausdorff measure of noncompactness of $PC([0,T];X)$.

Throughout this work, we suppose the following

(H_A) The linear operator $A : D(A) \subseteq X \to X$ generates an equicontinuous C_0-semigroup $\{S(t) : t \geq 0\}$. Hence, there exists a positive number M such that $\|S(t)\| \leq M$.

For further information about the theory of semigroup of operators, we may refer to some classic books, such as [24–26].

To discuss the problem (1), we also need the following lemma.

Lemma 5. *If $W \subseteq PC([0,T];X)$ is bounded, then one has*

$$\sup_{t\in[0,T]} \beta(W(t)) \leq \beta_{pc}(W), \tag{6}$$

where $W(t) = \{u(t); u \in W\} \subset X$.

Lemma 6 (see [27]). *If $W \subseteq C([0,T];X)$ is bounded, then for all $t \in [0,T]$,*

$$\beta(W(t)) \leq \beta_c(W). \tag{7}$$

Furthermore, if W is equicontinuous on $[0,T]$, then $\beta(W(t))$ is continuous on $[0,T]$ and

$$\beta_c(W) = \sup\{\beta(W(t)) : t \in [0,T]\}. \tag{8}$$

Since C_0-semigroup $S(t)$ is said to be equicontinuous, the following lemma is easily checked.

Lemma 7. *If the semigroup $S(t)$ is equicontinuous and $w \in L^1([0,T];R^+)$, then the set $\{\int_0^t S(t-s)u(s)ds, \|u(s)\| \leq w(s)$ for a.e. $s \in [0,T]\}$ is equicontinuous for $t \in [0,T]$.*

Definition 8. A function $u \in PC([0,T];X)$ is said to be a mild solution of the nonlocal problem (1), if it satisfies

$$\begin{aligned} u(t) &= S(t)g(u) + \int_0^t S(t-s) f(s,u(s))\, ds \\ &\quad + \sum_{0<t_i<t} S(t-t_i) I_i(u(t_i)), \quad 0 \leq t \leq T. \end{aligned} \tag{9}$$

In addition, let r be a finite positive constant, and set $B_r := \{x \in X : \|x\| \leq r\}$ and $W_r := \{u \in C([0,T];X) : u(t) \in B_r$, for all $t \in [0,T]\}$.

3. g Is Compact

In this section, we state and prove the existence theorems for the nonlocal impulsive problem (1). First, we give the following hypotheses:

(H_f)

(1) $f : [0,T] \times X \to X$ is a Carathéodory function; that is, for all $x \in X$, $f(\cdot,x) : [0,T] \to X$ is measurable and for a.e. $t \in [0,T]$, $f(t,\cdot) : X \to X$ is continuous;

(2) for finite positive constant $r > 0$, there exists a function $\alpha_r \in L^1(0,T;R)$ such that

$$\|f(t,x)\| \leq \alpha_r(t) \tag{10}$$

for a.e. $t \in [0,T]$ and $x \in B_r$;

(3) there exists a function $l(t) \in L^1(0,T;R)$ such that

$$\beta(f(t,D)) \leq l(t)\beta(D) \tag{11}$$

for a.e. $t \in [0,T]$ and every bounded subset $D \subset B_r$;

(H_g) $g : PC([0,T];X) \to X$ is a continuous and compact mapping; furthermore, there exists a positive number N_1 such that $\|g(u)\| \leq N_1$, for any $u \in W_r$;

(H_I) $I_i : X \to X$ is a continuous and compact mapping for every $i = 1,2,\ldots,p$;

(H_r) $M(N_1 + \|\alpha_r\|_{L^1} + \sup_{u\in W_r}\sum_{i=1}^{p}\|I_i(u(t_i))\|) \leq r$.

Remark 9. The mapping f is said to be L^1-Carathéodory if the assumption $(H_f)(1)(2)$ is satisfied.

Theorem 10. *If the hypotheses (H_A), $(H_f)(1)(2)(3)$, (H_g), (H_I), and (H_r) are satisfied, then the nonlocal problem* (1) *has at least one mild solution on* $[0,T]$.

To prove the above theorem, we need the following lemma.

Lemma 11. *If the condition (H_r) holds, then for arbitrary bounded set $B \subset W_r$, we have*

$$\begin{aligned} &\beta\left(\int_0^t S(t-s) f(s,B(s))\, ds\right) \\ &\quad \leq 4M\int_0^t \beta(f(s,B(s)))\, ds, \quad t \in [0,T]. \end{aligned} \tag{12}$$

This proof is quite similar to that of Lemma 3.1 in [20]; we omit it.

Proof of Theorem 10. We consider the operator $Q : PC([0,T];X) \to PC([0,T];X)$ defined by

$$\begin{aligned} (Qu)(t) &= S(t)g(u) + \int_0^t S(t-s) f(s,u(s))\, ds \\ &\quad + \sum_{0<t_i<t} S(t-t_i) I_i(u(t_i)), \quad 0 \leq t \leq T. \end{aligned} \tag{13}$$

It is easy to see that the fixed points of Q are the mild solutions of nonlocal impulsive semilinear differential equation (1). Subsequently, we shall prove that Q has a fixed point by using Lemma 4.

We shall first prove that Q is continuous on $PC([0,T];X)$. In fact, let $\{u_n\}_{n=1}^{+\infty} \subset PC([0,T];X)$ be an arbitrary sequence satisfying $\lim_{n\to+\infty}u_n = u$ in $PC([0,T];X)$. It follows from Definition 8 that

$$\|Qu_n - Qu\| \le M\left(\|g(u_n) - g(u)\| + \int_0^T \|f(s,u_n(s)) - f(s,u(s))\| ds + \sum_{i=1}^{p} \|I_i(u_n(t_i)) - I_i(u(t_i))\| \right). \tag{14}$$

According to the continuity of f in its second argument, for each $s \in [0,T]$, we have the following:

$$\lim_{n\to+\infty} f(s,u_n(s)) = f(s,u(s)). \tag{15}$$

In addition, g and I_i are all continuous for each $i = 1,2,\ldots,p$, and hence, the Lebesgue dominated convergence theorem implies

$$\lim_{n\to+\infty} \|Qu_n - Qu\| = 0. \tag{16}$$

Namely, Q is continuous on $PC([0,T];X)$.

Subsequently, we claim that $QW_r \subseteq W_r$. Actually, by (H_r), we obtain

$$\begin{aligned}\|(Qu)(t)\| &\le \|S(t)g(u)\| + \int_0^t \|S(t-s)f(s,u(s))\| ds + \sum_{0<t_i<t} \|S(t-t_i)I_i(u(t_i))\| \\ &\le M\left(N_1 + \|\alpha_r\|_{L^1} + \sup_{u\in W_r}\sum_{i=1}^{p} \|I_i(u(t_i))\|\right) \\ &\le r,\end{aligned} \tag{17}$$

for any $u \in W_r \subseteq PC([0,T];X)$. Thus, $QW_r \subseteq W_r$.

Now we demonstrate that QW_r is equicontinuous for any $t > 0$. Let $t \in (0,T]$, $\varepsilon > 0$. Since g is compact, $g(W_r)$ is relatively compact; that is, there is a finite family $\{x_j\}_{j=1}^{m} \subset g(W_r)$ such that for any $u \in W_r$, there exists some $j \in \{1,2,\ldots,m\}$ such that

$$2M\|g(u) - x_j\| \le \frac{\varepsilon}{5}. \tag{18}$$

On the other hand, as $S(t)$ is equicontinuous at $t > 0$, we can choose $\delta_1 \in (0,t)$ such that

$$\begin{aligned}&\|(S(t+\delta') - S(t))x_j\| \le \frac{\varepsilon}{5}, \\ &\sum_{i=1}^{p} \|(S(t+\delta'-t_i) - S(t-t_i))I_i(u(t_i))\| \le \frac{\varepsilon}{5},\end{aligned} \tag{19}$$

for each $\delta' \in R$, $|\delta'| < \delta_1$, uniformly for $u \in W_r$, and $j \in \{1,2,\ldots,m\}$. By $(H_f(2))$, it can be obtained that there exists $\delta_2 \in (0,t)$ such that

$$M\int_t^{t+\delta'} \|f(s,u(s))\| ds \le \frac{\varepsilon}{5}, \tag{20}$$

for each $\delta' \in R$, $|\delta'| < \delta_2$, uniformly for $u \in W_r$. Furthermore, by Lemma 7, we get that there exists $\delta_3 \in (0,t)$ such that

$$\int_0^t \|(S(t+\delta'-s) - S(t-s))f(s,u(s))\| ds \le \frac{\varepsilon}{5}, \tag{21}$$

for each $\delta' \in R$, $|\delta'| < \delta_3$, uniformly for $u \in W_r$. Thus, there exists $h = \min\{\delta_1,\delta_2,\delta_3\}$ such that

$$\begin{aligned}&\|(Qu)(t+h') - (Qu)(t)\| \\ &\le \|(S(t+h') - S(t))g(u)\| + \sum_{i=1}^{p} \|(S(t+h'-t_i) - S(t-t_i))I_i(u(t_i))\| \\ &\quad + \int_0^t \|(S(t+h'-s) - S(t-s))f(s,u(s))\| ds + \int_t^{t+h'} \|S(t+h'-s)f(s,u(s))\| ds \\ &\le \|(S(t+h') - S(t))(g(u) - x_j)\| + \|(S(t+h') - S(t))x_j\| \\ &\quad + \sum_{i=1}^{p} \|(S(t+h'-t_i) - S(t-t_i))I_i(u(t_i))\| \\ &\quad + \int_0^t \|(S(t+h'-s) - S(t-s))f(s,u(s))\| ds + \int_t^{t+h'} \|S(t+h'-s)f(s,u(s))\| ds \\ &\le 2M\|g(u) - x_j\| + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + M\int_t^{t+h'} \|f(s,u(s))\| ds \\ &\le \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} \\ &= \varepsilon,\end{aligned} \tag{22}$$

for each $h' \in R$, $|h'| < h$, uniformly for $u \in W_r$. Therefore, QW_r is equicontinuous at $t > 0$.

Similarly, we can conclude that QW_r is also equicontinuous at $t = 0$. Thus, QW_r is equicontinuous on $[0,T]$.

Set $W = \overline{\text{co}}(QW_r)$. It is obvious that W is equicontinuous on $[0,T]$ and Q maps W into itself.

Next, we shall prove that $Q : W \to W$ is a convex-power condensing operator. Take $x_0 \in W$; by the definition of convex-power condensing operator, we shall show that there exists a positive integral n_0 such that

$$\beta_{pc}\left(Q^{(n_0,x_0)}B\right) < \beta_{pc}(B) \tag{23}$$

if $B \subset W$ is not relatively compact. In fact, by using the conditions (H_g) and (H_I), we get from Lemma 11 that

$$\begin{aligned}
\beta\left(\left(Q^{(1,x_0)}B\right)(t)\right) &= \beta\left((QB)(t)\right) \\
&\leq \beta\left(S(t)\,g(B)\right) \\
&\quad + \beta\left(\int_0^t S(t-s)\,f(s,B(s))\,ds\right) \\
&\quad + \beta\left(\sum_{i=1}^{p} S(t-t_i)\,I_i(B(t_i))\right) \\
&= 4M\int_0^t \beta\left(f(s,B(s))\right)ds \\
&\leq 4M\int_0^t l(s)\,\beta(B(s))\,ds \\
&\leq 4M\int_0^t l(s)\,ds\beta_{pc}(B).
\end{aligned} \tag{24}$$

Since $l(t) \in L^1(0,T;R^+)$, there exists a continuous function $\omega : [0,T] \rightarrow R^1$ such that for any $0 < \varepsilon < 1$,

$$\int_0^T |l(s)-\omega(s)|\,ds < \varepsilon. \tag{25}$$

Then

$$\begin{aligned}
&\beta\left(\left(Q^{(1,x_0)}B\right)(t)\right) \\
&\leq 4M\left(\int_0^t |l(s)-\omega(s)|\,ds + \int_0^t |\omega(s)|\,ds\right)\beta_{pc}(B) \\
&\leq 4M\left(\varepsilon + N_2 t\right)\beta_{pc}(B),
\end{aligned} \tag{26}$$

where $N_2 = \max\{|\omega(t)| : t \in [0,T]\}$. Hence,

$$\begin{aligned}
&\beta\left(\left(Q^{(2,x_0)}B\right)(t)\right) \\
&\leq \beta\left(S(t)\,g\left(\overline{\mathrm{co}}\left\{\left(Q^{(1,x_0)}B\right),x_0\right\}\right)\right) \\
&\quad + \beta\left(\int_0^t S(t-s)\,f\left(s,\overline{\mathrm{co}}\left\{\left(Q^{(1,x_0)}B\right)(s),x_0(s)\right\}\right)ds\right) \\
&\quad + \beta\left(\sum_{i=1}^{p} S(t-t_i)\,I_i\left(\overline{\mathrm{co}}\left\{\left(Q^{(1,x_0)}B\right)(t_i),x_0(t_i)\right\}\right)\right) \\
&= \beta\left(\int_0^t S(t-s)\,f\left(s,\overline{\mathrm{co}}\left\{\left(Q^{(1,x_0)}B\right)(s),x_0(s)\right\}\right)ds\right) \\
&\leq 4M\int_0^t \beta\left(f\left(s,\overline{\mathrm{co}}\left\{\left(Q^{(1,x_0)}B\right)(s),x_0(s)\right\}\right)\right)ds \\
&\leq 4M\int_0^t l(s)\,\beta\left(\left(Q^{(1,x_0)}B\right)(s)\right)ds \\
&\leq 4M\int_0^t \left(|l(s)-\omega(s)| + |\omega(s)|\right)4M\left(\varepsilon + N_2 s\right)\beta_{pc}(B)\,ds \\
&\leq 4^2M^2\left[\varepsilon\left(\varepsilon + N_2 t\right) + N_2\left(t\varepsilon + N_2\frac{t^2}{2}\right)\right]\beta_{pc}(B) \\
&= 4^2M^2\left[\varepsilon^2 + C_2^1\varepsilon\left(N_2 t\right) + \frac{\left(N_2 t\right)^2}{2!}\right]\beta_{pc}(B).
\end{aligned} \tag{27, 28}$$

Thus,

$$\begin{aligned}
&\beta\left(\left(Q^{(3,x_0)}B\right)(t)\right) \\
&\leq \beta\left(S(t)\,g\left(\overline{\mathrm{co}}\left\{\left(Q^{(2,x_0)}B\right),x_0\right\}\right)\right) \\
&\quad + \beta\left(\int_0^t S(t-s)\,f\left(s,\overline{\mathrm{co}}\left\{\left(Q^{(2,x_0)}B\right)(s),x_0(s)\right\}\right)ds\right) \\
&\quad + \beta\left(\sum_{i=1}^{p} S(t-t_i)\,I_i\left(\overline{\mathrm{co}}\left\{\left(Q^{(2,x_0)}B\right)(t_i),x_0(t_i)\right\}\right)\right) \\
&= \beta\left(\int_0^t S(t-s)\,f\left(s,\overline{\mathrm{co}}\left\{\left(Q^{(2,x_0)}B\right)(s),x_0(s)\right\}\right)ds\right) \\
&\leq 4M\int_0^t \beta\left(f\left(s,\overline{\mathrm{co}}\left\{\left(Q^{(2,x_0)}B\right)(s),x_0(s)\right\}\right)\right)ds \\
&\leq 4M\int_0^t l(s)\,\beta\left(\left(Q^{(2,x_0)}B\right)(s)\right)ds \\
&\leq 4M\int_0^t \left(|l(s)-\omega(s)| + |\omega(s)|\right) \\
&\quad \times 4^2M^2\left[\varepsilon^2 + C_2^1\varepsilon\left(N_2 s\right) + \frac{\left(N_2 s\right)^2}{2!}\right]ds\beta_{pc}(B) \\
&\leq 4^3M^3\left[\varepsilon\left(\varepsilon^2 + C_2^1\varepsilon\left(N_2 t\right) + \frac{\left(N_2 t\right)^2}{2!}\right)\right. \\
&\qquad \left. + N_2\int_0^t\left(\varepsilon^2 + C_2^1\varepsilon\left(N_2 s\right) + \frac{\left(N_2 s\right)^2}{2!}\right)ds\right]\beta_{pc}(B) \\
&= 4^3M^3\left[\varepsilon^3 + C_3^1\varepsilon^2\left(N_2 t\right) + C_3^2\varepsilon\frac{\left(N_2 t\right)^2}{2!}\right. \\
&\qquad \left. + \frac{\left(N_2 t\right)^3}{3!}\right]\beta_{pc}(B),
\end{aligned} \tag{29}$$

and hence, by the method of mathematical induction, for any positive integer n and $t \in [0,T]$, we obtain

$$\begin{aligned}
&\beta\left(\left(Q^{(n,x_0)}B\right)(t)\right) \\
&\leq 4^nM^n\left[\varepsilon^n + C_n^1\varepsilon^{n-1}\left(N_2 t\right)\right. \\
&\qquad \left. + C_n^2\varepsilon^{n-2}\frac{\left(N_2 t\right)^2}{2!} + \cdots + \frac{\left(N_2 t\right)^n}{n!}\right]\beta_{pc}(B).
\end{aligned} \tag{30}$$

Therefore, for any positive integer n, we have

$$\begin{aligned}\beta_{pc}\left(Q^{(n,x_0)}B\right) &\leq 4^n M^n \left[\varepsilon^n + C_n^1 \varepsilon^{n-1}\left(N_2 T\right)\right.\\ &\quad \left. + C_n^2 \varepsilon^{n-2}\frac{\left(N_2 T\right)^2}{2!} + \cdots + \frac{\left(N_2 T\right)^n}{n!}\right]\beta_{pc}(B).\end{aligned} \tag{31}$$

Since $\lim_{n\to+\infty}[\varepsilon^{n-1} n(n/(n-1))^{n-1}]^{1/n} = \varepsilon$, it follows from the Stirling Formula (see [28]) that

$$\beta_{pc}\left(Q^{(n,x_0)}B\right) \leq o\left(\frac{1}{n^s}\right) 4^n M^n \beta_{pc}(B), \quad n \longrightarrow \infty,\ \forall s > 1, \tag{32}$$

and hence, there exists sufficiently large positive integer n_0 such that

$$\beta_{pc}\left(Q^{(n_0,x_0)}B\right) < \beta_{pc}(B), \tag{33}$$

which shows that $Q : W \to W$ is a convex-power condensing operator. From Lemma 4, we get that Q has at least one fixed point in W; that is, (1) has at least one mild solution $u \in W$. This completes the proof. □

Remark 12. The technique of constructing convex-power condensing operator plays a key role in the proof of Theorem 10, which enables us to get rid of the strict inequality restriction on the coefficient function $l(t)$ of f. However, in many previous articles, such as [9, 12, 13, 19, 20], the authors had to impose a strong inequality condition on the integrable function $l(t)$, as they used Darbo-Sadovskii's fixed point theorem only. Thus, our result extends and complements those obtained in [9, 12, 13, 19, 20] and has more broad applications.

Remark 13. If we use the following assumption instead of $(H_f)(3)$:

$(H_f)(3')$ there exists a constant $l > 0$ such that

$$\beta\left(f(t,D)\right) \leq l\beta(D) \tag{34}$$

for a.e. $t \in [0,T]$ and every bounded subset $D \subset B_r$,

we may use the same method to obtain

$$\beta_{pc}\left(Q^{(n,x_0)}B\right) \leq 4^n M^n l^n \frac{T^n}{n!}\beta_{pc}(B), \tag{35}$$

for any $B \subset W$. Thus, there exists a large enough positive integral n_0 such that

$$4^{n_0} M^{n_0} l^{n_0} \frac{T^{n_0}}{n_0!} < 1; \tag{36}$$

namely,

$$\beta_{pc}\left(Q^{(n,x_0)}B\right) \leq \beta_{pc}(B), \quad B \subset W. \tag{37}$$

Therefore, we can get the following consequence.

Theorem 14. *If the hypotheses (H_A), $(H_f)(1)(2)(3')$, (H_g), (H_I), and (H_r) are satisfied, then the nonlocal problem* (1) *has at least one mild solution on* $[0,T]$.

4. g Is Lipschitz Continuous

In this section, by applying the proof of Theorem 10 and Darbo-Sadovskii's fixed point theorem, we give the existence of mild solutions of the problem (1) when the nonlocal condition g is Lipscitz continuous in $PC([0,T];X)$.

We give the following hypotheses:

(H_g') there exists a constant $k > 0$ such that

$$\left\|g(u) - g(v)\right\| \leq k\left\|u - v\right\|, \quad \text{for } u, v \in PC([0,T];X); \tag{38}$$

(H_I') $I_i : X \to X$ is Lipschitz continuous with Lipschitz constant k_i, for $i = 1, 2, \ldots, p$.

Theorem 15. *If the hypotheses (H_A), $(H_f)(1)(2)(3)$, (H_g'), (H_I'), and (H_r) are satisfied, then the nonlocal problem* (1) *has at least one mild solution on* $[0,T]$ *provided that* $M(k + \sum_{i=1}^p k_i + 4\|l\|_{L^1}) < 1$.

Proof of Theorem 15. Given $x \in W$, let's first consider the following Cauchy initial problem:

$$\begin{gathered}u'(t) = Au(t) + f(t, u(t)), \quad t \in [0,T],\ t \neq t_i,\\ \Delta u(t_i) = I_i(x(t_i)), \quad i = 1, 2, \ldots, p,\\ 0 < t_1 < t_2 < \cdots < t_p < T,\\ u(0) = g(x).\end{gathered} \tag{39}$$

From the proof of Theorem 10, we can easily see that there exists at least one mild solution to (39). Define $G : W \to W$ by that Gx is the mild solution to (39). Then

$$\begin{aligned}(Gx)(t) &= S(t)g(x) + \sum_{0<t_i<t} S(t-t_i) I_i(x(t_i))\\ &\quad + \int_0^t S(t-s) f(s,(Gx)(s))\,ds.\end{aligned} \tag{40}$$

Now, we will show that G is β_{pc}-condensing on W. According to Lemma 11, for any bounded subset $B \subset W$, we deduce

$$\begin{aligned}&\beta((GB)(t))\\ &\quad\leq \beta\left(S(t)g(B)\right) + \sum_{0<t_i<t} S(t-t_i) I_i\left(B(t_i)\right)\\ &\qquad + \beta\left(\int_0^t S(t-s) f(s,(GB)(s))\,ds\right)\\ &\quad\leq Mk\beta_{pc}(B) + M\sum_{i=1}^p k_i \beta_{pc}(B)\\ &\qquad + 4M\int_0^t \beta\left(f(s,(GB)(s))\right) ds\end{aligned}$$

$$\begin{aligned}&\leq M\left(k+\sum_{i=1}^{p}k_i\right)\beta_{pc}(B)\\&\quad+4M\int_0^t l(s)\,\beta((GB)(s))\,ds\\&\leq M\left(k+\sum_{i=1}^{p}k_i\right)\beta_{pc}(B)\\&\quad+4M\|l\|_{L^1}\beta_{pc}(GB),\end{aligned}\tag{41}$$

which implies that

$$\beta_{pc}(GB)\leq\frac{M\left(k+\sum_{i=1}^{p}k_i\right)}{1-4M\|l\|_{L^1}}\beta_{pc}(B).\tag{42}$$

In addition, since $M(k+\sum_{i=1}^{p}k_i+4\|l\|_{L^1})<1$, it follows that the mapping K is a β_{pc}-condensing operator on W. In view of Lemma 2, the mapping G has at least one fixed point in W, which produces a mild solution for the nonlocal impulsive problem (1). □

Remark 16. Similarly, one can show that the conclusion of Theorem 15 remains valid provided that hypothesis $(H_f)(3)$ is replaced by condition $(H_f)(3')$.

Remark 17. In Theorem 15, we do not assume the compactness of nonlocal item g. Under the Lipschitz assumption, we make full use of the conclusion of Theorem 10, the properties of noncompact measure and the technique of fixed point to deal with the solution operator G.

Remark 18. Recently, the existence results for fractional differential equations have been widely studied in many papers. For more details on this theory one can refer to [29, 30] and references therein. It should be pointed out that the techniques and ideas in this paper can also be used to study fractional equations. In the future, we will also try to investigate to nonlocal controllability of impulsive differential equations by applying the similar techniques, methods, and compactness conditions. Further discussions on this topic will be in our consequent papers.

5. Examples

In this section, we shall give two examples to illustrate Theorems 10 and 15.

Example 1. Consider the following semilinear parabolic system:

$$\begin{aligned}&\frac{\partial}{\partial t}u(t,x)=-A(x,D)\,u(t,x)+F(t,u(t,x)),\\&\qquad\qquad t\in[0,T],\ x\in\Omega,\ t\neq t_i,\\&D^{\alpha}u(t,x)=0,\quad t\in[0,T],\ x\in\partial\Omega\quad\text{for }|\alpha|\leq m,\\&u(t_i^+,x)-u(t_i^-,x)=I_i(u(t_i,x)),\quad i=1,2,\ldots,p,\\&u(0,x)=\int_\Omega\int_0^T\Gamma\left(t,x,x',u\left(t,x'\right)\right)dt\,dx',\quad x\in\Omega,\end{aligned}\tag{43}$$

where Ω is a bounded domain in R^n $(n\geq 1)$ with smooth boundary $\partial\Omega$, $A(x,D)u=\Sigma_{|\alpha|\leq 2m}a_\alpha(x)D^\alpha u$ is strongly elliptic, $F:[0,T]\times R\to R$, and $\Gamma:[0,T]\times\Omega\times\Omega\times R\to R$.

Let $X=L^2(\Omega)$ and define the operator $A:D(A)\subseteq X\to X$ by

$$\begin{aligned}D(A)&=H^{2m}(\Omega)\cap H_0^m(\Omega),\\Au&=-A(x,D)\,u.\end{aligned}\tag{44}$$

Then the operator A is an infinitesimal generator of an equicontinuous C_0-semigroup $S(t)$ on X (see [26]).

Suppose that the function $\Gamma:[0,T]\times\Omega\times\Omega\times R\to R$ satisfies the following conditions:

(i) the Carathéodory condition, that is, $\Gamma(t,x,x',r)$, is a continuous function about r for a.e. $(t,x,x')\in[0,T]\times\Omega\times\Omega$; $\Gamma(t,x,x',r)$ is measurable about (t,x,x') for each fixed $r\in R$;

(ii) $|\Gamma(t,x,x',r)-\Gamma(t,\overline{x},x',r)|\leq\mu_k(t,x,\overline{x},x')$ for all $(t,x,x',r),(t,\overline{x},x',r)\in[0,T]\times\Omega\times\Omega\times R$ with $|r|\leq k$, where $\mu_k\in L^1([0,T]\times\Omega\times\Omega\times R;R^+)$ satisfies $\lim_{x\to\overline{x}}\int_\Omega\int_0^T\mu_k(t,x,\overline{x},x')dt\,dx'=0$, uniformly in $\overline{x}\in\Omega$;

(iii) $|\Gamma(t,x,x',r)|\leq(\delta/Tm(\Omega))|r|+\Phi(t,x,x')$ for all $r\in R$, where $\Phi\in L^2([0,T]\times\Omega\times\Omega;R^+)$ and $\delta>0$.

We assume the following.

(1) $f:[0,T]\times X\to X$ is defined by

$$f(t,z)(x)=F(t,z(x)),\quad x\in\Omega.\tag{45}$$

Moreover, for given $r>0$, there exist two integrable functions $\phi_r,\ \psi:[0,T]\to R$ such that $\|f(t,z)\|\leq\phi_r(t)$ and $\beta(f(t,D))\leq\psi(t)\beta(D)$ for a.e. $t\in[0,T]$, $z\in B_r$ and every bounded subset $D\subset B_r$;

(2) $g:C([0,T];X)\to X$ is defined by

$$g(u)(x)=\int_\Omega\int_0^T\Gamma\left(t,x,x',u\left(t,x'\right)\right)dt\,dx',\quad x\in\Omega.\tag{46}$$

From Theorem 4.2 in [31], we get directly that g is well defined and is a completely continuous operator by the above conditions about the function Γ.

(3) $I_i:X\to X$ is a continuous and compact function for each $i=1,2,\ldots,p$, defined by

$$I_i(u)(x)=I_i(u(x)).\tag{47}$$

Let us observe that the problem (43) may be reformulated as the abstract problem (1) under the above conditions. By using Theorem 10, the problem (43) has at least one mild solution $u\in C([0,T];L^2(\Omega))$ provided that the hypothesis (H_r) holds.

Example 2. Consider the following partial differential system:

$$\begin{gathered}\frac{\partial}{\partial t}u(t,x)=\frac{\partial}{\partial x}u(t,x)+F(t,u(t,x)),\\ t\in[0,T],\ x\in\Omega,\ t\neq t_i,\\ u(t,x)=0,\quad t\in[0,T],\ x\in\partial\Omega,\\ u(t_i^+,x)-u(t_i^-,x)=I_i(u(t_i,x)),\quad i=1,2,\ldots,p,\\ u(0,x)=\sum_{j=1}^{q}c_j u(s_j,x),\end{gathered}\tag{48}$$

where Ω is a bounded domain in R^n $(n\geq 1)$ with smooth boundary $\partial\Omega$, $F:[0,T]\times R\to R$, and c_j and s_j both are given real numbers for $j=1,2,\ldots,q$.

Let $X=C(\overline{\Omega})$, and define the operator $A:D(A)\subseteq X\to X$ by

$$\begin{gathered}D(A)=\left\{v\in X:v'\in X,v(x)=0,x\in\partial\Omega\right\},\\ Av=v'.\end{gathered}\tag{49}$$

As is known to all, the operator A is an infinitesimal generator of the semigroup $S(t)$ defined by $S(t)x(s)=x(t+s)$ for each $x\in X$. Here, $S(t)$ is equicontinuous but is not compact.

We now suppose the following.

(1) $f:[0,T]\times X\to X$ is defined by

$$f(t,y)(x)=F(t,y(x)),\quad x\in\Omega.\tag{50}$$

Moreover, for given $r>0$, there exist two integrable functions $\varphi_r,\omega:[0,T]\to R$ such that $\|f(t,y)\|\leq\varphi_r(t)$ and $\beta(f(t,D))\leq\varpi(t)\beta(D)$ for a.e. $t\in[0,T]$, $y\in B_r$, and every bounded subset $D\subset B_r$;

(2) $g:PC([0,T];X)\to X$ is defined by

$$\begin{gathered}g(u)(x)=\sum_{j=1}^{q}c_j u(s_j,x),\\ 0<s_1<s_2<\cdots<s_q<T,\ x\in\Omega.\end{gathered}\tag{51}$$

Then g is Lipschitz continuous with constant $k=\sum_{j=1}^{q}|c_j|$; that is, the assumption H_g' is satisfied.

(3) $I_i:X\to X$ is a continuous function for each $i=1,2,\ldots,p$, defined by

$$I_i(u)(x)=I_i(u(x)).\tag{52}$$

Here we take $I_i(u(x))=(\alpha_i|u(x)|+t_i)^{-1}$, $\alpha_i>0$, $i=1,2,\ldots,p$, $0<t_1<t_2<\cdots<t_q<T$, $x\in\Omega$. Then I_i is Lipschitz continuous with constant $k_i=\alpha_i/t_i^2$, $i=1,2,\ldots,p$; that is, the assumption H_I' is satisfied.

Let us observe that (48) may be rewritten as the abstract problem (1) under the above conditions. If the following inequality

$$M\left(\sum_{j=1}^{q}\left|c_j\right|+\sum_{i=1}^{p}\frac{\alpha_i}{t_i^2}+4\|\varpi\|_{L^1}\right)<1\tag{53}$$

holds, then according to Theorem 15, the impulsive problem (48) has at least one mild solution in $PC([0,T];X)$.

Acknowledgments

The authors would like to thank the referees for their careful reading and their valuable comments and suggestions to improve their results. This research is supported by the Natural Science Foundation of China (11201410 and 11271316), the Natural Science Foundation of Jiangsu Province (BK2012260), and the Natural Science Foundation of Jiangsu Education Committee (10KJB110012).

References

[1] L. Byszewski, "Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem," *Journal of Mathematical Analysis and Applications*, vol. 162, no. 2, pp. 494–505, 1991.

[2] L. Byszewski and V. Lakshmikantham, "Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space," *Applicable Analysis*, vol. 40, no. 1, pp. 11–19, 1991.

[3] S. K. Ntouyas and P. Tsamatos, "Global existence for semilinear evolution equations with nonlocal conditions," *Journal of Mathematical Analysis and Applications*, vol. 210, no. 2, pp. 679–687, 1997.

[4] L. Byszewski and H. Akca, "Existence of solutions of a semilinear functional-differential evolution nonlocal problem," *Nonlinear Analysis*, vol. 34, no. 1, pp. 65–72, 1998.

[5] M. Benchohra and S. K. Ntouyas, "Nonlocal Cauchy problems for neutral functional differential and integrodifferential inclusions in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 258, no. 2, pp. 573–590, 2001.

[6] S. Aizicovici and M. McKibben, "Existence results for a class of abstract nonlocal Cauchy problems," *Nonlinear Analysis*, vol. 39, no. 5, pp. 649–668, 2000.

[7] S. Aizicovici and H. Lee, "Existence results for nonautonomous evolution equations with nonlocal initial conditions," *Communications in Applied Analysis*, vol. 11, no. 2, pp. 285–297, 2007.

[8] S. Aizicovici and V. Staicu, "Multivalued evolution equations with nonlocal initial conditions in Banach spaces," *Nonlinear Differential Equations and Applications NoDEA*, vol. 14, no. 3-4, pp. 361–376, 2007.

[9] J. García-Falset, "Existence results and asymptotic behavior for nonlocal abstract Cauchy problems," *Journal of Mathematical Analysis and Applications*, vol. 338, no. 1, pp. 639–652, 2008.

[10] A. Paicu and I. I. Vrabie, "A class of nonlinear evolution equations subjected to nonlocal initial conditions," *Nonlinear Analysis*, vol. 72, no. 11, pp. 4091–4100, 2010.

[11] V. Obukhovski and P. Zecca, "Controllability for systems governed by semilinear differential inclusions in a Banach space with a noncompact semigroup," *Nonlinear Analysis*, vol. 70, no. 9, pp. 3424–3436, 2009.

[12] X. Xue, "Nonlocal nonlinear differential equations with a measure of noncompactness in Banach spaces," *Nonlinear Analysis*, vol. 70, no. 7, pp. 2593–2601, 2009.

[13] X. Xue, "Semilinear nonlocal problems without the assumptions of compactness in Banach spaces," *Analysis and Applications*, vol. 8, no. 2, pp. 211–225, 2010.

[14] T. Cardinali and P. Rubbioni, "Impulsive semilinear differential inclusions: topological structure of the solution set and solutions on non-compact domains," *Nonlinear Analysis*, vol. 69, no. 1, pp. 73–84, 2008.

[15] J. Liang, J. H. Liu, and T. J. Xiao, "Nonlocal impulsive problems for nonlinear differential equations in Banach spaces," *Mathematical and Computer Modelling*, vol. 49, no. 3-4, pp. 798–804, 2009.

[16] N. Abada, M. Benchohra, and H. Hammouche, "Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions," *Journal of Differential Equations*, vol. 246, no. 10, pp. 3834–3863, 2009.

[17] N. U. Ahmed, "Optimal feedback control for impulsive systems on the space of finitely additive measures," *Publicationes Mathematicae Debrecen*, vol. 70, no. 3-4, pp. 371–393, 2007.

[18] Z. Fan and G. Li, "Existence results for semilinear differential equations with nonlocal and impulsive conditions," *Journal of Functional Analysis*, vol. 258, no. 5, pp. 1709–1727, 2010.

[19] T. Cardinali and P. Rubbioni, "On the existence of mild solutions of semilinear evolution differential inclusions," *Journal of Mathematical Analysis and Applications*, vol. 308, no. 2, pp. 620–635, 2005.

[20] L. Zhu and G. Li, "On a nonlocal problem for semilinear differential equations with upper semicontinuous nonlinearities in general Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 660–675, 2008.

[21] B. Ahmad, K. Malar, and K. Karthikeyan, "A study of nonlocal problems of impulsive integrodifferential equations with measure of noncompactness," *Advances in Difference Equations*, vol. 2013, article 205, pp. 1–11, 2013.

[22] L. Zhu and G. Li, "Existence results of semilinear differential equations with nonlocal initial conditions in Banach spaces," *Nonlinear Analysis*, vol. 74, no. 15, pp. 5133–5140, 2011.

[23] J. X. Sun and X. Y. Zhang, "A fixed point theorem for convex-power condensing operators and its applications to abstract semilinear evolution equations," *Acta Mathematica Sinica*, vol. 48, no. 3, pp. 439–446, 2005.

[24] K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, vol. 194, Springer, New York, NY, USA, 2000.

[25] K. J. Engel and R. Nagel, *A Short Course on Operator Semigroups*, Universitext, Springer, New York, NY, USA, 2006.

[26] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, Berlin, Germany, 1983.

[27] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, vol. 60 of *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1980.

[28] L. Liu, F. Guo, C. Wu, and Y. Wu, "Existence theorems of global solutions for nonlinear Volterra type integral equations in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 309, no. 2, pp. 638–649, 2005.

[29] L. Hu, Y. Ren, and R. Sakthivel, "Existence and uniqueness of mild solutions for semilinear integro-differential equations of fractional order with nonlocal initial conditions and delays," *Semigroup Forum*, vol. 79, no. 3, pp. 507–514, 2009.

[30] Y. Ren, Y. Qin, and R. Sakthivel, "Existence results for fractional order semilinear integro-differential evolution equations with infinite delay," *Integral Equations and Operator Theory*, vol. 67, no. 1, pp. 33–49, 2010.

[31] R. H. Martin, Jr., *Nonlinear Operators and Differential Equations in Banach Spaces*, Wiley, New York, NY, USA, 1976.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 902563, 6 pages
http://dx.doi.org/10.1155/2013/902563

Research Article

On Lipschitz Perturbations of a Self-Adjoint Strongly Positive Operator

Dinu Teodorescu[1] and N. Hussain[2]

[1] *Department of Mathematics, Valahia University of Targoviste, Boulevard Unirii 18, 130024 Targoviste, Romania*
[2] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Dinu Teodorescu; dteodorescu2003@yahoo.com

Received 5 May 2013; Accepted 13 September 2013

Academic Editor: Satit Saejung

In this paper we study semilinear equations of the form $Au + \lambda F(u) = f$, where A is a linear self-adjoint operator, satisfying a strong positivity condition, and F is a nonlinear Lipschitz operator. As applications we develop Krasnoselskii and Ky Fan type approximation results for certain pair of maps and to illustrate the usability of the obtained results, the existence of solution of an integral equation is provided.

1. Introduction and Preliminaries

The study of abstract operator equations involving linear or nonlinear operators has generated over time useful instruments in the approach of some concrete equations. Therefore, we consider as interesting to present some aspects regarding the semilinear abstract operator equations in Hilbert spaces.

Let H be a real Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$.

In [1, 2] semilinear equations of the form $Au - F(u) = 0$ were studied, where $A : D(A) \subset H \to H$ is a self-adjoint linear operator with the resolvent set $R(A)$ and $F : H \to H$ is a Gateaux differentiable gradient operator. If there exist real numbers $a < b$ such that $[a, b] \subset R(A)$ and

$$a \leq \frac{\langle F(u) - F(v), u - v \rangle}{\|u - v\|^2} \leq b, \tag{1}$$

for all $u, v \in H$, $u \neq v$ (i.e., F interacts suitably with the spectrum of A), then it is proved in [2] that the equation $Au - F(u) = 0$ has exactly one solution.

The author in [3] presented an existence and uniqueness result for the semilinear equation $Au + F(u) = f$, where $A : D(A) \subset H \to H$ is a linear maximal monotone operator, satisfying a strong positivity condition, and the nonlinearity $F : H \to H$ is a Lipschitz operator.

Let X be a real Banach space, ordered by a cone K. A cone K is a closed convex subset of X with $\lambda K \subseteq K$ ($\lambda \geq 0$), and $K \cap (-K) = \{0\}$. As usual $x \leq y \Leftrightarrow y - x \in K$.

Definition 1. Let M be a nonempty subset of an ordered Banach space X with order $\leq$. Two mappings $S, T : M \to M$ are said to be weakly isotone increasing if $Sx \leq TSx$ and $Tx \leq STx$ hold for all $x \in M$. Similarly, we say that S and T are weakly isotone decreasing if $Tx \geq STx$ and $Sx \geq TSx$ hold for all $x \in M$. The mappings S and T are said to be weakly isotone if they are either weakly isotone increasing or weakly isotone decreasing.

In our considerations, the following definition will play an important role. Let $\mathscr{B}(X)$ denote the collection of all nonempty bounded subsets of X and $\mathscr{W}(X)$ the subset of $\mathscr{B}(X)$ consisting of all weakly compact subsets of X. Also, let B_r denote the closed ball centered at 0 with radius r.

Definition 2 (see [4]). A function $\psi : \mathscr{B}(X) \to \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions.

(1) The family $\ker(\psi) = \{M \in \mathscr{B}(X) : \psi(M) = 0\}$ is nonempty, and $\ker(\psi)$ is contained in the set of relatively weakly compact sets of X.

(2) $M_1 \subseteq M_2 \Rightarrow \psi(M_1) \leq \psi(M_2)$.

(3) $\psi(\overline{\text{co}}(M)) = \psi(M)$, where $\overline{\text{co}}(M)$ is the closed convex hull of M.

(4) $\psi(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\psi(M_1) + (1-\lambda)\psi(M_2)$ for $\lambda \in [0,1]$.

(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ such that $\lim_{n\to\infty}\psi(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.

The family $\ker\psi$ described in (1) is said to be the kernel of the measure of weak noncompactness ψ. Note that the intersection set M_∞ from (5) belongs to $\ker\psi$ since $\psi(M_\infty) \leq \psi(M_n)$ for every n, and $\lim_{n\to\infty}\psi(M_n) = 0$. Also, it can be easily verified that the measure ψ satisfies

$$\psi\left(\overline{M^w}\right) = \psi(M), \tag{2}$$

where $\overline{M^w}$ is the weak closure of M.

A measure of weak noncompactness ψ is said to be *regular* if

$$\psi(M) = 0 \quad \text{iff } M \text{ is relatively weakly compact}, \tag{3}$$

subadditive if

$$\psi(M_1 + M_2) \leq \psi(M_1) + \psi(M_2), \tag{4}$$

homogeneous if

$$\psi(\lambda M) = |\lambda|\,\psi(M), \quad \lambda \in \mathbb{R}, \tag{5}$$

and *set additive* (or *has the maximum property*) if

$$\psi(M_1 \cup M_2) = \max(\psi(M_1), \psi(M_2)). \tag{6}$$

The first important example of a measure of weak noncompactness has been defined by de Blasi [5] as follows:

$$\begin{aligned} w(M) = \inf\{r > 0 : &\text{ there exists } W \in \mathscr{W}(X) \\ &\text{with } M \subseteq W + B_r\}, \end{aligned} \tag{7}$$

for each $M \in \mathscr{B}(X)$.

Notice that $w(\cdot)$ is regular, homogeneous, subadditive, and set additive (see [5]).

By a measure of noncompactness on a Banach space X, we mean a map $\psi : \mathscr{B}(X) \to \mathbb{R}_+$ which satisfies conditions (1)–(5) in Definition 2 relative to the strong topology instead of the weak topology.

Definition 3. Let X be a Banach space and ψ a measure of (weak) noncompactness on X. Let $A : D(A) \subseteq X \to X$ be a mapping. If $A(D(A))$ is bounded and for every nonempty bounded subset M of $D(A)$ with $\psi(M) > 0$, we have $\psi(A(M)) < \psi(M)$; then A is called ψ-condensing. If there exists k, $0 \leq k \leq 1$, such that $A(D(A))$ is bounded and for each nonempty bounded subset M of $D(A)$, we have $\psi(A(M)) \leq k\psi(M)$; then A is called k-ψ-contractive.

Definition 4 (see [6]). A map $A : D(A) \to X$ is said to be ws-compact if it is continuous, and for any weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X.

Definition 5. A map $A : D(A) \to X$ is said to be ww-compact if it is continuous, and for any weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A)$ the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a weakly convergent subsequence in X.

Definition 6. Let X be a Banach space. A mapping $T : D(T) \subseteq X \to X$ is called a nonlinear contraction if there exists a continuous and nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|Tx - Ty\| \leq \varphi(\|x - y\|), \tag{8}$$

for all $x, y \in D(T)$, where $\varphi(r) < r$ for $r > 0$.

In this paper we consider the semilinear equation

$$Au + \lambda F(u) = f, \tag{9}$$

where $A : H \to H$ is a linear self-adjoint operator, satisfying a strong positivity condition, $F : H \to H$ is a nonlinear Lipschitz operator, and λ is a positive parameter. Using the Banach fixed point theorem, we prove an existence and uniqueness result about the considered equation. Thus, we obtain here the same type of result as in [2], replacing the maximal monotonicity of linear part A of the semilinear equation with the hypothesis that A is self-adjoint. So, the principal result of this paper can be applied in the study of nonlinear Lipschitz perturbations of a linear integral operator with symmetric kernel. Further a result of continuous dependence on the free term and a fixed point theorem are presented. As applications we present some common fixed point theorems and approximation results for a pair of nonlinear mappings. Finally, the existence of solution of an integral equation is provided to illustrate the usability of the obtained results.

2. Results

Theorem 7. *Let $A : H \to H$ be a linear self-adjoint operator and $F : H \to H$ nonlinear, satisfying the following conditions:*

(i) *F is a Lipschitz operator, that is, there is a constant $M > 0$ such that*

$$\|F(x) - F(y)\| \leq M\|x - y\|, \tag{10}$$

for all $x, y \in H$;

(ii) *A is a strongly positive operator, that is, there is a constant $c > 0$ such that*

$$\langle Ax, x\rangle \geq c\|x\|^2, \tag{11}$$

for all $x \in H$.

Then the equation $Au + \lambda F(u) = f$ has a unique solution for all $f \in H$ and $\lambda \in (0; c/M)$.

Proof. Let us choose s in the spectrum of A. We have

$$s \geq \inf \{\langle Ax, x\rangle \mid x \in H, \|x\| = 1\} \geq c, \tag{12}$$

and we obtain that every real number $\omega \in (-\infty; c)$ is in the resolvent set of the operator A. Consequently, we have

$$\operatorname{Rg}(A - \omega I) = H, \tag{13}$$

for all $\omega < c$, where I is the identity of H.

Let $\delta < 0$. We write (9) in the equivalent form

$$A_\delta u + F_\delta (u) = f, \tag{14}$$

where $A_\delta = A - \delta I$ and $F_\delta = \lambda F + \delta I$. We have $\operatorname{Rg}(A_\delta) = H$ and

$$\|F_\delta (x) - F_\delta (y)\| \leq (\lambda M + |\delta|) \|x - y\|, \tag{15}$$

for all $x, y \in H$.

Also

$$\begin{aligned}\langle A_\delta x, x\rangle &= \langle Ax, x\rangle - \delta\|x\|^2 \\ &\geq (c - \delta) \|x\|^2 = (c + |\delta|) \|x\|^2 \quad \forall x \in H.\end{aligned} \tag{16}$$

From (16) we obtain

$$\|A_\delta x\| \geq (c + |\delta|) \|x\| \quad \forall x \in H. \tag{17}$$

Consequently, there exists $A_\delta^{-1} : H \to H$ which is linear and continuous, that is $A_\delta^{-1} \in \mathscr{L}(H)$, the Banach space of all linear and bounded operators from H to H. Moreover, we have

$$\|A_\delta^{-1}\|_{\mathscr{L}(H)} \leq \frac{1}{c + |\delta|}. \tag{18}$$

Now (14) can be equivalently written as

$$u + A_\delta^{-1} F_\delta u = A_\delta^{-1} f. \tag{19}$$

We consider the operator $T : H \to H$ defined by

$$Tu = -A_\delta^{-1} F_\delta u + A_\delta^{-1} f. \tag{20}$$

Therefore (19) becomes

$$u = Tu, \tag{21}$$

and so, the problem of the solvability of (9) is reduced to the study of fixed points of the operator T. We have

$$\begin{aligned}\|Tx - Ty\| &= \|A_\delta^{-1} F_\delta x - A_\delta^{-1} F_\delta y\| \\ &= \|A_\delta^{-1} (F_\delta x - F_\delta y)\| \leq \|A_\delta^{-1}\|_{\mathscr{L}(H)} \|F_\delta x - F_\delta y\| \\ &\leq \frac{\lambda M + |\delta|}{c + |\delta|} \|x - y\| \quad \forall x, y \in H.\end{aligned} \tag{22}$$

It results that T is a strict contraction from H to H because $\lambda M < c$. According to the Banach fixed point theorem, T has a unique fixed point, and thus the proof of Theorem 7 is complete. □

Let us consider now the dependence of solution of (9) on the data f.

Theorem 8. *Under the assumptions from the hypothesis of Theorem 7, let $i \in \{1, 2\}$, and let u_i be the unique solution of the equation*

$$Au + \lambda F (u) = f_i, \quad i \in \{1, 2\};\ f_1, f_2 \in H. \tag{23}$$

Then

$$\|u_1 - u_2\| \leq \frac{1}{c - \lambda M} \|f_1 - f_2\|. \tag{24}$$

Proof. According to the equivalent form (19) of (9), we have

$$\begin{aligned}\|u_1 - u_2\| &= \|-A_\delta^{-1} F_\delta u_1 + A_\delta^{-1} f_1 + A_\delta^{-1} F_\delta u_2 - A_\delta^{-1} f_2\| \\ &\leq \|A_\delta^{-1} F_\delta u_1 - A_\delta^{-1} F_\delta u_2\| + \|A_\delta^{-1} f_1 - A_\delta^{-1} f_2\| \\ &\leq \frac{\lambda M + |\delta|}{c + |\delta|} \|u_1 - u_2\| + \frac{1}{c + |\delta|} \|f_1 - f_2\|.\end{aligned} \tag{25}$$

It results that

$$(c - \lambda M) \|u_1 - u_2\| \leq \|f_1 - f_2\|, \tag{26}$$

and thus our assertion is proved. □

In fact, Theorem 8 establishes the continuous dependence of the solution of (9) on the free term and signifies the stability of the solution.

3. Consequences of Principal Result

The previous results prove that, under the considered assumptions, the operator $A + \lambda F$ is invertible for all $\lambda \in (0; c/M)$ and the inverse $C = (A + \lambda F)^{-1}$ is a Lipschitz operator. From (24) it results that the operator C satisfies

$$\|C(f_1) - C(f_2)\| \leq \frac{1}{c - \lambda M} \|f_1 - f_2\| \quad \forall f_1, f_2 \in H. \tag{27}$$

Suppose now that $c > 1$ and $\lambda \in (0; (c-1)/M) \subset (0; c/M)$. We obtain $M\lambda < c - 1$ or $1/(c - \lambda M) < 1$, and, consequently, C is a strict contraction. It results that there exists an unique element $u^* \in H$ such that $C(u^*) = u^*$ which is equivalent to $(A + \lambda F)(u^*) = u^*$. So we obtained the following fixed point theorem.

Theorem 9. *Let $A : H \to H$ be a linear self-adjoint operator and $F : H \to H$ nonlinear, satisfying the following conditions:*

(i) *F is a Lipschitz operator, that is, there is a constant $M > 0$ such that*

$$\|F (x) - F (y)\| \leq M \|x - y\|, \tag{28}$$

for all $x, y \in H$;

(ii) *A is a strongly positive operator, that is, there is a constant $c > 0$ such that*

$$\langle Ax, x\rangle \geq c\|x\|^2, \tag{29}$$

for all $x \in H$.

If $c > 1$ and $\lambda \in (0; (c-1)/M)$, then the operator $A + \lambda F$ has a unique fixed point.

The way used in obtaining Theorem 9 can be applied in the study of the following problem: extracting operators which have the unique fixed point property from a family of operators $\{V_\lambda/\lambda \in \mathbf{R}\}$.

Let E be a Banach space, and $\mathcal{F} = \{V_\lambda : E \to E/\lambda \in \mathbf{R}\}$, V_λ satisfying some constraints for any $\lambda \in \mathbf{R}$. Our intention is to extract a subfamily $\mathcal{G} \subset \mathcal{F}$, so that V_λ can have a unique fixed point for all $V_\lambda \in \mathcal{G}$. It is easy to observe, using the same method as in obtaining Theorem 9, that the following result holds.

Theorem 10. *If*

(i) *there exists $J \subset \mathbf{R}$, $J \neq \phi$, so that V_λ is invertible and*

$$\left\|V_\lambda^{-1}x - V_\lambda^{-1}y\right\| \leq g(\lambda)\left\|x - y\right\|, \quad x, y \in E;\ g(\lambda) > 0, \tag{30}$$

for all $\lambda \in J$;

(ii) *there exists $I \subset J$, $I \neq \phi$, so that $g(\lambda) < 1$ for all $\lambda \in I$, then V_λ has a unique fixed point for all $\lambda \in I$.*

As an application of the observations established above, we develop here the Krasnoselskii and Ky Fan type approximation results for certain pair of maps.

Theorem 11. *Let A, F, λ, M and c be as in Theorem 9, $T = (A + \lambda F)^{-1}$, and let H be an ordered Hilbert space. Let D be a nonempty closed convex subset of H and ψ a set additive measure of noncompactness on H. Let $S : H \to H$ be a mapping satisfying the following:*

(i) *$T(D) \subseteq D$ and $S(D) \subseteq D$,*

(ii) *S is continuous and ψ-condensing,*

(iii) *T and S are weakly isotone.*

Then there exists a unique $x^* \in D$ such that $ASx^* + \lambda FSx^* = x^*$.

Proof. By Theorem 9, $T = (A + \lambda F)^{-1} : H \to H$ is a contraction with contractive constant $\alpha = 1/(c - \lambda M) < 1$. Thus T is a shrinking mapping. Now all of the conditions of Corollary 1.18 [7] are satisfied so there exists an $x^* \in D$ such that $Tx^* = Sx^* = x^*$ which implies that $(A + \lambda F)oSx^* = x^*$, and hence $ASx^* + \lambda FSx^* = x^*$. □

Theorem 12. *Let A, F, λ, M, and c be as in Theorem 9, $T = (A + \lambda F)^{-1}$, and let H be an ordered Hilbert space. Let D be a nonempty closed convex subset of H and ψ a set additive measure of noncompactness on H. Let $S : H \to H$ be a mapping satisfying the following:*

(i) *$T(D) \subseteq D$ and $S(D) \subseteq D$,*

(ii) *S is a nonlinear contraction,*

(iii) *T and S are weakly isotone.*

Then there exists a unique $x^* \in D$ such that $ASx^* + \lambda FSx^* = x^*$.

Proof. By Theorem 9, $T = (A + \lambda F)^{-1} : H \to H$ is a contraction with contractive constant $\alpha = 1/(c - \lambda M) < 1$. Thus T is a shrinking mapping. Now all of the conditions of Corollary 1.19 [7] are satisfied so there exists an $x^* \in D$ such that $Tx^* = Sx^* = x^*$ which implies $ASx^* + \lambda FSx^* = x^*$. □

As an application of Corollaries 1.24 or 1.25 and 1.30 or 1.31 [7], we obtain the following results, respectively

Theorem 13. *Let A, F, λ, M, and c be as in Theorem 9, $T = (A + \lambda F)^{-1}$, and let H be an ordered Hilbert space. Let D be a nonempty closed convex subset of H and ψ a set additive measure of weak noncompactness on H. Assume that $S, T : H \to H$ are sequentially weakly continuous mappings satisfying the following:*

(i) *$T(D) \subseteq D$ and $S(D) \subseteq D$,*

(ii) *S is ψ-condensing or S is a nonlinear contraction,*

(iii) *T and S are weakly isotone.*

Then there exists a unique point $x^* \in D$ such that $ASx^* + \lambda FSx^* = x^*$.

Theorem 14. *Let A, F, λ, M, and c be as in Theorem 9, $T = (A + \lambda F)^{-1}$, and let H be an ordered Hilbert space. Let D be a nonempty closed convex subset of H and ψ a set additive measure of weak noncompactness on H. Assume that S, T satisfy the following:*

(i) *T is a ww-compact mapping,*

(ii) *S is continuous ws-compact and ψ-condensing or S is continuous ws-compact and nonlinear contraction,*

(iii) *T and S are weakly isotone,*

(iv) *$T(D) \subseteq D$ and $S(D) \subseteq D$.*

Then there exists a unique $x^* \in D$ such that $ASx^* + \lambda FSx^* = x^*$.

Theorem 15. *Let H, A, F, λ, M, and c be as in Theorem 9 and $T = (A + \lambda F)^{-1}$ sequentially weakly continuous. Assume that D is nonempty closed bounded convex subset of H and $S : D \to H$ is sequentially weakly continuous mapping satisfying the following:*

(i) *$S(D)$ is relatively weakly compact,*

(ii) *$(x = Tx + Sy; y \in D) \Rightarrow x \in D$.*

Then there exists a unique point $x^* \in D$ such that $(A + \lambda F)o(I - S)x^* = x^*$.

Proof. By Theorem 9, $T = (A + \lambda F)^{-1} : H \to H$ is a strict contraction with contractive constant $\alpha = 1/(c - \lambda M) < 1$. Now Theorem 2.1 [6] implies that there exists an $x^* \in D$ such that $Sx^* + Tx^* = x^*$ which implies that $(A + \lambda F)o(I - S)x^* = x^*$. □

Theorem 16. *Let A, F, λ, M, and c be as in Theorem 9, $T = (A + \lambda F)^{-1}$, and let H be an ordered Hilbert space. Let D be a nonempty closed convex subset of H and ψ a set additive measure of noncompactness on H. Let $S : H \to H$ be a mapping satisfying the following:*

(i) *$S(D) \subseteq D$,*

(ii) *S is continuous and ψ-condensing or (S is a nonlinear contraction),*

(iii) *PT and S are weakly isotone,*

where P is the proximity map on D. Then there exists $x_0 \in D$ such that

$$\begin{aligned} \|x_0 - Tx_0\| &= \|Sx_0 - Tx_0\| \\ &= d(Tx_0, D) = d\left(Tx_0, \overline{I_D(x_0)}\right). \end{aligned} \tag{31}$$

More precisely, either

(1) S and T have a common fixed point $x_0 \in D$, or

(2) there exists $x_0 \in \partial D$ with

$$\begin{aligned} 0 < \|Tx_0 - Sx_0\| &= \|Tx_0 - x_0\| \\ &= d(Tx_0, D) = d\left(Tx_0, \overline{I_D(x_0)}\right). \end{aligned} \tag{32}$$

Proof. Let P be the proximity map on D; that is, for each $x \in H$, we have $\|Px - x\| = d(x, D)$. It is well known that P is non-expansive in H. As T is shrinking map, so $PT : D \to D$ is also shrinking mapping. By Theorem 4.1 (or 4.2), there exists $x_0 \in M$ such that $x_0 = Sx_0 = PTx_0$. Thus we obtain, as in Theorem 4.1 [8] the desired conclusion. □

Following the proof of Corollary 4.5 [8] and using Theorem 16, we obtain the following common fixed point theorem.

Theorem 17. *Let A, F, λ, M, and c be as in Theorem 9, $T = (A + \lambda F)^{-1}$, and let H be an ordered Hilbert space. Let D be a nonempty closed convex subset of H and ψ a set additive measure of noncompactness on H. Assume that $S : H \to H$ is a mapping satisfying the following:*

(i) *$S(D) \subseteq D$,*

(ii) *S is continuous and ψ-condensing or (S is a nonlinear contraction),*

(iii) *PT and S are weakly isotone,*

where P is the proximity map on D. Suppose that T satisfies one of the following conditions for each $x \in \partial D$, with $x \neq Tx$:

(i) *$\|Tx - y\| < \|Tx - x\|$ for some y in $\overline{I_D(x)}$;*

(ii) *there is a γ such that $|\gamma| < 1$ and $\gamma x + (1-\gamma)Tx \in \overline{I_D(x)}$;*

(iii) *$Tx \in \overline{I_D(x)}$;*

(iv) *for each $\gamma \in (0,1)$, $x \neq \gamma Tx$;*

(v) *there exists an $\alpha \in (1,\infty)$ such that, $\|Tx - x\|^{\alpha} \geq \|Tx\|^{\alpha} - r^{\alpha}$;*

(vi) *there exists $\beta \in (0,1)$ such that, $\|Tx - x\|^{\beta} \leq \|Tx\|^{\beta} - r^{\beta}$.*

Then $F(S) \cap F(T) \neq \emptyset$.

4. An Application

Fixed point theorems for certain operators have found various applications in differential and integral equations (see [7–10] and references therein). In this section, we present an application of our Theorem 7 to establish a solution of a nonlinear integral equation.

Let $K : [0,1] \times [0,1] \to \mathbf{R}$ be a continuous function, and suppose that K is symmetric (i.e., $K(x,y) = K(y,x)$ for all $x, y \in [0,1]$). We consider the linear operator $B : L^2_{\mathbf{R}}[0,1] \to L^2_{\mathbf{R}}[0,1]$ defined by $Bu(t) = \int_0^1 K(t,s)u(s)ds$. It is easy to observe that B is a self-adjoint operator. Now let $A : L^2_{\mathbf{R}}[0,1] \to L^2_{\mathbf{R}}[0,1]$ defined by $Au(t) = u(t) + \iint_0^1 K(t,s) K(s,z)u(z)\,dz\,ds$; that is, $A = I + B^2$ where I is the identity of $L^2_{\mathbf{R}}[0,1]$. A is a self-adjoint strongly positive operator satisfying $\langle Au, u\rangle_2 = \langle u, u\rangle_2 + \langle B^2u, u\rangle_2 \geq \|u\|_2^2$($\langle\cdot,\cdot\rangle_2$ and $\|\cdot\|_2$ signify the inner product and the norm in $L^2_{\mathbf{R}}[0,1]$).

Define $f : [0,1] \times \mathbf{R} \to \mathbf{R}$, $(t,u) \mapsto f(t,u)$, having partial derivative of first order in the second variable u and $|(\partial f/\partial u)(t,u)| \leq M$ for all $(t,u) \in [0,1] \times \mathbf{R}(M > 0)$. $F : L^2_{\mathbf{R}}[0,1] \to L^2_{\mathbf{R}}[0,1]$ defined by $Fu(t) = f(t, u(t))$ is a Lipschitz operator of constant M from $L^2_{\mathbf{R}}[0,1]$ into $L^2_{\mathbf{R}}[0,1]$.

Define nonlinear integral equation

$$u(t) + \int_0^1 \int_0^1 K(t,s)\,K(s,z)\,u(z)\,dz\,ds + \lambda f(t, u(t)) = g(t), \quad g \in L^2_{\mathbf{R}}[0,1], \tag{33}$$

where $\lambda \in (0; 1/M)$. All of the conditions of Theorem 7 are satisfied, so the above-mentioned integral equation has a unique solution for all $\lambda \in (0; 1/M)$. Moreover, by Theorem 8 the solution depends continuously on the free term g.

Acknowledgment

The work of first author was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0087.

References

[1] H. Amann, "On the unique solvability of semilinear operator equations in Hilbert spaces," *Journal de Mathématiques Pures et Appliquées*, vol. 61, no. 2, pp. 149–175, 1982.

[2] H. Amann, "Saddle points and multiple solutions of differential equations," *Mathematische Zeitschrift*, vol. 169, no. 2, pp. 127–166, 1979.

[3] D. Teodorescu, "A contractive method for a semilinear equation in Hilbert spaces," *Analele Universitatii Bucuresti Matematica*, vol. 54, no. 2, pp. 289–292, 2005.

[4] J. Banaś and J. Rivero, "On measures of weak noncompactness," *Annali di Matematica Pura ed Applicata*, vol. 151, pp. 213–224, 1988.

[5] F. S. de Blasi, "On a property of the unit sphere in a Banach space," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 21, no. 3-4, pp. 259–262, 1977.

[6] M. A. Taoudi, "Krasnosel'skii type fixed point theorems under weak topology features," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 478–482, 2010.

[7] R. P. Agarwal, N. Hussain, and M.-A. Taoudi, "Fixed point theorems in ordered Banach spaces and applications to nonlinear integral equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 245872, 15 pages, 2012.

[8] N. Hussain, A. R. Khan, and R. P. Agarwal, "Krasnosel'skii and Ky Fan type fixed point theorems in ordered Banach spaces," *Journal of Nonlinear and Convex Analysis*, vol. 11, no. 3, pp. 475–489, 2010.

[9] N. Hussain and M. A. Taoudi, "Krasnosel'skii-type fixed point theorems with applications to Volterra integral equations," *Fixed Point Theory and Applications*, vol. 2013, article 196, 2013.

[10] K. Latrach, M. A. Taoudi, and A. Zeghal, "Some fixed point theorems of the Schauder and the Krasnosel'skii type and application to nonlinear transport equations," *Journal of Differential Equations*, vol. 221, no. 1, pp. 256–271, 2006.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 704251, 10 pages
http://dx.doi.org/10.1155/2013/704251

Research Article

A Minimax Theorem for $\overline{L}^0$-Valued Functions on Random Normed Modules

Shien Zhao[1] and Yuan Zhao[2]

[1] *Elementary Educational College, Capital Normal University, Beijing 100048, China*
[2] *Department of Basic Sciences, Hebei Finance University, Baoding 071051, China*

Correspondence should be addressed to Shien Zhao; zsefh@cnu.edu.cn

Received 9 April 2013; Accepted 25 July 2013

Academic Editor: Pei De Liu

We generalize the well-known minimax theorems to $\overline{L}^0$-valued functions on random normed modules. We first give some basic properties of an L^0-valued lower semicontinuous function on a random normed module under the two kinds of topologies, namely, the (ε, λ)-topology and the locally L^0-convex topology. Then, we introduce the definition of random saddle points. Conditions for an L^0-valued function to have a random saddle point are given. The most greatest difference between our results and the classical minimax theorems is that we have to overcome the difficulty resulted from the lack of the condition of compactness. Finally, we, using relations between the two kinds of topologies, establish the minimax theorem of $\overline{L}^0$-valued functions in the framework of random normed modules and random conjugate spaces.

1. Introduction

The classical minimax theorem, which originated from game theory, is an important content of nonlinear analysis. It has been applied in many fields, such as optimization theory, different equations, and fixed point theory. The first mathematical formulation was established by Neumann in 1928 [1]. Since then, various generalizations of Neumann's minimax theorem have been given by several scholars; see [2–7]. The classical minimax theorems for extended real-valued functions $L : A \times B \to \overline{R}$ show that, under some suitable conditions of compactness, convexity, and continuity, the equality

$$\inf_{y\in B}\sup_{x\in A} f(x, y) = \sup_{x\in A}\inf_{y\in B} f(x, y) \tag{1}$$

holds. In 1980s, to meet the needs of vectorial optimization, minimax problems in this more general setting have been investigated; see [4–7]. In this paper, we generalize the well-known minimax theorems to $\overline{L}^0$-valued functions on random normed modules (briefly RN modules).

Random metric theory is based on the idea of randomizing the classical space theory of functional analysis. All the basic notions such as RN modules and random inner product modules (briefly RIP modules) and random locally convex modules (briefly RLC modules) together with their random conjugate spaces were naturally presented by Guo in the course of the development of random functional analysis, (cf. [8–12]). In the last ten years, random metric theory and its applications in the theory of conditional risk measures have undergone a systematic and deep development. Especially after 2009, in [13] Guo gives the relations between the basic results currently available derived from the two kinds of topologies, namely, the (ε, λ)-topology and the locally L^0-convex topology. In [14], Guo gives some basic results on L^0-convex analysis together with some applications to conditional risk measures and studies the relations among the three kinds of conditional convex risk measures. Furthermore, in [15] Guo et al. establish a complete random convex analysis over RN modules and RLC modules by simultaneously considering the two kinds of topologies in order to provide a solid analytic foundation for the module approach to conditional risk measures. These results pave the way for further research of the theory of random convex analysis and conditional risk measures.

Motivated by the recent applications of random metric theory to conditional risk measures [13, 16, 17], in this paper, we establish a minimax theorem for $\bar{L}^0$-valued functions on random normed modules. Theorem 1, which is the main result of this paper, can be seen as a natural extension of the classical minimax theorems and has potential applications in the further study of conditional risk measures.

To introduce the main result of this paper, let us first recall some notation and terminology as follows:

K: the scalar field R of real numbers or C of complex numbers;

$(\Omega, \mathcal{F}, P)$: a probability space;

$L^0(\mathcal{F}, K)$ = the algebra of equivalence classes of K-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$;

$L^0(\mathcal{F}) = L^0(\mathcal{F}, R)$;

$\bar{L}^0(\mathcal{F})$ = the set of equivalence classes of extended real-valued $\mathcal{F}$-measurable random variables on $(\Omega, \mathcal{F}, P)$.

Theorem 1. *Let $(E, \|\cdot\|)$ be a random strictly convex and random reflexive random normed module over R with base $(\Omega, \mathcal{F}, P)$, A and B $\mathcal{T}_c$-closed, $L^0(\mathcal{F})$-convex subset with the countable concatenation property of E and $L : E\times E \to \bar{L}^0(\mathcal{F})$. If L satisfies the following:*

(1) *for any fixed $p \in B$, $L(\cdot, p) : E \to \bar{L}^0(\mathcal{F})$ is proper $L^0(\mathcal{F})$-convex, $\mathcal{T}_c$-lower semicontinuous function on A and has the local property;*

(2) *for any fixed $u \in A$, $-L(u, \cdot) : E \to \bar{L}^0(\mathcal{F})$ is proper $L^0(\mathcal{F})$-convex, $\mathcal{T}_c$-lower semicontinuous function on B and has the local property;*

(3) *A and B are a.s. bounded,*

then there exists a random saddle point $(u_0, p_0) \in A \times B$ of L with respect to $A \times B$, namely,

$$\bigwedge_{u\in A} \bigvee_{p\in B} L(u, p) = L(u_0, p_0) = \bigvee_{p\in B} \bigwedge_{u\in A} L(u, p). \quad (2)$$

Theorem 1 has the same shape as the classical minimax theorems, and its proof follows a known pattern in [18], but it is not trivial since the complicated stratification structure in the random setting needs to be considered. Besides, the most greatest difference between our results and the classical minimax theorems is that we have to overcome the difficulty resulted from the lack of the condition of compactness. In order to overcome this obstacle, we make full use of the respective advantages of the (ε, λ)-topology and the locally L^0-convex topology. In [13], Guo pointed out that these two kinds of topologies can complement each other (see also Propositions 14 and 15 in this paper), and we can consider them simultaneously in some cases. Specifically, on one hand, in Theorem 1 we require the functions to be $\mathcal{T}_c$-lower semicontinuous; namely, the functions are lower semicontinuous under the locally L^0-convex topology, because we need a very important inequality to establish this theorem; see Definition 21 and Proposition 22 of this paper for details. On the other hand, in the process of the proof of Theorem 1 we must employ the (ε, λ)-topology also, because the (ε, λ)-topology is very natural from the viewpoint of probability theory, and under this type of topology we can use the relations between random normed modules and classical normed spaces to prove the main result; see the proof of Theorem 1 in Section 4.

The remainder of this paper is organized as follows: in Section 2 we will briefly collect some necessary known facts; in Section 3 we will give some basic properties of an $\bar{L}^0$-valued lower semicontinuous function on a random normed module under the two kinds of topologies, namely, Theorems 26 and 28; in Section 4 we will present the definition of random saddle points and prove our main result.

2. Preliminaries

It is well known from [19] that $\bar{L}^0(\mathcal{F})$ is a complete lattice under the ordering $\le$: $\xi \le \eta$ if and only if $\xi^0(\omega) \le \eta^0(\omega)$, for almost all ω in Ω (briefly, a.s.), where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η, respectively. Furthermore, every subset G of $\bar{L}^0(\mathcal{F})$ has a supremum, denoted by $\bigvee G$, and an infimum, denoted by $\bigwedge G$. Finally $L^0(\mathcal{F})$, as a sublattice of $\bar{L}^0(\mathcal{F})$, is also a complete lattice in the sense that every subset with upper bound has a supremum. The pleasant properties of $\bar{L}^0(\mathcal{F})$ are summarized as follows.

Proposition 2 (see [19]). *For every subset G of $\bar{L}^0(\mathcal{F})$, there exist countable subsets $\{a_n \mid n \in N\}$ and $\{b_n \mid n \in N\}$ of G such that $\bigvee G = \bigvee_{n\ge1} a_n$ and $\bigwedge G = \bigwedge_{n\ge1} b_n$. Further, if G is directed (dually directed) with respect to $\le$, then the above $\{a_n \mid n \in N\}$ (accordingly, $\{b_n \mid n \in N\}$) can be chosen as nondecreasing (correspondingly, nonincreasing) with respect to $\le$.*

Specially, $L^0_+ = \{\xi \in L^0(\mathcal{F}) \mid \xi \ge 0\}$, $L^0_{++} = \{\xi \in L^0(\mathcal{F}) \mid \xi > 0 \text{ on } \Omega\}$, where for $A \in \mathcal{F}$, "$\xi > \eta$" on A means $\xi^0(\omega) > \eta^0(\omega)$ a.s. on A for any chosen representatives ξ^0 and η^0 of ξ and η, respectively. As usual, $\xi > \eta$ means $\xi \ge \eta$ and $\xi \ne \eta$. For any $A \in \mathcal{F}$, A^c denotes the complement of A, and $\tilde{A} = \{B \in \mathcal{F} \mid P(A\Delta B) = 0\}$ denotes the equivalence class of A, where Δ is the symmetric difference operation, I_A is the characteristic function of A, and $\tilde{I}_A$ is used to denote the equivalence class of I_A; given two ξ and η in $L^0(\mathcal{F})$, and $A = \{\omega \in \Omega \mid \xi^0 \ne \eta^0\}$, where ξ^0 and η^0 are arbitrarily chosen representatives of ξ and η respectively, then we always write $[\xi \ne \eta]$ for the equivalence class of A and $I_{[\xi\ne\eta]}$ for $\tilde{I}_A$; one can also understand the implication of such notation as $I_{[\xi\le\eta]}$, $I_{[\xi<\eta]}$ and $I_{[\xi=\eta]}$.

For an arbitrarily chosen representative ξ^0 of $\xi \in L^0(\mathcal{F}, K)$, define the two random variables $(\xi^0)^{-1}$ and $|\xi^0|$ by $(\xi^0)^{-1}(\omega) = 1/\xi^0(\omega)$ if $\xi^0(\omega) \ne 0$, and $(\xi^0)^{-1}(\omega) = 0$ otherwise, and by $|\xi^0|(\omega) = |\xi^0(\omega)|$, for all $\omega \in \Omega$. Then the equivalent class ξ^{-1} of $(\xi^0)^{-1}$ is called the generalized inverse of ξ, and the equivalent class $|\xi|$ of $|\xi^0|$ is called the absolute value of ξ.

Now, we introduce the definition of a random normed module, which is a random generalization of an ordinary normed space, and give some important examples.

Definition 3 (see [11, 20]). An ordered pair $(E, \|\cdot\|)$ is called a random normed space (briefly, an RN space) over K with base $(\Omega, \mathcal{F}, P)$ if E is a linear space over K, and $\|\cdot\|$ is a mapping from E to $L^0_+(\mathcal{F})$ such that the following are satisfied:

(RN-1) $\|\alpha x\| = |\alpha|\|x\|$, for all $\alpha \in K$ and $x \in E$;

(RN-2) $\|x\| = 0$ implies $x = \theta$ (the null element of E);

(RN-3) $\|x + y\| \le \|x\| + \|y\|$, for all $x, y \in E$.

Here $\|\cdot\|$ is called the random norm on E and $\|x\|$ the random norm of $x \in E$ (if $\|\cdot\|$ only satisfies (RN-1) and (RN-3) above, it is called a random seminorm on E).

Furthermore, if, in addition, E is a left module over the algebra $L^0(\mathcal{F}, K)$ (briefly, an $L^0(\mathcal{F}, K)$-module) such that

(RNM-1) $\|\xi x\| = |\xi|\|x\|$, for all $\xi \in L^0(\mathcal{F}, K)$ and $x \in E$,

then $(E, \|\cdot\|)$ is called a random normed module (briefly, an RN module) over K with base $(\Omega, \mathcal{F}, P)$, and the random norm $\|\cdot\|$ with the property (RNM-1) is also called an L^0-norm on E (a mapping only satisfying (RN-3) and (RNM-1) above is called an L^0-seminorm on E).

