Abstract and Applied Analysis

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Editorial Board

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Editorial

Generalized Differential and Integral Equations

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The theory of differential equations is always and necessarily "under construction," because more accurate mathematical models usually demand new theoretical developments. For instance, discontinuities or singularities often occur in applications and they are often removed from models just for technical limitations.

On the other hand, the last few years have witnessed an increasing interest in different types of differential and integral equations as mathematical models for real life situations. Besides the classical examples of impulsive equations or equations with deviating arguments, many other types of revisions of the classical concepts of differential or integral equations are being intensively studied: set-valued equations, stochastic equations, fractional equations, fuzzy equations, and many more.

This diversity notwithstanding, it appears that generalized Stieltjes integration provides an unified framework for many of the above types of equations, thus simplifying and improving the theory at the same time.

The special issue succeeded in bringing together a number of papers on many different branches of the theory of differential equations which clearly deserve the adjective "generalized." The editors in charge of this special issue did not expect such a variety when they first proposed a special issue mainly focused on the following topics:

(i) generalized differential and integral equations (such as differential inclusions, stochastic equations, and fractional equations, but not restricted to these three),

(ii) discontinuous or singular equations,

(iii) fixed point theorems with applications to differential equations,

(iv) generalized integration with applications to differential equations.

While many of papers in the special issue fall inside at least one of the four previous categories, there are some others which do not and yet represent many other interesting topics in this area. For instance, to point out just a few, readers will find in this special issue papers on singular semigroups, fuzzy differential equations, homogeneization of parabolic equations, and interior field methods for Laplace's equation.

For more details on recent and future developments in generalized differential and integral equations, we refer the reader to the survey paper authored by the editors in this special issue.

Acknowledgments

Finally, the editors wish to express their gratitude to the many authors and reviewers who contributed so greatly to the success of this special issue.

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Review Article

A Survey of Recent Results for the Generalizations of Ordinary Differential Equations

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This is a review paper on recent results for different types of generalized ordinary differential equations. Its scope ranges from discontinuous equations to equations on time scales. We also discuss their relation with inclusion and highlight the use of generalized integration to unify many of them under one single formulation.

1. Existence Theory for Differential Equations and Inclusions

There was a series of results which progressively weakened the continuity in the state variable of the classical Carathéodory existence theorem for first-order differential equations; these include [1–7]. Biles and Schechter posed the open problem of proving an existence result for discontinuous systems of differential equations lacking a quasimonotonicity property; see [7, page 3352]. Motivated by that question, Cid and Pouso [8] explored an alternative approach to discontinuous equations which consisted, roughly speaking, of inserting the differential equation into a semicontinuous differential inclusion for which existence results were available, and then positing assumptions on the discontinuities of the former differential equation so that every solution of the inclusion is a solution of the equation. Besides getting an existence result for nonquasimonotone discontinuous systems, the approach in [8] came to unify and extend previous similar results for autonomous equations proven in [9] and for nonautonomous equations proven in [10].

Here and henceforth we work in a real interval $I = [t_0, t_0 + L]$ with $L > 0$.

Theorem 1 (see [8, Theorem 2.4]). *Assume that $f : I \times \mathbb{R}^m \to \mathbb{R}^m (m \in \mathbb{N})$ and the null set $N \subset I$ satisfy the following conditions.*

(i) *There exists $\psi \in L^1(I)$ such that for all $t \in I \setminus N$ and all $x \in \mathbb{R}^m$ one has $\|f(t,x)\| \le \psi(t)(1 + \|x\|)$, where $\|\cdot\|$ is a norm in $\mathbb{R}^m$.*

(ii) *For all $x \in \mathbb{R}^m$, $f(\cdot, x)$ is measurable.*

(iii) *For all $t \in I \setminus N$, $f(t,\cdot)$ is continuous in $\mathbb{R}^m \setminus K(t)$, where $K(t) = \cup_{n=1}^{\infty} K_n(t)$, and for each $n \in \mathbb{N}$ and $x \in K_n(t)$ one has*

$$\bigcap_{\varepsilon>0} \overline{co} f(t, x + \varepsilon B) \cap DK_n(t,x)(1) \subset \{f(t,x)\}, \tag{1}$$

where $\overline{co}$ denotes the closed convex hull, B is the unit ball centered at the origin, and DK_n is the contingent derivative of the multivalued map K_n (see [11] for details).

Then the set $\mathscr{C}$ of all Carathéodory solutions of the initial value problem

$$x'(t) = f(t, x(t)), \quad t \in I, \qquad x(t_0) = x_0 \tag{2}$$

is a nonempty, compact, and connected subset of $\mathscr{C}(I, \mathbb{R}^m)$.

Moreover, in the scalar case ($m = 1$), *one has the following.*

(1) $\mathscr{C}$ *has pointwise maximum*, x^*, *and minimum*, x_*, *which are the extremal solutions of the initial value problem. Furthermore for each* $t \in I$ *one has*

$$\begin{aligned} x^*(t) &= \max\{v(t) : v \in AC(I), \\ &\qquad v'(s) \le f(s, v(s))\ a.e., v(t_0) \le x_0\}, \\ x_*(t) &= \min\{v(t) : v \in AC(I), \\ &\qquad v'(s) \ge f(s, v(s))\ a.e., v(t_0) \ge x_0\}. \end{aligned} \tag{3}$$

(2) $\mathscr{C}$ *is a funnel; that is, for all* $\bar{t} \in I$ *and* $c \in [x_*(\bar{t}), x^*(\bar{t})]$ *there exists* $x \in \mathscr{C}$ *such that* $x(\bar{t}) = c$.

The simplest case of Theorem 1 occurs in the one-dimensional case, that is, $m = 1$, and when the discontinuity sets K_n are single-valued, that is, $K_n(t) = \{\gamma_n(t)\}$ for, let us say, absolutely continuous functions γ_n, $n \in \mathbb{N}$. In this situation we have

$$DK_n(t, \gamma_n(t))(1) = \{\gamma_n'(t)\}, \tag{4}$$

and then condition (1) reads simply as follows:

$$\begin{aligned} &\text{either } \gamma_n'(t) \notin \bigcap_{\varepsilon>0} \overline{co} f(t, \gamma_n(t) + \varepsilon B), \\ &\text{or } \quad \gamma_n'(t) = f(t, \gamma_n(t)), \end{aligned} \tag{5}$$

and it is helpful to note that, in the one-dimensional case, we have

$$\begin{aligned} \bigcap_{\varepsilon>0} \overline{co} f(t, x + \varepsilon B) = \Big[&\min\Big\{f(t,x), \liminf_{y\to x} f(t,y)\Big\}, \\ &\max\Big\{f(t,x), \limsup_{y\to x} f(t,y)\Big\}\Big]. \end{aligned} \tag{6}$$

The first alternative in (5) means that $\gamma_n'(t)$ coincides neither with $f(t, \gamma_n(t))$ nor with any limit value of f when the variables tend to $(t, \gamma_n(t))$. This is a sort of transversality condition between $f(t, x)$ and the discontinuity curve $\gamma_n(t)$, and it is immediately satisfied in case γ_n has a sufficiently big (or sufficiently small) slope.

The second alternative is much clearer: it simply means that γ_n solves the differential equation at the point t. The moral is that we do not have to worry about discontinuities of f when they are located over graphs of solutions of the differential equation (even though these solutions do not satisfy the initial condition or they are not defined on the whole interval I).

For simplicity, we have often called *admissible discontinuity curve* any function $\gamma(t)$ satisfying (5), and they have proven to be useful in other contexts; see [12] for singular and discontinuous problems and [13] for a revision of Perron's method using similar curves.

Let us now turn our attention to differential inclusions. The rest of this section is devoted to a somewhat inverse approach to that in the first part: one can get new results for inclusions by means of known results for discontinuous equations.

To start with, we quote [14] where we can find necessary and sufficient conditions for the existence of Carathéodory solutions to

$$x' \in F(x), \qquad x(0) = x_0, \tag{7}$$

where F is an arbitrary multifunction. Biles proves that the necessary and sufficient conditions for (7) to have at least one solution are that F have a selection f such that either $f(x_0) = 0$ or $\int_{x_0}^{\beta} dx/f(x)$ exists (in Lebesgue's sense) for some $\beta \ne x_0$. This uses and generalizes a theorem for differential equations by binding in [15].

We also used known results for equations to study nonautonomous first-order inclusions in [16]. Let us proceed to review the main ideas in that paper.

For a given multifunction $F : I \times \mathbb{R} \to P(\mathbb{R}) \setminus \{\emptyset\}$ we consider the initial value problem

$$\begin{aligned} x'(t) \in F(t, x(t)) \ \text{ for almost all } (a.a.)\ t \in I, \\ x(t_0) = x_0, \end{aligned} \tag{8}$$

and we look for solutions in the Carathéodory sense, that is, absolutely continuous solutions.

A very usual assumption on the multifunction F is that it assumes compact values for $a.a.$ $t \in I$ and all $x \in \mathbb{R}$, hence the set $F(t, x)$ has minimum and maximum. We simply impose the following condition.

(H1) For $a.a.$ $t \in I$ and all $x \in \mathbb{R}$ the set $F(t, x)$ has a minimum; and now we introduce the following definition.

Definition 2. A superfunction (or upper solution) of (8) is any $u \in AC(I)$ such that $u(t_0) \ge x_0$ and for $a.a.$ $t \in I$ one has $u'(t) \ge \min F(t, u(t))$.

We also impose the following.

(H2) There exists $\psi \in L^1(I)$ such that for $a.a.$ $t \in I$ and all $x \in \mathbb{R}$ we have

$$|\min F(t, x)| \le \psi(t), \tag{9}$$

and we restrict, for technical convenience, the set of superfunctions to the following one.

Definition 3. The set of admissible superfunctions of (8) is

$$\begin{aligned} U := \{&u \in AC(I) : u \text{ is a superfunction of (8)} \\ &\text{and } |u'| \le \psi + 1\ a.e. \text{ on } I\}. \end{aligned} \tag{10}$$

Notice that $u(t) := x_0 + \int_{t_0}^{t} \psi(r)dr$, $t \in I$, is an admissible superfunction of (8). Thus we can define

$$u_{\inf}(t) := \inf\{u(t) : u \in U\} \quad \forall t \in I. \tag{11}$$

Standard arguments reveal that $u_{\inf}(t_0) = x_0$ and that $u_{\inf} \in AC(I)$.

For simplicity of notation, we also define

$$f_m(t,x) := \min F(t,x) \quad \text{for } a.a.\ t \in I \text{ and all } x \in \mathbb{R}, \tag{12}$$

and we consider the ordinary problem

$$x'(t) = f_m(t,x(t)) \quad \text{for } a.a.\ t \in I, \qquad x(t_0) = x_0. \tag{13}$$

Plainly, solutions of (13) are solutions of (8) by virtue of (H1), but the converse is false in general. Moreover, superfunctions of (8) in the sense of Definition 2 are nothing but the usual superfunctions of (13), and so $u_{\inf}$ is a reasonable candidate for being a solution to (8). Note also that solutions of (8) need not be admissible superfunctions in the sense of Definition 3, so $u_{\inf}$ might not be the least solution of (8).

Definition 4. A lower admissible nonquasisemicontinuity curve for (8) (LAD curve, for short) is an absolutely continuous function $\gamma : [a,b] \subset I \to \mathbb{R}$ for which there exist disjoint sets $A, B \subset [a,b]$ such that $A \cup B = [a,b]$ and for $a.a.\ t \in A$ one has

$$\gamma'(t) \in F(t,\gamma(t)), \tag{14}$$

and for $a.a.\ t \in B$ one has

$$\begin{aligned} &\gamma'(t) \geq f_m(t,\gamma(t)) \ \text{ whenever } \gamma'(t) \geq \liminf_{y \to (\gamma(t))^+} f_m(t,y), \\ &\gamma'(t) \leq f_m(t,\gamma(t)) \ \text{ whenever } \gamma'(t) \leq \limsup_{y \to (\gamma(t))^-} f_m(t,y). \end{aligned} \tag{15}$$

Remark 5. The sets A or B in Definition 4 might be empty.

A particularly clear case of a LAD curve corresponds to $B = \emptyset$, which means that $A = [a,b]$, so in that case the LAD curve is nothing but a solution of the differential inclusion on $[a,b]$.

In turn, let us point out the following sufficient condition for an absolutely continuous function $\gamma : [a,b] \subset I \to \mathbb{R}$ to be a LAD curve with $B = [a,b]$: there exist $\varepsilon > 0$ and $\rho > 0$ such that for $a.a.\ t \in [a,b]$ we have

$$f_m(t,x) \geq \gamma'(t) + \rho \quad \forall x \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon], \tag{16}$$

or for $a.a.\ t \in [a,b]$ we have

$$f_m(t,x) \leq \gamma'(t) - \rho \quad \forall x \in [\gamma(t) - \varepsilon, \gamma(t) + \varepsilon]. \tag{17}$$

Notice that (16) (or (17)) implies that γ crosses each solution of $x' = f_m(t,x)$ at most once, so (16) (or (17)) is a transversality condition for γ with respect to the differential equation $x' = f_m(t,x)$.

We are now in a position to present the main result in [16].

Theorem 6 (see [16, Theorem 2.5]). *Assume that conditions (H1) and (H2) hold. Suppose moreover that the following condition is fulfilled.*

(H3) *Either for a.a. $t \in I$ and all $x \in \mathbb{R}$ one has*

$$\limsup_{y \to x^-} f_m(t,y) \leq f_m(t,x) \leq \liminf_{y \to x^+} f_m(t,y) \tag{18}$$

or there exist countably many LAD curves for (8), $\gamma_n : [a_n, b_n] \subset I \to \mathbb{R}$, $n \in \mathbb{N}$, such that for a.a. $t \in I$ and all $x \in \mathbb{R} \setminus \cup_{\{n \mid a_n \leq t \leq b_n\}} \{\gamma_n(t)\}$ one has (18).

Then one has the following results.

(a) *There exists a null measure set $N \subset I$ such that*

$$\{t \in I : u'_{\inf}(t) \notin F(t, u_{\inf}(t))\} \subset J \cup N, \tag{19}$$

where $J = \cup_{n,m \in \mathbb{N}} J_{n,m}$, and for each $n, m \in \mathbb{N}$ the set

$$\begin{aligned} J_{n,m} := \Big\{ t \in I : u'_{\inf}(t) - \frac{1}{n} \\ > \sup \Big\{ f_m(t,y) : u_{\inf}(t) - \frac{1}{m} < y < u_{\inf}(t) \Big\} \Big\} \end{aligned} \tag{20}$$

contains no positive measure subset.

(b) *The function $u_{\inf}$ is a solution of (8) provided that for all $n, m \in \mathbb{N}$ the set $J_{n,m}$ is measurable.*

(c) *If $J_{n,m}$ is measurable for every $n, m \in \mathbb{N}$, then $u_{\inf}$ is the least solution of (8) provided that one of the following conditions hold:*

either for a.a. $t \in I$, all $x \in \mathbb{R}$, and all $y \in F(t,x)$ one has $y \leq \psi(t) + 1$ or the first alternative in (H3) holds, which, furthermore, guarantees that $u_{\inf}$ is the least solution to (13).

The following result is Lemma 2 in [13], and it is very useful to prove that the $J_{n,m}$'s are measurable in practical situations.

Lemma 7. *Let $N \subset I$ be a null measure set and let $g : I \times \mathbb{R} \to \mathbb{R}$ be such that $g(\cdot, q)$ is measurable for each $q \in \mathbb{Q}$.*

If, moreover, for all $t \in I \setminus N$ and all $x \in \mathbb{R}$ one has

$$\max \Big\{ \liminf_{y \to x^-} g(t,y), \liminf_{y \to x^+} g(t,y) \Big\} \geq g(t,x), \tag{21}$$

then the mapping $t \in I \mapsto \sup\{g(t,y) : x_1(t) < y < x_2(t)\}$ is measurable for each pair $x_1, x_2 \in C(I)$ such that $x_1(t) < x_2(t)$ for all $t \in I$.

Notice that our multifunctions F need not satisfy the usual hypotheses such as monotonicity or upper/lower semicontinuity. Moreover, F need not assume closed or convex values.

An analogous result for the greatest solution to (13) is also given in [16] and existence of solution for a singular version of (8) is considered in [17].

Another example where we used known results for equations to deduce new result for inclusions is [18], which concerns second-order inclusions and relies on the results proven for equations in [19]. In order to present the main result in [18] we need some notations and preliminaries.

Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$, and

$$X = \{u \in \mathscr{C}([0,T]) : u(0) = x_0, u \ \text{ is nondecreasing}\}. \tag{22}$$

For each $u \in X$ we define its "pseudoinverse" $\widehat{u} : \mathbb{R} \to [0,T]$ as

$$\widehat{u}(x) = \begin{cases} 0, & x < x_0, \\ \min u^{-1}(\{x\}), & x_0 \le x \le u(T), \\ T, & u(T) < x. \end{cases} \quad (23)$$

We notice that $\widehat{u}$ is nondecreasing but not necessarily continuous. Moreover, if $u \in X$ is increasing in I, then $\widehat{u}(x) = u^{-1}(x)$ for all $x \in [x_0, u(T)]$.

Theorem 8 (see [18, Theorem 4.1]). *Suppose that for some $R > 0$ the following hypotheses hold.*

(F1) *For each $u \in X$ the multifunction $F_u : \mathbb{R} \to \mathscr{P}(\mathbb{R})\backslash\{\emptyset\}$ defined as $F_u(\cdot) = F(\widehat{u}(\cdot), \cdot)$ has an admissible selection on the right of x_0, that is, a selection $f_u : [x_0, x_0+R] \to \mathbb{R}$ such that*

(i) $f_u \in L^1(x_0, x_0 + R)$;

(ii) $x_1^2 + 2\int_{x_0}^{x} f_u(r)dr > 0$ *for a.a.* $x \in [x_0, x_0 + R]$;

(iii) $\max\{1, |f|\}/\sqrt{x_1^2 + 2\int_{x_0}^{\cdot} f_u(r)dr} \in L^1(x_0, x_0 + R)$;

(iv) $\int_{x_0}^{x_0+R}(dx/\sqrt{x_1^2 + 2\int_{x_0}^{x} f_u(r)dr}) \ge T$.

(F2) *There exists $M \in L^1(x_0, x_0 + R)$ such that for all $t \in I$ and all $x \in [x_0, x_0 + R]$ one has*

$$\sup\{y : y \in F(t,x)\} \le M(x). \quad (24)$$

(F3) *For every $u, v \in \widehat{X}$, the relation $u \le v$ on I implies $f_u \le f_v$ on $[x_0, x_0 + R]$.*

Then the initial value problem

$$x''(t) \in F(t, x(t)) \quad \text{for a.a. } t \in I := [0,T], \qquad x(0) = x_0, \quad x'(0) = x_1 \ge 0 \quad (25)$$

has an increasing solution in $W^{2,1}(0,T)$.

2. Dynamic Equations on Time Scales

The study of time scales was formalized in the Ph.D. thesis of Hilger in 1988 [20]. The notions of derivative from differential calculus and the forward jump operator from difference calculus are unified and extended to the delta derivative f^{Δ} on a time scale $\mathbb{T}$ (an arbitrary set on the real line). These lead to the study of dynamic equations on time scales, unifying differential and difference equations. In addition, these ideas can be applied in situations more general than those for differential and difference equations, such as population problems in which the species alternates between time frames in which they are active and periods of dormancy. The study of time scales yields interesting insight into the special cases. For example, one realizes that the only reason we have the simple derivative from elementary calculus of t^2 is $2t$ is because the graininess of real line is zero.

Much of the earlier history of time scales can be found in the books by Bohner and Peterson [21, 22] which are on the bookshelf of every time scales analyst. We refer the reader to these sources for the basic concepts and definitions for time scales. Reference [21] collects much of the information for the linear case. As an example, we will overview the first-order linear case. We define the cylinder transformation ξ_h on $\{z \in \mathbb{C} \mid z \ne -1/h\}$ by $\xi_h(z) = 1/h\log(1 + zh)$ for $h > 0$, where log is the principal logarithm function and $\xi_0(z) = z$. We call a function $p : \mathbb{T} \to \mathbb{R}$ regressive if $1 + \mu(t)p(t)$ for all $t \in \mathbb{T}^{\kappa}$, where μ is the graininess of the time scale. We can now define the time scales (or generalized) exponential function by $e_p(t,s) = \exp(\int_s^t \xi_{\mu(t)}(p(\tau))\Delta\tau)$, where $s, t \in \mathbb{T}$. The exponential function thus defined enjoys many properties analogous to that of the standard exponential function on the real line. The following can now be proven.

Theorem 9. *Suppose p is rd-continuous and regressive, and let $t_0 \in \mathbb{T}$. Then, $e_p(\cdot, t_0)$ is the unique solution to*

$$y^{\Delta} = p(t)\,y, \qquad y(t_0) = 1. \quad (26)$$

Note that this yields the corollaries that $y = e^{\alpha t}$ is the unique solution to $y' = \alpha y$, $y(0) = 1$ on the real line and $y = (1+\alpha)^t$ is the unique solution to $\Delta y(t) = \alpha y(t)$, $y(0) = 1$ on the integers, where Δy represents the forward difference operator from difference calculus.

We note that the nabla derivative on time scales was defined by Atici and Guseinov [23] in 2002, which generalizes the backward difference operator. One might think that the results for nabla derivatives mirror those for the delta case, but this is not true; see, for example, [24]. Recently, work has progressed for dynamic equations with the diamond-alpha derivative initiated in [25] and furthered in [26–28]. In the remainder of this section, without making a claim to being complete, we overview some of the recent work in dynamic equations on time scales to illustrate how many of the ideas from differential and difference equations have been generalized and extended.

Existence of solutions has been proven in a number of cases, such as [29] using fixed point theory, [30] proving a Nagumo-type existence result, and [31] using a fixed point theorem due to Avery and Peterson. (A number of other existence theorems are mentioned in specific contexts below.) As an example, here is the theorem from [30].

Theorem 10. *Assume there exist a lower solution α and an upper solution β with $\alpha \le \beta$ on $\mathbb{T}$ and*

(a) $f \in C([a,b] \times \mathbb{R}^2, \mathbb{R})$ *satisfies* $f(t,x,y) > 0$ *for all* $t \in \mathbb{T}$, $x \in [\alpha^{\sigma}(t), \beta^{\sigma}(t)]$ *and* $y \ne 0$,

(b) *there exists a $K > 0$ such that $f(t,x,y) \le K$ for all right scattered $t \in \mathbb{T}$, $x \in [\alpha^{\sigma}(t), \beta^{\sigma}(t)]$ and $y \in \mathbb{R}$,*

(c) *$f(t,x,\cdot)$ is nonincreasing for all right scattered $t \in \mathbb{T}$ and $x \in [\alpha^{\sigma}(t), \beta^{\sigma}(t)]$,*

(d) *$L_1 \in C(\mathbb{R}^4 \times C(\mathbb{T}), \mathbb{R})$ is nondecreasing in its third variable, nonincreasing in its fourth variable, and nondecreasing in its fifth variable,*

(e) $L_2 \in C(\mathbb{R}^2, \mathbb{R})$ *is nonincreasing in its first variable, and*

(f) f *satisfies a Nagumo condition with respect to the pair* α *and* β.

Then, there exists a solution $y \in [\alpha, \beta]$ *to the problem*

$$y^{\Delta\Delta}(t) = f\left(t, y^{\sigma}(t), y^{\Delta}(t)\right), \quad \text{for } t \in \mathbb{T}^{\kappa^2},$$
$$0 = L_1\left(y(a), y^{\Delta}(a), y\left(\sigma^2(b)\right), y^{\Delta}(\sigma(b)), y\right),$$
$$0 = L_2\left(y(a), y\left(\sigma^2(b)\right)\right). \tag{27}$$

Singular problems have been studied in [32, 33]. Green's functions have been considered in [23, 34]. A Sturm-Liouville eigenvalue problem was studied by [35]. Periodic solutions were investigated in [36]. OscillationS of solutions have been considered in [37–39] using the time scales Taylor formula, [40–45]. Asymptotic behavior of solutions has been studied in [46, 47] using Taylor monomials and in [47]. Laplace transforms on time scales were studied by [48].

Delay equations were studied in [40, 42, 44]. Impulsive problems have been studied in [37, 38, 49–51]. Functional dynamic equations have been studied in [50, 52] using Lyapunov functions [41, 46]. Fractional derivatives have been considered in [53, 54]. Problems in abstract spaces were studied in [55]. Dynamic inclusions have been studied in [24, 37, 50, 56]. Partial differentiation on time scales was introduced in [57] and was continued in [28].

Recently, work has begun on extending stochastic calculus to time scales, for example, [58] for the isolated time scale case, [59, 60] for the delta case, AND [61] for the nabla case.

3. Generalized Ordinary Differential Equations

In order to generalize some results on continuous dependence of solutions of ordinary differential equations with respect to the initial data, Jaroslav Kurzweil introduced, in 1957, the notion of generalized ordinary differential equations for functions taking values in Euclidean and Banach spaces. This generalization of the notion of ordinary differential equations uses the concept of the Perron generalized integral, also known as the Kurzweil integral. We refer to these equations as generalized ODEs. See [62–66].

The correspondence between generalized ODEs and classic ODEs is very simple. It is known that the ordinary system

$$\dot{x} = f(x, t), \tag{28}$$

where $\dot{x} = dx/dt$, $\Omega \subset \mathbb{R}^n$ is an open set and $f : \Omega \times \mathbb{R} \to \mathbb{R}^n$, has the integral representation

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau), \tau)\, d\tau, \quad t \geq t_0, \tag{29}$$

whenever the integral exists in some sense. It is also known that if the integral in (29) is in the sense of Riemann, Lebesgue (with the equivalent definition given by E. J. McShane), or Henstock-Kurzweil, for instance, then such an integral can be approximated by a Riemann-type sum of the form

$$\sum_{i=1}^{m} f(x(\tau_i), \tau_i)[s_i - s_{i-1}], \tag{30}$$

where $t_0 = s_0 \leq s_1 \leq \cdots \leq s_m = t$ is a fine partition of the interval $[t_0, t]$ and, for each $i = 1, 2, \ldots, m$, τ_i is sufficiently "*close*" to the interval $[s_{i-1}, s_i]$.

Alternatively, if we define

$$F(x, s) = \int_{s_0}^{s} f(x, \sigma)\, d\sigma, \qquad (x, t) \in \Omega \times \mathbb{R}, \tag{31}$$

then the integral in (29) can be approximated by

$$\sum_{i=1}^{m} \int_{s_{i-1}}^{s_i} f(x(\tau_i), \sigma)\, d\sigma = \sum_{i=1}^{m} [F(x(\tau_i), s_i) - F(x(\tau_i), s_{i-1})]. \tag{32}$$

In such a case, the right-hand side of (32) approximates the nonabsolute Kurzweil integral which, when considered in (29), gives rise to a "*differential equation*" of type (28), however in a wider sense. Such type of equation is known as generalized ordinary differential equation or *Kurzweil equation*. See [67, 68].

Let $[a, b] \subset \mathbb{R}$ be a compact interval and consider a function $\delta : [a, b] \to \mathbb{R}^+$ (called a gauge on $[a, b]$). A tagged partition of the interval $[a, b]$ with division points $a = s_0 \leq s_1 \leq \cdots \leq s_k = b$ and tags $\tau_i \in [s_{i-1}, s_i]$, $i = 1, \ldots, k$, is called δ-fine if

$$[s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)), \quad i = 1, \ldots, k. \tag{33}$$

Definition 11. Let X be a Banach space. A function $U(\tau, t) : [a, b] \times [a, b] \to X$ is called Kurzweil integrable over $[a, b]$, if there is an element $I \in X$ such that, given $\varepsilon > 0$, there is a gauge δ on $[a, b]$ such that

$$\left\| \sum_{i=1}^{k} [U(\tau_i, s_i) - U(\tau_i, s_{i-1})] - I \right\| < \varepsilon, \tag{34}$$

for every δ-fine tagged partition of $[a, b]$. In this case, I is called the Kurzweil integral of U over $[a, b]$ and it will be denoted by $\int_a^b DU(\tau, t)$.

The Kurzweil integral has the usual properties of linearity, additivity with respect to adjacent intervals, and integrability on subintervals. See, for instance, [68], for these and other interesting properties.

Now, consider a subset $O \subset X$ and a function $G : O \times [a, b] \to X$.

Any function $x : [a,b] \to O$ is called a solution of the generalized ordinary differential equation (we write simply generalized ODE)

$$\frac{dx}{d\tau} = DG(x,t) \tag{35}$$

on the interval $[a,b]$, provided

$$x(d) - x(c) = \int_c^d DG(x(\tau),t), \quad c,d \in [a,b], \tag{36}$$

where the integral is obtained by setting $U(\tau,t) = G(x(\tau),t)$ in the definition of the Kurzweil integral.

As it was done in [69, 70], but using different assumptions, namely, Carathéodory and Lipschitz-type conditions on the indefinite integral, we proved in [71] that retarded functional differential equations (we write RFDEs, for short) can be regarded as abstract generalized ODEs and some applications were investigated.

In [72], together with professor Štefan Schwabik, we proved that RFDEs subject to impulse effects can also be regarded as generalized ODEs taking values in a Banach space.

Recently, in [73], together with Federson et al., we proved that a solution of a measure RFDEs of the form

$$Dy = f(y_t,t)\,Dg, \qquad y_{t_0} = \phi, \tag{37}$$

where Dy and Dg are distributional derivatives in the sense of L. Schwartz with respect to y and g, respectively, can be related to a solution of an abstract generalized ODE. More precisely, we considered the integral form of (37) as follows:

$$\begin{gathered} x(t) = x(t_0) + \int_{t_0}^t f(x_s,s)\,dg(s), \quad t \ge t_0, \\ x_{t_0} = \phi, \end{gathered} \tag{38}$$

where t_0,σ,r are given real numbers, with $\sigma, r > 0$, and $y_t(\theta) = y(t+\theta)$, for $\theta \in [-r,0]$. We also considered $O \subset G([t_0-r,t_0+\sigma],\mathbb{R}^n)$ as being an open set and

$$P = \{y_t : y \in O, t \in [t_0,t_0+\sigma]\} \subset G([-r,0],\mathbb{R}^n), \tag{39}$$

where by $G([a,b],X)$ we mean the Banach space of all regulated functions $f : [a,b] \to X$ endowed with the usual supremum norm

$$\|f\|_\infty = \sup_{a\le t\le b} \|f(t)\|, \tag{40}$$

and we assumed that $f : P\times[t_0,t_0+\sigma] \to \mathbb{R}^n$ is a function such that, for each $y \in O$, the mapping $t \mapsto f(y_t,t)$ is Henstock-Kurzweil integrable (or Perron integrable) over $[t_0,t_0+\sigma]$ with respect to a nondecreasing function $g : [t_0,t_0+\sigma] \to \mathbb{R}$. Then, we defined a function $G : O\times[t_0,t_0+\sigma] \to G([t_0,t_0+\sigma],\mathbb{R}^n)$ by

$$G(x,t)(\vartheta) = \begin{cases} 0, & t_0 - r \le \vartheta \le t_0, \\ \displaystyle\int_{t_0}^{\vartheta} f(x_s,s)\,dg(s), & t_0 \le \vartheta \le t \le t_0+\sigma, \\ \displaystyle\int_{t_0}^{t} f(x_s,s)\,dg(s), & t \le \vartheta \le t_0+\sigma, \end{cases} \tag{41}$$

and proved the correspondence between a solution of (38) and a solution of the generalized ODE

$$\frac{dx}{d\tau} = DG(x(\tau),t), \tag{42}$$

with initial condition

$$x(t_0)(\vartheta) = \begin{cases} \phi(\vartheta - t_0), & t_0 - r \le \vartheta \le t_0, \\ x(t_0)(t_0), & t_0 \le \vartheta \le t_0+\sigma, \end{cases} \tag{43}$$

just by requiring the following conditions.

(A) The integral $\int_{t_0}^{t_0+\sigma} f(y_t,t)dg(t)$ exists, for every $y \in O$.

(B) There exists a function $M : [t_0,t_0+\sigma] \to \mathbb{R}^+$ which is Lebesgue integrable with respect to g such that, for all $y \in O$, $u_1,u_2 \in [t_0,t_0+\sigma]$, we have

$$\left|\int_{u_1}^{u_2} f(y_s,s)\,dg(s)\right| \le \int_{u_1}^{u_2} M(s)\,dg(s). \tag{44}$$

(C) There exists a function $L : [t_0,t_0+\sigma] \to \mathbb{R}^+$ which is Lebesgue integrable with respect to g, such that for all $y, x \in O$, $u_1,u_2 \in [t_0,t_0+\sigma]$, we have

$$\left|\int_{u_1}^{u_2} [f(x_s,s) - f(y_s,s)]\,dg(s)\right| \le \int_{u_1}^{u_2} L(s)\|x_s - y_s\|\,dg(s). \tag{45}$$

Under the above conditions, the paper [73] introduces new concepts of stability for the trivial solutions of (38), with $f(0,t) = 0$, for $t \in [t_0,t_0+\sigma]$, and new results which generalize those from [74–76], for instance.

In [77, 78], together with Federson et al., we proved that measure RFDEs are useful tools in the study of impulsive RFDEs and functional dynamic equations on time scales with or without impulse action. In other words, it was proved that the unique solution of the Cauchy problem for a measure RFDE of type

$$\begin{gathered} x(t) = x(t_0) + \int_{t_0}^t \tilde{f}(x_s,s)\,d\tilde{g}(s), \quad t \in [t_0,t_0+\sigma], \\ x_{t_0} = \phi \end{gathered} \tag{46}$$

can be regarded, in a one-to-one relation, with the unique solution of the Cauchy problem for the measure RFDE with impulses given by

$$\begin{gathered} x(t) = x(t_0) + \int_{t_0}^t f(x_s,s)\,dg(s) \\ + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k(x(t_k)), \quad t \in [t_0,t_0+\sigma], \\ x_{t_0} = \phi. \end{gathered} \tag{47}$$

Still in [77, 78], we related the solution of problem (47) to the solution of the following impulsive functional dynamic equation on time scales

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x_s^*, s)\,\Delta s + \sum_{\substack{k\in\{1,\dots,m\},\\ t_k<t}} I_k(x(t_k)), \quad t \in [t_0, t_0+\sigma]_{\mathbb{T}},$$
$$x(t) = \phi(t), \quad t \in [t_0 - r, t_0]_{\mathbb{T}}, \tag{48}$$

where x^* is defined as being the extension of x defined by $x^*(t) = x(t^*)$, for $t^* = \inf\{s \in \mathbb{T} : s \geq t\}$.

In order to obtain the correspondences presented in [77, 78], the requirement was mainly that f is Henstock-Kurzweil integrable with respect to a nondecreasing function g. Therefore many discontinuities are allowed. Moreover, f does not need to be rd-continuous nor regulated, and yet good results for impulsive functional dynamic equations on time scales can be obtained through these correspondences.

Even more recently, together with Federson et al., we studied, in [79], measure neutral functional differential equations (we write measure NFDEs) of type

$$D[N(y_t, t)] = f(y_t, t)\,Dg, \qquad y_{t_0} = \phi, \tag{49}$$

where N is a nonautonomous linear operator (i.e., $N(y_t, t) = N(t)y_t$). Besides, we assume that N admits a representation

$$N(t)\varphi = \varphi(0) - \int_{-r}^{0} d_\theta[\mu(t,\theta)]\,\varphi(\theta), \tag{50}$$

where $\mu : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ is a normalized measurable function satisfying

$$\mu(t,\theta) = 0, \quad \theta \geq 0; \qquad \mu(t,\theta) = \mu(-r), \quad \theta \leq -r, \tag{51}$$

which is continuous to the left on $\theta \in (-r, 0)$ of bounded variation in $\theta \in [-r, 0]$, and the variation of μ in $[s, 0]$, $\mathrm{var}_{[s,0]}\mu$ tends to zero as $s \to 0$.

In order to obtain a correspondence between solutions of measure NFDEs and solutions of a certain class of generalized ODEs of the form

$$\frac{dx}{d\tau} = DG(x(\tau), t), \tag{52}$$

whose right-hand side is given by

$$G(x,t) = F(x,t) + J(x,t), \quad x \in O,\ t \in [t_0, t_0+\sigma], \tag{53}$$

with F as (47) and J given by

$$J(x,t)(\vartheta) = \begin{cases} 0, & t_0 - r \leq \vartheta \leq t_0, \\ \displaystyle\int_{-r}^{0} d_\theta[\mu(\vartheta,\theta)]\,x(\vartheta+\theta) - \int_{-r}^{0} d_\theta[\mu(t_0,\theta)]\,x(t_0+\theta), & t_0 \leq \vartheta \leq t \leq t_0+\sigma, \\ \displaystyle\int_{-r}^{0} d_\theta[\mu(t,\theta)]\,x(t+\theta) - \int_{-r}^{0} d_\theta[\mu(t_0,\theta)]\,x(t_0+\theta), & t \leq \vartheta \leq t_0+\sigma. \end{cases} \tag{54}$$

Besides conditions (A), (B), and (C) presented above, we required that the normalized function μ satisfies the following:

(D) there exists a Lebesgue integrable function K: $[t_0, t_0+\sigma] \to \mathbb{R}^+$ such that

$$\left| \int_{-r}^{0} d_\theta\mu(s_2,\theta)\,x(s_2+\theta) - \int_{-r}^{0} d_\theta\mu(s_1,\theta)\,x(s_1+\theta) \right| \leq \int_{s_1}^{s_2} K(s) \int_{-r}^{0} d_\theta\mu(s,\theta)\,|x(s+\theta)|. \tag{55}$$

Thus, in [79], results on the local existence and uniqueness of solutions, as well as continuous dependence of solutions on the initial data, were established.

Clearly there is still much to do to develop the theory of abstract generalized ODEs and to apply the results to other types of differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] Z-Q. Wu, "The ordinay differential equations with discontinuous right members and the discontinuous solutions of the quasilinear partial differential equations," *Scientia Sinica*, vol. 13, pp. 1901–1917, 1964.

[2] D. V. V. Wend, "Existence and uniqueness of solutions of ordinary differential equations," *Proceedings of the American Mathematical Society*, vol. 23, pp. 27–33, 1969.

[3] D. C. Biles, "Existence of solutions for discontinuous differential equations," *Differential and Integral Equations*, vol. 8, no. 6, pp. 1525–1532, 1995.

[4] W. Rzymowski and D. Walachowski, "One-dimensional differential equations under weak assumptions," *Journal of Mathematical Analysis and Applications*, vol. 198, no. 3, pp. 657–670, 1996.

[5] D. C. Biles and P. A. Binding, "On Carathéodory's conditions for the initial value problem," *Proceedings of the American Mathematical Society*, vol. 125, no. 5, pp. 1371–1376, 1997.

[6] E. R. Hassan and W. Rzymowski, "Extremal solutions of a discontinuous scalar differential equation," *Nonlinear Analysis—Theory, Methods & Applications*, vol. 37, no. 8, pp. 997–1017, 1999.

[7] D. C. Biles and E. Schechter, "Solvability of a finite or infinite system of discontinuous quasimonotone differential equations," *Proceedings of the American Mathematical Society*, vol. 128, no. 11, pp. 3349–3360, 2000.

[8] J. Á. Cid and R. L. Pouso, "Ordinary differential equations and systems with time-dependent discontinuity sets," *Proceedings of the Royal Society of Edinburgh*, vol. 134, no. 4, pp. 617–637, 2004.

[9] S. C. Hu, "Differential equations with discontinuous right-hand sides," *Journal of Mathematical Analysis and Applications*, vol. 154, no. 2, pp. 377–390, 1991.

[10] R. L. Pouso, "Upper and lower solutions for first-order discontinuous ordinary differential equations," *Journal of Mathematical Analysis and Applications*, vol. 244, no. 2, pp. 466–482, 2000.

[11] J. P. Aubin and A. Cellina, *Differential Inclusions*, Elsevier, Amsterdam, The Netherlands, 2006.

[12] R. L. Pouso and D. C. Biles, "First-order singular and discontinuous differential equations," *Boundary Value Problems*, vol. 2009, Article ID 507671, 25 pages, 2009.

[13] R. L. Pouso, "Necessary conditions for solving initial value problems with infima of superfunctions," *Mathematical Inequalities & Applications*, vol. 8, no. 4, pp. 633–641, 2005.

[14] D. C. Biles, "A necessary and sufficient condition for existence of solutions for differential inclusions," *Nonlinear Analysis—Theory, Methods & Applications*, vol. 31, no. 3-4, pp. 311–315, 1998.

[15] P. Binding, "The differential equation $\dot{x} = f \circ x$," *Journal of Differential Equations*, vol. 31, no. 2, pp. 183–199, 1979.

[16] D. C. Biles and R. L. Pouso, "Extremal solutions to first-order differential inclusions under weak assumptions," *Dynamics of Continuous, Discrete & Impulsive Systems*, vol. 18, no. 3, pp. 341–352, 2011.

[17] D. C. Biles, "Existence of solutions for singular differential inclusions," *Dynamics of Continuous, Discrete & Impulsive Systems*, vol. 14, no. 4, pp. 557–564, 2007.

[18] D. C. Biles, J. Á. Cid, and R. L. Pouso, "Initial value problems for singular and nonsmooth second order differential inclusions," *Mathematische Nachrichten*, vol. 280, no. 12, pp. 1335–1343, 2007.

[19] R. L. Pouso, "Necessary and sufficient conditions for existence and uniqueness of solutions of second-order autonomous differential equations," *Journal of the London Mathematical Society*, vol. 71, no. 2, pp. 397–414, 2005.

[20] S. Hilger, *Ein masskettenkalkül mit anwendung auf zentrumsmannigfaltigkeiten [Ph.D. thesis]*, Universität Würzburg, Würzburg, Germany, 1988.

[21] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2001.

[22] M. Bohner and A. Peterson, Eds., *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, Mass, USA, 2003.

[23] F. M. Atici and G. Sh. Guseinov, "On Green's functions and positive solutions for boundary value problems on time scales," *Journal of Computational and Applied Mathematics*, vol. 141, no. 1-2, pp. 75–99, 2002.

[24] F. M. Atici and D. C. Biles, "First order dynamic inclusions on time scales," *Journal of Mathematical Analysis and Applications*, vol. 292, no. 1, pp. 222–237, 2004.

[25] Q. Sheng, M. Fadag, J. Henderson, and J. M. Davis, "An exploration of combined dynamic derivatives on time scales and their applications," *Nonlinear Analysis—Real World Applications*, vol. 7, no. 3, pp. 395–413, 2006.

[26] J. W. Rogers Jr. and Q. Sheng, "Notes on the diamond-α dynamic derivative on time scales," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 228–241, 2007.

[27] Q. Sheng, "Hybrid approximations via second-order crossed dynamic derivatives with the diamond-α derivative," *Nonlinear Analysis—Real World Applications*, vol. 9, no. 2, pp. 628–640, 2008.

[28] U. M. Özkan and B. Kaymakçalan, "Basics of diamond-α partial dynamic calculus on time scales," *Mathematical and Computer Modelling*, vol. 50, no. 9-10, pp. 1253–1261, 2009.

[29] Q. Dai and C. C. Tisdell, "Existence of solutions to first-order dynamic boundary value problems," *International Journal of Difference Equations*, vol. 1, no. 1, pp. 1–17, 2006.

[30] F. M. Atici, A. Cabada, C. J. Chyan, and B. Kaymakçalan, "Nagumo type existence results for second-order nonlinear dynamic BVPS," *Nonlinear Analysis—Theory, Methods & Applications*, vol. 60, no. 2, pp. 209–220, 2005.

[31] J. Fan and L. Li, "Existence of positive solutions for p-Laplacian dynamic equations with derivative on time scales," *Journal of Applied Mathematics*, vol. 2013, Article ID 736583, 7 pages, 2013.

[32] L. Erbe, A. Peterson, and J. Weiss, "Positive solutions to a singular second order boundary value problem," *Advances in Dynamical Systems and Applications*, vol. 3, no. 1, pp. 89–105, 2008.

[33] Z. Liu, "Existence and uniqueness of solutions for singular boundary value problem on time scales," *International Journal of Mathematics*, vol. 23, no. 7, Article ID 1250070, 11 pages, 2012.

[34] P. W. Eloe, "Sign properties of Green's functions for disconjugate dynamic equations on time scales," *Journal of Mathematical Analysis and Applications*, vol. 287, no. 2, pp. 444–454, 2003.

[35] G. Sh. Guseinov, "Eigenfunction expansions for a Sturm-Liouville problem on time scales," *International Journal of Difference Equations*, vol. 2, no. 1, pp. 93–104, 2007.

[36] S-P. Wang, "Existence of periodic solutions for higher order dynamic equations on time scales," *Taiwanese Journal of Mathematics*, vol. 16, no. 6, pp. 2259–2273, 2012.

[37] M. Benchohra, S. Hamani, and J. Henderson, "Oscillatory and nonoscillatory solutions for first order impulsive dynamic inclusions on time scales," *Archivum Mathematicum*, vol. 43, no. 4, pp. 237–250, 2007.

[38] M. Benchohra, S. Hamani, and J. Henderson, "Oscillation and nonoscillation for impulsive dynamic equations on certain time scales," *Advances in Difference Equations*, vol. 2006, Article ID 60860, 2006.

[39] S. R. Grace, M. Bohner, and S. Sun, "Oscillation of fourth-order dynamic equations," *Hacettepe Journal of Mathematics and Statistics*, vol. 39, no. 4, pp. 545–553, 2010.

[40] S. R. Grace, J. R. Graef, S. Panigrahi, and E. Tunc, "On the oscillatory behavior of even order neutral delay dynamic equations on time-scales," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 96, pp. 1–12, 2012.

[41] S. H. Saker and S. R. Grace, "Oscillation criteria for quasi-linear functional dynamic equations on time scales," *Mathematica Slovaca*, vol. 62, no. 3, pp. 501–524, 2012.

[42] S. R. Grace, S. Pinelas, and R. P. Agarwal, "Oscillatory behaviour of odd order neutral delay dynamic equations on time-scales," *International Journal of Dynamical Systems and Differential Equations*, vol. 4, no. 3, pp. 187–197, 2012.

[43] S. H. Saker and D. O'Regan, "New oscillation criteria for second-order neutral dynamic equations on time scales via Riccati substitution," *Hiroshima Mathematical Journal*, vol. 42, no. 1, pp. 77–98, 2012.

[44] C. Zhang, R. P. Agarwal, M. Bohner, and T. Li, "New oscillation results for second-order neutral delay dynamic equations," *Advances in Difference Equations*, vol. 2012, article 227, 14 pages, 2012.

[45] S. R. Grace, "On the oscillation of nth order dynamic equations on time-scales," *Mediterranean Journal of Mathematics*, vol. 10, no. 1, pp. 147–156, 2013.

[46] I. Kubiaczyk and S. H. Saker, "Asymptotic properties of third order functional dynamic equations on time scales," *Annales Polonici Mathematici*, vol. 100, no. 3, pp. 203–222, 2011.

[47] L. Erbe, G. Hovhannisyan, and A. Peterson, "Asymptotic behavior of n-th order dynamic equations," *Nonlinear Dynamics and Systems Theory*, vol. 12, no. 1, pp. 63–80, 2012.

[48] M. Bohner, G. Sh. Guseinov, and B. Karpuz, "Properties of the Laplace transform on time scales with arbitrary graininess," *Integral Transforms and Special Functions*, vol. 22, no. 11, pp. 785–800, 2011.

[49] F. M. Atici and D. C. Biles, "First- and second-order dynamic equations with impulse," *Advances in Difference Equations*, vol. 2005, no. 2, pp. 119–132, 2005.

[50] J. R. Graef and A. Ouahab, "Nonresonance impulsive functional dynamic boundary value inclusions on time scales," *Nonlinear Studies*, vol. 15, no. 4, pp. 339–354, 2008.

[51] J. Zhou, Y. Wang, and Y. Li, "Existence and multiplicity of solutions for some second-order systems on time scales with impulsive effects," *Boundary Value Problems*, vol. 2012, article 148, 26 pages, 2012.

[52] E. Akin-Bohner, Y. N. Raffoul, and C. Tisdell, "Exponential stability in functional dynamic equations on time scales," *Communications in Mathematical Analysis*, vol. 9, no. 1, pp. 93–108, 2010.

[53] F. M. Atici and P. W. Eloe, "Fractional q-calculus on a time scale," *Journal of Nonlinear Mathematical Physics*, vol. 14, no. 3, pp. 333–344, 2007.

[54] A. Ahmadkhanlu and M. Jahanshahi, "On the existence and uniqueness of solution of initial value problem for fractional order differential equations on time scales," *Bulletin of the Iranian Mathematical Society*, vol. 38, no. 1, pp. 241–252, 2012.

[55] M. Cichoń, I. Kubiaczyk, A. Sikorska-Nowak, and A. Yantir, "Existence of solutions of the dynamic Cauchy problem in Banach spaces," *Demonstratio Mathematica*, vol. 45, no. 3, pp. 561–573, 2012.

[56] I. L. D. Santos, G. N. Silva, and L. Barbanti, "Dynamical inclusions in time scales: existence of solutions under a lower semicontinuity condition," *Trends in Applied and Computational Mathematics* , vol. 13, no. 2, pp. 109–120, 2012.

[57] M. Bohner and G. Sh. Guseinov, "Partial differentiation on time scales," *Dynamic Systems and Applications*, vol. 13, no. 3-4, pp. 351–379, 2004.

[58] S. Sanyal, *Stochastic dynamic equations [Ph.D. thesis]*, Applied Mathematics, Missouri University of Science and Technology, Rolla, Mo, USA, 2008.

[59] F. M. Atici and D. C. Biles, "The stochastic Itô integral on time scales," *PanAmerican Mathematical Journal*, vol. 20, no. 4, pp. 45–56, 2010.

[60] C. Lungan and V. Lupulescu, "Random dynamical systems on time scales," *Electronic Journal of Differential Equations*, vol. 2012, no. 86, pp. 1–14, 2012.

[61] N. H. Du and N. T. Dieu, "The first attempt on the stochastic calculus on time scale," *Stochastic Analysis and Applications*, vol. 29, no. 6, pp. 1057–1080, 2011.

[62] J. Kurzweil, "Generalized ordinary differential equations and continuous dependence on a parameter," *Czechoslovak Mathematical Journal*, vol. 7, no. 82, pp. 418–449, 1957.

[63] J. Kurzweil, "Generalized ordinary differential equations," *Czechoslovak Mathematical Journal*, vol. 8, no. 83, pp. 360–388, 1958.

[64] J. Kurzweil, "Unicity of solutions of generalized differential equations," *Czechoslovak Mathematical Journal*, vol. 8, no. 83, pp. 502–509, 1958.

[65] J. Kurzweil, "Generalized ordinary differential equations and continuous dependence on a parameter," *Czechoslovak Mathematical Journal*, vol. 9, no. 84, pp. 564–573, 1959.

[66] J. Kurzweil, "Problems which lead to a generalization of the concept of an ordinary nonlinear differential equation," in *Differential Equations and their Applications*, pp. 65–76, House of the Czechoslovak Academy of Sciences, Prague,Czech Republic, Academic Press, New York, NY, USA, 1963.

[67] Z. Artstein, "Topological dynamics of ordinary differential equations and kurzweil equations," *Journal of Differential Equations*, vol. 23, no. 2, pp. 224–243, 1977.

[68] Š. Schwabik, *Generalized Ordinary Differential Equations*, vol. 5 of *Series in Real Analysis*, World Scientific Publishing, River Edge, NJ, USA, 1992.

[69] C. Imaz and Z. Vorel, "Generalized ordinary differential equations in Banach space and applications to functional equations," *Boletin Sociedad Matemática Mexicana*, vol. 11, pp. 47–59, 1966.

[70] F. Oliva and Z. Vorel, "Functional equations and generalized ordinary differential equations," *Boletin Sociedad Matemática Mexicana*, vol. 11, pp. 40–46, 1966.

[71] M. Federson and P. Táboas, "Topological dynamics of retarded functional differential equations," *Journal of Differential Equations*, vol. 195, no. 2, pp. 313–331, 2003.

[72] M. Federson and Š. Schwabik, "Generalized ODE approach to impulsive retarded functional differential equations," *Differential and Integral Equations*, vol. 19, no. 11, pp. 1201–1234, 2006.

[73] M. Federson, J. Mesquita, and E. Toon, "Stability results for measure functional differential equations via Kurzweil-equations," submitted.

[74] M. Federson and Š. Schwabik, "A new approach to impulsive retarded differential equations: stability results," *Functional Differential Equations*, vol. 16, no. 4, pp. 583–607, 2009.

[75] M. Federson and Š. Schwabik, "Stability for retarded functional differential equations," *Ukrains'kyi Matematychnyi Zhurnal*, vol. 60, no. 1, pp. 107–126, 2008.

[76] Š. Schwabik, "Variational stability for generalized ordinary differential equations," *Časopis Pro Pěstování Matematiky*, vol. 109, no. 4, pp. 389–420, 1984.

[77] M. Federson, J. G. Mesquita, and A. Slavík, "Basic results for functional differential and dynamic equations involving impulses," *Mathematische Nachrichten*, vol. 286, no. 2-3, pp. 181–204, 2013.

[78] M. Federson, J. G. Mesquita, and A. Slavík, "Measure functional differential equations and functional dynamic equations on time scales," *Journal of Differential Equations*, vol. 252, no. 6, pp. 3816–3847, 2012.

[79] M. Federson, M. Frasson, J. Mesquita, and P. Tacuri, "Neutral measure functional differential equations as generalized ODEs," submitted.

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Research Article

Fuzzy Integral Equations and Strong Fuzzy Henstock Integrals

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By using the strong fuzzy Henstock integral and its controlled convergence theorem, we generalized the existence theorems of solution for initial problems of fuzzy discontinuous integral equation.

1. Introduction

The fuzzy differential and integral equations are important part of the fuzzy analysis theory and they have the important value of theory and application in control theory.

The Cauchy problems for fuzzy differential equations have been studied by several authors [1–6] on the metric space (E^n, D) of normal fuzzy convex set with the distance D given by the maximum of the Hausdorff distance between the corresponding level sets. Seikkala in [7] defined the fuzzy derivative and then some generalizations of that have been investigated in [8, 9]. Consequently, the fuzzy integral which is the same as that of Dubois and Prade in [10], by means of the extension principle of Zadeh, showed that the fuzzy initial value problem $x'(t) = f(t, x(t))$, $x(0) = x_0$, has a unique fuzzy solution when f satisfies the generalized Lipschitz condition which guarantees a unique solution of the deterministic initial value problem. Kaleva [1] studied the Cauchy problem of fuzzy differential equation and characterized those subsets of fuzzy sets in which the Peano theorem is valid. Park et al. in [11–14] have considered the existence of solution of fuzzy integral equation in Banach space. In 2002, Xue and Fu [15] established solutions to fuzzy differential equations with right-hand side functions satisfying Caratheodory conditions on a class of Lipschitz fuzzy sets.

However, there are discontinuous systems in which the right-hand side functions $f : [a, b] \times E^n \to E^n$ are not integrable in the sense of Kaleva [1] on certain intervals and their solutions are not absolute continuous functions. To illustrate, we consider the following example.

Example 1. Consider the following discontinuous system:

$$\begin{gathered} x'(t) = h(t), \qquad x(0) = \widetilde{A}, \\ g(t) = \begin{cases} 2t \sin \dfrac{1}{t^2} - \dfrac{2}{t} \cos \dfrac{1}{t^2}, & t \neq 0, \\ 0, & t = 0, \end{cases} \\ \widetilde{A}(s) = \begin{cases} s, & 0 \le s \le 1, \\ 2 - s, & 1 < s \le 2, \\ 0, & \text{others}, \end{cases} \\ h(t) = \chi_{|g(t)|} + \widetilde{A}. \end{gathered} \tag{1}$$

Then $h(t) = \chi_{|g(t)|} + \widetilde{A}$ is not integrable in the sense of Kaleva. However, the above system has the following solution:

$$x(t) = \chi_{|G(t)|} + \widetilde{A}t, \tag{2}$$

where

$$G(t) = \begin{cases} t^2 \sin \dfrac{1}{t^2}, & t \neq 0, \\ 0, & t = 0. \end{cases} \tag{3}$$

It is well known that the Henstock integral is designed to integrate highly oscillatory functions which the Lebesgue integral fails to do. It is known as nonabsolute integral and it is a powerful tool. It is well known that the Henstock integral includes the Riemann, improper Riemann, Lebesgue, and

Newton integrals. Though such an integral was defined by Denjoy in 1912 and also by Perron in 1914, it was difficult to handle using their definitions. But with the Riemann-type definition introduced more recently by Henstock in 1963 and also independently by Kurzweil, the definition is now simple and furthermore the proof involving the integral also turns out to be easy. For more detailed results about the Henstock integral, we refer to [16]. Recently, Wu and Gong [17, 18] have combined the fuzzy set theory and nonabsolute integral theory and discussed the fuzzy Henstock integrals of fuzzy-number-valued functions which extended Kaleva [1] integration. In order to complete the theory of fuzzy calculus and to transfer a fuzzy differential equation into a fuzzy integral equation, we [19, 20] have defined the strong fuzzy Henstock integrals and discussed some of their properties and the controlled convergence theorem.

In this paper, according to the idea of [6, 21, 22] and using the concept of generalized differentiability [8], we will deal with the Cauchy problem of discontinuous fuzzy systems as follows:

$$x(t) = \tilde{f}(t) + \int_0^a k_1(t,s)\, x(s)\, \mathrm{d}s + \int_0^a k_2(t,s)\, \tilde{g}(s, x(s))\, \mathrm{d}s, \tag{4}$$

where $t \in I_a = [0,a]$, $a \in R^+$, and $x, \tilde{f}, \tilde{g} : I_\alpha \to E^n$ are fuzzy-number-valued function and integrals which are taken in sense of strong fuzzy Henstock integration, and $k_1, k_2 : I_a \times I_a \to R^+$ are measurable functions such that $k_1(t,\cdot)$, $k_2(t,\cdot)$ are continuous.

2. Preliminaries

2.1. Fuzzy Number Theory. Let $P_k(R^n)$ denote the family of all nonempty compact convex subset of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Let A and B be two nonempty bounded subsets of R^n. The distance between A and B is defined by the Hausdorff metric [10]

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|b-a\|\right\}. \tag{5}$$

Denote $E^n = \{u : R^n \to [0,1] \mid u \text{ satisfies (1)–(4) below}\}$, where

(1) u is normal; that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,

(2) u is fuzzy convex; that is, $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

(3) u is upper semicontinuous,

(4) $[u]^0 = \mathrm{cl}\{x \in R^n \mid u(x) > 0\}$ is compact.

Then it is easy to see that E^n is a fuzzy number space.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n \mid u(x) \geq \alpha\}$. Then from the above conditions (1)–(4), it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha < 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in the fuzzy number space E^n as follows [10]:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [ku]^\alpha = k[u]^\alpha, \tag{6}$$

where $u, v \in E^n$ and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0,\infty)$

$$D(u,v) = \sup\left\{d_H([u]^\alpha, [v]^\alpha) : \alpha \in [0,1]\right\}, \tag{7}$$

where d is the Hausdorff metric defined in $P_k(R^n)$. Then it is easy to see that D is a metric in E^n. Using the results in [23], we know that

(1) (E^n, D) is a complete metric space;

(2) $D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^n$;

(3) $D(\lambda u, \lambda v) = |\lambda| D(u,v)$ for all $u, v, w \in E^n$ and $\lambda \in R$.

The metric space (E^n, D) has a linear structure; it can be embedded isomorphically as a cone in a Banach space of function $u^* : I \times S^{n-1} \to R$, where S^{n-1} is the unit sphere in R^n, with an embedded function $u^* = j(u)$ defined by

$$u^*(r,x) = \sup_{\alpha\in[u]^\alpha} \langle \alpha, x\rangle \tag{8}$$

for all $\langle r, x\rangle \in I \times S^{n-1}$ (see [23]).

Theorem 2 (see [24]). *There exists a real Banach space X such that E^n can be embed as a convex cone C with vertex 0 into X. Furthermore the following conditions hold true:*

(1) *the embedding j is isometric;*

(2) *addition in X induces addition in E^n;*

(3) *multiplication by nonnegative real number in X induces the corresponding operation in E^n;*

(4) *$C - C$ is dense in X;*

(5) *C is closed.*

It is well known that the H-derivative for fuzzy-number-functions was initially introduced by Puri and Ralescu [5] and it is based on the condition (H) of sets. We note that this definition is fairly strong, because the family of fuzzy-number-valued functions H-differentiable is very restrictive. For example, the fuzzy-number-valued function $\tilde{f} : [a,b] \to E^n$ defined by $\tilde{f}(x) = C \cdot g(x)$, where C is a fuzzy number, $\cdot$ is the scalar multiplication (in the fuzzy context), and $g : [a,b] \to R^+$, with $g'(t_0) < 0$, is not H-differentiable in t_0 (see [8, 9]). To avoid the above difficulty, in this paper we consider a more general definition of a derivative for fuzzy-number-valued functions enlarging the class of differentiable fuzzy-number-valued functions, which has been introduced in [8].

Definition 3 (see [8]). Let $\tilde{f} : (a,b) \to E^n$ and $x_0 \in (a,b)$. We say that $\tilde{f}$ is differentiable at x_0, if there exists an element $\tilde{f}'(t_0) \in E^n$, such that,

(1) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)$, $\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits (in the metric D)

$$\lim_{h\to 0} \frac{\tilde{f}(x_0+h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0-h)}{h} = \tilde{f}'(x_0) \tag{9}$$

or

(2) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)$, $\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0+h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0-h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0) \tag{10}$$

or

(3) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0 + h) -_H \tilde{f}(x_0)$, $\tilde{f}(x_0 - h) -_H \tilde{f}(x_0)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0+h) -_H \tilde{f}(x_0)}{h} = \lim_{h\to 0} \frac{\tilde{f}(x_0-h) -_H \tilde{f}(x_0)}{-h} = \tilde{f}'(x_0) \tag{11}$$

or

(4) for all $h > 0$ sufficiently small, there exists $\tilde{f}(x_0) -_H \tilde{f}(x_0 + h)$, $\tilde{f}(x_0) -_H \tilde{f}(x_0 - h)$ and the limits

$$\lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0+h)}{-h} = \lim_{h\to 0} \frac{\tilde{f}(x_0) -_H \tilde{f}(x_0-h)}{h} = \tilde{f}'(x_0) \tag{12}$$

(h and $-h$ at denominators mean $(1/h)\cdot$ and $-(1/h)\cdot$, resp.).

2.2. The Strong Henstock Integrals of Fuzzy-Number-Valued Functions in E^n. In this section we define the strong Henstock integrals of fuzzy-number-valued functions in the fuzzy number space E^n and we give some properties of this integral.

Definition 4 (see [20]). A fuzzy-number-valued function $\tilde{f}$ will be termed piecewise additive on $[a,b]$ if there exists a division $T : a = a_0 < a_1 < \cdots < a_n = b$, such that $\tilde{f}(x)$ is additive on each $[a_i, a_{i+1}]$ $(i = 0, 1, \ldots, n-1)$.

Definition 5 (see [19, 20]). A fuzzy-number-valued function $\tilde{f}$ is said to be strong Henstock integrable on $[a,b]$ if there exists a piecewise additive fuzzy-number-valued function $\tilde{F}$ on $[a,b]$ such that for every $\varepsilon > 0$ there exists a function $\delta(\xi) > 0$ and for any δ-fine division $P = \{[x_{i-1}, x_i], \xi_i\}_{i=1}^n$ of $[a,b]$ we have

$$\begin{aligned}(P)\sum_{i\in K_n} D\left(\tilde{f}(\xi_i)(x_i - x_{i-1}), \tilde{F}([x_{i-1}, x_i])\right) &\\ + (P)\sum_{j\in I_n} D\left(\tilde{f}(\xi_j)(x_j - x_{j-1}),\right. &\\ \left.(-1)\cdot \tilde{F}\left(\left[x_j, x_{j-1}\right]\right)\right) &< \varepsilon,\end{aligned} \tag{13}$$

where $K_n = \{i \in \{1, 2, \ldots, n\}$ such that $\tilde{F}([x_{i-1}, x_i])$ is a fuzzy number and $I_n = \{j \in \{1, 2, \ldots, n\}$ such that $\tilde{F}([x_j, x_{j-1}])$ is a fuzzy number. We write $\tilde{f} \in SFH[a,b]$.

Definition 6 (see [20]). A fuzzy-number-valued function $\tilde{F}$ defined on $X \subset [a,b]$ is said to be $AC^*(X)$ if for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every finite sequence of nonoverlapping intervals $\{[a_i, b_i]\}$, satisfying $\sum_{i=1}^n |b_i - a_i| < \eta$ where $a_i, b_i \in X$ for all i, we have

$$\sum \omega\left(\tilde{F}, [a_i, b_i]\right) < \varepsilon, \tag{14}$$

where ω denotes the oscillation of $\tilde{F}$ over $[a_i, b_i]$; that is,

$$\omega\left(\tilde{F}, [a_i, b_i]\right) = \sup\left\{D\left(\tilde{F}(y), \tilde{F}(x)\right); x, y \in [a_i, b_i]\right\}. \tag{15}$$

Definition 7 (see [20]). A fuzzy-number-valued function $\tilde{F}$ is said to be ACG^* on X if X is the union of a sequence of closed sets $\{X_i\}$ such that, on each X_i, $\tilde{F}$ is $AC^*(X_i)$.

For the strong fuzzy Henstock integrable we have the following theorems.

Theorem 8. *Let $\tilde{f} : [a,b] \to E^n$. If $\tilde{f} = 0$ a.e on $[a,b]$, then $\tilde{f}$ is SFH integrable on $[a,b]$ and $\int_a^b \tilde{f}(t)\mathrm{d}t = 0$.*

Theorem 9. *Let $\tilde{f} : [a,b] \to E^n$ be SFH integrable on $[a,b]$ and let $\tilde{F}(x) = \int_a^x \tilde{f}(t)\mathrm{d}t$ for each $x \in [a,b]$. Then*

(a) *the function $\tilde{F}$ is continuous on $[a,b]$;*

(b) *the function $\tilde{F}$ is differentiable a.e on $[a,b]$ and $\tilde{F}' = f$;*

(c) *$\tilde{f}$ is measurable.*

Theorem 10 (controlled convergence theorem; see [20]). *Suppose that $\{\tilde{f}_n\}$ is a sequence of SFH integrable functions on $[a,b]$ satisfying the following conditions:*

(1) *$\tilde{f}_n(x) \to \tilde{f}(x)$ a.e. in $[a,b]$ as $n \to \infty$;*

(2) *the primitives $\tilde{F}_n$ of $\tilde{f}_n$ are ACG^* uniformly in n;*

(3) *the primitives $\tilde{F}_n$ converge uniformly on $[a,b]$;*

then $\tilde{f}$ is also SFH integrable on $[a,b]$ and

$$\lim_{n\to\infty} \int_a^b \tilde{f}_n(x)\,\mathrm{d}x = \int_a^b \tilde{f}(x)\,\mathrm{d}x. \tag{16}$$

3. Main Results

In this section we prove some existence theorems for the problem (4).

For any bounded subset A of the Banach space X, we denote by $\alpha(A)$ the Kuratowski measure of noncompactness of A; that is, the infimum of all $\varepsilon > 0$ such that there exists a finite covering of A by sets of diameter less than ε. For the properties of α we refer to [25], for example.

Lemma 11 (see [25]). *Let $H \subset C(I_\gamma, X)$ be a family of strong equicontinuous functions; then*

$$\alpha_c(H) = \sup_{t\in I_\gamma} \alpha(H(t)) = \alpha\left(H\left(I_\gamma\right)\right), \tag{17}$$

where $\alpha_c(H)$ denotes the Kuratowski measure of noncompactness in $C(I_\gamma, X)$ and the function $t \to \alpha(H(t))$ is continuous.

Theorem 12 (see [25]). *Let D be a closed convex subset of X and let F be a continuous function from D into itself. If, for $x \in D$,*

$$\overline{V} = \overline{con}(\{x\} \cup F(V)) \Longrightarrow V \tag{18}$$

is relatively compact, then F has a fixed point.

Theorem 13. *If the fuzzy-number-valued function $\tilde{f} : I_a \to E^n$ is (SFH) integrable, then*

$$\int_I \tilde{f}(t)\,dt \in |I| \cdot \overline{conv}\tilde{f}(I), \tag{19}$$

where $\overline{conv}\tilde{f}(I)$ is the convex hull of $\tilde{f}(I)$, I is an arbitrary subinterval of I_a, and $|I|$ is the length of I.

Proof. Because $j \circ \tilde{f}$ is abstract (H) integrable in a Banach Space, by using the mean valued theorem of (H) integrals, we have

$$(H)\int_I j \circ \tilde{f}(t)\,dt \in |I| \cdot \overline{\mathrm{conv}}\, j \circ \tilde{f}(I) = |I| \cdot j \circ \overline{\mathrm{conv}}\tilde{f}(t). \tag{20}$$

On the other hand, there exists $(H)\int_I j\circ\tilde{f}(t)dt = j\circ\int_I \tilde{f}(t)dt$.

So, we have $j \circ \int_I \tilde{f}(t)dt \in |I| \cdot \overline{\mathrm{conv}}\, j \circ \tilde{f}(I)$. And the set $\{|I| \cdot \overline{\mathrm{conv}}\tilde{f}(I)\}$ is a closed convex set; we have

$$\int_I \tilde{f}(t)\,dt \in |I| \cdot \overline{\mathrm{conv}}\tilde{f}(I). \tag{21}$$

□

Definition 14. A fuzzy valued function $\tilde{f} : I_\alpha \times E^n \to E^n$ is a Caratheodory function if, for each $x \in E^n$, the fuzzy valued function $\tilde{f}(t,x)$ is measurable in $t \in I_\alpha$, and for almost all $t \in I_\alpha$, the fuzzy valued function $\tilde{f}(t,x)$ is continuous with respect to x.

For $x \in C(I_a, E^n)$, we define the norm of x by

$$H\left(x,\tilde{0}\right) = \sup_{t\in I_a} D\left(x,\tilde{0}\right). \tag{22}$$

Let

$$\begin{aligned} B(p) = \{&x \in C(I_a, E^n) \mid \\ &H\left(x,\tilde{0}\right) \le H\left(\tilde{f}(\cdot),\tilde{0}\right) + p, p > 0\}. \end{aligned} \tag{23}$$

Obviously, the set $B(p)$ is closed and convex in E^n.

We define the operator $F : C(I_a, E^n) \to C(I_a, E^n)$ by

$$\begin{aligned} F(x)(t) = {}& \tilde{f}(t) + \int_0^a k_1(t,s)\,x(s)\,ds \\ &+ \int_0^a k_2(t,s)\,\tilde{g}(s,x(s))\,ds, \quad t \in I_a,\ x \in B(p), \end{aligned} \tag{24}$$

where integrals are taken in the sense of *SFH*. Moreover, let $\Gamma(p) = \{F(x) \in C(I_a, E^n) \mid x \in B(p)\}$.

Definition 15. A continuous function $x : I_a \to E^n$ is said to be a solution of the problem (4), if $x(t)$ satisfies

$$x(t) = \tilde{f}(t) + \int_0^a k_1(t,s)\,x(s)\,ds + \int_0^a k_2(t,s)\,\tilde{g}(s,x(s))\,ds \tag{25}$$

or

$$\begin{aligned} x(t) = {}& \tilde{f}(t) + (-1) \cdot \int_0^a k_1(t,s)\,x(s)\,ds \\ &+ (-1) \cdot \int_0^a k_2(t,s)\,\tilde{g}(s,x(s))\,ds, \quad t \in I_a. \end{aligned} \tag{26}$$

Theorem 16. *Assume that, for each continuous function $x(t)$, $\tilde{g}(\cdot, x(\cdot))$ is (SFH) integrable, and $\tilde{g}$ is a Caratheodory function. Let $k_1, k_2 : I_a \times I_a \to R^+$ be measure functions such that $k_1(t,\cdot)$, $k_2(t,\cdot)$ are continuous. Moreover, there exists $P_0 > 0$ and a Caratheodory function $\omega : I_a \times R^+ \to R^+$, with*

$$\begin{gathered} \alpha\left(j \circ \tilde{g}(s,X)\right) \le \omega\left(s, \alpha\left(j \circ X\right)\right), \\ \text{a.e. } s \in I_a, \quad X \subset B\left(p_0\right), \end{gathered} \tag{27}$$

such that the zero function is the unique continuous solution of the inequality

$$q(t) \le 2\left[\int_0^c k_1(t,s)\,q(t,s)\,ds + \int_0^c k_2(t,s)\,\omega\left(s,q(s)\right)ds\right]. \tag{28}$$

Suppose that $\Gamma(p_0)$ is equicontinuous, equibounded, and uniformly ACG^* on I_a. Then there exists at least a solution of the problem (4) on I_a for some $0 < c \le a$ with continuous initial function $\tilde{f}$.

Proof. By equicontinuity and equiboundedness of $\Gamma(p_0)$, there exist some numbers c $(0 < c \leq a)$ such that

$$\begin{aligned} H&\left(\int_0^c \left[k_1(t,s)x(s) + k_2(t,s)\tilde{g}(s,x(s))\right]\mathrm{d}s, \tilde{0}\right) \\ &= \sup_{t\in I_c} D\left(\int_0^c \left[k_1(t,s)x(s) + k_2(t,s)\tilde{g}(s,x(s))\right]\mathrm{d}s, \tilde{0}\right) \\ &= \sup_{t\in I_c}\max_{r\in[0,1]}\left\{\left|\int_0^c \left[k_1(t,s)x_r^+(s)\right.\right.\right. \\ &\qquad\left.\left. + k_2(t,s)\tilde{g}_r^+(s,x(s))\right]\mathrm{d}s - \tilde{0}_r^+\right|, \\ &\qquad\left|\int_0^c \left[k_1(t,s)x_r^-(s)\right.\right. \\ &\qquad\left.\left.\left. + k_2(t,s)\tilde{g}_r^-(s,x(s))\right]\mathrm{d}s - \tilde{0}_r^-\right|\right\} \leq p_0 \end{aligned} \tag{29}$$

for $t \in I_c$ and $x \in B(p_0)$.

Next, we will prove that the operator F is continuous. In fact, let $x_n \to x$. Because the function $\tilde{g}$ is a Caratheodory function, by the following equality

$$\begin{aligned} H&(F(x_n), F(x)) \\ &= H\left(\int_0^c (k_1(t,s)(x_n(s) - x(s)) + k_2(t,s)\right. \\ &\qquad\left.\times(\tilde{g}(s,x_n(s)) - \tilde{g}(s,x(s))))\mathrm{d}s, \tilde{0}\right) \\ &= \sup_{t\in I_c} D\left(\int_0^c (k_1(t,s)(x_n(s) - x(s)) + k_2(t,s)\right. \\ &\qquad\left.\times(\tilde{g}(s,x_n(s)) - \tilde{g}(s,x(s))))\mathrm{d}s, \tilde{0}\right) \end{aligned} \tag{30}$$

and Theorem 10, we have $F(x_n) \to F(x)$.

Observe that a fixed point of F is the solution of the problem (4). Now we prove that F has a fixed point using Theorem 12.

Suppose that $V(t) = \{v(t) \in E^n \mid v \in V\} \subset B(p_0)$ satisfies condition $\overline{V} = \overline{\operatorname{conv}}(\{x\} \cup F(V))$ for some $x \in B(p_0)$, $t \in I_c$. Let $V \subset B(p_0)$, $F(V) \subset \Delta(p_0)$; then $V \subset \overline{V}$ is equicontinuous. By Lemma 11, $t \to v(t) = \alpha(j \circ V(t))$ is continuous on I_c.

Let $\int_0^c Z(s)\mathrm{d}s = \{\int_0^c x(s)\mathrm{d}s \mid x \in Z\}$ for any $Z \in C(I_c, E^n)$ and let $\tilde{h}$ denote the mapping defined by $\tilde{h}(x(s)) = \tilde{g}(s, x(s))$, for each $x \in B(p_0)$, $s \in I_c$. Obviously, $\tilde{h}(V(s)) = \tilde{g}(s, V(s))$, and

$$F(V(t)) = \tilde{f}(t) + \int_0^c \left[k_1(t,s)V(s) + k_2(t,s)\tilde{h}(V(s))\right]\mathrm{d}s \tag{31}$$

holds ture.

Using (27), Lemma 11, and the properties of measure of noncompactness α, we have

$$\begin{aligned} \alpha&(j \circ F(V(t))) \\ &= \alpha\left(j \circ \left(\tilde{f}(t) + \int_0^c \left[k_1(t,s)V(s)\right.\right.\right. \\ &\qquad\left.\left.\left. + k_2(t,s)\tilde{h}(V(s))\right]\mathrm{d}s\right)\right) \\ &\leq 2\alpha\left(j \circ \left(\int_0^c \left[k_1(t,s)V(s) + k_2(t,s)\tilde{h}(V(s))\right]\mathrm{d}s\right)\right) \\ &\leq 2\int_0^c \left[k_1(t,s)\alpha(j \circ V(s))\right. \\ &\qquad\left. + k_2(t,s)\alpha(j \circ \tilde{g}(s,V(s)))\right]\mathrm{d}s \\ &\leq 2\int_0^c \left[k_1(t,s)\alpha(j \circ V(s))\right. \\ &\qquad\left. + k_2(t,s)\omega(s,\alpha(j \circ V(s)))\right]\mathrm{d}s. \end{aligned} \tag{32}$$

Because $V = \overline{\operatorname{conv}}(\{x\} \cup F(V))$, we have

$$v(t) \leq 2\left[\int_0^c k_1(t,s)v(s)\mathrm{d}s + \int_0^c k_2(t,s)\omega(s,v(s))\mathrm{d}s\right]. \tag{33}$$

By assumption, because the zero function is unique continuous solution of the last inequality, so we have $v(t) = \alpha(j \circ V(t)) = 0$. By Arzelá-Ascoli Theorem, V is relatively compact. So, by Theorem 12, F has a fixed point which is a solution of problem (4). □

Next, we give another existence theorem for problem (4).

Let $\gamma(K)$ be the spectral radius of the integral operator K defined by

$$\begin{gathered} K(u)(t) = \int_0^a (k_1(t,s) + k_2(t,s))u(s)\mathrm{d}s, \\ u \in B(p_0), \quad t \in I_\alpha. \end{gathered} \tag{34}$$

Theorem 17. *Assume that, for each continuous function $x(t)$, $\tilde{g}(\cdot, x(\cdot))$ is (SFH) integrable, and $\tilde{g}$ is a Caratheodory function and $k_1, k_2 : I_a \times I_a \to R^+$ are measure functions such that $k_1(t,\cdot)$, $k_2(t,\cdot)$ are continuous. Moreover, there exists $P_0 > 0$ and $L > 0$ such that*

$$\alpha(j \circ \tilde{g}(I, X)) \leq L\alpha(j \circ X) \tag{35}$$

for each $I \subset I_a$, $X \subset B(p_0)$. Suppose that $\Gamma(p_0)$ is equicontinuous, equibounded, and uniformly ACG* on I_a and $(1 + L)\gamma(K) < 1$. Then there exists at least a solution of the problem (4) on I_a for some $0 < c \leq a$ with continuous initial function $\tilde{f}$.

Proof. By equicontinuity and equiboundedness of $\Gamma(p_0)$, there exist some numbers c $(0 < c \leq a)$ such that

$$H\left(\int_0^c (k_1(t,s) + k_2(t,s))u(s)\mathrm{d}s, \tilde{0}\right) \leq p_0 \tag{36}$$

for $t \in p_0$ and $x \in B(p_0)$. By assumption, the operator F is well defined and maps $B(p_0)$ into $B(p_0)$. Now, we show that the operator F is continuous. In fact, let $x_n \to x$. Because the function $\widetilde{g}$ is a Caratheodory function, by following equality

$$\begin{aligned}
&H\left(F\left(x_n\right), F(x)\right) \\
&\quad = H\left(\int_0^c \left(k_1(t,s)\left(x_n(s) - x(s)\right) + k_2(t,s)\right.\right. \\
&\qquad \left.\left. \times \left(\widetilde{g}\left(s, x_n(s)\right) - \widetilde{g}(s, x(s))\right)\right) ds, \widetilde{0}\right) \\
&\quad = \sup_{t \in I_c} D\left(\int_0^c \left(k_1(t,s)\left(x_n(s) - x(s)\right) + k_2(t,s)\right.\right. \\
&\qquad \left.\left. \times \left(\widetilde{g}\left(s, x_n(s)\right) - \widetilde{g}(s, x(s))\right)\right) ds, \widetilde{0}\right)
\end{aligned} \tag{37}$$

and Theorem 10, we have $F(x_n) \to F(x)$.

Observe that a fixed point of F is the solution of the problem (4). Now we prove that F has a fixed point using Theorem 12.

Suppose that $V(t) = \{v(t) \in E^n \mid v \in V\} \subset B(p_0)$ satisfies condition $\overline{V} = \overline{\text{conv}}(\{x\} \cup F(V))$ for some $x \in B(p_0)$, $t \in I_c$. Let $V \subset B(p_0)$, $F(V) \subset \Delta(p_0)$; then $V \subset \overline{V}$ is equicontinuous. By Lemma 11, $t \to v(t) = \alpha(j \circ V(t))$ is continuous on I_c.

We divide the interval $I_c : 0 = t_0 < t_1 < t_2 < \cdots < t_m = c$, where $t_i = ic/m$, $i = 0, 1, \ldots, m$. Let $V([t_i, t_{i+1}]) = \{u(s) \in E^n : u \in V, t_i \leq s \leq t_{t+1}, i = 0, 1, \ldots, m-1\}$. By Lemma 11 and the continuity of v there exists $s_i \in T_i = [t_i, t_{i+1}]$ such that

$$\begin{aligned}
\alpha\left(j \circ V\left(\left[t_i, t_{i+1}\right]\right)\right) &= \sup\left\{\alpha(j \circ V(s)) : t_i \leq s \leq t_{t+1}\right\} \\
&= v\left(s_i\right).
\end{aligned} \tag{38}$$

In addition, by the definition of operator F and Theorem 16 we have

$$\begin{aligned}
&F(u)(t) \\
&\quad = \widetilde{f}(t) + \sum_{i=0}^{m-1} \int_{t_i}^{t_i+1} \left[k_1(t,s) u(s)\right. \\
&\qquad \left. + k_2(t,s) \widetilde{g}(s, u(s))\right] ds \\
&\quad \in \widetilde{f}(t) + \sum_{i=0}^{m-1} \left(t_{i-1} - t_i\right) \\
&\qquad \times \overline{\text{conv}}\left[k_1\left(t, T_i\right) V\left(T_i\right)\right. \\
&\qquad \left. + k_2\left(t, T_i\right) \widetilde{g}\left(T_i, V\left(T_i\right)\right)\right]
\end{aligned} \tag{39}$$

for all $u \in V$, where $k_m(t, T_i) = \{k_m(t,s), t, s \in T_i\}$ and $\widetilde{g}(T_i, V(T_i)) = \{\widetilde{g}(t, x(t)) : t \in T_i, x \in V\}$. So, we have

$$\begin{aligned}
&F(V)(t) \\
&\quad \subset \widetilde{f}(t) + \sum_{i=0}^{m-1} \left(t_{i-1} - t_i\right) \overline{\text{conv}}\left[k_1\left(t, T_i\right) V\left(T_i\right)\right. \\
&\qquad \left. + k_2\left(t, T_i\right) \widetilde{g}\left(T_i, V\left(T_i\right)\right)\right].
\end{aligned} \tag{40}$$

Using (35), (38) and the properties of measure of noncompactness α, we have

$$\begin{aligned}
&\alpha(j \circ F(V)(t)) \\
&\leq \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right)\left[k_1\left(t, T_i\right) j \circ v\left(s_i\right) + k_2\left(t, T_i\right) L \cdot j \circ v\left(s_i\right)\right] \\
&= \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right) k_1\left(t, T_i\right) j \circ v\left(s_i\right) \\
&\quad + L \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right) k_2\left(t, T_i\right) j \circ v\left(s_i\right) \\
&\leq \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right) \sup_{s_i \in T_i} k_1\left(t, s_i\right) j \circ v\left(s_i\right) \\
&\quad + L \cdot \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right) \sup_{s_i \in T_i} k_2\left(t, s_i\right) j \circ v\left(s_i\right) \\
&= \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right) k_1\left(t, p_i\right) j \circ v\left(s_i\right) \\
&\quad + L \cdot \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right) k_2\left(t, q_i\right) j \circ v\left(s_i\right),
\end{aligned} \tag{41}$$

where $s_i, p_i, q_i \in T_i$; so we get

$$\begin{aligned}
&\alpha(j \circ F(V)(t)) \\
&\quad \leq \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right) k_1\left(t, p_i\right) j \circ v\left(p_i\right) \\
&\qquad + \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right)\left[k_1\left(t, p_i\right) j \circ \left(v\left(s_i\right) - v\left(p_i\right)\right)\right] \\
&\qquad + L \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right) k_2\left(t, q_i\right) j \circ v\left(q_i\right) \\
&\qquad + L \cdot \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right)\left[k_2\left(t, q_i\right) j \circ \left(v\left(s_i\right) - v\left(q_i\right)\right)\right] \\
&\quad = \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right) k_1\left(t, p_i\right) j \circ v\left(p_i\right) \\
&\qquad + \frac{c}{m} \sum_{i=0}^{m-1} \left[k_1\left(t, p_i\right) j \circ \left(v\left(s_i\right) - v\left(p_i\right)\right)\right] \\
&\qquad + L \cdot \sum_{i=0}^{m-1} \left(t_{i+1} - t_i\right) k_2\left(t, q_i\right) j \circ v\left(q_i\right) \\
&\qquad + \frac{L \cdot c}{m} \sum_{i=0}^{m-1} \left[k_2\left(t, q_i\right) j \circ \left(v\left(s_i\right) - v\left(q_i\right)\right)\right].
\end{aligned} \tag{42}$$

By continuity of v we have $v(s_i) \rightarrow v(p_i) < \varepsilon_1$ and $v(s_i) \rightarrow v(q_i) < \varepsilon_2$ as $m \rightarrow \infty$. So, we have

$$\begin{aligned}\alpha\left(j \circ F(V)(t)\right) &< \int_0^c k_1(t,s)\, v(s)\, ds + c \cdot \sup_{p \in I_c} k_1(t,p)\, \varepsilon_1 \\ &\quad + L \cdot \int_0^c k_2(t,s)\, j \circ v(s)\, ds + L \cdot c \cdot \sup_{q \in I_c} k_2(t,q)\, \varepsilon_2.\end{aligned} \tag{43}$$

Therefore, we have

$$\alpha\left(j \circ F(V)(t)\right) \leq (1+L) \cdot j \circ \int_0^c \left[k_1(t,s) + k_2(t,s)\right] v(s)\, ds \tag{44}$$

for $t \in I_c$. Since $V = \overline{\text{conv}}(\{u\} \cup F(V))$, by the properties of measure of noncompactness α, we have

$$\alpha\left(j \circ V(t)\right) \leq \alpha\left(j \circ (F(V)(t))\right), \tag{45}$$

and so in view of (44) it follows that

$$v(t) \leq (1+L) \int_0^c \left[k_1(t,s) + k_2(t,s)\right] v(s)\, ds \tag{46}$$

for $t \in I_c$. Because this inequality holds for all $t \in I_c$ and $(1+L)\gamma(K) < 1$, by applying Gronwall's inequality, we get that $\alpha(j \circ V(t)) = 0$ for $t \in I_c$. By Arzelá-Ascoli Theorem, V is relatively compact. So, by Theorem 12, F has a fixed point which is a solution of problem (4). □

4. Conclusion

In this paper, we deal with the existence problems of discontinuous fuzzy integral equations involving the strong fuzzy Henstock integral in fuzzy number space. The functions of the equations are supposed to be discontinuous with respect to some variables and satisfy nonabsolute fuzzy integrability. Our result improves the result given in [15, 26] (where uniform continuity was required), as well as those referred to therein.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] O. Kaleva, "Fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 301–319, 1987.

[2] Z. Gong and Y. Shao, "Global existence and uniqueness of solutions for fuzzy differential equations under dissipative-type conditions," *Computers & Mathematics with Applications*, vol. 56, no. 10, pp. 2716–2723, 2008.

[3] O. Kaleva, "The Cauchy problem for fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 35, no. 3, pp. 389–396, 1990.

[4] J. J. Nieto, "The Cauchy problem for continuous fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 102, no. 2, pp. 259–262, 1999.

[5] M. L. Puri and D. A. Ralescu, "Differentials of fuzzy functions," *Journal of Mathematical Analysis and Applications*, vol. 91, no. 2, pp. 552–558, 1983.

[6] C. Wu and S. Song, "Existence theorem to the Cauchy problem of fuzzy differential equations under compactness-type conditions," *Information Sciences*, vol. 108, no. 1–4, pp. 123–134, 1998.

[7] S. Seikkala, "On the fuzzy initial value problem," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 319–330, 1987.

[8] B. Bede and S. G. Gal, "Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 151, no. 3, pp. 581–599, 2005.

[9] Y. Chalco-Cano and H. Román-Flores, "On new solutions of fuzzy differential equations," *Chaos, Solitons and Fractals*, vol. 38, no. 1, pp. 112–119, 2008.

[10] D. Dubois and H. Prade, "Towards fuzzy differential calculus. I: integration of fuzzy mappings," *Fuzzy Sets and Systems*, vol. 8, no. 1, pp. 1–17, 1982.

[11] T. Allahviranloo, M. Khezerloo, O. Sedaghatfar, and S. Salahshour, "Toward the existence and uniqueness of solutions of second-order fuzzy volterra integro-differential equations with fuzzy kernel," *Neural Computing and Applications*, vol. 22, no. 1, supplement, pp. 133–141, 2013.

[12] J. Y. Park, Y. C. Kwun, and J. U. Jeong, "Existence of solutions of fuzzy integral equations in Banach spaces," *Fuzzy Sets and Systems*, vol. 72, no. 3, pp. 373–378, 1995.

[13] J. Y. Park and J. U. Jeong, "On the existence and uniqueness of solutions of fuzzy Volterra-Fredholm integral equations," *Fuzzy Sets and Systems*, vol. 115, no. 3, pp. 425–431, 2000.

[14] D. Zhang, W. Feng, and J. Qiu, "Global existence of solutions to fuzzy volterra integral equations," *ICIC Express Letters*, vol. 3, no. 3, pp. 707–711, 2009.

[15] X. Xue and Y. Fu, "Carathéodory solutions of fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 125, no. 2, pp. 239–243, 2002.

[16] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, 1989.

[17] C. Wu and Z. Gong, "On Henstock integrals of interval-valued functions and fuzzy-valued functions," *Fuzzy Sets and Systems*, vol. 115, no. 3, pp. 377–391, 2000.

[18] C. Wu and Z. Gong, "On Henstock integral of fuzzy-number-valued functions. I," *Fuzzy Sets and Systems*, vol. 120, no. 3, pp. 523–532, 2001.

[19] Z. Gong, "On the problem of characterizing derivatives for the fuzzy-valued functions. II. Almost everywhere differentiability and strong Henstock integral," *Fuzzy Sets and Systems*, vol. 145, no. 3, pp. 381–393, 2004.

[20] Z. Gong and Y. Shao, "The controlled convergence theorems for the strong Henstock integrals of fuzzy-number-valued functions," *Fuzzy Sets and Systems*, vol. 160, no. 11, pp. 1528–1546, 2009.

[21] T. S. Chew and F. Flordeliza, "On $x' = f(t, x)$ and Henstock-Kurzweil integrals," *Differential and Integral Equations*, vol. 4, no. 4, pp. 861–868, 1991.

[22] A. Sikorska-Nowak, "Existence of solutions of nonlinear integral equations and Henstock-Kurzweil integrals," *Annales Societatis Mathematicae Polonae*, vol. 47, no. 2, pp. 227–238, 2007.

[23] P. Diamond and P. Kloeden, *Metric Space of Fuzzy Fets: Theory and Applications*, World Scientific, Singapore, 1994.

[24] C. X. Wu and M. Ma, "Embedding problem of fuzzy number space. II," *Fuzzy Sets and Systems*, vol. 45, no. 2, pp. 189–202, 1992.

[25] J. Banaś and K. Goebel, *Measure of Noncompactness in Banach Space*, vol. 60 of *Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1980.

[26] V. Lakshmikantham and R. N. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor & Francis, London, UK, 2003.

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Research Article

On the Behaviour of Singular Semigroups in Intermediate and Interpolation Spaces and Its Applications to Maximal Regularity for Degenerate Integro-Differential Evolution Equations

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For those semigroups, which may have power type singularities and whose generators are abstract multivalued linear operators, we characterize the behaviour with respect to a certain set of intermediate and interpolation spaces. The obtained results are then applied to provide maximal time regularity for the solutions to a wide class of degenerate integro- and non-integro-differential evolution equations in Banach spaces.

1. Introduction

Let X be a complex Banach space and let $\{\mathcal{T}_A(t)\}_{t\geq 0}$ be a semigroup of operators on X, which is generated by a multivalued linear operator $A : \mathcal{D}(A) \subseteq X \to X$ and which may have a power type singularity at the origin $t = 0$, that is,

$$\begin{gathered} \|\mathcal{T}_A(t)\|_{\mathcal{L}(X)} \leq C_0 t^{\nu}, \quad \forall t > 0, \\ \mathcal{T}_A(0)\, x = x, \quad \forall x \in X, \end{gathered} \tag{1}$$

for some nonnegative constant C_0 and nonpositive exponent ν, where $\mathcal{L}(X)$ denotes the Banach algebra of all endomorphisms of X endowed with the uniform operator norm. In this context our aim here is twofold. The first is to characterize the behaviour of $\{\mathcal{T}_A(t)\}_{t\geq 0}$ with respect to some intermediate and interpolation spaces between X and the domain $\mathcal{D}(A)$ of A. The second is to investigate how this behaviour reflects on the question of maximal time regularity for the solutions to a class of degenerate integro- and non-integrodifferential initial value problems in X.

The class of operators we will deal with consists precisely of those multivalued linear operators A whose single-valued resolvents satisfy the following estimate:

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathcal{L}(X)} \leq C(|\lambda| + 1)^{-\beta}, \quad \forall \lambda \in \Sigma_\alpha. \tag{2}$$

Here, I is the identity operator, C is a positive constant, $\beta \in (0, 1]$, and Σ_α is the complex region $\{z \in \mathbf{C} : \mathfrak{Re}\, z \geq -c(|\mathfrak{Im}\, z|+1)^{\alpha}, \mathfrak{Im}\, z \in \mathbf{R}\}, c > 0, \alpha \in [\beta, 1]$. It thus happens (cf. [1–3]) that A is the infinitesimal generator of a semigroup of linear bounded operators in X satisfying (1) with $\nu = \nu_{\alpha,\beta}$, where $\nu_{\alpha,\beta} = (\beta - 1)/\alpha$.

To outline the motivations of our research, let us assume for a moment that A is a single-valued linear operator satisfying (2). It is well known that if $\beta = 1$, then A is the infinitesimal generator of a bounded analytic semigroup. For this case, an extensive literature exists concerning the behaviour of $\{\mathcal{T}_A(t)\}_{t\geq 0}$ with respect to the real interpolation spaces $(X, \mathcal{D}(A))_{\gamma,p}$, $\gamma \in (0, 1)$, $p \in [1, \infty]$, and its application to questions of maximal regularity for the solutions to nondegenerate (possibly nonautonomous) integro- and non-integrodifferential abstract Cauchy problems. See, for instance, [4–11]. Due to (1) with $\nu = \nu_{1,\beta}$, the case of $\alpha = 1$ and $\beta \in (0, 1)$ is definitely worsened and the literature for it is considerably less conspicuous, although estimate of type (2), with $(\mathfrak{Re}\, \lambda + |\mathfrak{Im}\, \lambda|^{\beta})^{-1}$ in place of $(|\lambda| + 1)^{-\beta}$, goes back even to [12, Remark p. 383] in the ambit of Abel summable semigroups admitting uniform derivatives of all orders. One of the main problems with the case $\beta \in (0, 1)$ is that some equivalent characterizations of $(X, \mathcal{D}(A))_{\gamma,p}$ begin to fail (cf. [13]), so that some spaces which were just real interpolation spaces between X and $\mathscr{D}(A)$ in the case $\beta = 1$ become only intermediate spaces in the case $\beta \in (0,1)$. However, avoiding questions of interpolation theory and of maximal regularity, a quite satisfactorily semigroup theory for the single-valued case with $\beta \in (0,1)$ and its application to the unique solvability of some concrete partial (non-integro-) differential equations have been developed in [14–18]. Since the multivalued case embraces the single-valued one, our contribution in this field is to fill this gap, supplying a theory for the behaviour of singular semig n intermediate and interpolation spaces which, in the case $\beta = 1$, reduces to that in [9, 11]. As an effect of this theory, there is the possibility of investigating questions of maximal time regularity for an entire class of nondegenerate evolution equations which does not fall within the case $\beta = 1$.

The case when A is really a multivalued linear operator arises naturally when we shift our attention to degenerate evolution equations of the type considered in [1–3]. There, a semigroup theory for multivalued linear operators was introduced as a tool to handle degenerate equations by means of analogous techniques of the nondegenerate ones. Such a theory has been then successfully applied to questions of maximal regularity for the solutions to a wide class of degenerate integro- and non-integrodifferential equations. We quote [2, 19–23] where, in general and unless $\beta = 1$, it is shown that the time regularity of the solutions decreases with respect to that of the data. In this respect, we mention the recent results in [20] where, under an additional condition of space regularity on the data and provided that α and β are large enough, the loss of time regularity is restored. Regrettably (cf. the appendix below), we have found some inaccuracies in [20, Section 4], and for this reason we must indicate some changes to that paper. On the other side, fortunately, the basic idea in [20] is correct and remedy can be applied to all the inappropriate items. Furthermore, unexpectedly, we will see that the more delicate approach followed in this paper not only corrects the mistakes in [20], but also gives rise to an effective improvement of the achievable results. In fact, here, we will straighten out, refine, and extend [20], enlarging the class of the admissible spaces to which the data may belong, weakening the assumption for the pair (α, β), and complicating the structure of the underlying equations. This is why we will first analyze the behaviour of the semigroup generated by A with respect to some intermediate and interpolation spaces which turn out to be equivalent only in the case $\beta = 1$. Indeed, the phenomena exhibited in [13] for the single-valued case extend to the multivalued one (cf. [24]), and, until now, for the mentioned behaviour there exist no more than some partial results obtained in [2, 19, 24].

We now give the detailed plan of the paper. In Section 2, for a multivalued linear operator A having domain $\mathscr{D}(A)$ and satisfying (2), we introduce the corresponding generated semigroup $\{e^{tA}\}_{t\geq 0}$. This leads us to define also the linear bounded operators $[(-A)^{\theta}]^{\circ} e^{tA}$, $\Re e\,\theta \geq 0$, $t > 0$, ($[(-A)^{0}]^{\circ} e^{tA} = e^{tA}$) and to recall the fundamental estimates for their $\mathscr{L}(X)$-norm. For the operators $[(-A)^{\theta}]^{\circ} e^{tA}$ a semigroup type property is proven in Proposition 1. We then introduce the spaces we will deal with in this paper, that is, the interpolation spaces $(X, \mathscr{D}(A))_{\gamma,p}$ and the spaces $X_A^{\gamma,p}$, $\gamma \in (0,1)$, $p \in [1,\infty]$. Special attention is given to the embeddings linking these two classes of spaces which, in general, are equivalent only in the case $\beta = 1$. Some relations existing between the spaces $X_A^{\gamma,p}$ for different values of γ and p are proven in Proposition 2 and discussed in Remarks 3–5. We conclude the section recalling the estimates proven in [19, 24] for the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(X;(X,\mathscr{D}(A))_{\gamma,p})}$, $\Re e\,\theta \geq 0$, and $\|[(-A)^{1}]^{\circ} e^{tA}\|_{\mathscr{L}(Y_\gamma^p;X)}$, $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$. In Remarks 7 and 8 we explain why, unless we renounce to optimality, in the case $\beta < 1$ these estimates can not be directly extended to the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(X;X_A^{\gamma,p})}$ and $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(Y_\gamma^p;X)}$, $\Re e\,\theta \geq 1$, respectively.

In Section 3, we investigate the behaviour of the operators $[(-A)^{\theta}]^{\circ} e^{tA}$ with respect to both of the spaces $(X, \mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$. First, in Proposition 9, we deal with the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(X;X_A^{\gamma,p})}$, $\Re e\,\theta \geq 0$, and we show that, except for replacing $(X, \mathscr{D}(A))_{\gamma,p}$ with $X_A^{\gamma,\infty}$ if $p = \infty$ and with $X_A^{\beta\gamma,p}$ if $p \in [1,\infty)$, the *same* estimates of [19] for the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(X;(X,\mathscr{D}(A))_{\gamma,p})}$ continue to hold. The second significant result is Proposition 12 where, extending those in [24] to values of θ other than one, we establish estimates for the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(Y_\gamma^p;X)}$, $\Re e\,\theta \geq 1$, $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$. As a byproduct we deduce the basic Corollary 14, which in Section 5 will be a key tool in proving the equivalence between the following problem (3) and the fixed-point equation (179). The estimates in Proposition 12 are then merged together with those in [19] to achieve estimates for the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma,p},(X,\mathscr{D}(A))_{\delta,p})}$, $\Re e\,\theta \geq 1$. In particular, two different estimates are obtained, if $\gamma + \delta < 1$ or not. For if $\gamma + \delta < 1$, then (cf. the proof of Proposition 16) we can take advantage of the reiteration theorem for interpolation spaces and obtain estimates that, unless $\beta = 1$, are better than those rougher estimates derived in the general case $\gamma, \delta \in (0,1)$ (see Remarks 17 and 18). We stress that if $\beta = 1$, $\theta \in \mathbf{N}$ and A is single-valued, then we restore the estimates in [9]. Finally, in Proposition 20, a combination of Propositions 9 and 12 yields the estimate for the norms $\|[(-A)^{\theta}]^{\circ} e^{tA}\|_{\mathscr{L}(X_A^{\gamma,p}, X_A^{\delta,p})}$, $\Re e\,\theta \geq 1$. Since $\beta < 1$, the spaces $X_A^{\sigma,q}$ are, in general, only intermediate spaces between X and $\mathscr{D}(A)$ for $\sigma \in (0,\beta)$; here the reiteration theorem does not apply and a weaker result is obtained (cf. (101)–(103)).

The estimates of Section 3 are applied in Section 4 to study the time regularity of those operator functions Q_j, $j = 1, \ldots, 6$, that we will need in Section 5. In particular (cf. formula (106)), we modify the definition of Q_2 in [20, Section 4] in order that it is well defined, at least when acting on functions $g \in C^{\delta}([0,T];X)$, $\delta \in ((2-\alpha-\beta)/\alpha, 1)$ (cf. Corollary 26). Consequently, operators Q_3 and Q_4 in [20] change too, and the new Q_5 and Q_6 should be introduced (cf. formulae (107)–(110)). The Hölder in time regularity of the Q_j's is characterized in Lemmas 22, 24, 30, and 32 and Propositions 29 and 36. The main feature of these results is to show that the loss of regularity produced by Q_2 and

Q_5 can be restored, in Q_3 and Q_6 respectively, employing the regularization property established in [20, Section 3] for a wide range of general convolution operators.

In Section 5 we analyze the maximal time regularity of the strict solutions v to the following class of degenerate integrodifferential equations in a complex Banach space X:

$$\frac{\mathrm{d}}{\mathrm{d}t}(Mv(t)) = [\lambda_0 M + L]\, v(t) + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1} v\right)(t) + \sum_{i_2=1}^{n_2} h_{i_2}(t)\, y_{i_2} + f(t), \quad t \in I_T, \qquad Mv(0) = Mv_0. \tag{3}$$

Here, $I_T = [0,T]$, $\lambda_0 \in \mathbf{C}$, $n_1, n_2 \in \mathbf{N}$, $h_{i_2} : I_T \to \mathbf{C}$, $y_{i_2} \in X$, $i_2 = 1,\dots,n_2$, whereas, Z being another complex Banach space and $\mathscr{P} : Z\times X \to X$ being a bilinear bounded operator, $k_{i_1} : I_T \to Z$, and $\mathscr{K}(k_{i_1}, L_{i_1}v)(t) = \int_0^t \mathscr{P}(k_{i_1}(t-s), L_{i_1}v(s))\mathrm{d}s$, $i_1 = 1,\dots,n_1$. Of course, if $Z = \mathbf{C}$, then $\mathscr{P}$ may be the scalar multiplication in X. As M, L, and L_{i_1}, $i_1 = 1,\dots,n_1$, we take closed single-valued linear operators from X to itself, whose domains fulfill the relation $\mathscr{D}(L) \subseteq \bigcap_{i_1=1}^{n_1}[\mathscr{D}(M) \cap \mathscr{D}(L_{i_1})]$, and we require L to have a bounded inverse, allowing M to be *not* invertible. Hence, in general, $A = LM^{-1}$ is only a multivalued linear operator in X having domain $\mathscr{D}(A) = M(\mathscr{D}(L))$. Assuming that A satisfies (2) and that the data k_{i_1}, h_{i_2}, y_{i_2} and f, $i_l = 1,\dots,n_l$, $l = 1,2$, are suitably chosen, problem (3) is then reduced to an equivalent fixed point-equation for the new unknown $w = L(v - v_0)$, $v_0 \in \mathscr{D}(L)$. It is here that the results of Sections 3 and 4 play their role, leading us to Theorem 48. In that theorem, provided that $5\alpha + 2\beta > 6$, we will prove that if $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$, $y_{i_2} \in Y^r_{\gamma_{i_2}}$, $Y^r_{\gamma_{i_2}} \in \{(X,\mathscr{D}(A))_{\gamma_{i_2},r}, X_A^{\gamma_{i_2},r}\}$, and $f \in C^{\mu}(I_T; X)$ for opportunely chosen η_{i_1}, σ_{i_2}, γ_{i_2}, and μ, $i_l = 1,\dots,n_l$, $l = 1,2$, then problem (3) has a unique strict solution $v \in C^{\tau}(I_T; \mathscr{D}(L))$ satisfying $v(0) = v_0$ and Lv, $\mathrm{d}Mv/\mathrm{d}t \in C^{\tau}(I_T; X)$, where $\tau = \min_{i_l=1,\dots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}\}$ (cf. Remark 51). Section 5 concludes with applications of Theorem 48 to integral and nonintegral subcases of (3), (cf. Theorems 52–54 and 56). We stress that Theorem 48 repairs, generalizes, and improves [20, Theorems 5.6 and 5.7], where similar results were proven only for the case $(n_1, n_2, Y^p_{\psi}) = (1, 1, X_A^{\psi,p})$ and under the stronger condition $3\alpha + 8\beta > 10$.

In Section 6, we give an application of Theorem 48 to a concrete case of problem (3) arising in the theory of heat conduction for materials with memory. In particular, we show how Theorem 48 characterizes the appropriate functional framework where to search for the solution of the inverse problem of recovering both v and the vector $(k_1, \dots, k_{r_1})$, $r_1 \le n_1$, in (3) with $(i_2, n_2) = (i_1, n_1)$ and $h_{i_1} = k_{i_1}$, $i_1 = 1,\dots,n_1$.

Finally, in the Appendix we explain how to amend [20, Theorems 5.6 and 5.7] in accordance to Theorem 48.

2. Multivalued Linear Operators, Singular Semigroups, and the Spaces $(X,\mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$

Let X be a complex Banach space endowed with norm $\|\cdot\|_X$ and let $\mathscr{P}(X)$ be the collection of all the subsets of X. For a number $\lambda \in \mathbf{C}$ and elements $\mathscr{U}, \mathscr{V}, \mathscr{W} \in \mathscr{P}(X) \setminus \emptyset$, $\lambda\mathscr{U}$, and $\mathscr{V} + \mathscr{W}$ denote the subsets of X defined by $\{\lambda u : u \in \mathscr{U}\}$ and $\{v + w : v \in \mathscr{V}, w \in \mathscr{W}\}$, respectively. Then, a mapping A from X into $\mathscr{P}(X)$ is called a *multivalued linear operator* in X if its domain $\mathscr{D}(A) = \{x \in X : Ax \neq \emptyset\}$ is a linear subspace of X and A satisfies the following: (i) $Ax + Ay \subset A(x + y)$, for all $x, y \in \mathscr{D}(A)$; (ii) $\lambda Ax \subset A(\lambda x)$, for all $\lambda \in \mathbf{C}$, for all $x \in \mathscr{D}(A)$. From now on, the shortening m. l. will be always used for multivalued linear.

The set $\mathscr{R}(A) = \bigcup_{x\in\mathscr{D}(A)} Ax$ is called the range of A. If $\mathscr{R}(A) = X$, then A is said to be surjective. The following properties of a m. l. operator A are immediate consequences of its definition (cf. [1, Theorems 2.1 and 2.2]): (iii) $Ax + Ay = A(x + y)$, for all $x, y \in \mathscr{D}(A)$; (iv) $\lambda Ax = A(\lambda x)$, for all $\lambda \in \mathbf{C} \setminus \{0\}$, for all $x \in \mathscr{D}(A)$; (v) $A0$ is a linear subspace of X and $Ax = y + A0$ for any $y \in Ax$, $x \in \mathscr{D}(A)$. In particular, A is single-valued if and only if $A0 = \{0\}$.

If A is an m. l. operator in X, then its inverse A^{-1} is defined to be the operator having domain $\mathscr{D}(A^{-1}) = \mathscr{R}(A)$ such that $A^{-1}y = \{x \in \mathscr{D}(A) : y \in Ax\}$, $y \in \mathscr{D}(A^{-1})$. A^{-1} is an m. l. operator in X too, and $(A^{-1})^{-1} = A$. The set $A^{-1}0 = \{x \in \mathscr{D}(A) : 0 \in Ax\}$ is called the kernel of A and denoted by $\mathscr{N}(A)$. If $\mathscr{N}(A) = \{0\}$; that is, if A^{-1} is single-valued, then A is said to be injective. Observe that (v) yields $Ax = A0$ if and only if $x \in \mathscr{N}(A)$.

Given $\mathscr{U} \in \mathscr{P}(X) \setminus \emptyset$, we write $A(\mathscr{U}) = \bigcup_{u\in\mathscr{U}\cap\mathscr{D}(A)} Au$, so that, in particular, $A(X) = A(\mathscr{D}(A)) = \mathscr{R}(A)$. If A_j, $j = 1,2$ are m. l. operators in X and $\lambda \in \mathbf{C}$, then the scalar multiplication λA_1, the sum $A_1 + A_2$, and the product A_1A_2 are defined by

$$\begin{gathered}
\mathscr{D}(\lambda A_1) = \mathscr{D}(A_1), \\
(\lambda A_1)x = \lambda A_1 x, \quad x \in \mathscr{D}(\lambda A_1), \\
\mathscr{D}(A_1 + A_2) = \mathscr{D}(A_1) \cap \mathscr{D}(A_2), \\
(A_1 + A_2)x = A_1x + A_2x, \quad x \in \mathscr{D}(A_1 + A_2), \\
\mathscr{D}(A_1A_2) = \{x \in \mathscr{D}(A_2) : A_1(A_2x) \neq \emptyset\}, \\
(A_1A_2)x = A_1(A_2x), \quad x \in \mathscr{D}(A_1A_2),
\end{gathered} \tag{4}$$

where λA_1, $A_1 + A_2$ and A_1A_2 are m. l. operators in X and $(A_1A_2)^{-1} = A_2^{-1}A_1^{-1}$.

Let A and B be m. l. operators in X. We write $A \subset B$ if $\mathscr{D}(A) \subseteq \mathscr{D}(B)$ and $Ax \subseteq Bx$ for every $x \in \mathscr{D}(A)$. Clearly, $A \subset B \subset A$ if and only if $A = B$. If $A \subset B$ and $Ax = Bx$ for every $x \in \mathscr{D}(A)$, then B is called an extension of A. If a linear single-valued operator S has domain $\mathscr{D}(S) = \mathscr{D}(A)$ and $S \subset A$, that is, $Sx \in Ax$ for every $x \in \mathscr{D}(A)$, then S is called a section of A. With an arbitrary section S, it holds $Ax = Sx + A0$, $x \in \mathscr{D}(A)$, and $\mathscr{R}(A) = \mathscr{R}(S) + A0$, but this latter sum may or may not be direct (cf. [25, p. 14]). A method for constructing sections is provided in [25, Proposition I.5.2].

If X_j, $j = 1, 2$, are two complex Banach spaces, then the linear space of all bounded *single-valued* linear operators L from $X_1 = \mathscr{D}(L)$ to X_2 is denoted by $\mathscr{L}(X_1; X_2)$ ($\mathscr{L}(X_1)$ if $X_1 = X_2$) and it is equipped with the uniform operator norm $\|L\|_{\mathscr{L}(X_1;X_2)} = \sup_{\|x\|_{X_1}\le 1}\|Lx\|_{X_2} = \inf_{K\ge 0}\{\|Lx\|_{X_2} \le K\|x\|_{X_1} : x \in X_1\}$. Then the resolvent set $\rho(A)$ of a m. l. operator A is defined to be the set $\{z \in \mathbf{C} : (zI - A)^{-1} \in \mathscr{L}(X)\}$, with I being the identity operator in X. The basic properties of the resolvent set of single-valued linear operators hold the same for m. l. operators. First, if $\rho(A) \ne \emptyset$, then A is closed; that is, its graph $\{(x, y) \in X \times X : x \in \mathscr{D}(A), y \in Ax\}$ is closed (cf. [25, p. 43]). Further (cf. [1, Theorem 2.6]), $\rho(A)$ is an open set and the operator function $z \in \rho(A) \to (zI - A)^{-1} \in \mathscr{L}(X)$ is holomorphic. Finally (cf. [1, formula (2.1)]), the resolvent equation $(\lambda_2-\lambda_1)(\lambda_1 I-A)^{-1}(\lambda_2 I-A)^{-1} = (\lambda_1 I-A)^{-1}-(\lambda_2 I-A)^{-1}$, $\lambda_1, \lambda_2 \in \rho(A)$, is satisfied, too. Unlike the single-valued case, instead, for $z \in \rho(A)$ the following inclusions hold (cf. [1, Theorem 2.7]):

$$(zI - A)^{-1}A \subset z(zI - A)^{-1} - I \subset A(zI - A)^{-1}. \tag{5}$$

Then, in general, $z(zI - A)^{-1} - I$, $z \in \rho(A)$, is only a bounded section of the m. l. operator $A(zI - A)^{-1}$. Throughout this paper, we denote this bounded section by $A^\circ(zI - A)^{-1}$, but we warn the reader that here A° does not necessarily denote a section of A itself. Of course, if A is single-valued, then $A^\circ(zI - A)^{-1}$ reduces to $A(zI - A)^{-1}$. Notice that (5) implies that $(zI-A)^{-1}A$, $z \in \rho(A)$, is single-valued on $\mathscr{D}(A)$ and $(zI-A)^{-1}Ax = (zI - A)^{-1}y$ with any $y \in Ax$, $x \in \mathscr{D}(A)$. Another difference with the single-valued case is that for every $z \in \rho(A)$ it holds $\mathscr{N}((zI - A)^{-1}) = A0$. Indeed, $((zI - A)^{-1})^{-1}0 = (zI - A)0 = A0$. Therefore, in the m. l. case, $\{0\} \subsetneq \mathscr{N}((zI - A)^{-1})$, $z \in \rho(A)$. However (cf. [24, Lemma 2.1]), if $0 \in \rho(A)$, then $\mathscr{N}(A^\circ(zI - A)^{-1}) = \{0\}$, and, in addition, $x \notin A0$ if and only if $A^\circ(zI - A)^{-1}x \notin A0$, $z \in \rho(A)$. We also recall that for every $\lambda_1, \lambda_2 \in \rho(A)$ the following slight variants of the resolvent equation hold (cf. [24, Lemma 2.2]):

$$\begin{aligned}
&(\lambda_2 - \lambda_1)(\lambda_1 I - A)^{-1}A^\circ(\lambda_2 I - A)^{-1}\\
&\quad = A^\circ(\lambda_1 I - A)^{-1} - A^\circ(\lambda_2 I - A)^{-1},\\
&(\lambda_2 - \lambda_1)A^\circ(\lambda_1 I - A)^{-1}(\lambda_2 I - A)^{-1}\\
&\quad = A^\circ(\lambda_1 I - A)^{-1} - A^\circ(\lambda_2 I - A)^{-1}.
\end{aligned} \tag{6}$$

In particular, if $0 \in \rho(A)$, then, since $A^\circ(0I - A)^{-1} = -I$, the first in (6) with $(\lambda_1, \lambda_2) = (0, \lambda)$ yields $\lambda(-A)^{-1}A^\circ(\lambda I-A)^{-1} = -I - A^\circ(\lambda I - A)^{-1} = -\lambda(\lambda I - A)^{-1}$; that is,

$$A^{-1}A^\circ(\lambda I - A)^{-1} = (\lambda I - A)^{-1}, \quad \lambda \in \rho(A). \tag{7}$$

Let $(A, \mathscr{D}(A))$ be a m. l. operator in X satisfying the following resolvent condition:

(H1) $\rho(A)$ contains a region $\Sigma_\alpha = \{z \in \mathbf{C} : \mathfrak{Re}\, z \ge -c(|\mathfrak{Im}\, z| + 1)^\alpha, \mathfrak{Im}\, z \in \mathbf{R}\}$, $\alpha \in (0, 1]$, $c > 0$, and for some exponent $\beta \in (0, \alpha]$ and constant $C > 0$ the following estimate holds:

$$\left\|(\lambda I - A)^{-1}\right\|_{\mathscr{L}(X)} \le C(|\lambda| + 1)^{-\beta}, \quad \forall\lambda \in \Sigma_\alpha. \tag{8}$$

Introduce the family $\{e^{tA}\}_{t\ge 0} \in \mathscr{L}(X)$ defined by $e^{0A} = I$ and

$$e^{tA} = \frac{1}{2\pi i}\int_\Gamma e^{t\lambda}(\lambda I - A)^{-1}d\lambda, \quad t > 0, \tag{9}$$

where $\Gamma \subsetneq \Sigma_\alpha \setminus \{z \in \mathbf{C} : \mathfrak{Re}\, z \ge 0\}$ is the contour parametrized by $\lambda = -c(|\eta| + 1)^\alpha + i\eta$, $\eta \in (-\infty, \infty)$. Then (cf. [1, pp. 360, 361]), $\{e^{tA}\}_{t\ge 0}$ is a semigroup on X, infinitely many times strongly differentiable for $t > 0$ with

$$\begin{aligned}
D_t^k e^{tA} &= \frac{1}{2\pi i}\int_\Gamma \lambda^k e^{t\lambda}(\lambda I - A)^{-1}d\lambda,\\
&\quad t > 0, \quad k \in \mathbf{N} = \{1, 2, \ldots\},
\end{aligned} \tag{10}$$

where $D_t^k = d^k/dt^k$. In general, no analyticity should be expected for e^{tA}. For if $\alpha < 1$ in (H1), then Σ_α does not contain any sector $\Lambda_{\omega+\pi/2} = \{z \in \mathbf{C} \setminus \{0\} : |\arg z| < \omega + \pi/2\}$, $\omega \in (0, \pi/2)$, and [15, Theorem 5.3], which extends e^{tA} analytically to the sector Λ_ω containing the positive real axis, is not applicable. We stress that (9) and $\mathscr{N}((zI - A)^{-1}) = A0$, $z \in \rho(A)$, imply $A0 \subseteq \mathscr{N}(e^{tA})$ for every $t > 0$, whereas $\mathscr{N}(e^{0A}) = \mathscr{N}(I) = \{0\}$. Hence, if A is really an m. l. operator, then $\{0\} \subsetneq A0 \subseteq \bigcap_{t>0}\mathscr{N}(e^{tA})$. From the semigroup property it also follows that $\mathscr{N}(e^{t_0 A}) \subseteq \mathscr{N}(e^{t_1 A})$ for $t_1 \ge t_0 \ge 0$.

Now, for every $\theta \in \mathbf{C}$ such that $\mathfrak{Re}\,\theta \ge 0$ we set

$$\left[(-A)^\theta\right]^\circ e^{tA} = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^\theta e^{t\lambda}(\lambda I - A)^{-1}d\lambda, \quad t > 0. \tag{11}$$

Here, for the multivalued function $(-\lambda)^\theta = e^{\theta \mathrm{Ln}(-\lambda)}$ we choose the principal branch holomorphic in the region $\mathbf{C} \setminus \{z \in \mathbf{C} : \mathfrak{Re}\, z \ge 0\}$, where for principal branch we mean the principal determination $\ln|z| + i\arg(z)$ of $\mathrm{Ln}(z)$. We briefly recall the main properties of operators $[(-A)^\theta]^\circ e^{tA}$. Of course, $[(-A)^0]^\circ e^{tA} = e^{tA}$, $t > 0$. As shown in [26, p. 426], $[(-A)^k]^\circ e^{tA}$, $k \in \mathbf{N}$, $t > 0$, is a section of $(-A)^k e^{tA}$, so that from (10) we get

$$(-1)^k D_t^k e^{tA} = \left[(-A)^k\right]^\circ e^{tA} \subset (-A)^k e^{tA}, \quad t > 0,\ k \in \mathbf{N}. \tag{12}$$

Moreover (cf. [19, formula (22)] with $\theta \ge 0$ being replaced by $\mathfrak{Re}\,\theta \ge 0$), we get

$$\begin{aligned}
&\left[(-A)^\theta\right]^\circ e^{tA} - \left[(-A)^\theta\right]^\circ e^{sA} = -\int_s^t \left[(-A)^{\theta+1}\right]^\circ e^{\xi A}d\xi,\\
&\qquad \mathfrak{Re}\,\theta \ge 0, \quad 0 < s < t.
\end{aligned} \tag{13}$$

Finally, (H1) implies the following estimates (cf. [1, 24, Section 3]):

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(X)} \le \tilde{c}_{\alpha,\beta,\theta}t^{(\beta-\mathfrak{Re}\,\theta-1)/\alpha}, \quad \mathfrak{Re}\,\theta \ge 0,\ t > 0, \tag{14}$$

where the $\tilde{c}_{\alpha,\beta,\theta}$'s are positive constants depending on α, β, and θ. Thus, letting $\theta = 0$ in (14), we see that if $\beta \in (0,1)$, then the operator function $t \in (0,\infty) \to \mathrm{e}^{tA} \in \mathscr{L}(X)$ may be singular at the origin and the semigroup is not necessarily strongly continuous in the X-norm on the closure $\overline{\mathscr{D}(A)}$ of $\mathscr{D}(A)$ in X. Notice that if $\alpha + \beta > 1$, then the singularity is a *weak* one, in the sense that $\{\mathrm{e}^{tA}\}_{t\geq 0}$ is integrable in norm in any interval $[0,\tau]$, $\tau > 0$. Further (cf. [24, Lemma 3.9]), if $\alpha + \beta > 1$, then $A0 = \bigcap_{t>0} \mathscr{N}(\mathrm{e}^{tA})$, and if $\alpha = 1$, then $A0 = \mathscr{N}(\mathrm{e}^{tA})$ for every $t > 0$.

Observe that $A0 \subseteq \mathscr{N}([(-A)^\theta]^\circ \mathrm{e}^{tA})$, $\Re\mathrm{e}\,\theta \geq 0$, $t > 0$, so that $A0 \subseteq \bigcap_{t>0} \mathscr{N}([(-A)^\theta]^\circ \mathrm{e}^{tA})$, $\Re\mathrm{e}\,\theta \geq 0$. The operators $[(-A)^\theta]^\circ \mathrm{e}^{tA}$ satisfy the following semigroup type property.

Proposition 1. *Let $\theta_j \in \mathbf{C}$, $\Re\mathrm{e}\theta_j \geq 0$, and let $t_j > 0$, $j = 1,2$. Then*

$$\left[(-A)^{\theta_1}\right]^\circ e^{t_1 A}\left[(-A)^{\theta_2}\right]^\circ e^{t_2 A} = \left[(-A)^{\theta_1+\theta_2}\right]^\circ e^{(t_1+t_2)A}. \tag{15}$$

Proof. First, the function $\lambda \in \rho(A) \to (-\lambda)^\theta \mathrm{e}^{t\lambda}(\lambda I - A)^{-1} \in \mathscr{L}(X)$ being holomorphic for every $\Re\mathrm{e}\,\theta \geq 0$ and $t > 0$, and the contour Γ in (11) with $(\theta,t) = (\theta_2, t_2)$ can be replaced with the contour $\Gamma' \subsetneq \Sigma_\alpha \setminus \{z \in \mathbf{C} : \Re\mathrm{e}\, z \geq 0\}$ parametrized by $\mu = -c'(|\eta|+1)^\alpha + i\eta$, $\eta \in (-\infty,\infty)$, $c' \in (0,c)$, and lies to the right of Γ. Then, for every $x \in X$, from the resolvent equation we obtain

$$\begin{aligned}
&\left[(-A)^{\theta_1}\right]^\circ \mathrm{e}^{t_1 A}\left[(-A)^{\theta_2}\right]^\circ \mathrm{e}^{t_2 A} x \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma (-\lambda)^{\theta_1}\mathrm{e}^{t_1\lambda} \\
&\qquad \times \left[\int_{\Gamma'} (-\mu)^{\theta_2}\mathrm{e}^{t_2\mu}(\lambda I - A)^{-1}(\mu I - A)^{-1} x\, \mathrm{d}\mu\right]\mathrm{d}\lambda \\
&= \left(\frac{1}{2\pi i}\right)^2 \int_\Gamma (-\lambda)^{\theta_1}\mathrm{e}^{t_1\lambda}(\lambda I - A)^{-1} \\
&\qquad \times \left[\left(\int_{\Gamma'} (-\mu)^{\theta_2}\mathrm{e}^{t_2\mu}(\mu-\lambda)^{-1}\mathrm{d}\mu\right)x\right]\mathrm{d}\lambda \\
&\quad - \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} (-\mu)^{\theta_2}\mathrm{e}^{t_2\mu}(\mu I - A)^{-1} \\
&\qquad \times \left[\left(\int_{\Gamma} (-\lambda)^{\theta_1}\mathrm{e}^{t_1\lambda}(\lambda-\mu)^{-1}\mathrm{d}\lambda\right)x\right]\mathrm{d}\mu.
\end{aligned} \tag{16}$$

Now, after having enclosed Γ and Γ' on the left with an arc Δ_R of the circle $\{z \in \mathbf{C} : |z + c'| = R\}$, $R > c - c'$, we apply the residue theorem and let R go to infinity. To this purpose, we observe that since the contours Γ and Γ' both lie in the half-plane $\{z \in \mathbf{C} : \Re\mathrm{e}\, z \leq -c'\}$, the arc Δ_R may be parametrized in polar coordinates by $\Re\mathrm{e}\, z = -c' + R\cos\varphi$, $\Im\mathrm{m}\, z = R\sin\varphi$, $\varphi \in (\pi/2, 3\pi/2)$. Then, for every $z \in \Delta_R$ we have

$$\begin{aligned}
\left|(-z)^\theta \mathrm{e}^{tz}\right| &= |z|^{\Re\mathrm{e}\,\theta}\mathrm{e}^{-\Im\mathrm{m}\,\theta\arg(-z)}\mathrm{e}^{t\Re\mathrm{e}\, z} \\
&\leq \left(R + c'\right)^{\Re\mathrm{e}\,\theta}\mathrm{e}^{(\pi/2)|\Im\mathrm{m}\,\theta|}\mathrm{e}^{-tc'}\mathrm{e}^{tR\cos\varphi}.
\end{aligned} \tag{17}$$

Since $t > 0$ and $\varphi \in (\pi/2, 3\pi/2)$, the right-hand side of the latter inequality goes to zero as R goes to infinity, so that $\lim_{R\to\infty, z\in\Delta_R}(-z)^\theta \mathrm{e}^{tz} = 0$ for every $\Re\mathrm{e}\,\theta \geq 0$ and $t > 0$. The residue theorem together with the fact that Γ' lies to the right of Γ thus yields $\int_{\Gamma'}(-\mu)^{\theta_2}\mathrm{e}^{t_2\mu}(\mu-\lambda)^{-1}\mathrm{d}\mu = 2\pi i(-\lambda)^{\theta_2}\mathrm{e}^{t_2\lambda}$ and $\int_\Gamma (-\lambda)^{\theta_1}\mathrm{e}^{t_1\lambda}(\lambda-\mu)^{-1}\mathrm{d}\lambda = 0$. Replacing these identities in (16) and using the equality $(-\lambda)^{\theta_1}(-\lambda)^{\theta_2} = (-\lambda)^{\theta_1+\theta_2}$ which is satisfied for the principal branch of the function $(-\lambda)^\theta = \mathrm{e}^{\theta \mathrm{Ln}(-\lambda)}$, we finally find

$$\begin{aligned}
&\left[(-A)^{\theta_1}\right]^\circ \mathrm{e}^{t_1 A}\left[(-A)^{\theta_2}\right]^\circ \mathrm{e}^{t_2 A} x \\
&\quad = \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{\theta_1+\theta_2}\mathrm{e}^{(t_1+t_2)\lambda}(\lambda I - A)^{-1} x\, \mathrm{d}\lambda.
\end{aligned} \tag{18}$$

The right-hand side being precisely $[(-A)^{\theta_1+\theta_2}]^\circ \mathrm{e}^{(t_1+t_2)A}x$, the proof is complete. □

For an m. l. operator A satisfying (H1) we introduce now the spaces $(X, \mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$. We first specify a topology on $\mathscr{D}(A)$ equipping it with the norm $\|x\|_{\mathscr{D}(A)} = \inf_{y\in Ax}\|y\|_X$, $x \in \mathscr{D}(A)$. Since $A^{-1} \in \mathscr{L}(X)$, this norm is equivalent to the graph norm and makes $\mathscr{D}(A)$ a complex Banach space (cf. [2, Proposition 1.11]). As X_1 and X_2 being given normed complex linear spaces, we will write $X_1 \hookrightarrow X_2$ if $X_1 \subseteq X_2$ and there exists a positive constant C_0 such that $\|x\|_{X_2} \leq C_0\|x\|_{X_1}$ for every $x \in X_1$. If $X_1 \hookrightarrow X_2 \hookrightarrow X_1$, that is, if $X_1 = X_2$ and the norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$ are equivalent, then we will write $X_1 \cong X_2$. Of course, $\mathscr{D}(A)$ with the norm $\|\cdot\|_{\mathscr{D}(A)}$ satisfies $\mathscr{D}(A) \hookrightarrow X$. In fact, if $x \in \mathscr{D}(A)$, then for every $y \in Ax$ we have $x = A^{-1}y$, so that $\|x\|_X \leq \|A^{-1}\|_{\mathscr{L}(X)}\|y\|_X \leq C\|y\|_X$. Taking the infimum with respect to $y \in Ax$, we thus find $\|x\|_X \leq C\|x\|_{\mathscr{D}(A)}$ for every $x \in \mathscr{D}(A)$. If Y is a Banach space, we denote by $C((0,\infty);Y)$ the set of all continuos functions from $(0,\infty)$ to Y, and for a Y-valued strongly measurable function $g(\xi)$, $\xi \in (0,\infty)$, we set $\|g(\xi)\|_{L_q^*(Y)} = (\int_0^\infty \|g(\xi)\|_Y^q(\mathrm{d}\xi/\xi))^{1/q}$, $q \in [1,\infty)$, and $\|g(\xi)\|_{L_\infty^*(Y)} = \sup_{\xi\in(0,\infty)}\|g(\xi)\|_Y$. Let $p_0, p_1 \in [1,\infty)$ or let $p_0 = p_1 = \infty$, and for $\gamma \in (0,1)$ define $p^{-1} = (1-\gamma)p_0^{-1} + \gamma p_1^{-1}$ if $p_0, p_1 \in [1,\infty)$ and $p = \infty$ if $p_0 = p_1 = \infty$. Let us set

$$\begin{aligned}
&(X, \mathscr{D}(A))_{\gamma,p} \\
&\quad = \Big\{x \in X : x = v_0(\xi) + v_1(\xi), \xi \in (0,\infty), \\
&\qquad v_0 \in C((0,\infty);X), v_1 \in C((0,\infty);\mathscr{D}(A)), \\
&\qquad \left\|\xi^\gamma v_0(\xi)\right\|_{L_{p_0}^*(X)} + \left\|\xi^{\gamma-1}v_1(\xi)\right\|_{L_{p_1}^*(\mathscr{D}(A))} < \infty\Big\}, \\
&\|x\|_{(X,\mathscr{D}(A))_{\gamma,p}} \\
&\quad = \inf_{v_0,v_1}\left\{\left\|\xi^\gamma v_0(\xi)\right\|_{L_{p_0}^*(X)} + \left\|\xi^{\gamma-1}v_1(\xi)\right\|_{L_{p_1}^*(\mathscr{D}(A))}\right\}.
\end{aligned} \tag{19}$$

This characterization of the spaces $(X, \mathscr{D}(A))_{\gamma,p}$ is that obtained by the so-called "mean-methods", and it is equivalent to that performed by the "K-method" (cf. [27, Theorem 1.5.2 and Remark 1.5.2/2]) and the "trace-method" (cf. [27, Theorem 1.8.2]). Then, due to [27, Theorem 1.3.3], for every $\gamma \in (0,1)$ and $p \in [1,\infty]$ the space $(X,\mathscr{D}(A))_{\gamma,p}$ is an exact real interpolation space of exponent γ between X and $\mathscr{D}(A)$. Observe that by exchanging the role of X and $\mathscr{D}(A)$ and performing the transformation $\xi = \tau^{-1}$, we get $(X,\mathscr{D}(A))_{\gamma,p} = (\mathscr{D}(A),X)_{1-\gamma,p}$. Also, if $\mathscr{D}(A) = X$, then $(X,\mathscr{D}(A))_{\gamma,p} \cong X$ (cf. [27, Theorem 1.3.3(f)]). The definition of the spaces $(X,\mathscr{D}(A))_{\gamma,p}$ is meaningful even for the limiting cases $(\gamma,p) = (i,\infty)$, $i = 0,1$, whereas $(X,\mathscr{D}(A))_{i,p}$, $i = 0,1$, $p \in [1,\infty)$, reduces to the zero element of X. In particular (cf. [28, pp. 10–15]), denoting by $\widetilde{Y}^X$ the completion of $\mathscr{D}(A)$ relative to X and endowing it with the norm $\|\cdot\|_{\widetilde{Y}^X}$ in [28, p. 14], we get $(X,\mathscr{D}(A))_{0,\infty} \cong X$ and $(X,\mathscr{D}(A))_{1,\infty} \cong \widetilde{Y}^X$. Let $\gamma_1 \in (0,1)$ and let $p_j \in [1,\infty]$, $j = 1,2$. Then, for $\gamma_2 \in (0,\gamma_1)$ and $q_j \in [1,p_j]$, $j = 1,2$, the following chain of embeddings holds:

$$\begin{aligned}\mathscr{D}(A) &\hookrightarrow (X,\mathscr{D}(A))_{1,\infty} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_1,1} \\ &\hookrightarrow (X,\mathscr{D}(A))_{\gamma_1,q_1} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_1,p_1} \\ &\hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,1} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,q_2} \\ &\hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,p_2} \hookrightarrow \overline{\mathscr{D}(A)}.\end{aligned} \tag{20}$$

Let $\gamma \in [0,1]$. Recall that a Banach space E is said to be of class $J(\gamma,X,\mathscr{D}(A)) \cap K(\gamma,X,\mathscr{D}(A))$ and shortened to $E \in J(\gamma) \cap K(\gamma)$, if E is an intermediate space between $(X,\mathscr{D}(A))_{\gamma,\infty}$ and $(X,\mathscr{D}(A))_{\gamma,1}$, that is, if $(X,\mathscr{D}(A))_{\gamma,1} \hookrightarrow E \hookrightarrow (X,\mathscr{D}(A))_{\gamma,\infty}$. From (20) it thus follows that $(X,\mathscr{D}(A))_{\gamma,p} \in J(\gamma) \cap K(\gamma)$, for every $\gamma \in (0,1)$ and $p \in [1,\infty]$. Moreover, since $(X,\mathscr{D}(A))_{i,1} = \{0\}$, $i = 0,1$, and $(X,\mathscr{D}(A))_{0,\infty} \cong X$, we have $\mathscr{D}(A) \in J(1) \cap K(1)$ and $X \in J(0) \cap K(0)$. Then (cf. [28, p. 12], [27, Theorem 1.10.2], and [9, Section 1.2.3]), for $\gamma_j \in (0,1)$ and $p_j \in [1,\infty]$, $j = 0,1,2$, the reiteration theorem yields

$$\begin{aligned}&\left((X,\mathscr{D}(A))_{\gamma_1,p_1},(X,\mathscr{D}(A))_{\gamma_2,p_2}\right)_{\gamma_0,p_0} \\ &\quad\cong (X,\mathscr{D}(A))_{(1-\gamma_0)\gamma_1+\gamma_0\gamma_2,p_0}, \\ &\left((X,\mathscr{D}(A))_{\gamma_1,p_1},\mathscr{D}(A)\right)_{\gamma_0,p_0} \cong (X,\mathscr{D}(A))_{(1-\gamma_0)\gamma_1+\gamma_0,p_0}, \\ &\left(X,(X,\mathscr{D}(A))_{\gamma_2,p_2}\right)_{\gamma_0,p_0} \cong (X,\mathscr{D}(A))_{\gamma_0\gamma_2,p_0}.\end{aligned} \tag{21}$$

Finally (cf. [29, Theorem 1.II and Remark 1.III]), we recall that if X_1 and X_2 are two complex Banach spaces and $T \in \mathscr{L}(X_1;X_2)$ is such that $T \in \mathscr{L}(Y_{1_k};Y_{2_k})$, $Y_{j_k} \subseteq X_j$, $j,k = 1,2$, then $T \in \mathscr{L}((Y_{1_1},Y_{1_2})_{\gamma_0,p_0};(Y_{2_1},Y_{2_2})_{\gamma_0,p_0})$, $\gamma_0 \in (0,1)$, $p_0 \in [1,\infty]$, and

$$\|T\|_{\mathscr{L}((Y_{1_1},Y_{1_2})_{\gamma_0,p_0};(Y_{2_1},Y_{2_2})_{\gamma_0,p_0})} \leq \|T\|^{1-\gamma_0}_{\mathscr{L}(Y_{1_1};Y_{2_1})}\|T\|^{\gamma_0}_{\mathscr{L}(Y_{1_2};Y_{2_2})}. \tag{22}$$

As a consequence of this general result and the identity

$$\left((X,\mathscr{D}(A))_{\gamma_1,p_1},X\right)_{\gamma_0,p_0} = \left(X,(X,\mathscr{D}(A))_{\gamma_1,p_1}\right)_{1-\gamma_0,p_0}, \tag{23}$$

from the third in (21) we find that if $T \in \mathscr{L}(X)$ is such that $T \in \mathscr{L}(X;(X,\mathscr{D}(A))_{\gamma_1,p_1})$ and $T \in \mathscr{L}((X,\mathscr{D}(A))_{\gamma_2,p_2};X)$, then $T \in \mathscr{L}((X,\mathscr{D}(A))_{\gamma_0\gamma_2,p_0};(X,\mathscr{D}(A))_{(1-\gamma_0)\gamma_1,p_0})$, $\gamma_j \in (0,1)$, $p_j \in [1,\infty]$, $j = 0,1,2$, and the following estimate holds:

$$\begin{aligned}&\|T\|_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma_0\gamma_2,p_0};(X,\mathscr{D}(A))_{(1-\gamma_0)\gamma_1,p_0})} \\ &\quad\leq \|T\|^{1-\gamma_0}_{\mathscr{L}(X;(X,\mathscr{D}(A))_{\gamma_1,p_1})}\|T\|^{\gamma_0}_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma_2,p_2};X)}.\end{aligned} \tag{24}$$

Notice that here $\gamma_0\gamma_2+(1-\gamma_0)\gamma_1 \in (\min\{\gamma_1,\gamma_2\},\max\{\gamma_1,\gamma_2\}) \subsetneq (0,1)$ for every $\gamma_0 \in (0,1)$. Therefore, if we let $\gamma = \gamma_0\gamma_2$ and let $\delta = (1-\gamma_0)\gamma_1$, then $\gamma+\delta < 1$, $\gamma_1 = \delta/(1-\gamma_0) > \delta$, and $\gamma_2 = \gamma/\gamma_0 > \gamma$. Hence, in order that the additional inequalities $\gamma_j < 1$, $j = 1,2$, are satisfied, we have to choose $\gamma_0 \in (\gamma,1-\delta)$. As we will see this simple observation will be the key for the proof of the second estimates (90) in the following Proposition 16.

We recall that for every fixed $x \in \mathscr{D}(A)$ the map $T(\lambda) = \lambda x$ satisfies $\|T\|_{\mathscr{L}(\mathbf{C};X)} = \|x\|_X$, $\|T\|_{\mathscr{L}(\mathbf{C};\mathscr{D}(A))} = \|x\|_{\mathscr{D}(A)}$ and $\|T\|_{\mathscr{L}(\mathbf{C},(X,\mathscr{D}(A))_{\gamma,p})} = \|x\|_{(X,\mathscr{D}(A))_{\gamma,p}}$. Then (22) with $X_1 = Y_{1_1} = Y_{1_2} = \mathbf{C}$, $X_2 = Y_{2_1} = X$ and $Y_{2_2} = \mathscr{D}(A)$ yields the interpolation inequality:

$$\begin{aligned}&\|x\|_{(X,\mathscr{D}(A))_{\gamma,p}} \leq c_0\|x\|^{1-\gamma}_X\|x\|^{\gamma}_{\mathscr{D}(A)}, \\ &x \in \mathscr{D}(A), \quad \gamma \in (0,1), \quad p \in [1,\infty],\end{aligned} \tag{25}$$

with c_0 being the positive constant depending on γ and p such that $\|\lambda\|_{(\mathbf{C},\mathbf{C})_{\gamma,p}} \leq c_0|\lambda|$.

As another application of (22) and for further needs, we also recall that if A satisfies (H1), then $A^{\circ}(zI-A)^{-1}$ satisfies the estimate (cf. [24, formulae (4.16) and (4.17)]).

Consider

$$\begin{aligned}&\left\|A^{\circ}(zI-A)^{-1}\right\|_{\mathscr{L}(X)} \leq (C+1)(|z|+1)^{1-\beta}, \quad \forall z \in \Sigma_\alpha, \\ &\left\|A^{\circ}(zI-A)^{-1}\right\|_{\mathscr{L}(\mathscr{D}(A);X)} \leq C(|z|+1)^{-\beta}, \quad \forall z \in \Sigma_\alpha.\end{aligned} \tag{26}$$

From (26), using (22) with $X_j = Y_{j_1} = Y_{2_2} = X$, $j = 1,2$, and $Y_{1_2} = \mathscr{D}(A)$, it then follows for every $\gamma \in (0,1)$ and $p \in [1,\infty]$

$$\begin{aligned}&\left\|A^{\circ}(zI-A)^{-1}\right\|_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma,p};X)} \\ &\quad\leq c_1(C+1)^{1-\gamma}C^{\gamma}(|z|+1)^{1-\beta-\gamma}, \quad \forall z \in \Sigma_\alpha,\end{aligned} \tag{27}$$

where c_1 is the positive constant depending on γ and p such that $\|x\|_X \leq c_1\|x\|_{(X,X)_{\gamma,p}}$.

For $\gamma \in (0,1)$ and $p \in [1,\infty]$ we now define the Banach spaces $X^{\gamma,p}_A$ by

$$\begin{aligned}&X^{\gamma,p}_A = \left\{x \in X : [x]_{X^{\gamma,p}_A} := \left\|\xi^{\gamma}A^{\circ}(\xi I-A)^{-1}x\right\|_{L^*_p(X)} < \infty\right\}, \\ &\|x\|_{X^{\gamma,p}_A} = \|x\|_X + [x]_{X^{\gamma,p}_A}.\end{aligned} \tag{28}$$

It is a well-known fact that if A is single-valued and $\beta = 1$ in (H1), then $(X,\mathscr{D}(A))_{\gamma,p} \cong X^{\gamma,p}_A$ (cf. [30, Theorem 3.1] and [27, Theorem 1.14.2]). On the contrary, if $\beta \in (0,1)$, then such an equivalence is no longer true, as first observed in [13, Theorem 2] for single-valued operators and, in the case $p = \infty$, in [2, Theorem 1.12] for the m. l. ones. Recently, extending [13] to m. l. operators and [2] to $p \in [1,\infty]$, in [24, Proposition 4.3] it has been shown that the following embedding relations hold:

$$X_A^{\gamma,p} \hookrightarrow (X,\mathscr{D}(A))_{\gamma,p}, \quad \gamma \in (0,1), \ p \in [1,\infty], \tag{29}$$

$$(X,\mathscr{D}(A))_{\gamma,p} \hookrightarrow X_A^{\gamma+\beta-1,p}, \quad \gamma \in (1-\beta,1), \ p \in [1,\infty]. \tag{30}$$

Then, as in the single-valued case, $(X,\mathscr{D}(A))_{\gamma,p} \cong X_A^{\gamma,p}$ if $\beta = 1$ in (H1). More precisely (see the proof of [24, Proposition 4.3]), if $x \in X_A^{\gamma,p}$, $\gamma \in (0,1)$, $p \in [1,\infty]$, then

$$\|x\|_{(X,\mathscr{D}(A))_{\gamma,p}} \le 2\|x\|_{X_A^{\gamma,p}}, \tag{31}$$

whereas if $x \in (X,\mathscr{D}(A))_{\gamma,p}$, $\gamma \in (1-\beta,1)$, $p \in [1,\infty]$, then

$$\|x\|_{X_A^{\gamma+\beta-1,p}} \le c_2\|x\|_{(X,\mathscr{D}(A))_{\gamma,p}}, \tag{32}$$

with c_2 being a positive constant depending on β, γ and p.

By setting $\delta = \gamma+\beta-1$, $\gamma \in (1-\beta,1)$, from (30) it follows

$$\begin{gathered}\mathscr{D}(A) \hookrightarrow (X,\mathscr{D}(A))_{1+\delta-\beta,p} \hookrightarrow X_A^{\delta,p} \hookrightarrow X,\\ \delta \in (0,\beta), \quad p \in [1,\infty].\end{gathered} \tag{33}$$

Then, if $\beta \in (0,1)$, the spaces $X_A^{\delta,p}$, $\delta \in (0,1)$, $p \in [1,\infty]$, are intermediate spaces between X and $\mathscr{D}(A)$ only for $\delta \in (0,\beta)$, whereas, when $\delta \in [\beta,1)$, they may be smaller than $\mathscr{D}(A)$. In any case, when $\beta \in (0,1)$, it is not known if the spaces $X_A^{\delta,p}$, $\delta \in (0,\beta)$, $p \in [1,\infty]$, are only intermediate or just interpolation spaces between X and $\mathscr{D}(A)$.

Notice that $[X_A^{\gamma,p} \cap A0] = \{0\}$, $\gamma \in (0,1)$, $p \in [1,\infty]$. Indeed, assume that there exists $x \ne 0$ such that $x \in [X_A^{\gamma,p} \cap A0]$ for some $\gamma \in (0,1)$ and $p \in [1,\infty]$. Then, since $x \in A0 = \mathscr{N}((zI-A)^{-1})$, $z \in \rho(A)$, we have $A^\circ(\xi I-A)^{-1}x = \xi(\xi-A)^{-1}x - x = -x$ for every $\xi > 0$ and $[x]_{X_A^{\gamma,p}} = \|\xi^\gamma\|_{L_p^*(X)}\|x\|_X = \infty$, contradicting $x \in X_A^{\gamma,p}$. This property plays a key role in the proof of many of the results in [24]. Further, due to (30), it implies that $[\mathscr{D}(A) \cap A0] = [(X,\mathscr{D}(A))_{\gamma,p} \cap A0] = \{0\}$, $\gamma \in (1-\beta,1)$, $p \in [1,\infty]$. On the contrary, since $\{0\}$ may be a proper subset of $[(X,\mathscr{D}(A))_{\gamma,p} \cap A0]$ for $\gamma \in (0,1-\beta]$, $\beta < 1$, in general it is not true that $[\overline{\mathscr{D}(A)} \cap A0] = \{0\}$. This is true, instead, if $\beta = 1$. In this case the topological direct sum $X_0 = \overline{\mathscr{D}(A)} \oplus A0$ is a closed subspace of X, and if X is reflexive, it coincides with the whole X (cf. [3, Theorems 2.4 and 2.6]).

For every $\gamma \in (0,1)$ and $p \in [1,\infty]$ from (27), (29), and (31) it follows

$$\begin{aligned}&\left\|A^\circ(zI-A)^{-1}\right\|_{\mathscr{L}(X_A^{\gamma,p};X)}\\ &\quad \le 2c_1(C+1)^{1-\gamma}C^\gamma(|z|+1)^{1-\beta-\gamma}, \quad \forall z \in \Sigma_\alpha.\end{aligned} \tag{34}$$

Hence, for $\gamma \in (0,1)$ and $p \in [1,\infty]$ we may rewrite (27) and (34) more compactly as

$$\left\|A^\circ(zI-A)^{-1}\right\|_{\mathscr{L}(Y_\gamma^p;X)} \le c_3(|z|+1)^{1-\beta-\gamma}, \quad \forall z \in \Sigma_\alpha, \tag{35}$$

where $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$ and c_3 is equal to $c_1(C+1)^{1-\gamma}C^\gamma$ or $2c_1(C+1)^{1-\gamma}C^\gamma$ according that $Y_\gamma^p = (X,\mathscr{D}(A))_{\gamma,p}$ or $Y_\gamma^p = X_A^{\gamma,p}$.

With the exception of the case $\beta = 1$, in general it is not clear if embeddings analogous to (20) hold even for the spaces $X_A^{\gamma,p}$. In fact, using (20), (29), and (30) we can only prove that if $\gamma \in (1-\beta,1)$ and $1 \le q \le p \le \infty$, then

$$X_A^{\gamma,q} \hookrightarrow (X,\mathscr{D}(A))_{\gamma,q} \hookrightarrow (X,\mathscr{D}(A))_{\gamma,q} \hookrightarrow X_A^{\gamma+\beta-1,p}, \tag{36}$$

whereas if $1-\beta < \gamma_2 < \gamma_1 < 1$ and $p_1, p_2 \in [1,\infty]$, then

$$X_A^{\gamma_1,p_1} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_1,p_1} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,p_2} \hookrightarrow X_A^{\gamma_2+\beta-1,p_2}. \tag{37}$$

What can be proved without invoking (20), (29), and (30) and using only the definition of the norm $\|\cdot\|_{X_A^{\gamma,p}}$ is instead the following result, which extends to the spaces $X_A^{\gamma,p}$ the embeddings $(X,\mathscr{D}(A))_{\gamma_1,p} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,p}$, and $(X,\mathscr{D}(A))_{\gamma_1,\infty} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,p}$, $0 < \gamma_2 < \gamma_1 < 1$, $p \in [1,\infty]$ (cf. (20) with $(p_1,p_2) = (p,p)$ and $(p_1,p_2) = (\infty,p)$).

Proposition 2. *Let A be an m. l. operator satisfying the resolvent condition (H1). Then the following embeddings hold for every $0 < \gamma_2 < \gamma_1 < 1$ and $p \in [1,\infty]$:*

$$X_A^{\gamma_1,p} \hookrightarrow X_A^{\gamma_2,p}, \tag{38}$$

$$X_A^{\gamma_1,\infty} \hookrightarrow X_A^{\gamma_2,p}. \tag{39}$$

Proof. If $\beta = 1$ in (H1), then there is nothing to prove since $(X,\mathscr{D}(A))_{\gamma,p} \cong X_A^{\gamma,p}$ and both (38) and (39) follow from (20). Therefore, without loss of generality, we assume that $\beta \in (0,\alpha]$ is such that $\beta < \alpha$ if $\alpha = 1$. We begin by proving (38). Let first $p \in [1,\infty)$. For every $x \in X_A^{\gamma_1,p}$, $0 < \gamma_2 < \gamma_1 < 1$, we write

$$[x]^p_{X_A^{\gamma_2,p}} = I_1 + I_2, \tag{40}$$

where

$$I_j = \int_{a_j}^{b_j} \left\|\xi^{\gamma_2}A^\circ(\xi I-A)^{-1}x\right\|_X^p \frac{d\xi}{\xi}, \quad j = 1,2, \tag{41}$$

$(a_1,b_1,a_2,b_2) = (0,1,1,\infty)$. Using the first inequality in (26) we find

$$\begin{aligned}I_1 &\le (C+1)^p\|x\|_X^p \int_0^1 \xi^{\gamma_2 p-1}(\xi+1)^{(1-\beta)p}d\xi\\ &\le 2^{(1-\beta)p}(C+1)^p\|x\|_X^p \int_0^1 \xi^{\gamma_2 p-1}d\xi \le \left[c_4\|x\|_{X_A^{\gamma_1,p}}\right]^p,\end{aligned} \tag{42}$$

where $c_4 = 2^{1-\beta}(C+1)(\gamma_2 p)^{-1/p}$. Concerning I_2, instead, using $\gamma_2 - \gamma_1 < 0$, we get

$$\begin{aligned}I_2 &= \int_1^\infty \xi^{(\gamma_2-\gamma_1)p}\left\|\xi^{\gamma_1}A^\circ(\xi I-A)^{-1}x\right\|_X^p \frac{d\xi}{\xi}\\ &\le \int_1^\infty \left\|\xi^{\gamma_1}A^\circ(\xi I-A)^{-1}x\right\|_X^p \frac{d\xi}{\xi} \le [x]^p_{X_A^{\gamma_1,p}} \le \|x\|^p_{X_A^{\gamma_1,p}}.\end{aligned} \tag{43}$$

Summing up (40)–(43) and setting $c_5 = [(c_4)^p + 1]^{1/p}$, it thus follows $\|x\|_{X_A^{\gamma_2,p}} = \|x\|_X + [x]_{X_A^{\gamma_2,p}} \le (1+c_5)\|x\|_{X_A^{\gamma_1,p}}$, completing the proof of (38) in the case $p \in [1,\infty)$. Let $p = \infty$. For every $x \in X_A^{\gamma_1,\infty}$, $0 < \gamma_2 < \gamma_1 < 1$, we write

$$[x]_{X_A^{\gamma_2,\infty}} = \max\{I_3, I_4\}, \tag{44}$$

where $I_j = \sup_{\xi\in U_j} \|\xi^{\gamma_2} A^\circ(\xi I - A)^{-1}x\|_X$, $j = 3,4$, $U_3 = (0,1)$, $U_4 = [1,\infty)$. Again, the first inequality in (26) yields

$$\begin{aligned} I_3 &\le (C+1)\|x\|_X \sup_{\xi\in(0,1)} \left[\xi^{\gamma_2}(\xi+1)^{1-\beta}\right] \\ &\le 2^{1-\beta}(C+1)\|x\|_{X_A^{\gamma_1,\infty}}. \end{aligned} \tag{45}$$

Instead, using $\gamma_2 - \gamma_1 < 0$, we have

$$\begin{aligned} I_4 &= \sup_{\xi\in[1,\infty)} \xi^{\gamma_2-\gamma_1} \left\|\xi^{\gamma_1} A^\circ(\xi I - A)^{-1}x\right\|_X \le [x]_{X_A^{\gamma_1,\infty}} \\ &\le \|x\|_{X_A^{\gamma_1,\infty}}. \end{aligned} \tag{46}$$

Summing up (44)–(46) and setting $c_6 = 2^{1-\beta}(C+1)$, we thus find $\|x\|_{X_A^{\gamma_2,\infty}} = \|x\|_X + [x]_{X_A^{\gamma_2,\infty}} \le (1+c_6)\|x\|_{X_A^{\gamma_1,\infty}}$. This completes the proof of (38) for the case $p = \infty$. We now prove (39). Due to (38) with $p = \infty$, it suffices to assume that $p \in [1,\infty)$. As above, for every $x \in X_A^{\gamma_1,\infty}$, $0 < \gamma_2 < \gamma_1 < 1$, we write $[x]^p_{X_A^{\gamma_2,p}} = I_1 + I_2$, where I_1 and I_2 are defined by (41). Hence, the same computations as in (42) yield

$$I_1 \le \left[c_4\|x\|_{X_A^{\gamma_1,\infty}}\right]^p. \tag{47}$$

As far as I_2 is concerned, instead, we have

$$I_2 \le [x]^p_{X_A^{\gamma_1,\infty}} \int_1^\infty \xi^{(\gamma_2-\gamma_1)p-1} d\xi \le \left[c_7\|x\|_{X_A^{\gamma_1,\infty}}\right]^p, \tag{48}$$

where $c_7 = [(\gamma_1-\gamma_2)p]^{-1/p}$. Summing up (47) and (48) and setting $c_8 = [(c_4)^p + (c_7)^p]^{1/p}$, we deduce $\|x\|_{X_A^{\gamma_2,p}} \le (1+c_8)\|x\|_{X_A^{\gamma_1,\infty}}$. The proof is complete. □

Remark 3. Notice that (37) with $p_1 = p_2 = p$ yields $X_A^{\gamma_1,p} \hookrightarrow X_A^{\gamma_2+\beta-1,p}$, $1-\beta < \gamma_2 < \gamma_1 < 1$, and this latter embedding is less accurate than (38).

Remark 4. The main problem for extending (20) to the spaces $X_A^{\gamma,p}$ in the case $\beta < 1$ is that it is not clear if it holds $X_A^{\gamma,q} \hookrightarrow X_A^{\gamma,p}$, $1 \le q < p \le \infty$. In fact, the embedding

$$\begin{gathered} (X,\mathscr{D}(A))_{\gamma,q} \hookrightarrow (X,\mathscr{D}(A))_{\gamma,q}, \\ \gamma\in(0,1), \quad 1 \le q < p \le \infty, \end{gathered} \tag{49}$$

is a consequence of the property of the functional K entering the definition of the interpolation spaces $(X,\mathscr{D}(A))_{\gamma,p}$ through the "K-method", and in particular of its monotonicity (see the proof of [27, Theorem 1.3.3(c), (d)]). With embedding (49) at hands, to derive (20) it thus suffices to prove that $(X,\mathscr{D}(A))_{\gamma_1,\infty} \hookrightarrow (X,\mathscr{D}(A))_{\gamma_2,1}$, $0 < \gamma_2 < \gamma_1 < 1$ (see the proof of [27, Theorem 1.3.3(e)] taking there $(A_0, A_1, \theta, \widetilde{\theta}) = (\mathscr{D}(A), X, 1-\gamma_1, 1-\gamma_2)$ and using $(\mathscr{D}(A), X)_{1-\gamma,p} = (X,\mathscr{D}(A))_{\gamma,p}$). If we try to repeat the proof of (49) for the spaces $X_A^{\gamma,p}$, we will be faced with two problems. The first is that we do not know if the function $g(\xi) = \|A^\circ(\xi I - A)^{-1}x\|_X$, $\xi\in(0,\infty)$, $x \in X$, is monotone decreasing, which would allow us to prove $X_A^{\gamma,p} \hookrightarrow X_A^{\gamma,\infty}$, $\gamma\in(0,1)$, $p\in[1,\infty)$. For if $g(\xi)$ was monotone decreasing, then for every $\xi\in(0,\infty)$ and $x \in X_A^{\gamma,p}$, $\gamma\in(0,1)$, $p\in[1,\infty)$, we would find

$$\begin{aligned} \xi^\gamma g(\xi) &= c_9\left(\int_0^\xi \mu^{\gamma p}\frac{d\mu}{\mu}\right)^{1/p} g(\xi) \\ &\le c_9\left(\int_0^\xi [\mu^\gamma g(\mu)]^p \frac{d\mu}{\mu}\right)^{1/p} \le c_9[x]_{X_A^{\gamma,p}}, \end{aligned} \tag{50}$$

where $c_9 = (\gamma p)^{-1/p}$. Taking the supremum with respect to $\xi\in(0,\infty)$ in the latter inequality, we would get $[x]_{X_A^{\gamma,\infty}} \le c_9[x]_{X_A^{\gamma,p}}$, proving $X_A^{\gamma,p} \hookrightarrow X_A^{\gamma,\infty}$, $\gamma\in(0,1)$, $p\in[1,\infty)$. The second problem is that the function $\xi^\gamma g(\xi)$ is not necessarily bounded for $x \in X_A^{\gamma,p}$, $\gamma\in(0,1)$, $p\in[1,\infty)$, precluding us to prove $X_A^{\gamma,q} \hookrightarrow X_A^{\gamma,p}$, $\gamma\in(0,1)$, $q\in[1,p)$. Indeed, from (35) we can only find $\xi^\gamma g(\xi) \le c_3\xi^\gamma(\xi+1)^{1-\beta-\gamma}\|x\|_{X_A^{\gamma,p}}$, and when $\beta < 1$, the right-hand side of this inequality goes to infinity as ξ goes to infinity. On the contrary, if $\xi^\gamma g(\xi)$ were bounded, then for every $1 \le q < p < \infty$ we would obtain

$$\begin{aligned} [x]^p_{X_A^{\gamma,p}} &= \int_0^\infty [\xi^\gamma g(\xi)]^p \frac{d\xi}{\xi} \\ &\le \left(\sup_{\xi\in(0,\infty)} \xi^\gamma g(\xi)\right)^{p-q} \int_0^\infty [\xi^\gamma g(\xi)]^q \frac{d\xi}{\xi} \\ &= [x]^{p-q}_{X_A^{\gamma,\infty}} [x]^p_{X_A^{\gamma,p}}. \end{aligned} \tag{51}$$

If now in addition $g(\xi)$ were also monotone decreasing, in order that $[x]_{X_A^{\gamma,\infty}} \le c_9[x]_{X_A^{\gamma,q}}$, from the latter inequality we would get $[x]_{X_A^{\gamma,p}} \le (c_9)^{(p-q)/p}[x]_{X_A^{\gamma,q}}$, completing the proof of $X_A^{\gamma,q} \hookrightarrow X_A^{\gamma,p}$, $\gamma\in(0,1)$, $1 \le q < p < \infty$. Due to the former computations, we can thus conclude that in the case $\beta < 1$ the quoted problems are the main obstacles which prevent us to extend (49) and, as its consequence, (20) to the spaces $X_A^{\gamma,p}$.

Remark 5. Let $0 < \gamma_2 < \gamma_1 < 1$ be fixed and for every $p \in [1,\infty]$ and let us set $A_p = X_A^{\gamma_2,p}$ and $B_p = X_A^{\gamma_1,p}$. We thus have the two families of sets $\mathscr{A} = \{A_p\}_{p\in[1,\infty]}$ and $\mathscr{B} = \{B_p\}_{p\in[1,\infty]}$. Let first $\beta = 1$. In this case, since $(X,\mathscr{D}(A))_{\gamma,p} \cong X_A^{\gamma,p}$, from (20) we deduce that the sets A_p and B_p are related by the following inclusions in which $1 < q_1 < q_2 < \infty$:

$$B_1 \subseteq B_{q_1} \subseteq B_{q_2} \subseteq B_\infty \subseteq A_1 \subseteq A_{q_1} \subseteq A_{q_2} \subseteq A_\infty. \tag{52}$$

Now let $\beta < 1$. As observed in Remark 4, in this case the embedding $X_A^{\gamma,q} \hookrightarrow X_A^{\gamma,p}$, $1 \le q < p \le \infty$, may be not satisfied and the chain of inclusions (52) could not take place. However, (38) and (39) hold true and for every $p \in [1,\infty]$, and we have $B_p \subseteq A_p$ and $B_\infty \subseteq A_p$.

We have already pointed out that $\{e^{tA}\}_{t\geq 0}$ may be not strongly continuous in the X-norm on $\overline{\mathscr{D}(A)}$. On the contrary, the following result (cf. [24, Proposition 5.2] for the proof) shows that the things are finer on $(X,\mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$. Later, we will need this fact.

Proposition 6. *Let A be as in Proposition 2. If $\gamma \in (1-\beta,1)$; then $\{e^{tA}\}_{t\geq 0}$ is strongly continuous in the X-norm on $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$ for every $p \in [1,\infty]$.*

We conclude the section listing some estimates for the operators $[(-A)^\theta]^\circ e^{tA}$ defined by (11) with respect to the spaces $(X,\mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$. First, in [19, Lemma 3.1] it is shown that $[(-A)^\theta]^\circ e^{tA}x \in \mathscr{D}(A)$ for every $x \in X$ and that the estimate $\|[(-A)^\theta]^\circ e^{tA}x\|_{\mathscr{D}(A)} \leq \|[(-A)^{\theta+1}]^\circ e^{tA}x\|_X$ is satisfied. Hence, using (14), we get

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(X;\mathscr{D}(A))} \leq \widetilde{c}_{\alpha,\beta,\theta+1} t^{(\beta-\Re\mathrm{e}\,\theta-2)/\alpha}, \quad \Re\mathrm{e}\,\theta \geq 0, \quad t>0. \tag{53}$$

Combining (14) and (53) with (25) and letting $c_{10} = c_0 (c_{\alpha,\beta,\theta})^{1-\gamma}(c_{\alpha,\beta,\theta+1})^\gamma$, it thus follows (cf. [19, Proposition 3.1]) that for every $\gamma \in (0,1)$ and $p \in [1,\infty]$ the following estimate holds:

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(X;(X,\mathscr{D}(A))_{\gamma,p})} \leq c_{10} t^{(\beta-\gamma-\Re\mathrm{e}\,\theta-1)/\alpha}, \quad \Re\mathrm{e}\,\theta \geq 0, \quad t>0. \tag{54}$$

Remark 7. We stress that if $\beta < 1$, then we can not derive an estimate for the $\mathscr{L}(X;X_A^{\gamma,p})$-norm of $[(-A)^\theta]^\circ e^{tA}$ simply by replacing $(X,\mathscr{D}(A))_{\gamma,p}$ with $X_A^{\gamma,p}$ in (54). This is for two reasons. First, when $\gamma \in [\beta,1)$, we are not assured that $[(-A)^\theta]^\circ e^{tA}x \in X_A^{\gamma,p}$ for every $x \in X$. For if $\gamma \in [\beta,1)$, then the space $X_A^{\gamma,p}$ may be smaller than the domain $\mathscr{D}(A)$ to which $[(-A)^\theta]^\circ e^{tA}x$ belongs by virtue of [19, Lemma 3.1]. The second reason is that, even limiting to $\gamma \in (0,\beta)$ in order that $\mathscr{D}(A) \hookrightarrow X_A^{\gamma,p}$, from (31) we only get $\|[(-A)^\theta]^\circ e^{tA} x\|_{(X,\mathscr{D}(A))_{\gamma,p}} \leq 2\|[(-A)^\theta]^\circ e^{tA}x\|_{X_A^{\gamma,p}}$, $x \in X$, and we do not know if the right-hand side can be bounded from above by some constant times $t^{(\beta-\gamma-\Re\mathrm{e}\,\theta-1)/\alpha}\|x\|_X$. Of course, we can employ (32), but in this way all that we can reach is the estimate

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(X;X_A^{\gamma+\beta-1,p})} \leq c_{11} t^{(\beta-\gamma-\Re\mathrm{e}\,\theta-1)/\alpha}, \quad \Re\mathrm{e}\,\theta \geq 0, \quad t>0, \tag{55}$$

where $c_{11} = c_2 c_{10}$, $\gamma \in (1-\beta,1)$ and $p \in [1,\infty]$. Letting $\delta = \gamma+\beta-1$, (55) can be rewritten equivalently as

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(X;X_A^{\delta,p})} \leq c_{11} t^{(2\beta-\delta-\Re\mathrm{e}\,\theta-2)/\alpha}, \quad \Re\mathrm{e}\,\theta \geq 0, \quad t>0, \tag{56}$$

where $\delta \in (0,\beta)$ and $p \in [1,\infty]$. When $\beta < 1$, there are good motivations to believe that estimate (56) is not the best one. In fact, for instance, when $(\theta,p) = (0,\infty)$, (56) leads us to an estimate which is rougher than the estimate

$$\left\|e^{tA}\right\|_{\mathscr{L}(X;X_A^{\delta,\infty})} \leq c_{12} t^{(\beta-\delta-1)/\alpha}, \quad \delta \in (0,1), \; t>0, \tag{57}$$

as shown in [2, Proposition 3.2], with c_{12} being a positive constant depending on α, β, and δ. Also, (57) ensures that $e^{tA}x$, $x \in X$, belongs to $X_A^{\delta,\infty}$ for every $\delta \in (0,1)$ and not only for $\delta \in (0,\beta)$ as (56) suggests. Furthermore, due to (31), estimate (57) yields (54) with $(\theta,\gamma,p) = (0,\delta,\infty)$. This leads us to believe that (57) can be improved and that estimate (54) holds the same if $X_A^{\gamma,\infty}$ is taken in place of $(X,\mathscr{D}(A))_{\gamma,\infty}$.

Now let $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$, $\gamma \in (0,1)$, $p \in [1,\infty]$. As far as the estimates for the $\mathscr{L}(Y_\gamma^p;X)$-norm of operators $[(-A)^\theta]^\circ e^{tA}$ are concerned, instead, at the moment only the following estimates for the case $\theta = 1$ are available (cf. [24, Lemma 5.1]):

$$\left\|\left[(-A)^1\right]^\circ e^{tA}\right\|_{\mathscr{L}(Y_\gamma^p;X)} \leq c_{13} t^{(\beta+\gamma-2)/\alpha}, \quad t>0, \quad \gamma \in (0,1), \quad p \in [1,\infty], \tag{58}$$

with c_{13} being a positive constant depending on α, β, γ, and p. Estimates (58) are successfully applied in [24, Corollary 5.4] to prove that if $\alpha+\beta > 1$, then the map $t \to e^{tA}$ is Hölder continuous from $[0,\infty)$ to $\mathscr{L}(Y_\gamma^p;X)$, $\gamma \in (2-\alpha-\beta,1)$, $p \in [1,\infty]$, with Hölder exponent $\sigma = (\alpha+\beta+\gamma-2)/\alpha$. In Section 3 we will extend (58), proving some estimates for the $\mathscr{L}(Y_\gamma^p;X)$-norm of $[(-A)^\theta]^\circ e^{tA}$, $\Re\mathrm{e}\,\theta \geq 1$, which reduce to (58) in the case $\theta = 1$.

Remark 8. Observe that an estimate for the norm $\|[(-A)^\theta]^\circ e^{tA}\|_{\mathscr{L}(Y_\gamma^p;X)}$, $\Re\mathrm{e}\,\theta \geq 1$, $t>0$, $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$, $\gamma \in (0,1)$, $p \in [1,\infty]$, can be obtained combining (14), (15), and (58). Indeed, using (15), for every $\Re\mathrm{e}\,\theta \geq 1$, $t>0$ and $x \in Y_\gamma^p$, we have

$$\begin{aligned}\left\|\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_X &= \left\|\left[(-A)^{\theta-1}\right]^\circ e^{(t/2)A}\left[(-A)^1\right]^\circ e^{(t/2)A}x\right\|_X \\ &\leq \left\|\left[(-A)^{\theta-1}\right]^\circ e^{(t/2)A}\right\|_{\mathscr{L}(X)}\left\|\left[(-A)^1\right]^\circ e^{(t/2)A}x\right\|_X.\end{aligned} \tag{59}$$

Therefore, due to (14) and (58), from (59) we deduce that

$$\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(Y_\gamma^p;X)} \leq c_{14} t^{(2\beta+\gamma-\Re\mathrm{e}\,\theta-2)/\alpha}, \quad \Re\mathrm{e}\,\theta \geq 1, \quad t>0, \tag{60}$$

where $\gamma \in (0,1)$, $p \in [1,\infty]$ and $c_{14} = 2^{(2+\Re e\,\theta-\gamma-2\beta)/\alpha}\tilde{c}_{\alpha,\beta,\theta-1}c_{13}$. As we will see in the next section estimate (60) is not optimal, in the sense that the negative exponent $(2\beta+\gamma-\Re e\,\theta-2)/\alpha$ can be refined; of course, unless $\beta=1$. The main reason to believe that (60) can be improved is that its derivation consists of two steps: the first in which $[(-A)^{\theta}]^{\circ}e^{tA}$ is decomposed with the help of (15), and the second in which (60) is obtained combining estimates of very different nature, such as (14) and (58). It is thus to be expected that in this double step derivation some regularity goes missing and that a better result can be reached analyzing more detailedly $[(-A)^{\theta}]^{\circ}e^{tA}x$ for $x \in Y_{\gamma}^{p}$.

3. Behaviour of $[(-A)^{\theta}]^{\circ}e^{tA}$ in $(X,\mathscr{D}(A))_{\gamma,p}$ and $X_A^{\gamma,p}$

According to Remark 7 we begin by improving (54), showing that the same estimate holds with $(X,\mathscr{D}(A))_{\gamma,p}$ being replaced by $X_A^{\gamma,\infty}$ if $p=\infty$ and by $X_A^{\beta\gamma,p}$ if $p\in[1,\infty)$. Throughout this and the next section, A will be an m. l. operator in X having nonempty domain $\mathscr{D}(A)$ and satisfying the resolvent condition (H1) of Section 2.

Proposition 9. *Let $\Re e\theta \ge 0$, $\gamma\in(0,1)$ and let $p\in[1,\infty]$. Then, there exist positive constants c_j, $j=15,16$, depending on α, β, γ, θ, and p such that*

$$\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}\right\|_{\mathscr{L}(X;X_A^{\gamma,\infty})} \le c_{15}t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \quad t>0,\quad p=\infty, \tag{61}$$

$$\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}\right\|_{\mathscr{L}(X;X_A^{\beta\gamma,p})} \le c_{16}t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \quad t>0,\quad p\in[1,\infty). \tag{62}$$

Proof. If $\beta=1$, then $(X,\mathscr{D}(A))_{\gamma,p}\cong X_A^{\gamma,p}$ and (61) and (62) with $c_j=c_2c_{10}$, $j=15,16$, follow by taking $\beta=1$ in (32) and (54). Therefore, without the loss of generality, we assume that $\beta\in(0,\alpha]$ is such that $\beta<\alpha$ if $\alpha=1$. Let $\theta\in\mathbf{C}$, $\Re e\,\theta\ge 0$, $\gamma\in(0,1)$, and $p\in[1,\infty)$ be fixed and let x be an arbitrary element of X. Then, for every $t>0$ we have

$$\begin{aligned}\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{X_A^{\gamma,\infty}} &= \left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_X + \left\|\xi^{\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{L_{\infty}^{*}(X)},\end{aligned} \tag{63}$$

$$\begin{aligned}\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{X_A^{\beta\gamma,p}} &= \left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_X + \left\|\xi^{\beta\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{L_{p}^{*}(X)}.\end{aligned} \tag{64}$$

Of course, from estimate (54) we find

$$\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_X \le c_{\gamma,p}c_{10}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \quad t>0, \tag{65}$$

with $c_{\gamma,p}$ being such that $\|y\|_X \le c_{\gamma,p}\|y\|_{(X,\mathscr{D}(A))_{\gamma,p}}$, $y\in(X,\mathscr{D}(A))_{\gamma,p}$, $p\in[1,\infty]$. It thus suffices to investigate only the second terms on the right-hand side of (63) and (64). We begin by proving (61). First, using the second identity in (6), for every $\xi\in(0,\infty)$ we get

$$\begin{aligned}&\xi^{\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\\ &= \frac{1}{2\pi i}\int_{\Gamma}\xi^{\gamma}(-\lambda)^{\theta}e^{t\lambda}A^{\circ}(\xi I-A)^{-1}(\lambda I-A)^{-1}x\,d\lambda\\ &= \xi^{\gamma}\left[\frac{1}{2\pi i}\int_{\Gamma}(-\lambda)^{\theta}e^{t\lambda}(\lambda-\xi)^{-1}d\lambda\right]A^{\circ}(\xi I-A)^{-1}x\\ &\quad - \frac{1}{2\pi i}\int_{\Gamma}\xi^{\gamma}(-\lambda)^{\theta}e^{t\lambda}(\lambda-\xi)^{-1}A^{\circ}(\lambda I-A)^{-1}x\,d\lambda\\ &= -\frac{1}{2\pi i}\int_{\Gamma}\xi^{\gamma}(-\lambda)^{\theta}e^{t\lambda}(\lambda-\xi)^{-1}\left[\lambda(\lambda I-A)^{-1}-I\right]x\,d\lambda\\ &= \frac{1}{2\pi i}\int_{\Gamma}\xi^{\gamma}(-\lambda)^{\theta+1}e^{t\lambda}(\lambda-\xi)^{-1}(\lambda I-A)^{-1}x\,d\lambda.\end{aligned} \tag{66}$$

Here we have used twice the equality $\int_{\Gamma}(-\lambda)^{\theta}e^{t\lambda}(\lambda-\xi)^{-1}d\lambda = 0$, $\xi\in(0,\infty)$, which follows from Cauchy's formula after having enclosed Γ on the left with an arc of the circle $\{z\in\mathbf{C}: |z+c|=R\}$, $R>0$, and letting R to infinity. From (66), using $\|(\lambda I-A)^{-1}\|_{\mathscr{L}(X)} \le C(|\lambda|+1)^{-\beta} \le C|\lambda|^{-\beta}$, $\lambda\in\Sigma_{\alpha}$, it follows that

$$\begin{aligned}&\left\|\xi^{\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_X\\ &\le C(2\pi)^{-1}\|x\|_X \times\int_{\Gamma}\xi^{\gamma}|\lambda|^{1+\Re e\,\theta-\beta}e^{-\Im m\,\theta\arg(-\lambda)}e^{t\Re e\,\lambda}|\lambda-\xi|^{-1}|d\lambda|\\ &\le C(2\pi)^{-1}e^{(\pi/2)|\Im m\,\theta|}\|x\|_X \times\int_{\Gamma}\left(\frac{\xi}{|\lambda|}\right)^{\gamma}|\lambda|^{\gamma+\Re e\,\theta-\beta}e^{t\Re e\,\lambda}\left|1-\left(\frac{\xi}{\lambda}\right)\right|^{-1}|d\lambda|.\end{aligned} \tag{67}$$

Now, since $\Re e\,\lambda\le -c<0$ for every $\lambda\in\Gamma$ and since $\xi\in(0,\infty)$, we have

$$\begin{aligned}\left|1-\left(\frac{\xi}{\lambda}\right)\right| &= \left|1-\left(\frac{\xi\bar{\lambda}}{|\lambda|^2}\right)\right|\\ &= \left[1+\left(\frac{\xi}{|\lambda|}\right)^2-\frac{2\xi\Re e\,\lambda}{|\lambda|^2}\right]^{1/2}\\ &\ge \left[1+\left(\frac{\xi}{|\lambda|}\right)^2\right]^{1/2}.\end{aligned} \tag{68}$$

Therefore, for every $\lambda \in \Gamma$ and $\xi \in (0,\infty)$ the following inequality holds:

$$\left(\frac{\xi}{|\lambda|}\right)^{\gamma}\left|1-\left(\frac{\xi}{\lambda}\right)\right|^{-1} \le \left(\frac{\xi}{|\lambda|}\right)^{\gamma}\left[1+\left(\frac{\xi}{|\lambda|}\right)^{2}\right]^{-1/2} \le \gamma^{\gamma/2}(1-\gamma)^{(1-\gamma)/2} =: c_{\gamma}, \tag{69}$$

where we have used the fact that the function $f(s) = s^{\gamma}(1+s^2)^{-1/2}$, $s \ge 0$, $\gamma \in (0,1)$, attains its maximum value c_γ at the point $s_\gamma = \gamma^{1/2}(1-\gamma)^{-1/2}$. Coming back to (67) and setting $c_{17} = C(2\pi)^{-1}e^{(\pi/2)|\Im m\,\theta|}c_\gamma$, we thus find (here we use also that on Γ it holds $|\lambda| \ge c$, so that $\Re e\,\lambda = -c(|\Im m\,\lambda|+1)^{\alpha} \ge -c(1+c^{-1})^{\alpha}|\lambda|^{\alpha}$):

$$\begin{aligned}\left\|\xi^{\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_X &\le c_{17}\|x\|_X\int_{\Gamma}|\lambda|^{\gamma+\Re e\,\theta-\beta}e^{t\Re e\,\lambda}\,|d\lambda| \\ &\le c_{17}\|x\|_X\int_{\Gamma}|\lambda|^{\gamma+\Re e\,\theta-\beta}e^{-c(1+c^{-1})^{\alpha}t|\lambda|^{\alpha}}\,|d\lambda| \\ &\le 2c_{17}\|x\|_X\int_0^{\infty}\mu^{\gamma+\Re e\,\theta-\beta}e^{-c_\alpha t\mu^{\alpha}}\,d\mu,\end{aligned} \tag{70}$$

where $c_\alpha = c(1+c^{-1})^{\alpha}$. Finally, taking the supremum with respect to $\xi \in (0,\infty)$ in (70) and performing the transformation $c_\alpha t\mu^{\alpha} = s$ in the integral on the right, we obtain

$$\left\|\xi^{\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{L_\infty^*(X)} \le c_{18}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \tag{71}$$

where $c_{18} = 2c_{17}\alpha^{-1}c_\alpha^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}E((\gamma+\Re e\,\theta+1-\beta)/\alpha)$, $E(\chi)$, $\chi > 0$, being the Euler gamma function $\int_0^\infty s^{\chi-1}e^{-s}ds$. Then, summing up (65) and (71), from (63) it follows that

$$\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{X_A^{\gamma,\infty}} \le \left(c_{\gamma,\infty}c_{10}+c_{18}\right)\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \quad \Re e\,\theta \ge 0,\quad t > 0. \tag{72}$$

Since $x \in X$ was arbitrary, this completes the proof of (61) with $c_{15} = c_{\gamma,\infty}c_{10} + c_{18}$. Let us now prove (62). For every $p \in [1,\infty)$ we write

$$\left\|\xi^{\beta\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{L_p^*(X)}^{p} = I_1 + I_2, \tag{73}$$

where $I_j = \int_{a_j}^{b_j}\|\xi^{\beta\gamma}A^{\circ}(\xi I-A)^{-1}[(-A)^{\theta}]^{\circ}e^{tA}x\|_X^p(d\xi/\xi)$, $j = 1,2$, $(a_1,b_1,a_2,b_2) = (0,1,1,\infty)$. First, (35) with $Y_\gamma^p = (X,\mathscr{D}(A))_{\gamma,p}$ yields

$$I_1 \le \left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{(X,\mathscr{D}(A))_{\gamma,p}}^{p}\int_0^1\xi^{\beta\gamma p-1}\left[c_3(\xi+1)^{1-\beta-\gamma}\right]^{p}d\xi. \tag{74}$$

Therefore, since $(\xi+1)^{1-\beta-\gamma} \le c_{\beta,\gamma}$ for every $\xi \in (0,1]$, where $c_{\beta,\gamma} = 2^{1-\beta-\gamma}$ or $c_{\beta,\gamma} = 1$ according that $\gamma \in (0,1-\beta)$ or $\gamma \in [1-\beta,1)$, from (54), we deduce that

$$\begin{aligned}I_1 &\le \left[c_{\beta,\gamma}c_3\right]^{p}\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{(X,\mathscr{D}(A))_{\gamma,p}}^{p}\int_0^1\xi^{\beta\gamma p-1}d\xi \\ &= \left[c_{19}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}\right]^{p},\end{aligned} \tag{75}$$

with $c_{19} = c_{\beta,\gamma}c_3c_{10}(\beta\gamma p)^{-1/p}$. As far as I_2 is concerned, exploiting (71) and recalling that we have assumed $\beta < 1$, we obtain

$$\begin{aligned}I_2 &= \int_1^\infty \xi^{(\beta-1)\gamma p}\left\|\xi^{\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_X^p\frac{d\xi}{\xi} \\ &\le \left[c_{18}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}\right]^{p}\int_1^\infty\xi^{(\beta-1)\gamma p-1}\,d\xi \\ &\le \left[c_{20}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}\right]^{p},\end{aligned} \tag{76}$$

where $c_{20} = c_{18}[(1-\beta)\gamma p]^{-1/p}$. Summing up (73)–(76), it thus follows that

$$\left\|\xi^{\beta\gamma}A^{\circ}(\xi I-A)^{-1}\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{L_p^*(X)} \le c_{21}\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \tag{77}$$

where $c_{21} = [(c_{19})^p + (c_{20})^p]^{1/p}$. Finally, (65) and (77) lead us to

$$\left\|\left[(-A)^{\theta}\right]^{\circ}e^{tA}x\right\|_{X_A^{\beta\gamma,p}} \le \left(c_{\gamma,p}c_{10}+c_{21}\right)\|x\|_X t^{(\beta-\gamma-\Re e\,\theta-1)/\alpha}, \quad \Re e\,\theta \ge 0,\quad t > 0. \tag{78}$$

Since $x \in X$ was arbitrary, this completes the proof of (62) with $c_{16} = c_{\gamma,p}c_{10} + c_{21}$. □

Remark 10. If $\theta = 0$, then (61) is precisely the estimate (57). In this sense our result improves [2] and shows that (54) holds the same with $(X,\mathscr{D}(A))_{\gamma,p}$ being replaced with $X_A^{\gamma,\infty}$ if $p = \infty$ and $X_A^{\beta\gamma,p}$ and if $p \in [1,\infty)$. Also, when $\beta < 1$, (61) and (62) are in two aspects better than the estimate (55) deduced from (54) with the help of (32). First, here we do not need to restrict γ to $(1-\beta,1)$. Further, despite limiting γ to $(1-\beta,1)$, (61) and (62) show that $[(-A)^{\theta}]^{\circ}e^{tA}x$, $\Re e\,\theta \ge 0$, $t > 0$, $x \in X$, enjoys more regularity than that predicted by (55). For, since when $\beta < 1$ it holds $0 < \gamma+\beta-1 < \beta\gamma < \gamma$, from (38) and (39) it follows $X_A^{\gamma,\infty} \hookrightarrow X_A^{\beta\gamma,p} \hookrightarrow X_A^{\gamma+\beta-1,p}$, $p \in [1,\infty]$.

Remark 11. We recall that when $\beta < 1$ the spaces $X_A^{\sigma,p}$, $\sigma \in (0,1)$, $p \in [1,\infty]$, are intermediate spaces between X and $\mathscr{D}(A)$ for $\sigma \in (0,\beta)$, but they may be contained in $\mathscr{D}(A)$ for $\sigma \in [\beta,1)$. Therefore, whereas (61) is satisfied for spaces $X_A^{\sigma,\infty}$ eventually smaller than $\mathscr{D}(A)$, for (62) to hold we have to consider only spaces $X_A^{\sigma,p}$, $p \in [1,\infty)$, bigger than $\mathscr{D}(A)$. In fact, letting $\sigma = \beta\gamma$, we have $\sigma \in (0,\beta)$ for every $\gamma \in (0,1)$.

In accordance with Remark 8 we now improve estimate (58).

Proposition 12. *Let $\Re e\theta \geq 1$, $\gamma \in (0,1)$, $p \in [1,\infty]$ and let $Y_\gamma^p \in \{(X,\mathscr{D}(A))_{\gamma,p}, X_A^{\gamma,p}\}$. Then, there exists a positive constant c_{22} depending on α, β, γ, θ, and p such that*

$$\left\| \left[(-A)^\theta\right]^\circ e^{tA} \right\|_{\mathscr{L}(Y_\gamma^p;X)} \leq c_{22} t^{(\beta+\gamma-\Re e\,\theta-1)/\alpha}, \quad t > 0. \tag{79}$$

Proof. First, using the identity $A^\circ(zI-A)^{-1} = z(zI-A)^{-1} - I$, $z \in \Sigma_\alpha$, for every $x \in X$, we rewrite $[(-A)^\theta]^\circ e^{tA}x$, $\Re e\,\theta \geq 0$, in the following way:

$$\begin{aligned} &\left[(-A)^\theta\right]^\circ e^{tA} x \\ &\quad = -\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{\theta-1} e^{t\lambda}\lambda(\lambda I - A)^{-1} x \, d\lambda \\ &\quad = -\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{\theta-1} e^{t\lambda}\left[A^\circ(\lambda I - A)^{-1} x + I\right] x \, d\lambda \\ &\quad = -\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{\theta-1} e^{t\lambda} A^\circ(\lambda I - A)^{-1} x \, d\lambda, \quad t > 0. \end{aligned} \tag{80}$$

Here we have used $\int_\Gamma (-\lambda)^{\theta-1} e^{t\lambda} d\lambda = 0$, which follows from the Cauchy formula applied to $(-\lambda)^\theta e^{t\lambda}$ after having enclosed Γ on the left with an arc of the circle $\{z \in \mathbf{C} : |z+c| = R\}$, $R > 0$, and letting R to infinity. Let now $\theta \in \mathbf{C}$, $\Re e\,\theta \geq 1$, $\gamma \in (0,1)$, and $p \in [1,\infty]$ be fixed and let x be an arbitrary element of Y_γ^p. From (35) it then follows that

$$\begin{aligned} &\left\| \left[(-A)^\theta\right]^\circ e^{tA} x \right\|_X \\ &\quad \leq c_{23} \|x\|_{Y_\gamma^p} \int_\Gamma |\lambda|^{\Re e\,\theta-1} e^{t\Re e\,\lambda} (|\lambda|+1)^{1-\beta-\gamma} |d\lambda|, \quad t > 0, \end{aligned} \tag{81}$$

where $c_{23} = (2\pi)^{-1} e^{(\pi/2)|\Im m\,\theta|} c_3$. Now, recalling that $|\lambda| \geq c > 0$ for every $\lambda \in \Gamma$, we have $|\lambda| \leq |\lambda| + 1 \leq (1+c^{-1})|\lambda|$, $\lambda \in \Gamma$. As a consequence, the following inequality holds:

$$(|\lambda|+1)^{1-\beta-\gamma} \leq \widetilde{c}_{\beta,\gamma} |\lambda|^{1-\beta-\gamma}, \quad \forall \lambda \in \Gamma, \tag{82}$$

where $\widetilde{c}_{\beta,\gamma} = (1+c^{-1})^{1-\beta-\gamma}$ or $\widetilde{c}_{\beta,\gamma} = 1$ according that $\gamma \in (0, 1-\beta]$ or $\gamma \in (1-\beta, 1)$ ($(0,1-\beta] = \emptyset$ if $\beta = 1$). Therefore, setting $c_{24} = 2\widetilde{c}_{\beta,\gamma} c_{23}$, (81) and (82) yield

$$\begin{aligned} &\left\| \left[(-A)^\theta\right]^\circ e^{tA} x \right\|_X \\ &\quad \leq c_{24} \|x\|_{Y_\gamma^p} \int_0^\infty \mu^{\Re e\,\theta-\beta-\gamma} e^{-c_\alpha t\mu^\alpha} d\mu, \quad t > 0, \end{aligned} \tag{83}$$

with c_α being as in (70). Finally, the transformation $c_\alpha t\mu^\alpha = s$ in the last integral leads us to the following estimate:

$$\left\| \left[(-A)^\theta\right]^\circ e^{tA} x \right\|_X \leq c_{25} \|x\|_{Y_\gamma^p} t^{(\beta+\gamma-\Re e\,\theta-1)/\alpha}, \quad t > 0, \tag{84}$$

where $c_{25} = c_{24}\alpha^{-1} c_\alpha^{(\beta+\gamma-\Re e\,\theta-1)/\alpha} E((\Re e\,\theta + 1 - \beta - \gamma)/\alpha)$, $E(\chi)$, $\chi > 0$, is the Euler's gamma function. Notice that here $\Re e\,\theta \geq 1$ implies $\Re e\,\theta + 1 - \beta - \gamma \geq 2 - \beta - \gamma > 0$ for every $\beta \in (0,1]$ and $\gamma \in (0,1)$, so that $E((\Re e\,\theta + 1 - \beta - \gamma)/\alpha)$ makes sense. Since (84) is satisfied for every arbitrary element $x \in Y_\gamma^p$, the proof is complete with $c_{22} = c_{25}$. □

Remark 13. Estimate (79) is better than (60) obtained in Remark 8 using (14), (15), and (58). In fact, for every $\beta \in (0,\alpha]$, $\alpha \in (0,1]$, $\gamma \in (0,1)$ and $\Re e\,\theta \geq 1$, the following inequality holds:

$$\begin{aligned} \rho_1 &:= \frac{(2\beta + \gamma - \Re e\,\theta - 2)}{\alpha} \\ &\leq \frac{(\beta + \gamma - \Re e\,\theta - 1)}{\alpha} := \rho_2 < 0. \end{aligned} \tag{85}$$

Then, $t^{\rho_2} \leq t^{\rho_1}$, $t \in (0,1]$, and (79) is more accurate than (60) for small values of t.

Estimate (79) with $\theta = 1$ yields the following result which we will need in Section 5 to prove the equivalence between problem (170) and the fixed-point equation (179).

Corollary 14. *Let $\alpha + \beta > 1$ in (H1). Then, for every $x \in X$ the following equalities hold:*

$$A^{-1}\left(e^{tA} - I\right)x = \left(e^{tA} - I\right)A^{-1}x = \int_0^t e^{(t-s)A} x \, ds, \quad t \geq 0. \tag{86}$$

Proof. The assertion is obvious for $t = 0$. Let $t > 0$ and let $x \in X$. Commuting $A^{-1} \in \mathscr{L}(X)$ with the integral sign, from (9) and the resolvent equation, we have $A^{-1}e^{tA}x = e^{tA}A^{-1}x$, which proves the first equality in (86). To prove the second equality, we first write

$$\begin{aligned} \left(e^{tA} - I\right)A^{-1}x &= \int_0^t \left[D_r e^{rA}\right]_{r=t-s} A^{-1} x \, ds \\ &= -\int_0^t \left[(-A)^1\right]^\circ e^{(t-s)A} A^{-1} x \, ds, \end{aligned} \tag{87}$$

and we show that the latter integral is convergent. Indeed, since $\alpha + \beta > 1$, we may consider $A^{-1}x \in \mathscr{D}(A)$ as an element of $(X,\mathscr{D}(A))_{\gamma,p}$, where $\gamma \in (2-\alpha-\beta, 1)$ and $p \in [1,\infty]$. With this choice for γ, from (79) with $\theta = 1$ and (25) we obtain (here we use also $\|A^{-1}x\|_{\mathscr{D}(A)} = \inf_{y \in A(A^{-1}x)} \|y\|_X = \inf_{y \in (AA^{-1})x} \|y\|_X = \|x\|_{\mathscr{D}(AA^{-1})} \leq \|x\|_X$, due to $I \subset AA^{-1}$. Then, $\|A^{-1}x\|_{(X,\mathscr{D}(A))_{\gamma,p}} \leq c_0 \|A^{-1}x\|_X^{1-\gamma} \|A^{-1}x\|_{\mathscr{D}(A)}^\gamma \leq c_0 \|A^{-1}\|_{\mathscr{L}(X)}^{1-\gamma} \|x\|_X$):

$$\begin{aligned} &\left\| \int_0^t \left[(-A)^1\right]^\circ e^{(t-s)A} A^{-1} x \, ds \right\|_X \\ &\quad \leq c_{22} \left\| A^{-1}x \right\|_{(X,\mathscr{D}(A))_{\gamma,p}} \int_0^t (t-s)^{(\beta+\gamma-2)/\alpha} ds \\ &\quad \leq c_{22} c_{\alpha,\beta,\gamma} c_0 \left\| A^{-1} \right\|_{\mathscr{L}(X)}^{1-\gamma} \|x\|_X t^{(\alpha+\beta+\gamma-2)/\alpha}, \end{aligned} \tag{88}$$

where $c_{\alpha,\beta,\gamma} = \alpha(\alpha+\beta+\gamma-2)^{-1}$. We now recall that (cf. [24, formula (3.21)])

$$\left[(-A)^1\right]^\circ e^{tA}(-A)^{-\zeta} = \left[(-A)^{1-\zeta}\right]^\circ e^{tA}, \quad \Re e\,\zeta \in (1-\beta, 1], \quad t>0, \tag{89}$$

with $(-A)^{-\zeta}$ being the negative fractional powers of $-A$ defined by (cf. [24, Section 3]) $(2\pi i)^{-1}\int_\Gamma (-\lambda)^{-\zeta}(\lambda I - A)^{-1} d\lambda$, $\Re e\,\zeta > 1-\beta$. To complete the proof it thus suffices to apply (89) with $\zeta = 1$ to (87) and to recall that $[(-A)^0]^\circ e^{tA} = e^{tA}$, $t>0$. Notice that the integral on the right-hand side of (86) is convergent, too. In fact, from (14), it follows that $\|\int_0^t e^{(t-s)A}x\, ds\|_X \le \tilde{c}_{\alpha,\beta,0}\|x\|_X \int_0^t (t-s)^{(\beta-1)/\alpha} ds = \alpha(\alpha+\beta-1)^{-1}\tilde{c}_{\alpha,\beta,0}\|x\|_X t^{(\alpha+\beta-1)/\alpha}$. □

Remark 15. In particular, from (86) it follows that if $\alpha+\beta>1$, then $\int_0^t e^{(t-s)A}x\, ds \in \mathscr{D}(A)$ for every $x \in X$ and $(e^{tA}-I)x \subseteq A\int_0^t e^{(t-s)A}x\, ds$. This extends to m. l. operators satisfying (H1) the well-known result for sectorial single-valued linear operators (see, for instance, [9, Proposition 2.1.4(ii)] and [11, Proposition 1.2(ii)]).

With the help of (54) and Proposition 12, we can now derive the following interpolation estimates (90) for the operators $[(-A)^\theta]^\circ e^{tA}$, $\Re e\,\theta \ge 1$, which are considered as operators from $(X, \mathscr{D}(A))_{\gamma,p}$ to $(X,\mathscr{D}(A))_{\delta,p}$. As we will see in the proof of Proposition 16, here the fact that the spaces $(X,\mathscr{D}(A))_{\sigma,p}$ are real interpolation spaces between X and $\mathscr{D}(A)$ plays a key role. For it allows us to exploit the interpolation inequality (24) in the derivation of our estimates in the case $\gamma+\delta<1$.

