Journal of Applied Mathematics

New Contribution to the Advancement of Fixed Point Theory, Equilibrium Problems, and Optimization Problems

Guest Editors: Wei-Shih Du, Erdal Karapınar, Lai-Jiu Lin, Gue Myung Lee, and Tamaki Tanaka

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Editorial Board

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 895439, 1 page
http://dx.doi.org/10.1155/2013/895439

Editorial

New Contribution to the Advancement of Fixed Point Theory, Equilibrium Problems, and Optimization Problems

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Received 13 November 2013; Accepted 13 November 2013

Fixed point theory has attracted much attention as a very powerful and important tool in nonlinear sciences. Various fields such as biology, chemistry, economics, engineering, game theory, physics, and computer sciences have been using fixed point techniques quite extensively. Recent investigations in the fixed point theory and its applications motivate the development of new fixed point techniques for solving practical problems arising in natural sciences. Recent generalizations of optimization theory and techniques to other formulations comprise a large area of applied mathematics.

The purpose of this special issue is to provide new contribution to the advancement of fixed point theory, equilibrium problems, optimization problems, and their applications in mathematics and quantitative sciences. This special issue includes 27 high-quality peer-reviewed papers related to different aspects of theory and applications of fixed points and optimization. These papers contain new original, creative, and outstanding ideas. We profoundly believe that all the papers published in this special issue will motivate and inspire further scientific activities in the field of fixed point theory, optimization, and their applications.

Acknowledgments

We would like to express our deepest gratitude to a lot of reviewers, whose professional comments guaranteed the high quality of the selected papers. In addition, we would also like to convey our appreciation to the editorial board members of this journal, for their kind assistance and support throughout the reviewing process and the preparation of this special issue. We sincerely hope that the readers will find this special issue helpful for them in their future studies.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 237428, 7 pages
http://dx.doi.org/10.1155/2013/237428

Research Article

Nonsmooth Multiobjective Fractional Programming with Local Lipschitz Exponential B-(p, r)-Invexity

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Received 3 June 2013; Accepted 4 September 2013

Academic Editor: Gue Lee

We study nonsmooth multiobjective fractional programming problem containing local Lipschitz exponential B-(p, r)-invex functions with respect to η and b. We introduce a new concept of nonconvex functions, called exponential B-(p, r)-invex functions. Base on the generalized invex functions, we establish sufficient optimality conditions for a feasible point to be an efficient solution. Furthermore, employing optimality conditions to perform Mond-Weir type duality model and prove the duality theorems including weak duality, strong duality, and strict converse duality theorem under exponential B-(p, r)-invexity assumptions. Consequently, the optimal values of the primal problem and the Mond-Weir type duality problem have no duality gap under the framework of exponential B-(p, r)-invexity.

1. Introduction

Convexity plays an important role in mathematical programming problems, some of which are sufficient optimality conditions or duality theorems. The sufficient optimality conditions and duality theorems are being studied by extending the concept of convexity. One of the most generalizations of convexity of differentiable function in optimality theory was introduced by Hanson [1]. Then the characteristics of invexity—an invariant convexity—were applied in mathematical programming (cf. [1–7]). Besides, the concept of invexity of differentiable functions has been extended to the case of nonsmooth functions (cf. [8–17]). After Clarke [18] defined generalized derivative and subdifferential on local Lipschitz functions, many practical problems are described under nonsmooth functions. For example, Reiland [17] used the generalized gradient of Clarke [18] to define nondifferentiable invexity for Lipschitz real valued functions. Later on, with generalized invex Lipschitz functions, optimality conditions and duality theorems were established in nonsmooth mathematical programming problems (cf. [8–17]). Indeed, problems of multiobjective factional programming have various types of optimization problems, for example, financial and economic problems, game theory, and all optimal decision problems. In multiobjective programming problems, when the necessary optimality conditions are established, the conditions for searching an optimal solution will be employed. That is, extra reasonable assumptions for the necessary optimality conditions are needed in order to prove the sufficient optimality conditions. Moreover, these reasonable assumptions are various (e.g., generalized convexity, generalized invexity, set-value functions, and complex functions). When the existence of optimality solution is approved in the sufficient optimality theorems, the optimality conditions to investigate the duality models could be employed. Then the duality theorems could be proved. The better condition is that there is no duality gap between primal problems and duality problems.

In this paper, we focus a system of nondifferentiable multiobjective nonlinear fractional programming problem as the following form:

$$(P)\ \text{Minimize}\ \phi(x) \equiv \frac{f(x)}{g(x)} \equiv \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_k(x)}{g_k(x)}\right) \equiv \left(\phi_1(x), \phi_2(x), \ldots, \phi_k(x)\right), \tag{1}$$

subject to $x \in X \subset \mathbb{R}^n$ with

$$\mathfrak{F} = \{x \in X \mid h(x) = (h_1, h_2, \ldots, h_m)(x) \in -\mathbb{R}^m_+\}, \quad (2)$$

where X is a separable reflexive Banach space in the Euclidean n-space $\mathbb{R}^n$, f_i, $g_i : X \to \mathbb{R}$, $i = 1, 2, \ldots, k$, and $h : X \to \mathbb{R}^m$ are locally Lipschitz functions on X. Without loss of generality, we may assume that $f_i(x) \geq 0$, $g_i(x) > 0$ for all $x \in X$, $i = 1, 2, \ldots, k$.

In this paper, we introduce a new class of Lipschitz functions, namely, exponential B-(p, r)-invex Lipschitz functions which are motivated from the results of Antczak [3], Clarke [18], and Reiland [17]. We employ this exponential B-(p, r)-invexity and necessary optimality conditions to establish the sufficient optimality conditions on a nondifferentiable multiobjective fractional programming problem (P). Using optimality conditions, we construct Mond-Weir duality model for the primal problem (P) and prove that the duality theorems have the same optimal value as the primal problem involving B-(p, r)-invexity.

2. Definitions and Preliminaries

Let $\mathbb{R}^n$ denote Euclidean space, and let $\mathbb{R}^n_+$ denote the order cone. For cone partial order, if $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, we define:

(1) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, \ldots, n$;

(2) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, \ldots, n$;

(3) $x \geqq y$ if and only if $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$;

(4) $x \geq y$ if and only if $x \geqq y$ and $x_i \neq y_i$ for some $i \in \{1, 2, \ldots, n\}$.

Definition 1. Let X be an open subset of $\mathbb{R}^n$. The function $\theta : X \to \mathbb{R}$ is said to be *locally Lipschitz* at $x \in X$ if there exists a positive real constant $\mathscr{C}$ and a neighborhood $\mathscr{N}$ of $x \in X$ such that

$$|\theta(y) - \theta(z)| \leqq \mathscr{C} \|y - z\|, \quad \forall z, y \in \mathscr{N}, \quad (3)$$

where $\|\cdot\|$ is an arbitrary norm in $\mathbb{R}^n$.

For any vector ν in $\mathbb{R}^n$, the *generalized directional derivative* of θ at x in the direction $\nu \in \mathbb{R}^n$ in Clarke's sense [18] is defined by

$$\theta^\circ(x; \nu) = \limsup_{\substack{y \to x \\ \lambda \to 0^+}} \frac{\theta(y + \lambda\nu) - \theta(y)}{\lambda}. \quad (4)$$

The *generalized subdifferential* of θ at $x \in X$ is defined by the set

$$\partial^\circ \theta(x) = \{\xi \in X^* : \theta^\circ(x; \nu) \geq \langle \xi, \nu \rangle \ \ \forall \nu \in X\}, \quad (5)$$

where X^* is the dual space of X and $\langle \xi; \nu \rangle$ stands for the dual pair of X and X^*.

Evidently, $\theta^\circ(x; \nu) = \max\{\langle \xi; \nu \rangle : \xi \in \partial^\circ \theta(x)\}$ for any x and ν in X. If θ is a convex function, then $\partial^\circ \theta$ is coincid with usual subdifferential $\partial\theta$.

Definition 2 (see [18]). θ is said to be regular at x if for any $\nu \in X$, the one-side directional derivative $\theta'(x; \nu)$ exists and $\theta'(x; \nu) = \theta^\circ(x; \nu)$.

Lemma 3 (see [18]). *Let f and g be Lipschitz near x, and suppose $g(x) \neq 0$. Then $f(x)/g(x)$ is Lipschitz near x and one has*

$$\partial^\circ \left(\frac{f}{g}\right)(x) \subset \frac{g(x)\partial^\circ f(x) - f(x)\partial^\circ g(x)}{g^2(x)}, \quad \text{provided } f(x) \geqq 0,\ g(x) > 0. \quad (6)$$

If f and $-g$ are regular at x, then equality holds to the above $\subset$, that is, the subdifferential is singleton and f/g is regular at x.

Let $h : X \to \mathbb{R}^m$ be a local Lipschitz function. For $x_0 \in X$, we define

$$J(x_0) = \{j \in J : h_j(x_0) = 0\}, \quad J = \{1, 2, \ldots, m\},$$
$$\Lambda = \{\nu \in X : h_j^\circ(x_0, \nu) < 0,\ j \in J(x_0)\}. \quad (7)$$

If $\Lambda \neq \emptyset$, we say that the problem (P) has *constraint qualification* at x_0 (cf. [19]).

On the basis of the definition for invex functions of Lipschitz functions in Reiland [17], we modified Antczak's generalized B-(p, r)-invex with respect to η and b for differentiable to nondifferentiable case for a class of locally Lipschitz exponential B-(p, r)-invex functions as follows.

Definition 4. Let p, r be arbitrary real numbers. A locally Lipschitz function $\theta : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be *exponential B-(p, r)-invex (strictly)* at $u \in X$ with respect to w.r.t. (for brevity) if there exists a function $\eta : X \times X \to \mathbb{R}^n$ with property $\eta(x, u) = 0$ only if $u = x$ in X and a function $b : X \times X \to \mathbb{R}_+ \setminus \{0\}$ such that for each $x \in X$, the following inequality holds for $\xi \in \partial^\circ f(u)$:

$$\frac{1}{r} b(x, u)\left(e^{r(\theta(x) - \theta(u))} - 1\right) \geq \frac{1}{p}\left\langle \xi, \left(e^{p\eta(x,u)} - \mathbf{1}\right)\right\rangle (> \text{if } x \neq u), \quad \text{for } p \neq 0,\ r \neq 0. \quad (8)$$

If p or r is zero, then (8) can give some modification by using the limit of $p \to 0$ or $r \to 0$.

(i) If $r \neq 0$, $p \to 0$ in (8), then we deduce that

$$\frac{1}{r} b(x, u)\left(e^{r(\theta(x) - \theta(u))} - 1\right) \geq \langle \xi, \eta(x, u)\rangle (> \text{if } x \neq u), \quad \text{for } p = 0,\ r \neq 0. \quad (9)$$

(ii) If $p \neq 0$, $r \to 0$, then (8) becomes

$$b(x, u)(\theta(x) - \theta(u)) \geq \frac{1}{p}\left\langle \xi, \left(e^{p\eta(x,u)} - \mathbf{1}\right)\right\rangle (> \text{if } x \neq u) \quad \text{for } p \neq 0,\ r = 0. \quad (10)$$

(iii) If $r = 0$, $p \to 0$, then (10)

$$b(x,u)\left(f(x) - f(u)\right) \geq \langle \xi, \eta(x,u)\rangle \ (> \text{if } x \neq u) \quad \text{for } p = 0,\ r = 0 \tag{11}$$

holds.

Remark 5. All theorems in our work will be described only in the case of $p \neq 0$ and $r \neq 0$. We omit the proof of other cases like in (i), (ii), and (iii).

A feasible solution $\overline{x}$ to (P) is said to be an *efficient solution* to (P) if there is no $x \in \mathfrak{F}$ such that $\phi(x) \leq \phi(\overline{x})$.

3. Optimality Conditions

In this section, we establish some sufficient optimality conditions. The necessary optimality conditions to the primal problem (P) given by [20] and the subproblems (SP_i) of (P), for $i \in \{1, 2, \ldots, k\}$, given by [8] are used in our theorem.

Lemma 6 (see [8]). *$\overline{x}$ is an optimal solution to problem (P) if and only if $\overline{x}$ solves (SP_i), where (SP_i) is as the following problem:*

$$\begin{aligned}
(SP_i) \quad & \textit{Minimize} \quad \frac{f_i(x)}{g_i(x)} \\
& \textit{subject to} \quad x \in M_i \\
& = \left\{x \in X : \frac{f_p(x)}{g_p(x)} \leqq \frac{f_p(\overline{x})}{g_p(\overline{x})} = \phi_p(\overline{x}),\ p \neq i,\ p = 1, 2, \ldots, k,\ h(x) \in -\mathbb{R}_+^m\right\} \\
& = \left\{x \in X : f_p(x) - \phi_p(\overline{x})\, g_p(x) \leqq 0,\ p \neq i,\ p = 1, 2, \ldots, k,\ h(x) \in -\mathbb{R}_+^m\right\}.
\end{aligned} \tag{12}$$

Theorem 7 (see [20], necessary optimality conditions). *If $\overline{x}$ is an optimal solution of (P) and has a constraint qualification, for (SP_i), $i = 1, 2, \ldots, k$, then, there exist $\alpha^* \in \mathbb{R}^k$ and $z^* \in \mathbb{R}^m$ such that*

$$0 \in \sum_{i=1}^{k} \alpha_i^* \partial^\circ f_i(\overline{x}) + \phi_i(\overline{x})\, \partial^\circ(-g_i)(\overline{x}) + \langle z^*, \partial^\circ h(\overline{x})\rangle_m, \tag{13}$$

$$z_j^* h_j(\overline{x}) = 0 \quad \forall j = 1, 2, \ldots, m, \tag{14}$$

$$f_i(\overline{x}) - \phi_i(\overline{x})\, g_i(\overline{x}) = 0, \quad \forall i = 1, 2, \ldots, k, \tag{15}$$

$$\alpha^* \in \mathbb{I}_+^k \setminus \{0\}, \qquad z^* \in \mathbb{R}_+^m, \tag{16}$$

where

$$\begin{gathered}
\mathbb{I}_+^k = \left\{\alpha^* \in \mathbb{R}_+^k \mid \alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_k^*),\ \sum_{i=1}^{k} \alpha_i^* = 1\right\}, \\
\langle z^*, \partial^\circ h(\overline{x})\rangle_m \equiv \sum_{j=1}^{m} z_j^* \partial^\circ h_j(\overline{x}).
\end{gathered} \tag{17}$$

For convenience, let

$$\begin{gathered}
\langle z^*, h(x)\rangle_m \equiv \sum_{j=1}^{m} z_j^* h_j(x), \\
\langle z^*, \rho\rangle_m \equiv \sum_{j=1}^{m} z_j^* \rho_j,
\end{gathered} \tag{18}$$

where $z^* \in \mathbb{R}_+^m \rho_j \in \partial^\circ h_j(\overline{x})$.

Now, we give a useful lemma whose simple proof is omitted in this paper.

Lemma 8. *If $(1/r)(e^{r\theta(x)} - 1) \geq 0$, where $\theta(x)$ is a real function, then $\theta(x) \geq 0$.*

The sufficient optimality conditions can be deduced from the converse of necessary optimality conditions with extra assumptions. Since the sufficient optimality theorem is various depending on extra assumptions, the duality model is also various. We establish the sufficient optimality conditions and duality theorems involving the exponential B-(p, r)-invexity.

Theorem 9. *Let $\overline{x} \in \mathfrak{F}$ be a feasible solution of (P) such that there exist y^*, z^* satisfying the conditions* (13)~(16) *at $\overline{x}$. Furthermore, suppose that any one of the conditions (a) and (b) hold:*

(a) *$A_1(x) = \sum_{i=1}^{k} \alpha_i^* [f_i(x) - \phi_i(\overline{x}) g_i(x)] + \langle z^*, h(x)\rangle_m$ is an exponential B-(p, r)-invex function at $\overline{x}$ in $\mathfrak{F}$ w.r.t. η and b_1,*

(b) *$A_2(x) = \sum_{i=1}^{k} \alpha_i^* [f_i(x) - \phi_i(\overline{x}) g_i(x)]$ is an exponential B-(p, r)-invex function at $\overline{x}$ in $\mathfrak{F}$ w.r.t. η and b_2, and $A_3(x) = \langle z^*, h(x)\rangle_m$ is an exponential B-(p, r)-invex function at $\overline{x}$ in $\mathfrak{F}$ w.r.t. the same function η and b_3 but not necessarily, equal to b_2.*

Then, $\overline{x}$ is an efficient solution to problem (P).

Proof. Suppose that $\overline{x}$ is (P)-feasible. By expression (13), there exist $\xi_i \in \partial^\circ f_i(\overline{x})$, $\zeta_i \in \partial^\circ(-g_i)(\overline{x})$, $i = 1, 2, \ldots, k$ and $\rho_j \in \partial^\circ h_j(\overline{x})$, $j = 1, 2, \ldots, m$ such that

$$\langle \widetilde{\mathbb{a}}_1 \rangle \equiv \sum_{i=1}^{k} \alpha_i^* \left(\xi_i + \phi_i(\overline{x})\, \zeta_i\right) + \langle z^*, \rho\rangle_m = 0 \quad \text{in } X^* \tag{19}$$

and that $\langle \widetilde{\mathbb{a}}_1 \rangle$ is a zero vector of X^*.

From the above expression, the dual pair of $\langle X^*, X\rangle$

$$\left\langle \langle \widetilde{\mathbb{a}}_1 \rangle, \left(e^{p\eta(x,\overline{x})} - \mathbf{1}\right)\right\rangle = 0. \tag{20}$$

If $\overline{x}$ is not an efficient solution to problem (P), then there exists $x \in (P)$-feasible such that

$$\begin{aligned} &\frac{f_i(x)}{g_i(x)} \leqq \frac{f_i(\overline{x})}{g_i(\overline{x})} \quad \text{for } i = 1, 2, \ldots, k, \\ &\frac{f_t(x)}{g_t(x)} < \frac{f_t(\overline{x})}{g_t(\overline{x})} \quad \text{for some } t \in \underline{k} = \{1, 2, \ldots, k\}; \end{aligned} \tag{21}$$

that is,

$$\begin{aligned} &f_i(x) - \phi_i(\overline{x}) g_i(x) \\ &\quad \leqq f_i(\overline{x}) - \phi_i(\overline{x}) g_i(\overline{x}) \quad \text{for } i = 1, 2, \ldots, k, \\ &f_t(x) - \phi_t(\overline{x}) g_t(x) \\ &\quad < f_t(\overline{x}) - \phi_t(\overline{x}) g_t(\overline{x}) \quad \text{for some } t \in \underline{k}. \end{aligned} \tag{22}$$

Thus, we have

$$\begin{aligned} A_2(x) &= \sum_{i=1}^{k} \alpha_i^* \left[f_i(x) - \phi_i(\overline{x}) g_i(x) \right] \\ &< \sum_{i=1}^{k} \alpha_i^* \left[f_i(\overline{x}) - \phi_i(\overline{x}) g_i(\overline{x}) \right] = A_2(\overline{x}). \end{aligned} \tag{23}$$

From relations $h(x) \in -\mathbb{R}_+^m$, (14), and (16), we obtain

$$A_3(x) = \langle z^*, h(x) \rangle_m \leqq \langle z^*, h(\overline{x}) \rangle_m = A_3(\overline{x}), \tag{24}$$

where $\langle z^*, h(\overline{x}) \rangle_m \equiv \sum_{j=1}^{m} z_j^* h(\overline{x})$.

If hypothesis (a) holds, $A_1(x)$ is an exponential B-(p, r)-invexity w.r.t. η and b_1 at $\overline{x}$ for all $x \in \mathfrak{F}$. Then by Definition 4, we have that the following inequality

$$\begin{aligned} &\frac{1}{r} b_1(x, \overline{x}) \left(e^{r(A_1(x) - A_1(\overline{x}))} - 1 \right) \\ &\quad \geqq \frac{1}{p} \left\langle \langle \widetilde{\mathrm{a}}_1 \rangle, \left(e^{p\eta(x, \overline{x})} - \mathbf{1} \right) \right\rangle \end{aligned} \tag{25}$$

holds. Because of equality (20) and inequality (25), we obtain

$$\frac{1}{r} b_1(x, \overline{x}) \left(e^{r(A_1(x) - A_1(\overline{x}))} - 1 \right) \geqq 0. \tag{26}$$

According to Lemma 8 and $b_1(x, \overline{x}) \in \mathbb{R}_+ \setminus \{0\}$, we have

$$A_1(x) \geqq A_1(\overline{x}). \tag{27}$$

Equation (23) along with (24) yields

$$\begin{aligned} A_1(x) &= \sum_{i=1}^{k} \alpha_i^* \left[f_i(x) - \phi_i(\overline{x}) g_i(x) \right] + \langle z^*, h(\overline{x}) \rangle_m \\ &< \sum_{i=1}^{k} \alpha_i^* \left[f_i(\overline{x}) - \phi_i(\overline{x}) g_i(\overline{x}) \right] + \left\langle z^*, h\left(\overline{\overline{x}}\right) \right\rangle_m \\ &= A_1(\overline{x}) \end{aligned} \tag{28}$$

which contradicts inequality (27).

If hypothesis (b) holds, $A_3(x)$ is an exponential B-(p, r)-invex function w.r.t. η and b_3 at $\overline{x}$ for all x, that is, (P)-feasible. Then by Definition 4, we have the following inequality:

$$\begin{aligned} &\frac{1}{r} b_3(x, \overline{x}) \left(e^{r(A_3(x) - A_3(\overline{x}))} - 1 \right) \\ &\quad \geqq \frac{1}{p} \left\langle \langle z^*, \rho \rangle_m, \left(e^{p\eta(x, \overline{x})} - \mathbf{1} \right) \right\rangle. \end{aligned} \tag{29}$$

From inequalities (24) and (29), we have

$$\frac{1}{p} \left\langle \langle z^*, \rho \rangle_m, \left(e^{p\eta(x, \overline{x})} - \mathbf{1} \right) \right\rangle \leq 0. \tag{30}$$

By inequality (30) and multiplying (20) by $1/p$, it yields that

$$\frac{1}{p} \left\langle \sum_{i=1}^{k} \alpha_i^* \left(\xi_i + \phi_i(\overline{x}) \zeta_i \right), \left(e^{p\eta(x, \overline{x})} - \mathbf{1} \right) \right\rangle \geq 0. \tag{31}$$

Since $A_2(x)$ is an exponential B-(p, r)-invex function w.r.t. η and b_2 at $\overline{x}$ for all x, that is, (P)-feasible then by Definition 4, we have

$$\begin{aligned} &\frac{1}{r} b_2(x, \overline{x}) \left(e^{r(A_2(x) - A_2(\overline{x}))} - 1 \right) \\ &\quad \geqq \frac{1}{p} \left\langle \sum_{i=1}^{k} \alpha_i^* \left(\xi_i + \phi_i(\overline{x}) \zeta_i \right), \left(e^{p\eta(x, \overline{x})} - \mathbf{1} \right) \right\rangle. \end{aligned} \tag{32}$$

From inequalities (31) and (32), we obtain

$$\frac{1}{r} b_2(x, \overline{x}) \left(e^{r(A_2(x) - A_2(\overline{x}))} - 1 \right) \geqq 0. \tag{33}$$

By Lemma 8 and $b_2(x, \overline{x}) \in \mathbb{R}_+ \setminus \{0\}$, we get

$$A_2(x) \geqq A_2(\overline{x}). \tag{34}$$

If $\overline{x}$ is not an efficient solution to problem (P), then we reduce inequality (23) in the same way. But inequality (34) contradicts inequality (23). Hence, the proof is complete. □

4. Mond-Weir Type Duality Model

In order to propose Mond-Weir type duality model, it is convenient to restate the necessary conditions in Theorem 7 as the following form. Mainly, we use the expressions (13) and (15) to get

$$0 \in \sum_{i=1}^{k} \alpha_i^* \left[\partial^\circ f_i(\overline{x}) + \frac{f_i(\overline{x})}{g_i(\overline{x})} \partial^\circ (-g_i)(\overline{x}) \right] + \langle z^*, \partial^\circ h(\overline{x}) \rangle_m. \tag{35}$$

Then putting $\alpha^* = \overline{\alpha}^* g(\overline{x}) \in \mathbb{I}_+^k$ in the above expression, we obtain

$$\begin{aligned} 0 \in &\sum_{i=1}^{k} \overline{\alpha}_i^* g_i(\overline{x}) \left[\partial^\circ f_i(\overline{x}) + \langle z^*, \partial^\circ h(\overline{x}) \rangle_m \right] \\ &+ \sum_{i=1}^{k} \overline{\alpha}_i^* f_i(\overline{x}) \partial^\circ (-g_i)(\overline{x}). \end{aligned} \tag{36}$$

Consequently, from inequality (14), it yields that

$$0 \in \sum_{i=1}^{k} \overline{\alpha}_i^* g_i(\overline{x}) \left[\partial^\circ f_i(\overline{x}) + \langle z^*, \partial^\circ h(\overline{x})\rangle_m\right] + \sum_{i=1}^{k} \overline{\alpha}_i^* \left[f_i(\overline{x}) + \langle z^*, h(\overline{x})\rangle_m\right] \partial^\circ(-g_i)(\overline{x}), \tag{37}$$

where $\langle z^*, h(\overline{x})\rangle_m \equiv \sum_{j=1}^{m} z_j^* h_j(\overline{x})$. For simplicity, we write $\overline{\alpha}_i^*$ still by α_i^*. Then the result of Theorem 7 can be restated as the following theorem.

Theorem 10 (necessary optimality conditions). *If $\overline{x}$ is an efficient solution to (P) and satisfyies constraint qualification in (SP_i), $i = 1, 2, \ldots, k$, then, there exist $\alpha^* \in \mathbb{R}^k$, $z^* \in \mathbb{R}^m$ such that*

$$0 \in \sum_{i=1}^{k} \alpha_i^* g_i(\overline{x}) \left[\partial^\circ f_i(\overline{x}) + \langle z^*, \partial^\circ h(\overline{x})\rangle_m\right] + \sum_{i=1}^{k} \alpha_i^* \left[f_i(\overline{x}) + \langle z^*, h(\overline{x})\rangle_m\right] \partial^\circ(-g_i)(\overline{x}), \tag{38}$$

$$z_j^* h_j(\overline{x}) = 0 \quad \forall j = 1, 2, \ldots, m, \tag{39}$$

$$\alpha^* \in \mathbb{I}_+^k, \qquad z^* \in \mathbb{R}_+^m. \tag{40}$$

For any $u \in \mathfrak{F}$, if we use $(\alpha, z) \in \mathbb{R}^k \times \mathbb{R}^m$ instead of $(\alpha^*, z^*) \in \mathbb{R}^k \times \mathbb{R}^m$ satisfying the necessary conditions (38)~(40) as the constraints of a new dual problem, namely, Mond-Weir type dual (D), then it constitutes by a maximization programming problem with the same objective function as the problem (P), and we use the necessary optimality conditions of (P) as the constraint of the new problem (D). Precisely, we can state this dual problem as the maximization problem as the following form:

$$(D) \text{ Maximize } \quad \Phi(u) \equiv \left(\frac{f_1(u)}{g_1(u)}, \frac{f_2(u)}{g_2(u)}, \ldots, \frac{f_k(u)}{g_k(u)}\right) \equiv \left(\Phi_1(u), \Phi_2(u), \ldots, \Phi_k(u)\right), \tag{41}$$

subject to the resultant of necessary condition in Theorem 10:

$$0 \in \sum_{i=1}^{k} \alpha_i g_i(u) \left[\partial^\circ f_i(u) + \langle z, \partial^\circ h(u)\rangle_m\right] + \sum_{i=1}^{k} \alpha_i \partial^\circ(-g_i)(u) \left[f_i(u) + \langle z, h(u)\rangle_m\right], \tag{42}$$

$$\langle z, h(u)\rangle_m \equiv \sum_{j=1}^{m} z_j h_j(u) = 0, \tag{43}$$

$$u \in X, \qquad \alpha \in \mathbb{I}_+^k, \qquad z \in \mathbb{R}_+^m. \tag{44}$$

Let $\mathscr{D}$ be the constraint set $\{u; \alpha, z\}$ of (D) satisfying (42)~(44) which are the necessary optimality conditions of (P). For convenience, we denote the projective-like set by:

$$pr_{\mathfrak{F}}\mathscr{D} = \{u \in \mathfrak{F} \mid (u; \alpha, z) \in \mathscr{D}\}. \tag{45}$$

Then we can derive the following weak duality theorem between (P) and (D).

Theorem 11 (weak duality). *Let x and $(u; \alpha, z)$ be (P)-feasible and (D)-feasible, respectively. Denote a function $A_4 : X \to \mathbb{R}$ by*

$$A_4(\cdot) = \sum_{i=1}^{k} \alpha_i g_i(u) \left[f_i(\cdot) + \langle z, h(\cdot)\rangle_m\right] - \sum_{i=1}^{k} \alpha_i g_i(\cdot) \left[f_i(u) + \langle z, h(u)\rangle_m\right], \tag{46}$$

with $A_4(u) = 0$. Suppose that $A_4(\cdot)$ is an exponential B-(p, r)-invex function at $u \in pr_{\mathfrak{F}}\mathscr{D}$ w.r.t. η and b_4.

Then $\phi(x) \nleq \Phi(u)$.

Proof. Let x and $(u; \alpha, z)$ be (P)- and (D)-feasible, respectively. From expression (38), there exist $\xi_i \in \partial^\circ f_i(u)$, $\zeta_i \in \partial^\circ(-g_i)(u)$, $i = 1, 2, \ldots, k$ and $\rho_j \in \partial^\circ h_j(u)$, $j = 1, 2, \ldots, m$ to satisfy

$$\langle \widetilde{\mathrm{a}}_4 \rangle \equiv \sum_{i=1}^{k} \alpha_i g_i(u) \left[\xi_i + \langle z, \rho\rangle_m\right] + \sum_{i=1}^{k} \alpha_i \left[f_i(u) + \langle z, h(u)\rangle_m\right] \zeta_i = 0 \in X^*, \tag{47}$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$.

Then, the dual pair of $\langle X^*, X\rangle$ yields

$$\left\langle \langle \widetilde{\mathrm{a}}_4 \rangle, \left(e^{p\eta(x,u)} - \mathbf{1}\right)\right\rangle = 0. \tag{48}$$

Since A_4 is an exponential B-(p, r)-invex function w.r.t. η and b_4 at $u \in pr_{\mathfrak{F}}\mathscr{D}$, we have the following inequality:

$$\frac{1}{r} b_4(x, u) \left(e^{r(A_4(x) - A_4(u))} - 1\right) \geq \frac{1}{p} \left\langle \langle \widetilde{\mathrm{a}}_4 \rangle, \left(e^{p\eta(x,u)} - \mathbf{1}\right)\right\rangle = 0. \tag{49}$$

By the above inequality and equality (48), we obtain

$$\frac{1}{r} b_4(x, u) \left(e^{r(A_4(x) - A_4(u))} - 1\right) \geq 0. \tag{50}$$

According to Lemma 8 and $b_4 \in \mathbb{R} \setminus \{0\}$, we have

$$A_4(x) \geq A_4(u) = 0. \tag{51}$$

We want to prove that $\phi(x) \nleq \Phi(u)$.

Suppose on the contrary that $\phi(x) \leq \Phi(u)$. Then

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(u)}{g_i(u)} \quad \forall i = 1, 2, \ldots, k, \tag{52}$$

and there is some index $t \in \underline{k}$ such that

$$\frac{f_t(x)}{g_t(x)} < \frac{f_t(u)}{g_t(u)}. \tag{53}$$

Then by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{I}_+^k$, we have

$$\sum_{i=1}^{k} \alpha_i f_i(x) g_i(u) < \sum_{i=1}^{k} \alpha_i g_i(x) f_i(u). \tag{54}$$

Since $h(x) \in -\mathbb{R}_+^m$, it follows from (43), (44), and (54) that

$$\begin{aligned} &\sum_{i=1}^{k} \alpha_i g_i(u) \left[f_i(x) + \langle z, h(x) \rangle_m \right] \\ &\quad < \sum_{i=1}^{k} \alpha_i g_i(x) \left[f_i(u) + \langle z, h(u) \rangle_m \right]. \end{aligned} \tag{55}$$

This implies that

$$\begin{aligned} A_4(x) = &\sum_{i=1}^{k} \alpha_i g_i(u) \left[f_i(x) + \langle z, h(x) \rangle_m \right] \\ &- \sum_{i=1}^{k} \alpha_i g_i(x) \left[f_i(u) + \langle z, h(u) \rangle_m \right] < 0, \end{aligned} \tag{56}$$

which contradicts inequality (51), and the proof of theorem is complete. □

Theorem 12 (strong duality). *Let $\overline{x}$ be the efficient solution of problem (P) satisfying the constraint qualification at $\overline{x}$ in (SP_i), $i = 1, 2, \ldots, k$. Then there exist $\alpha^* \in \mathbb{R}^k$ and $z^* \in \mathbb{R}^m$ such that $(\overline{x}; \alpha^*, z^*) \in (D)$-feasible. If the hypotheses of Theorem 11 are fulfilled, then $(\overline{x}; \alpha^*, z^*)$ is an efficient solution to problem (D). Furthermore, the efficient values of (P) and (D) are equal.*

Proof. Let $\overline{x}$ be an efficient solution to problem (P). Then there exist α^*, z^* such that $(\overline{x}; \alpha^*, z^*)$ satisfies (42)~(44) that is, $(\overline{x}; \alpha^*, z^*) \in \mathscr{D}$ is a feasible solution for the problem (D). Actually, $(\overline{x}; \alpha^*, z^*)$ is also an efficient solution of (D).

Suppose on the contrary that if $(\overline{x}; \alpha^*, z^*)$ were not an efficient solution to (D), then there exists a feasible solution $(x; \alpha, z)$ of (D) such that

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} \leqq \frac{f_i(x)}{g_i(x)} \quad \forall i = 1, 2, \ldots, k, \tag{57}$$

and there is a $t \in \underline{k}$,

$$\frac{f_t(\overline{x})}{g_t(\overline{x})} < \frac{f_t(x)}{g_t(x)}. \tag{58}$$

It follows that $\phi(\overline{x}) \leq \Phi(x)$ which contradicts the weak duality Theorem 11. Hence, $(\overline{x}; \alpha^*, z^*)$ is an efficient solution of (D) and the efficient values of (P) and (D) are clearly equal. □

Theorem 13 (strict converse duality). *Let $\overline{x}$ and $(u^*; \alpha^*, z^*)$ be the efficient solutions of (P) and (D), respectively. Denote a function $A_5 : X \to \mathbb{R}$ by*

$$\begin{aligned} A_5(\cdot) = &\sum_{i=1}^{k} \alpha_i^* g_i(u^*) \left[f_i(\cdot) + \langle z^*, h(\cdot) \rangle_m \right] \\ &- \sum_{i=1}^{k} \alpha_i^* g_i(\cdot) \left[f_i(u^*) + \langle z^*, h(u^*) \rangle_m \right], \end{aligned} \tag{59}$$

with $A_5(u^*) = 0$. If $A_5(\cdot)$ is a strictly exponential B-(p, r)-invex function at $u^* \in pr_{\mathfrak{F}}\mathscr{D}$ w.r.t. η and b_5 for all optimal vectors $\overline{x}$ in (P) and $(u^*; \alpha^*, z^*)$ in (D), respectively, then $\overline{x} = u^*$ and the efficient values of (P) and (D) are equal.

Proof. Suppose that $\overline{x} \neq u^*$. From expression (42), there exist $\xi_i \in \partial^\circ f_i(u^*)$, $\zeta_i \in \partial^\circ(-g_i)(u^*)$, $i = 1, 2, \ldots, k$ and $\rho_j \in \partial^\circ h_j(u^*)$, $j = 1, 2, \ldots, m$ such that

$$\begin{aligned} \langle \widetilde{\mathfrak{a}}_5 \rangle \equiv &\sum_{i=1}^{k} \alpha_i^* g_i(u^*) \left[\xi_i + \langle z^*, \rho \rangle_m \right] \\ &+ \sum_{i=1}^{k} \alpha_i^* \left[f_i(u^*) + \langle z^*, h(u^*) \rangle_m \right] \zeta_i = 0 \in X^*, \end{aligned} \tag{60}$$

where $\rho = (\rho_1, \rho_2, \ldots, \rho_m)$.

It follows that the dual pair in $\langle X^*, X \rangle$ becomes

$$\frac{1}{p} \left\langle \langle \widetilde{\mathfrak{a}}_5 \rangle, \left(e^{p\eta(\overline{x}, u^*)} - \mathbf{1} \right) \right\rangle = 0. \tag{61}$$

From Theorem 12, we see that there exist $\overline{\alpha}$ and $\overline{z}$ such that $(\overline{x}; \overline{\alpha}, \overline{z})$ is the efficient solution of (D) and

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} = \frac{f_i(u^*)}{g_i(u^*)} \quad \forall i = 1, 2, \ldots, k. \tag{62}$$

By inequality (43) and equality (62), it becomes

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} = \frac{f_i(u^*) + \langle z^*, h(u^*) \rangle_m}{g_i(u^*)}. \tag{63}$$

Eliminating the dominators in (63), we get

$$f_i(\overline{x}) g_i(u^*) = \left[f_i(u^*) + \langle z^*, h(u^*) \rangle_m \right] g_i(\overline{x}) \tag{64}$$

or

$$f_i(\overline{x}) g_i(u^*) - \left[f_i(u^*) + \langle z^*, h(u^*) \rangle_m \right] g_i(\overline{x}) = 0. \tag{65}$$

According to the above equality and by the property (44), $A_5(\overline{x})$ reduces to

$$\begin{aligned} &\sum_{i=1}^{k} \alpha_i^* g_i(u^*) \left[f_i(\overline{x}) + \langle z^*, h(\overline{x}) \rangle_m \right] \\ &\quad - \sum_{i=1}^{k} \alpha_i^* g_i(\overline{x}) \left[f_i(u^*) + \langle z^*, h(u^*) \rangle_m \right] \\ &\quad = A_5(\overline{x}) = \sum_{i=1}^{k} \alpha_i^* g_i(u^*) \langle z^*, h(\overline{x}) \rangle_m. \end{aligned} \tag{66}$$

From relations $h(\overline{x}) \in -\mathbb{R}_+^m$, (44), (66), and $g_i(u^*) > 0$, we obtain

$$A_5(\overline{x}) \leqq 0 = A_5(u^*). \tag{67}$$

Hence, we reduce

$$\frac{1}{r} b_5(\overline{x}, u^*)\left(e^{r(A_5(\overline{x})-A_5(u^*))} - 1\right) \leqq 0 \quad \text{for any } r \neq 0. \tag{68}$$

Since A_5 is a strictly exponential B-(p,r)-invex function w.r.t. η and b_5 at $u^* \in pr_{\mathfrak{F}}\mathcal{D}$, we have

$$\begin{aligned} &\frac{1}{r} b_5(\overline{x}, u^*)\left(e^{r(A_5(\overline{x})-A_5(u^*)} - 1\right) \\ &\quad > \left\langle \langle \widetilde{\mathrm{a}}_5 \rangle, \left(e^{p\eta(\overline{x},u^*)} - \mathbf{1}\right)\right\rangle. \end{aligned} \tag{69}$$

From (68) and (69), we obtain

$$\frac{1}{p}\left\langle \langle \widetilde{\mathrm{a}}_5 \rangle, \left(e^{p\eta(x^*,u^*)} - \mathbf{1}\right)\right\rangle < 0. \tag{70}$$

This contradicts equality (61). Hence, the proof of theorem is complete. □

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 902692, 14 pages
http://dx.doi.org/10.1155/2013/902692

Research Article

Caristi Type Coincidence Point Theorem in Topological Spaces

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Received 6 June 2013; Revised 7 August 2013; Accepted 13 August 2013

Academic Editor: Wei-Shih Du

A generalized Caristi type coincidence point theorem and its equivalences in the setting of topological spaces by using a kind of nonmetric type function are obtained. These results are used to establish variational principle and its equivalences in d-complete spaces, bornological vector space, seven kinds of completed quasi-semimetric spaces equipped with Q-functions, uniform spaces with q-distance, generating spaces of quasimetric family, and fuzzy metric spaces.

1. Introduction

Caristi's fixed-point theorem [1, 2] and its equivalences, Ekeland variational principle [3, 4], and Takahashi minimization theorem are forceful tools in nonlinear analysis, control theory, and global analysis; see, for example, [3–5]. In the last two decades, Caristi's fixed-point theorem and Ekeland variational principle have been generalized and extended in several directions. About these, one can refer to, for example, [1–32] and the references therein. In particular, in [25], a very general Ekeland variational principle and Caristi's fixed-point theorem are presented, which give a unified approach to three classes of Ekeland type variational principle: in the first class, the underlying space is a sequentially complete uniform space (or equivalently, a sequentially complete F-type topological space), and the perturbation involves a family of topology generating pseudometrics (or quasimetrics); in the second class, the underlying space is a locally complete locally convex space (resp., a locally complete locally p-convex space), and the perturbation involves a family of topology generating seminorms (resp., topology generating p-homogeneous F-pseudonorms) or involves a single Minkowski functional; in the third class, the underlying space is a complete metric space, and the perturbation involves a w-distance or a τ-function. On the other hand, Banach fixed-point theorem has been extended to large class of nonmetric spaces which included d-complete topological spaces, symmetric spaces, and quasimetric spaces (see, e.g., [33–35]). But to our knowledge, neither Ekeland's variational principle nor any of its equivalents have been established in such d-complete topological spaces.

Motivated by the aforementioned works, we attempt to give a unified approach to the previous works. A generalized Caristi type coincidence point theorem in the setting of topological spaces by using a kind of nonmetric type function is proved. As an application of this Caristi's coincidence point theorem, an Ekeland type variational principle and its equivalences in the setting of topological spaces are obtained. Also, these results present Caristi type coincidence point theorem, variational principle, and its equivalences in d-complete topological spaces. Moreover, these results are used to establish variational principle and its equivalences in bornological vector space, seven kinds of completed quasi-semimetric spaces equipped with Q-functions, uniform spaces with q-distance, generating spaces of quasimetric family, and fuzzy metric spaces. The results of this paper uniformly extend and generalize the corresponding results appeared in the literature [1–4, 6–13, 15, 25, 26, 28, 30, 32].

2. Caristi Type Coincidence Point Theorem

The primary goal of this section is to establish two equivalent generalized Caristi type coincidence point theorems in the setting of topological spaces by using a kind of nonmetric type function. As an application of these Caristi's coincidence

point theorems, equivalent generalized Caristi type common fixed point theorem, Caristi type fixed point theorem for set valued, Caristi type fixed point theorem for single-valued map, Ekeland type variational principles and its equivalences in the setting of topological spaces are obtained. To establish our main results, we need the following definitions.

Definition 1 (see [15]). Let (X,τ) be a topological space. An extended real-valued function $f : X \to (-\infty,+\infty]$ is said to be sequentially lower monotone if for every sequence $\{x_n\}$ converging to x and satisfying

$$f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots \tag{1}$$

we have $f(x) \leq f(x_n)$, for each n.

Definition 2. Let (X,τ) be a topological space and $p : X \times X \to [0,+\infty]$ a function. A proper function $f : X \to (-\infty,+\infty]$ (i.e., $f(x)$ is not identically to $+\infty$) is said to be sequentially lower monotone with respect to p (in short, sequentially lower monotone with respect to p) if for any sequence $\{x_n\}$ in X satisfying $\sum_{n=1}^{\infty} p(x_n,x_{n+1}) < +\infty$, $\lim_{n\to\infty} x_n = x$ and $f(x_{n+1}) \leq f(x_n)$ for each n, we have $f(x) \leq f(x_n)$ for each $n \in N$.

Definition 3. Let (X,τ) be a topological space and $p : X \times X \to [0,+\infty]$ a function.

(1) (X,τ) is said to be p-complete [33, 34] if any sequence $\{x_n\}$ with $\sum_{n=1}^{\infty} p(x_n,x_{n+1}) < +\infty$ implies that the sequence $\{x_n\}$ is convergent to some $x \in X$.

(2) (X,τ) is said to be sequentially lower complete with respect to p, if any sequence $\{x_n\}$ with $\sum_{n=1}^{\infty} p(x_n,x_{n+1}) < +\infty$ implies that the sequence $\{x_n\}$ is convergent to some $x \in X$, and

$$\liminf_{n\to\infty} p(x_n,y) \leq p(x,y), \tag{2}$$

for any $y \in X$.

(3) Let $f : X \to (-\infty,+\infty]$ be a proper function. The topological space (X,τ) is said to be sequentially lower complete with respect to p and f if any sequence $\{x_n\}$ in X satisfying $\sum_{n=1}^{\infty} p(x_n,x_{n+1}) < +\infty$ and $f(x_{n+1}) \leq f(x_n)$ for all n is convergent to some $x \in X$, and (2) holds for any $y \in X$.

Remark 4. It is clearly that if (X,τ) is sequentially lower complete w.r.t. p, then for any proper function $f : X \to (-\infty,+\infty]$, (X,τ) is sequentially lower complete w.r.t. p and f.

Now, we can prove the following Caristi type coincidence point theorem in the setting of topological spaces.

Theorem 5. *Let (X,τ) be a topological space, $p : X \times X \to [0,+\infty]$ a function, $f : X \to (-\infty,+\infty]$ a proper, bounded from below, sequentially lower monotone function with respect to p, and $\varphi : (-\infty,+\infty) \to (0,+\infty)$ a nondecreasing function. Assume that (X,τ) is sequentially lower complete with respect to p and f. Let D be a nonempty subset of X, $g : D \to X$ a surjective function, I an index set, and, for each $i \in I$, $T_i : D \to 2^X$ a multivalued map. Then the following conclusions hold and are equivalent.*

(1) *Suppose that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X$, such that*

$$\begin{aligned} &\text{either} \quad 0 = p(g(x),y) < \varphi(f(g(x)))(f(g(x)) - f(y)), \\ &\text{or} \quad 0 < p(g(x),y) \leq \varphi(f(g(x)))(f(g(x)) - f(y)). \end{aligned} \tag{3}$$

Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that

$$f(g(u)) \leq f(x_0). \tag{4}$$

(2) *Suppose that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists an $i_0 \in I$ and $y \in T_{i_0}(x)$, such that (3) holds. Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that (4) holds.*

Proof. (1) We take an $x_0 \in X$, since $f \not\equiv +\infty$; without loss of generality, we can assume that $f(x_0) < +\infty$. Since $g : D \to X$ is a surjective function, there exists $u_0 \in D$, such that $g(u_0) = x_0$. If $g(u_0) \in \bigcap_{i\in I} T_i(u_0)$, then the conclusion holds. Otherwise, by the supposition, there exists $y \in X$, such that

$$\begin{aligned} &\text{either} \quad 0 = p(g(u_0),y) < \varphi(f(g(u_0)))(f(g(u_0)) - f(y)), \\ &\text{or} \quad 0 < p(g(u_0),y) \leq \varphi(f(g(u_0)))(f(g(u_0)) - f(y)). \end{aligned} \tag{5}$$

Thus,

$$\begin{aligned} p(x_0,y) &= p(g(u_0),y) \\ &< 2\varphi(f(g(u_0)))(f(g(u_0)) - f(y)) \\ &= 2\varphi(f(x_0))(f(x_0) - f(y)). \end{aligned} \tag{6}$$

Hence,

$$S_1 = \{y \in X : p(x_0,y) < 2\varphi(f(x_0))(f(x_0) - f(y))\} \neq \emptyset. \tag{7}$$

Obviously, for any $y \in S_1$, $f(x_0) > f(y)$. Thus, we can take $y_1 \in S_1$ such that

$$f(y_1) < \frac{1}{2}\left(f(x_0) + \inf_{x\in S_1} f(x)\right) < f(x_0). \tag{8}$$

Assume that y_n has been taken, and $y_n = g(u_n)$ for $u_n \in D$. If $g(u_n) \in \bigcap_{i\in I} T_i(u_n)$, then the conclusion holds. Otherwise,

$$S_{n+1} = \{y \in X : p(y_n,y) < 2\varphi(f(y_n))(f(y_n) - f(y))\} \neq \emptyset. \tag{9}$$

Note that for any $y \in S_{n+1}$, we have $f(y_n) > f(y)$. Thus, we can take $y_{n+1} \in S_{n+1}$ such that

$$f(y_{n+1}) < \frac{1}{2}\left(f(y_n) + \inf_{x\in S_{n+1}} f(x)\right) < f(y_n), \tag{10}$$

$$p(y_n,y_{n+1}) < 2\varphi(f(y_n))(f(y_n) - f(y_{n+1})). \tag{11}$$

It remains to consider the case that there is an infinite sequence $\{y_n\}$ which satisfies (10) and (11). From (10), we know that $\{f(y_n)\}$ is a decreasing sequence. Since φ is nondecreasing, it follows from (11) that

$$\begin{aligned}\sum_{n=1}^{\infty} p(y_n, y_{n+1}) &\le \sum_{n=1}^{\infty} 2\varphi(f(y_n))(f(y_n) - f(y_{n+1})) \\ &\le 2\varphi(f(x_0)) \sum_{n=1}^{\infty} (f(y_n) - f(y_{n+1})) \\ &\le 2\varphi(f(x_0)) \left(f(x_0) - \inf_{x\in X} f(x)\right) \\ &< +\infty.\end{aligned} \tag{12}$$

Since (X, τ) is sequentially lower complete w.r.t. p and f, there exists $v \in X$ such that $\lim_{n\to\infty} y_n = v$ and $\liminf_{n\to\infty} p(y_n, x) \le p(v, x)$ for any $x \in X$. Assume that $v = g(u)$, $u \in D$. We claim that the conclusion holds for u. Since f is sequentially lower monotone, we have

$$f(g(u)) = f(v) \le f(y_n) < f(x_0). \tag{13}$$

That is (4) holds. If $g(u) \notin \bigcap_{i\in I} T_i(u)$, then there exists $y' \in X$, such that

$$\begin{aligned}p(v, y') &= p(g(u), y') \\ &< 2\varphi(f(g(u)))(f(g(u)) - f(y')) \\ &= 2\varphi(f(v))(f(v) - f(y')).\end{aligned} \tag{14}$$

It follows from $\liminf_{n\to\infty} p(y_n, y') \le p(v, y')$ that there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that

$$\lim_{k\to\infty} p(y_{n_k}, y') = \liminf_{n\to\infty} p(y_n, y'). \tag{15}$$

From this and (14) we know that there exists an m such that for all $k \ge m$,

$$p(y_{n_k}, y') < 2\varphi(f(v))(f(v) - f(y')). \tag{16}$$

From (13) we have

$$p(y_{n_k}, y') < 2\varphi(f(y_{n_k}))(f(y_{n_k}) - f(y')), \tag{17}$$

that is, $y' \in S_{n_k+1}$, $k \ge m$. From (10) we have

$$2f(y_{n_k+1}) - f(y_{n_k}) \le \inf_{y\in S_{n_k+1}} f(y) \le f(y'). \tag{18}$$

By letting $k \to +\infty$, we get that

$$\lim_{k\to\infty} f(y_{n_k}) = \lim_{n\to\infty} f(y_n) \le f(y'). \tag{19}$$

Combing with (13) we have $f(v) \le f(y')$ which contradicts (14). Thus the conclusion of Theorem 5(1) holds.

(2) It is clear that Theorem 5(1) $\Rightarrow$ Theorem 5(2). Now, we prove that Theorem 5(2) $\Rightarrow$ Theorem 5(1). Assume that the conditions of Theorem 5(1) are satisfied; then, for each $x \in D$, if $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X$, such that (3) holds. Then we get that

$$G(x) = \{y \in X : g(x), y \text{ satisfy } (3)\} \ne \emptyset \tag{20}$$

and $g(x) \notin G(x)$. For each $i \in I$, we define $\widetilde{T}_i : D \to 2^X$ by

$$\widetilde{T}_i(x) = \begin{cases} T_i(x), & \text{if } g(x) \in \bigcap_{i\in I} T_i(x); \\ T_i(x) \cup G(x), & \text{if } g(x) \notin \bigcap_{i\in I} T_i(x). \end{cases} \tag{21}$$

It is clear that $g(x) \in \bigcap_{i\in I} T_i(x)$ if and only if $g(x) \in \bigcap_{i\in I} \widetilde{T}_i(x)$. Also, $\{\widetilde{T}_i\}$ satisfies the condition of Theorem 5(2). Then by the conclusion of Theorem 5(2), there exists a coincidence point $u \in D$ of g and $\{\widetilde{T}_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} \widetilde{T}_i(u)$, such that (4) holds. Therefore, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that (4) holds. That is Theorem 5(1) holds. The proof is completed. □

Remark 6. If for any $x, y \in X$ with $x \ne y$ implies that $p(x, y) > 0$, then the conclusion (1) of Theorem 5 can be rewritten as: for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$ there exists $y \in X$, such that

$$0 < p(g(x), y) \le \varphi(f(g(x)))(f(g(x)) - f(y)). \tag{22}$$

In particular, if $D = X$ and $g : D \to X$ is the identity map in Theorem 5, then we obtain the following generalized Caristi type common fixed point theorem, Caristi type fixed point theorem for set-valued, and single-valued map.

Theorem 7. *Let (X, τ) be a topological space, $p : X \times X \to [0, +\infty]$ a function, $f : X \to (-\infty, +\infty]$ a proper, bounded from below, sequentially lower monotone function with respect to p, and $\varphi : (-\infty, +\infty) \to (0, +\infty)$ a nondecreasing function. Assume that (X, τ) is sequentially lower complete w.r.t. p and f. Then the following conclusions hold and are all equivalent to Theorem 5.*

(1) *Let I be an index set, and, for each $i \in I$, let $T_i : X \to 2^X$ be a multivalued map. Suppose further that for each $x \in X$ with $x \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X$, such that*

$$\begin{aligned}&\text{either} \quad 0 = p(x, y) < \varphi(f(x))(f(x) - f(y)), \\ &\text{or} \quad 0 < p(x, y) \le \varphi(f(x))(f(x) - f(y)).\end{aligned} \tag{23}$$

Then for any $x_0 \in X$, there exists a common fixed point $u \in X$ of $\{T_i\}_{i\in I}$; that is, $u \in \bigcap_{i\in I} T_i(u)$, such that

$$f(u) \le f(x_0). \tag{24}$$

(2) *Let $T : X \to 2^X$ be a multivalued map. Suppose further that for each $x \in X$ with $x \notin T(x)$, there exists $y \in X$, such that (23) holds. Then for any $x_0 \in X$, T has a fixed point $u \in X$, such that (24) holds.*

(3) *Let $T : X \to X$ be a map. Suppose further that for each $x \in X$ with $x \neq T(x)$, there exists $y \in X$, such that* (23) *holds. Then for any $x_0 \in X$, T has a fixed point $u \in X$, such that* (24) *holds.*

Proof. It is clear that the following implications hold: Theorem 5(1) $\Rightarrow$ Theorem 7(1) $\Rightarrow$ Theorem 7(2) $\Rightarrow$ Theorem 7(3).

Now, we prove that Theorem 7(3) $\Rightarrow$ Theorem 5(1). Assume that the conditions of Theorem 5 hold. It is similar to the proof of Lemma 2.1 in [36] that, by using the axiom of choice, we can prove that there exists a subset $E \subseteq D$ such that $g(E) = g(D) = X$ and $g : E \to X$ is one-to-one. Define a map $T : X \to X$ by

$$T(x) = \begin{cases} x, & \text{if } g(u) = x \in \bigcap_{i\in I} T_i(u),\ u \in E; \\ y, & \text{if } g(u) = x \notin \bigcap_{i\in I} T_i(u),\ u \in E, \end{cases} \tag{25}$$

where $y \in X$ such that either

$$\begin{aligned} 0 &= p(g(u), y) = p(x, y) \\ &< \varphi(f(g(u)))(f(g(u)) - f(y)) \\ &= \varphi(f(x))(f(x) - f(y)), \end{aligned} \tag{26}$$

or

$$\begin{aligned} 0 &< p(g(u), y) = p(x, y) \\ &\leq \varphi(f(g(u)))(f(g(u)) - f(y)) \\ &= \varphi(f(x))(f(x) - f(y)). \end{aligned} \tag{27}$$

Then T satisfies the condition of Theorem 7(3); thus for any $x_0 \in X$, T has a fixed point $v \in X$, such that $f(v) \leq f(x_0)$ holds. Since $g(E) = X$, there exists $u \in E \subset D$, such that $g(u) = v$. Then by the definition of T, we get that $g(u) = v \in \bigcap_{i\in I} T_i(u)$. That is, the conclusion of Theorem 5 holds. The proof is completed. □

The following corollary is an extension of the results in [19, 20]. In Corollary 8, we remove the condition that η is nondecreasing, which is used in [19, 20].

Corollary 8. *Let (X, d) be a completed metric space. Suppose that $\eta : [0, +\infty) \to (-\infty, +\infty)$ satisfies $\eta(0) = 0$ and that $f : X \to (-\infty, +\infty)$ is lower semicontinuous on X, and there exist $x_0 \in X$ and two real numbers $a, b \in (-\infty, +\infty)$, such that*

$$f(x) \geq ad(x, x_0) + b, \tag{28}$$

and one of the following conditions is satisfied:

(i) *$a \geq 0$, η is nonnegative on $W = \{d(x, y) : x, y \in X\}$, and there exist $c > 0$ and $\varepsilon > 0$ such that*

$$\eta(t) \geq ct, \quad \forall t \in \{t \geq 0 : \eta(t) \leq \varepsilon\} \cap W; \tag{29}$$

(ii) *$a < 0$, $\eta(t) + at$ is nonnegative on W, and there exist $c > 0$ and $\varepsilon > 0$ such that*

$$\eta(t) + at \geq ct, \quad \forall t \in \{t \geq 0 : \eta(t) + at \leq \varepsilon\} \cap W. \tag{30}$$

Then each Caristi type mapping $T : X \to X$ (i.e., satisfying $\eta(d(x, Tx)) \leq f(x) - f(Tx), \forall x \in X$) has a fixed point $v \in X$.

Proof.

Case (i). It follows from $a \geq 0$ and (28) that f is a bounded from below, lower semicontinuous function on X. Let

$$\alpha = \inf_{x\in X} f(x), \quad Q = \{x \in X : f(x) \leq \alpha + \varepsilon\}. \tag{31}$$

Then, as in the proof of the Theorem 1 in [20] that (Q, d) is a complete metric space, $TQ \subset Q$, and

$$0 \leq \eta(d(x, Tx)) \leq f(x) - f(Tx) \leq \varepsilon, \quad \forall x \in Q. \tag{32}$$

Thus we have, by (29),

$$cd(x, Tx) \leq \eta(d(x, Tx)) \leq f(x) - f(Tx) \leq \varepsilon, \quad \forall x \in Q. \tag{33}$$

Define a function $p : Q \times Q \to [0, +\infty]$ by $p(x, y) = cd(x, y)$. Let $\{x_n\}$ be a sequence in Q, such that

$$\sum_{n=1}^{\infty} p(x_n, x_{n+1}) = \sum_{n=1}^{\infty} cd(x_n, x_{n+1}) < +\infty. \tag{34}$$

This implies that $\{x_n\}$ is a Cauchy sequence in Q. Since (Q, d) is a complete metric space, $\{x_n\}$ is a convergent sequence in Q. If $\lim_{n\to\infty} x_n = x$, then for any $y \in Q$, $\lim_{n\to\infty} p(x_n, y) = \lim_{n\to\infty} cd(x_n, y) = cd(x, y) = p(x, y)$. Thus, (Q, d) is sequentially lower complete w.r.t. p. Clearly, all conditions of Theorem 7 are satisfied. Therefore, T has a fixed point in Q.

Case (ii). Let

$$\begin{aligned} \psi(x) &= f(x) - ad(x, x_0), \quad \forall x \in X, \\ \eta_1(t) &= \eta(t) + at, \quad \forall t \in [0, +\infty). \end{aligned} \tag{35}$$

Then we have

$$\eta_1(d(x, Tx)) \leq \psi(x) - \psi(Tx), \quad \forall x \in X. \tag{36}$$

Thus, the conclusion can be deduced by Case (i). The proof is completed. □

In Theorem 9, by using Theorem 5, we present a generalized Ekeland type variational principle, maximal element theorem for a family of multivalued maps, equilibrium theorem, and a generalized Takahashi minimization theorem in topological spaces and prove the equivalence among these results.

Theorem 9. *Let (X, τ) be a topological space, $p : X \times X \to [0, +\infty]$ a function, $f : X \to (-\infty, +\infty]$ a proper, bounded from below, sequentially lower monotone function with respect*

to p, and $\varphi : (-\infty, +\infty) \to (0, +\infty)$ a nondecreasing function. Assume that (X, τ) is sequentially lower complete with respect to p and f. Then for any $x_0 \in X$, the following conclusions hold, and they are equivalent to Theorem 5.

(I) *(Ekeland type variational principle in topological spaces) There exists $v \in X$, such that $f(v) \le f(x_0)$ and*

$$\begin{aligned} &p(v,x) = \varphi(f(v))(f(v) - f(x)), \\ &\qquad \forall x \in X \text{ with } p(v,x) = 0, \\ &p(v,x) > \varphi(f(v))(f(v) - f(x)), \\ &\qquad \forall x \in X \text{ with } p(v,x) > 0. \end{aligned} \tag{37}$$

(II) *(Maximal element for a family of multivalued maps in topological spaces) Let I be any index set, and, for each $i \in I$, let $T_i : X \to 2^X$ be a multivalued map. Assume that for each $(x, i) \in X \times I$ with $T_i(x) \ne \emptyset$, there exists $y = y(x, i) \in X$, such that (23) holds. Then there exists $v \in X$, such that $f(v) \le f(x_0)$ and $T_i(v) = \emptyset$ for each $i \in I$.*

(III) *(Equilibrium theorem in topological spaces) Let $F : X \times X \to (-\infty, +\infty]$ be a proper, bounded from below, sequentially lower monotone function in the first argument. Suppose that there exists $w \in X$ such that for each x with*

$$\{u \in X : F(x,u) > 0\} \ne \emptyset, \tag{38}$$

there exists $y = y(x) \in X$ such that

$$\begin{aligned} &\text{either} \quad 0 = p(x,y) < \varphi(F(x,w))(F(x,w) - F(y,w)), \\ &\text{or} \quad 0 < p(x,y) \le \varphi(F(x,w))(F(x,w) - F(y,w)). \end{aligned} \tag{39}$$

If (X, τ) is sequentially lower complete w.r.t. p and $F(\cdot, w)$, then there exists $v \in X$, such that $F(v, w) \le F(x_0, w)$ and $F(v, y) \le 0$, for all $y \in X$.

(IV) *(Generalized Takahashi minimization theorem in topological spaces) Suppose that for any x with $f(x) > \inf_{x \in X} f(x)$, there exists $y \in X$ such that (23) holds. Then there exists $v \in X$, such that $f(v) \le f(x_0)$ and $f(v) = \inf_{x \in X} f(x)$.*

Proof. "Theorem 5(1) ⇒Theorem 9(II)." If the conclusion of (II) does not hold, then for any $x \in X$, there exists i, such that $T_i(x) \ne \emptyset$. By the hypotheses of (II), there exists $y = y(x, i) \in X$, such that (23) holds; thus, $y \ne x$. Let $D = X$, $g = I_d$ (the identical map of X),

$$H_i(x) = \begin{cases} (T_i(x) \cup \{y(x,i)\}) \setminus \{x\}, & \text{if } T_i(x) \ne \emptyset, \\ \{x\}, & \text{if } T_i(x) = \emptyset. \end{cases} \tag{40}$$

Then the conditions of Theorem 5(1) are satisfied for $\{H_i\}_{i \in I}$, $D = X$ and $g = I_d$. Thus, from Theorem 5(1) there exists $v \in X$ such that $v \in \bigcap_{l \in I} H_l(v)$. This is a contradiction with the definition of H_i. Therefore, there exists $v \in X$ with $f(v) \le f(x_0)$ such that $T_i(v) = \emptyset$ for any $i \in I$.

"(II) ⇒ (III)" Let

$$T_y(x) = \begin{cases} \{x\}, & \text{if } F(x,y) > 0, \\ \emptyset, & \text{if } F(x,y) \le 0. \end{cases} \tag{41}$$

From this we know that if $T_y(x) \ne \emptyset$, then $F(x, y) > 0$. By the hypotheses of (III) there exists $z = z(x) \in X$, such that

$$\begin{aligned} &\text{either} \quad 0 = p(x,z) < \varphi(F(x,w))(F(x,w) - F(z,w)), \\ &\text{or} \quad 0 < p(x,z) \le \varphi(F(x,w))(F(x,w) - F(z,w)). \end{aligned} \tag{42}$$

By using (II) for $f(x) = F(x, w)$, there exists $v \in X$, such that $F(v, w) \le F(x_0, w)$ and $T_y(v) = \emptyset$ for any $y \in X$; that is $F(v, y) \le 0$, for any $y \in X$.

"(III) ⇒ (IV)" Let $F(x, y) = f(x) - \inf_{x \in X} f(x)$. If

$$\{u \in X : F(x,u) > 0\} \ne \emptyset, \tag{43}$$

then $f(x) > \inf_{x \in X} f(x)$. Fix $w \in X$; then $F(x, w) \le F(x_0, w)$ if and only if $f(x) \le f(x_0)$. By the hypothesis of (IV), there exists $y = y(x) \in X$, such that either

$$\begin{aligned} 0 &= p(x,y) < \varphi(f(x))(f(x) - f(y)) \\ &= \varphi(f(x))(F(x,w) - F(y,w)), \end{aligned} \tag{44}$$

or

$$\begin{aligned} 0 &< p(x,y) \le \varphi(f(x))(f(x) - f(y)) \\ &= \varphi(f(x))(F(x,w) - F(y,w)). \end{aligned} \tag{45}$$

Define $\psi : (-\infty, +\infty] \to (0, +\infty)$ by $\psi(t) = \varphi(t + \inf_{x \in X} f(x))$. Then $\psi(F(x, y)) = \varphi(f(x))$; thus, the hypotheses of (III) are satisfied for ψ and F. It follows from (III) that there exists $v \in X$, such that $F(v, w) \le F(x_0, w)$ and $F(v, y) \le 0$, for any $y \in X$. This implies that $f(v) \le f(x_0)$ and $f(v) \le \inf_{x \in X} f(x)$, that is, $f(v) = \inf_{x \in X} f(x)$.

"(IV) ⇒ (I)" If (I) does not hold, then for any $x \in X$ with $f(x) \le f(x_0)$, there exists $y = y(x)$, such that

$$\begin{aligned} &\text{either} \quad 0 = p(x,y) < \varphi(f(x))(f(x) - f(y)), \\ &\text{or} \quad 0 < p(x,y) \le \varphi(f(x))(f(x) - f(y)). \end{aligned} \tag{46}$$

This implies that condition of (IV) holds on $X_1 = \{x \in X : f(x) \le f(x_0)\}$. Then, by (IV) there exists $v \in X_1$, such that $f(v) \le f(x_0)$ and $f(v) = \inf_{x \in X_1} f(x) = \inf_{x \in X} f(x)$. This is a contradiction with (46). Thus, (I) holds.

"(I) ⇒ Theorem 5(1)" From (I), there exists $v \in X$, such that $f(v) \le f(x_0)$ and (37) holds. Since g is a surjective mapping, there exists a $u \in D$, such that $g(u) = v$. We claim that $g(u) \in \bigcap_{l \in I} T_l(u)$. If $g(u) \notin \bigcap_{l \in I} T_l(u)$, by the hypotheses of Theorem 5(1), there exists $y \in X$, such that (3) holds. This is a contradiction with (37). Thus, $g(u) \in \bigcap_{l \in I} T_l(u)$ and $f(g(u)) \le f(x_0)$. That is, Theorem 5(1) holds.

The proof is completed. □

Remark 10. Theorem 5–Theorem 9 also present Caristi type coincidence point theorem, Ekeland type variational principle, and their equivalences in p-complete topological spaces. Moreover, from [34] we know that d-complete topological spaces include d-complete symmetric (semimetric) spaces and complete quasimetric spaces.

3. Applications to Some Non-Metric Spaces

In this section, we show that our results in section two can be used with many nonmetric spaces. The reader may refer to the references [6, 13, 15, 25, 28, 37] for the notions and symbols in this section.

In [6], the authors introduce the concept of Q-function in quasimetric spaces which generalizes the notion of the τ-function and w-distance, and they also prove an Ekeland variational principle as well as its equivalences in such spaces.

For the convenience of the reader we present the main concept of quasimetric space in the following (refer to [38]).

Let X be a nonempty set. A real valued function $d : X \times X \to [0, +\infty)$ is said to be a quasi-semimetric on X if the following conditions are satisfied:

(QM1) $d(x, y) \geq 0$ and $d(x, x) = 0$ for all $x, y \in X$;

(QM2) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If further

(QM3) $d(x, y) = d(y, x) = 0$ implies $x = y$ for all $x, y \in X$,

then $d : X \times X \to [0, +\infty)$ is said to be a quasimetric on X. A nonempty set X together with a quasimetric d (or quasi-semimetric d) is called a quasimetric space (or quasi-semimetric space), and it is denoted by (X, d). If (X, d) is a quasi-semimetric space, for $x \in X$ and $r > 0$, we define the balls in X by the formula

$B(x, r) = \{y \in X : d(x, y) < r\}$—the open ball, and

$B[x, r] = \{y \in X : d(x, y) \leq r\}$—the closed ball.

The topology τ of a quasi-semimetric (X, d) can be defined starting from the family $V(x)$ of neighborhoods of an arbitrary point $x \in X$:

$V \in V(x) \Leftrightarrow \exists r > 0$ such that $B(x, r) \subset V \Leftrightarrow \exists r' > 0$ such that $B[x, r'] \subset V$.

The convergence of a sequence $\{x_n\}$ to x with respect to τ can be characterized by $d(x, x_n) \to 0$.

Definition 11. Let (X, d) be a quasi-semimetric space. A sequence $\{x_n\}$ in X is said to be

(i) left d-Cauchy if for each $\varepsilon > 0$ there is a point x in X and an integer k such that $d(x, x_m) < \varepsilon$ for all $m \geq k$;

(ii) right d-Cauchy if for each $\varepsilon > 0$ there is a point x in X and an integer k such that $d(x_m, x) < \varepsilon$ for all $m \geq k$;

(iii) d-Cauchy if for each $\varepsilon > 0$ there is an integer k such that $d(x_r, x_s) < \varepsilon$ for all $r, s \geq k$;

(iv) right K-Cauchy if for each $\varepsilon > 0$ there is an integer k such that $d(x_r, x_s) < \varepsilon$ for all $r \geq s \geq k$;

(v) left K-Cauchy if for each $\varepsilon > 0$ there is an integer k such that $d(x_r, x_s) < \varepsilon$ for all $s \geq r \geq k$;

(vi) weakly left (right) K-Cauchy if for each $\varepsilon > 0$ there is an integer k such that $d(x_k, x_m) < \varepsilon$ ($d(x_m, x_k) < \varepsilon$) for all $m \geq k$;

(vii) corresponding to the seven definitions of Cauchy sequence in a quasi-semimetric space, we have seven notions of completeness: X is said to be left (right) d-, [weakly] left (right) K-, or d-sequentially complete if every left (right) d-, [weakly] left (right) K-, or d- (resp.) Cauchy sequence in X converges to some point in X (with respect to the topology τ induced on X by d).

Remark 12. The implications between the seven notions of Cauchyness (refer to [38]) are as follows: d-Cauchy $\Rightarrow$ left and right K-Cauchy, left (right) K-Cauchy $\Rightarrow$ weakly left (right) K-Cauchy $\Rightarrow$ left (right) d-Cauchy.

Definition 13 (see [6]). Let (X, d) be a quasi-semimetric space. A function $q : X \times X \to [0, +\infty)$ is called a Q-function on X if the following conditions are satisfied:

(Q1) for all $x, y, z \in X$, $q(x, z) \leq q(x, y) + q(y, z)$;

(Q2) if $x \in X$, $\{y_n\}_{n \in N}$ is a sequence in X such that it converges to a point y (with respect to the quasi-semimetric) and $q(x, y_n) \leq M$ for some $M = M(x) > 0$, then $q(x, y) \leq M$;

(Q3) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$.

Lemma 14. *Let (X, d) be a quasi-semimetric space with one of seven completeness defined in Definition 11(vii). If q is a Q-function on X, then (X, d) is sequentially lower complete w.r.t. q.*

Proof. Assume that $\{x_n\}$ is a sequence in X and $\sum_{n=1}^{\infty} q(x_n, x_{n+1}) < +\infty$. Let $\lambda_n = \sum_{k=n+1}^{\infty} q(x_n, x_{n+1})$, then we have $\lim_{n \to \infty} \lambda_n = 0$ and for any $m > n$,

$$q(x_n, x_m) \leq \lambda_n. \tag{47}$$

By (Q3), for any $\varepsilon > 0$, there exists $\delta > 0$ such that $q(x, y) \leq \delta$ and $q(x, z) \leq \delta$ imply $d(y, z) \leq \varepsilon$. For the $\delta > 0$, there exists $K > 0$, such that $\lambda_n < \delta$ for any $n \geq K$. It follows from (47) that for any $r, s \geq n > K$, we have

$$q(x_n, x_r) \leq \lambda_n < \delta, \qquad q(x_n, x_s) \leq \lambda_n < \delta. \tag{48}$$

Thus (Q3) implies that $d(x_r, x_s) < \varepsilon$. That is, $\{x_n\}$ is a d-Cauchy sequence. Therefore, by Remark 12, $\{x_n\}$ is any one of seven Cauchy sequences in Definition 11. Thus, $\{x_n\}$ converges to some $x \in X$. Equation (47) and (Q2) imply that $q(x_n, x) \leq \lambda_n$. This shows that $\lim_{n \to \infty} q(x_n, x) = 0$. For any $y \in X$, it follows from (Q1) that

$$q(x_n, y) \leq q(x_n, x) + q(x, y). \tag{49}$$

Therefore

$$\liminf_{n \to \infty} q(x_n, y) \leq \lim_{n \to \infty} q(x_n, x) + q(x, y) = q(x, y). \tag{50}$$

Thus, (X, d) is sequentially lower complete w.r.t. q. The proof is completed. □

From Lemma 14 and Theorems 5, 7, and 9, we can get the following Ekeland type variational principle and its equivalences in quasi-semimetric spaces equipped with Q-functions, which also generalize the results in [6, 12].

Theorem 15. *Let (X, d) be a complete quasi-semimetric space with one of seven completeness defined in Definition 11(vii) and $q : X \times X \to [0, +\infty)$ a Q-function on X. Let $f : X \to (-\infty, +\infty]$ be a proper, bounded from below, sequentially lower monotone function and $\varphi : (-\infty, +\infty) \to (0, +\infty)$ a nondecreasing function. Let D be a nonempty subset of X, $g : D \to X$ a surjective function, I an index set and for each $i \in I$, $T_i : D \to 2^X$ a multivalued map. Then the following conclusions hold and are equivalent.*

(1) *Suppose that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X$, such that*

$$\begin{aligned} &\text{either} \quad 0 = q(g(x), y) < \varphi(f(g(x)))(f(g(x)) - f(y)), \\ &\text{or} \quad 0 < q(g(x), y) \le \varphi(f(g(x)))(f(g(x)) - f(y)). \end{aligned} \tag{51}$$

Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that

$$f(g(u)) \le f(x_0). \tag{52}$$

(2) *Suppose that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists an $i_0 \in I$ and $y \in T_{i_0}(x)$, such that (51) holds. Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that (52) holds.*

(3) *Let I be an index set, and for each $i \in I$, let $T_i : X \to 2^X$ be a multivalued map. Suppose further that for each $x \in X$ with $x \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X$, such that*

$$\begin{aligned} &\text{either} \quad 0 = q(x, y) < \varphi(f(x))(f(x) - f(y)), \\ &\text{or} \quad 0 < q(x, y) \le \varphi(f(x))(f(x) - f(y)). \end{aligned} \tag{53}$$

Then for any $x_0 \in X$, there exists a common fixed point $u \in X$ of $\{T_i\}_{i\in I}$; that is, $u \in \bigcap_{i\in I} T_i(u)$, such that

$$f(u) \le f(x_0). \tag{54}$$

(4) *(Ekeland type variational principle in quasi-semimetric spaces) For any $x_0 \in X$, there exists $v \in X$, such that $f(v) \le f(x_0)$ and*

$$\begin{aligned} &q(v, x) = \varphi(f(v))(f(v) - f(x)), \\ &\qquad \forall x \in X \text{ with } q(v, x) = 0, \\ &q(v, x) > \varphi(f(v))(f(v) - f(x)), \\ &\qquad \forall x \in X \text{ with } q(v, x) > 0. \end{aligned} \tag{55}$$

Moreover, the rest of corresponding equivalent principles in Theorem 9 hold.

In particularly, if (X, d) is a complete quasi-metric space, then from Theorem 15, we have the following results.

Theorem 16. *Let (X, d) be a complete quasimetric space with one of seven completeness defined in Definition 11(vii) and $q : X \times X \to [0, +\infty)$ a Q-function on X. Let $f : X \to (-\infty, +\infty]$ be a proper, bounded from below, sequentially lower monotone function and $\varphi : (-\infty, +\infty) \to (0, +\infty)$ a nondecreasing function. Let D be a nonempty subset of X, $g : D \to X$ a surjective function, I an index set and for each $i \in I$, $T_i : D \to 2^X$ a multivalued map. Then the following conclusions hold and are equivalent.*

(1) *Suppose that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X \setminus \{g(x)\}$, such that*

$$q(g(x), y) \le \varphi(f(g(x)))(f(g(x)) - f(y)). \tag{56}$$

Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that

$$f(g(u)) \le f(x_0). \tag{57}$$

(2) *Suppose that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists an $i_0 \in I$ and $y \in T_{i_0}(x) \setminus \{g(x)\}$, such that (56) holds. Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that (57) holds.*

(3) *Let I be an index set, and for each $i \in I$, let $T_i : X \to 2^X$ be a multivalued map. Suppose further that for each $x \in X$ with $x \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X \setminus \{x\}$, such that*

$$q(x, y) \le \varphi(f(x))(f(x) - f(y)). \tag{58}$$

Then for any $x_0 \in X$, there exists a common fixed point $u \in X$ of $\{T_i\}_{i\in I}$; that is, $u \in \bigcap_{i\in I} T_i(u)$, such that

$$f(u) \le f(x_0). \tag{59}$$

(4) *(Ekeland type variational principle in quasimetric spaces) For any $x_0 \in X$, there exists $v \in X$, such that $f(v) \le f(x_0)$ and*

$$q(v, x) > \varphi(f(v))(f(v) - f(x)), \quad \forall x \in X \text{ with } x \ne v. \tag{60}$$

Proof. The equivalence of the conclusions (1)–(4) is clear. We only prove (4). It follows from (4) of Theorem 15 that for any $x_0 \in X$, there exists $v \in X$, such that $f(v) \le f(x_0)$ and

$$\begin{aligned} &q(v, x) = \varphi(f(v))(f(v) - f(x)), \\ &\qquad \forall x \in X \text{ with } q(v, x) = 0, \\ &q(v, x) > \varphi(f(v))(f(v) - f(x)), \\ &\qquad \forall x \in X \text{ with } q(v, x) > 0. \end{aligned} \tag{61}$$

If v satisfies (60), then conclusion (4) is proved. Otherwise, there exists $x \in X \setminus \{v\}$, such that

$$0 = q(v, x) = \varphi(f(v))(f(v) - f(x)). \quad (62)$$

If x and y satisfy (62), then $q(v, x) = 0$ and $q(v, y) = 0$. By using (Q3) in Definition 13, we get that $d(x, y) = 0$. It follows from (QM1) that $d(x, x) = 0$, and then (QM3) implies that $x = y$. That is, there is only one point $\overline{x}$ which satisfies (62). Let

$$S(x) = \{y \in X : q(x, y) \le \varphi(f(x))(f(x) - f(y))\}. \quad (63)$$

Then $S(v) = \{\overline{x}\}$. Since $\overline{x} \in S(v)$, we can imply that $S(\overline{x}) \subset S(v) = \{\overline{x}\}$. This shows that for any $x \ne \overline{x}$, $x \notin S(\overline{x})$; that is, $\overline{x}$ satisfies (60). The proof is completed. □

Definition 17 (see, e.g., [37]). Let X be a real vector space; a collection $\mathscr{B}$ of subsets of X is called a vector bornology on X, if it satisfies the following conditions:

(B1) $x \in X$ implies that $\{x\} \in \mathscr{B}$;

(B2) $B_1 \subset B_2$ and $B_2 \in \mathscr{B}$ imply that $B_1 \in \mathscr{B}$;

(B3) $B_1, B_2 \in \mathscr{B}$ implies that $B_1 \cup B_2 \in \mathscr{B}$;

(B4) $B_1, B_2 \in \mathscr{B}$ implies that

$$B_1 + B_2 = \{x + y : x \in B_1, y \in B_2\} \in \mathscr{B}; \quad (64)$$

(B5) for any bounded interval $I \subset (-\infty, +\infty)$, $B \in \mathscr{B}$ implies that

$$I \cdot B = \{\alpha x : \alpha \in I, x \in B\} \in \mathscr{B}. \quad (65)$$

In view of (B5), if $B \in \mathscr{B}$, so is its balanced hull B_b which is defined by $B_b = [-1, 1] \cdot B$.

Definition 18. The ordered pair $(X, \mathscr{B})$ is called a bornological vector space (in short: BVS), and every element of $\mathscr{B}$ is called a bounded subset (with respect to $\mathscr{B}$).

Definition 19 (see, e.g., [28, 37]). Let $(X, \mathscr{B})$ be a bornological vector space.

(i) A sequence $\{x_n\}$ in X is said to be Mackey-convergent (or M-convergent) to a point x, denoted by $\lim^b_{n\to\infty} x_n = x$, if there is a balanced $B \in \mathscr{B}$ and a sequence of positive real numbers $\{\lambda_n\}$ such that $\lim_{n\to\infty}\lambda_n = 0$ and $x_n - x \in \lambda_n B$ for any $n \in N$. Also, we say that x is a bornological limit of $\{x_n\}$.

(ii) A sequence $\{x_n\}$ in X is said to be Mackey-Cauchy (or M-Cauchy) if there is a balanced $B \in \mathscr{B}$ and a double sequence of positive real numbers $\{\lambda_{mn}\}$ such that $\lim_{m,n\to\infty}\lambda_{mn} = 0$ and $x_m - x_n \in \lambda_{mn}B$ for any $m, n \in N$.

(iii) $A \subset X$ is said to be Mackey-closed (or M-closed) if it contains all bornological limits of any sequences in A.

(iv) $A \subset X$ is said to be Mackey-complete (or M-complete) if every M-Cauchy sequence in A will be M-convergent to some element in A.

(v) A BVS $(X, \mathscr{B})$ is said to be separated if every M-convergent sequence is M-convergent to exactly one bornological limit.

Remark 20. From Lemma 2.13 in [28] we know that if $A \subset X$ is a M-complete subset, then A is M-closed. On the other hand, if $(X, \mathscr{B})$ is M-complete and $A \subset X$ is M-closed, then A is M-complete. For the details about BVS, one can refer to [17, 28, 37].

The collection of all (complements of) M-closed subsets of X defines a topology on X, and we called it bornological topology. Therefore, $(X, \mathscr{B})$ endowed this topology is a topological space (but, from Remark 2.4 in [28], we can see that it is rarely a vector topology with respect to the algebraic structure of X). In the following, we will assume that $(X, \mathscr{B})$ is separated; that is (v) in Definition 19 holds.

Let X be a separated bornological vector space and $P : X \to (-\infty, +\infty]$ a positively homogeneous subadditive function. By Lemma 4.4 in [28], $P(x) > 0$ for any nonzero x if P satisfies the following condition:

(P1) the set $C = \{x \in X : P(x) \le 1\}$ is M-complete and bounded.