Example 4. Let $L^0(\mathcal{F}, B)$ be the $L^0(\mathcal{F}, K)$-module of equivalence classes of $\mathcal{F}$-random variables (or strongly $\mathcal{F}$-measurable functions) from $(\Omega, \mathcal{F}, P)$ to a normed space $(B, \|\cdot\|)$ over K. $\|\cdot\|$ induces an L^0-norm (still denoted by $\|\cdot\|$) on $L^0(\mathcal{F}, B)$ by $\|x\| :=$ the equivalence class of $\|x^0(\cdot)\|$ for all $x \in L^0(\mathcal{F}, B)$, where $x^0(\cdot)$ is a representative of x. Then $(L^0(\mathcal{F}, B), \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$. Specially, $L^0(\mathcal{F}, K)$ is an RN module, and the L^0-norm $\|\cdot\|$ on $L^0(\mathcal{F}, K)$ is still denoted by $|\cdot|$.

The next example of RN modules $L^p_{\mathcal{F}}(\mathcal{E})(1 \le p \le +\infty)$ is constructed by Filipović et al. in [16].

Example 5. Let $(\Omega, \mathcal{E}, P)$ be a probability space and $\mathcal{F}$ a σ-subalgebra of $\mathcal{E}$. Define $|||\cdot|||_p : L^0(\mathcal{E}) \to \overline{L}^0_+(\mathcal{F})$ by

$$|||x|||_p = \begin{cases} E\left[|x|^p \mid \mathcal{F}\right]^{1/p}, & \text{when } 1 \le p < \infty, \\ \bigwedge\left\{\xi \in \overline{L}^0_+(\mathcal{F}) \mid |x| \le \xi\right\}, & \text{when } p = +\infty, \end{cases} \tag{3}$$

for all $x \in L^0(\mathcal{E})$.

Denote $L^p_{\mathcal{F}}(\mathcal{E}) = \{x \in L^0(\mathcal{E}) | |||x|||_p \in L^0_+(\mathcal{F})\}$, then $(L^p_{\mathcal{F}}(\mathcal{E}), |||\cdot|||_p)$ is an RN module over R with base $(\Omega, \mathcal{F}, P)$ and $L^p_{\mathcal{F}}(\mathcal{E}) = L^0(\mathcal{F}) \cdot L^p(\mathcal{E}) = \{\xi x \mid \xi \in L^0(\mathcal{F})$ and $x \in L^p(\mathcal{E})\}$.

To put some important classes of stochastic processes into the framework of RN modules, Guo constructed a more general RN module $L^p_{\mathcal{F}}(S)$ in [13] for each $p \in [1, +\infty]$ as follows.

Example 6. Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{E}, P)$ and $\mathcal{F}$ a σ-subalgebra. Define $|||\cdot|||_p : E \to \overline{L}^0_+(\mathcal{F})$ by

$$|||\cdot|||_p = \begin{cases} E\left[\|x\|^p \mid \mathcal{F}\right]^{1/p}, & \text{when } 1 \le p < \infty, \\ \bigwedge\left\{\xi \in \overline{L}^0_+(\mathcal{F}) \mid \|x\| \le \xi\right\}, & \text{when } p = +\infty, \end{cases} \tag{4}$$

for all $x \in E$.

Denote $L^p_{\mathcal{F}}(E) = \{x \in S \mid |||x|||_p \in L^0_+(\mathcal{F})\}$; then $(L^p_{\mathcal{F}}(E), |||\cdot|||_p)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$. When $E = L^0(\mathcal{E})$, $L^p_{\mathcal{F}}(E)$ is exactly $L^p_{\mathcal{F}}(\mathcal{E})$.

Remark 7. For a given RN module $(E, \|\cdot\|)$ over K with base (Ω, F, P) and a given real or extended real number p such that $1 \le p \le +\infty$, define $\|\cdot\|_p : E \to [0, +\infty]$ by

$$\|g\|_p = \begin{cases} \left(\int_\Omega (\|g\|)^p\right)^{1/p}, & \text{if } 1 \le p < +\infty, \\ \text{the } P\text{-essential supremum}, & \text{if } p = +\infty. \end{cases} \tag{5}$$

Let $L^p(E) = \{g \in E : \|g\|_p < +\infty\}$. As mentioned in [13], $(L^p(E), \|\cdot\|_p)$ is a normed space over K and is further a Banach space if $(E, \|\cdot\|)$ is complete.

For each RN module $(E, \|\cdot\|)$ over K with base $(\Omega, \mathcal{F}, P)$, $\|\cdot\|$ can induce two kinds of topologies, namely, the (ε, λ)-topology and the locally L^0-convex topology.

Definition 8 (see [12–14]). Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. For any positive real numbers ε and λ such that $0 < \lambda < 1$, let $N_\theta(\varepsilon, \lambda) = \{x \in E \mid P\{\omega \in \Omega \mid \|x\|(\omega) < \varepsilon\} > 1 - \lambda\}$; then $\{N_\theta(\varepsilon, \lambda) \mid \varepsilon > 0, 0 < \lambda < 1\}$ is easily verified to be a local base at the null vector θ of some Hausdorff linear topology. The linear topology is called the (ε, λ)-topology for E induced by $\|\cdot\|$.

From now on, the (ε, λ)-topology for each RN module is always denoted by $\mathcal{T}_{\varepsilon,\lambda}$ when no confusion occurs.

Proposition 9 (see [12–14]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. Then one has the following statements.*

(1) *The (ε, λ)-topology for $L^0(\mathcal{F}, K)$ is exactly the topology of convergence in probability P, and $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})$ is a topological algebra over K.*

(2) *If $(E, \|\cdot\|)$ is an RN modules, then $(E, \mathcal{T}_{\varepsilon,\lambda})$ is a topological module over the topological algebra $L^0(F, K)$.*

(3) *A net $\{x_\delta, \delta \in \Gamma\}$ converges in the (ε, λ)-topology to some x in E if and only if $\{\|x_\delta - x\|, \delta \in \Gamma\}$ converges in probability P to 0.*

The following locally L^0-convex topology is easily seen to be much stronger than the (ε, λ)-topology and was first introduced by Filipović et al. in [16].

Definition 10 (see [14, 16]). Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. For any $\varepsilon \in L^0_{++}$, let $N_\theta(\varepsilon) = \{x \in E \mid \|x\| \le \varepsilon\}$. A subset G of E is called $\mathcal{T}_c$-open if for each $x \in G$ there exists some $N_\theta(\varepsilon)$ such that $x + N_\theta(\varepsilon) \subset G$, and $\mathcal{T}_c$ denotes the family of $\mathcal{T}_c$-open subsets of E. Then it is easy to see that $(E, \mathcal{T}_c)$ is a Hausdorff topological group with respect to the addition on E. $\mathcal{T}_c$ is called the locally L^0-convex topology for E induced by $\|\cdot\|$.

From now on, the locally L^0-convex topology for each random locally convex space is always denoted by $\mathcal{T}_c$ when no confusion occurs.

Proposition 11 (see [13, 14, 16]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. Then*

(1) *$L^0(\mathcal{F}, K)$ is a topological ring endowed with its locally L^0-convex topology;*

(2) *E is a topological module over the topological ring $L^0(\mathcal{F}, K)$ when E and $L^0(\mathcal{F}, K)$ are endowed with their respective locally L^0-convex topologies;*

(3) *a net $\{x_\alpha \mid \alpha \in \Gamma\}$ in E converges in the locally L^0-convex topology to $x \in E$ if and only if $\{\|x_\alpha - x\| \mid \alpha \in \Gamma\}$ converges in the locally L^0-convex topology of $L^0(\mathcal{F}, K)$ to* 0.

$\mathcal{T}_c$ is called locally L^0-convex because it has a striking local base $\mathcal{U}_\theta = \{B(\varepsilon) \mid \varepsilon \in L^0_{++}\}$, each member U of which is as follows:

(i) L^0-convex: $\xi \cdot x + (1 - \xi) \cdot y \in U$ for any $x, y \in U$ and $\xi \in L^0_+$ such that $0 \leqslant \xi \leqslant 1$;

(ii) L^0-absorbent: there is $\xi \in L^0_{++}$ for each $x \in E$ such that $x \in \xi \cdot U$;

(iii) L^0-balanced: $\xi \cdot x \in U$ for any $x \in U$ and any $\xi \in L^0(F, K)$ such that $|\xi| \leqslant 1$.

Remark 12. Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$ endowed with the locally L^0-convex topology $\mathcal{T}_c$. Although E can be viewed as a linear space over K with scalar multiplication $\alpha \cdot x := (\alpha \cdot 1) \cdot x$ for $\alpha \in K$, $1 \in L^0$ and $x \in E$, $(E, \mathcal{T}_c)$ is not a topological linear space since the map $K \to (E, \mathcal{T}_c)$, $\alpha \to \alpha \cdot x$, is not necessarily continuous for $x \neq \theta$; see [16] for details.

In the sequel of this paper, for a subset G of an RN module $(E, \|\cdot\|)$, $\overline{G}_{\varepsilon,\lambda}$ denotes the $\mathcal{T}_{\varepsilon,\lambda}$-closure of G, and $\overline{G}_c$ denotes the $\mathcal{T}_c$-closure of G.

For giving the relations of the two kinds of topologies, which Guo has studied the [13], we need to introduce the definition of the countable concatenation property.

Definition 13 (see [13]). Let E be a left module over the algebra $L^0(\mathcal{F}, K)$. A formal sum $\Sigma_{n\ge1}\tilde{I}_{A_n} x_n$ for some countable partition $\{A_n, n \in N\}$ of Ω to $\mathcal{F}$ and some sequence $\{x_n \mid n \in N\}$ in E is called a countable concatenation of $\{x_n \mid n \in N\}$ with respect to $\{A_n, n \in N\}$. Furthermore a countable concatenation $\Sigma_{n\ge1}\tilde{I}_{A_n} x_n$ is well defined or $\Sigma_{n\ge1}\tilde{I}_{A_n} x_n \in E$ if there is $x \in E$ such that $\tilde{I}_{A_n} x = \tilde{I}_{A_n} x_n$, for all $n \in N$. A subset G of E is said to have the countable concatenation property if every countable concatenation $\Sigma_{n\ge1}\tilde{I}_{A_n} x_n$ with $x_n \in G$ for each $n \in N$ still belongs to G; namely, $\Sigma_{n\ge1}\tilde{I}_{A_n} x_n$ is well defined and there exists $x \in G$ such that $x = \Sigma_{n\ge1}\tilde{I}_{A_n} x_n$.

Proposition 14 (see [13]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. Then E is $\mathcal{T}_{\varepsilon,\lambda}$-complete if and only if E is $\mathcal{T}_c$-complete and has the countable concatenation property.*

Proposition 15 (see [13]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$ and $G \subseteq E$ a subset with the countable concatenation property. Then $\overline{G}_{\varepsilon,\lambda} = \overline{G}_c$.*

Now, we introduce the definition of random conjugate spaces of RN modules.

Definition 16 (see [7, 10, 11, 13]). Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. A linear operator f from E to $L^0(\mathcal{F}, K)$ is said to be an a.s. bounded random linear functional on E if there exists some ξ in $L^0_+(\mathcal{F}, R)$ such that $|f(x)| \le \xi \cdot \|x\|$, for all $x \in E$. Denote by E^* the linear space of a.s. bounded random linear functionals on E with the pointwise addition and scalar multiplication on linear operators; define $\|\cdot\|^* : E^* \to L^0_+(\mathcal{F}, R)$ by $\|f\|^* = \bigwedge\{\xi \in L^0_+(\mathcal{F}) \mid |f(x)| \le \xi\|x\|, \text{for all } x \in E\}$ for all $f \in E^*$ and define $\cdot : L^0(\mathcal{F}, K) \times E^* \to E^*$ by $(\eta \cdot f)(x) = \eta(f(x))$ for all $\eta \in L^0(\mathcal{F}, K)$, $f \in E^*$, and $x \in E$; then it is easy to check that $(E^*, \|\cdot\|^*)$ is also an RN module over K with base $(\Omega, \mathcal{F}, P)$, called the random conjugate space of $(E, \|\cdot\|)$.

Guo et al. gave the topological characterizations of an a.s. bounded random linear functional in [10, 11, 16] as follows: let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$, $E^*_{\varepsilon,\lambda}$ the $L^0(\mathcal{F}, K)$-module of continuous module homomorphisms from $(E, \mathcal{T}_{\varepsilon,\lambda})$ to $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon,\lambda})$, and E^*_c the $L^0(F, K)$-module of continuous module homomorphisms from $(E, \mathcal{T}_c)$ to $(L^0(F, K), \mathcal{T}_c)$, then it was proved that $E^*_{\varepsilon,\lambda} = E^*_c$. In fact, Guo et al. also proved in [10, 11, 13] $\|f\|^* = \bigvee\{|f(x)| \mid x \in E \text{ and } \|x\| \le 1\}$ for any $f \in E$.

Let $(E, \|\cdot\|)$ be an RN module, E^{**} denotes $(E^*)^*$, and the canonical embedding mapping $J : E \to E^{**}$ defined by $(Jx)(f) = f(x)$, for all $x \in E$ and for all $f \in E^*$, is random-norm preserving. If J is subjective, then E is called random reflexive. In [13] Guo proved that the random reflexivity is independent of a special choice of $\mathcal{T}_{\varepsilon,\lambda}$ and $\mathcal{T}_c$. The following propositions are very essential relations, which are established by Guo in [12, 21], between classical reflexive spaces and random reflexive RN modules.

Proposition 17 (see [21]). *$L^0(\mathcal{F}, B)$ is random reflexive if and only if B is a reflexive Banach space.*

Proposition 18 (see [12]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$. Then E is random reflexive if and only if $(L^p(E), \|\cdot\|_p)$ is reflexive, where $1 < p < +\infty$.*

Proposition 19 (see [12]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$, $1 \le p < +\infty$ and $1 < q \le +\infty$ a pair of Hölder conjugate numbers. Then $(L^q(E^*), \|\cdot\|_q)$ is isometrically isomorphic with the classical conjugate space of $(L^p(E), \|\cdot\|)$, denoted by $(L^p(E))'$, under the canonical mapping $T : L^q(E^*) \to (L^p(E))'$ defined as follows. For each $f \in L^q(E^*)$, T_f (denoting $T(f)$): $L^p(E) \to K$ is defined by $T_f(g) = \int_\Omega f(g)dP$ for all $g \in L^p(E)$.*

3. Some Basic Properties of $\overline{L}^0$-Valued Lower Semicontinuous Functions

In this section, we give some basic properties of $\overline{L}^0$-valued lower semicontinuous functions. First, we recall the definition of $\overline{L}^0$-valued lower semicontinuous functions under two kinds of topologies, which was presented by Guo in [14] for the first time.

Let E be a left module over the algebra $L^0(\mathcal{F})$. The effective domain of function $f : E \to \overline{L}^0(\mathcal{F})$ is denoted by $\mathrm{dom}(f) := \{x \in E \mid f(x) \in L^0(\mathcal{F})\}$. The epigraph of f is denoted by $\mathrm{epi}(f) := \{(x, y) \in E \times L^0(\mathcal{F}) \mid f(x) \le y\}$. The function f is called proper if $f(x) > -\infty$ on Ω for every $x \in E$ and $\mathrm{dom}(f) \ne \emptyset$.

Definition 20 (see [14]). Let E be a left module over the algebra $L^0(\mathcal{F})$ and $f : E \to \overline{L}^0(\mathcal{F})$.

(1) f is $L^0(\mathcal{F})$-convex if $f(\xi x + (1-\xi)y) \le \xi f(x) + (1-\xi)f(y)$ for all x and y in E and $\xi \in L^0_+$ such that $0 \le \xi \le 1$ (here we make the convention that $0 \cdot (\pm\infty) = 0$ and $\infty - \infty = \infty$).

(2) f has the local property if $\tilde{I}_A f(x) = \tilde{I}_A f(\tilde{I}_A x)$ for all $x \in E$ and $A \in \mathcal{F}$.

(3) f is regular if $\tilde{I}_A f(x) = f(\tilde{I}_A x)$ for all $x \in E$ and $A \in \mathcal{F}$.

Definition 21 (see [14]). Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$. A function $f : E \to \overline{L}^0(\mathcal{F})$ is called $\mathcal{T}_c$-lower semicontinuous if $\mathrm{epi}(f)$ is closed in $(E, \mathcal{T}_c) \times (L^0(\mathcal{F}), \mathcal{T}_c)$. A function $f : E \to \overline{L}^0(\mathcal{F})$ is called $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous if $\mathrm{epi}(f)$ is closed in $(E, \mathcal{T}_{\varepsilon,\lambda}) \times (L^0(\mathcal{F}), \mathcal{T}_{\varepsilon,\lambda})$.

Proposition 22 (see [14]). *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property and $f : E \to \overline{L}^0(\mathcal{F})$ a function with the local property. Then the following are equivalent to each other:*

(1) *f is $\mathcal{T}_c$-lower semicontinuous;*

(2) *$\{x \in E \mid f(x) \le r\}$ is $\mathcal{T}_c$-closed for each $r \in L^0(\mathcal{F})$;*

(3) *$\underline{\lim}_\alpha f(x_\alpha) \ge f(x_0)$ for each $x_0 \in E$ and each net $\{x_\alpha, \alpha \in \Gamma\}$ in E such that $\{x_\alpha, \alpha \in \Gamma\}$ is $\mathcal{T}_c$-convergent to x_0, where $\underline{\lim}_\alpha f(x_\alpha) = \bigvee_{\alpha\in\Gamma}(\bigwedge_{\beta\ge\alpha} f(x_\beta))$.*

Remark 23. Proposition 22 first occurred in [16] where the countable concatenation property of E was not assumed, but this condition should be added (see [14] for details).

For $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous functions, we only have the following proposition.

Proposition 24 (see [14]). *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$ and $f : E \to \overline{L}^0(\mathcal{F})$ a function. Then one has the following statements:*

(1) *f is $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous if $\underline{\lim}_\alpha f(x_\alpha) \ge f(x_0)$ for each $x_0 \in E$ and each net $\{x_\alpha, x \in \Gamma\}$ in E such that $\{x_\alpha, \alpha \in \Gamma\}$ is $\mathcal{T}_{\varepsilon,\lambda}$-convergent to x_0;*

(2) *$\{x \in E \mid f(x) \le r\}$ is $\mathcal{T}_{\varepsilon,\lambda}$-closed for each $r \in L^0(\mathcal{F})$ if f is $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous.*

If we define f to be lower semicontinuous via "$\underline{\lim}_\alpha f(x_\alpha) \ge f(x)$ for all net $\{x_\alpha, \alpha \in \Lambda\}$ in E such that it converges in the (ε, λ)-topology to some $x \in E$", the notion is, however, meaningless in the random setting, since we can construct an RN module E and a $\mathcal{T}_{\varepsilon,\lambda}$-continuous L^0-convex function f from E to $L^0(\mathcal{F})$, whereas f is not a $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous function. Hence, we cannot use this inequality for $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous functions. Since this inequality is very important for the proof of Theorem 1 (see Section 4 for details), we can only establish $\overline{L}^0$-valued minimax theorems for $\mathcal{T}_c$-lower semicontinuous functions.

Proposition 25 (see [14]). *Let $(E, \|\cdot\|)$ be an RN module over K with base $(\Omega, \mathcal{F}, P)$ such that E has the countable concatenation property and $f : E \to \overline{L}^0(\mathcal{F})$ a function with the local property. Then f is $\mathcal{T}_{\varepsilon,\lambda}$-lower semicontinuous if and only if f is $\mathcal{T}_c$-lower semicontinuous, specially this is true for an $L^0(\mathcal{F})$-convex function f.*

Now, we give some important properties of $\overline{L}^0$-valued lower semicontinuous functions on RN modules. To pave the way for Theorem 26, we first introduce some notation: if $(E, \|\cdot\|)$ is an RN module over K with base $(\Omega, \mathcal{F}, P)$, Π denotes the set of all probability measures equivalent to P on $(\Omega, \mathcal{F})$, $L^p_Q(E) = \{x \in E \mid \int_\Omega \|x\|^p dQ < +\infty\}$, where $Q \in \Pi$, and $\|\cdot\|^Q_p$ denotes the norm on $L^p_Q(E)$, namely, $\|x\|^Q_p = (\int_\Omega \|x\|^p dQ)^{1/p}$ for any $x \in L^p_Q(E)$.

Theorem 26. *Let $(E, \|\cdot\|)$ be a random reflexive RN module over R with base $(\Omega, \mathcal{F}, P)$, $G \subset E$ a $\mathcal{T}_c$-closed, $L^0(\mathcal{F})$-convex, and a.s. bounded set with the countable concatenation property and $f : E \to \overline{L}^0(\mathcal{F})$ a $\mathcal{T}_c$-lower semicontinuous function with the local property. If $f|_G$ is proper, then f is bounded from below on G.*

Proof. Let $\eta = \vee\{\|x\| : x \in G\}$; then it is easy to see that $\eta \in L^0_+$. If f is not bounded from below on G, then there exists $B \in \mathcal{F}$ such that $P(B) > 0$ and

$$\bigwedge \{f(x) \mid x \in G\} = -\infty \tag{6}$$

on B. Since G is $L^0(\mathcal{F})$-convex and f has the local property, it is easy to see that $\{f(x) \mid x \in G\}$ is directed. Hence there exists a sequence $\{x_n, n \in N\}$ such that $\{f(x_n), n \in N\} \searrow \bigwedge\{f(x) \mid x \in G\}$. Let $G_n = \{x \in E \mid f(x) \leq f(x_n)\} \bigcap G$; then it is clear that G_n is $\mathcal{T}_c$-closed by Definition 21. Since f has the local property and G has the countable concatenation property, for any $n \in N$, we have that G_n has the countable concatenation property and G_n is $\mathcal{T}_{\varepsilon,\lambda}$-closed by Proposition 15. By the fact that G is a.s. bounded, we can define a probability measure Q on $(\Omega, \mathcal{F})$ by $dQ/dP = 1/c(1+\eta)^2$, where $c = E[1/(1+\eta)^2]$. Then Q is equivalent to P and $\int_\Omega \|x\|^2 dQ \leq \int_\Omega (\eta^2/c(1+\eta)^2) dP < +\infty$, for any $x \in G$, which means that G is bounded in $(L^2_Q(E), \|\cdot\|^Q_2)$. Noting that replacing the probability measure P of the base space $(\Omega, \mathcal{F}, P)$ with a probability measure Q does not change the (ε, λ)-topology of E, for any given $n \in N$, we can obtain that G_n is norm-closed and convex in $(L^2_Q(E), \|\cdot\|^Q_2)$. Since $(E, \|\cdot\|)$ is a random reflexive RN module, we have that $(L^2_Q(E), \|\cdot\|^Q_2)$ is reflexive normed space from Proposition 18. Hence G_n is compact under the weak topology of the normed space $L^2_Q(E)$. Let $\mathcal{O} = \{G_n \mid n \in N\}$, then one can obtain that $\mathcal{O}$ has the finite intersection property and $\bigcap \mathcal{O} \neq \emptyset$. Let $x^* \in \bigcap \mathcal{O}$; then

$$f(x^*) = -\infty \tag{7}$$

on B, which it contradicts to the fact that $f|_G$ is proper. □

For giving Theorem 28, we need to introduce the following Proposition 27, which was established by Guo and Yang in [22] for studying Ekeland's variational principle for $\overline{L}^0$-valued functions on RN modules.

Proposition 27 (see [22]). *Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$, $G \subset E$ a subset with the countable concatenation property and $f : E \to \overline{L}^0(\mathcal{F})$ have the local property. If $f|_G$ is proper and bounded from below on G (resp., bounded from above on G), then for each $\varepsilon \in L^0_{++}(\mathcal{F})$, there exists $x_\varepsilon \in G$ such that $f(x_\varepsilon) \leq \bigwedge f(G) + \varepsilon$ (accordingly, $f(x_\varepsilon) \geq \bigvee f(G) - \varepsilon$).*

Theorem 28. *Let $(E, \|\cdot\|)$ be a random reflexive RN module over R with base $(\Omega, \mathcal{F}, P)$, $G \subset E$ a $\mathcal{T}_c$-closed, $L^0(\mathcal{F})$-convex and a.s. bounded set with the countable concatenation property and $f : E \to \overline{L}^0(\mathcal{F})$ a $\mathcal{T}_c$-lower semicontinuous and $L^0(\mathcal{F})$-convex function with the local property. If $f|_G$ is proper, then there exists $x^* \in G$ such that $f(x^*) = \bigwedge f(G)$.*

Proof. Let $\xi = \bigvee\{\|x\| \mid x \in G\}$ and $\eta = \bigwedge f(G)$. It is clear that $\xi \in L^0_+$ and $\eta \in L^0(\mathcal{F})$ by Theorem 26. Take $B_j = [j-1 \leq \xi < j]$, for all $j \in N$, then $\{B_j \mid j = 1, 2, \ldots\}$ is a countable partition of Ω to $\mathcal{F}$. For any $j \in N$; define a function $f_j : \tilde{I}_{B_j} \cdot E \to \overline{L}^0(\mathcal{F})$ as follows:

$$f_j\left(\tilde{I}_{B_j} x\right) = \tilde{I}_{B_j} f(x), \quad \forall x \in E. \tag{8}$$

Since f has the local property, for any $j \in N$, we have that $f_j(\tilde{I}_{B_j} x) = \tilde{I}_{B_j} f(\tilde{I}_{B_j} x)$ and $\tilde{I}_{B_j} \cdot \bigwedge f(G) = \bigwedge f_j(\tilde{I}_{B_j} \cdot G)$. Because $f|_G$ is proper and f is $\mathcal{T}_c$-lower semicontinuous and $L^0(\mathcal{F})$-convex, it is clear that $f_j|_{\tilde{I}_{B_j} \cdot G}$ is proper, $\mathcal{T}_c$-lower semicontinuous, and $L^0(\mathcal{F})$-convex. Next, we prove that f_j has the local property. We need only to prove that

$$\tilde{I}_A f_j\left(\tilde{I}_A \tilde{I}_{B_j} x\right) = \tilde{I}_A f_j\left(\tilde{I}_{B_j} x\right), \quad \forall A \in \mathcal{F} \bigcap B_j. \tag{9}$$

In fact, since f has the local property, for any $A \in \mathcal{F} \bigcap B_j$, we have that

$$\begin{aligned} \tilde{I}_A f_j\left(\tilde{I}_{B_j} x\right) &= \tilde{I}_A \tilde{I}_{B_j} f(x) = \tilde{I}_A \tilde{I}_{B_j} f\left(\tilde{I}_{B_j} x\right) \\ &= \tilde{I}_A \tilde{I}_{B_j} f\left(\tilde{I}_A \tilde{I}_{B_j} x\right) \\ &= \tilde{I}_A f_j\left(\tilde{I}_A \tilde{I}_{B_j} x\right). \end{aligned} \tag{10}$$

Let $\eta_j = \tilde{I}_{B_j} \cdot \eta$ for any $j \in N$. It is easy to see that $\tilde{I}_j G \subseteq (L^2(E), \|\cdot\|_2)$ is a bounded and convex set. Since $\tilde{I}_{B_j} G$ is $L^0(\mathcal{F})$-convex, $\mathcal{T}_c$-closed, and a.s. bounded in E and has the countable concatenation, we can obtain that $\tilde{I}_{B_j} G$ is $\mathcal{T}_{\varepsilon,\lambda}$-closed in E by Proposition 15. It is easy to see that $\tilde{I}_{B_j} G$ is convex and $\|\cdot\|_2$-closed in $(L^2(E), \|\cdot\|_2)$ from the fact that the topology induced by $\|\cdot\|_2$ is stronger than the (ε, λ)-topology. Since $(E, \|\cdot\|)$ is random reflexive, $(L^2(E), \|\cdot\|_2)$ is reflexive normed space, and $\tilde{I}_{B_j} G$ is compact in $L^2(E)$ under the weak topology of $L^2(E)$. For any $\varepsilon \in L^0_{++}$, define $G_j(\varepsilon) = \{\tilde{I}_{B_j} x \mid f_j(\tilde{I}_{B_j} x) \leq \tilde{I}_{B_j} \eta + \varepsilon, x \in G\}$. It is clear that $G_j(\varepsilon) \neq \emptyset$ by Proposition 27. Since $f_j|_{\tilde{I}_{B_j} \cdot G}$ is $\mathcal{T}_c$-lower semicontinuous, we have that $G_j(\varepsilon)$ is $\mathcal{T}_c$-closed. Thus, we have that $G_j(\varepsilon)$ is $L^0(\mathcal{F})$-convex and $\|\cdot\|_2$-closed in $L^2(E)$ by the fact that f_j has the local property and Proposition 15. By Hahn-Banach theorem, we have that $G_j(\varepsilon)$ is closed under the weak topology of $L^2(E)$. Take

$$\mathcal{O} = \left\{G_j(\varepsilon) \mid \varepsilon \in L^0_{++}\right\}, \tag{11}$$

it is easy to prove that $\mathcal{O}$ has the finite intersection property. Since $\tilde{I}_{B_j} G$ is compact under the weak topology of $L^2(E)$, we have that $\bigcap \mathcal{O} \neq \emptyset$. Let $x_j \in \bigcap \mathcal{O}$ for any $j \in N$ and

$$x^* = \sum_{j=1}^{\infty} \tilde{I}_{B_j} \cdot x_j. \tag{12}$$

We have that $f_j(x_j) = \eta_j$ and

$$f(x^*) = \bigwedge f(G). \tag{13}$$

This completes the proof. □

Definition 29. Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$, and G a $L^0(\mathcal{F})$-convex subset in E. $f : G \to \overline{L}^0(\mathcal{F})$ is called strictly $L^0(\mathcal{F})$-convex if

$$f(\alpha x + (1-\alpha) y) < \alpha f(x) + (1-\alpha) f(y), \quad (14)$$

for any $x, y \in G$, $x \neq y$ and $0 < \alpha < 1$ on Ω.

Corollary 30. *Let $(E, \|\cdot\|)$ be a random reflexive RN module over R with base $(\Omega, \mathcal{F}, P)$, $G \subset E$ a $\mathcal{T}_c$-closed, $L^0(\mathcal{F})$-convex and a.s. bounded set with the countable concatenation property, and $f : E \to \overline{L}^0(\mathcal{F})$ a $\mathcal{T}_c$-lower semicontinuous and strictly $L^0(\mathcal{F})$-convex function with the local property. If $f|_G$ is proper, then there exists an unique $x^* \in G$ such that $f(x^*) = \bigwedge f(G)$.*

4. Main Results

Now, we give the definition of random saddle points.

Definition 31. Let A and B be any two nonempty sets, $(u_0, p_0) \in A \times B$ and $L : A \times B \to \overline{L}^0(\mathcal{F})$. Then (u_0, p_0) is called a random saddle point of f with respect to $A \times B$ if

$$L(u_0, p) \le L(u_0, p_0) \le L(u, p_0), \quad \forall u \in A, \ p \in B. \quad (15)$$

Remark 32. Let A and B be any two nonempty sets, $L : A \times B \to \overline{L}^0(\mathcal{F})$. It is easy to see that the following statements are equivalent:

(1) $(u_0, p_0) \in A \times B$ is a random saddle point of f with respect to $A \times B$;

(2) $\bigwedge_{u \in A} \bigvee_{p \in B} L(u, p) \le \bigvee_{p \in B} \bigwedge_{u \in A} L(u, p)$;

(3) $\bigwedge_{u \in A} \bigvee_{p \in B} L(u, p) = L(u_0, p_0) = \bigvee_{p \in B} \bigwedge_{u \in A} L(u, p)$.

Before giving the proof of main result in this paper, we first recall the definition of random strictly convex RN module, which is presented by Guo and Zeng in [23] for the first time.

Definition 33 (see [23]). An RN module $(E, \|\cdot\|)$ is said to be random strictly convex if for any x and $y \in E \setminus \{\theta\}$ such that $\|x + y\| = \|x\| + \|y\|$, then there exist $A \in \mathcal{F}$ and $\xi \in L^0_+$ such that $P(A) > 0$, $\xi > 0$ on A and $\tilde{I}_A x = \xi(\tilde{I}_A y)$.

Definition 34 (see [24]). Let $(E, \|\cdot\|)$ be an RN module over R with base $(\Omega, \mathcal{F}, P)$, $x, y \in E$ and $F \in \mathcal{F}$. Then x and y are called L^0-independent on F if $\xi \tilde{I}_F = \eta \tilde{I}_F = 0$ whenever $\xi, \eta \in L^0(\mathcal{F})$ such that $\xi \tilde{I}_F x + \eta \tilde{I}_F y = \theta$.

By Definitions 33 and 34, we can obtain the following lemma easily.

Lemma 35. *Let $(E, \|\cdot\|)$ be a random strictly convex RN module over R with base $(\Omega, \mathcal{F}, P)$. Then the mapping $f : E \to L^0(\mathcal{F})$*

$$f(x) = \|x\|^2 \quad (16)$$

is strictly $L^0(\mathcal{F})$-convex.

For giving the proof of Theorem 1, we need the following lemma and remark.

Lemma 36 (Mazur lemma). *Let $(X, \|\cdot\|)$ be a normed space, $\{u_n \in X, n \in N\}$ converge to $\overline{u}$ under the weak topology on X. Then there exists a sequence $\{v_n \in X, n \in N\}$ such that it converges to $\overline{u}$ in norm, where*

$$v_n = \sum_{k=n}^{N_n} \lambda_k u_k, \qquad \sum_{k=n}^{N_n} \lambda_k = 1, \quad \lambda_k \ge 0. \quad (17)$$

Remark 37. Let $\{\xi_n \in L^0(\mathcal{F}), n \in N\}$ converge to ξ uniformly. Then we can obtain a net $\{\xi_\varepsilon \in L^0(\mathcal{F}) \mid \varepsilon \in L^0_{++}, \varepsilon \le 1\}$ such that it converges to ξ under the locally L^0-convex topology of $(L^0(\mathcal{F}, R), |\cdot|)$. In fact, for any $\varepsilon \in L^0_{++}$ let $A_1 = [\varepsilon > 1]$, $A_i = [1/(i+1) < \varepsilon \le 1/i]$, for all $i \in N$. Then $\{A_n, n \in N\}$ is a countable partition of Ω to $\mathcal{F}$. Since the sequence $\{\xi_n, n \in N\}$ converges to ξ uniformly, thus for any number $k > 0$, there exists $N(k) \in N$ such that

$$|\xi_n - \xi| < \frac{1}{k} \quad (18)$$

for any $n > N(k)$. Let

$$\xi_\varepsilon = \sum_{i=1}^{\infty} \tilde{I}_{A_i} \xi_{N(i)+1}; \quad (19)$$

then it is easy to see that $|\xi_\varepsilon - \xi| < \varepsilon$. Set $\wedge = \{\varepsilon \in L^0_{++} \mid \varepsilon \le 1\}$; then $\wedge$ is directed with respect to $\le$, and one can easy to see that the net $\{\xi_\varepsilon, \varepsilon \in \wedge\}$ converges to ξ under the locally L^0-convex topology of $(L^0(\mathcal{F}), |\cdot|)$.

With the above preparations, we now give the proof of Theorem 1.

Proof of Theorem 1. First, let us assume that for any $p \in B$, $L(\cdot, p)$ is strictly $L^0(\mathcal{F})$-convex on A. Set $F(u) = \bigvee_{p \in B} L(u, p)$ and $G(p) = \bigwedge_{u \in A} L(u, p)$. We show that the functional F has the local property and F is $L^0(\mathcal{F})$-convex and $\mathcal{T}_c$-lower semicontinuous on A. By conditions (1) and (2), it is clear that F has the local property. For any $x_1, x_2 \in A$, $\alpha \in L^0_+$, we have that

$$\begin{aligned} F(\alpha x_1 + (1-\alpha) x_2) &= \bigvee_{p \in B} L(\alpha x_1 + (1-\alpha) x_2, p) \\ &\le \bigvee_{p \in B} [\alpha L(x_1, p) + (1-\alpha) L(x_2, p)] \\ &\le \alpha \bigvee_{p \in B} L(x_1, p) + (1-\alpha) \bigvee_{p \in B} L(x_2, p) \\ &= \alpha F(x_1) + (1-\alpha) F(x_2). \end{aligned} \quad (20)$$

Thus, F is $L^0(\mathscr{F})$-convex on A. For any $r \in L^0(\mathscr{F}, R)$, let $A_r = \{u \in A \mid F(u) \le r\}$. Since F has the local property, we have that A_r has the countable concatenation property. Let $\{x_\alpha, \alpha \in \wedge\} \subset A_r$ converge to x_0 under the locally L^0-convex topology of E. By $F(x_\alpha) \le r$, we can obtain that

$$L(x_\alpha, p) \le r, \quad \forall p \in B. \tag{21}$$

Since $L(\cdot, p)$ is $\mathscr{T}_c$-lower semicontinuous, it is easy to see that

$$L(x_0, p) \le r, \quad \forall p \in B. \tag{22}$$

Hence, $F(x_0) \le r$ and $x_0 \in A_r$. So we have that F is $\mathscr{T}_c$-lower semicontinuous. Similarly, replacing L with $-L$, we see that $-G$ has the local property and $-G$ is $L^0(\mathscr{F})$-convex and $\mathscr{T}_c$-lower semicontinuous on B.

By Theorems 26 and 28, we have that $\bigvee_{p\in B} G(p) \in L^0(\mathscr{F})$, and there exists $p_0 \in B$ such that

$$G(p_0) = \bigvee_{p\in B} G(p). \tag{23}$$

Since for all $p \in B$, $L(\cdot, p)$ is strictly $L^0(\mathscr{F})$-convex on A, according to Corollary 30 there exists an unique $u_p \in A$ such that

$$G(p) = \bigwedge_{u\in A} L(u, p) = L(u_p, p), \tag{24}$$

for any $p \in B$. Let $u_0 = u_{p_0}$, we need only to prove that

$$G(p_0) \ge L(u_0, p), \quad \forall p \in B. \tag{25}$$

For every $p \in B$, let $p_n = (1 - n^{-1})p_0 + n^{-1}p$ and $u_n = u_{p_n}$, for all $n \in N$. It is clear that

$$G(p_0) \ge G(p_n) = L(u_n, p_n). \tag{26}$$

By condition (2), we have that

$$\begin{gathered} G(p_0) \ge L(u_n, p_n) \ge (1 - n^{-1}) L(u_n, p_0) + n^{-1} L(u_n, p), \\ G(p_0) \ge (1 - n^{-1}) G(p_0) + n^{-1} L(u_n, p), \end{gathered} \tag{27}$$

namely, $G(p_0) \ge L(u_n, p)$. Let $\eta = \bigvee\{\|x\| \mid x \in A\}$ and $C_n = \{\omega \in \Omega \mid n - 1 \le \eta < n\}$, for all $n \in N$, according to the condition that A is a.s. bounded in E, and it is easy to see that $\{C_n, n \in N\}$ is a countable partition of Ω to $\mathscr{F}$.

For any fixed $i \in N$, we have that $\tilde{I}_{C_i} \cdot u_n \in L^2(E)$, for all $n \in N$. It is easy to see that $\tilde{I}_{C_i} \cdot A$ is a bounded and closed subset of $(L^2(E), \|\cdot\|_2)$. Thus, there exists a subsequence $\{\tilde{I}_{C_i} \cdot u_{n_k}, k \in N\}$ and $\omega_i \in \tilde{I}_{C_i} \cdot A$ such that $\{\tilde{I}_{C_i} \cdot u_{n_k}, k \in N\}$ converges to ω_i under the weak topology of $L^2(E)$. Without loss of generality, denote this subsequence by $\{\tilde{I}_{C_i} \cdot u_n, n \in N\}$. By Lemma 36, there exists a sequence $\{v_n \in \tilde{I}_{C_i} \cdot A, n \in N\}$ such that it converges to ω_i in norm, where

$$v_n = \sum_{k=n}^{N_n} \lambda_k u_k, \tag{28}$$

$\sum_{k=n}^{N_n} \lambda_k = 1, \lambda_k \ge 0$. Since $\|v_n - \omega_i\|_2 \to 0$, we have that $\{\|v_n - \omega_i\|, n \in N\}$ converges in probability P to θ. By Egoroff theorem, for any number $\delta > 0$, there exists $C_\delta \in \mathscr{F}$ such that $P(C_\delta) < \delta$ and $\{\|v_n - \omega_i\|, n \in N\}$ converges uniformly to 0 on C_δ^c. Then there is a net

$$\{\|v_\varepsilon - \omega_i\|, \varepsilon \in \wedge\} \tag{29}$$

as in Remark 37, which converges 0 on C_δ^c under the locally L^0-convex topology of $(L^0(\mathscr{F}), |\cdot|)$. By the construction of $\{\|v_\varepsilon - \omega_i\|, \varepsilon \in \wedge\}$ as in Remark 37, we have that

$$\underline{\lim}_{n\to\infty} L(v_n, p) \ge \underline{\lim}_{\varepsilon\in\wedge} L(v_\varepsilon, p). \tag{30}$$

Since $L(\cdot, p)$ is $L^0(\mathscr{F})$-convex for any $p \in B$, we have that

$$\begin{aligned} \tilde{I}_{C_\delta^c} \tilde{I}_{C_i} G(p_0) &\ge \tilde{I}_{C_\delta^c} \tilde{I}_{C_i} \underline{\lim}_{n\to\infty} \sum_{k=n}^{N_n} \lambda_k L(u_k, p) \\ &\ge \tilde{I}_{C_\delta^c} \tilde{I}_{C_i} \underline{\lim}_{n\to\infty} L(v_n, p) \\ &\ge \tilde{I}_{C_\delta^c} \tilde{I}_{C_i} \underline{\lim}_{\varepsilon\in\wedge} L(v_\varepsilon, p) \ge \tilde{I}_{C_\delta^c} \tilde{I}_{C_i} L(\omega_i, p). \end{aligned} \tag{31}$$

Hence, one can obtain that $\tilde{I}_{C_\delta^c} \tilde{I}_{C_i} G(p_0) \ge \tilde{I}_{C_\delta^c} \tilde{I}_{C_i} L(\omega_i, p)$. Because δ is an arbitrary nonnegative number and $L(\cdot, p)$ has the local property, we have that

$$\tilde{I}_{C_i} G(p_0) \ge \tilde{I}_{C_i} L(\omega_i, p), \tag{32}$$

for any $p \in B$.

Now, we prove that $\omega_i = \tilde{I}_{C_i} u_0$ for any $i \in N$. By the definition of u_n, it is clear that

$$L(u_n, p_n) \le L(u, p_n), \quad \forall u \in A. \tag{33}$$

By condition (2), we have that

$$(1 - n^{-1}) L(u_n, p_0) + n^{-1} L(u_n, p) \le L(u, p_n), \quad \forall p \in B. \tag{34}$$

Hence, we can obtainthat

$$\overline{\lim}_{n\to\infty} (1 - n^{-1}) L(u_n, p_0) \le \overline{\lim}_{n\to\infty} L(u, p_n), \quad \forall p \in B. \tag{35}$$

Since $L(u_n, p) \ge G(p)$, we can obtain that $(1 - n^{-1}) L(u_n, p_0) + n^{-1} G(p) \le L(u, p_n)$ and $(1 - n^{-1}) L(v_n, p_0) + n^{-1} G(p) \le \tilde{I}_{C_i} \cdot \overline{\lim}_{n\to\infty} ((1 - n^{-1}) L(u_n, p_0) + n^{-1} L(u_n, p))$. According to

$$\begin{aligned} \underline{\lim}_{n\to\infty} L(v_n, p_0) &= \underline{\lim}_{n\to\infty} (1 - n^{-1}) L(v_n, p_0) \\ &\le \underline{\lim}_{n\to\infty} (1 - n^{-1}) \sum_{k=n}^{N_n} \lambda_k L(u_k, p_0) \\ &\le \overline{\lim}_{n\to\infty} (1 - n^{-1}) L(u_n, p_0) \\ &\le \overline{\lim}_{n\to\infty} L(u, p_n), \end{aligned} \tag{36}$$

where $\sum_{k=n}^{N_n} \lambda_k = 1$, it is obvious that

$$\begin{aligned} \widetilde{I}_{C_i} L(\omega_i, p_0) &\le \varliminf_{n\to\infty} \widetilde{I}_{C_i} L(\nu_n, p_0) \le \varlimsup_{n\to\infty} \widetilde{I}_{C_i} L(u_n, p_0) \\ &\le \varlimsup_{n\to\infty} \widetilde{I}_{C_i} L(u, p_n). \end{aligned} \tag{37}$$

Since $\{\|p_n - p_0\|, n \in N\}$ converges in probability to 0, by Egoroff theorem, we can obtain that for any $\sigma > 0$, there exists $C_\sigma \in \mathscr{F}$ such that $P(C_\sigma) < \sigma$ and $\{\|p_n - p_0\|, n \in N\}$ converges to 0 uniformly on C_σ^c. By Remark 37, we can construct a net $\{p_\alpha \in B, \alpha \in \wedge\}$ as in Remark 37 such that $\|p_\alpha - p_0\| \to 0$ under the locally L^0-convex topology of E. Hence, by (2) we can obtain that

$$\widetilde{I}_{C_\sigma^c} \widetilde{I}_{C_i} L(\omega_i, p_0) \le \widetilde{I}_{C_\sigma^c} \widetilde{I}_{C_i} \varlimsup_{\alpha\in\wedge} L(u, p_\alpha) \le \widetilde{I}_{C_\sigma^c} \widetilde{I}_{C_i} L(u, p_0). \tag{38}$$

Since σ is an arbitrary nonnegative number, it is clear that

$$\widetilde{I}_{C_i} L(\omega_i, p_0) \le \widetilde{I}_{C_i} L(u, p_0). \tag{39}$$

Therefore, for any $i \in N$, $\omega_i = \widetilde{I}_{C_i} u_0$ and $G(p_0) \ge L(u_0, p)$, for all $p \in B$; namely, (u_0, p_0) is a random saddle point of L with respect to $A \times B$.

If there is $p \in B$ such that $L(\cdot, p)$ is not strictly $L^0(\mathscr{F})$-convex on A, define

$$L_n(u, p) = L(u, p) + n^{-1}\|u\|^2, \quad \forall n \in N. \tag{40}$$

Since E is random strictly convex RN module, we can obtain that L_n is strictly $L^0(\mathscr{F})$-convex from Lemma 35. By the similar method, we have that for any $n \in N$, there exists $(u_n, p_n) \in A \times B$ such that it is a saddle point of L_n with respect to $A \times B$.

Let $\zeta = \bigvee\{\|y\| \mid y \in B\}$ and D_m be any representation element of $[m - 1 \le \eta < m]$, for all $m \in N$, according to the condition that B is a.s. bounded in E, and it is easy to see that $\{D_m, m \in N\}$ is a countable partition of Ω to $\mathscr{F}$. It is easy to see that $\{C_i \bigcap D_j, i, j \in N\}$ is also a countable partition of Ω to $\mathscr{F}$. For any $i, j \in N$, let $H_{ij} = C_i \bigcap D_j$. We can suppose that, without loss of generality, $\widetilde{I}_{H_{ij}} u_n \in L^2(E)$ converge to μ_{ij} under the weak topology of $L^2(E)$. Then we have that there exists a net $\{\widetilde{I}_{H_{ij}} u_\alpha \in E \mid \alpha \in \Lambda\}$ such that it converges to μ_{ij} under the locally L^0-convex topology of E. Thus, we have that

$$\begin{aligned} \widetilde{I}_{H_{ij}} L(\mu_{ij}, p) &\le \varliminf_{\alpha} \widetilde{I}_{H_{ij}} L(\widetilde{I}_{H_{ij}} u_\alpha, p) \\ &\le \varliminf_{n\to\infty} \widetilde{I}_{H_{ij}} L(\widetilde{I}_{H_{ij}} u_n, p) + n^{-1}\|u_n\|^2 \\ &\le \varliminf_{n\to\infty} \left(\widetilde{I}_{H_{ij}} L(u, \widetilde{I}_{H_{ij}} p_n) + n^{-1}\|u\|^2\right) \end{aligned} \tag{41}$$

for all $u \in A$, $p \in B$. Similarly, for any $i, j \in N$, one can have a net $\{\widetilde{I}_{H_{ij}} p_\beta \in E \mid \beta \in \Gamma\}$ such that it converges to ν_{ij} under the locally L^0-convex topology of E and

$$\begin{aligned} &\varlimsup_{n\to\infty} \left(\widetilde{I}_{H_{ij}} L(u, \widetilde{I}_{H_{ij}} p_n) + n^{-1}\|u\|^2\right) \\ &\quad\le \varlimsup_{n\to\infty} \widetilde{I}_{H_{ij}} L(u, \widetilde{I}_{H_{ij}} p_n) + \varlimsup_{n\to\infty} n^{-1}\|u\|^2 \\ &\quad\le \varlimsup_{\beta\in\Gamma} \widetilde{I}_{H_{ij}} L(u, \widetilde{I}_{H_{ij}} p_\beta) + \varlimsup_{n\to\infty} n^{-1}\|u\|^2 \\ &\quad\le \widetilde{I}_{H_{ij}} L(u, \nu_{ij}). \end{aligned} \tag{42}$$

Let $u_0 = \sum_{i,j\in N} \widetilde{I}_{H_{ij}} \mu_{ij}$ and $p_0 = \sum_{i,j\in N} \widetilde{I}_{H_{ij}} \nu_{ij}$; it is easy to check that

$$L(u_0, p) \le L(u_0, p_0) \le L(u, p_0) \tag{43}$$

for all $u \in A, p \in B$; namely, (u_0, p_0) is a random saddle point of L with respect to $A \times B$.

This completes the proof. □

Acknowledgment

This work is supported by NNSF no. 11171015.

References

[1] J. V. Neumann, "Zur Theorie der Gesellschaftsspiele," *Mathematische Annalen*, vol. 100, no. 1, pp. 295–320, 1928.

[2] K. Fan, "Fixed-point and minimax theorems in locally convex topological linear spaces," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 38, pp. 121–126, 1952.

[3] H. Kneser, "Sur un théorème fondamental de la théorie des jeux," *Comptes Rendus de l'Académie des Sciences*, vol. 234, pp. 2418–2420, 1952.

[4] P. L. Yu, "Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives," *Journal of Optimization Theory and Applications*, vol. 14, pp. 319–377, 1974.

[5] H. W. Corley, "Duality theory for maximizations with respect to cones," *Journal of Mathematical Analysis and Applications*, vol. 84, no. 2, pp. 560–568, 1981.

[6] K. L. Chew, "Maximal points with respect to cone dominance in Banach spaces and their existence," *Journal of Optimization Theory and Applications*, vol. 44, no. 1, pp. 1–53, 1984.

[7] F. Ferro, "Minimax type theorems for n-valued functions," *Annali di Matematica Pura ed Applicata*, vol. 132, pp. 113–130, 1982.

[8] T. X. Guo, "Extension theorems of continuous random linear operators on random domains," *Journal of Mathematical Analysis and Applications*, vol. 193, no. 1, pp. 15–27, 1995.

[9] T. X. Guo, "Radon-Nikodym property of conjugate Banach spaces and $w^* - \mu$-equivalence theorems of $w^* - \mu$-measurable functions," *Science in China A*, vol. 39, no. 10, pp. 1034–1041, 1996.

[10] T. X. Guo, X. H. Xiao, and X. X. Chen, "A basic strict separation theorem in random locally convex modules," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 9, pp. 3794–3804, 2009.

[11] T. X. Guo, "Some basic theories of random normed linear spaces and random inner product spaces," *Acta Analysis Functionalis Applicata*, vol. 1, no. 2, pp. 160–184, 1999.

[12] T. X. Guo, "Survey of recent developments of random metric theory and its applications in China (II)," *Acta Analysis Functionalis Applicata*, vol. 3, no. 3, pp. 208–230, 2001.

[13] T. X. Guo, "Relations between some basic results derived from two kinds of topologies for a random locally convex module," *Journal of Functional Analysis*, vol. 258, no. 9, pp. 3024–3047, 2010.

[14] T. X. Guo, "Recent progress in random metric theory and its applications to conditional risk measures," *Science China Mathematics*, vol. 54, no. 4, pp. 633–660, 2011.

[15] T. X. Guo, S. E. Zhao, and X. L. Zeng, "On random convex analysis—the analytic foundation of the module approach to conditionalrisk measures," http://arxiv.org/abs/1210.1848.

[16] D. Filipović, M. Kupper, and N. Vogelpoth, "Separation and duality in locally L^0-convex modules," *Journal of Functional Analysis*, vol. 256, no. 12, pp. 3996–4029, 2009.

[17] M. Z. Wu, "The Bishop-Phelps theorem in complete random normed modules endowed with the (ε, λ)-topology," *Journal of Mathematical Analysis and Applications*, vol. 391, no. 2, pp. 648–652, 2012.

[18] E. Zeidler, *Applied Functional Analysis: Main Principles and Their Applications*, vol. 109 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1995.

[19] N. Dunford and J. T. Schwartz, *Linear Operators (I)*, Interscience, New York, NY, USA, 1957.

[20] T. X. Guo, *Random metric theory and its applications [Ph.D. thesis]*, Xi'an Jiaotong University, Shaanxi, China, 1992.

[21] T. X. Guo, "A characterization for a complete random normed module to be random reflexive," *Journal of Xiamen University (Natural Science)*, vol. 36, no. 4, pp. 499–502, 1997.

[22] T. X. Guo and Y. J. Yang, "Ekeland's variational principle for an $\overline{L}^0$-valued function on a complete random metric space," *Journal of Mathematical Analysis and Applications*, vol. 389, no. 1, pp. 1–14, 2012.

[23] T. X. Guo and X. L. Zeng, "Random strict convexity and random uniform convexity in random normed modules," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 5, pp. 1239–1263, 2010.

[24] T. X. Guo and X. L. Zeng, "An L^0-valued function's intermediate value theorem and its applications to random uniform convexity," *Acta Mathematica Sinica*, vol. 28, no. 5, pp. 909–924, 2012.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 601490, 10 pages
http://dx.doi.org/10.1155/2013/601490

Research Article

New Sequence Spaces and Function Spaces on Interval $[0, 1]$

Cheng-Zhong Xu[1] and Gen-Qi Xu[2]

[1] *Université de Lyon, LAGEP, Bâtiment CPE, Université Lyon 1, 43 boulevard du 11 Novembre 1918, 69622 Villeurbanne, France*
[2] *Department of Mathematics, Tianjin University, Tianjin 300072, China*

Correspondence should be addressed to Gen-Qi Xu; gqxu@tju.edu.cn

Received 27 April 2013; Accepted 17 August 2013

Academic Editor: Ji Gao

We study the sequence spaces and the spaces of functions defined on interval $[0, 1]$ in this paper. By a new summation method of sequences, we find out some new sequence spaces that are interpolating into spaces between ℓ^p and ℓ^q and function spaces that are interpolating into the spaces between the polynomial space $P[0, 1]$ and $C^\infty[0, 1]$. We prove that these spaces of sequences and functions are Banach spaces.

1. Introduction

With development of sciences and technologies, more and more information are obtaining and need to be reserved and transmitted in the form of data sequence, such as DNA sequence, protein structure [1], brain imaging data, optic spectral analysis, text retrieval, financial data, and climate data. These data have common features: (1) there are at most finite many nonzero elements in the sequence; (2) their dimensions have not bounded from above; (3) the sample size is relatively small. In particular, some elements in the sequence repeat many times, for instance, there are only four different elements in DNA sequence: T, A, C, and G. When the data have much greater dimension, their record and reserve also become a serious problem. On the other hand, we usually use the data to obtain some information, such as the image reconstruction, sequence comparison in medicine, and plant classification in biology. From application point view, the basic requirement is that one can draw easily information from the reservoir; the is to use this data to handle some things. When the data have lower dimension and the samples have larger size, the statistics method such as the covariance matrix can give a good treatment; for instance, see [2] for the semiparameter estimation, [3] for the sparse data estimation, and [4, 5] for the threshold sparse sample covariance matrix method. However, when the data have higher dimensional and the sample size is smaller, the statistics method shall lead to great errors. So, we need new methods to treat them.

Let us consider a simple example from a classification problem. Set S as a set of some class samples and a as a given data. Is a close to someone of S or a new class? A simpler approach is to consider problem $\inf_{s\in S}\|a - s\|_p$, where p denotes the norm in ℓ^p space. In most cases, there is at least one $s_0 \in S$ such that $\|a - s_0\|_p = \inf_{s\in S}\|a - s\|_p$. We denote by $F(a)$ the feasible set. Can we say that a is close to some $s_0 \in F(a)$? To see disadvantage, we divide sequence $s \in S$ into three segments (s_1, s_2, s_3); the first segment s_1 is composed of the first n_1 elements, the second segment s_2 is made of the next n_2 elements, and the third is composed of the others. Similarly, we also divide a into corresponding three parts (a_1, a_2, a_3). Now, we reconsider

$$\inf_{s_1}\|a_1 - s_1\|_p, \quad \inf_{s_2}\|a_2 - s_2\|_p, \quad \inf_{s_3}\|a_3 - s_3\|_p. \tag{1}$$

Perhaps we would find that $F(a_1) \cap F(a_2) \cap F(a_3) = \emptyset$. Can one say that a is a new class? From the above example, we see that we need a new definition of the norm to fit application.

Motivated by these questions, we revisit the sequence spaces and function spaces defined on $[0, 1]$ in this paper. We have observed recent studies on the sequence spaces, for instants, [6–8] for different requirements. Here, the sequence spaces we work on are different from the existing spaces, this is because the spaces are aimed to solve our problem. Now, let us introduce our idea and the resulted sequence space and function spaces.

Let $a = (a_1, a_2, a_3, \dots, a_n, \dots)$ be a DNA sequence. Obviously, there are at most finite many nonzero terms and $a_n \in \{A, C, T, G\}$. To shorten the representation, we can embed a into a polynomial. In this way, we can write a as

$$a(x) = Ap_1(x) + Tp_2(x) + Cp_3(x) + Gp_4(x). \quad (2)$$

For a different DNA sequence, we have different polynomial $p_j(x)$. Obviously, it is a simpler reserve form.

How do we extract original sequence from the polynomial? By the classical mathematics, we know that

$$a_k = \frac{1}{n!}\frac{d^{(k)}a(x)}{dx^k}\Big|_{x=0}, \quad k = 0, 1, 2, \dots, \quad (3)$$

so we recover the sequence.

To extend this form into a sequence of infinite many nonzero terms, we usually take $x \in [0,1]$; $a(x)$ is called the generation function in the classical queuing theory. Note that the generation function is not a continuous function defined on $[0,1]$. So, the differential operation is not fitting to such a function, although the formal differential is always feasible. To find out a feasible form of $a(x)$, let us consider the operations of integral and derivative. We denote by L the integral operation. Operating for the constant 1 leads to

$$\begin{aligned} L^1(1)(x) &= \int_0^x 1\,dx = x, \\ L^2(1)(x) &= \int_0^x L^1(1)\,x\,dx = \frac{x^2}{2}, \\ L^3(1)(x) &= \int_0^x L^2(1)(x)\,dx = \frac{x^3}{3!}. \end{aligned} \quad (4)$$

Generally, we have

$$L^n(1)(x) = \frac{x^n}{n!}, \quad \forall n \in \mathbb{N}. \quad (5)$$

For any polynomial of n-order, $p_n(x)$, it can be written as

$$\begin{aligned} p_n(x) &= a_0 1 + a_1 x + a_2\frac{x^2}{2!} + \cdots + a_n\frac{x^n}{n!} \\ &= \left[\sum_{k=0}^{n} a_k L^k\right](1)(x). \end{aligned} \quad (6)$$

Next, let us consider the differential operation $D(f) = f'(x)$. Taking deferential for function $x^n/n!$ leads to

$$D\left(\frac{x^n}{n!}\right) = \frac{x^{n-1}}{(n-1)!}, \qquad D^2\left(\frac{x^n}{n!}\right) = \frac{x^{n-2}}{(n-2)!}. \quad (7)$$

In general, for any $1 \le k \le n$, it holds that

$$D^k\left(\frac{x^n}{n!}\right) = \frac{x^{n-k}}{(n-k)!}. \quad (8)$$

Obviously, using D, we get once again the coefficients in (6):

$$\begin{aligned} &a_0 = p_n(0), \qquad a_1 = Dp_n(0), \dots, \\ &a_k = D^k p_n(0), \dots, \qquad a_n = D^n p_n(0). \end{aligned} \quad (9)$$

Therefore, the coefficient sequence is given by

$$(a_0, a_1, \dots, a_n) = \left(D^0, D^1, \dots, D^n\right) p_n|_{x=0}. \quad (10)$$

Clearly, we should take functions $x^n/n!$ as the basis functions.

Moreover, we note that in the polynomial space over $[0,1]$, denoted by $P[0,1]$, if the norm is defined as

$$\|p\|_\phi = \sup_{n\ge0}\{\|D^n p\|_\infty\}; \quad (11)$$

where $\|f\|_\infty = \max_{0\le x\le1}|f(x)|$, then it is a normed linear space. In this space, the integral and differential operations are bounded linear operators. To extend to an infinite sequence, we should choose such a function space in which the integral and differential operations are bounded linear operators. What is such a function space?

Consider a subset of $C^\infty[0,1]$ defined as

$$C_M^\infty[0,1] = \left\{f \in C^\infty[0,1] \mid \sup_{n\ge0}\|D^n f\|_\infty < \infty\right\}. \quad (12)$$

The set is in fact a linear space. We define a norm on it by

$$\|f\|_\phi = \sup_{n\ge0}\|D^n f\|_\infty, \quad (13)$$

then it becomes a Banach space. Now, for the function spaces over interval $[0,1]$, there are the following inclusion relations

$$\begin{aligned} P[0,1] &\subset C_M^\infty[0,1] \subset C^\infty[0,1] \subset C^k[0,1] \subset C[0,1] \\ &\subset L^\infty[0,1] \subset L^p[0,1] \subset L^1[0,1]. \end{aligned} \quad (14)$$

But the completion of $(P[0,1], \|\cdot\|_\phi)$ is not the space $(C_M^\infty[0,1], \|\cdot\|_\phi)$. Let $C_{\phi,0}[0,1]$ be the completion space of $(P[0,1], \|\cdot\|_\phi)$. Clearly,

$$P[0,1] \subset C_{\phi,0}[0,1] \subset C_M^\infty[0,1]. \quad (15)$$

And the integral and differential operations are bounded operators.

For space $C_{\phi,0}[0,1]$, we have the following representation theorem.

Theorem 1. *The set $C_{\phi,0}[0,1]$ has the following representation:*

$$C_{\phi,0}[0,1] = \left\{f(x) = \sum_{n=0}^{\infty} a_n\frac{x^n}{n!} \mid \lim_{n\to\infty} a_n = 0\right\}. \quad (16)$$

Moreover, the space $C_{\phi,0}[0,1]$ is isomorphic to the sequence space c_0, and the isomorphism mapping is given by

$$T\{a_n\} = f(x) = \sum_{n=0}^{\infty} a_n\frac{x^n}{n!}, \quad \forall\{a_n\} \in c_0. \quad (17)$$

Motivated by above result, for $p \geq 1$, we define a new norm on the space $P[0,1]$ by

$$\|f\|_{\phi,p} = \left(\sum_k \|D^k f\|_\infty^p\right)^{1/p}. \tag{18}$$

The completion of space $P[0,1]$ under this norm is denoted by $C_{\phi,p}$. Similarly, we have the following result.

Theorem 2. *The space $C_{\phi,p}[0,1]$ has a representation:*

$$C_{\phi,p}[0,1] = \left\{f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \mid \sum_{n=0}^{\infty} |a_n|^p < \infty\right\}. \tag{19}$$

Furthermore, $C_{\phi,p}[0,1]$ is isomorphic to the space ℓ^p, and the isomorphism operator is given by

$$T\{a_n\} = f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad \forall \{a_n\} \in \ell^p. \tag{20}$$

Theorem 3. *The space $C_{\phi,\infty}[0,1]$ has a representation:*

$$C_{\phi,\infty}[0,1] = \left\{f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \mid \sup_{n\geq 0} |a_n| < \infty\right\}. \tag{21}$$

Moreover, $C_{\phi,\infty}[0,1] = C_M^\infty[0,1]$ is isomorphic to ℓ^∞, and the isomorphism operator is given by

$$T\{a_n\} = f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \quad \forall \{a_n\} \in \ell^\infty. \tag{22}$$

By now, we have gotten a series spaces in which both the differential operator and integral operator are bounded linear operators. Obviously, these sets have the following inclusion relations:

$$\begin{aligned} P[0,1] &\subset C_{\phi,1}[0,1] \subset C_{\phi,p}[0,1] \subset C_{\phi,0}[0,1] \\ &\subset C_{\phi,\infty}[0,1] = C_M^\infty[0,1], \quad 1 \leq p < \infty. \end{aligned} \tag{23}$$

We observe that for $p \geq 1$, in definition of these spaces, the term

$$\sum_{n=0}^{\infty} \|D^n f\|_\infty^p < \infty, \tag{24}$$

means that the terms $\sum_{k=1}^{n} \|D^{n+k} f\|_\infty^p$ are small as n is large enough.

To insert a new space between $C_{\phi,p}[0,1]$ and $C_{\phi,0}[0,1]$, let us consider the sequence formed by $f \in C_{\phi,0}[0,1]$,

$$f(x), f'(x), f''(x), \ldots, f^{(n)}(x), \ldots; \tag{25}$$

the norm $\|f\|_\phi = \sup_{n\geq 0} \|D^n f\|_\infty$ is maximum of the function sequence under space $C[0,1]$. To define a new normed space, in Section 2, we shall study new summation approach for a sequence. Based on such a new summation approach, we shall define some new sequence spaces. In Section 3, we shall define some function spaces and discuss their completeness. Furthermore, we compare these spaces and give some inclusion relations.

2. New Summation Method

2.1. Summation of Absolutely Dominant Operator. Let $\{a_n\}$ be a sequence (real or complex number) and satisfy $\lim_{n\to\infty} a_n = 0$. We define an operator σ by

$$\sigma\{a_n\} = \{a_{\sigma(k)}; k \geq 0\}, \tag{26}$$

where $|a_{\sigma(k)}| \geq |a_{\sigma(k+1)}|$. It is called the absolutely dominant queuing operator.

Removing the first n terms, the remainder sequence $\{a_{n+k}; k \geq 0\}$ has a new queuing:

$$\sigma\{a_{n+k}\} = \{a_{n+\sigma(k)}; k \geq 0\}. \tag{27}$$

Using the queuing operator, for a sequence $\{a_n\}$ of zero limit (this is only used to ensure that we can take the maximal value for such a sequence), we define a positive number by

$$s_0 = \sup_{n\geq 0} |a_n| = |a_{\sigma(0)}|. \tag{28}$$

By removing the first term a_0 from $\{a_n\}_{n=0}^\infty$, we define the second number s_1, that is, the absolute summation of the first two terms in $\sigma\{a_{n+1}\}$, that is,

$$s_1 = \sum_2 \sigma\{a_{1+n}, n \geq 0\} = |a_{1+\sigma(0)}| + |\sigma_{1+\sigma(1)}|, \tag{29}$$

where the number in subscription of summation is the number of the terms σ denotes the absolutely dominant queuing operator.

After removing the first two terms a_0 and a_1 from $\{a_n\}_{n=0}^\infty$, we define the third positive number s_2, that is, the absolute summation of the first three terms in $\sigma\{\{a_{2+n}\}\}$, that is,

$$s_2 = \sum_3 \sigma\{a_{2+n}, n \geq 0\} = |a_{2+\sigma(0)}| + |a_{2+\sigma(1)}| + |a_{2+\sigma(2)}|. \tag{30}$$

Generally, we define positive number s_k as

$$\begin{aligned} s_k = \sum_{k+1} \sigma\{a_{k+n,\, n\geq 0}\} &= |a_{k+\sigma(0)}| + |a_{k+\sigma(1)}| \\ &+ |a_{k+\sigma(2)}| + \cdots + |a_{k+\sigma(k)}|. \end{aligned} \tag{31}$$

According to this rule, we get a new nonnegative sequence $\{s_k, ; k \geq 0\}$.

Example 4. Let $\{a_k\} = \{1,2,3,4,5,6\}$. Then, we have

$$\begin{gathered} s_0 = 6, \qquad s_1 = 6+5, \\ s_2 = 6+5+4, \qquad s_3 = 6+5+4, \\ s_4 = 6+5, \qquad s_5 = 6. \end{gathered} \tag{32}$$

2.2. Relationship between Summation and Order of Sequence. To explain the thing we concerned about, let us see an example.

Example 5. Let scalar group be $\{b_k\} = \{6, 5, 4, 3, 2, 1\}$. Then, we have

$$\begin{gathered} \widehat{s}_0 = 6, \qquad \widehat{s}_1 = 5 + 4, \\ \widehat{s}_2 = 4 + 3 + 2, \qquad \widehat{s}_3 = 3 + 2 + 1, \\ \widehat{s}_4 = 2 + 1, \qquad \widehat{s}_5 = 1. \end{gathered} \tag{33}$$

Comparing this with Example 4, we see that the summation of a sequence has a relationship with its order.

From Examples 4 and 5, we see that the sequences $\{\widehat{s}_n\}$ and $\{s_n\}$ generated by a new summation method have relation of order of a sequence. In the sequel, we mainly discuss the infinite sequence. If a sequence has only finite many terms, we shall complement zero after the last term so that it becomes an infinite sequence.

2.3. Distribution of s-Sequence. In this subsection, we shall consider the distribution of the sequence $\{s_k, k \ge 0\}$. We discuss it according to the different cases.

(1) Let $\{a_k\}$ be a positive and increasing sequence, that is,

$$0 < a_0 < a_1 < a_2 < a_3 < \cdots < a_{N-1} < a_N. \tag{34}$$

According to the absolutely dominant summation, we have

$$\begin{aligned} s_0 &= a_N, \\ s_1 &= a_N + a_{N-1}, \\ s_2 &= a_N + a_{N-1} + a_{N-2}, \\ &\ldots. \\ s_k &= \sum_{j=0}^{k} a_{N-j}, \quad k \le \frac{N}{2}, \end{aligned} \tag{35}$$

when $N = 2m + 1$, the sequence has even terms $2(m + 1)$; then

$$\begin{aligned} s_{m-1} &= \sum_{j=0}^{m-1} a_{N-j}, \\ s_m &= \sum_{j=0}^{m} a_{N-j}, \\ s_{m+1} &= \sum_{j=0}^{m} a_{N-j}, \\ s_{m+2} &= \sum_{j=0}^{m-1} a_{N-j}, \ldots, \\ s_{m+k} &= \sum_{j=0}^{m+1-k} a_{N-j} \\ &\ldots. \\ s_{N-1} &= a_N + a_{N-1}, \\ s_N &= a_N. \end{aligned} \tag{36}$$

In this case, the s-sequence has a U-type distribution shown as follows:

$$\begin{array}{ccccccccc} s_0 & & & & & & & & \\ & s_1 & & & & & & & s_N \\ & & s_2 & & & & & s_{N-1} & \\ & & & \ddots & & & s_{N-2} & & \\ & & & & s_m \ \ s_{m+1} & \cdots & & & \end{array}. \tag{37}$$

If $N = 2m$, the sequence has odd terms $(2m + 1)$; then

$$\begin{aligned} s_{m-1} &= \sum_{j=0}^{m-1} a_{N-j}, \\ s_m &= \sum_{j=0}^{m} a_{N-j}, \\ s_{m+1} &= \sum_{j=0}^{m-1} a_{N-j}, \ldots, \\ s_{m+k} &= \sum_{j=0}^{m-k} a_{N-j}, \ldots \\ s_{N-1} &= a_N + a_{N-1}, \\ s_N &= a_N. \end{aligned} \tag{38}$$

In this case, the s-sequence has V-type distribution as follows:

$$\begin{array}{ccccccccc} s_0 & & & & & & & & s_N \\ & s_1 & & & & & & s_{N-1} & \\ & & s_2 & & & & s_{N-2} & & \\ & & & \ddots & & \cdots & & & \\ & & & s_{m-1} & & s_{m+1} & & & \\ & & & & s_{m+1} & & & & \end{array}. \tag{39}$$

If $\{a_k\}$ is an increasing sequence, then $\{s_k\}$ has a symmetrical form; the s-data at the medial term is its maximal value.

(2) $\{a_n\}$ is a positive and decreasing sequence

$$a_N > a_{N-1} > a_{N-2} > \cdots > a_{N/2} > a_{(N/2)-1} > \cdots > a_1 > a_0. \tag{40}$$

According to the absolutely dominant summation, we have

$$\begin{aligned} s_0 &= a_N, \\ s_1 &= a_{N-1} + a_{N-2}, \\ s_2 &= a_{N-2} + a_{N-3} + a_{N-4}, \\ s_k &= \sum_{j=0}^{k} a_{N-k-j}, \quad 0 \le k \le \frac{N}{2}. \end{aligned} \tag{41}$$

If $N = 2m+1$, the sequence has an even term $2(m+1)$; then

$$\begin{aligned} s_{m-1} &= \sum_{j=0}^{m-1} a_{N-(m-1)-j}, \\ s_m &= \sum_{j=0}^{m} a_{N-m-j}, \\ s_{m+1} &= \sum_{j=0}^{m} a_{N-j} \\ s_{m+2} &= \sum_{j=0}^{m-1} a_{N-j}, \ldots, \\ s_{N-k} &= \sum_{j=0}^{k} a_j, \qquad s_N = a_0. \end{aligned} \tag{42}$$

If $N = 2m$, the sequence has odd terms $(2m+1)$, then

$$\begin{aligned} s_{m-1} &= \sum_{j=0}^{m-1} a_{N-j}, \\ s_m &= \sum_{j=0}^{m} a_j, \\ s_{m+1} &= \sum_{j=0}^{m-1} a_j, \ldots, \\ s_{N-k} &= \sum_{j=0}^{k} a_j, \\ s_N &= a_0. \end{aligned} \tag{43}$$

So, s-sequence has distribution as

$$\begin{array}{ccccccccc} & & & s_k & & & & & \\ & & \cdots & & s_{k+1} & & & & \\ & s_1 & & & & \ddots & & & \\ s_0 & & & & & & s_m & & . \\ & & & & & & & \ddots & \\ & & & & & & & & s_N \end{array} \tag{44}$$

In this case, the s-sequence has a character that the initial original data is the absolute largest; at the first several steps, s-sequence arrives at its maximum value; after then, s-sequence decreases until it arrives at its minimum value. The final value is minimal.

(3) Let $\{a_n\}$ be an arbitrary sequence.

In this case, it is difficult to give a general distribution. Although so, we have the following relation

$$s_0 = \sup_{k \ge 0} |a_k| = |a_{\sigma(0)}|, \quad s_k^m \le s_k \le s_k^M, \ \forall k, \tag{45}$$

where s_k^m denotes the value in decreasing queuing, s_k^M denotes the value in increasing queuing.

From above, we see that s-sequence undergoes a great contortion due to different queuing order of a sequence. Denote by $J(a_n)$ the maximum change of s-sequence for $\{a_n\}$ under different queuing order; then

$$J(a_n) = \frac{\max\{s_k^M\}}{\max\{s_k^m\}}. \tag{46}$$

For the sequence $\{a_n\} = \{1,2,3,4,5,6\}$, it is easy to see that $J(a_n) = 15/9$.

2.4. Generalized p-Summation. From the absolutely dominant summation, we get a new sequence $\{s_k\}$. Now, we define a new p-summation for $\{a_n\}$ sequence:

$$s_{0,p} = s_0 = \sup_{n \ge 0} |a_n| = |a_{\sigma(0)}|. \tag{47}$$

Removing the first term a_0, we define the second value $s_{1,p}$ by

$$s_{1,p} = \sum_{2,p} \sigma\{a_{1+n}, n \ge 0\} = \left(|a_{1+\sigma(0)}|^p + |a_{1+\sigma(1)}|^p\right)^{1/p}. \tag{48}$$

Again, removing the second term a_1, we define the summation of the first three absolute maximum value as $s_{2,p}$, that is,

$$\begin{aligned} s_{2,p} &= \sum_{3,p} \sigma\{a_{2+n}, n \ge 0\} \\ &= \left(|a_{2+\sigma(0)}|^p + |a_{2+\sigma(1)}|^p + |a_{2+\sigma(2)}|^p\right)^{1/p}. \end{aligned} \tag{49}$$

In general, we define

$$\begin{aligned} s_{k,p} &= \sum_{k+1,p} \sigma\{a_{k+n}, n \ge 0\} \\ &= \left(|a_{k+\sigma(0)}|^p + |a_{k+\sigma(1)}|^p + |a_{k+\sigma(2)}|^p \right. \\ &\quad \left. + \cdots + |a_{k+\sigma(k)}|^p\right)^{1/p}. \end{aligned} \tag{50}$$

According to this definition, we get a new nonnegative sequence $\{s_{k,p},; k \ge 0\}$.