Proposition 16. *Let $\Re e\theta \ge 1$, $\gamma,\delta \in (0,1)$, and $p \in [1,\infty]$. Then, there exist positive constants c_j, $j = 26, 27$, depending on α, β, γ, δ, θ, and p such that for every $t>0$*

$$\begin{aligned}&\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma,p};(X,\mathscr{D}(A))_{\delta,p})}\\ &\quad\le \begin{cases} c_{26}t^{(2\beta+\gamma-\delta-\Re e\,\theta-2)/\alpha}, & \gamma,\delta\in(0,1),\\ c_{27}t^{(\beta+\gamma-\delta-\Re e\,\theta-1)/\alpha}, & \text{if } \gamma+\delta<1.\end{cases}\end{aligned} \tag{90}$$

Proof. For brevity, we will use the shortenings $Y_\sigma^p = (X, \mathscr{D}(A))_{\sigma,p}$, $\sigma \in (0,1)$, $p\in[1,\infty]$. We begin by proving the first estimate in (90). Let $\theta \in \mathbf{C}$, $\Re e\,\theta \ge 1$, $\gamma,\delta\in(0,1)$ and $p\in[1,\infty]$ be fixed and let x be an arbitrary element of Y_γ^p. Moreover, let ζ and ζ' be two arbitrary complex numbers such that $\theta = \zeta+\zeta'$ and whose real parts satisfy $\Re e\,\zeta \ge 0$ and $\Re e\,\zeta' \ge 1$. From the decomposition formula (15) it then follows for every $t>0$:

$$\begin{aligned}&\left\|\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_{Y_\delta^p}\\ &\quad= \left\|\left[(-A)^\zeta\right]^\circ e^{(t/2)A}\left[(-A)^{\zeta'}\right]^\circ e^{(t/2)A}x\right\|_{Y_\delta^p}\\ &\quad\le \left\|\left[(-A)^\zeta\right]^\circ e^{(t/2)A}\right\|_{\mathscr{L}(X;Y_\delta^p)}\left\|\left[(-A)^{\zeta'}\right]^\circ e^{(t/2)A}x\right\|_X\\ &\quad\le \left\|\left[(-A)^\zeta\right]^\circ e^{(t/2)A}\right\|_{\mathscr{L}(X;Y_\delta^p)}\left\|\left[(-A)^{\zeta'}\right]^\circ e^{(t/2)A}\right\|_{\mathscr{L}(Y_\gamma^p;X)}\|x\|_{Y_\gamma^p}.\end{aligned} \tag{91}$$

Therefore, using (54) and (79) with the triplet (θ,γ,t) being equal to $(\zeta,\delta,t/2)$ and $(\zeta',\gamma,t/2)$, respectively, from (91) and $\Re e\,\theta = \Re e\,\zeta + \Re e\,\zeta'$, we deduce that

$$\begin{aligned}&\left\|\left[(-A)^\theta\right]^\circ e^{tA}x\right\|_{Y_\delta^p}\\ &\quad\le c_{10}c_{22}\left(\frac{t}{2}\right)^{(\beta-\delta-\Re e\,\zeta-1)/\alpha}\left(\frac{t}{2}\right)^{(\beta+\gamma-\Re e\,\zeta'-1)/\alpha}\|x\|_{Y_\gamma^p}\\ &\quad\le c_{26}t^{(2\beta+\gamma-\delta-\Re e\,\theta-2)/\alpha}\|x\|_{Y_\gamma^p}, \quad t>0,\end{aligned} \tag{92}$$

where $c_{26} = 2^{(2+\Re e\,\theta+\delta-\gamma-2\beta)/\alpha}c_{10}c_{22}$. This completes the proof of the first estimate in (90), due to the arbitrariness of $x \in Y_\gamma^p$. Let us now prove the second estimate in (90). Let $\theta\in\mathbf{C}$, $\Re e\,\theta \ge 1$, $\gamma,\delta\in(0,1)$, $\gamma+\delta<1$, and $p\in[1,\infty]$ be fixed. Using $\gamma+\delta<1$, we fix $\gamma_2 \in (\gamma/(1-\delta),1) \subsetneq (\gamma,1)$, and we let $\gamma_1 = (\gamma_2\delta)/(\gamma_2-\gamma)$. Clearly, since $\gamma_2\in(\gamma/(1-\delta),1)$, we have $\gamma_1\in(\delta,1)$. In addition, it holds:

$$1-\delta > \frac{\gamma_1-\delta}{\gamma_1} = \left(\frac{\gamma_2\delta}{\gamma_2-\gamma}-\delta\right)\left(\frac{\gamma_2-\gamma}{\gamma_2\delta}\right) = \frac{\gamma}{\gamma_2} > \gamma. \tag{93}$$

Due to (93), we now set $\gamma_0 = \gamma/\gamma_2 = (\gamma_1-\delta)/\gamma_1 \in (\gamma, 1-\delta)$, so that $\gamma = \gamma_0\gamma_2$ and $\delta = (1-\gamma_0)\gamma_1$. From (24) with $p_0 = p$ it thus follows that

$$\begin{aligned}&\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(Y_\gamma^p;Y_\delta^p)}\\ &\quad\le \left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(X;Y_{\gamma_1}^{p_1})}^{1-\gamma_0}\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(Y_{\gamma_2}^{p_2};X)}^{\gamma_0}, \quad t>0,\end{aligned} \tag{94}$$

where $p_j \in [1,\infty]$, $j=1,2$. Applying (54) and (79) with the pair (γ,p) being replaced with (γ_1,p_1) and (γ_2,p_2), respectively, from (94) we finally obtain

$$\begin{aligned}&\left\|\left[(-A)^\theta\right]^\circ e^{tA}\right\|_{\mathscr{L}(Y_\gamma^p;Y_\delta^p)}\\ &\quad\le \left[c_{10}t^{(\beta-\gamma_1-\Re e\,\theta-1)/\alpha}\right]^{1-\gamma_0}\left[c_{22}t^{(\beta+\gamma_2-\Re e\,\theta-1)/\alpha}\right]^{\gamma_0}\\ &\quad\le (c_{10})^{1-\gamma_0}(c_{22})^{\gamma_0}t^{[\beta+\gamma_0\gamma_2-(1-\gamma_0)\gamma_1-\Re e\,\theta-1]/\alpha}\\ &\quad= (c_{10})^{\delta/\gamma_1}(c_{22})^{\gamma/\gamma_2}t^{(\beta+\gamma-\delta-\Re e\,\theta-1)/\alpha}, \quad t>0.\end{aligned} \tag{95}$$

This completes the proof of the second estimate in (90) with $c_{27} = (c_{10})^{\delta/\gamma_1}(c_{22})^{\gamma/\gamma_2}$. □

Remark 17. We stress that if $\beta < 1$ and $\gamma + \delta < 1$, then the first estimate in (90) is rougher than the second one for small values of t, which justify our special attention to the case $\gamma + \delta < 1$. Indeed, if $\beta < 1$, then for every $\mathfrak{Re}\,\theta \geq 1$ the following inequality holds:

$$\rho_3 := \frac{(2\beta + \gamma - \delta - \mathfrak{Re}\,\theta - 2)}{\alpha} < \frac{(\beta + \gamma - \delta - \mathfrak{Re}\,\theta - 1)}{\alpha} =: \rho_4 < 0, \tag{96}$$

so that $t^{\rho_4} \leq t^{\rho_3}$ for $t \in (0,1]$. In other words, if β and $\gamma + \delta$ are both less than one, then the second estimate in (90) establishes that the norm $\|[(-A)^\theta]^\circ e^{tA}\|_{\mathscr{L}((X,\mathscr{D}(A))_{\gamma,p};(X,\mathscr{D}(A))_{\delta,p})}$, $\mathfrak{Re}\,\theta \geq 1$, may blow up as t goes to 0, but with an order of singularity lower than that predicted by the first estimate. In this sense, though less general, the second estimate in (90) is better than the first one.

Remark 18. The reason why the second estimate in (90) yields a better exponent than the first one is the same mentioned in Remark 8. That is, while the first estimate is obtained in two steps: decomposing $[(-A)^\theta]^\circ e^{tA}$ through (15) and then applying (54) and (79), the second estimate is essentially derived in a single step, using (24).

The following Remark 19 points out why, with the exception of the case when $\beta = 1$ and A is single-valued, to prove (90) we can not proceed as in [9, Proposition 2.2.9].

Remark 19. In the optimal case $\beta = 1$, the exponents in both estimates (90) coincide equals to $\nu = \gamma - \delta - \mathfrak{Re}\,\theta$. Hence, in this special case, the assumption $\gamma + \delta < 1$ does not give any enhancement. Also, if we further assume that $\theta \in \mathbf{N}$, then we restore the same estimates as in [9, Proposition 2.2.9(i)]. In this respect, our result extends [9] to the m. l. case, even though our proof really differs from that in [9]. For, there, the norms in the spaces $(X,\mathscr{D}(A))_{\sigma,p}$ are replaced with the norms in the spaces $\mathscr{D}_A(\sigma,p)$, with the latter being the spaces of all $x \in X$ such that $\|x\|_{\mathscr{D}_A(\sigma,p)} = \|x\|_X + [x]_{\mathscr{D}_A(\sigma,p)} < \infty$, where $[x]_{\mathscr{D}_A(\sigma,p)} = \|\xi^{(2-\beta-\sigma)/\alpha}[(-A)^1]^\circ e^{\xi A}\|_{L_p^*(X)}$. It is well known that if $\beta = 1$ and A is single-valued, then $(X,\mathscr{D}(A))_{\sigma,p} \cong \mathscr{D}_A(\sigma,p)$ (cf. [31, Theorem 3], [9, Proposition 2.2.2] and [27, Theorem 1.14.5]). On the contrary, if $(\alpha,\beta) \neq (1,1)$ and/or A is really an m. l. operator, such equivalence is no longer true and we have

$$\begin{aligned} X_A^{\sigma,p} &\hookrightarrow (X,\mathscr{D}(A))_{\sigma,p} \hookrightarrow \mathscr{D}_A(\alpha\sigma,p), \quad p \in [1,\infty), \\ X_A^{\sigma,\infty} &\hookrightarrow (X,\mathscr{D}(A))_{\sigma,\infty} \hookrightarrow \mathscr{D}_A(\sigma,\infty), \quad p = \infty. \end{aligned} \tag{97}$$

Differently from the spaces $X_A^{\sigma,p}$ and as a consequence of $A0 \subseteq \bigcap_{t>0} \mathscr{N}([(-A)^1]^\circ e^{tA})$, the spaces $\mathscr{D}_A(\sigma,p)$ contain $A0$. It can thus be shown that if $\alpha + \beta > 1$, then for every $\sigma \in (2-\alpha-\beta,1)$ and $\varphi \in (0,(\alpha+\beta+\sigma-2)/\alpha)$ (here $(\alpha+\beta+\sigma-2)/\alpha < 1$, since $\sigma < 1 \leq 2-\beta$) the following embeddings hold:

$$\begin{aligned} \{0\} \cup [\mathscr{D}_A(\sigma,p) \setminus A0] &\hookrightarrow X_A^{\varphi,p} \hookrightarrow (X,\mathscr{D}(A))_{\varphi,p}, \quad p \in [1,\infty), \\ \{0\} \cup [\mathscr{D}_A(\sigma,\infty) \setminus A0] &\hookrightarrow X_A^{(\alpha+\beta+\sigma-2)/\alpha,\infty} \hookrightarrow (X,\mathscr{D}(A))_{(\alpha+\beta+\sigma-2)/\alpha,\infty}, \end{aligned} \tag{98}$$

with $\{0\} \cup [\mathscr{D}_A(\sigma,p) \setminus A0]$ being endowed with the norm of $\mathscr{D}_A(\sigma,p)$. Obviously, due to (29), it suffices to prove the embeddings on the right of (97) and on the left of (98). It is out of the aims of this paper to go into the details of these proofs, and for them we refer the readers to [24, Proposition 6.3]. Here we want only to make clear that, with the exception of the case when $\beta = 1$ and A is single-valued, embeddings (97) and (98) prevent us from carrying out the proof of estimates (90) simply by repeating the computations in [9]. Notice that, due to the property $[X_A^{\sigma,p} \cap A0] = \{0\}$, from the second embeddings in (97) and (98) it follows that if $\alpha+\beta > 1$ and $\sigma \in (2-\alpha-\beta,1)$, then

$$X_A^{\sigma,\infty} \hookrightarrow \{0\} \cup [\mathscr{D}_A(\sigma,\infty) \setminus A0] \hookrightarrow X_A^{(\alpha+\beta+\sigma-2)/\alpha,\infty}. \tag{99}$$

Since $(\alpha+\beta+\sigma-2)/\alpha \leq \sigma$ (indeed, $\alpha \leq 1 \leq (2-\beta-\sigma)/(1-\sigma)$ implies $\alpha+\beta+\sigma-2 \leq \alpha\sigma$), (99) agrees with (38) for $p = \infty$. In addition, if $2\alpha+\beta > 2$ and $\sigma \in ((2-\alpha-\beta)/\alpha, 1)$, then the first embeddings in (97) and (98) yield for every $\varphi \in (0,(\alpha+\beta+\alpha\sigma-2)/\alpha)$ the following:

$$X_A^{\sigma,p} \hookrightarrow \{0\} \cup [\mathscr{D}_A(\alpha\sigma,p) \setminus A0] \hookrightarrow X_A^{\varphi,p}, \quad p \in [1,\infty). \tag{100}$$

Since $\varphi < (\alpha+\beta+\alpha\sigma-2)/\alpha \leq \sigma$, (100) agrees with (38) for $p \in [1,\infty)$. Furthermore, if $\beta = 1$, then from (29), (30), and (99) it follows that $(X,\mathscr{D}(A))_{\sigma,\infty} \cong X_A^{\sigma,\infty} \cong \{0\} \cup [\mathscr{D}_A(\sigma,\infty) \setminus A0]$, $\sigma \in (0,1)$. This confirms that in the real m. l. case the equivalence between $X_A^{\sigma,p}$, $(X,\mathscr{D}(A))_{\sigma,p}$ and $\mathscr{D}_A(\sigma,p)$ does not hold even when $\beta = 1$.

Using Propositions 9 and 12, we now obtain estimates for the operators $[(-A)^\theta]^\circ e^{tA}$, $\mathfrak{Re}\,\theta \geq 1$, considered as operators from $X_A^{\gamma,p}$ to $X_A^{\delta,p}$. Clearly, since $\beta < 1$ the spaces $X_A^{\sigma,p}$ may be not real interpolation spaces between X and $\mathscr{D}(A)$, we can not proceed as in the proof of the second estimate in (90) and a weaker result has to be expected.

Proposition 20. *Let $\mathfrak{Re}\theta \geq 1$, $\gamma, \delta \in (0,1)$, and $p \in [1,\infty]$. Then, there exist positive constants c_j, $j = 28, 29, 30$, depending on α, β, γ, δ, θ, and p such that*

$$\left\| [(-A)^\theta]^\circ e^{tA} \right\|_{\mathscr{L}(X_A^{\gamma,\infty};X_A^{\delta,\infty})} \leq c_{28} t^{(2\beta+\gamma-\delta-\mathfrak{Re}\theta-2)/\alpha}, \quad p = \infty, \quad t > 0, \tag{101}$$

$$\left\| [(-A)^\theta]^\circ e^{tA} \right\|_{\mathscr{L}(X_A^{\gamma,p};X_A^{\beta\delta,p})} \leq c_{29} t^{(2\beta+\gamma-\delta-\mathfrak{Re}\theta-2)/\alpha}, \quad p \in [1,\infty), \quad t > 0. \tag{102}$$

Moreover, if $\gamma \in (0,1)$ and $\delta \in (1-\beta,1)$ are such that $\gamma+\delta<1$, then

$$\left\|\left[(-A)^{\theta}\right]^{\circ} e^{tA}\right\|_{\mathscr{L}(X_A^{\gamma,p};X_A^{\delta+\beta-1,p})} \le c_{30} t^{(\beta+\gamma-\delta-\Re e-1)/\alpha}, \quad p \in [1,\infty], \quad t>0. \tag{103}$$

Proof. Due to (61) and (79), in order to prove (101) and (102) it suffices to repeat the same computations as in (91) and (92), with the pair $((X,\mathscr{D}(A))_{\gamma,p},(X,\mathscr{D}(A))_{\delta,p})$ being replaced with $(X_A^{\gamma,\infty},X_A^{\delta,\infty})$ or with $(X_A^{\gamma,p},X_A^{\beta\delta,p})$ provided that $p=\infty$ or $p\in[1,\infty)$. In this way we derive (101) and (102) with $c_{j+13} = 2^{(2+\Re e\,\theta+\delta-\gamma-2\beta)/\alpha} c_j c_{22}$, $j=15,16$. As far as (103) is concerned, we recall that if X_j, $j=1,\ldots,4$, are four Banach spaces such that $X_j \hookrightarrow X_{j+2}$, $j=1,2$, and $L \in \mathscr{L}(X_3;X_2)$, then $L\in\mathscr{L}(X_1;X_4)$ with $\|L\|_{\mathscr{L}(X_1;X_4)} \le C_1C_2\|L\|_{\mathscr{L}(X_3;X_2)}$, C_1 and C_2 being the positive constants such that $\|x\|_{X_{j+2}} \le C_j\|x\|_{X_j}$, $x\in X_j$, $j=1,2$. Applying this result to $L=[(-A)^{\theta}]^{\circ}e^{tA}$ with $(X_1,X_2,X_3,X_4) = (X_A^{\gamma,p},(X,\mathscr{D}(A))_{\delta,p},(X,\mathscr{D}(A))_{\gamma,p},X_A^{\delta+\beta-1,p})$, from (29)–(32) and the second estimate in (90) we deduce (103) with $c_{30}=2c_2c_{27}$. This completes the proof. □

Remark 21. The assumption $\gamma+\delta<1$ with $\gamma\in(0,1)$ and $\delta\in(1-\beta,1)$ implies that $\gamma\in(0,1-\delta)\subsetneq(0,\beta)$. Therefore (cf. Remark 11), we conclude that for (103) to hold we have to consider $[(-A)^{\theta}]^{\circ}e^{tA}$, $\Re e\,\theta \ge 1$, as an operator between the intermediate spaces $X_A^{\gamma,p}$ and $X_A^{\varepsilon,p}$, where $\gamma,\varepsilon\in(0,\beta)$, $\varepsilon=\delta+\beta-1$, $\delta\in(1-\beta,1)$, $\gamma+\delta<1$.

4. Hölder Regularity of Some Operator Functions

Here, we study the Hölder regularity of those operator functions that we will need in Section 5. From now on, with $(Z,\|\cdot\|_Z)$ being a complex Banach space, $C([a,b];Z) = C^0([a,b];Z)$ and $C^{\delta}([a,b];Z)$, $\delta\in(0,1)$, $a<b$, denote, respectively, the spaces of all continuous and δ-Hölder continuous functions from $[a,b]$ into Z endowed with the norms $\|g\|_{0,a,b;Z} = \sup_{t\in[a,b]}\|g(t)\|_Z$ and $\|g\|_{\delta,a,b;Z} = \|g\|_{0,a,b;Z} + |g|_{\delta,a,b;Z}$, where $|g|_{\delta,a,b;Z}$ is the seminorm $\sup_{a\le t_1<t_2\le b}(t_2-t_1)^{-\delta}\|g(t_2)-g(t_1)\|_Z$. We endow the subspace $C_0^{\delta}([a,b];Z) = \{g\in C^{\delta}([a,b];Z) : g(a)=0\}$, $\delta\in[0,1)$ with the norm $\|\cdot\|_{\delta,a,b;Z}$. Further, for $k\in\mathbf{N}$ and $\delta\in(0,1)$ we set $C^k([a,b];Z) = \{g\in C([a,b];Z) : D_t^k g\in C([a,b];Z)\}$, $\|g\|_{k,a,b;Z} = \sum_0^k\|D_t^j g\|_{0,a,b;Z}$ ($D_t^0=I$), and $C^{k+\delta}([a,b];Z) = \{g\in C^k([a,b];Z) : D_t^k g\in C^{\delta}([a,b];Z)\}$, $\|g\|_{k+\delta,a,b;Z} = \|g\|_{k,a,b;Z} + |D_t^k g|_{\delta,a,b;Z}$. Recall that if $0\le\delta_2\le\delta_1\le1$, then $C^{\delta_1}([a,b];Z)\hookrightarrow C^{\delta_2}([a,b];Z)$ and $\|g\|_{\delta_2,a,b;Z}\le\max\{1,(b-a)^{\delta_1-\delta_2}\}\|g\|_{\delta_1,a,b;Z}$, $g\in C^{\delta_1}([a,b];X)$. Finally, given three complex Banach spaces $(X_k,\|\cdot\|_{X_k})$, $k=1,2,3$, and a bilinear bounded operator $\mathscr{P}$ from $X_1\times X_2$ to X_3 with norm C_0, that is, $\mathscr{P}\in\mathscr{B}(X_1\times X_2;X_3)$ and $\|\mathscr{P}\|_{\mathscr{B}(X_1\times X_2;X_3)} = \sup_{\|x_k\|_{X_k}=1,k=1,2}\|\mathscr{P}(x_1,x_2)\|_{X_3} = C_0$, we denote by $\mathscr{K}$ the convolution operator

$$\mathscr{K}(v_1,v_2)(t) = \int_0^t \mathscr{P}(v_1(t-r),v_2(r))\,dr, \quad t\in[0,b], \quad b>0, \tag{104}$$

where $v_k:[0,b]\to X_k$, $k=1,2$. Of course, if $(X_1,X_2)=(\mathbf{C},X_3)$ and if $\mathscr{P}$ is the scalar multiplication in X_3, that is, $\mathscr{P}(z,x)=zx$, $z\in\mathbf{C}$, $x\in X_3$, then $C_0=1$ and $\mathscr{K}$ reduces to the usual convolution operator $\mathscr{K}(v_1,v_2)(t) = \int_0^t v_1(t-r)v_2(r)\,dr$. As usual, for every $q\in[1,\infty]$, we will denote by q' the conjugate exponent of q.

Now let $X_3=X$ and introduce the following linear operators Q_j, $j=1,\ldots,6$, where $g_j\in C^{\delta_j}([0,T];X)$, $j=1,2,5$, $g_{l_k}\in C^{\delta_{l_k}}([0,T],X_k)$, $l=3,6$, $k=1,2$, $g_4\in C^{\delta_4}([0,T];\mathbf{C})$, $y\in Y_\gamma^p$, $Y_\gamma^p\in\{(X,\mathscr{D}(A))_{\gamma,p},X_A^{\gamma,p}\}$, $p\in[1,\infty]$, and $t\in[0,T]$, $T>0$ as follows:

$$[Q_1g_1](t) := \int_0^t e^{(t-s)A}g_1(s)\,ds, \tag{105}$$

$$[Q_2g_2](t) := \int_0^t \left[(-A)^1\right]^{\circ} e^{(t-s)A}\left[g_2(s)-g_2(t)\right]ds, \tag{106}$$

$$\left[Q_3\left(g_{3_1},g_{3_2}\right)\right](t) := \left[Q_2\mathscr{K}\left(g_{3_1},g_{3_2}\right)\right](t), \tag{107}$$

$$\left[Q_4\left(g_4,y\right)\right](t) := \left[Q_2\left(g_4y\right)\right](t), \tag{108}$$

$$[Q_5g_5](t) := \left[e^{tA}-I\right]g_5(t), \tag{109}$$

$$\left[Q_6\left(g_{6_1},g_{6_2}\right)\right](t) := \left[Q_5\mathscr{K}\left(g_{6_1},g_{6_2}\right)\right](t), \tag{110}$$

with g_4y being the function from $[0,T]$ to Y_γ^p defined by $(g_4y)(t)=g_4(t)y$. We will find conditions on $\delta_j,\delta_{l_k},\delta_4,\gamma\in(0,1)$, $j=1,2,5$, $l=3,6$, $k=1,2$, in order that $Q_jg_j\in C^{\tau_j}([0,T];X)$, $Q_l(g_{l_1},g_{l_2})\in C^{\tau_l}([0,T];X)$ and $Q_4(g_4,y)\in C^{\tau_4}([0,T];X)$ for opportunely chosen $\tau_j,\tau_l,\tau_4\in(0,1)$. We emphasize of the presence of the increment $g_2(s)-g_2(t)$ inside the integral defining Q_2g_2. As we will see, and differently from Q_1, it is just this presence which makes Q_2g_2 well-defined for smooth enough functions g_2. This is the reason why the operator Q_2 as it was defined in [20, formula (4.12)] can make no sense and has to be replaced with that defined by the present (106) (cf. the appendix below). We begin our analysis on the Q_j's with the following result proven in [20, Lemma 4.1]. Since we will need it later, here, removing some misprints in [20], we report its short proof for the reader's convenience.

Lemma 22. *Let $\alpha+\beta>1$ in (H1). Then, for every $\delta_1\in(0,(\alpha+\beta-1)/\alpha)$, the operator Q_1 defined by (105) maps $C^{\delta_1}([0,T];X)$ into $C_0^{\delta_1}([0,T];X)$, and for every $t\in[0,T]$ satisfies the following estimate, where $p\in(\alpha/(\alpha+\beta-1-\alpha\delta_1),\infty)$ as follows:*

$$\|Q_1g_1\|_{\delta_1,0,t;X} \le C_1(t)\left(\int_0^t\|g_1\|_{\delta_1,0,s;X}^p\,ds\right)^{1/p}. \tag{111}$$

Here $C_1(t)$ is a nondecreasing function of t depending also on α, β, δ_1, and p'.

Proof. Let $g_1 \in C^{\delta_1}([0,T];X)$, $\delta_1 \in (0,(\alpha+\beta-1)/\alpha)$, and $t \in [0,T]$. From (14) and the Hölder inequality with $p \in (\alpha/(\alpha+\beta-1-\alpha\delta_1),\infty) \subsetneq (1,\infty)$, for any $\tau \in [0,t]$, we deduce that

$$\begin{aligned}
&\left\|[Q_1 g_1](\tau)\right\|_X \\
&\quad \leq \widetilde{c}_{\alpha,\beta,0}\int_0^\tau (\tau-s)^{(\beta-1)/\alpha}\|g_1\|_{0,0,s;X}\mathrm{d}s \\
&\quad \leq c_{31}\tau^{[\alpha-(1-\beta)p']/(\alpha p')}\left(\int_0^\tau \|g_1\|^p_{\delta_1,0,s;X}\mathrm{d}s\right)^{1/p} \\
&\quad \leq c_{31}\tau^{[\alpha-(1+\alpha\delta_1-\beta)p']/(\alpha p')}\tau^{\delta_1}\left(\int_0^\tau \|g_1\|^p_{\delta_1,0,s;X}\mathrm{d}s\right)^{1/p},
\end{aligned} \tag{112}$$

where $c_{31} = \widetilde{c}_{\alpha,\beta,0}\alpha^{1/p'}[\alpha-(1-\beta)p']^{-1/p'}$. Here $\alpha-(1+\alpha\delta_1-\beta)p' > 0$, since $p' \in (1,\alpha/(1+\alpha\delta_1-\beta))$. For $1-1/p > 1-(\alpha+\beta-1-\alpha\delta_1)/\alpha = (1+\alpha\delta_1-\beta)/\alpha$. passing to the supremum with respect to $\tau \in [0,t]$ in (112) we thus find

$$\begin{aligned}
&\left\|Q_1 g_1\right\|_{0,0,t;X} \\
&\quad \leq c_{31}\, t^{[\alpha-(1+\alpha\delta_1-\beta)p']/(\alpha p')}t^{\delta_1}\left(\int_0^t \|g_1\|^p_{\delta_1,0,s;X}\mathrm{d}s\right)^{1/p}.
\end{aligned} \tag{113}$$

Now let (since $[Q_1 g_1](0) = 0$, the case $t_1 = 0$ follows from (112) with $\tau = t_2$) $0 < t_1 < t_2 \leq t$. The change of variable $t-s=r$ in (105) leads us to $[Q_1 g_1](t_2) - [Q_1 g_1](t_1) = \sum_{k=1}^2 I_{k;t_1,t_2,g_1}$, where $I_{1;t_1,t_2,g_1} := \int_{t_1}^{t_2} \mathrm{e}^{rA} g_1(t_2-r)\,\mathrm{d}r$ and $I_{2;t_1,t_2,g_1} := \int_0^{t_1} \mathrm{e}^{rA}[g_1(t_2-r)-g_1(t_1-r)]\,\mathrm{d}r$. Reasoning as in (112) and using the inequality $t_2^\mu - t_1^\mu \leq (t_2-t_1)^\mu$, $\mu \in (0,1]$, we get

$$\begin{aligned}
&\left\|I_{1;t_1,t_2,g_1}\right\|_X \\
&\leq c_{31}(t_2-t_1)^{[\alpha-(1-\beta)p']/(\alpha p')}\left(\int_{t_1}^{t_2} \|g_1\|^p_{\delta_1,0,t_2-r;X}\mathrm{d}r\right)^{1/p} \\
&\leq c_{31}t_2^{[\alpha-(1+\alpha\delta_1-\beta)p']/(\alpha p')}(t_2-t_1)^{\delta_1}\left(\int_0^t \|g_1\|^p_{\delta_1,0,t-r;X}\mathrm{d}r\right)^{1/p}.
\end{aligned} \tag{114}$$

Similarly, but taking advantage from $g_1 \in C^{\delta_1}([0,T];X)$, we obtain

$$\begin{aligned}
&\left\|I_{2;t_1,t_2,g_1}\right\|_X \\
&\leq \widetilde{c}_{\alpha,\beta,0}(t_2-t_1)^{\delta_1}\int_0^{t_1} r^{(\beta-1)/\alpha}|g_1|_{\delta_1,0,t_2-r;X}\mathrm{d}r \\
&\leq c_{31}t_1^{[\alpha-(1+\alpha\delta_1-\beta)p']/(\alpha p')}t_1^{\delta_1}(t_2-t_1)^{\delta_1}\left(\int_0^t \|g_1\|^p_{\delta_1,0,t-r;X}\mathrm{d}r\right)^{1/p}.
\end{aligned} \tag{115}$$

Thus, letting $\widetilde{c}_1(t) = c_{31}t^{[\alpha-(1+\alpha\delta_1-\beta)p']/(\alpha p')}$ from (114) and (115) it follows that

$$\begin{aligned}
&\left\|[Q_1 g_1](t_2) - [Q_1 g_1](t_1)\right\|_X \\
&\quad \leq \widetilde{c}_1(t)\left(t^{\delta_1}+1\right)(t_2-t_1)^{\delta_1}\left(\int_0^t \|g_1\|^p_{\delta_1,0,t-r;X}\mathrm{d}r\right)^{1/p}.
\end{aligned} \tag{116}$$

Finally, summing up (113) and (116) and using $\int_0^t \|g_1\|^p_{\delta_1,0,t-r;X}\mathrm{d}r = \int_0^t \|g_1\|^p_{\delta_1,0,s;X}\mathrm{d}s$, we derive (111) with $C_1(t) = \widetilde{c}_1(t)(2t^{\delta_1}+1)$. This completes the proof. □

Remark 23. We stress that if we renounce to its Hölder regularity, then for $Q_1 g_1$ to be well-defined it suffices that α and β are as in Lemma 22 and that g_1 is merely in $C([0,T];X)$. In fact (see the last part of the proof of Corollary 14, replacing there x with $g_1(s)$), $\|[Q_1 g_1](t)\|_X \leq \alpha(\alpha+\beta-1)^{-1}\widetilde{c}_{\alpha,\beta,0}\|g_1\|_{0,0,t;X}t^{(\alpha+\beta-1)/\alpha}$, $t \in [0,T]$.

Lemma 24. *Let $3\alpha+\beta > 3$ in (H1). Then, for every $\delta_2 \in ((3-2\alpha-\beta)/\alpha,1)$, the operator Q_2 defined by (106) maps $C^{\delta_2}([0,T];X)$ into $C_0^{\nu_2}([0,T];X)$, $\nu_2 = (\alpha\delta_2+2\alpha+\beta-3)/\alpha \in (0,\delta_2]$, and for every $t \in [0,T]$ it satisfies the following estimate:*

$$\left\|Q_2 g_2\right\|_{\nu_2,0,t;X} \leq C_2(t)\left|g_2\right|_{\delta_2,0,t;X}. \tag{117}$$

Here $C_2(t)$ is a nondecreasing function of t depending also on α, β, and δ_2.

Proof. Denote by $\overline{\alpha}$ the number $(1-\alpha)/\alpha$. In particular, since $3\alpha+\beta > 3$ implies $\alpha \in (2/3,1]$, we have $\overline{\alpha} \in [0,1/2)$. Let $t \in [0,T]$, $g_2 \in C^{\delta_2}([0,T];X)$, $\delta_2 \in ((3-2\alpha-\beta)/\alpha,1)$, and $\nu_2 = (\alpha\delta_2+2\alpha+\beta-3)/\alpha \in (0,\delta_2]$. We notice that $(\alpha\delta_2+\beta-2)/\alpha = \nu_2+\overline{\alpha}-1$ and $(\alpha\delta_2+\beta-3)/\alpha = \nu_2-2$. Then, using (14) with $\theta = 1$, for every $\tau \in [0,t]$ we obtain

$$\begin{aligned}
&\left\|[Q_2 g_2](\tau)\right\|_X \\
&\quad \leq \widetilde{c}_{\alpha,\beta,1}\left|g_2\right|_{\delta_2,0,\tau;X}\int_0^\tau (\tau-s)^{(\alpha\delta_2+\beta-2)/\alpha}\mathrm{d}s \\
&\quad = c_{32}\left|g_2\right|_{\delta_2,0,\tau;X}\tau^{\nu_2+\overline{\alpha}},
\end{aligned} \tag{118}$$

where $c_{32} = \widetilde{c}_{\alpha,\beta,1}(\nu_2+\overline{\alpha})^{-1}$. Hence

$$\left\|Q_2 g_2\right\|_{0,0,t;X} \leq c_{32}\left|g_2\right|_{\delta_2,0,t;X}t^{\nu_2+\overline{\alpha}}. \tag{119}$$

Now let (since $[Q_2 g_2](0) = 0$, the case $t_1 = 0$ follows from (118) with $\tau = t_2$) $0 < t_1 < t_2 \leq t$. We have

$[Q_2 g_2](t_2) - [Q_2 g_2](t_1) = \sum_{k=1}^{3} J_{k;t_1,t_2,g_2}$, where for a function $g : [0,T] \to X$ we set

$$J_{1;t_1,t_2,g} := \int_0^{t_1} \left\{ \left[(-A)^1\right]^\circ e^{(t_2-s)A} - \left[(-A)^1\right]^\circ e^{(t_1-s)A} \right\} \times \left[g(s) - g(t_1)\right] ds,$$
$$J_{2;t_1,t_2,g} := \int_0^{t_1} \left[(-A)^1\right]^\circ e^{(t_2-s)A} \left[g(t_1) - g(t_2)\right] ds,$$
$$J_{3;t_1,t_2,g} := \int_{t_1}^{t_2} \left[(-A)^1\right]^\circ e^{(t_2-s)A} \left[g(s) - g(t_2)\right] ds. \tag{120}$$

First, using (13) with $(s,t,\theta) = (t_1 - s, t_2 - s, 1)$, $s \in (0, t_1)$, and (14) with $\theta = 2$, and letting $(c_{33}, c_{34}) = (\tilde{c}_{\alpha,\beta,2}(1-\nu_2)^{-1}, c_{33}\nu_2^{-1})$, we get

$$\begin{aligned}
&\left\| J_{1;t_1,t_2,g_2} \right\|_X \\
&\le \tilde{c}_{\alpha,\beta,2} |g_2|_{\delta_2,0,t_1;X} \int_0^{t_1} \left[\int_{t_1-s}^{t_2-s} \xi^{(\beta-3)/\alpha} d\xi \right] (t_1 - s)^{\delta_2} ds \\
&\le \tilde{c}_{\alpha,\beta,2} |g_2|_{\delta_2,0,t_1;X} \int_0^{t_1} \left[\int_{t_1-s}^{t_2-s} \xi^{(\alpha\delta_2+\beta-3)/\alpha} d\xi \right] ds \\
&= c_{33} |g_2|_{\delta_2,0,t_1;X} \int_0^{t_1} \left[(t_1 - s)^{\nu_2 - 1} - (t_2 - s)^{\nu_2-1} \right] ds \\
&= c_{34} |g_2|_{\delta_2,0,t_1;X} \left[t_1^{\nu_2} + (t_2 - t_1)^{\nu_2} - t_2^{\nu_2} \right] \\
&\le c_{34} |g_2|_{\delta_2,0,t_2;X} (t_2 - t_1)^{\nu_2}.
\end{aligned} \tag{121}$$

Let us turn to $J_{2;t_1,t_2,g_2}$. We first observe that the integral $\int_0^{t_1} [(-A)^1]^\circ e^{(t_2-s)A} ds$ is convergent. For, $\| \int_0^{t_1} [(-A)^1]^\circ e^{(t_2-s)A} ds \|_X \le \tilde{c}_{\alpha,\beta,1} \int_0^{t_1} (t_2 - s)^{(\beta-2)/\alpha} ds \le C_{\alpha,\beta,t_1,t_2}$, where C_{α,β,t_1,t_2} is equal to $\tilde{c}_{\alpha,\beta,1} \ln[t_2(t_2 - t_1)^{-1}]$ if $\beta = 1$ and to $\alpha(2-\alpha-\beta)^{-1} \tilde{c}_{\alpha,\beta,1}[(t_2 - t_1)^{(\alpha+\beta-2)/\alpha} - t_2^{(\alpha+\beta-2)/\alpha}]$ if $\beta \in (0,1)$. Thus, we may rewrite it as $-\int_{t_2}^{t_2-t_1} [(-A)^1]^\circ e^{rA} dr = \int_{t_2}^{t_2-t_1} D_r e^{rA} dr = e^{(t_2-t_1)A} - e^{t_2 A}$. Consequently,

$$\begin{aligned}
&\left\| J_{2;t_1,t_2,g_2} \right\|_X \\
&\le \tilde{c}_{\alpha,\beta,0} \left[(t_2 - t_1)^{(\beta-1)/\alpha} + t_2^{(\beta-1)/\alpha} \right] |g_2|_{\delta_2,0,t_2;X} (t_2 - t_1)^{\delta_2} \\
&\le \tilde{c}_{\alpha,\beta,0} \left\{ 1 + \left[t_2 (t_2 - t_1)^{-1} \right]^{(\beta-1)/\alpha} \right\} \\
&\quad \times |g_2|_{\delta_2,0,t_2;X} (t_2 - t_1)^{(\alpha\delta_2+\beta-1)/\alpha} \\
&\le 2\tilde{c}_{\alpha,\beta,0} |g_2|_{\delta_2,0,t_2;X} (t_2 - t_1)^{\nu_2 + 2\overline{\alpha}},
\end{aligned} \tag{122}$$

where we have used $[t_2(t_2 - t_1)^{-1}]^{(\beta-1)/\alpha} \le 1$ and $(\alpha\delta_2 + \beta - 1)/\alpha = \nu_2 + 2\overline{\alpha}$. As far as $J_{3;t_1,t_2,g_2}$ is concerned, instead, reasoning as in the derivation of (118) we find

$$\begin{aligned}
\left\| J_{3;t_1,t_2,g_2} \right\|_X &\le \tilde{c}_{\alpha,\beta,1} |g_2|_{\delta_2,0,t_2;X} \int_{t_1}^{t_2} (t_2 - s)^{\nu_2 + \overline{\alpha} - 1} ds \\
&= c_{32} |g_2|_{\delta_2,0,t_2;X} (t_2 - t_1)^{\nu_2 + \overline{\alpha}}.
\end{aligned} \tag{123}$$

Then, summing up (121)–(123) and letting $\tilde{c}_2(t) = c_{34} + 2\tilde{c}_{\alpha,\beta,0} t^{2\overline{\alpha}} + c_{32} t^{\overline{\alpha}}$, we obtain

$$\begin{aligned}
\left\| [Q_2 g_2](t_2) - [Q_2 g_2](t_1) \right\|_X &\le \sum_{k=1}^{3} \left\| J_{k;t_1,t_2,g_2} \right\|_X \\
&\le \tilde{c}_2(t) |g_2|_{\delta_2,0,t;X} (t_2 - t_1)^{\nu_2}.
\end{aligned} \tag{124}$$

Finally, (119) and (124) yield (117) with $C_2(t) = c_{32} t^{\nu_2 + \overline{\alpha}} + \tilde{c}_2(t)$. □

Remark 25. In particular, Lemma 24 establishes that, with the exception of the case $\beta = 1$ in which $\nu_2 = \delta_2$, Q_2 produces a loss of regularity equal to $\delta_2 - \nu_2 = (3 - 2\alpha - \beta)/\alpha$.

As Corollary 14, the next result will be needed to prove the equivalence between problem (170) and the fixed-point equation (179). From now on, if $A^{-1} \in \mathscr{L}(X)$ and $g \in C^\delta([0,T];X)$, $\delta \in [0,1)$, with $A^{-1}g$ we will always mean the function in $C^\delta([0,T];\mathscr{D}(A))$ defined by $(A^{-1}g)(t) = A^{-1}(g(t))$. Notice that $\|A^{-1}g\|_{\delta,0,t;\mathscr{D}(A)} \le \|g\|_{\delta,0,t;X}$, $t \in [0,T]$.

Corollary 26.

(i) *Let $2\alpha + \beta > 2$ in (H1). Then, for every $g \in C^\delta([0,T];X)$, $\delta \in ((2-\alpha-\beta)/\alpha, 1)$,*

$$A^{-1} [Q_2 g](t) = -\int_0^t e^{(t-s)A} \left[g(s) - g(t)\right] ds, \quad t \in [0,T]. \tag{125}$$

(ii) *Let $\alpha + \beta > 1$ in (H1). Then, for every $g \in C([0,T];X)$*

$$\left[Q_2 \left(A^{-1} g\right)\right](t) = -\int_0^t e^{(t-s)A} \left[g(s) - g(t)\right] ds, \quad t \in [0,T]. \tag{126}$$

Proof. Of course, it suffices to assume that $t \in (0,T]$. Let us first prove (i). So, let $2\alpha + \beta > 2$, $g \in C^\delta([0,T];X)$, $\delta \in ((2 - \alpha - \beta)/\alpha, 1)$, and $t \in (0,T]$, and we observe that both sides of (125) are well defined. Indeed, replacing the pair (g_2, δ_2) with (g, δ), from (118) we get

$$\begin{aligned}
&\left\| [Q_2 g](t) \right\|_X \\
&\quad \le \tilde{c}_{\alpha,\beta,1} \alpha (\alpha\delta + \alpha + \beta - 2)^{-1} |g|_{\delta,0,t;X} t^{(\alpha\delta+\alpha+\beta-2)/\alpha}.
\end{aligned} \tag{127}$$

On the other side, $I_{t,g} = \int_0^t e^{(t-s)A}[g(s) - g(t)]ds$ satisfies

$$\begin{aligned}\|I_{t,g}\|_X &\le \tilde{c}_{\alpha,\beta,0}|g|_{\delta,0,t;X}\int_0^t (t-s)^{(\alpha\delta+\beta-1)/\alpha}ds\\ &\le c_{35}|g|_{\delta,0,t;X}t^{(\alpha+\alpha\delta+\beta-1)/\alpha},\end{aligned} \tag{128}$$

where $c_{35} = \alpha(\alpha\delta + \alpha + \beta - 1)^{-1}\tilde{c}_{\alpha,\beta,0}$. Then, commuting $A^{-1} \in \mathscr{L}(X)$ with the integral signs, using (80) with $\theta = 1$, and taking into account (7), we find

$$\begin{aligned}&A^{-1}[Q_2 g_2](t)\\ &= A^{-1}\int_0^t\left[-\frac{1}{2\pi i}\int_\Gamma e^{(t-s)\lambda}A^\circ(\lambda I - A)^{-1}d\lambda\right][g_2(s) - g_2(t)]ds\\ &= -\int_0^t\left[\frac{1}{2\pi i}\int_\Gamma e^{(t-s)\lambda}A^{-1}A^\circ(\lambda I - A)^{-1}d\lambda\right][g_2(s) - g_2(t)]ds\\ &= -\int_0^t\left[\frac{1}{2\pi i}\int_\Gamma e^{(t-s)\lambda}(\lambda I - A)^{-1}d\lambda\right][g_2(s) - g_2(t)]ds.\end{aligned} \tag{129}$$

Since $(2\pi i)^{-1}\int_\Gamma e^{(t-s)\lambda}(\lambda I - A)^{-1}d\lambda = e^{(t-s)A}$, the proof of (125) is complete. We now prove (ii). Let $\alpha + \beta > 1$, $g \in C([0,T];X)$ and $t \in (0,T]$. Then, for every $\gamma \in (2-\alpha-\beta, 1)$, the same reasonings made to derive (88), except for replacing x with $g(s) - g(t)$, yield

$$\begin{aligned}&\|[Q_2(A^{-1}g)](t)\|_X\\ &\le 2c_{22}c_{\alpha,\beta,\gamma}c_0\|A^{-1}\|^{1-\gamma}_{\mathscr{L}(X)}\|g\|_{0,0,t;X}t^{(\alpha+\beta+\gamma-2)/\alpha}.\end{aligned} \tag{130}$$

Hence, $[Q_2(A^{-1}g)](t)$ being meaningful, we obtain (126) simply applying to it formula (89) with $\zeta = 1$ and then using $[(-A)^0]^\circ e^{(t-s)A} = e^{(t-s)A}$, $s \in (0,t)$. In particular, a better estimate than (130) holds. For, $[Q_2(A^{-1}g)](t) = -\int_0^t e^{(t-s)A}[g(s) - g(t)]ds$ satisfies

$$\begin{aligned}\|[Q_2(A^{-1}g)](t)\|_X &\le 2\tilde{c}_{\alpha,\beta,0}\|g\|_{0,0,t;X}\int_0^t (t-s)^{(\beta-1)/\alpha}ds\\ &\le 2c_{36}\|g\|_{0,0,t;X}t^{(\alpha+\beta-1)/\alpha},\end{aligned} \tag{131}$$

where $c_{36} = \alpha(\alpha+\beta-1)^{-1}\tilde{c}_{\alpha,\beta,0}$. The proof is complete. □

Let us now examine the operator Q_3 defined by (107). To this purpose we need the following result which is proved in [20, Corollary 3.2].

Lemma 27. *Let $\delta_{3_k} \in (0,1)$, $k = 1,2$, be such that $\sigma_3 = \delta_{3_1} + \delta_{3_2} \in (0, 1/p')$, $p \in (1/(1-\delta_{3_1}), \infty)$. Then the convolution operator $\mathscr{K}$ defined by* (104) *maps $C^{\delta_{3_1}}([0,T];X_1) \times C^{\delta_{3_2}}([0,T];X_2)$ into $C_0^{\sigma_3}([0,T];X)$, and for every $t \in [0,T]$ satisfies the following estimate:*

$$\begin{aligned}&\|\mathscr{K}(g_{3_1}, g_{3_2})\|_{\sigma_3,0,t;X}\\ &\le t^{-\sigma_3+1/p'}\tilde{c}_3(t)\|g_{3_1}\|_{\delta_{3_1},0,t;X_1}\left(\int_0^t \|g_{3_2}\|^p_{\delta_{3_2},0,s;X_2}ds\right)^{1/p}.\end{aligned} \tag{132}$$

Here $\tilde{c}_3(t)$ is a nondecreasing function of t depending also on δ_{3_1} and δ_{3_2}. Further, in the cases $\delta_{3_1} \in (0,1)$, $\delta_{3_2} = 0$, and $\delta_{3_1} = \delta_{3_2} = 0$, the following estimates hold, respectively, as follows:

$$\begin{aligned}&\|\mathscr{K}(g_{3_1}, g_{3_2})\|_{\delta_{3_1},0,t;X}\\ &\quad\le C_0 t^{1-\delta_{3_1}}(1 + t^{\delta_{3_1}})\|g_{3_1}\|_{\delta_{3_1},0,t;X_1}\|g_{3_2}\|_{0,0,t;X_2},\end{aligned} \tag{133}$$

$$\|\mathscr{K}(g_{3_1}, g_{3_2})\|_{0,0,t;X} \le C_0 t\|g_{3_1}\|_{0,0,t;X_1}\|g_{3_2}\|_{0,0,t;X_2}.$$

From Lemmas 24 and 27 we obtain the following Lemma 28.

Lemma 28. *Let α and β be as in Lemma 24. Then, for every $\delta_{3_1} \in ((3-2\alpha-\beta)/\alpha, 1)$ and $\delta_{3_2} \in (0,1)$ such that $\sigma_3 = \delta_{3_1} + \delta_{3_2} \in ((3-2\alpha-\beta)/\alpha, 1/p')$, $p \in (1/(1-\delta_{3_1}), \infty)$, the operator Q_3 defined by* (107) *maps $C^{\delta_{3_1}}([0,T];X_1) \times C^{\delta_{3_2}}([0,T];X_2)$ into $C_0^{\nu_3}([0,T];X)$, $\nu_3 = (\alpha\sigma_3 + 2\alpha + \beta - 3)/\alpha$, and for every $t \in [0,T]$ satisfies the following estimate:*

$$\begin{aligned}&\|Q_3(g_{3_1}, g_{3_2})\|_{\nu_3,0,t;X}\\ &\le t^{-\sigma_3+1/p'}C_2(t)\tilde{c}_3(t)\|g_{3_1}\|_{\delta_{3_1},0,t;X_1}\left(\int_0^t \|g_{3_2}\|^p_{\delta_{3_2},0,s;X_2}ds\right)^{1/p}.\end{aligned} \tag{134}$$

Proof. First, if $\delta_{3_1} \in ((3-2\alpha-\beta)/\alpha, 1)$ and $p \in (1/(1-\delta_{3_1}), \infty)$, then $1/p' \in (\delta_{3_1}, 1) \subsetneq ((3-2\alpha-\beta)/\alpha, 1)$. Consequently, the assumption $\sigma_3 = \delta_{3_1} + \delta_{3_2} \in ((3-2\alpha-\beta)/\alpha, 1/p')$, $\delta_{3_2} \in (0,1)$, makes sense. Now, Lemma 27 yields $\mathscr{K}(g_{3_1}, g_{3_2}) \in C_0^{\sigma_3}([0,T];X)$ for any pair $(g_{3_1}, g_{3_2}) \in C^{\delta_{3_1}}([0,T];X_1) \times C^{\delta_{3_2}}([0,T];X_2)$. Then, recalling that $Q_3(g_{3_1}, g_{3_2}) = Q_2\mathscr{K}(g_{3_1}, g_{3_2})$, the assertion follows from Lemma 24, with δ_2 and g_2 being replaced by σ_3 and $\mathscr{K}(g_{3_1}, g_{3_2})$, respectively. Finally, (134) follows from (117) and (132). □

We can now restore the loss of regularity produced by Q_2.

Proposition 29. *Let $5\alpha + 2\beta > 6$ in (H1). Then, for every $\delta_3 \in ((3-2\alpha-\beta)/\alpha, 1/2)$, the operator Q_3 defined by* (107) *maps $C^{\delta_3}([0,T];X_1) \times C^{\delta_3}([0,T];X_2)$ into $C_0^{\delta_3}([0,T];X)$, and* *for every* $t \in [0,T]$ *satisfies the following estimate, where* $p \in (1/(1-2\delta_3),\infty)$ *and* $C_3(t) = C_2(t)\tilde{c}_3(t)\max\{1, t^{(\alpha\delta_3+2\alpha+\beta-3)/\alpha}\}$*:*

$$\left\|Q_3\left(g_{3_1}, g_{3_2}\right)\right\|_{\delta_3,0,t;X} \leq t^{1-2\delta_3-1/p} C_3(t) \left\|g_{3_1}\right\|_{\delta_3,0,t;X_1} \left(\int_0^t \left\|g_{3_2}\right\|^p_{\delta_3,0,s;X_2} \mathrm{d}s\right)^{1/p}. \tag{135}$$

Proof. Let $\delta_3 \in ((3-2\alpha-\beta)/\alpha, 1/2)$ and let $p \in (1/(1-2\delta_3),\infty) \subsetneq (1/(1-\delta_3),\infty)$. Then, $2\delta_3 \in ((6-4\alpha-2\beta)/\alpha, 1/p') \subseteq ((3-2\alpha-\beta)/\alpha, 1/p')$. We are thus in position to apply Lemma 28 with $\delta_{3_1} = \delta_{3_2} = \delta_3$ from which we deduce that Q_3 maps $C^{\delta_3}([0,T];X_1) \times C^{\delta_3}([0,T];X_2)$ into $C_0^{\nu_3}([0,T];X)$, $\nu_3 = (2\alpha\delta_3 + 2\alpha + \beta - 3)/\alpha$. But, since our choice for δ_3 implies $\nu_3 > \delta_3$, we *a fortiori* have the fact that Q_3 maps $C^{\delta_3}([0,T];X_1) \times C^{\delta_3}([0,T];X_2)$ into $C_0^{\delta_3}([0,T];X)$. Finally, (135) follows from (134) and the estimate $\|g\|_{\gamma,0,t;X} \leq \max\{1, t^{\delta-\gamma}\}\|g\|_{\delta,0,t;X}$, $g \in C^{\delta}([0,T];X)$, $\delta \geq \gamma$. □

The next Lemma 30 concerns the operator Q_4. Its proof is similar to that of Lemma 24, but with the essential difference that the presence of $y \in Y_\gamma^p$ allows us to use estimate (79) in place of (14). As a consequence and provided to choose γ large enough, we will achieve a better result in which any loss of regularity is observed.

Lemma 30. *Let* $2\alpha + \beta > 2$ *in (H1) and* $r \in [1,\infty]$*. Then, for every* $\delta_4 \in (0,1)$ *and* $\gamma \in (3-2\alpha-\beta, 1)$ *the operator* Q_4 *defined by* (108) *maps* $C^{\delta_4}([0,T];\mathbf{C}) \times Y_\gamma^r$, $Y_\gamma^r \in \{(X,\mathscr{D}(A))_{\gamma,r}, X_A^{\gamma,r}\}$, *into* $C_0^{\delta_4}([0,T];X)$, *and for every* $t \in [0,T]$ *satisfies the following estimate:*

$$\left\|Q_4(g_4, y)\right\|_{\delta_4,0,t;X} \leq C_4(t)\, t^{(2\alpha+\beta+\gamma-3)/\alpha} |g_4|_{\delta_4,0,t;\mathbf{C}} \|y\|_{Y_\gamma^r}. \tag{136}$$

Here $C_4(t)$ *is a nondecreasing function of* t *depending on* α, β, δ_4, γ *and* r.

Proof. Let $t \in [0,T]$, $g_4 \in C^{\delta_4}([0,T];\mathbf{C})$, $\delta_4 \in (0,1)$, and $y \in Y_\gamma^r$, $\gamma \in (3-2\alpha-\beta, 1)$, $r \in [1,\infty]$. As in the proof of Lemma 24 we set $\overline{\alpha} = (1-\alpha)/\alpha$ and we observe that, since $2\alpha+\beta > 2$ implies $\alpha \in (1/2, 1]$, here $\overline{\alpha} \in [0,1)$. Furthermore, we denote by $\sigma_{\alpha,\beta,\gamma}$ the number $(2\alpha+\beta+\gamma-3)/\alpha \in (0,1)$, so that the exponents $(\beta+\gamma-2)/\alpha$ and $(\beta+\gamma-3)/\alpha$ appearing in (79) with $\theta = 1$ and $\theta = 2$ may be rewritten, as $\sigma_{\alpha,\beta,\gamma} + \overline{\alpha} - 1$ and $\sigma_{\alpha,\beta,\gamma} - 2$, respectively. Then, using (79) with $\theta = 1$, we obtain

$$\begin{aligned} \left\|\left[Q_4(g_4, y)\right](\tau)\right\|_X &\leq c_{22}|g_4|_{\delta_4,0,\tau;\mathbf{C}} \|y\|_{Y_\gamma^r} \int_0^\tau (\tau - s)^{\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}-1} \mathrm{d}s \\ &= c_{37}|g_4|_{\delta_4,0,\tau;\mathbf{C}} \|y\|_{Y_\gamma^r} \tau^{\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}}, \quad \forall \tau \in [0,t], \end{aligned} \tag{137}$$

where $c_{37} = c_{22}(\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha})^{-1}$. Hence, taking the supremum with respect to $\tau \in [0,t]$, one has

$$\left\|Q_4(g_4, y)\right\|_{0,0,t;X} \leq c_{37}|g_4|_{\delta_4,0,t;\mathbf{C}} \|y\|_{Y_\gamma^r} t^{\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}}. \tag{138}$$

Now, let (since $[Q_4(g_4,y)](0) = 0$, the case $t_1 = 0$ follows from (137) with $\tau = t_2$) $0 < t_1 < t_2 \leq t$. We have $[Q_4(g_4,y)](t_2) - [Q_4(g_4,y)](t_1) = \sum_{k=1}^3 J_{k;t_1,t_2,g_4 y}$, the $J_{k;t_1,t_2,g}$'s, $g : [0,T] \to X$, being as in (120). Using (13) with $(s,t,\theta) = (t_1 - s, t_2 - s, 1)$, $s \in (0,t_1)$, and (79) with $\theta = 2$, and letting $(c_{38}, c_{39}) = (c_{22}(1-\delta_4)^{-1}, c_{38}\delta_4^{-1})$, we get

$$\begin{aligned} \left\|J_{1;t_1,t_2,g_4 y}\right\|_X &\leq c_{22}|g_4|_{\delta_4,0,t_1;\mathbf{C}} \|y\|_{Y_\gamma^r} \times \int_0^{t_1} \left[\int_{t_1-s}^{t_2-s} \xi^{\sigma_{\alpha,\beta,\gamma}-2} \mathrm{d}\xi\right] (t_1-s)^{\delta_4} \mathrm{d}s \\ &\leq c_{22}|g_4|_{\delta_4,0,t_1;\mathbf{C}} \|y\|_{Y_\gamma^r} \int_0^{t_1} \left[\int_{t_1-s}^{t_2-s} \xi^{\delta_4+\sigma_{\alpha,\beta,\gamma}-2} \mathrm{d}\xi\right] \mathrm{d}s \\ &\leq c_{22}|g_4|_{\delta_4,0,t_1;\mathbf{C}} \|y\|_{Y_\gamma^r} \times \int_0^{t_1} (t_2-s)^{\sigma_{\alpha,\beta,\gamma}} \left[\int_{t_1-s}^{t_2-s} \xi^{\delta_4-2} \mathrm{d}\xi\right] \mathrm{d}s \\ &\leq c_{38}|g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y_\gamma^r} t_2^{\sigma_{\alpha,\beta,\gamma}} \times \int_0^{t_1} \left[(t_1-s)^{\delta_4-1} - (t_2-s)^{\delta_4-1}\right] \mathrm{d}s \\ &= c_{39}|g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y_\gamma^r} t_2^{\sigma_{\alpha,\beta,\gamma}} \left[t_1^{\delta_4} + (t_2-t_1)^{\delta_4} - t_2^{\delta_4}\right] \\ &\leq c_{39}|g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y_\gamma^r} t_2^{\sigma_{\alpha,\beta,\gamma}} (t_2-t_1)^{\delta_4}. \end{aligned} \tag{139}$$

Now, let us examine $J_{k;t_1,t_2,g_4 y}$, $k = 2,3$. First, using (79) with $\theta = 1$, we find

$$\begin{aligned} \left\|J_{2;t_1,t_2,g_4 y}\right\|_X &\leq c_{22}|g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y_\gamma^r} \left[\int_0^{t_1} (t_2-s)^{\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}-1} \mathrm{d}s\right] (t_2-t_1)^{\delta_4} \\ &= c_{40}|g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y_\gamma^r} \left[t_2^{\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}} - (t_2-t_1)^{\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}}\right] (t_2-t_1)^{\delta_4} \\ &\leq c_{40}|g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y_\gamma^r} t_2^{\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}} (t_2-t_1)^{\delta_4}. \end{aligned} \tag{140}$$

Instead, the same computations made to derive (137) yield

$$\begin{aligned} \left\|J_{3;t_1,t_2,g_4 y}\right\|_X &\leq c_{22}|g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y_\gamma^r} \int_{t_1}^{t_2} (t_2-s)^{\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}-1} \mathrm{d}s \\ &= c_{37}|g_4|_{\delta_4,0,t_2;\mathbf{C}} \|y\|_{Y_\gamma^r} (t_2-t_1)^{\delta_4+\sigma_{\alpha,\beta,\gamma}+\overline{\alpha}}. \end{aligned} \tag{141}$$

From (139)–(141) and $\|[Q_4(g_4,y)](t_2) - [Q_4(g_4,y)](t_1)\|_X \leq \sum_{k=1}^3 \|J_{k;t_1,t_2,g_4 y}\|_X$, it follows that

$$\begin{aligned} \left\|\left[Q_4(g_4,y)\right](t_2) - \left[Q_4(g_4,y)\right](t_1)\right\|_X \\ \leq \tilde{c}_4(t)\, t^{\sigma_{\alpha,\beta,\gamma}} |g_4|_{\delta_4,0,t;\mathbf{C}} \|y\|_{Y_\gamma^r} (t_2-t_1)^{\delta_4}, \end{aligned} \tag{142}$$

where $\widetilde{c}_4(t) = c_{39} + (c_{37} + c_{40})t^{\overline{\alpha}}$. Finally, summing up (138) and (142) we get (136) with $C_4(t) = c_{37}t^{\delta_4+\overline{\alpha}} + \widetilde{c}_4(t)$. The proof is complete. □

Remark 31. Notice that if $Y_\gamma^r = X_A^{\gamma,r}$, then in order to be sure that the conclusions of Lemma 30 hold with y which really belongs to some intermediate space between X and $\mathscr{D}(A)$ we have to choose $\gamma \in (3-2\alpha-\beta, \beta)$. This is possible, provided that the stronger assumption $2\alpha+\beta > 3-\beta \geq 2$ is satisfied. Otherwise, if $2\alpha+\beta \in (2, 3-\beta]$, $\beta < 1$, then $\gamma \in (3-2\alpha-\beta, 1) \subsetneq [\beta, 1)$ and y may be contained in $\mathscr{D}(A)$.

Finally, for the operator Q_5 we have the following result. Again a loss of regularity is exhibited, even though of an amount smaller than that in Lemma 24 (cf. Remark 33).

Lemma 32. *Let $2\alpha+\beta > 2$ in (H1). Then, for every $\delta_5 \in ((2-\alpha-\beta)/\alpha, 1)$, the operator Q_5 defined by* (109) *maps $C_0^{\delta_5}([0,T];X)$ into $C_0^{\nu_5}([0,T];X)$, $\nu_5 = (\alpha\delta_5+\alpha+\beta-2)/\alpha \in (0, \delta_5]$, and for every $t \in [0,T]$ satisfies the following estimate:*

$$\|Q_5 g_5\|_{\nu_5,0,t;X} \leq C_5(t)\,|g_5|_{\delta_5,0,t;X}. \tag{143}$$

Here $C_5(t)$ is a nondecreasing function of t depending also on α, β, and δ_5.

Proof. Let $g_5 \in C_0^{\delta_5}([0,T];X)$, $\delta_5 \in ((2-\alpha-\beta)/\alpha, 1)$, and $\nu_5 = (\alpha\delta_5+\alpha+\beta-2)/\alpha \in (0,\delta_5]$. We still let $\overline{\alpha} = (1-\alpha)/\alpha$ and as in Lemma 30 we have $\overline{\alpha} \in [0,1)$. Further, observe that $\delta_5 + (\beta-1)/\alpha = \nu_5 + \overline{\alpha} \in (0,\delta_5]$. Let $t \in [0,T]$. Then, using (14) and $g_5(0) = 0$, we get

$$\begin{aligned}
&\|Q_5 g_5\|_{0,0,t;X}\\
&\quad\leq \sup_{\tau\in[0,t]} \left[\widetilde{c}_{\alpha,\beta,0}\tau^{(\beta-1)/\alpha} + 1\right] |g_5|_{\delta_5,0,\tau;X}\tau^{\delta_5}\\
&\quad\leq \left[\widetilde{c}_{\alpha,\beta,0} + t^{(1-\beta)/\alpha}\right] |g_5|_{\delta_5,0,t;X} t^{\nu_5+\overline{\alpha}}.
\end{aligned} \tag{144}$$

Now, let (since $[Q_5 g_5](0) = 0$, the case $t_1 = 0$ follows from (144) and $\|[Q_5 g_5](t_2)\|_X \leq \|Q_5 g_5\|_{0,0,t_2;X}$) $0 < t_1 < t_2 \leq t$. We have $[Q_5 g_5](t_2) - [Q_5 g_5](t_1) = \sum_{k=1}^{3} U_{k;t_1,t_2,g_5}$, where for a function $g: [0,T] \to X$ we let

$$\begin{aligned}
U_{1;t_1,t_2,g} &:= e^{t_2A}\left[g(t_2) - g(t_1)\right],\\
U_{2;t_1,t_2,g} &:= \left(e^{t_2A} - e^{t_1A}\right) g(t_1),\\
U_{3;t_1,t_2,g} &:= g(t_1) - g(t_2).
\end{aligned} \tag{145}$$

First, since $t_2^{(\beta-1)/\alpha} \leq (t_2-t_1)^{(\beta-1)/\alpha}$ for every $\beta \in (0,1]$, we deduce that

$$\begin{aligned}
\left\|U_{1;t_1,t_2,g_5}\right\|_X &\leq \widetilde{c}_{\alpha,\beta,0} t_2^{(\beta-1)/\alpha} |g_5|_{\delta_5,0,t_2;X}(t_2-t_1)^{\delta_5}\\
&\leq \widetilde{c}_{\alpha,\beta,0} |g_5|_{\delta_5,0,t_2;X}(t_2-t_1)^{\nu_5+\overline{\alpha}}.
\end{aligned} \tag{146}$$

As far as $U_{2;t_1,t_2,g_5}$ is concerned, instead, rewriting $e^{t_2A} - e^{t_1A}$ as $-\int_{t_1}^{t_2}[(-A)^1]^\circ e^{rA}dr$ and using both $g_5(0) = 0$ and $(\alpha\delta_5 + \beta - 2)/\alpha = \nu_5 - 1$, it follows that

$$\begin{aligned}
\left\|U_{2;t_1,t_2,g_5}\right\|_X &\leq \widetilde{c}_{\alpha,\beta,1} |g_5|_{\delta_5,0,t_1;X} t_1^{\delta_5} \int_{t_1}^{t_2} r^{(\beta-2)/\alpha} dr\\
&\leq \widetilde{c}_{\alpha,\beta,1} |g_5|_{\delta_5,0,t_1;X} \int_{t_1}^{t_2} r^{\nu_5-1} dr\\
&\leq \widetilde{c}_{\alpha,\beta,1}\nu_5^{-1} |g_5|_{\delta_5,0,t_1;X}\left(t_2^{\nu_5} - t_1^{\nu_5}\right)\\
&\leq \widetilde{c}_{\alpha,\beta,1}\nu_5^{-1} |g_5|_{\delta_5,0,t_1;X}(t_2-t_1)^{\nu_5}.
\end{aligned} \tag{147}$$

Then, since $\|U_{3;t_1,t_2,g_5}\|_X \leq |g_5|_{\delta_5,0,t_2;X}(t_2-t_1)^{\delta_5}$, from (146) and (147) we find

$$\begin{aligned}
&\left\|[Q_5 g_5](t_2) - [Q_5 g_5](t_1)\right\|_X\\
&\quad\leq \sum_{k=1}^{3}\left\|U_{k;t_1,t_2,g_5}\right\|_X \leq \widetilde{c}_5(t)\,|g_5|_{\delta_5,0,t;X}(t_2-t_1)^{\nu_5},
\end{aligned} \tag{148}$$

where $\widetilde{c}_5(t) = \widetilde{c}_{\alpha,\beta,0}t^{\overline{\alpha}} + \widetilde{c}_{\alpha,\beta,1}\nu_5^{-1} + t^{\delta_5-\nu_5}$. Summing up (144) and (148) we obtain (143) with $C_5(t) = [\widetilde{c}_{\alpha,\beta,0} + t^{(1-\beta)/\alpha}]t^{\nu_5+\overline{\alpha}} + \widetilde{c}_5(t)$. This completes the proof. □

Remark 33. Thus, with the exception of $\beta = 1$, Q_5 produces a loss of regularity equal to $\delta_5 - \nu_5 = (2-\alpha-\beta)/\alpha \leq (3-2\alpha-\beta)/\alpha$. In this sense Q_5 behaves better than Q_2.

Remark 34. Notice that, under the weaker assumptions $\alpha+\beta > 1$ and $g_5 \in C([0,T];X)$, (86) with $x = g_5(t)$, $t \in [0,T]$, yields $A^{-1}[Q_5 g_5](t) = [Q_5(A^{-1}g_5)](t) = \int_0^t e^{(t-s)A} g_5(t)ds$.

Similarly as we have done in Proposition 29 for restoring the loss of regularity produced by Q_2, we now show how Lemma 27 allows to restore that produced by Q_5. We begin with the following version of Lemma 28 relative to Q_6, and which is obtained combining Lemma 27 with Lemma 32 instead of Lemma 24.

Lemma 35. *Let α and β be as in Lemma 32. Then, for every $\delta_{6_1} \in ((2-\alpha-\beta)/\alpha, 1)$ and $\delta_{6_2} \in (0,1)$ such that $\sigma_6 = \delta_{6_1}+\delta_{6_2} \in ((2-\alpha-\beta)/\alpha, 1/p')$, $p \in (1/(1-\delta_{6_1}), \infty)$, the operator Q_6 defined by* (110) *maps $C^{\delta_{6_1}}([0,T];X_1) \times C^{\delta_{6_2}}([0,T];X_2)$ into $C_0^{\nu_6}([0,T];X)$, $\nu_6 = (\alpha\sigma_6+\alpha+\beta-2)/\alpha$, and for every $t \in [0,T]$ satisfies the following estimate:*

$$\begin{aligned}
&\left\|Q_6\left(g_{6_1}, g_{6_2}\right)\right\|_{\nu_6,0,t;X}\\
&\quad\leq t^{-\sigma_6+1/p'} C_5(t)\,\widetilde{c}_3(t)\left\|g_{6_1}\right\|_{\delta_{6_1},0,t;X_1}\left(\int_0^t \left\|g_{6_2}\right\|^p_{\delta_{6_2},0,s;X_2} ds\right)^{1/p}.
\end{aligned} \tag{149}$$

Proof. First, if $\delta_{6_1} \in ((2-\alpha-\beta)/\alpha, 1)$ and $p \in (1/(1-\delta_{6_1}), \infty)$, then $1/p' \in (\delta_{6_1}, 1) \subsetneq ((2-\alpha-\beta)/\alpha, 1)$. Consequently, the assumption $\sigma_6 = \delta_{6_1}+\delta_{6_2} \in ((2-\alpha-\beta)/\alpha, 1/p')$ makes sense, provided to choose $\delta_{6_2} \in (0,1)$ small enough. Lemma 27 then yields $\mathscr{K}(g_{6_1}, g_{6_2}) \in C_0^{\sigma_6}([0,T];X)$ for any pair $(g_{6_1}, g_{6_2}) \in C^{\delta_{6_1}}([0,T];X_1) \times C^{\delta_{6_2}}([0,T];X_2)$. Then, since $Q_6(g_{6_1}, g_{6_2}) = Q_5\mathscr{K}(g_{6_1}, g_{6_2})$, the assertion follows from Lemma 32, with the pair (δ_5, g_5) being replaced by $(\sigma_6, \mathscr{K}(g_{6_1}, g_{6_2}))$. Finally, (149) follows from (143) and (132). □

From Lemma 35 we obtain the analogous of Proposition 29 for Q_6.

Proposition 36. *Let $3\alpha + 2\beta > 4$ in (H1). Then, for every $\delta_6 \in ((2-\alpha-\beta)/\alpha, 1/2)$, the operator Q_6 defined by* (110) *maps $C^{\delta_6}([0,T];X_1) \times C^{\delta_6}([0,T];X_2)$ into $C_0^{\delta_6}([0,T];X)$, and for every $t \in [0,T]$ satisfies the following estimate, where $p \in (1/(1-2\delta_6), \infty)$ and $C_6(t) = C_5(t)\widetilde{c}_3(t)\max\{1, t^{(\alpha\delta_6+\alpha+\beta-2)/\alpha}\}$:*

$$\begin{aligned}&\left\|Q_6\left(g_{6_1}, g_{6_2}\right)\right\|_{\delta_6,0,t;X}\\&\quad\leq t^{1-2\delta_6-1/p} C_6(t)\left\|g_{6_1}\right\|_{\delta_6,0,t;X_1}\left(\int_0^t \left\|g_{6_2}\right\|^p_{\delta_6,0,s;X_2} \mathrm{d}s\right)^{1/p}.\end{aligned} \tag{150}$$

Proof. Let $\delta_6 \in ((2-\alpha-\beta)/\alpha, 1/2)$ and $p \in (1/(1-2\delta_6), \infty) \subsetneq (1/(1-\delta_6), \infty)$. Then, $2\delta_6 \in ((4-2\alpha-2\beta)/\alpha, 1/p') \subseteq ((2-\alpha-\beta)/\alpha, 1/p')$ and we can apply Lemma 35 with $\delta_{6_k} = \delta_6$, $k = 1,2$. We thus deduce that Q_6 maps $C^{\delta_6}([0,T];X_1) \times C^{\delta_6}([0,T];X_2)$ into $C_0^{\nu_6}([0,T];X)$, $\nu_6 = (2\alpha\delta_6 + \alpha + \beta - 2)/\alpha$. But, since $\delta_6 > (2-\alpha-\beta)/\alpha$ implies $\nu_6 > \delta_6$, we *a fortiori* have the fact that Q_6 maps $C^{\delta_6}([0,T];X_1) \times C^{\delta_6}([0,T];X_2)$ into $C_0^{\delta_6}([0,T];X)$. Finally, (150) follows from (149) and $\|Q_6(g_{6_1}, g_{6_2})\|_{\delta_6,0,t;X} \leq \max\{1, t^{\nu_6-\delta_6}\}\|Q_6(g_{6_1}, g_{6_2})\|_{\nu_6,0,t;X}$. □

In Section 6 we will also encounter Q_5 acting on functions which enjoy some space regularity, that is, functions g_5 which are Hölder continuous in time with values on $Y^r_\gamma \in \{(X, \mathscr{D}(A))_{\gamma,r}, X_A^{\gamma,r}\}$. In this case Lemma 32 can be refined, and the loss of regularity produced by Q_5 is naturally restored by the additional condition of space regularity on g_5. In some sense, the forthcoming Corollary 38 is the analogous of Lemma 30, where the function $g_4 y$ involved in the definition of $Q_4(g_4, y)$ (cf. (108)) was of class $C^{\delta_4}([0,T];Y^r_\gamma)$.

Lemma 37. *Let $\alpha + \beta > 1$ in (H1) and $Y^r_\gamma \in \{(X, \mathscr{D}(A))_{\gamma,r}, X_A^{\gamma,r}\}$, $\gamma \in (2-\alpha-\beta, 1)$, $r \in [1,\infty]$. Then, for every $\delta_5 \in (0, (\alpha+\beta+\gamma-2)/\alpha]$, the operator Q_5 defined by* (109) *maps $C^{\delta_5}([0,T];Y^r_\gamma)$ into $C_0^{\delta_5}([0,T],X)$, and for every $t \in [0,T]$ satisfies the following estimate:*

$$\left\|Q_5 g_5\right\|_{\delta_5,0,t;X} \leq c_{41} t^{(\alpha+\beta+\gamma-2-\alpha\delta_5)/\alpha}\left(2t^{\delta_5}+1\right)\left\|g_5\right\|_{\delta_5,0,t;Y^r_\gamma}. \tag{151}$$

Here c_{41} is a positive constant depending on α, β, γ, and r.

Proof. Let $\gamma \in (2-\alpha-\beta, 1) \subseteq (1-\beta, 1)$ and let $\chi_{\alpha,\beta,\gamma}$ be the number $(\alpha+\beta+\gamma-2)/\alpha \in (0,1)$, so that the exponent $(\beta+\gamma-2)/\alpha$ in (79) with $\theta = 1$ is equal to $\chi_{\alpha,\beta,\gamma} - 1$. Let $g_5 \in C^{\delta_5}([0,T];Y^r_\gamma)$, $\delta_5 \in (0, \chi_{\alpha,\beta,\gamma}]$, $r \in [1,\infty]$. Since $[Q_5 g_5](0) = 0$, we assume that $t \in (0,T]$ and we observe that, due to Propositions 6 and 12, $[Q_5 g_5](t)$ is rewritten as follows:

$$\begin{aligned}\left[Q_5 g_5\right](t) &= \left[\mathrm{e}^{tA} - I\right] g_5(t) = \lim_{\varepsilon\to 0^+}\left[\mathrm{e}^{tA} - \mathrm{e}^{\varepsilon A}\right] g_5(t)\\&= \lim_{\varepsilon\to 0^+}\int_\varepsilon^t D_s \mathrm{e}^{sA} g_5(t)\,\mathrm{d}s\\&= -\lim_{\varepsilon\to 0^+}\int_\varepsilon^t \left[(-A)^1\right]^\circ \mathrm{e}^{sA} g_5(t)\,\mathrm{d}s\\&= -\int_0^t \left[(-A)^1\right]^\circ \mathrm{e}^{sA} g_5(t)\,\mathrm{d}s.\end{aligned} \tag{152}$$

Indeed, for every $\varepsilon \in [0,t)$ and $x \in Y^r_\gamma$, (79) with $\theta = 1$ yields

$$\begin{aligned}&\left\|\int_\varepsilon^t \left[(-A)^1\right]^\circ \mathrm{e}^{sA} x\,\mathrm{d}s\right\|_X\\&\quad\leq c_{22}\|x\|_{Y^r_\gamma}\int_\varepsilon^t s^{\chi_{\alpha,\beta,\gamma}-1}\mathrm{d}s \leq c_{41}\|x\|_{Y^r_\gamma}(t-\varepsilon)^{\chi_{\alpha,\beta,\gamma}},\end{aligned} \tag{153}$$

where $c_{41} = c_{22}\chi^{-1}_{\alpha,\beta,\gamma}$. From (152) and (153) with $(\varepsilon, t, x) = (0, \tau, g_5(\tau))$ we thus get

$$\begin{aligned}\left\|Q_5 g_5\right\|_{0,0,t;X} &= \sup_{\tau\in[0,t]}\left\|\left[Q_5 g_5\right](\tau)\right\|_X\\&\leq c_{41}\left\|g_5\right\|_{0,0,t;Y^r_\gamma} t^{\chi_{\alpha,\beta,\gamma}}.\end{aligned} \tag{154}$$

Now, let $0 \leq t_1 < t_2 \leq t$. From (152) it follows that $[Q_5 g_5](t_2) - [Q_5 g_5](t_1) = -\sum_{k=1}^2 V_{k;t_1,t_2,g_5}$, where for every function $g : [0,T] \to Y^p_\gamma$ we have set

$$\begin{aligned}V_{1;t_1,t_2,g} &:= \int_0^{t_1}\left[(-A)^1\right]^\circ \mathrm{e}^{sA}\left[g(t_2) - g(t_1)\right]\mathrm{d}s,\\V_{2;t_1,t_2,g} &:= \int_{t_1}^{t_2}\left[(-A)^1\right]^\circ \mathrm{e}^{sA} g(t_2)\,\mathrm{d}s.\end{aligned} \tag{155}$$

Hence, using (153) with the triplet (ε, t, x) being replaced by $(0, t_1, g_5(t_2) - g_5(t_1))$ and $(t_1, t_2, g_5(t_2))$, respectively, we deduce that

$$\begin{aligned}\left\|V_{1;t_1,t_2,g_5}\right\|_X &\leq c_{41}\left|g_5\right|_{\delta_5,0,t_2;Y^r_\gamma} t_1^{\chi_{\alpha,\beta,\gamma}}\left(t_2-t_1\right)^{\delta_5},\\\left\|V_{2;t_1,t_2,g_5}\right\|_X &\leq c_{41}\left\|g_5\right\|_{0,0,t_2;Y^r_\gamma}\left(t_2-t_1\right)^{\chi_{\alpha,\beta,\gamma}}.\end{aligned} \tag{156}$$

As a consequence, since $\delta_5 \in (0, \chi_{\alpha,\beta,\gamma}]$,

$$\begin{aligned}&\left\|\left[Q_5 g_5\right](t_2) - \left[Q_5 g_5\right](t_1)\right\|_X\\&\quad\leq c_{41} t^{\chi_{\alpha,\beta,\gamma}-\delta_5}\left(t^{\delta_5}+1\right)\left\|g_5\right\|_{\delta_5,0,t_2;Y^r_\gamma}\left(t_2-t_1\right)^{\delta_5}.\end{aligned} \tag{157}$$

Summing up (154) and (157), we obtain (151). The proof is complete. □

Since in Lemma 37 it is not required that $g_5(0) = 0$, the special case of the constant function $g_5(t) = x \in Y^p_\gamma$, $t \in [0,T]$, is admissible, and we obtain the following result.

Corollary 38. *Let α, β, and Y^r_γ be as in Lemma 37, and let $x \in Y^r_\gamma$, $\gamma \in (2-\alpha-\beta, 1)$, and $r \in [1,\infty]$. Then, for every $\delta_7 \in (0,(\alpha+\beta+\gamma-2)/\alpha]$, the function $[Q_7 x](\cdot) := (e^{\cdot A} - I)x$ belongs to $C^{\delta_7}_0([0,T];X)$, and for every $t \in [0,T]$ satisfies the estimate*

$$\|Q_7 x\|_{\delta_7,0,t;X} \le c_{41} t^{(\alpha+\beta+\gamma-2-\alpha\delta_7)/\alpha} \left(t^{\delta_7}+1\right) c_{41} \|x\|_{Y^r_\gamma}. \quad (158)$$

Proof. Let $g_5(t) = x$ in the proof of Lemma 37, and observe that V_{1,t_1,t_2,g_5} reduces to the zero element of X. Estimate (158) then follows from (154) and the second estimate in (156). □

For later purposes, we conclude the section with the following remark.

Remark 39. The condition $5\alpha + 2\beta > 6$ in (H1) required in Proposition 29 is the strongest among the conditions for the pair (α,β) required in Corollary 14 and the other results of this section. Indeed,

$$\begin{aligned} 5\alpha + 2\beta > 6 &\Longrightarrow 3\alpha + 2\beta > 6 - 2\alpha \ge 4 \\ &\Longrightarrow 3\alpha + \beta > 4 - \beta \ge 3 \\ &\Longrightarrow 2\alpha + \beta > 3 - \alpha \ge 2 \\ &\Longrightarrow \alpha + \beta > 2 - \alpha \ge 1. \end{aligned} \quad (159)$$

Hence, if $5\alpha + 2\beta > 6$, then Corollary 14 and all the results from Lemma 22 to Corollary 38 are applicable. Next we will make large usage of this fact, but we warn the reader that, for brevity and regarding it as acquired, we will not mention it anymore.

5. Application to Maximal Time Regularity

The results of the previous sections are here applied to correct, refine, and extend the results in [20] concerning the maximal time regularity of the solutions to a class of degenerate abstract evolution equations. Let $(X, \|\cdot\|_X)$ and $(Z, \|\cdot\|_Z)$ be two complex Banach spaces, and consider the following degenerate first-order integrodifferential Cauchy problem for $v : I_T \to X$, where $I_T = [0,T]$, $T > 0$, and $n_1, n_2 \in \mathbf{N}$:

$$\begin{aligned} D_t(Mv(t)) = {} & [\lambda_0 M + L] v(t) + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1} v\right)(t) \\ & + \sum_{i_2=1}^{n_2} h_{i_2}(t) y_{i_2} + f(t), \quad t \in I_T, \\ & Mv(0) = Mv_0. \end{aligned} \quad (160)$$

Here $\mathscr{K}$ is the convolution operator (104) in which $(X_1, X_2, X_3) = (Z, X, X)$, whereas M, L, and L_{i_1}, $i_1 = 1,\ldots,n_1$, are closed single-valued linear operators from X to itself, whose domains fulfill the relation $\mathscr{D}(L) \subseteq \bigcap_{i_1=1}^{n_1}[\mathscr{D}(M) \cap \mathscr{D}(L_{i_1})]$. Further, we assume that

$$L \text{ admits a continuous inverse operator } L^{-1} \in \mathscr{L}(X), \quad \text{i.e., } 0 \in \rho(L), \quad (161)$$

whereas we allow M to have *no* bounded inverse. Hence, in general, $A := LM^{-1}$ is only the m. l. operator defined by

$$\begin{aligned} \mathscr{D}(A) &= \left\{x \in \mathscr{D}\left(M^{-1}\right) : L\left(M^{-1}x\right) \ne \emptyset\right\} \\ &= \left\{x \in \mathscr{R}(M) : M^{-1}x \cap \mathscr{D}(L) \ne \emptyset\right\} \\ &= \{x \in \mathscr{R}(M) : \\ &\qquad \text{there exists } y \in \mathscr{D}(L) \text{ such that } y \in M^{-1}x\} \\ &= \left\{x \in \mathscr{R}(M) : x = My \text{ for some } y \in \mathscr{D}(L)\right\} \\ &= M(\mathscr{D}(L)), \\ Ax &= \bigcup_{y \in M^{-1}x \cap \mathscr{D}(L)} Ly \\ &= \left\{Ly : y \in \mathscr{D}(L) \text{ such that } x = My\right\}, \\ &\qquad x \in \mathscr{D}(A). \end{aligned} \quad (162)$$

Therefore, problem (160) can *not* be reduced, via the change of unknown $u = Mv$, to an integrodifferential problem related to single-valued linear operators. On the contrary, due to (161) and the closed graph theorem, $ML^{-1}, L_{i_1}L^{-1} \in \mathscr{L}(X)$, $i_1 = 1,\ldots,n_1$. As far as the data vector $(\lambda_0, v_0, k_1, \ldots, k_{n_1}, h_1, \ldots, h_{n_2}, y_1, \ldots, y_{n_2}, f)$ is concerned, at the moment, we only assume $\lambda_0 \in \mathbf{C}$, $v_0 \in \mathscr{D}(M)$, $k_{i_1} : I_T \to Z$, $h_{i_2} : I_T \to \mathbf{C}$, $y_{i_2} \in X$, $i_l = 1,\ldots,n_l$, $l = 1,2$, and $f : I_T \to X$, in order that (160) makes sense in X. This minimal assumptions will be refined later. In general, only *strict* solutions v to (160) shall be investigated, where (cf. [22, 23]) by a strict solution v to (160) we mean that, $\mathscr{D}(L)$ being endowed with the graph norm $\|\cdot\|_{\mathscr{D}(L)} = \|\cdot\|_X + \|L\cdot\|_X$, $v \in C(I_T;\mathscr{D}(L))$, $Mv \in C^1(I_T;X)$, and (160) holds. Clearly, if M^{-1} is really a m. l. operator, then $Mv(0) = Mv_0$ does not necessarily mean $v(0) = v_0$, but only $v(0) - v_0 \in M^{-1}0$. As we will see below, if $v_0 \in \mathscr{D}(L)$ and the data k_{i_1}, h_{i_2}, y_{i_2} and f, $i_l = 1,\ldots,n_l$, $l = 1,2$, satisfy suitable assumptions, then for a strict solution v to (160) it just holds $v(0) = v_0$. Throughout the section, Y^q_ψ, $\psi \in (0,1)$, $q \in [1,\infty]$, will always denote one between the spaces $(X,\mathscr{D}(A))_{\psi,q}$ and $X^{\psi,q}_A$, A being defined by (162). That is, $Y^q_\psi \in \{(X,\mathscr{D}(A))_{\psi,q}, X^{\psi,q}_A\}$. To avoid confusion, if more than a single Y^q_ψ is involved in some statement, that is, if we write $x_j \in Y^q_{\psi_j}$, $j = 1,\ldots,n$, $n \in \mathbf{N}$, then it is understood that the same choice has been made for all the $Y^q_{\psi_j}$ in the sense that $Y^q_{\psi_j} = (X,\mathscr{D}(A))_{\psi_j,q}$ or $Y^q_{\psi_j} = X^{\psi_j,q}_A$ for every $j = 1,\ldots,n$.

According to [2, Section 1.6], we recall that the M-modified resolvent set $\rho_M(L)$ of L is defined to be the set $\{z \in \mathbf{C} : (zM - L)^{-1} \in \mathscr{L}(X)\}$. The bounded operator $(zM - L)^{-1}$ is called the M modified resolvent of L. It is easy to prove that $\rho_M(L) \subseteq \rho(A)$ and that $M(zM - L)^{-1} = (zI - A)^{-1}$, $z \in \rho_M(L)$ (cf. [2, Theorem 1.14]). With the notion of M-modified resolvent of L at hand, we assume that

(H2) $\rho_M(L)$ contains a region $\Sigma_\alpha = \{z \in \mathbf{C} : \mathfrak{Re}\, z \ge -c(|\mathfrak{Im}\, z| + 1)^\alpha, \mathfrak{Im}\, z \in \mathbf{R}\}$, $\alpha \in (0,1]$, $c > 0$, and for some exponent $\beta \in (0, \alpha]$ and constant $C > 0$ the estimate $\|M(\lambda M - L)^{-1}\|_{\mathscr{L}(X)} \le C(|\lambda|+1)^{-\beta}$ holds for every $\lambda \in \Sigma_\alpha$.

Before we proceed with our analysis we remark that, due to the wide range of choices for the data vector, problem (160) contains many subcases at its interior. So, in spite of the case when at least one between the k_i's is different from zero and problem (160) is really an integrodifferential one, the choice $k_{i_1} = 0$, $i_1 = 1, \ldots, n_1$, yields to consider also various nonintegrodifferential degenerate problems. For instance, those corresponding to $\lambda_0 = k_{i_1} = h_{i_2} = 0$ and $\lambda_0 = k_{i_1} = f = 0$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, respectively:

$$D_t (Mv(t)) = Lv(t) + f(t), \quad t \in I_T, \qquad Mv(0) = Mv_0, \tag{163}$$

$$D_t (Mv(t)) = Lv(t) + \sum_{i_2=1}^{n_2} h_{i_2}(t) y_{i_2}, \quad t \in I_T, \qquad Mv(0) = Mv_0. \tag{164}$$

Although (164) differs from (163) only in the fact that f is replaced with $\sum_{i_2=1}^{n_2} h_{i_2}(t) y_{i_2}$; nevertheless a very different result is achieved when the y_{i_2}'s are assumed to belong to $Y^r_{\gamma_{i_2}}$, at least for opportunely chosen $\gamma_{i_2} \in (0,1)$, $i_2 = 1, \ldots, n_2$. As we will see (cf. Remark 51 and Theorem 56), in this situation the loss of time regularity for the pair $(Lv, D_t Mv)$ with respect to that of f, typical of the case $\beta < 1$ in (H2) (see [21, Theorem 9], [2, Theorem 3.26], and [22, Theorem 7.2]), can be restored in order that $(Lv, D_t Mv)$ possesses the maximal time regularity which is the minimal between the time regularities of the h_{i_2}'s. The same phenomenon is carried over into the integrodifferential case for the following problems, corresponding to $\lambda_0 = h_{i_2} = 0$, $i_2 = 1, \ldots, n_2$, and $\lambda_0 = f = 0$:

$$D_t (Mv(t)) = Lv(t) + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1} v\right)(t) + f(t), \qquad Mv(0) = Mv_0, \tag{165}$$

$$D_t (Mv(t)) = Lv(t) + \sum_{i_1=1}^{n_1} \mathscr{K}\left(k_{i_1}, L_{i_1} v\right)(t) + \sum_{i_2=1}^{n_2} h_{i_2}(t) y_{i_2}, \qquad Mv(0) = Mv_0, \tag{166}$$

$t \in I_T$. When $\beta < 1$, the loss of time regularity for the pair $(Lv, D_t Mv)$ with respect to that of the vector $(k_1, \ldots, k_{n_1}, f)$ in problem (165) (cf. [22, Theorem 7.1] and [23, Theorem 2.1] for $n_1 = 1$) can be restored in problem (166) assuming that $y_{i_2} \in Y^r_{\gamma_{i_2}}$, $i_2 = 1, \ldots, n_2$. In this context (cf. Remark 51 and Theorem 53) the pair $(Lv, D_t Mv)$ has the maximal time regularity which is the minimal between the time regularities of the k_{i_1}'s and h_{i_2}'s.

We stress that, if $\beta = 1$, then no loss of time regularity is observed and all the quoted results agree with the well-known theory of maximal regularity in spaces of continuous functions for the nondegenerate version of (160), corresponding to the case when $M = I$ and L generates an analytic semigroup. Hence, roughly speaking, one can verify the consistency of any result on problem (160) with condition (H2) simply by letting $\beta = 1$ on its statement, and then checking if it is compatible with those for the nondegenerate case. To this purpose, we recall that the question of maximal regularity for the nondegenerate (possibly nonautonomous) version of (160) has been deeply investigated by several authors. See, for instance, [4, 6–8, 10, 32] for problem (165) with $(M, \beta, n_1) = (I, 1, 1)$ and [9, 11] for problem (163) with $(M, \beta) = (I, 1)$.

Finally, assumption (161) excludes the case of $L = 0$ in (160), so that our results cannot be compared with those in [5, 33, 34]. There, assuming that the bilinear bounded operator $\mathscr{P}$ underlying the definition of $\mathscr{K}$ is the scalar multiplication in X, the problem

$$D_t v(t) = \mathscr{K}\left(k_1, L_1 v\right)(t) + f(t), \quad t \in I_T, \ v(0) = v_0 \tag{167}$$

is treated under the following assumptions: (i) L_1 is a closed densely defined linear operator generating an analytic semigroup; (ii) $k_1 : [0, \infty) \to \mathbf{R}$ is absolutely Laplace transformable. Observe that, if $(k_1, f) = (1, 0)$, then problem (167) reduces to the abstact wave equation $D_t^2 v(t) = L_1 v(t)$, $D_t v(0) = 0$, $v(0) = v_0$, whereas when $M = I$ and $\lambda_0 = k_{i_1} = h_{i_2} = f = 0$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, problem (160) reduces to the abstract heat equation $D_t v(t) = Lv(t)$, $v(0) = v_0$. In other words, whereas [5, 33, 34] are concerned with the hyperbolic case, here we are concerned with the *parabolic* one.

Let us now come back to problem (160). Of course, assumption (H2) implies that the operator A defined by (162) satisfies (H1), so that it generates a semigroup $\{e^{tA}\}_{t \ge 0}$ defined by $e^{0A} = I$ and (9) and satisfying (14). Assuming that $v_0 \in \mathscr{D}(L)$, we let

$$w = L\left(v - v_0\right) \Longleftrightarrow v = L^{-1} w + v_0. \tag{168}$$

Then, by setting

$$\begin{aligned} F_w(t) = {} & \lambda_0 A^{-1} w(t) \\ & + \sum_{i_1=1}^{n_1} \left[\mathscr{K}\left(k_{i_1}, S_{i_1} w\right)(t) + \mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right)(t)\right] \\ & + \sum_{i_2=1}^{n_2} h_{i_2}(t) y_{i_2} + v_1 + f(t), \quad t \in I_T, \end{aligned} \tag{169}$$

where $A^{-1} = ML^{-1} \in \mathscr{L}(X)$, $S_{i_1} = L_{i_1} L^{-1} \in \mathscr{L}(X)$, $i_1 = 1, \ldots, n_1$, and $v_1 = (\lambda_0 M + L) v_0$, we see that v is a strict solution to (160) if and only if w satisfies (indeed, if $v \in C(I_T; \mathscr{D}(L))$, then $\|w(t) - w(s)\|_X = \|L[v(t) - v(s)]\|_X \le \|v(t) - v(s)\|_{\mathscr{D}(L)} \to 0$ as $s \to t$, $t, s \in I_T$, that is, $w \in C(I_T; X)$. Conversely, if $w \in C(I_T; X)$, then $v = L^{-1} w + v_0 \in \mathscr{D}(L)$ and $\|v(t) - v(s)\|_{\mathscr{D}(L)} \le (\|L^{-1}\|_{\mathscr{L}(X)} + 1)\|w(t) - w(s)\|_X \to 0$ as $s \to t$, $t, s \in I_T$, that is, $v \in C(I_T; \mathscr{D}(L))$. Finally, since $Mv = A^{-1} w + Mv_0$, we have $Mv \in C^1(I_T; X)$

if and only if $A^{-1}w \in C^1(I_T;X))$ $w \in C(I_T;X)$, $A^{-1}w \in C^1(I_T;X)$, and solves to the following problem:

$$D_t\left(A^{-1}w(t)\right) = w(t) + F_w(t) \in A\left(A^{-1}w(t)\right) + F_w(t), \quad t \in I_T,$$

$$A^{-1}w(0) = 0 \quad (\text{i.e., } w(0) \in A0). \tag{170}$$

Now let $2\alpha + \beta > 2$, and assume that $k_{i_1} \in C^{\eta_{i_1}}(I_T;Z)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T;\mathbf{C})$, and $f \in C^{\mu}(I_T;X)$, where $\eta_{i_1}, \sigma_{i_2}, \mu \in (2-\alpha-\beta/\alpha, 1)$, $i_l = 1,\ldots,n_l$, $l = 1,2$. Then, if $w \in C(I_T;X)$ is a solution to (170) such that $A^{-1}w \in C^1(I_T;X)$, the function F_w satisfies

$$F_w \in C^{\delta}(I_T;X), \quad \delta = \min_{i_k=1,\ldots,n_k,\, k=1,2}\left\{\eta_{i_1}, \sigma_{i_2}, \mu\right\} \in \left(\frac{2-\alpha-\beta}{\alpha}, 1\right). \tag{171}$$

Indeed, δ being the smallest Hölder exponent, for every $i_l = 1,\ldots,n_l$, $l = 1,2$, we have $A^{-1}w, h_{i_2}y_{i_2}, f \in C^{\delta}(I_T;X)$ and $\mathscr{K}(k_{i_1}, S_{i_1}w)$, $\mathscr{K}(k_{i_1}, L_{i_1}v_0) \in C_0^{\eta_{i_1}}(I_T;X) \hookrightarrow C_0^{\delta}(I_T;X)$ (cf. Lemma 27 for the case $(\delta_{3_1}, \delta_{3_2}, X_1, X_2) = (\eta_{i_1}, 0, Z, X)$ with the pair (g_{3_1}, g_{3_2}) being replaced by (in fact, since $S_{i_1} = L_{i_1}L^{-1} \in \mathscr{L}(X)$, $i_1 = 1,\ldots,n_1$, if $w \in C(I_T;X)$, then $S_{i_1}w \in C(I_T;X)$, whereas the constant functions $\kappa_{i_1}(t) = L_{i_1}v_0$, $t \in I_T$, $i_1 = 1,\ldots,n_1$, obviously belong to $C(I_T;X)$) $(k_{i_1}, S_{i_1}w)$ and $(k_{i_1}, L_{i_1}v_0)$, resp.). Consequently (cf. [2, Theorem 3.7 and Remark p. 54] with $u_0 = 0$), the solution $A^{-1}w$ to the multivalued evolution problem $D_t(A^{-1}w) \in A(A^{-1}w) + F_w$, $A^{-1}w(0) = 0$ is necessarily of the form

$$A^{-1}w(t) = \left[Q_1F_w\right](t), \quad t \in I_T, \tag{172}$$

with Q_1 being the operator defined by (105). Further (cf. [2, Remark p. 55] with $u_0 = 0$, and where $A^{\circ}e^{tA}$ stands for $D_te^{tA} = -[(-A)^1]^{\circ}e^{tA}$) the derivative of $A^{-1}w$ is given by

$$D_t\left(A^{-1}w(t)\right) = e^{tA}F_w(t) - \left[Q_2F_w\right](t), \quad t \in I_T \setminus \{0\}, \tag{173}$$

with Q_2 being the operator in (106). Notice that Q_2F_w is well defined by virtue of (127) with $g = F_w$. Now let $y_{i_2} \in Y^r_{\gamma_{i_2}}$ and $v_1 + f(0) \in Y^r_{\varphi}$ where $\gamma_{i_2}, \varphi \in (1-\beta, 1)$, $i_2 = 1,\ldots,n_2$, and $r \in [1,\infty]$. Since $A^{-1}w(0) = \mathscr{K}(k_{i_1}, S_{i_1}w)(0) = \mathscr{K}(k_{i_1}, L_{i_1}v_0)(0) = 0$, $i_1 = 1,\ldots,n_1$, from (169) it thus follows that $F_w(0) := x_0$ is independent on w and

$$x_0 = \sum_{i_2=1}^{n_2} h_{i_2}(0)\, y_{i_2} + v_1 + f(0) \in Y^r_{\gamma}, \quad \gamma = \min_{i_2=1,\ldots,n_2}\left\{\gamma_{i_2}, \varphi\right\} \in (1-\beta, 1). \tag{174}$$

Indeed (cf. (20) or (38)), we have $Y^r_{\gamma_{i_2}} \hookrightarrow Y^r_{\gamma}$, $i_2 = 1,\ldots,n_2$, and $Y^r_{\varphi} \hookrightarrow Y^r_{\gamma}$, the embeddings being equalities for those between the numbers $\gamma_1,\ldots,\gamma_{n_2}$ and φ which are equal to γ. Then, under these assumptions on the data, formula (173) for $D_t(A^{-1}w(t))$ can be extended until $t = 0$. For, we have $\lim_{t\to 0^+} D_t(A^{-1}w(t)) = x_0 \in A0 + x_0$ and the differential equation in (170) is satisfied even at $t = 0$. To see this, we observe that

$$\left\|D_t\left(A^{-1}w(t)\right) - x_0\right\|_X \le I_1(t) + I_{2,w}(t) + I_{3,w}(t), \quad t \in I_T \setminus \{0\}, \tag{175}$$

where $I_1(t) = \|(e^{tA} - I)x_0\|_X$, $I_{2,w}(t) = \|e^{tA}[F_w(t) - x_0]\|_X$, and $I_{3,w}(t) = \|[Q_2F_w](t)\|_X$. First, from Proposition 6 we get $\lim_{t\to 0^+} I_1(t) = 0$. On the other side, using $F_w \in C^{\delta}(I_T;X)$, $\delta \in ((2-\alpha-\beta)/\alpha, 1) \subseteq ((1-\beta)/\alpha, 1)$, we obtain

$$I_{2,w}(t) \le \widetilde{c}_{\alpha,\beta,0}\left\|F_w\right\|_{\delta,0,t;X} t^{(\alpha\delta+\beta-1)/\alpha}, \quad t \in I_T \setminus \{0\}, \tag{176}$$

so that $\lim_{t\to 0^+} I_{2,w}(t) = 0$. Finally, (127) with $g = F_w$ yields $\lim_{t\to 0^+} I_{3,w}(t) = 0$, too. Formula (173) thus holds at $t = 0$ with $D_t(A^{-1}w(0)) = \lim_{t\to 0^+} D_t(A^{-1}w(t)) = x_0$.

Remark 40. In [2, Remark p. 55], formula (173) was extended until $t = 0$ only under the more restrictive assumption $x_0 \in X_A^{\gamma,\infty}$, $\gamma \in (1-\beta, 1)$. Indeed [24, Proposition 5.2] was not available at the time of [2] and only the strong continuity of $\{e^{tA}\}_{t\ge 0}$ in the X-norm on the spaces $X_A^{\gamma,\infty}$, $\gamma \in (1-\beta, 1)$, was known (cf. [2, Theorem 3.3]). Notice that in the case of problem (163) the element x_0 reduces to $Lv_0 + f(0)$, so that in the nondegenerate case $(M,\beta) = (I,1)$ we get back the classical assumption $Lv_0 + f(0) \in (X, \mathscr{D}(L))_{\gamma,r}$, $\gamma \in (0,1)$, $r \in [1,\infty]$ (see, for instance, [9, Theorem 4.3.1(iii)] and [11, Theorem 4.5]).

Since (170) implies that $w(t) = D_t(A^{-1}w(t)) - F_w(t)$, from (173) we thus find that

$$w(t) = \left[Q_7x_0\right](t) + \left(e^{tA} - I\right)\left[F_w(t) - x_0\right] - \left[Q_2F_w\right](t), \quad t \in I_T, \tag{177}$$

where, according to the notation in Corollary 38, we have set $[Q_7x_0](t) = (e^{tA} - I)x_0$. In particular, $w(0) = 0$. We conclude that, under the previous assumptions on the pair (α,β) and on the data vector $(k_1,\ldots,k_{n_1}, h_1,\ldots,h_{n_2}, y_1,\ldots,y_{n_2}, f, v_1)$, if $w \in C(I_T;X)$ solves (170), then necessarily $w \in C_0(I_T;X)$. As a consequence (cf. (168)), the strict solution v to (160) satisfies the initial condition just in the sense $v(0) = v_0$.

Introduce the functions $\widetilde{f} : I_T \to X$ and $\widetilde{h}_{i_2} : I_T \to Y^r_{\gamma_{i_2}}$, $i_2 = 1,\ldots,n_2$, defined by

$$\widetilde{f}(t) = f(t) - f(0), \qquad \widetilde{h}_{i_2}(t) = \left[h_{i_2}(t) - h_{i_2}(0)\right] y_{i_2}, \quad t \in I_T. \tag{178}$$

Then, replacing F_w with the right-hand side of (169), using (174), and recalling the definitions of the operators Q_j,

$j = 2, \ldots, 6$, in (106)–(110), from (177) we deduce that $w \in C_0(I_T; X)$ solves the fixed-point equation

$$w = w_0 + w_1 + Rw, \tag{179}$$

the functions w_l, $l = 0, 1$, and the operator Rw being defined by

$$w_0 := Q_7 x_0 + \sum_{i_1=1}^{n_1} Q_6\left(k_{i_1}, L_{i_1} v_0\right) + \sum_{i_2=1}^{n_2} Q_5 \widetilde{h}_{i_2} + Q_5 \widetilde{f}, \tag{180}$$

$$w_1 := -\sum_{i_1=1}^{n_1} Q_3\left(k_{i_1}, L_{i_1} v_0\right) - \sum_{i_2=1}^{n_2} Q_4\left(h_{i_2}, y_{i_2}\right) - Q_2 f, \tag{181}$$

$$\begin{aligned} Rw := {} & \lambda_0\left[Q_5\left(A^{-1}w\right) - Q_2\left(A^{-1}w\right)\right] \\ & + \sum_{i_1=1}^{n_1}\left[Q_6\left(k_{i_1}, S_{i_1} w\right) - Q_3\left(k_{i_1}, S_{i_1} w\right)\right]. \end{aligned} \tag{182}$$

Conversely, let $w \in C_0(I_T; X)$ be a solution to the fixed-point equation (179), and assume that the pair (α, β) and the data vector $(k_1, \ldots, k_{n_1}, h_1, \ldots, h_{n_2}, y_1, \ldots, y_{n_2}, f, v_1)$ satisfy the assumptions below (170) and (173). Then, as before, $\mathscr{K}(k_{i_1}, S_{i_1} w), \mathscr{K}(k_{i_1}, L_{i_1} v_0) \in C_0^{\delta}(I_T; X)$ and $h_{i_2} y_{i_2}, f \in C^{\delta}(I_T; X)$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, $\delta \in ((2-\alpha-\beta)/\alpha, 1)$ being as in (171). We apply $A^{-1} \in \mathscr{L}(X)$ to both sides of (179), and we show that $A^{-1}w$ satisfies (172) with $F_w \in C(I_T; X)$ as in (169), so that $A^{-1}w$ is a solution to problem (170). To this purpose, we take into account Corollaries 14 and 26. Let $t \in I_T$. First (cf. Remark 34 and recall that $Q_6(\cdot,\cdot) = Q_5\mathscr{K}(\cdot,\cdot)$), using (86), (174), and (178), we get

$$\begin{aligned} A^{-1}w_0(t) & = \int_0^t e^{(t-s)A}\left[x_0 + \sum_{i_1=1}^{n_1}\mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right)(t) + \sum_{i_2=1}^{n_2}\widetilde{h}_{i_2}(t) + \widetilde{f}(t)\right] ds \\ & = \int_0^t e^{(t-s)A}\left[v_1 + \sum_{i_1=1}^{n_1}\mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right)(t) + \sum_{i_2=1}^{n_2} h_{i_2}(t)\, y_{i_2} + f(t)\right] ds. \end{aligned} \tag{183}$$

Instead, due to the definition of Q_3 and Q_4, using (125) we obtain

$$\begin{aligned} A^{-1}w_1(t) = \int_0^t e^{(t-s)A} \Bigg\{ & \sum_{i_1=1}^{n_1}\left[\mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right)(s) - \mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right)(t)\right] \\ & + \sum_{i_2=1}^{n_2}\left[h_{i_2}(s) - h_{i_2}(t)\right] y_{i_2} + f(s) - f(t)\Bigg\} ds. \end{aligned} \tag{184}$$

Therefore, from (183), (184), and the definition (105) of Q_1 it follows that

$$A^{-1}\left[w_0 + w_1\right](t) = \left[Q_1\left(v_1 + \sum_{i_1=1}^{n_1}\mathscr{K}\left(k_{i_1}, L_{i_1} v_0\right) + \sum_{i_2=1}^{n_2} h_{i_2} y_{i_2} + f\right)\right](t), \tag{185}$$

the left-hand side being well-defined due to Remark 23. As far as $A^{-1}[Rw](t)$ is concerned, we first observe that, w being in $C(I_T; X)$, from formula (126) and Remark 34 it follows that $[Q_2(A^{-1}w)](t)$ and $[Q_5(A^{-1}w)](t)$ are both well defined and equal to $-\int_0^t e^{(t-s)A}[w(s) - w(t)]ds$ and $\int_0^t e^{(t-s)A} w(t)\, ds$, respectively. Consequently

$$\left[Q_5\left(A^{-1}w\right) - Q_2\left(A^{-1}w\right)\right](t) = \int_0^t e^{(t-s)A} w(s)\, ds = \left[Q_1 w\right](t). \tag{186}$$

Hence, commuting $A^{-1} \in \mathscr{L}(X)$ with both the integral sign and the semigroup, one has

$$A^{-1}\left[Q_5\left(A^{-1}w\right) - Q_2\left(A^{-1}w\right)\right](t) = \left[Q_1\left(A^{-1}w\right)\right](t). \tag{187}$$

Similarly, since Remark 34 and formula (125) yield

$$\begin{aligned} A^{-1}\left[Q_6\left(k_{i_1}, S_{i_1} w\right)\right](t) & = \int_0^t e^{(t-s)A}\mathscr{K}\left(k_{i_1}, S_{i_1} w\right)(t)\, ds, \\ A^{-1}\left[Q_3\left(k_{i_1}, S_{i_1} w\right)\right](t) & = -\int_0^t e^{(t-s)A}\left[\mathscr{K}\left(k_{i_1}, S_{i_1} w\right)(s) - \mathscr{K}\left(k_{i_1}, S_{i_1} w\right)(t)\right] ds, \end{aligned} \tag{188}$$

we find that

$$A^{-1}\left[Q_6\left(k_{i_1}, S_{i_1} w\right) - Q_3\left(k_{i_1}, S_{i_1} w\right)\right](t) = \left[Q_1\mathscr{K}\left(k_{i_1}, S_{i_1} w\right)\right](t), \tag{189}$$

$i_1 = 1, \ldots, n_1$. In conclusion, from (187) and (189) it follows that

$$A^{-1}[Rw](t) = \left[Q_1\left(\lambda_0 A^{-1}w + \sum_{i_1=1}^{n_1}\mathscr{K}\left(k_{i_1}, S_{i_1} w\right)\right)\right](t). \tag{190}$$

Summing up (185) and (190), we finally obtain $A^{-1}w(t) = [Q_1 F_w](t)$, F_w being as in (169). This completes the proof of the equivalence between problem (170) and the fixed point equation (179), provided that the data satisfy the mentioned assumptions.

Remark 41. We can summarize the previous reasonings as follows: problem (160) has been reduced to the fixed-point equation (179) for the new unknow $w = L(v - v_0)$, $v_0 \in \mathscr{D}(L)$. This fixed-point argument is similar to that first successfully applied in [4, 7, 8, 32] to problem (165) with $(M, \beta, n_1) = (I, 1, 1)$ and then generalized in [23] to the degenerate case. A different approach has been followed in [6, 10] for the nondegenerate case and in [22] for the degenerate one. There, assuming that k_1 is absolutely Laplace transformable (cf. [6, 22]) or of bounded variation (cf. [10]), problem (165) with $n_1 = 1$ is solved by constructing its relative resolvent operator. We quote also [35] where the method of constructing the fundamental solution for the equation without the integral term is applied to a class of *concrete* degenerate integrodifferential equations.

From now on, for $5\alpha + 2\beta > 6$, $\beta \in (0, \alpha]$, $\alpha \in (0, 1]$, and $\nu \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $I_{\alpha,\beta,\nu} \subseteq ((3 - 2\alpha - \beta)/\alpha, 1/2) \subseteq (0, 1/2)$ will denote the interval defined by

$$I_{\alpha,\beta,\nu} = \begin{cases} \left(\dfrac{3-2\alpha-\beta}{\alpha}, \nu\right], & \text{if } \nu \in \left(\dfrac{3-2\alpha-\beta}{\alpha}, \dfrac{1}{2}\right), \\ \left(\dfrac{3-2\alpha-\beta}{\alpha}, \dfrac{1}{2}\right), & \text{if } \nu \in \left[\dfrac{1}{2}, 1\right). \end{cases} \quad (191)$$

Clearly, if $\nu, \rho \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $\nu \le \rho$, then $I_{\alpha,\beta,\nu} \subseteq I_{\alpha,\beta,\rho}$.

Lemma 42. *Assume* (161), *and let* $5\alpha+2\beta > 6$ *in (H2). Assume that* $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$, $\eta_{i_1} \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $i_1 = 1, \dots, n_1$, *and let* $\eta = \min_{i_1=1,\dots,n_1} \eta_{i_1}$. *Then, for every fixed* $\delta \in I_{\alpha,\beta,\eta}$, *the operator R defined by* (182) *maps continuously* $C^\delta(I_T; X)$ *into* $C_0^\delta(I_T; X)$, *and for every* $t \in I_T$ *satisfies the following estimate, where* $p \in (1/(1 - 2\delta), \infty)$:

$$\|Rw\|_{\delta,0,t;X} \le c_{42}(T) \left(\int_0^t \|w\|_{\delta,0,s;X}^p \mathrm{d}s \right)^{1/p}, \quad w \in C^\delta(I_T; X). \quad (192)$$

Here $c_{42}(T)$ *is a positive constant depending only on* T, λ_0, α, β, η_{i_1}, δ, p, $\|k_{i_1}\|_{\eta_{i_1},0,T;Z}$ *and* $\|S_{i_1}\|_{\mathscr{L}(X)}$, $i_1 = 1, \dots, n_1$.

Proof. Let $k_{i_1} \in C^{\eta_i}(I_T; Z)$, $\eta_{i_1} \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $i_1 = 1, \dots, n_1$, and let us fix an arbitrary number $\delta \in I_{\alpha,\beta,\eta}$, where $\eta = \min_{i_1=1,\dots,n_1} \eta_i$. In particular, since $\delta \le \eta \le \eta_i$, we have $k_{i_1} \in C^\delta(I_T; Z)$ with $\|k_{i_1}\|_{\delta,0,t;Z} \le \max\{1, t^{\eta_{i_1}-\delta}\}\|k_{i_1}\|_{\eta_{i_1},0,t;Z}$, $i_1 = 1, \dots, n_1$. Now let $w \in C^\delta(I_T; X)$ and $t \in I_T$. First, formula (186) being applicable, we rewrite (182) as

$$Rw = \lambda_0 Q_1 w + \sum_{i_1=1}^{n_1} \left[Q_6 \left(k_{i_1}, S_{i_1} w\right) - Q_3 \left(k_i, S_i w\right) \right]. \quad (193)$$

Now, we notice that $5\alpha + 2\beta > 6$ implies that

$$\frac{\alpha + \beta - 1}{\alpha} = \frac{5\alpha + 2\beta - 3\alpha - 2}{2\alpha} > \frac{4 - 3\alpha}{2\alpha} \ge \frac{1}{2}. \quad (194)$$

Since $(1 - \beta)/\alpha \le (2 - \alpha - \beta)/\alpha \le (3 - 2\alpha - \beta)/\alpha$, from (194) it follows that $\delta \in I_{\alpha,\beta,\eta} \subseteq ((3 - 2\alpha - \beta)/\alpha, 1/2) \subseteq ((2 - \alpha - \beta)/\alpha, 1/2) \subsetneq ((1 - \beta)/\alpha, (\alpha + \beta - 1)/\alpha)$, and, consequently,

$$\frac{\alpha}{\alpha + \beta - 1 - \alpha\delta} < \frac{1}{1 - 2\delta}. \quad (195)$$

We conclude (cf. Remark 39) that Lemma 22 and Propositions 29 and 36 are applicable with $\delta \in I_{\alpha,\beta,\eta}$ and $p \in (1/(1 - 2\delta), \infty)$. Then, using estimates (111), (135), and (150) with the pair (g_1, δ_1) and the quintuplets $(g_{l_1}, g_{l_2}, \delta_l, X_1, X_2)$, $l = 3, 6$, being replaced, respectively, by (w, δ) and (indeed, since $S_{i_1} = L_{i_1} L^{-1} \in \mathscr{L}(X)$, if $w \in C^\delta(I_T; X)$, then $S_{i_1} w \in C^\delta(I_T; X)$ with $\|S_{i_1} w\|_{\delta,0,t;X} \le \|S_{i_1}\|_{\mathscr{L}(X)} \|w\|_{\delta,0,t;X}$, $i_1 = 1, \dots, n_1$) $(k_{i_1}, S_{i_1} w, \delta, Z, X)$, $i_1 = 1, \dots, n_1$, from (193) we finally obtain

$$\begin{aligned} \|Rw\|_{\delta,0,t;X} &\le \left\|\lambda_0 Q_1 w\right\|_{\delta,0,t;X} \\ &\quad + \sum_{l=3,6,\, i_1=1,\dots,n_1} \left\| Q_l \left(k_{i_1}, S_{i_1} w\right) \right\|_{\delta,0,t;X} \\ &\le c_{42}(T) \left(\int_0^t \|w\|_{\delta,0,s;X}^p \mathrm{d}s \right)^{1/p}. \end{aligned} \quad (196)$$

Here we have set $c_{42}(T) = |\lambda_0| C_1(T) + T^{1-2\delta-1/p} \sum_{l=3,6,\, i_1=1,\dots,n_1} C_l(T) \|k_{i_1}\|_{\delta,0,T;Z} \|S_{i_1}\|_{\mathscr{L}(X)}$, where $C_l(T)$, $l = 1, 3, 6$, are the values at $t = T$ of the functions $C_l(t)$ in Lemma 22 and Propositions 29 and 36. This completes the proof. □

Remark 43. Assume that in Lemma 42 the Hölder exponents $\eta_{i_1} \in ((3-2\alpha-\beta)/\alpha, 1)$ are such that $\eta = \min_{i_1=1,\dots,n_1} \eta_i$ belongs to $((3 - 2\alpha - \beta)/\alpha, 1/2)$. In this case (cf. (191)), the choice $\delta = \eta$ is admissible, and the meaning of Lemma 42 is that the operator R defined by (182) preserves the minimal of the time regularities of $k_1, \dots, k_{n_1}$.

Corollary 44. *Let the assumptions of Lemma 42 be satisfied, and let* η *and* R *be as there. Then, for every fixed* $\delta \in I_{\alpha,\beta,\eta}$, *the sequence* $\{R^n\}_{n=0}^\infty$ ($R^0 = I$, $R^n = RR^{n-1}$, $n \in \mathbf{N}$) *satisfies the following estimates, where* $w \in C^\delta(I_T; X)$ *and* $p \in (1/(1 - 2\delta), \infty)$:

$$\left\|R^n w\right\|_{\delta,0,t;X} \le \left[c_{42}(T)\right]^n \left(\frac{t^n}{n!}\right)^{1/p} \|w\|_{\delta,0,T;X}, \quad t \in I_T, \quad n \in \mathbf{N} \cup \{0\}. \quad (197)$$

Proof. Reasoning as in [23, p. 468], we prove (197) by induction. Since for every fixed $\delta \in I_{\alpha,\beta,\eta}$ the operator R maps $C^\delta(I_T; X)$ in $C_0^\delta(I_T; X)$, replacing w with $R^n w$ in (192) and introducing the sequence of scalar nonnegative nondecreasing functions $\{\varphi_n\}_{n=0}^\infty$ defined by $\varphi_n(t) = \|R^n w\|_{\delta,0,t;X}$, $t \in I_T$, from (192) we obtain

$$\varphi_{n+1}(t) \le c_{42}(T) \left(\int_0^t \left|\varphi_n(s)\right|^p \mathrm{d}s \right)^{1/p}, \quad t \in I_T, \quad n \in \mathbf{N} \cup \{0\}. \quad (198)$$

Then, applying to (198) an induction argument in which the first step of the induction follows from (192), we immediately deduce the following estimates:

$$\varphi_n(t) \le [c_{42}(T)]^n \left(\frac{t^n}{n!}\right)^{1/p} \|w\|_{\delta,0,T;X}, \qquad t \in I_T, \quad n \in \mathbf{N} \cup \{0\}. \tag{199}$$

The proof is complete. □

Lemma 45. *Let $5\alpha + 2\beta > 6$ in (H2) and $v_0 \in \bigcap_{i_1=1}^{n_1} \mathscr{D}(L_{i_1})$. Assume that $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$, and $y_{i_2} \in Y^r_{\gamma_{i_2}}$, where $\eta_{i_1}, \sigma_{i_2} \in ((3-2\alpha-\beta)/\alpha, 1)$, $\gamma_{i_2} \in (3-2\alpha-\beta, 1)$, $i_l = 1,\dots,n_l$, $l = 1,2$, and $r \in [1,\infty]$. Let $\tau_1 = \min_{i_l=1,\dots,n_l,\, l=1,2}\{\eta_{i_1}, \sigma_{i_2}\}$. Then, for every fixed $\delta \in I_{\alpha,\beta,\tau_1}$, the function w_1 defined by* (181) *belongs to $C^\delta_0(I_T; X)$, provided that $f \in C^\mu(I_T; X)$, $\mu \in [\delta + \mu_{\alpha,\beta}, 1)$, $\mu_{\alpha,\beta} = (3-2\alpha-\beta)/\alpha$.*

Proof. Let us fix $\delta \in I_{\alpha,\beta,\tau_1}$, $\tau_1 = \min_{i_l=1,\dots,n_l,\, l=1,2}\{\eta_{i_1}, \sigma_{i_2}\}$. Of course, $k_{i_1} \in C^\delta(I_T; Z)$ and $h_{i_2} \in C^\delta(I_T; \mathbf{C})$, $i_l = 1,\dots,n_l$, $l = 1,2$. Then, Proposition 29 and Lemma 30 applied with the quintuplets $(g_{3_1}, g_{3_2}, \delta_3, X_1, X_2)$ and the quadruplet $(g_4, y, \delta_4, \gamma)$ being replaced, respectively, by (the constant functions $\kappa_{i_1}(t) = L_{i_1} v_0$, $t \in I_T$, $i = 1,\dots,n_1$, being obviously of class $C^\delta(I_T; X)$) $(k_{i_1}, L_{i_1} v_0, \delta, Z, X)$ and $(h_{i_2}, y_{i_2}, \delta, \gamma_{i_2})$, imply that $Q_3(k_{i_1}, L_{i_1} v_0), Q_4(h_{i_2}, y_{i_2}) \in C^\delta_0(I_T; X)$, $i_l = 1,\dots,n_l$, $l = 1,2$. Now, since $\delta \in I_{\alpha,\beta,\tau_1} \subseteq ((3-2\alpha-\beta)/\alpha, 1/2) \subseteq (0,1/2)$, the number $\delta + \mu_{\alpha,\beta}$ satisfies

$$\frac{3-2\alpha-\beta}{\alpha} < \delta \le \delta + \mu_{\alpha,\beta} < \frac{6-3\alpha-2\beta}{2\alpha} < 1, \tag{200}$$

and assumption $f \in C^\mu(I_T; X)$, $\mu \in [\delta + \mu_{\alpha,\beta}, 1)$, is meaningful. Lemma 24 with $(g_2, \delta_2) = (f, \mu)$ then yields $Q_2 f \in C_0^{\nu_{\alpha,\beta,\mu}}(I_T; X)$, $\nu_{\alpha,\beta,\mu} = (\alpha\mu + 2\alpha + \beta - 3)/\alpha$. Since $\nu_{\alpha,\beta,\mu} \ge \nu_{\alpha,\beta,\delta+\mu_{\alpha,\beta}} = \delta$, we get $Q_2 f \in C^\delta_0(I_T; X)$, too. Summing up, we get the assertion. □

Before considering the function w_0 in (180), we introduce the following notation. In the sequel, for $3\alpha + 2\beta > 4$, $\beta \in (0,\alpha]$, $\alpha \in (0,1]$, and $\nu \in ((2-\alpha-\beta)/\alpha, 1)$, $J_{\alpha,\beta,\nu} \subseteq ((2-\alpha-\beta)/\alpha, 1/2) \subseteq (0,1/2)$ will denote the interval

$$J_{\alpha,\beta,\nu} = \begin{cases} \left(\dfrac{2-\alpha-\beta}{\alpha}, \nu\right], & \text{if } \nu \in \left(\dfrac{2-\alpha-\beta}{\alpha}, \dfrac{1}{2}\right), \\ \left(\dfrac{2-\alpha-\beta}{\alpha}, \dfrac{1}{2}\right), & \text{if } \nu \in \left[\dfrac{1}{2}, 1\right). \end{cases} \tag{201}$$

Notice that, since $(2-\alpha-\beta)/\alpha \le (3-2\alpha-\beta)/\alpha$, if the stronger condition $5\alpha + 2\beta > 6$ is satisfied, then (191) and (201) yield $I_{\alpha,\beta,\nu} \subseteq J_{\alpha,\beta,\nu}$ for every fixed $\nu \in ((3-2\alpha-\beta)/\alpha, 1)$. The introduction of the intervals $J_{\alpha,\beta,\nu}$ is justified by Lemma 46, which requires a weaker condition on the pair (α,β) than the one in Lemmas 42 and 45.

Lemma 46. *Let $3\alpha + 2\beta > 4$ in (H2), and let $v_0 \in \mathscr{D}(L)$. Assume that $k_{i_1} \in C^{\eta_{i_1}}(I_T; X)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$, $y_{i_2} \in Y^r_{\gamma_{i_2}}$, and $v_1 + f(0) \in Y^r_\varphi$, where $\eta_{i_1}, \sigma_{i_2} \in ((2-\alpha-\beta)/\alpha, 1)$, $\gamma_{i_2}, \varphi \in (4-2\alpha-2\beta, 1)$, $i_l = 1,\dots,n_l$, $l = 1,2$, $r \in [1,\infty]$, and $v_1 = (\lambda_0 M + L)v_0$. Let $\gamma = \min_{i_2=1,\dots,n_2}\{\gamma_{i_2}, \varphi\}$ and $\tau_0 = \min_{i_l=1,\dots,n_l,\, l=1,2}\{\eta_{i_1}, \sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\}$, where $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$. Then, for every fixed $\delta \in J_{\alpha,\beta,\tau_0}$, the function w_0 defined by* (180) *belongs to $C^\delta_0(I_T; X)$, provided that $f \in C^\mu(I_T; X)$, $\mu \in [\delta + \varrho_{\alpha,\beta}, 1)$, $\varrho_{\alpha,\beta} = (2-\alpha-\beta)/\alpha$.*

Proof. Observe that (cf. (159)) all the results from Lemma 32 to Corollary 38 will be applicable. First, since $2\alpha + 2\beta > 4 - \alpha \ge 3$, the choice $\gamma_{i_2}, \varphi \in (4-2\alpha-2\beta, 1)$, $i_2 = 1,\dots,n_2$, is meaningful. Moreover, since $\gamma = \min_{i_2=1,\dots,n_2}\{\gamma_{i_2}, \varphi\} \in (4-2\alpha-2\beta, 1)$, the number $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$ satisfies $\chi_{\alpha,\beta,\gamma} \in ((2-\alpha-\beta)/\alpha, 1)$. Hence, $\tau_0 = \min_{i_l=1,\dots,n_l,\, l=1,2}\{\eta_{i_1}, \sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\} \in ((2-\alpha-\beta)/\alpha, 1)$, too, and J_{α,β,τ_0} is well defined. Now, let $\delta \in J_{\alpha,\beta,\tau_0}$ be fixed. Due to (20) or (38), the element x_0 defined by (174) belongs to Y^r_γ, whereas the functions $\widetilde{h}_{i_2}$ defined by (178) are of class $C^\delta_0(I_T; Y^r_{\gamma_{i_2}}) \hookrightarrow C^\delta_0(I_T; Y^r_\gamma)$. Then, since $\gamma \in (4-2\alpha-2\beta, 1) \subseteq (2-\alpha-\beta, 1)$, from Lemma 37 and Corollary 38 applied with the pairs (g_5, δ_5) and (x, δ_7) being replaced by $(\widetilde{h}_{i_2}, \delta)$ and (x_0, δ), respectively, we deduce that $Q_5 \widetilde{h}_{i_2}, Q_7 x_0 \in C^\delta_0(I_T; X)$, $i_2 = 1,\dots,n_2$. In addition, since the k_{i_1}'s and the constant functions $\kappa_{i_1}(t) = L_{i_1} v_0$ belong to $C^\delta(I_T; X)$, from Proposition 36 applied with $(g_{6_1}, g_{6_2}, X_1, X_2) = (k_{i_1}, L_{i_1} v_0, Z, X)$, it follows that $Q_6(k_{i_1}, L_{i_1} v_0) \in C^\delta_0(I_T; X)$, $i_1 = 1,\dots,n_1$. Finally, since $\delta \in J_{\alpha,\beta,\tau_0} \subseteq ((2-\alpha-\beta)/\alpha, 1/2)$, the number $\delta + \varrho_{\alpha,\beta}$ satisfies

$$\frac{2-\alpha-\beta}{\alpha} < \delta \le \delta + \varrho_{\alpha,\beta} < \frac{4-\alpha-2\beta}{2\alpha} < 1, \tag{202}$$

and the assumption $f \in C^\mu(I_T; X)$, $\mu \in [\delta + \varrho_{\alpha,\beta}, 1)$, makes sense. Then, the function $\widetilde{f} = f - f(0)$ being of class $C^\mu_0(I_T; X)$, Lemma 32 applied with $(g_5, \delta_5) = (\widetilde{f}, \mu)$ yields $Q_5 \widetilde{f} \in C_0^{\widetilde{\nu}_{\alpha,\beta,\mu}}(I_T; X)$, $\widetilde{\nu}_{\alpha,\beta,\mu} = (\alpha\mu + \alpha + \beta - 2)/\alpha$. Since $\widetilde{\nu}_{\alpha,\beta,\mu} \ge \widetilde{\nu}_{\alpha,\beta,\delta+\varrho_{\alpha,\beta}} = \delta$, we conclude that $Q_5 \widetilde{f} \in C^\delta_0(I_T; X)$, too. Summing up, we get the assertion. □

Remark 47. We stress that, if $\beta \in (0,1)$ in (H2), then $0 < \rho_{\alpha,\beta} \le \mu_{\alpha,\beta}$, so that in both Lemmas 45 and 46 we have to assume that $f \in C^\mu(I_T; X)$ with $\mu > \delta$. This is necessary in order to restore the loss of regularity produced by the operators Q_2 and Q_5.

We can now prove the main results of the section.

Theorem 48. *Assume* (161) *and $v_0 \in \mathscr{D}(L)$, and let $5\alpha + 2\beta > 6$ in (H2). Assume that $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$, $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$, $y_{i_2} \in Y^r_{\gamma_{i_2}}$, and $v_1 + f(0) \in Y^r_\varphi$, where $\eta_{i_1}, \sigma_{i_2} \in ((3-2\alpha-\beta)/\alpha, 1)$, $\gamma_{i_2}, \varphi \in (5-3\alpha-2\beta, 1)$, $i_l = 1,\dots,n_l$, $l = 1,2$, $r \in [1,\infty]$, and $v_1 = (\lambda_0 M + L)v_0$. Let $\gamma = \min_{i_2=1,\dots,n_2}\{\gamma_{i_2}, \varphi\}$ and $\tau = \min_{i_l=1,\dots,n_l,\, l=1,2}\{\eta_{i_1}, \sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\}$, where $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$. Then, for every fixed $\delta \in I_{\alpha,\beta,\tau}$ problem* (160) *admits a unique strict solution $v \in C^\delta(I_T; \mathscr{D}(L))$ satisfying $v(0) = v_0$ and such that $Lv, D_t M v \in C^\delta(I_T; X)$, provided that $f \in C^\mu(I_T; X)$, $\mu \in [\delta + \mu_{\alpha,\beta}, 1)$, $\mu_{\alpha,\beta} = (3-2\alpha-\beta)/\alpha$.*

Proof. Of course, due to (159), the assumption $\gamma_{i_2}, \varphi \in (5 - 3\alpha - 2\beta, 1)$, $i_2 = 1, \ldots, n_2$, makes sense. In addition, since $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\} \in (5 - 3\alpha - 2\beta, 1)$, we have $\chi_{\alpha,\beta,\gamma} = (\alpha + \beta + \gamma - 2)/\alpha \in ((3 - 2\alpha - \beta)/\alpha, 1)$. Therefore, by virtue of the choice of the Hölder exponents η_{i_1} and σ_{i_2}, the number $\tau = \min_{i_l=1,\ldots,n_l, l=1,2}\{\eta_{i_1}, \sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\}$ belongs to $((3 - 2\alpha - \beta)/\alpha, 1)$ too, and the interval $I_{\alpha,\beta,\tau}$ is well defined. Further, the numbers η, τ_1, and τ_0 being as in the statements of Lemmas 42, 45, and 46, respectively, we have $\tau = \tau_0 \le \tau_1 \le \eta$. As a consequence, since $I_{\alpha,\beta,\tau} \subseteq I_{\alpha,\beta,\tau_1} \subseteq I_{\alpha,\beta,\eta}$ and $I_{\alpha,\beta,\tau} \subseteq J_{\alpha,\beta,\tau}$, all the mentioned lemmas are applicable with $\delta \in I_{\alpha,\beta,\tau}$. To this purpose, we stress that since $((3-2\alpha-\beta)/\alpha, 1) \subseteq ((2-\alpha-\beta)/\alpha, 1)$ and $(5-3\alpha-2\beta, 1) \subseteq (4-2\alpha-2\beta, 1) \subseteq (3-2\alpha-\beta, 1)$, the conditions for the applicability of both Lemmas 45 and 46 are fulfilled. Hence, now let $\delta \in I_{\alpha,\beta,\tau}$ being fixed. First, due to Lemma 42, the operator $\widetilde{R} = R|_{C_0^\delta(I_T;X)}$, $\widetilde{R}g = Rg$, $g \in C_0^\delta(I_T; X)$, *a fortiori* maps $C_0^\delta(I_T; X)$ into itself. Then, $C_0^\delta(I_T; X)$ being endowed with the same norm $\|\cdot\|_{\delta,0,T;X}$ of $C^\delta(I_T; X)$, from (197) we obtain the estimates

$$\left\|\widetilde{R}^n\right\|_{\mathscr{L}(C_0^\delta(I_T;X))} \le \left[c_{42}(T)\right]^n\left(\frac{T^n}{n!}\right)^{1/p}, \quad n \in \mathbf{N} \cup \{0\}, \quad p \in \left(\frac{1}{1-2\delta}, \infty\right). \tag{203}$$

In particular, (203) yields that $\sum_{n=0}^\infty \widetilde{R}^n$ converges in $\mathscr{L}(C_0^\delta(I_T; X))$. From generalized Neumann's Theorem it thus follows that $1 \in \rho(\widetilde{R})$, the inverse $(I - \widetilde{R})^{-1} \in \mathscr{L}(C_0^\delta(I_T; X))$ being precisely $\sum_{n=0}^\infty \widetilde{R}^n$. Since Lemmas 45 and 46 (both applied with (observe here that if $\mu \in [\delta + \mu_{\alpha,\beta}, 1)$, then the exponent $\widetilde{\nu}_{\alpha,\beta,\mu}$ in the last part of the proof of Lemma 46 satisfies $\widetilde{\nu}_{\alpha,\beta,\mu} \ge \widetilde{\nu}_{\alpha,\beta,\delta+\mu_{\alpha,\beta}} \ge \widetilde{\nu}_{\alpha,\beta,\delta+\varrho_{\alpha,\beta}} = \delta$. For, $\nu_{\alpha,\beta,\delta+\mu_{\alpha,\beta}} = (\alpha\delta + 1 - \alpha)/\alpha = \delta + (1-\alpha)/\alpha)f \in C^\mu(I_T; X)$, $\mu \in [\delta + \mu_{\alpha,\beta}, 1) \subseteq [\delta + \rho_{\alpha,\beta}, 1))$ imply that $w_0, w_1 \in C_0^\delta(I_T; X)$, we conclude that the fixed-point equation (179) admits the unique solution

$$w = \sum_{n=0}^\infty \widetilde{R}^n(w_0 + w_1) \in C_0^\delta(I_T; X). \tag{204}$$

Observe now that the data vector $(k_1, \ldots, k_{n_1}, h_1, \ldots, h_{n_2}, f, y_1, \ldots, y_{n_2}, v_1 + f(0))$ satisfies all the assumptions which were needed to show the equivalence between the fixed-point equation (179) and problem (170). Indeed, $\delta \le \tau$ and $\delta \le \delta + \mu_{\alpha,\beta} \le \mu$ imply, respectively, that $k_{i_1} \in C^\delta(I_T; Z)$, $h_{i_2} \in C^\delta(I_T; \mathbf{C})$ and $f \in C^\delta(I_T; X)$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, whereas, as in Lemma 46, $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\}$ implies that $y_{i_2}, v_1 + f(0) \in Y_\gamma^r$. Therefore, since $A^{-1} \in \mathscr{L}(X)$, if $w \in C_0^\delta(I_T; X)$ is the solution to the fixed-point equation (179), then $A^{-1}w \in C_0^\delta(I_T; X)$, too, and the function F_w defined by (169) satisfies

$$F_w \in C^\delta(I_T; X), \quad x_0 = F_w(0) = \sum_{i_2=1}^{n_2} h_{i_2}(0)\, y_{i_2} + v_1 + f(0) \in Y_\gamma^r, \tag{205}$$

where $\delta \in I_{\alpha,\beta,\tau} \subsetneq (2-\alpha-\beta)/\alpha, 1)$, $\gamma \in (5-3\alpha-2\beta, 1) \subsetneq (1-\beta, 1)$, and $r \in [1, \infty]$. Consequently, recalling (168), we have proved that problem (160) has a unique strict global solution $v = L^{-1}w + v_0 \in C^\delta(I_T; \mathscr{D}(L))$ satisfying $v(0) = L^{-1}w(0) + v_0 = v_0$ and such that $Lv = w + Lv_0 \in C^\delta(I_T; X)$. As far as the regularity of D_tMv is concerned, instead, it suffices to observe that (168), (170), $w \in C_0^\delta(I_T; X)$, and $F_w \in C^\delta(I_T; X)$ yield

$$D_tMv = D_tA^{-1}w = w + F_w \in C^\delta(I_T; X). \tag{206}$$

The proof is complete. □

Remark 49. Theorem 48 improves the faulty Thereoms 5.6 and 5.7 in [20] in two aspects. First, the assumption $3\alpha + 8\beta > 10$ is weakened to $5\alpha + 2\beta > 6$. In fact, $3\alpha + 8\beta > 10$ implies that $5\alpha + 2\beta = 3\alpha + 8\beta + 2\alpha - 6\beta > 10 - 4\alpha \ge 6$. Hence, in the special case $\alpha = 1$, the constraint $\beta > 7/8$ in [20] reduces to the definitely weaker $\beta > 1/2$. Second, in [20], only for $n_1 = n_2 = 1$ and opportunely chosen $\gamma < \beta$, the data y_1 and $v_1 + f(0)$ were assumed to belong to the intermediate spaces $X_A^{\gamma,r}$, whereas here, removing the assumption $\gamma < \beta$ and considering the general case $n_1, n_2 \in \mathbf{N}$, we allow $y_1, \ldots, y_{n_2}$ and $v_1 + f(0)$ to belong also to the interpolation spaces $(X, \mathscr{D}(A))_{\gamma,r}$. To emphasize how much these aspects are decisive, let $\alpha = 1$ in Theorem 48. Then, if $\beta \in (1/2, 2/3]$ and the choice $X_A^{\psi,r}$ is understood for Y_ψ^r, we have $\gamma_{i_2}, \varphi \in (2 - 2\beta, 1) \subsetneq [\beta, 1)$, and the spaces $X_A^{\gamma_{i_2},r}$ and $X_A^{\varphi,r}$, $i_2 = 1, \ldots, n_2$, may be smaller than $\mathscr{D}(A)$. However, the choice $Y_\psi^r = (X, \mathscr{D}(A))_{\psi,r}$ being admissible, in this situation too we can solve problem (160) with the data in spaces larger than $\mathscr{D}(A)$. Further, since $2/3 < 7/8$, in this case the results in [20] would not be applicable. These observations lead us to conclude that the more delicate approach followed in this paper with respect to that in [20, Sections 4 and 5], and especially the sharper results of the present Sections 3 and 4, yield a valuable refinement in the treatment of questions of maximal time regularity for the strict solutions to (160); of course, unless that the not too much significant case $\beta = 1$ is assumed in (H2).

Remark 50. The assumption $5\alpha + 2\beta > 6$ in (H2) implies that $\beta \in ((6 - 5\alpha)/2, \alpha] \subseteq (1/2, 1]$ and $\alpha \in (6/7, 1]$. In particular, if $\alpha = 1$, then Theorem 48 holds with $\beta \in (1/2, 1]$, $\eta_{i_1}, \sigma_{i_2} \in (1 - \beta, 1)$, $\gamma_{i_2}, \varphi \in (2 - 2\beta, 1)$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, and $\mu_{1,\beta} = 1 - \beta$. Hence, $\gamma \in (2 - 2\beta, 1)$, $\chi_{1,\beta,\gamma} = \beta + \gamma - 1 \in (1 - \beta, \beta)$, and $\delta \in I_{1,\beta,\tau}$ with $\tau \in (1 - \beta, \beta)$, where

$$I_{1,\beta,\tau} = (1 - \beta, \tau], \quad \text{if } \tau \in \left(1 - \beta, \frac{1}{2}\right), \qquad I_{1,\beta,\tau} = \left(1 - \beta, \frac{1}{2}\right), \quad \text{if } \tau \in \left[\frac{1}{2}, \beta\right). \tag{207}$$

Clearly, if $\beta = 1$, then $5\alpha + 2\beta > 6$ is redundant, and Theorem 48 holds with $\eta_{i_1}, \sigma_{i_2}, \gamma_{i_2}, \varphi \in (0, 1)$, $i_l = 1, \ldots, n_l$, $l = 1, 2$, $\mu_{1,1} = 0$, $\gamma = \chi_{1,1,\gamma} \in (0, 1)$, and $\delta \in I_{1,1,\tau}$, $\tau \in (0, 1)$, where $I_{1,1,\tau} = (0, \tau]$ if $\tau \in (0, 1/2)$ and $I_{1,1,\tau} = (0, 1/2)$ if $\tau \in [1/2, 1)$.

Remark 51. Observe that, if the η_{i_1}'s and σ_{i_2}'s are assumed to vary in the smaller interval $U_{\alpha,\beta} := ((3-2\alpha-\beta)/\alpha, (\alpha+\beta-1)/\alpha)$, then φ and the γ_{i_2}'s can be chosen such that $\tau = \min_{i_l=1,\ldots,n_l,\, l=1,2}\{\eta_{i_1}, \sigma_{i_2}\}$. To this purpose, letting $\rho = \max_{i_l=1,\ldots,n_l,\, l=1,2}\{\eta_{i_1}, \sigma_{i_2}\} \in U_{\alpha,\beta}$, it suffices to take $\gamma_{i_2}, \varphi \in V_{\alpha,\beta,\rho}$, $i_2 = 1,\ldots,n_2$, where $V_{\alpha,\beta,\rho} := [2+\alpha\rho-\alpha-\beta, 1) \subsetneq (5-3\alpha-2\beta, 1)$. Then $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\} \in V_{\alpha,\beta,\rho}$ and $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha \geq \rho$. In other words, provided that the data vector $(y_1,\ldots,y_{n_2}, v_1+f(0))$ is smooth enough, the pair (Lv, D_tMv) has the maximal time regularities which is the minimal between the time regularities of the k_{i_1}'s and h_{i_2}'s.

We conclude with the results which follow from Theorem 48 for problems (163)–(166).

Theorem 52. *Assume* (161) *and* $v_0 \in \mathscr{D}(L)$*, and let* $5\alpha+2\beta > 6$ *in (H2). Assume that* $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$ *and* $Lv_0 + f(0) \in Y^r_\gamma$*, where* $\eta_{i_1} \in ((3-2\alpha-\beta)/\alpha, 1)$*,* $i_1 = 1,\ldots,n_1$*,* $\gamma \in (5-3\alpha-2\beta, 1)$*, and* $r \in [1,\infty]$*. Let* $\tau = \min_{i_1=1,\ldots,n_1}\{\eta_{i_1}, \chi_{\alpha,\beta,\gamma}\}$*, where* $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$*. Then, for every fixed* $\delta \in I_{\alpha,\beta,\tau}$ *problem* (165) *admits a unique strict solution* $v \in C^\delta(I_T; \mathscr{D}(L))$ *satisfying* $v(0) = v_0$ *and such that* $Lv, D_tMv \in C^\delta(I_T; X)$*, provided that* $f \in C^\mu(I_T; X)$*,* $\mu \in [\delta+\mu_{\alpha,\beta}, 1)$*,* $\mu_{\alpha,\beta} = (3-2\alpha-\beta)/\alpha$*.*

Proof. Repeat the proofs of Lemmas 42, 45, and 46, Corollary 44, and Theorem 48, letting there $\lambda_0 = h_{i_2} = 0$, $i_2 = 1,\ldots,n_2$. To this purpose, observe that (169) and (174) reduce to $F_w(t) = \sum_{i_1=1}^{n_1}[\mathscr{K}(k_{i_1}, S_{i_1}w)(t) + \mathscr{K}(k_{i_1}, L_{i_1}v_0)(t)] + Lv_0 + f(t)$ and $x_0 = Lv_0 + f(0)$. Consequently, (180)–(182) change to $w_0 = Q_7x_0 + \sum_{i_1=1}^{n_1} Q_6(k_{i_1}, L_{i_1}v_0) + Q_5\tilde{f}$, $w_1 = -\sum_{i_1=1}^{n_1} Q_3(k_{i_1}, L_{i_1}v_0) - Q_2 f$, and $Rw = \sum_{i_1=1}^{n_1}[Q_6(k_{i_1}, S_{i_1}w) - Q_3(k_{i_1}, S_{i_1}w)]$. □

Theorem 53. *Assume* (161) *and* $v_0 \in \mathscr{D}(L)$*, and let* $5\alpha+2\beta > 6$ *in (H2). Assume that* $k_{i_1} \in C^{\eta_{i_1}}(I_T; Z)$*,* $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$*,* $y_{i_2} \in Y^r_{\gamma_{i_2}}$*, and* $Lv_0 \in Y^r_\varphi$*, where* $\eta_{i_1}, \sigma_{i_2} \in ((3-2\alpha-\beta)/\alpha, 1)$*,* $\gamma_{i_2}, \varphi \in (5-3\alpha-2\beta, 1)$*,* $i_l = 1,\ldots,n_l$*,* $l = 1,2$*, and* $r \in [1,\infty]$*. Let* $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\}$ *and* $\tau = \min_{i_l=1,\ldots,n_l,\, l=1,2}\{\eta_{i_1}, \sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\}$*, where* $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$*. Then, for every fixed* $\delta \in I_{\alpha,\beta,\tau}$ *problem* (166) *admits a unique strict solution* $v \in C^\delta(I_T; \mathscr{D}(L))$ *satisfying* $v(0) = v_0$ *and such that* $Lv, D_tMv \in C^\delta(I_T; X)$*.*

Proof. Let $\lambda_0 = f = 0$ in the proofs of Lemmas 42, 45, and 46, Corollary 44, and Theorem 48. In this case, (169) and (174) reduce to $F_w(t) = \sum_{i_1=1}^{n_1}[\mathscr{K}(k_{i_1}, S_{i_1}w)(t) + \mathscr{K}(k_{i_1}, L_{i_1}v_0)(t)] + \sum_{i_2=1}^{n_2} h_{i_2}(t)y_{i_2} + Lv_0$ and $x_0 = \sum_{i_2=1}^{n_2} h_{i_2}(0)y_{i_2} + Lv_0$. Hence, (180)–(182) change to $w_0 = Q_7x_0 + \sum_{i_1=1}^{n_1} Q_6(k_{i_1}, L_{i_1}v_0) + \sum_{i_2=1}^{n_2} Q_5\tilde{h}_{i_2}$, $w_1 = -\sum_{i_1=1}^{n_1} Q_3(k_{i_1}, L_{i_1}v_0) - \sum_{i_2=1}^{n_2} Q_4(h_{i_2}, y_{i_2})$, and $Rw = \sum_{i_1=1}^{n_1}[Q_6(k_{i_1}, S_{i_1}w) - Q_3(k_{i_1}, S_{i_1}w)]$. □

Let us now turn to the degenerate differential problems (163) and (164).

Theorem 54. *Assume* (161) *and* $v_0 \in \mathscr{D}(L)$*, and let* $5\alpha+2\beta > 6$ *in (H2). Assume that* $Lv_0 + f(0) \in Y^r_\gamma$*,* $\gamma \in (5-3\alpha-2\beta, 1)$*,* $r \in [1,\infty]$*, and let* $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$*. Then, for every fixed* $\delta \in I_{\alpha,\beta,\chi_{\alpha,\beta,\gamma}}$ *problem* (163) *admits a unique strict global solution* $v \in C^\delta(I_T; \mathscr{D}(L))$ *satisfying* $v(0) = v_0$ *and such that* $Lv, D_tMv \in C^\delta(I_T; X)$*, provided that* $f \in C^\mu(I_T; X)$*,* $\mu \in [\delta+\mu_{\alpha,\beta}, 1)$*,* $\mu_{\alpha,\beta} = (3-2\alpha-\beta)/\alpha$*.*

Proof. Let $\lambda_0 = k_{i_1} = h_{i_2} = 0$, $i_l = 1,\ldots,n_l$, $l = 1,2$, in problem (160) and formulae (169), (174) and, (179)–(182). Then, $F_w(t) = Lv_0 + f(t)$, $x_0 = Lv_0 + f(0)$ and $w = w_0 + w_1 = Q_7x_0 + Q_5\tilde{f} - Q_2 f$. Consequently, Lemma 42 and Corollary 44 are unneeded, and the proof of Theorem 48 simplifies as follows. First, due to $\gamma \in (5-3\alpha-2\beta, 1)$ we have $\chi_{\alpha,\beta,\gamma} \in ((3-2\alpha-\beta)/\alpha, 1)$, and the interval $I_{\alpha,\beta,\chi_{\alpha,\beta,\gamma}}$ is well defined. Hence, let $\delta \in I_{\alpha,\beta,\chi_{\alpha,\beta,\gamma}}$ being fixed. Since (cf. (200)) $f \in C^\mu(I_T; X)$, $\mu \in [\delta+\mu_{\alpha,\beta}, 1) \subsetneq ((3-2\alpha-\beta)/\alpha, 1)$, reasoning as in the last part of the proof of Lemma 45 we get $Q_2 f \in C_0^\delta(I_T; X)$. Moreover (see the proof of Lemma 46), since $x_0 \in Y^r_\gamma$, $\gamma \in (5-3\alpha-2\beta, 1) \subseteq (2-\alpha-\beta, 1)$ and $\tilde{f} \in C_0^\mu(I_T; X)$, $\mu \in [\delta+\mu_{\alpha,\beta}, 1) \subseteq [\delta+\varrho_{\alpha,\beta}, 1)$, $\varrho_{\alpha,\beta} = (2-\alpha-\beta)/\alpha$, Corollary 38 and Lemma 32 applied with $(x, \delta_7) = (x_0, \delta)$ and $(g_5, \delta_5) = (\tilde{f}, \delta+\varrho_{\alpha,\beta})$ yield Q_7x_0, $Q_5\tilde{f} \in C_0^\delta(I_T; X)$. Summing up, we find that $w \in C_0^\delta(I_T; X)$. The assertion then follows from $v = L^{-1}w + v_0$ and (cf. (206)) $D_tMv = w + Lv_0 + f$. □

Remark 55. We refer to [19, Theorem 5.3] for a result of both time *and* space regularity for problem (163). There, provided that ψ and δ are opportunely chosen and the data satisfy assumptions similar to those in Theorem 54, it is shown that $D_tMv \in C^\delta(I_T; (X, \mathscr{D}(A))_{\psi,r})$, and that the higher is the order ψ of the interpolation space where we look for space regularity, the lower is the Hölder exponent δ of regularity in time. Notice that $Lv = D_tMv - f$ has no space regularity, unless f has too.

Theorem 56. *Assume* (161) *and* $v_0 \in \mathscr{D}(L)$*, and let* $5\alpha+2\beta > 6$ *in (H2). Assume that* $h_{i_2} \in C^{\sigma_{i_2}}(I_T; \mathbf{C})$*,* $y_{i_2} \in Y^r_{\gamma_{i_2}}$*, and* $Lv_0 \in Y^r_\varphi$*, where* $\sigma_{i_2} \in ((3-2\alpha-\beta)/\alpha, 1)$*,* $\gamma_{i_2}, \varphi \in (5-3\alpha-2\beta, 1)$*,* $i_2 = 1,\ldots,n_2$*, and* $r \in [1,\infty]$*. Let* $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\}$ *and* $\tau = \min_{i_2=1,\ldots,n_2}\{\sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\}$*, where* $\chi_{\alpha,\beta,\gamma} = (\alpha+\beta+\gamma-2)/\alpha$*. Then, for every fixed* $\delta \in I_{\alpha,\beta,\tau}$*, problem* (164) *admits a unique strict global solution* $v \in C^\delta(I_T; \mathscr{D}(L))$ *satisfying* $v(0) = v_0$ *and such that* $Lv, D_tMv \in C^\delta(I_T; X)$*.*

Proof. Let $\lambda_0 = k_{i_1} = f = 0$, $i_1 = 1,\ldots,n_1$, in problem (160) and formulae (169), (174), and (179)–(182). Then, $F_w(t) = \sum_{i_2=1}^{n_2} h_{i_2}(t)y_{i_2} + Lv_0$, $x_0 = \sum_{i_2=1}^{n_2} h_{i_2}(0)y_{i_2} + Lv_0$ and $w = w_0 + w_1 = Q_7x_0 + \sum_{i_2=1}^{n_2} Q_5\tilde{h}_{i_2} - \sum_{i_2=1}^{n_2} Q_4(h_{i_2}, y_{i_2})$. Therefore, as in Theorem 54, we do not need Lemma 42 and Corollary 44, and the proof of Theorem 48 simplifies as follows. Again, $\gamma = \min_{i_2=1,\ldots,n_2}\{\gamma_{i_2}, \varphi\} \in (5-3\alpha-2\beta, 1)$ implies that

$\chi_{\alpha,\beta,\gamma} \in ((3-2\alpha-\beta)/\alpha, 1)$, so that $\tau = \min_{i_2=1,\ldots,n_2}\{\sigma_{i_2}, \chi_{\alpha,\beta,\gamma}\} \in ((3-2\alpha-\beta)/\alpha, 1)$, and the interval $I_{\alpha,\beta,\tau}$ is well defined. Let $\delta \in I_{\alpha,\beta,\tau}$ be fixed. First (see the proof of Lemma 45), since $\gamma_{i_2} \in (5-3\alpha-2\beta, 1) \subseteq (3-2\alpha-\beta, 1)$, Lemma 30 applied with $(g_4, y, \delta_4, \gamma) = (h_{i_2}, y_{i_2}, \delta, \gamma_{i_2})$ yields $Q_4(h_{i_2}, y_{i_2}) \in C_0^{\delta}(I_T; X)$, $i_2 = 1, \ldots, n_2$. On the other side (see the proof of Lemma 46), since $x_0 \in Y_\gamma^r$ and $\widetilde{h}_{i_2} \in C_0^{\delta}(I_T; Y_{\gamma_{i_2}}^r) \hookrightarrow C_0^{\delta}(I_T; Y_\gamma^r)$, $\gamma \in (5-3\alpha-2\beta, 1) \subseteq (2-\alpha-\beta, 1)$, from Lemma 37 and Corollary 38 applied with $(g_5, \delta_5) = (\widetilde{h}_{i_2}, \delta)$ and $(x, \delta_7) = (x_0, \delta)$ we deduce that $Q_5\widetilde{h}_{i_2}, Q_7x_0 \in C_0^{\delta}(I_T; X)$, $i_2 = 1, \ldots, n_2$. Summing up, we find that $w \in C_0^{\delta}(I_T; X)$, and the assertion again follows from $v = L^{-1}w + v_0$ and (cf. (206)) $D_tMv = w + Lv_0 + \sum_{i_2=1}^{n_2} h_{i_2} y_{i_2}$. □

6. An Application to a Concrete Case

Theorem 48 is here applied to determine the right functional framework where to search for the solution of an inverse problem arising in the theory of heat conduction for materials with memory. To this purpose, let $\Omega \subsetneq \mathbf{R}^N$, $N \in \mathbf{N}$, be a bounded domain with boundary $\partial\Omega$ of class $C^{1,1}$ (cf. [36, p. 94]). If Ω represents a rigid thermal body with memory, then the linearized theory of heat flow yields the following equations linking the internal energy e, the heat flux $\mathbf{q} = (q_1, \ldots, q_N)$, and the temperature Θ (cf. [32, 37–40]):

$$e(t,x) = e_0 + a(0,x)\Theta(t,x) + \int_0^t D_t a(t-s,x)\Theta(s,x)\,ds,$$

$$\begin{aligned} q_j(t,x) = &-\sum_{i=1}^{r_1} b_i(0) C_{i,j}(x; D_x)\Theta(t,x) \\ &- \sum_{i=1}^{r_1} \int_0^t D_t b_i(t-s) C_{i,j}(x; D_x)\Theta(s,x)\,ds, \\ &\qquad j = 1, \ldots, N, \end{aligned}$$

$$\begin{aligned} D_t e(t,x) &= -\mathrm{div}_x \mathbf{q}(t,x) + g(t,x) \\ &= -\sum_{j=1}^{N} D_{x_j} q_j(t,x) + g(t,x). \end{aligned} \tag{208}$$

Here $t \in I_T$, $I_T = [0,T]$, $T > 0$, $x = (x_1, \ldots, x_N) \in \Omega$, $r_1 \in \mathbf{N}$, $e_0 \in \mathbf{R}$, and $D_t = \partial/\partial t$, whereas the $C_{i,j}(x; D_x)$'s represent the first-order linear differential operators

$$\begin{gathered} C_{i,j}(\mathrm{x}; D_x) = \sum_{k=1}^{N} c_{i,j,k}(x) D_{x_k}, \quad x \in \overline{\Omega}, \\ i = 1, \ldots, r_1, \quad j = 1, \ldots, N, \end{gathered} \tag{209}$$

where $c_{i,j,k} \in C^1(\overline{\Omega}; \mathbf{R})$ and $D_{x_k} = \partial/\partial x_k$, $i = 1, \ldots, r_1$, $j, k = 1, \ldots, N$. According to the terminology of [39, 40], the functions a, b_i, $i = 1, \ldots, r_1$, and g are called, respectively, the *energy-temperature relaxation function*, the *heat conduction relaxation functions,* and the *heat supply function* and we assume that they satisfy the following conditions:

$$\begin{gathered} D_t^k a(\cdot, x) \in C(I_T; \mathbf{R}), \quad k = 0, 1, 2, \\ a(0, x) \geq 0, \quad x \in \Omega, \end{gathered} \tag{210}$$

$$\begin{gathered} D_t^k b_i \in C(I_T; \mathbf{R}), \quad k = 0, 1, \ i = 1, \ldots, r_1, \\ g \in C^1(I_T \times \Omega; \mathbf{R}). \end{gathered} \tag{211}$$

Notice that, different from [32, 37–40], here the energy-temperature relaxation function a is assumed to depend also on the spatial variable $x \in \Omega$. In physical terms, this is equivalent to say that Ω represents a rigid *inhomogeneous* material with memory. Furthermore, in contrast with the quoted papers where only the cases $r_1 = 1$ and $C_{1,j}(x; D_x) = D_{x_j}$ are treated, here we have assumed that the history record of Ω is kept by an arbitrary number $r_1 \in \mathbf{N}$ of heat conduction relaxation functions and that the $C_{i,j}$'s are the more general first-order differential operators defined in (209).

By setting

$$\widetilde{a}_{j,k} = \sum_{i=1}^{r_1} b_i(0) c_{i,j,k} \in C^1(\overline{\Omega}; \mathbf{R}), \quad j, k = 1, \ldots, N, \tag{212}$$

from (208) and (209), it thus follows that the temperature Θ must satisfy the following equation:

$$\begin{aligned} &a(0,x) D_t\Theta(t,x) + D_t a(0,x)\Theta(t,x) \\ &\quad + \int_0^t D_t^2 a(t-s,x)\Theta(s,x)\,ds - g(t,x) \\ &= \sum_{j,k=1}^{N} D_{x_j}\left[\widetilde{a}_{j,k}(x) D_{x_k}\Theta(t,x)\right] \\ &\quad + \sum_{i=1}^{r_1} \int_0^t D_t b_i(t-s) \sum_{j=1}^{N} D_{x_j} C_{i,j}(x; D_x)\Theta(s,x)\,ds. \end{aligned} \tag{213}$$

Let us now assume that a is of the following special form:

$$a(t,x) = \sum_{n=1}^{2} m_n(x) u_n(t), \quad (t,x) \in I_T \times \Omega, \tag{214}$$

where the functions m_n and u_n, $n = 1, 2$, satisfy the following conditions (cf. (210)):

$$\begin{gathered} m_n \in L_\infty(\Omega), \quad n = 1, 2, \\ m_1 \geq 0, \qquad m_2 > 0, \end{gathered} \tag{215}$$

$$\begin{gathered} u_n \in C^2(I_T; \mathbf{R}), \quad n = 1, 2, \\ u_2(0) = 0, \qquad u_1(0) > 0, \qquad D_t u_2(0) > 0. \end{gathered} \tag{216}$$

Here, $L_q(\Omega) = L_q(\Omega;\mathbf{R})$, $q \in [1,\infty]$, is the usual L_q space with norm $\|\cdot\|_{q;\Omega}$ (cf. [36, Chapter 7]). Using $m_2, u_1(0), D_t u_2(0) > 0$, for $t \in I_T$ and $x \in \Omega$ we now set

$$a_0(x) = -[u_1(0)]^{-1} m_2(x) D_t u_2(0) < 0, \quad (217)$$

$$a_{j,k}(x) = [u_1(0)]^{-1} \tilde{a}_{j,k}(x), \qquad j,k = 1,\ldots,N, \quad (218)$$

$$L(x;D_x) = \sum_{j,k=1}^{N} D_{x_j}\left[a_{j,k}(x) D_{x_k}\right] + a_0(x), \quad (219)$$

$$L_i(x;D_x) = [u_1(0)]^{-1} \sum_{j=1}^{N} D_{x_j} C_{i,j}(x;D_x), \qquad i = 1,\ldots,r_1, \quad (220)$$

$$L_{r_1+n}(x;D_x) = L_{r_1+n}(x) = [u_1(0)]^{-1} m_n(x), \qquad n = 1,2, \quad (221)$$

$$k_i(t) = D_t b_i(t), \quad i = 1,\ldots,r_1,$$
$$k_{r_1+n}(t) = -D_t^2 u_n(t), \quad n = 1,2, \quad (222)$$

$$\tilde{g}(t,x) = [u_1(0)]^{-1} g(t,x),$$
$$\lambda_0 = -[u_1(0)]^{-1} D_t u_1(0) \in \mathbf{R}. \quad (223)$$

Then, since (214)–(216) yield $a(0,x) = m_1(x)u_1(0)$ and $D_t^k a(t,x) = \sum_{n=1}^{2} m_n(x) D_t^k u_n(t)$, $k = 1,2$, if we multiply both sides of (213) by $[u_1(0)]^{-1}$ and use (218)–(223), we are led to the following basic differential equation for the temperature Θ, where $n_1 = r_1 + 2$:

$$\begin{aligned} D_t&\left[m_1(x)\Theta(t,x)\right] \\ &= \lambda_0 m_1(x)\Theta(t,x) + L(x;D_x)\Theta(t,x) + \tilde{g}(t,x) \\ &\quad + \sum_{i=1}^{n_1}\int_0^t k_i(t-s) L_i(x;D_x)\Theta(s,x)\,\mathrm{d}s, \\ &\qquad t \in I_T, \quad x \in \Omega. \end{aligned} \quad (224)$$

We endow this differential equation with the initial condition $\Theta(0,x) = \Theta_0(x)$, $x \in \Omega$, and the Dirichlet boundary condition $\Theta(t,x) = 0$, $t \in I_T$, $x \in \partial\Omega$.

We now suppress the dependence on $x \in \Omega$, and we transform (224) in a degenerate integrodifferential Cauchy problem in a Banach space X. To this purpose, for every fixed $q \in (1,\infty)$ and observing that $m_n \in L_\infty(\Omega)$ implies that $\|m_n u\|_{q;\Omega} \le \|m_n\|_{\infty;\Omega}\|u\|_{q;\Omega}$ for every $u \in L_q(\Omega)$, $n = 1,2$, we set

$$X = \mathscr{D}(M) = \mathscr{D}\left(L_{r_1+n}\right) = L_q(\Omega), \quad n = 1,2, \quad (225)$$

$$\mathscr{D}(L) = W_q^2(\Omega) \cap \overset{\circ}{W}{}_q^1(\Omega), \qquad \mathscr{D}(L_i) = W_q^2(\Omega), \qquad i = 1,\ldots,r_1, \quad (226)$$

$$M, L_{r_1+n} \in \mathscr{L}(X), \qquad Mu = m_1 u,$$
$$L_{r_1+n}u = L_{r_1+n}(x)u, \quad u \in X,\ n = 1,2, \quad (227)$$

$$L : \mathscr{D}(L) \subseteq X \longrightarrow X,$$
$$Lu = L(x;D_x)u,\ u \in \mathscr{D}(L), \quad (228)$$

$$L_i : \mathscr{D}(L_i) \subseteq X \longrightarrow X,$$
$$L_i u = L_i(x;D_x)u, \quad u \in \mathscr{D}(L_i), \quad i = 1,\ldots,r_1. \quad (229)$$

Here (cf. [36, Chapter 7]), $W_q^k(\Omega) = W_q^k(\Omega;\mathbf{R})$, $k \in \mathbf{N}\cup\{0\}$, $q \in (1,\infty)$, denotes the usual Sobolev space endowed with the norm $\|\cdot\|_{k,q;\Omega}$ $((W_q^0(\Omega), \|\cdot\|_{0,q;\Omega}) = (L_q(\Omega), \|\cdot\|_{q;\Omega}))$, whereas $\overset{\circ}{W}{}_q^k(\Omega)$ denotes the completion of $C_0^\infty(\Omega;\mathbf{R})$ in $W_q^k(\Omega)$, $C_0^\infty(\Omega;\mathbf{R})$ being the set of all real-valued infinitely differentiable functions having compact support in Ω. We further assume that there exists positive constant Λ_i, $i = 0,\ldots,r_1$, such that for every $(x,\xi) \in \overline{\Omega}\times\mathbf{R}^N$ the following inequalities hold:

$$\sum_{j,k=1}^{N} c_{i,j,k}(x)\xi_j\xi_k \ge \Lambda_i|\xi|^2, \quad i = 1,\ldots,r_1,$$
$$\sum_{j,k=1}^{r_1} b_i(0)\Lambda_i \ge \Lambda_0, \quad (230)$$

where $|\xi|^2 = \sum_{l=1}^{N}\xi_l^2$. Therefore, from (212), (218), and (230) we get

$$\begin{aligned} \sum_{j,k=1}^{N} a_{j,k}(x)\xi_j\xi_k &= [u_1(0)]^{-1}\sum_{i=1}^{r_1} b_i(0)\sum_{j,k=1}^{N} c_{i,j,k}\xi_j\xi_k \\ &\ge [u_1(0)]^{-1}\Lambda_0|\xi|^2. \end{aligned} \quad (231)$$

From (225)–(231) it follows that M, L, and L_i, $i = 1,\ldots,n_1$, are closed linear operators from X to itself, and the relation $\mathscr{D}(L) \subsetneq \bigcap_{i=1}^{n_1}[\mathscr{D}(M)\cap\mathscr{D}(L_i)] = W_q^2(\Omega)$ holds. In addition, due to (212), (217), (218), and (231), from [36, Theorem 9.15 and Lemma 9.17], it follows that for every fixed $q \in (1,\infty)$ the operator L admits an inverse operator $L^{-1} \in \mathscr{L}(X;W_q^2(\Omega))$. Hence, a fortiori, $L^{-1} \in \mathscr{L}(X)$ and so condition (161) is satisfied (observe also that $L^{-1} \in \mathscr{L}(X;W_q^2(\Omega))$ implies that the norms $\|\cdot\|_{2,q;\Omega}$ and $\|\cdot\|_{\mathscr{D}(L)} = \|\cdot\|_{q;\Omega} + \|L\cdot\|_{q;\Omega}$ are equivalent on $\mathscr{D}(L)$. In fact, if $v \in \mathscr{D}(L)$, then $\|v\|_{2,q;\Omega} = \|L^{-1}Lv\|_{2,q;\Omega} \le \|L^{-1}\|_{\mathscr{L}(X;W_q^2(\Omega))}\|v\|_{\mathscr{D}(L)} \le \widetilde{C}\|L^{-1}\|_{\mathscr{L}(X;W_q^2(\Omega))}\|v\|_{2,q;\Omega}$, $\widetilde{C}$ being a positive constant depending on $\max_{j,k=1,\dots,N}\|a_{j,k}\|_{C^1(\overline{\Omega};\mathbf{R})}$). The closed graph theorem then yield $ML^{-1}, L_iL^{-1} \in \mathscr{L}(X)$, $i = 1,\dots,n_1$. Moreover (cf. [19, formula (77)], and [41, formula (2.16)]), the following estimate holds (of course, here $X = L_q(\Omega;\mathbf{R})$ is replaced with the more general $X = L_q(\Omega;\mathbf{C})$):

$$\left\|M(\lambda M - L)^{-1}\right\|_{\mathscr{L}(X)} \le C(|\lambda| + 1)^{-\beta}, \quad \forall\lambda \in \Sigma_1,\ \beta = \frac{1}{q}, \tag{232}$$

where $\Sigma_1 = \{z \in \mathbf{C} : \Re\mathrm{e}\, z \ge -c(|\Im\mathrm{m}\, z| + 1),\ \Im\mathrm{m}\, z \in \mathbf{R}\}$, c being a suitable positive constant depending on q and $\|m_1\|_{\infty;\Omega}$. Hence, condition (H2) is satisfied with $X = L^q(\Omega)$ and $(\alpha,\beta) = (1, 1/q)$. Notice that, since m_1 may have zeros in Ω, M^{-1} is in general a m. l. operator, so that $A = LM^{-1}$ is determined by (cf. (162)):

$$\mathscr{D}(A) = M(\mathscr{D}(L)) = \{m_1 v : v \in \mathscr{D}(L)\},$$
$$Au = \{Lv : v \in \mathscr{D}(L) \text{ such that } u = m_1 v\}, \quad u \in \mathscr{D}(A). \tag{233}$$

Using the convolution operator $\mathscr{K}$ in (104) in which for the bilinear operator $\mathscr{P}$ we take the scalar multiplication in X, from (224)–(229) we finally obtain that the temperature $\Theta(t) = \Theta(t,\cdot)$ solves the following degenerate integrodifferential Cauchy problem in X:

$$D_t(M\Theta(t)) = [\lambda_0 M + L]\Theta(t) + \sum_{i=1}^{n_1}\mathscr{K}(k_i, L_i\Theta)(t) + \widetilde{g}(t), \quad t \in I_T, \tag{234}$$
$$\Theta(0) = \Theta_0.$$

Now, assume for a moment that we are interested in solving the *inverse* problem of recovering both the temperature Θ and the memory kernels $k_1,\dots,k_{r_1}$ in (234). Clearly, due to (222), if we recover $k_1,\dots,k_{r_1}$, then the heat conduction relaxation functions $b_1,\dots,b_{r_1}$ will be known too, unless of the r_1 arbitrary constants $b_i(0)$, $i = 1,\dots,r_1$. Indeed, $b_1(t) = b_i(0) + \int_0^t k_i(s)\mathrm{d}s$, $t \in I_T$. To solve such an inverse problem, we need r_1 additional informations other than the initial condition $\Theta(0) = \Theta_0$, which, in general, suffices only to guarantee the well-posedness of the *direct* problem of recovering Θ in (234). Suppose then that the following additional pieces of information are given:

$$\Psi_j[M\Theta(t)] = g_j(t), \quad t \in I_T,\ j = 1,\dots,r_1, \tag{235}$$

where $\Psi_j \in X^* = \mathscr{L}(X;\mathbf{R})$ and $g_j \in C^{2+\nu_j}(I_T;\mathbf{R})$, $\nu_j \in (0,1)$, $j = 1,\dots,r_1$. We will search for a solution vector $(\Theta, k_1,\dots,k_{r_1})$ of the inverse problem (234) and (235) such that $\Theta \in C^{1+\delta}(I_T;\mathscr{D}(L))$ and $k_j \in C^{\eta_j}(I_T;\mathbf{R})$, $j = 1,\dots,r_1$, with the Hölder exponents δ and η_j, $j = 1,\dots,r_1$, to be made precise in the sequel. We stress that here we will not solve completely the mentioned inverse problem. For, its detailed treatment would lead us out of the aims of this paper. Our intention here is only to highlight how the main results of Section 5 allow to determine the correct functional framework in which the solution of the inverse problem has to be searched. However, a complete treatment of the inverse problem will be the object of a future paper.

Assuming that $\Theta \in C^{1+\delta}(I_T;\mathscr{D}(L))$ solves (234), we introduce the new unknown

$$v(t,x) = D_t\Theta(t,x) \Longleftrightarrow \Theta(t,x) = \Theta_0(x) + \int_0^t v(s,x)\,\mathrm{d}s. \tag{236}$$

Then, differentiating (234) with respect to time and using

$$\begin{aligned} D_t\mathscr{K}(k_i, L_i\Theta)(t) &= D_t\int_0^t k_i(t-s)L_i\Theta(s)\,\mathrm{d}s = D_t\int_0^t k_i(s)L_i\Theta(t-s)\,\mathrm{d}s \\ &= k_i(t)L_i\Theta(0) + \int_0^t k_i(s)L_iD_t\Theta(t-s)\,\mathrm{d}s \\ &= k_i(t)L_i\Theta_0 + \int_0^t k_i(t-s)L_iv(s)\,\mathrm{d}s, \end{aligned} \tag{237}$$

we find that $v \in C^\delta(I_T;\mathscr{D}(L))$ solves the following degenerate integrodifferential problem:

$$\begin{aligned} D_t(Mv(t)) &= [\lambda_0 M + L]v(t) + \sum_{i=1}^{n_1}[\mathscr{K}(k_i, L_iv)(t) + k_i(t)y_i] + f(t), \\ &\qquad t \in I_T, \\ Mv(0) &= Mv_0, \end{aligned} \tag{238}$$

where $y_i = L_i\Theta_0$, $i = 1,\dots,n_1$, $f = D_t\widetilde{g}$ and $Mv_0 = [\lambda_0 M + L]\Theta_0 + \widetilde{g}(0,\cdot)$ (indeed, since M is the multiplication operator by the function m_1 independent of t, from the differential equation in (234) with $t = 0$ we get $Mv(0) = MD_t\Theta(0) = [\lambda_0 M + L]\Theta(0) + \widetilde{g}(0)$). Of course, (238) is the special case $(i_1, i_2, n_2) = (i, i, n_1)$, $h_i = k_i$, $i = 1\dots,n_1$, of problem (160).

Conversely, assume that $v \in C^\delta(I_T;\mathscr{D}(L))$ solves (238). Then, the function Θ defined by (236) belongs to $C^{1+\delta}(I_T;\mathscr{D}(L))$ and solves (234). Indeed, using the fact that m_1

does not depend on time and that M, L, and L_i, $i = 1, \dots, n_1$, are closed, we obtain

$$\begin{aligned}
&D_t\left(M\Theta(t)\right) - \left[\lambda_0 M + L\right]\Theta(t) \\
&\quad - \sum_{i=1}^{n_1}\mathscr{K}\left(k_i, L_i\Theta\right)(t) - \tilde{g}(t) \\
&= D_t\left[M\left(\Theta_0 + \int_0^t v(s)\,\mathrm{d}s\right)\right] \\
&\quad - \left[\lambda_0 M + L\right]\left[\Theta_0 + \int_0^t v(s)\,\mathrm{d}s\right] \\
&\quad - \sum_{i=1}^{n_1}\int_0^t k_i(t-s)L_i\left[\Theta_0 + \int_0^s v(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s \\
&\quad - \tilde{g}(0) - \int_0^t D_s\tilde{g}(s)\,\mathrm{d}s \\
&= Mv(t) - \left[\lambda_0 M + L\right]\Theta_0 \\
&\quad - \int_0^t\left[\lambda_0 M + L\right]v(s)\,\mathrm{d}s \\
&\quad - \sum_{i=1}^{n_1}\int_0^t k_i(t-s)L_i\Theta_0\mathrm{d}s \\
&\quad - \sum_{i=1}^{n_1}\int_0^t k_i(t-s)\left[\int_0^s L_i v(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s \\
&\quad - \tilde{g}(0) - \int_0^t f(s)\,\mathrm{d}s.
\end{aligned} \tag{239}$$

Now, observe that

$$\begin{aligned}
Mv(t) &= Mv_0 + \int_0^t D_s\left(Mv(s)\right)\mathrm{d}s \\
&= \left[\lambda_0 M + L\right]\Theta_0 + \tilde{g}(0) + \int_0^t D_s\left(Mv(s)\right)\mathrm{d}s,
\end{aligned} \tag{240}$$

$$\int_0^t k_i(t-s)L_i\Theta_0\mathrm{d}s = \int_0^t k_i(s)L_i\Theta_0\mathrm{d}s = \int_0^t k_i(s)y_i\mathrm{d}s, \quad i = 1, \dots, n_1, \tag{241}$$

whereas an application of Fubini's theorem combined with the changes of variables $\xi = s - r$, $r - s = \tau$ and $t - s = \zeta$ easily yields for every $i = 1, \dots, n_1$ the following:

$$\begin{aligned}
&\int_0^t k_i(t-s)\left[\int_0^s L_i v(\xi)\,\mathrm{d}\xi\right]\mathrm{d}s \\
&\quad = \int_0^t k_i(t-s)\left[\int_0^s L_i v(s-r)\,\mathrm{d}r\right]\mathrm{d}s \\
&\quad = \int_0^t\left[\int_s^t k_i(t-r)L_i v(r-s)\,\mathrm{d}r\right]\mathrm{d}s \\
&\quad = \int_0^t\left[\int_0^{t-s} k_i(t-s-\tau)L_i v(\tau)\,\mathrm{d}\tau\right]\mathrm{d}s \\
&\quad = \int_0^t \mathscr{K}\left(k_i, L_i v\right)(t-s)\,\mathrm{d}s \\
&\quad = \int_0^t \mathscr{K}\left(k_i, L_i v\right)(\zeta)\,\mathrm{d}\zeta = \int_0^t \mathscr{K}\left(k_i, L_i v\right)(s)\,\mathrm{d}s.
\end{aligned} \tag{242}$$

Therefore, replacing (240)–(242) in (239), it follows for every $t \in I_T$ that

$$\begin{aligned}
&D_t\left(M\Theta(t)\right) - \left[\lambda_0 M + L\right]\Theta(t) - \sum_{i=1}^{n_1}\mathscr{K}\left(k_i, L_i\Theta\right)(t) - \tilde{g}(t) \\
&\quad = \int_0^t\Big\{D_s\left(Mv(s)\right) - \left[\lambda_0 M + L\right]v(s) \\
&\qquad - \sum_{i=1}^{n_1}\left[\mathscr{K}\left(k_i, L_i v\right)(s) + k_i(s)y_i\right] - f(s)\Big\}\,\mathrm{d}s,
\end{aligned} \tag{243}$$

and the latter integral is equal to zero by virtue of (238). Since from (236) it follows that $\Theta(0) = \Theta_0$, we have thus shown that (234) and (238) are *equivalent*. Such an equivalence is the first step in solving the mentioned inverse problem of recovering the vector $(\Theta, k_1, \dots, k_{r_1})$ with the help of the additional information (235).

Let us now apply the linear functional Ψ_j, $j = 1 \dots, r_1$, to (238). Using

$$\begin{aligned}
\Psi_j\left[D_t^k\left(Mv(t)\right)\right] &= \Psi_j\left[MD_t^{k+1}\Theta(t)\right] = D_t^{k+1}\Psi_j\left[M\Theta(t)\right] \\
&= D_t^{k+1}g_j(t), \quad k = 0, 1,
\end{aligned} \tag{244}$$

we thus find the following system of r_1 equations for the r_1 unknown $k_1, \dots, k_{r_1}$:

$$\begin{aligned}
&\sum_{i=1}^{r_1}\Psi_j\left[y_i\right]k_i(t) \\
&\quad = N_j(t) - \Psi_j\left[Lv\right] - \sum_{i=1}^{n_1}\Psi_j\left[\mathscr{K}\left(k_i, L_i v\right)(t)\right], \\
&\qquad j = 1, \dots, r_1,
\end{aligned} \tag{245}$$

where we have set (recall that $k_{r_1+n} = -D_t u_n$, $n = 1, 2$, are known)

$$\begin{aligned}
N_j(t) = &\left(D_t - \lambda_0\right)D_t g_j(t) - \Psi_j\left[f(t)\right] \\
&- \sum_{n=1}^{2}\Psi_j\left[y_{r_1+n}\right]k_{r_1+n}(t), \quad j = 1, \dots, r_1.
\end{aligned} \tag{246}$$

Therefore, if the matrix $\mathscr{U} := \mathscr{U}^{y_1,\ldots,y_{r_1}}_{\Psi_1,\ldots,\Psi_{r_1}} = (\Psi_i[y_j])_{i,j=1,\ldots,r_1}$ has determinant $\det \mathscr{U} \neq 0$, then from Cramer's formula it follows that the solution $(k_1,\ldots,k_{r_1})$ of (245) is given by

$$\begin{aligned} k_j(t) &= [\det \mathscr{U}]^{-1} \sum_{k=1}^{r_1} \Big\{ N_k(t) - \Psi_k[Lv] \\ &\quad - \sum_{i=1}^{n_1} \Psi_k[\mathscr{K}(k_i, L_i v)(t)] \Big\} \mathscr{U}_{k,j} \\ &=: \widetilde{R}_j(v, k_1, \ldots, k_{r_1})(t), \quad j = 1, \ldots, r_1, \end{aligned} \tag{247}$$

with $\mathscr{U}_{k,j}$, $k, j = 1, \ldots, r_1$, being the cofactor of the element $\Psi_k[y_j]$ of $\mathscr{U}$ (with the convention that $\mathscr{U}_{1,1} = 1$ in the case of $r_1 = 1$). We have thus found a system of r_1 fixed-point equations for the r_1 unknown $k_1, \ldots, k_{r_1}$.

Now, let $Y^r_\psi \in \{(X, \mathscr{D}(A))_{\psi,r}, X^{\psi,r}_A\}$, $\psi \in (0,1)$, $r \in [1,\infty]$, where A is as in (233). Assume that v_0 in the initial condition $Mv(0) = Mv_0$ belongs to $\mathscr{D}(L)$ and that

$$\begin{aligned} &k_i \in C^{\eta_i}(I_T, \mathbf{R}), \qquad f \in C^{\mu}(I_T; X), \qquad \eta_i, \mu \in \left(\frac{1}{q'}, 1\right), \\ &\qquad\qquad i = 1, \ldots, n_1, \\ &y_i \in Y^p_{\gamma_i}, \qquad v_1 + f(0) \in Y^p_\varphi, \quad \gamma_i, \varphi \in \left(\frac{1}{q'}, 1\right), \\ &\qquad\qquad p \in [1,\infty], \quad i = 1, \ldots, n_1, \end{aligned} \tag{248}$$

where $v_1 = (\lambda_0 M + L)v_0$ and q' is the conjugate exponent of $q \in (1,\infty)$. Then (cf. (179) with $(i_1, i_2, n_2) = (i, i, n_1)$, $(\alpha, \beta, Z) = (1, 1/q, \mathbf{R})$, and $h_i = k_i$, $i = 1, \ldots, n_1$), problem (238) is equivalent to the fixed-point equation

$$\begin{aligned} w(t) &= R(w, k_1, \ldots, k_{r_1})(t) + \sum_{l=0}^{1} w_l(k_1, \ldots, k_{r_1})(t) \\ &=: T(w, k_1, \ldots, k_{r_1})(t), \end{aligned} \tag{249}$$

where $w = L(v - v_0)$ and

$$\begin{aligned} w_0(k_1, \ldots, k_{r_1}) &= Q_7 x_0 + \sum_{i=1}^{n_1} \left[Q_6(k_i, L_i v_0) + Q_5 \widetilde{k}_i \right] + Q_5 \widetilde{f}, \\ w_1(k_1, \ldots, k_{r_1}) &= -\sum_{i=1}^{n_1} \left[Q_3(k_i, L_i v_0) + Q_4(k_i, y_i) \right] - Q_2 f, \\ R(w, k_1, \ldots, k_{r_1}) &= \lambda_0 \left[Q_5(A^{-1}w) - Q_2(A^{-1}w) \right] \\ &\quad + \sum_{i=1}^{n_1} \left[Q_6(k_i, S_i w) - Q_3(k_i, S_i w) \right]. \end{aligned} \tag{250}$$

Here, the Q_j's, $j = 2, \ldots, 6$, are defined by (106)–(110), $S_i = L_i L^{-1}$, and the functions $\widetilde{f}, \widetilde{k}_i$ and $Q_7 x_0$ are defined by $\widetilde{f}(t) = f(t) - f(0)$, $\widetilde{k}_i(t) = [k_i(t) - k_i(0)] y_i$, and $[Q_7 x_0](t) = (e^{tA} - I)x_0$, respectively, where (cf. (174)) $x_0 = \sum_{i=1}^{n_1} k_i(0) y_i + v_1 + f(0)$.

Then, since $v = L^{-1}w + v_0$, if we set $R_j(w, k_1, \ldots, k_{r_1}) = \widetilde{R}_j(L^{-1}w + v_0, k_1, \ldots, k_{r_1})$, $j = 1, \ldots, r_1$, and

$$\begin{aligned} &\Xi(w, k_1, \ldots, k_{r_1}) \\ &\quad = \left(T(w, k_1, \ldots, k_{r_1}), R_1(w, k_1, \ldots, k_{r_1}), \right. \\ &\qquad \left. \ldots, R_{r_i}(w, k_1, \ldots, k_{r_1}) \right), \end{aligned} \tag{251}$$

from (247) and (249) we deduce that to solve the inverse problems (234) and (235) for the unknown vector $(\Theta, k_1, \ldots, k_{r_1})$, it suffices to show that the fixed-point equation

$$(w, k_1, \ldots, k_{r_1}) = \Xi(w, k_1, \ldots, k_{r_1}) \tag{252}$$

has a unique solution. In general, this will be done by proving that Ξ is a contraction map in the Banach space

$$\begin{aligned} &Z_{\delta,\eta_1,\ldots,\eta_{r_1}} = C^{\delta}(I_T; X) \times C^{\eta_1}(I_T; \mathbf{R}) \times \cdots \times C^{\eta_{r_1}}(I_T; \mathbf{R}), \\ &\left\| (f_0, f_1, \ldots, f_{r_1}) \right\|_{Z_{\delta,\eta_1,\ldots,\eta_{r_1}}} \\ &\quad = \|f_0\|_{\delta,0,T;X} + \|f_1\|_{\eta_1,0,T;\mathbf{R}} + \cdots + \|f_{r_1}\|_{\eta_{r_1},0,T;\mathbf{R}}, \end{aligned} \tag{253}$$

at least for opportunely chosen Hölder exponents $\delta \in (0,1)$ and $\eta_i \in (1/q', 1)$, $i = 1, \ldots, r_1$, and, eventually, sufficiently small values of $T > 0$. It is just in the choice of δ and the η_i's that the main result of Section 5 plays a key role. The Hölder exponents have to be chosen so that the direct problem (234) in which the k_i's are assumed to be known is well posed. Due to the shown equivalence between problems (234) and (238), the well-posedness of the direct problem (234) is then a consequence of Theorem 48 and formula (236). More precisely, recalling Remark 50 for the case $\alpha = 1$, an application of that theorem yields the following maximal time regularity result for the solution Θ of (234).

Theorem 57. *Let X, $\mathscr{D}(M)$, $\mathscr{D}(L)$, and $\mathscr{D}(L_i)$, $i = 1, \ldots, n_1$, $n_1 = r_1 + 2$, $r_1 \in \mathbf{N}$, be defined by* (225) *and* (226) *with $q \in (1,2)$. Let M, L, and L_i, $i = 1, \ldots, n_1$, be defined by* (227)–(229) *through* (209), (212), *and* (215)–(221), *and let* (230) *and* (231) *be satisfied. Further, let $(A, \mathscr{D}(A))$ be defined by* (233), *and let $Y^r_\psi \in \{(X, \mathscr{D}(A))_{\psi,r}, X^{\psi,r}_A\}$, $\psi \in (0,1)$, $r \in [1,\infty]$. Let $\eta_i \in (1/q', 1)$ and $\gamma_i, \varphi \in (2/q', 1)$, $i = 1, \ldots, n_1$, and assume that*

$$\begin{aligned} &k_i \in C^{\eta_i}(I_T; \mathbf{R}), \quad i = 1, \ldots, n_1, \qquad \Theta_0 \in \mathscr{D}(L), \\ &(\lambda_0 M + L)\Theta_0 + \widetilde{g}(0, \cdot) = Mv_0 \quad \text{for some } v_0 \in \mathscr{D}(L), \\ &L_i \Theta_0 \in Y^r_{\gamma_i}, \qquad v_1 + D_t \widetilde{g}(0, \cdot) \in Y^r_\varphi, \quad i = 1, \ldots, n_1, \\ &\qquad\qquad r \in [1,\infty], \end{aligned} \tag{254}$$

where k_i, $i = 1, \dots, n_1$, $\tilde{g}$ *and* λ_0 *are defined by* (222) *and* (223) *through* (211) *and* (216), *whereas* $v_1 = (\lambda_0 M + L)v_0$. *Let* $\gamma = \min_{i=1,\dots,n_1}\{\gamma_i, \varphi\}$ *and* $\tau = \min_{i=1,\dots,n_1}\{\eta_i, \gamma - 1/q'\}$, *and let* $I_{1,1/q,\tau} \subseteq (1/q', 1/2)$ *be the interval defined by (cf.* (207) *with* $\beta = 1/q$)

$$I_{1,1/q,\tau} = \left(\frac{1}{q'}, \tau\right], \quad \text{if } \tau \in \left(\frac{1}{q'}, \frac{1}{2}\right),$$
$$I_{1,1/q,\tau} = \left(\frac{1}{q'}, \frac{1}{2}\right), \quad \text{if } \tau \in \left[\frac{1}{2}, 1\right). \tag{255}$$

Then, for every fixed $\delta \in I_{1,1/q,\tau}$ *problem* (234), *or, equivalently, problem* (224), *admits a unique strict solution* $\Theta \in C^{1+\delta}(I_T; \mathscr{D}(L))$ *satisfying* $D_t\Theta(0) = v_0$ *and such that* $D_t M\Theta, L\Theta \in C^{1+\delta}(I_T; X)$, *provided that* $D_t\tilde{g} \in C^{\mu}(I_T; X)$, $\mu \in [\delta + 1/q', 1)$.

Proof. Apply Theorem 48 with $(i_1, i_2, n_2) = (i, i, n_1)$, $(\alpha, \beta, Z) = (1, 1/q, \mathbf{R})$, and $h_i = k_i$, $i = 1, \dots, n_1$, to the equivalent problem (238). Since M is the multiplication operator by the function m_1 independent of t, the assertion then follows from $D_t\Theta = v \in C^{\delta}(I_T; \mathscr{D}(L))$, $D_t\Theta(0) = v(0)$, $D_t L\Theta = Lv \in C^{\delta}(I_T; X)$ and $D_t^2 M\Theta = D_t M v \in C^{\delta}(I_T; X)$. □

Larger values of q in Theorem 57 can be obtained assuming more smoothness and some order of vanishing for the function m_1. In fact, let $m_1 \in C^1(\overline{\Omega})$ be such that the following estimate holds for some positive constant K:

$$\left|\nabla m_1(x)\right| := \left\{\sum_{j=1}^{N}\left[D_{x_j} m_1(x)\right]^2\right\}^{1/2} \le K[m_1(x)]^{\vartheta}, \quad x \in \overline{\Omega}, \quad \vartheta \in (0, 1). \tag{256}$$

Then (232) holds with $\beta = 1/q$ being replaced by (cf. [41, formulae (3.23) and (4.41)]):

$$\beta = \frac{1}{2-\vartheta}, \quad \text{if } q \in (2-\vartheta, 2),$$
$$\beta = \frac{2}{q(2-\vartheta)}, \quad \text{if } q \in [2, \infty). \tag{257}$$

(precisely, in [41, formula (3.23)] it is shown that $(|\lambda| + 1)\|Mu\|_{q;\Omega}^{q(2-\vartheta)/2} \le C_q[\|f\|_{q;\Omega}\|Mu\|_{q;\Omega}^{-1+q(2-\vartheta)/2} + \|f\|_{q;\Omega}^{q(2-\vartheta)/2}]$, where $u = (\lambda M - L)^{-1}f$ and $q \in [2, \infty)$. Using (cf. [41, formula (2.15)]) $\|Mu\|_{q;\Omega} \le \|m_1\|_{\infty;\Omega}\|u\|_{q;\Omega} \le C\|m_1\|_{\infty;\Omega}\|f\|_{q;\Omega}$, we thus find that $(|\lambda| + 1)\|Mu\|_{q;\Omega}^{q(2-\vartheta)/2} \le C_q[(C\|m_1\|_{\infty;\Omega})^{-1+q(2-\vartheta)/2} + 1]\|f\|_{q;\Omega}^{q(2-\vartheta)/2}$; that is, $\|M(\lambda M - L)^{-1}\|_{\mathscr{L}(X)} \le \{C_q[(C\|m_1\|_{\infty;\Omega})^{-1+q(2-\vartheta)/2} + 1]\}^{2/[q(2-\vartheta)]}(|\lambda| + 1)^{-2/[q(2-\vartheta)]}$). Under (256) we thus find the following better result, where q may be greater than two.

Theorem 58. *Let* (256) *holds, and let* X, $(M, \mathscr{D}(M))$, $(L, \mathscr{D}(L))$, $(L_i, \mathscr{D}(L_i))$, $i = 1, \dots, n_1$, *be as in Theorem 57, but with* $q \in (2-\vartheta, 2) \cup [2, 4/(2-\vartheta))$. *Let* (254) *be fulfilled, but with* $\eta_i \in (1-\beta, 1)$ *and* $\gamma_i, \varphi \in (2-2\beta, 1)$, $i = 1, \dots, n_1$, *where* β *is as in* (257). *Let* $\gamma = \min_{i=1,\dots,n_1}\{\gamma_i, \varphi\}$ *and* $\tau = \min_{i=1,\dots,n_1}\{\eta_i, \beta + \gamma - 1\}$, *and let* $I_{1,\beta,\tau}$ *be as in* (207). *Then, for every fixed* $\delta \in I_{1,\beta,\tau}$ *problem* (234), *or, equivalently, problem* (224), *admits a unique strict solution* $\Theta \in C^{1+\delta}(I_T; \mathscr{D}(L))$ *satisfying* $D_t\Theta(0) = v_0$ *and such that* $D_t M\Theta, L\Theta \in C^{1+\delta}(I_T; X)$, *provided that* $D_t\tilde{g} \in C^{\mu}(I_T; X)$, $\mu \in [\delta + 1 - \beta, 1)$.

Proof. It suffices to observe that for every $\vartheta \in (0,1)$ and $q \in (2-\vartheta, 2) \cup [2, 4/(2-\vartheta))$, the number β in (257) satisfies $\beta > 1/2$. Hence, proceeding as in the proofs of Theorem 57, except for replacing there $\beta = 1/q$ with β as in (257), we get the assertion. □

Appendix

Here we clarify why the definition of Q_2 in [20] has to be modified in accordance to that in this paper. To avoid confusion with the present notation, we will denote the operator Q_2 in [20] with S_2. Precisely, in [20, formula (4.12)], S_2 was defined as follows:

$$[S_2 g_2](t) := \int_0^t \left[(-A)^1\right]^{\circ} \mathrm{e}^{(t-s)A} g_2(s)\,\mathrm{d}s, \quad t \in [0, T], \tag{A.1}$$

and considered as acting on functions $g_2 \in C_0^{\delta_2}([0,T]; X)$, $\delta_2 \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $3\alpha + \beta > 3$. Even though $g_2(0) = 0$, formula (A.1) may have no sense, since

$$\|S_2 g_2(t)\|_X \le \tilde{c}_{\alpha,\beta,1}|g_2|_{\delta_2,0,t;X}\int_0^t (t-s)^{(\beta-2)/\alpha} s^{\delta_2}\,\mathrm{d}s, \tag{A.2}$$

and the integral on the right is *not* convergent, the exponent $(\beta - 2)/\alpha$ being less or equal than -1. It is for this reason that $g_2(s)$ in (A.1) has to be replaced with the increment $g_2(s) - g_2(t)$ as in formula (106) (see inequality (118)) and to introduce the operator Q_5 as in (109). Of course, as a consequence, the definitions of Q_3 and Q_4 in [20, Lemmas 4.6 and 4.8] as $S_2\mathscr{K}(g_{3_1}, g_{3_2})$ and $S_2(g_4 y)$, respectively, have to be changed too in accordance with the present formulae (107) and (108) containing the increments $\mathscr{K}(g_{3_1}, g_{3_2})(s) - \mathscr{K}(g_{3_1}, g_{3_2})(t)$ and $[g_4(s) - g_4(t)]y$. To this purpose, we want to make clear that, contrarily to [20, Lemma 4.4], the statement and the proof of [20, Lemma 4.8] is correct, since there the function inside the integral on the right-hand side of (A.1) takes its values in an opportune intermediate space $X_A^{\theta,r}$. However, the correctness of that lemma does not suffice to proceed as in [20, Section 5] to solve problem (160) with $n_1 = n_2 = 1$.

For the reader's convenience we thus now indicate how to change the definitions of the functions w_j, $j = 0, 1$, and the operator Rw in [20, formulae (5.8)–(5.10)], and we state the amended version of [20, Theorems 5.6 and 5.7]. First, according to [20] where only this case was treated, let $n_1 = n_2 = 1$ in problem (160), and write k, h, y in place of k_1, h_1 and y_1, respectively. Then, under the same assumptions on the vector (α, β, k, h, f) as those in the present Section 5, it can be shown that problem (160) with $n_1 = n_2 = 1$ is equivalent to the fixed-point equation (179), where (cf. (180)–(182))

$$\begin{aligned} w_0 &= Q_7 x_0 + Q_6(k, L_1 v_0) + Q_5 \tilde{h} + Q_5 \tilde{f}, \\ w_1 &= -Q_3(k, L_1 v_0) - Q_4(h, y) - Q_2 f, \\ Rw &:= \lambda_0 \left[Q_5\left(A^{-1} w\right) - Q_2\left(A^{-1} w\right)\right] \\ &\quad + Q_6(k, Sw) - Q_3(k, Sw). \end{aligned} \tag{A.3}$$

Here, $x_0 = v_1 + h(0)y + f(0)$, $v_1 = (\lambda_0 M + L)v_0$, is the value at $t = 0$ of the function F_w defined by (169) with $n_l = n_2 = 1$, $Q_7 x_0$, $\tilde{f}$ and $\tilde{h}$ are defined, respectively, by $(e^{tA} - I)x_0$, $f(t) - f(0)$ and $[h(t) - h(0)]y$, S is the operator $L_1 L^{-1} \in \mathscr{L}(X)$, and the Q_j's, $j = 2, \ldots, 6$, are as in (106)–(110). Formulae (A.3) replace the definitions of w_0, w_1 and Rw in [20, formulae (5.8)–(5.10)]. Therefore, from Lemmas 42, 45, and 46 and Corollary 44 with $n_1 = n_2 = 1$ we obtain the following version of Theorem 48.