Lemma 21. *Let $p : X \times X \to (-\infty, +\infty)$ be defined by $p(x, y) = P(x - y)$. If (P1) holds, then $(X, \mathscr{B})$ is sequentially lower complete with respect to p.*

Proof. Let $\{x_n\}$ be a sequence in X and $\sum_{n=1}^{\infty} p(x_n, x_{n+1}) = \sum_{n=1}^{\infty} P(x_n - x_{n+1}) < +\infty$. Then for any $0 < \delta < 1$, there exists a positive integer n_0, such that $\sum_{n=n_0}^{\infty} P(x_n - x_{n+1}) < \delta$. Since P is subadditive, we get that for any $m > n_0$,

$$P(x_{n_0} - x_m) \le \sum_{n=n_0}^{m-1} P(x_n - x_{n+1}) < \delta. \quad (66)$$

From this we have

$$P(x_{n_0} - x_m) \le \delta < 1. \quad (67)$$

So we have $x_{n_0} - x_m \in C$. For any $m > n$, let $\lambda_{mn} = \sum_{k=n}^{m-1} P(x_k - x_{k+1})$, and then we have $P(x_n - x_m) \le \lambda_{mn}$, and $\lim_{m,n\to\infty}\lambda_{mn} = 0$. It follows from

$$P((x_{n_0} - x_m) - (x_{n_0} - x_n)) = P(x_n - x_m) \le \lambda_{mn} \quad (68)$$

that $(x_{n_0} - x_m) - (x_{n_0} - x_n) \in \lambda_{mn}C \subset \lambda_{mn}C_b$ where C_b, is the (bounded) balanced hull of C. That is, $\{x_{n_0} - x_n\}$ is M-Cauchy. Since C is M-complete, $\{x_{n_0} - x_n\}$ is M-convergent. Thus, $\{x_n\}$ is also M-convergent. Assume that $\{x_n\}$ is M-convergent to a point x. If we set

$$\lambda_n = \sum_{k=n}^{\infty} P(x_k - x_{k+1}), \quad (69)$$

then $\lambda_{mn} \le \lambda_n$, and hence $\lambda_{mn}C \subset \lambda_n C$ whenever $m > n$. It follows from (68) that $x_n - x_m \in C$ for $m > n$. Since $\lambda_n C$ is M-closed by (P1), we have

$$\lim^b_{m\to\infty}(x_n - x_m) = x_n - x \in \lambda_n C. \quad (70)$$

Consequently, $P(x_n - x) \le \lambda_n$, and hence $\lim_{n\to\infty} P(x_n - x) = 0$. Since $P(x_n - y) \le P(x_n - x) + P(x - y)$, we have

$$\begin{aligned}\liminf_{n\to\infty} P(x_n - y) &\le \lim_{n\to\infty} P(x_n - x) + P(x - y)\\ &= P(x - y).\end{aligned} \tag{71}$$

That is,

$$\liminf_{n\to\infty} p(x_n, y) \le p(x, y). \tag{72}$$

Thus, $(X, \mathscr{B})$ is sequentially lower complete with respect to p. The proof is completed. □

From Lemma 21 and Theorem 9, we can get the following Ekeland type variational principle in bornological vector space, which is also proved in [17, 28].

Theorem 22 (Ekeland type variational principle in bornological vector space). *Let $(X, \mathscr{B})$ be a separated bornological vector space and $P : X \to (-\infty, +\infty]$ a positively homogeneous subadditive function satisfying the condition (P1). Let $f : X \to (-\infty, +\infty]$ be a proper, bounded from below, sequentially lower monotone function and $\varphi : (-\infty, +\infty) \to (0, +\infty)$ a nondecreasing function. Then for any $x_0 \in X$, there exists $v \in X$, such that $f(v) \le f(x_0)$ and*

$$P(v - x) > \varphi(f(v))(f(v) - f(x)), \quad \forall x \ne v. \tag{73}$$

Moreover, the corresponding equivalent principles in Theorem 9 hold.

Definition 23 (see [25]). Let X be a uniform space. An extended real-valued function $p : X \times X \to [0, +\infty]$ is called a q-distance on X if the following conditions are satisfied:

(q1) for any $x, y, z \in X$, $p(x, z) \le p(x, y) + p(y, z)$;

(q2) every sequence $\{y_n\} \subset X$ with $p(y_n, y_m) \to 0$ $(m > n \to \infty)$ is a Cauchy sequence and in the case $p(y_n, y) \to 0$ implies that $y_n \to y$ in X;

(q3) for $x, y, z \in X$, $p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.

Here $p(y_n, y_m) \to 0$ $(m > n \to \infty)$ means that for any $\varepsilon > 0$, there exists $n_0 \in N$ such that $p(y_n, y_m) < \varepsilon$ for all $m > n \ge n_0$.

Definition 24 (see [25]). Let $(X, \mathscr{U})$ be a uniform space and p a q-distance on X. A proper function $f : X \to (-\infty, +\infty]$ is said to be sequentially lower monotone with respect to p (in short, sequentially lower monotone with respect to p) if for any sequence $\{x_n\}$ in X satisfying $p(x_n, x_m) \to 0$ $(m > n \to \infty)$, $p(x_n, \overline{x}) \to 0$ $(n \to \infty)$ and $f(x_{n+1}) \le f(x_n)$, we have $f(\overline{x}) \le f(x_n)$ for each $n \in N$.

Definition 25 (see [25]). Let $(X, \mathscr{U})$ be a uniform space, p a q-distance on X, and $f : X \to (-\infty, +\infty]$ a proper function. $(X, \mathscr{U})$ is said to be sequentially complete with respect to $(p, f \downarrow)$ if for any sequence $\{x_n\}$ in X satisfying $p(x_n, x_m) \to 0$ $(m > n \to \infty)$ and $f(x_{n+1}) \le f(x_n)$ for each $n \in N$, there exists $\overline{x} \in X$ such that $p(x_n, \overline{x}) \to 0$ $(n \to \infty)$.

Lemma 26. *Let $(X, \mathscr{U})$ be a uniform space, p a q-distance on X, and $f : X \to (-\infty, +\infty]$ a sequentially lower monotone with respect to p, proper function, bounded from below. If $(X, \mathscr{U})$ is sequentially lower complete with respect to $(p, f \downarrow)$, then $(X, \mathscr{U})$ is sequentially lower complete with respect to p and f.*

Proof. Assume that $\{x_n\}$ is a sequence in X with $f(x_{n+1}) \le f(x_n)$ for each $n \in N$ and $\sum_{n=1}^{\infty} p(x_n, x_{n+1}) < +\infty$. Let $\lambda_n = \sum_{k=n+1}^{\infty} p(x_n, x_{n+1})$, then we have that $\lim_{n\to\infty} \lambda_n = 0$, and for any $m > n$,

$$p(x_n, x_m) \le \lambda_n. \tag{74}$$

This shows that $p(x_n, x_m) \to 0$ $(m > n \to \infty)$. By using Definition 23 and Definition 25, we get that there exists $\overline{x} \in X$ such that $p(x_n, \overline{x}) \to 0$ $(n \to \infty)$ and $\{x_n\}$ converges to $\overline{x}$. Then for any $y \in X$, by $p(x_n, y) \le p(x_n, \overline{x}) + p(\overline{x}, y)$ we have that

$$\liminf_{n\to\infty} p(x_n, y) \le \lim_{n\to\infty} p(x_n, \overline{x}) + p(\overline{x}, y) = p(\overline{x}, y). \tag{75}$$

Thus, $(X, \mathscr{U})$ is sequentially lower complete w.r.t. p and f. The proof is completed. □

From Theorems 5, 7, and 9, we can get the following Caristi type coincidence point theorem and Ekeland type variational principle in uniform space equipped with q-distance. From Lemma 26, we also see that this is a transformation of the results appeared in [25].

Theorem 27. *Let $(X, \mathscr{U})$ be a uniform space, p a q-distance on X, and $f : X \to (-\infty, +\infty]$ a sequentially lower monotone with respect to p, proper function, bounded from below. Let D be a nonempty subset of X, $g : D \to X$ a surjective function, I an index set, and, for each $i \in I$, $T_i : D \to 2^X$ a multivalued map. If $(X, \mathscr{U})$ is sequentially complete with repect to p and f and $\varphi : (-\infty, +\infty) \to (0, +\infty)$ is a nondecreasing function, then the following conclusions hold and are equivalent.*

(1) *Suppose that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X \setminus \{g(x)\}$, such that*

$$p(g(x), y) \le \varphi(f(g(x)))(f(g(x)) - f(y)). \tag{76}$$

Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that

$$f(g(u)) \le f(x_0). \tag{77}$$

(2) *Suppose that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists an $i_0 \in I$ and $y \in T_{i_0}(x) \setminus \{g(x)\}$, such that (76) holds. Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that (77) holds.*

(3) *Let I be an index set, and, for each $i \in I$, let $T_i : X \to 2^X$ be a multivalued map. Suppose further that for*

each $x \in X$ with $x \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X\backslash\{x\}$, such that

$$p(x,y) \le \varphi(f(x))(f(x)-f(y)). \tag{78}$$

Then for any $x_0 \in X$, there exists a common fixed point $u \in X$ of $\{T_i\}_{i\in I}$; that is, $u \in \bigcap_{i\in I} T_i(u)$, such that

$$f(u) \le f(x_0). \tag{79}$$

(4) *(Ekeland type variational principle in uniform spaces) For any $x_0 \in X$, there exists $v \in X$, such that $f(v) \le f(x_0)$ and*

$$p(v,x) > \varphi(f(v))(f(v)-f(x)), \quad \forall x \in X \text{ with } x \ne v. \tag{80}$$

Moreover, the rest of corresponding equivalent principles in Theorem 9 hold.

Proof. The equivalence of the conclusions (1)–(4) is clear. We only need to prove (4). The proof of (4) is similar to the proof of Theorem 16, by using (q3). So we delete the detail of the proof. □

Remark 28. If $\varphi : (-\infty,+\infty) \to (0,+\infty)$ is upper semicontinuous and (76) is replaced by

$$\begin{aligned} &p(g(x),y) \\ &\quad \le \max\{\varphi(f(g(x))),\varphi(f(y))\}\cdot(f(g(x))-f(y)), \end{aligned} \tag{81}$$

then the conclusions of Theorem 27 hold. In this case, the proof is similar to the proof of Theorem 2.1 in [11].

As noted in Remark 5.1 in [25], from our Theorem 27, we can deduce [13, Theorems 3.1 and 3.2, and Corollary 3.3] and [15, Theorems 4–6]. Furthermore, we will show that Theorem 27 improves some coincidence point theorems and their equivalences in F-type separated topological space (or equivalently, generating spaces of quasimetric family) and fuzzy metric spaces, which were proved in [8–11].

In the following, we will assume that $(X,\mathfrak{I})$ is an F-type separated topological space (or equivalently, a uniform space, see [13, 15]) whose topology is generated by a separated family $\{q_\lambda\}_{\lambda\in\Lambda}$ of quasimetrics, where $(\Lambda,\prec)$ is a directed set. Moreover, let $\alpha : \Lambda \to (0,+\infty)$ be a nondecreasing function; that is, $\lambda,\mu \in \Lambda$, $\lambda \prec \mu$ implies $\alpha(\lambda) \le \alpha(\mu)$. An extended real-valued function $p : X\times X \to [0,+\infty]$ is defined as follows:

$$p(x,y) = \sup_{\lambda\in\Lambda}\alpha(\lambda)\, q_\lambda(x,y), \quad \forall (x,y) \in X\times X. \tag{82}$$

Lemma 29. *Let $(X,\mathfrak{I})$ be an F-type separated topological space (or equivalently, a uniform space) whose topology is generated by a family $\{q_\lambda\}_{\lambda\in\Lambda}$ of quasimetrics, and let $p : X\times X \to [0,+\infty]$ be defined by (82). If $(X,\mathfrak{I})$ is sequentially complete, then $(X,\mathfrak{I})$ is sequentially lower complete w.r.t. p.*

Proof. Assume that a sequence $\{x_n\}$ in X satisfies $\sum_{n=1}^{\infty} p(x_n,x_{n+1}) < +\infty$. This implies that the sequence $\{x_n\}$ satisfies $p(x_n,x_m) \to 0$ $(m > n \to \infty)$. It follows from Examples 2.3 and 3.1 in [25] that p is a q-distance on X, and there exists an $x \in X$, such that $\{x_n\}$ converges to x and $p(x_n,x) \to 0$. Then for any $y \in X$,

$$\liminf_{n\to\infty} p(x_n,y) \le \liminf_{n\to\infty}[p(x_n,x)+p(x,y)] = p(x,y). \tag{83}$$

Thus, $(X,\mathfrak{I})$ is sequentially lower complete w.r.t. p. The proof is completed. □

By using Lemma 29 and Theorem 27, we have the following results.

Theorem 30. *Let $(X,\mathfrak{I})$ be a sequentially complete and separated F-type topological space (or equivalently, a uniform space) whose topology is generated by a family $\{q_\lambda\}_{\lambda\in\Lambda}$ of quasimetrics, $\varphi : (-\infty,+\infty) \to (0,+\infty)$ a nondecreasing function, and $f : X \to (-\infty,+\infty]$ a proper, bounded from below, sequentially lower monotone function. Let D be a nonempty subset of X, $g : D \to X$ a surjective function, and I an index set, and, for each $i \in I$, let $T_i : D \to 2^X$ be a multivalued map. Then the following conclusions hold and are equivalent.*

(1) *Suppose that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X\setminus\{g(x)\}$, such that*

$$\alpha(\lambda)\, q_\lambda(g(x),y) \le \varphi(f(g(x)))(f(g(x))-f(y)), \quad \forall\lambda \in \Lambda. \tag{84}$$

Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that

$$f(g(u)) \le f(x_0). \tag{85}$$

(2) *Suppose that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists an $i_0 \in I$ and $y \in T_{i_0}(x)\setminus\{g(x)\}$, such that (82) holds. Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that (84) holds.*

(3) *Let I be an index set and, for each $i \in I$, $T_i : X \to 2^X$ a multivalued map. Suppose further that for each $x \in X$ with $x \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X\setminus\{x\}$, such that*

$$\alpha(\lambda)\, q_\lambda(x,y) \le \varphi(f(x))(f(x)-f(y)), \quad \forall\lambda \in \Lambda. \tag{86}$$

Then for any $x_0 \in X$, there exists a common fixed point $u \in X$ of $\{T_i\}_{i\in I}$; that is, $u \in \bigcap_{i\in I} T_i(u)$, such that

$$f(u) \le f(x_0). \tag{87}$$

(4) *(Ekeland type variational principle in F-type topological spaces) For any $x_0 \in X$, there exists $v \in X$, such that $f(v) \le f(x_0)$ and for any $x \in X$ with $v \ne x$, there exists $\lambda_0 \in \Lambda$, such that*

$$\alpha(\lambda_0)\, q_{\lambda_0}(v,x) > \varphi(f(v))(f(v)-f(x)). \tag{88}$$

The following version of coincidence point theorem is an improvement for the coincidence point theorems proved in [8, 9].

Theorem 31. *Let $(X, d_\lambda : \lambda \in \Lambda)$ and $(Y, \delta_\lambda : \lambda \in \Lambda)$ be two sequentially complete and separated generating spaces of quasimetric family, $\alpha : \Lambda \to (0, +\infty)$ a nondecreasing function, D a nonempty subset of X, $g : D \to X$ a surjective function, $h : X \to Y$ a closed mapping, $f : h(X) \to (-\infty, +\infty]$ a proper, bounded from below, sequentially lower monotone function, and $\varphi : (-\infty, +\infty) \to (0, +\infty)$ a non-decreasing function. Let I be an index set, and, for each $i \in I$, let $T_i : D \to 2^X$ be a multivalued map. Suppose further that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X \setminus \{g(x)\}$, such that*

$$\begin{aligned}\alpha(\lambda) &\max\{d_\lambda(g(x), y), c\delta_\lambda(h(g(x)), h(y))\}\\ &\le \varphi(f(h(g(x))))(f(h(g(x))) - f(h(y))), \quad \forall\lambda \in \Lambda,\end{aligned} \tag{89}$$

where $c > 0$ is a given constant. Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that

$$f(h(g(u))) \le f(h(x_0)). \tag{90}$$

Proof. For each $\lambda \in \Lambda$, we define $q_\lambda : X \times X \to R$ by

$$q_\lambda(x, y) = \max\{d_\lambda(x, y), c\delta_\lambda(h(x), h(y))\}, \tag{91}$$

and then, by Definition 2 in [15], we can verify that the collection $\{q_\lambda\}_{\lambda\in\Lambda}$ defined by (91) is a family of quasimetrics on X. Since $(X, d_\lambda : \lambda \in \Lambda)$ and $(Y, \delta_\lambda : \lambda \in \Lambda)$ are sequentially complete generating spaces of quasimetric family and h is a closed mapping, we can deduce that $(X, q_\lambda : \lambda \in \Lambda)$ is also a sequentially complete generating space of quasimetric family. Next, we assume that $\{x_n\} \subset X$ is a sequence which converges to x in $(X, q_\lambda : \lambda \in \Lambda)$ and satisfies

$$f(h(x_1)) \ge f(h(x_2)) \ge \cdots \ge f(h(x_n)) \ge \cdots. \tag{92}$$

By (91) we know that $\{x_n\}$ converges to x in $(X, d_\lambda : \lambda \in \Lambda)$ and $\{h(x_n)\}$ converges to $h(x)$ in $(Y, \delta_\lambda : \lambda \in \Lambda)$. Since $f : h(X) \to (-\infty, +\infty]$ is a sequentially lower monotone function, we have $f(h(x)) \le f(h(x_n))$, for each n, that is, $f \circ h$ is a sequentially lower monotone function on $(X, q_\lambda : \lambda \in \Lambda)$. Then by using Theorem 27 (1) for $(X, q_\lambda : \lambda \in \Lambda)$ and $f \circ h$, we can get the conclusion of Theorem 31. The proof is completed. □

The following version of coincidence point theorem is an improvement for the coincidence point theorems proved in [11].

Theorem 32. *Let $(X, d_\lambda : \lambda \in \Lambda)$ and $(Y, \delta_\lambda : \lambda \in \Lambda)$ be two sequentially complete generating spaces of quasimetric family, $\alpha : \Lambda \to (0, +\infty)$ a nondecreasing function, D a non-empty subset of X, $g : D \to X$ a surjective function, $h : X \to Y$ a closed mapping, $f : h(X) \to (-\infty, +\infty]$ a proper, bounded from below, sequentially lower monotone function, and $\varphi : (-\infty, +\infty) \to (0, +\infty)$ an upper semicontinuous function. Let I be an index set, and, for each $i \in I$, let $T_i : D \to 2^X$ be a multivalued map. Suppose further that for each $x \in D$ with $g(x) \notin \bigcap_{i\in I} T_i(x)$, there exists $y \in X \setminus \{g(x)\}$, such that*

$$\begin{aligned}\alpha(\lambda) &\max\{d_\lambda(g(x), y), c\delta_\lambda(h(g(x)), h(y))\}\\ &\le \max\{\varphi(f(h(g(x)))), \varphi(f(h(y)))\}\\ &\quad \times (f(h(g(x))) - f(h(y))), \quad \forall\lambda \in \Lambda,\end{aligned} \tag{93}$$

where $c > 0$ is a given constant. Then for any $x_0 \in X$, there exists a coincidence point $u \in D$ of g and $\{T_i\}_{i\in I}$; that is, $g(u) \in \bigcap_{i\in I} T_i(u)$, such that

$$f(h(g(u))) \le f(h(x_0)). \tag{94}$$

Proof. It follows from the proof of Theorem 16 that $f \circ h$ is a sequentially lower monotone function on $(X, q_\lambda : \lambda \in \Lambda)$, where $\{q_\lambda\}_{\lambda\in\Lambda}$ is defined by (91). Then by using Remark 28 for $(X, q_\lambda : \lambda \in \Lambda)$ and $f \circ h$, we can get the conclusion of Theorem 32. The proof is completed. □

Remark 33. Similarly, under the conditions of Theorem 31 or Theorem 32, we can get corresponding common fixed point theorems, Ekeland's variational principle, and other equivalences of Theorems 31 and 32. On the other hand, a fuzzy metric space in the sense of Kaleva and Seikkala [39] is a generating space of quasimetric family (see, e.g., [8, 9, 11]); thus, by using our results in this section we can get coincidence point theorems and its equivalences in fuzzy metric spaces. Therefore, our results are also unified improvements of the results in [8–11].

Let $(X, \mathcal{U})$ be a uniform space, $q : X \times X \to [0, +\infty)$ a q-distance on X, and $h : [0, +\infty) \to [0, +\infty)$ a nondecreasing function, such that

$$\int_0^{+\infty} \frac{dr}{1 + h(r)} = +\infty, \tag{95}$$

and x_0 is a given point in X. Let $p : X \times X \to [0, +\infty)$ be defined by

$$p(x, y) = \int_{q(x_0,x)}^{q(x_0,x)+q(x,y)} \frac{dr}{1 + h(r)}. \tag{96}$$

Lemma 34. *Let $(X, \mathcal{U})$ be a uniform space, q a q-distance on X, $h : [0, +\infty) \to [0, +\infty)$ a nondecreasing function satisfying (95), and $f : X \to (-\infty, +\infty]$ a sequentially lower monotone with respect to q, proper function, bounded from below. If $(X, \mathcal{U})$ is sequentially lower complete with respect to $(q, f \downarrow)$, then $(X, \mathcal{U})$ is sequentially lower complete with respect to p and f, where p is defined by (96).*

Proof. Let $\{x_n\}$ be a sequence in X with $f(x_{n+1}) \le f(x_n)$ for each $n \in N$ and satisfy

$$\sum_{n=1}^{\infty} p(x_n, x_{n+1}) = \sum_{n=1}^{\infty} \int_{q(x_0,x_n)}^{q(x_0,x_n)+q(x_n,x_{n+1})} \frac{dr}{1 + h(r)} < +\infty. \tag{97}$$

For any $n > 1$, by noting that $h : [0,+\infty) \to [0,+\infty)$ is nondecreasing, we have

$$\begin{aligned}
&\int_{q(x_0,x_1)}^{\sum_{k=0}^{n-1} q(x_k,x_{k+1})} \frac{dr}{1+h(r)} \\
&\quad = \int_{q(x_0,x_1)}^{q(x_0,x_1)+q(x_1,x_2)} \frac{dr}{1+h(r)} + \cdots \\
&\qquad + \int_{q(x_0,x_1)+\cdots+q(x_{n-2},x_{n-1})}^{q(x_0,x_1)+\cdots+q(x_{n-1},x_n)} \frac{dr}{1+h(r)} \\
&\quad \le \int_{q(x_0,x_1)}^{q(x_0,x_1)+q(x_1,x_2)} \frac{dr}{1+h(r)} + \cdots \\
&\qquad + \int_{q(x_0,x_n)}^{q(x_0,x_n)+q(x_{n-1},x_n)} \frac{dr}{1+h(r)} \\
&\quad \le \sum_{n=1}^{\infty} \int_{q(x_0,x_n)}^{q(x_0,x_n)+q(x_n,x_{n+1})} \frac{dr}{1+h(r)} \\
&\quad < +\infty.
\end{aligned} \tag{98}$$

This shows that

$$\begin{aligned}
&\int_{q(x_0,x_1)}^{\sum_{n=1}^{\infty} q(x_n,x_{n+1})} \frac{dr}{1+h(r)} \\
&\quad \le \sum_{n=1}^{\infty} \int_{q(x_0,x_n)}^{q(x_0,x_n)+q(x_n,x_{n+1})} \frac{dr}{1+h(r)} < +\infty.
\end{aligned} \tag{99}$$

By (95) we obtain that $\sum_{n=1}^{\infty} q(x_n, x_{n+1}) < +\infty$. Since $(X, \mathcal{U})$ is sequentially lower complete w.r.t. $(q, f \downarrow)$, by Lemma 26, we get that there exists $\overline{x} \in X$ such that $q(x_n, \overline{x}) \to 0$ $(n \to \infty)$, $\{x_n\}$ converges to $\overline{x}$, and, for any $y \in X$,

$$\liminf_{n\to\infty} q(x_n, y) \le p(\overline{x}, y). \tag{100}$$

Then there exists a subsequence $\{x_{n_k}\}$ such that

$$\lim_{k\to\infty} q(x_{n_k}, y) = \liminf_{n\to\infty} q(x_n, y) \le q(\overline{x}, y). \tag{101}$$

Also, there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $\lim_{i\to\infty} q(x_0, x_{n_{k_i}}) = \liminf_{k\to\infty} q(x_0, x_{n_k})$. It follows from $q(x_n, \overline{x}) \to 0$ $(n \to \infty)$ and $q(x_0, \overline{x}) \le q(x_0, x_{n_{k_i}}) + q(x_{n_{k_i}}, \overline{x})$ that $q(x_0, \overline{x}) \le \lim_{i\to\infty} q(x_0, x_{n_{k_i}})$. Then, by noting that h is nondecreasing, we have

$$\begin{aligned}
\liminf_{n\to\infty} p(x_n, y) &\le \liminf_{k\to\infty} p(x_{n_k}, y) \\
&\le \liminf_{i\to\infty} p(x_{n_{k_i}}, y) \\
&= \liminf_{i\to\infty} \int_{q(x_0,x_{n_{k_i}})}^{q(x_0,x_{n_{k_i}})+q(x_{n_{k_i}},y)} \frac{dr}{1+h(r)} \\
&= \int_{\lim_{i\to\infty} q(x_0,x_{n_{k_i}})}^{\lim_{i\to\infty} q(x_0,x_{n_{k_i}})+\lim_{i\to\infty} q(x_{n_{k_i}},y)} \frac{dr}{1+h(r)} \\
&\le \int_{q(x_0,\overline{x})}^{q(x_0,\overline{x})+\lim_{i\to\infty} q(x_{n_{k_i}},y)} \frac{dr}{1+h(r)} \\
&\le \int_{q(x_0,\overline{x})}^{q(x_0,\overline{x})+q(\overline{x},y)} \frac{dr}{1+h(r)} = p(\overline{x}, y).
\end{aligned} \tag{102}$$

Thus, $(X, \mathcal{U})$ is sequentially lower complete with respect to p and f. The proof is completed. □

Remark 35. By using Lemma 34 and Theorem 27, we can get coincidence point theorems and its equivalences for $(X, \mathcal{U})$ and p, which improve the results in [29, 30].

Remark 36. Let $(X, \mathfrak{I})$ be a sequentially complete F-type separated topological space (or equivalently, a uniform space) whose topology is generated by a family $\{q_\lambda\}_{\lambda\in\Lambda}$ of pseudometrics (see [15]), $\alpha : \Lambda \to (0,+\infty)$ a nondecreasing function, and $h : [0,+\infty) \to [0,+\infty)$ a nondecreasing function satisfying (95). Let $p_\lambda : X \times X \to [0,+\infty)$ be defined by

$$p_\lambda(x, y) = \int_{q_\lambda(x_0,x)}^{q_\lambda(x_0,x)+q_\lambda(x,y)} \frac{dr}{1+h(r)}, \quad \lambda \in \Lambda, \tag{103}$$

and let an extended real-valued function $p : X \times X \to [0,+\infty]$ be defined by

$$p(x, y) = \sup_{\lambda\in\Lambda} \alpha(\lambda)\, p_\lambda(x, y), \quad \forall (x, y) \in X \times X. \tag{104}$$

Similar to the proof of Lemma 34, one can prove that $(X, \mathfrak{I})$ is sequentially lower complete w.r.t. p. Then, by using Theorem 27, we can get coincidence point theorems and its equivalences for $(X, \mathfrak{I})$ and p, which improve the results in [32].

Acknowledgments

First, the authors are very grateful to the referees for their careful reading of the manuscript and a lot of valuable comments and suggestions, which greatly improve this paper. Next, this paper was supported by the National Natural Science Foundation of China (11171286, 11126344).

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 206759, 5 pages
http://dx.doi.org/10.1155/2013/206759

Research Article

Common Fixed Points for Suzuki-Generalized Nonexpansive Maps

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Received 17 May 2013; Revised 27 July 2013; Accepted 27 July 2013

Academic Editor: E. Karapinar

A common fixed point theorem for a pair of maps satisfying condition (C) is proved under certain conditions. We extend the well-knownDeMarr's fixed point theorem to the case of noncommuting family of maps satisfying condition (C). As for an application, an invariant approximation theorem is also derived.

1. Introduction

Jungck [1] initiated the systematic study of finding a common fixed point of a pair of commuting maps. This problem of finding a common fixed point has been of significant interest in the area of fixed point theory and has been studied by many authors such as in [2–6]. At the first time, the commutativity for two maps was always assumed to find a common fixed point. Later, it was found that the two maps were not necessarily commutative at each point, and then weaker forms of commutativity were defined to obtain a common fixed point for maps on a metric space. For example, the notions of weakly commutative maps [2], compatible maps (weakly compatible maps) [7], biased maps [8], R-subweakly commuting maps [4], and occasionally weakly compatible [9] have been introduced and used to find common fixed points of maps.

Recently, Chen and Li [5] introduced the class of Banach operator pairs and, in [10], they investigated the common fixed point problem for nonexpansive maps where (I, T) is a Banach operator pair. Also, they extended the well-known De Marr's fixed point theorem to the noncommuting case.

More recently, Suzuki [11] introduced a condition on maps, called condition (C) (maps satisfying condition (C) are also known as Suzuki-generalized nonexpansive maps), and obtained some fixed point theorems and convergence theorems for such maps. Dhompongsa et al. [12] and Dhompongsa and Kaewcharoen [13] made significant contribution to fixed point theory for maps satisfying condition (C). For more results see [14].

In this paper, we discuss a common fixed point problem for a Banach operator pair satisfying condition (C). A family of maps satisfying condition (C) is also investigated. As for an application, an invariant approximation theorem is obtained.

2. Preliminaries

Let E be a Banach space. E is said to be

(i) strictly convex if $\|x + y\| < 2$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$,

(ii) uniformly convex in every direction (UCED) if, for $\varepsilon \in (0, 2]$ and $z \in E$ with $\|z\| = 1$, there exists $\delta(\varepsilon, z) > 0$ such that

$$\|x + y\| \leq 2 \quad (1 - \delta(\varepsilon, z)) \tag{1}$$

for all $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $x - y \in \{tz : t \in [-2, -\varepsilon] \cup [+\varepsilon, +2]\}$.

It is obvious that being UCED implies strict convexity.

Let K be a nonempty subset of E and let T be a self-map of K. We denote by $F(T)$ the set of fixed points of T; that is, $F(T) = \{x \in K : Tx = x\}$. Also, if I and T are self-maps of K, we denote by $F(I,T)$ the set of common fixed points of I and T; that is, $F(I,T) = \{x \in K : Ix = Tx = x\}$. If H is a nonempty family of self-maps of E, a point $x \in E$ is called common fixed point of H if it is the fixed point of each member of H.

The map T is called

(i) nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in K, \tag{2}$$

(ii) quasi-nonexpansive if $F(T) \neq \phi$ and

$$\|Tx - p\| \le \|x - p\|, \quad \text{for } x \in K, p \in F(T). \tag{3}$$

Suzuki [11] introduced a condition on maps, called condition (C), which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

Definition 1 (see [11]). A self-map T of K is said to satisfy condition (C) if

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \text{ implies } \|Tx - Ty\| \le \|x - y\| \tag{4}$$

for all $x, y \in K$.

Example 2 (see [13]). Define a map T on $[0, 3(1/2)]$ by

$$Tx = \begin{cases} 0 & \text{if } x \in [0,3] \\ 4x - 12 & \text{if } x \in \left[3, 3\frac{1}{4}\right] \\ -4x + 14 & \text{if } x \in \left[3\frac{1}{4}, 3\frac{1}{2}\right]. \end{cases} \tag{5}$$

Then T is a continuous map satisfying condition (C) and T is not nonexpansive.

Proposition 3 (see [11]). *Let K be a nonempty subset of a Banach space E. Assume that $T : K \to K$ is a nonexpansive map. Then T satisfies condition (C).*

Proposition 4 (see [11]). *Let K be a nonempty subset of a Banach space E. Assume that a map $T : K \to K$ satisfies condition (C) and has a fixed point. Then T is a quasi-nonexpansive map.*

Chen and Li [5] introduced the class of Banach operator pairs.

Definition 5 (see [5]). Let (X, d) be a metric space; the pair (I, T) of two self-maps I and T of X is called a Banach operator pair if the set $F(T)$ of fixed points of T is I-invariant; that is, $I(F(T)) \subseteq F(T)$.

A Banach operator pair (I, T) depends on the order of I and T; that is, if (I, T) is a Banach operator pair, (T, I) need not be such a pair. It is well known that for two self-maps I and T of a metric space X, the pair (I, T) is a Banach pair if and only if I and T commute on the set $F(T)$ [5].

Example 6 (see [5]). Let f and g be two self-maps of $X = \mathbb{R}^2$ defined by

$$\begin{aligned} f(s,t) &= \left(s^2 + t^2 + s - 1, s^2 + t^2 + t - 1\right), \\ g(s,t) &= \left((s-t)^2 + 2s - t, (s-t)^2 + s\right) \end{aligned} \tag{6}$$

for $(s,t) \in \mathbb{R}^2$. Directly, we have

$$\begin{aligned} F(f) &= \left\{(s,t) \in \mathbb{R}^2 : s^2 + t^2 - 1 = 0\right\}, \\ F(g) &= \left\{(s,t) \in \mathbb{R}^2 : s - t = 0 \text{ or } s - t + 1 = 0\right\}. \end{aligned} \tag{7}$$

The following assertions can be verified:

(i) $f(F(g)) \subseteq F(g)$, and hence (f, g) is a Banach operator pair on $\mathbb{R}^2$; equivalently, f and g commute on the set $F(g)$.

(ii) (g, f) is not a Banach operator pair, since for $(1,0) \in F(f)$, $g(1,0) = (3,2)$ is not in $F(f)$.

The following proposition for Banach operator pairs can be found in [10].

Proposition 7. *If $F(T)$ is a q-star shaped set (i.e., $tx+(1-t)q \in F(T)$ for any $x \in F(T)$ and $0 \le t \le 1$) with $q \in F(T)$, then (I, T) is a Banach operator pair if and only if the pairs $(I_k, T)'s$ are Banach operator pairs for all $k \in [0,1]$, where $I_k x = (1-k)Ix + kq$.*

Definition 8 (see [10]). Let T and I be two self-maps of a metric space X. The pair (I, T) is called symmetric Banach operator pair if both (T, I) and (I, T) are Banach operator pairs; that is, $T(F(I)) \subset F(I)$ and $I(F(T)) \subset F(T)$.

The pair (I, T) is a symmetric Banach operator pair if and only if T and I are commuting on $F(T) \cup F(I)$. It is easy to see that a Banach operator pair may not be a symmetric Banach operator pair; see [10].

Definition 9 (see [10]). Let H be a nonempty family of self-maps of a metric space X. H is called a Banach operator family if for all $I, T \in H$, (I, T) is a symmetrical Banach operator pair.

In 1963, DeMarr [15] stated the following well-known fixed point theorem for a family of commuting nonexpansive maps.

Theorem 10 (DeMarr [15]). *If K is a nonempty compact convex subset of a Banach space X and H is a nonempty family of commuting nonexpansive maps of K into itself, then the family H has a common fixed point in K.*

Recently Chen and Li [10] extended DeMarr's theorem to the noncommuting case.

Theorem 11 (see [10]). *Let K be a nonempty closed convex subset of a normed space E and let H be a nonempty family of nonexpansive maps of K into itself. If H is a Banach operator family and there exists a $T \in H$ such that $\overline{T(K)}$ is compact, then H has a common fixed point in K.*

We now collect some results about condition (C) which will be used in the sequel.

Lemma 12 (see [11]). *Let K be a nonempty closed subset of a Banach space E. Assume that $T : K \to K$ satisfies condition (C). Then $F(T)$ is closed. Moreover, if E is strictly convex and K is convex, then $F(T)$ is also convex.*

Theorem 13 (see [14]). *Let K be a closed bounded convex subset of a Banach space E. Assume that $T : K \to K$ is a map satisfying condition (C) and that $\overline{T(K)}$ is compact. Then T has a fixed point.*

Lemma 14 (see [11]). *Let K be a nonempty subset of a Banach space E. Assume that $T : K \to K$ is a map satisfying condition (C). Then for $x, y \in K$, the following hold:*

(i) $\|Tx - T^2x\| \le \|x - Tx\|$,

(ii) *either* $\frac{1}{2}\|x - Tx\| \le \|x - y\|$ *or* $\frac{1}{2}\|Tx - T^2x\| \le \|Tx - y\|$ *holds,*

(iii) *either* $\|Tx - Ty\| \le \|x - y\|$ *or* $\|T^2x - Ty\| \le \|Tx - y\|$ *holds.*

3. Main Results

Lemma 15 (see [16] or [15]). *Let M be a nonempty compact subset of a Banach space E. Let δ be the diameter of M. If $\delta > 0$, then there exists an element $u \in \overline{co}M$ such that*

$$\sup\{\|x - u\| : x \in M\} < \delta, \tag{8}$$

where $\overline{co}M$ is the smallest closed convex set containing M.

Following [15], we are able to prove the following lemma.

Lemma 16. *Let K be a nonempty closed convex subset of a Banach space E. Suppose that $T : K \to K$ satisfies condition (C) such that there exists a compact set $M \subset F(T)$ not reduced to a point. Then there exists a nonempty closed convex set K_1 such that*

(1) $K_1 \subset K$ *and* $T(K_1) \subset K_1$,

(2) $M \cap (K_1)^c \ne \emptyset$.

Proof. Let δ be the diameter of M. Since M is not reduced to a point, we have $\delta > 0$. According to Lemma 15, there is $u \in \overline{co}M$ such that

$$\delta_1 = \sup\{\|x - u\| : x \in M\} < \delta. \tag{9}$$

For each $x \in M$, define

$$U(x) = \{y : \|y - x\| \le \delta_1\}. \tag{10}$$

Since $u \in U(x)$ for each $x \in M$, it follows that $K_0 = \bigcap_{x\in M} U(x) \ne \emptyset$. It is easy to check that K_0 is closed and convex. Let $K_1 = K_0 \cap K$. Then K_1 is not empty since $u \in K_1$. For any $x \in K_1$ and any $z \in M$, we have $x \in U(z)$; that is, $\|x - z\| \le \delta_1$. Since

$$\frac{1}{2}\|z - Tz\| = 0 \le \|z - x\|, \tag{11}$$

we obtain that

$$\|z - T(x)\| = \|T(z) - T(x)\| \le \|z - x\| \le \delta_1. \tag{12}$$

That is, $T(x) \in U(z)$. This is true for any $z \in M$; thus $T(x) \in K_1$. This shows that $T(x) \in K_1$ for all $x \in K_1$. Recalling that M is compact, therefore, there exist $x_0, x_1 \in M$ such that $\|x_0 - x_1\| = \delta > \delta_1$. Thus, $x_1 \notin U(x_0) \supset K_1$; that is, $x_1 \in M \cap (K_1)^c \ne \emptyset$. □

Theorem 17. *Let K be a nonempty closed bounded convex subset of a Banach space E. Suppose that T and I are two self-maps on K satisfying condition (C). If (I, T) is a Banach operator pair, I is nonexpansive, and $\overline{T(K)}$ is compact, then $F(I, T) \ne \emptyset$.*

Proof. Let Γ be the set of all nonempty closed bounded convex subsets A of K such that $T(A) \subset A$ and $I(A) \subset A$ and $\overline{T(A)}$ is compact. Since $K \in \Gamma$, then Γ is nonempty. Define a partial order "$\le$" by set inclusion on the set Γ; that is, $A_i \le A_j$ whenever $A_i \subseteq A_j$.

Let Γ_0 be any total ordering subset of Γ and $A \in \Gamma_0$. Since A is closed, we have $\overline{T(A)} \subset A$, and since $\overline{T(A)}$ is compact, it follows that

$$\emptyset \ne \bigcap_{A\in\Gamma_0} \overline{T(A)} \subset \bigcap_{A\in\Gamma_0} A = A_0. \tag{13}$$

It is clear that $A_0 \in \Gamma$. By Zorn's lemma, Γ has a minimal set K_0.

Since T satisfies condition (C) and $\overline{T(K_0)}$ is compact, then, by Theorem 13, T has a nonempty fixed point set $F(T) \subset K_0$. It follows that $F(T)$ is a closed subset of $\overline{T(K_0)}$ and thus is compact. On the other hand, we have $T(F(T)) = F(T)$, and since (I, T) is a Banach operator pair, it implies that $I(F(T)) \subset F(T)$. Using Zorn's lemma again, there exists a minimal nonempty compact subset M of $F(T)$ which satisfies $T(M) = M$ and $I(M) \subseteq M$ (M is not necessarily convex).

Next, we show $I(M) = M$. If $I(M) \ne M$, then we have $I(I(M)) \subset I(M)$, and $I(M)$ is compact because I is continuous. Also, we have $T(I(M)) = I(M)$ since $I(M) \subset M \subset F(T)$. This contradicts the minimality of M.

If M has only one point, the proof is finished. Suppose that M has at least two points. By Lemma 16 there exists a set K_1 satisfying $T(K_1) \subset K_1$ and $M \cap (K_1)^c \ne \emptyset$. Since I is nonexpansive and $I(M) = M$, it follows that $K_1 \in \Gamma$ which implies that K_1 is a proper subset of K_0 and this contradicts the minimality of K_0. This completes the proof. □

Theorem 18. *Let K be a nonempty closed bounded convex subset of a strictly convex space E. Suppose that T and I are two self-maps on K satisfying condition (C). If (I, T) is a Banach operator pair and $\overline{T(K)}$ is compact, then $F(I, T) \ne \emptyset$.*

Proof. By Theorem 13 and Lemma 12, $F(T)$ is a nonempty closed bounded convex set. It is compact since $\overline{T(K)}$ is compact. Since $I(F(T)) \subset F(T)$, again by Theorem 13, I has a fixed point in $F(T)$; that is, $F(T) \cap F(I) \neq \emptyset$. □

Corollary 19. *Let K be a nonempty closed bounded convex subset of an UCED Banach space E. Suppose that T and I are two self-maps on K satisfying condition (C). If (I, T) is a Banach operator pair and $\overline{T(K)}$ is compact, then $F(I, T) \neq \emptyset$.*

Example 20. Consider $\mathbb{R}$ with the usual metric and let $K = [0, 3(1/2)]$. Define a map T on K by

$$Tx = \begin{cases} 0 & \text{if } x \in [0,3] \\ 4x - 12 & \text{if } x \in \left[3, 3\frac{1}{4}\right] \\ -4x + 14 & \text{if } x \in \left[3\frac{1}{4}, 3\frac{1}{2}\right] \end{cases} \tag{14}$$

and define a map I on K by $Ix = x^2/7$. Then T is a continuous map satisfying condition (C) and T is not nonexpansive (see [13]) and I is nonexpansive and hence satisfies condition (C). Also (I, T) is a Banach operator pair. Therefore, all conditions of Theorem 17 (and Theorem 18) are satisfied and I and T have a common fixed point. Note that Theorem 2.1 in Chen and Li [10] is not applicable here.

Next, we show a common fixed point theorem of a countable family of maps satisfying condition (C). We need first the following proposition which shows that for a given map I there are a lot of maps T such that (I, T) is a symmetric Banach operator pair.

Proposition 21 (see [10]). *Let I be a self-map on a convex subset K of a normed space E and let α be a map from K to $[0, 1]$ such that the set $\{x \in X : \alpha(x) = 0\}$ is I-invariant; that is, $\alpha(Ix) = 0$, $\forall x \in \{x \in X : \alpha(x) = 0\}$. Define*

$$T_\alpha x = \alpha(x) Ix + (1 - \alpha(x)) x. \tag{15}$$

Then (I, T_α) is a symmetric Banach operator pair.

Theorem 22. *Let K be a nonempty closed bounded convex subset of a Banach space E. Suppose that H is a nonempty family of self-maps on K satisfying condition (C). If H is a Banach operator family and there exists a $T_1 \in H$ such that $\overline{T_1(K)}$ is compact and every T_j (except T_1) in the family H is nonexpansive, then H has a common fixed point in K.*

Proof. Let T_1, T_2 and $T_3 \in H$ and let Γ be the set of all nonempty closed bounded convex subsets A of K such that $T_1(A) \subset A, T_2(A) \subset A$, and $T_3(A) \subset A$ and $\overline{T_1(A)}$ is compact. On the set Γ, define a partial order by set inclusion; then we can find a minimal set $K_0 \in \Gamma$.

As in the proof of Theorem 17, T_1 and T_2 have a nonempty compact common fixed point set $F = F(T_1, T_2)$ in K_0 satisfying $T_1(F) = F$ and $T_2(F) = F$. Since (T_3, T_1) and (T_3, T_2) are Banach operator pairs, we have $T_3(F) \subset F$. Using Zorn's lemma, there exists a minimal nonempty compact subset M of K_0 which satisfies $T_1(M) = M$, $T_2(M) = M$, and $T_3(M) \subset M$. Using an argument similar to that in Theorem 17, we can show that $T_3(M) = M$. If M reduces to a point, then $F(T_1, T_2, T_3) \neq \emptyset$. If M has at least two different points, then, by Lemma 16, this contradicts the minimality of K_0. Therefore we obtain that K_0 is a singleton and $F(T_1, T_2, T_3) \neq \emptyset$. □

For any finite maps $T_j \in H, j = 1, 2, \ldots, n$, we have by induction that $F(T_1, T_2, \ldots, T_n) \neq \emptyset$. We now let $\theta = \{F(T_1, T) : T \in H\}$. Thus for any $T \in H$, $F(T_1, T)$ is a nonempty compact set, and for each $T_j \in H$, $j = 2, \ldots, n$, we have

$$\bigcap_{j=2}^{n} F(T_1, T_j) = F(T_1, T_2, \ldots, T_n) \neq \emptyset. \tag{16}$$

This implies that the set family θ has the finite intersect property. Thus,

$$\bigcap_{T \in H} F(T_1, T) \neq \emptyset. \tag{17}$$

Therefore the family H has a common fixed point in K.

4. Applications

Let K be a subset of the normed space E and $\hat{x} \in E$; then the distance of a point $\hat{x}$ to the subset K is defined by

$$\operatorname{dist}(\hat{x}, K) = \inf\{\|y - \hat{x}\| : y \in K\}. \tag{18}$$

The set of best approximants of a point $\hat{x}$ in K is denoted by $P_K(\hat{x})$ and defined by

$$P_K(\hat{x}) = \{y \in K : \|y - \hat{x}\| = \operatorname{dist}(\hat{x}, K)\}. \tag{19}$$

It is well known that $P_K(\hat{x})$ is always a bounded subset of K and is a closed and convex set if K is so. Also, if K is compact, then $P_K(\hat{x})$ is nonempty. For more details, we refer to [17].

Let Ω_0 denote the class of closed convex subsets of E containing 0. For $K \in \Omega_0$ and $\hat{x} \in E$, let

$$K_{\hat{x}} = \{x \in K : \|x\| \leq 2\|\hat{x}\|\}. \tag{20}$$

It is clear that $P_K(\hat{x}) \subset K_{\hat{x}} \in \Omega_0$.

The following result provides a partial solution of an existence problem of approximation theory in the following result (see also [14]).

Theorem 23. *Let E be a Banach space and let T be a self-map of E with $\hat{x} \in F(T)$ and $K \in \Omega_0$ such that $T(K_{\hat{x}}) \subset K$. Assume that T satisfies condition (C) on $K_{\hat{x}} \cup \{\hat{x}\}$ and $\overline{T(K_{\hat{x}})}$ is compact. Then the set of best approximations $P_K(\hat{x})$ is nonempty.*

Proof. Without loss of generality we may assume that $\hat{x} \in E \setminus K$. If $x \in K \setminus K_{\hat{x}}$, then

$$\begin{aligned} \|x - \hat{x}\| &\geq \|x\| - \|\hat{x}\| \\ &> 2\|\hat{x}\| - \|\hat{x}\| \\ &= \|\hat{x}\| \end{aligned}$$

$$\begin{aligned} &\geq \operatorname{dist}(\hat{x}, K_{\hat{x}}) \\ &\geq \operatorname{dist}(\hat{x}, K). \end{aligned} \tag{21}$$

As a result

$$\operatorname{dist}(\hat{x}, K_{\hat{x}}) = \operatorname{dist}(\hat{x}, K). \tag{22}$$

Since $\overline{T(K_{\hat{x}})}$ is compact, we can find $y \in \overline{T(K_{\hat{x}})}$ such that

$$\operatorname{dist}\left(\hat{x}, \overline{T(K_{\hat{x}})}\right) = \|y - \hat{x}\|, \tag{23}$$

and so by Lemma 14,

$$\begin{aligned} \operatorname{dist}(\hat{x}, K_{\hat{x}}) &= \operatorname{dist}(\hat{x}, K) \\ &\leq \operatorname{dist}\left(\hat{x}, \overline{T(K_{\hat{x}})}\right) \\ &\leq \|Tx - T\hat{x}\| \\ &\leq \|x - \hat{x}\| \end{aligned} \tag{24}$$

for all $x \in K_{\hat{x}}$. Hence

$$\operatorname{dist}(\hat{x}, K) = \operatorname{dist}(\hat{x}, K_{\hat{x}}) = \operatorname{dist}\left(\hat{x}, \overline{T(K_{\hat{x}})}\right) = \|y - \hat{x}\| \tag{25}$$

and thus $y \in P_K(\hat{x})$. □

The following is an application of Theorem 17 to invariant approximations for convex sets.

Theorem 24. *Let E be a Banach space, I and T self-maps of E with $\hat{x} \in F(I,T)$, and $K \in \Omega_0$ with $I(K_{\hat{x}}) \subset K$ and $T(K_{\hat{x}}) \subset K$. If (I,T) is a Banach operator pair on $K_{\hat{x}}$, both I and T are maps satisfying condition (C) on $K_{\hat{x}} \cup \{\hat{x}\}$, I is nonexpansive, and $\overline{T(K_{\hat{x}})}$ is compact, then $F(I,T) \cap P_K(\hat{x}) \neq \emptyset$.*

Proof. By Theorem 23, $P_K(\hat{x})$ is a nonempty. Since K is closed and convex, then $P_K(\hat{x})$ is a closed convex set. We now show that $P_K(\hat{x})$ is T-invariant. Let $y \in P_K(\hat{x})$. Then $\|y - \hat{x}\| = \operatorname{dist}(\hat{x}, K)$. Since T satisfies condition (C) on $K_{\hat{x}} \cup \{\hat{x}\}$, by Lemma 14, we obtain that

$$\|T(y) - \hat{x}\| = \|T(y) - T(\hat{x})\| \leq \|y - \hat{x}\|, \tag{26}$$

and so

$$\operatorname{dist}(\hat{x}, K) \leq \|T(y) - \hat{x}\| \leq \|y - \hat{x}\| = \operatorname{dist}(\hat{x}, K). \tag{27}$$

This implies that $T(y) \in P_K(\hat{x})$. Consequently, we have $T(P_K(\hat{x})) \subset P_K(\hat{x})$, and, similarly, we can prove that $I(P_K(\hat{x})) \subset P_K(\hat{x})$. Since $\overline{T(P_K(\hat{x}))} \subset \overline{T(K_{\hat{x}})}$ and $\overline{T(K_{\hat{x}})}$ is compact, we have that $\overline{T(P_K(\hat{x}))}$ is compact. Now, Theorem 17 guarantees that $F(I,T) \cap P_K(\hat{x}) \neq \emptyset$. □

Acknowledgments

The authors are grateful to the referees for their valuable suggestions. This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The first and third authors acknowledge with thanks DSR for financial support. The second author is grateful to Chiang Mai University for the financial support.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 510691, 7 pages
http://dx.doi.org/10.1155/2013/510691

Research Article

A Common Fixed Point Theorem in Metric Space under General Contractive Condition

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Received 30 April 2013; Revised 12 July 2013; Accepted 4 August 2013

Academic Editor: E. Karapinar

We prove a common fixed point theorem for two pairs of compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous) mappings satisfying a general contractive condition in a metric space. Some illustrative examples are furnished to highlight the realized improvements. Our result improves the main result of Sedghi and Shobe (2007).

1. Introduction

Fixed point theory is one of the most fruitful and effective tools in mathematics which has enormous applications within as well as outside mathematics. Starting from the celebrated Banach contraction principle [1], many authors have obtained its several generalizations in different ways (see, e.g., [2–4]).

A metrical common fixed point theorem generally involves conditions on commutativity, continuity, and contraction of the given mappings, as well as completeness (or closedness) of the underlying space (or subspaces), along with conditions on suitable containment amongst the ranges of involved mappings. Hence, in order to prove a new metrical common fixed point theorem, one is always required to weaken one or more of these conditions. In order to weaken commutativity conditions in common fixed point theorems, Sessa [5] introduced the concept of weakly commuting mappings. Jungck [6] defined the notion of compatible mappings in order to generalize the concept of weak commutativity and showed that weakly commuting mappings are compatible, but the converse is not true. Afterwards Jungck and Rhoades [7] introduced the concept of weak compatibility to the setting of single-valued and multivalued mappings which is more general than compatibility. However, the study of common fixed points of noncompatible mappings is also equally interesting, and it was initiated by Pant [8] in metric spaces. Researchers of this domain introduced several definitions of weak commutativity such as compatible mappings, compatibility of type (A), (B), (C), and (P), and several others, whose systematic comparisons and illustrations are available in Murthy [9] and Singh and Tomar [10].

In 2002, Aamri and El Moutawakil [11] introduced the notion of property (E.A). Further, Al-Thagafi and Shahzad [12] introduced the notion of occasionally weakly compatible mappings. It was used further by many authors (see, e.g., [13]). However, in [14], Đorić et al. observed that the condition of occasionally weak compatibility reduces to weak compatibility in the presence of a unique point of coincidence of the given pair of mappings. Thus, no generalization can be obtained by replacing weak compatibility with occasionally weak compatibility (see also [15–17]).

In 2009, Bouhadjera and Godet-Thobie [18] further enlarged the class of compatible (reciprocally continuous) pairs by introducing the concept of subcompatibility (subsequential continuity) of pairs of mappings, which is substantially weaker than compatibility (reciprocal continuity). Since then, Imdad et al. [19] improved the results of Bouhadjera and Godet-Thobie and showed that these results can easily be recovered by replacing subcompatibility with compatibility

or subsequential continuity with reciprocal continuity. Very recently, Chauhan et al. [20] obtained some results of this kind using integral type contractive conditions. Many authors established a number of other fixed point results in metric and related spaces (see, e.g., [21–31]).

In 2007, Sedghi and Shobe [32] proved a common fixed point theorem for weakly compatible mappings satisfying a new general contractive type condition. The aim of this paper is to prove a common fixed point theorem for two pairs of self-mappings by using the notions of compatibility and subsequential continuity (alternately subcompatibility and reciprocal continuity) satisfying general contractive condition in a metric space. Some examples are furnished which demonstrate the validity of our result.

2. Preliminaries

Definition 1. Let $A, S : X \to X$ be two self-mappings on a metric space (X, d). The mappings A and S are said to be

(1) weakly commuting if $d(ASx, SAx) \leq d(Ax, Sx)$ for all $x \in X$ [5],

(2) compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ for each sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ [6],

(3) noncompatible if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$, but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent [8],

(4) weakly compatible if they commute at their coincidence points; that is, $ASu = SAu$ whenever $Au = Su$, for some $u \in X$ [7],

(5) occasionally weakly compatible if there is a point $x \in X$ which is a coincidence point of A and S at which A and S commute [12],

(6) with the property (E.A) if there exists a sequence $\{x_n\}$ in X and some $z \in X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ [11].

It can be noticed that arbitrary noncompatible self-mappings satisfy the property (E.A) but two mappings satisfying the property (E.A), need not be noncompatible (see [33, Example 1]). Also, weak compatibility and property (E.A) are independent of each other (see [31, Examples 2.1 and 2.2]).

Definition 2 (see [34]). A pair (A, S) of self-mappings on a metric space (X, d) is called reciprocally continuous if for a sequence $\{x_n\}$ in X, $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$, whenever $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$.

It is easy to see that if two self-mappings are continuous, then they are obviously reciprocally continuous, but the converse is not true. Moreover, in the setting of common fixed point theorems for compatible pairs of self-mappings satisfying contractive conditions, continuity of one of the mappings implies their reciprocal continuity but not conversely (see [35]).

Definition 3 (see [18]). A pair (A, S) of self-mappings on a metric space (X, d) is said to be subcompatible if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$.

A pair of subcompatible mappings satisfies the property (E.A). Obviously, compatible mappings which satisfy the property (E.A) are subcompatible, but the converse statement does not hold in general (see [36, Example 2.3]). Two occasionally weakly compatible mappings are subcompatible; however, the converse is not true in general (see [18, Example 1.2]).

Definition 4 (see [18]). A pair (A, S) of self-mappings on a metric space (X, d) is called subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ such that $\lim_{n\to\infty} ASx_n = Az$ and $\lim_{n\to\infty} SAx_n = Sz$.

One can easily check that if two self-mappings A and S are both continuous, hence also reciprocally continuous mappings but A and S are not subsequentially continuous (see [35, Example 1]).

Definition 5 (see [32]). By $\diamond : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, a binary operation will be denoted, satisfying the following conditions:

(1) $\diamond$ is associative and commutative,

(2) $\diamond$ is continuous.

Some typical examples of $\diamond$ are $a \diamond b = \max\{a, b\}$, $a \diamond b = a+b$, $a \diamond b = ab$, $a \diamond b = ab+a+b$, and $a \diamond b = ab/\max\{a, b, 1\}$, for each $a, b \in \mathbb{R}^+$.

Definition 6 (see [32]). The binary operation $\diamond$ is said to satisfy α-property if there exists a positive real number α such that

$$a \diamond b \leq \alpha \max\{a, b\}, \tag{1}$$

for all $a, b \in \mathbb{R}^+$.

Example 7 (see [32]). (1) If $a \diamond b = a + b$, for each $a, b \in \mathbb{R}^+$, then for $\alpha \geq 2$, we have $a \diamond b \leq \alpha \max\{a, b\}$.

(2) If $a \diamond b = ab/\max\{a, b, 1\}$, for each $a, b \in \mathbb{R}^+$, then for $\alpha \geq 1$, we have $a \diamond b \leq \alpha \max\{a, b\}$.

3. Main Results

In 2007, Sedghi and Shobe [32] proved the following result.

Theorem 8 (see [32, Theorem 2.1]). *Let (X, d) be a complete metric space such that $\diamond$ satisfies the α-property with $\alpha > 0$. Let A, B, S, and T be self-mappings on X satisfying the following conditions:*

(1) *$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, and $T(X)$ or $S(X)$ is a closed subset of X,*

(2) *the pairs (A, S) and (B, T) are weakly compatible,*

(3) *for all* $x, y \in X$,

$$d(Ax, By) \le k_1 (d(Sx, Ty) \diamond d(Ax, Sx)) + k_2 (d(Sx, Ty) \diamond d(By, Ty)) + k_3 \left(d(Sx, Ty) \diamond \frac{d(Sx, By) + d(Ax, Ty)}{2} \right), \tag{2}$$

where $k_1, k_2, k_3 > 0$ *and* $0 < \alpha(k_1 + k_2 + k_3) < 1$.

Then, A, B, S, and T have a unique common fixed point in X.

Now we prove our main result.

Theorem 9. *Let A, B, S, and T be four self-mappings on a metric space* (X, d)*, and let the operation* $\diamond$ *satisfy the* α*-property with* $\alpha > 0$*. Suppose that the pairs* (A, S) *and* (B, T) *are compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous), satisfying inequality* (2) *of Theorem 8. Then A, B, S, and T have a unique common fixed point in X.*

Proof. If the pair of mappings (A, S) is subsequentially continuous and compatible, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \tag{3}$$

for some $z \in X$, and

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = d(Az, Sz) = 0; \tag{4}$$

that is, $Az = Sz$. Similarly, with respect to the pair (B, T), there exists a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = w, \tag{5}$$

for some $w \in X$, and

$$\lim_{n\to\infty} d(BTy_n, TBy_n) = d(Bw, Tw) = 0; \tag{6}$$

that is, $Bw = Tw$. Hence z is a coincidence point of the pair (A, S) whereas w is a coincidence point of the pair (B, T).

Now we assert that $z = w$. If $z \ne w$ then using inequality (2) with $x = x_n$ and $y = y_n$, we have

$$\begin{aligned} d(Ax_n, By_n) &\le k_1 (d(Sx_n, Ty_n) \diamond d(Ax_n, Sx_n)) \\ &\quad + k_2 (d(Sx_n, Ty_n) \diamond d(By_n, Ty_n)) \\ &\quad + k_3 \left(d(Sx_n, Ty_n) \diamond \frac{d(Sx_n, By_n) + d(Ax_n, Ty_n)}{2} \right). \end{aligned} \tag{7}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(z, w) &\le k_1 (d(z, w) \diamond d(z, z)) \\ &\quad + k_2 (d(z, w) \diamond d(w, w)) \\ &\quad + k_3 \left(d(z, w) \diamond \frac{d(z, w) + d(z, w)}{2} \right). \end{aligned} \tag{8}$$

Since $\diamond$ satisfies the α-property, we obtain

$$\begin{aligned} d(z, w) &\le k_1 \alpha \max\{d(z, w), d(z, z)\} \\ &\quad + k_2 \alpha \max\{d(z, w), d(w, w)\} \\ &\quad + k_3 \alpha \max\left\{ d(z, w), \frac{d(z, w) + d(z, w)}{2} \right\} \\ &= k_1 \alpha \max\{d(z, w), 0\} + k_2 \alpha \max\{d(z, w), 0\} \\ &\quad + k_3 \alpha \max\{d(z, w), d(z, w)\} \\ &= \alpha (k_1 + k_2 + k_3) d(z, w) \\ &< d(z, w), \end{aligned} \tag{9}$$

which is a contradiction. Hence $z = w$. Now we prove that $Az = z$. If we suppose that $Az \ne z$, then from inequality (2) with $x = z$ and $y = y_n$, we have

$$\begin{aligned} d(Az, By_n) &\le k_1 (d(Sz, Ty_n) \diamond d(Az, Sz)) \\ &\quad + k_2 (d(Sz, Ty_n) \diamond d(By_n, Ty_n)) \\ &\quad + k_3 \left(d(Sz, Ty_n) \diamond \frac{d(Sz, By_n) + d(Az, Ty_n)}{2} \right). \end{aligned} \tag{10}$$

Taking the limit as $n \to \infty$, we get

$$d(Az, w) \le k_1 (d(Sz, w) \diamond d(Az, Az)) + k_2 (d(Az, w) \diamond d(w, w)) + k_3 \left(d(Az, w) \diamond \frac{d(Az, w) + d(Az, w)}{2} \right); \tag{11}$$

that is,

$$\begin{aligned} d(Az, z) &\le k_1 (d(Sz, z) \diamond d(Az, Az)) \\ &\quad + k_2 (d(Az, z) \diamond d(z, z)) \\ &\quad + k_3 \left(d(Az, z) \diamond \frac{d(Az, z) + d(Az, z)}{2} \right) \\ &\le k_1 \alpha \max\{d(Sz, z), d(Az, Az)\} \\ &\quad + k_2 \alpha \max\{d(Az, z), d(z, z)\} \\ &\quad + k_3 \alpha \max\left\{ d(Az, z), \frac{d(Az, z) + d(Az, z)}{2} \right\}. \end{aligned} \tag{12}$$

Then, simplifying, we obtain

$$\begin{aligned} d(Az, z) &\le \alpha (k_1 + k_2 + k_3) d(Az, z) \\ &< d(Az, z), \end{aligned} \tag{13}$$

a contradiction. Hence, $Az = z$. Therefore, $Az = Sz = z$. Now we show that $Bz = z$. If $Bz \neq z$ then using (2) with $x = x_n$ and $y = z$, we have

$$\begin{aligned} d(Ax_n, Bz) &\leq k_1 (d(Sx_n, Tz) \diamond d(Ax_n, Sx_n)) \\ &\quad + k_2 (d(Sx_n, Tz) \diamond d(Bz, Tz)) \\ &\quad + k_3 \left(d(Sx_n, Tz) \diamond \frac{d(Sx_n, Bz) + d(Ax_n, Tz)}{2} \right). \end{aligned} \tag{14}$$

As $n \to \infty$, we get

$$\begin{aligned} d(z, Bz) &\leq k_1 (d(z, Bz) \diamond d(z, z)) \\ &\quad + k_2 (d(z, Bz) \diamond d(Bz, Bz)) \\ &\quad + k_3 \left(d(z, Bz) \diamond \frac{d(z, Bz) + d(z, Bz)}{2} \right) \\ &\leq k_1 \alpha \max \{d(z, Bz), d(z, z)\} \\ &\quad + k_2 \alpha \max \{d(z, Bz), d(Bz, Bz)\} \\ &\quad + k_3 \alpha \max \left\{ d(z, Bz), \frac{d(z, Bz) + d(z, Bz)}{2} \right\}. \end{aligned} \tag{15}$$

Then, simplifying, we obtain

$$\begin{aligned} d(z, Bz) &\leq \alpha (k_1 + k_2 + k_3) d(z, Bz) \\ &< d(z, Bz), \end{aligned} \tag{16}$$

a contradiction. Hence, $Bz = z$. Therefore, $z = Az = Sz = Bz = Tz$; that is, z is a common fixed point of A, B, S, and T. The uniqueness of common fixed point is an easy consequence of inequality (2).

Now suppose that the mappings (A, S) (as well as (B, T)) are subcompatible and reciprocally continuous. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = z, \tag{17}$$

for some $z \in X$, and

$$\lim_{n \to \infty} d(ASx_n, SAx_n) = d(Az, Sz) = 0, \tag{18}$$

whereas in respect of the pair (B, T), there exists a sequence $\{y_n\}$ in X with

$$\lim_{n \to \infty} By_n = \lim_{n \to \infty} Ty_n = w, \tag{19}$$

for some $w \in X$, and

$$\lim_{n \to \infty} d(BTy_n, TBy_n) = d(Bw, Tw) = 0. \tag{20}$$

Therefore, $Az = Sz$ and $Bw = Tw$; that is, z is a coincidence point of the pair (A, S) whereas w is a coincidence point of the pair (B, T). The rest of the proof can be completed easily. □

Example 10. Let $X = [0, \infty)$, and let d be the usual metric on X. Define self-mappings A, B, S, and T by

$$\begin{aligned} Ax = Bx &= \begin{cases} \dfrac{x}{7}, & \text{if } x \in [0, 1]; \\ \dfrac{x+6}{7}, & \text{if } x \in (1, \infty), \end{cases} \\ Sx = Tx &= \begin{cases} \dfrac{x}{6}, & \text{if } x \in [0, 1]; \\ \dfrac{x+5}{6}, & \text{if } x \in (1, \infty). \end{cases} \end{aligned} \tag{21}$$

Consider the sequence $\{x_n\} = \{1/n\}_{n \in \mathbb{N}}$ in X. Then

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} \frac{1}{7n} = 0 = \lim_{n \to \infty} \frac{1}{6n} = \lim_{n \to \infty} Sx_n. \tag{22}$$

Next,

$$\begin{aligned} \lim_{n \to \infty} ASx_n &= \lim_{n \to \infty} A\left(\frac{1}{6n}\right) = \lim_{n \to \infty} \frac{1}{42n} = 0 = A0, \\ \lim_{n \to \infty} SAx_n &= \lim_{n \to \infty} S\left(\frac{1}{7n}\right) = \lim_{n \to \infty} \frac{1}{42n} = 0 = S0, \\ \lim_{n \to \infty} d(ASx_n, SAx_n) &= 0. \end{aligned} \tag{23}$$

Consider another sequence $\{x_n\} = \{1 + 1/n\}_{n \in \mathbb{N}}$ in X. Then

$$\begin{aligned} \lim_{n \to \infty} Ax_n &= \lim_{n \to \infty} \left(1 + \frac{1}{7n}\right) = 1 \\ &= \lim_{n \to \infty} \left(1 + \frac{1}{6n}\right) = \lim_{n \to \infty} Sx_n. \end{aligned} \tag{24}$$

However,

$$\begin{aligned} \lim_{n \to \infty} ASx_n &= \lim_{n \to \infty} A\left(1 + \frac{1}{6n}\right) = \lim_{n \to \infty} \left(1 + \frac{1}{42n}\right) = 1 \neq A1, \\ \lim_{n \to \infty} SAx_n &= \lim_{n \to \infty} S\left(1 + \frac{1}{7n}\right) = \lim_{n \to \infty} \left(1 + \frac{1}{42n}\right) = 1 \neq S1, \end{aligned} \tag{25}$$

but $\lim_{n \to \infty} d(ASx_n, SAx_n) = 0$. Thus, the pair (A, S) is compatible as well as subsequentially continuous but not reciprocally continuous (the same for the pair (B, T)). It is easy to check that condition (2) is satisfied with $\diamond = \max$, $\alpha = 1$, and $k_1 + k_2 + k_3 = 6/7$. Therefore, all the conditions of Theorem 9 are satisfied. Here, 0 is a coincidence as well as the unique common fixed point of the mappings A, B, S, and T.

It can be noted that this example cannot be covered by those fixed point theorems which assume both compatibility and reciprocal continuity or by involving conditions of closedness of respective ranges. Indeed, in this example $A(X) = [0, 1/7] \cup (1, \infty)$ and $S(X) = [0, 1/6] \cup (1, \infty)$; hence, neither of $A(X)$ and $S(X)$ is closed.

Example 11. Let $X = \mathbb{R}$ (set of real numbers), and let d be the usual metric on X. Define self-mappings A, B, S, and T by

$$\begin{aligned} Ax = Bx &= \begin{cases} \dfrac{x}{4}, & \text{if } x \in (-\infty, 1); \\ 4x - 3, & \text{if } x \in [1, \infty), \end{cases} \\ Sx = Tx &= \begin{cases} x + 3, & \text{if } x \in (-\infty, 1); \\ 5x - 4, & \text{if } x \in [1, \infty). \end{cases} \end{aligned} \tag{26}$$

Consider the sequence $\{x_n\} = \{1 + 1/n\}_{n\in\mathbb{N}}$ in X. Then

$$\begin{aligned}\lim_{n\to\infty} Ax_n &= \lim_{n\to\infty}\left(1+\frac{4}{n}\right) = 1\\ &= \lim_{n\to\infty}\left(1+\frac{5}{n}\right) = \lim_{n\to\infty} Sx_n.\end{aligned} \tag{27}$$

Also,

$$\begin{gathered}\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A\left(1+\frac{5}{n}\right) = \lim_{n\to\infty}\left(1+\frac{20}{n}\right) = 1 = A1,\\ \lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S\left(1+\frac{4}{n}\right) = \lim_{n\to\infty}\left(1+\frac{20}{n}\right) = 1 = S1,\\ \lim_{n\to\infty} d\left(ASx_n, SAx_n\right) = 0.\end{gathered} \tag{28}$$

Consider another sequence $\{x_n\} = \{(1/n) - 4\}_{n\in\mathbb{N}}$ in X. Then

$$\begin{aligned}\lim_{n\to\infty} A\left(x_n\right) &= \lim_{n\to\infty}\left(\frac{1}{4n}-1\right) = -1\\ &= \lim_{n\to\infty}\left(\frac{1}{n}-1\right) = \lim_{n\to\infty} S\left(x_n\right).\end{aligned} \tag{29}$$

Next,

$$\begin{aligned}\lim_{n\to\infty} ASx_n &= \lim_{n\to\infty} A\left(\frac{1}{n}-1\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{4n}-\frac{1}{4}\right) = -\frac{1}{4} = A\left(-1\right),\\ \lim_{n\to\infty} SAx_n &= \lim_{n\to\infty} S\left(\frac{1}{4n}-1\right)\\ &= \lim_{n\to\infty}\left(\frac{1}{4n}-1+3\right) = 2 = S\left(-1\right),\end{aligned} \tag{30}$$

and $\lim_{n\to\infty} d(ASx_n, SAx_n) \neq 0$. Thus, the pair (A, S) is reciprocally continuous as well as subcompatible but not compatible (the same for the pair (B, T)). It is easy to check that condition (2) is satisfied with $\diamond = \max$, $\alpha = 1$, and $k_1 + k_2 + k_3 = 4/5$. Therefore, all the conditions of Theorem 9 are satisfied. Thus, 1 is a coincidence as well as the unique common fixed point of the pair (A, S).

It can be noted that this example cannot be covered by those fixed point theorems which involve both compatibility and reciprocal continuity. Again, $A(X) = (-\infty, 1/4) \cup [1, +\infty)$ is not closed. Note also that the mappings A and S have two points of coincidence (-1 and 1), which are occasionally weakly compatible but not weakly compatible.

In the next example (taken from [18, Example 1.4]), we demonstrate the situation when conditions of Theorem 9 are not satisfied, and the given pairs have no common fixed points.

Example 12. Let $X = [0, +\infty)$ with the standard metric d, and let $A, B, S, T : X \to X$ be given by

$$\begin{aligned}Ax = Bx &= \begin{cases} x+1, & 0 \le x \le 1,\\ 2x-1, & x > 1,\end{cases}\\ Sx = Tx &= \begin{cases} 1-x, & 0 \le x < 1,\\ 3x-2, & x \ge 1.\end{cases}\end{aligned} \tag{31}$$

Then, as it was shown in [18, Example 1.4], the pairs (A, S) and (B, T) are subsequentially continuous and subcompatible. However, they are neither reciprocally continuous nor compatible (not even occasionally weakly compatible). We note that (A, S) has no common fixed points, although it has a unique point of coincidence $z = 1$.

We present an example of different kind, inspired by [20, Example 3].

Example 13. Let $X = \{0, 1, 2, \ldots, 10\}$, and define a metric d on X by

$$d\left(x, y\right) = \begin{cases} 0, & x = y,\\ \max\left\{x, y\right\}, & x \neq y.\end{cases} \tag{32}$$

Consider the mappings $A, B, S, T : X \to X$ given by

$$\begin{aligned}Ax = Bx &= \begin{cases} 0, & x = 0,\\ x-1, & x \ge 1;\end{cases}\\ Sx = Tx &= \begin{cases} 0, & x = 0,\\ x+1, & 1 \le x \le 9,\\ 10, & x = 10.\end{cases}\end{aligned} \tag{33}$$

Take $\diamond = \max$ (which satisfies α-condition with $\alpha = 1$). Then,

(1) the pair (A, S) (as well as (B, T)) is compatible and subsequentially continuous,

(2) condition (2) is satisfied with $k_1 = k_2 = k_3 = 0.3$.

Indeed, in order to prove (1), take $x_n = 0$ for all n but finitely many (which is the only possibility to obtain the same limit for (Ax_n) and (Sx_n)). Then $d(Ax_n, 0) \to 0$ and $d(Sx_n, 0) \to 0$; also $SAx_n \to 0 = S0$ and $ASx_n \to 0 = A0$. Hence, the pair (A, S) is compatible and subsequentially continuous.

In order to prove (2), suppose that $x, y \in X$, $x \neq y$ (the case $x = y$ is trivial). Since we have $A = B$, $S = T$, and $k_1 = k_2$, condition (2) is symmetric in x, y; hence, without loss of generality, we can suppose that $x \ge y$. Consider the following possible cases.

Case 1. One has $y = 0$ and $x = 1$. Then $Ax = By = 0$, $d(Ax, By) = 0$, and (2) is satisfied.

Case 2. One has $y \in \{0, 1\}$ and $2 \le x \le 9$. Then $Ax = x - 1$, $By = 0$, $d(Ax, By) = x-1$. The right-hand side of (2) becomes ($t \in \{0, 2\}$)

$$\begin{aligned} R &= k_1 \max\{x+1, t\} + k_2 \max\{x+1, x+1\} \\ &\quad + k_3 \max\left\{x+1, \frac{1}{2}\left[\max\{Sx, By\} + x + 1\right]\right\} \\ &= (k_1 + k_2 + k_3)(x+1) = 0.9(x+1) \\ &\ge 0.9 \cdot \frac{10}{8}(x-1) > x - 1 = d(Ax, By). \end{aligned} \tag{34}$$

Case 3. One has $y \in \{0, 1\}$ and $x = 10$. Then

$$d(Ax, By) = 9 = (k_1 + k_2 + k_3) \cdot 10 = R. \tag{35}$$

Case 4. One has $2 \le y < x \le 9$. Then $d(Ax, By) = d(x - 1, y - 1) = x - 1$ and

$$\begin{aligned} R &= k_1 \max\{x+1, x+1\} + k_2 \max\{x+1, y+1\} \\ &\quad + k_3 \max\left\{x+1, \frac{1}{2}\left[x + 1 + \max\{Ax, Ty\}\right]\right\} \\ &= (k_1 + k_2 + k_3)(x+1) = 0.9(x+1) \\ &\ge 0.9 \cdot \frac{10}{8}(x-1) > x - 1 = d(Ax, By). \end{aligned} \tag{36}$$

Case 5. One has $2 \le y < x = 10$. Then (2) again reduces to $9 = (k_1 + k_2 + k_3) \cdot 10$.

All the conditions of Theorem 9 are satisfied, and A, B, S, and T have a unique common fixed point (which is $z = 0$).

By choosing A, B, S, and T suitably in Theorem 9, we can deduce corollaries for two or three self-mappings. As a sample, we deduce the following corollary for two self-mappings.

Corollary 14. *Let A and S be two self-mappings on a metric space (X, d) such that $\diamond$ satisfies the α-property with $\alpha > 0$. If the pair (A, S) is compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous) satisfying*

$$\begin{aligned} d(Ax, Ay) &\le k_1 (d(Sx, Sy) \diamond d(Ax, Sx)) \\ &\quad + k_2 (d(Sx, Sy) \diamond d(Ay, Sy)) \\ &\quad + k_3 \left(d(Sx, Sy) \diamond \frac{d(Sx, Ay) + d(Ax, Sy)}{2}\right), \end{aligned} \tag{37}$$

for all $x, y \in X$, where $k_1, k_2, k_3 > 0$ and $0 < \alpha(k_1+k_2+k_3) < 1$, then A and S have a unique common fixed point in X.

Remark 15. The conclusion of Theorem 9 remains true if we replace inequality (2) by the following:

$$\begin{aligned} d(Ax, By) &\le k_1 (d(Sx, Ty) + d(Ax, Sx)) \\ &\quad + k_2 (d(Sx, Ty) + d(By, Ty)) \\ &\quad + k_3 \left(d(Sx, Ty) + \frac{d(Sx, By) + d(Ax, Ty)}{2}\right), \end{aligned} \tag{38}$$

for all $x, y \in X$, where $k_1, k_2, k_3 > 0$ and $0 < k_1+k_2+k_3 < 1/2$.

Similarly, other variants of contractive condition can be obtained by specifying operation $\diamond$.

Remark 16. Similar results can be obtained if condition (2) is replaced by the following one:

$$\begin{aligned} &d(Ax, By) \le \psi(u) \\ &\text{for some } u \in \{d(Sx, Ty), d(Ax, Sx), \\ &\qquad d(By, Ty), d(Sx, By), \\ &\qquad d(Ax, Ty)\}, \end{aligned} \tag{39}$$

for a suitable function $\psi : [0, +\infty) \to [0, +\infty)$.

4. Conclusion

Theorem 9 is proved for two pairs of compatible and subsequentially continuous (alternately subcompatible and reciprocally continuous) mappings satisfying a general contractive condition. Theorem 9 improves the main result of Sedghi and Shobe [32, Theorem 2.1] as we do not require any condition on the containment of ranges of involved mappings and completeness (or closedness) of the whole space (or any subspace). A natural result is defined in the form of a corollary (see Corollary 14). On the other hand, Remark 15 is developed for a particular case, $a \diamond b = a + b$, which also improves the result of Sedghi and Shobe [32, Corollary 2.2].

Acknowledgments

The authors would like to thank Professor Mohammad Imdad for the preprint of paper [36] and anonymous referees for their valuable comments that helped them to improve the exposition, in particular for Remark 16. The second author is thankful to the Ministry of Education, Science and Technological Development of Serbia.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 576379, 7 pages
http://dx.doi.org/10.1155/2013/576379

Research Article

H_∞ Control for Linear Positive Discrete-Time Systems

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Received 7 June 2013; Accepted 11 August 2013

Academic Editor: Wei-Shih Du

This paper is concerned with H_∞ control for linear positive discrete-time systems. Positive systems are characterized by nonnegative restriction on systems' variables. This restriction results in some remarkable results which are available only for linear positive systems. One of them is the celebrated diagonal positive definite matrix solutions to some existed well-known results for linear systems without nonnegative restriction. We provide an alternative proof for criterion of H_∞ norm by using separating hyperplane theorem and Perron-Frobenius theorem for nonnegative matrices. We also consider H_∞ control problem for linear positive discrete-time systems via state feedback. Necessary and sufficient conditions for such problem are presented under controller gain with and without nonnegative restriction, and then the desired controller gains can be obtained from the feasible solutions.

1. Introduction

Positive systems are widespread in many practical systems, such as economic systems [1], biology systems [2], and age-structured population models [3], whose variables are required to be nonnegative and have no meaning with negative values. The explicit definition of a positive system is that its state and output are always nonnegative for any nonnegative initial state and any nonnegative input. Due to the nonnegative restriction on systems' variables, positive systems are defined on cones rather than linear space. Hence, there are excellent and remarkable outcomes which are available only for positive systems. One of them is the existence of diagonal positive definite matrix solutions to some celebrated results for linear systems without nonnegative restriction. Therefore, the investigation of positive systems is interesting and challenging and developed a new branch in systems theory. Positive systems have been of great interest to many researchers over several decades. A great number of results have been reported in the literature; see, for instance, [3–16].

It is worth noting that convex optimization is a powerful tool for analysis of positive systems. In [9–14], some remarkable results for positive systems are studied using convex optimization. On the other hand, the problem of H_∞ control has been a topic of recurring interest for several decades. A great number of results on H_∞ control have been obtained, and different approaches have been proposed. In recent years, increasing attention has been paid to H_∞ norm analysis for positive systems. In [12], the KYP lemma for linear positive continuous-time systems is proved based on the semidefinite programming duality. The alternative proofs along the line of the rank-one separable property are given to several remarkable and peculiar results for positive systems in [13]. In [14], the KYP lemma for linear positive discrete-time systems is studied using a theorem of alternatives on the feasibility of linear matrix inequalities (LMIs).

This paper is organized as follows. Preliminaries are introduced in Section 2. Main results on H_∞ control for linear positive discrete-time systems are presented and proved in Section 3. Section 4 is devoted to illustrate the effectiveness of the obtained results by numerical examples. Section 5 concludes this paper.

2. Preliminaries

In this section, we introduce terminology, positive systems, various other definitions, and lemmas, which will be essentially used for proving our main results.

At first, the following notations will be used throughout this paper.

$\mathbb{Z}_+$ denotes the nonnegative integers. $x \geq 0$ denotes the vector $x \in \mathbb{R}^n$ with nonnegative entries. $A \geq 0$ denotes

the matrix $A \in \mathbb{R}^{n\times m}$ with nonnegative entries. $\mathbb{R}_+^n$ denotes the set of all vectors $x \in \mathbb{R}^n$ with $x \geq 0$. $\mathbb{R}_+^{n\times m}$ denotes the set of all matrices $A \in \mathbb{R}^{n\times m}$ with $A \geq 0$. S^n denotes the set of all symmetric matrices. $D_+^{n\times n}$ denotes the set of all diagonal positive definite matrices. $A \succeq 0$, $A \succ 0$, and $A \prec 0$ mean that A is a positive semidefinite, positive definite, and negative definite matrix, respectively. A_{ij} denotes the ijth entry of matrix A. x_i denotes the ith entry of vector x. For two matrices $A, B \in \mathbb{R}^{n\times m}$, $A \geq B$ means $A_{ij} \geq B_{ij}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. For two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ means $x_i \geq y_i$, $i = 1, 2, \ldots, n$. $\rho(A)$ denotes the spectral radius of matrix A which is defined as $\rho(A) := \max_{1\leq i\leq n}|\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. $D(A)$ denotes the vector which is composed of the diagonal entries of $A \in \mathbb{R}^{n\times n}$. $\langle X, Y\rangle = \operatorname{trace}(XY)$ is the inner product on S^n.

Consider the following linear discrete-time system:

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k) + Du(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the input, and $y(k) \in \mathbb{R}^p$ is the output. $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $D \in \mathbb{R}^{p\times m}$ are known matrices.

Definition 1 (see [3]). System (1) is said to be positive if and only if $x(k) \geq 0$, $y(k) \geq 0$, $k \in \mathbb{Z}_+$ for any $x(0) \geq 0$ and for any $u(k) \geq 0$, $k \in \mathbb{Z}_+$.

Definition 2 (see [3]). System (1) is said to be asymptotically stable if $\rho(A) < 1$.