Obviously, when $p = 1$, we recover the above summation sequence $s_k = s_{k,1}$, for all $k \ge 0$. For the sequence $\{a_n\}$, we define a positive number $\|\{a_n\}\|_{G,p}$ by

$$\|\{a_n\}\|_{G,p} = \sup_{k \ge 0} s_{k,p}, \tag{51}$$

where G denotes the generalized summation and p denotes the p-norm in finite-dimensional space.

In particular, when $\{a_n\} \in \ell^p$, $s_{k,p} \to 0$ as $k \to \infty$, and hence $\|\{a_n\}\|_{G,p} < (\sum_{n \ge 0}^{\infty} |a_n|^p)^{1/p} < \infty$.

Now, we define the sequence spaces by

$$\begin{aligned} c_{G,p,M} &= \left\{\{a_n\} \mid \sup_{k \ge 0} s_{k,p} < \infty\right\}, \\ c_{G,p} &= \left\{\{a_n\} \mid \lim_{k \to \infty} s_{k,p} = 0\right\}. \end{aligned} \tag{52}$$

We can prove the following result.

Theorem 6. *$(c_{G,p,M}, \|\cdot\|_{G,p})$ and $(c_{G,p}, \|\cdot\|_{G,p})$ are Banach spaces.*

3. $C_{G,p}[0,1]$-Type Spaces

3.1. Definition of Spaces $C_{G,p,M}[0,1]$. Let $p \ge 1$. For each $f \in C_{\phi,0}[0,1]$, we define positive number:

$$\begin{aligned} s_{k,p}(f) &= \sum_{k,p}\sigma\left\{D^{k+n}f, n \ge 0\right\} \\ &= \left(\left\|D^{k+\sigma(0)}(f)\right\|_\infty^p + \left\|D^{k+\sigma(1)}(f)\right\|_\infty^p \right. \\ &\qquad \left. + \left\|D^{k+\sigma(2)}(f)\right\|_\infty^p + \cdots + \left\|D^{k+\sigma(k)}(f)\right\|_\infty^p\right)^{1/p}. \end{aligned} \tag{53}$$

Example 7. To show the computation method, let us consider the case that $p = 1$ and discuss the function $e_n(x) = x^n/n!$.

By computing various derivative functions

$$\begin{aligned} \left\|D^0 e_n\right\|_\infty &= \frac{1}{n!}, \\ \left\|D^1 e_n\right\|_\infty &= \frac{1}{(n-1)!}, \\ \left\|D^2 e_n\right\|_\infty &= \frac{1}{(n-2)!} \\ &\cdots\cdot \\ \left\|D^k e_n\right\|_\infty &= \frac{1}{(n-k)!} \\ &\cdots\cdot \\ \left\|D^{n-2} e_n\right\|_\infty &= \frac{1}{2!} \\ \left\|D^{n-1} e_n\right\|_\infty &= 1, \\ \left\|D^{n} e_n\right\|_\infty &= 1 \\ \left\|D^{n+1} e_n\right\|_\infty &= 0, \\ \left\|D^{n+2} e_n\right\|_\infty &= 0, \\ &\cdots\cdot \end{aligned} \tag{54}$$

We calculate $s_k(e_n)$ as follows:

$$\begin{aligned} s_0(e_n) &= \sup_{k\ge 0}\left\|D^k e_n\right\|_\infty = 1, \\ s_1(e_n) &= \sum_2 \sigma\left\{D^{1+k}e_n, k \ge 0\right\} = 1 + 1, \\ s_2(e_n) &= \sum_3 \sigma\left\{D^{2+k}e_n, k \ge 0\right\} = 1 + 1 + \frac{1}{2!}, \\ &\cdots\cdot \\ s_j(e_n) &= \sum_k \sigma\left\{D^{j+k}e_n, k \ge 0\right\} \\ &= 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{j!}, \quad j \le \frac{n}{2}, \\ s_{(n/2)}(e_n) &= \sum_{n/2}\sigma\left\{D^{(n/2)+k}e_n, k \ge 0\right\} = \sum_{k=0}^{n/2}\frac{1}{k!}, \\ s_{(n/2)+1} &= \sum_{k=0}^{n/2}\frac{1}{k!}, \\ s_{(n/2)+2} &= \sum_{k=0}^{(n/2)-1}\frac{1}{k!}, \\ s_{n-2} &= 1 + 1 + \frac{1}{2!}, \\ s_{n-1} &= 1 + 1, \\ s_n &= 1, \\ s_{n+j} &= 0, \quad j \ge 1. \end{aligned} \tag{55}$$

According to the definition of $s_{k,p}(f)$, it has the following properties:

(1) for any $k \ge 0$,

$$\begin{aligned} s_{0,p}(f) &= \sup_{n\ge 0}\|D^n f\|_\infty, \\ s_{k,p}(f) &\ge \left\|D^k f\right\|_\infty \ge 0; \end{aligned} \tag{56}$$

(2) for each $\alpha \in \mathbb{C}$,

$$s_{k,p}(\alpha f) = |\alpha|\, s_{k,p}(f), \quad \forall f \in C_{\phi,0}[0,1]; \tag{57}$$

(3) for any $g, f \in C_{\phi,0}[0,1]$,

$$s_{k,p}(f+g) \le s_{k,p}(f) + s_{k,p}(g). \tag{58}$$

This can be obtained from the Minkovwski inequality.

These properties show that $s_{0,p}(f)$ satisfies the norm axiom, and hence it is a norm on space $C_{\phi,0}[0,1]$. But for $k \ge 1$, $s_{k,p}(f)$ only is a seminorm (or prenorm), it is continuous in the topology of $C_{\phi,0}[0,1]$.

For each $f \in C_{\phi,0}[0,1]$, we define the following:

$$\|f\|_{G,p} = \sup_{n\ge 0} s_{n,p}(f). \tag{59}$$

We define the following functions spaces:

$$C_{G,p,M}[0,1] = \left\{f \in C_{\phi,0}[0,1] \mid \|f\|_{G,p} < \infty\right\}, \tag{60}$$

$$C_{G,p}[0,1] = \left\{f \in C_{\phi,0}[0,1] \mid \lim_{k\to\infty} s_{k,p}(f) = 0\right\}. \tag{61}$$

Theorem 8. *For $f \in C_{\phi,0}[0,1]$, the scalar $\|f\|_{G,p}$ is defined as above. Then, $\|f\|_{G,p}$ is a norm on $C_{G,p,M}[0,1]$, and hence $(C_{G,p,M}[0,1], \|\cdot\|_{G,p})$ is a normed linear space. Moreover, $(C_{G,p,M}[0,1], \|\cdot\|_{G,p})$ is a Banach space, which is isomorphic to the sequence space $c_{G,p,M}$.*

Proof. Here, we only prove the second assertion.

Let $f_n \in C_{G,p,M}[0,1]$ be a Cauchy sequence, that is, for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that

$$\|f_n - f_m\|_{G,p} < \varepsilon, \quad \forall n, m > N(\varepsilon). \tag{62}$$

For any $k \geq 0$, it holds that

$$\left\|D^k(f_n - f_m)\right\|_\infty \leq s_{k,p}(f_n - f_m) \leq \|f_n - f_m\|_{G,p}. \tag{63}$$

This shows that $\{f_n\}$ also is a Cauchy sequence in $C_{\phi,0}[0,1]$. By the completeness of space $C_{\phi,0}[0,1]$, there is a $f_0 \in C_{\phi,0}[0,1]$ such that $\|f_n - f_0\|_\phi \to 0$.

For each $k \geq 0$, by the continuity of $s_{k,p}(f)$, we can get

$$\left\|D^k(f_n - f_0)\right\|_\infty \leq s_{k,p}(f_n - f_0) \leq \varepsilon. \tag{64}$$

Thus, we find that

$$\|f_n - f_0\|_{G,p} = \sup_{k\geq 0} s_{k,p}(f_n - f_0) \leq \varepsilon, \quad \forall n > N(\varepsilon). \tag{65}$$

This means that the sequence converges to f_0 in the sense of norm on $C_{G,p,M}[0,1]$. Since $s_{k,p}(f)$ is a seminorm, it holds that

$$\begin{aligned} s_{k,p}(f_0) &\leq s_{k,p}(f_n - f_0) + s_{k,p}(f_n) \\ &< \varepsilon + s_{k,p}(f_n), \quad \text{for } n > N(\varepsilon). \end{aligned} \tag{66}$$

Therefore, $f_0 \in C_{G,p,M}[0,1]$, which implies that $C_{\phi,p,M}[0,1]$ is a Banach space.

For each $f \in C_{G,p,M}[0,1]$,

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!},$$

$$|a_k| = \left|D^k f(0)\right| \leq \left\|D^k f\right\|_\infty \leq \sum_{n=0}^{\infty} |a_{n+k}| \frac{1}{n!} \leq e \sup_{n\geq 0} |a_{k+n}|. \tag{67}$$

Clearly,

$$\begin{aligned} \left|a_{k+\sigma(0)}\right| &\leq \left\|D^{k+\sigma(0)} f\right\|_\infty \leq e\left|a_{k+\sigma(0)}\right|, \\ \left|a_{k+\sigma(j)}\right| &\leq \left\|D^{k+\sigma(j)} f\right\|_\infty \leq e\left|a_{k+\sigma(j)}\right|. \end{aligned} \tag{68}$$

Thus, we have

$$s_{k,p} \leq s_{k,p}(f) \leq e\ s_{k,p}. \tag{69}$$

This implies that

$$\|\{a_n\}\|_{G,p} \leq \|f\|_{G,p} \leq e\|\{a_n\}\|_{G,p}. \tag{70}$$

Therefore, the mapping

$$T(f) = \left\{f^{(n)}(0)\right\}_{n=0}^{\infty}, \tag{71}$$

is a bounded invertible linear operator from $C_{G,p,M}[0,1]$ to $c_{G,p,M}$. $\square$

Theorem 9. *Let $C_{G,p}[0,1]$ be defined as (61). Then, $C_{G,p}[0,1]$ is the completion of space $(P[0,1], \|\cdot\|_{G,p})$, and*

$$C_{\phi,p}[0,1] \subset C_{G,p}[0,1] \subset C_{\phi,0}[0,1]. \tag{72}$$

Moreover, space $C_{G,p}[0,1]$ is isomorphic to $c_{G,p}$.

The proof of Theorem 9 is similar to that of Theorem 8; we omit the details.

3.2. Comparison of Spaces. So far, we have introduced some Banach spaces. The question is whether there is an inclusion relation $C_{\phi,q}[0,1] \subset C_{G,p}[0,1]$ for $q > p$. In general, the answer is negative.

Example 10. Let $q > p \geq 1$. Then, $C_{\phi,q}[0,1] \not\subset C_{G,p,M}[0,1]$

Take $p' \in (p,q)$. Define a function by

$$f(x) = \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{1/p'} \frac{x^k}{k!}. \tag{73}$$

Obviously, $f \in C_{\phi,q}[0,1]$, but $f \notin C_{\phi,p}[0,1]$. Furthermore, we also have $f \notin C_{G,p,M}[0,1]$.

In fact, for any $n \in \mathbb{N}$,

$$D^n f(0) = \left(\frac{1}{n}\right)^{1/p'}. \tag{74}$$

Clearly,

$$\sum_{n=1}^{\infty} |D^n f(0)|^q = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{q/p'} < \infty. \tag{75}$$

Due to

$$\begin{aligned} D^n f(x) &= \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)^{1/p'} \frac{x^{k-n}}{(k-n)!} = \sum_{r=0}^{\infty} \left(\frac{1}{n+r}\right)^{1/p'} \frac{x^r}{r!}, \\ \|D^n f\|_\infty &\leq \sum_{k=0}^{\infty} \frac{1}{(n+k)^{1/p'}} \frac{1}{k!}, \quad \forall n \geq 1, \end{aligned} \tag{76}$$

it holds that

$$\begin{aligned} \sum_{n=1}^{\infty} |D^n f(0)|^q &\leq \sum_{n=1}^{\infty} \|D^n f\|_\infty^q \leq \sum_{n=1}^{\infty} \left(\sum_{k=0}^{\infty} \left(\frac{1}{n+k}\right)^{1/p'} \frac{1}{k!}\right)^q \\ &< \infty. \end{aligned} \tag{77}$$

For any $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \left|D^{n+k} f(0)\right|^{p} = \sum_{k=1}^{n} \left(\frac{1}{n+k}\right)^{p/p'} = \frac{1}{n^{p/p'}} \sum_{k=1}^{n} \frac{1}{(1+(k/n))^{p/p'}}, \tag{78}$$

while

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(1+(k/n))^{p/p'}} = \int_0^1 \frac{dx}{(1+x)^{p/p'}}, \tag{79}$$

this shows that $\sup_{n\geq 1} \sum_{k=1}^{n} |D^{n+k} f(0)|^p = \infty$, and hence $\sup_{n\geq 0} s_{n,p}(f) = \infty$. So, $f \notin C_{G,p,M}[0,1]$.

Example 11. It holds that $C_{\phi,p}[0,1] \neq C_{G,p}[0,1]$.

Define a function as

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{1}{(1+n)\ln(1+n)}\right)^{1/p} \frac{x^n}{n!}. \tag{80}$$

Obviously,

$$\begin{gathered} D^k f(0) = \left(\frac{1}{(1+k)\ln(1+k)}\right)^{1/p}, \\ \sum_{k=1}^{n} \left|D^k f(0)\right|^p = \sum_{k=1}^{n} \frac{1}{(1+k)\ln(1+k)} \\ \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(1+k)\ln(1+k)} = \infty. \end{gathered} \tag{81}$$

So, $f \notin C_{\phi,p}[0,1]$.

Furthermore,

$$\begin{aligned} &\left|D^n f(0)\right| \leq \left\|D^n f\right\|_\infty \\ &\quad \leq \sum_{k=1}^{\infty} \left(\frac{1}{(1+n+k)\ln(1+n+k)}\right)^{1/p} \frac{1}{k!} \\ &\quad \leq e\left|D^n f(0)\right|, \\ &\quad \sum_{k=1}^{n} \left|D^{n+k} f(0)\right|^p \\ &\quad = \sum_{k=1}^{n} \frac{1}{(n+k)\ln(n+k)} \\ &\quad = \frac{1}{n\ln n} \sum_{k=1}^{n} \frac{1}{(1+(k/n))(1+(\ln(1+(k/n))/\ln n))}. \end{aligned} \tag{82}$$

Notice the identity

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{(1+(k/n))(1+(\ln(1+(k/n))/\ln n))} = \int_0^1 \frac{dx}{1+x}, \tag{83}$$

and so $f \in C_{G,p}[0,1]$.

Example 12. It holds that $C_{G,p}[0,1] \subset C_{G,p,M}[0,1]$.

Let us consider the following function:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{(1+n)^{1/p}} \frac{x^n}{n!}. \tag{84}$$

Since

$$\begin{gathered} D^n f(0) = \frac{1}{(1+n)^{1/p}}, \\ \left|D^n f(0)\right| \leq \left\|D^n f\right\|_\infty \leq e\left|D^n f(0)\right|, \end{gathered} \tag{85}$$

we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(1+k+n)} = \int_0^1 \frac{dx}{1+x}, \tag{86}$$

this means that $f \in C_{G,p,M}[0,1]$, but $f \notin C_{G,p}[0,1]$.

A new question is whether there is a inclusion relation between $C_{G,p,M}[0,1]$ and $C_{G,q}[0,1]$ for any $q > p$? The following result gives a positive answer.

Theorem 13. *For any $q > p \geq 1$, the inclusion $C_{G,p,M}[0,1] \subset C_{G,q}[0,1]$ holds true.*

Proof. Set $f \in C_{G,p,M}[0,1]$. Then,

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \tag{87}$$

and for any $n \in \mathbb{N}$,

$$\begin{aligned} \left|D^n f(0)\right| &\leq \left\|D^n f\right\|_\infty \leq \sum_{k=0}^{\infty} \left|a_{n+k}\right| \frac{1}{k!} \leq e \sup_{k\geq 0} \left|a_{n+k}\right| \\ &= e\left|a_{n+\sigma(0)}\right|. \end{aligned} \tag{88}$$

From above, we get that

$$\left|D^{n+\sigma(0)} f(0)\right| \leq \left\|D^{n+\sigma(0)} f\right\|_\infty \leq e\left|a_{n+\sigma(0)}\right|. \tag{89}$$

Similarly, for $k \in \{1, 2, \ldots, n\}$,

$$\left|D^{n+\sigma(k)} f(0)\right| \leq \left\|D^{n+\sigma(k)} f\right\|_\infty \leq e\left|a_{n+\sigma(k)}\right|. \tag{90}$$

Since $f \in C_{G,p,M}[0,1]$, it holds that

$$\sum_{k=1}^{n} \left\|D^{(n+\sigma(k))} f\right\|_\infty^p \leq \|f\|_{G,p}, \quad \forall n \in \mathbb{N}. \tag{91}$$

Let $q > p \geq 1$. Set $q = p + \delta$, then

$$\begin{aligned} &\sum_{k=1}^{n} \left\|D^{(n+\sigma(k))} f\right\|_\infty^q \\ &\quad = \sum_{k=1}^{n} \left\|D^{(n+\sigma(k))} f\right\|_\infty^\delta \left\|D^{(n+\sigma(k))} f\right\|_\infty^p \\ &\quad \leq \left\|D^{n+\sigma(0)} f\right\|_\infty^\delta \|f\|_{G,p}^p, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{92}$$

Since

$$\lim_{n\to\infty}\|D^n f\|_\infty = 0, \tag{93}$$

it holds that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left\|D^{(n+\sigma(k))}f\right\|_\infty^q = 0. \tag{94}$$

This gives that $f \in C_{G,q}[0,1]$. □

3.3. $C_{\phi,g,1}[0,1]$-Type Space. In the previous discussion, we insert the some spaces between $C_{\phi,p}[0,1]$ and $C_{\phi,q}[0,1]$. The questions is can one insert a Banach space between $P[0,1]$ and $C_{\phi,1}[0,1]$?

For each $f \in C_{\phi,1}[0,1]$, whose norm is

$$\|f\|_{\phi,1} = \sum_{n=0}^{\infty}\|D^n f\|_\infty, \tag{95}$$

the corresponding s-sequence is

$$s_k(f) = \sum_k \sigma\left\{D^{k+n}f, n \geq 0\right\} = \sum_{j=0}^{k}\left\|D^{k+\sigma(j)}f\right\|_\infty, \quad \forall k \geq 0. \tag{96}$$

Define a positive number

$$\|f\|_{g,1} = \sum_{k=0}^{\infty}s_k(f), \tag{97}$$

where $(g,1)$ denotes the ℓ^1 sum after the generalized 1-summation.

Define the function space by

$$C_{\phi,g,1}[0,1] = \left\{f \in C_{\phi,1}[0,1] \mid \|f\|_{g,1} < \infty\right\}. \tag{98}$$

Then, we have the following result.

Theorem 14. *Let $\|f\|_{g,1}$ be defined as before. Then, for any $f \in C_{\phi,g,1}[0,1]$,*

(1) *the following inequality holds*

$$\|f\|_{g,1} \geq \|f\|_{\phi,1}, \quad \forall f \in C_{\phi,g,1}[0,1]; \tag{99}$$

(2) *$\|f\|_{g,1}$ is a norm on $C_{\phi,g,1}[0,1]$. Further, $(C_{\phi,g,1}[0,1], \|\cdot\|_{g,1})$ is a Banach space;*

(3) *$P[0,1] \subset C_{\phi,g,1}[0,1]$ is a dense subspace.*

Proof. For any $f \in C_{\phi,1}[0,1]$, we have

$$s_0(f) \geq \|f\|_\infty, \qquad s_k(f) \geq \left\|D^k f\right\|_\infty, \quad \forall k \geq 1. \tag{100}$$

This leads to

$$\sum_{k=0}^{n}s_k(f) \geq \sum_{k=0}^{n}\left\|D^k f\right\|_\infty. \tag{101}$$

Obviously, when $f \in C_{\phi,g,1}[0,1]$, it holds that $\|f\|_{\phi,1} \leq \|f\|_{g,1}$.

Since $s_k(f)$ is a seminorm, the above relation shows that $\|f\|_{g,1}$ is a norm. In what follows, we shall prove that $C_{\phi,g,1}[0,1]$ is a Banach space.

Let $f_n \in C_{\phi,g,1}[0,1]$ be a Cauchy sequence, that is,

$$\lim_{m,n\to\infty}\|f_n - f_m\|_{g,1} = 0. \tag{102}$$

Since

$$\|f_n - f_n\|_{\phi,1} \leq \|f_n - f_m\|_{g,1}, \tag{103}$$

this implies that $\{f_n\} \subset C_{\phi,1}[0,1]$ also is a Cauchy sequence, so there exists a $f_0 \in C_{\phi,1}[0,1]$ such that

$$\lim_{m\to\infty}\|f_m - f_0\|_{\phi,1} = 0. \tag{104}$$

For any $k \in \mathbb{N}$, when N is large enough, we have

$$\sum_{j=0}^{k}s_j(f_n - f_m) \leq \|f_n - f_m\|_{g,1} < \varepsilon, \quad \forall n, m > N. \tag{105}$$

Using the continuity of $s_k(f)$ with respect to $\|f|_{\phi,1}$, and taking $m \to \infty$, we get that

$$\sum_{j=0}^{k}s_j(f_n - f_0) \leq \varepsilon, \quad \forall n \geq N. \tag{106}$$

Again taking $k \to \infty$, we get that

$$\|f_n - f_0\|_{g,1} \leq \varepsilon, \quad \forall n > N. \tag{107}$$

So, $f_0 \in C_{\phi,g,1}[0,1]$, and

$$\lim_{n\to\infty}\|f_n - f_0\|_{g,1} = 0. \tag{108}$$

The completeness of the space follows.

Let $p(x)$ be a n-order polynomial. Then,

$$\begin{aligned} &s_0(p) \geq \sup_{k\geq 0}|p(x)|, \quad s_k(p) \leq \|p\|_{\phi,1}, \\ &s_n(p) = \|D^n p\|_\infty, \quad s_{n+k}(p) = 0, \quad k \geq 1. \end{aligned} \tag{109}$$

Therefore,

$$\|p\|_{g,1} = \sum_{k=0}^{n}s_k(p) < \infty. \tag{110}$$

For any $f \in C_{\phi,g,1}[0,1]$,

$$f(x) = \sum_{j=0}^{\infty}a_j\frac{x^j}{j!}, \tag{111}$$

for $\varepsilon > 0$, there exists a N such that

$$\frac{e}{2}\sup_{j\geq n+1}\left|a_{j+k}\right| + \sum_{k=n}^{\infty}s_k(f) < \varepsilon, \quad \forall n > N. \tag{112}$$

Taking a polynomial

$$p_n(x) = \sum_{j=0}^{n} a_j \frac{x^j}{j!} \tag{113}$$

and noticing

$$\begin{aligned}\left\|(f-p_n)^{(k)}\right\|_\infty &\le \sum_{j=n}^{\infty} \left|a_j\right| \frac{1}{(j-k)!} \\ &\le \sup_{j\ge n}\left|a_j\right| \sum_{j=n}^{\infty} \frac{1}{(j-k)!}, \quad k \le n,\end{aligned} \tag{114}$$

and $(f-p_n)^{(k)} = f^{(k)}$ for $k > n$, we have

$$\|f-p_n\|_{g,1} \le \frac{e}{2}\sup_{j\ge n+1}\left|a_{j+k}\right| + \sum_{k=n}^{\infty} s_k(f) < \varepsilon. \tag{115}$$

This means that $P[0,1]$ is dense in $C_{\phi,g,1}[0,1]$. □

3.4. Some Open Questions. From the previous discussion we see that

$$\begin{aligned} C_{\phi,1}[0,1] &\subset C_{\phi,p}[0,1] \subset C_{\phi,q}[0,1] \\ &\subset C_{\phi,M}[0,1], \quad \forall 1 < p < q, \\ C_{\phi,p}[0,1] &\subset C_{G,p}[0,1] \subset C_{G,p,M}[0,1] \subset C_{G,q}[0,1] \\ &\subset C_{G,q,M}[0,1], \quad \forall 1 < p < q. \end{aligned} \tag{116}$$

Note that the first relation

$$C_{\phi,q}[0,1] \subset C_{G,q}[0,1] \subset C_{G,q,M}[0,1], \tag{117}$$

and the relation $C_{\phi,q}[0,1] \not\subset C_{G,p,M}[0,1]$ (see, Example 10). These spaces are based on the new summation approach.

Now, we propose some open questions in mathematics aspect.

Question 1. Does the inclusion $C_{G,p,M}[0,1] \subset C_{\phi,q}[0,1]$ hold? yes or no.

Question 2. What is the dual space of $C_{G,p}[0,1]$? Is it a reflexive space?

Question 3. Can one find out a Banach space $(\widetilde{P}[0,1], \|\cdot\|_*)$, that is, the least in the following sense: $\widetilde{P}[0,1]$ is the completion of $P[0,1]$ under the norm $\|\cdot\|_*$ such that, for every Banach space $\mathbb{X}$ of functions, $\widetilde{P}[0,1]$ embeds densely and continuously into $\mathbb{X}$?

Acknowledgment

The research is supported by the Natural Science Foundation of China (NSFC-61174080). The work has been completed during the visit of Gen-Qi Xu in the University of Lyon, April 2013.

References

[1] E. Bornberg-Bauer, “How are model protein structures distributed in sequence space?” *Biophysical Journal*, vol. 73, no. 5, pp. 2393–2403, 1997.

[2] J. Fan, T. Huang, and R. Li, “Analysis of longitudinal data with semiparametric estimation of convariance function,” *Journal of the American Statistical Association*, vol. 102, no. 478, pp. 632–641, 2007.

[3] E. Levina, A. J. Rothman, and J. Zhu, “Sparse estimation of large covariance matrices via a nested Lasso penalty,” *Annals of Applied Statistics*, vol. 2, no. 1, pp. 245–263, 2008.

[4] N. El Karoui, “Operator norm consistent estimation of large-dimensional sparse covariance matrices,” *The Annals of Statistics*, vol. 36, no. 6, pp. 2717–2756, 2008.

[5] A. J. Rothman, E. Levina, and J. Zhu, “A new approach to Cholesky-based covariance regularization in high dimensions,” *Biometrika*, vol. 97, no. 3, pp. 539–550, 2010.

[6] A. Delcroix, M. F. Hasler, S. Pilipović, and V. Valmorin, “Generalized function algebras as sequence space algebras,” *Proceedings of the American Mathematical Society*, vol. 132, no. 7, pp. 2031–2038, 2004.

[7] N. Subramanian and U. K. Misra, “The semi normed space defined by a double gai sequence of modulus function,” *Fasciculi Mathematici*, no. 45, pp. 111–120, 2010.

[8] M. Kirişçi and F. Başar, “Some new sequence spaces derived by the domain of generalized difference matrix,” *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1299–1309, 2010.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 656034, 4 pages
http://dx.doi.org/10.1155/2013/656034

Research Article

Semicommutators and Zero Product of Block Toeplitz Operators with Harmonic Symbols

Puyu Cui,[1] Yufeng Lu,[1] and Yanyue Shi[2]

[1] *School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China*
[2] *School of Mathematical Sciences, Ocean University of China, Qingdao 266100, China*

Correspondence should be addressed to Yanyue Shi; shiyanyue@gmail.com

Received 15 February 2013; Accepted 17 August 2013

Academic Editor: Gen-Qi Xu

We completely characterize the finite rank semicommutators, commutators, and zero product of block Toeplitz operators T_F and T_G with $F, G \in h^\infty \otimes M_{n\times n}$ on the vector valued Bergman space $L_a^2(\mathbb{D}, \mathbb{C}^n)$.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, and let dA be the normalized Lebesgue area measure on $\mathbb{D}$. Denote by $L^\infty(\mathbb{D}, dA)$ and $L^2(\mathbb{D}, dA)$ the space of essential bounded measurable functions and the space of square integral functions with respect to dA, respectively. The Bergman space L_a^2 consists of all analytic functions in $L^2(\mathbb{D}, dA)$. Let $L^2(\mathbb{D}, \mathbb{C}^n) = L^2(\mathbb{D}, dA) \otimes \mathbb{C}^n$ and $L_a^2(\mathbb{D}, \mathbb{C}^n) = L_a^2 \otimes \mathbb{C}^n$, where $\otimes$ means the Hilbert space tensor product. Let $M_{n\times n}$ be the set of all $n \times n$ complex matrixes in the complex plane $\mathbb{C}$, and let $L^\infty(\mathbb{D}, dA) \otimes M_{n\times n}$ be the space of matrix valued essential bounded Lebesgue measurable functions on $\mathbb{D}$.

Given a matrix valued function $\Phi(z) = [\varphi_{ij}(z)]_{n\times n} \in L^\infty(\mathbb{D}, dA) \otimes M_{n\times n}$, the Toeplitz operator T_Φ and the Hankel operator H_Φ on $L_a^2(\mathbb{D}, \mathbb{C}^n)$ with symbol Φ are defined by $T_\Phi h = P(\Phi h)$ and $H_\Phi h = (I - P)(\Phi h)$, respectively, where P is the orthogonal projection from $L^2(\mathbb{D}, \mathbb{C}^n)$ onto $L_a^2(\mathbb{D}, \mathbb{C}^n)$. Then the operators T_Φ and H_Φ have the following matrix representations:

$$T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \cdots & T_{\varphi_{1n}} \\ \vdots & & \vdots \\ T_{\varphi_{n1}} & \cdots & T_{\varphi_{nn}} \end{bmatrix}, \qquad H_\Phi = \begin{bmatrix} H_{\varphi_{11}} & \cdots & H_{\varphi_{1n}} \\ \vdots & & \vdots \\ H_{\varphi_{n1}} & \cdots & H_{\varphi_{nn}} \end{bmatrix}. \quad (1)$$

The problem of the finite rank semicommutator of two Toeplitz operators on the Hardy space has been completely solved in [1, 2]. The analogous problems on the Bergman space have been characterized in [3–5]. In the case of vector valued Hardy space, Gu and Zheng [6] have studied the problems of compact semicommutators and commutators of two block Toeplitz operators.

In this paper, we study the related problems of the two block Toeplitz operators T_F and T_G with $F, G \in h^\infty \otimes M_{n\times n}$ on the vector valued Bergman space $L_a^2(\mathbb{D}, \mathbb{C}^n)$, where h^∞ denotes the space of all bounded harmonic functions on $\mathbb{D}$. Our main idea here is to reduce the problems on $L_a^2(\mathbb{D}, \mathbb{C}^n)$ to the problems of the finite sum of some Toeplitz operator products or some Hankel operator products on the Bergman space L_a^2. In the following, we recall two useful theorems. For every $f \in h^\infty$, denote by f_+ (or $f_- = f - f_+$) the analytic part (or the coanalytic part) of f.

Theorem 1 (see [7]). *Suppose that f_j and g_j are bounded harmonic functions on $\mathbb{D}$ for $j = 1, \ldots, k$. Then the following are equivalent:*

(1) $\sum_{j=1}^k H^*_{\overline{g_j}} H_{f_j}$ *has finite rank,*

(2) $\sum_{j=1}^k H^*_{\overline{g_j}} H_{f_j} = 0$,

(3) $\sigma(f_1, \ldots, f_k; g_1, \ldots, g_k) \equiv 0$,

where $\sigma(f_1, \ldots, f_k; g_1, \ldots, g_k) = \widetilde{\Delta}[\sum_{i=1}^k (f_i)_-(g_i)_+] = (1 - |z|^2)^2 \sum_{i=1}^k (f_i)'_-(g_i)'_+$.

Theorem 2 (see [8]). *Let $u_1, \ldots u_n, v_1, \ldots, v_n$ be bounded harmonic functions on $\mathbb{D}$, and $x_1, \ldots, x_n, y_1, \ldots, y_n \in L^2_a$. Then $\sum_{j=1}^n T_{u_j} T_{v_j} = \sum_{j=1}^n x_j \otimes y_j$ if and only if the following two conditions hold:*

(a) $\sum_{j=1}^n u_j v_j = (1-|z|^2)^2 \sum_{j=1}^n x_j \overline{y_j}$;

(b) $\sum_{j=1}^n [\overline{Pu_j} - u_j(0)][Pv_j - v_j(0)] = 0$.

2. Semicommutators of Block Toeplitz Operators

In this section, we discuss the finite rank semicommutators and commutators of the block Toeplitz operators T_F, T_G on $L^2_a(\mathbb{D}, \mathbb{C}^n)$ with $F, G \in h^\infty \otimes M_{n\times n}$. In the following part of this paper, E_j denote the matrix unit with (j, j) th entry equal to one and all others equal to zero.

Theorem 3. *Let $F(z) = F_+(z) + F_-(z)$, and let $G(z) = G_+(z) + G_-(z) \in h^\infty \otimes M_{n\times n}$ such that $F_+(0) = G_-(0) = 0_{n\times n}$. Then the following statements are equivalent:*

(1) *$T_{FG} - T_F T_G$ has finite rank;*

(2) *$T_{FG} = T_F T_G$;*

(3) *for each $1 \le j \le n$, there exist a matrix $M_j \in M_{n\times n}$ and a permutation matrix Q_j, such that $M_j Q_j G_- E_j = 0_{n\times n}$ and $F_+ Q_j (I + M_j) = 0_{n\times n}$.*

Proof. (1) $\Rightarrow$ (2) Note that $T_{FG} - T_F T_G = H^*_{F^*} H_G$. If $H^*_{F^*} H_G$ has a finite rank, then $\sum_{p=1}^n H^*_{\overline{f_{ip}}} H_{g_{pj}}$ has a finite rank for $1 \le i, j \le n$. By Theorem 1, we have that $\sum_{p=1}^n H^*_{\overline{f_{ip}}} H_{g_{pj}} = 0$ for $1 \le i, j \le n$. Hence, $T_{FG} = T_F T_G$.

(2) $\Rightarrow$ (3) If $T_{FG} - T_F T_G = H^*_{F^*} H_G = 0$, then $[H^*_{F^*} H_G]_{i,j} = \sum_{p=1}^n H^*_{\overline{f_{ip}}} H_{g_{pj}} = 0$, for $1 \le i, j \le n$. By Theorem 1, we get $(1-|z|^2)^2 \sum_{p=1}^n [(f_{ip})'_+ (g_{pj})'_-] = 0$ on $\mathbb{D}$. Therefore, $\sum_{p=1}^n [(f_{ip})'_+ (g_{pj})'_-] = 0$ on $\mathbb{D}$ for $1 \le i, j \le n$.

It is known that if $H(z, w)$ is holomorphic in $\mathbb{D} \times \mathbb{D}$ and $H(z, \overline{z}) = 0$, then $H(z, w) \equiv 0$ for $(z, w) \in \mathbb{D} \times \mathbb{D}$ in several complex variables. So we can "complexificate" the above identity and have $\sum_{p=1}^n [(f_{ip})'_+(z) (g_{pj})'_-(w)] = 0$ for $1 \le i, j \le n$, $(z, w) \in \mathbb{D} \times \mathbb{D}$. Therefore,

$$\begin{aligned} 0 &= \sum_{p=1}^n \left[\left(f_{ip}\right)'_+ (u) \left(g_{pj}\right)'_- (v) \right] \\ &= \int_0^z \sum_{p=1}^n \left[\left(f_{ip}\right)'_+ (u) \left(g_{pj}\right)'_- (v) \right] du \\ &= \sum_{p=1}^n \left[\left(f_{ip}\right)_+ (z) \left(g_{pj}\right)'_- (v) \right] \\ &= \int_0^w \sum_{p=1}^n \left[\left(f_{ip}\right)_+ (z) \left(g_{pj}\right)'_- (v) \right] dv \\ &= \sum_{p=1}^n \left[\left(f_{ip}\right)_+ (z) \left(g_{pj}\right)_- (w) \right], \end{aligned} \quad (2)$$

which means that $F_+(z) G_-(w) = 0_{n\times n}$ on $\mathbb{D} \times \mathbb{D}$.

Note that $\det F_+(z)$ is the analytic function of z and $\det G_-(w)$ is the coanalytic function of w. If $\det F_+(z)$ is not identically zero, then $G_- = 0_{n\times n}$; that is, G is analytic. If $\det G_-(w)$ is not identically zero, then $F_+ = 0_{n\times n}$; that is, F is coanalytic.

In the following, we deal with the case that $\det F_+ \equiv 0$ and $\det G_- \equiv 0$. Suppose $G_- E_j = (g_1^j, \ldots, g_n^j)^T$ for $1 \le j \le n$. Since $\det G_- \equiv 0$, $G_- E_j$ is linearly dependent. Let Q_j be a permutation matrix such that

$$Q_j G_- E_j = \left(g^j_{\sigma(1)}, \ldots, g^j_{\sigma(k_0)}, \ldots, g^j_{\sigma(n)} \right)^T, \quad (3)$$

where $\{g^j_{\sigma(1)}, \ldots, g^j_{\sigma(k_0)}\}$ is maximally linearly independent of $\{g^j_{\sigma(1)}, \ldots, g^j_{\sigma(n)}\}$. And

$$\begin{aligned} g^j_{\sigma(k_0+1)} &= a_1^1 g^j_{\sigma(1)} + \cdots + a^1_{k_0} g^j_{\sigma(k_0)}, \\ &\vdots \\ g^j_{\sigma(n)} &= a_1^{n-k_0} g^j_{\sigma(1)} + \cdots + a^{n-k_0}_{k_0} g^j_{\sigma(k_0)}, \end{aligned} \quad (4)$$

with the matrix representation

$$\begin{bmatrix} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ a_1^1 & \cdots & a^1_{k_0} & -1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_1^{n-k_0} & \cdots & a^{n-k_0}_{k_0} & 0 & \cdots & -1 \end{bmatrix} \begin{bmatrix} g^j_{\sigma(1)} \\ \vdots \\ g^j_{\sigma(k_0)} \\ \vdots \\ g^j_{\sigma(n-1)} \\ g^j_{\sigma(n)} \end{bmatrix} = 0_{n\times n}, \quad (5)$$

that is, $M_j Q_j G_- E_j = 0_{n\times n}$.

Since $F_+(z)[G_-(w) E_j] = 0$, we have $[F_+(z) Q_j][Q_j G_- \times (w) E_j] = 0_{n\times n}$. Denote $F_+ = [f_{ij}]_{n\times n}$. It follows that

$$f_{k\sigma(1)} g^j_{\sigma(1)} + \cdots + f_{k\sigma(k_0)} g^j_{\sigma(k_0)} + \cdots + f_{k\sigma(n)} g^j_{\sigma(n)} = 0 \quad (6)$$

for $1 \le k \le n$. Note that $g^j_{\sigma(k_0+l)} = a_1^l g^j_{\sigma(1)} + \cdots + a^l_{k_0} g^j_{\sigma(k_0)}$, $1 \le l \le n - k_0$. Therefore

$$\begin{aligned} & f_{k\sigma(1)} g^j_{\sigma(1)} + \cdots + f_{k\sigma(k_0)} g^j_{\sigma(k_0)} \\ & \quad + \sum_{l=1}^{n-k_0} \left(a_1^l g^j_{\sigma(1)} + \cdots + a^l_{k_0} g^j_{\sigma(k_0)} \right) f_{k\sigma(k_0+l)} = 0, \end{aligned} \quad (7)$$

and hence, $(f_{k\sigma(1)} + \sum_{l=1}^{n-k_0} a_1^l f_{k\sigma(k_0+l)})g_{\sigma(1)}^j + (f_{k\sigma(2)} + \sum_{l=1}^{n-k_0} a_2^l \times f_{k\sigma(k_0+l)})g_{\sigma(2)}^j + \cdots + (f_{k\sigma(k_0)} + \sum_{l=1}^{n-k_0} a_{k_0}^l f_{k\sigma(k_0+l)})g_{\sigma(k_0)}^j = 0$. Since $\{g_{\sigma(1)}^j, \ldots, g_{\sigma(k_0)}^j\}$ is linearly independent, we have

$$
\begin{gathered}
f_{k\sigma(1)} + \sum_{l=1}^{n-k_0} a_1^l f_{k\sigma(k_0+l)} = 0, \\
\vdots \\
f_{k\sigma(k_0)} + \sum_{l=1}^{n-k_0} a_{k_0}^l f_{k\sigma(k_0+l)} = 0,
\end{gathered} \tag{8}
$$

with the matrix representation

$$
\begin{gathered}
\begin{bmatrix}
f_{1\sigma(1)} & \cdots & f_{1\sigma(k_0)} & \cdots & f_{1\sigma(n)} \\
f_{2\sigma(1)} & \cdots & f_{2\sigma(k_0)} & \cdots & f_{2\sigma(n)} \\
\vdots & & \vdots & & \vdots \\
\vdots & & \vdots & & \vdots \\
f_{n-1\sigma(1)} & \cdots & f_{n-1\sigma(k_0)} & \cdots & f_{n-1\sigma(n)} \\
f_{n\sigma(1)} & \cdots & f_{n\sigma(k_0)} & \cdots & f_{n\sigma(n)}
\end{bmatrix} \\
\times \begin{bmatrix}
1 & \ldots & 0 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
0 & \cdots & 1 & 0 & \cdots & 0 \\
a_1^1 & \cdots & a_{k_0}^1 & 0 & \cdots & 0 \\
\vdots & & \vdots & \vdots & & \vdots \\
a_1^{n-k_0} & \cdots & a_{k_0}^{n-k_0} & 0 & \cdots & 0
\end{bmatrix} = 0_{n\times n},
\end{gathered} \tag{9}
$$

that is $F_+Q_j(I+M_j) = 0_{n\times n}$.

In the following, we prove (3) $\Rightarrow$ (1).

Note that

$$
\begin{aligned}
T_F T_G - T_{FG} &= T_{(F_+ + F_-)} T_{(G_+ + G_-)} - T_{(F_+ + F_-)(G_+ + G_-)} \\
&= T_{(F_+G_+ + F_-G_- + F_-G_+)} + T_{F_+} T_{G_-} \\
&\quad - T_{(F_+G_+ + F_-G_- + F_-G_+)} - T_{F_+G_-} \\
&= T_{F_+} T_{G_-} - T_{F_+G_-}.
\end{aligned} \tag{10}
$$

By the hypothesis, we have

$$
\begin{aligned}
T_{F_+} T_{G_-E_j} &= T_{F_+Q_j(I+M_j-M_j)Q_j} T_{G_-E_j} \\
&= T_{F_+Q_j(I+M_j)Q_j} T_{G_-E_j} - T_{F_+Q_jM_jQ_j} T_{G_-E_j} \\
&= T_{F_+Q_j(I+M_j)Q_j} T_{G_-E_j} - T_{F_+Q_j} T_{M_jQ_jG_-E_j} = 0_{n\times n}, \\
F_+G_-E_j &= F_+Q_j\left[I + M_j - M_j\right] Q_jG_-E_j \\
&= \left[F_+Q_j\left(I + M_j\right)\right] Q_jG_-E_j \\
&\quad - F_+Q_j\left[M_jQ_jG_-E_j\right] = 0_{n\times n}.
\end{aligned} \tag{11}
$$

Hence, we obtain the desired result. □

Remark 4. If the assumption $F_+(0) = G_-(0) = 0_{n\times n}$ is removed and F_+, G_- are replaced by F_+', G_-' in statement (3), respectively, then the above theorem still holds.

Corollary 5. *Let $F(z) = F_+(z) + F_-(z)$, and let $G(z) = G_+(z) + G_-(z) \in h^\infty \otimes M_{n\times n}$ such that $F_+(0) = G_-(0) = 0_{n\times n}$. Suppose that $|\det F_+| + |\det G_-|$ is not identically zero. Then the following are equivalent:*

(1) *$T_{FG} - T_F T_G$ has finite rank;*

(2) *$T_{FG} = T_F T_G$;*

(3) *if $\det F_+$ is not identically zero, then G is analytic; if $\det G_-$ is not identically zero, then F is coanalytic.*

Proof. The proof is obvious. □

Let

$$
\Phi = \begin{bmatrix} F & -G \\ 0 & 0 \end{bmatrix}, \qquad \Psi = \begin{bmatrix} G & 0 \\ F & 0 \end{bmatrix}. \tag{12}
$$

As in [6], the commutator $T_F T_G - T_G T_F$ can be reduced to the semicommutator $T_{\Phi\Psi} - T_\Phi T_\Psi$.

It is easy to check that

$$
\begin{aligned}
T_{\Phi\Psi} - T_\Phi T_\Psi &= \begin{bmatrix} T_{FG-GF} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} T_F & -T_G \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_G & 0 \\ T_F & 0 \end{bmatrix} \\
&= \begin{bmatrix} T_{FG-GF} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} T_F T_G - T_G T_F & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{13}
$$

If $T_F T_G = T_G T_F$, Theorem 2 implies that $FG = GF$. Therefore, $T_\Phi T_\Psi = T_{\Psi\Phi}$. Combining Theorem 3, we get the following explicit theorem.

Theorem 6. *Suppose that $F = F_+ + F_- \in h^\infty \otimes M_{n\times n}$ and $G = G_+ + G_- \in h^\infty \otimes M_{n\times n}$. Then $T_F T_G = T_G T_F$ if and only if F and G satisfy the following two conditions:*

(1) *$FG = GF$;*

(2) *for each $1 \le j \le n$, there exist permutation matrixes $\widetilde{Q_j}$ and $\mathbb{M}_j \in M_{2n\times 2n}$ such that $\mathbb{M}_j \widetilde{Q_j} \left[\begin{smallmatrix} G_-' \\ F_-' \end{smallmatrix}\right] E_j = 0_{n\times n}$ and $[F_+' - G_+'] \widetilde{Q_j}(I + \mathbb{M}_j) = 0_{n\times n}$.*

3. Zero Product of Two Block Toeplitz Operators

In this section, we discuss the zero product of two block Toeplitz operators T_F and T_G with $F, G \in h^\infty \otimes M_{n\times n}$ on $L_a^2(\mathbb{D}, \mathbb{C}^n)$.

Theorem 7. *Let $F(z) = F_+(z) + F_-(z)$ and $G(z) = G_+(z) + G_-(z) \in h^\infty \otimes M_{n\times n}$ such that $F_+(0) = G_-(0) = F_-(0) = G_+(0) = 0_{n\times n}$. Then $T_F T_G = 0_{n\times n}$ if and only if*

(1) *$FG = 0_{n\times n}$;*

(2) *for each $1 \leq j \leq n$, there exist a matrix $N_j \in M_{n\times n}$ and a permutation matrix Q_j^1, such that $N_j Q_j^1 G_- E_j = 0_{n\times n}$ and $F_+ Q_j^1 (I + N_j) = 0_{n\times n}$.*

Proof. By Theorem 2, it is easy to know that $\sum_{p=1}^{n} f_{ip}(z) g_{pj}(z) = 0$ for $1 \leq i, j \leq n$; that is, $FG = 0_{n\times n}$. Since $T_F T_G = T_F T_G - T_{FG} = 0_{n\times n}$, we obtain the desired result by Theorem 3. □

Corollary 8. *Let $F(z) = F_+(z) + F_-(z)$ and let $G(z) = G_+(z) + G_-(z) \in h^\infty \otimes M_{n\times n}$ such that $F_+(0) = G_-(0) = F_-(0) = G_+(0) = 0_{n\times n}$.*

(1) *If neither* det G_+ *nor* det G_- *is identically zero, then $T_F T_G = 0$ if and only if $F = 0_{n\times n}$.*

(2) *If neither* det F_+ *nor* det F_- *is identically zero, then $T_F T_G = 0$ if and only if $G = 0_{n\times n}$.*

Proof. (1) We only need to prove the necessity. By the proof of Theorem 3, we have $F_+(z)G_-(w) = 0_{n\times n}$ on $\mathbb{D} \times \mathbb{D}$. By the theorem given above, we know that $FG = [F_+ + F_-][G_+ + G_-] = 0_{n\times n}$. Taking Laplace transform, it follows that $F'_- G'_+ = 0_{n\times n}$. As in Theorem 3, we can prove that $F_-(w)G_+(z) = 0_{n\times n}$ on $\mathbb{D} \times \mathbb{D}$. "Complexificate" the identity $FG = 0_{n\times n}$, and then $[F_+(z) + F_-(w)][G_+(z) + G_-(w)] = 0_{n\times n}$. Hence, $F_+(z)G_+(z) = -F_-(w)G_-(w)$ on $\mathbb{D} \times \mathbb{D}$. Since $F_+(0) = G_-(0) = 0_{n\times n}$, we have $F_+ G_+ = F_- G_- = 0_{n\times n}$.

It is clear that det $G_+(z)$ is the analytic function of z and det $G_-(w)$ is the coanalytic function of w. If det $G_+(z)$ is not identically zero, then $F_- \equiv 0_{n\times n}$ and $F_+ \equiv 0_{n\times n}$. Therefore, $F = 0_{n\times n}$. If det $G_-(w)$ is not identically zero, then $F_- \equiv 0_{n\times n}$ and $F_+ \equiv 0_{n\times n}$. Therefore, $F = 0_{n\times n}$.

(2) The proof is similar to the proof of (1). □

Acknowledgments

This research is supported by NSFC (11271059, 11201438), Research Fund for the Doctoral Program of Higher Education of China, Shandong Province Young Scientist Research Award Fund (BS2012SF031), and the Fundamental Research Funds for the Central Universities (201213011).

References

[1] S. Axler, S. Y. A. Chang, and D. Sarason, "Products of Toeplitz operators," *Integral Equations and Operator Theory*, vol. 1, no. 3, pp. 285–309, 1978.

[2] X. Ding and D. Zheng, "Finite rank commutator of Toeplitz operators or Hankel operators," *Houston Journal of Mathematics*, vol. 34, no. 4, pp. 1099–1119, 2008.

[3] P. Ahern and Ž. Čučković, "A theorem of Brown-Halmos type for Bergman space Toeplitz operators," *Journal of Functional Analysis*, vol. 187, no. 1, pp. 200–210, 2001.

[4] S. Axler and Ž. Čučković, "Commuting Toeplitz operators with harmonic symbols," *Integral Equations and Operator Theory*, vol. 14, no. 1, pp. 1–12, 1991.

[5] D. C. Zheng, "Hankel operators and Toeplitz operators on the Bergman space," *Journal of Functional Analysis*, vol. 83, no. 1, pp. 98–120, 1989.

[6] C. Gu and D. Zheng, "Products of block Toeplitz operators," *Pacific Journal of Mathematics*, vol. 185, no. 1, pp. 115–148, 1998.

[7] K. Guo, S. Sun, and D. Zheng, "Finite rank commutators and semicommutators of Toeplitz operators with harmonic symbols," *Illinois Journal of Mathematics*, vol. 51, no. 2, pp. 583–596, 2007.

[8] B. R. Choe, H. Koo, and Y. J. Lee, "Sums of Toeplitz products with harmonic symbols," *Revista Mathemática Iberoamericana*, vol. 24, no. 1, pp. 43–70, 2008.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 108535, 3 pages
http://dx.doi.org/10.1155/2013/108535

Research Article

On Supra-Additive and Supra-Multiplicative Maps

Jin Xi Chen[1,2] and Zi Li Chen[2]

[1] *College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, China*
[2] *Department of Mathematics, Southwest Jiaotong University, Chengdu 610031, China*

Correspondence should be addressed to Jin Xi Chen; jinxichen@home.swjtu.edu.cn

Received 9 May 2013; Revised 28 July 2013; Accepted 14 August 2013

Academic Editor: Gen-Qi Xu

Let A and B be ordered algebras over $\mathbb{R}$, where A has a generating positive cone and B satisfies the property that $b^2 > 0$ if $0 \neq b \in B$. We give some conditions for a map $T : A \rightarrow B$ which is supra-additive and supra-multiplicative for all positive and negative elements to be linear and multiplicative; that is, T is a homomorphism of algebras. Our results generalize some known results on supra-additive and supra-multiplicative maps between spaces of real functions.

Let X be a compact Hausdorff space. Rădulescu [1] proved that whenever a map $T : C(X) \rightarrow C(X)$ satisfies the conditions $T(f+g) \geq T(f) + T(g)$ (*supra-additive*) and $T(fg) \geq T(f)T(g)$ (*supra-multiplicative*) for all $f, g \in C(X)$, then T is linear and multiplicative. Ercan [2] generalized this result to arbitrary topological spaces. For arbitrary topological spaces X, Y, a supra-additive and supra-multiplicative map $T : C(X) \rightarrow C(Y)$ is both linear and multiplicative if and only if $\lim_n T(f^+ \wedge n - f^- \wedge n)(y) = T(f)(y)$ for each $f \in C(X)$ and $y \in Y$ [2, Theorem 2]. Similar results were obtained for rings (see [3, 4]). In [3] Dhombres showed that if A is a ring and R is an ordered ring in which nonzero elements have nonzero positive squares, the supra-additive and supra-multiplicative map $T : A \rightarrow R$ is both additive and multiplicative, that is, a homomorphism of rings. Recently, Gusić [5] considered supra-additive and supra-multiplicative maps between ordered fields. It was proved that on ordered fields every supra-additive and supra-multiplicative nonzero map is an injective homomorphism of fields.

Let A and B be ordered algebras over $\mathbb{R}$, where A has a generating positive cone and B satisfies the property that $b^2 > 0$ if $0 \neq b \in B$. In this short paper, we prove that if a positive map $T : A \rightarrow B$ is supra-additive and supra-multiplicative for all positive and negative elements in A, then T is indeed linear and multiplicative. In particular, if A is squareroot closed, then every map which is supra-additive and supra-multiplicative for all positive and negative elements is both linear and multiplicative. From our result, it follows that for arbitrary topological spaces X, Y, a supra-additive and supra-multiplicative map $T : C(X) \rightarrow C(Y)$ is indeed both linear and multiplicative. This generalizes the results of Rădulescu [1] and Ercan [2]. As a special case we consider the supra-additive and supra-multiplicative maps on $C_0(X)$ and obtain a Banach-Stone type result.

Recall that an ordered real vector space A under a multiplication is said to be an ordered algebra whenever the multiplication makes A an algebra, and in addition it satisfies the following property: if $a_1, a_2 \geq 0$, then $a_1 a_2 \geq 0$. A is called Archimedean if $x, y \in A$ and $nx \leq y$ for all $n \in \mathbb{N}$ implies that $x \leq 0$. The set $A^+ = \{a \in A : a \geq 0\}$ is called the positive cone of A. The positive cone of A is said to be generating (or A is positively generated) if $A = A^+ - A^+$. A map $T : A \rightarrow B$ between two ordered algebras is called positive whenever $T(A^+) \subseteq B^+$. Let $A^- := -A^+$ and $A^{\pm} := A^+ \cup A^-$.

Theorem 1. *Let A be an ordered algebra which has a generating positive cone. Let B be an Archimedean ordered algebra satisfying the property that $b^2 > 0$ if $0 \neq b \in B$. If a positive map $T : A \rightarrow B$ satisfies the following inequalities:*

(1) $T(a_1 + a_2) \geq T(a_1) + T(a_2)$ *(supra-additive),*

(2) $T(a_1 a_2) \geq T(a_1)T(a_2)$ *(supra-multiplicative),*

for all $a_1, a_2 \in A^{\pm}$, then T is both linear and multiplicative.

Proof. From the inequality $T(0) = T(0+0) \geq T(0) + T(0)$, it follows that $T(0) \leq 0$. On the other hand, $T(0) \geq T(0 \cdot 0) \geq T(0)T(0) \geq 0$. Therefore, $T(0) = 0$. It should be noted that $a^2 \geq 0$ for every $a \in A^{\pm}$. By the supra-additivity of T, we have

$$0 = T(0) = T\left(a^2 - a^2\right) \geq T\left(a^2\right) + T\left(-a^2\right), \quad (1)$$

for every $a \in A^{\pm}$. Thus, for every $a \in A^{\pm}$, from the following inequalities:

$$\begin{aligned} 0 &\leq [T(a) + T(-a)]^2 = T(a)^2 + T(a)T(-a) \\ &\quad + T(-a)T(a) + T(-a)^2 \\ &\leq T\left(a^2\right) + T\left(-a^2\right) + T\left(-a^2\right) + T\left(a^2\right) \\ &= 2\left[T\left(a^2\right) + T\left(-a^2\right)\right] \leq 0, \end{aligned} \quad (2)$$

it follows that $[T(a) + T(-a)]^2 = 0$ for every $a \in A^{\pm}$. By our hypothesis on B, we have $T(-a) = -T(a)$ for all $a \in A^{\pm}$. Now, for all $a_1, a_2 \in A^{\pm}$, we have

$$\begin{aligned} T(a_1 + a_2) &\geq T(a_1) + T(a_2) \\ &= -T(-a_1) - T(-a_2) \\ &= -[T(-a_1) + T(-a_2)] \\ &\geq -T(-a_1 - a_2) \\ &= T(a_1 + a_2). \end{aligned} \quad (3)$$

That is, $T(a_1 + a_2) = T(a_1) + T(a_2)$ for all $a_1, a_2 \in A^{\pm}$. Similarly, for all $a_1, a_2 \in A^{\pm}$, from

$$\begin{aligned} T(a_1 a_2) &\geq T(a_1)T(a_2) = -T(a_1)T(-a_2) \\ &\geq -T(-a_1 a_2) = T(a_1 a_2), \end{aligned} \quad (4)$$

it follows that $T(a_1 a_2) = T(a_1)T(a_2)$ for all $a_1, a_2 \in A^{\pm}$.

Next, because A is positively generated and B is Archimedean, by the positive additivity of T on A^+ and the Kantorovich theorem (cf. [6, Proposition 1, page 150] or [7, Theorem 1.7]) there exists a unique positive linear map $S : A \to B$ such that $S = T$ on A^+, where S is defined by $S(a_1 - a_2) = T(a_1) - T(a_2)$ for all $a_1, a_2 \in A^+$. For every $a \in A$, there exist $u, v \in A^+$ satisfying $a = u - v$ since A is positively generated. From

$$\begin{aligned} T(a) &= T(u - v) = T(u) + T(-v) \\ &= T(u) - T(v) \\ &= S(u - v) = S(a), \end{aligned} \quad (5)$$

it follows that $T = S$ on A. This implies that T is linear. On the other hand, let a_1 and a_2 be arbitrary elements in A. Since A^+ is positively generated, there exist $u_i, v_i \in A^+$ $(i = 1, 2)$, such that $a_1 = u_1 - v_1$ and $a_2 = u_2 - v_2$. We have

$$\begin{aligned} T(a_1 a_2) &= T[(u_1 - v_1)(u_2 - v_2)] \\ &= T(u_1 u_2) - T(u_1 v_2) - T(v_1 u_2) + T(v_1 v_2) \\ &= T(u_1)T(u_2) - T(u_1)T(v_2) \\ &\quad - T(v_1)T(u_2) + T(v_1)T(v_2) \\ &= [T(u_1) - T(v_1)][T(u_2) - T(v_2)] \\ &= T(a_1)T(a_2). \end{aligned} \quad (6)$$

That is, T is multiplicative, as desired. □

Remark 2. Let A and B be as in the above theorem. It may be asked whether the positive map $T : A \to B$ is linear and multiplicative whenever $T : A \to B$ is supra-additive and supra-multiplicative only on A^+ or only on A^-. Indeed, this is not the case. For instance, the positive nonlinear map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is additive and multiplicative on $\mathbb{R}^+$ (or $\mathbb{R}^-$, resp.).

Recall that an ordered algebra A is said to be squareroot closed and that whenever for any $a \in A^+$ there exists $x \in A^+$, such that $a = x^2$. When A is square-root closed, we have the following corollary.

Corollary 3. *Let A be a square-root closed ordered algebra with a generating positive cone. Let B be an Archimedean ordered algebra satisfying the property that $b^2 > 0$ if $0 \neq b \in B$. If a map $T : A \to B$ is supra-additive and supra-multiplicative on $A^{\pm}$, then T is both linear and multiplicative on A.*

Proof. By the above Theorem, to complete the proof, we need only to verify that T is positive. Since A is square-root closed for each $a \geq 0$ in A there exists $x \in A^+$, such that $a = x^2$. Hence, by our hypothesis on B, we have

$$T(a) = T\left(x^2\right) \geq T(x)^2 \geq 0. \quad (7)$$

This implies that T is positive. □

Remark 4. It should be noted that the space of all real functions (all real continuous functions) on a nonempty set (a topological space, resp.), with the pointwise algebraic operations and the pointwise ordering, is a square-root closed Archimedean lattice-ordered algebra with the property mentioned in Corollary 3. Thus, the results on supra-additive and supra-multiplicative maps between spaces of real functions obtained by Rădulescu [1], Volkmann [4], and Ercan [2] can now follow from Corollary 3. In their earlier proofs, the constant function or the multiplicative unit element plays an essential role.

Let X be a locally compact Hausdorff space, and let $C_0(X)$ be the Banach lattice of all continuous real functions defined on X and vanishing at infinity. Note that $C_0(X)$ does not necessarily contain the constant function or a unit element unless X is compact. The following result is an immediate consequence of Corollary 3.

Corollary 5. *Let X and Y be locally compact Hausdorff spaces. If $T : C_0(X) \rightarrow C_0(Y)$ is supra-additive and supra-multiplicative for all elements in $C_0(X)^{\pm}$, then T is an algebra and lattice homomorphism.*

Recall that $T : C_0(X) \rightarrow C_0(Y)$ for any f, g is said to be separating or disjointness preserving if, for any $f, g \in C_0(X)$, $fg = 0$ implies that $TfTg = 0$. Clearly, every multiplicative map is disjointness preserving. Combining Corollary 5 with Theorem 2 of [8] or Theorem 8 in [9], we obtain the following Banach-Stone type result.

Corollary 6. *Let X, Y be locally compact Hausdorff spaces. If there exists a bijection from $C_0(X)$ onto $C_0(Y)$ which is supra-additive and supra-multiplicative on $C_0(X)^{\pm}$, then X is topologically homeomorphic to Y.*

Acknowledgment

The authors would like to thank the reviewer for his/her kind comments and valuable suggestions which have improved this paper. The authors were supported in part by the Fundamental Research Funds for the Central Universities (SWJTU11CX154, SWJTU12ZT13) and NSFC (No. 11301285).

References

[1] M. Rădulescu, "On a supra-additive and supra-multiplicative operator of $C(X)$," *Bulletin Mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie*, vol. 24, no. 3, pp. 303–305, 1980.

[2] Z. Ercan, "A remark on supra-additive and supra-multiplicative operators on $C(X)$," *Mathematica Bohemica*, vol. 132, no. 1, pp. 55–58, 2007.

[3] J. Dhombres, "Sur les fonctions simultanément suradditives et surmultiplicatives," *Comptes Rendus Mathématiques. La Société Royale du Canada. L'Academie des Sciences*, vol. 5, no. 5, pp. 207–210, 1983.

[4] P. Volkmann, "Sur les fonctions simultanément suradditives et surmultiplicatives," *Bulletin Mathématique de la Société des Sciences Mathématiques de la République Socialiste de Roumanie*, vol. 28, no. 2, pp. 181–184, 1984.

[5] I. Gusić, "A note on certain maps between ordered fields," *Journal of Mathematical Inequalities*, vol. 3, no. 4, pp. 657–661, 2009.

[6] R. Cristescu, *Topological Vector Spaces*, Editura Academiei, Bucharest, Romania, 1977.

[7] C. D. Aliprantis and O. Burkinshaw, *Positive Operators*, vol. 119 of *Pure and Applied Mathematics*, Academic Press, Orlando, Fla, USA, 1985.

[8] J. J. Font and S. Hernández, "On separating maps between locally compact spaces," *Archiv der Mathematik*, vol. 63, no. 2, pp. 158–165, 1994.

[9] J.-S. Jeang and N.-C. Wong, "Weighted composition operators of $C_0(X)$'s," *Journal of Mathematical Analysis and Applications*, vol. 201, no. 3, pp. 981–993, 1996.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 394216, 7 pages
http://dx.doi.org/10.1155/2013/394216

Research Article

On the Aleksandrov-Rassias Problems on Linear n-Normed Spaces

Yumei Ma

Department of Mathematics, Dalian Nationality University, Dalian 116600, China

Correspondence should be addressed to Yumei Ma; mayumei1962@163.com

Received 11 May 2013; Accepted 17 July 2013

Academic Editor: Ji Gao

This paper generalizes T. M. Rassias' results in 1993 to n-normed spaces. If X and Y are two real n-normed spaces and Y is n-strictly convex, a surjective mapping $f : X \to Y$ preserving unit distance in both directions and preserving any integer distance is an n-isometry.

1. Introduction

Let X and Y be two metric spaces. A mapping $f : X \to Y$ is called an isometry if f satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X(\cdot,\cdot)$ and $d_Y(\cdot,\cdot)$ denote the metrics in the spaces X and Y, respectively. For some fixed number $r > 0$, suppose that f preserves distance r, that is, for all $x, y \in X$ with $d_X(x, y) = r$, we have $d_Y(f(x), f(y)) = r$, then r is called a conservative (or preserved) distance for the mapping f. In particular, we denote DOPP as f preserving the one distance property and SDOPP as f preserving the strong one distance property and also for f^{-1}.

In 1970 [1], Aleksandrov posed the following problem. *Examine whether the existence of a single conservative distance for some mapping T implies that T is an isometry.* This question is of great significance for the Mazur-Ulam Theorem [2].

In 1993, T. M. Rassias and P. Šemrl proved the following.

Theorem 1 (see [3]). *Let X and Y be two real normed linear spaces such that one of them has a dimension greater than one. Assume also that one of them is strictly convex. Suppose that $f : X \to Y$ is a surjective mapping that satisfies SDOPP. Then, f is an affine isometry (a linear isometry up to translation).*

Theorem 2 (see [3]). *Let X and Y be two real normed linear spaces such that one of them has a dimension greater than one. Suppose that $f : X \to Y$ is a Lipschitz mapping. Assume also that f is a surjective mapping satisfying (SDOPP). Then, f is an isometry.*

Since 2004, the Aleksandrov problem in n-normed spaces ($n \geq 2$) has been discussed, and some results are obtained [4–8].

Definition 3 (see [7]). Let X be a real linear space with $\dim X \geq n$ and $\|\cdot, \ldots, \cdot\| : X^n \to R$, a function, then $(X, \|\cdot, \ldots, \cdot\|)$ is called a linear n-normed space if for any $\alpha \in R$ and all $x, y, x_1, \ldots, x_n \in X$

nN_1: $\|x_1, \ldots, x_n\| = 0 \Leftrightarrow x_1, \ldots, x_n$ are linearly dependent,

nN_2: $\|x_1, \ldots, x_n\| = \|x_{j1}, \ldots, x_{jn}\|$ for every permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$,

nN_3: $\|\alpha x_1, \ldots, x_n\| = |\alpha|\|x_1, \ldots, x_n\|$,

nN_4: $\|x + y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$. The function $\|\cdot, \ldots, \cdot\|$ is called the n-norm on X.

Definition 4 (see [8]). Let X and Y be two real linear n-normed spaces.

(i) A mapping $f : X \to Y$ is defined to be an n-isometry if for all $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$,

$$\|f(x_1) - f(y_1), \ldots, f(x_n) - f(y_n)\| = \|x_1 - y_1, \ldots, x_n - y_n\|. \quad (1)$$

(ii) A mapping $f : X \to Y$ is called the n-distance one preserving property (n-DOPP) if for $x_1, \dots, x_n, y_1, \dots, y_n \in X$, $\|x_1 - y_1, \dots, x_n - y_n\| = 1$, it follows that $\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| = 1$.

(iii) A mapping $f : X \to Y$ is called the n-strong distance one preserving property (n-SDOPP) if for $x_1, \dots, x_n, y_1, \dots, y_n \in X$, $\|x_1 - y_1, \dots, x_n - y_n\| = 1$, it follows that $\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| = 1$ and conversely.

(iv) A mapping $f : X \to Y$ is called an n-Lipschitz if for all $x_1, \dots, x_n, y_1, \dots, y_n \in X$,

$$\begin{aligned}&\|f(x_1) - f(y_1), \dots, f(x_n) - f(y_n)\| \\ &\quad \le \|x_1 - y_1, \dots, x_n - y_n\|.\end{aligned} \tag{2}$$

Definition 5 (see [7]). The points $x_0, x_1, \dots, x_n$ of X are called n-collinear if for every i, $\{x_j - x_i : 0 \le j \ne i \le n\}$ is linearly dependent.

Definition 6. X is said to be n-strictly convex normed spaces if for any $x_0, x_1, x_2, \dots, x_n \in X$, $x_2, \dots, x_n \notin \operatorname{span}\{x_0, x_1\}$, and $\|x_0 + x_1, x_2, \dots, x_n\| = \|x_0 x_2, \dots, x_n\| + \|x_1 x_2, \dots, x_n\|$ imply that x_0 and x_1 are linearly dependent.

C. Park and T. M. Rassias obtained the following.

Theorem 7 (see [8]). *Let X and Y be real linear n-normed spaces. If a mapping $f : X \to Y$ satisfies the following conditions:*

(i) *f has the n-DOPP,*

(ii) *f is n-Lipschitz,*

(iii) *f preserves the 2-collinearity,*

(iv) *f preserves the n-collinearity,*

then f is an n-isometry.

In 2009, Gao [6] researched another n-isometry and gave the 2-strictly convex concept [6].

In this paper, we generalize T. M. Rassias Theorems 1 and 7 on n-strictly convex normed spaces ($n > 1$).

2. Main Results

The proof of the following lemma was presented in [9], to be published; the proof is given again for the convenience of readers.

Lemma 8. *Let X be an n-normed space such that X has dimension greater than n and $r > 0$. Suppose that $0 < \|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| \le 2r$ for $x_1, \dots, x_n, y_1, \dots, y_n \in X$. Then, there exists $\omega \in X$ such that*

$$\begin{aligned}\|x_1 - \omega, x_2 - y_2, \dots, x_n - y_n\| &= r, \\ \|\omega - y_1, x_2 - y_2, \dots, x_n - y_n\| &= r.\end{aligned} \tag{3}$$

Proof. Since $x_1 - y_1, x_2 - y_2, \dots, x_n - y_n$ are linearly independent and $\dim X > n$, then there exists $z_0 \in X \setminus \operatorname{span}\{x_1 - y_1, \dots, x_n - y_n\}$ with $\|z_0, x_2 - y_2, \dots, x_n - y_n\| = r$.

Set $y_0 = y_1 - x_1$. For any $\alpha \in R$, we have

$$\|z_0 + \alpha y_0, x_2 - y_2, \dots, x_n - y_n\| \ne 0. \tag{4}$$

Let us define $h(\alpha)$ by

$$h(\alpha) = \frac{r(z_0 + \alpha y_0)}{\|z_0 + \alpha y_0, x_2 - y_2, \dots, x_n - y_n\|}, \tag{5}$$

then, we obtain

$$\|h(\alpha), x_2 - y_2, \dots, x_n - y_n\| = r. \tag{6}$$

Set

$$\begin{aligned}z_1 &= \frac{-r(y_1 - x_1)}{\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\|}, \\ z_2 &= \frac{r(y_1 - x_1)}{\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\|}.\end{aligned} \tag{7}$$

Clearly, $z_0 \ne z_1, z_2$. And we have

$$\begin{aligned}\|z_1, x_2 - y_2, \dots, x_n - y_n\| &= r, \\ \|z_2, x_2 - y_2, \dots, x_n - y_n\| &= r.\end{aligned} \tag{8}$$

On the other hand,

$$\lim_{\alpha \to -\infty} h(\alpha) = z_1, \qquad \lim_{\alpha \to \infty} h(\alpha) = z_2. \tag{9}$$

Thus,

$$h(-\infty) = z_1, \qquad h(+\infty) = z_2. \tag{10}$$

Define $g : h(R) \to R$ by

$$g(z) = \|z - y_0, x_2 - y_2, \dots, x_n - y_n\|. \tag{11}$$

It follows that

$$\begin{aligned}g(z_1) &= \|z_1 - y_0, x_2 - y_2, \dots, x_n - y_n\| \\ &= \left(1 + \frac{r}{\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\|}\right) \\ &\quad \times \|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| \ge r, \\ g(z_2) & \\ &= \begin{cases}\left(1 - \dfrac{r}{\|x_1 - y_1, \dots, x_n - y_n\|}\right)\|x_1 - y_1, \dots, x_n - y_n\|, \\ \qquad \text{if } \|x_1 - y_1, \dots, x_n - y_n\| > r \\ \left(\dfrac{r}{\|x_1 - y_1, \dots, x_n - y_n\|} - 1\right)\|x_1 - y_1, \dots, x_n - y_n\|, \\ \qquad \text{if } \|x_1 - y_1, \dots, x_n - y_n\| < r.\end{cases}\end{aligned} \tag{12}$$

Thus, $g(z_2) \le r$.

Obviously, $g(h(\alpha))$ is continuous on R. Using the mean value theorem, there exists $\alpha_0 \in R$ such that $g(h(\alpha_0)) = r$.

Set $\omega_0 = h(\alpha_0)$, $\omega = \omega_0 + x_1$, we have

$$\|\omega_0 - y_0, x_2 - y_2, \ldots, x_n - y_n\| = r. \tag{13}$$

And from $\|h(\alpha), x_2 - y_2, \ldots, x_n - y_n\| = r$, we have

$$\begin{gathered}\|\omega - x_1, x_2 - y_2, \ldots, x_n - y_n\| = r, \\ \begin{aligned}\|\omega - y_1, x_2 - y_2, \ldots, x_n - y_n\| &\\ = \|\omega_0 + x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| &\\ = \|\omega_0 - y_0, x_2 - y_2, \ldots, x_n - y_n\| = r.&\end{aligned}\end{gathered} \tag{14}$$

□

Lemma 9. *Let X and Y be two real linear n-normed spaces whose dimensions are greater than n, and let Y be n-strictly convex normed space. Suppose that $f : X \to Y$ is a surjective mapping satisfying (n-SDOPP) with preserving distance k for any $k \in N$. Then, f preserves distance $1/k$ for any $k \in N$.*

Proof. Firstly, f is injective. Suppose, on the contrary, that there are $x_0, x_1 \in X$, $x_0 \neq x_1$, such that $f(x_0) = f(x_1)$. As $\dim X > n$, it follows that there exist vectors $x_2, \ldots, x_n \in X$ such that $x_1 - x_0, \ldots, x_n - x_0$ are linearly independent. Then, $\|x_1 - x_0, \ldots, x_n - x_0\| \neq 0$.

Set

$$z_2 := x_0 + \frac{x_2 - x_0}{\|x_1 - x_0, \ldots, x_n - x_0\|}. \tag{15}$$

Clearly,

$$\|x_1 - x_0, z_2 - x_0, x_3 - x_0, \ldots, x_n - x_0\| = 1. \tag{16}$$

Then

$$\begin{aligned}&\|f(x_1) - f(x_0), f(z_2) - f(x_0), \\ &\quad f(x_3) - f(x_0), \ldots, f(x_n) - f(x_0)\| = 1.\end{aligned} \tag{17}$$

This implies that $f(x_0) \neq f(x_1)$, which is a contradiction. Therefore, f is a bijective mapping.

Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ and $(k \in N \setminus \{1\})$ satisfying

$$\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| = \frac{1}{k}. \tag{18}$$

By Lemma 8, we can find $w_1 \in X$ with

$$\begin{gathered}\|x_1 - w_1, x_2 - y_2, \ldots, x_n - y_n\| = 1, \\ \|w_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| = 1.\end{gathered} \tag{19}$$

Set

$$u_1 = w_1 + k(y_1 - w_1), \qquad v_1 = w_1 + k(x_1 - w_1). \tag{20}$$

Clearly, we have

$$\begin{aligned}&\|x_1 - v_1, x_2 - y_2, \ldots, x_n - y_n\| \\ &\quad = \|(k-1)(x_1 - w_1), x_2 - y_2, \ldots, x_n - y_n\| \\ &\quad = k - 1 \\ &\qquad \|w_1 - v_1, x_2 - y_2, \ldots, x_n - y_n\| = k.\end{aligned} \tag{21}$$

It follows from the hypothesis of f preserving any integer k; then,

$$\begin{gathered}\|f(x_1) - f(w_1), f(x_2 - y_2), \ldots, f(x_n) - f(y_n)\| = 1, \\ \|f(x_1) - f(v_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| = k - 1, \\ \|f(w_1) - f(v_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| = k.\end{gathered} \tag{22}$$

Clearly, we have

$$\begin{aligned}&\|f(w_1) - f(v_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| \\ &\quad = \|f(x_1) - f(v_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| \\ &\qquad + \|f(x_1) - f(w_1), f(x_2 - y_2), \ldots, f(x_n) - f(y_n)\|.\end{aligned} \tag{23}$$

We conclude that

$$\begin{aligned}&f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n) \\ &\quad \notin \operatorname{span}\{f(x_1) - f(v_1), f(x_1) - f(w_1)\}.\end{aligned} \tag{24}$$

Otherwise, if for some $f(x_i) - f(y_i)$, we have $\mu_i, \lambda_i \in R$ with $\mu_i \neq 0$ or $\lambda_i \neq 0$ such that

$$f(x_i) - f(y_i) = \mu_i (f(x_1) - f(v_1)) + \lambda_i (f(x_1) - f(w_1)). \tag{25}$$

Suppose that $\lambda_i \neq 0$. Then,

$$\begin{aligned}k - 1 = {} & \|f(x_1) - f(v_1), \ldots, f(x_i) \\ & -f(y_i), \ldots, f(x_n) - f(y_n)\| \\ = {} & |\lambda_i| \, \|(x_1) - f(v_1), \ldots, f(x_1) \\ & -f(w_1), \ldots, f(x_n) - f(y_n)\|.\end{aligned} \tag{26}$$

Assume that

$$\begin{aligned}&\|f(x_1) - f(v_1), \ldots, f(x_1) - f(w_1), \ldots, f(x_n) - f(y_n)\| \\ &\quad \neq 0.\end{aligned} \tag{27}$$

Set

$$\begin{aligned}s_j = {} & f(x_j) \\ & + (f(x_j) - f(y_j)) \\ & \times (\|f(x_1) - f(v_1), \ldots, f(x_1) - f(w_1), \ldots, \\ & \quad f(x_n) - f(y_n)\|)^{-1}, \quad (j \geq 2).\end{aligned} \tag{28}$$

Then, for $j \neq i$,

$$\begin{aligned}&\|f(x_1) - f(v_1), \ldots, s_j - f(x_j), \ldots, f(x_1) \\ &\quad -f(w_1), \ldots, f(x_n) - f(y_n)\| = 1.\end{aligned} \tag{29}$$

Since f is bijective and preserves n-SDOPP on both directions. Then, there exists $t_j \in X$ with $f(t_j) = s_j$ which satisfies that

$$\left\| x_1 - v_1, t_j - x_j, \dots, x_1 - w_1, \dots, x_n - y_n \right\| = 1. \quad (30)$$

However, by (20), $x_1 - v_1 = (1-k)(x_1 - w_1)$, and thus $x_1 - v_1$, $x_1 - w_1$ are linear dependent. Then,

$$\left\| x_1 - v_1, t_j - x_j, \dots, x_1 - w_1, \dots, x_n - y_n \right\| = 0. \quad (31)$$

This contradiction implies that

$$\begin{aligned} &\| f(x_1) - f(v_1), \dots, f(x_1) - f(w_1), \dots, f(x_n) - f(y_n) \| \\ &\quad = 0. \end{aligned} \quad (32)$$

This also contradicts with (26). Since Y is n-strictly convex, then there exists $\alpha > 0$ such that

$$f(x_1) - f(v_1) = \alpha (f(x_1) - f(w_1)). \quad (33)$$

Then,

$$f(x_1) = \frac{1}{1+\alpha} f(v_1) + \frac{\alpha}{1+\alpha} f(w_1). \quad (34)$$

Since

$$\begin{aligned} &\| f(x_1) - f(v_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n) \| = k - 1, \\ &\quad \| f(x_1) - f(w_1), f(x_2 - y_2), \dots, f(x_n) - f(y_n) \| = 1, \end{aligned} \quad (35)$$

then $\alpha = k - 1$. Thus,

$$f(x_1) = \frac{1}{k} f(v_1) + \frac{k-1}{k} f(w_1), \quad (36)$$

Similarly,

$$f(y_1) = \frac{1}{k} f(u_1) + \frac{k-1}{k} f(w_1). \quad (37)$$

Hence,

$$\| f(x_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n) \| = \frac{1}{k}. \quad (38)$$

□

Lemma 10. *Let X and Y be real n-normed spaces such that $\dim X \geq n$. If a mapping $f : X \to Y$ preserves the distance $1/k$ for each $k \in \mathbb{N}$, then f preserves the distance zero.*

Proof. Choose $x_1, \dots, x_n, y_1, \dots, y_n \in X$ such that $\|x_1 - y_1, \dots, x_n - y_n\| = 0$; that is, $x_1 - y_1, \dots, x_n - y_n$ are linearly dependent. Assume that $\{x_{m+1} - y_{m+1}, \dots, x_n - y_n\}$ is a maximum linearly independent group of $\{x_1 - y_1, \dots, x_n - y_n\}$ $(m < n)$. As $\dim X \geq n$, we can find a finite sequence of vectors $\omega_1, \omega_2, \dots, \omega_m \in X$ such that $x_1 - \omega_1, \dots, x_m - \omega_m, x_{m+1} - y_{m+1}, \dots, x_n - y_n$ are linearly independent. Hence, it holds that

$$\| x_1 - \omega_1, \dots, x_m - \omega_m, x_{m+1} - y_{m+1}, \dots, x_n - y_n \| \neq 0. \quad (39)$$

We will prove that

$$\| f(x_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n) \| \leq \frac{1}{k}, \quad (40)$$

for every $k \in \mathbb{N}$. Let $m = 1$. We can find a vector $\omega_1 \in X$ such that $x_1 - \omega_1, x_2 - y_2, \dots, x_n - y_n$ are linearly independent. Set

$$v_1 = x_1 + \frac{x_1 - \omega_1}{2k \| x_1 - \omega_1, x_2 - y_2, \dots, x_n - y_n \|}, \quad (41)$$

for arbitrarily fixed $k \in \mathbb{N}$. Then,

$$\begin{aligned} &\| x_1 - v_1, x_2 - y_2, \dots, x_n - y_n \| = \frac{1}{2k}, \\ &\| v_1 - x_1, x_2 - y_2, \dots, x_n - y_n \| \\ &\quad - \| x_1 - y_1, x_2 - y_2, \dots, x_n - y_n \| \\ &\quad \leq \| (v_1 - x_1) + (x_1 - y_1), x_2 - y_2, \dots, x_n - y_n \| \\ &\quad \leq \| v_1 - x_1, x_2 - y_2, \dots, x_n - y_n \| \\ &\quad\quad + \| x_1 - y_1, x_2 - y_2, \dots, x_n - y_n \|. \end{aligned} \quad (42)$$

Since $\|x_1 - y_1, x_2 - y_2, \dots, x_n - y_n\| = 0$, we get

$$\| v_1 - y_1, x_2 - y_2, \dots, x_n - y_n \| = \frac{1}{2k}. \quad (43)$$

Since f preserves the distance $1/(2k)$, we see that

$$\begin{aligned} &\| f(x_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n) \| \\ &\quad \leq \| f(x_1) - f(v_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n) \| \\ &\quad\quad + \| f(v_1) - f(y_1), f(x_2) - f(y_2), \dots, f(x_n) - f(y_n) \| \\ &\quad = \frac{1}{2k} \cdot 2 = \frac{1}{k}. \end{aligned} \quad (44)$$

For $m \geq 2$, we set

$$\begin{aligned} v_1 &= x_1 + (x_1 - \omega_1) \\ &\quad \times (2^m k \| x_1 - \omega_1, \dots, x_m - \omega_m, x_{m+1} \\ &\quad - y_{m+1}, \dots, x_n - y_n \|)^{-1}, \end{aligned} \quad (45)$$

$$v_i = 2x_i - \omega_i, \quad (46)$$

for any $i \in \{2, 3, \dots, m\}$. Then, we have

$$x_i - v_i = \omega_i - x_i, \qquad v_i - y_i = (x_i - \omega_i) + (x_i - y_i), \quad (47)$$

for each $i \in \{2, 3, \dots, m\}$. Since $x_i - y_i, x_{m+1} - y_{m+1}, \dots, x_n - y_n$ are linearly dependent, we get

$$\| \dots, x_i - y_i, \dots, x_{m+1} - y_{m+1}, \dots, x_n - y_n \| = 0, \quad (48)$$

and hence,

$$\begin{aligned}
&\|\ldots, x_i-\omega_i, \ldots, x_{m+1}-y_{m+1}, \ldots, x_n-y_n\| \\
&\quad -\|\ldots, x_i-y_i, \ldots, x_{m+1}-y_{m+1}, \ldots, x_n-y_n\| \\
&\quad \leq \|\ldots, (x_i-\omega_i)+(x_i-y_i), \ldots, x_{m+1} \\
&\qquad -y_{m+1}, \ldots, x_n-y_n\| \\
&\quad \leq \|\ldots, x_i-\omega_i, \ldots, x_{m+1}-y_{m+1}, \ldots, x_n-y_n\| \\
&\qquad +\|\ldots, x_i-y_i, \ldots, x_{m+1}-y_{m+1}, \ldots, x_n-y_n\|,
\end{aligned} \tag{49}$$

which together with (48) implies that

$$\begin{aligned}
&\|\ldots, \nu_i-y_i, \ldots, x_{m+1}-y_{m+1}, \ldots, x_n-y_n\| \\
&\quad =\|\ldots, x_i-\omega_i, \ldots, x_{m+1}-y_{m+1}, \ldots, x_n-y_n\|,
\end{aligned} \tag{50}$$

for all $i \in \{2,3,\ldots,m\}$. By a similar argument, we further obtain that

$$\begin{aligned}
&\|\nu_1-y_1, \ldots, x_{m+1}-y_{m+1}, \ldots, x_n-y_n\| \\
&\quad =\|\nu_1-x_1, \ldots, x_{m+1}-y_{m+1}, \ldots, x_n-y_n\|.
\end{aligned} \tag{51}$$

In view of (45), (50), and (51), we conclude that

$$\begin{aligned}
&\|\nu_1-y_1, \mu_2, \ldots, \mu_m, x_{m+1}-y_{m+1}, \ldots, x_n-y_n\| \\
&\quad =\|x_1-\nu_1, x_2-\omega_2, \ldots, x_m-\omega_m, x_{m+1} \\
&\qquad -y_{m+1}, \ldots, x_n-y_n\| \\
&\quad =\frac{1}{2^m k},
\end{aligned} \tag{52}$$

where μ_i denotes either ν_i-y_i or $x_i-\nu_i$ for $i \in \{2,3,\ldots,m\}$.

Since f preserves the distance $1/(2^m k)$ for any $k \in \mathbb{N}$, it follows from (52) that

$$\begin{aligned}
&\|f(x_1)-f(y_1), f(x_2)-f(y_2), f(x_3) \\
&\quad -f(y_3), \ldots, f(x_n)-f(y_n)\| \\
&\quad \leq \|f(x_1)-f(\nu_1), \\
&\qquad f(x_2)-f(\nu_2), \ldots, f(x_{m-1})-f(\nu_{m-1}), \\
&\qquad f(x_m)-f(\nu_m), \\
&\qquad f(x_{m+1})-f(y_{m+1}), \ldots, f(x_n)-f(y_n)\| \\
&\qquad +\|f(x_1)-f(\nu_1), \\
&\qquad f(x_2)-f(\nu_2), \ldots, f(x_{m-1})-f(\nu_{m-1}), \\
&\qquad f(\nu_m)-f(y_m), \\
&\qquad f(x_{m+1})-f(y_{m+1}), \ldots, f(x_n)-f(y_n)\| \\
&\qquad +\|f(x_1)-f(\nu_1), \\
&\qquad f(x_2)-f(\nu_2), \ldots, f(\nu_{m-1})-f(y_{m-1}), \\
&\qquad f(x_m)-f(\nu_m), \\
&\qquad f(x_{m+1})-f(y_{m+1}), \ldots, f(x_n)-f(y_n)\| \\
&\qquad +\|f(x_1)-f(\nu_1), \\
&\qquad f(x_2)-f(\nu_2), \ldots, f(\nu_{m-1})-f(y_{m-1}), \\
&\qquad f(\nu_m)-f(y_m), \\
&\qquad f(x_{m+1})-f(y_{m+1}), \ldots, f(x_n)-f(y_n)\| \\
&\qquad +\cdots+ \\
&\qquad +\|f(\nu_1)-f(y_1), \\
&\qquad f(\nu_2)-f(y_2), \ldots, f(\nu_{m-1})-f(y_{m-1}), \\
&\qquad f(\nu_m)-f(y_m), \\
&\qquad f(x_{m+1})-f(y_{m+1}), \ldots, f(x_n)-f(y_n)\| \\
&\quad =\frac{1}{2^m k}\cdot 2^m=\frac{1}{k},
\end{aligned} \tag{53}$$

where k is an arbitrary positive integer. Hence, we conclude that

$$\|f(x_1)-f(y_1), f(x_2)-f(y_2), \ldots, f(x_n)-f(y_n)\|=0, \tag{54}$$

which implies that f preserves the distance zero. □

Remark 11. In ([9], Lemma 2.2 to be published), we give the same method under the condition of f preserving 2-colinear.

Theorem 12. *Let X and Y be real n-normed spaces such that $\dim X > n$ and Y is n-strictly convex. If a surjective mapping $f: X \to Y$ has the n-SDOPP and preserves the distance k for any $k \in \mathbb{N}$, then f is an affine n-isometry.*

Proof. Assume that $\|x_1-y_1, x_2-y_2, \ldots, x_n-y_n\| > 0$ for $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$.

Take positive integers k, m such that

$$\frac{m-1}{k} \leq \|x_1-y_1, x_2-y_2, \ldots, x_n-y_n\| \leq \frac{m}{k}. \tag{55}$$

Set

$$p_i = x_1 + \frac{i}{k}\cdot\frac{y_1-x_1}{\|x_1-y_1, x_2-y_2, \ldots, x_n-y_n\|}, \tag{56}$$

for $i = 0, 1, \ldots, m-2$, and

$$p_m = y_1. \tag{57}$$

Clearly, for $i = 1, \ldots, m-2$,

$$\begin{aligned}
&\left\|p_i - p_{i-1}, x_2 - y_2, \ldots, x_n - y_n\right\| = \frac{1}{k},\\
&0 < \left\|p_m - p_{m-2}, x_2 - y_2, \ldots, x_n - y_n\right\|\\
&= \left\|y_1 - x_1 - \frac{m-2}{k} \cdot \frac{y_1 - x_1}{\left\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\right\|},\right.\\
&\qquad \left. x_2 - y_2, \ldots, x_n - y_n\right\|\\
&= \left(1 - \frac{m-2}{k} \cdot \frac{1}{\left\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\right\|}\right)\\
&\quad \cdot \left\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\right\|\\
&= \left\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\right\| - \frac{m-2}{k}\\
&\le \frac{m}{k} - \frac{m-2}{k} = \frac{2}{k}.
\end{aligned} \tag{58}$$

According to Lemma 8, there exists $p_{m-1} \in X$ such that

$$\begin{aligned}
\left\|p_{m-1} - p_{m-2}, x_2 - y_2, \ldots, x_n - y_n\right\| &= \frac{1}{k},\\
\left\|p_{m-1} - y_1, x_2 - y_2, \ldots, x_n - y_n\right\| &= \frac{1}{k}.
\end{aligned} \tag{59}$$

It follows from Lemma 9 that we have

$$\left\|f(p_i) - f(p_{i-1}), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\right\| = \frac{1}{k}, \tag{60}$$

for $i = 0, 1, 2, \ldots, m$.

On the other hand,

$$\begin{aligned}
&\left\|f(x_1) - f(y_1), f(x_2) - f(y_2) \ldots, f(x_n) - f(y_n)\right\|\\
&\quad \le \sum_{i=1}^{m} \left\|f(p_i) - f(p_{i-1}), f(x_2) - f(y_2), \ldots, f(x_n)\right.\\
&\qquad \left. - f(y_n)\right\| = \frac{m}{k}.
\end{aligned} \tag{61}$$

Hence

$$\begin{aligned}
&\left\|f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\right\|\\
&\quad \le \left\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\right\|.
\end{aligned} \tag{62}$$

Suppose that

$$\begin{aligned}
&\left\|f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\right\|\\
&\quad < \left\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\right\|.
\end{aligned} \tag{63}$$

For any $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in X$, with

$$\left\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\right\| \ne 0, \tag{64}$$

find a positive integer k_0 satisfying $\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| < k_0$.

Set $z_1 = x_1 + k_0(y_1 - x_1)/\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|$. Clearly, $\|z_1 - x_1, x_2 - y_2, \ldots, x_n - y_n\| = k_0$, and $\|z_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\| = k_0 - \|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\|$.

It follows that $\|f(z_1) - f(x_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\| = k_0$ and

$$\begin{aligned}
k_0 &= \left\|f(z_1) - f(x_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\right\|\\
&\le \left\|f(z_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\right\|\\
&\quad + \left\|f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\right\|\\
&< k_0 - \left\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\right\|\\
&\quad + \left\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\right\| = k_0.
\end{aligned} \tag{65}$$

Then (63) is not valid. Hence,

$$\begin{aligned}
&\left\|f(x_1) - f(y_1), f(x_2) - f(y_2), \ldots, f(x_n) - f(y_n)\right\|\\
&\quad = \left\|x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n\right\|.
\end{aligned} \tag{66}$$

□

Corollary 13. *Let X and Y be two real linear n-normed spaces. Suppose that mapping $f : X \to Y$ preserves any positive integer k-distance and Lipschitz condition. Then, f is an n-isometry.*

Acknowledgments

This work is supported by the Fundamental Research Funds for the Central Universities in China and Education Department of Liaoning province in china.

References

[1] A. D. Aleksandrov, "Mappings of families of sets," *Soviet Mathematics Doklady*, vol. 11, pp. 116–120, 1970.

[2] S. Mazur and S. Ulam, "Sur les transformationes isométriques d'espaces vectoriels normés," *Comptes Rendus de L'académie des Sciences*, vol. 194, pp. 946–948, 1932.

[3] T. M. Rassias and P. Šemrl, "On the Mazur-Ulam theorem and the Aleksandrov problem for unit distance preserving mappings," *Proceedings of the American Mathematical Society*, vol. 118, no. 3, pp. 919–925, 1993.

[4] H.-Y. Chu, C.-G. Park, and W.-G. Park, "The Aleksandrov problem in linear 2-normed spaces," *Journal of Mathematical Analysis and Applications*, vol. 289, no. 2, pp. 666–672, 2004.

[5] H.-Y. Chu, S. K. Choi, and D. S. Kang, "Mappings of conservative distances in linear n-normed spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 3, pp. 1168–1174, 2009.

[6] J. Gao, "On the Alexandrov problem of distance preserving mapping," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 2, pp. 583–590, 2009.

[7] H.-Y. Chu, K. Lee, and C.-G. Park, "On the Aleksandrov problem in linear n-normed spaces," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 59, no. 7, pp. 1001–1011, 2004.

[8] C.-G. Park and T. M. Rassias, "Isometries on linear n-normed spaces," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 5, pp. 1–17, 2006.

[9] Y. Ma, "The Aleksandrov problem on linear n-normed spaces," *Acta Mathematica Scientia*. In press.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 163814, 7 pages
http://dx.doi.org/10.1155/2013/163814

Research Article

Construction of Frames for Shift-Invariant Spaces

Stevan Pilipović[1] and Suzana Simić[2]

[1] *Department of Mathematics and Informatics, Faculty of Science, University of Novi Sad, Trg Dositeja Obradovica 4, 21000 Novi Sad, Serbia*
[2] *Department of Mathematics and Informatics, Faculty of Science, University of Kragujevac, Radoja Domanovića 12, 34000 Kragujevac, Serbia*

Correspondence should be addressed to Suzana Simić; suzanasimic@kg.ac.rs

Received 9 May 2013; Accepted 25 July 2013

Academic Editor: Satit Saejung

We construct a sequence $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z},\ i = 1,\dots,r\}$ which constitutes a p-frame for the weighted shift-invariant space $V_\mu^p(\Phi) = \{\sum_{i=1}^r \sum_{j\in\mathbb{Z}} c_i(j)\phi_i(\cdot - j) \mid \{c_i(j)\}_{j\in\mathbb{Z}} \in \ell_\mu^p,\ i = 1,\dots,r\}$, $p \in [1,\infty]$, and generates a closed shift-invariant subspace of $L_\mu^p(\mathbb{R})$. The first construction is obtained by choosing functions ϕ_i, $i = 1,\dots,r$, with compactly supported Fourier transforms $\widehat{\phi}_i$, $i = 1,\dots,r$. The second construction, with compactly supported ϕ_i, $i = 1,\dots,r$, gives the Riesz basis.

1. Introduction and Preliminaries

The shift-invariant spaces $V_\mu^p(\Phi)$, $p \in [1,\infty]$, quoted in the abstract, are used in the wavelet analysis, approximation theory, sampling theory, and so forth. They have been extensively studied by many authors [1–18]. The aim of this paper is to construct $V_\mu^p(\Phi)$, $p \in [1,\infty]$, spaces with specially chosen functions ϕ_i, $i = 1,\dots,r$, which generate its p-frame. These results extend and correct the construction obtained in [19]. For the first construction, we take functions ϕ_i, $i = 1,\dots,r$, so that the Fourier transforms are compactly supported smooth functions. Also, we derive conditions for the collection $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z},\ i = 1,\dots,r\}$ to form a Riesz basis for $V_\mu^p(\Phi)$. We note that the properties of the constructed frame guarantee the feasibility of a stable and continuous reconstruction algorithm in $V_\mu^p(\Phi)$ [20]. We generalize these results for a shift-invariant subspace of $L_\mu^p(\mathbb{R}^d)$. The second construction is obtained by choosing compactly supported functions ϕ_i, $i = 1,\dots,r$. In this way, we obtain the Riesz basis.

This paper is organized as follows. In Section 2 we quote some basic properties of certain subspaces of the weighted L^p and ℓ^p spaces. In Section 3 we derive conditions for functions of the form $\widehat{\phi}_i(\cdot) = \theta(\cdot + k_i\pi)$, $k_i \in \mathbb{Z}$, $i = 1,2,\dots,r$, $r \in \mathbb{N}$, to form a Riesz basis for $V_\mu^p(\Phi)$. We also show that using functions of the form $\widehat{\phi}_i(\cdot) = \theta(\cdot + i\pi)$, $i = 1,\dots,r$, where θ is compactly supported smooth function whose length of support is less than or equal to 2π, we cannot construct a p-frame for the shift-invariant space $V_\mu^p(\Phi)$. In Section 4 we construct a sequence $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z},\ i = 0,\dots,r\}$, where $r \in 2\mathbb{N}$ or $r \in 3\mathbb{N}$, which constitutes a p-frame for the weighted shift-invariant space $V_\mu^p(\Phi)$. Our construction shows that the sampling and reconstruction problem in the shift-invariant spaces is robust in the sense of [1]. In Section 5 we construct p-Riesz basis by using compactly supported functions ϕ_i, $i = 1,\dots,r$.

2. Basic Spaces

Let a function ω be nonnegative, continuous, symmetric, and submultiplicative; that is, $\omega(x + y) \le \omega(x)\omega(y)$, $x, y \in \mathbb{R}^d$; let a function μ be ω-moderate; that is, $\mu(x + y) \le C\omega(x)\mu(y)$, $x, y \in \mathbb{R}^d$. Functions μ and ω are called weights. We consider the weighted function spaces L_μ^p and the weighted sequence spaces $\ell_\mu^p(\mathbb{Z}^d)$ with ω-moderate weights μ (see [19]). Let $p \in [1,\infty)$. Then (with obvious modification for $p=\infty$)

$$\mathscr{L}_\omega^p = \left\{ f \mid \|f\|_{\mathscr{L}_\omega^p} = \left(\int_{[0,1]^d} \left(\sum_{j\in\mathbb{Z}^d} \left| f(x+j) \right| \omega(x+j) \right)^p dx \right)^{1/p} < +\infty \right\},$$

$$W_\omega^p := \left\{ f \mid \|f\|_{W_\omega^p} = \left(\sum_{j\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} \left| f(x+j) \right|^p \omega(j)^p \right)^{1/p} < +\infty \right\}. \tag{1}$$

In what follows, we use the notation $\Phi = (\phi_1,\dots,\phi_r)^T$. Define $\|\Phi\|_{\mathscr{H}} = \sum_{i=1}^r \|\phi_i\|_{\mathscr{H}}$, where $\mathscr{H} = L_\omega^p$, $\mathscr{L}_\omega^p$ or W_ω^p, $p \in [1,\infty]$. With $\mathscr{F}\phi = \widehat{\phi}$ we denote the Fourier transform of the function ϕ; that is, $\widehat{\phi}(\xi) = \int_{\mathbb{R}^d} \phi(x) e^{-i\pi x\cdot\xi}\,\mathrm{d}x$, $\xi \in \mathbb{R}^d$.

Let $c = \{c_i\}_{i\in\mathbb{N}} \in \ell_\mu^p$ and $f, g \in L_\omega^p$, $p \in [1,\infty]$. We define, as in [1], the semiconvolution $f *' c$ as $(f *' c)(x) = \sum_{j\in\mathbb{Z}^d} c_j f(x-j)$, $x \in \mathbb{R}^d$, and $\langle f, g\rangle = \int_{\mathbb{R}^d} f(x)\overline{g(x)}\,\mathrm{d}x$.

The concept of a p-frame is introduced in [1].

It is said that a collection $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d,\ i = 1,\dots,r\}$ is a p-frame for $V_\mu^p(\Phi)$ if there exists a positive constant C (dependent upon Φ, p, and ω) such that

$$\begin{aligned} C^{-1}\|f\|_{L_\mu^p} &\le \sum_{i=1}^r \left\| \left\{ \int_{\mathbb{R}^d} f(x)\overline{\phi_i(x-j)}\,\mathrm{d}x \right\}_{j\in\mathbb{Z}^d} \right\|_{\ell_\mu^p} \\ &\le C\|f\|_{L_\mu^p}, \quad f \in V_\mu^p(\Phi). \end{aligned} \tag{2}$$

Recall [21] that the shift-invariant spaces are defined by

$$V_\mu^p(\Phi) := \left\{ f \in L_\mu^p \mid f(\cdot) = \sum_{i=1}^r \sum_{j\in\mathbb{Z}^d} c_j^i\, \phi_i(\cdot - j),\ \left\{c_j^i\right\}_{j\in\mathbb{Z}^d} \in \ell_\mu^p,\ i = 1,\dots,r \right\}. \tag{3}$$

Remark 1 (see [22]). Let $\Phi \in W_\omega^1$ and let μ be ω-moderate. Then $V_\mu^p(\Phi)$ is a subspace (not necessarily closed) of L_μ^p and W_μ^p for any $p \in [1,\infty]$. Clearly (2) implies that ℓ_μ^p and $V_\mu^p(\Phi)$ are isomorphic Banach spaces.

Let $\Phi = (\phi_1,\dots,\phi_r)^T$. Let

$$\left[\widehat{\Phi},\widehat{\Phi}\right](\xi) = \left[\sum_{k\in\mathbb{Z}^d} \widehat{\phi}_i(\xi + 2k\pi)\,\overline{\widehat{\phi}_j(\xi + 2k\pi)} \right]_{1\le i\le r, 1\le j\le r}, \tag{4}$$

where we assume that $\widehat{\phi}_i(\xi)\overline{\widehat{\phi}_j(\xi)}$ is integrable for any $1 \le i, j \le r$. Let $A = [a(j)]_{j\in\mathbb{Z}^d}$ be an $r\times\infty$ matrix and $A\overline{A^T} = [\sum_{j\in\mathbb{Z}^d} a_i(j)\overline{a_{i'}(j)}]_{1\le i,i'\le r}$. Then $\operatorname{rank} A = \operatorname{rank} A\overline{A^T}$.

We recall results from [1, 19] which are needed in the sequel.

Lemma 2 (see [1]). *The following statements are equivalent.*

(1) $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}^d}$ *is a constant function on* $\mathbb{R}^d$.

(2) $\operatorname{rank}[\widehat{\Phi},\widehat{\Phi}](\xi)$ *is a constant function on* $\mathbb{R}^d$.

(3) *There exists a positive constant C independent of* ξ *such that*

$$\begin{aligned} C^{-1}\left[\widehat{\Phi},\widehat{\Phi}\right](\xi) &\le \left[\widehat{\Phi},\widehat{\Phi}\right](\xi)\,\overline{\left[\widehat{\Phi},\widehat{\Phi}\right](\xi)^T} \\ &\le C\left[\widehat{\Phi},\widehat{\Phi}\right](\xi), \quad \xi \in [-\pi,\pi]^d. \end{aligned} \tag{5}$$

The next theorem [19] derives necessary and sufficient conditions for an indexed family $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d,\ i = 1,\dots,r\}$ to constitute a p-frame for $V_\mu^p(\Phi)$, which is equivalent with the closedness of this space in L_μ^p. Thus, it is shown that under appropriate conditions on the frame vectors, there is an equivalence between the concept of p-frames, Banach frames, and the closedness of the space they generate.

Theorem 3 (see [19]). *Let* $\Phi = (\phi_1,\dots,\phi_r)^T \in (W_\omega^1)^r$, $p_0 \in [1,\infty]$, *and let* μ *be* ω*-moderate. The following statements are equivalent.*

(i) $V_\mu^{p_0}(\Phi)$ *is closed in* $L_\mu^{p_0}$.

(ii) $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d,\ i = 1,\dots,r\}$ *is a* p_0*-frame for* $V_\mu^{p_0}(\Phi)$.

(iii) *There exists a positive constant C such that*

$$\begin{aligned} C^{-1}\left[\widehat{\Phi},\widehat{\Phi}\right](\xi) &\le \left[\widehat{\Phi},\widehat{\Phi}\right](\xi)\,\overline{\left[\widehat{\Phi},\widehat{\Phi}\right](\xi)^T} \\ &\le C\left[\widehat{\Phi},\widehat{\Phi}\right](\xi), \quad \xi \in [-\pi,\pi]^d. \end{aligned} \tag{6}$$

(iv) *There exist positive constants* C_1 *and* C_2 *(depending on* Φ *and* ω*) such that*

$$\begin{aligned} C_1\|f\|_{L_\mu^{p_0}} &\le \inf_{f=\sum_{i=1}^r \phi_i *' c^i} \sum_{i=1}^r \left\| \left\{c_j^i\right\}_{j\in\mathbb{Z}^d} \right\|_{\ell_\mu^{p_0}} \\ &\le C_2\|f\|_{L_\mu^{p_0}}, \quad f \in V_\mu^{p_0}(\Phi). \end{aligned} \tag{7}$$

(v) *There exists* $\Psi = (\psi_1, \ldots, \psi_r)^T \in (W^1_\omega)^r$, *such that*

$$f = \sum_{i=1}^{r} \sum_{j \in \mathbb{Z}^d} \langle f, \psi_i(\cdot - j)\rangle \phi_i(\cdot - j) = \sum_{i=1}^{r} \sum_{j \in \mathbb{Z}^d} \langle f, \phi_i(\cdot - j)\rangle \psi_i(\cdot - j), \quad f \in V^{p_0}_\mu(\Phi). \tag{8}$$

Corollary 4 (see [19]). *Let* $\Phi = (\phi_1, \ldots, \phi_r)^T \in (W^1_\omega)^r$, $p_0 \in [1, \infty]$, *and let* μ *be* ω*-moderate.*

(i) *If* $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d,\ i = 1, \ldots, r\}$ *is a* p_0*-frame for* $V^{p_0}_\mu(\Phi)$, *then the collection* $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z}^d,\ i = 1, \ldots, r\}$ *is a* p*-frame for* $V^p_\mu(\Phi)$ *for any* $p \in [1, \infty]$.

(ii) *If* $V^{p_0}_\mu(\Phi)$ *is closed in* $L^{p_0}_\mu$ *and* $W^{p_0}_\mu$, *then* $V^p_\mu(\Phi)$ *is closed in* L^p_μ *and* W^p_μ *for any* $p \in [1, \infty]$.

(iii) *If* (7) *holds for* p_0, *then it holds for any* $p \in [1, \infty]$.

3. Construction of Frames Using a Band-Limited Function

Considering the length of the support of a function θ, we have different cases for the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$.

First, we consider the next claim.

Let $\theta \in C^\infty_0(\mathbb{R})$ be a nonnegative function such that $\theta(x) > 0$, $x \in (-\pi, \pi)$, and $\operatorname{supp}\theta \subseteq [-\pi, \pi]$. Moreover, let

$$\widehat{\phi}_k(\xi) = \theta(\xi + k\pi), \quad k \in \mathbb{Z}, \tag{9}$$

and $\Phi = (\phi_i, \phi_{i+1}, \ldots, \phi_{i+r})^T$, $i \in \mathbb{Z}$, $r \in \mathbb{N}$.

Then the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ is not a constant function on $\mathbb{R}$ and it depends on $\xi \in \mathbb{R}$.

As a matter of fact, by the Paley-Wiener theorem, $\phi_i \in \mathcal{S}(\mathbb{R}) \subset W^1_\mu(\mathbb{R})$, $i \in \mathbb{Z}$. For any $i \in \mathbb{Z}$, matrix $[\widehat{\phi}_i, \widehat{\phi}_i](\xi) = \sum_{j\in\mathbb{Z}} |\theta(\xi + i\pi + 2j\pi)|^2$, $\xi \in \mathbb{R}$, has the rank 0 or 1, depending on ξ. Moreover, we have $[\widehat{\phi}_{2i}, \widehat{\phi}_{2i}](\pi) = 0$ and $[\widehat{\phi}_{2i}, \widehat{\phi}_{2i}](0) > 0$. Because of that, the rank of the matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ is not a constant function on $\mathbb{R}$ and it depends on $\xi \in \mathbb{R}$.

Theorem 5. *Let* $\theta \in C^\infty_0(\mathbb{R})$ *be a non-negative function such that* $\theta(x) > 0$, $x \in (-\pi - \varepsilon, \pi + \varepsilon)$, *and* $\operatorname{supp}\theta = [-\pi - \varepsilon, \pi + \varepsilon]$, *where* $0 < \varepsilon < 1/4$. *Moreover, let*

$$\widehat{\phi}_i(\xi) = \theta(\xi + k_i\pi), \quad k_i \in \mathbb{Z},\ i = 1, 2, \ldots, r, \tag{10}$$

and $\Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T$.

(1) *If* $|k_2 - k_1| = 2$ *and* $|k_i - k_j| \geq 2$ *for different* $i, j \leq r$, *then*

$$\operatorname{rank}\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = r, \quad \xi \in \mathbb{R}. \tag{11}$$

(2) *If* $|k_2 - k_1| = 2$ *and, at least for* k_{i_1} *and* k_{i_2}, *it holds that* $|k_{i_1} - k_{i_2}| = 1$, *where* $1 \leq i_1, i_2 \leq r$, *then* $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ *is not a constant function on* $\mathbb{R}$.

Proof. By the Paley-Wiener theorem, $\phi_i \in \mathcal{S}(\mathbb{R}) \subset W^1_\mu(\mathbb{R})$, $i = 1, \ldots, r$. All possible cases are described in the following lemmas.

Lemma 6. *Let* $\Phi = (\phi_{k_1}, \phi_{k_2})^T$, $k_2 - k_1 = 2$, $k_1, k_2 \in \mathbb{Z}$. *The rank of matrix* $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ *is a constant function on* $\mathbb{R}$ *and equals 2.*

Proof. We have the next two cases.

(1°) If $\xi \in (-\pi - \varepsilon - k_1\pi + 2\ell\pi, -\pi + \varepsilon - k_1\pi + 2\ell\pi)$, $\ell \in \mathbb{Z}$, for matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ we obtain $2 \times \infty$ matrix

$$\begin{bmatrix} \cdots & 0 & 0 & a^1 & b^1 & 0 & \cdots \\ \cdots & 0 & a^2 & b^2 & 0 & 0 & \cdots \end{bmatrix}, \tag{12}$$

for some $a^i, b^i > 0$, $i = 1, 2$. It is obvious that $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}} = 2$, $\xi \in (-\pi - \varepsilon - k_1\pi + 2\ell\pi, -\pi + \varepsilon - k_1\pi + 2\ell\pi)$, $\ell \in \mathbb{Z}$.

(2°) For $\xi \in [-\pi + \varepsilon - k_1\pi + 2\ell\pi, \pi - \varepsilon - k_1\pi + 2\ell\pi]$, $\ell \in \mathbb{Z}$, there are only two nonzero values a^1 and a^2 which are in different columns of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$. Since

$$\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & a^2 & 0 & 0 & \cdots \end{bmatrix}_{2\times\infty}, \tag{13}$$

it has the rank 2 for all $\xi \in [-\pi + \varepsilon - k_1\pi + 2\ell\pi, \pi - \varepsilon - k_1\pi + 2\ell\pi]$, $\ell \in \mathbb{Z}$.

We conclude that the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$, $\Phi = (\phi_{k_1}, \phi_{k_2})^T$, $k_2 - k_1 = 2$, $k_1, k_2 \in \mathbb{Z}$, is a constant function on $\mathbb{R}$ and equals 2. □

Lemma 7. *The rank of matrix* $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ *is not a constant function on* $\mathbb{R}$ *if* $\Phi = (\phi_{k_1}, \phi_{k_2})^T$, $k_2 - k_1 = 1$, $k_1, k_2 \in \mathbb{Z}$.

Proof. We have four different cases for matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$. Suppose, without losing generality, that $k_1 \in 2\mathbb{Z}$.

(1°) If $\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, then

$$\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & a^1 & b^1 & 0 & \cdots \\ \cdots & 0 & a^2 & 0 & 0 & \cdots \end{bmatrix}, \quad a^1, a^2, b^1 > 0, \tag{14}$$

and $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}} = 2$, for all $\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$.

(2°) For $\xi \in [-\pi + \varepsilon + 2\ell\pi, -\varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, nonzero values a^1 and a^2 are in the same column of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$. For any choice of a 2×2 matrix, we get that the determinant equals 0. So we obtain

$$\operatorname{rank}\begin{bmatrix} \cdots & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & a^2 & 0 & \cdots \end{bmatrix} = 1, \tag{15}$$

for all $\xi \in [-\pi + \varepsilon + 2\ell\pi, -\varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$.

(3°) If $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, then

$$\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & a^1 & 0 & \cdots \\ \cdots & 0 & b^2 & a^2 & 0 & \cdots \end{bmatrix}, \tag{16}$$

for some $a^1, a^2, b^2 > 0$, has the rank 2, for all $\xi \in (-\varepsilon+2\ell\pi, \varepsilon+2\ell\pi)$, $\ell \in \mathbb{Z}$.

(4°) For $\xi \in [\varepsilon+2\ell\pi, \pi-\varepsilon+2\ell\pi]$, $\ell \in \mathbb{Z}$, there are two non-zero values a^1 and b^2 in different columns of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ and the block with these elements determines the rank 2 for all $\xi \in [-\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$.

Considering possible cases, we conclude that $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$, $\Phi = (\phi_{k_1}, \phi_{k_2})^T$, $k_2 - k_1 = 1$, $k_1, k_2 \in \mathbb{Z}$, depends on $\xi \in \mathbb{R}$ and equals 1 or 2. This rank is a nonconstant function on $\mathbb{R}$. □

Proof of Theorem 5. (1) Using Lemmas 6 and 7, it is obvious that if $|k_2 - k_1| = 2$ and $|k_i - k_j| \geq 2$ for different $i, j \leq r$, then the position of the first non-zero element in each row of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ is unique for each row. So the rank of matrix $[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}}$ is a constant function on $\mathbb{R}$ and equals r for all $\xi \in \mathbb{R}$.

(2) If $|k_2 - k_1| = 2$ and, at least for k_{i_1} and k_{i_2}, it holds that $|k_{i_1} - k_{i_2}| = 1$, $1 \leq i_1, i_2 \leq r$, then, in the row with the index i_2 (suppose, without losing generality, that $i_2 \in 2\mathbb{Z} + 1$), we will have a new column with a non-zero element for $\xi \in (-\pi - \varepsilon + 2\ell\pi, -\pi + \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, but for $\xi \in [\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, the positions of all non-zero elements in that row will appear in the previous columns. It is obvious that the rank of the matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ depends on $\xi \in \mathbb{R}$ and is not the same for all $\xi \in \mathbb{R}$. □

As a consequence of Theorems 3 and 5(1), we have the following result.

Theorem 8. *Let the functions θ and Φ satisfy all the conditions of Theorem 5(1). Then space $V^p_\mu(\Phi)$ is closed in L^p_μ for any $p \in [1, \infty]$, and the family $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z},\ 1 \leq i \leq r\}$ is a p-Riesz basis for $V^p_\mu(\Phi)$ for any $p \in [1, \infty]$.*

The following theorem is a generalisation of Theorem 5 and can be proved in the same way, so we omit the proof.

Theorem 9. *Let $\theta \in C^\infty_0(\mathbb{R})$ be a positive function such that $\theta(x) > 0$, $x \in (a, b)$, $b > a$, and supported by $[a, b]$ where $b - a > 2\pi$. Moreover, let*

$$\widehat{\phi_i}(\xi) = \theta(\xi + k_i\pi), \quad k_i \in \mathbb{Z},\ i = 1, 2, \dots, r,\ r \in \mathbb{N}, \tag{17}$$

and $\Phi = (\phi_1, \phi_2, \dots, \phi_r)^T$.

(1) *If $|k_2 - k_1| = 2$ and $|k_i - k_j| \geq 2$ for different $i, j \leq r$, then*

$$\operatorname{rank}\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = r, \quad \xi \in \mathbb{R}. \tag{18}$$

(2) *If $|k_2 - k_1| = 2$ and, at least for k_{i_1} and k_{i_2}, it holds that $|k_{i_1} - k_{i_2}| = 1$, where $1 \leq i_1, i_2 \leq r$, then $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ is not a constant function on $\mathbb{R}$.*

4. Construction of Frames Using Several Band-Limited Functions

Firstly, we consider two smooth functions with proper compact supports.

Lemma 10. *Let $\theta \in C^\infty_0(\mathbb{R})$, $\psi \in C^\infty_0(\mathbb{R})$ be positive functions such that*

$$\begin{gathered} \theta(x) > 0, \quad x \in (-\varepsilon, 2\pi + \varepsilon), \\ \psi(x) > 0, \quad x \in (\varepsilon, 2\pi - \varepsilon), \\ \operatorname{supp}\theta = [-\varepsilon, 2\pi + \varepsilon], \\ \operatorname{supp}\psi = [\varepsilon, 2\pi - \varepsilon], \quad 0 < \varepsilon < \frac{1}{4}. \end{gathered} \tag{19}$$

Moreover, let $\widehat{\phi}_1(\xi) = \theta(\xi)$, $\widehat{\phi}_2(\xi) = \psi(\xi)$, $\xi \in \mathbb{R}$, and $\Phi = (\phi_1, \phi_2)^T$. Then $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}} = 1$, $\xi \in \mathbb{R}$.

Proof. Note that $\phi_i \in \mathcal{S}(\mathbb{R}) \subset W^1_\mu(\mathbb{R})$, $i = 1, 2$.

We have the following two cases.

(1°) If $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, then matrix

$$\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & a & b & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}, \quad a, b > 0, \tag{20}$$

has a constant rank equal to 1.

(2°) For $\xi \in (\varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, the rank of matrix

$$\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & c & 0 & \cdots \\ \cdots & 0 & d & 0 & \cdots \end{bmatrix}, \tag{21}$$

where c, d are non-zero values, is equal to 1. An equivalent matrix is obtained for $\xi = \varepsilon + 2\ell\pi$ and $\xi = -\varepsilon + 2\ell\pi$, so we conclude that $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}} = 1$, for $\xi \in [\varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$.

Considering these two cases, the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ is a constant function on $\mathbb{R}$, $\operatorname{rank}[\widehat{\Phi}(\xi+2j\pi)]_{j\in\mathbb{Z}} = 1$, $\xi \in \mathbb{R}$. □

Using functions θ and ψ from Lemma 10, in the next lemma we construct the p-frame with four functions.

Lemma 11. *Let the functions θ and ψ satisfy all the conditions of Lemma 10. Moreover, let*

$$\widehat{\phi}_k(\xi) = \theta(\xi + 2k\pi), \qquad \widehat{\phi}_{k+2}(\xi) = \psi(\xi + 2k\pi), \quad k = 0, 1, \tag{22}$$

and $\Phi = (\phi_0, \phi_1, \phi_2, \phi_3)^T$. Then $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}} = 2$, $\xi \in \mathbb{R}$.

Proof. The proof is similar to the proof of Lemma 10.

(1°) If $\xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi)$, $\ell \in \mathbb{Z}$, then matrix

$$\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & a^1 & b^1 & 0 & \cdots \\ \cdots & 0 & a^2 & b^2 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}, \tag{23}$$

where $a^i, b^i > 0$, $i = 1, 2$, has a constant rank equal to 2.

(2°) For $\xi \in [\varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi]$, $\ell \in \mathbb{Z}$, we have

$$\operatorname{rank}\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & 0 & c^1 & 0 & \cdots \\ \cdots & 0 & 0 & d^1 & 0 & \cdots \\ \cdots & 0 & c^2 & 0 & 0 & \cdots \\ \cdots & 0 & d^2 & 0 & 0 & \cdots \end{bmatrix} = 2, \tag{24}$$

where $c^i, d^i > 0$, $i = 1, 2$.

We conclude that the rank of matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ is a constant function on $\mathbb{R}$ equal to 2. □

Lemma 10 can be easily generalised for an even number of functions ϕ_i, $i = 0, \ldots, 2r-1$, with compactly supported $\widehat{\phi}_i$, $i = 0, \ldots, 2r-1$. The proof of the next theorem is similar to the previous proofs.

Theorem 12. *Let the functions θ and ψ satisfy all the conditions of Lemma 10. Moreover, let*

$$\widehat{\phi}_k(\xi) = \theta(\xi + 2k\pi), \qquad \widehat{\phi}_{k+r}(\xi) = \psi(\xi + 2k\pi), \qquad k = 0, \ldots, r-1, \ r \in \mathbb{N}, \tag{25}$$

and $\Phi = (\phi_0, \phi_1, \ldots, \phi_{2r-1})^T$.

The following statements hold.

(1°) $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}} = r$ *for all* $\xi \in \mathbb{R}$.

(2°) $V_\mu^p(\Phi)$ *is closed in* L_μ^p *for any* $p \in [1, \infty]$.

(3°) $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z},\ 0 \le i \le 2r-1\}$ *is a p-frame for* $V_\mu^p(\Phi)$ *for any* $p \in [1, \infty]$.

Now we consider three functions with compact supports.

Lemma 13. *Let the function θ satisfies all the conditions of Lemma 10, and let $\tau \in C_0^\infty(\mathbb{R})$ and $\omega \in C_0^\infty(\mathbb{R})$ be positive functions such that*

$$\begin{gathered} \tau(x) > 0, \quad x \in (\varepsilon, \pi - \varepsilon) \cup (\pi + \varepsilon, 2\pi - \varepsilon), \\ \omega(x) > 0, \quad x \in (-3\pi - \varepsilon, -\pi + \varepsilon), \\ \operatorname{supp}\tau = [\varepsilon, \pi - \varepsilon] \cup [\pi + \varepsilon, 2\pi - \varepsilon], \\ \operatorname{supp}\omega = [-3\pi - \varepsilon, -\pi + \varepsilon], \quad 0 < \varepsilon < \frac{1}{4}. \end{gathered} \tag{26}$$

Moreover, let $\widehat{\phi}_1(\xi) = \theta(\xi)$, $\widehat{\phi}_2(\xi) = \tau(\xi)$, $\widehat{\phi}_3(\xi) = \omega(\xi)$, $\xi \in \mathbb{R}$, and $\Phi = (\phi_1, \phi_2, \phi_3)^T$. Then $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}} = 2$, $\xi \in \mathbb{R}$.

Proof. We have four different forms for matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ and in all we show that the rank of matrix is equal to 2.

Now we will show all possible cases. Denote with a^i, $i = 1, 2, 3$, and b^i, $i = 1, 2$, some positive values.

(1°) Consider

$$\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & a^1 & b^1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & a^2 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}, \qquad \xi \in (-\varepsilon + 2\ell\pi, \varepsilon + 2\ell\pi). \tag{27}$$

(2°) Consider

$$\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & b^1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & a^2 & 0 & \cdots \\ \cdots & 0 & a^3 & 0 & 0 & \cdots \end{bmatrix}, \qquad \xi \in [\varepsilon + 2\ell\pi, \pi - \varepsilon + 2\ell\pi]. \tag{28}$$

(3°) Consider

$$\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & b^1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & a^2 & b^2 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}, \qquad \xi \in (\pi - \varepsilon + 2\ell\pi, \pi + \varepsilon + 2\ell\pi). \tag{29}$$

(4°) Consider

$$\left[\widehat{\Phi}(\xi + 2j\pi)\right]_{j\in\mathbb{Z}} = \begin{bmatrix} \cdots & 0 & b^1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & b^2 & 0 & \cdots \\ \cdots & 0 & a^3 & 0 & 0 & 0 & \cdots \end{bmatrix}, \qquad \xi \in [\pi + \varepsilon + 2\ell\pi, 2\pi - \varepsilon + 2\ell\pi]. \tag{30}$$

□

Remark 14. In Lemma 13 the support of the function ω must have an empty intersection with the supports of θ and τ. In the opposite case, that is, $\operatorname{supp}\theta \cap \operatorname{supp}\tau \cap \operatorname{supp}\omega \neq \emptyset$, the rank of the matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ is a non-constant function on $\mathbb{R}$.

Lemma 13 can be easily generalised for functions ϕ_i, $i = 0, \ldots, 3r-1$, with compactly supported $\widehat{\phi}_i$, $i = 0, \ldots, 3r-1$. The proof of the next theorem is similar to the previous proofs.

Theorem 15. *Let the functions θ, τ, and ω satisfy all the conditions of Lemma 13. Moreover, let*

$$\begin{gathered} \widehat{\phi}_k(\xi) = \theta(\xi + 2k\pi), \\ \widehat{\phi}_{k+r}(\xi) = \tau(\xi + 2k\pi), \\ \widehat{\phi}_{k+2r}(\xi) = \omega(\xi + 2k\pi), \end{gathered} \tag{31}$$

$k = 0, \ldots, r-1$, $r \in \mathbb{N}$, *and* $\Phi = (\phi_0, \phi_1, \ldots, \phi_{3r-1})^T$.

The following statements hold.

(1°) $\operatorname{rank}[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}} = 2r$ *for all* $\xi \in \mathbb{R}$.

(2°) $V_\mu^p(\Phi)$ *is closed in* L_μ^p *for any* $p \in [1, \infty]$.

(3°) $\{\phi_i(\cdot - j) \mid j \in \mathbb{Z},\ 0 \le i \le 3r-1\}$ *is a p-frame for* $V_\mu^p(\Phi)$ *for any* $p \in [1, \infty]$.

5. Construction of Frames of Functions with Finite Regularities and Compact Supports

We will recall the well-known construction of the B-spline functions in order to justify the rank properties of the corresponding matrices.

Let $H(x)$, $x \in \mathbb{R}$, be the characteristic function of the semiaxis $x \geq 0$; that is, $H(x) = 0$ if $x < 0$ and $H(x) = 1$ if $x \geq 0$ (Heaviside's function). We construct a sequence $\{\phi_n\}_{n\in\mathbb{N}}$ in the following way. Let $\phi_1(x) := (H(x) - H(x-a))/a$, $a > 0$, $\phi_2 := \phi_1 * \phi_1$, $\phi_3 := \phi_1 * \phi_1 * \phi_1, \ldots$; that is,

$$\phi_n := \underbrace{\phi_1 * \phi_1 * \cdots * \phi_1}_{n-1 \text{ times}}, \quad n \in \mathbb{N}, \tag{32}$$

where $*$ denotes the convolution of the functions.

We obtain

$$\begin{aligned}\phi_2(x) &= \frac{1}{a^2}(xH(x) - 2(x-a)H(x-a) \\ &\quad + (x-2a)H(x-2a)), \\ \phi_3(x) &= \frac{1}{2!a^3}\Big(x^2H(x) - 3(x-a)^2H(x-a) + 3(x-2a)^2 \\ &\quad \times H(x-2a) - (x-3a)^2H(x-3a)\Big), \\ \phi_4(x) &= \frac{1}{3!a^4}\Big(x^3H(x) - 4(x-a)^3H(x-a) \\ &\quad + 6(x-2a)^3H(x-2a) - 4(x-3a)^3 \\ &\quad \times H(x-3a) + (x-4a)^3H(x-4a)\Big).\end{aligned} \tag{33}$$

Continuing in this manner, for all $n \in \mathbb{N}$, we have

$$\begin{aligned}\phi_n(x) &= \frac{1}{a^n(n-1)!} \\ &\quad \times \left(\binom{n}{0}x^{n-1}H(x) - \binom{n}{1}(x-a)^{n-1}\right. \\ &\quad \times H(x-a) + \binom{n}{2}(x-2a)^{n-1} \\ &\quad \times H(x-2a) - \binom{n}{3}(x-3a)^{n-1}H(x-3a) \\ &\quad + \cdots + (-1)^{n-1}\binom{n}{n-1}(x-(n-1)a)^{n-1} \\ &\quad \times H(x-(n-1)a) + (-1)^n\binom{n}{n} \\ &\quad \left.\times (x-na)^{n-1}H(x-na)\right).\end{aligned} \tag{34}$$

Calculating the Fourier transform of functions ϕ_n, $n \in \mathbb{N}$, we get

$$\begin{aligned}\widehat{\phi}_1(\xi) &= \frac{-i}{a}\nu\cdot p\cdot\left(\frac{1}{\xi}\right)\left(e^{ia\xi} - 1\right), \\ \widehat{\phi}_2(\xi) &= \frac{(-i)^2}{a^2}\nu\cdot p\cdot\left(\frac{1}{\xi^2}\right)\left(e^{ia\xi} - 1\right)^2, \\ \widehat{\phi}_3(\xi) &= \frac{(-i)^3}{a^3}\nu\cdot p\cdot\left(\frac{1}{\xi^3}\right)\left(e^{ia\xi} - 1\right)^3.\end{aligned} \tag{35}$$

Continuing in this manner, we obtain $\widehat{\phi}_n(\xi) = (-i)^n/a^n \nu \cdot p \cdot (1/\xi^n)(e^{ia\xi} - 1)^n$, $n \in \mathbb{N}$, where $\nu \cdot p \cdot$ denotes the principal value.

Let $\Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T$, $r \in \mathbb{N}$. matrix $[\widehat{\Phi}(\xi + 2j\pi)]_{j\in\mathbb{Z}}$ has for all $\xi \in \mathbb{R}$ the same rank as matrix

$$R(\xi) = \begin{bmatrix} \cdots & \alpha_{-4\pi}\beta_{-4\pi} & \alpha_{-2\pi}\beta_{-2\pi} & \alpha_0\beta_0 & \alpha_{2\pi}\beta_{2\pi} & \alpha_{4\pi}\beta_{4\pi} & \cdots \\ \cdots & \alpha_{-4\pi}^2\beta_{-4\pi}^2 & \alpha_{-2\pi}^2\beta_{-2\pi}^2 & \alpha_0^2\beta_0^2 & \alpha_{2\pi}^2\beta_{2\pi}^2 & \alpha_{4\pi}^2\beta_{4\pi}^2 & \cdots \\ \cdots & \alpha_{-4\pi}^3\beta_{-4\pi}^3 & \alpha_{-2\pi}^3\beta_{-2\pi}^3 & \alpha_0^3\beta_0^3 & \alpha_{2\pi}^3\beta_{2\pi}^3 & \alpha_{4\pi}^3\beta_{4\pi}^3 & \cdots \\ \cdots & \alpha_{-4\pi}^4\beta_{-4\pi}^4 & \alpha_{-2\pi}^4\beta_{-2\pi}^4 & \alpha_0^4\beta_0^4 & \alpha_{2\pi}^4\beta_{2\pi}^4 & \alpha_{4\pi}^4\beta_{4\pi}^4 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \alpha_{-4\pi}^r\beta_{-4\pi}^r & \alpha_{-2\pi}^r\beta_{-2\pi}^r & \alpha_0^r\beta_0^r & \alpha_{2\pi}^r\beta_{2\pi}^r & \alpha_{4\pi}^r\beta_{4\pi}^r & \cdots \end{bmatrix}, \tag{36}$$

where $\alpha_k^m = \nu \cdot p \cdot (1/(\xi - k))^m$ and $\beta_k^m = (e^{ia(\xi-k)} - 1)^m$. Since rank $R(\xi) = r$, $\xi \in \mathbb{R}$, we have the next result.

Theorem 16. *Let $\Phi = (\phi_k, \phi_{k+1}, \ldots, \phi_{k+(r-1)})^T$, for $k \in \mathbb{Z}$, $r \in \mathbb{N}$. Then $V_\mu^p(\Phi)$ is closed in L_μ^p for any $p \in [1,\infty]$ and $\{\phi_{k+s}(\cdot - j) \mid j \in \mathbb{Z},\ 0 \leq s \leq r-1\}$ is a p-Riesz basis for $V_\mu^p(\Phi)$ for any $p \in [1,\infty]$.*

Remark 17. Let k be a positive integer. We refer to [23] for the L^p-approximation order k. Shift-invariant spaces generated by a finite number of compactly supported functions in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, were studied in [23] by Jia, who gave a characterization of the approximation order providing such shift-invariant spaces. Theorem 3 in [23] shows that the shift-invariant space generated with the family of splines, which we constructed in Section 5, provides L^p-approximation order k.

Remark 18. (1) We refer to [3, 20] for the γ-dense set $X = \{x_j \mid j \in J\}$. Let $\phi_k(x) = \mathcal{F}^{-1}(\theta(\cdot - k\pi))(x)$, $x \in \mathbb{R}$. Following the notation of [20], we put $\psi_{x_j} = \phi_{x_j}$, where $\{x_j \mid j \in J\}$ is γ-dense set determined by $f \in V^2(\phi) = V^2(\mathcal{F}^{-1}(\theta))$. Theorems 3.1, 3.2, and 4.1 in [20] give conditions and explicit form of $C_p > 0$ and $c_p > 0$ such that inequality $c_p\|f\|_{L_\mu^p} \leq (\sum_{j\in J}|\langle f, \psi_{x_j}\rangle\mu(x_j)|^p)^{1/p} \leq C_p\|f\|_{L_\mu^p}$ holds. This inequality guarantees the feasibility of a stable and continuous reconstruction algorithm in the signal spaces $V_\mu^p(\Phi)$ [20].

(2) Since the spectrum of the Gram matrix $[\widehat{\Phi}, \widehat{\Phi}](\xi)$, where Φ is defined in Theorem 16, is bounded and bounded away from zero (see [7]), it follows that the family $\{\Phi(\cdot - j) \mid j \in \mathbb{Z}\}$ forms a p-Riesz basis for $V_\mu^p(\Phi)$.

(3) Frames of the above sections may be useful in applications since they satisfy assumptions of Theorems 3.1 and 3.2 in [4]. They show that error analysis for sampling and reconstruction can be tolerated or that the sampling and reconstruction problem in shift-invariant space is robust with respect to appropriate set of functions $\phi_{k_1}, \ldots, \phi_{k_r}$.

Acknowledgments

The authors are indebted to the referee for the valuable suggestions, which have contributed to the improvement of the presentation of the paper. The authors were supported in part by the Serbian Ministry of Science and Technological Developments (Project no. 174024).

References

[1] A. Aldroubi, Q. Sun, and W.-S. Tang, "p-frames and shift invariant subspaces of L^p," *The Journal of Fourier Analysis and Applications*, vol. 7, no. 1, pp. 1–21, 2001.

[2] A. Aldroubi, Q. Sun, and W.-S. Tang, "Non-uniform sampling in multiply generated shift-invariant subspaces of $L^p(\mathbb{R}^d)$," in *Wavelet Analysis and Applications (Guangzhou, 1999)*, vol. 25 of *AMS/IP Stud. Adv. Math.*, pp. 1–8, American Mathematical Society, Providence, RI, USA, 2002.

[3] A. Aldroubi, "Non-uniform weighted average sampling and reconstruction in shift-invariant and wavelet spaces," *Applied and Computational Harmonic Analysis*, vol. 13, no. 2, pp. 151–161, 2002.

[4] A. Aldroubi and I. Krishtal, "Robustness of sampling and reconstruction and Beurling-Landau-type theorems for shift-invariant spaces," *Applied and Computational Harmonic Analysis*, vol. 20, no. 2, pp. 250–260, 2006.

[5] A. Aldroubi and M. Unser, "Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory," *Numerical Functional Analysis and Optimization*, vol. 15, no. 1-2, pp. 1–21, 1994.

[6] A. Aldroubi, A. Baskakov, and I. Krishtal, "Slanted matrices, Banach frames, and sampling," *Journal of Functional Analysis*, vol. 255, no. 7, pp. 1667–1691, 2008.

[7] C. de Boor, R. A. DeVore, and A. Ron, "The structure of finitely generated shift-invariant spaces in $L^2(\mathbb{R}^d)$," *Journal of Functional Analysis*, vol. 119, no. 1, pp. 37–78, 1994.

[8] R. A. DeVore, B. Jawerth, and B. J. Lucier, "Image compression through wavelet transform coding," *IEEE Transactions on Information Theory*, vol. 38, no. 2, pp. 719–746, 1992.

[9] H. G. Feichtinger, "Banach convolution algebras of Wiener type," in *Functions, Series, Operators, Vol. I, II (Budapest, 1980)*, vol. 35 of *Colloq. Math. Soc. János Bolyai*, pp. 509–524, North-Holland, Amsterdam, The Netherlands, 1983.

[10] H. G. Feichtinger and K. Gröchenig, "A unified approach to atomic decompositions via integrable group representations," in *Function Spaces and Applications (Lund, 1986)*, vol. 1302 of *Lecture Notes in Math.*, pp. 52–73, Springer, Berlin, Germany, 1988.

[11] H. G. Feichtinger and K. H. Gröchenig, "Banach spaces related to integrable group representations and their atomic decompositions. I," *Journal of Functional Analysis*, vol. 86, no. 2, pp. 307–340, 1989.

[12] H. G. Feichtinger, "Generalized amalgams, with applications to Fourier transform," *Canadian Journal of Mathematics*, vol. 42, no. 3, pp. 395–409, 1990.

[13] H. G. Feichtinger and K. Gröchenig, "Iterative reconstruction of multivariate band-limited functions from irregular sampling values," *SIAM Journal on Mathematical Analysis*, vol. 23, no. 1, pp. 244–261, 1992.

[14] H. G. Feichtinger and K. Gröchenig, "Theory and practice of irregular sampling," in *Wavelets: Mathematics and Applications*, J. J. Benedetto and W. Frazier, Eds., Stud. Adv. Math., pp. 305–363, CRC, Boca Raton, Fla, USA, 1994.

[15] H. G. Feichtinger, K. Gröchenig, and T. Strohmer, "Efficient numerical methods in non-uniform sampling theory," *Numerische Mathematik*, vol. 69, no. 4, pp. 423–440, 1995.

[16] H. G. Feichtinger and T. Strohmer, Eds., *Gabor Analysis and Algorithms*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, Mass, USA, 1998.

[17] K. Gröchenig, "Describing functions: atomic decompositions versus frames," *Monatshefte für Mathematik*, vol. 112, no. 1, pp. 1–42, 1991.

[18] R. Q. Jia and C. A. Micchelli, "On linear independence for integer translates of a finite number of functions," *Proceedings of the Edinburgh Mathematical Society*, vol. 36, no. 1, pp. 69–85, 1993.

[19] S. Pilipović and S. Simić, "Frames for weighted shift-invariant spaces," *Mediterranean Journal of Mathematics*, vol. 9, no. 4, pp. 897–912, 2012.

[20] J. Xian and S. Li, "Sampling set conditions in weighted multiply generated shift-invariant spaces and their applications," *Applied and Computational Harmonic Analysis*, vol. 23, no. 2, pp. 171–180, 2007.

[21] A. Aldroubi and K. Gröchenig, "Non-uniform sampling and reconstruction in shift-invariant spaces," *SIAM Review*, vol. 43, no. 4, pp. 585–620, 2001.

[22] S. Simić, "Fréchet frames for shift invariant weighted spaces," *Novi Sad Journal of Mathematics*, vol. 39, no. 2, pp. 119–128, 2009.

[23] R.-Q. Jia, "Shift-invariant spaces on the real line," *Proceedings of the American Mathematical Society*, vol. 125, no. 3, pp. 785–793, 1997.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 143686, 9 pages
http://dx.doi.org/10.1155/2013/143686

Research Article

Suzuki-Type Fixed Point Results in Metric-Like Spaces

Nabiollah Shobkolaei,[1] Shaban Sedghi,[2] Jamal Rezaei Roshan,[2] and Nawab Hussain[3]

[1] *Department of Mathematics, Babol Branch, Islamic Azad University, Babol, Iran*
[2] *Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr, Iran*
[3] *Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Nawab Hussain; nhusain@kau.edu.sa

Received 16 May 2013; Accepted 17 July 2013

Academic Editor: Pei De Liu

We demonstrate a fundamental lemma for the convergence of sequences in metric-like spaces, and by using it we prove some Suzuki-type fixed point results in the setup of metric-like spaces. As an immediate consequence of our results we obtain certain recent results in partial metric spaces as corollaries. Finally, three examples are presented to verify the effectiveness and applicability of our main results.

1. Introduction

There are a lot of generalizations of Banach fixed-point principle in the literature. So far several authors have studied the problem of existence and uniqueness of a fixed point for mappings satisfying different contractive conditions (e.g., [1–20]). In 2008, Suzuki introduced an interesting generalization of Banach fixed-point principle. This interesting fixed-point result is as follows.

Theorem 1 (see [19]). *Let (X, d) be a complete metric space, and let T be a mapping on X. Define a nonincreasing function θ from $[0, 1]$ into $[1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1, & 0 \le r \le \dfrac{\sqrt{5}-1}{2} \\ \dfrac{1-r}{r^2}, & \dfrac{\sqrt{5}-1}{2} \le r \le \dfrac{1}{\sqrt{2}} \\ \dfrac{1}{1+r}, & \dfrac{1}{\sqrt{2}} \le r < 1. \end{cases} \tag{1}$$

Assume that there exists $r \in [0, 1]$, such that

$$\theta(r)\, d(x, Tx) \le d(x, y) \Longrightarrow d(Tx, Ty) \le r d(x, y), \tag{2}$$

for all $x, y \in X$, then there exists a unique fixed-point z of T. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in X$.

Suzuki proved also the following version of Edelstein's fixed point theorem.

Theorem 2. *Let (X, d) be a compact metric space. Let $T : X \to X$ be a self-map, satisfying for all $x, y \in X$, $x \ne y$ the condition*

$$\frac{1}{2} d(x, Tx) \le d(x, y) \Longrightarrow d(Tx, Ty) < d(x, y). \tag{3}$$

Then T has a unique fixed point in X.

This theorem was generalized in [3].

In addition to the above results, Kikkawa and Suzuki [8] provided a Kannan type version of the theorems mentioned before. In [14], Chatterjea type version is provided. Popescu [15] presented a Cirić type version. Recently, Kikkawa and Suzuki also provided multivalued versions which can be found in [9, 10].

Very recently Hussain et al. [4] have extended Suzuki's Theorems 1 and 2, as well as Popescu's results from [15] to the case of metric type spaces and cone metric type spaces (see also [5–7, 11]).

The aim of this paper is to generalize the above-mentioned results. Indeed we prove a fixed point theorem in the set up of metric-like spaces and derive certain new results as corollaries. Finally, three examples are presented to verify the effectiveness and applicability of our main results.

In the rest of this section, we recall some definitions and facts which will be used throughout the paper. First, we present some known definitions and propositions in partial metric and metric-like spaces.

A partial metric on a nonempty set X is a mapping $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

(p_1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$,

(p_2) $p(x,x) \le p(x,y)$,

(p_3) $p(x,y) = p(y,x)$,

(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that if $p(x,y) = 0$, then from (p_1) and (p_2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$.

Lemma 3 (see [17]). *Let (X,d) and (X,p) be a metric space and partial metric space, respectively. Then*

(i) *the function $\rho : X \times X \to \mathbb{R}^+$ defined by $\rho(x,y) = d(x,y) + p(x,y)$ is a partial metric;*

(ii) *let $\rho : X \times X \to \mathbb{R}^+$ be defined by $\rho(x,y) = d(x,y) + \max\{\omega(x), \omega(y)\}$; then ρ is a partial metric on X, where $\omega : X \to \mathbb{R}^+$ is an arbitrary function;*

(iii) *Let $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $\rho(x,y) = \max\{2^x, 2^y\}$; then ρ is a partial metric on $\mathbb{R}$;*

(iv) *Let $\rho : X \times X \to \mathbb{R}^+$ be defined by $\rho(x,y) = d(x,y) + a$; then ρ is a partial metric on X, where $a \ge 0$.*

Other examples of the partial metric spaces which are interesting from a computational point of view may be found in [7, 11, 12, 18].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Let (X,p) be a partial metric.

A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n,x_m)$.

A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Suppose that $\{x_n\}$ is a sequence in partial metric space (X,p); then we define $L(x_n) = \{x \mid x_n \to x\}$.

The following example shows that every convergent sequence $\{x_n\}$ in a partial metric space (X,p) may not be a Cauchy sequence. In particular, it shows that the limit is not unique.

Example 4 (see [17]). Let $X = [0,\infty)$ and $p(x,y) = \max\{x,y\}$. Let

$$x_n = \begin{cases} 0, & n = 2k, \\ 1, & n = 2k+1. \end{cases} \tag{4}$$

Then clearly it is a convergent sequence, and for every $x \ge 1$ we have $\lim_{n\to\infty} p(x_n,x) = p(x,x)$, hence $L(x_n) = [1,\infty)$. But $\lim_{n,m\to\infty} p(x_n,x_m)$ does not exist; that is, it is not a Cauchy sequence.

Definition 5 (see [2]). A metric-like on a nonempty set X is a mapping $\sigma : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

($\sigma 1$) $\sigma(x,y) = 0 \Rightarrow x = y$,

($\sigma 2$) $\sigma(x,y) = \sigma(y,x)$,

($\sigma 3$) $\sigma(x,y) \le \sigma(x,z) + \sigma(z,y)$.

The pair (X,σ) is called a metric-like space. Then a metric-like on X satisfies all of the conditions of a metric except that $\sigma(x,x)$ may be positive for $x \in X$. Each metric-like σ on X generates a topology τ_σ on X whose base is the family of open σ-ball, $\{B_\sigma(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_\sigma(x,\varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

A sequence $\{x_n\}$ in a metric-like space (X,σ) converges to a point $x \in X$ if and only if $\lim_{n\to\infty} \sigma(x,x_n) = \sigma(x,x)$.

A sequence $\{x_n\}$ in a metric-like space (X,σ) is called a σ-Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} \sigma(x_n,x_m)$.

A metric-like space (X,σ) is said to be complete if every σ-Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_σ, to a point $x \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n,x) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_n,x_m). \tag{5}$$

Every partial metric space is a metric-like space. Below we give some examples of a metric-like space.

Example 6. Let $X = [0,1]$; then mapping $\sigma_1 : X \times X \to \mathbb{R}^+$ defined by $\sigma_1(x,y) = x + y - xy$ is a metric-like on X.

Example 7. Let $X = \mathbb{R}$; then mappings $\sigma_i : X \times X \to \mathbb{R}^+$ ($i \in \{2,3,4\}$) defined by

$$\begin{gathered} \sigma_2(x,y) = |x| + |y| + a, \\ \sigma_3(x,y) = |x-b| + |y-b|, \\ \sigma_4(x,y) = x^2 + y^2 \end{gathered} \tag{6}$$

are metric-like space on X, where $a \ge 0$ and $b \in \mathbb{R}$.

2. Main Results

We start our work by proving the following crucial lemma.

Lemma 8. *Let (X,σ) be a metric-like space, and suppose that $\{x_n\}$ is σ-convergent to x. Then for every $y \in X$, one has*

$$\sigma(x,y) - \sigma(x,x) \le \liminf_{n\to\infty} \sigma(x_n,y)$$

$$\leq \limsup_{n\to\infty} \sigma(x_n, y) \leq \sigma(x, y) + \sigma(x, x). \tag{7}$$

In particular, if $\sigma(x,x) = 0$*, then one has* $\lim_{n\to\infty}\sigma(x_n, y) = \sigma(x, y)$.

Proof. Using the triangle inequality in a metric-like space, it is easy to see that

$$\begin{aligned} \sigma(x_n, y) &\leq \sigma(x_n, x) + \sigma(x, y), \\ \sigma(x, y) &\leq \sigma(x, x_n) + \sigma(x_n, y). \end{aligned} \tag{8}$$

Taking the upper limit as $n \to \infty$ in the first inequality and the lower limit as $n \to \infty$ in the second inequality, we obtain the desired result. □

Theorem 9. *Let* (X, σ) *be a complete metric-like space. Let* $T : X \to X$ *be a self-map, and let* $\theta =: [0, 1) \to (1/2, 1]$ *be defined by*

$$\theta(r) = \begin{cases} 1, & 0 \leq r \leq \dfrac{\sqrt{5}-1}{2}, \\ \dfrac{1-r}{r^2}, & \dfrac{\sqrt{5}-1}{2} \leq r \leq \dfrac{1}{\sqrt{2}}, \\ \dfrac{1}{1+r}, & \dfrac{1}{\sqrt{2}} \leq r < 1. \end{cases} \tag{9}$$

If there exists $r \in [0, 1)$ *such that for each* $x, y \in X$

$$\theta(r)\,\sigma(x, Tx) \leq \sigma(x, y) \Longrightarrow \sigma(Tx, Ty) \leq r\sigma(x, y). \tag{10}$$

Then T *has a unique fixed point* $z \in X$*, and for each* $x \in X$*, the sequence* $\{T^n x\}$ *converges to* z.

Proof. Putting $y = Tx$ in (10), hence from

$$\theta(r)\,\sigma(x, Tx) \leq \sigma(x, Tx), \tag{11}$$

it follows

$$\sigma(Tx, T^2x) \leq r\sigma(x, Tx), \tag{12}$$

for every $x \in X$. Let $x_0 \in X$ be arbitrary and form the sequence $\{x_n\}$ by $x_1 = Tx_0$ and $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. By (12), we have

$$\begin{aligned} \sigma(x_n, x_{n+1}) &= \sigma(Tx_{n-1}, T^2x_{n-1}) \\ &\leq r\sigma(x_{n-1}, Tx_{n-1}) \\ &= r\sigma(x_{n-1}, x_n) \\ &\;\vdots \\ &\leq r^n\sigma(x_0, x_1). \end{aligned} \tag{13}$$

Also, by the condition $\sigma3$ of the definition of metric-like space, for all $m \geq n$, we have

$$\begin{aligned} \sigma(x_n, x_m) &\leq \sigma(x_n, x_{n+1}) \\ &\quad + \sigma(x_{n+1}, x_{n+2}) \\ &\quad + \cdots + \sigma(x_{m-1}, x_m) \\ &\leq r^n\sigma(x_0, x_1) + r^{n+1}\sigma(x_0, x_1) \\ &\quad + \cdots + r^{m-1}\sigma(x_0, x_1) \\ &= \frac{r^n - r^m}{1-r}\sigma(x_0, x_1) \\ &< \frac{r^n}{1-r}\sigma(x_0, x_1) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{14}$$

Hence, $\{x_n\}$ is a σ-Cauchy sequence.
Since X is σ-complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} \sigma(x_n, z) = \sigma(z, z) = \lim_{m,n\to\infty} \sigma(x_m, x_n) = 0. \tag{15}$$

That is, $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = z$. We prove that $Tz = z$. Putting $x = T^{n-1}z$ in (12), we get that

$$\sigma(T^n z, T^{n+1}z) \leq r\sigma(T^{n-1}z, T^n z) \tag{16}$$

holds for each $n \in \mathbb{N}$ (where $T^0 z = z$). It follows by induction that

$$\sigma(T^n z, T^{n+1}z) \leq r^n\sigma(z, Tz). \tag{17}$$

Let us prove now that

$$\sigma(z, Tx) \leq r\sigma(z, x) \tag{18}$$

holds for each $x \neq z$. Since $\sigma(x_n, Tx_n) \to 0$ and by Lemma 8 $\sigma(x_n, x) \to \sigma(z, x) \neq 0$, it follows that there exists $n_0 \in \mathbb{N}$ such that

$$\theta(r)\,\sigma(x_n, Tx_n) \leq \sigma(x_n, x) \tag{19}$$

holds for every $n \geq n_0$. Assumption (10) implies that for such $n\sigma(Tx_n, Tx) \leq r\sigma(x_n, x)$, thus as $n \to \infty$, we get that

$$\sigma(z, Tx) \leq r\sigma(z, x). \tag{20}$$

We will prove that

$$\sigma(T^n z, z) \leq \sigma(Tz, z), \tag{21}$$

for each $n \in \mathbb{N}$. For $n = 1$, this relation is obvious. Suppose that it holds for some $m \in \mathbb{N}$. If $T^m z = z$, then $T^{m+1}z = Tz$ and $\sigma(T^{m+1}z, z) = \sigma(Tz, z) \leq \sigma(Tz, z)$. If $T^m z \neq z$, then applying (18) and the induction hypothesis; we get that

$$\begin{aligned} \sigma(T^{m+1}z, z) &\leq r\sigma(T^m z, z) \\ &\leq r\sigma(Tz, z) \leq \sigma(Tz, z), \end{aligned} \tag{22}$$

and (21) is proved by induction.