Theorem A.1. *Assume* (161) *and* $v_0 \in \mathscr{D}(L)$, *and let* $5\alpha + 2\beta > 6$ *in (H2). Assume that* $k \in C^{\eta}(I_T; Z)$, $h \in C^{\sigma}(I_T; \mathbf{C})$, $y \in Y^r_{\theta}$, *and* $(\lambda_0 M + L)v_0 + f(0) \in Y^r_{\varphi}$, *where* $\eta, \sigma \in ((3 - 2\alpha - \beta)/\alpha, 1)$, $\theta, \varphi \in (5 - 3\alpha - 2\beta, 1)$, *and* $r \in [1, \infty]$. *Let* $\gamma = \min\{\theta, \varphi\}$ *and* $\tau = \min\{\eta, \sigma, (\alpha + \beta + \gamma - 2)/\alpha\}$. *Then, for every fixed* $\delta \in I_{\alpha,\beta,\tau}$ *the problem*

$$\begin{aligned} D_t(Mv(t)) &= [\lambda_0 M + L] v(t) + \mathscr{K}(k, L_1 v)(t) \\ &\quad + h(t) y + f(t), \quad t \in I_T, \\ Mv(0) &= Mv_0 \end{aligned} \tag{A.4}$$

admits a unique strict solution $v \in C^{\delta}(I_T; \mathscr{D}(L))$ *satisfying* $v(0) = v_0$ *and such that* $Lv, D_t Mv \in C^{\delta}(I_T; X)$, *provided that* $f \in C^{\mu}(I_T; X)$, $\mu \in [\delta + (3 - 2\alpha - \beta)/\alpha, 1)$.

Theorem A.1 substitutes [20, Theorem 5.6 and 5.7]. Notice that, differently than [20], here only one statement occurs. In fact, the more suitable procedure followed in this paper makes the separation in [20] of two distinct intervals in which γ may vary totally unneeded. Finally, letting $n_1 = n_2 = 1$ in Theorems 52, 5.14, 54, and 56, we obtain the correct versions of [20, Theorems 5.11, 53, and 5.16] for the subcases of (A.4) corresponding to the choices $\lambda_0 = h = 0$, $\lambda_0 = f = 0$, $\lambda_0 = k = h = 0$, and $\lambda_0 = k = f = 0$, respectively. For saving space, we leave this easy task to the reader.

References

[1] A. Favini and A. Yagi, "Multivalued linear operators and degenerate evolution equations," *Annali di Matematica Pura ed Applicata*, vol. 163, pp. 353–384, 1993.

[2] A. Favini and A. Yagi, *Degenerate Differential Equations in Banach Spaces*, vol. 215, Marcel Dekker, New York, NY, USA, 1999.

[3] A. Yagi, "Generation theorem of semigroup for multivalued linear operators," *Osaka Journal of Mathematics*, vol. 28, no. 2, pp. 385–410, 1991.

[4] P. Acquistapace and B. Terreni, "Existence and sharp regularity results for linear parabolic nonautonomous integro-differential equations," *Israel Journal of Mathematics*, vol. 53, no. 3, pp. 257–303, 1986.

[5] G. Da Prato, M. Iannelli, and E. Sinestrari, "Regularity of solutions of a class of linear integro-differential equations in Banach spaces," *Journal of Integral Equations*, vol. 8, no. 1, pp. 27–40, 1985.

[6] G. Da Prato and M. Iannelli, "Existence and regularity for a class of integro-differential equations of parabolic type," *Journal of Mathematical Analysis and Applications*, vol. 112, no. 1, pp. 36–55, 1985.

[7] G. Da Prato, "Abstract differential equations, maximal regularity and linearization," in *Proceedings of the Symposia in Pure Mathematics*, vol. 45, part 1, pp. 359–370, 1986.

[8] A. Lunardi and E. Sinestrari, "C^{α}-regularity for nonautonomous linear integro-differential equations of parabolic type," *Journal of Differential Equations*, vol. 63, no. 1, pp. 88–116, 1986.

[9] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, Switzerland, 1995.

[10] J. Prüss, "On linear Volterra equations of parabolic type in Banach spaces," *Transactions of the American Mathematical Society*, vol. 301, no. 2, pp. 691–721, 1987.

[11] E. Sinestrari, "On the abstract Cauchy problem of parabolic type in spaces of continuous functions," *Journal of Mathematical Analysis and Applications*, vol. 107, no. 1, pp. 16–66, 1985.

[12] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, American Mathematical Society, Providence, RI, USA, 1957.

[13] C. Wild, "Semi-groupes de croissance $\alpha < 1$ holomorphes," *Comptes Rendus de l'Académie des Sciences*, vol. 285, no. 6, pp. A437–A440, 1977 (French).

[14] K. Taira, "On a degenerate oblique derivative problem with interior boundary conditions," *Proceedings of the Japan Academy*, vol. 52, no. 9, pp. 484–487, 1976.

[15] K. Taira, "The theory of semigroups with weak singularity and its applications to partial differential equations," *Tsukuba Journal of Mathematics*, vol. 13, no. 2, pp. 513–562, 1989.

[16] W. von Wahl, "Gebrochene Potenzen eines elliptischen Operators und parabolische Differentialgleichungen in Räumen Höldersteiger Funktionen," *Nachrichten der Akademie der Wissenschaften in Göttingen. II*, vol. 11, pp. 231–258, 1972 (German).

[17] W. von Wahl, "Lineare und semilineare parabolische Differentialgleichungen in Räumen hölderstetiger Funktionen," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 43, pp. 234–262, 1975 (German).

[18] W. von Wahl, "Neue Resolventenabschätzungen für elliptische Differentialoperatoren und semilineare parabolische Gleichungen," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 46, pp. 179–204, 1977 (German).

[19] A. Favaron, "Optimal time and space regularity for solutions of degenerate differential equations," *Central European Journal of Mathematics*, vol. 7, no. 2, pp. 249–271, 2009.

[20] A. Favaron and A. Favini, "Maximal time regularity for degenerate evolution integro-differential equations," *Journal of Evolution Equations*, vol. 10, no. 2, pp. 377–412, 2010.

[21] A. Favini and A. Yagi, "Space and time regularity for degenerate evolution equations," *Journal of the Mathematical Society of Japan*, vol. 44, no. 2, pp. 331–350, 1992.

[22] A. Favini, A. Lorenzi, and H. Tanabe, "Singular integro-differential equations of parabolic type," *Advances in Differential Equations*, vol. 7, no. 7, pp. 769–798, 2002.

[23] A. Favini, A. Lorenzi, and H. Tanabe, "Singular evolution integro-differential equations with kernels defined on bounded intervals," *Applicable Analysis*, vol. 84, no. 5, pp. 463–497, 2005.

[24] A. Favaron and A. Favini, "Fractional powers and interpolation theory for multivalued linear operators and applications to degenerate differential equations," *Tsukuba Journal of Mathematics*, vol. 35, no. 2, pp. 259–323, 2011.

[25] R. Cross, *Multivalued Linear Operators*, vol. 213, Marcel Dekker, New York, NY, USA, 1998.

[26] A. Favini and A. Yagi, "Quasilinear degenerate evolution equations in Banach spaces," *Journal of Evolution Equations*, vol. 4, no. 3, pp. 421–449, 2004.

[27] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, vol. 18, North-Holland Publishing, Amsterdam, The Netherlands, 1978.

[28] H. Berens, *Interpolationsmethoden zur Behandlung von Approximationsprozessen auf Banachräumen*, Lecture Notes in Mathematics, Springer, Berlin, Germany, 1968 (German).

[29] E. Gagliardo, "Interpolation d'espaces de Banach et applications. III," *Comptes Rendus de l'Académie des Sciences*, vol. 248, pp. 3517–3518, 1959 (French).

[30] P. Grisvard, "Commutativité de deux foncteurs d'interpolation et applications," *Journal de Mathématiques Pures et Appliquées*, vol. 45, pp. 143–206, 1966 (French).

[31] H. Berens and P. L. Butzer, "Approximation theorems for semi-group operators in intermediate spaces," *Bulletin of the American Mathematical Society*, vol. 70, pp. 689–692, 1964.

[32] A. Belleni-Morante, "An integro-differential equation arising from the theory of heat conduction in rigid materials with memory," *Unione Matematica Italiana*, vol. 15, no. 2, pp. 470–482, 1978.

[33] G. Da Prato and M. Iannelli, "Linear integro-differential equations in Banach spaces," *Rendiconti del Seminario Matematico dell'Università di Padova*, vol. 62, pp. 207–219, 1980.

[34] G. Da Prato, M. Iannelli, and E. Sinestrari, "Temporal regularity for a class of integro-differential equations in Banach spaces," *Bollettino Unione Matematica Italiana*, vol. 2, no. 1, pp. 171–185, 1983.

[35] A. Favini, A. Lorenzi, and H. Tanabe, "Degenerate integrodifferential equations of parabolic type with Robin boundary conditions: L^2-theory," *Journal of the Mathematical Society of Japan*, vol. 61, no. 1, pp. 133–176, 2009.

[36] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, Germany, 2001, Reprint of the 1998 edition.

[37] B. D. Coleman and M. E. Gurtin, "Equipresence and constitutive equations for rigid heat conductors," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 18, pp. 199–208, 1967.

[38] M. E. Gurtin, "On the thermodynamics of materials with memory," *Archive for Rational Mechanics and Analysis*, vol. 28, no. 1, pp. 40–50, 1968.

[39] R. K. Miller, "An integro-differential equation for rigid heat conductors with memory," *Journal of Mathematical Analysis and Applications*, vol. 66, no. 2, pp. 313–332, 1978.

[40] J. W. Nunziato, "On heat conduction in materials with memory," *Quarterly of Applied Mathematics*, vol. 29, pp. 187–204, 1971.

[41] A. Favini, A. Lorenzi, H. Tanabe, and A. Yagi, "An L^p-approach to singular linear parabolic equations in bounded domains," *Osaka Journal of Mathematics*, vol. 42, no. 2, pp. 385–406, 2005.

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Research Article

Solving a System of Linear Volterra Integral Equations Using the Modified Reproducing Kernel Method

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A numerical technique based on reproducing kernel methods for the exact solution of linear Volterra integral equations system of the second kind is given. The traditional reproducing kernel method requests that operator a satisfied linear operator equation $Au = f$, is bounded and its image space is the reproducing kernel space $W_2^1[a,b]$. It limits its application. Now, we modify the reproducing kernel method such that it can be more widely applicable. The n-term approximation solution obtained by the modified method is of high accuracy. The numerical example compared with other methods shows that the modified method is more efficient.

1. Introduction

The purpose of this paper is to solve a system of linear Volterra integral equations

$$F(s) = Gs + \int_a^b K(s,t)\,F(t)\,dt, \quad s \in [0,1], \tag{1}$$

where

$$\begin{aligned} F(s) &= [f_1(s), f_2(s), \ldots, f_n(s)]^T, \\ G(s) &= [g_1(s), g_2(s), \ldots, g_n(s)]^T, \\ K(s,t) &= [k_{i,j}], \quad i,j = 1,2,\ldots,n. \end{aligned} \tag{2}$$

In (1), the functions K and G are given, and F is the solution to be determined. We assume that (1) has a unique solution. Volterra integral equation arises in many physical applications, for example, potential theory and Dirichlet problems, electrostatics, mathematical problems of radiative equilibrium, the particle transport problems of astrophysics and reactor theory, and radiative heat transfer problems [1–5]. Several valid methods for solving Volterra integral equation have been developed in recent years, including power series method [6], Adomain's decomposition method [7], homotopy perturbation method [8, 9], block by block method [10], and expansion method [11].

Since the reproducing kernel space $W_2^1[a,b]$, which is a special Hilbert space, is constructed in 1986 [12], the reproducing kernel theory has been applied successfully to many linear and nonlinear problems, such as differential equation, population model, and many other equations appearing in physics and engineering [12–21]. The traditional reproducing kernel method is limited, because it requires that the image space of operator A in linear operator equation $Au = f$ is $W_2^1[a,b]$ and operator A must be bounded. In order to enlarge its application range, the MRKM removes the boundedness of A and weakens its image space to $L^2[a,b]$. Subsequently, we apply the MRKM to obtain the series expression of the exact solution for (1). The n-term approximation solution is provided by truncating the series. The final numerical comparisons between our method and other methods show the efficiency of the proposed method. It is worth to mention that the MRKM can be generalized to solve other system of linear equations.

2. Preliminaries

2.1. The Reproducing Kernel Space $W_2^1[0,1]$. The reproducing kernel space $W_2^1[0,1]$ consists of all absolute continuous real-valued functions, which defined on the closed interval $[0,1]$, and the first derivative functions belong to $L^2[0,1]$.

The inner product and the norm are equipped with

$$(u,v)_{w_2^1} = u(0)\,v(0) + \int_0^1 u'(x)\,v'(x)\,dx, \quad \forall u,v \in w_2^1,$$
$$\|u\|_{W_2^1} = \sqrt{(u,v)_{w_2^1}}. \tag{3}$$

Theorem 1. *$W_2^1[0,1]$ is a reproducing kernel space with reproducing kernel [22]*

$$R_x(y) = \begin{cases} 1+y, & y \le x \\ 1+x, & y > x; \end{cases} \tag{4}$$

that is, for every $x \in [0,1]$ and $u \in W_2^1$, it follows that

$$(u(y), R_x(y))_{w_2^1} = u(x). \tag{5}$$

2.2. The Reproducing Kernel Space $W_2^2[0,1]$. The reproducing kernel space $W_2^2[0,1]$ consists of all real-valued functions in which the first derivative functions are absolute continuous on the closed interval $[0,1]$ and the second derivative functions belong to $L^2[0,1]$.

The inner product and the norm are equipped with

$$(u,v)_{W_2^2} = \sum_{k=0}^{1} u^{(k)}(0)\,v^{(k)}(0) + \int_0^1 u''(x)\,v''(x)\,dx, \quad \forall u,v \in W_2^2[0,1], \tag{6}$$
$$\|u\|_{W_2^2} = \sqrt{(u,u)_{w_2^2}}.$$

Theorem 2. *$W_2^2[0,1]$ is a reproducing kernel space with reproducing kernel [22]*

$$Q(x,y) = \begin{cases} 1 + x \times y + \dfrac{x \times y^2}{2} - \dfrac{y^3}{6} & y \le x \\ 1 + x \times y + \dfrac{x^2 \times y}{2} - \dfrac{x^3}{6}, & y > x; \end{cases} \tag{7}$$

that is, for every $x \in [0,1]$ and $u \in W_2^2$, it follows that

$$(u(y), Q(x,y))_{w_2^2} = u(x). \tag{8}$$

The proof of Theorems 1 and 2 can be found in [23].

2.3. Hilbert Space E. Hilbert space E is defined by

$$E = \bigoplus_{i=1}^{n} W_2^1 = \left\{(u_1,\ldots,u_n)^T \mid u_i \in w_2^1, i = 1,\ldots,n\right\}. \tag{9}$$

The inner product and the norm are given by

$$(u,v)_E = \sum_{i=1}^{n} (u_i, v_i)_{w_2^1},$$
$$\|u\|_E = \sqrt{(u,u)_E}. \tag{10}$$

It is easy to prove that E is a Hilbert space.

3. The Exact Solution of (1)

3.1. Identical Transformation of (1). Consider the ith equation of (1):

$$f_i(s) - \sum_{j=1}^{n} \int_0^s K_{ij}(s,t)\,f_j(t)\,dt = g_i(s). \tag{11}$$

Define operator $A_{ij} : W_2^1 \to L^2[0,1]$, $j = 1,\ldots,n$,

$$A_{ij} = \begin{cases} u(s) - \int_0^1 k_{ij}(s,t)\,u(t)\,dt, & j = i \\ -\displaystyle\int_0^s k_{ij}(s,t)\,u(t)\,dt, & j \ne i, \end{cases} \tag{12}$$

where $u \in W_2^1$. Then, (1) can be turned into

$$\begin{gathered} A_{11}f_1 + A_{12}f_2 + \cdots + A_{1n}f_{1n} = g_1(s) \\ A_{21}f_1 + A_{22}f_2 + \cdots + A_{2n}f_{1n} = g_2(s) \\ \vdots \\ A_{n1}f_1 + A_{n2}f_2 + \cdots + A_{nn}f_{1n} = g_n(s), \end{gathered} \tag{13}$$

where $F(s) = [f_1(s), f_2(s), \ldots, f_n(s)]^T \in E$.

3.2. The Exact solution of (1). Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of interval $[0,1]$, and define

$$\Psi_{ij}(x) = \left(A_{j_1,y} R_x(y)\big|_{y=x_i}, A_{j_2,y} R_x(y)\big|_{y=x_i}, \ldots, A_{j_n,y} R_x(y)\big|_{y=x_i} \right)^T \tag{14}$$

for every $j = 1,2,\ldots,n$, $i = 1,2,\ldots$; the subscript y of $A_{ij,y}$ means that the operator A_{ij} acts on the function of y. It is easy to prove that $\Psi_{ij} \in E$.

Theorem 3. *$\{\Psi_{i1}, \Psi_{i2}, \ldots, \Psi_{in}\}_{i=1}^{\infty}$ is complete in E.*

Proof. Take $u = (u_1, u_2, \ldots, u_n)^T \in E$ such that $(u(x), \Psi_{ij}(x)) = 0$ for every $j = 1,2,\ldots,n$, $i = 1,2,\ldots$.

From this fact, it holds that

$$\begin{aligned} &(u(x), \Psi_{ij}(x)) \\ &= \Big((u_1, u_2, \ldots, u_n)^T, \\ &\quad \Big(A_{j1,y} R_x(y)\big|_{y=x_i}, A_{j2,y} R_x(y)\big|_{y=x_i}, \ldots, A_{jn,y} R_x(y)\big|_{y=x_i} \Big)^T \Big) \\ &= \sum_{k=1}^{n} A_{jk,y} (u_k(x), R_x(y))_{w_2^1}\big|_{y=x_i} \\ &= \sum_{k=1}^{n} A_{jk} u_k(x_i) = 0, \end{aligned} \tag{15}$$

for every $j = 1, 2, \ldots, n$. The dense $\{x_i\}_{i=1}^{\infty}$ assumes that

$$\begin{gathered} A_{11}u_1 + A_{12}u_2 + \cdots + A_{1n}u_n = 0 \\ A_{21}u_1 + A_{22}u_2 + \cdots + A_{2n}u_n = 0 \\ \vdots \\ A_{n1}u_1 + A_{n2}u_2 + \cdots + A_{nn}u_n = 0. \end{gathered} \tag{16}$$

Since (16) has a unique solution, it follows that $u = u_1, u_2, \ldots, u_n{}^T = 0$. This completes the proof. □

We arrange $\Psi_{11}, \Psi_{12}, \ldots, \Psi_{1n}, \Psi_{21}, \Psi_{22}, \ldots, \Psi_{2n}, \ldots, \Psi_{i1}, \Psi_{i2}, \ldots, \Psi_{in}, \ldots$, denoted by $\{r_i\}_{i=1}^{\infty}$; that is, $r_1 = \Psi_{11}, r_2 = \Psi_{12}, \ldots, r_n = \Psi_{1n}, r_{n+1} = \Psi_{21}, r_{n+2} = \Psi_{22}, \ldots, r_{n+n} = \Psi_{2n}, \ldots$. In a general way, $r_{(i-1)n+j} = \Psi_{ij}$, $i = 1, 2, 3, \ldots$; $j = 1, 2, \ldots, n$. The orthogonal basis $\{\overline{r}_i\}_{i=1}^{\infty}$ in E from Gram-Schmidt orthogonalization of $\{r_i\}_{i=1}^{\infty}$ is as follows:

$$\overline{r}_i = \sum_{k=1}^{i} \beta_{ik} r_k, \quad i = 1, 2, \ldots. \tag{17}$$

Theorem 4. *The exact solution of* (1) *can be expressed by*

$$F(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \rho_k \overline{r}_i(x), \tag{18}$$

where $\rho_k = (F(x), r_k)_E$; if $r_k = \Psi_{jl}$, then $\rho_k = g_l(x_j)$.

Proof. Assume that $F(x)$ is the exact solution of (1). $F(x)$ can be expanded to Fourier series in terms of normal orthogonal basis $\{\overline{r}_i(x)\}_{i=1}^{\infty}$ in E:

$$F(x) = \sum_{i=1}^{\infty} (F, \overline{r}_i)_E \overline{r}_i(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} (F, r_k)_E \overline{r}_i(x); \tag{19}$$

if $\rho_k = (F, r_k)_E$, then

$$F(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ij} \rho_k \overline{r}_i(x). \tag{20}$$

When $r_k = \Psi_{jl}$, it holds that

$$\rho_k = (F, \Psi_{jl}) = \sum_{k=1}^{n} A_{lk} u_k(x_j) = g_l(x_j). \tag{21}$$

□

Corollary 5. *The approximate solution of* (1) *is*

$$F_m(x) = \sum_{i=1}^{m} \sum_{k=1}^{i} \beta_{ik} \rho_k \overline{r}_i(x) = (f_{1,m}, f_{2,m}, \ldots, f_{n,m})^T, \tag{22}$$

and $f_{i,m}(x)$ converges uniformly to $f_i(x)$ on $[0,1]$ as $m \to \infty$ for every $i = 1, 2, \ldots, n$.

Proof. Obviously, $\|F_m - F\|_E^2 \to 0$ holds as $m \to \infty$; that is, $F_m(x)$ is the approximate solution of (1).

Note that $\sum_{i=1}^{n} \|f_{i,m} - f_i\|_{W_2^1}^2 = \|F_m - F\|_E^2 \to 0$. Combining with the expression of $R_x(y)$, we have

$$\begin{aligned} |f_{i,m} - f_i| &= \left|(f_{i,m}(y) - f_i(y), R_x(y))_{W_2^1}\right| \\ &\le \|f_{i,m} - f_i\|_{W_2^1} \cdot \|R_x(y)\|_{W_2^1} \\ &= \|f_{i,m} - f_i\|_{W_2^1} \sqrt{R_x(x)} \\ &\le \sqrt{2} \|f_{i,m} - f_i\|_{W_2^1}, \quad \forall x \in [0,1]. \end{aligned} \tag{23}$$

It shows that $f_{i,m}$ converges uniformly to f_i on $[0,1]$ as $m \to \infty$ for every $i = 1, 2, \ldots, n$. So the proof is complete. □

Remark 6. If $k_{ij}(s,t) \in C([0,1] \times [0,1])$ and $g_i \in W_2^2$ in (1), then it is reasonable to regard the unknown functions as the elements of W_2^2.

4. Numerical Examples

Taking nodes $\{x_i = (i-1)/(N-1)\}_{i=1}^{N}$, $f_{i,N}$ is the approximate solutions of f_i, and $e(f_{i,N})$ denotes the absolute errors of f_i, $i = 1, 2, \ldots, n$. According to Remark 6, we solve the following two examples appearing in [11] in W_2^2.

Example 7. Consider the following system of Volterra integral equations of the second kind [11]:

$$\begin{aligned} f_1(s) &= g_1(s) + \int_0^s (s-t)^3 f_1(t)\,dt + \int_0^s (s-t)^2 f_2(t)\,dt, \\ f_2(s) &= g_2(s) + \int_0^s (s-t)^4 f_1(t)\,dt \\ &\quad + \int_0^s (s-t)^3 f_2(t)\,dt, \end{aligned} \tag{24}$$

where $g_1(s)$ and $g_2(s)$ are chosen such that the exact solution is $f_1(s) = 1 + s^2$, $f_2(s) = 1 + s - s^3$. The numerical results obtained by using the present method are compared with [11] in Table 1.

Example 8. Consider the following system of linear Volterra integral equations of the second kind [11]:

$$\begin{aligned} f_1(s) &= g_1(s) + \int_0^s (\sin(s-t) - 1) f_1(t)\,dt \\ &\quad + \int_0^s (1 - t\cos s) f_2(t)\,dt, \\ f_2(s) &= g_2(s) + \int_0^s (f_1(t))\,dt + \int_0^s (s-t) f_2(t)\,dt, \end{aligned} \tag{25}$$

where $g_1(s)$ and $g_2(s)$ are chosen such that the exact solution is $f_1(s) = \cos s, f_2(s) = \sin s$. The numerical results obtained by using the present method are compared with [11] in Table 2.

Table 1: Absolute errors for Example 7.

Nodes x_i	Errors $e(f_1)$ [11]	Errors $e(f_{1,100})$	Errors $e(f_2)$ [11]	Errors $e(f_{2,100})$ [11]
0.0	0	$1.58309E-10$	0	$3.98245E-10$
0.1	$2.63472E-7$	$3.92220E-12$	$2.11685E-8$	$3.94493E-10$
0.2	$1.62592E-5$	$3.21563E-10$	$2.61132E-6$	$4.72710E-10$
0.3	$1.74905E-4$	$5.95890E-10$	$4.18979E-5$	$6.19366E-10$
0.4	$8.93799E-4$	$5.11051E-10$	$2.86285E-4$	$6.95422E-10$
0.5	$3.00491E-3$	$2.46104E-10$	$1.19940E-3$	$4.19959E-10$
0.6	$7.47528E-3$	$1.98685E-9$	$3.56141E-3$	$5.90035E-10$
0.7	$1.40733E-2$	$5.01512E-9$	$7.74239E-3$	$2.83080E-9$
0.8	$1.78384E-2$	$9.62848E-9$	$1.09171E-2$	$6.94058E-9$
0.9	$4.97756E-3$	$1.61180E-8$	$2.27326E-3$	$1.36984E-8$
1.0	$3.84378E-2$	$2.49043E-8$	$3.32111E-2$	$2.42565E-8$

Table 2: Absolute errors for Example 8.

Nodes x_i	Errors $e(f_1)$ [11]	Errors $e(f_{1,100})$	Errors $e(f_2)$ [11]	Errors $e(f_{2,100})$ [11]
0.0	0	$6.93348E-11$	0	$3.60316E-11$
0.1	$1.37735E-4$	$4.53518E-09$	$1.52721E-4$	$2.75123E-08$
0.2	$9.27188E-4$	$8.84879E-09$	$1.14715E-3$	$3.10611E-08$
0.3	$2.67117E-3$	$1.28253E-08$	$3.71248E-3$	$3.53307E-08$
0.4	$5.45507E-3$	$1.65442E-08$	$8.57201E-3$	$4.03402E-08$
0.5	$9.22670E-3$	$2.00881E-08$	$1.64412E-2$	$4.61209E-08$
0.6	$1.38644E-2$	$2.35657E-09$	$2.78243E-2$	$5.27214E-08$
0.7	$1.92960E-2$	$2.71160E-08$	$4.25337E-2$	$6.02041E-08$
0.8	$2.56349E-2$	$3.09302E-08$	$5.91212E-2$	$6.86601E-08$
0.9	$3.31574E-2$	$3.52645E-08$	$7.48883E-2$	$7.82029E-08$
1.0	$4.19808E-2$	$3.67322E-08$	$8.70896E-2$	$1.02387E-07$

5. Conclusion

In this paper, we modify the traditional reproducing kernel method to enlarge its application range. The new method named MRKM is applied successfully to solve a system of linear Volterra integral equations. The numerical results show that our method is effective. It is worth to be pointed out that the MRKM is still suitable for solving other systems of linear equations.

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References

[1] Y. Ren, B. Zhang, and H. Qiao, "A simple Taylor-series expansion method for a class of second kind integral equations," *Journal of Computational and Applied Mathematics*, vol. 110, no. 1, pp. 15–24, 1999.

[2] K. Maleknejad and Y. Mahmoudi, "Numerical solution of linear Fredholm integral equation by using hybrid Taylor and block-pulse functions," *Applied Mathematics and Computation*, vol. 149, no. 3, pp. 799–806, 2004.

[3] S. Yalçınbaş and M. Sezer, "The approximate solution of high-order linear Volterra-Fredholm integro-differential equations in terms of Taylor polynomials," *Applied Mathematics and Computation*, vol. 112, no. 2-3, pp. 291–308, 2000.

[4] S. Yalçinbaş, "Taylor polynomial solutions of nonlinear Volterra-Fredholm integral equations," *Applied Mathematics and Computation*, vol. 127, no. 2-3, pp. 195–206, 2002.

[5] E. Deeba, S. A. Khuri, and S. Xie, "An algorithm for solving a nonlinear integro-differential equation," *Applied Mathematics and Computation*, vol. 115, no. 2-3, pp. 123–131, 2000.

[6] A. Tahmasbi and O. S. Fard, "Numerical solution of linear Volterra integral equations system of the second kind," *Applied Mathematics and Computation*, vol. 201, no. 1-2, pp. 547–552, 2008.

[7] E. Babolian, J. Biazar, and A. R. Vahidi, "On the decomposition method for system of linear equations and system of linear Volterra integral equations," *Applied Mathematics and Computation*, vol. 147, no. 1, pp. 19–27, 2004.

[8] J. Biazar and H. Ghazvini, "He's homotopy perturbation method for solving systems of Volterra integral equations of the second kind," *Chaos, Solitons & Fractals*, vol. 39, no. 2, pp. 770–777, 2009.

[9] E. Yusufoğlu (Agadjanov), "A homotopy perturbation algorithm to solve a system of Fredholm-Volterra type integral equations," *Mathematical and Computer Modelling*, vol. 47, no. 11-12, pp. 1099–1107, 2008.

[10] R. Katani and S. Shahmorad, "Block by block method for the systems of nonlinear Volterra integral equations," *Applied Mathematical Modelling*, vol. 34, no. 2, pp. 400–406, 2010.

[11] M. Rabbani, K. Maleknejad, and N. Aghazadeh, "Numerical computational solution of the Volterra integral equations system of the second kind by using an expansion method," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 1143–1146, 2007.

[12] M. G. Cui and Z. X. Deng, "On the best operator of interpolation in $W_2^1[a,b]$," *Mathematica Numerica Sinica*, vol. 2, pp. 209–216, 1986.

[13] M. Cui and F. Geng, "A computational method for solving one-dimensional variable-coefficient Burgers equation," *Applied Mathematics and Computation*, vol. 188, no. 2, pp. 1389–1401, 2007.

[14] M. Cui and F. Geng, "A computational method for solving third-order singularly perturbed boundary-value problems," *Applied Mathematics and Computation*, vol. 198, no. 2, pp. 896–903, 2008.

[15] M. Cui and Z. Chen, "The exact solution of nonlinear age-structured population model," *Nonlinear Analysis. Real World Applications*, vol. 8, no. 4, pp. 1096–1112, 2007.

[16] F. Z. Geng and M. G. Cui, "Solving singular nonlinear two-point boundary value problems in the reproducing kernel space," *Journal of the Korean Mathematical Society*, vol. 45, no. 3, pp. 631–644, 2008.

[17] M. G. Cui and Y. B. Yan, "The representation of the solution of a kind operator equation $Au = f$," *Journal of Chinese Universities*, vol. 3, pp. 82–86, 1995.

[18] O. Abu Arqub, M. Al-Smadi, and S. Momani, "Application of reproducing kernel method for solving nonlinear Fredholm-Volterra integrodifferential equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 839836, 16 pages, 2012.

[19] O. A. Arqub, M. Al-Smadi, and N. Shawagfeh, "Solving Fredholm integro-differential equations using reproducing kernel Hilbert space method," *Applied Mathematics and Computation*, vol. 219, no. 17, pp. 8938–8948, 2013.

[20] M. Al-Smadi, O. Abu Arqub, and S. Momani, "A computational method for two-point boundary value problems of fourth-order mixed integrodifferential equations," *Mathematical Problems in Engineering*, vol. 2013, Article ID 832074, 10 pages, 2013.

[21] M. Al-Smadi, O. Abu Arqub, and N. Shawagfeh, "Approximate solution of BVPs for 4th-order IDEs by using RKHS method," *Applied Mathematical Sciences*, vol. 6, no. 49–52, pp. 2453–2464, 2012.

[22] Z. Chen and Y. Lin, "The exact solution of a linear integral equation with weakly singular kernel," *Journal of Mathematical Analysis and Applications*, vol. 344, no. 2, pp. 726–734, 2008.

[23] M. Cui and Y. Lin, *Nonlinear Numerical Analysis in the Reproducing Kernel Space*, Nova Science Publishers, New York, NY, USA, 2009.

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Research Article

The Method of Coupled Fixed Points and Coupled Quasisolutions When Working with ODE's with Arguments of Bounded Variation

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The aim of this paper is to show the use of the coupled quasisolutions method as a useful technique when dealing with ordinary differential equations with functional arguments of bounded variation. We will do this by looking for solutions for a first-order ordinary differential equation with an advanced argument of bounded variation. The main trick is to use the Jordan decomposition of this argument in a nondecreasing part and a nonincreasing one. As a necessary step, we will also talk about coupled fixed points of multivalued operators.

1. Introduction

In the paper [1], we proved a new result on the existence of coupled fixed points for multivalued operators, and then we used it to guarantee the existence of coupled quasisolutions and solutions to a certain first-order ordinary differential equation with state-dependent delay. In that paper, the nonlinearity was allowed to have both nondecreasing and nonincreasing arguments and the existence of solutions was obtained under strong Lipschitz conditions. We pointed out there that this tool could be useful when working with arguments of bounded variation, but no literature about this was written since then. So, the main goal in the present paper is to develop the application of the coupled quasisolutions technique in the framework of arguments of bounded variation, and we do it in an appropriate way, in order to take advantage of the Jordan decomposition and avoid the use of strong assumptions, as Lipschitz-continuity.

To show the application of this technique, we will study throughout this paper the existence of solutions for the following first-order problem:

$$\begin{gathered} x'(t) = f(t, x(t), x(\tau(t))), \quad \text{for a.a. } t \in I = [a, b], \\ x(t) = \phi(t), \quad \forall t \in [b, b + r], \end{gathered} \tag{1}$$

where $r \geq 0$, τ is a measurable function such that $\tau(t) \geq t$ for a.a. t; that is, τ is an advanced argument, and ϕ is a bounded function which represents the final state of the solution. By a solution of (1), we mean a function $x \in \mathscr{C}[a, b + r]$ such that $x_{|I} \in AC(I)$ and x satisfies both the differential equation (a.e. on I) and the final condition. We refer the readers to papers [2–4] to see more results on the existence of solutions and some applications of first-order problems with advance.

This paper is organized as follows. In Section 2, we gather some preliminary concepts and results involving functions of bounded variation and coupled fixed points of multivalued operators. These preliminaries are used later, in Section 3, to prove the existence of quasisolutions and solutions for problem (1). In Section 4, we show how our results can be adapted to deal with delay problems. Finally, in Section 5, some examples of application are available.

2. Preliminaries on Bounded Variation and Coupled Fixed Points of Multivalued Operators

In this section, we introduce some preliminaries that we will use throughout this work. First, we remember some concepts about functions of bounded variation. The reader can see more about this in the monographs [5, 6].

Definition 1. Given a function $f : I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ and a partition $P = \{x_0, \ldots, x_n\}$ of I, one defines the variation of f relative to the partition P as the number

$$V(f,P) = \sum_{i=1}^{n} \left| f(x_i) - f(x_{i-1}) \right|, \tag{2}$$

and one defines the total variation of f on I as

$$V_a^b(f) = \sup_{P \in \mathscr{P}} V(f,P), \tag{3}$$

where $\mathscr{P} = \{P : P \text{ is a partition of } I\}$.

One says that f is a function of bounded variation on I if $V_a^b(f) < +\infty$. In that case, one writes $f \in \mathrm{BV}(I)$.

Functions of bounded variation satisfy the following well-known result, which becomes essential now for our purposes.

Proposition 2 (Jordan decomposition). *A function f is of bounded variation on I if and only if there exist a nondecreasing function, g, and a nonincreasing one, h, such that*

$$f(t) = g(t) + h(t), \quad \forall t \in I. \tag{4}$$

The proof of Proposition 2 uses the fact that the function $t \in I \to V_a^t(f)$ is nondecreasing and $t \to f(t) - V_a^t(f)$ is nonincreasing, and thus the desired decomposition is

$$f(t) = V_a^t(f) + f(t) - V_a^t(f). \tag{5}$$

We remark that this decomposition is not unique. Finally, notice that, as a consequence of this result, every function of bounded variation is a.e. differentiable.

The set $\mathrm{BV}(I)$ is an algebra which is included neither in the set of continuous functions nor in its complementary. Indeed, if f is monotone on $[a,b]$, then $V_a^b(f) = |f(b) - f(a)|$, and thus $f \in \mathrm{BV}(I)$. Then, there exist discontinuous functions which are of bounded variation (e.g., step function). In fact, it is also a well-known fact that if $f \in \mathrm{BV}(I)$, then f has only "jump" discontinuities. On the other hand, there exist continuous functions which are not of bounded variation, as (see [5, Example 6.3.1])

$$f(t) = \begin{cases} t \cos\left(\dfrac{\pi}{2t}\right) & \text{if } 0 < t \leq 1, \\ 0 & \text{if } t = 0. \end{cases} \tag{6}$$

To obtain our main result, we will use a generalized monotone method in presence of lower and upper solutions. This is a very well-known tool which is extensively used in the literature of ordinary differential equations. The classical version of this technique uses a pair of monotone sequences which will converge to the extremal solutions of the problem. The generalized version of this technique was developed in [7], and it is used when the nonlinearity has discontinuous arguments and therefore the pair of monotone sequences is replaced by a monotone operator. As a novelty which respect to the method developed in [7] and related references, we will use here a multivalued operator (i.e., a set-valued mapping) defined in a product space and then we will look for coupled fixed points. We concrete this idea in the following lines.

Definition 3. A metric space X equipped with a partial ordering $\leq$ is an ordered metric space if the intervals $[x) = \{y \in X : x \leq y\}$ and $(x] = \{y \in X : y \leq x\}$ are closed for every $x \in X$. Let P be a subset of an ordered metric space. An operator $A : P \times P \to P$ is said to be mixed monotone if $A(\cdot, x)$ is nondecreasing and $A(x, \cdot)$ is nonincreasing for each $x \in P$. One says that A satisfies the mixed monotone convergence property (m.m.c.p.) if $(A(v_j, w_j))_{j=1}^{\infty}$ converges in X whenever $(v_j)_{j=1}^{\infty}$ and $(w_j)_{j=1}^{\infty}$ are sequences in P, one being nondecreasing and the other nonincreasing.

Definition 4. Let $\overline{X}$ be a subset of an ordered metric space X. One defines a multivalued operator in the product $\overline{X} \times \overline{X}$ as a mapping

$$\mathscr{A} : \overline{X} \times \overline{X} \longrightarrow 2^{\overline{X}} \setminus \emptyset. \tag{7}$$

We say that $v, w \in \overline{X}$ are coupled fixed points of $\mathscr{A}$ if $v \in \mathscr{A}(v,w)$ and $w \in \mathscr{A}(w,v)$. We say that $v_*, w^* \in \overline{X}$ are the extremal coupled fixed points of $\mathscr{A}$ in $\overline{X}$ if v_*, w^* are coupled fixed points of $\mathscr{A}$ and if $v, w \in \overline{X}$ are another pair of coupled fixed points of $\mathscr{A}$; then $v_* \leq v$ and $w \leq w^*$.

Theorem 5 (see [1, Theorem 2.1]). *Let Y be a subset of an ordered metric space X, $[\alpha, \beta]$ be a nonempty closed interval in Y, and $\mathscr{A} : [\alpha,\beta] \times [\alpha,\beta] \to 2^{[\alpha,\beta]} \setminus \emptyset$ be a multivalued operator.*

If for all $v, w \in [\alpha, \beta]$, there exist

$$\begin{aligned} A_*(v,w) &= \min \mathscr{A}(v,w) \in [\alpha,\beta], \\ A^*(v,w) &= \max \mathscr{A}(v,w) \in [\alpha,\beta], \end{aligned} \tag{8}$$

and the (single-valued) operators A_* and A^* are mixed monotone and satisfy the m.m.c.p., then $\mathscr{A}$ has the extremal coupled fixed points in $[\alpha,\beta]$, v_*, v^*. Moreover, they satisfy the following characterization:

$$(v_*, w^*) = \min_{\preceq} \left\{ (v,w) : \left(A_*(v,w), A^*(w,v)\right) \preceq (v,w) \right\}, \tag{9}$$

where

$$(v,w) \preceq (\overline{v}, \overline{w}) \Longleftrightarrow v \leq \overline{v}, w \geq \overline{w}. \tag{10}$$

3. Main Result

Now, we develop our generalized monotone method applied to problem (1). To do this, throughout this section, we will assume the following.

(H_1) There exists a closed interval $J \subset \mathbb{R}$, such that for a.a. $t \in I$ and all $x \in \mathbb{R}$ the function $f(t, x, \cdot)$ is of bounded variation on J.

Assumption (H_1) implies that there exists a nondecreasing function, g, and a nonincreasing one, h, such that

$$f(t,x,\cdot)=g(t,x,\cdot)+h(t,x,\cdot), \tag{11}$$

for all $(t,x)\in I\times\mathbb{R}$.

Now, we define what we mean by lower and upper solutions for problem (1).

Definition 6. One says that $\alpha,\beta\in\mathscr{C}[a,b+r]$ are, respectively, a lower and upper solutions for problem (1), and if $\alpha_{|I},\beta_{|I}\in AC(I)$,

$$\left[\min_{t\in[a,b+r]}\alpha(t),\max_{t\in[a,b+r]}\beta(t)\right]\subset J, \tag{12}$$

the compositions

$$t\longmapsto f(t,\alpha(t),y),\qquad t\longmapsto f(t,\beta(t),y) \tag{13}$$

are measurable for all $y\in J$ and the following inequalities hold:

$$\begin{gathered}\alpha'(t)\geq g(t,\alpha(t),\beta(\tau(t)))+h(t,\alpha(t),\alpha(\tau(t))),\\ \text{for a.a. } t\in I,\\ \alpha(t)\leq\phi(t),\quad\forall t\in[b,b+r],\\ \beta'(t)\leq g(t,\beta(t),\alpha(\tau(t)))+h(t,\beta(t),\beta(\tau(t))),\\ \text{for a.a. } t\in I,\\ \beta(t)\geq\phi(t),\quad\forall t\in[b,b+r].\end{gathered} \tag{14}$$

Remark 7. Notice that, under the previous definition, the lower and the upper solutions appear "coupled." On the other hand, it is assumed that

$$\min_{t\in[a,b+r]}\alpha(t)\leq\max_{t\in[a,b+r]}\beta(t). \tag{15}$$

This is not a strong assumption, taking into account that, as usual, we will ask the lower and the upper solutions to be well ordered in the whole interval $[a,b+r]$.

On the other hand, the fact that $t\mapsto f(t,\alpha(t),y)$ and $t\in I\mapsto f(t,\beta(t),y)$ are being measurable for all $y\in J$ implies that the compositions

$$\begin{gathered}t\in I\longmapsto g(t,\alpha(t),\beta(\tau(t)))+h(t,\alpha(t),\alpha(\tau(t))),\\ t\in I\longmapsto g(t,\beta(t),\alpha(\tau(t)))+h(t,\beta(t),\beta(\tau(t)))\end{gathered} \tag{16}$$

are measurable too, because g and h are being monotone with respect to their last variables.

As we said in the Introduction, an essential tool in our work is the use of coupled quasisolutions. So, we introduce now this concept.

Definition 8. One says that two functions $x_*, x^*\in\mathscr{C}[a,b+r]$are coupled quasisolutions of problem (1), and if $x_{*|I},x^*_{|I}\in AC(I)$, $x_*(t)=x^*(t)=\phi(t)$ for all $t\in[b,b+r]$ and for a.a. $t\in I$, they satisfy

$$\begin{gathered}x_*'(t)=g(t,x_*(t),x^*(\tau(t)))+h(t,x_*(t),x_*(\tau(t))),\\ x^{*\prime}(t)=g(t,x^*(t),x_*(\tau(t)))+h(t,x^*(t),x^*(\tau(t))).\end{gathered} \tag{17}$$

We say that these coupled quasisolutions are extremal in a subset $\overline{X}\subset\mathscr{C}[a,b+r]$ if $x_*,x^*\in\overline{X}$ and $x_*(t)\leq x_1(t),x_2(t)\leq x^*(t)$ whenever $x_1,x_2\in\overline{X}$ is another pair of quasisolutions.

We need the following maximum principle related to problems with advance, as an auxiliar tool, for proving our main result. Compare it with [3, Lemma 3.2], [4, Lemma 1].

Lemma 9. *Let $\tau:I\to[a,b+r]$ be a measurable function such that $\tau(t)\geq t$ for a.a. $t\in I$ and assume that $p\in\mathscr{C}[a,b+r]$ such that $p_{|I}\in AC(I)$ and satisfies*

$$\begin{gathered}p'(t)\geq K(t)p(t)-L(t)p(\tau(t))\quad\text{for a.a. } t\in I,\\ p(t)=0,\quad\forall t\in[b,b+r],\end{gathered} \tag{18}$$

where $K, L\in L^1(I)$ and $L\geq 0$ a.e.

If

$$\int_a^b(K_-(t)+L(t))\,dt<1, \tag{19}$$

where $K_-=\max\{-K,0\}$, then $p(t)\leq 0$ for all $t\in[a,b+r]$.

Proof. Let $t_1\in[a,b+r]$ such that

$$p(t_1)=\max_{t\in[a,b+r]}p(t) \tag{20}$$

and assume by contradiction that $p(t_1)>0$. Then, $t_1\in[a,b)$. Now, let $t_2\in(t_1,b]$ such that $p(t_2)=0$ and $p(t)\geq 0$ for all $t\in[t_1,t_2]$. Now, integrating t_1 and t_2, we obtain

$$\begin{aligned}p(t_1)&=-\int_{t_1}^{t_2}p'(t)\,dt\\ &\leq-\int_{t_1}^{t_2}K(t)p(t)\,dt+\int_{t_1}^{t_2}L(t)p(\tau(t))\,dt\\ &\leq p(t_1)\int_{t_1}^{t_2}(K_-(t)+L(t))\,dt,\end{aligned} \tag{21}$$

and then condition (49) provides the contradiction $p(t_1)<p(t_1)$. □

The main result on this paper concerns the existence of extremal quasisolutions and solutions for problem (1). It is as follows.

Theorem 10. *Assume (H_1) and that there exist $\alpha,\beta\in\mathscr{C}[a,b+r]$ which are, respectively, lower and upper solutions for problem (1) such that $\alpha(t)\leq\beta(t)$ for all $t\in[a,b+r]$ and*

$$E=\left[\min_{t\in[a,b+r]}\alpha(t),\max_{t\in[a,b+r]}\beta(t)\right]\subset J. \tag{22}$$

Assume moreover that the following conditions hold:

(H_2) *for each* $\gamma_1, \gamma_2 \in [\alpha, \beta] = \{\gamma \in \mathscr{C}[a, b+r] : \alpha(t) \leq \gamma(t) \leq \beta(t)$ *for all* $t \in [a, b+r]\}$, *the final value problem*

$$\left(P_{\gamma_1,\gamma_2}\right) \begin{cases} x'(t) = F_{\gamma_2,\gamma_1}(t, x(t)) \\ \qquad := g(t, x(t), \gamma_2(\tau(t))) \\ \qquad\quad + h(t, x(t), \gamma_1(\tau(t))), \quad \text{for a.a. } t \in I, \\ x(b) = \phi(b) \end{cases} \tag{23}$$

has the extremal solutions in $[\alpha, \beta]$;

(H_3) *there exists* $\psi \in L^1(I, [0, +\infty))$ *such that for a.a.* $t \in I$, *all* $x \in [\alpha(t), \beta(t)]$, *and all* y_1, $y_2 \in [\alpha(\tau(t)), \beta(\tau(t))]$, *one has*

$$\left|g(t, x, y_1) + h(t, x, y_2)\right| \leq \psi(t); \tag{24}$$

(H_4) *there exists* K_1, K_2, L_1, $L_2 \in L^1(I)$ *such that* L_1, $L_2 \geq 0$ *a.e. and*

$$\begin{aligned} g(t, \overline{x}, y) - g(t, x, \overline{y}) &\geq K_1(t)(\overline{x} - x) - L_1(t)(\overline{y} - y), \\ h(t, \overline{x}, \overline{y}) - h(t, x, y) &\geq K_2(t)(\overline{x} - x) - L_2(t)(\overline{y} - y) \end{aligned} \tag{25}$$

whenever $\alpha(t) \leq x \leq \overline{x} \leq \beta(t)$ *and*

$$\begin{aligned} &\min_{s\in[b,b+r]} \phi(s) - \int_t^b \psi(s)\, ds \\ &\quad \leq y \leq \overline{y} \leq \max_{s\in[b,b+r]} \phi(s) + \int_t^b \psi(s)\, ds. \end{aligned} \tag{26}$$

Moreover,

$$\int_a^b \left(K_-(t) + L(t)\right) dt < 1, \tag{27}$$

where $K = K_1 + K_2$, $L = L_1 + L_2$ *and* $K_-(t) = \max\{-K(t), 0\}$.

In these conditions, problem (1) *has a unique solution in* $[\alpha, \beta]$.

Proof. We consider the space $X = \mathscr{C}[a, b + r]$ endowed with the ordering

$$\gamma_1 \leq \gamma_2 \Longleftrightarrow \gamma_1(t) \leq \gamma_2(t), \quad \forall t \in [a, b + r], \tag{28}$$

and we define a multivalued operator

$$\mathscr{A} : [\alpha, \beta] \times [\alpha, \beta] \subset X \times X \longrightarrow 2^{[\alpha,\beta]} \setminus \emptyset \tag{29}$$

as follows: for each $\gamma_1, \gamma_2 \in [\alpha, \beta]$, we have $x \in \mathscr{A}(\gamma_1, \gamma_2)$ if and only if $x \in [\alpha, \beta]$, x_I is a solution of (P_{γ_1,γ_2}) and $x_{|[b,b+r]} = \phi$.

Step 1. Operator $\mathscr{A}$ *has the extremal coupled fixed points in* $[\alpha, \beta]$. By virtue of condition (H_2), operator $\mathscr{A}$ is well defined and there exist

$$A_* = \min \mathscr{A}(\gamma_1, \gamma_2), \qquad A^* = \max \mathscr{A}(\gamma_1, \gamma_2). \tag{30}$$

We will show now that A_*, A^* are mixed monotone and satisfy m.m.c.p. So, let

$$\gamma_1, \overline{\gamma}_1, \gamma_2, \overline{\gamma}_2 \in [\alpha, \beta] \tag{31}$$

such that $\gamma_1 \leq \overline{\gamma}_1$, $\gamma_2 \leq \overline{\gamma}_2$ and put

$$x_1 = A_*(\gamma_1, \gamma_2), \quad \overline{x}_1 = A_*(\overline{\gamma}_1, \gamma_2), \quad \overline{x}_2 = A_*(\gamma_1, \overline{\gamma}_2). \tag{32}$$

Then, for all $t \in [b, b + r]$, we have that $x_1(t) = \overline{x}_1(t) = \overline{x}_2(t)$, and for a.a. $t \in I$, we have

$$\begin{aligned} \overline{x}_1'(t) &= g(t, \overline{x}_1(t), \gamma_2(\tau(t))) + h(t, \overline{x}_1, \overline{\gamma}_1(\tau(t))) \\ &\leq g(t, \overline{x}_1(t), \gamma_2(\tau(t))) + h(t, \overline{x}_1, \gamma_1(\tau(t))). \end{aligned} \tag{33}$$

And so, $\overline{x}_1$ is an upper solution for problem (P_{γ_1,γ_2}). The fact that x_1 is being the least solution of this problem in $[\alpha, \beta]$ implies that $\overline{x}_1 \geq x_1$ and then $A_*(\cdot, \gamma_2)$ is nondecreasing. On the other hand,

$$\begin{aligned} x_1'(t) &= g(t, x_1(t), \gamma_2(\tau(t))) + h(t, x_1(t), \gamma_1(\tau(t))) \\ &\leq g(t, x_1(t), \overline{\gamma}_2(\tau(t))) + h(t, x_1(t), \gamma_1(\tau(t))), \end{aligned} \tag{34}$$

and therefore x_1 is an upper solution for problem $(P_{\gamma_1,\overline{\gamma}_2})$. Then, $x_1 \geq \overline{x}_2$, and so the mapping $A_*(\gamma_1, \cdot)$ is nonincreasing. In the same way, we show that A^* is mixed monotone.

To see that A_*, A^* satisfy the m.m.c.p., let $(v_j)_{j=1}^{\infty}$, $(w_j)_{j=1}^{\infty}$ be sequences in $[\alpha, \beta]$, one being nondecreasing and the other being nonincreasing. As A_*, A^* are mixed monotone and bounded, we obtain that the sequences $(A_*(v_j, w_j))_{j=1}^{\infty}$, $(A^*(v_j, w_j))_{j=1}^{\infty}$ have their pointwise limit; say z_*, z^*. As $(A_*(v_j, w_j))_{j=1}^{\infty}$, $(A^*(v_j, w_j))_{j=1}^{\infty}$ are constant in $[b, b + r]$, the convergence is uniform in this interval. On the other hand, for $t, s \in I$, $s < t$, and $j \in \mathbb{N}$, we have

$$\begin{aligned} &\left|z_j^*(t) - z_j^*(s)\right| \\ &\leq \int_s^t \left|g\left(r, z_j^*(r), w_j(\tau(r))\right) + h\left(t, z_j^*(r), v_j(\tau(r))\right)\right| dr \\ &\leq \int_s^t \psi(r)\, dr, \end{aligned} \tag{35}$$

and thus $(z_j^*)_{j=1}^{\infty}$ converges to z^* uniformly on I. The same argument is valid for z_*.

By application of Theorem 5, operator $\mathscr{A}$ has the extremal coupled fixed points in $[\alpha, \beta]$; say x_*, x^*.

Step 2. Problem (1) *has the extremal quasisolutions in* $[\alpha, \beta]$. Indeed, we will show that the extremal coupled fixed points of operators $\mathscr{A}$, x_*, and x^* correspond with these extremal quasisolutions. First, it is clear that if $x, \overline{x} \in [\alpha, \beta]$ are coupled fixed points of $\mathscr{A}$, then they are coupled quasisolutions of problem (1). On the other hand, if x, $\overline{x}$ are quasisolutions of problem (1), then $A_*(x, \overline{x}) \leq x$ and $A^*(\overline{x}, x) \geq \overline{x}$, and then characterization (9) implies that $x_* \leq x$ and $\overline{x} \leq x^*$. This shows that x_*, x^* are the extremal quasisolutions of problem (1) in $[\alpha, \beta]$.

Step 3. Problem (1) *has a unique solution in* $[\alpha, \beta]$. We will prove this by showing that the extremal quasisolutions x_*, x^* are, in fact, the same functions, and thus defining a solution of the problem. This solution must be unique in $[\alpha, \beta]$ because if $\overline{x} \in [\alpha, \beta]$ is a solution of (1), then the pair $\overline{x}, \overline{x}$ is also a quasisolution, and then $x_* \leq \overline{x} \leq x^*$.

To see that $x_* = x^*$, first notice that as (x_*, x^*) is a pair of quasisolutions; then, the reversed pair, (x^*, x_*), is quasisolutions too, and then, extremality implies $x_* \leq x^*$. Moreover, condition (H_3) implies that for a.a. $t \in I$

$$x_*(t), x^*(t) \in \left[\phi(b) - \int_t^b \psi(s)\, ds, \phi(b) + \int_t^b \psi(s)\, ds\right]. \tag{36}$$

Now, define the function $p(t) = x^*(t) - x_*(t) \geq 0$. On the one hand, $p(t) = 0$ for all $t \in [b, b+r]$. On the other hand, condition (H_4) implies for a.a. $t \in I$ that

$$\begin{aligned} p'(t) &= g(t, x^*(t), x_*(\tau(t))) - g(t, x_*(t), x^*(\tau(t))) \\ &\quad + h(t, x^*(t), x^*(\tau(t))) - h(t, x_*(t), x_*(\tau(t))) \\ &\geq K(t)(x^*(t) - x_*(t)) - L(t)(x^*(\tau(t)) - x_*(\tau(t))), \end{aligned} \tag{37}$$

and then by virtue of Lemma 9, we obtain that $p(t) \leq 0$ on I. We conclude that $p(t) = 0$ for all $t \in [a, b+r]$; that is, $x_* = x^*$. This ends the proof. □

Remark 11. Now, we point out some remarks related to Theorem 10.

(1) Condition (H_2) could be replaced by any result on the existence of extremal solutions between lower and upper solutions for problem (P_{γ_1,γ_2}). For example, as it is well known, if F_{γ_1,γ_2} is a Carathéodory function, then (H_3) implies that (P_{γ_1,γ_2}) has the extremal solutions between α and β. Moreover, there exists a very extensive literature about the existence of extremal solutions for problem (P_{γ_1,γ_2}) for discontinuous F_{γ_1,γ_2}. The reader is referred to [1, 8–10] and references therein for some results of this type. Notice that although most of these references deal with initial value problems, these results can easily be adapted for final value problems. Finally, notice that (H_2) implies, in particular, measurability of the composition $t \in I \mapsto F_{\gamma_1,\gamma_2}(t, x(t))$ for all $x \in [\alpha, \beta]$.

(2) As we said in Section 2, a function of bounded variation has only "jump" discontinuities. Although condition (H_4) implies that for a.a. $t \in I$ the function f is continuous with respect to its third variable in the interval

$$\left[\min_{s\in[b,b+r]} \phi(s) - \int_t^b \psi(s)\, ds, \max_{s\in[b,b+r]} \phi(s) + \int_t^b \psi(s)\, ds\right], \tag{38}$$

a countable number of discontinuities are allowed to exist outside this interval. Moreover, notice that this interval can be improved if we find another function $\widetilde{\psi}$ satisfying (H_3) and such that $\widetilde{\psi}(t) \leq \psi(t)$ for a.a. t.

(3) For almost all $t \in I$ and all $x \in [\alpha(t), \beta(t)]$ the function $f_{t,x}(\cdot) = f(t, x, \cdot)$ is of bounded variation in $[\alpha(\tau(t)), \beta(\tau(t))]$, and thus there exists in this interval a decomposition $f_{t,x}(\cdot) = g_{t,x}(\cdot) + h_{t,x}(\cdot)$, with g nondecreasing and h nonincreasing. Although all conditions in Theorem 10 are stated for an arbitrary Jordan decomposition of this type, all of them can be rewritten with

$$\begin{aligned} g_{t,x}(y) &= V_A^y(f), \\ h_{t,x}(y) &= f_{t,x}(y) - V_A^y(f), \end{aligned} \tag{39}$$

for any choice of $A \leq \min\{\alpha(t) : t \in [a, b+r]\}$, $A \geq \min J$.

Theorem 10 provides, in particular, a new result on the existence of extremal solutions for problem (1) in the case that function f is nonincreasing with respect to its third variable. In this case, the nondecreasing part of the Jordan decomposition of f does not exist, and therefore the lower and upper solutions introduced in Definition 6 appear uncoupled. Moreover, a pair of quasisolutions in the sense of Definition 8 becomes, in fact, a pair of solutions, and then extremal quasisolutions provided by Theorem 10 reduce to extremal solutions. We specify these ideas in the following corollary.

Corollary 12. *Assume that there exist* $\alpha, \beta \in \mathscr{C}[a, b+r]$ *such that* $\alpha_I, \beta_I \in AC(I)$, $\alpha \leq \beta$ *on* $[a, b+r]$ *and the following inequalities hold:*

$$\begin{aligned} &\alpha'(t) \geq f(t, \alpha(t), \alpha(\tau(t))), \quad \textit{for a.a. } t \in I, \ \alpha(t) \leq \phi(t) \quad \forall t \in [b, b+r], \\ &\beta'(t) \leq f(t, \beta(t), \beta(\tau(t))), \quad \textit{for a.a. } t \in I, \ \beta(t) \geq \phi(t) \quad t \in [b, b+r]. \end{aligned} \tag{40}$$

Assume moreover that the following conditions hold:

$(H_2)'$ *for all* $\gamma \in [\alpha, \beta]$, *the final value problem*

$$\begin{aligned} x'(t) &= f(t, x(t), \gamma), \quad \textit{for a. a. } t \in I, \\ x(b) &= \phi(b) \end{aligned} \tag{41}$$

has the extremal solutions in $[\alpha, \beta]$;

$(H_3)'$ *there exists* $\psi \in L^1(I,[0,+\infty))$ *such that for a.a.* $t \in I$, *all* $x \in [\alpha(t), \beta(t)]$, *and all* $y \in [\alpha(\tau(t)), \beta(\tau(t))]$, *one has*

$$\left|f\left(t,x,y\right)\right| \le \psi\left(t\right); \tag{42}$$

$(H_4)'$ *for a.a.* $t \in I$ *and all* $x \in [\alpha(t), \beta(t)]$, *the function* $f(t,x,\cdot)$ *is nonincreasing.*

In these conditions problem (1) *has the extremal solutions in* $[\alpha, \beta]$.

4. Delay Problems

The results obtained in the previous section can be easily reformulated in order to deal with problems with delay. We concrete this idea in the following lines.

Consider the following problem:

$$\begin{aligned} x'\left(t\right) &= f\left(t,x\left(t\right),x\left(\tau\left(t\right)\right)\right), \quad \text{for a.a. } t \in I=[a,b], \\ x\left(t\right) &= \phi\left(t\right), \quad \forall t \in [a-r,a], \end{aligned} \tag{43}$$

where $r \ge 0$, τ is a measurable function such that $\tau(t) \le t$ for a.a. t; that is, τ is a delayed argument and ϕ is a bounded function which represents the initial state of the solution. Now, by a solution of (43), we mean a function $x \in \mathscr{C}[a-r,b]$ such that $x_{|I} \in AC(I)$ and x satisfies both the differential equation (a.e. on I) and the initial condition.

As we said, we will show now that we can use our technique to obtain a new result on the existence of solutions for problem (43) in the case that function $f(t,x,\cdot)$ is of bounded variation. We begin by reformulating the concept of lower and upper solutions and coupled quasisolutions in order to adapt them to our new problem. As in previous section, we assume (H_1).

Definition 13. One says that $\alpha, \beta \in \mathscr{C}[a-r,b]$ are, respectively, lower and upper solutions for problem (43), and if $\alpha_{|I}, \beta_{|I} \in AC(I)$,

$$\left[\min_{t\in[a-r,b]} \alpha\left(t\right), \max_{t\in[a-r,b]} \beta\left(t\right)\right] \subset J, \tag{44}$$

the compositions

$$t \longmapsto f\left(t,\alpha\left(t\right),y\right), \qquad t \longmapsto f\left(t,\beta\left(t\right),y\right) \tag{45}$$

are measurable for all $y \in J$ and the following inequalities hold:

$$\begin{aligned} \alpha'\left(t\right) &\le g\left(t,\alpha\left(t\right),\alpha\left(\tau\left(t\right)\right)\right) + h\left(t,\alpha\left(t\right),\beta\left(\tau\left(t\right)\right)\right), \\ &\qquad \text{for a.a. } t \in I, \\ \alpha\left(t\right) &\le \phi\left(t\right), \quad \forall t \in [a-r,a], \\ \beta'\left(t\right) &\ge g\left(t,\beta\left(t\right),\beta\left(\tau\left(t\right)\right)\right) + h\left(t,\beta\left(t\right),\alpha\left(\tau\left(t\right)\right)\right), \\ &\qquad \text{for a.a. } t \in I, \\ \beta\left(t\right) &\ge \phi\left(t\right), \quad \forall t \in [a-r,a]. \end{aligned} \tag{46}$$

Definition 14. One says that two functions $x_*, x^* \in \mathscr{C}[a-r,b]$ are coupled quasisolutions of problem (43) if $x_{*|I}, x^*_{|I} \in AC(I)$, $x_*(t) = x^*(t) = \phi(t)$ for all $t \in [a-r,a]$ and for a.a. $t \in I$, they satisfy

$$\begin{aligned} x_*'\left(t\right) &= g\left(t,x_*\left(t\right),x_*\left(\tau\left(t\right)\right)\right) + h\left(t,x_*\left(t\right),x^*\left(\tau\left(t\right)\right)\right), \\ x^{*\prime}\left(t\right) &= g\left(t,x^*\left(t\right),x^*\left(\tau\left(t\right)\right)\right) + h\left(t,x^*\left(t\right),x_*\left(\tau\left(t\right)\right)\right). \end{aligned} \tag{47}$$

We say that these coupled quasisolutions are extremal in a subset $\overline{X} \subset \mathscr{C}[a-r,b]$; if $x_*, x^* \in \overline{X}$ and $x_*(t) \le x_1(t), x_2(t) \le x^*(t)$ whenever $x_1, x_2 \in \overline{X}$ is another pair of quasisolutions.

Before introducing our main result for problem (43), we need a maximum principle for problems with delay, which is as follows. Its proof is analogous to that done in Lemma 9, so we omit it.

Lemma 15. *Let* $\tau : I \to [a-r,b]$ *be a measurable function such that* $\tau(t) \le t$ *for a.a.* $t \in I$ *and assume that* $p \in \mathscr{C}[a-r,b]$ *is such that* $p_{|I} \in AC(I)$ *and satisfies*

$$\begin{aligned} p'\left(t\right) &\le K\left(t\right)p\left(t\right) + L\left(t\right)p\left(\tau\left(t\right)\right), \quad \text{for a.a. } t \in I, \\ p\left(t\right) &= 0, \quad \forall t \in [a-r,a], \end{aligned} \tag{48}$$

where K, $L \in L^1(I)$ *and* $L \ge 0$ *a.e.*

If

$$\int_a^b \left(K_+\left(t\right) + L\left(t\right)\right) dt < 1, \tag{49}$$

where $K_+ = \max\{K, 0\}$, *then* $p(t) \le 0$ *for all* $t \in [a-r,b]$.

Now, we state our main result in this Section.

Theorem 16. *Assume* (H_1) *and that there exist* $\alpha, \beta \in \mathscr{C}[a-r,b]$ *which are, respectively, lower and upper solutions for problem* (43) *such that* $\alpha(t) \le \beta(t)$ *for all* $t \in [a-r,b]$ *and*

$$\widehat{E} = \left[\min_{t\in[a-r,b]} \alpha\left(t\right), \max_{t\in[a-r,b]} \beta\left(t\right)\right] \subset J. \tag{50}$$

Assume moreover that the following conditions hold:

$(\widehat{H}_2)$ *for each* $\gamma_1, \gamma_2 \in [\alpha, \beta] = \{\gamma \in \mathscr{C}[a-r,b] : \alpha(t) \le \gamma(t) \le \beta(t)$ *for all* $t \in [a-r,b]\}$, *the initial value problem*

$$\left(\widehat{P}_{\gamma_1,\gamma_2}\right) \begin{cases} x'\left(t\right) = F_{\gamma_1,\gamma_2}\left(t,x\left(t\right)\right) \\ \qquad := g\left(t,x\left(t\right),\gamma_1\left(\tau\left(t\right)\right)\right) \\ \qquad\quad + h\left(t,x\left(t\right),\gamma_2\left(\tau\left(t\right)\right)\right), \quad \text{for a.a. } t \in I, \\ x\left(a\right) = \phi\left(a\right) \end{cases} \tag{51}$$

has the extremal solutions in $[\alpha, \beta]$;

(H_3) *there exists* $\psi \in L^1(I,[0,+\infty))$ *such that for a.a.* $t \in I$, *all* $x \in [\alpha(t), \beta(t)]$, *and all* y_1, $y_2 \in [\alpha(\tau(t)), \beta(\tau(t))]$, *one has*

$$\left|g\left(t,x,y_1\right) + h\left(t,x,y_2\right)\right| \le \psi\left(t\right); \tag{52}$$

$(\widehat{H}_4)$ *there exists* $K_1, K_2, L_1, L_2 \in L^1(I)$ *such that* $L_1, L_2 \geq 0$ *a.e. and*

$$\begin{aligned} & g(t,\overline{x},\overline{y}) - g(t,x,y) \\ & \quad \leq K_1(t)(\overline{x}-x) + L_1(t)(\overline{y}-y), \\ & h(t,\overline{x},y) - h(t,x,\overline{y}) \\ & \quad \leq K_2(t)(\overline{x}-x) + L_2(t)(\overline{y}-y) \end{aligned} \tag{53}$$

whenever $\alpha(t) \leq x \leq \overline{x} \leq \beta(t)$ *and*

$$\begin{aligned} & \min_{s\in[a-r,a]} \phi(s) - \int_a^t \psi(s)\,ds \\ & \quad \leq y \leq \overline{y} \leq \max_{s\in[a-r,a]} \phi(s) + \int_a^t \psi(s)\,ds. \end{aligned} \tag{54}$$

Moreover,

$$\int_a^b (K_+(t) + L(t))\,dt < 1, \tag{55}$$

where $K = K_1 + K_2$, $L = L_1 + L_2$ *and* $K_+(t) = \max\{K(t), 0\}$.

In these conditions, problem (43) *has a unique solution in* $[\alpha, \beta]$.

Proof. The proof is analogous to that done in Theorem 10, but now, redefining operator $\mathscr{A}$ in this way, first, we consider the space $\widehat{X} = \mathscr{C}[a-r,b]$ endowed with the ordering

$$\gamma_1 \leq \gamma_2 \Longleftrightarrow \gamma_1(t) \leq \gamma_2(t), \quad \forall t \in [a-r,b]. \tag{56}$$

Then, we consider the operator

$$\widehat{\mathscr{A}} : [\alpha,\beta] \times [\alpha,\beta] \subset \widehat{X} \times \widehat{X} \longrightarrow 2^{[\alpha,\beta]} \setminus \emptyset \tag{57}$$

as follows: for each $\gamma_1, \gamma_2 \in [\alpha, \beta]$, we have $x \in \mathscr{A}(\gamma_1, \gamma_2)$ if and only if $x \in [\alpha,\beta]$, x_I is a solution of $(\widehat{P}_{\gamma_1,\gamma_2})$ and $x_{|[a-r,a]} = \phi$.

The rest of the proof is analogous, with obvious changes. □

Now, Theorem 16 provides, in particular, a new result on the existence of extremal solutions in the case that function f is nondecreasing with respect to its third variable. For the sake of completeness, we concrete this idea in the following Corollary, which is the analogous to Corollary 12.

Corollary 17. *Assume that there exist* $\alpha, \beta \in \mathscr{C}[a-r,b]$ *such that* $\alpha_I, \beta_I \in AC(I)$, $\alpha \leq \beta$ *on* $[a-r,b]$ *and the following inequalities hold:*

$$\begin{aligned} & \alpha'(t) \leq f(t,\alpha(t),\alpha(\tau(t))), \quad \text{for a.a. } t \in I,\ \alpha(t) \leq \phi(t) \\ & \qquad \forall t \in [a-r,a], \\ & \beta'(t) \geq f(t,\beta(t),\beta(\tau(t))), \quad \text{for a.a. } t \in I,\ \beta(t) \geq \phi(t) \\ & \qquad \forall t \in [a-r,a]. \end{aligned} \tag{58}$$

Assume moreover that the following conditions hold:

$(\widehat{H}_2)'$ *for all* $\gamma \in [\alpha,\beta]$, *the initial value problem*

$$\begin{aligned} x'(t) &= f(t,x(t),\gamma) \quad \text{for a.a. } t \in I, \\ x(a) &= \phi(a) \end{aligned} \tag{59}$$

has the extremal solutions in $[\alpha,\beta]$;

$(H_3)'$ *there exists* $\psi \in L^1(I,[0,+\infty))$ *such that for a.a.* $t \in I$, *all* $x \in [\alpha(t),\beta(t)]$, *and all* $y \in [\alpha(\tau(t)),\beta(\tau(t))]$, *one has*

$$|f(t,x,y)| \leq \psi(t); \tag{60}$$

$(\widehat{H}_4)'$ *for a.a.* $t \in I$ *and all* $x \in [\alpha(t),\beta(t)]$, *the function* $f(t,x,\cdot)$ *is nondecreasing.*

In these conditions, problem (43) *has the extremal solutions in* $[\alpha,\beta]$.

5. Examples of Application

We finish this work with two applications of our main results.

Example 1. Consider the following problem with advance:

$$\begin{aligned} x'(t) &= f(x(4t)), \quad \text{for a.a. } t \in I = \left[0, \frac{\pi}{8}\right], \\ x(t) = \phi(t) &= \frac{1}{2}\left(x - \frac{\pi}{2}\right)\sin\left(\frac{1}{x-\pi/2}\right), \quad \forall t \in \left[\frac{\pi}{8}, \frac{\pi}{2}\right], \end{aligned} \tag{61}$$

where f is defined as follows: for each $n \in \{1,2,\ldots\}$, we have

$$f(y) = \begin{cases} \dfrac{1}{10}y, & \text{if } y \in (2n-2, 2n-1], \\ \dfrac{4n-1}{10} - \dfrac{1}{10}y, & \text{if } y \in (2n-1, 2n], \end{cases} \tag{62}$$

and for $y \leq 0$, we define $f(y) = -f(-y)$.

Defined that way, f is a function of bounded variation in any bounded interval of $\mathbb{R}$. Moreover, f has a countable number of both downwards and upwards discontinuities. We will construct later a pair (α,β) of coupled lower and upper solutions for problem (1) such that for all $t \in [0,\pi/2]$, we have

$$-\frac{\pi}{2} \leq \alpha(t) \leq \beta(t) \leq \frac{\pi}{2}. \tag{63}$$

And then, it suffices to consider a Jordan decomposition of f in the interval $[-2,2]$. So, we put $f = g + h$, with

$$g(y) = V_{-2}^{y}(f) = \begin{cases} \dfrac{1}{10}y + \dfrac{2}{10}, & \text{if } y \in [-2,-1), \\ \dfrac{1}{10}y + \dfrac{3}{10}, & \text{if } y \in [-1,1], \\ \dfrac{1}{10}y + \dfrac{4}{10}, & \text{if } y \in (1,2], \end{cases}$$

$$h(y) = f(y) - V_{-2}^{y}(f) = \begin{cases} -\frac{2}{10}y - \frac{5}{10}, & \text{if } y \in [-2,-1), \\ -\frac{3}{10}, & \text{if } y \in [-1,1], \\ -\frac{2}{10}y - \frac{1}{10}, & \text{if } y \in (1,2]. \end{cases} \tag{64}$$

We will show now that the functions $\alpha(t) = t - \pi/2 = -\beta(t)$ are coupled lower and upper solutions for problem (1). First, we have that $\alpha(t) \le \phi(t) \le \beta(t)$ for all $t \in [\pi/8, \pi/2]$. On the other hand, for a.a. $t \in I$ we have the following.

(i) If $(\pi/2) - 4t \in (1,2]$, then $4t - (\pi/2) \in [-2,-1)$ and then

$$\begin{aligned} &V_{-2}^{\beta(4t)}(f) + \alpha(4t) - V_{-2}^{\alpha(4t)}(f) \\ &\quad = \frac{8}{10}\left(\frac{\pi}{2} - 4t\right) = \frac{-8}{10}\left(\frac{\pi}{2} - 4t\right) + \frac{2}{10} \le 1 = \alpha'(t), \\ &V_{-2}^{\alpha(4t)}(f) + \beta(4t) - V_{-2}^{\beta(4t)}(f) \\ &\quad = \frac{8}{10}\left(\frac{\pi}{2} - 4t\right) - \frac{2}{10} \ge -1 = \beta'(t). \end{aligned} \tag{65}$$

(ii) If $(\pi/2) - 4t \in [0,1]$, then $4t - (\pi/2) \in [-1,0]$ and then

$$\begin{aligned} &V_{-2}^{\beta(4t)}(f) + \alpha(4t) - V_{-2}^{\alpha(4t)}(f) \\ &\quad = \frac{8}{10}\left(\frac{\pi}{2} - 4t\right) = \frac{-8}{10}\left(\frac{\pi}{2} - 4t\right) \le 1 = \alpha'(t), \\ &V_{-2}^{\alpha(4t)}(f) + \beta(4t) - V_{-2}^{\beta(4t)}(f) \\ &\quad = \frac{8}{10}\left(\frac{\pi}{2} - 4t\right) \ge -1 = \beta'(t). \end{aligned} \tag{66}$$

Then, α and β are coupled lower and upper solutions for problem (61), satisfying $\alpha \le \beta$ on $[0, \pi/2]$.

Now, we check condition (H_3). We have for a.a. $t \in I$, all $x \in [\alpha(t), \beta(t)]$, and all $y_1, y_2 \in [\alpha(4t), \beta(4t)]$,

$$\left|g(t, x, y_1) + h(t, x, y_2)\right| \le \frac{6}{10} + \frac{5}{10}, \tag{67}$$

and thus condition (H_2) is satisfied with $\psi \equiv 11/10$.

Finally, notice that for a.a. $t \in I$, it is

$$\begin{aligned} &\left[\min_{s\in[\pi/8,\pi/2]} \phi(s) - \int_t^{\pi/8} \psi(s)\,ds,\right. \\ &\quad \left.\max_{s\in[\pi/8,\pi/2]} \phi(s) + \int_t^{\pi/8} \psi(s)\,ds\right] \subset [-1,1], \end{aligned} \tag{68}$$

and thus condition (H_4) is satisfied with

$$L_1 \equiv \frac{1}{10}, \qquad L_2 \equiv 0. \tag{69}$$

By application of Theorem 10, we conclude that problem (61) has exactly one solution in the functional interval

$$\left[4t - \frac{\pi}{2}, \frac{\pi}{2} - 4t\right]. \tag{70}$$

In the following example, we consider a practical application of Corollary 17. It involves a modified logistic-type model with delay.

Example 2. Consider a bacterial culture governed by a logistic-type equation of the form

$$x'(t) = r_1 x(t)(K - x(t)), \tag{71}$$

where x represents the population in thousands.

To counteract the effects of saturation term, we introduce an electronic mechanism which acts as follows. It counts the number of individuals and then it provides some food that makes the population grow. The amount of food supplied by the machine is proportional to the number of individuals. Moreover, the machine can distinguish only thousands of individuals and it supplies the food with a delay τ which also depends on time; as time goes by, this delay increases. Therefore, we can model this process with a differential equation of the form

$$x'(t) = r_1 x(t)(K - x(t)) + r_2[x(t - \tau(t))], \quad t \in [0,1], \tag{72}$$

where $r_1, r_2 \ge 0$, $K > 1$, $[\cdot]$ means integer part and $\tau : t \in [0,1] \to \tau(t) \in [0,t]$ is measurable. We consider the normalized time interval $[0,1]$ for simplicity.

Finally, we consider an initial population of one thousand individuals. Therefore, we deal with the following initial value problem with delay:

$$\begin{aligned} &x'(t) = r_1 x(t)(K - x(t)) + r_2[x(t - \tau(t))], \\ &\qquad \text{for a.a. } t \in [0,1], \ x(0) = 1. \end{aligned} \tag{73}$$

We will show now that problem (73) has extremal solutions between suitable lower and upper solutions.

First, notice that $\alpha \equiv 1$ and $\beta \equiv L$, for large enough L, are, respectively, lower and upper solutions for the problem. Indeed, for all $t \in [0,1]$, we have

$$\begin{aligned} &0 = \alpha'(t) \le r_1(K-1) + r_2, \\ &0 = \beta'(t) \ge r_1 L(K - L) + r_2 L, \quad \text{for large enough } L. \end{aligned} \tag{74}$$

Then, α and β are, respectively, lower and upper solutions for problem (73), which moreover satisfy $\alpha(t) \leq \beta(t)$ for all $t \in [0,1]$.

To check conditions $(\widehat{H}_2)'$, $(\widehat{H}_3)'$, and $(\widehat{H}_4)'$, notice that the differential equation in (73) is defined by the function

$$f(t,x,y) = r_1 x(K-x) + r_2[y]. \tag{75}$$

First, for each continuous γ such that $1 \leq \gamma(t) \leq L$ for all $t \in [0,1]$, the function

$$f_\gamma(t,x) = f(t,x,\gamma) \tag{76}$$

is the classical logistic function and then the initial value problem

$$x'(t) = f_\gamma(t,x(t)), \quad \text{for a.a. } t \in [0,1], \; x(0) = 1 \tag{77}$$

has extremal solutions (in fact, a unique solution) between α and β. Therefore, condition $(\widehat{H}_2)'$ is satisfied.

Finally, as for x, $y \in [1,L]$, the function f is bounded and, moreover, f is nondecreasing with respect to its third variable; we conclude that $(\widehat{H}_3)'$ and $(\widehat{H}_4)'$ hold.

Therefore, we can apply Corollary 17 to ensure that problem (73) has the extremal solutions between α and β.

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References

[1] R. Figueroa and R. L. Pouso, "Coupled fixed points of multivalued operators and first-order ODEs with state-dependent deviating arguments," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 18, pp. 6876–6889, 2011.

[2] A. Dyki and T. Jankowski, "Boundary value problems for ordinary differential equations with deviated arguments," *Journal of Optimization Theory and Applications*, vol. 135, no. 2, pp. 257–269, 2007.

[3] R. Figueroa, "Discontinuous functional differential equations with delayed or advanced arguments," *Applied Mathematics and Computation*, vol. 218, no. 19, pp. 9882–9889, 2012.

[4] T. Jankowski, "Advanced differential equations with nonlinear boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 304, no. 2, pp. 490–503, 2005.

[5] T. M. Apostol, *Mathematical Analysis*, Addison-Wesley, Reading, Mass, USA, 2nd edition, 1974.

[6] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer, New York, NY, USA, 3rd edition, 1975.

[7] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, vol. 181 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1994.

[8] E. R. Hassan and W. Rzymowski, "Extremal solutions of a discontinuous scalar differential equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 37, no. 8, pp. 997–1017, 1999.

[9] R. L. Pouso, "Nonordered discontinuous upper and lower solutions for first-order ordinary differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 45, no. 4, pp. 391–406, 2001.

[10] R. L. Pouso, "Necessary conditions for solving initial value problems with infima of superfunctions," *Mathematical Inequalities & Applications*, vol. 8, no. 4, pp. 633–641, 2005.

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Research Article

Null Field and Interior Field Methods for Laplace's Equation in Actually Punctured Disks

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For solving Laplace's equation in circular domains with circular holes, the null field method (NFM) was developed by Chen and his research group (see Chen and Shen (2009)). In Li et al. (2012) the explicit algebraic equations of the NFM were provided, where some stability analysis was made. For the NFM, the conservative schemes were proposed in Lee et al. (2013), and the algorithm singularity was fully investigated in Lee et al., submitted to Engineering Analysis with Boundary Elements, (2013). To target the same problems, a new interior field method (IFM) is also proposed. Besides the NFM and the IFM, the collocation Trefftz method (CTM) and the boundary integral equation method (BIE) are two effective boundary methods. This paper is devoted to a further study on NFM and IFM for three goals. The first goal is to explore their intrinsic relations. Since there exists no error analysis for the NFM, the second goal is to drive error bounds of the numerical solutions. The third goal is to apply those methods to Laplace's equation in the domains with extremely small holes, which are called actually punctured disks. By NFM, IFM, BIE, and CTM, numerical experiments are carried out, and comparisons are provided. This paper provides an in-depth overview of four methods, the error analysis of the NFM, and the intriguing computation, which are essential for the boundary methods.

1. Introduction

For circular domains with circular holes, there exist a number of papers of boundary methods. In Barone and Caulk [1, 2] and Caulk [3], the Fourier functions are used for the circular holes for boundary integral equations. In Bird and Steele [4], the simple algorithms as the collocation Trefftz method (CTM) in [5, 6] are used. In Ang and Kang [7], complex boundary elements are studied. Recently, Chen and his research group have developed the null filed method (NFM), in which the field nodes Q are located outside of the solution domain S. The fundamental solutions (FS) can be expanded as the convergent series, and the Fourier functions are also used to approximate the Dirichlet and Neumann boundary conditions. Numerous papers have been published for different physical problems. Since error analysis and numerical experiments for four boundary methods are our main concern, we only cite [8–14]. More references of NFM are also given in [10–12, 14–17].

In [17], explicit algebraic equations of the NFM are derived, stability analysis is first made for the simple annular domain with concentric circular boundaries, and numerical experiments are performed to find the optimal field nodes. The field nodes can be located on the domain boundary: $Q \in \partial S$, if the solutions are smooth enough to satisfy $u \in H^2(\partial S)$ and $u_\nu \in H^1(\partial S)$, where u_ν is the normal derivative and $H^k(\partial S)$ $(k = 1, 2)$ are the Sobolev spaces; see the proof in [17]. It is discovered numerically that when the field nodes $Q \in \partial S$, the NFM provides small errors and the smallest condition numbers, compared with all $Q \in S^c$. Moreover for the NFM, the conservative schemes are proposed in [15], and the algorithm singularity is fully investigated in [16]. In fact, the explicit algebraic equations can also be derived from the Green representation formula with the field nodes inside the solution domain. This method is called the interior field method (IFM).

In addition to the NFM and IFM, the collocation Trefftz method (CTM) and the boundary integral equation method (BIE) are effective boundary methods too. Three goals are motivated in this paper. The first goal is to explore the intrinsic relations of NFM, IFM, CTM, and BIE with an in-depth overview. So far, there exists no error analysis for the NFM. The second goal is to derive error bounds of the numerical solutions by the NFM. The optimal convergence (or exponential) rates can be achieved. The third goal is to solve a challenging problem: Laplace's equation in the circular domains with extremely small holes, which are called the actually punctured disks in this paper. Four boundary methods, NFM, IFM, CTM, and BIE, are employed. Numerical experiments are carried out, and comparisons are provided. It is observed that the CTM is more advantageous in the applications than the others.

Besides, the method of fundamental solutions (MFS) is also popular in boundary methods, which originated from Kupradze and Aleksidze [18] in 1964. For the MFS, numerous computations are reviewed in Fairweather and Karageorghis [19] and Chen et al. [20], but the error and stability analysis is developed by Li et al. in [21, 22]. Both the CTM and the MFS can be applied to arbitrary solution domains. However, the MFS incurs a severe numerical instability for very elongated domains [22]. Since the performance of the CTM is better than that of the MFS, reported elsewhere, we do not carry out the numerical computation of the MFS in this paper. Moreover, the null-field method with discrete source (NFM-DS) is effective and popular in light scattering (see Wriedt [23]), where the transition (T) matrix is provided in Doicu and Wriedt [24]. In fact, the null field equation (NFE) of the Green representation formula in (9) can be employed on a source outside the solution domain S, without a need of the FS expansions, called the T matrix method [24]. Hence, the T matrix method is valid for arbitrary solution domains. There also occurs a severe numerical instability for very elongated holes (i.e., particles). To improve the stability for this case, different sources (i.e., discrete sources) may be utilized in the NFM-DS, by means of the idea of the MFS. The techniques for improving the stability by the NFM-DS are reported in many papers; we only cite [23, 25].

This paper is organized as follows. In the next section, the explicit discrete equations of NFM, IFM, CTM, and BIE are given, and their relations and overviews are explored. In Section 3, for the NFM some analysis is studied for circular domains with concentric circular boundaries. In Section 4, error bounds are provided without proof for the NFM with eccentric circular boundaries of simple annular domains. In Section 5, numerical experiments are carried out for Laplace's equation in the actually punctured disks. The results are reported with comparisons. In the last section, a few concluding remarks are addressed.

2. The Null Field Method and Other Algorithms

2.1. The Null Field Method. For simplicity in description of the NFM, we confine ourselves to Laplace's equation and choose the circular domain with one circular hole in this paper. Denote the disks S_R and S_{R_1} with radii R and R_1, respectively. Let $S_{R_1} \subset S_R$, and the eccentric circular domains S_R and S_{R_1} may have different origins. Hence $2R_1 < R$. Choose the annular solution domain $S = S_R \setminus S_{R_1}$ with the exterior and the interior boundaries ∂S_R and ∂S_{R_1}, respectively. The following Dirichlet problems are discussed by Palaniappan [26]:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } S,$$
$$u = 1 \text{ on } \partial S_R, \quad u = 0 \quad \text{on } \partial S_{R_1}. \tag{1}$$

In [11], $R = 2.5$ and $R_1 = 1$ and the origins of S_R and S_{R_1} are located at $(0,0)$ and $(-R_1, 0)$, respectively. In this paper, we fix $R = 2.5$, while R_1 may be infinitesimal; that is, $R_1 \ll 1$.

On the exterior boundary ∂S_R, there exist the approximations of Fourier expansions:

$$u = u_0 := a_0 + \sum_{k=1}^{M} \{a_k \cos k\theta + b_k \sin k\theta\} \quad \text{on } \partial S_R, \tag{2}$$

$$\frac{\partial u}{\partial \nu} = q_0 := p_0 + \sum_{k=1}^{M} \{p_k \cos k\theta + q_k \sin k\theta\} \quad \text{on } \partial S_R, \tag{3}$$

where a_k, b_k, p_k, and q_k are coefficients. On the interior boundary ∂S_{R_1}, we have similarly

$$\overline{u} = \overline{u}_0 := \overline{a}_0 + \sum_{k=1}^{N} \{\overline{a}_k \cos k\overline{\theta} + \overline{b}_k \sin k\overline{\theta}\} \quad \text{on } \partial S_{R_1}, \tag{4}$$

$$\frac{\partial \overline{u}}{\partial \overline{\nu}} = -\frac{\partial \overline{u}}{\partial \overline{r}} = \overline{q}_0 := \overline{p}_0 + \sum_{k=1}^{N} \{\overline{p}_k \cos k\overline{\theta} + \overline{q}_k \sin k\overline{\theta}\} \quad \text{on } \partial S_{R_1}, \tag{5}$$

where $\overline{a}_k, \overline{b}_k, \overline{p}_k$, and $\overline{q}_k$ are coefficients. In (2)–(5), θ and $\overline{\theta}$ are the polar coordinates of S_R and S_{R_1} with the origins $(0,0)$ and $(-R_1, 0)$, respectively, and ν and $\overline{\nu}$ are the exterior normals of ∂S_R and ∂S_{R_1}, respectively. The Dirichlet, the Neumann conditions, and their mixed types on ∂S_R may be given with known coefficients.

In S, denote two nodes $\mathbf{x} = Q = (x, y) = (\rho, \theta)$ and $\mathbf{y} = P = (\xi, \eta) = (r, \phi)$, where $x = \rho\cos\theta$, $y = \rho\sin\theta$, $\xi = r\cos\phi$, and $\eta = r\sin\phi$. Then $\rho = \sqrt{x^2 + y^2}$ and $R = r = \sqrt{\xi^2 + \eta^2}$. The FS of Laplace's equation is given by $\ln \overline{PQ} =$

$\ln\{\sqrt{\rho^2 - 2\rho r\cos(\theta-\phi) + r^2}\}$. From the Green representation formula, we have different formulas for different locations of the field nodes $Q(\mathbf{x})$:

$$\int_{\partial S}\left\{\ln|PQ|\frac{\partial u(\mathbf{y})}{\partial \nu} - u(\xi)\frac{\partial \ln|PQ|}{\partial \nu}\right\} d\sigma_\xi = \begin{cases} -2\pi u(Q), & Q \in S, \\ -\pi u(Q), & Q \in \partial S, \\ 0, & \text{otherwise}, \end{cases} \tag{6}$$

where $P(\mathbf{y}) \in (S \cup \partial S)$ and two kinds of series expansions of the FS $\ln|PQ|$ are given by (see [27])

$$\begin{aligned} \ln|PQ| &= \ln|P(\mathbf{y}) - Q(\mathbf{x})| \\ &= \ln|P(r,\phi) - Q(\rho,\theta)| \\ &= \begin{cases} U^i(\mathbf{x},\mathbf{y}) = \ln r - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{\rho}{r}\right)^n \cos n(\theta-\phi), & \rho < r, \\ U^e(\mathbf{x},\mathbf{y}) = \ln \rho - \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{\rho}\right)^n \cos n(\theta-\phi), & \rho > r, \end{cases} \end{aligned} \tag{7}$$

where $\mathbf{x} = (\rho,\theta)$ and $\mathbf{y} = (r,\phi)$. Then we have two kinds of derivative expansions of FS

$$\begin{aligned} \frac{\partial U^i(\mathbf{x},\mathbf{y})}{\partial r} &= \frac{1}{r} + \sum_{n=1}^{\infty}\left(\frac{\rho^n}{r^{n+1}}\right)\cos n(\theta-\phi), \quad \rho < r, \\ \frac{\partial U^e(\mathbf{x},\mathbf{y})}{\partial r} &= -\sum_{n=1}^{\infty}\left(\frac{r^{n-1}}{\rho^n}\right)\cos n(\theta-\phi), \quad \rho > r, \end{aligned} \tag{8}$$

where the superscripts "e" and "i" designate the exterior and interior field nodes $\mathbf{x}$, respectively. Note that the boundary element method (BEM) is based on the second equation of the Green formula (6), but the NFM is based on the third equation (i.e., the null field equation (NFE)) by using the FS expansions. We have

$$\begin{aligned} &\int_{\partial S_R \cup \partial S_{R_1}} U(\mathbf{x},\mathbf{y})\frac{\partial u(\mathbf{y})}{\partial \nu} d\sigma_{\mathbf{y}} \\ &\quad = \int_{\partial S_R \cup \partial S_{R_1}} u(\mathbf{y})\frac{\partial U(\mathbf{x},\mathbf{y})}{\partial \nu} d\sigma_{\mathbf{y}}, \quad \mathbf{x} \in \overline{S}^c, \end{aligned} \tag{9}$$

where $\overline{S}^c$ is the complementary domain of $S \cup \partial S$. Substituting the Fourier expansions (7)–(8) into (9) yields the basic algorithms of NFM, where the exterior normal of ∂S_{R_1} is given by $\partial U(\mathbf{x},\mathbf{y})/\partial\nu = -\partial U(\mathbf{x},\mathbf{y})/\partial r$. In the Green formula (9), the field point $\mathbf{x} = (\rho,\theta)$ is supposed to locate outside of the solution domain $S \cup \partial S$ only, so the algorithm of Chen is called the null field method (NFM) [8, 9, 11]. The field nodes can also be located on the domain boundary: $Q \in \partial S$, if the solutions are smooth enough to satisfy $u \in H^2(\partial S)$ and $u_\nu \in H^1(\partial S)$, where u_ν is the normal derivative and $H^k(\partial S)$ ($k = 1, 2$) are the Sobolev spaces; see the rigorous proof in [17]. It is discovered numerically that when the field nodes $Q \in \partial S$, the NFM provides small errors and condition numbers and has been widely implemented in many engineering problems.

Denote two systems of polar coordinates by (ρ,θ) and $(\overline{\rho},\overline{\theta})$ with the origins $(0,0)$ and (x_1,y_1) for S_R and S_{R_1}, respectively. There exist the following conversion formulas:

$$\begin{aligned} \rho &= \sqrt{\left(\overline{\rho}\cos\overline{\theta} + x_1\right)^2 + \left(\overline{\rho}\sin\overline{\theta} + y_1\right)^2}, \\ \tan\theta &= \frac{\overline{\rho}\sin\overline{\theta} + y_1}{\overline{\rho}\cos\overline{\theta} + x_1}, \end{aligned} \tag{10}$$

$$\begin{aligned} \overline{\rho} &= \sqrt{\left(\rho\cos\theta - x_1\right)^2 + \left(\rho\sin\theta - y_1\right)^2}, \\ \tan\overline{\theta} &= \frac{\rho\sin\theta - y_1}{\rho\cos\theta - x_1}. \end{aligned} \tag{11}$$

First, consider the exterior field nodes $\mathbf{x} = (\rho,\theta)$ with $\rho > r = R$. The first explicit algebraic equations of the NFM are given for the exterior field nodes (see [17])

$$\begin{aligned} &\mathscr{L}_{\text{ext}}\left(\rho,\theta;\overline{\rho},\overline{\theta}\right) \\ &:= -R\pi\sum_{k=1}^{M}\left(\frac{R^{k-1}}{\rho^k}\right)\left(a_k\cos k\theta + b_k\sin k\theta\right) \\ &\quad + R_1\pi\sum_{k=1}^{N}\left(\frac{R_1^{k-1}}{\overline{\rho}^k}\right)\left(\overline{a}_k\cos k\overline{\theta} + \overline{b}_k\sin k\overline{\theta}\right) \\ &\quad - \Bigg\{2\pi R(\ln\rho)\,p_0 - R\pi\sum_{k=1}^{M}\frac{1}{k}\left(\frac{R}{\rho}\right)^k \\ &\qquad \times\left(p_k\cos k\theta + q_k\sin k\theta\right) + 2\pi R_1\left(\ln\overline{\rho}\right)\overline{p}_0 \\ &\qquad - R_1\pi\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\overline{\rho}}\right)^k\left(\overline{p}_k\cos k\overline{\theta} + \overline{q}_k\sin k\overline{\theta}\right)\Bigg\} = 0. \end{aligned} \tag{12}$$

Next, consider the interior field nodes $\mathbf{x} = (\overline{\rho},\overline{\theta})$ with $\overline{\rho} < \overline{r} = R_1$. The second explicit algebraic equations of the NFM are given for the interior field nodes (see [17])

$$\begin{aligned} &\mathscr{L}_{\text{int}}\left(\rho,\theta;\overline{\rho},\overline{\theta}\right) \\ &:= -2\pi\overline{a}_0 - R_1\pi\sum_{k=1}^{N}\left(\frac{\overline{\rho}^k}{R_1^{k+1}}\right)\left(\overline{a}_k\cos k\overline{\theta} + \overline{b}_k\sin k\overline{\theta}\right) \\ &\quad + 2\pi a_0 + R\pi\sum_{k=1}^{M}\left(\frac{\rho^k}{R^{k+1}}\right)\left(a_k\cos k\theta + b_k\sin k\theta\right) \end{aligned}$$

$$-\left\{2\pi R_1 \ln R_1 \overline{p}_0 - R_1\pi\sum_{k=1}^{N}\frac{1}{k}\left(\frac{\overline{\rho}}{R_1}\right)^k \times\left(\overline{p}_k\cos k\overline{\theta}+\overline{q}_k\sin k\overline{\theta}\right)+2\pi R\ln R p_0 - R\pi\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho}{R}\right)^k \times\left(p_k\cos k\theta+q_k\sin k\theta\right)\right\}=0. \tag{13}$$

Since one of Dirichlet or Neumann conditions is given on ∂S_R and ∂S_{R_1}, only $2(M+N)+2$ coefficients in (2)–(5) are unknowns. We choose $2M+1$ and $2N+1$ field nodes located uniformly on the exterior and the interior circles, respectively,

$$\begin{aligned}(\rho,\theta)&=(R+\epsilon, i\Delta\theta), \quad i=0,1,\ldots,2M,\\ \left(\overline{\rho},\overline{\theta}\right)&=\left(R_1-\overline{\epsilon}, i\Delta\overline{\theta}\right), \quad i=0,1,\ldots,2N,\end{aligned} \tag{14}$$

where $\epsilon\ge 0$, $0\le\overline{\epsilon}<R_1$, $\Delta\theta=2\pi/(2M+1)$, and $\Delta\overline{\theta}=2\pi/(2N+1)$. Denote the explicit equations (12) and (13) by

$$\mathscr{L}_{\text{ext}}\left(\rho,\theta;\overline{\rho},\overline{\theta}\right)=0, \qquad \mathscr{L}_{\text{int}}\left(\rho,\theta;\overline{\rho},\overline{\theta}\right)=0. \tag{15}$$

We obtain $2(M+N)+2$ discrete equations from (15)

$$\begin{aligned}&\mathscr{L}_{\text{ext}}\left(R+\epsilon, i\Delta\theta;\overline{\rho}_i,\overline{\theta}_i\right)=0, \quad i=0,1,\ldots,2M,\\ &\mathscr{L}_{\text{int}}\left(\rho_i,\theta_i;R_1-\overline{\epsilon}, i\Delta\overline{\theta}\right)=0, \quad i=0,1,\ldots,2N,\end{aligned} \tag{16}$$

where the corresponding coordinates (ρ_i,θ_i) and $(\overline{\rho}_i,\overline{\theta}_i)$ are obtained from (10) and (11). Hence from (16), we obtain the following linear algebraic equations:

$$\mathbf{F}\mathbf{x}=\mathbf{b}, \tag{17}$$

where the matrices $\mathbf{F}(\in R^{n\times n})$, the vector $\mathbf{x}(\in R^n)$, and $n=2(M+N)+2$. The unknown coefficients can be obtained from (17), if the matrix $\mathbf{F}$ is nonsingular. In this paper, we confine the Dirichlet problems. The study of the Neumann problems will be reported in a subsequent paper.

Once all the coefficients are known, based on the first equation of the Green formula (6), the solution at the interior nodes: $\mathbf{x}=(\rho,\theta)\in S$ is expressed by

$$\begin{aligned}u(\mathbf{x})&=u(\rho,\theta)=\\ &-\frac{1}{2\pi}\int_{\partial S_R\cup\partial S_{R_1}}\left\{U(\mathbf{x},\mathbf{y})\frac{\partial u(\mathbf{y})}{\partial\nu}-u(\xi)\frac{\partial U(\mathbf{x},\mathbf{y})}{\partial r}\right\}d\sigma_{\mathbf{y}}\\ &=-\frac{1}{2\pi}\left\{\int_{\partial S_R}\left\{U^i(\mathbf{x},\mathbf{y})\frac{\partial u(\mathbf{y})}{\partial\nu}-u(\mathbf{y})\frac{\partial U^i(\mathbf{x},\mathbf{y})}{\partial r}\right\}d\sigma_{\mathbf{y}}\right.\\ &\qquad\left.+\int_{\partial S_{R_1}}\left\{U^e(\mathbf{x},\mathbf{y})\frac{\partial u(\mathbf{y})}{\partial\overline{\nu}}+u(\mathbf{y})\frac{\partial U^e(\mathbf{x},\mathbf{y})}{\partial\overline{r}}\right\}d\sigma_{\mathbf{y}}\right\}\\ &\qquad \mathbf{x}\in S.\end{aligned} \tag{18}$$

For $(\rho,\theta)\in S$, from (2)–(5) and (7)-(8), (2.20) leads to (see [17])

$$\begin{aligned}u_{M-N}&=u_{M-N}(\rho,\theta)=u_{M-N}\left(\overline{\rho},\overline{\theta}\right)\\ &=a_0-R\ln R p_0-R_1\ln\overline{\rho}\overline{p}_0+\frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho}{R}\right)^k\\ &\quad\times\left(p_k\cos k\theta+q_k\sin k\theta\right)+\frac{R}{2}\sum_{k=1}^{M}\left(\frac{\rho^k}{R^{k+1}}\right)\\ &\quad\times\left(a_k\cos k\theta+b_k\sin k\theta\right)+\frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\overline{\rho}}\right)^k\\ &\quad\times\left(\overline{p}_k\cos k\overline{\theta}+\overline{q}_k\sin k\overline{\theta}\right)+\frac{R_1}{2}\sum_{k=1}^{N}\left(\frac{R_1^{k-1}}{\overline{\rho}^k}\right)\\ &\quad\times\left(\overline{a}_k\cos k\overline{\theta}+\overline{b}_k\sin k\overline{\theta}\right), \quad (r,\theta)\in S,\end{aligned} \tag{19}$$

where $(\overline{\rho},\overline{\theta})$ are also given from (11).

2.2. Conservative Schemes. For some physical problems, the flux conservation is imperative and essential. The conservative schemes of NFM can be designed to satisfy exactly the flux conservation [15]

$$\int_{S_R}(u_M)_\nu+\int_{S_{R_1}}(u_N)_{\overline{\nu}}=0. \tag{20}$$

Substituting (3) and (5) into (20) yields directly

$$Rp_0+R_1\overline{p}_0=0. \tag{21}$$

We may use (21) to remove one coefficient, say $\overline{p}_0$,

$$\overline{p}_0=-\frac{R}{R_1}p_0. \tag{22}$$

By using (22), (12) and (13) lead to

$$\begin{aligned}&\mathscr{L}^C_{\text{ext}}\left(\rho,\theta;\overline{\rho},\overline{\theta}\right)\\ &:=-R\pi\sum_{k=1}^{M}\left(\frac{R^{k-1}}{\rho^k}\right)\left(a_k\cos k\theta+b_k\sin k\theta\right)\\ &\quad+R_1\pi\sum_{k=1}^{N}\left(\frac{R_1^{k-1}}{\overline{\rho}^k}\right)\left(\overline{a}_k\cos k\overline{\theta}+\overline{b}_k\sin k\overline{\theta}\right)\end{aligned}$$

$$
\begin{aligned}
&-\left\{2\pi R\left(\ln\left(\frac{\rho}{\overline{\rho}}\right)\right)p_0 - R\pi\sum_{k=1}^{M}\frac{1}{k}\left(\frac{R}{\rho}\right)^k\right.\\
&\quad\times\left(p_k\cos k\theta + q_k\sin k\theta\right) - R_1\pi\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\overline{\rho}}\right)^k\\
&\quad\left.\times\left(\overline{p}_k\cos k\overline{\theta} + \overline{q}_k\sin k\overline{\theta}\right)\right\} = 0,\\
&\mathscr{L}^{C}_{\text{int}}\left(\rho,\theta;\overline{\rho},\overline{\theta}\right)\\
&:= -2\pi\overline{a}_0 - R_1\pi\sum_{k=1}^{N}\left(\frac{\overline{\rho}^k}{R_1^{k+1}}\right)\\
&\quad\times\left(\overline{a}_k\cos k\overline{\theta} + \overline{b}_k\sin k\overline{\theta}\right) + 2\pi a_0\\
&\quad + R\pi\sum_{k=1}^{M}\left(\frac{\rho^k}{R^{k+1}}\right)\left(a_k\cos k\theta + b_k\sin k\theta\right)\\
&\quad -\left\{-R_1\pi\sum_{k=1}^{N}\frac{1}{k}\left(\frac{\overline{\rho}}{R_1}\right)^k\right.\\
&\quad\times\left(\overline{p}_k\cos k\overline{\theta} + \overline{q}_k\sin k\overline{\theta}\right)\\
&\quad + 2\pi R\left(\ln\frac{R}{R_1}\right)p_0 - R\pi\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho}{R}\right)^k\\
&\quad\left.\times\left(p_k\cos k\theta + q_k\sin k\theta\right)\right\} = 0.
\end{aligned}\tag{23}
$$

Also the interior solution (19) leads to

$$
\begin{aligned}
u^{C}_{M-N} &= u^{C}_{M-N}\left(\rho,\theta\right) = u^{C}_{M-N}\left(\overline{\rho},\overline{\theta}\right)\\
&= a_0 - R\left(\ln\frac{R}{R_1}\right)p_0 + \frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho}{R}\right)^k\\
&\quad\times\left(p_k\cos k\theta + q_k\sin k\theta\right) + \frac{R}{2}\sum_{k=1}^{M}\left(\frac{\rho^k}{R^{k+1}}\right)\\
&\quad\times\left(a_k\cos k\theta + b_k\sin k\theta\right) + \frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\overline{\rho}}\right)^k\\
&\quad\times\left(\overline{p}_k\cos k\overline{\theta} + \overline{q}_k\sin k\overline{\theta}\right) + \frac{R_1}{2}\sum_{k=1}^{N}\left(\frac{R_1^{k-1}}{\overline{\rho}^k}\right)\\
&\quad\times\left(\overline{a}_k\cos k\overline{\theta} + \overline{b}_k\sin k\overline{\theta}\right), \quad (r,\theta)\in S.
\end{aligned}\tag{24}
$$

Hence, the total number of unknown coefficients is reduced to $2(M+N)+1$. Based on the analysis in [15], to remain good stability, we still choose $2(M+N)+2$ collocation nodes as in (16):

$$
\begin{aligned}
&w_i\mathscr{L}^{C}_{\text{ext}}\left(R+\varepsilon, i\Delta\theta;\overline{\rho}_i,\overline{\theta}_i\right) = 0, \quad i = 0,1,\ldots,2M,\\
&w_i\mathscr{L}^{C}_{\text{int}}\left(\rho_i,\theta_i;R_1-\overline{\varepsilon}, i\Delta\overline{\theta}\right) = 0, \quad i = 1,2,\ldots,2N,
\end{aligned}\tag{25}
$$

where the weights $w_0 = 1$, $w_i = \sqrt{2}$ for $i \geq 1$, $\Delta\theta = 2\pi/(2M+1)$, and $\Delta\overline{\theta} = 2\pi/(2N+1)$. Equation (25) form an overdetermined system, which can be solved by the QR method or the singular value decomposition.

2.3. The Interior Field Method. In [17], we prove that when $u \in H^2(\partial S)$ and $u_\nu \in H^1(\partial S)$, the NFM remains valid for the field nodes $Q \in \partial S$; that is, $\rho = R$ on ∂S_R and $\overline{\rho} = R_1$ on ∂S_{R_1} and (23) and (24) hold. In fact, we may use (24) only, because (23) is obtained directly from the Dirichlet conditions on ∂S_R and ∂S_{R_1}, respectively. Interestingly, (24) is obtained from the interior (i.e., the first) Green formula in (6) only. For this reason, the interior field method (IFM) is named. Evidently, the IFM is equivalent to the special NFM. Based on this linkage, the new error analysis in Section 4 is explored.

2.4. The First Kind Boundary Integral Equations. We may also apply the series expansions of FS to the first kind boundary integral equations. Consider the Dirichlet problem

$$
\begin{aligned}
\Delta u &= 0, \quad \text{in } \Omega = R^2 \setminus \Gamma,\\
u &= f, \quad \text{on } \Gamma,\\
u(x) &= O\left(\log|\mathbf{x}|\right), \quad \text{as } |\mathbf{x}| \longrightarrow \infty,
\end{aligned}\tag{26}
$$

where $|\mathbf{x}|$ is the Euclidean distance. In (26), $\Gamma(=\cup_{m=1}^{d}\Gamma_m)$ is an open arc, and each of its edges, Γ_m $(m = 1,\ldots,d)$, is assumed to be smooth. Let C_Γ be the logarithmic capacity of Γ. From the single layer potential theory [28–30], if $C_\Gamma \neq 1$, (26) can be converted to the first kind boundary integral equation (BIE),

$$
-\frac{1}{2\pi}\int_\Gamma v(\mathbf{x})\ln|\mathbf{x}-y|\,ds_{\mathbf{x}} = f(\mathbf{y}) \quad (\mathbf{y}\in\Gamma),\tag{27}
$$

where $v(\mathbf{x})(= (\partial u(\mathbf{x})/\partial n^-) - (\partial u(\mathbf{x})/\partial n^+))$ is the unknown function and $\partial u/\partial n^\pm$ denote the normal derivatives along the positive and negative sides of Γ. If $C_\Gamma \neq 1$, there exists a unique solution of (27), see [28]. As soon as $v(\mathbf{x})$ is solved from (27), the solution $u(\mathbf{x})$ $(\mathbf{x}\in\Omega)$ of (26) can be evaluated by

$$
u(\mathbf{x}) = -\frac{1}{2\pi}\int_\Gamma v(\mathbf{x})\ln|\mathbf{x}-\mathbf{y}|\,ds_{\mathbf{x}} \quad (\mathbf{y}\in\Omega).\tag{28}
$$

For the smooth solution u, we have $\nu(\mathbf{x}) = 2(\partial u/\partial \nu)$, where ν is the normal of Γ. We may assume the Fourier expansions of ν on Γ

$$\begin{aligned} \nu(s) &= \nu^{+}(s) = q_0^{\star} \\ &:= p_0^{\star} + \sum_{k=1}^{M} \{p_k^{\star} \cos k\theta + q_k^{\star} \sin k\theta\} \quad \text{on } \partial S_R, \\ \nu(s) &= \nu^{-}(s) = \overline{q}_0^{\star} \\ &:= \overline{p}_0^{\star} + \sum_{k=1}^{N} \{\overline{p}_k^{\star} \cos k\overline{\theta} + \overline{q}_k^{\star} \sin k\overline{\theta}\} \quad \text{on } \partial S_{R_1}, \end{aligned} \tag{29}$$

where $p_k^{\star}, q_k^{\star}, \overline{p}_k^{\star}$, and $\overline{q}_k^{\star}$ are the coefficients. We have from [17]

$$\begin{aligned} u(\mathbf{x}) &= u(\rho,\theta) \\ &= -\frac{1}{2\pi} \int_{\partial S_R \cup \partial S_{R_1}} U(\mathbf{x},\mathbf{y})\, \nu(\mathbf{y})\, d\sigma_{\mathbf{y}} \\ &= -\frac{1}{2\pi} \left\{ \int_{\partial S_R} U^{i}(\mathbf{x},\mathbf{y})\, \nu(\mathbf{y})\, d\sigma_{\mathbf{y}} \right. \\ &\qquad \left. + \int_{\partial S_{R_1}} U^{e}(\mathbf{x},\mathbf{y})\, \nu(\mathbf{y})\, d\sigma_{\mathbf{y}} \right\}, \quad \mathbf{x} \in S, \end{aligned} \tag{30}$$

to give

$$\begin{aligned} u_{M-N}(\rho,\theta) &= -R \ln R p_0^{\star} - R_1 \ln \overline{\rho}\, \overline{p}_0^{\star} + \frac{R}{2} \sum_{k=1}^{M} \frac{1}{k} \left(\frac{\rho}{R}\right)^{k} \\ &\quad \times \left(p_k^{\star} \cos k\theta + q_k^{\star} \sin k\theta\right) + \frac{R_1}{2} \sum_{k=1}^{N} \frac{1}{k} \left(\frac{R_1}{\overline{\rho}}\right)^{k} \\ &\quad \times \left(\overline{p}_k^{\star} \cos k\overline{\theta} + \overline{q}_k^{\star} \sin k\overline{\theta}\right), \quad (r,\theta) \in S. \end{aligned} \tag{31}$$

Note that the derivation of (31) in the first kind BIE is simpler, because we do not need the series expansions of $\partial U^{i}(\mathbf{x},\mathbf{y})/\partial r$ and $\partial U^{e}(\mathbf{x},\mathbf{y})/\partial r$. This advantage is very important for elasticity problems, because the displacement conditions are much simpler than the traction ones.

2.5. The Collocation Trefftz Method. We also use the collocation Trefftz method (CTM). For (1), the particular solutions of CTM are given by (see [6])

$$\begin{aligned} u_{M-N}\left(\rho,\theta;\overline{\rho},\overline{\theta}\right) &= a_0 + \sum_{i=1}^{M} \left(\frac{\rho}{R}\right)^{i} \left(a_i \cos i\theta + b_i \sin i\theta\right) \\ &\quad + \overline{a}_0 \ln \overline{\rho} + \sum_{i=1}^{N} \left(\frac{R_1}{\overline{\rho}}\right)^{i} \\ &\quad \times \left(\overline{a}_i \cos i\overline{\theta} + \overline{b}_i \sin i\overline{\theta}\right), \\ &\quad \rho \le R, \quad \overline{\rho} \ge R_1, \end{aligned} \tag{32}$$

where $a_i, b_i, \overline{a}_i$, and $\overline{b}_i$ are the coefficients. Evidently, the admissible functions (19) of the IFM and (31) of the first kind BIE are the special cases of (32). Equation (31) may be written as (32) with the following relations of coefficients:

$$\begin{aligned} a_0 &:= -R \ln R p_0^{\star}, & \overline{a}_0 &:= -R_1 \overline{p}_0^{\star}, \\ a_k &:= \frac{R}{2k} p_k^{\star}, & b_k &:= \frac{R}{2k} q_k^{\star}, \\ \overline{a}_k &:= \frac{R_1}{2k} \overline{p}_k^{\star}, & \overline{b}_k &:= \frac{R_1}{2k} \overline{q}_k^{\star}. \end{aligned} \tag{33}$$

Equation (19) can also be written as (32) with

$$\begin{aligned} a_0 &:= a_0^{\mathrm{IFM}} - R \ln R p_0^{\mathrm{IFM}}, & \overline{a}_0 &:= -R_1 \overline{p}_0^{\mathrm{IFM}}, \\ a_k &:= \frac{R}{2k} p_k^{\mathrm{IFM}} + \frac{1}{2} a_k^{\mathrm{IFM}}, & b_k &:= \frac{R}{2k} q_k^{\mathrm{IFM}} + \frac{1}{2} b_k^{\mathrm{IFM}}, \\ \overline{a}_k &:= \frac{R_1}{2k} \overline{p}_k^{\mathrm{IFM}} + \frac{1}{2} \overline{a}_k^{\mathrm{IFM}}, & \overline{b}_k &:= \frac{R_1}{2k} \overline{q}_k^{\mathrm{IFM}} + \frac{1}{2} \overline{b}_k^{\mathrm{IFM}}, \end{aligned} \tag{34}$$

where $p_k^{\mathrm{IFM}}, q_k^{\mathrm{IFM}}, \ldots$ are the coefficients in (19) of the IFM.

Therefore, we may classify the IFM and the first kind BIE into the TM family, and their analysis may follow the framework in [6]. However, the particular solutions (32) can be applied to arbitrary shaped domains, for example, simply or multiple-connected domains, but the functions (19) and (31) are confined themselves to the circular domains with circular holes only. The four boundary methods, NFM, IFM, BIE, and CTM, are described together, with their explicit algebraic equations. The relations of their expansion coefficients are discovered at the first time. Moreover, Figure 1 shows clear relations among NFM, IFM, BIE, and CTM. The intrinsic relations have been provided to fulfill the first goal of this paper.

To close this section, we describe the CTM. Denote V_{M-N} the set of (32), and define the energy

$$I(u) = \int_{\Gamma} (\nu - f)^2, \tag{35}$$

where $\Gamma = \partial S$ and f is the known function of Dirichlet boundary conditions. Then the solution u_{M-N} of the Trefftz methods (TM) can be obtained by

$$I(u_{M-N}) = \min_{\nu \in V_{M-N}} I(\nu). \tag{36}$$

The TM solution u_{M-N} also satisfies

$$\|u - u_{M-N}\|_{0,\Gamma} = \min_{\nu \in V_{M-N}} \|u - \nu\|_{0,\Gamma}. \tag{37}$$

When the integral in (35) involves numerical approximation, the modified energy is defined as

$$\widehat{I}(\nu) = \widehat{\int}_{\Gamma} (\nu - f)^2, \tag{38}$$

where $\widehat{\int}_{\Gamma}$ is the numerical approximations of $\int_{\Gamma}$ by some quadrature rules, such as the central or the Gaussian rule. Hence, the numerical solution $\widehat{u}_{M-N} \in V_{M-N}$ is obtained by

$$\widehat{I}(\widehat{u}_{M-N}) = \min_{\nu \in V_{M-N}} \widehat{I}(\nu). \tag{39}$$

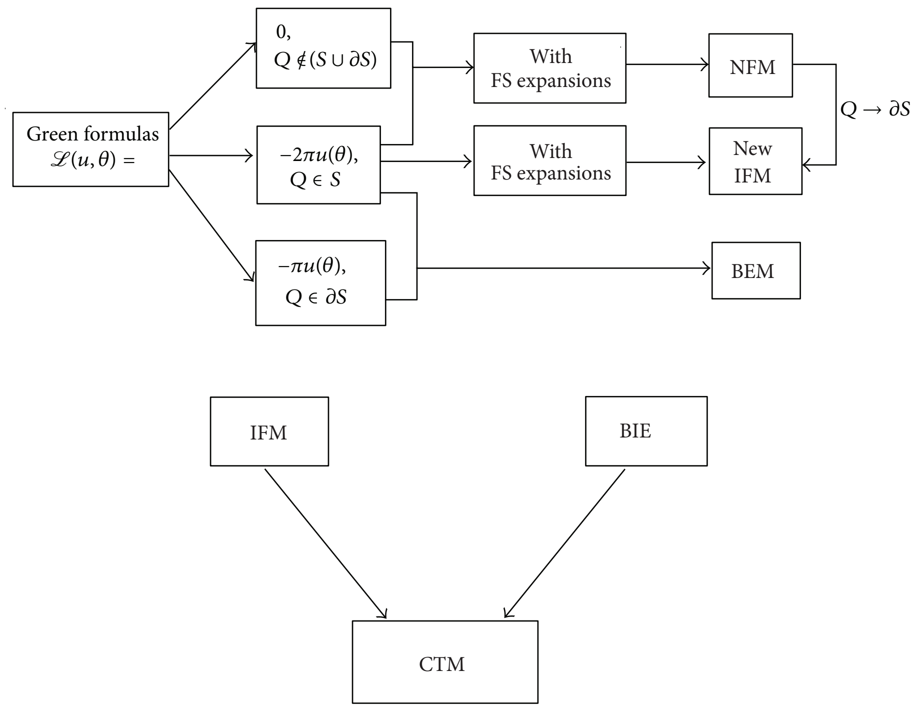

Figure 1: Relations among NFM, IFM, BIE, and CTM.

We may also establish the collocation equations directly from the Dirichlet condition to yield

$$\hat{u}_{M-N}\left(P_j\right) = f_{M-N}\left(P_j\right), \quad P_j \in \Gamma. \tag{40}$$

Following [6], (40) is just equivalent to (38).

3. Preliminary Analysis of the NFM

In this section, a preliminary analysis of the NFM is made for concentric circular boundaries. In the next section, error analysis of the NFM with $\epsilon = \bar{\epsilon} = 0$ is explored for eccentric circular boundaries. Consider the simple domains of $S = S_R \setminus S_{R_1}$, where S_R and S_{R_1} have the same origin. For the same origin O of S_R and S_{R_1}, the same polar coordinates (ρ,θ) are used, and the general solutions in $S_R \setminus S_{R_1}$ can be denoted by

$$\begin{aligned} u(\rho,\theta) = {} & a_0^* + \sum_{k=1}^{\infty} \rho^k \left\{a_k^* \cos k\theta + b_k^* \sin k\theta\right\} + \bar{a}_0^* \ln \rho \\ & + \sum_{k=1}^{\infty} \rho^{-k} \left\{\bar{a}_k^* \cos k\theta + \bar{b}_k^* \sin k\theta\right\}, \end{aligned} \tag{41}$$

where $a_i^*, b_i^*, \bar{a}_i^*, \bar{b}_i^*$ are true coefficients and $R_1 \le \rho \le R$. Then their derivatives are given by

$$\begin{aligned} \frac{\partial}{\partial \rho} u(\rho,\theta) = {} & \sum_{k=1}^{\infty} k\rho^{k-1} \left\{a_k^* \cos k\theta + b_k^* \sin k\theta\right\} + \bar{a}_0^* \frac{1}{\rho} \\ & - \sum_{k=1}^{\infty} k\rho^{-k-1} \left\{\bar{a}_k^* \cos k\theta + \bar{b}_k^* \sin k\theta\right\}. \end{aligned} \tag{42}$$

When $\rho = R$, from (41) and (42), we have

$$\begin{aligned} u(\rho,\theta)\big|_{\rho=R} = {} & a_0^* + \sum_{k=1}^{\infty} R^k \left\{a_k^* \cos k\theta + b_k^* \sin k\theta\right\} + \bar{a}_0^* \ln R \\ & + \sum_{k=1}^{\infty} R^{-k} \left\{\bar{a}_k^* \cos k\theta + \bar{b}_k^* \sin k\theta\right\}, \\ \frac{\partial}{\partial \rho} u(\rho,\theta)\bigg|_{\rho=R} = {} & \sum_{k=1}^{\infty} kR^{k-1} \left\{a_k^* \cos k\theta + b_k^* \sin k\theta\right\} + \bar{a}_0^* \frac{1}{R} \\ & - \sum_{k=1}^{\infty} kR^{-k-1} \left\{\bar{a}_k^* \cos k\theta + \bar{b}_k^* \sin k\theta\right\}. \end{aligned} \tag{43}$$

Comparing (43) with (2) and (3), we have the following equalities of coefficients:

$$\begin{aligned} a_0 &= a_0^* + \bar{a}_0^* \ln R, \\ a_k &= R^k a_k^* + R^{-k} \bar{a}_k^*, \\ b_k &= R^k b_k^* + R^{-k} \bar{b}_k^*, \end{aligned} \tag{44}$$

$$\begin{aligned} p_0 &= \bar{a}_0^* \frac{1}{R}, \\ p_k &= k\left\{R^{k-1} a_k^* - R^{-k-1} \bar{a}_k^*\right\}, \\ q_k &= k\left\{R^{k-1} b_k^* - R^{-k-1} \bar{b}_k^*\right\}, \end{aligned} \tag{45}$$

where a_k, b_k, p_k, and q_k are the coefficients of the NFM in Section 2.1.

Also, when $\rho = R_1$, from (41) and (42), we have

$$\begin{aligned} u(\rho,\theta)\big|_{\rho=R_1} &= a_0^* + \sum_{k=1}^{\infty} R_1^k \{a_k^* \cos k\theta + b_k^* \sin k\theta\} \\ &\quad + \overline{a}_0^* \ln R_1 \\ &\quad + \sum_{k=1}^{\infty} R_1^{-k} \left\{\overline{a}_k^* \cos k\theta + \overline{b}_k^* \sin k\theta\right\}, \\ \frac{\partial}{\partial\rho} u(\rho,\theta)\bigg|_{\rho=R_1} &= \sum_{k=1}^{\infty} k R_1^{k-1} \{a_k^* \cos k\theta + b_k^* \sin k\theta\} + \overline{a}_0^* \frac{1}{R_1} \\ &\quad - \sum_{k=1}^{\infty} k R_1^{-k-1} \left\{\overline{a}_k^* \cos k\theta + \overline{b}_k^* \sin k\theta\right\}. \end{aligned} \tag{46}$$

Comparing (46) with (4) and (5), we have

$$\begin{aligned} \overline{a}_0 &= a_0^* + \overline{a}_0^* \ln R_1, \\ a_k &= R_1^k a_k^* + R_1^{-k} \overline{a}_k^*, \\ \overline{b}_k &= R_1^k b_k^* + R_1^{-k} \overline{b}_k^*, \end{aligned} \tag{47}$$

$$\begin{aligned} \overline{p}_0 &= -\overline{a}_0^* \frac{1}{R_1}, \\ \overline{p}_k &= -k \left\{R_1^{k-1} a_k^* - R_1^{-k-1} \overline{a}_k^*\right\}, \\ \overline{q}_k &= -k \left\{R_1^{k-1} b_k^* - R_1^{-k-1} \overline{b}_k^*\right\}, \end{aligned} \tag{48}$$

where $\overline{a}_k, \overline{b}_k, \overline{p}_k$, and $\overline{q}_k$ are also the coefficients of the NFM in Section 2.1.

On the other hand, when $(\overline{\rho}, \overline{\theta}) = (\rho, \theta)$, we have from the first original equation (12)

$$\begin{aligned} &- R\pi \sum_{k=1}^{M} \left(\frac{R^{k-1}}{\rho^k}\right) (a_k \cos k\theta + b_k \sin k\theta) \\ &\quad + R_1 \pi \sum_{k=1}^{N} \left(\frac{R_1^{k-1}}{\rho^k}\right) \left(\overline{a}_k \cos k\theta + \overline{b}_k \sin k\theta\right) \\ &= 2\pi R (\ln \rho)\, p_0 - R\pi \sum_{k=1}^{M} \frac{1}{k} \left(\frac{R}{\rho}\right)^k \\ &\quad \times (p_k \cos k\theta + q_k \sin k\theta) + 2\pi R_1 (\ln \rho)\, \overline{p}_0 \\ &\quad - R_1 \pi \sum_{k=1}^{N} \frac{1}{k} \left(\frac{R_1}{\rho}\right)^k (\overline{p}_k \cos k\theta + \overline{q}_k \sin k\theta). \end{aligned} \tag{49}$$

Then for $\rho \geq R$, we obtain the following equalities, based on the orthogonality of trigonometric functions:

$$(\ln \rho)(R p_0 + R_1 \overline{p}_0) = 0, \tag{50}$$

$$R^k a_k - R_1^k \overline{a}_k = \frac{1}{k} R^{k+1} p_k + \frac{1}{k} R_1^{k+1} \overline{p}_k, \tag{51}$$

$$R^k b_k - R_1^k \overline{b}_k = \frac{1}{k} R^{k+1} q_k + \frac{1}{k} R_1^{k+1} \overline{q}_k. \tag{52}$$

Similarly, from the second equation (13),

$$\begin{aligned} &- 2\pi \overline{a}_0 - R_1 \pi \sum_{k=1}^{N} \left(\frac{\rho^k}{R_1^{k+1}}\right) \left(\overline{a}_k \cos k\theta + \overline{b}_k \sin k\theta\right) \\ &\quad + 2\pi a_0 + R\pi \sum_{k=1}^{M} \left(\frac{\rho^k}{R^{k+1}}\right) (a_k \cos k\theta + b_k \sin k\theta) \\ &= 2\pi R_1 \ln R_1 \overline{p}_0 - R_1 \pi \sum_{k=1}^{N} \frac{1}{k} \left(\frac{\rho}{R_1}\right)^k \\ &\quad \times (\overline{p}_k \cos k\theta + \overline{q}_k \sin k\theta) + 2\pi R \ln R p_0 \\ &\quad - R\pi \sum_{k=1}^{M} \frac{1}{k} \left(\frac{\rho}{R}\right)^k (p_k \cos k\theta + q_k \sin k\theta). \end{aligned} \tag{53}$$

Then for $\rho \leq R_1$, we obtain

$$a_0 - \overline{a}_0 = R \ln R p_0 + R_1 \ln R_1 \overline{p}_0, \tag{54}$$

$$\frac{1}{R^k} a_k - \frac{1}{R_1^k} \overline{a}_k = -\frac{1}{k} \frac{1}{R^{k-1}} p_k - \frac{1}{k} \frac{1}{R_1^{k-1}} \overline{p}_k, \tag{55}$$

$$\frac{1}{R^k} b_k - \frac{1}{R_1^k} \overline{b}_k = -\frac{1}{k} \frac{1}{R^{k-1}} q_k - \frac{1}{k} \frac{1}{R_1^{k-1}} \overline{q}_k. \tag{56}$$

Below, we prove that the true coefficients can be obtained directly from the NFM based on (50)–(52) for $\rho \geq R$ and on (54)–(56) for $\rho \leq R_1$. Outline of the proof is as follows. We will prove that the true solutions satisfy (50)–(52) and (54)–(56) of the NFM. Based on the analysis in [16], when $R \neq 1$, there exists a unique solution of the special NFM with $\epsilon = \overline{\epsilon} = 0$. Therefore, the true coefficients can be determined by the IFM uniquely.

First to show (50). The consistent condition is given by

$$\int_{\partial S_R} \frac{\partial u}{\partial \nu} + \int_{\partial S_{R_1}} \frac{\partial u}{\partial \nu} = 2\pi R p_0 + 2\pi R_1 \overline{p}_0 = 0. \tag{57}$$

Equation (57) can also be obtained from (45) and (48). Equations (57) and (50) are equivalent if $\ln \rho \neq 0$ (i.e., $\rho \neq 1$), which is also the necessary condition of nonsingularity of matrix $\mathbf{F}$ in (17) [16]. Based on (57), the conservative schemes are proposed in [15]. Equation (54) is shown next. We have from (44) and (47)

$$\begin{aligned} a_0 - \overline{a}_0 &= \overline{a}_0^* \ln R - \overline{a}_0^* \ln R_1 \\ &= R \ln R \left(\frac{\overline{a}_0^*}{R}\right) + R_1 \ln R_1 \left(-\frac{\overline{a}_0^*}{R_1}\right) \\ &= R \ln R p_0 + R_1 \ln R_1 \overline{p}_0, \end{aligned} \tag{58}$$

where we have used (45) and (48).

Equations (51) and (55) are shown below. Denote them in matrix form

$$\begin{pmatrix} R^k & -R_1^k \\ \dfrac{1}{R^k} & -\dfrac{1}{R_1^k} \end{pmatrix} \begin{pmatrix} a_k \\ \overline{a}_k \end{pmatrix} = \frac{1}{k} \begin{pmatrix} R^{k+1} & R_1^{k+1} \\ -\dfrac{1}{R^{k-1}} & -\dfrac{1}{R_1^{k-1}} \end{pmatrix} \begin{pmatrix} p_k \\ \overline{p}_k \end{pmatrix}, \tag{59}$$

and denote from (44) and (47)

$$\begin{pmatrix} a_k \\ \overline{a}_k \end{pmatrix} = \begin{pmatrix} R^k & R^{-k} \\ R_1^k & R_1^{-k} \end{pmatrix} \begin{pmatrix} a_k^* \\ \overline{a}_k^* \end{pmatrix}, \tag{60}$$

where a_k^* and $\overline{a}_k^*$ are true expansion coefficients. Also denote from (45) and (48)

$$\begin{pmatrix} p_k \\ \overline{p}_k \end{pmatrix} = k \begin{pmatrix} R^{k-1} & -R^{-k-1} \\ -R_1^{k-1} & R_1^{-k-1} \end{pmatrix} \begin{pmatrix} a_k^* \\ \overline{a}_k^* \end{pmatrix}. \tag{61}$$

By substituting (60) and (61) into (59), its left-hand side leads to

$$\begin{aligned} \begin{pmatrix} R^k & -R_1^k \\ \dfrac{1}{R^k} & -\dfrac{1}{R_1^k} \end{pmatrix} \begin{pmatrix} a_k \\ \overline{a}_k \end{pmatrix} &= \begin{pmatrix} R^k & -R_1^k \\ \dfrac{1}{R^k} & -\dfrac{1}{R_1^k} \end{pmatrix} \begin{pmatrix} R^k & R^{-k} \\ R_1^k & R_1^{-k} \end{pmatrix} \begin{pmatrix} a_k^* \\ \overline{a}_k^* \end{pmatrix} \\ &= \begin{pmatrix} R^{2k} - R_1^{2k} & 0 \\ 0 & R^{-2k} - R_1^{-2k} \end{pmatrix} \begin{pmatrix} a_k^* \\ \overline{a}_k^* \end{pmatrix}. \end{aligned} \tag{62}$$

The right-hand side of (59) leads to

$$\begin{aligned} &\frac{1}{k} \begin{pmatrix} R^{k+1} & R_1^{k+1} \\ -\dfrac{1}{R^{k-1}} & -\dfrac{1}{R_1^{k-1}} \end{pmatrix} \begin{pmatrix} p_k \\ \overline{p}_k \end{pmatrix} \\ &\quad = \frac{1}{k} \begin{pmatrix} R^{k+1} & R_1^{k+1} \\ -\dfrac{1}{R^{k-1}} & -\dfrac{1}{R_1^{k-1}} \end{pmatrix} k \begin{pmatrix} R^{k-1} & -R^{-k-1} \\ -R_1^{k-1} & R_1^{-k-1} \end{pmatrix} \begin{pmatrix} a_k^* \\ \overline{a}_k^* \end{pmatrix} \\ &\quad = \begin{pmatrix} R^{2k} - R_1^{2k} & 0 \\ 0 & R^{-2k} - R_1^{-2k} \end{pmatrix} \begin{pmatrix} a_k^* \\ \overline{a}_k^* \end{pmatrix}. \end{aligned} \tag{63}$$

The second equality of the right-hand sides of (62) and (63) yield (59). The proof for the validity of (52) and (56) is similar. We write these important results as a proposition.

Proposition 1. *For the concentric circular domains, when $\rho = R + \epsilon \neq 1$, the leading coefficients are exact by the NFM, and the solution errors result only from the truncations of their Fourier expansions.*

4. Error Bounds of the NFM with $\epsilon = \overline{\epsilon} = 0$

The NFM with the field nodes $Q \in \partial S$ (i.e., $\epsilon = \overline{\epsilon} = 0$) located on the domain boundary is the most important application for Chen's publications (see [8–14]). We will provide the errors bounds under the Sobolev norms of this special NFM for circular domains with eccentric circular boundaries without proof. Based on the equivalence of the special NFM and the CTM, we may follow the framework of analysis of Treffez method in [6]. The Sobolev norms for Fourier functions are provided in Kreiss and Oliger [31], Pasciak [32], and Canuto and Quarteroni [33].

Let the domain S be divided into two subdomains S^{ext} and S^{int} with an interface boundary $\Gamma_0 \in S$. We have $S = S^{\text{ext}} \cup S^{\text{int}} \cup \Gamma_0$ and $S^{\text{ext}} \cap S^{\text{int}} = \emptyset$, where $\partial S^{\text{ext}} = \partial S_R \cup \Gamma_0$ and $\partial S^{\text{int}} = \partial S_{R_1} \cup \Gamma_0$. We assume that the true solutions have different regularities

$$u \in H^{p+(1/2)}\left(S^{\text{ext}}\right), \qquad u \in H^{\sigma+(1/2)}\left(S^{\text{int}}\right), \tag{64}$$

where $p \geq 2$ and $\sigma \geq 2$. Then there are different regularities on the boundary

$$\begin{aligned} u &\in H^{p}\left(\partial S^{\text{ext}}\right), \qquad u_\nu \in H^{p-1}\left(\partial S^{\text{ext}}\right), \\ u &\in H^{\sigma}\left(\partial S^{\text{int}}\right), \qquad u_{\overline{\nu}} \in H^{\sigma-1}\left(\partial S^{\text{int}}\right), \end{aligned} \tag{65}$$

where ν and $\overline{\nu}$ are the exterior normal to ∂S^{ext} and ∂S^{int}, respectively. Therefore, the true solutions can be expressed by the Fourier expansions on ∂S_R

$$u(\rho, \theta)\big|_{\partial S_R} = a_0^\circ + \sum_{k=1}^{\infty} \left(a_k^\circ \cos k\theta + b_k^\circ \sin k\theta\right), \tag{66}$$

$$\frac{\partial}{\partial \rho} u(\rho, \theta)\bigg|_{\partial S_R} = p_0^\circ + \sum_{k=1}^{\infty} \left(p_k^\circ \cos k\theta + q_k^\circ \sin k\theta\right), \tag{67}$$

where $a_k^\circ, b_k^\circ, p_k^\circ, q_k^\circ$, are the true boundary coefficients. Similarly, we have

$$\begin{aligned} u\left(\overline{\rho}, \overline{\theta}\right)\big|_{\partial S_{R_1}} &= \overline{a}_0^\circ + \sum_{k=1}^{\infty} \left(\overline{a}_k^\circ \cos k\overline{\theta} + \overline{b}_k^\circ \sin k\overline{\theta}\right), \\ \frac{\partial}{\partial \overline{\nu}} u\left(\overline{\rho}, \overline{\theta}\right)\bigg|_{\partial S_{R_1}} &= \overline{p}_0^\circ + \sum_{k=1}^{\infty} \left(\overline{p}_k^\circ \cos k\overline{\theta} + \overline{q}_k^\circ \sin k\overline{\theta}\right), \end{aligned} \tag{68}$$

where $\overline{a}_k^\circ, \overline{b}_k^\circ, \overline{p}_k^\circ, \overline{q}_k^\circ$, are the true boundary coefficients.

Denote finite terms of the Fourier expansions on ∂S_R in (66) and (67) by

$$\begin{aligned} \overline{u}^M &= \overline{u}^M(\rho, \theta)\big|_{\partial S_R} = a_0^\circ + \sum_{k=1}^{M} \left(a_k^\circ \cos k\theta + b_k^\circ \sin k\theta\right), \\ \overline{u}_\rho^M &= \frac{\partial}{\partial \rho} \overline{u}^M(\rho, \theta)\bigg|_{\partial S_R} = p_0^\circ + \sum_{k=1}^{M} \left(p_k^\circ \cos k\theta + q_k^\circ \sin k\theta\right); \end{aligned} \tag{69}$$

Table 1: The errors and condition numbers by the conservative schemes of the IFM, where $R = 2.5$ and $\delta = u - u_{M-N}$.

R_1	(M, N)	$\|\delta\|_{\infty,\partial S}$	$\|\delta\|_{0,\partial S}$	Cond	Cond_eff
1	**(24, 12)**	**1.52 (−8)**	**1.54 (−8)**	**8.48 (1)**	**2.70 (1)**
0.5	**(24, 12)**	**4.90 (−11)**	**4.70 (−11)**	**2.64 (2)**	**1.30 (2)**
0.1	(24, 8)	6.81 (−12)	6.67 (−12)	1.62 (3)	9.21 (2)
0.1	(24, 7)	6.81 (−12)	6.67 (−12)	1.39 (3)	7.89 (2)
0.1	**(24, 6)**	**6.81 (−12)**	**6.83 (−12)**	**1.16 (3)**	**6.58 (2)**
0.1	(24, 5)	2.40 (−10)	9.34 (−11)	9.34 (2)	5.26 (2)
0.1	(24, 4)	1.58 (−8)	6.10 (−9)	7.05 (2)	3.95 (2)
10^{-2}	(24, 6)	3.69 (−12)	3.62 (−12)	1.97 (4)	1.16 (4)
10^{-2}	(24, 5)	6.69 (−12)	3.62 (−12)	1.57 (4)	9.26 (3)
10^{-2}	**(24, 4)**	**3.69 (−12)**	**3.62 (−12)**	**1.18 (4)**	**6.94 (3)**
10^{-2}	(24, 3)	6.10 (−10)	7.63 (−11)	7.91 (3)	4.63 (3)
10^{-2}	(24, 2)	4.80 (−7)	6.00 (−8)	3.98 (3)	2.31 (3)
10^{-3}	(24, 5)	2.58 (−12)	2.53 (−12)	2.22 (5)	1.32 (5)
10^{-3}	(24, 4)	2.58 (−12)	2.53 (−12)	1.67 (5)	9.93 (4)
10^{-3}	**(24, 3)**	**2.58 (−12)**	**2.53 (−12)**	**1.11 (5)**	**6.62 (4)**
10^{-3}	(24, 2)	3.35 (−9)	1.33 (−10)	5.58 (4)	3.31 (4)
10^{-3}	(24, 1)	3.52 (−5)	1.39 (−6)	1.09 (2)	7.90 (1)
10^{-4}	(24, 5)	1.98 (−12)	1.94 (−12)	2.87 (6)	1.72 (6)
10^{-4}	(24, 4)	1.98 (−12)	1.94 (−12)	2.15 (6)	1.29 (6)
10^{-4}	**(24, 3)**	**1.98 (−12)**	**1.95 (−12)**	**1.44 (6)**	**8.62 (5)**
10^{-4}	(24, 2)	2.57 (−11)	1.97 (−12)	7.20 (5)	4.31 (5)
10^{-4}	(24, 1)	2.70 (−6)	3.38 (−8)	1.40 (2)	1.03 (2)

Table 2: The leading coefficients by the conservative schemes of the IFM, where $R = 2.5$.

R_1	(M, N)	p_0	p_1	$\overline{p}_1$
1	**(24, 12)**	**5.770780163555825 (−1)**	**−5.770780163555787 (−1)**	**7.213475204444549 (−1)**
0.5	**(24, 12)**	**2.806196474263354 (−1)**	**−2.358350543983467 (−1)**	**2.834834114319612 (−1)**
0.1	(24, 8)	1.313991926276972 (−1)	−1.053200366713576 (−1)	1.254266057498590 (−1)
0.1	(24, 7)	1.313991926276972 (−1)	−1.053200366713574 (−1)	1.254266057498546 (−1)
0.1	**(24, 6)**	**1.313991926276971 (−1)**	**−1.053200366713574 (−1)**	**1.254266057498543 (−1)**
0.1	(24, 5)	1.313991926276971 (−1)	−1.053200366713575 (−1)	1.254266057498583 (−1)
0.1	(24, 4)	1.313991926276965 (−1)	−1.053200366713555 (−1)	1.254266057489581 (−1)
10^{-2}	(24, 6)	7.480685008050482 (−2)	−5.984662000415889 (−2)	7.124623469104319 (−2)
10^{-2}	(24, 5)	7.480685008050478 (−2)	−5.984662000415886 (−2)	7.124623469102030 (−2)
10^{-2}	**(24, 4)**	**7.480685008050478 (−2)**	**−5.984662000415891 (−2)**	**7.124623469102347 (−2)**
10^{-2}	(24, 3)	7.480685008050478 (−2)	−5.984662000415888 (−2)	7.124623469085235 (−2)
10^{-2}	(24, 2)	7.480685008030524 (−2)	−5.984662000262044 (−2)	7.124614851570256 (−2)
10^{-3}	(24, 5)	5.228968294193471 (−2)	−4.183175432150129 (−2)	4.979970933244982 (−2)
10^{-3}	(24, 4)	5.228968294193472 (−2)	−4.183175432150129 (−2)	4.979970933242725 (−2)
10^{-3}	**(24, 3)**	**5.228968294193472 (−2)**	**−4.183175432150130 (−2)**	**4.979970933271582 (−2)**
10^{-3}	(24, 2)	5.228968294193472 (−2)	−4.183175432150118 (−2)	4.979970873003448 (−2)
10^{-3}	(24, 1)	5.228968256993316 (−2)	−4.183174605594654 (−2)	
10^{-4}	(24, 5)	4.019180446935835 (−2)	−3.215344363673135 (−2)	3.827790910656165 (−2)
10^{-4}	(24, 4)	4.019180446935836 (−2)	−3.215344363673130 (−2)	3.827790910851808 (−2)
10^{-4}	**(24, 3)**	**4.019180446935834 (−2)**	**−3.215344363673132 (−2)**	**3.827790910677455 (−2)**
10^{-4}	(24, 2)	4.019180446935835 (−2)	−3.215344363673135 (−2)	3.827790909840129 (−2)
10^{-4}	(24, 1)	4.019180446716058 (−2)	−3.215344357372844 (−2)	

Table 3: The errors and condition numbers by the original IFM, where $R = 2.5$ and $\delta = u - u_{M-N}$.

R_1	(M, N)	$\|\delta\|_{\infty,\partial S}$	$\|\delta\|_{0,\partial S}$	Cond	Cond_eff
1	**(24, 12)**	**8.94 (−9)**	**8.87 (−9)**	**1.38 (2)**	**1.85 (1)**
0.5	**(24, 12)**	**4.90 (−11)**	**4.70 (−11)**	**2.67 (2)**	**4.61 (1)**
0.1	**(24, 6)**	**6.81 (−12)**	**6.67 (−12)**	**8.32 (2)**	**5.19 (1)**
10^{-2}	**(24, 3)**	**3.69 (−12)**	**3.63 (−12)**	**5.20 (3)**	**4.54 (1)**
10^{-4}	**(24, 2)**	**1.98 (−12)**	**1.94 (−12)**	**3.88 (5)**	**5.57 (1)**

also denote the circle $\ell_r = \{(\rho, \theta) \mid \rho = r, 0 \le \theta \le 2\pi\}$. For $\partial S_R = \ell_R$, for the solution (66), the Sobolev norms are defined as

$$|u|_{0,\ell_R} = \pi R \left\{ (\mathring{a}_0)^2 + \sum_{k=1}^{\infty} \left[(\mathring{a}_k)^2 + (\mathring{b}_k)^2 \right] \right\}^{1/2},$$

$$|u|_{p,\ell_R} = \pi R \left\{ \sum_{k=1}^{\infty} k^{2p} \left[(\mathring{a}_k)^2 + (\mathring{b}_k)^2 \right] \right\}^{1/2}, \quad p \ge 1, \tag{70}$$

$$\|u\|_{p,\ell_R} = \left\{ \sum_{k=0}^{p} |u|^2_{k,\ell_R} \right\}^{1/2}.$$

We have the following lemma, whose proof can be found in Canuto et al. [33, 34].

Lemma 2. *Let* (64) *be given, for* $\partial S_R = \ell_R$*; there exist the bounds of the remainders of* (69)

$$\begin{gathered} \left\| u - \overline{u}^M \right\|_{q,\partial S_R} \le C \frac{1}{M^{p-q}} |u|_{p,\partial S_R}, \quad 0 \le q \le p, \\ \left\| u_\rho - \overline{u}^M_\rho \right\|_{q,\partial S_R} \le C \frac{1}{M^{p-q-1}} \left| u_\rho \right|_{p-1,\partial S_R}, \quad 0 \le q \le p-1, \end{gathered} \tag{71}$$

where C *is a constant independent of* M.

Also denote the finite terms of the Fourier expansions on ∂S_{R_1} in (68) by

$$\begin{aligned} \overline{u}^N &= \overline{u}^N (\overline{\rho}, \overline{\theta}) \Big|_{\partial S_{R_1}} = \mathring{\overline{a}}_0 + \sum_{k=1}^{N} \left(\mathring{\overline{a}}_k \cos k\overline{\theta} + \mathring{\overline{b}}_k \sin k\overline{\theta} \right), \\ \overline{u}^N_{\overline{\nu}} &= \frac{\partial}{\partial \overline{\nu}} u (\overline{\rho}, \overline{\theta}) \Big|_{\partial S_{R_1}} = \mathring{\overline{p}}_0 + \sum_{k=1}^{N} \left(\mathring{\overline{p}}_k \cos k\overline{\theta} + \mathring{\overline{q}}_k \sin k\overline{\theta} \right). \end{aligned} \tag{72}$$

We can prove the following lemma similarly.

Lemma 3. *Let* (64) *be given, for* $\partial S_{R_1} = \ell_{R_1}$*; there exist the bounds of the remainders of* (72)

$$\begin{gathered} \left\| u - \overline{u}^N \right\|_{q,\partial S_{R_1}} \le C \frac{1}{N^{p-q}} |u|_{p,\partial S_{R_1}}, \quad 0 \le q \le p, \\ \left\| u_{\overline{\nu}} - \overline{u}^N_{\overline{\nu}} \right\|_{q,\partial S_{R_1}} \le C \frac{1}{N^{p-q-1}} \left| u_{\overline{\nu}} \right|_{p-1,\partial S_{R_1}}, \quad 0 \le q \le p-1, \end{gathered} \tag{73}$$

where C *is a constant independent of* N.

We have the following theorem.

Theorem 4. *Let* (64) *and* $R \ne 1$ *hold. For the solution* $u_{N,M}$ *from the TM in* (36)*, there exists the error bound*

$$\left\| u - u_{N,M} \right\|_{0,\partial S} \le C \left\{ \frac{1}{M^p} |u|_{p,\partial S_R} + \frac{1}{N^\sigma} |u|_{\sigma,\partial S_{R_1}} \right\}, \tag{74}$$

where C *is a constant independent of* N *and* M.

Next, we study the errors of the interpolant solutions from (16) of the NFM with $\epsilon = \overline{\epsilon} = 0$,

$$\begin{gathered} \mathscr{L}_{\text{ext}} \left(R, i\Delta\theta; \overline{\rho}_i, \overline{\theta}_i \right) = 0, \quad i = 0, 1, \ldots, 2M, \\ \mathscr{L}_{\text{int}} \left(\rho_i, \theta_i; R_1, i\Delta\overline{\theta} \right) = 0, \quad i = 0, 1, \ldots, 2N, \end{gathered} \tag{75}$$

where the uniform nodes $\Delta\theta = 2\pi/(2M+1)$ and $\Delta\overline{\theta} = 2\pi/(2N+1)$. Equation (75) is equivalent to

$$\begin{gathered} \hat{u}_{M-N} \left(R, i\Delta\theta; \overline{\rho}_i, \overline{\theta}_i \right) = u_0 \left(\theta_i \right), \quad i = 0, 1, \ldots, 2M, \\ \hat{u}_{M-N} \left(\rho_i, \theta_i; R_1, i\Delta\overline{\theta} \right) = \overline{u}_0 \left(\overline{\theta}_i \right), \quad i = 0, 1, \ldots, 2N, \end{gathered} \tag{76}$$

where u_0 and $\overline{u}_0$ are given in (2) and (4). We have the following theorem.

Theorem 5. *Let* (64) *and* $R \ne 1$ *hold. For the NFM with* $\epsilon = \overline{\epsilon} = 0$ *and the uniform nodes, the interpolant solutions* $\hat{u}_{M-N}$ *from* (76) *have the same error bound of* (74)

$$\left\| u - \hat{u}_{N,M} \right\|_{0,\partial S} \le C \left\{ \frac{1}{M^p} |u|_{p,\partial S_R} + \frac{1}{N^\sigma} |u|_{\sigma,\partial S_{R_1}} \right\}, \tag{77}$$

where C *is a constant independent of* N *and* M.

5. Numerical Experiments

5.1. IFM and Its Conservative Schemes. In this paper, we choose the NFM with $\epsilon = \overline{\epsilon} = 0$, which is equivalent to the IFM, and its conservative schemes of [15]. For (1) with symmetry, the explicit interior solution (24) is simplified as

$$\begin{aligned} u^C_{M-N} (\rho, \theta) = a_0 - R \left(\ln \frac{R}{R_1} \right) p_0 + \frac{R}{2} \sum_{k=1}^{M} \frac{1}{k} \left(\frac{\rho}{R} \right)^k p_k \cos k\theta \\ + \frac{R_1}{2} \sum_{k=1}^{N} \frac{1}{k} \left(\frac{R_1}{\overline{\rho}} \right)^k \overline{p}_k \cos k\overline{\theta}, \quad (\rho, \theta) \in S. \end{aligned} \tag{78}$$

In [17], when $u \in H^2(S)$, we may choose the field nodes to be located on the solution boundary for (78): $(\rho, \theta) \in \partial S_R$ and

Table 4: The leading coefficients by the original IFM, where $R = 2.5$.

R_1	(M,N)	p_0	p_1	$\overline{p}_0$	$\overline{p}_1$
1	**(24, 12)**	**5.770780163555844 (−1)**	**−5.770780163555829 (−1)**	**−1.442695040888961 (0)**	**7.213475204444735 (−1)**
0.5	**(24, 12)**	**2.806196474263354 (−1)**	**−2.358350543983468 (−1)**	**−1.403098237131676 (0)**	**2.834834114319622 (−1)**
0.1	**(24, 6)**	**1.313991926276971 (−1)**	**−1.053200366713573 (−1)**	**−3.284979815692429 (0)**	**1.254266057498605 (−1)**
10^{-2}	**(24, 3)**	**7.480685008050476 (−2)**	**−5.984662000415886 (−2)**	**−1.870171252012620 (1)**	**7.124623469101694 (−2)**
10^{-4}	**(24, 2)**	**4.019180446935834 (−2)**	**−3.215344363673133 (−2)**	**−1.004795111733959 (3)**	**3.827790910389037 (−2)**

Table 5: The errors and condition numbers by the simple particular solutions of the CTM, where $R = 2.5$ and $\delta = u - u_{M-N}$.

R_1	(M,N)	$\\|\delta\\|_{\infty,\partial S}$	$\\|\delta\\|_{0,\partial S}$	Cond	Cond_eff
1	**(24, 12)**	**1.58 (−9)**	**1.57 (−9)**	**7.62**	**3.03**
0.5	**(24, 12)**	**8.68 (−12)**	**8.32 (−12)**	**4.59**	**3.93**
0.1	**(24, 5)**	**1.21 (−12)**	**1.19 (−12)**	**1.31 (1)**	**1.21 (1)**
10^{-2}	**(24, 3)**	**6.54 (−13)**	**6.41 (−13)**	**5.17 (1)**	**4.39 (1)**
10^{-3}	**(24, 2)**	**4.57 (−13)**	**4.48 (−13)**	**1.93 (2)**	**1.57 (2)**
10^{-4}	**(24, 1)**	**3.51 (−13)**	**3.44 (−13)**	**6.44 (2)**	**5.11 (2)**

$(\overline{\rho},\overline{\theta}) \in \partial S_{R_1}$. Then we obtain two boundary equations of the conservative schemes of the IFM from (78), (2), and (4)

$$\begin{aligned}
\mathscr{L}^{C}_{\text{ext}}\left(R,\theta;\overline{\rho},\overline{\theta}\right) &= -R\left(\ln\frac{R}{R_1}\right)p_0 + \frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}p_k\cos k\theta \\
&\quad + \frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\overline{\rho}}\right)^k\overline{p}_k\cos k\overline{\theta} \\
&= 0, \quad (r,\theta)\in\partial S_R, \\
\mathscr{L}^{C}_{\text{int}}\left(\rho,\theta;R_1,\overline{\theta}\right) &:= a_0 - \overline{a}_0 - R\left(\ln\frac{R}{R_1}\right)p_0 \\
&\quad + \frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho}{R}\right)^k p_k\cos k\theta \\
&\quad + \frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\overline{p}_k\cos k\overline{\theta} \\
&= 0, \quad \left(\overline{r},\overline{\theta}\right)\in\partial S_{R_1}.
\end{aligned} \tag{79}$$

The coefficients $p_0, p_k, \overline{p}_k$ are unknowns, and the total number of unknowns is $M + N + 1$. Based on [15], to bypass the pseudosingularity, we still choose $M + N + 2$ equations from (79)

$$\begin{aligned}
w_i\mathscr{L}^{C}_{\text{ext}}\left(R, i\Delta\theta;\overline{\rho}_i,\overline{\theta}_i\right) &= 0, \quad i = 0,1,\ldots,M, \\
w_i\mathscr{L}^{C}_{\text{int}}\left(\rho_i,\theta_i;R_1,i\Delta\overline{\theta}\right) &= 0, \quad i = 0,1,\ldots,N,
\end{aligned} \tag{80}$$

where $\epsilon \geq 0$, $0 \leq \overline{\epsilon} < R_1$, $\Delta\theta = 2\pi/(2M+1)$ and $\Delta\overline{\theta} = 2\pi/(2N+1)$. The weights $w_0 = 1$ and $w_i = \sqrt{2}$ are defined for $i \geq 1$, based on the stability analysis in [17].

The overdetermined system of (80) is denoted by the linear algebraic equations

$$\mathbf{F}\mathbf{x} = \mathbf{b}, \tag{81}$$

where $\mathbf{F} \in R^{m\times n}$ with $n = M + N + 1$ and $m = M + N + 2$. The traditional condition number and the effective condition number in [35] are defined by

$$\text{Cond} = \frac{\sigma_{\max}}{\sigma_{\min}}, \qquad \text{Cond\_eff} = \frac{\|\mathbf{b}\|}{\sigma_{\min}\|\mathbf{x}\|}, \tag{82}$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the maximal and the minimal singular values of the matrix $\mathbf{F}$ in (81), respectively.

Next, we use the original IFM (i.e., the original NFM with $\epsilon = \overline{\epsilon} = 0$). The particular solutions (78) are replaced by

$$\begin{aligned}
u_{M-N}(\rho,\theta) &= a_0 - R\ln Rp_0 - R_1\ln\overline{\rho}\,\overline{p}_0 \\
&\quad + \frac{R}{2}\sum_{k=1}^{M}\frac{1}{k}\left(\frac{\rho}{R}\right)^k p_k\cos k\theta \\
&\quad + \frac{R_1}{2}\sum_{k=1}^{N}\frac{1}{k}\left(\frac{R_1}{\overline{\rho}}\right)^k\overline{p}_k\cos k\overline{\theta}, \quad (r,\theta)\in S.
\end{aligned} \tag{83}$$

In (83), both p_0, $\overline{p}_0$ are also unknown variables, and the total number of unknowns is now $M + N + 2$. Then $m = n = M + N + 2$ in (81).

Consider the model problem with $R = 2.5$ and $R_1 = 1$ and then shrink the interior hole S_{R_1} by decreasing radius R_1 from 1 down to 10^{-4}. This reflects that Laplace's equation may occur in an actually punctured disk, where there may be a very small hole but not as a solitary point. For the conservative schemes of the IFM, the errors, condition numbers, and the leading coefficients are listed in Tables 1 and 2, where $\delta = u - u_{M-N}$. For $R_1 = 0.1, 0.01, 0.001, 0.0001$, the optimal results are marked in bold. We also note that when R_1 decreases, the errors decrease and the condition numbers increase. Table 2 lists the leading coefficients, p_0, p_1, and $\overline{p}_1$. All tables are computed by MATLAB with double precision.

As for the computations by the original IFM, the errors, condition numbers, and the leading coefficients are listed in Tables 3 and 4, where only the optimal results are listed. Comparing Table 3 with Table 1, the differences in terms of errors and condition number are insignificant, but the effective condition numbers are much smaller by the original IFM. Strictly speaking, the conservative schemes satisfy the flux conservative law exactly, but the original IFM does not.

Table 6: The leading coefficients by the CTM, where $R = 2.5$.

R_1	(M,N)	a_0	a_1	$\overline{a}_0$	$\overline{a}_1$
1	**(24, 12)**	**−3.219280948873607 (−1)**	**−7.213475204444795 (−1)**	**1.442695040888962 (0)**	**3.606737602222373 (−1)**
0.5	**(24, 12)**	**3.571770447036389 (−1)**	**−2.947938179979336 (−1)**	**7.015491185658381 (−1)**	**7.087085285799036 (−2)**
0.1	**(24, 5)**	**6.990003440487363 (−1)**	**−1.316500458391969 (−1)**	**3.284979815692430 (−1)**	**6.271330287492891 (−3)**
10^{-2}	**(24, 3)**	**8.286379414763349 (−1)**	**−7.480827500519865 (−2)**	**1.870171252012619 (−1)**	**3.562311734550295 (−4)**
10^{-3}	**(24, 2)**	**8.802186203691674 (−1)**	**−5.228969290187686 (−2)**	**1.307242073548369 (−1)**	**2.489985466613060 (−5)**
10^{-4}	**(24, 1)**	**9.079315551685712 (−1)**	**−4.019180454591526 (−2)**	**1.004795111733959 (−1)**	**1.913895455111127 (−6)**

Table 7: The errors and condition numbers by the BIE, where $R = 2.5$ and $\delta = u - u_{M-N}$.

R_1	(M,N)	$\|\delta\|_{\infty,\partial S}$	$\|\delta\|_{0,\partial S}$	Cond	Cond_eff
1	**(24, 12)**	**8.94 (−9)**	**8.87 (−9)**	**1.38 (2)**	**2.66 (1)**
0.5	**(24, 12)**	**4.90 (−11)**	**4.70 (−11)**	**2.67 (2)**	**6.41 (1)**
0.1	**(24, 5)**	**7.27 (−12)**	**7.00 (−12)**	**7.36 (2)**	**8.71 (1)**
10^{-2}	**(24, 3)**	**3.69 (−12)**	**3.63 (−12)**	**5.20 (3)**	**1.12 (2)**
10^{-3}	**(24, 2)**	**2.58 (−12)**	**2.53 (−12)**	**3.88 (4)**	**1.22 (2)**
10^{-4}	**(24, 2)**	**1.98 (−12)**	**1.94 (−12)**	**3.88 (5)**	**1.59 (2)**

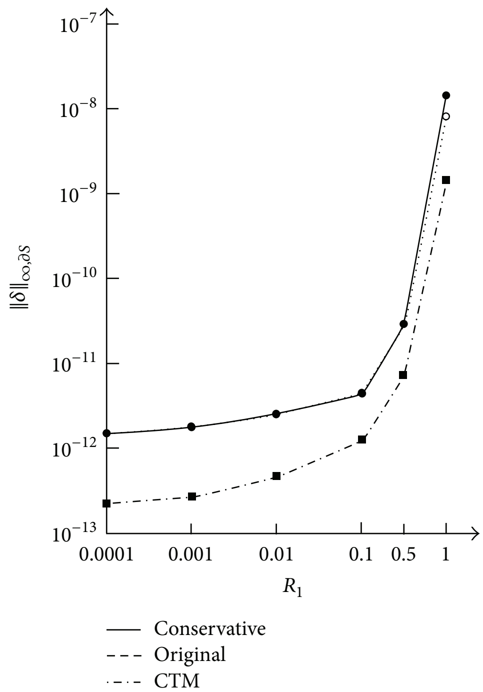

Figure 2: The curves of $\|\delta\|_{\infty,\partial S}$ via R_1 by the conservative schemes, the original IFM, and the CTM.

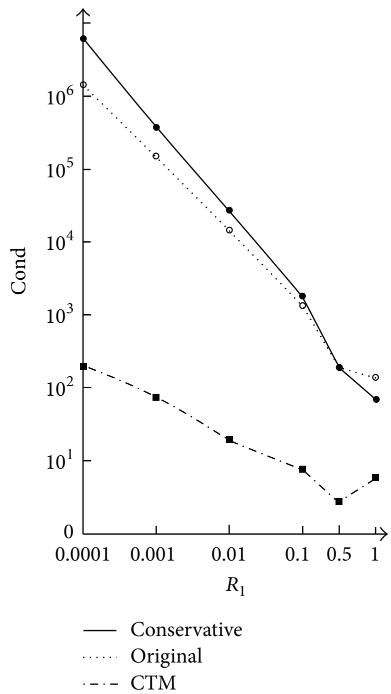

Figure 3: The curves of Cond via R_1 by the conservative schemes, the original IFM, and the CTM.

5.2. The CTM and the BIE. By means of symmetry, we choose the simple particular solutions in the CTM

$$u_{M-N}\left(\rho,\theta;\overline{\rho},\overline{\theta}\right) = a_0 + \sum_{i=1}^{M}\left(\frac{\rho}{R}\right)^i a_i \cos i\theta + \overline{a}_0 \ln \overline{\rho} + \sum_{i=1}^{N}\left(\frac{R_1}{\overline{\rho}}\right)^i \overline{a}_i \cos i\overline{\theta}, \quad \rho \le R,\ \overline{\rho} \ge R_1, \tag{84}$$

where a_i and $\overline{a}_i$ are the true coefficients and (ρ,θ) and $(\overline{\rho},\overline{\theta})$ are the polar coordinates with the origins $(0,0)$ and $(-1,0)$, respectively. We have also carried out the computation by CTM and BIE and have given their results in Tables 5, 6, 7, and 8. Comparing Table 7 of the BIE with Table 3 of the original IFM, the errors and the condition numbers are the same, but the effective condition numbers are slightly different. Then we conclude that the performance of the original IFM and BIE is the same. For comparisons of different methods, we draw their curves of errors and condition numbers in Figures 2 and 3, and it is clear that CTM is the best.

Table 8: The leading coefficients by the BIE, where $R = 2.5$.

R_1	(M,N)	$p_0^\star$	$p_1^\star$	$\overline{p}_0^\star$	$\overline{p}_1^\star$
1	**(24, 12)**	**1.405353491806680 (−1)**	**−5.770780163555832 (−1)**	**−1.442695040888961 (0)**	**7.213475204444746 (−1)**
0.5	**(24, 12)**	**−1.559230197485811 (−1)**	**−2.358350543983469 (−1)**	**−1.403098237131677 (0)**	**2.834834114319625 (−1)**
0.1	**(24, 5)**	**−3.051434745472195 (−1)**	**−1.053200366713573 (−1)**	**−3.284979815692429 (0)**	**1.254266057498587 (−1)**
10^{-2}	**(24, 3)**	**−3.617358170944118 (−1)**	**−5.984662000415891 (−2)**	**−1.870171252012620 (1)**	**7.124623469100554 (−2)**
10^{-3}	**(24, 2)**	**−3.842529842329820 (−1)**	**−4.183175432150123 (−2)**	**−1.307242073548368 (2)**	**4.979970933294560 (−2)**
10^{-4}	**(24, 2)**	**−3.963508627055583 (−1)**	**−3.215344363673110 (−2)**	**−1.004795111733959 (3)**	**3.827790910431746 (−2)**

6. Concluding Remarks

To close this paper, let us make a few concluding remarks.

(1) By following [17] for the NFM, we propose the interior field method (IFM). Since all boundary methods can be applied to any annular domains, they may be used for circular domains with circular holes; in this paper, we employ the first kind boundary integral equation (BIE) in [30] and the collocation Trefftz method (CTM) in [6]. The relations of expansion coefficients among NFM, IFM, BIE, and CTM are found. The intrinsic relations among them are discovered, to show that the IFM and the BIE are special cases of CTM. Section 2 yields an in-depth overview of four methods for circular domains with circular holes.

(2) For the NFM, some stability analysis in [17] was made for concentric circular boundaries. The error analysis of the NFM is challenging. Sections 3 and 4 are devoted to the error analysis of the NFM. In Section 3, a preliminary analysis is provided. In Section 4, for the special NFM with $\epsilon = \overline{\epsilon} = 0$, the error bounds are provided without proof. The optimal convergence rates can be achieved. The error analysis is important and valid in wide applications, because the special NFM offers the best numerical performance in convergence and stability; see [17].

(3) Numerical experiments are carried out for a challenging problem of the actually punctured disks. We choose NFM, IFM, CTM, and BIE and their conservative schemes. Numerical results are reported from $R_1 = 1$ down to $R_1 = 10^{-4}$. Note that the popular methods, such as the finite element method (FEM), the finite difference method (FDM), and the boundary element method (BEM), may fail to handle this problem. The actually punctured disks may be regarded as a kind of singularity problems, and the local mesh refinements and other innovations of FEM, FDM, and BEM are indispensable. However, their algorithms are complicated and troublesome; see [5]. Consequently, the computation of this paper enriches the boundary methods [6].

(4) Numerical comparisons of different methods are *imperative* in real application. Though their numerical performances are basically the same, the CTM is best in accuracy, stability, and simplicity of algorithms. Moreover, the CTM can always circumvent the degenerate scale problems encountered in NFM, IFM, and BIE. More importantly, the CTM can be applied to any shape domains and singularity problems (see [5, 6]). In summary, three goals motivated have been fulfilled.

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References

[1] M. R. Barone and D. A. Caulk, "Special boundary integral equations for approximate solution of Laplace's equation in two-dimensional regions with circular holes," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 34, no. 3, pp. 265–286, 1981.

[2] M. R. Barone and D. A. Caulk, "Special boundary integral equations for approximate solution of potential problems in three-dimensional regions with slender cavities of circular cross-section," *IMA Journal of Applied Mathematics*, vol. 35, no. 3, pp. 311–325, 1985.

[3] D. A. Caulk, "Analysis of steady heat conduction in regions with circular holes by a special boundary-integral method," *IMA Journal of Applied Mathematics*, vol. 30, pp. 231–246, 1983.

[4] M. D. Bird and C. R. Steele, "A solution procedure for Laplace's equation on multiply connected circular domains," *Journal of Applied Mechanics*, vol. 59, pp. 398–3404, 1992.

[5] Z. C. Li, *Combined Methods for Elliptic Equations with Singularities, Interfaces and Infinities*, Kluwer Academic Publishers, Boston, Mass, USA, 1998.

[6] Z.-C. Li, T.-T. Lu, H.-Y. Hu, and A. H.-D. Cheng, *Trefftz and Collocation Methods*, WIT Press, Boston, Mass, USA, 2008.

[7] W. T. Ang and I. Kang, "A complex variable boundary element method for elliptic partial differential equations in a multiple-connected region," *International Journal of Computer Mathematics*, vol. 75, no. 4, pp. 515–525, 2000.

[8] J. T. Chen, S. R. Kuo, and J. H. Lin, "Analytical study and numerical experiments for degenerate scale problems in the boundary element method for two-dimensional elasticity," *International Journal for Numerical Methods in Engineering*, vol. 54, no. 12, pp. 1669–1681, 2002.

[9] J. T. Chen, C. F. Lee, J. L. Chen, and J. H. Lin, "An alternative method for degenerate scale problem in boundary element methods for two-dimensional Laplace equation," *Engineering Analysis with Boundary Elements*, vol. 26, pp. 559–569, 2002.

[10] J.-T. Chen and W.-C. Shen, "Degenerate scale for multiply connected Laplace problems," *Mechanics Research Communications*, vol. 34, no. 1, pp. 69–77, 2007.

[11] J.-T. Chen and W.-C. Shen, "Null-field approach for Laplace problems with circular boundaries using degenerate kernels,"

Numerical Methods for Partial Differential Equations, vol. 25, no. 1, pp. 63–86, 2009.

[12] J. T. Chen, H. C. Shieh, Y. T. Lee, and J. W. Lee, "Bipolar coordinates, image method and the method of fundamental solutions for Green's functions of Laplace problems containing circular boundaries," *Engineering Analysis with Boundary Elements*, vol. 35, no. 2, pp. 236–243, 2011.

[13] J. T. Chen, C. S. Wu, and K. H. Chen, "A study of free terms for plate problems in the dual boundary integral equations," *Engineering Analysis with Boundary Elements*, vol. 29, pp. 435–446, 2005.

[14] S. R. Kuo, J. T. Chen, and S. K. Kao, "Linkage between the unit logarithmic capacity in the theory of complex variables and the degenerate scale in the BEM/BIEMs," *Applied Mathematics Letters*, vol. 29, pp. 929–938, 2013.

[15] M.-G. Lee, Z.-C. Li, H.-T. Huang, and J. Y. Chiang, "Conservative schemes and degenerate scale problems in the null-field method for Dirichlet problems of Laplace's equation in circular domains with circular holes," *Engineering Analysis with Boundary Elements*, vol. 37, no. 1, pp. 95–106, 2013.

[16] M.-G. Lee, Z.-C. Li, L. P. Zhang, H.-T. Huang, and J. Y. Chiang, "Algorithm singularity of the null-field method for Dirichlet problems of Laplace's equation in annular and circulardomains," submitted to. *Engineering Analysis with Boundary Elements*.

[17] Z.-C. Li, H.-T. Huang, C.-P. Liaw, and M.-G. Lee, "The null-field method of Dirichlet problems of Laplace's equation on circular domains with circular holes," *Engineering Analysis with Boundary Elements*, vol. 36, no. 3, pp. 477–491, 2012.

[18] V. D. Kupradze and M. A. Aleksidze, "The method of functional equations for the approximate solution of certain boundary-value problems," *USSR Computational Mathematics and Mathematical Physics*, vol. 4, pp. 82–126, 1964.

[19] G. Fairweather and A. Karageorghis, "The method of fundamental solutions for elliptic boundary value problems," *Advances in Computational Mathematics*, vol. 9, no. 1-2, pp. 69–95, 1998.

[20] C. S. Chen, Y. C. Hon, and R. A. Schaback, *Scientific computing with radial basis functions [Ph.D. thesis]*, Department of Mathematics, University of Southern Mississippi, Hattiesburg, Miss, USA, 2005.

[21] Z.-C. Li, H.-T. Huang, M.-G. Lee, and J. Y. Chiang, "Error analysis of the method of fundamental solutions for linear elastostatics," *Journal of Computational and Applied Mathematics*, vol. 251, pp. 133–153, 2013.

[22] Z.-C. Li, J. Huang, and H.-T. Huang, "Stability analysis of method of fundamental solutions for mixed boundary value problems of Laplace's equation," *Computing*, vol. 88, no. 1-2, pp. 1–29, 2010.

[23] T. Wriedt, "Review of the null-field method with discrete source," *Journal of Quantitative Spectroscopy and Radiative Transfer*, vol. 106, pp. 534–545, 2007.

[24] A. Doicu and T. Wried, "Calculation of the T matrix in the null-field method with discret sources," *The Journal of the Optical Society of America*, vol. 16, pp. 2539–2544, 1999.

[25] J. Hellmers, V. Schmidt, and T. Wriedt, "Improving the numerical instability of T-matrix light scattering calculations for extreme articles shapes using the nullfield method with discrete sources," *Journal of Quantitative Spectroscopy and Radiative Transfer*, vol. 112, pp. 1679–1686, 2011.

[26] D. Palaniappan, "Electrostatics of two intersecting conducting cylinders," *Mathematical and Computer Modelling*, vol. 36, no. 7-8, pp. 821–830, 2002.

[27] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover Publications, New York, NY, USA, 1964.

[28] K. E. Atkinson, *A Survey of Numerical Methods for the Solutions of Fredholm Integral Equations of the Second Kind*, Cambridge University Press, 1997.

[29] G. C. Hsiao and W. L. Wendland, *Boundary Integral Equations*, Springer, Berlin, Germany, 2008.

[30] C. B. Liem, T. Lü, and T. M. Shih, *The Splitting Extrapolation Method*, World Scientific, Singapore, 1995.

[31] H.-O. Kreiss and J. Oliger, "Stability of the Fourier method," *SIAM Journal on Numerical Analysis*, vol. 16, no. 3, pp. 421–433, 1979.

[32] J. E. Pasciak, "Spectral and pseudospectral methods for advection equations," *Mathematics of Computation*, vol. 35, no. 152, pp. 1081–1092, 1980.

[33] C. Canuto and A. Quarteroni, "Approximation results for orthogonal polynomials in Sobolev spaces," *Mathematics of Computation*, vol. 38, no. 157, pp. 67–86, 1982.

[34] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods, Fundamentals in Single Domains*, Springer, New York, NY, USA, 2006.

[35] Z.-C. Li, H.-T. Huang, Y. Wei, and A. H.-D. Cheng, *Effective Condition Number for Numerical Partial Differential Equations*, Science Press, Beijing, China, 2013.

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Research Article

Oscillation Criteria of First Order Neutral Delay Differential Equations with Variable Coefficients

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Some new oscillation criteria are given for first order neutral delay differential equations with variable coefficients. Our results generalize and extend some of the well-known results in the literature. Some examples are considered to illustrate the main results.

1. Introduction

In recent years, oscillation of neutral delay differential equations (or NDDEs for short) has received great attention and has been studied extensively. It is a relatively new field with interesting applications from the real world. In fact, NDDEs appear in modeling of the problems as transformation of information, population dynamics, the networks containing lossless transmission lines, and in the theory of automatic control (see, e.g., [1–4] and references cited therein).

Consider the first order NDDE of the form

$$[r(t)(x(t)+p(t)x(t-\tau))]'+q(t)x(t-\sigma)=0, \quad t\geq t_0, \tag{1}$$

where

$$p\in C[[t_0,\infty),\mathbb{R}], \quad r,q\in C[[t_0,\infty),\mathbb{R}^+], \quad \tau,\sigma\in\mathbb{R}^+. \tag{2}$$

Let $m=\max\{\tau,\sigma\}$. By a solution of (1), we mean a function $x\in C[[t_1-m,\infty),\mathbb{R}]$ for some $t_1\geq t_0$ such that $x(t)+p(t)x(t-\tau)$ is continuously differentiable, and (1) is satisfied identically for $t_1\geq t_0$. Such a solution of (1) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative. The NDDE (1) is called oscillatory if all its solutions are oscillatory; otherwise, it is called nonoscillatory.

Recently, some investigations such as [5–7] have appeared which are concerned with the oscillation as well as the nonoscillation behaviour of NDDE (1). In fact, Zahariev and Baĭnov [8] is the first work dealing with oscillation of neutral equations. A systematic development of oscillation theory of NDDEs was initiated by Ladas and Sficas [9]. For the oscillation of (1) when $r(t)=1$ and $p(t)$ and $q(t)$ are constants, we refer the readers to the articles by Ladas and Schults [10], Sficas and Stavroulakis [11], Grammatikopoulos et al. [12], Zhang [13], and Gopalsamy and Zhang [14]. For the oscillation of (1) when $r(t)=1$ and $p(t)$ is equal to a constant, we refer the readers to the papers by Grammatikopoulos et al. [15], Zhang [13], Gopalsamy and Zhang [14], and Saker and Elabbasy [16] and the references cited therein. Grammatikopoulos et al. [6], Ladas and Schults [10], Chuanxi and Ladas [17, 18], Kubiaczyk and Saker [19], and Karpuz and Ocalan [20] considered the NDDE (1) when $r(t)=1$ and established some new oscillation results sorted by the value of function $p(t)$. For further oscillation results on the oscillatory behaviour of solutions of (1), we refer the readers to the monographs by Győri and Ladas [21] and Erbe et al. [22] as well as the papers of Yu et al. [23], Choi and Koo [24], Ocalan [25], and Candan and Dahiya [26].

The purpose of this work is to find some sufficient conditions for the oscillation of all solutions of the first order NDDE (1).

Remark 1. (i) When we write a functional inequality we assume that it holds for all sufficiently large t.

(ii) Without loss of generality, we will deal only with the positive solutions of (1).

In the proof of our main results, we need the following well-known lemmas which can be found in Chuanxi and Ladas [17], Győri and Ladas [21], and Kulenović et al. [27].

Lemma 2. *Assume that ρ is a positive constant. Let $h \in C[[t_0, \infty), \mathbb{R}^+]$, and suppose that*

$$\liminf_{t\to\infty} \int_{t-\rho}^{t} h(s)\,ds > \frac{1}{e}. \tag{3}$$

Then

(i) *the delay differential inequality*

$$x'(t) + h(t)\,x(t-\rho) \le 0, \quad t \ge t_0, \tag{4}$$

has no eventually positive solution;

(ii) *the delay differential inequality*

$$x'(t) + h(t)\,x(t-\rho) \ge 0, \quad t \ge t_0, \tag{5}$$

has no eventually negative solution;

(iii) *the advanced differential inequality*

$$x'(t) - h(t)\,x(t+\rho) \le 0, \quad t \ge t_0, \tag{6}$$

has no eventually negative solution;

(iv) *the advanced differential inequality*

$$x'(t) - h(t)\,x(t+\rho) \ge 0, \quad t \ge t_0, \tag{7}$$

has no eventually positive solution.

Lemma 3. *Consider the NDDE*

$$\left(x(t) + p(t)\,x(t-\tau)\right)' + q(t)\,x(t-\sigma) = 0, \quad t \ge t_0, \tag{8}$$

where p, q, τ, and σ are as in (2). Assume that

$$\int_{t_0}^{\infty} q(s)\,ds = \infty. \tag{9}$$

Let $x(t)$ be an eventually positive solution of equation and set

$$z(t) = x(t) + p(t)\,x(t-\tau). \tag{10}$$

Then the following statements are true:

(i) *$z(t)$ is an eventually decreasing function;*

(ii) *if $p(t) \le -1$ then $z(t) < 0$;*

(iii) *if $-1 \le p(t) \le 0$ then $z(t) > 0$ and $\lim_{t\to\infty} z(t) = 0$.*

Lemma 4. *Assume that (9) holds and let $x(t)$ be an eventually positive solution of NDDE*

$$\left[(x(t) + px(t-\tau))\right]' + q(t)\,x(t-\sigma) = 0, \quad t \ge t_0, \tag{11}$$

where $p \ne 1$, $q \in C[[t_0,\infty), \mathbb{R}^+]$, and $\tau, \sigma \in \mathbb{R}^+$.

Set

$$z(t) = x(t) + px(t-\tau). \tag{12}$$

Then

(a) *$z(t)$ is a decreasing function and either*

$$\lim_{t\to\infty} z(t) = -\infty \tag{13}$$

or

$$\lim_{t\to\infty} z(t) = 0. \tag{14}$$

(b) *The following statements are equivalent:*

(i) *(13) holds;*
(ii) *$p < -1$;*
(iii) *$\lim_{t\to\infty} x(t) = \infty$;*
(iv) *$w(t) > 0$, $w'(t) > 0$.*

(c) *The following statements are equivalent:*

(i) *(14) holds;*
(ii) *$p > -1$;*
(iii) *$\lim_{t\to\infty} x(t) = 0$;*
(iv) *$w(t) > 0$, $w'(t) < 0$.*

2. Main Results

In this section we give some new sufficient conditions for all solutions of NDDE (1) to be oscillatory.

Theorem 5. *Assume that (2) and (9) hold, $p(t) \le -1$, $\tau > \sigma$, and*

$$\liminf_{t\to\infty} \int_{t+\sigma}^{t+\tau} \left[\frac{q(s-\tau)}{-r(s-\sigma)\,p(s-\sigma)}\right] ds > \frac{1}{e}. \tag{15}$$

Then every solution of NDDE (1) is oscillatory.

Proof. Assume, for the sake of a contradiction, that (1) has an eventually positive solution $x(t) > 0$ for all $t \ge t_0 > 0$. Set

$$z(t) = x(t) + p(t)\,x(t-\tau). \tag{16}$$

Then by Lemma 3 we have

$$z(t) < 0. \tag{17}$$

Observe that

$$z(t) > p(t)\, x(t-\tau). \tag{18}$$

From which we find eventually

$$\frac{1}{p(t+\tau-\sigma)} q(t) z(t+\tau-\sigma) < q(t) x(t-\sigma) = -(r(t)\, z(t))', \tag{19}$$

and hence

$$z'(t) + \frac{r'(t)}{r(t)} z(t) + \frac{q(t)}{r(t)\, p(t+\tau-\sigma)} z(t+\tau-\sigma) < 0. \tag{20}$$

Set

$$z(t) = e^{-\int_{t_0}^{t} (r'(s)/r(s))ds} y(t). \tag{21}$$

This implies that $y(t) < 0$.

Substituting in (20) yields for all $t \geq t_0$

$$y'(t) + \frac{q(t)}{r(t+\tau-\sigma)\, p(t+\tau-\sigma)} y(t+\tau-\sigma) < 0, \tag{22}$$

or

$$y'(t) - \left[\frac{q(t)}{-r(t+\tau-\sigma)\, p(t+\tau-\sigma)}\right] y(t+(\tau-\sigma)) < 0. \tag{23}$$

In view of (15) and Lemma 2(iii), it is impossible for (23) to have an eventually negative solution. This contradicts the fact that $y(t) < 0$ and the proof is complete. □

Example 6. Consider NDDE

$$\left[\frac{e^{t+1}}{t+1}\left(x(t) - \frac{t+1}{t} x(t-2)\right)\right]' + e^{t+2} x(t-1) = 0, \quad t > 0. \tag{24}$$

Here we have

$$\begin{aligned} p(t) &= -\frac{t+1}{t} \leq -1, \qquad q(t) = e^{t+2}, \\ r(t) &= \frac{e^{t+1}}{t+1}, \qquad \tau = 2, \qquad \sigma = 1. \end{aligned} \tag{25}$$

Then all the hypotheses of Theorem 5 are satisfied where

$$\begin{aligned} &\liminf_{t\to\infty} \int_{t+\sigma}^{t+\tau} \frac{q(s-\tau)}{-r(s-\sigma)\, p(s-\sigma)} ds \\ &= \liminf_{t\to\infty} \int_{t+1}^{t+2} (s-1)\, ds = \liminf_{t\to\infty} \left(t + \frac{9}{2}\right) = \infty > \frac{1}{e}. \end{aligned} \tag{26}$$

Hence every solution of (24) is oscillatory.

Remark 7. Theorem 5 is an extent of [17, Theorem 2], [15, Theorem 7], and [21, Theorem 6.4.3].

Theorem 8. *Assume that* (2) *and* (9) *hold,* $-1 \leq p(t) \leq 0$, *and*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \frac{q(s)}{r(s-\sigma)} ds > \frac{1}{e}. \tag{27}$$

Then every solution of NDDE (1) *oscillates.*

Proof. Assume, for the sake of contradiction, that (1) has an eventually positive solution $x(t) > 0$ for all $t \geq t_0 > 0$. Set

$$z(t) = x(t) + p(t)\, x(t-\tau). \tag{28}$$

Then by Lemma 3, it follows that

$$z(t) > 0. \tag{29}$$

As $x(t) > z(t)$, it follows from (1) that

$$(r(t)\, z(t))' + q(t)\, z(t-\sigma) \leq 0. \tag{30}$$

Dividing the last inequality by $r(t) > 0$, we obtain

$$z'(t) + \frac{r'(t)}{r(t)} z(t) + \frac{q(t)}{r(t)} z(t-\sigma) \leq 0. \tag{31}$$

Let

$$z(t) = e^{-\int_{t_0}^{t} (r'(s)/r(s))ds} y(t). \tag{32}$$

This implies that $y(t) > 0$.

Substituting in (31) yields for all $t \geq t_0$

$$y'(t) + \frac{q(t)}{r(t-\sigma)} y(t-\sigma) \leq 0, \quad t \geq t_0. \tag{33}$$

In view of Lemma 2(i) and (27), it is impossible for (33) to have an eventually positive solution. This contradicts the fact that $y(t) > 0$ and the proof is complete. □

Example 9. Consider the NDDE

$$\left[\frac{1}{t}\left(x(t) - \frac{t}{t+1} x(t-\tau)\right)\right]' + \frac{1}{t-(5\pi/2)} x\left(t - \frac{5\pi}{2}\right) = 0, \quad t > \frac{5\pi}{2}. \tag{34}$$

Note that all the hypotheses of Theorem 8 are satisfied:

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \frac{q(s)}{r(s-\sigma)} ds = \liminf_{t\to\infty} \int_{t-(5\pi/2)}^{t} ds = \frac{5\pi}{2} > \frac{1}{e}. \tag{35}$$

Therefore every solution of (34) is oscillatory.

Remark 10. Theorem 8 is an extent of [17, Theorem 3] and [21, Theorem 6.4.2].

Theorem 11. *Assume that* (2) *holds with* $p(t) \equiv p \neq \pm 1$, $r(t) \equiv r > 0$, $q(t)$ *being* τ *periodic, and*

$$\frac{1}{r(1+p)} \liminf_{t\to\infty} \int_{t-\sigma}^{t-\tau} q(s)\, ds > \frac{1}{e}. \tag{36}$$

Then every solution of NDDE

$$[r(x(t)+px(t-\tau))]'+q(t)x(t-\sigma)=0, \quad t\ge t_0, \tag{37}$$

is oscillatory.

Proof. Assume, for the sake of contradiction, that (37) has an eventually positive solution $x(t)>0$ for all $t\ge t_0>0$. Set

$$\begin{aligned} z(t)&=x(t)+px(t-\tau),\\ w(t)&=z(t)+pz(t-\tau). \end{aligned} \tag{38}$$

It is easily seen, by direct substituting, that $z(t)$ and $w(t)$ are also solutions of (37). That is,

$$rz'(t)+prz'(t-\tau)+q(t)z(t-\sigma)=0, \tag{39}$$

$$rw'(t)+prw'(t-\tau)+q(t)w(t-\sigma)=0. \tag{40}$$

By Lemma 4, $z(t)$ is decreasing and either (13) or (14) holds. In either case we claim that

$$w'(t-\tau)\ge w'(t). \tag{41}$$

Indeed,

$$\begin{aligned} w'(t)&=-\frac{1}{r}q(t)z(t-\sigma)\le-\frac{1}{r}q(t)z(t-\sigma-\tau)\\ &=-\frac{1}{r}q(t-\tau)z(t-\sigma-\tau)=w'(t-\tau). \end{aligned} \tag{42}$$

Furthermore, we have by Lemma 4 that as long as $p\ne\pm1$,

$$w(t)>0. \tag{43}$$

Using (41) in (40) implies

$$r(1+p)w'(t-\tau)+q(t)w(t-\sigma)\le0 \tag{44}$$

or

$$w'(t-\tau)+\frac{1}{r(1+p)}q(t)w(t-\sigma)\le0. \tag{45}$$

Since $q(t)$ is periodic of period τ, we find

$$w'(t)+\frac{1}{r(1+p)}q(t)w(t-(\sigma-\tau))\le0, \quad \text{if } 1+p>0, \tag{46}$$

or

$$w'(t)-\left[\frac{1}{-r(1+p)}\right]q(t)w(t+(\tau-\sigma))\ge0, \quad \text{if } 1+p<0. \tag{47}$$

In view of Lemma 2((i) and (iv)) and (36), it is impossible for (46) and (47) to have eventually positive solutions. This contradicts the fact that $w(t)>0$ and the proof is complete. □

Remark 12. Theorem 11 extends [15, Theorems 8 and 10]. See also [21, Theorem 6.4.4].

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References

[1] H. A. Agwo, "On the oscillation of delay differential equations with real coefficients," *International Journal of Mathematics and Mathematical Sciences*, vol. 22, no. 3, pp. 573–578, 1999.

[2] R. D. Driver, "A mixed neutral system," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 8, no. 2, pp. 155–158, 1984.

[3] J. K. Hale, *Theory of Functional Differential Equations*, vol. 3 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1977.

[4] N. Parhi and R. N. Rath, "On oscillation and asymptotic behaviour of solutions of forced first order neutral differential equations," *Indian Academy of Sciences*, vol. 111, no. 3, pp. 337–350, 2001.

[5] R. P. Agarwal, M. Bohner, and W. T. Li, *Nonoscillation and Oscillation: Theory for Functional Differential Equations*, vol. 267 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2004.

[6] M. K. Grammatikopoulos, G. Ladas, and Y. G. Sficas, "Oscillation and asymptotic behavior of neutral equations with variable coefficients," *Radovi Matematicki*, vol. 2, no. 2, pp. 279–303, 1986.

[7] S. Tanaka, "Oscillation of solutions of first-order neutral differential equations," *Hiroshima Mathematical Journal*, vol. 32, no. 1, pp. 79–85, 2002.

[8] A. I. Zahariev and D. D. Baĭnov, "Oscillating properties of the solutions of a class of neutral type functional-differential equations," *Bulletin of the Australian Mathematical Society*, vol. 22, no. 3, pp. 365–372, 1980.

[9] G. Ladas and Y. G. Sficas, "Oscillations of neutral delay differential equations," *Canadian Mathematical Bulletin*, vol. 29, no. 4, pp. 438–445, 1986.

[10] G. Ladas and S. W. Schults, "On oscillations of neutral equations with mixed arguments," *Hiroshima Mathematical Journal*, vol. 19, no. 2, pp. 409–429, 1989.

[11] Y. G. Sficas and I. P. Stavroulakis, "Necessary and sufficient conditions for oscillations of neutral differential equations," *Journal of Mathematical Analysis and Applications*, vol. 123, no. 2, pp. 494–507, 1987.

[12] M. K. Grammatikopoulos, E. A. Grove, and G. Ladas, "Oscillation and asymptotic behavior of neutral differential equations with deviating arguments," *Applicable Analysis*, vol. 22, no. 1, pp. 1–19, 1986.

[13] B. G. Zhang, "Oscillation of first order neutral functional-differential equations," *Journal of Mathematical Analysis and Applications*, vol. 139, no. 2, pp. 311–318, 1989.

[14] K. Gopalsamy and B. G. Zhang, "Oscillation and nonoscillation in first order neutral differential equations," *Journal of Mathematical Analysis and Applications*, vol. 151, no. 1, pp. 42–57, 1990.

[15] M. K. Grammatikopoulos, E. A. Grove, and G. Ladas, "Oscillations of first-order neutral delay differential equations," *Journal of Mathematical Analysis and Applications*, vol. 120, no. 2, pp. 510–520, 1986.

[16] S. H. Saker and E. M. Elabbasy, "Oscillation of first order neutral delay differential equations," *Kyungpook Mathematical Journal*, vol. 41, no. 2, pp. 311–321, 2001.

[17] Q. Chuanxi and G. Ladas, "Oscillations of neutral differential equations with variable coefficients," *Applicable Analysis*, vol. 32, no. 3-4, pp. 215–228, 1989.

[18] Q. Chuanxi and G. Ladas, "Oscillations of first-order neutral equations with variable coefficients," *Monatshefte für Mathematik*, vol. 109, no. 2, pp. 103–111, 1990.

[19] I. Kubiaczyk and S. H. Saker, "Oscillation of solutions to neutral delay differential equations," *Mathematica Slovaca*, vol. 52, no. 3, pp. 343–359, 2002.

[20] B. Karpuz and O. Ocalan, "Oscillation criteria for some classes of linear delay differential equations of first-order," *Bulletin of the Institute of Mathematics*, vol. 3, no. 2, pp. 293–314, 2008.

[21] I. Győri and G. Ladas, *Oscillation Theory of Delay Differential Equations*, Oxford Mathematical Monographs, Clarendon Press, New York, NY, USA, 1991.

[22] L. H. Erbe, Q. Kong, and B. G. Zhang, *Oscillation Theory for Functional-Differential Equations*, vol. 190 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1995.

[23] J. S. Yu, Z. C. Wang, and C. X. Qian, "Oscillation of neutral delay differential equations," *Bulletin of the Australian Mathematical Society*, vol. 45, no. 2, pp. 195–200, 1992.

[24] S. K. Choi and N. J. Koo, "Oscillation theory for delay and neutral differential equations," *Trends in Mathematics, Information Center for Mathematical Sciences*, vol. 2, pp. 170–176, 1999.

[25] O. Ocalan, "Existence of positive solutions for a neutral differential equation with positive and negative coefficients," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 84–90, 2009.

[26] T. Candan and R. S. Dahiya, "Positive solutions of first-order neutral differential equations," *Applied Mathematics Letters*, vol. 22, no. 8, pp. 1266–1270, 2009.

[27] M. R. S. Kulenović, G. Ladas, and A. Meimaridou, "Necessary and sufficient condition for oscillations of neutral differential equations," *Journal of the Australian Mathematical Society B*, vol. 28, no. 3, pp. 362–375, 1987.

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Research Article

A Numerical Method for Fuzzy Differential Equations and Hybrid Fuzzy Differential Equations

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Numerical algorithms for solving first-order fuzzy differential equations and hybrid fuzzy differential equations have been investigated. Sufficient conditions for stability and convergence of the proposed algorithms are given, and their applicability is illustrated with some examples.

1. Introduction

Hybrid systems are devoted to modeling, design, and validation of interactive systems of computer programs and continuous systems. That is, control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics can be modeled by hybrid systems. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named hybrid fuzzy differential systems.

The Hukuhara derivative of a fuzzy-number-valued function was introduced in [1]. Under this setting, the existence and uniqueness of the solution of a fuzzy differential equation are studied by Kaleva [2, 3], Seikkala [4], and Kloeden [5]. This approach has the disadvantage that it leads to solutions which have an increasing length of their support [2]. A generalized differentiability was studied in [6–8]. This concept allows us to resolve the previously mentioned shortcoming. Indeed, the generalized derivative is defined for a larger class of fuzzy-number-valued functions than the Hukuhara derivative. Some applications of numerical methods in FDE and hybrid fuzzy differential equation (HFDE) are presented in [9–19]. Some other approaches to study FDE and fuzzy dynamical systems have been investigated in [20–22].

In engineering and physical problems, Trapezoidal rule is a simple and powerful method to solve numerically related ODEs. Trapezoidal rule has a higher convergence order in comparison to other one step methods, for instance, Euler method.

In this work, we concentrate on numerical procedure for solving FDEs and HFDEs, whenever these equations possess unique fuzzy solutions.

In Section 2, we briefly present the basic definitions. Trapezoidal rule for solving fuzzy differential equations is introduced in Section 3, and convergence and stability of the mentioned method are proved. The proposed algorithm is illustrated by solving two examples. In Section 4 we present Trapezoidal rule for solving hybrid fuzzy differential equations.

2. Preliminary Notes

In this section the most basic definition of ordinary differential equations (ODEs) and notation used in fuzzy calculus are introduced. See, for example, [23].

Consider the first-order ordinary differential equation

$$y'(t) = f(t, y(t)), \qquad y(t_0) = y_0, \tag{1}$$

where $f : [t_0, t_N] \times \mathbb{R}^n \to \mathbb{R}^n$ and $t_0 \in \mathbb{R}$. A linear multistep method applied to (1) is

$$\sum_{i=0}^{k} \alpha_i y_{m+i} = h \sum_{i=0}^{k} \beta_i f(t_{m+i}, y_{m+i}), \tag{2}$$

with $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_k \neq 0$, given starting values $y_0, y_1, \ldots, y_{k-1}$. In the case $\beta_k = 0$, the corresponding methods (2) are explicit and are implicit otherwise. The constant step size $h > 0$ leads to time discretizations with respect to the grid points $t_m := t_0 + mh$. The value y_{m+i} is an approximation of the exact solution at t_{m+i}. The special case of explicit methods, $m = 2$, $\alpha_0 = -1$, $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_0 = \beta_2 = 0$, and $\beta_1 = 2$, corresponds to the Midpoint rule:

$$y_{m+2} = y_m + 2hf(t_{m+1}, y_{m+1}), \tag{3}$$

and the especial case of implicit methods, $m = 1$, $\alpha_0 = -1$, $\alpha_1 = 1$, and $\beta_0 = \beta_1 = 1/2$, corresponds to the Trapezoidal rule:

$$y_{m+1} = y_m + \frac{h}{2}\left[f(t_m, y_m), f(t_{m+1}, y_{m+1})\right]. \tag{4}$$

For an explicit method, (2) yields the current value y_{m+k} directly in terms of y_{m+j}, f_{m+j}, $j = 0, 1, \ldots, k-1$, which, at this stage of the computation, have already been calculated. An implicit method will call for the solution, at each stage of computation, of the the equation

$$y_{m+k} = h\beta_k f(t_{m+k}, y_{m+k}) + g, \tag{5}$$

where g is a known function of previously calculated values y_{m+j}, f_{m+j}, $j = 0, 1, \ldots, k-1$. When the original differential equation in (1) is linear, then (5) is linear in y_{m+k}, and there is no problem in solving it. When f is nonlinear, for finding solution of (1), we can use the following iteration:

$$y_{m+k}^{[s+1]} = h\beta_k f\left(t_{m+k}, y_{m+k}^{[s]}\right) + g. \tag{6}$$

Definition 1. Associated with the multistep method (2), we define the first characteristic polynomial as follows:

$$\rho(\xi) := \sum_{i=0}^{k} \alpha_i \xi^i. \tag{7}$$

Theorem 2. *A multistep method is stable if the first characteristic polynomial satisfies the root condition, that is, the roots of $\rho(\xi)$ lie on or within the unit circle, and further the roots on the unit circle are simple.*

According to Theorem 2, we know the Midpoint rule and Trapezoidal rule are stable.