Definition 3 (see [13]). For a given matrix $H \in S^n$ with $H \succeq 0$, we define $h \in \mathbb{R}_+^n$ by

$$h_i := \sqrt{H_{ii}}, \quad i = 1, 2, \ldots, n. \tag{2}$$

Lemma 4 (see [3], Perron-Frobenius theorem for nonnegative matrices). *Let $A \in \mathbb{R}_+^{n\times n}$; then $\rho(A)$ is an eigenvalue of A and A has a nonnegative eigenvector v corresponding to $\rho(A)$.*

A matrix $A \in \mathbb{R}^{n\times n}$ is called a Metzler matrix if $A_{ij} \geq 0$, for all i, j with $i \neq j$.

Lemma 5 (see [13]). *For a given Metzler matrix $A \in \mathbb{R}^{n\times n}$ and $H \in S^n$ with $H \succeq 0$, the following conditions hold:*

(i) $(hh^T)_{ii} = H_{ii}$, $(hh^T)_{ij} \geq H_{ij}$, $i \neq j$,

(ii) $D(hh^T A) \geq D(HA)$,

where $h \in \mathbb{R}_+^n$ is defined from H as in Definition 3.

Lemma 6 (see [3]). *System* (1) *is positive if and only if $A \geq 0$, $B \geq 0$, $C \geq 0$, and $D \geq 0$.*

Lemma 7 (see [17], separating hyperplane theorem). *Suppose that C_1 and C_2 are two convex sets that do not intersect; that is, $C_1 \cap C_2 = \varnothing$. Then, there exist $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C_1$ and $a^T x \geq b$ for all $x \in C_2$.*

For $A \in \mathbb{R}_+^{n\times n}$, we consider $B := sI - A$. If $s > 0$ and $s \geq \rho(A)$, then B is called an M-matrix. If $s > \rho(A)$, then B is a nonsingular M-matrix.

Lemma 8 (see [4]). *A nonsingular matrix $A \in \mathbb{R}^{n\times n}$ is an M-matrix if and only if $A^{-1} \geq 0$.*

Lemma 9 (see [18], Schur complement). *Given any real matrices Q, S, and R with $Q = Q^T$ and $R = R^T$, the following statement holds:*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \prec 0 \tag{3}$$

if and only if

$$R < 0, \quad Q - SR^{-1}S^T < 0, \tag{4}$$

or, equivalently,

$$Q < 0, \quad R - S^T Q^{-1} S < 0. \tag{5}$$

The transfer function matrix of system (1) is given by

$$G(z) = C(zI - A)^{-1}B + D, \tag{6}$$

and its H_∞ norm is defined as

$$\|G\|_\infty := \sup_{\theta\in(-\pi,\pi]} \overline{\sigma}\left(G\left(e^{j\theta}\right)\right), \tag{7}$$

where $\overline{\sigma}(G(e^{j\theta}))$ denotes the maximum singular value of $G(e^{j\theta})$. In [14], it has been pointed out that $\|G\|_\infty = \|G(1)\|$, where $\|G(1)\| = \overline{\sigma}(G(1))$, if system (1) is positive and asymptotically stable.

3. H_∞ Control

In this section, we give an alternative proof for the existed result of H_∞ norm for positive discrete-time systems and investigate the H_∞ control under state feedback.

At first, we propose the following theorem which is helpful for the alternative proof.

Theorem 10. *Suppose that system* (1) *is positive; the following conditions are equivalent:*

(i) *There exists a nonzero $h \in \mathbb{R}_+^n$ such that $(A - I)h \geq 0$.*

(ii) *System* (1) *is not asymptotically stable.*

Proof. (i)⇒(ii). Since $A \geq 0$, from Perron-Frobenius theorem for nonnegative matrices, it follows that $A^T v = \rho(A)v \geq 0$, where $\rho(A) \geq 0$ is the spectral radius of matrix A^T and $v \geq 0$ is an eigenvector corresponding to $\rho(A)$. Then it is obtained from condition (i) that

$$(\rho(A) - 1)\, v^T h \geq 0, \tag{8}$$

which implies $\rho(A) \geq 1$; namely, system (1) is not asymptotically stable.

(ii)⇒(i). System (1) is not asymptotically stable; that is, $\rho(A) \geq 1$. From Perron-Frobenius theorem, we immediately obtain the following result:

$$\rho(A)h = Ah \geq h, \tag{9}$$

where $h \geq 0$ is an eigenvector corresponding to $\rho(A)$. This completes the proof. □

The following theorem was firstly presented and proved in the light of alternatives on the feasibility of LMIs by Federico Najson in the literature [14]. Now we will give another proof using separating hyperplane theorem and Theorem 10.

Theorem 11. *Suppose that system* (1) *is positive; the following conditions are equivalent.*

(i) *System* (1) *is asymptotically stable and* $\|G\|_\infty < 1$.

(ii) *There exists a diagonal positive definite matrix P such that*

$$\begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD - I \end{bmatrix} \prec 0. \tag{10}$$

(iii) *There exists a diagonal positive definite matrix P such that*

$$\begin{bmatrix} -P & 0 & A^TP & C^T \\ 0 & -I & B^TP & D^T \\ PA & PB & -P & 0 \\ C & D & 0 & -I \end{bmatrix} \prec 0. \tag{11}$$

Proof. We only prove (i)⇒(ii) since the implication (ii)⇒(i) is obvious from the existed criterion for linear discrete-time systems, and the equivalence between (ii) and (iii) is immediately obtained using Schur complement.

To the contrary, suppose that condition (10) does not hold for any diagonal positive definite matrix P. Define the following two sets:

$$C_1 := \left(\begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD - I \end{bmatrix} \mid P \in D_+^{n\times n} \right),$$
$$C_2 := \{Q \mid Q \prec 0, Q \in S^{n+m}\}; \tag{12}$$

then it is easy to check that sets C_1 and C_2 are nonempty and convex. By the assumption, we have $C_1 \cap C_2 = \varnothing$. Then from the separating hyperplane theorem, there exists a nonzero $H \in S^{n+m}$ such that

$$\langle H, S\rangle \geq 0, \quad \forall S \in C_1, \tag{13}$$

$$\langle H, S\rangle \leq 0, \quad \forall S \in C_2. \tag{14}$$

By condition (14), we can conclude that $\langle H, S\rangle = \operatorname{trace}(HS) \leq 0$, for all $S \prec 0$, from which it is easy to verify that $H \succeq 0$. Thus it follows from condition (13) that there exists a nonzero $H \succeq 0$ such that

$$\operatorname{trace}\left(H \begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD - I \end{bmatrix} \right) \geq 0, \quad \forall P \in D_+^{n\times n}, \tag{15}$$

which is equivalent to

$$\operatorname{trace}\left(H \begin{bmatrix} A^TPA - P & A^TPB \\ B^TPA & B^TPB \end{bmatrix} \right) + \operatorname{trace}\left(H \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - I \end{bmatrix} \right) \geq 0, \quad \forall P \in D_+^{n\times n}. \tag{16}$$

Let $H := \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix}$, $H_{11} \succeq 0$, $H_{22} \succeq 0$; then the above condition can be rewritten as

$$\operatorname{trace}\left(H_{11}\left(A^TPA - P\right) + H_{12}B^TPA + H_{12}^TA^TPB + H_{22}B^TPB \right) + \operatorname{trace}\left(H \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - I \end{bmatrix} \right) \geq 0, \quad \forall P \in D_+^{n\times n}, \tag{17}$$

or, equivalently,

$$\operatorname{trace}\left(P\left(AH_{11}A^T - H_{11} + AH_{12}B^T + BH_{12}^TA^T + BH_{22}B^T \right) \right) + \operatorname{trace}\left(H \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - I \end{bmatrix} \right) \geq 0, \quad \forall P \in D_+^{n\times n}, \tag{18}$$

which implies that

(a) $D(AH_{11}A^T - H_{11} + AH_{12}B^T + BH_{12}^TA^T + BH_{22}B^T) \geq 0$,

(b) $\operatorname{trace}(H \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - I \end{bmatrix}) \geq 0$.

From condition (a), it follows that

$$\begin{aligned} D\left(AH_{11}A^T + AH_{12}B^T + BH_{12}^TA^T + BH_{22}B^T \right) &= D\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix} \begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix} \right) \\ &\geq D\left(\begin{pmatrix} H_{11} & 0 \\ 0 & 0 \end{pmatrix} \right). \end{aligned} \tag{19}$$

Since system (1) is positive, then

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - I \end{bmatrix} \tag{20}$$

are nonnegative matrix and Metzler matrix, respectively. Define the nonzero $h \in \mathbb{R}_+^n$ from H as in Definition 3 then from Lemma 5, we obtain

$$\begin{aligned} D\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} hh^T \begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix} \right) &\geq D\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix} \begin{bmatrix} A^T & 0 \\ B^T & 0 \end{bmatrix} \right) \\ &\geq D\left(\begin{pmatrix} H_{11} & 0 \\ 0 & 0 \end{pmatrix} \right), \end{aligned} \tag{21}$$

$$\operatorname{trace}\left(hh^T \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - I \end{bmatrix} \right) \geq \operatorname{trace}\left(H \begin{bmatrix} C^TC & C^TD \\ D^TC & D^TD - I \end{bmatrix} \right) \geq 0. \tag{22}$$

Set $h := \left[h_1^T \;\; h_2^T\right]^T$, $h_1 \in \mathbb{R}_+^n$, $h_2 \in \mathbb{R}_+^m$; conditions (21) and (22) can be rewritten as

$$\begin{aligned} &D\left(Ah_1h_1^TA^T + Ah_1h_2^TB^T + Bh_2h_1^TA^T + Bh_2h_2^TB^T\right) \\ &\quad \geq D\left(h_1h_1^T\right), \end{aligned} \tag{23}$$

$$\begin{aligned} &h_1^TC^TCh_1 + h_2^TD^TCh_1 + h_1^TC^TDh_2 + h_2^TD^TDh_2 \\ &\quad = \left(Ch_1 + Dh_2\right)^T\left(Ch_1 + Dh_2\right) \geq h_2^Th_2. \end{aligned} \tag{24}$$

Since H is nonzero, then we have the following three cases.

(1) Consider that $h_1 = 0, h_2 \neq 0$. By condition (24), $h_2^TD^TDh_2 \geq h_2^Th_2$, which contradicts $\|G\|_\infty < 1$.

(2) Consider that $h_1 \neq 0, h_2 = 0$. We observe from (23) that $D(Ah_1h_1^TA^T) \geq D(h_1h_1^T)$, which implies that $Ah_1 \geq h_1$.

From Theorem 10, this is a contradiction.

(3) Consider that $h_1 \neq 0, h_2 \neq 0$. From condition (24), it yields that $Ch_1 + Dh_2 \neq 0$. Define matrix

$$\Delta = \frac{h_2\left(Ch_1 + Dh_2\right)^T}{\left(Ch_1 + Dh_2\right)^T\left(Ch_1 + Dh_2\right)}, \tag{25}$$

which is well-defined and satisfies $\overline{\sigma}(\Delta) \leq 1$. Note that $h_2 = \Delta(Ch_1 + Dh_2)$. On the other hand, $\overline{\sigma}(D) < 1$; otherwise, this contradicts $\|G\|_\infty < 1$. It is known that for matrices $M \in \mathbb{C}^{l\times m}$ and $N \in \mathbb{C}^{m\times l}$ with $\overline{\sigma}(M) \leq 1$ and $\overline{\sigma}(N) < 1$, $\det(I - MN) \neq 0$. Thus, $I - \Delta D$ is invertible and $(I - \Delta D)^{-1} \geq 0$ since $I - \Delta D$ is a nonsingular M-matrix. Therefore, $h_2 = (I - \Delta D)^{-1}\Delta Ch_1$ holds. Then from (23) we have

$$\begin{aligned} &D\left(Ah_1h_1^TA^T + Ah_1h_2^TB^T + Bh_2h_1^TA^T + Bh_2h_2^TB^T\right) \\ &= D\left(Ah_1h_1^TA^T + Ah_1h_1^TC^T\Delta^T(I - \Delta D)^{-T}B^T\right. \\ &\qquad + B(I - \Delta D)^{-1}\Delta Ch_1h_1^TA^T \\ &\qquad \left. + B(I - \Delta D)^{-1}\Delta Ch_1h_1^TC^T\Delta^T(I - \Delta D)^{-T}B^T\right) \\ &= D\left(\left(A + B(I - \Delta D)^{-1}\Delta C\right)h_1h_1^T\left(A + B(I - \Delta D)^{-1}\Delta C\right)^T\right) \\ &= D\left(\widetilde{A}h_1h_1^T\widetilde{A}^T\right) \\ &\geq D\left(h_1h_1^T\right), \end{aligned} \tag{26}$$

where $\widetilde{A} := A + B(I - \Delta D)^{-1}\Delta C \geq 0$. From Theorem 10, $\rho(\widetilde{A}) \geq 1$, which contradicts $\|G\|_\infty < 1$. □

Corollary 12. *Suppose that system* (1) *is positive; then the following conditions are equivalent.*

(i) *System* (1) *is asymptotically stable and* $\|G\|_\infty < \gamma$.

(ii) *There exists a diagonal positive definite matrix P such that*

$$\begin{bmatrix} A^TPA - P + C^TC & A^TPB + C^TD \\ B^TPA + D^TC & B^TPB + D^TD - \gamma^2 I \end{bmatrix} \prec 0. \tag{27}$$

(iii) *There exists a diagonal positive definite matrix P such that*

$$\begin{bmatrix} -P & 0 & A^TP & C^T \\ 0 & -\gamma^2 I & B^TP & D^T \\ PA & PB & -P & 0 \\ C & D & 0 & -I \end{bmatrix} \prec 0. \tag{28}$$

Now, our purpose is to design a state feedback controller given by

$$u(k) = Kx(k) + v(k), \tag{29}$$

where $K \in \mathbb{R}^{m\times n}$ is the controller gain to be designed, and $v(k) \in \mathbb{R}_+^n$, such that the closed-loop system described as

$$\begin{aligned} x(k+1) &= (A + BK)x(k) + Bv(k), \\ y(k) &= (C + DK)x(k) + Dv(k), \end{aligned} \tag{30}$$

is positive, asymptotically stable, and $\|\overline{G}\|_\infty < 1$, where $\overline{G}(z) = (C + DK)(zI - (A + BK))^{-1} + D$.

At first, we focus on nonnegative control gain, as it has practical importance in many cases. For instance, for a chemical system whose variables represent concentrations of reactants and reaction speed is impacted by concentrations, in order to improve the speed of reaction, it is natural to consider such controller for increasing concentrations.

Theorem 13. *For the given positive system* (1), *there exists a nonnegative controller of the form in* (29) *such that the closed-loop system* (30) *is asymptotically stable and* $\|\overline{G}\|_\infty < 1$ *if and only if there exist* $X \in D_+^{n\times n}$ *and* $Y \geq 0$ *satisfying*

$$\begin{bmatrix} -X & 0 & XA^T + Y^TB^T & XC^T + Y^TD^T \\ 0 & -I & B^T & D^T \\ AX + BY & B & -X & 0 \\ CX + DY & D & 0 & -I \end{bmatrix} \prec 0. \tag{31}$$

Under the above condition, the desired nonnegative controller gain is obtained as

$$K = YX^{-1}. \tag{32}$$

Proof. Necessity. From Theorem 11, there exists a diagonal positive definite matrix P such that

$$\begin{bmatrix} -P & 0 & (A + BK)^TP & (C + DK)^T \\ 0 & -I & B^TP & D^T \\ P(A + BK) & PB & -P & 0 \\ C + DK & D & 0 & -I \end{bmatrix} \prec 0. \tag{33}$$

Multiplying on both sides of inequality (33) by $T = \operatorname{diag}(P^{-1}, I, P^{-1}, I)$, it follows that

$$\begin{bmatrix} -P^{-1} & 0 & P^{-1}A^T + P^{-1}K^TB^T & P^{-1}C^T + P^{-1}K^TD^T \\ 0 & -I & B^T & D^T \\ AP^{-1} + BKP^{-1} & B & -P^{-1} & 0 \\ CP^{-1} + DKP^{-1} & D & 0 & -I \end{bmatrix} \prec 0. \tag{34}$$

By defining $X = P^{-1}$, $Y = KP^{-1}$, inequality (31) is immediately obtained. On the other hand, since $X \in D_+^{n\times n}$ and $Y \geq 0$, it is easy to see that $K \geq 0$.

Sufficiency. Positivity of the closed-loop system (30) is obvious. From (32), $Y = KX$; substituting it into inequality (31) leads to

$$\begin{bmatrix} -X & 0 & XA^T + XK^TB^T & XC^T + XK^TD^T \\ 0 & -I & B^T & D^T \\ AX + BKX & B & -X & 0 \\ CX + DKX & D & 0 & -I \end{bmatrix} \prec 0. \tag{35}$$

Multiplying on both sides of inequality (35) by $T = \operatorname{diag}(X^{-1}, I, X^{-1}, I)$, we have

$$\begin{bmatrix} -X^{-1} & 0 & (A+BK)^T X^{-1} & (C+DK)^T \\ 0 & -I & B^T X^{-1} & D^T \\ X^{-1}(A+BK) & X^{-1}B & -X^{-1} & 0 \\ C+DK & D & 0 & -I \end{bmatrix} \prec 0. \tag{36}$$

Therefore, from Theorem 11, the closed-loop system (30) is asymptotically stable and $\|\overline{G}\|_\infty < 1$. □

Remark 14. Under the assumption that system (1) is positive, it is worth noting that there does not exist any $X \in D_+^{n\times n}$ or $Y \geq 0$ satisfying condition (31) if there exists $A_{ii} > 1$, $i = 1, 2, \ldots, n$. It is easy to verify in the light of the following facts.

(1) A linear positive discrete-time system is unstable if at least one diagonal entry of matrix A is greater than 1 which is presented in the literature [3].

(2) $\rho(B) \leq \rho(A)$ if $0 \leq B \leq A$ which has been pointed out in the literature [19].

On the other hand, it is known that the maximal eigenvalue $\rho(A)$ of $A \geq 0$ belongs to the interval

$$\max\{\min c_i, \min r_i\} \leq \rho(A) \leq \min\{\max c_i \ \max r_i\}, \tag{37}$$

where c_i and r_i denote the sum of the elements of the ith column and the ith row of matrix A, respectively. Therefore, there also does not exist $X \in D_+^{n\times n}$ or $Y \geq 0$ satisfying condition (31) if $\max\{\min c_i, \min r_i\} \geq 1$.

Remark 15. If $0 \leq B \leq A$, then $\overline{\sigma}(A) \geq \overline{\sigma}(B)$ which is obtained immediately due to the fact (2) in Remark 14. Suppose that system (1) is positive, $\|G\|_\infty \geq 1$, and there exists a nonnegative controller (29) such that system (30) is asymptotically stable; it is obvious that system (1) is also asymptotically stable. Thus,

$$\begin{aligned} \|\overline{G}\|_\infty &= \|\overline{G}(1)\| \\ &= \|(C+DK)(I-(A+BK))^{-1}B + D\| \\ &= \left\|(C+DK)\left(\sum_{i=0}^{\infty}(A+BK)^i\right)B + D\right\| \\ &= \overline{\sigma}\left((C+DK)\left(\sum_{i=0}^{\infty}(A+BK)^i\right)B + D\right) \\ &\geq \overline{\sigma}\left(C\left(\sum_{i=0}^{\infty}A^i\right)B + D\right) = \|G\|_\infty \geq 1. \end{aligned} \tag{38}$$

Therefore, if system (1) is positive, asymptotically stable, but $\|G\|_\infty \geq 1$, there does not exist any nonnegative state feedback (29) such that system (30) is asymptotically stable and $\|\overline{G}\|_\infty < 1$.

Corollary 16. *For the given positive system* (1)*, there exists a nonnegative controller of the form in* (29) *such that the closed-loop system* (30) *is asymptotically stable and* $\|\overline{G}\|_\infty < \gamma$ *if and only if there exist* $X \in D_+^{n\times n}$ *and* $Y \geq 0$ *satisfying*

$$\begin{bmatrix} -X & 0 & XA^T + Y^TB^T & XC^T + Y^TD^T \\ 0 & -\gamma^2 I & B^T & D^T \\ AX + BY & B & -X & 0 \\ CX + DY & D & 0 & -I \end{bmatrix} \prec 0. \tag{39}$$

Under the above condition, the desired controller gain is obtained as (32).

From Remarks 14 and 15, for some positive systems, there is no nonnegative state feedback (29) such that system (30) is asymptotically stable and $\|\overline{G}\|_\infty < 1$. Hence, we are obligated to pay attention to state feedback without nonnegative restriction. It is also natural to consider such controller. For example, for an ecosystem whose variables represent population of animals in a forest; population cannot exceed the ecological capacity of the forest, otherwise, the ecosystem may be destroyed. Therefore, we must decrease the number of animals by means of harvesting or other methods when population of certain animals exceeds their ecological capacity.

Now we are looking for a state feedback without nonnegative restriction having form in (29) such that the closed-loop system (30) is positive, asymptotically stable, and $\|\overline{G}\|_\infty < 1$.

Theorem 17. *For the given positive system* (1)*, there exists a controller of the form in* (29) *such that the closed-loop system* (30) *is positive, asymptotically stable, and* $\|\overline{G}\|_\infty < 1$ *if and only if there exist* $X \in D_+^{n\times n}$ *and* $Y \in \mathbb{R}^{m\times n}$ *satisfying* (39) *and*

$$AX + BY \geq 0, \tag{40}$$

$$CX + DY \geq 0. \tag{41}$$

Under the above conditions, the desired controller gain is given by (32).

Proof. We only prove conditions (40) and (41).

Necessity. The close-loop system (30) is positive then it follows that

$$A + BK \geq 0, \qquad C + DK \geq 0. \tag{42}$$

Since X is a diagonal positive definite matrix, then it is easy to check that

$$AX + BKX \geq 0, \qquad CX + DKX \geq 0. \tag{43}$$

Conditions (40) and (41) are immediately obtained.

Sufficiency. The proof is similar to necessity. □

Corollary 18. *For the given positive system* (1)*, there exists a controller of the form in* (29) *such that the closed-loop system* (30) *is positive, asymptotically stable, and* $\|\overline{G}\|_\infty < \gamma$ *if and only if there exist* $X \in D_+^{n\times n}$ *and* $Y \in \mathbb{R}^{m\times n}$ *satisfying conditions* (39), (40)*, and* (41)*. Under these conditions, the desired controller gain is obtained as* (32).

4. Examples

In this section, we give some examples to illustrate the validity of the obtained results.

Example 1. Consider a linear positive discrete-time system with the following parameter matrices:

$$A = \begin{bmatrix} 0.6048 & 0 \\ 0.0861 & 0.4187 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.0952 & 0 \\ 0.0457 & 0.1813 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.0921 & 0 \\ 0 & 0.1286 \end{bmatrix}. \tag{44}$$

Solving the conditions in Theorem 13, one feasible solution is obtained as

$$X = \begin{bmatrix} 0.3033 & 0 \\ 0 & 0.4477 \end{bmatrix}, \qquad Y = \begin{bmatrix} 0.0854 & 0.1142 \\ 0.0743 & 0.1150 \end{bmatrix}. \tag{45}$$

Then the desired controller gain is given by

$$K = \begin{bmatrix} 0.2816 & 0.2551 \\ 0.2551 & 0.2551 \end{bmatrix}. \tag{46}$$

Example 2. Consider a linear positive discrete-time system with the following parameter matrices:

$$A = \begin{bmatrix} 1.2148 & 0 \\ 0.0861 & 0.9187 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.1702 \\ 0.0457 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.816 & 0 \end{bmatrix}, \qquad D = 0.3157. \tag{47}$$

We observe that $A_{11} > 1$ and check the condition in Theorem 13; as pointed out in Remark 14, there is no feasible solution.

Then solving the conditions in Theorem 17, one feasible solution is obtained as

$$X = \begin{bmatrix} 2.1747 & 0 \\ 0 & 810.9260 \end{bmatrix}, \qquad Y = \begin{bmatrix} -4.0062 & 5.0132 \end{bmatrix}. \tag{48}$$

The desired controller gain is

$$K = \begin{bmatrix} -1.8421 & 0.0062 \end{bmatrix}. \tag{49}$$

Example 3. Consider a linear positive discrete-time system with the following parameter matrices:

$$A = \begin{bmatrix} 0.9187 & 0.1295 \\ 0.0861 & 0.9187 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.1702 \\ 0.1570 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.5700 & 0.3850 \end{bmatrix}, \qquad D = 0.2103. \tag{50}$$

By calculating, $\max\{\min c_i, \min r_i\} = 1.0048$. We check the condition in Theorem 13; there is no feasible solution.

Solving the conditions in Theorem 17, one feasible solution is obtained as

$$X = \begin{bmatrix} 0.4315 & 0 \\ 0 & 1.6751 \end{bmatrix}, \qquad Y = \begin{bmatrix} -0.2126 & -1.2241 \end{bmatrix}. \tag{51}$$

Then the controller gain is given by

$$K = \begin{bmatrix} -0.4926 & -0.7308 \end{bmatrix}. \tag{52}$$

5. Conclusion

In this paper, we are interested in H_∞ control for linear positive discrete-time systems. We present a necessary and sufficient condition to check the stability of linear positive discrete-time systems using Perron-Frobenius theorem for nonnegative matrices, which is the key point for the alternative proof. We believe that it is useful for checking the existence of diagonal positive definite matrices to some other results given for linear discrete-time systems without nonnegative restriction. The alternative proof is along the line of separating hyperplane theorem and Theorem 10 given in this paper. In addition, we investigate the H_∞ control for such systems under state feedback. Necessary and sufficient conditions for such problem are presented under controller gain with and without nonnegative restriction, and then the desired controller gains can be obtained from the feasible solutions. However, in this paper, we have restricted our attention to the case of state feedback, but in practice, it is not always possible to have access to all of the state variables. The case of static and dynamic output feedback is left for future research. Robust H_∞ analysis and synthesis for positive uncertain systems are also open problems.

Acknowledgments

The work was supported by the National Natural Science Foundation of China under Grant 61273008 and China Scholarship Council.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 487273, 12 pages
http://dx.doi.org/10.1155/2013/487273

Research Article

Convergence Analysis of an Iterative Method for Nonlinear Partial Differential Equations

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Received 22 May 2013; Accepted 20 July 2013

Academic Editor: E. Karapinar

We will combine linear successive overrelaxation method with nonlinear monotone iterative scheme to obtain a new iterative method for solving nonlinear equations. The basic idea of this method joining traditional monotone iterative method (known as the method of lower and upper solutions) which depends essentially on the monotone parameter is that by introducing an acceleration parameter one can construct a sequence to accelerate the convergence. The resulting increase in the speed of convergence is very dramatic. Moreover, the sequence can accomplish monotonic convergence behavior in the iterative process when some suitable acceleration parameters are chosen. Under some suitable assumptions in aspect of the nonlinear function and the matrix norm generated from this method, we can prove the boundedness and convergence of the resulting sequences. Application of the iterative scheme is given to a logistic model problem in ecology, and numerical results for a test problem with known analytical solution are given to demonstrate the accuracy and efficiency of the present method.

1. Introduction

In terms of solving linear equations, we usually use two different iterative methods, namely, the Jacobi and Gauss-Seidel methods [1–3]. The monotone iterative (MI) schemes which combine linear iterative techniques, respectively, are presented and analyzed in [4–8] for solving nonlinear equations. The method of monotone iterations is a classical tool for the study of the existence of solutions of semilinear PDEs of certain types [9–12]. It is also useful for numerical solutions of these types of problems approximated, for instance, by the finite difference [5, 6, 13–15], finite element [16], or boundary element [17, 18] method. It is a constructive method that depends essentially on only one parameter, called the monotone parameter herein, which determines the convergent behavior of the iterative process. Besides, the block Picard, block Jacobi, and block Gauss-Seidel MI methods are also developed and compared the rates of convergence with the point MI schemes [6]. The block MI methods accelerate the rate of convergence more than the point MI methods. In particular, Ortega and Rheinboldt [19, page 456] mention an analysis of the Newton-SOR methods to research some properties of convergence for relaxation factor $0 < \omega < 1$. The MI methods have been widely used in the treatment of certain nonlinear parabolic and elliptic differential equations. For instance, in the study of certain subsonic flows and molecular interactions, the equation $\Delta u = u^2$ is of fundamental importance [18]. For parabolic problems with time delays we refer to [20]. In addition, we also utilize MI schemes to handle nonlinear problems on analysis of numerical results for semiconductor equations [21–23] and the Poisson Boltzmann equation [24].

Consider the nonlinear boundary-value problem:

$$\begin{gathered} -\left[\left(\mathrm{D}^{(1)}u_x\right)_x + \left(\mathrm{D}^{(2)}u_y\right)_y\right] = f(x,y,u), \quad \text{in } \Omega, \\ \alpha\frac{\partial u}{\partial \nu} + \beta u = g(x,y), \quad \text{on } \partial\Omega, \end{gathered} \tag{1}$$

in a two-dimensional domain Ω with boundary $\partial\Omega$, where $\partial u/\partial \nu$ is the outward normal derivative of u on $\partial\Omega$, $\mathrm{D}^{(l)} \equiv \mathrm{D}^{(l)}(x,y)$, $l = 1, 2$, are positive functions on $\overline{\Omega} \equiv \Omega \cup \partial\Omega$, $\alpha \equiv \alpha(x,y)$ and $\beta \equiv \beta(x,y)$ are nonnegative functions on $\partial\Omega$ with $\alpha + \beta > 0$, and f and g are given functions in their

respective domains. For the nonlinear function f, we give two assumptions:

(i)

$$f \text{ is uniformly bounded for } -\infty < u < \infty; \tag{H1}$$

(ii) if $|u|, |v| \le c$, then there exists a function $\mathscr{H}(c)$, such that for all $x, y \in \Omega$ we have

$$\left|f(x,y,u) - f(x,y,v)\right| \le \mathscr{H}(c) \cdot |u - v|. \tag{H2}$$

Applying the finite difference method to (1), we obtain a system of nonlinear algebraic equations in a compact form:

$$AU = F(U) + G^*. \tag{2}$$

Suppose that A can be written in the splitting form $A = \mathscr{D} - \mathscr{L} - \mathscr{U}$, where $\mathscr{D}$, $-\mathscr{L}$, and $-\mathscr{U}$ are the diagonal, lower-off-diagonal, and upper-off-diagonal matrices of A, respectively. We consider that linear SOR method can be combined with nonlinear MI scheme to obtain a nonlinear SOR monotone iterative method for solving nonlinear equations which gives rise to the terminology "SORMI". The basic idea of this method joining MI method which depends essentially on the monotone parameter Γ is that by introducing an acceleration parameter one can construct a sequence to accelerate the convergence. The algorithm is similar to the SOR method. Roughly speaking, given an initial vector $U^{(0)}$, the SORMI method generates a sequence of iterates $\{U^{(m)}\}$, $m = 0, 1, \ldots$, by solving the equation:

$$\begin{aligned}(\mathscr{D} + \Gamma - \omega\mathscr{L})\, U^{(m+1)} &= [(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}]\, U^{(m)} \\ &\quad + \omega\left[\Gamma U^{(m)} + F\left(U^{(m)}\right) + G^*\right],\end{aligned} \tag{3}$$

where ω is a relaxation factor. Under some suitable assumptions in aspect of the nonlinear function and the matrix norm generated from this method, we can prove the boundedness and convergence of the resulting sequences. Moreover, the sequences can accomplish monotonic convergence in the iterative process when some suitable relaxation factors are chosen.

The structure of the paper is as follows. In Section 2, we briefly make a description for discretization process to obtain algebraic equations for model (1) and state some properties of the matrix. Section 3 deals with the monotone parameter and constructs the SORMI scheme. We show the boundedness and convergence of the SORMI sequence in Sections 4 and 5. Moreover, we offer another proof for the convergence of the SORMI sequence in the case $0 < \omega < 1$. In Section 6, we solve a one dimensional problem, and a logistic model in population growth problem and numerical results of the method are also given to verify the theoretical analysis. The final section is for some concluding remarks.

2. A Finite Difference Discretization

We discuss problem (1) in a rectangular domain $\Omega = (0, l_1) \times (0, l_2)$. Let $h = l_1/n$, $k = l_2/m$, and let $x_i = ih$, $y_j = jk$ for $i = 0, 1, 2, \ldots, n$, $j = 0, 1, 2, \ldots, m$. The set of points (x_i, y_j) in Ω and $\overline{\Omega} \equiv \Omega \cup \partial\Omega$ are defined, respectively, by Υ and $\overline{\Upsilon}$. When no confusion arises we write a point (x_i, y_j) in $\overline{\Upsilon}$ by $(i, j) \in \overline{\Upsilon}$. Define

$$\begin{aligned} u_{i,j} &= u\left(x_i, y_j\right), \\ f_{i,j}\left(u_{i,j}\right) &= f\left(x_i, y_j, u\left(x_i, y_j\right)\right), \\ g_{i,j} &= g\left(x_i, y_j\right). \end{aligned} \tag{4}$$

The finite difference method for differential and boundary operators in (1) leads to a discrete system in the form

$$\begin{aligned} a_{ij}u_{ij} - b_{ij}u_{i-1,j} - b'_{ij}u_{i+1,j} - c_{ij}u_{i,j-1} - c'_{ij}u_{i,j+1} \\ = hkf_{ij}\left(u_{ij}\right) + g^*_{ij}, \end{aligned} \tag{5}$$

for all $(i, j) \in \overline{\Upsilon}$, where the coefficients $a_{ij}, b_{ij}, b'_{ij}, c_{ij}$, and c'_{ij} are associated with the diffusion coefficients $\mathrm{D}^{(l)}_{ij} \equiv \mathrm{D}^{(l)}(x_i, y_j)$, $l = 1, 2$, as well as the boundary coefficients $\alpha_{ij} \equiv \alpha(x_i, y_j)$ and $\beta_{ij} \equiv \beta(x_i, y_j)$, g^*_{ij} is associated with the boundary functions g_{ij}, and

$$\begin{gathered} b_{0j} = b'_{nj} = c_{i0} = c'_{im} = 0 \\ \text{for } i = 0, 1, \ldots, n, \ \ j = 0, 1, \ldots, m, \\ g^*_{ij} = 0 \quad \text{for } i = 1, 2, \ldots, n-1, \ \ j = 1, 2, \ldots, m-1. \end{gathered} \tag{6}$$

Typical choice of the coefficients in (5) for the interior mesh points is given by

$$\begin{aligned} b_{ij} &= \left(\frac{k}{h}\right)\mathrm{D}^{(1)}\left(x_i - \frac{h}{2}, y_j\right), \\ b'_{ij} &= \left(\frac{k}{h}\right)\mathrm{D}^{(1)}\left(x_i + \frac{h}{2}, y_j\right), \\ c_{ij} &= \left(\frac{h}{k}\right)\mathrm{D}^{(2)}\left(x_i, y_j - \frac{k}{2}\right), \\ c'_{ij} &= \left(\frac{h}{k}\right)\mathrm{D}^{(2)}\left(x_i, y_j + \frac{k}{2}\right), \\ a_{ij} &= b_{ij} + b'_{ij} + c_{ij} + c'_{ij}, \quad \left((i,j) \in \Upsilon\right) \end{aligned} \tag{7}$$

(e.g., see [25]). For the boundary points (x_i, y_j), where $i = 0, n$ or $j = 0, m$, the above coefficients are associated with the boundary coefficients α_{ij} and β_{ij}, and possess the property (for the case $\alpha > 0$),

$$a_{ij} \ge b_{ij} + b'_{ij} + c_{ij} + c'_{ij} \quad \text{for } i = 0, n \text{ or } j = 0, m. \tag{8}$$

Furthermore, strict inequality in (8) holds for at least one (i, j) when the boundary condition is not of pure Neumann condition. In either case, these coefficients satisfy the condition

$$\begin{gathered} b_{ij} \ge 0, \quad b'_{ij} \ge 0, \quad b_{ij} + b'_{ij} > 0, \\ c_{ij} \ge 0, \quad c'_{ij} \ge 0, \\ a_{ij} \ge b_{ij} + b'_{ij} + c_{ij} + c'_{ij}, \quad \left((i,j) \in \overline{\Upsilon}\right). \end{gathered} \tag{9}$$

Condition (9) is our basic hypothesis for the boundedness and the convergence of the SORMI sequence. Now we rewrite system (5) in a compact form:

$$AU = F(U) + G^*. \tag{10}$$

Definition 1. A real $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} \le 0$ for all $i \ne j$ and $a_{ii} > 0$ for all $1 \le i \le n$ is an $\mathscr{M}$-matrix if A is nonsingular, and $A^{-1} \ge 0$ [2].

Obviously, A is a diagonally dominant with strict inequality for at least one (i, j). Since the domain $\overline{\Upsilon}$ is connected, the above property implies that A is nonsingular, and $A^{-1} \ge 0$. Hence, A is an $\mathscr{M}$-matrix. This implies that for any nonnegative diagonal matrix $\mathbf{D}$, $(A + \mathbf{D})^{-1}$ exists and is nonnegative.

Remark 2. Nonnegative matrices play a crucial role in the theory of matrices. They are important in the study of convergence of iterative methods and arise in many applications including economics, queuing theory, and chemical engineering. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two real $n \times r$ matrices. Then, $A \ge B$ $(A > B)$ if $a_{ij} \ge b_{ij}$ $(> b_{ij})$ for all $1 \le i \le n$, $1 \le j \le r$. If O is the null matrix and $A \ge O$ $(> O)$, we say that A is a nonnegative (positive) matrix. Since column vectors are $n \times 1$ matrices, we will use the terms nonnegative and positive vector throughout. A theorem which has important consequences on the analysis of iterative methods should be stated. Let B be a nonnegative matrix. Then $\rho(B) < 1$ if and only if $I - B$ is nonsingular and $(I-B)^{-1}$ is nonnegative, where $\rho(B)$ is the spectral radius of B.

Remark 3. In reality, the four conditions in the definition of $\mathscr{M}$-matrix are somewhat redundant, and equivalent conditions that are more rigorous will be (i) $a_{ij} \le 0$ for all $i \ne j$, (ii) A is nonsingular, and (iii) $A^{-1} \ge 0$. The condition, $a_{ii} > 0$ for all i, is implied by the other three. Moreover, let D be the diagonal of A, and $B \equiv I - D^{-1}A$. We can also obtain $\rho(B) < 1$. A comparison theorem is as follows.

Let A and B be two $n \times n\mathscr{M}$-matrices, with $A \ge B$. Then we have $B^{-1} \ge A^{-1}$.

Remark 4. Let us look in more detail at the algebraic system (10) [26, 27]. The connectedness assumption of Ω ensures that A is irreducible. Condition (9) implies that A is irreducibly diagonally dominant [28]. Let $A = \mathscr{D} - \mathscr{B}$, where $\mathscr{D}$ is the diagonal matrix of A. It can be shown that $0 < \rho(\mathscr{B}) < 1$, using Perron-Frobenius theorem and the theory of regular splittings. A theorem states the following.

If $A = (a_{ij})$ is a real $n \times n$ matrix with $a_{ij} \le 0$ for all $i \ne j$, then the following are equivalent.

(i) A is nonsingular, and $A^{-1} > 0$.

(ii) The diagonal entries of A are positive real numbers. $\mathscr{B}$ is nonnegative, irreducible, and convergent.

Thus, we know that A is a diagonally dominant $\mathscr{M}$-matrix.

3. The SORMI Method

We now arrive to construct the SORMI sequence.

Definition 5. A vector $\widetilde{U} \equiv (\widetilde{u}_{00}, \widetilde{u}_{10}, \ldots, \widetilde{u}_{n0}, \ldots, \widetilde{u}_{0m}, \widetilde{u}_{1m}, \ldots, \widetilde{u}_{nm})^T$ with $(n+1)\times(m+1)$ components is called an upper solution of (10) if

$$A\widetilde{U} \ge F\left(\widetilde{U}\right) + G^*, \tag{11}$$

and $\widehat{U} \equiv (\widehat{u}_{00}, \widehat{u}_{10}, \ldots, \widehat{u}_{n0}, \ldots, \widehat{u}_{0m}, \widehat{u}_{1m}, \ldots, \widehat{u}_{nm})^T$ is called a lower solution of (10) if

$$A\widehat{U} \le F\left(\widehat{U}\right) + G^*. \tag{12}$$

We say that $\widetilde{U}$ and $\widehat{U}$ are ordered if $\widetilde{U} \ge \widehat{U}$. Given any ordered upper and lower solutions $\widetilde{U}, \widehat{U}$, we set

$$\left\langle \widehat{U}, \widetilde{U} \right\rangle \equiv \left\{ U : \widehat{U} \le U \le \widetilde{U} \right\}. \tag{13}$$

Define

$$\begin{gathered} \gamma_{ij} \equiv \max\left\{ -\frac{\partial f_{ij}}{\partial u}\left(u_{ij}\right) : \widehat{u}_{ij} \le u_{ij} \le \widetilde{u}_{ij} \right\}, \\ \gamma_{ij}^{+} \equiv \max\left\{0, \gamma_{ij}\right\}, \\ \Gamma \equiv hk \cdot \operatorname{diag}\left(\overline{\gamma}_{00}, \overline{\gamma}_{10}, \ldots, \overline{\gamma}_{n0}, \overline{\gamma}_{01}, \right. \\ \left. \overline{\gamma}_{11}, \ldots, \overline{\gamma}_{n1}, \ldots, \overline{\gamma}_{0m}, \overline{\gamma}_{1m}, \ldots, \overline{\gamma}_{nm}\right), \end{gathered} \tag{14}$$

where $\overline{\gamma}_{ij}$ is any nonnegative scalar satisfying $\overline{\gamma}_{ij} \ge \gamma_{ij}^{+}$, and $\widetilde{u}_{ij}$ and $\widehat{u}_{ij}$ are the components of $\widetilde{U}$ and $\widehat{U}$, respectively. Then problem (10) is equivalent to

$$(A + \Gamma)U = \Gamma U + F(U) + G^*. \tag{15}$$

Suppose that A can be written in the splitting form $A = \mathscr{D} - \mathscr{L} - \mathscr{U}$, where $\mathscr{D}$, $-\mathscr{L}$, and $-\mathscr{U}$ are the diagonal, lower-off-diagonal, and upper-off-diagonal matrices of A, respectively. The elements of $\mathscr{D}$ are positive, and those of $\mathscr{L}$ and $\mathscr{U}$ are nonnegative. Given an initial iterate vector $U^{(0)}$, the SOR method for solving the linear system $AU = b$ is

$$(\mathscr{D} - \omega\mathscr{L})U_L^{(m+1)} = [(1-\omega)\mathscr{D} + \omega\mathscr{U}]U_L^{(m)} + \omega b, \tag{16}$$

where ω is a relaxation factor, and $U_L^{(0)} = U^{(0)}$. Moreover, the Gauss-Seidel MI method for solving the nonlinear system (15) is defined by

$$(\mathscr{D} + \Gamma - \mathscr{L})U_{GS}^{(m+1)} = (\Gamma + \mathscr{U})U_{GS}^{(m)} + F\left(U_{GS}^{(m)}\right) + G^*. \tag{17}$$

Thus, we define the SORMI method for solving the nonlinear system (15) by

$$\begin{aligned} (\mathscr{D} + \Gamma - \omega\mathscr{L})U_s^{(m+1)} &= [(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}]U_s^{(m)} \\ &\quad + \omega\left[\Gamma U_s^{(m)} + F\left(U_s^{(m)}\right) + G^*\right]. \end{aligned} \tag{18}$$

4. The Boundedness of the SORMI Sequences

Before the convergence analysis of the method, we want to ask whether the SORMI sequences are bounded. Now we consider the property.

Lemma 6. *Given a pair of upper and lower solutions $\widetilde{U}$, $\widehat{U}$ of (10), let U, V be two vectors with $(n+1)\times(m+1)$ components, and $\widetilde{U} \geq U \geq V \geq \widehat{U}$. Then*

$$\Gamma U + F(U) \geq \Gamma V + F(V). \tag{19}$$

Proof. Let u_{ij} and v_{ij} be the components of U and V, respectively. By the mean value theorem,

$$-\left[\frac{f(x_i, y_j, u_{ij}) - f(x_i, y_j, v_{ij})}{u_{ij} - v_{ij}}\right] = -\frac{\partial f_{ij}}{\partial u}(w_{ij}), \tag{20}$$

where w_{ij} lies between v_{ij} and u_{ij}. From (14), we have $\overline{\gamma}_{ij} \geq \gamma_{ij}^{+}$. Hence

$$\begin{gathered} -hk\left[\frac{f(x_i, y_j, u_{ij}) - f(x_i, y_j, v_{ij})}{u_{ij} - v_{ij}}\right] \leq hk\gamma_{ij}^{+} \leq hk\overline{\gamma}_{ij}, \\ hk\overline{\gamma}_{ij}\cdot u_{ij} + hkf_{ij}(u_{ij}) \geq hk\overline{\gamma}_{ij}\cdot v_{ij} + hkf_{ij}(v_{ij}). \end{gathered} \tag{21}$$

This completes the proof. □

In [2, page 83], the theorem is stated as follows.

Theorem 7. *If $A \geq O$ is an $n\times n$ matrix, then $\kappa > \rho(A)$ if and only if $\kappa I - A$ is nonsingular, and $(\kappa I - A)^{-1} \geq O$, where $\rho(A)$ is a spectral radius of A.*

Hence, we quote the above theorem to obtain the following lemma.

Lemma 8. *The matrix $(\mathscr{D} + \Gamma - \omega\mathscr{L})$ in (18) is nonsingular, and $(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1}$ is nonnegative for $0 < \omega < 2$.*

Proof. From the splitting form of A and (14), we have $(\mathscr{D}+\Gamma-\omega\mathscr{L}) = (a_{ij}+hk\overline{\gamma}_{ij})I-\omega\mathscr{L}$, $i = 0, 1, 2, \ldots, n$, $j = 0, 1, 2, \ldots, m$, where $a_{ij} + hk\overline{\gamma}_{ij} > 0$. Since $0 < \omega < 2$, $\omega\mathscr{L} \geq O$, and $\omega\mathscr{L}$ is a strictly lower triangle matrix. It follows that all eigenvalues of $\omega\mathscr{L}$ are zeros, and thus $\rho(\omega\mathscr{L}) = 0$. Hence, $a_{ij} + hk\overline{\gamma}_{ij} > \rho(\omega\mathscr{L})$. By Theorem 7, $(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} = [(a_{ij} + hk\overline{\gamma}_{ij})I - \omega\mathscr{L}]^{-1} \geq O$. □

Definition 9. Let the vector $U = (u_{00}, u_{10}, \ldots, u_{n0}, \ldots, u_{0m}, u_{1m}, \ldots, u_{nm})^T$. We define $|U| = (|u_{00}|, |u_{10}|, \ldots, |u_{n0}|, \ldots, |u_{0m}|, |u_{1m}|, \ldots, |u_{nm}|)^T$.

Definition 10. Let $U = (u_1, u_2, \ldots, u_n)^T \in \mathfrak{R}^n$. The vector 2-norm is usually defined by $\|U\| = (\Sigma_{i=1}^{n} u_i^2)^{1/2}$.

Definition 11. Let $\|\cdot\|$ be a vector 2-norm. The induced matrix 2-norm of an $n\times n$ matrix A is defined by

$$\|A\| = \sup_{U\neq 0}\frac{\|AU\|}{\|U\|}. \tag{22}$$

Let $\{U_s^{(m)}\}_{m=0}^{\infty}$ be a sequence generated by the SORMI method with initial vector $U_s^{(0)} \in \langle\widehat{U}, \widetilde{U}\rangle$, where $\widetilde{U}, \widehat{U}$ are upper and lower solutions of (10), respectively. We have the following.

Lemma 12. *$\|U\| \leq \||\widehat{U}| + |\widetilde{U}|\|$ for any $U \in \langle\widehat{U}, \widetilde{U}\rangle$. In fact, $\|U\| = \||U|\|$.*

Proof. Let $U = (u_{00}, u_{10}, \ldots, u_{n0}, \ldots, u_{0m}, u_{1m}, \ldots, u_{nm})^T$, $\widehat{U} = (\widehat{u}_{00}, \widehat{u}_{10}, \ldots, \widehat{u}_{n0}, \ldots, \widehat{u}_{0m}, \widehat{u}_{1m}, \ldots, \widehat{u}_{nm})^T$, $\widetilde{U} = (\widetilde{u}_{00}, \widetilde{u}_{10}, \ldots, \widetilde{u}_{n0}, \ldots, \widetilde{u}_{0m}, \widetilde{u}_{1m}, \ldots, \widetilde{u}_{nm})^T$. Since $U \in \langle\widehat{U}, \widetilde{U}\rangle$, $\widehat{u}_{ij} \leq u_{ij} \leq \widetilde{u}_{ij}$ for all $i = 0, 1, 2, \ldots, n$, $j = 0, 1, 2, \ldots, m$. Hence,

$$\begin{gathered} u_{ij} \leq |u_{ij}| \leq \max\{|\widehat{u}_{ij}|, |\widetilde{u}_{ij}|\} \leq |\widehat{u}_{ij}| + |\widetilde{u}_{ij}|, \\ \left(\sum_{i=0}^{n}\sum_{j=0}^{m} u_{ij}^2\right)^{1/2} \leq \left[\sum_{i=0}^{n}\sum_{j=0}^{m}\left(|\widehat{u}_{ij}| + |\widetilde{u}_{ij}|\right)^2\right]^{1/2}; \end{gathered} \tag{23}$$

that is, $\|U\| \leq \||\widehat{U}| + |\widetilde{U}|\|$ for any $U \in \langle\widehat{U}, \widetilde{U}\rangle$. □

To prove that the SORMI sequences are bounded, we must define several values about matrix and vector norms.

Notations and Assumptions. (a) Let $M_1 \overset{\text{def}}{=} \|\widehat{U}\| + \|\widetilde{U}\|$. By Lemma 12,

$$\left\|U_s^{(0)}\right\| \leq \left\||\widehat{U}| + |\widetilde{U}|\right\| \leq \left\|\widehat{U}\right\| + \left\|\widetilde{U}\right\| = M_1. \tag{24}$$

(b) In (H1), we assume that $f_{ij}(u_{ij})$ is uniformly bounded for $-\infty < u_{ij} < \infty$, and g_{ij}^{*} is known boundary value; that is, there exists $\delta > 0$ such that $|hkf_{ij}(u_{ij}) + g_{ij}^{*}| \leq \delta$ for $-\infty < u_{ij} < \infty$, where h and k are mesh sizes. By Definition 10, (5) and (10), we have

$$\left\|F(U) + G^{*}\right\| \leq \left[(n+1)\times(m+1)\times\delta^2\right]^{1/2} = \delta\sqrt{N}. \tag{25}$$

(c) Define

$$\begin{gathered} M_2 = \|E\| = \left\|(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1}\right\|, \\ \|B\| = \left\|(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1}\left[(1-\omega)\mathscr{D} + \Gamma + \omega\mathscr{U}\right]\right\|, \end{gathered} \tag{26}$$

and assume that

$$\|B\| < 1. \tag{H3}$$

(d) Consider two vectors:

$$\begin{gathered} U_1 \overset{\text{def}}{=} \widehat{U} - \omega(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1}(\mathscr{D} + \Gamma)\left(\widetilde{U} - \widehat{U}\right), \\ U_2 \overset{\text{def}}{=} \widetilde{U} + \omega(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1}(\mathscr{D} + \Gamma)\left(\widetilde{U} - \widehat{U}\right). \end{gathered} \tag{27}$$

Since $0 < \omega < 2$, $(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} \geq O$, $\mathscr{D} + \Gamma \geq O$, and $\widetilde{U} \geq \widehat{U}$, we know that $[\omega(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1}(\mathscr{D} + \Gamma)(\widetilde{U} - \widehat{U})]$ is a nonnegative vector. It follows that $U_1 \leq \widehat{U} \leq \widetilde{U} \leq U_2$. Let

$$\begin{gathered} \langle U_1, U_2\rangle \overset{\text{def}}{=} \{U : U_1 \leq U \leq U_2\}, \\ M_3 \overset{\text{def}}{=} \left\||U_1| + |U_2|\right\|. \end{gathered} \tag{28}$$

Then

$$\begin{aligned} M_3 &= \left\| |U_1| + |U_2| \right\| \le \|U_1\| + \|U_2\| \\ &= \left\| \widehat{U} - \omega(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} (\mathscr{D} + \Gamma) \left(\widetilde{U} - \widehat{U}\right) \right\| \\ &\quad + \left\| \widetilde{U} + \omega(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} (\mathscr{D} + \Gamma) \left(\widetilde{U} - \widehat{U}\right) \right\| \\ &\le \left\| \widehat{U} \right\| + \omega \left\| (\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} (\mathscr{D} + \Gamma) \right\| \left(\left\| \widetilde{U} \right\| + \left\| \widetilde{U} \right\| \right) \\ &\quad + \left\| \widetilde{U} \right\| + \omega \left\| (\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} (\mathscr{D} + \Gamma) \right\| \left(\left\| \widetilde{U} \right\| + \left\| \widetilde{U} \right\| \right) \\ &< \infty. \end{aligned} \tag{29}$$

(e) Define

$$M_4 = \max\left\{ M_1, 2\delta\sqrt{N} M_2 \right\}. \tag{30}$$

Theorem 13. *Let $\widetilde{U}$, $\widehat{U}$ be a pair of ordered upper and lower solutions of* (10)*, respectively, and let $\{U_s^{(m)}\}_{m=0}^{\infty}$ be a sequence generated by* (18) *with initial vector $U_s^{(0)} \in \langle \widehat{U}, \widetilde{U} \rangle$. Then, the iterative sequence $\{U_s^{(m)}\}_{m=0}^{\infty}$ is bounded for $0 < \omega < 2$.*

Proof. By (28) and (30), we can choose a constant $M = \max\{M_3, M_4/(1 - \|B\|)\}$.

Consider two cases of the sequence $\{U_s^{(m)}\}$.

Case 1. Let $U_s^{(m-1)} \in \langle \widehat{U}, \widetilde{U} \rangle$. Then

$$\begin{aligned} (\mathscr{D} + \Gamma - \omega\mathscr{L}) U_s^{(m)} &= [(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}] U_s^{(m-1)} \\ &\quad + \omega \left[\Gamma U_s^{(m-1)} + F\left(U_s^{(m-1)}\right) + G^* \right] \\ &\le \mathscr{D} U_s^{(m-1)} + \Gamma U_s^{(m-1)} - \omega\mathscr{D} U_s^{(m-1)} \\ &\quad - \omega\Gamma U_s^{(m-1)} + \omega\mathscr{U} U_s^{(m-1)} \\ &\quad + \omega \left[\Gamma\widetilde{U} + F\left(\widetilde{U}\right) + G^* \right] \\ &\le \mathscr{D} U_s^{(m-1)} + \Gamma U_s^{(m-1)} - \omega\mathscr{D} U_s^{(m-1)} \\ &\quad - \omega\Gamma U_s^{(m-1)} + \omega\mathscr{U} U_s^{(m-1)} \\ &\quad + \omega \left[\Gamma\widetilde{U} + A\widetilde{U} \right] \\ &\le \mathscr{D}\widetilde{U} + \Gamma\widetilde{U} - \omega\mathscr{D}\widehat{U} - \omega\Gamma\widehat{U} + \omega\mathscr{U}\widetilde{U} \\ &\quad + \omega\Gamma\widetilde{U} + \omega\mathscr{D}\widetilde{U} - \omega\mathscr{L}\widetilde{U} - \omega\mathscr{U}\widetilde{U} \\ &= \omega(\mathscr{D} + \Gamma)\left(\widetilde{U} - \widehat{U}\right) \\ &\quad + (\mathscr{D} + \Gamma - \omega\mathscr{L})\widetilde{U}. \end{aligned} \tag{31}$$

By (27) and $(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} \ge 0$, we obtain

$$U_s^{(m)} \le \widetilde{U} + \omega(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} (\mathscr{D} + \Gamma)\left(\widetilde{U} - \widehat{U}\right) = U_2. \tag{32}$$

Similarly,

$$(D + \Gamma - \omega\mathscr{L}) U_s^{(m)} \ge (\mathscr{D} + \Gamma - \omega\mathscr{L})\widehat{U} - \omega(\mathscr{D} + \Gamma)\left(\widetilde{U} - \widehat{U}\right), \tag{33}$$

and thus

$$U_s^{(m)} \ge \widehat{U} - \omega(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} (\mathscr{D} + \Gamma)\left(\widetilde{U} - \widehat{U}\right) = U_1. \tag{34}$$

So $U_1 \le U_s^{(m)} \le U_2$. By Lemma 12 and (28), we obtain

$$\left\| U_s^{(m)} \right\| \le \left\| |U_1| + |U_2| \right\| = M_3 \le M. \tag{35}$$

Case 2. Let $U_s^{(m-1)} \notin \langle \widehat{U}, \widetilde{U} \rangle$. Then

$$\begin{aligned} (\mathscr{D} + \Gamma - \omega\mathscr{L}) U_s^{(m)} &= [(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}] U_s^{(m-1)} \\ &\quad + \omega \left[\Gamma U_s^{(m-1)} + F\left(U_s^{(m-1)}\right) + G^* \right] \\ &= [(1-\omega)\mathscr{D} + \Gamma + \omega\mathscr{U}] U_s^{(m-1)} \\ &\quad + \omega \left[F\left(U_s^{(m-1)}\right) + G^* \right]. \end{aligned} \tag{36}$$

By (26), we have

$$\begin{aligned} U_s^{(m)} &= (\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} [(1-\omega)\mathscr{D} + \Gamma + \omega\mathscr{U}] U_s^{(m-1)} \\ &\quad + \omega(\mathscr{D} + \Gamma - \omega\mathscr{L})^{-1} \left[F\left(U_s^{(m-1)}\right) + G^* \right] \\ &= BU_s^{(m-1)} + \omega E \left[F\left(U_s^{(m-1)}\right) + G^* \right]. \end{aligned} \tag{37}$$

Consider the iterative process. An induction argument gives

$$\begin{aligned} U_s^{(1)} &= BU_s^{(0)} + \omega E \left[F\left(U_s^{(0)}\right) + G^* \right], \\ U_s^{(2)} &= BU_s^{(1)} + \omega E \left[F\left(U_s^{(1)}\right) + G^* \right] \\ &= B\left\{ BU_s^{(0)} + \omega E \left[F\left(U_s^{(0)}\right) + G^* \right] \right\} \\ &\quad + \omega E \left[F\left(U_s^{(1)}\right) + G^* \right] \\ &= B^2 U_s^{(0)} + \omega BE \left[F\left(U_s^{(0)}\right) + G^* \right] \\ &\quad + \omega E \left[F\left(U_s^{(1)}\right) + G^* \right], \\ &\vdots \\ U_s^{(m)} &= B^m U_s^{(0)} + \omega B^{m-1} E \left[F\left(U_s^{(0)}\right) + G^* \right] \\ &\quad + \omega B^{m-2} E \left[F\left(U_s^{(1)}\right) + G^* \right] \\ &\quad + \cdots + \omega BE \left[F\left(U_s^{(m-2)}\right) + G^* \right] \\ &\quad + \omega E \left[F\left(U_s^{(m-1)}\right) + G^* \right]. \end{aligned} \tag{38}$$

Hence, from Definition 11, (H3), (24), (25), (26), (30), and $0 < \omega < 2$, we obtain

$$\begin{aligned}
\left\|U_s^{(1)}\right\| &\le \|B\| \cdot \left\|U_s^{(0)}\right\| + \omega \cdot \|E\| \cdot \left\|F\left(U_s^{(0)}\right) + G^*\right\| \\
&\le M_1 \|B\| + 2\delta\sqrt{N} M_2 \\
&\le M_4 (\|B\| + 1) \le \frac{M_4}{1 - \|B\|} \le M, \\
\left\|U_s^{(2)}\right\| &\le \|B\|^2 \left\|U_s^{(0)}\right\| + \omega \|B\| \cdot \|E\| \cdot \left\|F\left(U_s^{(0)}\right) + G^*\right\| \\
&\quad + \omega \|E\| \cdot \left\|F\left(U_s^{(1)}\right) + G^*\right\| \\
&\le M_1 \|B\|^2 + 2\delta\sqrt{N} M_2 \|B\| + 2\delta\sqrt{N} M_2 \\
&\le M_4 \left(\|B\|^2 + \|B\| + 1\right) \le \frac{M_4}{1 - \|B\|} \le M, \\
&\vdots \\
\left\|U_s^{(m)}\right\| &\le \|B\|^m \left\|U_s^{(0)}\right\| + \omega \|B\|^{m-1} \|E\| \cdot \left\|F\left(U_s^{(0)}\right) + G^*\right\| \\
&\quad + \omega \|B\|^{m-2} \|E\| \cdot \left\|F\left(U_s^{(1)}\right) + G^*\right\| \\
&\quad + \cdots + \omega \|B\| \cdot \|E\| \cdot \left\|F\left(U_s^{(m-2)}\right) + G^*\right\| \\
&\quad + \omega \|E\| \cdot \left\|F\left(U_s^{(m-1)}\right) + G^*\right\| \\
&\le M_4 \left(\|B\|^m + \cdots + \|B\| + 1\right) \le \frac{M_4}{1 - \|B\|} \le M.
\end{aligned} \tag{39}$$

Thus, by Cases 1 and 2, there exists $M > 0$, such that $\|U_s^{(m)}\| \le M$ for all $m \in \{0\} \cup \mathcal{N}$, and the proof is completed. □

5. The Convergence of the SORMI Sequences

In (H2), we assume that if $|u|, |v| \le c$, then there is a function $\mathcal{H}(c)$, such that $|f(x, y, u) - f(x, y, v)| \le \mathcal{H}(c) \cdot |u - v|$. From Theorem 13, we have shown the boundedness of the SORMI sequences; that is, there exists a constant M such that $\|U_s^{(k)}\| < M = \max\{M_3, M_4/(1 - \|B\|)\}$. Let c_k satisfy $\max_{0 \le i \le m,\, 0 \le j \le n} |u_{ij}^{(k)}| \le c_k$. So $0 \le c_k < M$ for all k. Let $c_0 = \sup_{k=1,2,\ldots}\{c_k\}$. c_0 exists and $0 \le c_0 < M$. Hence, we can find the constant $\lambda \overset{\text{def}}{=} \mathcal{H}(c_0)$. So we have

$$\begin{aligned}
&\left|f\left(x_i, y_j, u_{ij}\right) - f\left(x_i, y_j, v_{ij}\right)\right| \le \lambda \left|u_{ij} - v_{ij}\right|, \\
&\left|hkf\left(x_i, y_j, u_{ij}\right) - hkf\left(x_i, y_j, v_{ij}\right)\right| \le \lambda hk \left|u_{ij} - v_{ij}\right|, \\
&|F(U) - F(V)| \le \lambda hk |U - V|.
\end{aligned} \tag{40}$$

Then we obtain the inequality:

$$\begin{aligned}
\|F(U) - F(V)\| &= \||F(U) - F(V)|\| \\
&\le \lambda hk \||U - V|\| = \lambda hk \|U - V\|.
\end{aligned} \tag{41}$$

Theorem 14. *Let $\widetilde{U}$, $\widehat{U}$ be a pair of ordered upper and lower solutions of* (10), *respectively, and let $\{U_s^{(m)}\}_{m=0}^{\infty}$ be a sequence generated by the SORMI method with initial vector $U_s^{(0)} \in \langle \widehat{U}, \widetilde{U} \rangle$. Suppose that*

$$\eta \overset{\text{def}}{=} \|B\| + \omega\lambda hk \|E\| < 1, \quad 0 < \omega < 2, \tag{H4}$$

where B and E are defined by (26). *Then the sequence $\{U_s^{(m)}\}_{m=0}^{\infty}$ is convergent for $0 < \omega < 2$.*

Proof. By Theorem 13, $\{U_s^{(m)}\}_{m=0}^{\infty}$ is bounded for $0 < \omega < 2$. Hence, $\exists M > 0$, such that $\|U_s^{(m)}\| \le M$ for all $m \in \{0\} \cup \mathcal{N}$. Since $\eta = \|B\| + \omega\lambda hk\|E\| < 1$, and $0 < \omega < 2$, then, for every $\varepsilon > 0$, $\exists N \in \mathcal{N}$, such that $\eta^n < \varepsilon/2M$ for all $n \ge N$. Let $k \ge m \ge N$. By (37), we have

$$\begin{aligned}
U_s^{(k)} - U_s^{(m)} &= \left\{BU_s^{(k-1)} + \omega E\left[F\left(U_s^{(k-1)}\right) + G^*\right]\right\} \\
&\quad - \left\{BU_s^{(m-1)} + \omega E\left[F\left(U_s^{(m-1)}\right) + G^*\right]\right\} \\
&= B\left(U_s^{(k-1)} - U_s^{(m-1)}\right) \\
&\quad + \omega E\left[F\left(U_s^{(k-1)}\right) - F\left(U_s^{(m-1)}\right)\right].
\end{aligned} \tag{42}$$

Hence, from $0 < \omega < 2$ and (41), we obtain

$$\begin{aligned}
\left\|U_s^{(k)} - U_s^{(m)}\right\| &= \left\|B\left(U_s^{(k-1)} - U_s^{(m-1)}\right)\right. \\
&\quad \left. + \omega E\left[F\left(U_s^{(k-1)}\right) - F\left(U_s^{(m-1)}\right)\right]\right\| \\
&\le \|B\| \cdot \left\|U_s^{(k-1)} - U_s^{(m-1)}\right\| \\
&\quad + \omega \|E\| \cdot \left\|F\left(U_s^{(k-1)}\right) - F\left(U_s^{(m-1)}\right)\right\| \\
&\le \|B\| \cdot \left\|U_s^{(k-1)} - U_s^{(m-1)}\right\| \\
&\quad + \omega\lambda hk \|E\| \cdot \left\|U_s^{(k-1)} - U_s^{(m-1)}\right\| \\
&= (\|B\| + \omega\lambda hk \|E\|) \cdot \left\|U_s^{(k-1)} - U_s^{(m-1)}\right\| \\
&= \eta \left\|U_s^{(k-1)} - U_s^{(m-1)}\right\|.
\end{aligned} \tag{43}$$

Inductively we have

$$\left\|U_s^{(k)} - U_s^{(m)}\right\| \le \eta^m \left\|U_s^{(k-m)} - U_s^{(0)}\right\|. \tag{44}$$

Hence,

$$\begin{aligned}
\left\|U_s^{(k)} - U_s^{(m)}\right\| &\le \eta^m \left\|U_s^{(k-m)} - U_s^{(0)}\right\| \\
&\le \eta^m \left(\left\|U_s^{(k-m)}\right\| + \left\|U_s^{(0)}\right\|\right) \\
&\le \eta^m \times 2M < \frac{\varepsilon}{2M} \times 2M = \varepsilon.
\end{aligned} \tag{45}$$

We have proved that $\{U_s^{(m)}\}_{m=0}^{\infty}$ is a Cauchy sequence for $0 < \omega < 2$. It implies that $\{U_s^{(m)}\}_{m=0}^{\infty}$ is convergent for $0 < \omega < 2$. □

Furthermore, we provide another proof about the convergence of the SORMI sequences for $0 < \omega < 1$ without the assumptions (H3) and (H4). Denote the sequence by $\{\overline{U}_s^{(m)}\}_{m=0}^{\infty}$ when $U^{(0)} = \widetilde{U}$ and by $\{\underline{U}_s^{(m)}\}_{m=0}^{\infty}$ when $U^{(0)} = \widehat{U}$, and refer to them as the maximal and minimal sequences, respectively. The following theorem gives some monotone property of these sequences.

Theorem 15. *The maximal and minimal sequences $\{\overline{U}_s^{(m)}\}_{m=0}^{\infty}$ and $\{\underline{U}_s^{(m)}\}_{m=0}^{\infty}$ given by (18) with $\widetilde{U} = \overline{U}_s^{(0)}$ and $\widehat{U} = \underline{U}_s^{(0)}$ possess the monotone property*

$$\widehat{U} \le \underline{U}_s^{(m)} \le \underline{U}_s^{(m+1)} \le \overline{U}_s^{(m+1)} \le \overline{U}_s^{(m)} \le \widetilde{U}, \quad m = 1, 2, \ldots. \tag{46}$$

Moreover for each m, $\overline{U}_s^{(m)}$ and $\underline{U}_s^{(m)}$ are ordered upper and lower solutions.

Proof. We will use induction to complete the proof of monotone property. First, let $W^{(0)} = \overline{U}_s^{(0)} - \overline{U}_s^{(1)} = \widetilde{U} - \overline{U}_s^{(1)}$. From (11), (18), and $0 < \omega < 1$,

$$\begin{aligned}
(\mathscr{D} + \Gamma - \omega\mathscr{L}) W^{(0)} &= (\mathscr{D} + \Gamma - \omega\mathscr{L}) \widetilde{U} - (\mathscr{D} + \Gamma - \omega\mathscr{L}) \overline{U}_s^{(1)} \\
&= (\mathscr{D} + \Gamma - \omega\mathscr{L}) \widetilde{U} \\
&\quad - \Big\{ [(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}] \overline{U}_s^{(0)} \\
&\qquad + \omega \left[\Gamma \overline{U}_s^{(0)} + F\left(\overline{U}_s^{(0)}\right) + G^* \right] \Big\} \\
&= \omega (\mathscr{D} - \mathscr{L} - \mathscr{U}) \widetilde{U} - \omega \left[F\left(\widetilde{U}\right) + G^* \right] \\
&= \omega \left\{ A\widetilde{U} - \left[F\left(\widetilde{U}\right) + G^* \right] \right\} \ge 0.
\end{aligned} \tag{47}$$

By Lemma 8, we obtain $W^{(0)} \ge 0$. This leads to $\overline{U}_s^{(1)} \le \overline{U}_s^{(0)}$. Similarly let $V^{(0)} = \underline{U}_s^{(0)} - \underline{U}_s^{(1)} = \widehat{U} - \underline{U}_s^{(1)}$, and use (12) to obtain

$$(\mathscr{D} + \Gamma - \omega\mathscr{L}) V^{(0)} = \omega \left\{ A\widehat{U} - \left[F\left(\widehat{U}\right) + G^* \right] \right\} \le 0. \tag{48}$$

Since $(\mathscr{D}+\Gamma-\omega\mathscr{L})^{-1} \ge 0$, it implies that $\underline{U}_s^{(1)} \ge \underline{U}_s^{(0)}$. Secondly, let $W^{(1)} = \overline{U}_s^{(1)} - \underline{U}_s^{(1)}$. By (18),

$$\begin{aligned}
(\mathscr{D} + \Gamma - \omega\mathscr{L}) W^{(1)} &= (\mathscr{D} + \Gamma - \omega\mathscr{L}) \overline{U}_s^{(1)} \\
&\quad - (\mathscr{D} + \Gamma - \omega\mathscr{L}) \underline{U}_s^{(1)} \\
&= \Big\{ [(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}] \overline{U}_s^{(0)} \\
&\qquad + \omega \left[\Gamma \overline{U}_s^{(0)} + F\left(\overline{U}_s^{(0)}\right) + G^* \right] \Big\} \\
&\quad - \Big\{ [(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}] \underline{U}_s^{(0)} \\
&\qquad + \omega \left[\Gamma \underline{U}_s^{(0)} + F\left(\underline{U}_s^{(0)}\right) + G^* \right] \Big\} \\
&= \{(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}\} \\
&\quad \times \left(\overline{U}_s^{(0)} - \underline{U}_s^{(0)} \right) \\
&\quad + \omega \Big[\Gamma \left(\overline{U}_s^{(0)} - \underline{U}_s^{(0)} \right) + F\left(\overline{U}_s^{(0)}\right) \\
&\qquad - F\left(\underline{U}_s^{(0)}\right) \Big].
\end{aligned} \tag{49}$$

We have from Lemma 6, $\overline{U}_s^{(0)} \ge \underline{U}_s^{(0)}$, and the nonnegative property of $[(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}]$ that $(\mathscr{D} + \Gamma - \omega\mathscr{L})W^{(1)} \ge 0$. It follows from Lemma 8 again that $W^{(1)} \ge 0$. The above conclusions imply that

$$\underline{U}_s^{(0)} \le \underline{U}_s^{(1)} \le \overline{U}_s^{(1)} \le \overline{U}_s^{(0)}. \tag{50}$$

We finally assume that $\underline{U}_s^{(m-1)} \le \underline{U}_s^{(m)} \le \overline{U}_s^{(m)} \le \overline{U}_s^{(m-1)}$ for some $m > 1$. Let $W^{(m)} = \overline{U}_s^{(m)} - \overline{U}_s^{(m+1)}$, and by (18) we have

$$\begin{aligned}
(\mathscr{D} + \Gamma - \omega\mathscr{L}) W^{(m)} &= \{(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}\} \\
&\quad \times \left(\overline{U}_s^{(m-1)} - \overline{U}_s^{(m)} \right) \\
&\quad + \omega \Big[\Gamma \left(\overline{U}_s^{(m-1)} - \overline{U}_s^{(m)} \right) \\
&\qquad + F\left(\overline{U}_s^{(m-1)}\right) - F\left(\overline{U}_s^{(m)}\right) \Big].
\end{aligned} \tag{51}$$

Since $\overline{U}_s^{(m)} \le \overline{U}_s^{(m-1)}$, we have $(\mathscr{D} + \Gamma - \omega\mathscr{L})W^{(m)} \ge 0$. So $W^{(m)} \ge 0$ which shows that $\overline{U}_s^{(m+1)} \le \overline{U}_s^{(m)}$. Similarly, let $V^{(m)} = \underline{U}_s^{(m)} - \underline{U}_s^{(m+1)}$ and $W^{(m+1)} = \overline{U}_s^{(m+1)} - \underline{U}_s^{(m+1)}$ simultaneously. We see that

$$\begin{aligned}
(\mathscr{D} + \Gamma - \omega\mathscr{L}) V^{(m)} &= \{(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}\} \\
&\quad \times \left(\underline{U}_s^{(m-1)} - \underline{U}_s^{(m)} \right) \\
&\quad + \omega \Big[\Gamma \left(\underline{U}_s^{(m-1)} - \underline{U}_s^{(m)} \right) \\
&\qquad + F\left(\underline{U}_s^{(m-1)}\right) - F\left(\underline{U}_s^{(m)}\right) \Big] \\
&\le 0, \\
(\mathscr{D} + \Gamma - \omega\mathscr{L}) W^{(m+1)} &= \{(1-\omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}\} \\
&\quad \times \left(\overline{U}_s^{(m)} - \underline{U}_s^{(m)} \right) \\
&\quad + \omega \Big[\Gamma \left(\overline{U}_s^{(m)} - \underline{U}_s^{(m)} \right) \\
&\qquad + F\left(\overline{U}_s^{(m)}\right) - F\left(\underline{U}_s^{(m)}\right) \Big] \\
&\ge 0.
\end{aligned} \tag{52}$$

By $(\mathscr{D}+\Gamma-\omega\mathscr{L})^{-1}\geq 0$, we have $V^{(m)}\leq 0$, and $W^{(m+1)}\geq 0$. Hence, $\underline{U}_s^{(m)}\leq \underline{U}_s^{(m+1)}$, and $\overline{U}_s^{(m+1)}\geq \underline{U}_s^{(m+1)}$. The proof of monotone property (46) is completed.

To show that $\overline{U}_s^{(m)}$ is an upper solution for each m, we observe from (18) that

$$\begin{aligned}
(\mathscr{D}+\Gamma-\omega\mathscr{L})\,\overline{U}_s^{(m)} &= \left\{[(1-\omega)(\mathscr{D}+\Gamma)+\omega\mathscr{U}]\,\overline{U}_s^{(m-1)}\right.\\
&\quad\left.+\omega\left[\Gamma\overline{U}_s^{(m-1)}+F\left(\overline{U}_s^{(m-1)}\right)+G^*\right]\right\},\\
\omega\mathscr{D}\overline{U}_s^{(m)}-\omega\mathscr{L}\overline{U}_s^{(m)} &= \omega\mathscr{D}\overline{U}_s^{(m)}-\mathscr{D}\overline{U}_s^{(m)}\\
&\quad+\mathscr{D}\overline{U}_s^{(m-1)}-\omega\mathscr{D}\overline{U}_s^{(m-1)}\\
&\quad+\Gamma\overline{U}_s^{(m-1)}-\Gamma\overline{U}_s^{(m)}+\omega\mathscr{U}\overline{U}_s^{(m-1)}\\
&\quad+\omega F\left(\overline{U}_s^{(m-1)}\right)+\omega G^*,\\
\omega\mathscr{D}\overline{U}_s^{(m)}-\omega\mathscr{L}\overline{U}_s^{(m)} &= (1-\omega)\,\mathscr{D}\left(\overline{U}_s^{(m-1)}-\overline{U}_s^{(m)}\right)\\
&\quad+\omega\mathscr{U}\overline{U}_s^{(m-1)}+\Gamma\left(\overline{U}_s^{(m-1)}-\overline{U}_s^{(m)}\right)\\
&\quad+\omega F\left(\overline{U}_s^{(m-1)}\right)+\omega G^*.
\end{aligned}\tag{53}$$

By $0<\omega<1$ and (46), we obtain

$$\begin{aligned}
\omega\mathscr{D}\overline{U}_s^{(m)}-\omega\mathscr{L}\overline{U}_s^{(m)} &\geq \omega\mathscr{U}\overline{U}_s^{(m)}+\omega\Gamma\left(\overline{U}_s^{(m-1)}-\overline{U}_s^{(m)}\right)\\
&\quad+\omega F\left(\overline{U}_s^{(m-1)}\right)+\omega G^*,\\
\omega(\mathscr{D}-\mathscr{L}-\mathscr{U})\,\overline{U}_s^{(m)} &\geq \omega\left[\Gamma\left(\overline{U}_s^{(m-1)}-\overline{U}_s^{(m)}\right)\right.\\
&\quad\left.+F\left(\overline{U}_s^{(m-1)}\right)+G^*\right],\\
A\overline{U}_s^{(m)} &\geq \Gamma\left(\overline{U}_s^{(m-1)}-\overline{U}_s^{(m)}\right)+F\left(\overline{U}_s^{(m-1)}\right)+G^*.
\end{aligned}\tag{54}$$

By Lemma 6, we have

$$A\overline{U}_s^{(m)}\geq F\left(\overline{U}_s^{(m)}\right)+G^*.\tag{55}$$

This shows that $\overline{U}_s^{(m)}$ is an upper solution for each m. Similarly, we have

$$A\underline{U}_s^{(m)}\leq F\left(\underline{U}_s^{(m)}\right)+G^*;\tag{56}$$

that is, for each m, $\underline{U}_s^{(m)}$ is a lower solution, and thus the proof is completed. □

Theorem 16. *Let $\widetilde{U}$, $\widehat{U}$ be a pair of ordered upper and lower solutions of* (10). *Then the sequences $\{\overline{U}_s^{(m)}\}_{m=0}^{\infty}$, $\{\underline{U}_s^{(m)}\}_{m=0}^{\infty}$ given by* (18) *with $\overline{U}_s^{(0)}=\widetilde{U}$ and $\underline{U}_s^{(0)}=\widehat{U}$ converge monotonically to solutions $\overline{U}_s$ and $\underline{U}_s$ of* (10), *respectively, where*

$$\begin{aligned}
\overline{U}_s &\stackrel{\text{def}}{=} \left(\overline{u}_{00},\overline{u}_{10},\ldots,\overline{u}_{n0},\ldots,\overline{u}_{0m},\overline{u}_{1m},\ldots,\overline{u}_{nm}\right)^T,\\
\underline{U}_s &\stackrel{\text{def}}{=} \left(\underline{u}_{00},\underline{u}_{10},\ldots,\underline{u}_{n0},\ldots,\underline{u}_{0m},\underline{u}_{1m},\ldots,\underline{u}_{nm}\right)^T.
\end{aligned}\tag{57}$$

Moreover,

$$\begin{aligned}
\widehat{U}\leq \underline{U}_s^{(m)}\leq \underline{U}_s^{(m+1)}\leq \underline{U}_s\leq \overline{U}_s\leq \overline{U}_s^{(m+1)}\leq \overline{U}_s^{(m)}\leq \widetilde{U},\\
m=1,2,\ldots,
\end{aligned}\tag{58}$$

and if U^* is any solution in $\langle\widehat{U},\widetilde{U}\rangle$, then $\underline{U}_s\leq U^*\leq \overline{U}_s$.

Proof. By Theorem 15, the limits $\lim\overline{U}^{(m)}=\overline{U}_s$ and $\lim\underline{U}^{(m)}=\underline{U}_s$ as $m\to\infty$ exist, and the relation (58) also holds. Letting $m\to\infty$ in (18) shows that $\overline{U}_s$ and $\underline{U}_s$ are solutions of (15). The equivalence between (10) and (15) ensures that $\overline{U}_s$ and $\underline{U}_s$ are solutions of (10). Now if U^* is a solution in $\langle\widehat{U},\widetilde{U}\rangle$, then $\widetilde{U}$ and U^* are ordered upper and lower solutions. Using $\overline{U}_s^{(0)}=\widetilde{U}$ and $\underline{U}_s^{(0)}=U^*$, Theorem 15 implies that $\overline{U}_s^{(m)}\geq U^*$ for every m. Letting $m\to\infty$ gives $\overline{U}_s\geq U^*$. A similar argument using U^* and $\widehat{U}$ as ordered upper and lower solutions yields $U^*\geq\underline{U}_s$. This proves the theorem. □

In Theorem 16, $\overline{U}_s$ and $\underline{U}_s$ are often called maximal and minimal solutions in $\langle\widehat{U},\widetilde{U}\rangle$, respectively. In general, these two solutions are not necessarily the same. Let A be symmetric. Then A has real and positive eigenvalues [2]. However, if $\sigma<\underline{\mu}$, where $\underline{\mu}$ is the smallest positive eigenvalue of A and

$$\sigma\equiv\max\left\{\frac{\partial f_{ij}}{\partial u}\left(u_{ij}\right):\widehat{u}_{ij}\leq u_{ij}\leq\widetilde{u}_{ij},\ (i,j)\in\overline{\Upsilon}\right\},\tag{59}$$

then the following theorem holds.

Theorem 17. *Let the conditions in Theorem 16 hold. If either $\sigma\leq 0$ or $\sigma<\underline{\mu}$ and A is symmetric, then $\overline{U}_s=\underline{U}_s$ and is the unique solution of* (10).

Proof. Let $W=\overline{U}_s-\underline{U}_s$ and $\overline{u}_{ij}$ and $\underline{u}_{ij}$ be the components of $\overline{U}_s$ and $\underline{U}_s$, respectively. By (58), we have $W\geq 0$. On the other hand, by the mean value theorem, we have

$$\frac{f\left(x_i,y_j,\overline{u}_{ij}\right)-f\left(x_i,y_j,\underline{u}_{ij}\right)}{\overline{u}_{ij}-\underline{u}_{ij}}=\frac{\partial f_{ij}}{\partial u}\left(u_{ij}\right)\leq\sigma,\tag{60}$$

where u_{ij} lies between $\underline{u}_{ij}$ and $\overline{u}_{ij}$. Hence,

$$\begin{aligned} AW &= A\left(\overline{U}_s - \underline{U}_s\right) = A\overline{U}_s - A\underline{U}_s \\ &= F\left(\overline{U}_s\right) + G^* - F\left(\underline{U}_s\right) - G^* \\ &= F\left(\overline{U}_s\right) - F\left(\underline{U}_s\right) \\ &\le \sigma\left(\overline{U}_s - \underline{U}_s\right) \\ &= \sigma W, \end{aligned} \tag{61}$$

and thus

$$(A - \sigma I)\, W \le 0. \tag{62}$$

Case 1. If $\sigma \le 0$. From the form of A, $(A - \sigma I)$ is strictly diagonally dominant. It follows that $(A - \sigma I)^{-1} \ge 0$, and thus $W \le 0$.