In order to prove that $Tz = z$, we consider two possible cases.

Case I. $0 \le r < 1/\sqrt{2}$ (and hence $\theta(r) \le (1-r)/r^2$). We will prove first that

$$\sigma\left(T^n z, Tz\right) \le r\sigma\left(Tz, z\right) \tag{23}$$

for $n \ge 2$. For $n = 2$, it follows from (16). Suppose that (23) holds for some $n > 2$. Then

$$\begin{aligned}\sigma\left(Tz, z\right) &\le \sigma\left(z, T^n z\right) + \sigma\left(T^n z, Tz\right)\\ &\le \sigma\left(z, T^n z\right) + r\sigma\left(z, Tz\right),\end{aligned} \tag{24}$$

which implies $(1-r)\sigma(z, Tz) \le \sigma(z, T^n z)$. Using (17) we obtain

$$\begin{aligned}\theta(r)\sigma\left(T^n z, T^{n+1} z\right) &\le \frac{1-r}{r^n}\sigma\left(T^n z, T^{n+1} z\right)\\ &\le \frac{1-r}{r^n}\cdot r^n\sigma\left(z, Tz\right)\\ &= (1-r)\sigma\left(z, Tz\right) \le \sigma\left(z, T^n z\right).\end{aligned} \tag{25}$$

Assumption (10) and relation (21) imply that

$$\begin{aligned}\sigma\left(Tz, T^{n+1} z\right) &\le r\sigma\left(z, T^n z\right)\\ &\le r\sigma\left(z, Tz\right).\end{aligned} \tag{26}$$

So relation (23) is proved by induction.

Now $Tz \ne z$ and (23) implies that $T^n z \ne z$ for each $n \in \mathbb{N}$. Hence, (18) imply that

$$\begin{aligned}\sigma\left(z, T^{n+1} z\right) &\le r\sigma\left(z, T^n z\right)\\ &\le r^2\sigma\left(z, T^{n-1} z\right)\\ &\le r^n\sigma\left(z, Tz\right).\end{aligned} \tag{27}$$

Hence $\lim_{n\to\infty}\sigma(z, T^{n+1}z) = 0 = \sigma(z,z)$, thus $T^n z \to z$ and; using Lemma 8 in (23), we have $\sigma(z, Tz) \le r\sigma(Tz, z)$ as $n \to \infty$ which implies that $\sigma(z, Tz) = 0$, a contradiction.

Case II. $1/\sqrt{2} \le r < 1$ (and so $\theta(r) = 1/(1+r)$). We will prove that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\theta(r)\sigma\left(x_{n_k}, Tx_{n_k}\right) = \theta(r)\sigma\left(x_{n_k}, x_{n_k+1}\right) \le \sigma\left(x_{n_k}, z\right) \tag{28}$$

holds for each $k \in \mathbb{N}$. From (12) we know that $\sigma(x_n, x_{n+1}) \le r\sigma(x_{n-1}, x_n)$ holds for each $n \in \mathbb{N}$. Suppose that

$$\begin{aligned}&\frac{1}{1+r}\sigma\left(x_{n-1}, x_n\right) > \sigma\left(x_{n-1}, z\right),\\ &\frac{1}{1+r}\sigma\left(x_n, x_{n+1}\right) > \sigma\left(x_n, z\right)\end{aligned} \tag{29}$$

hold for some $n \in \mathbb{N}$. Then

$$\begin{aligned}\sigma\left(x_{n-1}, x_n\right) &\le \sigma\left(x_{n-1}, z\right) + \sigma\left(z, x_n\right)\\ &< \frac{1}{1+r}\sigma\left(x_{n-1}, x_n\right) + \sigma\left(x_n, z\right)\\ &< \frac{1}{1+r}\sigma\left(x_{n-1}, x_n\right) + \frac{1}{1+r}\sigma\left(x_n, x_{n+1}\right)\\ &\le \frac{1}{1+r}\sigma\left(x_{n-1}, x_n\right) + \frac{r}{1+r}\sigma\left(x_{n-1}, x_n\right)\\ &= \sigma\left(x_{n-1}, x_n\right),\end{aligned} \tag{30}$$

which is impossible. Hence one of the following holds for each n:

$$\theta(r)\sigma\left(x_{n-1}, x_n\right) \le \sigma\left(x_{n-1}, z\right), \tag{31}$$

or

$$\theta(r)\sigma\left(x_n, x_{n+1}\right) \le \sigma\left(x_n, z\right). \tag{32}$$

In particular,

$$\theta(r)\sigma\left(x_{2n-1}, x_{2n}\right) \le \sigma\left(x_{2n-1}, z\right), \tag{33}$$

or

$$\theta(r)\sigma\left(x_{2n}, x_{2n+1}\right) \le \sigma\left(x_{2n}, z\right). \tag{34}$$

In other words, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that (28) holds for each $k \in \mathbb{N}$. But then assumption (10) implies that

$$\sigma\left(Tx_{n_k}, Tz\right) \le r\sigma\left(x_{n_k}, z\right), \tag{35}$$

or

$$\sigma\left(Tx_{n_{k-1}}, Tz\right) \le r\sigma\left(x_{n_{k-1}}, z\right). \tag{36}$$

Passing to the limit when $k \to \infty$, we get that $\sigma(z, Tz) \le 0$, which is possible only if $Tz = z$.

Thus, we have proved that z is a fixed point of T. The uniqueness of the fixed point follows easily from (10). Indeed, if y and z are two fixed points of T such that $y \ne z$, then from (18) we have

$$\begin{aligned}\sigma\left(y, z\right) &= \sigma\left(y, Tz\right)\\ &\le r\sigma\left(y, z\right),\end{aligned} \tag{37}$$

which is a contradiction. □

According to Theorem 9, we get the following result.

Corollary 10 (see [19]). *Let (X, d) be a complete metric space, and let T be a mapping on X. Define a nonincreasing function θ from $[0,1]$ into $[1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1, & 0 \le r \le \dfrac{\sqrt{5}-1}{2},\\ \dfrac{1-r}{r^2}, & \dfrac{\sqrt{5}-1}{2} \le r \le \dfrac{1}{\sqrt{2}},\\ \dfrac{1}{1+r}, & \dfrac{1}{\sqrt{2}} \le r < 1.\end{cases} \tag{38}$$

Assume that there exists $r \in [0,1]$, *such that*

$$\theta(r)\,d(x,Tx) \le d(x,y) \Longrightarrow d(Tx,Ty) \le r d(x,y), \quad (39)$$

for all $x, y \in X$*; then there exists a unique fixed-point* z *of* T*. Moreover,* $\lim_{n\to\infty} T^n x = z$ *for all* $x \in X$.

Proof. Using a similar argument given in Theorem 9 for $\sigma(x,y) = d(x,y)$, the desired result is obtained. □

Now, in order to support the useability of our results, let us introduce the following example.

Example 11. Let $X = [0,\infty)$. Define $\sigma : X \times X \to \mathbb{R}^+$ by

$$\sigma(x,y) = x + y \quad (40)$$

for all $x, y \in X$. Then (X,σ) is a complete metric-like space. Define a map $T : X \to X$ by

$$T(x) = \ln\left(1 + \frac{1}{\sqrt{2}}x\right) \quad (41)$$

for $x \in X$. Then for each $x, y \in X$, we have

$$\begin{aligned} \frac{1}{1+1/\sqrt{2}}\sigma(x,Tx) &= \frac{\sqrt{2}}{\sqrt{2}+1}\left(x + \ln\left(1+\frac{1}{\sqrt{2}}x\right)\right) \\ &\le \frac{\sqrt{2}}{\sqrt{2}+1}\left(x + \frac{1}{\sqrt{2}}x\right) = x \\ &\le x + y = \sigma(x,y). \end{aligned} \quad (42)$$

On the other hand, we have

$$\begin{aligned} \sigma(Tx,Ty) &= \ln\left(1+\frac{1}{\sqrt{2}}x\right) + \ln\left(1+\frac{1}{\sqrt{2}}y\right) \\ &\le \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \\ &= \frac{1}{\sqrt{2}}\sigma(x,y). \end{aligned} \quad (43)$$

Thus T satisfies all the hypotheses of Theorem 9, and hence T has a unique fixed point. Indeed, $r = 1/\sqrt{2}$, $\theta(r) = 1/(1+r)$, and 0 is the unique fixed point of T.

Theorem 12. *Let* (X,σ) *be a complete metric-like space. Let* S, $T : X \to X$ *be two self-mappings. Suppose that there exists* $r \in [0,1)$ *such that*

$$\begin{aligned} &\max\{\sigma(S(x),TS(x)),\sigma(T(x),ST(x))\} \\ &\quad\le r\min\{\sigma(x,S(x)),\sigma(x,T(x))\} \end{aligned} \quad (44)$$

for every $x \in X$ *and that*

$$\begin{aligned} \alpha(y) = \ &\inf\{\sigma(x,y) + \min\{\sigma(x,S(x)),\sigma(x,T(x))\} : \\ &x \in X\} > 0 \end{aligned} \quad (45)$$

for every $y \in X$ *with* y *that is not a common fixed point of* S *and* T*. Then there exists* $z \in X$ *such that* $z = S(z) = T(z)$*. Moreover, if* $v = S(v) = T(v)$*, then* $\sigma(v,v) = 0$.

Proof. Let $x_0 \in X$ be arbitrary, and define a sequence $\{x_n\}$ by

$$\begin{aligned} x_n &= S(x_{n-1}), \quad \text{if } n \text{ is odd} \\ &= T(x_{n-1}), \quad \text{if } n \text{ is even}. \end{aligned} \quad (46)$$

Then if $n \in \mathbb{N}$ is odd, we have

$$\begin{aligned} &\sigma(x_n,x_{n+1}) \\ &\quad= \sigma(S(x_{n-1}),T(x_n)) \\ &\quad= \sigma(S(x_{n-1}),TS(x_{n-1})) \\ &\quad\le \max\{\sigma(S(x_{n-1}),TS(x_{n-1})), \\ &\qquad\quad \sigma(T(x_{n-1}),ST(x_{n-1}))\} \\ &\quad\le r\min\{\sigma(x_{n-1},S(x_{n-1})),\sigma(x_{n-1},T(x_{n-1}))\} \\ &\quad\le r\sigma(x_{n-1},S(x_{n-1})) \\ &\quad= r\sigma(x_{n-1},x_n). \end{aligned} \quad (47)$$

If n is even, then by (44), we have

$$\begin{aligned} &\sigma(x_n,x_{n+1}) \\ &\quad= \sigma(T(x_{n-1}),S(x_n)) \\ &\quad= \sigma(T(x_{n-1}),ST(x_{n-1})) \\ &\quad\le \max\{\sigma(T(x_{n-1}),ST(x_{n-1})), \\ &\qquad\quad \sigma(S(x_{n-1}),TS(x_{n-1}))\} \\ &\quad\le r\min\{\sigma(x_{n-1},T(x_{n-1})),\sigma(x_{n-1},S(x_{n-1}))\} \\ &\quad\le r\sigma(x_{n-1},T(x_{n-1})) \\ &\quad= r\sigma(x_{n-1},x_n). \end{aligned} \quad (48)$$

Thus for any positive integer n, it must be the case that

$$\sigma(x_n,x_{n+1}) \le r\sigma(x_{n-1},x_n). \quad (49)$$

Repeating (49), we obtain

$$\sigma(x_n,x_{n+1}) \le r^n\sigma(x_0,x_1). \quad (50)$$

So, if $m > n$, then

$$\begin{aligned} \sigma(x_n,x_m) &\le \sigma(x_n,x_{n+1}) \\ &\quad+ \sigma(x_{n+1},x_{n+2}) + \cdots + \sigma(x_{m-1},x_m) \\ &\le \left[r^n + r^{n+1} + \cdots + r^{m-1}\right]\sigma(x_0,x_1) \\ &\le \frac{r^n}{1-r}\sigma(x_0,x_1). \end{aligned} \quad (51)$$

Thus $\lim_{n,m\to\infty}\sigma(x_n,x_m) = 0$.

That is, $\{x_n\}$ is a σ-Cauchy sequence in the metric-like space (X,σ). Since (X,σ) is σ-complete, there exist $z \in X$ such that

$$\sigma(z,z) = \lim_{n\to\infty}\sigma(x_n,z) = \lim_{n,m\to\infty}\sigma(x_n,x_m) = 0. \quad (52)$$

Assume that z is not a common fixed point of S and T. Then by hypothesis

$$\begin{aligned} 0 &< \inf\{\sigma(x,z) + \min\{\sigma(x,S(x)), \sigma(x,T(x))\} : x \in X\} \\ &\leq \inf\{\sigma(x_n,z) + \min\{\sigma(x_n,S(x_n)), \sigma(x_n,T(x_n))\} : \\ &\qquad n \in \mathbb{N}\} \\ &\leq \inf\left\{\frac{r^n}{1-r}\sigma(x_0,x_1) + \sigma(x_n,x_{n+1}) : n \in \mathbb{N}\right\} \\ &\leq \inf\left\{\frac{r^n}{1-r}\sigma(x_0,x_1) + r^n\sigma(x_0,x_1) : n \in \mathbb{N}\right\} = 0 \end{aligned} \tag{53}$$

which is a contradiction. Therefore, $z = S(z) = T(z)$.

If $v = S(v) = T(v)$ for some $v \in X$, then

$$\begin{aligned} \sigma(v,v) &= \max\{\sigma(S(v), TS(v)), \sigma(T(v), ST(v))\} \\ &\leq r\min\{\sigma(v,S(v)), \sigma(v,T(v))\} \\ &= r\min\{\sigma(v,v), \sigma(v,v)\} \\ &= r\sigma(v,v) \end{aligned} \tag{54}$$

which gives that $\sigma(v,v) = 0$. □

Example 13. Let (X,σ) be a metric-like space where $X = \{1/n\}_{n=1}^{\infty} \cup \{0\}$ and $\sigma(x,y) = x + y$. Define $S : X \to X$ by $S(0) = 0$, $S(1/2n) = 1/(4n+3)$, and $S(1/(2n-1)) = 0$ and $T(0) = 0$, $T(1/(2n-1)) = 1/(4n+4)$, and $T(1/2n) = 0$. Then for $x = 1/2n$, we have

$$\begin{aligned} &\max\{\sigma(S(x), TS(x)), \sigma(T(x), ST(x))\} \\ &\quad = \max\left\{\sigma\left(S\left(\frac{1}{2n}\right), T\left(S\left(\frac{1}{2n}\right)\right)\right),\right. \\ &\qquad \left.\sigma\left(T\left(\frac{1}{2n}\right), S\left(T\left(\frac{1}{2n}\right)\right)\right)\right\} \\ &\quad = \max\left\{\frac{1}{4n+3} + \frac{1}{8n+12}, 0\right\} = \frac{1}{4n+3} + \frac{1}{8n+12} \\ &\quad \leq r\min\{\sigma(x,S(x)), \sigma(x,T(x))\} \\ &\quad = r\min\left\{\frac{1}{2n} + \frac{1}{4n+3}, \frac{1}{2n} + 0\right\} = r\frac{1}{2n}. \end{aligned} \tag{55}$$

It is easy to see that the above inequality is true for $x = 1/(2n-1)$ and for $3/4 \leq r < 1$. Also,

$$\begin{aligned} \alpha(y) = &\inf\{\sigma(x,y) + \min\{\sigma(x,S(x)), \sigma(x,T(x))\} : \\ &x \in X\} > 0 \end{aligned} \tag{56}$$

for every $y \in X$ with y is not a common fixed point of S and T. This shows that all conditions of Theorem 12 are satisfied and 0 is a common fixed point for S and T.

Corollary 14. *Let (X,σ) be a complete metric-like space, and let $T : X \to X$ be a mapping. Suppose that there exists $r \in [0,1)$ such that*

$$\sigma\left(T(x), T^2(x)\right) \leq r\sigma(x,T(x)) \tag{57}$$

for every $x \in X$ and that

$$\alpha(y) = \inf\{\sigma(x,y) + \sigma(x,T(x)) : x \in X\} > 0 \tag{58}$$

for every $y \in X$ with $y \neq T(y)$. Then there exists $z \in X$ such that $z = T(z)$. Moreover, if $v = T(v)$, then $\sigma(v,v) = 0$.

Proof. Taking $S = T$ in Theorem 12, the conclusion of the corollary follows. □

Theorem 15. *Let (X,σ) be a complete metric-like space. Let S, T be mappings from X onto itself. Suppose that there exists $r > 1$ such that*

$$\begin{aligned} &\min\{\sigma(TS(x), S(x)), \sigma(ST(x), T(x))\} \\ &\qquad \geq r\max\{\sigma(Sx,x), \sigma(Tx,x)\} \end{aligned} \tag{59}$$

for every $x \in X$ and that

$$\begin{aligned} \alpha(y) = &\inf\{\sigma(x,y) + \min\{\sigma(x,S(x)), \sigma(x,T(x))\} : \\ &x \in X\} > 0 \end{aligned} \tag{60}$$

for every $y \in X$ with y that is not a common fixed point of S and T. Then there exists $z \in X$ such that $z = S(z) = T(z)$. Moreover, if $v = S(v) = T(v)$, then $\sigma(v,v) = 0$.

Proof. Let $x_0 \in X$ be arbitrary. Since S is onto, there is an element x_1 satisfying $x_1 \in S^{-1}(x_0)$. Since T is also onto, there is an element x_2 satisfying $x_2 \in T^{-1}(x_1)$. Proceeding in the same way, we can find that $x_{2n+1} \in S^{-1}(x_{2n})$ and $x_{2n+2} \in T^{-1}(x_{2n+1})$ for $n = 1,2,3,\ldots$. Therefore, $x_{2n} = Sx_{2n+1}$ and $x_{2n+1} = Tx_{2n+2}$ for $n = 0,1,2,\ldots$. If $n = 2m$, then using (59)

$$\begin{aligned} &\sigma(x_{n-1}, x_n) \\ &\quad = \sigma(x_{2m-1}, x_{2m}) \\ &\quad = \sigma(Tx_{2m}, Sx_{2m+1}) \\ &\quad = \sigma(TSx_{2m+1}, Sx_{2m+1}) \\ &\quad \geq \min\{\sigma(TS(x_{2m+1}), S(x_{2m+1})), \\ &\qquad \sigma(ST(x_{2m+1}), T(x_{2m+1}))\} \\ &\quad \geq r\max\{\sigma(Sx_{2m+1}, x_{2m+1}), \sigma(Tx_{2m+1}, x_{2m+1})\} \\ &\quad \geq r\sigma(Sx_{2m+1}, x_{2m+1}) \\ &\quad = r\sigma(x_{2m}, x_{2m+1}) \\ &\quad = r\sigma(x_n, x_{n+1}). \end{aligned} \tag{61}$$

If $n = 2m+1$, then using (59)

$$\begin{aligned}\sigma(x_{n-1}, x_n) &= \sigma(x_{2m}, x_{2m+1}) \\ &= \sigma(Sx_{2m+1}, Tx_{2m+2}) \\ &= \sigma(STx_{2m+2}, Tx_{2m+2}) \\ &\geq \min\{\sigma(TS(x_{2m+2}), S(x_{2m+2})), \\ &\qquad \sigma(ST(x_{2m+2}), T(x_{2m+2}))\} \\ &\geq r\max\{\sigma(Sx_{2m+2}, x_{2m+2}), \sigma(Tx_{2m+2}, x_{2m+2})\} \\ &\geq r\sigma(Tx_{2m+2}, x_{2m+2}) \\ &= r\sigma(x_{2m+1}, x_{2m+2}) \\ &= r\sigma(x_n, x_{n+1}).\end{aligned} \tag{62}$$

Thus for any positive integer n, it must be the case that

$$\sigma(x_{n-1}, x_n) \geq r\sigma(x_n, x_{n+1}) \tag{63}$$

which implies that

$$\sigma(x_n, x_{n+1}) \leq \frac{1}{r}\sigma(x_{n-1}, x_n) \leq \cdots \leq \left(\frac{1}{r}\right)^n \sigma(x_0, x_1). \tag{64}$$

Let $\alpha = 1/r$; then $0 < \alpha < 1$ since $r > 1$.

Now, (64) becomes

$$\sigma(x_n, x_{n+1}) \leq \alpha^n \sigma(x_0, x_1). \tag{65}$$

So, if $m > n$, then

$$\begin{aligned}\sigma(x_n, x_m) &\leq \sigma(x_n, x_{n+1}) \\ &\quad + \sigma(x_{n+1}, x_{n+2}) + \cdots + \sigma(x_{m-1}, x_m) \\ &\leq \left[\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}\right]\sigma(x_0, x_1) \\ &\leq \frac{\alpha^n}{1-\alpha}\sigma(x_0, x_1).\end{aligned} \tag{66}$$

Thus $\lim_{n,m\to\infty}\sigma(x_n, x_m) = 0$. That is, $\{x_n\}$ is a σ-Cauchy sequence in the metric-like space (X, σ). Since (X, σ) is σ-complete, there exists $z \in X$ such that

$$\sigma(z, z) = \lim_{n\to\infty}\sigma(x_n, z) = \lim_{n,m\to\infty}\sigma(x_n, x_m) = 0. \tag{67}$$

Assume that z is not a common fixed point of S and T. Then by hypothesis

$$\begin{aligned}0 &< \inf\{\sigma(x, z) + \min\{\sigma(x, S(x)), \sigma(x, T(x))\} : x \in X\} \\ &\leq \inf\{\sigma(x_n, z) + \min\{\sigma(x_n, S(x_n)), \sigma(x_n, T(x_n))\} : \\ &\qquad n \in \mathbb{N}\} \\ &\leq \inf\left\{\frac{\alpha^n}{1-\alpha}\sigma(x_0, x_1) + \sigma(x_{n-1}, x_n) : n \in \mathbb{N}\right\} \\ &\leq \inf\left\{\frac{\alpha^n}{1-\alpha}\sigma(x_0, x_1) + \alpha^{n-1}\sigma(x_0, x_1) : n \in \mathbb{N}\right\} = 0\end{aligned} \tag{68}$$

which is a contradiction. Therefore, $z = S(z) = T(z)$.

If $v = S(v) = T(v)$ for some $v \in X$, then

$$\begin{aligned}\sigma(v, v) &= \min\{\sigma(TS(v), S(v)), \sigma(ST(v), T(v))\} \\ &\geq r\max\{\sigma(S(v), v), \sigma(T(v), v)\} \\ &= r\max\{\sigma(v, v), \sigma(v, v)\} \\ &= r\sigma(v, v)\end{aligned} \tag{69}$$

which gives that $\sigma(v, v) = 0$. □

Corollary 16. *Let (X, σ) be a complete metric-like space, and let $T : X \to X$ be an onto mapping. Suppose that there exists $r \in [0, 1)$ such that*

$$\sigma(T^2(x), T(x)) \geq r\sigma(T(x), x) \tag{70}$$

for every $x \in X$ and that

$$\alpha(y) = \inf\{\sigma(x, y) + \sigma(T(x), x) : x \in X\} > 0 \tag{71}$$

for every $y \in X$ with $y \neq T(y)$. Then there exists $z \in X$ such that $z = T(z)$. Moreover, if $v = T(v)$, then $\sigma(v, v) = 0$.

Proof. Taking $S = T$ in Theorem 15, we have the desired result. □

Definition 17. Let (X, σ) and (Y, τ) be metric-like spaces. Then $f : X \to Y$ is said to be a continuous mapping, if $\lim_{n\to\infty}x_n = x$ implies that $\lim_{n\to\infty}f(x_n) = f(x)$.

Corollary 18. *Let (X, σ) be a complete metric-like space, and let T be a mapping of X into itself. If there is a real number r with $r > 1$ satisfying*

$$\sigma(T^2(x), T(x)) \geq r\sigma(T(x), x) \tag{72}$$

for every $x \in X$ and T is onto and continuous, then T has a fixed point.

Proof. Assume that there exists $y \in X$ with $y \neq T(y)$ and

$$\inf\{\sigma(x, y) + \sigma(T(x), x) : x \in X\} = 0. \tag{73}$$

Then there exists a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty}\{\sigma(x_n, y) + \sigma(T(x_n), x_n)\} = 0. \tag{74}$$

So, we have $\sigma(x_n, y) \to 0$ and $\sigma(T(x_n), x_n) \to 0$ as $n \to \infty$. Since, $\sigma(y, y) \leq \sigma(y, x_n) + \sigma(x_n, y)$, hence $\sigma(y, y) \to 0$ as $n \to \infty$. Now,

$$\sigma(T(x_n), y) \leq \sigma(T(x_n), x_n) + \sigma(x_n, y) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{75}$$

Since T is continuous, we have

$$T(y) = T\left(\lim_{n\to\infty}x_n\right) = \lim_{n\to\infty}T(x_n) = y. \tag{76}$$

This is a contradiction. Hence if $y \neq T(y)$, then

$$\inf\{\sigma(x, y) + \sigma(T(x), x) : x \in X\} > 0, \tag{77}$$

which is condition (71) of Corollary 16. By Corollary 16, there exists $z \in X$ such that $z = T(z)$. □

Now we give an example to support our result.

Example 19. Let $X = [0, \infty)$ and $\sigma(x, y) = x + y$. Define $T : X \to X$ by $T(x) = 2x$.

Obviously T is onto and continuous. Also for each $x, y \in X$, we have

$$\sigma\left(T^2 x, Tx\right) = 4x + 2x = 6x \geq r3x = r\sigma(Tx, x), \quad (78)$$

where $r = 2$. Thus T satisfies the conditions given in Corollary 18, and 0 is the fixed point of T.

Corollary 20. *Let (X, σ) be a complete metric-like space, and T be a mapping of X into itself. If there is a real number r with $r > 1$ satisfying*

$$\sigma(T(x), T(y)) \geq r \min\{\sigma(x, T(x)), \sigma(T(y), y), \sigma(x, y)\} \quad (79)$$

for every $x, y \in X$ and T is onto and continuous, then T has a fixed point.

Proof. Replacing y by $T(x)$ in (79), we obtain

$$\sigma\left(T(x), T^2(x)\right) \geq r \min\left\{\sigma(x, T(x)), \sigma\left(T^2(x), T(x)\right), \sigma(x, T(x))\right\} \quad (80)$$

for all $x \in X$.

Without loss of generality, we may assume that $T(x) \neq T^2(x)$. Otherwise T has a fixed point. Since $r > 1$, it follows from (80) that

$$\sigma\left(T^2(x), T(x)\right) \geq r\sigma(T(x), x) \quad (81)$$

for every $x \in X$. By the argument similar to that used in Corollary 18, we can prove that if $y \neq T(y)$, then

$$\inf\{\sigma(x, y) + \sigma(T(x), x) : x \in X\} > 0, \quad (82)$$

which is condition (71) of Corollary 16. So, Corollary 16 applies to obtain a fixed point of T. □

According to Theorem 12, we get the following result.

Corollary 21 (see [17, Theorem 1]). *Let (X, p) be a complete partial metric space. Let $S, T : X \to X$ be two self-mappings. Suppose that there exists $r \in [0, 1)$ such that*

$$\max\{p(S(x), TS(x)), p(T(x), ST(x))\} \leq r \min\{p(x, S(x)), p(x, T(x))\} \quad (83)$$

for every $x \in X$ and that

$$\alpha(y) = \inf\{p(x, y) + \min\{p(x, S(x)), p(x, T(x))\} : x \in X\} > 0 \quad (84)$$

for every $y \in X$ with y that is not a common fixed point of S and T. Then there exists $z \in X$ such that $z = S(z) = T(z)$. Moreover, if $v = S(v) = T(v)$, then $p(v, v) = 0$.

Proof. Using a similar argument given in the Theorem 12 for $\sigma(x, y) = p(x, y)$, the desired result is obtained, where p is a partial metric on X. □

Also, according to Theorem 15, we get Theorem 2 from [17].

References

[1] A. Aghajani, S. Radenović, and J. R. Roshan, "Common fixed point results for four mappings satisfying almost generalized (S, T)-contractive condition in partially ordered metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5665–5670, 2012.

[2] A. Amini-Harandi, "Metric-like spaces, partial metric spaces and fixed points," *Fixed Point Theory and Applications*, vol. 2012, article 204, 10 pages, 2012.

[3] D. Đorić, Z. Kadelburg, and S. Radenović, "Edelstein-Suzuki-type fixed point results in metric and abstract metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 1927–1932, 2012.

[4] N. Hussain, D. Đorić, Z. Kadelburg, and S. Radenović, "Suzuki-type fixed point results in metric type spaces," *Fixed Point Theory and Applications*, vol. 2012, article 126, 12 pages, 2012.

[5] N. Hussain and M. H. Shah, "KKM mappings in cone b-metric spaces," *Computers & Mathematics with Applications*, vol. 62, no. 4, pp. 1677–1684, 2011.

[6] N. Hussain, M. H. Shah, A. Amini-Harandi, and Z. Akhtar, "Common fixed point theorems for generalized contractive mappings with applications," *Fixed Point Theory and Applications*, vol. 2013, article 169, 17 pages, 2013.

[7] N. Hussain, Z. Kadelburg, S. Radenović, and F. Al-Solamy, "Comparison functions and fixed point results in partial metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 605781, 15 pages, 2012.

[8] M. Kikkawa and T. Suzuki, "Some similarity between contractions and Kannan mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 649749, 8 pages, 2008.

[9] M. Kikkawa and T. Suzuki, "Some notes on fixed point theorems with constants," *Bulletin of the Kyushu Institute of Technology*, no. 56, pp. 11–18, 2009.

[10] M. Kikkawa and T. Suzuki, "Three fixed point theorems for generalized contractions with constants in complete metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 2942–2949, 2008.

[11] M. A. Kutbi, J. Ahmad, N. Hussain, and M. Arshad, "Common fixed point results for mappings with rational expressions," *Abstract and Applied Analysis*, vol. 2013, Article ID 549518, 11 pages, 2013.

[12] S. G. Matthews, "Partial metric topology," in *Papers on General Topology and Applications*, vol. 728 of *Annals of the New York Academy of Sciences*, pp. 183–197, New York Academy of Sciences, New York, NY, USA, 1994.

[13] V. Parvaneh, J. R. Roshan, and S. Radenović, "Existence of tripled coincidence points in ordered b-metric spaces and an application to a system of integral equations," *Fixed Point Theory and Applications*, vol. 2013, article 130, 19 pages, 2013.

[14] O. Popescu, "Fixed point theorem in metric spaces," *Bulletin of the Transilvania University of Brașov*, vol. 150, pp. 479–482, 2008.

[15] O. Popescu, "Two fixed point theorems for generalized contractions with constants in complete metric space," *Central European Journal of Mathematics*, vol. 7, no. 3, pp. 529–538, 2009.

[16] J. R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei, and W. Shatanawi, "Common fixed points of almost generalized $(\psi, \varphi)_s$-contractive mappings in ordered b-metric spaces," *Fixed Point Theory and Applications*, vol. 2013, article 159, 23 pages, 2013.

[17] S. Sedghi and N. Shobkolaei, "Common fixed point of maps in complete partial metric spaces," *East Asian Mathematical Journal*, vol. 29, no. 1, pp. 1–12, 2013.

[18] N. Shobkolaei, S. Sedghi, J. R. Roshan, and I. Altun, "Common fixed point of mappings satisfying almost generalized (S, T)-contractive condition in partially ordered partial metric spaces," *Applied Mathematics and Computation*, vol. 219, no. 2, pp. 443–452, 2012.

[19] T. Suzuki, "A generalized Banach contraction principle that characterizes metric completeness," *Proceedings of the American Mathematical Society*, vol. 136, no. 5, pp. 1861–1869, 2008.

[20] T. Suzuki, "A new type of fixed point theorem in metric spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 11, pp. 5313–5317, 2009.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 167514, 13 pages
http://dx.doi.org/10.1155/2013/167514

Research Article

Maximal and Area Integral Characterizations of Bergman Spaces in the Unit Ball of $\mathbb{C}^n$

Zeqian Chen[1] **and Wei Ouyang**[1,2]

[1] *Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, 30 West District, Xiao-Hong-Shan, Wuhan 430071, China*
[2] *Graduate University of Chinese Academy of Sciences, Beijing 100049, China*

Correspondence should be addressed to Zeqian Chen; zqchen@wipm.ac.cn

Received 15 May 2013; Accepted 17 July 2013

Academic Editor: Pei De Liu

We present maximal and area integral characterizations of Bergman spaces in the unit ball of $\mathbb{C}^n$. The characterizations are in terms of maximal functions and area integral functions on Bergman balls involving the radial derivative, the complex gradient, and the invariant gradient. As an application, we obtain new maximal and area integral characterizations of Besov spaces. Moreover, we give an atomic decomposition of real-variable type with respect to Carleson tubes for Bergman spaces.

1. Introduction and Main Results

Let $\mathbb{C}$ denote the set of complex numbers. Throughout the paper we fix a positive integer n, and let

$$\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C} \tag{1}$$

denote the Euclidean space of complex dimension n. Addition, scalar multiplication, and conjugation are defined on $\mathbb{C}^n$ component wise. For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write

$$\langle z, w \rangle = z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n, \tag{2}$$

where $\overline{w}_k$ is the complex conjugate of w_k. We also write

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}. \tag{3}$$

The open unit ball in $\mathbb{C}^n$ is the set

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}. \tag{4}$$

The boundary of $\mathbb{B}_n$ will be denoted by $\mathbb{S}_n$ and is called the unit sphere in $\mathbb{C}^n$, that is,

$$\mathbb{S}_n = \{z \in \mathbb{C}^n : |z| = 1\}. \tag{5}$$

Also, we denote by $\overline{\mathbb{B}}_n$ the closed unit ball, that is,

$$\overline{\mathbb{B}}_n = \{z \in \mathbb{C}^n : |z| \le 1\} = \mathbb{B}_n \cup \mathbb{S}_n. \tag{6}$$

The automorphism group of $\mathbb{B}^n$, denoted by $\mathrm{Aut}(\mathbb{B}^n)$, consists of all biholomorphic mappings of $\mathbb{B}^n$. Traditionally, biholomorphic mappings are also called automorphisms.

For $\alpha \in \mathbb{R}$, the weighted Lebesgue measure dv_α on $\mathbb{B}_n$ is defined by

$$dv_\alpha(z) = c_\alpha \left(1 - |z|^2\right)^\alpha dv(z), \tag{7}$$

where $c_\alpha = 1$ for $\alpha \le -1$ and $c_\alpha = \Gamma(n + \alpha + 1)/[n!\Gamma(\alpha + 1)]$ if $\alpha > -1$, which is a normalizing constant so that dv_α is a probability measure on $\mathbb{B}_n$. In the case of $\alpha = -(n + 1)$, we denote the resulting measure by

$$d\tau(z) = \frac{dv}{\left(1 - |z|^2\right)^{n+1}} \tag{8}$$

and call it the invariant measure on $\mathbb{B}^n$, since $d\tau = d\tau \circ \varphi$ for any automorphism φ of $\mathbb{B}^n$.

For $\alpha > -1$ and $p > 0$, the (weighted) Bergman space $\mathcal{A}^p_\alpha$ consists of holomorphic functions f in $\mathbb{B}_n$ with

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \right)^{1/p} < \infty, \tag{9}$$

where the weighted Lebesgue measure dv_α on $\mathbb{B}_n$ is defined by

$$dv_\alpha(z) = c_\alpha \left(1 - |z|^2\right)^\alpha dv(z), \tag{10}$$

and $c_\alpha = \Gamma(n+\alpha+1)/[n!\Gamma(\alpha+1)]$ is a normalizing constant so that dv_α is a probability measure on $\mathbb{B}_n$. Thus,

$$\mathcal{A}^p_\alpha = \mathcal{H}(\mathbb{B}_n) \cap L^p(\mathbb{B}_n, dv_\alpha), \tag{11}$$

where $\mathcal{H}(\mathbb{B}_n)$ is the space of all holomorphic functions in $\mathbb{B}_n$. When $\alpha = 0$, we simply write $\mathcal{A}^p$ for $\mathcal{A}^p_0$. These are the usual Bergman spaces. Note that for $1 \le p < \infty$, $\mathcal{A}^p_\alpha$ is a Banach space under the norm $\|\ \|_{p,\alpha}$. If $0 < p < 1$, the space $\mathcal{A}^p_\alpha$ is a quasi-Banach space with p-norm $\|f\|^p_{p,\alpha}$.

Recall that $D(z,\gamma)$ denotes the Bergman metric ball at z

$$D(z,\gamma) = \{w \in \mathbb{B}_n : \beta(z,w) < \gamma\}, \tag{12}$$

with $\gamma > 0$, where β is the Bergman metric on $\mathbb{B}_n$. It is known that

$$\beta(z,w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}, \quad z, w \in \mathbb{B}_n, \tag{13}$$

whereafter φ_z is the bijective holomorphic mapping in $\mathbb{B}_n$, which satisfies $\varphi_z(0) = z$, $\varphi_z(z) = 0$, and $\varphi_z \circ \varphi_z = \mathrm{id}$.

As is well known, maximal functions play a crucial role in the real-variable theory of Hardy spaces (cf. [1]). In this paper, we first establish a maximal-function characterization for the Bergman spaces. To this end, we define for each $\gamma > 0$ and $f \in \mathcal{H}(\mathbb{B}_n)$:

$$\left(M_\gamma f\right)(z) = \sup_{w \in D(z,\gamma)} |f(w)|, \quad \forall z \in \mathbb{B}_n. \tag{14}$$

Then we have the following result.

Theorem 1. *Suppose $\gamma > 0$ and $\alpha > -1$. Let $0 < p < \infty$. Then for any $f \in \mathcal{H}(\mathbb{B}_n)$, $f \in \mathcal{A}^p_\alpha$ if and only if $M_\gamma f \in L^p(\mathbb{B}_n, dv_\alpha)$. Moreover,*

$$\|f\|_{p,\alpha} \approx \left\|M_\gamma f\right\|_{p,\alpha}, \tag{15}$$

where "≈" depends only on γ, α, p, and n.

The norm appearing on the right-hand side of (15) can be viewed as an analogue of the so-called nontangential maximal function in Hardy spaces. The proof of Theorem 1 is fairly elementary (see Section 2), using some basic facts and estimates on the Bergman balls.

In order to state the area integral characterizations of the Bergman spaces, we require some more notation. For any $f \in \mathcal{H}(\mathbb{B}_n)$ and $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$, we define

$$\mathcal{R} f(z) = \sum_{k=1}^n z_k \frac{\partial f(z)}{\partial z_k} \tag{16}$$

and call it the radial derivative of f at z. The complex and invariant gradients of f at z are, respectively, defined as

$$\nabla f(z) = \left(\frac{\partial f(z)}{\partial z_1}, \ldots, \frac{\partial f(z)}{\partial z_n} \right), \qquad \widetilde{\nabla} f(z) = \nabla (f \circ \varphi_z)(0). \tag{17}$$

Now, for fixed $1 < q < \infty$ and $\gamma > 0$, we define for each $f \in \mathcal{H}(\mathbb{B}_n)$ and $z \in \mathbb{B}_n$:

(1) the radial area integral function

$$A^{\gamma,q}_{\mathcal{R}}(f)(z) = \left(\int_{D(z,\gamma)} |(1 - |w|^2) \mathcal{R} f(w)|^q d\tau(w) \right)^{1/q}, \tag{18}$$

(2) the complex gradient area integral function

$$A^{\gamma,q}_{\nabla}(f)(z) = \left(\int_{D(z,\gamma)} |(1 - |w|^2) \nabla f(w)|^q d\tau(w) \right)^{1/q}, \tag{19}$$

(3) the invariant gradient area integral function

$$A^{\gamma,q}_{\widetilde{\nabla}}(f)(z) = \left(\int_{D(z,\gamma)} |\widetilde{\nabla} f(w)|^q d\tau(w) \right)^{1/q}. \tag{20}$$

We state the second main result of this paper as follows.

Theorem 2. *Suppose $1 < q < \infty$, $\gamma > 0$, and $\alpha > -1$. Let $0 < p < \infty$. Then, for any $f \in \mathcal{H}(\mathbb{B}_n)$, the following conditions are equivalent:*

(a) $f \in \mathcal{A}^p_\alpha$,
(b) *$A^{\gamma,q}_{\mathcal{R}}(f)$ is in $L^p(\mathbb{B}_n, dv_\alpha)$,*
(c) *$A^{\gamma,q}_{\nabla}(f)$ is in $L^p(\mathbb{B}_n, dv_\alpha)$,*
(d) *$A^{\gamma,q}_{\widetilde{\nabla}}(f)$ is in $L^p(\mathbb{B}_n, dv_\alpha)$.*

Moreover, the quantities

$$\left\|A^{\gamma,q}_{\mathcal{R}}(f)\right\|_{p,\alpha}, \qquad \left\|A^{\gamma,q}_{\nabla}(f)\right\|_{p,\alpha}, \qquad \left\|A^{\gamma,q}_{\widetilde{\nabla}}(f)\right\|_{p,\alpha} \tag{21}$$

are all comparable to $\|f - f(0)\|_{p,\alpha}$, where the comparable constants depend only on q, γ, α, p, and n.

For $0 < p < \infty$ and $-\infty < \alpha < \infty$, we fix a nonnegative integer k with $pk + \alpha > -1$ and define the generalized Bergman space $\mathcal{A}^p_\alpha$ as introduced in [2] to be the space of all $f \in \mathcal{H}(\mathbb{B}_n)$ such that $(1-|z|^2)^k \mathcal{R}^k f \in L^p(\mathbb{B}_n, dv_\alpha)$. One then easily observes that $\mathcal{A}^p_\alpha$ is independent of the choice of k and consistent with the traditional definition when $\alpha > -1$. Let N

be the smallest nonnegative integer such that $pN + \alpha > -1$. Put

$$\|f\|_{p,\alpha} = |f(0)| + \left(\int_{\mathbb{B}_n} \left(1-|z|^2\right)^{pN} \left|\mathscr{R}^N f(z)\right|^p dv_\alpha(z)\right)^{1/p}, \quad f \in \mathscr{A}^p_\alpha. \tag{22}$$

Equipped with (22), $\mathscr{A}^p_\alpha$ becomes a Banach space when $p \geq 1$ and a quasi-Banach space for $0 < p < 1$.

It is known that the family of the generalized Bergman spaces $\mathscr{A}^p_\alpha$ covers most of the spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$, such as the classical diagonal Besov space B^s_p and the Sobolev space $W^p_{k,\beta}$, which has been extensively studied before in the literature under different names (see e.g., [2] for an overview).

There are various characterizations for B^s_p or $W^p_{k,\beta}$ involving complex-variable quantities in terms of radical derivatives, complex and invariant gradients, and fractional differential operators (for a review and details see [2] and references therein). However, as an application of Theorems 1 and 2, we obtain new maximal and area integral characterizations of the Besov spaces as follows, which can be considered as a unified characterization for such spaces involving real-variable quantities.

Corollary 3. *Suppose $\gamma > 0$ and $\alpha \in \mathbb{R}$. Let $0 < p < \infty$ and k a positive integer such that $pk + \alpha > -1$. Then for any $f \in \mathscr{H}(\mathbb{B}_n)$, $f \in \mathscr{A}^p_\alpha$ if and only if $M^{(k)}_\gamma(f) \in L^p(\mathbb{B}_n, dv_\alpha)$, where*

$$M^{(k)}_\gamma(f)(z) = \sup_{w \in D(z,\gamma)} \left|\left(1-|w|^2\right)^k \mathscr{R}^k f(w)\right|, \quad z \in \mathbb{B}_n. \tag{23}$$

Moreover,

$$\|f - f(0)\|_{p,\alpha} \approx \left\|M_\gamma(\mathscr{R}^k f)\right\|_{p,\alpha}, \tag{24}$$

where "≈" depends only on γ, α, p, k, and n.

Corollary 4. *Suppose $1 < q < \infty$, $\gamma > 0$, and $\alpha \in \mathbb{R}$. Let $0 < p < \infty$ and k a nonnegative integer such that $pk+\alpha > -1$. Then for any $f \in \mathscr{H}(\mathbb{B}_n)$, $f \in \mathscr{A}^p_\alpha$ if and only if $A^{\gamma,q}_{\mathscr{R}^{k+1}}(f)$ is in $L^p(\mathbb{B}_n, dv_\alpha)$, where*

$$A^{\gamma,q}_{\mathscr{R}^{k+1}}(f)(z) = \left(\int_{D(z,\gamma)} \left|\left(1-|w|^2\right)^{k+1} \mathscr{R}^{k+1} f(w)\right|^q d\tau(w)\right)^{1/q}. \tag{25}$$

Moreover,

$$\|f - f(0)\|_{p,\alpha} \approx \left\|A^{\gamma,q}_{\mathscr{R}^{k+1}}(f)\right\|_{p,\alpha}, \tag{26}$$

where "≈" depends only on q, γ, α, p, k, and n.

To prove Corollaries 3 and 4, one merely notices that $f \in \mathscr{A}^p_\alpha$ if and only if $\mathscr{R}^k f \in L^p(\mathbb{B}_n, dv_{\alpha+pk})$ and applies Theorems 1 and 2, respectively, to $\mathscr{R}^k f$ with the help of Lemma 5 below.

The paper is organized as follows. In Section 2 we will prove Theorems 1 and 2. An atomic decomposition of real-variable type with respect to Carleson tubes for Bergman spaces will be presented in Section 3 via duality method. Finally, in Section 4, we will prove Theorem 2 through using the real-variable atomic decomposition of Bergman spaces established in the preceding section.

In what follows, C always denotes a constant depending (possibly) on n, q, p, γ, or α but not on f, which may be different in different places. For two nonnegative (possibly infinite) quantities X and Y, by $X \lesssim Y$ we mean that there exists a constant $C > 0$ such that $X \leq CY$ and by $X \approx Y$ that $X \lesssim Y$ and $Y \lesssim X$. Any notation and terminology not otherwise explained are as used in [3] for spaces of holomorphic functions in the unit ball of $\mathbb{C}^n$.

2. Proofs of Theorems 1 and 2

For the sake of convenience, we collect some elementary facts on the Bergman metric and holomorphic functions in the unit ball of $\mathbb{C}^n$ as follows.

Lemma 5 (cf. [3, Lemma 2.20]). *For each $\gamma > 0$,*

$$1 - |a|^2 \approx 1 - |z|^2 \approx |1 - \langle a, z\rangle|, \tag{27}$$

for all a and z in $\mathbb{B}_n$ with $\beta(a,z) < \gamma$.

Lemma 6 (cf. [3, Lemma 2.24]). *Suppose $\gamma > 0$, $p > 0$, and $\alpha > -1$. Then there exists a constant $C > 0$ such that for any $f \in \mathscr{H}(\mathbb{B}_n)$,*

$$|f(z)|^p \leq \frac{C}{\left(1-|z|^2\right)^{n+1+\alpha}} \int_{D(z,\gamma)} |f(w)|^p dv_\alpha(w), \quad \forall z \in \mathbb{B}_n. \tag{28}$$

Lemma 7 (cf. [3, Lemma 2.27]). *For each $\gamma > 0$,*

$$|1 - \langle z, u\rangle| \approx |1 - \langle z, v\rangle|, \tag{29}$$

for all z in $\overline{\mathbb{B}}_n$ and u, v in $\mathbb{B}_n$ with $\beta(u,v) < \gamma$.

2.1. Proof of Theorem 1. One needs the following result (cf. [4, Lemma 5]).

Lemma 8. *For fixed $\gamma > 0$, there exist a positive integer N and a sequence $\{a_k\}$ in $\mathbb{B}_n$ such that*

(1) *$\mathbb{B}_n = \bigcup_k D(a_k, \gamma)$, and*

(2) *each $z \in \mathbb{B}_n$ belongs to at most N of the sets $D(a_k, 3\gamma)$.*

Proof of Theorem 1. Let $p > 0$. By Lemmas 8, 6, and 5, one has

$$\begin{aligned}
&\int_{\mathbb{B}_n} \left|M_\gamma(f)(z)\right|^p dv_\alpha(z) \\
&\leq \sum_k \int_{D(a_k,\gamma)} \left|M_\gamma(f)(z)\right|^p dv_\alpha(z) \\
&= \sum_k \int_{D(a_k,\gamma)} \sup_{w\in D(z,\gamma)} |f(w)|^p dv_\alpha(z) \\
&\lesssim \sum_k \int_{D(a_k,\gamma)} \sup_{w\in D(z,\gamma)} \frac{1}{\left(1-|w|^2\right)^{n+1+\alpha}} \\
&\qquad \times \int_{D(w,\gamma)} |f(u)|^p dv_\alpha(u)\, dv_\alpha(z) \\
&\lesssim \sum_k \int_{D(a_k,\gamma)} \left(\frac{1}{\left(1-|a_k|^2\right)^{n+1+\alpha}}\right. \\
&\qquad\qquad \left.\times \int_{D(a_k,3\gamma)} |f(u)|^p dv_\alpha(u)\right) dv_\alpha(z) \\
&\lesssim \sum_k \int_{D(a_k,3\gamma)} |f(u)|^p dv_\alpha(u) \\
&\lesssim N \int_{\mathbb{B}_n} |f(u)|^p dv_\alpha(u),
\end{aligned} \tag{30}$$

where N is the constant in Lemma 8 depending only on γ and n. □

2.2. Proof of Theorem 2. Recall that $\mathscr{B}(\mathbb{B}_n)$ is defined as the space of all $f \in \mathscr{H}(\mathbb{B}_n)$ so that

$$\|f\|_{\mathscr{B}} = \sup_{z\in\mathbb{B}_n} \left|\widetilde{\nabla} f(z)\right| < \infty. \tag{31}$$

$\mathscr{B}(\mathbb{B}_n)$ with the norm $\|f\| = |f(0)| + \|f\|_{\mathscr{B}}$ is a Banach space and called the Bloch space. Then, the following interpolation result holds.

Lemma 9 (cf. [3, Theorem 3.25]). *Let $1 < p < \infty$. Suppose $\alpha > -1$ and*

$$\frac{1}{p} = \frac{1-\theta}{p'}, \tag{32}$$

for $0 < \theta < 1$ and $1 \leq p' < \infty$. Then

$$\mathscr{A}^p_\alpha(\mathbb{B}_n) = \left[\mathscr{A}^{p'}_\alpha(\mathbb{B}_n), \mathscr{B}(\mathbb{B}_n)\right]_\theta, \tag{33}$$

with equivalent norms.

Moreover, to prove Theorem 2 for the case $0 < p \leq 1$, one will use atom decomposition for Bergman spaces due to Coifman and Rochberg [5] (see also [3, Theorem 2.30]) as follows.

Proposition 10. *Suppose $p > 0$, $\alpha > -1$, and $b > n\max\{1, 1/p\} + (\alpha+1)/p$. Then there exists a sequence $\{a_k\}$ in $\mathbb{B}_n$ such that $\mathscr{A}^p_\alpha$ consists exactly of functions of the form*

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{\left(1-|a_k|^2\right)^{(pb-n-1-\alpha)/p}}{\left(1-\langle z, a_k\rangle\right)^b}, \quad z \in \mathbb{B}_n, \tag{34}$$

where $\{c_k\}$ belongs to the sequence space ℓ^p and the series converges in the norm topology of $\mathscr{A}^p_\alpha$. Moreover,

$$\int_{\mathbb{B}_n} |f(z)|^p dv_\alpha(z) \approx \inf\left\{\sum_k |c_k|^p\right\}, \tag{35}$$

where the infimum runs over all the above decompositions.

Also, we need a characterization of Carleson type measures for Bergman spaces as follows, which can be found in [2, Theorem 45].

Proposition 11. *Suppose $n+1+\alpha > 0$ and μ is a positive Borel measure on $\mathbb{B}_n$. Then, there exists a constant $C > 0$ such that*

$$\mu(Q_r(\zeta)) \leq C r^{2(n+1+\alpha)}, \quad \forall \zeta \in \mathbb{S}_n,\ r > 0, \tag{36}$$

if and only if for each $s > 0$, there exists a constant $C > 0$ such that

$$\int_{\mathbb{B}_n} \frac{\left(1-|z|^2\right)^s}{|1-\langle z, w\rangle|^{n+1+\alpha+s}} d\mu(w) \leq C, \tag{37}$$

for all $z \in \mathbb{B}_n$.

We are now ready to prove Theorem 2. Note that for any $f \in \mathscr{H}(\mathbb{B}_n)$,

$$\begin{aligned}
\left(1-|z|^2\right)|\mathscr{R}f(z)| &\leq \left(1-|z|^2\right)|\nabla f(z)| \\
&\leq \left|\widetilde{\nabla} f(z)\right|, \quad \forall z \in \mathbb{B}_n
\end{aligned} \tag{38}$$

(cf. [3, Lemma 2.14]). We have that (d) implies (c) and (c) implies (b) in Theorem 2. Then, it remains to prove that (b) implies (a) and (a) implies (d).

Proof of (b) ⇒ (a). Since $\mathscr{R}f(z)$ is holomorphic, by Lemma 6 we have

$$\begin{aligned}
|\mathscr{R}f(z)|^q &\leq \frac{C}{\left(1-|z|^2\right)^{n+1}} \int_{D(z,\gamma)} |\mathscr{R}f(w)|^q dv(w) \\
&\leq C_\gamma \int_{D(z,\gamma)} |\mathscr{R}f(w)|^q d\tau(w).
\end{aligned} \tag{39}$$

Then,

$$\begin{aligned}&\left(1-|z|^2\right)|\mathscr{R}f(z)|\\&\quad\le C\left(1-|z|^2\right)\left(\int_{D(z,\gamma)}|\mathscr{R}f(w)|^q d\tau(w)\right)^{1/q}\\&\quad\le C_\gamma\left(\int_{D(z,\gamma)}\left|(1-|w|^2)\mathscr{R}f(w)\right|^q d\tau(w)\right)^{1/q}\\&\quad=C_\gamma A_{\mathscr{R}}^{\gamma,q}(f)(z).\end{aligned}\tag{40}$$

Hence, for any $p>0$, if $A_{\mathscr{R}}^{\gamma,q}(f)\in L^p(\mathbb{B}_n,dv_\alpha)$ then $(1-|z|^2)|\mathscr{R}f(z)|$ is in $L^p(\mathbb{B}_n,dv_\alpha)$, which implies that $f\in\mathscr{A}_\alpha^p$ (cf. [3, Theorem 2.16]). □

The proof of (a) ⇒ (d) is divided into two steps. We first prove the case $0<p\le 1$ using the atomic decomposition and then the remaining case via complex interpolation.

Proof of (a) ⇒ (d) for $0<p\le 1$. To this end, we write

$$f_k(z)=\frac{\left(1-|a_k|^2\right)^{(pb-n-1-\alpha)/p}}{\left(1-\langle z,a_k\rangle\right)^b}.\tag{41}$$

An immediate computation yields that

$$\begin{aligned}\nabla f_k(z)&=\frac{b\overline{a}_k\left(1-|a_k|^2\right)^{(pb-n-1-\alpha)/p}}{\left(1-\langle z,a_k\rangle\right)^{b+1}},\\\mathscr{R}f_k(z)&=\frac{b\langle z,a_k\rangle\left(1-|a_k|^2\right)^{(pb-n-1-\alpha)/p}}{\left(1-\langle z,a_k\rangle\right)^{b+1}}.\end{aligned}\tag{42}$$

Then we have

$$\begin{aligned}&\left|\widetilde{\nabla}f_k(z)\right|^2\\&\quad=\left(1-|z|^2\right)\left(|\nabla f_k(z)|^2-|\mathscr{R}f_k(z)|^2\right)\\&\quad=b^2\left(1-|z|^2\right)\left(1-|a_k|^2\right)^{2(pb-n-1-\alpha)/p}\frac{|a_k|^2-|\langle z,a_k\rangle|^2}{|1-\langle z,a_k\rangle|^{2(b+1)}}.\end{aligned}\tag{43}$$

By Lemmas 5 and 7, one has

$$\begin{aligned}A_{\widetilde{\nabla}}^{\gamma,q}(f_k)(z)&=\left(\int_{D(z,\gamma)}\left|\widetilde{\nabla}f_k(w)\right|^q d\tau(w)\right)^{1/q}\\&\le b\left(1-|a_k|^2\right)^{(pb-n-1-\alpha)/p}\\&\quad\times\left(\int_{D(z,\gamma)}\frac{1}{|1-\langle w,a_k\rangle|^{qb}}d\tau(w)\right)^{1/q}\\&\le\frac{C_\gamma b\left(1-|a_k|^2\right)^{(pb-n-1-\alpha)/p}}{|1-\langle z,a_k\rangle|^b},\end{aligned}\tag{44}$$

where we have used **a** the fact $v(D(z,\gamma))\approx(1-|z|^2)^{n+1}$. Note that $v_\alpha(Q_r)\approx r^{2(n+1+\alpha)}$ (cf. [3, Corollary 5.24]); by Proposition 11 we have

$$\begin{aligned}&\int_{\mathbb{B}_n}\left|A_{\widetilde{\nabla}}^{\gamma,q}(f_k)(z)\right|^p dv_\alpha(z)\\&\quad\le Cb^p\int_{\mathbb{B}_n}\frac{\left(1-|a_k|^2\right)^{(pb-n-1-\alpha)}}{|1-\langle z,a_k\rangle|^{pb}}dv_\alpha(z)\\&\quad\le C_{p,\alpha}.\end{aligned}\tag{45}$$

Hence, for $0<p\le 1$, we have for $f=\sum_{k=1}^\infty c_kf_k$ with $\sum_k|c_k|^p<\infty$,

$$\begin{aligned}&\int_{\mathbb{B}_n}\left|A_{\widetilde{\nabla}}^{\gamma,q}(f)(z)\right|^p dv_\alpha\\&\quad\le\sum_{k=1}^\infty|c_k|^p\int_{\mathbb{B}_n}\left|A_{\widetilde{\nabla}}^{\gamma,q}(f_k)(z)\right|^p dv_\alpha\\&\quad\le C_{p,\alpha}\sum_{k=1}^\infty|c_k|^p.\end{aligned}\tag{46}$$

This concludes that

$$\begin{aligned}&\int_{\mathbb{B}_n}\left|A_{\widetilde{\nabla}}^{\gamma,q}(f)(z)\right|^p dv_\alpha\\&\quad\le C_{p,\alpha}\inf\left\{\sum_{k=1}^\infty|c_k|^p\right\}\\&\quad\le C_{p,\alpha}\int_{\mathbb{B}_n}|f(z)|^p dv_\alpha(z).\end{aligned}\tag{47}$$

The proof is complete. □

Proof of (a) ⇒ (d) for $p>1$. Set $E=L^q(\mathbb{B}_n,\chi_{D(0,\gamma)}d\tau;\mathbb{C}^n)$. Consider the operator

$$T(f)(z,w)=\left(\widetilde{\nabla}f\right)(\varphi_z(w)),\quad f\in\mathscr{H}(\mathbb{B}_n).\tag{48}$$

Note that $\varphi_z(D(0,\gamma))=D(z,\gamma)$ and the measure $d\tau$ is invariant under any automorphism of $\mathbb{B}_n$ (cf. [3, Proposition 1.13]); we have

$$\begin{aligned}&\|T(f)(z)\|_E\\&\quad=\left(\int_{\mathbb{B}_n}\left|\left(\widetilde{\nabla}f\right)(\varphi_z(w))\right|^q\chi_{D(0,\gamma)}(w)\,d\tau(w)\right)^{1/q}\\&\quad=\left(\int_{\mathbb{B}_n}\left|\widetilde{\nabla}f(w)\right|^q\chi_{D(z,\gamma)}(w)d\tau(w)\right)^{1/q}\\&\quad=A_{\widetilde{\nabla}}^{\gamma,q}(f)(z).\end{aligned}\tag{49}$$

On the other hand,

$$A_{\tilde{\nabla}}^{\gamma,q}(f)(z) \le \left[C_\gamma\left(1-|z|^2\right)^{-n-1}\nu(D(z,\gamma))\right]^{1/2}\|f\|_{\mathscr{B}} \le C\|f\|_{\mathscr{B}}. \tag{50}$$

This follows that T is bounded from $\mathscr{B}$ into $L_\alpha^\infty(\mathbb{B}_n, E)$. Notice that we have proved that T is bounded from $\mathscr{A}_\alpha^1$ to $L_\alpha^1(\mathbb{B}_n, E)$. Thus, by Lemma 9 and the well known fact that

$$L_p(\mathbb{B}_n, E) = \left(L_\alpha^1(\mathbb{B}_n, E), L_\alpha^\infty(\mathbb{B}_n, E)\right)_\theta \quad \text{with } \theta = 1 - \frac{1}{p}, \tag{51}$$

we conclude that T is bounded from $\mathscr{A}_\alpha^p$ into $L_\alpha^p(\mathbb{B}_n, E)$ for any $1 < p < \infty$; that is,

$$\left\|A_{\tilde{\nabla}}^{\gamma,q}(f)\right\|_{p,\alpha} \le C\|f\|_{p,\alpha}, \quad \forall f \in \mathscr{A}_\alpha^p, \tag{52}$$

where C depends only on q, γ, n, p, and α. The proof is complete. □

Remark 12. From the proofs of that (b) ⇒ (a) and that (a) ⇒ (d) for $p > 1$, we find that Theorem 2 still holds true for the Bloch space. That is, for any $f \in \mathscr{H}(\mathbb{B}_n)$, $f \in \mathscr{B}$ if and only if one (or equivalently, all) of $A_{\mathscr{R}}^{\gamma,q}(f)$, $A_{\nabla}^{\gamma,q}(f)$, and $A_{\tilde{\nabla}}^{\gamma,q}(f)$ is (or, are) in $L^\infty(\mathbb{B}_n)$. Moreover,

$$\begin{aligned}\|f\|_{\mathscr{B}} &\approx \left\|A_{\mathscr{R}}^{\gamma,q}(f)\right\|_{L^\infty(\mathbb{B}_n)} \\ &\approx \left\|A_{\nabla}^{\gamma,q}(f)\right\|_{L^\infty(\mathbb{B}_n)} \\ &\approx \left\|A_{\tilde{\nabla}}^{\gamma,q}(f)\right\|_{L^\infty(\mathbb{B}_n)},\end{aligned} \tag{53}$$

where "≈" depends only on q, γ, and n.

3. Atomic Decomposition for Bergman Spaces

We let

$$d(z,w) = |1 - \langle z, w\rangle|^{1/2}, \quad z, w \in \overline{\mathbb{B}}_n. \tag{54}$$

It is known that d satisfies the triangle inequality and the restriction of d to $\mathbb{S}_n$ is a metric. As usual, d is called the nonisotropic metric.

For any $\zeta \in \mathbb{S}_n$ and $r > 0$, the set

$$Q_r(\zeta) = \{z \in \mathbb{B}_n : d(z,\zeta) < r\} \tag{55}$$

is called a Carleson tube with respect to the nonisotropic metric d. We usually write $Q = Q_r(\zeta)$ in short.

As usual, we define the atoms with respect to the Carleson tube as follows: for $1 < q < \infty$, $a \in L^q(\mathbb{B}_n, dv_\alpha)$ is said to be a $(1,q)_\alpha$-atom if there is a Carleson tube Q such that

(1) a is supported in Q;

(2) $\|a\|_{L^q(\mathbb{B}_n, dv_\alpha)} \le v_\alpha(Q)^{(1/q)-1}$;

(3) $\int_{\mathbb{B}_n} a(z)\, dv_\alpha(z) = 0$.

The constant function (1) is also considered to be a $(1,q)_\alpha$-atom. Note that for any $(1,q)_\alpha$-atom a,

$$\|a\|_{1,\alpha} = \int_Q |a|\, dv_\alpha \le v_\alpha(Q)^{1-1/q}\|a\|_{q,\alpha} \le 1. \tag{56}$$

Recall that P_α is the orthogonal projection from $L^2(\mathbb{B}_n, dv_\alpha)$ onto $\mathscr{A}_\alpha^2$, which can be expressed as

$$P_\alpha f(z) = \int_{\mathbb{B}_n} K^\alpha(z,w) f(w)\, dv_\alpha(w), \quad \forall f \in L^1(\mathbb{B}_n, dv_\alpha),\ \alpha > -1, \tag{57}$$

where

$$K^\alpha(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}, \quad z, w \in \mathbb{B}_n. \tag{58}$$

P_α extends to a bounded projection from $L^p(\mathbb{B}_n, dv_\alpha)$ onto $\mathscr{A}_\alpha^p$ $(1 < p < \infty)$.

We have the following useful estimates.

Lemma 13. *For $\alpha > -1$ and $1 < q < \infty$, there exists a constant $C_{q,\alpha,n} > 0$ such that*

$$\left\|P_\alpha(a)\right\|_{1,\alpha} \le C_{q,\alpha,n}, \tag{59}$$

for any $(1,q)_\alpha$-atom a.

To prove Lemma 13, we need first to show an inequality for reproducing kernel K^α associated with d, which is essentially borrowed from [6, Proposition 2.13].

Lemma 14. *For $\alpha > -1$, there exists a constant $\delta > 0$ such that for all $z, w \in \mathbb{B}_n$, $\zeta \in \mathbb{S}_n$ satisfying $d(z,\zeta) > \delta d(w,\zeta)$, we have*

$$\left|K^\alpha(z,w) - K^\alpha(z,\zeta)\right| \le C_{\alpha,n}\frac{d(w,\zeta)}{d(z,\zeta)^{2(n+1+\alpha)+1}}. \tag{60}$$

Proof. Note that

$$K^\alpha(z,w) - K^\alpha(z,\zeta) = \int_0^1 \frac{d}{dt}\left(\frac{1}{(1-\langle z,\zeta\rangle - t\langle z, w-\zeta\rangle)^{n+1+\alpha}}\right)dt. \tag{61}$$

We have

$$\left|K^\alpha(z,w) - K^\alpha(z,\zeta)\right| \le \int_0^1 \frac{(n+1+\alpha)\left|\langle z, w-\zeta\rangle\right|}{\left|1-\langle z,\zeta\rangle - t\langle z, w-\zeta\rangle\right|^{n+2+\alpha}}dt. \tag{62}$$

Write $z = z_1 + z_2$ and $w = w_1 + w_2$, where z_1 and w_1 are parallel to ζ, while z_2 and w_2 are perpendicular to ζ. Then

$$\langle z, w\rangle - \langle z,\zeta\rangle = \langle z_2, w_2\rangle - \langle z_1, w_1 - \zeta\rangle \tag{63}$$

and so

$$\left|\langle z,w\rangle-\langle z,\zeta\rangle\right|\le\left|z_2\right|\left|w_2\right|+\left|w_1-\zeta\right|. \tag{64}$$

Since $|w_1-\zeta|=|1-\langle w,\zeta\rangle|$,

$$\begin{aligned}\left|z_2\right|^2&=|z|^2-\left|z_1\right|^2\\&<1-\left|z_1\right|^2\\&<\left(1+\left|z_1\right|\right)\left(1+\left|z_1\right|\right)\\&\le\left|1-\langle z_1,\zeta\rangle\right|=2\left|1-\langle z,\zeta\rangle\right|,\end{aligned} \tag{65}$$

and similarly

$$\left|w_2\right|^2\le2\left|1-\langle w,\zeta\rangle\right|, \tag{66}$$

we have

$$\begin{aligned}&\left|\langle z,w\rangle-\langle z,\zeta\rangle\right|\\&\quad\le2\left|1-\langle z,\zeta\rangle\right|^{1/2}\left|1-\langle w,\zeta\rangle\right|^{1/2}+\left|1-\langle w,\zeta\rangle\right|\\&\quad=2d(w,\zeta)\left[d(z,\zeta)+d(w,\zeta)\right]\\&\quad\le2\left(1+\frac{1}{\delta}\right)\frac{1}{\delta}d^2(z,\zeta).\end{aligned} \tag{67}$$

This concludes that there is $\delta>1$ such that

$$\left|\langle z,w-\zeta\rangle\right|<\frac{1}{2}\left|1-\langle z,\zeta\rangle\right|,\quad\forall z,w\in\mathbb{B}_n,\ \zeta\in\mathbb{S}_n, \tag{68}$$

whenever $d(z,\zeta)>\delta d(w,\zeta)$. Then, we have

$$\begin{aligned}&\left|1-\langle z,\zeta\rangle-t\langle z,w-\zeta\rangle\right|\\&\quad>\left|1-\langle z,\zeta\rangle\right|-t\left|\langle z,\zeta-w\rangle\right|\\&\quad>\frac{1}{2}\left|1-\langle z,\zeta\rangle\right|.\end{aligned} \tag{69}$$

Therefore,

$$\begin{aligned}&\left|K^\alpha(z,w)-K^\alpha(z,\zeta)\right|\\&\quad\le\frac{2^{n+3+\alpha}(n+1+\alpha)(1+1/\delta)\,d(w,\zeta)\,d(z,\zeta)}{\left|1-\langle z,\zeta\rangle\right|^{n+2+\alpha}}\\&\quad\le C_{\alpha,n}\frac{d(w,\zeta)}{d(z,\zeta)^{2(n+1+\alpha)+1}},\end{aligned} \tag{70}$$

and the lemma is proved. □

Proof of Lemma 13. When a is the constant function 1, the result is clear. Thus we may suppose that a is a $(1,q)_\alpha$-atom. Let a be supported in a Carleson tuber $Q_r(\zeta)$ and $\delta r\le\sqrt{2}$, where δ is the constant in Lemma 14. Since P_α is a bounded operator on $L^q(\mathbb{B}_n,dv_\alpha)$, we have

$$\begin{aligned}&\int_{Q_{\delta r}}\left|P_\alpha(a)\right|dv_\alpha(z)\\&\quad\le v_\alpha\left(Q_{\delta r}\right)^{1-(1/q)}\left\|P_\alpha(a)\right\|_{q,\alpha}\\&\quad\le\left\|P_\alpha\right\|_{L^q}v_\alpha\left(Q_{\delta r}\right)^{1-(1/q)}\|a\|_{q,\alpha}\\&\quad\le\left\|P_\alpha\right\|_{L^q}.\end{aligned} \tag{71}$$

Next, if $d(z,\zeta)>\delta r$ then

$$\begin{aligned}&\left|\int_{\mathbb{B}_n}\frac{a(w)}{(1-\langle z,w\rangle)^{n+1+\alpha}}dv_\alpha(w)\right|\\&\quad=\left|\int_{Q_r(\zeta)}a(w)\left[\frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}\right.\right.\\&\qquad\left.\left.-\frac{1}{\left(1-\langle z,\zeta\rangle\right)^{n+1+\alpha}}\right]dv_\alpha(w)\right|\\&\quad\le C\int_{Q_r(\zeta)}|a(w)|\frac{d(w,\zeta)}{d(z,\zeta)^{2(n+1+\alpha)+1}}dv_\alpha(w)\\&\quad\le Cr\int_{Q_r(\zeta)}|a(w)|\,dv_\alpha(w)\frac{1}{d(z,\zeta)^{2(n+1+\alpha)+1}}\\&\quad\le\frac{Cr}{d(z,\zeta)^{2(n+1+\alpha)+1}}.\end{aligned} \tag{72}$$

Then

$$\begin{aligned}&\int_{d(z,\zeta)>\delta r}\left|P_\alpha(a)\right|dv_\alpha(z)\\&\quad\le Cr\int_{d(z,\zeta)>\delta r}\frac{1}{d(z,\zeta)^{2(n+1+\alpha)+1}}dv_\alpha(z)\\&\quad=Cr\sum_{k\ge0}\int_{2^k\delta r<d(z,\zeta)\le2^{k+1}\delta r}\frac{1}{d(z,\zeta)^{2(n+1+\alpha)+1}}dv_\alpha(z)\\&\quad\le Cr\sum_{k\ge0}\frac{v_\alpha\left(Q_{2^{k+1}\delta r}\right)}{\left(2^k\delta r\right)^{2(n+1+\alpha)+1}}\\&\quad\le Cr\sum_{k=0}^{\infty}\frac{\left(2^{k+1}\delta r\right)^{2(n+1+\alpha)}}{\left(2^k\delta r\right)^{2(n+1+\alpha)+1}}\\&\quad\le C,\end{aligned} \tag{73}$$

where we have used the fact that $v_\alpha(Q_r) \approx r^{2(n+1+\alpha)}$ in the third inequality (cf. [3, Corollary 5.24]). Thus, we get

$$\begin{aligned}\int_{\mathbb{B}_n} |P_\alpha(a)|\, dv_\alpha(z) &= \int_{Q_{\delta r}} |P_\alpha(a)|\, dv_\alpha(z) + \int_{d(z,\zeta)>\delta r} |P_\alpha(h)|\, dv_\alpha(z) \\ &\le C,\end{aligned} \tag{74}$$

where C depends only on q, n, and α. □

Now we turn to the atomic decomposition of $\mathscr{A}_\alpha^1$ ($\alpha > -1$) with respect to the Carleson tubes. Recall that $\|a\|_{1,\alpha} \le 1$ for any $(1,q)_\alpha$-atom a. Then, we define $\mathscr{A}_\alpha^{1,q}$ as the space of all $f \in \mathscr{A}_\alpha^1$ which admits a decomposition

$$f = \sum_i \lambda_i P_\alpha a_i, \qquad \sum_i |\lambda_i| \le C_q \|f\|_{1,\alpha}, \tag{75}$$

where for each i, a_i is an $(1,q)_\alpha$-atom and $\lambda_i \in \mathbb{C}$ so that $\sum_i |\lambda_i| < \infty$. We equip this space with the norm

$$\|f\|_{\mathscr{A}_\alpha^{1,q}} = \inf\left\{\sum_i |\lambda_i| : f = \sum_i \lambda_i P_\alpha a_i\right\}, \tag{76}$$

where the infimum is taken over all decompositions of f described above.

It is easy to see that $\mathscr{A}_\alpha^{1,q}$ is a Banach space. By Lemma 13, we have the contractive inclusion $\mathscr{A}_\alpha^{1,q} \subset \mathscr{A}_\alpha^1$. We will prove in what follows that these two spaces coincide. That establishes the "real-variable" atomic decomposition of the Bergman space $\mathscr{A}_\alpha^1$. In fact, we will show the remaining inclusion $\mathscr{A}_\alpha^1 \subset \mathscr{A}_\alpha^{1,q}$ by duality.

Theorem 15. *Let $1 < q < \infty$ and $\alpha > -1$. For every $f \in \mathscr{A}_\alpha^1$ there exist a sequence $\{a_i\}$ of $(1,q)_\alpha$-atoms and a sequence $\{\lambda_i\}$ of complex numbers such that*

$$f = \sum_i \lambda_i P_\alpha a_i, \qquad \sum_i |\lambda_i| \le C_q \|f\|_{1,\alpha}. \tag{77}$$

Moreover,

$$\|f\|_{1,\alpha} \approx \inf \sum_i |\lambda_i|, \tag{78}$$

where the infimum is taken over all decompositions of f described above and "≈" depends only on α, n, and q.

Recall that the dual space of $\mathscr{A}_\alpha^1$ is the Bloch space $\mathscr{B}$ (we refer to [3] for details). The Banach dual of $\mathscr{A}_\alpha^1$ can be identified with $\mathscr{B}$ (with equivalent norms) under the integral pairing

$$\langle f, g\rangle_\alpha = \lim_{r\to 1^-} \int_{\mathbb{B}_n} f(rz)\overline{g(z)}\, dv_\alpha(z), \qquad f \in \mathscr{A}_\alpha^1,\ g \in \mathscr{B}. \tag{79}$$

(cf. [3, Theorem 3.17].)