Definition 3. The difference operator

$$\mathfrak{L}[y(t); h] = \sum_{j=0}^{k} \left[\alpha_j y(t + jh) - h\beta_j y'(t + jh)\right] \tag{8}$$

and the associated linear multistep method (2) are said to be of order p if for the following equation:

$$\mathfrak{L}[y(t); h] = C_0 y(t) + C_1 h y^{(1)}(t) + \cdots + C_q h^q y^{(q)}(t) + \cdots, \tag{9}$$

we have $C_0 = C_1 = \cdots = C_p = 0, C_{p+1} \neq 0$, where $C_0 = \sum_{j=0}^{k} \alpha_j$ and $C_i = (1/i!)(\sum_{j=0}^{k} \alpha_j j^i - i \sum_{j=0}^{k} \beta_j j^{i-1})$, for $i \geq 1$.

According to Definition 3, Midpoint rule and Trapezoidal rule are second-order methods.

We now recall some general concepts of fuzzy set theory; see, for example, [2, 24].

Definition 4. Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u : X \to [0, 1]$, and $u(x)$ is interpreted as the degree of membership of an element x in fuzzy set u for each $x \in X$.

Let us denote by $\mathbb{R}_F$ the class of fuzzy subsets of the real axis, that is,

$$u : \mathbb{R} \longrightarrow [0, 1], \tag{10}$$

satisfying the following properties:

(i) u is normal, that is, there exists $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$,

(ii) u is a convex fuzzy set (i.e., $u(ts + (1-t)r) \geq \min\{u(s), u(r)\}, \forall t \in [0, 1],\ s, r \in \mathbb{R}$),

(iii) u is upper semicontinuous on $\mathbb{R}$,

(iv) $\mathrm{cl}\{s \in \mathbb{R} \mid u(s) > 0\}$ is compact, where cl denotes the closure of a subset.

The space $\mathbb{R}_F$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_F$. For $0 < \alpha \leq 1$, we denote

$$[u]^\alpha = \{s \in \mathbb{R} \mid u(s) \geq \alpha\}, \qquad [u]^0 = \mathrm{cl}\{s \in \mathbb{R} \mid u(s) > 0\}. \tag{11}$$

Then from (i)–(iv), it follows that the α-level set $[u]^\alpha$ is a nonempty compact interval for all $0 \leq \alpha \leq 1$. The notation

$$[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha] \tag{12}$$

denotes explicitly the α-level set of u. The following remark shows when $[\underline{u}^\alpha, \overline{u}^\alpha]$ is a valid α-level set.

Remark 5. The sufficient conditions for $[\underline{u}^\alpha, \overline{u}^\alpha]$ to define the parametric form of a fuzzy number are as follows:

(i) $\underline{u}^\alpha$ is a bounded monotonic increasing (nondecreasing) left-continuous function on $(0, 1]$ and right-continuous for $\alpha = 0$,

(ii) $\overline{u}^\alpha$ is a bounded monotonic decreasing (nonincreasing) left-continuous function on $(0, 1]$ and right-continuous for $\alpha = 0$.

(iii) $\underline{u}^\alpha \leq \overline{u}^\alpha$, $0 \leq \alpha \leq 1$.

For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined by $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda[u]^\alpha$, $\forall\alpha \in [0, 1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$, and $\lambda[u]^\alpha$ means the usual product between a scaler and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance

$$D : \mathbb{R}_F \times \mathbb{R}_F \longrightarrow \mathbb{R}_+ \cup \{0\}, \tag{13}$$

by

$$D(u, v) = \sup_{\alpha\in[0,1]} \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}. \tag{14}$$

The following properties are well known:

$$D(u + w, v + w) = D(u, v), \forall u, v, w \in \mathbb{R}_F,$$
$$D(ku, kv) = |k|D(u, v), \forall k \in \mathbb{R}, u, v \in \mathbb{R}_F,$$
$$D(u + v, w + e) \le D(u, w) + D(v, e), \forall u, v, w, e \in \mathbb{R}_F,$$

and $(\mathbb{R}_F, D)$ is complete metric spaces.

Let I be a real interval. A mapping $y : I \to \mathbb{R}_F$ is called a fuzzy process and its α-level set is denoted by

$$[y(t)]^\alpha = [\underline{y}^\alpha(t), \overline{y}^\alpha(t)], \quad t \in I,\ \alpha \in (0, 1]. \tag{15}$$

A triangular fuzzy number N is defined by an ordered triple $(x^l, x^c, x^r) \in \mathbb{R}^3$ with $x^l \le x^c \le x^r$, where the graph of $N(s)$ is a triangle with base on the interval $[x^l, x^r]$ and vertex at $s = x^c$. An α-level of N is always a closed, bounded interval. We write $N = (x^l, x^c, x^r)$; then

$$[N]^\alpha = [x^c - (1-\alpha)(x^c - x^l), x^c + (1-\alpha)(x^r - x^c)], \tag{16}$$

for any $0 \le \alpha \le 1$.

Definition 6. Let $x, y \in \mathbb{R}_F$. If there exists $z \in \mathbb{R}_F$ such that $x = y + z$, then z is called the H-difference of x and y, and it is denoted by $x \ominus y$.

In this paper the sign "$\ominus$" stands always for H-difference, and let us remark that $x \ominus y \ne x + (-1)y$. Usually we denote $x + (-1)y$ by $x - y$, while $x \ominus y$ stands for the H-difference.

Definition 7. Let $F : I \to \mathbb{R}_F$ be a fuzzy function. We say F is Hukuhara differentiable at $t_0 \in I$ if there exists an element $F'(t_0) \in \mathbb{R}_F$ such that the limits

$$\lim_{h\to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h}, \qquad \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h} \tag{17}$$

exist and are equal to $F'(t_0)$. Here the limits are taken in the metric space $(\mathbb{R}_F, D)$.

Definition 8. Let $[a, b] \subset I$. The fuzzy integral $\int_a^b y(t)\,dt$ is defined by

$$\left[\int_a^b y(t)dt\right]^\alpha = \left[\int_a^b \underline{y}^\alpha(t)\,dt, \int_a^b \overline{y}^\alpha(t)\,dt\right], \tag{18}$$

provided the Lebesgue integrals on the right exist.

Remark 9. Let $[a, b] \subset I$. If $F : I \to \mathbb{R}_F$ is Hukuhara differentiable and its Hukuhara derivative F' is integrable over $[a, b]$, then

$$F(t) = F(t_0) + \int_{t_0}^t F'(s)\,ds, \tag{19}$$

for all values of t_0, t, where $a \le t_0 \le t \le b$.

Theorem 10. *Let (t_i, u_i), $i = 0, 1, \dots, n$, be the observed data, and suppose that each of the $u_i = (u_i^l, u_i^c, u_i^r)$ is a triangular fuzzy number. Then for each $t \in [t_0, t_n]$, the fuzzy polynomial interpolation is a fuzzy-value continuous function $f : \mathbb{R} \to \mathbb{R}_F$, where $f(t_i) = u_i$, $f(t) = (f^l(t), f^c(t), f^r(t)) \in \mathbb{R}_F$, and*

$$f^l(t) = \sum_{L_i(t)\ge 0} L_i(t)\,u_i^l + \sum_{L_i(t)<0} L_i(t)\,u_i^r,$$
$$f^c(t) = \sum_{i=0}^n L_i(t)\,u_i^c, \tag{20}$$
$$f^r(t) = \sum_{L_i(t)\ge 0} L_i(t)\,u_i^r + \sum_{L_i(t)<0} L_i(t)\,u_i^l,$$

such that $L_i(t) = \prod_{i\ne j}((t - t_j)/(t_i - t_j))$.

Proof. See [25]. □

3. Fuzzy Differential Equations

Consider the first-order fuzzy differential equation $y' = f(t, y)$, where y is a fuzzy function of t, $f(t, y)$ is a fuzzy function of crisp variable t and fuzzy variable y, and y' is Hukuhara fuzzy derivative of y. If an initial value $y(t_0) = y_0 \in \mathbb{R}_F$ is given, a fuzzy Cauchy problem of first order will be obtained as follows:

$$y'(t) = f(t, y(t)), \quad t_0 \le t \le T,$$
$$y(t_0) = y_0. \tag{21}$$

By Theorem 5.2 in [11] we may replace (21) by equivalent system

$$\underline{y}'(t) = \underline{f}(t, y) = F(t, \underline{y}, \overline{y}), \qquad \underline{y}(t_0) = \underline{y}_0,$$
$$\overline{y}'(t) = \overline{f}(t, y) = G(t, \underline{y}, \overline{y}), \qquad \overline{y}(t_0) = \overline{y}_0. \tag{22}$$

The parametric form of (22) is given by

$$\underline{y}'(t;\alpha) = F(t, \underline{y}(t;\alpha), \overline{y}(t;\alpha)), \qquad \underline{y}(t_0;\alpha) = \underline{y}_0^\alpha,$$
$$\overline{y}'(t;\alpha) = G(t, \underline{y}(t;\alpha), \overline{y}(t;\alpha)), \qquad \overline{y}(t_0;\alpha) = \overline{y}_0^\alpha, \tag{23}$$

for $0 \le \alpha \le 1$. In some cases the system given by (23) can be solved analytically. In most cases analytical solutions may not be found, and a numerical approach must be considered. Some numerical methods such as the fuzzy Euler method, Nyström method, and predictor-corrector method presented in [7, 10, 11, 13, 15]. In the following, we present a new method to numerical solution of FDE.

3.1. Trapezoidal Rule for Fuzzy Differential Equations. In the interval $I = [t_0, T]$ we consider a set of discrete equally spaced grid points $t_0 < t_1 < t_2 < \cdots < t_N = T$. The exact and approximate solutions at t_n, $0 \le n \le N$, are denoted by $[y(t_n)]^\alpha = [\underline{y}^\alpha(t_n), \overline{y}^\alpha(t_n)]$ and $[y_n]^\alpha = [\underline{y}_n^\alpha, \overline{y}_n^\alpha]$, respectively. The grid points at which the solution is calculated are

$$t_n = t_0 + nh, \quad h = \frac{T - t_0}{N}, \ 0 \le n \le N. \tag{24}$$

Let $y_p = [\underline{y}, \overline{y}]$, $0 \le p < N$ which $f(t_p, y_p)$ is triangular fuzzy number. We have

$$y(t_{p+1}) = y(t_p) + \int_{t_p}^{t_{p+1}} f(t, y(t))\, dt. \tag{25}$$

By fuzzy interpolation, Theorem 10, we get

$$f_I^l(t, y(t)) = l_0(t) f^l(t_p, y_p) + l_1(t) f^l(t_{p+1}, y_{p+1}), \tag{26}$$

$$f_I^c(t, y(t)) = l_0(t) f^c(t_p, y_p) + l_1(t) f^c(t_{p+1}, y_{p+1}), \tag{27}$$

$$f_I^r(t, y(t)) = l_0(t) f^r(t_p, y_p) + l_1(t) f^r(t_{p+1}, y_{p+1}), \tag{28}$$

where $f_I(t, y(t)) = (f_I^l(t, y(t)), f_I^c(t, y(t)), f_I^r(t, y(t)))$, interpolates $f(t, y(t))$ with the interpolation data given by the value $f(t_p, y_p)$, and $l_0(t) = (t - t_{p+1})/(t_p - t_{p+1})$, $l_1(t) = (t - t_p)/(t_{p+1} - t_p)$.

For $t_p \le t \le t_{p+1}$ we have

$$l_0(t) = \frac{t - t_{p+1}}{t_p - t_{p+1}} \ge 0, \qquad l_1(t) = \frac{t - t_p}{t_{p+1} - t_p} \ge 0. \tag{29}$$

From (16) and (25) it follows that

$$[y(t_{p+1})]^\alpha = [\underline{y}^\alpha(t_{p+1}), \overline{y}^\alpha(t_{p+1})], \tag{30}$$

where

$$\underline{y}^\alpha(t_{p+1}) = \underline{y}^\alpha(t_p) + \int_{t_p}^{t_{p+1}} \{\alpha f^c(t, y(t)) + (1-\alpha) f^l(t, y(t))\}\, dt, \tag{31}$$

$$\overline{y}^\alpha(t_{p+1}) = \overline{y}^\alpha(t_p) + \int_{t_p}^{t_{p+1}} \{\alpha f^c(t, y(t)) + (1-\alpha) f^r(t, y(t))\}\, dt. \tag{32}$$

According to (25), if (26) and (27) are situated in (31), (27) and (28) in (32), we obtain

$$\begin{aligned}\underline{y}_{p+1}^\alpha = \underline{y}_p^\alpha &+ \int_{t_p}^{t_{p+1}} \{\alpha [l_0(t) f^c(t_p, y_p) + l_1(t) f^c(t_{p+1}, y_{p+1})] \\ &+ (1-\alpha) \times [l_0(t) f^l(t_p, y_p) + l_1(t) f^l(t_{p+1}, y_{p+1})]\}\, dt.\end{aligned} \tag{33}$$

By integration we have

$$\begin{aligned}\underline{y}_{p+1}^\alpha = \underline{y}_p^\alpha &+ \frac{h}{2} \times [\alpha f^c(t_p, y_p) + (1-\alpha) f^l(t_p, y_p) \\ &+ \alpha f^c(t_{p+1}, y_{p+1}) + (1-\alpha) f^l(t_{p+1}, y_{p+1})].\end{aligned} \tag{34}$$

By (16) deduce

$$\underline{y}_{p+1}^\alpha = \underline{y}_p^\alpha + \frac{h}{2} [\underline{f}^\alpha(t_p, y_p) + \underline{f}^\alpha(t_{p+1}, y_{p+1})]. \tag{35}$$

Similarly we obtain

$$\overline{y}_{p+1}^\alpha = \overline{y}_p^\alpha + \frac{h}{2} [\overline{f}^\alpha(t_p, y_p) + \overline{f}^\alpha(t_{p+1}, y_{p+1})]. \tag{36}$$

Therefore, Trapezoidal rule is obtained as follows:

$$\begin{aligned}\underline{y}_{p+1}^\alpha &= \underline{y}_p^\alpha + \frac{h}{2} [\underline{f}^\alpha(t_p, y_p) + \underline{f}^\alpha(t_{p+1}, y_{p+1})], \\ \overline{y}_{p+1}^\alpha &= \overline{y}_p^\alpha + \frac{h}{2} [\overline{f}^\alpha(t_p, y_p) + \overline{f}^\alpha(t_{p+1}, y_{p+1})], \\ \underline{y}_p^\alpha &= \underline{y}, \qquad \overline{y}_p^\alpha = \overline{y},\end{aligned} \tag{37}$$

for $0 \le p < N$.

3.2. Convergence and Stability. Suppose the exact solution $(\underline{Y}(t; \alpha), \overline{Y}(t; \alpha))$ is approximated by some $(\underline{y}(t; \alpha), \overline{y}(t; \alpha))$. The exact and approximate solutions at t_n, $0 \le n \le N$, are denoted by $[Y_n]^\alpha = [\underline{Y}_n^\alpha, \overline{Y}_n^\alpha]$ and $[y_n]^\alpha = [\underline{y}_n^\alpha, \overline{y}_n^\alpha]$, respectively. Our next result determines the pointwise convergence of the Trapezoidal approximates to the exact solution. The following lemma will be applied to show convergence of these approximates; that is,

$$\lim_{h \to 0} \underline{y}(t; h; \alpha) = \underline{Y}(t; \alpha), \qquad \lim_{h \to 0} \overline{y}(t; h; \alpha) = \overline{Y}(t; \alpha). \tag{38}$$

Lemma 11. *Let a sequence of numbers* $\{w_n\}_{n=0}^N$ *satisfy*

$$|w_{n+1}| \le A|w_n| + B, \qquad 0 \le n \le N-1, \tag{39}$$

for some given positive constant A and B. Then

$$|w_n| \le A^N |w_0| + B\frac{A^n - 1}{A-1}, \qquad 0 \le n \le N-1. \tag{40}$$

Proof. See [15]. □

Let $F(t,u,v)$ and $G(t,u,v)$ be the functions F and G of (22), where u and v are constants and $u \le v$. The domain where F and G are defined is therefore

$$K = \{(t,u,v) \mid t_0 \le t \le T,\ -\infty < v < \infty,\ -\infty < u \le v\}. \tag{41}$$

Theorem 12. *Let* $F(t,u,v)$ *and* $G(t,u,v)$ *belong to* $C^2(K)$*, and let the partial derivatives of F, G be bounded over K. Then for arbitrary fixed* $\alpha : 0 \le \alpha \le 1$*, the Trapezoidal rule approximate of* (37) *converges to the exact solutions* $\underline{Y}(t;\alpha)$*,* $\overline{Y}(t;\alpha)$ *uniformly in t, for* $\underline{Y}, \overline{Y} \in C^3[t_0,T]$*.*

Proof. It is sufficient to show that

$$\lim_{h\to 0} \underline{y}_N^\alpha = \underline{Y}(T;\alpha), \qquad \lim_{h\to 0} \overline{y}_N^\alpha = \overline{Y}(T;\alpha). \tag{42}$$

By using Taylor's theorem, we get

$$\begin{aligned} \underline{Y}_{p+1}^\alpha &= \underline{Y}_p^\alpha + \frac{h}{2} \\ &\quad \times \left[F\left(t_p, \underline{Y}_p^\alpha, \overline{Y}_p^\alpha\right) + F\left(t_{p+1}, \underline{Y}_{p+1}^\alpha, \overline{Y}_{p+1}^\alpha\right)\right] \\ &\quad + \frac{h^3}{12}\underline{Y}'''\left(\underline{\xi}_p\right), \\ \overline{Y}_{p+1}^\alpha &= \overline{Y}_p^\alpha + \frac{h}{2} \\ &\quad \times \left[G\left(t_p, \underline{Y}_p^\alpha, \overline{Y}_p^\alpha\right) + G\left(t_{p+1}, \underline{Y}_{p+1}^\alpha, \overline{Y}_{p+1}^\alpha\right)\right] \\ &\quad + \frac{h^3}{12}\overline{Y}'''\left(\overline{\xi}_p\right), \end{aligned} \tag{43}$$

where $t_p < \underline{\xi}_p, \overline{\xi}_p < t_{p+1}$. Consequently,

$$\begin{aligned} &\underline{Y}_{p+1}^\alpha - \underline{y}_{p+1}^\alpha \\ &= \underline{Y}_p^\alpha - \underline{y}_p^\alpha + \frac{h}{2} \\ &\quad \times \left\{F\left(t_p, \underline{Y}_p^\alpha, \overline{Y}_p^\alpha\right) - F\left(t_p, \underline{y}_p^\alpha, \overline{y}_p^\alpha\right) + F\left(t_{p+1}, \underline{Y}_{p+1}^\alpha, \overline{Y}_{p+1}^\alpha\right)\right. \\ &\quad \left. - F\left(t_{p+1}, \underline{y}_{p+1}^\alpha, \overline{y}_{p+1}^\alpha\right)\right\} + \frac{h^3}{12}\underline{Y}'''\left(\underline{\xi}_p\right), \\ &\overline{Y}_{p+1}^\alpha - \overline{y}_{p+1}^\alpha \\ &= \overline{Y}_p^\alpha - \overline{y}_p^\alpha + \frac{h}{2} \\ &\quad \times \left\{G\left(t_p, \underline{Y}_p^\alpha, \overline{Y}_p^\alpha\right) - G\left(t_p, \underline{y}_p^\alpha, \overline{y}_p^\alpha\right)\right. \\ &\quad + G\left(t_{p+1}, \underline{Y}_{p+1}^\alpha, \overline{Y}_{p+1}^\alpha\right) \\ &\quad \left. - G\left(t_{p+1}, \underline{y}_{p+1}^\alpha, \overline{y}_{p+1}^\alpha\right)\right\} + \frac{h^3}{12}\underline{Y}'''\left(\underline{\xi}_p\right). \end{aligned} \tag{44}$$

Denote $w_n = \underline{Y}_n^\alpha - \underline{y}_n^\alpha$ and $v_n = \overline{Y}_n^\alpha - \overline{y}_n^\alpha$. Then

$$\begin{aligned} |w_{p+1}| \le |w_p| &+ h \\ &\times \left[L_1 \max\left\{|w_p|, |v_p|\right\} + L_2 \max\left\{|w_{p+1}|, |v_{p+1}|\right\}\right] \\ &+ \frac{h^3}{12}\underline{M}, \end{aligned} \tag{45}$$

$$\begin{aligned} |v_{p+1}| \le |v_p| &+ h \\ &\times \left[L_1 \max\left\{|w_p|, |v_p|\right\} + L_2 \max\left\{|w_{p+1}|, |v_{p+1}|\right\}\right] \\ &+ \frac{h^3}{12}\overline{M}, \end{aligned} \tag{46}$$

where $\underline{M} = \max_{t_0\le t\le T} |\underline{Y}'''(t;\alpha)|$ and $\overline{M} = \max_{t_0\le t\le T} |\overline{Y}'''(t;\alpha)|$, and $L_1, L_2 > 0$ is a bound for partial derivatives of F and G in t_p, t_{p+1}. Thus,

$$\begin{aligned} &|w_{p+1}| + |v_{p+1}| \\ &\le |w_p| + |v_p| + 2h \\ &\quad \times \left[L_1 \max\left\{|w_p|, |v_p|\right\} + L_2 \max\left\{|w_{p+1}|, |v_{p+1}|\right\}\right] \\ &\quad + \frac{h^3}{12}\left(\underline{M} + \overline{M}\right) \\ &\le |w_p| + |v_p| + 2h \\ &\quad \times \left[L_1\left(|w_p| + |v_p|\right) + L_2\left(|w_{p+1}| + |v_{p+1}|\right)\right] \\ &\quad + \frac{h^3}{12}\left(\underline{M} + \overline{M}\right). \end{aligned} \tag{47}$$

Table 1

α	$\underline{y}$	$\underline{y}_{\text{Mid}}$	$\underline{Y}$	$\overline{y}$	$\overline{y}_{\text{Mid}}$	$\overline{Y}$
0	0.9636348	0.9686955	0.9636356	1.0188934	1.0138372	1.0188941
0.1	0.9677550	0.9723098	0.9677558	1.0174878	1.0129374	1.0174885
0.2	0.9718752	0.9759241	0.9718760	1.0160820	1.0120376	1.0160828
0.3	0.9759954	0.9795385	0.9759961	1.0146763	1.0111377	1.0146772
0.4	0.9801155	0.9831529	0.9801163	1.0132707	1.0102379	1.0132715
0.5	0.9842358	0.9867672	0.9842365	1.0118650	1.0093381	1.0118657
0.6	0.9883559	0.9903815	0.9883567	1.0104593	1.0084382	1.0104601
0.7	0.9924761	0.9939959	0.9924769	1.0090537	1.0075384	1.0090544
0.8	0.9965963	0.9976103	0.9965971	1.0076480	1.0066386	1.0076487
0.9	1.0007164	1.0012246	1.0007173	1.0062424	1.0057387	1.0062431
1	1.0048367	1.0048389	1.0048374	1.0048367	1.0048389	1.0048374

If we put $|u_p| = |w_p| + |v_p|$ and $L = \max\{L_1, L_2\} < 1/2h$, then

$$\begin{aligned} \left|u_{p+1}\right| &\le (1+2hL)\left|u_p\right| + 2hL\left|u_{p+1}\right| + \frac{h^3}{12}\left(\underline{M}+\overline{M}\right) \\ &\le \left(\frac{1+2hL}{1-2hL}\right)\left|u_p\right| + \frac{h^3}{12(1-2hL)}\left(\underline{M}+\overline{M}\right). \end{aligned} \tag{48}$$

Then by Lemma 11 and $w_0 = v_0 = 0$, we have

$$\left|u_p\right| \le \frac{h^3}{12(1-2hL)}\left(\underline{M}+\overline{M}\right)\frac{((1+2hL)/(1-2hL))^n - 1}{((1+2hL)/(1-2hL)) - 1}. \tag{49}$$

If $h \to 0$, then $w_n \to 0$, $v_n \to 0$ which concludes the proof. □

Remark 13. According to Definition 3, Trapezoidal rule is a second-order method. In fact we may consider the definition of convergence order given in Definition 3 for system of ODEs.

Theorem 14. *Trapezoidal rule is stable.*

Proof. For Trapezoidal rule exists only one characteristic polynomial $\rho(\xi) = \xi - 1$, and it is clear that satisfies the root condition. Then by Theorem 2, the Trapezoidal rule is stable. □

3.3. Numerical Results. In this section we apply Trapezoidal rule for numerical solution of two linear fuzzy differential equations. We compare our results with Midpoint rule. The authors in [13] have presented the Midpoint rule for numerical solution of FDEs as follows:

$$\begin{gathered} \underline{y}^{\alpha}_{p+1} = \underline{y}^{\alpha}_{p-1} + 2h\underline{f}^{\alpha}\left(t_p, y_p\right), \\ \overline{y}^{\alpha}_{p+1} = \overline{y}^{\alpha}_{p-1} + 2h\overline{f}^{\alpha}\left(t_p, y_p\right), \\ \underline{y}^{\alpha}_{p-1} = \alpha_0, \qquad \underline{y}^{\alpha}_{p} = \alpha_1, \qquad \overline{y}^{\alpha}_{p} = \alpha_2, \qquad \overline{y}^{\alpha}_{p-1} = \alpha_3. \end{gathered} \tag{50}$$

The Midpoint rule is a second-order and stable method [13].

In the following two examples, the implicit nature of Trapezoidal rule for solving linear fuzzy differential equation is implemented by solving a linear system at each stage of computation.

Example 15 (see [13]). Consider the initial value problem

$$\begin{gathered} y'(t) = -y(t) + t + 1, \\ y(0) = [0.96 + 0.04\alpha, 1.01 - 0.01\alpha]. \end{gathered} \tag{51}$$

The exact solution at $t = 0.1$ for $0 \le \alpha \le 1$ is given by

$$\begin{aligned} \underline{Y}(0.1;\alpha) &= 0.1 + (0.985 + 0.015\alpha)e^{-0.1} \\ &\quad - (1-\alpha)\,0.025e^{0.1}, \\ \overline{Y}(0.1;\alpha) &= 0.1 + (0.985 + 0.015\alpha)e^{-0.1} + (1-\alpha)\,0.025e^{0.1}. \end{aligned} \tag{52}$$

A comparison between the exact solution, $Y(t;\alpha)$, and the approximate solutions by Midpoint method [13], $y_{\text{Mid}}(t;\alpha)$, and Trapezoidal method, $y(t;\alpha)$, at $t = 0.1$ with $N = 10$, is shown in Table 1 and Figure 1.

Example 16. Let us consider the first-order fuzzy differential equation

$$y'(t) = -y(t), \qquad y(0) = y_0, \tag{53}$$

where $y_0 = [0.96 + 0.04\alpha, 1.01 - 0.01\alpha]$.

The exact solution at $t = 0.1$ is given by

$$\begin{aligned} \underline{Y}(0.1;\alpha) &= (0.985 + 0.015\alpha)e^{-0.1} - (1-\alpha)\,0.025e^{0.1}, \\ \overline{Y}(0.1;\alpha) &= (0.985 + 0.015\alpha)e^{-0.1} + (1-\alpha)\,0.025e^{0.1}. \end{aligned} \tag{54}$$

A comparison between the exact solution, $Y(t;\alpha)$, and the approximate solutions by Midpoint method, $y_{\text{Mid}}(t;\alpha)$, and Trapezoidal method, $y(t;\alpha)$, at $t = 0.1$ with $N = 10$, is shown in Table 2 and Figure 2.

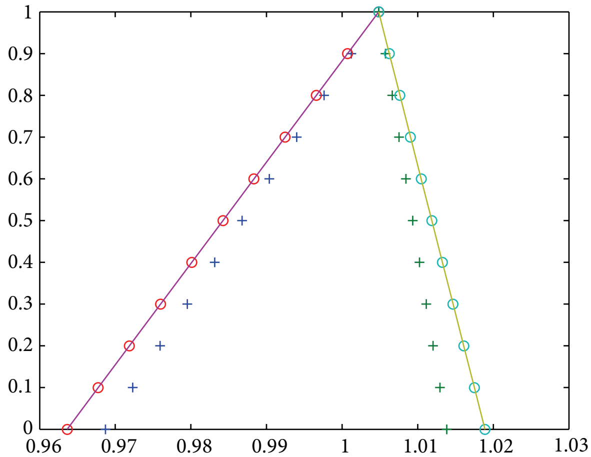

FIGURE 1: (-) Exact solution, (∘) Trapezoidal, and (+) Midpoint approximated points.

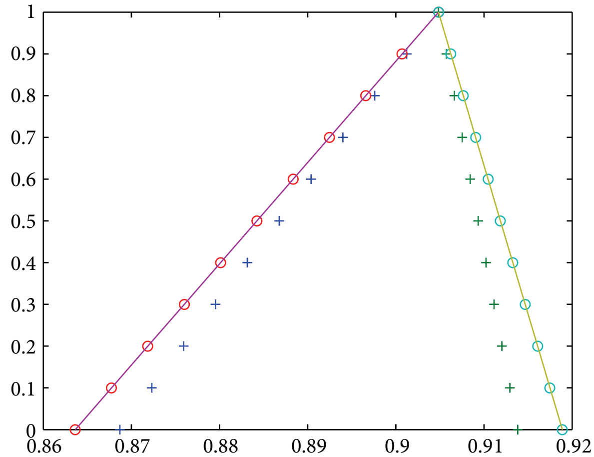

FIGURE 2: (-) Exact solution, (∘) Trapezoidal, and (+) Midpoint approximated points.

4. Hybrid Fuzzy Differential Equations

Consider the hybrid fuzzy differential equation

$$\begin{gathered} y'(t) = f(t, y(t), \lambda_k(y_k)), \qquad t \in [t_k, t_{k+1}], \\ k = 0, 1, 2, \ldots, \\ y(t_0) = y_0, \end{gathered} \tag{55}$$

where $\{t_k\}_{k=0}^{\infty}$ is strictly increasing and unbounded, y_k denotes $y(t_k)$, $f : [t_0, \infty) \times \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_F$ is continuous, and each $\lambda_k : \mathbb{R}_F \to \mathbb{R}_F$ is a continuous function. A solution y to (55) will be a function $y : [t_0, \infty) \to \mathbb{R}_F$ satisfying (55). For $k = 0, 1, 2, \ldots$, let $f_k : [t_k, t_{k+1}] \times \mathbb{R}_F \to \mathbb{R}_F$, where $f_k(t, y_k(t)) = f(t, y(t), \lambda_k(y_k))$. The hybrid fuzzy differential equation in (55) can be written in expanded form as

$$y'(t) = \begin{cases} y_0'(t) = f(t, y_0(t), \lambda_0(y_0)) \equiv f_0(t, y_0(t)), \\ \qquad y_0(t_0) = y_0, \quad t_0 \le t \le t_1, \\ y_1'(t) = f(t, y_1(t), \lambda_1(y_1)) \equiv f_1(t, y_1(t)), \\ \qquad y_1(t_1) = y_1, \quad t_1 \le t \le t_2, \\ \qquad \vdots \\ y_k'(t) = f(t, y_k(t), \lambda_k(y_k)) \equiv f_k(t, y_k(t)), \\ \qquad y_k(t_k) = y_k, \quad t_k \le t \le t_{k+1}, \\ \qquad \vdots \end{cases} \tag{56}$$

and a solution of (55) can be expressed as

$$y(t) = \begin{cases} y_0(t), & t_0 < t \le t_1, \\ y_1(t), & t_1 < t \le t_2, \\ \vdots \\ y_k(t), & t_k < t \le t_{k+1}, \\ \vdots \end{cases} \tag{57}$$

We note that the solution y of (55) is continuous and piecewise differentiable over $[t_0, \infty)$ and differentiable on each interval (t_k, t_{k+1}) for any fixed $y_k \in \mathbb{R}_F$ and $k = 0, 1, 2, \ldots$.

Theorem 17. *Suppose for $k = 0, 1, 2, \ldots$ that each $f_k : [t_k, t_{k+1}] \times \mathbb{R}_F \to \mathbb{R}_F$ is such that*

$$[f_k(t, y)]^{\alpha} = \left[\underline{f_k}^{\alpha}(t, \underline{y}^{\alpha}, \overline{y}^{\alpha}), \overline{f_k}^{\alpha}(t, \underline{y}^{\alpha}, \overline{y}^{\alpha})\right]. \tag{58}$$

If for each $k = 0, 1, 2, \ldots$ there exists $L_k > 0$ such that

$$\begin{aligned} &\left|\underline{f_k}^{\alpha}(t_1, x_1, y_1) - \underline{f_k}^{\alpha}(t_2, x_2, y_2)\right| \\ &\quad \le L_k \max\{|t_2 - t_1|, |x_2 - x_1|, |y_2 - y_1|\}, \\ &\left|\overline{f_k}^{\alpha}(t_1, x_1, y_1) - \overline{f_k}^{\alpha}(t_2, x_2, y_2)\right| \\ &\quad \le L_k \max\{|t_2 - t_1|, |x_2 - x_1|, |y_2 - y_1|\}, \end{aligned} \tag{59}$$

for all $\alpha \in [0, 1]$, then (55) and the hybrid system of ODEs

$$\begin{gathered} \left(\underline{y}_k^{\alpha}(t)\right)' = \underline{f_k}^{\alpha}\left(t, \underline{y}_k^{\alpha}(t), \overline{y}_k^{\alpha}(t)\right), \\ \left(\overline{y}_k^{\alpha}(t)\right)' = \overline{f_k}^{\alpha}\left(t, \underline{y}_k^{\alpha}(t), \overline{y}_k^{\alpha}(t)\right), \\ \underline{y}_k^{\alpha}(t_k) = \underline{y}_{k-1}^{\alpha}(t_k), \quad \text{if } k > 0, \ \underline{y}_0^{\alpha}(t_0) = \underline{y}_0^{\alpha}, \\ \overline{y}_k^{\alpha}(t_k) = \overline{y}_{k-1}^{\alpha}(t_k), \quad \text{if } k > 0, \ \overline{y}_0^{\alpha}(t_0) = \overline{y}_0^{\alpha} \end{gathered} \tag{60}$$

are equivalent.

Proof. See [19]. □

4.1. Trapezoidal Rule for Hybrid Fuzzy Differential Equations. For each $\alpha \in [0, 1]$, to numerically solve (55) in $[t_0, t_1], [t_1, t_2], \ldots, [t_k, t_{k+1}], \ldots$, replace each interval $[t_k, t_{k+1}]$, $k = 0, 1, \ldots$ by a set of N_{k+1} regularly spaced grid points (including the endpoints). The grid point on

Table 2

α	$\underline{y}$	$\underline{y}_{\text{Mid}}$	$\underline{Y}$	$\overline{y}$	$\overline{y}_{\text{Mid}}$	$\overline{Y}$
0	0.8636348	0.8686954	0.8636356	0.9188934	0.9138373	0.9188941
0.1	0.8677550	0.8723098	0.8677558	0.9174877	0.9129374	0.9174885
0.2	0.8718752	0.8759242	0.8718759	0.9160821	0.9120376	0.9160828
0.3	0.8759954	0.8795385	0.8759961	0.9146764	0.9111378	0.9146771
0.4	0.8801156	0.8831528	0.8801163	0.9132707	0.9102379	0.9132714
0.5	0.8842357	0.8867672	0.8842365	0.9118651	0.9093381	0.9118658
0.6	0.8883559	0.8903816	0.8883567	0.9104593	0.9084383	0.9104601
0.7	0.8924761	0.8939959	0.8924769	0.9090537	0.9075384	0.9090545
0.8	0.8965963	0.8976102	0.8965970	0.9076480	0.9066386	0.9076487
0.9	0.9007165	0.9012246	0.9007173	0.9062423	0.9057388	0.9062431
1	0.9048367	0.9048389	0.9048374	0.9048367	0.9048389	0.9048374

$[t_k, t_{k+1}]$ will be $t_{k,n} = t_k + nh_k$, $h_k = (t_{k+1} - t_k)/N_k$, $0 \le n \le N_k$ at which the exact solution $(\underline{y}^\alpha(t_{k,n}), \overline{y}^\alpha(t_{k,n}))$ will be approximated by some $(\underline{y}^\alpha_{k,n}, \overline{y}^\alpha_{k,n})$. We set $\underline{y}^\alpha_{0,0} = \underline{y}^\alpha_0$, $\overline{y}^\alpha_{0,0} = \overline{y}^\alpha_0$ and $\underline{y}^\alpha_{k,0} = \underline{y}^\alpha_{k-1,N_{k-1}}$, $\overline{y}^\alpha_{k,0} = \overline{y}^\alpha_{k-1,N_{k-1}}$ if $k \ge 1$.

According to Section 3, by similar computation we obtain the Trapezoidal rule for solving (60) as follows:

$$\begin{aligned}
&\underline{y}^\alpha_{k,n+1} \\
&\quad = \underline{y}^\alpha_{k,n} + \frac{h}{2} \\
&\qquad \times \left[\underline{f_k}^\alpha\left(t_{k,n}, \underline{y}^\alpha_{k,n}, \overline{y}^\alpha_{k,n}\right) + \underline{f_k}^\alpha\left(t_{k,n+1}, \underline{y}^\alpha_{k,n+1}, \overline{y}^\alpha_{k,n+1}\right)\right], \\
&\overline{y}^\alpha_{k,n+1} \\
&\quad = \overline{y}^\alpha_{k,n} + \frac{h}{2} \\
&\qquad \times \left[\overline{f_k}^\alpha\left(t_{k,n}, \underline{y}^\alpha_{k,n}, \overline{y}^\alpha_{k,n}\right) + \overline{f_k}^\alpha\left(t_{k,n+1}, \underline{y}^\alpha_{k,n+1}, \overline{y}^\alpha_{k,n+1}\right)\right], \\
&\qquad\qquad \underline{y}^\alpha_{k,n} = \underline{y}_k, \qquad \overline{y}^\alpha_{k,n} = \overline{y}_k,
\end{aligned} \tag{61}$$

for $0 \le n < N_k$, $k = 0, 1, 2, \ldots$.

Next, we give the algorithm to numerically solve (55) in $[t_0, t_1], [t_1, t_2], \ldots, [t_k, t_{k+1}], \ldots$.

First Step. $\{[(\underline{y}^\alpha_{0,n}, \overline{y}^\alpha_{0,n})]\}_{n=0}^{N_0}$ will be a numerical solution generated by (61) for $k = 0$ as follows:

$$\begin{aligned}
\left(\underline{y}^\alpha_0(t)\right)' &= \underline{f_0}^\alpha\left(t, \underline{y}^\alpha_0(t), \overline{y}^\alpha_0(t)\right), \\
\left(\overline{y}^\alpha_0(t)\right)' &= \overline{f_0}^\alpha\left(t, \underline{y}^\alpha_0(t), \overline{y}^\alpha_0(t)\right), \\
\underline{y}^\alpha_0(t_0) &= \underline{y}^\alpha_{0,0}, \qquad \overline{y}^\alpha_0(t_0) = \overline{y}^\alpha_{0,0}.
\end{aligned} \tag{62}$$

$\{[(\underline{y}^\alpha_{0,n}, \overline{y}^\alpha_{0,n})]\}_{n=0}^{N_0}$ is a numerical solution of (60) over $[t_0, t_1]$.

Second Step. For each $k \ge 1$, $\{[(\underline{y}^\alpha_{k,n}, \overline{y}^\alpha_{k,n})]\}_{n=0}^{N_k}$ will be numerical solution generated by (61) for

$$\begin{aligned}
\left(\underline{y}^\alpha_k(t)\right)' &= \underline{f_k}^\alpha\left(t, \underline{x}^\alpha_k(t), \overline{x}^\alpha_k(t)\right), \\
\left(\overline{y}^\alpha_k(t)\right)' &= \overline{f_k}^\alpha\left(t, \underline{x}^\alpha_k(t), \overline{x}^\alpha_k(t)\right), \\
\underline{y}^\alpha_k(t_k) &= \underline{y}^\alpha_{k,0}, \qquad \overline{y}^\alpha_k(t_k) = \overline{y}^\alpha_{k,0},
\end{aligned} \tag{63}$$

where $\underline{y}^\alpha_{k,0} = \underline{y}_{k-1,N^\alpha_{k-1}}$, $\overline{y}^\alpha_{k,0} = \underline{y}_{k-1,N^\alpha_{k-1}}$. $\{[(\underline{y}^\alpha_{k,n}, \overline{y}^\alpha_{k,n})]\}_{n=0}^{N_k}$ is a numerical solution of (60) over $[t_k, t_{k+1}]$ for each $k \ge 1$.

For arbitrary fixed $\alpha \in [0,1]$ and k, we can prove that the numerical solution of (55) converges to the exact solution; that is,

$$\lim_{h_0,\ldots,h_k \to 0} \underline{y}^\alpha_{k,N_k} = \underline{y}(t_{k+1}), \qquad \lim_{h_0,\ldots,h_k \to 0} \overline{y}^\alpha_{k,N_k} = \overline{y}(t_{k+1}). \tag{64}$$

The Trapezoidal rule is a one-step method as the Euler method. Therefore, the proof of the convergence closely follows the idea of the proof of Theorem 3.2 in [18] and Theorem 4.1 in [19].

Theorem 18. *Consider the system of* (55). *Suppose for some fixed k and* $\alpha \in [0,1]$ *that* $\{[(\underline{y}^\alpha_{i,n_i}, \overline{y}^\alpha_{i,n_i})]\}_{i=0}^{k}$, *where* $0 \le n_i \le N_i$ *is obtained by* (61). *Then*

$$\lim_{h_0,\ldots,h_k \to 0} \underline{y}^\alpha_{k,N_k} = \underline{y}(t_{k+1}), \qquad \lim_{h_0,\ldots,h_k \to 0} \overline{y}^\alpha_{k,N_k} = \overline{y}(t_{k+1}). \tag{65}$$

Proof. See [19]. □

Example 19. Consider the following hybrid fuzzy system:

$$\begin{aligned}
y'(t) &= y(t) + m(t)\lambda_k(y(t_k)), \quad t \in [t_k, t_{k+1}], \\
&\qquad t_k = k, \; k = 0, 1, 2, \ldots, \\
y(0) &= \gamma,
\end{aligned} \tag{66}$$

Table 3

α	$\underline{y}$	$\underline{y}_{\mathrm{Mid}}$	$\underline{Y}$	$\overline{y}$	$\overline{y}_{\mathrm{Mid}}$	$\overline{Y}$
0	7.2644238	7.2370696	7.2577319	10.8966360	10.8556042	10.8865976
0.1	7.5065713	7.4783049	7.4996562	10.7755623	10.7349863	10.7656355
0.1	7.7487187	7.7195406	7.7415805	10.6544886	10.6143684	10.6446733
0.1	7.9908662	7.9607763	7.9835048	10.5334148	10.4937506	10.5237112
0.1	8.2330141	8.2020121	8.2254295	10.4123411	10.3731327	10.4027491
0.1	8.4751616	8.4432478	8.4673538	10.2912674	10.2525148	10.2817869
0.1	8.7173090	8.6844835	8.7092781	10.1701937	10.1318970	10.1608248
0.1	8.9594564	8.9257193	8.9512024	10.0491199	10.0112791	10.0398626
0.1	9.2016039	9.1669550	9.1931267	9.9280462	9.8906612	9.9189005
0.1	9.4437513	9.4081898	9.4350510	9.8069725	9.7700434	9.7979374
1	9.6858988	9.6494255	9.6769753	9.6858988	9.6494255	9.6769753

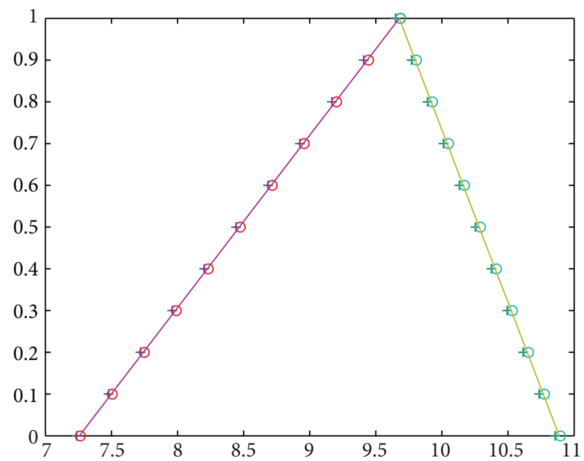

Figure 3: (-) Exact solution, (∘) Trapezoidal, and (+) Midpoint approximated points.

where γ is a triangular fuzzy number having α-level sets $[\gamma]^\alpha = [0.75 + 0.25\alpha, 1.125 - 0.125\alpha]$,

$$m(t) = \begin{cases} 2(t \,(\mathrm{mod}\, 1)), & \text{if } t \,(\mathrm{mod}\, 1) \le 0.5, \\ 2(1 - t \,(\mathrm{mod}\, 1)), & \text{if } t \,(\mathrm{mod}\, 1) > 0.5, \end{cases}$$
$$\lambda_k(\mu) = \begin{cases} \widehat{0}, & \text{if } k = 0, \\ \mu, & \text{if } k \in \{1, 2, \ldots\}. \end{cases} \tag{67}$$

By [19, Example 1], we know (66) has a unique solution and the exact solution on $[0, 2]$ is given by

$$[y(t)]^\alpha = \left[(0.75 + 0.25\alpha)e^t, (1.125 - 0.125\alpha)e^t\right], \quad t \in [0, 1],$$
$$[y(t)]^\alpha = \begin{cases} y(1)\left(3e^{t-1} - 2t\right), & t \in [1, 1.5], \\ y(1)\left(2t - 2 + e^{t-1.5}\left(3\sqrt{e} - 4\right)\right), & t \in [1.5, 2]. \end{cases} \tag{68}$$

To numerically solve the hybrid fuzzy initial value problem (66) we apply the Trapezoidal rule for hybrid fuzzy differential equations.

A comparison between the exact solution and the approximate solutions by Midpoint method and Trapezoidal method at $t = 2$ with $N = 10$ is shown in Table 3 and Figure 3.

5. Conclusion

We have presented Trapezoidal rule for numerical solution of first-order fuzzy differential equations and hybrid fuzzy differential equations. Also convergence and stability of the method are studied. To illustrate the efficiency of the new method, we have compared our method with the Midpoint rule in some examples. We have shown the global error in Trapezoidal rule is much less than in Midpoint rule.

For future research, we will apply Trapezoidal rule to fuzzy differential equations and hybrid fuzzy differential equations under generalized Hukuhara differentiability. Also one can apply Trapezoidal rule and Midpoint rule as a predictor-corrector method to solve FDE and HFDE.

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References

[1] M. L. Puri and D. A. Ralescu, "Differentials of fuzzy functions," *Journal of Mathematical Analysis and Applications*, vol. 91, no. 2, pp. 552–558, 1983.

[2] O. Kaleva, "Fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 301–317, 1987.

[3] O. Kaleva, "The Cauchy problem for fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 35, no. 3, pp. 389–396, 1990.

[4] S. Seikkala, "On the fuzzy initial value problem," *Fuzzy Sets and Systems*, vol. 24, no. 3, pp. 319–330, 1987.

[5] P. E. Kloeden, "Remarks on Peano-like theorems for fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 44, no. 1, pp. 161–163, 1991.

[6] B. Bede, I. J. Rudas, and A. L. Bencsik, "First order linear fuzzy differential equations under generalized differentiability," *Information Sciences*, vol. 177, no. 7, pp. 1648–1662, 2007.

[7] J. J. Nieto, A. Khastan, and K. Ivaz, "Numerical solution of fuzzy differential equations under generalized differentiability," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 700–707, 2009.

[8] A. Khastan, J. J. Nieto, and R. Rodríguez-López, "Variation of constant formula for first order fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 177, pp. 20–33, 2011.

[9] S. Abbasbandy, T. Allahviranloo, and P. Darabi, "Numerical soluion of *n*th-order fuzzy differential equations by Runge-Kutta method," *Journal of Mathematical and Computational Applications*, vol. 16, no. 4, pp. 935–946, 2011.

[10] T. Allahviranloo, N. Ahmady, and E. Ahmady, "Numerical solution of fuzzy differential equations by predictor-corrector method," *Information Sciences*, vol. 177, no. 7, pp. 1633–1647, 2007.

[11] M. Friedman, M. Ma, and A. Kandel, "Numerical solutions of fuzzy differential and integral equations," *Fuzzy Sets and Systems*, vol. 106, no. 1, pp. 35–48, 1999.

[12] B. Ghazanfari and A. Shakerami, "Numerical solutions of fuzzy differential equations by extended Runge-Kutta-like formulae of order 4," *Fuzzy Sets and Systems*, vol. 189, pp. 74–91, 2012.

[13] A. Khastan and K. Ivaz, "Numerical solution of fuzzy differential equations by Nyström method," *Chaos, Solitons & Fractals*, vol. 41, no. 2, pp. 859–868, 2009.

[14] H. Kim and R. Sakthivel, "Numerical solution of hybrid fuzzy differential equations using improved predictor-corrector method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 10, pp. 3788–3794, 2012.

[15] M. Ma, M. Friedman, and A. Kandel, "Numerical solutions of fuzzy differential equations," *Fuzzy Sets and Systems*, vol. 105, no. 1, pp. 133–138, 1999.

[16] N. Parandin, "Numerical solution of fuzzy differential equations of *n*th-orderby Runge-Kutta method," *Neural Computing and Applications*, vol. 21, no. 1, Supplement, pp. 347–355, 2012.

[17] P. Prakash and V. Kalaiselvi, "Numerical solution of hybrid fuzzy differential equations by predictor-corrector method," *International Journal of Computer Mathematics*, vol. 86, no. 1, pp. 121–134, 2009.

[18] S. Pederson and M. Sambandham, "Numerical solution to hybrid fuzzy systems," *Mathematical and Computer Modelling*, vol. 45, no. 9-10, pp. 1133–1144, 2007.

[19] S. Pederson and M. Sambandham, "Numerical solution of hybrid fuzzy differential equation IVPs by a characterization theorem," *Information Sciences*, vol. 179, no. 3, pp. 319–328, 2009.

[20] J. Li, A. Zhao, and J. Yan, "The Cauchy problem of fuzzy differential equations under generalized differentiability," *Fuzzy Sets and Systems*, vol. 200, pp. 1–24, 2012.

[21] M. T. Malinowski, "Random fuzzy differential equations under generalized Lipschitz condition," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 2, pp. 860–881, 2012.

[22] J. Xu, Z. Liao, and J. J. Nieto, "A class of linear differential dynamical systems with fuzzy matrices," *Journal of Mathematical Analysis and Applications*, vol. 368, no. 1, pp. 54–68, 2010.

[23] P. Henrici, *Discrete Variable Methods in Ordinary Differential Equations*, John Wiley & Sons, New York, NY, USA, 1962.

[24] P. Diamond and P. Kloeden, *Metric Spaces of Fuzzy Sets*, World Scientific, Singapore, 1994.

[25] O. Kaleva, "Interpolation of fuzzy data," *Fuzzy Sets and Systems*, vol. 61, no. 1, pp. 63–70, 1994.

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Research Article

A Note on Parabolic Homogenization with a Mismatch between the Spatial Scales

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We consider the homogenization of the linear parabolic problem $\rho(x/\varepsilon_2)\partial_t u^\varepsilon(x,t) - \nabla\cdot(a(x/\varepsilon_1, t/\varepsilon_1^2)\nabla u^\varepsilon(x,t)) = f(x,t)$ which exhibits a mismatch between the spatial scales in the sense that the coefficient $a(x/\varepsilon_1, t/\varepsilon_1^2)$ of the elliptic part has one frequency of fast spatial oscillations, whereas the coefficient $\rho(x/\varepsilon_2)$ of the time derivative contains a faster spatial scale. It is shown that the faster spatial microscale does not give rise to any corrector term and that there is only one local problem needed to characterize the homogenized problem. Hence, the problem is not of a reiterated type even though two rapid scales of spatial oscillation appear.

1. Introduction

The field of homogenization has its main source of inspiration in the problem of finding the macroscopic properties of strongly heterogeneous materials. Mathematically, the approach is to study a sequence of partial differential equations where a parameter ε associated with the length scales of the heterogeneities tends to zero. The sequence of solutions u^ε converges to the solution u to a so-called homogenized problem governed by a coefficient b, where b gives the effective property of the material and can be characterized by certain auxiliary problems called the local problems.

In this paper, we study the homogenization of the linear parabolic problem

$$\begin{gathered}\rho\left(\frac{x}{\varepsilon_2}\right)\partial_t u^\varepsilon(x,t) - \nabla\cdot\left(a\left(\frac{x}{\varepsilon_1},\frac{t}{\varepsilon_1^2}\right)\nabla u^\varepsilon(x,t)\right)\\ = f(x,t) \quad \text{in } \Omega_T,\\ u^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T),\\ u^\varepsilon(x,0) = u^0(x) \quad \text{in } \Omega,\end{gathered} \tag{1}$$

where $\Omega_T = \Omega\times(0,T)$, Ω is an open, bounded set in $\mathbb{R}^N$ with locally Lipschitz boundary, where both a and ρ possess unit periodicity in their respective arguments and the scales ε_1, ε_2, and ε_1^2 fulfill a certain separatedness assumption.

The problem exhibits rapid spatial oscillations in ρ and spatial as well as temporal oscillations in a. Furthermore, there is a "mismatch" between the spatial scales in the sense that the frequency of the spatial oscillations in $\rho(x/\varepsilon_2)$ is higher than that of $a(x/\varepsilon_1, t/\varepsilon_1^2)$. Since there are two spatial microscales represented in (1), one might expect two local problems with respect to one corrector each, see, for example, [1]. However, it is shown that no corrector corresponding to the scale emanating from $\rho(x/\varepsilon_2)$ appears in the local and homogenized problem and accordingly there is only one local problem appearing in the formulated theorem. Hence, the problem is not of a reiterated type. We prove by means of very weak multiscale convergence [2] that the corrector u_2 associated with the gradient for the second rapid spatial scale y_2 actually vanishes. Already, in [3, 4], it was observed that having more than one rapid temporal scale in parabolic problems does not generate a reiterated problem and in this paper we can see that nor does the addition of a spatial scale if it is contained in a coefficient that is multiplied with the time derivative of u^ε.

Thinking in terms of heat conduction, our result means that the heat capacity ρ may oscillate with completely different periodic patterns without making any difference for the homogenized coefficient as long as the arithmetic mean over one period is the same.

Parabolic homogenization problems for $\rho \equiv 1$ have been studied for different combinations of spatial and temporal scales in several papers by means of techniques of two-scale convergence type with approaches related to the one first introduced in [5], see, for example, [2, 3, 6–8], and in, for example, [9–11], techniques not of two-scale convergence type are applied. Concerning cases where, as in (1) above, we do not have $\rho \equiv 1$, Nandakumaran and Rajesh [12] studied a nonlinear parabolic problem with the same frequency of oscillation in time and space, respectively, in the elliptic part of the equation and an operator oscillating in space with the same frequency appearing in the temporal differentiation term. Recently, a number of papers have addressed various kinds of related problems where the temporal scale is not assumed to be identical with the spatial scale, see for example, [13, 14]. Up to the authors' knowledge, this is the first study of this type of problems where the oscillations of the coefficient of the term including the time derivative do not match the spatial oscillations of the elliptic part.

Notation. We denote $Y_k = (0,1)^N$ for $k = 1,\ldots,n$, $Y^n = Y_1 \times \cdots \times Y_n$, $y^n = (y_1,\ldots,y_n)$, $dy^n = dy_1 \cdots dy_n$, $S_j = S = (0,1)$ for $j = 1,\ldots,m$, $S^m = S_1 \times \cdots \times S_m$, $s^m = (s_1,\ldots,s_m)$, and $ds^m = ds_1 \cdots ds_m$. Let $\varepsilon_k(\varepsilon)$, $k = 1,\ldots,n$, and $\varepsilon'_j(\varepsilon)$, $j = 1,\ldots,m$, be positive and go to zero when ε does. Furthermore, let $F_\sharp((0,1)^M)$ be the space of all functions in $F_{\text{loc}}(\mathbb{R}^M)$ that are $(0,1)^M$-periodic repetitions of some function in $F((0,1)^M)$.

2. Multiscale Convergence

A two-scale convergence was invented by Nguetseng [15] as a new approach for the homogenization of problems with fast oscillations in one scale in space. The method was further developed by Allaire [16] and generalized to multiple scales by Allaire and Briane [1]. To homogenize problem (1), we use the further generalization in the definition below, adapted to evolution settings, see, for example, [8].

Definition 1. A sequence $\{u^\varepsilon\}$ in $L^2(\Omega_T)$ is said to $(n+1, m+1)$-scale converge to $u_0 \in L^2(\Omega_T \times Y^n \times S^m)$ if

$$\begin{aligned}\int_{\Omega_T} u^\varepsilon(x,t)\, v\left(x, t, \frac{x}{\varepsilon_1}, \ldots, \frac{x}{\varepsilon_n}, \frac{t}{\varepsilon'_1}, \ldots, \frac{t}{\varepsilon'_m}\right) dx\, dt \\ \longrightarrow \int_{\Omega_T}\int_{Y^n}\int_{S^m} u_0(x,t,y^n,s^m) \\ \times v(x,t,y^n,s^m)\, dy^n ds^m dx\, dt,\end{aligned} \tag{2}$$

for any $v \in L^2(\Omega_T; C_\sharp(Y^n \times S^m))$. We write

$$u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} u_0(x,t,y^n,s^m). \tag{3}$$

Usually, some assumptions are made about how the scales are related to each other. We say that the scales in a list $\{\varepsilon_1,\ldots,\varepsilon_n\}$ are separated if

$$\lim_{\varepsilon\to 0}\frac{\varepsilon_{k+1}}{\varepsilon_k} = 0, \tag{4}$$

for $k = 1,\ldots,n-1$ and that the scales are well-separated if there exists a positive integer l such that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon_k}\left(\frac{\varepsilon_{k+1}}{\varepsilon_k}\right)^l = 0, \tag{5}$$

for $k = 1,\ldots,n-1$.

The concept in the following definition is used as an assumption in the proofs of the compactness results in Theorems 3 and 7. For a more technically formulated definition and some examples, see [17, Section 2.4].

Definition 2. Let $\{\varepsilon_1,\ldots,\varepsilon_n\}$ and $\{\varepsilon'_1,\ldots,\varepsilon'_m\}$ be lists of well-separated scales. Consider all elements from both lists. If from possible duplicates, where by duplicates we mean scales which tend to zero equally fast, one member of each pair is removed and the list in order of magnitude of all the remaining elements is well separated, the lists $\{\varepsilon_1,\ldots,\varepsilon_n\}$ and $\{\varepsilon'_1,\ldots,\varepsilon'_m\}$ are said to be jointly well separated.

In the theorem below, which will be used in the homogenization procedure in Section 3, $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$ denotes all functions $u \in L^2(0,T;H_0^1(\Omega))$ such that $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$, see, for example, [18, Chapter 23].

Theorem 3. *Let $\{u^\varepsilon\}$ be a bounded sequence in $W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$, and suppose that the lists $\{\varepsilon_1,\ldots,\varepsilon_n\}$ and $\{\varepsilon'_1,\ldots,\varepsilon'_m\}$ are jointly well separated. Then there exists a subsequence such that*

$$\begin{aligned} u^\varepsilon(x,t) &\longrightarrow u(x,t) \quad \text{in } L^2(\Omega_T), \\ u^\varepsilon(x,t) &\rightharpoonup u(x,t) \quad \text{in } L^2\left(0,T;H_0^1(\Omega)\right),\end{aligned} \tag{6}$$

$$\nabla u^\varepsilon(x,t) \overset{n+1,m+1}{\rightharpoonup} \nabla u(x,t) + \sum_{j=1}^n \nabla_{y_j} u_j\left(x,t,y^j,s^m\right), \tag{7}$$

where $u \in W_2^1(0,T;H_0^1(\Omega),L^2(\Omega))$, $u_1 \in L^2(\Omega_T \times S^m; H_\sharp^1(Y_1)/\mathbb{R})$, and $u_j \in L^2(\Omega_T \times Y^{j-1} \times S^m; H_\sharp^1(Y_j)/\mathbb{R})$ for $j = 2,\ldots,n$.

Proof. See [17, Theorem 2.74]. □

To treat evolution problems with fast time oscillations, such as (1), we also need the concept of very weak multiscale convergence, see, for example, [2, 5].

Definition 4. A sequence $\{g^\varepsilon\}$ in $L^1(\Omega_T)$ is said to $(n+1, m+1)$-scale converge very weakly to $g_0 \in L^1(\Omega_T \times Y^n \times S^m)$ if

$$\int_{\Omega_T} g^\varepsilon(x,t)\, v\left(x, \frac{x}{\varepsilon_1}, \dots, \frac{x}{\varepsilon_{n-1}}\right) \times c\left(t, \frac{t}{\varepsilon_1'}, \dots, \frac{t}{\varepsilon_m'}\right) \varphi\left(\frac{x}{\varepsilon_n}\right) dx\, dt \longrightarrow \int_{\Omega_T}\int_{Y^n}\int_{S^m} g_0(x,t,y^n,s^m)\, v(x,y^{n-1}) \times c(t,s^m)\,\varphi(y_n)\, dy^n ds^m dx\, dt, \tag{8}$$

for any $v \in D(\Omega, C_\sharp^\infty(Y^{n-1}))$, $c \in D(0,T; C_\sharp^\infty(S^m))$, and $\varphi \in C_\sharp^\infty(Y_n)/\mathbb{R}$, where

$$\int_{Y_n} g_0(x,t,y^n,s^m)\, dy_n = 0. \tag{9}$$

We write

$$g^\varepsilon(x,t) \underset{vw}{\overset{n+1,m+1}{\rightharpoonup}} g_0(x,t,y^n,s^m). \tag{10}$$

Remark 5. The requirement (9) is imposed in order to ensure the uniqueness of the limit. For details, see [17, Proposition 2.26].

Remark 6. The convergence in Definition 1 may take place only if $\{u^\varepsilon\}$ is bounded in $L^2(\Omega_T)$ and hence also is a weakly convergent in $L^2(\Omega_T)$, at least up to suitable subsequences. For very weak multiscale convergence, this is not so. The main intention with the concept is to study sequences of the type $\{u^\varepsilon/\varepsilon_n\}$, which are in general not bounded in $L^2(\Omega_T)$. This requires a more restrictive class of test functions.

The theorem below is a key result for the homogenization procedure in Section 3.

Theorem 7. *Let $\{u^\varepsilon\}$ be a bounded sequence in $W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$, and assume that the lists $\{\varepsilon_1, \dots, \varepsilon_n\}$ and $\{\varepsilon_1', \dots, \varepsilon_m'\}$ are jointly well separated. Then there exists a subsequence such that*

$$\frac{u^\varepsilon(x,t)}{\varepsilon_n} \underset{vw}{\overset{n+1,m+1}{\rightharpoonup}} u_n(x,t,y^n,s^m), \tag{11}$$

where, for $n = 1$, $u_1 \in L^2(\Omega_T \times S^m; H_\sharp^1(Y_1)/\mathbb{R})$ and, for $n = 2, 3, \dots$, $u_n \in L^2(\Omega_T \times Y^{n-1} \times S^m; H_\sharp^1(Y_n)/\mathbb{R})$ are the same as those in Theorem 3.

Proof. See [17, Theorem 2.54]. □

Remark 8. For a sequence of solutions $\{u^\varepsilon\}$ to (1), we may replace the requirement that $\{\partial_t u^\varepsilon\}$ should be bounded in $L^2(0,T; H^{-1}(\Omega))$ by the assumption that $\{u^\varepsilon\}$ is bounded in $L^\infty(\Omega_T)$ and still obtain (6), see [12, Lemmas 3.3 and (4.1)] and thereby also (7) and (11). The only difference is that u will belong to $L^2(0,T; H_0^1(\Omega))$ instead of the space $W_2^1(0,T; H_0^1(\Omega), L^2(\Omega))$. See also [13].

3. Homogenization

Let us now investigate the heat conduction problem

$$\begin{gathered}\rho\left(\frac{x}{\varepsilon_2}\right)\partial_t u^\varepsilon(x,t) - \nabla\cdot\left(a\left(\frac{x}{\varepsilon_1}, \frac{t}{\varepsilon_1^2}\right)\nabla u^\varepsilon(x,t)\right) = f(x,t) \quad \text{in } \Omega_T,\\ u^\varepsilon(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T),\\ u^\varepsilon(x,0) = u^0(x) \quad \text{in } \Omega,\end{gathered} \tag{12}$$

which takes into consideration heat capacity oscillations. We assume that $\rho \in C_\sharp^\infty(Y_2)$, is positive, $f \in L^2(\Omega_T)$, $u^0 \in L^2(\Omega)$, and

$$a(y_1,s)\,\xi\cdot\xi \ge \alpha|\xi|^2 \tag{13}$$

for some $\alpha > 0$, all $(y_1,s) \in \mathbb{R}^{N+1}$, and all $\xi \in \mathbb{R}^N$, where $a \in C_\sharp(Y_1 \times S)^{N\times N}$. Moreover, we assume that $\{u^\varepsilon\}$ is bounded in $L^\infty(\Omega_T)$, see Remark 8, and that the lists $\{\varepsilon_1, \varepsilon_2\}$ and $\{\varepsilon_1^2\}$ are jointly well separated. Note that this separatedness assumption implies, for example, that ε_2 tends to zero faster than ε_1, which means that we have a mismatch between the spatial scales in (12).

We give a homogenization result for this problem in the theorem below. In the proof, it is shown that the local problem associated with the slower spatial microscale is enough to characterize the homogenized problem; that is, the fastest spatial scale does not give rise to any corrector involved in the homogenization. We also prove that the second corrector u_2 actually vanishes.

Theorem 9. *Let $\{u^\varepsilon\}$ be a sequence of solutions to (12). Then*

$$u^\varepsilon(x,t) \rightharpoonup u(x,t) \quad \text{in } L^2\left(0,T; H_0^1(\Omega)\right), \tag{14}$$

$$\nabla u^\varepsilon(x,t) \overset{3,2}{\rightharpoonup} \nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s) + \nabla_{y_2} u_2(x,t,y^2,s), \tag{15}$$

where u is the unique solution to

$$\begin{gathered}\left(\int_{Y_2}\rho(y_2)\,dy_2\right)\partial_t u(x,t) - \nabla\cdot(b\nabla u(x,t)) = f(x,t) \quad \text{in } \Omega_T,\\ u(x,t) = 0 \quad \text{on } \partial\Omega\times(0,T),\\ u(x,0) = u^0(x) \quad \text{in } \Omega,\end{gathered} \tag{16}$$

with

$$b\nabla u(x,t) = \int_S\int_{Y_1} a(y_1,s)\left(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s)\right) dy_1 ds. \tag{17}$$

Here, $u_1 \in L^2(\Omega_T \times S; H^1_\sharp(Y_1)/\mathbb{R})$ *uniquely solves*

$$\left(\int_{Y_2} \rho(y_2)\, dy_2\right) \partial_s u_1(x,t,y_1,s) - \nabla_{y_1} \cdot \left(a(y_1,s)\left(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s)\right)\right) = 0. \tag{18}$$

Furthermore, the corrector u_2 vanishes.

Remark 10. After a separation of variables, we can write the local problem as

$$\left(\int_{Y_2} \rho(y_2)\, dy_2\right) \partial_s z_k(y_1,s) - \sum_{i,j=1}^{N} \partial_{y_i}\left(a_{ij}(y_1,s)\left(\delta_{jk} + \partial_{y_j} z_k(y_1,s)\right)\right) = 0 \tag{19}$$

and the homogenized coefficient as

$$b_{ik} = \int_S \int_{Y_1} \sum_{j=1}^{N} \left(a_{ij}(y_1,s)\left(\delta_{jk} + \partial_{y_j} z_k(y_1,s)\right)\right) dy_1\, ds, \tag{20}$$

where $k = 1, \ldots, N$ and

$$u_1(x,t,y_1,s) = \sum_{k=1}^{N} \partial_{x_k} u(x,t) \cdot z_k(y_1,s). \tag{21}$$

Remark 11. Periodic homogenization problems of, for example, elliptic or parabolic type may be seen as special cases of the more general concepts of G-convergence, which gives a characterization of the limit problem but no suggestion of how to compute the homogenized matrix. Essential features of G-convergence for parabolic problems are that boundary conditions, and initial conditions are preserved in the limit. G-convergence for linear parabolic problems were studied already in [19] by Spagnolo and extended to the monotone case by Svanstedt in [20]. A treatment of this problem in a quite general setting is found in the recent work [21] by Paronetto.

Proof of Theorem 9. Following the procedure in Section 23.9 in [18], we obtain that $\{u^\varepsilon\}$ is bounded in $L^2(0,T; H^1_0(\Omega))$, see also [22]. Hence, (14) holds up to a subsequence. We proceed by studying the weak form of (12); that is,

$$\int_{\Omega_T} -\rho\left(\frac{x}{\varepsilon_2}\right) u^\varepsilon(x,t)\, v(x)\, \partial_t c(t) + a\left(\frac{x}{\varepsilon_1}, \frac{t}{\varepsilon_1^2}\right) \nabla u^\varepsilon(x,t)\, \nabla v(x)\, c(t)\, dx\, dt = \int_{\Omega_T} f(x,t)\, v(x)\, c(t)\, dx\, dt, \tag{22}$$

for all $v \in H^1_0(\Omega)$ and $c \in D(0,T)$. We pass to the limit by applying (6), taking into consideration Remark 8, and (7) with $n = 1$ and $m = 1$ and arrive at the homogenized problem

$$\begin{aligned}\int_{\Omega_T} \int_S \int_{Y_1} & -\left(\int_{Y_2} \rho(y_2)\, dy_2\right) u(x,t)\, v(x)\, \partial_t c(t) \\ & + a(y_1,s)\left(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s)\right) \\ & \times \nabla v(x)\, c(t)\, dy_1 ds\, dx\, dt \\ = \int_{\Omega_T} & f(x,t)\, v(x)\, c(t)\, dx\, dt.\end{aligned} \tag{23}$$

To find the local problem associated with u_1, let us again consider (22) in which we choose

$$v(x) = \varepsilon_1 v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right); \quad v_1 \in D(\Omega),\ v_2 \in \frac{C^\infty_\sharp(Y_1)}{\mathbb{R}}, \tag{24}$$

$$c(t) = c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right); \quad c_1 \in D(0,T);\ c_2 \in C^\infty_\sharp(S); \tag{25}$$

that is, we study

$$\begin{aligned}\int_{\Omega_T} & -\rho\left(\frac{x}{\varepsilon_2}\right) u^\varepsilon(x,t)\, v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) \\ & \times \left(\varepsilon_1 \partial_t c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) + \varepsilon_1^{-1} c_1(t)\, \partial_s c_2\left(\frac{t}{\varepsilon_1^2}\right)\right) \\ & + a\left(\frac{x}{\varepsilon_1}, \frac{t}{\varepsilon_1^2}\right) \nabla u^\varepsilon(x,t) \\ & \cdot \left(\varepsilon_1 \nabla v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) + v_1(x)\, \nabla_{y_1} v_2\left(\frac{x}{\varepsilon_1}\right)\right) \\ & \times c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\, dt \\ = \int_{\Omega_T} & f(x,t)\, \varepsilon_1 v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\, dt.\end{aligned} \tag{26}$$

We first investigate the second term of the part of the expression containing time derivatives. We have

$$\int_{\Omega_T} -\rho\left(\frac{x}{\varepsilon_2}\right) u^\varepsilon(x,t)\, v_1(x)\, v_2 \times \left(\frac{x}{\varepsilon_1}\right) \varepsilon_1^{-1} c_1(t)\, \partial_s c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\, dt$$

$$= \int_{\Omega_T} -\varepsilon_1^{-1} u^\varepsilon (x,t)\, v_1(x)\, v_2 \times \left(\frac{x}{\varepsilon_1}\right)\left(\rho\left(\frac{x}{\varepsilon_2}\right) - \int_{Y_2} \rho(y_2)\, dy_2\right) \times c_1(t)\, \partial_s c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\, dt$$
$$+ \int_{\Omega_T} -\varepsilon_1^{-1} u^\varepsilon (x,t)\, v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) \times \left(\int_{Y_2} \rho(y_2)\, dy_2\right) c_1(t)\, \partial_s c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\, dt$$
$$\longrightarrow \int_{\Omega_T}\int_S\int_{Y_1} -\left(\int_{Y_2} \rho(y_2)\, dy_2\right) u_1(x,t,y_1,s) \times v_1(x)\, v_2(y_1)\, c_1(t) \times \partial_s c_2(s)\, dy_1\, ds\, dx\, dt, \tag{27}$$

where we have applied (11) with $n = 2$ and $m = 1$ and with $n = 1$ and $m = 1$, respectively, in the last step. The passage to the limit in the remaining part of (26) is a simple application of (7) with $n = 1$ and $m = 1$. This provides us with the weak form,

$$\int_{\Omega_T}\int_S\int_{Y_1} -\left(\int_{Y_2} \rho(y_2)\, dy_2\right) u_1(x,t,y_1,s) \times v_1(x)\, v_2(y_1)\, c_1(t)\, \partial_s c_2(s) + a(y_1,s)\left(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s)\right) \cdot v_1(x)\, \nabla_{y_1} v_2(y_1)\, c_1(t)\, c_2(s)\, dy_1\, ds\, dx\, dt = 0, \tag{28}$$

of the local problem (18). This means that u_1, and thus also u, is uniquely determined and hence the entire sequence $\{u^\varepsilon\}$ converges and not just the extracted subsequence.

This far, we have only used test functions oscillating with a period ε_1, and hence we have not given the coefficient $\rho(x/\varepsilon_2)$ a fair chance to produce a second corrector u_2. In order to do so, we use a slightly different set of test functions in (22). Again, we let c be as in (25), whereas v is chosen according to

$$v(x) = \varepsilon_2 v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) \tilde{v}\left(\frac{x}{\varepsilon_2}\right); \quad v_1 \in D(\Omega), \quad v_2 \in C_\sharp^\infty(Y_1), \tag{29}$$

where

$$\tilde{v}(y_2) = v_3(y_2) - \frac{K}{\rho(y_2)}; \quad v_3 \in C_\sharp^\infty(Y_2), \tag{30}$$

with

$$K = \int_{Y_2} \rho(y_2)\, v_3(y_2)\, dy_2. \tag{31}$$

Note that

$$\int_{Y_2} \rho(y_2)\, \tilde{v}(y_2)\, dy_2 = 0, \tag{32}$$

which means that $\rho\tilde{v} \in C_\sharp^\infty(Y_2)/\mathbb{R}$. We get

$$\int_{\Omega_T} -\rho\left(\frac{x}{\varepsilon_2}\right) u^\varepsilon(x,t)\, v_1(x)\, v_2\left(\frac{x}{\varepsilon}\right) \tilde{v}\left(\frac{x}{\varepsilon_2}\right) \times \left(\varepsilon_2 \partial_t c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) + \frac{\varepsilon_2}{\varepsilon_1^2} c_1(t)\, \partial_s c_2\left(\frac{t}{\varepsilon_1^2}\right)\right)$$
$$+ a\left(\frac{x}{\varepsilon_1}, \frac{t}{\varepsilon_1^2}\right) \nabla u^\varepsilon(x,t) \cdot \left(\varepsilon_2 \nabla v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) \tilde{v}\left(\frac{x}{\varepsilon_2}\right) + \frac{\varepsilon_2}{\varepsilon_1} v_1(x)\, \nabla_{y_1} v_2\left(\frac{x}{\varepsilon_1}\right) \tilde{v}\left(\frac{x}{\varepsilon_2}\right) + v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) \nabla_{y_2} \tilde{v}\left(\frac{x}{\varepsilon_2}\right)\right)$$
$$\times c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\, dt = \int_{\Omega_T} f(x,t)\, \varepsilon_2 v_1(x)\, v_2\left(\frac{x}{\varepsilon_1}\right) \times \tilde{v}\left(\frac{x}{\varepsilon_2}\right) c_1(t)\, c_2\left(\frac{t}{\varepsilon_1^2}\right) dx\, dt, \tag{33}$$

and applying (11) with $n = 2$ and $m = 1$ together with (15), that is, (7) with $n = 2$ and $m = 1$, we achieve

$$\int_{\Omega_T}\int_S\int_{Y^2} a(y_1,s) \times \left(\nabla u(x,t) + \nabla_{y_1} u_1(x,t,y_1,s) + \nabla_{y_2} u_2(x,t,y^2,s)\right) \cdot v_1(x)\, v_2(y_1)\, \nabla_{y_2} \tilde{v}(y_2)\, c_1(t)\, c_2(s)\, dy^2\, ds\, dx\, dt = 0. \tag{34}$$

Noting that a, u, and u_1 are all independent of y_2, (34) reduces to

$$\int_{\Omega_T}\int_S\int_{Y^2} a(y_1,s)\, \nabla_{y_2} u_2(x,t,y^2,s) \cdot v_1(x)\, v_2(y_1)\, \nabla_{y_2} \tilde{v}(y_2)\, c_1(t)\, c_2(s)\, dy^2\, ds\, dx\, dt = 0. \tag{35}$$

Recalling (30), we have

$$\int_{\Omega_T}\int_S\int_{Y^2} a(y_1,s)\, \nabla_{y_2} u_2(x,t,y^2,s) \cdot v_1(x)\, v_2(y_1)\, \nabla_{y_2}\left(v_3(y_2) - \frac{K}{\rho(y_2)}\right) \times c_1(t)\, c_2(s)\, dy^2\, ds\, dx\, dt = 0, \tag{36}$$

which after rearranging can be written as

$$\int_{\Omega_T}\int_S\int_{Y^2} a(y_1,s)\nabla_{y_2}u_2(x,t,y^2,s)\cdot v_1(x)v_2(y_1)\nabla_{y_2}v_3(y_2)c_1(t)c_2(s)\,dy^2ds\,dx\,dt$$
$$= K\int_{\Omega_T}\int_S\int_{Y_1}\left(\int_{Y_2} a(y_1,s)\nabla_{y_2}u_2(x,t,y^2,s)\cdot\nabla_{y_2}\left(\frac{1}{\rho(y_2)}\right)dy_2\right)\times v_1(x)v_2(y_1)c_1(t)c_2(s)\,dy_1ds\,dx\,dt. \tag{37}$$

If we replace $\tilde{v}$ with $1/\rho$ in (33), let $\varepsilon\to 0$, and use (6) and (7) with $n=2$ and $m=1$, we find that

$$\int_{\Omega_T}\int_S\int_{Y^2} a(y_1,s)\nabla_{y_2}u_2(x,t,y^2,s)\cdot v_1(x)v_2(y_1)\nabla_{y_2}\left(\frac{1}{\rho(y_2)}\right)c_1(t)\times c_2(s)\,dy^2ds\,dx\,dt=0. \tag{38}$$

This means that the right-hand side in (37) is zero. Applying several times the variational lemma on the remaining part, we obtain

$$\int_{Y_2} a(y_1,s)\nabla_{y_2}u_2(x,t,y^2,s)\cdot\nabla_{y_2}v_3(y_2)\,dy_2=0, \tag{39}$$

and hence the corrector u_2 is zero. □

Remark 12. That u_2 vanishes means that $u^\varepsilon/\varepsilon_2$ tends to zero in the sense of very weak $(3,2)$-scale convergence. However, there might still be oscillations originating from the oscillations of $\rho(x/\varepsilon_2)$ that have an impact on u^ε. The possibility is that their amplitude is so small that the magnification by $1/\varepsilon_2$ is not enough for the oscillations to be recognized in the limit. In this sense, the concept of very weak multiscale convergence gives us a more precise idea of what a corrector equals zero means.

References

[1] G. Allaire and M. Briane, "Multiscale convergence and reiterated homogenisation," *Proceedings of the Royal Society of Edinburgh A*, vol. 126, no. 2, pp. 297–342, 1996.

[2] L. Flodén, A. Holmbom, M. Olsson, and J. Persson, "Very weak multiscale convergence," *Applied Mathematics Letters*, vol. 23, no. 10, pp. 1170–1173, 2010.

[3] J. L. Woukeng, "Periodic homogenization of nonlinear non-monotone parabolic operators with three time scales," *Annali di Matematica Pura ed Applicata*, vol. 189, no. 3, pp. 357–379, 2010.

[4] L. Flodén and M. Olsson, "Homogenization of some parabolic operators with several time scales," *Applications of Mathematics*, vol. 52, no. 5, pp. 431–446, 2007.

[5] A. Holmbom, "Homogenization of parabolic equations: an alternative approach and some corrector-type results," *Applications of Mathematics*, vol. 42, no. 5, pp. 321–343, 1997.

[6] G. Nguetseng and J. L. Woukeng, "Σ-convergence of nonlinear parabolic operators," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 66, no. 4, pp. 968–1004, 2007.

[7] J. L. Woukeng, "Σ-convergence and reiterated homogenization of nonlinear parabolic operators," *Communications on Pure and Applied Analysis*, vol. 9, no. 6, pp. 1753–1789, 2010.

[8] L. Flodén, A. Holmbom, M. O. Lindberg, and J. Persson, "Detection of scales of heterogeneity and parabolic homogenization applying very weak multiscale convergence," *Annals of Functional Analysis*, vol. 2, no. 1, pp. 84–99, 2011.

[9] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, vol. 5 of *Studies in Mathematics and its Applications*, North-Holland Publishing Co., Amsterdam, The Netherlands, 1978.

[10] A. Piatnitski, "A parabolic equation with rapidly oscillating coefficients," *Moscow University Mathematics Bulletin*, vol. 35, no. 3, pp. 35–42, 1980.

[11] J. Garnier, "Homogenization in a periodic and time-dependent potential," *SIAM Journal on Applied Mathematics*, vol. 57, no. 1, pp. 95–111, 1997.

[12] A. K. Nandakumaran and M. Rajesh, "Homogenization of a nonlinear degenerate parabolic differential equation," *Electronic Journal of Differential Equations*, vol. 2001, no. 17, pp. 1–19, 2001.

[13] N. Svanstedt and J. L. Woukeng, "Periodic homogenization of strongly nonlinear reaction-diffusion equations with large reaction terms," *Applicable Analysis*, vol. 92, no. 7, pp. 1–22, 2012.

[14] L. Flodén, A. Holmbom, and M. Olsson Lindberg, "A strange term in the homogenization of parabolic equations with two spatial and two temporal scales," *Journal of Function Spaces and Applications*, vol. 2012, Article ID 643458, 9 pages, 2012.

[15] G. Nguetseng, "A general convergence result for a functional related to the theory of homogenization," *SIAM Journal on Mathematical Analysis*, vol. 20, no. 3, pp. 608–623, 1989.

[16] G. Allaire, "Homogenization and two-scale convergence," *SIAM Journal on Mathematical Analysis*, vol. 23, no. 6, pp. 1482–1518, 1992.

[17] J. Persson, *Selected topics in homogenization [Doctoral thesis]*, Mid Sweden University, Östersund, Sweden, 2012.

[18] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, Springer, New York, NY, USA, 1990.

[19] S. Spagnolo, "Convergence of parabolic equations," *Bollettino della Unione Matematica Italiana. Serie VIII. Sezione B*, vol. 14, no. 2, pp. 547–568, 1977.

[20] N. Svanstedt, *G-convergence and homogenization of sequences of linear and nonlinear partial differential operators [Doctoral thesis]*, Department of Mathematics, Luleå University of Technology, Luleå, Sweden, 1992.

[21] F. Paronetto, "G-convergence of mixed type evolution operators," *Journal de Mathématiques Pures et Appliquées*, vol. 93, no. 4, pp. 361–407, 2010.

[22] L. E. Persson, L. Persson, N. Svanstedt, and J. Wyller, *The homogenization Method. An Introduction*, Studentlitteratur, Lund, Sweden, Chartwell-Bratt, Bromley, UK, 1993.

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Research Article

Existence Results for Constrained Quasivariational Inequalities

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We deal with a constrained quasivariational inequality under a general form. We study existence of solutions in two situations depending on whether the set of constraints is bounded or possibly unbounded.

1. Introduction and Statement of Main Results

Let X be a real reflexive and separable Banach space assumed to be compactly embedded in a Banach space Y. We denote by X^* the dual space of X, by Y^* the dual space of Y, by $\langle\cdot,\cdot\rangle_X$ the duality brackets between X^* and X, by $\langle\cdot,\cdot\rangle_Y$ the duality brackets between Y^* and Y, by $\|\cdot\|_X$ the norm of X, and by $\|\cdot\|_Y$ the norm of Y. Given a function $\psi : X \to \mathbb{R}\cup\{+\infty\}$, we denote by $D(\psi) := \{x \in X : \psi(x) < +\infty\}$ the effective domain of ψ.

In this paper we deal with the following problem

$$\begin{gathered}\text{Find } u \in K \text{ such that } (u,u) \in D(\Phi), \\ \langle Au, v-u\rangle_X + \Phi(u,v) - \Phi(u,u) + J^0(u; v-u) \\ \geq \langle f, v-u\rangle_X, \quad \forall v \in K.\end{gathered} \tag{1}$$

We describe the data entering problem (1):

(i) $K \subset X$ is a nonempty, convex, closed subset;

(ii) $A : X \to X^*$ is a (possibly nonlinear) operator;

(iii) $\Phi : X \times X \to \mathbb{R}\cup\{+\infty\}$ is such that, for all $\eta \in K$, the function $\Phi(\eta,\cdot) : X \to \mathbb{R}\cup\{+\infty\}$ is convex with $K \cap D(\Phi(\eta,\cdot)) \neq \emptyset$; moreover, we will denote by $\partial\Phi(\eta,\cdot)$ the convex subdifferential of $\Phi(\eta,\cdot)$; that is,

$$\begin{aligned}\partial\Phi(\eta,u) = \{w \in X^* : \Phi(\eta,v) - \Phi(\eta,u) \\ \geq \langle w, v-u\rangle_X, \forall v \in X\};\end{aligned} \tag{2}$$

(iv) $J : Y \to \mathbb{R}$ is a locally Lipschitz function, and the notation J^0 stands for its generalized directional derivative in the sense of Clarke [1]; that is,

$$\begin{aligned}&J^0(u; v) \\ &\quad = \limsup_{\substack{w \to u \\ \lambda \to 0^+}} \frac{J(w+\lambda v) - J(w)}{\lambda}, \quad \forall u, v \in Y.\end{aligned} \tag{3}$$

In addition, we will denote by ∂J the generalized gradient of J; that is,

$$\begin{aligned}&\partial J(u) \\ &\quad = \left\{w \in Y^* : J^0(u; v) \geq \langle w, v\rangle_Y, \forall v \in Y\right\}, \quad \forall u \in Y;\end{aligned} \tag{4}$$

(v) $f \in X^*$.

Problem (1) is called a constrained quasivariational problem. Typically, we can choose X to be the Sobolev space $(H_0^1(\Omega), \|\nabla\cdot\|_{L^2(\Omega)})$ defined as the closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$ for a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$, Y to be the Lebesgue space $L^p(\Omega)$ for $1 \leq p < 2^*$ (where $2^* = +\infty$ if $N \in \{1,2\}$ and $2^* = 2N/(N-2)$ if $N \geq 3$), $K = \{u \in H_0^1(\Omega) : u \geq 0 \text{ a.e. in } \Omega\}$, $A = -\Delta$ (the negative Laplacian operator), $\Phi(u,v) = \int_\Omega g(u,v)dx$ where $g : \mathbb{R}^2 \to \mathbb{R}_+$ is convex in the second variable (then $D(\Phi) = \{(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega) : g(u,v) \in L^1(\Omega)\}$), and

$J(u) = \int_\Omega j(x, u(x))dx$ where $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz in the second variable. Constrained quasivariational problems were extensively studied; we refer, for example, to [2–5] and to the references therein. We point out three aspects which make our approach natural and general. First, we deal with the general setting of a pair of Banach spaces (X, Y) instead of focusing on spaces of functions; in particular, our results can be applied to problems with different boundary conditions. Second, the set of constraints K may be unbounded. Third, the form of the studied problem allows both variational and hemivariational constraints as it involves both a convex term $\Phi(u, \cdot)$ and a generalized directional derivative J^0; this type of problems models important processes in mechanics and engineering (see [6, 7]).

In this paper, we consider the following hypotheses on the data described above:

(H_1) for every sequence $\{u_n\}_{n\geq 1} \subset K$ with $u_n \rightharpoonup u$ in X, for some $u \in K$, one has

$$\langle Au, u - v\rangle_X \leq \limsup_{n\to\infty} \langle Au_n, u_n - v\rangle_X, \quad \forall v \in K; \tag{5}$$

(H_2) whenever $\{(\eta_n, u_n)\}_{n\geq 1} \subset (K \times K) \cap D(\Phi)$, $\eta_n \rightharpoonup \eta$ in X, $u_n \rightharpoonup u$ in X, one has $(\eta, u) \in (K \times K) \cap D(\Phi)$ and

$$\limsup_{n\to\infty} (\Phi(\eta_n, v) - \Phi(\eta_n, u_n)) \leq \Phi(\eta, v) - \Phi(\eta, u), \quad \forall v \in K; \tag{6}$$

(H_3) given $\eta \in K$, if $u_1, u_2 \in K$ satisfy $(\eta, u_1) \in D(\Phi)$, $(\eta, u_2) \in D(\Phi)$ and

$$J^0(\eta; u_2 - u_1) + J^0(\eta; u_1 - u_2) \geq \langle Au_2 - Au_1, u_2 - u_1\rangle_X, \tag{7}$$

then $u_1 = u_2$.

Remark 1. We emphasize certain situations when hypotheses (H_1)–(H_3) are satisfied.

(a) Hypothesis (H_1) is satisfied, for instance, if A is weakly strongly continuous, that is, A is continuous from X endowed with the weak topology to X^* endowed with the norm topology.

(b) Note that (H_1) is satisfied, for instance, for $X = H_0^1(\Omega)$, any closed, convex subset $K \subset X$, and $A : H_0^1(\Omega) \to H_0^1(\Omega)^*$ defined by $A = -\Delta$, where $\Delta : H_0^1(\Omega) \to H_0^1(\Omega)^*$ is the Laplacian operator, with $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ a bounded domain. Indeed, let a sequence $\{u_n\}_{n\geq 1} \subset K$ with $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, for some $u \in K$. Using the weak lower semicontinuity of the norm, we can write

$$\begin{aligned}\limsup_{n\to\infty} \langle -\Delta u_n, u_n - v\rangle &= \limsup_{n\to\infty} \left(\|u_n\|^2_{H_0^1(\Omega)} - (u_n, v)_{H_0^1(\Omega)} \right) \\ &\geq \liminf_{n\to\infty} \|u_n\|^2_{H_0^1(\Omega)} - (u, v)_{H_0^1(\Omega)} \\ &\geq \|u\|^2_{H_0^1(\Omega)} - (u, v)_{H_0^1(\Omega)} \\ &= \langle -\Delta u, u - v\rangle\end{aligned} \tag{8}$$

for all $v \in H_0^1(\Omega)$. Here, $\langle \cdot, \cdot \rangle$ are the duality brackets for the pair $(H_0^1(\Omega)^*, H_0^1(\Omega))$ and $(u, v)_{H_0^1(\Omega)} = \int_\Omega \nabla u \cdot \nabla v dx$ denotes the scalar product on $H_0^1(\Omega)$. Whence (H_1) holds in this case.

(c) Hypothesis (H_2) is fulfilled in the case where Φ is sequentially weakly lower semicontinuous, $D(\Phi)$ is weakly closed, and $\Phi(\cdot, u)$ is weakly strongly continuous on its effective domain for all $u \in X$.

(d) If A is strongly monotone, that is, there exists a constant $m > 0$ such that

$$\langle Au_2 - Au_1, u_2 - u_1\rangle_X \geq m\|u_1 - u_2\|_X^2, \quad \forall u_1, u_2 \in K, \tag{9}$$

and ∂J is bounded on K in the sense that

$$\|\zeta\|_{Y^*} \leq c\|u\|_Y, \quad \forall \zeta \in \partial J(u), \ \forall u \in K, \tag{10}$$

with a positive constant $c < m/(2\bar{c})$, where $\bar{c} > 0$ is the best constant satisfying $\|u\|_Y \leq \bar{c}\|u\|_X$, for all $u \in X$ (which exists by the continuity of the embedding of X in Y), then condition (H_3) is satisfied.

(e) If A is strictly monotone and J is Gâteaux differentiable and regular (see [1, Definition 2.3.4]), then condition (H_3) is satisfied. In particular, if A is strictly monotone and J is continuously differentiable, then (H_3) is satisfied.

In this paper, we distinguish two cases depending on whether the set K is bounded or not necessarily bounded. The following result concerns the former situation.

Theorem 2. *Assume that conditions* (H_1) – (H_3) *are satisfied and that the closed, convex set K is bounded in X. Then problem* (1) *has at least one solution.*

Remark 3. Note that the existence of a solution of problem (1), which is the conclusion of Theorem 2, forces the intersection $\operatorname{diag}(K) \cap D(\Phi)$ to be nonempty, where the notation $\operatorname{diag}(K)$ stands for the diagonal of the set K; that is, $\operatorname{diag}(K) = \{(v, v) : v \in K\}$. The nonemptiness of this intersection is not directly implied by the hypotheses (H_1)–(H_3), nor by the assumption made that $K \cap D(\Phi(\eta, \cdot)) \neq \emptyset$ for all $\eta \in K$. However, Theorem 4 below incorporates hypothesis (H_4) which assumes in particular that $\operatorname{diag}(K) \cap D(\Phi) \neq \emptyset$.

Now, we deal with the case where K is not assumed to be bounded. In this case, we additionally suppose the following:

(H_4) there exist an element $v_0 \in K$ with $(\eta, v_0) \in D(\Phi)$ for all $\eta \in K$ and a real $p \geq 1$ such that

$$\limsup_{\|w\|_X \to \infty} \frac{\langle Aw, w - v_0 \rangle_X}{\|w\|_X^p} = +\infty; \tag{11}$$

(H_5) there exists a constant $c_0 > 0$ such that we have

$$\begin{aligned} &\langle z, v_0 - u \rangle_X \\ &\qquad \leq c_0 \left(1 + \|u\|_X^p\right), \quad \forall z \in \partial\Phi(u, \cdot)(v_0), \\ &\|z\|_{Y^*} \leq c_0 \left(1 + \|u\|_Y^{p-1}\right), \quad \forall z \in \partial J(u), \end{aligned} \tag{12}$$

for all $u \in K$ with $(u, u) \in D(\Phi)$, where v_0 and $p \geq 1$ are as in (H_4) .

We state now our main result for problem (1) dealing with the case where the set K is possibly unbounded.

Theorem 4. *Assume that conditions (H_1)–(H_5) are satisfied. Then problem* (1) *has at least a solution.*

The rest of the paper is organized as follows. In Section 2, we present the proof of Theorem 2, where we apply a version of the Schauder fixed point theorem. In Section 3, we give the proof of Theorem 4, which is actually based on Theorem 2.

2. Proof of Theorem 2

For each $\eta \in K$, we consider the auxiliary problem

$$\begin{aligned} &\text{Find } u \in K \text{ such that } (\eta, u) \in D(\Phi), \\ &\langle Au, v - u \rangle_X + \Phi(\eta, v) - \Phi(\eta, u) + J^0(\eta; v - u) \\ &\qquad \geq \langle f, v - u \rangle_X, \quad \forall v \in K. \end{aligned} \tag{13}$$

Our first purpose, accomplished in Lemma 6 below, is to show that problem (13) has a unique solution. To do this, we need Fan's lemma (see [8, page 208]) which we recall in the following statement.

Theorem 5. *Let W be a Hausdorff topological vector space, let Z be a nonempty subset of W, and let $F : Z \to 2^W$ be such that*

(i) *$F(x)$ is a nonempty, closed subset of W, for all $x \in Z$;*

(ii) $\operatorname{conv}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$ *for all* $\{x_1, \ldots, x_n\} \subset Z$;

(iii) *there is $\overline{x} \in Z$ for which $F(\overline{x})$ is compact.*

Then $\bigcap_{x \in Z} F(x) \neq \emptyset$.

Lemma 6. *Assume that hypotheses (H_1)–(H_3) are fulfilled and that the closed, convex set K is bounded in X. Then, for every $\eta \in K$, problem* (13) *has a unique solution.*

Proof. Fix $\eta \in K$. Consider the set-valued mapping $G : K \cap D(\Phi(\eta, \cdot)) \to 2^X$ defined by

$$\begin{aligned} G(v) = \{u \in K \cap D(\Phi(\eta, \cdot)) : &\langle Au - f, u - v \rangle_X \\ &- J^0(\eta; v - u) \\ &+ \Phi(\eta, u) - \Phi(\eta, v) \leq 0\} \end{aligned} \tag{14}$$

for all $v \in K \cap D(\Phi(\eta, \cdot))$. We show that the assumptions of Theorem 5 are satisfied for $W = X$ endowed with the weak topology, $Z = K \cap D(\Phi(\eta, \cdot))$, and $F = G$.

For every $v \in K \cap D(\Phi(\eta, \cdot))$, we clearly have $v \in G(v)$; hence $G(v)$ is nonempty.

We check that $G(v)$ is weakly compact for every $v \in K \cap D(\Phi(\eta, \cdot))$. To this end, we first prove that $G(v)$ is sequentially weakly closed in X. Let a sequence $\{u_n\}_{n\geq 1} \subset G(v)$ with $u_n \rightharpoonup u$ in X, for some $u \in X$. Taking into account that X is compactly embedded in Y it follows that $u_n \to u$ in Y. Using the first part of assumption (H_2), we have that $u \in K \cap D(\Phi(\eta, \cdot))$. As $u_n \in G(v)$, we know that

$$\begin{aligned} &\langle Au_n, u_n - v \rangle_X \\ &\qquad \leq \langle f, u_n - v \rangle_X + J^0(\eta; v - u_n) + \Phi(\eta, v) - \Phi(\eta, u_n). \end{aligned} \tag{15}$$

Passing to the lim sup as $n \to \infty$, we find

$$\begin{aligned} &\limsup_{n \to \infty} \langle Au_n, u_n - v \rangle_X \\ &\qquad \leq \langle f, u - v \rangle_X + J^0(\eta; v - u) + \Phi(\eta, v) - \Phi(\eta, u). \end{aligned} \tag{16}$$

Here we made use of the weak convergence $u_n \rightharpoonup u$ in X, the continuity of $J^0(\eta; \cdot)$ on Y, and the second part of (H_2). Combining with (H_1), we obtain that $u \in G(v)$, thereby $G(v)$ is sequentially weakly closed in X.

Using that X is reflexive and separable and K is bounded, convex, and closed, we deduce that K is metrizable and weakly compact (see, e.g., [9, pages 44–50]). Since $G(v) \subset K$ and using that $G(v)$ is sequentially weakly closed, we derive that $G(v)$ is weakly compact whenever $v \in K \cap D(\Phi(\eta, \cdot))$. Therefore conditions (i) and (iii) in Theorem 5 are fulfilled.

We focus now on the verification of condition (ii) in Theorem 5. Arguing by contradiction, we suppose that there exist $v_1, \ldots, v_n \in K \cap D(\Phi(\eta, \cdot))$ and $u_0 \in \operatorname{conv}\{v_1, \ldots, v_n\}$ such that $u_0 \notin \bigcup_{i=1}^n G(v_i)$. The convexity of the set K and of the function $\Phi(\eta, \cdot)$ ensures that $u_0 \in K \cap D(\Phi(\eta, \cdot))$. Then the assertion that $u_0 \notin \bigcup_{i=1}^n G(v_i)$ reads as

$$\begin{aligned} &\langle Au_0 - f, u_0 - v_i \rangle_X - J^0(\eta; v_i - u_0) \\ &\qquad + \Phi(\eta, u_0) - \Phi(\eta, v_i) > 0, \quad \forall i \in \{1, \ldots, n\}. \end{aligned} \tag{17}$$

Let

$$\Lambda := \{v \in D(\Phi(\eta,\cdot)) : \langle Au_0 - f, u_0 - v\rangle_X - J^0(\eta; v - u_0) + \Phi(\eta,u_0) - \Phi(\eta,v) > 0\}. \tag{18}$$

It is clear that $v_i \in \Lambda$ for all $i \in \{1,\dots,n\}$. The convexity of the functions $\Phi(\eta,\cdot)$ and $J^0(\eta;\cdot)$ implies that Λ is a convex subset in X. We infer that $\mathrm{conv}\{v_1,\dots,v_n\} \subset \Lambda$, so $u_0 \in \Lambda$, which is obviously impossible. This contradiction justifies condition (ii) in Theorem 5. Thus all the assumptions of Theorem 5 are satisfied.

Applying Theorem 5, we obtain

$$\bigcap_{v\in K\cap D(\Phi(\eta,\cdot))} G(v) \neq \emptyset. \tag{19}$$

This ensures the existence of an element $u \in K \cap D(\Phi(\eta,\cdot))$ satisfying

$$\langle Au, v-u\rangle_X + \Phi(\eta,v) - \Phi(\eta,u) + J^0(\eta; v-u) \geq \langle f, v-u\rangle_X \tag{20}$$

for all $v \in K \cap D(\Phi(\eta,\cdot))$. The above inequality being also satisfied if $v \notin D(\Phi(\eta,\cdot))$, we conclude that u is a solution of problem (13).

It remains to show that the solution of problem (13) is unique. If $u_1, u_2 \in K$ are solutions of (13), then we have that $(\eta,u_1) \in D(\Phi)$, $(\eta,u_2) \in D(\Phi)$, and

$$\begin{aligned}&\langle Au_1, v-u_1\rangle_X + \Phi(\eta,v) - \Phi(\eta,u_1) + J^0(\eta; v-u_1) \geq \langle f, v-u_1\rangle_X, \quad \forall v \in K,\\ &\langle Au_2, v-u_2\rangle_X + \Phi(\eta,v) - \Phi(\eta,u_2) + J^0(\eta; v-u_2) \geq \langle f, v-u_2\rangle_X, \quad \forall v \in K.\end{aligned} \tag{21}$$

Letting $v = u_2$ in the first inequality and $v = u_1$ in the second one and then adding the obtained relations, we arrive at

$$\langle Au_1 - Au_2, u_2 - u_1\rangle_X + J^0(\eta; u_2 - u_1) + J^0(\eta; u_1 - u_2) \geq 0. \tag{22}$$

By assumption (H_3), we conclude that $u_1 = u_2$. The proof is complete. □

Denote by $u_\eta \in K$ the unique solution of problem (13) corresponding to $\eta \in K$. Lemma 6 guarantees that u_η exists and is unique. We define $\pi : K \to K$ by

$$\pi(\eta) = u_\eta, \quad \forall \eta \in K. \tag{23}$$

Lemma 7. *Assume that hypotheses (H_1)–(H_3) are fulfilled and that the closed, convex set K is bounded in X. Then, the map $\pi : K \to K$ given in (23) is sequentially weakly continuous.*

Proof. Let a sequence $\{\eta_n\}_{n\geq 1} \subset K$ such that $\eta_n \rightharpoonup \eta$ in X for some $\eta \in K$. We need to show that $\pi(\eta_n) \rightharpoonup \pi(\eta)$ as $n \to \infty$. To do this, it suffices to check that, for any relabeled subsequence $\{\eta_n\}_{n\geq 1}$, there is a subsequence of $\{\pi(\eta_n)\}_{n\geq 1}$ weakly converging to $\pi(\eta)$.

By the compactness of the embedding of X in Y, we have that $\eta_n \to \eta$ in Y. Denote, for simplicity, $\pi(\eta_n) = u_n$. The definition of π yields $(\eta_n, u_n) \in D(\Phi)$ and

$$\langle Au_n, u_n - v\rangle_X \leq \Phi(\eta_n, v) - \Phi(\eta_n, u_n) + J^0(\eta_n; v - u_n) + \langle f, u_n - v\rangle_X, \quad \forall v \in K. \tag{24}$$

Since K is bounded, $\{u_n\}_{n\geq 1} \subset K$ and X is reflexive, we know that along a subsequence, denoted again by $\{u_n\}_{n\geq 1}$, we have

$$u_n \rightharpoonup w \quad \text{in } X \text{ as } n \longrightarrow \infty, \tag{25}$$

for some $w \in X$. The first part of (H_2) yields $(\eta, w) \in (K \times K) \cap D(\Phi)$. Moreover, the compactness of the embedding of X in Y implies that $u_n \to w$ in Y. Letting $n \to \infty$ in (24), by means of (H_1), (H_2), the convergences $\eta_n \to \eta$ and $u_n \to w$ in Y, and the upper semicontinuity of $J^0(\cdot;\cdot)$ on $Y \times Y$, we get

$$\begin{aligned}\langle Aw, w - v\rangle_X &\leq \limsup_{n\to\infty} \langle Au_n, u_n - v\rangle_X\\ &\leq \limsup_{n\to\infty} (\Phi(\eta_n, v) - \Phi(\eta_n, u_n)) + \limsup_{n\to\infty} J^0(\eta_n; v - u_n) + \langle f, w - v\rangle_X\\ &\leq \Phi(\eta, v) - \Phi(\eta, w) + J^0(\eta; v - w) + \langle f, w - v\rangle_X, \quad \forall v \in K.\end{aligned} \tag{26}$$

This means that $w \in K$ is a solution of problem (13). Lemma 6 ensures that w is the unique solution of (13). Thus, by (23), we have $\pi(\eta) = w$. Taking into account (25), it follows that $\pi(\eta_n) \rightharpoonup \pi(\eta)$ as $n \to \infty$ up to a subsequence. This completes the proof. □

Remark 8. As noted in the proof of Lemma 6, the closed, bounded, convex subset $K \subset X$ is metrizable for the weak topology. Therefore, Lemma 7 implies that π is weakly continuous.

We need the following version of the Schauder fixed point theorem (see [10, page 452]).

Theorem 9. *Suppose that*

(i) *X is a reflexive, separable Banach space;*

(ii) *the map $T : M \subset X \to M$ is sequentially weakly continuous;*

(iii) *the set M is nonempty, closed, bounded, and convex.*

Then T has a fixed point.

We are now in position to prove Theorem 2.

Proof of Theorem 2. In view of Lemma 7 and the assumptions on X and K, we may apply Theorem 9 which shows that the map $\pi : K \to K$ admits a fixed point $u \in K$; that is, $\pi(u) = u$. Using the definition of π (see (23)), we deduce that $u \in K$ is a solution of problem (1). □

3. Proof of Theorem 4

It suffices to prove Theorem 4 when the set K is unbounded because for a bounded set K the result is true according to Theorem 2. Let $K_m = \{x \in K : \|x\|_X \leq m\}$. Let $m_0 \geq 1$ be an integer such that $\|v_0\|_X \leq m_0$, where v_0 is the element entering (H_4). We claim that Theorem 2 can be applied with K replaced by K_m whenever $m \geq m_0$.

Note that $v_0 \in K_{m_0}$, so $v_0 \in K_m \cap D(\Phi(\eta,\cdot))$ for all $\eta \in K$, all $m \geq m_0$ (using the first part of (H_4)). Thus, $K_m \cap D(\Phi(\eta,\cdot)) \neq \emptyset$ for all $\eta \in K_m$, all $m \geq m_0$. Since K is convex and closed in X, it turns out that K_m is convex, closed, and bounded in X, for all $m \geq m_0$.

We check that assumptions (H_1)–(H_3) of Theorem 2 remain valid when K is replaced by K_m with $m \geq m_0$. Towards this, we fix some $m \geq m_0$. If $\{(\eta_n, u_n)\}_{n\geq 1} \subset (K_m \times K_m) \cap D(\Phi)$ satisfies $\eta_n \rightharpoonup \eta$ in X and $u_n \rightharpoonup u$ in X, then assumption (H_2) (for K) implies $(\eta, u) \in (K \times K) \cap D(\Phi)$. On the other hand, the weak convergences ensure that

$$\|\eta\|_X \leq \liminf_{n\to\infty} \|\eta_n\|_X \leq m, \qquad \|u\|_X \leq \liminf_{n\to\infty} \|u_n\|_X \leq m. \tag{27}$$

Hence, $(\eta, u) \in (K_m \times K_m) \cap D(\Phi)$. The second part of (H_2) for K_m and conditions (H_1) and (H_3) for K_m hold because (H_1), (H_2), and (H_3) have been imposed for K, which contains K_m. Thus it is permitted to apply Theorem 2 for K_m in place of K, with any $m \geq m_0$.

Applying Theorem 2, we find a sequence $\{u_m\}_{m\geq m_0}$ in X such that $u_m \in K_m$, $(u_m, u_m) \in D(\Phi)$, and

$$\begin{aligned} &\langle A u_m, v - u_m \rangle_X + \Phi(u_m, v) - \Phi(u_m, u_m) \\ &\quad + J^0(u_m; v - u_m) \geq \langle f, v - u_m \rangle_X \end{aligned} \tag{28}$$

for all $v \in K_m$, all $m \geq m_0$. Letting $v = v_0$ (see (H_4)) in (28), we obtain

$$\begin{aligned} \langle A u_m, u_m - v_0 \rangle_X &\leq \Phi(u_m, v_0) - \Phi(u_m, u_m) \\ &\quad + J^0(u_m; v_0 - u_m) + \langle f, u_m - v_0 \rangle_X \end{aligned} \tag{29}$$

for all $m \geq m_0$. By the definition of the convex subdifferential $\partial\Phi(u_m,\cdot)$, we have

$$\begin{aligned} &\Phi(u_m, v_0) - \Phi(u_m, u_m) \\ &\quad \leq \langle z, v_0 - u_m \rangle_X, \quad \forall z \in \partial\Phi(u_m,\cdot)(v_0), \ \forall m \geq m_0. \end{aligned} \tag{30}$$

Then, invoking the growth condition for $\partial\Phi(u_m,\cdot)(v_0)$ in (H_5), we see that

$$\Phi(u_m, v_0) - \Phi(u_m, u_m) \leq c_0 \left(1 + \|u_m\|_X^p\right), \quad \forall m \geq m_0. \tag{31}$$

Recall that

$$J^0(u; v) = \max_{w\in\partial J(u)} \langle w, v \rangle_Y, \quad \forall u, v \in Y \tag{32}$$

(see [1, Proposition 2.1.2(b)]). This fact combined with the growth condition for the generalized gradient $\partial J(u_m)$ as stated in (H_5) enables us to write

$$\begin{aligned} J^0(u_m; v_0 - u_m) &= \max_{w\in\partial J(u_m)} \langle w, v_0 - u_m \rangle_Y \\ &\leq c_0 \left(1 + \|u_m\|_Y^{p-1}\right) \|v_0 - u_m\|_Y \end{aligned} \tag{33}$$

for all $m \geq m_0$. By the continuity of the embedding $X \subset Y$, the inequality above leads to

$$\begin{aligned} &J^0(u_m; v_0 - u_m) \\ &\quad \leq c_1 \left(1 + \|u_m\|_X^{p-1}\right) \|v_0 - u_m\|_X, \quad \forall m \geq m_0, \end{aligned} \tag{34}$$

where $c_1 > 0$ is a constant. Combining (29), (31), and (34) yields

$$\begin{aligned} &\langle A u_m, u_m - v_0 \rangle_X \\ &\quad \leq c_0 \left(1 + \|u_m\|_X^p\right) + \left[c_1 \left(1 + \|u_m\|_X^{p-1}\right) + \|f\|_{X^*}\right] \|v_0 - u_m\|_X \end{aligned} \tag{35}$$

for all $m \geq m_0$. Relation (35) ensures that the sequence $\{u_m\}_{m\geq m_0}$ is bounded in X; indeed, if we suppose that we have $\|u_m\|_X \to +\infty$ along a (relabeled) subsequence, then it is seen from (35) that there is a constant $c > 0$ such that

$$\limsup_{m\to\infty} \frac{\langle A u_m, u_m - v_0 \rangle_X}{\|u_m\|_X^p} \leq c, \tag{36}$$

which contradicts hypothesis (H_4).

By the reflexivity of X, there exists a subsequence of $\{u_m\}_{m\geq m_0}$, denoted again by $\{u_m\}_{m\geq m_0}$, such that

$$u_m \rightharpoonup u \quad \text{in } X \text{ as } m \longrightarrow \infty, \tag{37}$$

for some $u \in X$. Using hypothesis (H_2) with $\eta_m = u_m$, we derive that $(u, u) \in (K \times K) \cap D(\Phi)$.

It remains to show that u verifies the inequality in problem (1). Let an arbitrary element $v \in K$ and let $m_1 = m_1(v) \in \mathbb{N}$ such that $m_1 \geq \max\{m_0, \|v\|_X\}$. Then $v \in K_m$ for each $m \geq m_1$ and so from (28), we have that

$$\begin{aligned} \langle A u_m, u_m - v \rangle_X &\leq \Phi(u_m, v) - \Phi(u_m, u_m) \\ &\quad + J^0(u_m; v - u_m) + \langle f, u_m - v \rangle_X. \end{aligned} \tag{38}$$

The compactness of the embedding $X \subset Y$ and (37) guarantee that $u_m \to u$ in Y as $m \to \infty$. Then the upper semicontinuity of $J^0(\cdot;\cdot)$ on $Y \times Y$ implies

$$\limsup_{m\to\infty} J^0 (u_m; v - u_m) \le J^0 (u; v - u) . \quad (39)$$

Assumptions (H_1) and (H_2) ensure that

$$\begin{gathered} \langle Au, u - v\rangle_X \le \limsup_{m\to\infty} \langle Au_m, u_m - v\rangle_X, \\ \limsup_{m\to\infty} (\Phi(u_m, v) - \Phi(u_m, u_m)) \le \Phi(u, v) - \Phi(u, u) . \end{gathered} \quad (40)$$

Passing to the lim sup as $m \to \infty$ in (38) and using (39) and (40), we get that $u \in K$ satisfies the inequality in (1). Since v was chosen arbitrarily in K, we conclude that u solves problem (1). The proof of Theorem 4 is complete.

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References

[1] F. H. Clarke, *Optimization and Nonsmooth Analysis*, vol. 5 of *Classics in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 2nd edition, 1990.

[2] P. Jebelean, D. Motreanu, and V. V. Motreanu, "A unified approach for a class of problems involving a pseudo-monotone operator," *Mathematische Nachrichten*, vol. 281, no. 9, pp. 1283–1293, 2008.

[3] B. Jadamba, A. A. Khan, and M. Sama, "Generalized solutions of quasi variational inequalities," *Optimization Letters*, vol. 6, no. 7, pp. 1221–1231, 2012.

[4] Z. Liu, "Generalized quasi-variational hemi-variational inequalities," *Applied Mathematics Letters*, vol. 17, no. 6, pp. 741–745, 2004.

[5] D. Motreanu and M. Sofonea, "Quasivariational inequalities and applications in frictional contact problems with normal compliance," *Advances in Mathematical Sciences and Applications*, vol. 10, no. 1, pp. 103–118, 2000.

[6] D. Goeleven and D. Motreanu, *Variational and Hemivariational Inequalities: Theory, Methods and Applications. Vol. II. Unilateral Problems*, vol. 70 of *Nonconvex Optimization and its Applications*, Kluwer Academic Publishers, Boston, Mass, USA, 2003.

[7] D. Motreanu and P. D. Panagiotopoulos, *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*, vol. 29 of *Nonconvex Optimization and its Applications*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.

[8] C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities. Applications to Free Boundary Problems*, John Wiley & Sons, New York, NY, USA, 1984.

[9] H. Brezis, *Analyse Fonctionnelle. Théorie et Applications*, Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris, 1983.

[10] E. Zeidler, *Nonlinear Functional Analysis and Its Applications. Vol. I. Fixed-Point Theorems*, Springer, New York, NY, USA, 1986.

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Research Article

Classical Solvability of a Free Boundary Problem for an Incompressible Viscous Fluid with a Surface Density Equation

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We investigate a mathematical model introduced by Shikhmurzaev to remove singularities that arise when classical hydrodynamic models are applied to certain physical phenomena. The model is described as a free boundary problem consisting of the Navier-Stokes equations and a surface mass balance equation. We prove the local-in-time solvability in Hölder spaces.

1. Introduction

Let a time-dependent bounded domain $\Omega_t \subset \mathbf{R}^3$ with the outer boundary $\Gamma_t \equiv \partial\Omega_t$ be filled with an incompressible viscous fluid, and let Γ_t represent the interface. In Ω_t, we assume that the flow is governed by the Navier-Stokes equations:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p - \nu\Delta\mathbf{v} = \mathbf{0}, \quad \nabla\cdot\mathbf{v} = 0, \tag{1}$$

where $\mathbf{v}$ is the velocity, p is the pressure, and ν, which is assumed to be a positive constant, is the kinematic viscosity.

On Γ_t, we assume the following equations:

$$\Pi\mathbf{T}(\mathbf{v},p)\,\mathbf{n} = \overline{\nabla}\sigma, \qquad \mathbf{n}\cdot\mathbf{T}(\mathbf{v},p)\,\mathbf{n} = \sigma H, \tag{2}$$

$$\frac{D\rho^s}{Dt} + \rho^s\overline{\nabla}\cdot\mathbf{v}^s = \rho(\mathbf{v}-\mathbf{v}^s)\cdot\mathbf{n}, \tag{3}$$

$$(\mathbf{v}-\mathbf{v}^s)\cdot\mathbf{n} = -\frac{\rho_e^s-\rho^s}{\rho\tau}, \qquad \Pi(\mathbf{v}-\mathbf{v}^s) = -\chi\overline{\nabla}\sigma, \tag{4}$$

$$\sigma = \gamma(\overline{\rho}-\rho^s). \tag{5}$$

Here $\mathbf{v}^s$ and ρ^s are the velocity and the density of surface layer, respectively. $\mathbf{T}(\mathbf{v},p) = \nu\mathbf{D}(\mathbf{v}) - pI$ is the stress tensor, where $\mathbf{D}(\mathbf{v}) = ((\partial v_i/\partial x_j) + (\partial v_i/\partial x_j))_{i,j=1,2,3}$ is the velocity deformation tensor. H is the twice mean curvature of Γ_t at the point x, which is negative if Ω_t is convex in the neighborhood of x. $\mathbf{n}$ is the unit outward normal to Γ_t at the point x. Π is the projection operator onto the tangent plane at the point x on Γ_t. D/Dt denotes the derivative along the trajectory of particle on Γ_t. $\overline{\nabla}$ is the gradient restricted to the surface. ρ, ρ_e^s, τ, χ, γ, $\overline{\rho}$ are positive constants; in particular, ρ is the density of the bulk and τ is the characteristic time scale over which the surface density ρ^s relaxes to its equilibrium value ρ_e^s.

Finally, to complete the problem, we give the initial conditions:

$$\mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{on } \overline{\Omega} \equiv \overline{\Omega}_0, \qquad \rho^s\big|_{t=0} = \rho_0 \quad \text{on } \Gamma \equiv \Gamma_0. \tag{6}$$

It is known that singularities arise when the the classical hydrodynamic equations and modeling assumptions are applied to certain physical phenomena. For example, the application of the classical no-slip boundary condition to the spreading of a drop on a plate gives rise to a nonintegrable shear stress, and the application of the classical kinematic condition at the free boundary to the formation of a cusp on a free surface of a viscous fluid leads to an infinite energy dissipation in the fluid (e.g., refer to [1] and the references therein).

To remove the above mentioned singularities, we are required to modify classical boundary conditions by taking into account molecular interaction near interfaces. The molecule in the liquid region which is very close to another phases experiences an asymmetric force due to the presence of another materials. This gives rise to the variation in the density in the liquid region near to the adjacent phase, and the surface tension occurs as a result of this variation in density. The thin layer in the liquid region in which the above mentioned density variation occurs is called the surface layer.

Through [1–4], Shikhmurzaev developed a theory to remove the above mentioned singularities by introducing a surface layer which is treated as a separate phase. In this theory, the no-slip condition assumed in classical models for dynamic wetting processes is modified as the Navier-slip condition through thermodynamic considerations on the surface layer (refer to [2, 3]). The formation of a free surface cusp associated with fluid flow is also investigated in [4]. In [4], the cusp formation is modeled as an interface disappearance process. In this model, an internal surface stretching from the cusp, which is referred to as "the surface-tension-relaxation tail", is introduced. The above mentioned singularity associated with the modeling of cusp formation arises owing to the absence of viscous stress at the cusp with which the surface tension acting from the liquid surface is balanced. In this model, the surface tension at the cusp can be balanced by shear stresses acting on this tail.

The problem (1)–(6) is a model describing the behavior of an isolated liquid drop in which the interface is modeled as a surface layer based on Shikhmurzaev's theory. The dynamics of the liquid in this layer are governed by (3) which represents conservation of mass. The right-hand side of (3) represents the source consisting of a flow of molecules from the bulk. Equations in (4) are conditions that minimize the rate of entropy production in the surface layer. Equation (5) represents a linearized state equation in the surface layer (refer to [1] for details). In Shikhmurzaev's theory, the surface layer is modeled as a sharp interface as a result of a continuum approximation. Thus, in the above problem, the surface layer is described by the equations given in (3)–(5) defined on a geometric surface, and the behavior of the surface layer is related to (1) in the bulk through the boundary conditions given in (2).

In the present paper, we prove the local-in-time classical solvability of problem (1)–(6). As is mentioned above, this model is important as a basic model to describe the above mentioned physical phenomena; however, as far as the author knows, any rigorous proofs on its solvability have not been given. In the present paper, we consider the case where the mass exchange between the interface and the bulk does not occur. As is seen in Section 2, under such an assumption, we can reformulate our problem as a problem defined in a domain with a fixed known boundary by introducing Lagrangian coordinates, and in Section 3, we construct a unique solution of the reformulated problem in Hölder spaces with the aid of the method of successive approximations.

2. Reformulation of the Problem

In this section, we reformulate our problem in Lagrangian coordinates. By Lagrangian coordinates we mean the initial coordinates of the fluid particles. In the case where no exchange of molecules occurs between the surface and the bulk, $(4)^1$ is reduced to $\mathbf{v}\cdot\mathbf{n} = \mathbf{v}^s\cdot\mathbf{n}$. This relation indicates that the following kinematic condition at the interface is satisfied: the interface consists of the particles located at the interface at the initial time. This circumstance enables us to relate each point $x \in \overline{\Omega}_t$ to its initial point $\xi \in \overline{\Omega}$ by relation (10) given below.

Before reformulating our problem, we rewrite (3) as a nonlinear parabolic equation on Γ_t with the time derivative $\widetilde{D}\rho^s/\widetilde{D}t$, where $\widetilde{D}\rho^s/\widetilde{D}t$ denotes the derivative along the trajectory of particle on the interface with velocity $\mathbf{v}$. Noting the following relation (e.g., see [5]):

$$\left(\frac{D\rho^s}{Dt}\right)_n = \frac{D\rho^s}{Dt} - \mathbf{v}^s\cdot\overline{\nabla}\rho^s = \frac{\widetilde{D}\rho^s}{\widetilde{D}t} - \mathbf{v}\cdot\overline{\nabla}\rho^s, \tag{7}$$

where $(D\rho^s/Dt)_n$ represents the derivative along the trajectory which is normal to the interface, (3) can be written as

$$\frac{\widetilde{D}\rho^s}{\widetilde{D}t} + \rho^s\overline{\nabla}\cdot\mathbf{v}^s = (\mathbf{v}-\mathbf{v}^s)\cdot\overline{\nabla}\rho^s. \tag{8}$$

Then eliminating $\mathbf{v}^s$ from the above equation with the aid of the relation $(4)^2$, we obtain the following equation:

$$\begin{aligned}\frac{\widetilde{D}\rho^s}{\widetilde{D}t} - \chi\gamma\rho_0^s\overline{\nabla}^2\rho^s &= -\rho^s\overline{\nabla}\cdot\mathbf{v}\\ &\quad + \chi\gamma\left\{\left(\rho^s-\rho_0^s\right)\overline{\nabla}^2\rho^s + \overline{\nabla}\rho^s\cdot\overline{\nabla}\rho^s\right\}.\end{aligned} \tag{9}$$

Now let us reformulate our problem. The Lagrangian and Eulerian coordinates are related by

$$x = X_u(\xi,t) \equiv \xi + \int_0^t \mathbf{u}(\xi,\tau)\,d\tau, \tag{10}$$

where $\mathbf{u}(\xi,\tau)$ is the velocity at time t of the particle which was located at ξ at $t=0$. By changing the variables from x to ξ by relation (10), problem (1)–(6) is reformulated as the following problem defined in the cylindrical domain $\Omega_{0T} = \Omega\times(0,T)$ with the lateral boundary $\Gamma_{0T} \equiv \Gamma\times(0,T)$:

$$\frac{\partial\mathbf{u}}{\partial t} - \nu\nabla_u^2\mathbf{u} + \nabla_u q = \mathbf{0}, \quad \nabla_u\cdot\mathbf{u} = 0 \quad \text{in } \Omega_{0T}, \tag{11}$$

$$\begin{aligned}\Pi\Pi_u\mathbf{T}_u(\mathbf{u},q)\,\mathbf{n}_u &= \Pi\nabla_{\Gamma_t}\theta,\\ \mathbf{n}\cdot\mathbf{T}_u(\mathbf{u},q)\,\mathbf{n}_u &= \theta\nabla_{\Gamma_t}^2 X_u(\xi,t)\big|_{\xi\in\Gamma}\cdot\mathbf{n} + \mathbf{n}\cdot\nabla_{\Gamma_t}\theta,\\ \frac{\partial r^s}{\partial t} - \chi\gamma\rho_0^s\nabla_{\Gamma_t}^2 r^s &\\ = -r^s\nabla_{\Gamma_t}\cdot\mathbf{u} &+ \chi\gamma\left\{\left(r^s-\rho_0^s\right)\nabla_{\Gamma_t}^2 r^s + \nabla_{\Gamma_t}r^s\cdot\nabla_{\Gamma_t}r^s\right\},\\ \theta &= \gamma\left(\overline{\rho}-r^s\right) \quad \text{on } \Gamma_{0T},\end{aligned} \tag{12}$$

$$\mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{on } \overline{\Omega}, \qquad r^s\big|_{t=0} = \rho_0^s \quad \text{on } \Gamma. \tag{13}$$

In (11)–(13), $\mathbf{u}$, q and r^s are $\mathbf{v}(X_u(\xi,t),t)$, $p(X_u(\xi,t),t)$, and $\rho^s(X_u(\xi,t),t)$, respectively. Consider $\nabla_u = (\mathcal{J}_u^{-1})^t\nabla \equiv \mathcal{J}_u^*\nabla$; here $\mathcal{J}_u$ denotes the Jacobian matrix of X_u, and the notation

A^t means the transpose of the matrix A. $\mathbf{n}$ is the outward unit normal to Γ at the point ξ, $\mathbf{n}_u = \mathcal{J}_u^*\mathbf{n}/|\mathcal{J}_u^*\mathbf{n}|$, and Π and Π_u are the operators defined by $\Pi\mathbf{f} = \mathbf{f} - (\mathbf{f}\cdot\mathbf{n})\mathbf{f}$ and $\Pi_u\mathbf{f} = \mathbf{f} - (\mathbf{f}\cdot\mathbf{n}_u)\mathbf{f}$, respectively. $\mathbf{T}_u(\mathbf{u}, q)$ is the tensor with the elements $\nu\sum_{k=1}^3(A_{jk}(\partial u_i/\partial\xi_k) + A_{ik}(\partial u_j/\partial\xi_k)) - p\delta_{ij}$, where A_{ij} is the (i, j)-element of $\mathcal{J}_u^*$, and δ_{ij} is Kronecker's delta. The operators $\nabla_{\Gamma_t} f$ and $\nabla_{\Gamma_t}\cdot\mathbf{A}$ are defined by

$$\nabla_{\Gamma_t} f = \sum_{\alpha,\beta=1,2} g^{\alpha\beta}\frac{\partial f}{\partial s_\beta}\frac{\partial X_u(s)}{\partial s_\alpha},$$
$$\nabla_{\Gamma_t}\cdot\mathbf{A} = \frac{1}{\sqrt{g}}\sum_{\alpha=1,2}\frac{\partial}{\partial s_\alpha}\sqrt{g}A_\alpha, \tag{14}$$

where $g = \det(g_{\alpha\beta})_{\alpha,\beta=1,2}$, $g_{\alpha\beta} = (\partial X_u(s)/\partial s_\alpha)\cdot(\partial X_u(s)/\partial s_\beta)$, $g^{\alpha\beta}$ denote the components of the inverse matrix of $(g_{\alpha\beta})_{\alpha,\beta=1,2}$, $X_u(s) = X_u(\xi(s), t)$, $s = (s_1, s_2)$ denotes the local coordinates on Γ, and A_α denotes the components of the vector $\mathbf{A}$ with respect to the basis $(\partial X_u(s)/\partial s_\alpha)$, $\alpha = 1, 2$. Finally, the operator $\nabla_{\Gamma_t}^2 f$ is defined as

$$\nabla_{\Gamma_t}^2 f = \nabla_{\Gamma_t}\cdot\nabla_{\Gamma_t} f = \frac{1}{\sqrt{g}}\sum_{\alpha,\beta=1,2}\frac{\partial}{\partial s_\alpha}\sqrt{g}g^{\alpha\beta}\frac{\partial f}{\partial s_\beta}. \tag{15}$$

Note that in derivation of $(12)^2$, we have used the formula $H\mathbf{n} = \overline{\nabla}^2 x \equiv \overline{\nabla}^2 X_u(\xi, t)$. Note also that although $(12)^{1,2}$ are different from the following formulas which are obtained directly from (2):

$$\Pi_u\mathbf{T}_u(\mathbf{u}, q)\mathbf{n}_u = \nabla_{\Gamma_t}\theta,$$
$$\mathbf{n}_u\cdot\mathbf{T}_u(\mathbf{u}, q)\mathbf{n}_u = \theta\nabla_{\Gamma_t}^2 X_u(\xi, t)\Big|_{\xi\in\Gamma}\cdot\mathbf{n}_u, \tag{16}$$

problem (11)–(13) is equivalent to problem (1)–(6) as far as the condition $\mathbf{n}\cdot\mathbf{n}_u > 0$, which is valid for sufficiently small t, is satisfied.

We now introduce some function spaces. Let D be a domain in $\mathbf{R}^n$, let T be a positive constant, let D_T be a cylindrical domain $D\times(0, T)$, let l be a nonnegative integer, and let $\alpha, \gamma \in (0, 1)$.

By $C^{l+\alpha}(D)$, we define the space of functions $f(x)$, $x \in D$, with the norm

$$|f|_D^{(l+\alpha)} \equiv \sum_{|m|\le l}|\partial_x^m f|_D + [f]_D^{(l+\alpha)}, \qquad |f|_D \equiv \sup_{x\in D}|f(x)|,$$
$$[f]_D^{(l+\alpha)} \equiv \sum_{|m|=l}[\partial_x^m f]_D^{(\alpha)} \equiv \sup_{x,y\in D, x\ne y}\sum_{m=|l|}\frac{|\partial_x^m f(x) - \partial_y^m f(y)|}{|x-y|^\alpha},$$
$$|m| = \sum_{i=1}^n m_i, \qquad \partial_x^m = \frac{\partial^{|m|}}{\partial_{x_1}^{m_1}\cdots\partial_{x_n}^{m_n}}, \tag{17}$$

for a multi-index $m = (m_i)$ $(m_i \ge 0, i = 1, \ldots, n)$.

By $C^{l+\alpha,((l+\alpha)/2)}(D_T)$ we denote an anisotropic Hölder space of functions whose norm is defined by

$$|f|_{D_T}^{(l+\alpha,((l+\alpha)/2))} \equiv \sum_{2r+|m|=0}^{l}|\partial_t^r\partial_x^m f|_{D_T} + [f]_{D_T}^{(l+\alpha,((l+\alpha)/2))}, \tag{18}$$

where

$$|f|_{D_T} \equiv \sup_{(x,t)\in D_T}|f(x,t)|,$$
$$[f]_{D_T}^{(l+\alpha,((l+\alpha)/2))} \equiv \sum_{2r+|m|=l-1}^{l}[\partial_t^r\partial_x^m f]_{D_T}^{(0,((l+\alpha-(2r+|m|))/2))} + \sum_{2r+|m|=l}[\partial_t^r\partial_x^m f]_{D_T}^{(\alpha,0)}. \tag{19}$$

Here,

$$[f]_{D_T}^{(0,(\alpha/2))} \equiv \sup_{(x,t),(x,t')\in D_T, t\ne t'}\frac{|f(x,t) - f(x,t')|}{|t-t'|^{(\alpha/2)}},$$
$$[f]_{D_T}^{(\alpha,0)} \equiv \sup_{(x,t),(x',t)\in D_T, x\ne x'}\frac{|f(x,t) - f(x',t)|}{|x-x'|^\alpha}. \tag{20}$$

Finally, we introduce the function space $C^{1+\alpha,\gamma}(D_T)$ equipped with the norm

$$\|f\|_{D_T}^{(1+\alpha,\gamma)} \equiv |f|_{D_T} + |\nabla f|_{D_T}^{(\alpha,(\alpha/2))} + |f|_{D_T}^{(1+\alpha,\gamma)}, \tag{21}$$

where

$$|f|_{D_T}^{(1+\alpha,\gamma)} \equiv \sup_{\tau,t\in(0,T),\tau\ne t}\frac{|f(\cdot,t) - f(\cdot,\tau)|_D^{(\gamma)}}{|t-\tau|^{((1+\alpha-\gamma)/2)}}. \tag{22}$$

Now, let us state our main result.

Theorem 1. *Let α, γ be constants satisfying $0 < \alpha, \gamma < 1$. Assume that*

$$\mathbf{v}_0 \in C^{2+\alpha}(\overline{\Omega}), \qquad \rho_0^s \in C^{2+\alpha}(\Gamma), \qquad \Gamma \in C^{3+\alpha}. \tag{23}$$

Assume that there exist positive constants Δ_1 and Δ_2 such that $\overline{\rho} - \rho_0^s \ge \Delta_1 > 0$ and $\rho_0^s \ge \Delta_2 > 0$ on Γ. In addition, assume that the following compatibility conditions are satisfied:

$$\nabla\cdot\mathbf{v}_0 = \mathbf{0}, \qquad (\nu\Pi\mathbf{D}(\mathbf{v}_0)\mathbf{n} + \gamma\nabla_\Gamma\rho_0^s)|_\Gamma = \mathbf{0}, \tag{24}$$

where ∇_Γ is the operator corresponding to ∇_{Γ_t} with $t = 0$; namely, ∇_Γ is given by the formula in (14) with $X_u(s) = \xi(s)$.

Then, for a positive constant T, problem (11)–(13) has a unique solution $(\mathbf{u}, q, r^s)$ with the following smoothness:

$$\mathbf{u} \in C^{2+\alpha,1+(\alpha/2)}(\Omega_{0T}),$$
$$q \in C^{1+\alpha,\gamma}(\Omega_{0T}) \cap C^{1+\alpha,((1+\alpha)/2)}(\Gamma_{0T}), \tag{25}$$
$$r^s \in C^{2+\alpha,1+(\alpha/2)}(\Gamma_{0T}).$$

3. Proof of the Main Result

In this section, we will prove Theorem 1.

We begin with preparing estimates of solutions to some linear problems. For the following problem:

$$\begin{gathered}
\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla_w^2 \mathbf{u} + \nabla_w q = \mathbf{F}_1, \qquad \nabla_w \cdot \mathbf{u} = F_2 \quad \text{in } \Omega_{0T}, \\
\nu \Pi \Pi_w \mathbf{D}_w(\mathbf{u}) \mathbf{n}_w = \mathbf{F}_3, \\
\mathbf{n} \cdot \mathbf{T}_w(\mathbf{u}, q) \mathbf{n}_w - \overline{\Theta} \mathbf{n} \cdot \nabla_{\widetilde{\Gamma}_t}^2 \int_0^t \mathbf{u} d\tau = b + \int_0^t B d\tau \quad \text{on } \Gamma_{0T}, \\
\mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{on } \overline{\Omega},
\end{gathered} \tag{26}$$

the following result is given in [6]. In (26), ∇_w, Π_w, $\mathbf{T}_w(\mathbf{u}, q)$ $(= \nu \mathbf{D}_w(\mathbf{u}) - qI)$, $\mathbf{n}_w$ are defined for a given vector $\mathbf{w}$ in the same manner as ∇_u, Π_u, $\mathbf{T}_u(\mathbf{u}, q)$, $\mathbf{n}_u$ are defined, and $\nabla_{\widetilde{\Gamma}_t}$ is defined by (14) with $X_u = X_w$.

Theorem 2. *Let $T > 0$, and let α, γ be positive constants satisfying $0 < \alpha, \gamma < 1$. Assume that*

$$\begin{gathered}
\Gamma \in C^{2+\alpha}, \qquad \mathbf{v}_0 \in C^{2+\alpha}(\overline{\Omega}), \\
\mathbf{F}_1 \in C^{\alpha,(\alpha/2)}(\Omega_{0T}), \qquad F_2 \in C^{1+\alpha,((1+\alpha)/2)}(\Omega_{0T}), \\
\mathbf{F}_3 \in C^{1+\alpha,((1+\alpha)/2)}(\Gamma_{0T}), \qquad b \in C^{1+\alpha,((1+\alpha)/2)}(\Gamma_{0T}), \\
B \in C^{\alpha,(\alpha/2)}(\Gamma_{0T}), \qquad \overline{\Theta} \in C^{\alpha}(\Gamma).
\end{gathered} \tag{27}$$

Assume that there exists a positive constant Δ_1 such that $\overline{\Theta} \geq \Delta_1 > 0$ on Γ. Assume that the following compatibility conditions are satisfied:

$$\nabla \cdot \mathbf{v}_0 = F_2(\xi, 0), \qquad \nu \Pi \mathbf{D}(\mathbf{v}_0) \mathbf{n}|_{\Gamma} = \mathbf{F}_3(\xi, 0). \tag{28}$$

Assume that there exist functions $\mathbf{h} \in C^{\alpha,(\alpha/2)}(\Omega_{0T})$, $\mathbf{H}_k$, $k = 1, 2, 3$, with a finite norm $|\mathbf{H}_k|_{\Omega_{0T}}^{(1+\alpha,\gamma)}$ satisfying the relation

$$\frac{\partial F_2}{\partial t} - \nabla_w \cdot \mathbf{F}_1 = \nabla \cdot \mathbf{h}, \quad \mathbf{h} = \sum_{k=1}^{3} \partial_{\xi_k} \mathbf{H}_k, \tag{29}$$

in the sense of distribution. Furthermore, assume that $\mathbf{w} \in C^{2+\alpha,1+(\alpha/2)}(\Omega_{0T})$ satisfies the inequality

$$\left(T + T^{1/2}\right) |\mathbf{w}|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))} + T^{((1-\alpha+\gamma)/2)} \left|\partial_\xi \mathbf{w}\right|_{\Omega_{0T}} \leq \delta, \tag{30}$$

for a sufficiently small positive constant δ.

Then problem (26) has a unique solution $(\mathbf{u}, q)$ satisfying the following inequality:

$$\begin{aligned}
&|\mathbf{u}|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))} + \|q\|_{\Omega_{0T}}^{(1+\alpha,\gamma)} + |q|_{\Gamma_{0T}}^{(1+\alpha,((1+\alpha)/2))} \\
&\quad \leq \overline{C}_1(T) \Bigg\{ |\mathbf{F}_1|_{\Omega_{0T}}^{(\alpha,(\alpha/2))} + |F_2|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))} \\
&\qquad + |\mathbf{F}_3|_{\Gamma_{0T}}^{(1+\alpha,((1+\alpha)/2))} + |b|_{\Gamma_{0T}}^{(1+\alpha,((1+\alpha)/2))} \\
&\qquad + |B|_{\Gamma_{0T}}^{(\alpha,(\alpha/2))} + |\mathbf{v}_0|_{\Omega}^{(2+\alpha)} + |\mathbf{h}|_{\Omega_{0T}}^{(\alpha,(\alpha/2))} \\
&\qquad + \sum_{k=1}^{3} |\mathbf{H}_k|_{\Omega_{0T}}^{(1+\alpha,\gamma)} + P_T(\mathbf{w}) \left(|\mathbf{v}_0|_{\Omega}^{(1)} + |b(\cdot, 0)|_{\Gamma} \right) \Bigg\},
\end{aligned} \tag{31}$$

where $P_T(\mathbf{w}) = T^{((1-\alpha)/2)} |\mathbf{w}|_{\Omega_{0T}}^{(1,0)} + |\partial_\xi \mathbf{w}|_{\Omega_{0T}}^{(\alpha,(\alpha/2))} + [\partial_\xi \mathbf{w}]_{\Omega_{0T}}^{(0,((1+\alpha-\gamma)/2))}$ and $\overline{C}_1(T)$ is a nondecreasing function of T.

For the following problem:

$$\begin{gathered}
\frac{\partial r^s}{\partial t} - \chi \gamma \rho_0^s \nabla_{\widetilde{\Gamma}_t}^2 r^s = G \quad \text{on } \Gamma_{0T}, \\
r^s|_{t=0} = \rho_0^s \quad \text{on } \Gamma,
\end{gathered} \tag{32}$$

we have the following theorem. The assertion of the theorem immediately follows from the Hölder estimates for linear parabolic equations (e.g., see [7]).

Theorem 3. *Let $T > 0$, and let α be a positive constant satisfying $0 < \alpha < 1$. Assume that*

$$\Gamma \in C^{2+\alpha}, \qquad \rho_0^s \in C^{2+\alpha}(\Gamma), \qquad G \in C^{\alpha,(\alpha/2)}(\Gamma_{0T}). \tag{33}$$

Assume that there exists a positive constant Δ_2 such that $\rho_0^s \geq \Delta_2 > 0$ on Γ. Further assume the same assumptions for $\mathbf{w}$ stated in Theorem 2.

Then, problem (32) has a unique solution r^s satisfying the following inequality:

$$|r^s|_{\Gamma_{0T}}^{(2+\alpha,1+(\alpha/2))} \leq \overline{C}_2(T) \left(|G|_{\Gamma_{0T}}^{(\alpha,(\alpha/2))} + |\rho_0^s|_{\Gamma}^{(2+\alpha)} \right), \tag{34}$$

where $\overline{C}_2(T)$ is a nondecreasing function of T.

Combining the above results, we can easily obtain Theorem 4 given below for the following problem:

$$\begin{gathered}
\frac{\partial \mathbf{u}}{\partial t} - \nu \nabla_w^2 \mathbf{u} + \nabla_w q = \mathbf{F}_1, \qquad \nabla_w \cdot \mathbf{u} = F_2 \quad \text{in } \Omega_{0T}, \\
\nu \Pi \Pi_w \mathbf{D}_w(\mathbf{u})\mathbf{n}_w + \gamma \Pi \nabla_{\widetilde{\Gamma}_t} r^s = \mathbf{F}_3, \\
\mathbf{n} \cdot \mathbf{T}_w(\mathbf{u}, q)\mathbf{n}_w - \overline{\Theta}\mathbf{n} \cdot \nabla_{\widetilde{\Gamma}_t}^2 \int_0^t \mathbf{u}\, d\tau + \gamma \mathbf{n} \cdot \nabla_{\widetilde{\Gamma}_t} r^s \\
= b + \int_0^t B\, d\tau \quad \text{on } \Gamma_{0T}, \\
\mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{on } \overline{\Omega}, \\
\frac{\partial r^s}{\partial t} - \chi \gamma \rho_0^s \nabla_{\widetilde{\Gamma}_t}^2 r^s = G \quad \text{on } \Gamma_{0T}, \\
r^s\big|_{t=0} = \rho_0^s \quad \text{on } \Gamma.
\end{gathered} \tag{35}$$

The estimate given in the theorem will be essentially used in the later argument to prove Theorem 1.

Theorem 4. *Under the same assumptions given in Theorem 2 where only compatibility condition is replaced by*

$$\nabla \cdot \mathbf{v}_0 = F_2(\xi, 0), \qquad \left(\nu \Pi \mathbf{D}(\mathbf{v}_0)\mathbf{n} + \gamma \Pi \nabla_\Gamma \rho_0^s\right)\big|_\Gamma = \mathbf{F}_3(\xi, 0), \tag{36}$$

and Theorem 3, problem (35) *has a unique solution* $(\mathbf{u}, q, r^s)$ *satisfying the following inequality:*

$$\begin{aligned}
&|\mathbf{u}|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))} + \|q\|_{\Omega_{0T}}^{(1+\alpha,\gamma)} + |q|_{\Gamma_{0T}}^{(1+\alpha,((1+\alpha)/2))} + |r^s|_{\Gamma_{0T}}^{(2+\alpha,1+(\alpha/2))} \\
&\quad \le \overline{C}_3(T)\Big\{ |\mathbf{F}_1|_{\Omega_{0T}}^{(\alpha,(\alpha/2))} + |F_2|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))} \\
&\qquad + |\mathbf{F}_3|_{\Gamma_{0T}}^{(1+\alpha,((1+\alpha)/2))} + |b|_{\Gamma_{0T}}^{(1+\alpha,((1+\alpha)/2))} \\
&\qquad + |B|_{\Gamma_{0T}}^{(\alpha,(\alpha/2))} + |\mathbf{h}|_{\Omega_{0T}}^{(\alpha,(\alpha/2))} + \sum_{k=1}^{3} |\mathbf{H}_k|_{\Omega_{0T}}^{(1+\alpha,\gamma)} \\
&\qquad + |\mathbf{v}_0|_{\Omega}^{(2+\alpha)} + |G|_{\Gamma_{0T}}^{(\alpha,(\alpha/2))} + |\rho_0^s|_{\Gamma}^{(2+\alpha)} \\
&\qquad + P_T(\mathbf{w})\left(|\mathbf{v}_0|_{\Omega}^{(1)} + |b(\cdot,0)|_\Gamma\right)\Big\},
\end{aligned} \tag{37}$$

where $\overline{C}_3(T)$ *is a nondecreasing function of* T.

In addition, we prepare estimates for $\mathcal{J}_u^*$, which are used later.

Lemma 5. *Let* $\mathcal{J}_u$ *and* $\mathcal{J}_{u'}$ *be the Jacobian matrices of the mappings* X_u *and* $X_{u'}$*, respectively. Let us assume that* $\mathbf{u}$ *and* $\mathbf{u}'$ *satisfy condition* (30) *for sufficiently small* $\delta > 0$*. Then, the following inequalities hold:*

$$\left|\mathcal{J}_u^*\right|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))}, \left|\mathcal{J}_u^*\right|_{\Omega_{0T}}^{(1+\alpha,\gamma)} \le C, \tag{38}$$

where C *is a positive constant independent of* δ*, and*

$$\begin{aligned}
&\left|\mathcal{J}_u^* - \mathcal{J}_{u'}^*\right|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))}, \left|\mathcal{J}_u^* - \mathcal{J}_{u'}^*\right|_{\Omega_{0T}}^{(1+\alpha,\gamma)} \\
&\quad \le \epsilon \left|\mathbf{u} - \mathbf{u}'\right|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))} + C(\epsilon) \int_0^T \left|\mathbf{u} - \mathbf{u}'\right|_{\Omega_{0\tau}}^{(2+\alpha,1+(\alpha/2))} d\tau,
\end{aligned} \tag{39}$$

for arbitrary $0 < \epsilon < 1$*, where* $C(\epsilon)$ *is a positive constant depending only on* ϵ*.*

Proof. In the following proof, c_1, c_2, and c_3 are positive constants independent of δ and ϵ. Let a_{ij} be the (i, j)-component of $\mathcal{J}_u$.

Then, we have

$$\delta_{ij} - \left|\int_0^t \partial_{\xi_j} u_i \, d\tau\right| \le a_{ij} \le \delta_{ij} + \left|\int_0^t \partial_{\xi_j} u_i \, d\tau\right|, \tag{40}$$

where δ_{ij} denotes Kronecker's delta. Then, using the inequality

$$\left|\int_0^t \partial_{\xi_j} u_i \, d\tau\right|_{\Omega_{0T}} \le T|\mathbf{u}|_{\Omega_{0T}}^{(1,0)} \le \delta, \tag{41}$$

from (40), we have

$$\delta_{ij} - \delta \le \left|a_{ij}\right|_{\Omega_{0T}} \le \delta_{ij} + \delta. \tag{42}$$

This inequality implies that $\det \mathcal{J}_u > 0$ holds for sufficiently small $\delta > 0$.

Now, let a'_{ij} be the (i, j)-components of $\mathcal{J}_{u'}$, and let A_{ij} and A'_{ij} be the cofactors of a_{ij} and a'_{ij}, respectively.

Then from the inequalities

$$\left|a_{ij}\right|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))}, \left|a'_{ij}\right|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))} \le c_1(1+\delta), \tag{43}$$

we have

$$\left|A_{ij}\right|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))}, \left|A_{ij}\right|_{\Omega_{0T}}^{(1+\alpha,\gamma)} \le c_2. \tag{44}$$

On the other hand, from the inequality

$$\begin{aligned}
&\left|a_{ij} - a'_{ij}\right|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))} \\
&\quad \le \epsilon \left|\mathbf{u} - \mathbf{u}'\right|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))} \\
&\qquad + c_3\left(1 + \epsilon^{-((1+\alpha)/(2-\alpha))}\right) \int_0^T \left|\mathbf{u} - \mathbf{u}'\right|_{\Omega_{0\tau}}^{(2+\alpha,1+(\alpha/2))} d\tau,
\end{aligned} \tag{45}$$

which are obtained with the aid of the following inequality which holds for arbitrary $0 < \epsilon < 1$:

$$\begin{aligned}
&\left|\int_0^t f(\xi,\tau)d\tau\right|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))} \\
&\quad \le \int_0^T |f|_{\Omega_{0\tau}}^{(1+\alpha,0)} d\tau \\
&\qquad + \sup_{|t'-t''|<\epsilon^{(2/(1-\alpha))}} \left(\left|t'-t''\right|^{-((1+\alpha)/2)} \int_{t''}^{t'} |f(\xi,\tau)|_\Omega d\tau \right. \\
&\qquad\qquad \left. + \left|t'-t''\right|^{-(\alpha/2)} \int_{t''}^{t'} \left|\partial_\xi f(\xi,\tau)\right|_\Omega d\tau \right) \\
&\qquad + \sup_{\epsilon^{(2/(1-\alpha))}<|t'-t''|\le T} \left(\left|t'-t''\right|^{-((1+\alpha)/2)} \int_{t''}^{t'} |f(\xi,\tau)|_\Omega d\tau \right. \\
&\qquad\qquad \left. + \left|t'-t''\right|^{-(\alpha/2)} \int_{t''}^{t'} \left|\partial_\xi f(\xi,\tau)\right|_\Omega d\tau \right) \\
&\quad \le \epsilon |f|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))} \\
&\qquad + c_3 \left(1+\epsilon^{-((1+\alpha)/(1-\alpha))}\right) \int_0^T |f|_{\Omega_{0\tau}}^{(1+\alpha,((1+\alpha)/2))} d\tau,
\end{aligned} \tag{46}$$

we have

$$\begin{aligned}
&\left|A_{ij}-A'_{ij}\right|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))}, \left|A_{ij}-A'_{ij}\right|_{\Omega_{0T}}^{(1+\alpha,\gamma)} \\
&\quad \le \epsilon \left|\mathbf{u}-\mathbf{u}'\right|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))} + C_1(\epsilon) \int_0^T \left|\mathbf{u}-\mathbf{u}'\right|_{\Omega_{0\tau}}^{(2+\alpha,1+(\alpha/2))} d\tau,
\end{aligned} \tag{47}$$

where $C_1(\epsilon)$ is a positive constant depending only on ϵ. From (44) and (47), estimates (38) and (39) immediately follow. Thus, the proof is completed. □

Now, let us prove Theorem 1 by the method of successive approximations. We take $(\mathbf{u}_0, q_0, r_0^s) = (\mathbf{0}, 0, 0)$, and, for the known nth approximation, we define the $(n+1)$th approximation by the solutions of the following problem:

$$\begin{gathered}
\frac{\partial \mathbf{u}_{n+1}}{\partial t} - \nu \nabla_{u_n}^2 \mathbf{u}_{n+1} + \nabla_{u_n} q_{n+1} = \mathbf{0}, \\
\nabla_{u_n} \cdot \mathbf{u}_{n+1} = 0 \quad \text{in } \Omega_{0T}, \\
\nu \Pi \Pi_{u_n} \mathbf{D}_{u_n}(\mathbf{u}_{n+1}) \mathbf{n}_{u_n} + \gamma \Pi \nabla_{\Gamma_{n,t}} r_{n+1}^s = \mathbf{0}, \\
\mathbf{n} \cdot \mathbf{T}_{u_n}(\mathbf{u}_{n+1}, q_{n+1}) \mathbf{n}_{u_n} - \overline{\Theta} \mathbf{n} \cdot \nabla_{\Gamma_{n,t}}^2 \int_0^t \mathbf{u}_{n+1} d\tau \\
+ \gamma \mathbf{n} \cdot \nabla_{\Gamma_{n,t}} r_{n+1}^s \\
= b(\mathbf{u}_n, r_n^s) + \int_0^t B(\mathbf{u}_n, r_n^s) d\tau \quad \text{on } \Gamma_{0T}, \\
\mathbf{u}_{n+1}\big|_{t=0} = \mathbf{v}_0 \quad \text{on } \overline{\Omega}, \\
\frac{\partial r_{n+1}^s}{\partial t} - \chi\gamma\rho_0^s \nabla_{\Gamma_{n,t}}^2 r_{n+1}^s = G(r_n^s, \mathbf{u}_n) \quad \text{on } \Gamma_{0T}, \\
r_{n+1}^s\big|_{t=0} = \rho_0^s \quad \text{on } \Gamma,
\end{gathered} \tag{48}$$

where

$$\begin{aligned}
b(\mathbf{u}_n, r_n^s) &= \theta_n \left(H_0 + \mathbf{n} \cdot \int_0^t \left(\nabla_{\Gamma_{n,\tau}}^2\right)_\tau \xi d\tau \right), \\
H_0 &= \mathbf{n} \cdot \nabla_\Gamma^2 \xi \\
B(\mathbf{u}_n, r_n^s) &= \left\{ \frac{\partial}{\partial \tau} \left(\theta_n - \overline{\Theta}\right) \right\} \mathbf{n} \cdot \nabla_{\Gamma_{n,\tau}}^2 \int_0^\tau \mathbf{u}_n(\xi, s) ds \\
&\quad + \left(\theta_n - \overline{\Theta}\right) \mathbf{n} \cdot \left(\nabla_{\Gamma_{n,\tau}}^2\right)_\tau \int_0^\tau \mathbf{u}_n(\xi, s) ds \\
&\quad + \left(\theta_n - \overline{\Theta}\right) \mathbf{n} \cdot \nabla_{\Gamma_{n,\tau}}^2 \mathbf{u}_n, \\
\theta_n &= \gamma\left(\overline{\rho} - r_n^s\right), \qquad \overline{\Theta} = \gamma\left(\overline{\rho} - \rho_0^s\right), \\
G(\mathbf{u}_n, r_n^s) &\equiv -r_n^s \nabla_{\Gamma_{n,t}} \cdot \mathbf{u}_n \\
&\quad + \chi\gamma \left\{ \left(r_n^s - \rho_0^s\right) \nabla_{\Gamma_{n,t}}^2 r_n^s + \nabla_{\Gamma_{n,t}} r_n^s \cdot \nabla_{\Gamma_{n,t}} r_n^s \right\}.
\end{aligned} \tag{49}$$

In the above formulas, $\nabla_{\Gamma_{n,t}}$ is the operator corresponding to ∇_{Γ_t} with $X_u = X_{u_n}$, and $(\nabla_{\Gamma_{n,\tau}}^2)_\tau$ denotes the operator obtained by differentiating the coefficients of $\nabla_{\Gamma_{n,\tau}}^2$ with respect to τ.

Now, let us verify that all terms of the sequence $\{(\mathbf{u}_n, q_n, r_n^s)\}$ are defined on some time interval independent of n. We begin with the following lemma.

Lemma 6. *Let T_n be a constant satisfying $0 < T_n \le 1$. Then there exist positive constants δ and β such that if $\mathbf{u}_n$ and r_n^s satisfy the following conditions:*

$$\begin{gathered}
\left(T_n + T_n^{1/2}\right) |\mathbf{u}_n|_{\Omega_{0T_n}}^{(2+\alpha,1+(\alpha/2))} + T_n^{((1-\alpha+\gamma)/2)} \left|\partial_\xi \mathbf{u}_n\right|_{\Omega_{0T_n}} \le \delta, \\
\left(T_n + T_n^\beta\right) \left(|\mathbf{u}_n|_{\Omega_{0T_n}}^{(2+\alpha,1+(\alpha/2))} + |r_n^s|_{\Gamma_{0T_n}}^{(2+\alpha,1+(\alpha/2))} \right) \le \delta,
\end{gathered} \tag{50}$$

then the following inequality holds for a positive constant $\overline{C}$ independent of $\mathbf{u}_n$, r_n^s and T_n:

$$\begin{aligned}
&\left|b(\mathbf{u}_n, r_n^s)\right|_{\Gamma_{0T_n}}^{(1+\alpha,((1+\alpha)/2))} + \left|B(\mathbf{u}_n, r_n^s)\right|_{\Gamma_{0T_n}}^{(\alpha,(\alpha/2))} \\
&\quad + \left|G(\mathbf{u}_n, r_n^s)\right|_{\Gamma_{0T_n}}^{(\alpha,(\alpha/2))} + P_{T_n}(\mathbf{u}_n) \le \overline{C}.
\end{aligned} \tag{51}$$

Proof. In the proof, $c_1, \ldots, c_9$ are positive constants independent of $\mathbf{u}_n$, r_n^s, and T_n.

Choosing $T_n^{((1-\alpha)/2\alpha)}$ as ϵ in the following interpolation inequality:

$$
\begin{aligned}
&\left|\theta_n - \left(\theta_n\big|_{t=0}\right)\right|^{(1+\alpha,((1+\alpha)/2))}_{\Gamma_{0T_n}} \\
&\quad\le \epsilon\left|r_n^s - \rho_0^s\right|^{(2,1)}_{\Gamma_{0T_n}} + c_1\epsilon^{-(\alpha/(1-\alpha))}\left|r_n^s - \rho_0^s\right|_{\Gamma_{0T_n}} \\
&\quad\le \epsilon\left|r_n^s - \rho_0^s\right|^{(2,1)}_{\Gamma_{0T_n}} + c_1\epsilon^{-(\alpha/(1-\alpha))}\int_0^{T_n}\left|\partial_\tau\left(r_n^s - \rho_0^s\right)\right|_{\Gamma_{0\tau}} d\tau,
\end{aligned}
\tag{52}
$$

we have the following estimate:

$$
\begin{aligned}
&\left|\theta_n\right|^{(1+\alpha,((1+\alpha)/2))}_{\Gamma_{0T_n}} \\
&\quad\le c_2\left(\left|\rho_0^s\right|^{(1+\alpha)}_{\Gamma} + \left|\theta_n - \left(\theta_n\big|_{t=0}\right)\right|^{(1+\alpha,((1+\alpha)/2))}_{\Gamma_{0T_n}}\right) \\
&\quad\le c_3\left\{\left|\rho_0^s\right|^{(1+\alpha)}_{\Gamma} + \left(T_n^{((1-\alpha)/2\alpha)} + T_n^{(1/2)}\right)\left|r_n^s - \rho_0^s\right|^{(2,1)}_{\Gamma_{0T_n}}\right\} \\
&\quad\le c_4\left(1+\delta\right).
\end{aligned}
\tag{53}
$$

With the aid of the above estimate, we can easily obtain the desired estimate for $b(\mathbf{u}_n, r_n^s)$.

$B(\mathbf{u}_n, r_n^s)$ is estimated as follows. With the aid of the inequality

$$
\begin{aligned}
&\left|\int_0^t \partial_\xi^2 f\left(\xi,\tau\right) d\tau\right|^{(\alpha,(\alpha/2))}_{\Gamma_{0T}} \\
&\qquad\le c_5\left(T + T^{1-(\alpha/2)}\right)\left|f\right|^{(2+\alpha,1+(\alpha/2))}_{\Gamma_{0T}},
\end{aligned}
\tag{54}
$$

we have

$$
\begin{aligned}
&\left|\left\{\frac{\partial}{\partial t}\left(\theta_n - \overline{\Theta}\right)\right\}\mathbf{n}\cdot\nabla^2_{\Gamma_{n,t}}\int_0^t \mathbf{u}_n\left(\xi,\tau\right) d\tau\right. \\
&\quad\left.+\left(\theta_n - \overline{\Theta}\right)\mathbf{n}\cdot\left(\nabla^2_{\Gamma_{n,t}}\right)_t\int_0^t \mathbf{u}_n\left(\xi,\tau\right) d\tau\right|^{(\alpha,(\alpha/2))}_{\Gamma_{0T_n}} \\
&\quad\le c_6\left(T_n + T_n^{1-(\alpha/2)}\right)\left|r_n^s - \rho_0^s\right|^{(2+\alpha,1+(\alpha/2))}_{\Gamma_{0T_n}}\left|\mathbf{u}_n\right|^{(2+\alpha,1+(\alpha/2))}_{\Gamma_{0T_n}} \\
&\quad\le c_7\left(1+\delta^2\right).
\end{aligned}
\tag{55}
$$

On the other hand, with the aid of

$$
\begin{aligned}
\left|\theta_n - \overline{\Theta}\right|^{(\alpha,(\alpha/2))}_{\Gamma_{0T_n}} &\le \int_0^{T_n}\left|\partial_\tau\left(\theta_n - \overline{\Theta}\right)\right|^{(\alpha,(\alpha/2))}_{\Gamma_{0\tau}} d\tau \\
&\le T_n\left|r_n^s - \rho_0^s\right|^{(2+\alpha,1+(\alpha/2))}_{\Gamma_{0T_n}},
\end{aligned}
\tag{56}
$$

we have

$$
\begin{aligned}
&\left|\left(\theta_n - \overline{\Theta}\right)\mathbf{n}\cdot\nabla^2_{\Gamma_{n,t}}\mathbf{u}_n\right|^{(\alpha,(\alpha/2))}_{\Gamma_{0T_n}} \\
&\quad\le c_8 T_n\left|r_n^s - \rho_0^s\right|^{(2+\alpha,1+(\alpha/2))}_{\Gamma_{0T_n}}\left|\mathbf{u}_n\right|^{(2+\alpha,1+(\alpha/2))}_{\Gamma_{0T_n}} \\
&\quad\le c_9\left(1+\delta^2\right).
\end{aligned}
\tag{57}
$$

From (55) and (57), we have the desired estimate for $B(\mathbf{u}_n, r_s^n)$. $G(\mathbf{u}_n, r_n^s)$ and $P_{T_n}(\mathbf{u}_n)$ are estimated in a similar manner. Thus, we have proved the lemma. □

From this lemma, if $\mathbf{u}_n$ and r_n^s satisfy conditions (50), by applying Theorem 4 to problem (3) we have the following estimate of $\mathbf{u}_{n+1}$ and r_{n+1}^s:

$$
\left|\mathbf{u}_{n+1}\right|^{(2+\alpha,1+(\alpha/2))}_{\Omega_{0T_n}} + \left|r_{n+1}^s\right|^{(2+\alpha,1+(\alpha/2))}_{\Gamma_{0T_n}} \le \overline{C}_3\left(1\right)\overline{C},
\tag{58}
$$

where $\overline{C}_3(\cdot)$ is the function given in Theorem 4.

Now, let us take T satisfying the following conditions:

$$
0 < T \le 1, \quad \overline{C}_3\left(1\right)\overline{C}\left(T + T^{\beta}\right) \le \delta.
\tag{59}
$$

Here β is assumed to be chosen, so that $\beta < 1/2$ and $\beta < (1-\alpha+\gamma)/2$. Since $(\mathbf{u}_0, r_0^s) = (\mathbf{0}, 0)$, the zeroth approximation $(\mathbf{u}_0, r_0^s)$ obviously satisfies conditions (50) for the above T. Hence, from (58), $(\mathbf{u}_1, r_1^s)$ satisfies

$$
\left|\mathbf{u}_1\right|^{(2+\alpha,1+(\alpha/2))}_{\Omega_{0T}} + \left|r_1^s\right|^{(2+\alpha,1+(\alpha/2))}_{\Gamma_{0T}} \le \overline{C}_3\left(1\right)\overline{C}.
\tag{60}
$$

From (59), this inequality indicates that $(\mathbf{u}_1, r_1^s)$ satisfies conditions (50), and hence we can obtain the same estimate as (60) for $(\mathbf{u}_2, r_2^s)$. Thus, repeating this argument, we can construct a sequence $\{(\mathbf{u}_n, q_n, r_n^s)\}$ such that each term is defined on $(0, T)$.

Let us proceed to the proof of the convergence of the sequence $\{(\mathbf{u}_n, q_n, r_n^s)\}$. In the following argument, $C_1, \dots, C_{10}$ denote positive constants independent of n and $C(\epsilon)$ represents various positive constants depending only on ϵ.

Let us set

$$
\begin{aligned}
&U^{(n+1)} \equiv \mathbf{u}_{n+1} - \mathbf{u}_n, \\
&Q^{(n+1)} \equiv q_{n+1} - q_n, \qquad R^{(n+1)} \equiv r_{n+1}^s - r_n^s.
\end{aligned}
\tag{61}
$$

Subtracting (3) with index n from that with index $n+1$, we have

$$
\begin{gathered}
\frac{\partial U^{(n+1)}}{\partial t} - \nu\nabla^2_{u_n}U^{(n+1)} + \nabla_{u_n}Q^{(n+1)} = \mathscr{F}_1^{(n)}, \\
\nabla_{u_n}\cdot U^{(n+1)} = \mathscr{F}_2^{(n)} \quad \text{in } \Omega_{0T}, \\
\nu\Pi\Pi_{u_n}\mathbf{D}_{u_n}\left(U^{(n+1)}\right)\mathbf{n}_{u_n} + \gamma\Pi\nabla_{\Gamma_{n,t}}R^{(n+1)} = \mathscr{F}_3^{(n)}, \\
\mathbf{n}\cdot\mathbf{T}_{u_n}\left(U^{(n+1)}, Q^{(n+1)}\right)\mathbf{n}_{u_n} - \overline{\Theta}\mathbf{n}\cdot\nabla^2_{\Gamma_{n,t}}\int_0^t U^{(n+1)} d\tau \\
+ \gamma\mathbf{n}\cdot\nabla_{\Gamma_{n,t}}R^{(n+1)} \\
= b^{(n)} + \int_0^t \mathscr{B}^{(n)} d\tau \quad \text{on } \Gamma_{0T},
\end{gathered}
$$

$$U^{(n+1)}\big|_{t=0} = \mathbf{0} \quad \text{on } \overline{\Omega},$$
$$\frac{\partial R^{(n+1)}}{\partial t} - \chi\gamma\rho_0^s \nabla^2_{\Gamma_{n,t}} R^{(n+1)} = \mathscr{G}^{(n)} \quad \text{in } \Gamma_{0T},$$
$$R^{(n+1)}\big|_{t=0} = 0 \quad \text{on } \Gamma, \tag{62}$$

where

$$\mathscr{F}_1^{(n)} \equiv \nu\left(\nabla^2_{u_n} - \nabla^2_{u_{n-1}}\right)\mathbf{u}_n - \left(\nabla_{u_n} - \nabla_{u_{n-1}}\right) q_n,$$
$$\mathscr{F}_2^{(n)} \equiv -\left(\nabla_{u_n} - \nabla_{u_{n-1}}\right)\cdot \mathbf{u}_n,$$
$$\mathscr{F}_3^{(n)} \equiv -\nu\Pi\left\{\Pi_{u_n}\mathbf{D}_{u_n}(\mathbf{u}_n)\mathbf{n}_{u_n} - \Pi_{u_{n-1}}\mathbf{D}_{u_{n-1}}(\mathbf{u}_n)\mathbf{n}_{u_{n-1}}\right\}$$
$$- \gamma\Pi\left(\nabla_{\Gamma_{n,t}} - \nabla_{\Gamma_{n-1,t}}\right) r_n^s,$$
$$b^{(n)} \equiv -\mathbf{n}\cdot\left\{\mathbf{T}_{u_n}(\mathbf{u}_n, q_n)\mathbf{n}_{u_n} - \mathbf{T}_{u_{n-1}}(\mathbf{u}_n, q_n)\mathbf{n}_{u_{n-1}}\right\}$$
$$- \gamma\mathbf{n}\cdot\left(\nabla_{\Gamma_{n,t}} - \nabla_{\Gamma_{n-1,t}}\right) r_n^s + b\left(\mathbf{u}_n, r_n^s\right) - b\left(\mathbf{u}_{n-1}, r_{n-1}^s\right),$$
$$\mathscr{B}^{(n)} \equiv \overline{\Theta}\mathbf{n}\cdot\left(\nabla^2_{\Gamma_{n,\tau}} - \nabla^2_{\Gamma_{n-1,\tau}}\right)_\tau \int_0^\tau \mathbf{u}_n(\xi, s)\, ds$$
$$+ \overline{\Theta}\mathbf{n}\cdot\left(\nabla^2_{\Gamma_{n,\tau}} - \nabla^2_{\Gamma_{n-1,\tau}}\right)\mathbf{u}_n + B\left(\mathbf{u}_n, r_n^s\right) - B\left(\mathbf{u}_{n-1}, r_{n-1}^s\right),$$
$$\mathscr{G}^{(n)} \equiv \chi\gamma\rho_0^s\left(\nabla^2_{\Gamma_{n,t}} - \nabla^2_{\Gamma_{n-1,t}}\right) r_n^s + G\left(\mathbf{u}_n, r_n^s\right) - G\left(\mathbf{u}_{n-1}, r_{n-1}^s\right). \tag{63}$$

Now, noting that the relations $\sum_{j=1}^3 \partial_{\xi_j} A_{ij} = 0$ hold for the cofactors A_{ij} of the Jacobian matrix of any transformation from ξ to x, by direct calculations, we can verify that the following relations hold:

$$\frac{\partial \mathscr{F}_2^{(n)}}{\partial t} - \nabla_{u_n}\cdot\mathscr{F}_1^{(n)} = \nabla\cdot\mathbf{h}^{(n)}, \quad \mathbf{h}^{(n)} = \sum_{k=1}^3 \partial_{\xi_k}\mathbf{H}_k^{(n)}, \tag{64}$$

where

$$\mathbf{H}_k^{(n)} = -\left(\mathscr{J}_{u_n}^{-1} - \mathscr{J}_{u_{n-1}}^{-1}\right)\mathbf{L}_k^{(n)} - \mathscr{J}_{u_n}^{-1}\mathbf{M}_k^{(n)}$$
$$+ \frac{1}{4\pi}\partial_{\xi_k}\int_\Omega \frac{\mathbf{N}_k^{(n)}(\eta, t)}{|\xi - \eta|} d\eta, \tag{65}$$

with

$$\mathbf{L}_k^{(n)} \equiv \nu A_{ik}^{(n-1)} A_{il}^{(n-1)} \partial_{\xi_l}\mathbf{u}_n - \mathscr{J}_{u_{n-1}}^{*}\mathbf{e}_k q_n,$$
$$\mathbf{M}_k^{(n)} \equiv \nu\left(A_{ik}^{(n)} A_{il}^{(n)} - A_{ik}^{(n-1)} A_{il}^{(n-1)}\right)\partial_{\xi_l}\mathbf{u}_n$$
$$- \left(\mathscr{J}_{u_n}^{*} - \mathscr{J}_{u_{n-1}}^{*}\right)\mathbf{e}_k q_n, \tag{66}$$
$$\mathbf{N}_k^{(n)} \equiv \left\{\partial_t\left(\mathscr{J}_{u_n}^{-1} - \mathscr{J}_{u_{n-1}}^{-1}\right)\right\}\mathbf{u}_n$$
$$- \left\{\partial_{\eta_k}\left(\mathscr{J}_{u_n}^{-1} - \mathscr{J}_{u_{n-1}}^{-1}\right)\right\}\mathbf{L}_k^{(n)} - \left(\partial_{\eta_k}\mathscr{J}_{u_n}^{-1}\right)\mathbf{M}_k^{(n)}.$$

In (66), the Einstein summation convention is used, $A_{ij}^{(n)}$ denotes the (i, j)-component of $\mathscr{J}_n^{*}$, and $\mathbf{e}_k$, $k = 1, 2, 3$, denote fundamental unit vectors in $\mathbf{R}^3$.

For the terms in (63), $\mathbf{h}^{(n)}$, and $\mathbf{H}_k^{(n)}$, we have the following estimate for arbitrary $0 < \epsilon < 1$:

$$\left|\mathscr{F}_1^{(n)}\right|_{\Omega_{0T}}^{(\alpha,(\alpha/2))} + \left|\mathscr{F}_2^{(n)}\right|_{\Omega_{0T}}^{(1+\alpha,((1+\alpha)/2))} + \left|\mathscr{F}_3^{(n)}\right|_{\Gamma_{0T}}^{(1+\alpha,((1+\alpha)/2))}$$
$$+ \left|b^{(n)}\right|_{\Gamma_{0T}}^{(1+\alpha,((1+\alpha)/2))} + \left|\mathscr{B}^{(n)}\right|_{\Gamma_{0T}}^{(\alpha,(\alpha/2))} + \left|\mathbf{h}^{(n)}\right|_{\Omega_{0T}}^{(\alpha,(\alpha/2))}$$
$$+ \sum_{k=1}^3 \left|\mathbf{H}_k^{(n)}\right|_{\Omega_{0T}}^{(1+\alpha,\gamma)} + \left|\mathscr{G}^{(n)}\right|_{\Gamma_{0T}}^{(\alpha,(\alpha/2))}$$
$$\le C_1\left\{\epsilon\left(\left|U^{(n)}\right|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))} + \left|R^{(n)}\right|_{\Gamma_{0T}}^{(2+\alpha,1+(\alpha/2))}\right)\right.$$
$$+ C(\epsilon)\int_0^T\left(\left|U^{(n)}\right|_{\Omega_{0\tau}}^{(2+\alpha,1+(\alpha/2))}\right.$$
$$\left.\left. + \left|R^{(n)}\right|_{\Gamma_{0\tau}}^{(2+\alpha,1+(\alpha/2))}\right) d\tau\right\}. \tag{67}$$

We will derive here the estimate only of $\mathbf{H}_k^{(n)}$ because the other terms can be estimated in a similar and simpler manner. Noting that the following estimates hold for $\mathbf{u}_n$ and q_n:

$$\left|\mathbf{u}_n\right|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))} + \left|q_n\right|_{\Omega_{0T}}^{(1+\alpha,\gamma)} \le C_2, \tag{68}$$

and with the aid of estimates (38) and (39), we have

$$\left|\left(\mathscr{J}_{u_n}^{-1} - \mathscr{J}_{u_{n-1}}^{-1}\right)\mathbf{L}_k^{(n)}\right|_{\Omega_{0T}}^{(1+\alpha,\gamma)}, \left|\mathscr{J}_{u_n}^{-1}\mathbf{M}_k^{(n)}\right|_{\Omega_{0T}}^{(1+\alpha,\gamma)}, \left|\mathbf{N}_k^{(n)}\right|_{\Omega_{0T}}^{(0,((1+\alpha)/2))}$$
$$\le C_3\left(\epsilon\left|U^{(n)}\right|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))}\right.$$
$$\left. + C(\epsilon)\int_0^T \left|U^{(n)}\right|_{\Omega_{0\tau}}^{(2+\alpha,1+(\alpha/2))} d\tau\right). \tag{69}$$

Noting also the following inequality:

$$\left|\partial_{\xi_k}\int_\Omega \frac{\mathbf{N}_k^{(n)}(\eta, t)}{|\xi - \eta|} d\eta\right|_{\Omega_{0T}}^{(1+\alpha,\gamma)} \le C_4\left|\mathbf{N}_k^{(n)}\right|_{\Omega_{0T}}^{(0,((1+\alpha-\gamma)/2))}, \tag{70}$$

from estimate (69) for $\mathbf{N}_k^{(n)}$, we have

$$\left|\partial_{\xi_k}\int_\Omega \frac{\mathbf{N}_k^{(n)}(\eta, t)}{|\xi - \eta|} d\eta\right|_{\Omega_{0T}}^{(1+\alpha,\gamma)}$$
$$\le C_5\left(\epsilon\left|U^{(n)}\right|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))}\right.$$
$$\left. + C(\epsilon)\int_0^T \left|U^{(n)}\right|_{\Omega_{0\tau}}^{(2+\alpha,1+(\alpha/2))} d\tau\right). \tag{71}$$

Thus, from estimates (69) and (71), we have the desired estimate for $\mathbf{H}_k^{(n)}$.

Now, applying Theorem 4 to problem (62), we have the following estimate:

$$\begin{aligned}&\left|U^{(n+1)}\right|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))}+\left\|Q^{(n+1)}\right\|_{T}+\left|R^{(n+1)}\right|_{\Gamma_{0T}}^{(2+\alpha,1+(\alpha/2))}\\&\quad\leq C_6\left\{\epsilon\left(\left|U^{(n)}\right|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))}+\left|R^{(n)}\right|_{\Gamma_{0T}}^{(2+\alpha,1+(\alpha/2))}\right)\right.\\&\qquad\left.+C(\epsilon)\int_0^T\left(\left|U^{(n)}\right|_{\Omega_{0\tau}}^{(2+\alpha,1+(\alpha/2))}+\left|R^{(n)}\right|_{\Gamma_{0\tau}}^{(2+\alpha,1+(\alpha/2))}\right)d\tau\right\},\end{aligned}\tag{72}$$

where the norm $\|f\|_T$ is defined as $\|f\|_T \equiv |f|_{\Omega_{0T}}^{(1+\alpha,\gamma)} + |\nabla f|_{\Omega_{0T}}^{(\alpha,(\alpha/2))} + |f|_{\Gamma_{0T}}^{(1+\alpha,((1+\alpha)/2))}$. Fixing ϵ so that $C_6\epsilon \leq 1/2$ and summing the above inequalities from $n = 1$ to $n = m$, we have

$$S_m(T) \leq C_7\left(S_1(T) + \int_0^T S_m(\tau)\,d\tau\right),\tag{73}$$

where

$$S_m(T) \equiv \sum_{n=1}^{m}\left(\left|U^{(n)}\right|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))}+\left\|Q^{(n)}\right\|_{T}+\left|R^{(n)}\right|_{\Gamma_{0T}}^{(\alpha,(\alpha/2))}\right).\tag{74}$$

Then, using Gronwall's inequality, from the above inequality, we have

$$S_m(T) \leq C_8\left(1 + Te^{C_9T}\right).\tag{75}$$

Noting that the right-hand side in (75) is independent of m, we can conclude that the sequence $\{(\mathbf{u}_n, q_n, r_n^s)\}$ is convergent in $C^{2+\alpha,1+(\alpha/2)}(\Omega_{0T})\times(C^{1+\alpha,\gamma}(\Omega_{0T})\cap C^{1+\alpha,((1+\alpha)/2)}(\Gamma_{0T}))\times C^{2+\alpha,1+(\alpha/2)}(\Gamma_{0T})$.

Now, let us prove Theorem 1. Taking the limit as n tends to infinity in problem (3), we can easily see that the limit of the sequence $\{(\mathbf{u}_n, q_n, r_n^s)\}$ is a solution of problem (11)–(13). The uniqueness can be proved as follows. Let $(\mathbf{u}, q, r^s)$ and $(\tilde{\mathbf{u}}, \tilde{q}, \tilde{r}^s)$ be two solutions of problem (11)–(13). By subtracting one equation from the other, we obtain the equations for the differences $V \equiv \mathbf{u} - \tilde{\mathbf{u}}$, $P \equiv q - \tilde{q}$, and $R \equiv r^s - \tilde{r}^s$, the form of which is similar to (62). Then, in a similar manner to obtain (72), we can obtain the following estimate:

$$\begin{aligned}&|V|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))}+|R|_{\Gamma_{0T}}^{(2+\alpha,1+(\alpha/2))}\\&\quad\leq C_{10}\left\{\epsilon\left(|V|_{\Omega_{0T}}^{(2+\alpha,1+(\alpha/2))}+|R|_{\Gamma_{0T}}^{(2+\alpha,1+(\alpha/2))}\right)\right.\\&\qquad\left.+C(\epsilon)\int_0^T\left(|V|_{\Omega_{0\tau}}^{(2+\alpha,1+(\alpha/2))}+|R|_{\Gamma_{0\tau}}^{(2+\alpha,1+(\alpha/2))}\right)d\tau\right\}.\end{aligned}\tag{76}$$

This inequality implies that $V = \mathbf{0}$ and $R = 0$, and as a consequence, $P = 0$ follows. Thus, the proof of Theorem 1 is completed.

References

[1] Y. D. Shikhmurzaev, *Capillary Flows with Forming Interfaces*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2008.

[2] Y. D. Shikhmurzaev, "The moving contact line on a smooth solid surface," *International Journal of Multiphase Flow*, vol. 19, no. 4, pp. 589–610, 1993.

[3] Y. D. Shikhmurzaev, "Moving contact lines in liquid/liquid/ solid systems," *Journal of Fluid Mechanics*, vol. 334, pp. 211–249, 1997.

[4] Y. D. Shikhmurzaev, "On cusped interfaces," *Journal of Fluid Mechanics*, vol. 359, pp. 313–328, 1998.

[5] H. Wong, D. Rumschitzki, and C. Maldarelli, "On the surfactant mass balance at a deforming fluid interface," *Physics of Fluids*, vol. 8, no. 11, pp. 3203–3204, 1996.

[6] I. Sh. Mogilevskiĭ and V. A. Solonnikov, "On the solvability of an evolution free boundary problem for the Navier-Stokes equations in Hölder spaces of functions," in *Mathematical Problems Relating to the Navier-Stokes Equation*, vol. 11 of *Advances in Mathematics for Applied Sciences*, pp. 105–181, World Science Publications, River Edge, NJ, USA, 1992.

[7] O. A. Lady ženskaja, V. A. Solonnikov, and N. N. Ural'ceva, "Linear and quasi-linear equations of parabolic type," *Translations of Mathematical Monographs*, 1968.

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Research Article

Inverse Coefficient Problem of the Parabolic Equation with Periodic Boundary and Integral Overdetermination Conditions

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This paper investigates the inverse problem of finding a time-dependent diffusion coefficient in a parabolic equation with the periodic boundary and integral overdetermination conditions. Under some assumption on the data, the existence, uniqueness, and continuous dependence on the data of the solution are shown by using the generalized Fourier method. The accuracy and computational efficiency of the proposed method are verified with the help of the numerical examples.

1. Introduction

Denote the domain D_T by

$$D_T = \{(x,t) : 0 < x < 1,\ 0 < t \le T\}. \tag{1}$$

Consider the equation

$$u_t = a(t) u_{xx} + F(x,t), \tag{2}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad 0 \le x \le 1, \tag{3}$$

the periodic boundary condition

$$u(0,t) = u(1,t), \qquad u_x(0,t) = u_x(1,t), \qquad 0 \le t \le T, \tag{4}$$

and the overdetermination condition

$$\int_0^1 x u(x,t)\, dx = E(t), \quad 0 \le t \le T. \tag{5}$$

The problem of finding a pair $\{a(t), u(x,t)\}$ in (2)–(5) will be called an inverse problem.

Definition 1. The pair $\{a(t), u(x,t)\}$ from the class $C[0,T] \times C^{2,1}(D_T) \cap C^{1,0}(\overline{D}_T)$ for which conditions (2)–(5) is satisfied and $a(t) > 0$ on the interval $[0,T]$, is called a classical solution of the inverse problem (2)–(5).

The parameter identification in a parabolic differential equation from the data of integral overdetermination condition plays an important role in engineering and physics [1–7]. This integral condition in parabolic problems is also called heat moments [5].

Boundary value problems for parabolic equations in one or two local classical conditions are replaced by heat moments [8–13]. These kinds of conditions such as (5) arise from many important applications in heat transfer, thermoelasticity, control theory, life sciences, and so forth. For example, in heat propagation in a thin rod, the law of variation $E(t)$ of the total quantity of heat in the rod is given in [8]. In [12], a physical-mechanical interpretation of the integral conditions was also given.

Various statements of inverse problems on determination of thermal coefficient in one-dimensional heat equation were studied in [4, 5, 7, 14]. In papers [4, 5, 7], the time-dependent thermal coefficient is determined from the heat moment.

Boundary value problems and inverse problems for parabolic equations with periodic boundary conditions are investigated in [15, 16].

In the present work, one heat moment is used with periodic boundary condition for the determination of thermal coefficient. The existence and uniqueness of the classical solution of the problem (2)–(5) is reduced to fixed point principles by applying the Fourier method.

This paper organized as follows. In Section 2, the existence and uniqueness of the solution of inverse problem (2)–(5) are proved by using the Fourier method. In Section 3, the continuous dependence on the solution of the inverse problem is shown. In Section 4, the numerical procedure for the solution of the inverse problem using the Crank-Nicolson scheme combined with an iteration method is given. Finally, in Section 5, numerical experiments are presented and discussed.

2. Existence and Uniqueness of the Solution of the Inverse Problem

We have the following assumptions on the data of the problem (2)–(5).

(A_1) $E(t) \in C^1[0,T]$, $E'(t) > 0$, for all $t \in [0,T]$;

(A_2) $\varphi(x) \in C^4[0,1]$;

(1) $\varphi(0) = \varphi(1)$, $\varphi'(0) = \varphi'(1)$, $\varphi''(0) = \varphi''(1)$, $\int_0^1 x\varphi(x)dx = E(0)$;

(2) $\varphi_n \geq 0$, $n = 1, 2, \ldots$;

(A_3) $F(x,t) \in C(\overline{D}_T)$; $F(x,t) \in C^4[0,1]$ for arbitrary fixed $t \in [0,T]$;

(1) $F(0,t) = F(1,t)$, $F_x(0,t) = F_x(1,t)$, $F_{xx}(0,t) = F_{xx}(1,t)$;

(2) $F_n(t) \geq 0$, $n = 1, 2, \ldots$,

where $\varphi_n = \int_0^1 \varphi(x)\sin(2\pi nx)dx$, $F_n(t) = \int_0^1 F(x,t)\sin(2\pi nx)dx$, $n = 0, 1, 2, \ldots$.

Theorem 2. *Let the assumptions (A_1)–(A_3) be satisfied. Then the following statements are true.*

(1) *The inverse problem* (2)–(5) *has a solution in D_T.*

(2) *The solution of inverse problem* (2)–(5) *is unique in D_{T_0}, where the number T_0 $(0 < T_0 < T)$ is determined by the data of the problem.*

Proof. By applying the standard procedure of the Fourier method, we obtain the following representation for the solution of (2)–(4) for arbitrary $a(t) \in C[0,T]$:

$$u(x,t) = \sum_{n=1}^{\infty} \left[\varphi_n e^{-(2\pi n)^2 \int_0^t a(s)ds} + \int_0^t F_n(\tau) e^{-(2\pi n)^2 \int_\tau^t a(s)ds} d\tau \right] \times \sin(2\pi nx). \tag{6}$$

The assumptions $\varphi(0) = \varphi(1)$, $\varphi'(0) = \varphi'(1)$, $F(0,t) = F(1,t)$, and $F_x(0,t) = F_x(1,t)$ are consistent conditions for the representation (2) of the solution $u(x,t)$ to be valid. Furthermore, under the smoothness assumptions $\varphi(x) \in C^4[0,1]$, $F(x,t) \in C(\overline{D}_T)$, and $F(x,t) \in C^4[0,1]$ for all $t \in [0,T]$, the series (6) and its x-partial derivative converge uniformly in $\overline{D}_T$ since their majorizing sums are absolutely convergent. Therefore, their sums $u(x,t)$ and $u_x(x,t)$ are continuous in $\overline{D}_T$. In addition, the t-partial derivative and the xx-second-order partial derivative series are uniformly convergent for $t \geq \varepsilon > 0$ (ε is an arbitrary positive number). Thus, $u(x,t) \in C^{2,1}(D_T) \cap C^{1,0}(\overline{D}_T)$ and satisfies the conditions (2)–(4). In addition, $u_t(x,t)$ is continuous in $\overline{D}_T$ because the majorizing sum of t-partial derivative series is absolutely convergent under the condition $\varphi''(0) = \varphi''(1)$ and $f_{xx}(0,t) = f_{xx}(1,t)$ in $\overline{D}_T$. Equation (6) can be differentiated under the condition (A_1) to obtain

$$\int_0^1 x u_t(x,t)\, dx = E'(t), \tag{7}$$

and this yields

$$a(t) = P[a(t)], \tag{8}$$

where

$$P[a(t)] = \frac{E'(t) + \sum_{n=1}^{\infty} (1/2\pi n) F_n(t)}{\sum_{n=1}^{\infty} 2\pi n \left(\varphi_n e^{-(2\pi n)^2 \int_0^t a(s)ds} + \int_0^t F_n(\tau) e^{-(2\pi n)^2 \int_\tau^t a(s)ds} d\tau \right)}. \tag{9}$$

Denote

$$C_0 = \min_{t\in[0,T]} E'(t) + \min_{t\in[0,T]} \left(\sum_{n=1}^{\infty} \frac{1}{2\pi n} F_n(t) \right),$$
$$C_1 = \max_{t\in[0,T]} E'(t) + \max_{t\in[0,T]} \left(\sum_{n=1}^{\infty} \frac{1}{2\pi n} F_n(t) \right),$$
$$C_2 = E'(0), \qquad C_3 = \sum_{k=1}^{\infty} 2\pi n \left(\varphi_n + \int_0^T F_n(\tau)\, d\tau \right). \tag{10}$$

Using the representation (8), the following estimate is true:

$$0 < \frac{C_0}{C_3} \leq a(t) \leq \frac{C_1}{C_2}. \tag{11}$$

Introduce the set M as

$$M = \left\{ a(t) \in C[0,T] : \frac{C_0}{C_3} \leq a(t) \leq \frac{C_1}{C_2} \right\}. \tag{12}$$

It is easy to see that

$$P : M \longrightarrow M. \tag{13}$$

Compactness of P is verified by analogy to [7]. By virtue of Schauder's fixed-point theorem, we have a solution $a(t) \in C[0,T]$ of (8).

Now let us show that there exists D_{T_0} $(0 < T_0 \leq T)$ for which the solution (a, u) of the problem (2)–(5) is unique in

D_{T_0}. Suppose that (b, v) is also a solution pair of the problem (2)–(5). Then from the representations (6) and (8) of the solution, we have

$$\begin{aligned} & u(x,t) - v(x,t) \\ & = \sum_{n=1}^{\infty} \varphi_n \left(e^{-(2\pi n)^2 \int_0^t a(s)ds} - e^{-(2\pi n)^2 \int_0^t b(s)ds} \right) \sin 2\pi n(x) \\ & + \sum_{n=1}^{\infty} \left(\int_0^t F_n(\tau) \left(e^{-(2\pi n)^2 \int_\tau^t a(s)ds} - e^{-(2\pi n)^2 \int_\tau^t b(s)ds} \right) d\tau \right) \\ & \times \sin 2\pi n(x), \\ & a(t) - b(t) = P[a(t)] - P[b(t)], \end{aligned} \tag{14}$$

where

$$\begin{aligned} & P[a(t)] - P[b(t)] \\ & = \frac{E'(t) + \sum_{n=1}^{\infty} (1/2\pi n) F_n(t)}{\sum_{n=1}^{\infty} 2\pi n \left(\varphi_n e^{-(2\pi n)^2 \int_0^t a(s)ds} + \int_0^t F_n(\tau) e^{-(2\pi n)^2 \int_\tau^t a(s)ds} d\tau \right)} \\ & - \frac{E'(t) + \sum_{n=1}^{\infty} (1/2\pi n) F_n(t)}{\sum_{n=1}^{\infty} 2\pi n \left(\varphi_n e^{-(2\pi n)^2 \int_0^t b(s)ds} + \int_0^t F_n(\tau) e^{-(2\pi n)^2 \int_\tau^t b(s)ds} d\tau \right)}. \end{aligned} \tag{15}$$

The following estimate is true:

$$\begin{aligned} & |P[a(t)] - P[b(t)]| \\ & \leq \frac{\left(E'(t) + \sum_{n=1}^{\infty} (1/2\pi n) F_n(t)\right)}{C_2^2} \\ & \cdot \left(\sum_{n=1}^{\infty} 2\pi n \varphi_n \left(e^{-(2\pi n)^2 \int_0^t a(s)ds} - e^{-(2\pi n)^2 \int_0^t b(s)ds} \right) + \sum_{n=1}^{\infty} 2\pi n \right. \\ & \left. \times \left(\int_0^t F_n(\tau) \left(e^{-(2\pi n)^2 \int_\tau^t a(s)ds} - e^{-(2\pi n)^2 \int_\tau^t b(s)ds} \right) d\tau \right) \right). \end{aligned} \tag{16}$$

Using the estimates

$$\begin{aligned} & \left| e^{-(2\pi n)^2 \int_0^t a(s)ds} - e^{-(2\pi n)^2 \int_0^t b(s)ds} \right| \\ & \leq (2\pi n)^2 T \max_{0 \leq t \leq T} |a(t) - b(t)|, \\ & \left| e e^{-(2\pi n)^2 \int_\tau^t a(s)ds} - e^{-(2\pi n)^2 \int_\tau^t b(s)ds} \right| \\ & \leq (2\pi n)^2 T \max_{0 \leq t \leq T} |a(t) - b(t)|, \end{aligned} \tag{17}$$

we obtain

$$\max_{0 \leq t \leq T} |P[a(t)] - P[b(t)]| \leq \alpha \max_{0 \leq t \leq T} |a(t) - b(t)|. \tag{18}$$

Let $\alpha \in (0,1)$ be arbitrary fixed number. Fix a number T_0, $0 < T_0 \leq T$, such that

$$\frac{C_1(C_4 + C_5)}{C_2^2} T_0 \leq \alpha. \tag{19}$$

Then from the equality (10), we obtain

$$\|a - b\|_{C[0,T_0]} \leq \alpha \|a - b\|_{C[0,T_0]}, \tag{20}$$

which implies that $a = b$. By substituting $a = b$ in (9), we have $u = v$. □

3. Continuous Dependence of (a, u) on the Data

Theorem 3. *Under assumptions (A_1)–(A_3), the solution (a, u) of the problem (2)–(5) depends continuously on the data for small T.*

Proof. Let $\Phi = \{\varphi, F, E\}$ and $\overline{\Phi} = \{\overline{\varphi}, \overline{F}, \overline{E}\}$ be two sets of the data, which satisfy the assumptions (A_1)–(A_3). Then there exist positive constants M_i, $i = 1,2,3$ such that

$$\begin{aligned} & \|\varphi\|_{C^4[0,1]} \leq M_1, \\ & \|F\|_{C^{4,0}(\overline{D}_T)} \leq M_2, \\ & \|E\|_{C^1[0,T]} \leq M_3, \\ & \|\overline{\varphi}\|_{C^4[0,1]} \leq M_1, \\ & \|\overline{F}\|_{C^{4,0}(\overline{D}_T)} \leq M_2, \\ & \|\overline{E}\|_{C^1[0,T]} \leq M_3. \end{aligned} \tag{21}$$

Let (a, u) and $(\overline{a}, \overline{u})$ be solutions of the inverse problem (2)–(5) corresponding to the data Φ and $\overline{\Phi}$, respectively. According to (8),

$$\begin{aligned} & a(t) \\ & = \frac{E'(t) + \sum_{n=1}^{\infty} (1/2\pi n) F_n(t)}{\sum_{n=1}^{\infty} 2\pi n \left(\varphi_n e^{-(2\pi n)^2 \int_0^t a(s)ds} + \int_0^t F_n(\tau) e^{-(2\pi n)^2 \int_\tau^t a(s)ds} d\tau \right)}, \\ & \overline{a}(t) \\ & = \frac{\overline{E}'(t) + \sum_{n=1}^{\infty} (1/2\pi n) \overline{F}_n(t)}{\sum_{n=1}^{\infty} 2\pi n \left(\overline{\varphi}_n e^{-(2\pi n)^2 \int_0^t \overline{a}(s)ds} + \int_0^t \overline{F}_n(\tau) e^{-(2\pi n)^2 \int_\tau^t \overline{a}(s)ds} d\tau \right)}. \end{aligned} \tag{22}$$

First let us estimate the difference $a - \overline{a}$. It is easy to compute that

$$
\begin{aligned}
&\left| E'(t) \sum_{n=1}^{\infty} 2\pi n \overline{\varphi}_n e^{-(2\pi n)^2 \int_0^t \overline{a}(s)ds} \right. \\
&\quad \left. - \overline{E}'(t) \sum_{n=1}^{\infty} 2\pi n \varphi_n e^{-(2\pi n)^2 \int_0^t a(s)ds} \right| \\
&\quad \leq M_4 \left\| E - \overline{E} \right\|_{C^1[0,T]} + M_5 \left\| \varphi - \overline{\varphi} \right\|_{C^4[0,1]} \\
&\qquad + M_6 \left\| a - \overline{a} \right\|_{C[0,T]}, \\
&\left| E'(t) \sum_{n=1}^{\infty} 2\pi n \int_0^t \overline{F}_n(\tau) e^{-(2\pi n)^2 \int_\tau^t \overline{a}(s)ds} d\tau \right. \\
&\quad \left. - \overline{E}'(t) \sum_{n=1}^{\infty} 2\pi n \int_0^t F_n(\tau) e^{-(2\pi n)^2 \int_\tau^t a(s)ds} d\tau \right| \\
&\quad \leq M_7 T \left\| E - \overline{E} \right\|_{C^1[0,T]} + M_5 T \left\| F - \overline{F} \right\|_{C^{4,0}(\overline{D_T})} \\
&\qquad + M_8 \left\| a - \overline{a} \right\|_{C[0,T]}, \\
&\left| \sum_{n=1}^{\infty} \frac{1}{2\pi n} F_n(t) \sum_{n=1}^{\infty} 2\pi n \overline{\varphi}_n e^{-(2\pi n)^2 \int_0^t \overline{a}(s)ds} \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \frac{1}{2\pi n} \overline{F}_n(t) \sum_{n=1}^{\infty} 2\pi n \varphi_n e^{-(2\pi n)^2 \int_0^t a(s)ds} \right| \\
&\quad \leq 2\sqrt{6} M_4 \left\| F - \overline{F} \right\|_{C^{4,0}(\overline{D_T})} + 2\sqrt{6} M_7 \left\| \varphi - \overline{\varphi} \right\|_{C^4[0,1]} \\
&\qquad + M_9 \left\| a - \overline{a} \right\|_{C[0,T]}, \\
&\left| \sum_{n=1}^{\infty} \frac{1}{2\pi n} F_n(t) \sum_{n=1}^{\infty} 2\pi n \int_0^t \overline{F}_n(\tau) e^{-(2\pi n)^2 \int_\tau^t \overline{a}(s)ds} \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \frac{1}{2\pi n} \overline{F}_n(t) \sum_{n=1}^{\infty} 2\pi n \int_0^t F_n(\tau) e^{-(2\pi n)^2 \int_\tau^t a(s)ds} d\tau \right| \\
&\quad \leq \sqrt{6} T M \left\| F - \overline{F} \right\|_{C^{4,0}(\overline{D_T})} + M_{10} \left\| a - \overline{a} \right\|_{C[0,T]},
\end{aligned}
\tag{23}
$$

where M_k, $k = 4, \ldots, 10$, are some constants.