Case 2. If $\sigma < \underline{\mu}$. Since $\underline{\mu}$ is the smallest positive eigenvalue of A, then $\underline{\mu}^{-1}$ is the biggest positive eigenvalue of A^{-1}. From Theorem 7 and $A^{-1} \ge 0$, we have

$$\begin{gathered} \sigma^{-1} > \underline{\mu}^{-1} \ge \rho\left(A^{-1}\right), \\ \left(\sigma^{-1} I - A^{-1}\right)^{-1} \ge 0, \\ \sigma^{-1} A^{-1}\left(\sigma^{-1} I - A^{-1}\right)^{-1} \ge 0, \\ (A - \sigma I)^{-1} \ge 0. \end{gathered} \tag{63}$$

Hence, $W \le 0$. So we know that $W = 0$. This proves $\overline{U}_s = \underline{U}_s$. The uniqueness follows from the relation $\overline{U}_s \ge U^* \ge \underline{U}_s$ for any solution $U^* \in \langle \widehat{U}, \widetilde{U} \rangle$. □

Remark 18. For system (1), the well-known method of upper and lower solutions with SORMI is applied for the case $0 < \omega < 1$ (see Theorems 15, 16, and 17). However, the nonnegative property of $(1 - \omega)(\mathscr{D} + \Gamma) + \omega\mathscr{U}$ is not available when $\omega > 1$. A new approach for solving this problem by the boundedness of the SORMI sequences and Cauchy sequence property is proposed. To make sure of the convergence of the SORMI sequences, the assumptions (H1), (H2), and (H3) are necessary. But it should be pointed out that these constraints are not easy to be verified. It is important to weaken these constraints when the SORMI method is applied to realistic problems. Fixed point theory is a powerful tool to overcome this problem for further study.

6. Numerical Results

Assume that the matrix A of (10) is an $n \times n$ matrix. The componentwise SORMI algorithm is given as follows:

$$\begin{aligned} u_i^{(m+1)} = \omega \Bigg[\Bigg(& \sum_{j=1}^{i-1} a_{ij} u_j^{(m+1)} + \sum_{j=i+1}^{n} a_{ij} u_j^{(m)} \\ & + \overline{\gamma}_i u_i^{(m)} + hkf_i\left(u_i^{(m)}\right) + g_i^* \Bigg) \left(a_{ii} + \overline{\gamma}_i\right)^{-1} \Bigg] \\ & + (1 - \omega)\, u_i^{(m)}, \quad i = 1, \ldots, n. \end{aligned} \tag{64}$$

Another equivalent form is

$$\begin{aligned} u_i^{(m+1)} = \omega \Bigg[\Bigg(& \sum_{j=1}^{i-1} a_{ij} u_j^{(m+1)} + \sum_{j=i+1}^{n} a_{ij} u_j^{(m)} \\ & + hkf_i\left(u_i^{(m)}\right) + g_i^* \Bigg) \left(a_{ii} + \overline{\gamma}_i\right)^{-1} \Bigg] \\ & + \left(1 - \frac{\omega a_{ii}}{a_{ii} + \overline{\gamma}_i}\right) u_i^{(m)}, \quad i = 1, \ldots, n. \end{aligned} \tag{65}$$

The main requirement for the application of the various MI schemes is the existence of a pair of ordered upper and lower solutions. To ensure the existence, the nonlinear function f must have some necessary conditions. Hence, in Section 1, we require that $f_{ij}(u_{ij})$ is uniformly bounded in $\mathfrak{R}$. Now we present some numerical results with two test problems.

Example 19. Consider the one-dimensional boundary value problem:

$$\begin{gathered} -u'' = -\frac{1}{2}u + \frac{5}{2}\pi \sin\left(\sqrt{2}x\right) - \frac{1}{2}x \sin\left(\sqrt{2}\pi\right), \\ 0 < x < \pi, \quad u(0) = u(\pi) = 0. \end{gathered} \tag{66}$$

The exact solution is

$$u(x) = \pi \sin\left(\sqrt{2}x\right) - x \sin\left(\sqrt{2}\pi\right). \tag{67}$$

Let $f(x, u) = (-1/2)u + (5/2)\pi \sin(\sqrt{2}x) - (1/2)x \sin(\sqrt{2}\pi)$, and choose $\Gamma = \operatorname{diag}(2, 2, \ldots, 2)$, $U_s^{(0)} = (0, 0, \ldots, 0)^T$. Then,

$$\begin{aligned} \left|f(x, u_1) - f(x, u_2)\right| &= \frac{1}{2}\left|u_1 - u_2\right| \\ &\stackrel{\text{def}}{=} \mathscr{H}(c)\left|u_1 - u_2\right|. \end{aligned} \tag{68}$$

Hence we choose $\lambda = 1/2$. We examine the assumptions (H3) and (H4) and the convergence of the SORMI sequences. The numerical results are given in Table 1, and the exact and approximate solutions are shown in Figure 1. Moreover, we sketch the relation between the numbers of iterations and relaxation factors ω by Figure 2.

Table 1: Numbers of iterations and η values for Example 19.

ω	0.2	0.25	0.3	0.4	0.5	0.6
Number of iterations	18330	14745	12268	9046	7024	5626
$\|B\|$	0.9999	0.9998	0.9998	0.9997	0.9997	0.9996
η	0.9999	0.9999	0.9999	0.9999	0.9998	0.9998
ω	0.8	1.0	1.2	1.6	1.8	1.9
Number of iterations	3800	2645	1837	751	342	127
$\|B\|$	0.9994	0.9992	0.9990	0.9985	0.9982	0.9980
η	0.9997	0.9996	0.9995	0.9993	0.9992	0.9991
ω	1.92	1.95	1.96	1.97	1.98	1.99
Number of iterations	151	236	295	381	535	940
$\|B\|$	0.9979	0.9979	0.9979	0.9978	0.9978	0.9978
η	0.9991	0.9990	0.9990	0.9990	0.9990	0.9990

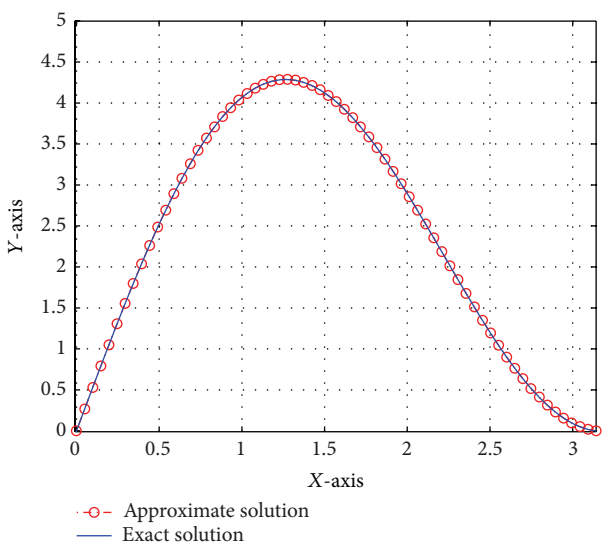

Figure 1: The exact and approximate solutions for Example 19.

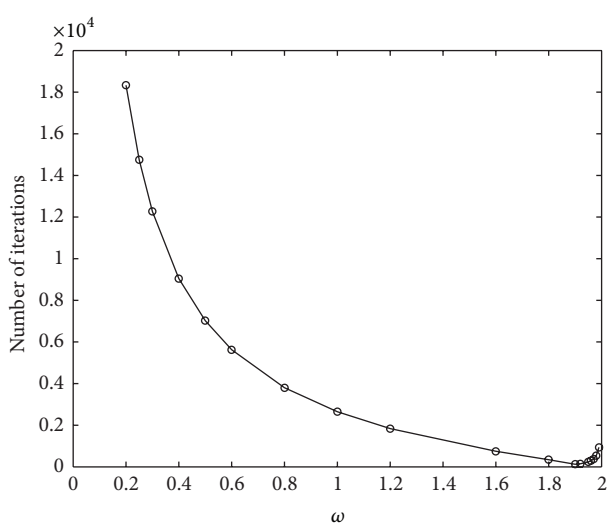

Figure 2: Numbers of iterations and ω values for Example 19.

The numbers of iterations for the SORMI method are listed in Table 1. We focus on the SORMI method with $\omega = 1.9$. The number of iterations is 127. Compared with it, the Jacobi and Gauss-Seidel MI methods require 4905 and 2645 iterations, respectively. The resulting increase in the speed of convergence is very dramatic. Moreover, the values of $\|B\|$ and η in Table 1 are smaller than 1 which verify our theory of the boundedness and convergence for the SORMI method.

Example 20. As the second test problem, consider the logistic model in population growth problem with nonlinear function

$$f(x, y, u) = \sigma u(1-u) + q(x, y), \tag{69}$$

where σ is a positive constant, and $q(x, y)$ is a possible internal source [6]. The discretized function is given by

$$f_{ij}(u_{ij}) = \sigma u_{ij}(1-u_{ij}) + q_{ij}. \tag{70}$$

For physical reasons, we suppose that $f(x, y, u) \geq 0, q(x, y) \geq 0$, and $g(x, y) \geq 0$, such that $\widehat{U} = \mathbf{0}$ is a lower solution of (10). For upper solution, consider the following.

Case 1. $q(x, y) > 0$ and $g(x, y) > 0$. (i) If the upper solution is dependent on σ, define $K^{(1)} = \sigma/4 + \overline{q} + \overline{g}^*, V = (K^{(1)}, K^{(1)}, \ldots, K^{(1)})$, and $A\widetilde{U} = V$. Then $\widetilde{U}$ is an upper solution, where $\overline{q}$ and $\overline{g}^*$ are any upper bounds of $q_{i,j}$ and $g^*_{i,j}$, respectively. (ii) If upper solution is independent of σ, then $\widetilde{U} = P + V^*$ is an upper solution, where $P \stackrel{\text{def}}{=} (1, 1, \ldots, 1)^T$, and Q is the vector with components $q_{i,j}$ which satisfies $Q + G^* = AV^*$.

Case 2. (i) If $g = 0$, then $\widetilde{U} = (K, K, \ldots, K)^T$ is an upper solution, where K is a constant which satisfies $\sigma K(K-1) \geq \overline{q}$. (ii) If $q = g = 0$, then $\widetilde{U} = P$ is an upper solution.

We now give a model problem where the exact solution is known explicitly [6]. This problem is given by

$$-\left(u_{xx}+u_{yy}\right)=\sigma u\left(1-u\right)+q\left(x,y\right),\quad 0<x<1,\ 0<y<2,\qquad (71)$$
$$u_x\left(0,y\right)=u\left(1,y\right)=u\left(x,0\right)=u\left(x,2\right)=0.$$

It is easy to verify that when $\sigma=\pi^2/4$ and

$$q\left(x,y\right)=2\sin\left(\frac{\pi y}{2}\right)+\frac{\pi^2}{4}\left(1-x^2\right)^2\sin^2\left(\frac{\pi y}{2}\right),\qquad (72)$$

the exact solution of (71) is given by

$$u\left(x,y\right)=\left(1-x^2\right)\sin\left(\frac{\pi y}{2}\right).\qquad (73)$$

Since $0\le q(x,y)\le 2+\pi^2/4$ for $(x,y)\in[0,1]\times[0,2]$, the constant pair $\widetilde{U}=(K,K,\ldots,K)^T$ and $\widehat{U}=\mathbf{0}$ are ordered upper and lower solutions whenever $(\pi^2/4)K(K-1)\ge 2+\pi^2/4$ with $K=2$. Hence, (71) is case 2 whenever $g=0$, and we have the upper and lower solutions $\widehat{U}=(0,0,\ldots,0)^T$ and $\widetilde{U}=(2,2,\ldots,2)^T$. Let $f(x,y,u)=(\pi^2/4)u(1-u)+q(x,y)$. We define

$$\gamma=\max_{0\le u\le 2}\left|\frac{\partial f\left(x,y,u\right)}{\partial u}\right|=\frac{3\pi^2}{4},\quad \Gamma=\operatorname{diag}\left(\gamma,\gamma,\ldots,\gamma\right)^T.\qquad (74)$$

If $0\le u_1,u_2\le c$, then

$$\begin{aligned}\left|f\left(x,y,u_1\right)-f\left(x,y,u_2\right)\right|&=\left|\sigma u_1\left(1-u_1\right)+q\left(x,y\right)\right.\\&\quad\left.-\sigma u_2\left(1-u_2\right)-q\left(x,y\right)\right|\\&=\left|\sigma u_1-\sigma u_1^2-\sigma u_2+\sigma u_2^2\right|\\&=\left|\sigma\left(u_2^2-u_1^2\right)-\sigma\left(u_2-u_1\right)\right|\\&=\left|\sigma\left(u_2+u_1\right)\left(u_2-u_1\right)\right.\\&\quad\left.-\sigma\left(u_2-u_1\right)\right|\\&=\left|\sigma\left(u_2+u_1-1\right)\left(u_2-u_1\right)\right|\\&=\sigma\left|u_2+u_1-1\right|\left|u_1-u_2\right|\\&\le\sigma\left|2c-1\right|\left|u_1-u_2\right|\\&\overset{\text{def}}{=}\mathscr{H}\left(c\right)\left|u_1-u_2\right|.\end{aligned}\qquad (75)$$

The SORMI method takes 94 iterations with $\omega=1.92$ when $U^{(0)}=0$ is applied. The approximate solution is similar to the exact solution as shown in Figure 3. Since they are similar, we omit to sketch the exact solution figure. In comparison with the SORMI method, the Jacobi and Gauss-Seidel MI methods require 2243 and 1249 iterations, respectively. The reduction of iterations is significant. The numerical results are listed in Table 2. The relation between the numbers of iterations and relaxation factors ω is shown in Figure 4. They verify our theory of the boundedness and convergence for the SORMI method again.

Table 2: Numbers of iterations and η values for Example 20.

ω	0.05	0.1	0.2	0.3
Number of iterations	28503	15706	8344	5612
$\lVert B\rVert$	0.9999	0.9997	0.9995	0.9992
η	0.9999	0.9998	0.9996	0.9994
ω	0.4	0.5	0.8	1.0
Number of iterations	4155	3239	1775	1249
$\lVert B\rVert$	0.9989	0.9986	0.9976	0.9969
η	0.9992	0.9989	0.9982	0.9976
ω	1.2	1.6	1.8	1.9
Number of iterations	881	386	200	104
$\lVert B\rVert$	0.9961	0.9943	0.9933	0.9928
η	0.9970	0.9955	0.9947	0.9943
ω	1.92	1.95	1.97	1.99
Number of iterations	94	129	185	307
$\lVert B\rVert$	0.9927	0.9925	0.9924	0.9923
η	0.9942	0.9942	0.9942	0.9944

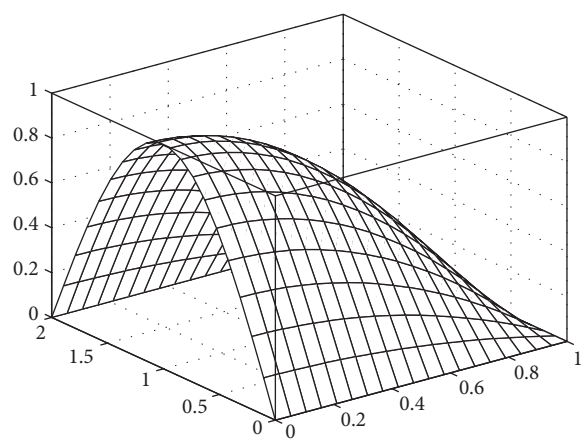

Figure 3: The approximate solution for Example 20.

7. Conclusions

A new iterative method for solving nonlinear equations is developed in this paper. It combines SOR method with MI scheme and therefore gives rise to the terminology "SORMI." The boundedness and convergence of the SORMI sequence are proven under some suitable assumptions. Some numerical examples are given to verify our theory of the boundedness and convergence for the SORMI method. Moreover, the reduction of iterations is quite significant in comparison with the Jacobi and Gauss-Seidel MI methods.

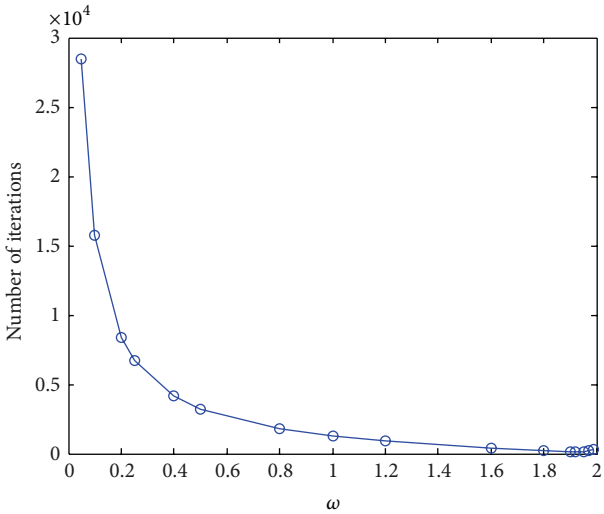

Figure 4: Numbers of iterations and ω values for Example 20.

Acknowledgments

This work was supported by the National Science Council of Taiwan under the Grants 100-2115-M-017-004-MY2. The authors would like to thank the referees for their valuable comments and suggestions which improved the presentation of the paper.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 959464, 8 pages
http://dx.doi.org/10.1155/2013/959464

Research Article

New Quasi-Coincidence Point Polynomial Problems

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Received 30 April 2013; Accepted 10 July 2013

Academic Editor: Tamaki Tanaka

Let $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a real-valued polynomial function of the form $F(x, y) = a_s(x)y^s + a_{s-1}(x)y^{s-1} + \cdots + a_0(x)$, where the degree s of y in $F(x, y)$ is greater than or equal to 1. For arbitrary polynomial function $f(x) \in \mathbb{R}[x]$, $x \in \mathbb{R}$, we will find a polynomial solution $y(x) \in \mathbb{R}[x]$ to satisfy the following equation: $(*)$: $F(x, y(x)) = af(x)$, where $a \in \mathbb{R}$ is a constant depending on the solution $y(x)$, namely, a quasi-coincidence (point) solution of $(*)$, and a is called a quasi-coincidence value. In this paper, we prove that (i) the leading coefficient $a_s(x)$ must be a factor of $f(x)$, and (ii) each solution of $(*)$ is of the form $y(x) = -a_{s-1}(x)/sa_s(x) + \lambda p(x)$, where λ is arbitrary and $p(x) = c(f(x)/a_s(x))^{1/s}$ is also a factor of $f(x)$, for some constant $c \in \mathbb{R}$, provided the equation $(*)$ has infinitely many quasi-coincidence (point) solutions.

1. Introduction and Preliminaries

Let $F : \mathbb{Q}(\alpha) \times \mathbb{Q}(\alpha) \to \mathbb{Q}(\alpha)$ (where α is an algebraic number) be a polynomial function. Lenstra [1] investigated that $F(x, y(x)) = 0$. He solved a polynomial function $y = y(x) \in \mathbb{Q}(\alpha)[x]$ and derived to find a polynomial $y = y(x)$ satisfying an $x \in \mathbb{Q}(\alpha)[x]$ as a fixed point of the polynomial equation. That is,

$$F(x, y(x)) = x \tag{1}$$

has a polynomial solution $y(x) \in \mathbb{Q}(\alpha)[x]$.

Further, Tung [2, 3] extended (1) to solve $y(x) \in \mathbb{K}[x]$ ($\mathbb{K}$ is a field) for the following equation:

$$F(x, y(x)) = ax^m, \tag{2}$$

where $a \in \mathbb{K}$ is a constant depending on the polynomial solution $y(x)$ and $m \in \mathbb{N}$ a given positive integer.

Recently, Lai and Chen [4, 5] extended (2) to solve $y(x) \in \mathbb{R}[x]$ to satisfy the polynomial equation as the form:

$$F(x, y(x)) = ap^m(x), \quad x \in \mathbb{R}, \tag{3}$$

where $a \in \mathbb{R}$, $p(\cdot)$ is an irreducible polynomial in $x \in \mathbb{R}$ and the polynomial function $F(x, y) : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is written by

$$F(x, y) = \sum_{i=0}^{s} a_i(x)y^i \quad \text{with } s \geq 1, \tag{4}$$

where $s = \deg_y F$ denotes the degree of y in $F(x, y)$.

Definition 1 (see [4]). A polynomial function $y = y(x)$ satisfying (3) is called a quasi-fixed solution corresponding to some real number a. This number a is called a quasi-fixed value corresponding to the polynomial solutions $y = y(x)$.

In mathematics, a coincidence point (or simply coincidence) of two mappings is a point in their domain having the same image point under both mappings. Coincidence theory (the study of coincidence points) is, in most settings, a generalization of fixed point theory.

In this paper, we define a more general coincidence (point) problem in which the $f(x) \in \mathbb{R}[x]$ is replaced by the irreducible polynomial power $p^m(x) \in \mathbb{R}[x]$ throughout this

paper, where $f(x)$ is an arbitrary polynomial. Then, we restate (3) as the following equation:

$$F(x, y) = af(x). \tag{5}$$

It is a new development coincidence point-like problem. We call the polynomial solution $y = y(x)$ for (5) as a quasi-coincidence (point) solution. Precisely, we give the following definition like Definition 1.

Definition 2. A polynomial function $y = y(x)$ satisfying (5) is called a quasi-coincidence (point) solution corresponding to some real number a. This number a is called a quasi-coincidence value corresponding to the polynomial solutions $y = y(x)$.

The number of all solutions in (5) may be infinitely many, or finitely many, or not solvable.

Since there may have many solutions corresponding to the number a, for convenience, we use the following notations to represent different situations:

(1) Qcs_F, the set of all quasi-coincidence solutions satisfying (5),

(2) Qcv_F, the set of all quasi-coincidence values satisfying (5),

(3) $\mathrm{Qcs}_F(a)$, the set of all quasi-coincidence solutions $y(x)$ corresponding to a fixed quasi-coincidence value a.

Evidently,

$$\mathrm{Qcs}_F = \bigcup_{a \in \mathrm{Qcv}_F} \mathrm{Qcs}_F(a), \tag{6}$$

$$\mathrm{Qcs}_F(a) \bigcap \mathrm{Qcs}_F(b) = \emptyset \tag{7}$$

for any $a \neq b$ in Qcv_F. Moreover, for each $a \in \mathbb{R}$, the cardinal number of $\mathrm{Qcs}_F(a)$, denoted by $|\mathrm{Qcs}_F(a)|$, satisfies the following condition:

$$\left|\mathrm{Qcs}_F(a)\right| \leq \deg_y F(x, y). \tag{8}$$

In Section 2, we derive some properties of quasi-coincidence solutions of $F(x, y)$. If (5) has infinitely many quasi-coincidence solutions, the concerned properties are described in Section Section 3.

Throughout the paper, we denote the polynomial function by

$$\begin{aligned} F(x, y) &= a_s(x) y^s + a_{s-1}(x) y^{s-1} \\ &\quad + \cdots + a_1(x) y + a_0(x) \\ &= \sum_{i=0}^{s} a_i(x) y^i. \end{aligned} \tag{9}$$

2. Auxiliary Lemmas

For convenience, we explain some interesting properties of quasi-coincidence point solutions as the following lemmas. Throughout this paper, we consider (5) for polynomial function (9) and arbitrary polynomial $f(x)$ in $\mathbb{R}[x]$.

Lemma 3. *Let $y_1(x) \in Qcs_F(a)$, $y_2(x) \in Qcs_F(b)$, $a \neq b$ in Qcv_F. Then,*

$$y_1(x) - y_2(x) = dp(x) \quad \text{for some } d \in \mathbb{R}, \tag{10}$$

and this $p(x)$ is divisible $f(x)$; that is, $p(x) \mid f(x)$.

Proof. Since $y_1(x) \neq y_2(x)$ in Qcs_F correspond to $a \neq b$ in Qcv_F, respectively, thus

$$\begin{aligned} F(x, y_1(x)) &= af(x), \\ F(x, y_2(x)) &= bf(x). \end{aligned} \tag{11}$$

Subtracting the above two equations and using binomial formula, it yields

$$\begin{aligned} (a-b) f(x) &= F(x, y_1(x)) - F(x, y_2(x)) \\ &= a_s(x)\left[y_1^s(x) - y_2^s(x)\right] + a_{s-1}(x) \\ &\quad \times \left[y_1^{s-1}(x) - y_2^{s-1}(x)\right] \\ &\quad + \cdots + a_1(x)\left[y_1(x) - y_2(x)\right] \\ &= \left[y_1(x) - y_2(x)\right] \\ &\quad \times \left[a_s(x) G_s(y_1(x), y_2(x)) \right. \\ &\qquad + a_{s-1}(x) G_{s-1}(y_1(x), y_2(x)) \\ &\qquad \left. + \cdots + a_1(x)\right] \\ &= \left[y_1(x) - y_2(x)\right] Q(x, y_1(x), y_2(x)), \end{aligned} \tag{12}$$

where $G_j(y_1(x), y_2(x)) = y_1^{j-1}(x) + y_1^{j-2}(x) y_2(x) + \cdots + y_2^{j-1}(x)$, for $j = s, s-1, \ldots, 2, 1$. Evidently, the factor $y_1(x) - y_2(x)$ is divisible to the term $(a-b) f(x)$.

Since $a \neq b$,

$$y_1(x) - y_2(x) = dp(x) \tag{13}$$

for a real number $d \in \mathbb{R}$ and some factor $p(x)$ of $f(x)$. □

In Lemma 3, the difference of any two distinct quasi-coincidence solutions corresponding to distinct values is a factor of $f(x)$. Thus, we define a class of this factor as follows.

Notation. (i) Let $p(x)$ be a factor of $f(x)$, and we denote $\Phi(p(x)) = \{\alpha p(x) : \alpha \in \mathbb{R}\}$.

(ii) Let $y(x)$ be an arbitrary polynomial in $\mathbb{R}[x]$, and we denote $y(x) + \Phi(p(x)) = \{y(x) + \alpha p(x) : \alpha \in \mathbb{R}\}$.

It is obvious that for any $y(x)$, $p(x) \in \mathbb{R}[x]$, then the cardinal number

$$\left|y(x) + \Phi(p(x))\right| = \infty. \tag{14}$$

For convenience, we explain the relations of Qcs_F and $\Phi(p(x))$ in the following lemma.

Lemma 4. *Let $y(x) \in Qcs_F(a)$ for some $a \in \mathbb{R}$, then*

$$Qcs_F = Qcs_F(a) \bigcup \left(\bigcup_{p(x)|f(x)} \{(y(x) + \Phi(p(x))) \cap Qcs_F\} \right). \quad (15)$$

Proof. For any $y_1(x) \in \mathrm{Qcs}_F \setminus \mathrm{Qcs}_F(a)$, then $y_1(x) \in \mathrm{Qcs}(b)$ for some $b \in \mathrm{Qcv}_F$. By Lemma 3, we have

$$y_1(x) - y(x) \in \Phi(p(x)) \quad (16)$$

for some factor $p(x)$ of $f(x)$. Then,

$$y_1(x) \in \bigcup_{p(x)|f(x)} \{y(x) + \Phi(p(x))\}. \quad (17)$$

That is,

$$\mathrm{Qcs}_F \subseteq \mathrm{Qcs}_F(a) \bigcup \left(\bigcup_{p(x)|f(x)} \{y(x) + \Phi(p(x))\} \right). \quad (18)$$

Moreover, by (6), $\mathrm{Qcs}_F(a) \subseteq \mathrm{Qcs}_F$, then it follows that

$$\mathrm{Qcs}_F = \mathrm{Qcs}_F(a) \bigcup \left(\bigcup_{p(x)|f(x)} \{(y(x) + \Phi(p(x))) \cap \mathrm{Qcs}_F\} \right). \quad (19)$$

□

We will use the definitions of "the pigeonhole principle;" it could concert to Grimaldi [6], and the relation can be explained as the following lemma.

Lemma 5. *Suppose that the cardinal number $|Qcs_F| = \infty$. For any $y(x) \in Qcs_F$, there exists a factor $p(x)$ of $f(x)$ such that the cardinal number*

$$|(y(x) + \Phi(p(x))) \cap Qcs_F| = \infty. \quad (20)$$

Proof. Let $y(x) \in \mathrm{Qcs}_F$, then $y(x) \in \mathrm{Qcs}_F(a)$ for some $a \in \mathbb{R}$. Since $|\mathrm{Qcs}_F| = \infty$ and $|\mathrm{Qcs}_F(a)| \le s$, by Lemma 4, we obtain

$$\left| \bigcup_{p(x)|f(x)} (\{y(x) + \Phi(p(x))\} \cap \mathrm{Qcs}_F) \right| = \infty, \quad (21)$$

it yields

$$\sum_{p(x)|f(x)} |(y(x) + \Phi(p(x))) \cap \mathrm{Qcs}_F| = \infty. \quad (22)$$

Moreover, the number of all factor $p(x)$ of $f(x)$ is at most $2^{\deg f(x)}$, by pigeonhole's principle, it leads to

$$|(y(x) + \Phi(p(x))) \cap \mathrm{Qcs}_F| = \infty \quad (23)$$

for some factor $p(x)$ of $f(x)$. □

In order to know if the intersection of two sets still has infinite solutions, we state the following result to give an explanation.

Lemma 6. *Suppose that the cardinal number $|Qcs_F| = \infty$, for any $y_1(x) \neq y_2(x) \in Qcs_F$, there exist some factors $p_1(x)$ and $p_2(x)$ of $f(x)$ such that*

$$|(y_1(x) + \Phi(p_1(x))) \cap (y_2(x) + \Phi(p_2(x))) \cap Qcs_F| = \infty. \quad (24)$$

Proof. Let $y_1(x) \in \mathrm{Qcs}_F$ and since $|\mathrm{Qcs}_F| = \infty$, by Lemma 5, there exists a factor $p_1(x)$ of $f(x)$ such that

$$|(y_1(x) + \Phi(p_1(x))) \cap \mathrm{Qcs}_F| = \infty. \quad (25)$$

Moreover, $y_2(x) \in \mathrm{Qcs}_F$, by Lemma 4,

$$\mathrm{Qcs}_F \subseteq \mathrm{Qcs}_F(b) \bigcup \left(\bigcup_{p_2(x)|f(x)} \{y_2(x) + \Phi(p_2(x))\} \right) \quad (26)$$

for some constant $b \in \mathbb{R}$ and some factor $p_2(x)$ of $f(x)$. Thus,

$$(\{y_1(x) + \Phi(p_1(x))\} \cap \mathrm{Qcs}_F) \subseteq \mathrm{Qcs}_F \subseteq \mathrm{Qcs}_F(b) \bigcup \left(\bigcup_{p_2(x)|f(x)} \{y_2(x) + \Phi(p_2(x))\} \right). \quad (27)$$

Since $|\mathrm{Qcs}_F(b)| \le s$ and the number of all factor to $f(x)$ is at most $2^{\deg f(x)}$, by pigeonhole's principle and (25), we have

$$\begin{aligned} &|(y_2(x) + \Phi(p_2(x))) \cap \mathrm{Qcs}_F| = \infty, \\ &|(y_1(x) + \Phi(p_1(x))) \\ &\quad \cap (y_2(x) + \Phi(p_2(x))) \cap \mathrm{Qcs}_F| = \infty \end{aligned} \quad (28)$$

for some factor $p_2(x)$ of $f(x)$. □

Up to now, we have not shown that the factor $p(x)$ is uniquely existed. Eventually, if the number of all solutions is infinitely many, then the factor $p(x)$ of $f(x)$ is unique up to the choice of the solution $y(x)$.

Lemma 7. *Assume that the cardinal number $|Qcs_F| = \infty$, then for any $y_1(x)$, $y_2(x) \in Qcs_F$, one has*

$$y_1(x) - y_2(x) = \lambda p(x) \quad (29)$$

for some constant $\lambda \in \mathbb{R}$ and some factor $p(x)$ of $f(x)$ (this $p(x)$ is independent to the choice of $y_1(x)$ and $y_2(x)$).

Proof. Let $y_1(x) \neq y_2(x) \in \mathrm{Qcs}_F$, by Lemma 6, we have

$$\begin{aligned} &|(y_1(x) + \Phi(p_1(x))) \\ &\quad \cap (y_2(x) + \Phi(p_2(x))) \cap \mathrm{Qcs}_F| = \infty \end{aligned} \quad (30)$$

for some factors $p_1(x)$, $p_2(x)$ of $f(x)$.

Let $g_1(x) \neq g_2(x) \in (y_1(x) + \Phi(p_1(x))) \cap (y_2(x) + \Phi(p_2(x))) \cap \mathrm{Qcs}_F - \{y_1(x), y_2(x)\}$, then

$$\begin{aligned} g_1(x) &\in y_1(x) + \Phi(p_1(x)), \\ g_2(x) &\in y_1(x) + \Phi(p_1(x)), \\ g_1(x) &\in y_2(x) + \Phi(p_2(x)), \\ g_2(x) &\in y_2(x) + \Phi(p_2(x)). \end{aligned} \quad (31)$$

By Lemma 6, it yields

$$\begin{aligned} g_1(x) - y_1(x) &= \lambda_1 p_1(x), \\ g_2(x) - y_1(x) &= \lambda_2 p_1(x), \\ g_1(x) - y_2(x) &= \lambda_3 p_2(x), \\ g_2(x) - y_2(x) &= \lambda_4 p_2(x) \end{aligned} \tag{32}$$

for some constants λ_1, λ_2, λ_3, and $\lambda_4 \in \mathbb{R}$, and consequently

$$\begin{aligned} g_2(x) - g_1(x) &= (g_2(x) - y_1(x)) - (g_1(x) - y_1(x)) \\ &= (\lambda_2 - \lambda_1) p_1(x), \\ g_2(x) - g_1(x) &= (g_2(x) - y_2(x)) \\ &\quad - (g_1(x) - y_2(x)) = (\lambda_4 - \lambda_3) p_2(x). \end{aligned} \tag{33}$$

This implies that

$$(\lambda_2 - \lambda_1) p_1(x) = (\lambda_4 - \lambda_3) p_2(x) \tag{34}$$

and $p_1(x) = p_2(x)$. Therefore,

$$\begin{aligned} y_1(x) - y_2(x) &= (g_1(x) - y_2(x)) - (g_1(x) - y_1(x)) \\ &= (\lambda_3 - \lambda_1) p_1(x). \end{aligned} \tag{35}$$

Consequently, the factor $p(x)$ is uniquely existed. □

By the above preparations, at first we consider the polynomial function $F(x, y)$ with $\deg_y F = 1$ as the form

$$F(x, y) = a_1(x) y + a_0(x). \tag{36}$$

Then, we consider the theorem of problem as

$$F(x, y) = af(x). \tag{37}$$

In the following theorem, we integrate the above type as follows.

Theorem 8. *Let $F(x, y)$ be a polynomial function with $\deg_y F = 1$ as the form $F(x, y) = a_1(x)y + a_0(x) = af(x)$ for some $a \in \mathbb{R}$ (where polynomial function $f(x)$ is given). If the cardinal number $|Qcv_F| \geq 2 (= \deg_y F + 1)$, then*

(i) *$a_1(x)$ is some factor of $f(x)$,*

(ii) *any solution of (5) is of the form:*

$$y(x) = -\frac{a_0(x)}{a_1(x)} + \lambda p(x) \tag{38}$$

for some $\lambda \in \mathbb{R}$ and some factor $p(x)$ of $f(x)$,

(iii) *the cardinal number $|Qcs_F| = \infty$.*

Proof. Since $|\mathrm{Qcv}_F| \geq 2$, we see that there are two distinct quasi-fixed values $a, b \in \mathrm{Qcv}_F$ corresponding to two distinct solutions $y_1(x)$, $y_2(x)$ in Qcs_F such that

$$\begin{aligned} F(x, y_1(x)) &= af(x), \\ F(x, y_2(x)) &= bf(x). \end{aligned} \tag{39}$$

(i) It follows that

$$F(x, y_1(x)) = a_1(x) y_1(x) + a_0(x) = af(x), \tag{40}$$

$$F(x, y_2(x)) = a_1(x) y_2(x) + a_0(x) = bf(x). \tag{41}$$

By (40)-(41), we get

$$a_1(x)(y_1(x) - y_2(x)) = (a - b) f(x). \tag{42}$$

It follows that $a_1(x)$ must be a factor of $f(x)$ and

$$y_1(x) - y_2(x) = \frac{(a - b) f(x)}{a_1(x)} \in \mathbb{R}[x]. \tag{43}$$

(ii) By (40), we have

$$a_0(x) = af(x) - a_1(x) y_1(x) \tag{44}$$

$$= a_1(x)\left(a\frac{f(x)}{a_1(x)} - y_1(x)\right). \tag{45}$$

Thus, (i) and (45) imply $a_1(x) \mid a_0(x)$, and by (44), $y_1(x)$ can be written as

$$y_1(x) = \frac{af(x) - a_0(x)}{a_1(x)}. \tag{46}$$

Moreover, we derive

$$\begin{aligned} F(x, y) &= a_1(x) y + a_0(x) \\ &= a_1(x)(y - y_1(x)) + (a_1(x) y_1(x) + a_0(x)) \\ \text{by (40)} &= a_1(x)(y - y_1(x)) + af(x). \end{aligned} \tag{47}$$

For any $y(x) \in \mathrm{Qcs}_F$, we have

$$F(x, y(x)) = \tilde{a} f(x) \tag{48}$$

for some $\tilde{a} \in \mathbb{R}$. By (47) and (48), it follows that

$$a_1(x)(y(x) - y_1(x)) + af(x) = \tilde{a} f(x). \tag{49}$$

Hence,

$$y(x) - y_1(x) = \frac{(\tilde{a} - a) f(x)}{a_1(x)}. \tag{50}$$

Then,

$$\begin{aligned} y(x) &= y_1(x) + \frac{(\tilde{a}-a)f(x)}{a_1(x)} \\ \text{by (46)} &= \frac{af(x)-a_0(x)}{a_1(x)} + \frac{(\tilde{a}-a)f(x)}{a_1(x)} \\ &= \frac{-a_0(x)}{a_1(x)} + \frac{\tilde{a}f(x)}{a_1(x)}. \end{aligned} \tag{51}$$

Therefore,

$$y(x) = \frac{-a_0(x)}{a_1(x)} + \lambda p(x) \quad \text{for some factor } p(x) \text{ of } f(x). \tag{52}$$

(Note that this $p(x) = f(x)/a_1(x)$ is only dependent on the choice of $F(x, y)$ and $f(x)$.)

(iii) Actually in (ii), for any $\lambda \in \mathbb{R}$, $y(x) = -a_0(x)/a_1(x) + \lambda f(x)/a_1(x)$ is also a quasi-coincidence solution for $F(x, y)$. The reason is

$$\begin{aligned} F(x, y(x)) &= a_1(x)\, y(x) + a_0(x) \\ &= a_1(x)\left(\frac{-a_0(x)}{a_1(x)} + \lambda\frac{f(x)}{a_1(x)}\right) + a_0(x) \\ &= \lambda f(x). \end{aligned} \tag{53}$$

This shows that $(*)$ has infinitely many solutions (i.e., $|\mathrm{Qcs}_F| = \infty$). □

Remark 9. Notice that in the case of $\deg_y F = 1$ and $|\mathrm{Qcs}_F| < \infty$, the number of all quasi-coincidence values cannot be larger than 1. Otherwise, it will contract the result of Theorem 8; the case (iii) means that "$|\mathrm{Qcv}_F| \geq 2$, and then $|\mathrm{Qcs}_F| = \infty$".

3. Main Theorems

In this section, we consider (5) for polynomial function $F(x, y)$ in (9); that is,

$$F(x, y) = \sum_{i=0}^{s} a_i(x)\, y^i \quad \text{with } s \geq 2. \tag{54}$$

A given polynomial function $f(x)$ in $\mathbb{R}[x]$ and $F(x, y)$ has at least $s + 1$ distinct quasi-coincidence solutions satisfying some conditions, that is, $y_1(x), y_2(x), y_3(x), \ldots, y_{s+1}(x), \ldots$. According to the above assumptions, we could derive the following theorem.

Theorem 10. *Suppose that the cardinal number $|Qcs_F| = \infty$ and for each $y(x) \in Qcs_F$ can be represented as the form*

$$y(x) = y_1(x) + \lambda p(x), \quad \lambda \in \mathbb{R} \tag{55}$$

for some $y_1(x)$, $p(x) \in \mathbb{R}[x]$. Then, $p^s(x) \mid f(x)$, and so the polynomial $F(x, y)$ can be represented as

$$F(x, y) = \sum_{i=0}^{s} c_i \frac{f(x)}{p^i(x)}\left(y - y_1(x)\right)^i \tag{56}$$

for constants $c_i \in \mathbb{R}$.

Proof. Let $y_i(x)$ be distinct quasi-coincidence solutions of $F(x, y)$ corresponding to quasi-coincidence values a_i, $1 \leq i \leq s + 1$ such that

$$F(x, y_i(x)) = a_i f(x). \tag{57}$$

Choose $i = 1$, $F(x, y_1(x)) = a_1 f(x)$. When $y - y_1(x)$ divides the function $F(x, y)$, we get

$$F(x, y) = (y - y_1(x))\, F_1(x, y) + a_1 f(x), \tag{58}$$

where $F_1(x, y)$ is the quotient and $a_1 f(x)$ is the remainder. From the above identity, taking $y = y_2(x)$, it becomes

$$\begin{aligned} F(x, y_2(x)) &= (y_2(x) - y_1(x))\, F_1(x, y_2(x)) + a_1 f(x) \\ &= a_2 f(x). \end{aligned} \tag{59}$$

Then,

$$(y_2(x) - y_1(x))\, F_1(x, y_2(x)) = (a_2 - a_1)\, f(x). \tag{60}$$

By (55), $y_2(x) - y_1(x) = \lambda_2 p(x)$, it yields

$$\begin{aligned} F_1(x, y_2(x)) &= \left(\frac{(a_2 - a_1)}{\lambda_2}\right)\frac{f(x)}{p(x)} \\ &= d_2\frac{f(x)}{p(x)} \in \mathbb{R}[x] \quad \text{for } d_2 = \frac{(a_2 - a_1)}{\lambda_2}. \end{aligned} \tag{61}$$

Hence,

$$F_1(x, y) = (y - y_2(x))\, F_2(x, y) + d_2\frac{f(x)}{p(x)}. \tag{62}$$

Continuing this process from $i = 2$ to $s - 1$, we obtain

$$F_i(x, y) = (y - y_{i+1}(x))\, F_{i+1}(x, y) + d_{i+1}\frac{f(x)}{p^i(x)} \tag{63}$$

for some $d_{i+1} \in \mathbb{R}$, $i = 1, 2, \ldots, s - 1$. Finally, we could get

$$F_{s-1}(x, y) = (y - y_s(x))\, F_s(x) + d_s\frac{f(x)}{p^{s-1}(x)}. \tag{64}$$

$F_s(x)$ does not contain the variable y since $\deg_y F = s$. By the assumption (57), $F(x, y_{s+1}(x)) = a_{s+1} f(x)$. It follows that

$$F_s(x) = \lambda\frac{f(x)}{p^s(x)} \in \mathbb{R}[x] \quad \textit{for some constant } \lambda \in \mathbb{R}. \tag{65}$$

Consequently,

$$\begin{aligned}
F(x,y) &= (y-y_1(x))F_1(x,y)+a_1f(x)\\
&= (y-y_1(x))\\
&\quad\times\left((y-y_2(x))F_2(x,y)+d_2\frac{f(x)}{p(x)}\right)+a_1f(x)\\
&= \cdots\\
&= (y-y_1(x))\Bigg((y-y_2(x))\\
&\qquad\times\Bigg(\cdots\Bigg((y-y_s(x))F_s(x)\\
&\qquad\qquad+d_s\frac{f(x)}{p^{s-1}(x)}\Bigg)\cdots\Bigg)\\
&\qquad+d_2\frac{f(x)}{p(x)}\Bigg)+a_1f(x)\\
&= (y-y_1(x))\Bigg((y-y_2(x))\\
&\qquad\times\Bigg(\cdots\Bigg((y-y_s(x))\lambda\frac{f(x)}{p^s(x)}\\
&\qquad\qquad+d_s\frac{f(x)}{p^{s-1}(x)}\Bigg)\cdots\Bigg)\\
&\qquad+d_2\frac{f(x)}{p(x)}\Bigg)+a_1f(x).
\end{aligned}\tag{66}$$

By (55), we have $y_i(x) = y_1(x)+\lambda_i p(x)$, $i = 2,3,\ldots,s+1$. Then, $F(x,y)$ can be expanded to a power series in the expression:

$$\begin{aligned}
F(x,y) &= (y-y_1(x))\\
&\quad\times\Bigg((y-y_1(x)-\lambda_2p(x))\\
&\qquad\times\Bigg(\cdots\Bigg((y-y_1(x)-\lambda_sp(x))\lambda\frac{f(x)}{p^s(x)}\\
&\qquad\qquad+d_s\frac{f(x)}{p^{s-1}(x)}\Bigg)\cdots\Bigg)\\
&\qquad+d_2\frac{f(x)}{p(x)}\Bigg)+a_1f(x)\\
&= \sum_{i=0}^{s}c_i\frac{f(x)}{p^i(x)}(y-y_1(x))^i
\end{aligned}\tag{67}$$

for some real numbers c_j, $j = 0,1,\ldots,s$. Moreover, the leading coefficient of $F(x,y)$, $c_s(f(x)/p^s(x))$ is contained to $\mathbb{R}[x]$, and it follows $p^s(x)\mid f(x)$. □

Conversely, if $F(x,y)$ is expressed as in Theorem 10, then the cardinal number $|\mathrm{Qcs}_F| = \infty$, this is the same as the sufficient conditions.

Theorem 11. *The following two conditions are equivalent:*

(i) $F(x,y) = \sum_{i=0}^{s}c_i(f(x)/p^i(x))(y-y_1(x))^i$ *for some* $y_1(x)\in\mathbb{R}[x]$, $p(x)$ *is a factor of* $f(x)$, *and* $c_i\in\mathbb{R}$ *for* $i = 0,1,\ldots,s$,

(ii) $|Qcs_F| = \infty$.

(In fact, if $|\mathrm{Qcs}_F| = \infty$, then $|\mathrm{Qcs}_F|$ = the cardinal number of $\mathbb{R}$.)

Proof. (i)⇒(ii) Suppose that (i) holds. Then,

$$\begin{aligned}
F(x,y_1(x)+\lambda p(x)) &= \sum_{i=0}^{s}c_i\frac{f(x)}{p^i(x)}(\lambda p(x))^i\\
&= \left(\sum_{i=0}^{s}c_i\lambda^i\right)f(x)\\
&= ap^m(x)\quad\text{for } a=\sum_{i=0}^{s}c_i\lambda^i\in\mathbb{R}.
\end{aligned}\tag{68}$$

This means that $y_1(x)+\lambda p(x)\in\mathrm{Qcs}_F$ for each $\lambda\in\mathbb{R}$. It follows that the cardinal $|\mathrm{Qcs}_F| = \infty$.

(ii)⇒(i) For any $y(x), y_1(x)\in\mathrm{Qcs}_F$, since $|\mathrm{Qcs}_F| = \infty$ and by Lemma 7, we obtain

$$y(x)-y_1(x) = dp(x)\tag{69}$$

for some factor $p(x)$ of $f(x)$. By Theorem 10, we have

$$F(x,y) = \sum_{i=0}^{s}c_i\frac{f(x)}{p^i(x)}(y-y_1(x))^i\tag{70}$$

for some $y_1(x), p(x)\in\mathbb{R}[x]$, and $c_i\in\mathbb{R}$ for $i = 0,1,\ldots,s$. □

If the $F(x,y)$ can be represented as the form of (71) in the following lemma, then any quasi-coincidence solution can be determined.

Lemma 12. *Suppose that*

$$F(x,y) = \sum_{i=0}^{s}c_i\frac{f(x)}{p^i(x)}(y-y(x))^i,\tag{71}$$

where $y(x)\in\mathbb{R}[x]$, $c_i\in\mathbb{R}$, $i = 0,1,\ldots,s$ *and* $p(x)$ *is a factor of* $f(x)$. *Then,* $h(x)\in\mathbb{R}[x]$ *is a quasi-coincidence solution of* $F(x,y)$, *if and only if*

$$h(x) = y(x)+dp(x)\quad\text{for some } d\in\mathbb{R}.\tag{72}$$

Proof. At first, we assume that $h(x)\in\mathbb{R}[x]$ is a quasi-coincidence solution of $F(x,y)$, and we consider

$$\begin{aligned}
F(x,y(x)) &= \sum_{i=0}^{s}c_i\frac{f(x)}{p^i(x)}(y(x)-y(x))^i\\
&= c_0f(x).
\end{aligned}\tag{73}$$

This means $y(x)\in\mathrm{Qcs}_F$.

By Theorem 11,

$$F(x,y)=\sum_{i=0}^{s}c_i\frac{f(x)}{p^i(x)}(y-y(x))^i,\quad \text{then } |\mathrm{Qcs}_F|=\infty. \tag{74}$$

It follows from Lemma 7 that for any quasi-coincidence solution $h(x)$, we obtain

$$h(x)=y(x)+dp(x)\quad \text{for some } d\in\mathbb{R}. \tag{75}$$

Conversely, suppose $h(x)=y(x)+dp(x)$ for some factor $p(x)$ of $f(x)$ and some constant $d\in\mathbb{R}$. Substituting this $h(x)$ as y in (71), we have

$$\begin{aligned}F(x,h(x))&=F(x,y(x)+dp(x))\\&=\sum_{i=0}^{s}c_i\frac{f(x)}{p^i(x)}(dp(x))^i\\&=\left(\sum_{i=0}^{s}c_id^i\right)f(x).\end{aligned} \tag{76}$$

Therefore, $h(x)\in\mathrm{Qcs}_F$.

Note that not all polynomial functions $F(x,y)$ can be written as (71). Actually, almost all $F(x,y)$ are expressed as the form of the next theorem. In that situation, any solution can be written as the next form $(\star)$ if the cardinal number $|\mathrm{Qcs}_F|$ is in_nitely many in this theorem. □

Theorem 13. *Let $F(x,y)$ be a polynomial function with*

$$F(x,y)=a_s(x)y^s+a_{s-1}(x)y^{s-1}+\cdots+a_0(x), \tag{77}$$

and $f(x)$ a polynomial. If the cardinal number $|Qcs_F|$ is infinitely many, then for each quasi-coincidence point solution of (5) *must be of the form*

$$-\frac{a_{s-1}(x)}{sa_s(x)}+\lambda p(x) \tag{$\star$}$$

for arbitrary $\lambda\in\mathbb{R}$, where $p(x)=c(f(x)/a_s(x))^{1/s}$ is a factor of $f(x)$ and c is a constant.

Proof. Assume $|\mathrm{Qcs}_F|=\infty$. By Theorem 11, we have

$$\begin{aligned}F(x,y)&=a_s(x)y^s+a_{s-1}(x)y^{s-1}+\cdots+a_0(x)\\&=\sum_{i=0}^{s}c_i\frac{f(x)}{p^i(x)}(y-y(x))^i\end{aligned} \tag{78}$$

for some $c_i\in\mathbb{R}$, and $y(x)\in\mathrm{Qcs}_F$. Comparing the coefficients of y^s and y^{s-1} in both sides, we get

$$a_s(x)=c_s\frac{f(x)}{p^s(x)}, \tag{79}$$

$$a_{s-1}(x)=-sa_s(x)y(x)+c_{s-1}\frac{f(x)}{p^{s-1}(x)}. \tag{80}$$

Consequently, by (79) and (80), we get

$$\begin{aligned}p^s(x)&=c_s\frac{f(x)}{a_s(x)},\\y(x)&=\frac{c_{s-1}}{sc_s}p(x)-\frac{a_{s-1}(x)}{sa_s(x)}\in\mathbb{R}[x].\end{aligned} \tag{81}$$

By Lemma 12 and (81), for any $d\in\mathbb{R}$, we have that any quasi-coincidence solution is represented by

$$\begin{aligned}y(x)+dp(x)&=\frac{c_{s-1}}{sc_s}p(x)-\frac{a_{s-1}(x)}{sa_s(x)}+dp(x)\\&=-\frac{a_{s-1}(x)}{sa_s(x)}+\left(d-\frac{c_{s-1}}{sc_s}\right)p(x)\\&=-\frac{a_{s-1}(x)}{sa_s(x)}+\lambda p(x),\end{aligned} \tag{82}$$

where $p(x)=(c_s)^{1/s}(f(x)/a_s(x))^{1/s}$ (note that since d is arbitrary, then λ is arbitrary).

This completes the proof. □

Finally, we provide two examples. Example 1 explains the case of all cardinal number $|\mathrm{Qcs}_F|=4$.

Example 1. Let

$$\begin{aligned}F(x,y)&=(x^2+x+1)y^2-x^6\\&\quad-3x^5-6x^4-7x^3-10x^2-7x-5,\\f(x)&=(x^2+x+1)^2.\end{aligned} \tag{83}$$

Then,

$$\begin{aligned}F(x,y)&=(x^2+x+1)\\&\quad\times\left[y^2-(x^2+x+1)^2-4\right],\quad \deg_yF=s=2.\end{aligned} \tag{84}$$

This polynomial The polynomial equation $F(x,y)=a(x^2+x+1)^2$ for some $a\in\mathbb{R}$ has exactly $4=(s+2)$ quasi-coincidence solutions as follows:

$$\begin{aligned}F(x,x^2+x+3)&=4(x^2+x+1)^2,\\F(x,-x^2-x-3)&=4(x^2+x+1)^2,\\F(x,x^2+x-1)&=-4(x^2+x+1)^2,\\F(x,-x^2-x+1)&=-4(x^2+x+1)^2.\end{aligned} \tag{85}$$

The next example explains that the number of all quasi-coincidence solutions of (5) is infinitely many.

Example 2. Let $x \in \mathbb{R}$, $f(x) = x^4(x-1)^4$, and

$$\begin{aligned} F(x,y) &= a_3(x)y^3 + a_2(x)y^2 + a_1(x)y + a_0(x) \\ &= x(x-1)y^3 + 0y^2 + x^3(x-1)^3 y + 0. \end{aligned} \tag{86}$$

We will solve all quasi-coincidence solutions of $F(x,y) = ax^4(x-1)^4$ for some $a \in \mathbb{R}$. This polynomial function has at least $6(\geq s+3$, since $s=3)$ quasi-coincidence solutions as follows:

$$\begin{aligned} F\left(x_1, x_2, x^2 - x\right) &= 2x^4(x-1)^4, \\ F\left(x_1, x_2, 2x^2 - 2x\right) &= 10x^4(x-1)^4, \\ F\left(x_1, x_2, -x^2 + x\right) &= -2x^4(x-1)^4, \\ F\left(x_1, x_2, -2x^2 + 2x\right) &= -10x^4(x-1)^4, \\ F\left(x_1, x_2, \frac{x^2}{2} - \frac{x}{2}\right) &= \frac{5}{8x^4(x-1)^4}, \\ F\left(x_1, x_2, -\frac{x^2}{2} + \frac{x}{2}\right) &= -\frac{5}{8x^4(x-1)^4}. \end{aligned} \tag{87}$$

In fact, we have $|\mathrm{Qcs}_F| = \infty$, and by (79), we obtain

$$\begin{aligned} p(x) &= c\left(\frac{f(x)}{a_s(x)}\right)^{1/s} \\ &= c\left(\frac{x^4(x-1)^4}{x(x-1)}\right)^{1/3} \\ &= cx(x-1), \end{aligned} \tag{88}$$

for some real number c.

By Theorem 13, any quasi-coincidence solution is written as

$$\begin{aligned} -\frac{a_2(x)}{sa_3(x)} + \lambda p(x) &= \frac{0}{3x(x-1)} + \lambda cx(x-1) \\ &= \mu x(x-1), \end{aligned} \tag{89}$$

where $\mu = \lambda c \in \mathbb{R}$ is arbitrary. This shows that the quasi-coincidence (point) solutions have cardinal $|\mathrm{Qcs}_F| = \infty$.

We would like to provide one open problem as follows.

Further Development. For a real-valued polynomial function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Can we find all rational quasi-coincidence solutions $y = b(x)/a(x)$ with coprime polynomials $a(x)$, $b(x) \in \mathbb{R}[x]$ to satisfy

$$F(x,y) = af(x) \tag{90}$$

for some polynomials $f(x) \in \mathbb{R}[x]$?

Acknowledgments

The authors wish to express their deep gratitude to Professor Tamaki Tanaka for his valuable comments on this paper and thank the referees for the very useful suggestions and remarks that contributed to the improvement of the paper.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 507805, 8 pages
http://dx.doi.org/10.1155/2013/507805

Research Article

Coincidence Points of Weaker Contractions in Partially Ordered Metric Spaces

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Received 6 June 2013; Accepted 4 August 2013

Academic Editor: E. Karapinar

We prove new coincidence point theorems for the $(\varphi, \psi, \phi, \xi)$-contractions and generalized Meir-Keeler-type α-ψ-contractions in partially ordered metric spaces. Our results generalize many recent coincidence point theorems in the literature.

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space, D a subset of X, and $f : D \to X$ a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \tag{1}$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Also, this principle has many generalizations. For instance, a mapping $f : X \to X$ is called a quasicontraction if there exists $k < 1$ such that

$$\begin{aligned} d(fx, fy) \leq k \cdot \max\{&d(x, y), d(x, fx), \\ &d(y, fy), d(x, fy), d(y, fx)\}, \end{aligned} \tag{2}$$

for any $x, y \in X$. In 1974, Ćirić [2] introduced these maps and proved an existence and uniqueness fixed point theorem.

Recently, Eslamian and Abkar proved the following theorem.

Theorem 1 (see [3]). *Let (X, d) be a complete metric space and $f : X \to X$ be such that*

$$\begin{aligned} \psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x, y)), \\ \text{for each } x, y \in X, \end{aligned} \tag{3}$$

where $\psi, \alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are as follows: ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, and

$$\begin{gathered} \psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall t > 0, \\ \psi(t) = 0 \quad \text{iff } t = 0, \quad \alpha(0) = \beta(0) = 0. \end{gathered} \tag{4}$$

Then f has a fixed point in X.

Recently, fixed point theory has developed rapidly in partially ordered metric spaces (e.g., [4–8]).

In 2012, Choudhury and Kundu [9] proved the following coincidence theorem as a generalization of Theorem 1.

Theorem 2 (see [9]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space and $f, g : X \to X$ be such that $fX \subset gX$, f is g-nondecreasing, gX is closed, and*

$$\begin{aligned} \psi(d(fx, fy)) \leq \alpha(d(gx, gy)) - \beta(d(gx, gy)), \\ \text{for each } x, y \in X \text{ such that } gx \sqsubseteq gy, \end{aligned} \tag{5}$$

where $\psi, \alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, and

$$\begin{gathered}\psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall t > 0, \\ \psi(t) = 0 \quad iff\ t = 0, \quad \alpha(0) = \beta(0) = 0.\end{gathered} \tag{6}$$

Also, if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then we assume that

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{7}$$

If there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X.

In this paper, we prove new coincidence point theorems for the $(\varphi, \psi, \phi, \xi)$-contractions and generalized Meir-Keeler-type α-ψ-contractions in partially ordered metric spaces. Our results generalize many recent coincidence point theorems in the literature.

2. Main Results

We start with the following definition.

Definition 3 (g-nondecreasing mapping [4]). Let $(X, \sqsubseteq)$ be a partially ordered set and $f, g : X \to X$. Then f is said to be g-nondecreasing if, for $x, y \in X$,

$$gx \sqsubseteq gy \Longrightarrow fx \sqsubseteq fy. \tag{8}$$

In the sequel, we denote by Ψ the class of functions $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ satisfying the following conditions:

(ψ_1) ψ is an increasing, continuous function in each coordinate,

(ψ_2) for all $t \in \mathbb{R}^+$, $\psi(t, t, t, 0, 2t) \le t$, $\psi(t, t, t, 2t, 0) \le t$, $\psi(0, 0, t, t, 0) \le t$, and $\psi(t, 0, 0, t, t) \le t$,

(ψ_3) $\psi(t_1, t_2, t_3, t_4, t_5) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = t_5 = 0$.

Next, we denote by Φ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(ϕ_1) ϕ is a continuous function and monotone nondecreasing;

(ϕ_2) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;

(ϕ_3) ϕ is subadditive, that is, $\phi(t_1 + t_2) \le \phi(t_1) + \phi(t_2)$ for all $t_1, t_2 > 0$.

And, we denote the following sets of functions:

$$\begin{gathered}\Theta = \{\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ such that } \varphi \text{ is continuous}\}, \\ \Xi = \{\xi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \text{ such that } \xi \text{ is lower continuous}\}.\end{gathered} \tag{9}$$

Let X be a nonempty set and $(X, \sqsubseteq)$ be a partially ordered set endowed with a metric d. Then, the triple $(X, \sqsubseteq, d)$ is called a partially ordered metric space.

We now state the $(\varphi, \psi, \phi, \xi)$-contraction and the main fixed point theorem for the $(\varphi, \psi, \phi, \xi)$-contraction in partially ordered metric spaces, as follows.

Definition 4. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \to X$. Then the pair (f, g) is called a $(\varphi, \psi, \phi, \xi)$-contraction if the following inequality holds:

$$\begin{aligned}&\varphi(d(fx, fy)) \\ &\quad \le \psi(\phi(d(gx, gy)), \phi(d(gx, fx)), \\ &\qquad \phi(d(gy, fy)), \phi(d(gx, fy)), \phi(d(gy, fx))) \\ &\qquad - \xi(\max\{d(gx, gy), d(gx, fx), d(gy, fy)\}),\end{aligned} \tag{10}$$

for all $x, y \in X$ with $gx \sqsubseteq gy$, where $\varphi \in \Theta$, $\psi \in \Psi$, $\phi \in \Phi$ and $\xi \in \Xi$.

We now state the main fixed point theorem for the $(\varphi, \psi, \phi, \xi)$-contraction in partially ordered metric spaces, as follows.

Theorem 5. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \to X$ be such that $fX \subset gX$, f is g-nondecreasing and gX is closed. Suppose the pair (f, g) is a $(\varphi, \psi, \phi, \xi)$-contraction, and*

$$\begin{gathered}\varphi(t) - \phi(t) + \xi(t) > 0 \quad \forall t > 0, \\ \varphi(t) = 0 \quad iff\ t = 0, \quad \phi(0) = \xi(0) = 0.\end{gathered} \tag{11}$$

Also, if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then we assume that

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{12}$$

If there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X.

Proof. Since $fX \subset gX$ and there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Since f is g-nondecreasing, we have $fx_0 \sqsubseteq fx_1$. In this process, we construct the sequence $\{x_n\}$ recursively as

$$fx_n = gx_{n+1} \quad \forall n \in \mathbb{N}. \tag{13}$$

Thus, we also conclude that

$$\begin{aligned}gx_0 \sqsubseteq fx_0 = gx_1 \sqsubseteq fx_1 = gx_2 \sqsubseteq \cdots \sqsubseteq fx_{n-1} \\ = gx_n \sqsubseteq fx_n = gx_{n+1} \sqsubseteq \cdots.\end{aligned} \tag{14}$$

If any two consecutive terms in (14) are equal, then the conclusion of the theorem follows. So we may assume that

$$d(fx_{n-1}, fx_n) \ne 0, \quad \forall n \in \mathbb{N}. \tag{15}$$

Now, we claim that $d(fx_n, fx_{n+1}) \le d(fx_{n-1}, fx_n)$ for all $n \in \mathbb{N}$. If not, we assume that $d(fx_{n-1}, fx_n) < d(fx_n, fx_{n+1})$ for

some $n \in \mathbb{N}$, substituting $x = x_n$ and $y = x_{n+1}$ in (10) and using the definition of the function ψ, we have

$$\begin{aligned}
&\psi(\phi(d(gx_n, gx_{n+1})), \phi(d(gx_n, fx_n)), \\
&\quad \phi(d(gx_{n+1}, fx_{n+1})), \phi(d(gx_n, fx_{n+1})), \\
&\quad \phi(d(gx_{n+1}, fx_n))) \\
&= \psi(\phi(d(fx_{n-1}, fx_n)), \phi(d(fx_{n-1}, fx_n)), \\
&\quad \phi(d(fx_n, fx_{n+1})), \phi(d(fx_{n-1}, fx_{n+1})), \\
&\quad \phi(d(fx_n, fx_n))) \\
&\le \psi(\phi(d(fx_n, fx_{n+1})), \phi(d(fx_n, fx_{n+1})), \\
&\quad \phi(d(fx_n, fx_{n+1})), 2\phi(d(fx_n, fx_{n+1})), \phi(0)) \\
&\le \phi(d(fx_n, fx_{n+1})), \\
&\xi(\max\{d(gx_n, gx_{n+1}), d(gx_n, fx_n), \\
&\quad d(gx_{n+1}, fx_{n+1})\}) \\
&= \xi(\max\{d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), \\
&\quad d(fx_n, fx_{n+1})\}) \\
&= \xi(d(fx_n, fx_{n+1})),
\end{aligned} \tag{16}$$

and hence

$$\varphi(d(fx_n, fx_{n+1})) \le \phi(d(fx_n, fx_{n+1})) - \xi(d(fx_n, fx_{n+1})). \tag{17}$$

Since $\varphi(t) - \phi(t) + \xi(t) > 0$ for all $t > 0$, we have that $d(fx_n, fx_{n+1}) = 0$, which contradicts to (15). Therefore, we conclude that

$$d(fx_n, fx_{n+1}) \le d(fx_{n-1}, x_n) \quad \forall n \in \mathbb{N}. \tag{18}$$

From above argument, we also have that for each $n \in \mathbb{N}$

$$\varphi(d(fx_n, fx_{n+1})) \le \phi(d(fx_{n-1}, fx_n)) - \xi(d(fx_{n-1}, fx_n)). \tag{19}$$

It follows (18) that the sequence $\{d(fx_n, fx_{n+1})\}$ is monotone decreasing, it must converge to some $\eta \ge 0$. Taking limit as $n \to \infty$ in (19) and using the continuities of φ and ϕ and the lower semicontinuity of ξ, we get

$$\varphi(\eta) \le \phi(\eta) - \xi(\eta), \tag{20}$$

which implies that $\eta = 0$. So we conclude that

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0. \tag{21}$$

We next claim that $\{fx_n\}$ is a Cauchy sequence, that is, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \ge n$, then $d(fx_p, fx_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \ge n$ satisfying

$$d(fx_{q_n}, fx_{p_n}) \ge \epsilon. \tag{22}$$

Further, corresponding to $q_n \ge n$, we can choose p_n in such a way that it is the smallest integer with $p_n > q_n \ge n$ and $d(fx_{q_n}, fx_{p_n}) \ge \epsilon$. Therefore $d(fx_{q_n}, fx_{p_n-1}) < \epsilon$. Now we have that for all $n \in \mathbb{N}$

$$\begin{aligned}
\epsilon &\le d(fx_{p_n}, fx_{q_n}) \\
&\le d(fx_{p_n}, fx_{p_n-1}) + d(fx_{p_n-1}, fx_{q_n}) \\
&< d(fx_{p_n}, fx_{p_n-1}) + \epsilon.
\end{aligned} \tag{23}$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d(fx_{p_n}, fx_{q_n}) = \epsilon. \tag{24}$$

On the other hand, we have

$$\begin{aligned}
d(fx_{p_n}, fx_{q_n}) &\le d(fx_{p_n}, fx_{p_n-1}) + d(fx_{p_n-1}, fx_{q_n-1}) \\
&\quad + d(fx_{q_n-1}, fx_{q_n}), \\
d(fx_{p_n-1}, fx_{q_n-1}) &\le d(fx_{p_n-1}, fx_{p_n}) + d(fx_{p_n}, fx_{q_n}) \\
&\quad + d(fx_{q_n}, fx_{q_n-1}).
\end{aligned} \tag{25}$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d(fx_{p_n-1}, fx_{q_n-1}) = \epsilon. \tag{26}$$

By (14), we have that the elements gx_{p_n} and gx_{q_n} are comparable. Substituting $x = x_{p_n}$ and $y = x_{q_n}$ in (10), we have that for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\psi(\phi(d(gx_{p_n}, gx_{q_n})), \phi(d(gx_{p_n}, fx_{p_n})), \\
&\quad \phi(d(gx_{q_n}, fx_{q_n})), \phi(d(gx_{p_n}, fx_{q_n})), \\
&\quad \phi(d(gx_{q_n}, fx_{p_n}))) \\
&\le \psi(\phi(d(fx_{p_n-1}, fx_{q_n-1})), \phi(d(fx_{p_n-1}, fx_{p_n})), \\
&\quad \phi(d(fx_{q_n-1}, fx_{q_n})), \phi(d(fx_{p_n-1}, fx_{q_n})), \\
&\quad \phi(d(fx_{q_n-1}, fx_{p_n}))) \\
&\le \psi(\phi(d(fx_{p_n-1}, fx_{q_n-1})), \phi(d(fx_{p_n-1}, fx_{p_n})), \\
&\quad \phi(d(fx_{q_n-1}, fx_{q_n})), \phi(d(fx_{p_n-1}, fx_{p_n})) \\
&\quad + \phi(d(fx_{p_n}, fx_{q_n})), \phi(d(fx_{q_n-1}, fx_{q_n})) \\
&\quad + \phi(d(fx_{q_n}, fx_{p_n}))), \\
&M(x_{p_n}, x_{q_n}) = \max\{d(gx_{p_n}, gx_{q_n}), \\
&\quad d(gx_{p_n}, fx_{p_n}), d(gx_{q_n}, fx_{q_n})\} \\
&\quad = \max\{d(fx_{p_n-1}, fx_{q_n-1}), d(fx_{p_n-1}, fx_{p_n}), \\
&\quad d(fx_{q_n-1}, fx_{q_n})\}.
\end{aligned} \tag{27}$$

By above argument and using inequality (10), we can conclude that

$$\begin{aligned}\varphi(\epsilon) &\le \psi(\phi(\epsilon), 0, 0, \phi(\epsilon), \phi(\epsilon)) - \xi(\epsilon)\\ &\le \phi(\epsilon) - \xi(\epsilon),\end{aligned} \tag{28}$$

which implies that $\epsilon = 0$, a contradiction. Therefore, the sequence $\{fx_n\}$ is a Cauchy sequence.

Since X is complete and gX is closed, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = g\nu. \tag{29}$$

Later, we prove that ν is a coincidence point of f and g. From (14) and (29), we deduce that

$$gx_n \sqsubseteq g\nu, \quad \forall n \in \mathbb{N}. \tag{30}$$

Substituting $x = x_n$ and $y = \nu$ in (10), we have that

$$\begin{aligned}&\varphi(d(fx_n, f\nu))\\ &\quad\le \psi(\phi(d(gx_n, g\nu)), \phi(d(gx_n, fx_n)),\\ &\qquad \phi(d(g\nu, f\nu)), \phi(d(gx_n, f\nu)),\\ &\qquad \phi(d(g\nu, fx_n)))\\ &\qquad - \xi(\max\{d(gx_n, g\nu), d(gx_n, fx_n), d(g\nu, f\nu)\}).\end{aligned} \tag{31}$$

Taking $n \to \infty$ in the above inequality, we have

$$\begin{aligned}\varphi(d(g\nu, f\nu)) &\le \psi(0, 0, \phi(d(g\nu, f\nu)), \phi(d(g\nu, f\nu)), 0)\\ &\quad - \xi(d(g\nu, f\nu)) \le \phi(d(g\nu, f\nu))\\ &\quad - \xi(d(g\nu, f\nu)),\end{aligned} \tag{32}$$

which implies that $d(g\nu, f\nu) = 0$, that is, $g\nu = f\nu$. So we complete the proof. □

We give the following example to illustrate Theorem 5.

Example 6. Let $X = [0, 1]$. We define a partial order "$\sqsubseteq$" on X as $x \sqsubseteq y$ if and only if $x \ge y$ for all $x, y \in X$. We take the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let $f, g : X \to X$ be defined as

$$f(x) = \frac{1}{16}x^2, \qquad g(x) = \frac{1}{4}x^2. \tag{33}$$

Let $\varphi, \phi, \xi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined as

$$\varphi(t) = \phi(t) = t, \qquad \xi(t) = \frac{t}{8} \quad \forall t \in [0, 1], \tag{34}$$

and let $\psi : \mathbb{R}^{+5} \to \mathbb{R}^+$ denote

$$\psi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2} \cdot \max\left\{t_1, t_2, t_3, \frac{t_4}{2}, \frac{t_5}{2}\right\}. \tag{35}$$

Without loss of generality, we assume that $x > y$ and verity inequality (10).

For all $x, y \in [0, 1]$ with $x > y$, we have

$$\begin{aligned}\varphi(d(fx, fy)) &= \frac{1}{16}(x^2 - y^2),\\ \phi(d(gx, gy)) &= \frac{1}{4}(x^2 - y^2),\\ \phi(d(gx, fx)) &= \frac{1}{4}x^2 - \frac{1}{16}x^2 = \frac{3}{16}x^2,\\ \phi(d(gy, fy)) &= \frac{1}{4}y^2 - \frac{1}{16}y^2 = \frac{3}{16}y^2,\\ \phi(d(gx, fy)) &= \frac{1}{4}x^2 - \frac{1}{16}y^2 > \frac{3}{16}x^2,\\ \phi(d(fx, gy)) &= \left|\frac{1}{16}x^2 - \frac{1}{4}y^2\right|,\\ &\xi(\max d(gx, gy), d(gx, fx), d(gy, fy))\\ &= \begin{cases}\frac{1}{4}(x^2 - y^2), & \text{if } x > 2y,\\ \frac{3}{16}x^2, & \text{if } x \le 2y,\end{cases}\\ &\psi(\phi(d(gx, gy)), \phi(d(gx, fx)), \phi(d(gy, fy)),\\ &\quad \phi(d(gx, fy)), \phi(d(fx, gy)))\\ &= \frac{1}{8}x^2 - \frac{1}{32}y^2.\end{aligned} \tag{36}$$

Therefore, inequality (10) is satisfied and all the conditions of Theorem 5 are satisfied, and we obtained that 0 is a coincidence point of f and g.

Applying Definition 4, Theorem 5, and Example 6, if we let

$$\begin{aligned}&\psi(\phi(d(gx, gy)), \phi(d(gx, fx)), \phi(d(gy, fy)),\\ &\quad \phi(d(gx, fy)), \phi(d(gy, fx)))\\ &= \max\left\{\phi(d(gx, gy)), \phi(d(gx, fx)), \phi(d(gy, fy)),\right.\\ &\qquad \left.\frac{1}{2}\phi(d(gx, fy)), \frac{1}{2}\phi(d(fx, gy)\right\},\end{aligned} \tag{37}$$

we are easy to get the following theorem.

Theorem 7. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \to X$ be such that $fX \subset gX$, f is g-nondecreasing, gX is closed, and*

$$\begin{aligned}&\varphi(d(fx, fy))\\ &\quad\le \max\{\phi(d(gx, gy)), \phi(d(gx, fx)),\\ &\qquad \phi(d(gy, fy)), \phi(d(gx, fy)), \psi(d(fx, gy))\}\\ &\qquad - \xi(\max\{d(gx, gy), d(gx, fx), d(gy, fy)\}),\end{aligned} \tag{38}$$

for all $x, y \in X$ such that $gx \sqsubseteq gy$, where $\varphi \in \Theta$, $\psi \in \Psi$, $\phi \in \Phi$ and $\xi \in \Xi$, and

$$\begin{gathered}\varphi(t) - \phi(t) + \xi(t) > 0 \quad \forall t > 0, \\ \varphi(t) = 0 \quad iff \ t = 0, \quad \phi(0) = \xi(0) = 0.\end{gathered} \tag{39}$$

Also, if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{40}$$

If there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X.

In the other research of this paper, we recall the Meir-Keeler-type contraction [10] and α-admissible mapping [11]. In 1969, Meir and Keeler [10] introduced the following notion of Meir-Keeler-type contraction in a metric space (X, d).

Definition 8. Let (X, d) be a metric space, $f : X \to X$. Then f is called a Meir-Keeler-type contraction whenever for each $\eta > 0$ there exists $\gamma > 0$ such that

$$\eta \le d(x, y) < \eta + \gamma \Longrightarrow d(fx, fy) < \eta. \tag{41}$$

And, the following definition was introduced in [11].

Definition 9. Let $f : X \to X$ be a self-mapping of a set X and $\alpha : X \times X \to \mathbb{R}^+$. Then f is called a α-admissible mapping if

$$x, y \in X, \qquad \alpha(x, y) \ge 1 \Longrightarrow \alpha(fx, fy) \ge 1. \tag{42}$$

We introduce the notion of α-g-admissible mapping, as follows.

Definition 10. Let $f, g : X \to X$ be a self-mapping of a set X and $\alpha : X \times X \to \mathbb{R}^+$. Then f is called a α-g-admissible mapping if

$$x, y \in X, \qquad \alpha(gx, gy) \ge 1 \Longrightarrow \alpha(fx, fy) \ge 1. \tag{43}$$

We give the following example to illustrate Definition 10.

Example 11. Let $X = \mathbb{R}^+$ and we define

$$\begin{gathered} g(x) = x + \frac{1}{2}, \qquad f(x) = g(x) + \frac{1}{x+1}, \\ \alpha(x, y) = x + y. \end{gathered} \tag{44}$$

Then f is a α-g-admissible mapping.

We now state the new notions of generalized Meir-Keeler-type ψ-contractions and generalized Meir-Keeler-type α-ψ-contractions in partially ordered complete metric spaces, as follows.

Definition 12. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \to X$. Then the pair (f, g) is called a generalized Meir-Keeler-type ψ-contraction whenever for each $\eta > 0$, there exists $\delta > 0$ such that

$$\begin{aligned} \eta \le \psi(&d(gx, gy), d(gx, fx), d(gy, fy), \\ &d(gx, fy), d(gy, fx)) \\ &< \eta + \delta \Longrightarrow d(fx, fy) < \eta, \end{aligned} \tag{45}$$

for all $x, y \in X$ with $gx \sqsubseteq gy$, where $\psi \in \Psi$.

Definition 13. Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, let $f, g : X \to X$, and $\alpha : X \times X \to \mathbb{R}^+$. Then (f, g) is called a generalized Meir-Keeler-type α-ψ-contraction if the following conditions hold:

(1) f is α-g-admissible;

(2) for each $\eta > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} \eta \le \psi(&d(gx, gy), d(gx, fx), d(gy, fy), \\ &d(gx, fy), d(gy, fx)) \\ &< \eta + \delta \Longrightarrow \alpha(fx, fx)\alpha(gy, gy)d(fx, fy) < \eta, \end{aligned} \tag{46}$$

for all $x, y \in X$ with $gx \sqsubseteq gy$, where $\psi \in \Psi$.

Remark 14. Note that if f is a generalized Meir-Keeler-type α-ψ-contraction, then we have that for all $x, y \in X$

$$\begin{aligned} &\alpha(fx, fx)\alpha(gy, gy)d(fx, fy) \\ &\quad \le \psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ &\qquad d(gx, fy), d(gy, fx)). \end{aligned} \tag{47}$$

Further, if

$$\begin{aligned} &\psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ &\quad d(gx, fy), d(gy, fx)) = 0, \end{aligned} \tag{48}$$

then $d(fx, fy) = 0$.

On the other hand, if

$$\begin{aligned} &\psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ &\quad d(gx, fy), d(gy, fx)) > 0, \end{aligned} \tag{49}$$

then

$$\begin{aligned} &\alpha(fx, fx)\alpha(gy, gy)d(fx, fy) \\ &\quad < \psi(d(gx, gy), d(gx, fx), d(gy, fy), \\ &\qquad d(gx, fy), d(gy, fx)). \end{aligned} \tag{50}$$

We now state our main result for the generalized Meir-Keeler-type α-ψ-contraction, as follows.

Theorem 15. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, let $\alpha : X \times X \to \mathbb{R}^+$ be continuous in each coordinate, and let $f, g : X \to X$ be such that $fX \subset gX$, f is g-nondecreasing, and gX is closed. Suppose the pair (f, g) is a generalized Meir-Keeler-type α-ψ-contraction and the following conditions hold.*

(i) *If any nondecreasing sequence $\{x_n\}$ in X converges to ν, then we assume that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{51}$$

(ii) *There exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$ and $\alpha(fx_0, fx_0) \ge 1$.*

(iii) *If* $\alpha(fx_n, fx_n) \geq 1$ *for all* $n \in \mathbb{N}$, *then* $\lim_{n\to\infty}\alpha(fx_n, fx_n) \geq 1$.

Then f *and* g *have a coincidence point in* X.

Proof. Since $fX \subset gX$ and by (ii), there exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$ and $\alpha(fx_0, fx_0) \geq 1$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Since f is g-nondecreasing, we have $fx_0 \sqsubseteq fx_1$. In this process, we construct the sequence $\{x_n\}$ recursively as

$$fx_n = gx_{n+1} \quad \forall n \in \mathbb{N}. \tag{52}$$

Thus, we also conclude that

$$\begin{aligned} gx_0 \sqsubseteq fx_0 = gx_1 \sqsubseteq fx_1 = gx_2 \sqsubseteq \cdots \sqsubseteq fx_{n-1} \\ = gx_n \sqsubseteq fx_n = gx_{n+1} \sqsubseteq \cdots . \end{aligned} \tag{53}$$

If any two consecutive terms in (53) are equal, then the conclusion of the theorem follows. So we may assume that

$$d(fx_{n-1}, fx_n) \neq 0, \quad \forall n \in \mathbb{N}. \tag{54}$$

On the other hand, since f is α-g-admissible and $\alpha(fx_0, fx_0) = \alpha(gx_1, gx_1) \geq 1$, we have

$$\alpha(fx_1, fx_1) = \alpha(gx_2, gx_2) \geq 1. \tag{55}$$

By continuing this process, we get

$$\alpha(fx_n, fx_n) = \alpha(gx_{n+1}, gx_{n+1}) \geq 1 \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{56}$$

By (53), (54), and (56), substituting $x = x_n$ and $y = x_{n+1}$ in (50), we have

$$\begin{aligned} &d(fx_n, fx_{n+1}) \\ &\quad \leq \alpha(fx_n, fx_n)\,\alpha(gx_{n+1}, gx_{n+1})\,d(fx_n, fx_{n+1}) \\ &\quad < \psi(d(gx_n, gx_{n+1}), d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1}), \\ &\qquad (gx_n, fx_{n+1}), d(gx_{n+1}, fx_n)) \\ &\quad = \psi(d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\qquad d(fx_{n-1}, fx_{n+1}), d(fx_n, fx_n)). \end{aligned} \tag{57}$$

If $d(fx_{n-1}, fx_n) \leq d(fx_n, fx_{n+1})$, then the inequality (57) becomes

$$\begin{aligned} &d(fx_n, fx_{n+1}) \\ &\quad < \psi(d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ &\qquad d(fx_{n-1}, fx_{n+1}), d(fx_n, fx_n)) \\ &\quad \leq \psi(d(fx_n, fx_{n+1}), d(fx_n, fx_{n+1}), d(fx_n, fx_{n+1}), \\ &\qquad 2d(fx_n, fx_{n+1}), 0) \\ &\quad \leq d(fx_n, fx_{n+1}), \end{aligned} \tag{58}$$

which implies a contradiction, and we get that $d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n)$.

From the argument above, we have that the sequence $\{d(fx_n, fx_{n+1})\}$ is decreasing, and it must converge to some $\eta \geq 0$, that is,

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = \eta. \tag{59}$$

It follow from that (57) and (59), we have

$$\begin{aligned} \lim_{n\to\infty} \psi(d(fx_{n-1}, fx_n), d(fx_{n-1}, fx_n), d(fx_n, fx_{n+1}), \\ d(fx_{n-1}, fx_{n+1}), d(fx_n, fx_n)) = \eta. \end{aligned} \tag{60}$$

Notice that $\eta = \inf\{d(fx_n, fx_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$. We claim that $\eta = 0$. Suppose, to the contrary, that $\eta > 0$. Since (f, g) is a generalized Meir-Keeler-type α-ψ-contraction, corresponding to η use, and taking into account the above inequality (60), there exist $\delta > 0$ and a natural number k such that

$$\begin{aligned} \eta &\leq \psi(d(fx_{k-1}, fx_k), d(fx_{k-1}, fx_k), \\ &\qquad d(fx_k, fx_{k+1}), d(fx_{k-1}, fx_{k+1}), \\ &\qquad d(fx_k, fx_k)) \\ &< \eta + \delta \Longrightarrow \alpha(fx_k, fx_k)\,\alpha(gx_{k+1}, gx_{k+1}) \\ &\qquad \times d(fx_k, fx_{k+1}) < \eta, \end{aligned} \tag{61}$$

which implies

$$\begin{aligned} d(fx_k, fx_{k+1}) &\leq \alpha(fx_k, fx_k) \\ &\quad \times \alpha(gx_{k+1}, gx_{k+1})\,d(fx_k, fx_{k+1}) < \eta. \end{aligned} \tag{62}$$

So we get a contradiction, since $\eta = \inf\{d(fx_n, fx_{n+1}) : n \in \mathbb{N} \cup \{0\}\}$. Thus we have that

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0. \tag{63}$$

We next claim that $\{fx_n\}$ is a Cauchy sequence, that is, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \geq n$, then $d(fx_p, fx_q) < \varepsilon$.