In order to prove Theorem 15, we need the following result, which can be found in [4] (see also [3, Theorem 5.25]).

Lemma 16. *Suppose $\alpha > -1$ and $1 \le p < \infty$. Then, for any $f \in \mathscr{H}(\mathbb{B}_n)$, f is in $\mathscr{B}$ if and only if there exists a constant $C > 0$ depending only on α and p such that*

$$\frac{1}{v_\alpha(Q_r(\zeta))} \int_{Q_r(\zeta)} \left|f - f_{\alpha,Q_r(\zeta)}\right|^p dv_\alpha \le C, \tag{80}$$

for all $r > 0$ and all $\zeta \in \mathbb{S}_n$, where

$$f_{\alpha,Q_r(\zeta)} = \frac{1}{Q_r(\zeta)} \int_{Q_r(\zeta)} f(z)\, dv_\alpha(z). \tag{81}$$

Moreover,

$$\|f\|_{\mathscr{B}} \approx \sup_{r>0,\ \zeta\in\mathbb{S}} \left(\frac{1}{v_\alpha(Q_r(\zeta))} \int_{Q_r(\zeta)} \left|f - f_{\alpha,Q_r(\zeta)}\right|^p dv_\alpha\right)^{1/p}, \tag{82}$$

where "≈" depends only on α, p, and n.

As noted above, we will prove Theorem 15 via duality. To this end, we first prove the following duality theorem.

Proposition 17. *For any $1 < q < \infty$ and $\alpha > -1$, one has $(\mathscr{A}_\alpha^{1,q})^* = \mathscr{B}$ isometrically. More precisely,*

(i) *every $g \in \mathscr{B}$ defines a continuous linear functional φ_g on $\mathscr{A}_\alpha^{1,q}$ by*

$$\varphi_g(f) = \lim_{r\to 1^-} \int_{\mathbb{B}_n} f(rz)\overline{g(z)}\, dv_\alpha(z), \quad \forall f \in \mathscr{A}_\alpha^{1,q}, \tag{83}$$

(ii) *conversely, each $\varphi \in (\mathscr{A}_\alpha^{1,q})^*$ is given as (83) by some $g \in \mathscr{B}$.*

Moreover, we have

$$\|\varphi_g\| \approx |g(0)| + \|g\|_{\mathscr{B}}, \quad \forall g \in \mathscr{B}. \tag{84}$$

Proof. Let p be the conjugate index of q, that is, $1/p + 1/q = 1$. We first show $\mathscr{B} \subset (\mathscr{A}_\alpha^{1,q})^*$. Let $g \in \mathscr{B}$. For any $(1,q)_\alpha$-atom a, by Lemma 16 one has

$$\begin{aligned}&\left|\int_{\mathbb{B}_n} P_\alpha a(z)\overline{g(z)}\, dv_\alpha(z)\right| \\ &= \left|\langle P_\alpha(a_j), g\rangle_\alpha\right| \\ &= \left|\int_{\mathbb{B}_n} a\overline{g}\, dv_\alpha\right| \\ &= \left|\int_{\mathbb{B}_n} a\overline{(g - g_Q)}\, dv_\alpha\right| \\ &\le \left(\int_Q |a|^q dv_\alpha\right)^{1/q} \left(\int_Q |g - g_Q|^p dv_\alpha\right)^{1/p} \\ &\le \left(\frac{1}{v_\alpha(Q)} \int_Q |g - g_Q|^p dv_\alpha\right)^{1/p} \\ &\le C\|g\|_{\mathscr{B}}.\end{aligned} \tag{85}$$

On the other hand, for the constant function (1), we have $P_\alpha 1 = 1$ and so

$$\left|\int_{\mathbb{B}_n} P_\alpha 1(z)\overline{g(z)}dv_\alpha(z)\right| = \left|\int_{\mathbb{B}_n} g(z)\,dv_\alpha(z)\right| = |g(0)|. \tag{86}$$

Thus, we deduce that

$$\left|\int_{\mathbb{B}_n} f\overline{g}dv_\alpha\right| \le C\|f\|_{\mathscr{A}_\alpha^{1,q}}\left(|g(0)| + \|g\|_{\mathscr{B}}\right), \tag{87}$$

for any finite linear combination f of $(1,q)_\alpha$-atoms. Hence, g defines a continuous linear functional φ_g on a dense subspace of $\mathscr{A}_\alpha^{1,q}$, and φ_g extends to a continuous linear functional on $\mathscr{A}_\alpha^{1,q}$ such that

$$\left|\varphi_g(f)\right| \le C\left(|g(0)| + \|g\|_{\mathscr{B}}\right)\|f\|_{\mathscr{A}_\alpha^{1,q}}, \tag{88}$$

for all $f \in \mathscr{A}_\alpha^{1,q}$.

Next let φ be a bounded linear functional on $\mathscr{A}_\alpha^{1,q}$. Note that

$$\mathscr{H}^q(\mathbb{B}_n, dv_\alpha) = \mathscr{H}(\mathbb{B}_n) \cap L^q(\mathbb{B}_n, dv_\alpha) \subset \mathscr{A}_\alpha^{1,q}. \tag{89}$$

Then, φ is a bounded linear functional on $\mathscr{H}^q(\mathbb{B}_n, dv_\alpha)$. By duality there exists $g \in \mathscr{H}^p(\mathbb{B}_n, dv_\alpha)$ such that

$$\varphi(f) = \int_{\mathbb{B}_n} f\overline{g}dv_\alpha, \quad \forall f \in \mathscr{H}^q(\mathbb{B}_n, dv_\alpha). \tag{90}$$

Let $Q = Q_r(\zeta)$ be a Carleson tube. For any $f \in L^q(\mathbb{B}_n, dv_\alpha)$ supported in Q, it is easy to check that

$$a_f = \frac{(f - f_Q)\chi_Q}{\left[\|f\|_{L^q} v_\alpha(Q)^{1/p}\right]} \tag{91}$$

is a $(1,q)$-atom. Then, $|\varphi(P_\alpha a_f)| \le \|\varphi\|$ and so

$$\left|\varphi\left(P_\alpha\left[(f - f_Q)\chi_Q\right]\right)\right| \le \|\varphi\|\,\|f\|_{L^q} v_\alpha(Q)^{1/p}. \tag{92}$$

Hence, for any $f \in L^q(\mathbb{B}_n, dv_\alpha)$, we have

$$\begin{aligned}
&\left|\int_Q f\overline{(g - g_Q)}dv_\alpha\right| \\
&\quad = \left|\int_Q (f - f_Q)\,\overline{g}dv_\alpha\right| \\
&\quad = \left|\int_{\mathbb{B}_n} (f - f_Q)\,\chi_Q\overline{g}dv_\alpha\right| \\
&\quad = \left|\int_{\mathbb{B}_n} P_\alpha\left[(f - f_Q)\chi_Q\right]\overline{g}dv_\alpha\right| \\
&\quad = \left|\varphi\left(P_\alpha\left[(f - f_Q)\chi_Q\right]\right)\right| \\
&\quad \le \|\varphi\|\,\left\|(f - f_Q)\chi_Q\right\|_{L^q(\mathbb{B}_n, dv_\alpha)} v_\alpha(Q)^{1/p} \\
&\quad \le 2\|\varphi\|\,\|f\|_{L^q(Q, dv_\alpha)} v_\alpha(Q)^{1/p}.
\end{aligned} \tag{93}$$

This concludes that

$$\left(\frac{1}{v_\alpha(Q)}\int_Q \left|g - g_Q\right|^p dv_\alpha\right)^{1/p} \le 2\|\varphi\|. \tag{94}$$

By Lemma 16, we have that $g \in \mathscr{B}$ and $\|g\|_{\mathscr{B}} \le C\|\varphi\|$. Therefore, φ is given as (83) by g with $|g(0)| + \|g\|_{\mathscr{B}} \le C\|\varphi\|$. □

Now we are ready to prove Theorem 15.

Proof of Theorem 15. By Lemma 13 we know that $\mathscr{A}_\alpha^{1,q} \subset \mathscr{A}_\alpha^1$. On the other hand, by Proposition 17 we have $(\mathscr{A}_\alpha^1)^* = (\mathscr{A}_\alpha^{1,q})^*$. Hence, by duality we have $\|f\|_{1,q} \approx \|f\|_{\mathscr{A}_\alpha^{1,q}}$. □

Remark 18. (1) One would like to expect that when $0 < p < 1$, $\mathscr{A}_\alpha^p$ also admits an atomic decomposition in terms of atoms with respect to Carleson tubes. However, the proof of Theorem 15 via duality cannot be extended to the case $0 < p < 1$. At the time of this writing, this problem is entirely open.

(2) The real-variable atomic decomposition of Bergman spaces should be known to specialists in the case $p = 1$. Indeed, based on their theory of harmonic analysis on homogeneous spaces, Coifman and Weiss [7] claimed that the Bergman space $\mathscr{A}^1$ admits an atomic decomposition in terms of atoms with respect to $(\mathbb{B}_n, \varrho, dv)$, where

$$\begin{aligned}
&\varrho(z, w) \\
&\quad = \begin{cases} \left||z| - |w|\right| + \left|1 - \dfrac{1}{|z|\,|w|}\langle z, w\rangle\right|, & \text{if } z, w \in \mathbb{B}_n \setminus \{0\}, \\ |z| + |w|, & \text{otherwise.} \end{cases}
\end{aligned} \tag{95}$$

This also applies to $\mathscr{A}_\alpha^1$ because $(\mathbb{B}_n, \varrho, dv_\alpha)$ is a homogeneous space for $\alpha > -1$ (see e.g. [6]). However, the approach of Coifman and Weiss is again based on duality and therefore not constructive and cannot be applied to the case $0 < p < 1$. Recently, the present authors [8] extend this result to the case $0 < p < 1$ through using a constructive method.

4. Area Integral Inequalities: Real-Variable Methods

In this section, we will prove the area integral inequality for the Bergman space $\mathscr{A}_\alpha^1$ via atomic decomposition established in Section 3.

Theorem 19. *Suppose* $1 < q < \infty,\ \gamma > 0$, *and* $\alpha > -1$. *Then,*

$$\left\|A_{\widetilde{\nabla}}^{\gamma,q}(f)\right\|_{1,\alpha} \lesssim \|f\|_{1,\alpha}, \quad \forall f \in \mathscr{H}(\mathbb{B}_n). \tag{96}$$

This is the assertion (a) $\Rightarrow$ (d) of Theorem 2 in the case $p = 1$. The novelty of the proof here is to involve a real-variable method.

The following lemma is elementary.

Lemma 20. *Suppose* $1 < q < \infty$, $\gamma > 0$ *and* $\alpha > -1$. *If* $f \in \mathscr{A}_\alpha^q$, *then*

$$\int_{\mathbb{B}_n} \left|A_{\widetilde{\nabla}}^{\gamma,q}(f)(z)\right|^q dv_\alpha \approx \int_{\mathbb{B}_n} |f(z) - f(0)|^q dv_\alpha, \tag{97}$$

where "≈" depends only on $q,\ \gamma,\ \alpha$, *and* n.

Proof. Note that $v_\alpha(D(z,\gamma)) \approx (1-|z|^2)^{n+1+\alpha}$. Then

$$\begin{aligned}
&\int_{\mathbb{B}_n} \left|A_{\widetilde{\nabla}}^{\gamma,q}(f)(z)\right|^q dv_\alpha \\
&= \int_{\mathbb{B}_n} \int_{D(z,\gamma)} \left(1-|w|^2\right)^{-1-n} \left|\widetilde{\nabla} f(w)\right|^q dv(w)\, dv_\alpha(z) \\
&= \int_{\mathbb{B}_n} v_\alpha(D(w,\gamma)) \left(1-|w|^2\right)^{-1-n} \left|\widetilde{\nabla} f(w)\right|^q dv(w) \\
&\approx \int_{\mathbb{B}_n} \left|\widetilde{\nabla} f(w)\right|^q dv_\alpha(w) \\
&\approx \int_{\mathbb{B}_n} |f(w) - f(0)|^q dv_\alpha(w).
\end{aligned} \tag{98}$$

In the last step, we have used [3, Theorem 2.16(b)]. □

Proof of (96). By Theorem 15, it suffices to show that for $1 < q < \infty,\ \gamma > 0$, and $\alpha > -1$, there exists $C > 0$ such that

$$\left\|A_{\widetilde{\nabla}}^{\gamma,q}(P_\alpha a)\right\|_{1,\alpha} \leq C, \tag{99}$$

for all $(1,q)_\alpha$-atoms a. Given an $(1,q)_\alpha$-atom a supported in $Q = Q_r(\zeta)$, by Lemma 20, we have

$$\begin{aligned}
&\int_{2Q} A_{\widetilde{\nabla}}^{\gamma,q}(P_\alpha a)\, dv_\alpha \\
&\leq v_\alpha(2Q)^{1-(1/q)} \left(\int_{2Q} \left[A_{\widetilde{\nabla}}^{\gamma,q}(P_\alpha a)\right]^q dv_\alpha\right)^{1/q} \\
&\leq C v_\alpha(Q)^{1-(1/q)} \left(\int_{\mathbb{B}_n} \left|P_\alpha a(z) - P_\alpha a(0)\right|^q dv_\alpha\right)^{1/q} \\
&\leq C v_\alpha(Q)^{1-(1/q)} \|a\|_{q,\alpha} \\
&\leq C,
\end{aligned} \tag{100}$$

where $2Q = Q_{2r}(\zeta)$. On the other hand,

$$\begin{aligned}
&\int_{(2Q)^c} A_{\widetilde{\nabla}}^{\gamma,q}(P_\alpha a)\, dv_\alpha \\
&= \int_{(2Q)^c} \left(\int_{D(z,\gamma)} \left|\widetilde{\nabla} P_\alpha a(w)\right|^q d\tau(w)\right)^{1/q} dv_\alpha(z) \\
&= \int_{(2Q)^c} \left(\int_{D(z,\gamma)} \left|\int_Q \widetilde{\nabla}_w [K^\alpha(w,u) - K^\alpha(w,\zeta)] a(u) dv_\alpha(u)\right|^q d\tau(w)\right)^{1/q} dv_\alpha(z) \\
&\leq \|a\|_{q,\alpha} \int_{(2Q)^c} \left(\int_{D(z,\gamma)} \left(\int_Q \left|\widetilde{\nabla}_w [K^\alpha(w,u) - K^\alpha(w,\zeta)]\right|^{q/(q-1)} dv_\alpha(u)\right)^{q-1} d\tau(w)\right)^{1/q} dv_\alpha(z) \\
&\leq \int_{(2Q)^c} \left(\int_{D(z,\gamma)} \sup_{u\in Q} \left|\widetilde{\nabla}_w [K^\alpha(w,u) - K^\alpha(w,\zeta)]\right|^q d\tau(w)\right)^{1/q} dv_\alpha(z),
\end{aligned} \tag{101}$$

where $(2Q)^c = \mathbb{B}_n \setminus 2Q$.

An immediate computation yields that

$$\begin{aligned}
&\nabla_w [K^\alpha(w,u) - K^\alpha(w,\zeta)] \\
&= (n+1+\alpha) \left[\frac{\overline{u}}{(1-\langle w,u\rangle)^{n+2+\alpha}} - \frac{\overline{\zeta}}{(1-\langle w,\zeta\rangle)^{n+2+\alpha}}\right] \\
&= (n+1+\alpha) \frac{\overline{u}(1-\langle w,\zeta\rangle)^{n+2+\alpha} - \overline{\zeta}(1-\langle w,u\rangle)^{n+2+\alpha}}{(1-\langle w,u\rangle)^{n+2+\alpha}(1-\langle w,\zeta\rangle)^{n+2+\alpha}},
\end{aligned}$$

$$\begin{aligned}
&\mathscr{R}_w [K^\alpha(w,u) - K^\alpha(w,\zeta)] \\
&= (n+1+\alpha) \left[\frac{\langle w,u\rangle}{(1-\langle w,u\rangle)^{n+2+\alpha}} - \frac{\langle w,\zeta\rangle}{(1-\langle w,\zeta\rangle)^{n+2+\alpha}}\right] \\
&= (n+1+\alpha) \\
&\quad\times \frac{\langle w,u\rangle(1-\langle w,\zeta\rangle)^{n+2+\alpha} - \langle w,\zeta\rangle(1-\langle w,u\rangle)^{n+2+\alpha}}{(1-\langle w,u\rangle)^{n+2+\alpha}(1-\langle w,\zeta\rangle)^{n+2+\alpha}}.
\end{aligned} \tag{102}$$

Moreover,

$$
\begin{aligned}
&\left|\nabla_w [K^\alpha(w,u) - K^\alpha(w,\zeta)]\right|^2 \\
&= (n+1+\alpha)^2 \\
&\quad \times \left\{ \frac{|u|^2 \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)} + \left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)}}{\left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)} \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)}} \right. \\
&\qquad - \frac{(1 - \langle w,\zeta\rangle)^{n+2+\alpha} (1 - \langle u,w\rangle)^{n+2+\alpha} \langle \zeta,u\rangle}{\left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)} \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)}} \\
&\qquad \left. - \frac{(1 - \langle w,u\rangle)^{n+2+\alpha} (1 - \langle \zeta,w\rangle)^{n+2+\alpha} \langle u,\zeta\rangle}{\left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)} \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)}} \right\}, \\
&\left|\mathscr{R}_w [K^\alpha(w,u) - K^\alpha(w,\zeta)]\right|^2 \\
&= (n+1+\alpha)^2 \\
&\quad \times \left\{ \frac{|\langle w,u\rangle|^2 \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)} + |\langle w,\zeta\rangle|^2 \left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)}}{\left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)} \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)}} \right. \\
&\qquad - \frac{\langle w,u\rangle \langle \zeta,w\rangle (1 - \langle w,\zeta\rangle)^{n+2+\alpha} (1 - \langle u,w\rangle)^{n+2+\alpha}}{\left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)} \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)}} \\
&\qquad \left. - \frac{\langle w,\zeta\rangle \langle u,w\rangle (1 - \langle w,u\rangle)^{n+2+\alpha} (1 - \langle \zeta,w\rangle)^{n+2+\alpha}}{\left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)} \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)}} \right\}.
\end{aligned}
\tag{103}
$$

Then, we have

$$
\begin{aligned}
&\left|\nabla_w [K^\alpha(w,u) - K^\alpha(w,\zeta)]\right|^2 - \left|\mathscr{R}_w [K^\alpha(w,u) - K^\alpha(w,\zeta)]\right|^2 \\
&= \frac{(n+1+\alpha)^2}{\left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)} \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)}} \\
&\quad \times \left\{ \left(|u|^2 - |\langle w,u\rangle|^2\right) \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)} \right. \\
&\qquad + \left(1 - |\langle w,\zeta\rangle|^2\right) \left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)} \\
&\qquad + (\langle w,u\rangle \langle \zeta,w\rangle - \langle \zeta,u\rangle) \\
&\qquad \times (1 - \langle w,\zeta\rangle)^{n+2+\alpha} (1 - \langle u,w\rangle)^{n+2+\alpha} \\
&\qquad + (\langle w,\zeta\rangle \langle u,w\rangle - \langle u,\zeta\rangle) \\
&\qquad \left. \times (1 - \langle w,u\rangle)^{n+2+\alpha} (1 - \langle \zeta,w\rangle)^{n+2+\alpha} \right\}.
\end{aligned}
\tag{104}
$$

Note that for any $f \in \mathscr{H}(\mathbb{B}_n)$,

$$
\left|\widetilde{\nabla} f(z)\right|^2 = \left(1 - |z|^2\right) \left(\left|\nabla f(z)\right|^2 - \left|\mathscr{R} f(z)\right|^2\right), \quad z \in \mathbb{B}_n
\tag{105}
$$

(cf. [3, Lemma 2.13]). It is concluded that

$$
\begin{aligned}
&\left|\widetilde{\nabla}_w [K^\alpha(w,u) - K^\alpha(w,\zeta)]\right| \\
&\leq \frac{(n+1+\alpha)\left(1 - |w|^2\right)^{1/2}}{\left|1 - \langle w,u\rangle\right|^{n+2+\alpha} \left|1 - \langle w,\zeta\rangle\right|^{n+2+\alpha}} \\
&\quad \times \left\{ \left(1 - |\langle w,u\rangle|^2\right) \left|1 - \langle w,\zeta\rangle\right|^{2(n+2+\alpha)} \right. \\
&\qquad + \left(1 - |\langle w,\zeta\rangle|^2\right) \left|1 - \langle w,u\rangle\right|^{2(n+2+\alpha)} \\
&\qquad + \left[\left(\langle w,u-\zeta\rangle \langle \zeta,w\rangle + \left(|\langle w,\zeta\rangle|^2 - 1\right) + (1 - \langle \zeta,u\rangle)\right]\right. \\
&\qquad \times (1 - \langle w,\zeta\rangle)^{n+2+\alpha} (1 - \langle u,w\rangle)^{n+2+\alpha} \\
&\qquad + \left[\left(\langle w,\zeta-u\rangle \langle u,w\rangle + \left(|\langle w,u\rangle|^2 - 1\right) + (1 - \langle u,\zeta\rangle)\right]\right. \\
&\qquad \left. \times (1 - \langle w,u\rangle)^{n+2+\alpha} (1 - \langle \zeta,w\rangle)^{n+2+\alpha} \right\}^{1/2} \\
&\leq \frac{(n+1+\alpha)\left(1 - |w|^2\right)^{1/2} (M_1 + M_2 + M_3 + M_4)^{1/2}}{\left|1 - \langle w,u\rangle\right|^{n+2+\alpha} \left|1 - \langle w,\zeta\rangle\right|^{n+2+\alpha}},
\end{aligned}
\tag{106}
$$

where

$$
\begin{aligned}
M_1 &= \left|1 - \langle w,\zeta\rangle\right|^{n+2+\alpha} \left|1 - \langle u,w\rangle\right|^{n+2+\alpha} \\
&\quad \times \left|\langle w,u-\zeta\rangle \langle \zeta,w\rangle + (1 - \langle \zeta,u\rangle)\right|, \\
M_2 &= \left|1 - \langle w,u\rangle\right|^{n+2+\alpha} \left|1 - \langle \zeta,w\rangle\right|^{n+2+\alpha} \\
&\quad \times \left|\langle w,\zeta-u\rangle \langle u,w\rangle + (1 - \langle u,\zeta\rangle)\right|, \\
M_3 &= \left(1 - |\langle w,u\rangle|^2\right) \left|1 - \langle \zeta,w\rangle\right|^{n+2+\alpha} \\
&\quad \times \left|(1 - \langle w,\zeta\rangle)^{n+2+\alpha} - (1 - \langle w,u\rangle)^{n+2+\alpha}\right|, \\
M_4 &= \left(1 - |\langle w,\zeta\rangle|^2\right) \left|1 - \langle u,w\rangle\right|^{n+2+\alpha} \\
&\quad \times \left|(1 - \langle w,u\rangle)^{n+2+\alpha} - (1 - \langle w,\zeta\rangle)^{n+2+\alpha}\right|,
\end{aligned}
\tag{107}
$$

for $w \in D(z,\gamma)$, $u \in Q_r(\zeta)$, and $z \in \mathbb{B}_n$, $\zeta \in \mathbb{S}_n$.

Hence,

$$
\begin{aligned}
&\int_{(2Q)^c} A^{\gamma,q}_{\widetilde{\nabla}} (P_\alpha a)\, dv_\alpha \\
&\leq \int_{(2Q)^c} \left(\int_{D(z,\gamma)} \sup_{u \in Q} \left|\widetilde{\nabla}_w [K^\alpha(w,u) - K^\alpha(w,\zeta)]\right|^q d\tau(w) \right)^{1/q} dv_\alpha(z) \\
&\leq (n+1+\alpha) \int_{(2Q)^c} (I_1 + I_2 + I_3 + I_4)\, dv_\alpha(z),
\end{aligned}
\tag{108}
$$

where

$$I_1 = \left(\int_{D(z,\gamma)} \sup_{u\in Q} \frac{\left(1-|w|^2\right)^{q/2} M_1^{q/2}}{|1-\langle w,u\rangle|^{q(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{q(n+2+\alpha)}} d\tau(w)\right)^{1/q},$$
$$I_2 = \left(\int_{D(z,\gamma)} \sup_{u\in Q} \frac{\left(1-|w|^2\right)^{q/2} M_2^{q/2}}{|1-\langle w,u\rangle|^{q(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{q(n+2+\alpha)}} d\tau(w)\right)^{1/q},$$
$$I_3 = \left(\int_{D(z,\gamma)} \sup_{u\in Q} \frac{\left(1-|w|^2\right)^{q/2} M_3^{q/2}}{|1-\langle w,u\rangle|^{q(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{q(n+2+\alpha)}} d\tau(w)\right)^{1/q},$$
$$I_4 = \left(\int_{D(z,\gamma)} \sup_{u\in Q} \frac{\left(1-|w|^2\right)^{q/2} M_4^{q/2}}{|1-\langle w,u\rangle|^{q(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{q(n+2+\alpha)}} d\tau(w)\right)^{1/q}. \tag{109}$$

We first estimate I_1. Note that

$$\begin{aligned}
M_1 \le\ & \left(\left|\langle w,u-\zeta\rangle\right| + \left|1-\langle \zeta,u\rangle\right|\right) \\
& \times \left|1-\langle w,\zeta\rangle\right|^{n+2+\alpha}|1-\langle w,u\rangle|^{n+2+\alpha} \\
\le\ & \left(2\left|1-\langle u,\zeta\rangle\right|^{1/2}\left(\left|1-\langle w,\zeta\rangle\right|^{1/2} + \left|1-\langle u,\zeta\rangle\right|^{1/2}\right)\right. \\
& \left. + \left|1-\langle \zeta,u\rangle\right|\right) \\
& \times \left|1-\langle w,\zeta\rangle\right|^{n+2+\alpha}|1-\langle w,u\rangle|^{n+2+\alpha} \\
\le\ & \left(2\left|1-\langle u,\zeta\rangle\right|^{1/2}\left(C_\gamma\left|1-\langle z,\zeta\rangle\right|^{1/2} + \frac{1}{2}\left|1-\langle z,\zeta\rangle\right|^{1/2}\right)\right. \\
& \left. + \left|1-\langle \zeta,u\rangle\right|\right) \\
& \times \left|1-\langle w,\zeta\rangle\right|^{n+2+\alpha}|1-\langle w,u\rangle|^{n+2+\alpha} \\
\le\ & \left(C_\gamma r\left|1-\langle z,\zeta\rangle\right|^{1/2} + r^2\right) \\
& \times \left|1-\langle w,\zeta\rangle\right|^{n+2+\alpha}|1-\langle w,u\rangle|^{n+2+\alpha},
\end{aligned} \tag{110}$$

where the second inequality is the consequence of the following fact which has appeared in the proof of Lemma 14:

$$\begin{aligned}
&\left|\langle w,u-\zeta\rangle\right| \\
&\quad \le 2\left|1-\langle u,\zeta\rangle\right|^{1/2}\left(\left|1-\langle w,\zeta\rangle\right|^{1/2} + \left|1-\langle u,\zeta\rangle\right|^{1/2}\right);
\end{aligned} \tag{111}$$

the third inequality is obtained by Lemma 7, and the fact

$$\left|1-\langle u,\zeta\rangle\right|^{1/2} < r < \frac{1}{2}\left|1-\langle z,\zeta\rangle\right|^{1/2}, \tag{112}$$

for $u \in Q$ and $z \in (2Q)^c$. Since

$$\begin{aligned}
|1-\langle z,u\rangle|^{1/2} &\ge \left|1-\langle z,\zeta\rangle\right|^{1/2} - \left|1-\langle u,\zeta\rangle\right|^{1/2} \\
&\ge \left|1-\langle z,\zeta\rangle\right|^{1/2} - \frac{1}{2}\left|1-\langle z,\zeta\rangle\right|^{1/2} \\
&\ge \frac{1}{2}\left|1-\langle z,\zeta\rangle\right|^{1/2},
\end{aligned} \tag{113}$$

by Lemmas 5 and 7, one has

$$\begin{aligned}
I_1 &\le \left(\int_{D(z,\gamma)} \sup_{u\in Q} \frac{\left(1-|w|^2\right)^{q/2}\left[Cr|1-\langle z,\zeta\rangle|^{1/2}+r^2\right]^{q/2}}{|1-\langle w,u\rangle|^{(q/2)(n+2+\alpha)}|1-\langle w,\zeta\rangle|^{(q/2)(n+2+\alpha)}} d\tau(w)\right)^{1/q} \\
&\le C_\gamma\left(\int_{D(z,\gamma)} \sup_{u\in Q} \frac{\left(1-|z|^2\right)^{q/2}\left(Cr|1-\langle z,\zeta\rangle|^{1/2}+r^2\right)^{q/2}}{|1-\langle z,u\rangle|^{(q/2)(n+2+\alpha)}|1-\langle z,\zeta\rangle|^{(q/2)(n+2+\alpha)}} d\tau(w)\right)^{1/q} \\
&\le C_\gamma\left(\frac{\left(1-|z|^2\right)^{q/2}\left(r|1-\langle z,\zeta\rangle|^{1/2}+r^2\right)^{q/2}}{|1-\langle z,\zeta\rangle|^{q(n+2+\alpha)}}\right)^{1/q} \\
&\le C_\gamma \frac{\left(r|1-\langle z,\zeta\rangle|^{1/2}+r^2\right)^{1/2}}{|1-\langle z,\zeta\rangle|^{n+(3/2)+\alpha}} \\
&\le C_\gamma\left(\frac{r^{1/2}}{d(z,\zeta)^{2(n+1+\alpha)+(1/2)}} + \frac{r}{d(z,\zeta)^{2(n+1+\alpha)+1}}\right).
\end{aligned} \tag{114}$$

Hence,

$$\begin{aligned}
&\int_{(2Q)^c} I_1 d\nu_\alpha(z) \\
&\quad \lesssim \int_{(2Q)^c} \frac{r^{1/2}}{d(z,\zeta)^{2(n+1+\alpha)+(1/2)}} d\nu_\alpha(z) \\
&\qquad + \int_{(2Q)^c} \frac{r}{d(z,\zeta)^{2(n+1+\alpha)+1}} d\nu_\alpha(z).
\end{aligned} \tag{115}$$

The second term on the right hand side has been estimated in the proof of Lemma 13. The first term can be estimated as follows:

$$\begin{aligned}
&\int_{d(z,\zeta)>2r} \frac{r^{1/2}}{d(z,\zeta)^{2(n+1+\alpha)+(1/2)}} d\nu_\alpha(z) \\
&\quad = r^{1/2}\sum_{k\ge 0}\int_{2^k r<d(z,\zeta)\le 2^{k+1}r} \frac{1}{d(z,\zeta)^{2(n+1+\alpha)+(1/2)}} d\nu_\alpha(z) \\
&\quad \le r^{1/2}\sum_{k\ge 0}\frac{\nu_\alpha\left(Q_{2^{k+1}r}\right)}{\left(2^k r\right)^{2(n+1+\alpha)+(1/2)}} \\
&\quad \le Cr^{1/2}\sum_{k=0}^{\infty}\frac{\left(2^{k+1}r\right)^{2(n+1+\alpha)}}{\left(2^k r\right)^{2(n+1+\alpha)+(1/2)}} \\
&\quad \le C,
\end{aligned} \tag{116}$$

where we have used the fact that $\nu_\alpha(Q_r) \approx r^{2(n+1+\alpha)}$ in the third inequality (cf. [3, Corollary 5.24]).

By the same argument we can estimate I_2 and omit the details.

Next, we estimate I_3. Note that

$$\begin{aligned}
M_3 \leq\ & \left(1-|\langle w,u\rangle|^2\right)|1-\langle w,\zeta\rangle|^{n+2+\alpha} \\
& \times\left|(1-\langle w,\zeta\rangle)^{n+2+\alpha}-(1-\langle w,u\rangle)^{n+2+\alpha}\right| \\
\leq\ & 2\,|1-\langle w,u\rangle|\,|1-\langle w,\zeta\rangle|^{n+2+\alpha} \\
& \times\left|\int_0^1 \frac{d}{dt}(1-\langle w,t\zeta+(1-t)\,u\rangle)^{n+2+\alpha}dt\right| \\
=\ & 2\,(n+2+\alpha)\,|1-\langle w,u\rangle|\,|1-\langle w,\zeta\rangle|^{n+2+\alpha} \\
& \times\left|\langle w,\zeta-u\rangle\int_0^1(1-\langle w,t\zeta+(1-t)\,u\rangle)^{n+1+\alpha}dt\right| \\
\leq\ & C_\gamma\,|1-\langle w,u\rangle|\,|1-\langle w,\zeta\rangle|^{n+2+\alpha}r|1-\langle z,\zeta\rangle|^{n+3/2+\alpha},
\end{aligned} \tag{117}$$

where the last inequality is achieved by the following estimates:

$$\begin{aligned}
& |1-\langle w,t\zeta+(1-t)\,u\rangle| \\
& \quad\leq C_\gamma\,|1-\langle z,t\zeta+(1-t)\,u\rangle| \\
& \quad\leq C_\gamma\,|1-\langle z,u\rangle|+|\langle z,\zeta-u\rangle| \\
& \quad\leq C_\gamma\,|1-\langle z,\zeta\rangle|, \\
& |\langle w,\zeta-u\rangle|\leq C_\gamma r|1-\langle z,\zeta\rangle|^{1/2},
\end{aligned} \tag{118}$$

for any $w\in D(z,\gamma)$ and $u\in Q_r(\zeta)$. Thus, by Lemmas 5 and 7

$$\begin{aligned}
I_3 \leq\ & C_\gamma\left(\int_{D(z,\gamma)}\sup_{u\in Q}\frac{\left(1-|w|^2\right)^{q/2}r^{q/2}|1-\langle z,\zeta\rangle|^{(q/2)(n+(3/2)+\alpha)}}{|1-\langle w,u\rangle|^{q(n+1+\alpha)+(q/2)}\,1-\langle w,\zeta\rangle\,|^{(q/2)(n+2+\alpha)}}d\tau(w)\right)^{1/q} \\
\leq\ & C_\gamma\left(\int_{D(z,\gamma)}\sup_{u\in Q}\frac{(1-|z|^2)^{q/2}r^{q/2}|1-\langle z,\zeta\rangle|^{(q/2)(n+(3/2)+\alpha)}}{|1-\langle z,u\rangle|^{q(n+1+\alpha)+(q/2)}|1-\langle z,\zeta\rangle|^{(q/2)(n+2+\alpha)}}d\tau(w)\right)^{1/q} \\
\leq\ & C_\gamma\left(\frac{(1-|z|^2)^{q/2}r^{q/2}}{|1-\langle z,\zeta\rangle|^{q(n+1+\alpha)+(3/4)q}}\right)^{1/q} \\
\leq\ & C_\gamma\frac{r^{1/2}}{d(z,\zeta)^{2(n+1+\alpha)+(1/2)}}.
\end{aligned} \tag{119}$$

Hence,

$$\int_{(2Q)^c} I_3\,dv_\alpha(z)\leq C_\gamma\int_{(2Q)^c}\frac{r^{1/2}}{d(z,\zeta)^{2(n+1+\alpha)+(1/2)}}dv_\alpha(z)\leq C_\gamma, \tag{120}$$

as shown above.

Similarly, we can estimate I_4 and omit the details. Therefore, combining the above estimates we conclude that

$$\int_{(2Q)^c} A_\gamma\left(\tilde{\nabla}P_\alpha a\right)dv_\alpha\leq C, \tag{121}$$

where C depends only on q, γ, n, and α. □

Remark 21. We remark that whenever $\mathscr{A}^p_\alpha$ has an atomic decomposition in terms of atoms with respect to Carleson tubes for $0<p<1$, the argument of Theorem 2 works as well in this case. However, as noted in Remark 18(1), the problem of the atomic decomposition of $\mathscr{A}^p_\alpha$ with respect to Carleson tubes in $0<p<1$ is entirely open.

Remark 22. The area integral inequality in case $1<p<\infty$ can be also proved through using the method of vector-valued Calderón-Zygmund operators for Bergman spaces. This has been done in [9].

Acknowledgment

This research was supported in part by the NSFC under Grant no. 11171338.

References

[1] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univer-Sity Press, Princeton, NJ, USA, 1993.

[2] R. Zhao and K. Zhu, "Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$," *Mémoires de la Société Mathématique de France*, vol. 115, p. 103, 2008.

[3] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer, New York, NY, USA, 2005.

[4] D. Békollé, C. A. Berger, L. A. Coburn, and K. H. Zhu, "BMO in the Bergman metric on bounded symmetric domains," *Journal of Functional Analysis*, vol. 93, no. 2, pp. 310–350, 1990.

[5] R. Coifman and R. Rochberg, "Representation theorems for holomorphic and harmonic functions in L^p," *Asterisque*, vol. 77, pp. 11–66, 1980.

[6] E. Tchoundja, "Carleson measures for the generalized Bergman spaces via a $T(1)$-type theorem," *Arkiv for Matematik*, vol. 46, no. 2, pp. 377–406, 2008.

[7] R. Coifman and G. Weiss, "Extension of Hardy spaces and their use in analysis," *Bulletin of the American Mathematical Society*, vol. 83, pp. 569–643, 1977.

[8] Z. Chen and W. Ouyang, "Atomic decomposition of real-variable type for Bergman spaces in the unit ball of $\mathbb{C}^n$," http://arxiv.org/abs/1303.2182 .

[9] Z. Chen and W. Ouyang, "Real-variable characterizations of Bergman spaces," *Acta Analysis Functionalis Applicata*, vol. 13, pp. 246–259, 2011.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 247628, 9 pages
http://dx.doi.org/10.1155/2013/247628

Research Article

On a Class of Anisotropic Nonlinear Elliptic Equations with Variable Exponent

Guoqing Zhang[1,2] and Hongtao Zhang[1]

[1] *College of Sciences, University of Shanghai for Science and Technology, Shanghai 200093, China*
[2] *Department of Mathematical Sciences, University of Nevada Las Vegas, Las Vegas, NV 89154-4020, USA*

Correspondence should be addressed to Hongtao Zhang; zhanght2007@sina.com

Received 13 May 2013; Accepted 16 July 2013

Academic Editor: Gen-Qi Xu

Based on truncation technique and priori estimates, we prove the existence and uniqueness of weak solution for a class of anisotropic nonlinear elliptic equations with variable exponent $\overrightarrow{p(x)}$ growth. Furthermore, we also obtain that the weak solution is locally bounded and regular; that is, the weak solution is $L^{\infty}_{\mathrm{loc}}(\Omega)$ and $C^{1,\alpha}(\Omega)$.

1. Introduction

Since the variable exponent spaces have reflected into a various range of applications such as non-Newtonian fluids, plasticity, image processing, and nonlinear elasticity [1–4], some authors began to study the various properties of variable exponent space and some nonlinear problems with variable exponent growth. Edmunds et al. [5], Fan and Zhao [6] obtained that the variable exponent space $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ are reflexive Banach spaces under suitable conditions on $p(x)$. Later, Edmunds and Rákosník [7], Fan et al. [8] proved some continuous and compact Sobolev embedding theorems for the variable exponent spaces $W^{k,p(x)}(\Omega)$. For the anisotropic variable exponents spaces, in 2008, Mihăilescu et al. [9] studied the eigenvalue problems for a class of anisotropic quasilinear elliptic equations with variable exponents. In 2011, Boureanu et al. [10] proved the existence of multiple solutions for a class of anisotropic elliptic equations with variable exponents. Recently, Stancu-Dumitru [11, 12] has studied the existence of nontrivial solutions for a class of nonhomogeneous anisotropic problem. In particular, Fan [13] established some embedding theorems for anisotropic variable exponent Sobolev spaces.

In this paper, we investigate the following anisotropic nonlinear elliptic equation:

$$\begin{aligned} -\partial_{x_i} a_i(x,u,\nabla u) + \sum_{i=1}^{N} T_i(x,\nabla u) &= f - \partial_{x_i} h_i, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ ($N > 2$) with Lipschitz conditions boundary, a_i and T_i ($i = 1,2,\ldots,N$) are Carathéodory functions, h_i and f satisfy suitable conditions, and $\partial_{x_i} a_i(x,u,\nabla u)$ and $\partial_{x_i} h_i(x)$ are Einstein Sum; that is, $\partial_{x_i} a_i(x,u,\nabla u) = \sum_{i=1}^{N}(\partial a_i(x,u,\nabla u)/\partial x_i)$, $\partial_{x_i} h_i(x) = \sum_{i=1}^{N}(\partial h_i(x)/\partial x_i)$. We usually use critical theory to obtain the existence of weak solutions. However, since the problem (1) has no variational structure, we cannot define the energy functional for the problem (1). Therefore, based on truncation technique and priori estimates, we prove the existence and uniqueness of weak solutions for the problem (1) in $W_0^{1,\overrightarrow{p(x)}}(\Omega)$. Furthermore, we obtain that the weak solution for the problem (1) is locally bounded.

In particular, for the special case

$$-\partial_{x_i} a_i(x, \nabla u) = f - \partial_{x_i} h_i, \quad x \in \Omega, \qquad u = 0, \quad x \in \partial\Omega, \tag{2}$$

we obtain that the weak solution is $C^{1,\alpha}(\Omega)$. To our knowledge, the above two problems have not been deeply studied in the anisotropic variable exponent Sobolev spaces.

The paper is organized as follows. In Section 2, we recall some results on anisotropic variable exponent Sobolev spaces and state our main results. In Section 3, we prove the existence, uniqueness and locally bounded of weak solution for the problem (1). In Section 4, the regularity of weak solutions for the problem (2) is proved.

2. Preliminary and Main Results

This section is dedicated to a general overview on the $W^{1,\overrightarrow{p(x)}}(\Omega)$ and $L^{p(x)}(\Omega)$; for a deeper treatment on these spaces, see [5, 7, 8, 13, 14].

Let

$$C_+(\overline{\Omega}) := \left\{h : h \in C(\overline{\Omega}), h(x) > 1 \ \forall x \in \overline{\Omega}\right\}. \tag{3}$$

Set

$$h^+ = \sup_{x\in\overline{\Omega}} h(x) \ \text{ and } \ h^- = \inf_{x\in\overline{\Omega}} h(x), \tag{4}$$

for any $p(x) \in C_+(\overline{\Omega})$, and we define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \left\{u : \text{a measurable real} - \text{valued function such that} \int_\Omega |u(x)|^{p(x)} dx < \infty\right\}. \tag{5}$$

We define the norm of $L^{p(x)}(\Omega)$:

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{\mu > 0 \ ; \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}. \tag{6}$$

From [6], we have the following:

(1) $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is a Banach space;

(2) $L^{p(x)}(\Omega)$ is reflexive, if and only if $1 < p^- \le p^+ < \infty$;

(3) Hölder inequality.

For all $u, v \in L^{p(x)}(\Omega)$,

$$\left|\int_\Omega uv\, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(x)}(\Omega)} \|v\|_{L^{p'(x)}(\Omega)}, \tag{7}$$

where $1/p(x) + 1/p'(x) = 1$, $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(x)}(\Omega)$.

Now, we recall some results on anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p(x)}}(\Omega)$ [13]; set

$$L_+^\infty(\Omega) := \left\{p \in L^\infty(\Omega) \ : p(x) \ge 1 \quad \text{for a.e. } x \in \Omega\right\}. \tag{8}$$

Denote $\overrightarrow{p(x)} = (p_1(x), p_2(x), \dots, p_N(x)) \in (L_+^\infty(\Omega))^N$ and define

$$\forall x \in \Omega, \quad p_-(x) = \min\{p_1(x), p_2(x), \dots, p_N(x)\}, \quad p_+(x) = \max\{p_1(x), p_2(x), \dots, p_N(x)\}. \tag{9}$$

The anisotropic variable exponent Sobolev space

$$W^{1,\overrightarrow{p(x)}}(\Omega) = \left\{u \in L^{p_+(x)}(\Omega) : D_i u \in L^{p_i(x)}(\Omega) \quad \text{for } i = 1, 2, \dots, N\right\} \tag{10}$$

is a Banach space with respect to the norm

$$\|u\|_{W^{1,\overrightarrow{p(x)}}(\Omega)} = \|u\|_{p_+(x)} + \sum_{i=1}^N \|D_i u\|_{p_i(x)}. \tag{11}$$

If $p_i(x) \in L_+^\infty(\Omega)$, for $i = 1, 2, \dots, N$, $p_i^- > 1$, then $W^{1,\overrightarrow{p(x)}}(\Omega)$ is reflexive. We define $W_0^{1,\overrightarrow{p(x)}}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,\overrightarrow{p(\cdot)}}(\Omega)} = \sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i(x)}(\Omega)}, \tag{12}$$

and $W_0^{1,\overrightarrow{p(x)}}(\Omega)$ is a reflexive Banach space (see [9]).

Let

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^N (1/p_i(x))}, \quad \overline{p}^*(x) = \begin{cases} \dfrac{N\overline{p}(x)}{N - \overline{p}(x)} & \text{if } \overline{p}(x) < N, \\ +\infty & \text{if } \overline{p}(x) \ge N, \end{cases} \quad p_\infty(x) = \max\{\overline{p}^*(x), p_+(x)\}. \tag{13}$$

Hence, we have the following embedding theorem for $W_0^{1,\overrightarrow{p(x)}}(\Omega)$.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p(\cdot)} = (p_1(\cdot), p_2(\cdot), \dots, p_N(\cdot)) \in (C_+^0(\overline{\Omega}))^N$.

Theorem 1 (see [13, Theorem 2.5]).

(i) *If $q \in C_+^0(\overline{\Omega})$ and*

$$q(x) < p_\infty(x) \quad \forall x \in \overline{\Omega}, \tag{14}$$

then $W_0^{1,\overrightarrow{p(\cdot)}}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$. *The embedding is compact.*

(ii) *If* $\overline{p}(x) > N$ *for all* $x \in \overline{\Omega}$, *then there exists* $\beta \in (0,1)$ *such that* $W_0^{1,\overrightarrow{p(\cdot)}}(\Omega) \hookrightarrow\hookrightarrow C^{0,\beta}(\overline{\Omega})$.

The embedding is also compact.

Theorem 2 (see [13, Theorem 2.6]). *Let* $\Omega \subset \mathbb{R}^N$ *be a bounded domain and* $\overrightarrow{p(\cdot)} = (p_1(\cdot), p_2(\cdot), \ldots, p_N(\cdot)) \in (C_+^0(\overline{\Omega}))^N$. *Suppose that*

$$p_+(x) < \overline{p}^*(x) \quad \forall x \in \overline{\Omega}. \tag{15}$$

Then one has

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \le C\sum_{i=1}^{N}\left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i(\cdot)}(\Omega)} \quad \forall u \in W_0^{1,\overrightarrow{p(\cdot)}}(\Omega), \tag{16}$$

where C is a positive constant independent of $u \in W_0^{1,\overrightarrow{p(\cdot)}}(\Omega)$.

Remark 3. From [13], we know that if $p_+(x) < \overline{p}^*(x)$, for all $u \in W_0^{1,\overrightarrow{p(\cdot)}}(\Omega)$, for every $i = 1, 2, \ldots, N$, then $\|u\|_{L^{p_i(x)}(\Omega)} \le C\|\partial_{x_i} u\|_{L^{p_i(x)}(\Omega)}$, where C is a positive constant independent of $u \in W_0^{1,\overrightarrow{p(\cdot)}}(\Omega)$.

Assume that $a_i : \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ and $T_i : \Omega\times\mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions and satisfy the following:

(A1) for a.e. $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$, we have

$$\lambda\sum_{i=1}^{N}|\xi_i|^{p_i(x)} \le \sum_{i=1}^{N} a_i(x,s,\xi)\,\xi_i \tag{17}$$

and $|a_i(x,s,\xi)| \le \beta[|s|^{p_\infty(x)/p_i'(x)} + |\xi_i|^{p_i(x)-1}]$, where $\lambda, \beta > 0$;

(A2) for a.e. $x \in \overline{\Omega}$, for all $\xi_i \ne \xi_i'$, $\gamma > 0$ and $\varepsilon > 0$, a_i satisfies

$$\begin{aligned}&\left[a_i(x,s,\xi) - a_i\left(x,s,\xi'\right)\right]\left(\xi_i - \xi_i'\right)\\ &\quad \ge \gamma\left(\varepsilon + |\xi_i| + |\xi_i'|\right)^{p_i(x)-2}|\xi_i - \xi_i'|^2;\end{aligned} \tag{18}$$

(A3) for a.e. $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N$, $\zeta \in \mathbb{R}^N$, $i, j = 1, 2, \ldots, N$,

$$c_1|\zeta|^2 \le \sum_{i=1}^{N}\sum_{j=1}^{N} a_{\xi_j}^i(x,s,\xi)\,\zeta_i\zeta_j \le c_2\left(1 + \sum_{i=1}^{N}|\xi_i|^{p_i(x)-2}\right)|\zeta|^2, \tag{19}$$

where $a_{\xi_j}^i(x,s,\xi) = \partial a_i(x,s,\xi)/\partial\xi_j$;

(A4) for a.e. $x \in \overline{\Omega}$, $i = 1, 2, \ldots, N$, for some $k \in \{1, 2, \ldots, N\}$, for all $\zeta \in \mathbb{R}^N$, $s \in \mathbb{R}$,

$$c_3|\zeta|^2 \le \sum_{i=1}^{N} a_{x_k}^i(x,s,\xi)\,\zeta_i \le c_4\left(|\zeta|^2 + |s|^{q(x)}\right), \tag{20}$$

where $a_{x_k(x,s,\xi)}^i = \partial a_i(x,s,\xi)/\partial x_k$;

(T1) for a.e. $x \in \overline{\Omega}$, for all $\xi \in \mathbb{R}^N$, $s \in \mathbb{R}$,

$$\sum_{i=1}^{N}|T_i(x,\xi)| \le c_5|s|^{q(x)/2} \le \sum_{i=1}^{N} m_i|\xi_i|^{p_i(x)-1}, \tag{21}$$

where $m_i \in L^{b_i(x)}(\Omega)$ with $1/b_i(x) = 1/p_\infty'(x) - 1/p_i'(x)$, $c_1, \ldots, c_5$ are positive constants.

We enumerate the hypotheses concerning f and h_i,

(F1) $f \in L^{p_\infty'(x)}(\Omega)$;

(F2) for a.e. $x \in \overline{\Omega}$, $s \in \mathbb{R}$, for some $k \in \{1, 2, \ldots, N\}$, $|f_{x_k}| \le c_6|s|^{q(x)/2}$, where $f_{x_k} = \partial f/\partial x_k$;

(H1) for $i = 1, 2, \ldots, N$, $h_i \in L^{p_i'(x)}(\Omega)$;

(H2) for a.e. $x \in \overline{\Omega}$, $i = 1, 2, \ldots, N$, for some $k \in \{1, 2, \ldots, N\}$, $\zeta \in \mathbb{R}^N$, $s \in \mathbb{R}$,

$$c_7|\zeta|^2 \le \sum_{i=1}^{N} h_{x_k}^i\zeta_i \le c_8\left(|\zeta|^2 + |s|^{q(x)}\right), \tag{22}$$

where $h_{x_k}^i = \partial h_i/\partial x_k$, c_6, c_7, c_8 are positive constants.

Remark 4. Now, we give a simple example. Let $k = 1$, $x_0 = (3,0)$, $x = (x_1, x_2) \in \Omega = B(x_0, 1) \subset \mathbb{R}^2$, $u(x) = x_1^2 + x_2^2$, $\zeta = x_1^{3/2} + x_2^{3/2}$, $h_1(x) = x_1^2 + 2x_1x_2$, $h_2(x) = (1/2)x_1^2 + x_1x_2 + x_2^2$, $T_1(x,\xi) = x_1 + x_2 + \sum_{i=1}^{2}\xi_i$, $T_2(x,\xi) = x_1x_2 + |\xi|^2$, $f(x) = (1/2)x_1^2 + x_1x_2$. By a simple calculation, we obtain that T_i satisfies (T1), f satisfies (F2), and h_i satisfies (H2), where $i = 1, 2$.

Now, we define the weak solution of the problem (1). A function $u \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ is a weak solution of the problem (1), if for all $\varphi \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$,

$$\sum_{i=1}^{N}\int_\Omega\left[a_i(x,u,\nabla u)\,\varphi_{x_i} + T_i(x,\nabla u)\,\varphi\right] = \int_\Omega\left[f\varphi + \sum_{i=1}^{N} h_i\varphi_{x_i}\right], \tag{23}$$

where $\varphi_{x_i} = \partial\varphi/\partial x_i$.

Theorem 5. *Let* $\Omega \subset \mathbb{R}^N$ $(N > 2)$ *be a bounded open subset, for every* $i = 1, 2, \ldots, N$, $x \in \Omega$, *assume* $\overline{p(x)} < N$, $q \in C_+^0(\Omega)$ *and* $q(x) < p_\infty = \overline{p}^*(x)$, $p_i(x)$ *satisfies*

$$2 \le p_i(x) \le \frac{2Nr_i(x)}{Nr_i(x) - 2r_i(x) - 2N} \tag{24}$$

and $\sum_{i=1}^{N}(1/p_i(x)) < 2$, *where* $r_i(x) \ge \max\{N, p_\infty(x)p_i(x)/(p_\infty(x) - p_i(x))\}$. a_i *satisfies the hypotheses (A1), (A2), the hypotheses (F1) and (H1) hold, and* T_i *satisfies (T1) and the following Lipschitz condition:*

$$\left|T_i(x,\xi) - T_i\left(x,\xi'\right)\right| \le k_i(x)\left(|\xi_i| + |\xi_i'|\right)^{\delta_i(x)}\left|\xi_i - \xi_i'\right|, \tag{25}$$

where $k_i(x) \in L^{r_i(x)}(\Omega)$, $0 \le \delta_i(x) \le p_i(x)/N - p_i(x)/r_i(x) + (p_i(x) - 2)/2$. *Then there exists a unique weak solution u for the problem* (1). *Furthermore,* $u \in L_{\mathrm{loc}}^\infty(\Omega)$.

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$ $(N > 2)$ be a bounded open subset. Suppose (A3), (A4) and (F2), (H2) hold, for $i = 1, 2, \ldots, N$, $2 \le p_i(x) \le 2N/(N-2)$, $q \in C_+^0(\Omega)$ and $q(x) < \overline{p}^*(x)$. If u is a weak solution of the problem* (2)*, then $u \in C^{1,\alpha}(\Omega)$, for all $0 < \alpha < 1$.*

3. The Proof of Theorem 5

We consider the following problem:

$$-\partial_{x_i} a_i (x, v_n, \nabla v_n) + \sum_{i=1}^{N} T_n^i (x, \nabla v_n) = f - \partial_{x_i} h_i, \quad x \in \Omega,$$
$$u = 0, \quad x \in \partial\Omega, \tag{26}$$

where $T_n^i(x, \nabla v)$ is the truncation at levels $\pm n$ of T_i. Due to [15], we obtain that there exists a weak solution $v_n \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ for the problem (26).

Lemma 7. *If $\overline{p(x)} < N$, (A1), (A2), (T1), (F1), and (H1) hold. Assume $v_n \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ is a weak solution of the problem* (26)*; then one has*

$$\sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} v_n\right|^{p_i(x)} \le C. \tag{27}$$

Proof. Let $K \in \mathbb{R}^+$; K will be chosen later. There exist m measurable subsets $\Omega_1, \ldots, \Omega_m$ of Ω and m functions $v_1, \ldots, v_m$ such that if $i \ne j$, $\Omega_i \bigcap \Omega_j = \emptyset$. For $t \in \{1, 2, \ldots, m-1\}$, $|\Omega_t| = K$, if $t = m$, $|\Omega_t| \le K$. Let $\{x \in \Omega : |\nabla v_t| \ne 0\} \subset \Omega_t$, $\nabla v = \nabla v_t$ a.e. in Ω_t, $\nabla(v_1 + v_2 + \cdots + v_t)v_t = (\nabla v)v_t$, $v_1 + v_2 + \cdots + v_t = v$ in Ω. For $t \in \{1, 2, \ldots, m\}$, if $v_t \ne 0$, $\operatorname{sign}(v) = \operatorname{sign}(v_t)$.

Choose v_t as test function of the problem (26) and fix $t \in \{1, 2, \ldots, m\}$. Using Young inequality, Hölder inequality, the embedding theorem of the $W_0^{1,\overrightarrow{p(x)}}(\Omega)$, and the hypotheses (A1), (F1), and (H1), we have

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} v_t\right|^{p_i(x)}\\ &\quad\le c_1 \left(\|f\|_{L^{p'_\infty(x)}} A_t^{1/N} + \sum_{i=1}^{N} \int_\Omega \left|T^i (x, \nabla v)\right| |v_t| \right.\\ &\qquad \left. + \sum_{i=1}^{N} \int_\Omega |h_i|^{p'_i(x)} \right),\end{aligned} \tag{28}$$

where $c_1 > 0$ is a constant, $A_t = \prod_{i=1}^{N} \|\partial_{x_i} v_t\|_{L^{p_i(x)}(\Omega)}$. By (T1), Young inequality and Hölder inequality, we obtain, for $c_2 > 0$,

$$\begin{aligned}&\left|\sum_{i=1}^{N} \int_\Omega T^i (x, \nabla v)\, v_t\right|\\ &\quad\le \sum_{i=1}^{N} m_i \int_\Omega \left|\partial_{x_i} v\right|^{p_i(x)-1} |v_t|\\ &\quad\le \sum_{i=1}^{N} m_i \sum_{s=1}^{t} \int_{\Omega_s} \left|\partial_{x_i} v_s\right|^{p_i(x)-1} |v_t|\\ &\quad\le c_2 \sum_{i=1}^{N} \sum_{s=1}^{t} \times \left[\int_{\Omega_s} \left|\partial_{x_i} v_s\right|^{p_i(x)} \left|\Omega_s\right|^{1/p_i^- - 1/p_\infty^+} \right.\\ &\qquad \left. + A_t^{p_+^+/N} \left|\Omega_s\right|^{1/p_i^- - 1/p_\infty^+} \right]\\ &\quad\le c_2 \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} \times \left[\int_{\Omega_t} \left|\partial_{x_i} v_t\right|^{p_i(x)} \right.\\ &\qquad \left. + \sum_{s=1}^{t-1} \int_{\Omega_s} \left|\partial_{x_i} v_s\right|^{p_i(x)} + A_t^{p_+^+/N} \right],\end{aligned} \tag{29}$$

where $p_i^- = \inf_{x\in\overline{\Omega}} p_i(x)$, $p_+^+ = \sup_{x\in\overline{\Omega}} p_+(x)$, $p_\infty^+ = \sup_{x\in\overline{\Omega}} p_\infty(x)$.

Combining (28) with (29), we have

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} v_t\right|^{p_i(x)}\\ &\quad\le c_1 \left\{ \|f\|_{L^{p'_\infty(x)}} A_t^{1/N} + \sum_{i=1}^{N} \int_\Omega |h_i|^{p'_i(x)} \right.\\ &\qquad + c_2 \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} \times \left[\int_{\Omega_t} \left|\partial_{x_i} v_t\right|^{p_i(x)} \right.\\ &\qquad \left.\left. + \sum_{s=1}^{t-1} \int_{\Omega_s} \left|\partial_{x_i} v_s\right|^{p_i(x)} + A_t^{p_+^+/N} \right] \right\}.\end{aligned} \tag{30}$$

Let K satisfy $c_1 c_2 \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} < 1$; then we have

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega \left|\partial_{x_i} v_t\right|^{p_i(x)}\\ &\quad\le c_3 \left\{ \|f\|_{L^{p'_\infty(x)}} A_t^{1/N} + \sum_{i=1}^{N} \int_\Omega |h_i|^{p'_i(x)} \right.\\ &\qquad + \sum_{s=1}^{t-1} \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} \left(\sum_{j=1}^{N} \int_{\Omega_s} \left|\partial_{x_i} v_s\right|^{p_j(x)} \right)\\ &\qquad \left. + \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} A_t^{p_+^+/N} \right\},\end{aligned} \tag{31}$$

where $c_3 > 0$ is a constant. Let $t = 1$; we obtain

$$\begin{aligned}&\int_\Omega \left|\partial_{x_i} v_1\right|^{p_i(x)}\\&\leq \sum_{i=1}^{N}\int_\Omega \left|\partial_{x_i} v_1\right|^{p_i(x)}\\&\leq c_3\left[\|f\|_{L^{p'_\infty(x)}} A_1^{1/N} + \sum_{i=1}^{N}\int_\Omega |h_i|^{p'_i(x)}\right.\\&\left.\quad + \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} A_1^{p_+^+/N}\right].\end{aligned} \tag{32}$$

Now, we recall the following classical inequality, for $a_1, a_2, \ldots, a_N$ are positive numbers,

$$\prod_{i=1}^{N} a_i^{1/N} \leq \frac{1}{N}\sum_{i=1}^{N} a_i. \tag{33}$$

From (33), we have

$$A_1 = \prod_{i=1}^{N}\left\|\partial_{x_i} v_1\right\|_{L^{p_i(x)}(\Omega)} \leq \left(\frac{1}{N}\sum_{i=1}^{N}\left\|\partial_{x_i} v_1\right\|_{L^{p_i(x)}(\Omega)}\right)^N. \tag{34}$$

Combining (32) with (34), there exists a constant c_4 such that

$$\sum_{i=1}^{N}\int_\Omega \left|\partial_{x_i} v_1\right|^{p_i(x)} \leq c_4. \tag{35}$$

Furthermore, put (35) in (31) and iterate on t; we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega \left|\partial_{x_i} v_t\right|^{p_i(x)}\\&\leq c_5\left[\|f\|_{L^{p'_\infty(x)}} A_t^{1/N} + \sum_{i=1}^{N}\int_\Omega |h_i|^{p'_i(x)}\right.\\&\left.\quad + 1 + \sum_{i=1}^{N} K^{1/p_i^- - 1/p_\infty^+} A_1^{p_+^+/N}\right].\end{aligned} \tag{36}$$

Therefore, we obtain

$$\sum_{i=1}^{N}\int_\Omega \left|\partial_{x_i} v_t\right|^{p_i(x)} \leq c_6, \tag{37}$$

for some constant $c_6 > 0$. □

Lemma 8. *If $\overline{p(x)} < N$, and a_i satisfies the hypotheses (A1), (A2), the hypotheses (T1), (F1), and (H1) hold. Then there exists at least a weak solution for the problem* (1).

Proof. By (27), we obtain that there exists a sequence $\partial_{x_i} u_n$ which is bounded in $L^{p_i(x)}(\Omega)$; we have for $i = 1, 2, \ldots, N$, $\partial_{x_i} u_n \rightharpoonup \partial_{x_i} u$ weakly in $L^{p_i(x)}(\Omega)$ and $u_n \to u$ strongly in $L^{p_i(x)}(\Omega)$. We argue as in [16] to prove

$$\partial_{x_i} u_n \longrightarrow \partial_{x_i} u \quad \text{a.e. in } \Omega \text{ for } i = 1, 2, \ldots, N. \tag{38}$$

From (38), we obtain

$$\begin{aligned}&a_i\left(x, u_n, \nabla u_n\right) \longrightarrow a_i\left(x, u, \nabla u\right) \quad \text{a.e. in } \Omega,\\&T_n^i\left(x, \nabla u_n\right) \longrightarrow T_n^i\left(x, \nabla u\right) \quad \text{a.e. in } \Omega.\end{aligned} \tag{39}$$

By the hypotheses (A1) and (T1), for $\kappa_i(x) \in [1, p'_i(x)]$ and $c > 0$, we have

$$\begin{aligned}&\int_\Omega \left|a_i\left(x, u_n, \nabla u_n\right)\right|^{\kappa_i(x)}\\&\leq c\left[\left(\int_\Omega |u|^{p_\infty(x)}\right)^{\kappa_i(x)/p'_i(x)}\right.\\&\left.\quad + \left(\int_\Omega \left|\partial_{x_i} u\right|^{p_i(x)}\right)^{\kappa_i(x)/p'_i(x)}\right] |\Omega|^{(p_i'^+ - \kappa_i^-)/p_i'^-},\\&\int_\Omega \left|T_n^i\left(x, \nabla u_n\right)\right|^{\kappa_i(x)}\\&\leq c\left[\int_\Omega \left|\partial_{x_i} u\right|^{p_i(x)}\right]^{\kappa_i(x)/p'_i(x)} |\Omega|^{(p_i'^+ - \kappa_i^-)/p_i'^-}.\end{aligned} \tag{40}$$

By Vitali Theorem, we have

$$\begin{aligned}&a_i\left(x, u_n, \nabla u_n\right) \longrightarrow a_i\left(x, u, \nabla u\right) \quad \text{strongly in } L^{\kappa_i(x)}(\Omega),\\&T_n^i\left(x, \nabla u_n\right) \longrightarrow T_i\left(x, \nabla u\right) \quad \text{strongly in } L^{\kappa_i(x)}(\Omega),\end{aligned} \tag{41}$$

for any $\kappa_i(x) \in [1, p'_i(x)]$. Hence, we complete the proof of Lemma 8. □

Lemma 9. *If $\overline{p(x)} < N$, $p_\infty(x) = \overline{p}^*(x)$, a_i satisfies the hypotheses (A1), (A2), the hypotheses (F1) and (H1) hold, for $i = 1, 2, \ldots, N$, $x \in \Omega$, $p_i(x)$ satisfies*

$$2 \leq p_i(x) \leq \frac{2Nr_i(x)}{Nr_i(x) - 2r_i(x) - 2N}, \tag{42}$$

where $r_i(x) \geq \max\{N, p_\infty(x)p_i(x)/(p_\infty(x) - p_i(x))\}$, T_i satisfies (T1) and (25). Then there exists a unique weak solution for the problem (1).

Proof. Suppose that there exist two distinct weak solutions u and v for the problem (1). Denote $\varphi = \max\{u - v, 0\}$, and for $t \in [1, \sup\varphi]$, let $\Omega_t = \{x \in \Omega : t < \varphi < \sup\varphi\}$. Otherwise, if $t \in [0, 1)$, we choose $m = 1/t$ and let $\Omega_m = \{x \in \Omega : m < \varphi < \sup\varphi\}$. We choose

$$\varphi_t = \begin{cases}\varphi - t, & \varphi > t,\\ 0, & \text{otherwise},\end{cases} \tag{43}$$

as test function. From (A2) and (25), we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega_t}\left|\partial_{x_i}\varphi_t\right|^2\left(\varepsilon + \left|\partial_{x_i} u\right| + \left|\partial_{x_i} v\right|\right)^{p_i(x)-2}\\&\leq \frac{1}{\gamma}\sum_{i=1}^{N}\int_{\Omega_t} k_i(x)\left(\left|\partial_{x_i} u\right| + \left|\partial_{x_i} v\right|\right)^{\delta_i(x)}\left|\partial_{x_i}\varphi_t\right|\varphi_t\end{aligned} \tag{44}$$

when $\delta_+(x) \geq (p_i(x)-2)/2$. By Young inequality, for c independent on t, we obtain

$$\sum_{i=1}^{N}\int_{\Omega_t}\left|\partial_{x_i}\varphi_t\right|^2\left(\varepsilon+\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{p_i(x)-2} \leq c\sum_{i=1}^{N}\int_{\Omega_t}k_i(x)^2\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{2\delta_+(x)-p_i(x)+2}\varphi_t^2. \quad (45)$$

Then we have

$$\begin{aligned}\frac{1}{C}\left(\int_{\Omega_t}\varphi_t^{2^*}\right)^{2/2^*} &\leq \prod_{i=1}^{N}\left(\int_{\Omega_t}\left|\partial_{x_i}\varphi_t\right|^2\right)^{1/N}\\ &\leq c_1\prod_{i=1}^{N}\left(\int_{\Omega_t}\left|\partial_{x_i}\varphi_t\right|^2\left(\varepsilon+\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{p_i(x)-2}\right)^{1/N}\\ &\leq c_1\sum_{i=1}^{N}\int_{\Omega_t}k_i(x)^2\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{2\delta_+(x)-p_i(x)+2}\varphi_t^2\\ &\leq c_1\left(\int_{\Omega_t}\varphi_t^{2^*}\right)^{2/2^*}\\ &\quad\times\sum_{i=1}^{N}\left(\int_{\Omega_t}k_i(x)^N\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{(2\delta_+(x)-p_i(x)+2)(N/2)}\right)^{2/N},\end{aligned} \quad (46)$$

where c_1 is a constant independent on t. Hence, we have

$$\frac{1}{C} \leq \sum_{i=1}^{N}\left(\int_{\Omega_t}k_i(x)^N\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{(2\delta_+(x)-p_i(x)+2)(N/2)}\right)^{2/N}. \quad (47)$$

Since $N/r_i(x)+((2\delta_i(x)-p_i(x)+2)/p_i(x))(N/2) \leq 1$, we have

$$\lim_{t\to\sup\varphi}\int_{\Omega_t}k_i(x)^N\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{(2\delta_+(x)-p_i(x)+2)(N/2)} = 0. \quad (48)$$

Therefore, (48) leads to a contradiction.

On the other hand, if $\delta_+(x) < (p_i(x)-2)/2$, from (44), and Young inequality, for some constant c_2 (independent on t), we have

$$\int_{\Omega_t}\left|\partial_{x_i}\varphi_t\right|^2 \leq c_2\sum_{i=1}^{N}\int_{\Omega_t}k_i(x)^2\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{2\delta_+(x)}\varphi_t^2. \quad (49)$$

Argue as in (46), for some constant c_3 (independent on t), we have

$$\frac{1}{C} \leq c_3\sum_{i=1}^{N}\left(\int_{\Omega_t}k_i(x)^N\left(\left|\partial_{x_i}u\right|+\left|\partial_{x_i}v\right|\right)^{N\delta_+(x)}\right)^{2/N}. \quad (50)$$

Since $N/r_i(x)+N\delta_i(x)/p_i(x) \leq 1$, we obtain a contradiction. □

Lemma 10. *If (A1), (T1), (F1), and (H1) hold, for $i=1,2,\ldots,N$, $2 \leq p_i(x)$ and $\sum_{i=1}^{N}(1/p_i(x)) < 2$, $q \in C_+^0(\Omega)$ and $q(x) < p_\infty = \overline{p}^*(x)$, and u is a weak solution of the problem* (1), *then $u \in L^\infty_{\mathrm{loc}}(\Omega)$.*

Proof. We denote $E = W_0^{1,\overrightarrow{p(x)}}(\Omega)$. Let $t>0$, $\varphi = \max\{u-t,0\}$, and choose φ as test function to the problem (1). We have

$$\sum_{i=1}^{N}\int_{\Omega}\left[a_i(x,u,\nabla u)\varphi_{x_i}+T_i(x,\nabla u)\varphi\right] = \int_{\Omega}\left[f\varphi+\sum_{i=1}^{N}h_i\varphi_{x_i}\right]. \quad (51)$$

Let $x_0 \in \Omega$, fix $k>1$, $B(x_0,R)$ be a ball, and let $\delta \in (0,1)$, $\delta R < s < t < R$, and $A_{k,t} = \{x \in B(x_0,t),\ u(x) > k\}$. Using (51), (A1), (F1), (H1), and Hölder inequality, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{A_{k,s}}\left|\partial_{x_i}u\right|^{p_i(x)} &\leq \sum_{i=1}^{N}\int_{A_{k,t}}\left|\partial_{x_i}u\right|^{p_i(x)}\\ &\leq c\Bigg[\|f\|_{L^{p'_\infty}(A_{k,t})}\|u-k\|_{L^{p_\infty}(A_{k,t})}\\ &\quad+\sum_{i=1}^{N}\int_{A_{k,t}}\left|T_i(x,\nabla u)\right||u-k|\\ &\quad+\sum_{i=1}^{N}\int_{A_{k,t}}h_i\partial_{x_i}u\Bigg].\end{aligned} \quad (52)$$

For $2 \leq p_i(x)$ and (T1), by Young inequality, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{A_{k,t}}\left|T_i(x,\nabla u)\right||u-k| &\leq \sum_{i=1}^{N}\left(\frac{1}{2}\int_{A_{k,t}}|u-k|^2+\frac{1}{2}\int_{A_{k,t}}\left|T_i(x,\nabla u)\right|^2\right)\\ &\leq \frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}|u-k|^{p_i(x)}+\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}|u|^{q(x)}\\ &\leq c\int_{A_{k,t}}\left|\frac{u-k}{t-s}\right|^{\overline{p(x)}^*}+\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}|u|^{q(x)},\\ \sum_{i=1}^{N}\int_{A_{k,t}}h_i\partial_{x_i}u &\leq \frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}h_i^2+\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}\left|\partial_{x_i}u\right|^2\\ &\leq \frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}h_i^2+\frac{1}{2}\sum_{i=1}^{N}\int_{A_{k,t}}\left|\partial_{x_i}u\right|^{p_i(x)}.\end{aligned} \quad (53)$$

As $2 \leq p_i(x)$ and $\sum_{i=1}^{N}(1/p_i(x)) < 2$, if $\overline{p(x)} < N$, there exists $\varepsilon = 1/2 - \overline{p(x)}/\overline{p(x)}^* > 0$ such that

$$c\left(\int_{A_{k,t}}h_i^2\right)^{1/2}\left|A_{k,t}\right|^{1/2} \leq c\left|A_{k,t}\right|^{\overline{p(x)}/\overline{p(x)}^*+\varepsilon}. \quad (54)$$

If $\overline{p(x)} \geq N$, a similar estimate is true; just only choose a suitable $\overline{p(x)}^*$.

Due to $q \in C_+^0(\Omega)$ and $q(x) < p_\infty(x) = \overline{p(x)}^*$, we have

$$\|f\|_{L^{p'_\infty}} \|u-k\|_{L^{p_\infty}} \leq c \int_{A_{k,t}} |u|^{q(x)}. \tag{55}$$

From

$$\begin{aligned} \int_{A_{k,t}} |u|^{q(x)} &= \int_{A_{k,t}} |(u-k)+k|^{q(x)} \\ &\leq c \int_{A_{k,t}} |u-k|^{q(x)} + ck^{q^+} |A_{k,t}| \\ &\leq c \int_{A_{k,t}} \left|\frac{u-k}{t-s}\right|^{\overline{p(x)}^*} + ck^{q^+} |A_{k,t}|, \end{aligned} \tag{56}$$

and (52)–(55), we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{A_{k,s}} \left|\partial_{x_i} u\right|^{p_i(x)} \\ &\leq \frac{1}{2} \sum_{i=1}^{N} \int_{A_{k,t}} \left|\partial_{x_i} u\right|^{p_i(x)} + c \int_{A_{k,t}} \left|\frac{u-k}{t-s}\right|^{\overline{p(x)}^*} \\ &\quad + ck^{q^+} |A_{k,t}| + c|A_{k,t}|^{\overline{p(x)}/\overline{p(x)}^*+\varepsilon}. \end{aligned} \tag{57}$$

Using Lemma 3.1 in [17], and (57), we have

$$\begin{aligned} &\sum_{i=1}^{N} \int_{A_{k,\delta R}} \left|\partial_{x_i} u\right|^{p_i(x)} \\ &\leq c \int_{A_{k,R}} \left|\frac{u-k}{R-\delta R}\right|^{\overline{p(x)}^*} + ck^{q^+} |A_{k,R}| \\ &\quad + c|A_{k,R}|^{\overline{p(x)}/\overline{p(x)}^*+\varepsilon}. \end{aligned} \tag{58}$$

By Lemma 2.4 in [18], (58), we obtain that u is bounded from above on $B(x_0, R/2)$. Note that $-u$ is also a weak solution of the problem (1), where $\tilde{a}_i(x,u,\nabla u) = a_i(x,-u,-\nabla u)$ and $\tilde{T}_i(x,\nabla u) = T_i(x,-\nabla u)$. Hence, $-u$ is also bounded from above on $B(x_0, R/2)$, and $u \in L^\infty(B(x_0, R/2))$. This implies that $u \in L^\infty_{\mathrm{loc}}(\Omega)$. □

Proof of Theorem 5. From Lemmas 8, 9, and 10, we obtain that there exists a uniqueness weak solution $u \in W_0^{1,\overrightarrow{p(x)}}(\Omega)$ for the problem (1), and what is more, $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Hence we complete the proof of Theorem 5. □

4. The Proof of Theorem 6

In this section, we prove the regularity of weak solutions for the problem (2).