If we consider these estimates in $a - \overline{a}$, we obtain

$$
\begin{aligned}
&\left(1 - M_{11}\right) \left\| a - \overline{a} \right\|_{C[0,T]} \\
&\quad \leq M_{12} \left(\left\| E - \overline{E} \right\|_{C^1[0,T]} + \left\| \varphi - \overline{\varphi} \right\|_{C^4[0,1]} + \left\| F - \overline{F} \right\|_{C^{4,0}(\overline{D_T})} \right).
\end{aligned}
\tag{24}
$$

The inequality $M_{11} < 1$ holds for small T. Finally, we obtain

$$
\left\| a - \overline{a} \right\|_{C[0,T]} \leq M_{13} \left\| \Phi - \overline{\Phi} \right\|, \quad M_{13} = \frac{M_{12}}{\left(1 - M_{11}\right)},
\tag{25}
$$

where $\left\| \Phi - \overline{\Phi} \right\| = \left\| E - \overline{E} \right\|_{C^1[0,T]} + \left\| \varphi - \overline{\varphi} \right\|_{C^4[0,1]} + \left\| F - \overline{F} \right\|_{C^{4,0}(\overline{D_T})}$.

From (6), a similar estimate is also obtained for the difference $u - \overline{u}$ as

$$
\left\| u - \overline{u} \right\|_{C(\overline{D_T})} \leq M_{14} \left\| \Phi - \overline{\Phi} \right\|.
\tag{26}
$$

□

4. Numerical Method

We use the finite difference method with a predictor-corrector-type approach, that is suggested in [2]. Apply this method to the problem (2)–(5).

We subdivide the intervals $[0, 1]$ and $[0, T]$ into N_x and N_t subintervals of equal lengths $h = (1/N_x)$ and $\tau = (T/N_t)$, respectively. Then we add two lines $x = 0$ and $x = (N_x + 1)h$ to generate the fictitious points needed for dealing with the second boundary condition. We choose the Crank-Nicolson scheme, which is absolutely stable and has a second-order accuracy in both h and τ [15]. The Crank-Nicolson scheme for (2)–(5) is as follows:

$$
\begin{gathered}
\begin{aligned}
&\frac{1}{\tau} \left(u_i^{j+1} - u_i^j \right) \\
&\quad = \frac{1}{2} \left(a^{j+1} + a^j \right) \frac{1}{2h^2} \\
&\qquad \times \left[\left(u_{i-1}^j - 2u_i^j + u_{i+1}^j \right) + \left(u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1} \right) \right] \\
&\qquad + \frac{1}{2} \left(F_i^{j+1} + F_i^j \right),
\end{aligned} \\
u_i^0 = \phi_i, \\
u_0^j = u_{N_x}^j, \\
u_1^j = u_{N_x+1}^j,
\end{gathered}
\tag{27}
$$

where $1 \leq i \leq N_x$ and $0 \leq j \leq N_t$ are the indices for the spatial and time steps, respectively, $u_i^j = u(x_i, t_j)$, $\phi_i = \varphi(x_i)$, $F_i^j = F(x_i, t_j)$, and $x_i = ih$, $t_j = j\tau$. At the $t = 0$ level, adjustment should be made according to the initial condition and the compatibility requirements.

Equation (27) form an $N_x \times N_x$ linear system of equations

$$
AU^{j+1} = b,
\tag{28}
$$

where

$$
\begin{gathered}
U^j = \left(u_1^j, u_2^j, \ldots, u_{N_x}^j \right)^{\mathrm{tr}}, \quad 1 \leq j \leq N_t, \\
b = \left(b_1, b_2, \ldots, b_{N_x} \right)^{\mathrm{tr}},
\end{gathered}
$$

$$
A = \begin{bmatrix}
-2(1+R) & 1 & 0 & \cdots & & & 0 & 1 \\
1 & -2(1+R) & 1 & 0 & \cdots & & & 0 \\
0 & 1 & -2(1+R) & 1 & 0 & \cdots & & 0 \\
\vdots & & & \ddots & & & & \\
 & & & & 0 & 1 & -2(1+R) & 1 \\
1 & & & & & 0 & 1 & -2(1+R)
\end{bmatrix},
$$

$$R = \frac{2h^2}{\tau\left(a^{j+1} + a^j\right)}, \quad j = 0, 1, \ldots, N_t,$$

$$b_1 = 2(1-R)u_1^j - u_2^j - u_{N_x}^j - R\tau\left(F_1^{j+1} + F_1^j\right), \quad j = 0, 1, \ldots, N_t,$$

$$b_{N_x} = -u_{N_x-1}^j + 2(1-R)u_{N_x}^j - u_1^j - R\tau\left(F_{N_x}^{j+1} + F_{N_x}^j\right), \quad j = 0, 1, \ldots, N_t,$$

$$b_i = -u_{i-1}^j + 2(1-R)u_i^j - u_{i+1}^j - R\tau\left(F_i^{j+1} + F_i^j\right), \quad i = 2, 3, \ldots, N_x - 1, \; j = 0, 1, \ldots, N_t. \tag{29}$$

Now, let us construct the predicting-correcting mechanism. First, multiplying (2) by x from 0 to 1 and using (4) and (5), we obtain

$$a(t) = \frac{E'(t) - \int_0^1 xF(x,t)\,dx}{u_x(1,t)}. \tag{30}$$

The finite difference approximation of (30) is

$$a^j = \frac{\left[\left(\left(E^{j+1} - E^j\right)/\tau\right) - (\mathrm{Fin})^j\right]h}{u_{N_x+1}^j - u_{N_x}^j}, \tag{31}$$

where $E^j = E(t_j)$, $(\mathrm{Fin})^j = \int_0^1 xF(x,t_j)dx$, $j = 0, 1, \ldots, N_t$. For $j = 0$,

$$a^0 = \frac{\left[\left(\left(E^1 - E^0\right)/\tau\right) - (\mathrm{Fin})^0\right]h}{\phi_{N_x+1} - \phi_{N_x}}, \tag{32}$$

and the values of ϕ_i help us to start our computation. We denote the values of a^j, u_i^j at the sth iteration step $a^{j(s)}$, $u_i^{j(s)}$, respectively. In numerical computation, since the time step is very small, we can take $a^{j+1(0)} = a^j$, $u_i^{j+1(0)} = u_i^j$, $j = 0, 1, 2, \ldots N_t$, $i = 1, 2, \ldots, N_x$. At each $(s+1)$th iteration step, we first determine $a^{j+1(s+1)}$ from the formula

$$a^{j+1(s+1)} = \frac{\left[\left(\left(E^{j+2} - E^{j+1}\right)/\tau\right) - (\mathrm{Fin})^{j+1}\right]h}{u_{N_x+1}^{j+1(s)} - u_{N_x}^{j+1(s)}}. \tag{33}$$

Then from (27) we obtain

$$\begin{aligned} &\frac{1}{\tau}\left(u_i^{j+1(s+1)} - u_i^{j+1(s)}\right) \\ &\quad = \frac{1}{4h^2}\left(a^{j+1(s+1)} + a^{j+1(s)}\right) \\ &\qquad \times \left[\left(u_{i-1}^{j+1(s+1)} - 2u_i^{j+1(s+1)} + u_{i+1}^{j+1(s+1)}\right)\right. \\ &\qquad\quad \left. + \left(u_{i-1}^{j+1(s)} - 2u_i^{j+1(s)} + u_{i+1}^{j+1(s)}\right)\right] \\ &\qquad + \frac{1}{2}\left(F_i^{j+1} + F_i^j\right), \\ &\qquad\qquad u_0^{j+1(s)} = u_{N_x}^{j+1(s)}, \\ &\quad u_1^{j+1(s)} = u_{N_x+1}^{j+1(s)}, \quad s = 0, 1, 2, \ldots. \end{aligned} \tag{34}$$

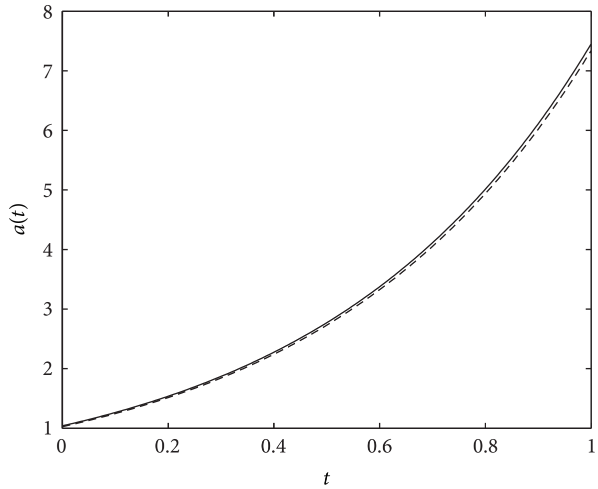

FIGURE 1: The analytical and numerical solutions of $a(t)$ when $T = 1$. The analytical solution is shown with dashed line.

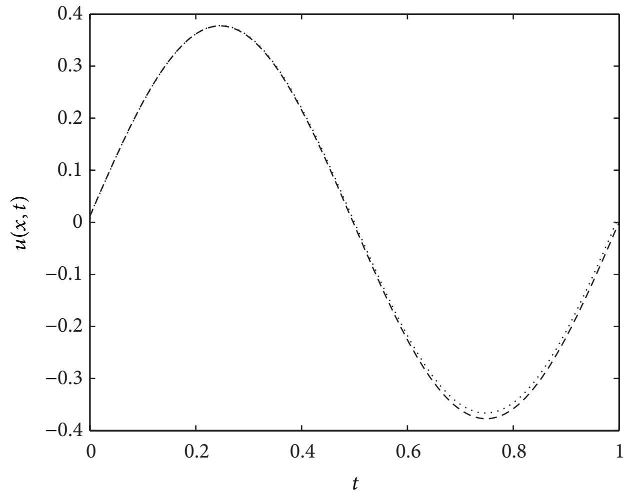

FIGURE 2: The analytical and numerical solutions of $u(x,t)$ at the $T = 1$. The analytical solution is shown with dashed line.

The system of (34) can be solved by the Gauss elimination method and $u_i^{j+1(s+1)}$ is determined. If the difference of values between two iterations reaches the prescribed tolerance, the iteration is stopped and we accept the corresponding values $a^{j+1(s+1)}$, $u_i^{j+1(s+1)}$ $(i = 1, 2, \ldots, N_x)$ as a^{j+1}, u_i^{j+1} $(i = 1, 2, \ldots, N_x)$, on the $(j+1)$th time step, respectively. In virtue of this iteration, we can move from level j to level $j+1$.

5. Numerical Examples and Discussions

Example 1. Consider the inverse problem (2)–(5), with

$$\begin{gathered} F(x,t) = (2\pi)^2 \sin(2\pi x)\exp(t), \\ \varphi(x) = \sin(2\pi x), \\ E(t) = -\frac{1}{2\pi}\exp(-t), \\ x \in [0,1], \; t \in [0,T]. \end{gathered} \tag{35}$$

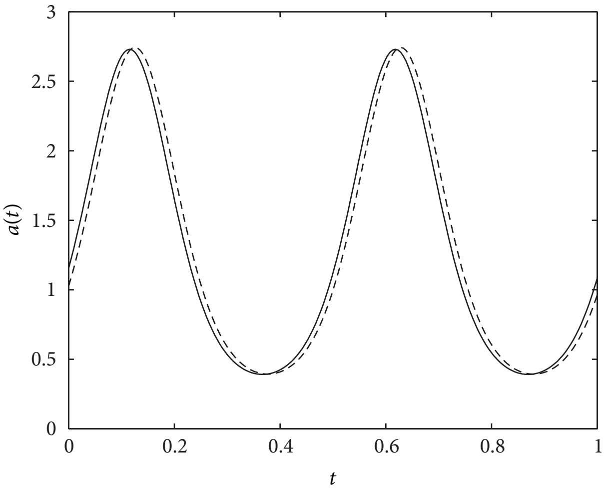

FIGURE 3: The analytical and numerical solutions of $a(t)$ when $T = 1$. The analytical solution is shown with dashed line.

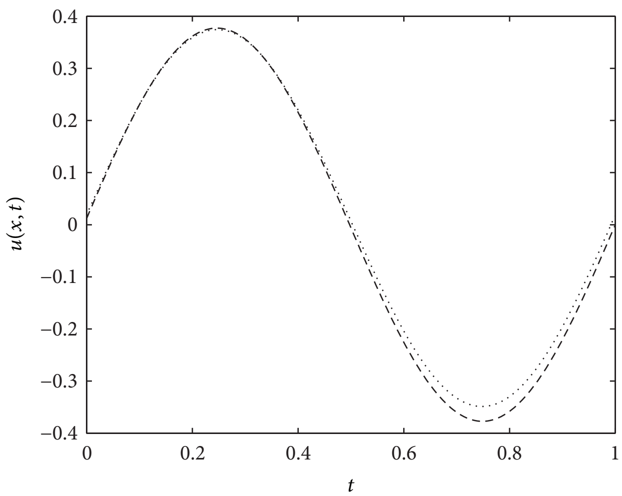

FIGURE 4: The analytical and numerical solutions of $u(x,t)$ at the $T = 1$. The analytical solution is shown with dashed line.

It is easy to check that the analytical solution of the problem (2)–(5) is

$$\{a(t), u(x,t)\} = \left\{ \frac{1}{(2\pi)^2} + \exp(2t),\ \sin(2\pi x)\exp(-t) \right\}. \tag{36}$$

Let us apply the scheme which was explained in the previous section for the step sizes $h = 0.005$, $\tau = 0.005$.

In the case when $T = 1$, the comparisons between the analytical solution (36) and the numerical finite difference solution are shown in Figures 1 and 2.

Example 2. Consider the inverse problem (2)–(5), with

$$\begin{gathered} F(x,t) = (2\pi)^2 \sin(2\pi x)\exp(-t + \sin(4\pi t)), \\ \varphi(x) = \sin(2\pi x), \\ E(t) = -\frac{1}{2\pi}\exp(-t), \\ x \in [0,1],\ t \in [0,T]. \end{gathered} \tag{37}$$

It is easy to check that the analytical solution of the problem (2)–(5) is

$$\{a(t), u(x,t)\} = \left\{ \frac{1}{(2\pi)^2} + \exp(\sin(4\pi t)),\ \sin(2\pi x)\exp(-t) \right\}. \tag{38}$$

Let us apply the scheme which was explained in the previous section for the step sizes $h = 0.01$, $\tau = h/8$.

In the case when $T = 1$, the comparisons between the analytical solution (38) and the numerical finite difference solution are shown in Figures 3 and 4.

6. Conclusions

The inverse problem regarding the simultaneously identification of the time-dependent thermal diffusivity and the temperature distribution in one-dimensional heat equation with periodic boundary and integral overdetermination conditions has been considered. This inverse problem has been investigated from both theoretical and numerical points of view. In the theoretical part of the paper, the conditions for the existence, uniqueness, and continuous dependence on the data of the problem have been established. In the numerical part, the sensitivity of the Crank-Nicolson finite-difference scheme combined with an iteration method with the examples has been illustrated.

References

[1] J. R. Cannon, Y. P. Lin, and S. Wang, "Determination of a control parameter in a parabolic partial differential equation," *The Journal of the Australian Mathematical Society B*, vol. 33, no. 2, pp. 149–163, 1991.

[2] J. R. Cannon, Y. Lin, and S. Wang, "Determination of source parameter in parabolic equations," *Meccanica*, vol. 27, no. 2, pp. 85–94, 1992.

[3] A. G. Fatullayev, N. Gasilov, and I. Yusubov, "Simultaneous determination of unknown coefficients in a parabolic equation," *Applicable Analysis*, vol. 87, no. 10-11, pp. 1167–1177, 2008.

[4] M. I. Ivanchov, "Inverse problems for the heat-conduction equation with nonlocal boundary condition," *Ukrainian Mathematical Journal*, vol. 45, no. 8, pp. 1186–1192, 1993.

[5] M. I. Ivanchov and N. V. Pabyrivska, "Simultaneous determination of two coefficients in a parabolic equation in the case of nonlocal and integral conditions," *Ukrainian Mathematical Journal*, vol. 53, no. 5, pp. 674–684, 2001.

[6] M. I. Ismailov and F. Kanca, "An inverse coefficient problem for a parabolic equation in the case of nonlocal boundary and overdetermination conditions," *Mathematical Methods in the Applied Sciences*, vol. 34, no. 6, pp. 692–702, 2011.

[7] F. Kanca and M. I. Ismailov, "The inverse problem of finding the time-dependent diffusion coefficient of the heat equation from integral overdetermination data," *Inverse Problems in Science and Engineering*, vol. 20, no. 4, pp. 463–476, 2012.

[8] J. R. Cannon, "The solution of the heat equation subject to the specification of energy," *Quarterly of Applied Mathematics*, vol. 21, pp. 155–160, 1963.

[9] L. I. Kamynin, "A boundary value problem in the theory of heat conduction with a nonclassical boundary condition," *Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki*, vol. 4, no. 6, pp. 1006–1024, 1964.

[10] N. I. Ionkin, "Solution of a boundary-value problem in heat conduction with a nonclassical boundary condition," *Differential Equations*, vol. 13, pp. 204–211, 1977.

[11] N. I. Yurchuk, "Mixed problem with an integral condition for certain parabolic equations," *Differential Equations*, vol. 22, pp. 1457–1463, 1986.

[12] V. M. Vīgak, "Construction of a solution of the heat conduction problem with integral conditions," *Doklady Akademii Nauk Ukrainy*, vol. 8, pp. 57–60, 1994.

[13] N. I. Ivanchov, "Boundary value problems for a parabolic equation with integral conditions," *Differential Equations*, vol. 40, no. 4, pp. 591–609, 2004.

[14] W. Liao, M. Dehghan, and A. Mohebbi, "Direct numerical method for an inverse problem of a parabolic partial differential equation," *Journal of Computational and Applied Mathematics*, vol. 232, no. 2, pp. 351–360, 2009.

[15] I. Sakinc, "Numerical solution of a quasilinear parabolic problem with periodic boundary condition," *Hacettepe Journal of Mathematics and Statistics*, vol. 39, no. 2, pp. 183–189, 2010.

[16] J. Choi, "Inverse problem for a parabolic equation with space-periodic boundary conditions by a Carleman estimate," *Journal of Inverse and Ill-Posed Problems*, vol. 11, no. 2, pp. 111–135, 2003.

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Research Article

Measure Functional Differential Equations in the Space of Functions of Bounded Variation

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We establish general conditions for the unique solvability of nonlinear measure functional differential equations in terms of properties of suitable linear majorants.

1. Introduction, Motivation, and Problem Setting

Let $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n \ni x = (x_k)_{k=1}^n \mapsto \|x\| := \max_{1\le k\le n}|x_k|$ be the norm in $\mathbb{R}^n$, and let $\mathrm{BV}([a,b],\mathbb{R}^n)$ be the Banach space of functions of bounded variation with the standard norm $\mathrm{BV}([a,b],\mathbb{R}^n) \ni u \mapsto \|u\|_{\mathrm{BV}} := |u(a)| + \mathrm{Var}_{[a,b]}u$, where $-\infty < a < b < \infty$.

Our aim is to examine the solvability of the equation

$$u(t) = \varphi(u) + \int_a^t (fu)(s)\,dg(s), \quad t \in [a,b]; \tag{1}$$

$f : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ is a, generally speaking, nonlinear operator and $\varphi : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is a nonlinear vector functional. The integral on the right-hand side of (1) is the Kurzweil-Stieltjes integral with respect to a nondecreasing function $g : [a,b] \to \mathbb{R}$. We refer to [1–5] for the definition and properties of this kind of an integral, recalling only that (1) is a particular case of a generalised differential equation [2, 6]. It is important to note that, for any $u \in \mathrm{BV}([a,b],\mathbb{R}^n)$, the Kurzweil-Stieltjes integral in (1) exists (see, e.g., [4, 7]) and, therefore, the equation itself makes sense.

By a *solution* of (1), we mean a vector function $u : [a,b] \to \mathbb{R}^n$ which has bounded variation and satisfies (1) on the interval $[a,b]$.

Equation (1) is an extension of a measure differential equation studied systematically, for example, in [2, 8–10]. It is a fairly general object which includes many other types of equations such as differential equations with impulses [11] or functional dynamic equations on time scales [12] (see, e.g., [13, 14]). In particular, if $g(s) = s$, $s \in [a,b]$, (1) takes the form

$$u(t) = \varphi(u) + \int_a^t (fu)(s)\,ds, \quad t \in [a,b], \tag{2}$$

and, thus, in the absolutely continuous case, reduces to the nonlocal boundary value problem for a functional differential equation

$$u'(t) = (fu)(t), \quad t \in [a,b], \qquad u(a) = \varphi(u), \tag{3}$$

whose various particular types are the object of investigation of many authors (see, e.g., [15–19]). A more general choice of g in (1) allows one to cover further cases where solutions lose their absolute continuity at some points. For example, consider the impulsive functional differential equation [16, page 191]

$$\begin{aligned} u'(t) &= (fu)(t), \quad t \in [a,b] \setminus \{t_1, t_2, \dots, t_m\}, \\ \Delta u(t) &= I_i(u(t)) \quad \text{for } t = t_i,\ i \in \{1, 2, \dots, m\}, \end{aligned} \tag{4}$$

where $\Delta u(t) := u(t+) - u(t-)$ for any function u from $\mathrm{BV}([a,b],\mathbb{R}^n)$ (in fact, $\Delta u(t) = u(t+) - u(t)$ if, as is customary [11] in that context, a solution is assumed to be left continuous). Here, $f = (f_k)_{k=1}^n : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, the jumps may occur at the preassigned times $t_1, t_2, \ldots, t_m$, and their action is described by the operators $I_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, m$. By the usual integration argument [11], one can represent (4) alternatively in the form

$$u(t) = u(a) + \int_a^t (fu)(s)\,ds + \sum_{k: a<t_k<t,} I_k(u(t_k)), \quad t \in [a,b]. \tag{5}$$

It follows, in particular, from [14, Lemma 2.4] that (5) is equivalent to the measure functional differential equation

$$u(t) = u(a) + \int_a^t \left(\tilde{f}u\right)(s)\,dg(s), \quad t \in [a,b], \tag{6}$$

with $g(s) = s + \sum_{i=1}^m \chi_{(t_i,b]}(s)$, $s \in [a,b]$, and $\tilde{f} : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ defined by the relation

$$\left(\tilde{f}u\right)(s) = \begin{cases} (fu)(s) & \text{if } s \in [a,b] \setminus \{t_1, t_2, \ldots, t_m\}, \\ I_i(u(s)) & \text{if } s \in \{t_1, t_2, \ldots, t_m\}. \end{cases} \tag{7}$$

Thus, system (4) can be considered as a particular case of (1). Likewise, an appropriate construction [13, 20] allows one to regard differential equations on time scales [12] as measure differential equations. The same is true for equations involving functional components; in the case of a differential equation on a time scale with retarded argument, by choosing g suitably [13], one arrives at the equation

$$u(t) = u(a) + \int_a^t h(u_s, s)\,dg(s), \quad t \in [a,b], \qquad u_a = \phi. \tag{8}$$

In (8), $h : C([-r,0],\mathbb{R}^n) \times [a,b] \to \mathbb{R}^n$ is a functional in the first variable, ϕ is from the space $C([-r,0],\mathbb{R}^n)$ of continuous functions on $[-r,0]$, and the Krasovsky notation $u_t : [-r,0] \ni s \mapsto u(t+s)$, $r > 0$, is used [21, Chapter VI]. Finally, eliminating the initial function ϕ from (8) in a standard way by transforming it to a forcing term (see [15]), we conclude that the resulting equation falls into the class of equations of form (1).

Note that, by measure functional differential equations, the Volterra type equations of form (8) are usually meant in the existing bibliography on the subject (see, e.g., [8, 13, 22]), whereas equations with more general types of argument deviation are rather scarce (we can cite, perhaps, only [4, page 217]). Comparing (8) with (1), we find that the latter includes non-Volterra cases as well.

This list of examples can be continued. It is interesting to observe that solutions of problems of type (3) studied in the literature up to now are always assumed, at least locally, to be absolutely continuous [16], or even continuously differentiable [23]. In contrast to this, the gauge integral involved in (1) allows one to deal with a considerably wider class of solutions, which are, in fact, assumed to be of bounded variation only. A possible noteworthy consequence for systems with impulses may be that the unpleasant effect of the so-called *pulsation* phenomenon [11, page 5] might be more natural to be dealt with in the framework of the space $\mathrm{BV}([a,b],\mathbb{R}^n)$. Our interest in (1), originally motivated by a relation to problems of type (3), has strengthened still further due to the last observation.

The general character of the object represented by (1) suggests a natural idea to examine its solvability by comparing it to simpler linear equations with suitable properties. Here, we show that such statements can indeed be obtained rather easily by analogy to [24–26]. The key assumption is that certain linear operators associated with the nonlinear operator f appearing in (1) possess the following property.

Definition 1. Let $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ be a linear mapping. One says that a linear operator $p : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ belongs to the set $\mathcal{S}_h([a,b],\mathbb{R}^n)$ if the equation

$$u(t) = h(u) + \int_a^t (pu)(s)\,dg(s) + r(t), \qquad t \in [a,b] \tag{9}$$

has a unique solution u for any r from $\mathrm{BV}([a,b],\mathbb{R}^n)$, and, moreover, the solution u is nonnegative for any nonnegative r.

The property described by Definition 1, in fact, means that the linear operator associated with (9) is positively invertible on $\mathrm{BV}([a,b],\mathbb{R}^n)$, and thus it corresponds to the existence and positivity of Green's operator for a boundary value problem [15].

Remark 2. The inclusion $p \in \mathcal{S}_h([a,b],\mathbb{R}^n)$, generally speaking, does not imply that $\lambda p \in \mathcal{S}_h([a,b],\mathbb{R}^n)$ for $\lambda \neq 1$!

The question on the unique solvability of (1) is thus reduced to estimating the nonlinearities suitably, so that the appropriate majorants generate linear equations with a unique solution depending monotonously on forcing terms. The problem of finding such majorants is a separate topic not discussed here. We only note that, in a number of cases, the existing results on differential inequalities can be applied (see, e.g., [17–19]).

Note that, due to the nature of the techniques used, statements of this kind available in the literature on problems of type (3), as a rule, are established separately in every concrete case (see, e.g., [27–29]). Here, we provide a simple unified proof, which is, in a sense, independent on the character of the equation and also allows one to gain a considerable degree of generality. The results may be useful in studies of the solvability of various measure functional differential equations and, in particular, of problem (3) and its generalisations (note that, e.g., rather complicated neutral-type functional differential equations [23] can be formulated in form (1); see also [4, 30]).

2. Unique Solvability Conditions

We are going to show that the knowledge of the property $p \in \mathcal{S}_h([a,b],\mathbb{R}^n)$ for certain *linear* operators p and h associated with (1) allows one to guarantee its unique solvability.

2.1. Nonlinear Equations. The following statement is true.

Theorem 3. *Assume that there exist certain linear operators $p_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1, 2$, such that*

$$p_2(u-v)(t) \le (fu)(t) - (fv)(t) \le p_1(u-v)(t), \quad t \in [a,b], \tag{10}$$

for arbitrary functions $u : [a,b] \to \mathbb{R}^n$, $v : [a,b] \to \mathbb{R}^n$ with the property

$$u(t) \ge v(t) \quad \forall t \in [a,b]. \tag{11}$$

Furthermore, let the inclusions

$$p_1 \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \qquad \frac{1}{2}(p_1 + p_2) \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n) \tag{12}$$

be fulfilled with some linear functionals $h_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1, 2$. Then (1) *has a unique solution for an arbitrary φ such that*

$$h_2(u-v) \le \varphi(u) - \varphi(v) \le h_1(u-v) \tag{13}$$

whenever (11) *holds.*

The inequality sign and modulus for vectors in (10), (11), (13), and similar relations below are understood componentwise. The theorem as well as the other statements formulated below will be proved later.

Theorem 4. *Let there exist certain linear operators $l_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1, 2$, and linear functionals $h_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1, 2$ satisfying the inclusions*

$$l_1 + l_2 \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \qquad l_1 \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n), \tag{14}$$

and such that (13) *and the inequality*

$$\left|(fu)(t) - (fv)(t) - l_1(u-v)(t)\right| \le l_2(u-v)(t), \quad t \in [a,b], \tag{15}$$

is true for arbitrary functions u and v of bounded variation with property (11). *Then* (1) *is uniquely solvable.*

Theorem 4 is, in fact, an alternative form of Theorem 3, where the estimate of a "linear part" is more visible.

In other statements, we need the following natural notion of positivity of a linear operator in the space $\mathrm{BV}([a,b],\mathbb{R}^n)$.

Definition 5. A linear operator $q : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ will be called *positive* if qu is a nonnegative function for an arbitrary nonnegative u from $\mathrm{BV}([a,b],\mathbb{R}^n)$.

Note that no monotonicity assumptions are imposed on l_1 in Theorem 4. In the cases where the positivity of certain linear majorants is known, the following statement may be of use.

Corollary 6. *Assume that there exist some positive linear operators $q_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1, 2$, such that the inequalities*

$$\left|(fu)(t) - (fv)(t) + q_2(u-v)(t)\right| \le q_1(u-v)(t) \tag{16}$$

hold on $[a,b]$ for any u and v from $\mathrm{BV}([a,b],\mathbb{R}^n)$ with property (11). *Moreover, let one can specify linear functionals $h_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1, 2$, satisfying* (13), *and such that the inclusions*

$$q_1 + (1-2\theta)q_2 \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \qquad -\theta q_2 \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n) \tag{17}$$

hold for a certain $\theta \in (0,1)$. Then (1) *has a unique solution.*

Corollary 6 allows one to obtain, in particular, the following statements.

Corollary 7. *Assume that, for arbitrary u and v from $\mathrm{BV}([a,b],\mathbb{R}^n)$ with property* (11), *f and φ satisfy estimates* (13) *and* (16) *with some linear functionals $h_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$, $i = 1, 2$ and positive linear operators $q_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1, 2$. Then the inclusions*

$$q_1 \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \qquad -\frac{1}{2}q_2 \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n) \tag{18}$$

guarantee that (1) *is uniquely solvable.*

Corollary 8. *The assertion of Corollary 7 is true with* (18) *replaced by the condition*

$$q_1 + \frac{1}{2}q_2 \in \mathcal{S}_{h_1}([a,b],\mathbb{R}^n), \qquad -\frac{1}{4}q_2 \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b],\mathbb{R}^n). \tag{19}$$

The statements formulated above express fairly general properties of (1) and extend, in particular, the corresponding results of [25, 27, 29, 31].

2.2. Linear Equations. Let us now assume that $f : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ in (1) is an affine mapping, and, therefore, (1) has the form

$$u(t) = h(u) + \int_a^t (lu)(s)\,dg(s) + y(t), \quad t \in [a,b], \tag{20}$$

where $l : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ and $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ are linear, and $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$ is a given function.

Corollary 9. *Assume that there exist certain linear operators $p_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 0,1$, and a linear mapping $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ such that the inclusions*

$$p_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \qquad p_0 + p_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n) \tag{21}$$

hold, and the estimate

$$\left|(lu)(t) - (p_1 u)(t)\right| \le (p_0 u)(t), \quad t \in [a,b] \tag{22}$$

is satisfied for any nonnegative $u \in \mathrm{BV}([a,b],\mathbb{R}^n)$. Then (20) *has a unique solution.*

We also have the following.

Corollary 10. *Let there exist positive linear operators $q_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 0,1$, satisfying the inclusions*

$$q_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \qquad -\frac{1}{2}q_2 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \tag{23}$$

and such that the inequalities

$$\left|(lu)(t) + (q_2 u)(t)\right| \le (q_1 u)(t), \quad t \in [a,b] \tag{24}$$

are true for an arbitrary nonnegative function $u : [a,b] \to \mathbb{R}^n$ of bounded variation. Then (20) *has a unique solution for any $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$.*

We conclude this note by considering the case where l in (20) is a linear mapping admitting a decomposition into the sum of its positive and negative parts; that is,

$$l = l_0 - l_1, \tag{25}$$

where $l_k : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $k = 0,1$, are linear and positive. In that case, for the equation of the form

$$u(t) = h(u) + \int_a^t \left[(l_0 u)(s) - (l_1 u)(s)\right] dg(s) + y(t), \quad t \in [a,b], \tag{26}$$

where $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ is linear and $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$, the following result is obtained.

Corollary 11. *Let the linear vector functional $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}^n$ and the linear positive operators $l_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1,2$, be such that the inclusions*

$$l_0 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \qquad \frac{1}{2}(l_0 - l_1) \in \mathcal{S}_h([a,b],\mathbb{R}^n) \tag{27}$$

are satisfied. Then (26) *has a unique solution for any $y \in \mathrm{BV}([a,b],\mathbb{R}^n)$.*

It is interesting to observe the second condition in (27); it thus turns out that property $\mathcal{S}_h([a,b],\mathbb{R}^n)$ for *one half* of the operator under the integral sign in (26) ensures the unique solvability of the *original* equation (26).

3. Proofs

Let $\langle E, \|\cdot\|\rangle$ be real Banach space, let $z \in E$ be a given vector, and let $F : E \to E$ be a mapping. Let $K_i \subset E$, $i = 1,2$, be closed cones inducing the corresponding partial orderings $\leqq_{K_i}$, so that $x \leqq_{K_i} y$ if and only if $y - x \in K_i$. The following statement [32, 33] on the abstract equation

$$Fu = z \tag{28}$$

will be used below.

Theorem 12 (see [33], Theorem 49.4). *Let the cone K_2 be normal and generating. Furthermore, let $B_k : E \to E$, $k = 1,2$, be linear operators such that B_1^{-1} and $(B_1 + B_2)^{-1}$ exist and possess the properties*

$$B_1^{-1}(K_2) \subset K_1, \qquad (B_1 + B_2)^{-1}(K_2) \subset K_1, \tag{29}$$

and, furthermore, let the order relation

$$B_1(x - y) \leqq_{K_2} Fx - Fy \leqq_{K_2} B_2(x - y) \tag{30}$$

be satisfied for any pair (x, y) such that $y \leqq_{K_1} x$. Then (28) *has a unique solution for an arbitrary element $z \in E$.*

Recall that K_2 is normal if all the sets order bounded with respect to $\leqq_{K_2}$ are also norm bounded and that K_1 is generating if and only if $\{u - v \mid \{u, v\} \subset K_1\} = E$ (see, e.g., [33, 34]).

Let $\mathrm{BV}^+([a,b],\mathbb{R}^n)$ (resp., $\mathrm{BV}^{++}([a,b],\mathbb{R}^n)$) be the set of all the nonnegative (resp., nonnegative and nondecreasing) functions from $\mathrm{BV}([a,b],\mathbb{R}^n)$.

Lemma 13. *(1) The set $\mathrm{BV}^+([a,b],\mathbb{R}^n)$ is a cone in the space $\mathrm{BV}([a,b],\mathbb{R}^n)$.*

(2) The set $\mathrm{BV}^{++}([a,b],\mathbb{R}^n)$ is a normal and generating cone in $\mathrm{BV}([a,b],\mathbb{R}^n)$.

Proof. The first assertion of the lemma being obvious, only the second one should be verified.

It follows directly from the definition of the set $\mathrm{BV}^{++}([a,b],\mathbb{R}^n)$ that it is a cone in $\mathrm{BV}([a,b],\mathbb{R}^n)$, which is also generating due to the Jordan decomposition of a function of bounded variation (see, e.g., [3]). In order to verify its normality, it will be sufficient to show [32, Theorem 4.1] that the set

$$A(\alpha,\beta) := \{x \in \mathrm{BV}([a,b],\mathbb{R}^n) : \{x - \alpha, \beta - x\} \subset \mathrm{BV}^{++}([a,b],\mathbb{R}^n)\} \tag{31}$$

is bounded for any $\{\alpha,\beta\} \subset \mathrm{BV}([a,b],\mathbb{R}^n)$. Indeed, if $x \in A(\alpha,\beta)$, then the functions $x - \alpha$ and $\beta - x$ are both nonnegative and nondecreasing. Therefore,

$$\operatorname*{Var}_{[a,b]}(x - \alpha) = \alpha(a) - \alpha(b) + x(b) - x(a), \tag{32}$$

and, hence,

$$\begin{aligned}\|x\|_{\mathrm{BV}} &\leq \|\alpha\|_{\mathrm{BV}} + \|x-\alpha\|_{\mathrm{BV}} \\ &= \|\alpha\|_{\mathrm{BV}} + |x(a)-\alpha(a)| + \operatorname*{Var}_{[a,b]}(x-\alpha) \\ &= \|\alpha\|_{\mathrm{BV}} + x(b) - \alpha(b) \leq \|\alpha\|_{\mathrm{BV}} + \beta(b) - \alpha(b).\end{aligned} \tag{33}$$

The last estimate shows that the norms of all such x are uniformly bounded. □

Let $p : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ be a linear operator and $h : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathbb{R}$ a linear functional. Let us put

$$V_{p,h}u := u - \int_a^{\cdot} (pu)(\xi)\,dg(\xi) - h(u) \tag{34}$$

for any u from $\mathrm{BV}([a,b],\mathbb{R}^n)$. It follows immediately from Definition 1 that the linear operator $V_{p,h} : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$ defined by (34) has the following property.

Lemma 14. *If p is a linear operator such that*

$$p \in \mathcal{S}_h([a,b],\mathbb{R}^n), \tag{35}$$

then $V_{p,h}$ is invertible and, moreover, its inverse $V_{p,h}^{-1}$ satisfies the inclusion

$$V_{p,h}^{-1}(\mathrm{BV}^{++}([a,b],\mathbb{R}^n)) \subset \mathrm{BV}^{+}([a,b],\mathbb{R}^n). \tag{36}$$

We will also use the obvious identity

$$V_{p_1,h_1} + V_{p_2,h_2} = 2V_{(1/2)(p_1+p_2),(1/2)(h_1+h_2)}, \tag{37}$$

which is valid for any linear $p_i : \mathrm{BV}([a,b],\mathbb{R}^n) \to \mathrm{BV}([a,b],\mathbb{R}^n)$, $i = 1,2$.

3.1. Proof of Theorem 3. Let us set $E = \mathrm{BV}([a,b],\mathbb{R}^n)$ and put

$$(Fu)(t) := u(t) - \int_a^t (fu)(s)\,dg(s) - \varphi(u), \quad t \in [a,b], \tag{38}$$

for any u from $\mathrm{BV}([a,b],\mathbb{R}^n)$. Then (1) takes the form of (28) with $z = 0$. Since fu and g are both from $\mathrm{BV}([a,b],\mathbb{R}^n)$, it follows (see, e.g., [30]) that the function

$$[a,b] \ni t \longmapsto \int_a^t (fu)(s)\,dg(s) \tag{39}$$

also belongs to $\mathrm{BV}([a,b],\mathbb{R}^n)$. Therefore, F given by (38) is an operator acting in E.

Note that relation (10) is equivalent to inequalities

$$-p_1(u-v)(t) \leq -(fu)(t) + (fv)(t) \leq -p_2(u-v)(t), \tag{40}$$

for any $t \in [a,b]$ and $\{u,v\}$ from $\mathrm{BV}([a,b],\mathbb{R}^n)$ with properties (11). Integrating (40) with respect to g, we obtain

$$\begin{aligned}-\int_a^t p_1(u-v)(s)\,dg(s) &\leq -\int_a^t (fu)(s)\,dg(s) \\ &\quad + \int_a^t (fv)(s)\,dg(s) \\ &\leq -\int_a^t p_2(u-v)(s)\,dg(s),\end{aligned} \tag{41}$$

and, therefore, according to (38),

$$\begin{aligned}&u(t) - v(t) - \int_a^t p_1(u-v)(s)\,dg(s) - \varphi(u) + \varphi(v) \\ &\quad \leq (Fu)(t) - (Fv)(t) \\ &\quad \leq u(t) - v(t) - \int_a^t p_2(u-v)(s)\,dg(s) - \varphi(u) + \varphi(v),\end{aligned} \tag{42}$$

for all $t \in [a,b]$. Taking assumption (13) into account and using notation (34), we get

$$V_{p_1,h_1}(u-v)(t) \leq (Fu)(t) - (Fv)(t) \leq V_{p_2,h_2}(u-v)(t), \tag{43}$$

for all $t \in [a,b]$ and u and v from $\mathrm{BV}([a,b],\mathbb{R}^n)$ with properties (11). Furthermore, it follows immediately from (34) and (38) that, for any $t \in [a,b]$,

$$\begin{aligned}&(Fu)(t) - (Fv)(t) - V_{p_1,h_1}(u-v)(t) \\ &\quad = \varphi(v) - \varphi(u) \\ &\qquad + \int_a^t [p_1(u-v)(s) - (fu)(s) + (fv)(s)]\,dg(s).\end{aligned} \tag{44}$$

Therefore, by virtue of inequality (43) and assumption (10), the function $Fu - Fv - V_{p_1,h_1}(u-v)$ is nonnegative and nondecreasing and, hence,

$$Fu - Fv - V_{p_1,h_1}(u-v) \in \mathrm{BV}^{++}([a,b],\mathbb{R}^n). \tag{45}$$

In the same manner, one shows that

$$V_{p_2,h_2}(u-v) - Fu + Fv \in \mathrm{BV}^{++}([a,b],\mathbb{R}^n). \tag{46}$$

Considering (45) and (46), we conclude that F satisfies condition (30) with

$$B_i = V_{p_i,h_i}, \tag{47}$$

$i = 1,2$, and

$$\begin{aligned}K_1 &= \mathrm{BV}^{+}([a,b],\mathbb{R}^n), \\ K_2 &= \mathrm{BV}^{++}([a,b],\mathbb{R}^n).\end{aligned} \tag{48}$$

By virtue of Lemma 13, K_2 is a normal and generating cone in $BV([a,b],\mathbb{R}^n)$.

Since, by assumption (12), $p_1 \in \mathcal{S}_{h_1}$, it follows that V_{p_1,h_1} is invertible and the inclusion

$$V_{p_1,h_1}^{-1}(K_2) \subset K_1 \tag{49}$$

holds. Furthermore, by (12) and Lemma 14, the operator $(1/2)V^{-1}_{(1/2)(p_1+p_2),(1/2)(h_1+h_2)}$ exists and coincides with the inverse operator to $V_{p_1,h_1} + V_{p_2,h_2}$. It is moreover positive in the sense that

$$\left(V_{p_1,h_1} + V_{p_2,h_2}\right)^{-1}(K_2) \subset K_1. \tag{50}$$

Combining (49) and (50), we see that the inverse operators B^{-1} and $(B_1 + B_2)^{-1}$ exist and possess properties (29) with respect to cones (48). Applying now Theorem 12, we prove the unique solvability of (28) and, hence, that of (1).

3.2. Proof of Theorem 4. Rewriting relations (15) in the form

$$\begin{aligned} l_1(u-v)(t) - l_2(u-v)(t) &\le (fu)(t) - (fv)(t) \\ &\le l_2(u-v)(t) + l_1(u-v)(t), \quad t \in [a,b], \end{aligned} \tag{51}$$

and putting

$$p_i := l_1 - (-1)^i l_2, \quad i = 1,2, \tag{52}$$

we find that f admits estimate (10) with p_1 and p_2 defined by (52). Therefore, it remains only to note that assumption (14) ensures the validity of inclusions (12), and to apply Theorem 3.

3.3. Proof of Corollary 6. It turns out that, under assumptions (16) and (17), the operators $l_i : BV([a,b],\mathbb{R}^n) \to BV([a,b],\mathbb{R}^n)$, $i = 1,2$, defined by the formulae

$$l_1 := -\theta q_2, \qquad l_2 := q_1 + (1-\theta) q_2 \tag{53}$$

with $\theta \in (0,1)$, satisfy conditions (14) and (15) of Theorem 4. Indeed, estimate (16) and the positivity of the operator q_2 imply that, for any u and v with properties (11) and all $t \in [a,b]$, the relations

$$\begin{aligned} &\left|(fu)(t) - (fv)(t) + \theta q_2(u-v)(t)\right| \\ &\quad = \left|(fu)(t) - (fv)(t) + q_2(u-v)(t) - (1-\theta) q_2(u-v)(t)\right| \\ &\quad \le q_1(u-v)(t) + \left|(1-\theta) q_2(u-v)(t)\right| \\ &\quad = q_1(u-v)(t) + (1-\theta) q_2(u-v)(t) \end{aligned} \tag{54}$$

are true. This means that f admits estimate (15) with the operators l_1 and l_2 of form (53). It is easy to verify that assumption (17) ensures the validity of inclusions (14) for operators (53), and, therefore, Theorem 4 can be applied.

3.4. Proof of Corollaries 7 and 8. The results follow directly from Corollary 6 if one puts $\theta = (1/2)$ and $\theta = (1/4)$, respectively.

3.5. Proof of Corollary 9. If $y = 0$, one should apply Theorem 4 with $f = l$, $l_1 = p_1$, $l_2 = p_0$, and $h_1 = h$, $h_2 = h$. For a nonzero $y \in BV([a,b],\mathbb{R}^n)$, one can modify the theorem slightly by incorporating the forcing term y directly into (1) similarly to (20). Then we find that the argument of Section 3.1 remains almost unchanged.

3.6. Proof of Corollary 10. Corollary 7 with $f = l$, $h_1 = h$, and $h_2 = h$ is applied.

3.7. Proof of Corollary 11. It is sufficient to note that, under these assumptions, the linear operators $p_i : BV([a,b],\mathbb{R}^n) \to BV([a,b],\mathbb{R}^n)$, $i = 1,2$, defined by the formulae

$$p_0 := \frac{1}{2}(l_0 + l_1), \qquad p_1 := \frac{1}{2}(l_0 - l_1), \tag{55}$$

satisfy conditions (21) and (22) of Corollary 9.

4. Comments

The following can be pointed out in relation to the above said.

4.1. Remark on Constants. The conditions presented in Sections 2.1 and 2.2 are, in a sense, optimal and cannot be improved. For example, it follows from [26] that assumption (14) of Corollary 7 can be replaced neither by the condition

$$(1-\varepsilon) l_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \qquad l_0 + l_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n) \tag{56}$$

nor by the condition

$$l_1 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \qquad (1-\varepsilon)(l_0 + l_1) \in \mathcal{S}_h([a,b],\mathbb{R}^n), \tag{57}$$

no matter how small $\varepsilon \in (0,\infty)$ may be. Likewise, counterexamples show that the assertion of Corollary 11 is not true any more if condition (27) is replaced by either of its weaker versions

$$\begin{aligned} &(1-\varepsilon) l_0 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \\ &\frac{1}{2}(l_0 - l_1) \in \mathcal{S}_h([a,b],\mathbb{R}^n), \end{aligned} \tag{58}$$

and

$$\begin{aligned} &l_0 \in \mathcal{S}_h([a,b],\mathbb{R}^n), \\ &\frac{1}{2+\varepsilon}(l_0 - l_1) \in \mathcal{S}_h([a,b],\mathbb{R}^n) \end{aligned} \tag{59}$$

with a positive ε. The same holds for the other inequalities and constants.

4.2. Equations with Matrix-Valued Functions. It is clear from the proofs given above that similar statements can also be obtained in the case where the integrals of matrix-valued functions are considered in (1), as described, for example, in [3, 4].

4.3. The Case of a Nonmonotone Measure. Results similar to those stated above can also be formulated in the case where the function g involved in (1) is of bounded variation only and not necessarily nondecreasing. For this purpose, one should use the representation

$$g = g_1 - g_2, \tag{60}$$

where g_k, $k = 1, 2$, are nondecreasing functions, and modify the definition of the set $\mathcal{S}_h([a,b], \mathbb{R}^n)$ in the following way.

Definition 15. A pair of operators (q_1, q_2) is said to belong to $\mathcal{S}_h([a,b], \mathbb{R}^n)$ if the equation

$$\begin{aligned} u(t) = h(u) &+ \int_a^t (q_1 u)(s)\, dg_1(s) \\ &- \int_a^t (q_2 u)(s)\, dg_2(s) + r(t), \quad t \in [a,b], \end{aligned} \tag{61}$$

has a unique solution u for any r from $\mathrm{BV}([a,b], \mathbb{R}^n)$ and, moreover, the solution u is nonnegative for nonnegative r.

In that case, an analogue of the assertion of Theorem 3 is obtained if assumption (12) is replaced by the pair of conditions

$$\begin{gathered} (p_1, p_2) \in \mathcal{S}_{h_1}([a,b], \mathbb{R}^n), \\ \left(\frac{1}{2}(p_1 + p_2), \frac{1}{2}(p_1 + p_2)\right) \in \mathcal{S}_{(1/2)(h_1+h_2)}([a,b], \mathbb{R}^n). \end{gathered} \tag{62}$$

The proof of this fact is pretty similar to the argument given in Section 3.1 and uses Theorem 12 with the operators $B_k : \mathrm{BV}([a,b], \mathbb{R}^n) \to \mathrm{BV}([a,b], \mathbb{R}^n)$, $k = 1, 2$,

$$\begin{aligned} (B_k u)(t) := u(t) &- \int_a^t (p_k u)(s)\, dg_1(s) \\ &+ \int_a^t (p_{3-k} u)(s)\, dg_2(s) - h_k(u), \quad t \in [a,b], \end{aligned} \tag{63}$$

instead of those defined by (47).

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References

[1] J. Kurzweil, "Generalized ordinary differential equations and continuous dependence on a parameter," *Czechoslovak Mathematical Journal*, vol. 7, no. 82, pp. 418–449, 1957.

[2] S. Schwabik, *Generalized Ordinary Differential Equations*, vol. 5 of *Series in Real Analysis*, World Scientific, River Edge, NJ, USA, 1992.

[3] S. Schwabik, M. Tvrdý, and O. Vejvoda, *Differential and Integral Equations*, D. Reidel Publishing, Dordrecht, The Netherlands, 1979, Boundary value problems and adjoints.

[4] M. Tvrdý, "Differential and integral equations in the space of regulated functions," *Memoirs on Differential equations and Mathematical Physics*, vol. 25, pp. 1–104, 2002.

[5] M. Tvrdý, *Stieltjesův integrál. Kurzweilova teorie*, Palacky University, Olomouc, Czech Republic, 2012, Czech.

[6] J. Kurzweil, *Generalized Ordinary Differential Equations*, vol. 11 of *Series in Real Analysis*, World Scientific, Hackensack, NJ, USA, 2012, Not absolutely continuous solutions.

[7] M. Tvrdý, "Regulated functions and the Perron-Stieltjes integral," *Časopis pro Pěstování Matematiky*, vol. 114, pp. 187–209, 1989.

[8] P. C. Das and R. R. Sharma, "Existence and stability of measure differential equations," *Czechoslovak Mathematical Journal*, vol. 22, no. 97, pp. 145–158, 1972.

[9] S. G. Pandit and S. G. Deo, *Differential Systems Involving Impulses*, vol. 954 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1982.

[10] R. R. Sharma, "An abstract measure differential equation," *Proceedings of the American Mathematical Society*, vol. 32, pp. 503–510, 1972.

[11] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, vol. 14 of *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*, World Scientific, River Edge, NJ, USA, 1995.

[12] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales*, An Introduction with Applications, Birkhäauser, Boston, Mass, USA, 2001.

[13] M. Federson, J. G. Mesquita, and A. Slavík, "Measure functional differential equations and functional dynamic equations on time scales," *Journal of Differential Equations*, vol. 252, no. 6, pp. 3816–3847, 2012.

[14] M. Federson, J. G. Mesquita, and A. Slavík, "Basic results for functional differential and dynamic equations involving impulses," *Mathematische Nachrichten*, vol. 286, pp. 181–204, 2013.

[15] N. Azbelev, V. Maksimov, and L. Rakhmatullina, *Introduction to the Theory of Linear Functional-Differential Equations*, vol. 3 of *Advanced Series in Mathematical Science and Engineering*, World Federation, Atlanta, Ga, USA, 1995.

[16] N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, *Introduction to the Theory of Functional Differential Equations: Methods and Applications*, vol. 3 of *Contemporary Mathematics and Its Applications*, Hindawi, Cairo, Egypt, 2007.

[17] E. I. Bravyi, *Solvability of Boundary Value Problems for Linear Functional Differential Equations*, R&C Dynamics, Izhevsk, Russia, 2011, (Russian).

[18] R. Hakl, A. Lomtatidze, and J. Šremr, *Some Boundary Value Problems for First Order Scalar Functional Differential Equations*, Masaryk University, Brno, Czech Republic, 2002.

[19] I. Kiguradze and B. Půža, *Boundary Value Problems for Systems of Linear Functional Differential Equations*, Masaryk University, Brno, Czech Republic, 2003.

[20] A. Slavík, "Dynamic equations on time scales and generalized ordinary differential equations," *Journal of Mathematical Analysis and Applications*, vol. 385, no. 1, pp. 534–550, 2012.

[21] N. N. Krasovskii, *Stability of Motion. Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay*, Translated by J. L. Brenner, Stanford University Press, Stanford, Calif, USA, 1963.

[22] M. H. Chang and P. T. Liu, "Maximum principles for control systems described by measure functional differential equations," *Journal of Optimization Theory and Applications*, vol. 15, no. 5, pp. 517–531, 1975.

[23] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, vol. 99 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1993.

[24] N. Dilna and A. Rontó, "General conditions guaranteeing the solvability of the Cauchy problem for functional differential equations," *Mathematica Bohemica*, vol. 133, pp. 435–445, 2008.

[25] N. Dilna and A. Rontó, "Unique solvability of a non-linear non-local boundary-value problem for systems of non-linear functional differential equations," *Mathematica Slovaca*, vol. 60, no. 3, pp. 327–338, 2010.

[26] A. Rontó, V. Pylypenko, and N. Dilna, "On the unique solvability of a non-local boundary value problem for linear functional differential equations," *Mathematical Modelling and Analysis*, vol. 13, no. 2, pp. 241–250, 2008.

[27] R. Hakl, A. Lomtatidze, and B. Půža, "On a boundary value problem for first-order scalar functional differential equations," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 53, no. 3-4, pp. 391–405, 2003.

[28] A. Lomtatidze and H. Štěpánková, "On sign constant and monotone solutions of second order linear functional differential equations," *Memoirs on Differential equations and Mathematical Physics*, vol. 35, pp. 65–90, 2005.

[29] J. Šremr, "On the Cauchy type problem for systems of functional differential equations," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 67, no. 12, pp. 3240–3260, 2007.

[30] M. Tvrdý, "Linear integral equations in the space of regulated functions," *Mathematica Bohemica*, vol. 123, pp. 177–212, 1998.

[31] N. Z. Dilna and A. M. Rontó, "General conditions for the unique solvability of initial-value problem for nonlinear functional differential equations," *Ukrainian Mathematical Journal*, vol. 60, no. 2, pp. 191–198, 2008.

[32] M. A. Krasnoselskii, E. A. Lifshits, Y. V. Pokornyi, and V. Y. Stetsenko, "Positively invertible linear operators and the solvability of non-linear equations," *Doklady Akademii Nauk Tadzhikskoj SSR*, vol. 17, pp. 12–14, 1974.

[33] M. A. Krasnoselskii and P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Springer, New York, NY, USA, 1984.

[34] M. A. Krasnoselskii, *Positive Solutions of Operator Equations*, Translated from the Russian by R. E. Flaherty, P. Noordhoff, Groningen, The Netherlands, 1964.

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Research Article

Variational Approximate Solutions of Fractional Nonlinear Nonhomogeneous Equations with Laplace Transform

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A novel modification of the variational iteration method is proposed by means of Laplace transform and homotopy perturbation method. The fractional lagrange multiplier is accurately determined by the Laplace transform and the nonlinear one can be easily handled by the use of He's polynomials. Several fractional nonlinear nonhomogeneous equations are analytically solved as examples and the methodology is demonstrated.

1. Introduction

Recently, systems of fractional nonlinear partial differential equations [1–3] have attracted much attention in a variety of applied sciences. With the development of nonlinear sciences, some numerical [4–6], semianalytical [7–12], and analytical methods [13–15] have been developed for fractional differential equations. So, the semianalytical methods have largely been used to solve fractional equations. Most of these methods have their inbuilt deficiencies like the calculation of Adomian's polynomials, the Lagrange multiplier, divergent results, and huge computational work. Recently, some improved homotopy perturbation methods [16, 17] and improved variational iteration methods, [18, 19] have been used by many researches.

The variational iteration method (VIM) [8, 9, 20] was extended to initial value problems of differential equations and has been one of the methods used most often. The key problem of the VIM is the correct determination of the Lagrange multiplier when the method is applied to fractional equations; combined with the Laplace transform, the crucial point of this method is solved efficiently by Wu and Baleanu [21, 22]. Laplace transform overcomes principle drawbacks in application of the VIM to fractional equations.

Motivated and inspired by the ongoing research in this field, we give a new modification of variational iteration method, combined with the Laplace transform and the homotopy perturbation method. The fractional Lagrange multiplier is accurately determined by the Laplace transform and the nonlinear one can be easily handed by the use of He's polynomials. In this work, we will use this new method to obtain approximate solutions of the fractional nonlinear equations, and the fractional derivatives are described in the Caputo sense.

2. Description of the Method

In order to illustrate the basic idea of the technique, consider the following general nonlinear system:

$$\frac{\partial^m u(x,t)}{\partial t^m} + R[u(x,t)] + N[u(x,t)] = g(x,t), \quad (1)$$

$$u^k(x,0^+) = a_k, \quad (2)$$

where $k = 0,\dots,m-1$, $\partial^m u(x,t)/\partial t^m$ is the term of the highest-order derivative, $g(x,t)$ is the source term, N represents the general nonlinear differential operator, and R is the linear differential operator.

Now, we consider the application of the modified VIM [21, 22]. Taking the above Laplace transform to both sides of (1) and (2), then the linear part is transformed into an algebraic equation as follows:

$$s^m U(x,s) - u^{(m-1)}(x,0) - \cdots - s^{m-1}u(x,0) + L[R[u]] + L[N[u]] - L[g(x,t)] = 0, \tag{3}$$

where $U(x,s) = L[u(x,t)] = \int_0^\infty e^{-st}u(x,t)dt$. The iteration formula of (3) can be used to suggest the main iterative scheme involving the Lagrange multiplier as

$$U_{n+1}(x,s) = U_n(x,s) + \lambda(s) \times \left[s^m U_n(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+)s^{m-1-k} + L[R[u_n(x,t)] + N[u_n(x,t)] - g(x,t)]\right]. \tag{4}$$

Considering $L[R[u_n(x,t)] + N[u_n(x,t)]]$ as restricted terms, one can derive a Lagrange multiplier as

$$\lambda = -\frac{1}{s^m}. \tag{5}$$

With (5) and the inverse-Laplace transform L^{-1}, the iteration formula (4) can be explicitly given as

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - L^{-1} \times \left[\frac{1}{s^m}\left[s^m U_n(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+)s^{m-1-k} + L[R[u_n(x,t)] + N[u_n(x,t)] - g(x,t)]\right]\right] \\ &= u_0(x,t) - L^{-1}\left[\frac{1}{s^m}[L[R[u_n(x,t)] + N[u_n(x,t)]]]\right]; \end{aligned} \tag{6}$$

$u_0(x,t)$ is an initial approximation of (1), and

$$\begin{aligned} u_0(x,t) &= L^{-1}\left(\sum_{k=0}^{m-1} u^k(x,0^+)s^{m-1-k}\right) + L^{-1}\left[\frac{1}{s^m}L[g(x,t)]\right] \\ &= u(x,0) + u'(x,0)t + \cdots + \frac{u^{m-1}(x,0)t^{m-1}}{(m-1)!} + L^{-1}\left[\frac{1}{s^m}L[g(x,t)]\right]. \end{aligned} \tag{7}$$

In order to deal with the nonlinear term in the iteration formula (6), combining with the homotopy perturbation method, we give a new modification of the above method [21, 22]. In the homotopy method, the basic assumption is that the solutions can be written as a power series in p:

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} p^n u_n(x,t) \\ &= u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots, \end{aligned} \tag{8}$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n \mathscr{H}_n(u), \tag{9}$$

where $p \in [0,1]$ is an embedding parameter. $\mathscr{H}_n(u)$ is He's polynomials [16, 23] can be generated by

$$\mathscr{H}_n(u_0,\ldots,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{n} p^i u_i\right)\right]_{p=0}, \quad n = 0,1,2,\ldots. \tag{10}$$

This new modified method is obtained by the elegant coupling of correction function (6) of variational iteration method with He's polynomials and is given by

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) - p\left(L^{-1}\left[\frac{1}{s^m}L\left[R\sum_{n=0}^{\infty}p^n u_n(x,t)\right] + \frac{1}{s^m}L\left[\sum_{n=0}^{\infty}p^n\mathscr{H}_n(u)\right]\right]\right), \tag{11}$$

$u_0(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Equating the terms with identical powers in p, we obtain the following approximations:

$$\begin{aligned} p^0 &: u_0(x,t) = u(x,0) + u'(x,0)t + \cdots + \frac{u^{m-1}(x,0)t^{m-1}}{(m-1)!} + L^{-1}\left[\frac{1}{s^m}L[g(x,t)]\right], \\ p^1 &: u_1(x,t) = -L^{-1}\left[\frac{1}{s^m}L[Ru_0(x,t)] + \frac{1}{s^m}L[\mathscr{H}_0(u)]\right], \\ p^2 &: u_2(x,t) = -L^{-1}\left[\frac{1}{s^m}L[Ru_1(x,t)] + \frac{1}{s^m}L[\mathscr{H}_1(u)]\right], \\ &\vdots \end{aligned} \tag{12}$$

The best approximations for the solution are

$$u(x,t)=\sum_{n=0}^{\infty}u_n. \tag{13}$$

This new modified method here transfers the problem into the partial differential equation in the Laplace s-domain, removes the differentiation with respect to time, and uses He's polynomials to deal with the nonlinear term. This new method basically illustrates how three powerful algorithms, variational iteration method, Laplace transform method, and homotopy perturbation method, can be combined and used to approximate the solutions of nonlinear equation. In this work, we will use this method to solve fractional nonlinear equations.

3. Illustrative Examples

We will apply the new modified VIM to both PDEs and FDEs. All the results are calculated by using the symbolic calculation software Mathematica.

3.1. Partial Differential Equations

Example 1. Consider the following nonhomogeneous nonlinear Gas Dynamic equation [24]

$$\frac{\partial u}{\partial t}+\frac{1}{2}\frac{\partial u^2}{\partial x}-u(1-u)=-e^{t-x} \tag{14}$$

with the initial condition

$$u(x,0)=1-e^{-x}. \tag{15}$$

After taking the Laplace transform to both sides of (14) and (15), we get the following iteration formula:

$$U_{n+1}(x,s)=U_n(x,s)+\lambda(s)\left[sU_n(x,s)-u(x,0)+L\left[\frac{1}{2}\frac{\partial u_n^2}{\partial x}-u_n+u_n^2+e^{t-x}\right]\right]. \tag{16}$$

Considering $L[(1/2)(\partial u_n^2/\partial x)-u_n+u_n^2]$ as restricted terms, Lagrange multiplier can be defined as $\lambda(s)=-1/s$; with the inverse-Laplace transform, the approximate solution of (16) can be given as

$$\begin{aligned}u_{n+1}(x,t)&=u_n(x,t)-L^{-1}\\&\quad\times\left[\frac{1}{s}\left[sU_n(x,s)-u(x,0)+L\left[\frac{1}{2}\frac{\partial u_n^2}{\partial x}-u_n+u_n^2+e^{t-x}\right]\right]\right]\\&=u_0(x,t)-L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[\frac{1}{2}\frac{\partial u_n^2}{\partial x}-u_n+u_n^2\right]\right]\right],\end{aligned} \tag{17}$$

where $u_0(x,t)$ is an initial approximation of (14), and

$$u_0(x,t)=u(x,0)-L^{-1}\left[\frac{1}{s}L\left[e^{t-x}\right]\right]. \tag{18}$$

Combining with the homotopy perturbation method, one has

$$\begin{aligned}\sum_{n=0}^{\infty}p^n u_n(x,t)&=u_0(x,t)-p\left[L^{-1}\left[\frac{1}{s}L\left[\frac{1}{2}\frac{\partial\left(\sum_{n=0}^{\infty}p^n\mathscr{H}_n(u)\right)}{\partial x}\right.\right.\right.\\&\qquad\left.\left.\left.-\sum_{n=0}^{\infty}p^n u_n+\sum_{n=0}^{\infty}p^n\mathscr{H}_n(u)\right]\right]\right],\end{aligned} \tag{19}$$

where $\mathscr{H}_n(u)$ is He's polynomials that represent nonlinear term u^2; we have a few terms of the He's polynomials for u^2 which are given by

$$\begin{aligned}\mathscr{H}_0(u)&=u_0^2,\\\mathscr{H}_1(u)&=2u_0u_1,\\\mathscr{H}_2(u)&=u_1^2+2u_0u_2,\\&\vdots\end{aligned} \tag{20}$$

Comparing the coefficient with identical powers in p,

$$\begin{aligned}u_0(x,t)&=1-e^{t-x},\\u_1&=-L^{-1}\left[\frac{1}{s}\left[L\left[\frac{1}{2}\frac{\partial u_0^2}{\partial x}-u_0+u_0^2\right]\right]\right]=0,\\u_2&=-L^{-1}\left[\frac{1}{s}\left[L\left[\frac{1}{2}\frac{\partial(2u_0u_1)}{\partial x}-u_1+2u_0u_1\right]\right]\right]\\&=e^{-x}\frac{t^{2\alpha}}{\Gamma[1+2\alpha]}=0,\\&\vdots\end{aligned} \tag{21}$$

and so on; in this manner the rest of component of the solution can be obtained. The solution of (14) and (15) in series form is given by

$$u(x,t)=1-e^{t-x}, \tag{22}$$

which is the exact solution. For this equation, the first-order approximate solution is justly the exact solution, and this proposed new method provides the solution in a rapid convergent. Furthermore, the new modified method can be easily extended to FDEs and this is the main purpose of our work.

3.2. Fractional Differential Equations. Let us consider the time fractional equation as follows:

$$ {}_0^C D_t^{\alpha} u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{23} $$

$$ u^k(x, 0^+) = a_k, \tag{24} $$

where $k = 0, \dots, m-1$, $m = [\alpha] + 1$, $g(x,t)$ is the source term, N represents the general nonlinear differential operator, and R is the linear differential operator. And the Caputo timefractional derivative operator of order $\alpha > 0$ is defined as

$$ {}_0^C D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, \quad m = [\alpha] + 1,\ m \in N, \tag{25} $$

where $\Gamma(\cdot)$ denotes the Gamma function.

Now, we consider the application of the modified VIM [21, 22]. The following Laplace transform of the term ${}_0^C D_t^{\alpha} u(x,t)$ holds:

$$ L\left[{}_0^C D_t^{\alpha} u(x,t)\right] = s^{\alpha} U(x,s) - \sum_{k=0}^{m-1} u^k(x, 0^+) s^{\alpha-1-k}, \quad m-1 < \alpha \le m, \tag{26} $$

where $U(x,s) = L[u(x,t)] = \int_0^{\infty} e^{-st} u(x,t) dt$. The detailed properties of fractional calculus and Laplace transform can be found in [1, 2]; we have chosen to the Caputo fractional derivative because it allows traditional initial and boundary conditions to be included in the formulation of the problem. Taking the above Laplace transform to both sides of (23) and (24), the iteration formula of (23) can be constructed as

$$ U_{n+1}(x,s) = U_n(x,s) + \lambda(s) \times \left[s^{\alpha} U_n(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k} + L\left[R\left[u_n(x,t)\right] + N\left[u_n(x,t)\right] - g(x,t)\right] \right]. \tag{27} $$

Considering $L[R[u_n(x,t)] + N[u_n(x,t)]]$ as restricted terms, one can derive a Lagrange multiplier as

$$ \lambda = \frac{-1}{s^{\alpha}}. \tag{28} $$

With (28) and the inverse-Laplace transform L^{-1}, the iteration formula (27) can be explicitly given as

$$ \begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - L^{-1} \times \left[\frac{1}{s^{\alpha}} \left[s^{\alpha} U_n(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k} + L\left[R\left[u_n(x,t)\right] + N\left[u_n(x,t)\right] - g(x,t)\right] \right] \right] \\ &= u_0(x,t) - L^{-1} \times \left[\frac{1}{s^{\alpha}} \left[L\left[R\left[u_n(x,t)\right] + N\left[u_n(x,t)\right] \right] \right] \right]; \end{aligned} \tag{29} $$

$u_0(x,t)$ is an initial approximation of (23), and

$$ \begin{aligned} u_0(x,t) &= L^{-1}\left(\sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k} \right) + L^{-1}\left[\frac{1}{s^{\alpha}} L\left[g(x,t)\right] \right] \\ &= u(x,0) + u'(x,0)t + \cdots + \frac{u^{m-1}(x,0) t^{m-1}}{(m-1)!} + L^{-1}\left[\frac{1}{s^{\alpha}} L\left[g(x,t)\right] \right]. \end{aligned} \tag{30} $$

In the homotopy method, the basic assumption is that the solutions can be written as a power series in p:

$$ \begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} p^n u_n(x,t) \\ &= u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots, \end{aligned} \tag{31} $$

and the nonlinear term can be decomposed as

$$ Nu(x,t) = \sum_{n=0}^{\infty} p^n \mathscr{H}_n(u), \tag{32} $$

where $p \in [0,1]$ is an embedding parameter. $\mathscr{H}_n(u)$ is He's polynomials [16, 23] that can be generated by

$$ \mathscr{H}_n(u_0, \dots, u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[N\left(\sum_{i=0}^{n} p^i u_i \right) \right]_{p=0}, \quad n = 0, 1, 2, \dots. \tag{33} $$

The variational homotopy perturbation method is obtained by the elegant coupling of correction function (29) of variational iteration method with He's polynomials and is given by

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) - p\left(L^{-1}\left[\frac{1}{s^\alpha}L\left[R\sum_{n=0}^{\infty} p^n u_n(x,t)\right] + \frac{1}{s^\alpha}L\left[\sum_{n=0}^{\infty} p^n \mathscr{H}_n(u)\right]\right]\right), \tag{34}$$

$u_0(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Equating the terms with identical powers in p, we obtain the following approximations:

$$\begin{aligned} p^0 &: u_0(x,t) = u(x,0) + u'(x,0)t + \cdots + \frac{u^{m-1}(x,0)t^{m-1}}{(m-1)!} + L^{-1}\left[\frac{1}{s^\alpha}L[g(x,t)]\right], \\ p^1 &: u_1(x,t) = -L^{-1}\left[\frac{1}{s^\alpha}L[Ru_0(x,t)] + \frac{1}{s^\alpha}L[\mathscr{H}_0(u)]\right], \\ p^2 &: u_2(x,t) = -L^{-1}\left[\frac{1}{s^\alpha}L[Ru_1(x,t)] + \frac{1}{s^\alpha}L[\mathscr{H}_1(u)]\right], \\ &\vdots \end{aligned} \tag{35}$$

The best approximations for the solution are $u(x,t) = \sum_{n=0}^{\infty} u_n$. Let us apply the above method to solve fractional nonlinear equations of Caputo type.

Example 2. Consider the following nonlinear space time fractional equation [25]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u\frac{\partial^\beta u(x,t)}{\partial x^\beta} = x + xt^2, \tag{36}$$

$$u(x,0) = 0, \tag{37}$$

where $0 < \alpha,\ \beta \leqslant 1$, and the time-space fractional derivatives defined here are in Caputo sense. The Caputo space-fractional derivative operator of order $\beta > 0$ is defined as

$${}_0^C D_x^\beta u(x,t) = \frac{1}{\Gamma(m-\beta)}\int_0^x (x-\xi)^{m-\beta-1}\frac{\partial^m u(\xi,t)}{\partial \xi^m}d\xi, \quad m = [\beta]+1,\ m \in N. \tag{38}$$

After taking the Laplace transform on both sides of (36) and (37), we get the following iteration formula:

$$U_{n+1} = U_n + \lambda(s)\left[s^\alpha U_n(x,s) - s^{\alpha-1}u(x,0) + L\left[u_n\frac{\partial^\beta u_n(x,t)}{\partial x^\beta} - \left(x + xt^2\right)\right]\right]. \tag{39}$$

As a result, after the identification of a Lagrange multiplier $\lambda(s) = -1/s^\alpha$, and with the inverse-Laplace transform, one can derive

$$u_{n+1}(x,y,t) = u_0(x,y,t) - L\left[u_n\frac{\partial^\beta u_n(x,t)}{\partial x^\beta}\right] \tag{40}$$

$u_0(x,y,t)$ is an initial approximation of (36), and

$$u_0(x,t) = L^{-1}\left[\frac{1}{s^\alpha}\left[L\left[x + xt^2\right]\right]\right]. \tag{41}$$

Applying the variational homotopy perturbation method, one has

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = u_0(x,t) - p\left[L^{-1}\left[\frac{1}{s^\alpha}\left[L\left[\sum_{n=0}^{\infty} p^n \mathscr{H}_n(u)\right]\right]\right]\right], \tag{42}$$

where $\mathscr{H}_n(u)$ is He's polynomials that represent nonlinear term $u(\partial^\beta u(x,t)/\partial x^\beta)$; we have a few terms of the He's polynomials for $u(\partial^\beta u(x,t)/\partial x^\beta)$ which are given by

$$\begin{aligned} \mathscr{H}_0(u) &= u_0\frac{\partial^\beta u_0}{\partial x^\beta}, \\ \mathscr{H}_1(u) &= u_0\frac{\partial^\beta u_1}{\partial x^\beta} + u_1\frac{\partial^\beta u_0}{\partial x^\beta}, \\ \mathscr{H}_2(u) &= u_0\frac{\partial^\beta u_2}{\partial x^\beta} + u_1\frac{\partial^\beta u_1}{\partial x^\beta} + u_2\frac{\partial^\beta u_0}{\partial x^\beta}, \\ &\vdots \end{aligned} \tag{43}$$

Comparing the coefficient with identical powers in p, one has

$$u_0(x,t) = \frac{xt^{\alpha}}{\Gamma(1+\alpha)} + \frac{2xt^{\alpha+2}}{\Gamma(3+\alpha)},$$

$$\begin{aligned}
u_1 &= -L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[u_0\frac{\partial^{\beta}u_0}{\partial x^{\beta}}\right]\right]\right] \\
&= -\frac{t^{3\alpha}x^{2-\beta}\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(2-\beta)} \\
&\quad - \frac{4t^{4+3\alpha}x^{2-\beta}\Gamma(5+2\alpha)}{\Gamma^2(3+\alpha)\Gamma(5+3\alpha)\Gamma(2-\beta)} \\
&\quad - \frac{4t^{2+3\alpha}x^{2-\beta}\Gamma(3+2\alpha)}{\Gamma(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)\Gamma(2-\beta)},
\end{aligned}$$

$$\begin{aligned}
u_2 &= -L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[u_0\frac{\partial^{\beta}u_1}{\partial x^{\beta}} + u_1\frac{\partial^{\beta}u_0}{\partial x^{\beta}}\right]\right]\right] \\
&= \frac{t^{5\alpha}x^{3-2\beta}\Gamma(1+2\alpha)\Gamma(1+4\alpha)}{\Gamma^3(1+\alpha)\Gamma(1+3\alpha)\Gamma(1+5\alpha)\Gamma^2(2-\beta)} \\
&\quad + \frac{2t^{2+5\alpha}x^{3-2\beta}\Gamma(1+2\alpha)\Gamma(3+4\alpha)}{\Gamma^2(1+\alpha)\Gamma(3+\alpha)\Gamma(1+3\alpha)\Gamma(3+5\alpha)\Gamma^2(2-\beta)} \\
&\quad + \frac{4t^{2+5\alpha}x^{3-2\beta}\Gamma(3+2\alpha)\Gamma(3+4\alpha)}{\Gamma^2(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)\Gamma(3+5\alpha)\Gamma^2(2-\beta)} \\
&\quad + \frac{8t^{4+5\alpha}x^{3-2\beta}\Gamma(3+2\alpha)\Gamma(5+4\alpha)}{\Gamma(1+\alpha)\Gamma^2(3+\alpha)\Gamma(3+3\alpha)\Gamma(5+5\alpha)\Gamma^2(2-\beta)} \\
&\quad + \frac{4t^{4+5\alpha}x^{3-2\beta}\Gamma(5+2\alpha)\Gamma(5+4\alpha)}{\Gamma(1+\alpha)\Gamma^2(3+\alpha)\Gamma(5+3\alpha)\Gamma(5+5\alpha)\Gamma^2(2-\beta)} \\
&\quad + \frac{8t^{6+5\alpha}x^{3-2\beta}\Gamma(5+2\alpha)\Gamma(7+4\alpha)}{\Gamma^3(3+\alpha)\Gamma(5+3\alpha)\Gamma(7+5\alpha)\Gamma^2(2-\beta)} \\
&\quad + \left(4t^{2+5\alpha}x^{3-2\beta}\Gamma(3+2\alpha)\Gamma(3+4\alpha)\Gamma(3-\beta)\right) \\
&\quad \times \left(\Gamma^2(1+\alpha)\Gamma(3+\alpha)\Gamma(3+3\alpha)\right. \\
&\qquad \left. \times \Gamma(3+5\alpha)\Gamma(3-2\beta)\Gamma(2-\beta)\right)^{-1} \\
&\quad + \left(8t^{4+5\alpha}x^{3-2\beta}\Gamma(3+2\alpha)\Gamma(5+4\alpha)\Gamma(3-\beta)\right) \\
&\quad \times \left(\Gamma(1+\alpha)\Gamma^2(3+\alpha)\Gamma(3+3\alpha)\right. \\
&\qquad \left. \times \Gamma(5+5\alpha)\Gamma(3-2\alpha)\Gamma(2-\beta)\right)^{-1} \\
&\quad + \left(4t^{4+5\alpha}x^{3-2\beta}\Gamma(5+2\alpha)\Gamma(5+4\alpha)\Gamma(3-\beta)\right) \\
&\quad \times \left(\Gamma(1+\alpha)\Gamma^2(3+\alpha)\Gamma(5+3\alpha)\right. \\
&\qquad \left. \times \Gamma(5+5\alpha)\Gamma(3-2\beta)\Gamma(2-\beta)\right)^{-1} \\
&\quad + \left(8t^{6+5\alpha}x^{3-2\beta}\Gamma(5+2\alpha)\Gamma(7+4\alpha)\Gamma(3-\beta)\right) \\
&\quad \times \left(\Gamma^3(1+\alpha)\Gamma(5+3\alpha)\Gamma(7+5\alpha)\right. \\
&\qquad \left. \times \Gamma(3-2\alpha)\Gamma(2-\beta)\right)^{-1}, \\
&\ \ \vdots
\end{aligned} \tag{44}$$

The solution of (36) and (37) is given as $u(x,t) = u_0 + u_1 + u_2 + \cdots$. If we take $\alpha = \beta = 1$, one has

$$\begin{aligned}
u_0 &= xt + \frac{t^3x}{3}, \\
u_1 &= -\frac{t^3x}{3} - \frac{2t^5x}{15} - \frac{t^7x}{63}, \\
u_2 &= \frac{2t^5x}{15} + \frac{22t^7x}{315} + \frac{38t^9x}{2835} + \frac{2t^{11}x}{2079}, \\
&\ \ \vdots
\end{aligned} \tag{45}$$

The noise terms $-(t^3x/3)$ between the components u_0 and u_1 can be canceled and the remaining term of u_0 still satisfies the equation. For this special case, the exact solution is therefore $u(x,t) = tx$ which was given in [25].

Example 3. Consider the following timefractional nonlinear system arising in thermoelasticity [26]:

$$\begin{aligned}
&\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} - a(u_x,\theta)u_{xx} + b(u_x,\theta)\theta_x = f(x,t), \\
&c(u_x,\theta)\frac{\partial^{\beta}v(x,t)}{\partial t^{\beta}} + b(u_x,\theta)u_{xt} - d(u_x,\theta)\theta_{xx} = g(x,t),
\end{aligned} \tag{46}$$

where $t > 0$, $x \in R^1$, $1 < \alpha \le 2$, $0 < \beta \le 1$, and the time fractional derivatives defined here are in Caputo sense. a, b, c, and d are defined by

$$\begin{aligned}
a(u_x,\theta) &= 2 - u_x\theta, \qquad b(u_x,\theta) = 2 + u_x\theta, \\
c(u_x,\theta) &= 1, \qquad d(u_x,\theta) = \theta,
\end{aligned} \tag{47}$$

and the right-hand side of (46) is replaced by

$$\begin{aligned}
f(x,t) &= \frac{2}{1+x^2} - \frac{2(1+t^2)(3x^2-1)}{(1+x^2)^3}a(w,v) \\
&\quad - \frac{2x(1+t)}{(1+x^2)^2}b(w,v), \\
g(x,t) &= \frac{1}{1+x^2}c(w,v) - \frac{4xt}{(1+x^2)^2 b(w,v)} \\
&\quad - \frac{2(1+t)(3x^2-1)}{(1+x^2)^3}d(w,v),
\end{aligned} \tag{48}$$

where a, b, c, and d are defined above and

$$w \equiv w(x,t) = \frac{2x\left(1+t^2\right)}{\left(1+x^2\right)^2}, \qquad w \equiv w(x,t) = \frac{1+t}{1+x^2}, \tag{49}$$

with the initial conditions

$$u(x,0) = \frac{1}{1+x^2}, \qquad u_t(x,0) = 0, \qquad v(x,0) = \frac{1}{1+x^2}; \tag{50}$$

thus the exact solution of system is $u(x,t) = (1+t^2)/(1+x^2)$, $\theta = (1+t)/(1+x^2)$. After taking the Laplace transform to both sides of (46) and (50), we get the following iteration formula:

$$\begin{aligned}
U_{n+1}(x,s) &= U_n(x,s) + \lambda_1(s) \\
&\quad \times \left[s^{\alpha} U_n(x,s) - s^{\alpha-1} u(x,0) - s^{\alpha-2} u_t(x,0)\right. \\
&\quad\quad - L\left[2u_{nxx} - 2\theta_{nx}\right] \\
&\quad\quad \left. - L\left[u_{nx}\theta_n u_{nxx} + u_{nx}\theta_n\theta_{nx}\right]\right], \\
\Theta_{n+1}(x,s) &= \Theta_n(x,s) + \lambda_2(s) \\
&\quad \times \left[s^{\beta} U_n(x,s) - s^{\beta-1} u(x,0)\right. \\
&\quad\quad + L\left[-2u_{nxt}\right] \\
&\quad\quad \left. - L\left[u_{nx}\theta_n u_{nxt} - \theta_n\theta_{nxx}\right]\right],
\end{aligned} \tag{51}$$

where $\Theta(x,s) = L[\theta(x,t)] = \int_0^{\infty} e^{-st}\theta(x,t)dt$. As a result, after the identification of a Lagrange multiplier $\lambda_1(s) = -1/s^{\alpha}$, $\lambda_2(s) = -1/s^{\beta}$ and with the inverse-Laplace transform, one can derive the following iteration formula:

$$\begin{aligned}
u_{n+1} &= u_0 + L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[2u_{nxx} - 2\theta_{nx}\right]\right.\right. \\
&\qquad \left.\left. - L\left[u_{nx}\theta_n u_{nxx} + u_{nx}\theta_n\theta_{nx}\right]\right]\right], \\
\theta_{n+1} &= \theta_0 + L^{-1}\left[\frac{1}{s^{\beta}}\left[L\left[-2u_{nxt}\right]\right.\right. \\
&\qquad \left.\left. - L\left[u_{nx}\theta_n u_{nxt} - \theta_n\theta_{nxx}\right]\right]\right],
\end{aligned} \tag{52}$$

$u_0(x,t), v_0(x,t)$ is an initial approximation of (46), and

$$\begin{aligned}
u_0(x,t) &= u(x,0) + L^{-1}\left[\frac{1}{s^{\alpha}} L\left[f(x,t)\right]\right], \\
\theta_0(x,t) &= \theta(x,0) + L^{-1}\left[\frac{1}{s^{\beta}} L\left[g(x,t)\right]\right].
\end{aligned} \tag{53}$$

Applying the variational homotopy perturbation method, one has

$$\begin{aligned}
\sum_{n=0}^{\infty} p^n u_n &= u_0 + p \\
&\quad \times \left[L^{-1}\left[\frac{1}{s^{\alpha}}\left[L\left[2\sum_{n=0}^{\infty} p^n u_{nxx} - 2\sum_{n=0}^{\infty} p^n \theta_{nx}\right]\right.\right.\right. \\
&\qquad - L\left[\sum_{n=0}^{\infty} p^n \mathscr{H}_{1n}(u,\theta)\right. \\
&\qquad\quad \left.\left.\left.\left. + \sum_{n=0}^{\infty} p^n \mathscr{H}_{2n}(u,\theta)\right]\right]\right]\right], \\
\sum_{n=0}^{\infty} p^n \theta_n &= \theta_0 + p\left[L^{-1}\left[\frac{1}{s^{\beta}}\left[L\left[-2\sum_{n=0}^{\infty} p^n u_{nxt}\right]\right.\right.\right. \\
&\qquad - L\left[\sum_{n=0}^{\infty} p^n \mathscr{H}_{3n}(u,\theta)\right. \\
&\qquad\quad \left.\left.\left.\left. - \sum_{n=0}^{\infty} p^n \mathscr{H}_{4n}(u,\theta)\right]\right]\right]\right],
\end{aligned} \tag{54}$$

where $\mathscr{H}_{in}(u,\theta)$, $i = 1, 2, 3, 4$, is He's polynomials that represent nonlinear terms $u_x\theta u_{xx}, u_x\theta\theta_x, u_x\theta u_{xt}, \theta\theta_{xx}$, respectively; we have a few terms of the He's polynomials for these nonlinear terms which are given by

$$\begin{gathered}
\mathscr{H}_{10}(u,\theta) = u_{0x}\theta_0 u_{0xx}, \\
\mathscr{H}_{11}(u,\theta) = u_{0x}\theta_0 u_{1xx} + u_{0x}\theta_1 u_{0xx} + u_{1x}\theta_0 u_{0xx}, \\
\vdots \\
\mathscr{H}_{20}(u,\theta) = u_{0x}\theta_0\theta_{0x}, \\
\mathscr{H}_{21}(u,\theta) = u_{0x}\theta_1\theta_{0x} + u_{0x}\theta_0\theta_{1x} + u_{1x}\theta_0 u_{0x}, \\
\vdots \\
\mathscr{H}_{30}(u,\theta) = u_{0x}\theta_0 u_{0xt}, \\
\mathscr{H}_{31}(u,\theta) = u_{0x}\theta_0 u_{1xt} + u_{0x}\theta_1 u_{0xt} + u_{1x}\theta_0 u_{0xt}, \\
\vdots \\
\mathscr{H}_{40}(u,\theta) = \theta_0\theta_{0xx}, \\
\mathscr{H}_{41}(u,\theta) = \theta_0\theta_{1xx} + u_{0x}\theta_1 u_{0xx}, \\
\vdots
\end{gathered} \tag{55}$$

Comparing the coefficient with identical powers in p, one has

$$\begin{aligned}
u_0(x,t) &= \frac{1}{1+x^2} \\
&\quad + \left(\frac{4x-12x^3}{(1+x^2)^6} + \frac{4x^2}{(1+x^2)^5} + \frac{4-12x^2}{(1+x^2)^3} + \frac{4x}{(1+x^2)^2} + \frac{2}{1+x^2}\right)\frac{t^\alpha}{\Gamma(\alpha)} \\
&\quad + \left(\frac{4x-12x^3}{(1+x^2)^6} + \frac{8x^2}{(1+x^2)^5} - \frac{4}{(1+x^2)^5}\right) \times \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \left(\frac{480x-1440x^3}{(1+x^2)^6}\right)\frac{t^{5+\alpha}}{\Gamma(6+\alpha)} \\
&\quad + \left(\frac{16x-48x^3}{(1+x^2)^6} + \frac{16x^2}{(1+x^2)^5} + \frac{8+24x^2}{(1+x^2)^3}\right) \times \frac{t^{2+\alpha}}{\Gamma(3+\alpha)} \\
&\quad + \left(\frac{48-144x^3}{(1+x^2)^6} + \frac{48x^2}{(1+x^2)^5}\right)\frac{t^{3+\alpha}}{\Gamma(4+\alpha)} \\
&\quad + \left(\frac{96x-288x^3}{(1+x^2)^6} + \frac{96x^2}{(1+x^2)^5}\right)\frac{t^{4+\alpha}}{\Gamma(5+\alpha)}, \\
\theta_0(x,t) &= \frac{1}{1+x^2} + \left(\frac{2-6x^2}{(1+x^2)^4} + \frac{1}{1+x^2}\right)\frac{t^\beta}{\Gamma(1+\beta)} \\
&\quad + \left(\frac{8x^2}{(1+x^2)^5} + \frac{4-12x^2}{(1+x^2)^4} - \frac{8x}{(1+x^2)^2}\right) \times \frac{t^{1+\beta}}{\Gamma(2+\beta)} \\
&\quad + \left(\frac{16x^2}{(1+x^2)^5} + \frac{4-12x^2}{(1+x^2)^4}\right)\frac{t^{2+\beta}}{\Gamma(3+\beta)} \times \frac{48x^2}{(1+x^2)^5}\frac{t^{3+\beta}}{\Gamma(4+\beta)} \\
&\quad + \frac{192x^2}{(1+x^2)^5}\frac{t^{4+\beta}}{\Gamma(5+\beta)}, \\
u_1(x,t) &= L^{-1}\left[\frac{1}{s^\alpha}\left[L\left[2u_{0xx} - 2\theta_{0x}\right] - L\left[u_{0x}\theta_0 u_{0xt} + u_{0x}\theta_0\theta_{0x}\right]\right]\right], \\
\theta_1(x,t) &= L^{-1}\left[\frac{1}{s^\beta}\left[L\left[-2u_{0xt}\right] - L\left[u_{0x}\theta_0 u_{0xt} - \theta_0\theta_{0xx}\right]\right]\right], \\
u_2 &= L^{-1}\left[\frac{1}{s^\alpha}\left[L\left[2u_{1xx} - 2\theta_{1x}\right] - L\left[u_{0x}\theta_0 u_{1xx} + u_{0x}\theta_1 u_{0xx} + u_{1x}\theta_0 u_{0xx} + u_{0x}\theta_1\theta_{0x} + u_{0x}\theta_0\theta_{1x} + u_{1x}\theta_0 u_{0x}\right]\right]\right], \\
\theta_2 &= L^{-1}\left[\frac{1}{s^\beta}\left[L\left[-2u_{1xt}\right] - L\left[u_{0x}\theta_0 u_{1xt} + u_{0x}\theta_1 u_{0xt} + u_{1x}\theta_0 u_{0xt} - \theta_0\theta_{1xx} + u_{0x}\theta_1 u_{0xx}\right]\right]\right], \\
&\ \vdots
\end{aligned} \tag{56}$$

and so on; in this manner the rest of components of the solution can be obtained using the Mathematica symbolic computation software for purpose of simlification, the approximate solutions are not listed here.

4. Conclusion

In this paper, a new modification of variational iteration method is considered, which is based on Laplace transform and homotopy perturbation method. The fractional lagrange multiplier is accurately determined by the Laplace transform and the nonlinear one can be easily handled by the use of He's polynomials. Several fractional nonlinear nonhomogeneous equations are analytically solved as examples and the methodology is demonstrated. Examples 1, 2, and 3 have been successfully solved. And the results show that this method is a powerful and reliable method for finding the solution of the fractional nonlinear equations.

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References

[1] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[2] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing, River Edge, NJ, USA, 2000.

[3] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Physics Reports*, vol. 339, no. 1, p. 77, 2000.

[4] K. Diethelm and N. J. Ford, "Multi-order fractional differential equations and their numerical solution," *Applied Mathematics and Computation*, vol. 154, no. 3, pp. 621–640, 2004.

[5] F. W. Liu, V. Anh, and I. Turner, "Numerical solution of the space fractional Fokker-Planck equation," *Journal of Computational and Applied Mathematics*, vol. 166, no. 1, pp. 209–219, 2004.

[6] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, vol. 3 of *Series on Complexity, Nonlinearity and Chaos*, World Scientific, Boston, Mass, USA, 2012.

[7] J. S. Duan, R. Rach, D. Buleanu, and A. M. Wazwaz, "A review of the Adomian decomposition method and its applicaitons to fractional differential equations," *Communications in Fractional Calculus*, vol. 3, pp. 73–99, 2012.

[8] J.-H. He, "Variational iteration method—a kind of non-linear analytical technique: some examples," *International Journal of Non-Linear Mechanics*, vol. 34, no. 4, pp. 699–708, 1999.

[9] A.-M. Wazwaz, "The variational iteration method for analytic treatment of linear and nonlinear ODEs," *Applied Mathematics and Computation*, vol. 212, no. 1, pp. 120–134, 2009.

[10] V. S. Erturk, S. Momani, and Z. Odibat, "Application of generalized differential transform method to multi-order fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 8, pp. 1642–1654, 2008.

[11] J.-H. He, "Homotopy perturbation method: a new nonlinear analytical technique," *Applied Mathematics and Computation*, vol. 135, no. 1, pp. 73–79, 2003.

[12] Z. Odibat and S. Momani, "Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order," *Chaos, Solitons & Fractals*, vol. 36, no. 1, pp. 167–174, 2008.

[13] X. Y. Jiang and H. T. Qi, "Thermal wave model of bioheat transfer with modified Riemann-Liouville fractional derivative," *Journal of Physics*, vol. 45, no. 48, Article ID 485101, 2012.

[14] J. H. Ma and Y. Q. Liu, "Exact solutions for a generalized nonlinear fractional Fokker-Planck equation," *Nonlinear Analysis. Real World Applications*, vol. 11, no. 1, pp. 515–521, 2010.

[15] Y.-Q. Liu and J.-H. Ma, "Exact solutions of a generalized multi-fractional nonlinear diffusion equation in radical symmetry," *Communications in Theoretical Physics*, vol. 52, no. 5, pp. 857–861, 2009.

[16] Y. Q. Liu, "Approximate solutions of fractional nonlinear equations using homotopy perturbation transformation method," *Abstract and Applied Analysis*, vol. 2012, Article ID 752869, 14 pages, 2012.

[17] Y. Q. Liu, "Study on space-time fractional nonlinear biological equation in radial symmetry," *Mathematical Problems in Engineering*, vol. 2013, Article ID 654759, 6 pages, 2013.

[18] Y. Q. Liu, "Variational homotopy perturbation method for solving fractional initial boundary value problems," *Abstract and Applied Analysis*, vol. 2012, Article ID 727031, 10 pages, 2012.

[19] S. M. Guo, L. Q. Mei, and Y. Li, "Fractional variational homotopy perturbation iteration method and its application to a fractional diffusion equation," *Applied Mathematics and Computation*, vol. 219, no. 11, pp. 5909–5917, 2013.

[20] J.-H. He, "Variational iteration method—some recent results and new interpretations," *Journal of Computational and Applied Mathematics*, vol. 207, no. 1, pp. 3–17, 2007.

[21] G.-C. Wu and D. Baleanu, "Variational iteration method for fractional calculus—a universal approach by Laplace transform," *Advances in Difference Equations*, vol. 2013, article 18, 9 pages, 2013.

[22] G. C. Wu, "Laplace tranform overcoming principle drawbacks in applicaiton of the variation iteration method to fractional heat equations," *Thermal Science*, vol. 16, no. 4, pp. 1257–1261, 2012.

[23] A. Ghorbani, "Beyond Adomian polynomials: He polynomials," *Chaos, Solitons and Fractals*, vol. 39, no. 3, pp. 1486–1492, 2009.

[24] H. Jafari, M. Alipour, and H. Tajadodi, "Two-dimensional differential transform method for solving nonlinear partial differential equaitons," *International Journal of Research and Reviews in Applied Sciences*, vol. 2, no. 1, pp. 47–52, 2010.

[25] S. Momani and Z. Odibat, "A novel method for nonlinear fractional partial differential equations: combination of DTM and generalized Taylor's formula," *Journal of Computational and Applied Mathematics*, vol. 220, no. 1-2, pp. 85–95, 2008.

[26] N. H. Sweilam and M. M. Khader, "Variational iteration method for one dimensional nonlinear thermoelasticity," *Chaos, Solitons & Fractals*, vol. 32, no. 1, pp. 145–149, 2007.

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Research Article

Extremal Solutions and Relaxation Problems for Fractional Differential Inclusions

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We present the existence of extremal solution and relaxation problem for fractional differential inclusion with initial conditions.

1. Introduction

Differential equations with fractional order have recently proved to be valuable tools in the modeling of many physical phenomena [1–9]. There has also been a significant theoretical development in fractional differential equations in recent years; see the monographs of Kilbas et al. [10], Miller and Ross [11], Podlubny [12], and Samko et al. [13] and the papers of Kilbas and Trujillo [14], Nahušev [15], Podlubny et al. [16], and Yu and Gao [17].

Recently, some basic theory for initial value problems for fractional differential equations and inclusions involving the Riemann-Liouville differential operator was discussed, for example, by Lakshmikantham [18] and Chalco-Cano et al. [19].

Applied problems requiring definitions of fractional derivatives are those that are physically interpretable for initial conditions containing $y(0)$, $y'(0)$, and so forth. The same requirements are true for boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both Riemann-Liouville and Caputo types, see Podlubny [12].

Fractional calculus has a long history. We refer the reader to [20].

Recently fractional functional differential equations and inclusions and impulsive fractional differential equations and inclusions with standard Riemann-Liouville and Caputo derivatives with differences conditions were studied by Abbas et al. [21, 22], Benchohra et al. [23], Henderson and Ouahab [24, 25], Jiao and Zhou [26], and Ouahab [27–29] and in the references therein.

In this paper, we will be concerned with the existence of solutions, Filippov's theorem, and the relaxation theorem of abstract fractional differential inclusions. More precisely, we will consider the following problem:

$$\begin{gathered} {}^cD^{\alpha} y(t) \in F(t, y(t)), \quad \text{a.e. } t \in J := [0,b], \\ y(0) = y_0, \qquad y'(0) = y_1, \end{gathered} \tag{1}$$

$$\begin{gathered} {}^cD^{\alpha} y(t) \in \operatorname{ext} F(t, y(t)), \quad \text{a.e. } t \in J := [0,b], \\ y(0) = y_0, \qquad y'(0) = y_1, \end{gathered} \tag{2}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivatives, $\alpha \in (1,2]$, $F : J \times \mathbb{R}^N \to \mathscr{P}(\mathbb{R}^N)$ is a multifunction, and $\operatorname{ext} F(t,y)$ represents the set of extreme points of $F(t,y)$. ($\mathscr{P}(\mathbb{R}^N)$ is the family of all nonempty subsets of $\mathbb{R}^N$.

During the last couple of years, the existence of extremal solutions and relaxation problem for ordinary differential inclusions was studied by many authors, for example, see [30–34] and the references therein.

The paper is organized as follows. We first collect some background material and basic results from multivalued analysis and give some results on fractional calculus in Sections 2 and 3, respectively. Then, we will be concerned with the existence of solution for extremal problem. This is the aim of Section 4. In Section 5, we prove the relaxation problem.

2. Preliminaries

The reader is assumed to be familiar with the theory of multivalued analysis and differential inclusions in Banach spaces, as presented in Aubin et al. [35, 36], Hu and Papageorgiou [37], Kisielewicz [38], and Tolstonogov [32].

Let $(X, \|\cdot\|)$ be a real Banach space, $[0,b]$ an interval in R, and $C([0,b],X)$ the Banach space of all continuous functions from J into X with the norm

$$\|y\|_\infty = \sup\{\|y(t)\| : 0 \le t \le b\}. \tag{3}$$

A measurable function $y : [0,b] \to X$ is Bochner integrable if $\|y\|$ is Lebesgue integrable. In what follows, $L^1([0,b],X)$ denotes the Banach space of functions $y : [0,b] \to X$, which are Bochner integrable with norm

$$\|y\|_1 = \int_0^b \|y(t)\|\,dt. \tag{4}$$

Denote by $L^1_w([0,b],X)$ the space of equivalence classes of Bochner integrable function $y : [0,b] \to X$ with the norm

$$\|y\|_w = \sup_{t\in[0,t]} \left\| \int_0^t y(s)\,ds \right\|. \tag{5}$$

The norm $\|\cdot\|_w$ is weaker than the usual norm $\|\cdot\|_1$, and for a broad class of subsets of $L^1([0,b],X)$, the topology defined by the weak norm coincides with the usual weak topology (see [37, Proposition 4.14, page 195]). Denote by

$$\begin{gathered} \mathscr{P}(X) = \{Y \subset X : Y \neq \emptyset\}, \\ \mathscr{P}_{\mathrm{cl}}(X) = \{Y \in \mathscr{P}(X) : Y \text{ closed}\}, \\ \mathscr{P}_b(X) = \{Y \in \mathscr{P}(X) : Y \text{ bounded}\}, \\ \mathscr{P}_{\mathrm{cv}}(X) = \{Y \in \mathscr{P}(X) : Y \text{ convex}\}, \\ \mathscr{P}_{\mathrm{cp}}(X) = \{Y \in \mathscr{P}(X) : Y \text{ compact}\}. \end{gathered} \tag{6}$$

A multivalued map $G : X \to \mathscr{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x \in X$. We say that G is *bounded on bounded sets* if $G(B)$ is bounded in X for each bounded set B of X (i.e., $\sup_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

Definition 1. A multifunction $F : X \to \mathscr{P}(Y)$ is said to be upper semicontinuous at the point $x_0 \in X$, if, for every open $W \subseteq Y$ such that $F(x_0) \subset W$, there exists a neighborhood $V(x_0)$ of x_0 such that $F(V(x_0)) \subset W$.

A multifunction is called *upper semicontinuous* (*u.s.c.* for short) on X if for each $x \in X$ it is u.s.c. at x.

Definition 2. A multifunction $F : X \to \mathscr{P}(Y)$ is said to be lower continuous at the point $x_0 \in X$, if, for every open $W \subseteq Y$ such that $F(x_0) \cap W \neq \emptyset$, there exists a neighborhood $V(x_0)$ of x_0 with property that $F(x) \cap W \neq \emptyset$ for all $x \in V(x_0)$.

A multifunction is called *lower semicontinuous* (*l.s.c.* for short) provided that it is lower semicontinuous at every point $x \in X$.

Lemma 3 (see [39, Lemma 3.2]). *Let $F : [0,b] \to \mathscr{P}(Y)$ be a measurable multivalued map and $u : [a,b] \to Y$ a measurable function. Then for any measurable $v : [a,b] \to (0,+\infty)$, there exists a measurable selection f_v of F such that for a.e. $t \in [a,b]$,*

$$\|u(t) - f_v(t)\| \le d(u(t), F(t)) + v(t). \tag{7}$$

First, consider the Hausdorff pseudometric

$$H_d : \mathscr{P}(E) \times \mathscr{P}(E) \longrightarrow \mathbb{R}^+ \cup \{\infty\}, \tag{8}$$

defined by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}, \tag{9}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. $(\mathscr{P}_{b,\mathrm{cl}}(E), H_d)$ is a metric space and $(\mathscr{P}_{\mathrm{cl}}(X), H_d)$ is a generalized metric space.

Definition 4. A multifunction $F : Y \to \mathscr{P}(X)$ is called Hausdorff lower semicontinuous at the point $y_0 \in Y$, if for any $\epsilon > 0$ there exists a neighbourhood $U(y_0)$ of the point y_0 such that

$$F(y_0) \subset F(y) + \epsilon B(0,1), \quad \text{for every } y \in U(y_0), \tag{10}$$

where $B(0,1)$ is the unite ball in X.

Definition 5. A multifunction $F : Y \to \mathscr{P}(X)$ is called Hausdorff upper semicontinuous at the point $y_0 \in Y$, if for any $\epsilon > 0$ there exists a neighbourhood $U(y_0)$ of the point y_0 such that

$$F(y) \subset F(y_0) + \epsilon B(0,1), \quad \text{for every } y \in U(y_0). \tag{11}$$

F is called continuous, if it is Hausdorff lower and upper semicontinuous.

Definition 6. Let X be a Banach space; a subset $A \subset L^1([0,b],X)$ is decomposable if, for all $u, v \in A$ and for every Lebesgue measurable set $I \subset J$, one has

$$u\chi_I + v\chi_{[0,b]\setminus I} \in A, \tag{12}$$

where χ_A stands for the characteristic function of the set A. We denote by $\mathrm{Dco}(L^1([0,b],X))$ the family of decomposable sets.

Let $F : [0,b] \times X \to \mathcal{P}(X)$ be a multivalued map with nonempty closed values. Assign to F the multivalued operator $\mathcal{F} : C([0,b],X) \to \mathcal{P}(L^1([0,b],X))$ defined by

$$\mathcal{F}(y) = \Big\{ v \in L^1([0,b],X) : v(t) \in F(t,y(t)), \quad \text{a.e. } t \in [0,b] \Big\}. \tag{13}$$

The operator $\mathcal{F}$ is called the Nemyts'kiĭ operator associated to F.

Definition 7. Let $F : [0,b] \times X \to \mathcal{P}(X)$ be a multivalued map with nonempty compact values. We say that F is of lower semicontinuous type (l.s.c. type) if its associated Nemyts'kiĭ operator $\mathcal{F}$ is lower semicontinuous and has nonempty closed and decomposable values.

Next, we state a classical selection theorem due to Bressan and Colombo.

Lemma 8 (see [40]). *Let X be a separable metric space and let E be a Banach space. Then every l.s.c. multivalued operator $N : X \to \mathcal{P}_{cl}(L^1([0,b],E))$ with closed decomposable values has a continuous selection; that is, there exists a continuous single-valued function $f : X \to L^1([0,b],E)$ such that $f(x) \in N(x)$ for every $x \in X$.*

Let us introduce the following hypothesis.

($\overline{\mathcal{H}}_1$) $F : [0,b] \times X \to \mathcal{P}(X)$ is a nonempty compact valued multivalued map such that

(a) the mapping $(t,y) \mapsto F(t,y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;

(b) the mapping $y \mapsto F(t,y)$ is lower semicontinuous for a.e. $t \in [0,b]$.

Lemma 9 (see, e.g., [41]). *Let $F : J \times X \to \mathcal{P}_{cp}(E)$ be an integrably bounded multivalued map satisfying ($\overline{\mathcal{H}}_1$). Then F is of lower semicontinuous type.*

Define

$$F(K) = \Big\{ f \in L^1([0,b],X) : f(t) \in K \text{ a.e. } t \in [0,b] \Big\}, \quad K \subset X, \tag{14}$$

where X is a Banach space.

Lemma 10 (see [37]). *Let $K \subset X$ be a weakly compact subset of X. Then $F(K)$ is relatively weakly compact subset of $L^1([0,b],X)$. Moreover if K is convex, then $F(K)$ is weakly compact in $L^1([0,b],X)$.*

Definition 11. A multifunction $F : [0,b] \times Y \to \mathcal{P}_{wcpcv}(X)$ possesses the Scorza-Dragoni property (S-D property) if for each $\epsilon > 0$, there exists a closed set $J_\epsilon \subset [0,b]$ whose Lebesgue measure $\mu(J_\epsilon) \le \epsilon$ and such that $F : [0,b] \setminus J_\epsilon \times Y \to X$ is continuous with respect to the metric $d_X(\cdot,\cdot)$.

Remark 12. It is well known that if the map $F : [0,b] \times Y \to \mathcal{P}_{wcpcv}(X)$ is continuous with respect to y for almost every $t \in [0,b]$ and is measurable with respect to t for every $y \in Y$, then it possesses the S-D property.

In what follows, we present some definitions and properties of extreme points.

Definition 13. Let A be a nonempty subset of a real or complex linear vector space. An extreme point of a convex set A is a point $x \in A$ with the property that if $x = \lambda y + (1-\lambda)z$ with $y, z \in A$ and $\lambda \in [0,1]$, then $y = x$ and/or $z = x$. $\mathrm{ext}(A)$ will denote the set of extreme points of A.

In other words, an extreme point is a point that is not an interior point of any line segment lying entirely in A.

Lemma 14 (see [42]). *A nonempty compact set in a locally convex linear topological space has extremal points.*

Let $\{x_n'\}_{n\in\mathbb{N}}$ be a denumerable, dense (in $\sigma(X',X)$ topology) subset of the set $\{x \in X : \|x\| \le 1\}$. For any $A \in \mathcal{P}_{cpcv}(X)$ and x_n' define the function

$$d^n(A,u) = \max\left\{ \left\langle y - z, x_n' \right\rangle : y, z \in A,\ u = \frac{y+z}{2} \right\}. \tag{15}$$

Lemma 15 (see [33]). *$u \in \mathrm{ext}(A)$ if and only if $d^n(A,u) = 0$ for all $n \ge 1$.*

In accordance with Krein-Milman and Trojansky theorem [43], the set $\mathrm{ext}(S_F)$ is nonempty and $\overline{\mathrm{co}}(\mathrm{ext}(S_F)) = S_F$.

Lemma 16 (see [33]). *Let $F : [0,b] \to \mathcal{P}_{wcpcv}(X)$ be a measurable, integrably bounded map. Then*

$$\overline{\mathrm{ext}}(S_F) \subseteq S_F, \tag{16}$$

where $\overline{\mathrm{ext}}(S_F)$ is the closure of set $\mathrm{ext}(S_F)$ in the topology of the space $L^1([0,b],X)$.

Theorem 17 (see [33]). *Let $F : [0,b] \times Y \to \mathcal{P}_{wcpcv}(X)$ be a multivalued map that has the S-D property and let it be integrable bounded on compacts from Y. Consider a compact subset $K \subset C([0,b],X)$ and define the multivalued map $G : K \to L^1([0,b],X)$, by*

$$G(y(\cdot)) = \Big\{ f \in L^1([0,b],X) : f(t) \in F(t,y(t)) \text{ a.e. on } [0,b] \Big\}, \quad y \in K. \tag{17}$$

Then for every K compact in $C([0,b],X)$, $\epsilon > 0$ and any continuous selection $f : K \to L^1([0,b],X)$, there exists a continuous selector $g : K \to L^1([0,b],X)$ of the map $\mathrm{ext}(G) : K \to L^1([0,b],X)$ such that for all $y \in C([0,b],X)$ one has

$$\sup_{t\in[0,b]} \left\| \int_0^t ((fy)(s) - (gy)(s))\, ds \right\| \le \epsilon. \tag{18}$$

For a background of extreme point of $F(t, y(t))$ see Dunford-Schwartz [42, Chapter 5, Section 8] and Florenzano and Le Van [44, Chapter 3].

3. Fractional Calculus

According to the Riemann-Liouville approach to fractional calculus, the notation of fractional integral of order α ($\alpha > 0$) is a natural consequence of the well known formula (usually attributed to Cauchy) that reduces the calculation of the n-fold primitive of a function $f(t)$ to a single integral of convolution type. In our notation the Cauchy formula reads

$$I^n f(t) := \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)\,ds, \quad t > 0,\ n \in \mathbb{N}. \tag{19}$$

Definition 18 (see [13, 45]). The fractional integral of order $\alpha > 0$ of a function $f \in L^1([a,b],\mathbb{R})$ is defined by

$$I^{\alpha}_{a^+} f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds, \tag{20}$$

where Γ is the gamma function. When $a = 0$, we write $I^{\alpha} f(t) = f(t) * \phi_{\alpha}(t)$, where $\phi_{\alpha}(t) = t^{(\alpha-1)}/\Gamma(\alpha)$ for $t > 0$, and we write $\phi_{\alpha}(t) = 0$ for $t \leq 0$ and $\phi_{\alpha} \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function and Γ is the Euler gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t}\,dt, \quad \alpha > 0. \tag{21}$$

For consistency, $I^0 = \mathrm{Id}$ (identity operator), that is, $I^0 f(t) = f(t)$. Furthermore, by $I^{\alpha} f(0^+)$ we mean the limit (if it exists) of $I^{\alpha} f(t)$ for $t \to 0^+$; this limit may be infinite.

After the notion of fractional integral, that of fractional derivative of order α ($\alpha > 0$) becomes a natural requirement and one is attempted to substitute α with $-\alpha$ in the above formulas. However, this generalization needs some care in order to guarantee the convergence of the integral and preserve the well known properties of the ordinary derivative of integer order. Denoting by D^n, with $n \in \mathbb{N}$, the operator of the derivative of order n, we first note that

$$D^n I^n = \mathrm{Id}, \quad I^n D^n \neq \mathrm{Id}, \quad n \in \mathbb{N}, \tag{22}$$

that is, D^n is the left inverse (and not the right inverse) to the corresponding integral operator J^n. We can easily prove that

$$I^n D^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(a^+)\frac{(t-a)^k}{k!}, \quad t > 0. \tag{23}$$

As a consequence, we expect that D^{α} is defined as the left inverse to I^{α}. For this purpose, introducing the positive integer n such that $n-1 < \alpha \leq n$, one defines the fractional derivative of order $\alpha > 0$.

Definition 19. For a function f given on interval $[a,b]$, the αth Riemann-Liouville fractional-order derivative of f is defined by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{-\alpha+n-1} f(s)\,ds, \tag{24}$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

Defining for consistency, $D^0 = I^0 = \mathrm{Id}$, then we easily recognize that

$$D^{\alpha} I^{\alpha} = \mathrm{Id}, \quad \alpha \geq 0, \tag{25}$$

$$D^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha}, \quad \alpha > 0, \quad \gamma \in (-1,0) \cup (0,+\infty), \quad t > 0. \tag{26}$$

Of course, properties (25) and (26) are a natural generalization of those known when the order is a positive integer.

Note the remarkable fact that the fractional derivative $D^{\alpha} f$ is not zero for the constant function $f(t) = 1$, if $\alpha \notin \mathbb{N}$. In fact, (26) with $\gamma = 0$ illustrates that

$$D^{\alpha} 1 = \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad \alpha > 0,\ t > 0. \tag{27}$$

It is clear that $D^{\alpha} 1 = 0$, for $\alpha \in \mathbb{N}$, due to the poles of the gamma function at the points $0, -1, -2, \ldots$.

We now observe an alternative definition of fractional derivative, originally introduced by Caputo [46, 47] in the late sixties and adopted by Caputo and Mainardi [48] in the framework of the theory of Linear Viscoelasticity (see a review in [4]).

Definition 20. Let $f \in AC^n([a,b])$. The Caputo fractional-order derivative of f is defined by

$$\left({}^{c}D^{\alpha} f\right)(t) := \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1} f^{n}(s)\,ds. \tag{28}$$

This definition is of course more restrictive than Riemann-Liouville definition, in that it requires the absolute integrability of the derivative of order m. Whenever we use the operator D^{α}_{*} we (tacitly) assume that this condition is met. We easily recognize that in general

$$D^{\alpha} f(t) := D^m I^{m-\alpha} f(t) \neq J^{m-\alpha} D^m f(t) := D^{\alpha}_{*} f(t), \tag{29}$$

unless the function $f(t)$, along with its first $n-1$ derivatives, vanishes at $t = a^+$. In fact, assuming that the passage of the m-derivative under the integral is legitimate, we recognize that, for $m-1 < \alpha < m$ and $t > 0$,

$$D^{\alpha} f(t) = {}^{c}D^{\alpha} f(t) + \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+), \tag{30}$$

and therefore, recalling the fractional derivative of the power function (26), one has

$$D^{\alpha}\left(f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a^+)\right) = D^{\alpha}_{*} f(t). \tag{31}$$

The alternative definition, that is, Definition 20, for the fractional derivative thus incorporates the initial values of the function and of lower order. The subtraction of the Taylor polynomial of degree $n-1$ at $t=a^+$ from $f(t)$ means a sort of regularization of the fractional derivative. In particular, according to this definition, the relevant property for which the fractional derivative of a constant is still zero:

$$^cD^\alpha 1 = 0, \quad \alpha > 0. \tag{32}$$

We now explore the most relevant differences between the two fractional derivatives given in Definitions 19 and 20. From Riemann-Liouville fractional derivatives, we have

$$D^\alpha (t-a)^{\alpha-j} = 0, \quad \text{for } j = 1, 2, \ldots, [\alpha]+1. \tag{33}$$

From (32) and (33) we thus recognize the following statements about functions which, for $t > 0$, admit the same fractional derivative of order α, with $n-1 < \alpha \leq n$, $n \in \mathbb{N}$:

$$D^\alpha f(t) = D^\alpha g(t) \Longleftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j (t-a)^{\alpha-j},$$
$$^cD^\alpha f(t) = {}^cD^\alpha g(t) \Longleftrightarrow f(t) = g(t) + \sum_{j=1}^{m} c_j (t-a)^{n-j}. \tag{34}$$

In these formulas, the coefficients c_j are arbitrary constants. For proving all main results we present the following auxiliary lemmas.

Lemma 21 (see [10]). *Let $\alpha > 0$ and let $y \in L^\infty(a,b)$ or $C([a,b])$. Then*

$$\left({}^cD^\alpha I^\alpha y\right)(t) = y(t). \tag{35}$$

Lemma 22 (see [10]). *Let $\alpha > 0$ and $n = [\alpha]+1$. If $y \in AC^n[a,b]$ or $y \in C^n[a,b]$, then*

$$\left(I^\alpha {}^cD^\alpha y\right)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k. \tag{36}$$

For further readings and details on fractional calculus, we refer to the books and papers by Kilbas [10], Podlubny [12], Samko [13], and Caputo [46–48].

4. Existence Result

Definition 23. A function $y \in C([0,b],\mathbb{R}^N)$ is called mild solution of problem (1) if there exist $f \in L^1(J,\mathbb{R}^N)$ such that

$$y(t) = y_0 + ty_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{1-\alpha} f(s)\,ds, \quad t \in [0,b], \tag{37}$$

where $f \in S_{F,y} = \{v \in L^1([0,b],\mathbb{R}^N) : f(t) \in F(t,y(t))$ a.e. on $[0,b]\}$.

We will impose the following conditions on F.

($\mathcal{H}_1$) The function $F : J \times \mathbb{R}^N \to \mathcal{P}_{cpcv}(\mathbb{R}^N)$ such that

(a) for all $x \in \mathbb{R}^N$, the map $t \mapsto F(t,x)$ is measurable,

(b) for every $t \in [0,b]$, the multivalued map $x \to F(t,x)$ is H_d continuous

($\mathcal{H}_2$) There exist $p \in L^1(J,\mathbb{R}^+)$ and a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\|F(t,x)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t,x)\} \leq p(t)\,\psi(\|x\|),$$
$$\text{for a.e. } t \in [0,b] \text{ and each } x \in \mathbb{R}^N, \tag{38}$$

with

$$\int_0^b p(s)\,ds < \int_{\|y_0\|+b\|y_1\|}^{\infty} \frac{du}{\psi(u)}. \tag{39}$$

Theorem 24. *Assume that the conditions ($\mathcal{H}_1$)-($\mathcal{H}_2$) and then the problem (2) have at least one solution.*

Proof. From ($\mathcal{H}_2$) there exists $M > 0$ such that $\|y\|_\infty \leq M$ for each $y \in S_c$.

Let

$$F_1(t,y) = \begin{cases} F(t,y) & \text{if } \|y\| \leq M, \\ F\left(t, \dfrac{My}{\|y\|}\right) & \text{if } \|y\| \geq M. \end{cases} \tag{40}$$

We consider

$$^cD^\alpha y(t) \in F_1(t,y(t)), \quad \text{a.e. } t \in [0,b],$$
$$y(0) = y_0, \qquad y'(0) = y_1. \tag{41}$$

It is clear that all the solutions of (41) are solutions of (2).

Set

$$V = \left\{f \in L^1\left([0,b],\mathbb{R}^N\right) : \|f(t)\| \leq \psi_*(t)\right\},$$
$$\psi_*(t) = p(t)\,\psi(M). \tag{42}$$

It is clear that V is weakly compact in $L^1([0,b],\mathbb{R}^N)$. Remark that for every $f \in V$, there exists a unique solution $L(f)$ of the following problem:

$$^cD^\alpha y(t) = f(t), \quad \text{a.e. } t \in [0,b],$$
$$y(t) = y_0, \qquad y'(0) = y_1; \tag{43}$$

this solution is defined by

$$L(f)(t) = y_0 + ty_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds,$$
$$\text{a.e. } t \in [0,b]. \tag{44}$$

We claim that L is continuous. Indeed, let $f_n \to f$ converge in $L^1([0,b],\mathbb{R}^N)$, as $n \to \infty$, set $y_n = L(f_n)$, $n \in \mathbb{N}$. It is clear that $\{y_n : n \in \mathbb{N}\}$ is relatively compact in $C([0,b],\mathbb{R}^N)$ and y_n converge to $y \in C([0,b],\mathbb{R}^N)$. Let

$$z(t) = y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t \in [0,b]. \tag{45}$$

Then

$$\|y_n - z\|_\infty \leq \frac{b^\alpha}{\Gamma(\alpha)} \int_0^b \|f_n(s) - f(s)\|\, ds \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{46}$$

Hence $K = L(V)$ is compact and convex subset of $C([0,b],\mathbb{R}^N)$. Let $S_F : K \to \mathscr{P}_{\text{clcv}}(L^1([0,b],\mathbb{R}^N))$ be the multivalued Nemitsky operator defined by

$$S_{F_1}(y) = \left\{ f \in L^1([0,b],\mathbb{R}^N) : f(t) \in F_1(t, y(t)), \text{ a.e. } t \in [0,b] \right\} := S_{F_1,y}. \tag{47}$$

It is clear that $F_1(\cdot,\cdot)$ is H_d continuous and $F_1(\cdot,\cdot) \in \mathscr{P}_{wk\text{cpcv}}(\mathbb{R}^N)$ and is integrably bounded, then by Theorem 17 (see also Theorem 6.5 in [32] or Theorem 1.1 in [34]), we can find a continuous function $g : K \to L^1_w([0,b],\mathbb{R}^N)$ such that

$$g(x) \in \operatorname{ext} S_{F_1}(y) \quad \forall y \in K. \tag{48}$$

From Benamara [49] we know that

$$\operatorname{ext} S_{F_1}(y) = S_{\operatorname{ext} F_1(\cdot,y(\cdot))} \quad \forall y \in K. \tag{49}$$

Setting $N = L \circ g$ and letting $y \in K$, then

$$g(y) \in F_1(\cdot, y(\cdot)) \Longrightarrow g(y) \in V \Longrightarrow N(y) = L(g(y)) \in K. \tag{50}$$

Now, we prove that N is continuous. Indeed, let $y_n \in K$ converge to y in $C([0,b],\mathbb{R}^N)$.

Then

$$g(y_n) \text{ converge weakly to } g(y) \quad \text{as } n \longrightarrow \infty. \tag{51}$$

Since $N(y_n) = L(g(y_n)) \in K$ and $g(y_n)(\cdot) \in F(t, y_n(t))$, then

$$g(y_n)(\cdot) \in F(\cdot, \overline{B}_M) \in \mathscr{P}_{\text{cp}}(\mathbb{R}^N). \tag{52}$$

From Lemma 10, $g(y_n)$ converge weakly to y in $L^1([0,b],\mathbb{R}^N)$ as $n \to \infty$. By the definition of N, we have

$$\begin{aligned} N(y_n) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(y_n)(s)\, ds, \quad t \in [0,b], \\ N(y) &= y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(y)(s)\, ds, \quad t \in [0,b]. \end{aligned} \tag{53}$$

Since $\{N(y_n) : n \in \mathbb{N}\} \subset K$, then there exists subsequence of $N(y_n)$ converge in $C([0,b],\mathbb{R}^N)$. Then

$$N(y_n)(t) \longrightarrow N(y)(t), \quad \forall t \in [0,b], \text{ as } n \longrightarrow \infty. \tag{54}$$

This proves that N is continuous. Hence by Schauder's fixed point there exists $y \in K$ such that $y = N(y)$. □

5. The Relaxed Problem

In this section, we examine whether the solutions of the extremal problem are dense in those of the convexified one. Such a result is important in optimal control theory whether the relaxed optimal state can be approximated by original states; the relaxed problems are generally much simpler to build. For the problem for first-order differential inclusions, we refer, for example, to [35, Theorem 2, page 124] or [36, Theorem 10.4.4, page 402]. For the relaxation of extremal problems we see the following recent references [30, 50].

Now we present our main result of this section.

Theorem 25. *Let $F : [0,b] \times \mathbb{R}^N \to \mathscr{P}(\mathbb{R}^N)$ be a multifunction satisfying the following hypotheses.*

($\mathscr{H}_3$) *The function $F : [0,b] \times \mathbb{R}^N \to \mathscr{P}_{cpcv}(\mathbb{R}^N)$ such that, for all $x \in \mathbb{R}^N$, the map*

$$t \longmapsto F(t,x) \tag{55}$$

is measurable.

($\mathscr{H}_4$) *There exists $p \in L^1(J,\mathbb{R}^+)$ such that*

$$\begin{aligned} &H_d(F(t,x), F(t,y)) \leq p(t)\|x-y\|, \\ &\text{for a.e. } t \in [0,b] \text{ and each } x, y \in \mathbb{R}^N, \\ &H_d(F(t,0), 0) \leq p(t) \quad \text{for a.e. } t \in [0,b]. \end{aligned} \tag{56}$$

Then $\overline{S_e} = S_c$.

Proof. By Coviz and Nadlar fixed point theorem, we can easily prove that $S_c \neq \emptyset$, and since F has compact and convex valued, then S_c is compact in $C([0,b],\mathbb{R}^N)$. For more information we see [25, 27–29, 51, 52].

Let $y \in S_c$; then there exists $f \in S_{F,y}$ such that

$$y(t) = y_0 + y_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad \text{a.e. } t \in [0,b]. \tag{57}$$

Let K be a compact and convex set in $C([0,b],\mathbb{R}^N)$ such that $S_c \subset K$. Given that $y_* \in K$ and $\epsilon > 0$, we define the following multifunction $U_\epsilon : [0,b] \to \mathscr{P}(\mathbb{R}^N)$ by

$$U_\epsilon(t) = \left\{ u \in \mathbb{R}^N : \|f(t) - u\| < d(f(t), F(t, y(t))) + \epsilon, \ u \in F(t, y_*(t)) \right\}. \tag{58}$$

The multivalued map $t \to F(t,\cdot)$ is measurable and $x \to F(\cdot,x)$ is H_d continuous. In addition, if $F(\cdot,\cdot)$ has compact values, then $F(\cdot,\cdot)$ is graph measurable, and the mapping $t \to F(t,y(t))$ is a measurable multivalued map for fixed $y \in C([0,b],\mathbb{R}^N)$. Then by Lemma 3, there exists a measurable selection $v_1(t) \in F(t,y(t))$ a.e. $t \in [0,b]$ such that

$$\|f(t) - v_1(t)\| < d(f(t), F(t,y(t))) + \epsilon; \tag{59}$$

this implies that $U_\epsilon(\cdot) \neq \emptyset$. We consider $G_\epsilon : K \to \mathscr{P}(L^1(J,\mathbb{R}^N)$ defined by

$$\begin{aligned} G_\epsilon(y) = \{f_* \in \mathscr{F}(y) : \|f(t) - f_*(t)\| \\ < \epsilon + d(f_*(t), F(t,y(t)))\}. \end{aligned} \tag{60}$$

Since the measurable multifunction F is integrable bounded, Lemma 9 implies that the Nemyts'kiĭ operator $\mathscr{F}$ has decomposable values. Hence $y \to \overline{G_\epsilon(y)}$ is l.s.c. with decomposable values. By Lemma 8, there exists a continuous selection $f_\epsilon : C([0,b],\mathbb{R}^N) \to L^1(J,\mathbb{R}^N)$ such that

$$f_\epsilon(y) \in \overline{G_\epsilon(y)} \quad \forall y \in C([0,b],\mathbb{R}^N). \tag{61}$$

From Theorem 17, there exists function $g_\epsilon : K \to L_w([0,b],\mathbb{R}^N)$ such that

$$\begin{aligned} g_\epsilon(y) \in \operatorname{ext} S_F(y) = S_{\operatorname{ext} F(\cdot,y(\cdot))} \quad \forall y \in K, \\ \| g_\epsilon(y) - f_\epsilon(y)\|_w \leq \epsilon, \quad \forall y \in K. \end{aligned} \tag{62}$$

From $(\mathscr{H}_3)$ we can prove that there exists $M > 0$ such that

$$\|y\|_\infty \leq M \quad \forall y \in S_c. \tag{63}$$

Consider the sequence $\epsilon_n \to 0$, as $n \to \infty$, and set $g_n = g_{\epsilon_n}$, $f_n = f_{\epsilon_n}$. Set

$$\begin{aligned} V = \{f \in L^1([0,b],\mathbb{R}^N) : \|f(t)\| \leq \psi(t) \text{ a.e. } t \in [0,b]\}, \\ \psi(t) = (1+M)p(t). \end{aligned} \tag{64}$$

Let $L : V \to C([0,b],\mathbb{R}^N)$ be the map such that each $f \in V$ assigns the unique solution of the problem

$$\begin{aligned} {}^cD^\alpha y(t) = f(t), \quad \text{a.e. } t \in [0,b], \\ y(0) = y_0, \qquad y'(0) = y_1. \end{aligned} \tag{65}$$

As in Theorem 24, we can prove that $L(V)$ is compact in $C([0,b],\mathbb{R}^N)$ and the operator $N_n = L \circ g_n : K \to K$ is compact; then by Schauder's fixed point there exists $\tilde{y}_n \in K$ such that $\tilde{y}_n \in S_e$ and

$$\begin{aligned} \tilde{y}_n(t) = y_0 + ty_1 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g_n(y_n)(s)\,ds, \\ \text{a.e. } t \in [0,b], \; n \in \mathbb{N}. \end{aligned} \tag{66}$$

Hence

$$\begin{aligned} &\|y(t) - \tilde{y}_n(t)\| \\ &\leq \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}[g_n(\tilde{y}_n)(s) - f(s)]\,ds\right\| \\ &\leq \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}[g_n(\tilde{y}_n)(s) - f_n(\tilde{y}_n)(s)]\,ds\right\| \\ &\quad + \frac{b^\alpha}{\Gamma(\alpha)}\int_0^t \|f_n(\tilde{y}_n)(s) - f(s)\|\,ds \\ &\leq \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}[g_n(\tilde{y}_n)(s) - f_n(\tilde{y}_n)(s)]\,ds\right\| \\ &\quad + \frac{b^\alpha}{\Gamma(\alpha)}\int_0^t (\epsilon_n + d(f(s), f_n(\tilde{y}_n)(s)))\,ds \\ &\leq \frac{1}{\Gamma(\alpha)}\left\|\int_0^t (t-s)^{\alpha-1}[g_n(\tilde{y}_n)(s) - f_n(\tilde{y}_n)(s)]\,ds\right\| \\ &\quad + \frac{b^\alpha}{\Gamma(\alpha)}\int_0^t (\epsilon_n + H_d(F(s,y(s)), F(s,\tilde{y}_n(s))))\,ds \\ &\leq \frac{b^{\alpha+1}}{\Gamma(\alpha+1)}\epsilon_n + \frac{b^{\alpha+1}}{\Gamma(\alpha)}\epsilon_n + \int_0^t p(s)\|y(s) - \tilde{y}_n(s)\|. \end{aligned} \tag{67}$$

Let $\tilde{y}(\cdot)$ be a limit point of the sequence $\tilde{y}_n(\cdot)$. Then, it follows that from the above inequality, one has

$$\|y(t) - \tilde{y}(t)\| \leq \int_0^t p(s)\|y(s) - \tilde{y}(s)\|\,ds, \tag{68}$$

which implies $y(\cdot) = \tilde{y}(\cdot)$. Consequently, $y \in S_c$ is a unique limit point of $\tilde{y}_n(\cdot) \in S_e$. □

Example 26. Let $F : J \times \mathbb{R}^N \to \mathscr{P}_{\text{cpcv}}(\mathbb{R}^N)$ with

$$F(t,y) = \overline{B}(f_1(t,y), f_2(t,y)), \tag{69}$$

where $f_1, f_2 : J \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory functions and bounded.

Then (2) is solvable.

Example 27. If, in addition to the conditions on F of Example 26, f_1 and f_2 are Lipschitz functions, then $\overline{S_e} = S_c$.

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References

[1] K. Diethelm and A. D. Freed, "On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity," in *Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties*, F. Keil, W. Mackens, H. Voss, and J. Werther, Eds., pp. 217–224, Springer, Heidelberg, Germany, 1999.

[2] L. Gaul, P. Klein, and S. Kemple, "Damping description involving fractional operators," *Mechanical Systems and Signal Processing*, vol. 5, no. 2, pp. 81–88, 1991.

[3] W. G. Glockle and T. F. Nonnenmacher, "A fractional calculus approach to self-similar protein dynamics," *Biophysical Journal*, vol. 68, no. 1, pp. 46–53, 1995.

[4] F. Mainardi, "Fractional calculus: some basic problems in continuum and statistical mechanics," in *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi, Eds., pp. 291–348, Springer, Wien, Germany, 1997.

[5] A. B. Malinowska and D. F. M. Torres, "Towards a combined fractional mechanics and quantization," *Fractional Calculus and Applied Analysis*, vol. 15, no. 3, pp. 407–417, 2012.

[6] R. Metzler, W. Schick, H. Kilian, and T. F. Nonnenmacher, "Relaxation in filled polymers: a fractional calculus approach," *The Journal of Chemical Physics*, vol. 103, no. 16, pp. 7180–7186, 1995.

[7] S. P. Näsholm and S. Holm, "On a fractional Zener elastic wave equation," *Fractional Calculus and Applied Analysis*, vol. 16, no. 1, pp. 26–50, 2013.

[8] L. Vázquez, J. J. Trujillo, and M. P. Velasco, "Fractional heat equation and the second law of thermodynamics," *Fractional Calculus and Applied Analysis*, vol. 14, no. 3, pp. 334–342, 2011.

[9] B. J. West and D. West, "Fractional dynamics of allometry," *Fractional Calculus and Applied Analysis*, vol. 15, no. 1, pp. 70–96, 2012.

[10] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, The Netherland, 2006.

[11] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, NY, USA, 1993.

[12] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[13] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives*, Theory and Applications, Gordon and Breach Science, Yverdon, Switzerland, 1993.

[14] A. A. Kilbas and J. J. Trujillo, "Differential equations of fractional order: methods, results and problems. II," *Applicable Analysis*, vol. 81, no. 2, pp. 435–493, 2002.

[15] A. M. Nahušev, "The Sturm-Liouville problem for a second order ordinary differential equation with fractional derivatives in the lower terms," *Doklady Akademii Nauk SSSR*, vol. 234, no. 2, pp. 308–311, 1977.

[16] I. Podlubny, I. Petráš, B. M. Vinagre, P. O'Leary, and L. Dorčák, "Analogue realizations of fractional-order controllers," *Nonlinear Dynamics*, vol. 29, no. 1–4, pp. 281–296, 2002.

[17] C. Yu and G. Gao, "Existence of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 310, no. 1, pp. 26–29, 2005.

[18] V. Lakshmikantham, "Theory of fractional functional differential equations," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 69, no. 10, pp. 3337–3343, 2008.

[19] Y. Chalco-Cano, J. J. Nieto, A. Ouahab, and H. Román-Flores, "Solution set for fractional differential equations with Riemann-Liouville derivative," *Fractional Calculus and Applied Analysis*, vol. 16, no. 3, pp. 682–694, 2013.

[20] J. T. Machado, V. Kiryakova, and F. Mainardi, "Recent history of fractional calculus," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1140–1153, 2011.

[21] S. Abbas and M. Benchohra, "Fractional order Riemann-Liouville integral inclusions with two independent variables and multiple delay," *Opuscula Mathematica*, vol. 33, no. 2, pp. 209–222, 2013.

[22] S. Abbas, M. Benchohra, and J. J. Nieto, "Global uniqueness results for fractional order partial hyperbolic functional differential equations," *Advances in Difference Equations*, vol. 2011, Article ID 379876, 25 pages, 2011.

[23] M. Benchohra, J. Henderson, S. K. Ntouyas, and A. Ouahab, "Existence results for fractional functional differential inclusions with infinite delay and applications to control theory," *Fractional Calculus & Applied Analysis*, vol. 11, no. 1, pp. 35–56, 2008.

[24] J. Henderson and A. Ouahab, "Fractional functional differential inclusions with finite delay," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 70, no. 5, pp. 2091–2105, 2009.

[25] J. Henderson and A. Ouahab, "Impulsive differential inclusions with fractional order," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1191–1226, 2010.

[26] F. Jiao and Y. Zhou, "Existence of solutions for a class of fractional boundary value problems via critical point theory," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1181–1199, 2011.

[27] A. Ouahab, "Some results for fractional boundary value problem of differential inclusions," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 69, no. 11, pp. 3877–3896, 2008.

[28] A. Ouahab, "Filippov's theorem for impulsive differential inclusions with fractional order," *Electronic Journal of Qualitative Theory of Differential Equations*, no. 23, pp. 1–23, 2009.

[29] A. Ouahab, "Fractional semilinear differential inclusions," *Computers & Mathematics with Applications*, vol. 64, no. 10, pp. 3235–3252, 2012.

[30] F. S. de Blasi and G. Pianigiani, "Non-convex valued differential inclusions in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 128, pp. 541–555, 1996.

[31] N. S. Papageorgiou, "On the "bang-bang" principle for nonlinear evolution inclusions," *Aequationes Mathematicae*, vol. 45, no. 2-3, pp. 267–280, 1993.

[32] A. Tolstonogov, *Differential Inclusions in a Banach Space*, Kluwer Academic, Dordrecht, The Netherlands, 2000.

[33] A. A. Tolstonogov, "Extreme continuous selectors of multivalued maps and their applications," *Journal of Differential Equations*, vol. 122, no. 2, pp. 161–180, 1995.

[34] A. A. Tolstonogov, "Extremal selectors of multivalued mappings and the, "bangbang" principle for evolution inclusions," *Doklady Akademii Nauk SSSR*, vol. 317, no. 3, pp. 589–593, 1991 (Russian), translation in *Soviet Mathematics Doklady*, vol. 43, no. 2, 481–485, 1991.

[35] J.-P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin, Germany, 1984.

[36] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, Mass, USA, 1990.

[37] S. Hu and N. S. Papageorgiou, *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer Academic Publishers, London, UK, 1997.

[38] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Academic, Dordrecht, The Netherlands, 1991.

[39] Q. J. Zhu, "On the solution set of differential inclusions in Banach space," *Journal of Differential Equations*, vol. 93, no. 2, pp. 213–237, 1991.

[40] A. Bressan and G. Colombo, "Extensions and selections of maps with decomposable values," *Studia Mathematica*, vol. 90, no. 1, pp. 70–85, 1988.

[41] M. Frigon and A. Granas, "Théorèmes d'existence pour des inclusions différentielles sans convexité," *Comptes Rendus de l'Académie des Sciences. Série I*, vol. 310, no. 12, pp. 819–822, 1990.

[42] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, With the Assistance of W. G. Bade and R. G. Bartle, Pure and Applied Mathematics, Interscience Publishers, New York, NY, USA, 1958.

[43] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon-Nikody'm Property*, vol. 993 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1983.

[44] M. Florenzano and C. Le Van, *Finite Dimensional Convexity and Optimization*, vol. 13 of *Studies in Economic Theory*, Springer, Berlin, Germany, 2001, In Cooperation with Pascal Gourdel.

[45] S. Abbas, M. Benchohra, and G. M. N'Guérékata, *Topics in Fractional Differential Equations*, vol. 27 of *Developments in Mathematics*, Springer, New York, NY, USA, 2012.

[46] M. Caputo, *Elasticita e Dissipazione*, Zanichelli, Bologna, Italy, 1969.

[47] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent, part II," *Geophysical Journal of the Royal Astronomical Society*, vol. 13, pp. 529–529, 1967.

[48] M. Caputo and F. Mainardi, "Linear models of dissipation in anelastic solids," *La Rivista del Nuovo Cimento*, vol. 1, no. 2, pp. 161–198, 1971.

[49] M. Benamara, *Point extremaux, multi-applications et fonctionnelles integrales [Thése de 3éme Cycle]*, Universite de Grenoble, Grenoble, France, 1975.

[50] F. S. de Blasi and G. Pianigiani, "Baire's category and the bang-bang property for evolution differential inclusions of contractive type," *Journal of Mathematical Analysis and Applications*, vol. 367, no. 2, pp. 550–567, 2010.

[51] B. Ahmad and J. J. Nieto, "A study of impulsive fractional differential inclusions with anti-periodic boundary conditions," *Fractional Differential Calculus*, vol. 2, no. 1, pp. 1–15, 2012.

[52] B. Ahmad and J. J. Nieto, "Existence results for higher order fractional differential inclusions with nonlocal boundary conditions," *Nonlinear Studies*, vol. 17, no. 2, pp. 131–138, 2010.

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Research Article

Stability in a Simple Food Chain System with Michaelis-Menten Functional Response and Nonlocal Delays

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This paper is concerned with the asymptotical behavior of solutions to the reaction-diffusion system under homogeneous Neumann boundary condition. By taking food ingestion and species' moving into account, the model is further coupled with Michaelis-Menten type functional response and nonlocal delay. Sufficient conditions are derived for the global stability of the positive steady state and the semitrivial steady state of the proposed problem by using the Lyapunov functional. Our results show that intraspecific competition benefits the coexistence of prey and predator. Furthermore, the introduction of Michaelis-Menten type functional response positively affects the coexistence of prey and predator, and the nonlocal delay is harmless for stabilities of all nonnegative steady states of the system. Numerical simulations are carried out to illustrate the main results.

1. Introduction

The overall behavior of ecological systems continues to be of great interest to both applied mathematicians and ecologists. Two species predator-prey models have been extensively investigated in the literature. But recently more and more attention has been focused on systems with three or more trophic levels. For example, the predator-prey system for three species with Michaelis-Menten type functional response was studied by many authors [1–4]. However, the systems in [1–4] are either with discrete delay or without delay or without diffusion. In view of individuals taking time to move, spatial dispersal was dealt with by introducing diffusion term to corresponding delayed ODE model in previous literatures, namely, adding a Laplacian term to the ODE model. In recent years, it has been recognized that there are modelling difficulties with this approach. The difficulty is that diffusion and time delay are independent of each other, since individuals have not been at the same point at previous times. Britton [5] made a first comprehensive attempt to address this difficulty by introducing a nonlocal delay; that is, the delay term involves a weighted-temporal average over the whole of the infinite domain and the whole of the previous times.

There are many results for reaction-diffusion equations with nonlocal delays [5–18]. The existence and stability of traveling wave fronts were studied in reaction-diffusion equations with nonlocal delay [5–9]. The stability of impulsive cellular neural networks with time varying was discussed in [10] by means of new Poincare integral inequality. The asymptotic behavior of solutions of the reaction-diffusion equations with nonlocal delay was investigated in [11, 12] by using an iterative technique and in [13–15] by the Lyapunov functional. The stability and Hopf bifurcation were discussed in [16] for a diffusive logistic population model with nonlocal delay effect.

Motivated by the work above, we are concerned with the following food chain model with Michaelis-Menten type functional response:

$$\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 = u_1 \left(a_1 - a_{11} u_1 - \frac{a_{12} u_2}{m_1 + u_1} \right),$$

$$\begin{aligned}
&\frac{\partial u_2}{\partial t} - d_2 \Delta u_2 \\
&\quad = u_2 \left(-a_2 + a_{21} \int_\Omega \int_{-\infty}^{t} \frac{u_1(s,y)}{m_1 + u_1(s,y)} \right. \\
&\qquad \times K_1(x,y,t-s)\, ds\, dy \\
&\qquad \left. - a_{22} u_2 - \frac{a_{23} u_3}{m_2 + u_2} \right), \\
&\frac{\partial u_3}{\partial t} - d_3 \Delta u_3 \\
&\quad = u_3 \left(-a_3 + a_{32} \int_\Omega \int_{-\infty}^{t} \frac{u_2(s,y)}{m_2 + u_2(s,y)} \right. \\
&\qquad \times K_2(x,y,t-s)\, ds\, dy \\
&\qquad \left. - a_{33} u_3 \right),
\end{aligned} \tag{1}$$

for $t > 0$, $x \in \Omega$ with homogeneous Neumann boundary conditions

$$\frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial u_3}{\partial \nu} = 0, \quad t > 0, \ x \in \partial\Omega, \tag{2}$$

and initial conditions

$$\begin{aligned}
&u_i(t,x) = \phi_i(t,x) \geq 0 \quad (i = 1,2), \ (t,x) \in (-\infty, 0] \times \Omega, \\
&u_3(0,x) = \phi_3(x) \geq 0, \quad x \in \Omega,
\end{aligned} \tag{3}$$

where ϕ_i is bounded, Hölder continuous function and satisfies $\partial\phi_i/\partial\nu = 0$ $(i = 1,2,3)$ on $(-\infty, 0] \times \partial\Omega$. Here, Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. $u_i(t,x)$ represents the density of the ith species (prey, predator, and top predator resp.) at time t and location x and thus only nonnegative $u_i(t,x)$ is of interest. The parameter a_1 is the intrinsic growth rate of the prey, and a_2 and a_3 are the death rates of the predator and top-predator. a_{ii} is the intraspecific competitive rate of the ith species. a_{12} is the maximum predation rate. a_{23} and a_{32} are the efficiencies of food utilization of the predator and top predator, respectively. We assume the predator and top predator show the Michaelis-Menten (or Holling type II) functional response with $u_1/(m_1 + u_1)$ and $u_2/(m_2 + u_2)$, respectively, where m_1 and m_2 are half-saturation constants. For a through biological background of similar models, see [18, 19]. As our most knowledge, the tritrophic food chain model has been found to have many interesting biological properties, such as the coexistence and the Hopf bifurcation. However, the effect of nonlocal time delays on the coexistence has not been reported. Our paper mainly concerns this perspective.

Additionally, $\int_\Omega \int_{-\infty}^{t} (u_i(s,y)/(m_i + u_i\ (s,y)))K_i(x,y,t-s)ds\,dy$ $(i = 1,2)$ represents the nonlocal delay due to the ingestion of predator; that is, mature adult predator can only contribute to the production of predator biomass. The boundary condition in (2) implies that there is no migration across the boundary of Ω.

The main purpose of this paper is to study the global asymptotic behavior of the solution of system (1)–(3). The preliminary results are presented in Section 2. Section 3 contains sufficient conditions for the global asymptotic behaviors of the equilibria of system (1)–(3) by means of the Lyapunov functional. Numerical simulations are carried out to show the feasibility of the conditions in Theorems 8–10 in Section 4. Finally, a brief discussion is given to conclude this work.

2. Preliminary Results

In this section, we present several preliminary results that will be employed in the sequel.

Lemma 1 (see [3]). *Let a and b be positive constants. Assume that ϕ, $\varphi \in C^1(a, +\infty)$, $\varphi(t) \geq 0$, and ϕ is bounded from below. If $\phi'(t) \leq -b\varphi(t)$ and $\varphi'(t) \leq K$ in $[a, +\infty)$ for some positive constant K, then $\lim_{t \to +\infty} \varphi(t) = 0$.*

The following lemma is the Positivity Lemma in [20].

Lemma 2. *Let $u_i \in C([0,T] \times \overline{\Omega}) \bigcap C^{1,2}((0,T] \times \Omega)$ $(i = 1,2,3)$ and satisfy*

$$\begin{aligned}
&u_{it} - d_i \Delta u_i \geq \sum_{j=1}^{3} b_{ij}(t,x) u_j(t,x) \\
&\qquad + \sum_{j=1}^{3} c_{ij} u_j(t - \tau_j, x) \quad \text{in } (0,T] \times \Omega, \\
&\frac{\partial u_i(t,x)}{\partial \nu} \geq 0 \quad \text{on } (0,T] \times \partial\Omega, \\
&u_i(t,x) \geq 0 \quad \text{in } [-\tau_i, 0] \times \Omega,
\end{aligned} \tag{4}$$

where $b_{ij}, c_{ij} \in C([0,T] \times \overline{\Omega})$, $u_{it} = (u_i)_t$. If $b_{ij} \geq 0$ for $j \neq i$ and $c_{ij} \geq 0$ for all $i, j = 1,2,3$. Then $u_i(t,x) \geq 0$ on $[0,T] \times \overline{\Omega}$. Moreover, if the initial function is nontrivial, then $u_i > 0$ in $(0,T] \times \overline{\Omega}$.

Lemma 3 (see [20]). *Let $\widehat{\mathbf{c}}$ and $\widetilde{\mathbf{c}}$ be a pair of constant vector satisfying $\widetilde{\mathbf{c}} \geq \widehat{\mathbf{c}}$ and let the reaction functions satisfy local Lipschitz condition with $\Lambda = \langle \widehat{\mathbf{c}}, \widetilde{\mathbf{c}} \rangle$. Then system (1)–(3) admits a unique global solution $\mathbf{u}(t,x)$ such that*

$$\widehat{\mathbf{c}} \leq \mathbf{u}(t,x) \leq \widetilde{\mathbf{c}}, \quad \forall t > 0, \ x \in \overline{\Omega}, \tag{5}$$

whenever $\widehat{\mathbf{c}} \leq \phi(t,x) \leq \widetilde{\mathbf{c}}$, $(t,x) \in (-\infty, 0] \times \Omega$.

The following result was obtained by the method of upper and lower solutions and the associated iterations in [21].

Lemma 4 (see [21]). *Let $u(t,x) \in C([0,\infty)\times\overline{\Omega}) \cap C^{2,1}((0,\infty)\times\Omega)$ be the nontrivial positive solution of the system*

$$\begin{aligned}
&u_t - d\Delta u = Bu(t-\tau,x) \pm A_1 u(t,x) \\
&\qquad - A_2 u^2(t,x), \quad (t,x) \in (0,\infty)\times\Omega, \\
&\frac{\partial u}{\partial \nu} = 0, \quad (t,x) \in (0,\infty)\times\partial\Omega, \\
&u(t,x) = \phi(t,x) \ge 0, \quad (t,x) \in (-\tau,0)\times\Omega,
\end{aligned} \tag{6}$$

where $A_1 \ge 0$, $B, A_2, \tau > 0$. The following results hold:

(1) *if $B \pm A_1 > 0$, then $u \to (B\pm A_1)/A_2$ uniformly on $\overline{\Omega}$ as $t \to \infty$;*

(2) *if $B \pm A_1 \le 0$, then $u \to 0$ uniformly on $\overline{\Omega}$ as $t \to \infty$.*

Throughout this paper, we assume that

$$\begin{aligned}
&K_i(x,y,t) = G_i(x,y,t)\,k_i(t), \quad x,y \in \Omega,\ k_i(t) \ge 0; \\
&\int_\Omega G_i(x,y,t)\,dx = \int_\Omega G_i(x,y,t)\,dy = 1, \quad t \ge 0; \\
&\int_0^{+\infty} k_i(t)\,dt = 1, \quad tk_i(t) \in L^1((0,+\infty);R),
\end{aligned} \tag{7}$$

where $G_i(x,y,t)$ is a nonnegative function, which is continuous in $(x,y) \in \overline{\Omega}\times\overline{\Omega}$ for each $t \in [0,+\infty)$ and measurable in $t \in [0,+\infty)$ for each pair $(x,y) \in \overline{\Omega}\times\overline{\Omega}$.

Now we prove the following propositions which will be used in the sequel.

Proposition 5. *For any nonnegative initial function, the corresponding solution of system* (1)–(3) *is nonnegative.*

Proof. Suppose that (u_1,u_2,u_3) is a solution and satisfies $u_i \in C([0,T)\times\overline{\Omega}) \cap C^{1,2}((0,T)\times\Omega)$ with $T \le +\infty$. Choose $0 < \tau < T$. Then

$$\begin{aligned}
&u_{1t} - d_1\Delta u_1 = A_1 u_1, \quad 0<t<\tau,\ x \in \Omega, \\
&u_{2t} - d_2\Delta u_2 = A_2 u_2, \quad 0<t<\tau,\ x \in \Omega, \\
&u_{3t} - d_3\Delta u_3 = A_3 u_3, \quad 0<t<\tau,\ x \in \Omega, \\
&\frac{\partial u_i}{\partial \nu} = 0, \quad 0<t<\tau,\ x \in \partial\Omega, \\
&u_i(t,x) \ge 0 \quad (i=1,2,3),\ t \le 0,\ x \in \overline{\Omega},
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
A_1 &= a_1 - a_{11}u_1 - \frac{a_{12}u_2}{m_1+u_1}, \\
A_2 &= -a_2 + a_{21}\int_\Omega\int_{-\infty}^t \frac{u_1(s,y)}{m_1+u_1(s,y)} \\
&\qquad \times K_1(x,y,t-s)\,ds\,dy \\
&\qquad - a_{22}u_2 - \frac{a_{23}u_3}{m_2+u_3}, \\
A_3 &= -a_3 + a_{32}\int_\Omega\int_{-\infty}^t \frac{u_2(s,y)}{m_2+u_2(s,y)} \\
&\qquad \times K_2(x,y,t-s)\,ds\,dy - a_{33}u_3.
\end{aligned} \tag{9}$$

It follows from Lemma 2 that $u_i \ge 0, (t,x) \in [0,\tau]\times\overline{\Omega}$. Due to the arbitrariness of τ, we have $u_i \ge 0$ for $(t,x) \in [0,T)\times\overline{\Omega}$. □

Proposition 6. *System* (1)–(3) *with initial functions ϕ_i admits a unique global solution (u_1,u_2,u_3) satisfying $0 \le u_i(t,x) \le M_i$ for $i = 1,2,3$, where M_i is defined as*

$$\begin{aligned}
M_1 &= \max\left\{\frac{a_1}{a_{11}}, \|\phi_1(t,x)\|_{L^\infty((-\infty,0]\times\overline{\Omega})}\right\}, \\
M_2 &= \max\left\{\frac{(a_{21}-a_2)M_1 - m_1a_2}{a_{22}(m_1+M_1)}, \|\phi_2(t,x)\|_{L^\infty((-\infty,0]\times\overline{\Omega})}\right\}, \\
M_3 &= \max\left\{\frac{(a_{32}-a_3)M_2 - m_2a_3}{a_{22}(m_2+M_2)}, \|\phi_3(x)\|_{L^\infty(\overline{\Omega})}\right\}.
\end{aligned} \tag{10}$$

Proof. It follows from standard PDE theory that there exists a $T > 0$ such that problem (1)–(3) admits a unique solution in $[0,T)\times\overline{\Omega}$. From Lemma 2 we know that $u_i(t,x) \ge 0$ for $(t,x) \in [0,T)\times\overline{\Omega}$. Considering the first equation in (1), we have

$$\frac{\partial u_1}{\partial t} - d_1\Delta u_1 \le u_1(a_1 - a_{11}u_1). \tag{11}$$

Using the maximum principle gives that $u_1 \le M_1$. In a similar way, we have $u_i \le M_i$ for $i = 2,3$. It is easy to see that $(0,0,0)$ and (M_1,M_2,M_3) are a pair of coupled upper and lower solutions to system (1)–(3) from the direct computation. In virtue of Lemma 3, system (1)–(3) admits a unique global solution (u_1,u_2,u_3) satisfying $0 \le u_i(t,x) \le M_i$ for $i = 1,2,3$. In addition, if $\phi_1(t,x), \phi_2(t,x), \phi_3(x)$, $(t,x) \in (-\infty,0]\times\Omega$ is nontrivial, it follows from the strong maximum principle that $u_i(t,x) > 0$ $(i = 1,2,3)$ for all $t > 0, x \in \overline{\Omega}$. □

3. Global Stability

In this section, we study the asymptotic behavior of the equilibrium of system (1)–(3). In the beginning, we show the existence and uniqueness of positive steady state.

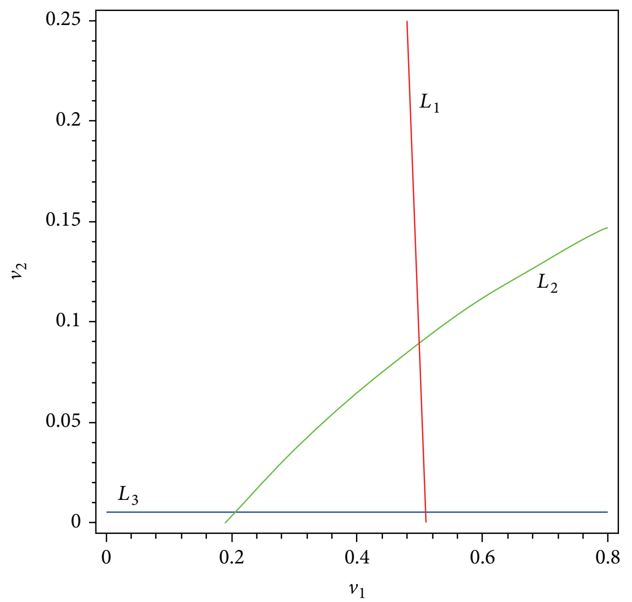

Figure 1: Graphs of the equations in (12). L_3 is the boundary condition ($a_3 = a_{32}v_2/(m_2+v_2)$) and L_2 corresponds to $-a_2 + a_{21}(v_1/(m_1+v_1)) - a_{22}v_2 = 0$, while L_1 corresponds to $a_1 - a_{11}v_1 - a_{12}v_2/(m_1+v_1) = 0$. Intersection point between L_1 and L_2 gives the positive solution (v_1^*, v_2^*) to (12). Parameter values are listed in the example in Section 4.

Let us consider the following equations:

$$\begin{gathered} a_1 - a_{11}v_1 - \frac{a_{12}v_2}{m_1+v_1} = 0, \\ -a_2 + \frac{a_{21}v_1}{m_1+v_1} - a_{22}v_2 = 0. \end{gathered} \tag{12}$$

A direct computation shows that the above equations have only one positive solution (v_1^*, v_2^*) if and only if

$$H_1 : a_1(a_{21}-a_2) > m_1 a_2 a_{11}, \tag{13}$$

is satisfied. Taking H_1 into account, we consider the equation $a_3 = a_{32}v_2^*/(m_2+v_2^*)$, which corresponds to L_3 in Figure 1.

It is clear that if $a_3 < a_{32}v_2^*/(m_2+v_2^*)$, the intersection point (v_1^*, v_2^*) always lies in L_3. Let

$$H_2 : a_3 m_2 < (a_{32}-a_3)v_2^*. \tag{14}$$

Suppose that H_1 and H_2 hold. We take u_3 as a parameter and consider the following system:

$$\begin{gathered} a_1 - a_{11}u_1 - \frac{a_{12}u_2}{m_1+u_1} = 0, \\ -a_2 + \frac{a_{21}u_1}{m_1+u_1} - a_{22}u_2 - \frac{a_{23}u_3}{m_2+u_2} = 0, \\ -a_3 + \frac{a_{32}u_2}{m_2+u_2} - a_{33}u_3 = 0. \end{gathered} \tag{15}$$

When the parameter u_3 is sufficiently small, the first two equations in (15) can be approximated by (12). Moreover, by continuously increasing the value of u_3, L_3 goes up, and meanwhile the intersection point between L_1 and L_2 also goes up. However, L_3 goes up faster than L_2, while L_1 keeps still. In other words, there exists a critical value u_3 such that the intersection point lies in L_3, which implies that there is a unique positive solution $E^*(u_1^*, u_2^*, u_3^*)$ to (15), or equivalently (1)–(3).

Lemma 7. *Assume that H_2 and G_2 hold, and then the positive steady state E^* of system* (1)–(3) *is locally asymptotically stable.*

Proof. We get the local stability of E^* by performing a linearization and analyzing the corresponding characteristic equation. Similarly as in [22], let $0 < \mu_1 < \mu_2 < \mu_3 < \dots$ be the eigenvalues of $-\Delta$ on Ω with the homogeneous Neumann boundary condition. Let $E(\mu_i)$ be the eigenspace corresponding to μ_i in $C^1(\overline{\Omega})$, for $i = 1, 2, 3, \dots$. Let

$$\mathbf{X} = \left\{ \mathbf{u} = (u_1, u_2, u_3) \in \left[C^1\left(\overline{\Omega}\right)\right]^3 \partial_\eta \mathbf{u} = 0, x \in \partial\Omega \right\}, \tag{16}$$

$\{\varphi_{ij}, j = 1, \dots, \dim E(\mu_i)\}$ be an orthonormal basis of $E(\mu_i)$, and $\mathbf{X}_{ij} = \{c \cdot \varphi_{ij} \mid c \in R^3\}$. Then

$$\mathbf{X}_i = \bigoplus_{j=1}^{\dim E(\mu_i)} \mathbf{X}_{ij}, \quad \mathbf{X} = \bigoplus_{i=1}^{\infty} \mathbf{X}_i. \tag{17}$$

Let $D = \operatorname{diag}(D_1, D_2, D_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, $v_i = u_i - u_i^*$, $(i = 1, 2, 3)$. Then the linearization of (1) is $\mathbf{v}_t = L\mathbf{v} = D\Delta\mathbf{v} + \mathbf{F}_\mathbf{v}(\mathbf{E}^*)\mathbf{v}$, $\mathbf{F}_\mathbf{v}(\mathbf{E}^*)\mathbf{v} = \{c_{ij}\}$, where $c_{ii} = b_{ii}v_i, i = 1, 2, 3$, $c_{12} = b_{12}v_2$, $c_{23} = b_{23}v_3$, $c_{21} = b_{21}\int_\Omega \int_{-\infty}^t v_1(s, y)K_1(x, y, t-s)\,ds\,dy$, and $c_{32} = b_{32}\int_\Omega \int_{-\infty}^t v_2(s, y)K_2(x, y, t-s)\,ds\,dy$. The coefficients b_{ij} are defined as follows:

$$\begin{gathered} b_{11} = -a_{11}u_1^* + \frac{a_{12}u_1^*u_2^*}{(m_1+u_1^*)^2}, \\ b_{12} = -\frac{a_{12}u_1^*}{m_1+u_1^*}, \qquad b_{13} = 0, \\ b_{21} = \frac{a_{21}m_1u_2^*}{(m_1+u_1^*)^2}, \\ b_{22} = -a_{22}u_2^* + \frac{a_{23}u_2^*u_3^*}{(m_2+u_2^*)^2}, \qquad b_{23} = -\frac{a_{23}u_2^*}{m_2+u_2^*}, \\ b_{31} = 0, \qquad b_{32} = \frac{a_{32}m_2u_3^*}{(m_2+u_2^*)^2}, \qquad b_{33} = -a_{33}u_3^*. \end{gathered} \tag{18}$$

Since $\mathbf{X}_i$ is invariant under the operator L for each $i \geq 1$, then the operator L on $\mathbf{X}_i$ is $\mathbf{v}_t = L\mathbf{v} = D\mu_i\mathbf{v} + \mathbf{F}_\mathbf{v}(\mathbf{E}^*)\mathbf{v}$. Let $v_i = c_ie^{\lambda t}(i = 1, 2, 3)$ and we can get the characteristic equation

$\varphi_i(\lambda) = \lambda^3 + A_i\lambda^2 + B_i\lambda + C_i = 0$, where A_i, B_i, and C_i are defined as follows:

$$\begin{aligned}
A_i &= (d_1 + d_2 + d_3)\,\mu_i - b_{11} - b_{22} - b_{33},\\
B_i &= (d_1d_2 + d_2d_3 + d_3d_1)\,\mu_i^2\\
&\quad + [d_3(-b_{11} - b_{22}) + d_1(-b_{33} - b_{22})\\
&\qquad + d_2(-b_{11} - b_{33})]\,\mu_i\\
&\quad + b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33}\\
&\quad - b_{12}b_{21}\int_0^{\infty} k_1(s)\,e^{-\lambda s}ds - b_{23}b_{32}\int_0^{\infty} k_2(s)\,e^{-\lambda s}ds,\\
C_i &= d_1d_2d_3\mu_i^3 + [-b_{33}d_1d_2 - b_{11}d_2d_3 - b_{22}d_1d_3]\,\mu_i^2\\
&\quad + \Big[b_{11}b_{22}d_3 + b_{11}b_{33}d_2 + b_{22}b_{33}d_1 + b_{22}b_{33}d_1\\
&\qquad - d_3b_{12}b_{21}\int_0^{\infty} k_1(s)\,e^{-\lambda s}ds\\
&\qquad - d_1b_{23}b_{32}\int_0^{\infty} k_2(s)\,e^{-\lambda s}ds\Big]\,\mu_i\\
&\quad + \Big[-b_{11}b_{22}b_{33} + b_{12}b_{21}b_{33}\int_0^{\infty} k_1(s)\,e^{-\lambda s}ds\\
&\qquad + b_{11}b_{23}b_{32}\int_0^{\infty} k_2(s)\,e^{-\lambda s}ds\Big].
\end{aligned} \tag{19}$$

It is easy to see that b_{12}, b_{23}, $b_{33} < 0$ and b_{21}, $b_{32} > 0$. It follows from assumption G_1 and G_2 that $a_{11} < 0$, $a_{22} < 0$. So $A_i > 0$, $B_i > 0$, $C_i > 0$ and $A_iB_i - C_i > 0$ for $i \geq 1$ from the direct calculation. According to the Routh-Hurwitz criterion, the three roots $\lambda_{i,1}$, $\lambda_{i,2}$, $\lambda_{i,3}$ of $\varphi_i(\lambda) = 0$ all have negative real parts.

By continuity of the roots with respect to μ_i and Routh-Hurwitz criterion, we can conclude that there exists a positive constant ε such that

$$\operatorname{Re}\{\lambda_{i,1}\}, \operatorname{Re}\{\lambda_{i,2}\}, \operatorname{Re}\{\lambda_{i,3}\} \leq -\varepsilon, \quad i \geq 1. \tag{20}$$

Consequently, the spectrum of L, consisting only of eigenvalues, lies in $\{\operatorname{Re}\lambda \leq -\varepsilon\}$. It is easy to see that $\mathbf{E}^*$ is locally asymptotically stable and follows from Theorem 5.1.1 of [23]. □

Theorem 8. *Assume that*

$$H_3: \frac{a_1(a_{21} - a_2)}{m_1a_2} > a_{11} > \frac{a_{12}u_2^*}{m_1^2}, \tag{21}$$

H_2 and G_2 hold, and the positive steady state E^* of system (1)–(3) *with nontrivial initial function is globally asymptotically stable.*

Proof. It is easy to see that the equations in (1) can be rewritten as

$$\begin{aligned}
\frac{\partial u_1}{\partial t} - d_1\Delta u_1 &= u_1\Big[-a_{11}(u_1 - u_1^*)\\
&\quad - \frac{a_{12}(u_2 - u_2^*)}{m_1 + u_1} + \frac{a_{12}u_2^*(u_1 - u_1^*)}{(m_1 + u_1)(m_1 + u_1^*)}\Big],\\
\frac{\partial u_2}{\partial t} - d_2\Delta u_2 &= u_2\Big[\int_{\Omega}\int_{-\infty}^{t} K_1(x, y, t - s)\\
&\quad \times \frac{a_{21}m_1(u_1(s, y) - u_1^*)}{(m_1 + u_1(s, y))(m_1 + u_1^*)}ds\,dy\\
&\quad - a_{22}(u_2 - u_2^*) - \frac{a_{23}(u_3 - u_3^*)}{m_2 + u_2}\\
&\quad + \frac{a_{23}u_3^*(u_2 - u_2^*)}{(m_2 + u_2)(m_2 + u_2^*)}\Big],\\
\frac{\partial u_3}{\partial t} - d_3\Delta u_3 &= u_3\Big[\int_{\Omega}\int_{-\infty}^{t} K_2(x, y, t - s)\\
&\quad \times \frac{a_{32}m_2(u_2(s, y) - u_2^*)}{(m_2 + u_2(s, y))(m_2 + u_2^*)}ds\,dy\\
&\quad - a_{33}(u_3 - u_3^*)\Big].
\end{aligned} \tag{22}$$

Define

$$\begin{aligned}
V(t) &= \alpha\int_{\Omega}\left(u_1 - u_1^* - u_1^*\ln\frac{u_1}{u_1^*}\right)dx\\
&\quad + \int_{\Omega}\left(u_2 - u_2^* - u_2^*\ln\frac{u_2}{u_2^*}\right)dx\\
&\quad + \beta\int_{\Omega}\left(u_2 - u_3^* - u_3^*\ln\frac{u_3}{u_3^*}\right)dx,
\end{aligned} \tag{23}$$

where α and β are positive constants to be determined. Calculating the derivatives $V(t)$ along the positive solution to the system (1)–(3) yields

$$V'(t) = \Phi_1(t) + \Phi_2(t), \tag{24}$$

where

$$\begin{aligned}
\Phi_1(t) &= -\int_{\Omega}\left(\frac{\alpha d_1u_1^*}{u_1^2}|\nabla u_1|^2 + \frac{d_2u_2^*}{u_2^2}|\nabla u_2|^2\right.\\
&\quad \left. + \frac{\beta d_3u_3^*}{u_3^2}|\nabla u_3|^2\right)dx,
\end{aligned}$$

$$\begin{aligned}
\Phi_2(t) = &-\int_\Omega \alpha\left[a_{11}-\frac{a_{12}u_2^*}{(m_1+u_1)(m_1+u_1^*)}\right](u_1-u_1^*)^2dx \\
&-\int_\Omega\left[a_{22}-\frac{a_{23}u_3^*}{(m_2+u_2)(m_2+u_2^*)}\right](u_2-u_2^*)^2dx \\
&-\int_\Omega a_{33}\beta(u_3-u_3^*)^2dx \\
&-\int_\Omega \frac{a_{23}(u_3-u_3^*)(u_2-u_2^*)}{m_2+u_2}dx \\
&-\int_\Omega \frac{\alpha a_{12}(u_2-u_2^*)(u_1-u_1^*)}{m_1+u_1}dx \\
&+\iint_\Omega\int_{-\infty}^t K_1(x,y,t-s) \\
&\times\frac{a_{21}m_1(u_1(s,y)-u_1^*)(u_2(t,x)-u_2^*)}{(m_1+u_1(s,y))(m_1+u_1^*)}ds\,dy\,dx \\
&+\beta\iint_\Omega\int_{-\infty}^t K_2(x,y,t-s) \\
&\times\frac{a_{32}m_2(u_2(s,y)-u_2^*)(u_3(t,x)-u_3^*)}{(m_2+u_2(s,y))(m_2+u_2^*)}ds\,dy\,dx.
\end{aligned} \tag{25}$$

Applying the inequality $ab \le \epsilon a^2 + (1/4\epsilon)b^2$, we derive from (25) that

$$\begin{aligned}
\Phi_2(t) \le &-\int_\Omega \alpha\left[a_{11}-\frac{a_{12}u_2^*}{(m_1+u_1)(m_1+u_1^*)}\right](u_1-u_1^*)^2dx \\
&-\int_\Omega a_{33}\beta(u_3-u_3^*)^2dx \\
&-\int_\Omega\left[a_{22}-\frac{a_{23}u_3^*}{(m_2+u_2)(m_2+u_2^*)}\right](u_2-u_2^*)^2dx \\
&+\int_\Omega \frac{\alpha a_{12}}{m_1+u_1}\left[\epsilon_1(u_1-u_1^*)^2+\frac{1}{4\epsilon_1}(u_2-u_2^*)^2\right]dx \\
&+\int_\Omega \frac{a_{23}}{m_2+u_2}\left[\frac{1}{4\epsilon_2}(u_2-u_2^*)^2+\epsilon_2(u_3-u_3^*)^2\right]dx \\
&+\iint_\Omega\int_{-\infty}^t K_1(x,y,t-s) \\
&\times\frac{a_{21}m_1}{(m_1+u_1(s,y))(m_1+u_1^*)} \\
&\times\left[\epsilon_3(u_1(s,y)-u_1^*)^2\right. \\
&\left.+\frac{1}{4\epsilon_3}(u_2(t,x)-u_2^*)^2\right]ds\,dy\,dx \\
&+\iint_\Omega\int_{-\infty}^t K_2(x,y,t-s) \\
&\times\frac{a_{32}\beta m_2}{(m_2+u_2(s,y))(m_2+u_2^*)} \\
&\times\left[\frac{1}{4\epsilon_4}(u_2(s,y)-u_2^*)^2\right. \\
&\left.+\epsilon_4(u_3(t,x)-u_3^*)^2\right]ds\,dy\,dx.
\end{aligned} \tag{26}$$

According to the property of the Kernel functions $K_i(x,y,t)$, $(i=1,2)$, we know that

$$\begin{aligned}
\Phi_2(t) \le &-\int_\Omega\left[\alpha a_{11}-\frac{\alpha a_{12}u_2^*}{m_1^2}-\frac{\alpha a_{12}\epsilon_1}{m_1}\right](u_1-u_1^*)^2dx \\
&-\int_\Omega\left[a_{22}-\frac{a_{23}u_3^*}{m_2^2}-\frac{\alpha a_{12}}{4m_1\epsilon_1}-\frac{a_{23}}{4m_2\epsilon_2}\right. \\
&\left.-\frac{a_{21}}{4m_1\epsilon_3}\right](u_2-u_2^*)^2dx \\
&-\int_\Omega\left[a_{33}\beta-\frac{a_{23}\epsilon_2}{m_2}-\frac{a_{32}\beta\epsilon_4}{m_2}\right](u_3-u_3^*)^2dx \\
&+\frac{a_{21}\epsilon_3}{m_1}\iint_\Omega\int_{-\infty}^t K_1(x,y,t-s) \\
&\times(u_1(s,y)-u_1^*)^2ds\,dy\,dx \\
&+\frac{a_{32}\beta}{4m_2\epsilon_4}\iint_\Omega\int_{-\infty}^t K_2(x,y,t-s) \\
&\times(u_2(s,y)-u_2^*)^2ds\,dy\,dx.
\end{aligned} \tag{27}$$

Define a new Lyapunov functional as follows:

$$\begin{aligned}
E(t) = &V(t) \\
&+\frac{a_{21}\epsilon_3}{m_1}\iint_\Omega\int_0^{+\infty}\int_{t-r}^t K_1(x,y,r) \\
&\times(u_1(s,y)-u_1^*)^2ds\,dr\,dy\,dx \\
&+\frac{a_{32}\beta}{4m_2\epsilon_4}\iint_\Omega\int_0^{+\infty}\int_{t-r}^t K_2(x,y,r) \\
&\times(u_2(s,y)-u_2^*)^2ds\,dr\,dy\,dx.
\end{aligned} \tag{28}$$

It is derived from (27) and (28) that

$$\begin{aligned} E'(t) \le &-\int_\Omega \left(\frac{\alpha d_1 u_1^*}{u_1^2}|\nabla u_1|^2 + \frac{d_2 u_2^*}{u_2^2}|\nabla u_2|^2 + \frac{\beta d_3 u_3^*}{u_3^2}|\nabla u_3|^2 \right) dx \\ &- \left[\alpha a_{11} - \frac{\alpha a_{12} u_2^*}{m_1^2} - \frac{\alpha a_{12}\epsilon_1}{m_1} - \frac{a_{21}\epsilon_3}{m_1} \right] \int_\Omega (u_1 - u_1^*)^2 dx \\ &- \left[a_{22} - \frac{a_{23} u_3^*}{m_2^2} - \frac{\alpha a_{12}}{4m_1\epsilon_1} - \frac{a_{23}}{4m_2\epsilon_2} - \frac{a_{21}}{4m_1\epsilon_3} - \frac{a_{32}\beta}{4m_2\epsilon_4} \right] \int_\Omega (u_2 - u_2^*)^2 dx \\ &- \left[a_{33}\beta - \frac{a_{23}\epsilon_2}{m_2} - \frac{\beta a_{32}\epsilon_4}{m_2} \right] \int_\Omega (u_3 - u_3^*)^2 dx. \end{aligned} \quad (29)$$

Since (u_1, u_2, u_3) is the unique positive solution of system (1). Using **Proposition 5**, there exists a constant C which does not depend on $x \in \overline{\Omega}$ or $t \ge 0$ such that $\|u_i(\cdot, t)\|_\infty \le C$ $(i = 1, 2, 3)$ for $t \ge 0$. By Theorem A_2 in [24], we have

$$\|u_i(\cdot, t)\|_{C^{2,\alpha}(\overline{\Omega})} \le C, \quad \forall t \ge 1. \quad (30)$$

Assume that

$$\begin{aligned} &G_1 : a_{11} m_1^2 > a_{12} u_2^*, \\ &G_2 : a_{22} m_2^2 > a_{23} u_3^* + \frac{2a_{23}a_{32}}{a_{33}} + \frac{2a_{12}a_{21}m_2^2}{a_{11}m_1^2 - a_{12}u_2^*}, \end{aligned} \quad (31)$$

and denote

$$\begin{aligned} l_1 &= \alpha a_{11} - \frac{\alpha a_{12} u_2^*}{m_1^2} - \frac{\alpha a_{12}\epsilon_1}{m_1} - \frac{a_{21}\epsilon_3}{m_1}, \\ l_2 &= a_{22} - \frac{a_{23} u_3^*}{m_2^2} - \frac{\alpha a_{12}}{4m_1\epsilon_1} - \frac{a_{23}}{4m_2\epsilon_2} - \frac{a_{21}}{4m_1\epsilon_3} - \frac{a_{32}\beta}{4m_2\epsilon_4}, \\ l_3 &= a_{33}\beta - \frac{a_{23}\epsilon_2}{m_2} - \frac{\beta a_{32}\epsilon_4}{m_2}. \end{aligned} \quad (32)$$

Then (29) is transformed into

$$\begin{aligned} E'(t) \le &-\int_\Omega \left(\frac{\alpha d_1 u_1^*}{u_1^2}|\nabla u_1|^2 + \frac{d_2 u_2^*}{u_2^2}|\nabla u_2|^2 + \frac{\beta d_3 u_3^*}{u_3^2}|\nabla u_3|^2 \right) dx \\ &- \sum_{i=1}^3 l_i \int_\Omega (u_i - u_i^*)^2 dx. \end{aligned} \quad (33)$$

If we choose

$$\begin{aligned} &\alpha = \frac{a_{21}}{a_{12}}, \qquad \beta = \frac{a_{23}}{a_{32}}, \qquad \epsilon_1 = \epsilon_3 = \frac{a_{11}m_1^2 - a_{12}u_2^*}{4m_1 a_{12}}, \\ &\epsilon_2 = \epsilon_4 = \frac{m_2 a_{33}}{4a_{32}}, \end{aligned} \quad (34)$$

then $l_i > 0$ $(i = 1, 2, 3)$. Therefore, we have

$$E'(t) \le -\sum_{i=1}^3 l_i \int_\Omega (u_i - u_i^*)^2 dx. \quad (35)$$

From **Proposition 6** we can see that the solution of system (1) and (3) is bounded, and so are the derivatives of $(u_i - u_i^*)$ $(i = 1, 2, 3)$ by the equations in (1). Applying **Lemma 1**, we obtain that

$$\lim_{t\to\infty} \int_\Omega (u_i - u_i^*)\, dx = 0, \quad (i = 1, 2, 3). \quad (36)$$

Recomputing $E'(t)$ gives

$$\begin{aligned} E'(t) &\le -\int_\Omega \left(\frac{\alpha d_1 u_1^*}{u_1^2}|\nabla u_1|^2 + \frac{d_2 u_2^*}{u_2^2}|\nabla u_2|^2 + \frac{\beta d_3 u_3^*}{u_3^2}|\nabla u_3|^2 \right) dx \\ &\le -c \int_\Omega \left(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2 \right) dx = -g(t), \end{aligned} \quad (37)$$

where $c = \min\{\alpha d_1 u_1^*/M_1^2, d_2 u_2^*/M_2^2, \beta d_3 u_3^*/M_3^2\}$. Using (30) and (1), we obtain that the derivative of $g(t)$ is bounded in $[1, +\infty)$. From **Lemma 1**, we conclude that $g(t) \to 0$ as $t \to \infty$. Therefore

$$\lim_{t\to\infty} \int_\Omega \left(|\nabla u_1|^2 + |\nabla u_2|^2 + |\nabla u_3|^2 \right) dx = 0. \quad (38)$$

Applying the Poincaré inequality

$$\int_\Omega \lambda |u - \overline{u}|^2 dx \le \int_\Omega |\nabla u|^2 dx, \quad (39)$$

leads to

$$\lim_{t\to\infty} \int_\Omega (u_i - \overline{u}_i)^2 dx = 0, \quad (i = 1, 2, 3), \quad (40)$$

where $\overline{u}_i = (1/|\Omega|) \int_\Omega u_i dx$ and λ is the smallest positive eigenvalue of $-\Delta$ with the homogeneous Neumann condition. Therefore,

$$\begin{aligned} |\Omega| \left(\overline{u}_1(t) - u_1^*\right)^2 &= \int_\Omega \left(\overline{u}_1(t) - u_1^*\right)^2 dx \\ &= \int_\Omega \left(\overline{u}_1(t) - u_1(t,x) + u_1(t,x) - u_1^*\right)^2 dx \\ &\le 2\int_\Omega \left(\overline{u}_1(t) - u_1(t,x)\right)^2 dx \\ &\quad + 2\int_\Omega \left(u_1(t,x) - u_1^*\right)^2 dx. \end{aligned} \quad (41)$$

So we have $\overline{u}_1(t) \to u_1^*$ as $t \to \infty$. Similarly, $\overline{u}_2(t) \to u_2^*$ and $\overline{u}_3(t) \to u_3^*$ as $t \to \infty$. According to (30), there exists a subsequence t_m, and non-negative functions $w_i \in C^2(\overline{\Omega})$, such that

$$\lim_{m\to\infty} \left\| u_i(\cdot, t_m) - w_i(\cdot) \right\|_{C^2(\overline{\Omega})} = 0 \quad (i = 1, 2, 3). \tag{42}$$

Applying (40) and noting that $\overline{u}_i(t) \to u_i^*$, we then have $w_i = u_i^*$, $(i = 1, 2, 3)$. That is,

$$\lim_{m\to\infty} \left\| u_i(\cdot, t_m) - u_i^* \right\|_{C^2(\overline{\Omega})} = 0 \quad (i = 1, 2, 3). \tag{43}$$

Furthermore, the local stability of $\mathbf{E}^*$ combining with (43) gives the following global stability. □

Theorem 9. *Assume that*

$$H_5 : \frac{a_1(a_{21} - a_2)}{m_1 a_2} > a_{11} > \frac{a_{12} v_2^*}{m_1^2}, \tag{44}$$

H_4 and G_4 hold, and then the semi-trivial steady state E_2 of system (1)–(3) *with non-trivial initial functions is globally asymptotically stable.*

Proof. It is obvious that system (1)–(3) always has two non-negative equilibria (u_1, u_2, u_3) as follows: $E_0 = (0, 0, 0)$ and $E_1 = (a_1/a_{11}, 0, 0)$. If H_1 and H_4 are satisfied, system (1) has the other semitrivial solution denoted by $E_2(v_1^*, v_2^*, 0)$, where

$$H_4 : a_3 m_2 > (a_{32} - a_3) v_2^*. \tag{45}$$

We consider the stability of E_2 under condition H_1 and H_4. Equation (1) can be rewritten as

$$\begin{aligned}
\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 &= u_1 \left[-a_{11}(u_1 - v_1^*) - \frac{a_{12}(u_2 - v_2^*)}{m_1 + u_1} \right. \\
&\qquad \left. + \frac{a_{12} v_2^* (u_1 - v_1^*)}{(m_1 + u_1)(m_1 + v_1^*)} \right], \\
\frac{\partial u_2}{\partial t} - d_2 \Delta u_2 & \\
&= u_2 \left[\int_\Omega \int_{-\infty}^t K_1(x, y, t - s) \right. \\
&\qquad \times \frac{a_{21} m_1 (u_1(s, y) - v_1^*)}{(m_1 + u_1(s, y))(m_1 + v_1^*)} ds\, dy \\
&\qquad \left. - a_{22}(u_2 - v_2^*) - \frac{a_{23} u_3}{m_2 + u_2} \right], \\
\frac{\partial u_3}{\partial t} - d_3 \Delta u_3 & \\
&= u_3 \left[\int_\Omega \int_{-\infty}^t K_2(x, y, t - s) \right. \\
&\qquad \times \frac{a_{32} m_2 (u_2(s, y) - v_2^*)}{(m_2 + u_2(s, y))(m_2 + v_2^*)} ds\, dy \\
&\qquad \left. - a_{33} u_3 \right].
\end{aligned} \tag{46}$$

Similar to the argument of Theorem 8, we have $u_1(x, t) \to v_1^*$ and $u_2(x, t) \to v_2^*$ as $t \to \infty$ uniformly on $\overline{\Omega}$ provided that the following additional condition holds:

$$\begin{aligned}
G_3 &: \quad a_{11} m_1^2 > a_{12} v_2^*, \\
G_4 &: \quad a_{22} m_2^2 > \frac{a_{23} a_{32}}{a_{33}} + \frac{2 a_{12} a_{21} m_2^2}{a_{11} m_1^2 - a_{12} v_2^*}.
\end{aligned} \tag{47}$$

Next, we consider the asymptotic behavior of $u_3(t, x)$. Let

$$\theta = \frac{a_3 - (a_{32} - a_3) v_2^*}{2\delta}, \quad \delta = |a_{32} - a_3|. \tag{48}$$

Then there exists $t_1 > 0$ such that

$$v_2^* - \theta < u_2(t, x) < v_2^* + \theta, \quad \forall t > t_1, \ x \in \overline{\Omega}. \tag{49}$$

Consider the following two systems:

$$\begin{aligned}
& w_{3t} - d_3 \Delta w_3 = w_3 \left[-a_3 + \frac{a_{32}(v_2^* + \theta)}{m_2 + v_2^* + \theta} - a_{33} w_3 \right], \\
& \qquad (t, x) \in [t_1, +\infty) \times \Omega, \\
& \frac{\partial w_3}{\partial \nu} = 0, \quad (t, x) \in [t_1, +\infty) \times \partial\Omega, \\
& w_3 = u_3, \quad (t, x) \in (-\infty, t_1] \times \Omega, \\
& W_{3t} - d_3 \Delta W_3 = W_3 \left[-a_3 + \frac{a_{32}(v_2^* - \theta)}{m_2 + v_2^* - \theta} - a_{33} v_3 \right], \\
& \qquad (t, x) \in [t_1, +\infty) \times \Omega, \\
& \frac{\partial W_3}{\partial \nu} = 0, \quad (t, x) \in [t_1, +\infty) \times \partial\Omega, \\
& W_3 = u_3, \quad (t, x) \in (-\infty, t_1] \times \Omega.
\end{aligned} \tag{50}$$

Combining comparison principle with (50), we obtain that

$$W_3(t, x) \le u_3(t, x) \le w_3(t, x), \quad \forall t > t_1, \ x \in \overline{\Omega}. \tag{51}$$

By Lemma 4, we obtain

$$\begin{aligned}
0 = \lim_{t\to\infty} W_3(t, x) &\le \inf u_3(t, x) \le \sup u_3(t, x) \\
&\le \lim_{t\to\infty} w_3(t, x) = 0, \quad \forall x \in \overline{\Omega},
\end{aligned} \tag{52}$$

which implies that $\lim_{t\to\infty} u_3(t, x) = 0$ uniformly on $\overline{\Omega}$. □

Theorem 10. *Suppose that G_5 and G_6 hold, and then the semi-trivial steady state E_1 of system* (1)–(3) *with non-trivial initial functions is globally asymptotically stable.*

Proof. We study the stability of the semi-trivial solution $E_1 = (\tilde{u}_1, 0, 0)$. Similarly, the equations in (1) can be written as

$$\begin{aligned}
\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 &= u_1 \left[-a_{11} (u_1 - \tilde{u}_1) - \frac{a_{12} u_2}{m_1 + u_1} \right], \\
\frac{\partial u_2}{\partial t} - d_2 \Delta u_2 & \\
&= u_2 \left[-a_2 + \int_\Omega \int_{-\infty}^t K_1 (x, y, t-s) \right. \\
&\quad \times \frac{a_{21} m_1 (u_1 (s, y) - \tilde{u}_1)}{(m_1 + u_1 (s, y)) (m_1 + \tilde{u}_1)} ds\, dy \\
&\quad \left. + \frac{a_{21} \tilde{u}_1}{m_1 + \tilde{u}_1} - a_{22} u_2 - \frac{a_{23} u_3}{m_2 + u_2} \right], \\
\frac{\partial u_3}{\partial t} - d_3 \Delta u_3 & \\
&= u_3 \left[-a_3 + \int_\Omega \int_{-\infty}^t K_2 (x, y, t-s) \right. \\
&\quad \left. \times \frac{a_{32} u_2 (s, y)}{m_2 + u_2 (s, y)} ds\, dy - a_{33} u_3 \right].
\end{aligned} \tag{53}$$

Define

$$\begin{aligned}
V(t) &= \alpha \int_\Omega \left[u_1 - \tilde{u}_1 - \tilde{u}_1 \log \frac{u_1}{\tilde{u}_1} \right] dx + \int_\Omega u_2 dx \\
&\quad + \beta \int_\Omega u_3 dx.
\end{aligned} \tag{54}$$

Calculating the derivative of $V(t)$ along E_1, we get from (54) that

$$\begin{aligned}
V'(t) &\le - \int_\Omega \left[\alpha \left(a_{11} - \frac{a_{12} \epsilon_1}{m_1} \right) \right] (u_1 - \tilde{u}_1)^2 dx \\
&\quad + \int_\Omega \left[a_{22} - \frac{\alpha a_{12}}{4 m_1 \epsilon_1} - \frac{a_{23}}{4 \epsilon_2 m_2} - \frac{a_{21}}{4 m_1 \epsilon_3} \right] u_2^2 dx \\
&\quad + \int_\Omega \left[\beta a_{33} - \frac{a_{23} \epsilon_2}{m_2} \right] u_3^2 dx \\
&\quad - \int_\Omega \left[\left(a_2 - \frac{a_{21} \tilde{u}_1}{m_1 + \tilde{u}_1} \right) u_2 + a_3 u_3 \right] dx \\
&\quad + \frac{a_{21} \epsilon_3}{m_1} \int_\Omega \int_{-\infty}^t K_1 (x, y, t-s) \\
&\qquad \times (u_1 (s, y) - \tilde{u}_1)^2 ds\, dy\, dx \\
&\quad + \frac{a_{32} \beta}{4 \epsilon_4 m_2} \iint_\Omega \int_{-\infty}^t K_2 (x, y, t-s) \\
&\qquad \times u_2^2 (s, y)\, ds\, dy\, dx.
\end{aligned} \tag{55}$$

Define

$$\begin{aligned}
E(t) &= V(t) \\
&\quad + \frac{a_{21} \epsilon_3}{m_1} \iint_\Omega \int_0^{+\infty} \int_{t-r}^t K_1 (x, y, r) \\
&\qquad \times (u_1 (s, y) - u_1^*)^2 ds\, dr\, dy\, dx \\
&\quad + \frac{a_{32} \beta}{4 \epsilon_4 m_2} \iint_\Omega \int_0^{+\infty} \int_{t-r}^t K_2 (x, y, r) \\
&\qquad \times (u_2 (s, y) - u_2^*)^2 ds\, dr\, dy\, dx.
\end{aligned} \tag{56}$$

It is easy to see that

$$\begin{aligned}
E'(t) &\le -\alpha d_1 \tilde{u}_1 \int_\Omega \frac{|\nabla u_1|^2}{u_1^2} dx \\
&\quad - \int_\Omega \left[\alpha \left(a_{11} - \frac{a_{12} \epsilon_1}{m_1} \right) - \frac{a_{21} \epsilon_3}{m_1} \right] (u_1 - \tilde{u}_1)^2 dx \\
&\quad + \int_\Omega \left[a_{22} - \frac{\alpha a_{12}}{4 m_1 \epsilon_1} - \frac{a_{23}}{4 m_2 \epsilon_2} - \frac{a_{21}}{4 m_1 \epsilon_3} \right. \\
&\qquad \left. - \frac{a_{32}}{4 m_2 \epsilon_4} \right] u_2^2 dx \\
&\quad + \int_\Omega \left[\beta a_{33} - \frac{a_{23} \epsilon_2}{m_2} - \frac{a_{32} \beta \epsilon_4}{m_2} \right] u_3^2 dx \\
&\quad - \int_\Omega \left[\left(a_2 - \frac{a_{21} \tilde{u}_1}{m_1 + \tilde{u}_1} \right) u_2 - a_3 \beta u_3 \right] dx.
\end{aligned} \tag{57}$$

Assume that

$$\begin{aligned}
&G_5: \quad a_1 (a_{21} - a_2) < m_1 a_2 a_{11}, \\
&G_6: a_{22} m_2^2 > \frac{a_{23} a_{32}}{a_{33}} + \frac{2 a_{12} a_{21} m_2^2}{a_{11} m_1^2}.
\end{aligned} \tag{58}$$

Let

$$\begin{aligned}
l_1 &= \alpha a_{11} - \frac{\alpha a_{12} \epsilon_1}{m_1} - \frac{a_{21} \epsilon_3}{m_1}, \\
l_2 &= a_{22} - \frac{\alpha a_{12}}{4 m_1 \epsilon_1} - \frac{a_{23}}{4 m_2 \epsilon_2} - \frac{a_{21}}{4 m_1 \epsilon_3} - \frac{a_{32} \beta}{4 m_2 \epsilon_4}, \\
l_3 &= a_{33} \beta - \frac{a_{23} \epsilon_2}{m_2} - \frac{\beta a_{32} \epsilon_4}{m_2}.
\end{aligned} \tag{59}$$

Choose

$$\begin{aligned}
\alpha &= \frac{a_{21}}{a_{12}}, \qquad \beta = \frac{a_{23}}{a_{32}}, \quad \epsilon_1 = \epsilon_3 = \frac{a_{11} m_1}{4 a_{12}}, \\
&\qquad \epsilon_2 = \epsilon_4 = \frac{m_2 a_{33}}{4 a_{32}}.
\end{aligned} \tag{60}$$

Then we get $l_i > 0$ $(i = 1, 2, 3)$. Therefore we have

$$\lim_{t \to \infty} u_1 (t, x) = \tilde{u}_1, \tag{61}$$

uniformly on $\overline{\Omega}$. □

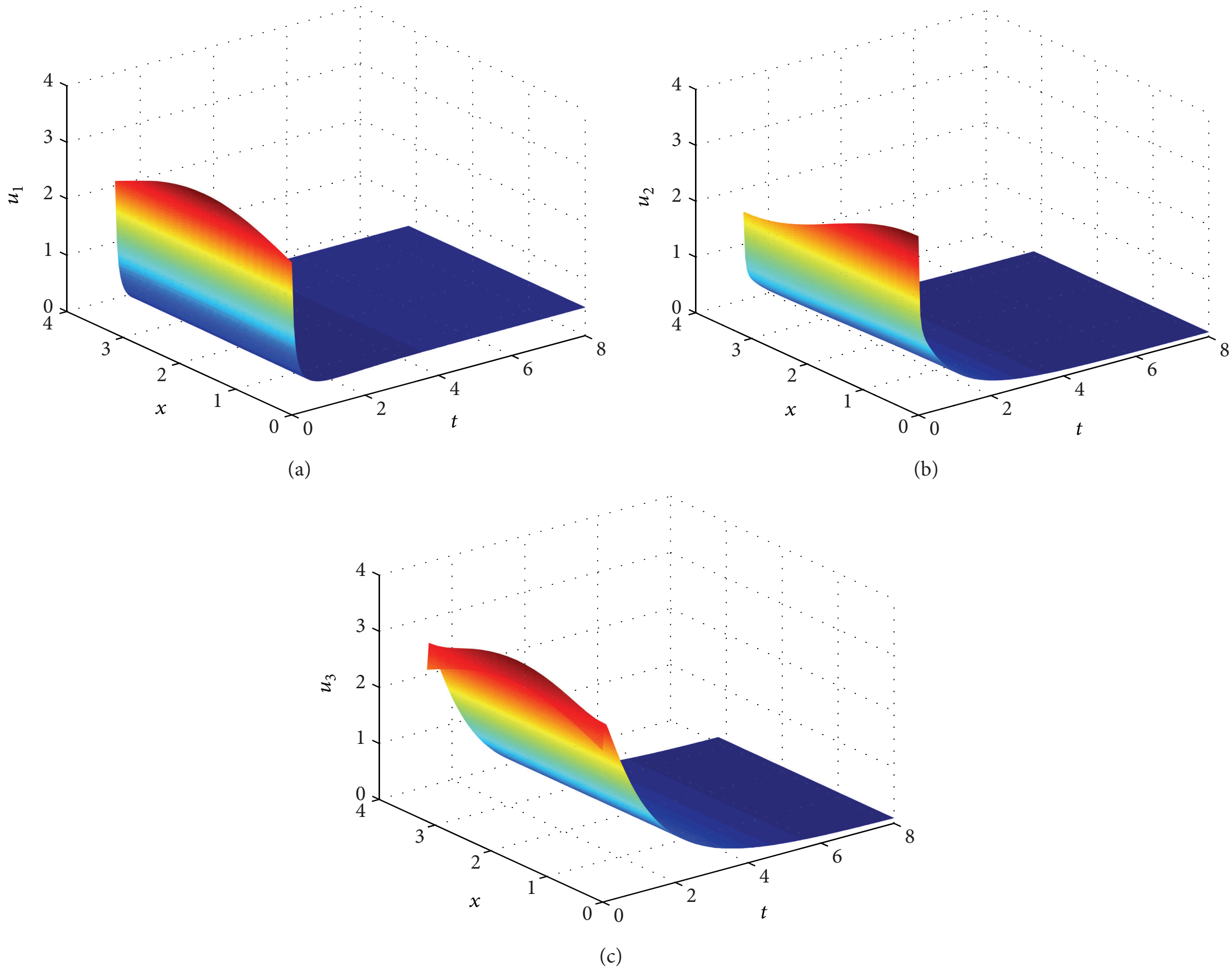

FIGURE 2: Global stability of the positive equilibrium E^* with an initial condition $(\phi_1, \phi_2, \phi_3) = (2.7 + 0.5\sin x, 2.7 + 0.5\cos x, 2.7 + 0.5\sin x)$.