Suppose the above statement is false. Then there exists $\epsilon > 0$ such that for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \geq n$ satisfying

$$d(fx_{q_n}, fx_{p_n}) \geq \epsilon. \tag{64}$$

Further, corresponding to $q_n \geq n$, we can choose p_n in such a way that is it the smallest integer with $p_n > q_n \geq n$ and $d(fx_{q_n}, fx_{p_n}) \geq \epsilon$. Therefore $d(fx_{q_n}, fx_{p_n-1}) < \epsilon$. Now we have that for all $n \in \mathbb{N}$

$$\begin{aligned} \epsilon &\leq d(fx_{p_n}, fx_{q_n}) \leq d(fx_{p_n}, fx_{p_n-1}) \\ &\quad + d(fx_{p_n-1}, fx_{q_n}) \\ &< d(fx_{p_n}, fx_{p_n-1}) + \epsilon. \end{aligned} \tag{65}$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d\left(fx_{p_n}, fx_{q_n}\right) = \epsilon. \tag{66}$$

On the other hand, we have

$$\begin{aligned} d\left(fx_{p_n}, fx_{q_n}\right) &\le d\left(fx_{p_n}, fx_{p_n-1}\right) + d\left(fx_{p_n-1}, fx_{q_n-1}\right) \\ &\quad + d\left(fx_{q_n-1}, fx_{q_n}\right), \\ d\left(fx_{p_n-1}, fx_{q_n-1}\right) &\le d\left(fx_{p_n-1}, fx_{p_n}\right) + d\left(fx_{p_n}, fx_{q_n}\right) \\ &\quad + d\left(fx_{q_n}, fx_{q_n-1}\right). \end{aligned} \tag{67}$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d\left(fx_{p_n-1}, fx_{q_n-1}\right) = \epsilon. \tag{68}$$

By (53), we have that the elements gx_{p_n} and gx_{q_n} are comparable. Substituting $x = x_{p_n}$ and $y = x_{q_n}$ in (50), we have that for all $n \in \mathbb{N}$,

$$\begin{aligned} &d\left(fx_{p_n}, fx_{q_n}\right) \\ &\quad \le \alpha\left(fx_{p_n}, fx_{p_n}\right)\alpha\left(gx_{q_n}, gx_{q_n}\right) d\left(fx_{p_n}, fx_{q_n}\right) \\ &\quad < \psi\left(d\left(gx_{p_n}, gx_{q_n}\right), d\left(gx_{p_n}, fx_{p_n}\right), d\left(gx_{q_n}, fx_{q_n}\right),\right. \\ &\qquad \left. d\left(gx_{p_n}, fx_{q_n}\right), d\left(gx_{q_n}, fx_{p_n}\right)\right) \\ &\quad \le \psi\left(d\left(fx_{p_n-1}, fx_{q_n-1}\right), d\left(fx_{p_n-1}, fx_{p_n}\right),\right. \\ &\qquad d\left(fx_{q_n-1}, fx_{q_n}\right), d\left(fx_{p_n-1}, fx_{q_n}\right), \\ &\qquad \left. d\left(fx_{q_n-1}, fx_{p_n}\right)\right) \\ &\quad \le \psi\left(d\left(fx_{p_n-1}, fx_{q_n-1}\right), d\left(fx_{p_n-1}, fx_{p_n}\right),\right. \\ &\qquad d\left(fx_{q_n-1}, fx_{q_n}\right), d\left(fx_{p_n-1}, fx_{p_n}\right) + d\left(fx_{p_n}, fx_{q_n}\right), \\ &\qquad \left. d\left(fx_{q_n-1}, fx_{q_n}\right) + d\left(fx_{q_n}, fx_{p_n}\right)\right). \end{aligned} \tag{69}$$

Letting $n \to \infty$ in (69), then we get

$$\epsilon < \psi(\epsilon, 0, 0, \epsilon, \epsilon) \le \epsilon, \tag{70}$$

which implies a contradiction. Thus, $\{fx_n\}$ is a Cauchy sequence.

Since X is complete and gX is closed, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = g\nu. \tag{71}$$

Since α is continuous in each coordinate and by the condition (iii), we have

$$\alpha(g\nu, g\nu) = \lim_{n\to\infty} \alpha\left(fx_n, fx_n\right) \ge 1. \tag{72}$$

Later, we prove that ν is a coincidence point of f and g. From (53) and (71), we deduce that

$$gx_n \sqsubseteq g\nu, \quad \forall n \in \mathbb{N}. \tag{73}$$

By (72) and substituting $x = x_n$ and $y = \nu$ in (50), we have that

$$\begin{aligned} d\left(fx_n, f\nu\right) &\le \alpha\left(fx_n, fx_n\right)\alpha\left(g\nu, g\nu\right) d\left(fx_n, f\nu\right) \\ &< \psi\left(d\left(gx_n, g\nu\right), d\left(gx_n, fx_n\right),\right. \\ &\qquad \left. d\left(g\nu, f\nu\right), d\left(gx_n, f\nu\right), d\left(g\nu, fx_n\right)\right). \end{aligned} \tag{74}$$

Taking $n \to \infty$ in the above inequality, we have

$$\begin{aligned} d\left(g\nu, f\nu\right) &< \psi\left(d\left(g\nu, g\nu\right), d\left(g\nu, g\nu\right),\right. \\ &\qquad \left. d\left(g\nu, f\nu\right), d\left(g\nu, f\nu\right), d\left(g\nu, g\nu\right)\right) \\ &\le d\left(g\nu, f\nu\right). \end{aligned} \tag{75}$$

This implies that $g\nu = f\nu$. So we complete the proof. □

Apply Theorem 15, we are easy to get the following theorem.

Theorem 16. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f, g : X \to X$ be such that $fX \subset gX$, f is g-nondecreasing, and gX is closed. Suppose the pair (f, g) is a generalized Meir-Keeler-type ψ-contraction and the following conditions hold.*

(i) *If any nondecreasing sequence $\{x_n\}$ in X converges to ν, then we assume that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{76}$$

(ii) *There exists $x_0 \in X$ with $gx_0 \sqsubseteq fx_0$.*

Then f and g have a coincidence point in X.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 583813, 6 pages
http://dx.doi.org/10.1155/2013/583813

Research Article

Dual Quaternion Functions and Its Applications

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Received 7 June 2013; Accepted 23 July 2013

Academic Editor: Gue Lee

A dual quaternion is associated with two quaternions that have basis elements e_0, e_1, e_2, e_3, and ε. Dual numbers are often written in the form $z = \zeta + \varepsilon\zeta^*$, where ε is the dual identity and has the properties $\varepsilon^2 = 0$ $(\varepsilon \neq 0)$. We research the properties of some regular functions with values in dual quaternion and give applications of the extension problem for dual quaternion functions.

1. Introduction

Let $\mathscr{T}$ be the quaternion algebra constructed over a real anti-Euclidean quadratic four-dimensional vector space. Brackx [1], Deavours [2], and Sudbery [3] researched properties of theories of a quaternion function. Naser [4] gave properties of hyperholomorphic functions, and Nôno [5, 6] gave properties of various hyperholomorphic functions. They obtained basic theorems such as Cauchy Theorem, Morera's Theorem, and Cauchy Integral Formula with respect to Clifford analysis. Also, we [7–10] have investigated certain properties of hyperholomorphic functions and some regular functions in Clifford analysis.

A dual quaternion algebra $\mathscr{DH}$ is an ordered pair of quaternions and is constructed from real eight-dimensional vector spaces. A dual quaternion can be represented in the form $z = \zeta + \varepsilon\zeta^*$, where ζ and ζ^* are ordinary quaternions and ε is the dual symbol. The quaternion can represent only rotation, while the dual quaternion can do both rotation and translation. So, the dual quaternion is used in applications to 3D computer graphics, robotics, and computer vision. Kenwright [11] gave characteristics of dual quaternions; Pennestrì and Stefanelli [12] researched some properties using dual, and Kula and Yayli [13] investigated dual split quaternions and screw motion in Minkowski 3-space.

Son [14–16] gave the extension problem for the solutions of partial differential equations in $\mathbf{R}^n$ and it is generalized for the solutions of the Riesz system. In this paper, we give some regular functions with values in dual quaternions and research the extension problem for regular functions with values in dual quaternions. Also, we give some applications for these problems.

2. Preliminaries

We consider associated Pauli matrices

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$
$$e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1}$$

Then the associated Pauli matrices satisfy the triple rule as follows:

$$e_j^2 = -1, \quad e_j e_k + e_k e_j = -\delta_{jk} \quad (j, k = 1, 2, 3), \tag{2}$$

where δ_{jk} is Kronecker delta. And we let the dual symbol

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{3}$$

be a nonzero and satisfy $0\varepsilon = \varepsilon 0 = 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$, $\varepsilon^2 = 0$. The element e_0 is the identity and the element ε is the dual identity of $\mathscr{T}$.

The dual quaternion algebra $\mathscr{DH}$ is a noncommutative and associative one of the quaternion algebra. Then

$$\mathscr{DH} := \left\{ z = \sum_{j=0}^{3} \left(e_j x_j + e_j x_j^* \varepsilon \right) \mid x_j, x_j^* \in \mathbf{R} \quad (j = 0, 1, 2, 3) \right\} = \left\{ z = \zeta + \zeta^* \varepsilon \mid \zeta, \zeta^* \in \mathscr{T} \right\} \cong \mathscr{T} \times \mathscr{T}, \tag{4}$$

where $\zeta = \sum_{j=0}^{3} e_j x_j$, $\zeta^* = \sum_{j=0}^{3} e_j x_j^*$, and x_j^* is a dual quaternion component of x_j. We can identify $\mathscr{DH}$ with $\mathbf{C}^4$. The numbers of the skew field $\mathscr{DH}$ of dual quaternions are

$$\begin{aligned} z &= \sum_{j=0}^{3} \left(e_j x_j + e_j x_j^* \varepsilon \right) = \sum_{j=0}^{3} e_j \xi_j = \begin{pmatrix} \xi_0 + i\xi_1 & \xi_2 + i\xi_3 \\ -\xi_2 + i\xi_3 & \xi_0 - i\xi_1 \end{pmatrix}, \\ w &= \sum_{j=0}^{3} \left(e_j y_j + e_j y_j^* \varepsilon \right) = \sum_{j=0}^{3} e_j \eta_j = \begin{pmatrix} \eta_0 + i\eta_1 & \eta_2 + i\eta_3 \\ -\eta_2 + i\eta_3 & \eta_0 - i\eta_1 \end{pmatrix}, \end{aligned} \tag{5}$$

where $\xi_j = x_j + \varepsilon x_j^*$ and $\eta_j = y_j + \varepsilon y_j^*$ $(j = 0, 1, 2, 3)$. The dual quaternion conjugate z^* of z is

$$\begin{aligned} z^* &= \sum_{j=0}^{3} \left(\overline{e_j} x_j + \overline{e_j} x_j^* \varepsilon \right) = \sum_{j=0}^{3} \overline{e_j} \xi_j = \begin{pmatrix} \xi_0 - i\xi_1 & -\xi_2 - i\xi_3 \\ \xi_2 - i\xi_3 & \xi_0 + i\xi_1 \end{pmatrix}, \end{aligned} \tag{6}$$

where $\overline{e_j} = -e_j$. The absolute value $|z|$ of z and the inverse z^{-1} of z are, respectively,

$$\begin{aligned} |z| &= \sqrt{zz^*} = \sqrt{\sum_{j=0}^{3} \xi_j^2}, \\ z^{-1} &= \frac{z^*}{|z|^2} \quad (z \neq 0). \end{aligned} \tag{7}$$

Let Ω be an open subset of $\mathbf{C}^2 \times \mathbf{C}^2$ and let the dual quaternion function

$$f : \Omega \longrightarrow \mathscr{DH} \tag{8}$$

satisfy

$$\begin{aligned} z \in \Omega &\longmapsto f(z) = \sum_{j=0}^{3} e_j f_j(\zeta, \zeta^*) \\ &= \begin{pmatrix} f_0(\zeta, \zeta^*) + i f_1(\zeta, \zeta^*) & f_2(\zeta, \zeta^*) + i f_3(\zeta, \zeta^*) \\ -f_2(\zeta, \zeta^*) + i f_3(\zeta, \zeta^*) & f_0(\zeta, \zeta^*) - i f_1(\zeta, \zeta^*) \end{pmatrix} \in \mathscr{DH}, \end{aligned} \tag{9}$$

where $f_j(\zeta, \zeta^*) = u_j(\zeta, \zeta^*) + \varepsilon u_j^*(\zeta, \zeta^*)$ and u_j, u_j^* $(j = 0, 1, 2, 3)$ are real-valued functions.

We use the following dual quaternion differential operators in $\mathscr{DH}$:

$$D = \sum_{j=0}^{3} \overline{e_j} \frac{\partial}{\partial q_j} = \begin{pmatrix} \dfrac{\partial}{\partial q_0} - i\dfrac{\partial}{\partial q_1} & -\dfrac{\partial}{\partial q_2} - i\dfrac{\partial}{\partial q_3} \\ \dfrac{\partial}{\partial q_2} - i\dfrac{\partial}{\partial q_3} & \dfrac{\partial}{\partial q_0} + i\dfrac{\partial}{\partial q_1} \end{pmatrix}, \tag{10}$$

and the dual quaternion conjugates differential operators

$$D^* = \sum_{j=0}^{3} e_j \frac{\partial}{\partial q_j} = \begin{pmatrix} \dfrac{\partial}{\partial q_0} + i\dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} + i\dfrac{\partial}{\partial q_3} \\ -\dfrac{\partial}{\partial q_2} + i\dfrac{\partial}{\partial q_3} & \dfrac{\partial}{\partial q_0} - i\dfrac{\partial}{\partial q_1} \end{pmatrix}, \tag{11}$$

where $\partial/\partial q_j = \partial/\partial x_j + \varepsilon(\partial/\partial x_j^*)$ $(j = 0, 1, 2, 3)$ and $q_j = x_j + (1/\varepsilon)x_j^*$. Then we have

$$DD^* = \sum_{j=0}^{3} \frac{\partial^2}{\partial q_j^2} = \Delta_q. \tag{12}$$

Definition 1. Let Ω be an open set in $\mathbf{C}^2 \times \mathbf{C}^2$. A function $f(z)$ is said to be ε-regular in Ω if the following two conditions are satisfied:

(a) f_j $(j = 0, 1, 2, 3)$ are continuously differential functions in Ω,

(b) $D^* f(z) = 0$ in Ω.

Definition 2. Let Ω be an open set in $\mathbf{C}^2\times\mathbf{C}^2$. A function $f(z)$ is said to be ε-biregular in Ω if the following two conditions are satisfied:

(a) f_j $(j = 0,1,2,3)$ are continuously differential functions in Ω,

(b) $D^* f(z) = 0$ and $f(z)D^* = 0$ in Ω.

The operators act for a function $f(z)$ on $\mathscr{D}\mathscr{H}$ as follows:

$$D^* f(z) = \left(\sum_{j=0}^{3} e_j \frac{\partial}{\partial q_j}\right)\left(\sum_{j=0}^{3} e_j f_j\right) = \begin{pmatrix} D_1^* & D_2^* \\ D_3^* & D_4^* \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned}
D_1^* &= \left(\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \frac{\partial f_2}{\partial q_2} - \frac{\partial f_3}{\partial q_3}\right) + i\left(\frac{\partial f_1}{\partial q_0} + \frac{\partial f_0}{\partial q_1} + \frac{\partial f_3}{\partial q_2} - \frac{\partial f_2}{\partial q_3}\right),\\
D_2^* &= \left(\frac{\partial f_2}{\partial q_0} - \frac{\partial f_3}{\partial q_1} + \frac{\partial f_0}{\partial q_2} + \frac{\partial f_1}{\partial q_3}\right) + i\left(\frac{\partial f_3}{\partial q_0} + \frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} + \frac{\partial f_0}{\partial q_3}\right),\\
D_3^* &= \left(-\frac{\partial f_2}{\partial q_0} + \frac{\partial f_3}{\partial q_1} - \frac{\partial f_0}{\partial q_2} - \frac{\partial f_1}{\partial q_3}\right) + i\left(\frac{\partial f_3}{\partial q_0} + \frac{\partial f_2}{\partial q_1} - \frac{\partial f_1}{\partial q_2} + \frac{\partial f_0}{\partial q_3}\right),\\
D_4^* &= \left(\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \frac{\partial f_2}{\partial q_2} - \frac{\partial f_3}{\partial q_3}\right) + i\left(-\frac{\partial f_1}{\partial q_0} - \frac{\partial f_0}{\partial q_1} - \frac{\partial f_3}{\partial q_2} + \frac{\partial f_2}{\partial q_3}\right),
\end{aligned} \tag{14}$$

$$f(z)D^* = \left(\sum_{j=0}^{3} e_j f_j\right)\left(\sum_{j=0}^{3} e_j \frac{\partial}{\partial q_j}\right) = \begin{pmatrix} D_5^* & D_6^* \\ D_7^* & D_8^* \end{pmatrix},$$

where

$$\begin{aligned}
D_5^* &= \left(\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \frac{\partial f_2}{\partial q_2} - \frac{\partial f_3}{\partial q_3}\right) + i\left(\frac{\partial f_1}{\partial q_0} + \frac{\partial f_0}{\partial q_1} - \frac{\partial f_3}{\partial q_2} + \frac{\partial f_2}{\partial q_3}\right),\\
D_6^* &= \left(\frac{\partial f_2}{\partial q_0} + \frac{\partial f_3}{\partial q_1} + \frac{\partial f_0}{\partial q_2} - \frac{\partial f_1}{\partial q_3}\right) + i\left(\frac{\partial f_3}{\partial q_0} - \frac{\partial f_2}{\partial q_1} + \frac{\partial f_1}{\partial q_2} + \frac{\partial f_0}{\partial q_3}\right),\\
D_7^* &= \left(-\frac{\partial f_2}{\partial q_0} - \frac{\partial f_3}{\partial q_1} - \frac{\partial f_0}{\partial q_2} + \frac{\partial f_1}{\partial q_3}\right) + i\left(\frac{\partial f_3}{\partial q_0} - \frac{\partial f_2}{\partial q_1} + \frac{\partial f_1}{\partial q_2} + \frac{\partial f_0}{\partial q_3}\right),\\
D_8^* &= \left(\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \frac{\partial f_2}{\partial q_2} - \frac{\partial f_3}{\partial q_3}\right) + i\left(-\frac{\partial f_1}{\partial q_0} - \frac{\partial f_0}{\partial q_1} + \frac{\partial f_3}{\partial q_2} - \frac{\partial f_2}{\partial q_3}\right).
\end{aligned} \tag{15}$$

Remark 3. Equations (b) of Definition 2 are equivalent to the following system:

$$\begin{aligned}
&\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \frac{\partial f_2}{\partial q_2} - \frac{\partial f_3}{\partial q_3} = 0,\\
&\frac{\partial f_1}{\partial q_0} = -\frac{\partial f_0}{\partial q_1}, \qquad \frac{\partial f_3}{\partial q_2} = \frac{\partial f_2}{\partial q_3},\\
&\frac{\partial f_2}{\partial q_0} = -\frac{\partial f_0}{\partial q_2}, \qquad \frac{\partial f_3}{\partial q_1} = \frac{\partial f_1}{\partial q_3},\\
&\frac{\partial f_3}{\partial q_0} = -\frac{\partial f_0}{\partial q_3}, \qquad \frac{\partial f_2}{\partial q_1} = \frac{\partial f_1}{\partial q_2}.
\end{aligned} \tag{16}$$

3. Extension Problem for the Dual Quaternion Functions

Definition 4. Let Ω be a domain in $\mathbf{C}^n \times \mathbf{C}^n$ $(n \geq 1)$. A function $f(z) = \sum_{j=0}^{n-1} e_j f_j(z)$ is said to be regular in Ω if

$$\widetilde{D} f(z) = 0, \tag{17}$$

where $\widetilde{D} = \sum_{j=0}^{n-1} e_j(\partial/\partial q_j)$ on Ω.

Theorem 5 (uniqueness theorem for regular functions). *If two regular functions f and g in a domain $\Omega \subset \mathbf{C}^n \times \mathbf{C}^n$ $(n \geq 1)$ and coincide on a nonempty open set $G \subset \Omega$, then $f \equiv g$ in Ω.*

Rocha-Chávez et al. [17] obtained the following remark.

Remark 6. For a regular function f in the domain $\Omega \subset \mathbf{C}^n \times \mathbf{C}^n$ $(n \geq 1)$ and a bounded domain G with smooth boundary bG, such that $\overline{G} \subset \Omega$, one has

$$f(z) = \frac{1}{a_n}\int_{bG} \frac{\overline{\zeta} - \overline{z}}{|\zeta - z|^n} d\sigma_\zeta f(\zeta), \quad z \in G, \tag{18}$$

with a_n the area of the unit sphere in $\mathbf{C}^n$ and $d\sigma_\zeta$ a Clifford algebra valued differential form of order $n-1$.

Let $\Omega = \Omega_1 \times \Omega_2$ be a domain in $\mathbf{C}^4 \times \mathbf{C}^{n-4}$ $(n \geq 5)$ where Ω_1 is a domain in $\mathbf{C}^4(\xi_0, \xi_1, \xi_2, \xi_3)$ and Ω_2 is a domain in $\mathbf{C}^{n-4}(\xi_4, \xi_5, \ldots, \xi_{n-1})$. Let U be an open connected neighborhood of $b\Omega$.

Proposition 7. *If $f(z)$ is a regular function in $U \subset \mathbf{C}^4 \times \mathbf{C}^{n-4}$ $(n \geq 5)$ which satisfies the condition*

$$D^* f(z) = 0, \tag{19}$$

then $f(z)$ can be extended continuously to a regular function in the whole domain of Ω. That is, there exists a regular function $\widetilde{f}(z)$ in Ω such that $\widetilde{f}(z) = f(z)$ in U.

Proof. By Remark 6 and the proof of the main extension theorem of Son [15], it is proved. □

We consider the system of an extension of the system (16)

$$\begin{gathered}\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \cdots - \frac{\partial f_{n-1}}{\partial q_{n-1}} = 0, \\ \frac{\partial f_j}{\partial q_0} = -\frac{\partial f_0}{\partial q_j}, \qquad \frac{\partial f_l}{\partial q_k} = \frac{\partial f_k}{\partial q_l} \\ (j, k, l = 1, \ldots, n-1),\end{gathered} \tag{20}$$

where $f(z) = \{f_0(z), f_1(z), \ldots, f_{n-1}(z)\}$ are the unknown functions.

By using the same technique as in Son [15], we have the following theorem.

Theorem 8. *Let $f(z) = \{f_0(z), f_1(z), \ldots, f_{n-1}(z)\}$ be a given $\mathscr{C}^2$-solution of the system (20) in $U \subset \mathbf{C}^4 \times \mathbf{C}^{n-4}$ $(n \geq 5)$, which satisfies the system (16) in Remark 3. If the functions $f_4(z), f_5(z), \ldots, f_{n-1}(z)$ depend only on ξ_0, ξ_1, ξ_2, ξ_3, and U is an open neighborhood of the boundary of the domain $\Omega \subset \mathbf{C}^4 \times \mathbf{C}^{n-4}$ $(n \geq 5)$, then $f(z)$ can be extended to a solution of the system (20) in the whole domain of Ω.*

Proof. Let the function $\phi(z)$ with values in Clifford algebra be defined by

$$\phi(z) = \sum_{j=0}^{n-1} f_j e_j. \tag{21}$$

Then we have $\widetilde{D}\phi(z) = 0$ and $D^*\phi(z) = 0$. By Proposition 7, the result follows. □

We consider the following system:

$$\sum_{j=0}^{n-1} \sum_{k=0}^{m-1} H_{jk}^{(P)}(z) \frac{\partial f_j}{\partial q_k} = 0 \quad (P = 1, \ldots, p), \tag{22}$$

where $H_{jk}^{(P)}(z)$ are holomorphic functions and $f = \{f_0, f_1, \ldots, f_{n-1}\}$ are the unknown functions of the system (22).

Let Ω be an open set in $\mathbf{C}^n$ $(n \geq 2)$ and let K be a compact subset of Ω such that $\Omega \setminus K$ is simply connected. We consider the system

$$\sum_{j=0}^{n-1} \sum_{k=0}^{m-1} H_{jk}^{(P)}(z) \frac{\partial \omega_j}{\partial q_k} = \varphi_{(z)}^{(P)} \quad (P = 1, \ldots, p), \tag{23}$$

where $\varphi_{(z)}^{(P)} \in \mathscr{C}^\infty(\Omega)$.

By using the same technique as in Son [16], we have the following theorem and example.

Theorem 9. *If every $\varphi = \{\varphi^{(1)}, \ldots, \varphi^{(p)}\} \in \mathscr{C}^\infty(\Omega)$ in the inhomogeneous system (23) has a solution*

$$\omega = \{\omega_0, \ldots, \omega_{n-1}\} \in \mathscr{C}^\infty(\Omega), \tag{24}$$

then every solution f of (22) given in $\Omega \setminus K$ can be extended to a solution of this system (23) in the whole domain of Ω.

Proof. This result follows from the theorem in [18, page 30]. □

Example 10. We give an application of Theorem 9 to the system (20) and recall the system (16) as follows:

$$\begin{gathered}\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \cdots - \frac{\partial f_{n-1}}{\partial q_{n-1}} = 0, \\ \frac{\partial f_j}{\partial q_0} + \frac{\partial f_0}{\partial q_j} = 0, \qquad \frac{\partial f_l}{\partial q_k} - \frac{\partial f_k}{\partial q_l} = 0 \\ (j, k, l = 1, \ldots, n-1).\end{gathered} \tag{25}$$

Assume that

$$\frac{\partial f_{n-1}}{\partial q_{n-1}} = 0. \tag{26}$$

Then we have the following form:

$$\begin{gathered}\frac{\partial f_0}{\partial q_0} - \frac{\partial f_1}{\partial q_1} - \cdots - \frac{\partial f_{n-2}}{\partial q_{n-2}} = 0, \\ \frac{\partial f_j}{\partial q_0} + \frac{\partial f_0}{\partial q_j} = 0, \qquad \frac{\partial f_l}{\partial q_k} - \frac{\partial f_k}{\partial q_l} = 0 \\ (j, k, l = 1, \ldots, n-2).\end{gathered} \tag{27}$$

The corresponding inhomogeneous system of (27) has the following form:

$$\begin{gathered}\frac{\partial \omega_0}{\partial q_0} - \frac{\partial \omega_1}{\partial q_1} - \cdots - \frac{\partial \omega_{n-2}}{\partial q_{n-2}} = \varphi, \\ \frac{\partial \omega_j}{\partial q_0} + \frac{\partial \omega_0}{\partial q_j} = \varphi_{j,0}, \\ \frac{\partial \omega_j}{\partial q_k} - \frac{\partial \omega_k}{\partial q_j} = \varphi_{j,k}, \\ \frac{\partial \omega_k}{\partial q_0} + \frac{\partial \omega_0}{\partial q_k} = \varphi_{k,0}, \\ \frac{\partial \omega_j}{\partial q_l} - \frac{\partial \omega_l}{\partial q_j} = \varphi_{j,l}, \\ \frac{\partial \omega_k}{\partial q_l} - \frac{\partial \omega_l}{\partial q_k} = \varphi_{k,l} \\ (j, k, l = 1, \ldots, n-2),\end{gathered} \tag{28}$$

where $\varphi, \varphi_{j,0}, \varphi_{j,k}, \varphi_{k,0}, \varphi_{j,l}, \varphi_{k,l} \in \mathscr{C}^{\infty}(\Omega)$. Then we can get the system from (28) as

$$\frac{\partial \omega_k}{\partial q_j} = \frac{\partial \omega_j}{\partial q_k} - \varphi_{j,k}, \qquad \frac{\partial \omega_k}{\partial q_0} = -\frac{\partial \omega_0}{\partial q_k} + \varphi_{k,0},$$
$$\frac{\partial \omega_l}{\partial q_j} = \frac{\partial \omega_j}{\partial q_l} - \varphi_{j,l}, \qquad \frac{\partial \omega_l}{\partial q_k} = \frac{\partial \omega_k}{\partial q_l} - \varphi_{k,l}, \tag{29}$$

$$\frac{\partial^2 \omega_k}{\partial q_j \partial q_0} + \frac{\partial \varphi_{j,k}}{\partial q_0} = -\frac{\partial^2 \omega_0}{\partial q_j \partial q_k} + \frac{\partial \varphi_{j,0}}{\partial q_k},$$
$$\frac{\partial^2 \omega_j}{\partial q_l \partial q_k} - \frac{\partial \varphi_{j,l}}{\partial q_k} = \frac{\partial^2 \omega_k}{\partial q_j \partial q_l} - \frac{\partial \varphi_{k,l}}{\partial q_j}. \tag{30}$$

From (29), we have

$$\frac{\partial}{\partial q_j}\left(-\frac{\partial \omega_0}{\partial q_k} + \varphi_{k,0}\right) + \frac{\partial \varphi_{j,k}}{\partial q_0} = -\frac{\partial^2 \omega_0}{\partial q_j \partial q_k} + \frac{\partial \varphi_{j,0}}{\partial q_k},$$
$$\frac{\partial}{\partial q_l}\left(\frac{\partial \omega_k}{\partial q_j} + \varphi_{j,k}\right) - \frac{\partial \varphi_{j,l}}{\partial q_k} = \frac{\partial^2 \omega_k}{\partial q_j \partial q_l} - \frac{\partial \varphi_{k,l}}{\partial q_j}. \tag{31}$$

Thus, we can have the system

$$\frac{\partial \varphi_{k,0}}{\partial q_j} + \frac{\partial \varphi_{j,k}}{\partial q_0} = \frac{\partial \varphi_{j,0}}{\partial q_k}, \qquad \frac{\partial \varphi_{j,k}}{\partial q_l} - \frac{\partial \varphi_{j,l}}{\partial q_k} = -\frac{\partial \varphi_{k,l}}{\partial q_j}. \tag{32}$$

We let

$$\widetilde{\omega} = \sum_{j=0}^{n-2} e_j \omega_j, \qquad \widetilde{z} = \sum_{j=0}^{n-2} \overline{e_j} \xi_j. \tag{33}$$

The system (29) has the form

$$\widetilde{D}\widetilde{\omega} = F(\widetilde{z}, \xi_{n-1}), \qquad \frac{\partial \omega_0}{\partial q_k} + \frac{\partial \omega_k}{\partial q_0} = \varphi_{k,0}, \tag{34}$$

$$\frac{\partial \omega_j}{\partial q_l} - \frac{\partial \omega_l}{\partial q_j} = \varphi_{j,l}, \qquad \frac{\partial \omega_k}{\partial q_l} - \frac{\partial \omega_l}{\partial q_k} = \varphi_{k,l}, \tag{35}$$

where

$$F(\widetilde{z}, \xi_{n-1}) = \varphi + \sum_{j=1}^{n-2} e_j \varphi_{j,0} + \left(\frac{n}{2} - 1\right) \sum_{\alpha=1}^{n-2} \sum_{\substack{j,k=1 \\ j,k \neq \alpha}}^{n-2} e_\alpha \varphi_{j,k}. \tag{36}$$

We put

$$F_1 := \varphi + \sum_{j=1}^{n-2} e_j \varphi_{j,0},$$
$$F_2 := \left(\frac{n}{2} - 1\right) \sum_{\alpha=1}^{n-2} \sum_{\substack{j,k=1 \\ j,k \neq \alpha}}^{n-2} e_\alpha \varphi_{j,k}. \tag{37}$$

From the system (28), we have

$$\omega_0 = \int \left(\varphi + \frac{\partial \omega_1}{\partial q_1} + \frac{\partial \omega_2}{\partial q_2} + \cdots + \frac{\partial \omega_{n-2}}{\partial q_{n-2}}\right) dq_0,$$
$$\omega_k = \int \left(\varphi_{k,0} - \frac{\partial \omega_0}{\partial q_k}\right) dq_0 \quad (k = 1, \ldots, n-2). \tag{38}$$

By the systems (29) and (38), we get

$$\begin{aligned} \widetilde{\omega} &= \int F\, dq_0 + \int \sum_{j=1}^{n-2} \frac{\partial \omega_j}{\partial q_j}\, dq_0 \\ &\quad - \int \sum_{j=1}^{n-2} e_j \frac{\partial \omega_0}{\partial q_j}\, dq_0 - \int F_2 dq_0 \\ &= \int F\, dq_0 - \int \sum_{j=1}^{n-2} e_j \frac{\partial \widetilde{\omega}}{\partial q_j}\, dq_0. \end{aligned} \tag{39}$$

By Cauchy Integral Formula,

$$F(Z, \xi_{n-1}) = \frac{1}{(2\pi i)^n} \int \frac{F(Z, \xi_{n-1})}{Z - \zeta} dZ, \tag{40}$$

where $Z = \sum_{k=0}^{n-2} e_k Z_k$. Thus we have

$$\widetilde{\omega} = \frac{1}{(2\pi i)^n} \iint \frac{F(Z, \xi_{n-1})}{Z - \zeta} dZ_0 dZ, \tag{41}$$

and $\widetilde{\omega} = 0$ when $|\xi_{n-1}|$ is large enough. Also, $\{\widetilde{\omega}, 0\} = \{\omega_0, \omega_1, \ldots, \omega_{n-2}, 0\}$ is a solution of the system (22) outside a compact set K of Ω. From Theorem 5, it follows that $\widetilde{\omega} = 0$ is outside the compact set K of Ω or $\omega \in \mathscr{C}^{\infty}(\Omega)$. That is, $\omega_0, \omega_1, \ldots, \omega_{n-2} \in \mathscr{C}^{\infty}(\Omega)$. It follows from the system (35) that

$$\omega_{n-1} = \int \left(\frac{\partial \omega_{m-1}}{\partial q_{n-1}} - \varphi_{m-1,n-1}\right) dq_{m-1}. \tag{42}$$

Since $\omega_{m-1} \in \mathscr{C}^{\infty}(\Omega)$, we get $\partial \omega_{m-1}/\partial q_{n-1} \in \mathscr{C}^{\infty}(\Omega)$. We can choose $\omega_{n-1} \in \mathscr{C}^{\infty}(\Omega)$ which satisfy the system (35). From (42), we find that

$$\begin{aligned} &\frac{\partial \omega_{n-1}}{\partial q_P} \\ &= \int \left(\frac{\partial^2 \omega_{m-1}}{\partial q_P \partial q_{n-1}} - \frac{\partial \varphi_{m-1,n-1}}{\partial q_P}\right) dq_{m-1} \\ &= \int \left(\frac{\partial}{\partial q_{n-1}}\left(\frac{\partial \omega_P}{\partial q_{m-1}}\right) + \frac{\partial \varphi_{m-1,P}}{\partial q_{n-1}} - \frac{\partial \varphi_{m-1,n-1}}{\partial q_P}\right) dq_{m-1} \\ &= \int \left(\frac{\partial}{\partial q_{n-1}}\left(\frac{\partial \omega_P}{\partial q_{m-1}}\right) - \frac{\partial \varphi_{P,n-1}}{\partial q_{m-1}}\right) dq_{m-1} \\ &= \frac{\partial \omega_P}{\partial q_{n-1}} - \varphi_{P,n-1}. \end{aligned} \tag{43}$$

Hence, ω_{n-1} satisfies the system (29). This means that the function $\omega = (\widetilde{\omega}, \omega_{n-1}) = (\omega_0, \ldots, \omega_{n-1})$ is a solution of the system (29) and $\omega \in \mathscr{C}^{\infty}(\Omega)$.

Acknowledgment

The second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT and Future Planning (2013R1A1A2008978).

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 382620, 12 pages
http://dx.doi.org/10.1155/2013/382620

Research Article

An Equilibrium Chance-Constrained Multiobjective Programming Model with Birandom Parameters and Its Application to Inventory Problem

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Received 14 May 2013; Revised 25 July 2013; Accepted 25 July 2013

Academic Editor: Wei-Shih Du

An equilibrium chance-constrained multiobjective programming model with birandom parameters is proposed. A type of linear model is converted into its crisp equivalent model. Then a birandom simulation technique is developed to tackle the general birandom objective functions and birandom constraints. By embedding the birandom simulation technique, a modified genetic algorithm is designed to solve the equilibrium chance-constrained multiobjective programming model. We apply the proposed model and algorithm to a real-world inventory problem and show the effectiveness of the model and the solution method.

1. Introduction

As a key issue in supply chain management, inventory management plays an important role in controlling operation cost and improving management efficiency. As a result, inventory problems have attracted many researchers from various fields. Classical inventory problem research discussed single-item cases [1–5]. However, multiitem inventory problems are more common in the real-world. In fact, for many enterprises, especially supermarkets, it is necessary to prepare hundreds of goods to meet the needs of different consumers. Therefore, multiitem inventory problem research has become more attractive for researchers [6–9].

Apart from the multiitem inventory, uncertainty is another characteristic of inventory problems. For example, the price of a product usually varies due to the market fluctuation. One kind of common uncertainty is randomness. Many researchers have studied random inventory models. Baker and Ehrhardt analyzed a periodic-review, random-demand inventory model under the assumption that replenishment quantities are random fractions of the amounts ordered [10]. Bera et al. dealt with a multiitem mixture inventory model in which both demand and lead time are random [8]. Fuzziness is another type of uncertainty in inventory problems. Wang et al. [11] developed a novel joint replenishment problem model with fuzzy minor replenishment cost and fuzzy inventory holding cost. Roy et al. [12] considered an inventory model for a deteriorating item (seasonal product) with linearly displayed stock dependent demand in an imprecise environment (involving both fuzzy and random parameters) under inflation and the time value of money. M. K. Maiti and M. Maiti [13] proposed a multiitem inventory model with advertising costs, price, and displayed inventory level-dependent demand in a fuzzy environment (purchase cost, investment amount, and storehouse capacity were considered imprecise). Besides these, twofold uncertain inventory models have also been studied as more complex; imprecise information is considered. Dutta et al. [14] presented a single-period inventory problem in an imprecise and uncertain mixed environment, in which the demand is assumed as a fuzzy random variable. Xu and Zhao [15, 16] formulated inventory models under fuzzy rough environments.

However, an inventory model with twofold random phenomenon is seldom discussed. In reality, inventory problems

in the real-world may be subjected to twofold randomness with incomplete or uncertain information. Consequently, developing an inventory strategy in a more complete stochastic environment takes place. As a case in point, it is widely accepted that the price of a product is a normal distributed variable, denoted by $N(\mu, \sigma^2)$ from the viewpoint of probability theory, but the values of μ and σ may be still uncertain variables. If there is statistical information about μ and σ, it is possible to specify realistic distributions for μ and σ by utilizing statistical methods. When the values of μ and σ are provided as random variables, the price then is not a conventional random variable but a so-called birandom variable. A birandom variable, which plays a role analogous to a random variable in probability theory, is appropriate to describe this kind of twofold randomness. In this paper, we discuss a class of multiobjective programming models with birandom parameters. Chance-constrained programming proposed by Charnes and Cooper [17] is an effective technique to deal with uncertain optimization problem [18]. By using the equilibrium chance-constrained programming technique, the initial model is meaningful mathematically.

The rest of the paper is organized as follows. Section 2 develops a general equilibrium chance-constrained programming model with birandom parameters. A linear model is converted into its crisp equivalent model. In order solve a general model, the birandom simulation-based genetic algorithm is deigned in Section 3. Section 4 applies the theoretical results into a real-world inventory problem. Further discussions in Section 5 illustrate the effectiveness of the proposed model and algorithm. Finally, concluding remarks are outlined in Section 6.

2. Equilibrium Chance-Constrained Multiobjective Programming Model with Birandom Parameters

In order construct a general equilibrium chance-constrained multiobjective programming model with birandom parameters, we first state some basic concepts and theorems on birandom theory which is presented firstly.

2.1. Birandom Variable and Equilibrium Chance. Birandom variable, which is proposed by Peng and Liu [19], is a mathematical tool to describe twofold random phenomena. An n-dimensional birandom vector ξ is a map from the probability space $(\Omega, \mathscr{A}, \Pr)$ to a collection of n-dimensional random vectors such that $\Pr\{\xi(\omega) \in \mathscr{B}\}$ is a measurable function with respect to ω for any Borel set $\mathscr{B}$ of the real space $\mathscr{R}^n$. In particular, ξ is called a birandom variable as $n = 1$.

Example 1. A birandom variable ξ is said to be uniform, if for each ω, $\xi(\omega)$ is a random variable with uniform distribution, denoted by $\mathscr{U}[a(\omega), b(\omega)]$, where $a(\omega)$ and $b(\omega)$ are random variables defined on the probability space $(\Omega, \mathscr{A}, \Pr)$.

Example 2. A birandom variable ξ is said to be normal, if for each ω, $\xi(\omega)$ is a random variable with normal distribution, denoted by $\mathscr{N}(\mu(\omega), \sigma^2(\omega))$, where $\mu(\omega)$ and $\sigma(\omega)$ are random variables defined on the probability space $(\Omega, \mathscr{A}, \Pr)$.

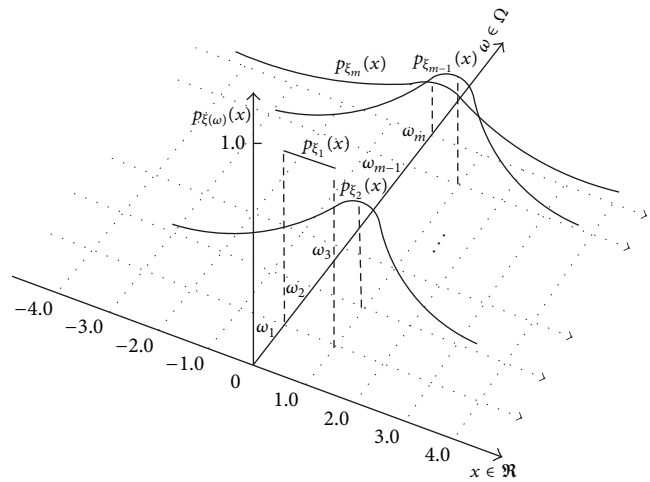

Figure 1: A birandom variable.

Example 3. A birandom variable ξ is said to be exponential, if for each ω, $\xi(\omega)$ is a random variable with exponential distribution, denoted by $\exp(\lambda(\omega))$, where $\lambda(\omega)$ are random variables defined on the probability space $(\Omega, \mathscr{A}, \Pr)$.

Example 4. Let $(\Omega, \mathscr{A}, \Pr)$ be a probability space. If $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$,

$$\xi(\omega) = \begin{cases} \xi_1, & \omega = \omega_1 \\ \xi_2, & \omega = \omega_2 \\ \vdots, & \vdots, \\ \xi_n, & \omega = \omega_n, \end{cases} \tag{1}$$

where $\xi_1, \xi_2, \ldots, \xi_n$ are random variables with density function $p_1(x), p_1(x), \ldots, p_n(x)$, respectively. Birandom variable ξ is illustrated by Figure 1.

By birandom event we mean that $g(\xi) \leq 0$. In order to compare the degrees of occurrence of two birandom events, quantitative measures of the chance of a birandom event are necessary. In the literature, the first attempt to develop the definition of the chance of a birandom event is primitive chance, which is a function from $[0, 1]$ to $[0, 1]$.

Definition 5 (see [19]). Let $\xi = (\xi_1, \xi_2, \ldots \xi_n)$ be a birandom vector defined on Ω, and $g : \mathfrak{R}^n \mapsto \mathfrak{R}$ is Borel measurable function. Then the primitive chance of a birandom event characterized by $g(\xi) \leq 0$ is a function from $[0, 1]$ to $[0, 1]$, defined as

$$\begin{aligned} &\mathrm{Ch}\{g(\xi) \leq 0\}(\alpha) \\ &\quad = \sup_{\alpha \in [0,1]} \left\{\beta \mid \Pr\left\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \leq 0\} \geq \beta\right\} \geq \alpha\right\}, \end{aligned} \tag{2}$$

where α is a prescribed probability level. The value of primitive chance at α is called α-chance.

Remark 6. It should be noted that the symbol Pr appears twice in the right side of (2). In fact, they represent different meanings. In other words, the overloading allows us to use the same symbol Pr for different probability measures, because we can deduce the exact meaning in the context.

In the case of fuzzy random programming problems, the equilibrium chance of a fuzzy random event was introduced to measure the degree of the occurrence of a fuzzy random event [20]. Rather than a function, the equilibrium chance is a scalar value, like the probability of a random event and the possibility of a fuzzy event. Thus it is easy for the decision maker to rank the decisions via equilibrium chance using the natural order of real numbers, rather than requiring a preference order from the decision maker. Motivated by the idea, we introduce the equilibrium chance of a birandom event as follows.

Definition 7. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a birandom vector, and $g : \mathfrak{R}^n \mapsto \mathfrak{R}$ is a Borel measurable function. Then the equilibrium chance of birandom event $g(\xi) \le 0$ is defined as

$$\begin{aligned} &\mathrm{Ch}^e \{g(\xi) \le 0\} \\ &\quad = \sup_{\alpha \in [0,1]} \{\alpha \wedge \Pr\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \ge \alpha\}\}. \end{aligned} \tag{3}$$

Remark 8. Peng and Liu [19] introduced the equilibrium chance of a birandom event as

$$\mathrm{Ch}^e \{g(\xi) \le 0\} = \sup_{\alpha \in [0,1]} \{\alpha \mid \mathrm{Ch}\{g(\xi) \le 0\}(\alpha) \ge \alpha\}, \tag{4}$$

where Ch is the primitive chance. It is easy to verify the forms (3) and (4) are equivalent.

Remark 9. If ξ degenerates to a random vector, $\Pr\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \ge \alpha\}$ implies $\Pr\{g(\xi(\omega)) \le 0\} \ge \alpha$, and then (3) is equivalent to $\Pr\{g(\xi(\omega)) \le 0\}$, which is the probability measure.

For any $\alpha \in [0,1]$, denote

$$\begin{aligned} F(\alpha) &= \Pr\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \ge \alpha\}, \\ F(\alpha+) &= \Pr\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} > \alpha\}. \end{aligned} \tag{5}$$

It follows from the definition of birandom variable that $\Pr\{(\xi(\omega)) \le 0\}$ is a random variable. Then $F(\alpha)$ is a left-continuous and nonincreasing function with respect to α, and for any given α, $F(\alpha)$ represents the following random event occurring:

$$\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \ge \alpha\}. \tag{6}$$

Therefore, the equilibrium chances defined above measure the twofold maximum probability simultaneously.

Generally, if

$$F(\alpha) \le \alpha \le F(\alpha+), \tag{7}$$

then α is just the value of the equilibrium chance. As $\Pr\{g(\xi(\omega)) \le 0\}$ is a continuous random variable, the left side and right side of relation (7) are equal. Thus the equilibrium chance is available at the fixed point of F; that is, $F(\alpha) = \alpha$.

It follows from the definitions of equilibrium chance and primitive chance of a birandom event that they have the following differences.

(i) The equilibrium chance of a birandom chance of a birandom event is a scalar value, just like the probability of a random event and the possibility of a fuzzy event, while the primitive chance of a birandom event is a function from $[0,1]$ to $[0,1]$.

(ii) The equilibrium chance measures the twofold probabilities at the same time, while the value of the primitive chance at α measures the single maximum probability of a birandom event occurring under the given value of the other probability.

However, the connections of the equilibrium chance and the primitive chance can be summarized as follows.

(i) Equilibrium chance (3) can be represented as

$$\mathrm{Ch}^e \{g(\xi) \le 0\} = \sup_{\alpha \in [0,1]} \{\alpha \wedge \mathrm{Ch}\{g_r(\xi(\omega)) \le 0\}(\alpha)\}, \tag{8}$$

where $\mathrm{Ch}\{g(\xi(\omega)) \le 0\}(\alpha)$ is the value of primitive chance at α.

(ii) The primitive chance of a birandom event is the pseudo-inverse function of the function $F(\alpha)$.

In order to generate the notation of the equilibrium chance of a birandom event defined by (3), we give another equivalent form as follows:

$$\begin{aligned} &\mathrm{Ch}^e \{g(\xi) \le 0\} \\ &\quad = \sup_{(\alpha,\beta) \in [0,1]^2} \{\alpha \wedge \beta \mid \Pr\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \\ &\qquad \ge \alpha\} \ge \beta\}, \end{aligned} \tag{9}$$

where the parameters α and β represent two kinds of probability. It is easy to see from (3) that we use the min/max operator, which is a special triangular norm, to define the equilibrium chance of a birandom event. In fact, by using a common triangular norm T, the equilibrium chance can be extended as

$$\begin{aligned} &\mathrm{Ch}_T^e \{g(\xi) \le 0\} \\ &\quad = \sup_{(\alpha,\beta) \in [0,1]^2} \{T(\alpha,\beta) \mid \\ &\qquad \Pr\{\omega \in \Omega \mid \Pr\{f_j(\xi(\omega)) \le 0\} \ge \alpha\} \ge \beta\}, \end{aligned} \tag{10}$$

where $T : [0,1] \mapsto [0,1]$ is a triangular norm. In practice, we may also use various triangular norms such as $T_1(\alpha,\beta) = \alpha\beta$, $T_2(\alpha,\beta) = \alpha\beta/[1+(1-\alpha)(1-\beta)]$, $T_3(\alpha,\beta) = \alpha\beta/(\alpha+\beta-\alpha\beta)$,

or $T_4 = \max\{0, \alpha + \beta + 1\}$ according to a decision maker's philosophy of modeling uncertainty.

The following theorem implies a basic property of the equilibrium chance.

Theorem 10. *Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a birandom vector, and let $g : \Re^n \mapsto \Re$ be Borel measurable function, and $\alpha \in [0, 1]$. Then*

$$\begin{aligned} &\mathrm{Ch}^e \{g(\xi) \le 0\} \\ &\quad \ge \alpha \Longleftrightarrow \Pr\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \ge \alpha\} \ge \alpha. \end{aligned} \tag{11}$$

Proof. "$\Rightarrow$" Assume that $\mathrm{Ch}^e\{g(\xi) \le 0\} \ge \alpha$. It follows from the definition of $\mathrm{Ch}^e\{\cdot\}$ that there exists a real number $\alpha_0 \in [0, 1]$ such that

$$\alpha_0 \wedge \Pr\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \ge \alpha_0\} \ge \alpha. \tag{12}$$

Thus we have

$$\begin{gathered} \alpha_0 \ge \alpha, \\ \Pr\{g(\xi(\omega)) \le 0\} \ge \alpha_0 \ge \alpha. \end{gathered} \tag{13}$$

So we get that

$$\begin{aligned} &\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \ge \alpha\} \\ &\quad \supseteq \{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \ge \alpha_0\}. \end{aligned} \tag{14}$$

It follows that

$$\begin{aligned} &\Pr\{\omega \in \Omega \{g(\xi(\omega)) \le 0\} \ge \alpha\} \\ &\quad \ge \Pr\{\omega \in \Omega \{g(\xi(\omega)) \le 0\} \ge \alpha_0\} \ge \alpha. \end{aligned} \tag{15}$$

"$\Leftarrow$" If $\Pr\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \ge \alpha\} \ge \alpha$, then we have

$$\begin{aligned} \alpha &= \alpha \wedge \Pr\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\}\} \\ &\le \sup_{\beta \in [0,1]} \{\beta \wedge \Pr\{\omega \in \Omega \mid \Pr\{g(\xi(\omega)) \le 0\} \ge \beta\}\} \\ &= \mathrm{Ch}^e \{g(\xi(\omega)) \le 0\}. \end{aligned} \tag{16}$$

The theorem is proved. □

The independence of birandom variables is defined as follows.

Definition 11. Birandom variables $\xi_1, \xi_2, \ldots, \xi_n$, which are defined on the probability space $(\Omega, \mathscr{A}, \Pr)$, are said to be independent if and only if $\xi_1(\omega), \xi_2(\omega), \ldots, \xi_n(\omega)$ are independent random variables for all $\omega \in \Omega$.

2.2. Model Formulation. A general equilibrium chance-constrained multiobjective programming model with birandom parameters is formulated as

$$\begin{aligned} &\max \quad \{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_m\} \\ &\text{s.t.} \quad \mathrm{Ch}^e \{f_i(x, \xi) \ge \overline{f}_i\} \ge \alpha_i, \quad i = 1, 2, \ldots, m, \\ &\qquad \mathrm{Ch}^e \{g_r(x, \xi) \le 0\} \ge \beta_r, \quad r = 1, 2, \ldots, p, \\ &\qquad x \in D, \end{aligned} \tag{17}$$

where x is an n-ary decision vector, ξ m-ary is a birandom vector, D is a fixed set that is usually determined by a finite number of inequalities involving functions of x, f_i and g_r are $(m+n)$-ary real-valued continuous functions, and α_i and β_r are predetermined confidence levels, $i = 1, 2, \ldots, m$, $r = 1, 2, \ldots, p$.

It follows from Remark 9 that if ξ degenerates to random vector, then model (17) degenerates to

$$\begin{aligned} &\max \quad \{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_m\} \\ &\text{s.t.} \quad \Pr\{f_i(x, \xi) \ge \overline{f}_i\} \ge \alpha_i, \quad i = 1, 2, \ldots, m, \\ &\qquad \Pr\{g_r(x, \xi) \le 0\} \ge \beta_r, \quad r = 1, 2, \ldots, p, \\ &\qquad x \in D, \end{aligned} \tag{18}$$

which is a stochastic chance-constrained programming model.

Definition 12 (birandom efficient solution at α_i-levels). Suppose a feasible solution x^* of problem (17) satisfies

$$\mathrm{Ch}^e \{f_i(x^*, \xi) \ge \overline{f}_i(x^*)\} \ge \alpha_i, \quad i = 1, 2, \ldots, m, \tag{19}$$

where confidence levels $\alpha_i \in [0, 1]$. x^* is said to be a birandom efficient solution at α_i-levels to problem (17) if and only if there exists no other feasible solution x such that

$$\mathrm{Ch}^e \{f_i(x, \xi) \ge \overline{f}_i(x)\} \ge \alpha_i, \quad i = 1, 2, \ldots, m, \tag{20}$$

and $\overline{f}_i(x) \ge \overline{f}_i(x^*)$ for all i and $\overline{f}_{i_0}(x) > \overline{f}_{i_0}(x^*)$ for at least one $i_0 \in \{1, 2, \ldots, m\}$.

Specially, we consider the linear form of model (17):

$$\begin{aligned} &\max \quad \{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_m\} \\ &\text{s.} \quad \text{t. } \mathrm{Ch}^e \{\tilde{\tilde{c}}_i^T x \ge \overline{f}_i\} \ge \alpha_i, \quad i = 1, 2, \ldots, m, \\ &\qquad \mathrm{Ch}^e \{\tilde{\tilde{a}}_r^T x \le \tilde{\tilde{b}}_r\} \ge \beta_r, \quad r = 1, 2, \ldots, p, \\ &\qquad x \in D, \end{aligned} \tag{21}$$

where $\tilde{\tilde{c}}_i = (\tilde{\tilde{c}}_{i1}, \tilde{\tilde{c}}_{i2}, \ldots, \tilde{\tilde{c}}_{in})^T$ and $\tilde{\tilde{a}}_r = (\tilde{\tilde{a}}_{r1}, \tilde{\tilde{a}}_{r2}, \ldots, \tilde{\tilde{a}}_{rn})^T$ are birandom vectors, and $\tilde{\tilde{b}}_r$ are birandom variables, $i = 1, 2, \ldots, m$, $r = 1, 2, \ldots, p$.

It follows from Theorem 10 that model (21) can be rewritten as

$$\begin{aligned} &\max \quad \{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_m\} \\ &\text{s.t.} \quad \Pr\{\omega \in \Omega \mid \Pr\{\tilde{\tilde{c}}_i^T(\omega) x \ge \overline{f}_i\} \ge \alpha_i\} \ge \alpha_i, \\ &\qquad\qquad i = 1, 2, \ldots, m, \\ &\qquad \Pr\{\omega \in \Omega \mid \Pr\{\tilde{\tilde{a}}_r^T(\omega) x \ge \tilde{\tilde{b}}_r\} \ge \beta_r\} \ge \beta_r, \\ &\qquad\qquad r = 1, 2, \ldots, p, \\ &\qquad\qquad x \in D. \end{aligned} \tag{22}$$

Lemma 13. *Assume that birandom vector* $\tilde{\tilde{c}}_i(\omega) = (\tilde{\tilde{c}}_{i1}(\omega), \tilde{\tilde{c}}_{i2}(\omega), \ldots, \tilde{\tilde{c}}_{in}(\omega))^T$ *follows normal distribution with mean vector* $\tilde{c}_i(\omega)$ *and positive definite covariance matrix* V_i^c, *denoted by* $\tilde{\tilde{c}}_i(\omega) \sim \mathcal{N}(\tilde{c}_i(\omega), V_i^c)$. $\tilde{c}_i(\omega)$ *is a normal random vector with mean vector* μ_i^c *and positive definite covariance matrix* $V_i'^c$, *written as* $\tilde{c}_i(\omega) \sim \mathcal{N}(\mu_i^c, V_i'^c)$. *If* $\tilde{\tilde{c}}_{i1}(\omega), \tilde{\tilde{c}}_{i2}(\omega), \ldots, \tilde{\tilde{c}}_{in}(\omega)$ *are independent birandom variables, then* $\Pr\{\omega \mid \Pr(\tilde{\tilde{c}}_i(\omega)^T x \geq \overline{f}_i) \geq \alpha_i\} \geq \alpha_i$ *holds if and only if*

$$\mu_i^{cT} x + \Phi^{-1}(1-\alpha_i)\sqrt{x^T V_i^c x} + \Phi^{-1}(1-\alpha_i)\sqrt{x^T V_i^{c'} x} \geq \overline{f}_i, \tag{23}$$

where Φ *is the standardized normal distribution.*

Proof. Let $y_i = \tilde{\tilde{c}}_i^T(\omega)x - \overline{f}_i$; then y_i follows normal distribution with the following expected value and variance:

$$E[y_i] = \tilde{c}_i(\omega)^T x - \overline{f}_i, \qquad \operatorname{Var}[y_i] = x^T V_i^c x, \tag{24}$$

where $\tilde{c}_i(\omega) = (\tilde{c}_{i1}(\omega), \tilde{c}_{i2}(\omega), \ldots, \tilde{c}_{in}(\omega))^T$. We note that

$$\frac{\left(\tilde{\tilde{c}}_i(\omega)^T x - \overline{f}_i\right) - \left(\tilde{c}_i(\omega)^T x - \overline{f}_i\right)}{\sqrt{x^T V_i^c x}} \tag{25}$$

must be standardized normally distributed. The inequality $\tilde{\tilde{c}}_i(\omega)^T x \geq \overline{f}_i$ is equivalent to

$$\frac{\left(\tilde{\tilde{c}}_i(\omega)^T x - \overline{f}_i\right) - \left(\tilde{c}_i(\omega)^T x - \overline{f}_i\right)}{\sqrt{x^T V_i^c x}} \geq -\frac{c_i(\omega)^T x - \overline{f}_i}{\sqrt{x^T V_i^c x}}. \tag{26}$$

Let $\eta = ((\tilde{\tilde{c}}_i(\omega)^T x - \overline{f}_i) - (\tilde{c}_i(\omega)^T x - \overline{f}_i))/\sqrt{x^T V_i^c x}$, which is standardized normally distributed. Then we have

$$\begin{aligned}
&\Pr\left\{\tilde{\tilde{c}}_{ij}(\omega)^T x \geq \overline{f}_i\right\} \geq \alpha_i \\
&\iff \Pr\left\{\eta \geq \frac{\overline{f}_i - \tilde{c}_i(\omega)^T x}{\sqrt{x^T V_i^c x}}\right\} \geq \alpha_i \\
&\iff 1 - \Pr\left\{\eta \leq \frac{\overline{f}_i - \tilde{c}_i(\omega)^T x}{\sqrt{x^T V_i^c x}}\right\} \geq \alpha_i \\
&\iff \Pr\left\{\eta \leq \frac{\overline{f}_i - \tilde{c}_i(\omega)^T x}{\sqrt{x^T V_i^c x}}\right\} \leq 1 - \alpha_i \\
&\iff \Phi\left(\frac{\overline{f}_i - \tilde{c}_i(\omega)^T x}{\sqrt{x^T V_i^c x}}\right) \leq 1 - \alpha_i \\
&\iff \frac{\overline{f}_i - \tilde{c}_i(\omega)^T x}{\sqrt{x^T V_i x}} \leq \Phi^{-1}(1-\alpha_i) \\
&\iff \tilde{c}_i(\omega)^T x + \Phi^{-1}(1-\alpha_i)\sqrt{x^T V_i^c x} \geq \overline{f}_i.
\end{aligned} \tag{27}$$

Thus

$$\begin{aligned}
&\Pr\left\{\omega \mid \Pr\left\{\tilde{\tilde{c}}_{ij}(\omega)^T x \geq \overline{f}_i\right\} \geq \alpha_i\right\} \geq \alpha_i \\
&\iff \Pr\left\{\tilde{c}_i(\omega)^T x + \Phi^{-1}(1-\alpha_i)\sqrt{x^T V_i^c x} \geq \overline{f}_i\right\} \geq \alpha_i \\
&\iff \Pr\left\{\left(\left(\tilde{c}_i(\omega)^T x + \Phi^{-1}(1-\alpha_i)\sqrt{x^T V_i^c x} - \overline{f}_i\right)\right.\right. \\
&\qquad \left. - \left(\mu_i^{cT} x + \Phi^{-1}(1-\alpha_i)\sqrt{x^T V_i^c x} - \overline{f}_i\right)\right) \\
&\qquad \times \left(\sqrt{x^{cT} V_i^{c'} x}\right)^{-1} \\
&\qquad \left. \geq -\frac{\mu_i^{cT} x + \Phi^{-1}(1-\alpha_i)\sqrt{x^T V_i^c x} - \overline{f}_i}{\sqrt{x^T V_i^{c'} x}}\right\} \geq \alpha_i \\
&\iff \mu_i^{cT} x + \Phi^{-1}(1-\alpha_i)\sqrt{x^T V_i^c x} + \Phi^{-1}(1-\alpha_i) \\
&\qquad \times \sqrt{x^T V_i^{c'} x} \geq \overline{f}_i.
\end{aligned} \tag{28}$$

This completes the proof. □

Lemma 14 (see [21]). *Assume that birandom vector* $\tilde{\tilde{a}}_r(\omega) = (\tilde{\tilde{a}}_{r1}(\omega), \tilde{\tilde{a}}_{r2}(\omega), \ldots, \tilde{\tilde{a}}_{in}(\omega))^T$ *follows normal distribution with mean vector* $\tilde{a}_r(\omega)$ *and positive definite covariance matrix* V_r^a, *written as* $\tilde{\tilde{a}}_r(\omega) \sim \mathcal{N}(\tilde{a}_r(\omega), V_r^a)$. $\tilde{a}_r(\omega)$ *is a normally distributed random variable, written as* $\tilde{a}_r(\omega) \sim \mathcal{N}(\mu_r^a, V_r'^a)$. *Birandom variable* $\tilde{\tilde{b}}_r(\omega)$ *follows normal distribution with mean value* $\tilde{b}_r(\omega)$ *and variance* $(\sigma_r^b)^2$, *denoted by* $\tilde{\tilde{b}}_r(\omega) \sim \mathcal{N}(\tilde{b}_r(\omega), (\sigma_r^b)^2)$. $\tilde{b}_r(\omega)$ *is a normally distributed random variable, written as* $\tilde{b}_r(\omega) \sim \mathcal{N}(\mu_r^b, (\sigma_r'^b)^2)$. *If* $\tilde{\tilde{a}}_{r1}(\omega), \tilde{\tilde{a}}_{r2}(\omega), \ldots, \tilde{\tilde{a}}_{rn}(\omega), \tilde{\tilde{b}}_r(\omega)$ *are independent birandom variables, then* $\operatorname{Ch}^e\{\tilde{\tilde{a}}_r^T x \leq \tilde{\tilde{b}}_r\} \geq \beta_r$ *holds if and only if*

$$\begin{aligned}
&u_r^{aT} x + \Phi^{-1}(\beta_r)\sqrt{x^T V_r^a x + (\sigma_r^b)^2} \\
&\quad + \Phi^{-1}(\beta_r)\sqrt{x^T V_r^{a'} x + \left(\sigma_r^{b'}\right)^2} \leq \mu_r^b, \quad r = 1, 2, \ldots, p,
\end{aligned} \tag{29}$$

where β_r *are predetermined confidence levels,* $r = 1, 2, \ldots, p$.

It follows from Lemmas 13 and 14 that model (21) is equivalent to the following crisp multiobjective programming model:

$$\begin{aligned} &\max \quad \left\{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_m\right\} \\ &\text{s.t.} \quad \overline{f}_i \le \mu_i^{cT} x + \Phi^{-1}\left(1-\alpha_i\right)\sqrt{x^T V_i^c x} \\ &\qquad + \Phi^{-1}\left(1-\alpha_i\right)\sqrt{x^T V_i^{c'} x}, \quad i = 1, 2, \ldots, m, \\ &\qquad\qquad x \in X, \end{aligned} \tag{30}$$

or equivalently

$$\max_{x \in X} \left\{F_1(x), F_2(x), \ldots, F_m(x)\right\}, \tag{31}$$

where $F_i(x) := \mu_i^{cT} x + \Phi^{-1}(1-\alpha_i)\sqrt{x^T V_i^c x} + \Phi^{-1}(1-\alpha_i)\sqrt{x^T V_i^{c'} x}$, $i = 1, 2, \ldots, m$, and $X := \{x \mid u_r^{aT} x + \Phi^{-1}(\beta_r)\sqrt{x^T V_r^a x + (\sigma_r^b)^2} + \Phi^{-1}(\beta_r)\sqrt{x^T V_r^{a'} x + (\sigma_r^{b'})^2} \le \mu_r^b$, $x \in D, r = 1, 2, \ldots, p\}$.

3. Solution Method

If model (17) satisfies the conditions for Lemmas 13 and 14, then model (17) can be converted into a crisp multiobjective programming model. Many classical methods, such as the goal programming method, the interactive method, and the weighted sum method, can be used to solve it. However, for a general case, it is a difficult to convert the general model into its deterministic equivalent for the predetermined confidence levels. In order to handle the birandom objective functions and to check the birandom equilibrium chance constraints, we used a birandom simulation technique as this method is similar to a stochastic simulation but more complicated and time consuming.

For the following constraints,

$$\Pr\left\{\omega \in \Omega \mid \Pr\left\{\tilde{\tilde{c}}_i(\omega)^T x \ge \overline{f}_i\right\} \ge \alpha_i\right\} \ge \alpha_i, \quad i = 1, 2, \ldots, m; \tag{32}$$

in view of the purpose of maximizing $\overline{f}_i$, we should find the maximal $\overline{f}_i$ such that (32) holds for a given x. It suffices to estimate the maximal value of $\overline{f}_i$ such that the probability of the following random event

$$\left\{\omega \in \Omega \mid \Pr\left\{\tilde{\tilde{c}}_i(\omega)^T x \ge \overline{f}_i\right\} \ge \alpha_i\right\} \tag{33}$$

is not less than α_i.

First, we generate N independent vectors $\omega^k = (\omega_1^k, \omega_2^k, \ldots, \omega_n^k)$ from Ω according to distribution function. Then $\tilde{\tilde{c}}_i(\omega^k)$ are random vectors, $k = 1, 2, \ldots, N$. So we can apply the stochastic simulation to handle

$$\Pr\left\{\tilde{\tilde{c}}_i\left(\omega^k\right)^T x\right\} \ge \overline{f}_i, \quad i = 1, 2, \ldots, m. \tag{34}$$

We define

$$h\left(x, \tilde{\tilde{c}}_i\left(\omega^k\right)\right) = \begin{cases} 1, & \text{if } \tilde{\tilde{c}}_i\left(\omega^k\right)^T x \ge \overline{f}_i, \\ 0, & \text{otherwise}, \end{cases} \tag{35}$$

for $k = 1, 2, \ldots, N$, which are random variables, and the expected value $E[h(x, \tilde{\tilde{c}}_i(\omega^k))] = \alpha_i$ for all k. By the strong law of large numbers, we obtain

$$\frac{\sum_{k=1}^N h\left(x, \tilde{\tilde{c}}_i\left(\omega^k\right)\right)}{N} \longrightarrow \alpha_i \tag{36}$$

as $N \to \infty$. Note that $\sum_{k=1}^N h(x, \tilde{\tilde{c}}_i(\omega^k))$ is just the number of $\tilde{\tilde{c}}_i(\omega^k)$ satisfying $\tilde{\tilde{c}}_i(\omega^k)^T x \ge \overline{f}_i$ for $i = 1, 2, \ldots, N$. Thus $\overline{f}_i$ is just the N'th largest element in the sequence $\{\tilde{\tilde{c}}_i(\omega^1)^T x, \tilde{\tilde{c}}_i(\omega^2)^T x, \ldots, \tilde{\tilde{c}}_i(\omega^N)^T x\}$, where N' is the integer part of $\alpha_i N$. The process to estimate the maximal $\overline{f}_i$ such that (32) holds is summarized as follows.

Step 1. Sample $\omega^1, \omega^2, \ldots, \omega^N$ from Ω according to distribution function.

Step 2. Find the maximal values $\overline{f}_{in}$ such that $\Pr\{\tilde{\tilde{c}}_i^T(\omega^k) x \ge \overline{f}_{in}\} \ge \alpha_i$ for $n = 1, 2, \ldots, N$, respectively, by stochastic simulation.

Step 3. Set N' as the integer part of $\alpha_i N$.

Step 4. Return the N'th largest element in $\{\overline{f}_{i1}, \overline{f}_{i2}, \ldots, \overline{f}_{iN}\}$ as $\overline{f}_i$, $i = 1, 2, \ldots, m$.

For a fixed x, we check whether the constraint

$$\Pr\left\{\omega \mid \Pr\left\{\tilde{\tilde{a}}_r(\omega)^T x \le \tilde{\tilde{b}}_r(\omega)\right\} \ge \beta_r\right\} \ge \beta_r, \quad r = 1, 2, \ldots, p, \tag{37}$$

holds by following process.

Step 1. Set $N' = 0$.

Step 2. Generate $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ from Ω according to distribution function.

Step 3. Calculate $\Pr\{\tilde{\tilde{a}}_r(\omega)^T x \le \tilde{\tilde{b}}_r(\omega)\}$ by stochastic simulation.

Step 4. If $\Pr\{\tilde{\tilde{a}}_r(\omega)^T x \le \tilde{\tilde{b}}_r(\omega)\} \ge \beta_r$, then $N' = N' + 1$.

Step 5. Repeat Step 2 to Step 4 N times.

Step 6. If $N_r'/N \ge \beta_r$, return x is feasible, or else x is infeasible.

Genetic algorithms (GAs) are stochastic search methods for optimization problems based on the mechanics of natural selection and natural genetics. They have been applied to different sectors for both technical and management problems and have shown good performance [22–26]. In this paper, the birandom simulation technique is embedded into a GA to develop the birandom simulation-based GA. The overall procedure of the birandom simulation-based GA for solving the birandom programming models is shown in Figure 2. The main parts of the algorithm are stated in more detail as follows.

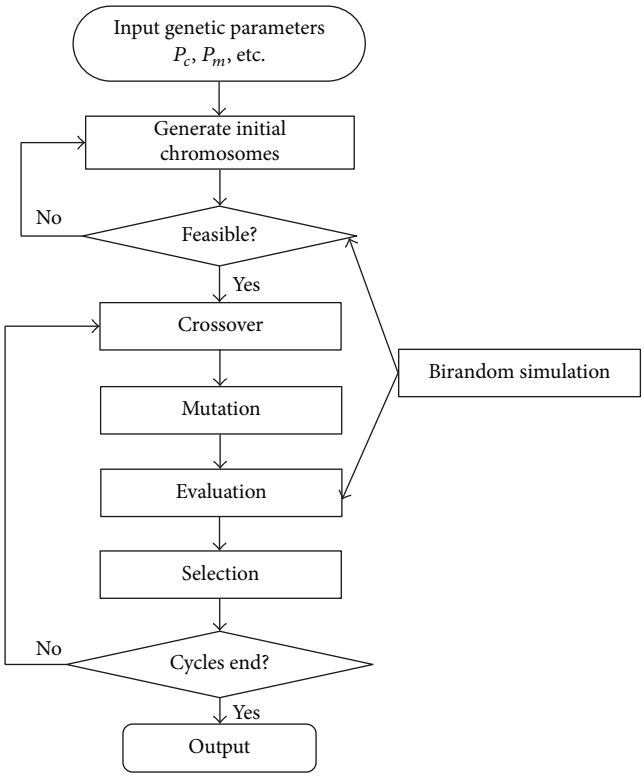

Figure 2: The flowchart of the birandom simulation-based GA.

(1) Representation: a vector $x = (x_1, x_2, \ldots, x_n)$ is chosen as a chromosome to represent a solution to the optimization problem.

(2) Handling the constraints: to ensure the chromosomes generated by genetic operators are feasible, we can use the technique of birandom simulation to check them.

(3) Initializing process: suppose that the decision maker is able to predetermine a region which contains the feasible set. Randomly generate a vector x from this region until a feasible one is accepted as a chromosome. Repeat the above process $N_{\text{pop-size}}$ times; then we have $N_{\text{pop-size}}$ initial feasible chromosomes $x^1, x^2, \ldots, x^{N_{\text{pop-size}}}$. In general, the larger the number of population size $N_{\text{pop-size}}$ is, the better the result is. However, the computation time will increase with $N_{\text{pop-size}}$ increasing. In the paper, we set $N_{\text{pop-size}}$ from 20 to 40.

(4) Evaluation function: let $\text{eval}(x)$ equal weighted sum of all objectives; that is,

$$\text{eval}(x) = \sum_{i=1}^{m} w_i f_i, \tag{38}$$

where w_i are random numbers in $[0, 1]$, representing the weight for objective f_i, $i = 1, 2, \ldots, m$. The random weight-sum approach explores the entire solution space in order to avoid local optima and thus gives a uniform chance to search all the possible solutions. All chromosomes are arranged from large to small according to their objective function values. In other words, after rearrangement, x^1 is the best chromosome, and $x^{N_{\text{pop-size}}}$ is the worst one.

(5) Selection process: the selection process is based on spinning the roulette wheel $N_{\text{pop-size}}$ times. Each time a single chromosome for a new population is selected in the following way: calculate the cumulative probability q_i for each chromosome x^i:

$$q_0 = 0, \qquad q_i = \sum_{j=1}^{i} \text{eval}\left(x^j\right), \quad i = 1, 2, \ldots, N_{\text{pop-size}}. \tag{39}$$

Generate a random number r in $[0, q_{N_{\text{pop-size}}}]$ and select the ith chromosome x_i such that $q_{i-1} < r \leq q_i$, $1 \leq i \leq N_{\text{pop-size}}$. Repeat the above process $N_{\text{pop-size}}$ times and we obtain $N_{\text{pop-size}}$ copies of chromosomes.

(6) Crossover operation: generate a random number c from the open interval (0, 1) and the chromosome x^i is selected as a parent provided that $c < P_c$, where parameter P_c is the probability of crossover operation. Repeat this process $N_{\text{pop-size}}$ times and $P_c \cdot N_{\text{pop-size}}$ chromosomes are expected to be selected to undergo the crossover operation. The crossover operator on x^1 and x^2 will produce two children y^1 and y^2 as follows:

$$y^1 = cx^1 + (1 - c)\, x^2, \qquad y^2 = cx^2 + (1 - c)\, x^2. \tag{40}$$

If both children are feasible, then we replace the parents with them, or else we keep the feasible one if it exists. Repeat the above operation until two feasible children are obtained or a given number of cycles is finished.

Let us consider the setting of probability of crossover P_c. It is obvious that larger crossover probability results in reaching larger solution space, and then it helps reduce the chance of stopping at nonoptimal solution. However, too large P_c will result in considerable time consuming because of too much searching in unnecessary solution space. In the paper, we set P_c from 0.2 to 0.4.

(7) Mutation operation: similar to the crossover process, the chromosome x^i is selected as a parent to undergo the mutation operation provided that random number $m < P_m$, where parameter P_m is the probability of mutation operation. $P_{\lambda_i} \cdot N_{\text{pop-size}}$ are expected to be selected after repeating the process $N_{\text{pop-size}}$ times. Suppose that x^1 is chosen as a parent. Choose a mutation direction $\mathbf{d} \in \mathbf{R^n}$ randomly. Replace x with $x + M \cdot \mathbf{d}$ if $x + M \cdot \mathbf{d}$ is feasible; otherwise we set M as a random number between 0 and M until it is feasible or a given number of cycles are finished. Here, M is a sufficiently large positive number.

P_m controls the proportion of new genes generating in population. If P_m is too small, it will be difficult for some effective genes to be selected. On the contrary, if P_m is too large, that is, there exists too much random change, then offsprings may lose good characteristics inherited from their parents. Thus, the algorithm will lose the learning ability from the past searching. In the paper, we set P_m from 0.2 to 0.4.

After running a given number of cycles of above birandom simulation-based genetic algorithm, the best chromosome can be regarded as the optimal solution.

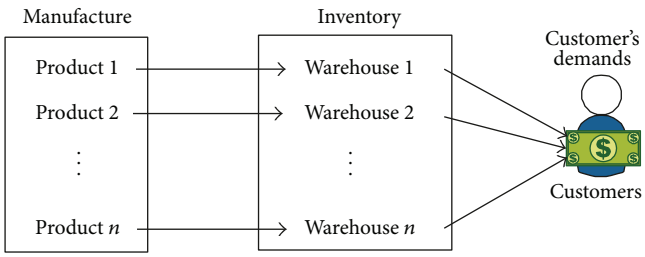

FIGURE 3: The inventory system of H Chain Co., Ltd.

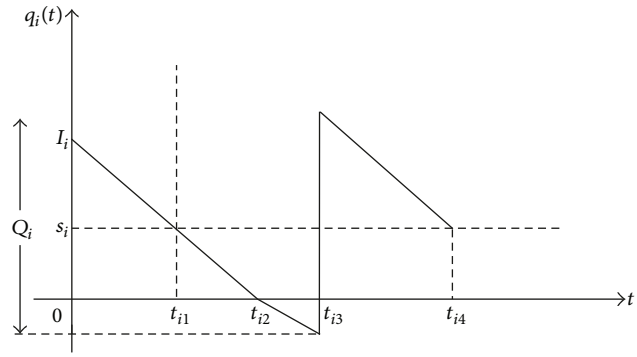

FIGURE 4: Inventory level of product i at time t.

4. Practical Application for H Chain Co., Ltd.

A real-world inventory problem is discussed in this section to illustrate the effectiveness of the theoretical results in this paper.

4.1. Key Problem Statement and Model Formulation. The problem is from H Chain Co., Ltd., which is one of largest chain enterprises in southwest of China. It operates a variety of products, such as eggs, grains, milk, and meat. Inventory management is an important part of company management. The inventory system of H Chain Co., Ltd. is illustrated by Figure 3.

A new executive is assigned to the management of four product items: eggs, dairy products, grains, and meat products. The executive needs to make an effective inventory strategy to reduce costs and increase profit.

4.1.1. Assumptions. In order to develop the mathematical model for the inventory problem, there are some assumptions made, which are as follows. (1) This is an inventory model with a finite time horizon. (2) There are multiple items in this inventory system. (3) Average demand rates are continuous and constant. (4) Deterioration rates are constant. (5) Shortage is allowed. (6) Initial inventory quantities are not zeros; (7) The inventory system uses a continuous review fixed-order-quantity strategy. (8) The lead time is proportional to the order quantity. (9) Fixed-ordering costs are constant. (10) Unit price, unit acquisition cost, holding cost per unit per unit time and shortage cost per unit per unit time are birandom variables. (11) Available total storage space is limited. (12) Available total budget is limited. (13) The inventory system uses nonintegrated management.

4.1.2. Notations. The following notations are used to describe this inventory model:

n is the number of items;

W_i is the available storage space of warehouse of product i, $i = 1, 2, \ldots, n$;

B is the available total budgetary cost;

K_i is the required storage area per unit quantity i, $i = 1, 2, \ldots, n$;

A_i is the fixed-ordering cost of product i, $i = 1, 2, \ldots, n$;

D_i is the average demand rate of product i, $i = 1, 2, \ldots, n$;

I_i is the initial demand rate of product i, $i = 1, 2, \ldots, n$;

a_i is the average deteriorating rate of product i, $i = 1, 2, \ldots, n$;

l_i is the lead time for unit product i, $i = 1, 2, \ldots, n$;

$\tilde{\tilde{p}}_i$ is the unit selling price of product i (birandom variable), $i = 1, 2, \ldots, n$;

$\tilde{\tilde{c}}_{i1}$ is the unit acquisition cost of product i (birandom variable), $i = 1, 2, \ldots, n$;

$\tilde{\tilde{c}}_{i2}$ is the unit shortage cost of product i (birandom variable), $i = 1, 2, \ldots, n$;

s_i is the reorder point of product i (decision variable), $i = 1, 2, \ldots, n$;

Q_i is the order quantity of product i (decision variable), $i = 1, 2, \ldots, n$;

$\tilde{\tilde{N}}_i$ is the net revenue of product i, $i = 1, 2, \ldots, n$;

$\tilde{\tilde{C}}_{i1}$ is the accquisition cost of product i, $i = 1, 2, \ldots, n$;

$\tilde{\tilde{C}}_{i2}$ is the holding cost of product i, $i = 1, 2, \ldots, n$;

$\tilde{\tilde{C}}_{i3}$ is the shortage cost of product i, $i = 1, 2, \ldots, n$;

$\tilde{\tilde{TP}}_i$ is the average profit of product i, $i = 1, 2, \ldots, n$.

4.2. Model Formulation. It follows from the assumptions above that the inventory level $q_i(t)$ of product i at time t can be illustrated by Figure 4.

In Figure 4, t_{i1} is the time of order point for product i; t_{i2} is the time when inventory level of product i reduces to zero; t_{i3} is the time when replenishment of product i arrives; t_{i4} is the time when inventory level of product i reduces to reorder point.