Proof of Theorem 6. We denote $E = W_0^{1,\overrightarrow{p(x)}}(\Omega)$. For understanding, we write $a^i(u, Du) = a_i(u, \nabla u)$, and $h^i = h_i$. If g is any function with compact support in Ω, for $1 \leq k \leq N$, for small enough m, we define

$$\Delta_m^k g(x) = \frac{g(x+me_k) - g(x)}{m}, \tag{59}$$

where $e_k = (0,0,\ldots,\underset{k}{1},\ldots,0)$. Hence, if for any $\phi \in E$, ϕ has compact support in Ω and $|m| < (1/2)\operatorname{dist}(\operatorname{supp}\phi, \partial\Omega)$; from the definition of weak solution for the problem (1), we have

$$\sum_{i=1}^{N} \int_\Omega \Delta_m^k a^i(x,Du)\phi_{x_i} - \int_\Omega \Delta_m^k f \phi - \sum_{i=1}^{N} \int_\Omega \Delta_m^k h^i \phi_{x_i} = 0. \tag{60}$$

For the definition of Δ_m^k and [19], we have

$$\begin{aligned} &\Delta_m^k a^i(x,Du) \\ &\quad = \frac{1}{m}\int_0^1 \frac{d}{dt}\left[a^i\left(x+tme_k, Du + tm\Delta_m^k Du\right)\right] dt \\ &\quad = \int_0^1 \left[a^i_{x_k} + \sum_{j=1}^{N} a^i_{\xi_j} D_j\left(\Delta_m^k u\right)\right] dt \\ &\quad \doteq e^i + g_{ij} D_j\left(\Delta_m^k u\right), \end{aligned} \tag{61}$$

where $e^i = \int_0^1 a^i_{x_k} dt$, $g_{ij} = \int_0^1 \sum_{j=1}^N a^i_{\xi_j} dt$, and

$$\begin{aligned} \Delta_m^k f &= \frac{1}{m}\int_0^1 \frac{d}{dt} f(x+tme_k)\, dt \\ &= \int_0^1 f_{x_k} dt \doteq F, \\ \Delta_m^k h^i &= \frac{1}{m}\int_0^1 \frac{d}{dt} h^i(x+tme_k)\, dt \\ &= \int_0^1 h^i_{x_k} dt \doteq H^i. \end{aligned} \tag{62}$$

Then, combining (60) with (61)-(62), we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega e^i D_i\phi + \sum_{i,j=1}^{N} \int_\Omega g_{ij} D_j\left(\Delta_m^k u\right) D_i\phi \\ &\quad - \int_\Omega F\phi - \sum_{i=1}^{N} \int_\Omega H^i D_i\phi = 0. \end{aligned} \tag{63}$$

Due to (A4), (F2), (H2), (63), Young inequality, we have

$$\begin{aligned} &\sum_{i,j=1}^{N} \int_\Omega g_{ij} D_j\left(\Delta_m^k u\right) D_i\phi \\ &\quad = \int_\Omega F\phi + \sum_{i=1}^{N} \int_\Omega H^i D_i\phi - \sum_{i=1}^{N} \int_\Omega e^i D_i\phi \\ &\quad \leq c\int_\Omega |D\phi|^2 + c\int_\Omega |u|^{q(x)} + c\int_\Omega |u|^{q(x)/2}\phi \\ &\quad \leq c\int_\Omega |D\phi|^2 + c\int_\Omega |u|^{q(x)} + c\int_\Omega \phi^2; \end{aligned} \tag{64}$$

that is,

$$\sum_{i,j=1}^{N}\int_{\Omega} g_{ij}D_j\left(\Delta_m^k u\right)D_i\phi \le c\int_{\Omega}\left|D\phi\right|^2 + c\int_{\Omega}|u|^{q(x)} + c\int_{\Omega}\phi^2. \tag{65}$$

Choose $\phi = \psi^2\Delta_m^k u$, where $\psi \in C_0^1(\overline{\Omega})$, $|\psi| \le 1$, and (65), we obtain

$$\sum_{i,j=1}^{N}\int_{\Omega} g_{ij}\psi^2 D_i\left(\Delta_m^k u\right)D_j\left(\Delta_m^k u\right) \le -2\sum_{i,j=1}^{N}\int_{\Omega}\psi u g_{ij}D_i\psi D_j\left(\Delta_m^k u\right) + c\int_{\Omega}\left|D\psi^2\Delta_m^k u\right|^2 + c\int_{\Omega}|u|^{q(x)} + c\int_{\Omega}\left(\psi^2\Delta_m^k u\right)^2. \tag{66}$$

From (66), (A3), Cauchy inequality and Young inequality, we have

$$\begin{aligned}
&\sum_{i,j=1}^{N}\int_{\Omega}\left|g_{ij}\psi^2 D_i\left(\Delta_m^k u\right)D_j\left(\Delta_m^k u\right)\right| \\
&\le 2\sum_{i,j=1}^{N}\int_{\Omega}\left|\psi u g_{ij}D_i\psi D_j\left(\Delta_m^k u\right)\right| \\
&\qquad + c\int_{\Omega}\left|D\psi^2\Delta_m^k u\right|^2 + c\int_{\Omega}|u|^{q(x)} + c\int_{\Omega}|\psi|^4\left|\Delta_m^k u\right|^2 \\
&\le 2\sum_{i,j=1}^{N}\left[\int_{\Omega} g_{ij}\psi^2 D_i\left(\Delta_m^k u\right)D_i\left(\Delta_m^k u\right)\right]^{1/2} \\
&\qquad \times\left[\int_{\Omega} g_{ij}\left(\Delta_m^k u\right)^2 D_i\psi D_j\psi\right]^{1/2} + c\int_{\Omega}\left|D\psi^2\Delta_m^k u\right|^2 \\
&\qquad + c\int_{\Omega}|u|^{q(x)} + c\int_{\Omega}|\psi|^4\left|\Delta_m^k u\right|^2 \\
&\le 2\sum_{i,j=1}^{N}\left[\frac{\varepsilon}{2}\left(\int_{\Omega} g_{ij}\psi^2 D_i\left(\Delta_m^k u\right)D_i\left(\Delta_m^k u\right)\right) + \frac{1}{2\varepsilon}\right. \\
&\qquad \left.\times\left(\int_{\Omega} g_{ij}\left(\Delta_m^k u\right)^2 D_i\psi D_j\psi\right)\right] + c\int_{\Omega}|u|^{q(x)} \\
&\quad + c\int_{\Omega}\psi^2\left|D\left(\Delta_m^k u\right)\right|^2 \\
&\quad + c\int_{\Omega}\left|\Delta_m^k u\right|^2\left|D\psi\right|^2 + c\int_{\Omega}\left|\Delta_m^k u\right|^2.
\end{aligned} \tag{67}$$

By (A3), we obtain

$$\begin{aligned}
&\int_{\Omega}\psi^2\left|D\left(\Delta_m^k u\right)\right|^2 \\
&\le c\int_{\Omega}\left[1 + \sum_{i=1}^{N}\left(\left|D_i u(x)\right|^{p_i(x)-2}\right.\right. \\
&\qquad\qquad \left.\left. + \left|D_i u\left(x + me_k\right)\right|^{p_i(x)-2}\right)\right] \\
&\qquad \times\left|\Delta_m^k u\right|^2\left|D\psi\right|^2 + c\int_{\Omega}|u|^{q(x)}.
\end{aligned} \tag{68}$$

Assume $p_N = p_+^+ = \sup_{x\in\overline{\Omega}} p_+(x)$; we have for $i = 1, 2, \ldots, N$, $p_i(x) \le p_N$. For $q(x) < \overline{p}^*(x)$, let $k = N$ and take $m \to 0$; then we have

$$\int_{\Omega}\psi^2\left|D\left(D_N u\right)\right|^2 \le c\int_{\Omega}\left(1 + \sum_{i=1}^{N}\left|D_i u\right|^{p_i(x)-2}\right)\left|D\psi\right|^2\left|D_N u\right|^2. \tag{69}$$

Hence, $D_N u \in W_{\mathrm{loc}}^{1,2}(\Omega)$. If we take $\phi = \psi^2(\Delta_m^N u - r)^+$, where $r > 0$, $|\psi| < 1$, $(\Delta_m^N u - r)^+ = \max\{\Delta_m^N u - r, 0\}$. Since $D_N u \in W_{\mathrm{loc}}^{1,2}(\Omega)$, $D_N u \in L_{\mathrm{loc}}^{2N/(N-2)}(\Omega)$, we have

$$\begin{aligned}
&\int_{\Omega}\psi^2\left|D(D_N u - r)^+\right|^2 \\
&\le c\int_{\Omega}\left(1 + \sum_{i=1}^{N}\left|D_i u\right|^{p_i(x)-2}\right)\left|D\psi\right|^2\left|(D_N u - r)^+\right|^2 \\
&\le c\int_{\Omega}\left(1 + \sum_{i=1}^{N-1}\left|D_i u\right|^{p_i(x)-2}\right)\left|D\psi\right|^2\left|(D_N u - r)^+\right|^2 \\
&\quad + c\int_{\Omega}\left|D_N u\right|^{p_N-2}\left|D\psi\right|^2\left|(D_N u - r)^+\right|^2 \\
&\le c\int_{\Omega}\left|(D_N u - r)^+\right|^{p_N}\left|D\psi\right|^2 + cr^{p_N-2}\int_{\Omega}\left|(D_N u - r)^+\right|^2 \\
&\quad + c\sum_{i=1}^{N}\int_{\Omega}\left(\left|(D_N u - r)^+\right|^{2N/(N-2)}\left|D\psi\right|^{2N/(N-2)}\right)^{(N-2)/N} \\
&\qquad \times\left(\int_{\{D_N u > k\}\bigcap \operatorname{supp}\psi}\left(1 + \left|D_i u\right|^{p_i(x)-2}\right)^{N/2}\right)^{2/N}.
\end{aligned} \tag{70}$$

For $2 \le p_i(x) \le 2N/(N-2)$, we have

$$\left(\int_{\{D_N u > k\}\bigcap \operatorname{supp}\psi}\left(1 + \left|D_i u\right|^{p_i(x)-2}\right)^{N/2}\right)^{2/N} \le c\left|\{D_N u > k\}\bigcap \operatorname{supp}\psi\right|^{2/N-2/p_-^-+1}. \tag{71}$$

If we fix $B(x_0, R)$, supp $\psi \subset B(x_0, R)$, where $R < 1$; let $r \geq 1$, $0 < \delta < 1$, and take $\psi \in C_0^1(B(x_0, R))$, $\psi \equiv 1$ on $B(x_0, \delta R)$; $0 \leq \psi \leq 1$, $|D\psi| \leq c/(R - \delta R)$, then we have

$$\begin{aligned}
&\int_{A_{r,\delta R}} \left|D(D_N u - r)^+\right|^2 \\
&\quad \leq c \int_{A_{r,R}} \left(\frac{|D_N u - r|}{R - \delta R}\right)^{2N/(N-2)} \\
&\qquad + c\left(\int_{A_{r,R}} \left(\frac{|D_N u - r|}{R - \delta R}\right)^{2N/(N-2)}\right)^{(N-2)/N} \\
&\qquad \times \left|A_{r,R}\right|^{2/N-2/p_-^-+1} + cr^{2N/(N-2)} \left|A_{r,R}\right|.
\end{aligned} \tag{72}$$

Hence, (72) implies that $D_N u$ is in $L^\infty_{\text{loc}}(\Omega)$. So we obtain $u \in W^{1,\infty}_{\text{loc}}(\Omega) \bigcap W^{2,2}_{\text{loc}}(\Omega)$. By De Giorgi-Moser regularity theorem, for any $\omega \subset\subset \Omega$ and $0 < \alpha < 1$, we have $D_N u \in C^{0,\alpha}(\omega)$; then $u \in C^{1,\alpha}(\Omega)$, for $0 < \alpha < 1$. □

Acknowledgments

The authors express their sincere thanks to the referees for their valuable suggestions. This paper was supported by Shanghai Natural Science Foundation Project (no. 11ZR1424500) and Shanghai Leading Academic Discipline Project (no. XTKX2012).

References

[1] Y. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration," *SIAM Journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006.

[2] M. Mihăilescu and V. Rădulescu, "A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids," *Proceedings of The Royal Society of London A*, vol. 462, no. 2073, pp. 2625–2641, 2006.

[3] M. Ružička, *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer, Berlin, Germany, 2000.

[4] V. V. Zhikov, "Averaging of functionals of the calculus of variations and elasticity theory, Math," *USSR-Izvestiya*, vol. 29, pp. 33–66, 1987.

[5] D. E. Edmunds, J. Lang, and A. Nekvinda, "On $L^{p(x)}$ norms," *The Royal Society of London A*, vol. 455, no. 1981, pp. 219–225, 1999.

[6] X. Fan and D. Zhao, "On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 424–446, 2001.

[7] D. E. Edmunds and J. Rákosník, "Sobolev embeddings with variable exponent," *Studia Mathematica*, vol. 143, no. 3, pp. 267–293, 2000.

[8] X. Fan, J. Shen, and D. Zhao, "Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$," *Journal of Mathematical Analysis and Applications*, vol. 262, no. 2, pp. 749–760, 2001.

[9] M. Mihăilescu, P. Pucci, and V. Rădulescu, "Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 687–698, 2008.

[10] M.-M. Boureanu, P. Pucci, and V. D. Rădulescu, "Multiplicity of solutions for a class of anisotropic elliptic equations with variable exponent," *Complex Variables and Elliptic Equations*, vol. 56, no. 7-9, pp. 755–767, 2011.

[11] D. Stancu-Dumitru, "Multiplicity of solutions for a nonlinear degenerate problem in anisotropic variable exponent spaces," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 36, no. 1, pp. 117–130, 2013.

[12] D. Stancu-Dumitru, "Two nontrivial solutions for a class of anisotropic variable exponent problems," *Taiwanese Journal of Mathematics*, vol. 16, no. 4, pp. 1205–1219, 2012.

[13] X. Fan, "Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p(x)}$-Laplacian equations," *Complex Variables and Elliptic Equations*, vol. 56, no. 7–9, pp. 623–642, 2011.

[14] L. Diening, P. Harjulehto, P. Hästö, and M. Ružička, *Lebesgue and Sobolev Spaces with Variable Exponents*, vol. 2017 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2011.

[15] J. Leray and J.-L. Lions, "Quelques résulatats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder," *Bulletin de la Société Mathématique de France*, vol. 93, pp. 97–107, 1965.

[16] L. Boccardo and F. Murat, "Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 19, no. 6, pp. 581–597, 1992.

[17] G. Mariano, *Multiple Integrals in the Calculus of Variations and Nonlinear elliptic Systems*, Princeton University Press, Princeton, NJ, USA, 1983.

[18] N. Fusco and C. Sbordone, "Some remarks on the regularity of minima of anisotropic integrals," *Communications in Partial Differential Equations*, vol. 18, no. 1-2, pp. 153–167, 1993.

[19] P. Marcellini, "Regularity and existence of solutions of elliptic equations with p, q-growth conditions," *Journal of Differential Equations*, vol. 90, no. 1, pp. 1–30, 1991.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 182569, 7 pages
http://dx.doi.org/10.1155/2013/182569

Research Article

Linear Transformations between Multipartite Quantum Systems That Map the Set of Tensor Product of Idempotent Matrices into Idempotent Matrix Set

Jinli Xu,[1] Baodong Zheng,[1] and Hongmei Yao[2]

[1] *Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China*
[2] *College of Science, Harbin Engineering University, Harbin 150001, China*

Correspondence should be addressed to Baodong Zheng; zbd@hit.edu.cn

Received 12 May 2013; Accepted 16 July 2013

Academic Editor: Gen-Qi Xu

Let H_n be the set of $n \times n$ complex Hermitian matrices and $\mathscr{P}_n$ (resp., $\mathscr{T}_n$) be the set of all idempotent (resp., tripotent) matrices in H_n. In l-partite quantum system $H_{m_1 \cdots m_l} = \otimes_1^l H_{m_i}$, $\otimes_1^l \mathscr{P}_{m_i}$ (resp., $\otimes_1^l \mathscr{T}_{m_i}$) denotes the set of all decomposable elements $\otimes_1^l A_i$ such that $A_i \in \mathscr{P}_{m_i}$ (resp., $A_i \in \mathscr{T}_{m_i}$). In this paper, linear maps ϕ from $H_{m_1 \cdots m_l}$ to H_n with $n \le m_1 \cdots m_l$ such that $\phi(\otimes_1^l \mathscr{P}_{m_i}) \in \mathscr{P}_n$ are characterized. As its application, the structure of linear maps ϕ from $H_{m_1 \cdots m_l}$ to H_n with $n \le m_1 \cdots m_l$ such that $\phi(\otimes_1^l \mathscr{T}_{m_i}) \in \mathscr{T}_n$ is also obtained.

1. Introduction

Let M_n be the vector space of $n \times n$ complex matrices and $H_n \subseteq M_n$ be the vector space of $n \times n$ complex Hermitian matrices. In quantum information theory, a l-partite system can be represented as the tensor product space $H_{m_1 \cdots m_l} \equiv \otimes_1^l H_{m_i}$, where $\otimes$ is the usual Kronecker product of matrices; a quantum state is represented as a positive semidefinite with trace one in H_n, see [1]. In quantum information science and quantum computing, it is important to understand, characterize, and construct different classes of maps on quantum states [2]. For example, entanglement is one of the main concepts in quantum information theory, and entangled state involves at least bipartite system or multipartite system [3]; to study entangled states, one should construct entanglement witnesses, which are special types of positive maps, see [4]. On this background, the research on the characterizations of maps leaving invariant, some important subsets or quantum properties attracted more and more researchers' attention, see ([1, 3, 5]). Especially, in [5], Lim characterized the linear and additive maps on tensor products of spaces of Hermitian matrices that carry the set of tensor product of rank one matrices into itself.

Preserver problem is a hot area in Banach algebra; there are many results about this area, see ([6–11]). Specially, the idempotent preservers and the rank one preservers play an important role; therefore, it is meaningful to study the two preservers. Chan et al. [12] first characterized linear transformations on M_n preserving idempotent matrices. In [13], the authors obtained the following result: linear map $T : H_n \to H_m$ with $n \le m$ satisfies $T(\mathscr{P}_n) \subset \mathscr{P}_m$ if and only if $T = 0$ or

$$T(X) = QXQ^{-1} \quad \text{or} \quad T(X) = QX^TQ^{-1}, \quad \forall X \in H_n, \quad (1)$$

where Q is unitary matrix.

Since the study of linear transformations, preserving idempotents is important in many aspects of mathematics and physics, see [14–17], and inspired by the above, the purpose of this paper is to study linear maps from l-partite system to the space of Hermitian matrices that carry the set of tensor product of idempotent matrices into the set of idempotent matrices in the space of Hermitian matrices, that is, $\phi : H_{m_1 \cdots m_l} \to H_n$ with $n \le m_1 \cdots m_l$ satisfying

$$\phi\left(\otimes_1^l \mathscr{P}_{m_i}\right) \subset \mathscr{P}_n, \quad (2)$$

where $\mathscr{P}_n$ is the set of all idempotent matrices in H_n, that is, $\mathscr{P}_n = \{A \in H_n | A^2 = A\}$ and $\otimes_1^l \mathscr{P}_{m_i}$ denotes the set of all decomposable elements $\otimes_1^l A_i$ such that $A_i \in \mathscr{P}_{m_i}$. As application, the forms of linear maps from l-partite system to the space of Hermitian matrices that carry the set of tensor product of tripotent matrices into the set of tripotent matrices in the space of Hermitian matrices are obtained, that is, $\phi : H_{m_1 \cdots m_l} \to H_n$ with $n \le m_1 \cdots m_l$ satisfying

$$\phi\left(\otimes_1^l \mathscr{T}_{m_i}\right) \subset \mathscr{T}_n, \tag{3}$$

where $\mathscr{T}_n$ is the set of all tripotent matrices in H_n, that is, $\mathscr{T}_n = \{A \in H_n | A^3 = A\}$ and $\otimes_1^l \mathscr{T}_{m_i}$ denotes the set of all decomposable elements $\otimes_1^l A_i$ such that $A_i \in \mathscr{T}_{m_i}$.

Throughout this paper, we always assume integers $l \ge 1$ and $n, m_1, \ldots, m_l \ge 2$. Let I_k be the $k \times k$ identity matrix, 0 be the zero matrix which order is omitted in different matrices just for simplicity, and X^T (resp., X^*, rank X) be the transpose (resp., conjugate transpose, rank) of X. $E_{ij}^{(n)}$ $(1 \le i, j \le n)$ stands for the $n \times n$ matrix with 1 at the (i, j)th entry and 0 otherwise. $D_{ij}^{(n)} = E_{ij}^{(n)} + E_{ji}^{(n)}$ and $K_{ij}^{(n)} = \sqrt{-1}E_{ij}^{(n)} - \sqrt{-1}E_{ji}^{(n)}$, where $1 \le i \ne j \le n$. For real numbers a and b with $a \le b$, let $[a, b]$ be the set of all integers between a and b. $\oplus$ is the usual direct sum of matrices. A linear map $\pi = \otimes_1^l \pi_i : H_{m_1 \cdots m_l} \to H_{m_1 \cdots m_l}$ is canonical if π satisfies $\pi(\otimes_1^l A_i) = \otimes_1^l \pi_i(A_i)$, where $\pi_i : H_{m_i} \to H_{m_i}$ satisfies $\pi_i(A_i) = A_i$ or $\pi_i(A_i) = A_i^T$ for all $A_i \in H_{m_i}$.

2. Main Results

Lemma 1 (see [18]). *Suppose $P_1, \ldots, P_k \in \mathscr{P}_n$ such that $P_i + P_j \in \mathscr{P}_n$, $\forall 1 \le i < j \le k$. Let $r_i = \operatorname{rank} P_i$. Then, there exists a unitary matrix $U \in M_n$ such that*

$$P_i = U \operatorname{diag}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0) U^*, \tag{4}$$

where $\operatorname{diag}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ *is the diagonal matrix in which all diagonal entries are zero except those in the $(r_1 + \cdots + r_{i-1} + 1)$st to the $(r_1 + \cdots + r_i)$th rows.*

Lemma 2. *Let $P \in \mathscr{P}_r$. Suppose $X, 0_s \oplus P \oplus 0_{n-r-s} - X \in \mathscr{P}_n$. Then, there exists $X_r \in \mathscr{P}_r$ such that $X = 0_s \oplus X_r \oplus 0_{n-r-s}$, where $0 \le r, s \le n$.*

Proof. Let $T \in M_r$ be a unitary matrix satisfying $P = T(I_t \oplus 0)T^*$. Set

$$\begin{bmatrix} I_r & 0 & 0 \\ 0 & T^* & 0 \\ 0 & 0 & I_{n-r-s} \end{bmatrix} X \begin{bmatrix} I_r & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & I_{n-r-s} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}, \tag{5}$$

where $X_{11} \in H_s$, $X_{22} \in H_t$. Then,

$$\begin{gathered} \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \in \mathscr{P}_n, \\ \begin{bmatrix} 0_r & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix} \in \mathscr{P}_n. \end{gathered} \tag{6}$$

A straightforward computation shows that $X = 0_s \oplus [T(X_{22} \oplus 0_{r-t})T^*] \oplus 0_{n-r-s}$; therefore, the result holds. □

Lemma 3. *Let $A = P_r \oplus 0_s \oplus 0_{n-r-s}$ and $B = 0_r \oplus P_s \oplus 0_{n-r-s}$ with $P_r \in \mathscr{P}_r$, $P_s \in \mathscr{P}_s$. If*

$$\alpha A + \beta B + \gamma X \in \mathscr{P}_n \quad \text{for any real } \gamma \text{ with } 0 \ne |\gamma| \le \frac{1}{2}, \tag{7}$$

where $X \in M_n$, α and β are all roots of $x^2 - x + \gamma^2 = 0$, then, $X^2 = AB$, and X is of the form

$$\begin{bmatrix} 0_r & X_{12} \\ X_{21} & 0_s \end{bmatrix} \oplus 0_{n-r-s}, \tag{8}$$

where $X_{12} \in M_{r \times s}$, $X_{21} \in M_{s \times r}$.

Proof. It is clear that $A, B \in \mathscr{P}_n$ and $AB = BA = 0$. By a direct computation, one can obtain that

$$X = \alpha(AX + XA) + \beta(BX + XB) + \gamma\left(X^2 - A - B\right). \tag{9}$$

Replacing γ by $-\gamma$, we have $X^2 = A + B$ and

$$X = \alpha(AX + XA) + \beta(BX + XB). \tag{10}$$

Choosing $\gamma = 2/5$, then $\alpha = 1/5$, $\beta = 4/5$ or $\alpha = 4/5$, $\beta = 1/5$, this yields that

$$\begin{gathered} X = \frac{1}{5}(AX + XA) + \frac{4}{5}(BX + XB), \\ X = \frac{4}{5}(AX + XA) + \frac{1}{5}(BX + XB). \end{gathered} \tag{11}$$

Thus, $X = AX + XA$ and $X = BX + XB$. This, together with the form of A and B, implies that X has the form $\left[\begin{smallmatrix} 0_r & X_{12} \\ X_{21} & 0_s \end{smallmatrix}\right] \oplus 0_{n-r-s}$, where $X_{12} \in M_{r \times s}$, $X_{21} \in M_{s \times r}$. □

Lemma 4. *Let $J = P_p \oplus -P_q \oplus 0_{n-p-q}$ with $P_p \in \mathscr{P}_p$ and $P_q \in \mathscr{P}_q$. Suppose $X \in \mathscr{T}_n$ satisfying $J - X \in \mathscr{T}_n$ and $J - 2X \in \mathscr{T}_n$. Then, $X = X_p \oplus -X_q \oplus 0_{n-p-q}$ with $X_p \in \mathscr{P}_p$ and $X_q \in \mathscr{P}_q$.*

Proof. By a direct computation, we have

$$3X = J^2 X + JXJ + XJ^2, \tag{12}$$

$$3X = JX^2 + XJX + X^2 J. \tag{13}$$

Let $U \in M_p$ and $V \in M_q$ be unitary matrices satisfying $P_p = U(I_r \oplus 0_{p-r})U^*$ and $P_q = V(I_s \oplus 0_{q-s})V^*$, respectively. Then,

$$\begin{bmatrix} U^* & 0 & 0 \\ 0 & V^* & 0 \\ 0 & 0 & I_{n-p-q} \end{bmatrix} J \begin{bmatrix} U & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & I_{n-p-q} \end{bmatrix} = \begin{bmatrix} I_r \oplus 0_{p-r} & 0 & 0 \\ 0 & -I_s \oplus 0_{q-s} & 0 \\ 0 & 0 & 0_{n-p-q} \end{bmatrix}. \quad (14)$$

By (12), one can assume that

$$\begin{bmatrix} U^* & 0 & 0 \\ 0 & V^* & 0 \\ 0 & 0 & I_{n-p-q} \end{bmatrix} X \begin{bmatrix} U & 0 & 0 \\ 0 & V & 0 \\ 0 & 0 & I_{n-p-q} \end{bmatrix} = \begin{bmatrix} X_r \oplus 0_{p-r} & 0 & 0 \\ 0 & -X_s \oplus 0_{q-s} & 0 \\ 0 & 0 & 0_{n-p-q} \end{bmatrix}, \quad (15)$$

where $X_r \in H_r$ and $X_s \in H_s$. This, together with (13), implies that $X_r \in \mathscr{P}_r$ and $X_s \in \mathscr{P}_s$. Thus,

$$X = \left[U\left(X_r \oplus 0_{p-r}\right)U^*\right] \oplus -\left[V\left(X_s \oplus 0_{q-s}\right)V^*\right] \oplus 0_{n-p-q}, \quad (16)$$

as desired. □

Lemma 5. *Let ψ_1, ψ_2 be canonical maps on $H_{m_1\cdots m_l}$, and U, V are invertible matrices of order $m_1\cdots m_l$. If $\psi_1(X)U = V\psi_2(X)$ for all $X \in H_{m_1\cdots m_l}$, then $\psi_1 = \psi_2$ and $U = V = \lambda I_{m_1\cdots m_l}$ for some $\lambda \neq 0$.*

Proof. Let us first point out a simple observation which will be used in our proof. If a matrix $U \in M_{mn}$ commutes with $I_m \otimes S$ for all real symmetric $S \in M_n$, then U has the form $W \otimes I_n$ with $W \in M_m$.

Since ψ_1, ψ_2 are canonical maps on $H_{m_1\cdots m_l}$, we assume that

$$\psi_1 = \tau_1 \otimes \cdots \otimes \tau_l, \qquad \psi_2 = \eta_1 \otimes \cdots \otimes \eta_l, \quad (17)$$

where τ_i and η_i are the identity maps $X \mapsto X$ or the transposition maps $X \mapsto X^T$ for $i \in [1, l]$.

Clearly, $\psi_i(I_{m_1\cdots m_l}) = I_{m_1\cdots m_l}$, $i = 1, 2$, hence $U = V$.

We prove $U = \lambda I_{m_1\cdots m_l}$ by induction on l. The case of $l = 1$ is the well-known fact. We assume that the statement holds true for $l-1$. We will prove it for l. For any real symmetric $S \in M_{m_l}$, since $\psi_i(I_{m_1\cdots m_{l-1}} \otimes S) = I_{m_1\cdots m_{l-1}} \otimes S$, $i = 1, 2$, it follows that $U = W \otimes I_{m_l}$ for some matrix $W \in M_{m_1\cdots m_{l-1}}$. We define linear maps

$$\varphi_1 = \tau_1 \otimes \cdots \otimes \tau_{l-1}, \qquad \varphi_2 = \eta_1 \otimes \cdots \otimes \eta_{l-1}. \quad (18)$$

It is easy to see that φ_1 and φ_2 are canonical maps on $H_{m_1\cdots m_{l-1}}$. For any $Y \in H_{m_1\cdots m_{l-1}}$, since

$$\begin{aligned}(\varphi_1(Y)W) \otimes I_{m_l} &= \psi_1\left(Y \otimes I_{m_l}\right)U \\ &= U\psi_2\left(Y \otimes I_{m_l}\right) \\ &= (W\varphi_2(Y)) \otimes I_{m_l},\end{aligned} \quad (19)$$

we have $\varphi_1(Y)W = W\varphi_2(Y)$, thus, by induction hypothesis, $W = \lambda I_{m_1\cdots m_{l-1}}$. Since U is invertible, we have $\lambda \neq 0$. Thus, $U = \lambda I_{m_1\cdots m_l}$ and $\psi_1(X) = (1/\lambda)\psi_1(X)U = (1/\lambda)U\psi_2(X) = \psi_2(X)$ for all $X \in H_{m_1\cdots m_l}$. □

Theorem 6. *Suppose ϕ is a linear map from $H_{m_1\cdots m_l}$ to H_n with $n \le m_1\cdots m_l$. Then, $\phi(\otimes_1^l \mathscr{P}_{m_i}) \subset \mathscr{P}_n$ if and only if either $\phi = 0$ or $n = m_1\cdots m_l$, there exists a unitary matrix $U \in M_n$ and a canonical map π on $H_{m_1\cdots m_l}$ such that*

$$\phi(X) = U\pi(X)U^*, \quad \forall X \in H_{m_1\cdots m_l}. \quad (20)$$

Proof. The sufficiency part is obvious. We prove the necessity part by induction on l. When $l = 1$, it is the result in [13], which has been given in the above introduction. We assume now that the result holds true for $l-1$ and give the proof of the case l by the following five steps.

Step 1. Suppose $P \in \mathscr{P}_{m_1}$ and $\phi(P \otimes I_{m_2\cdots m_l}) = 0_s \oplus I_r \oplus 0_{n-r-s}$. Then,

$$\phi(P \otimes X) = 0_s \oplus \psi(X) \oplus 0_{n-r-s}, \quad \forall X \in H_{m_2\cdots m_l}, \quad (21)$$

where ψ is a linear map from $H_{m_2\cdots m_l}$ to H_r satisfying $\psi(\otimes_2^l \mathscr{P}_{m_i}) \subset \mathscr{P}_r$.

Proof of Step 1. Set

$$\begin{aligned}\Gamma_0 &= \left\{I_{m_2} \otimes \cdots \otimes I_{m_l}\right\}, \\ \Gamma_1 &= \left\{P_2 \otimes I_{m_3\cdots m_l} : P_2 \in \mathscr{P}_{m_2}\right\}, \\ \Gamma_2 &= \left\{P_2 \otimes P_3 \otimes I_{m_4\cdots m_l} : P_i \in \mathscr{P}_{m_i}, i \in [2,3]\right\}, \\ &\cdots \\ \Gamma_{l-1} &= \left\{P_2 \otimes \cdots \otimes P_{m_l} : P_i \in \mathscr{P}_{m_i}, i \in [2,l]\right\}.\end{aligned} \quad (22)$$

We prove by induction on k that $\phi(P \otimes X) = 0_s \oplus \psi(X) \oplus 0_{n-r-s}$, $\forall X \in \Gamma_k, k \in [0, l-1]$. Assume for a moment that we have already proved this. Then, when ϕ is linear, we can complete the proof of Step 1.

Now, we prove the assertion. The case of $k = 0$ is just the assumption. Then, we assume that our statement holds true for $k-1$, and we consider the image of $P \otimes P_2 \otimes \cdots \otimes P_{k+1} \otimes I_{m_{k+2}\cdots m_l}$ for $P_i \in \mathscr{P}_{m_i}$, $i \in [2, k+1]$. Because

$$\begin{aligned}&P \otimes P_2 \otimes \cdots \otimes P_k \otimes I_{m_{k+1}} \otimes I_{m_{k+2}\cdots m_l} \in \otimes_1^l \mathscr{P}_{m_i}, \\ &P \otimes P_2 \otimes \cdots \otimes P_k \otimes P_{k+1} \otimes I_{m_{k+2}\cdots m_l} \in \otimes_1^l \mathscr{P}_{m_i}, \\ &P \otimes P_2 \otimes \cdots \otimes P_k \otimes \left(I_{m_{k+1}} - P_{k+1}\right) \otimes I_{m_{k+2}\cdots m_l} \in \otimes_1^l \mathscr{P}_{m_i},\end{aligned} \quad (23)$$

we obtain using the property of ϕ that

$$\begin{aligned}&\phi\left(P \otimes P_2 \otimes \cdots \otimes P_k \otimes I_{m_{k+1}} \otimes I_{m_{k+2}\cdots m_l}\right) \in \mathscr{P}_n, \\ &\phi\left(P \otimes P_2 \otimes \cdots \otimes P_k \otimes P_{k+1} \otimes I_{m_{k+2}\cdots m_l}\right) \in \mathscr{P}_n, \\ &\phi\left(P \otimes P_2 \otimes \cdots \otimes P_k \otimes I_{m_{k+1}} \otimes I_{m_{k+2}\cdots m_l}\right), \\ &\quad - \phi\left(P \otimes P_2 \otimes \cdots \otimes P_{k+1} \otimes I_{m_{k+2}\cdots m_l}\right) \in \mathscr{P}_n.\end{aligned} \quad (24)$$

We obtain by induction hypothesis and Lemma 2 that $\phi(P \otimes P_2 \otimes \cdots \otimes P_{k+1} \otimes I_{m_{k+2}\cdots m_l}) = 0_s \oplus X_r \oplus 0_{n-r-s}$, where $X_r \in \mathscr{P}_r$, as desired.

Step 2. If $m_1 \cdots m_l > n$. Then, $\phi = 0$.

Proof of Step 2. Since $E_{ii}^{(m_1)} \otimes I_{m_2\cdots m_l} \in \otimes_1^l \mathscr{P}_{m_i}, \forall i \in [1, m_1]$ and $(E_{ii}^{(m_1)} + E_{jj}^{(m_1)}) \otimes I_{m_2\cdots m_l} \in \otimes_1^l \mathscr{P}_{m_i}, \forall i \neq j \in [1, m_1]$, we obtain using the property of ϕ that

$$\phi\left(E_{ii}^{(m_1)} \otimes I_{m_2\cdots m_l}\right) \in \mathscr{P}_n, \quad \forall i \in [1, m_1],$$
$$\phi\left(E_{ii}^{(m_1)} \otimes I_{m_2\cdots m_l}\right) + \phi\left(E_{jj}^{(m_1)} \otimes I_{m_2\cdots m_l}\right) \in \mathscr{P}_n, \quad \forall i \neq j \in [1, m_1]. \tag{25}$$

Denote by $r_i = \operatorname{rank} \phi(E_{ii}^{(m_1)} \otimes I_{m_2\cdots m_l})$, $\forall i \in [1, m_1]$. Using Lemma 1 and composing ϕ by a similarity transformation, we may obtain that

$$\phi\left(E_{ii}^{(m_1)} \otimes I_{m_2\cdots m_l}\right) = \operatorname{diag}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0), \tag{26}$$

where $\operatorname{diag}(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$ is the diagonal matrix in which all diagonal entries are zero except those in the $(r_1 + \cdots + r_{i-1} + 1)$st to the $(r_1 + \cdots + r_i)$th rows.

Since $n < m_1 \cdots m_l$, there exists some $r_i < m_2 \cdots m_l$. Without loss of generality, we may assume $r_1 < m_2 \cdots m_l$. Thus, $\phi(E_{11}^{(m_1)} \otimes I_{m_2\cdots m_l}) = I_{r_1} \oplus 0_{n-r_1}$. By Step 1, we see that there exists a linear map ψ from $H_{m_2\cdots m_l}$ to H_{r_1} satisfying $\psi(\otimes_2^l \mathscr{P}_{m_i}) \subset \mathscr{P}_{r_1}$ such that $\phi(E_{11}^{(m_1)} \otimes X) = \psi(X) \oplus 0_{n-r_1}$ for all $X \in H_{m_2\cdots m_l}$. We obtain by induction hypothesis $\psi = 0$. Thus,

$$\phi\left(E_{11}^{(m_1)} \otimes X\right) = 0, \quad \forall X \in H_{m_2\cdots m_l}. \tag{27}$$

For any $j \in [2, m_1]$, $P \in \otimes_2^l \mathscr{P}_{m_i}$ and real γ with $0 \neq |\gamma| \leq 1/2$, we have $(\alpha E_{11}^{(m_1)} + \beta E_{jj}^{(m_1)} + \gamma D_{1j}^{(m_1)}) \otimes P \in \otimes_1^l \mathscr{P}_{m_i}$, where α and β are all roots of $x^2 - x + \gamma^2 = 0$. We obtain using the property of ϕ, (27), and Lemma 3 that $\phi(E_{jj}^{(m_1)} \otimes P) = 0$ and $\phi(D_{1j}^{(m_1)} \otimes P) = 0$. By using the similar argument, we see that $\phi(D_{ij}^{(m_1)} \otimes P) = 0$, $\phi(K_{ij}^{(m_1)} \otimes P) = 0$, $\forall i \neq j \in [1, m_1]$. Thus,

$$\phi(A \otimes P) = 0, \quad \forall A \in H_{m_1}. \tag{28}$$

By the arbitrariness of $P \in \otimes_2^l \mathscr{P}_{m_i}$, we have

$$\phi(A \otimes X) = 0, \quad \forall A \in H_{m_1}, \ X \in H_{m_2\cdots m_l}. \tag{29}$$

Therefore, $\phi = 0$.

We next always assume $m_1 \cdots m_l = n$ and $\phi \neq 0$.

Step 3. Suppose $F_1, \ldots, F_{m_1} \in \mathscr{P}_{m_1}$ are of rank one with $F_i + F_j \in \mathscr{P}_{m_1}, \forall i \neq j \in [1, m_1]$. Then, there exists a unitary matrix U of order $m_1 \cdots m_l$ such that

$$\phi(F_k \otimes X) = U\left[E_{kk}^{(m_1)} \otimes \pi_k(X)\right]U^*, \quad \forall X \in H_{m_2\cdots m_l}, \ k \in [1, m_1], \tag{30}$$

where π_k is a canonical map on $H_{m_2\cdots m_l}, \forall k \in [1, m_1]$.

Proof of Step 3. Using the similar approach as in the proof of Step 2, we may show that $\operatorname{rank} \phi(F_k \otimes I_{m_2\cdots m_l}) = m_2 \cdots m_l$, $\forall k \in [1, m_1]$. In fact, if $\operatorname{rank} \phi(F_k \otimes I_{m_2\cdots m_l}) > m_2 \cdots m_l$ for some k, then, by Lemma 1, there exists $\operatorname{rank} \phi(F_i \otimes I_{m_2\cdots m_l}) < m_2 \cdots m_l$ for some i. Then, using the similar approach as in the proof of Step 2, we have $\phi = 0$, which is a contradiction, and there exist a unitary matrix Q such that $\phi(F_k \otimes I_{m_2\cdots m_l}) = Q[E_{kk}^{(m_1)} \otimes I_{m_2\cdots m_l}]Q^*$. Thus, applying Step 1, we obtain that

$$\phi(F_k \otimes X) = Q\left[E_{kk}^{(m_1)} \otimes \psi_k(X)\right]Q^*, \quad \forall X \in H_{m_2\cdots m_l}, \tag{31}$$

where linear map $\psi_k : H_{m_2\cdots m_l} \to H_{m_2\cdots m_l}$ satisfying $\psi_k(\otimes_2^l \mathscr{P}_{m_i}) \subset \mathscr{P}_{m_2\cdots m_l}$, for any $k \in [1, m_1]$. We obtain by induction hypothesis that there exists unitary matrices $U_k \in \mathscr{M}_{m_2\cdots m_l}$ and canonical maps π_k on $H_{m_2\cdots m_l}$ such that

$$\psi_k(X) = U_k \pi_k(X) U_k^*, \quad \forall X \in H_{m_2\cdots m_l}, \ k \in [1, m_1]. \tag{32}$$

Set $U = Q \operatorname{diag}(U_1, \ldots, U_{m_1})$, we complete the proof of Step 3.

By Step 3, we may assume that

$$\phi\left(E_{kk}^{(m_1)} \otimes X\right) = E_{kk}^{(m_1)} \otimes \pi_k(X), \quad \forall X \in H_{m_2\cdots m_l}, \ k \in [1, m_1], \tag{33}$$

where π_k is canonical map on $H_{m_2\cdots m_l}, \forall k \in [1, m_1]$.

Step 4. For $\forall i \neq j \in [1, m_1]$, $\pi_i = \pi_j$ and there exists λ_{ij}, μ_{ij} with $|\lambda_{ij}| = |\mu_{ij}| = 1$ such that

$$\phi\left(D_{ij}^{(m_1)} \otimes X\right) = \left(\lambda_{ij} E_{ij}^{(m_1)} + \overline{\lambda}_{ij} E_{ji}^{(m_1)}\right) \otimes \pi_1(X), \quad \forall X \in H_{m_2\cdots m_l},$$
$$\phi\left(K_{ij}^{(m_1)} \otimes X\right) = \left(\mu_{ij} E_{ij}^{(m_1)} + \overline{\mu}_{ij} E_{ji}^{(m_1)}\right) \otimes \pi_1(X), \quad \forall X \in H_{m_2\cdots m_l}. \tag{34}$$

Proof of Step 4. Without loss of generality, we only prove the case of $i = 1$, $j = 2$.

Set $F_1 = (1/2)E_{11}^{(m_1)} + (1/2)E_{22}^{(m_1)} + (1/2)D_{12}^{(m_1)}$, $F_2 = (1/2)E_{11}^{(m_1)} + (1/2)E_{22}^{(m_1)} - (1/2)D_{12}^{(m_1)}$, $F_k = E_{kk}^{(m_1)}$ for $k \in [3, m_1]$ (if $m_1 \geq 3$). By Step 3, there exists a unitary matrix V and canonical maps η_k on $H_{m_2\cdots m_l}$ such that

$$\phi(F_k \otimes X) = V\left[E_{kk}^{(m_1)} \otimes \eta_k(X)\right]V^*, \quad \forall X \in H_{m_2\cdots m_l}, \ k \in [1, m_1]. \tag{35}$$

This, together with (33), implies that

$$E_{kk}^{(m_1)} \otimes I_{m_2\cdots m_l} = V\left[E_{kk}^{(m_1)} \otimes I_{m_2\cdots m_l}\right]V^*, \quad \forall k \in [3, m_1]. \tag{36}$$

Thus, we may set

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \oplus \operatorname{diag}\left(V_{33}, \ldots, V_{m_1 m_1}\right), \tag{37}$$

where $V_{kk} \in M_{m_2\cdots m_l}, k \in [1, m_1]$.

On one hand, noting that

$$D_{12}^{(m_1)} = 2F_1 - E_{11}^{(m_1)} - E_{22}^{(m_1)},$$
$$D_{12}^{(m_1)} = E_{11}^{(m_1)} + E_{22}^{(m_1)} - 2F_2, \tag{38}$$

we obtain using (33) and (35) that for all $X \in H_{m_2\cdots m_l}$,

$$\phi\left(D_{12}^{(m_1)} \otimes X\right) = \begin{bmatrix} 2V_{11}\eta_1(X)V_{11}^* - \pi_1(X) & 2V_{11}\eta_1(X)V_{21}^* \\ 2V_{21}\eta_1(X)V_{11}^* & 2V_{21}\eta_1(X)V_{21}^* - \pi_2(X) \end{bmatrix} \oplus 0, \tag{39}$$

$$\phi\left(D_{12}^{(m_1)} \otimes X\right) = \begin{bmatrix} \pi_1(X) - 2V_{12}\eta_2(X)V_{12}^* & -2V_{12}\eta_2(X)V_{22}^* \\ -2V_{22}\eta_2(X)V_{12}^* & \pi_2(X) - 2V_{22}\eta_2(X)V_{22}^* \end{bmatrix} \oplus 0. \tag{40}$$

On the other hand, for any $P \in \otimes_2^l \mathcal{P}_{m_i}$ and real γ with $0 \neq |\gamma| \leq 1/2$, we have $(\alpha E_{11}^{(m_1)} + \beta E_{22}^{(m_1)} + \gamma D_{12}^{(m_1)}) \otimes P \in \otimes_1^l \mathcal{P}_{m_i}$, where α and β are all roots of $x^2 - x + \gamma^2 = 0$. We obtain using the property of ϕ, (33), and Lemma 3 that

$$\left[\phi\left(D_{12}^{(m_1)} \otimes I_{m_2\cdots m_l}\right)\right]^2 = \phi\left(E_{11}^{(m_1)} \otimes I_{m_2\cdots m_l}\right) + \phi\left(E_{22}^{(m_1)} \otimes I_{m_2\cdots m_l}\right), \tag{41}$$

$$\phi\left(D_{12}^{(m_1)} \otimes X\right) = \begin{bmatrix} 0_{m_1} & Y_{12} \\ Y_{12}^* & 0_{m_1} \end{bmatrix} \oplus 0, \quad \forall X \in H_{m_2\cdots m_l}, \tag{42}$$

where $Y_{12} \in M_{m_1}$. Choosing $X = I_{m_2\cdots m_l}$, we get using (39), (40), and (42) that $2V_{11}V_{11}^* = 2V_{22}V_{22}^* = 2V_{21}V_{21}^* = 2V_{12}V_{12}^* = I_{m_2\cdots m_l}$. Hence, by (39), (40), and (42) again, one can obtain that

$$\pi_1(X)\left(V_{11}^*\right)^{-1}V_{21}^* = 2V_{11}\eta_1(X)V_{21}^* = -2V_{12}\eta_2(X)V_{22}^* = -V_{12}V_{22}^{-1}\pi_2(X). \tag{43}$$

By Lemma 5, we have $\pi_1 = \pi_2$ and $(V_{11}^*)^{-1}V_{21}^* = \lambda_{12} I_{m_2\cdots m_l}$ with $\lambda_{12} \neq 0$. Hence,

$$\phi\left(D_{12}^{(m_1)} \otimes X\right) = \left(\lambda_{12}E_{12}^{(m_1)} + \overline{\lambda}_{12}E_{21}^{(m_1)}\right) \otimes \pi_1(X), \quad \forall X \in H_{m_2\cdots m_l},\ \lambda_{12} \neq 0. \tag{44}$$

This, together with (41), implies that $|\lambda_{12}| = 1$. Similarly,

$$\phi\left(K_{12}^{(m_1)} \otimes X\right) = \left(\mu_{12}E_{12}^{(m_1)} + \overline{\mu}_{12}E_{21}^{(m_1)}\right) \otimes \pi_1(X), \quad \forall X \in H_{m_2\cdots m_l},\ |\mu_{12}| = 1. \tag{45}$$

This completes the proof of Step 4.

Step 5. There exists $\varepsilon \in \{-1, 1\}$ such that $\mu_{ij} = \varepsilon\sqrt{-1}\lambda_{ij}$, $\forall i \neq j \in [1, m_1]$. If $m_1 \geq 3$, then $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$, for distinct $i, j, k \in [1, m_1]$.

Proof of Step 5. Since

$$\left(\frac{1}{2}E_{ii}^{(m_1)} + \frac{1}{2}E_{jj}^{(m_1)} + \frac{\sqrt{2}}{4}D_{ij}^{(m_1)} + \frac{\sqrt{2}}{4}K_{ij}^{(m_1)}\right) \otimes I_{m_2\cdots m_l} \in \otimes_1^l \mathcal{P}_{m_i}, \tag{46}$$

we obtain using the property of ϕ, (33), and Step 4 that

$$\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4}\left(\lambda_{ij} + \mu_{ij}\right) \\ \frac{\sqrt{2}}{4}\left(\overline{\lambda}_{ij} + \overline{\mu}_{ij}\right) & \frac{1}{2} \end{bmatrix} \in \mathcal{P}_2. \tag{47}$$

Hence, $(\lambda_{ij} + \mu_{ij})(\overline{\lambda}_{ij} + \overline{\mu}_{ij}) = 2$. It follows from $|\lambda_{ij}| = |\mu_{ij}| = 1$ that $\lambda_{ij}^2 = -\mu_{ij}^2$. Thus, there exists $\varepsilon_{ij} \in \{-1, 1\}$ such that $\mu_{ij} = \varepsilon_{ij}\sqrt{-1}\lambda_{ij}$.

If $m_1 \geq 3$, since

$$\frac{1}{3}\left(E_{ii}^{(m_1)} + E_{jj}^{(m_1)} + E_{kk}^{(m_1)} + D_{ij}^{(m_1)} + D_{ik}^{(m_1)} + D_{jk}^{(m_1)}\right) \otimes I_{m_2\cdots m_l} \in \otimes_1^l \mathcal{P}_{m_i},$$
$$\frac{1}{3}\left(E_{ii}^{(m_1)} + E_{jj}^{(m_1)} + E_{kk}^{(m_1)} + K_{ij}^{(m_1)} + K_{ik}^{(m_1)} + D_{jk}^{(m_1)}\right) \otimes I_{m_2\cdots m_l} \in \otimes_1^l \mathcal{P}_{m_i}, \tag{48}$$
$$\frac{1}{3}\left(E_{ii}^{(m_1)} + E_{jj}^{(m_1)} + E_{kk}^{(m_1)} + D_{ij}^{(m_1)} + K_{ik}^{(m_1)} + K_{jk}^{(m_1)}\right) \otimes I_{m_2\cdots m_l} \in \otimes_1^l \mathcal{P}_{m_i},$$

we obtain using the property of ϕ, (33), and Step 4 that

$$\frac{1}{3}\begin{bmatrix} 1 & \lambda_{ij} & \lambda_{ik} \\ \overline{\lambda}_{ij} & 1 & \lambda_{jk} \\ \overline{\lambda}_{ik} & \overline{\lambda}_{jk} & 1 \end{bmatrix} \in \mathcal{P}_3, \tag{49}$$

$$\frac{1}{3}\begin{bmatrix} 1 & \varepsilon_{ij}\lambda_{ij}\sqrt{-1} & \varepsilon_{ik}\lambda_{ik}\sqrt{-1} \\ -\varepsilon_{ij}\overline{\lambda}_{ij}\sqrt{-1} & 1 & \lambda_{jk} \\ -\varepsilon_{ik}\overline{\lambda}_{ik}\sqrt{-1} & \overline{\lambda}_{jk} & 1 \end{bmatrix} \in \mathcal{P}_3, \tag{50}$$

$$\frac{1}{3}\begin{bmatrix} 1 & \lambda_{ij} & \varepsilon_{ik}\lambda_{ik}\sqrt{-1} \\ \overline{\lambda}_{ij} & 1 & \varepsilon_{jk}\lambda_{jk}\sqrt{-1} \\ -\varepsilon_{ik}\overline{\lambda}_{ik}\sqrt{-1} & -\varepsilon_{jk}\overline{\lambda}_{jk}\sqrt{-1} & 1 \end{bmatrix} \in \mathcal{P}_3. \tag{51}$$

It follows from (49) that $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$. This, together with (50), implies that $\varepsilon_{ij} = \varepsilon_{ik}$. Similarly, (51) implies that $\varepsilon_{ik} = \varepsilon_{jk}$. Thus, all numbers ε_{ij} have to be the same for any $i \neq j \in [1, m_1]$. This completes the proof of this step.

By Steps 3–5, we set $Q = \operatorname{diag}(1, \lambda_{12}, \ldots, \lambda_{1m_1})$, then Q is unitary such that

$$\begin{aligned}\phi\left(E_{kk}^{(m_1)} \otimes X\right) &= \left[QE_{kk}^{(m_1)}Q^*\right] \otimes \pi_1(X), \\ &\qquad \forall X \in H_{m_2\cdots m_l},\ k \in [1, m_1], \\ \phi\left(D_{ij}^{(m_1)} \otimes X\right) &= \left[QD_{ij}^{(m_1)}Q^*\right] \otimes \pi_1(X), \\ &\qquad \forall X \in H_{m_2\cdots m_l},\ i \neq j \in [1, m_1], \\ \phi\left(K_{ij}^{(m_1)} \otimes X\right) &= \varepsilon\left[QK_{ij}^{(m_1)}Q^*\right] \otimes \pi_1(X), \\ &\qquad \forall X \in H_{m_2\cdots m_l},\ i \neq j \in [1, m_1].\end{aligned} \tag{52}$$

Thus, set $U = Q \otimes I_{m_2\cdots m_l}$, we have

$$\begin{aligned}\phi(A \otimes X) &= U\left[A \otimes \pi_1(X)\right]U^*, \\ &\qquad \forall A \in H_{m_1},\ X \in H_{m_2\cdots m_l}\end{aligned} \tag{53}$$

or

$$\begin{aligned}\phi(A \otimes X) &= U\left[A^T \otimes \pi_1(X)\right]U^*, \\ &\qquad \forall A \in H_{m_1},\ X \in H_{m_2\cdots m_l}.\end{aligned} \tag{54}$$

This completes the proof of Theorem 6. □

Remark 7. When $l \geq 2$, the linear transformation that maps the set of tensor product of idempotent matrices into idempotent matrix set does not necessarily preserve idempotent matrices. For example, let

$$\begin{aligned}\phi\left(A_1 \otimes A_2 \otimes \cdots \otimes A_l\right) &= A_1^T \otimes A_2 \otimes \cdots \otimes A_l, \\ &\qquad \forall A_i \in \mathscr{H}_{m_i},\ i \in [1, l], \\ C = \frac{1}{2}\Big(E_{11}^{(m_1)} &\otimes E_{11}^{(m_2)} + E_{22}^{(m_1)} \otimes E_{22}^{(m_2)} + E_{12}^{(m_1)} \\ &\otimes E_{12}^{(m_2)} + E_{21}^{(m_1)} \otimes E_{21}^{(m_2)}\Big) \otimes I_{m_3\cdots m_l}.\end{aligned} \tag{55}$$

Then $\phi(\otimes_1^l \mathscr{P}_{m_i}) \subset \mathscr{P}_{m_1\cdots m_l}$ and $C \in \mathscr{P}_{m_1\cdots m_l}$, but

$$\begin{aligned}\phi(C) = \frac{1}{2}\Big(E_{11}^{(m_1)} &\otimes E_{11}^{(m_2)} + E_{22}^{(m_1)} \otimes E_{22}^{(m_2)} + E_{21}^{(m_1)} \\ &\otimes E_{12}^{(m_2)} + E_{12}^{(m_1)} \otimes E_{21}^{(m_2)}\Big) \otimes I_{m_3\cdots m_l} \notin \mathscr{P}_{m_1\cdots m_l}.\end{aligned} \tag{56}$$

As the application of Theorem 6, we give the following theorem.

Theorem 8. *Suppose ϕ is a linear map from $H_{m_1\cdots m_l}$ to H_n with $n \leq m_1 \cdots m_l$. Then, $\phi(\otimes_1^l \mathscr{T}_{m_i}) \subset \mathscr{T}_n$ if and only if either $\phi = 0$ or $n = m_1 \cdots m_l$, there exist a unitary matrix $U \in M_n$, $\varepsilon \in \{-1, 1\}$ and a canonical map π on $H_{m_1\cdots m_l}$ such that*

$$\phi(X) = \varepsilon U\pi(X)U^*, \quad \forall X \in H_{m_1\cdots m_l}. \tag{57}$$

Proof. The sufficiency part is clear. We give the proof of the necessity part.

Set

$$\begin{aligned}\Gamma_0 &= \left\{I_{m_1} \otimes \cdots \otimes I_{m_l}\right\}, \\ \Gamma_1 &= \left\{P_1 \otimes I_{m_2\cdots m_l} : P_1 \in \mathscr{P}_{m_1}\right\}, \\ \Gamma_2 &= \left\{P_1 \otimes P_2 \otimes I_{m_3\cdots m_l} : P_i \in \mathscr{P}_{m_i},\ i \in [1, 2]\right\}, \\ &\cdots \\ \Gamma_l &= \left\{P_1 \otimes \cdots \otimes P_{m_l} : P_i \in \mathscr{P}_{m_i},\ i \in [1, l]\right\}.\end{aligned} \tag{58}$$

It is obvious that $\Gamma_l = \otimes_1^l \mathscr{P}_{m_i}$ and $\Gamma_k \subset \otimes_1^l \mathscr{T}_{m_i}$, $\forall k \in [0, l]$.

Since $I_{m_1\cdots m_l} \in \otimes_1^l \mathscr{T}_{m_i}$, we have $\phi(I_{m_1\cdots m_l}) \in \mathscr{T}_n$. Without loss of generality, we may assume that $\phi(I_{m_1\cdots m_l}) = I_p \oplus -I_q \oplus 0_{n-p-q}$. We next prove by induction on k that

$$\phi(X) = \psi_p(X) \oplus -\psi_q(X) \oplus 0_{n-p-q}, \quad \forall X \in \Gamma_k, \tag{59}$$

where $\psi_p(X) \in \mathscr{P}_p$ and $\psi_q(X) \in \mathscr{P}_q$.

We assume that our statement holds true for $k - 1$ and prove it for k. For any $X = P_1 \otimes \cdots \otimes P_k \otimes I_{m_{k+1}\cdots m_l} \in \Gamma_k$, choosing $Y = P_1 \otimes \cdots \otimes P_{k-1} \otimes I_{m_k} \otimes I_{m_{k+1}\cdots m_l} \in \Gamma_{k-1}$, we have by induction hypothesis that

$$\phi(Y) = \psi_p(Y) \oplus -\psi_q(Y) \oplus 0_{n-p-q}, \tag{60}$$

with $\psi_p(Y) \in \mathscr{P}_p$ and $\psi_q(Y) \in \mathscr{P}_q$. Since $X \in \otimes_1^l \mathscr{T}_{m_i}$, $Y - X \in \otimes_1^l \mathscr{T}_{m_i}$ and $Y - 2X \in \otimes_1^l \mathscr{T}_{m_i}$. Using the property of ϕ, we have $\phi(X) \in \mathscr{T}_n$, $\phi(Y) - \phi(X) \in \mathscr{T}_n$ and $\phi(Y) - 2\phi(X) \in \mathscr{T}_n$. By Lemma 4, we have

$$\phi(X) = \psi_p(X) \oplus -\psi_q(X) \oplus 0_{n-p-q}, \tag{61}$$

with $\psi_p(X) \in \mathscr{P}_p$ and $\psi_q(X) \in \mathscr{P}_q$. This implies that (59) holds.

As ϕ is linear, we can expend ψ_p to be a linear map from $H_{m_1\cdots m_l}$ to H_p and ψ_q to be a linear map from $H_{m_1\cdots m_l}$ to H_q, then

$$\phi(X) = \psi_p(X) \oplus -\psi_q(X) \oplus 0_{n-p-q}, \quad \forall X \in H_{m_1\cdots m_l}, \tag{62}$$

with $\psi_p(\otimes_1^l \mathscr{P}_{m_i}) \subset \mathscr{P}_p$ and $\psi_q(\otimes_1^l \mathscr{P}_{m_i}) \subset \mathscr{P}_q$, we obtain using Theorem 6 that

(a) if $n < m_1 \cdots m_l$, then $p, q < m_1 \cdots m_l$, $\psi_p = 0$, $\psi_q = 0$, thus, $\phi = 0$, and

(b) if $\phi \neq 0$, then $n = m_1 \cdots m_l$ and $p = n$ or $q = n$, thus $\phi = \psi_p$ or $\phi = -\psi_q$.

This completes the proof of Theorem 8. □

Acknowledgments

The authors show great thanks to the referee for his/her careful reading of the paper and valuable comments which greatly improved the readability of the paper. Jinli Xu is supported in part by NSFC (11171294), Natural Science Foundation of Heilongjiang Province of China (Grant no. A201013). Baodong Zheng is supported by the National Natural Science Foundation Grants of China (Grant no. 10871056). Hongmei Yao is supported by the Fundamental Research Funds for the Central Universities.

References

[1] A. Fošner, Z. Haung, C. K. Li, and N. S. Sze, "Linear preservers and quantum information science," *Linear and Multilinear Algebra*, 2012.

[2] K. He, J. C. Hou, and C. K. Li, "A geometric characterization of invertible quantum measurement maps," *Journal of Functional Analysis*, vol. 264, no. 2, pp. 464–478, 2013.

[3] S. Friedland, C. K. Li, Y. T. Poon, and N. S. Sze, "The automorphism group of separable states in quantum information theory," *Journal of Mathematical Physics*, vol. 52, no. 4, Article ID 042203, 8 pages, 2011.

[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, UK, 2000.

[5] M. H. Lim, "Additive preservers of tensor product of rank one Hermitian matrices," *Electronic Journal of Linear Algebra*, vol. 23, pp. 356–374, 2012.

[6] Z. H. Zhou and M. Zhu, "Extended Cesáro operators between generalized Besov spaces and Bloch type spaces in the unit ball," *Journal of Function Spaces and Applications*, vol. 7, no. 3, pp. 209–223, 2009.

[7] H. Yao and B. Zheng, "Zero triple product determined matrix algebras," *Journal of Applied Mathematics*, vol. 2012, Article ID 925092, 18 pages, 2012.

[8] A. El-Sayed Ahmed and M. A. Bakhit, "Properties of Toeplitz operators on some holomorphic Banach function spaces," *Journal of Function Spaces and Applications*, vol. 2012, Article ID 517689, 18 pages, 2012.

[9] Z. Liao, S. Hu, D. Sun, and W. Chen, "Enclosed Laplacian operator of nonlinear anisotropic diffusion to preserve singularities and delete isolated points in image smoothing," *Mathematical Problems in Engineering*, vol. 2011, Article ID 749456, 15 pages, 2011.

[10] O. Agratini, "Linear operators that preserve some test functions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 94136, 11 pages, 2006.

[11] D. Huo, B. Zheng, and H. Liu, "Characterizations of nonlinear Lie derivations of $B(X)$," *Abstract and Applied Analysis*, vol. 2013, Article ID 245452, 7 pages, 2013.

[12] G. H. Chan, M. H. Lim, and K. K. Tan, "Linear preservers on matrices," *Linear Algebra and Its Applications*, vol. 93, pp. 67–80, 1987.

[13] C. H. Sheng, C. G. Cao, and J. Hu, "Linear preservers on complex Hermitian matrices," *Journal of Natural Science of Heilongjiang University*, vol. 25, pp. 358–362, 2008 (Chinese).

[14] L. Molnár, "Transformations on the set of all n-dimensional subspaces of a Hilbert space preserving principal angles," *Communications in Mathematical Physics*, vol. 217, no. 2, pp. 409–421, 2001.

[15] L. Molnár, "Orthogonality preserving transformations on indefinite inner product spaces: generalization of Uhlhorn's version of Wigner's theorem," *Journal of Functional Analysis*, vol. 194, no. 2, pp. 248–262, 2002.

[16] P. Šemrl, "Applying projective geometry to transformations on rank one idempotents," *Journal of Functional Analysis*, vol. 210, no. 1, pp. 248–257, 2004.

[17] P. Šemrl, "Maps on idempotent matrices over division rings," *Journal of Algebra*, vol. 298, no. 1, pp. 142–187, 2006.

[18] N. Jacobson, *Lectures in Abstract Algebra*, vol. 2, Van Nostrand, Toronto, Canada, 1953.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 274857, 9 pages
http://dx.doi.org/10.1155/2013/274857

Research Article

Exponential Stability of a Linear Distributed Parameter Bioprocess with Input Delay in Boundary Control

Fu Zheng,[1] LingLing Fu,[1] and MingYan Teng[2]

[1] *Department of Mathematics, Bohai University, Jinzhou 121013, China*
[2] *Beijing Institute of Information and Control, Beijing 100037, China*

Correspondence should be addressed to Fu Zheng; fz555888@yahoo.com.cn

Received 9 April 2013; Accepted 20 July 2013

Academic Editor: Gen-Qi Xu

We consider a linear distributed parameter bioprocess with boundary control input possessing a time delay. Using a simple boundary feedback law, we show that the closed-loop system generates a uniformly bounded C_0-semigroup of linear operators under a certain condition with respect to the feedback gain. After analyzing the spectrum configuration of closed-loop system and verifying the spectrum determined growth assumption, we show that the closed-loop system is exponentially stable. Thus, we demonstrate that the linear distributed parameter bioprocess preserves the exponential stability for arbitrary time delays.

1. Introduction

In a practical control system, there is often a time delay between the controller to be implemented and the information via the observation of the system. These hereditary effects are sometime unavoidable because they might turn a well-behaved system into a wild one. A simple example can be found in Gumowski and Mira [1], where they demonstrated that the occurrence of delays could destroy the stability and cause periodic oscillations in a system governed by differential equation. Datko [2, 3] illustrated that an arbitrary small time delay in the control could destabilize a boundary feedback hyperbolic control system as well. On the other side, the inclusion of an appropriate time delay effect can sometime improve the performance of the system (e.g., see [3–7]). When the time delay appears, redesigning a stabilizing controller becomes thereby sometimes necessary because the stabilization by the PI output feedback becomes defective or the stabilization is not robust to time delay. The stabilization with time delay in observation or control represents difficult mathematical challenges in the control of distributed parameter systems. However, this does not mean that there is no stabilizing controller in the presence of time delay. You can refer to [8–12] for some successful examples.

Motivated by these works, we will introduce time delays to a linear distributed parameter bioprocess and investigate the effect of time delays on exponential stability of the system. The linear distributed parameter bioprocess treated here was firstly discussed by Bourrel and Dochain in [13]. They showed that the system with zero boundary input is exponentially stable. Following [13], Sano considered the linear distributed parameter bioprocess from the feedback control point of view in [14]. Namely, the control input and the measured output were imposed on the boundaries, and a simply proportional feedback controller was designed. By using Huang's result in [15], He showed that the closed-loop system is exponentially stable under a certain condition with respect to the feedback gain and further that the exponential decay rate of the system with zero input was derived by letting the feedback gain tend to zero. However, if time delays in the boundary input arise in this linear distributed parameter bioprocess, we want to pose a question. Is the stabilization robust to time delays for the proportional feedback controller? The present paper is devoted to answering this question.

The content of this paper is organized as follows. In Section 2 we will introduce the linear distributed parameter bioprocess mentioned previously and formulate our problem in a suitable Hilbert space. We show that the closed-loop system generates a uniformly bounded C_0-semigroup of linear operators and obtain the wellposedness of the system as well. In Section 3, we carry out a spectral analysis and obtain the spectrum configuration of the closed-loop system. From verifying the spectrum determined growth assumption, we show that the closed-loop system is exponentially stable. In the last section, a concise conclusion is given.

2. System Description and Wellposedness of the System

We will consider the following type of linear distributed parameter bioprocess model in which time delays occur in boundary control input:

$$\begin{gathered}\frac{\partial z_1}{\partial t}(t,x) = -\nu\frac{\partial z_1}{\partial x}(t,x) - a_1 z_1(t,x) - a_2 z_2(t,x),\\ (t,x)\in(0,\infty)\times(0,1),\\ \frac{\partial z_2}{\partial t}(t,x) = a_3 z_1(t,x) - a_4 z_2(t,x),\\ (t,x)\in(0,\infty)\times(0,1),\\ z_1(t,0) = u(t-\tau),\quad t\in(0,\infty),\\ z_1(0,x) = z_{10}(x),\quad z_2(0,x) = z_{20}(x),\quad x\in(0,1),\\ y(t) = z_1(t,1),\quad t\in(0,\infty),\end{gathered}\tag{1}$$

where $z_1(t,x), z_2(t,x) \in \mathbb{R}$ are the deviations of substrate and biomass concentrations from steady-state values at the time t and at the point $x \in (0,1)$, respectively. And $u(t) \in \mathbb{R}$ is the control input, $y(t) \in \mathbb{R}$ is the measured output, $\nu > 0$ is the fluid superficial velocity, a_1, a_2, a_3, and a_4 are positive constants, and $\tau \geq 0$ is the length of time delay.

As usual, we adopt the simple feedback control law $u(t) = -ky(t)$ with $k > 0$ which results in the following closed-loop system:

$$\begin{gathered}\frac{\partial z_1}{\partial t}(t,x) = -\nu\frac{\partial z_1}{\partial x}(t,x) - a_1 z_1(t,x) - a_2 z_2(t,x),\\ (t,x)\in(0,\infty)\times(0,1),\\ \frac{\partial z_2}{\partial t}(t,x) = a_3 z_1(t,x) - a_4 z_2(t,x),\\ (t,x)\in(0,\infty)\times(0,1),\\ z_1(t,0) = -kz_1(t-\tau,1),\quad t\in(0,\infty),\\ z_1(0,x) = z_{10}(x),\quad z_2(0,x) = z_{20}(x),\quad x\in(0,1).\end{gathered}\tag{2}$$

Setting $z_3(t,x) = z_1(t - x\tau, 1)$, (2) is equivalent to

$$\begin{gathered}\frac{\partial z_1}{\partial t}(t,x) = -\nu\frac{\partial z_1}{\partial x}(t,x) - a_1 z_1(t,x) - a_2 z_2(t,x),\\ (t,x)\in(0,\infty)\times(0,1),\\ \frac{\partial z_2}{\partial t}(t,x) = a_3 z_1(t,x) - a_4 z_2(t,x),\\ (t,x)\in(0,\infty)\times(0,1),\\ \frac{\partial z_3}{\partial t}(t,x) = -\tau^{-1}\frac{\partial z_3}{\partial x}(t,x),\quad (t,x)\in(0,\infty)\times(0,1),\\ z_3(t,0) = z_1(t,1),\quad z_1(t,0) = -kz_3(t,1),\quad t\in(0,\infty),\\ z_1(0,x) = z_{10}(x),\quad z_2(0,x) = z_{20}(x),\\ z_3(0,x) = z_{30}(-\tau x),\quad x\in(0,1).\end{gathered}\tag{3}$$

We take the state Hilbert space $\mathscr{H}$,

$$\mathscr{H} = L^2[0,1]\times L^2[0,1]\times L^2[0,1] = \left(L^2[0,1]\right)^3,\tag{4}$$

equipped with inner product

$$\begin{aligned}\langle f,g\rangle_{\mathscr{H}} = &\int_0^1 f_1(x)\overline{g_1(x)}dx + \int_0^1 f_2(x)\overline{g_2(x)}dx\\ &+\int_0^1 f_3(x)\overline{g_3(x)}dx,\quad f,g\in\mathscr{H}.\end{aligned}\tag{5}$$

Define the operator $A : D(A) \subset \mathscr{H} \to \mathscr{H}$ as

$$\begin{gathered}Af = \begin{pmatrix} -\nu\dfrac{d}{dx} - a_1 & -a_2 & 0\\ a_3 & -a_4 & 0\\ 0 & 0 & -\tau^{-1}\dfrac{d}{dx}\end{pmatrix}\begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix},\\ f = (f_1, f_2, f_3)^{\perp}\in D(A),\\ D(A) = \{f\in H^1(0,1)\times L^2[0,1]\times H^1(0,1) \mid f_1(0) = -kf_3(1),\ f_1(1) = f_3(0)\}.\end{gathered}\tag{6}$$

Then the system (3) can be written as

$$\frac{d}{dt}\begin{pmatrix} z_1(t)\\ z_2(t)\\ z_3(t)\end{pmatrix} = A\begin{pmatrix} z_1(t)\\ z_2(t)\\ z_3(t)\end{pmatrix},\quad \begin{pmatrix} z_1(0)\\ z_2(0)\\ z_3(0)\end{pmatrix} = \begin{pmatrix} z_{10}\\ z_{20}\\ z_{30}\end{pmatrix},\tag{7}$$

where z_{10}, z_{20}, $z_{30} \in L^2[0,1]$. Therefore, if the operator A generates a C_0-semigroup $T(t)$ on $\mathscr{H}$, then a unique solution of (7) is expressed as

$$\begin{pmatrix} z_1(t)\\ z_2(t)\\ z_3(t)\end{pmatrix} = T(t)\begin{pmatrix} z_{10}\\ z_{20}\\ z_{30}\end{pmatrix},\quad \forall t\geq 0,\tag{8}$$

which means that the unique solution to (2) or (3) exists.

Let us define the operator $M \in L(\mathscr{H})$ as

$$M = \begin{pmatrix} \sqrt{a_2} & 0 & 0\\ 0 & \sqrt{a_3} & 0\\ 0 & 0 & 1\end{pmatrix}\tag{9}$$

and consider the properties of a semigroup generated by the operator $M^{-1}AM$. The operator $M^{-1}AM$ is expressed as

$$M^{-1}AMf = \begin{pmatrix} -\nu\dfrac{d}{dx} - a_1 & -\sqrt{a_2a_3} & 0 \\ \sqrt{a_2a_3} & -a_4 & 0 \\ 0 & 0 & -\tau^{-1}\dfrac{d}{dx} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

$$f = (f_1, f_2, f_3)^{\perp} \in D\left(M^{-1}AM\right),$$

$$D\left(M^{-1}AM\right) = \left\{f \in H^1(0,1) \times L^2[0,1] \times H^1(0,1) \mid f_1(0) = -kf_3(1), f_1(1) = f_3(0)\right\}. \tag{10}$$

Firstly, we have the following result.