Next, we consider the asymptotic behavior of $u_2(t,x)$ and $u_3(t,x)$. For any $T > 0$, integrating (57) over $[0,T]$ yields

$$\begin{aligned} E(T) + \alpha d_1 \left|\left|\left| \frac{\nabla u_1}{u_1} \right|\right|\right|_2^2 + l_1 |||u_1 - \tilde{u}_1|||_2^2 \\ + l_2 |||u_2|||_2^2 + l_3 |||u_3|||_2^2 \le E(0), \end{aligned} \tag{62}$$

where $|||u_i|||_2^2 = \int_0^T \int_\Omega u_i^2 dx\, dt$. It implies that $|||u_i|||_2 \le C_i$ $(i = 2, 3)$ for the constant C_i which is independent of T. Now we consider the boundedness of $|||\nabla u_2|||_2$ and $|||\nabla u_3|||_2$. From the Green's identity, we obtain

$$\begin{aligned} & d_2 \int_0^T \int_\Omega |\nabla u_2(t,x)|^2 dx\, dt \\ & = -d_2 \int_0^T \int_\Omega u_2(t,x) \Delta u_2(t,x)\, dx\, dt \\ & = -\int_0^T \int_\Omega u_2 \frac{\partial u_2}{\partial t} dx\, dt - a_2 \int_0^T \int_\Omega u_2^2 dx\, dt \\ & \quad - a_{22} \int_0^T \int_\Omega u_2^3 dx\, dt - a_{23} \int_0^T \int_\Omega \frac{u_2^2 u_3}{m_2 + u_3} dx\, dt \\ & \quad + a_{21} \int_0^T \iint_\Omega \int_{-\infty}^t \frac{u_1(s,y) u_2^2(t,x)}{m_1 + u_1(s,y)} \\ & \qquad \times K_1(x,y,t-s)\, ds\, dy\, dx\, dt. \end{aligned} \tag{63}$$

Note that

$$\begin{aligned} & \int_0^T \int_\Omega u_2(t,x) \frac{\partial u_2(t,x)}{\partial t} dx\, dt \\ & = \int_0^T \int_\Omega \frac{1}{2} \frac{\partial u_2^2}{\partial t} dt\, dx \\ & = \frac{1}{2} \int_\Omega u_2^2(T,x)\, dx - \frac{1}{2} \int_\Omega u_2^2(0,x)\, dx \le M_2^2 |\Omega|, \\ & \int_0^T \int_\Omega u_2^2 u_3 dx\, dt \le M_3 \int_0^T \int_\Omega u_2^2 dx\, dt, \end{aligned}$$

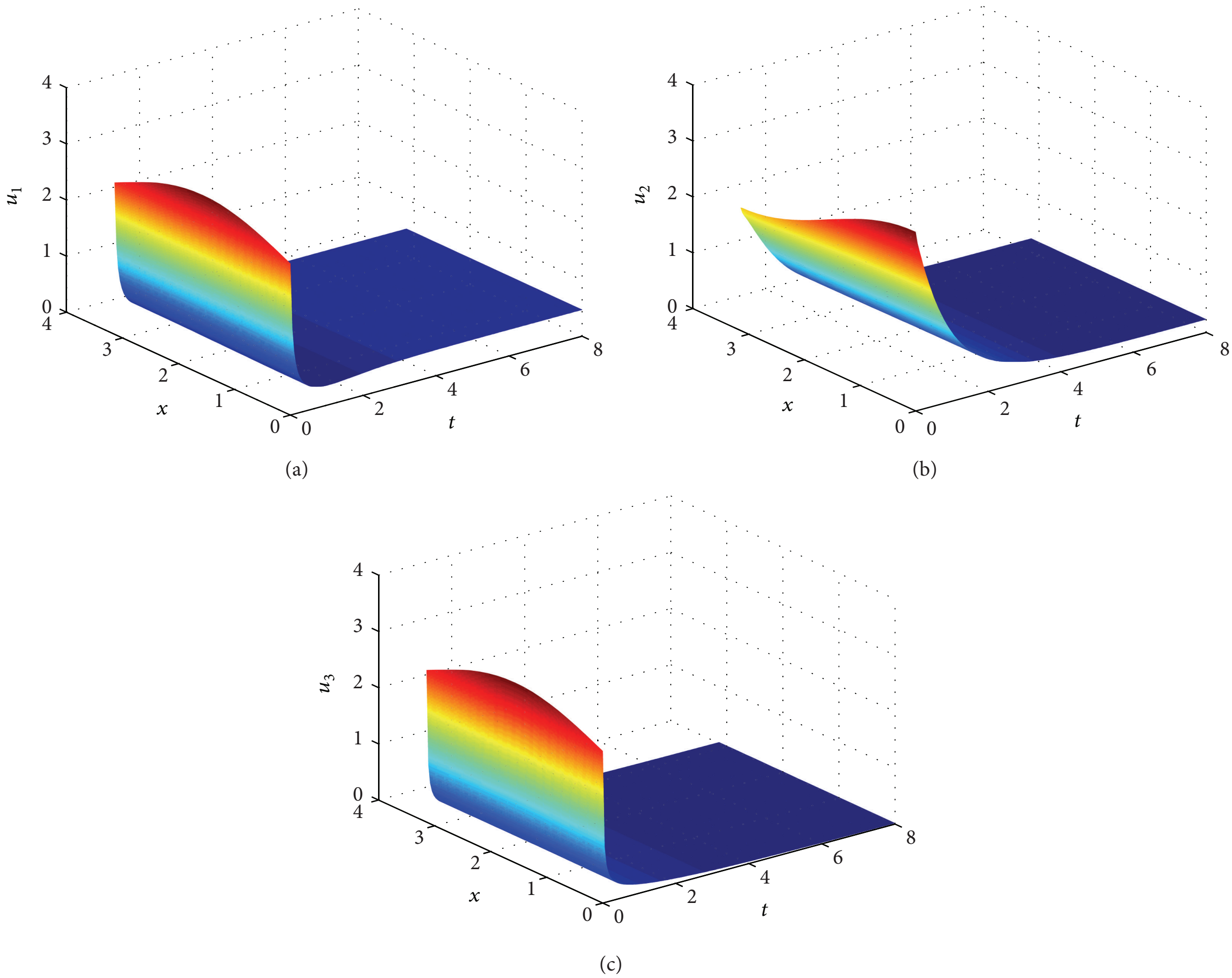

FIGURE 3: Global stability of the positive equilibrium E_2 with an initial condition $(\phi_1, \phi_2, \phi_3) = (2.7 + 0.5\sin x, 2.7 + 0.5\cos x, 2.7 + 0.5\sin x)$.

$$\int_0^T \int_\Omega u_2^3 dx\, dt \le M_2 \int_0^T \int_\Omega u_2^2 dx\, dt,$$

$$\int_0^T \iint_\Omega \int_{-\infty}^t \frac{u_1(s,y)\, u_2^2(t,x)}{m_1 + u_1(s,y)} K_1(x,y,t-s)\, ds\, dy\, dx\, dt \le \int_0^T \int_\Omega u_2^2(t,x)\, dx\, dt. \tag{64}$$

Thus, we get $\||\nabla u_2|\|_2 \le C_4$. In a similar way, we have $\||\nabla u_3|\|_2 \le C_5$. Here C_4 and C_5 are independent of T.

It is easy to see that $u_2(t,x), u_3(t,x) \in L^2((0,\infty); W^{1,2}(\Omega))$. These imply that

$$\lim_{t\to\infty} \|u_i(\cdot,t)\|_{W^{1,2}(\Omega)} = 0, \quad i = 2, 3. \tag{65}$$

From the Sobolev compact embedding theorem, we know

$$\lim_{t\to\infty} \|u_i(\cdot,t)\|_{C(\overline{\Omega})} = 0, \quad i = 2, 3. \tag{66}$$

In the end, we show that the trivial solution E_0 is an unstable equilibrium. Similarly to the local stability to E^*, we can get the characteristic equation of E_0 as

$$(\lambda + \mu_i D_1 - a_1)(\lambda + \mu_i D_2 + a_2)(\lambda + \mu_i D_3 + a_3) = 0. \tag{67}$$

If $i = 1$, then $\mu_1 = 0$. It is easy to see that this equation admits a positive solution $\lambda = a_1$. According to Theorem 5.1 in [23], we have the following result.

Theorem 11. *The trivial equilibrium E_0 is an unstable equilibrium of system* (1)–(3).

4. Numerical Illustrations

In this section, we perform numerical simulations to illustrate the theoretical results given in Section 3.

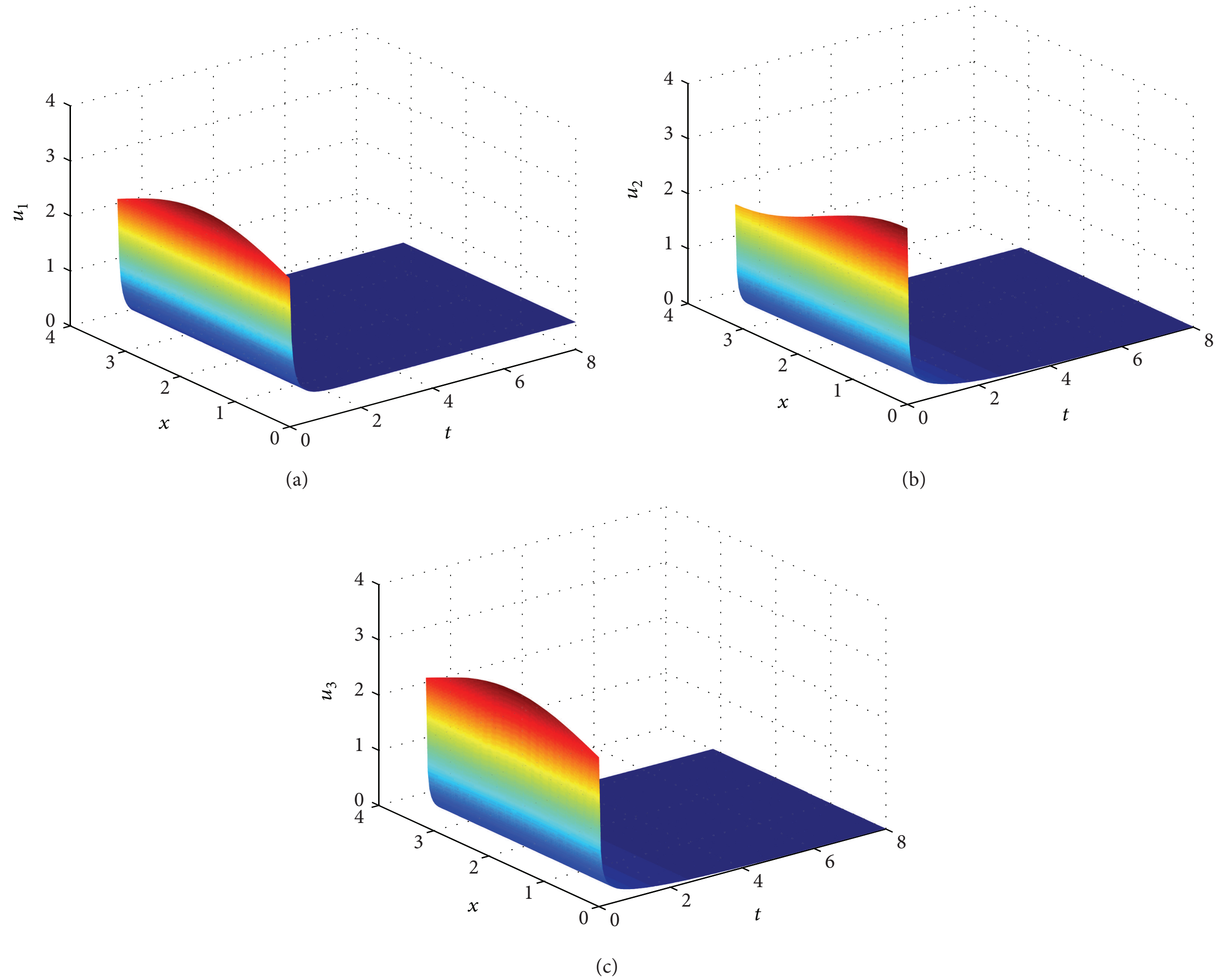

FIGURE 4: Global stability of the positive equilibrium E_1 with an initial condition $(\phi_1, \phi_2, \phi_3) = (2.7 + 0.5 \sin x, 2.7 + 0.5 \cos x, 2.7 + 0.5 \sin x)$.

In the following, we always take $\Omega = [0, \pi]$, $K_i(x, y, t) = G_i(x, y, t) k_i(t)$, where $k_i(t) = (1/\tau^*) e^{-t/\tau^*}$ $(i = 1, 2)$ and

$$G_1(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-d_2 n^2 t} \cos nx \cos ny,$$
$$G_2(x, y, t) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} e^{-d_3 n^2 t} \cos nx \cos ny. \tag{68}$$

However, it is difficult for us to simulate our results directly because of the nonlocal term. Similar to [25], the equations in (1) can be rewritten as follows:

$$\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 = u_1 \left(a_1 - a_{11} u_1 - \frac{a_{12} u_2}{m_1 + u_1} \right),$$
$$\frac{\partial u_2}{\partial t} - d_2 \Delta u_2 = u_2 \left(-a_2 + a_{21} v_1 - a_{22} u_2 - \frac{a_{23} u_3}{m_2 + u_3} \right),$$
$$\frac{\partial u_3}{\partial t} - d_3 \Delta u_3 = u_3 \left(-a_3 + a_{32} v_2 - a_{33} u_3 \right),$$
$$\frac{\partial v_1}{\partial t} - d_2 \Delta v_1 = \frac{1}{\tau^*} \left(\frac{u_1}{m_1 + u_1} - v_1 \right),$$
$$\frac{\partial v_2}{\partial t} - d_3 \Delta v_2 = \frac{1}{\tau^*} \left(\frac{u_2}{m_2 + u_2} - v_2 \right), \tag{69}$$

where

$$v_i = \int_0^{\pi} \int_{-\infty}^{t} G_i(x, y, t - s) \frac{1}{\tau^*} e^{-(t-s)/\tau^*} \frac{u_i(s, y)}{m_i + u_i(s, y)} dy\, ds, \quad i = 1, 2. \tag{70}$$

Each component is considered with homogeneous Neumann boundary conditions, and the initial condition of v_i is

$$v_i(0, x) = \int_0^{\pi} \int_{-\infty}^{0} G_i(x, y, -s) \frac{1}{\tau^*} e^{s/\tau^*} \frac{u_i(s, y)}{m_i + u_i(s, y)} dy\, ds, \quad i = 1, 2. \tag{71}$$

In the following examples, we fix some coefficients and assume that $d_1 = d_2 = d_3 = 1$, $a_1 = 3$, $a_{12} = 1$, $m_1 = m_2 = 1$, $a_2 = 1$, $a_{22} = 12$, $a_{23} = 1$, $a_{33} = 12$, and $\tau^* = 1$. The asymptotic behaviors of system (1)–(3) are shown by choosing different coefficients a_{11}, a_{21}, a_3, and a_{32}.

Example 12. Let $a_{11} = 53/9$, $a_{21} = 81/13$, $a_3 = 1/13$ and $a_{32} = 14$. Then it is easy to see that the system admits a unique positive equilibrium $E^*(1/2, 1/12, 1/12)$. By Theorem 8, we see that the positive solution $(u_1(t,x), u_2(t,x), u_3(t,x))$ of system (1)–(3) converges to E^* as $t \to \infty$. See Figure 2.

Example 13. Let $a_{11} = 6$, $a_{21} = 12$, $a_3 = 1/4$ and $a_{32} = 1$. Clearly, H_2 does not hold. Hence, the positive steady state is not feasible. System (1) admits two semi-trivial steady state $E_2(0.4732, 0.2372, 0)$ and $E_1(1/2, 0, 0)$. According to Theorem 9, we know that the positive solution $(u_1(t,x),\ u_2(t,x),\ u_3(t,x))$ of system (1)–(3) converges to E_2 as $t \to \infty$. See Figure 3.

Example 14. Let $a_{11} = 6$, $a_{21} = 2$, $a_3 = 1/4$ and $a_{32} = 1$. Clearly, H_1 does not hold. Hence, E^* and E_2 are not feasible. System (1) has a unique semi-trivial steady state $E_1(1/2, 0, 0)$. According to Theorem 10 we know that the positive solution $(u_1(t,x),\ u_2(t,x),\ u_3(t,x))$ of system (1)–(3) converges to E_1 as $t \to \infty$. See Figure 4.

5. Discussion

In this paper, we incorporate nonlocal delay into a three-species food chain model with Michaelis-Menten functional response to represent a delay due to the gestation of the predator. The conditions, under which the spatial homogeneous equilibria are asymptotically stable, are given by using the Lyapunov functional.

We now summarize the ecological meanings of our theoretical results. Firstly, the positive equilibrium E^* of system (1)–(3) exists under the high birth rate of the prey (a_1) and low death rates (a_2 and a_3) of predator and top predator. E^* is globally stable if the intraspecific competition a_{11} is neither too big nor too small and the maximum harvest rates a_{12}, a_{23} are small enough. Secondly, the semi-trivial equilibrium E_2 of system (1)–(3) exists if the birth rate of the prey (a_1) is high, death rate of predator a_2 is low, and the death rate (a_3) exceeds the conversion rate from predator to top predator (a_{32}). E_2 is globally stable if the maximum harvest rates a_{12}, a_{23} are small and the intra-specific competition a_{11} is neither too big nor too small. Thirdly, system (1)–(3) has only one semi-trivial equilibrium E_1 when the death rate (a_2) exceeds the conversion rate from prey to predator (a_{21}). E_1 is globally stable if intra-specific competitions (a_{11}, a_{22}, and a_{33}) are strong. Finally, E_0 is unstable and the non-stability of trivial equilibrium tells us that not all of the populations go to extinction. Furthermore, our main results imply that the nonlocal delay is harmless for stabilities of all non-negative steady states of system (1)–(3).

There are still many interesting and challenging problems with respect to system (1)–(3), for example, the permanence and stability of periodic solution or almost periodic solution. These problems are clearly worthy for further investigations.

Acknowledgments

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References

[1] R. Xu and M. A. J. Chaplain, "Persistence and global stability in a delayed predator-prey system with Michaelis-Menten type functional response," *Applied Mathematics and Computation*, vol. 130, no. 2-3, pp. 441–455, 2002.

[2] H. F. Huo and W. T. Li, "Periodic solution of a delayed predator-prey system with Michaelis-Menten type functional response," *Journal of Computational and Applied Mathematics*, vol. 166, no. 2, pp. 453–463, 2004.

[3] Z. Lin and M. Pedersen, "Stability in a diffusive food-chain model with Michaelis-Menten functional response," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 57, no. 3, pp. 421–433, 2004.

[4] B. X. Dai, N. Zhang, and J. Z. Zou, "Permanence for the Michaelis-Menten type discrete three-species ratio-dependent food chain model with delay," *Journal of Mathematical Analysis and Applications*, vol. 324, no. 1, pp. 728–738, 2006.

[5] N. F. Britton, "Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model," *SIAM Journal on Applied Mathematics*, vol. 50, no. 6, pp. 1663–1688, 1990.

[6] S. A. Gourley and N. F. Britton, "Instability of travelling wave solutions of a population model with nonlocal effects," *IMA Journal of Applied Mathematics*, vol. 51, no. 3, pp. 299–310, 1993.

[7] X. Zhang and R. Xu, "Traveling waves of a diffusive predator-prey model with nonlocal delay and stage structure," *Journal of Mathematical Analysis and Applications*, vol. 373, no. 2, pp. 475–484, 2011.

[8] Z. C. Wang, W. T. Li, and S. G. Ruan, "Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay," *Journal of Differential Equations*, vol. 238, no. 1, pp. 153–200, 2007.

[9] S. A. Gourley and S. G. Ruan, "Convergence and travelling fronts in functional differential equations with nonlocal terms: a competition model," *SIAM Journal on Mathematical Analysis*, vol. 35, no. 3, pp. 806–822, 2003.

[10] X. H. Lai and T. X. Yao, "Exponential stability of impulsive delayed reaction-diffusion cellular neural networks via Poincaré integral inequality," *Abstract and Applied Analysis*, vol. 2013, Article ID 131836, 10 pages, 2013.

[11] R. Xu and Z. E. Ma, "Global stability of a reaction-diffusion predator-prey model with a nonlocal delay," *Mathematical and Computer Modelling*, vol. 50, no. 1-2, pp. 194–206, 2009.

[12] Z. Li, F. D. Chen, and M. X. He, "Asymptotic behavior of the reaction-diffusion model of plankton allelopathy with nonlocal delays," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 3, pp. 1748–1758, 2011.

[13] Y. Kuang and H. L. Smith, "Global stability in diffusive delay Lotka-Volterra systems," *Differential and Integral Equations*, vol. 4, no. 1, pp. 117–128, 1991.

[14] Z. G. Lin, M. Pedersen, and L. Zhang, "A predator-prey system with stage-structure for predator and nonlocal delay," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 2019–2030, 2010.

[15] R. Xu, "A reaction-diffusion predator-prey model with stage structure and nonlocal delay," *Applied Mathematics and Computation*, vol. 175, no. 2, pp. 984–1006, 2006.

[16] S. S. Chen and J. P. Shi, "Stability and Hopf bifurcation in a diffusive logistic population model with nonlocal delay effect," *Journal of Differential Equations*, vol. 253, no. 12, pp. 3440–3470, 2012.

[17] H. I. Freedman, *Deterministic Mathematical Models in Population Ecology*, vol. 57 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1980.

[18] P. A. Abrams and L. R. Ginzburg, "The nature of predation: prey depedent, ratio dependent or neither," *Trends in Ecology and Evolution*, vol. 15, no. 8, pp. 337–341, 2000.

[19] D. L. DeAngelis, R. A. Goldstein, and R. V. O'Neill, "A model for trophic iteration," *Ecology*, vol. 56, no. 4, pp. 881–892, 1975.

[20] C. V. Pao, "Dynamics of nonlinear parabolic systems with time delays," *Journal of Mathematical Analysis and Applications*, vol. 198, no. 3, pp. 751–779, 1996.

[21] F. Xu, Y. Hou, and Z. G. Lin, "Time delay parabolic system in a predator-prey model with stage structure," *Acta Mathematica Sinica*, vol. 48, no. 6, pp. 1121–1130, 2005 (Chinese).

[22] P. Y. H. Pang and M. X. Wang, "Strategy and stationary pattern in a three-species predator-prey model," *Journal of Differential Equations*, vol. 200, no. 2, pp. 245–273, 2004.

[23] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, vol. 840 of *Lecture Notes in Mathematics*, Springer, New York, NY, USA, 1981.

[24] K. J. Brown, P. C. Dunne, and R. A. Gardner, "A semilinear parabolic system arising in the theory of superconductivity," *Journal of Differential Equations*, vol. 40, no. 2, pp. 232–252, 1981.

[25] S. A. Gourley and J. W. H. So, "Dynamics of a food-limited population model incorporating nonlocal delays on a finite domain," *Journal of Mathematical Biology*, vol. 44, no. 1, pp. 49–78, 2002.

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Research Article

Final State Problem for the Dirac-Klein-Gordon Equations in Two Space Dimensions

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We study the final state problem for the Dirac-Klein-Gordon equations (DKG) in two space dimensions. We prove that if the nonresonance mass condition is satisfied, then the wave operator for DKG is well defined from a neighborhood at the origin in lower order weighted Sobolev space to some Sobolev space.

1. Introduction

We study the final state problem for the Dirac-Klein-Gordon equations (DKG) in two space dimensions:

$$\begin{aligned} (\partial_t + \alpha\cdot\nabla + iM\beta)\psi &= \phi\beta\psi, \\ \left(\partial_t^2 - \Delta + m^2\right)\phi &= \psi^*\beta\psi, \end{aligned} \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^2, \tag{DKG}$$

where (ψ,ϕ) is a $\mathbb{C}^2\times\mathbb{R}$-valued unknown function of (t,x), $\psi = (\psi_1,\psi_2)^t$ stands a spinor field and ϕ denotes a scalar field, $M, m > 0$ denote masses of the spinor field and the scalar field, respectively, and ψ^* denotes a transposed conjugate to ψ. The operators $\alpha\cdot\nabla$ and Δ are defined by $\alpha\cdot\nabla = \sum_{j=1}^2 \alpha_j\partial_{x_j}$ and $\Delta = \sum_{j=1}^2 \partial_{x_j}^2$, respectively. Here, α_j $(j = 1,2)$ and β are Dirac matrices, that is, 2×2 self-adjoint matrices with constant elements such that

$$\begin{gathered} \alpha_j^2 = \beta^2 = I, \quad \alpha_j\beta + \beta\alpha_j = O, \quad \text{for } j = 1,2, \\ \alpha_j\alpha_k + \alpha_k\alpha_j = O, \quad \text{for } j,k = 1,2,\ j\neq k. \end{gathered} \tag{1}$$

Our aim in the present paper is to show existence of the wave operator for the DKG system (DKG) under the nonresonance mass condition $m \neq 2M$ in two space dimensions.

First, we recall some well-posedness results for (DKG). Many local well-posedness results in low-order Sobolev spaces have been obtained for these ten years (for recent information see, e.g., [1, 2] and references therein). Global well-posedness results in 2d case were also obtained (see, e.g., [3]). Moreover, very recently, unconditional uniqueness in 2d case was discussed in [4, 5]. On the other hand, there are few results about scattering for (DKG) in 2d case.

In [6, 7], the asymptotic behavior of solutions for DKG system was studied in 3d case by reducing it to a nonlinear Klein-Gordon system (KG). Denote $\mathscr{D}_\pm \equiv \partial_t \pm (\alpha\cdot\nabla + iM\beta)$. In view of the properties (1), we have

$$\mathscr{D}_-\mathscr{D}_+ = \partial_t^2 - (\alpha\cdot\nabla + iM\beta)(\alpha\cdot\nabla + iM\beta) = \partial_t^2 + \langle\nabla\rangle_M^2, \tag{2}$$

where $\langle\nabla\rangle_M \equiv \sqrt{M^2-\Delta}$. Hence, multiplying both sides of the Dirac part by $\mathscr{D}_-$, we obtain

$$\begin{aligned} \left(\partial_t^2 + \langle\nabla\rangle_M^2\right)\psi &= \mathscr{D}_-(\phi\beta\psi) \\ &= (\mathscr{D}_-\phi)\beta\psi - iM\phi I\psi + \phi\beta\mathscr{D}_+\psi \\ &= \left((\mathscr{D}_-\phi)\beta - iM\phi I + \lambda\phi^2 I\right)\psi, \end{aligned} \tag{3}$$

where we have used the fact that ψ is the solution of the DKG system. Thus, the solution of the DKG system satisfies the following KG one:

$$\begin{aligned} \left(\partial_t^2 + \langle\nabla\rangle_M^2\right)\psi &= \left((\mathscr{D}_-\phi)\beta - iM\phi I + \lambda\phi^2 I\right)\psi, \\ \left(\partial_t^2 + \langle\nabla\rangle_m^2\right)\phi &= \psi^*\beta\psi. \end{aligned} \tag{4}$$

If we want to obtain a priori estimates to the local solution for the DKG system, we can use estimates to solutions for the above KG one. Moreover, in the present two-dimensional case, the initial value problem for nonlinear KG systems including (4) was studied in [8] (see also [9]). In [8], Sunagawa proved existence of a unique global asymptotically free solution under the nonresonance mass conditions, if the initial data are sufficiently small, smooth and decay fast at infinity. However, asymptotic behavior of solutions for DKG is not clear because (DKG) is not equivalent to (4) in general. In this paper, we will consider the DKG system itself without reducing it into (4) such as in [10]. Though the initial value problem for DKG was treated in [11], the final value problem which will be discussed in this paper is more delicate because of the derivative loss difficulties.

In [10], the wave operator for the DKG system has been obtained in a three-dimensional case. They dealt with the DKG system itself. Nevertheless, from a point of time decay property for the free solutions of the DKG system, two dimensional-case is critical, that is, borderline case between the long range scattering and the short range one. Therefore, their argument cannot be applicable to the two-dimensional case. To overcome the lack of time decay property, we will use the *algebraic normal form transformation* developed in paper [8] and the *decomposition of the Klein-Gordon operator*, that is,

$$\partial_t^2 + \langle\nabla\rangle_M^2 = \mathscr{D}_+\mathscr{D}_-. \tag{5}$$

By this combination, we will find a suitable second approximate solution to ψ (given by (42)). We note that the implicit null structure for (DKG) was discovered in [12], and it was used to prove local well-posedness in low regular setting in [2]. On the other hand, in this paper, by explicit null structure, wave operator for (DKG) will be constructed.

Next, we recall the problem of existence of the wave operator for (DKG). We define the free-Dirac-and Klein-Gordon evolution groups as follows:

$$\begin{gathered}\mathscr{V}_D(t) \equiv I\cos(t\langle\nabla\rangle_M) - (\alpha\cdot\nabla + iM\beta)\langle\nabla\rangle_M^{-1}\sin(t\langle\nabla\rangle_M),\\ \mathscr{V}_K(t) \equiv \begin{pmatrix}\cos(\langle\nabla\rangle_m t) & \sin(\langle\nabla\rangle_m t)\\ -\sin(\langle\nabla\rangle_m t) & \cos(\langle\nabla\rangle_m t)\end{pmatrix}.\end{gathered} \tag{6}$$

For given final data $(\psi^+, (\langle\nabla\rangle_m\phi_1^+, \phi_2^+)) \in (\mathbf{X})^4$ with some Banach spaces $\mathbf{X}$ defined explicitly later, we put

$$\begin{gathered}\psi_0(t) \equiv \mathscr{V}_D(t)\psi^+,\\ \begin{pmatrix}\phi_0(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi_0(t)\end{pmatrix} \equiv \mathscr{V}_K(t)\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}.\end{gathered} \tag{7}$$

We will look for a unique time local solution of (DKG) which satisfies the final state conditions as follows:

$$\lim_{t\to\infty}\|\psi(t) - \psi_0(t)\|_{\widetilde{\mathbf{X}}} = 0, \tag{8}$$

$$\lim_{t\to\infty}\left\|\begin{pmatrix}\langle\nabla\rangle_m^{1/2}\phi(t)\\ \langle\nabla\rangle_m^{-1/2}\partial_t\phi(t)\end{pmatrix} - \begin{pmatrix}\langle\nabla\rangle_m^{1/2}\phi_0(t)\\ \langle\nabla\rangle_m^{-1/2}\partial_t\phi_0(t)\end{pmatrix}\right\|_{\widetilde{\mathbf{X}}} = 0, \tag{9}$$

where $\widetilde{\mathbf{X}}$ is also a suitable Banach space. If there exist $T > 0$ and a unique solution $(\psi, \langle\nabla\rangle_m^{1/2}\phi, \langle\nabla\rangle_m^{-1/2}\partial_t\phi) \in (\mathbf{C}([T,\infty); \widetilde{\mathbf{X}}))^4$ for (DKG) satisfying (8)-(9), then the wave operator $\mathscr{W}^+$ for (DKG) is defined by the mapping as follows:

$$\begin{gathered}\mathscr{W}^+ : (\mathbf{X})^2 \times (\langle\nabla\rangle^{-1}\mathbf{X}\times\mathbf{X})\\ \longrightarrow (\widetilde{\mathbf{X}})^2 \times (\langle\nabla\rangle^{-1/2}\widetilde{\mathbf{X}}\times\langle\nabla\rangle^{1/2}\widetilde{\mathbf{X}}),\\ (\psi(t), (\phi(t), \partial_t\phi(t))) = \mathscr{W}^+(\psi^+, (\phi_1^+, \phi_2^+)),\\ \text{for } t\in[T,\infty),\end{gathered} \tag{10}$$

where $\langle\nabla\rangle^{-s}\mathbf{X} \equiv \{\phi; \|\langle\nabla\rangle^s\phi\|_{\mathbf{X}} < \infty\}$.

2. Several Notations and Main Results

We introduce several notations to state our main results. For $m, k \in \mathbb{R}$, and $1 \le p \le \infty$, we introduce the weighted Sobolev space as follows:

$$H_p^{m,k} = \left\{\phi; \|\phi\|_{H_p^{m,k}} \equiv \left\|\langle x\rangle^k\langle\nabla\rangle^m\phi\right\|_{L^p} < \infty\right\}, \tag{11}$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$, $\langle\nabla\rangle = (1-\Delta)^{1/2}$. We also write for simplicity $H^{m,k} = H_2^{m,k}$, $H^m = H_2^{m,0}$, and $H_p^m = H_p^{m,0}$, and so we usually omit the index 0 and $p = 2$ if it does not cause a confusion.

We now state our main results in this paper. We introduce the function space as follows:

$$D_q \equiv H_{q/(q-1)}^{4-4/q} \cap H^{5/2,1} \cap H_1^2. \tag{12}$$

Theorem 1. *Let $m, M > 0$, $m \ne 2M$, $4 < q \le \infty$ and $(\psi^+, (\langle\nabla\rangle\phi_1^+, \phi_2^+)) \in (D_q)^4$. If the norm $\rho \equiv \|(\psi^+, (\langle\nabla\rangle\phi_1^+, \phi_2^+))\|_{H_1^2}$ is sufficiently small, then there exist a positive constant $T > 0$ and a unique solution*

$$\left(\psi(t), \begin{pmatrix}\langle\nabla\rangle_m^{1/2}\phi(t)\\ \langle\nabla\rangle_m^{-1/2}\partial_t\phi(t)\end{pmatrix}\right) \in \left(C\left([T,\infty); H^{1/2}\right)\right)^4, \tag{13}$$

for the system (DKG). Moreover, there exists a positive constant $C > 0$ such that the following estimate

$$\begin{gathered}\|\psi(t) - \psi_0(t)\|_{H^{1/2}} + \left\|\begin{pmatrix}\phi(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi(t)\end{pmatrix} - \begin{pmatrix}\phi_0(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi_0(t)\end{pmatrix}\right\|_{H^1}\\ \le Ct^{-\mu}\end{gathered} \tag{14}$$

is true for all $t \ge T$, where $1/2 < \mu < 1 - 2/q$ and (ψ_0, ϕ_0) is given by (7).

By Theorem 1, we can get existence of the wave operator for (DKG) as follows.

Corollary 2. *Let $m, M > 0$, $m \ne 2M$, and $4 \le q < \infty$. Then the wave operator $\mathscr{W}^+$ for (DKG) is well defined from a neighborhood at the origin in the space $(D_q)^2 \times (\langle\nabla\rangle^{-1}D_q \times D_q)$ to the space $(H^{1/2})^2 \times (H^1 \times L^2)$.*

The rest of this paper is organized as follows. In Section 3, we state some basic estimates for free solutions of the DKG system and we introduce "null forms" and state their properties. In Section 4, we decompose two harmful terms by the algebraic normal form transformation and we find a second approximation for ψ through the decomposition of the Klein-Gordon operator by the Dirac one. In Section 5, following paper [10], we will also change the transformed DKG system into another form in order to apply the Strichartz type estimates to the Dirac part. In Section 6, we will prove Theorem 1 by an iteration scheme based on paper [13].

3. Elementary Estimates and Null Forms

Through the paper, we write $A \simeq B$ if there exist some positive constants $C_1, C_2 > 0$ such that $C_1 B \le A \le C_2 B$, and we also write $A \lesssim B$ if there exists a positive constant $C > 0$ such that $A \le CB$.

We introduce the free evolution groups as follows:

$$\mathscr{U}_{\pm,m}(t) \equiv e^{\pm it\langle\nabla\rangle_m} = \mathscr{F}^{-1} e^{\pm it\langle\xi\rangle_m}\mathscr{F}. \tag{15}$$

Then, we have the following decomposition:

$$\mathscr{V}_D(t) = \sum_{\pm} \mathscr{U}_{\pm,M}(t)\mathscr{A}^D_{\pm}, \tag{16}$$

where

$$\mathscr{A}^D_{\pm} \equiv \frac{1}{2}\left(I \pm i\langle\nabla\rangle_M^{-1}(\alpha\cdot\nabla + iM\beta)\right) \tag{17}$$

is 0th order matrix operator. We note that for any $\mathbb{C}^2$-valued function ψ, the following equivalency is valid:

$$\left\|\mathscr{A}^D_{\pm}\psi\right\|_{H^{m,k}_p} \simeq \|\psi\|_{H^{m,k}_p}. \tag{18}$$

Now, we state $L^p - L^q$ time decay estimates through the free evolution groups $\mathscr{U}_{\pm,m}(t)$ obtained in paper [14].

Lemma 3. *Let $m \neq 0$ and $2 \le p \le \infty$. Then the estimate*

$$\left\|\mathscr{U}_{\pm,m}(t)\phi\right\|_{L^p} \lesssim t^{2/p-1}\|\phi\|_{H^{2(1-2/p)}_q} \tag{19}$$

is true for any $t > 0$, where q is a conjugate exponent of p: $1/p + 1/q = 1$.

By the lemma, we can easily get $L^p - L^q$ time decay estimates to free solutions for the DKG system.

Corollary 4. *Under the same assumption of Lemma 3 and $M > 0$, the following estimates*

$$\begin{gathered}\left\|\mathscr{V}_D(t)\psi^+\right\|_{L^p} \lesssim t^{2/p-1}\left\|\psi^+\right\|_{H^{2(1-2/p)}_q},\\ \left\|\mathscr{V}_K(t)\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{L^p} \lesssim t^{2/p-1}\left\|\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{H^{2(1-2/p)}_q}\end{gathered} \tag{20}$$

are valid for any $t > 0$, where q is a conjugate exponent of p: $1/p + 1/q = 1$.

Remark 5. Let $\kappa \in \mathbb{R}$, $M, m \neq 0$, and $2 \le p < \infty$. Then the following estimates

$$\begin{gathered}\left\|\mathscr{V}_D(t)\psi^+\right\|_{H^\kappa_p} \lesssim t^{2/p-1}\left\|\psi^+\right\|_{H^{\kappa+2-4/p,1}},\\ \left\|\mathscr{V}_K(t)\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{H^\kappa_p} \lesssim t^{2/p-1}\left\|\begin{pmatrix}\phi_1^+\\ \langle\nabla\rangle_m^{-1}\phi_2^+\end{pmatrix}\right\|_{H^{\kappa+2-4/p,1}}\end{gathered} \tag{21}$$

hold for any $t > 0$.

Next, we introduce the Strichartz estimates, which enable us to treat the problem in lower order Sobolev spaces. Denote the space-time norm

$$\|\phi\|_{L^r_t(I;L^q_x)} \equiv \left\|\|\phi(t)\|_{L^q_x}\right\|_{L^r_t(I)}, \tag{22}$$

where I is a bounded or unbounded time interval. We define the integral operator as follows:

$$\mathscr{G}_{\pm,m}[g](t) \equiv \int_T^t \mathscr{U}_{\pm,m}(t-\tau)\langle\nabla\rangle_m^{-1} g(\tau)\,d\tau \tag{23}$$

for any $T \in \bar{I}$, where $m > 0$. By the duality argument of [15] along with Lemma 3, we have the following (see also [10, 13]).

Lemma 6. *Let $2 \le q < \infty$ and $2/r = 1 - (2/q)$. Then for any time interval I, the following estimates are true:*

$$\begin{gathered}\left\|\mathscr{G}_{\pm,m}[g]\right\|_{L^r_t(I;L^q_x)} \lesssim \|g\|_{L^{r'}_t(I;H^{2\gamma-1}_{q'})},\\ \left\|\mathscr{G}_{\pm,m}[g]\right\|_{L^\infty_t(I;L^2_x)} \lesssim \|g\|_{L^{r'}_t(I;H^{\gamma-1}_{q'})},\\ \left\|\mathscr{G}_{\pm,m}[g]\right\|_{L^r_t(I;L^q_x)} \lesssim \|g\|_{L^1_t(I;H^{\gamma-1})},\end{gathered} \tag{24}$$

where $r' = r/(r-1)$, $q' = q/(q-1)$ and $\gamma = 1-(2/q)$.

Next, we introduce the Leibniz rule for fractional derivatives.

Lemma 7. *Let $\kappa > 0$, $1 < p$, $q_1, q_2 < \infty$, $1 < r_1$, $r_2 \le \infty$, and $1/p = 1/q_1 + 1/r_1 = 1/q_2 + 1/r_2$. Then the following estimate holds:*

$$\|uv\|_{H^\kappa_p} \lesssim \|u\|_{H^\kappa_{q_1}}\|v\|_{L^{r_1}} + \|v\|_{H^\kappa_{q_2}}\|u\|_{L^{r_2}}. \tag{25}$$

For the proof of (25) see, for example, [16].

We introduce the operator $\mathscr{Z} = (\mathscr{Z}_1, \mathscr{Z}_2)$, where $\mathscr{Z}_k \equiv x_k\partial_t + t\partial_k$ for $k = 1, 2$. Let $\mathscr{Z}^\alpha = \mathscr{Z}_1^{\alpha_1}\mathscr{Z}_2^{\alpha_2}$ for a multi-index $\alpha = (\alpha_1, \alpha_2) \in (\mathbb{N}\cup\{0\})^2$. We can see the commutation relations (see [6, 17]) as follows:

$$\begin{gathered}\left[\mathscr{D}_+, \mathscr{Z}_k - \left(\frac{1}{2}\right)\alpha_k\right] = \alpha_k\mathscr{D}_+,\\ \left[\partial_t^2 - \Delta + m^2, \mathscr{Z}_k\right] = 0,\end{gathered} \tag{26}$$

for $k = 1, 2$, where $[A, B] \equiv AB - BA$.

We introduce the quadratic null forms as follows:

$$\begin{gathered}\mathscr{Q}_0(f, g) \equiv (\partial_t f)(\partial_t g) - (\nabla f)\cdot(\nabla g),\\ \mathscr{Q}_{j,k}(f, g) \equiv \left(\partial_j f\right)(\partial_k g) - (\partial_k f)\left(\partial_j g\right),\end{gathered} \tag{27}$$

for $0 \le j < k \le 2$, where $\partial \equiv (\partial_0, \nabla) \equiv (i\partial_t, \partial_1, \partial_2)$. In particular, $\mathcal{Q}_{j,k}$ is called a strong null form and has an additional time decay property through the operator $\mathcal{Z}_k$, obtained in [18] (see also [8, 13, 19], etc.).

Lemma 8. *Let $j, k = 1, 2$. Then, for any smooth function f, g, the identities*

$$\begin{aligned}\mathcal{Q}_{0,j}(f,g) &= t^{-1}(\partial_0 f)(\mathcal{Z}_j g) - t^{-1}(\mathcal{Z}_j f)(\partial_0 g),\\ \mathcal{Q}_{j,k}(f,g) &= t^{-2}(\mathcal{Z}_j g)(\mathcal{Z}_k f)\\ &\quad - t^{-2}(\mathcal{Z}_j f)(\mathcal{Z}_k g) + t^{-1}(\partial_j f)(\mathcal{Z}_k g)\\ &\quad - t^{-1}(\partial_j g)(\mathcal{Z}_k f) + t^{-1}(\mathcal{Z}_j f)(\partial_k g)\\ &\quad - t^{-1}(\mathcal{Z}_j g)(\partial_k f)\end{aligned} \tag{28}$$

are valid for any $t \in \mathbb{R} \setminus \{0\}$.

4. Decomposition of Critical Terms

We study a structure of some harmful terms of (DKG). By the difference of (DKG) and the free DKG system, it follows that

$$\begin{aligned}\mathcal{D}_+(\psi - \psi_0) &= (\phi - \phi_0)\beta\psi\\ &\quad + \phi_0\beta(\psi - \psi_0) + \phi_0\beta\psi_0,\\ (\Box + m^2)(\phi - \phi_0) &= (\psi - \psi_0)^*\beta\psi\\ &\quad + \psi_0^*\beta(\psi - \psi_0) + \psi_0^*\beta\psi_0,\end{aligned} \tag{29}$$

where $\Box = \partial_t^2 - \Delta$. The last two terms $\phi_0\beta\psi_0$ and $\psi_0^*\beta\psi_0$ are critical, both of which have the worst time decay property. Especially, since

$$\phi_0\beta\psi_0, \psi_0^*\beta\psi_0 = O(t^{-1}) \quad \text{in } L^2 \text{ as } t \longrightarrow +\infty \tag{30}$$

(see Corollary 4), the L^2-norm of these terms is not integrable with respect to time t over $[1, \infty)$. Therefore, it can not be expected that usual perturbation technique is applicable to (29). To overcome this lack of time decay property, we will decompose them into an image of a Klein-Gordon operator and a remainder term following paper [8], based on papers [19–21].

Let (v_1, v_2) be a solution for the following homogeneous KG system with masses $M_1, M_2 > 0$,

$$(\Box + M_j^2)v_j = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \quad \text{for } j = 1, 2. \tag{31}$$

By the masses M_1, M_2, we introduce the symmetric matrix as follows:

$$\mathcal{M} = \mathcal{M}(M_1, M_2) = \begin{pmatrix} M_1^2 + M_2^2 & 2M_1M_2 \\ 2M_1M_2 & M_1^2 + M_2^2 \end{pmatrix}. \tag{32}$$

We have the following.

Lemma 9 (see [8]). *Let $\widetilde{m} > 0$ with $\det(\widetilde{m}^2 I - \mathcal{M}) \ne 0$. Then the quadratic term v_1v_2 can be decomposed as*

$$v_1v_2 = \frac{1}{\det(\widetilde{m}^2 I - \mathcal{M})}\left\{(\Box + \widetilde{m}^2) f - 4\mathcal{R}\right\}, \tag{33}$$

where

$$\begin{aligned}f &= f(v_1, v_2) \equiv (-M_1^2 - M_2^2 + \widetilde{m}^2)v_1v_2 - 2\mathcal{Q}_0(v_1, v_2),\\ \mathcal{R} &= \mathcal{R}(v_1, v_2) \equiv \sum_{m=1}^{2} \mathcal{Q}_{0,m}(\partial_t v_1, \partial_m v_2)\\ &\quad + \sum_{m=1}^{2} \mathcal{Q}_{0,m}(\partial_t v_2, \partial_m v_1) - \mathcal{Q}_{1,2}(\partial_1 v_1, \partial_2 v_2)\\ &\quad - \mathcal{Q}_{2,1}(\partial_2 v_1, \partial_1 v_2).\end{aligned} \tag{34}$$

Under the nonresonance mass condition $m, M > 0$, and $m \ne 2M$, we can apply Lemma 9 to the critical terms $\phi_0\beta\psi_0$ and $\psi_0^*\beta\psi_0$. Before doing so, we prepare for several notations. We put

$$\widetilde{\mathcal{M}} \equiv \frac{1}{m^2(2M + m)(m - 2M)} \tag{35}$$

which is well defined if $m, M > 0$ and $m \ne 2M$. For a real-valued function ϕ and a $\mathbb{C}^2$-valued function $\psi = (\psi_1, \psi_2)^t$, we define $\mathbb{C}^2$-valued functions of bilinear form:

$$\begin{aligned}f_D &= f_D(\phi, \psi) \equiv (f(\phi, \psi_1), f(\phi, \psi_2))^t,\\ \mathcal{R}_D &= \mathcal{R}_D(\phi, \psi) \equiv (\mathcal{R}(\phi, \psi_1), \mathcal{R}(\phi, \psi_2))^t,\\ \mathcal{Q}_0^D &= \mathcal{Q}_0^D(\phi, \psi) \equiv (\mathcal{Q}_0(\phi, \psi_1), \mathcal{Q}_0(\phi, \psi_2))^t,\end{aligned} \tag{36}$$

Moreover, for $\mathbf{C}^2$-valued functions $\varphi = (\varphi_1, \varphi_2)^t$, $\psi = (\psi_1, \psi_2)^t$, we put the following bilinear forms:

$$\begin{aligned}f_K &= f_K(\varphi^t, \psi) \equiv \sum_{j=1}^{2} f(\varphi_j, \psi_j),\\ \mathcal{R}_K &= \mathcal{R}_K(\varphi^t, \psi) \equiv \sum_{j=1}^{2} \mathcal{R}(\varphi_j, \psi_j),\\ \mathcal{Q}_0^K &= \mathcal{Q}_0^K(\varphi^t, \psi) \equiv \sum_{j=1}^{2} \mathcal{Q}_0(\varphi_j, \psi_j).\end{aligned} \tag{37}$$

We have the following.

Corollary 10. *Let $m, M > 0$, $m \ne 2M$, and (ψ_0, ϕ_0) be a free solution for the Dirac-Klein-Gordon equations. Then the quadratic terms $\phi_0\beta\psi_0$, $\psi_0^*\beta\psi_0$ can be expressed as*

$$\begin{aligned}\phi_0\beta\psi_0 &= \widetilde{\mathcal{M}}\left\{(\Box + M^2) f_D(\phi_0, \beta\psi_0) - 4\mathcal{R}_D(\phi_0, \beta\psi_0)\right\},\\ \psi_0^*\beta\psi_0 &= \widetilde{\mathcal{M}}\left\{(\Box + m^2) f_K(\psi_0^*, \beta\psi_0) - 4\mathcal{R}_K(\psi_0^*, \beta\psi_0)\right\}.\end{aligned} \tag{38}$$

Proof. We consider the Dirac part of (38). Multiplying by $\mathscr{D}_-$ both hand sides of $\mathscr{D}_+\psi_0 = 0$, we get

$$\mathscr{D}_-\mathscr{D}_+\psi_0 = \left(\square + M^2\right)\psi_0 = 0, \tag{39}$$

which implies that $\psi_0 = (\psi_{0,1}, \psi_{0,2})^t$ is also a solution of the free KG equation. Note that by the condition $m, M > 0$ and $m \neq 2M$, we can apply Lemma 9 with $\widetilde{m} = M$, $v_1 = \phi_0$, and $v_2 = \psi_{0,k}$ to get, for $k = 1, 2$,

$$\phi_0\psi_{0,k} = \widetilde{\mathscr{M}}\left\{\left(\square + M^2\right) f\left(\phi_0, \psi_{0,k}\right) - 4\mathscr{R}\left(\phi_0, \psi_{0,k}\right)\right\}. \tag{40}$$

Thus, by a simple calculation, we obtain (38). Next, note that from equality (39), we see that $\overline{\psi_0}$ satisfies the free KG equation. Thus in the same manner as the proof of the Dirac part, we can prove the KG part, which completes the proof of the corollary. □

Next, we will change the DKG equations into another form without critical nonlinearities. We introduce a new unknown function (Ψ, Φ) as follows:

$$\begin{gathered}\Psi \equiv \psi - \psi_0 - \widetilde{f}_D \equiv \widetilde{\psi} - \widetilde{f}_D, \\ \Phi \equiv \phi - \phi_0 - \widetilde{f}_K \equiv \widetilde{\phi} - \widetilde{f}_K,\end{gathered} \tag{41}$$

where (ψ_0, ϕ_0) is defined by (7) and

$$\begin{aligned}\widetilde{f}_D = \widetilde{f}_D\left(\phi_0, \psi_0\right) &\equiv \widetilde{\mathscr{M}}\mathscr{D}_- f_D\left(\phi_0, \beta\psi_0\right) \\ &= \widetilde{\mathscr{M}}\left(f_D\left(\mathscr{D}_-\phi_0, \beta\psi_0\right) - iMf_D\left(\phi_0, \psi_0\right)\right),\end{aligned} \tag{42}$$

$$\widetilde{f}_K = \widetilde{f}_K\left(\psi_0\right) \equiv \widetilde{\mathscr{M}} f_K\left(\psi_0^*, \beta\psi_0\right) \tag{43}$$

are the second approximate solution to (ψ, ϕ), where we have used the identities $\alpha_j\beta + \beta\alpha_j = O$, $\beta^2 = I$ and $\mathscr{D}_+\psi_0 = 0$ to obtain the third equality in (42).

Here, we remember that by the anticommutation relations (1) of the Dirac matrices, we can decompose the KG operator as follows:

$$\square + M^2 = \mathscr{D}_+\mathscr{D}_-. \tag{44}$$

By combining Corollary 10 and this decomposition, we can rewrite (DKG) as follows.

Lemma 11. *Let $m, M > 0$ and $m \neq 2M$. Then (ψ, ϕ) satisfies (DKG) if and only if the new variable (Ψ, Φ) defined by (41) is a solution of*

$$\begin{gathered}\mathscr{D}_+\Psi = F, \\ \left(\square + m^2\right)\Phi = G,\end{gathered} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^2, \tag{45}$$

where

$$\begin{aligned}F &= F\left(\widetilde{\phi}, \widetilde{\psi}\right) \\ &\equiv \widetilde{\phi}\beta\widetilde{\psi} + \widetilde{\phi}\beta\psi_0 + \phi_0\beta\widetilde{\psi} - 4\widetilde{\mathscr{M}}\mathscr{R}_D\left(\phi_0, \beta\psi_0\right), \\ G &= G\left(\widetilde{\psi}\right) \\ &\equiv \widetilde{\psi}^*\beta\widetilde{\psi} + \widetilde{\psi}^*\beta\psi_0 + \psi_0^*\beta\widetilde{\psi} - 4\widetilde{\mathscr{M}}\mathscr{R}_K\left(\psi_0^*, \beta\psi_0\right),\end{aligned} \tag{46}$$

and $\widetilde{\mathscr{M}}$, $\mathscr{R}_D$ and $\mathscr{R}_K$ are defined by (35), (36), and (37), respectively.

This lemma enables us to treat the Dirac-Klein-Gordon equations (DKG) as well as the reduced KG system (4) in two space dimensions.

Proof. From (29), we see that (ψ, ϕ) is a solution of (DKG) if and only if the new variable $(\widetilde{\psi}, \widetilde{\phi})$ satisfies the following DKG equations:

$$\begin{gathered}\mathscr{D}_+\widetilde{\psi} = \widetilde{\phi}\beta\widetilde{\psi} + \widetilde{\phi}\beta\psi_0 + \phi_0\beta\widetilde{\psi} + \phi_0\beta\psi_0, \\ \left(\square + m^2\right)\widetilde{\phi} = \widetilde{\psi}^*\beta\widetilde{\psi} + \widetilde{\psi}^*\beta\psi_0 + \psi_0^*\beta\widetilde{\psi} + \psi_0^*\beta\psi_0.\end{gathered} \tag{47}$$

We consider the Dirac part of (47) only, since it is easier to handle the KG part. Note that by the assumption $m, M > 0$ and $m \neq 2M$, we can apply Corollary 10 to $\phi_0\beta\psi_0$. Thus, we have

$$\phi_0\beta\psi_0 = \widetilde{\mathscr{M}}\left\{\left(\square + M^2\right) f_D\left(\phi_0, \beta\psi_0\right) - 4\mathscr{R}_D\left(\phi_0, \beta\psi_0\right)\right\}. \tag{48}$$

Moreover, by the decomposition (44), we can transform the first term of the right hand side of (48) as follows:

$$\begin{aligned}&\lambda\widetilde{\mathscr{M}}\left(\square + M^2\right) f_D\left(\phi_0, \beta\psi_0\right) \\ &\quad = \widetilde{\mathscr{M}}\mathscr{D}_+\mathscr{D}_- f_D\left(\phi_0, \beta\psi_0\right) = \mathscr{D}_+\widetilde{f}_D,\end{aligned} \tag{49}$$

where we have used the definition of $\widetilde{f}_D$ given by (42). Inserting (48) and (49) into the Dirac part of (47), we obtain the Dirac part of (45), which completes the proof of the lemma. □

Remark 12. The null structure of (DKG) was characterized in [12] by using Fourier space. On the other hand, we note that in the above argument, Fourier space does not appear at all.

5. Reduction to Some First Order System

To construct a solution for the final value problem of the DKG system, we will use the Strichartz type estimates (Lemma 6). However, it seems difficult to apply these estimates to the Dirac part for (45) due to a derivative loss difficulty. To gain first order differentiability properties of nonlinear term, we use the matrix operators

$$\begin{gathered}\mathscr{B}_\pm^D \equiv \frac{1}{2}I\left(1 \mp i\langle\nabla\rangle_M^{-1}\partial_t\right) = \mp\frac{i}{2}\langle\nabla\rangle_M^{-1}\mathscr{L}_\mp^D, \\ \mathscr{L}_\pm^D \equiv \left(\partial_t \mp i\langle\nabla\rangle_m\right) I,\end{gathered} \tag{50}$$

though we do not necessarily need the operator $\mathscr{B}$ in dealing with the initial value problem for the DKG system (see [11]). We will construct the desired solution (ψ, ϕ) for the DKG system by the iteration scheme. Let $\{(\psi^l, \phi^l)\}_{l\geq 0}$ be a sequence such that

$$\begin{gathered}\mathscr{D}_+\psi^{l+1} = \phi^l\beta\psi^l, \\ \left(\square + m^2\right)\phi^{l+1} = \left(\psi^l\right)^*\beta\psi^l, \quad l \geq 0, \\ \left(\psi^0, \phi^0\right) = \left(\psi_0, \phi_0\right),\end{gathered} \tag{51}$$

under the final conditions

$$\lim_{t\to\infty}\left\|\psi^l(t)-\psi_0(t)\right\|_{H^{1/2}}=0, \tag{52}$$

$$\lim_{t\to\infty}\left\|\begin{pmatrix}\phi^l(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi^l(t)\end{pmatrix}-\begin{pmatrix}\phi_0(t)\\ \langle\nabla\rangle_m^{-1}\partial_t\phi_0(t)\end{pmatrix}\right\|_{H^1}=0, \tag{53}$$

for $l\geq 0$, where (ψ_0,ϕ_0) is given by (7). It suffices to prove that the sequence $\{\psi^l,(\langle\nabla\rangle_m^{1/2}\phi^l,\langle\nabla\rangle_m^{-1/2}\partial_t\phi^l)\}_{l\geq 0}$ is a Cauchy one in the Banach space $(C([T,\infty);H^{1/2}))^4$ for some $T>0$.

As the previous section, we introduce the new sequence $\{(\Psi^l,\Phi^l)\}$ as follows:

$$\begin{aligned}\Psi^l&\equiv\psi^l-\psi_0-\tilde{f}_D\equiv\tilde{\psi}^l-\tilde{f}_D,\\ \Phi^l&\equiv\phi^l-\phi_0-\tilde{f}_K\equiv\tilde{\phi}^l-\tilde{f}_K.\end{aligned} \tag{54}$$

By Lemma 11, the sequence $\{(\psi^l,\phi^l)\}$ is a solution of (51) if and only if the new one $\{(\Psi^l,\Phi^l)\}$ satisfies the transformed DKG equations as follows:

$$\begin{aligned}\mathscr{D}_+\Psi^{l+1}&=F^l,\\ \left(\Box+m^2\right)\Phi^{l+1}&=G^l,\end{aligned}\qquad l\geq 1, \tag{55}$$

$$\left(\Psi^0,\Phi^0\right)=-\left(\tilde{f}_D,\tilde{f}_K\right), \tag{56}$$

where

$$F^l\equiv F\left(\tilde{\phi}^l,\tilde{\psi}^l\right),\qquad G^l\equiv G\left(\tilde{\psi}^l\right), \tag{57}$$

for $l\geq 0$ ($\tilde{f}_D$, $\tilde{f}_K$, and F and G are defined by (42)-(43) and (46), resp.).

By the decomposition of the Klein-Gordon operator by the Dirac operator, we have

$$\mathscr{L}_\pm^D\mathscr{B}_\pm^D=\mp\frac{i}{2}\langle\nabla\rangle_M^{-1}I\left(\partial_t^2+\langle\nabla\rangle_M^2\right)=\mp\frac{i}{2}\langle\nabla\rangle_M^{-1}\mathscr{D}_-\mathscr{D}_+. \tag{58}$$

Thus, from the Dirac part for (55), we can deduce the following:

$$\mathscr{L}_\pm^D\mathscr{B}_\pm^D\Psi^{l+1}=\mp\frac{i}{2}\langle\nabla\rangle_M^{-1}\mathscr{D}_-\mathscr{D}_+\Psi^{l+1}=\langle\nabla\rangle_M^{-1}F_\pm^l, \tag{59}$$

for $l\geq 0$, where $F_\pm^l\equiv\mp(i/2)\mathscr{D}_-F^l$. Therefore, from Dirac part of (55), we have

$$\begin{aligned}\mathscr{L}_\pm^D\mathscr{B}_\pm^D\Psi^{l+1}&=\langle\nabla\rangle_M^{-1}F_\pm^l,\quad l\geq 0,\\ \mathscr{B}_\pm^D\Psi^0&=-\mathscr{B}_\pm^D\tilde{f}_D.\end{aligned} \tag{60}$$

Remark 13. By properties (1) of the Dirac matrices, we can transform $F_\pm^l$ into another form without any derivatives of $\tilde{\psi}$ or the free solution ψ_0 (see (78)-(79), precisely). This fact enables us to use the Strichartz estimates for (60).

Next we will also transform the KG part of (55) as in [10, 13]. We also use the operator (1-component version of the Dirac part) as follows:

$$\mathscr{B}_\pm^K\equiv\frac{1}{2}\left(1\mp i\langle\nabla\rangle_m^{-1}\partial_t\right),\qquad\mathscr{L}_\pm^K\equiv\partial_t\mp i\langle\nabla\rangle_m. \tag{61}$$

We can see that the sequence $\{\Phi^l\}$ is a solution of the KG part for (55) if and only if the sequence $\{\mathscr{B}_\pm^K\Phi^l\}$ satisfies

$$\begin{aligned}\mathscr{L}_\pm^K\mathscr{B}_\pm^K\Phi^{l+1}&=\langle\nabla\rangle_m^{-1}G_\pm^l,\quad\text{for } l\geq 0,\\ \mathscr{B}_\pm^K\Phi^0&=-\mathscr{B}_\pm^K\tilde{f}_K,\end{aligned} \tag{62}$$

where $G_\pm^l\equiv G_\pm^l(\tilde{\psi}^l)\equiv\mp(i/2)G^l$.

Therefore, by (60) and (62), we get

$$\begin{aligned}\mathscr{L}_\pm^D\mathscr{B}_\pm^D\Psi^{l+1}&=\langle\nabla\rangle_M^{-1}F_\pm^l,\\ \mathscr{L}_\pm^K\mathscr{B}_\pm^K\Phi^{l+1}&=\langle\nabla\rangle_m^{-1}G_\pm^l,\end{aligned}\qquad\text{for } l\geq 0, \tag{63}$$

$$\left(\mathscr{B}_\pm^D\Psi^0,\mathscr{B}_\pm^K\Phi^0\right)=-\left(\mathscr{B}_\pm^D\tilde{f}_D,\mathscr{B}_\pm^K\tilde{f}_K\right). \tag{64}$$

Remark 14. The identity $\sum_\pm\mathscr{B}_\pm^*=I$ holds, which enables us to reconstruct a solution (Ψ,Φ) for (45) from $(\mathscr{B}_\pm^D\Psi,\mathscr{B}_\pm^K\Phi)$.

Inserting the identities

$$\tilde{\psi}^l=\sum_\pm\mathscr{B}_\pm^D\Psi^l+\tilde{f}_D,\qquad\tilde{\phi}^l=\sum_\pm\mathscr{B}_\pm^K\Phi^l+\tilde{f}_K, \tag{65}$$

into the nonlinearities $F_\pm^l$, $G_\pm^l$, we can express (63) by the new variable $(\mathscr{B}_\pm^D\Psi^l,\mathscr{B}_\pm^K\Phi^l)$ only without $(\tilde{\phi}^l,\tilde{\psi}^l)$.

At the end of this section, we will lead the integral equations associated with (63). We introduce a new unknown function sequence $\{v^l\}$ whose components are defined by

$$v^l\equiv\left(\mathscr{B}_+^D\Psi^l,\mathscr{B}_-^D\Psi^l,\langle\nabla\rangle_m^{1/2}\mathscr{B}_+^K\Phi^l,\langle\nabla\rangle_m^{1/2}\mathscr{B}_-^K\Phi^l\right)^t, \tag{66}$$

a nonlinear term

$$\mathscr{N}=\mathscr{N}\left(v^l\right)\equiv\left(\langle\nabla\rangle_M^{-1}F_+^l,\langle\nabla\rangle_M^{-1}F_-^l,\langle\nabla\rangle_m^{-1/2}G_+^l,\langle\nabla\rangle_m^{-1/2}G_-^l\right)^t \tag{67}$$

for $l\geq 0$, and a matrix-operator $\mathscr{L}\equiv\operatorname{diag}(\mathscr{L}_+^D,\mathscr{L}_-^D,\mathscr{L}_+^K,\mathscr{L}_-^K)$. Then by using these notations, (63) can be simplified as

$$\mathscr{L}v^{l+1}=\mathscr{N}\left(v^l\right)\quad\text{for } l\geq 0. \tag{68}$$

To lead the integral equations for (68), we need to study the asymptotic behavior of the new variable v^l. We can obtain the following.

Lemma 15. *Let $(\psi^+,(\langle\nabla\rangle\phi_1^+,\phi_2^+))\in(H^{5/2,1})^4$. The function (ψ^l,ϕ^l) defined by (51) satisfies (52)-(53) for any $l\geq 0$ if and only if the new function v^l satisfies (68) and*

$$\lim_{t\to\infty}\left\|v^l\right\|_{H^{1/2}}=0,\quad\textit{for } l\geq 0. \tag{69}$$

The proof of the lemma will be given in Appendix.

We introduce a matrix evolution operator as follows:

$$\mathscr{U}(t)\equiv\operatorname{diag}\left(\mathscr{U}_{+,M}(t),\mathscr{U}_{-,M}(t),\mathscr{U}_{+,m}(t),\mathscr{U}_{-,m}(t)\right). \tag{70}$$

From Lemma 15, we can lead the integral equations associated with (68) as follows:

$$v^{l+1}(t)=-\int_t^\infty\mathscr{U}(t-s)\mathscr{N}\left(v^l\right)ds. \tag{71}$$

6. Proof of Theorem 1

In this section, we give a proof of Theorem 1. Note that the identities

$$\begin{aligned} \partial_t \Psi^l &= i\langle\nabla\rangle_M \left(v_1^l - v_2^l\right), \\ \partial_t \Phi^l &= i\langle\nabla\rangle_m^{1/2} \left(v_3^l - v_4^l\right) \end{aligned} \tag{72}$$

hold; the nonlinearity $\mathcal{N}(v^l)$ can be expressed in terms of the space derivatives of v^l (so excluding the time derivatives).

For $T > 1$, where T is sufficiently large, we introduce the following function space:

$$\mathbf{X}_T = \left\{ v \in \left(C\left([T,\infty); H^{1/2} \right) \right)^6 ; \|v\|_{\mathbf{X}_T} < \infty \right\}, \tag{73}$$

with the norm

$$\|v\|_{\mathbf{X}_T} \equiv \sup_{t\in[T,\infty)} t^{\mu} \left(\|v\|_{L_t^4(I;L^4)} + \|v\|_{L_t^\infty(I;H^{1/2})} \right), \tag{74}$$

where $1/2 < \mu < 1 - 2/q$, $4 < q \le \infty$, and $I = [t,\infty)$. We define

$$A \equiv C\left\| \left(\psi^+, \left(\langle\nabla\rangle \phi_1^+, \phi_2^+ \right) \right) \right\|_{H^{4-4/q}_{q/(q-1)} \cap H^{5/2,1}}. \tag{75}$$

In order to obtain the theorem, we will show that the sequence $\{v^l\}$ is a Cauchy one in a closed ball $\mathbf{X}_{T,A}$ for appropriate T and ρ, where $\mathbf{X}_{T,A} \equiv \{v \in \mathbf{X}_T; \|v\|_{\mathbf{X}_T} \le A\}$.

Hereafter, we will use the notation $L_t^r X = L_t^r(I;X)$, $\mathcal{D} = \mathcal{D}_-$ and

$$\mathcal{B}\Psi = \mathcal{B}_\pm^D \Psi, \qquad \mathcal{B}\Phi = \mathcal{B}_\pm^K \Phi, \tag{76}$$

for simplicity if it does not cause a confusion.

Proof. We will prove that $v^l \in \mathbf{X}_{T,A}$ for any $l \ge 0$ by induction. In the case of $l = 0$, it is easy to see that $v^0 \in \mathbf{X}_{T,A}$ for some T and ρ. We omit the details. For $l \ge 1$, we assume that $v^k \in \mathbf{X}_{T,A}$ for $0 \le k \le l$. We will show that $v^{l+1} \in \mathbf{X}_{T,A}$ for some T and ρ.

First, by the identities $\mathcal{D}_+\psi_0 = 0$ and $\mathcal{D}_+\psi^l = \lambda\phi^{l-1}\beta\psi^{l-1}$ for $l \ge 1$, we get, for $l \ge 1$,

$$\begin{aligned} \mathcal{D}_-\left(\widetilde{\phi}^l \beta \widetilde{\psi}^l\right) &= \left(\mathcal{D}_-\widetilde{\phi}^l\right)\beta\widetilde{\psi}^l \\ &\quad - iM\widetilde{\phi}^l I \widetilde{\psi}^l + \lambda\widetilde{\phi}^l\widetilde{\phi}^{l-1} I \widetilde{\psi}^{l-1} \\ &\quad + \lambda\widetilde{\phi}^l\widetilde{\phi}^{l-1} I \psi_0 + \lambda\widetilde{\phi}^l \phi_0 I \widetilde{\psi}^{l-1} \\ &\quad + \lambda\widetilde{\phi}^l \phi_0 I \psi_0, \\ \mathcal{D}_-\mathcal{R}_D\left(\phi_0, \beta\psi_0\right) &= \mathcal{R}_D\left(\mathcal{D}_-\phi_0, \beta\psi_0\right) - iM\mathcal{R}_D\left(\phi_0, \beta\psi_0\right). \end{aligned} \tag{77}$$

From these identities, we can express $F_\pm^l$ as follows:

$$F_\pm^l = \mp\frac{i}{2}\sum_{j=1}^{3} F_j^l + \text{“remainder”} \quad \text{for } l \ge 1, \tag{78}$$

where

$$F_1^l \equiv \left(\mathcal{D}_-\widetilde{\phi}^l\right)\beta\widetilde{\psi}^l, \qquad F_2^l \equiv \left(\mathcal{D}_-\phi_0\right)\beta\widetilde{\psi}^l + \left(\mathcal{D}_-\widetilde{\phi}^l\right)\beta\psi_0, \tag{79}$$

$$F_3^l \equiv 4i\widetilde{\mathcal{M}}\mathcal{R}_D\left(\mathcal{D}_-\phi_0, \beta\psi_0\right). \tag{80}$$

Here, we note that “remainder” (given by (78)) can be handled in the same manner as F_j^l ($j = 1, 2,$ or 3). Thus, we will omit the estimate of them. We also decompose $G_\pm^l$ as $G_\pm^l = \mp(i/2)\sum_{j=1}^3 G_j^l$, where

$$\begin{aligned} G_1^l &= \left(\widetilde{\psi}^l\right)^*\beta\widetilde{\psi}^l, \qquad G_2^l = \left(\widetilde{\psi}^l\right)^*\beta\psi_0 + \psi_0^*\beta\widetilde{\psi}^l, \\ G_3^l &= 4i\widetilde{\mathcal{M}}\mathcal{R}_K\left(\psi_0^*, \beta\psi_0\right). \end{aligned} \tag{81}$$

Taking $L_t^4L_x^4$-norm and $L_t^\infty H^{1/2}$-norm of (71) and applying Lemma 6 with $(q, r, \gamma) = (4, 4, 1/2)$ and $(2, \infty, 0)$, we have

$$\begin{aligned} &\left\|v^{l+1}\right\|_{L_t^4L_x^4} + \left\|v^{l+1}\right\|_{L_t^\infty H^{1/2}} \\ &\quad \lesssim \left\|F_1^l\right\|_{L_t^{4/3}L_x^{4/3}} + \left\|G_1^l\right\|_{L_t^{4/3}H^{1/2}_{4/3}} \\ &\qquad + \sum_{j=2,3}\left(\left\|F_j^l\right\|_{L_t^1H^{-1/2}} + \left\|G_j^l\right\|_{L_t^1L_x^2}\right). \end{aligned} \tag{82}$$

Moreover, we remember that $(\widetilde{\phi}^l, \widetilde{\psi}^l)$ is expressed as (65).

Now, we will estimate F_1^l. By the Hölder inequality, we have

$$\begin{aligned} \left\|\left(\mathcal{D}\mathcal{B}\Phi^l\right)\mathcal{B}\Psi^l\right\|_{L_t^{4/3}L_x^{4/3}} &\lesssim \left\|\left\|\mathcal{B}\Phi^l(s)\right\|_{H^1}\left\|\mathcal{B}\Psi^l(s)\right\|_{L_x^4}\right\|_{L_t^{4/3}} \\ &\lesssim A\left\|s^{-\mu}\left\|\mathcal{B}\Psi^l(s)\right\|_{L_x^4}\right\|_{L_t^{4/3}} \\ &\lesssim A\left\|\mathcal{B}\Psi^l\right\|_{L_t^4L_x^4}\left\|s^{-\mu}\right\|_{L_t^2(I)} \\ &\lesssim A^2t^{1/2-2\mu}, \end{aligned} \tag{83}$$

for any $t \ge T$ since $v^k \in \mathbf{X}_{T,A}$ for $0 \le k \le l$. By the Hölder inequality and Remark 5 with $p = 8$, we obtain

$$\left\|\widetilde{f}_D(s)\right\|_{L^4} \lesssim \left\|\phi_0(s)\right\|_{H_8^2}\left\|\psi_0(s)\right\|_{H_8^1} \lesssim A^2s^{-3/2}, \tag{84}$$

for any $s \ge t$. In the same manner as the proof of the estimate (83), we also obtain

$$\left\|\left(\mathcal{D}\mathcal{B}\Phi^l\right)\widetilde{f}_D\right\|_{L_t^{4/3}L_x^{4/3}} \lesssim A^3t^{-3/4-\mu}, \tag{85}$$

for all $t \ge T$, due to $v^k \in \mathbf{X}_{T,A}$ for $0 \le k \le l$ and (84). By the Hölder inequality and Remark 5 with $p = 8/3, 8$, we obtain

$$\begin{aligned} \left\|\mathcal{D}\widetilde{f}_K(s)\right\|_{L^2} &\lesssim \left\|\psi_0(s)\right\|_{H^2_{8/3}}\left\|\psi_0(s)\right\|_{H^1_8} \\ &\lesssim s^{-1}\left\|\psi^+\right\|_{H^{5/2}_{8/5}}\left\|\psi^+\right\|_{H^{5/2}_{8/7}} \lesssim A^2s^{-1} \end{aligned} \tag{86}$$

for any $s \geq t$, where we have used properties (1) of α, β, and $\mathscr{D}_+\psi_0 = 0$. Thus, in the same manner as the proof of the estimate (83), we obtain

$$\left\|\left(\mathscr{D}\tilde{f}_K\right)\mathscr{B}\Psi^l\right\|_{L_t^{4/3}L_x^{4/3}} \lesssim A^3 t^{-1/2-\mu}, \tag{87}$$

for all $t \geq T$ due to $v^l \in \mathbf{X}_{T,A}$ and (86). By the Hölder inequality and estimates (84) and (86), we get

$$\begin{aligned}&\left\|\left(\mathscr{D}\tilde{f}_K\right)\tilde{f}_D\right\|_{L_t^{4/3}L_x^{4/3}}\\&\quad\lesssim\left\|\left\|\mathscr{D}\tilde{f}_K(s)\right\|_{L_x^2}\left\|\tilde{f}_D(s)\right\|_{L_x^4}\right\|_{L_t^{4/3}(I)}\lesssim A^4t^{-7/4},\end{aligned} \tag{88}$$

for all $t \geq T$. Thus by combining (83), (85), and (87)-(88), we obtain

$$\left\|F_1^l\right\|_{L_t^{4/3}L_x^{4/3}} \lesssim A^2 t^{1/2-2\mu}, \tag{89}$$

for $t \geq T \geq 1$ since $\mu < 1$. Next, we consider F_2^l. We have

$$\left\|F_2^l\right\|_{L_t^1H^{-1/2}} \leq \left\|\left(\mathscr{D}\tilde{\phi}^l\right)\psi_0\right\|_{L_t^1L_x^2} + \left\|\left(\mathscr{D}\phi_0\right)\tilde{\psi}^l\right\|_{L_t^1L_x^2}. \tag{90}$$

By Corollary 4 with $p = \infty$, we have

$$\begin{aligned}&\left\|\left(\mathscr{D}\mathscr{B}\Phi^l\right)\psi_0\right\|_{L_t^1L_x^2}\\&\quad\lesssim\left\|\left\|\mathscr{B}\Phi^l(s)\right\|_{H^1}\left\|\psi_0(s)\right\|_{L_x^\infty}\right\|_{L_t^1}\lesssim\rho A t^{-\mu},\end{aligned} \tag{91}$$

for all $t \geq T$ since $v^k \in \mathbf{X}_{T,A}$ for $0 \leq k \leq l$. In the same manner as the estimate (91), we get

$$\begin{aligned}\left\|\left(\mathscr{D}\tilde{f}_K\right)\psi_0\right\|_{L_t^1L_x^2}&\lesssim\left\|\left\|\mathscr{D}\tilde{f}_K(s)\right\|_{L_x^2}\left\|\psi_0(s)\right\|_{L_x^\infty}\right\|_{L_t^1}\\&\leq\rho A^2t^{-1},\end{aligned} \tag{92}$$

for any $t \geq T$, where we have used the estimate (86). Moreover, we also have

$$\begin{aligned}\left\|\left(\mathscr{D}\phi_0\right)\mathscr{B}\Psi^l\right\|_{L_t^1L_x^2}&\lesssim\left\|\left\|\phi_0(s)\right\|_{H_\infty^1}\left\|\mathscr{B}\Psi^l(s)\right\|_{L_x^2}\right\|_{L_t^1}\\&\leq A\rho t^{-\mu},\end{aligned} \tag{93}$$

for all $t \geq T$ since $v^l \in \mathbf{X}_{T,A}$. In the same proof as the estimate (84), by the Hölder inequality and Remark 5 with $p = 4$, we get

$$\left\|\tilde{f}_D(s)\right\|_{L^2} \lesssim \left\|\phi_0(s)\right\|_{H_4^2}\left\|\psi_0(s)\right\|_{H_4^1} \lesssim A^2 s^{-1}, \tag{94}$$

for any $s \geq t$. By estimate (94) and Corollary 4 with $p = \infty$, we have

$$\left\|\left(\mathscr{D}\phi_0\right)\tilde{f}_D\right\|_{L_t^1L_x^2} \lesssim \left\|\left\|\phi_0(s)\right\|_{H_\infty^1}\left\|\tilde{f}_D(s)\right\|_{L_x^2}\right\|_{L_t^1} \lesssim \rho A^2 t^{-1}, \tag{95}$$

for all $t \geq T$. Therefore, by combining estimates (90)–(93) and (95), we obtain

$$\left\|F_2^l\right\|_{L_t^1H^{-1/2}} \lesssim \rho A t^{-\mu}, \tag{96}$$

for any $t \geq T \geq 1$ since $\mu < 1$. Next, we consider F_3^l. By the definition of $\mathscr{R}_D$, we have

$$\left\|\mathscr{R}_D\left(\mathscr{D}\phi_0,\psi_0\right)\right\|_{L_t^1H^{-1/2}} \lesssim \sum_{j=1,2}\left\|\mathscr{R}\left(\mathscr{D}\phi_0,\psi_{0,j}\right)\right\|_{L_t^1L_x^2}, \tag{97}$$

where we put $\psi_0 = (\psi_{0,1},\psi_{0,2})^t$. By Lemma 8, we can express $\mathscr{R}$ as

$$\mathscr{R}\left(\mathscr{D}\phi_0,\psi_{0,j}\right) \equiv s^{-1}Z_1 + s^{-2}Z_2, \tag{98}$$

for $s \in \mathbb{R} \setminus \{0\}$, where

$$\begin{aligned}Z_1 &\equiv \left(\partial_0\partial_t\mathscr{D}\phi_0\right)\left(\mathscr{Z}_1\partial_1\psi_{0,j}\right)\\&\quad-\left(\mathscr{Z}_1\partial_t\mathscr{D}\phi_0\right)\left(\partial_0\partial_1\psi_{0,j}\right)+\text{similar},\\Z_2 &\equiv -\left(\mathscr{Z}_1\partial_2\psi_{0,j}\right)\left(\mathscr{Z}_2\partial_1\mathscr{D}\phi_0\right)\\&\quad+\left(\mathscr{Z}_1\partial_1\mathscr{D}\phi_0\right)\left(\mathscr{Z}_2\partial_2\psi_{0,j}\right)+\text{similar}.\end{aligned} \tag{99}$$

By applying the Hölder inequality, we have

$$\begin{aligned}\left\|s^{-1}Z_1\right\|_{L_t^1L_x^2} \lesssim \int_t^\infty s^{-1}&\Big(\left\|\phi_0\right\|_{H_q^3}\left\|\mathscr{Z}\psi_0\right\|_{H_{2q/(q-2)}^1}\\&+\left\|\psi_0\right\|_{H_q^2}\left\|\mathscr{Z}\phi_0\right\|_{H_{2q/(q-2)}^2}\Big)ds.\end{aligned} \tag{100}$$

By Corollary 4 with $p = q$, we get

$$\begin{aligned}\left\|\phi_0(s)\right\|_{H_q^3} &\lesssim s^{-1+2/q}\left\|\left(\langle\nabla\rangle\phi_1^+,\phi_2^+\right)\right\|_{H_{q/(q-1)}^{4-4/q}} \lesssim As^{-1+2/q},\\\left\|\psi_0(s)\right\|_{H_q^2} &\lesssim s^{-1+2/q}\left\|\psi^+\right\|_{H_{q/(q-1)}^{4-4/q}} \lesssim As^{-1+2/q},\end{aligned} \tag{101}$$

for any $s \geq t$. On the other hand, note that the commutation relations (26) hold. By applying the Sobolev inequality and the charge and energy conservation laws, we obtain

$$\begin{aligned}\left\|\mathscr{Z}\psi_0\right\|_{H_{2q/(q-2)}^1} &\lesssim \left\|\mathscr{Z}\psi_0\right\|_{H^{1+2/q}}\\&\lesssim\left\|\mathscr{Z}\psi_0\right\|_{H^{3/2}}\lesssim\left\|\left(\mathscr{Z}\psi_0\right)(0)\right\|_{H^{3/2}}\lesssim A,\\\left\|\mathscr{Z}\phi_0\right\|_{H_{2q/(q-2)}^2} &\lesssim \left\|\mathscr{Z}\phi_0\right\|_{H^{2+2/q}}\\&\lesssim\left\|\mathscr{Z}\phi_0\right\|_{H^{5/2}}\lesssim\left\|\left(\mathscr{Z}\phi_0\right)(0)\right\|_{H^{5/2}}\lesssim A,\end{aligned} \tag{102}$$

since $q > 4$. Thus, by combining (100)–(102), we get

$$\left\|s^{-1}Z_1\right\|_{L_t^1L_x^2} \lesssim A^2 t^{-1+2/q}, \tag{103}$$

for any $t \geq T$. By the Hölder inequality, we have

$$\begin{aligned}\left\|s^{-2}Z_2\right\|_{L_t^1L_x^2} &\lesssim \int_t^\infty s^{-2}\left\|\mathscr{Z}\psi_0(s)\right\|_{H_4^1}\left\|\mathscr{Z}\phi_0(s)\right\|_{H_4^2}ds\\&\lesssim A^2t^{-1},\end{aligned} \tag{104}$$

since in the same manner as the proof of estimates (102), we obtain

$$\left\|\mathscr{Z}\psi_0(s)\right\|_{H_4^1} + \left\|\mathscr{Z}\phi_0(s)\right\|_{H_4^2} \lesssim A, \tag{105}$$

for any $s \geq t$. Therefore, combining (97)-(98), (103), and (104), we have

$$\left\|F_3^l\right\|_{L_t^1 H^{-1/2}} \lesssim A^2 t^{-1+2/q}, \tag{106}$$

for all $t \geq T \geq 1$ since $q > 4$.

Next, we will estimate G_1^l. By the Leibniz formula (25) with $\kappa = 1/2$, $p = 4/3$, $q_1 = q_2 = 2$, and $r_1 = r_2 = 4$ and the Hölder inequality, we obtain

$$\begin{aligned} \left\|\left(\mathscr{B}\Psi^l\right)^* \mathscr{B}\Psi^l\right\|_{L_t^{4/3} H_{4/3}^{1/2}} &\lesssim \left\|\left\|\mathscr{B}\Psi^l(s)\right\|_{H^{1/2}}\left\|\mathscr{B}\Psi^l(s)\right\|_{L_x^4}\right\|_{L_t^{4/3}} \\ &\lesssim A\left\|s^{-\mu}\left\|\mathscr{B}\Psi^l(s)\right\|_{L_x^4}\right\|_{L_t^{4/3}} \\ &\lesssim A\left\|s^{-\mu}\right\|_{L_t^2(I)}\left\|\mathscr{B}\Psi^l\right\|_{L_t^4 L_x^4} \\ &\lesssim A^2 t^{1/2-2\mu}, \end{aligned} \tag{107}$$

for any $t \geq T$ since $v^l \in \mathbf{X}_{T,A}$. By the fractional Leibniz rule (25) again and Remark 5 with $p = 4$, we have

$$\left\|\tilde{f}_D(s)\right\|_{H^{1/2}} \lesssim \left\|\phi_0(s)\right\|_{H_4^{5/2}}\left\|\psi_0(s)\right\|_{H_4^{3/2}} \lesssim A^2 s^{-3/2}, \tag{108}$$

for any $s \geq t$. In the same manner as the proof of the estimate (107), we obtain

$$\begin{aligned} \left\|\left(\mathscr{B}\Psi^l\right)^* \tilde{f}_D\right\|_{L_t^{4/3} H_{4/3}^{1/2}} &\lesssim \left\|\left\|\mathscr{B}\Psi^l(s)\right\|_{H^{1/2}}\left\|\tilde{f}_D(s)\right\|_{H^{1/2}}\right\|_{L_t^{4/3}} \\ &\lesssim A^3 t^{-3/4-\mu}, \end{aligned} \tag{109}$$

for any $t \geq T$ due to $v^l \in \mathbf{X}_{T,A}$ and (108). In the same manner as the proof of the estimate (109), we get

$$\left\|\left(\tilde{f}_D\right)^* \tilde{f}_D\right\|_{L_t^{4/3} H_{4/3}^{1/2}} \lesssim A^4 t^{-7/4}, \tag{110}$$

for all $t \geq T$. Thus, by combining the estimates (107) and (109)-(110), we obtain

$$\left\|G_1^l\right\|_{L_t^{4/3} H_{4/3}^{1/2}} \lesssim A^2 t^{1/2-2\mu}, \tag{111}$$

for $t \geq T \geq 1$ since $\mu < 1$. In the same manner as the proof of the estimates (96) and (106), we obtain

$$\left\|G_2^l\right\|_{L_t^1 L_x^2} \lesssim \rho A t^{-\mu}, \qquad \left\|G_3^l\right\|_{L_t^1 L_x^2} \lesssim A^2 t^{-1+2/q}, \tag{112}$$

for any $t \geq T$. Finally, by combining (82), (89), (96), (106), and (111)-(112), we obtain

$$\left\|v^{l+1}\right\|_{\mathbf{X}_T} \lesssim A\left(AT^{1/2-\mu} + \rho + AT^{-1+\mu+2/q}\right), \tag{113}$$

for $T \geq 1$. By the estimate (113) and $1/2 < \mu < 1 - 2/q$, there exist a large $T > 0$ and a small $\rho > 0$ such that $v^{l+1} \in \mathbf{X}_{T,A}$. In the same manner as the proof of (113), we can prove the estimate

$$\left\|v^{l+1} - v^l\right\|_{\mathbf{X}_T} \leq \frac{1}{2}\left\|v^l - v^{l-1}\right\|_{\mathbf{X}_T}, \tag{114}$$

for $l \geq 1$ if $T > 1$ is sufficiently large and $\rho > 0$ is sufficiently small, which implies that $\{v^l\}_{l\geq 0}$ is a Cauchy sequence in $\mathbf{X}_{T,A}$. Theorem 1 is proved. □

Appendix

In this section, we give a proof of Lemma 15. First, we prepare the following.

Lemma 16 (see [10]). *Let $\kappa \in \mathbb{R}$ and let $\psi^+ = \psi^+(x)$ be a $\mathbb{C}^2$-valued given function. Then, for any $\mathbf{C}^2$-valued function $\psi = \psi(t,x)$, the equivalency*

$$\left\|\psi(t) - \mathscr{V}_D(t)\psi^+\right\|_{H^\kappa} \simeq \sum_{\pm}\left\|\mathscr{A}_\pm^D \psi(t) - \mathscr{U}_{\pm,M}(t)\mathscr{A}_\pm^D \psi^+\right\|_{H^\kappa} \tag{A.1}$$

holds for all $t \in \mathbb{R}$.

For the proof of the lemma, see [10].

By the lemma and a decay property of $\tilde{f}_D$ given by (42), we also have the following.

Corollary 17. *Let $(\psi^+, (\langle\nabla\rangle\phi_1^+, \phi_2^+)) \in (H^{5/2,1})^4$. The final state condition* (8) *with $\mathbf{X} = H^{1/2}$ holds if and only if the identity*

$$\lim_{t\to\infty}\sum_{\pm}\left\|\mathscr{A}_\pm^D \Psi(t)\right\|_{H^\kappa} = 0 \tag{A.2}$$

is valid, where Ψ is defined by (41).

Before proving the corollary, we remember some properties of the operators $\mathscr{A}_\pm^D$ given by (17) (see [10] in detail). We note that the identity

$$(\alpha\cdot\nabla + iM\beta)^2 = -\langle\nabla\rangle_M^2 I \tag{A.3}$$

holds due to properties (1) of Dirac matrices. Hence, by a direct calculation, we get the following identities:

$$\mathscr{A}_\pm^D \mathscr{A}_\mp^D = O, \qquad \sum_{\pm}\mathscr{A}_\pm^D = I, \qquad \left(\mathscr{A}_\pm^D\right)^2 = \mathscr{A}_\pm^D. \tag{A.4}$$

We put $B = \|(\psi^+, (\langle\nabla\rangle\phi_1^+, \phi_2^+))\|_{H^{5/2,1}}$.

Proof. By Lemma 16, we see that (8) with $\mathbf{X} = H^{1/2}$ is equivalent to

$$\lim_{t\to\infty}\sum_{\pm}\left\|\mathscr{A}_\pm^D \psi(t) - \mathscr{U}_{\pm,M}(t)\mathscr{A}_\pm^D \psi^+\right\|_{H^{1/2}} = 0. \tag{A.5}$$

By decomposition (16) and identities (A.4), we have

$$\left\|\mathscr{A}_\pm^D \Psi(t)\right\|_{H^{1/2}} = \left\|\mathscr{A}_\pm^D \psi(t) - \mathscr{U}_{\pm,M}(t)\mathscr{A}_\pm^D \psi^+ - \mathscr{A}_\pm^D \tilde{f}_D\right\|_{H^{1/2}}. \tag{A.6}$$

By estimate (18), the fractional Leibniz rule (25) with $p = 2$ and $q_i = r_i = 4$ $(i = 1, 2)$, and Remark 5 with $p = 4$, we get

$$\left\|\mathscr{A}_\pm^D \tilde{f}_D\right\|_{H^{1/2}} \lesssim \left\|\phi_0\right\|_{H_4^{5/2}}\left\|\psi_0\right\|_{H_4^{3/2}} \lesssim t^{-1}B^2, \tag{A.7}$$

for all $t > 0$, which completes the proof of the corollary. □

Next we will prove Lemma 15.

Proof of Lemma 15. First we prove the Dirac part. By Corollary 17, we see that (52) is equivalent to

$$\lim_{t\to\infty}\sum_{\pm}\left\|\mathscr{A}_{\pm}^{D}\Psi^{l}(t)\right\|_{H^{1/2}}=0\quad\text{for } l\geq 0. \tag{A.8}$$

Note that the identity

$$\mathscr{A}_{\pm}^{D}-\mathscr{B}_{\pm}^{D}=\pm\frac{i}{2}\langle\nabla\rangle_{M}^{-1}\mathscr{D}_{+} \tag{A.9}$$

holds. From the Dirac part of (55), we have

$$\mathscr{B}_{\pm}^{D}\Psi^{l+1}=\mathscr{A}_{\pm}^{D}\Psi^{l+1}-\langle\nabla\rangle_{M}^{-1}F^{l}\quad\text{for } l\geq 0. \tag{A.10}$$

Thus, it is sufficient to show that

$$\lim_{t\to\infty}\left\|F^{l}\right\|_{H^{-1/2}}=0\quad\text{for } l\geq 0. \tag{A.11}$$

By the Sobolev inequality and the Hölder inequality, we have, for $l\geq 1$,

$$\begin{aligned}\left\|F^{l}\right\|_{H^{-1/2}}&\lesssim\left\|\widetilde{\phi}^{l}\right\|_{H^{1/2}}\left\|\widetilde{\psi}^{l}\right\|_{H^{1/2}}+\left\|\widetilde{\phi}^{l}\right\|_{H^{1/2}}\left\|\psi^{+}\right\|_{H^{1/2}}\\&\quad+\left(\left\|\phi_{1}^{+}\right\|_{H^{1/2}}+\left\|\phi_{2}^{+}\right\|_{H^{-1/2}}\right)\left\|\widetilde{\psi}^{l}\right\|_{H^{1/2}}\\&\quad+\left\|\phi_{0}\right\|_{H_{8}^{2}}\left\|\psi_{0}\right\|_{H_{8/3}^{2}}.\end{aligned} \tag{A.12}$$

By Remark 5 with $p=8,8/3$, we get

$$\left\|\phi_{0}\right\|_{H_{8}^{2}}\lesssim t^{-3/4}B,\qquad\left\|\psi_{0}\right\|_{H_{8/3}^{2}}\lesssim t^{-1/2}B. \tag{A.13}$$

Thus, by assumptions and estimates (A.12)-(A.13), we obtain (A.11) for $l\geq 1$. In the case of $l=0$, it is easy to see (69). We omit the details. Conversely, assume (69) and will prove (52). By the decomposition $I=\sum_{\pm}\mathscr{B}_{\pm}^{D}$, we have only to show that

$$\lim_{t\to\infty}\sum_{\pm}\left\|\mathscr{B}_{\pm}^{D}\widetilde{f}_{D}\right\|_{H^{1/2}}=0. \tag{A.14}$$

We have

$$\left\|\mathscr{B}\widetilde{f}_{D}\right\|_{H^{1/2}}\lesssim\left\|\mathscr{B}\mathscr{Q}_{0}^{D}\left(\mathscr{D}\phi_{0},\psi_{0}\right)\right\|_{H^{1/2}}+\text{remainder}, \tag{A.15}$$

$$\left\|\mathscr{B}\mathscr{Q}_{0}^{D}\left(\mathscr{D}\phi_{0},\psi_{0}\right)\right\|_{H^{1/2}}\lesssim\left\|\mathscr{Q}_{0}^{D}\right\|_{H^{1/2}}+\left\|\partial_{t}\mathscr{Q}_{0}^{D}\right\|_{H^{1/2}}. \tag{A.16}$$

By the Hölder inequality and Remark 5 with $p=8,8/3$, we obtain

$$\begin{aligned}&\left\|\partial_{t}\mathscr{Q}_{0}\left(\mathscr{D}\phi_{0},\psi_{0,j}\right)\right\|_{H^{-1/2}}\\&\quad\lesssim\left\|\phi_{0}\right\|_{H_{8/3}^{3}}\left\|\psi_{0}\right\|_{H_{8}^{1}}+\left\|\phi_{0}\right\|_{H_{8}^{2}}\left\|\psi_{0}\right\|_{H_{8/3}^{2}}\\&\quad\lesssim t^{-1}B^{2}.\end{aligned} \tag{A.17}$$

Since the remainder terms in (A.15) can be estimated in the same manner as the proof of (A.17), we obtain

$$\left\|\mathscr{B}\widetilde{f}_{D}\right\|_{H^{1/2}}\lesssim t^{-1}B^{2}, \tag{A.18}$$

from which (A.14) follows.

Next, we consider the KG part. By the identity

$$\|f+g\|_{H^{\kappa}}^{2}+\|f-g\|_{H^{\kappa}}^{2}=2\left(\|f\|_{H^{\kappa}}^{2}+\|g\|_{H^{\kappa}}^{2}\right), \tag{A.19}$$

we can see that (53) is equivalent to

$$\sum_{\pm}\left\|\mathscr{B}_{\pm}^{K}\left(\phi^{l}(t)-\phi_{0}(t)\right)\right\|_{H^{1}}. \tag{A.20}$$

In the same manner as the proof of estimate (A.18), we can obtain

$$\left\|\mathscr{B}\widetilde{f}_{K}\right\|_{H^{1}}\lesssim t^{-1}B^{2}, \tag{A.21}$$

which completes the proof of the lemma. □

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References

[1] N. Bournaveas, "Low regularity solutions of the Dirac Klein-Gordon equations in two space dimensions," *Communications in Partial Differential Equations*, vol. 26, no. 7-8, pp. 1345–1366, 2001.

[2] P. D'Ancona, D. Foschi, and S. Selberg, "Local well-posedness below the charge norm for the Dirac-Klein-Gordon system in two space dimensions," *Journal of Hyperbolic Differential Equations*, vol. 4, no. 2, pp. 295–330, 2007.

[3] A. Grünrock and H. Pecher, "Global solutions for the Dirac-Klein-Gordon system in two space dimensions," *Communications in Partial Differential Equations*, vol. 35, no. 1, pp. 89–112, 2010.

[4] H. Pecher, "Unconditional well-posedness for the Dirac-Klein-Gordon system in two space dimensions," http://arxiv.org/abs/1001.3065.

[5] S. Selberg and A. Tesfahun, "Unconditional uniqueness in the charge class for the Dirac-Klein-Gordon equations in two space dimensions," *Nonlinear Differential Equations and Applications*, vol. 20, no. 3, pp. 1055–1063, 2013.

[6] A. Bachelot, "Problème de Cauchy global pour des systèmes de Dirac-Klein-Gordon," *Annales de l'Institut Henri Poincaré*, vol. 48, no. 4, pp. 387–422, 1988.

[7] R. B. E. Wibowo, "Scattering problem for a system of nonlinear Klein-Gordon equations related to Dirac-Klein-Gordon equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 3-4, pp. 881–890, 2009.

[8] H. Sunagawa, "On global small amplitude solutions to systems of cubic nonlinear Klein-Gordon equations with different mass terms in one space dimension," *Journal of Differential Equations*, vol. 192, no. 2, pp. 308–325, 2003.

[9] Y. Kawahara and H. Sunagawa, "Global small amplitude solutions for two-dimensional nonlinear Klein-Gordon systems in the presence of mass resonance," *Journal of Differential Equations*, vol. 251, no. 9, pp. 2549–2567, 2011.

[10] N. Hayashi, M. Ikeda, and P. I. Naumkin, "Wave operator for the system of the Dirac-Klein-Gordon equations," *Mathematical Methods in the Applied Sciences*, vol. 34, no. 8, pp. 896–910, 2011.

[11] M. Ikeda, A. Shimomura, and H. Sunagawa, "A remark on the algebraic normal form method applied to the Dirac-Klein-Gordon system in two space dimensions," *RIMS Kôkyûroku Bessatsu B*, vol. 33, pp. 87–96, 2012.

[12] P. D'Ancona, D. Foschi, and S. Selberg, "Null structure and almost optimal local regularity for the Dirac-Klein-Gordon system," *Journal of the European Mathematical Society*, vol. 9, no. 4, pp. 877–899, 2007.

[13] N. Hayashi and P. I. Naumkin, "Wave operators to a quadratic nonlinear Klein-Gordon equation in two space dimensions," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 9, pp. 3826–3833, 2009.

[14] B. Marshall, W. Strauss, and S. Wainger, "$L^p - L^q$ estimates for the Klein-Gordon equation," *Journal de Mathématiques Pures et Appliquées*, vol. 59, no. 4, pp. 417–440, 1980.

[15] K. Yajima, "Existence of solutions for Schrödinger evolution equations," *Communications in Mathematical Physics*, vol. 110, no. 3, pp. 415–426, 1987.

[16] T. Kato and G. Ponce, "Commutator estimates and the Euler and Navier-Stokes equations," *Communications on Pure and Applied Mathematics*, vol. 41, no. 7, pp. 891–907, 1988.

[17] Y. Tsutsumi, "Global solutions for the Dirac-Proca equations with small initial data in 3 + 1 space time dimensions," *Journal of Mathematical Analysis and Applications*, vol. 278, no. 2, pp. 485–499, 2003.

[18] S. Klainerman, "The null condition and global existence to nonlinear wave equations," in *Nonlinear Systems of Partial Differential Equations in Applied Mathematics*, vol. 23 of *Lectures in Applied Mathematics*, pp. 293–326, American Mathematical Society, Providence, RI, USA, 1986.

[19] S. Katayama, "A note on global existence of solutions to nonlinear Klein-Gordon equations in one space dimension," *Journal of Mathematics of Kyoto University*, vol. 39, no. 2, pp. 203–213, 1999.

[20] R. Kosecki, "The unit condition and global existence for a class of nonlinear Klein-Gordon equations," *Journal of Differential Equations*, vol. 100, no. 2, pp. 257–268, 1992.

[21] Y. Tsutsumi, "Stability of constant equilibrium for the Maxwell-Higgs equations," *Funkcialaj Ekvacioj*, vol. 46, no. 1, pp. 41–62, 2003.

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Research Article

Existence and Uniqueness of Positive Solution for a Fractional Dirichlet Problem with Combined Nonlinear Effects in Bounded Domains

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We prove the existence and uniqueness of a positive continuous solution to the following singular semilinear fractional Dirichlet problem $(-\Delta)^{\alpha/2}u = a_1(x)u^{\sigma_1} + a_2(x)u^{\sigma_2}$, in D $\lim_{x\to z\in\partial D}(\delta(x))^{1-(\alpha/2)}u(x) = 0$, where $0 < \alpha < 2, \sigma_1, \sigma_2 \in (-1, 1)$, D is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, $n \geq 2$, and $\delta(x)$ denotes the Euclidian distance from x to the boundary of D. The nonnegative weight functions a_1, a_2 are required to satisfy certain hypotheses related to the Karamata class. We also investigate the global behavior of such solution.

1. Introduction

In the last two decades, several studies have been performed for the so-called fractional Laplacian, $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, which can be defined by the integral representation

$$(-\Delta)^{\alpha/2}u(x) = c_{n,\alpha}\lim_{\varepsilon\searrow 0}\int_{(|x-y|>\varepsilon)}\frac{u(x)-u(y)}{|x-y|^{n+\alpha}}dy, \quad (1)$$

where $c_{n,\alpha} = (\alpha 2^{\alpha-1}/\pi^{n/2})(\Gamma((n+\alpha)/2)/\Gamma(1-(\alpha/2)))$ is a normalization constant; see, for instance, [1, 2]. From a probabilistic point of view, the fractional Laplacian appears as the infinitesimal generator of the stable Lévy process [3, 4]; see also [5]. The fractional powers of the Laplacian arise in a numerous variety of equations in mathematical physics and related fields (see, for instance, [6–11] and the references therein). Motivation from mechanics appears in the Signorini problem (cf. [12, 13]). And there are applications in fluid mechanics, (cf. [14]). The systematic study of the corresponding PDE models is more recent and many of the results have arisen in the last decade. The linear or quasilinear elliptic theory has been actively studied recently in the works of Caffarelli and collaborators [15, 16], Kassmann [17], Silvestre [18], and many others. The standard linear evolution equation involving fractional diffusion is

$$\frac{\partial u}{\partial t} + (-\Delta)^{\alpha/2}u = 0. \quad (2)$$

This is a model of the so-called anomalous diffusion, a much studied topic in physics, probability, and finance (see [19–23] and their references). For more applications, we refer the reader to the survey papers [24, 25].

Throughout this paper, we consider a bounded $C^{1,1}$-domain D in $\mathbb{R}^n$, $n \geq 2$, and we denote by $\delta(x)$ the Euclidian distance from x to the boundary of D. For two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that $(1/c)f(x) \leq g(x) \leq cf(x)$, for all $x \in S$.

Recently, in [26], the authors considered the following problem:

$$\begin{cases} (-\Delta)^{\alpha/2}u = \varphi(\cdot, u) & \text{in } D \text{ (in the sense of distributions)},\\ u > 0 & \text{in } D,\\ \lim_{x\to z\in\partial D} (\delta(x))^{1-(\alpha/2)} u(x) = 0, \end{cases} \tag{3}$$

where $0 < \alpha < 2$ and φ is a positive measurable function in $D \times (0,\infty)$ satisfying the following:

(A_1) the map $t \to \varphi(x,t)$ is continuous and nonincreasing in $(0,\infty)$, for $x \in D$;

(A_2) for each $c > 0$, the function $x \to (\delta(x))^{1-(\alpha/2)}\varphi(x, c(\delta(x))^{(\alpha/2)-1})$ is in $K^\alpha(D)$ (see Definition 1 below).

They have proved that problem (3) has a positive continuous solution u in D satisfying, for each $x \in D$,

$$u(x) = \int_D G_D^\alpha(x,y)\varphi(y,u(y))\,dy, \tag{4}$$

where $G_D^\alpha(x,y)$ denotes the Green function of the fractional Laplacian $(-\Delta)^{\alpha/2}$ in D. However they have not investigated the asymptotic behavior of such solution.

As a typical example of function φ satisfying (A_1) and (A_2), we quote $\varphi(x,u) = a(x)u^\sigma$, where $\sigma \le 0$ and a is a positive measurable function in D such that the function

$$x \longrightarrow (\delta(x))^{((\alpha/2)-1)(\sigma-1)} a(x) \tag{5}$$

belongs to the Kato class $K^\alpha(D)$ defined as follows.

Definition 1 (see [26]). A Borel measurable function q in D belongs to the Kato class $K^\alpha(D)$ if

$$\lim_{r\to 0}\left(\sup_{x\in D}\int_{(|x-y|\le r)\cap D}\left(\frac{\delta(y)}{\delta(x)}\right)^{\alpha/2} G_D^\alpha(x,y)|q(y)|\,dy\right) = 0. \tag{6}$$

It has been proved in [26] that the function

$$x \longrightarrow (\delta(x))^{-\lambda} \text{ belongs to } K^\alpha(D) \quad \text{iff } \lambda < \alpha. \tag{7}$$

For more examples of functions belonging to $K^\alpha(D)$, we refer to [26]. Note that for the classical case (i.e., $\alpha = 2$) the class $K^2(D)$ was introduced and studied in [27].

On the other hand, Chemmam et al. considered in [28] the following semilinear fractional Dirichlet problem:

$$\begin{cases} (-\Delta)^{\alpha/2}u = a(x)u^\sigma & \text{in } D \text{ (in the sense of distributions)},\\ u > 0 & \text{in } D,\\ \lim_{x\to z\in\partial D} (\delta(x))^{1-(\alpha/2)} u(x) = 0, \end{cases} \tag{8}$$

where $0 < \alpha < 2$, $\sigma < 1$, and a satisfies the following hypothesis:

(H_0) $a \in C_{\text{loc}}^\gamma(D)$, $0 < \gamma < 1$, satisfying D,

$$a(x) \approx (\delta(x))^{-\lambda} L(\delta(x)), \tag{9}$$

where $\lambda < (\alpha/2)(1+\sigma) + 1 - \sigma$ and L belongs to the Karamata class $\mathcal{K}$ defined as follows.

Definition 2. The class $\mathcal{K}$ is the set of all the Karamata functions L defined on $(0,\eta]$ by

$$L(t) := c\exp\left(\int_t^\eta \frac{z(s)}{s}\,ds\right), \tag{10}$$

where $\eta > \operatorname{diam}(D)$, $c > 0$, and $z \in C([0,\eta])$ such that $z(0) = 0$.

As a typical example of a function belonging to the class $\mathcal{K}$ (see [29–31]), we quote

$$L(t) = \prod_{k=1}^m \left(\log_k\left(\frac{\omega}{t}\right)\right)^{-\xi_k}, \tag{11}$$

where ξ_k are real numbers, $\log_k x = \log\circ\log\circ\cdots\log x$ (k times), and ω is a sufficiently large positive real number such that L is defined and positive on $(0,\eta]$.

Using a fixed-point argument, the authors have proved in [28] the existence and uniqueness of a positive continuous solution u for (8) satisfying, for $x \in D$,

$$u(x) \approx (\delta(x))^{\min(\alpha/2,(\alpha-\lambda)/(1-\sigma))}\Psi_{L,\lambda,\sigma}(\delta(x)), \tag{12}$$

where the function $\Psi_{L,\lambda,\sigma}$ is defined on $(0,\eta)$ by

$$\Psi_{L,\lambda,\sigma}(t) := \begin{cases} (L(t))^{1/(1-\sigma)}, & \frac{\alpha}{2}(1+\sigma) < \lambda < \frac{\alpha}{2}(1+\sigma) + 1 - \sigma,\\ \left(\int_t^\eta \frac{L(s)}{s}\,ds\right)^{1/(1-\sigma)}, & \text{if } \lambda = \frac{\alpha}{2}(1+\sigma),\\ 1, & \text{if } \lambda < \frac{\alpha}{2}(1+\sigma). \end{cases} \tag{13}$$

In particular, they have extended the results of [32, 33].

In the present paper, we aim at studying the following fractional nonlinear problem involving both singular and sublinear nonlinearities with the reformulated Dirichlet boundary condition:

$$\begin{cases} (-\Delta)^{\alpha/2}u = a_1(x)u^{\sigma_1} + a_2(x)u^{\sigma_2} & \text{in } D \text{ (in the sense of distributions)},\\ u > 0 & \text{in } D,\\ \lim_{x\to z\in\partial D} (\delta(x))^{1-(\alpha/2)} u(x) = 0, \end{cases} \tag{14}$$

where $0 < \alpha < 2$ and $\sigma_1, \sigma_2 \in (-1,1)$. We will address the question of existence, uniqueness, and global behavior of a positive continuous solution to problem (14).

In the elliptic case (i.e., $\alpha = 2$), problems related to (14) have been studied by several authors (see, e.g., [34–39] and references therein). Using the subsupersolution method, the authors in [36] have established the existence and uniqueness of a positive continuous solution to (14) for $\alpha = 2$, $\sigma_1, \sigma_2 < 1$, where the functions a_1, a_2 are required to satisfy some adequate assumptions related to the Karamata class $\mathcal{K}$.

Here, our goal is to study problem (14) for $0 < \alpha < 2$. To this end, we assume that the potential functions a_1, a_2 satisfy the following hypothesis.

(H) for $i \in \{1, 2\}$, $a_i \in C^{\gamma}_{\mathrm{loc}}(D)$, $0 < \gamma < 1$, and satisfies, for $x \in D$,

$$a_i(x) \approx (\delta(x))^{-\lambda_i} L_i(\delta(x)), \tag{15}$$

where $\lambda_i < (\alpha/2)(1+\sigma_i) + 1 - \sigma_i$ and $L_i \in \mathcal{K}$ defined on $(0, \eta]$ with $\eta > \mathrm{diam}(D)$.

As it turns out, estimates (12) depend closely on $\min(\alpha/2, (\alpha-\lambda)/(1-\sigma))$. Also, as it will be seen, the numbers

$$\beta_1 := \min\left(\frac{\alpha}{2}, \frac{\alpha-\lambda_1}{1-\sigma_1}\right), \qquad \beta_2 := \min\left(\frac{\alpha}{2}, \frac{\alpha-\lambda_2}{1-\sigma_2}\right) \tag{16}$$

play an important role in the combined effect of singular and superlinear nonlinearities in (14) and lead to a competition. It is not obvious which wins, essentially in the estimates of solution. From here on and without loss of generality, we may assume that $(\alpha - \lambda_1)/(1 - \sigma_1) \le (\alpha - \lambda_2)/(1 - \sigma_2)$ and we introduce the function θ defined on $(0, \eta)$ by

$$\theta(t) = \begin{cases} t^{\beta_1}\Psi_{L_1,\lambda_1,\sigma_1}(t) & \text{if } \beta_1 < \beta_2, \\ t^{\beta_1}\left(\Psi_{L_1,\lambda_1,\sigma_1}(t) + \Psi_{L_2,\lambda_2,\sigma_2}(t)\right) & \text{if } \beta_1 = \beta_2. \end{cases} \tag{17}$$

For an explicit form of the function θ, see (36).

Throughout this paper, we define the potential kernel G^{α}_D by

$$G^{\alpha}_D f(x) := \int_D G^{\alpha}_D(x, y) f(y)\, dy, \quad \text{for } x \in D,\ f \in B^+(D), \tag{18}$$

where $B^+(D)$ denotes the set of the nonnegative Borel measurable functions in D.

Our main results are the following.

Theorem 3. *Let $\sigma_1, \sigma_2 \in (-1, 1)$ and assume (H). Then one has, for $x \in D$,*

$$G^{\alpha}_D\left[a_1\theta^{\sigma_1}(\delta(\cdot)) + a_2\theta^{\sigma_2}(\delta(\cdot))\right](x) \approx \theta(\delta(x)). \tag{19}$$

Using Theorem 3 and the Schauder fixed-point theorem, we will prove the following.

Theorem 4. *Let $\sigma_1, \sigma_2 \in (-1, 1)$ and assume (H). Then problem (14) has a unique positive continuous solution u in D satisfying, for $x \in D$,*

$$u(x) \approx \theta(\delta(x)). \tag{20}$$

In particular, we generalize the result obtained in [36] to the fractional setting and we recover the result obtained in [28].

The content of this paper is organized as follows. In Section 2, we collect some properties of functions belonging to the Karamata class $\mathcal{K}$ and the Kato class $K^{\alpha}(D)$, which are useful to establish our results. In Section 3, we prove our main results.

As usual, we denote by $C_0(D)$ the set of continuous functions in $\overline{D}$ vanishing continuously on ∂D. Note that $C_0(D)$ is a Banach space with respect to the uniform norm $\|u\|_{\infty} = \sup_{x\in D}|u(x)|$. As in the elliptic case, if $f \in B^+(D)$ satisfies $\int_D (\delta(y))^{\alpha/2} f(y)dy < \infty$, then the functions f and $G^{\alpha}_D f$ are in $L^1_{\mathrm{loc}}(D)$ and we have in the distributional sense

$$(-\Delta)^{\alpha/2} G^{\alpha}_D f = f, \quad \text{in } D. \tag{21}$$

2. The Karamata Class $\mathcal{K}$ and the Kato Class $K^{\alpha}(D)$

We collect in this paragraph some properties of the Karamata class $\mathcal{K}$ and the Kato class $K^{\alpha}(D)$. We recall that a function L defined on $(0, \eta]$ belongs to the class $\mathcal{K}$ if

$$L(t) := c \exp\left(\int_t^{\eta} \frac{z(s)}{s} ds\right), \tag{22}$$

where $\eta > \mathrm{diam}(D)$, $c > 0$, and $z \in C([0, \eta])$ such that $z(0) = 0$.

Proposition 5 (see [30, 31]). (i) *A function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1((0, \eta])$ such that*

$$\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0. \tag{23}$$

(ii) *Let $L_1, L_2 \in \mathcal{K}$, $p \in \mathbb{R}$. Then one has*

$$L_1 + L_2 \in \mathcal{K}, \quad L_1L_2 \in \mathcal{K}, \quad L_1^p \in \mathcal{K}. \tag{24}$$

(iii) *Let $L \in \mathcal{K}$ and $\varepsilon > 0$. Then one has*

$$\lim_{t\to 0^+} t^{\varepsilon} L(t) = 0. \tag{25}$$

Applying Karamata's theorem (see [30, 31]), we get the following.

Lemma 6. *Let $\mu \in \mathbb{R}$ and let L be a function in $\mathcal{K}$. One has the following:*

(i) *if $\mu < -1$, then $\int_0^{\eta} s^{\mu}L(s)ds$ diverges and $\int_t^{\eta} s^{\mu}L(s)ds \sim_{t\to 0^+} (-t^{1+\mu}L(t))/(\mu + 1)$;*

(ii) if $\mu > -1$, then $\int_0^\eta s^\mu L(s)ds$ converges and $\int_0^t s^\mu L(s)\, ds \sim_{t\to 0^+} (t^{1+\mu}L(t))/(\mu+1)$.

Lemma 7 (see [36]). *Let L be a function in $\mathcal{K}$. Then one has*

$$\lim_{t\to 0^+} \frac{L(t)}{\int_t^\eta (L(s)/s)\,ds} = 0. \tag{26}$$

In particular

$$t \longrightarrow \int_t^\eta \frac{L(s)}{s} ds \in \mathcal{K}. \tag{27}$$

Proposition 8 (see [40, 41]). *For $(x,y) \in D\times D$, one has*

$$G_D^\alpha(x,y) \approx |x-y|^{\alpha-n} \min\left(1, \frac{(\delta(x)\delta(y))^{\alpha/2}}{|x-y|^\alpha}\right). \tag{28}$$

Proposition 9 (see [26, Corollary 6]). *Let q be a nonnegative function in $K^\alpha(D)$; then the family of functions*

$$\Lambda_q = \left\{x \longrightarrow \int_D \left(\frac{\delta(y)}{\delta(x)}\right)^{(\alpha/2)-1} G_D^\alpha(x,y) f(y)\,dy, |f| \le q\right\} \tag{29}$$

is uniformly bounded and equicontinuous in $\overline{D}$. Consequently Λ_q is relatively compact in $C_0(D)$.

3. Proofs of the Main Results

In this section we aim at proving Theorems 3 and 4. To this end, we need the following lemmas.

3.1. Technical Lemmas

Lemma 10. *For $r, s > 0$, one has*

$$\begin{aligned} &2^{-\max(1-\sigma_1,1-\sigma_2)}(r+s) \\ &\quad \le r^{1-\sigma_1}(r+s)^{\sigma_1} + s^{1-\sigma_2}(r+s)^{\sigma_2} \le 2(r+s). \end{aligned} \tag{30}$$

Proof. Let $r, s > 0$ and put $t = r/(r+s)$. Since $0 \le t \le 1$, then we get obviously

$$2^{-\max(1-\sigma_1,1-\sigma_2)} \le t^{1-\sigma_1} + (1-t)^{1-\sigma_2} \le 2. \tag{31}$$

□

Lemma 11 provides sharp estimates on some Riesz potential functions.

Lemma 11 (see [28, Proposition 3.1]). *Let $\mu \le (\alpha/2)+1$ and let L be a function in $\mathcal{K}$ such that $\int_0^\eta t^{(\alpha/2)-\mu}L(t)dt < \infty$. Let q be a positive measurable function in D such that, for $x \in D$,*

$$q(x) \approx (\delta(x))^{-\mu} L(\delta(x)). \tag{32}$$

Then, for $x \in D$, one has

$$G_D^\alpha q(x) \approx \psi(\delta(x)), \tag{33}$$

where ψ is the function defined on $(0,\eta)$ by

$$\psi(t) := \begin{cases} t^{(\alpha/2)-1}\displaystyle\int_0^t \frac{L(s)}{s}ds, & \text{if } \mu = \dfrac{\alpha}{2}+1, \\ t^{\alpha-\mu}L(t), & \text{if } \dfrac{\alpha}{2} < \mu < \dfrac{\alpha}{2}+1, \\ t^{\alpha/2}\displaystyle\int_t^\eta \frac{L(s)}{s}ds, & \text{if } \mu = \dfrac{\alpha}{2}, \\ t^{\alpha/2}, & \text{if } \mu < \dfrac{\alpha}{2}. \end{cases} \tag{34}$$

Lemma 12. *Assume (H). Let u be a continuous function in D such that, for $x \in D$, $u(x) \approx \theta(\delta(x))$. Then u is a solution of problem (14) if and only if*

$$u(x) = \int_D G_D^\alpha(x,y)\left[a_1(y)u^{\sigma_1}(y) + a_2(y)u^{\sigma_2}(y)\right]dy, \quad x \in D. \tag{35}$$

Proof. Assume (H). First we will give an explicit form of the function θ. We recall that, for $i \in \{1,2\}$, $\lambda_i < (\alpha/2)(1+\sigma_i) + 1 - \sigma_i$ and $\beta_i := \min(\alpha/2, (\alpha-\lambda_i)/(1-\sigma_i))$. Since $\beta_1 < \beta_2$ is equivalent to $(\alpha-\lambda_1)/(1-\sigma_1) < (\alpha-\lambda_2)/(1-\sigma_2)$ and $(\alpha/2)(1+\sigma_1) < \lambda_1$, we deduce that, for $t \in (0,\eta)$, we have

$$\theta(t) = \begin{cases} t^{(\alpha-\lambda_1)/(1-\sigma_1)}\left(L_1(t)\right)^{1/(1-\sigma_1)}, & \text{if } \dfrac{\alpha-\lambda_1}{1-\sigma_1} < \dfrac{\alpha-\lambda_2}{1-\sigma_2},\ \dfrac{\alpha}{2}(1+\sigma_1) < \lambda_1, \\ t^{(\alpha-\lambda_1)/(1-\sigma_1)}L(t), & \text{if } \dfrac{\alpha-\lambda_1}{1-\sigma_1} = \dfrac{\alpha-\lambda_2}{1-\sigma_2},\ \dfrac{\alpha}{2}(1+\sigma_1) < \lambda_1, \\ t^{\alpha/2}M(t), & \text{if } \lambda_1 = \dfrac{\alpha}{2}(1+\sigma_1),\ \lambda_2 = \dfrac{\alpha}{2}(1+\sigma_2), \\ t^{\alpha/2}\left(\displaystyle\int_t^\eta \frac{L_1(s)}{s}ds\right)^{1/(1-\sigma_1)}, & \text{if } \lambda_1 = \dfrac{\alpha}{2}(1+\sigma_1),\ \lambda_2 < \dfrac{\alpha}{2}(1+\sigma_2), \\ t^{\alpha/2} & \text{if } \lambda_1 < \dfrac{\alpha}{2}(1+\sigma_1), \end{cases} \tag{36}$$

where

$$\begin{aligned} L(t) &:= \left(L_1(t)\right)^{1/(1-\sigma_1)} + \left(L_2(t)\right)^{1/(1-\sigma_2)}, \\ M(t) &:= \left(\int_t^\eta \frac{L_1(s)}{s}ds\right)^{1/(1-\sigma_1)} + \left(\int_t^\eta \frac{L_2(s)}{s}ds\right)^{1/(1-\sigma_2)}. \end{aligned} \tag{37}$$

Now using the fact that $u(x) \approx \theta(\delta(x))$, we deduce by simple computation from hypothesis (H), (36), and Proposition 5 that

$$a_1(x)u^{\sigma_1}(x) + a_2(x)u^{\sigma_2}(x) \approx (\delta(x))^{(\alpha/2)-1}h(\delta(x)), \tag{38}$$

where h is defined in $(0,\eta)$ by

$$h(t) := \begin{cases} t^{1-(\alpha/2)-((\lambda_1-\alpha\sigma_1)/(1-\sigma_1))}(L_1(t))^{1/(1-\sigma_1)}, & \text{if } \dfrac{\alpha-\lambda_1}{1-\sigma_1} < \dfrac{\alpha-\lambda_2}{1-\sigma_2},\ \dfrac{\alpha}{2}(1+\sigma_1) < \lambda_1, \\ t^{1-(\alpha/2)-((\lambda_1-\alpha\sigma_1)/(1-\sigma_1))}\left(L_1L^{\sigma_1}+L_2L^{\sigma_2}\right)(t), & \text{if } \dfrac{\alpha-\lambda_1}{1-\sigma_1} = \dfrac{\alpha-\lambda_2}{1-\sigma_2},\ \dfrac{\alpha}{2}(1+\sigma_1) < \lambda_1, \\ t^{1-\alpha}\left(L_1M^{\sigma_1}+L_2M^{\sigma_2}\right)(t), & \text{if } \lambda_1 = \dfrac{\alpha}{2}(1+\sigma_1),\ \lambda_2 = \dfrac{\alpha}{2}(1+\sigma_2), \\ t^{1-\alpha}L_1(t)\left(\displaystyle\int_t^\eta \frac{L_1(s)}{s}ds\right)^{\sigma_1/(1-\sigma_1)}, & \text{if } \lambda_1 = \dfrac{\alpha}{2}(1+\sigma_1),\ \lambda_2 < \dfrac{\alpha}{2}(1+\sigma_2), \\ t^{1+(\alpha/2)(\sigma_1-1)-\lambda_1}L_1(t)+t^{1+(\alpha/2)(\sigma_2-1)-\lambda_2}L_2(t), & \text{if } \lambda_1 < \dfrac{\alpha}{2}(1+\sigma_1). \end{cases} \tag{39}$$

We point out that for each case, the function $h(t)$ can be written as a sum of terms of the form $t^{-\mu}\widetilde{L}(t)$, where $\mu < \alpha$. By Proposition 5 and Lemma 7, we have $\widetilde{L} \in \mathcal{K}$. On the other hand, since by Proposition 5, the function $x \to (\delta(x))^{(\alpha-\mu)/2}\widetilde{L}(\delta(x))$ is positive and belongs to $C_0(D)$, then there exists $c > 0$ such that for each $x \in D$

$$0 < (\delta(x))^{-\mu}\widetilde{L}(\delta(x)) \le c(\delta(x))^{-(\alpha+\mu)/2}. \tag{40}$$

Hence we deduce from (7) that the function $x \to h(\delta(x))$ is in $K^\alpha(D)$.

Now using Proposition 9, we obtain that $x \to (\delta(x))^{1-(\alpha/2)}G_D^\alpha[a_1u^{\sigma_1}+a_2u^{\sigma_2}](x)$ is in $C_0(D)$. In particular, we have

$$\lim_{x\to z\in\partial D}(\delta(x))^{1-(\alpha/2)}G_D^\alpha\left(a_1u^{\sigma_1}+a_2u^{\sigma_2}\right)(x) = 0,$$
$$(-\Delta)^{\alpha/2}G_D^\alpha\left(a_1u^{\sigma_1}+a_2u^{\sigma_2}\right) = a_1(x)u^{\sigma_1}+a_2(x)u^{\sigma_2} \tag{41}$$
$$\text{in } D \text{ (in the sense of distributions)}.$$

Consequently, it follows by (41) that u is a weak continuous solution of problem (14) if and only if u satisfies

$$\begin{cases} (-\Delta)^{\alpha/2}\left(u - G_D^\alpha\left(a_1u^{\sigma_1}+a_2u^{\sigma_2}\right)\right) = 0 & \text{in } D \\ \lim\limits_{x\to z\in\partial D}(\delta(x))^{1-(\alpha/2)}\left(u - G_D^\alpha\left(a_1u^{\sigma_1}+a_2u^{\sigma_2}\right)\right) = 0. \end{cases} \tag{42}$$

We deduce by [26, Theorem 6] that $u - G_D^\alpha(a_1u^{\sigma_1}+a_2u^{\sigma_2}) = 0$ in D. The proof is complete. □

Lemma 13. *For $i \in \{1,2\}$, let $L_i \in \mathcal{K}$ defined on $(0,\eta]$ with $\eta > \operatorname{diam}(D)$ and let M be the function given by (37). Then one has, for $t \in (0,\eta)$,*

$$\int_t^\eta \frac{\left(L_1M^{\sigma_1}+L_2M^{\sigma_2}\right)(s)}{s}ds \approx M(t). \tag{43}$$

Proof. The proof can be found in [36]. □

Now we are ready to prove our main results.

3.2. Proof of Theorem 3. Assume (H). For $i \in \{1,2\}$, let $L_i \in \mathcal{K}$ defined on $(0,\eta]$ with $\eta > \operatorname{diam}(D)$ and define the nonnegative functions b_i in $(0,\eta)$ by

$$b_i(t) = \left(\int_t^\eta \frac{L_i(s)}{s}ds\right)^{1/(1-\sigma_i)}. \tag{44}$$

Let θ be the function given by (36). To prove Theorem 3, we distinguish the following cases.

Case 1. $(\alpha-\lambda_1)/(1-\sigma_1) < (\alpha-\lambda_2)/(1-\sigma_2)$ and $(\alpha/2)(1+\sigma_1) < \lambda_1 < (\alpha/2)(1+\sigma_1)+1-\sigma_1$.

Since $\theta(t) = t^{(\alpha-\lambda_1)/(1-\sigma_1)}(L_1(t))^{1/(1-\sigma_1)}$, then we have

$$a_1(x)\theta^{\sigma_1}(\delta(x)) \approx (\delta(x))^{(\alpha\sigma_1-\lambda_1)/(1-\sigma_1)}(L_1(\delta(x)))^{1/(1-\sigma_1)},$$
$$a_2(x)\theta^{\sigma_2}(\delta(x))$$
$$\approx (\delta(x))^{((\alpha-\lambda_1)/(1-\sigma_1))\sigma_2-\lambda_2}\left(L_2L_1^{\sigma_2/(1-\sigma_1)}\right)(\delta(x)). \tag{45}$$

Using the fact that $(\alpha\sigma_1-\lambda_1)/(1-\sigma_1) < ((\alpha-\lambda_1)/(1-\sigma_1))\sigma_2-\lambda_2$, we deduce by Proposition 5 that

$$a_1(x)\theta^{\sigma_1}(\delta(x)) + a_2(x)\theta^{\sigma_2}(\delta(x))$$
$$\approx (\delta(x))^{(\alpha\sigma_1-\lambda_1)/(1-\sigma_1)}(L_1(\delta(x)))^{1/(1-\sigma_1)}. \tag{46}$$

Since, for $\mu = (\lambda_1-\alpha\sigma_1)/(1-\sigma_1) \in (\alpha/2,(\alpha/2)+1)$, we have $\int_0^\eta t^{(\alpha/2)-\mu}(L_1(t))^{1/(1-\sigma_1)}dt < \infty$, then applying Lemma 11, we deduce that

$$G_D^\alpha\left[a_1\theta^{\sigma_1}(\delta(\cdot)) + a_2\theta^{\sigma_2}(\delta(\cdot))\right](x)$$
$$\approx G_D^\alpha\left[(\delta(\cdot))^{(\alpha\sigma_1-\lambda_1)/(1-\sigma_1)}(L_1(\delta(\cdot)))^{1/(1-\sigma_1)}\right](x) \tag{47}$$
$$\approx (\delta(x))^{(\alpha-\lambda_1)/(1-\sigma_1)}(L_1(\delta(x)))^{1/(1-\sigma_1)} = \theta(\delta(x)).$$

Case 2. $(\alpha-\lambda_1)/(1-\sigma_1) = (\alpha-\lambda_2)/(1-\sigma_2)$ and $(\alpha/2)(1+\sigma_1) < \lambda_1 < (\alpha/2)(1+\sigma_1)+1-\sigma_1$.

In this case $\theta(t) = t^{(\alpha-\lambda_1)/(1-\sigma_1)}L(t)$. Therefore

$$a_1(x)\theta^{\sigma_1}(\delta(x)) \approx (\delta(x))^{(\alpha\sigma_1-\lambda_1)/(1-\sigma_1)}\left(L_1L^{\sigma_1}\right)(\delta(x)). \tag{48}$$

So we obtain by Proposition 5 and Lemma 11 with $\mu = (\lambda_1 - \alpha\sigma_1)/(1-\sigma_1) \in (\alpha/2, (\alpha/2)+1)$,

$$G_D^\alpha \left[a_1\theta^{\sigma_1}(\delta(\cdot))\right](x) \approx (\delta(x))^{(\alpha-\lambda_1)/(1-\sigma_1)} \left(L_1L^{\sigma_1}\right)(\delta(x)). \tag{49}$$

Similarly, since $(\alpha/2)(1+\sigma_2) < \lambda_2 < (\alpha/2)(1+\sigma_2) + 1 - \sigma_2$, we obtain

$$\begin{aligned} &G_D^\alpha \left[a_2\theta^{\sigma_2}(\delta(\cdot))\right](x) \\ &\quad\approx (\delta(x))^{(\alpha-\lambda_2)/(1-\sigma_2)} \left(L_2L^{\sigma_2}\right)(\delta(x)) \\ &\quad\approx (\delta(x))^{(\alpha-\lambda_1)/(1-\sigma_1)} \left(L_2L^{\sigma_2}\right)(\delta(x)). \end{aligned} \tag{50}$$

Hence by using (30), we deduce that

$$\begin{aligned} &G_D^\alpha \left[a_1\theta^{\sigma_1}(\delta(\cdot)) + a_2\theta^{\sigma_2}(\delta(\cdot))\right](x) \\ &\quad\approx (\delta(x))^{(\alpha-\lambda_1)/(1-\sigma_1)} \left(L_1L^{\sigma_1} + L_2L^{\sigma_2}\right)(\delta(x)) \\ &\quad\approx (\delta(x))^{(\alpha-\lambda_1)/(1-\sigma_1)} L(\delta(x)) = \theta(\delta(x)). \end{aligned} \tag{51}$$

Case 3. If $\lambda_1 = (\alpha/2)(1+\sigma_1)$ and $\lambda_2 = (\alpha/2)(1+\sigma_2)$ and since $\theta(t) = t^{\alpha/2}M(t)$, then we have

$$\begin{aligned} &a_1(x)\theta^{\sigma_1}(\delta(x)) + a_2(x)\theta^{\sigma_2}(\delta(x)) \\ &\quad\approx (\delta(x))^{-\alpha/2} \left(L_1M^{\sigma_1} + L_2M^{\sigma_2}\right)(\delta(x)). \end{aligned} \tag{52}$$

So by Proposition 5, Lemma 11 with $\mu = \alpha/2$, and Lemma 13, we deduce that

$$\begin{aligned} &G_D^\alpha \left[a_1\theta^{\sigma_1}(\delta(\cdot)) + a_2\theta^{\sigma_2}(\delta(\cdot))\right](x) \\ &\quad\approx (\delta(x))^{\alpha/2} \int_{\delta(x)}^{\eta} \frac{\left(L_1M^{\sigma_1} + L_2M^{\sigma_2}\right)(s)}{s} ds \\ &\quad\approx (\delta(x))^{\alpha/2} M(\delta(x)) = \theta(\delta(x)). \end{aligned} \tag{53}$$

Case 4. $\lambda_1 = (\alpha/2)(1+\sigma_1)$ and $\lambda_2 < (\alpha/2)(1+\sigma_2)$.

In this case $\theta(t) = t^{\alpha/2}b_1(t)$. Since $\lambda_2 - (\alpha\sigma_2/2) < (\alpha/2)$, we deduce by Proposition 5 that

$$\begin{aligned} &a_1(x)\theta^{\sigma_1}(\delta(x)) + a_2(x)\theta^{\sigma_2}(\delta(x)) \\ &\quad\approx (\delta(x))^{-\alpha/2} \left(L_1b_1^{\sigma_1}\right)(\delta(x)) \\ &\qquad + (\delta(x))^{(\alpha\sigma_2/2)-\lambda_2} \left(L_2b_1^{\sigma_2}\right)(\delta(x)) \\ &\quad\approx (\delta(x))^{-\alpha/2} \left(L_1b_1^{\sigma_1}\right)(\delta(x)). \end{aligned} \tag{54}$$

Hence applying Lemma 11 with $\mu = \alpha/2$, we obtain

$$\begin{aligned} &G_D^\alpha \left[a_1\theta^{\sigma_1}(\delta(\cdot)) + a_2\theta^{\sigma_2}(\delta(\cdot))\right](x) \\ &\quad\approx G_D^\alpha \left[(\delta(\cdot))^{-\alpha/2} \left(L_1b_1^{\sigma_1}\right)(\delta(\cdot))\right](x) \\ &\quad\approx (\delta(x))^{\alpha/2} \int_{\delta(x)}^{\eta} \frac{\left(L_1b_1^{\sigma_1}\right)(s)}{s} ds \\ &\quad\approx (\delta(x))^{\alpha/2} b_1(\delta(x)) = \theta(\delta(x)). \end{aligned} \tag{55}$$

Case 5. $\lambda_1 < (\alpha/2)(1+\sigma_1)$.

We have $\theta(t) = t^{\alpha/2}$. So

$$a_1(x)\theta^{\sigma_1}(\delta(x)) \approx (\delta(x))^{-(\lambda_1 - (\alpha\sigma_1/2))} L_1(\delta(x)). \tag{56}$$

Applying again Lemma 11 with $\mu = \lambda_1 - (\alpha\sigma_1/2) < (\alpha/2)$, we obtain

$$G_D^\alpha \left[a_1\theta^{\sigma_1}(\delta(\cdot))\right](x) \approx (\delta(x))^{\alpha/2}. \tag{57}$$

On the other hand, since $(\alpha/2) < (\alpha - \lambda_1)/(1-\sigma_1) \le (\alpha - \lambda_2)/(1-\sigma_2)$, then $\lambda_1 < (\alpha/2)(1+\sigma_1)$ and therefore

$$G_D^\alpha \left[a_2\theta^{\sigma_2}(\delta(\cdot))\right](x) \approx (\delta(x))^{\alpha/2}. \tag{58}$$

Hence

$$G_D^\alpha \left[a_1\theta^{\sigma_1}(\delta(\cdot)) + a_2\theta^{\sigma_2}(\delta(\cdot))\right](x) \approx (\delta(x))^{\alpha/2} = \theta(\delta(x)). \tag{59}$$

The proof is complete.

3.3. Proof of Theorem 4. Let $\sigma_1, \sigma_2 \in (-1, 1)$, assume (H), and consider $v := G_\alpha^D[a_1\theta^{\sigma_1}(\delta(\cdot)) + a_2\theta^{\sigma_2}(\delta(\cdot))]$. Using Theorem 3, there exists $m > 1$ such that

$$\frac{1}{m} v(x) \le \theta(\delta(x)) \le m v(x). \tag{60}$$

Put $\sigma := \max(|\sigma_1|, |\sigma_2|)$, $c := m^{\sigma/(1-\sigma)}$ and consider the set

$$\begin{aligned} \Gamma := \Big\{\omega \in C_0(D) : &\frac{1}{c}(\delta(x))^{1-(\alpha/2)} v(x) \le \omega(x) \\ &\le c(\delta(x))^{1-(\alpha/2)} v(x), x \in D\Big\}. \end{aligned} \tag{61}$$

Let h be the function given by (39). Since $a_1(x)\theta^{\sigma_1}(\delta(x)) + a_2(x)\theta^{\sigma_2}(\delta(x)) \approx (\delta(x))^{(\alpha/2)-1}h(\delta(x))$ and the function $x \to h(\delta(x))$ is in $K_\alpha(D)$, it follows by Proposition 9 that $x \to (\delta(x))^{1-(\alpha/2)}v(x)$ is in $C_0(D)$. So Γ is a nonempty, closed, bounded, and convex set in $C_0(D)$. Define the operator T on Γ by

$$\begin{aligned} T\omega(x) := (\delta(x))^{1-(\alpha/2)} G_\alpha^D \Big(&\left((\delta(\cdot))^{(\alpha/2)-1}\omega\right)^{\sigma_1} a_1 \\ &+ \left((\delta(\cdot))^{(\alpha/2)-1}\omega\right)^{\sigma_2} a_2\Big)(x). \end{aligned} \tag{62}$$

We will prove that T has a fixed point. Since there exists a constant $c > 0$ such that for all $\omega \in \Gamma$ we have

$$\begin{aligned} \Big|&a_1(x)\left((\delta(x))^{(\alpha/2)-1}\omega(x)\right)^{\sigma_1} \\ &+ a_2(x)\left((\delta(x))^{(\alpha/2)-1}\omega(x)\right)^{\sigma_2}\Big| \le c(\delta(x))^{(\alpha/2)-1} h(\delta(x)), \end{aligned} \tag{63}$$

where the function $x \to h(\delta(x))$ is in $K_\alpha(D)$, it follows that $T(\Gamma) \subset \Lambda_{h(\delta(\cdot))}$, where $\Lambda_{h(\delta(\cdot))}$ is given by (29). Therefore by Proposition 9, the family of functions $\{x \to T\omega(x), \omega \in \Gamma\}$ is relatively compact in $C_0(D)$.

Next, we will prove that T maps Γ into itself.

Indeed, by using (60) we have for all $\omega \in \Gamma$

$$\begin{aligned} G_\alpha^D &\left(a_1\left((\delta(\cdot))^{(\alpha/2)-1}\omega\right)^{\sigma_1} + a_2\left((\delta(\cdot))^{(\alpha/2)-1}\omega\right)^{\sigma_2}\right) \\ &\leq G_\alpha^D\left(a_1 c^\sigma m^\sigma \theta^{\sigma_1}(\delta(\cdot)) + a_2 c^\sigma m^\sigma \theta^{\sigma_2}(\delta(\cdot))\right) \\ &= cv. \end{aligned} \tag{64}$$

On the other hand, we have

$$\begin{aligned} G_\alpha^D &\left(a_1\left((\delta(\cdot))^{(\alpha/2)-1}\omega\right)^{\sigma_1} + a_2\left((\delta(\cdot))^{(\alpha/2)-1}\omega\right)^{\sigma_2}\right) \\ &\geq G_\alpha^D\left(a_1 c^{-\sigma} m^{-\sigma} \theta^{\sigma_1}(\delta(\cdot)) + a_2 c^{-\sigma} m^{-\sigma} \theta^{\sigma_2}(\delta(\cdot))\right) \\ &= \frac{1}{c} v. \end{aligned} \tag{65}$$

This implies that $T(\Gamma) \subset \Gamma$.

Now, we will prove the continuity of the operator T in Γ in the supremum norm. Let $(\omega_k)_{k\in\mathbb{N}}$ be a sequence in Γ which converges uniformly to a function ω in Γ. Then, for each $x \in D$, we have

$$\begin{aligned} &\left|T\omega_k(x) - T\omega(x)\right| \\ &\leq (\delta(x))^{1-(\alpha/2)} G_\alpha^D\Big(a_1(\delta(.))^{((\alpha/2)-1)\sigma_1}\left|\omega_k^{\sigma_1} - \omega^{\sigma_1}\right| \\ &\qquad + a_2(\delta(\cdot))^{((\alpha/2)-1)\sigma_2}\left|\omega_k^{\sigma_2} - \omega^{\sigma_2}\right|\Big)(x). \end{aligned} \tag{66}$$

On the other hand, by similar arguments to the previous ones, we have

$$\begin{aligned} &a_1(x)(\delta(x))^{((\alpha/2)-1)\sigma_1}\left|\omega_k^{\sigma_1} - \omega^{\sigma_1}\right|(x) \\ &\quad + a_2(x)(\delta(x))^{((\alpha/2)-1)\sigma_2}\left|\omega_k^{\sigma_2} - \omega^{\sigma_2}\right|(x) \\ &\leq c(\delta(x))^{(\alpha/2)-1} h(\delta(x)). \end{aligned} \tag{67}$$

We conclude by Proposition 9 and the dominated convergence theorem that, for all $x \in D$,

$$T\omega_k(x) \longrightarrow T\omega(x) \quad \text{as } k \longrightarrow +\infty. \tag{68}$$

Consequently, as $T(\Gamma)$ is relatively compact in $C_0(D)$, we deduce that the pointwise convergence implies the uniform convergence; namely,

$$\left\|T\omega_k - T\omega\right\|_\infty \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty. \tag{69}$$

Therefore, T is a compact operator from Γ into itself. So the Schauder fixed-point theorem implies the existence of $\omega \in \Gamma$ such that

$$\begin{aligned} \omega(x) = (\delta(x))^{1-(\alpha/2)} G_\alpha^D &\left(a_1\left((\delta(\cdot))^{(\alpha/2)-1}\omega\right)^{\sigma_1}\right. \\ &\left.+ a_2\left((\delta(\cdot))^{(\alpha/2)-1}\omega\right)^{\sigma_2}\right)(x). \end{aligned} \tag{70}$$

Put $u(x) = (\delta(x))^{(\alpha/2)-1}\omega(x)$. Then u is continuous and satisfies

$$u(x) = G_\alpha^D\left(a_1 u^{\sigma_1} + a_2 u^{\sigma_2}\right)(x). \tag{71}$$

Hence by Lemma 12 and Theorem 3, u is a required solution.

Next, we aim at proving the uniqueness in the cone

$$S := \{u \in C(D) : u(x) \approx \theta(\delta(x))\}. \tag{72}$$

Let u and v be two solutions of (14) in S. Then there exists a constant $m > 1$ such that

$$\frac{1}{m} \leq \frac{u}{v} \leq m. \tag{73}$$

This implies that the set

$$J = \left\{t \in (1,\infty), \frac{1}{t} v \leq u \leq tv\right\} \tag{74}$$

is not empty. Let $c_0 := \inf J$ and put $w = v - c_0^{-\sigma} u$ with $\sigma = \max(|\sigma_1|, |\sigma_2|)$.

We claim that $c_0 = 1$. Indeed, assume that $c_0 > 1$; then by using Lemma 12, we deduce that

$$\begin{aligned} w &= G_\alpha^D\left(a_1\left(v^{\sigma_1} - c_0^{-\sigma} u^{\sigma_1}\right) + a_2\left(v^{\sigma_2} - c_0^{-\sigma} u^{\sigma_2}\right)\right) \\ &\geq 0 \quad \text{in } D, \end{aligned} \tag{75}$$

which implies that

$$v \geq c_0^{-\sigma} u. \tag{76}$$

By symmetry, we deduce that

$$v \leq c_0^{\sigma} u. \tag{77}$$

So $c_0^\sigma \in J$. Since $\sigma < 1$, then we have $c_0^\sigma < c_0$. This is a contradiction to the fact that $c_0 := \inf J$. Hence $c_0 = 1$ and so $u = v$. This completes the proof.

Example 14. Let $\sigma_1 \in (-1, 0)$, let $\sigma_2 \in (0, 1)$, and put $d = \operatorname{diam}(D)$. For $i \in \{1, 2\}$, let $a_i \in C_{\mathrm{loc}}^\gamma(D)$, $0 < \gamma < 1$, satisfying

$$\begin{aligned} a_1(x) &\approx (\delta(x))^{-\lambda_1}\left(\log\left(\frac{3d}{\delta(x)}\right)\right)^{-1}, \\ a_2(x) &\approx (\delta(x))^{-\lambda_2}, \end{aligned} \tag{78}$$

where $\lambda_i < (\alpha/2)(1+\sigma_i) + 1 - \sigma_i$, such that $(\alpha - \lambda_1)/(1-\sigma_1) \leq (\alpha - \lambda_2)/(1 - \sigma_2)$. Then using Theorem 4, problem (14) has a unique positive continuous solution u in D satisfying the following estimates:

(i) if $(\alpha - \lambda_1)/(1 - \sigma_1) < (\alpha - \lambda_2)/(1 - \sigma_2)$ and $(\alpha/2)(1 + \sigma_1) < \lambda_1$, then, for $x \in D$,

$$u(x) \approx (\delta(x))^{(\alpha-\lambda_1)/(1-\sigma_1)}\left(\log\left(\frac{3d}{\delta(x)}\right)\right)^{-1/(1-\sigma_1)}; \tag{79}$$

(ii) if $(\alpha - \lambda_1)/(1 - \sigma_1) = (\alpha - \lambda_2)/(1 - \sigma_2)$ and $(\alpha/2)(1 + \sigma_1) < \lambda_1$, then, for $x \in D$,

$$u(x) \approx (\delta(x))^{(\alpha-\lambda_1)/(1-\sigma_1)}; \tag{80}$$

(iii) if $\lambda_1 = (\alpha/2)(1 + \sigma_1)$ and $\lambda_2 = (\alpha/2)(1 + \sigma_2)$, then, for $x \in D$,

$$u(x) \approx (\delta(x))^{\alpha/2}\left(\log\left(\frac{3d}{\delta(x)}\right)\right)^{1/(1-\sigma_2)}; \tag{81}$$

(iv) if $\lambda_1 = (\alpha/2)(1+\sigma_1)$ and $\lambda_2 < (\alpha/2)(1+\sigma_2)$, then, for $x \in D$,

$$u(x) \approx (\delta(x))^{\alpha/2}\left(\log \circ \log\left(\frac{3d}{\delta(x)}\right)\right)^{1/(1-\sigma_1)}; \tag{82}$$

(v) if $\lambda_1 < (\alpha/2)(1+\sigma_1)$, then, for $x \in D$,

$$u(x) \approx (\delta(x))^{(\alpha/2)}. \tag{83}$$

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References

[1] N. S. Landkof, *Foundations of Modern Potential Theory*, Springer, New York, NY, USA, 1972.

[2] E. M. Stein, *Singular integrals and differentiability properties of functions*, vol. 30 of *Princeton Mathematical Series*, Princeton University Press, Princeton, NJ, USA, 1970.

[3] D. Applebaum, *Lévy Processes and Stochastic Calculus*, vol. 116 of *Cambridge Studies in Advanced Mathematics*, Cambridge University Press, Cambridge, UK, 2nd edition, 2009.

[4] J. Bertoin, *Lévy Processes*, vol. 121 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 1996.

[5] E. Valdinoci, "From the long jump random walk to the fractional Laplacian," *Boletín de la Sociedad Española de Matemática Aplicada*, no. 49, pp. 33–44, 2009.

[6] L. Abdelouhab, J. L. Bona, M. Felland, and J.-C. Saut, "Nonlocal models for nonlinear, dispersive waves," *Physica D*, vol. 40, no. 3, pp. 360–392, 1989.

[7] K. Bogdan and T. Byczkowski, "Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains," *Studia Mathematica*, vol. 133, no. 1, pp. 53–92, 1999.

[8] K. Bogdan, "Representation of α-harmonic functions in Lipschitz domains," *Hiroshima Mathematical Journal*, vol. 29, no. 2, pp. 227–243, 1999.

[9] L. Caffarelli and L. Silvestre, "An extension problem related to the fractional Laplacian," *Communications in Partial Differential Equations*, vol. 32, no. 7-9, pp. 1245–1260, 2007.

[10] A. Elgart and B. Schlein, "Mean field dynamics of boson stars," *Communications on Pure and Applied Mathematics*, vol. 60, no. 4, pp. 500–545, 2007.

[11] J. Fröhlich and E. Lenzmann, "Blowup for nonlinear wave equations describing boson stars," *Communications on Pure and Applied Mathematics*, vol. 60, no. 11, pp. 1691–1705, 2007.

[12] L. A. Caffarelli, "Further regularity for the Signorini problem," *Communications in Partial Differential Equations*, vol. 4, no. 9, pp. 1067–1075, 1979.

[13] A. Signorini, "Questioni di elasticità non linearizzata e semilinearizzata," vol. 18, pp. 95–139, 1959.

[14] L. A. Caffarelli and A. Vasseur, "Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation," *Annals of Mathematics*, vol. 171, no. 3, pp. 1903–1930, 2010.

[15] I. Athanasopoulos, L. A. Caffarelli, and S. Salsa, "The structure of the free boundary for lower dimensional obstacle problems," *American Journal of Mathematics*, vol. 130, no. 2, pp. 485–498, 2008.

[16] L. A. Caffarelli, S. Salsa, and L. Silvestre, "Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian," *Inventiones Mathematicae*, vol. 171, no. 2, pp. 425–461, 2008.

[17] M. Kassmann, "A priori estimates for integro-differential operators with measurable kernels," *Calculus of Variations and Partial Differential Equations*, vol. 34, no. 1, pp. 1–21, 2009.

[18] L. Silvestre, "Regularity of the obstacle problem for a fractional power of the Laplace operator," *Communications on Pure and Applied Mathematics*, vol. 60, no. 1, pp. 67–112, 2007.

[19] S. Abe and S. Thurner, "Anomalous diffusion in view of Einstein's 1905 theory of Brownian motion," *Physica A*, vol. 356, no. 2-4, pp. 403–407, 2005.

[20] M. Jara, "Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps," *Communications on Pure and Applied Mathematics*, vol. 62, no. 2, pp. 198–214, 2009.

[21] A. Mellet, S. Mischler, and C. Mouhot, "Fractional diffusion limit for collisional kinetic equations," *Archive for Rational Mechanics and Analysis*, vol. 199, no. 2, pp. 493–525, 2011.

[22] L. Vlahos, H. Isliker, Y. Kominis, and K. Hizonidis, "Normal and anomalous diffusion: a tutorial," in *Order and Chaos*, T. Bountis, Ed., vol. 10, Patras University Press, Patras, Greece, 2008.

[23] H. Weitzner and G. M. Zaslavsky, "Some applications of fractional equations. Chaotic transport and complexity in classical and quantum dynamics," *Communications in Nonlinear Science and Numerical Simulation*, vol. 8, no. 3-4, pp. 273–281, 2003.

[24] J. L. Vazquez, "Nonlinear diffusion with fractional Laplacian operators," *Non-Linear Partial Differential Equations*, vol. 7, pp. 271–298, 2012.

[25] E. Di Nezza, G. Palatucci, and E. Valdinoci, "Hitchhiker's guide to the fractional Sobolev spaces," *Bulletin des Sciences Mathématiques*, vol. 136, no. 5, pp. 521–573, 2012.

[26] R. Chemmam, H. Mâagli, and S. Masmoudi, "On a new Kato class and positive solutions of Dirichlet problems for the fractional Laplacian in bounded domains," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 5, pp. 1555–1576, 2011.

[27] H. Mâagli and M. Zribi, "On a new Kato class and singular solutions of a nonlinear elliptic equation in bounded domains of $\mathbb{R}^n$," *Positivity*, vol. 9, no. 4, pp. 667–686, 2005.

[28] R. Chemmam, H. Mâagli, and S. Masmoudi, "Boundary behavior of positive solutions of a semilinear fractional Dirichlet problem," *Journal of Abstract Differential Equations and Applications*, vol. 3, no. 2, pp. 75–90, 2012.

[29] F. C. Cîrstea and V. Rădulescu, "Nonlinear problems with boundary blow-up: a Karamata regular variation theory approach," *Asymptotic Analysis*, vol. 46, no. 3-4, pp. 275–298, 2006.

[30] V. Marić, *Regular Variation and Differential Equations*, vol. 1726 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2000.

[31] E. Seneta, *Regularly Varying Functions*, vol. 508, Springer, Berlin, Germany, 1976.

[32] S. Gontara, H. Mâagli, S. Masmoudi, and S. Turki, "Asymptotic behavior of positive solutions of a singular nonlinear Dirichlet problem," *Journal of Mathematical Analysis and Applications*, vol. 369, no. 2, pp. 719–729, 2010.

[33] H. Mâagli, "Asymptotic behavior of positive solutions of a semilinear Dirichlet problem," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 74, no. 9, pp. 2941–2947, 2011.

[34] A. Ambrosetti, H. Brezis, and G. Cerami, "Combined effects of concave and convex nonlinearities in some elliptic problems," *Journal of Functional Analysis*, vol. 122, no. 2, pp. 519–543, 1994.

[35] L. Boccardo, "A Dirichlet problem with singular and supercritical nonlinearities," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 75, no. 12, pp. 4436–4440, 2012.

[36] R. Chemmam, H. Mâagli, S. Masmoudi, and M. Zribi, "Combined effects in nonlinear singular elliptic problems in a bounded domain," *Advances in Nonlinear Analysis*, vol. 1, no. 4, pp. 301–318, 2012.

[37] V. Rădulescu and D. Repovš, "Combined effects in nonlinear problems arising in the study of anisotropic continuous media," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 75, no. 3, pp. 1524–1530, 2012.

[38] S. Yijing and L. Shujie, "Some remarks on a superlinear-singular problem: estimates of λ^*," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 69, no. 8, pp. 2636–2650, 2008.

[39] Y. Sun, S. Wu, and Y. Long, "Combined effects of singular and superlinear nonlinearities in some singular boundary value problems," *Journal of Differential Equations*, vol. 176, no. 2, pp. 511–531, 2001.

[40] Z.-Q. Chen and R. Song, "Estimates on Green functions and Poisson kernels for symmetric stable processes," *Mathematische Annalen*, vol. 312, no. 3, pp. 465–501, 1998.

[41] T. Kulczycki, "Properties of Green function of symmetric stable processes," *Probability and Mathematical Statistics*, vol. 17, pp. 339–364, 1997.

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Research Article

The Boundedness and Exponential Stability Criterions for Nonlinear Hybrid Neutral Stochastic Functional Differential Equations

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Neutral differential equations have been used to describe the systems that not only depend on the present and past states but also involve derivatives with delays. This paper considers hybrid nonlinear neutral stochastic functional differential equations (HNSFDEs) without the linear growth condition and examines the boundedness and exponential stability. Two illustrative examples are given to show the effectiveness of our theoretical results.

1. Introduction

Many dynamic systems not only depend on the present and past states but also involve derivatives with delays. Neutral differential equations have been used to model such systems. Deterministic neutral differential equations were introduced by Hale and Meyer [1] and discussed in Hale and Lunel (see [2]) and Kolmanovskii et al. (for details see also [3, 4]), among others. Such equations were used to study two or more simple oscillatory systems with some interconnections between them, such as Brayton [5], Rubanik [6], and Driver [7].

Generally speaking, many practical systems commonly encounter stochastic perturbations and may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs and abrupt environmental disturbances. Of course, there is no exception to neutral systems, mentioned previous. Taking these stochastic factors into account, Mao and Yuan developed hybrid systems driven by Brownian motion and continuous-time Markovian chain to cope with such a situation (see [8]). Hu et al. [9] investigated the stability and boundedness of stochastic differential delay equations with Markovian switching. Kolmanovskii et al. [10] discussed the neutral stochastic delay differential equations with Markovian switching, also known as hybrid neutral stochastic delay differential equations (HNSDDEs).

The boundedness and stability analysis of the neutral stochastic systems without switching has attracted much attention; see [11–18] to mention a few. For hybrid neutral systems, studying boundedness and stability of the solutions is also a challenging and interesting work. Kolmanovskii et al. [10] established a fundamental theorem of HNSDDEs and discussed the boundedness and exponential stability of the solutions. They also gave an example to show that Markovian can average the subsystems; that is, when some subsystems are stable and others are not stable, the overall system formed by the Markovian switching may be stable. Bao et al. [19] discussed stability in distribution of the HNSDDEs. Hu and Wang [20] studied the stability in distribution for the general HNSFDEs. The stability of HNSDDEs with interval uncertainty was investigated in [21]. Mao et al. [22] gave a criterion related to almost surely asymptotic stability of HNSDDEs. These results are undoubtedly remarkable.

However, there are few publications on the boundedness and exponential stability of the general HNSFDEs with highly nonlinear terms. To fill in this gap, this work gives the boundedness and exponential stability criterions for such

HNSFDEs. Moreover, when this HNSFDE degenerates to the HNSDDE, our stability criterions improve the related results in [10]. Further, these stability criterions can also be used to investigate the exponential stability of NSFDEs or NSDDEs with more accurate Lyapunov exponent bound than that obtained in [23, 24].

The rest of the paper is arranged as follows. The next section provides necessary notations and definitions for the use of this paper. Section 3 establishes the boundedness and exponential stability criterions of the solutions to HNSFDEs. Section 4 further gives the generalized results for the HNSDDEs with variable time delay. Finally, two illustrative examples are provided to show the effectiveness of our theoretical results.

2. Notations and Definitions

Throughout this paper, unless otherwise specified, we use the following notations. $|\cdot|$ denotes both the Euclidean vector norm in $\mathbb{R}^n$ and Frobenius matrix norm in $\mathbb{R}^{n\times d}$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^TA)}$. Let $(\Omega, \mathfrak{F}, P)$ be a complete probability space with a filtration $\{\mathfrak{F}_t\}_{t\geq 0}$ satisfying the usual conditions; that is, it is right continuous and increasing while $\mathfrak{F}_0$ contains all P-null sets. Let $w(t) = (w_1(t), \dots, w_d(t))$ be a d-dimensional Brownian motion defined on this probability space. Let $\mathbb{R}_+ = [0,\infty)$ and $\tau > 0$. Denote by $C([-\tau,0],\mathbb{R}^n)$ the family of continuous functions from $[-\tau,0]$ to $\mathbb{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\leq\theta\leq 0}|\varphi(\theta)|$. Let $C^b_{\mathfrak{F}_0}([-\tau,0],\mathbb{R}^n)$ be the family of all $\mathfrak{F}_0$-measurable bounded $C([-\tau,0],\mathbb{R}^n)$ valued random variables $\xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\}$. $a \vee b$ represents $\max\{a, b\}$, and $a \wedge b$ denotes $\min\{a, b\}$.

Let $r(t)$ be a Markov chain (independent of $w(t)$) taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, m\}$. Assume the generator of $r(t)$ is denoted by $\Gamma = (\gamma_{ij})_{m\times m}$, so that

$$\mathbb{P}\{r(t+\delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta), & \text{if } i = j, \end{cases} \tag{1}$$

where $\delta > 0$. Here γ_{ij} is the transition rate from i to j and $\gamma_{ij} > 0$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. Let us consider the following n-dimensional nonlinear HNSFDE:

$$\begin{aligned} d\left[x(t) - u(x_t, r(t))\right] \\ = f(x_t, r(t))\,dt + g(x_t, r(t))\,dw(t), \quad t \geq 0, \end{aligned} \tag{2}$$

with initial data $x_0 = \xi \in C^b_{\mathfrak{F}_0}([-\tau,0];\mathbb{R}^n)$ and $r(0) = r_0 \in \mathbb{S}$, where

$$x_t = \{x(t+\theta); -\tau \leq \theta \leq 0\} \in C([-\tau,0];\mathbb{R}^n), \tag{3}$$

$u : C([-\tau,0],\mathbb{R}^n)\times\mathbb{S} \to \mathbb{R}^n$, $f : C([-\tau,0],\mathbb{R}^n)\times\mathbb{S} \to \mathbb{R}^n$, and $g : C([-\tau,0],\mathbb{R}^n)\times\mathbb{S} \to \mathbb{R}^{n\times d}$. In order to guarantee the existence and uniqueness of the solution to (2), we give the following assumptions for the functionals u, f, and g.

Assumption 1 (local Lipschitz condition). f and g satisfy the local Lipschitz condition; that is, for each $j > 0$ there exists a positive constant C_j such that for any maps ϕ, $\varphi \in C([-\tau,0];\mathbb{R}^n)$ with $\|\phi\| \vee \|\varphi\| \leq j$

$$\begin{aligned} &\left|f(\phi,i) - f(\varphi,i)\right| \vee \left|g(\phi,i) - g(\varphi,i)\right| \\ &\quad \leq C_j\left\|\phi - \varphi\right\|, \quad \forall i \in \mathbb{S}, \end{aligned} \tag{4}$$

where $\|\phi - \varphi\| \leq \sup_{-\tau\leq\theta\leq 0}|\phi(\theta) - \varphi(\theta)|$.

Assumption 2 (contractive mapping). There exists a positive constant $\kappa \in (0,1)$ such that for all ϕ, $\varphi \in C([-\tau,0];\mathbb{R}^n)$ and $i \in \mathbb{S}$

$$\left|u(\phi,i) - u(\varphi,i)\right| \leq \kappa\left\|\phi - \varphi\right\| \tag{5}$$

and $u(0,i) = 0$.

Note that the previous assumptions are standard for the existence and uniqueness of the local solutions (see [19, 22]). Additional conditions should be imposed for the local solution to be global. In view of this, we need a few more notations. Let $C^2(\mathbb{R}^n\times\mathbb{S};\mathbb{R}_+)$ denote the family of all nonnegative functions $V(x,i)$ on $\mathbb{R}^n\times\mathbb{S}$ which are continuously twice differentiable in x. For each $V(x,i) \in C^2(\mathbb{R}^n\times\mathbb{S};\mathbb{R}_+)$, define an operator $\mathscr{L}V$ from $C([-\tau,0];\mathbb{R}^n)\times\mathbb{S}$ to $\mathbb{R}$:

$$\begin{aligned} &\mathscr{L}V(\varphi,i) \\ &\quad = V_x(\varphi(0) - u(\varphi,i), i)\,f(\varphi,i) \\ &\qquad + \sum_{j\in\mathbb{S}}\gamma_{ij}V(\varphi(0) - u(\varphi,i), j) \\ &\qquad + \frac{1}{2}\text{trace}\left[g^T(\varphi,i)\,V_{xx}(\varphi(0) - u(\varphi,i), i)\,g(\varphi,i)\right], \end{aligned} \tag{6}$$

where

$$\begin{aligned} V_x(x,i) &= \left(\frac{\partial V(x,i)}{\partial x_1}, \dots, \frac{\partial V(x,i)}{\partial x_n}\right), \\ V_{xx}(x,i) &= \left(\frac{\partial^2 V(x,i)}{\partial x_j\partial x_l}\right)_{n\times n}. \end{aligned} \tag{7}$$

In the following sections, we will impose the some conditions on the diffusion operator $\mathscr{L}V$ for the global solution and its asymptotic behavior.

3. The Boundedness and Exponential Stability of HNSFDEs

The following theorem gives the boundedness and exponential stability criterions of the solution to (2).

Theorem 3. *Let Assumptions 1 and 2 hold. Assume that there are two functions $V \in C^2(\mathbb{R}^n\times\mathbb{S};\mathbb{R}_+)$, $U \in C(\mathbb{R}^n;\mathbb{R}_+)$, three probability measures η, μ, $\overline{\mu}$ on $[-\tau,0]$, and a number* *of positive constants* $\kappa \in (0,1)$, c, c_1, c_2, $p \geq 1$, λ_1, λ_2, λ_3, λ_4 *such that for any* $x \in \mathbb{R}^n$ *and* $(\varphi, i) \in C([-\tau,0],\mathbb{R}_+) \times \mathbb{S}$

$$|u(\varphi, i)| \leq \kappa \int_{-\tau}^{0} \varphi(\theta)\, \eta(d\theta), \tag{8}$$

$$c_1 |x|^p \leq V(x,i) \leq c_2 |x|^p, \tag{9}$$

$$\begin{aligned}\mathscr{L}V(\varphi, i) &\leq c - \lambda_1 |\varphi(0)|^p + \lambda_2 \int_{-\tau}^{0} |\varphi(\theta)|^p \mu(d\theta) - \lambda_3 U(\varphi(0)) \\ &\quad + \lambda_4 \int_{-\tau}^{0} U(\varphi(\theta))\, \overline{\mu}(d\theta).\end{aligned} \tag{10}$$

If $\lambda_1 > \lambda_2$ *and* $\lambda_3 > \lambda_4$, *then we have the following assertions:*

(i) *for any given initial data* $\xi \in C^b_{\mathfrak{F}_0}([-\tau,0],\mathbb{R}^n)$, *there is a unique global solution* $x(t) = x(t;\xi)$ *to the hybrid system* (2) *on* $t \in [-\tau, \infty)$;

(ii) *the solution* $x(t)$ *obeys*

$$\limsup_{t\to\infty} \mathbb{E}|x(t) - u(x_t, r(t))|^p \leq \frac{c}{\Lambda}, \tag{11}$$

$$\limsup_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{E}U(x(s))\, ds \leq \frac{c}{\lambda_3 - \lambda_4}, \tag{12}$$

where $\Lambda := \overline{\gamma} \wedge (1/\tau)\log(\lambda_3/\lambda_4) \wedge r$ *with* $\overline{\gamma}$ *and* r *defined by*

$$\begin{aligned}\overline{\gamma} = \max\Big\{ q > 0;\, & c_2 q \left(1 + \varepsilon^{1/(p-1)}\right)^{p-1} - \lambda_1 \\ & + \left[c_2 q \left(1 + \varepsilon^{1/(p-1)}\right)^{p-1} \frac{\kappa^p}{\varepsilon} + \lambda_2 \right] e^{q\tau} = 0,\ \varepsilon > 0 \Big\}\end{aligned} \tag{13}$$

and $r := (p/\tau)\log(1/\kappa) - \ell$ *for sufficiently small* $\ell > 0$.

(iii) *If, in addition,* $c = 0$, *then the solution of* (2) *has properties that*

$$\limsup_{t\to\infty} \frac{\log(\mathbb{E}|x(t)|^p)}{t} \leq -\Lambda, \tag{14}$$

$$\begin{aligned}\int_0^\infty \mathbb{E}U(x(s))\, ds &\leq \frac{1}{\lambda_3 - \lambda_4} \left[\mathbb{E}V(y_0, r(0)) + \lambda_2 \int_{-\tau}^{0} \mathbb{E}|x(s)|^p\, ds \right. \\ &\quad \left. + \lambda_4 \mathbb{E} \int_{-\tau}^{0} U(x(s))\, ds \right],\end{aligned} \tag{15}$$

where $y_0 = x(0) - u(x_0, r(0))$.

Proof. We prove these three assertions, separately. For any given initial data $\xi \in C^b_{\mathfrak{F}_0}([-\tau,0],\mathbb{R}^n)$, by Assumptions 1 and 2, there exists a unique maximal local strong solution $x(t)$ to (2) on $t \in [-\tau, \rho_e)$, where ρ_e is the explosion time. To show that this solution is global, we only need to prove that $\rho_e = \infty$ a.s. Define $y_t = x(t) - u(x_t, r(t))$, then by Assumption 2, we have

$$|y_0| \leq |x(0)| + |u(x_0, r(0))| \leq (1+\kappa)\|\xi\|. \tag{16}$$

Let k_0 be sufficiently lager positive number, such that $\|\xi\| < k_0$. For each $k > (1+\kappa)k_0$, define the stopping time $\rho_k = \inf\{t \in [0,\rho_e) : |y_t| \geq k\}$. Clearly, ρ_k is increasing as $k \to \infty$ and $\rho_k \to \rho_\infty \leq \rho_e$ a.s. If we can show $\rho_\infty = \infty$ a.s., then $\rho_e = \infty$, which implies that the solution $x(t)$ is actually global. By the generalized Itô formula (see [20]) and condition (10), we can obtain that, for any $k > k_0$ and $t \geq 0$,

$$\begin{aligned}\mathbb{E}V&\left(y_{(t\wedge\rho_k)}, r(t\wedge\rho_k)\right) \\ &= \mathbb{E}V(y_0, r(0)) + \mathbb{E}\int_0^{t\wedge\rho_k} \mathscr{L}V(x_s, r(s))\, ds \\ &\leq ct + \mathbb{E}V(y_0, r(0)) - \lambda_1 \mathbb{E}\int_0^{t\wedge\rho_k} |x(s)|^p\, ds \\ &\quad + \lambda_2 \mathbb{E}\int_0^{t\wedge\rho_k} \int_{-\tau}^{0} |x(s+\theta)|^p \mu(d\theta)\, ds \\ &\quad - \lambda_3 \mathbb{E}\int_0^{t\wedge\rho_k} U(x(s))\, ds \\ &\quad + \lambda_4 \mathbb{E}\int_0^{t\wedge\rho_k} \int_{-\tau}^{0} U(x(s+\theta))\, \overline{\mu}(d\theta)\, ds.\end{aligned} \tag{17}$$

By the Fubini theorem, we compute

$$\begin{aligned}\mathbb{E}&\int_0^{t\wedge\rho_k} \int_{-\tau}^{0} |x(s+\theta)|^p d\mu(\theta)\, ds \\ &= \mathbb{E}\int_{-\tau}^{0} \int_0^{t\wedge\rho_k} |x(s+\theta)|^p ds\, d\mu(\theta) \\ &\leq \int_{-\tau}^{0} \mathbb{E}|x(s)|^p ds + \mathbb{E}\int_0^{t\wedge\rho_k} |x(s)|^p\, ds.\end{aligned} \tag{18}$$

Similarly,

$$\begin{aligned}\mathbb{E}&\int_0^{t\wedge\rho_k} \int_{-\tau}^{0} U(x(s+\theta))\, \overline{\mu}(d\theta)\, ds \\ &\leq \int_{-\tau}^{0} \mathbb{E}U(x(s))\, ds + \mathbb{E}\int_0^{t\wedge\rho_k} U(x(s))\, ds.\end{aligned} \tag{19}$$

Substituting (18) and (19) into (17) yields

$$\begin{aligned}\mathbb{E}V\left(y_{(t\wedge\rho_k)}, r(t\wedge\rho_k)\right) &\le \mathbb{E}V\left(y_0, r(0)\right) \\ &\quad + \lambda_2\int_{-\tau}^{0}\mathbb{E}|x(s)|^p ds + \lambda_4\mathbb{E}\int_{-\tau}^{0} U(x(s))\,ds \\ &\quad + ct - (\lambda_1-\lambda_2)\,\mathbb{E} \\ &\quad \times \int_0^{t\wedge\rho_k}|x(s)|^p ds - (\lambda_3-\lambda_4)\,\mathbb{E}\int_0^{t\wedge\rho_k} U(x(s))\,ds \\ &\le H_0 + ct,\end{aligned} \tag{20}$$

where

$$\begin{aligned}H_0 = \mathbb{E}V\left(y_0, r(0)\right) + \lambda_2\int_{-\tau}^{0}\mathbb{E}|x(s)|^p ds \\ + \lambda_4\mathbb{E}\int_{-\tau}^{0} U(x(s))\,ds.\end{aligned} \tag{21}$$

Note that

$$\begin{aligned}\mathbb{E}V\left(y_{(t\wedge\rho_k)}, r(t\wedge\rho_k)\right) \\ \ge \mathbb{E}\left[V\left(y_{(t\wedge\rho_k)}, r(t\wedge\rho_k)\right)1_{\{\rho_k\le t\}}\right] \ge c_1 k^p\,\mathbb{P}\{\rho_k\le t\}.\end{aligned} \tag{22}$$

Hence,

$$c_1 k^p\,\mathbb{P}\{\rho_k\le t\} \le H_0 + ct. \tag{23}$$

Then, for any $t>0$,

$$\lim_{k\to\infty}\mathbb{P}\{\rho_k\le t\} = 0, \tag{24}$$

which together with the arbitrariness of t implies that $\rho_\infty = \infty$ a.s. Therefore, the solution $x(t)$ is global, and assertion (i) follows.

Then by Itô's formula and condition (10), we have, for any $\gamma>0$,

$$\begin{aligned}\mathbb{E}e^{\gamma(t\wedge\rho_k)}V\left(y_{(t\wedge\rho_k))}, r(t\wedge\rho_k)\right) &= \mathbb{E}V\left(y_0, r(0)\right) \\ &\quad + \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}\left[\gamma V\left(y_s, r(s)\right) + \mathscr{L}V\left(x_s, r(s)\right)\right] ds \\ &\le \mathbb{E}V\left(y_0, r(0)\right) + \gamma\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}V\left(y_s, r(s)\right) ds \\ &\quad - \lambda_1\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}|x(s)|^p ds \\ &\quad + \lambda_2\mathbb{E}\int_0^{t\wedge\rho_k}\int_{-\tau}^{0} e^{\gamma s}|x(s+\theta)|^p\mu(d\theta)\,ds \\ &\quad + c\int_0^{t} e^{\gamma s} ds \\ &\quad - \lambda_3\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}U(x(s))\,ds \\ &\quad + \lambda_4\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}\int_{-\tau}^{0} U(x(s+\theta))\,\overline{\mu}(d\theta)\,ds.\end{aligned} \tag{25}$$

For $p\ge 1$ and any $\varepsilon>0$, we have

$$\begin{aligned}V\left(y_s, r(s)\right) &\le c_2\left|x(s) - u\left(x_s, r(s)\right)\right|^p \\ &\le c_2\left[1+\varepsilon^{1/(p-1)}\right]^{p-1}\left(|x(s)|^p + \frac{1}{\varepsilon}\left|u\left(x_s, r(s)\right)\right|^p\right) \\ &\le c_2\left[1+\varepsilon^{1/(p-1)}\right]^{p-1}\left[|x(s)|^p + \frac{\kappa^p}{\varepsilon}\int_{-\tau}^{0}|x(s+\theta)|^p d\eta(\theta)\right],\end{aligned} \tag{26}$$

where we used the Hölder inequality and condition (8). Substituting (26) into (25), we obtain

$$\begin{aligned}e^{\gamma(t\wedge\rho_k)}\mathbb{E}V\left(y_{(t\wedge\rho_k)}, r(t\wedge\rho_k)\right) &\le \mathbb{E}V\left(y_0\right) + \left[c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1} - \lambda_1\right] \\ &\quad \times \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}V(x(s))\,ds \\ &\quad + c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon} \\ &\quad \times \mathbb{E}\int_0^{t\wedge\rho_k}\int_{-\tau}^{0} e^{\gamma s}|x(s+\theta)|^p\eta(d\theta)\,ds \\ &\quad + \lambda_2\mathbb{E}\int_0^{t\wedge\rho_k}\int_{-\tau}^{0} e^{\gamma s}|x(s+\theta)|^p\mu(d\theta)\,ds + c\int_0^{t} e^{\gamma s}\,ds \\ &\quad - \lambda_3\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}U(x(s))\,ds \\ &\quad + \lambda_4\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}\int_{-\tau}^{0} U(x(s+\theta))\,\overline{\mu}(d\theta)\,ds.\end{aligned} \tag{27}$$

Define a probability measure ν on $[-\tau, 0]$

$$d\nu(\theta) = \frac{c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}(\kappa^p/\varepsilon)\,\eta(\theta) + \lambda_2\mu(\theta)}{c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}(\kappa^p/\varepsilon) + \lambda_2}; \tag{28}$$

then from (27), we have

$$\begin{aligned}
&\mathbb{E}e^{\gamma(t\wedge\rho_k)}V\left(y_{(t\wedge\rho_k)}, r\left(t\wedge\rho_k\right)\right)\\
&\quad\leq \mathbb{E}V\left(y_0\right)+c\int_0^t e^{\gamma s}\,ds\\
&\qquad+\left[c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}-\lambda_1\right]\\
&\qquad\times\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}V\left(x\left(s\right)\right)ds\\
&\qquad+\left[c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon}+\lambda_2\right]\\
&\qquad\times\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}\int_{-\tau}^0 \left|x\left(s+\theta\right)\right|^p\nu\left(d\theta\right)ds\\
&\qquad-\lambda_3\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}U\left(x\left(s\right)\right)ds\\
&\qquad+\lambda_4\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}\int_{-\tau}^0 U\left(x\left(s+\theta\right)\right)\overline{\mu}\left(d\theta\right)ds.
\end{aligned}\tag{29}$$

By the Fubini theorem

$$\begin{aligned}
&\int_0^t e^{\gamma s}\int_{-\tau}^0 \left|x\left(s+\theta\right)\right|^p d\nu\left(\theta\right)ds\\
&\quad=\int_{-\tau}^0\int_0^t e^{\gamma(s+\theta)}\left|x\left(s+\theta\right)\right|^p ds\,d\nu\left(\theta\right)\\
&\quad\leq e^{\gamma\tau}\int_{-\tau}^0 e^{\gamma s}\left|x\left(s\right)\right|^p ds+e^{\gamma\tau}\int_0^t e^{\gamma s}\left|x\left(s\right)\right|^p ds,
\end{aligned}\tag{30}$$

we have from (29)

$$\begin{aligned}
&\mathbb{E}e^{\gamma(t\wedge\rho_k)}V\left(y_{(t\wedge\rho_k)}, r\left(t\wedge\rho_k\right)\right)\\
&\quad\leq \mathbb{E}V\left(y_0\right)+\left[c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon}+\lambda_2\right]\\
&\qquad\times e^{\gamma\tau}\int_{-\tau}^0 e^{\gamma s}\mathbb{E}V\left(x\left(s\right)\right)ds+c\int_0^t e^{\gamma s}\,ds\\
&\qquad+\Bigg(c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\\
&\qquad\qquad-\lambda_1+\left[c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon}+\lambda_2\right]e^{\gamma\tau}\Bigg)\\
&\qquad\times\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}V\left(x\left(s\right)\right)ds\\
&\qquad+\lambda_4 e^{\gamma\tau}\int_{-\tau}^0 e^{\gamma s}\mathbb{E}U\left(x\left(s\right)\right)ds-\left[\lambda_3-\lambda_4e^{\gamma\tau}\right]\\
&\qquad\times\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}U\left(x\left(s\right)\right)ds.
\end{aligned}\tag{31}$$

Denote

$$\begin{aligned}
l\left(\gamma\right)&=\lambda_3-\lambda_4e^{\gamma\tau},\\
h\left(\gamma,\varepsilon\right)&=c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\\
&\quad-\lambda_1+\left[c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon}+\lambda_2\right]e^{\gamma\tau}.
\end{aligned}\tag{32}$$

Let ε be fixed; then it is easy to obtain $h'_\gamma(\gamma,\varepsilon)>0$ and $h(0,\varepsilon)=-\lambda_1+\lambda_2<0$, which implies that for any fixed $\varepsilon>0$, function $h(\cdot,\varepsilon)$ has a unique positive root, denoted by q. Choose a $\varepsilon=\varepsilon^*>0$ such that

$$\overline{\gamma}=\sup_{\varepsilon>0,h(q,\varepsilon)=0} q=\sup_{h(q,\varepsilon^*)=0} q.\tag{33}$$

Noting that for any $\gamma\in(0,\Lambda]$, $h(\gamma,\varepsilon^*)\leq 0$ and $l(\gamma)\geq 0$, we therefore have

$$\begin{aligned}
&\mathbb{E}e^{\gamma(t\wedge\rho_k)}V\left(y_{(t\wedge\rho_k)}, r\left(t\wedge\rho_k\right)\right)\\
&\quad\leq \mathbb{E}V\left(y_0\right)+\lambda_4e^{\gamma\tau}\int_{-\tau}^0 e^{\gamma s}\mathbb{E}U\left(x\left(s\right)\right)ds+c\int_0^t e^{\gamma s}\,ds\\
&\qquad+\left[c_2\gamma\left(1+\varepsilon^{*(1/(p-1))}\right)^{p-1}\frac{\kappa^p}{\varepsilon^*}+\lambda_2\right]e^{\gamma\tau}\\
&\qquad\times\int_{-\tau}^0 e^{\gamma s}\mathbb{E}V\left(\xi\left(s\right)\right)ds\\
&\quad\leq c_1C_0\sup_{-\tau\leq\theta\leq0}\mathbb{E}\left|\xi\left(\theta\right)\right|^p+c\frac{e^{\gamma t}}{\gamma}
\end{aligned}\tag{34}$$

for some positive constant $C_0>1$. Letting $k\to\infty$, we have

$$e^{\gamma t}\mathbb{E}V\left(y_t, r\left(t\right)\right)\leq c_1C_0\sup_{-\tau\leq\theta\leq0}\mathbb{E}\left|\xi\left(\theta\right)\right|^p+c\frac{e^{\gamma t}}{\gamma},\tag{35}$$

which implies the desired assertion (11). Assertion (12) can be obtained from (20) by letting $k\to\infty$. Hence assertion (ii) follows.

Let $c=0$. For any $\epsilon>0$, we have that

$$\begin{aligned}
\left|x\left(s\right)\right|^p&\leq\left[1+\epsilon^{1/(p-1)}\right]^{p-1}\\
&\qquad\times\left[\left|y_s\right|^p+\frac{\left|u\left(x_s, r\left(s\right)\right)\right|^p}{\epsilon}\right]\\
&\leq\left[1+\epsilon^{1/(p-1)}\right]^{p-1}\\
&\qquad\times\left[\left|y_s\right|^p+\frac{\kappa^p}{\epsilon}\int_{-\tau}^0\left|x\left(s+\theta\right)\right|^p d\eta\left(\theta\right)\right].
\end{aligned}\tag{36}$$

By (34) and (36), we have for $t > s > 0$

$$\begin{aligned} e^{\gamma s}\mathbb{E}|x(s)|^p &\le \left[1+\epsilon^{1/(p-1)}\right]^{p-1} \\ &\quad \times \left[e^{\gamma s}\mathbb{E}|y_s|^p + \frac{\kappa^p}{\epsilon}e^{\gamma s}\int_{-\tau}^{0}\mathbb{E}|x(s+\theta)|^p d\eta(\theta)\right] \\ &\le \left[1+\epsilon^{1/(p-1)}\right]^{p-1} \\ &\quad \times \left[e^{\gamma s}\mathbb{E}|y_s|^p + \frac{\kappa^p}{\epsilon}e^{\gamma\tau}\sup_{s-\tau\le\theta\le s}\left[e^{\gamma\theta}\mathbb{E}|x(\theta)|^p\right]\right] \\ &\le \left[1+\epsilon^{1/(p-1)}\right]^{p-1} \\ &\quad \times \left[C_0\sup_{-\tau\le\theta\le 0}\mathbb{E}|\xi(\theta)|^p + \frac{\kappa^p}{\epsilon}e^{\gamma\tau}\sup_{-\tau\le\theta\le t}\left[e^{\gamma\theta}\mathbb{E}|x(\theta)|^p\right]\right]. \end{aligned} \tag{37}$$

This inequality also holds for all $-\tau \le s \le 0$. In view of $\gamma \le r < (p/\tau)\log(1/\kappa)$, there exists a positive number $\epsilon_0 > 0$ such that

$$\overline{a}(\epsilon_0) := \left[1+\epsilon_0^{1/(p-1)}\right]^{p-1}\frac{\kappa^p}{\epsilon_0}e^{\gamma\tau} = \left[1+\frac{1}{\epsilon_0^{1/(p-1)}}\right]^{p-1}\kappa^p e^{\gamma\tau} < 1. \tag{38}$$

Therefore,

$$\begin{aligned} \sup_{-\tau\le s\le t} e^{\gamma s}\mathbb{E}|x(s)|^p &\le \left[1+\epsilon_0^{1/(p-1)}\right]^{p-1}C_0\sup_{-\tau\le\theta\le 0}\mathbb{E}|\xi(\theta)|^p \\ &\quad + \overline{a}(\epsilon_0)\sup_{-\tau\le s\le t}e^{\gamma s}\mathbb{E}|x(s)|^p, \end{aligned} \tag{39}$$

which implies

$$e^{\gamma t}\mathbb{E}|x(t)|^p \le \frac{\left[1+\epsilon_0^{1/(p-1)}\right]^{p-1}C_0}{1-\overline{a}(\epsilon_0)}\sup_{-\tau\le t\le 0}\mathbb{E}|\xi(t)|^p. \tag{40}$$

Finally, the required inequality (14) follows by taking logarithm and limitation. Inequality (15) can be also obtained from (20) by letting $k \to \infty$. Hence assertion (iii) follows. □

Remark 4. If the Markovian switching vanishes, Theorem 3 is also true and gives the pth moment exponential stability with the decay rate bigger than that in [24, Theorem 2]. Since the decay rate in [24] is the special case of Theorem 3 with $\varepsilon = 1$ in (13).

If $\lambda_3 = \lambda_4 = 0$, then we directly obtain the following corollary.

Corollary 5. *Let Assumptions 1 and 2 hold. Assume that there exist a function $V \in C^2(\mathbb{R}^n \times \mathbb{S};\mathbb{R}_+) \times \mathbb{S}$, two probability measures η, μ on $[-\tau, 0]$, and a number of positive constants $\kappa \in (0,1)$, c_1, c_2, $p \ge 1$, λ_1, λ_2 such that for any $x \in \mathbb{R}^n$ and $(\varphi, i) \in C([-\tau,0],\mathbb{R}_+)\times\mathbb{S}$*

$$\begin{gathered} \left|u(\varphi,i)\right| \le \kappa\int_{-\tau}^{0}\varphi(\theta)\eta(d\theta), \\ c_1|x|^p \le V(x,i) \le c_2|x|^p, \\ \mathscr{L}V(\varphi,i) \le -\lambda_1|\varphi(0)|^p + \lambda_2\int_{-\tau}^{0}|\varphi(\theta)|^p\mu(d\theta). \end{gathered} \tag{41}$$

If $\lambda_1 > \lambda_2$, then for any given initial data $\xi \in C^b_{\mathfrak{F}_0}([-\tau,0],\mathbb{R}^n)$, the solution of (2), denoted by $x(t) = x(t;\xi)$, has property that

$$\limsup_{t\to\infty}\frac{\log\left(\mathbb{E}|x(t)|^p\right)}{t} \le -\left(\overline{\gamma}\wedge r\right), \tag{42}$$

where $\overline{\gamma}$ and r satisfy

$$\begin{aligned} \overline{\gamma} = \max\Big\{ & q > 0; c_2q\left(1+\varepsilon^{1/(p-1)}\right)^{p-1} - \lambda_1 \\ & + \left[c_2q\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon} + \lambda_2\right]e^{q\tau} = 0, \varepsilon > 0\Big\} \end{aligned} \tag{43}$$

and $r := (p/\tau)\log(1/\kappa) - \ell$ for sufficiently small $\ell > 0$.

Although the pth moment exponential stability and almost sure exponential stability of the exact solution do not imply each other in general, under a restrictive condition the pth moment exponential stability implies almost sure exponential stability (cf. [11]). Here, we give the following theorem about the almost sure exponential stability of the exact solution to (2).

Theorem 6. *Let $p \ge 1$. Assume that there exists a constant $K > 0$ such that*

$$\left|f(\varphi,i)\right| + \left|g(\varphi,i)\right| \le K\|\varphi\|, \quad \forall(\varphi,i)\in C([-\tau,0],\mathbb{R}_+)\times\mathbb{S}. \tag{44}$$

Then (42) implies

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \le -\frac{\overline{\gamma}\wedge r}{p} \quad a.s. \tag{45}$$

In other words, the pth moment exponential stability implies almost sure exponential stability.

Remark 7. One may question that whether the semimartingale technique can be used to obtain the almost sure exponential stability directly. In fact, semimartingale technique may fail, since it may not be true to transfer the almost sure exponential stability from $x(t) - u(x_t, r(t))$ to $x(t)$.

4. The Boundedness and Exponential Stability of HNSDDEs

In this section, we investigate the exponential stability of the hybrid NSDDE with varying delay

$$\begin{aligned} d\left[x(t)-N(x(t-\tau(t)), r(t))\right] &= F(x(t), x(t-\tau(t)), r(t))\, dt \\ &\quad + G(x(t), x(t-\tau(t)), r(t))\, dw(t), \end{aligned} \tag{46}$$

where $\tau(t) : \mathbb{R}^{+} \to [0,\tau]$ is a continuously differentiable function such that

$$\frac{d\tau(t)}{dt} \leq \bar{\tau} \tag{47}$$

for some constant $\bar{\tau} < 1$, while

$$\begin{aligned} &N : \mathbb{R}^{n} \times \mathbb{S} \to \mathbb{R}^{n}, \\ &F : \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{S} \to \mathbb{R}^{n}, \qquad G : \mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{S} \to \mathbb{R}^{n\times d}. \end{aligned} \tag{48}$$

For (46), we impose the following assumptions.

Assumption 8 (local Lipschitz condition). F and G satisfy the local Lipschitz condition; that is, for each $j > 0$ there exists a positive constant C_j such that

$$\begin{aligned} &\left|F(x,y,i) - F(\bar{x},\bar{y},i)\right| \vee \left|G(x,y,i) - G(\bar{x},\bar{y},i)\right| \\ &\quad \leq C_j\left(|x-\bar{x}| + |y-\bar{y}|\right) \end{aligned} \tag{49}$$

for all $i \in \mathbb{S}$ and $x, y, \bar{x}, \bar{y} \in \mathbb{R}^{n}$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq j$.

Assumption 9 (contractive mapping). N is a contractive mapping; that is, there exists a positive constant $\kappa \in (0,1)$ such that for all $x,\ y \in \mathbb{R}^{n}$ and $i \in \mathbb{S}$

$$\left|N(x,i) - N(y,i)\right| \leq \kappa\left|x-y\right|. \tag{50}$$

Under the previous two assumptions, HNSDDE (46) admits a unique local solution. We also need more conditions to guarantee that the local solution is actually global. So we introduce an operator LV from $\mathbb{R}^{n} \times \mathbb{R}^{n} \times \mathbb{S}$ to $\mathbb{R}$ by

$$\begin{aligned} LV(x,y,i) &= V_x(x-N(y,i),i)\, F(x,y,i) \\ &\quad + \sum_{j\in\mathbb{S}} \gamma_{ij} V(x-N(y,i),j) \\ &\quad + \frac{1}{2}\operatorname{trace}\left[G^{T}(x,y,i)\, V_{xx}(x-N(y,i),i)\, G(x,y,i)\right] \end{aligned} \tag{51}$$

for each $V(x,i) \in C^{2}(\mathbb{R}^{n} \times \mathbb{S}; \mathbb{R}_{+})$, and we will impose the same conditions on the diffusion operator LV for the global solution and its asymptotic behavior.

Although HNSDDE can be regarded as the special case of HNSFDEs, we still establish the boundedness and exponential stability criterions of the solution for (46) so as to obtain more accurate results.

Theorem 10. *Let Assumptions 8 and 9 hold. Assume that there are functions $V \in C^{2}(\mathbb{R}^{n} \times \mathbb{S}; \mathbb{R}_{+})$, $U \in C(\mathbb{R}^{n}; \mathbb{R}_{+})$ as well as a number of positive constants c, c_1, c_2, $p \geq 1$, λ_1, λ_2, λ_3, λ_4 such that for any $x,\ y \in \mathbb{R}^{n}$ and $i \in \mathbb{S}$,*

$$c_1|x|^{p} \leq V(x,i) \leq c_2|x|^{p}, \tag{52}$$

$$\begin{aligned} &LV(x,y,i) \\ &\quad \leq c - \lambda_1|x|^{p} + \lambda_2|y|^{p} - \lambda_3 U(x) + \lambda_4 U(y). \end{aligned} \tag{53}$$

If $\lambda_1 > \lambda_2/(1-\bar{\tau})$ and $\lambda_3 > \lambda_4/(1-\bar{\tau})$, then for any given initial data $\xi \in C^{b}_{\mathfrak{F}_0}([-\tau,0],\mathbb{R}^{n})$, (46) admits a unique global solution $x(t) = x(t;\xi)$. Moreover, we have the following assertions:

(i) *the solution $x(t)$ obeys*

$$\limsup_{t\to\infty} \mathbb{E}|x(t) - N(x(t-\tau(t)), r(t))|^{p} \leq \frac{c}{\Lambda}, \tag{54}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{E}U(x(s))\, ds \leq \frac{c}{\lambda_3 - \lambda_4/(1-\bar{\tau})}, \tag{55}$$

where $\Lambda := \bar{\gamma} \wedge (1/\tau)\log(\lambda_3/\lambda_4) \wedge r$ with $\bar{\gamma}$ and r defined by

$$\begin{aligned} \bar{\gamma} = \max\Big\{ &q > 0; c_2 q\left(1+\varepsilon^{1/(p-1)}\right)^{p-1} - \lambda_1 \\ &+ \left[c_2 q\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^{p}}{\varepsilon} + \lambda_2\right] e^{q\tau} = 0,\ \varepsilon > 0 \Big\} \end{aligned} \tag{56}$$

and $r := (p/\tau)\log(1/\kappa) - \ell$ for sufficiently small $\ell > 0$.

(ii) *If, in addition, $c = 0$, then the solution to (46) has properties that*

$$\limsup_{t\to\infty} \frac{\log\left(\mathbb{E}|x(t)|^{p}\right)}{t} \leq -\Lambda, \tag{57}$$

$$\begin{aligned} &\int_0^{\infty} \mathbb{E}U(x(s))\, ds \\ &\quad \leq \frac{1}{\lambda_3 - \lambda_4/(1-\bar{\tau})} \\ &\qquad \times \Big[\mathbb{E}V(x(0) - N(x(-\tau(0)), r(0)), r(0)) \\ &\qquad + \frac{\lambda_2}{1-\bar{\tau}}\int_{-\tau}^{0} \mathbb{E}|x(s)|^{p} ds + \frac{\lambda_4}{1-\bar{\tau}}\mathbb{E}\int_{-\tau}^{0} U(x(s))\, ds\Big]. \end{aligned} \tag{58}$$

Proof. The proof is similar to that of Theorem 3, so we only give an outlined one. Denote $y_t = x(t) - N(x(t-\tau(t)), r(t))$. Let ρ_k be the stopping time defined similarly in the proof of Theorem 3. By the generalized Itô formula (see [10]) and condition (53), we can obtain that, for any $k > k_0$ and $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}V\left(y_{(t\wedge\rho_k)}, r(t\wedge\rho_k)\right) \\
&\quad \leq ct + \mathbb{E}V\left(y_0, r(0)\right) - \lambda_1 \mathbb{E}\int_0^{t\wedge\rho_k} |x(s)|^p\, ds \\
&\qquad + \lambda_2 \mathbb{E}\int_0^{t\wedge\rho_k} |x(s-\tau(s))|^p\, ds \\
&\qquad - \lambda_3 \mathbb{E}\int_0^{t\wedge\rho_k} U(x(s))\, ds \\
&\qquad + \lambda_4 \mathbb{E}\int_0^{t\wedge\rho_k} U(x(s-\tau(s)))\, ds.
\end{aligned} \tag{59}$$

Noting that

$$\begin{aligned}
\int_0^t |x(s-\tau(s))|^p\, ds
&\leq \frac{1}{1-\bar{\tau}}\int_{-\tau}^0 |x(s)|^p\, ds \\
&\quad + \frac{1}{1-\bar{\tau}}\int_0^t |x(s)|^p\, ds,
\end{aligned} \tag{60}$$

then we have

$$\begin{aligned}
&\mathbb{E}V\left(y_{(t\wedge\rho_k)}, r(t\wedge\rho_k)\right) \\
&\quad \leq ct + \mathbb{E}V\left(y_0, r(0)\right) + \frac{\lambda_2}{1-\bar{\tau}}\mathbb{E}\int_{-\tau}^0 |x(s)|^p\, ds \\
&\qquad + \frac{\lambda_4}{1-\bar{\tau}}\int_{-\tau}^0 \mathbb{E}U(x(s))\, ds \\
&\qquad - \left(\lambda_1 - \frac{\lambda_2}{1-\bar{\tau}}\right)\mathbb{E}\int_0^{t\wedge\rho_k} |x(s)|^p\, ds \\
&\qquad - \left(\lambda_3 - \frac{\lambda_4}{1-\bar{\tau}}\right)\mathbb{E}\int_0^{t\wedge\rho_k} U(x(s))\, ds.
\end{aligned} \tag{61}$$

Then by the similar arguments used in the proof of Theorem 3, we easily obtain $\rho_k \to \infty$ as $k \to \infty$; that is, the solution $x(t)$ is global. The desired assertions (55) and (58) follow from (61) by letting $k \to \infty$.