From Figure 4, we have

$$\begin{aligned}
t_{i1} &= \frac{I_i - s_i}{D_i + a_i}, \\
t_{i2} &= \frac{I_i}{D_i + a_i}, \\
t_{i3} &= t_{i1} + l_i Q_i = \frac{I_i - s_i}{D_i + a_i} + l_i Q_i, \\
t_{i4} &= t_{i3} + \frac{Q_i - s_i - (t_{i3} - t_{i2}) D_i}{D_i + a_i} \\
&= \frac{I_i - s_i}{D_i + a_i} + l_i Q_i + \frac{Q_i - s_i - (-s_i/(D_i + a_i) + l_i Q_i) D_i}{D_i + a_i} \\
&= \frac{I_i + Q_i - 2s_i - (-s_i/(D_i + a_i) + l_i Q_i) D_i}{D_i + a_i} + l_i Q_i.
\end{aligned} \tag{41}$$

For product i, the total profit from initial time 0 to t_{i4} equals the net revenue minus total cost, including fixed-ordering cost, acquisition cost, holding cost, and shortage cost. The net revenue is equal to unit price multiplied by sale quantity, that is,

$$\widetilde{\widetilde{N}}_i = \widetilde{\widetilde{p}}_i (t_{i2} + t_{i4} - t_{i3}) D_i = \widetilde{\widetilde{p}}_i \left(\frac{I_i + Q_i - s_i - l_i Q_i D_i}{D_i + a_i} \right) D_i. \tag{42}$$

The acquisition cost is equal to unit acquisition cost multiplied ordering quantity; that is,

$$\widetilde{\widetilde{C}}_{i1} = \widetilde{\widetilde{c}}_{i1} Q_i. \tag{43}$$

The total holding cost consists of holding cost from 0 to t_{i2} and holding cost from t_{i3} to t_{i4}; that is,

$$\begin{aligned}
\widetilde{\widetilde{C}}_{i2} &= \widetilde{\widetilde{c}}_{i2} \left(\int_0^{t_{i2}} q_i(t)\, dt + \int_{t_{i3}}^{t_{i4}} q_i(t)\, dt \right) \\
&= \widetilde{\widetilde{c}}_{i2} \left(\frac{I_i}{2} t_{i2} + \frac{Q_i - (t_{i3} - t_{i2}) D_i + s_i}{2} (t_{i4} - t_{i3}) \right) \\
&= \widetilde{\widetilde{c}}_{i2} \left(\frac{I_i^2 + (Q_i - l_i Q_i D_i)^2 - s_i^2}{2(D_i + a_i)} \right).
\end{aligned} \tag{44}$$

The shortage cost occurs from t_{i2} to t_{i3}; that is,

$$\widetilde{\widetilde{C}}_{i3} = \widetilde{\widetilde{c}}_{i3} \int_{t_{i3}}^{t_{i3}} D_i dt = \widetilde{\widetilde{c}}_{i3} \frac{(t_{i3} - t_{i2}) D_i}{2} = \frac{\widetilde{\widetilde{c}}_{i3} l_i Q_i D_i}{2}. \tag{45}$$

The total profit of product i from t_{i2} to t_{i4} is $\widetilde{\widetilde{N}}_{i0} - A_i - \widetilde{\widetilde{C}}_{i1} - \widetilde{\widetilde{C}}_{i2} - \widetilde{\widetilde{C}}_{i3}$. Then the average total profit of product i is formulated as

$$\widetilde{\widetilde{TP}}_i = \frac{\widetilde{\widetilde{N}}_i - A_i - \widetilde{\widetilde{C}}_{i1} - \widetilde{\widetilde{C}}_{i2} - \widetilde{\widetilde{C}}_{i3}}{t_{i4}}. \tag{46}$$

It is noted that $\widetilde{\widetilde{TP}}_i$ can not be maximized due to birandomness. By utilizing the idea of Charnes and Cooper [17], $\overline{f}_i$ is maximized which satisfies

$$\mathrm{Ch}^e \left\{ \widetilde{\widetilde{TP}}_i \geq \overline{f}_i \right\} \geq \alpha_i, \tag{47}$$

where $\mathrm{Ch}^e\{\cdot\}$ is the equilibrium chance and α_i is given confidence level.

The space of product i ordered should not exceed the available storage space of warehouse; that is,

$$K_i Q_i \leq W_i. \tag{48}$$

All the costs should not exceed the available total budgetary cost; that is, $\sum_{i=1}^{m} (\widetilde{\widetilde{C}}_{i1} + \widetilde{\widetilde{C}}_{i2} + \widetilde{\widetilde{C}}_{i3} + \widetilde{\widetilde{C}}_{i4}) \leq B$. It is noted that the constraint is meaningless mathematically due to the existence of birandom variable. We use equilibrium chance-constrained technique to make it meaningful mathematically, that is,

$$\mathrm{Ch}^e \left\{ \sum_{i=1}^{n} \left(A_i + \widetilde{\widetilde{C}}_{i1} + \widetilde{\widetilde{C}}_{i2} + \widetilde{\widetilde{C}}_{i3} \right) \leq B \right\} \leq \beta, \tag{49}$$

where β is the confidence level predetermined by the decision maker.

Since shortage is allowed, the order quantity of product i should not be less than reorder point s_i; that is,

$$Q_i \geq s_i. \tag{50}$$

In addition, It is natural to require the nonnegativity of decision variable; that is,

$$Q_i \geq 0, \qquad s_i \geq 0. \tag{51}$$

By integration of (47)–(51), we can formulate the programming model of the inventory problem as

$$\begin{aligned}
\max \quad & \left\{ \overline{f}_1, \overline{f}_2, \ldots, \overline{f}_m \right\} \\
\text{s.t.} \quad & \mathrm{Ch}^e \left\{ \widetilde{\widetilde{TC}}_i \geq \overline{f}_i \right\} \geq \alpha_i, \quad i = 1, 2, \ldots, n, \\
& K_i Q_i \leq W_i, \quad i = 1, 2, \ldots, n, \\
& Q_i \geq s_i, \quad i = 1, 2, \ldots, n, \\
& \mathrm{Ch}^e \left\{ \sum_{i=1}^{n} \left(A_i + \widetilde{\widetilde{C}}_{i1} + \widetilde{\widetilde{C}}_{i2} + \widetilde{\widetilde{C}}_{i3} \right) \leq B \right\} \leq \beta, \\
& Q_i \geq 0, \quad s_i \geq 0, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{52}$$

4.3. Data and Compaction Result. All the data are from the marketing section of H Chain Co., Ltd. These data can be divided into two different groups: determined parameters and birandom parameters. Determined parameters include W_i, B, K_i, A_i, D_i, I_i, a_i, and l_i. The available total budgetary cost $B = 250000$ yuan. Except for B, all determined parameters are listed in Table 1. Birandom variables including

Table 1: Determined parameters.

	W_i (m^3)	K_i (m^3/kg)	A_i (yuan)	D_i (kg/day)	I_i (kg)	a_i (kg/day)	l_i (day/kg)
Eggs	5.00	0.50×10^{-3}	500	500	1200	10	1×10^{-3}
Dairy products	5.00	0.80×10^{-3}	1000	500	2000	20	2×10^{-3}
Grains	20.00	0.40×10^{-3}	2000	1000	3000	10	2×10^{-3}
Meat products	10.00	0.50×10^{-3}	1000	1000	2000	50	4×10^{-3}

Table 2: Birandom variables.

	$\tilde{\tilde{p}}_i$ (yuan/kg)	$\tilde{\tilde{c}}_{i1}$ (yuan/kg)	$\tilde{\tilde{c}}_{i2}$ (yuan/kg)	$\tilde{\tilde{c}}_{i3}$ (yuan/kg)
Eggs	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(16, 1)$	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(12, 1)$	$\mathcal{N}(\tilde{\mu}, 0.01), \tilde{\mu} \sim \mathcal{N}(0.10, 0.01)$	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(15, 1)$
Dairy products	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(20, 1)$	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(16, 1)$	$\mathcal{N}(\tilde{\mu}, 0.01), \tilde{\mu} \sim \mathcal{N}(0.10, 0.01)$	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(18, 1)$
Grains	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(8, 1)$	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(4, 1)$	$\mathcal{N}(\tilde{\mu}, 0.01), \tilde{\mu} \sim \mathcal{N}(0.10, 0.01)$	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(6, 1)$
Meat products	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(40, 1)$	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(30, 1)$	$\mathcal{N}(\tilde{\mu}, 0.01), \tilde{\mu} \sim \mathcal{N}(0.10, 0.01)$	$\mathcal{N}(\tilde{\mu}, 1), \tilde{\mu} \sim \mathcal{N}(36, 1)$

Table 3: Optimal order quantities and reorder points.

	Q_i	s_i
Eggs	2405.6	1288.0
Dairy products	3423.8	2288.0
Grains	5400.5	1288.0
Meat products	2436.9	1280.5

$\tilde{\tilde{p}}_i$ (unit selling price of product i), $\tilde{\tilde{c}}_{i1}$ (unit acquisition cost of product i), and $\tilde{\tilde{c}}_{i2}$ (unit shortage cost of product i). These birandom variables are listed in Table 2. The decision maker set confident levels $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \beta = 0.8$ and weighs $w_1 = w_2 = w_3 = w_4 = 0.25$.

We implement the birandom simulation-based genetic algorithm in Matlab 7.0 (the population size $N_{\text{pop-size}} = 30$, the probability of crossover $P_c = 0.3$, and the probability of mutation $P_m = 0.2$) and ran on a PC, 2.40 GHz with 1024 MB memory. After running 500 generations, the final solution is listed in Table 3. In addition, the objective function values are $(\overline{f}_1^*, \overline{f}_2^*, \overline{f}_3^*, \overline{f}_4^*) = (1878.8, 1256.6, 2890.3, 3097.4)$.

5. Further Discussions

To show the effectiveness of the model and algorithm better, evaluation, analysis, and comparison from various aspects are presented.

5.1. Algorithm Evaluation. We evaluate the birandom simulation-based algorithm from two aspects.

First, we will show that the birandom simulation-based algorithm is robust to genetic parameters. We compare the results by the algorithm with different parameters, including population size $N_{\text{pop-size}}$, probability of crossover P_c, probability of mutation P_m, cycles in birandom simulation, and generations in GA. These results are listed in Table 4. It appears that almost all the objective values differ little from each other. In fact, each relative error does not exceed 3% when different parameters are selected, which implies that the proposed algorithm is robust to the parameters setting and effective to solve the problem in this paper.

Second, since the functions in model (52) are linear, by Lemmas 13 and 14, model (52) can be transformed into its crisp equivalent model. Assume that all the parameters are the same and the final objective function values are $(\overline{f}_1^*, \overline{f}_2^*, \overline{f}_3^*, \overline{f}_4^*) = (1881.2, 1253.6, 2895.3, 3093.8)$ by using the weighted sum method. The results obtained by the birandom simulation genetic algorithm and by the traditional weighted sum method are close to each other, which implies the effectiveness of the birandom simulation-based genetic algorithm proposed in this paper.

5.2. Sensitivity Analysis. However, under a different decision making environment and different conditions, the decision maker may alter the predetermined confidence levels. Then the optimal solution will alter accordingly. In order to show how the optimal solutions change along with the predetermined confidence levels, we conducted a sensitivity analysis as shown in Table 5. As shown from Table 5, a bigger β induces a better result, because a bigger β extends the feasible region. Conversely, a bigger α_i induces a worse result since a bigger α_i narrows the feasible region.

5.3. Model Comparison. Birandom system is a generation for stochastic system. For illustrating the advantage of birandom system, all the birandom variables in (52) are assumed to be random variables, and other parameters keep original values. Each birandom variable in (52) has the following form:

$$\tilde{\tilde{\xi}} \sim \mathcal{N}\left(\tilde{\xi}, \sigma^2\right), \qquad \tilde{\xi} \sim \mathcal{N}\left(\mu, \sigma'^2\right). \tag{53}$$

Replace $\tilde{\xi}$ with μ, which is a constant. Then $\tilde{\tilde{\xi}}$ degenerates into a random variable. For example, we use $\tilde{p}_{11} (\sim \mathcal{N}(16, 1))$ to replace $\tilde{\tilde{p}}_{11}$. As all the birandom variables are replaced by

Table 4: Comparison of the results obtained with different parameters.

	$N_{\text{pop-size}}$	P_c	P_m	Cycles	Generations	$(\overline{f}_1^*, \overline{f}_2^*, \overline{f}_3^*, \overline{f}_4^*)$
1	20	0.2	0.2	400	400	(1878.8, 1256.6, 2890.3, 3097.4)
2	30	0.3	0.3	400	400	(1865.9, 1259.8, 2887.2, 3104.5)
3	40	0.4	0.4	400	400	(1872.4, 1257.6, 2891.1, 3099.2)
4	20	0.2	0.2	500	500	(1873.8, 1257.2, 2891.3, 3099.4)
5	30	0.3	0.3	500	500	(1873.8, 1256.6, 2893.3, 3098.6)
6	40	0.4	0.4	500	500	(1877.5, 1256.8, 2891.6, 3099.7)
7	20	0.2	0.2	600	600	(1877.7, 1257.7, 2892.3, 3098.5)
8	30	0.3	0.3	600	600	(1876.8, 1259.6, 2888.3, 3099.4)
9	40	0.4	0.4	600	600	(1879.7, 1259.6, 2891.2, 3099.5)
10	20	0.2	0.2	700	700	(1880.8, 1257.6, 2891.3, 3096.4)
11	30	0.3	0.3	700	700	(1874.7, 1256.5, 2892.3, 3093.5)
12	40	0.4	0.4	700	700	(1874.8, 1254.6, 2890.9, 3097.2)

"Cycles" are the cycles in simulation. "Generations" are the generations in GA.

Table 5: Sensitivity analysis.

	$(\overline{f}_1^*, \overline{f}_2^*, \overline{f}_3^*, \overline{f}_4^*)$
$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.8, \beta = 0.75$	(1873.8, 1250.1, 2878.2, 3067.2)
$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.8, \beta = 0.85$	(1892.5, 1266.6, 2896.7, 3110.5)
$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.75, \beta = 0.80$	(1885.8, 1258.4, 2882.1, 3097.6)
$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0.85, \beta = 0.80$	(1866.8, 1254.6, 2885.3, 3091.9)

random variables, equilibrium chance Ch^e degenerates into probability Pr. Then (52) can be written as

$$\begin{aligned}
&\max \quad \left\{\overline{f}_1, \overline{f}_2, \ldots, \overline{f}_m\right\} \\
&\text{s.t.} \quad \Pr\left\{\widetilde{TC}_i \geq \overline{f}_i\right\} \geq \alpha_i, \quad i = 1, 2, \ldots, n, \\
&\qquad K_i Q_i \leq W_i, \quad i = 1, 2, \ldots, n, \\
&\qquad Q_i \geq s_i, \quad i = 1, 2, \ldots, n, \\
&\qquad \Pr\left\{\sum_{i=1}^{n}\left(A_i + \widetilde{C}_{i1} + \widetilde{C}_{i2} + \widetilde{C}_{i3}\right) \leq B\right\} \leq \beta, \\
&\qquad Q_i \geq 0, \quad s_i \geq 0, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{54}$$

This stochastic system is a special case of a birandom system, which can be solved using a stochastic simulation-based GA. Similar to the birandom simulation-based GA, the stochastic simulation-based GA is formed by embedding stochastic simulation into a GA. The robustness of the stochastic simulation-based GA can be verified as similar to the birandom simulation-based GA.

In order to make a comparison with the birandom system, we solve problem (54) with following parameter: $N_{\text{pop-size}} = 30$, $P_c = 0.3$, and $P_m = 0.2$, 500 cycles in stochastic simulation, 500 generations in GA. The final result is $(\overline{f}_1^*, \overline{f}_2^*, \overline{f}_3^*, \overline{f}_4^*) = (1873.1, 1231.5, 2790.2, 3016.9)$, which is worse than the result of the birandom system. Thus it shows that birandom system is more flexible and can generate better objective values than the corresponding stochastic system.

5.4. Managerial Insight. Some managerial insights can be derived from the above discussions. From the sensitivity analysis, we can see the following. (1) A pessimistic decision maker will set lower confidence levels to avoid higher risk. As a result, the values for the objective functions are worse than those at higher confidence levels. (2) An optimistic decision maker will set higher confidence levels to obtain better values for the objective functions. However, there will be the cost of higher risk. To sum up, the values for the objective functions depend on the attitude of the decision maker towards risk. From a comparison between the stochastic model and the birandom model, the decision maker can obtain more flexible results if they can construct a birandom model rather than a stochastic model in a complex stochastic decision environment. However, depicting the complex stochastic decision environment as birandom variables costs more, so the decision maker must make a trade-off between the cost and the benefit.

6. Conclusions

In this paper, we have formulated an equilibrium chance-constrained multiobjective programming model with birandom parameters, which extends the general multiobjective programming model. We converted a special linear model into a crisp multiobjective programming model which can be solved using traditional techniques. A modified genetic algorithm was designed by embedding the birandom simulation technique to deal with the general model. By assuming that some parameters (unit selling price, unit acquisition cost, etc.) are birandom variables, we developed an equilibrium chance-constrained multiobjective programming model with birandom parameters for an inventory problem and solved it using the birandom simulation-based GA. Further discussions show the effectiveness of the proposed model and algorithm. Some managerial insights may help decision makers improve their decision making.

Though the inventory problem considered is specific, the theoretical results can be extended to various inventory problems. In the future, detailed analysis, further research, and

practical application of the model and algorithm proposed in this paper will be considered.

Acknowledgments

The authors would like to give their great appreciates to the editors and anonymous referees for their helpful comments and valuable suggestions which improved the content and composition substantially. This research was supported by the Programs of NSFC (Grant nos. 70831005 and 71273036), Projects of International Cooperation and Exchanges NSFC (Grant no. 71011140076), "985" Program of Sichuan University (Innovative Research Base for Economic Development and Management), Sichuan University Young Teachers Scientific Research Start Funds (Grant no. 2012SCU11016), and the Key Program of Sichuan System Science and Enterprise Development Research Center (Grant no. Xq13B04).

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 139123, 8 pages
http://dx.doi.org/10.1155/2013/139123

Research Article

New Hybrid Steepest Descent Algorithms for Equilibrium Problem and Infinitely Many Strict Pseudo-Contractions in Hilbert Spaces

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Received 4 May 2013; Accepted 17 June 2013

Academic Editor: Gue Myung Lee

We propose an explicit iterative scheme for finding a common element of the set of fixed points of infinitely many strict pseudo-contractive mappings and the set of solutions of an equilibrium problem by the general iterative method, which solves the variational inequality. In the setting of real Hilbert spaces, strong convergence theorems are proved. The results presented in this paper improve and extend the corresponding results reported by some authors recently. Furthermore, two numerical examples are given to demonstrate the effectiveness of our iterative scheme.

1. Introduction

Let H be a real Hilbert space with inner product $\langle , \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H.

Let $A : C \to H$ be a nonlinear mapping; we consider the problem of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1}$$

It is known as the variational inequality problem (denoted by $\mathrm{VI}(A, C)$).

Generally, A is assumed to be Lipschitzian and strongly monotone. The relative definitions are listed as follows.

(i) A is called k-Lipschitzian on C, if there exists a constant $k > 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in C. \tag{2}$$

(ii) A is said to be η-strongly monotone on C, if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C. \tag{3}$$

(iii) A mapping S of C is said to be a κ-strict pseudo-contraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2 \tag{4}$$

for all $x, y \in C$; see [1].

(iv) A mapping S of C is said to be a nonexpansive mapping if it is strictly pseudo-contractive with constant $\kappa = 0$.

Obviously, the class of strict pseudo-contractions strictly includes the class of nonexpansive mappings. We denote the set of fixed points of S by $F_{ix}(S)$ (i.e., $F_{ix}(S) = \{x \in C : Sx = x\}$).

Let F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers.

The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to determine its equilibrium points, that is, the set

$$\{x \in C : F(x, y) \geq 0, \ \forall y \in C\}. \tag{5}$$

The set of such solutions is denoted by $\mathrm{EP}(F)$.

Many problems in applied sciences such as physics, optimization, and economics reduce into finding some element of $\mathrm{EP}(F)$. Some methods have been proposed to solve the

equilibrium problem (5); see, for instance, [2–6]. In particular, Combettes and Hirstoaga [7] proposed several methods for solving the equilibrium problem. On the other hand, Mann [8] and Shimoji and Takahashi [9] considered iterative schemes for finding a fixed point of a nonexpansive mapping. Further, Acedo and Xu [10] projected new iterative methods for finding a fixed point of strict pseudo-contractions.

In 2006, Marino and Xu [5] proposed a general iterative method and proved that the algorithm converged strongly. Recently, Tian [11] revealed the inner contact of Yamada's algorithm [12] and viscosity iterative algorithm and then introduced a new general iterative algorithm combining a k-Lipschitzian and η-strong monotone operator. On this basis, Wang [13] considered a general composite iterative method for infinitely many strict pseudo-contractions in 2010. However, the W-mapping used in Wang's paper requires many composite operations. Very recently, He and Sun [14] proposed a new operator L_n to replace the W-mapping for infinite family nonexpansive mappings.

The mapping L_n is defined as follows:

$$L_n = \sum_{i=1}^{n} \frac{\omega_i}{s_n} T_i \quad (n = 1, 2, \ldots), \tag{6}$$

where $\{\omega_i\} \subset (0,1)$ such that $\sum_{i=1}^{\infty} \omega_i = 1$, $s_n = \sum_{i=1}^{n} \omega_i$, and $\{T_i\}$ are infinite nonexpansive mappings. Because it does not contain many composite computations, it is more simple and easy to realize.

In this paper, we combine the operator L_n and the general iterative algorithm to propose a new explicit iterative scheme involving equilibrium problem (5) and an infinite family of strict pseudo-contractions. Under certain assumptions, we will prove that the sequence converges strongly. Further an example will be given to demonstrate the effectiveness of our iterative scheme and another will be given to compare numerical results and convergence rate of the algorithm in this paper and [15].

2. Preliminaries

In the sequel, we will make use of the following lemmas in a real Hilbert space H.

Lemma 1. *Let H be a real Hilbert space. There hold the following identities:*

(i)

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \quad \forall x, y \in H, \tag{7}$$

(ii)

$$\begin{aligned} &\|tx + (1-t)y\|^2 \\ &\quad = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \\ &\qquad \forall t \in [0,1], \quad \forall x, y \in H. \end{aligned} \tag{8}$$

Lemma 2 (see [13]). *Let $A : H \to H$ be a k-Lipschitzian and η-strongly monotone operator on a Hilbert space H with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$, and $0 < t < 1$. Then $S = (I - t\mu A) : H \to H$ is a contraction with contractive coefficient $1 - t\tau$ and $\tau = \mu(2\eta - (\mu k^2/2))$.*

Lemma 3 (see [1]). *Let $S : C \to C$ be a κ-strict pseudo-contraction. Define $T : C \to C$ by $Tx = \lambda x + (1-\lambda)Sx$ for each $x \in C$. Then, as $\lambda \in [\kappa, 1)$, T is a nonexpansive mapping such that $F_{ix}(T) = F_{ix}(S)$.*

Lemma 4. *Let $V : C \to H$ be an l-Lipschitz mapping with coefficient $l \geq 0$ and $A : C \to H$ a k-Lipschitzian continuous operator and η-strongly monotone operator with $k > 0$, $\eta > 0$. Then, for $0 < \gamma < \mu\eta/l$,*

$$\begin{aligned} &\langle x - y, (\mu A - \gamma V)x - (\mu A - \gamma V)y\rangle \\ &\quad \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad x, y \in H. \end{aligned} \tag{9}$$

That is, $\mu A - \gamma V$ is strongly monotone with coefficient $\mu\eta - \gamma l$.

Proof. Since A is l-Lipschitz and η-strongly monotone, it is easy to get

$$\begin{aligned} &\langle x - y, (\mu A - \gamma V)x - (\mu A - \gamma V)y\rangle \\ &\quad = \mu\langle x - y, Ax - Ay\rangle - \gamma\langle x - y, Vx - Vy\rangle \\ &\quad \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad x, y \in C. \end{aligned} \tag{10}$$

□

Lemma 5 (see [16]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \tag{11}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i)

$$\sum_{n=1}^{\infty} \gamma_n = \infty; \tag{12}$$

(ii)

$$\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0 \quad \text{or} \quad \sum_{n=1}^{\infty} |\delta_n| < \infty. \tag{13}$$

Then, $\lim_{n\to\infty} a_n = 0$.

Let $\{S_n\}$ be a sequence of κ_n-strict pseudo-contractions. Define $S_n' = \theta_n I + (1 - \theta_n)S_n$, $\theta_n \in [\kappa_n, 1)$. Then, by Lemma 3, S_n' is nonexpansive. In order to find the common fixed point set of infinite mappings, W-mapping is often used; see [9, 13, 15, 17, 18] and references therein. The mapping W_n is defined by

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= t_n S_n' U_{n,n+1} + (1 - t_n)I, \\ U_{n,n-1} &= t_{n-1} S_{n-1}' U_{n,n} + (1 - t_{n-1})I, \\ &\vdots \end{aligned}$$

$$\begin{aligned} U_{n,i} &= t_i S_i' U_{n,i+1} + (1-t_i) I, \\ &\vdots \\ U_{n,2} &= t_2 S_2' U_{n,3} + (1-t_2) I, \\ W_n = U_{n,1} &= t_1 S_1' U_{n,2} + (1-t_1) I, \end{aligned} \tag{14}$$

where $t_1, t_2, \ldots$ are real numbers such that $0 \le t_n < 1$. Such a mapping W_n is called a W-mapping generated by $S_1', S_2', \ldots$ and $t_1, t_2, \ldots$. As we have seen, W-mapping contains many composite computation of S_n', and it is complicated and needs a large number of complex operations. In [14], He and Sun proposed a new hybrid steepest descent method for solving fixed point problem defined on the common fixed point set of infinite nonexpansive mappings.

Lemma 6 (see [14]). *Let H be a real Hilbert and $T_i : H \to H$ $(i = 1, 2, \ldots)$ all nonexpansive mappings with $\bigcap_{i=1}^{\infty} F_{ix}(T_i) \neq \emptyset$. Let $T = \sum_{i=1}^{\infty} \omega_i T_i$ $(i = 1, 2, \ldots)$, where $\{\omega_i\} \subset (0,1)$ such that $\sum_{i=1}^{\infty} \omega_i = 1$. Then T is a nonexpansive mapping with $F_{ix}(T) = \bigcap_{i=1}^{\infty} F_{ix}(T_i)$.*

Lemma 7 (see [14]). *Let H be a real Hilbert and $T_i : H \to H$ $(i = 1, 2, \ldots)$ all nonexpansive mappings with $\bigcap_{i=1}^{\infty} F_{ix}(T_i) \neq \emptyset$. Let $T = \sum_{i=1}^{\infty} \omega_i T_i$, where $\{\omega_i\} \subset (0,1)$ such that $\sum_{i=1}^{\infty} \omega_i = 1$. Assume $L_n = \sum_{i=1}^{n} \omega_i T_i / s_n$, where $s_n = \sum_{i=1}^{n} \omega_i$. Then L_n uniformly converges to T in each bounded subset in H.*

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone; that is, $F(x, y) + F(y, x) \le 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t \to 0} F(tz + (1-t)x, y) \le F(x, y)$;

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

We recall some lemmas which will be needed in the rest of this paper.

Lemma 8 (see [2]). *Let C be a nonempty closed convex subset of H, let F be bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4), and let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \quad \forall y \in C. \tag{15}$$

Lemma 9 (see [7]). *For $r > 0$, $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$\begin{aligned} &T_r(x) \\ &\quad = \left\{ z \in C \mid F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \ \forall y \in C \right\} \end{aligned} \tag{16}$$

for all $x \in H$. Then, the following statements hold:

(i) *T_r is single valued;*

(ii) *T_r is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y \rangle; \tag{17}$$

(iii) *$F_{ix}(T_r) = \mathrm{EP}(F)$;*

(iv) *$\mathrm{EP}(F)$ is closed and convex.*

Lemma 10 (see [19]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space and $\{\beta_n\}$ a sequence of real numbers such that $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$ for all $n = 0, 1, 2, \ldots$ Suppose that $x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n$ for all $n = 0, 1, 2, \ldots$ and $\limsup_{n\to\infty} \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 11 (see [6]). *Let C, H, F, and $T_r x$ be as in Lemma 9. Then the following holds:*

$$\|T_s x - T_t x\|^2 \le \frac{s-t}{s} \langle T_s x - T_t x, T_s x - x \rangle \tag{18}$$

for all $s, t > 0$ and $x \in H$.

Lemma 12 (see [13]). *Let H be a Hilbert space, C a nonempty closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F_{ix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

We adopt the following notations:

(1) $x_n \rightharpoonup x$ stands for the weak convergence of $\{x_n\}$ to x,

(2) $x_n \to x$ stands for the strong convergence of $\{x_n\}$ to x.

3. Main Result

Recall that, given a nonempty closed convex subset C of a real Hilbert space H, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\| \tag{19}$$

for all $y \in C$. Such a P_C is called the metric (or the nearest point) projection of H onto C. As we all know, $y = P_C x$ if and only if there holds the following relation:

$$\langle x - y, y - z \rangle \ge 0 \quad \forall z \in C. \tag{20}$$

Throughout the rest of this paper, we always assume that V is an l-Lipschitzian mapping of H into itself with coefficient $l \ge 0$ and A is a k-Lipschitzian continuous operator and η-strongly monotone on H with $k > 0$, $\eta > 0$. Assume that $0 < \mu < 2\eta/k^2$ and $0 < \gamma < \mu(\eta - (\mu k^2/2))/\alpha = \tau/l$.

Define a mapping $U_n = \beta_n I + (1 - \beta_n) L_n T_{r_n}$. Since both L_n and T_{r_n} are nonexpansive, it is easy to get that U_n is also nonexpansive. Consider the mapping G_n on H defined by

$$G_n x = \alpha_n \gamma V(x) + (I - \alpha_n \mu A) U_n x, \quad \forall x \in H, n \in N, \tag{21}$$

where $\alpha_n \in (0,1)$. By Lemmas 2 and 9, we have

$$\begin{aligned}\|G_n x - G_n y\| &\le \alpha_n \gamma \|Vx - Vy\| + (1-\alpha_n\tau)\|U_n x - U_n y\| \\ &\le \alpha_n \gamma l \|x-y\| + (1-\alpha_n\tau)\|x-y\| \\ &= (1-\alpha_n(\tau-\gamma l))\|x-y\|.\end{aligned} \tag{22}$$

Since $0 < 1-\alpha_n(\tau-\gamma l) < 1$, it follows that G_n is a contraction. Therefore, by the Banach contraction principle, G_n has a unique fixed point $x_n^V \in H$ such that

$$x_n^V = \alpha_n \gamma V(x_n^V) + (I - \alpha_n \mu A) U_n x_n^V. \tag{23}$$

For simplicity, we will write x_n for x_n^V provided no confusion occurs. Next we prove the sequence $\{x_n\}$ converges strongly to a $x^* \in \Omega = \bigcap_{i=1}^{\infty} F_{ix}(S_i) \cap \mathrm{EP}(F)$ which solves the variational inequality

$$\langle (\gamma V - \mu A) x^*, p - x^* \rangle \le 0, \quad \forall p \in \Omega. \tag{24}$$

By the property of the projection, we can get $x^* = P_\Omega(I-\mu A+\gamma V)x^*$ equivalently.

Theorem 13. *Let C be a nonempty closed convex subset of a real Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $S_i : C \to C$ be family κ_i-strict pseudo-contractions for some $0 \le \kappa_i < 1$. Assume the set $\Omega = \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{EP}(F) \neq \emptyset$. Let V be an l-Lipschitzian mapping of H into itself with $l \ge 0$, and let A be a k-Lipschitzian continuous operator and η-strongly monotone on H with $k > 0$, $\eta > 0$, $0 < \mu < 2\eta/k^2$, and $0 < \gamma < \mu(\eta - (\mu k^2/2))/l = \tau/l$. For every $n \in \mathbb{N}$, let L_n be the mapping generated by S_i' and $0 < \omega_i < 1$ with $\sum_{i=1}^{\infty} \omega_i = 1$ according to (6). Given $x_1 \in H$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by the following algorithm:*

$$\begin{aligned} u_n &= T_{r_n} x_n, \\ y_n &= \beta_n x_n + (1-\beta_n) L_n u_n, \\ x_{n+1} &= \alpha_n \gamma V x_n + (I - \mu\alpha_n A) y_n. \end{aligned} \tag{25}$$

If $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$, *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$,

then, $\{x_n\}$ converges strongly to $x^* \in \Omega$, which solves the variational inequality (24).

Proof. The proof is divided into several steps.

Step 1. Show first that $\{x_n\}$ is bounded.

Taking any $p \in \Omega$, by Lemma 9, we have

$$\|u_n - p\| = \|T_{r_n} x_n - T_{r_n} p\| \le \|x_n - p\|. \tag{26}$$

It follows from (25) that

$$\begin{aligned}&\|y_n - p\| \\ &\quad = \|\beta_n(x_n - p) + (1-\beta_n)(L_n u_n - L_n p)\| \\ &\quad \le \beta_n\|x_n - p\| + (1-\beta_n)\|u_n - p\| \\ &\quad \le \|x_n - p\|.\end{aligned} \tag{27}$$

Further we get

$$\begin{aligned}&\|x_{n+1} - p\| \\ &\quad = \|\alpha_n(\gamma V x_n - \mu A p) + (I - \mu\alpha_n A) y_n \\ &\qquad - (I - \mu\alpha_n A) p\| \\ &\quad \le \alpha_n(\|\gamma V x_n - \gamma V p\| + \|\gamma V p - \mu A p\|) \\ &\qquad + (1-\alpha_n\tau)\|y_n - p\| \\ &\quad \le \alpha_n l\gamma\|x_n - p\| + \alpha_n\|\gamma V p - \mu A p\| \\ &\qquad + (1-\alpha_n\tau)\|y_n - p\| \\ &\quad = (1-\alpha_n(\tau - l\gamma))\|x_n - p\| \\ &\qquad + \alpha_n(\tau - l\gamma)\frac{\|\gamma V p - \mu A p\|}{\tau - l\gamma} \\ &\quad \le \max\left\{\|x_n - p\|, \frac{\|\gamma V p - \mu A p\|}{\tau - l\gamma}\right\}.\end{aligned} \tag{28}$$

By induction, we obtain $\|x_n - p\| \le \max\{\|x_1 - p\|, \|\gamma V p - \mu A p\|/(\tau - l\gamma)\}$, $n \ge 1$. Hence, $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{y_n\}$. It follows from the Lipschitz continuity of A and V that $\{Ax_n\}$, $\{Au_n\}$, and $\{Vx_n\}$ are also bounded. From the nonexpansivity of L_n, it follows that $\{L_n x_n\}$ is also bounded.

Step 2. Show that

$$\|x_{n+1} - x_n\| \longrightarrow 0. \tag{29}$$

Suppose $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$, then $z_n = (x_{n+1} - \beta_n x_n)/(1-\beta_n) = (\alpha_n \gamma V x_n + (I - \mu\alpha_n A) y_n - \beta_n x_n)/(1-\beta_n)$.

Hence, we have

$$\begin{aligned} z_{n+1} - z_n &= \frac{\alpha_{n+1}\gamma V x_{n+1} + (I - \mu\alpha_{n+1} A) y_{n+1} - \beta_{n+1} x_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n \gamma V x_n + (I - \mu\alpha_n A) y_n - \beta_n x_n}{1-\beta_n} \\ &= \frac{\alpha_{n+1}(\gamma V x_{n+1} - \mu A y_{n+1})}{1-\beta_{n+1}} + \frac{y_{n+1} - \beta_{n+1} x_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n(\gamma V x_n - \mu A y_n)}{1-\beta_n} - \frac{y_n - \beta_n x_n}{1-\beta_n}\end{aligned}$$

$$\begin{aligned}
&= \frac{\alpha_{n+1}(\gamma V x_{n+1} - \mu A y_{n+1})}{1-\beta_{n+1}} \\
&\quad + \frac{\beta_{n+1} x_{n+1} + (1-\beta_{n+1}) L_{n+1} u_{n+1} - \beta_{n+1} x_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n(\gamma V x_n - \mu A y_n)}{1-\beta_n} - \frac{\beta_n x_n + (1-\beta_n) L_n u_n - \beta_n x_n}{1-\beta_n} \\
&\le \frac{\alpha_{n+1}(\gamma V x_{n+1} - \mu A y_{n+1})}{1-\beta_{n+1}} - \frac{\alpha_n(\gamma V x_n - \mu A y_n)}{1-\beta_n} \\
&\quad + L_{n+1} u_{n+1} - L_n u_n.
\end{aligned} \tag{30}$$

Observe that

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|T_{r_{n+1}} x_{n+1} - T_{r_n} x_n\| \\
&\le \|T_{r_{n+1}} x_{n+1} - T_{r_{n+1}} x_n\| + \|T_{r_{n+1}} x_n - T_{r_n} x_n\| \\
&\le \|x_{n+1} - x_n\| + \|T_{r_{n+1}} x_n - T_{r_n} x_n\|.
\end{aligned} \tag{31}$$

By the definition of L_n, we have

$$\begin{aligned}
&\|L_{n+1} u_{n+1} - L_n u_n\| \\
&\quad = \|L_{n+1} u_{n+1} - L_{n+1} u_n\| + \|L_{n+1} u_n - L_n u_n\| \\
&\quad \le \|u_{n+1} - u_n\| + \left\| \sum_{i=1}^{n+1} \frac{\omega_i}{s_{n+1}} S_i' u_n - \sum_{i=1}^{n} \frac{\omega_i}{s_n} S_i' u_n \right\| \\
&\quad \le \|x_{n+1} - x_n\| + \|T_{r_{n+1}} x_n - T_{r_n} x_n\| \\
&\qquad + \left\| \frac{\omega_{n+1}}{s_{n+1}} S_{n+1}' u_n + \sum_{i=1}^{n} \left(\frac{\omega_i}{s_{n+1}} - \frac{\omega_i}{s_n} \right) S_i' u_n \right\| \\
&\quad \le \|x_{n+1} - x_n\| + \|T_{r_{n+1}} x_n - T_{r_n} x_n\| \\
&\qquad + \left\| \frac{\omega_{n+1}}{s_{n+1}} S_{n+1}' u_n \right\| + \left\| \sum_{i=1}^{n} \frac{-\omega_{n+1}\omega_i}{s_{n+1} s_n} S_i' u_n \right\| \\
&\quad \le \|x_{n+1} - x_n\| + \|T_{r_{n+1}} x_n - T_{r_n} x_n\| + 2M_1 \frac{\omega_{n+1}}{s_{n+1}},
\end{aligned} \tag{32}$$

where $M_1 = \sup_{i,n\ge 1}\{\|S_i' u_n\|\}$.

It follows from (30) and (32) that

$$\begin{aligned}
&\|z_{n+1} - z_n\| \\
&\quad \le \frac{\alpha_{n+1}}{1-\beta_{n+1}} (\|\gamma V x_{n+1}\| + \|\mu A y_{n+1}\|) \\
&\qquad + \frac{\alpha_n}{1-\beta_n} (\|\gamma V x_n\| + \|\mu A y_n\|) + \|L_{n+1} u_{n+1} - L_n u_n\| \\
&\quad \le \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n} \right) M_2 + \|x_{n+1} - x_n\| \\
&\qquad + \|T_{r_{n+1}} x_n - T_{r_n} x_n\| + 2M_1 \frac{\omega_{n+1}}{s_{n+1}},
\end{aligned} \tag{33}$$

where $M_2 = \sup_n\{\|\gamma V x_n\| + \|\mu A y_n\|\}$.

Hence we get $\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \le \|T_{r_{n+1}} x_n - T_{r_n} x_n\| + ((\alpha_{n+1}/(1-\beta_{n+1})) + (\alpha_n/(1-\beta_n)))M_2 + M_1(\omega_{n+1}/S_{n+1})$. Since $\sum_{n=1}^{\infty} \omega_n = 1$ is convergent, it is easy to see that $\sum_{n=1}^{\infty} \omega_n/s_n$ is also convergent. Thus we have $\omega_n/s_n \to 0$ $(n \to \infty)$.

From conditions (i) and (iii) and Lemma 11, we obtain

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{34}$$

By Lemma 10, we have $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Thus

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|z_n - x_n\| = 0. \tag{35}$$

By Lemma 11 and (30) and (29), we obtain

$$\|u_{n+1} - u_n\| \longrightarrow 0. \tag{36}$$

Step 3. Show that

$$\|x_n - L x_n\| \longrightarrow 0, \tag{37}$$

where $L = \sum_{i=1}^{\infty} \omega_i S_i'$ $(i = 1, 2, \ldots)$.

Observe that

$$\begin{aligned}
\|x_n - L_n x_n\| &\le \|x_n - L_n u_n\| + \|L_n u_n - L_n x_n\| \\
&\le \|x_n - L_n u_n\| + \|u_n - x_n\|, \\
\|x_n - L_n u_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - L_n u_n\| \\
&= \|x_n - x_{n+1}\| + \alpha_n \|\gamma V x_n - \mu A y_n\| \\
&\quad + \beta_n \|x_n - L_n u_n\|.
\end{aligned} \tag{38}$$

From condition (i) and (25), we can obtain

$$\begin{aligned}
&(1-\beta_n)\|x_n - L_n u_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \alpha_n \|\gamma V x_n - \mu A y_n\| \longrightarrow 0.
\end{aligned} \tag{39}$$

It follows from condition (ii) that

$$\|x_n - L_n u_n\| \longrightarrow 0. \tag{40}$$

By Lemma 9, we get

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n} x_n - T_{r_n} p\|^2 \\
&\le \langle T_{r_n} x_n - T_{r_n} p, x_n - p \rangle \\
&= \frac{1}{2} \left(\|u_n - p\|^2 + \|x_n - p\|^2 + \|x_n - u_n\|^2 \right).
\end{aligned} \tag{41}$$

This implies that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{42}$$

By nonexpansivity of L_n, we have

$$\begin{aligned}
\|y_n - p\|^2 &\le \beta_n \|x_n - p\|^2 + (1-\beta_n)\|u_n - p\|^2 \\
&\le \|x_n - p\|^2 - (1-\beta_n)\|x_n - u_n\|^2.
\end{aligned} \tag{43}$$

It follows from (25) that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad = \|\alpha_n(\gamma V x_n - p) + (I-\mu\alpha_n A)y_n - (I-\mu\alpha_n A)p \\
&\qquad + \alpha_n(p-\mu A p)\|^2 \\
&\quad \le \alpha_n\|\gamma V x_n - p\|^2 + (1-\alpha_n\tau)\|y_n - p\|^2 \\
&\qquad + \alpha_n\|p-\mu A p\|^2 \\
&\quad \le \alpha_n\|\gamma V x_n - p\|^2 \\
&\qquad + (1-\alpha_n\tau)\left(\|x_n-p\|^2 - (1-\beta_n)\|x_n-u_n\|^2\right) \\
&\qquad + \alpha_n\|p-\mu A p\|^2 \\
&\quad \le \alpha_n\|\gamma V x_n - p\|^2 + \|x_n-p\|^2 \\
&\qquad - (1-\beta_n)\|x_n-u_n\|^2 + \alpha_n\|p-\mu A p\|^2.
\end{aligned} \tag{44}$$

This implies that

$$\begin{aligned}
&(1-\beta_n)\|x_n-u_n\|^2 \\
&\quad \le \alpha_n\left(\|\gamma V x_n - p\|^2 + \|p-\mu A p\|^2\right) \\
&\qquad + \|x_n-p\|^2 - \|x_{n+1}-p\|^2 \\
&\quad \le \alpha_n\left(\|\gamma V x_n - p\|^2 + \|p-\mu A p\|^2\right) \\
&\qquad + (\|x_n-p\| + \|x_{n+1}-p\|)\|x_{n+1}-x_n\|.
\end{aligned} \tag{45}$$

From conditions (i) and (ii) and (29), we have

$$\|x_n - u_n\| \longrightarrow 0. \tag{46}$$

Thus, we get

$$\|x_n - L_n x_n\| \longrightarrow 0. \tag{47}$$

On the other hand, we have

$$\|x_n - Lx_n\| \le \|x_n - L_n x_n\| + \|L_n x_n - Lx_n\|. \tag{48}$$

Combining (47) and Lemma 7, we obtain (37).

Step 4. Show that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu A)x^*, x_n - x^*\rangle \le 0, \tag{49}$$

where $x^* = P_\Omega(I-\mu A+\gamma V)x^*$ is a unique solution of the variational inequality (24). Indeed, take a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle(\gamma V-\mu A)x^*, x_n - x^*\rangle \\
&\quad = \lim_{j\to\infty}\left\langle(\gamma V-\mu A)x^*, x_{n_j} - x^*\right\rangle.
\end{aligned} \tag{50}$$

Since $\{x_{n_j}\}$ is bounded, there exists a subsequence $\{x_{n_{j_k}}\}$ of $\{x_{n_j}\}$ which converges weakly to q. Without loss of generality, we can assume $x_{n_j} \rightharpoonup q$. From (37), we obtain $Lx_{n_j} \rightharpoonup q$.

By the same argument as in the proof of Theorem 13, we have $q \in \Omega$. Since $x^* = P_\Omega(I-\mu A+\gamma V)x^*$, it follows that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle(\gamma V-\mu A)x^*, x_n - x^*\rangle \\
&\quad = \lim_{j\to\infty}\left\langle(\gamma V-\mu A)x^*, x_{n_j} - x^*\right\rangle \\
&\quad = \langle(\gamma V-\mu A)x^*, q - x^*\rangle \le 0.
\end{aligned} \tag{51}$$

Step 5. Show that

$$x_n \longrightarrow x^*. \tag{52}$$

Since

$$\begin{aligned}
&\langle(\gamma V-\mu A)x^*, x_{n+1} - x^*\rangle \\
&\quad = \langle(\gamma V-\mu A)x^*, x_{n+1} - x_n\rangle + \langle(\gamma V-\mu A)x^*, x_n - x^*\rangle \\
&\quad \le \|(\gamma V-\mu A)x^*\|\,\|x_{n+1}-x_n\| + \langle(\gamma V-\mu A)x^*, x_n - x^*\rangle.
\end{aligned} \tag{53}$$

It follows from (29) and (51) that

$$\limsup_{n\to\infty}\langle(\gamma V-\mu A)x^*, x_{n+1} - x^*\rangle \le 0,$$

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad = \|\alpha_n\gamma V x_n + (I-\mu\alpha_n A)y_n - x^*\|^2 \\
&\quad = \|(I-\mu\alpha_n A)y_n - (I-\mu\alpha_n A)x^* \\
&\qquad + \alpha_n(\gamma V x_n - \mu A x^*)\|^2 \\
&\quad \le \|(I-\mu\alpha_n A)y_n - (I-\mu\alpha_n A)x^*\|^2 \\
&\qquad + 2\alpha_n\langle\gamma V x_n - \mu A x^*, x_{n+1} - x^*\rangle \\
&\quad \le (1-\alpha_n\tau)^2\|y_n - x^*\|^2 + 2\alpha_n\langle\gamma V x_n - \gamma V x^*, x_{n+1} - x^*\rangle \\
&\qquad + 2\alpha_n\langle(\gamma V-\mu A)x^*, x_{n+1} - x^*\rangle \\
&\quad \le (1-\alpha_n\tau)^2\|x_n - x^*\|^2 \\
&\qquad + \alpha_n l\gamma\left(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\right) \\
&\qquad + 2\alpha_n\langle(\gamma V-\mu A)x^*, x_{n+1} - x^*\rangle.
\end{aligned} \tag{54}$$

This implies that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad \le \frac{(1-\alpha_n\tau)^2 + \alpha_n l\gamma}{1-\alpha_n l\gamma}\|x_n - x^*\|^2
\end{aligned}$$

Table 1: $x_1 = 1/2$.

n (iterative number)	$x^{(1)}$ (initial guess)	Errors (n)
94	0.9969	3.1×10^{-3}
150	0.9981	1.9×10^{-3}
450	0.9994	6.3×10^{-4}

$$\begin{aligned} &+ \frac{2\alpha_n}{1-\alpha_n l\gamma} \langle (\gamma V - \mu A) x^*, x_{n+1} - x^* \rangle \\ &\leq \left(1 - \frac{2\alpha_n(\tau - l\gamma)}{1-\alpha_n l\gamma}\right) \|x_n - x^*\|^2 \\ &+ \frac{2\alpha_n}{1-\alpha_n l\gamma} \langle (\gamma V - \mu A) x^*, x_{n+1} - x^* \rangle \\ &+ \frac{(\alpha_n \tau)^2}{1-\alpha_n l\gamma} M_3, \end{aligned} \tag{55}$$

where $M_3 = \sup_n \|x_n - x^*\|^2$, $n \geq 1$. Put $\gamma_n = 2\alpha_n(\tau - l\gamma)/(1 - \alpha_n l\gamma)$, $\delta_n = (2\alpha_n/(1 - \alpha_n l\gamma))\langle(\gamma V - \mu A)x^*, x_{n+1} - x^*\rangle + ((\alpha_n\tau)^2/(1 - \alpha_n l\gamma))M_3$. It is easy to see that $\limsup_{n\to\infty}\delta_n/\gamma_n \leq 0$. Hence, by Lemma 5, the sequence $\{x_n\}$ converges strongly to x^*. □

Remark 14. If we extend the equilibrium problem to be system of equilibrium problems, we still obtain the desired result by the similar proof of Theorem 13.

4. Numerical Result

In this section, we consider the following two simple examples to demonstrate the effectiveness, realization, and convergence of the algorithm in Theorem 13. Further, we compare convergence rates of the algorithm in this paper and [15].

First, we give an example as follows.

Example 15. In Theorem 13, let $H = R$, $C = [1/4, +\infty)$, $F \equiv 0$, for all $x, y \in C$. Define $S_0 : x \mapsto \sqrt{x}$, $S_1 : x \mapsto x + (\pi/4) - \arctan x$, and let $S_n = S_{\text{nmod2}}$, $n = 1, 2, \ldots$. Take $A = I$ with Lipschitz constant $k = 1$ and strongly monotone constant $\eta = 1$, $Vx = 2x$, for all $x \in H$ with Lipschitz coefficient $l = 2$. Give the parameters $\alpha_n = 1/20\sqrt{n}$, $\beta_n = 1/4$ for every $n \geq 1$, and fix $\mu = 1$ and $\gamma = 1/8$. Then $\{x_n\}$ is the sequence generated by

$$\begin{aligned} y_n &= \frac{1}{4}x_n + \frac{3}{4}L_n x_n, \\ x_{n+1} &= \frac{1}{8}\frac{1}{20\sqrt{n}}2x_n + \left(1 - \frac{1}{20\sqrt{n}}\right) y_n. \end{aligned} \tag{56}$$

As $n \to \infty$, we have $\{x_n\} \to x^* = 1$.

Let $\omega_i = 1/2^i$, $i = 1, 2, \ldots$; then we have $\sum_{i=1}^{\infty} \omega_i = 1$. Take the initial guess $x_1 = 1/2$, using software MATLAB R2012, we obtain the numerical experiment results in Table 1.

Let R^2 be the two-dimensional Euclidean space with usual inner product $\langle x^{(1)}, x^{(2)} \rangle = x_1^{(1)}x_1^{(2)} + x_2^{(1)}x_2^{(2)}$ (for all $x^{(1)} = (x_1^{(1)}, x_2^{(1)})^\top, x^{(2)} = (x_1^{(2)}, x_2^{(2)})^\top \in R^2$) and induced norm $\|x\| = \sqrt{x_1^2 + x_2^2}$ (for all $x = (x_1, x_2)^\top \in R^2$).

Table 2: (a) $x^{(1)} = (1, 0)^\top$. (b) $x^{(1)} = (1, 0)^\top$.

(a)

n (iterative number)	$x^{(1)}$ (initial guess)	Errors (n)
10	(0.8289, 0.5521)	4.6×10^{-3}
50	(0.8308, 0.5559)	3.8×10^{-4}
100	(0.8310, 0.5561)	1.9×10^{-4}

(b)

n (iterative number)	$x^{(1)}$ (initial guess)	Errors (n)
10	(0.8271, 0.5521)	5.6×10^{-3}
50	(0.8308, 0.5558)	4.7×10^{-4}
100	(0.8309, 0.5561)	2.2×10^{-4}

Next, we consider another simple example.

Example 16. In Theorem 13, let $H = R^2$, $C = [0, 1] \times [0, 1]$, $F \equiv 0$, for all $x, y \in C$. Give $S_1 = I, S_0 : x = (x_1, x_2)^\top \mapsto (\sin((x_1 + x_2)/\sqrt{2}), \cos((x_1 + x_2)/\sqrt{2}))^\top$, and let $S_n = S_{\text{nmod2}}$, $n = 1, 2, \ldots \omega_n = 1/2^n \in (0, 1)$, $n \geq 1$. Take $A = I$ with Lipschitz constant $k = 1$ and strongly monotone constant $\eta = 1$, $f(x) = ((1/4)x_1, -(1/4)x_2)^\top$, for all $x \in H$ with contraction coefficient $\rho = 1/4$. Give the parameters $\alpha_n = 1/10n$, $\beta_n = 1/2$ for every $n \geq 1$, and fix $\mu = 1$ and $\gamma = 1$. Then $\{x^{(n)}\}$ is the sequence generated by

$$\begin{aligned} y^{(n)} &= \frac{1}{2}x^{(n)} + \frac{1}{2}L_n x^{(n)}, \\ x^{(n+1)} &= \frac{1}{10n}\left(\frac{1}{4}x_1^{(n)}, -\frac{1}{4}x_2^{(n)}\right) + \frac{10n-1}{10n}y^{(n)}, \end{aligned} \tag{57}$$

As $n \to \infty$, we have $\{x^{(n)}\} \to x^* \approx (0.8310, 0.5562)^\top$.

For analysis of the rate of convergence, we use the concept introduced by Rhoades [20] as follows.

Definition 17. Let E be a closed interval on the real line and $f : E \to E$ a continuous function. Suppose that $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two iterations which converge to the fixed point p of f. Then, $\{x_n\}_{n=1}^{\infty}$ is said to converge faster than $\{y_n\}_{n=1}^{\infty}$ if

$$\|x_n - p\| \leq \|y_n - p\|, \quad \forall n \geq 1. \tag{58}$$

Now we turn to numerical simulation using the algorithm (57). Take the initial guess $x^{(1)} = (1, 0)^\top$ and $x^{(1)} = (1, 1)^\top$, respectively. All the numerical experiment results are given in Tables 2(a) and 3(a). Then we realize the algorithm in [15], and the W-mapping is used in the paper. Further we obtain the corresponding numerical results which can be found in Tables 2(b) and 3(b).

It is easy to see that the approximation values obtained by the algorithm (25) in this paper are more close to the common fixed point x^* at the same iterative number. And from the computer programming point of view, the algorithm is easier to implement in this paper.

Table 3: (a) $x^{(1)} = (1, 1)^{\top}$. (b) $x^{(1)} = (1, 1)^{\top}$.

(a)

n (iterative number)	$x^{(1)}$ (initial guess)	Errors (n)
10	(0.8341, 0.5534)	9.7×10^{-4}
50	(0.8308, 0.5559)	3.8×10^{-4}
100	(0.8310, 0.5561)	1.9×10^{-4}

(b)

n (iterative number)	$x^{(1)}$ (initial guess)	Errors (n)
10	(0.8359, 0.5531)	2.3×10^{-3}
50	(0.8308, 0.5558)	4.7×10^{-4}
100	(0.8309, 0.5561)	2.1×10^{-4}

Acknowledgments

The author would like to thank the referee for valuable suggestions to improve the paper and the Fundamental Research Funds for the Central Universities (Grant ZXH2012K001).

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 234681, 4 pages
http://dx.doi.org/10.1155/2013/234681

Research Article

Note on the Regularity of Nonadditive Measures

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Received 6 June 2013; Revised 25 July 2013; Accepted 25 July 2013

Academic Editor: Wei-Shih Du

We consider the regularity for nonadditive measures. We prove that the non-additive measures which satisfy Egoroff's theorem and have pseudometric generating property possess Radon property (strong regularity) on a complete or a locally compact, separable metric space.

1. Introduction

The relations of continuity and regularity of nonadditive measures are considered in several papers [1–4]. In [5], Li et al. investigated the regularity in nonadditive measures. They proved that the null-additive fuzzy measures possess a Radon property (strong regularity) on a complete metric space. In [6], Kawabe also investigated the regularity in fuzzy measures taking value in Riesz spaces. He proved that every weakly null-additive Riesz space valued fuzzy measure on a complete or a locally compact, separable metric space is Radon, provided that the Riesz space has the multiple Egoroff property.

On the other hand Li and Mesiar [7] proved the regularity of nonadditive monotone measures. They proved that the equivalence condition of Egoroff's theorem implies regularity for the nonadditive measures by using pseudometric generating property of a set function. For information on real valued nonadditive measures, see [8–10].

In this paper, as notes, we prove that Egoroff's theorem implies Radon property (strong regularity) for nonadditive measures which have pseudometric generating property on a complete or a locally compact, separable metric space.

2. Preliminaries

Let R be the set of real numbers and N the set of natural numbers. In what follows, let $(X, \mathscr{F})$ be a measurable space.

Definition 1. A set function $\mu : \mathscr{F} \to R$ is called a nonadditive measure if it satisfies the following two conditions:

(1) $\mu(\emptyset) = 0$,

(2) if $A, B \in \mathscr{F}$ and $A \subset B$, then $\mu(A) \leq \mu(B)$.

Definition 2. Let $\mu : \mathscr{F} \to R$ be a nonadditive measure.

(1) μ is said to be continuous from above if for any $\{A_n\} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfying $A_n \searrow A$ and there exists n_0 with $\mu(A_{n_0}) < \infty$ it holds that $\lim_{n\to\infty}\mu(A_n) = \mu(A)$.

(2) μ is said to be continuous from below if for any $\{A_n\} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfying $A_n \nearrow A$ it holds that $\lim_{n\to\infty}\mu(A_n) = \mu(A)$.

(3) μ is said to be fuzzy measure if it is continuous from above and below.

(4) μ is said to be strongly order continuous if it is continuous from above at measurable sets of measure 0; that is, for any $\{A_n\} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfying $A_n \searrow A$ and $\mu(A) = 0$ it holds that $\lim_{n\to\infty}\mu(A_n) = 0$.

(5) μ is said to be weakly null-additive if $\mu(A \cup B) = 0$ whenever $A, B \in \mathscr{F}$ and $\mu(A) = \mu(B) = 0$.

(6) μ has property (S) if for any sequence $\{A_n\} \subset \mathscr{F}$ with $\lim_{n\to\infty}\mu(A_n) = 0$ there exists a subsequence $\{A_{n_k}\}$ such that $\mu(\cap_{i=1}^{\infty}\cup_{k=i}^{\infty}A_{n_k}) = 0$; see [11].

(7) μ is said to be autocontinuous from above if $\lim_{n\to\infty}\mu(A \cup B_n) = \mu(A)$ for each $A \in \mathscr{F}$ and $\{B_n\} \subset \mathscr{F}$ with $\lim_{n\to\infty}\mu(B_n) = 0$.

(8) μ is said to be autocontinuous from below if $\lim_{n\to\infty}\mu(A \setminus B_n) = \mu(A)$ for each $A \in \mathscr{F}$ and $\{B_n\} \subset \mathscr{F}$ with $\lim_{n\to\infty}\mu(B_n) = 0$.

(9) μ is said to be autocontinuous if it is autocontinuous from above and below.

Definition 3. Let $\mu : \mathscr{F} \to R$ be a nonadditive measure.

(1) A double sequence $\{A_{m,n}\} \subset \mathscr{F}$ is said to be a μ-regulator if it satisfies the following two conditions:

(D1) $A_{m,n} \supset A_{m,n'}$ whenever $n \le n'$,
(D2) $\mu(\cup_{m=1}^{\infty}\cap_{n=1}^{\infty}A_{m,n}) = 0$.

(2) μ satisfies the Egoroff condition if for any μ-regulator $\{A_{m,n}\}$ and for every $\varepsilon > 0$ there exists a sequence $\{n_m\}$ of natural numbers such that $\mu(\cup_{m=1}^{\infty}A_{m,n_m}) < \varepsilon$.

Remark 4. A nonadditive measure μ satisfies the Egoroff condition if (and only if) for any double sequence $\{A_{m,n}\} \subset \mathscr{F}$ satisfying (D2) and the following (D1$'$) it holds that for every $\varepsilon > 0$ there exists a sequence $\{n_m\}$ of natural numbers such that $\mu(\cup_{m=1}^{\infty}A_{m,n_m}) < \varepsilon$:

(D1$'$) $A_{m,n} \supset A_{m',n'}$ whenever $m \ge m'$ and $n \le n'$.

3. Compact Measure and Regularity of Measure

In this section, we pick up several results for compact nonadditive measures and regularity of measures.

Definition 5. Let $\mu : \mathscr{F} \to R$ be a nonadditive measure.

(1) A nonempty family $\mathscr{K}$ of subsets of X is called a compact system if for any sequence $\{K_n\} \subset \mathscr{K}$ with $\cap_{n=1}^{\infty}K_n = \emptyset$ there is $n_0 \in N$ such that $\cap_{n=1}^{n_0}K_n = \emptyset$; see [12].

(2) We say that μ is compact if there exists a compact system $\mathscr{K}$ such that for each $A \in \mathscr{F}$ there are sequences $\{K_n\} \subset \mathscr{K}$ and $\{B_n\} \subset \mathscr{F}$ such that $B_n \subset K_n \subset A$ for all $n \in N$ and $\lim_{n\to\infty}\mu(A \setminus B_n) = 0$.

Remark 6. (1) The family of all compact subsets of a Hausdorff space is a compact system.

(2) The family of all finite unions of sets in a compact system is also compact [13, Lemma 1.4]. Therefore, in (2) of the above definition, the compact system $\mathscr{K}$ and the sequences $\{K_n\} \subset \mathscr{K}$ and $\{B_n\} \subset \mathscr{F}$ may be chosen so that $\mathscr{K}$ is closed for finite unions and both $\{K_n\}$ and $\{B_n\}$ are increasing.

By [6, Theorem 1], the following result follows.

Theorem 7. *Let $\mu : \mathscr{F} \to R$ be a nonadditive measure. If μ is compact and autocontinuous, then it is continuous from above and below.*

Proof. Since μ is compact and autocontinuous, by [6, Theorem 1], the assertion follows. □

In what follows, let (X, d) be a metric space. Denote by $\mathscr{B}(X)$ the σ-field of all Borel subsets of X, that is, the σ-field generated by the open subsets of X. A nonadditive measure defined on $\mathscr{B}(X)$ is called a nonadditive Borel measure on X.

Definition 8. μ is said to have pseudometric generating property if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $A, B \in \mathscr{B}(X)$, $\mu(A) \vee \mu(B) < \delta$ implies $\mu(A \cup B) < \varepsilon$.

Proposition 9. *If μ satisfies pseudometric generating property, then it is weakly null-additive.*

Proof. It is easy to see from the definition. □

Definition 10. Let $\mu : \mathscr{B}(X) \to R$ be a nonadditive Borel measure on X.

μ is called regular if for any $\varepsilon > 0$ and $A \in \mathscr{B}(X)$, there exist a closed set F_ε and an open set G_ε such that $F_\varepsilon \subset A \subset G_\varepsilon$ and $\mu(G_\varepsilon \setminus F_\varepsilon) < \varepsilon$.

Li and Mesiar [7] also investigated the regularity on monotone measures. The following follows.

Lemma 11. *Let X be a metric space and $\mu : \mathscr{B}(X) \to R$ a nonadditive Borel measure on X. If μ has the Egoroff condition and pseudometric generating property, then μ is regular.*

Corollary 12. *Let X be a metric space and $\mu : \mathscr{B}(X) \to R$ a nonadditive Borel measure on X. If μ has property (S), is strong order continuous, and is weakly null-additive, then μ is regular.*

By Li and Yasuda [14, Theorem 1], we also have the following.

Corollary 13. *Let X be a metric space. If $\mu : \mathscr{B}(X) \to R$ is weakly null-additive fuzzy Borel measure on X, then it is regular. Moreover if μ is null-additive, we have*

$$\begin{aligned}\mu(A) &= \sup\{\mu(F) \mid F \subset A, F \text{ is closed set}\} \\ &= \inf\{\mu(G) \mid G \supset A, G \text{ is open set}\}.\end{aligned} \tag{1}$$

Corollary 13 above is a special case of [6, Theorem 5] and [15, Theorem 3].

For more information on regularity of nonadditive measures, see [5, 6].

4. Radon Measure

In this section, as main results, if we assume that a nonadditive Borel measure satisfies the equivalence condition of Egoroff's theorem and pseudometric generating property on a complete or a locally compact, separable metric space, then it is Radon.

Definition 14. Let μ be a nonadditive Borel measure on X.

(1) μ is said to be Radon (strongly regular) if for each $A \in \mathscr{B}(X)$ there are sequences $\{K_n\}_{n\in N}$ of compact

sets and $\{G_n\}_{n\in N}$ of open sets such that $K_n \subset A \subset G_n$ for all $n \in N$ and $\lim_{n\to\infty}\mu(G_n \setminus K_n) = 0$.

(2) μ is said to be tight if there is a sequence $\{K_n\}_{n\in N}$ of compact sets such that $\lim_{n\to\infty}\mu(X \setminus K_n) = 0$.

Remark 15. Sequences of sets in the above definition may be chosen so that $\{G_n\}_{n\in N}$ is decreasing, while $\{F_n\}_{n\in N}$ and $\{K_n\}_{n\in N}$ are increasing.

Proposition 16. *Let X be a Hausdorff space. Let μ be a nonadditive Borel measure on X which is weakly null-additive and strongly order continuous. Then, the following two conditions are equivalent:*

(i) *μ is Radon (strongly regular),*

(ii) *μ is regular and tight.*

Proof. See [6, Proposition 2]. □

It is known that every finite Borel measure on a complete or a locally compact, separable metric space is Radon; see [16, Theorem 3.2] and [17, Theorems 6 and 9, Chapter II, Part I]. Its counterpart in nonadditive measure theory can be found in [5, 9, Theorem 1, Lemma 2], which states that every Borel fuzzy measure on a complete separable metric space is tight, so that it is Radon if it is null-additive; see also [3, Theorem 2.3]. The following two theorems contain those previous results; see also [18, Theorem 12].

Theorem 17. *Let X be a complete separable metric space and $\mu : \mathscr{B}(X) \to R$ a nonadditive Borel measure on X. If μ is weakly null-additive and satisfies the Egoroff condition, then it is tight. Moreover, if μ has pseudometric generating property and satisfies the Egoroff condition, then it is Radon.*

To prove the theorem, we need the following; see [7, Proposition 3.7].

Proposition 18. *Let $\mu : \mathscr{F} \to R$ be a nonadditive measure. Then (i) implies (ii).*

(i) *μ is weakly null-additive and satisfies the Egoroff condition.*

(ii) *For each $\varepsilon > 0$ and double sequence $\{A_{m,n}\} \subset \mathscr{F}$ satisfying $A_{m,n} \searrow \emptyset$ as $n \to \infty$ for each $m \in N$, there exists a sequence $\{n_m\}$ of natural numbers such that $\mu(\cup_{m=1}^{\infty} A_{m,n_m}) < \varepsilon$.*

Proof of Theorem 17. Since μ satisfies the Egoroff condition, by [19, Proposition 3], it is strongly order continuous. By Proposition 16 and Lemma 11, we have only to prove that μ is tight. Let $\{s_i\}_{i\in N}$ be a countable dense subset of X. For each $m, i \in N$, denote by $\overline{B_m(s_i)}$ the closed ball with center s_i and radius $1/m$. For each $m, n \in N$, put $A_{m,n} := X \setminus \cup_{i=1}^{n}\overline{B_m(s_i)}$. Then, for any $\varepsilon > 0$ and $m \in N$, we have $A_{m,n} \searrow \emptyset$, so that by Proposition 18, there exists a sequence $\{n_m\}$ of natural numbers such that

$$\mu\left(\cup_{m=1}^{\infty} A_{m,n_m}\right) < \varepsilon. \tag{2}$$

Put $P_\varepsilon := \cap_{m=1}^{\infty}\cup_{i=1}^{n_m}\overline{B_m(s_i)}$. Then, each P_ε is closed and totally bounded, so that it is compact. Since $X \setminus P_\varepsilon = \cup_{m=1}^{\infty} A_{m,n_m}$, it follows from (2) that $\mu(X \setminus P_\varepsilon) < \varepsilon$. Thus μ is tight. □

Corollary 19. *Let X be a complete separable metric space and $\mu : \mathscr{B}(X) \to R$ a nonadditive Borel measure on X. If μ is weakly null-additive, strongly order continuous, and has property (S), then it is Radon.*

Proof. It follows from Theorem 17 since μ has pseudometric generating property [7, Proposition 5.1] and satisfies the Egoroff condition [19, Proposition 2]. □

Corollary 20. *Let X be a complete separable metric space and $\mu : \mathscr{B}(X) \to R$ a fuzzy measure on X. If μ is weakly null-additive, then it is Radon.*

Proof. It follows from Theorem 17 since μ satisfies the Egoroff condition [7, Proposition 3.1] and it is regular [14, Theorem 1]. □

Remark 21. Corollary 20 above is a special case of [6, Theorem 5] and [15, Theorem 3].

Theorem 22. *Let X be a locally compact, separable metric space and $\mu : \mathscr{B}(X) \to R$ a nonadditive Borel measure on X. If μ is weakly null-additive and satisfies the Egoroff condition, then it is tight. Moreover, if μ has pseudometric generating property and satisfies Egoroff condition, then it is Radon.*

Proof. By Lemma 11 and Proposition 16, we have only to prove the tightness of μ. Denote by $\mathscr{H}$ the family of all open and relatively compact subsets of X. The local compactness of X implies that $\mathscr{H}$ is an open cover of X. Since X is strongly Lindelöf, that is, every open cover of any open subset of X has a countable subcover [17, Proposition 3 and Theorem 6, Chapter II, Part I], there is a sequence $\{H_m\}_{m\in N} \subset \mathscr{H}$ such that $X = \cup_{m=1}^{\infty} H_m$. Put $K_n := \cup_{m=1}^{n}\overline{H_m}$ for all $n \in N$, where $\overline{A}$ denotes the closure of a set A. Then K_n is compact and $X \setminus K_n \searrow \emptyset$. Since μ is strongly order continuous [19, Proposition 3], $\lim_{n\to\infty}\mu(X \setminus K_n) = 0$. Thus μ is tight. □

Corollary 23. *Let X be a locally compact, separable metric space and $\mu : \mathscr{B}(X) \to R$ a nonadditive Borel measure on X. If μ is weakly-null-additive, strongly order continuous, and has property (S), then μ is Radon.*

Corollary 24. *Let X be a locally compact, separable metric space and $\mu : \mathscr{B}(X) \to R$ a fuzzy Borel measure on X. If μ is weaklly null-additive, then μ is Radon.*

Remark 25. Corollary 24 above is a special case of [6, Theorem 6] and [15, Theorem 4].

Acknowledgments

The authors are grateful to the referees for their valuable suggestions and careful reading to revise this paper. The first author would like to express his hearty thanks to Professor

Shizu Nakanishi. The first author also would like to express his hearty thanks to Professor Toshiharu Kawasaki and Professor Ichiro Suzuki for many valuable suggestions.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 218402, 6 pages
http://dx.doi.org/10.1155/2013/218402

Research Article

The Existence and Stability of Solutions for Vector Quasiequilibrium Problems in Topological Order Spaces

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Received 4 June 2013; Accepted 18 July 2013

Academic Editor: Wei-Shih Du

In a topological sup-semilattice, we established a new existence result for vector quasiequilibrium problems. By the analysis of essential stabilities of maximal elements in a topological sup-semilattice, we prove that for solutions of each vector quasi-equilibrium problem, there exists a connected minimal essential set which can resist the perturbation of the vector quasi-equilibrium problem.

1. Introduction

Vector equilibrium problems can unify many nonlinear problems such as vector optimization, vector variational inequality [1], and vector complementarity problems [2]. Recently, not only vector equilibrium problems [3–7] but also vector quasiequilibrium problems [8–14] and the system of vector quasi-equilibrium problems have attracted much attention [15–19].

Topological vector spaces provide the usual mathematical framework in the study of many problems. To avoid the linear feature, sup-semilattices may be good choices. In fact, some results like the existence of KKM points in topological spaces were established in topological sup-semilattices [20], where a two-tuple $(X, \leq)$ is said to be a sup-semilattice, if X is a partially ordered set with the partial ordering $\leq$, in which every pair (x, x') has a least upper bound $x \vee x'$.

The aim of this paper is to study the existence and essential stability of vector quasi-equilibrium problems in topological sup-semilattices. In order to achieve this, firstly, we give a stability result in relation to maximal elements in a topological sup-semilattice. Secondly, a new existence result for vector quasi-equilibrium problems is established, and we show that each vector quasi-equilibrium problem has a connected minimal essential set in its solution set.

2. Preliminaries

Let $(X, \leq)$ be a sup-semilattice. If x and x' are two elements in $(X, \leq)$ and $x \leq x'$, the set $[x, x'] = \{y \in X : x \leq y \leq x'\}$ is called an order interval. Let A, A' be two nonempty finite subsets of X. Then the set $\Delta A = \cup_{x \in A}[x, \sup A]$ is well defined and has the properties: $A \subseteq \Delta A$ and $\Delta A \subseteq \Delta A'$ if $A \subseteq A'$.

Definition 1 (see [20]). A subset $E \subseteq X$ is Δ-convex, if for any nonempty finite subset $A \subseteq E$, we have $\Delta\ A \subseteq E$. E being a Δ-convex set is equivalent to the following conditions:

(a) if $x, x' \in E$, then its least upper bound $x \vee x' \in E$.

(b) if $x, x' \in E$ and $x \leq x'$, then the order interval $[x, x'] \subset E$.

It is easy to check that the intersection of two Δ-convex sets is Δ-convex as well.

A topological space X is said to be a topological sup-semilattice if X is equipped with a sup-semilattice as its partial ordering denoted by $\leq$, for which $f : X \times X \to X$ with $(x, x') \to x \vee x'$ is a continuous function.

Let Y be a topological vector space and θ the zero element in Y. A subset $C \subset Y$ is called a cone if, for any $y \in C$ and real number $t > 0$, $ty \in C$. A cone C is convex if C is a convex set. If $C \cap -C = \{\theta\}$, it is called a pointed cone.

Definition 2 (see [18]). Let X be a topological sup-semilattice, Y a topological vector space with a cone $C \subset Y$, $\varphi : X \to Y$ a vector-valued function.

(a) $\varphi : X \to Y$ is C_Δ-quasiconcave if, for any nonempty two points subset $A = \{x_1, x_2\} \subset X$ and $y \in Y, \varphi(A) \subset y + C \Rightarrow \varphi(\Delta A) \subset y + C$.

(b) $\varphi : X \to Y$ is said to be C_Δ-quasiconcave-like if, for any $x_1, x_2 \in X$, $\varphi(\Delta\{x_1, x_2\}) \in \varphi(x_1) + C$ or $\varphi(\Delta\{x_1, x_2\}) \in \varphi(x_2) + C$.

Remark 3. In general cases, C_Δ-quasiconcave, C_Δ-quasiconcave-like, and usual quasiconcave functions are independent of each other. See examples in [18]. Let $\varphi : \mathbb{R} \to \mathbb{R}$, $C = -\mathbb{R}^+$. Then the partial order on $\mathbb{R}$ is "$\leq$" (less than or equal to);, hence, the C_Δ-quasiconcave, C_Δ-quasiconcave-like, and usual quasiconcave property of φ coincide (the usual quasiconcave function φ means that for any x_1, x_2, $y \in \mathbb{R}$, $\varphi(x_1) \leq y$ and $\varphi(x_2) \leq y \Rightarrow \varphi(\lambda x_1 + (1-\lambda)x_2) \leq y$, for all $\lambda \in [0,1]$).

Lemma 4 (see [18]). *Let X be a topological sup-semilattice, Y a Hausdorff locally convex topological vector space with a closed, convex, and pointed cone $C \subset Y$. If the vector-valued function $\varphi : X \to Y$ is C_Δ-quasiconcave or C_Δ-quasiconcave-like, then the set $A = \{x : \varphi(x) \in \operatorname{int} C\}$ is Δ-convex.*

Now we introduce the vector quasiequilibrium problem (VQEP) that we will consider in this paper.

Let X be a topological sup-semilattice and Y a topological vector space. $C \subset Y$ is a closed, convex, and pointed cone with $\operatorname{int} C \neq \emptyset$. $\varphi : X \times X \to Y$ is a vector-valued function, and G is a multivalued mapping on X. The vector quasi-equilibrium problem ϕ with $\phi = (X, Y, C, \varphi, G)$ is to find $\overline{x} \in X$, such that

$$\overline{x} \in G(\overline{x}) : \varphi(\overline{x}, y) \notin \operatorname{int} C, \quad \forall y \in G(\overline{x}). \tag{1}$$

Let $G(x) = X$, for all $x \in X$; then the VQEP is just a vector equilibrium problem (X, Y, C, φ) (VEP). That is to find $\overline{x} \in X$, such that

$$\varphi(\overline{x}, y) \notin \operatorname{int} C, \quad \forall y \in X. \tag{2}$$

Definition 5 (see [21, 22]). A vector-valued function $\varphi : X \to Y$ is said to be C-continuous on X if, for each $x \in X$ and any open neighborhood $V(\theta)$ of θ in Y, there exists an open neighborhood $O(x)$ of x in X such that

$$\forall x' \in O(x), \quad \varphi(x') \in \varphi(x) + V(\theta) + C. \tag{3}$$

Remark 6. For a function $\varphi : X \to \mathbb{R}$, φ is $\mathbb{R}_+$-*continuous* on X if and only if φ is lower semicontinuous on X.

A maximal element version of the Browder fixed point theorem in a topological sup-semilattice can be found in [18]. We limit it in a metric space as the following lemma.

Lemma 7 (see [18]). *Let (X, d) be a compact sup-semilattice with path connected interval, where d is the metric on X, $S : X \to 2^X$ a multivalued map on X with the conditions: (i) for all $x \in X$, $S(x)$ is Δ-convex; (ii) for all $y \in X$, $S^{-1}(y) = \{x \in X : y \in S(x)\}$ is open in X; (iii) for all $x \in X$, $x \notin S(x)$. Then there exists an $\overline{x} \in X$, such that $S(\overline{x}) = \emptyset$.*

Remark 8. The existence of a metric space with a sup-semilattice can be guaranteed. For instance, let $x^j = (x_1^j, \ldots, x_i^j, \ldots, x_n^j) \in \mathbb{R}^n$, $j = 1, 2$, if $x^1 \leq x^2$ means that $x^2 \in x^1 + \mathbb{R}_+^n$, then $x^1 \vee x^2 = \overline{x}$, where $\overline{x}_i = \max\{x_i^1, x_i^2\}$. Clearly, $(\mathbb{R}^n, \leq)$ with the usual Euclidean metric is a topological sup-semilattice.

Let M denote the collection of S satisfying all the conditions of Lemma 7. For any $S_1, S_2 \in M$, define the metric between S_1 and S_2 as

$$\rho(S_1, S_2) = \sup_{y \in X} h\left(X \setminus S_1^{-1}(y), X \setminus S_2^{-1}(y)\right), \tag{4}$$

where h is the Hausdorff metric induced by d. For each $S \in M$ and each $y \in X$, since $y \notin S(y)$, we have $y \in X \setminus S^{-1}(y)$, that is, $X \setminus S^{-1}(y) \neq \emptyset$. Noting that $X \setminus S^{-1}(y)$ is closed, the metric ρ on M is well defined. Then (M, ρ) is a metric space.

For each $S \in M$, denote by $F(S)$ the set of all maximal elements of S. Then F defines a multivalued mapping from M to X and $F(S) = \cap_{y \in X}(X \setminus S^{-1}(y))$.

Definition 9. For each $S \in M$, a set $e(S)$ is called an essential set of $F(S)$ if it satisfies the following conditions:

(1) $e(S)$ is closed subset of $F(S)$.

(2) For any open set $U \supset e(S)$, there exists an open neighborhood $O(S)$ of $S \in M$ such that $U \cap F(S') \neq \emptyset$, for any $S' \in O(S)$.

A set $m(S)$ is called a minimal essential set of $F(S)$ if it is a minimal element of all essential sets ordered by set inclusion in $F(S)$. A connected component in $F(S)$ is called an essential component, if it includes at least one minimal essential set of $F(S)$.

We recall some notions about multi-valued mappings. Let $G : Y \to 2^P$ be a multi-valued mapping, where Y, P are two topological vector spaces. Then (i) G is said to be upper semicontinuous at $y \in Y$, if for each open set $U \supset G(y)$, there exists an open neighborhood $O(y)$ of y such that $U \supset G(y')$ for any $y' \in O(y)$. (ii) G is lower semi-continuous at $y \in Y$, if for each open set $U \cap G(y) \neq \phi$, there exists an open neighborhood $O(y)$ of y such that $U \cap G(y') \neq \phi$ for any $y' \in O(y)$.

Remark 10. For each $S \in M$, a set $e(S) \subset F(S)$ is essential if F is lower semi-continuous at S. If F is upper semi-continuous at S, then $F(S)$ itself is an essential set. For any two closed sets $A, B \subset F(S)$ with $A \subset B$, if A is essential, then B is also essential. For each $S \in M$ and each $y \in X$, if $S^{-1}(y)$ is open, then $X \setminus S^{-1}(y)$ is closed; hence, $F(S)$ is closed because we have $F(S) = \cap_{y \in X} X \setminus S^{-1}(y)$; consequently, $F(S)$ is compact.

Lemma 11 (see [23]). *Let (X, d) be a metric space, K_1 and K_2 two nonempty compact subsets of X, U_1, and U_2 two nonempty disjoint open subsets of X. If $h(K_1, K_2) < d(U_1, U_2)$, then $h(K_1, (K_1 \setminus U_2) \cup (K_2 \setminus U_1)) \leq h(K_1, K_2)$, where h is the Hausdorff metric defined on X.*

3. The Stability of Maximal Elements on Topological Semilattices

Theorem 12. *$F : M \to 2^X$ is an upper semi-continuous mapping with compact values.*

Proof. For each $S \in M$, by Remark 10, $F(S)$ is compact. Suppose that F is not upper semi-continuous. Then there is a $S \in M$, an open set U with $U \supset F(S)$ and S_n, such that $S_n \to S$ and $F(S_n) \not\subset U$, $n = 1, 2, \ldots$. That is, there exists a point $x_n \in F(S_n)$ such that $x_n \notin U$. Without loss of generality, we may assume that $x_n \to x^*$. Since $S_n \to S$, it holds that $X \setminus S_n^{-1}(y) \to X \setminus S^{-1}(y)$, for all $y \in X$. Since $x_n \in F(S_n)$, we have $x_n \in X \setminus S_n^{-1}(y)$, for all $y \in X$. As n gets close to infinity, we can obtain that $x^* \in X \setminus S^{-1}(y)$, for all $y \in X$, that is, $x^* \in F(S) \subset U$. This results in the fact that $x_n \in U$ while n is large enough, a contradiction with $x_n \notin U$. Therefore, F is definitely upper semi-continuous. □

Theorem 13. *For each $S \in M$, there exists at least a minimal essential set of $F(S)$. If $m(S)$ is a minimal essential set of $F(S)$, then $m(S)$ is connected.*

Proof. For the existence, by Remark 10, each decreasing chain, consisting of essential subsets of $F(S)$, has a minimal element, which is the intersection of the chain. By the Zorn's lemma, the minimal element is just a minimal essential set. For the connectedness, by way of contradiction, suppose that $m(S)$ is not connected. There exist two disjoint closed sets $C_1(S), C_2(S)$ such that $m(S) = C_1(S) \cup C_2(S)$.

Since $C_i(S)$ is not essential, there is an open set W_i with $W_i \supset C_i(S)$ such that for any $\varepsilon > 0$, there exists a $S_i \in M$ with $\rho(S, S_i) < \varepsilon$ and $F(S_i) \cap W_i = \emptyset$, $i = 1, 2$. Clearly, $C_i(S)$ is compact, then there is an open set U_i with $C_i(S) \subset U_i \subset W_i$, $i = 1, 2$, such that $U_1 \cap U_2 = \emptyset$. For $U_1 \cup U_2 \supset m(S)$, because $m(S)$ is essential, there is a number $\delta < 2d(U_1, U_2)$, such that $F(T) \cap (U_1 \cup U_2) \neq \emptyset$ for each T satisfying $\rho(T, S) < \delta$. Therefore, we can select a $S_i \in M$ such that $\rho(S, S_i) < \delta/4$ and $F(S_i) \cap U_i = \emptyset$, $i = 1, 2$. Then $\rho(S_1, S_2) < \rho(S_1, S) + \rho(S, S_2) < \delta/2 < d(U_1, U_2)$.

Define a multi-valued mapping $S' : M \to 2^X$ as

$$S' = \begin{cases} S_1(x), & x \in U_1, \\ S_2(x), & x \in U_2, \\ S_1(x) \cap S_2(x), & x \in X \setminus (U_1 \cup U_2). \end{cases} \tag{5}$$

We show that $S' \in M$.

(a) For each $x \in X$, since $x \notin S_1(x)$ and $x \notin S_2(x)$, we have $x \notin S'(x)$;

(b) for each $x \in X$, because $S_1(x)$ and $S_2(x)$ are Δ-convex sets, it follows that $S'(x)$ is Δ-convex;

(c) for each $y \in X$, we have

$$S'^{-1}(y) = \left(S_1^{-1}(y) \cap S_2^{-1}(y)\right) \cup \left(S_2^{-1}(y) \cap U_2\right) \cup \left(S_1^{-1}(y) \cap U_1\right). \tag{6}$$

Noting that $S_1^{-1}(y)$, $S_2^{-1}(y)$, U_1, and U_2 are open sets, it follows that $S'^{-1}(y)$ is open.