Theorem 1. *Suppose that the feedback gain k is chosen such that $0 < k < 1$. Then, the operator A defined by (6) generates a uniformly bounded C_0-semigroup $T(t)$ on $\mathscr{H}$.*

Proof. In order to prove that A generates a uniformly bounded C_0-semigroup, we introduce a new equivalent inner product in $\mathscr{H}$:

$$\langle f, g\rangle_{\mathscr{H}_1} = \frac{1}{\nu}\int_0^1 f_1(x)\overline{g_1(x)}dx + \frac{1}{\nu}\int_0^1 f_2(x)\overline{g_2(x)}dx + \tau\int_0^1 f_3(x)\overline{g_3(x)}dx, \quad f, g \in \mathscr{H}. \tag{11}$$

From the domain of the operator $M^{-1}AM$ it follows that the identities

$$\begin{aligned} \operatorname{Re}\left\langle M^{-1}AMf, f\right\rangle_{\mathscr{H}_1} &= -\frac{a_1}{\nu}\|f_1\|_{L^2(0,1)} - \frac{a_4}{\nu}\|f_2\|_{L^2(0,1)} - \left(|f_1(1)|^2 - |f_1(0)|^2\right) - \left(|f_3(1)|^2 - |f_3(0)|^2\right) \\ &= -\frac{a_1}{\nu}\|f_1\|_{L^2(0,1)} - \frac{a_4}{\nu}\|f_2\|_{L^2(0,1)} - \left(1-k^2\right)|f_3(1)|^2 \end{aligned} \tag{12}$$

hold for all $f \in D(M^{-1}AM)$. If the feedback gain k satisfies $0 < k < 1$, then it is easy to see

$$\operatorname{Re}\left\langle M^{-1}AMf, f\right\rangle_{\mathscr{H}_1} \le 0, \quad \forall f \in D\left(M^{-1}AM\right). \tag{13}$$

Next, for all $f \in D(M^{-1}AM)$ and $g \in \mathscr{H}$, we have

$$\begin{aligned} &\left\langle M^{-1}AMf, g\right\rangle_{\mathscr{H}_1} \\ &= \frac{1}{\nu}\int_0^1 \left(-\nu f_1'(x) - a_1 f_1(x)\right)\overline{g_1(x)}dx \\ &\quad - \frac{a_4}{\nu}\int_0^1 f_2(x)\overline{g_2(x)}dx - \frac{\sqrt{a_2a_3}}{\nu}\int_0^1 f_2(x)\overline{g_1(x)}dx \\ &\quad + \frac{\sqrt{a_2a_3}}{\nu}\int_0^1 f_1(x)\overline{g_2(x)}dx - \int_0^1 f_3'(x)\overline{g_3(x)}dx \\ &= \frac{1}{\nu}\int_0^1 f_1(x)\left(\nu\overline{g_1'(x)} - a_1\overline{g_1(x)} + \sqrt{a_2a_3}\overline{g_2(x)}\right)dx \\ &\quad - \frac{1}{\nu}\int_0^1 f_2(x)\left(a_4\overline{g_2(x)} + \sqrt{a_2a_3}\overline{g_1(x)}\right)dx \\ &\quad + \frac{1}{\tau}\int_0^1 f_3(x)\overline{g_3'(x)}dx \\ &\quad - \left[f_1(x)\overline{g_1'(x)}\right]_0^1 - \frac{1}{\tau}\left[f_3(x)\overline{g_3(x)}\right]_0^1. \end{aligned} \tag{14}$$

From the definition of the adjoint operator and the conditions in the domain of the $M^{-1}AM$, we know that

$$\left(M^{-1}AM\right)^* f = \begin{pmatrix} \nu\dfrac{d}{dx} - a_1 & \sqrt{a_2a_3} & 0 \\ -\sqrt{a_2a_3} & -a_4 & 0 \\ 0 & 0 & \tau^{-1}\dfrac{d}{dx} \end{pmatrix}\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix},$$

$$f = (f_1, f_2, f_3)^{\perp} \in D\left(\left(M^{-1}AM\right)^*\right),$$

$$D\left(\left(M^{-1}AM\right)^*\right) = \left\{f \in H^1(0,1) \times L^2[0,1] \times H^1(0,1) \mid f_3(1) = -kf_1(0), f_3(0) = f_1(1)\right\} \tag{15}$$

because it is easily verified that $M^{-1}AM$ is a closed and densely defined linear operator. Thus, by similar arguments as previously, we obtain that, for all $f \in D((M^{-1}AM)^*)$,

$$\begin{aligned} \operatorname{Re}\left\langle \left(M^{-1}AM\right)^* f, f\right\rangle_{\mathscr{H}_1} &= -\frac{a_1}{\nu}\|f_1\|_{L^2(0,1)} - \frac{a_4}{\nu}\|f_2\|_{L^2(0,1)} + \left(|f_1(1)|^2 - |f_1(0)|^2\right) + \left(|f_3(1)|^2 - |f_3(0)|^2\right) \\ &= -\frac{a_1}{\nu}\|f_1\|_{L^2(0,1)} - \frac{a_4}{\nu}\|f_2\|_{L^2(0,1)} - \left(1-k^2\right)|f_1(0)|^2 \le 0. \end{aligned} \tag{16}$$

It follows from $\operatorname{Re}\langle M^{-1}AMf, f\rangle_{\mathscr{H}_1} \le 0$ and $\operatorname{Re}\langle (M^{-1}AM)^* f, f\rangle_{\mathscr{H}_1} \le 0$ that the operators $M^{-1}AM$

and $(M^{-1}AM)^*$ are dissipative. According to Proposition 3.1.11 of [16], the closed operator $M^{-1}AM$ is m-dissipative. Therefore, Lumer-Philips theorem implies that $M^{-1}AM$ generates contraction semigroups $S(t)$ on state space $\mathscr{H}$. For all $t \geq 0$, if we define $T(t)$ by $T(t) = MS(t)M^{-1}$, then the semigroups $T(t)$ and $S(t)$ are similar. This means that the C_0-semigroups $T(t)$ are uniformly bounded (i.e., a C_0-semigroup with the operator norm bound $\| T(t)\|_{\mathscr{L}(\mathscr{H}_1)} \leq M$, for some $M > 0$ and $\forall t \geq 0$) and their generator is A. Thus, the proof of the theorem is complete since the new inner product is equivalent to the original one. □

3. Exponential Stability of the System (7)

In order to show the exponential stability of the system (7), we will verify that the operator A satisfies the conditions of Theorem 1.1 of [14], which is a summarized edition of Huang's result on the spectrum determined growth assumption in [15]. To this end, we should analyze the spectrum configuration of the operator A and show that the norm of the resolvent is uniformly bounded in any given right half-plane. All these results are collected in the following two lemmas.

Lemma 2. *Suppose that the assumption of Theorem 1 is satisfied. Then, the following inequality holds:*

$$\sup \{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\} \leq \mu(k,\tau), \tag{17}$$

where $\mu(k,\tau)$ is defined by

$$\mu(k,\tau) := \begin{cases} \max\left\{-a_4, -\dfrac{a_1 - \nu\log k}{\nu\tau+1}, \beta_1(k,\tau), \beta_2(k,\tau)\right\} \\ \quad \text{if } \dfrac{(a_1+a_4-\nu\log k)^2}{\nu\tau+1} > 4\left[a_2a_3 + a_4(a_1-\nu\log k)\right], \\ \max\left\{-a_4, -\dfrac{a_1 - \nu\log k}{\nu\tau+1}, -\dfrac{(a_1+a_4-\nu\log k)^2}{2(\nu\tau+1)}\right\} \\ \quad \text{if } \dfrac{(a_1+a_4-\nu\log k)^2}{\nu\tau+1} = 4\left[a_2a_3 + a_4(a_1-\nu\log k)\right], \end{cases}$$

$$\mu(k,\tau) := \max\left\{-a_4, \frac{-a_1+\nu\log k}{\nu\tau+1}\right\} \quad \text{if } \frac{(a_1+a_4-\nu\log k)^2}{\nu\tau+1} < 4\left[a_2a_3 + a_4(a_1-\nu\log k)\right], \tag{18}$$

with $\beta_1(k,\tau)$ and $\beta_2(k,\tau)$ being

$$\begin{aligned} \beta_1(k,\tau) &= \frac{-(a_1+a_4-\nu\log k) + \sqrt{(a_1+a_4-\nu\log k)^2 - 4(\nu\tau+1)\left[a_2a_3+a_4(a_1-\nu\log k)\right]}}{2(\nu\tau+1)}, \\ \beta_2(k,\tau) &= \frac{-(a_1+a_4-\nu\log k) - \sqrt{(a_1+a_4-\nu\log k)^2 - 4(\nu\tau+1)\left[a_2a_3+a_4(a_1-\nu\log k)\right]}}{2(\nu\tau+1)}. \end{aligned} \tag{19}$$

Proof. First, let us calculate the eigenvalues of the operator A. It is easy to see that, for $\lambda \in \mathbb{C}$ and $f = (f_1(x), f_2(x), f_3(x)) \in D(A)$, $Af = \lambda f$ is equivalent to

$$-\nu f_1'(x) - a_1 f_1(x) - a_2 f_2(x) = \lambda f_1(x), \tag{20}$$

$$a_3 f_1(x)(x) - a_4 f_2(x) = \lambda f_2(x), \tag{21}$$

$$-\tau^{-1} f_3'(x) = \lambda f_3(x), \tag{22}$$

$$f_1(0) = -k f_3(1), \qquad f_3(0) = f_1(1). \tag{23}$$

By using similar argument of the appendix of [14], it is easy to know that $\lambda = -a_4$ belongs to the continuous spectrum $\sigma_C(A)$ of A. When $\lambda \neq -a_4$, solving (21) and (22), we have

$$f_2(x) = \frac{a_3}{\lambda + a_4} f_1(x), \tag{24}$$

$$f_3(x) = f_3(0)\, e^{-\tau\lambda x}. \tag{25}$$

Set

$$\alpha(\lambda) = -\frac{1}{\nu}\left(\lambda + a_1 + \frac{a_2a_3}{\lambda+a_4}\right), \qquad \beta(\lambda) = \tau\lambda - \alpha(\lambda). \tag{26}$$

Substituting (24) to (20), we have

$$f_1(x) = f_1(0)\, e^{\alpha(\lambda)x}. \tag{27}$$

It follows from (23), (25), and (27) that

$$e^{\beta(\lambda)} = -k. \tag{28}$$

In order to solve (28) with respect to λ, let us set $\lambda = x + iy$, $x, y \in \mathbb{R}$, and

$$\begin{aligned} u(x,y) &= (\nu\tau+1)x + a_1 + \frac{a_2a_3(x+a_4)}{(x+a_4)^2 + y^2}, \\ v(x,y) &= y\left[1 + \nu\tau - \frac{a_2a_3}{(x+a_4)^2+y^2}\right]. \end{aligned} \tag{29}$$

Then, (28) becomes

$$e^{\beta(\lambda)} = e^{(1/\nu)[u(x,y)+iv(x,y)]} = -k, \tag{30}$$

which is equivalent to

$$\begin{aligned} e^{(1/\nu)u(x,y)} \cos \frac{1}{\nu} v(x,y) &= -k, \\ \sin \frac{1}{\nu} v(x,y) &= 0. \end{aligned} \tag{31}$$

Thus, it follows from the previous equations that

$$u(x,y) = \nu \log k, \quad v(x,y) = (2n+1)\nu\pi, \quad n \in \mathbb{Z}, \tag{32}$$

which are equivalent to

$$(\nu\tau + 1)x + a_1 + \frac{a_2 a_3 (x + a_4)}{(x+a_4)^2 + y^2} = \nu \log k, \tag{33}$$

$$y\left[1 + \nu\tau - \frac{a_2 a_3}{(x+a_4)^2 + y^2}\right] = (2n+1)\nu\pi. \tag{34}$$

Combining (33) with (34), we get

$$y = \frac{(2n+1)\nu\pi (x + a_4)}{2(\nu\tau+1)x + a_1 + a_4(\nu\tau+1) - \nu\log k}, \quad n \in \mathbb{Z}. \tag{35}$$

On the other hand, solving (33) with respect to y, we have

$$y = \pm\sqrt{\frac{a_2 a_3 (x + a_4)}{\nu\log k - (\nu\tau+1)x - a_1} - (x+a_4)^2}. \tag{36}$$

As a result, introducing two sets

$$\begin{aligned} S_1 &= \left\{ x + iy : y = \frac{(2n+1)\nu\pi(x+a_4)}{2(\nu\tau+1)x + a_1 + a_4(\nu\tau+1) - \nu\log k}, \right. \\ &\qquad \left. x, y \in \mathbb{R}, x \neq a_4, n \in \mathbb{Z} \right\}, \\ S_2 &= \left\{ x + iy : y \right. \\ &\qquad = \pm\sqrt{\frac{a_2 a_3(x+a_4)}{\nu\log k - (\nu\tau+1)x - a_1} - (x+a_4)^2}, \\ &\qquad \left. x, y \in \mathbb{R}, x \neq a_4 \right\}, \end{aligned} \tag{37}$$

we see that the point spectrum $\sigma_p(A)$ of A is given by $\sigma_p(A) = S_1 \cap S_2$. But we remark that the resolvent set $\rho(A)$ of A is $(S_1^c \cup S_2^c) \setminus \{-a_4\}$ (see the Appendix). This means that

$$\sigma(A) = (S_1 \cap S_2) \cup \{-a_4\}. \tag{38}$$

When $\lambda = x + iy \in \sigma_p(A)$, from the definition of the set S_2, x must satisfy the inequality

$$\frac{a_2 a_3 (x+a_4)}{\nu\log k - (\nu\tau+1)x - a_1} - (x+a_4)^2 \geq 0, \tag{39}$$

which is equivalent to

$$\begin{aligned} &(x+a_4)\left(x - \frac{\nu\log k - a_1}{\nu\tau+1}\right) \\ &\quad \times \left(x^2 + \frac{a_1 - \nu\log k + a_4}{\nu\tau+1} x \right. \\ &\qquad \left. + \frac{a_2 a_3 + a_4(a_1 - \nu\log k)}{\nu\tau+1}\right) \leq 0. \end{aligned} \tag{40}$$

From the inequality and the definition of the $\mu(k,\tau)$, it is obvious that

$$\sup\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\} \leq \mu(k,\tau). \tag{41}$$

□

Lemma 3. *Suppose that the assumption of Theorem 1 is satisfied. Then, for any $\varepsilon > 0$, the following holds:*

$$\sup\left\{\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} : \lambda \in \mathbb{C} \text{ and } \operatorname{Re}\lambda \geq \mu(k,\tau) + \varepsilon\right\} < \infty, \tag{42}$$

in which $\mu(k,\tau)$ is the number defined in Lemma 2.

Proof. In Theorem 1, it is shown that the operator A generates a uniformly bounded C_0-semigroup $T(t)$ on $\mathscr{H}$ when the feedback gain k is chosen such that $k^2 < 1/(\nu\tau) < 1$. Then it follows from Theorem 5.3 and Remark 5.4 of [17] that, for any $\varepsilon > 0$, there exists some constant M such that

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} \leq \frac{M}{\operatorname{Re}(\lambda)} \leq \frac{M}{\varepsilon} \tag{43}$$

holds for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}(\lambda) \geq \varepsilon$.

Now, let the subset E_1 of the complex domain $\mathbb{C}$ be given by

$$E_1 = \{\lambda \in \mathbb{C} : \mu(k,\tau) + \varepsilon \leq \operatorname{Re}(\lambda) \leq \varepsilon\}. \tag{44}$$

In order to apply Theorem 1.1 of [14], it must be shown that

$$\sup\left\{\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} : \lambda \in E_1\right\} < \infty. \tag{45}$$

First, for each $\lambda \in E_1$ and each $g = (g_1, g_2, g_3)^T \in \mathscr{H}$, we consider the resolvent equation $(\lambda I - A)f = g$, which is equivalent to

$$\nu f_1'(x) + (\lambda + a_1) f_1(x) + a_2 f_2(x) = g_1(x), \tag{46}$$

$$-a_3 f_1(x)(x) + (\lambda + a_4) f_2(x) = g_2(x), \tag{47}$$

$$f_3'(x) + \tau\lambda f_3(x) = \tau g_3(x). \tag{48}$$

Solving (47) and (48), we have

$$f_2(x) = \frac{a_3}{\lambda + a_4} f_1(x) + \frac{1}{\lambda + a_4} g_2(x), \quad (49)$$

$$f_3(x) = f_3(0) e^{-\tau\lambda x} + \int_0^x \tau e^{-\tau\lambda(x-s)} g_3(s)\, ds. \quad (50)$$

Substituting (49) with (46) and solving it, we have

$$\begin{aligned} f_1(x) = f_1(0) e^{\alpha(\lambda)x} + \int_0^x \tau e^{\alpha(\lambda)(x-s)} \\ \times \left[\frac{1}{\nu} g_1(s) - \frac{a_2}{\nu(\lambda + a_4)} g_2(s)\right] ds. \end{aligned} \quad (51)$$

Since $f = (f_1, f_2, f_3)^T$ belongs to the domain $D(A)$ of A, $f_1(x)$ and $f_3(x)$ should satisfy the relations

$$f_1(0) = -k f_3(1), \qquad f_1(1) = f_3(0). \quad (52)$$

Putting $x = 1$ in (50) and (51), respectively, we obtain

$$\begin{aligned} f_3(1) &= \frac{e^{-\tau\lambda}}{1 + k e^{-\beta(\lambda)}} \\ &\quad \times \left[\int_0^1 e^{\alpha(\lambda)(1-s)} \left(\frac{1}{\nu} g_1(s) - \frac{a_2}{\nu(\lambda + a_4)} g_2(s)\right) ds \right. \\ &\quad \left. + \int_0^1 \tau e^{\tau\lambda s} g_3(s)\, ds\right], \\ f_3(0) &= e^{\tau\lambda} f_3(1) - \int_0^1 \tau e^{\tau\lambda s} g_3(s)\, ds. \end{aligned} \quad (53)$$

If substitute them into (50) and (51), then we have

$$\begin{aligned} f_1(x) &= \frac{-k e^{-\tau\lambda} e^{\alpha(\lambda)x}}{1 + k e^{-\beta(\lambda)}} \\ &\quad \times \left[\int_0^1 e^{\alpha(\lambda)(1-s)} \left(\frac{1}{\nu} g_1(s) - \frac{a_2}{\nu(\lambda + a_4)} g_2(s)\right) ds \right. \\ &\quad \left. + \int_0^1 \tau e^{\tau\lambda s} g_3(s)\, ds\right] \\ &\quad + \int_0^x \tau e^{\alpha(\lambda)(x-s)} \left[\frac{1}{\nu} g_1(s) - \frac{a_2}{\nu(\lambda + a_4)} g_2(s)\right] ds, \\ f_3(x) &= \frac{e^{\tau\lambda x}}{1 + k e^{-\beta(\lambda)}} \\ &\quad \times \left[\int_0^1 e^{\alpha(\lambda)(1-s)} \left(\frac{1}{\nu} g_1(s) - \frac{a_2}{\nu(\lambda + a_4)} g_2(s)\right) ds \right. \\ &\quad \left. + \int_0^1 \tau e^{-\tau\lambda(1-s)} g_3(s)\, ds\right] \\ &\quad - \int_x^1 \tau e^{-\tau\lambda(x-s)} g_3(s)\, ds. \end{aligned} \quad (54)$$

Also, $f_2(x)$ can be obtained from the previous equations and (49).

Next, we will estimate a bound of $\mathscr{H}$ norm of $f = (f_1, f_2, f_3)^T$. It follows from

$$\begin{aligned} \varepsilon &\le \mu(k,\tau) + \varepsilon + a_4 \\ &= \left|\mu(k,\tau) + \varepsilon + a_4\right| \le \operatorname{Re}\lambda + a_4 = \left|\lambda + a_4\right| \end{aligned} \quad (55)$$

that

$$\begin{aligned} &\left|e^{-(1/\nu)(a_2 a_3/(\lambda + a_4))x}\right| \\ &\quad = e^{-(1/\nu) a_2 a_3 (\operatorname{Re}\lambda + a_4) x/((\operatorname{Re}\lambda + a_4)^2 + (\operatorname{Im}\lambda)^2)} \\ &\quad \le e^{-(1/\nu) a_2 a_3 \varepsilon x/((\operatorname{Re}\lambda + a_4)^2 + (\operatorname{Im}\lambda)^2)} \le 1. \end{aligned} \quad (56)$$

Noting that

$$\begin{aligned} \nu \log k + (1 + \nu\tau)\varepsilon &\le a_1 + (1 + \nu\tau)\left[\mu(k,\tau) + \varepsilon\right] \\ &\le a_1 + (1 + \nu\tau)\operatorname{Re}\lambda, \\ \nu \log k + \varepsilon \le \nu \log k + \varepsilon &+ \nu\tau(\varepsilon - \operatorname{Re}\lambda) \le a_1 + \operatorname{Re}\lambda, \end{aligned} \quad (57)$$

we have

$$\begin{aligned} &\left|e^{-\beta(\lambda)x}\right| \\ &\quad = e^{-(1/\nu)[(1+\nu\tau)\operatorname{Re}\lambda + a_1]} \left|e^{-(1/\nu)(a_2 a_3/(\lambda + a_4))x}\right| \\ &\quad \le e^{-(1/\nu)[\nu \log k + (1+\nu\tau)\varepsilon]x} \\ &\quad = k^{-1} e^{-(1/\nu)(1+\nu\tau)\varepsilon x} < k^{-1}, \end{aligned} \quad (58)$$

$$\begin{aligned} &\left|e^{\alpha(\lambda)x}\right| \\ &\quad = e^{-(1/\nu)[\operatorname{Re}\lambda + a_1]} \left|e^{-(1/\nu)(a_2 a_3/(\lambda + a_4))x}\right| \\ &\quad \le e^{-(1/\nu)[\nu \log k + \varepsilon]x} = k^{-1} e^{-(1/\nu)\varepsilon x} < k^{-1}, \end{aligned} \quad (59)$$

for all $x \in [0,1]$. Putting $x = 1$ in (58), we have

$$k\left|e^{-\beta(\lambda)}\right| \le e^{-(1/\nu)(1+\nu\tau)\varepsilon} < 1. \quad (60)$$

Moreover, the continuous function $h(x,y) := e^{-\tau xy}$ defined on the compact set $[0,1] \times [\mu(k,\tau) + \varepsilon, \varepsilon]$ has absolute maximum and absolute minimum, which are denoted by L and l, respectively. Thus, we have

$$\begin{aligned} \left|f_1(x)\right| &\le \frac{k\left|e^{-\tau\lambda}\right|\left|e^{\alpha(\lambda)x}\right|}{\left|1 + k e^{-\beta(\lambda)}\right|} \\ &\quad \times \left[\int_0^1 \left|e^{\alpha(\lambda)(1-s)}\right| \left(\frac{1}{\nu}\left|g_1(s)\right| + \frac{a_2}{\nu\left|\lambda + a_4\right|}\left|g_2(s)\right|\right) ds \right. \\ &\quad \left. + \int_0^1 \tau\left|e^{\tau\lambda s}\right|\left|g_3(s)\right| ds\right] \\ &\quad + \int_0^x \tau\left|e^{\alpha(\lambda)(x-s)}\right| \end{aligned}$$

$$\times\left[\frac{1}{\nu}\left|g_1(s)\right|+\frac{a_2}{\nu\left|\lambda+a_4\right|}\left|g_2(s)\right|\right]ds$$
$$\leq\frac{e^{-\tau\operatorname{Re}\lambda}}{1-\left|ke^{-\beta(\lambda)}\right|}$$
$$\times\left[\int_0^1\frac{1}{k\nu}\left|g_1(s)\right|+\frac{a_2}{k\nu\varepsilon}\left|g_2(s)\right|ds\right.$$
$$\left.+\int_0^1\tau e^{\tau s\operatorname{Re}\lambda}\left|g_3(s)\right|ds\right]$$
$$+\int_0^1\frac{\tau}{k\nu}\left|g_1(s)\right|+\frac{\tau a_2}{k\nu\varepsilon}\left|g_2(s)\right|ds$$
$$\leq\frac{L}{1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}}$$
$$\times\left[\int_0^1\frac{1}{k\nu}\left|g_1(s)\right|+\frac{a_2}{k\nu\varepsilon}\left|g_2(s)\right|ds\right.$$
$$\left.+\int_0^1\frac{\tau}{l}\left|g_3(s)\right|ds\right]$$
$$+\int_0^1\frac{\tau}{k\nu}\left|g_1(s)\right|+\frac{\tau a_2}{k\nu\varepsilon}\left|g_2(s)\right|ds$$
$$\leq\left(\frac{L}{k\nu\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}+\frac{\tau}{k\nu}\right)\sqrt{\int_0^1\left|g_1(s)\right|^2dx}$$
$$+\frac{a_2L}{k\nu\varepsilon\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}\sqrt{\int_0^1\left|g_2(s)\right|^2dx}$$
$$+\left(\frac{\tau L}{l\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}+\frac{\tau a_2}{k\nu\varepsilon}\right)\sqrt{\int_0^1\left|g_3(s)\right|^2dx}. \tag{61}$$

The Cauchy-Schwarz inequality is applied in the last step. This means that

$$\left|f_1(x)\right|\leq M_1\|g\|_{\mathscr{H}}\quad\text{or}\quad\left\|f_1(x)\right\|_{L^2[0,1]}\leq M_1\|g\|_{\mathscr{H}}, \tag{62}$$

in which

$$M_1=\max\left\{\frac{L}{k\nu\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}+\frac{\tau}{k\nu},\right.$$
$$\frac{a_2L}{k\nu\varepsilon\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}, \tag{63}$$
$$\left.\frac{\tau L}{l\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}+\frac{\tau a_2}{k\nu\varepsilon}\right\}.$$

Similarly, we have

$$\left|f_3(x)\right|\leq\frac{\left|e^{\tau\lambda x}\right|}{\left|1+ke^{-\beta(\lambda)}\right|}$$
$$\times\left[\int_0^1\left|e^{\alpha(\lambda)(1-s)}\right|\right.$$
$$\times\left(\frac{1}{\nu}\left|g_1(s)\right|+\frac{a_2}{\nu\left|\lambda+a_4\right|}\left|g_2(s)\right|\right)ds$$
$$\left.+\int_0^1\tau\left|e^{-\tau\lambda(1-s)}\right|\left|g_3(s)\right|ds\right]$$
$$+\int_x^1\tau\left|e^{-\tau\lambda(x-s)}\right|\left|g_3(s)\right|ds$$
$$\leq\frac{e^{\tau x\operatorname{Re}\lambda}}{1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}}$$
$$\times\left[\int_0^1\left(\frac{1}{\nu}\left|g_1(s)\right|+\frac{a_2}{\nu\varepsilon}\left|g_2(s)\right|\right)ds\right.$$
$$\left.+\int_0^1\tau e^{-\tau\operatorname{Re}\lambda(1-s)}\left|g_3(s)\right|ds\right]$$
$$+\int_x^1\tau e^{-\tau\operatorname{Re}\lambda(x-s)}\left|g_3(s)\right|ds$$
$$\leq\frac{1}{l\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}$$
$$\times\left[\int_0^1\left(\frac{1}{\nu}\left|g_1(s)\right|+\frac{a_2}{\nu\varepsilon}\left|g_2(s)\right|\right)ds\right.$$
$$\left.+\int_0^1\frac{\tau L}{l}\left|g_3(s)\right|ds\right]$$
$$+\int_0^1\frac{\tau L}{l}\left|g_3(s)\right|ds$$
$$\leq\frac{1}{l\nu\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}\sqrt{\int_0^1\left|g_1(s)\right|^2dx}$$
$$+\frac{a_2}{l\nu\varepsilon\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}\sqrt{\int_0^1\left|g_2(s)\right|^2dx}$$
$$+\left(\frac{\tau L}{l^2\left[1-e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}+\frac{\tau L}{l}\right)\sqrt{\int_0^1\left|g_3(s)\right|^2dx}. \tag{64}$$

This means that

$$\left|f_3(x)\right|\leq M_3\|g\|_{\mathscr{H}}\quad\text{or}\quad\left\|f_3(x)\right\|_{L^2[0,1]}\leq M_3\|g\|_{\mathscr{H}}, \tag{65}$$

in which

$$M_3 = \max\left\{\frac{1}{l\nu\left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}, \frac{a_2}{l\nu\varepsilon\left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]}, \frac{\tau L}{l^2\left[1 - e^{-(1/\nu)(1+\nu\tau)\varepsilon}\right]} + \frac{\tau L}{l}\right\}. \tag{66}$$

It follows from (49) and (62) that

$$|f_2(x)| \le \frac{a_3}{\varepsilon}|f_1(x)| + \frac{1}{\varepsilon}|g_2(x)| \le \frac{a_3 M_1}{\varepsilon}\|g\|_{\mathscr{H}} + \frac{1}{\varepsilon}|g_2(x)|. \tag{67}$$

This implies that

$$\|f_2(x)\|_{L^2[0,1]} \le M_2\|g\|_{\mathscr{H}} \tag{68}$$

in which

$$M_2 = \frac{a_3 M_1}{\varepsilon} + \frac{1}{\varepsilon}. \tag{69}$$

Therefore, by using (62), (65), and (68), we can estimate a bound of $\mathscr{H}$ norm of $f = (f_1, f_2, f_3)^T$ as follows:

$$\begin{aligned}\|f\|^2_{\mathscr{H}} &= \|f_1(x)\|^2_{L^2[0,1]} + \|f_2(x)\|^2_{L^2[0,1]} + \|f_3(x)\|^2_{L^2[0,1]} \\ &\le \left(M_1^2 + M_2^2 + M_3^2\right)\|g\|_{\mathscr{H}},\end{aligned} \tag{70}$$

which is equivalent to

$$\left\|(\lambda I - A)^{-1}\right\|_{L(\mathscr{H})} \le \sqrt{M_1^2 + M_2^2 + M_3^2}. \tag{71}$$

Since the inequality holds for all $g \in \mathscr{H}$ and for all $\lambda \in E_1$, we have

$$\sup_{\lambda\in E_1}\left\|(\lambda I - A)^{-1}\right\|_{L(\mathscr{H})} \le \sqrt{M_1^2 + M_2^2 + M_3^2} < \infty. \tag{72}$$

This shows that (45) holds. In this way, we finally obtain

$$\sup\left\{\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(\mathscr{H})} : \lambda \in \mathbb{C} \text{ and } \operatorname{Re}\lambda \ge \mu(k,\tau) + \varepsilon\right\} < \infty. \tag{73}$$

□

Theorem 4. *Suppose that the assumption of Theorem 1 is satisfied. Then, for any $\varepsilon > 0$, there exists a constant $M_{\varepsilon,\tau}$ such that*

$$\|T(t)\|_{\mathscr{L}(\mathscr{H})} \le M_{\varepsilon,\tau} e^{[\mu(k,\tau)+\varepsilon]t} \tag{74}$$

in which $\mu(k,\tau)$ is defined in Lemma 2.

Proof. According to Theorem 1.1 of [14], Theorem 4 is direct consequence of Lemmas 2 and 3. □

4. Conclusion

In the present paper, we have considered a linear distributed parameter bioprocess with boundary control input possessing a time delay. Using a simple boundary feedback law, we have shown that the closed-loop system generates a uniformly bounded C_0-semigroup of linear operators if the feedback gain k satisfies $0 < k < 1$. After analyzing the spectrum configuration of the closed-loop system and verifying the spectrum determined growth assumption, we have demonstrated that the closed-loop system becomes exponentially stable. Our main result implies that the linear distributed parameter bioprocess preserves the exponential stability for arbitrary time delay. This means that the answer to the question posed in Section 1 is positive.

Appendix

Let S_1 and S_2 be the sets defined in the proof of Lemma 2. To show that the resolvent set $\rho(A)$ of A is $(S_1^c \cup S_2^c) \setminus \{-a_4\}$, we have to prove that the operator $\lambda I - A$ is bijective for each $\lambda \in (S_1^c \cup S_2^c) \setminus \{-a_4\}$. Thus, for each $\lambda \in (S_1^c \cup S_2^c) \setminus \{-a_4\}$ and each $g = (g_1, g_2, g_3)^T \in \mathscr{H}$, we consider the resolvent equation $(\lambda I - A)f = g$, which is equivalent to

$$\begin{aligned}\nu f_1'(x) + (\lambda + a_1) f_1(x) + a_2 f_2(x) &= g_1(x), \\ -a_3 f_1(x)(x) + (\lambda + a_4) f_2(x) &= g_2(x), \\ f_3'(x) + \tau\lambda f_3(x) &= \tau g_3(x).\end{aligned} \tag{A.1}$$

It follows from the proof of Lemma 3 that

$$\begin{aligned}f_1(x) &= f_1(0)e^{\alpha(\lambda)x} + \int_0^x \tau e^{\alpha(\lambda)(x-s)}\left[\frac{1}{\nu}g_1(s) - \frac{a_2}{\nu(\lambda + a_4)}g_2(s)\right]ds, \\ f_2(x) &= \frac{a_3}{\lambda + a_4}f_1(x) + \frac{1}{\lambda + a_4}g_2(x), \\ f_3(x) &= f_3(0)e^{-\tau\lambda x} + \int_0^x \tau e^{-\tau\lambda(x-s)}g_3(s)\,ds, \\ \left(1 + ke^{-\beta(\lambda)}\right)f_3(1) &= e^{-\tau\lambda}\left[\int_0^1 e^{\alpha(\lambda)(1-s)}\left(\frac{1}{\nu}g_1(s) - \frac{a_2}{\nu(\lambda + a_4)}g_2(s)\right)ds + \int_0^1 \tau e^{\tau\lambda s}g_3(s)\,ds\right].\end{aligned} \tag{A.2}$$

It is easy to see that $\lambda I - A$ is bijective if and only if $1 + ke^{-\beta(\lambda)} \ne 0$. From the proof of Lemma 3, we know that $e^{\beta(\lambda)} = -k$ for $\lambda \in S_1 \cap S_2$. This implies that $\rho(A) = (S_1^c \cup S_2^c) \setminus \{-a_4\}$.

Acknowledgment

This work was supported by the National Natural Science Foundation of China (11201037).

References

[1] I. Gumowski and C. Mira, *Optimization in Control Theory and Practice*, Cambridge University Press, Cambridge, UK, 1968.

[2] R. Datko, J. Lagnese, and M. P. Polis, "An example on the effect of time delays in boundary feedback stabilization of wave equations," *SIAM Journal on Control and Optimization*, vol. 24, no. 1, pp. 152–156, 1986.

[3] R. Datko, "Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks," *SIAM Journal on Control and Optimization*, vol. 26, no. 3, pp. 697–713, 1988.

[4] I. H. Suh and Z. Bien, "Use of time delay action in the controller design," *IEEE Transactions on Automatic Control*, vol. 25, no. 3, pp. 600–603, 1980.

[5] W. H. Kwon, G. W. Lee, and S. W. Kim, "Performance improvement using time delays in multivariable controller design," *International Journal of Control*, vol. 52, no. 6, pp. 1455–1473, 1990.

[6] N. Jalili and N. Olgac, "Optimum delayed feedback vibration absorber for MDOF mechanical structures," in *Proceedings of the 37th IEEE Conference on Decision and Control (CDC '98)*, pp. 4734–4739, Tampa, Fla, USA, December 1998.

[7] W. Aernouts, D. Roose, and R. Sepulchre, "Delayed control of a Moore-Greitzer axial compressor model," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 10, no. 5, pp. 1157–1164, 2000.

[8] H. Logemann, R. Rebarber, and G. Weiss, "Conditions for robustness and nonrobustness of the stability of feedback systems with respect to small delays in the feedback loop," *SIAM Journal on Control and Optimization*, vol. 34, no. 2, pp. 572–600, 1996.

[9] B.-Z. Guo and K.-Y. Yang, "Dynamic stabilization of an Euler-Bernoulli beam equation with time delay in boundary observation," *Automatica*, vol. 45, no. 6, pp. 1468–1475, 2009.

[10] B.-Z. Guo, C.-Z. Xu, and H. Hammouri, "Output feedback stabilization of a one-dimensional wave equation with an arbitrary time delay in boundary observation," *European Series in Applied and Industrial Mathematics*, vol. 18, no. 1, pp. 22–35, 2012.

[11] S. Nicaise and C. Pignotti, "Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks," *SIAM Journal on Control and Optimization*, vol. 45, no. 5, pp. 1561–1585, 2006.

[12] G. Q. Xu, S. P. Yung, and L. K. Li, "Stabilization of wave systems with input delay in the boundary control," *European Series in Applied and Industrial Mathematics* , vol. 12, no. 4, pp. 770–785, 2006.

[13] S. Bourrel and D. Dochain, "Stability analysis of two linear distributed parameter bioprocess models," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 6, no. 3, pp. 267–281, 2000.

[14] H. Sano, "Boundary control of a linear distributed parameter bioprocess," *Journal of the Franklin Institute*, vol. 340, no. 5, pp. 293–306, 2003.

[15] F. L. Huang, "Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces," *Annals of Differential Equations*, vol. 1, no. 1, pp. 43–56, 1985.

[16] M. Tucsnak and G. Weiss, *Observation and Control for Operator Semigroups*, Birkhäuser Verlag, Basel, Switzerland, 2009.

[17] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.

Hindawi Publishing Corporation
Journal of Function Spaces and Applications
Volume 2013, Article ID 130495, 8 pages
http://dx.doi.org/10.1155/2013/130495

Research Article

On the Gauss Map of Surfaces of Revolution with Lightlike Axis in Minkowski 3-Space

Minghao Jin,[1,2] Donghe Pei,[1] and Shu Xu[3]

[1] *School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, China*
[2] *Department of Mathematics, Heilongjiang Institute of Technology, Harbin 150050, China*
[3] *Rear Services Office, Chongqing Police College, Chongqing 401331, China*

Correspondence should be addressed to Donghe Pei; peidh340@nenu.edu.cn

Received 29 March 2013; Accepted 2 July 2013

Academic Editor: Ji Gao

By studying the Gauss map G and Laplace operator Δ^h of the second fundamental form h, we will classify surfaces of revolution with a lightlike axis in 3-dimensional Minkowski space and also obtain the surface of Enneper of the 2nd kind, the surface of Enneper of the 3rd kind, the de Sitter pseudosphere, and the hyperbolic pseudosphere that satisfy condition $\Delta^h G = \Lambda G$, Λ being a 3×3 real matrix.

1. Introduction

The Gauss map is a useful tool for studying surfaces in Euclidean space and pseudo-Euclidean space.

Suppose that M is a connected surface in $\mathbb{R}^3$ and G is the Gauss map on M. According to a theorem proved by Ruh and Vilms [1], M has constant mean curvature if and only if

$$\Delta G = \|dG\|^2 G, \tag{1}$$

where Δ is the Laplace operator on M that corresponds to the metric induced on M from $\mathbb{R}^3$. A special case of (1) is given by

$$\Delta G = \lambda G, \tag{2}$$

where the Gauss map G is an eigenfunction of the Laplacian Δ on M. As a more general form of (1), Dillen et al. [2] proved that a surface of revolution M in $\mathbb{R}^3$ satisfies the condition

$$\Delta G = \Lambda G, \quad \Lambda \in \operatorname{Mat}(3, \mathbb{R}), \tag{3}$$

if and only if M is a plane, sphere, or cylinder. Baikoussis and Blair [3] proved that a ruled surface M in $\mathbb{R}^3$ satisfies condition (3) if and only if M is a plane, helicoidal surface, or spiral surface in $\mathbb{R}^3$. Additionally, Choi and Alías et al. [4–6] completely classified the surfaces of revolution and ruled surfaces in 3-dimensional Minkowski space that satisfy condition (3). Kim and Yoon [7] studied ruled surfaces in $\mathbb{R}_1^m$ such that

$$\Delta G = \Lambda G, \quad \Lambda \in \operatorname{Mat}(N, \mathbb{R}), \quad N = \binom{m}{2}. \tag{4}$$

Recently, an interesting question was raised: what surfaces of revolution without parabolic points in Euclidean or pseudo-Euclidean space satisfy the following condition?

$$\Delta^h G = \Lambda G, \quad \Lambda \in \operatorname{Mat}(3, \mathbb{R}), \tag{5}$$

where Δ^h is the Laplace operator with respect to the second fundamental form h of the surface. This operator is formally defined by

$$\Delta^h = -\frac{1}{\sqrt{|\mathscr{H}|}} \sum_{i,j=1}^{2} \frac{\partial}{\partial x^i} \left(\sqrt{|\mathscr{H}|} h^{ij} \frac{\partial}{\partial x^j} \right) \tag{6}$$

for the components h_{ij} $(i, j = 1, 2)$ of the second fundamental form h on M, and we denote by (h^{ij}) (resp., $\mathscr{H}$) the inverse matrix (resp., the determinant) of the matrix (h_{ij}).

In [8], the authors studied surfaces of revolution without parabolic points in Euclidean 3-space $\mathbb{R}^3$ and presented some classification theorems. In this paper, we will consider surfaces of revolution with lightlike axis in $\mathbb{R}^3_1$ and present some classification results.

2. Preliminaries

Let $\mathbb{R}^3_1$ be a 3-dimensional Minkowski space with the scalar product and Lorentz cross-product defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0^2 + x_1^2 + x_2^2, \quad \mathbf{x} \times \mathbf{y} = (x_2 y_1 - x_1 y_2, x_2 y_0 - x_0 y_2, x_0 y_1 - x_1 y_0) \tag{7}$$

for every vector $\mathbf{x} = (x_0, x_1, x_2)$ and $\mathbf{y} = (y_0, y_1, y_2)$ in $\mathbb{R}^3_1$.

A vector $\mathbf{x}$ of $\mathbb{R}^3_1$ is said to be spacelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ or $\mathbf{x} = \mathbf{0}$, timelike if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ and lightlike or null if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ and $\mathbf{x} \neq \mathbf{0}$. A timelike or lightlike vector in $\mathbb{R}^3_1$ is said to be causal. Let $\gamma : I \to \mathbb{R}^3_1$ be a smooth curve in $\mathbb{R}^3_1$, where I is an interval in $\mathbb{R}$. We call γ spacelike, timelike, or lightlike curve if the tangent vector γ' at any point is spacelike, timelike, or lightlike, respectively.

Let I be an open interval and $\gamma : I \to \Pi$ a plane curve lying in a plane Π of $\mathbb{R}^3_1$ and l a straight line in Π which does not intersect with the curve γ. A surface of revolution M with axis l in $\mathbb{R}^3_1$ is defined to be invariant under the group of motions in $\mathbb{R}^3_1$, which fixes each point of the line l [9]. Because the present paper discusses the case of lightlike axis, without loss of generality, we may assume that the axis is the line spanned by vector $(1, 1, 0)$ in the plane Ox_0x_1.

So, we choose the line spanned by the vector $(1, 1, 0)$ as axis and express the suppose curve γ as follows:

$$\gamma(u) = (f(u), g(u), 0), \tag{8}$$

where $f(u)$ is a smooth positive function and $g(u)$ is a smooth function such that $h(u) = f(u) - g(u) \neq 0$. Then, the surface of revolution M with such axis may be given by

$$x(u, v) = \left(f(u) + \frac{v^2}{2} h(u), g(u) + \frac{v^2}{2} h(u), h(u) v \right). \tag{9}$$

Now, let us consider the Gauss map G on a surface M in $\mathbb{R}^3_1$. The map $G : M \to Q^2(\varepsilon) \subset \mathbb{R}^3_1$, which sends each point of M to the unit normal vector to M at that point, is called the Gauss map of surface M. Here, $\varepsilon(= \pm 1)$ denotes the sign of the vector field G and $Q^2(\varepsilon)$ is a 2-dimensional space form as follows:

$$Q^2(\varepsilon) = \begin{cases} S^2_1(1) & \text{in } \mathbb{R}^3_1 \text{ if } \varepsilon = 1, \\ H^2(-1) & \text{in } \mathbb{R}^3_1 \text{ if } \varepsilon = -1. \end{cases} \tag{10}$$

A surface $M \subset \mathbb{R}^3_1$ is called minimal if and only if its mean curvature H is zero. As de Woestijne ([10]) proved, we have the following theorems.

Theorem 1 (see [10]). *Every minimal, spacelike surface of revolution $M \subset \mathbb{R}^3_1$ is congruent to a part of one of the following surfaces:*

(1) *a spacelike plane;*
(2) *the catenoid of the 1st kind;*
(3) *the catenoid of the 2nd kind;*
(4) *the surface of Enneper of the 2nd kind.*

Theorem 2 (see [10]). *Every minimal, timelike surface of revolution $M \subset \mathbb{R}^3_1$ is congruent to a part of one of the following surfaces:*

(1) *a Lorentzian plane;*
(2) *the catenoid of the 3rd kind;*
(3) *the catenoid of the 4th kind;*
(4) *the catenoid of the 5th kind;*
(5) *the surface of Enneper of the 3rd kind.*

Now, we consider some examples of surfaces of revolution which are mentioned in our theorems.

Example 1 (The surface of Enneper of the 2nd kind is shown in Figure 1). The surface of Enneper of the 2nd kind is parameterized by

$$x(u, v) = \left(u^3 - u - v^2 u, u^3 + u - v^2 u, -2uv\right) \tag{11}$$

for $u < 0$. Then, the components of the first and the second fundamental forms are given by

$$g_{11} = 12u^2, \quad g_{12} = g_{21} = 0, \quad g_{22} = 4u^2,$$
$$h_{11} = \frac{-24u^2}{|x_u \times x_v|}, \quad h_{12} = h_{21} = 0, \quad h_{22} = \frac{8u^2}{|x_u \times x_v|}. \tag{12}$$

So, the mean curvature H on the surface is

$$H = \frac{(-24u^2)(4u^2) + (8u^2)(12u^2)}{2(12u^2)(4u^2)|x_u \times x_v|} = 0. \tag{13}$$

Therefore, the surface of Enneper of the 2nd kind is minimal.

Example 2 (The surface of Enneper of the 3rd kind is shown in Figure 2). The surface of Enneper of the 3rd kind is parameterized by

$$x(u, v) = \left(-u^3 - u - v^2 u, -u^3 + u - v^2 u, -2uv\right) \tag{14}$$

for $u < 0$. Then, the components of the first and the second fundamental forms are given by

$$g_{11} = -12u^2, \quad g_{12} = g_{21} = 0, \quad g_{22} = 4u^2,$$
$$h_{11} = \frac{24u^2}{|x_u \times x_v|}, \quad h_{12} = h_{21} = 0, \quad h_{22} = \frac{8u^2}{|x_u \times x_v|}. \tag{15}$$

So, the mean curvature H on the surface is

$$H = \frac{(24u^2)(4u^2) + (8u^2)(-12u^2)}{2(-12u^2)(4u^2)|x_u \times x_v|} = 0. \tag{16}$$

Therefore, the surface of Enneper of the 3rd kind is minimal.

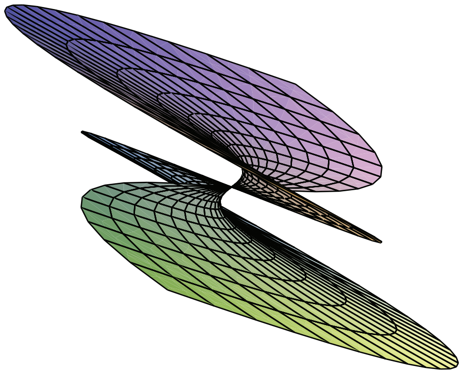


Figure 1: The surface of Enneper of the 2nd kind.

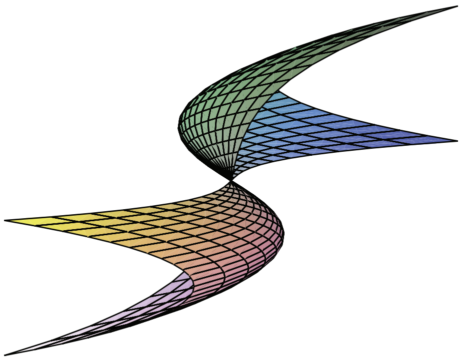


Figure 2: The surface of Enneper of the 3rd kind.

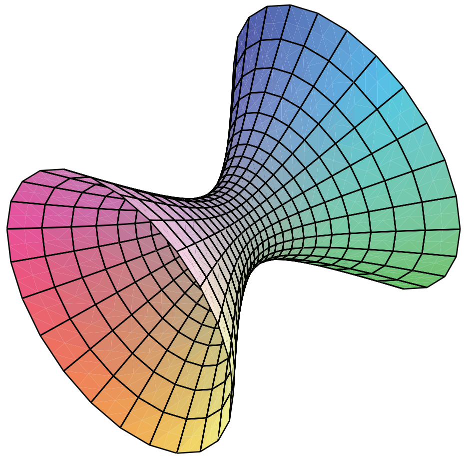


Figure 3: The de Sitter pseudosphere.

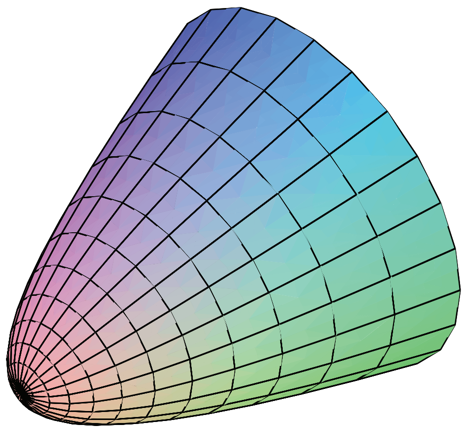


Figure 4: The future hyperbolic pseudosphere.

Example 3 (The de Sitter pseudosphere is shown in Figure 3). The de Sitter pseudosphere with radius 1 can be expressed as

$$x(u,v) = (\sinh u, \cosh u \cos v, \cosh u \sin v). \tag{17}$$

Then, its Gauss map G and Laplacian are given by

$$\begin{aligned} G &= (-\sinh u, -\cosh u \cos v, -\cosh u \sin v), \\ \Delta^h &= \frac{\partial^2}{\partial u^2} - \frac{1}{\cosh^2 u}\frac{\partial^2}{\partial v^2} + \frac{\sinh u}{\cosh u}\frac{\partial}{\partial u}. \end{aligned} \tag{18}$$

By a straight computation, we get

$$\Delta^h G = (-2\sinh u, -2\cosh u \cos v, -2\cosh u \sin v), \tag{19}$$

which means

$$\Delta^h G = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} G; \tag{20}$$

that is, the de Sitter pseudosphere satisfies condition (1).

Example 4 (The hyperbolic pseudosphere is shown in Figure 4). The hyperbolic pseudosphere with radius 1 is parameterized by

$$x(u,v) = (\cosh u, \sinh u \cos v, \sinh u \sin v). \tag{21}$$

Then, its Gauss map G and Laplacian are given by

$$\begin{aligned} G &= (-\cosh u, -\sinh u \cos v, -\sinh u \sin v), \\ \Delta^h &= -\frac{\partial^2}{\partial u^2} - \frac{1}{\sinh^2 u}\frac{\partial^2}{\partial v^2} - \frac{\cosh u}{\sinh u}\frac{\partial}{\partial u}. \end{aligned} \tag{22}$$

By a straight computation, we get

$$\Delta^h G = (2\cosh u, 2\sinh u \cos v, 2\sinh u \sin v). \tag{23}$$

So, we have

$$\Delta^h G = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} G; \tag{24}$$

that is, the hyperbolic pseudosphere satisfies condition (1).

3. The Surface of Revolution with Lightlike Axis

In this section, we will classify the surfaces of revolution with lightlike axis in $\mathbb{R}^3_1$ that satisfy condition (5).

Theorem 3. *The only surfaces of revolution with lightlike axis in $\mathbb{R}^3_1$, whose Gauss map G satisfies*

$$\Delta^h G = \Lambda G, \quad \Lambda \in Mat\,(3,\mathbb{R}), \tag{25}$$

are locally the surface of Enneper of the 2nd kind, the surface of Enneper of the 3rd kind, the de Sitter pseudosphere, and the hyperbolic pseudosphere.

Proof. Let M be a surface of revolution with lightlike axis as (9); then we may assume that the profile curve γ is of unit speed; thus

$$\langle \gamma', \gamma' \rangle = -f'^2(u) + g'^2(u) = \varepsilon\,(\pm 1). \tag{26}$$

Without lost of generality, we assume that $h = f(u) - g(u) > 0$ and give a detailed proof just for the case $\varepsilon = 1$.

Then, we may put

$$f'(u) = \sinh t, \qquad g'(u) = \cosh t \tag{27}$$

for the smooth function $t = t(u)$. Using the natural frame $\{x_u, x_v\}$ of M defined by

$$\begin{aligned} x_u &= \left(f' + \frac{v^2}{2}h', g' + \frac{v^2}{2}h', h'v\right), \qquad x_v = (vh, vh, h), \\ x_{uu} &= \left(f'' + \frac{v^2}{2}h'', g'' + \frac{v^2}{2}h'', h''v\right), \\ x_{uv} &= (vh', vh', 0), \qquad x_{vv} = (h, h, 0), \end{aligned} \tag{28}$$

we obtain the components of the first and the second fundamental forms of the surface as follows:

$$\begin{aligned} g_{11} &= \langle x_u, x_u \rangle = 1, \\ g_{12} &= g_{21} = \langle x_u, x_v \rangle = 0, \\ g_{22} &= \langle x_v, x_v \rangle = h^2, \\ h_{11} &= \langle x_{uu}, G \rangle = f''g' - f'g'' = t', \\ h_{12} &= h_{21} = \langle x_{uv}, G \rangle = 0, \\ h_{22} &= \langle x_{vv}, G \rangle = -hh', \end{aligned} \tag{29}$$

where Gauss map G is defined by $(x_u \times x_v)/|x_u \times x_v| = (-g' + (v^2/2)h', -f' + (v^2/2)h', vh')$.

So, the matrix (h_{ij}) is composed by second fundamental form h as follows:

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} t' & 0 \\ 0 & -hh' \end{pmatrix}. \tag{30}$$

Since $\mathscr{H} = h_{11}h_{22} - h_{12}^2 = 0$ makes Laplacian Δ^h degenerate, so we can assume that $\mathscr{H} \neq 0$ for every t. Then, the mean curvature H on M is given by

$$H = \frac{h^2 t' - hh'}{2h^2} = \frac{1}{2}\left(t' - \frac{h'}{h}\right). \tag{31}$$

By a straightforward computation, the Laplacian Δ^h of the second fundamental form h on M with the help of (2), (27), and (29) turns out to be

$$\begin{aligned} \Delta^h = &-\frac{1}{t'}\frac{\partial^2}{\partial u^2} + \frac{1}{hh'}\frac{\partial^2}{\partial v^2} \\ &+ \left(\frac{t''}{2t'^2} - \frac{h'}{2ht'} - \frac{h''}{2h't'}\right)\frac{\partial}{\partial u}. \end{aligned} \tag{32}$$

Accordingly, we get

$$\Delta^h G = \begin{pmatrix} \left(-\frac{h'''}{2t'} + \frac{t''h''}{4t'^2} - \frac{h'h''}{4t'h} - \frac{h''^2}{4t'h'}\right)v^2 + \frac{g'''}{t'} - \frac{t''g''}{2t'^2} + \frac{h'g''}{2t'h} + \frac{h''g''}{2t'h'} + \frac{1}{h} \\ \left(-\frac{h'''}{2t'} + \frac{t''h''}{4t'^2} - \frac{h'h''}{4t'h} - \frac{h''^2}{4t'h'}\right)v^2 + \frac{f'''}{t'} - \frac{t''f''}{2t'^2} + \frac{h'f''}{2t'h} + \frac{h''f''}{2t'h'} + \frac{1}{h} \\ \left(-\frac{h'''}{t'} + \frac{t''h''}{2t'^2} - \frac{h'h''}{2t'h} - \frac{h''^2}{2t'h'}\right)v \end{pmatrix}. \tag{33}$$

By the assumption (25) and the above equation, we get the following system of differential equations:

$$\begin{aligned} &\left(-\frac{h'''}{2t'} + \frac{t''h''}{4t'^2} - \frac{h'h''}{4t'h} - \frac{h''^2}{4t'h'} - \frac{a_{11} + a_{12}}{2}h'\right)v^2 \\ &- a_{13}h'v + \frac{g'''}{t'} - \frac{t''g''}{2t'^2} + \frac{h'g''}{2t'h} + \frac{h''g''}{2t'h'} \\ &+ \frac{1}{h} + a_{11}g' + a_{12}f' = 0, \\ &\left(-\frac{h'''}{2t'} + \frac{t''h''}{4t'^2} - \frac{h'h''}{4t'h} - \frac{h''^2}{4t'h'} - \frac{a_{21} + a_{22}}{2}h'\right)v^2 \\ &- a_{23}h'v + \frac{f'''}{t'} - \frac{t''f''}{2t'^2} + \frac{h'f''}{2t'h} + \frac{h''f''}{2t'h'} \end{aligned}$$

$$+\frac{1}{h}+a_{21}g'+a_{22}f'=0,$$
$$-\frac{a_{31}+a_{32}}{2}h'v^2+\left(-\frac{h'''}{t'}+\frac{t''h''}{2t'^2}-\frac{h'h''}{2t'h}-\frac{h''^2}{2t'h'}-a_{33}h'\right)v+a_{31}g'+a_{32}f'=0, \tag{34}$$

where a_{ij} $(i,j=1,2,3)$ denote the components of the matrix Λ given by (25).

In order to prove the theorem, we have to solve the above system of ordinary differential equations. So, we get three systems of ODE, equivalently:

$$\begin{gathered}
-\frac{h'''}{2t'}+\frac{t''h''}{4t'^2}-\frac{h'h''}{4t'h}-\frac{h''^2}{4t'h'}-\frac{a_{11}+a_{12}}{2}h'=0,\\
-a_{13}h'=0,\\
\frac{g'''}{t'}-\frac{t''g''}{2t'^2}+\frac{h'g''}{2t'h}+\frac{h''g''}{2t'h'}+\frac{1}{h}+a_{11}g'+a_{12}f'=0,\\
-\frac{h'''}{2t'}+\frac{t''h''}{4t'^2}-\frac{h'h''}{4t'h}-\frac{h''^2}{4t'h'}-\frac{a_{21}+a_{22}}{2}h'=0\\
-a_{23}h'=0,\\
\frac{f'''}{t'}-\frac{t''f''}{2t'^2}+\frac{h'f''}{2t'h}+\frac{h''f''}{2t'h'}+\frac{1}{h}+a_{21}g'+a_{22}f'=0,\\
-\frac{a_{31}+a_{32}}{2}h'=0,\\
-\frac{h'''}{t'}+\frac{t''h''}{2t'^2}-\frac{h'h''}{2t'h}-\frac{h''^2}{2t'h'}-a_{33}h'=0,\\
a_{31}g'+a_{32}f'=0.
\end{gathered} \tag{35}$$

From (35), we easily deduce that $a_{13}=a_{23}=a_{31}=a_{32}=0$ and $a_{33}=(a_{11}+a_{22})/2=a_{11}+a_{12}=a_{21}+a_{22}$. We put $a_{11}=\lambda$ and $a_{22}=\mu$. Therefore, the matrix Λ satisfies

$$\Lambda=\begin{pmatrix}\lambda & \frac{1}{2}(\mu-\lambda) & 0\\ \frac{1}{2}(\lambda-\mu) & \mu & 0\\ 0 & 0 & \frac{1}{2}(\lambda+\mu)\end{pmatrix}. \tag{36}$$

Then, three systems (35) now reduce to the following equations:

$$\frac{g'''}{t'}-\frac{t''g''}{2t'^2}+\frac{h'g''}{2t'h'}+\frac{h''g''}{2t'h'}+\frac{1}{h}=-\lambda g'-\frac{\mu-\lambda}{2}f', \tag{37}$$
$$\frac{f'''}{t'}-\frac{t''f''}{2t'^2}+\frac{h'f''}{2t'h'}+\frac{h''f''}{2t'h}+\frac{1}{h}=-\mu f'-\frac{\lambda-\mu}{2}g', \tag{38}$$
$$-\frac{h'''}{t'}+\frac{t''h''}{2t'^2}-\frac{h'h''}{2t'h}-\frac{h''^2}{2t'h'}=\frac{\mu+\lambda}{2}h'. \tag{39}$$

By the computation (37) $\times\cosh t-$ (38) $\times\sinh t$ and using $f'=\sinh t$, $f''=t'\cosh t$, $f'''=t'^2\sinh t+t''\cosh t$, $g'=\cosh t$, $g''=t'\sinh t$, and $g'''=t'^2\cosh t+t''\sinh t$, we easily get

$$t'-\frac{h'}{h}=-\lambda\cosh^2 t+\mu\sinh^2 t+(\lambda-\mu)\sinh t\cosh t. \tag{40}$$

On the other hand, substituting $h''=-h't'$ and $h'''=h'(t'^2-t'')$ into (39) equivalently, we get the following equation:

$$t''-3t'^2+\frac{h'}{h}t'=(\lambda+\mu)t'. \tag{41}$$

Now, we discuss five cases according to the constants λ and μ.

Case 1 ($\lambda=\mu=0$). In this case, we easily get $t'-(h'/h)=0$, which implies that the mean curvature H vanishes identically because of (31). Therefore, the surface is minimal; from Theorem 1 it is the surface of Enneper of the 2nd kind. Furthermore, a surface of Enneper of the 2nd kind satisfies the condition (25).

Case2 ($\lambda=\mu\neq 0$). By (40), we get

$$t'=\frac{h'}{h}-\lambda. \tag{42}$$

Differentiating (42) with respect to u, we have

$$t''=-\frac{h'}{h}t'-\left(\frac{h'}{h}\right)^2. \tag{43}$$

Substituting (42) and (43) into (41), we get

$$4\left(\frac{h'}{h}\right)^2+4\lambda\frac{h'}{h}+\lambda^2=0 \tag{44}$$

from which

$$\frac{h'}{h}=\frac{\lambda}{2}. \tag{45}$$

Furthermore, (45) together with (42) becomes $t'=-(\lambda/2)$; that is,

$$t(u)=-\frac{\lambda}{2}u+k,\quad k\in\mathbb{R}. \tag{46}$$

On the other hand, by (27), (45), and (46), we have

$$\begin{gathered}
f(u)=-\frac{2}{\lambda}\cosh\left(-\frac{\lambda}{2}u+k\right)+c,\\
g(u)=-\frac{2}{\lambda}\sinh\left(-\frac{\lambda}{2}u+k\right)+c,\quad c\in\mathbb{R}.
\end{gathered} \tag{47}$$

Then, the surface M has the following expression:

$$\begin{aligned}
x(u,v)=\Bigg(&-\frac{2}{\lambda}\cosh\left(-\frac{\lambda}{2}u+k\right)+\frac{v^2}{2}h+c,\\
&-\frac{2}{\lambda}\sinh\left(-\frac{\lambda}{2}+k\right)+\frac{v^2}{2}h+c,hv\Bigg),
\end{aligned} \tag{48}$$

where $h = f - g = -(2/\lambda)e^{(\lambda/2)-k}$, $c, k \in \mathbb{R}$. From this, we easily get

$$\langle x(u,v) - \mathbf{C}, x(u,v) - \mathbf{C}\rangle = -\left(\frac{2}{\lambda}\right)^2, \quad \mathbf{C} = (c, c, 0). \quad (49)$$

This equation means that the surface M is contained in the hyperbolic pseudosphere $H^2(-(2/|\lambda|))$ centered at $\mathbf{C}$ with radius $2/|\lambda|$. Also, the hyperbolic pseudosphere satisfies condition (25).

Case 3 ($\lambda \neq 0, \mu = 0$). In this case, (40) becomes $t' - (h'/h) = -\lambda\cosh^2 t + \lambda\sinh t\cosh t$; that is,

$$t' = \frac{h'}{h} - \lambda\cosh^2 t + \lambda\sinh t\cosh t \quad (50)$$

and thus

$$\begin{aligned} t'' = \frac{h''}{h} - \left(\frac{h'}{h}\right)^2 - 2\lambda t'\sinh t\cosh t \\ + \lambda t'\sinh^2 t + \lambda t'\cosh^2 t. \end{aligned} \quad (51)$$

Substituting (50) and (51) into (41), we get

$$\Phi_1 h^2 + \Phi_2 h + \Phi_3 = 0, \quad (52)$$

where we put

$$\begin{aligned} \Phi_1 &= \lambda^2 \left(-3\cosh^4 t + 8\sinh t\cosh^3 t \right. \\ &\quad \left. -7\sinh^2 t\cosh^2 t + 2\sinh^3 t\cosh t\right), \\ \Phi_2 &= \lambda \left(-6\cosh^3 t + 14\sinh t\cosh^2 t \right. \\ &\quad \left. -10\sinh^2 t\cosh t + 2\sinh^3 t\right), \\ \Phi_3 &= -4\sinh^2 t + 8\sinh t\cosh t - 4\cosh^2 t. \end{aligned} \quad (53)$$

Differentiating (52) and using (50), we find

$$\Psi_1 f^2 + \Psi_2 f + \Psi_3 = 0, \quad (54)$$

where

$$\begin{aligned} \Psi_1 &= \lambda^2 \left(-4\sinh^8 t\cosh t + 40\sinh^7 t\cosh^2 t \right. \\ &\quad - 182\sinh^6 t\cosh^3 t + 474\sinh^5 t\cosh^4 t \\ &\quad - 760\sinh^4 t\cosh^5 t \\ &\quad + 764\sinh^3 t\cosh^6 t - 470\sinh^2 t\cosh^7 t \\ &\quad \left. +162\sinh t\cosh^8 t - 24\cosh^9 t\right), \\ \Psi_2 &= \lambda \left(-4\sinh^8 t + 44\sinh^7 t\cosh t \right. \\ &\quad - 220\sinh^6 t\cosh^2 t + 612\sinh^5 t\cosh^3 t \\ &\quad - 1020\sinh^4 t\cosh^4 t + 1044\sinh^3 t\cosh^5 t \\ &\quad - 644\sinh^2 t\cosh^6 t + 220\sinh t\cosh^7 t \\ &\quad \left. -32\cosh^8 t\right), \\ \Psi_3 &= 8\sinh^7 t - 64\sinh^6 t\cosh t \\ &\quad + 208\sinh^5 t\cosh^2 t - 360\sinh^4 t\cosh^3 t \\ &\quad + 360\sinh^3 t\cosh^4 t - 208\sinh^2 t\cosh^5 t \\ &\quad + 64\sinh t\cosh^6 t - 8\cosh^7 t. \end{aligned} \quad (55)$$

Combining (52) and (54), we show that

$$\chi_1 f + \chi_2 = 0, \quad (56)$$

where $\chi_1 = \Phi_2\Psi_1 - \Phi_1\Psi_2$, $\chi_2 = \Phi_3\Psi_1 - \Phi_1\Psi_3$.

Differentiating once again this equation and using the same algebraic techniques above, we find the following trigonometric polynomial in $\sinh t$ and $\cosh t$ satisfying

$$\mu^2 \left(\sum_{i=1}^{31} c_i \sinh^{31-i} t\cosh^{5+i} t\right) = 0, \quad (57)$$

where $c_1 = 1024$, $c_2 = -24064, \ldots$, and $c_{31} = 170496$ are nonzero coefficients of the function $\sinh^{31-i} t\cosh^{5+i} t$. Since this polynomial is equal to zero for every t, all its coefficients must be zero. Thus, we have $\mu = 0$, which is a contradiction. Consequently, there are no surfaces of revolution with lightlike axis in this case.

Case 4 ($\lambda = 0, \mu \neq 0$). In this case, (40) becomes $t' - (h'/h) = \mu\sinh^2 t - \mu\sinh t\cosh t$; that is,

$$t' = \frac{h'}{h} + \mu\sinh^2 t - \mu\sinh t\cosh t \quad (58)$$

and thus

$$\begin{aligned} t'' = \frac{h''}{h} - \left(\frac{h'}{h}\right)^2 + 2\mu t'\sinh t\cosh t \\ - \mu t'\sinh^2 t - \mu t'\cosh^2 t. \end{aligned} \quad (59)$$

Substituting (58) and (59) into (41), we get

$$\iota_1 h^2 + \iota_2 h + \iota_3 = 0, \quad (60)$$

where we put

$$\begin{aligned} \iota_1 &= \mu^2\left(-3\sinh^4 t + 8\sinh^3 t\cosh t \right. \\ &\quad \left. -7\sinh^2 t\cosh^2 t + 2\sinh t\cosh^3 t\right), \\ \iota_2 &= \mu\left(-6\sinh^3 t + 14\sinh^2 t\cosh t \right. \\ &\quad \left. -10\sinh t\cosh^2 t + 2\cosh^3 t\right), \\ \iota_3 &= -4\sinh^2 t + 8\sinh t\cosh t - 4\cosh^2 t. \end{aligned} \tag{61}$$

Differentiating (60) and using (58), we find

$$\kappa_1 f^2 + \kappa_2 f + \kappa_3 = 0, \tag{62}$$

where

$$\begin{aligned} \kappa_1 &= \mu^2\left(108\sinh^9 t - 246\sinh^8 t\cosh t \right. \\ &\quad + 106\sinh^7 t\cosh^2 t - 64\sinh^6 t\cosh^3 t \\ &\quad + 424\sinh^5 t\cosh^4 t - 530\sinh^4 t\cosh^5 t \\ &\quad + 238\sinh^3 t\cosh^6 t - 40\sinh^2 t\cosh^7 t \\ &\quad \left. +4\sinh t\cosh^8 t\right), \\ \kappa_2 &= \mu\left(116\sinh^8 t - 220\sinh^7 t\cosh t \right. \\ &\quad - 28\sinh^6 t\cosh^2 t + 76\sinh^5 t\cosh^3 t \\ &\quad + 432\sinh^4 t\cosh^4 t - 612\sinh^3 t\cosh^5 t \\ &\quad + 276\sinh^2 t\cosh^6 t \\ &\quad \left. -44\sinh t\cosh^7 t + 4\cosh^8 t\right), \\ \kappa_3 &= 8\sinh^7 t - 64\sinh^6 t\cosh t \\ &\quad + 208\sinh^5 t\cosh^2 t - 360\sinh^4 t\cosh^3 t \\ &\quad + 360\sinh^3 t\cosh^4 t - 208\sinh^2 t\cosh^5 t \\ &\quad + 64\sinh t\cosh^6 t - 8\cosh^7 t. \end{aligned} \tag{63}$$

Combining (60) and (62), we show that

$$\omega_1 f + \omega_2 = 0, \tag{64}$$

where $\omega_1 = \iota_2\kappa_1 - \iota_1\kappa_2$, $\omega_2 = \iota_3\kappa_1 - \iota_1\kappa_3$.

Differentiating once again this equation and using the same method above, we find the following trigonometric polynomial in $\sinh t$ and $\cosh t$ satisfying

$$\mu^2\left(\sum_{i=1}^{31} c_i \sinh^{37-i} t\cosh^{i-1} t\right) = 0, \tag{65}$$

where $c_1 = 86420736$, $c_2 = -4471635456, \ldots$, and $c_{31} = -8192$ are nonzero coefficients of the function $\sinh^{37-i} t\cosh^{i-1} t$. Since this polynomial is equal to zero for every t, all its coefficients must be zero. Thus, we have $\mu = 0$, which is a contradiction. Consequently, there are no surfaces of revolution with lightlike axis.

Case 5 ($\lambda \neq 0$, $\mu \neq 0, \lambda \neq \mu$). In this case, (40) is unchanged; that is,

$$t' = \frac{h'}{h} - \lambda\cosh^2 t + \mu\sinh^2 t + (\lambda - \mu)\sinh t\cosh t \tag{66}$$

and thus

$$\begin{aligned} t'' &= \frac{h''}{h} - \left(\frac{h'}{h}\right)^2 - 2(\lambda - \mu)t'\sinh t\cosh t \\ &\quad + (\lambda - \mu)t'\sinh^2 t + (\lambda - \mu)t'\cosh^2 t. \end{aligned} \tag{67}$$

Substituting (66) and (67) into (41), we get

$$P_1 h^2 + P_2 h + P_3 = 0, \tag{68}$$

where we put

$$\begin{aligned} P_1 &= \left(2\lambda\mu - 5\mu^2\right)\sinh^4 t \\ &\quad + \left(2\lambda^2 - 12\lambda\mu + 10\mu^2\right)\sinh^3 t\cosh t \\ &\quad + \left(-7\lambda^2 + 18\lambda\mu - 5\mu^2\right)\sinh^2 t\cosh^2 t \\ &\quad + \left(8\lambda^2 - 8\lambda\mu\right)\sinh t\cosh^3 t - 3\lambda^2\cosh^4 t, \\ P_2 &= (-4\lambda - 2\mu)\sinh^3 t \\ &\quad + (-4\lambda + 10\mu)\sinh^2 t\cosh t \\ &\quad + (14\lambda - 8\mu)\sinh t\cosh^2 t - 6\lambda\cosh^3 t \\ P_3 &= -4\sinh^2 t + 8\sinh t\cosh t - 4\cosh^2 t. \end{aligned} \tag{69}$$

Differentiating (68) and using (66), we find

$$Q_1 f^2 + Q_2 f + Q_3 = 0, \tag{70}$$

where

$$\begin{aligned} Q_1 &= \left(8\lambda^3\mu - 44\lambda^2\mu^2 + 16\lambda\mu^3 + 20\mu^4\right)\sinh^9 t \\ &\quad + \left(8\lambda^4 - 100\lambda^3\mu + 144\lambda^2\mu^2 \right. \\ &\quad \left. +118\lambda\mu^3 - 170\mu^4\right)\sinh^8 t\cosh t \\ &\quad + \cdots + \left(-48\lambda^4 + 48\lambda^3\mu + 42\lambda^2\mu^2 \right. \\ &\quad \left. -24\lambda\mu^3 - 18\mu^4\right)\cosh^9 t, \end{aligned}$$

$$\begin{aligned} Q_2 = & \left(8\lambda^3 - 52\lambda^2\mu + 4\lambda\mu^2 - 20\mu^3\right)\sinh^8 t \\ & + \left(-64\lambda^3 + 180\lambda^2\mu - 28\lambda\mu^2 - 40\mu^3\right) \\ & \times \sinh^7 t \cosh t + \cdots \\ & + \left(-80\lambda^3 + 80\lambda^2\mu + 48\lambda\mu^2 - 24\mu^3\right) \\ & \times \cosh^8 t, \\ Q_3 = & \left(8\lambda^2 - 32\lambda\mu\right)\sinh^7 t \\ & + \left(-64\lambda^2 + 176\lambda\mu - 40\mu^2\right)\sinh^6 t \cosh t \\ & + \cdots + \left(-32\lambda^2 + 32\lambda\mu + 24\mu^2\right)\cosh^7 t. \end{aligned} \tag{71}$$

Combining (68) and (70), we show that

$$R_1 f + R_2 = 0, \tag{72}$$

where $R_1 = P_2Q_1 - P_1Q_2$, $R_2 = P_3Q_1 - P_1Q_3$.

Differentiating once again this equation and using the same algebraic techniques above, we find the following trigonometric polynomial in $\sinh t$ and $\cosh t$ satisfying

$$\sum_{i=1}^{37} c_i(\lambda,\mu)\sinh^{37-i} t \cosh^{i-1} t = 0, \tag{73}$$

where

$$\begin{aligned} c_1(\lambda,\mu) = & -331776\lambda^{11}\mu^3 + 6819840\lambda^{10}\mu^4 \\ & + \cdots - 4352000\mu^{14}, \\ & \vdots \\ c_{37}(\lambda,\mu) = & -18874368\lambda^{14} + 54263808\lambda^{13}\mu^1 \\ & + \cdots + 2985984\mu^{14}, \end{aligned} \tag{74}$$

where $c_i(\lambda,\mu)$ $(i = 1,\dots,37)$ are the known polynomials in λ and μ. Since this polynomial is equal to zero for every t, all its coefficients must be zero. Therefore, $\lambda = \mu = 0$, which is a contradiction. Consequently, there are no surfaces of revolution with lightlike axis in this case.

When $\varepsilon = -1$, we can assume that $f'(u) = \cosh t$ and $g'(u) = \sinh t$. Using the same algebraic techniques as for $\varepsilon = 1$, we easily prove from theorem (9) that the surfaces of Enneper of the 3rd kind and the de Sitter pseudosphere satisfy condition (25). This completes the proof. □

Acknowledgments

The authors were supported by NSF of China (no. 11271063) and NSF of Heilongjiang Institute of Technology (no. 2012QJ19).

References

[1] E. A. Ruh and J. Vilms, "The tension field of the Gauss map," *Transactions of the American Mathematical Society*, vol. 149, pp. 569–573, 1970.

[2] F. Dillen, J. Pas, and L. Verstraelen, "On the Gauss map of surfaces of revolution," *Bulletin of the Institute of Mathematics. Academia Sinica*, vol. 18, no. 3, pp. 239–246, 1990.

[3] C. Baikoussis and D. E. Blair, "On the Gauss map of ruled surfaces," *Glasgow Mathematical Journal*, vol. 34, no. 3, pp. 355–359, 1992.

[4] L. J. Alías, A. Ferrández, P. Lucas, and M. A. Meroño, "On the Gauss map of *B*-scrolls," *Tsukuba Journal of Mathematics*, vol. 22, no. 2, pp. 371–377, 1998.

[5] S. M. Choi, "On the Gauss map of surfaces of revolution in a 3-dimensional Minkowski space," *Tsukuba Journal of Mathematics*, vol. 19, no. 2, pp. 351–367, 1995.

[6] S. M. Choi, "On the Gauss map of ruled surfaces in a 3-dimensional Minkowski space," *Tsukuba Journal of Mathematics*, vol. 19, no. 2, pp. 285–304, 1995.

[7] Y. H. Kim and D. W. Yoon, "Classification of ruled surfaces in Minkowski 3-spaces," *Journal of Geometry and Physics*, vol. 49, no. 1, pp. 89–100, 2004.

[8] Y. H. Kim, C. W. Lee, and D. W. Yoon, "On the Gauss map of surfaces of revolution without parabolic points," *Bulletin of the Korean Mathematical Society*, vol. 46, no. 6, pp. 1141–1149, 2009.

[9] C. C. Beneki, G. Kaimakamis, and B. J. Papantoniou, "Helicoidal surfaces in three-dimensional Minkowski space," *Journal of Mathematical Analysis and Applications*, vol. 275, no. 2, pp. 586–614, 2002.

[10] I. V. de Woestijne, *Minimal Surfaces in the 3-Dimensional Minkowski Space, Geometry and Topology of Submanifolds: II*, World Scientific, Singapore, 1990.