Applying Itô's formula to $e^{\gamma t}V(y_t, r(t))$ and using condition (53), we have for any $\gamma \in (0, \Lambda]$

$$\begin{aligned}
&\mathbb{E}e^{\gamma(t\wedge\rho_k)}V\left(y_t, r(t)\right) \\
&\quad = \mathbb{E}V\left(y_0, r(0)\right) \\
&\qquad + \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}\left[\gamma V\left(y_s, r(s)\right)\right. \\
&\qquad\qquad \left. + LV\left(x(s), x(s-\tau(s)), r(s)\right)\right] ds \\
&\quad \leq \mathbb{E}V\left(y_0, r(0)\right) + \gamma\mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s} V\left(y_s, r(s)\right) ds \\
&\qquad - \lambda_1 \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}|x(s)|^p\, ds \\
&\qquad + \lambda_2 \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}|x(s-\tau(s))|^p\, ds \\
&\qquad - \lambda_3 \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}U(x(s))\, ds \\
&\qquad + \lambda_4 \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}U(x(s-\tau(s)))\, ds.
\end{aligned} \tag{62}$$

For $p \geq 1$ and any $\varepsilon > 0$, we have

$$\begin{aligned}
&V\left(y_s, r(s)\right) \\
&\quad \leq c_2 |x(s) - N(x(s-\tau(s)), r(s))|^p \\
&\quad \leq c_2\left[1 + \varepsilon^{1/(p-1)}\right]^{p-1} \\
&\qquad \times \left(|x(s)|^p + \frac{1}{\varepsilon}|N(x(s-\tau(s)), r(s))|^p\right) \\
&\quad \leq c_2\left[1 + \varepsilon^{1/(p-1)}\right]^{p-1} \\
&\qquad \times \left[|x(s)|^p + \frac{\kappa^p}{\varepsilon}|x(s-\tau(s))|^p\right],
\end{aligned} \tag{63}$$

where we used the Hölder inequality and Assumption 9. Substituting (63) into (62), we therefore obtain

$$\begin{aligned}
&\mathbb{E}e^{\gamma(t\wedge\rho_k)}V\left(y_{(t\wedge\rho_k)}, r(t\wedge\rho_k)\right) \\
&\quad \leq \mathbb{E}V\left(y_0\right) + \left[c_2\gamma\left(1 + \varepsilon^{1/(p-1)}\right)^{p-1} - \lambda_1\right] \\
&\qquad \times \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s} V\left(x(s)\right) ds \\
&\qquad + c_2\gamma\left(1 + \varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon}\mathbb{E} \\
&\qquad \times \int_0^{t\wedge\rho_k} e^{\gamma s}|x(s-\tau(s))|^p\, ds \\
&\qquad + \lambda_2 \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}|x(s-\tau(s))|^p\, ds \\
&\qquad - \lambda_3 \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}U(x(s))\, ds \\
&\qquad + \lambda_4 \mathbb{E}\int_0^{t\wedge\rho_k} e^{\gamma s}U(x(s-\tau(s)))\, ds.
\end{aligned} \tag{64}$$

Noting that

$$\begin{aligned}\int_0^t e^{\gamma s}|x(s-\tau(s))|^p\,ds &\le e^{\gamma\tau}\int_0^t e^{\gamma(s-\tau(s))}|x(s-\tau(s))|^p\,ds\\ &\le \frac{e^{\gamma\tau}}{1-\overline{\tau}}\int_{-\tau}^0 e^{\gamma s}|x(s)|^p\,ds\\ &\quad+\frac{e^{\gamma\tau}}{1-\overline{\tau}}\int_0^t e^{\gamma s}|x(s)|^p\,ds,\end{aligned}\tag{65}$$

then we have from (64)

$$\begin{aligned}&\mathbb{E}e^{\gamma(t\wedge\rho_k)}V\left(y_{(t\wedge\rho_k)},r(t\wedge\rho_k)\right)\\ &\quad\le \mathbb{E}V(y_0)+\left[c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon}+\lambda_2\right]\\ &\qquad\times e^{\gamma\tau}\int_{-\tau}^0 e^{\gamma s}\mathbb{E}V(x(s))\,ds\\ &\qquad+h(\gamma,\varepsilon)\,\mathbb{E}\int_0^{t\wedge\rho_k}e^{\gamma s}V(x(s))\,ds\\ &\qquad-l(\gamma)\,\mathbb{E}\int_0^{t\wedge\rho_k}e^{\gamma s}U(x(s))\,ds\\ &\qquad+\lambda_4\frac{e^{\gamma\tau}}{1-\overline{\tau}}\int_{-\tau}^0 e^{\gamma s}\mathbb{E}U(x(s))\,ds,\end{aligned}\tag{66}$$

where

$$\begin{aligned}l(\gamma)&=\lambda_3-\lambda_4\frac{e^{\gamma\tau}}{1-\overline{\tau}},\\ h(\gamma,\varepsilon)&=c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\\ &\quad-\lambda_1+\left[c_2\gamma\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon}+\lambda_2\right]e^{\gamma\tau}.\end{aligned}\tag{67}$$

Let ε be fixed; then it is easy to obtain $h'_\gamma(\gamma,\varepsilon)>0$ and $h(0,\varepsilon)=-\lambda_1+\lambda_2<0$, which implies that for any fixed $\varepsilon>0$ function $h(\cdot,\varepsilon)$ has a unique positive root denoted by q. Choose a $\varepsilon=\varepsilon^*>0$ such that

$$\overline{\gamma}=\sup_{\varepsilon>0,h(q,\varepsilon)=0}q=\sup_{h(q,\varepsilon^*)=0}q.\tag{68}$$

Noting that for any $\gamma\in(0,\Lambda]$, $h(\gamma,\varepsilon^*)\le 0$ and $l(\gamma)\ge 0$. We therefore have

$$\begin{aligned}&\mathbb{E}e^{\gamma(t\wedge\rho_k)}V\left(y_{(t\wedge\rho_k)},r(t\wedge\rho_k)\right)\\ &\quad\le \mathbb{E}V(y_0)+\lambda_4\frac{e^{\gamma\tau}}{1-\overline{\tau}}\int_{-\tau}^0 e^{\gamma s}\mathbb{E}|x(s)|^q\,ds\\ &\qquad+\left[c_2\gamma\left(1+\varepsilon^{*(1/(p-1))}\right)^{p-1}\frac{\kappa^p}{\varepsilon^*}+\lambda_2\right]\\ &\qquad\times e^{\gamma\tau}\int_{-\tau}^0 e^{\gamma s}\mathbb{E}V(\xi(s))\,ds\\ &\quad\le c_1C_0\sup_{-\tau\le\theta\le0}\mathbb{E}\left|\xi(\theta)\right|^p\end{aligned}\tag{69}$$

for some positive constant $C_0>1$. By the similar skills used in the proof of Theorem 3, we can easily obtain the desired assertions (54) and (57). □

If the delay $\tau(t)=\tau$ is a fixed constant, then $\overline{\tau}=0$. Hybrid system (46) becomes the following HNSDDE:

$$\begin{aligned}d\left[x(t)-N(x(t-\tau),r(t))\right]&=F(x(t),x(t-\tau),r(t))\,dt\\ &\quad+G(x(t),x(t-\tau),r(t))\,dw(t).\end{aligned}\tag{70}$$

Resorting to Theorem 10, we have the following corollary.

Corollary 11. *Let Assumptions 8 and 9 hold. Assume that there are functions $V\in C^2(\mathbb{R}^n\times\mathbb{S};\mathbb{R}_+)$, $U\in C(\mathbb{R}^n;\mathbb{R}_+)$ as well as a number of positive constants c, c_1, c_2, $p\ge1$, λ_1, λ_2, λ_3, λ_4 such that for any $x,y\in\mathbb{R}^n$ and $i\in\mathbb{S}$*

$$\begin{aligned}&c_1|x|^p\le V(x,i)\le c_2|x|^p,\\ &LV(x,y,i)\le c-\lambda_1|x|^p\\ &\qquad+\lambda_2|y|^p-\lambda_3U(x)+\lambda_4U(y).\end{aligned}\tag{71}$$

If $\lambda_1>\lambda_2$ and $\lambda_3>\lambda_4$, then for any given initial data $\xi\in C^b_{\mathfrak{F}_0}([-\tau,0],\mathbb{R}^n)$, (70) admits a unique global solution $x(t)$. Moreover, we have the following assertions:

(i) *the solution $x(t)$ obeys*

$$\limsup_{t\to\infty}\mathbb{E}|x(t)-N(x(t-\tau),r(t))|^p\le\frac{c}{\Lambda},\tag{72}$$

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbb{E}U(x(s))\,ds\le\frac{c}{\lambda_3-\lambda_4},\tag{73}$$

where $\Lambda:=\overline{\gamma}\wedge(1/\tau)\log(\lambda_3/\lambda_4)\wedge r$ with $\overline{\gamma}$ and r defined by

$$\begin{aligned}\overline{\gamma}=\max\Big\{&q>0;c_2q\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}-\lambda_1\\ &+\left[c_2q\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon}+\lambda_2\right]e^{q\tau}=0,\varepsilon>0\Big\}\end{aligned}\tag{74}$$

and $r:=(p/\tau)\log(1/\kappa)-\ell$ for sufficiently small $\ell>0$.

(ii) *If, in addition, $c = 0$, then the solution of (70) has properties that*

$$\limsup_{t\to\infty}\frac{\log\left(\mathbb{E}|x(t)|^p\right)}{t}\le -\Lambda,$$
$$\int_0^\infty \mathbb{E}U(x(s))\,ds \le \frac{1}{\lambda_3-\lambda_4}\times\left[\mathbb{E}V\left(y_0,r(0)\right)+\lambda_2\int_{-\tau}^0\mathbb{E}|x(s)|^p\,ds + \lambda_4\mathbb{E}\int_{-\tau}^0 U(x(s))\,ds\right]. \quad (75)$$

Further, if $U(x)\equiv 0$, Corollary 11 implies.

Corollary 12. *Let Assumptions 8 and 9 hold. Assume that there is a function $V\in C^2(\mathbb{R}^n\times\mathbb{S};\mathbb{R}_+)$ and a number of positive constants c_1, c_2, $p\ge 1$, λ_1, λ_2 such that for any $x,y\in\mathbb{R}^n$ and $i\in\mathbb{S}$*

$$c_1|x|^p\le V(x,i)\le c_2|x|^p,$$
$$LV(x,y,i)\le -\lambda_1|x|^p+\lambda_2|y|^p. \quad (76)$$

If $\lambda_1>\lambda_2$, then for any given initial data $\xi\in C^b_{\mathfrak{F}_0}([-\tau,0],\mathbb{R}^n)$, (70) admits a unique global solution $x(t)$. Moreover, the solution $x(t)$ obeys

$$\limsup_{t\to\infty}\frac{\log\left(\mathbb{E}|x(t)|^p\right)}{t}\le -\Lambda, \quad (77)$$

where $\Lambda := \overline{\gamma}\wedge r$ with $\overline{\gamma}$ and r defined by

$$\overline{\gamma}=\max\left\{q>0; c_2q\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}-\lambda_1 + \left[c_2q\left(1+\varepsilon^{1/(p-1)}\right)^{p-1}\frac{\kappa^p}{\varepsilon}+\lambda_2\right]e^{q\tau}=0, \varepsilon>0\right\} \quad (78)$$

and $r := (p/\tau)\log(1/\kappa)-\ell$ for sufficiently small $\ell>0$.

Remark 13. Corollary 12 improves Theorem 5.2 in [10, Chap. 5, pp. 838]. In [10], Theorem 5.2 states that if the assumptions and conditions in Corollary 12 hold, then

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(\mathbb{E}|x(t)|^p\right)\le -\Lambda^*, \quad (79)$$

where

$$\Lambda^*=\overline{\gamma}^*\wedge\frac{1}{2\tau}\log\left(\frac{1}{\kappa}\right) \quad (80)$$

with $\overline{\gamma}^*>0$ being the unique root to equation

$$\overline{\gamma}^*c_2(1+\kappa)^{p-1}+e^{\overline{\gamma}^*\tau}\left[\lambda_2+\overline{\gamma}^*c_2\kappa(1+\kappa)^{p-1}\right]=\lambda_1. \quad (81)$$

It is easy to see $\overline{\gamma}\ge\overline{\gamma}^*$. Moreover, $(p/\tau)\log(1/\kappa) > (1/2\tau)\log(1/\kappa)$. That means $\Lambda^*\le\Lambda$ for sufficiently small $\ell>0$, where Λ is defined in Corollary 12.

5. Examples

In this section, we give an example to illustrate the usefulness and flexibility of the theorems developed previously. Let $w(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markov chain value in $\mathbb{S}=\{1,2\}$ with generator

$$\Gamma=\left(\gamma_{ij}\right)_{2\times 2}=\begin{pmatrix}-2 & 2\\ 1 & -1\end{pmatrix}. \quad (82)$$

Assume that $w(t)$ and $r(t)$ are independent.

Example 1. Consider the one-dimensional linear HNSDDEs

$$d\left[x(t)-\kappa(r(s))\int_{-\tau}^0 x(s+\theta)\,d\theta\right] = \left[\mu(r(s))x(t)-2x(t)^3+\int_{-\tau}^0 x(t+\theta)\,d\theta\right]dt + \left[\sigma(r(s))\int_{-\tau}^0|x(t+\theta)|^2d\theta+c\right]dw(t), \quad (83)$$

where $\kappa(1)=1/8$, $\kappa(2)=1/4$, $\mu_1=-3$, $\mu_2=-4$, $\sigma(1)=1/\sqrt{8}$, $\sigma(2)=1/\sqrt{2}$, and $\tau=1$. To find out whether (83) is mean-square exponential stability, we use the Lyapunov function

$$V(x,i)=q_i|x|^2, \quad (84)$$

where $q_1=1$ and $q_2=0.5$. One can show that

$$\begin{aligned}\mathscr{L}V(\varphi,i) &= 2q_i\left[\varphi(0)+\kappa(i)\int_{-1}^0\varphi(\theta)\,d\theta\right]\times\left[\mu(i)\varphi(0)-3\varphi(0)^3+\int_{-1}^0\varphi(\theta)\,d\theta\right]\\ &\quad+\sum_{j=1,2}\gamma_{ij}q_j\left|\varphi(0)+\frac{1}{8}\int_{-1}^0\varphi(\theta)\,d\theta\right|^2 + 2q_i\sigma(i)^2\left|\int_{-1}^0|\varphi(\theta)|^2d\theta\right|^2+2c^2.\end{aligned} \quad (85)$$

By the elementary inequalities $a^\alpha b^\beta \le (\alpha/(\alpha+\beta))a^{\alpha+\beta} + (\beta/(\alpha+\beta))b^{\alpha+\beta}$, we have

$$\begin{aligned}
&\mathscr{L}V(\varphi,1) \\
&\quad\le 2\Bigg[-3|\varphi(0)|^2 - 3|\varphi(0)|^4 \\
&\qquad + \frac{1}{2}\left(|\varphi(0)|^2 + \int_{-1}^{0}|\varphi(\theta)|^2 d\theta\right) \\
&\qquad + \frac{3}{8}\frac{1}{2}\left(|\varphi(0)|^2 + \int_{-1}^{0}|\varphi(\theta)|^2 d\theta\right) \\
&\qquad + \frac{3}{8}\left(\frac{3}{4}|\varphi(0)|^4 + \frac{1}{4}\int_{-1}^{0}|\varphi(\theta)|^4 d\theta\right)\Bigg] \\
&\qquad - \varphi(0) + \frac{1}{8}\int_{-1}^{0}|\varphi(\theta)\,d\theta|^2 + \frac{1}{4}\int_{-1}^{0}|\varphi(\theta)|^4 d\theta + 2c^2 \\
&\quad\le 2c^2 - \frac{45}{8}|\varphi(0)|^2 \\
&\qquad + \frac{97}{64}\int_{-1}^{0}|\varphi(\theta)|^2 d\theta - \frac{87}{16}|\varphi(0)|^4 + \frac{7}{16}\int_{-1}^{0}|\varphi(\theta)|^4 d\theta.
\end{aligned} \tag{86}$$

Similarly,

$$\begin{aligned}
&\mathscr{L}V(\varphi,2) \\
&\quad\le 2c^2 - \frac{29}{8}|\varphi(0)|^2 \\
&\qquad + \frac{13}{32}\int_{-1}^{0}|\varphi(\theta)|^2 d\theta - \frac{39}{16}|\varphi(0)|^4 \\
&\qquad + \frac{23}{16}\int_{-1}^{0}|\varphi(\theta)|^4 d\theta.
\end{aligned} \tag{87}$$

Let $\lambda_1 = 29/8$, $\lambda_2 = 97/64$, $\lambda_3 = 39/16$, and $\lambda_4 = 23/16$. Then we have from (86), and (87) for each $i \in \mathbb{S}$,

$$\begin{aligned}
&\mathscr{L}V(\varphi,i) \\
&\quad\le 2c^2 - \lambda_1|\varphi(0)|^2 \\
&\qquad + \lambda_2\int_{-1}^{0}|\varphi(\theta)|^2 d\theta - \lambda_3|\varphi(0)|^4 + \lambda_4\int_{-1}^{0}|\varphi(\theta)|^4 d\theta.
\end{aligned} \tag{88}$$

Then one can compute $\Lambda = 0.5281$ by Theorem 3. If $c \ne 0$, then

$$\begin{aligned}
&\limsup_{t\to\infty}\mathbb{E}|x(t) - u(x_t, r(t))|^2 \le \frac{2c^2}{0.5281}, \\
&\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathbb{E}|x(s)|^4 ds \le \frac{2c^2}{\lambda_3 - \lambda_4}.
\end{aligned} \tag{89}$$

If $c = 0$, the solution to HNSDDE (83) has the property

$$\limsup_{t\to\infty}\frac{\log\left(\mathbb{E}|x(t)|^2\right)}{t} \le -0.5281. \tag{90}$$

Example 2. Consider the HNSDDE:

$$\begin{aligned}
d[x(t) - N(x(t-\tau), r(t))] &= F(x(t), x(t-\tau), r(t))\,dt \\
&\quad + G(x(t), x(t-\tau), r(t))\,dw(t),
\end{aligned} \tag{91}$$

where the functions $N(y,1) = 1/3y$, $N(y,2) = 1/4y$, $F(x,y,1) = -2x - 3x^3 + y/4$, $F(x,y,2) = -3x - 4x^3 + y/2$, $G(x,y,1) = 1/2y^2$, and $G(x,y,2) = y/2$. Let $V(x,i) = q_i|x|^2$ for $q_1 = 1$, $q_2 = 0.5$. Then one can compute

$$\begin{aligned}
LV(x,y,1) &\le -\frac{41}{2}x^2 + \frac{23}{24}y^2 - \frac{21}{4}x^4 + \frac{1}{6}y^4, \\
LV(x,y,2) &\le -\frac{29}{16}x^2 + \frac{31}{64}y^2 - \frac{13}{4}x^4 + \frac{1}{4}y^4.
\end{aligned} \tag{92}$$

Then we have from (92) that for $i \in \mathbb{S}$

$$LV(x,y,i) \le -\frac{29}{16}x^2 + \frac{23}{24}y^2 - \frac{13}{4}x^4 + \frac{1}{4}y^4. \tag{93}$$

One can compute $\Lambda = 0.2767$ by Corollary 11. Then the solution to HNSDDE (91) has the property

$$\limsup_{t\to\infty}\frac{\log\left(\mathbb{E}|x(t)|^2\right)}{t} \le -0.2767. \tag{94}$$

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References

[1] J. K. Hale and K. R. Meyer, *A Class of Functional Equations of Neutral Type*, Memoirs of the American Mathematical Society, no. 76, American Mathematical Society, Providence, RI, USA, 1967.

[2] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*, Springer, Berlin, Germany, 1991.

[3] V. B. Kolmanovskii and A. Myshkis, *Applied Theory of Functional Differential Equations*, Kluwer Academic Publishers, Norwell, Mass, USA, 1992.

[4] V. B. Kolmanovskiĭ and V. R. Nosov, *Stability of Functional-Differential Equations*, vol. 180 of *Mathematics in Science and Engineering*, Academic Press, London, UK, 1986.

[5] R. Brayton, "Nonlinear oscillations in a distributed network," *Quarterly of Applied Mathematics*, vol. 24, pp. 289–301, 1976.

[6] V. P. Rubanik, *Oscillations of Quasilinear Systems with Retardation*, Nauk, Moscow, Russia, 1969.

[7] R. D. Driver, "A functional-differential system of neutral type arising in a two-body problem of classical electrodynamics," in *Nonlinear Differential Equations and Nonlinear Mechanics*, pp. 474–484, Academic Press, New York, NY, USA, 1963.

[8] X. Mao and C. Yuan, *Stochastic Differential Equations with Markovian Switching*, Imperial College Press, London, UK, 2006.

[9] L. Hu, X. Mao, and Y. Shen, "Stability and boundedness of nonlinear hybrid stochastic differential delay equations," *Systems & Control Letters*, vol. 62, no. 2, pp. 178–187, 2013.

[10] V. Kolmanovskii, N. Koroleva, T. Maizenberg, X. Mao, and A. Matasov, "Neutral stochastic differential delay equations with Markovian switching," *Stochastic Analysis and Applications*, vol. 21, no. 4, pp. 819–847, 2003.

[11] X. Mao, *Stochastic Differential Equations and Applications*, Horwood, Chichester, UK, 2007.

[12] S. Janković and M. Jovanović, "The *p*th moment exponential stability of neutral stochastic functional differential equations," *Filomat*, no. 20, part 1, pp. 59–72, 2006.

[13] S. Janković, J. Randjelović, and M. Jovanović, "Razumikhin-type exponential stability criteria of neutral stochastic functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 355, no. 2, pp. 811–820, 2009.

[14] X. Mao, "Exponential stability in mean square of neutral stochastic differential-functional equations," *Systems & Control Letters*, vol. 26, no. 4, pp. 245–251, 1995.

[15] X. Mao, "Razumikhin-type theorems on exponential stability of neutral stochastic functional-differential equations," *SIAM Journal on Mathematical Analysis*, vol. 28, no. 2, pp. 389–401, 1997.

[16] X. Mao, "Razumikhin-type theorems on exponential stability of neutral stochastic functional-differential equations," *SIAM Journal on Mathematical Analysis*, vol. 28, no. 2, pp. 389–401, 1997.

[17] J. Randjelović and S. Janković, "On the *p*th moment exponential stability criteria of neutral stochastic functional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 266–280, 2007.

[18] F. Wu, S. Hu, and X. Mao, "Razumikhin-type theorem for neutral stochastic functional differential equations with unbounded delay," *Acta Mathematica Scientia. Series B. English Edition*, vol. 31, no. 4, pp. 1245–1258, 2011.

[19] J. Bao, Z. Hou, and C. Yuan, "Stability in distribution of neutral stochastic differential delay equations with Markovian switching," *Statistics & Probability Letters*, vol. 79, no. 15, pp. 1663–1673, 2009.

[20] G. Hu and K. Wang, "Stability in distribution of neutral stochastic functional differential equations with Markovian switching," *Journal of Mathematical Analysis and Applications*, vol. 385, no. 2, pp. 757–769, 2012.

[21] R. Ravi Kumar and K. T. Chong, "The stability of distributed neutral delay differential systems with Markovian switching," *Chaos, Solitons & Fractals*, vol. 45, no. 4, pp. 416–425, 2012.

[22] X. Mao, Y. Shen, and C. Yuan, "Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching," *Stochastic Processes and Their Applications*, vol. 118, no. 8, pp. 1385–1406, 2008.

[23] Q. Luo, X. Mao, and Y. Shen, "New criteria on exponential stability of neutral stochastic differential delay equations," *Systems & Control Letters*, vol. 55, no. 10, pp. 826–834, 2006.

[24] Y. Song and Y. Shen, "New criteria on asymptotic behavior of neutral stochastic functional differential equations," *Automatica*, vol. 49, no. 2, pp. 626–632, 2013.

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Research Article

Large Time Behavior of the Vlasov-Poisson-Boltzmann System

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The motion of dilute charged particles can be modeled by Vlasov-Poisson-Boltzmann system. We study the large time stability of the VPB system. To be precise, we prove that when time goes to infinity, the solution of VPB system tends to global Maxwellian state in a rate $O(t^{-\infty})$, by using a method developed for Boltzmann equation without force in the work of Desvillettes and Villani (2005). The improvement of the present paper is the removal of condition on parameter λ as in the work of Li (2008).

1. Introduction

Large time behavior for the Boltzmann equation and related systems is an important topic for both physicists and mathematicians. We consider the Cauchy problem for Vlasov-Poisson-Boltzmann system in a torus $\mathbb{T}^N$:

$$f_t + v\cdot\nabla_x f + \nabla_x\phi\cdot\nabla_v f = Q(f,f), \quad \text{on } \mathbb{T}^N, \tag{1}$$

$$\Delta\phi = \int_{\mathbb{R}^N} f\,dv - \rho_0, \quad \text{on } \mathbb{T}^N, \tag{2}$$

$$f(0,x,v) = f_0(x,v), \tag{3}$$

$$Q(f,f) = \int_{\mathbb{R}^N}\int_{\mathbb{S}^{N-1}} \left(f' f'_* - f f_*\right) q\left(v-v_*,\sigma\right) d\sigma\, dv_*. \tag{4}$$

$f = (t,x,v)$, which represents the distribution of particles, is a function of time $t \in \mathbb{R}^+$, particle velocity $v \in \mathbb{R}^N$, and position $x \in \mathbb{T}^N$. The force $\nabla\phi$ in (1) is controlled by Poisson equation (2), which comes intrinsically by the nonequilibrium distribution of particles.

The quadratic term $Q(f,f)$ is the collision operator and $q(v-v_*,\sigma)$ is the corresponding cross-section. It is well-known by the conservation of mass that $\rho_0 = \int_{\mathbb{T}^N\times\mathbb{R}^N} f_0 dx\,dv$ is a fixed constant which represents the background charge.

Without loss of generality, we can assume $|\mathbb{T}^N| = 1$, $\rho_0 = 1$. Define ρ, u, T, which are functions of t and x by

$$\rho = \int_{\mathbb{R}^N} f\,dv, \qquad \rho u = \int_{\mathbb{R}^N} f v\,dv,$$
$$\rho|u|^2 + N\rho T = \int_{\mathbb{R}^N} f|v|^2 dv. \tag{5}$$

Physically, they represent the macroscopic quantities: density, bulk velocity, and temperature, respectively. It is well known that the conservation of mass, momentum, and energy holds:

$$\frac{d}{dt}\int_{\mathbb{T}^N}\rho\,dx = 0,$$
$$\frac{d}{dt}\int_{\mathbb{T}^N}\rho u\,dx = 0,$$
$$\frac{d}{dt}\int_{\mathbb{T}^N}\left(\frac{\rho|u|^2}{2} + \frac{N\rho T}{2} + \frac{|\nabla\phi|^2}{2}\right)dx = 0. \tag{6}$$

Here, the total energy consists of the kinetic energy $\rho|u|^2/2$, the internal heat energy $N\rho T/2$, and the electric potential energy $|\nabla\phi|^2/2$. By simple translation and dilation, ρ, u, T can be normalized as

$$\int_{\mathbb{T}^N\times\mathbb{R}^N} f\,dv\,dx = \int_{\mathbb{T}^N} \rho\,dx = \rho_0 = 1,$$
$$\int_{\mathbb{T}^N\times\mathbb{R}^N} f\,v\,dv\,dx = \int_{\mathbb{T}^N} \rho u\,dx = 0,$$
$$\int_{\mathbb{T}^N\times\mathbb{R}^N} \left(f\frac{|v|^2}{2} + \frac{|\nabla\phi|^2}{2} \right) dv\,dx \tag{7}$$
$$= \int_{\mathbb{T}^N} \left(\frac{\rho|u|^2}{2} + \frac{N\rho T}{2} + \frac{|\nabla\phi|^2}{2} \right) dx = \frac{N}{2}.$$

If the initial datum f_0 satisfies the conservation laws (7), then the stationary solution is a global Maxwellian M, in the form of

$$M = M_{[1,0,1]} = \frac{1}{(2\pi)^{N/2}} \exp\left\{ -\frac{|v|^2}{2} \right\}, \tag{8}$$

where the subscript $[1, 0, 1]$ represents the corresponding macroscopic quantities: density, bulk velocity, and temperature, respectively.

Traditional method for studying the asymptotic behavior is using linearization around local or global Maxwellian state. Without external force, Ukai [1] proved an exponential decay rate for the cutoff hard potential in a torus in 1974. In 1980, Caflisch [2] obtained a rate like $O(e^{-t^{\beta}})$ for the cutoff soft potential with $\gamma \geq -1$ in a torus, where $\beta = 2/(2-\gamma) \in [0, 1]$. Strain and Guo [3] extend Caflisch's result in 2008 and get a convergence rate like $O(e^{-t^{P}})$ $(0 < P < 1)$ for the very soft potential case $(\gamma < -1)$. The previous results all make use of the linearization.

However, by using some estimates on systems of second-order differential inequalities, Desvillettes and Villani [4] obtain an almost exponential convergence rate like $O(t^{-\infty})$. The result is weaker than using linearization, but the smallness assumption on initial data $f_0 - M$ is removed and the conclusion holds for noncutoff collision kernels as well.

Our work is inspired by the work of Desvillettes and Villani [4]. We extend their result for Boltzmann equation without external force to the Vlasov-Poisson-Boltzmann system.

In a previous work [5], the Vlasov-Poisson-Boltzmann system with (2) replaced by

$$\lambda\Delta\phi = \int_{\mathbb{R}^N} f\,dv - \rho_0 \tag{9}$$

is proved to satisfy the following theorem.

Theorem 1. *Let $q(v - v_*, \sigma)$ satisfy*

$$q \geq K_B \min\left(|v - v_*|^{\gamma_-}, |v - v_*|^{-\beta_-} \right), \tag{10}$$

and let the collision operator satisfy

$$\|Q(g,h)\|_{L^2(\mathbb{R}^N_v)} \leq C_B \|g\|_{H^{k_0}_{s_0}(\mathbb{R}^N_v)} \|h\|_{H^{k_0}_{s_0}(\mathbb{R}^N_v)}, \tag{11}$$

for some k_0, $s_0 \geq 0$, where K_B and C_B are positive constants. Let $(f)_{t\geq 0}$ be a smooth solution of the problem (1), (9), *and* (3), *such that, for all k, $s > 0$,*

$$\sup_{t\geq 0} \|f\|_{H^k_s(\mathbb{T}^N\times\mathbb{R}^N)} \leq C_{k,s} < +\infty, \tag{12}$$

and for all $t > 0$, $x \in \mathbb{T}^N$, and $v \in \mathbb{R}^N$,

$$f(x,v) \geq K_0 e^{-A_0|v|^{q_0}} \quad (A_0, K_0 > 0; q_0 \geq 2). \tag{13}$$

Then $\exists\lambda_0$, such that, for all $\lambda > \lambda_0$, the solution f converges to M in an almost exponential rate; that is, for any small positive constant $\epsilon > 0$,

$$\|f - M\| = O(1)\, t^{-1/700\epsilon}, \tag{14}$$

where $O(1)$ depends on K_B, γ_-, β_-, C_B, k_0, s_0, $C_{k,s}$, K_0, A_0, q_0, and ϵ.

The present paper extends the result of [5] by removing the condition on λ and considers system (1)–(3). To be precise, the main result of this paper is as follows.

Theorem 2. *Under condition* (10)–(13), *the solution f of problem* (1)–(3) *converges to M in an almost exponential rate; that is, for any small positive constant $\epsilon > 0$,*

$$\|f - M\| = O(1)\, t^{-1/700\epsilon}, \tag{15}$$

where $O(1)$ depends on the constants in (10)–(13) *and ϵ.*

Now, we state some results on the existence of solutions of VPB system. The global existence of solutions is proved in [6] in a torus and [7–9] in the whole space with small perturbed initial data. The existence result in [7] also holds for a more general case, like the Vlasov-Maxwell-Boltzmann system.

The following is devoted to the proof of Theorem 2. Section 2 gives some lemmas which will be used later. Proof of the main result is given in Section 3.

2. Preliminaries

First, denote some local Maxwellian states in forms of ρ, u, T. Define $M_{[\rho,u,T]}$, $M_{[\rho,u,\langle T\rangle]}$, $M_{[\rho,0,\langle T\rangle]}$, $M_{[\rho,0,1]}$ as follows:

$$M_{[\rho,u,T]}(v) = \frac{\rho}{(2\pi T)^{N/2}} \exp\left\{ -\frac{|v-u|^2}{2T} \right\},$$
$$M_{[\rho,u,\langle T\rangle]}(v) = \frac{\rho}{(2\pi\langle T\rangle)^{N/2}} \exp\left\{ -\frac{|v-u|^2}{2\langle T\rangle} \right\},$$
$$M_{[\rho,0,\langle T\rangle]}(v) = \frac{\rho}{(2\pi\langle T\rangle)^{N/2}} \exp\left\{ -\frac{|v|^2}{2\langle T\rangle} \right\}, \tag{16}$$
$$M_{[\rho,0,1]}(v) = \frac{\rho}{(2\pi)^{N/2}} \exp\left\{ -\frac{|v|^2}{2} \right\},$$

where $\langle T\rangle = \int \rho T\,dx$ stands for the mean temperature.

As we will show in Section 3, the gradient of temperature prevents f from being close to $M_{[\rho,u,T]}$ for too long;

the symmetric gradient of velocity prevents f from being close to $M_{[\rho,u,\langle T\rangle]}$ for long, that is, the local Maxwellians with constant temperature; and finally, the gradient of ρ and ϕ prevents f from being close to $M_{[\rho,0,\langle T\rangle]}$ and $M_{[\rho,0,1]}$ for long. In order to estimate the distance between two distributions, we need to define H functional and relative information (or relative entropy) between two distributions, which is the main measure of the distance between f and the local Maxwellians.

Definition 3. Suppose f and g are two distributions on $\mathbb{T}^N \times \mathbb{R}^N$, s.t.:

$$\int_{\mathbb{T}^N\times\mathbb{R}^N} f = \int_{\mathbb{T}^N\times\mathbb{R}^N} g. \tag{17}$$

Define the H *functional* (negative of the entropy) and the *Kullback relative information* by

$$H(f) = \int_{\mathbb{T}^N\times\mathbb{R}^N} f\log f, \qquad H(f \mid g) = \int_{\mathbb{T}^N\times\mathbb{R}^N} f\log\frac{f}{g}. \tag{18}$$

Proposition 4. *The well-known Csiszár-Kullback inequality asserts*

$$H(fg) \ge \frac{1}{4}\|f-g\|^2_{L^1(x,v)}, \tag{19}$$

if f and g are two distributions on $\mathbb{T}^N \times \mathbb{R}^N$. Moreover, if f is the solution of (1), (2) *and satisfies* (7), *then*

$$H(f \mid M) = H(f) - H(M) - \frac{1}{2}\int_{\mathbb{T}^N} |\nabla\phi|^2. \tag{20}$$

Proof. Define $\varphi(h) = h\log(h)$; then since $\int f = \int g = 1$, we have

$$\begin{aligned} H(f \mid g) &= \int f\log\frac{f}{g} = \int f\log f - f\log g - f + g \\ &= \int f\log f - g\log g - (\log g + 1)(f-g) \\ &= \int \varphi(f) - \varphi(g) - \varphi'(g)(f-g) \\ &= \frac{1}{2}\int \varphi''(h)|f-g|^2 \\ &= \frac{1}{2}\int \frac{1}{h}|f-g|^2, \end{aligned} \tag{21}$$

where h stands for a positive function between f and g. The last equality is obtained by using second-order Taylor expansion. By Hölder's inequality, we have

$$\begin{aligned} &\int |f-g|\, h^{-1/2}h^{1/2} \\ &\quad\le \left(\int \frac{1}{h}|f-g|^2\right)^{1/2}\left(\int h\right)^{1/2}, \quad \forall h > 0. \end{aligned} \tag{22}$$

Since h lies between f and g, notice that distributions f, g are nonnegative; thus $h \le f + g$. We have

$$\int \frac{1}{h}|f-g|^2 \ge \frac{\left(\int |f-g|\right)^2}{\int (f+g)} = \frac{1}{2}\left(\int |f-g|\right)^2, \tag{23}$$

and (19) is obtained. Equation (20) follows directly from (7). □

We now state the quantitative version of H-theorem. See [10] for the proof.

Theorem 5 (Quantitative H-Theorem). *If $(f)_{t\ge 0}$ is a smooth solution of the VPB equation* (1), (2), *then the H functional $H(f)$ is nonincreasing as a function of t, and the decreasing rate*

$$\frac{d}{dt}H(f) = -\int_{\mathbb{T}^N} D(f(x,\cdot))\,dx, \tag{24}$$

where

$$D(f) = \frac{1}{4}\int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} \left(f'f'_* - ff_*\right)\log\frac{f'f'_*}{ff_*} B\,d\sigma\,dv\,dv_* \tag{25}$$

is a positive definite functional.

Moreover, if the collision kernel q satisfies (10), *and f complies with* (12), *then*

$$\begin{aligned} D(f) &\ge K_\epsilon\left(\int_{\mathbb{R}^N} f\log\frac{f}{M_{[\rho,u,T]}}\right)^{1+\epsilon}, \\ -\frac{d}{dt}H(f) &\ge K_H H\left(f \mid M_{[\rho,u,T]}\right)^{1+\epsilon}. \end{aligned} \tag{26}$$

The only set that can make D vanish is the local Maxwellian state.

We state some notations here for the fluency of description. Let A and B be matrices; let the operation $A : B = \sum_{ij} A_{ij}B_{ij}$. For a vector-valued function u, the divergence is

$$\nabla_x \cdot u = \sum_i \frac{\partial u_i}{\partial x_i}, \tag{27}$$

the elements of gradient matrix $\nabla_x u$ satisfy

$$(\nabla_x u)_{ij} = \frac{\partial u_j}{\partial x_i}, \tag{28}$$

the symmetric part of ∇u is

$$\nabla_x^{\mathrm{sym}} u = \frac{\nabla_x u + (\nabla_x u)^T}{2}, \tag{29}$$

and the traceless part of $\nabla_x^{\mathrm{sym}} u$ is symbolized by $\{\nabla_x u\}$:

$$\{\nabla_x u\} = \nabla_x^{\mathrm{sym}} u - \frac{\nabla_x \cdot u}{N} I_N. \tag{30}$$

We expect to estimate decay rate of the distance between f and M, and the distance is measured by Kullback relative information. By using conservation laws, a direct computation will show that the relative information between f and M can be decomposed into a purely hydrodynamic part and a purely kinetic part:

$$\begin{aligned} H(f \mid M) &= \mathscr{H}(\rho,u,T) + H\left(f \mid M_{[\rho,u,T]}\right), \\ \mathscr{H}(\rho,u,T) &= \int_{T^N} \rho\log\rho + \frac{N}{2}\int_{T^N}\rho\left(T - \log T - 1\right) \\ &\quad + \int_{T^N}\rho\frac{|u|^2}{2} \\ &=: \mathscr{H}(\rho \mid 1) + \mathscr{H}(T \mid 1) + \mathscr{H}(u \mid 0), \end{aligned} \tag{31}$$

where

$$\begin{aligned} \mathscr{H}(u \mid 0) &= \int \rho\frac{|u|^2}{2}, \\ \mathscr{H}(\rho \mid 1) &= \int_{T^N}\rho\log\rho = \int_{T^N}\rho\log\rho - \rho + 1 \end{aligned} \tag{32}$$

are nonnegative since $\rho\log\rho - \rho + 1$ is convex with the minimum zero at $\rho = 1$.

Moreover, denote $\Psi(X) = (N/2)(X - \ln X - 1)$; we can further decompose $\mathscr{H}(T \mid 1)$ into

$$\mathscr{H}(T \mid 1) = \mathscr{H}(T \mid \langle T\rangle) + \mathscr{H}(\langle T\rangle \mid 1), \tag{33}$$

where

$$\begin{aligned} \mathscr{H}(T \mid \langle T\rangle) &= \int \rho\Psi(T) - \Psi(\langle T\rangle), \\ \mathscr{H}(\langle T\rangle \mid 1) &= \Psi(\langle T\rangle). \end{aligned} \tag{34}$$

It is easy to check that each of the previous terms is nonnegative by using Jensen's inequality and convexity of functions $\Psi(X)$.

It is easy to verify the following.

Lemma 6. *Use the previously mentioned notations; then one has the following additivity roles:*

$$\begin{aligned} &H\left(f \mid M_{[\rho,u,T]}\right) + \mathscr{H}(T \mid \langle T\rangle) = H\left(f \mid M_{[\rho,u,\langle T\rangle]}\right), \\ &H\left(f \mid M_{[\rho,u,\langle T\rangle]}\right) + \frac{1}{\langle T\rangle}\mathscr{H}(u \mid 0) = H\left(f \mid M_{[\rho,0,\langle T\rangle]}\right), \\ &H\left(f \mid M_{[\rho,0,\langle T\rangle]}\right) + \mathscr{H}(\langle T\rangle \mid 1) + \left(1 - \frac{1}{\langle T\rangle}\right)\mathscr{H}(u \mid 0) \\ &\quad = H\left(f \mid M_{[\rho,0,1]}\right), \\ &H\left(f \mid M_{[\rho,0,1]}\right) + \mathscr{H}(\rho \mid 1) = H(f \mid M). \end{aligned} \tag{35}$$

Moreover, one has

$$\begin{aligned} H\left(f \mid M_{[\rho,u,T]}\right) &\geq K_I\left\|f - M_{[\rho,u,T]}\right\|_{L^2}^{2(1+\epsilon)}, \\ H\left(f \mid M_{[\rho,u,\langle T\rangle]}\right) &\geq K_I\left\|f - M_{[\rho,u,\langle T\rangle]}\right\|_{L^2}^{2(1+\epsilon)}, \\ H\left(f \mid M_{[\rho,0,\langle T\rangle]}\right) &\geq K_I\left\|f - M_{[\rho,0,\langle T\rangle]}\right\|_{L^2}^{2(1+\epsilon)}, \\ H\left(f \mid M_{[\rho,0,1]}\right) &\geq K_I\left\|f - M_{[\rho,0,1]}\right\|_{L^2}^{2(1+\epsilon)}. \end{aligned} \tag{36}$$

Here nonnegative terms $\mathscr{H}(\rho \mid 1)$, $\mathscr{H}(u \mid 0)$, $\mathscr{H}(T \mid \langle T\rangle)$, $\mathscr{H}(\langle T\rangle \mid 1)$ are parts of the relative entropy, $K_I > 0$.

Proof. Additivity rules can be verified by direct computation. By using Csiszár-Kullback inequality and the interpolation from L^2 into L^1, we can get (36). See [4] or [5] for more details. □

Now we assert the key lemma of the paper, which asserts the instability of hydrodynamic descriptions for f.

Lemma 7. *The following four second-order differential inequalities hold:*

$$\begin{aligned} &\frac{d^2}{dt^2}\left\|f - M_{[\rho,u,T]}\right\|_{L^2(x,v)}^2 \\ &\quad \geq K_1\left[\int_{T^N}|\nabla T(x)|^2dx + \int_{T^N}|\{\nabla u(x)\}|^2dx\right] \\ &\qquad - \frac{C_1}{\delta_1^{1-\epsilon}}\left(\left\|f - M_{[\rho,u,T]}\right\|_{L^2}^2\right)^{1-\epsilon} - \delta_1 H(f \mid M), \end{aligned} \tag{37}$$

$$\begin{aligned} &\frac{d^2}{dt^2}\left\|f - M_{[\rho,u,\langle T\rangle]}\right\|_{L^2(x,v)}^2 \\ &\quad \geq K_2\int_{T^N}\left|\nabla^{sym}u\right|^2dx \\ &\qquad - \frac{C_2}{\delta_2^{1-\epsilon}}\left(\left\|f - M_{[\rho,u,\langle T\rangle]}\right\|_{L^2}^2\right)^{1-\epsilon} - \delta_2 H(f \mid M), \end{aligned} \tag{38}$$

$$\begin{aligned} &\frac{d^2}{dt^2}\left\|f - M_{[\rho,0,\langle T\rangle]}\right\|_{L^2(x,v)}^2 \\ &\quad \geq K_3\left[\int_{T^N}|\nabla\phi|^2dx + \int_{T^N}|\nabla\rho|^2dx\right] \\ &\qquad - \frac{C_3}{\delta_3^{1-\epsilon}}\left(\left\|f - M_{[\rho,0,\langle T\rangle]}\right\|_{L^2}^2\right)^{1-\epsilon} - \delta_3 H(f \mid M), \end{aligned} \tag{39}$$

$$\begin{aligned} &\frac{d^2}{dt^2}\left\|f - M_{[\rho,0,1]}\right\|_{L^2(x,v)}^2 \\ &\quad \geq K_4\left[\int_{T^N}|\nabla\phi|^2dx + \int_{T^N}|\nabla\rho|^2dx\right] \\ &\qquad - \frac{C_4}{\delta_4^{1-\epsilon}}\left(\left\|f - M_{[\rho,0,1]}\right\|_{L^2}^2\right)^{1-\epsilon} - \delta_4 H(f \mid M). \end{aligned} \tag{40}$$

Here δ_1, δ_2, δ_3, δ_4 are small enough constants, and all constants are positive.

Roughly speaking, the previous inequalities show that f cannot stay near local Maxwellian states. The gradient of T prevents f from staying close to $M_{[\rho,u,T]}$ for long; the symmetric gradient of u prevents f from staying close to $M_{[\rho,u,\langle T\rangle]}$ for long; finally, the gradient of ρ prevents f from staying close to $M_{[\rho,0,\langle T\rangle]}$ and $M_{[\rho,0,1]}$. It left $M = M_{[1,0,1]}$ as the only stable state.

To prove Lemma 7, the following lemma is needed, whose proof can be found in [4].

Lemma 8. *Let h be a smooth function of x, v. Then, for all multi-indexes α, β, and for all $\eta < 1$,*

$$\int \left(v^{\alpha}\partial^{\beta}_{x,v}h\right)^2 dv\,dx \le \|h\|^{2\eta}_{H^{|\beta|}_{|\alpha|/\eta}} \|h\|^{2\eta(1-\eta)}_{H^{|\beta|/\eta}} \|h\|^{2(1-\eta)^2}_{L^2}. \quad (41)$$

Proof of Lemma 7. Most of the proof is similar to that in [4, 5]; the only difference is in estimating terms with ϕ. We will only prove (39) as an example of how to estimate terms with ϕ.

We have

$$\begin{aligned} &\frac{d^2}{dt^2}\left\|f - M_{[\rho,0,\langle T\rangle]}\right\|^2_{L^2} \\ &\quad = 2\int\left(\frac{\partial}{\partial t}\left(f - M_{[\rho,0,\langle T\rangle]}\right)\right)^2 dv\,dx \\ &\qquad + 2\int\left(f - M_{[\rho,0,\langle T\rangle]}\right)\frac{\partial^2}{\partial t^2}\left(f-g\right)dv\,dx \\ &\quad = A + B. \end{aligned} \quad (42)$$

At the moment when $f = M_{[\rho,0,\langle T\rangle]}$, B vanishes, so we only need to estimate A:

$$\begin{aligned} \frac{\partial}{\partial t}\left(f - M_{[\rho,0,\langle T\rangle]}\right) &= -v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f + Q(f,f) \\ &\quad - \frac{\partial}{\partial t}M_{[\rho,0,\langle T\rangle]} \\ &= -\left(\frac{\partial}{\partial t} + v\cdot\nabla_x + \nabla_x\phi\cdot\nabla_v\right)M_{[\rho,0,\langle T\rangle]}. \end{aligned} \quad (43)$$

From (1) we have

$$\begin{aligned} &\rho_t + \nabla_x\cdot(\rho u) = 0, \\ &(\rho u)_t + \nabla_x\cdot\left(\rho u\otimes u + \rho T I_N + D\right) - \rho\nabla_x\phi = 0, \\ &\left(\rho|u|^2 + N\rho T\right)_t + \nabla_x \\ &\quad\cdot\left(\rho|u|^2 u + (N+2)\,\rho u T + 2Du + 2R\right) \\ &\quad - 2\rho u\cdot\nabla_x\phi = 0. \end{aligned} \quad (44)$$

Here, D and R are matrix-valued and vector-valued functions, respectively, defined by

$$\begin{aligned} D_{ij}(x) &= \int_{R^N} f(x,v) \\ &\quad\times\left[(v-u)_i(v-u)_j - \frac{|v-u|^2}{N}\delta_{ij}\right]dv, \\ R(x) &= \int_{R^N} f(x,v)\,\frac{|v-u|^2}{2}\,(v-u)\,dv. \end{aligned} \quad (45)$$

Then, we obtain

$$\begin{aligned} &(\partial_t + u\cdot\nabla)\,\rho + \rho\nabla\cdot u = 0, \\ &(\partial_t + u\cdot\nabla)\,u + \nabla T + \frac{T\nabla\rho}{\rho} + \frac{\nabla\cdot D}{\rho} - \nabla_x\phi = 0, \\ &(\partial_t + u\cdot\nabla)\,T + \frac{2T}{N}\nabla\cdot u + \frac{2}{\rho N}\left(\nabla u : D + \nabla\cdot R\right) = 0. \end{aligned} \quad (46)$$

Also, we get

$$\begin{aligned} \partial_t\langle T\rangle &= \left[\int(\partial_t\rho)\,T + \int\rho\,(\partial_t T)\right] \\ &= \left[-\int\nabla\cdot(\rho u)\,T - \int\rho u\cdot\nabla T - \frac{2}{N}\int\rho T\nabla\cdot u\right. \\ &\qquad\left. - \frac{2}{N}\int\nabla u : D - \frac{2}{N}\int\nabla\cdot R\right] \\ &= \left[-\frac{2}{N}\int\rho T\nabla\cdot u - \frac{2}{N}\int\nabla u : D\right]. \end{aligned} \quad (47)$$

Then the equations of $M_{[\rho,0,\langle T\rangle]}$ can be stated as follows:

$$\begin{aligned} &\left(\partial_t + v\cdot\nabla_x + \nabla_x\phi\cdot\nabla_v\right)M_{[\rho,0,\langle T\rangle]} \\ &\quad = M_{[\rho,0,\langle T\rangle]} \\ &\qquad\times\left\{\left[\frac{\partial_t\rho}{\rho} - \frac{N}{2}\frac{\partial_t\langle T\rangle}{\langle T\rangle}\right]\right. \\ &\qquad\quad\left. + v\cdot\left[\frac{\nabla\rho}{\rho} - \frac{\nabla_x\phi}{\langle T\rangle}\right] + |v|^2\left[\frac{\partial_t\langle T\rangle}{2\langle T\rangle^2}\right]\right\}. \end{aligned} \quad (48)$$

From (46) and (47), we have

$$\begin{aligned} &\partial_t\left(f - M_{[\rho,0,\langle T\rangle]}\right) \\ &\quad = -M_{[\rho,0,\langle T\rangle]} \\ &\qquad\times\left\{\left[\frac{\partial_t\rho}{\rho} - \frac{N}{2}\frac{\partial_t\langle T\rangle}{\langle T\rangle}\right]\right. \\ &\qquad\quad\left. + v\cdot\left[\frac{\nabla\rho}{\rho} - \frac{\nabla_x\phi}{\langle T\rangle}\right] + |v|^2\left[\frac{\partial_t\langle T\rangle}{2\langle T\rangle^2}\right]\right\}. \end{aligned} \quad (49)$$

Note that $M_{[\rho,0,\langle T\rangle]}$, $|v|^2 M_{[\rho,0,\langle T\rangle]}$, $|v|^2 M_{[\rho,0,\langle T\rangle]}$ are linearly independent in weighted $L^2((1/M_{[\rho,0,\langle T\rangle]})dv)$ space. Therefore,

$$\begin{aligned} &\frac{d^2}{dt^2}\left\|f - M_{[\rho,0,\langle T\rangle]}\right\|_{L^2}^2\Big|_{f=M_{[\rho,0,\langle T\rangle]}} \\ &\geq \kappa \int_{T^N}\left|\frac{\nabla\rho}{\rho} - \frac{\nabla_x\phi}{\langle T\rangle}\right|^2 dx \\ &= \kappa\left[\int\left|\frac{\nabla\rho}{\rho}\right|^2 + \int\left|\frac{\nabla_x\phi}{\langle T\rangle}\right|^2 - 2\int\frac{\nabla\rho\cdot\nabla\phi}{\rho\langle T\rangle}\right], \end{aligned} \tag{50}$$

where

$$\begin{aligned} -2\int\frac{\nabla\rho\cdot\nabla\phi}{\rho\langle T\rangle} &= -\frac{2}{\langle T\rangle}\int\nabla\ln\rho\cdot\nabla\phi \\ &= \frac{2}{\langle T\rangle}\int\ln\rho\Delta\phi = \frac{2}{\langle T\rangle}\int\ln\rho\left(\rho-1\right). \end{aligned} \tag{51}$$

It is easy to verify the convexity and nonnegativity of $\ln\rho(\rho-1)$. Therefore,

$$\begin{aligned} &\frac{d^2}{dt^2}\left\|f - M_{[\rho,0,\langle T\rangle]}\right\|_{L^2}^2\Big|_{f=M_{[\rho,0,\langle T\rangle]}} \\ &\geq \kappa\left[\int\left|\frac{\nabla\rho}{\rho}\right|^2 + \int\left|\frac{\nabla_x\phi}{\langle T\rangle}\right|^2\right] \\ &\geq K_3\left[\int_{T^N}|\nabla\rho|^2 dx + \int_{T^N}|\nabla_x\phi|^2 dx\right]. \end{aligned} \tag{52}$$

When f does not coincide with $M_{[\rho,0,\langle T\rangle]}$, we need to estimate two terms A and B of (42) separately. The detailed calculation can be found in [4, 5]. Also, we just emphasize the estimates for terms with ϕ here.

Notice that, when estimating B, we need to control $\|\partial^2 f/\partial t^2\|_{L^2}$ by $\|f-M\|_{L^2}^{1-\alpha}$. Substitute the Vlasov-Poisson-Boltzmann equation (1) into $\partial^2 f/\partial t^2$; we get terms of ϕ.

(a) L^2 norm estimate of $\nabla_x\phi_t\cdot\nabla_v f$.

It is obvious that

$$\begin{aligned} &\left\|\nabla_x\phi_t\cdot\nabla_v f\right\|_{L^2} \\ &\leq \left\|\nabla_x\phi_t\cdot\nabla_v(f-M)\right\|_{L^2} + \left\|\nabla_x\phi_t\cdot\nabla_v M\right\|_{L^2} \\ &\leq C\left(\int\left|\nabla_v\left(f-M\right)\right|^2 dx\,dv\right)^{1/2} \\ &\quad + C\left(\int\left|\nabla_x\phi_t\right|^2 dx\right)^{1/2}. \end{aligned} \tag{53}$$

The first term is bounded by $C\|f-M\|_{L^2}^{1-\alpha}$ by interpolation lemma. As for the second term, since

$$\begin{aligned} \int\left|\nabla_x\phi_t\right|^2 dx &= -\int\Delta\phi_t\cdot\phi_t dx = -\int\rho_t\phi_t dx \\ &= \int\nabla_x\cdot\left(\rho u\right)\phi_t dx = -\int\rho u\cdot\left(\nabla_x\phi_t\right)dx \\ &\leq \left(\int\rho^2u^2dx\right)^{1/2}\left(\int\left|\nabla_x\phi_t\right|^2 dx\right)^{1/2}, \end{aligned} \tag{54}$$

we have

$$\begin{aligned} \int\left|\nabla_x\phi_t\right|^2 dx &\leq C\int\rho^2u^2dx \leq C\mathscr{H}\left(u\mid 0\right) \\ &\leq CH\left(f\mid M\right) \leq C\|f-M\|_{L^2}^2. \end{aligned} \tag{55}$$

Hence, $\|\nabla_x\phi_t\cdot\nabla_v f\|_{L^2} \leq C\|f-M\|_{L^2}^{1-\alpha}$.

(b) L^2 norm estimate of $(v\otimes\nabla_v M):(\nabla_x^2\phi)$.

Note that M is a Gaussian distribution, so that M^2 times any polynomials of v is integrable:

$$\begin{aligned} &\left\|(v\otimes\nabla_v M):(\nabla_x^2\phi)\right\|_{L^2}^2 \\ &= \left\|(v\otimes v):(\nabla_x^2\phi)M\right\|_{L^2}^2 \\ &= \sum_{i,j,k,l}\int v_iv_jv_kv_lM^2\partial_{x_ix_j}\phi\partial_{x_kx_l}\phi dv\,dx \\ &\leq C\sum_{i,j}\int\partial_{x_ix_i}\phi\partial_{x_jx_j}\phi dx \\ &= C\int\left(\Delta\phi\right)^2 dx = C\int\left|\rho-1\right|^2 dx \\ &\leq C\mathscr{H}\left(\rho\mid 1\right) \leq CH\left(f\mid M\right) \leq C\|f-M\|_{L^2}^2. \end{aligned} \tag{56}$$

(c) L^2 norm estimate of $(\nabla_x\phi\otimes\nabla_x\phi):\nabla_v^2 M$.

Similarly as in the previous argument, M^2 times any polynomials of v is integrable. Also, ϕ and $\partial\phi$ are bounded by Schauder estimate because it is constrained by a Poisson equation.

Note that $\Delta\phi = \rho-1$; we have

$$\begin{aligned} \int_{T^N}\left|\nabla\phi\right|^2 dx &= -\int\Delta\phi\left(\phi-\overline{\phi}\right)dx \\ &= -\int\left(\rho-1\right)\left(\phi-\overline{\phi}\right)dx \\ &\leq \left(\int\left(\rho-1\right)^2\right)^{1/2}\left(\int\left(\phi-\overline{\phi}\right)^2\right)^{1/2} \\ &\leq K_P^{1/2}\left(\int\left(\rho-1\right)^2\right)^{1/2}\left(\int\left|\nabla\phi\right|^2\right)^{1/2}. \end{aligned} \tag{57}$$

Here, K_P is the constant appearing in the Poincaré inequality, which is only relevant to the domain T^N. Thus,

$$\int\left|\nabla\phi\right|^2 \leq K_P\int\left(\rho-1\right)^2 \leq K_P^2\int\left|\nabla\rho\right|^2. \tag{58}$$

Therefore,

$$\begin{aligned}
&\left\|(\nabla_x\phi\otimes\nabla_x\phi):\nabla_v^2 M\right\|_{L^2}^2\\
&=\left\|(\nabla_x\phi\otimes\nabla_x\phi):(v\otimes v-I)M\right\|_{L^2}^2\\
&=\sum_{i,j,k,l}\int\partial_i\phi\partial_j\phi\partial_k\phi\partial_l\phi\left(v_iv_j-\delta_{ij}\right)\\
&\qquad\times\left(v_kv_l-\delta_{kl}\right)M^2\,dv\,dx\\
&\leq C\sum_{i,j}\int\left(\partial_i\phi\partial_j\phi\right)^2dx\leq C\sum_i\int\left(\partial_i\phi\right)^2dx\\
&=C\int|\nabla\phi|^2dx\leq C\int|\rho-1|^2dx\leq C\mathscr{H}(\rho\mid 1)\\
&\leq CH(f\mid M)\leq C\|f-M\|_{L^2}^2.
\end{aligned}\tag{59}$$

(d) L^2 norm estimate of $Q^{\mathrm{sym}}(f,\nabla_x\phi\cdot\nabla_v M)$.

From the momentum and energy conservation of particle collisions, it is easy to verify that

$$Q^{\mathrm{sym}}\left(M,v_iM\right)=0.\tag{60}$$

Thus,

$$Q^{\mathrm{sym}}\left(f,\nabla_x\phi\cdot\nabla_v M\right)=Q^{\mathrm{sym}}\left(f-M,\nabla_x\phi\cdot vM\right).\tag{61}$$

Then, using our continuity assumption (11) on $Q(g,h)$ and the interpolation Lemma 8, we can estimate L^2 norm of $Q^{\mathrm{sym}}(f,\nabla_x\phi\cdot\nabla_v M)$ by $\|f-M\|_{L^2}^{1-\alpha}$. Therefore, we have

$$\begin{aligned}
&\forall 0<\alpha<\eta<1,\\
&\left\|\frac{\partial^2 f}{\partial t^2}\right\|_{L^2}\leq C_\alpha\|f-M\|_{L^2}^{1-\alpha}\leq C_\eta\|f-M\|_{L^1}^{1-\eta}.
\end{aligned}\tag{62}$$

The rest of the proof is similar to that in [5]. Now we complete the proof of the lemma. □

Notice that there is the symmetric gradient of u in (38); the next lemma can provide a method to control this term.

Lemma 9. *One has the Korn-type inequality:*

$$\int_{T^N}\left|\nabla^{sym}u\right|^2dx\geq K_K\int_\Omega|\nabla u|^2dx\tag{63}$$

and the following Poincaré-type inequalities:

$$\begin{aligned}
&\int_{T^N}|\nabla T|^2dx\geq K_T\mathscr{H}(T\mid\langle T\rangle),\\
&\int_{T^N}|\nabla u|^2dx\geq K_u\mathscr{H}(u\mid 0),\\
&\int_{T^N}|\nabla\rho|^2dx\geq K_\rho\mathscr{H}(\rho\mid 1).
\end{aligned}\tag{64}$$

Here all constants are positive.

Lemma 10. *One has estimates on damping of hydrodynamic oscillations with $C_S>0$,*

$$\begin{aligned}
&\left|\frac{d}{dt}\mathscr{H}(\rho\mid 1),\frac{d}{dt}\mathscr{H}(u\mid 0),\frac{d}{dt}\mathscr{H}(\langle T\rangle\mid 1),\right.\\
&\qquad\left.\frac{d}{dt}\mathscr{H}(T\mid\langle T\rangle)\right|\\
&\leq C_S H(f\mid M)^{1-\epsilon}.
\end{aligned}\tag{65}$$

See [4] or [5] for the proof of the previous two lemmas. Inequalities of Lemma 9 provide estimates of the right-hand side of second-order differential inequalities in Lemma 7. Lemma 10 provides the decay rate for hydrodynamic oscillations.

3. Proof of the Main Result

Use the previous lemmas; we are now ready to prove Theorem 2. The main idea is similar to that in [5]; for convenience of the reader, we restate the sketch of the proof and make it more complete by proving Lemma 11.

From H-theorem (Theorem 5), the convergence rate of $H(f)$ to $H(M)$ is determined by entropy production functional $D(f)$. But there are many local Maxwellians, which make our entropy production functional $D(f)$ vanish. Therefore it is impossible to get a uniform lower bound on the entropy production. To overcome this difficulty, it is natural to estimate the average value of entropy production. Suppose that

$$\alpha_0=H(f)-H(M)\big|_{t=t_0}.\tag{66}$$

We wish to find an upper bound on a duration T_0 (it is possible since $H(f)$ is monotone nonincreasing), such that

$$H(f)-H(M)\big|_{t=t_0+T_0}=\sigma\alpha_0,\tag{67}$$

where $\sigma\in(0,1)$ is fixed; say $\sigma=4/5$. Therefore, we have

$$\frac{4}{5}\alpha_0\leq H(f)-H(M)\leq\alpha_0.\tag{68}$$

Lemma 11. *Choose that $\epsilon>0$ is small enough, like $\epsilon<0.01$, if one can show*

$$T_0\leq C_0(\epsilon)\,\alpha_0^{-699\epsilon},\tag{69}$$

where C_0 depends on ϵ and the various constants appearing in lemmas of Section 2. Then

$$H(f)-H(M)=O\left(t^{-1/700\epsilon}\right).\tag{70}$$

Proof. Fix $\epsilon>0$ sufficiently small. Denote $H(f)-H(M)$ by $g(t)$. It is not hard to prove the continuity of $g(t)$. From the boundedness of initial data f_0, we can denote $t_0:=0$, $g(t)|_{t=0}=g(0)=:\beta_0$. It is sufficient to prove that, for all $t>0$, $t^{1/700\epsilon}g(t)$ or equivalently $tg(t)^{700\epsilon}$ is uniformly bounded.

Define a sequence $\{t_i\}$, such that

$$g(t)\big|_{t=t_i}=\sigma^i\beta_0.\tag{71}$$

Correspondingly, we can define $\{T_i\}$, $T_i = t_i - t_{i-1}$. From the estimate of T_0 in (69), we have

$$T_i \leq C_0(\epsilon)\left(\sigma^{i-1}\beta_0\right)^{-699\epsilon}. \quad (72)$$

Therefore,

$$t_i = \sum_{k=1}^{i} T_k \leq C_0(\epsilon)(\beta_0)^{-699\epsilon}\frac{\sigma^{-699\epsilon(i-1)} - 1}{\sigma^{-699\epsilon} - 1}. \quad (73)$$

It is obvious that $t_i \to \infty$, as $i \to \infty$.

For any $t > 0$, we can find an interval such that $t \in [t_{i-1}, t_i]$. Now we are ready to estimate $tg(t)^{700\epsilon}$. From the monotonicity of $g(t)$, we have

$$\begin{aligned} tg(t)^{700\epsilon} &\leq t_i g(t_{i-1})^{700\epsilon} \\ &\leq C_0(\epsilon)(\beta_0)^{-699\epsilon}\frac{\sigma^{-699\epsilon(i-1)} - 1}{\sigma^{-699\epsilon} - 1} \\ &\quad \times \left(\sigma^{i-1}\beta_0\right)^{700\epsilon} \leq C, \end{aligned} \quad (74)$$

where the constant is independent of i, since $\sigma < 1$ is fixed and $\epsilon > 0$ can be chosen to be sufficiently small. □

Once condition (69) is proved, the main theorem is a direct consequence of $H(f \mid M) = O(H(f) - H(M))$. Indeed, from (58) and (20), we have

$$\begin{aligned} &\int |\nabla\phi|^2 \leq K_P \int (\rho - 1)^2 \leq K' \mathscr{H}(\rho \mid 1) \leq K'' H(f \mid M), \\ &H(f \mid M) \leq H(f) - H(M) = H(f \mid M) + \int \frac{1}{2}|\nabla\phi|^2 \\ &\qquad \leq \left(1 + \frac{K''}{2}\right) H(f \mid M). \end{aligned} \quad (75)$$

Therefore, it remains to prove condition (69). Detailed proof can be found in the last part of [5] for Vlasov-Poisson-Boltzmann equations; we only describe the idea of the proof for the completion of this paper. Consider

$$\frac{4}{5}\alpha_0 \leq H(f) - H(M) \leq \alpha_0, \quad (76)$$

on interval $I := [t_0, t_0 + T_0]$; that is, $H(f) - H(M)$ has variation $\alpha_0/5$. In order to prove (69), it is sufficient to prove that the average value of $-(d/dt)H(f)$ on interval I satisfies

$$\left\langle -\dot{H}(f) \right\rangle_I \geq \frac{C\alpha_0}{C_0(\epsilon)\alpha_0^{-699\epsilon}} = C\alpha_0^{1+699\epsilon}. \quad (77)$$

Now we proceed the proof of Theorem 2 step by step.

(1) I_G: *Subinterval of I Where $H(f \mid M_{[\rho,u,T]})$ Is Large*. From quantitative H-Theorem 5, $\langle -\dot{H}(f)\rangle_{I_G}$ can be estimated directly on subinterval of I where $H(f \mid M_{[\rho,u,T]})$ is large. The subinterval can be called I_G, which means good interval. Other interval is called I_B, bad interval.

Notice the entropy additivity rules in Lemma 6; we actually have

$$\begin{aligned} H(f) - H(M) &= H(f \mid M) + \int \frac{1}{2}|\nabla\phi|^2 \\ &\leq \left(1 + \frac{K'}{2}\right)\mathscr{H}(\rho \mid 1) + \mathscr{H}(u \mid 0) \\ &\quad + \mathscr{H}(T \mid \langle T\rangle) \\ &\quad + \mathscr{H}(\langle T\rangle \mid 1) + H\left(f \mid M_{[\rho,u,T]}\right). \end{aligned} \quad (78)$$

(2) I_{BG}: *Subinterval of I_B Where $\mathscr{H}(T \mid \langle T\rangle)$ Is Large*. On interval I_B, $H(f \mid M_{[\rho,u,T]})$ is small, while $H(f) - H(M)$ has lower bound $(4/5)\alpha_0$. Then by entropy additivity rules, we must have that $\mathscr{H}(\rho \mid 1) + \mathscr{H}(u \mid 0) + \mathscr{H}(T \mid \langle T\rangle) + \mathscr{H}(\langle T\rangle \mid 1)$ cannot be small.

Denote the subinterval of I_B by I_{BG} where $\mathscr{H}(T \mid \langle T\rangle)$ is large. Then from the Poincaré-type inequalities of Lemma 9, we have that $\int_{T^N} |\nabla T|^2 dx$ is large. Therefore, the right hand side of (37) is large. By an argument for second-order differential inequalities (Lemma 12 of Desvillettes and Villani in [4]), we can conclude that either the average value of $\| f - M_{[\rho,u,T]}\|_{L^2}$ is large (so is $H(f|M_{[\rho,u,T]})$) or the length of interval is small enough to be absorbed. H-theorem then asserts that average value $\langle -\dot{H}(f)\rangle_{I_{BG}}$ is large.

(3) I_{BBG}: *Subinterval of I_{BB} Where $\mathscr{H}(u \mid 0)$ Is Large*. On interval I_{BB}, $H(f \mid M_{[\rho,u,T]})$, $\mathscr{H}(T \mid \langle T\rangle)$ is small, while $H(f) - H(M)$ has lower bound $(4/5)\alpha_0$. Then similarly, $\mathscr{H}(\rho \mid 1) + \mathscr{H}(u \mid 0) + \mathscr{H}(\langle T\rangle \mid 1)$ cannot be small.

Denote the subinterval of I_{BB} by I_{BBG} where $\mathscr{H}(u \mid 0)$ is large. Then from the Poincaré-type and Korn-type inequalities of Lemma 9, we have that $\int |\nabla^{\mathrm{sym}} u|^2 dx$ is large. Therefore, the right-hand side of (38) is large. By an argument for second-order differential inequalities (Lemma 12 of Desvillettes and Villani in [4]), we can conclude that either the average value of $\|f - M_{[\rho,u,\langle T\rangle]}\|_{L^2}$ is large (so is $H(f \mid M_{[\rho,u,\langle T\rangle]})$) or the length of interval is small enough to be absorbed. But the first line of (35) shows that $H(f \mid M_{[\rho,u,T]})$ must be large in average. H-theorem then asserts that average value $\langle -\dot{H}(f)\rangle_{I_{BBG}}$ is large.

(4) I_{BBBG}: *subinterval of I_{BBB} where $\mathscr{H}(\langle T\rangle | 1)$ is large*. On interval I_{BBB}, $H(f \mid M_{[\rho,u,T]})$, $\mathscr{H}(T \mid \langle T\rangle)$, $\mathscr{H}(u \mid 0)$ is small, while $H(f) - H(M)$ has lower bound $(4/5)\alpha_0$. Then similarly, $\mathscr{H}(\rho \mid 1) + \mathscr{H}(\langle T\rangle \mid 1)$ cannot be small.

Denote the subinterval of I_{BBB} by I_{BBBG} where $\mathscr{H}(\langle T\rangle \mid 1)$ is large. From the conservation of energy, we have

$$|\langle T\rangle - 1| = \frac{2}{N}\mathscr{H}(u \mid 0) + \frac{1}{N}\int_{T^N} |\nabla\phi|^2 dx. \quad (79)$$

Because of the Lipschitz continuity of $\mathscr{H}(\langle T\rangle \mid 1) = \Psi(\langle T\rangle)$,

$$\mathscr{H}(\langle T\rangle \mid 1) \leq L|\langle T\rangle - 1|. \quad (80)$$

Since $\mathscr{H}(u \mid 0)$ is sufficiently small in I_{BBBG}, therefore, (79) turns to be

$$\mathscr{H}\left(\langle T\rangle \mid 1\right) \leq L\left|\langle T\rangle - 1\right| \leq C\int_{T^N}\left|\nabla\phi\right|^2 dx \leq C\mathscr{H}\left(\rho \mid 1\right). \tag{81}$$

Therefore, the right-hand side of (40) and (39) is large. By a similar argument as in previous subintervals, we can also show that average value $\langle -\dot{H}(f)\rangle_{I_{BBBG}}$ is large. By a careful calculation to absorb all the bad intervals into good ones, we can prove that average value $\langle -\dot{H}(f)\rangle_I$ is large on interval I. Thus, the whole proof is complete.

To conclude the paper, we remove the condition in Theorem 1 by making a crucial estimates on terms with ϕ. The main differences with previous works [5] are in proving Lemma 7. We also complete the gap in the last part of [4, 5] by proving Lemma 11.

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References

[1] S. Ukai, "On the existence of global solutions of mixed problem for non-linear Boltzmann equation," *Proceedings of the Japan Academy*, vol. 50, pp. 179–184, 1974.

[2] R. E. Caflisch, "The Boltzmann equation with a soft potential. I. Linear, spatially-homogeneous," *Communications in Mathematical Physics*, vol. 74, no. 1, pp. 71–95, 1980.

[3] R. M. Strain and Y. Guo, "Exponential decay for soft potentials near Maxwellian," *Archive for Rational Mechanics and Analysis*, vol. 187, no. 2, pp. 287–339, 2008.

[4] L. Desvillettes and C. Villani, "On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation," *Inventiones Mathematicae*, vol. 159, no. 2, pp. 245–316, 2005.

[5] L. Li, "On the trend to equilibrium for the Vlasov-Poisson-Boltzmann equation," *Journal of Differential Equations*, vol. 244, no. 6, pp. 1467–1501, 2008.

[6] Y. Guo, "The Vlasov-Poisson-Boltzmann system near Maxwellians," *Communications on Pure and Applied Mathematics*, vol. 55, no. 9, pp. 1104–1135, 2002.

[7] R. M. Strain, "The Vlasov-Maxwell-Boltzmann system in the whole space," *Communications in Mathematical Physics*, vol. 268, no. 2, pp. 543–567, 2006.

[8] T. Yang, H. J. Yu, and H. J. Zhao, "Cauchy problem for the Vlasov-Poisson-Boltzmann system," *Archive for Rational Mechanics and Analysis*, vol. 182, no. 3, pp. 415–470, 2006.

[9] T. Yang and H. J. Zhao, "Global existence of classical solutions to the Vlasov-Poisson-Boltzmann system," *Communications in Mathematical Physics*, vol. 268, no. 3, pp. 569–605, 2006.

[10] C. Villani, "Cercignani's conjecture is sometimes true and always almost true," *Communications in Mathematical Physics*, vol. 234, no. 3, pp. 455–490, 2003.

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Research Article

Global Asymptotic Behavior of a Nonautonomous Competitor-Competitor-Mutualist Model

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The global asymptotic behavior of a nonautonomous competitor-competitor-mutualist model is investigated, where all the coefficients are time-dependent and asymptotically approach periodic functions, respectively. Under certain conditions, it is shown that the limit periodic system of this asymptotically periodic model admits two positive periodic solutions (u_1^T, u_{2T}, u_3^T), (u_{1T}, u_2^T, u_{3T}) such that $u_{iT} \le u_i^T$ $(i = 1, 2, 3)$, and the sector $\{(u_1, u_2, u_3) : u_{iT} \le u_i \le u_i^T,\ i = 1, 2, 3\}$ is a global attractor of the asymptotically periodic model. In particular, we derive sufficient conditions that guarantee the existence of a positive periodic solution which is globally asymptotically stable.

1. Introduction

In this paper, we investigate the global asymptotic behavior of solutions for the following competitor-competitor-mutualist diffusion model:

$$
\begin{aligned}
& u_{1t} - d_1 \Delta u_1 \\
& \quad = g_1 u_1 \left(1 - \frac{u_1}{a_1} - \frac{a_2 u_2}{1 + a_3 u_3}\right) \quad \text{in } \Omega \times (0, \infty), \\
& u_{2t} - d_2 \Delta u_2 \\
& \quad = g_2 u_2 \left(1 - b_1 u_1 - \frac{u_2}{b_2}\right) \quad \text{in } \Omega \times (0, \infty), \\
& u_{3t} - d_3 \Delta u_3 \\
& \quad = g_3 u_3 \left(1 - \frac{u_3}{c_1 + c_2 u_1}\right) \quad \text{in } \Omega \times (0, \infty),
\end{aligned}
\tag{1}
$$

$$\frac{\partial u_i}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty), \quad i = 1, 2, 3, \tag{2}$$

$$u_i(x, 0) = u_{i0}(x) \quad \text{on } \Omega,\ i = 1, 2, 3, \tag{3}$$

where $u_1(x,t)$, $u_2(x,t)$, and $u_3(x,t)$ are the densities of a mutualist-competitor, a competitor, and a mutualist population, respectively. $\Omega \subset \mathbf{R}^N$ is a bounded smooth domain, $\partial/\partial n$ is an outward normal derivative on $\partial\Omega$.

In 1983, Rai et al. [1] firstly presented and studied a general competitor-competitor-mutualist ordinary differential equation (ODE) model. Zheng [2] studied the problem (1)–(3) in the case where all coefficients are positive constants. He proved the local stability of the unique positive constant steady-state solution under suitable condition on the reaction rates by the method of spectral analysis for linearized operator. Xu [3] investigated the global asymptotic stability of the unique positive constant steady-state solution under some assumptions by the iteration method. Pao [4] considered the model with time delays, and, under a very simple condition on the reaction rates, proved that the time-dependent solution with any nontrivial initial function converges to the positive steady-state solution by the method of upper and lower solutions. Chen and Peng [5] proved some existence results concerning nonconstant positive steady-states for the model with cross-diffusion and demonstrated that the cross-diffusion can create patterns when the corresponding model without cross-diffusion fails. Li et al. [6] proved that this model with cross-diffusion possesses at least one coexistence state if cross-diffusions and cross-reactions are weak by the Schauder fixed point theory and the method of upper and lower solutions and its associated monotone iterations. Fu et al. [7] investigated the global asymptotic behavior and the global existence of time-dependent solutions for the model with cross-diffusion when the space dimension is at most 5. Very recently, Tian and Ling [8] proved that, under some conditions, a corresponding predator-prey-mutualist model with cross-diffusion admits at least a nonhomogeneous stationary solution by the stability analysis for the positive uniform solution and the Leray-Schauder degree theory and carried out numerical simulations for a Turing pattern.

For the model (1) with T-periodic coefficients, Tineo [9] studied the asymptotic behavior of positive solutions by the method of upper and lower solutions. Du [10] investigated the existence of positive T-periodic solutions by using the degree and bifurcation theories. Pao [11] proved the existence of maximal and minimal T-periodic solutions by the method of upper and lower solutions. Wang et al. [12] considered the local asymptotic behavior of the time-dependent solutions and the existence of periodic solutions to the model in an unbounded domain. Zhou and Fu [13] investigated the global asymptotic behavior of the time-dependent solutions and the existence of periodic solutions for the model with discrete delays. Very recently, replacing the usual $-\Delta u$ term by a degenerate elliptic operator as $-\Delta u^m$, Wang and Yin [14] proved the existence of maximal and minimal T-periodic solutions to the model with time delays by the Schauder fixed point theorem. It is important to note that the uniqueness of positive periodic solution is not considered in the previous references.

When $a_3 = 0$, (1) reduces to the competition diffusion system

$$\begin{aligned} u_t - d_1 \Delta u &= u\left(a - bu - cv\right), \\ v_t - d_2 \Delta v &= v\left(d - eu - fv\right), \end{aligned} \tag{4}$$

where $a = g_1$, $b = g_1/a_1$, $c = g_1/a_2$, $d = g_2$, $e = g_2 b_1$, and $f = g_2/b_2$. The system (4) is a diffusion extension of the well-known Lotka-Volterra system

$$\begin{aligned} \frac{du}{dt} &= u\left(a - bu - cv\right), \\ \frac{dv}{dt} &= v\left(d - eu - fv\right). \end{aligned} \tag{5}$$

In the case that a, b, c, d, e, and f are positive T-periodic functions, the existence and asymptotic stability of periodic solutions for (5) was studied by Gopalsamy [15], Alvarez and Lazer [16], and Ahmad [17] in the 1980's. The global asymptotic behavior of (5) was studied by Ahmad and Lazer [18] and Tineo [19]. Denote $f_L = \inf_{x\in X} f(x)$ and $f_M = \sup_{x\in X} f(x)$ for any function $f : X \to \mathbf{R}$. If a, b, c, d, e, and f are positive asymptotically T-periodic functions on $\mathbf{R}$, Peng and Chen [20] proved that if the conditions

$$\frac{a_L - \varepsilon_0}{d_M + \varepsilon_0} > \frac{c_M + \varepsilon_0}{f_L - \varepsilon_0}, \qquad \frac{d_L - \varepsilon_0}{a_M + \varepsilon_0} > \frac{e_M + \varepsilon_0}{b_L - \varepsilon_0} \tag{6}$$

are satisfied for a certain sufficiently small $\varepsilon_0 > 0$, then any positive solutions of (5) asymptotically approach the unique positive periodic solution for the limit periodic system of (5).

It is well known that periodic reaction diffusion equations are of particular interests since they can take into account seasonal fluctuations occurring in the phenomena appearing in the models, and they have been extensively studied by many researchers (see, e.g., [9–14, 19, 21]). However, so far, the research work on asymptotically periodic systems is much fewer than on the periodic ones. In fact, asymptotically periodic systems describe our world more realistically and more accurately than periodic ones to some extent. Therefore, for asymptotically periodic systems, studying the dynamics behavior is important and necessary (see, e.g., [22–27]).

In this paper, we study the global asymptotic behavior of positive solutions for the asymptotically periodic system (1). Under some conditions, it is shown that any positive solutions of the models asymptotically approach the unique strictly positive periodic solutions of the corresponding periodic system. This means that the results in Tineo [9] and the results for ODE model in Peng and Chen [20] can be extended to the asymptotically periodic reaction diffusion system and the 3-species diffusion system, respectively. Furthermore, using the method of the present paper, we note that the corresponding conclusions hold for the time-dependent n-species Lotka-Volterra systems. More specifically, we provide a way of how to use the method of upper and lower solutions to study asymptotic behavior of solutions for asymptotically periodic reaction diffusion systems. As one can see, the optimal bounds and uniqueness of positive periodic solutions will play an important role in the study of the global asymptotic behavior of periodic solutions.

2. Permanence and Extinction

For the sake of convenience, we introduce the two signs $\sim$ and $\prec$ for functions $u, v : \overline{\Omega} \times \mathbf{R} \to [0, \infty)$. u is said to approach v asymptotically in notation, $u \sim v$, if $\lim_{t\to\infty} |u(x,t) - v(x,t)| = 0$ uniformly for x in $\overline{\Omega}$. Furthermore, if $(\varphi_1, \varphi_2, \dots, \varphi_n)$ and $(\psi_1, \psi_2, \dots, \psi_n)$ are vector functions, then $(\varphi_1, \varphi_2, \dots, \varphi_n) \sim (\psi_1, \psi_2, \dots, \psi_n)$ if and only if $\varphi_i \sim \psi_i$ $(i = 1, 2, \dots, n)$. We say that $u(x,t)$ is asymptotically smaller than $v(x,t)$ and write $u(x,t) \prec v(x,t)$ if $\overline{\lim}_{t\to\infty}(u(x,t) - v(x,t)) \leq 0$ uniformly for $x \in \overline{\Omega}$. It is clear that $u(x,t) \prec v(x,t)$ if and only, if for any $\varepsilon > 0$, there exists a corresponding $t_1 > 0$ such that $u(x,t) < v(x,t) + \varepsilon$ on $\Omega \times [t_1, \infty)$.

Assume the following.

(H_1) d_i, A_i, B_i, C_i, and G_i are positive smooth and T-periodic functions on $\overline{\Omega} \times \mathbf{R}$.

(H_2) a_i, b_i, c_i, and g_i are positive smooth functions on $\overline{\Omega} \times \mathbf{R}$, and

$$\left(a_i, b_i, c_i, g_i\right) \sim \left(A_i, B_i, C_i, G_i\right). \tag{7}$$

By (H_2), the limit periodic system of (1), (2) is given as follows:

$$\begin{aligned} & u_{1t}-d_1\Delta u_1 \\ &\quad = G_1u_1\left(1-\frac{u_1}{A_1}-\frac{A_2u_2}{1+A_3u_3}\right) \quad \text{in } \Omega\times(0,\infty), \\ & u_{2t}-d_2\Delta u_2 \\ &\quad = G_2u_2\left(1-B_1u_1-\frac{u_2}{B_2}\right) \quad \text{in } \Omega\times(0,\infty), \\ & u_{3t}-d_3\Delta u_3 \\ &\quad = G_3u_3\left(1-\frac{u_3}{C_1+C_2u_1}\right) \quad \text{in } \Omega\times(0,\infty), \\ & \frac{\partial u_i}{\partial n}=0 \quad \text{on } \partial\Omega\times(0,\infty) \ (i=1,2,3). \end{aligned} \tag{8}$$

As a complement, we state the following main result which comes from [9, Theorem 0.3].

Theorem 1. *Assume that* (H_1) *holds, and*

$$A_{2M}B_{2M} < 1 + A_{3L}C_{1L}, \tag{9}$$

$$A_{1M}B_{1M} < 1. \tag{10}$$

Then (8) *has the periodic solutions* (u_1^T, u_{2T}, u_3^T) *and* (u_{1T}, u_2^T, u_{3T}) *such that* $u_i^T \ge u_i \ge u_{iT} > 0$ $(i=1,2,3)$ *for any positive* T*-periodic solution* (u_1,u_2,u_3) *of* (8)*. Moreover, given that* $\varepsilon>0$ *and a solution* (u_1,u_2,u_3) *of* (8) *with* $u_i(x,0)\ge(\not\equiv)0$*, there exists* $t_1>0$ *such that* $u_{iT}(x,t)-\varepsilon<u_i(x,t)<u_i^T(x,t)+\varepsilon$ *on* $\Omega\times(t_1,\infty)$.

In order to get the conditions for the permanence of (1)–(3), we need to make the following optimal bounds.

Lemma 2. *If* (9) *and* (10) *hold and* (u_1,u_2,u_3) *is a positive smooth* T*-periodic solution of* (8)*, then*

$$\varepsilon_{u_i} \le u_i \le \delta_{u_i} \quad (i=1,2,3), \tag{11}$$

where δ_{u_1} *is the unique positive root of* $p_1x^2+q_1x+r_1=0$ *and*

$$\begin{aligned} p_1 &= A_{3M}C_{2M}, \\ r_1 &= A_{1M}\left(A_{2L}B_{2L}-1-A_{3M}C_{1M}\right), \\ q_1 &= 1+A_{3M}C_{1M}-A_{1M}A_{3M}C_{2M}-A_{1M}A_{2L}B_{1M}B_{2L}. \end{aligned} \tag{12}$$

ε_{u_1} *is the unique positive root of* $p_2x^2+q_2x+r_2=0$*, and*

$$\begin{aligned} p_2 &= A_{3L}C_{2L}, \\ r_2 &= A_{1L}\left(A_{2M}B_{2M}-1-A_{3L}C_{1L}\right), \\ q_2 &= 1+A_{3L}C_{1L}-A_{1L}A_{3L}C_{2L}-A_{1L}A_{2M}B_{1L}B_{2M}, \\ \delta_{u_2} &= B_{2M}-B_{1L}B_{2M}\varepsilon_{u_1}, \\ \varepsilon_{u_2} &= B_{2L}-B_{1M}B_{2L}\delta_{u_1}, \\ \delta_{u_3} &= C_{1M}+C_{2M}\delta_{u_1}, \qquad \varepsilon_{u_3}=C_{1L}+C_{2L}\varepsilon_{u_1}. \end{aligned} \tag{13}$$

Proof. By the maximum principle (see Lemma 1.2 of [18]), we have

$$\begin{aligned} & 1-\frac{u_{1M}}{A_{1M}}-\frac{A_{2L}u_{2L}}{1+A_{3M}u_{3M}} \ge 0, \\ & 1-\frac{u_{1L}}{A_{1L}}-\frac{A_{2M}u_{2M}}{1+A_{3L}u_{3L}} \le 0, \\ & 1-B_{1L}u_{1L}-\frac{u_{2M}}{B_{2M}} \ge 0, \\ & 1-B_{1M}u_{1M}-\frac{u_{2L}}{B_{2L}} \le 0, \\ & 1-\frac{u_{3M}}{C_{1M}+C_{2M}u_{1M}} \ge 0, \\ & 1-\frac{u_{3L}}{C_{1L}+C_{2L}u_{1L}} \le 0. \end{aligned} \tag{14}$$

Hence, $p_1u_{1M}^2+q_1u_{1M}+r_1\le 0$. Since $p_1>0$ and $r_1<0$ (by (9)), we can see immediately that $u_{1M}\le\delta_{u_1}$. Similarly, if ε_{u_1} is the unique positive root of $p_2x^2+q_2x+r_2=0$, then $u_{1L}\ge\varepsilon_{u_1}$. So,

$$\begin{aligned} & \varepsilon_{u_2} \le u_{2L} \le u_{2M} \le \delta_{u_2}, \\ & \varepsilon_{u_3} \le u_{3L} \le u_{3M} \le \delta_{u_3}. \end{aligned} \tag{15}$$

Evidently, $\varepsilon_{u_3}>0$. By (9) and (10), we have

$$\frac{p_1}{B_{1M}^2}+\frac{q_1}{B_{1M}}+r_1>0, \tag{16}$$

from which it follows that $\varepsilon_{u_2}>0$. This completes the proof. □

Corollary 3. *Assume that* (9) *and* (10) *hold. If* A_i, B_i*, and* C_i *are positive constants, then* (9) *has the unique positive periodic solution* $(r, B_2(1-B_1r), C_1+C_2r)$*, where* r *is the unique positive root of* $p_1x^2+q_1x+r_1=0$.

The main results in this section are the following theorems.

Theorem 4 (permanence). *Assume that* (H_1), (H_2), (9), *and* (10) *hold. Then* (8) *has the positive T-periodic solutions* (u_{1T}, u_2^T, u_{3T}) *and* (u_1^T, u_{2T}, u_3^T) *such that* $u_{iT} \le u_i^T$ $(i = 1, 2, 3)$. *Moreover, if* (u_1, u_2, u_3) *is the solution of* (1)–(3) *with smooth initial values* $u_{i0}(x) \ge (\not\equiv) 0$, *then*

$$u_{iT} \prec u_i \prec u_i^T \quad (i = 1, 2, 3). \tag{17}$$

Remark 5. Under the assumptions of Theorem 4, the system (1), (2) is permanent, the sector $\langle u_T, u^T \rangle = \{u \in C(\overline{\Omega} \times \mathbf{R}) : u_T \le u \le u^T\}$ is a global periodic attractor of (1), (2), and its trivial and semitrivial periodic solutions are unstable. Furthermore, if A_i, B_i, and C_i are positive constants, then $(r, B_2(1 - B_1 r), C_1 + C_2 r)$ is the unique globally asymptotically stable solution of (8).

Theorem 6. *Assume that* (H_1) *and* (H_2) *hold. Then one has the following conclusions.*

(1) *(Extinction of u_2) Assume that* (9) *holds and that* $A_{1L} B_{1L} \ge 1$. *Then* (8) *has a T-periodic solution* (U_1, U_2, U_3) *such that* $U_1 > 0$, $U_2 = 0$, $U_3 > 0$, *and*

$$\lim_{t \to \infty} \left| u_i(x, t) - U_i(x, t) \right| = 0 \quad (i = 1, 2, 3) \tag{18}$$

uniformly on $\overline{\Omega}$, *for any positive solution* (u_1, u_2, u_3) *of* (1)–(3).

(2) *(Extinction of u_1) Assume that* (10) *holds and that*

$$A_{2L} B_{2L} \ge 1 + A_{3M} C_{1M} + A_{3M} A_{1M} C_{2M}. \tag{19}$$

Then (8) *has a T-periodic solution* (U_1, U_2, U_3) *with* $U_1 = 0$, $U_2 > 0$, *and* $U_3 > 0$ *satisfying* (18), *where* (u_1, u_2, u_3) *is any positive solution of* (1)–(3).

Proof of Theorem 4. By (9) and (10), there exists a sufficiently small $\varepsilon_0 > 0$ such that, for $\delta \in (0, \varepsilon_0)$,

$$\begin{aligned} &\left[\frac{(G_1 + \delta)(A_2 + \delta)}{G_1 - \delta} \right]_M \left[\frac{(G_2 + \delta)(B_2 + \delta)}{G_2 - \delta} \right]_M \\ &\quad < 1 + (A_{3L} - \delta) \left[\frac{(G_3 - \delta)(C_1 - \delta)}{G_3 + \delta} \right]_L, \end{aligned} \tag{20}$$

$$\left[\frac{(G_1 + \delta)(A_1 + \delta)}{G_1 - \delta} \right]_M \left[\frac{(G_2 + \delta)(B_1 + \delta)}{G_2 - \delta} \right]_M < 1. \tag{21}$$

Consider two auxiliary systems as follows:

$$\begin{aligned} &u_{1t} - d_1 \Delta u_1 \\ &\quad = u_1 \left[(G_1 + \delta) - \frac{(G_1 - \delta) u_1}{A_1 + \delta} - \frac{(G_1 - \delta)(A_2 - \delta) u_2}{1 + (A_3 + \delta) u_3} \right] \quad \text{in } \Omega \times (0, \infty), \\ &u_{2t} - d_2 \Delta u_2 \\ &\quad = u_2 \left[(G_2 - \delta) - (G_2 + \delta)(B_1 + \delta) u_1 - \frac{(G_2 + \delta) u_2}{B_2 - \delta} \right] \quad \text{in } \Omega \times (0, \infty), \\ &u_{3t} - d_3 \Delta u_3 \\ &\quad = u_3 \left[(G_3 + \delta) - \frac{(G_3 - \delta) u_3}{(C_1 + \delta) + (C_2 + \delta) u_1} \right] \quad \text{in } \Omega \times (0, \infty), \\ &\frac{\partial u_i}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty) \ (i = 1, 2, 3), \end{aligned} \tag{22}$$

$$\begin{aligned} &u_{1t} - d_1 \Delta u_1 \\ &\quad = u_1 \left[(G_1 - \delta) - \frac{(G_1 + \delta) u_1}{A_1 - \delta} - \frac{(G_1 + \delta)(A_2 + \delta) u_2}{1 + (A_3 - \delta) u_3} \right] \quad \text{in } \Omega \times (0, \infty), \\ &u_{2t} - d_2 \Delta u_2 \\ &\quad = u_2 \left[(G_2 + \delta) - (G_2 - \delta)(B_1 - \delta) u_1 - \frac{(G_2 - \delta) u_2}{B_2 + \delta} \right] \quad \text{in } \Omega \times (0, \infty), \\ &u_{3t} - d_3 \Delta u_3 \\ &\quad = u_3 \left[(G_3 - \delta) - \frac{(G_3 + \delta) u_3}{(C_1 - \delta) + (C_2 - \delta) u_1} \right] \quad \text{in } \Omega \times (0, \infty), \\ &\frac{\partial u_i}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty) \ (i = 1, 2, 3). \end{aligned} \tag{23}$$

By (20), (21), and Theorem 1, (22) has the positive T-periodic solutions $(U_{1\delta}, u_2^\delta, U_{3\delta})$ and $(U_1^\delta, u_{2\delta}, U_3^\delta)$ such that $U_{i\delta} \le u_i \le U_i^\delta$ $(i = 1, 3)$ and $u_{2\delta} \le u_2 \le u_2^\delta$, for any positive T-periodic solution (u_1, u_2, u_3) of (22). Moreover, if

(u_1, u_2, u_3) is a solution of (22) with nontrivial nonnegative initial values, then, for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$\begin{gathered} U_{i\delta}(x,t) - \varepsilon < u_i(x,t) < U_i^\delta(x,t) + \varepsilon \quad (i = 1,3), \\ u_{2\delta}(x,t) - \varepsilon < u_2(x,t) < u_2^\delta(x,t) + \varepsilon, \end{gathered} \tag{24}$$

for all $x \in \overline{\Omega}$ and $t > t_\varepsilon$. Similarly, (23) has the positive T-periodic solutions $(u_{1\delta}, U_2^\delta, u_{3\delta})$ and $(u_1^\delta, U_{2\delta}, u_3^\delta)$ such that $u_{i\delta} \le u_i \le u_i^\delta$ $(i = 1,3)$ and $U_{2\delta} \le u_2 \le U_2^\delta$, for any positive T-periodic solution (u_1, u_2, u_3) of (23). Furthermore, if (u_1, u_2, u_3) is a solution of (23) with nontrivial nonnegative initial values, then, for the previous $\varepsilon > 0$, there exists $t'_\varepsilon > 0$ such that, for all $x \in \overline{\Omega}$ and $t > t'_\varepsilon$,

$$\begin{gathered} u_{i\delta}(x,t) - \varepsilon < u_i(x,t) < u_i^\delta(x,t) + \varepsilon \quad (i = 1,3), \\ U_{2\delta}(x,t) - \varepsilon < u_2(x,t) < U_2^\delta(x,t) + \varepsilon. \end{gathered} \tag{25}$$

Now we prove

$$u_{i\delta} \le u_{iT} \le U_{i\delta}, \qquad u_i^\delta \le u_i^T \le U_i^\delta, \tag{26}$$

where (u_{1T}, u_2^T, u_{3T}) and (u_1^T, u_{2T}, u_3^T) are positive T-periodic solutions of (8) (see Theorem 1). Let (U_1, u_2, U_3), (p_1, p_2, p_3), and (u_1, U_2, u_3) be the solutions of (22), (8), and (23), respectively, which all satisfy the same initial conditions. It is easily testified that (U_1, U_2, U_3), (u_1, u_2, u_3) are the upper and lower solutions of (8) and (3), respectively. So from [28, Corollary 5.2.10], we see that

$$u_i \le p_i \le U_i \quad (i = 1,2,3). \tag{27}$$

For sufficiently small $m > 0$ and $\delta > 0$, define

$$\begin{gathered} r_1 = \left[\frac{(G_1+\delta)(A_1+\delta)}{G_1-\delta}\right]_M, \\ r_2 = \left[\frac{(G_2+\delta)(B_2+\delta)}{G_2-\delta}\right]_M, \\ r_3 = \left[\frac{(G_3+\delta)(C_1+\delta)}{G_3-\delta}\right]_M + \left[\frac{(G_3+\delta)(C_2+\delta)}{G_3-\delta}\right]_M r_1, \\ s_0 = \left[\frac{(G_3-\delta)(C_1-\delta)}{G_3+\delta}\right]_L + \left[\frac{(G_3-\delta)(C_2-\delta)}{G_3+\delta}\right]_L m. \end{gathered} \tag{28}$$

Choose $(u_{10}(x), u_{20}(x), u_{30}) = (r_1, m, r_2)$. Then (r_1, r_2, r_3) and (m, m, s) are the ordered upper and lower solutions of (22) and (3) (also of (23) and (3) and of (8) and (3)). Applying the same technique from [18, Theorem 4.1], we can prove that the solution (U_1, u_2, U_3) of (22) and (3), the solution (u_1, U_2, u_3) of (23) and (3), and the solution (p_1, p_2, p_3) of (8) and (3) satisfy, respectively,

$$\begin{gathered} \lim_{n\to\infty}\left(U_1(x,t+nT), u_2(x,t+nT), U_3(x,t+nt)\right) \\ = \left(U_1^\delta(x,t), u_{2\delta}(x,t), U_3^\delta(x,t)\right), \\ \lim_{n\to\infty}\left(u_1(x,t+nt), U_2(x,t+nT), u_3(x,t+nT)\right) \\ = \left(u_1^\delta(x,t), U_{2\delta}(x,t), u_3^\delta(x,t)\right), \\ \lim_{n\to\infty}\left(p_1(x,t+nT), p_2(x,t+nT), p_3(x,t+nT)\right) \\ = \left(u_1^T(x,t), u_{2T}(x,t), u_3^T(x,t)\right). \end{gathered} \tag{29}$$

It follows from (27) that $u_i^\delta \le u_i^T \le U_i^\delta$ $(i = 1,3)$ and that $u_{2\delta} \le u_{2T} \le U_{2\delta}$. Similarly, choose $(u_{10}(x), u_{20}(x), u_{30}(x)) = (m, r_2, s)$; we can prove the other inequalities of (26).

Denote by $\varepsilon_{1u_i}, \delta_{1u_i}, \varepsilon_{2u_i}$, and δ_{2u_i} the optimal bounds for the positive periodic solutions of problems (22) and (23), respectively, (see Lemma 2). Then

$$\begin{gathered} \delta_{2u_i} \le \delta_{u_i} \le \delta_{1u_i}, \\ \varepsilon_{2u_i} \le \varepsilon_{u_i} \le \varepsilon_{1u_i} \quad (i = 1,3), \\ \delta_{1u_2} \le \delta_{u_2} \le \delta_{2u_2}, \qquad \varepsilon_{1u_2} \le \varepsilon_{u_2} \le \varepsilon_{2u_2}. \end{gathered} \tag{30}$$

Moreover,

$$\begin{gathered} \varepsilon_{2u_i} \le u_{i\delta} \le U_i^\delta \le \delta_{1u_i} \quad (i = 1,3), \\ \varepsilon_{1u_2} \le u_{2\delta} \le U_2^\delta \le \delta_{2u_2}, \\ u_{i\delta_1} < u_{i\delta_2}, \qquad U_i^{\delta_1} > U_i^{\delta_2} \end{gathered} \tag{31}$$

for $0 < \delta_2 < \delta_1 < \varepsilon_0$. By using the dominated convergence theorem and the bootstrap arguments (see [18]), we have

$$\begin{gathered} \lim_{\delta\to 0+}\left(U_1^\delta, u_{2\delta}, U_3^\delta\right) = \left(u_1^s, u_{2s}, u_3^s\right), \\ \lim_{\delta\to 0+}\left(u_{1\delta}, U_2^\delta, u_{3\delta}\right) = \left(u_{1s}, u_2^s, u_{3s}\right) \end{gathered} \tag{32}$$

uniformly for (x,t) on $\overline{\Omega}\times\mathbf{R}$, and (u_{1s}, u_2^s, u_{3s}) and (u_1^s, u_{2s}, u_3^s) are the positive T-periodic solutions of (8).

From Theorem 1, we see that $u_{iT} \le u_{is}$ and $u_i^s \le u_i^T$ $(i = 1,2,3)$. It follows from (26) and (32) that $u_{is} \le u_{iT}, u_i^T \le u_i^s$ $(i = 1,2,3)$. So,

$$u_i^s = u_i^T, \qquad u_{is} = u_{iT} \quad (i = 1,2,3). \tag{33}$$

Therefore, given that $\varepsilon > 0$, by (32) and (33), there exists $\delta_0 \in (0, \varepsilon_0)$ such that

$$u_{iT} - \frac{\varepsilon}{2} < u_{i\delta_0} \le U_i^{\delta_0} < u_i^T + \frac{\varepsilon}{2} \quad (i = 1,2,3). \tag{34}$$

Since $(a_i, b_i, c_i, g_i) \sim (A_i, B_i, C_i, G_i)$, for the previous δ_0, there exists $T_{\delta_0} > 0$ such that, for $t > T_{\delta_0}$,

$$A_i - \delta_0 < a_i < A_i + \delta_0, \ldots, G_i - \delta_0 < g_i < G_i + \delta_0. \tag{35}$$

Denote by (u_1, u_2, u_3) the solution of (1)–(3), and denote by (u_1^*, u_{2*}, u_3^*), (U_1, U_2, U_3), and (u_{1*}, u_2^*, u_{3*}) the solutions of problems (22), (8), and (23), respectively, which all satisfy the same initial conditions $u_i(x, T_{\delta_0}+1) = u_{i0}(x) \geq (\not\equiv) 0$ $(i = 1, 2, 3)$. Analogous to (27), we have $u_{i*} \leq u_i \leq u_i^*$ $(i = 1, 2, 3)$ for $t > T_{\delta_0}$. By (24), there exists $T_1 > T_{\delta_0} + 1$ such that

$$\begin{aligned} U_{i\delta_0} - \frac{\varepsilon}{2} < u_i^* < U_i^{\delta_0} + \frac{\varepsilon}{2} \quad (i = 1, 3), \\ u_{2\delta_0} - \frac{\varepsilon}{2} < u_{2*} < u_2^{\delta_0} + \frac{\varepsilon}{2} \end{aligned} \tag{36}$$

for $t > T_1$. Similarly, by (25), there exists $T_2 > T_{\delta_0} + 1$ such that

$$\begin{aligned} u_{i\delta_0} - \frac{\varepsilon}{2} < u_{i*} < u_i^{\delta_0} + \frac{\varepsilon}{2} \quad (i = 1, 3), \\ U_{2\delta_0} - \frac{\varepsilon}{2} < u_2^* < U_2^{\delta_0} + \frac{\varepsilon}{2} \end{aligned} \tag{37}$$

for $t > T_2$. Hence, by (34)–(37), we have

$$u_{iT} - \varepsilon < u_{i*} \leq u_i \leq u_i^* < u_i^T + \varepsilon \tag{38}$$

for $t > \max\{T_1, T_2\}$. This completes the proof. □

3. Global Stability

In order to get conditions of global stability for (1)–(3), we need the following result.

Lemma 7. *Let (u, v, w) be a T-periodic solution for the linear problem*

$$\begin{aligned} u_t - d_1 \Delta u - \Sigma a_i u_{x_i} &= M_1 (-Au + Bv - Cw), \\ v_t - d_2 \Delta v - \Sigma b_i v_{x_i} &= M_2 (Du - Ev), \\ w_t - d_3 \Delta w - \Sigma c_i w_{x_i} &= M_3 (Fu - Gw), \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} &= 0 \quad \text{on } \partial\Omega \times \mathbf{R}, \end{aligned} \tag{39}$$

where d_i, M_i, A, B, C, D, E, F, and G are positive smooth T-periodic functions on $\overline{\Omega} \times \mathbf{R}$ and where a_i, b_i, and c_i are smooth T-periodic functions. If

$$B\left(\frac{D}{E}\right)_M + C\left(\frac{F}{G}\right)_M < A, \tag{40}$$

then $u = v = w = 0$.

Proof. Let (u, v, w) be a smooth T-periodic solution of (39), and let positive constants ε, m, k, and l be chosen so that

$$\begin{aligned} M_{1L}(A - Bk - Cl) \geq \varepsilon, \qquad M_{2L}(Ek - D) \geq \varepsilon k, \\ M_{3L}(Gl - F) \geq \varepsilon l, \qquad u_M \leq m, \\ v_M \leq mk, \quad w_M \leq ml. \end{aligned} \tag{41}$$

Such choices are obviously possible because (40) holds.

It is easy to verify that $m(1, k, l)e^{-\varepsilon t}$ and $-m(1, k, l)e^{-\varepsilon t}$ are a pair of ordered upper and lower solutions of (39). Thus, $-me^{-\varepsilon t} \leq u(x, t) \leq me^{-\varepsilon t}$. This implies that $u(x, t) = 0$ because $u(x, t)$ is T-periodic in t.

Similarly, $v = 0 = w$. This completes the proof. □

Lemma 8 (uniqueness). *Assume that* (9) *and* (10) *hold. If*

$$\begin{aligned} &\frac{B_{1M} B_{2M}}{\varepsilon_{u_2}} + \frac{A_{3M} C_{2M} \delta_{u_3}^2}{\left(1 + A_{3L} \varepsilon_{u_3}\right) \varepsilon_{u_3}^2} \\ &\quad < \frac{\varepsilon_{u_1}\left(1 + A_{3L} \varepsilon_{u_3}\right)}{A_{1M} A_{2M} \delta_{u_1} \delta_{u_2}}, \end{aligned} \tag{42}$$

then the problem (8) *has a unique positive T-periodic solution.*

Proof. Let $\alpha = u_1^T / u_{1T} - 1$, $\beta = 1 - u_{2T}/u_2^T$ and $\gamma = u_3^T / u_{3T} - 1$. By Theorem 1, we have

$$\begin{aligned} &\alpha_t - d_1 \Delta\alpha - \frac{2d_1}{U_{1T}} \Sigma u_{1T x_i} \alpha_{x_i} \\ &\quad = \frac{g_1 u_1^T}{u_{1T}} \left[\frac{-u_{1T}\alpha}{A_1} + \frac{A_2 u_2^T \beta}{1 + A_3 u_3^T} \right. \\ &\qquad \left. - \frac{A_2 A_3 u_2^T u_{3T} \gamma}{\left(1 + A_3 u_3^T\right)\left(1 + A_3 u_{3T}\right)} \right], \\ &\beta_t - d_2 \Delta\beta - \frac{2d_2}{U_2^T} \Sigma u_{2x_i}^T \beta_{x_i} \\ &\quad = \frac{g_2 u_{2T}}{u_2^T} \left[B_1 u_{1T} \alpha - \frac{u_2^T \beta}{B_2} \right], \\ &\gamma_t - d_3 \Delta\gamma - \frac{2d_3}{U_{3T}} \Sigma u_{3T x_i} \gamma_{x_i} \\ &\quad = \frac{g_3 u_3^T}{\left(C_1 + C_2 u_{1T}\right) u_{3T}} \left[\frac{C_2 u_{1T} u_3^T \alpha}{C_1 + C_2 u_1^T} - u_{3T} \gamma \right], \\ &\frac{\partial\alpha}{\partial n} = \frac{\partial\beta}{\partial n} = \frac{\partial\gamma}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, \infty). \end{aligned} \tag{43}$$

It follows from Lemmas 2 and 7 and the conditions (9), (10), and (42) that $\alpha = \beta = \gamma = 0$. This completes the proof. □

Theorem 9. *If all conditions of Theorem 4 and* (42) *are satisfied, then*

$$\left(u_{1T}, u_2^T, u_{3T}\right) = \left(u_1^T, u_{2T}, u_3^T\right) \sim \left(u_1, u_2, u_3\right) \tag{44}$$

for any positive solution (u_1, u_2, u_3) of (1)–(3).

Proof. By some elementary calculations, we know that Theorem 9 is an immediate corollary of Theorem 4 and Lemma 8. □

Example 10. Consider the following asymptotically periodic system:

$$\begin{aligned}
&u_{1t} - (2+\sin t)\, u_{1xx} \\
&\quad = \left(1+\sin^2(t+x)\, e^{-t^2}\right) u_1 \left(1 - u_1 - \frac{u_2}{1+u_3}\right), \\
&u_{2t} - (2-\sin t)\, u_{2xx} \\
&\quad = u_2 \left(1 - \left(\frac{3}{8} + \frac{1}{24}\cos^2(t+x)\right) u_1 - u_2\right) \quad \text{in } (0,1)\times \mathbf{R}, \\
&u_{3t} - u_{3xx} \\
&\quad = u_3 \left(1 - \frac{u_3}{5/12 + (7/12)\sin^2(t-x) + u_1}\right), \\
&u_{ix}(0,t) = u_{ix}(1,t) = 0 \quad \text{on } \mathbf{R} \ (i=1,2,3).
\end{aligned} \tag{45}$$

It is not hard to verify that all conditions of Theorem 9 are satisfies. Thus, any positive solution of (45) asymptotically approach the unique positive periodic solution of the limit periodic system of (45).

4. Case $a_3 = 0$

The following results are natural generalizations of the main results in [20] which can be proved in the similar way as to prove Theorems 4 and 9.

Theorem 11. *Assume the following.*

(A_1) *d_1, d_2, A, B, C, D, E, and F are positive smooth T-periodic functions on $\overline{\Omega}\times \mathbf{R}$.*

(A_2) *a, b, c, d, e, and f are positive smooth functions on $\overline{\Omega}\times \mathbf{R}$:*

$$\begin{gathered}
(a,b,c,d,e,f) \sim (A,B,C,D,E,F), \\
\left(\frac{F}{D}\right)_L > \left(\frac{C}{A}\right)_M, \qquad \left(\frac{B}{A}\right)_L > \left(\frac{E}{D}\right)_M.
\end{gathered} \tag{46}$$

Then the limit periodic system of (4)

$$\begin{gathered}
U_t - k_1 \Delta U = U(A - BU - CV) \quad \text{in } \Omega\times\mathbf{R}, \\
V_t - k_2 \Delta V = V(D - EU - FV) \quad \text{in } \Omega\times\mathbf{R}, \\
\frac{\partial U}{\partial n} = 0 = \frac{\partial V}{\partial n} \quad \text{on } \partial\Omega\times\mathbf{R}
\end{gathered} \tag{47}$$

has the positive T-periodic maxmini solution (U^T, V_T) and minimax solution (U_T, V^T). Moreover, if (u,v) is any positive solution of (4) *with smooth initial value (u_0, v_0), then $U_T \prec u \prec U^T$ and $V_T \prec v \prec V^T$. In addition, if*

$$\begin{aligned}
&\left(\frac{C}{A}\right)_M \left(\frac{E}{D}\right)_M \left[\left(\frac{F}{D}\right)_M - \left(\frac{C}{A}\right)_L\right] \left[\left(\frac{B}{A}\right)_M - \left(\frac{E}{D}\right)_L\right] \\
&\quad < \left(\frac{B}{A}\right)_L \left(\frac{F}{D}\right)_L \left[\left(\frac{F}{D}\right)_L - \left(\frac{C}{A}\right)_M\right] \left[\left(\frac{B}{A}\right)_L - \left(\frac{E}{D}\right)_M\right],
\end{aligned} \tag{48}$$

then (47) *has the unique positive T-periodic solution (U,V) and*

$$(u(\cdot,t), v(\cdot,t)) \sim (U(\cdot,t), V(\cdot,t)). \tag{49}$$

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References

[1] B. Rai, H. I. Freedman, and J. F. Addicott, "Analysis of three-species models of mutualism in predator-prey and competitive systems," *Mathematical Biosciences*, vol. 65, no. 1, pp. 13–50, 1983.

[2] S. N. Zheng, "A reaction-diffusion system of a competitor-competitor-mutualist model," *Journal of Mathematical Analysis and Applications*, vol. 124, no. 1, pp. 254–280, 1987.

[3] S. Xu, "Global stability of a reaction-diffusion system of a competitor-competitor-mutualist model," *Taiwanese Journal of Mathematics*, vol. 15, no. 4, pp. 1617–1627, 2011.

[4] C. V. Pao, "The global attractor of a competitor-competitor-mutualist reaction-diffusion system with time delays," *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 67, no. 9, pp. 2623–2631, 2007.

[5] W. Chen and R. Peng, "Stationary patterns created by cross-diffusion for the competitor-competitor-mutualist model," *Journal of Mathematical Analysis and Applications*, vol. 291, no. 2, pp. 550–564, 2004.

[6] M. Li, Z. Lin, and J. Liu, "Coexistence in a competitor-competitor-mutualist model," *Applied Mathematical Modelling*, vol. 34, no. 11, pp. 3400–3407, 2010.

[7] S. M. Fu, H. Y. Gao, and S. B. Cui, "Global solutions for the competitor-competitor-mutualist model with cross-diffusion," *Acta Mathematica Sinica A*, vol. 51, no. 1, pp. 153–164, 2008.

[8] C. Tian and Z. Ling, "Turing pattern formation in a predator-prey-mutualist system," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3224–3237, 2011.

[9] A. Tineo, "Asymptotic behavior of solutions of a periodic reaction-diffusion system of a competitor-competitor-mutualist model," *Journal of Differential Equations*, vol. 108, no. 2, pp. 326–341, 1994.

[10] Y. Du, "Positive periodic solutions of a competitor-competitor-mutualist model," *Differential and Integral Equations*, vol. 9, no. 5, pp. 1043–1066, 1996.

[11] C. V. Pao, "Periodic solutions of parabolic systems with nonlinear boundary conditions," *Journal of Mathematical Analysis and Applications*, vol. 234, no. 2, pp. 695–716, 1999.

[12] R.-N. Wang, T.-J. Xiao, and J. Liang, "Asymptotic behavior of solutions for systems of periodic reaction-diffusion equations in unbounded domains," *International Journal of Evolution Equations*, vol. 1, no. 3, pp. 281–298, 2005.

[13] L. Zhou and Y. Fu, "Existence and stability of periodic quasisolutions in nonlinear parabolic systems with discrete delays," *Journal of Mathematical Analysis and Applications*, vol. 250, no. 1, pp. 139–161, 2000.

[14] Y. Wang and J. Yin, “Periodic solutions of a class of degenerate parabolic system with delays,” *Journal of Mathematical Analysis and Applications*, vol. 380, no. 1, pp. 57–68, 2011.

[15] K. Gopalsamy, “Exchange of equilibria in two-species Lotka-Volterra competition models,” *Australian Mathematical Society Journal B*, vol. 24, no. 2, pp. 160–170, 1982.

[16] C. Alvarez and A. C. Lazer, “An application of topological degree to the periodic competing species problem,” *Australian Mathematical Society Journal B*, vol. 28, no. 2, pp. 202–219, 1986.

[17] S. Ahmad, “Convergence and ultimate bounds of solutions of the nonautonomous Volterra-Lotka competition equations,” *Journal of Mathematical Analysis and Applications*, vol. 127, no. 2, pp. 377–387, 1987.

[18] S. Ahmad and A. C. Lazer, “Asymptotic behaviour of solutions of periodic competition diffusion system,” *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 13, no. 3, pp. 263–284, 1989.

[19] A. Tineo, “Existence of global coexistence state for periodic competition diffusion systems,” *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 19, no. 4, pp. 335–344, 1992.

[20] Q. L. Peng and L. S. Chen, “Asymptotic behavior of the nonautonomous two-species Lotka-Volterra competition models,” *Computers & Mathematics with Applications*, vol. 27, no. 12, pp. 53–60, 1994.

[21] S. Fu and S. Cui, “Persistence in a periodic competitor-competitor-mutualist diffusion system,” *Journal of Mathematical Analysis and Applications*, vol. 263, no. 1, pp. 234–245, 2001.

[22] W. Dimbour and G. M. N’Guérékata, “S-asymptotically ω-periodic solutions to some classes of partial evolution equations,” *Applied Mathematics and Computation*, vol. 218, no. 14, pp. 7622–7628, 2012.

[23] H. Gao, K. Wang, F. Wei, and X. Ding, “Massera-type theorem and asymptotically periodic logistic equations,” *Nonlinear Analysis: Real World Applications*, vol. 7, no. 5, pp. 1268–1283, 2006.

[24] H. R. Henríquez, “Asymptotically periodic solutions of abstract differential equations,” *Nonlinear Analysis: Theory, Methods & Applications A*, vol. 80, pp. 135–149, 2013.

[25] J. Wang, M. Feckan, and Y. Zhou, “Nonexistence of periodic solutions and asymptotically periodic solutions for fractional differential equations,” *Communications in Nonlinear Science and Numerical Simulation*, vol. 18, no. 2, pp. 246–256, 2013.

[26] F. Wei and K. Wang, “Asymptotically periodic solution of N-species cooperation system with time delay,” *Nonlinear Analysis: Real World Applications*, vol. 7, no. 4, pp. 591–596, 2006.

[27] F. Wei and K. Wang, “Uniform persistence of asymptotically periodic multispecies competition predator-prey systems with Holling III type functional response,” *Applied Mathematics and Computation*, vol. 170, no. 2, pp. 994–998, 2005.

[28] Q. X. Ye and Z. Y. Li, *Introduction to Reaction-Diffusion Equations*, Science Press, Beijing, China, 1990.

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Research Article

A New Approach for a Class of the Blasius Problem via a Transformation and Adomian's Method

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The main feature of the boundary layer flow problems is the inclusion of the boundary conditions at infinity. Such boundary conditions cause difficulties for any of the series methods when applied to solve such problems. To the best of the authors' knowledge, two procedures were used extensively in the past two decades to deal with the boundary conditions at infinity, either the Padé approximation or the direct numerical codes. However, an intensive work is needed to perform the calculations using the Padé technique. Regarding this point, a new idea is proposed in this paper. The idea is based on transforming the unbounded domain into a bounded one by the help of a transformation. Accordingly, the original differential equation is transformed into a singular differential equation with classical boundary conditions. The current approach is applied to solve a class of the Blasius problem and a special class of the Falkner-Skan problem via an improved version of Adomian's method (Ebaid, 2011). In addition, the numerical results obtained by using the proposed technique are compared with the other published solutions, where good agreement has been achieved. The main characteristic of the present approach is the avoidance of the Padé approximation to deal with the infinity boundary conditions.

1. Introduction

During the past two decades much effort has been spent in the numerical treatment of boundary value problems over an unbounded domain. In fact, these problems arise very frequently in many fields such as in fluid dynamics, aerodynamics, and quantum mechanics. A few notable examples are the Blasius and Falkner-Skan equations. The Blasius equation is one of the basic equations in fluid dynamics. It describes the velocity profile of the fluid in the boundary layer theory [1, 2] on a half-infinite interval. Several analytical and numerical methods have been proposed in [1–11] to handle this problem. The two forms of the Blasius problem are represented by the same differential equation with different sets of boundary conditions, as will be indicated later. The main feature of the Blasius problem is the existence of the boundary conditions at infinity. Such conditions at infinity cause difficulties for any of the series methods, such as the Adomian decomposition method [12–14] and the differential transformation method (or the Taylor series method) [15, 16]. This is because the infinity boundary condition cannot be imposed directly in the series, where the Padé approximation should be established before applying the boundary condition at infinity. It was observed in the past two decades that many authors [17–25] have been resorted to either the Padé technique or some numerical methods to treat the boundary conditions at infinity. Although the results obtained by using the Padé technique were accurate in many cases, a massive computational work was needed to obtain accurate approximate solutions. A possible way to avoid the Padé technique is to change the boundary conditions at infinity into classical conditions. Therefore, a suggestion is proposed in this paper to transform the domain of the problem from an unbounded domain into a bounded one with the help of a simple transformation.

According to the suggested transformation, the original Blasius equation is transformed into a system of two singular differential equations. Hence, the two mentioned forms are described by this system with two different sets of boundary conditions at classical point. The transformed singular system will be solved by a recent version of the ADM [26]. The first form of the original Blasius problem is given by [6]

$$f'''(\eta) + \frac{1}{2} f(\eta) f''(\eta) = 0, \quad (1)$$

subject to the following boundary conditions:

$$f(0) = 0, \qquad f'(0) = 1, \qquad f'(\infty) = 0, \quad (2)$$

while the second form is given by (1) with the following boundary conditions:

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(\infty) = 1. \quad (3)$$

A class of Blasius problem is given by

$$f'''(\eta) + \gamma f(\eta) f''(\eta) = 0, \quad (4)$$

subject to the following boundary conditions:

$$f(0) = 0, \qquad f'(0) = 1 - \epsilon, \qquad f'(\infty) = \epsilon, \quad (5)$$

where γ and ϵ are finite constants. This class will be studied for $\epsilon \in [0, 1]$. Here, it is noted that (2) and (3) are special examples of (5) for $\epsilon = 0$ and $\epsilon = 1$, respectively. In addition, the class (4)-(5) reduces to the two forms of the Blasius problem when $(\gamma = 1/2, \epsilon = 0)$ and $(\gamma = 1/2, \epsilon = 1)$, respectively. At the same time, when $\gamma = 1$, the suggested class reduces to a special class of the Falkner-Skan problem, at $\delta = 0$, which is well known as [27]

$$f'''(\eta) + f(\eta) f''(\eta) + \delta\left[\epsilon^2 - \left(f'(\eta)\right)^2\right] = 0, \quad (6)$$

with the class of boundary conditions (5), where δ refers to the pressure gradient parameter, while ϵ refers to the velocity ratio parameter, $\epsilon = U_\infty/(U_\infty + U_w)$. Equation (6) with the boundary conditions (5) is a new version of the Falkner-Skan equation relating free stream velocity U_∞ to composite reference velocity, that is, sum of the velocities of stretching boundary U_w and free stream U_∞. In order to use the improved Adomian's method [26] to solve the class (4)-(5), we first transform the governing equation (4) into the following system of differential equations:

$$\begin{gathered} f'(\eta) = u(\eta), \\ u''(\eta) + \gamma f(\eta) u'(\eta) = 0. \end{gathered} \quad (7)$$

Here, we may indicate that in the theory of the boundary layer, it is usually important to get information about three quantities: the skin-friction coefficient $f''(0)$, the fluid velocity $f'(\eta)$, and the stream function $f(\eta)$. Also, it is well known that at $\epsilon = 1$ the problem reduces to one of the two forms of the Blasius problem which has been studied extensively during the past decades.

2. A Transformation and a New System

The unbounded domain of the independent variable $\eta \in [0, \infty)$ can be changed into a bounded one by using a new independent variable t (say) $\in [0, 1)$ using the transformation $t = 1 - e^{-\eta}$. Accordingly, the governing system should be expressed in terms of the new variable t. In order to do that, we introduce the following relations between the derivatives with respect to η and the derivatives with respect to t:

$$\begin{gathered} \frac{d}{d\eta}(\square) = (1-t)\frac{d}{dt}(\square), \\ \frac{d^2}{d\eta^2}(\square) = (1-t)^2\frac{d^2}{dt^2}(\square) - (1-t)\frac{d}{dt}(\square). \end{gathered} \quad (8)$$

The relations given by (8) are obtained by using the chain rule in the differential calculus. Therefore, the system (7) becomes

$$f'(t) = \left(\frac{1}{1-t}\right) u(t), \quad (9)$$

$$u''(t) = \left(\frac{1}{1-t}\right) u'(t) - \gamma\left(\frac{1}{1-t}\right) f(t) u'(t), \quad (10)$$

subject to the following set of boundary conditions:

$$f(0) = 0, \qquad u(0) = 1 - \epsilon, \qquad u(1) = \epsilon. \quad (11)$$

Equation (9) with the initial condition $f(0) = 0$ can be easily integrated as an initial value problem, while (10) with the boundary conditions given in (11) should be solved as a two-point boundary value problem. In this regard, the improved Adomian decomposition method is suggested to deal with such a singular two-point boundary value problem. Before launching into the the main idea of this paper, we give an analysis for the improved Adomian decomposition method in the next section to solving (10) with general two-point boundary conditions $u(a) = \alpha$ and $u(b) = \beta$.

3. The Improved Adomian Decomposition Method

Consider the second order differential equation:

$$u''(t) + p(t) u'(t) + q(t) f(t) u'(t) = 0, \quad (12)$$

subject to the boundary conditions

$$u(a) = \alpha, \qquad u(b) = \beta, \quad (13)$$

where at least one of the functions $p(t)$ and $q(t)$ has a singular point and $f(t)$ is an unspecified function. In order to apply the approach suggested in [26], we first rewrite (12) as

$$u''(t) = -p(t) u'(t) - q(t) f(t) u'(t). \quad (14)$$

Now, suppose that $p(t)$ and $q(t)$ have the same singular point ($t = t_0$, say), Ebaid [26] proposed the following inverse operator to solve (14) with the boundary conditions (13):

$$\begin{gathered} L^{-1}[\cdot] = \int_a^t dt' \int_c^{t'} [\cdot]\, dt'' - \frac{t-a}{b-a} \int_a^b dt' \int_c^{t'} [\cdot]\, dt'', \\ a \neq b, \quad c \text{ (arbitrary)} \neq t_0. \end{gathered} \quad (15)$$

Operating both sides of (14) with this inverse operator, we have

$$u(t) - u(a) - \frac{t-a}{b-a}[u(b) - u(a)] \\ = -L^{-1}\left[p(t)u'(t) + q(t)f(t)u'(t)\right], \tag{16}$$

which can be rewritten as

$$u(t) = \alpha + \frac{t-a}{b-a}(\beta - \alpha) \\ - L^{-1}\left[p(t)u'(t) + q(t)f(t)u'(t)\right]. \tag{17}$$

Based on Adomian's method, the solutions $u(t)$ and $f(t)$ of system (9)-(10) are assumed in the following form:

$$u(t) = \sum_{n=0}^{\infty} u_n(t), \qquad f(t) = \sum_{n=0}^{\infty} f_n(t). \tag{18}$$

Inserting these series into (17), we obtain

$$\sum_{n=0}^{\infty} u_n(t) = \alpha + \frac{t-a}{b-a}(\beta-\alpha) \\ - L^{-1}\left[p(t)\sum_{n=0}^{\infty} u_n'(t) + q(t)\sum_{n=0}^{\infty}\sum_{i=0}^{n} f_i(t)u_{n-i}'(t)\right]. \tag{19}$$

Substituting $p(t) = -(1/(1-t))$ and $q(t) = \gamma(1/(1-t))$ into the last equation yields

$$\sum_{n=0}^{\infty} u_n(t) = \alpha + \frac{t-a}{b-a}(\beta-\alpha) \\ + L^{-1}\left[\left(\frac{1}{1-t}\right)\sum_{n=0}^{\infty} u_n'(t) \\ - \gamma\left(\frac{1}{1-t}\right)\sum_{n=0}^{\infty}\sum_{i=0}^{n} f_i(t)u_{n-i}'(t)\right]. \tag{20}$$

To overcome the difficulty of the singular point, we may replace the function $1/(1-t)$ with the series form $\sum_{n=0}^{\infty} t^n$, where $t \in [0,1)$. Thus, we have

$$\sum_{n=0}^{\infty} u_n(t) = \alpha + \frac{t-a}{b-a}(\beta-\alpha) \\ + L^{-1}\left[\sum_{n=0}^{\infty}\sum_{i=0}^{n} t^{n-i}u_i'(t) \\ - \gamma\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\sum_{i=0}^{j} t^{n-j} f_i(t)u_{j-i}'(t)\right)\right]. \tag{21}$$

According to the modified decomposition method [18], the solution $u(t)$ can be evaluated by using the recurrence scheme:

$$u_0(t) = \alpha, \\ u_1(t) = \frac{t-a}{b-a}(\beta-\alpha) + L^{-1}\left[u_0'(t) - \gamma f_0(t)u_0'(t)\right], \\ u_{n+1}(t) = L^{-1}\left[\sum_{i=0}^{n} t^{n-i}u_i'(t) - \gamma\left(\sum_{j=0}^{n}\sum_{i=0}^{j} t^{n-j} f_i(t)u_{j-i}'(t)\right)\right], \\ n \geq 1. \tag{22}$$

On integrating (9) with respect to t form 0 to t, it then follows that

$$f(t) = f(0) + \int_0^t \left(\frac{1}{1-z}\right)u(z)\,dz. \tag{23}$$

Hence, $f(t)$ is given by the recurrence scheme:

$$f_0(t) = 0, \qquad f_{n+1}(t) = \int_0^t z^{n-i}u_i(z)\,dz, \quad n \geq 0. \tag{24}$$

The algorithms given by (22) and (24) are applied in the next section to construct the approximate solutions.

4. Applications

4.1. A Class of the Blasius Problem. Here, we show how to implement (22) and (24) to solve the class of the Blasius problem. On substituting $\gamma = 1/2$, $a = 0$, $b = 1$, $\alpha = 1-\epsilon$, and $\beta = \epsilon$ into (22), and using (24) we obtain

$$u_0(t) = 1-\epsilon, \\ f_0(t) = 0, \\ u_1(t) = (2\epsilon - 1)t, \\ f_1(t) = (1-\epsilon)t, \\ u_{n+1}(t) = L^{-1}\left[\sum_{i=0}^{n} t^{n-i}u_i'(t) - \frac{1}{2}\left(\sum_{j=0}^{n}\sum_{i=0}^{j} t^{n-j} f_i(t)u_{j-i}'(t)\right)\right], \\ f_{n+1}(t) = \int_0^t \left(\frac{1}{1-z}\right)u_n(z)\,dz, \quad n \geq 1, \\ L^{-1}[\cdot] = \int_0^t dt' \int_c^{t'} [\cdot]\,dt'' - t\int_0^1 dt' \int_c^{t'} [\cdot]\,dt'', \\ c \neq 1 \ (c = 0, \text{ for simplicity}). \tag{25}$$

The first few terms of the stream function $f(t)$ are evaluated by implementing the previous algorithm and are listed in the following:

$$\begin{aligned}
f_0(t) &= 0,\\
f_1(t) &= (1-\epsilon)t,\\
f_2(t) &= \left(\frac{\epsilon}{2}\right)t^2,\\
f_3(t) &= \left(\frac{1}{4}-\frac{\epsilon}{2}\right)t^2+\left(-\frac{1}{6}+\frac{2\epsilon}{3}\right)t^3,\\
f_4(t) &= \left(\frac{\epsilon}{24}-\frac{\epsilon^2}{12}\right)t^2+\left(\frac{1}{4}-\frac{\epsilon}{2}\right)t^3\\
&\quad+\left(-\frac{3}{16}+\frac{29\epsilon}{28}+\frac{\epsilon^2}{24}\right)t^4,\\
f_5(t) &= \left(\frac{1}{96}-\frac{\epsilon}{48}\right)t^2+\left(\frac{\epsilon}{24}-\frac{\epsilon^2}{12}\right)t^3\\
&\quad+\left(\frac{7}{32}-\frac{41\epsilon}{96}-\frac{\epsilon^2}{48}\right)t^4\\
&\quad+\left(-\frac{43}{240}+\frac{41\epsilon}{40}+\frac{\epsilon^2}{15}\right)t^5.
\end{aligned}\tag{26}$$

The desired mth order approximate solution $\phi_m(\eta)$ obtained by Adomian's method is expressed as

$$\phi_m(\eta)=\sum_{n=0}^{m-1}f_n(\eta).\tag{27}$$

Hence, the approximate solutions $\phi_3(\eta)$, $\phi_5(\eta)$, and $\phi_7(\eta)$ are, respectively, given in terms of the original variable η as

$$\begin{aligned}
\phi_3(\eta) &= (1-\epsilon)(1-e^{-\eta})+\frac{1}{4}(1-e^{-\eta})^2\\
&\quad+\left(-\frac{1}{6}+\frac{2\epsilon}{3}\right)(1-e^{-\eta})^3,\\
\phi_5(\eta) &= (1-\epsilon)(1-e^{-\eta})+\left(\frac{25}{96}+\frac{\epsilon}{48}-\frac{\epsilon^2}{12}\right)(1-e^{-\eta})^2\\
&\quad+\left(\frac{1}{12}+\frac{5\epsilon}{24}-\frac{\epsilon^2}{12}\right)(1-e^{-\eta})^3\\
&\quad+\left(\frac{1}{32}+\frac{17\epsilon}{96}+\frac{\epsilon^2}{48}\right)(1-e^{-\eta})^4\\
&\quad+\left(-\frac{43}{240}+\frac{41\epsilon}{40}+\frac{\epsilon^2}{15}\right)(1-e^{-\eta})^5,\\
\phi_7(\eta) &= (1-\epsilon)(1-e^{-\eta})\\
&\quad+\left(\frac{171}{640}+\frac{\epsilon}{72}-\frac{553\epsilon^2}{5760}-\frac{\epsilon^3}{960}\right)(1-e^{-\eta})^2\\
&\quad+\left(\frac{47}{480}+\frac{11\epsilon}{60}-\frac{53\epsilon^2}{576}+\frac{\epsilon^3}{1440}\right)(1-e^{-\eta})^3\\
&\quad+\left(\frac{31}{768}+\frac{451\epsilon}{2304}-\frac{59\epsilon^2}{1152}-\frac{\epsilon^3}{288}\right)(1-e^{-\eta})^4\\
&\quad+\left(\frac{11}{960}+\frac{137\epsilon}{720}-\frac{7\epsilon^2}{288}-\frac{\epsilon^3}{240}\right)(1-e^{-\eta})^5\\
&\quad+\left(\frac{31}{5760}+\frac{367\epsilon}{2880}+\frac{65\epsilon^2}{1152}+\frac{\epsilon^3}{960}\right)(1-e^{-\eta})^6\\
&\quad+\left(-\frac{13}{8960}+\frac{5921\epsilon}{40320}-\frac{83\epsilon^2}{24192}+\frac{89\epsilon^3}{34560}-\frac{19\epsilon^4}{120960}\right)\\
&\quad\times(1-e^{-\eta})^7.
\end{aligned}\tag{28}$$

Here, we refer to that the series solution obtained previous leads to an exact solution at $\epsilon=0.5$. In this case, the approximate solutions become

$$\begin{aligned}
\phi_3(\eta) &= \frac{1}{2}(1-e^{-\eta})+\frac{1}{4}(1-e^{-\eta})^2+\frac{1}{6}(1-e^{-\eta})^3\\
&= \frac{1}{2}\sum_{r=1}^{3}\frac{1}{r}(1-e^{-\eta})^r,\\
\phi_5(\eta) &= \frac{1}{2}(1-e^{-\eta})+\frac{1}{4}(1-e^{-\eta})^2+\frac{1}{6}(1-e^{-\eta})^3\\
&\quad+\frac{1}{8}(1-e^{-\eta})^4+\frac{1}{10}(1-e^{-\eta})^5\\
&= \frac{1}{2}\sum_{r=1}^{5}\frac{1}{r}(1-e^{-\eta})^r,\\
\phi_7(\eta) &= \frac{1}{2}(1-e^{-\eta})+\frac{1}{4}(1-e^{-\eta})^2+\frac{1}{6}(1-e^{-\eta})^3\\
&\quad+\frac{1}{8}(1-e^{-\eta})^4+\frac{1}{10}(1-e^{-\eta})^5\\
&\quad+\frac{1}{12}(1-e^{-\eta})^6+\frac{1}{14}(1-e^{-\eta})^7\\
&= \frac{1}{2}\sum_{r=1}^{7}\frac{1}{r}(1-e^{-\eta})^r.
\end{aligned}\tag{29}$$

Therefore, the m-term series solution is given by

$$\phi_m(\eta)=\frac{1}{2}\sum_{r=1}^{m}\frac{1}{r}(1-e^{-\eta})^r,\tag{30}$$

and thus, the following exact solution is obtained as $m \to \infty$:

$$\begin{aligned} f(\eta) &= \lim_{m\to\infty} \phi_m(\eta) \\ &= \frac{1}{2}\sum_{r=1}^{\infty}\frac{1}{r}\left(1-e^{-\eta}\right)^r \\ &= -\frac{1}{2}\ln\left[1-\left(1-e^{-\eta}\right)\right] = \frac{\eta}{2}. \end{aligned} \tag{31}$$

This exact solution satisfies the boundary conditions and can be easily verified by direct substitution. For more validation, the results obtained by the present technique are checked here via a comparison with those published in the literature. It is well known that at $\epsilon = 1$, the problem reduces to one of the two forms of the Blasius problem. In that case, the skin-friction coefficient is computed by many authors as discussed in Section 5.

4.2. Special Class of the Falkner-Skan Problem. Here, the proposed approach is applied to a special class of the Falkner-Skan problem. As mentioned before, this special class is given by (4)-(5) at $\gamma = 1$. Proceeding as in the previous example, the approximate solution can be obtained by using the recurrence scheme:

$$\begin{aligned} u_0(t) &= 1-\epsilon, \\ f_0(t) &= 0, \\ u_1(t) &= (2\epsilon-1)\,t, \\ f_1(t) &= (1-\epsilon)\,t, \\ u_{n+1}(t) &= L^{-1}\left[\sum_{i=0}^{n} t^{n-i}u_i'(t) - \sum_{j=0}^{n}\sum_{i=0}^{j} t^{n-j} f_i(t)\,u_{j-i}'(t)\right], \\ f_{n+1}(t) &= \int_0^t \left(\frac{1}{1-z}\right)u_n(z)\,dz, \quad n \ge 1, \\ L^{-1}[\cdot] &= \int_0^t dt' \int_c^{t'} [\cdot]\,dt'' - t\int_0^1 dt' \int_c^{t'} [\cdot]\,dt'', \quad c \ne 1. \end{aligned} \tag{32}$$

The first few terms of the stream function $f(t)$ are evaluated by implementing the algorithm given in (32) and are listed in the following:

$$\begin{aligned} f_0(t) &= 0, \\ f_1(t) &= (1-\epsilon)\,t, \\ f_2(t) &= \left(\frac{\epsilon}{2}\right)t^2, \\ f_3(t) &= \left(\frac{1}{4}-\frac{\epsilon}{2}\right)t^2 + \left(-\frac{1}{6}+\frac{2\epsilon}{3}\right)t^3, \\ f_4(t) &= \left(-\frac{1}{24}+\frac{\epsilon}{6}-\frac{\epsilon^2}{6}\right)t^2 + \left(\frac{1}{4}-\frac{\epsilon}{2}\right)t^3 \\ &\quad + \left(-\frac{1}{6}+\frac{13\epsilon}{24}+\frac{\epsilon^2}{12}\right)t^4, \\ f_5(t) &= \left(-\frac{1}{24}+\frac{\epsilon}{6}-\frac{\epsilon^2}{6}\right)t^3 + \left(\frac{5}{24}-\frac{19\epsilon}{48}-\frac{\epsilon^2}{24}\right)t^4 \\ &\quad + \left(-\frac{17}{120}+\frac{5\epsilon}{12}+\frac{2\epsilon^2}{15}\right)t^5. \end{aligned} \tag{33}$$

Hence, the approximate solutions $\phi_3(\eta)$, $\phi_5(\eta)$, and $\phi_7(\eta)$ are, respectively, given in terms of the original variable η as

$$\begin{aligned} \phi_3(\eta) &= (1-\epsilon)\left(1-e^{-\eta}\right) + \frac{1}{4}\left(1-e^{-\eta}\right)^2 \\ &\quad + \left(-\frac{1}{6}+\frac{2\epsilon}{3}\right)\left(1-e^{-\eta}\right)^3, \\ \phi_5(\eta) &= (1-\epsilon)\left(1-e^{-\eta}\right) + \left(\frac{5}{24}+\frac{\epsilon}{6}-\frac{\epsilon^2}{6}\right)\left(1-e^{-\eta}\right)^2 \\ &\quad + \left(\frac{1}{24}+\frac{\epsilon}{3}-\frac{\epsilon^2}{6}\right)\left(1-e^{-\eta}\right)^3 \\ &\quad + \left(\frac{1}{24}+\frac{7\epsilon}{48}+\frac{\epsilon^2}{24}\right)\left(1-e^{-\eta}\right)^4 \\ &\quad + \left(-\frac{17}{120}+\frac{5\epsilon}{12}+\frac{2\epsilon^2}{15}\right)\left(1-e^{-\eta}\right)^5, \\ \phi_7(\eta) &= (1-\epsilon)\left(1-e^{-\eta}\right) \\ &\quad + \left(\frac{191}{960}+\frac{19\epsilon}{96}-\frac{91\epsilon^2}{480}-\frac{\epsilon^3}{240}\right)\left(1-e^{-\eta}\right)^2 \\ &\quad + \left(\frac{13}{360}+\frac{17\epsilon}{48}-\frac{3\epsilon^2}{16}+\frac{\epsilon^3}{360}\right)\left(1-e^{-\eta}\right)^3 \\ &\quad + \left(\frac{1}{144}+\frac{9\epsilon}{32}-\frac{\epsilon^2}{12}-\frac{\epsilon^3}{72}\right)\left(1-e^{-\eta}\right)^4 \\ &\quad + \left(\frac{1}{480}+\frac{17\epsilon}{80}-\frac{\epsilon^2}{40}-\frac{\epsilon^3}{60}\right)\left(1-e^{-\eta}\right)^5 \\ &\quad + \left(\frac{1}{30}+\frac{13\epsilon}{288}+\frac{31\epsilon^2}{288}+\frac{\epsilon^3}{240}\right)\left(1-e^{-\eta}\right)^6 \\ &\quad + \left(-\frac{67}{630}+\frac{25\epsilon}{84}+\frac{181\epsilon^2}{1680}+\frac{41\epsilon^3}{2520}\right)\left(1-e^{-\eta}\right)^7. \end{aligned} \tag{34}$$

The effectiveness of the present technique is used here not only to obtain the exact solution of the Falkner-Skan equation at $\delta = 0$ and $\epsilon = 0.5$ but also to get numerical solutions with good accuracy. On inserting $\epsilon = 0.5$ into the approximate solutions given by (34), we have

$$\begin{aligned} \phi_3(\eta) &= \frac{1}{2}\left(1-e^{-\eta}\right) + \frac{1}{4}\left(1-e^{-\eta}\right)^2 + \frac{1}{6}\left(1-e^{-\eta}\right)^3 \\ &= \frac{1}{2}\sum_{r=1}^{3}\frac{1}{r}\left(1-e^{-\eta}\right)^r, \end{aligned}$$

$$\phi_5(\eta) = \frac{1}{2}(1-e^{-\eta}) + \frac{1}{4}(1-e^{-\eta})^2 + \frac{1}{6}(1-e^{-\eta})^3 + \frac{1}{8}(1-e^{-\eta})^4 + \frac{1}{10}(1-e^{-\eta})^5 = \frac{1}{2}\sum_{r=1}^{5}\frac{1}{r}(1-e^{-\eta})^r,$$

$$\phi_7(\eta) = \frac{1}{2}(1-e^{-\eta}) + \frac{1}{4}(1-e^{-\eta})^2 + \frac{1}{6}(1-e^{-\eta})^3 + \frac{1}{8}(1-e^{-\eta})^4 + \frac{1}{10}(1-e^{-\eta})^5 + \frac{1}{12}(1-e^{-\eta})^6 + \frac{1}{14}(1-e^{-\eta})^7 = \frac{1}{2}\sum_{r=1}^{7}\frac{1}{r}(1-e^{-\eta})^r. \tag{35}$$

As indicated in Section 4.1, these approximate solutions lead to the same exact solution given by (31): $f(\eta) = \eta/2$ in the limit.

5. Results and Discussion

At $\epsilon = 1$, Bairstow [29] found that $f''(0) = 0.335$ using a power series, whereas Goldstein [30] obtained $f''(0) = 0.332$. Besides, using a finite difference method, Falkner [31] computed that $f''(0) = 0.3325765$, and Horwarth [32] yields that $f''(0) = 0.332057$. In [33], Fazio computed that $f''(0) = 0.332057336215$. Also, in [34] Boyd used Töpfer's algorithm to obtain the accurate value $f''(0) = 0.33205733621519630$. Adomain's method was implemented in [35] by Abbasbandy, and it was found that $f''(0) = 0.333329$, whereas a variational iteration method with the Padé approximants allows Wazwaz [6] to calculate the value $f''(0) = 0.3732905625$. Tajvidi et al. [28] apply the modified rational Legendre functions to get a value of $f''(0) = 0.33209$. The values of the skin-friction coefficient are given in Table 1 at $\epsilon = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$, and 1 using 11, 13, and 15 terms of the series (27). The current method finds that the skin-friction at $\epsilon = 1$ approximately equals $\phi''_{15}(0) = 0.331775$, which is very close to those values discussed previously.

Regarding the stream function $f(\eta)$, it is plotted in Figure 1 using 15 terms, and the fluid velocity $f'(\eta)$ is depicted in Figure 2 using the approximate solutions $\phi_7(\eta)$, $\phi_9(\eta)$, $\phi_{11}(\eta)$, $\phi_{13}(\eta)$, and $\phi_{15}(\eta)$ at ϵ = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, and 1. It is observed from Figure 2 that the approximate solutions using a few terms of Adomian's series converge rapidly to a certain curve at some values of the parameter ϵ.

The exact solution $f(\eta) = \eta/2$ obtained in Section 4.2 for the Falkner-Skan equation at $\epsilon = 0.5$ has been reported very recently by Kudenatti [27]. He has derived the exact solution to the Falkner-Skan equation for general values of the pressure gradient parameter δ. In order to check the accuracy of our approach, the values of the skin-friction coefficient are compared in Table 2 with those exactly obtained by Kudenatti [27] in the range $0 < \epsilon < 0.5$. The results reveal that a good agreement has been achieved via the present approach. In addition, the stream function $f(\eta)$ is graphed in Figure 3 at several values of the parameter ϵ by using 15 terms of the decomposition series. At the same values and in Figure 4,

Table 1: The approximate values of the skin-friction coefficient $f''(0)$ for the class of the Blasius problem using 11, 13, and 15 terms of Adomian's series.

ϵ	$\phi''_{11}(0)$	$\phi''_{13}(0)$	$\phi''_{15}(0)$
0.0	−0.456523	−0.454506	−0.453122
0.1	−0.356731	−0.354995	−0.353838
0.2	−0.261082	−0.259650	−0.258715
0.3	−0.169677	−0.168611	−0.167920
0.4	−0.082616	−0.082014	−0.081624
0.5	0.000000	0.000000	0.000000
0.6	0.078072	0.077292	0.076775
0.7	0.151503	0.149723	0.148522
0.8	0.220194	0.217151	0.215060
0.9	0.284048	0.279437	0.276206
1.0	0.342969	0.336441	0.331775

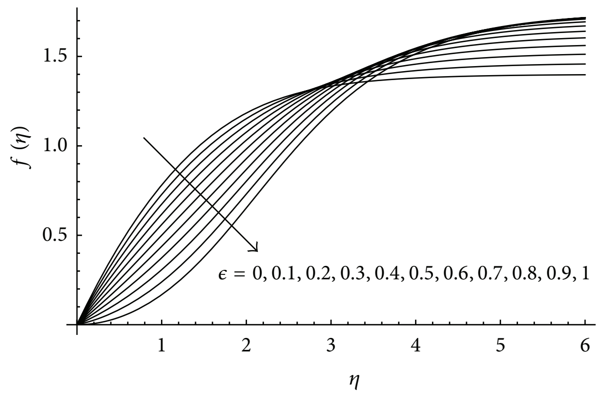

Figure 1: The stream function for the class of the Blasius problem at different values of ϵ using 15 terms of the current method.

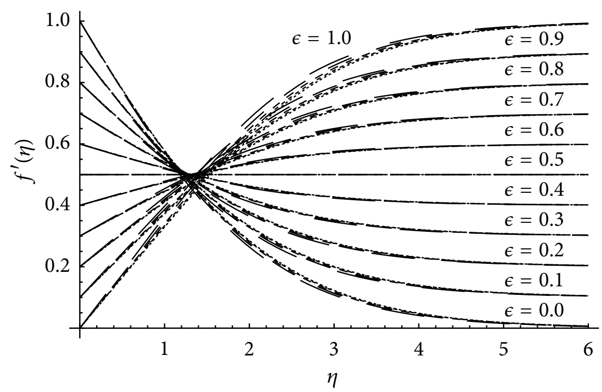

Figure 2: The fluid velocity for the class of the Blasius problem at different values of ϵ using 7, 9, 11, 13, and 15 terms of the current method.

TABLE 2: The approximate values of the skin-friction coefficient $f''(0)$ for the class of the Falkner-Skan problem using 11, 15, and 33 terms of Adomian's series.

ϵ	$\phi''_{11}(0)$	$\phi''_{15}(0)$	$\phi''_{33}(0)$	Exact values Reference [28]
0.0	−0.617661	−0.622494	−0.625945	−0.627504
0.1	−0.479542	−0.485036	−0.490729	−0.492625
0.2	−0.348276	−0.353437	−0.360126	−0.363901
0.3	−0.224304	−0.228296	−0.234485	−0.237219
0.4	−0.108067	−0.110256	−0.114249	−0.115811
0.5	0.000000	0.000000	0.000000	0.000000
0.6	0.099467	0.101753	0.107483	
0.7	0.189909	0.194253	0.207152	
0.8	0.270907	0.276719	0.297613	
0.9	0.342046	0.34835	0.377038	
1.0	0.402921	0.408321	0.443088	

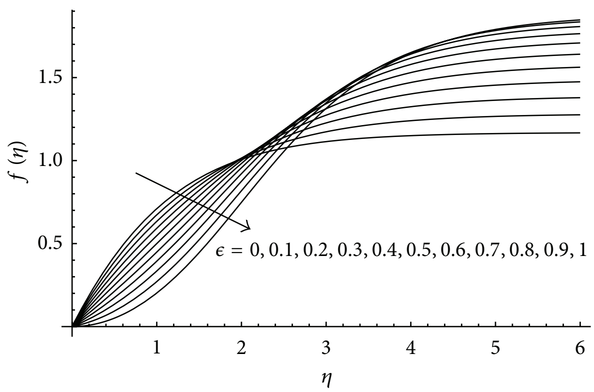

FIGURE 3: The stream function for the class of the Falkner-Skan problem at different values of ϵ using 15 terms of the current method.

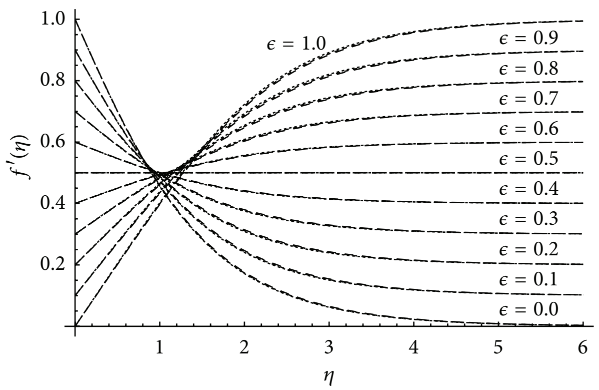

FIGURE 4: The fluid velocity for the class of the Falkner-Skan problem at different values of ϵ using 11, 13, and 15 terms of the current method.

the fluid velocity is depicted by using the approximate solutions $\phi_{11}(\eta)$, $\phi_{13}(\eta)$, and $\phi_{15}(\eta)$. It can be concluded from Figure 4 that our results are a coincidence with those exactly obtained in [27] at the values $\epsilon = 0.1, 0.2, 0.3, 0.4, 0.5, 0.7$, and 1, while the fluid velocity at the other values $\epsilon = 0, 0.6, 0.8$, and 0.9 was not discussed by Kudenatti [27].

6. Conclusion

An approach is presented in this paper to treat the boundary condition at infinity which is the main feature of the boundary layer equations. The suggested approach is based on changing the boundary condition at infinity to a classical one by the help of a transformation. The current approach is applied to solve a class of the Blasius problem and a special class of the Falkner-Skan problem via an improved version of Adomian's method. Moreover, exact solutions are deduced at a certain value of the velocity ratio parameter ϵ. In addition, the current numerical results are compared with the other published solutions, where good agreement is found. One of the main advantages of the present approach is the avoidance of the Padé approximation to deal with the infinity boundary condition.

References

[1] Z. Belhachmi, B. Brighi, and K. Taous, "On the concave solutions of the Blasius equation," *Acta Mathematica Universitatis Comenianae*, vol. 69, no. 2, pp. 199–214, 2000.

[2] B. K. Datta, "Analytic solution for the Blasius equation," *Indian Journal of Pure and Applied Mathematics*, vol. 34, no. 2, pp. 237–240, 2003.

[3] H. K. Kuiken, "On boundary layers in fluid mechanics that decay algebraically along stretches of wall that are not vanishingly small," *IMA Journal of Applied Mathematics*, vol. 27, no. 4, pp. 387–405, 1981.

[4] H. K. Kuiken, "A "backward" free-convective boundary layer," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 34, no. 3, pp. 397–413, 1981.

[5] J. He, "Approximate analytical solution of Blasius' equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 3, no. 4, pp. 260–263, 1998.

[6] A.-M. Wazwaz, "The variational iteration method for solving two forms of Blasius equation on a half-infinite domain," *Applied Mathematics and Computation*, vol. 188, no. 1, pp. 485–491, 2007.

[7] B. I. Yun, "Intuitive approach to the approximate analytical solution for the Blasius problem," *Applied Mathematics and Computation*, vol. 215, no. 10, pp. 3489–3494, 2010.

[8] S. Abbasbandy and C. Bervillier, "Analytic continuation of Taylor series and the boundary value problems of some nonlinear ordinary differential equations," *Applied Mathematics and Computation*, vol. 218, no. 5, pp. 2178–2199, 2011.

[9] M. Khan and M. A. Gondal, "Homotopy perturbation Padé transform method for Blasius flow equation using He's polynomials," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 12, no. 1-8, pp. 1–7, 2011.

[10] M. Khan, M. A. Gondal, I. Hussain, and S. K. Vanani, "A new comparative study between homotopy analysis transform method and homotopy perturbation transform method on a semi infinite domain," *Mathematical and Computer Modelling*, vol. 55, no. 3-4, pp. 1143–1150, 2012.

[11] R. Fazio, "Blasius problem and Falkner-Skan model: Töpfer's algorithm and its extension," *Computers and Fluids*, vol. 73, pp. 202–209, 2013.

[12] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, Mass, USA, 1994.

[13] A.-M. Wazwaz, R. Rach, and J.-S. Duan, "Adomian decomposition method for solving the Volterra integral form of the Lane-Emden equations with initial values and boundary conditions," *Applied Mathematics and Computation*, vol. 219, no. 10, pp. 5004–5019, 2013.

[14] A. M. Wazwaz, R. Rach, and J. S. Duan, "A study on the systems of the Volterra integral forms of the Lane-Emden equations by the Adomian decomposition method," *Mathematical Methods in the Applied Sciences*, 2013.

[15] A. E.-H. Ebaid, "Approximate periodic solutions for the non-linear relativistic harmonic oscillator via differential transformation method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 7, pp. 1921–1927, 2010.

[16] A. E. Ebaid, "A reliable aftertreatment for improving the differential transformation method and its application to nonlinear oscillators with fractional nonlinearities," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 1, pp. 528–536, 2011.

[17] S. N. Venkatarangan and K. Rajalakshmi, "Modification of Adomian's decomposition method to solve equations containing radicals," *Computers & Mathematics with Applications*, vol. 29, no. 6, pp. 75–80, 1995.

[18] A.-M. Wazwaz, "The modified decomposition method and Padé approximants for solving the Thomas-Fermi equation," *Applied Mathematics and Computation*, vol. 105, no. 1, pp. 11–19, 1999.

[19] A.-M. Wazwaz, "The modified decomposition method and Padé approximants for a boundary layer equation in unbounded domain," *Applied Mathematics and Computation*, vol. 177, no. 2, pp. 737–744, 2006.

[20] S. A. Kechil and I. Hashim, "Non-perturbative solution of free-convective boundary-layer equation by Adomian decomposition method," *Physics Letters A*, vol. 363, no. 1-2, pp. 110–114, 2007.

[21] S. Awang Kechil and I. Hashim, "Series solution of flow over nonlinearly stretching sheet with chemical reaction and magnetic field," *Physics Letters A*, vol. 372, no. 13, pp. 2258–2263, 2008.

[22] E. Alizadeh, K. Sedighi, M. Farhadi, and H. R. Ebrahimi-Kebria, "Analytical approximate solution of the cooling problem by Adomian decomposition method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 2, pp. 462–472, 2009.

[23] T. Hayat, Q. Hussain, and T. Javed, "The modified decomposition method and Padé approximants for the MHD flow over a non-linear stretching sheet," *Nonlinear Analysis. Real World Applications*, vol. 10, no. 2, pp. 966–973, 2009.

[24] M. M. Rashidi and S. A. Mohimanian Pour, "Explicit solution of axisymmetric stagnation flow towards a shrinking sheet by DTM-Padé," *Applied Mathematical Sciences*, vol. 4, no. 53–56, pp. 2617–2632, 2010.

[25] M. M. Rashidi and M. Keimanesh, "Using differential transform method and padé approximant for solving mhd flow in a laminar liquid film from a horizontal stretching surface," *Mathematical Problems in Engineering*, vol. 2010, Article ID 491319, 14 pages, 2010.

[26] A. Ebaid, "A new analytical and numerical treatment for singular two-point boundary value problems via the Adomian decomposition method," *Journal of Computational and Applied Mathematics*, vol. 235, no. 8, pp. 1914–1924, 2011.

[27] R. B. Kudenatti, "A new exact solution for boundary layer flow over a stretching plate," *International Journal of Non-Linear Mechanics*, vol. 47, pp. 727–733, 2012.

[28] T. Tajvidi, M. Razzaghi, and M. Dehghan, "Modified rational Legendre approach to laminar viscous flow over a semi-infinite flat plate," *Chaos, Solitons and Fractals*, vol. 35, no. 1, pp. 59–66, 2008.

[29] L. Bairstow, "Skin friction," *Journal of the Royal Aeronautical Society*, vol. 29, pp. 3–23, 1925.

[30] S. Goldstein, "Concerning some solutions of the boundary layer equations in hydrodynamics," *Proceedings of the Cambridge Philosophical Society*, vol. 26, pp. 1–30, 1930.

[31] V. M. Falkner, "A method of numerical solution of differential equations," *Philosophical Magazine*, vol. 21, pp. 624–640, 1936.

[32] L. Horwarth, "On the solution of the laminar boundary layer equations," *Proceedings of the Royal Society of London A*, vol. 164, pp. 547–579, 1938.

[33] R. Fazio, "The Blasius problem formulated as a free boundary value problem," *Acta Mechanica*, vol. 95, no. 1–4, pp. 1–7, 1992.

[34] J. P. Boyd, "The Blasius function in the complex plane," *Experimental Mathematics*, vol. 8, no. 4, pp. 381–394, 1999.

[35] S. Abbasbandy, "A numerical solution of Blasius equation by Adomian's decomposition method and comparison with homotopy perturbation method," *Chaos, Solitons and Fractals*, vol. 31, no. 1, pp. 257–260, 2007.

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Research Article

Solving a Class of Singular Fifth-Order Boundary Value Problems Using Reproducing Kernel Hilbert Space Method

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We use the reproducing kernel Hilbert space method to solve the fifth-order boundary value problems. The exact solution to the fifth-order boundary value problems is obtained in reproducing kernel space. The approximate solution is given by using an iterative method and the finite section method. The present method reveals to be more effective and convenient compared with the other methods.

1. Introduction

The reproducing kernel Hilbert space method has been shown [1–7] to solve effectively, easily, and accurately a large class of linear and nonlinear, ordinary, partial differential equations. However, in [1–7], it cannot be used directly boundary value problems with mixed boundary conditions, since it is very difficult to obtain a reproducing kernel function satisfying mixed nonlinear boundary conditions. The aim of this work is to fill this gap. In [8], we give a new reproducing kernel Hilbert space for solving singular linear fourth-order boundary value problems with mixed boundary conditions. In this paper, we use the new reproducing kernel Hilbert function space method to solve the nonlinear fifth-order boundary value problems.

Singular fifth-order boundary value problems arise in the fields of gas dynamics, Newtonian fluid mechanics, fluid mechanics, fluid dynamics, elasticity, reaction-diffusion processes, chemical kinetics, and other branches of applied mathematics.

Let us consider the following class of singular fifth-order mixed boundary value problems:

$$\begin{gathered} u^{(5)}(x) + \frac{p_1(x)}{x^{\alpha_1}(1-x)^{\beta_1}} u^{(4)}(x) + \cdots + \frac{p_4(x)}{x^{\alpha_4}(1-x)^{\beta_4}} u'(x) + \frac{p_5(x)}{x^{\alpha_5}(1-x)^{\beta_5}} u(x) = f(x), \quad x \in (0,1), \\ B_i u = r_i, \quad (i = 1, 2, \ldots, 5), \end{gathered} \tag{1}$$

where $p_j(x), f(x) \in L^2[0,1]$ $(j = 1,\ldots,5)$ are known functions. $B_i u$ $(i = 1,2,\ldots,5)$ are linear conditions. We assume that (1) has a unique solution which belongs to $W_2^6[0,1]$, where $W_2^6[0,1]$ is a reproducing kernel space.

Remark 1. If $B_i u = u^{(i)}(0)$ $(i = 1,\ldots,m)$, then (1) is an initial value problem. If $B_i u = u(x_i)$ $(i = 1,2,\ldots,m)$, then (1) is a multipoint problem and so on. That is, problem (1) has a rather general form.

Let $\alpha = \max_{1\le i\le 5}\{\alpha_i\}$ and $\beta = \max_{1\le i\le 5}\{\beta_i\}$, $F(x) = x^\alpha (1-x)^\beta f(x)$.

Consider

$$\begin{aligned} (Lu)(x) = {} & x^\alpha (1-x)^\beta u^{(5)}(x) \\ & + x^{\alpha-\alpha_1}(1-x)^{\beta-\beta_1} p_1(x) u^{(4)}(x) \\ & + \cdots + x^{\alpha-\alpha_4}(1-x)^{\beta-\beta_4} p_4(x) u'(x) \\ & + x^{\alpha-\alpha_5}(1-x)^{\beta-\beta_5} p_5(x) u(x). \end{aligned} \tag{2}$$

It is easy to prove that $L : W_2^6[0,1] \rightarrow L^2[0,1]$ is a bounded linear operator. On the other hand, we suppose that the linear conditions can also always be homogenized; after homogenization of these conditions, we put these conditions into the reproducing kernel space $W_2^6[0,1]$ constructed in the following section. Equation (1) can be transformed into the following form in $W_2^6[0,1]$:

$$(Lu)(x) = F(x). \tag{3}$$

2. Reproducing Kernel Hilbert Space

Definition 2. Let H be a real Hilbert space of functions $f : \Omega \rightarrow R$. A function $K : \Omega \times \Omega \rightarrow R$ is called reproducing kernel for H if

(i) $K(x,\cdot) \in H$ for all $x \in \Omega$,

(ii) $f(x) = \langle f, K(\cdot,x)\rangle_H$ for all $f \in H$ and all $x \in \Omega$.

Definition 3. A real Hilbert space H of functions on a set Ω is called a reproducing kernel Hilbert space if there exists a reproducing kernel K of H.

One defines that the inner product space $\overline{W}_2^{m+1}[0,1] = \{u \mid u, u', \ldots, u^{(m)}$ are absolutely continuous function, $u^{(m+1)} \in L^2[0,1]\}$.

The inner product in $\overline{W}_2^{m+1}[0,1]$ is given by

$$\langle u(x), v(x)\rangle = \sum_{i=0}^{m} u^{(i)}(0)\, v^{(i)}(0) + \int_0^1 u^{(m+1)}(t)\, v^{(m+1)}(t)\, dt. \tag{4}$$

Theorem 4 (see [8]). *The space $\overline{W}_2^{m+1}[0,1]$ is a reproducing kernel space, and its reproducing kernel is*

$$R^{\{m+1\}}(x,y) = \begin{cases} \displaystyle\sum_{i=0}^{m} \frac{1}{(i!)^2} x^i y^i + \frac{1}{(m!)^2}\int_0^x (x-t)^m (y-t)^m dt, & x<y, \\ \displaystyle\sum_{i=0}^{m} \frac{1}{(i!)^2} x^i y^i + \frac{1}{(m!)^2}\int_0^y (x-t)^m (y-t)^m dt, & y<x. \end{cases} \tag{5}$$

For studying the solution of (1) in the homogenized form, we give space (6) as follows:

$$W_2^6[0,1] = \left\{u \mid u \in \overline{W}_2^6[0,1],\ B_i u = 0,\ i = 1,2,\ldots,5\right\}. \tag{6}$$

$W_2^6[0,1]$ is equipped with the same inner product $\overline{W}_2^6[0,1]$. In the following, we construct a reproducing kernel for the space $W_2^6[0,1]$, and we give Lemmas 5 and 6.

Lemma 5. *Let $A : H[a,b] \rightarrow L^2[a,b]$ be a bounded linear operator; function $R_x(y)$ is the reproducing kernel of space $H[a,b]$. Let $g_z(x) = (A_s R_x(s))(z)$; then $\|g_z(x)\|^2 = (A_s(A_t R_s(t))(z))(z)$, where $H[a,b]$ denotes any reproducing kernel space of functions over $[a,b]$, the symbol A_s indicates that the operator A applies to functions of the variable s, and the symbol $(A_s R_x(s))(z)$ indicates that the operator A applies to function $R_x(s)$ of the variable s and $s = z$.*

Proof. Consider

$$\begin{aligned} \|g_z(x)\|^2 &= \langle (A_s R_x(s))(z), (A_t R_x(t))(z)\rangle \\ &= (A_s (A_t \langle R_x(s), R_x(t)\rangle)(z))(z) \\ &= (A_s (A_t R_s(t))(z))(z). \end{aligned} \tag{7}$$

□

Lemma 6. *If A, $g_z(x)$, and $R_x(y)$ are defined as in Lemma 5, let $K_x(y) = R_x(y) - g_z(x)g_z(y)/\|g_z(x)\|^2$; consider the space $H_1 = \{u(y) \mid u(y) \in H[a,b]$, and $(A_y u(y))(z) = 0\}$, then, $K_x(y)$ is the reproducing kernel of space H_1.*

Proof. For any $u(y) \in H_1$, next, we will prove $\langle u(y), K_x(y)\rangle = u(x)$.

Consider

$$\begin{aligned} \langle u(y), K_x(y)\rangle &= \left\langle u(y), R_x(y) - \frac{g_z(x)\, g_z(y)}{\|g_z(x)\|^2}\right\rangle \\ &= \langle u(y), R_x(y)\rangle - \left\langle u(y), \frac{g_z(x)\, g_z(y)}{\| g_z(x)\|^2}\right\rangle \\ &= u(x) - g_z(x)\frac{\langle u(y), (A_s R_y(s))(z)\rangle}{\|g_z(x)\|^2} \\ &= u(x) - g_z(x)\frac{(A_s \langle u(y), R_y(s)\rangle)(z)}{\|g_z(x)\|^2} \\ &= u(x) - \frac{g_z(x)(A_s u(s))(z)}{\|g_z(x)\|^2} = u(x). \end{aligned} \tag{8}$$

□

Let $h_1(x) = B_{1y}R^{\{6\}}(x,y)$, $h_2(x) = B_{2y}(R^{\{6\}}(x,y) - h_1(x)h_1(y)/\|h_1(x)\|^2)$, $h_3(x) = B_{3y}(R^{\{6\}}(x,y) - h_1(x)h_1(y)/\|h_1(x)\|^2 - h_2(x)h_2(y)/\|h_2(x)\|^2)$, $h_4(x) = B_{4y}(R^{\{6\}}(x,y) - h_1(x)h_1(y)/\|h_1(x)\|^2 - h_2(x)h_2(y)/\|h_2(x)\|^2 - h_3(x)h_3(y)/\|h_3(x)\|^2)$, and $h_5(x) = B_{5y}(R^{\{6\}}(x,y) - h_1(x)h_1(y)/\|h_1(x)\|^2 - h_2(x)h_2(y)/\|h_2(x)\|^2 - h_3(x)h_3(y)/\|h_3(x)\|^2 - h_4(x)h_4(y)/\|h_4(x)\|^2)$, where the symbol B_{iy} $(i = 1,2,3,4,5)$ indicates that the operator B_i $(i = 1,2,3,4,5)$ applies to functions of the variable y. Using Lemma 6, we get Theorem 7.

Table 1: The numerical results of Example 12.

x	$u_T(x)$	$u_{100}(x)$	$\lvert u_{20}(x) - u_T(x)\rvert$	$\lvert u_{40}(x) - u_T(x)\rvert$	$\lvert u_{100}(x) - u_T(x)\rvert$	$\lvert u'_{100}(x) - u'_T(x)\rvert$
0	0	0	0	0	0	0
0.08	−0.00150556	−0.00149854	3.11533×10^{-4}	2.12134×10^{-4}	7.01613×10^{-6}	4.26411×10^{-5}
0.16	−0.00300419	−0.00299716	5.86206×10^{-4}	4.03105×10^{-4}	7.02601×10^{-6}	3.96519×10^{-5}
0.24	−0.000739712	−0.000738437	1.36893×10^{-4}	1.08655×10^{-4}	1.27471×10^{-6}	1.06360×10^{-4}
0.32	0.00926282	0.00925363	1.38316×10^{-3}	9.02756×10^{-4}	9.18729×10^{-6}	1.59425×10^{-4}
0.4	0.0316161	0.0315926	4.00907×10^{-3}	2.65016×10^{-3}	2.34247×10^{-5}	1.93257×10^{-4}
0.48	0.0717954	0.0717548	7.49348×10^{-3}	4.96714×10^{-3}	4.06485×10^{-5}	2.28313×10^{-4}
0.56	0.136337	0.136276	1.13395×10^{-3}	7.52215×10^{-3}	6.03524×10^{-5}	2.58664×10^{-4}
0.64	0.23304	0.232959	1.48395×10^{-3}	9.84568×10^{-3}	8.01808×10^{-5}	2.77257×10^{-4}
0.72	0.371189	0.371084	1.71134×10^{-3}	1.13541×10^{-3}	1.04886×10^{-4}	2.93384×10^{-4}
0.8	0.561801	0.561673	1.71481×10^{-3}	1.13757×10^{-3}	1.27883×10^{-4}	3.40704×10^{-4}
0.88	0.817902	0.817739	1.38345×10^{-3}	9.1769×10^{-3}	1.63033×10^{-4}	6.89140×10^{-4}
0.96	1.15484	1.15458	5.99608×10^{-3}	3.97735×10^{-3}	2.60877×10^{-4}	1.51714×10^{-3}

Theorem 7. *The space $W_2^6[0,1]$ is a reproducing kernel space, and its reproducing kernel is*

$$K(x,y) = R^{\{6\}}(x,y) - \frac{h_1(x)h_1(y)}{\|h_1(x)\|^2} - \frac{h_2(x)h_2(y)}{\|h_2(x)\|^2} - \frac{h_3(x)h_3(y)}{\|h_3(x)\|^2} - \frac{h_4(x)h_4(y)}{\|h_4(x)\|^2} - \frac{h_5(x)h_5(y)}{\|h_5(x)\|^2}. \quad (9)$$

3. Analytical Solution

Let $\psi_i(x) = (L_y K(x,y))(x_i)$, $i = 1, 2, \ldots$. Via Gram-Schmidt orthonormalization for $\{\psi_i(x)\}_{i=1}^{\infty}$, we get

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (10)$$

where the β_{ik} are the coefficients resulting from Gram-Schmid orthonormalization.

Lemma 8. *If $\{x_i\}_{i=1}^{\infty}$ are distinct points dense in $[0,1]$ and L^{-1} is existent, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is a complete function system in $W_2^{m+1}[0,1]$.*

Proof. For each fixed $u(x) \in W_2^{m+1}[0,1]$, if $\langle u(x), \psi_i(x)\rangle = 0$, then

$$\langle u(x), \psi_i(x)\rangle = \left(L_y \langle u(x), K(x,y)\rangle\right)(x_i) = \left(L_y u(y)\right)(x_i) = 0. \quad (11)$$

Taking into account the density of $\{x_i\}_{i=1}^{\infty}$, it results in $(L_y u(y))(x) = 0$. It follows that $u(x) \equiv 0$ from the existence of L^{-1}. □

Theorem 9. *If $\{x_i\}_{i=1}^{\infty}$ are distinct points dense in $[0,1]$ and L^{-1} is existent, then*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k)\overline{\psi}_i(x) \quad (12)$$

is an analytical solution of (3).

Proof. $u(x)$ can be expanded to Fourier series in terms of normal orthogonal basis $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ in $W_2^{m+1}[0,1]$ as follows:

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x)\rangle \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle u(x), \psi_k(x)\rangle \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \langle u(x), (L_s K_x(s))(x_k)\rangle \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left(L_s \langle u(x), K_x(s)\rangle\right)(x_k) \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left(L_s u(s)\right)(x_k) \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k) \overline{\psi}_i(x). \end{aligned} \quad (13)$$

□

4. Numerical Solution

We define an approximate solution $u_n(x)$ by

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} F(x_k)\overline{\psi}_i(x). \quad (14)$$

Table 2: Comparison of the absolute error of Example 13.

x	Solution			Absolute error		
	$u_T(t,x)$	Reference [9]	$u_{100}(x)$	$\lvert u_{50}(x)-u_T(x)\rvert$	Reference [9]	$\lvert u_{100}(x)-u_T(x)\rvert$
0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.1249	0.0000752	0.0000754	0.0000754	8.11621×10^{-7}	2×10^{-7}	2.74983×10^{-7}
0.2431	0.0013039	0.0013043	0.0013037	3.70493×10^{-6}	4×10^{-7}	1.13843×10^{-7}
0.3806	0.0080242	0.0080249	0.0080244	7.8819×10^{-6}	7×10^{-7}	2.32339×10^{-7}
0.4195	0.0116531	0.0116538	0.0116533	8.85202×10^{-6}	7×10^{-7}	2.59035×10^{-7}
0.5	0.0220970	0.0220978	0.0220972	1.01895×10^{-5}	8×10^{-7}	2.94629×10^{-7}
0.6923	0.0588207	0.0588201	0.0588209	8.5994×10^{-6}	6×10^{-7}	2.44068×10^{-7}
0.7854	0.0723723	0.0723726	0.0723724	5.68292×10^{-6}	4×10^{-7}	1.60302×10^{-7}
0.8917	0.0646361	0.0646363	0.0646366	1.96562×10^{-6}	2×10^{-7}	5.51306×10^{-7}
1.0	0.0	0.0	0.0	0.0	0.0	0.0

Table 3: The numerical results of Example 13.

x	$u'_T(x)$	$u'_{100}(x)$	$\lvert u'_{100}(x)-u'_T(x)\rvert$	$u''_T(x)$	$u''_{100}(x)$	$\lvert u''_{100}(x)-u''_T(x)\rvert$
0.0	0	0	0	0	0	0
0.1	0.0012491	0.00125306	3.95623×10^{-6}	0.0419792	0.0420291	4.98366×10^{-5}
0.2	0.0121642	0.0121721	7.93764×10^{-6}	0.193196	0.193223	2.63540×10^{-5}
0.3	0.0421473	0.0421562	8.95432×10^{-6}	0.410381	0.410375	6.05312×10^{-6}
0.4	0.0930975	0.0931043	6.87917×10^{-6}	0.591978	0.591944	3.40825×10^{-5}
0.5	0.15468	0.154682	2.49761×10^{-6}	0.596621	0.59657	5.13938×10^{-5}
0.6	0.200775	0.200773	2.91343×10^{-6}	0.250969	0.250915	5.41587×10^{-5}
0.7	0.186533	0.186526	7.76184×10^{-6}	−0.645692	−0.645732	3.97775×10^{-5}
0.8	0.0457947	0.0457844	1.023370×10^{-5}	−2.31836	−2.31836	6.35516×10^{-6}
0.9	−0.311216	−0.311224	8.34947×10^{-6}	−5.01403	−5.01398	4.75597×10^{-5}
1.0	−1	−1	6.12843×10^{-14}	−9	−8.99988	1.23122×10^{-5}

Theorem 10. *Let $\varepsilon_n^2 = \|u(x)-u_n(x)\|^2$, where $u(x)$ and $u_n(x)$ are given by* (12) *and* (14)*; then the sequence of real numbers $\varepsilon_n(x)$ is monotonously decreasing and $\varepsilon_n(x) \to 0$.*

Proof. We have

$$\varepsilon_n^2 = \|u(x)-u_n(x)\|^2 = \left\|\sum_{i=n+1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x)\right\|^2 = \sum_{i=n+1}^{\infty}\left(\langle u(x),\overline{\psi}_i(x)\rangle\right)^2. \tag{15}$$

Clearly, $\varepsilon_{n-1} \geq \varepsilon_n$ and consequently $\{\varepsilon_n\}$ is monotonously decreasing in the sense of $\|\cdot\|$. By Theorem 9, we know that $\sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x)$ is convergent in the norm of $\|\cdot\|$; then we have

$$\varepsilon_n^2 = \|u(x)-u_n(x)\|^2 \longrightarrow 0. \tag{16}$$

Hence, $\varepsilon_n \to 0$. □

Theorem 11 (convergence analysis). *$u_n(x)$ and $u_n^{(k)}(x)$ are uniformly convergent to $u(x)$ and $u^{(k)}(x)$, $k = 0,1,2,\ldots,m$, where $u(x)$ and $u_n(x)$ are given by* (12) *and* (14).

Proof. For any $x\in[0,1]$, $k=0,1,2,\ldots,5$,

$$\left|u_n^{(k)}(x)-u^{(k)}(x)\right| = \left|\left\langle u_n(t)-u(t),\frac{\partial^k K(x,t)}{\partial x^k}\right\rangle\right| \leq \|u_n(t)-u(t)\|\cdot\left\|\frac{\partial^k K(x,t)}{\partial x^k}\right\|. \tag{17}$$

Then there exists $C_k>0$ such that

$$\left|u_n^{(k)}(x)-u^{(k)}(x)\right| \leq C_k\|u_n(t)-u(t)\| = C_k\varepsilon_n \longrightarrow 0. \tag{18}$$

□

The numerical solution to (3) can be obtained using the following method:

$$u_n(x) = \sum_{i=1}^{n} d_i\psi_i(x), \tag{19}$$

where the coefficients d_i, $i=1,\ldots,m$, are determined by the equation

$$\sum_{i=1}^{n} d_i L\psi_i(x)\,|_{x=x_j} = F\left(x_j\right),\quad j=1,2,\ldots,n. \tag{20}$$

Using (19) and (20), we have $(Lu_m)(x_j) = F(x_j)$, $j = 1,2,\ldots,n$. So, $u_n(x)$ is the approximation solution of (3).

Table 4: The numerical results of Example 13.

x	$u_T^{(3)}(x)$	$u_{100}^{(3)}(x)$	$\lvert u_{100}^{(3)}(x)-u_T^{(3)}(x)\rvert$	$u_T^{(4)}(x)$	$u_{100}^{(4)}(x)$	$\lvert u_{100}^{(4)}(x)-u_T^{(4)}(x)\rvert$
0.1	0.971215	0.971117	9.72528×10^{-5}	11.8289	11.8244	4.50235×10^{-3}
0.2	1.97221	1.9719	3.12435×10^{-4}	7.04361	7.0429	7.15499×10^{-4}
0.3	2.19979	2.19947	3.16007×10^{-4}	−3.23499	−3.2345	4.83768×10^{-4}
0.4	1.19534	1.19511	2.34524×10^{-4}	−17.4321	−17.431	1.09059×10^{-3}
0.5	−1.39212	−1.39222	1.05489×10^{-4}	−34.8029	−34.8014	1.46428×10^{-3}
0.6	−5.85595	−5.8559	5.44543×10^{-5}	−54.8995	−54.8978	1.72026×10^{-3}
0.7	−12.4526	−12.4524	2.36294×10^{-4}	−77.4172	−77.4153	1.90772×10^{-3}
0.8	−21.4126	−21.4122	4.34553×10^{-4}	−102.132	−102.13	2.05168×10^{-3}
0.9	−32.9466	−32.9459	6.45658×10^{-4}	−128.873	−128.871	2.16643×10^{-3}
1.0	−47.25	−47.2491	9.67158×10^{-4}	−157.5	−157.498	2.26074×10^{-3}

5. Numerical Experiment

In this section, two numerical examples are studied to demonstrate the accuracy of the present method.

Example 12. Consider the following fifth-order boundary value problem with nonclassical side condition (the right-hand side of this problem has a singularity at $x=0$, $x=1$):

$$\begin{gathered} u^{(5)}(x)-e^{x}\frac{x}{1-x}u''(x)+e^{x}\frac{x}{1-x}u(x)=f(x), \\ 0<x<1, \\ u(0)=u'(0)=u\left(\frac{1}{4}\right)=0, \\ 5u''(0)+42\int_0^1 e^{-x}u(x)\,dx=0, \\ 4u'''(1)+u'(1)=10u''(1), \end{gathered} \tag{21}$$

where $f(x)=e^{x}(-45+195x-750x^{2}+320x^{5/2}-600x^{3}-32(5+e^{x})x^{7/2}+40(20+3e^{x})x^{4}-16(9+4e^{x})x^{9/2}+16(23+10e^{x})x^{5}-16x^{11/2}+32x^{6})/32(-1+x)x^{5/2}$. The exact solution is $u_T(x)=x^{2}(\sqrt{x}-1/2)e^{x}$. The numerical results are presented in Table 1.

Example 13 (see [9]). Consider the following fifth-order boundary value problem (the right-hand side of this problem has a singularity at $x=0$):

$$\begin{gathered} u^{(5)}(x)-e^{-x}u(x) \\ =-e^{-x}x^{\frac{9}{2}}(1-x)+\frac{945(1-11x)}{32\sqrt{x}},\quad 0<x<1, \\ u(0)=u'(0)=u''(0)=u(1)=0,\qquad u'(1)=-1, \end{gathered} \tag{22}$$

where the exact solution is $u_T(x)=x^{9/2}(1-x)$. By the homogeneous process of the boundary condition, letting $v(x)=u(x)-x^{3}(1-x)$, the problem can be transformed into the equivalent form

$$\begin{gathered} v^{(5)}(x)-e^{-x}v(x) \\ =-e^{-x}\left(x^{\frac{9}{2}}-x^{3}\right)(1-x)+\frac{945(1-11x)}{32\sqrt{x}},\quad 0\le x\le 1, \\ v(0)=v'(0)=v''(0)=v(1)=v'(1)=0. \end{gathered} \tag{23}$$

The numerical results are presented in Tables 2, 3, and 4.

6. Conclusions and Remarks

In this paper, a new reproducing kernel space satisfying mixed boundary value conditions is constructed skillfully. This makes it easy to solve such kind of problems. Furthermore, the exact solution of the problem can be expressed in series form. The numerical results demonstrate that the new method is quite accurate and efficient for singular problems of fifth-order ordinary differential equations. All computations have been performed using the Mathematica 7.0 software package.

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References

[1] R. Mokhtari, F. T. Isfahani, and M. Mohammadi, "Reproducing kernel method for solving nonlinear differential-difference equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 514103, 10 pages, 2012.

[2] Y.-l. Wang and L. Chao, "Using reproducing kernel for solving a class of partial differential equation with variable-coefficients,"

Applied Mathematics and Mechanics, vol. 29, no. 1, pp. 129–137, 2008.

[3] Y. Wang, T. Chaolu, and Z. Chen, "Using reproducing kernel for solving a class of singular weakly nonlinear boundary value problems," *International Journal of Computer Mathematics*, vol. 87, no. 2, pp. 367–380, 2010.

[4] F. Z. Geng and X. M. Li, "A new method for Riccati differential equations based on reproducing kernel and quasilinearization methods," *Abstract and Applied Analysis*, vol. 2012, Article ID 603748, 8 pages, 2012.

[5] M. Mohammadi and R. Mokhtari, "Solving the generalized regularized long wave equation on the basis of a reproducing kernel space," *Journal of Computational and Applied Mathematics*, vol. 235, no. 14, pp. 4003–4014, 2011.

[6] Z. Li, Y. Wang, F. Tan, X. Wan, and T. Nie, "The solution of a class of singularly perturbed two-point boundary value problems by the iterative reproducing kernel method," *Abstract and Applied Analysis*, vol. 2012, Article ID 984057, 7 pages, 2012.

[7] X. Y. Li and B. Y. Wu, "A novel method for nonlinear singular fourth order four-point boundary value problems," *Computers & Mathematics with Applications*, vol. 62, no. 1, pp. 27–31, 2011.

[8] Y. Wang, X. Cao, and X. Li, "A new method for solving singular fourth-order boundary value problems with mixed boundary conditions," *Applied Mathematics and Computation*, vol. 217, no. 18, pp. 7385–7390, 2011.

[9] M. El-Gamel, "Sinc and the numerical solution of fifth-order boundary value problems," *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 1417–1433, 2007.

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Research Article

Three Solutions for Inequalities Dirichlet Problem Driven by $p(x)$-Laplacian-Like

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A class of nonlinear elliptic problems driven by $p(x)$-Laplacian-like with a nonsmooth locally Lipschitz potential was considered. Applying the version of a nonsmooth three-critical-point theorem, existence of three solutions of the problem is proved.

1. Introduction

Since many free boundary problems and obstacle problems may be reduced to partial differential equations with discontinuous nonlinearities, the existence of multiple solutions of the problems with discontinuous nonlinearities has been widely investigated in recent years. In 1981, Chang [1] extended the variational methods to a class of nondifferentiable functionals and directly applied the variational methods for nondifferentiable functionals to prove some existence theorems for PDE with discontinuous nonlinearities. Soon thereafter, Kourogenis and Papageorgiou [2] extend the nonsmooth critical point theory of Chang [1], by replacing the compactness and the boundary conditions. In [3], by using the Ekeland variational principle and a deformation theorem, Kandilakis et al. obtained the local linking theorem for locally Lipschitz functions. In the celebrated work [4], Ricceri elaborated a Ricceri-type variational principle for Gateaux differentiable functionals. Later, Marano and Motreanu [5] extended Ricceri's result to a large class of nondifferentiable functionals and gave an application to a Neumann-type problem involving the p-Laplacian with discontinuous nonlinearities.

In this paper, we consider a nonlinear elliptic problem driven by $p(x)$-Laplacian-like with a nonsmooth locally Lipschitz potential (hemivariational inequality):

$$-\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)\in\lambda\partial F(x,u),\quad \text{a.e. in } \Omega,$$
$$u=0,\quad \text{on } \partial\Omega, \tag{P}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with C^1-boundary $\partial\Omega$. $p\in C(\Omega)$, $2\le N<p^-:=\inf_{x\in\Omega}p(x)\le p^+:=\sup_{x\in\Omega}p(x)<+\infty$, $F\in C(\overline{\Omega}\times\mathbb{R})$, and $F:\Omega\times\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz with respect to the second variable. By $\partial F(x,u)$, we denote the generalized subdifferential of the locally Lipschitz function $u\to F(x,u)$. Our goal is to establish the same results under different assumptions.

The study of differential equations and variational problems with variable exponent has been a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, and so forth (see [6, 7]). The study on variable exponent problems attracts more and more interest in recent years. Many results have been obtained on this kind of problems, for example, [8–14]. Neumann-type problems involving the $p(x)$-Laplacian have been studied, for instance, in [15–18].

Recently, Rodrigues [19] has considered the existence of nontrivial solution for the Dirichlet problem involving the $p(x)$-Laplacian-like of the type

$$-\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), \quad \text{a.e. in } \Omega,$$
$$u=0, \quad \text{on } \partial\Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p \in C(\overline{\Omega})$ with $p(x) > 2$, for all $x \in \Omega$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Caratheodory condition. We emphasize that, in our approach, no continuity hypothesis will be required for the function f with respect to the second argument. So, (P) need not have a solution. To avoid this situation, we consider such function $f(x,\cdot)$ which is locally essentially bounded and fill the discontinuity gap of $f(x,\cdot)$, replacing f by the interval $[f_1, f_2]$, where

$$f_1(x,t) := \lim_{s\to 0^+} \operatorname{ess}\inf_{|s-t|<\delta} f(x,s),$$
$$f_2(x,t) := \lim_{s\to 0^+} \operatorname{ess}\sup_{|s-t|<\delta} f(x,s). \tag{2}$$

On the other hand, it is well known that if $F(x,u) = \int_0^u f(x,t)dt$, then F become locally Lipschitz and $\partial F(x,u) = [f_1(x,u), f_2(x,u)]$ (see [1, 20]).

The aim of the present paper is to establish a three-solution theorem for the nonlinear elliptic problem driven by $p(x)$-Laplacian-like with nonsmooth potential (see Theorem 6) by using a consequence (see Theorem 4) of the three-critical-point theorem established firstly by Marano and Motreanu in [20], which is a non-smooth version of Ricceri's three-critical-point theorem (see [21]). The paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces and the generalized gradient of the locally Lipschitz function. In Section 3, we give the main result of this paper and use the non-smooth three-critical-point theorem to prove it.

2. Preliminary

In order to discuss problem (P), we need some theories on $W_0^{1,p(x)}(\Omega)$ and the generalized gradient of the locally Lipschitz function. Firstly we state some basic properties of space $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [10–12]). Denote by $S(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $S(\Omega)$ are considered as the same element of $S(\Omega)$ when they are equal almost everywhere.

Put $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1, \ \forall x \in \overline{\Omega}\}$.

If $p \in C(\overline{\Omega})$, then write

$$L^{p(x)}(\Omega) = \left\{u \in S(\Omega) : \int_\Omega |u(x)|^{p(x)}dx < +\infty\right\}, \tag{3}$$

with the norm $|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |u(x)/\lambda|^{p(x)}dx \le 1\}$, and

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\right\}, \tag{4}$$

with the norm $\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}$. Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

We remember that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. Denote by $L^{q(x)}(\Omega)$ the conjugate Lebesgue space of $L^{p(x)}(\Omega)$ with $1/p(x) + 1/q(x) = 1$; then the Hölder-type inequality

$$\int_\Omega |uv|\,dx \le \left(\frac{1}{p^-}+\frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)}, \quad u \in L^{p(x)}(\Omega),\ v \in L^{q(x)}(\Omega) \tag{5}$$

holds. Furthermore, if we define the mapping $\rho : L^{p(x)}(\Omega) \to \mathbb{R}$ by

$$\rho(u) = \int_\Omega |u(x)|^{p(x)}dx, \tag{6}$$

then the following relations hold:

$$|u|_{p(x)} > 1 \Longrightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+},$$
$$|u|_{p(x)} < 1 \Longrightarrow |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}. \tag{7}$$

Proposition 1 (see [12]). *In $W_0^{1,p(x)}(\Omega)$ Poincare's inequality holds; that is, there exists a positive constant C_0 such that*

$$|u|_{p(x)} \le C_0|\nabla u|_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega). \tag{8}$$

So $|\nabla u|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$.

We will use the equivalent norm in the following discussion and write $\|u\| = |\nabla u|_{p(x)}$ for simplicity.

Proposition 2 (see [10]). *If $q \in C_+(\overline{\Omega})$ and $q(x) < p^*(x)$ for any $x \in \overline{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}(\Omega)$ is compact and continuous.*

Consider the following function:

$$J(u) = \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx, \quad u \in W_0^{1,p(x)}(\Omega). \tag{9}$$

We know that (see [1]).

If one denotes $A = J' : W_0^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$, then

$$\langle A(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)(\nabla u, \nabla v)_{\mathbb{R}^N}dx, \tag{10}$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$.

Proposition 3 (see [19]). *Set $X = W_0^{1,p(x)}(\Omega)$; A is as shown, then*

(1) *$A : X \to X^*$ is a convex, bounded previously; and strictly monotone operator;*

(2) *$A : X \to X^*$ is a mapping of type $(S)_+$; that is, $u_n \xrightarrow{w} u$ in X and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$ implies $u_n \to u$ in X;*

(3) *$A : X \to X^*$ is a homeomorphism.*

Let $(X, \|\cdot\|)$ be a real Banach space, and let X^* be its topological dual. A function $f : X \to \mathbb{R}$ is called locally Lipschitz if each point $u \in X$ possesses a neighborhood Ω_u such that $|f(u_1) - f(u_2)| \le L\,\|u_1 - u_2\|$ for all $u_1, u_2 \in \Omega_u$, for a positive constant L depending on Ω_u. The generalized directional derivative of f at the point $u \in X$ in the direction $h \in X$ is

$$f^0(u;h) = \limsup_{v\to u;\, t\downarrow 0} \frac{f(v+th) - f(v)}{t}. \tag{11}$$

The generalized gradient of f at $u \in X$ is defined by

$$\partial f(u) = \left\{u^* \in X^* : \langle u^*, h\rangle \le f^0(u;h)\ \ \forall h \in X\right\}, \tag{12}$$

which is a nonempty, convex, and w^*-compact subset of X, where $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X. One says that $u \in X$ is a critical point of f if $0 \in \partial f(u)$.

For further details, we refer the reader to the work of Chang [1].

Finally, for proving our results in the next section, we introduce the following theorem.

Theorem 4 (see [22, 23]). *Let X be a separable and reflexive real Banach space, and let $\Phi, \Psi : X \to \mathbb{R}$ be two locally Lipschitz functions. Assume that there exists $u_0 \in X$ such that $\Phi(u_0) = \Psi(u_0) = 0$ and $\Phi(u) \ge 0$ for every $u \in X$ and that there exist $u_1 \in X$ and $r > 0$ such that*

(1) *$r < \Phi(u_1)$;*

(2) *$\sup_{\Phi(u)<r}\Psi(u) < r(\Psi(u_1)/\Phi(u_1))$, and further, one assumes that function $\Phi - \lambda\Psi$ is sequentially lower semicontinuous and satisfies the (PS)-condition;*

(3) *$\lim_{\|u\|\to\infty}(\Phi(u) - \lambda\Psi(u)) = +\infty$ for every $\lambda \in [0, \bar{a}]$, where*

$$\bar{a} = \frac{hr}{r(\Psi(u_1)/\Phi(u_1)) - \sup_{\Phi(u)<r}\Psi(u)}, \quad \text{with } h > 1. \tag{13}$$

Then, there exist an open interval $\Lambda_1 \subseteq [0, \bar{a}]$ and a positive real number σ such that, for every $\lambda \in \Lambda_1$, the function $\Phi(u) - \lambda\Psi(u)$ admits at least three critical points whose norms are less than σ.

3. Existence Results

In this part, we will prove that there exist three solutions for problem (P) under certain conditions.

Definition 5. We say that I satisfies $(\mathrm{PS})_c$-condition if any sequence $\{u_n\} \subset W_0^{1,p(x)}(\Omega)$, such that $I(u_n) \to c$ and $m(u_n) \to 0$, as $n \to +\infty$, has a strongly convergent subsequence, where $m(u_n) = \inf\{\|u^*\|_{X^*} : u^* \in \partial I(u_n)\}$.

By a solution of (P), we mean a function $u \in W_0^{1,p(x)}(\Omega)$ to which there corresponds a mapping $\Omega \ni x \to w(x)$ with $w(x) \in \partial F(x,u)$ for almost every $x \in \Omega$ having the property that, for every $v \in W_0^{1,p(x)}(\Omega)$, the function $x \to w(x)v(x) \in L^1(\Omega)$ and

$$\begin{aligned}\int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)(\nabla u, \nabla v)_{\mathbb{R}^N}\,dx \\ = \lambda\int_\Omega w(x)\,v(x)\,dx.\end{aligned} \tag{14}$$

We know that $W_0^{1,p(x)}(\Omega)$ is compactly embedded into $C(\overline{\Omega})$ (by $N < p^- < p^*(x)$). So there is a constant $c_0 > 0$ such that $|u|_\infty \le c_0\|u\|$, for all $u \in W_0^{1,p(x)}(\Omega)$.

Set $\Phi(u) = \int_\Omega (1/p(x))(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}})dx$, $\Psi(u) = \int_\Omega F(x,u)dx$, $u \in W_0^{1,p(x)}(\Omega)$ and $\varphi(u) = \Phi(u) - \lambda\Psi(u)$, for all $u \in W_0^{1,p(x)}(\Omega)$.

We know that the critical points of φ are just the weak solutions of (P).

We consider a non-smooth potential function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ such that $F(x,0) = 0$ a.e. on Ω satisfying the following conditions:

H(j):

($\mathbf{h_1}$) $F(\cdot,t)$ is measurable for all $t \in \mathbb{R}$;
($\mathbf{h_2}$) $F(x,\cdot)$ is locally Lipschitz for a.e. $x \in \Omega$;
($\mathbf{h_3}$) there exist $a \in L^\infty(\Omega)_+$, $c > 0$ such that

$$|w| \le a(x) + c|t|^{\alpha(x)-1}, \quad \text{a.e. } x \in \Omega,\ \forall t \in \mathbb{R}, \tag{15}$$

where $w \in \partial F(x,t)$ and $1 < \alpha^- \le \alpha^+ < p^-$;
($\mathbf{h_4}$) there exists $q \in C(\overline{\Omega})$ with $p^+ < q^- \le q(x) < p^*(x)$, such that $\lim_{|t|\to 0}(F(x,t)/|t|^{q(x)}) = 0$ uniformly a.e. $x \in \Omega$;
($\mathbf{h_5}$) $\sup_{t\in\mathbb{R}}F(x,t) > 0$, for all $x \in \overline{\Omega}$.

Theorem 6. *Let ($\mathbf{h_1}$)–($\mathbf{h_5}$) hold. Then, there are an open interval $\Lambda \subseteq [0,+\infty)$ and a number σ such that, for every λ belonging to Λ, problem (P) possesses at least three solutions in $W_0^{1,p(x)}(\Omega)$ whose norms are less than σ.*

Proof. We observe that $\Psi(u)$ is Lipschitz on $L^{\alpha(x)}(\Omega)$ and, taking into account that $\alpha(x) < p^*(x)$, Ψ is also locally Lipschitz on $W_0^{1,p(x)}(\Omega)$ (see Proposition 2.2 of [15]). Moreover it results in $\partial\Psi(u) \subseteq \int_\Omega \partial F(x,u)dx$ (see [24]). The interpretation of $\partial\Psi(u) \subseteq \int_\Omega \partial F(x,u)dx$ is as follows: to every $w \in \partial\Psi(u)$ there corresponds a mapping $w(x) \in \partial F(x,u)$ for almost all $x \in \Omega$ having the property that for every $v \in W_0^{1,p(x)}(\Omega)$ the function $w(x)v(x) \in L^1(\Omega)$ and $\langle w, v\rangle = \int_\Omega w(x)v(x)dx$ (see [24]). The proof is divided into the following five steps.

Step 1. We show that φ is coercive.

By ($\mathbf{h_2}$), for almost all $x \in \Omega$, $t \mapsto F(x,t)$ is differentiable almost everywhere on $\mathbb{R}$ and we have

$$\frac{d}{dt}F(x,t) \in \partial F(x,t). \tag{16}$$

From ($\mathbf{h_3}$), there exist positive constants a_1, a_2 such that

$$\begin{aligned} F(x,t) &= F(x,0) + \int_0^t \frac{d}{ds}F(x,s)\,ds \\ &\leq a(x)t + \frac{c}{\alpha(x)}|t|^{\alpha(x)} \leq a_1 + a_2|t|^{\alpha(x)} \end{aligned} \tag{17}$$

for a.e. $x \in \Omega$ and $t \in \mathbb{R}$.

Note that $1 < \alpha(x) \leq \alpha^+ < p^- < p^*(x)$; then by Proposition 2, we have $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{\alpha(x)}(\Omega)$ (compact embedding). Furthermore, there exists c_1 such that $|u|_{\alpha(x)} \leq c_1\|u\|$.

So, for $|u|_{\alpha(x)} > 1$ and $\|u\| > 1$, we have $\int_\Omega |u|^{\alpha(x)}dx \leq |u|_{\alpha(x)}^{\alpha^+} \leq c_1^{\alpha^+}\|u\|^{\alpha^+}$.

Hence,

$$\begin{aligned} &\varphi(u) \\ &\quad= \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx - \lambda \\ &\qquad\times \int_\Omega F(x,u)\,dx \\ &\quad\geq \frac{2}{p^+}\int_\Omega |\nabla u|^{p(x)}dx - \lambda\int_\Omega F(x,u)\,dx \\ &\quad\geq \frac{2}{p^+}\|u\|^{p^-} - \lambda a_1 \operatorname{meas}(\Omega) - \lambda a_2 c_1^{\alpha^+}\|u\|^{\alpha^+} \longrightarrow +\infty, \end{aligned} \tag{18}$$

as $\|u\| \to +\infty$.