Through a direct calculation, $X \setminus S'^{-1}(y)$ can be written as

$$\begin{aligned} &\left(\left(X \setminus S_1^{-1}(y)\right) \cap \left(X \setminus S_2^{-1}(y)\right)\right) \\ &\quad \cup \left(\left(X \setminus S_1^{-1}(y)\right) \cap (X \setminus U_2)\right) \\ &\quad \cup \left(\left(X \setminus S_2^{-1}(y)\right) \cap (X \setminus U_1)\right). \end{aligned} \tag{7}$$

Take any $x \in X \setminus S'^{-1}(y)$. Note that if $x \in U_1$, then $x \in X \setminus U_2$; if $x \in U_2$, then $x \in X \setminus U_1$; if $x \in X \setminus (U_1 \cup U_2)$, then $x \in (X \setminus U_1) \cap (X \setminus U_2)$. Consequently, we can obtain that if $x \in (X \setminus S_1^{-1}(y)) \cap (X \setminus S_2^{-1}(y))$, then

$$\begin{aligned} x \in &\left(\left(X \setminus S_1^{-1}(y)\right) \cap (X \setminus U_2)\right) \\ &\cup \left(\left(X \setminus S_2^{-1}(y)\right) \cap (X \setminus U_1)\right). \end{aligned} \tag{8}$$

Therefore, we have

$$\begin{aligned} &X \setminus S'^{-1}(y) \\ &\quad = \left(\left(X \setminus S_1^{-1}(y)\right) \cap (X \setminus U_2)\right) \\ &\qquad \cup \left(\left(X \setminus S_2^{-1}(y)\right) \cap (X \setminus U_1)\right). \end{aligned} \tag{9}$$

Since $\rho(S_1, S_2) < d(U_1, U_2)$, by Lemma 11, we have

$$\begin{aligned} &h\left(X \setminus S'^{-1}(y), X \setminus S_1^{-1}(y)\right) \\ &\quad \le h\left(X \setminus S_2^{-1}(y), X \setminus S_1^{-1}(y)\right) \\ &\quad \le \rho(S_2, S) + \rho(S, S_1) < \frac{\delta}{2}. \end{aligned} \tag{10}$$

That is, $\rho(S', S_1) < \delta/2$. This results in the fact that

$$\rho(S', S) < \rho(S', S_1) + \rho(S_1, S) < \delta. \tag{11}$$

Consequently, we have $F(S') \cap (U_1 \cup U_2) \neq \emptyset$. If there is a point $x \in U_i \cap F(S')$, then $x \in U_i$ and $x \in X \setminus S_i^{-1}(y)$, for all $y \in X$, that is, $x \in U_i$ and $x \in F(S_i)$ which contradicts with $F(S_i) \cap U_i = \emptyset$, $i = 1, 2$. Therefore, $m(S)$ is connected. □

4. The Existence and Stability of Solutions for VQEP

This section gives an existence result in relation to VQEP in topological sup-semilattices and induces the existence of minimal essentially stable sets for each VQEP in the set of its solutions.

Theorem 14. *Let $\phi = (X, Y, C, \varphi, G)$ be a VQEP, where X is a compact topological sup-semilattice with path connected intervals, Y is a Hausdorff locally convex topological vector space, and $G : X \to 2^X$ is a multi-valued mapping with nonempty and Δ-convex values. If the VQEP satisfies that*

(i) *for all* $x \in X$, $\varphi(x,x) \notin \operatorname{int} C$;

(ii) *for all* $y \in X$, $x \to \varphi(x,y)$ *is* C*-continuous;*

(iii) *for all* $x \in X$, $y \to \varphi(x,y)$ *is* C_Δ*-quasiconcave-like or* C_Δ*-quasiconcave;*

(iv) *for all* $y \in X$, $\{x \in X : y \in G(x)\}$ *is open in* X,

(v) $\{x \in X : x \in G(x)\}$ *is closed in* X,

then the VQEP has a solution.

Proof. Denote $\{x \in X : x \in G(x)\}$ by K. Let $B : X \to 2^X$ such that $B(x) = \{y \in X : \varphi(x,y) \in \operatorname{int} C\}$, for all $x \in X$. Define

$$S(x) = \begin{cases} B(x) \cap G(x), & x \in K, \\ G(x), & x \in X \setminus K. \end{cases} \tag{12}$$

Then for each $x \in X$, if $x \in K$, we have $S(x) = B(x) \cap G(x)$, by the condition (i), $x \notin B(x)$, hence, $x \notin S(x)$; if $x \in X \setminus K$, from the definition of K, we have $x \notin G(x) = S(x)$.

Since $y \to \varphi(x,y)$ is C_Δ-quasiconcave-like or C_Δ-quasiconcave, by Lemma 4, we have that $B(x)$ is Δ-convex. Then $B(x) \cap G(x)$ is a Δ-convex set, noting that G has Δ-convex values, we have that $S(x)$ is also Δ-convex.

For each $y \in X$, we can check that

$$S^{-1}(y) = \left(B^{-1}(y) \cap G^{-1}(y)\right) \cup \left(G^{-1}(y) \cap (X \setminus K)\right). \tag{13}$$

Take a point $x \in B^{-1}(y) = \{x \in X : \varphi(x,y) \in \operatorname{int} C\}$, since $\operatorname{int} C$ is open, there is an open set $V(\theta)$ such that $V(\theta) + \varphi(x,y) \subset \operatorname{int} C$, then, by the condition (ii), there exists an open neighborhood $O(x)$ in X such that for all $x' \in O(x)$,

$$\varphi(x', y) \in \varphi(x,y) + V(\theta) + C \subset \operatorname{int} C + C \subset \operatorname{int} C. \tag{14}$$

That is, $O(x) \subset B^{-1}(y)$; hence, $B^{-1}(y)$ is open. Noting that $X \setminus K$ and $G^{-1}(y)$ are open sets in X. We can obtain that $S^{-1}(y)$ is also open in X.

Thus, there is an $\overline{x} \in X$ such that $S(\overline{x}) = \emptyset$ by Lemma 7. If $\overline{x} \in X \setminus K$, then $G(\overline{x}) = \emptyset$, a contradiction to the fact that G has nonempty values. Therefore, $\overline{x} \in K$ and $B(\overline{x}) \cap G(\overline{x}) = \emptyset$, that is, $\overline{x} \in G(\overline{x})$, $\varphi(\overline{x}, y) \notin \operatorname{int} C$, for all $y \in X$. □

By Theorem 14 and its proof, we can also obtain the existence result for VQEP as the following.

Corollary 15. *Let* $\phi = (X, Y, C, \varphi, G)$ *be a VQEP, where* X *is a compact topological sup-semilattice with path connected intervals,* Y *is a Hausdorff topological vector space, and* $G : X \to 2^X$ *is a multi-valued mapping with nonempty and* Δ*-convex values. If the VQEP satisfies that*

(i) *for all* $x \in X$, $\varphi(x,x) \notin \operatorname{int} C$;

(ii) *for all* $y \in X$, $\{x \in X : \varphi(x,y) \in \operatorname{int} C\}$ *is open in* X;

(iii) *for all* $x \in X$, $\{y \in X : \varphi(x,y) \in \operatorname{int} C\}$ *is* Δ*-convex;*

(iv) *for all* $y \in X$, $\{x \in X : y \in G(x)\}$ *is open in* X,

(v) $\{x \in X : x \in G(x)\}$ *is closed in* X,

then the VQEP has a solution.

By Theorem 14, for the special case of VQEP without the feasible mapping G, we can obtain the existence result concerning VEP as the following.

Corollary 16. *Let* (X, Y, C, φ) *be a vector equilibrium problem, where* X *is a compact topological sup-semilattice with path connected intervals,* Y *is a Hausdorff locally convex topological vector space. If the VEP satisfies the following conditions:*

(i) *for all* $x \in X$, $y \to \varphi(x,y)$ *is* C_Δ*-quasiconcave or* C_Δ*-quasiconcave-like;*

(ii) *for all* $y \in X$, $x \to \varphi(x,y)$ *is* C*-continuous;*

(iii) *for all* $x \in X$, $\varphi(x,x) \notin \operatorname{int} C$,

then this VEP has a solution.

Example 17. Let $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, $C = -\mathbb{R}_+$. The $(X, \leq)$ is a sup-semilattice, in which $x^1 \leq x^2$ means that $x^2 \in x^1 + \mathbb{R}^2_+$, for all $x^1, x^2 \in X$.

(a) For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$, the function φ is defined as

$$\varphi(x,y) = (1 - y_1)(1 - y_2) - (1 - x_1)(1 - x_2). \tag{15}$$

It can be easily checked that for each $x \in X$, $\varphi(x, \cdot)$ is C_Δ quasiconcave and C_Δ quasiconcave-like but not a usual quasiconcave function.

Denote by D the set $(1 \times [0,1]) \cup ([0,1] \times 1)$. For each $x = (x_1, x_2) \in X$, the multi-valued mapping G satisfies that

$$G(x) = \begin{cases} (x_1, 1] \times [0,1] \cup [0,1] \times (x_2, 1], & x \in X \setminus D, \\ (1,1), & x \in D. \end{cases} \tag{16}$$

Note that G is not a usual convex but a Δ-convex multi-valued mapping. For each $y = (y_1, y_2) \in X$, if $y \in X \setminus (1,1)$, then $G^{-1}(y) = [0, y_1) \times [0, y_2)$; if $y = (1,1)$, then $G^{-1}(y) = X$. Thus, $G^{-1}(y)$ is open in X for each $y \in X$. Then φ and G satisfy all the conditions in Theorem 14. We can find that $\overline{x} = (1,1)$ is the unique solution for the VQEP, $(X, \mathbb{R}, C, \varphi, G)$.

(b) For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in X$, let $T(x) = B_\varepsilon(x)$, where $B_\varepsilon(x)$ is the ε-neighborhood of x in X,

$$\varphi(x,y) = (1 + x_1 - y_1)(1 + x_2 - y_2), \tag{17}$$

and G is the same as the setting in (a). Then the function φ and the mappings G and T meet all the conditions in Theorem 14. The set of solutions for the VEP $(X, \mathbb{R}, C, \varphi)$ is the overall X which is also the set of solutions for $(X, \mathbb{R}, C, \varphi, T)$. The solution of $(X, \mathbb{R}, C, \varphi, G)$ is just one point $\overline{x} = (1,1)$.

To study the stability of vector quasi-equilibrium problems, let (X, d) be a metric space and define the set M' as

$$M' = \{\phi = (X, Y, C, \varphi, G) : \phi \text{ satisfies all the conditions in Theorem 14}\} \tag{18}$$

For each $\phi \in M'$, by the proof of Theorem 14, we can find that a point $\overline{x} \in X$ is a solution of ϕ if and only if $\overline{x}$ is a maximal element of S defined in the proof. Let $F'(\phi)$ denote

all the solutions of ϕ. Then F' is a multi-valued mapping from M' to X. For any two ϕ_1, ϕ_2, define the metric $\rho'(\phi_1, \phi_2)$ between ϕ_1 and ϕ_2 as

$$\rho'(\phi_1, \phi_2) = \rho(S_1, S_2), \tag{19}$$

where S_1 and S_2 are multi-valued mappings corresponding to ϕ_1 and ϕ_2 in the proof of Theorem 14. Then (M', ρ') is a metric space. Instead of M, S and $F(S)$ by M', ϕ and $F'(\phi)$ in Definition 9, we can also define essential sets $e(\phi)$, minimal essential sets $m(\phi)$ of $F'(\phi)$, and essential component in $F'(\phi)$. If an essential set $e(\phi)$ is singleton set $\{x^*\}$, x^* is called an essential solution of ϕ.

From Theorems 12 and 13, we have the following results.

Theorem 18. *$F' : M' \to 2^X$ is an upper semi-continuous mapping with compact values. For each VQEP $\phi \in M'$, there exists at least a connected minimal essential set $m(\phi)$ of $F'(\phi)$.*

Remark 19. For each $\phi \in M'$, $y \in X$, let

$$A_\phi(y) = \{x : y \notin G(x), \text{or } \varphi(x, y) \notin \operatorname{int} C \text{ and } x \in G(x)\}. \tag{20}$$

For any $\phi_1, \phi_2 \in M'$, from the definition of the metric between S_1 and S_2, then

$$\rho'(\phi_1, \phi_2) = \sup_{y \in X} h\left(A_{\phi_1}(y), A_{\phi_2}(y)\right), \tag{21}$$

which gives an overall consideration of φ and G. If ϕ_1 and ϕ_2 are two VEP, then

$$\begin{aligned} &\rho'(\phi_1, \phi_2) \\ &\quad = \sup_{y \in X} h(\{x : \varphi_1(x, y) \notin \operatorname{int} C\}, \\ &\qquad\qquad x\{x : \varphi_2(x, y) \notin \operatorname{int} C\}). \end{aligned} \tag{22}$$

For the essential stability of solutions for VQEP, clearly, the class of perturbations induced by the metric ρ' is different from the perturbation of uniform topology in [3, 14] and also different from the perturbation of best response defined in [16]. For example, the existence of essential sets of solutions for VQEP in topological vector spaces is proved in [3], and the uniform metric for two VQEP $\phi_1 = (X, X, C, \varphi_1, G_1)$ and $\phi_2 = (X, X, C, \varphi_2, G_2)$ is defined as

$$\begin{aligned} \rho''(\phi_1, \phi_2) &= \sup_{(x,y) \in X \times X} \|\varphi_1(x, y) - \varphi_2(x, y)\| \\ &\quad + \sup_{x \in X} h(G_1(x), G_2(x)), \end{aligned} \tag{23}$$

where X is a compact convex subset of a Banach space. Naturally, the feasible mapping G requires closed values, which is not a requirement in Theorem 14, however, where each inverse image being open is necessary.

By Theorem 18, each connected component including a connected minimal essential set of solutions is essential; that is, the existence of essential components can be induced.

Corollary 20. *Let $\phi \in M'$. There is an essential component in $F'(\phi)$. If $F'(\phi) = \{x^*\}$ is a singleton, then x^* is an essential solution of ϕ.*

Acknowledgments

This project is supported by Guangxi Natural Science Foundation (2012GXNSFBA053013, 2013GXNSFBA19004), NNSF (61164020), and Doctoral Research Fund of GUT.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 621614, 6 pages
http://dx.doi.org/10.1155/2013/621614

Research Article

Fixed Points of Multivalued Nonself Almost Contractions

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Received 29 March 2013; Accepted 20 May 2013

Academic Editor: Wei-Shih Du

We consider multivalued nonself-weak contractions on convex metric spaces and establish the existence of a fixed point of such mappings. Presented theorem generalizes results of M. Berinde and V. Berinde (2007), Assad and Kirk (1972), and many others existing in the literature.

1. Introduction

The study of fixed points of single-valued self-mappings or multivalued self-mappings satisfying certain contraction conditions has a great majority of results in metric fixed point theory. All these results are mainly generalizations of Banach contraction principle.

The Banach contraction principle guarantees the existence and uniqueness of fixed points of certain self-maps in complete metric spaces. This result has various applications to operator theory and variational analysis. So, it has been extended in many ways until now. One of these is related to multivalued mappings. Its starting point is due to Nadler Jr. [1].

The fixed point theory for multivalued nonself-mappings developed rapidly after the publication of Assad and Kirk's paper [2] in which they proved a non-self-multivalued version of Banach's contraction principle. Further results for multivalued non-self-mappings were proved in, for example, [3–7]. For other related results, see also [8–38].

On the other hand, Berinde [11–13] introduced a new class of self-mappings (usually called weak contractions or almost contractions) that satisfy a simple but general contraction condition that includes most of the conditions in Rhoades' classification [39]. He obtained a fixed point theorem for such mappings which generalized the results of Kannan [40], Chatterjea [41], and Zamfirescu [42]. As shown in [43], the weakly contractive metric-type fixed point result in [12] is "almost" covered by the related altering metric one due to Khan et al. [21].

In [9], M. Berinde and V. Berinde extended Theorem 8 to the case of multivalued weak contractions.

Definition 1. Let (X, d) be a metric space and K a nonempty subset of X. A map $T : K \to CB(X)$ is called a multivalued almost contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$H(Tx, Ty) \leq \delta \cdot d(x, y) + LD(y, Tx), \quad \forall x, y \in K. \quad (1)$$

Theorem 2 (see [9]). *Let X be a complete metric space and $T : X \to CB(X)$ a multivalued almost contraction. Then T has a fixed point.*

The aim of this paper is to prove a fixed point theorem for multivalued nonself almost contractions on convex metric spaces. This theorem extends several important results (including the above) in the fixed point theory of self-mappings to the case on nonself-mappings and generalizes several fixed point theorems for nonself-mappings.

2. Preliminaries

We recall some basic definitions and preliminaries that will be needed in this paper.

Let (X,d) be a metric space and $CB(X)$ the set of all nonempty bounded and closed subsets of X. For $A,B \in CB(X)$, define

$$D(x,A) = \inf\{d(x,y) : y \in A\},$$
$$D(A,B) = \inf\{d(x,y) : x \in A, y \in B\}, \tag{2}$$
$$H(A,B) = \max\left\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\right\}.$$

It is known that H is a metric on $CB(X)$ and H is called the Hausdorff metric or Pompeiu-Hausdorff metric induced by d. It is also known that $(CB(X),H)$ is a complete metric space whenever (X,d) is a complete metric space.

Definition 3. Let $T : X \to CB(X)$ be a multivalued map. An element $x \in X$ is said to be a fixed point of T if $x \in Tx$.

In this paper we assume that (X,d) is a convex metric space which is defined as follows.

Definition 4. A metric space (X,d) is convex if for each $x, y \in X$ with $x \neq y$ there exists $z \in X$, $x \neq z \neq y$, such that

$$d(x,y) = d(x,z) + d(z,y). \tag{3}$$

This notion is similar to the definition of metric space of hyperbolic type. The class of metric spaces of hyperbolic type includes all normed linear spaces and all spaces with hyperbolic metric.

It is known that in a convex metric space each two points are the endpoints of at least one metric segment (see [2]).

Proposition 5 (see [2]). *Let K be a closed subset of a complete and convex metric space X. If $x \in K$ and $y \notin K$, then there exists a point $z \in \partial K$ (the boundary of K) such that*

$$d(x,y) = d(x,z) + d(z,y). \tag{4}$$

The following lemma will be required in the sequel.

Lemma 6 (see [1, 2]). *Let (X,d) be a metric space and $A,B \in CB(X)$. If $x \in A$, then, for each positive number α, there exists $y \in B$ such that*

$$d(x,y) \le H(A,B) + \alpha. \tag{5}$$

The definition of an almost contraction given by Berinde [12] is as follows.

Definition 7. Let (X,d) be a metric space. A map $T : X \to X$ is called almost contraction if there exist a constant $\delta \in (0,1)$ and some $L \ge 0$ such that

$$d(Tx,Ty) \le \delta \cdot d(x,y) + Ld(y,Tx), \quad \forall x,y \in X. \tag{6}$$

Theorem 8 (see [12]). *Let (X,d) be a complete metric space and $T : X \to X$ an almost contraction. Then*

(1) $\mathrm{Fix}(T) = \{x \in X : Tx = x\} \neq \emptyset$;

(2) *for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in \mathrm{Fix}(T)$;*

(3) *the following estimate holds*

$$d(x_{n+i-1}, x^*) \le \frac{\delta^i}{1-\delta} d(x_n, x_{n-1}), \quad n = 1,2,\ldots;\ i = 1,2,\ldots. \tag{7}$$

Let us recall (see [30]) that a mapping T possessing properties (1) and (2) is called a *weakly Picard operator*.

In fact, Theorem 8 generalizes some important fixed point theorems in the literature such as Banach contraction principle, Kannan fixed point theorem [40], Chatterjea fixed point theorem [41], and Zamfirescu fixed point theorem [42].

3. Main Results

Theorem 9. *Let (X,d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that $T : K \to CB(X)$ is a multivalued almost contraction, that is,*

$$H(Tx,Ty) \le \delta \cdot d(x,y) + LD(y,Tx), \quad \forall x,y \in K, \tag{8}$$

with $\delta \in (0,1)$ and some $L \ge 0$ such that $\delta(1+L) < 1$. If T satisfies Rothe's type condition, that is, $x \in \partial K \Rightarrow Tx \subset K$, then there exists $z \in K$ such that $z \in Tz$; that is, T has a fixed point in K.

Proof. We construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way. Let $x_0 \in K$ and $y_1 \in Tx_0$. If $y_1 \in K$, let $x_1 = y_1$. If $y_1 \notin K$, then there exists $x_1 \in \partial K$ such that

$$d(x_0,x_1) + d(x_1,y_1) = d(x_0,y_1). \tag{9}$$

Thus $x_1 \in K$, and, by Lemma 6 and $\alpha = \delta$, we can choose $y_2 \in Tx_1$ such that

$$d(y_1,y_2) \le H(Tx_0,Tx_1) + \delta. \tag{10}$$

If $y_2 \in K$, let $x_2 = y_2$. If $y_2 \notin K$, then there exists $x_2 \in \partial K$ such that

$$d(x_1,x_2) + d(x_2,y_2) = d(x_1,y_2). \tag{11}$$

Thus $x_2 \in K$, and, by Lemma 6 and $\alpha = \delta^2$, we can choose $y_3 \in Tx_2$ such that

$$d(y_2,y_3) \le H(Tx_1,Tx_2) + \delta^2. \tag{12}$$

Continuing the arguments we construct two sequences $\{x_n\}$ and $\{y_n\}$ such that

(i) $y_{n+1} \in Tx_n$;

(ii) $d(y_n,y_{n+1}) \le H(Tx_{n-1},Tx_n) + \delta^n$, where

(iii) $y_n \in K \Rightarrow y_n = x_n$;

(iv) $y_n \neq x_n$ whenever $y_n \notin K$, and then $x_n \in \partial K$ is such that

$$d(x_{n-1},x_n) + d(x_n,y_n) = d(x_{n-1},y_n). \tag{13}$$

Now we claim that $\{x_n\}$ is a Cauchy sequence. Suppose that

$$P = \{x_i \in \{x_n\} : x_i = y_i\},$$
$$Q = \{x_i \in \{x_n\} : x_i \neq y_i\}. \tag{14}$$

Obviously, if $x_n \in Q$, then x_{n-1} and x_{n+1} belong to P. Now, we conclude that there are three possibilities.

Case 1. If $x_n, x_{n+1} \in P$, then $y_n = x_n$, $y_{n+1} = x_{n+1}$. Thus

$$\begin{aligned} d(x_n, x_{n+1}) &= d(y_n, y_{n+1}) \\ &\leq H(Tx_{n-1}, Tx_n) + \delta^n \\ &\leq \delta \cdot d(x_{n-1}, x_n) + LD(x_n, Tx_{n-1}) + \delta^n \\ &= \delta \cdot d(x_{n-1}, x_n) + \delta^n \end{aligned} \tag{15}$$

since $y_n \in Tx_{n-1}$.

Case 2. If $x_n \in P, x_{n+1} \in Q$, then $y_n = x_n, y_{n+1} \neq x_{n+1}$. We have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) \\ &= d(x_n, y_{n+1}) \\ &= d(y_n, y_{n+1}) \\ &\leq H(Tx_{n-1}, Tx_n) + \delta^n \\ &\leq \delta \cdot d(x_{n-1}, x_n) + LD(x_n, Tx_{n-1}) + \delta^n \\ &= \delta \cdot d(x_{n-1}, x_n) + \delta^n. \end{aligned} \tag{16}$$

Case 3. If $x_n \in Q, x_{n+1} \in P$, then $x_{n-1} \in P, y_n \neq x_n, y_{n+1} = x_{n+1}, y_{n-1} = x_{n-1}$, and $y_n \in Tx_{n-1}$. We have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, y_{n+1}) \\ &\leq d(x_n, y_n) + d(y_n, y_{n+1}) \\ &\leq d(x_n, y_n) + H(Tx_{n-1}, Tx_n) + \delta^n \\ &\leq d(x_n, y_n) + \delta \cdot d(x_{n-1}, x_n) \\ &\quad + LD(x_n, Tx_{n-1}) + \delta^n. \end{aligned} \tag{17}$$

Since $\delta < 1$, then

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, y_n) + d(x_{n-1}, x_n) \\ &\quad + LD(x_n, Tx_{n-1}) + \delta^n \\ &= d(x_{n-1}, y_n) + LD(x_n, Tx_{n-1}) + \delta^n \\ &\leq d(x_{n-1}, y_n) + Ld(x_n, y_n) + \delta^n \\ &= d(x_{n-1}, y_n) + Ld(x_{n-1}, y_n) \\ &\quad - Ld(x_{n-1}, x_n) + \delta^n \\ &\leq (1+L)\, d(y_{n-1}, y_n) + \delta^n \\ &\leq (1+L)\, H(Tx_{n-2}, Tx_{n-1}) \\ &\quad + (1+L)\, \delta^{n-1} + \delta^n \\ &\leq (1+L)\, \delta \cdot d(x_{n-2}, x_{n-1}) \\ &\quad + (1+L)\, LD(x_{n-1}, Tx_{n-2}) \\ &\quad + (1+L)\, \delta^{n-1} + \delta^n \\ &= (1+L)\, \delta \cdot d(x_{n-2}, x_{n-1}) + (1+L)\, \delta^{n-1} + \delta^n. \end{aligned} \tag{18}$$

Since

$$h = (1+L)\, \delta < 1, \tag{19}$$

then

$$d(x_n, x_{n+1}) < hd(x_{n-2}, x_{n-1}) + h\delta^{n-2} + \delta^n. \tag{20}$$

Thus, combining Cases 1, 2, and 3, it follows that

$$d(x_n, x_{n+1}) \leq \begin{cases} \alpha \cdot d(x_{n-1}, x_n) + \alpha^n \\ \alpha d(x_{n-2}, x_{n-1}) + \alpha^{n-1} + \alpha^n, \end{cases} \tag{21}$$

where

$$\alpha = \max\{\delta, h\} = h. \tag{22}$$

Following [2], by induction it follows that for $n > 1$

$$d(x_n, x_{n+1}) \leq h^{(n-1)/2}\omega + h^{n/2} n, \tag{23}$$

where

$$\omega = \max\{d(x_0, x_1), d(x_1, x_2)\}. \tag{24}$$

Now, for $n > m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) \\ &\quad + \cdots + d(x_{m-1}, x_m) \\ &\leq \left(h^{(n-1)/2} + h^{(n-2)/2} + \cdots + h^{(m-1)/2}\right)\omega \\ &\quad + \alpha^{n/2} n + \alpha^{(n-1)/2}(n-1) + \cdots + \alpha^{m/2} m. \end{aligned} \tag{25}$$

This implies that the sequence $\{x_n\}$ is a Cauchy sequence. Since X is complete and K is closed, it follows that there exists $z \in K$ such that

$$z = \lim_{n \to \infty} x_n. \tag{26}$$

By construction of $\{x_n\}$, there is a subsequence $\{x_q\}$ such that

$$y_q = x_q \in Tx_{q-1}. \tag{27}$$

We will prove that $z \in Tz$. In fact, by (i), $x_q \in Tx_{q-1}$. Since $x_q \to z$ as $q \to \infty$, we have

$$D(z, Tx_{q-1}) \longrightarrow 0, \tag{28}$$

as $q \to \infty$. Note that

$$D(z, Tz) \le d(z, x_q) + d(x_q, Tz) \tag{29}$$

$$\le d(z, x_q) + H(Tx_{q-1}, Tz) \tag{30}$$

$$\le d(z, x_q) + \delta d(x_{q-1}, z) + LD(z, Tx_{q-1}), \tag{31}$$

which on letting $q \to \infty$ implies that $D(z, Tz) = 0$; it, then, follows that $z \in Tz$. □

By Theorem 9 we obtain as a particular case, a fixed point theorem for multivalued nonself-contractions due to Assad and Kirk [2] that appears to be the first fixed point result for nonself-mappings in the literature.

Corollary 10 (see [2]). *Let (X, d) be a complete convex metric space and K a nonempty closed subset of X. Suppose that $T : K \to CB(X)$ is a multivalued contraction; that is,*

$$H(Tx, Ty) \le \delta d(x, y), \quad \forall x, y \in K, \tag{32}$$

with $\delta \in (0, 1)$. If T satisfies Rothe's type condition, that is, $x \in \partial K \Rightarrow Tx \subset K$, then there exists $z \in K$ such that $z \in Tz$; that is, T has a fixed point in K.

Example 11. Let X be the set of real numbers with the usual norm, $K = [0, 1]$ the unit interval, and $T : K \to CB(X)$ be given by $Tx = \{(1/9)x\}$, for $x \in [0, 1/2)$, $T(1/2) = \{-1\}$, and $Tx = [17/18, (1/9)x + 8/9]$, for $x \in (1/2, 1]$.

In order to show that T is a multivalued almost contraction, we have to discuss 8 possible cases.

Case 1. Consider $(x, y) \in \Omega_1 = [0, 1/2) \times (1/2, 1]$. Then condition (8) reduces to

$$\left|\frac{1}{9}x - \frac{1}{9}y - \frac{8}{9}\right| \le \delta |x - y| + L\left|y - \frac{1}{9}x\right|, \quad (x, y) \in \Omega_1. \tag{33}$$

Since, for $(x, y) \in \Omega_1$, one has $|(1/9)x - (1/9)y - 8/9| \le 1$ and $|y - (1/9)x| > 4/9$, in order to have the previous inequality satisfied, it suffices to take $L \ge 9/4$ and $0 < \delta < 4/13$ arbitrarily.

Case 2. Consider $(x, y) \in \Omega_2 = (1/2, 1] \times [0, 1/2)$. Then condition (8) reduces to

$$\left|\frac{1}{9}x + \frac{8}{9} - \frac{1}{9}y\right| \le \delta |x - y| + L\left|y - \frac{1}{9}x - \frac{8}{9}\right|, \quad (x, y) \in \Omega_2. \tag{34}$$

Since, for $(x, y) \in \Omega_2$, one has $|(1/9)x + 8/9 - (1/9)y| \le 1$ and $|y - (1/9)x - 8/9| > 4/9$, in order to have the previous inequality satisfied, it suffices to take $L \ge 9/4$ and $0 < \delta < 4/13$ arbitrarily.

Case 3. Take $(x, y) \in \Omega_3 = [0, 1/2)^2$. In this case we have

$$H(Tx, Ty) = d\left(\frac{1}{9}x, \frac{1}{9}y\right) = \left|\frac{1}{9}x - \frac{1}{9}y\right|, \tag{35}$$

and so condition (8) is satisfied with $\delta = 1/9$ and $L \ge 0$ arbitrarily.

Case 4. Consider $(x, y) \in \Omega_4 = (1/2, 1]^2$. In this case we have

$$\begin{aligned} H(Tx, Ty) &= H\left(\left[\frac{1}{9}x + \frac{8}{9}, 1\right], \left[\frac{1}{9}y + \frac{8}{9}, 1\right]\right) \\ &= \left|\frac{1}{9}x - \frac{1}{9}y\right|, \end{aligned} \tag{36}$$

and so condition (8) is satisfied with $\delta = 1/9$ and $L \ge 0$ arbitrarily.

Case 5. Take $(x, y) \in \Omega_5 = \{1/2\} \times [0, 1/2)$. Then condition (8) reduces to

$$\left|1 + \frac{1}{9}y\right| \le \delta\left|\frac{1}{2} - y\right| + L|y + 1|, (x, y) \in \Omega_5. \tag{37}$$

Since for $(x, y) \in \Omega_5$, one has $|1+(1/9)y| < 19/18$ and $|1+y| \ge 1$, in order to have the previous inequality satisfied, it suffices to take $L \ge 19/18$ and $0 < \delta < 18/37$ arbitrarily.

Case 6. Consider $(x, y) \in \Omega_6 = [0, 1/2) \times \{1/2\}$. Then condition (8) reduces to

$$\left|1 + \frac{1}{9}x\right| \le \delta\left|x - \frac{1}{2}\right| + L\left|\frac{1}{2} - \frac{1}{9}x\right|, \quad (x, y) \in \Omega_6. \tag{38}$$

Since, for $(x, y) \in \Omega_6$, one has $|1 + (1/9)x| \le 19/18$ and $|1/2 - (1/9)x| \ge 4/9$, in order to have the previous inequality satisfied, it suffices to take $L \ge 19/8$ and $0 < \delta < 8/27$ arbitrarily.

Case 7. Take $(x, y) \in \Omega_7 = \{1/2\} \times (1/2, 1]$. Then condition (8) reduces to

$$\left|1 + \frac{1}{9}y + \frac{8}{9}\right| \le \delta\left|\frac{1}{2} - y\right| + L|y + 1|, \quad (x, y) \in \Omega_7. \tag{39}$$

Since, for $(x, y) \in \Omega_7$, one has $|1 + (1/9)y + 8/9| \le 2$ and $|y+1| \ge 3/2$, in order to have the previous inequality satisfied, it suffices to take $L \ge 4/3$ and $0 < \delta < 3/7$.

Case 8. Consider $(x, y) \in \Omega_8 = (1/2, 1] \times \{1/2\}$. Then condition (8) reduces to

$$\left|1 + \frac{1}{9}x + \frac{8}{9}\right| \le \delta\left|x - \frac{1}{2}\right| + L\left|\frac{1}{2} - \frac{1}{9}x - \frac{8}{9}\right|, \quad (x, y) \in \Omega_8. \tag{40}$$

Since, for $(x, y) \in \Omega_8$, one has $|1 + (1/9)x + 8/9| \le 2$ and $|1/2 - (1/9)x - 8/9| > 4/9$, in order to have the previous inequality satisfied, it suffices to take $L \ge 9/2$ and $0 < \delta < 2/11$ arbitrarily.

Now, by summarizing all cases, we conclude that condition (8) is satisfied with $\delta = 1/9$ and $L = 9/2$. Note that the additional condition $\delta(1 + L) < 1$ is also satisfied.

Hence, T is a multivalued almost contraction that satisfies all assumptions in Theorem 9, and T has two fixed points; that is, $\mathrm{Fix}(T) = \{0, 1\}$.

Note that Corollary 10 cannot be applied to T in Example 11. Indeed, if we take $x = 1$ and $y = 1/2$ in (32), then one obtains

$$H\left(T1, T\frac{1}{2}\right) \leq \delta \left|1 - \frac{1}{2}\right|. \tag{41}$$

That is, $|1 + 1| \leq \delta|1/2|$, which leads to the contradiction $4 \leq \delta < 1$.

Acknowledgments

The authors thank the referees for the very useful suggestions and remarks that contributed to the improvement of the paper and especially for drawing attention to [21]. The second author's research was partially supported by the Grant PN-II-RU-TE-2011-3-239 of the Romanian Ministry of Education and Research. The research of the first and third authors was partially supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 527417, 5 pages
http://dx.doi.org/10.1155/2013/527417

Research Article

Note on the Hahn-Banach Theorem in a Partially Ordered Vector Space

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Received 17 May 2013; Accepted 25 June 2013

Academic Editor: Wei-Shih Du

Using a fixed point theorem in a partially ordered set, we give a new proof of the Hahn-Banach theorem in the case where the range space is a partially ordered vector space.

1. Introduction

The Hahn-Banach theorem is one of the most fundamental theorems in the functional analysis theory. This theorem is well known in the case where the range space is the real number system as follows.

Let p be a sublinear mapping from a vector space X into the real number system R, Y a subspace of X, and q a linear mapping from Y into R such that $q \leq p$ on Y. Then there exists a linear mapping g from X into R such that $g = q$ on Y and $g \leq p$ on X.

It is known that this theorem is established in the case where the range space is a Dedekind complete Riesz space as follows [1–3].

Let p be a sublinear mapping from a vector space X into a Dedekind complete Riesz space E, Y a subspace of X and q a linear mapping from Y into E such that $q \leq p$ on Y. Then there exists a linear mapping g from X into E such that $g = q$ on Y and $g \leq p$ on X.

On the other hand, Hirano et al. [4] showed the Hahn-Banach theorem by using the Markov-Kakutani fixed point theorem [5] in the case where the range space is the real number system.

In this paper, motivated by Hirano et al. [4], we give a proof of the Hahn-Banach theorem using a fixed point theorem. We show the Hahn-Banach theorem in the case where the range space is a Dedekind complete partially ordered vector space (Theorem 10). Moreover, we show the Mazur-Orlicz theorem in a Dedekind complete partially ordered vector space (Theorem 11).

2. Preliminaries

Let $(E, \leq)$ be a partially ordered set and F a subset of E. The set F is called a *chain* if any two elements are comparable; that is, $x \leq y$ or $y \leq x$ for any $x, y \in F$. An element $x \in E$ is called a *lower bound* of F if $x \leq y$ for any $y \in F$. An element $x \in E$ is called the *minimum* of F if x is a lower bound of F and $x \in F$. If there exists a lower bound of F, then F is said to be *bounded from below*. An element $x \in E$ is called an *upper bound* of F if $y \leq x$ for any $y \in F$. An element $x \in E$ is called the *maximum* of F if x is an upper bound and $x \in F$. If there exists an upper bound of F, then F is said to be *bounded from above*. If the set of all lower bounds of F has the maximum, then the maximum is called an *infimum* of F and denoted by $\inf F$. If the set of all upper bounds of F has the minimum, then the minimum is called a *supremum* of F and denoted by $\sup F$. An element $x \in F$ is called a *minimal* of F if $y \leq x$ and $y \in F$ implies $y = x$. A partially ordered set E is said to be *complete* if every nonempty chain of E has an infimum; E is said to be *chain complete* if every nonempty chain of E

which is bounded from below has an infimum; E is said to be *Dedekind complete* if every nonempty subset of E which is bounded from below has an infimum. A mapping f from E into E is said to be *decreasing* if $f(x) \leq x$ for any $x \in E$. For further information of a partially ordered set, see [1, 2, 6–10].

In a complete partially ordered set, the following theorem is obtained; see [11–14].

Theorem 1 (Bourbaki-Kneser). *Let E be a complete partially ordered set. Let f be a decreasing mapping from E into E. Then f has a fixed point.*

A partially ordered set E is called a partially ordered vector space if E is a vector space and $x+z \leq y+z$ and $\alpha x \leq \alpha y$ hold whenever $x, y, z \in E$, $x \leq y$ and α is a nonnegative real number. If a partially ordered vector space E is a lattice, that is, any two elements in E have a supremum and an infimum, then E is called a *Riesz space*.

Let X be a vector space and E a partially ordered vector space. A mapping f from X into E is said to be *concave* if $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$ for any $x, y \in X$ and $t \in [0, 1]$. A mapping p from X into E is said to be *sublinear* if the following conditions are satisfied.

(S1) For any $x, y \in X$, $p(x + y) \leq p(x) + p(y)$.

(S2) For any $x \in X$ and nonnegative real number α, $p(\alpha x) = \alpha p(x)$.

Let E^X be the set of mappings from X into E. Throughout this paper, E^X is ordered as follows. For $f, g \in E^X$, let $f \leq g$ mean that $f(x) \leq g(x)$ for any $x \in X$. It is easy to check that E^X is also a partially ordered vector space.

The following lemmas are useful for the proof of our main results.

Lemma 2. *Let X be a vector space, E a chain complete partially ordered vector space, and Z a nonempty chain of E^X which is bounded from below. Then there exists $\inf\{h(x) \mid h \in Z\}$ for any $x \in X$. Moreover, if $p \in E^X$ is defined by $p(x) = \inf\{h(x) \mid h \in Z\}$ for any $x \in X$, then $p = \inf Z$; that is, E^X is chain complete.*

Proof. Let $x \in X$ be fixed. Since Z is a nonempty chain, so is $\{h(x) \mid h \in Z\}$. Let f be a lower bound of Z. Since $f(x) \leq h(x)$ for any $h \in Z$, $\{h(x) \mid h \in Z\}$ is bounded from below. Therefore, since E is chain complete, there exists $\inf\{h(x) \mid h \in Z\}$.

Define $p \in E^X$ by $p(x) = \inf\{h(x) \mid h \in Z\}$ for any $x \in X$. Then it is clear that $p \leq h$ for any $h \in Z$; that is, p is a lower bound of Z. Let q be a lower bound of Z. Since $q(x) \leq h(x)$ for any $x \in X$ and $h \in Z$, $q(x)$ is a lower bound of $\{h(x) \mid h \in Z\}$ for any $x \in X$. Therefore, $q(x) \leq \inf\{h(x) \mid h \in Z\} = p(x)$ for any $x \in X$ and thus $p = \inf Z$. □

Lemma 3. *Let X be a vector space, E a Dedekind complete partially ordered vector space, and Z a nonempty subset in E^X which is bounded from below. Then there exists $\inf\{h(x) \mid h \in Z\}$ for any $x \in X$. Moreover, if $p \in E^X$ is defined by $p(x) = \inf\{h(x) \mid h \in Z\}$ for any $x \in X$, then $p = \inf Z$; that is, E^X is Dedekind complete.*

Proof. The proof is similar to that of Lemma 2. □

Lemma 4. *Let X, E, E^X, Z, and p be the same as in Lemma 2. Suppose that*

(1) *for any $h \in Z$, $x \in X$ and $\alpha > 0$, there exists $h' \in Z$ such that $h(\alpha x) = \alpha h'(x)$;*

(2) *$p(0) = 0$;*

(3) *for any $h_1, h_2 \in Z$ and $x, y \in X$, there exists $h \in Z$ such that $h(x + y) \leq h_1(x) + h_2(y)$.*

Then p is sublinear.

Proof. Let $x \in X$ and $\alpha > 0$ be fixed. It is clear from (1) that $\{h(\alpha x) \mid h \in Z\} \subset \{\alpha h'(x) \mid h' \in Z\}$. Since $\alpha x \in X$ and $1/\alpha > 0$, by (1), for any $h' \in Z$ there exists $h \in Z$ such that

$$\alpha h'(x) = \alpha h'\left(\frac{1}{\alpha}\alpha x\right) = \alpha\frac{1}{\alpha}h(\alpha x) = h(\alpha x) \tag{1}$$

and hence $\{\alpha h'(x) \mid h' \in Z\} \subset \{h(\alpha x) \mid h \in Z\}$. Therefore, we conclude that $\{h(\alpha x) \mid h \in Z\} = \{\alpha h'(x) \mid h' \in Z\}$. Thus we obtain that

$$\begin{aligned} p(\alpha x) &= \inf\{h(\alpha x) \mid h \in Z\} = \inf\left\{\alpha h'(x) \mid h' \in Z\right\} \\ &= \alpha \inf\left\{h'(x) \mid h' \in Z\right\} = \alpha p(x). \end{aligned} \tag{2}$$

Moreover, (2) shows that $p(0x) = p(0) = 0 = 0p(x)$. Therefore, (S2) holds.

Let $x, y \in X$ be fixed. By (3), for any $h_1, h_2 \in Z$, there exists $h \in Z$ such that $h(x + y) \leq h_1(x) + h_2(y)$. Thus we have

$$p(x + y) \leq h_1(x) + h_2(y) \tag{3}$$

for any $h_1, h_2 \in Z$. This shows that $p(x + y) - h_2(y)$ is a lower bound of $\{h(x) \mid h \in Z\}$ for any $h_2 \in Z$ and hence we have

$$p(x + y) - h_2(y) \leq p(x) \tag{4}$$

for any $h_2 \in Z$. This shows that $p(x+y)-p(x)$ is a lower bound of $\{h(y) \mid h \in Z\}$ and hence we have $p(x + y) - p(x) \leq p(y)$. Therefore, (S1) holds. This completes the proof. □

Lemma 5. *Let X, E, E^X, Z, and p be the same as in Lemma 3. Suppose that*

(1) *for any $h \in Z$, $x \in X$ and $\alpha > 0$, there exists $h' \in Z$ such that $h(\alpha x) = \alpha h'(x)$;*

(2) *$p(0) = 0$;*

(3) *for any $h_1, h_2 \in Z$ and $x, y \in X$, there exists $h \in Z$ such that $h(x + y) \leq h_1(x) + h_2(y)$.*

Then p is sublinear.

Proof. The proof is similar to that of Lemma 4. □

3. Main Results

To obtain our main results, we need the following.

Lemma 6. *Let g be a sublinear mapping from a vector space X into a chain complete partially ordered vector space E and $y \in X$. Let ϕ be a mapping from X into E defined by*

$$\phi(x) = \inf\{g(x+ty) - g(ty) \mid t \geq 0\} \tag{5}$$

for any $x \in X$. Then ϕ is sublinear and $g^* \leq \phi \leq g$ on X, where g^* is a mapping from X into E defined by $g^*(x) = -g(-x)$ for $x \in X$.

Proof. For any $x \in X$ and $t \geq 0$, put $\tau_t(x) = g(x+ty) - g(ty)$. Then $Z = \{\tau_t \mid t \geq 0\}$ is a nonempty chain and bounded from below in E^X. Indeed, since $g = \tau_0 \in Z$, Z is nonempty. If $s \leq t$, then

$$\begin{aligned}\tau_s(x) - \tau_t(x) &= g(x+sy) - g(sy) - (g(x+ty) - g(ty)) \\ &= g(x+sy) + (g(ty) - g(sy)) - g(x+ty) \\ &= g(x+sy) + (t-s)g(y) - g(x+ty) \\ &= g(x+sy) + g((t-s)y) - g(x+ty) \\ &\geq g(x+sy+(t-s)y) - g(x+ty) = 0\end{aligned} \tag{6}$$

for any $x \in X$. Thus Z is a chain in E^X. Since

$$\begin{aligned}\tau_t(x) &= g(x+ty) - g(ty) \geq g(ty) \\ &\quad - g(-x) - g(ty) = -g(-x) = g^*(x)\end{aligned} \tag{7}$$

for any $x \in X$ and $t \geq 0$, g^* is a lower bound of Z. Hence Z is bounded from below in E^X. Lemma 2 shows that $\phi(x) = \inf Z$ is well defined.

We next check (1), (2), and (3) in Lemma 4. Let $t \geq 0$, $x \in X$, and $\alpha > 0$. We have

$$\begin{aligned}\tau_t(\alpha x) &= g(\alpha x + ty) - g(ty) \\ &= \alpha\left(g\left(x + \frac{t}{\alpha}y\right) - g\left(\frac{t}{\alpha}y\right)\right) \\ &= \alpha\tau_{t/\alpha}(x).\end{aligned} \tag{8}$$

Clearly, $\tau_{t/\alpha} \in Z$ and hence (1) in Lemma 4 holds. Since $\phi(0) = \inf\{0 \mid t \geq 0\} = 0$, (2) in Lemma 4 holds. Let $t_1, t_2 \geq 0$ and $x_1, x_2 \in X$. Since we have

$$\begin{aligned}\tau_{t_1+t_2}(x_1+x_2) &= g(x_1+x_2+(t_1+t_2)y) \\ &\quad - g((t_1+t_2)y) \leq g(x_1+t_1y) \\ &\quad + g(x_2+t_2y) - (t_1+t_2)g(y) \\ &= g(x_1+t_1y) - g(t_1y) \\ &\quad + g(x_2+t_2y) - g(t_2y) \\ &= \tau_{t_1}(x_1) + \tau_{t_2}(x_2),\end{aligned} \tag{9}$$

(3) in Lemma 4 holds. Therefore, Lemma 4 implies that ϕ is sublinear.

Finally, it is clear that $\phi \leq g$. This inequality and (7) imply that $g^* \leq \phi \leq g$ on X. □

By Theorem 1 and Lemma 6, we obtain the following. For the case that E is a Dedekind complete Riesz space, see [2].

Theorem 7. *Let f be a sublinear mapping from a vector space X into a chain complete partially ordered vector space E. Then there exists a linear mapping g from X into E such that $g \leq f$ on X.*

Proof. Let Y be a subset of E^X defined by

$$Y = \left\{h \in E^X \mid h \text{ is sublinear}, \ f^* \leq h \leq f\right\}, \tag{10}$$

where f^* is defined by $f^*(x) = -f(-x)$ for any $x \in X$. Then it is clear that $f \in Y$ and hence Y is nonempty. Moreover Y is complete. In fact, let $Z \subset Y$ be a nonempty chain. Since for any $h \in Z$, $f^* \leq h$, Z is bounded from below. It follows from Lemma 2 that there exists $\inf Z \in E^X$. By Lemma 4, $\inf Z$ is sublinear. Since $f^* \leq h \leq f$ for any $h \in Z$, we have $f^* \leq \inf Z \leq f$. Thus $\inf Z \in Y$ and hence Y is complete. Furthermore Y has a minimal. In fact, we suppose that Y does not have a minimal element. Then, for any $h \in Y$, there exists $\widehat{h} \in Y$ such that $\widehat{h} \leq h$ and $\widehat{h} \neq h$. We define a mapping T from Y into Y by $Th = \widehat{h}$. Since the mapping T is decreasing, there exists $h_0 \in Y$ satisfying $Th_0 = h_0$ by Theorem 1. This is a contradiction.

Let g be a minimal in Y. Let $x \in X$. Let ϕ be a mapping from X into E defined by

$$\phi(z) = \inf\{g(z+tx) - g(tx) \mid t \geq 0\} \tag{11}$$

for any $z \in X$, then ϕ is sublinear and $g^* \leq \phi \leq g$ on X by Lemma 6. Moreover $\phi \in Y$. In fact, since $g \leq f$ and $f^* \leq g^*$, we have $f^* \leq g^* \leq \phi \leq g \leq f$ for any $f \in Z$. This shows that $\phi \in Y$. Since g is minimal, $\phi = g$. Then we have

$$\begin{aligned}g(-x) &= \phi(-x) \\ &= \inf\{g(-x+tx) - g(tx) \mid t \geq 0\} \\ &\leq g(-x+x) - g(x) \\ &= g(0) - g(x) = -g(x).\end{aligned} \tag{12}$$

Since g is sublinear and $0 = g(0) \leq g(x+z) + g(-x-z)$, we have

$$\begin{aligned}-g(x+z) &\leq g(-x-z) \\ &\leq g(-x) + g(-z) \\ &\leq -g(x) - g(z).\end{aligned} \tag{13}$$

Thus $g(x) + g(z) \leq g(x+z)$. Since g is sublinear, we also have $g(x+z) \leq g(x) + g(z)$ for any $x, z \in X$. Then we obtain that for any $x, z \in X$, $g(x+z) = g(x) + g(z)$. Let $x \in X$ and $\alpha > 0$. Since

$$0 = g(\alpha x - \alpha x) = \alpha g(x) + g(-\alpha x), \tag{14}$$

we have $g(-\alpha x) = -\alpha g(x)$. Then for any real number α, we have $g(\alpha x) = \alpha g(x)$. Thus g is linear. Therefore, g is a linear mapping from X into E such that $g \le f$ on X. □

Since Dedekind completeness implies chain completeness, we obtain the following.

Corollary 8. *Let f be a sublinear mapping from a vector space X into a Dedekind complete partially ordered vector space E. Then there exists a linear mapping g from X into E such that $g \le f$ on X.*

To give the Hahn-Banach Theorem in the case where the range space is a Dedekind complete partially ordered vector space, we need the following.

Lemma 9. *Let p be a sublinear mapping from a vector space X into a Dedekind complete partially ordered vector space E, K a nonempty convex subset of X, and q a concave mapping from K into E such that $q \le p$ on K. For any $x \in X$, let*

$$\phi(x) = \inf\{p(x+ty) - tq(y)\, t \ge 0,\ \ y \in K\}. \tag{15}$$

Then ϕ is a sublinear mapping such that $\phi \le p$ on X. Moreover, if g is a linear mapping from X into E, then $g \le \phi$ on X is equivalent to $g \le p$ on X and $q \le g$ on K.

Proof. First, we show that ϕ is well defined and $\phi(x) \ge -p(-x)$ for any $x \in X$. Let $Z = \{\tau_{t,y} \mid t \ge 0 \text{ and } y \in K\}$, where

$$\tau_{t,y}(x) = p(x+ty) - tq(y) \tag{16}$$

for any $x \in X$ and $t \ge 0$. For any $\tau_{t,y} \in Z$ and $x \in X$,

$$\begin{aligned}\tau_{t,y}(x) &= p(x+ty) - tq(y)\\ &\ge p(ty) - p(-x) - tq(y) \ge -p(-x),\end{aligned} \tag{17}$$

and thus $\phi(x) \ge -p(-x)$ and Z is bounded from below in E^X. Since E is Dedekind complete, ϕ is well defined by Lemma 3.

We next check (1), (2), and (3) in Lemma 5.

(1) Let $\tau_{t,y} \in Z$. For any $x \in X$ and $\alpha > 0$, we have

$$\begin{aligned}\tau_{t,y}(\alpha x) &= p(\alpha x + ty) - tq(y)\\ &= \alpha\left(p\left(x + \frac{t}{\alpha}y\right) - \frac{t}{\alpha}q(y)\right)\\ &= \alpha\tau_{t/\alpha,y}(x).\end{aligned} \tag{18}$$

(2) By the definition of ϕ, $\phi(x) \le p(x)$ for any $x \in X$. Therefore $\phi(0) \le p(0) = 0$. Since $p \ge q$ on K, we have

$$\begin{aligned}\phi(0) &= \inf\{p(ty) - tq(y)\ \ t \ge 0,\ \ y \in K\}\\ &= \inf\{tp(y) - tq(y)\, t \ge 0,\ \ y \in K \ge 0\}.\end{aligned} \tag{19}$$

Hence we have $\phi(0) = 0$.

(3) Let $\tau_{t_1,y_1}, \tau_{t_2,y_2} \in Z$ satisfying $t_1 + t_2 \ne 0$. Let $x_1, x_2 \in X$. Since K is convex and q is concave, we have

$$\begin{aligned}&\tau_{t_1,y_1}(x_1) + \tau_{t_2,y_2}(x_2)\\ &\quad= p(x_1 + t_1y_1) - t_1q(y_1) + p(x_2 + t_2y_2) - t_2q(y_2)\\ &\quad\ge p(x_1 + x_2 + (t_1+t_2)w) - (t_1+t_2)q(w)\\ &\quad= \tau_{t_1+t_2,w}(x_1+x_2),\end{aligned} \tag{20}$$

where $w = (1/(t_1+t_2))(t_1y_1 + t_2y_2) \in K$. Since p is sublinear, we have

$$\begin{aligned}\tau_{0,w}(x_1+x_2) &= p(x_1+x_2) \le p(x_1) + p(x_2)\\ &= \tau_{0,y_1}(x_1) + \tau_{0,y_2}(x_2).\end{aligned} \tag{21}$$

Therefore, for any $x_1, x_2 \in X$ and $t_1, t_2 \ge 0$, we have $\tau_{t_1,y_1}(x_1) + \tau_{t_2,y_2}(x_2) \ge \tau_{t_1+t_2,w}(x_1+x_2)$.

Thus by Lemma 5, ϕ is sublinear. Moreover, by the definition of ϕ, we have $\phi \le p$ on X.

Let g be a linear mapping from X into E. Suppose that $g \le \phi$ on X. Since $\phi \le p$ on X, we have $g \le p$ on X. Moreover, since for any $y \in K$,

$$\begin{aligned}-g(y) &= g(-y) \le \phi(-y)\\ &\le p(-y+y) - q(y) = -q(y),\end{aligned} \tag{22}$$

we have $g \ge q$ on K. To prove the converse, suppose that $g \le p$ on X and $q \le g$ on K. For any $x \in X$, $y \in K$ and $t \ge 0$, we have

$$g(x) = g(x+ty) - tg(y) \le p(x+ty) - tq(y). \tag{23}$$

This implies that $g \le \phi$ on X. □

By Corollary 8 and Lemma 9, we have the Hahn-Banach theorem in the case where the range space is a Dedekind complete partially ordered vector space. For the case that E is a Dedekind complete Riesz space, see [2].

Theorem 10. *Let p be a sublinear mapping from a vector space X into a Dedekind complete partially ordered vector space E, Y a subspace of X, and q a linear mapping from Y into E such that $q \le p$ on Y. Then there exists a linear mapping g from X into Y such that $g = q$ on Y and $g \le p$ on X.*

Proof. Let ϕ be a mapping from X into E defined by

$$\phi(x) = \inf\{p(x+ty) - tq(y)\, t \ge 0,\ \ y \in K\} \tag{24}$$

for any $x \in X$. By Lemma 9, ϕ is a sublinear mapping such that $\phi \le p$ on X. By Corollary 8, there exists a linear mapping g such that $g \le \phi$ on X. Then putting $K = Y$ in Lemma 9, we have $g \le p$ on X and $q \le g$ on Y. Since Y is a subspace, for any $y \in Y$, we have $-y \in Y$. Then $q(-y) \le g(-y)$. Since q and g are linear, we have $-q(y) \le -g(y)$. Then $g \le q$ on Y. Thus $g = q$ on Y. □

Moreover, by Corollary 8, we obtain the Mazur-Orlicz theorem in a Dedekind complete partially ordered vector space. For the case that E is a Dedekind complete Riesz space, see [1, 15].

Theorem 11. *Let p be a sublinear mapping from a vector space X into a Dedekind complete partially ordered vector space E. Let $\{x_j \mid j \in J\}$ be a family of elements of X and $\{y_j \mid j \in J\}$ a family of elements of E. Then the following (1) and (2) are equivalent.*

(1) *There exists a linear mapping g from X into E such that $g \le p$ on X and $y_j \le g(x_j)$ for any $j \in J$.*

(2) *For any natural number n, nonnegative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n \ge 0$ and $j_1, j_2, \ldots, j_n \in J$, one has*

$$\sum_{i=1}^{n} \alpha_i y_{j_i} \le p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right). \tag{25}$$

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_n \ge 0$ and $j_1, j_2, \ldots, j_n \in J$ for a natural number n. By (1), we have

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i y_{j_i} &\le \sum_{i=1}^{n} \alpha_i g\left(x_{j_i}\right) \\ &= g\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right) \le p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right).\end{aligned} \tag{26}$$

Thus (2) is established.

Next by (2), for any $x \in X$, we have

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i y_{j_i} &\le p\left(\sum_{i=1}^{n} \alpha_i x_{j_i}\right) = p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i} - x\right) \\ &\le p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) + p(-x), \\ -p(-x) &\le p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i}\end{aligned} \tag{27}$$

for any natural number n, $\alpha_1, \alpha_2, \ldots, \alpha_n \ge 0$ and $j_1, j_2, \ldots, j_n \in J$. Put

$$\begin{aligned}&p_0(x) \\ &= \inf\left\{p\left(x + \sum_{i=1}^{n} \alpha_i x_{j_i}\right) - \sum_{i=1}^{n} \alpha_i y_{j_i} \mid n \in N,\right. \\ &\qquad \left.\alpha_i \ge 0, j_i \in J,\ i = 1, 2, \ldots, n\right\}\end{aligned} \tag{28}$$

for $x \in X$, where N is the set of all natural numbers. By Lemma 3, p_0 is well defined. Since p is sublinear, p_0 is also sublinear. Thus by Corollary 8, there exists a linear mapping g from X into E such that $g \le p_0$ on X. Since $p_0(-x_j) \le p(-x_j + x_j) - y_j = -y_j$, we have

$$y_j \le -p_0\left(-x_j\right) \le -g\left(-x_j\right) = g\left(x_j\right). \tag{29}$$

Since $p_0(x) \le p(x)$ for any $x \in X$, we have $g(x) \le p(x)$ for any $x \in X$. Thus (1) is established. □

Acknowledgment

The authors would like to express their hearty thanks to Professor Koji Aoyama for careful reading and many valuable suggestions.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 761864, 21 pages
http://dx.doi.org/10.1155/2013/761864

Research Article

Composite Iterative Algorithms for Variational Inequality and Fixed Point Problems in Real Smooth and Uniformly Convex Banach Spaces

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Received 30 April 2013; Accepted 6 June 2013

Academic Editor: Wei-Shih Du

We introduce composite implicit and explicit iterative algorithms for solving a general system of variational inequalities and a common fixed point problem of an infinite family of nonexpansive mappings in a real smooth and uniformly convex Banach space. These composite iterative algorithms are based on Korpelevich's extragradient method and viscosity approximation method. We first consider and analyze a composite implicit iterative algorithm in the setting of uniformly convex and 2-uniformly smooth Banach space and then another composite explicit iterative algorithm in a uniformly convex Banach space with a uniformly Gâteaux differentiable norm. Under suitable assumptions, we derive some strong convergence theorems. The results presented in this paper improve, extend, supplement, and develop the corresponding results announced in the earlier and very recent literatures.

1. Introduction

Let X be a real Banach space whose dual space is denoted by X^*. The normalized duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \left\{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}, \quad \forall x \in X, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach theorem that $J(x)$ is nonempty for each $x \in X$. Let C be a nonempty, closed, and convex subset of X. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for every $x, y \in C$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. We use the notation $\rightharpoonup$ to indicate the weak convergence and the one $\to$ to indicate the strong convergence. A mapping $A : C \to X$ is said to be accretive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \ge 0. \tag{2}$$

It is said to be α-strongly accretive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \ge \alpha \|x - y\|^2, \tag{3}$$

for some $\alpha \in (0,1)$. The mapping is called β-inverse strongly-accretive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \ge \beta \|Ax - Ay\|^2, \tag{4}$$

for some $\beta > 0$ and is said to be λ-strictly pseudocontractive if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y) \rangle \le \|x - y\|^2 - \lambda \|x - y - (Ax - Ay)\|^2 \tag{5}$$

for some $\lambda \in (0,1)$.

Let $U = \{x \in X : \| x \| = 1\}$ denote the unite sphere of X. A Banach space X is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \Longrightarrow \frac{\|x + y\|}{2} \leq 1 - \delta. \tag{6}$$

It is known that a uniformly convex Banach space is reflexive and strict convex. A Banach space X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\| x + ty \| - \| x \|}{t} \tag{7}$$

exists for all $x, y \in U$; in this case, X is also said to have a Gâteaux differentiable norm. X is said to have a uniformly Gâteaux differentiable norm if for each $y \in U$, the limit is attained uniformly for $x \in U$. Moreover, it is said to be uniformly smooth if this limit is attained uniformly for $x, y \in U$. The norm of X is said to be the Fréchet differential if for each $x \in U$, this limit is attained uniformly for $y \in U$. In addition, we define a function $\rho : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of X as follows:

$$\begin{aligned} \rho(\tau) = \sup \Big\{ & \frac{1}{2} \left(\|x + y\| + \|x - y\| \right) \\ & - 1 : x, y \in X, \|x\| = 1, \|y\| = \tau \Big\}. \end{aligned} \tag{8}$$

It is known that X is uniformly smooth if and only if $\lim_{\tau \to 0} \rho(\tau)/\tau = 0$. Let q be a fixed real number with $1 < q \leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. As pointed out in [1], no Banach space is q-uniformly smooth for $q > 2$. In addition, it is also known that J is single-valued if and only if X is smooth, whereas if X is uniformly smooth, then the mapping J is norm-to-norm uniformly continuous on bounded subsets of X. If X has a uniformly Gâteaux differentiable norm, then the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X.

Very recently, Cai and Bu [2] considered the following general system of variational inequalities (GSVI) in a real smooth Banach space X, which involves finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \left\langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*) \right\rangle \geq 0, \quad \forall x \in C, \\ \left\langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*) \right\rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{9}$$

where C is a nonempty, closed, and convex subset of X, B_1, and $B_2 : C \to X$ are two nonlinear mappings, and μ_1 and μ_2 are two positive constants. Here the set of solutions of GSVI (9) is denoted by GSVI(C, B_1, B_2). In particular, if $X = H$, a real Hilbert space, then GSVI (9) reduces to the following GSVI of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, \quad \forall x \in C, \end{aligned} \tag{10}$$

which μ_1 and μ_2 are two positive constants. The set of solutions of problem (10) is still denoted by GSVI(C, B_1, B_2). It is clear that the problem (10) covers as special case the classical variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\left\langle Ax^*, x - x^* \right\rangle \geq 0, \quad \forall x \in C. \tag{11}$$

The solution set of the VIP (11) is denoted by VI(C, A).

Recently, Ceng et al. [3] transformed problem (10) into a fixed point problem in the following way.

Lemma 1 (see [3]). *For given $\overline{x}, \overline{y} \in C$, $(\overline{x}, \overline{y})$ is a solution of problem (10) if and only if $\overline{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$\begin{aligned} G(x) = P_C \big[& P_C (x - \mu_2 B_2 x) - \mu_1 B_1 P_C \\ & \times (x - \mu_2 B_2 x) \big], \quad \forall x \in C, \end{aligned} \tag{12}$$

where $\overline{y} = P_C(\overline{x} - \mu_2 B_2 \overline{x})$ and P_C is the the projection of H onto C.

In particular, if the mappings $B_i : C \to H$ is β_i-inverse strongly monotone for $i = 1, 2$, then the mapping G is nonexpansive provided $\mu_i \in (0, 2\beta_i)$ for $i = 1, 2$.

Let C be a nonempty, closed, and convex subset of a real smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C, and let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. In this paper we introduce composite implicit and explicit iterative algorithms for solving GSVI (9) and the common fixed point problem of an infinite family $\{S_n\}$ of nonexpansive mappings of C into itself. These composite iterative algorithms are based on Korpelevich's extragradient method [4] and viscosity approximation method [5]. Let the mapping G be defined by

$$G(x) := \Pi_C (I - \mu_1 B_1) \Pi_C (I - \mu_2 B_2) x, \quad \forall x \in C. \tag{13}$$

We first propose a composite implicit iterative algorithm in the setting of uniformly convex and 2-uniformly smooth Banach space X:

$$\begin{aligned} y_n &= \alpha_n f(y_n) + (1 - \alpha_n) S_n G(x_n), \\ x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S_n G(y_n), \quad \forall n \geq 0, \end{aligned} \tag{14}$$

where $B_i : C \to X$ is α_i-inverse-strongly accretive with $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ such that $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. It is proven that under appropriate conditions, $\{x_n\}$ converges strongly to $q \in F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega$, which solves the following VIP:

$$\left\langle q - f(q), J(q - p) \right\rangle \leq 0, \quad \forall p \in F. \tag{15}$$

On the other hand, we also propose another composite explicit iterative algorithm in a uniformly convex Banach space X with a uniformly Gateaux differentiable norm:

$$\begin{aligned} y_n &= \alpha_n G(x_n) + (1 - \alpha_n) S_n G(x_n), \\ x_{n+1} &= \beta_n f(x_n) + \gamma_n y_n + \delta_n S_n G(y_n), \quad \forall n \geq 0, \end{aligned} \tag{16}$$

where $B_i : C \to X$ is λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ such that $\beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. It is proven that under mild conditions, $\{x_n\}$ also converges strongly to $q \in F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(S_i) \cap \Omega$, which solves the VIP (15). The results presented in this paper improve, extend, supplement, and develop the corresponding results announced in the earlier and very recent literatures.

2. Preliminaries

We list some lemmas that will be used in the sequel. Lemma 2 can be found in [6]. Lemma 3 is an immediate consequence of the subdifferential inequality of the function $(1/2)\|\cdot\|^2$.

Lemma 2. *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n) s_n + \alpha_n \beta_n + \gamma_n, \quad \forall n \geq 0, \tag{17}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the conditions:

(i) $\{\alpha_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$,

(iii) $\gamma_n \geq 0$, $\forall n \geq 0$, *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\limsup_{n\to\infty} s_n = 0$.

Lemma 3. *In a smooth Banach space X, there holds the inequality*

$$\|x + y\|^2 \leq \|x\|^2 + 2 \langle y, J(x + y)\rangle, \quad \forall x, y \in X. \tag{18}$$

Lemma 4 (see [7]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X, and let $\{\alpha_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1. \tag{19}$$

Suppose that $x_{n+1} = \alpha_n x_n + (1 - \alpha_n) z_n$, $\forall n \geq 0$, and $\limsup_{n\to\infty}(\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \|) \leq 0$. Then $\lim_{n\to\infty} \| z_n - x_n \| = 0$.

Let D be a subset of C, and let Π be a mapping of C into D. Then Π is said to be sunny if

$$\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x), \tag{20}$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$ where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonexpansive retraction.

Lemma 5 (see [8]). *Let C be a nonempty, closed, and convex subset of a real smooth Banach space X. Let D be a nonempty subset of C. Let Π be a retraction of C onto D. Then the following are equivalent:*

(i) Π *is sunny and nonexpansive;*

(ii) $\| \Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle$, $\forall x, y \in C$;

(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0$, $\forall x \in C$, $y \in D$.

It is well known that if $X = H$ a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C; that is, $\Pi_C = P_C$. If C is a nonempty, closed, and convex subset of a strictly convex and uniformly smooth Banach space X and if $T : C \to C$ is a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) \neq \emptyset$, then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C.

Lemma 6 (see [9]). *Given a number $r > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$, $g(0) = 0$, such that*

$$\begin{aligned} &\|\lambda x + (1 - \lambda) y\|^2 \\ &\quad \leq \lambda \|x\|^2 + (1 - \lambda) \|y\|^2 - \lambda (1 - \lambda) g(\|x - y\|) \end{aligned} \tag{21}$$

for all $\lambda \in [0, 1]$ and $x, y \in X$ such that $\| x \| \leq r$ and $\| y \| \leq r$.

Lemma 7 (see [10]). *Let C be a nonempty, closed, and convex subset of a Banach space X. Let $S_0, S_1, \ldots$, be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1} x\| : x \in C\} < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\| Sx - S_n x \| : x \in C\} = 0$.*

Let C be a nonempty, closed, and convex subset of a Banach space X, and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. As previous, let Ξ_C be the set of all contractions on C. For $t \in (0, 1)$ and $f \in \Xi_C$, let $x_t \in C$ be the unique fixed point of the contraction $x \mapsto tf(x) + (1 - t)Tx$ on C; that is,

$$x_t = tf(x_t) + (1 - t) Tx_t. \tag{22}$$

Lemma 8 (see [11, 12]). *Let X be a uniformly smooth Banach space or a reflexive and strictly convex Banach space with a uniformly Gateaux differentiable norm. Let C be a nonempty, closed, and convex subset of X, let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and $f \in \Xi_C$. Then the net $\{x_t\}$ defined by $x_t = tf(x_t) + (1 - t)Tx_t$ converges strongly to a point in $\mathrm{Fix}(T)$. If we define a mapping $Q : \Xi_C \to \mathrm{Fix}(T)$ by $Q(f) := s - \lim_{t\to 0} x_t$, $\forall f \in \Xi_C$, then $Q(f)$ solves the VIP:*

$$\begin{aligned} &\langle (I - f) Q(f), J(Q(f) - p)\rangle \leq 0, \\ &\qquad \forall f \in \Xi_C, \ p \in \mathrm{Fix}(T). \end{aligned} \tag{23}$$

Lemma 9 (see [13]). *Let C be a nonempty, closed, and convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$. Then a mapping S on C defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$ for $x \in C$ is defined well, nonexpansive, and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ holds.*

3. Implicit Iterative Schemes

In this section, we introduce our implicit iterative schemes and show the strong convergence theorems. We will use the following useful lemmas in the sequel.

Lemma 10 (see [2, Lemma 2.8]). *Let C be a nonempty, closed, and convex subset of a real 2-uniformly smooth Banach space X. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive. Then, one has*

$$\begin{aligned}\|(I-\mu_i B_i)x-(I-\mu_i B_i)y\|^2 &\le \|x-y\|^2+2\mu_i(\mu_i\kappa^2-\alpha_i)\\ &\quad\times\|B_i x-B_i y\|^2, \quad \forall x,y\in C,\end{aligned} \tag{24}$$

for $i=1,2$, where $\mu_i>0$. In particular, if $0<\mu_i\le\alpha_i/\kappa^2$, then $I-\mu_i B_i$ is nonexpansive for $i=1,2$.

Lemma 11 (see [2, Lemma 2.9]). *Let C be a nonempty, closed, and convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i=1,2$. Let $G : C \to C$ be the mapping defined by*

$$\begin{aligned}G(x)=\Pi_C[\Pi_C(x-\mu_2B_2x)\\ -\mu_1B_1\Pi_C(x-\mu_2B_2x)], \quad \forall x\in C.\end{aligned} \tag{25}$$

If $0<\mu_i\le\alpha_i/\kappa^2$ for $i=1,2$, then $G : C \to C$ is nonexpensive.

Lemma 12 (see [2, Lemma 2.10]). *Let C be a nonempty, closed, and convex subset of a real 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $B_1, B_2 : C \to X$ be two nonlinear mappings. For given $x^*, y^* \in C, (x^*, y^*)$ is a solution of GSVI (9) if and only if $x^*=\Pi_C(y^*-\mu_1B_1y^*)$ where $y^*=\Pi_C(x^*-\mu_2B_2x^*)$.*

Remark 13. By Lemma 12, we observe that

$$x^*=\Pi_C[\Pi_C(x^*-\mu_2B_2x^*)-\mu_1B_1\Pi_C(x^*-\mu_2B_2x^*)], \tag{26}$$

which implies that x^* is a fixed point of the mapping G.

We now state and prove our first result on the implicit iterative scheme.

Theorem 14. *Let C be a nonempty, closed, and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i=1,2$. Let $f : C \to C$ be a contraction with coefficient $\rho\in(0,1)$. Let $\{S_n\}_{n=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $F=\bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)\cap\Omega\neq\emptyset$, where Ω is the fixed point set of the mapping $G=\Pi_C(I-\mu_1B_1)\Pi_C(I-\mu_2B_2)$. For arbitrarily given $x_0\in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}y_n&=\alpha_n f(y_n)+(1-\alpha_n)S_nG(x_n),\\ x_{n+1}&=\beta_nx_n+\gamma_ny_n+\delta_nS_nG(y_n), \quad \forall n\ge0,\end{aligned} \tag{27}$$

where $0<\mu_i<\alpha_i/\kappa^2$ for $i=1,2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0,1)$ such that $\beta_n+\gamma_n+\delta_n=1$, $\forall n\ge0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$,

(ii) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}(\beta_n+\gamma_n)<1$ *and* $\liminf_{n\to\infty}\gamma_n>0$,

(iii) $\lim_{n\to\infty}|\gamma_n/(1-\beta_n)-\gamma_{n-1}/(1-\beta_{n-1})|=0$.

Assume that $\sum_{n=1}^{\infty}\sup_{x\in D}\|S_nx-S_{n-1}x\|<\infty$ for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx=\lim_{n\to\infty}S_nx$ for all $x\in C$. Suppose that $\mathrm{Fix}(S)=\bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. Then $\{x_n\}$ converges strongly to $q\in F$, which solves the following VIP:

$$\langle q-f(q),J(q-p)\rangle\le0, \quad \forall p\in F. \tag{28}$$

Proof. Take a fixed $p\in F$ arbitrarily. Then by Lemma 12, we know that $p=G(p)$ and $p=S_np$ for all $n\ge0$. Moreover, by Lemma 11, we have

$$\begin{aligned}\|y_n-p\|&=\|\alpha_n(f(y_n)-p)+(1-\alpha_n)(S_nG(x_n)-p)\|\\ &\le\alpha_n\|f(y_n)-f(p)\|+\alpha_n\|f(p)-p\|\\ &\quad+(1-\alpha_n)\|S_nG(x_n)-p\|\\ &\le\alpha_n\rho\|y_n-p\|+\alpha_n\|f(p)-p\|\\ &\quad+(1-\alpha_n)\|G(x_n)-p\|\\ &\le\alpha_n\rho\|y_n-p\|+\alpha_n\|f(p)-p\|\\ &\quad+(1-\alpha_n)\|x_n-p\|,\end{aligned} \tag{29}$$

which hence implies that

$$\begin{aligned}\|y_n-p\|&\le\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n-p\|\\ &\quad+\frac{1}{1-\alpha_n\rho}\alpha_n\|f(p)-p\|\\ &\le\|x_n-p\|+\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|.\end{aligned} \tag{30}$$

Thus, from (27), we have

$$\begin{aligned}\|x_{n+1}-p\|&=\|\beta_n(x_n-p)+\gamma_n(y_n-p)+\delta_n(S_nG(y_n)-p)\|\\ &\le\beta_n\|x_n-p\|+\gamma_n\|y_n-p\|+\delta_n\|G(y_n)-p\|\\ &\le\beta_n\|x_n-p\|+\gamma_n\|y_n-p\|+\delta_n\|y_n-p\|\\ &=\beta_n\|x_n-p\|+(1-\beta_n)\|y_n-p\|\end{aligned}$$

$$\begin{aligned}
&\le \beta_n \|x_n - p\| + (1-\beta_n) \\
&\quad \times \left\{ \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \|x_n - p\| \right. \\
&\qquad \left. + \frac{1}{1-\alpha_n\rho}\alpha_n \|f(p) - p\| \right\} \\
&= \left[1 - \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right] \|x_n - p\| \\
&\quad + \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n \frac{\|f(p) - p\|}{1-\rho} \\
&\le \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\rho}\right\}.
\end{aligned} \tag{31}$$

It immediately follows that $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{G(x_n)\}$, and $\{G(y_n)\}$ due to (30) and the nonexpansivity of G.

Let us show that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. As a matter of fact, from (27), we have

$$\begin{aligned}
y_n &= \alpha_n f(y_n) + (1-\alpha_n) S_n G(x_n), \\
y_{n-1} &= \alpha_{n-1} f(y_{n-1}) + (1-\alpha_{n-1}) S_{n-1} G(x_{n-1}), \quad \forall n \ge 1.
\end{aligned} \tag{32}$$

Simple calculations show that

$$\begin{aligned}
y_n - y_{n-1} &= \alpha_n (f(y_n) - f(y_{n-1})) \\
&\quad + (\alpha_n - \alpha_{n-1})(f(y_{n-1}) - S_{n-1}G(x_{n-1})) \\
&\quad + (1-\alpha_n)(S_n G(x_n) - S_{n-1}G(x_{n-1})).
\end{aligned} \tag{33}$$

It follows that

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \le \alpha_n \|f(y_n) - f(y_{n-1})\| + |\alpha_n - \alpha_{n-1}| \\
&\qquad \times \|f(y_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + (1-\alpha_n)\|S_n G(x_n) - S_{n-1}G(x_{n-1})\| \\
&\quad \le \alpha_n\rho \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \\
&\qquad \times \|f(y_{n-1}) - S_{n-1}G(x_{n-1})\| + (1-\alpha_n) \\
&\qquad \times (\|S_n G(x_n) - S_n G(x_{n-1})\| \\
&\qquad\quad + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\|) \\
&\quad \le \alpha_n\rho \|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \\
&\qquad \times \|f(y_{n-1}) - S_{n-1}G(x_{n-1})\| + (1-\alpha_n) \\
&\qquad \times (\| x_n - x_{n-1} \| + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\|),
\end{aligned} \tag{34}$$

which hence yields

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \le \frac{1-\alpha_n}{1-\alpha_n\rho}(\|x_n - x_{n-1}\| \\
&\qquad\qquad + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\|) \\
&\qquad + \frac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n\rho}\|f(y_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad \le \|x_n - x_{n-1}\| + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + \frac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n\rho}\|f(y_{n-1}) - S_{n-1}G(x_{n-1})\|.
\end{aligned} \tag{35}$$

Now, we write $x_n = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})z_{n-1}$, $\forall n \ge 1$, where $z_{n-1} = (x_n - \beta_{n-1}x_{n-1})/(1-\beta_{n-1})$. It follows that for all $n \ge 1$,

$$\begin{aligned}
z_n - z_{n-1} &= \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} - \frac{x_n - \beta_{n-1}x_{n-1}}{1-\beta_{n-1}} \\
&= \frac{\gamma_n y_n + \delta_n S_n G(y_n)}{1-\beta_n} \\
&\quad - \frac{\gamma_{n-1}y_{n-1} + \delta_{n-1}S_{n-1}G(y_{n-1})}{1-\beta_{n-1}} \\
&= \frac{\gamma_n(y_n - y_{n-1}) + \delta_n(S_n G(y_n) - S_{n-1}G(y_{n-1}))}{1-\beta_n} \\
&\quad + \left(\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right) y_{n-1} \\
&\quad + \left(\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\right) S_{n-1}G(y_{n-1}).
\end{aligned} \tag{36}$$

This together with (35) implies that

$$\begin{aligned}
&\|z_n - z_{n-1}\| \\
&\quad \le \frac{\|\gamma_n(y_n - y_{n-1}) + \delta_n(S_n G(y_n) - S_{n-1}G(y_{n-1}))\|}{1-\beta_n} \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| \|y_{n-1}\| \\
&\qquad + \left|\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\right| \|S_{n-1}G(y_{n-1})\| \\
&\quad \le (\gamma_n \|y_n - y_{n-1}\| \\
&\qquad + \delta_n(\|S_n G(y_n) - S_n G(y_{n-1})\| \\
&\qquad\qquad + \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\|))
\end{aligned}$$

$$
\begin{aligned}
&\times (1-\beta_n)^{-1} \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \|y_{n-1}\| \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \|S_{n-1}G(y_{n-1})\| \\
\leq\ & \frac{\gamma_n \|y_n - y_{n-1}\| + \delta_n \|y_n - y_{n-1}\|}{\gamma_n + \delta_n} \\
&+ \frac{\delta_n}{\gamma_n + \delta_n} \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
\leq\ & \|y_n - y_{n-1}\| + \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
\leq\ & \|x_n - x_{n-1}\| + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&+ \frac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n\rho} \|f(y_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&+ \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
\leq\ & \|x_n - x_{n-1}\| + \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&+ \frac{|\alpha_n - \alpha_{n-1}|}{1-\alpha_n\rho} M + \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&+ \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{\mathrm{n}-1}} \right| M \\
=\ & \|x_n - x_{n-1}\| \\
&+ M \left(\frac{1}{1-\alpha_n\rho} |\alpha_n - \alpha_{n-1}| + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \right) \\
&+ \|S_n G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&+ \|S_n G(y_{n-1}) - S_{n-1}G(y_{n-1})\|,
\end{aligned}
\tag{37}
$$

where $\sup_{n\geq 0}\{\|f(y_n)\| + \|S_nG(x_n)\| + \|y_n\| + \|S_nG(y_n)\|\} \leq M$ for some $M > 0$. So, from $\alpha_n \to 0$, condition (iii), and the assumption on $\{S_n\}$, it immediately follows that

$$
\limsup_{n\to\infty} (\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\|) \leq 0. \tag{38}
$$

In terms of condition (ii) and Lemma 4, we get

$$
\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{39}
$$

Hence we obtain

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n) \|z_n - x_n\| = 0. \tag{40}
$$

Next we show that $\| x_n - G(x_n) \| \to 0$ as $n \to \infty$.