Step 2. We show that φ is weakly lower semicontinuous.

Let $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$, and by Proposition 2, we obtain the following results:

$$\begin{gathered} W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega); \quad u_n \longrightarrow u \text{ in } L^{p(x)}(\Omega); \\ u_n \longrightarrow u \quad \text{for a.a. } x \in \Omega; \\ F(x,u_n(x)) \longrightarrow F(x,u(x)) \quad \text{for a.a. } x \in \Omega. \end{gathered} \tag{19}$$

By Fatou's lemma, we have

$$\limsup_{n\to\infty}\int_\Omega F(x,u_n(x))\,dx \leq \int_\Omega F(x,u(x))\,dx. \tag{20}$$

Thus,

$$\begin{aligned} &\liminf_{n\to\infty}\varphi(u_n) \\ &\quad= \int_\Omega \frac{1}{p(x)}\left(|\nabla u_n|^{p(x)} + \sqrt{1+|\nabla u_n|^{2p(x)}}\right)dx \\ &\qquad- \lambda\limsup_{n\to\infty}\int_\Omega F(x,u_n)\,dx \\ &\quad\geq \int_\Omega \frac{1}{p(x)}\left(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\right)dx \\ &\qquad- \lambda\int_\Omega F(x,u)\,dx = \varphi(u). \end{aligned} \tag{21}$$

Step 3. We show that (PS)-condition holds.

Suppose $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p(x)}(\Omega)$ such that $|\varphi(u_n)| \leq c$ and $m(u_n) \to 0$ as $n \to +\infty$. If $u_n^* \in \partial\varphi(u_n)$ is such that $m(u_n) = \|u_n^*\|_{(W_0^{1,p(x)})^*}$, $n \geq 1$, then we know that

$$u_n^* = \Phi'(u_n) - \lambda w_n, \tag{22}$$

where the nonlinear operator $\Phi' : W_0^{1,p(x)} \to (W_0^{1,p(x)})^*$ is defined as

$$\begin{aligned} &\langle \Phi'(u), v\rangle \\ &\quad= \int_\Omega \left(|\nabla u|^{p(x)-2} + \frac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)(\nabla u, \nabla v)_{\mathbb{R}^N}dx, \end{aligned} \tag{23}$$

for all $u, v \in W_0^{1,p(x)}(\Omega)$. From the work of Chang [1], we know that if $w_n \in \partial\Psi(u_n)$, then $w_n \in L^{\alpha'(x)}(\Omega)$, where $1/\alpha'(x) + 1/\alpha(x) = 1$.

Since φ is coercive, $\{u_n\}_{n\geq 1}$ is bounded in $W_0^{1,p(x)}(\Omega)$ and there exists $u \in W_0^{1,p(x)}(\Omega)$ such that a subsequence of $\{u_n\}_{n\geq 1}$, which is still denoted as $\{u_n\}_{n\geq 1}$, satisfies $u_n \rightharpoonup u$ weakly in $W_0^{1,p(x)}(\Omega)$. Next we will prove that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$.

By $W_0^{1,p(x)}(\Omega) \to L^{\alpha(x)}(\Omega)$, we have $u_n \to u$ in $L^{\alpha(x)}(\Omega)$. Moreover, since $\|u_n^*\|_* \to 0$, we get $|\langle u_n^*, u_n\rangle| \leq \varepsilon_n$.

Since $u_n^* = \Phi'(u_n) - \lambda w_n$, we obtain

$$\langle \Phi'(u_n), u_n - u\rangle - \lambda\int_\Omega w_n(u_n - u)\,dx \leq \varepsilon_n, \quad \forall n \geq 1. \tag{24}$$

Moreover, since $u_n \to u$ in $L^{\alpha(x)}(\Omega)$ and $\{w_n\}_{n\geq 1}$ are bounded in $L^{\alpha'(x)}(\Omega)$, where $1/\alpha(x) + 1/\alpha'(x) = 1$, one has $\int_\Omega w_n(u_n - u)dx \to 0$. Therefore,

$$\limsup_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle \leq 0. \tag{25}$$

But we know that Φ' is a mapping of type (S_+) (by Proposition 3). Thus we obtain

$$u_n \longrightarrow u \quad \text{in } W_0^{1,p(x)}(\Omega). \tag{26}$$

Step 4. There exists a $u_1 \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that $\Psi(u_1) > 0$.

By $(\mathbf{h_5})$, for each $x \in \overline{\Omega}$, there is $t_x \in \mathbb{R}$ such that $F(x,t_x) > 0$.

For $x \in \mathbb{R}^N$, denote by N_x a neighborhood of x which is the product of N compact intervals. From $(\mathbf{h_5})$ and $F(x,t) \in C(\overline{\Omega} \times \mathbb{R})$, for any $x_0 \in \overline{\Omega}$, there are $N_{x_0} \subset \mathbb{R}^N$, $t_{x_0} \in \mathbb{R}$ and $\delta_0 > 0$, such that $F(x,t_{x_0}) > \delta_0 > 0$ for all $x \in N_{x_0} \bigcap \overline{\Omega}$.

Since $\Omega \subseteq \mathbb{R}^N$ is bounded, $\overline{\Omega}$ is compact. Then we can find $N_{x_1}, N_{x_2}, \ldots, N_{x_n}$ such that $\Omega \subset \bigcup_{i=1}^n N_{x_i}$ and $N_{x_i} \bigcap N_{x_j} = \partial N_{x_i} \bigcap \partial N_{x_j}$, $(i \neq j)$ and, also, we can find $t_{x_1}, t_{x_2}, \ldots, t_{x_n} \in \mathbb{R}$, and n positive numbers $\delta_1, \delta_2, \ldots, \delta_n$ such that

$$F\left(x, t_{x_i}\right) > \delta_i > 0 \text{ uniformly for } x \in N_{x_i} \bigcap \overline{\Omega}, \quad i = 1, 2, \ldots, n. \tag{27}$$

Now, set $\delta_0 = \min\{\delta_1, \delta_2, \ldots, \delta_n\}$, and $t_0 = \max\{t_{x_1}, t_{x_2}, \ldots, t_{x_n}\}$, and

$$\sup_{|t|<|t_0|;\, x\in\overline{\Omega}} |F(x,t)| = M. \tag{28}$$

Then, we can find a closed set $\Omega_{x_i} \subset \operatorname{int}(N_{x_i} \bigcap \Omega)$ such that

$$\operatorname{meas}\left(\Omega_{x_i}\right) > \frac{M \operatorname{meas}\left(N_{x_i} \bigcap \overline{\Omega}\right)}{\delta_0 + M}, \tag{29}$$

where $\operatorname{meas}(A)$ denote the Lebesgue measure of set A. We consider a function $u_1 \in W_0^{1,p(x)}(\Omega)$ such that $|u_1(x)| \in [0, t_0]$ and $u_1(x) \equiv t_{x_i}$ for all $x \in \Omega_{x_i}$. For instance, we can set $u_1(x) = \sum_{i=1}^n u_1^i(x)$, where $u_1^i \in C_0^\infty(N_{x_i} \bigcap \Omega)$ and

$$u_1^i(x) = \begin{cases} t_{x_i}, & x \in \Omega_{x_i}, \\ 0 \le u_1^i(x) < t_{x_i}, & x \in \left(N_{x_i} \cap \Omega\right) \setminus \Omega_{x_i}. \end{cases} \tag{30}$$

Then, from (27)–(29), we have

$$\begin{aligned}
\Psi(u_1) &= \int_\Omega F(x,u_1)\,dx = \int_{\bigcup_{i=1}^n N_{x_i} \cap \Omega} F(x,u_1)\,dx \\
&= \int_{\bigcup_{i=1}^n \Omega_{x_i}} F(x,u_1)\,dx \\
&\quad + \int_{(\bigcup_{i=1}^n N_{x_i} \cap \Omega) \setminus \bigcup_{i=1}^n \Omega_{x_i}} F(x,u_1)\,dx \\
&\ge \sum_{i=1}^n \delta_i \operatorname{meas}\left(\Omega_{x_i}\right) \\
&\quad - \sum_{i=1}^n M\left[\operatorname{meas}\left(N_{x_i} \bigcap \Omega\right) - \operatorname{meas}\left(\Omega_{x_i}\right)\right] \\
&> \sum_{i=1}^n \left[(\delta_0 + M)\operatorname{meas}\left(\Omega_{x_i}\right) - M \operatorname{meas}\left(N_{x_i} \bigcap \overline{\Omega}\right)\right] \\
&> 0.
\end{aligned} \tag{31}$$

Step 5. We show that Φ, Ψ satisfy conditions (1) and (2) of Theorem 4.

Let $u_0 = 0$; then we can easily find $\Phi(u_0) = \Psi(u_0) = 0$.

From (7) and Proposition 1, we have the following:

if $\|u\| \ge 1$, then

$$\frac{2}{p^+}\|u\|^{p^-} \le \Phi(u) \le \frac{2 + |\Omega|}{p^-}\|u\|^{p^+}; \tag{32}$$

if $\|u\| < 1$, then

$$\frac{2}{p^+}\|u\|^{p^+} \le \Phi(u) \le \frac{2 + |\Omega|}{p^-}. \tag{33}$$

From $(\mathbf{h_4})$, there exist $\eta \in]0,1[$ and $C_3 > 0$ such that

$$F(x,t) \le C_3|t|^{q(x)} \le C_3|t|^{q^-}, \quad \forall t \in [-\eta, \eta], \; x \in \Omega. \tag{34}$$

In view of $(\mathbf{h_3})$, if we put

$$C_4 = \max\left\{C_3, \sup_{\eta \le |t| < 1} \frac{a_1 + a_2|t|^{\alpha^-}}{|t|^{q^-}}, \sup_{|t|\ge 1} \frac{a_1 + a_2|t|^{\alpha^+}}{|t|^{q^-}}\right\}, \tag{35}$$

then we have

$$F(x,t) \le C_4|t|^{q^-}, \quad \forall t \in \mathbb{R}, \; x \in \Omega. \tag{36}$$

Fix r such that $0 < r < 1$. And when $(2/p^+)\max\{\|u\|^{p^-}, \|u\|^{p^+}\} < r < 1$, by Sobolev Embedding Theorem $(W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q^-}(\Omega))$, we have (for suitable positive constants C_5, C_6)

$$\begin{aligned}
\Psi(u) &= \int_\Omega F(x,u)\,dx \le C_4 \int_\Omega |u|^{q^-}dx \le C_5\|u\|^{q^-} \\
&< C_6 r^{q^-/p^-} \left(\text{or} \quad C_6 r^{q^-/p^+}\right).
\end{aligned} \tag{37}$$

Since $q^- > p^+ \ge p^-$, we have

$$\lim_{r \to 0^+} \frac{\sup_{(2/p^+)\max\{\|u\|^{p^-}, \|u\|^{p^+}\} < r} \Psi(u)}{r} = 0. \tag{38}$$

And so, taking into account (32) and (33),

$$\lim_{r \to 0^+} \frac{\sup_{\Phi(u)<r} \Psi(u)}{r} = 0. \tag{39}$$

From Step 4, there exists $u_1 \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that $\Psi(u_1) > 0$. Thanks to (32) and (33), we have

$$0 < \frac{2}{p^+}\max\left\{\|u_1\|^{p^-}, \|u_1\|^{p^+}\right\} \le \Phi(u_1), \tag{40}$$

and so

$$\frac{\Psi(u_1)}{\Phi(u_1)} > 0. \tag{41}$$

By (32), (33), and (39), there exists $r_0 < (2/p^+)\max\{\|u_1\|^{p^-}, \|u_1\|^{p^+}\} \le \Phi(u_1)$ such that, for each $r \in]0, r_0[$,

$$\sup_{\Phi(u)<r} \Psi(u) < r\frac{\Psi(u_1)}{\Phi(u_1)}. \tag{42}$$

By choosing $r \in]0, r_0[$, conditions (1) and (2) requested in Theorem 4 are verified and so the proof is complete. □

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References

[1] K. C. Chang, "Variational methods for nondifferentiable functionals and their applications to partial differential equations," *Journal of Mathematical Analysis and Applications*, vol. 80, no. 1, pp. 102–129, 1981.

[2] N. C. Kourogenis and N. S. Papageorgiou, "Nonsmooth critical point theory and nonlinear elliptic equations at resonance," *Australian Mathematical Society Journal A*, vol. 69, no. 2, pp. 245–271, 2000.

[3] D. Kandilakis, N. C. Kourogenis, and N. S. Papageorgiou, "Two nontrivial critical points for nonsmooth functionals via local linking and applications," *Journal of Global Optimization*, vol. 34, no. 2, pp. 219–244, 2006.

[4] B. Ricceri, "A general variational principle and some of its applications," *Journal of Computational and Applied Mathematics*, vol. 113, no. 1-2, pp. 401–410, 2000.

[5] S. A. Marano and D. Motreanu, "Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian," *Journal of Differential Equations*, vol. 182, no. 1, pp. 108–120, 2002.

[6] M. Ruzicka, *Electrortheological Fluids: Modeling and Mathematical Theory*, Springer, Berlin, 2000.

[7] V. V. Zhikov, "Averaging of functionals of the calculus of variations and elasticity theory," *Mathematics of the USSR-Izvestiya*, vol. 29, no. 1, pp. 33–66, 1987.

[8] X. Fan, "On the sub-supersolution method for $p(x)$-Laplacian equations," *Journal of Mathematical Analysis and Applications*, vol. 330, no. 1, pp. 665–682, 2007.

[9] X. Fan, Q. Zhang, and D. Zhao, "Eigenvalues of $p(x)$-Laplacian Dirichlet problem," *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 306–317, 2005.

[10] X.-L. Fan and Q.-H. Zhang, "Existence of solutions for $p(x)$-Laplacian Dirichlet problem," *Nonlinear Analysis: Theory, Methods and Applications A*, vol. 52, no. 8, pp. 1843–1852, 2003.

[11] X. L. Fan and D. Zhao, "On the generalized Orlicz-sobolev spaces $W^{k,p(x)}(\Omega)$," *Journal of Gansu Education College*, vol. 12, no. 1, pp. 1–6, 1998.

[12] X. Fan and D. Zhao, "On the spaces $L^{p(x)}$ and $W^{m,p(x)}$," *Journal of Mathematical Analysis and Applications*, vol. 263, no. 2, pp. 424–446, 2001.

[13] S. Liu, "Multiple solutions for coercive p-Laplacian equations," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 1, pp. 229–236, 2006.

[14] G. Dai, "Three solutions for a Neumann-type differential inclusion problem involving the $p(x)$-Laplacian," *Nonlinear Analysis: Theory, Methods and Applications A*, vol. 70, no. 10, pp. 3755–3760, 2009.

[15] G. Bonanno and A. Chinnì, "Discontinuous elliptic problems involving the $p(x)$-Laplacian," *Mathematische Nachrichten*, vol. 284, no. 5-6, pp. 639–652, 2011.

[16] G. Bonanno and A. Chinnì, "Multiple solutions for elliptic problems involving the $p(x)$-Laplacian," *Le Matematiche*, vol. 66, no. 1, pp. 105–113, 2011.

[17] G. Bonanno and A. Chinnì, "Existence results of infinitely many solutions for $p(x)$-Laplacian elliptic Dirichlet problems," *Complex Variables and Elliptic Equations*, vol. 57, no. 11, pp. 1233–1246, 2012.

[18] A. Chinnì and R. Livrea, "Multiple solutions for a Neumann-type differential inclusion problem involving the $p(\cdot)$-Laplacian," *Discrete and Continuous Dynamical Systems. Series S*, vol. 5, no. 4, pp. 753–764, 2012.

[19] M. M. Rodrigues, "Multiplicity of solutions on a nonlinear eigenvalue problem for $p(x)$-Laplacian-like operators," *Mediterranean Journal of Mathematics*, vol. 9, no. 1, pp. 211–223, 2012.

[20] S. A. Marano and D. Motreanu, "On a three critical points theorem for non-differentiable functions and applications to nonlinear boundary value problems," *Nonlinear Analysis: Theory, Methods and Applications A*, vol. 48, no. 1, pp. 37–52, 2002.

[21] B. Ricceri, "On a three critical points theorem," *Archiv der Mathematik*, vol. 75, no. 3, pp. 220–226, 2000.

[22] G. Bonanno and N. Giovannelli, "An eigenvalue Dirichlet problem involving the p-Laplacian with discontinuous nonlinearities," *Journal of Mathematical Analysis and Applications*, vol. 308, no. 2, pp. 596–604, 2005.

[23] G. Bonanno and P. Candito, "On a class of nonlinear variational-hemivariational inequalities," *Applicable Analysis*, vol. 83, no. 12, pp. 1229–1244, 2004.

[24] A. Kristály, "Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$," *Journal of Differential Equations*, vol. 220, no. 2, pp. 511–530, 2006.

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Research Article

Blow-Up in a Slow Diffusive p-Laplace Equation with the Neumann Boundary Conditions

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We study a slow diffusive p-Laplace equation in a bounded domain with the Neumann boundary conditions. A natural energy is associated to the equation. It is shown that the solution blows up in finite time with the nonpositive initial energy, based on an energy technique. Furthermore, under some assumptions of initial data, we prove that the solutions with bounded initial energy also blow up.

1. Introduction

In this paper, we consider a slow diffusive p-Laplace equation:

$$\begin{gathered} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^{q-1}u - \fint_\Omega |u|^{q-1}u\,dx, \\ (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega \end{gathered} \tag{1}$$

with $\fint_\Omega u_0 dx = 0$, where Ω is a bounded smooth domain $\Omega \subset \mathbb{R}^N$, $p > 2$, $q > p-1$, and $u_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, $u_0 \not\equiv 0$, and denote $\fint_\Omega f\,dx = (1/|\Omega|)\int_\Omega f\,dx$. It is easy to check that $\int_\Omega u\,dx = 0$; that is, the mass of u is conserved.

The problem (1) with $p = 2$ can be used to model phenomena in population dynamics and biological sciences where the total mass of a chemical or an organism is conserved [1, 2]. If $p > 2$, the problem (1) is the degenerate parabolic equation and appears to be relevant in the theory of non-Newtonian fluids (see [3]). Here, we are mainly interested in the case $p > 2$, namely, the degenerate one. When $p = 2$, (1) becomes the heat equation which has been deeply studied in [4, 5]. When $1 < p < 2$, (1) is singular, which can be handled similar to that of [6].

As an important feature of many evolutionary equations, the properties of blow-up solution have been the subject of intensive study during the last decades. Among those investigations in this area, it was Fujita [7] who first established the so-called theory of critical blow-up exponents for the heat equation with reaction sources in 1966, which can be, of course, regarded as the elegant description for either blow-up or global existence of solutions. From then on, there has been increasing interest in the study of critical Fujita exponents for different kinds of evolutionary equations; see [8, 9] for a survey of the literature. In recent years, special attention has been paid to the blow-up property to nonlinear degenerate or singular diffusion equations with different nonlinear sources, including the inner sources, boundary flux, or multiple sources; see, for example, [3, 10, 11].

In some situations, we have to deal with changing sign solutions. For instance, the changing sign solutions were considered in [1] for the nonlocal and quadratic equation

$$u_t = \Delta u + u^2 - \fint_\Omega u^2 dx \tag{2}$$

with the Neumann boundary condition. The study in [5] for

$$u_t = \Delta u + |u|^p - \fint_\Omega |u|^p dx, \tag{3}$$

a natural generalization of (2), proposed with $1 < p \leq 2$ a global existence result (for small initial data) and a new blow-up criterion (based on the partial maximum principle and a Gamma-convergence argument). The authors also conjectured that the solutions blow up when $p > 2$, which was then proved with a positive answer [4]. The changing sign solutions to the reaction-diffusion equation

$$u_t = \Delta u + f(u, k(t)) \tag{4}$$

were discussed in [2], with such as $f(u,k(t)) = |u|^{p-1}u - k(t)$. The blow-up of solutions was obtained even under the source with $\fint_\Omega f\,dx = 0$. The semilinear parabolic equation [12]

$$u_t = \Delta u + |u|^{p-1}u - \fint_\Omega |u|^{p-1}u\,dx \tag{5}$$

with a homogeneous Neumann's boundary condition is studied. A blow-up result for the changing sign solution with positive initial energy is established. In [6], a fast diffusive p-Laplace equation with the nonlocal source

$$\begin{gathered} u_t - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = |u|^q - \fint_\Omega |u|^q dx, \\ (x,t) \in \Omega \times (0,T), \\ \frac{\partial u}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{6}$$

was considered. The authors showed that a critical blow-up criterion was determined for the changing sign weak solutions, depending on the size of q and the sign of the natural energy associated. The relationship between the finite time blow-up and the nonpositivity of initial energy was discussed, based on an energy technique.

Notice that (1) is degenerate if $p > 2$ at points where $\nabla u = 0$; therefore, there is no classical solution in general. For this, a weak solution for problem (1) is defined as follows.

Definition 1. A function $u \in L^\infty(\Omega \times (0,T)) \cap L^p(0,T, W^{1,p}(\Omega))$ with $u_t \in L^2(\Omega \times (0,T))$ is called a weak solution of (1) if

$$\begin{aligned} &\int_0^t \int_\Omega \left[u\frac{\partial \varphi}{\partial s} - |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi + \left(|u|^q - \fint_\Omega |u|^q\right)\varphi\right] dx\,ds \\ &\quad = \int_\Omega u(x,t)\,\varphi(x,t)\,dx - \int_\Omega u_0(x)\,\varphi(x,0)\,dx \end{aligned} \tag{7}$$

holds for all $\varphi \in C^1(\overline{\Omega} \times [0,T])$.

The local existence of the weak solutions can be obtained via the standard procedure of regularized approximations [10]. Throughout the paper, we always assume that the weak solution is appropriately smooth for convenience of arguments, instead of considering the corresponding regularized problems.

This paper is organized as follows. In Section 2, we show that the solutions to (1) blow up with nonpositive initial energy. In Section 3, under some assumptions of initial data, we prove that the solutions with bounded initial energy also blow up in finite time.

2. Nonpositive Initial Energy Case

The technique used here is the same as in [4]; define the energy functional by

$$E(t) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{q+1}\int_\Omega |u|^{q+1} dx. \tag{8}$$

and denote

$$M(t) = \frac{1}{2}\int_\Omega u^2(x,t)\,dx, \qquad H(t) = \int_0^t M(s)\,ds. \tag{9}$$

Theorem 2. *Assume that $p > 2$, $q > p-1$, and $u_0 \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$, $u_0 \not\equiv 0$, and let the initial energy*

$$E(0) = \frac{1}{p}\int_\Omega |\nabla u_0|^p dx - \frac{1}{q+1}\int_\Omega |u_0|^{q+1} dx \tag{10}$$

be nonpositive. Then, there exists T_0 with $0 < T_0 < \infty$, such that

$$\lim_{t \to T_0} M(t) = +\infty. \tag{11}$$

We need three lemmas for the functionals $E(t)$, $M(t)$, and $H(t)$, respectively.

Lemma 3. *The energy $E(t)$ is a nonincreasing function and*

$$E(t) = E(0) - \int_0^t \int_\Omega (u_t)^2 dx\,ds. \tag{12}$$

Proof. A direct computation using (1) and by parts yields

$$\begin{aligned} \frac{d}{dt}E(t) &= \int_\Omega \left(|\nabla u|^{p-2}\nabla u \cdot \nabla u_t - |u|^{q-1}uu_t\right) dx \\ &= \int_\Omega \left(-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) - |u|^{q-1}u\right) u_t dx \\ &= \int_\Omega \left(-u_t - \fint_\Omega |u|^{q-1}u\,dx\right) u_t dx \\ &= -\int_\Omega (u_t)^2 dx. \end{aligned} \tag{13}$$

Integrate from 0 to t to get (12). □

Lemma 4. *Assume that $p > 2$, $q > p-1$, and $E(0) \leq 0$. Then, $M(t)$ satisfies the following inequality:*

$$M'(t) \geq (q+1)\int_0^t \int_\Omega (u_t)^2 dx\,ds. \tag{14}$$

Proof. An easy computation using (1) and the fact $\int_\Omega u\,dx = 0$ and by parts shows that

$$\begin{aligned} M'(t) &= \int_\Omega u u_t dx \\ &= \int_\Omega u\left(\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{q-1}u - \fint_\Omega |u|^{q-1}u\,dx\right) \\ &= -\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^{q+1} dx \\ &= -(q+1)E(t) + \frac{q+1-p}{p}\int_\Omega |\nabla u|^p dx. \end{aligned} \tag{15}$$

The last equality implies

$$\begin{aligned} M'(t) &\geq -(q+1)E(t) \\ &= -(q+1)E(0) + (q+1)\int_0^t\int_\Omega (u_t)^2 dx\,ds \\ &\geq (q+1)\int_0^t\int_\Omega (u_t)^2 dx\,ds, \end{aligned} \tag{16}$$

because of (12) of Lemma 3 and the assumption $E(0) \leq 0$. □

Lemma 5. *Assume that $p > 2$, $q > p-1$, and $E(0) \leq 0$. Then, $H(t)$ satisfies*

$$\frac{q+1}{2}\left(H'(t) - H'(0)\right)^2 \leq H(t)H''(t). \tag{17}$$

Proof. Note the definition of $M(t)$ and $H(t)$, and a simple calculation shows that

$$\begin{aligned} &H'(t) - H'(0) \\ &\quad= M(t) - M(0) \\ &\quad= \int_0^t M'(s)\,ds = \int_0^t\int_\Omega u u_t dx\,ds \\ &\quad\leq \left(\int_0^t\int_\Omega u^2 dx\,ds\right)^{1/2}\left(\int_0^t\int_\Omega (u_t)^2 dx\,ds\right)^{1/2} \\ &\quad\leq \left(\frac{2}{q+1}\right)^{1/2}(H(t))^{1/2}\left(M'(t)\right)^{1/2} \\ &\quad= \left(\frac{2}{q+1}\right)^{1/2}(H(t))^{1/2}\left(H''(t)\right)^{1/2}. \end{aligned} \tag{18}$$

Furthermore,

$$\begin{aligned} H'(t) - H'(0) &= \int_0^t M'(s)\,ds \\ &\geq (q+1)\,t\int_0^t\int_\Omega (u_t)^2 dx\,ds \geq 0. \end{aligned} \tag{19}$$

Therefore,

$$\frac{q+1}{2}\left(H'(t) - H'(0)\right)^2 \leq H(t)H''(t). \tag{20}$$

□

Proof of Theorem 2. Assume for contradiction that the solution u exists for all $t > 0$. We claim that

$$\int_0^{t_0}\int_\Omega (u_t)^2 dx\,ds > 0 \tag{21}$$

for any $t_0 > 0$. Otherwise, there exists $t_0 > 0$ such that

$$\int_0^{t_0}\int_\Omega (u_t)^2 dx\,ds = 0, \tag{22}$$

and hence $u_t = 0$ for a.e. $(x,t) \in \Omega \times (0, t_0]$. Therefore, noticing $E(t) \leq 0$ by Lemma 3, we have from (15) that

$$\int_\Omega |\nabla u|^p dx = 0 \tag{23}$$

for a.e. $t \in (0, t_0]$. Using the Poincaré inequality with $\int_\Omega u\,dx = 0$, we have $u = 0$ for a.e. $(x,t) \in \Omega \times (0, t_0]$. This contradicts $u_0 \not\equiv 0$.

Integrating (14) from t_0 to t, we have

$$M(t) \geq M(t_0) + (q+1)\int_{t_0}^t\int_0^\tau\int_\Omega (u_t)^2 dx\,ds\,d\tau, \tag{24}$$

which implies that

$$\lim_{t\to\infty} H'(t) = \lim_{t\to\infty} M(t) = +\infty. \tag{25}$$

Thus, there exists $t^* \geq t_0$ such that for all $t \geq t^*$

$$\frac{3q+5}{4}\left(H'(t)\right)^2 \leq (q+1)\left[H'(t) - H'(0)\right]^2. \tag{26}$$

Thus, combining (17), we further have

$$\frac{3q+5}{4}\left(H'(t)\right)^2 \leq 2H(t)H''(t) \tag{27}$$

for all $t \geq t^*$. Now, we consider the function $G(t) = (H(t))^{-((q-1)/4)}$. Combining with the above inequality and a simple calculation shows that

$$\begin{aligned} G''(t) &= \frac{q-1}{4}(H(t))^{(-q-7)/4} \\ &\quad\times\left(\frac{q+3}{4}\left(H'(t)\right)^2 - H(t)H''(t)\right) \\ &\leq -\frac{(q-1)^2}{32}(H(t))^{(-q-7)/4}\left(H'(t)\right)^2 \leq 0 \end{aligned} \tag{28}$$

for all $t \geq t^*$. However, since

$$\lim_{t\to\infty} H(t) = \lim_{t\to\infty} M(t) = \infty, \tag{29}$$

we also have

$$\lim_{t\to\infty} G(t) = 0, \tag{30}$$

which is a contradiction. □

3. Bounded Initial Energy Case

Define

$$W(\Omega)=\left\{u\in W^{1,p}(\Omega)\mid\int_\Omega u\,dx=0\right\}\tag{31}$$

with the norm $\|u\|=(\int_\Omega|\nabla u|^p dx)^{1/p}$. Let B be the optimal constant of the embedding inequality

$$\|u\|_{q+1}\le B\|\nabla u\|_p,\tag{32}$$

where $p-1<q\le(Np/(N-p)_+)-1$. Set

$$\begin{aligned}\alpha_1&=B^{-(q+1)/(q-p+1)},\\ E_1&=\left(\frac{1}{p}-\frac{1}{q+1}\right)B^{-p(q+1)/(q-p+1)}>0.\end{aligned}\tag{33}$$

Theorem 6. *Assume that $p>2$, $p-1<q\le(Np/(N-p)_+)-1$. Let the initial data u_0 satisfying $E(0)\le E_1$ and $\|\nabla u_0\|_p>\alpha_1$. Then, there exists T_1 with $0<T_1<\infty$, such that*

$$\lim_{t\to T_1}M(t)=+\infty.\tag{34}$$

First, we prove the following two Lemmas, similar to the idea in [13].

Lemma 7. *Assume that u is a solution of the system* (1). *If $E(0)<E_1$ and $\|\nabla u_0\|_p>\alpha_1$. Then, there exists a positive constant $\alpha_2>\alpha_1$, such that*

$$\|\nabla u\|_p\ge\alpha_2,\quad \textit{for any } t\ge0,\tag{35}$$

$$\|u\|_{q+1}\ge B\alpha_2,\quad \textit{for any } t\ge0.\tag{36}$$

Proof. Let $\|\nabla u\|_p=\alpha$ and by (32), we have

$$\begin{aligned}E(t)&=\frac{1}{p}\int_\Omega|\nabla u|^p dx-\frac{1}{q+1}\int_\Omega|u|^{q+1}dx\\ &\ge\frac{1}{p}\|\nabla u\|_p^p-\frac{1}{q+1}B^{q+1}\|\nabla u\|_p^{q+1}\\ &=\frac{1}{p}\alpha^p-\frac{1}{q+1}B^{q+1}\alpha^{q+1}.\end{aligned}\tag{37}$$

For convenience, we define

$$g(\alpha)=\frac{1}{p}\alpha^p-\frac{1}{q+1}B^{q+1}\alpha^{q+1}.\tag{38}$$

It is easy to find that g increases if $0<\alpha<\alpha_1$ and decreases if $\alpha>\alpha_1$. Moreover, $g(\alpha)\to-\infty$ as $\alpha\to\infty$ and $g(\alpha_1)=E_1$. Due to $E(0)<E_1$, there exists $\alpha_2>\alpha_1$ such that $g(\alpha_2)=E(0)$. Let $\|\nabla u_0\|_p=\alpha_0$; thus $\alpha_0>\alpha_1$. Then by (37) and (38), we have $g(\alpha_0)\le E(0)=g(\alpha_2)$, which implies that $\alpha_0\ge\alpha_2$. For contradiction to establish (35), we assume that there exists $t_0>0$ such that

$$\alpha_1<\|\nabla u(\cdot,t_0)\|_p<\alpha_2.\tag{39}$$

It follows from (37) and (38) that

$$E(t_0)\ge g\left(\|\nabla u(\cdot,t_0)\|_p\right)>g(\alpha_2)=E(0),\tag{40}$$

which is in contradiction with Lemma 3. Hence, (35) is established.

Next to prove (36),

$$E(t)=\frac{1}{p}\int_\Omega|\nabla u|^p dx-\frac{1}{q+1}\int_\Omega|u|^{q+1}dx\le E(0),\tag{41}$$

which implies that

$$\frac{1}{q+1}\int_\Omega|u|^{q+1}dx\ge\frac{1}{p}\int_\Omega|\nabla u|^p dx-E(0)\ge\frac{1}{p}\alpha_2^p-g(\alpha_2).\tag{42}$$

Therefore, (36) is concluded. □

Define

$$F(t)=E_1-E(t),\quad \text{for any } t\ge0.\tag{43}$$

Then, we have the following.

Lemma 8. *Assume that u is a solution of the system* (1). *If $E(0)<E_1$ and $\|\nabla u_0\|_p>\alpha_1$. Then for all $t\ge0$,*

$$0<F(0)\le F(t)\le\frac{1}{q+1}\int_\Omega|u|^{q+1}dx.\tag{44}$$

Proof. By Lemma 3, we know that $F'(t)\ge0$. Thus,

$$F(t)\ge F(0)=E_1-E(0)>0.\tag{45}$$

According to (35) of Lemma 7, a simple computation shows that

$$\begin{aligned}F(t)&=E_1-\frac{1}{p}\int_\Omega|\nabla u|^p dx+\frac{1}{q+1}\int_\Omega|u|^{q+1}dx\\ &\le E_1-\frac{1}{p}B^{-p(q+1)/(q-p+1)}+\frac{1}{q+1}\int_\Omega|u|^{q+1}dx\\ &=-\frac{1}{q+1}B^{-p(q+1)/(q-p+1)}+\frac{1}{q+1}\int_\Omega|u|^{q+1}dx\\ &\le\frac{1}{q+1}\int_\Omega|u|^{q+1}dx,\end{aligned}\tag{46}$$

which guarantees the conclusion of the lemma. □

At the end, let us finish the proof of Theorem 6.

Proof of Theorem 6. According to (15), we have

$$\begin{aligned}M'(t)&=-\int_\Omega|\nabla u|^p dx+\int_\Omega|u|^{q+1}dx\\ &=\int_\Omega|u|^{q+1}dx-pE(t)-\frac{p}{q+1}\int_\Omega|u|^{q+1}dx\\ &=\frac{q+1-p}{q+1}\int_\Omega|u|^{q+1}dx-pE_1+pF(t).\end{aligned}\tag{47}$$

By using (33) and (36), we obtain

$$\begin{aligned}pE_1 &= \left(1 - \frac{p}{q+1}\right) B^{-p(q+1)/(q+1-p)} \\ &= \frac{\alpha_1^{q+1}}{\alpha_2^{q+1}} \frac{q+1-p}{q+1} B^{q+1} \alpha_2^{q+1} \\ &\le \frac{\alpha_1^{q+1}}{\alpha_2^{q+1}} \frac{q-p+1}{q+1} \int_\Omega |u|^{q+1} dx.\end{aligned} \tag{48}$$

Combining (47) and (48), we get

$$\begin{aligned}M'(t) &\ge \left(1 - \frac{\alpha_1^{q+1}}{\alpha_2^{q+1}}\right) \frac{q+1-p}{q+1} \int_\Omega |u|^{q+1} dx + pF(t) \\ &\ge \left(1 - \frac{\alpha_1^{q+1}}{\alpha_2^{q+1}}\right) \frac{q+1-p}{q+1} |\Omega|^{(1-q)/2} M^{(q+1)/2}.\end{aligned} \tag{49}$$

Since $q > p-1 > 1$, $M(t)$ blows up at a finite time. The proof of Theorem 6 is complete. □

Remark 9 (behavior of the energy $E(t)$). Similar to Theorem 1.3 of [5], it is easy to be proved. Let $p > 2$, $p-1 < q \le (Np/(N-p)_+) - 1$, and let u be a weak solution of (1). If there exists a constant $C_0 > 0$ and a time $T_0' > 0$, such that the solution u exists on $[0, T_0')$ and satisfies $E(t) \ge -C_0$ on $[0, T_0')$, then $F(t)$ is bounded on $[0, T_0')$. Thus, the above result and Theorem 6 reveal that even though the initial energy could be chosen as positive, the energy $E(t)$ needs to become negative at a certain time and then goes to $-\infty$. Otherwise, $E(t)$ has a lower bound on $[0, +\infty)$; thus $F(t)$ is bounded on $[0, +\infty)$. It is in contradiction with Theorem 6.

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References

[1] C. Budd, B. Dold, and A. Stuart, "Blowup in a partial differential equation with conserved first integral," *SIAM Journal on Applied Mathematics*, vol. 53, no. 3, pp. 718–742, 1993.

[2] B. Hu and H.-M. Yin, "Semilinear parabolic equations with prescribed energy," *Rendiconti del Circolo Matematico di Palermo. Serie II*, vol. 44, no. 3, pp. 479–505, 1995.

[3] N. D. Alikakos and L. C. Evans, "Continuity of the gradient for weak solutions of a degenerate parabolic equation," *Journal de Mathématiques Pures et Appliquées. Neuvième Série*, vol. 62, no. 3, pp. 253–268, 1983.

[4] M. Jazar and R. Kiwan, "Blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 25, no. 2, pp. 215–218, 2008.

[5] A. El Soufi, M. Jazar, and R. Monneau, "A gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions," *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire*, vol. 24, no. 1, pp. 17–39, 2007.

[6] C. Y. Qu, X. L. Bai, and S. N. Zheng, "Blow-up and extinction in a nonlocal p-Laplace equation with Neumann boundary conditions," *Annales de l'Institut Henri Poincare (C) Non Linear Analysis*, vol. 25, no. 2, pp. 215–218, 2008.

[7] H. Fujita, "On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$," *Journal of the Faculty of Science. University of Tokyo. Section IA. Mathematics*, vol. 13, pp. 109–124, 1966.

[8] K. Deng and H. A. Levine, "The role of critical exponents in blow-up theorems: the sequel," *Journal of Mathematical Analysis and Applications*, vol. 243, no. 1, pp. 85–126, 2000.

[9] Y. X. Li and C. H. Xie, "Blow-up for p-Laplace parabolic equations," *Electronic Journal of Differential Equations*, vol. 2003, pp. 1–12, 2005.

[10] J. N. Zhao, "Existence and nonexistence of solutions for $u_t = div(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$," *Journal of Mathematical Analysis and Applications*, vol. 172, no. 1, pp. 130–146, 1993.

[11] S. Zheng, X. Song, and Z. Jiang, "Critical Fujita exponents for degenerate parabolic equations coupled via nonlinear boundary flux," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 308–324, 2004.

[12] W. Gao and Y. Han, "Blow-up of a nonlocal semilinear parabolic equation with positive initial energy," *Applied Mathematics Letters*, vol. 24, no. 5, pp. 784–788, 2011.

[13] E. Vitillaro, "Global nonexistence theorems for a class of evolution equations with dissipation," *Archive for Rational Mechanics and Analysis*, vol. 149, no. 2, pp. 155–182, 1999.

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Review Article

A Priori Bounds in L^p and in $W^{2,p}$ for Solutions of Elliptic Equations

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We give an overview on some recent results concerning the study of the Dirichlet problem for second-order linear elliptic partial differential equations in divergence form and with discontinuous coefficients, in unbounded domains. The main theorem consists in an L^p-a priori bound, $P > 1$. Some applications of this bound in the framework of non-variational problems, in a weighted and a non-weighted case, are also given.

1. Introduction

The aim of this work is to give an overview on some recent results dealing with the study of a certain kind of the Dirichlet problem in the framework of unbounded domains. To be more precise, given an unbounded open subset Ω of $\mathbb{R}^n$, $n \geq 2$, we are concerned with the elliptic second-order linear differential operator in variational form

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i} + d_j\right) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c, \tag{1}$$

with coefficients $a_{ij} \in L^\infty(\Omega)$ and with the associated Dirichlet problem

$$\begin{gathered} u \in \overset{\circ}{W}{}^{1,2}(\Omega), \\ Lu = f, \quad f \in W^{-1,2}(\Omega). \end{gathered} \tag{2}$$

As far as we know, were Bottaro and Marina the first to approach this kind of problem who proved, in [1], an existence and uniqueness theorem for the solution of problem (2), for $n \geq 3$, assuming that

$$a_{ij} \in L^\infty(\Omega), \quad i, j = 1, \ldots, n, \tag{3}$$

$$\begin{gathered} b_i, d_i \in L^n(\Omega), \quad i = 1, \ldots, n, \\ c \in L^{n/2}(\Omega) + L^\infty(\Omega), \end{gathered} \tag{4}$$

$$c - \sum_{i=1}^{n} (d_i)_{x_i} \geq \mu, \quad \mu \in \mathbb{R}_+. \tag{5}$$

The study was later on generalized in [2] weakening the hypothesis (4) by considering coefficients b_i, d_i, and c satisfying (4) only locally and for $n \geq 2$. Further improvements have been achieved in [3], for $n \geq 3$, since the b_i, d_i, and c are taken in suitable Morrey type spaces with lower summabilities.

In [1–3], the authors also provide the bound

$$\|u\|_{W^{1,2}(\Omega)} \leq C\|f\|_{W^{-1,2}(\Omega)}, \tag{6}$$

giving explicit description of the dependence of the constant C on the data of the problem.

In two recent works, [4, 5], considering a more regular set Ω and supposing that the lower order terms coefficients are as in [3] for $n \geq 3$ and as in [2] for $n = 2$, we prove that if $f \in L^2(\Omega) \cap L^\infty(\Omega)$, then there exists a constant C, whose dependence is completely described, such that

$$\|u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \tag{7}$$

for any bounded solution u of (2) and for every $p > 2$. This can be done taking into account two different sign hypotheses, namely, (5) and the less common

$$c - \sum_{i=1}^{n} (b_i)_{x_i} \geq \mu, \quad \mu \in \mathbb{R}_+. \tag{8}$$

Successively, in [6], we deepen the study begun in [4, 5] showing that to a bounded datum $f \in L^2(\Omega)$ it corresponds a bounded solution u. This allows us to prove, by means of an approximation argument, that if f belongs to $L^2(\Omega) \cap L^p(\Omega)$, $p > 2$, then the solution is in $L^p(\Omega)$ too and verifies (7). Putting together the two preliminary L^p-estimates, $p > 2$, obtained under the different sign assumptions and adding the further hypothesis that the a_{ij} are also symmetric, by means of a duality argument, we finally obtain (7) for $p > 1$, for each sign hypothesis, assuming no boundedness of the solution and for $f \in L^2(\Omega) \cap L^p(\Omega)$.

To conclude, we provide two applications of our final L^p-bound, $p > 1$, recalling the results of [7, 8] where our estimate plays a fundamental role in the study of certain weighted and non-weighted non-variational problems with leading coefficients satisfying hypotheses of Miranda's type (see [9]). The nodal point in this analysis is the existence of the derivatives of the leading coefficients that allows us to rewrite the involved operator in variational form and avail ourselves of the above-mentioned a priori bound.

Always in the framework of unbounded domains, the study of different variational problems can be found in [10, 11]. Quasilinear elliptic equations with quadratic growth have been considered in [12]. In [13–15] a very general weighted case, with principal coefficients having vanishing mean oscillation, has been taken into account.

2. A Class of Spaces of Morrey Type

In this section we recall the definitions and the main properties of a certain class of spaces of Morrey type where the coefficients of our operators belong. These spaces generalize the classical notion of Morrey spaces to unbounded domains and were introduced for the first time in [3]; see also [16] for some details. Thus, from now on, let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$. By $\Sigma(\Omega)$ we denote the σ-algebra of all Lebesgue measurable subsets of Ω. For $E \in \Sigma(\Omega)$, χ_E is its characteristic function, $|E|$ its Lebesgue measure, and $E(x,r) = E \cap B(x,r)$ $(x \in \mathbb{R}^n,\ r \in \mathbb{R}_+)$, where $B(x,r)$ is the open ball with center in x and radius r. The class of restrictions to $\overline{\Omega}$ of functions $\zeta \in C_\circ^\infty(\mathbb{R}^n)$ is $\mathfrak{D}(\overline{\Omega})$. For $q \in [1,+\infty[$, $L^q_{\text{loc}}(\overline{\Omega})$ is the class of all functions $g : \Omega \to \mathbb{R}$ such that $\zeta\, g \in L^q(\Omega)$ for any $\zeta \in \mathfrak{D}(\overline{\Omega})$.

For $q \in [1,+\infty[$ and $\lambda \in [0,n[$, the space of Morrey type $M^{q,\lambda}(\Omega)$ is made up of all the functions g in $L^q_{\text{loc}}(\overline{\Omega})$ such that

$$\|g\|_{M^{q,\lambda}(\Omega)} = \sup_{\substack{\tau \in]0,1] \\ x \in \Omega}} \tau^{-\lambda/q} \|g\|_{L^q(\Omega(x,\tau))} < +\infty, \tag{9}$$

equipped with the norm defined in (9).

The closures of $C_\circ^\infty(\Omega)$ and $L^\infty(\Omega)$ in $M^{q,\lambda}(\Omega)$ are denoted by $M_\circ^{q,\lambda}(\Omega)$ and $\widetilde{M}^{q,\lambda}(\Omega)$, respectively.

The following inclusion holds true:

$$M_\circ^{q,\lambda}(\Omega) \subset \widetilde{M}^{q,\lambda}(\Omega). \tag{10}$$

Moreover,

$$M^{q,\lambda}(\Omega) \subseteq M^{q_0,\lambda_0}(\Omega) \quad \text{if } q_0 \leq q, \qquad \frac{\lambda_0 - n}{q_0} \leq \frac{\lambda - n}{q}. \tag{11}$$

We put $M^q(\Omega) = M^{q,0}(\Omega)$, $\widetilde{M}^q(\Omega) = \widetilde{M}^{q,0}(\Omega)$, and $M_\circ^q(\Omega) = M_\circ^{q,0}(\Omega)$.

Now, let us define the moduli of continuity of functions belonging to $\widetilde{M}^{q,\lambda}(\Omega)$ or $M_\circ^{q,\lambda}(\Omega)$. For $h \in \mathbb{R}_+$ and $g \in M^{q,\lambda}(\Omega)$, we set

$$F[g](h) = \sup_{\substack{E \in \Sigma(\Omega) \\ \sup_{x \in \Omega} |E(x,1)| \leq 1/h}} \|g\chi_E\|_{M^{q,\lambda}(\Omega)}. \tag{12}$$

Given a function $g \in M^{q,\lambda}(\Omega)$, the following characterizations hold:

$$\begin{gathered} g \in \widetilde{M}^{q,\lambda}(\Omega) \Longleftrightarrow \lim_{h \to +\infty} F[g](h) = 0, \\ g \in M_\circ^{q,\lambda}(\Omega) \\ \Longleftrightarrow \lim_{h \to +\infty} \left(F[g](h) + \|(1-\zeta_h)\, g\|_{M^{q,\lambda}(\Omega)} \right) = 0, \end{gathered} \tag{13}$$

where ζ_h denotes a function of class $C_o^\infty(R^n)$ such that

$$\begin{gathered} 0 \leq \zeta_h \leq 1, \\ \zeta_{h|_{\overline{B(0,h)}}} = 1, \\ \operatorname{supp} \zeta_h \subset B(0,2h). \end{gathered} \tag{14}$$

Thus, if g is a function in $\widetilde{M}^{q,\lambda}(\Omega)$, a *modulus of continuity* of g in $\widetilde{M}^{q,\lambda}(\Omega)$ is a map $\widetilde{\sigma}^{q,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{gathered} F[g](h) \leq \widetilde{\sigma}^{q,\lambda}[g](h), \\ \lim_{h \to +\infty} \widetilde{\sigma}^{q,\lambda}[g](h) = 0. \end{gathered} \tag{15}$$

While if g belongs to $M_o^{q,\lambda}(\Omega)$, a *modulus of continuity* of g in $M_o^{q,\lambda}(\Omega)$ is an application $\sigma_o^{q,\lambda}[g] : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{gathered} F[g](h) + \|(1-\zeta_h)\, g\|_{M^{q,\lambda}(\Omega)} \leq \sigma_o^{q,\lambda}[g](h), \\ \lim_{h \to +\infty} \sigma_o^{q,\lambda}[g](h) = 0. \end{gathered} \tag{16}$$

We finally recall two results of [4, 7], obtained adapting to our needs a more general theorem proved in [17], providing the boundedness and some embedding estimates for the multiplication operator

$$u \longrightarrow gu, \tag{17}$$

where the function g belongs to suitable spaces of Morrey type.

Theorem 1. *If $g \in M^{q,\lambda}(\Omega)$, with $q > 2$ and $\lambda = 0$ if $n = 2$, and $q \in]2, n]$ and $\lambda = n - q$ if $n > 2$, then the operator in* (17) *is bounded from $\overset{\circ}{W}{}^{1,2}(\Omega)$ to $L^2(\Omega)$. Moreover, there exists a constant $C \in \mathbb{R}_+$ such that*

$$\|gu\|_{L^2(\Omega)} \le C\|g\|_{M^{q,\lambda}(\Omega)}\|u\|_{W^{1,2}(\Omega)} \quad \forall u \in \overset{\circ}{W}{}^{1,2}(\Omega), \quad (18)$$

with $C = C(n, q)$.

Let $p > 1$ and $r, t \in [p, +\infty[$. If Ω is an open subset of $\mathbb{R}^n$ having the cone property and $g \in M^r(\Omega)$, with $r > p$ if $p = n$, then the operator in (17) *is bounded from $W^{1,p}(\Omega)$ to $L^p(\Omega)$. Moreover, there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|gu\|_{L^p(\Omega)} \le c\|g\|_{M^r(\Omega)}\|u\|_{W^{1,p}(\Omega)} \quad \forall u \in W^{1,p}(\Omega), \quad (19)$$

with $c = c(\Omega, n, p, r)$.

If $g \in M^t(\Omega)$, with $t > p$ if $p = n/2$, then the operator in (17) *is bounded from $W^{2,p}(\Omega)$ to $L^p(\Omega)$. Moreover, there exists a constant $c' \in \mathbb{R}_+$ such that*

$$\|gu\|_{L^p(\Omega)} \le c'\|g\|_{M^t(\Omega)}\|u\|_{W^{2,p}(\Omega)} \quad \forall u \in W^{2,p}(\Omega) \quad (20)$$

with $c' = c'(\Omega, n, p, t)$.

3. The Variational Problem

Consider, in an unbounded open subset Ω of $\mathbb{R}^n$, $n \ge 2$, the second-order linear differential operator in divergence form

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i} + d_j\right) + \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i} + c. \quad (21)$$

Assume that the leading coefficients satisfy the hypotheses

$$\begin{gathered} a_{ij} \in L^\infty(\Omega), \quad i, j = 1, \dots, n, \\ \exists \nu > 0 : \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge \nu|\xi|^2 \quad \text{a.e. in } \Omega, \ \forall \xi \in \mathbb{R}^n. \end{gathered} \quad (h_1)$$

For the lower order terms coefficients suppose that

$$\begin{gathered} b_i, d_i \in M_o^{2t,\lambda}(\Omega), \quad i = 1, \dots, n, \\ c \in M^{t,\lambda}(\Omega), \\ \text{with } t > 1 \text{ and } \lambda = 0 \text{ if } n = 2, \\ \text{with } t \in \left]1, \frac{n}{2}\right] \text{ and } \lambda = n - 2t \text{ if } n > 2. \end{gathered} \quad (h_2)$$

Furthermore, let one of the following sign assumptions hold true:

$$c - \sum_{i=1}^{n} (d_i)_{x_i} \ge \mu, \quad (h_3)$$

or

$$c - \sum_{i=1}^{n} (b_i)_{x_i} \ge \mu, \quad (h_4)$$

in the distributional sense on Ω, with μ positive constant.

We are interested in the study of the Dirichlet problem

$$\begin{gathered} u \in \overset{\circ}{W}{}^{1,2}(\Omega), \\ Lu = f, \quad f \in W^{-1,2}(\Omega), \end{gathered} \quad (22)$$

(h_1)–(h_3) or (h_1), (h_2), and (h_4) being satisfied.

It is natural to associate to L the bilinear form

$$a(u, v) = \int_\Omega \left(\sum_{i,j=1}^{n}\left(a_{ij}u_{x_i} + d_j u\right)v_{x_j} + \left(\sum_{i=1}^{n} b_i u_{x_i} + cu\right)v\right)dx, \quad (23)$$

$u, v \in \overset{\circ}{W}{}^{1,2}(\Omega)$, and observe that, in view of Theorem 1, the form a is continuous on $\overset{\circ}{W}{}^{1,2}(\Omega) \times \overset{\circ}{W}{}^{1,2}(\Omega)$ and so the operator $L : \overset{\circ}{W}{}^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ is continuous too.

Let us start collecting some preliminary results concerning the existence and uniqueness of the solution of problem (22), as well as some a priori estimates. For the case where assumptions (h_1)–(h_3) are taken into account and for $n = 2$, we refer to [2] while for $n \ge 3$ details can be found in [3]. If (h_1), (h_2), and (h_4) hold true, the results are proved in the more recent [5].

Theorem 2. *Under hypotheses (h_1)–(h_3) (or (h_1), (h_2), and (h_4)), problem* (22) *is uniquely solvable and its solution u satisfies the estimate*

$$\|u\|_{W^{1,2}(\Omega)} \le C\|f\|_{W^{-1,2}(\Omega)}, \quad (24)$$

where C is a constant depending on $n, t, \nu, \mu, \|b_i - d_i\|_{M^{2t,\lambda}(\Omega)}$, $i = 1, \dots, n$.

The next step in our analysis is to achieve an L^p-estimate, $p > 2$, for the solution of (22) (see Theorem 8). This requires some additional hypotheses on the regularity of the set and on the datum f, and some preparatory results that essentially rely on the introduction of certain auxiliary functions u_s, used for the first time by Bottaro and Marina in [1] and employed in the framework of Morrey type spaces in [3]. Let us give their definition and recall some useful properties.

Let $h \in \mathbb{R}_+ \cup \{+\infty\}$ and $k \in \mathbb{R}$, with $0 \le k \le h$. For each $t \in \mathbb{R}$ we set

$$\begin{gathered} G_{kh}(t) = \begin{cases} t - k & \text{if } t > k, \\ 0 & \text{if } -k \le t \le k, \\ t + k & \text{if } t < -k, \end{cases} \quad \text{if } h = +\infty, \\ G_{kh}(t) = G_{k\infty}(t) - G_{h\infty}(t), \quad \text{if } h \in \mathbb{R}_+. \end{gathered} \quad (25)$$

Lemma 3. *Let $g \in M_o^{q,\lambda}(\Omega)$, $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$, and $\varepsilon \in \mathbb{R}_+$. Then there exist $r \in \mathbb{N}$ and $k_1, \dots, k_r \in \mathbb{R}$, with $0 = k_r < k_{r-1} < \cdots < k_1 < k_0 = +\infty$, such that set*

$$u_s = G_{k_s k_{s-1}}(u), \quad s = 1, \dots, r, \quad (26)$$

one has $u_1, \ldots, u_r \in \overset{\circ}{W}{}^{1,2}(\Omega)$ *and*

$$\left\| g\chi_{\operatorname{supp}(u_s)_x} \right\|_{M^{q,\lambda}(\Omega)} \leq \varepsilon, \quad s = 1, \ldots, r, \tag{27}$$

$$|u_s| \leq |u|, \quad s = 1, \ldots, r, \tag{28}$$

$$u_1 + \cdots + u_r = u, \tag{29}$$

$$r \leq c, \tag{30}$$

with $c = c(\varepsilon, q, \|g\|_{M^{q,\lambda}(\Omega)})$ *positive constant.*

In order to prove a fundamental preliminary estimate, obtained for $p > 2$ (see Theorem 7), we need to take products involving the above defined functions u_s as test functions in the variational formulation of our problem (23). To be more precise, in the first set of hypotheses ((h_1)–(h_3)), the test functions needed are $|u|^{p-2}u_s$. The following result ensures that these functions effectively belong to $\overset{\circ}{W}{}^{1,2}(\Omega)$.

Lemma 4. *If* Ω *has the uniform* C^1*-regularity property, then for every* $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$ *and for any* $p \in\,]2, +\infty[$ *one has*

$$|u|^{p-2}u_s \in \overset{\circ}{W}{}^{1,2}(\Omega), \quad s = 1, \ldots, r. \tag{31}$$

Lemma 4, whose rather technical proof can be found in [4], is a generalization of a known result by Stampacchia (see [18], or [19] for details), obtained within the framework of the generalization of the study of certain elliptic equations in divergence form with discontinuous coefficients on a bounded open subset of $\mathbb{R}^n$ to some problems arising for harmonic or subharmonic functions in the theory of potential.

Once achieved (31), always in [4], we could prove the next lemma. Let u_s be the functions of Lemma 3 obtained in correspondence of a given $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$, of $g = \sum_{i=1}^n |b_i - d_i|$ and of a positive real number ε specified in the proof of Lemma 4.1 of [4]. One has the following.

Lemma 5. *Let a be the bilinear form defined in* (23). *If* Ω *has the uniform* C^1*-regularity property, under hypotheses* (h_1)–(h_3), *there exists a constant* $C \in \mathbb{R}_+$ *such that*

$$\int_\Omega |u|^{p-2}\left((u_s)_x^2 + u_s^2\right)dx \leq C\sum_{h=1}^{s} a\left(u, |u|^{p-2}u_h\right), \quad s = 1, \ldots, r, \quad \forall p \in\,]2, +\infty[, \tag{32}$$

where C *depends on* s, ν, μ.

If we consider the second set of hypotheses ((h_1), (h_2), and (h_4)), the test functions required in (23) are the products $|u_s|^{p-2}u_s$, obtained in correspondence of a fixed $u \in \overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$, of $g = \sum_{i=1}^n |d_i - b_i|$ and of a positive real number ε specified in the proof of Lemma 4.1 of [5]. In this last case and if Ω has the uniform C^1-regularity property, a result of [20] applies giving that $|u_s|^{p-2}u_s \in \overset{\circ}{W}{}^{1,2}(\Omega)$, for any $p > 2$, $s = 1, \ldots, r$. Hence, in [5] we could show the result.

Lemma 6. *Let a be the bilinear form in* (23). *If* Ω *has the uniform* C^1*-regularity property, under hypotheses* (h_1), (h_2), *and* (h_4), *there exists a constant* $C \in \mathbb{R}_+$ *such that*

$$\int_\Omega |u_s|^{p-2}\left((u_s)_x^2 + u_s^2\right)dx \leq C\sum_{h=s}^{r} a\left(u, |u_h|^{p-2}u_h\right), \quad s = 1, \ldots, r, \ \forall p \in\,]2, +\infty[, \tag{33}$$

where C *depends on* s, r, ν, μ.

The two lemmas just stated put us in a position to prove the following preliminary L^p-a priori estimate, $p > 2$, in both sets of hypotheses; see also [4, 5]. We stress that here we require that both the datum f and the solution u are bounded.

Theorem 7. *Under hypotheses* (h_1)–(h_3) *or* (h_1), (h_2), *and* (h_4) *and if* Ω *has the uniform* C^1*-regularity property,* f *is in* $L^2(\Omega) \cap L^\infty(\Omega)$ *and the solution* u *of* (22) *is in* $\overset{\circ}{W}{}^{1,2}(\Omega) \cap L^\infty(\Omega)$, *then* $u \in L^p(\Omega)$ *and*

$$\|u\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)}, \quad \forall p \in\,]2, +\infty[, \tag{34}$$

where C *is a constant depending on* $n, t, p, \nu, \mu, \|b_i - d_i\|_{M^{2t,\lambda}(\Omega)}$, $i = 1, \ldots, n$.

Proof. Fix $p \in\,]2, +\infty[$. We provide two different proofs in the cases that hypotheses (h_3) or (h_4) hold true.

Let (h_1)–(h_3) be satisfied. We consider the functions u_s, $s = 1, \ldots, r$, obtained in correspondence of the solution u and of $g = \sum_{i=1}^n |d_1 - b_i|$ and of ε as in Lemma 4.1 of [4]. In view of (29) we get

$$\int_\Omega |u|^{p-2}\left(u_x^2 + u^2\right)dx \leq c_0\int_\Omega |u|^{p-2}\sum_{s=1}^{r}\left((u_s)_x^2 + u_s^2\right)dx, \tag{35}$$

with $c_0 = c_0(r)$.

Hence, (32) entails that

$$\begin{aligned}\int_\Omega |u|^{p-2}\left(u_x^2 + u^2\right)dx &\leq c_0\sum_{s=1}^{r} C_s\sum_{h=1}^{s} a\left(u, |u|^{p-2}u_h\right) \\ &\leq C\sum_{s=1}^{r} a\left(u, |u|^{p-2}u_s\right),\end{aligned} \tag{36}$$

with $C_s = C_s(s, \nu, \mu)$ and $C = C(r, \nu, \mu)$.

From the linearity of a, (29), and (30), we have then

$$\int_\Omega |u|^{p-2}\left(u_x^2 + u^2\right)dx \leq Ca\left(u, |u|^{p-2}u\right), \tag{37}$$

with $C = C(n, t, p, \nu, \mu, \|b_i - d_i\|_{M^{2t,\lambda}(\Omega)})$.

Using this last inequality and Hölder inequality we conclude our proof, since

$$\begin{aligned}\|u\|^p_{L^p(\Omega)} &\le \int_\Omega |u|^{p-2}\left(u_x^2+u^2\right)dx \\ &\le Ca\left(u,|u|^{p-2}u\right) = C\int_\Omega f|u|^{p-2}u\,dx \\ &\le C\int_\Omega |f|\,|u|^{p-1}dx \le C\|f\|_{L^p(\Omega)}\|u\|^{p-1}_{L^p(\Omega)}.\end{aligned} \tag{38}$$

If (h_1), (h_2), and (h_4) hold, we consider again the functions u_s, $s = 1,\dots,r$, obtained in correspondence of the solution u and of g as in the previous case, and of ε as in Lemma 4.1 of [5]. In this second case, easy computations together with (29) give

$$\int_\Omega |u|^p dx \le \bar{c}_0 \sum_{s=1}^{r}\int_\Omega |u_s|^p dx, \tag{39}$$

with $\bar{c}_0 = \bar{c}_0(r,p)$.

Thus, from (33), we deduce that

$$\begin{aligned}\int_\Omega |u|^p dx &\le \bar{c}_0 \sum_{s=1}^{r}\overline{C}_s\sum_{h=s}^{r} a\left(u,|u_h|^{p-2}u_h\right) \\ &\le \bar{c}_1\sum_{s=1}^{r} a\left(u,|u_s|^{p-2}u_s\right),\end{aligned} \tag{40}$$

with $\overline{C}_s = \overline{C}_s(s,r,\nu,\mu)$ and $\bar{c}_1 = \bar{c}_1(r,p,\nu,\mu)$.

Hence, by (28) and Hölder inequality we obtain

$$\begin{aligned}\|u\|^p_{L^p(\Omega)} &\le \bar{c}_1\sum_{s=1}^{r}\int_\Omega f|u_s|^{p-2}u_s dx \\ &\le r\bar{c}_1\int_\Omega |f|\,|u|^{p-1}dx \\ &\le r\bar{c}_1\|f\|_{L^p(\Omega)}\|u\|^{p-1}_{L^p(\Omega)}.\end{aligned} \tag{41}$$

This ends the proof, in view of (30). □

In the later paper [6], estimate (34) has been improved dropping the hypotheses on the boundedness of f and u, by means of the theorem below.

Theorem 8. *Assume that hypotheses (h_1)–(h_3) or (h_1), (h_2), and (h_4) are satisfied. If the set Ω has the uniform C^1-regularity property and the datum $f \in L^2(\Omega)\cap L^p(\Omega)$, for some $p \in\,]2,+\infty[$, then the solution u of problem (22) is in $L^p(\Omega)$ and*

$$\|u\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}, \tag{42}$$

where C is a constant depending on $n, t, p, \nu, \mu, \|b_i - d_i\|_{M^{2t,\lambda}(\Omega)}$, $i = 1,\dots,n$.

The proof, which is different according to hypothesis (h_3) or (h_4), is essentially performed into two steps. In the first step, we show some regularity results, exploiting a technique introduced by Miranda in [21]. Namely, we prove that if $u \in \overset{\circ}{W}{}^{1,2}(\Omega)$ is the solution of (22) with $f \in L^2(\Omega)\cap L^\infty(\Omega)$, then, the datum f being more regular, one also has $u \in L^\infty(\Omega)$. Thus Theorem 7 applies giving that $u \in L^p(\Omega)$ and satisfies (34). The second step consists in considering a datum $f \in L^2(\Omega)\cap L^p(\Omega)$ and then one can conclude by means of some approximation arguments; see also [16].

Finally, in [6], we prove the main result, that is, the claimed L^p-bound, $p > 1$. To this aim, a further assumption on the leading coefficients is required:

$$a_{ij} = a_{ji}, \quad i,j = 1,\dots,n. \tag{h_0}$$

Then one has the following.

Theorem 9. *Assume that hypotheses (h_0)–(h_3) or (h_0), (h_2), and (h_4) are satisfied. If the set Ω has the uniform C^1-regularity property and the datum $f \in L^2(\Omega)\cap L^p(\Omega)$, for some $p \in\,]1,+\infty[$, then the solution u of problem (22) is in $L^p(\Omega)$ and*

$$\|u\|_{L^p(\Omega)} \le C\|f\|_{L^p(\Omega)}, \tag{43}$$

where C is a constant depending on $n, t, p, \nu, \mu, \|b_i - d_i\|_{M^{2t,\lambda}(\Omega)}$, $i = 1,\dots,n$.

Proof. For $p \ge 2$, Theorems 2 and 8 already prove the result. It remains to show it for $1 < p < 2$.

We assume that hypotheses (h_0)–(h_3) hold true. Under hypotheses (h_0), (h_2), and (h_4), a similar argument, with suitable modifications, can be used (we refer the reader to [6] for the details).

Let us define the bilinear form

$$a^*(w,v) = a(v,w), \quad w,v \in \overset{\circ}{W}{}^{1,2}(\Omega). \tag{44}$$

By (h_0) one has

$$a^*(w,v) = \int_\Omega \left(\sum_{i,j=1}^{n}\left(a_{ij}w_{x_i} + b_j w\right)v_{x_j} + \left(\sum_{i=1}^{n} d_i w_{x_i} + cw\right)v\right)dx. \tag{45}$$

Now consider the problem

$$\begin{gathered} w \in \overset{\circ}{W}{}^{1,2}(\Omega), \\ a^*(w,v) = \int_\Omega gv\,dx, \quad g \in L^2(\Omega)\cap L^{p'}(\Omega),\end{gathered} \tag{46}$$

where, since $1 < p < 2$, one gets $p' = p/(p-1) > 2$.

As a consequence of Theorem 2 (in the second set of hypotheses) the solution w of (46) exists and is unique. Furthermore, by Theorem 8 (in the second set of hypotheses) one also has

$$\|w\|_{L^{p'}(\Omega)} \le C\|g\|_{L^{p'}(\Omega)}. \tag{47}$$

Hence, if we denote by u the solution of

$$\begin{gathered} u \in \overset{\circ}{W}{}^{1,2}(\Omega), \\ Lu = f, \quad f \in L^2(\Omega)\cap L^p(\Omega),\end{gathered} \tag{48}$$

which exists and is unique in view of Theorem 2 (in the first set of hypotheses), we obtain

$$\int_\Omega gu\,dx = a^*(w,u) = a(u,w) = \int_\Omega fw\,dx \le \|f\|_{L^p(\Omega)}\|w\|_{L^{p'}(\Omega)} \le C\|f\|_{L^p(\Omega)}\|g\|_{L^{p'}(\Omega)}. \quad (49)$$

Finally, taking $g = |u|^{p-1}\operatorname{sign} u$ in (49), we get the claimed result. □

4. Non-Variational Problems

In this section, we show two applications of our main estimate (43).

To this aim, let $p > 1$ and assume that

$$\Omega \text{ has the uniform } C^{1,1}\text{-regularity property.} \quad (h'_0)$$

Consider, then, the non-variational differential operator

$$\overline{L} = -\sum_{i,j=1}^{n} a_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a, \quad (50)$$

with the following conditions on the leading coefficients:

$$\begin{gathered} a_{ij} = a_{ji} \in L^\infty(\Omega), \quad i,j = 1,\dots,n, \\ \exists \nu > 0 : \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge \nu|\xi|^2 \quad \text{a.e in } \Omega,\ \forall \xi \in \mathbb{R}^n, \\ (a_{ij})_{x_h} \in M_o^{q,\lambda}(\Omega), \quad i,j,h = 1,\dots,n, \text{ with} \\ q > 2, \qquad \lambda = 0 \quad \text{for } n = 2, \\ q \in\ ]2,n[, \qquad \lambda = n - q \quad \text{for } n > 2. \end{gathered} \quad (h'_1)$$

Suppose that the lower order terms are such that

$$\begin{gathered} a_i \in M_o^r(\Omega), \quad i = 1,\dots,n, \text{ with} \\ r > 2 \text{ if } p \le 2, \qquad r = p \text{ if } p > 2 \quad \text{for } n = 2, \\ r \ge p,\ r \ge n, \text{ with } r > p \text{ if } p = n \quad \text{for } n > 2, \end{gathered} \quad (h'_2)$$

$$\begin{gathered} a \in \widetilde{M}^t(\Omega), \text{ with} \\ t = p \quad \text{for } n = 2, \\ t \ge p,\ t \ge \frac{n}{2}, \text{ with } t > p \text{ if } p = \frac{n}{2} \quad \text{for } n > 2, \\ \operatorname*{ess\,inf}_\Omega a = a_0 > 0. \end{gathered} \quad (h'_3)$$

In view of Theorem 1, under the assumptions (h'_0)–(h'_3), the operator $\overline{L} : W^{2,p}(\Omega) \to L^p(\Omega)$ is bounded.

The first application is contained in Theorem 3.2 and Corollary 3.3 of [7] (see also [22] where the case $p = 2$ is considered) and reads as follows.

Theorem 10. *Let $\overline{L}$ be defined in (50). If hypotheses (h'_0)–(h'_3) are satisfied, then there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W^{2,p}(\Omega)} \le c\|\overline{L}u\|_{L^p(\Omega)} \quad \forall u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \quad (51)$$

with $c = c(\Omega, n, \nu, p, r, t, \|a_{ij}\|_{L^\infty(\Omega)}, \sigma_o^{q,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \widetilde{\sigma}^t[a], a_0)$.

Moreover, the problem

$$\begin{gathered} u \in W^{2,p}(\Omega) \cap \overset{\circ}{W}{}^{1,p}(\Omega), \\ \overline{L}u = f, \quad f \in L^p(\Omega) \end{gathered} \quad (52)$$

is uniquely solvable.

The nodal point in achieving these results consists in the existence of the derivatives of the a_{ij}. Indeed, this consents to rewrite the operator $\overline{L}$ in divergence form and exploit (43) in order to obtain an estimate as that in (51) but for more regular functions. Then, one can prove (51) by means of an approximation argument. Estimate (51) immediately takes to the solvability of problem (52) via a straightforward application of the method of continuity along a parameter, see, for instance, [23], and by the already known solvability of an opportune auxiliary problem.

As second application of (43), we obtain, in [8], an analogous of Theorem 10, in a weighted framework. Namely, we consider a weight function ρ^s that is a power of a function ρ of class $C^2(\overline{\Omega})$ such that $\rho : \overline{\Omega} \to \mathbb{R}_+$ and

$$\begin{gathered} \sup_{x\in\Omega}\frac{|\partial^\alpha \rho(x)|}{\rho(x)} < +\infty, \quad \forall |\alpha| \le 2, \\ \lim_{|x|\to+\infty}\left(\rho(x) + \frac{1}{\rho(x)}\right) = +\infty, \\ \lim_{|x|\to+\infty}\frac{\rho_x(x) + \rho_{xx}(x)}{\rho(x)} = 0. \end{gathered} \quad (53)$$

For instance, one can think of ρ as the function

$$\rho(x) = \left(1 + |x|^2\right)^t, \quad t \in \mathbb{R} \setminus \{0\}. \quad (54)$$

For $k \in \mathbb{N}_0$, $p \in [1,+\infty[$ and $s \in \mathbb{R}$, and given ρ satisfying (53), we define the weighted Sobolev space $W_s^{k,p}(\Omega)$ as the space of distributions u on Ω such that

$$\|u\|_{W_s^{k,p}(\Omega)} = \sum_{|\alpha|\le k}\|\rho^s\partial^\alpha u\|_{L^p(\Omega)} < +\infty, \quad (55)$$

endowed with the norm in (55). Furthermore, we denote the closure of $C_\circ^\infty(\Omega)$ in $W_s^{k,p}(\Omega)$ by $\overset{\circ}{W}{}_s^{k,p}(\Omega)$ and put $W_s^{0,p}(\Omega) = L_s^p(\Omega)$.

In Theorems 4.2 and 5.2 of [8] we showed the following.

Theorem 11. *Let $\overline{L}$ be defined in (50). If hypotheses (h'_0)–(h'_3) are satisfied, then there exists a constant $c \in \mathbb{R}_+$ such that*

$$\|u\|_{W_s^{2,p}(\Omega)} \le c\|\overline{L}u\|_{L_s^p(\Omega)} \quad \forall u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,2}(\Omega), \quad (56)$$

with $c = c(\Omega, n, s, \nu, p, r, t, \|a_{ij}\|_{L^\infty(\Omega)}, \|a_i\|_{M^r(\Omega)}, \sigma_o^{q,\lambda}[(a_{ij})_{x_h}], \sigma_o^r[a_i], \widetilde{\sigma}^t[a], a_0)$.

Moreover, the problem

$$u \in W_s^{2,p}(\Omega) \cap \overset{\circ}{W}{}_s^{1,2}(\Omega), \qquad \overline{L}u = f, \quad f \in L_s^p(\Omega) \tag{57}$$

is uniquely solvable.

One of the main tools in the proof of Theorem 11 is given by the existence of a topological isomorphism from $W_s^{k,p}(\Omega)$ to $W^{k,p}(\Omega)$ and from $\overset{\circ}{W}{}_s^{k,p}(\Omega)$ to $\overset{\circ}{W}{}^{k,p}(\Omega)$. This isomorphism consents to deduce by the non-weighted bound in (51) the corresponding weighted estimate in (56), taking into account also the imbedding results of Theorem 1. The existence and uniqueness of the solution of problem (57) follow then, as in the previous case, from a direct application of the method of continuity along a parameter by the solvability of a suitable auxiliary problem.

References

[1] G. Bottaro and M. E. Marina, "Problema di Dirichlet per equazioni ellittiche di tipo variazionale su insiemi non limitati," *Bollettino dell'Unione Matematica Italiana. Serie 4*, vol. 8, pp. 46–56, 1973.

[2] M. Transirico and M. Troisi, "Equazioni ellittiche del secondo ordine a coefficienti discontinui e di tipo variazionale in aperti non limitati," *Bollettino dell'Unione Matematica Italiana. Serie 7*, vol. 2, no. 1, pp. 385–398, 1988.

[3] M. Transirico, M. Troisi, and A. Vitolo, "Spaces of Morrey type and elliptic equations in divergence form on unbounded domains," *Bollettino dell'Unione Matematica Italiana. Serie 7*, vol. 9, no. 1, pp. 153–174, 1995.

[4] S. Monsurrò and M. Transirico, "An L^p-estimate for weak solutions of elliptic equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 376179, 15 pages, 2012.

[5] S. Monsurrò and M. Transirico, "Dirichlet problem for divergence form elliptic equations with discontinuous coefficients," *Boundary Value Problems*, vol. 2012, 20 pages, 2012.

[6] S. Monsurrò and M. Transirico, "A priori bounds in L^p for solutions of elliptic equations in divergence form," *Bulletin des Sciences Mathématiques*. In press.

[7] S. Monsurrò and M. Transirico, "A $W^{2,p}$-estimate for a class of elliptic operators," *International Journal of Pure and Applied Mathematics*, vol. 83, no. 4, pp. 489–499, 2013.

[8] S. Monsurrò and M. Transirico, "A weighted $W^{2,p}$-bound for a class of elliptic operators," *Journal of Inequalities and Applications*, vol. 2013, article 263, 2013.

[9] C. Miranda, "Sulle equazioni ellittiche del secondo ordine di tipo non variazionale, a coefficienti discontinui," *Annali di Matematica Pura ed Applicata*, vol. 63, no. 1, pp. 353–386, 1963.

[10] P.-L. Lions, "Remarques sur les équations linéaires elliptiques du second ordre sous forme divergence dans les domaines non bornés," *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali. Serie VIII*, vol. 78, no. 5, pp. 205–212, 1985.

[11] P.-L. Lions, "Remarques sur les équations linéaires elliptiques du second ordre sous forme divergence dans les domaines non bornés. II," *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali. Serie VIII*, vol. 79, no. 6, pp. 178–183, 1985.

[12] P. Donato and D. Giachetti, "Quasilinear elliptic equations with quadratic growth in unbounded domains," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 10, no. 8, pp. 791–804, 1986.

[13] L. Caso, P. Cavaliere, and M. Transirico, "An existence result for elliptic equations with VMO-coefficients," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 2, pp. 1095–1102, 2007.

[14] S. Boccia, S. Monsurrò, and M. Transirico, "Elliptic equations in weighted Sobolev spaces on unbounded domains," *International Journal of Mathematics and Mathematical Sciences*, vol. 2008, Article ID 582435, 12 pages, 2008.

[15] S. Boccia, S. Monsurrò, and M. Transirico, "Solvability of the Dirichlet problem for elliptic equations in weighted Sobolev spaces on unbounded domains," *Boundary Value Problems*, vol. 2008, Article ID 901503, 13 pages, 2008.

[16] L. Caso, R. D'Ambrosio, and S. Monsurrò, "Some remarks on spaces of Morrey type," *Abstract and Applied Analysis*, vol. 2010, Article ID 242079, 22 pages, 2010.

[17] P. Cavaliere, M. Longobardi, and A. Vitolo, "Imbedding estimates and elliptic equations with discontinuous coefficients in unbounded domains," *Le Matematiche*, vol. 51, no. 1, pp. 87–104, 1996.

[18] G. Stampacchia, "Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus," *Annales de l'Institut Fourier*, vol. 15, pp. 189–258, 1965.

[19] G. Stampacchia, *Èquations Elliptiques du Second Ordre à Coefficients Discontinus*, Séminaire de Mathématiques Supérieures, no. 16 (Été, 1965), Les Presses de l'Université de Montréal, Montreal, Canada, 1966.

[20] L. Caso, P. Cavaliere, and M. Transirico, "Solvability of the Dirichlet problem in $W^{2,p}$ for elliptic equations with discontinuous coefficients in unbounded domains," *Le Matematiche*, vol. 57, no. 2, pp. 287–302, 2002.

[21] C. Miranda, "Alcune osservazioni sulla maggiorazione in L_p^ν delle soluzioni deboli delle equazioni ellittiche del secondo ordine," *Annali di Matematica Pura ed Applicata*, vol. 61, pp. 151–169, 1963.

[22] S. Monsurrò, M. Salvato, and M. Transirico, "$W^{2,2}$ a priori bounds for a class of elliptic operators," *International Journal of Differential Equations*, vol. 2011, Article ID 572824, 17 pages, 2011.

[23] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, vol. 224, Springer, Berlin, Germany, 2nd edition, 1983.

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Research Article

Existence and Controllability Results for Fractional Impulsive Integrodifferential Systems in Banach Spaces

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We firstly study the existence of PC-mild solutions for impulsive fractional semilinear integrodifferential equations and then present controllability results for fractional impulsive integrodifferential systems in Banach spaces. The method we adopt is based on fixed point theorem, semigroup theory, and generalized Bellman inequality. The results obtained in this paper improve and extend some known results. At last, an example is presented to demonstrate the applications of our main results.

1. Introduction

Fractional calculus is an area having a long history whose infancy dates back to three hundred years. However, at the beginning of fractional calculus, it develops slowly due to the disadvantage of technology. In recent decades, as the ancient mathematicians expected, fractional differential equations have been found to be a powerful tool in many fields, such as viscoelasticity, electrochemistry, control, porous media, and electromagnetic. For basic facts about fractional derivative and fractional calculus, one can refer to the books [1–4]. Since the fractional theory has played a very significant role in engineering, science, economy, and many other fields, during the past decades, fractional differential equations have attracted many authors, and there has been a great deal of interest in the solutions of fractional differential equations in analytical and numerical sense (see, e.g., [5–10] and references therein).

On the other hand, the impulsive differential systems are used to describe processes which are subjected to abrupt changes at certain moments [11–13]. The study of dynamical systems with impulsive effects has been an object of intensive investigations. It is well known that controllability is a key topic for control theory. Controllability means that it is possible to steer any initial state of the system to any final state in some finite time using an admissible control. We refer the readers to the survey [14] and the reference therein for controllability of nonlinear systems in Banach spaces. The sufficient controllability conditions for fractional impulsive integrodifferential systems in Banach spaces have already been obtained in [15–18].

Balachandran and Park [17] studied the controllability of fractional integrodifferential systems in Banach spaces without impulse

$$\begin{aligned} \frac{d^q x(t)}{dt^q} &= Ax(t) + f\left(t, x(t), \int_0^t h(t,s,x(s))\,ds\right) \\ &\quad + Bu(t), \quad t \in J = [0,b], \\ x(0) &= x_0 \in \mathbb{X}, \end{aligned} \tag{1}$$

where $0 < q < 1$, the state $x(\cdot)$ takes values in the Banach space $\mathbb{X}$, $f : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, $h : \Delta \times \mathbb{X} \to \mathbb{X}$ are continuous functions, and here $\Delta = \{(t,s) : 0 \le s \le t \le b\}$. The control function $u \in L^2[J,U]$, a Banach space of admissible control functions with U as a Banach space, and $B : U \to \mathbb{X}$ is a bounded linear operator.

In [19], Mophou considered the existence and uniqueness of a mild solution for impulsive fractional semilinear differential equation

$$\begin{aligned} D_t^{\alpha} x(t) &= Ax(t) + f(t, x(t)), \quad t \in I = [0,T], \ t \ne t_k, \\ x(0) &= x_0 \in \mathbb{X}, \\ \Delta x|_{t=t_k} &= I_k\left(x\left(t_k^-\right)\right), \quad k = 1,2,\ldots,m, \end{aligned} \tag{2}$$

where D_t^α is the Caputo fractional derivative, and $0 < \alpha < 1$. The operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a generator of $\mathscr{C}_0$-semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathbb{X}$, and $I_k : \mathbb{X} \to \mathbb{X}$ are impulsive functions.

To consider fractional systems in the infinite dimensional space, the first important step is to define a new concept of the mild solution. Unfortunately, By Hernández et al. [20], we know that the concept of mild solutions used in [15–17, 19], inspired by Jaradat et al. [21], was not suitable for fractional evolution systems at all. Therefore, it is necessary to restudy this interesting and hot topic again.

Recently, in Wang and Zhou [18], a suitable concept of mild solutions was introduced, using Krasnoselskii's fixed point theorem and Sadovskii's fixed point theorem, investigating complete controllability of fractional evolution systems in the infinite dimensional spaces

$$\begin{gathered}
{}^cD_t^q\, x(t) = Ax(t) + f(t, x(t)) + Bu(t), \\
t \in J = [0, b], \\
x(0) = x_0 \in \mathbb{X},
\end{gathered} \tag{3}$$

where ${}^cD_t^q$ is the Caputo fractional derivative of the order $0 < q \leq 1$ with the lower limit zero, the state $x(\cdot)$ takes values in Banach space $\mathbb{X}$, and the control function $u(\cdot)$ is given in $L^2[J, U]$, with U as a Banach space. $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ in $\mathbb{X}$, B is a bounded linear operator from U to $\mathbb{X}$, and $f : J \times \mathbb{X} \to \mathbb{X}$ is given $\mathbb{X}$-value functions. Some sufficient conditions for complete controllability of the previous system were obtained.

Inspired by the work of the previous papers and many known results in [22–24], we study the existence of mild solutions for impulsive fractional semilinear integrodifferential equation

$$\begin{gathered}
D_t^q x(t) = Ax(t) + f(t, x(t), (Hx)(t)), \\
t \in I = [0, b], \ t \neq t_k, \\
x(0) = x_0 \in \mathbb{X}, \\
\Delta x|_{t=t_k} = I_k\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, m,
\end{gathered} \tag{4}$$

where D_t^q is the Caputo fractional derivative, $0 < q < 1$, the state $x(\cdot)$ takes values in Banach space $\mathbb{X}$. $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ of a uniformly bounded operator on $\mathbb{X}$, and A is a bounded linear operator. $f : J \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is given $\mathbb{X}$-value functions, H is defined as

$$(Hx)(t) = \int_0^t h(t, s, x(s))\, ds, \tag{5}$$

where $h : \Delta \times \mathbb{X} \to \mathbb{X}$ are continuous, here $\Delta = \{(t, s) : 0 \leq s \leq t \leq b\}$, $I_k : \mathbb{X} \to \mathbb{X}$ are impulsive functions, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, and $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h\to 0^-} x(t_k + h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively.

We also define a control u and present controllability results for fractional integrodifferential systems in Banach spaces

$$\begin{gathered}
D_t^q x(t) = Ax(t) + f(t, x(t), (Hx)(t)) + Bu(t), \\
t \in I = [0, b], \ t \neq t_k, \\
x(0) = x_0 \in \mathbb{X}, \\
\Delta x|_{t=t_k} = I_k\left(x\left(t_k^-\right)\right), \quad k = 1, 2, \ldots, m,
\end{gathered} \tag{6}$$

where B is a bounded linear operator from U to $\mathbb{X}$, and the control function $u(\cdot)$ is given in $L^2[J, U]$, with U as a Banach space. The method we adopt is based on the ideas in [17–19, 22–24]. Compared with the previous results, this paper has three advantages. Firstly, we add operator H in the nonlinear term f and introduce a suitable concept of mild solutions of (4) and (6). Secondly, we not only study the existence of PC-mild solutions for impulsive fractional semilinear integrodifferential equation (4) but also present controllability results for fractional impulsive integrodifferential systems (6), and the results in [17, 19] could be seen as the special cases. Thirdly, our method avoids the compactness conditions on the semigroup $(T(t))_{t\geq 0}$, and some other hypotheses are more general compared with the previous research (see the conditions (H_1)–(H_3) and (H_5)–(H_8)).

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that are to be used later to prove our main results. In Section 3, the existence of PC-mild solutions for (4) is discussed. In Section 4, by introducing a class of controls, we present the controllability results for fractional impulsive integrodifferential systems (6). In Section 5, an example is given to illustrate the theory.

2. Preliminaries and Lemmas

Let us consider the set of functions $\mathrm{PC}[I, \mathbb{X}] = \{x : I \to \mathbb{X} : x \in C[(t_k, t_{k+1}), \mathbb{X}]$, and there exist $x(t_k^-)$ and $x(t_k^+)$, $k = 0, 1, 2, \ldots, m$ with $x(t_k^-) = x(t_k)\}$. Endowed with the norm $\|x\|_{\mathrm{PC}} = \sup_{t\in I}\|x(t)\|$, it is easy to know that $(\mathrm{PC}[I, \mathbb{X}], \|\cdot\|_{\mathrm{PC}})$ is a Banach space. Throughout this paper, let A be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ of a uniformly bounded operators on $\mathbb{X}$. Let $L_B(\mathbb{X})$ be the Banach space of all linear and bounded operator on $\mathbb{X}$. For a C_0-semigroup $(T(t))_{t\geq 0}$, we set $M_1 = \sup_{t\in I}\|T(t)\|_{L_B(\mathbb{X})}$. For each positive constant r, set $B_r = \{x \in \mathrm{PC}[I, \mathbb{X}] : \|x\| \leq r\}$.

Definition 1. The fractional integral of order γ with the lower limit zero for a function f is defined as

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t - s)^{1-\gamma}} ds, \qquad t > 0, \ \gamma > 0, \tag{7}$$

provided that the right side is point-wise defined on $[0, +\infty)$, where $\Gamma(\cdot)$ is the gamma function.

Definition 2. The Riemann-Liouville derivative of the order γ with the lower limit zero for a function $f : [0, \infty] \to R$ can be written as

$$ {}^{L}D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \times \int_0^t \frac{f(s)}{(t-s)^{1-n+\gamma}} ds, \quad t > 0,\ n-1 < \gamma < n. \tag{8} $$

Definition 3. The Caputo derivative of the order γ for a function $f : [0, \infty] \to R$ can be written as

$$ D^{\gamma} f(t) = {}^{L}D^{\gamma} \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0,\ n-1 < \gamma < n. \tag{9} $$

Remark 4. (1) If $f(t) \in C^n[0, \infty)$, then

$$ D^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1-n+\gamma}} ds = I^{n-r} f^{(n)}(t), \quad t > 0,\ n-1 < \gamma < n. \tag{10} $$

(2) The Caputo derivative of a constant is equal to zero.

(3) If f is an abstract function with values in $\mathbb{X}$, then integrals which appear in Definitions 1, 2, and 3 are taken in Bochner's sense.

Definition 5 (see [22]). A mild solution of the following nonhomogeneous impulsive linear fractional equation of the form

$$ \begin{gathered} D_t^q x(t) = Ax(t) + h(t), \quad t \in I = [0, b],\ 0 < q < 1,\ t \neq t_k, \\ x(0) = x_0 \in \mathbb{X}, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \end{gathered} \tag{11} $$

is given by

$$ x(t) = \begin{cases} \mathcal{T}(t) x_0 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) h(s) ds, & t \in [0, t_1], \\ \mathcal{T}(t) x_0 + \mathcal{T}(t-t_1) I_1(x(t_1^-)) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) h(s) ds, & t \in (t_1, t_2], \\ \vdots & \\ \mathcal{T}(t) x_0 + \sum_{k=1}^{m} \mathcal{T}(t-t_k) I_k(x(t_k^-)) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) h(s) ds, & t \in (t_m, b], \end{cases} \tag{12} $$

where $\mathcal{T}(\cdot)$ and $\mathcal{S}(\cdot)$ are called characteristic solution operators and given by

$$ \begin{gathered} \mathcal{T}(t) = \int_0^\infty \xi_q(\theta) T(t^q \theta) d\theta, \\ \mathcal{S}(t) = q \int_0^\infty \theta \xi_q(\theta) T(t^q \theta) d\theta, \end{gathered} \tag{13} $$

and for $\theta \in (0, \infty)$,

$$ \begin{gathered} \xi_q(\theta) = \frac{1}{q} \theta^{-1-(1/q)} \varpi_q(\theta^{-1/q}) \geq 0, \\ \varpi_q(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-qn-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \end{gathered} \tag{14} $$

where ξ_q is a probability density function defined on $(0, \infty)$; that is,

$$ \xi_q(\theta) \geq 0,\ \theta \in (0, \infty), \quad \int_0^\infty \xi_q(\theta) d\theta = 1. \tag{15} $$

Definition 6. By a PC-mild solution of (4), we mean that a function $x \in \mathrm{PC}[I, \mathbb{X}]$, which satisfies the following integral equation:

$$ x(t) = \begin{cases} \mathcal{T}(t) x_0 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) \times f(s, x(s), (Hx)(s)) ds, & t \in [0, t_1], \\ \mathcal{T}(t) x_0 + \mathcal{T}(t-t_1) I_1(x(t_1^-)) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) \times f(s, x(s), (Hx)(s)) ds, & t \in (t_1, t_2], \\ \vdots & \\ \mathcal{T}(t) x_0 + \sum_{k=1}^{m} \mathcal{T}(t-t_k) I_k(x(t_k^-)) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) \times f(s, x(s), (Hx)(s)) ds, & t \in (t_m, b]. \end{cases} \tag{16} $$

Definition 7. By a PC-mild solution of the system (6), we mean that a function $x \in \mathrm{PC}[I, \mathbb{X}]$, which satisfies the following integral equation:

$$ x(t) = \begin{cases} \mathcal{T}(t) x_0 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) \times [f(s, x(s), (Hx)(s)) + Bu(s)] ds, & t \in [0, t_1], \\ \mathcal{T}(t) x_0 + \mathcal{T}(t-t_1) I_1(x(t_1^-)) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) \times [f(s, x(s), (Hx)(s)) + Bu(s)] ds, & t \in (t_1, t_2], \\ \vdots & \\ \mathcal{T}(t) x_0 + \sum_{k=1}^{m} \mathcal{T}(t-t_k) I_k(x(t_k^-)) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) \times [f(s, x(s), (Hx)(s)) + Bu(s)] ds, & t \in (t_m, b]. \end{cases} \tag{17} $$

Definition 8. The system (6) is said to be controllable on the interval J if, for every $x_0, x_1 \in \mathbb{X}$, there exists a control $u \in L^2(J, U)$ such that a mild solution x of (6) satisfies $x(b) = x_1$.

Definition 9 (see [25]). Let $\mathbb{X}$ be a Banach space, and a one parameter family $T(t)$, $0 \le t < +\infty$, of bounded linear operators from $\mathbb{X}$ to $\mathbb{X}$ is a semigroup of bounded linear operators on $\mathbb{X}$ if

(1) $T(0) = I$ (here, I is the identity operator on $\mathbb{X}$);

(2) $T(t+s) = T(t)T(s)$ for every $t, s \ge 0$ (the semigroup property).

A semigroup of bounded linear operator, $T(t)$, is uniformly continuous if $\lim_{t\downarrow 0}\|T(t) - I\| = 0$.

Lemma 10 (see [25]). *Linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Lemma 11 (see [19]). *Let T be a continuous and compact mapping of a Banach space $\mathbb{X}$ into itself, such that*

$$\{x \in \mathbb{X} : x = \lambda T x \text{ for some } 0 \le \lambda \le 1\} \tag{18}$$

is bounded. Then, T has a fixed point.

Lemma 12. *The operators $\mathcal{T}(t)$ and $\mathcal{S}(t)$ have the following properties.*

(i) *For any fixed $t \ge 0$, $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are linear and bounded operators; that is, for any $x \in \mathbb{X}$,*

$$\|\mathcal{T}(t)x\| \le M_1\|x\|, \qquad \|\mathcal{S}(t)x\| \le \frac{qM_1}{\Gamma(1+q)}\|x\|. \tag{19}$$

(ii) *$\{\mathcal{T}(t), t \ge 0\}$ and $\{\mathcal{S}(t), t \ge 0\}$ are strongly continuous.*

(iii) *$\{\mathcal{T}(t), t \ge 0\}$ and $\{\mathcal{S}(t), t \ge 0\}$ are uniformly continuous; that is, for each fixed $t > 0$, and $\epsilon > 0$, there exists $h > 0$ such that*

$$\begin{aligned} \|\mathcal{T}(t+\epsilon) - \mathcal{T}(t)\| &\le \varepsilon, \quad \text{for } t+\epsilon \ge 0,\ |\epsilon| < h, \\ \|\mathcal{S}(t+\epsilon) - \mathcal{S}(t)\| &\le \varepsilon, \quad \text{for } t+\epsilon \ge 0,\ |\epsilon| < h. \end{aligned} \tag{20}$$

Proof. For the proof of (i) and (ii), the reader can refer to [23, Lemma 2.9] and [24, Lemmas 3.2–3.5]. For each fixed $t > 0$, and $h > \epsilon > 0$, one can obtain

$$\begin{aligned} &\|\mathcal{T}(t+\epsilon) - \mathcal{T}(t)\| \\ &\quad \le \int_0^\infty \xi_q(\theta)\|T((t+\epsilon)^q\theta) - T(t^q\theta)\|\,d\theta \\ &\quad \le M_1\int_0^\infty \xi_q(\theta)\|T((t+\epsilon)^q\theta - t^q\theta) - I\|\,d\theta, \\ &\|\mathcal{S}(t+\epsilon) - \mathcal{S}(t)\| \\ &\quad \le qM_1\int_0^\infty \theta\xi_q(\theta)\|T((t+\epsilon)^q\theta - t^q\theta) - I\|\,d\theta. \end{aligned} \tag{21}$$

Because A is a bounded linear operator, from Lemma 10 and Definition 9, we know that A is the infinitesimal generator of a uniformly continuous semigroup. Thus, by the properties of uniformly continuous semigroup $(T(t))_{t\ge 0}$, we get

$$\begin{aligned} \|\mathcal{T}(t+\epsilon) - \mathcal{T}(t)\| &\le \varepsilon, \\ \|\mathcal{S}(t+\epsilon) - \mathcal{S}(t)\| &\le \varepsilon; \end{aligned} \tag{22}$$

that is, $\{\mathcal{T}(t), t \ge 0\}$ and $\{\mathcal{S}(t), t \ge 0\}$ are uniformly continuous. □

We list here the hypotheses to be used later.

(H_1) $f : I \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is continuous and there exist functions $\mu_1, \mu_2 \in L[I, \mathbb{R}^+]$ such that

$$\begin{aligned} &\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \\ &\quad \le \mu_1(t)\|x_1 - x_2\| + \mu_2(t)\|y_1 - y_2\|, \\ &\quad\quad x_1, x_2, y_1, y_2 \in \mathbb{X}. \end{aligned} \tag{23}$$

(H_2) $h : \Delta \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is continuous and there exist function $\nu_1 \in C[I, \mathbb{R}^+]$ such that

$$\|h(t, s, x_1) - h(t, s, x_2)\| \le \nu_1(t)\|x_1 - x_2\|, \quad x_1, x_2 \in \mathbb{X}. \tag{24}$$

(H_3) There exist $\omega_k \in C[I, \mathbb{R}^+]$ such that

$$\begin{aligned} &\|I_k(x_1) - I_k(x_2)\| \le \omega_k(t)\|x_1 - x_2\|, \\ &x_1, x_2 \in \mathbb{X},\ k = 1, 2, \ldots, m. \end{aligned} \tag{25}$$

(H_4) The function $\Omega_m(t) : I \to \mathbb{R}^+$ is defined by

$$\begin{aligned} \Omega_m(t) = {}& \frac{qM_1}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\left(\mu_1(s) + \nu_1^0 b\mu_2(s)\right)ds \\ &+ m\omega_0 M_1, \end{aligned} \tag{26}$$

where $\nu_1^0 = \max\{\nu_1(t) \mid t \in I\}$, $\omega_0 = \max\{\omega_k(t) t \in I, k = 1, 2, \ldots, m\}$, and $0 < \Omega_m(t) < 1$, $t \in I$.

(H_4') The constants Ω_u and $\Omega_m'(t) : I \to \mathbb{R}^+$ are defined by

$$\begin{aligned} \Omega_u = {}& \frac{qM_1K}{\Gamma(1+q)}\int_0^b (b-s)^{q-1}\left(\mu_1(s) + \nu_1^0 b\mu_2(s)\right)ds \\ &+ \omega_0 m M_1, \\ \Omega_m'(t) = {}& \frac{qM_1}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}\left(\mu_1(s) + \nu_1^0 b\mu_2(s)\right)ds \\ &+ \frac{qM_1\Omega_u}{\Gamma(1+q)}\int_0^t (t-s)^{q-1}ds + \omega_0 m M_1, \end{aligned} \tag{27}$$

and $0 < \Omega_m'(t) < 1$, $t \in I$.

3. Existence of Mild Solutions

Theorem 13. *If the hypotheses* (H_1)*–*(H_4) *are satisfied, then the fractional impulsive integrodifferential equation* (4) *has a unique mild solution* $x \in PC[I, \mathbb{X}]$.

Proof. Define an operator Q on $\mathrm{PC}[I, \mathbb{X}]$ by

$$(Qx)(t) = \begin{cases} \mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \times f(s, x(s), (Hx)(s))\,ds, & t \in [0, t_1], \\ \mathcal{T}(t)x_0 + \mathcal{T}(t-t_1)I_1(x(t_1^-)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \times f(s, x(s), (Hx)(s))\,ds, & t \in (t_1, t_2], \\ \vdots \\ \mathcal{T}(t)x_0 + \sum_{k=1}^m \mathcal{T}(t-t_k)I_k(x(t_k^-)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \times f(s, x(s), (Hx)(s))\,ds, & t \in (t_m, b]. \end{cases} \tag{28}$$

We will show that Q is well defined on $\mathrm{PC}[I, \mathbb{X}]$. For $0 \le \tau < t \le t_1$, applying (28), we obtain

$$\begin{aligned} &\|(Qx)(t) - (Qx)(\tau)\| \\ &\quad \le \|\mathcal{T}(t) - \mathcal{T}(\tau)\| \|x_0\| \\ &\qquad + \Big\| \int_\tau^t (t-s)^{q-1}\mathcal{S}(t-s) f(s, x(s), (Hx)(s))\,ds \\ &\qquad + \int_0^\tau (t-s)^{q-1}\mathcal{S}(t-s) f(s, x(s), (Hx)(s))\,ds \\ &\qquad + \int_0^\tau (t-s)^{q-1}\mathcal{S}(\tau-s) f(s, x(s), (Hx)(s))\,ds \\ &\qquad - \int_0^\tau (t-s)^{q-1}\mathcal{S}(\tau-s) f(s, x(s), (Hx)(s))\,ds \\ &\qquad - \int_0^\tau (\tau-s)^{q-1}\mathcal{S}(\tau-s) f(s, x(s), (Hx)(s))\,ds \Big\| \\ &\quad \le \|\mathcal{T}(t) - \mathcal{T}(\tau)\| \|x_0\| \\ &\qquad + \Big\| \int_\tau^t (t-s)^{q-1}\mathcal{S}(t-s) f(s, x(s), (Hx)(s))\,ds \Big\| \\ &\qquad + \Big\| \int_0^\tau (t-s)^{q-1} [\mathcal{S}(t-s) - \mathcal{S}(\tau-s)] \times f(s, x(s), (Hx)(s))\,ds \Big\| \\ &\qquad + \Big\| \int_0^\tau \left[(t-s)^{q-1} - (\tau-s)^{q-1}\right] \times \mathcal{S}(\tau-s) f(s, x(s), (Hx)(s))\,ds \Big\|. \end{aligned} \tag{29}$$

From the well-known inequality $|t^\sigma - \tau^\sigma| \le (t-\tau)^\sigma$ for $\sigma \in (0,1]$ and $0 < \tau \le t$ and Lemma 12, it is obvious that $\|(Qx)(t) - (Qx)(\tau)\| \to 0$ as $t \to \tau$. Thus, we deduce that $Qx \in C[[0, t_1], \mathbb{X}]$.

For $t_1 < \tau < t \le t_2$, we have

$$\begin{aligned} &\|(Qx)(t) - (Qx)(\tau)\| \\ &\quad \le \|\mathcal{T}(t) - \mathcal{T}(\tau)\| \|x_0\| \\ &\qquad + \|\mathcal{T}(t-t_1) - \mathcal{T}(\tau-t_1)\| \|I_1(x(t_1^-))\| \\ &\qquad + \Big\| \int_\tau^t (t-s)^{q-1}\mathcal{S}(t-s) f(s, x(s), (Hx)(s))\,ds \Big\| \\ &\qquad + \Big\| \int_0^\tau (t-s)^{q-1} [\mathcal{S}(t-s) - \mathcal{S}(\tau-s)] \times f(s, x(s), (Hx)(s))\,ds \Big\| \\ &\qquad + \Big\| \int_0^\tau \left[(t-s)^{q-1} - (\tau-s)^{q-1}\right] \times \mathcal{S}(\tau-s) f(s, x(s), (Hx)(s))\,ds \Big\|. \end{aligned} \tag{30}$$

It is easy to get that, as $t \to \tau$, the right-hand side of the previous inequality tends to zero. Thus, we can deduce that $Qx \in C[(t_1, t_2], \mathbb{X}]$. By repeating the same procedure, we can also obtain that $Qx \in C[(t_2, t_3], \mathbb{X}], \dots, Qx \in C[(t_m, b], \mathbb{X}]$. That is, $Qx \in \mathrm{PC}[I, \mathbb{X}]$.

Take $t \in [0, t_1]$; then,

$$\begin{aligned} &\|(Qx)(t) - (Qy)(t)\| \\ &\quad \le \int_0^t (t-s)^{q-1} \|\mathcal{S}(t-s) \times (f(s, x(s), (Hx)(s)) - f(s, y(s), (Hy)(s)))\|\,ds \\ &\quad \le \frac{qM_1}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \times (\mu_1(s)\|x(s) - y(s)\| + \mu_2(s)\|(Hx)(s) - (Hy)(s)\|)\,ds. \end{aligned} \tag{31}$$

From (H_2) and (H_4), we obtain

$$\begin{aligned} &\|(Qx)(t) - (Qy)(t)\| \\ &\quad \le \frac{qM_1}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \left(\mu_1(s) + \nu_1^0 b \mu_2(s)\right) \times \|x(s) - y(s)\|\,ds. \end{aligned} \tag{32}$$

So we deduce that

$$\begin{aligned} &\|(Qx)(t) - (Qy)(t)\|_{\mathrm{PC}} \\ &\quad \le \frac{qM_1}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \left(\mu_1(s) + \nu_1^0 b \mu_2(s)\right) ds \times \|x - y\|_{\mathrm{PC}}. \end{aligned} \tag{33}$$

In general, for each $t \in (t_i, t_{i+1}]$, $1 \le i \le m$, using the assumptions,

$$\begin{aligned}
&\|(Qx)(t) - (Qy)(t)\|_{PC} \\
&\quad \le \frac{qM_1}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \left(\mu_1(s) + \nu_1^0 b \mu_2(s)\right) ds \|x-y\|_{PC} \\
&\qquad + \left\| \sum_{k=1}^{i} \mathcal{T}(t-t_k) I_k(x(t_k^-)) - \sum_{k=1}^{i} \mathcal{T}(t-t_k) I_k(y(t_k^-)) \right\| \\
&\quad \le \left(\frac{qM_1}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \left(\mu_1(t) + \nu_1^0 b \mu_2(t)\right) ds + i\omega_0 M_1 \right) \|x-y\|_{PC} \\
&\quad \le \Omega_i(t) \|x-y\|_{PC};
\end{aligned} \tag{34}$$

when $i = m$, obviously

$$\begin{aligned}
&\|(Qx)(t) - (Qy)(t)\|_{PC} \\
&\quad \le \left(\frac{qM_1}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \left(\mu_1(s) + \nu_1^0 b \mu_2(s)\right) ds + m\omega_0 M_1 \right) \|x-y\|_{PC} \\
&\quad \le \Omega_m(t) \|x-y\|_{PC}.
\end{aligned} \tag{35}$$

Noting that $\Omega_i(t) \le \Omega_m(t)$, with assumption (H_4) and in the view of the contraction mapping principle, we know that Q has a unique fixed point $x \in PC[I, \mathbb{X}]$; that is,

$$x(t) = \begin{cases}
\mathcal{T}(t)x_0 + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) \times f(s, x(s), (Hx)(s))\, ds, & t \in [0, t_1], \\
\mathcal{T}(t)x_0 + \mathcal{T}(t-t_1) I_1(x(t_1^-)) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) \times f(s, x(s), (Hx)(s))\, ds, & t \in (t_1, t_2], \\
\vdots \\
\mathcal{T}(t)x_0 + \sum_{k=1}^{m} \mathcal{T}(t-t_k) I_k(x(t_k^-)) + \int_0^t (t-s)^{q-1} \mathcal{S}(t-s) \times f(s, x(s), (Hx)(s))\, ds, & t \in (t_m, b],
\end{cases} \tag{36}$$

is a PC-mild solution of (4). □

In order to obtain results by the Schaefer fixed point theorem, let us list the following hypotheses.

(H_5) $f : I \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is continuous and there exist functions $\mu_3, \mu_4, \mu_5 \in L[I, \mathbb{R}^+]$ such that

$$\|f(t, x, y)\| \le \mu_3(t) + \mu_4(t)\|x\| + \mu_5(t)\|y\|, \quad t \in I,\ x, y \in \mathbb{X}. \tag{37}$$

(H_6) $h : \Delta \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is continuous and there exist functions $\nu_2, \nu_3 \in C[I, \mathbb{R}^+]$ such that

$$\|h(t, s, x)\| \le \nu_2(s) + \nu_3(s)\|x\|, \quad x \in \mathbb{X}. \tag{38}$$

(H_7) There exist $\psi_k \in C[I, \mathbb{R}^+]$ such that

$$\|I_k(x)\| \le \psi_k(t)\|x\|, \quad x \in \mathbb{X}. \tag{39}$$

(H_8) For all bounded subsets B_r, the set

$$\Pi_{h,\delta}(t) = \left\{ \mathcal{T}_\delta(t)x_0 + \int_0^{t-h} (t-s)^{q-1} \mathcal{S}_\delta(t-s) F(s)\, ds + \sum_{k=1}^{m} \mathcal{T}_\delta(t-t_k) I_k(x(t_k^-)) : x \in B_r \right\} \tag{40}$$

is relatively compact in $\mathbb{X}$ for arbitrary $h \in (0, t)$ and $\delta > 0$, where

$$\begin{aligned}
\mathcal{T}_\delta(t) &= \int_\delta^\infty \xi_q(\theta) T(t^q \theta)\, d\theta, \\
\mathcal{S}_\delta(t) &= q \int_\delta^\infty \theta \xi_q(\theta) T(t^q \theta)\, d\theta.
\end{aligned} \tag{41}$$

(H_8') For all bounded subsets B_r, the set

$$\Pi'_{h,\delta}(t) = \left\{ \mathcal{T}_\delta(t)x_0 + \int_0^{t-h} (t-s)^{q-1} \mathcal{S}_\delta(t-s) [F(s) + Bu(s)]\, ds + \sum_{k=1}^{m} \mathcal{T}_\delta(t-t_k) I_k(x(t_k^-)) : x \in B_r \right\} \tag{42}$$

is relatively compact in $\mathbb{X}$ for arbitrary $h \in (0, t)$ and $\delta > 0$.

Theorem 14. *If the hypotheses (H_5)–(H_8) are satisfied, the fractional impulsive integrodifferential equation (4) has at least one mild solution $x \in PC[I, \mathbb{X}]$.*

Proof. From Theorem 13, we know that operator Q is defined as follows:

$$(Qx)(t)=\begin{cases}\mathcal{T}(t)x_0+\int_0^t(t-s)^{q-1}\mathcal{S}(t-s)\\ \quad\times f(s,x(s),\\ \quad (Hx)(s))\,ds, & t\in[0,t_1],\\ \mathcal{T}(t)x_0+\mathcal{T}(t-t_1)I_1(x(t_1^-))\\ \quad+\int_0^t(t-s)^{q-1}\mathcal{S}(t-s)\\ \quad\times f(s,x(s),(Hx)(s))\,ds, & t\in(t_1,t_2],\\ \quad\vdots\\ \mathcal{T}(t)x_0+\sum_{k=1}^{m}\mathcal{T}(t-t_k)I_k(x(t_k^-))\\ \quad+\int_0^t(t-s)^{q-1}\mathcal{S}(t-s)\\ \quad\times f(s,x(s),(Hx)(s))\,ds, & t\in(t_m,b].\end{cases}\tag{43}$$

We will prove the results in five steps.

Step 1 (continuity of Q on $(t_i,t_{i+1}]$ $(i=0,1,2,\ldots,m)$). Let $x_n,x\in\mathrm{PC}[I,\mathbb{X}]$ such that $\|x_n-x^*\|_{\mathrm{PC}}\to 0$ $(n\to+\infty)$, and then $r=\sup_n\|x_n\|_{\mathrm{PC}}<\infty$ and $\|x^*\|_{\mathrm{PC}}<r$; for every $t\in(t_i,t_{i+1}]$ $(i=0,1,2,\ldots,m)$, we have

$$\begin{aligned}&\|(Qx_n)(t)-(Qx)(t)\|\\ &\le\frac{qM_1}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}\\ &\qquad\times\|f(s,x_n(s),(Hx_n)(s))\\ &\qquad\quad-f(s,x(s),(Hx)(s))\|\,ds\\ &\quad+\left\|\sum_{k=1}^{m}\mathcal{T}(t-t_k)I_k(x_n(t_k^-))-\sum_{k=1}^{m}\mathcal{T}(t-t_k)I_k(x(t_k^-))\right\|.\end{aligned}\tag{44}$$

Since the functions f and I_k are continuous,

$$\begin{aligned}&f(s,x_n(s),(Hx_n)(s))\longrightarrow f(s,x(s),(Hx)(s)),\\ &I_k(x_n(t_k^-))\longrightarrow I_k(x(t_k^-))\qquad n\longrightarrow\infty.\end{aligned}\tag{45}$$

By conditions (H_5) and (H_6), we know that

$$\begin{aligned}&\|f(s,x_n(s),(Hx_n)(s))-f(s,x(s),(Hx)(s))\|\\ &\le 2\mu_3(s)+\mu_4(s)(\|x\|+\|x_n\|)\\ &\quad+\mu_5(s)(\|Hx\|+\|Hx_n\|)\\ &\le 2\mu_3(s)+2\mu_5(s)\int_0^s\nu_2(\theta)\,d\theta\\ &\quad+\left(\mu_4(s)+\mu_5(s)\int_0^s\nu_3(\theta)\,d\theta\right)(\|x\|+\|x_n\|)\\ &\le 2\mu_3(s)+2\mu_5(s)\int_0^s\nu_2(\theta)\,d\theta\\ &\quad+\left(2\mu_4(s)+2\mu_5(s)\int_0^s\nu_3(\theta)\,d\theta\right)r.\end{aligned}\tag{46}$$

Hence,

$$\begin{aligned}&(t-s)^{q-1}\|f(s,x_n(s),(Hx_n)(s))\\ &\qquad-f(s,x(s),(Hx)(s))\|\in L^1[I,\mathbb{R}^+].\end{aligned}\tag{47}$$

By the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}&\int_0^t(t-s)^{q-1}\|f(s,x_n(s),(Hx_n)(s))\\ &\qquad-f(s,x(s),(Hx)(s))\|\,ds\longrightarrow 0.\end{aligned}\tag{48}$$

It is easy to obtain that

$$\lim_{n\to\infty}\|(Qx_n)(t)-(Qx)(t)\|_{\mathrm{PC}}=0.\tag{49}$$

Thus, Q is continuous on $(t_i,t_{i+1}]$ $(i=0,1,2,\ldots,m)$.

Step 2 (Q maps bounded sets into bounded sets in $\mathrm{PC}[I,\mathbb{X}]$). From (43), we get

$$\begin{aligned}&\|(Qx)(t)\|\\ &\le\|\mathcal{T}(t)x_0\|+\frac{qM_1}{\Gamma(1+q)}\\ &\quad\times\int_0^t(t-s)^{q-1}\|f(s,x(s),(Hx)(s))\|\,ds\\ &\quad+m\|\mathcal{T}(t-t_k)I_k(x(t_k^-))\|,\end{aligned}\tag{50}$$

and we know that

$$\begin{aligned}&\|f(s,x(s),(Hx)(s))\|\\ &\le\mu_3(s)+\mu_5(s)\int_0^s\nu_2(\theta)\,d\theta\\ &\quad+\left(\mu_4(s)+\mu_5(s)\int_0^s\nu_3(\theta)\,d\theta\right)\|x\|\\ &\le\varphi_1(s)+\varphi_2(s)\|x\|.\end{aligned}\tag{51}$$

From (50) and (51), we obtain

$$\begin{aligned}\|(Qx)(t)\|&\le M_1\|x_0\|+mM_1\psi_0\|x\|\\ &\quad+\frac{qb^qM_1}{\Gamma(1+q)}\int_0^t(\varphi_1(s)+\varphi_2(s)\|x\|)\,ds,\end{aligned}\tag{52}$$

where $\psi_0=\max\{\psi_k(t)\mid t\in I,k=1,2,\ldots,m\}$. Thus, for any $x\in B_r=\{x\in\mathrm{PC}[I,\mathbb{X}]:\|x\|_{\mathrm{PC}}\le r\}$,

$$\begin{aligned}&\|(Qx)(t)\|\\ &\le M_1\|x_0\|+\frac{qb^qM_1}{\Gamma(1+q)}\int_0^b\varphi_1(s)\,ds\\ &\quad+\left(\frac{qb^qM_1}{\Gamma(1+q)}\int_0^t\varphi_2(s)\,ds+mM\psi_0\right)r=\gamma_1.\end{aligned}\tag{53}$$

Hence, we deduce that $\|(Qx)(t)\|\le\gamma_1$; that is, Q maps bounded sets to bounded sets in $\mathrm{PC}[I,\mathbb{X}]$.

Step 3. ($Q(B_r)$ is equicontinuous with B_r on $(t_i, t_{i+1}]$ $(i = 0, 1, 2, \ldots, m)$). For any $x \in B_r, t', t'' \in (t_i, t_{i+1}]$ $(i = 0, 1, 2, \ldots, m)$, we obtain

$$\begin{aligned}
&\left\|(Qx)\left(t''\right) - (Qx)\left(t'\right)\right\| \\
&\quad \le \left\|\mathcal{T}\left(t''\right) x_0 - \mathcal{T}\left(t'\right) x_0\right\| \\
&\qquad + \left\|\int_0^{t''} \left(t'' - s\right)^{q-1} \mathcal{S}\left(t'' - s\right) F(s)\, ds \right. \\
&\qquad\quad \left. - \int_0^{t'} \left(t' - s\right)^{q-1} \mathcal{S}\left(t' - s\right) F(s)\, ds\right\| \\
&\qquad + \left\|\sum_{k=1}^{m} \mathcal{T}\left(t'' - t_k\right) I_k\left(x\left(t_k^-\right)\right) \right. \\
&\qquad\quad \left. - \sum_{k=1}^{m} \mathcal{T}\left(t' - t_k\right) I_k\left(x\left(t_k^-\right)\right)\right\|;
\end{aligned} \tag{54}$$

after some elementary computation, we have

$$\begin{aligned}
&\left\|(Qx)\left(t''\right) - (Qx)\left(t'\right)\right\| \\
&\quad \le \left\|\mathcal{T}\left(t''\right) - \mathcal{T}\left(t'\right)\right\| \left\|x_0\right\| \\
&\qquad + \left\|\int_{t'}^{t''} \left(t'' - s\right)^{q-1} \mathcal{S}\left(t'' - s\right) F(s)\, ds\right\| \\
&\qquad + \left\|\int_0^{t'} \left[\left(t'' - s\right)^{q-1} - \left(t' - s\right)^{q-1}\right] \mathcal{S}\left(t'' - s\right) F(s)\, ds\right\| \\
&\qquad + \left\|\int_0^{t'} \left(t' - s\right)^{q-1} \left[\mathcal{S}\left(t'' - s\right) - \mathcal{S}\left(t' - s\right)\right] F(s)\, ds\right\| \\
&\qquad + m \left\|\mathcal{T}\left(t'' - t'\right)\right\| \left\|I_k\left(x\left(t_k^-\right)\right)\right\|.
\end{aligned} \tag{55}$$

Using the fact that $\mathcal{T}(t)$ and $\mathcal{S}(t)$ are uniformly continuous, and the well-known inequality $|t'^{\sigma} - t''^{\sigma}| \le (t'' - t')^{\sigma}$ for $\sigma \in (0, 1]$ and $0 < t' \le t''$, we can conclude that $\lim_{t'' \to t'} \|(Qx)(t'') - (Qx)(t')\| = 0$. Thus $Q(B_r)$ is equicontinuous with B_r on $(t_i, t_{i+1}]$ $(i = 0, 1, 2, \ldots, m)$.

Step 4 (Q maps B_r into a precompact set in $\mathbb{X}$). We define $\Pi = QB_r$ and $\Pi(t) = \{(Qx)(t) : x \in B_r\}$ for $t \in I$. Set

$$\Pi_{h,\delta}(t) = \left\{\left(Q_{h,\delta} x\right)(t) : x \in B_r\right\}, \tag{56}$$

where

$$\begin{aligned}
\Pi_{h,\delta}(t) = \Bigg\{ &\mathcal{T}_{\delta}(t) x_0 + \int_0^{t-h} (t - s)^{q-1} \mathcal{S}_{\delta}(t - s) F(s)\, ds \\
&+ \sum_{k=1}^{m} \mathcal{T}_{\delta}\left(t - t_k\right) I_k\left(x\left(t_k^-\right)\right) : x \in B_r \Bigg\}.
\end{aligned} \tag{57}$$

From Lemma 12(ii)-(iii), (H_8), and the same method used in Theorem 3.2 of [18], we can verify that the set $\Pi(t)$ can be arbitrary approximated by the relatively compact set $\Pi_{h,\delta}(t)$. Thus, $Q(B_r)(t)$ is relatively compact in $\mathbb{X}$.

Step 5 (the set $E = \{x \in \mathrm{PC}[I, \mathbb{X}] : x = \lambda Qx$ for some $0 < \lambda < 1\}$ is bounded). Let $x \in E$, and then

$$x(t) = \begin{cases}
\lambda \mathcal{T}(t) x_0 + \lambda \displaystyle\int_0^t (t - s)^{q-1} \mathcal{S}(t - s) \\
\qquad \times f(s, x(s), (Hx)(s))\, ds, & t \in [0, t_1], \\
\lambda \mathcal{T}(t) x_0 + \lambda \mathcal{T}\left(t - t_1\right) I_1\left(x\left(t_1^-\right)\right) \\
\quad + \lambda \displaystyle\int_0^t (t - s)^{q-1} \mathcal{S}(t - s) \\
\qquad \times f(s, x(s), (Hx)(s))\, ds, & t \in \left(t_1, t_2\right], \\
\quad \vdots \\
\lambda \mathcal{T}(t) x_0 + \lambda \displaystyle\sum_{k=1}^{m} \mathcal{T}\left(t - t_k\right) I_k\left(x\left(t_k^-\right)\right) \\
\quad + \lambda \displaystyle\int_0^t (t - s)^{q-1} \mathcal{S}(t - s) \\
\qquad \times f(s, x(s), (Hx)(s))\, ds, & t \in \left(t_m, b\right].
\end{cases} \tag{58}$$

Similar to the results of (53), we know that

$$\begin{aligned}
\|x(t)\| \le{} & \lambda M_1 \left\|x_0\right\| + \frac{\lambda q b^q M_1}{\Gamma(1 + q)} \int_0^b \varphi_1(s)\, ds \\
& + \lambda \left(\frac{q b^q M_1}{\Gamma(1 + q)} \int_0^t \varphi_2(s)\, ds + m M_1 \psi_0 \right) \|x(t)\|.
\end{aligned} \tag{59}$$

Obviously there exists λ sufficiently small such that $\rho = 1 - \lambda m M_1 \psi_0 > 0$, and then we get

$$\begin{aligned}
\|x(t)\| \le{} & \frac{\lambda M_1}{\rho} \left\|x_0\right\| + \frac{\lambda q b^q M_1}{\rho \Gamma(1 + q)} \int_0^b \varphi_1(s)\, ds \\
& + \frac{\lambda q b^q M_1}{\rho \Gamma(1 + q)} \int_0^t \varphi_2(s) \|x(s)\|\, ds.
\end{aligned} \tag{60}$$

Let

$$\begin{aligned}
N_3 &= \frac{\lambda M_1}{\rho} \left\|x_0\right\| + \frac{\lambda q b^q M_1}{\rho \Gamma(1 + q)} \int_0^b \varphi_1(s)\, ds, \\
f(t) &= \frac{\lambda q b^q M_1}{\rho \Gamma(1 + q)} \int_0^t \varphi_2(s)\, ds.
\end{aligned} \tag{61}$$

It is clear that $f(t)$ is nonnegative continuous function on $[0, +\infty)$, and generalized Bellman inequality implies that

$$\|x(t)\| \le N_3 e^{\int_0^t f(s) ds} \le N_3 e^{\int_0^b f(s) ds} = C_0, \tag{62}$$

where C_0 is a constant. Obviously, the set E is bounded on $(t_i, t_{i+1}]$ $(i = 0, 1, 2, \ldots, m)$. Since Q is continuous and compact, thanks to Schaefer's fixed point Theorem, Q has a fixed point (36) which is a PC-mild solution of (4). □

4. Controllability Results

By introducing a class of controls, we present the controllability results for fractional impulsive integrodifferential systems (6).

(H_9) The linear operator W_i from $L^2[(t_{i-1},t_i],U]$ into $\mathbb{X}$ defined by

$$W_i u = \int_0^{t_i} (t_i - s)^{q-1} \mathcal{S}(t_i - s) Bu(s)\, ds, \quad i = 1,2,\ldots,m,m+1, \tag{63}$$

induces an invertible operator $\widetilde{W}_i^-$ defined on $L^2[(t_{i-1},t_i],U]/\operatorname{Ker} W_i$, and there exists a positive constant $K > 0$ such that $\|B\widetilde{W}_i^-\| \le K$.

Theorem 15. *If the hypotheses* (H_1)–(H_3), (H_4'), *and* (H_9) *are satisfied, then the fractional impulsive integrodifferential system* (6) *is controllable on* I.

Proof. Using the condition (H_9), for an arbitrary function $x(\cdot)$, define the control

$$u(t) = \begin{cases} \widetilde{W}_1^- \Big[x_0 + \dfrac{x_1 - x_0}{m+1} - \mathcal{T}(t_1)x_0 - \displaystyle\int_0^{t_1} (t_1 - s)^{q-1}\mathcal{S}(t_1 - s) \times f(s,x(s),(Hx)(s))\,ds \Big](t), & t \in [0,t_1], \\ \widetilde{W}_2^- \Big[x_0 + \dfrac{2(x_1 - x_0)}{m+1} - \mathcal{T}(t_2)x_0 - \displaystyle\int_0^{t_2} (t_2 - s)^{q-1}\mathcal{S}(t_2 - s) \times f(s,x(s),(Hx)(s))\,ds - \mathcal{T}(t_2 - t_1) I_1(x(t_1^-)) \Big](t), & t \in (t_1,t_2], \\ \vdots & \\ \widetilde{W}_{m+1}^- \Big[x_1 - \mathcal{T}(b)x_0 - \displaystyle\int_0^{b} (b - s)^{q-1}\mathcal{S}(b - s) \times f(s,x(s),(Hx)(s))\,ds - \sum_{k=1}^{m} \mathcal{T}(b - t_k) I_k(x(t_k^-)) \Big](t), & t \in (t_m,b]. \end{cases} \tag{64}$$

Define the operator $Q : \mathrm{PC}[I,\mathbb{X}] \to \mathrm{PC}[I,\mathbb{X}]$, where

$$(Qx)(t) = \begin{cases} \mathcal{T}(t)x_0 + \displaystyle\int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \times [f(s,x(s),(Hx)(s)) + Bu(s)]\,ds, & t \in [0,t_1] \\ \mathcal{T}(t)x_0 + \mathcal{T}(t-t_1)I_1(x(t_1^-)) + \displaystyle\int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \times [f(s,x(s),(Hx)(s)) + Bu(s)]\,ds, & t \in (t_1,t_2], \\ \vdots & \\ \mathcal{T}(t)x_0 + \displaystyle\sum_{k=1}^{m}\mathcal{T}(t-t_k)I_k(x(t_k^-)) + \int_0^t (t-s)^{q-1}\mathcal{S}(t-s) \times [f(s,x(s),(Hx)(s)) + Bu(s)]\,ds, & t \in (t_m,b]. \end{cases} \tag{65}$$

By Theorem 13, we know that Q is well defined, and we will prove that when using the previous control, operator Q has a fixed point. Clearly, this fixed point is a PC-mild solution of the control problem (6) and $x(b) = x_1$; that is, the control we defined steers the system (6) from initial x_0 to x_1 in the time b.

For any $x_1, x_2 \in C[(t_i,t_{i+1}],\mathbb{X}]$ ($i = 0,1,2,\ldots,m$), by conditions (H_1)–(H_3), (H_4'), and (H_9), we get

$$\begin{aligned} \|Bu_1(t) - Bu_2(t)\| &\le \left(\frac{qM_1K}{\Gamma(1+q)} \int_0^b (b-s)^{q-1} \times \left(\mu_1(s) + \nu_1^0 b\mu_2(s)\right) ds + \omega_0 m M_1 \right) \times \|x_1(s) - x_2(s)\|_{\mathrm{PC}} \\ &\le \Omega_u \|x_1(s) - x_2(s)\|_{\mathrm{PC}}, \end{aligned} \tag{66}$$

$$\begin{aligned} \|(Qx_1)(t) - (Qx_2)(t)\| &\le \int_0^t (t-s)^{q-1} \|\mathcal{S}(t-s) \times [f(s,x_1(s),(Hx_1)(s)) - f(s,x_2(s),(Hx_2)(s))]\|\,ds \\ &\quad + \int_0^t (t-s)^{q-1} \times \|\mathcal{S}(t-s)[Bu_1(s) - Bu_2(s)]\|\,ds \\ &\quad + \sum_{k=1}^{m} \|\mathcal{T}(t-t_k)(I_k(x_1(t_k^-)) - I_k(x_2(t_k^-)))\|. \end{aligned} \tag{67}$$

Therefore,

$$\begin{aligned} \|(Qx_1)(t) - (Qx_2)(t)\| &\le \left(\frac{qM_1}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} \left(\mu_1(s) + \nu_1^0 b\mu_2(s)\right) ds + \frac{qM_1\Omega_u}{\Gamma(1+q)} \int_0^t (t-s)^{q-1} ds + \omega_0 m M_1 \right) \\ &\quad \times \|x_1(s) - x_2(s)\|_{\mathrm{PC}} \\ &\le \Omega_m'(t) \|x_1(s) - x_2(s)\|_{\mathrm{PC}}. \end{aligned} \tag{68}$$

Since $0 < \Omega_m'(t) < 1$, then Q is contraction mapping. Any fixed point of Q is a PC-mild solution of (6) which satisfies $x(b) = x_1$. Thus, the system (6) is controllable on I. □

Theorem 16. *If the hypotheses* (H_5)–(H_7), (H_8'), *and* (H_9) *are satisfied, the fractional impulsive integrodifferential system* (6) *is controllable on* I.

Proof. Using the condition (H_9), for an arbitrary function $x(\cdot)$, define the control

$$u(t)=\begin{cases}\widetilde{W}_1^{-}\left[x_0+\dfrac{x_1-x_0}{m+1}-\mathcal{T}(t_1)x_0\right.\\ \qquad -\displaystyle\int_0^{t_1}(t_1-s)^{q-1}\mathcal{S}(t_1-s)\\ \qquad\quad \left.\times f(s,x(s),(Hx)(s))\,ds\right](t), & t\in[0,t_1],\\ \widetilde{W}_2^{-}\left[x_0+\dfrac{2(x_1-x_0)}{m+1}-\mathcal{T}(t_2)x_0\right.\\ \qquad -\displaystyle\int_0^{t_2}(t_2-s)^{q-1}\mathcal{S}(t_2-s)\\ \qquad\quad \times f(s,x(s),(Hx)(s))\,ds\\ \qquad \left.-\mathcal{T}(t_2-t_1)I_1(x(t_1^-))\right](t), & t\in(t_1,t_2],\\ \qquad\vdots\\ \widetilde{W}_{m+1}^{-}\left[x_1-\mathcal{T}(b)x_0\right.\\ \qquad -\displaystyle\int_0^{b}(b-s)^{q-1}\mathcal{S}(b-s)\\ \qquad\quad \times f(s,x(s),(Hx)(s))\,ds\\ \qquad \left.-\displaystyle\sum_{k=1}^{m}\mathcal{T}(b-t_k)I_k(x(t_k^-))\right](t), & t\in(t_m,b].\end{cases}\tag{69}$$

We will prove that when using the previous control, operator Q defined in (65) has a fixed point.

We discuss that in five steps.

Step 1 (continuity of Q on $(t_i,t_{i+1}]$ $(i=0,1,2,\ldots,m)$). Let $x_n,x\in \mathrm{PC}[I,\mathbb{X}]$ such that $\|x_n-x^*\|_{\mathrm{PC}}\to 0$ $(n\to+\infty)$, and then $r=\sup_n\|x_n\|_{\mathrm{PC}}<\infty$ and $\|x^*\|_{\mathrm{PC}}<r$. For every $t\in(t_i,t_{i+1}]$ $(i=0,1,2,\ldots,m)$, we have

$$\begin{aligned}&\left\|Qx_n(t)-Qx(t)\right\|\\&\quad\le\frac{qM_1}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}\\&\qquad\times\left\|f(s,x_n(s),(Hx_n)(s))\right.\\&\qquad\quad\left.-f(s,x(s),(Hx)(s))\right\|ds\\&\quad+\frac{qM_1}{\Gamma(1+q)}\int_0^t(t-s)^{q-1}\left\|Bx_n(s)-Bx(s)\right\|ds\\&\quad+\psi_0M_1\sum_{k=1}^{m}\left\|I_k(x_n(t_k^-))-I_k(x(t_k^-))\right\|.\end{aligned}\tag{70}$$

Since

$$\begin{aligned}&\left\|Bx_n(s)-Bx(s)\right\|\\&\quad\le\left(\frac{qM_1K}{\Gamma(1+q)}\int_0^b(b-s)^{q-1}\right.\\&\qquad\times\left\|f(s,x_n(s),(Hx_n)(s))\right.\\&\qquad\quad\left.\left.-f(s,x(s),(Hx)(s))\right\|ds\right)\\&\quad+\psi_0M\sum_{k=1}^{m}\left\|x_n(t_k^-)-x(t_k^-)\right\|,\end{aligned}\tag{71}$$

by (47), (71), and the Lebesgue dominated convergence theorem, it is easy to know that

$$\lim_{n\to\infty}\left\|(Qx_n)(t)-(Qx)(t)\right\|_{\mathrm{PC}}=0.\tag{72}$$

Consequently, Q is continuous on $(t_i,t_{i+1}]$ $(i=0,1,2,\ldots,m)$.

Step 2. (Q maps bounded sets into bounded sets in $\mathrm{PC}[I,\mathbb{X}]$). Since

$$\begin{aligned}\|Bu(s)\|&\le\left\|B\widetilde{W}_i^{-}\right\|\\&\quad\times\left(\|x_0\|+2\|x_1\|+M_1\|x_0\|+\frac{qM_1}{\Gamma(1+q)}\right.\\&\qquad\times\int_0^b(b-s)^{q-1}\left\|f(s,x(s),(Hx)(s))\right\|ds\\&\qquad\left.+\psi_0M\sum_{k=1}^{m}\left\|x(t_k^-)\right\|\right)\\&\le\left\|B\widetilde{W}_i^{-}\right\|\times\left(\frac{qM_1}{\Gamma(1+q)}\int_0^b(b-s)^{q-1}\varphi_1(s)\,ds\,\|x_0\|\right.\\&\qquad+2\|x_1\|+M_1\|x_0\|+\frac{qM_1\|x\|}{\Gamma(1+q)}\\&\qquad\times\int_0^b(b-s)^{q-1}\varphi_2(s)\,ds\\&\qquad\left.+\psi_0M\sum_{k=1}^{m}\left\|x(t_k^-)\right\|\right)\\&\le N_1+N_2\|x\|,\end{aligned}\tag{73}$$

thus, from (65), we get, for any $x\in B_r=\{x\in\mathrm{PC}[I,\mathbb{X}]:\|x\|_{\mathrm{PC}}\le r\}$,

$$\begin{aligned}&\|(Qx)(t)\|\\&\quad\le M_1\|x_0\|+\frac{qb^qM_1}{\Gamma(1+q)}\int_0^b(\varphi_1(s)+N_1)\,ds\\&\quad+\left(\frac{qb^qM_1}{\Gamma(1+q)}\int_0^b(\varphi_2(s)+N_2)\,ds+mM\psi_0\right)r=\gamma_2.\end{aligned}\tag{74}$$

Hence, we deduce that $\|(Qx)(t)\|\le\gamma_2$; that is, Q maps bounded sets to bounded sets in $\mathrm{PC}[I,\mathbb{X}]$. Using the same method used in Theorem 14, we can verify that $Q(B_r)$ is equicontinuous with B_r on $(t_i,t_{i+1}]$ $(i=0,1,2,\ldots,m)$, Q maps B_r into a precompact set in $\mathbb{X}$, and $Q(B_r)(t)$ is relatively compact in $\mathbb{X}$. Steps 3 and 4 are omitted.

Step 5 (the set $E=\{x\in\mathrm{PC}[I,\mathbb{X}]:x=\lambda Qx$ for some $0<\lambda<1\}$ is bounded). Let $x\in E$, and similar to the results (74) we know that

$$\begin{aligned}\|x(t)\| \le{}& \lambda M_1 \|x_0\| \\ &+ \frac{\lambda q b^q M_1}{\Gamma(1+q)} \int_0^t (\varphi_1(s) + N_1)\, ds \\ &+ \left(\frac{\lambda q b^q M_1}{\Gamma(1+q)} \int_0^t (\varphi_2(s) + N_2)\, ds + \lambda m M \psi_0\right) \\ &\times \|x(t)\|.\end{aligned} \tag{75}$$

There exists a λ sufficiently small such that $\rho_2 = 1 - \lambda m M \psi_0 > 0$, and then

$$\begin{aligned}\|x(t)\| \le{}& \frac{\lambda M_1}{\rho_2} \|x_0\| + \frac{\lambda q b^q M_1}{\rho_2 \Gamma(1+q)} \int_0^b (\varphi_1(s) + N_1)\, ds \\ &+ \frac{\lambda q b^q M_1}{\rho_2 \Gamma(1+q)} \int_0^t (\varphi_1(s) + N_1) \|x(s)\|\, ds.\end{aligned} \tag{76}$$

Let

$$\begin{aligned}N_4 &= \frac{\lambda M_1}{\rho_2} \|x_0\| + \frac{\lambda q b^q M_1}{\rho_2 \Gamma(1+q)} \int_0^b (\varphi_1(s) + N_1)\, ds, \\ f(s) &= \frac{\lambda q b^q M_1}{\rho_2 \Gamma(1+q)} \int_0^t (\varphi_2(s) + N_2)\, ds.\end{aligned} \tag{77}$$

It is clear that $f(s)$ is nonnegative continuous function on $[0, +\infty)$, and generalized Bellman inequality implies that

$$\|x(t)\| \le N_4 e^{\int_0^t f(s) ds} \le N_4 e^{\int_0^b f(s) ds} = C_1, \tag{78}$$

where C_1 is a constant. Thus the set E is bounded. Since Q is continuous and compact, thanks to Schaefer's fixed point Theorem, Q has a fixed point (36), and this fixed point is a PC-mild solution of (6) which satisfies $x(b) = x_1$. Hence, the system (6) is controllable on I. □

5. An Example

Consider the following nonlinear partial integrodifferential equation of the form

$$\begin{aligned}\frac{\partial^{2/3}}{\partial t^{2/3}} z(t,y) &= \int_0^1 (y-s) z(s,y)\, ds \\ &\quad + f(t, z(t,y), Hz(t,y)) \\ &\quad + \mu(t,y), \quad t \in J = [0,1], \\ z(t,0) &= z(t,1) = 0, \\ z(0,y) &= 0, \quad 0 < y < 1, \\ \Delta z|_{t=1/2} &= I_1\left(z\left(\frac{1}{2}^-, y\right)\right),\end{aligned} \tag{79}$$

where $0 < q < 1$, $\mu : J \times (0,1) \to (0,1)$ is continuous. Let us take $\mathbb{X} = C([0,1])$. Consider the operator $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ defined by

$$(Aw)(t) = \int_0^1 (y-s) w(s)\, ds. \tag{80}$$

It is easy to get

$$\|Aw\| = \|w\| \int_0^1 |y-s|\, ds = \left(\frac{1}{2} - y(1-y)\right) \|w\| \le \frac{1}{2} \|w\|; \tag{81}$$

clearly A is the infinitesimal generator of a uniformly continuous semigroup $(T(t))_{t\ge 0}$ on $\mathbb{X}$. Put $x(t) = z(t,\cdot)$ and $u(t) = \mu(t,\cdot)$, and take

$$\begin{aligned}f(t,x,Hx) &= e^t + a(t)\left(\frac{\|x\|}{1+\|x\|}\right) \\ &\quad + \int_0^t k(t,s)\left(\frac{\|x\|}{1+\|x\|}\right) ds, \\ I_1(x) &= \|x\|,\end{aligned} \tag{82}$$

where $a(t) \in C[0,1]$, $k(t,s) \in C([0,1] \times [0,1])$. Then clearly, $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $I_1 : \mathbb{R} \to \mathbb{R}$ are continuous functions. f, I_1, and h satisfy (H_5)–(H_8'), respectively. Equations (79) are an abstract formulation of (6). For $y \in (0,1)$, we define

$$\begin{aligned}W_1 u &= \int_0^{1/2} \left(\frac{1}{2} - s\right)^{-1/3} \mathcal{S}\left(\frac{1}{2} - s\right) Bu(s)\, ds, \\ W_2 u &= \int_0^1 (1-s)^{-1/3} \mathcal{S}(1-s) Bu(s)\, ds,\end{aligned} \tag{83}$$

where

$$\begin{aligned}\mathcal{T}(t) w(s) &= \int_0^\infty \xi_{2/3}(\theta) w\left(t^{2/3}\theta + s\right) d\theta, \\ \mathcal{S}(t) w(s) &= \frac{2}{3} \int_0^\infty \theta \xi_{2/3}(\theta) w\left(t^{2/3}\theta + s\right) d\theta,\end{aligned} \tag{84}$$

and for $\theta \in (0,\infty)$,

$$\begin{aligned}\xi_{2/3}(\theta) &= \frac{3}{2} \theta^{-5/2} \varpi_{2/3}\left(\theta^{-3/2}\right), \\ \varpi_{2/3}(\theta) &= \frac{1}{\pi} \sum_{n=1}^\infty (-1)^{n-1} \theta^{-((2n+3)/3)} \frac{\Gamma((2n+3)/3)}{n!} \sin\left(\frac{2n\pi}{3}\right).\end{aligned} \tag{85}$$

Assume that the linear operator W_i from $L^2[(t_{i-1}, t_i], U]$ $(i = 1,2)$ into $\mathbb{X}$ induces an invertible operator $\widetilde{W_i}^-$ defined on $L^2[(t_{i-1}, t_i], U] / \operatorname{Ker} W_i$ and there exists a positive constant $K > 0$ such that $\|B\widetilde{W_i}^-\| \le K$. Moreover, (H_9) is satisfied. All conditions of Theorem 16 are now fulfilled, so we deduce that (79) is controllable on I. On the other hand, we have

$$\begin{aligned}&\|f(t,x,Hx) - f(t,y,Hy)\| \\ &\quad \le a(t)\|x-y\| + \int_0^t k(t,s)\|x-y\|\, ds, \\ &k(t,s)\|x\| - k(t,s)\|y\| \le k_0(\|x\| - \|y\|), \\ &k_0 = \max\{k(t,s) \mid (t,s) \in I \times I\}, \\ &\|I_1(x) - I_1(y)\| \le \|x-y\|.\end{aligned} \tag{86}$$

Further, other conditions (H_1)–(H_3) are satisfied and it is possible to choose $a(t)$, $k(t,s)$ in such a way that condition (H_4') is satisfied. Hence, by Theorem 15, the system (79) is controllable on I.

References

[1] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier Science B.V., Amsterdam, The Netherlands, 2006.

[3] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 1993.

[4] V. Lakshmikantham, S. Leela, and J. Vasundhara Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cambridge, UK, 2009.

[5] Z.-W. Lv, J. Liang, and T.-J. Xiao, "Solutions to the Cauchy problem for differential equations in Banach spaces with fractional order," *Computers & Mathematics with Applications*, vol. 62, no. 3, pp. 1303–1311, 2011.

[6] C. Kou, H. Zhou, and Y. Yan, "Existence of solutions of initial value problems for nonlinear fractional differential equations on the half-axis," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 17, pp. 5975–5986, 2011.

[7] X. Su, "Solutions to boundary value problem of fractional order on unbounded domains in a Banach space," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 8, pp. 2844–2852, 2011.

[8] N. Kosmatov, "Integral equations and initial value problems for nonlinear differential equations of fractional order," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 7, pp. 2521–2529, 2009.

[9] S. Liang and J. Zhang, "Positive solutions for boundary value problems of nonlinear fractional differential equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 11, pp. 5545–5550, 2009.

[10] Y. Wang, L. Liu, and Y. Wu, "Positive solutions for a nonlocal fractional differential equation," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 11, pp. 3599–3605, 2011.

[11] D. Baĭnov and P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, vol. 66 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*, Longman Scientific & Technical, Harlow, UK, 1993.

[12] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol. 6 of *Series in Modern Applied Mathematics*, World Scientific Publishing, Teaneck, NJ, USA, 1989.

[13] M. Benchohra, J. Henderson, and S. Ntouyas, *Impulsive Differential Equations and Inclusions*, vol. 2 of *Contemporary Mathematics and Its Applications*, Hindawi Publishing Corporation, New York, NY, USA, 2006.

[14] K. Balachandran and J. P. Dauer, "Controllability of nonlinear systems in Banach spaces: a survey," *Journal of Optimization Theory and Applications*, vol. 115, no. 1, pp. 7–28, 2002.

[15] Z. Tai and X. Wang, "Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces," *Applied Mathematics Letters*, vol. 22, no. 11, pp. 1760–1765, 2009.

[16] Z. Yan, "Controllability of fractional-order partial neutral functional integrodifferential inclusions with infinite delay," *Journal of the Franklin Institute*, vol. 348, no. 8, pp. 2156–2173, 2011.

[17] K. Balachandran and J. Y. Park, "Controllability of fractional integrodifferential systems in Banach spaces," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 363–367, 2009.

[18] J. Wang and Y. Zhou, "Complete controllability of fractional evolution systems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 11, pp. 4346–4355, 2012.

[19] G. M. Mophou, "Existence and uniqueness of mild solutions to impulsive fractional differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1604–1615, 2010.

[20] E. Hernández, D. O'Regan, and K. Balachandran, "On recent developments in the theory of abstract differential equations with fractional derivatives," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 10, pp. 3462–3471, 2010.

[21] O. K. Jaradat, A. Al-Omari, and S. Momani, "Existence of the mild solution for fractional semilinear initial value problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 9, pp. 3153–3159, 2008.

[22] J. Wang, M. Fečkan, and Y. Zhou, "On the new concept of solutions and existence results for impulsive fractional evolution equations," *Dynamics of Partial Differential Equations*, vol. 8, no. 4, pp. 345–361, 2011.

[23] J. Wang and Y. Zhou, "A class of fractional evolution equations and optimal controls," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 262–272, 2011.

[24] Y. Zhou and F. Jiao, "Existence of mild solutions for fractional neutral evolution equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1063–1077, 2010.

[25] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.

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Research Article

Ψ-Stability of Nonlinear Volterra Integro-Differential Systems with Time Delay

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We give some sufficient conditions for Ψ-uniform stability of the trivial solutions of a nonlinear differential system and of nonlinear Volterra integro-differential systems with time delay.

1. Introduction

Akinyele [1] introduced the notion of Ψ-stability of the degree k with respect to a function $\Psi \in C(R_+\text{-}R_+)$, increasing and differentiable on R and such that $\Psi(t) \geq 1$ for $t \geq 0$ and $\lim_{t\to\infty}\Psi(t) = b$, $b \in [1,\infty)$. Constantin [2] introduced the notions of degree of stability and degree of boundedness of solutions of an ordinary differential equation, with respect to a continuous positive and nondecreasing function $\Psi : R_+ \to R_+$; some criteria for these notions are proved there too.

Morchało [3] introduced the notions of Ψ-stability, Ψ-uniform stability, and Ψ-asymptotic stability of trivial solution of the nonlinear system $x' = f(t,x)$. Several new and sufficient conditions for the mentioned types of stability are proved for the linear system $x' = A(t)x$; in this paper Ψ is a scalar continuous function. In [4, 5], Diamandescu gives some sufficient conditions for Ψ-asymptotic stability and Ψ-(uniform) stability of the nonlinear Volterra integro-differential system $x' = A(t)x + \int_0^t F(t,s,x(s))ds$; in these papers Ψ is a matrix function. Furthermore, in [6], sufficient conditions are given for the uniform Lipschitz stability of the system $x' = f(t,x) + g(t,x)$.

In paper [7], for the nonlinear system

$$y' = f(t,y) + g(t,y) \tag{1}$$

and the nonlinear Volterra integro-differential system

$$z' = f(t,z) + \int_0^t F(t,s,z(s))\,ds, \tag{2}$$

by using the knowledge of fundamental matrix and nonlinear variation of constants, we give some sufficient conditions for Ψ-(uniform) stability of trivial solution for the system. The purpose of this paper is to provide sufficient conditions for Ψ-uniform stability of trivial solutions for the nonlinear delayed system

$$x'(t) = f(t,x(t)) + g(t,x(t-\tau(t))) \tag{3}$$

and the nonlinear delayed Volterra integro-differential systems

$$\begin{aligned} x'(t) &= f(t,x(t)) + g(t,x(t-\tau(t))) \\ &\quad + p(t,x(t))\int_0^t q(s,x(s-\tau(s)))\,ds, \end{aligned} \tag{4}$$

$$\begin{aligned} x'(t) &= f(t,x(t)) + g(t,x(t-\tau(t))) \\ &\quad + p(t,x(t-\tau(t)))\int_0^t q(s,x(s))\,ds, \end{aligned} \tag{5}$$

where $f,g,p,q \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, $f(t,0) = g(t,0) = p(t,0) = q(t,0) = 0$ for $t \in \mathbb{R}_+$, and $\tau \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ with

$\tau(t) \le t$ on $\mathbb{R}_+$. The systems studied in [7] do not include time delay, whereas all the systems studied in this paper have time delay.

In this paper, we investigate conditions on the functions f, g, p, q under which the trivial solutions of systems (3), (4), and (5) are Ψ-stability on $\mathbb{R}_+$; the main tool used is the integral inequalities and the integral technique. Here Ψ is a matrix function whose introduction allows us to obtain a mixed behavior for the components of solutions.

Let $\mathbb{R}^n$ denote the Euclidean n-space. For $x = (x_1, x_2, x_3, \ldots, x_n)^T \in \mathbb{R}^n$, let $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$ be the norm of x. For an $n \times n$ matrix $A = (a_{ij})$, we define the norm $|A| = \sup_{\|x\|\le 1}\|Ax\|$. It is well known that

$$|A| = \max_{1\le i\le n}\sum_{j=1}^{n}\left|a_{ij}\right|. \quad (6)$$

Let $\Psi_i : \mathbb{R}_+ \to (0,\infty)$, $i = 1, 2, \ldots, n$, be continuous functions and $\Psi = \operatorname{diag}[\Psi_1, \Psi_2, \ldots, \Psi_n]$.

Now we give the definitions of Ψ-(uniform) stability that we will need in the sequel.

Definition 1 (see [4, 8]). The trivial solution of (3) ((4) or (5)) is said to be Ψ-stable on $\mathbb{R}_+$ if for every $\varepsilon > 0$ and any $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that any solution $x(t)$ of (3) ((4) or (5)), which satisfies the inequality $\|\Psi(t_0)x(t_0)\| < \delta$, exists and satisfies the inequality $\|\Psi(t)x(t)\| < \varepsilon$ for all $t \ge t_0$.

Definition 2 (see [4, 8]). The trivial solution of (3) ((4) or (5)) is said to be Ψ-uniformly stable on $\mathbb{R}_+$ if it is Ψ-stable on $\mathbb{R}_+$ and the previous δ is independent of t_0.

2. Ψ-Stability of the Systems

To prove our theorems, we need the following lemmas.

Lemma 3. *Let $h, k, p, q \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ with $(t,s) \mapsto \partial_t h(t,s)$, $\partial_t k(t,s)$, $\partial_t p(t,s)$, $\partial_t q(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$. Assume, in addition, that $b \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $\alpha \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions and $\alpha(t) \le t$ for $t \ge 0$. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$\begin{aligned} u(t) \le{}& b(t) + \int_0^t h(t,s)\,u(s)\,ds + \int_0^{\alpha(t)} k(t,s)\,u(s)\,ds \\ &+ \int_0^t p(t,s)\,u(s)\left(\int_0^{\alpha(s)} q(s,v)\,u(v)\,dv\right)ds, \end{aligned} \quad (7)$$

for $t \ge 0$, and $b(t)\int_0^t R(s)Q(s)ds < 1$, then

$$u(t) \le \frac{b(t)\,Q(t)}{1 - b(t)\int_0^t R(s)\,Q(s)\,ds}, \quad t \ge 0, \quad (8)$$

where $Q(t) = \exp(\int_0^t h(t,s)ds + \int_0^{\alpha(t)} k(t,s)ds)$, $R(t) = (d/dt)\int_0^t p(t,s)(\int_0^{\alpha(s)} q(s,v)dv)ds$.

Proof. Let $T \ge 0$ be fixed and denote

$$\begin{aligned} x(t) ={}& \int_0^t h(t,s)\,u(s)\,ds + \int_0^{\alpha(t)} k(t,s)\,u(s)\,ds \\ &+ \int_0^t p(t,s)\,u(s)\left(\int_0^{\alpha(s)} q(s,v)\,u(v)\,dv\right)ds, \quad t \ge 0, \end{aligned} \quad (9)$$

then $u(t) \le b(t) + x(t)$, and x is nondecreasing on $\mathbb{R}_+$. For $t \in [0,T]$, by calculations we get the following:

$$\begin{aligned} x'(t) ={}& \left[h(t,t)\,u(t) + \int_0^t \partial_t h(t,s)\,u(s)\,ds\right] \\ &+ \left[k(t,\alpha(t))\,u(\alpha(t))\,\alpha'(t) + \int_0^{\alpha(t)} \partial_t k(t,s)\,u(s)\,ds\right] \\ &+ \Bigg[p(t,t)\,u(t)\int_0^{\alpha(t)} q(t,v)\,u(v)\,dv \\ &\quad + \int_0^t \partial_t p(t,s)\,u(s)\left(\int_0^{\alpha(s)} q(s,v)\,u(v)\,dv\right)ds\Bigg] \\ \le{}& [b(T)+x(t)]\left[\frac{d}{dt}\left(\int_0^t h(t,s)\,ds + \int_0^{\alpha(t)} k(t,s)\,ds\right)\right] \\ &+ [b(T)+x(t)]^2\frac{d}{dt}\int_0^t p(t,s)\left(\int_0^{\alpha(s)} q(s,v)\,dv\right)ds. \end{aligned} \quad (10)$$

Suppose that $b(0) > 0$ (if $b(0) = 0$, carry out the following arguments with $b(t) + \varepsilon$ instead of $b(t)$, where $\varepsilon > 0$ is an arbitrary small constant, and subsequently pass to the limit as $\varepsilon \to 0$ to complete the proof), then we get

$$\begin{aligned} &\frac{x'(t)}{[b(T)+x(t)]^2} \\ &\quad - \frac{1}{b(T)+x(t)}\frac{d}{dt}\left(\int_0^t h(t,s)\,ds + \int_0^{\alpha(t)} k(t,s)\,ds\right) \\ &\le \frac{d}{dt}\int_0^t p(t,s)\left(\int_0^{\alpha(s)} q(s,v)\,dv\right)ds. \end{aligned} \quad (11)$$

Let

$$\begin{aligned} z(t) &= \frac{1}{b(T)+x(t)}, \\ q(t) &= \int_0^t h(t,s)\,ds + \int_0^{\alpha(t)} k(t,s)\,ds, \\ Q(t) &= \exp(q(t)) \\ &= \exp\left(\int_0^t h(t,s)\,ds + \int_0^{\alpha(t)} k(t,s)\,ds\right), \\ R(t) &= \frac{d}{dt}\int_0^t p(t,s)\left(\int_0^{\alpha(s)} q(s,v)\,dv\right)ds, \end{aligned} \quad (12)$$

then, we have

$$z'(t) + z(t)\left(\frac{d}{dt}q(t)\right) \geq -R(t). \tag{13}$$

Multiplying the above inequality by $e^{q(t)} = Q(t)$, we get

$$\frac{d}{dt}(z(t)Q(t)) \geq -Q(t)R(t). \tag{14}$$

Consider now the integral on the interval $[0, t]$ to obtain

$$z(t)Q(t) \geq z(0) - \int_0^t Q(s)R(s)\,ds, \quad 0 \leq t \leq T, \tag{15}$$

so

$$\begin{aligned} z(t) &= \frac{1}{b(T) + x(t)} \\ &\geq \left[\frac{1}{b(T)} - \int_0^t Q(s)R(s)\,ds\right]\frac{1}{Q(t)} \\ &= \frac{1 - b(T)\int_0^t Q(s)R(s)\,ds}{b(T)Q(t)} \end{aligned} \tag{16}$$

for $0 \leq t \leq T$. Let $t = T$, since $b(T)\int_0^T Q(s)R(s)ds < 1$, then we have

$$b(T) + x(T) \leq \frac{b(T)Q(T)}{1 - b(T)\int_0^T Q(s)R(s)\,ds}. \tag{17}$$

Since $T \geq 0$ was arbitrarily chosen, considering $u(t) \leq b(t) + x(t)$, we get (8). □

Lemma 4. *Let h, k, p, q, b, α be as in Lemma 3. If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfies*

$$\begin{aligned} u(t) \leq{}& b(t) + \int_0^t h(t,s)u(s)\,ds + \int_0^{\alpha(t)} k(t,s)u(s)\,ds \\ &+ \int_0^{\alpha(t)} p(t,s)u(s)\left(\int_0^s q(s,v)u(v)\,dv\right)ds, \end{aligned} \tag{18}$$

for $t \geq 0$, and $b(t)\int_0^t R(s)Q(s)ds < 1$, then

$$u(t) \leq \frac{b(t)Q(t)}{1 - b(t)\int_0^t R(s)Q(s)\,ds}, \quad t \geq 0, \tag{19}$$

where $Q(t) = \exp(\int_0^t h(t,s)ds + \int_0^{\alpha(t)} k(t,s)ds)$, $R(t) = (d/dt)\int_0^{\alpha(t)} p(t,s)(\int_0^s q(s,v)dv)ds$.

The proof is similar to the proof of Lemma 3, we omit the details.

Theorem 5. *If there exist functions $a(t,s), b(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ with $(t,s) \mapsto \partial_t a(t,s), \partial_t b(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ such that*

$$\begin{aligned} \|\Psi(t)f(s,x)\| &\leq a(t,s)\|\Psi(s)x\|, \\ \|\Psi(t)g(s,x)\| &\leq b(t,s)\|\Psi(s)x\|, \end{aligned} \tag{20}$$

for $0 \leq s \leq t$ and for all $x \in \mathbb{R}^n$. Moreover,

$$\begin{aligned} &\limsup_{t\to\infty} \int_0^t (a(t,s) + b(t,s))\,ds = L_1, \\ &\left|\Psi(t)\Psi^{-1}(s)\right| \leq L_2 \quad \text{for } 0 \leq s \leq t, \end{aligned} \tag{21}$$

and $|\Psi(t)x(\alpha(t))| \leq |\Psi(\alpha(t))x(\alpha(t))|$, where L_1, L_2 are nonnegative constants. If $\alpha(t) = t - \tau(t)$ is an increasing diffeomorphism of $\mathbb{R}_+$. Then, the trivial solution of system (3) is Ψ-uniformly stable on $\mathbb{R}_+$.

Proof. Suppose that $x(t, t_0, x_0) := x(t)$ is the unique solution of system (3) which satisfies $x(t_0) = x_0$, since

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^t f(s, x(s))\,ds + \int_{t_0}^t g(s, x(\alpha(s)))\,ds \\ &= x_0 + \int_{t_0}^t f(s, x(s))\,ds + \int_{\alpha(t_0)}^{\alpha(t)} \frac{g(\alpha^{-1}(r), x(r))}{\alpha'(\alpha^{-1}(r))}dr, \end{aligned} \tag{22}$$

after performing the change of variables $r = \alpha(s)$ in the second integral, and α^{-1} is the inverse of the diffeomorphism α then, it follows that

$$\begin{aligned} \|\Psi(t)x(t)\| \leq{}& \left\|\Psi(t)\Psi^{-1}(t_0)\Psi(t_0)x_0\right\| \\ &+ \int_{t_0}^t \|\Psi(t)f(s,x(s))\|\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \left\|\Psi(t)\frac{g(\alpha^{-1}(r), x(r))}{\alpha'(\alpha^{-1}(r))}\right\| ds \\ \leq{}& L_2\|\Psi(t_0)x_0\| + \int_{t_0}^t a(t,s)\|\Psi(s)x(s)\|\,ds \\ &+ \int_{\alpha(t_0)}^{\alpha(t)} \frac{b(t,\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}\|\Psi(r)x(r)\|\,dr, \end{aligned} \tag{23}$$

this implies by Lemma 3 that

$$\begin{aligned} \|\Psi(t)x(t)\| \leq{}& L_2\|\Psi(t_0)x_0\| \exp \\ &\times \left(\int_{t_0}^t a(t,s)\,ds + \int_{\alpha(t_0)}^{\alpha(t)} \frac{b(t,\alpha^{-1}(r))}{\alpha'(\alpha^{-1}(r))}dr\right) \\ ={}& L_2\|\Psi(t_0)x_0\| \exp\left(\int_{t_0}^t (a(t,s) + b(t,s))\,ds\right) \\ \leq{}& L_2 e^{L_1}\|\Psi(t_0)x_0\|, \end{aligned} \tag{24}$$

so for every $\varepsilon > 0$, choose $\delta = \varepsilon/(L_2 e^{L_1})$, then

$$\|\Psi(t)x(t)\| \leq L_2 e^{L_1}\|\Psi(t_0)x_0\| < \varepsilon \tag{25}$$

for $\|\Psi(t_0)x_0\| < \delta$ and for all $0 \leq t_0 \leq t < \infty$. Hence, the conclusion of the theorem follows. □

Theorem 6. *Let all the conditions in Theorem 5 hold. Suppose further that there exist functions* $m(t,s), n(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ *with* $(t,s) \mapsto \partial_t m(t,s)$, $\partial_t n(t,s) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ *such that*

$$\begin{gathered} \left\|\Psi(t)\, p(s,x)\, \Psi^{-1}(s)\right\| \le m(t,s)\, \|\Psi(s)\, x\|, \\ \left\|\Psi(t)\, q(s,x)\right\| \le n(t,s)\, \|\Psi(s)\, x\|, \end{gathered} \tag{26}$$

for $0 \le s \le t$ *and for all* $x \in \mathbb{R}^n$*, moreover,*

$$\limsup_{t\to\infty} \int_0^t m(t,s) \left(\int_0^s n(s,u)\, du \right) ds = L_3, \tag{27}$$

where L_3 *is a nonnegative constant. Then, the trivial solutions of systems* (4) *and* (5) *are* Ψ*-uniformly stable on* $\mathbb{R}_+$.

Proof. For that system (4), suppose $x(t,t_0,x_0) := x(t)$ is the unique solution of system (4) which satisfies $x(t_0) = x_0$, since

$$\begin{aligned} x(t) &= x_0 + \int_{t_0}^t f(s,x(s))\, ds + \int_{t_0}^t g(s, x(\alpha(s)))\, ds \\ &\quad + \int_{t_0}^t p(s,x(s)) \int_0^s q(u, x(\alpha(u)))\, du\, ds, \quad 0 \le t_0 \le t, \end{aligned} \tag{28}$$

it follows that

$$\begin{aligned} \|\Psi(t)\, x(t)\| &\le \left\|\Psi(t)\, \Psi^{-1}(t_0)\, \Psi(t_0)\, x_0\right\| \\ &\quad + \int_{t_0}^t \|\Psi(t)\, f(s,x(s))\|\, ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} \frac{\left\|\Psi(t)\, g\left(\alpha^{-1}(r), x(r)\right)\right\|}{\alpha'\left(\alpha^{-1}(r)\right)} dr \\ &\quad + \int_{t_0}^t \left\|\Psi(t)\, p(s,x(s))\, \Psi^{-1}(s)\right\| \\ &\quad \times \left(\int_0^{\alpha(s)} \frac{\left\|\Psi(s)\, q\left(\alpha^{-1}(r), x(r)\right)\right\|}{\alpha'\left(\alpha^{-1}(r)\right)} dr \right) ds \\ &\le L_2 \|\Psi(t_0)\, x_0\| + \int_{t_0}^t a(t,s)\, \|\Psi(s)\, x(s)\|\, ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} \frac{b\left(t, \alpha^{-1}(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)} \|\Psi(r)\, x(r)\|\, dr \\ &\quad + \int_{t_0}^t m(t,s)\, \|\Psi(s)\, x(s)\| \\ &\quad \times \left(\int_0^{\alpha(s)} \frac{n\left(s, \alpha^{-1}(r)\right) \|\Psi(r)\, x(r)\|}{\alpha'\left(\alpha^{-1}(r)\right)} dr \right) ds \end{aligned} \tag{29}$$

after performing the change of variables $r = \alpha(s)$ (or $r = \alpha(u)$) at some intermediate step, and α^{-1} is the inverse of the diffeomorphism α. Denote

$$\begin{aligned} Q(t) &= \exp\left(\int_{t_0}^t a(t,s)\, ds + \int_{\alpha(t_0)}^{\alpha(t)} \frac{b\left(t, \alpha^{-1}(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)} dr \right) \\ &= \exp\left(\int_{t_0}^t \left(a(t,s) + b(t,s)\right) ds \right), \\ R(t) &= \frac{d}{dt} \left[\int_{t_0}^t m(t,s) \left(\int_0^{\alpha(s)} \frac{n\left(s, \alpha^{-1}(r)\right)}{\alpha'\left(\alpha^{-1}(r)\right)} dr \right) ds \right] \\ &= \frac{d}{dt} \left[\int_{t_0}^t m(t,s) \left(\int_0^s n(s,u)\, du \right) ds \right]. \end{aligned} \tag{30}$$

This implies by Lemma 3 that

$$\begin{aligned} &\|\Psi(t)\, x(t)\| \\ &\quad \le L_2 \|\Psi(t_0)\, x_0\| \frac{Q(t)}{1 - L_2 \|\Psi(t_0)\, x_0\| \int_0^t Q(v)\, R(v)\, dv} \\ &\quad \le \|\Psi(t_0)\, x_0\| \frac{L_2 e^{L_1}}{1 - L_2 \|\Psi(t_0)\, x_0\| e^{L_1} \int_0^t R(v)\, dv} \\ &\quad = \|\Psi(t_0)\, x_0\| \\ &\qquad \times \frac{L_2 e^{L_1}}{1 - L_2 \|\Psi(t_0)\, x_0\| e^{L_1} \int_{t_0}^t m(t,s) \left(\int_0^s n(s,u)\, du \right) ds} \\ &\quad \le \|\Psi(t_0)\, x_0\| \frac{L_2 e^{L_1}}{1 - L_2 L_3 \|\Psi(t_0)\, x_0\| e^{L_1}} \end{aligned} \tag{31}$$

for $L_2 L_3 \|\Psi(t_0) x_0\| e^{L_1} < 1$ and $0 \le t_0 \le t$. So, for every $\varepsilon > 0$ and $t_0 \ge 0$, let $0 < q < 1/L_2 L_3 e^{L_1}$ be a constant and choose $\delta = \min\{q, ((1 - q L_2 L_3 e^{L_1})\varepsilon)/L_2 e^{L_1}\}$, then

$$\|\Psi(t)\, x(t)\| < \frac{\left(1 - q L_2 L_3 e^{L_1}\right)\varepsilon}{L_2 e^{L_1}} \times \frac{L_2 e^{L_1}}{1 - q L_2 L_3 e^{L_1}} = \varepsilon \tag{32}$$

for $\|\Psi(t_0) x_0\| < \delta$ and for all $0 \le t_0 \le t < \infty$. This proves that the trivial solution of system (4) is Ψ-uniformly stable on $\mathbb{R}_+$.

Using Lemma 4, the proof of system (5) is similar to that of system (4) and the details are left to the readers. □

Remark 7. For $\Psi_i = 1$, $i = 1, 2, \ldots, n$, we obtain the theorems of classical stability and uniform stability.

3. Examples

Example 8. Consider the nonlinear differential system

$$\begin{aligned} x_1'(t) &= x_1(t) + x_1\left(\frac{t}{2}\right) \sin t, \\ x_2'(t) &= -x_2(t) + x_2\left(\frac{t}{2}\right) \cos t. \end{aligned} \tag{33}$$

In (33), $f(t, x(t)) = (x_1(t), -x_2(t))^T$, $g(t, x(t/2)) = (x_1(t/2)\sin t, x_2(t/2)\cos t)^T$. Let $\Psi(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}$, then $a(t,s) = b(t,s) = e^{-(t-s)}$ for $0 \le s \le t \le \infty$, it is easy to verify that $L_1 = 2$, $L_2 = 1$, and all the assumptions in Theorem 5 satisfied, so the trivial solution of system (33) is ψ-uniformly stable on $\mathbb{R}_+$.

Example 9. Consider the nonlinear Volterra integro-differential system as follows:

$$\begin{aligned} x_1'(t) &= x_1(t) + x_1(t)e^{-t}\int_0^t x_1\left(\frac{s}{2}\right)\cos s\, ds, \\ x_2'(t) &= -x_2(t) + x_2(t)e^{-t}\int_0^t x_2\left(\frac{s}{2}\right)\sin s\, ds. \end{aligned} \tag{34}$$

In (34), $f(t, x(t)) = (x_1(t), -x_2(t))^T$, $g \equiv 0$, $p(t, x(t)) = (x_1(t)e^{-t}, x_2(t)e^{-t})^T$, $q(s, x(s/2)) = (x_1(s/2)\cos s, x_2(s/2)\sin s)^T$. Choose the same matrix function $\Psi(t)$, then $a(t,s) = n(t,s) = e^{-(t-s)}$, $b(t,s) \equiv 0$, $m(t,s) = e^{-2(t-s)}$ for $0 \le s \le t \le \infty$, it is easy to verify that $L_1 = L_2 = 1$, $L_3 = 1/2$, and all the assumptions in Theorem 6 are satisfied, so the trivial solution of system (34) is ψ-uniformly stable on $\mathbb{R}_+$.

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References

[1] O. Akinyele, "On partial stability and boundedness of degree k," *Atti della Accademia Nazionale dei Lincei. Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali*, vol. 65, no. 6, pp. 259–264, 1978.

[2] A. Constantin, *Asymptotic Properties of Solutions of Differential Equations*, vol. 3 of *Seria Stiinte Matematice*, Analele Universitãtii Din Timisoara, 1992.

[3] J. Morchało, "On (ψ, L_p)-stability of nonlinear systems of differential equations, Analele Stiintifice ale Universitàtii "AI. I. Cuza" Iasi, Tomul XXXVI, s. I-a,," *Matematicã*, vol. 36, no. 4, pp. 353–360, 1990.

[4] A. Diamandescu, "On the Ψ-asymptotic stability of a nonlinear Volterra integro-differential system," *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, vol. 46(94), no. 1-2, pp. 39–60, 2003.

[5] A. Diamandescu, "On the ψ-stability of a nonlinear Volterra integro-differential system," *Electronic Journal of Differential Equations*, vol. 2005, no. 56, pp. 1–14, 2005.

[6] F. M. Dannan and S. Elaydi, "Lipschitz stability of nonlinear systems of differential equations," *Journal of Mathematical Analysis and Applications*, vol. 113, no. 2, pp. 562–577, 1986.

[7] F. W. Meng, L. Z. Li, and Y. Z. Bai, "Ψ-stability of nonlinear Volterra integro-differential systems," *Dynamic Systems and Applications*, vol. 20, no. 4, pp. 563–574, 2011.

[8] V. Lakshmikantham and M. R. M. Rao, "Stability in variation for nonlinear integro-differential equations," *Applicable Analysis*, vol. 24, no. 3, pp. 165–173, 1987.

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Research Article

A Uniqueness Theorem for Bessel Operator from Interior Spectral Data

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Inverse problem for the Bessel operator is studied. A set of values of eigenfunctions at some internal point and parts of two spectra are taken as data. Uniqueness theorems are obtained. The approach that was used in investigation of problems with partially known potential is employed.

1. Introduction

Inverse spectral analysis involves the problem of restoring a linear operator from some of its spectral parameters. Currently, inverse problems are being studied for certain special classes of ordinary differential operators. The simplest of these is the Sturm-Liouville operator $Ly = -y'' + q(x)y$. For the case where it is considered on the whole line or half line, the Sturm-Liouville operator together with the function $q(x)$ has been called a potential. In this direction, Borg [1] gave important results. He showed that, in general, one spectrum does not determine a Sturm-Liouville operator, so the result of Ambarzumyan [2] is an exception to the general rule. In the same paper, Borg showed that two spectra of a Sturm-Liouville operator determine it uniquely. Later, Levinson [3], Levitan [4], and Hochstadt [5] showed that when the boundary condition and one possible reduced spectrum are given, then the potential is uniquely determined. Using spectral data, that is, the spectral function, spectrum, and norming constant, different methods have been proposed for obtaining the potential function in a Sturm-Liouville problem. Such problems were subsequently investigated by other authors [4–6]. On the other hand, inverse problems for regular and singular Sturm-Liouville operators have been extensively studied by [7–15].

The inverse problem for interior spectral data of the differential operator consists in reconstruction of this operator from the known eigenvalues and some information on eigenfunctions at some internal point. Similar problems for the Sturm-Liouville operator and discontinuous Sturm-Liouville problem were formulated and studied in [16, 17].

The main goal of the present work is to study the inverse problem of reconstructing the singular Sturm-Liouville operator on the basis of spectral data of a kind: one spectrum and some information on eigenfunctions at the internal point.

Consider the following singular Sturm-Liouville operator L satisfying (1)–(3):

$$Ly = -y'' + \left[\frac{\ell(\ell+1)}{x^2} + q(x)\right] y = \lambda y, \quad 0 < x < 1 \tag{1}$$

with boundary conditions,

$$y(0) = 0, \tag{2}$$

$$y'(1,\lambda) + Hy(1,\lambda) = 0, \tag{3}$$

where $q(x)$ is a real-valued function and $q \in L_2(0,1)$, λ spectral parameter, $\ell \in \mathbb{N}_0$, $H \in \mathbb{R}$. The operator L is self adjoint on the $L_2(0,1)$ and has a discrete spectrum $\{\lambda_n\}$.

Let us introduce the second singular Sturm-Liouville operator $\widetilde{L}$ satisfying

$$\widetilde{L}y = -y'' + \left[\frac{\ell(\ell+1)}{x^2} + \widetilde{q}(x)\right]y = \lambda y, \quad 0 < x < 1, \quad (4)$$

subject to the same boundary conditions (2), (3), where $\widetilde{q}(x)$ is a real-valued function and $\widetilde{q} \in L_2(0,1)$. The operator $\widetilde{L}$ is self adjoint on the $L_2(0,1)$ and has a discrete spectrum $\{\widetilde{\lambda}_n\}$.

2. Main Results

Before giving some results concerning the Bessel equation, we should give its physical properties. The total energy of the particle is given by $E = p^2/2M = \hbar^2 k^2/2M = k^2$, where p is its initial or final momentum, and k the corresponding wave number, $\hbar$ planck constant, M particle's mass, and E energy. The reduced radial Schrödinger equation for the partial wave of angular momentum ℓ then reads [18]

$$\frac{d^2}{dr^2}\Psi_1(k,r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)\Psi_1(k,r) = V(r)\Psi_1(k,r). \quad (5)$$

When $V = 0$, the above equation reduces to the classical Bessel equation in the form

$$\frac{d^2}{dr^2}\Psi_1(k,r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right)\Psi_1(k,r) = 0. \quad (6)$$

This equation has the solution $J_\ell(r)$, called the Bessel function.

Eigenvalues of the problem (1)–(3) are the roots of (3). This spectral characteristic satisfies the following asymptotic expression [19, 20]:

$$\lambda_n = \left(n + \frac{\ell}{2}\right)^2 \pi^2 + \int_0^1 q(x)\,dx - l(l+1) + a_n, \quad (7)$$

where the series $\sum_{n=1}^{\infty} a_n^2 < \infty$. Next, we present the main results in this paper. When $b = 1/2$, we get the following uniqueness Theorem 1.

Theorem 1. *If for every $n \in \mathbb{N}$ one has*

$$\lambda_n = \widetilde{\lambda}_n, \quad \frac{y_n'(1/2)}{y_n(1/2)} = \frac{\widetilde{y}_n'(1/2)}{\widetilde{y}_n(1/2)} \quad (8)$$

then

$$q(x) = \widetilde{q}(x) \quad \text{a.e on the interval } (0,1). \quad (9)$$

In the case $b \neq 1/2$, the uniqueness of $q(x)$ can be proved if we require the knowledge of a part of the second spectrum.

Let $\{m(n)\}_{n\in\mathbb{N}}$ be a sequence of natural numbers with a property

$$m(n) = \frac{n}{\sigma}(1 + \varepsilon_n), \quad 0 < \sigma \le 1, \ \varepsilon_n \longrightarrow 0. \quad (10)$$

Lemma 2. *Let $\{m(n)\}_{n\in\mathbb{N}}$ be a sequence of natural numbers satisfying* (10) *and $b \in (0,1/2)$ are so chosen that $\sigma > 2b$. If for any $n \in \mathbb{N}$*

$$\lambda_{m(n)} = \widetilde{\lambda}_{m(n)}, \quad \frac{y_{m(n)}'(b)}{y_{m(n)}(b)} = \frac{\widetilde{y}_{m(n)}'(b)}{\widetilde{y}_{m(n)}(b)} \quad (11)$$

then

$$q(x) = \widetilde{q}(x) \quad \text{a.e on } (0,b]. \quad (12)$$

Let $\{l(n)\}_{n\in\mathbb{N}}$ and $\{r(n)\}_{n\in\mathbb{N}}$ be a sequence of natural numbers such that

$$l(n) = \frac{n}{\sigma_1}(1 + \varepsilon_{1,n}), \quad 0 < \sigma_1 \le 1, \ \varepsilon_{1,n} \longrightarrow 0, \quad (13)$$

$$r(n) = \frac{n}{\sigma_2}(1 + \varepsilon_{2,n}), \quad 0 < \sigma_2 \le 1, \ \varepsilon_{2,n} \longrightarrow 0 \quad (14)$$

and let μ_n be the eigenvalues of (1), (2), and (15) and $\widetilde{\mu}_n$ be the eigenvalues of (4), (2), and (15)

$$y'(1,\lambda) + H_1 y(1,\lambda) = 0, \quad H \neq H_1. \quad (15)$$

Using Mochizuki and Trooshin's method from Lemma 2 and Theorem 1, we will prove that the following Theorem 3 holds.

Theorem 3. *Let $\{l(n)\}_{n\in\mathbb{N}}$ and $\{r(n)\}_{n\in\mathbb{N}}$ be a sequence of natural numbers satisfying* (13) *and* (14)*, and $1/2 < b < 1$ are so chosen that $\sigma_1 > 2b - 1$, $\sigma_2 > 2 - 2b$. If for any $n \in \mathbb{N}$ one has*

$$\lambda_n = \widetilde{\lambda}_n, \quad \mu_{l(n)} = \widetilde{\mu}_{l(n)}, \quad \frac{y_{r(n)}'(b)}{y_{r(n)}(b)} = \frac{\widetilde{y}_{r(n)}'(b)}{\widetilde{y}_{r(n)}(b)} \quad (16)$$

then

$$q(x) = \widetilde{q}(x) \quad \text{a.e on } (0,1). \quad (17)$$

3. Proof of the Main Results

In this section, we present the proofs of main results in this paper.

Proof of Theorem 1. Before proving Theorem 1, we will mention some results, which will be needed later. We get the initial value problems

$$-y'' + \left[\frac{\ell(\ell+1)}{x^2} + q(x)\right]y = \lambda y, \quad 0 < x < 1, \quad (18)$$

$$y(0) = 0, \quad (19)$$

$$-\widetilde{y}'' + \left[\frac{\ell(\ell+1)}{x^2} + \widetilde{q}(x)\right]\widetilde{y} = \lambda \widetilde{y}, \quad 0 < x < 1, \quad (20)$$

$$\widetilde{y}(0) = 0. \quad (21)$$

As known from [18], Bessel's functions of the first kind of order $\nu = \ell - 1/2$ are

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{\nu+2k}}{2^{\nu+2k} k!\,\Gamma(\nu+k+1)} \quad (22)$$

and asymptotic formulas for large argument

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\left\{\cos\left[x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] + O\left(\frac{1}{x}\right)\right\},$$
$$J'_\nu(x) = -\sqrt{\frac{2}{\pi x}}\left\{\sin\left[x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] + O\left(\frac{1}{x}\right)\right\}. \tag{23}$$

It can be shown [19] that there exists a kernel $H(x,t)(\widetilde{H}(x,t))$ continuous in the triangle $0 \le t \le x \le 1$ such that by using the transformation operator every solution of (18), (19) and (20), (21) can be expressed in the form [8, 21],

$$y(x,\lambda) = \frac{\sqrt{x}}{(\sqrt{\lambda})^\nu}J_\nu(\sqrt{\lambda}x) + \int_0^x H(x,t)\frac{\sqrt{t}}{(\sqrt{\lambda})^\nu}J_\nu(\sqrt{\lambda}t)\,dt, \tag{24}$$

$$\widetilde{y}(x,\lambda) = \frac{\sqrt{x}}{(\sqrt{\lambda})^\nu}J_\nu(\sqrt{\lambda}x) + \int_0^x \widetilde{H}(x,t)\frac{\sqrt{t}}{(\sqrt{\lambda})^\nu}J_\nu(\sqrt{\lambda}t)\,dt, \tag{25}$$

respectively, where the kernel $H(x,t)$ $(\widetilde{H}(x,t))$ is the solution of the equation

$$\frac{\partial^2 H(x,t)}{\partial x^2} + \frac{\ell(\ell+1)}{x^2}H(x,t) = \frac{\partial^2 H(x,t)}{\partial t^2} + \left(\frac{\ell(\ell+1)}{t^2} + q(t)\right)H(x,t) \tag{26}$$

subject to the boundary conditions

$$2\frac{dH(x,x)}{dx} = q(x),$$
$$\lim_{t\to 0} H(x,t)\,t^{\nu-1/2} = 0, \qquad \left[J'_\nu(t,\lambda) = O\left(t^{\nu-1/2}\right)\right]. \tag{27}$$

After the transformations

$$\xi = \frac{1}{4}(x+t)^2, \qquad \eta = \frac{1}{4}(x-t)^2,$$
$$H(x,t) = (\xi-\eta)^{-\nu+1/2}U(\xi,\eta), \tag{28}$$

we obtain the following problem:

$$\frac{\partial^2 U}{\partial\xi\partial\eta} - \frac{1}{4(\xi-\eta)}\frac{\partial U}{\partial\xi} + \frac{1}{4(\xi-\eta)}\frac{\partial U}{\partial\eta} = \frac{1}{4\sqrt{\xi\eta}}q\left(\sqrt{\xi}+\sqrt{\eta}\right)U,$$
$$U(\xi,\xi) = 0,$$
$$\frac{\partial U}{\partial\xi} + \frac{\alpha}{\xi}U = \frac{1}{4}q\left(\sqrt{\xi}\right)\xi^{\nu-1}, \quad \alpha = -\nu + \frac{1}{2}. \tag{29}$$

This problem can be solved by using the Riemann method [21].

Multiplying (18) by $\widetilde{y}(x,\lambda)$ and (20) by $y(x,\lambda)$, subtracting and integrating from 0 to 1/2, we obtain

$$\int_0^{1/2}(q(x)-\widetilde{q}(x))\,y(x,\lambda)\,\widetilde{y}(x,\lambda)\,dx = \left[\widetilde{y}(x,\lambda)y'(x,\lambda) - y(x,\lambda)\widetilde{y}'(x,\lambda)\right]\Big|_0^{1/2}. \tag{30}$$

The functions $y(x,\lambda)$ and $\widetilde{y}(x,\lambda)$ satisfy the same initial conditions (19) and (21), that is,

$$\widetilde{y}(0,\lambda)\,y'(0,\lambda) - y(0,\lambda)\,\widetilde{y}'(0,\lambda) = 0. \tag{31}$$

Let

$$Q(x) = q(x) - \widetilde{q}(x), \tag{32}$$

$$K(\lambda) = \int_0^{1/2} Q(x)\,y(x,\lambda)\,\widetilde{y}(x,\lambda)\,dx. \tag{33}$$

If the properties of $y(x,\lambda)$ and $\widetilde{y}(x,\lambda)$ are considered, the function $K(\lambda)$ is an entire function.

Therefore the condition of Theorem 1 implies

$$\widetilde{y}\left(\frac{1}{2},\lambda_n\right)y'\left(\frac{1}{2},\lambda_n\right) - y\left(\frac{1}{2},\lambda_n\right)\widetilde{y}'\left(\frac{1}{2},\lambda_n\right) = 0 \tag{34}$$

and hence

$$K(\lambda_n) = 0, \quad n \in \mathbb{N}. \tag{35}$$

In addition, using (24) and (33) for $0 < x < 1$,

$$|K(\lambda)| \le M\frac{1}{\lambda^\nu}, \tag{36}$$

where M is constant.

Introduce the function

$$W(\lambda) = y'(1,\lambda) + Hy(1,\lambda). \tag{37}$$

By using the asymptotic forms of y and y', we obtain

$$W(\lambda) = \sqrt{\lambda}\sin\left(\sqrt{\lambda} - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(1). \tag{38}$$

The zeros of $W(\lambda)$ are the eigenvalues of L and hence it has only simple zeros λ_n because of the seperated boundary conditions. From (38), $W(\lambda)$ is an entire function of order $1/2$ of λ. Since the set of zeros of the entire function $W(\lambda)$ is contained in the set of zeros of $K(\lambda)$, we see that the function

$$\Psi(\lambda) = \frac{K(\lambda)}{W(\lambda)} \tag{39}$$

is an entire function on the parameter λ. From (36), (38), and (39), we get

$$|\Psi(\lambda)| = O\left(\frac{1}{\lambda^{\nu+1/2}}\right). \tag{40}$$

So, for all λ, from the Liouville theorem,

$$\Psi(\lambda) = 0, \tag{41}$$

or

$$K(\lambda) = 0. \tag{42}$$

It was proved in [19] that there exists absolutely continuous function $\widetilde{\widetilde{H}}(x,\tau)$ such that we have

$$\begin{aligned} y(x,\lambda)\widetilde{y}(x,\lambda) = \frac{1}{2}\Big\{ & 1+\cos 2\left[\sqrt{\lambda}x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] \\ & + \int_0^x \widetilde{\widetilde{H}}(x,\tau) \\ & \times \cos 2\left[\sqrt{\lambda}\tau - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] d\tau \Big\}, \end{aligned} \tag{43}$$

where

$$\begin{aligned} \widetilde{\widetilde{H}}(x,t) = {} & 2\left[H(x,x-2\tau) + \widetilde{H}(x,x-2\tau)\right] \\ & + 2\left[\int_{-x+2\tau}^{x} H(x,s)\widetilde{H}(x,s-2\tau)\,ds \right. \\ & \left. + \int_{-x}^{x-2\tau} H(x,s)\widetilde{H}(x,s+2\tau)\,ds\right]. \end{aligned} \tag{44}$$

We are now going to show that $Q(x) = 0$ a.e. on $(0,1/2]$. From (33), (43) we have

$$\begin{aligned} \frac{1}{2}\int_0^{1/2} Q(x)\Big\{ & 1+\cos 2\left[\sqrt{\lambda}x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] \\ & + \int_0^x \widetilde{\widetilde{H}}(x,\tau) \\ & \times \cos 2\left[\sqrt{\lambda}\tau - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] d\tau\Big\}\,dx = 0. \end{aligned} \tag{45}$$

This can be written as

$$\begin{aligned} \int_0^{1/2} Q(x)\,dx + \int_0^{1/2} & \cos 2\left[\sqrt{\lambda}\tau - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] \\ & \times \left[Q(\tau) + \int_0^{1/2} Q(x) \right. \\ & \left. \times \widetilde{\widetilde{H}}(x,\tau)\,dx\right] d\tau = 0. \end{aligned} \tag{46}$$

Let $\lambda \to \infty$ along the real axis, by the Riemann-Lebesgue lemma, one should have

$$\begin{gathered} \int_0^{1/2} Q(x)\,dx = 0, \\ \int_0^{1/2} \cos 2\left[\sqrt{\lambda}\tau - \frac{\nu\pi}{2} - \frac{\pi}{4}\right] \\ \times \left[Q(\tau) + \int_\tau^{1/2} Q(x)\widetilde{\widetilde{H}}(x,\tau)\,dx\right] d\tau = 0. \end{gathered} \tag{47}$$

Thus from the completeness of the functions cos, it follows that

$$Q(\tau) + \int_\tau^{1/2} Q(x)\widetilde{\widetilde{H}}(x,\tau)\,dx = 0, \quad 0 < x < \frac{1}{2}. \tag{48}$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution. Thus we have obtained

$$Q(x) = q(x) - \widetilde{q}(x) = 0, \tag{49}$$

or

$$\widetilde{q}(x) = q(x) \tag{50}$$

almost everywhere on $(0,1/2]$. Therefore Theorem 1 is proved. □

Theorem 4. *To prove that $q(x) = 0$ on $[1/2,1)$ almost everywhere, we should repeat the above arguments for the supplementary problem*

$$Ly = -y'' + \left[\frac{\ell(\ell+1)}{(1-x)^2} + q(1-x)\right]y, \quad 0 < x < 1 \tag{51}$$

subject to the boundary conditions

$$\begin{gathered} y(1) = 0, \\ y'(0,\lambda) + Hy(0,\lambda) = 0. \end{gathered} \tag{52}$$

Consequently

$$q(x) = \widetilde{q}(x) \quad \text{a.e on the interval } (0,1). \tag{53}$$

Next, we show that Lemma 2 holds.

Proof of Lemma 2. As in the proof of Theorem 1 we can show that

$$\begin{aligned} G(\rho) &= \int_0^b Q(x)\,y(x,\lambda)\,\widetilde{y}(x,\lambda)\,dx \\ &= \left[\widetilde{y}(x,\lambda)\,y'(x,\lambda) - y(x,\lambda)\,\widetilde{y}'(x,\lambda)\right]\Big|_{x=b}, \end{aligned} \tag{54}$$

where $\rho = \sqrt{\lambda} = re^{i\theta}$ and $Q(x) = q(x) - \widetilde{q}(x)$. From the assumption

$$\frac{y'_{m(n)}(b)}{y_{m(n)}(b)} = \frac{\widetilde{y}'_{m(n)}(b)}{\widetilde{y}_{m(n)}(b)} \tag{55}$$

together with the initial condition at 0 it follows that,

$$G(\rho_{m(n)}) = 0, \quad n \in \mathbb{N}. \tag{56}$$

Next, we will show that $G(\rho) = 0$ on the whole ρ plane. The asymptotics (23) imply that the entire function $G(\rho)$ is a function of exponential type $\le 2b$.

Define the indicator of function $G(\rho)$ by

$$h(\theta) = \lim_{r\to\infty}\sup\frac{\ln\left|G\left(re^{i\theta}\right)\right|}{r}. \tag{57}$$

Since $|\operatorname{Im}\sqrt{\lambda}| = r|\sin\theta|, \theta = \arg\sqrt{\lambda}$ from (23) it follows that

$$h(\theta) \leq 2b|\sin\theta|. \tag{58}$$

Let us denote by $n(r)$ the number of zeros of $G(\rho)$ in the disk $\{|\rho| \leq r\}$. According to [22] set of zeros of every entire function of the exponential type, not identically zero, satisfies the inequality

$$\lim_{r\to\infty}\inf\frac{n(r)}{r} \leq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta, \tag{59}$$

where $n(r)$ is the number of zeros of $G(\rho)$ in the disk $|\rho| \leq r$. By (58),

$$\frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta \leq \frac{b}{\pi}\int_0^{2\pi}|\sin\theta|\,d\theta = \frac{4b}{\pi}. \tag{60}$$

From the assumption and the known asymptotic expression (7) of the eigenvalues $\sqrt{\lambda_n}$ we obtain

$$n(r) \geq 2\sum_{(\pi n/\sigma)[1+O(1/n)]<r} 1 = \frac{2}{\pi}\sigma r(1+o(1)), \quad r \longrightarrow \infty. \tag{61}$$

For the case $\sigma > 2b$,

$$\lim_{r\to\infty}\frac{n(r)}{r} \geq \frac{2}{\pi}\sigma > \frac{4b}{\pi} = 2b\int_0^{2\pi}|\sin\theta|\,d\theta \geq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\,d\theta. \tag{62}$$

The inequalities (59) and (62) imply that $G(\rho) = 0$ on the whole ρ plane.

Similar to the proof of Theorem 1, we have

$$q(x) = \tilde{q}(x) \quad \text{a.e on the interval } (0,b]. \tag{63}$$

This completes the proof of Lemma 2. □

Now we prove that Theorem 3 is valid.

Proof of Theorem 3. From

$$\lambda_{r(n)} = \tilde{\lambda}_{r(n)}, \qquad \frac{y'_{r(n)}(b)}{y_{r(n)}(b)} = \frac{\tilde{y}'_{r(n)}(b)}{\tilde{y}_{r(n)}(b)}, \tag{64}$$

where $\{r(n)\}_{n\in\mathbb{N}}$ satisfies (14) and $\sigma_2 > 2 - 2b$. Similar to the proof of Lemma 2, we get

$$q(x) = \tilde{q}(x) \quad \text{a.e on } [b,1). \tag{65}$$

Thus, it needs to be proved that $q(x) = \tilde{q}(x)$ a.e on $(0,b]$. The eigenfunctions $y_n(x,\lambda_n)$ and $\tilde{y}_n(x,\lambda_n)$ satisfy the same boundary condition at 1. It means that

$$y_n(x,\lambda_n) = \xi_n\tilde{y}_n(x,\lambda_n) \tag{66}$$

on $[b,1]$ for any $n \in \mathbb{N}$ where ξ_n are constants.

Let $\rho_n = \sqrt{\lambda_n}$, $s_n = \sqrt{\mu_n}$. From (54) and (66) we obtain

$$G(\rho_n) = 0, \quad n \in \mathbb{N}, \qquad G(s_{l_n}) = 0, \quad n \in \mathbb{N}. \tag{67}$$

We are going to show that inequality (59) fails and consequently, the entire function of exponential type $G(\rho)$ vanishes on the whole ρ-plane. The ρ_n and s_n have the same asymptotics (7). Counting the number of ρ_n and s_n located inside the disc of radius r, we have

$$1 + \frac{2}{\pi}r\left[1 + O\left(\frac{1}{n}\right)\right] \tag{68}$$

of ρ_n's and

$$1 + \frac{2}{\pi}r\sigma_1\left[1 + O\left(\frac{1}{n}\right)\right]. \tag{69}$$

of s_n's.

This means that

$$n(r) = 2 + \frac{2}{\pi}\left[r(\sigma_1+1) + O\left(\frac{1}{n}\right)\right], \qquad \lim_{r\to\infty}\frac{n(r)}{r} = \frac{2}{\pi}(\sigma_1+1). \tag{70}$$

Repeating the last part of the proof of Lemma 2, and considering the condition $\sigma_1 > 2b - 1$, we can show that $G(\rho) = 0$ identically on the whole ρ-plane which implies that

$$q(x) = \tilde{q}(x) \quad \text{a.e on } (0,b] \tag{71}$$

and consequently

$$q(x) = \tilde{q}(x) \quad \text{a.e on } (0,1). \tag{72}$$

Hence the proof of Theorem 3 is completed. □

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References

[1] G. Borg, "Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. Bestimmung der Differentialgleichung durch die Eigenwerte," *Acta Mathematica*, vol. 78, pp. 1–96, 1946.

[2] V. A. Ambarzumyan, "Über eine frage der eigenwerttheorie," *Zeitschrift Für Physik*, vol. 53, pp. 690–695, 1929.

[3] N. Levinson, "The inverse Sturm-Liouville problem," *Mathematica Tidsskrift B*, vol. 1949, pp. 25–30, 1949.

[4] B. M. Levitan, "On the determination of the Sturm-Liouville operator from one and two spectra," *Mathematics of the USSR-Izvestiya*, vol. 12, pp. 179–193, 1978.

[5] H. Hochstadt, "The inverse Sturm-Liouville problem," *Communications on Pure and Applied Mathematics*, vol. 26, pp. 715–729, 1973.

[6] V. A. Marchenko, "Certain problems of the theory of one dimensional linear di¤erential operators of the second order," *Trudy Moskovskogo Matematicheskogo Obshchestva*, vol. 1, pp. 327–340, 1952.

[7] R. Carlson, "A Borg-Levinson theorem for Bessel operators," *Pacific Journal of Mathematics*, vol. 177, no. 1, pp. 1–26, 1997.

[8] V. V. Stashevskaya, "On inverse problems of spectral analysis for a certain class of differential equations," *Doklady Akademii Nauk SSSR*, vol. 93, pp. 409–412, 1953.

[9] S. Albeverio, R. Hryniv, and Ya. Mykytyuk, "Inverse spectral problems for coupled oscillating systems: reconstruction from three spectra," *Methods of Functional Analysis and Topology*, vol. 13, no. 2, pp. 110–123, 2007.

[10] G. Freiling and V. Yurko, "Inverse problems for differential operators with singular boundary conditions," *Mathematische Nachrichten*, vol. 278, no. 12-13, pp. 1561–1578, 2005.

[11] V. A. Yurko, "On the reconstruction of Sturm-Liouville differential operators with singularities inside the interval," *Matematicheskie Zametki*, vol. 64, no. 1, pp. 143–156, 1998, English translation in *Mathematical Notes*, vol. 64, no.1, pp. 121–132, 1998.

[12] V. A. Yurko, "An inverse problem for differential equations with a singularity," *Differentsial'nye Uravneniya*, vol. 28, no. 8, pp. 1355–1362, 1992 (Russian), English translation in *Differential Equations*, vol. 28, pp. 1100–1107, 1992.

[13] V. A. Yurko, "On higher-order differential operators with a singular point," *Inverse Problems*, vol. 9, no. 4, pp. 495–502, 1993.

[14] F. Gesztesy and B. Simon, "Inverse spectral analysis with partial information on the potential. II. The case of discrete spectrum," *Transactions of the American Mathematical Society*, vol. 352, no. 6, pp. 2765–2787, 2000.

[15] E. S. Panakhov and M. Sat, "On the determination of the singular Sturm-Liouville operator from two spectra," *CMES. Computer Modeling in Engineering & Sciences*, vol. 84, no. 1, pp. 1–11, 2012.

[16] K. Mochizuki and I. Trooshin, "Inverse problem for interior spectral data of the Sturm-Liouville operator," *Journal of Inverse and Ill-Posed Problems*, vol. 9, no. 4, pp. 425–433, 2001.

[17] C.-F. Yang and X.-P. Yang, "An interior inverse problem for the Sturm-Liouville operator with discontinuous conditions," *Applied Mathematics Letters*, vol. 22, no. 9, pp. 1315–1319, 2009.

[18] G. A. Watson, *Treatise on the Theory of the Bessel Functions*, Cambridge University Press, Cambridge, UK, 1962.

[19] H. Koyunbakan, "Inverse spectral problem for some singular differential operators," *Tamsui Oxford Journal of Mathematical Sciences*, vol. 25, no. 3, pp. 277–283, 2009.

[20] W. Rundell and P. E. Sacks, "Reconstruction of a radially symmetric potential from two spectral sequences," *Journal of Mathematical Analysis and Applications*, vol. 264, no. 2, pp. 354–381, 2001.

[21] V. Ya. Volk, "On inversion formulas for a differential equation with a singularity at $x = 0$," *Uspekhi Matematicheskikh Nauk*, vol. 8, no. 4, pp. 141–151, 1953.

[22] B. Ja. Levin, *Distribution of Zeros of Entire Functions*, vol. 5, American Mathematical Society, Providence, RI, USA, 1964.