For simplicity, put $q = \Pi_C(p - \mu_2 B_2 p)$, $u_n = \Pi_C(x_n - \mu_2 B_2 x_n)$, and $v_n = \Pi_C(u_n - \mu_1 B_1 u_n)$. Then $v_n = G(x_n)$. From Lemma 10, we have

$$
\begin{aligned}
\|u_n - q\|^2 &= \|\Pi_C(x_n - \mu_2 B_2 x_n) - \Pi_C(p - \mu_2 B_2 p)\|^2 \\
&\leq \|x_n - p - \mu_2 (B_2 x_n - B_2 p)\|^2 \\
&\leq \|x_n - p\|^2 - 2\mu_2 (\alpha_2 - \kappa^2 \mu_2) \|B_2 x_n - B_2 p\|^2,
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
\|v_n - p\|^2 &= \|\Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(q - \mu_1 B_1 q)\|^2 \\
&\leq \|u_n - q - \mu_1 (B_1 u_n - B_1 q)\|^2 \\
&\leq \|u_n - q\|^2 - 2\mu_1 (\alpha_1 - \kappa^2 \mu_1) \|B_1 u_n - B_1 q\|^2.
\end{aligned}
\tag{42}
$$

Substituting (41) into (42), we obtain

$$
\begin{aligned}
\|v_n - p\|^2 \leq\ & \|x_n - p\|^2 - 2\mu_2 (\alpha_2 - \kappa^2 \mu_2) \|B_2 x_n - B_2 p\|^2 \\
&- 2\mu_1 (\alpha_1 - \kappa^2 \mu_1) \|B_1 u_n - B_1 q\|^2.
\end{aligned}
\tag{43}
$$

According to Lemma 3, we have from (27)

$$
\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \|\alpha_n (f(y_n) - f(p)) + (1-\alpha_n)(S_n v_n - p) \\
&\qquad + \alpha_n (f(p) - p)\|^2 \\
&\quad \leq \|\alpha_n (f(y_n) - f(p)) + (1-\alpha_n)(S_n v_n - p)\|^2 \\
&\qquad + 2\alpha_n \langle f(p) - p, J(y_n - p) \rangle \\
&\quad \leq \alpha_n \|f(y_n) - f(p)\|^2 + (1-\alpha_n) \|S_n v_n - p\|^2 \\
&\qquad + 2\alpha_n \langle f(p) - p, J(y_n - p) \rangle \\
&\quad \leq \alpha_n \rho \|y_n - p\|^2 + (1-\alpha_n) \|v_n - p\|^2 \\
&\qquad + 2\alpha_n \|f(p) - p\| \|y_n - p\|,
\end{aligned}
\tag{44}
$$

which hence yields

$$
\begin{aligned}
&\|y_n - p\|^2 \\
&\quad \leq \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \|v_n - p\|^2 \\
&\qquad + \frac{2\alpha_n}{1-\alpha_n\rho} \|f(p) - p\| \|y_n - p\|.
\end{aligned}
\tag{45}
$$

This together with (43) and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad= \|\beta_n(x_n-p)+\gamma_n(y_n-p)+\delta_n(S_nG(y_n)-p)\|^2 \\
&\quad\le \beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2+\delta_n\|S_nG(y_n)-p\|^2 \\
&\quad\le \beta_n\|x_n-p\|^2+\gamma_n\|y_n-p\|^2+\delta_n\|y_n-p\|^2 \\
&\quad= \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2 \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n) \\
&\qquad\times\left\{\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|v_n-p\|^2\right. \\
&\qquad\quad\left.+\frac{2\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\|y_n-p\|\right\} \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n) \\
&\qquad\times\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|v_n-p\|^2+\alpha_nM_1 \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\qquad\times\left[\|x_n-p\|^2-2\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_n-B_2p\|^2\right. \\
&\qquad\quad\left.-2\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_n-B_1q\|^2\right]+\alpha_nM_1 \\
&\quad= \left(1-\frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right)\|x_n-p\|^2-2(1-\beta_n) \\
&\qquad\times\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\qquad\times\left[\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_n-B_2p\|^2\right. \\
&\qquad\quad\left.+\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_n-B_1q\|^2\right]+\alpha_nM_1 \\
&\quad\le \|x_n-p\|^2-2(1-\beta_n)\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\qquad\times\left[\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_n-B_2p\|^2\right. \\
&\qquad\quad\left.+\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_n-B_1q\|^2\right]+\alpha_nM_1,
\end{aligned} \tag{46}$$

where $\sup_{n\ge0}\{2(1-\beta_n)/(1-\alpha_n\rho)\|f(p)-p\|\|y_n-p\|\}\le M_1$ for some $M_1>0$. So, it follows that

$$\begin{aligned}
&2(1-\beta_n)\left(1-\frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\qquad\times\left[\mu_2(\alpha_2-\kappa^2\mu_2)\|B_2x_n-B_2p\|^2\right. \\
&\qquad\quad\left.+\mu_1(\alpha_1-\kappa^2\mu_1)\|B_1u_n-B_1q\|^2\right] \\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_nM_1 \\
&\quad\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\alpha_nM_1.
\end{aligned} \tag{47}$$

Since $0<\mu_i<\alpha_i/\kappa^2$ for $i=1,2$, from conditions (i), (ii), and (40), we obtain

$$\lim_{n\to\infty}\|B_2x_n-B_2p\|=0,\qquad \lim_{n\to\infty}\|B_1u_n-B_1q\|=0. \tag{48}$$

Utilizing [14, Proposition 1] and Lemma 5, we have

$$\begin{aligned}
&\|u_n-q\|^2 \\
&\quad= \|\Pi_C(x_n-\mu_2B_2x_n)-\Pi_C(p-\mu_2B_2p)\|^2 \\
&\quad\le \langle x_n-\mu_2B_2x_n-(p-\mu_2B_2p),J(u_n-q)\rangle \\
&\quad= \langle x_n-p,J(u_n-q)\rangle \\
&\qquad+\mu_2\langle B_2p-B_2x_n,J(u_n-q)\rangle \\
&\quad\le \frac{1}{2}\left[\|x_n-p\|^2+\|u_n-q\|^2-g_1(\|x_n-u_n-(p-q)\|)\right] \\
&\qquad+\mu_2\|B_2p-B_2x_n\|\|u_n-q\|,
\end{aligned} \tag{49}$$

which implies that

$$\begin{aligned}
\|u_n-q\|^2 \le{}& \|x_n-p\|^2-g_1(\|x_n-u_n-(p-q)\|) \\
&+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|.
\end{aligned} \tag{50}$$

In the same way, we derive

$$\begin{aligned}
&\|v_n-p\|^2 \\
&\quad= \|\Pi_C(u_n-\mu_1B_1u_n)-\Pi_C(q-\mu_1B_1q)\|^2 \\
&\quad\le \langle u_n-\mu_1B_1u_n-(q-\mu_1B_1q),J(v_n-p)\rangle \\
&\quad= \langle u_n-q,J(v_n-p)\rangle+\mu_1\langle B_1q-B_1u_n,J(v_n-p)\rangle \\
&\quad\le \frac{1}{2}\left[\|u_n-q\|^2+\|v_n-p\|^2\right. \\
&\qquad\left.-g_2(\|u_n-v_n+(p-q)\|)\right] \\
&\qquad+\mu_1\|B_1q-B_1u_n\|\|v_n-p\|,
\end{aligned} \tag{51}$$

which implies that

$$\begin{aligned}
\|v_n-p\|^2 \le{}& \|u_n-q\|^2-g_2(\|u_n-v_n+(p-q)\|) \\
&+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|.
\end{aligned} \tag{52}$$

Substituting (50) into (52), we get

$$\begin{aligned}
\|v_n-p\|^2 \le{}& \|x_n-p\|^2-g_1(\|x_n-u_n-(p-q)\|) \\
&-g_2(\|u_n-v_n+(p-q)\|) \\
&+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\| \\
&+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|.
\end{aligned} \tag{53}$$

From (46) and (53), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n M_1 + \beta_n \|x_n - p\|^2 \\
&\quad + (1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\quad \times \left[\|x_n - p\|^2 - g_1(\|x_n - u_n - (p-q)\|)\right. \\
&\qquad - g_2(\|u_n - v_n + (p-q)\|) \\
&\qquad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\qquad \left. + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|\right] \\
&\le \alpha_n M_1 + \left(1 - \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right)\|x_n - p\|^2 \\
&\quad - (1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\quad \times [g_1(\|x_n - u_n - (p-q)\|) \\
&\qquad + g_2(\|u_n - v_n + (p-q)\|)] \\
&\quad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\quad + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\| \\
&\le \alpha_n M_1 + \|x_n - p\|^2 - (1-\beta_n) \\
&\quad \times \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\quad \times [g_1(\|x_n - u_n - (p-q)\|) \\
&\qquad + g_2(\|u_n - v_n + (p-q)\|)] \\
&\quad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\quad + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|,
\end{aligned} \tag{54}$$

which implies that

$$\begin{aligned}
&(1-\beta_n)\left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right) \\
&\quad \times [g_1(\|x_n - u_n - (p-q)\|) \\
&\qquad + g_2(\|u_n - v_n + (p-q)\|)] \\
&\le \alpha_n M_1 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\quad + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\| \\
&\le \alpha_n M_1 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\
&\quad + 2\mu_2 \|B_2 p - B_2 x_n\| \|u_n - q\| \\
&\quad + 2\mu_1 \|B_1 q - B_1 u_n\| \|v_n - p\|.
\end{aligned} \tag{55}$$

Utilizing conditions (i), (ii), from (40) and (48), we have

$$\begin{aligned}
\lim_{n\to\infty} g_1(\|x_n - u_n - (p-q)\|) &= 0, \\
\lim_{n\to\infty} g_2(\|u_n - v_n + (p-q)\|) &= 0.
\end{aligned} \tag{56}$$

Utilizing the properties of g_1 and g_2, we deduce that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - u_n - (p-q)\| &= 0, \\
\lim_{n\to\infty} \|u_n - v_n + (p-q)\| &= 0.
\end{aligned} \tag{57}$$

From (57), we obtain

$$\begin{aligned}
\|x_n - v_n\| &\le \|x_n - u_n - (p-q)\| \\
&\quad + \|u_n - v_n + (p-q)\| \longrightarrow 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{58}$$

That is,

$$\lim_{n\to\infty} \|x_n - G(x_n)\| = 0. \tag{59}$$

On the other hand, since $\{y_n\}$ and $\{S_n G(y_n)\}$ are bounded, by Lemma 6, there exists a continuous strictly increasing function $g_3 : [0,\infty) \to [0,\infty)$, $g_3(0) = 0$ such that for $p \in F$

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|\beta_n(x_n - p) + \gamma_n(y_n - p) + \delta_n(S_n G(y_n) - p)\|^2 \\
&\quad = \left\|(\gamma_n + \delta_n)\left[\frac{\gamma_n}{\gamma_n + \delta_n}(y_n - p)\right.\right. \\
&\qquad\qquad \left.\left. + \frac{\delta_n}{\gamma_n + \delta_n}(S_n G(y_n) - p)\right] + \beta_n(x_n - p)\right\|^2 \\
&\quad \le (\gamma_n + \delta_n)\left\|\frac{\gamma_n}{\gamma_n + \delta_n}(y_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(S_n G(y_n) - p)\right\|^2 \\
&\qquad + \beta_n \|x_n - p\|^2 \\
&\quad \le (\gamma_n + \delta_n)\left[\frac{\gamma_n}{\gamma_n + \delta_n}\|y_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|S_n G(y_n) - p\|^2\right. \\
&\qquad\qquad \left. - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_3(\|y_n - S_n G(y_n)\|)\right] \\
&\qquad + \beta_n \|x_n - p\|^2 \\
&\quad \le \gamma_n \|y_n - p\|^2 + \delta_n \|y_n - p\|^2 \\
&\qquad - \frac{\gamma_n \delta_n}{\gamma_n + \delta_n} g_3(\|y_n - S_n G(y_n)\|) + \beta_n \|x_n - p\|^2 \\
&\quad = (1-\beta_n)\|y_n - p\|^2 - \frac{\gamma_n \delta_n}{\gamma_n + \delta_n} \\
&\qquad \times g_3(\|y_n - S_n G(y_n)\|) + \beta_n \|x_n - p\|^2,
\end{aligned} \tag{60}$$

which together with (30) implies that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le (1-\beta_n)\left(\|x_n-p\|+\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\right)^2 \\
&\qquad -\frac{\gamma_n\delta_n}{\gamma_n+\delta_n}g_3(\|y_n-S_nG(y_n)\|)+\beta_n\|x_n-p\|^2 \\
&\quad\le \left(\|x_n-p\|+\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\right)^2 \\
&\qquad -\frac{\gamma_n\delta_n}{\gamma_n+\delta_n}g_3(\|y_n-S_nG(y_n)\|).
\end{aligned} \tag{61}$$

It immediately follows that

$$\begin{aligned}
&\gamma_n\delta_n g_3(\|y_n-S_nG(y_n)\|) \\
&\quad\le \frac{\gamma_n\delta_n}{\gamma_n+\delta_n}g_3(\|y_n-S_nG(y_n)\|) \\
&\quad\le \left(\|x_n-p\|+\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\right)^2 \\
&\qquad -\|x_{n+1}-p\|^2 \\
&\quad\le \Big(\|x_n-p\|+\|x_{n+1}-p\| \\
&\qquad +\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\Big) \\
&\qquad \times\left(\|x_n-x_{n+1}\|+\frac{\alpha_n}{1-\alpha_n\rho}\|f(p)-p\|\right).
\end{aligned} \tag{62}$$

According to condition (ii), we get

$$\begin{aligned}
\liminf_{n\to\infty}\delta_n &= \liminf_{n\to\infty}(1-\beta_n-\gamma_n) \\
&= 1-\limsup_{n\to\infty}(\beta_n+\gamma_n)>0.
\end{aligned} \tag{63}$$

Since $\alpha_n \to 0$, $\| x_{n+1}-x_n \| \to 0$, and $\liminf_{n\to\infty}\gamma_n>0$, we conclude that

$$\lim_{n\to\infty} g_3(\|y_n-S_nG(y_n)\|)=0. \tag{64}$$

Utilizing the property of g_3, we have

$$\lim_{n\to\infty}\|y_n-S_nG(y_n)\|=0. \tag{65}$$

We note that

$$\begin{aligned}
x_{n+1}-x_n+x_n-y_n &= x_{n+1}-y_n \\
&= \beta_n(x_n-y_n)+\delta_n(S_nG(y_n)-y_n).
\end{aligned} \tag{66}$$

So,

$$\begin{aligned}
&(1-\beta_n)\|x_n-y_n\| \\
&\quad= \|\delta_n(S_nG(y_n)-y_n)-(x_{n+1}-x_n)\| \\
&\quad\le \delta_n\|S_nG(y_n)-y_n\|+\|x_{n+1}-x_n\| \\
&\quad\le \|S_nG(y_n)-y_n\| \\
&\qquad +\|x_{n+1}-x_n\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{67}$$

That is,

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{68}$$

We observe that

$$\begin{aligned}
&\|G(x_n)-S_nG(x_n)\| \\
&\quad\le \|G(x_n)-x_n\|+\|x_n-y_n\|+\|y_n-S_nG(y_n)\| \\
&\qquad +\|S_nG(y_n)-S_nG(x_n)\| \\
&\quad\le \|G(x_n)-x_n\|+2\|x_n-y_n\|+\|y_n-S_nG(y_n)\|.
\end{aligned} \tag{69}$$

Thus, from (59)–(68), we obtain that

$$\lim_{n\to\infty}\|G(x_n)-S_nG(x_n)\|=0. \tag{70}$$

By (70) and Lemma 7, we have

$$\begin{aligned}
&\|SG(x_n)-G(x_n)\| \\
&\quad\le \|SG(x_n)-S_nG(x_n)\| \\
&\qquad +\|S_nG(x_n)-G(x_n)\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{71}$$

In terms of (59) and (71), we have

$$\begin{aligned}
\|x_n-Sx_n\| &\le \|x_n-G(x_n)\|+\|G(x_n)-SG(x_n)\| \\
&\quad +\|SG(x_n)-Sx_n\| \\
&\le 2\|x_n-G(x_n)\| \\
&\quad +\|G(x_n)-SG(x_n)\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{72}$$

Define a mapping $Wx=(1-\theta)Sx+\theta G(x)$, where $\theta\in(0,1)$ is a constant. Then by Lemma 9, we have that $\mathrm{Fix}(W)=\mathrm{Fix}(S)\cap\mathrm{Fix}(G)=F$. We observe that

$$\begin{aligned}
\|x_n-Wx_n\| &= \|(1-\theta)(x_n-Sx_n)+\theta(x_n-G(x_n))\| \\
&\le (1-\theta)\|x_n-Sx_n\|+\theta\|x_n-G(x_n)\|.
\end{aligned} \tag{73}$$

From (59) and (72), we obtain

$$\lim_{n\to\infty}\|x_n-Wx_n\|=0. \tag{74}$$

Now, we claim that

$$\limsup_{n\to\infty}\langle f(q)-q, J(x_n-q)\rangle\le 0, \tag{75}$$

where $q = s - \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \longmapsto tf(x) + (1-t)Wx. \tag{76}$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Thus we have

$$\|x_t - x_n\| = \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|. \tag{77}$$

By Lemma 3, we conclude that

$$\begin{aligned}
&\|x_t - x_n\|^2 \\
&\quad = \|(1-t)(Wx_t - x_n) + t(f(x_t) - x_n)\|^2 \\
&\quad \le (1-t)^2\|Wx_t - x_n\|^2 \\
&\qquad + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\quad \le (1-t)^2(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|)^2 \\
&\qquad + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\quad \le (1-t)^2(\|x_t - x_n\| + \|Wx_n - x_n\|)^2 \\
&\qquad + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\quad = (1-t)^2\left[\|x_t - x_n\|^2 + 2\|x_t - x_n\| \right. \\
&\qquad\qquad \left. \times \|Wx_n - x_n\| + \|Wx_n - x_n\|^2\right] \\
&\qquad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle \\
&\qquad + 2t\langle x_t - x_n, J(x_t - x_n)\rangle \\
&\quad = \left(1 - 2t + t^2\right)\|x_t - x_n\|^2 + f_n(t) \\
&\qquad + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2,
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
f_n(t) &= (1-t)^2(2\|x_t - x_n\| + \|x_n - Wx_n\|) \\
&\quad \times \|x_n - Wx_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{79}$$

It follows from (78) that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{80}$$

Letting $n \to \infty$ in (80) and noticing (79), we derive

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}M_2, \tag{81}$$

where $M_2 > 0$ is a constant such that $\|x_t - x_n\|^2 \le M_2$ for all $t \in (0,1)$ and $n \ge 0$. Taking $t \to 0$ in (81), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0. \tag{82}$$

On the other hand, we have

$$\begin{aligned}
&\langle f(q) - q, J(x_n - q)\rangle \\
&\quad = \langle f(q) - q, J(x_n - q)\rangle - \langle f(q) - q, J(x_n - x_t)\rangle \\
&\qquad + \langle f(q) - q, J(x_n - x_t)\rangle - \langle f(q) - x_t, J(x_n - x_t)\rangle \\
&\qquad + \langle f(q) - x_t, J(x_n - x_t)\rangle - \langle f(x_t) - x_t, J(x_n - x_t)\rangle \\
&\qquad + \langle f(x_t) - x_t, J(x_n - x_t)\rangle \\
&\quad = \langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle \\
&\qquad + \langle x_t - q, J(x_n - x_t)\rangle + \langle f(q) - f(x_t), J(x_n - x_t)\rangle \\
&\qquad + \langle f(x_t) - x_t, J(x_n - x_t)\rangle.
\end{aligned} \tag{83}$$

It follows that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \\
&\quad \le \limsup_{n\to\infty}\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle \\
&\qquad + \|x_t - q\|\limsup_{n\to\infty}\|x_n - x_t\| \\
&\qquad + \rho\|q - x_t\|\limsup_{n\to\infty}\|x_n - x_t\| \\
&\qquad + \limsup_{n\to\infty}\langle f(x_t) - x_t, J(x_n - x_t)\rangle.
\end{aligned} \tag{84}$$

Taking into account that $x_t \to q$ as $t \to 0$, we have from (82)

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \\
&\quad = \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \\
&\quad \le \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle.
\end{aligned} \tag{85}$$

Since X has a uniformly Fréchet differentiable norm, the duality mapping J is norm-to-norm uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable, and hence (75) holds. From (68), we get $(y_n - q) - (x_n - q) \to 0$. Noticing that J is norm-to-norm uniformly continuous on bounded subsets of X, we deduce from (75) that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(q) - q, J(y_n - q)\rangle \\
&\quad = \limsup_{n\to\infty}(\langle f(q) - q, J(x_n - q)\rangle \\
&\qquad\qquad + \langle f(q) - q, J(y_n - q) - J(x_n - q)\rangle) \\
&\quad = \limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \le 0.
\end{aligned} \tag{86}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. We observe that

$$\begin{aligned}
&\|y_n - q\|^2 \\
&\quad = \|\alpha_n (f(y_n) - f(q)) + (1-\alpha_n) \\
&\qquad \times (S_n G(x_n) - q) + \alpha_n (f(q) - q)\|^2 \\
&\quad \le \|\alpha_n (f(y_n) - f(q)) \\
&\qquad + (1-\alpha_n)(S_n G(x_n) - q)\|^2 \\
&\qquad + 2\alpha_n \langle f(q) - q, J(y_n - q)\rangle \\
&\quad \le \alpha_n \|f(y_n) - f(q)\|^2 \\
&\qquad + (1-\alpha_n)\|S_n G(x_n) - q\|^2 \\
&\qquad + 2\alpha_n \langle f(q) - q, J(y_n - q)\rangle \\
&\quad \le \alpha_n \rho \|y_n - q\|^2 + (1-\alpha_n)\|x_n - q\|^2 \\
&\qquad + 2\alpha_n \langle f(q) - q, J(y_n - q)\rangle,
\end{aligned} \tag{87}$$

which implies that

$$\begin{aligned}
\|y_n - q\|^2 &\le \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n - q\|^2 \\
&\quad + \frac{\alpha_n(1-\rho)}{1-\alpha_n\rho} \cdot \frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho}.
\end{aligned} \tag{88}$$

By (27) and the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \le \beta_n \|x_n - q\|^2 + \gamma_n \|y_n - q\|^2 + \delta_n \|S_n G(y_n) - q\|^2 \\
&\quad \le \beta_n \|x_n - q\|^2 + \gamma_n \|y_n - q\|^2 + \delta_n \|y_n - q\|^2 \\
&\quad = \beta_n \|x_n - q\|^2 + (1-\beta_n)\|y_n - q\|^2,
\end{aligned} \tag{89}$$

which together with (88) leads to

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad \le \beta_n \|x_n - q\|^2 + (1-\beta_n) \\
&\qquad \times \left\{ \left(1 - \frac{1-\rho}{1-\alpha_n\rho}\alpha_n\right)\|x_n - q\|^2 \right. \\
&\qquad \left. + \frac{\alpha_n(1-\rho)}{1-\alpha_n\rho} \cdot \frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho} \right\} \\
&\quad = \left[1 - \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n\right]\|x_n - q\|^2 \\
&\qquad + \frac{(1-\beta_n)(1-\rho)}{1-\alpha_n\rho}\alpha_n \cdot \frac{2\langle f(q) - q, J(y_n - q)\rangle}{1-\rho}.
\end{aligned} \tag{90}$$

Applying Lemma 2 to (88), we obtain that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 15. *Let C be a nonempty, closed, and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be α_i-inverse-strongly accretive for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let S be a nonexpansive mapping of C into itself such that $F = \mathrm{Fix}(S) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
y_n &= \alpha_n f(y_n) + (1-\alpha_n) SG(x_n), \\
x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n SG(y_n), \quad \forall n \ge 0,
\end{aligned} \tag{91}$$

where $0 < \mu_i < \alpha_i/\kappa^2$ for $i = 1, 2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0,1)$ such that $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \ge 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}(\beta_n + \gamma_n) < 1$ *and* $\liminf_{n\to\infty}\gamma_n > 0$,

(iii) $\lim_{n\to\infty}|(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| = 0$.

Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p)\rangle \le 0, \quad \forall p \in F. \tag{92}$$

Further, we illustrate Theorem 14 by virtue of an example, that is, the following corollary.

Corollary 16. *Let C be a nonempty, closed, and convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let T be an η-strictly pseudocontractive mapping of C into itself, and let S be a nonexpansive mapping of C into itself such that $F = \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
y_n &= \alpha_n f(y_n) + (1-\alpha_n) S(I - \lambda(I - T)) x_n, \\
x_{n+1} &= \beta_n x_n + \gamma_n y_n + \delta_n S(I - \lambda(I - T)) y_n, \quad \forall n \ge 0,
\end{aligned} \tag{93}$$

where $0 < \lambda < \max\{1, \eta/\kappa^2\}$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0,1)$ such that $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \ge 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}(\beta_n + \gamma_n) < 1$ *and* $\liminf_{n\to\infty}\gamma_n > 0$,

(iii) $\lim_{n\to\infty}|(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| = 0$.

Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p)\rangle \le 0, \quad \forall p \in F. \tag{94}$$

Proof. In Corollary 15, put $B_1 = I - T$, $B_2 = 0$, $\mu_1 = \lambda$, and $\alpha_1 = \eta$. Since T is an η-strictly pseudocontractive mapping, it is clear that $B_1 = I - T$ is an η-inverse strongly accretive mapping. Hence, the GSVI (9) is equivalent to the following VIP of finding $x^* \in C$ such that

$$\left(B_1 x^*, J\left(x - x^*\right)\right) \geq 0, \quad \forall x \in C, \tag{95}$$

which leads to $\Omega = \mathrm{VI}(C, B_1)$. In the meantime, we have

$$\begin{aligned} Gx_n &= \Pi_C\left(I - \mu_1 B_1\right)\Pi_C\left(I - \mu_2 B_2\right)x_n \\ &= \Pi_C\left(I - \mu_1 B_1\right)x_n \\ &= \Pi_C\left[(1-\lambda)x_n + \lambda T x_n\right] \\ &= x_n - \lambda\left(I - T\right)x_n. \end{aligned} \tag{96}$$

In the same way, we get $Gy_n = y_n - \lambda(I - T)y_n$. In this case, it is easy to see that (91) reduces to (93). We claim that $\mathrm{Fix}(T) = \mathrm{VI}(C, B_1)$. As a matter of fact, we have, for $\lambda > 0$,

$$\begin{aligned} & u \in \mathrm{VI}\left(C, B_1\right) \\ & \quad \Longleftrightarrow \langle B_1 u, J\left(y - u\right)\rangle \geq 0 \quad \forall y \in C \\ & \quad \Longleftrightarrow \langle u - \lambda B_1 u - u, J\left(u - y\right)\rangle \geq 0 \quad \forall y \in C \\ & \quad \Longleftrightarrow u = \Pi_C\left(u - \lambda B_1 u\right) \\ & \quad \Longleftrightarrow u = \Pi_C\left(u - \lambda u + \lambda T u\right) \\ & \quad \Longleftrightarrow \langle u - \lambda u + \lambda T u - u, J\left(u - y\right)\rangle \geq 0 \quad \forall y \in C \\ & \quad \Longleftrightarrow \langle u - T u, J\left(u - y\right)\rangle \leq 0 \quad \forall y \in C \\ & \quad \Longleftrightarrow u = T u \\ & \quad \Longleftrightarrow u \in \mathrm{Fix}\left(T\right). \end{aligned} \tag{97}$$

So, we conclude that $F = \mathrm{Fix}(S) \cap \Omega = \mathrm{Fix}(S) \cap \mathrm{Fix}(T)$. Therefore, the desired result follows from Corollary 15. □

Remark 17. Theorem 14 improves, extends, supplements, and develops Cai and Bu [2, Theorem 3.1 and Corollary 3.2] and Jung [5, Theorem 3.1] in the following aspects.

(i) The problem of finding a point $q \in \bigcap_n \mathrm{Fix}(S_n) \cap \Omega$ in Theorem 14 is more general and more subtle than the problem of finding a point $q \in \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ in Jung [5, Theorem 3.1].

(ii) The iterative scheme in [2, Theorem 3.1] is extended to develop the iterative scheme (27) of Theorem 14 by virtue of the iterative scheme of [5, Theorem 3.1]. The iterative scheme (27) of Theorem 14 is more advantageous and more flexible than the iterative scheme of [2, Theorem 3.1] because it involves several parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$.

(iii) The iterative scheme (27) in Theorem 14 is very different from everyone in both [2, Theorem 3.1] and [5, Theorem 3.1] because the mappings S_n and G in the iterative scheme of [2, Theorem 3.1] and the mapping $SP_C(I - \lambda_n A)$ in the iterative scheme of [5, Theorem 3.1] are replaced by the same composite mapping $S_n G$ in the iterative scheme (27) of Theorem 14.

(iv) The proof in [2, Theorem 3.1] depends on the argument techniques in [3], the inequality in 2-uniformly smooth Banach spaces ([9]), and the inequality in smooth and uniform convex Banach spaces ([14, Proposition 1]). Because the composite mapping $S_n G$ appears in the iterative scheme (27) of Theorem 14, the proof of Theorem 14 depends on the argument techniques in [3], the inequality in 2-uniformly smooth Banach spaces, the inequality in smooth and uniform convex Banach spaces, and the inequality in uniform convex Banach spaces (Lemma 6).

(v) The iterative scheme in [2, Corollary 3.2] is extended to develop the new iterative scheme in Corollary 15 because the mappings S and G are replaced by the same composite mapping SG in Corollary 15.

4. Explicit Iterative Schemes

In this section, we introduce our explicit iterative schemes and show the strong convergence theorems. First, we give several useful lemmas.

Lemma 18. *Let C be a nonempty, closed, and convex subset of a smooth Banach space X, and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Then, for $\mu_i \in (0, 1]$, we have*

$$\begin{aligned} & \left\|\left(I - \mu_i B_i\right)x - \left(I - \mu_i B_i\right)y\right\| \\ & \quad \leq \left\{\sqrt{\frac{1-\alpha_i}{\lambda_i}} + \left(1 - \mu_i\right)\left(1 + \frac{1}{\lambda_i}\right)\right\}\left\|x - y\right\|, \\ & \qquad \forall x, y \in C, \end{aligned} \tag{98}$$

for $i = 1, 2$. In particular, if $1 - (\lambda_i/(1+\lambda_i))(1 - \sqrt{(1-\alpha_i)/\lambda_i}) \leq \mu_i \leq 1$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Proof. Taking into account the λ_i-strict pseudocontractivity of B_i, we derive for every $x, y \in C$

$$\begin{aligned} & \lambda_i\left\|\left(I - B_i\right)x - \left(I - B_i\right)y\right\|^2 \\ & \quad \leq \left\langle\left(I - B_i\right)x - \left(I - B_i\right)y, J\left(x - y\right)\right\rangle \\ & \quad \leq \left\|\left(I - B_i\right)x - \left(I - B_i\right)y\right\|\left\|x - y\right\|, \end{aligned} \tag{99}$$

which implies that

$$\left\|\left(I - B_i\right)x - \left(I - B_i\right)y\right\| \leq \frac{1}{\lambda_i}\left\|x - y\right\|. \tag{100}$$

Hence,

$$\begin{aligned}&\|B_i x - B_i y\| \\ &\quad \le \|(I - B_i)x - (I - B_i)y\| + \|x - y\| \\ &\quad \le \left(1 + \frac{1}{\lambda_i}\right)\|x - y\|.\end{aligned} \tag{101}$$

Utilizing the α_i-strong accretivity and λ_i-strict pseudocontractivity of B_i, we get

$$\begin{aligned}&\lambda_i\|(I - B_i)x - (I - B_i)y\|^2 \\ &\quad \le \|x - y\|^2 - \langle B_i x - B_i y, J(x - y)\rangle \\ &\quad \le (1 - \alpha_i)\|x - y\|^2.\end{aligned} \tag{102}$$

So, we have

$$\|(I - B_i)x - (I - B_i)y\| \le \sqrt{\frac{1 - \alpha_i}{\lambda_i}}\|x - y\|. \tag{103}$$

Therefore, for $\mu_i \in (0, 1]$, we have

$$\begin{aligned}&\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\| \\ &\quad \le \|(I - B_i)x - (I - B_i)y\| + (1 - \mu_i)\|B_i x - B_i y\| \\ &\quad \le \sqrt{\frac{1 - \alpha_i}{\lambda_i}}\|x - y\| + (1 - \mu_i)\left(1 + \frac{1}{\lambda_i}\right)\|x - y\| \\ &\quad = \left\{\sqrt{\frac{1 - \alpha_i}{\lambda_i}} + (1 - \mu_i)\left(1 + \frac{1}{\lambda_i}\right)\right\}\|x - y\|.\end{aligned} \tag{104}$$

Since $1 - (\lambda_i/(1 + \lambda_i))(1 - \sqrt{(1 - \alpha_i)/\lambda_i}) \le \mu_i \le 1$, it follows immediately that

$$\sqrt{\frac{1 - \alpha_i}{\lambda_i}} + (1 - \mu_i)\left(1 + \frac{1}{\lambda_i}\right) \le 1. \tag{105}$$

This implies that $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$. □

Lemma 19. *Let C be a nonempty, closed, and convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C, and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \ge 1$ for $i = 1, 2$. Let $G : C \to C$ be the mapping defined by*

$$\begin{aligned}G(x) = \Pi_C[&\Pi_C(x - \mu_2 B_2 x) \\ &- \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)], \quad \forall x \in C.\end{aligned} \tag{106}$$

If $1-(\lambda_i/(1+\lambda_i))(1-\sqrt{(1-\alpha_i)/\lambda_i}) \le \mu_i \le 1$, then $G : C \to C$ is nonexpansive.

Proof. According to Lemma 10, we know that $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$. Hence, for all $x, y \in C$, we have

$$\begin{aligned}&\|G(x) - G(y)\| \\ &\quad = \|\Pi_C[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)] \\ &\qquad - \Pi_C[\Pi_C(y - \mu_2 B_2 y) - \mu_1 B_1 \Pi_C(y - \mu_2 B_2 y)]\| \\ &\quad = \|\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x \\ &\qquad - \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)y\| \\ &\quad \le \|(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x \\ &\qquad - (I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)y\| \\ &\quad \le \|\Pi_C(I - \mu_2 B_2)x - \Pi_C(I - \mu_2 B_2)y\| \\ &\quad \le \|(I - \mu_2 B_2)x - (I - \mu_2 B_2)y\| \\ &\quad \le \|x - y\|.\end{aligned} \tag{107}$$

This shows that $G : C \to C$ is nonexpansive. This completes the proof. □

Lemma 20. *Let C be a nonempty, closed, and convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C, and let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive for $i = 1, 2$. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of GSVI (9) if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$ where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

Proof. We can rewrite GSVI (9) as

$$\begin{aligned}&\langle x^* - (y^* - \mu_1 B_1 y^*), J(x - x^*)\rangle \ge 0, \quad \forall x \in C, \\ &\langle y^* - (x^* - \mu_2 B_2 x^*), J(x - y^*)\rangle \ge 0, \quad \forall x \in C,\end{aligned} \tag{108}$$

which is obviously equivalent to

$$\begin{aligned}x^* &= \Pi_C(y^* - \mu_1 B_1 y^*), \\ y^* &= \Pi_C(x^* - \mu_2 B_2 x^*),\end{aligned} \tag{109}$$

because of Lemma 5. This completes the proof. □

Remark 21. By Lemma 20, we observe that

$$x^* = \Pi_C[\Pi_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 \Pi_C(x^* - \mu_2 B_2 x^*)], \tag{110}$$

which implies that x^* is a fixed point of the mapping G. Throughout this paper, the set of fixed points of the mapping G is denoted by Ω.

We are now in a position to state and prove our result on the explicit iterative scheme.

Theorem 22. *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X which has a uniformly*

Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let $\{S_n\}_{n=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $F = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \Omega \neq \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by

$$\begin{aligned} y_n &= \alpha_n G(x_n) + (1 - \alpha_n) S_n G(x_n), \\ x_{n+1} &= \beta_n f(x_n) + \gamma_n y_n + \delta_n S_n G(y_n), \quad \forall n \geq 0, \end{aligned} \tag{111}$$

where $1 - (\lambda_i/(1 + \lambda_i))(1 - \sqrt{(1 - \alpha_i)/\lambda_i}) \leq \mu_i \leq 1$ for $i = 1, 2$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ such that $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \geq 0$. Suppose that the following conditions hold:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\beta_n = 0$,

(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \beta_{n-1}/\beta_n = 1$,

(v) $\sum_{n=1}^{\infty} |(\gamma_n/(1 - \beta_n)) - (\gamma_{n-1}/(1 - \beta_{n-1}))| < \infty$ *or* $\lim_{n\to\infty} (1/\beta_n)|(\gamma_n/(1 - \beta_n)) - (\gamma_{n-1}/(1 - \beta_{n-1}))| = 0$,

(vi) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \| S_n x - S_{n-1} x \| < \infty$ for any bounded subset D of C, and let S be a mapping of C into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$. Suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n)$. Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p) \rangle \leq 0, \quad \forall p \in F. \tag{112}$$

Proof. Take a fixed $p \in F$ arbitrarily. Then by Lemma 20, we know that $p = G(p)$ and $p = S_n p$ for all $n \geq 0$. Moreover, by Lemma 19, we have

$$\begin{aligned} &\|y_n - p\| \\ &\quad \leq \alpha_n \|G(x_n) - p\| + (1 - \alpha_n) \|S_n G(x_n) - p\| \\ &\quad \leq \alpha_n \|x_n - p\| + (1 - \alpha_n) \|x_n - p\| \\ &\quad = \|x_n - p\|. \end{aligned} \tag{113}$$

From (113) we obtain

$$\begin{aligned} &\|x_{n+1} - p\| \\ &\quad \leq \beta_n \|f(x_n) - p\| + \gamma_n \|y_n - p\| \\ &\qquad + \delta_n \|S_n G(y_n) - p\| \\ &\quad \leq \beta_n (\|f(x_n) - f(p)\| + \|f(p) - p\|) \\ &\qquad + \gamma_n \|y_n - p\| + \delta_n \|y_n - p\| \\ &\quad \leq \beta_n \rho \|x_n - p\| + \beta_n \|f(p) - p\| \\ &\qquad + (1 - \beta_n) \|y_n - p\| \\ &\quad \leq \beta_n \rho \|x_n - p\| + \beta_n \|f(p) - p\| \\ &\qquad + (1 - \beta_n) \|x_n - p\| \\ &\quad = (1 - \beta_n (1 - \rho)) \|x_n - p\| \\ &\qquad + \beta_n \|f(p) - p\| \\ &\quad = (1 - \beta_n (1 - \rho)) \|x_n - p\| \\ &\qquad + \beta_n (1 - \rho) \cdot \frac{\|f(p) - p\|}{1 - \rho} \\ &\quad \leq \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1 - \rho}\right\}, \end{aligned} \tag{114}$$

which implies that $\{x_n\}$ is bounded. By Lemma 19 we know from (113) that $\{y_n\}$, $\{G(x_n)\}$, and $\{G(y_n)\}$ are bounded.

Let us show that $\| x_{n+1} - x_n \| \to 0$ and $\| x_n - y_n \| \to 0$ as $n \to \infty$. As a matter of fact, from (113), we have

$$\begin{aligned} y_n &= \alpha_n G(x_n) + (1 - \alpha_n) S_n G(x_n), \\ y_{n-1} &= \alpha_{n-1} G(x_{n-1}) + (1 - \alpha_{n-1}) S_{n-1} G(x_{n-1}), \quad \forall n \geq 1. \end{aligned} \tag{115}$$

Simple calculations show that

$$\begin{aligned} y_n - y_{n-1} &= \alpha_n (G(x_n) - G(x_{n-1})) \\ &\quad + (\alpha_n - \alpha_{n-1}) (G(x_{n-1}) - S_{n-1} G(x_{n-1})) \\ &\quad + (1 - \alpha_n) (S_n G(x_n) - S_{n-1} G(x_{n-1})). \end{aligned} \tag{116}$$

It follows that

$$\begin{aligned} &\|y_n - y_{n-1}\| \\ &\quad \leq \alpha_n \|G(x_n) - G(x_{n-1})\| \\ &\qquad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1} G(x_{n-1})\| \\ &\qquad + (1 - \alpha_n) \|S_n G(x_n) - S_{n-1} G(x_{n-1})\| \end{aligned}$$

$$\begin{aligned}
&\le \alpha_n \|G(x_n) - G(x_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + (1-\alpha_n)(\|S_nG(x_n) - S_nG(x_{n-1})\| \\
&\qquad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\|) \\
&\le \alpha_n \|G(x_n) - G(x_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + (1-\alpha_n)(\|G(x_n) - G(x_{n-1})\| \\
&\qquad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\|) \\
&\le \|G(x_n) - G(x_{n-1})\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\le \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \\
&\quad \times \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\|.
\end{aligned} \tag{117}$$

Now, we write $x_n = \beta_{n-1}f(x_{n-1}) + (1-\beta_{n-1})v_{n-1}$, $\forall n \ge 1$, where $v_{n-1} = (x_n - \beta_{n-1}f(x_{n-1}))/(1-\beta_{n-1})$. It follows that, for all $n \ge 1$,

$$\begin{aligned}
&v_n - v_{n-1} \\
&\quad = \frac{x_{n+1} - \beta_n f(x_n)}{1-\beta_n} - \frac{x_n - \beta_{n-1}f(x_{n-1})}{1-\beta_{n-1}} \\
&\quad = \frac{\gamma_n y_n + \delta_n S_n G(y_n)}{1-\beta_n} \\
&\qquad - \frac{\gamma_{n-1}y_{n-1} + \delta_{n-1}S_{n-1}G(y_{n-1})}{1-\beta_{n-1}} \\
&\quad = \frac{\gamma_n(y_n - y_{n-1}) + \delta_n(S_nG(y_n) - S_{n-1}G(y_{n-1}))}{1-\beta_n} \\
&\qquad + \left(\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right) y_{n-1} \\
&\qquad + \left(\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\right) S_{n-1}G(y_{n-1}).
\end{aligned} \tag{118}$$

This together with (117) implies that

$$\begin{aligned}
&\|v_n - v_{n-1}\| \\
&\quad \le \frac{\|\gamma_n(y_n - y_{n-1}) + \delta_n(S_nG(y_n) - S_{n-1}G(y_{n-1}))\|}{1-\beta_n} \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| \|y_{n-1}\| \\
&\qquad + \left|\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\right| \|S_{n-1}G(y_{n-1})\| \\
&\quad \le (\gamma_n \|y_n - y_{n-1}\| \\
&\qquad + \delta_n(\|S_nG(y_n) - S_nG(y_{n-1})\| \\
&\qquad\quad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\|)) \\
&\qquad \times (1-\beta_n)^{-1} \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| \|y_{n-1}\| \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| \|S_{n-1}G(y_{n-1})\| \\
&\quad \le \frac{\gamma_n \|y_n - y_{n-1}\| + \delta_n \|y_n - y_{n-1}\|}{\gamma_n + \delta_n} \\
&\qquad + \frac{\delta_n}{\gamma_n + \delta_n} \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| \\
&\qquad \times (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
&\quad \le \|y_n - y_{n-1}\| + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
&\quad \le \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\qquad + \left|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|).
\end{aligned} \tag{119}$$

Furthermore, we note that

$$\begin{aligned}
x_{n+1} &= \beta_n f(x_n) + (1-\beta_n) v_n, \\
x_n &= \beta_{n-1} f(x_{n-1}) + (1-\beta_{n-1}) v_{n-1}, \quad \forall n \ge 1.
\end{aligned} \tag{120}$$

Also, simple calculations show that

$$\begin{aligned}
x_{n+1} - x_n &= \beta_n(f(x_n) - f(x_{n-1})) + (\beta_n - \beta_{n-1}) \\
&\quad \times (f(x_{n-1}) - v_{n-1}) + (1-\beta_n)(v_n - v_{n-1}).
\end{aligned} \tag{121}$$

This together with (119) implies that

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \beta_n \|f(x_n) - f(x_{n-1})\| + |\beta_n - \beta_{n-1}| \\
&\qquad \times \|f(x_{n-1}) - v_{n-1}\| + (1-\beta_n)\|v_n - v_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&\le \beta_n \rho \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|f(x_{n-1}) - v_{n-1}\| + (1-\beta_n) \\
&\quad \times \Big[\|x_n - x_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\qquad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\qquad + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \\
&\qquad \times (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \Big] \\
&\le (1 - \beta_n(1-\rho)) \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| \|f(x_{n-1}) - v_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}| \|G(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\quad + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| (\|y_{n-1}\| + \|S_{n-1}G(y_{n-1})\|) \\
&\le (1 - \beta_n(1-\rho)) \|x_n - x_{n-1}\| \\
&\quad + |\beta_n - \beta_{n-1}| M + |\alpha_n - \alpha_{n-1}| M \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\| \\
&\quad + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| M \\
&= (1 - \beta_n(1-\rho)) \|x_n - x_{n-1}\| \\
&\quad + M \Big(|\alpha_n - \alpha_{n-1}| \\
&\qquad + |\beta_n - \beta_{n-1}| + \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| \Big) \\
&\quad + \|S_nG(x_{n-1}) - S_{n-1}G(x_{n-1})\| \\
&\quad + \|S_nG(y_{n-1}) - S_{n-1}G(y_{n-1})\|,
\end{aligned} \tag{122}
$$

where $\sup_{n\ge 0}\{\| f(x_n) \| + \| v_n \| + \| G(x_n) \| + \| S_nG(x_n) \| + \| y_n \| + \| S_nG(y_n) \|\} \le M$ for some $M > 0$. Utilizing Lemma 2, from conditions (ii)–(v) and the assumption on $\{S_n\}$, we deduce that

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{123}
$$

Taking into account the boundedness of $\{G(x_n)\}$ and $\{S_nG(x_n)\}$, by Lemma 6, we know that there exists a continuous strictly increasing function $g_1 : [0,\infty) \to [0,\infty)$, $g_1(0) = 0$ such that for $p \in F$

$$
\begin{aligned}
&\|y_n - p\|^2 \\
&\quad \le \alpha_n \|G(x_n) - p\|^2 + (1-\alpha_n) \|S_nG(x_n) - p\|^2 \\
&\qquad - \alpha_n(1-\alpha_n) g_1(\|G(x_n) - S_nG(x_n)\|) \\
&\quad \le \alpha_n \|x_n - p\|^2 + (1-\alpha_n) \|x_n - p\|^2 \\
&\qquad - \alpha_n(1-\alpha_n) g_1(\|G(x_n) - S_nG(x_n)\|) \\
&\quad = \|x_n - p\|^2 - \alpha_n(1-\alpha_n) g_1(\|G(x_n) - S_nG(x_n)\|).
\end{aligned} \tag{124}
$$

Since $\{y_n\}$ and $\{S_nG(y_n)\}$ are bounded, by Lemma 6, there exists a continuous strictly increasing function $g_2 : [0,\infty) \to [0,\infty)$, $g_2(0) = 0$ such that for $p \in F$

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|\beta_n (f(x_n) - p) + \gamma_n (y_n - p) \\
&\qquad + \delta_n (S_nG(y_n) - p)\|^2 \\
&\quad = \Big\| (\gamma_n + \delta_n) \\
&\qquad \times \Big[\frac{\gamma_n}{\gamma_n + \delta_n} (y_n - p) \\
&\qquad\quad + \frac{\delta_n}{\gamma_n + \delta_n} (S_nG(y_n) - p) \Big] \\
&\qquad + \beta_n (f(x_n) - p) \Big\|^2 \\
&\quad \le (\gamma_n + \delta_n)^2 \Big\| \frac{\gamma_n}{\gamma_n + \delta_n} (y_n - p) \\
&\qquad\qquad + \frac{\delta_n}{\gamma_n + \delta_n} (S_nG(y_n) - p) \Big\|^2 \\
&\qquad + 2\beta_n \langle f(x_n) - p, J(x_{n+1} - p) \rangle \\
&\quad \le (\gamma_n + \delta_n)^2 \Big[\frac{\gamma_n}{\gamma_n + \delta_n} \|y_n - p\|^2 \\
&\qquad\qquad + \frac{\delta_n}{\gamma_n + \delta_n} \|S_nG(y_n) - p\|^2 \\
&\qquad\qquad - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2(\|y_n - S_nG(y_n)\|) \Big] \\
&\qquad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\|
\end{aligned}
$$

$$\begin{aligned}
&\le \gamma_n \|y_n - p\|^2 + \delta_n \|y_n - p\|^2 \\
&\quad - \gamma_n \delta_n g_2 (\|y_n - S_n G(y_n)\|) \\
&\quad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\| \\
&\le \|y_n - p\|^2 - \gamma_n \delta_n g_2 (\|y_n - S_n G(y_n)\|) \\
&\quad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\|,
\end{aligned} \tag{125}$$

which together with (124) implies that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \|x_n - p\|^2 - \alpha_n (1 - \alpha_n) g_1 (\|G(x_n) - S_n G(x_n)\|) \\
&\quad - \gamma_n \delta_n g_2 (\|y_n - S_n G(y_n)\|) \\
&\quad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\|.
\end{aligned} \tag{126}$$

It immediately follows that

$$\begin{aligned}
&\alpha_n (1 - \alpha_n) g_1 (\|G(x_n) - S_n G(x_n)\|) \\
&\quad + \gamma_n \delta_n g_2 (\|y_n - S_n G(y_n)\|) \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\| \\
&\quad \le (\|x_n - p\| + \|x_{n+1} - p\|) \|x_n - x_{n+1}\| \\
&\quad + 2\beta_n \|f(x_n) - p\| \|x_{n+1} - p\|.
\end{aligned} \tag{127}$$

According to condition (vi), we get

$$\liminf_{n\to\infty} \delta_n = \liminf_{n\to\infty} (1 - \beta_n - \gamma_n) = 1 - \limsup_{n\to\infty} (\beta_n + \gamma_n) > 0. \tag{128}$$

Since $\beta_n \to 0$ and $\| x_{n+1} - x_n \| \to 0$, we conclude from conditions (i) and (vi) that

$$\begin{aligned}
&\lim_{n\to\infty} g_1 (G(x_n) - S_n G(x_n)) = 0, \\
&\lim_{n\to\infty} g_2 (\|y_n - S_n G(y_n)\|) = 0.
\end{aligned} \tag{129}$$

Utilizing the properties of g_1 and g_2, we have

$$\begin{aligned}
&\lim_{n\to\infty} \|G(x_n) - S_n G(x_n)\| = 0, \\
&\lim_{n\to\infty} \|y_n - S_n G(y_n)\| = 0.
\end{aligned} \tag{130}$$

Note that

$$\begin{aligned}
&\|y_n - x_n\| \\
&\quad = \|x_{n+1} - x_n - \beta_n (f(x_n) - y_n) \\
&\quad - \delta_n (S_n G(y_n) - y_n)\| \\
&\quad \le \|x_{n+1} - x_n\| + \beta_n \|f(x_n) - y_n\| \\
&\quad + \delta_n \|S_n G(y_n) - y_n\|.
\end{aligned} \tag{131}$$

Thus, from (123), (130), and $\beta_n \to 0$, it follows that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{132}$$

On the other hand, from (130), we get

$$\begin{aligned}
\lim_{n\to\infty} \|y_n - G(x_n)\| &= \lim_{n\to\infty} (1 - \alpha_n) \\
&\quad \times \|S_n G(x_n) - G(x_n)\| = 0.
\end{aligned} \tag{133}$$

This together with (132) implies that

$$\lim_{n\to\infty} \|x_n - G(x_n)\| = 0. \tag{134}$$

By (130) and Lemma 7, we have

$$\begin{aligned}
&\|SG(x_n) - G(x_n)\| \\
&\quad \le \|SG(x_n) - S_n G(x_n)\| \\
&\quad + \|S_n G(x_n) - G(x_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{135}$$

In terms of (134) and (135), we have

$$\begin{aligned}
&\|x_n - Sx_n\| \\
&\quad \le \|x_n - G(x_n)\| + \|G(x_n) - SG(x_n)\| \\
&\quad + \|SG(x_n) - Sx_n\| \\
&\quad \le 2\|x_n - G(x_n)\| \\
&\quad + \|G(x_n) - SG(x_n)\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{136}$$

Define a mapping $Wx = (1-\theta)Sx + \theta G(x)$, where $\theta \in (0, 1)$ is a constant. Then by Lemma 9, we have that $\mathrm{Fix}(W) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) = F$. We observe that

$$\begin{aligned}
&\|x_n - Wx_n\| \\
&\quad = \|(1 - \theta)(x_n - Sx_n) + \theta (x_n - G(x_n))\| \\
&\quad \le (1 - \theta) \|x_n - Sx_n\| + \theta \|x_n - G(x_n)\|.
\end{aligned} \tag{137}$$

From (134) and (136), we obtain

$$\lim_{n\to\infty} \| x_n - Wx_n \| = 0. \tag{138}$$

Now, we claim that

$$\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q) \rangle \le 0, \tag{139}$$

where $q = s - \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \longmapsto tf(x) + (1 - t) Wx. \tag{140}$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1 - t)Wx_t$. Thus we have

$$\|x_t - x_n\| = \|(1 - t)(Wx_t - x_n) + t(f(x_t) - x_n)\|. \tag{141}$$

By Lemma 3, we conclude that

$$\begin{aligned}
&\|x_t - x_n\|^2 \\
&\quad \le (1-t)^2 \|Wx_t - x_n\|^2 \\
&\qquad + 2t \langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\quad \le (1-t)^2 (\|Wx_t - Wx_n\| + \|Wx_n - x_n\|)^2 \\
&\qquad + 2t \langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\quad \le (1-t)^2 \left[\|x_t - x_n\|^2 + 2\|x_t - x_n\| \right. \\
&\qquad\qquad \left. \times \|Wx_n - x_n\| + \|Wx_n - x_n\|^2\right] \\
&\qquad + 2t \langle f(x_t) - x_t, J(x_t - x_n)\rangle \\
&\qquad + 2t \langle x_t - x_n, J(x_t - x_n)\rangle \\
&\quad = \left(1 - 2t + t^2\right) \|x_t - x_n\|^2 + f_n(t) \\
&\qquad + 2t \langle f(x_t) - x_t, J(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2,
\end{aligned} \tag{142}$$

where

$$\begin{aligned}
f_n(t) &= (1-t)^2 (2\|x_t - x_n\| + \|x_n - Wx_n\|) \\
&\quad \times \|x_n - Wx_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{143}$$

It follows from (142) that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t} f_n(t). \tag{144}$$

Letting $n \to \infty$ in (144) and noticing (143), we derive

$$\limsup_{n\to\infty} \langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2} M_2, \tag{145}$$

where $M_2 > 0$ is a constant such that $\| x_t - x_n\|^2 \le M_2$ for all $t \in (0,1)$ and $n \ge 0$. Taking $t \to 0$ in (145), we have

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0. \tag{146}$$

On the other hand, we have

$$\begin{aligned}
&\langle f(q) - q, J(x_n - q)\rangle \\
&\quad = \langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle \\
&\qquad + \langle x_t - q, J(x_n - x_t)\rangle \\
&\qquad + \langle f(q) - f(x_t), J(x_n - x_t)\rangle \\
&\qquad + \langle f(x_t) - x_t, J(x_n - x_t)\rangle.
\end{aligned} \tag{147}$$

Hence it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \\
&\quad \le \limsup_{n\to\infty} \langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle \\
&\qquad + \|x_t - q\| \limsup_{n\to\infty} \|x_n - x_t\| \\
&\qquad + \rho \|q - x_t\| \limsup_{n\to\infty} \|x_n - x_t\| \\
&\qquad + \limsup_{n\to\infty} \langle f(x_t) - x_t, J(x_n - x_t)\rangle.
\end{aligned} \tag{148}$$

Taking into account that $x_t \to q$ as $t \to 0$, we have from (146)

$$\begin{aligned}
&\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \\
&\quad = \limsup_{t\to 0} \limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \\
&\quad \le \limsup_{t\to 0} \limsup_{n\to\infty} \langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle.
\end{aligned} \tag{149}$$

Since X has a uniformly Gâteaux differentiable norm, the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable, and hence (139) holds. From (123), we get $(x_{n+1} - q) - (x_n - q) \to 0$. Noticing the norm-to-weak* uniform continuity of J on bounded subsets of X, we deduce from (139) that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad = \limsup_{n\to\infty} (\langle f(q) - q, J(x_{n+1} - q) - J(x_n - q)\rangle \\
&\qquad\qquad + \langle f(q) - q, J(x_{n+1} - q)\rangle) \\
&\quad = \limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \le 0.
\end{aligned} \tag{150}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. We observe that

$$\begin{aligned}
\|y_n - q\| &= \|\alpha_n (G(x_n) - q) \\
&\qquad + (1-\alpha_n)(S_n G(x_n) - q)\| \\
&\le \alpha_n \|x_n - q\| + (1-\alpha_n)\|x_n - q\| \\
&= \|x_n - q\|,
\end{aligned} \tag{151}$$

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \beta_n \langle f(x_n)-f(q)+f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + \langle \gamma_n (y_n-q) \\
&\qquad + \delta_n (S_n G(y_n)-q), J(x_{n+1}-q)\rangle \\
&\le \beta_n \|f(x_n)-f(q)\| \|x_{n+1}-q\| \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + \|\gamma_n (y_n-q)+\delta_n (S_n G(y_n)-q)\| \|x_{n+1}-q\| \\
&\le \beta_n \rho \|x_n-q\| \|x_{n+1}-q\| \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + (\gamma_n \|y_n-q\| + \delta_n \|y_n-q\|) \|x_{n+1}-q\| \\
&= \beta_n \rho \|x_n-q\| \|x_{n+1}-q\| \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + (\gamma_n+\delta_n) \|y_n-q\| \|x_{n+1}-q\| \\
&\le \beta_n \rho \|x_n-q\| \|x_{n+1}-q\| \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad + (1-\beta_n) \|x_n-q\| \|x_{n+1}-q\| \\
&= (1-\beta_n(1-\rho)) \|x_n-q\| \|x_{n+1}-q\| \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\le \frac{1-\beta_n(1-\rho)}{2} \left(\|x_n-q\|^2 + \|x_{n+1}-q\|^2\right) \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\le \frac{1-\beta_n(1-\rho)}{2} \|x_n-q\|^2 + \frac{1}{2}\|x_{n+1}-q\|^2 \\
&\quad + \beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle .
\end{aligned} \tag{152}$$

So, we have

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
&\quad \le (1-\beta_n(1-\rho)) \|x_n-q\|^2 \\
&\qquad + 2\beta_n \langle f(q)-q, J(x_{n+1}-q)\rangle \\
&\quad = (1-\beta_n(1-\rho)) \|x_n-q\|^2 \\
&\qquad + \beta_n (1-\rho) \frac{2\langle f(q)-q, J(x_{n+1}-q)\rangle}{1-\rho}.
\end{aligned} \tag{153}$$

Since $\sum_{n=0}^{\infty} \beta_n = \infty$ and $\limsup_{n\to\infty} \langle f(q)-q, J(x_{n+1}-q)\rangle \le 0$, by Lemma 2, we conclude from (153) that $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 23. *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let the mapping $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \ge 1$ for $i = 1, 2$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let S be a nonexpansive mapping of C into itself such that $F = \mathrm{Fix}(S) \cap \Omega \ne \emptyset$, where Ω is the fixed point set of the mapping $G = \Pi_C(I-\mu_1 B_1)\Pi_C(I-\mu_2 B_2)$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
y_n &= \alpha_n G(x_n) + (1-\alpha_n) SG(x_n), \\
x_{n+1} &= \beta_n f(x_n) + \gamma_n y_n + \delta_n SG(y_n), \quad \forall n \ge 0,
\end{aligned} \tag{154}$$

where $1-(\lambda_i/(1+\lambda_i))(1-\sqrt{(1-\alpha_i)/\lambda_i}) \le \mu_i \le 1$ for $i = 1, 2$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ satisfying the following conditions:

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\beta_n = 0$,

(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \beta_{n-1}/\beta_n = 1$,

(v) $\sum_{n=1}^{\infty} |(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| < \infty$ *or* $\lim_{n\to\infty} (1/\beta_n)|(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| = 0$,

(vi) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$.

Then $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q-p)\rangle \le 0, \quad \forall p \in F. \tag{155}$$

Further, we illustrate Theorem 22 by virtue of an example, that is, the following corollary.

Corollary 24. *Let C be a nonempty, closed, and convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0, 1)$. Let $T : C \to C$ be a self-mapping on C such that $I - T$ is ζ-strictly pseudocontractive and θ-strongly accretive with $\zeta + \theta \ge 1$, and let S be a nonexpansive mapping of C into itself such that $F = \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \ne \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{aligned}
y_n &= \alpha_n G(x_n) + (1-\alpha_n) S(I-\lambda(I-T)) x_n, \\
x_{n+1} &= \beta_n f(x_n) + \gamma_n y_n + \delta_n S(I-\lambda(I-T)) y_n, \quad \forall n \ge 0,
\end{aligned} \tag{156}$$

where $1-(\zeta/(1+\zeta))(1-\sqrt{(1-\theta)/\zeta}) \le \lambda \le 1$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$ are the sequences in $(0, 1)$ satisfying the following conditions:

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

(ii) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} |\alpha_n - \alpha_{n-1}|/\beta_n = 0$,

(iv) $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \beta_{n-1}/\beta_n = 1$,

(v) $\sum_{n=1}^{\infty} |(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| < \infty$ *or* $\lim_{n\to\infty}(1/\beta_n)|(\gamma_n/(1-\beta_n)) - (\gamma_{n-1}/(1-\beta_{n-1}))| = 0$,

(vi) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$.

Then $\{x_n\}$ *converges strongly to* $q \in F$, *which solves the following VIP:*

$$\langle q - f(q), J(q-p)\rangle \le 0, \quad \forall p \in F. \tag{157}$$

Proof. Utilizing the arguments similar to those in the proof of Corollary 16, we can obtain the desired result. □

Remark 25. As previous, we emphasize that our composite iterative algorithms (i.e., the iterative schemes (27) and (111)) are based on Korpelevich's extragradient method and viscosity approximation method. It is well known that the so-called viscosity approximation method must contain a contraction f on C. In the meantime, it is worth pointing out that our proof of Theorems 14 and 22 must make use of Lemma 8 for implicit viscosity approximation method; that is, Lemma 8 plays a key role in our proof of Theorems 14 and 22. Therefore, there is no doubt that the contraction f in Theorems 14 and 22 cannot be replaced by a general k-Lipschitzian mapping with constant $k \ge 0$.

Remark 26. Theorem 22 improves, extends, supplements, and develops [2, Theorem 3.1 and Corollary 3.2] and [5, Theorems 3.1] in the following aspects.

(i) The problem of finding a point $q \in \bigcap_n \mathrm{Fix}(S_n) \cap \Omega$ in Theorem 22 is more general and more subtle than the problem of finding a point $q \in \mathrm{Fix}(S) \cap \mathrm{VI}(C, A)$ in Jung [5, Theorem 3.1].

(ii) The iterative scheme in [2, Theorem 3.1] is extended to develop the iterative scheme (111) of Theorem 22 by virtue of the iterative scheme of [5, Theorem 3.1]. The iterative scheme (111) in Theorem 22 is more advantageous and more flexible than the iterative scheme in [2, Theorem 3.1] because it involves several parameter sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\}$.

(iii) The iterative scheme (111) in Theorem 22 is very different from everyone in both [2, Theorem 3.1] and [5, Theorem 3.1] because the mappings S_n and G in the iterative scheme of [2, Theorem 3.1] and the mapping $SP_C(I - \lambda_n A)$ in the iterative scheme of [5, Theorem 3.1] are replaced by the same composite mapping S_nG in the iterative scheme (111) of Theorem 22.

(iv) The proof in [2, Theorem 3.1] depends on the argument techniques in [3], the inequality in 2-uniformly smooth Banach spaces, and the inequality in smooth and uniform convex Banach spaces. However, the proof of Theorem 22 does not depend on the argument techniques in [3], the inequality in 2-uniformly smooth Banach spaces, and the inequality in smooth and uniform convex Banach spaces. It depends on only the inequality in uniform convex Banach spaces.

(v) The assumption of the uniformly convex and 2-uniformly smooth Banach space X in [2, Theorem 3.1] is weakened to the one of the uniformly convex Banach space X having a uniformly Gateaux differentiable norm in Theorem 22.

(vi) The iterative scheme in [2, Corollary 3.2] is extended to develop the new iterative scheme in Corollary 15 because the mappings S and G are replaced by the same composite mapping SG in Corollary 23.

Finally, we observe that related results can be found in recent papers, for example, [15–24] and the references therein.

Acknowledgments

This research was partially supported by the National Science Foundation of China (11071169), Innovation Program of Shanghai Municipal Education Commission (09ZZ133), and Ph.D. Program Foundation of Ministry of Education of China (20123127110002). This research was partially supported by a Grant from NSC 101-2115-M-037-001.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 721909, 6 pages
http://dx.doi.org/10.1155/2013/721909

Research Article

Positive Solutions for Third-Order Boundary-Value Problems with the Integral Boundary Conditions and Dependence on the First-Order Derivatives

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Received 8 April 2013; Revised 2 June 2013; Accepted 9 June 2013

Academic Editor: Wei-Shih Du

By using a fixed point theorem in a cone and the nonlocal third-order BVP's Green function, the existence of at least one positive solution for the third-order boundary-value problem with the integral boundary conditions $x'''(t) + f(t, x(t), x'(t)) = 0$, $t \in J$, $x(0) = 0$, $x''(0) = 0$, and $x(1) = \int_0^1 g(t)x(t)dt$ is considered, where f is a nonnegative continuous function, $J = [0, 1]$, and $g \in L[0, 1]$. The emphasis here is that f depends on the first-order derivatives.

1. Introduction

Third-order boundary-value problems for differential equation play a very important role in a variety of different areas of applied mathematics and physics. Recently, third-order boundary-value problems have been many scholars' research object. For example, heat conduction, chemical engineering, underground water flow, thermoelasticity, and plasma physics can produce boundary-value problems with integral boundary conditions [1–3]. For more information about the general theory of integral equations and their relation with boundary-value problems, we refer readers to the books of Corduneanu [4] and Agarwal and O'Regan [5].

Moreover, boundary-value problems with integral boundary conditions constitute a very interesting and important class of problems. They include two, three, multipoint, and nonlocal boundary-value problems as special cases. Such kind of BVPs in Banach space has been studied by some researchers [6–8].

By the fixed point index theory in cones [9], Zhang et al. [10] investigated the multiplicity of positive solutions for a class of nonlinear boundary-value problems of fourth-order differential equations with integral boundary conditions in ordered Banach spaces. Feng et al. [11] investigated the existence and multiplicity of positive solutions for a class of nonlinear boundary-value problems of second-order differential equations with integral boundary conditions in ordered Banach spaces. Guo et al. [12] investigated the existence of positive solutions for the third-order boundary-value problems with integral boundary conditions and dependence on the second derivatives. In [13], by using the fixed point theorem of cone expansion and compression of norm type, Zhang and Ge proved the existence and multiplicity of symmetric positive solutions for the fourth-order boundary-value problems with integral boundary conditions. By using Krasnoselskii's fixed point theorem, Wang et al. [14] investigated the existence and nonexistence of positive solutions for a class of fourth-order nonlinear differential equation with integral boundary conditions

$$\begin{gathered} x^{(4)}(t) = \omega(t) f\left(t, x(t), x''(t)\right), \quad 0 < t < 1, \\ x(0) = \int_0^1 h_1(s) x(s) ds, \\ x(1) = \int_0^1 k_1(s) x(s) ds, \\ x''(0) = \int_0^1 h_2(s) x''(s) ds, \\ x''(1) = \int_0^1 k_2(s) x''(s) ds, \end{gathered} \tag{1}$$

where the arguments are based on Krasnoselskii's fixed point theorem for operators on a cone.

However, Zhao et al. [15] investigated the following third-order boundary-value problem with integral boundary conditions:

$$\begin{gathered} x'''(t) + f(t, x(t)) = \theta, \quad t \in J, \\ x(0) = \theta, \qquad x''(0) = \theta, \\ x(1) = \int_0^1 g(t)\, x(t)\, dt, \end{gathered} \tag{2}$$

under the assumptions

(1) $J = [0,1]$, and θ is the zero element of E,

(2) $f : C([0,1] \times P, P)$, and $g \in L[0,1]$ is nonnegative,

where P is a cone in the real Banach E.

All the above works were done under the assumption that the first-order derivative x' is not involved explicitly in the nonlinear term f. In this paper, we are concerned with the existence of positive solutions for the third-order boundary-value problem with the integral boundary conditions

$$\begin{gathered} x'''(t) + f\left(t, x(t), x'(t)\right) = 0, \quad t \in J, \\ x(0) = 0, \qquad x''(0) = 0, \\ x(1) = \int_0^1 g(t)\, x(t)\, dt. \end{gathered} \tag{3}$$

Throughout, we assume

(H_1) $J = [0,1]$, $f : [0,1] \times R^2 \to R^+$ is continuous, $g \in L[0,1]$, $g(t) \geq 0$, and $\sigma \in [0,1)$, where $\sigma = \int_0^1 sg(s)ds$.

To show the existence of positive solutions for (3), we define two positive continuous convex functionals. Then, by using the fixed point theorem [16] in a cone and the nonlocal third-order BVP's Green function, we give some new criteria for the existence of positive solutions for (3).

2. Preliminaries

Let $Y = C[0,1]$ be the Banach space equipped with the norm $\|x\|_0 = \max_{t\in[0,1]}|x(t)|$.

Lemma 1 (see [15]). *Suppose (H_1) holds. Then for any $y(t) \in C[0,1]$, the problem*

$$\begin{gathered} x'''(t) + y(t) = 0, \quad t \in J, \\ x(0) = 0, \qquad x''(0) = 0, \\ x(1) = \int_0^1 g(t)\, x(t)\, dt \end{gathered} \tag{4}$$

has a unique solution

$$x(t) = \int_0^1 H(t,s)\, y(s)\, ds, \tag{5}$$

where

$$\begin{gathered} H(t,s) = G(t,s) + \frac{t}{1-\sigma}\int_0^1 G(\tau,s)\, g(\tau)\, d\tau, \\ G(t,s) = \begin{cases} \frac{1}{2}t(1-s)^2 - \frac{1}{2}(t-s)^2, & 0 \leq s \leq t \leq 1, \\ \frac{1}{2}t(1-s)^2, & 0 \leq t \leq s \leq 1. \end{cases} \end{gathered} \tag{6}$$

Lemma 2 (see [15]). *For $t, s \in [0,1]$, one has $0 \leq G(t,s) \leq \max_{0\leq t,s\leq 1} G(t,s) \leq 1/8$.*

Remark 3. When $t, s \in (0,1)$, it is easy to check that $G(t,s) > 0$.

In addition, for $0 \leq s \leq t \leq 1$, the maximum of $G(t,s)$ occurs at $t = (1+s^2)/2$.

Lemma 4 (see [15]). *Choose $\delta \in (0, 1/2)$ and $J_\delta = [\delta, 1-\delta]$; then for all $t \in J_\delta$, $v, s \in [0,1]$, one has*

$$G(t,s) \geq \rho G(v,s), \tag{7}$$

where $\rho = 4\delta^2(1-\delta)$.

Remark 5. For $0 \leq s \leq t \leq 1$, denote $G(t,s) = G_1(t,s)$. Notice that $G_1(t,s)$ is concave with respect to t; we have

$$\begin{aligned} \min_{t\in J_\delta, 0\leq s\leq t} G_1(t,s) &= \min\{G_1(\delta,s), G_1(1-\delta,s)\} \\ &= \frac{1}{2}\delta^2(1-\delta). \end{aligned} \tag{8}$$

Lemma 6 (see [15]). *Assume that (H_1) holds; then*

(i) $H(t,s) \leq (1/2)\gamma$, $t \in [0,1]$,

(ii) $H(t,s) \geq \rho H(v,s)$, $t \in J_\delta$, $v, s \in [0,1]$,

where $\gamma = (1 + \int_0^1 (1-s)g(s)ds)/(1-\sigma)$.

Lemma 7. *If $y \in C[0,1]$, $y(t) \geq 0$, then the unique solution $x(t)$ of problem (4) satisfies*

$$\min_{t\in J_\delta} x(t) \geq \rho\|x\|_0. \tag{9}$$

Proof. By Lemmas 4 and 6 and (5), we get

$$\begin{aligned} \min_{t\in J_\delta} x(t) &= \min_{t\in J_\delta}\int_0^1 H(t,s)\, y(s)\, ds \\ &\geq \rho\int_0^1 H(v,s)\, y(s)\, ds \\ &\geq \rho x(v). \end{aligned} \tag{10}$$

For $v \in [0,1]$, we have

$$\min_{t\in J_\delta} x(t) \geq \rho x(v). \tag{11}$$

So,

$$\min_{t\in J_\delta} x(t) \geq \rho \max_{v\in[0,1]} x(v) = \rho \max_{v\in[0,1]} |x(v)| = \rho\|x\|_0. \tag{12}$$

The proof is completed. □

Let X be a Banach space and $K \subset X$ a cone. Suppose $\alpha, \beta : X \to R^+$ are two continuous convex functionals satisfying $\alpha(\lambda x) = |\lambda|\alpha(x)$, $\beta(\lambda x) = |\lambda|\beta(x)$, for $x \in X$, $\lambda \in R$, $\|x\| \le M \max\{\alpha(x), \beta(x)\}$, for $x \in X$, and $\alpha(x) \le \alpha(y)$ for $x, y \in K$, $x \le y$, where $M > 0$ is a constant.

Theorem 8 (see [16]). *Let $r_2 > r_1 > 0$, $L > 0$ be constants and*

$$\Omega_i = \{x \in X : \alpha(x) < r_i, \beta(x) < L\}, \quad i = 1, 2, \tag{13}$$

two bounded open sets in X. Set

$$D_i = \{x \in X : \alpha(x) = r_i\}, \quad i = 1, 2. \tag{14}$$

Assume $T : K \to K$ is a completely continuous operator satisfying

(A_1) $\alpha(Tx) < r_1$, $x \in D_1 \cap K$; $\alpha(Tx) > r_2$, $x \in D_2 \cap K$;

(A_2) $\beta(Tx) < L$, $x \in K$;

(A_3) *there is a $p \in (\Omega_2 \cap K) \setminus \{0\}$ such that $\alpha(p) \ne 0$ and $\alpha(x + \lambda p) \ge \alpha(x)$, for all $x \in K$ and $\lambda \ge 0$.*

Then T has at least one fixed point in $(\Omega_2 \setminus \overline{\Omega}_1) \cap K$.

3. Main Results

Let $X = C^1[0, 1]$ be the Banach space equipped with the norm $\|x\| = \max_{t\in[0,1]}|x(t)| + \max_{t\in[0,1]}|x'(t)|$, and $K = \{x \in X : x(t) \ge 0, \min_{t\in J_\delta} x(t) \ge \rho\|x\|_0\}$ is a cone in X.

Define two continuous convex functionals $\alpha(x) = \max_{t\in[0,1]}|x(t)|$ and $\beta(x) = \max_{t\in[0,1]}|x'(t)|$, for each $x \in X$; then $\|x\| \le 2 \max\{\alpha(x), \beta(x)\}$ and $\alpha(\lambda x) = |\lambda|\alpha(x)$, $\beta(\lambda x) = |\lambda|\beta(x)$, for $x \in X$, $\lambda \in R$; $\alpha(x) \le \alpha(y)$ for $x, y \in K$, $x \le y$.

In the following, we denote

$$\begin{aligned}
\eta_0 &= \frac{1}{8} + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau, s)\, g(\tau)\, d\tau\right] ds,\\
\eta_1 &= \max_{v\in[0,1]} \int_\delta^{1-\delta} H(v, s)\, ds,\\
\eta_2 &= \frac{2}{3} + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau, s)\, g(\tau)\, d\tau\right] ds.
\end{aligned} \tag{15}$$

We will suppose that there are $L > b > \rho b > c > 0$ such that $f(t, x, y)$ satisfies the following growth conditions:

(H_2) $f(t, x, y) < c/\eta_0$, for $(t, x, y) \in [0, 1] \times [0, c] \times [-L, L]$,

(H_3) $f(t, x, y) \ge b/\rho\eta_1$, for $(t, x, y) \in [\delta, 1 - \delta] \times [\rho b, b] \times [-L, L]$,

(H_4) $f(t, x, y) < L/\eta_2$, for $(t, x, y) \in [0, 1] \times [0, b] \times [-L, L]$.

Let

$$\begin{aligned}
&f^*(t, x, y)\\
&\quad = \begin{cases} f(t, x, y), (t, x, y) \in [0, 1] \times [0, b] \times (-\infty, \infty),\\ f(t, b, y), (t, x, y) \in [0, 1] \times (b, \infty) \times (-\infty, \infty), \end{cases}\\
&f_1(t, x, y)\\
&\quad = \begin{cases} f^*(t, x, y), (t, x, y) \in [0, 1] \times [0, \infty) \times [-L, L],\\ f^*(t, x, -L), (t, x, y) \in [0, 1] \times [0, \infty) \times (-\infty, -L],\\ f^*(t, x, L), (t, x, y) \in [0, 1] \times [0, \infty) \times [L, \infty). \end{cases}
\end{aligned} \tag{16}$$

We denote

$$\begin{aligned}
(Tx)(t) &= \int_0^1 H(t, s)\, f_1(s, x, x')\, ds,\\
(Tx)'(t) &= \int_0^1 \frac{\partial H(t, s)}{\partial t} f_1(s, x, x')\, ds.
\end{aligned} \tag{17}$$

Lemma 9. *Suppose (H_1) holds. Then $T : K \to K$ is completely continuous.*

Proof. For $x \in K$, by Lemmas 2 and 4, it is obviously that $Tx \ge 0$.

By Lemma 7, we have

$$\min_{t\in J_\delta} Tx(t) \ge \rho\|Tx\|_0. \tag{18}$$

So, we can get $T(K) \subset K$.

In the following, we will show that $T : K \to K$ is completely continuous.

At first we show that $T : K \to K$ is continuous.

Let $x_n, x^* \in K$, it satisfies $\|x_n - x^*\| \to 0$, $(n \to \infty)$, and then there is a constant $M_0 > 0$, such that $\max_{t\in[0,1]} \{|x_n(t)|, |x^*(t)|, |x_n'(t)|, |x^{*'}(t)|\} \le M_0$; then

$$\begin{aligned}
&|(Tx_n)(t) - (Tx^*)(t)|\\
&\quad = \left|\int_0^1 H(t, s)\, f_1(s, x_n, x_n')\, ds\right.\\
&\qquad \left. - \int_0^1 H(t, s)\, f_1(s, x^*, x^{*'})\, ds\right|\\
&\quad \le \int_0^1 H(t, s) \left|f_1(s, x_n, x_n') - f_1(s, x^*, x^{*'})\right| ds,\\
&|(Tx_n)'(t) - (Tx^*)'(t)|\\
&\quad = \left|\int_0^1 \frac{\partial H(t, s)}{\partial t} f_1(s, x, x_n')\, ds\right.\\
&\qquad \left. - \int_0^1 \frac{\partial H(t, s)}{\partial t} f_1(s, x^*, x^{*'})\, ds\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^1 \left|\frac{\partial H(t,s)}{\partial t}\right| \left|f_1\left(s,x,x_n'\right) - f_1\left(s,x^*,x^{*\prime}\right)\right| ds \\
&< \int_0^1 \left[\frac{1}{2}(1-s)^2 + (1-s)\right] \\
&\quad \times \left|f_1\left(s,x,x_n'\right) - f_1\left(s,x^*,x^{*\prime}\right)\right| ds \\
&\quad + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau,s)\, g(\tau)\, d\tau\right] \\
&\qquad \times \left|f_1\left(s,x,x_n'\right) - f_1\left(s,x^*,x^{*\prime}\right)\right| ds.
\end{aligned} \tag{19}$$

By f which is uniformly continuous on $[0,1]\times[-M_0, M_0]\times[-M_0,M_0]$, we get

$$\|Tx_n - Tx^*\| \longrightarrow 0, \quad (n \longrightarrow \infty). \tag{20}$$

Next we show that $T: K \to K$ is compact.

Let $B \subset K$ be bounded; then there is $M > 0$, such that $\|x\| \leq M$. For $x \in B$, we have

$$\begin{aligned}
|(Tx)(t)| &= \left|\int_0^1 H(t,s)\, f_1\left(s,x,x'\right) ds\right| \\
&\leq \int_0^1 \frac{1}{2}\gamma f_1\left(s,x,x'\right) ds \\
&= \frac{1}{2}\int_0^1 \frac{1+\int_0^1 (1-s)\, g(s)\, ds}{1-\sigma}\, ds \times C^*,
\end{aligned} \tag{21}$$

where $C^* = \max\{|f_1(t,x,x')|; t\in[0,1], x\in B\}$.

Consider

$$\begin{aligned}
&\left|(Tx)'(t)\right| \\
&\quad = \left|\int_0^1 \frac{\partial H(t,s)}{\partial t} f_1\left(s,x,x'\right) ds\right| \\
&\quad = \left|\int_0^1 \left[\frac{\partial G(t,s)}{\partial t} + \frac{1}{1-\sigma}\int_0^1 G(\tau,s)\, g(\tau)\, d\tau\right]\right. \\
&\qquad \left. \times f_1\left(s,x,x'\right) ds\right| \\
&\quad < \left|\int_0^1 \left[\frac{1}{2}(1-s)^2 + (1-s)\right] ds\right. \\
&\qquad \left. + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau,s)\, g(\tau)\, d\tau\right] ds\right| \times C^* \\
&\quad = \left|\frac{2}{3} + \int_0^1 \left[\frac{1}{1-\sigma}\int_0^1 G(\tau,s)\, g(\tau)\, d\tau\right] ds\right| \times C^*.
\end{aligned} \tag{22}$$

It is clear that $T(B)$ is a bounded set in K, because $H(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$, for $\varepsilon > 0$, there exists $\delta \in (0,\varepsilon)$, such that $|H(t_1,s) - H(t_2,s)| < \varepsilon$, and for t_1, $t_2 \in [0,1]$, $|t_1 - t_2| < \delta$.

For $x \in B$, we have

$$\begin{aligned}
&\left|(Tx)(t_1) - (Tx)(t_2)\right| \\
&\quad = \left|\int_0^1 H(t_1,s)\, f_1\left(s,x,x'\right) ds\right. \\
&\qquad \left. - \int_0^1 H(t_2,s)\, f_1\left(s,x,x'\right) ds\right| \\
&\quad \leq \int_0^1 \left|H(t_1,s) - H(t_2,s)\right| ds \times C^* \leq \varepsilon C^*, \\
&\left|(Tx)'(t_1) - (Tx)'(t_2)\right| \\
&\quad = \left|\int_0^1 \left.\frac{\partial H(t,s)}{\partial t}\right|_{t=t_1} f_1\left(s,x,x'\right) ds\right. \\
&\qquad \left. - \int_0^1 \left.\frac{\partial H(t,s)}{\partial t}\right|_{t=t_2} f_1\left(s,x,x'\right) ds\right| \\
&\quad = \left|\int_0^1 \left.\frac{\partial G(t,s)}{\partial t}\right|_{t=t_1} f_1\left(s,x,x'\right) ds\right. \\
&\qquad \left. - \int_0^1 \left.\frac{\partial G(t,s)}{\partial t}\right|_{t=t_2} f_1\left(s,x,x'\right) ds\right| \\
&\quad = \left|\int_0^{t_1} (s-t_1)\, f_1\left(s,x,x'\right) ds\right. \\
&\qquad \left. + \int_0^{t_2} (t_2-s)\, f_1\left(s,x,x'\right) ds\right| \\
&\quad \leq \frac{1}{2}\left|(t_1-t_2)(t_1+t_2)\right| \times C^* \leq \varepsilon C^*.
\end{aligned} \tag{23}$$

Therefore $T(B)$ is equicontinuous. Using the Arzela-Ascoli theorem, a standard proof yields $T: K \to K$ which is completely continuous. □

Theorem 10. *Suppose (H_1)–(H_4) hold. Then BVP (3) has at least one positive solution $x(t)$ satisfying*

$$c < \alpha(x) < b, \qquad \beta(x) < L. \tag{24}$$

Proof. Take

$$\begin{aligned}
\Omega_1 &= \left\{x \in X : |x(t)| < c, \left|x(t)'\right| < L\right\}, \\
\Omega_2 &= \left\{x \in X : |x(t)| < b, \left|x(t)'\right| < L\right\},
\end{aligned} \tag{25}$$

two bounded open sets in X, and

$$D_1 = \{x \in X : \alpha(x) = c\}, \qquad D_2 = \{x \in X : \alpha(x) = b\}. \tag{26}$$

By Lemma 9, $T: K \to K$ is completely continuous, and there is a $p \in (\Omega_2 \cap K)\setminus\{0\}$ such that $\alpha(p) \neq 0$ and $\alpha(x+\lambda p) \geq \alpha(x)$ for all $u \in K$ and $\lambda \geq 0$.

By (H_2), for $x \in D_1 \cap K$ and $\alpha(x) = c$, we get

$$\begin{aligned}
\alpha(Tx) &= \max_{t\in[0,1]} \left| \int_0^1 H(t,s) f_1(s,x,x') ds \right| \\
&= \max_{t\in[0,1]} \left| \int_0^1 \left[G(t,s) + \frac{t}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right] \right. \\
&\qquad \left. \times f_1(s,x,x') ds \right| \\
&< \int_0^1 \left[\max_{t\in[0,1]} G(t,s) + \frac{t}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right] \\
&\qquad \times f_1(s,x,x') ds \\
&< \left[\int_0^1 \frac{1}{8} ds + \int_0^1 \left(\frac{t}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right) ds \right] \\
&\qquad \times \frac{c}{\eta_0} \\
&= \left[\frac{1}{8} + \int_0^1 \left(\frac{t}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right) ds \right] \\
&\qquad \times \frac{c}{\eta_0} = c.
\end{aligned} \tag{27}$$

By Lemma 7, for $x \in D_2 \cap K$ and $\alpha(x) = b$, there is $x(t) \geq \rho\alpha(x) = \rho b$, $t \in J_\delta$.

So, by (H_3), we get

$$\begin{aligned}
\alpha(Tx) &= \max_{t\in[0,1]} \left| \int_0^1 H(t,s) f_1(s,x,x') ds \right| \\
&> \int_\delta^{1-\delta} H(t,s) f_1(s,x,x') ds \\
&> \rho \int_\delta^{1-\delta} H(v,s) ds \times \frac{b}{\rho\eta_1}.
\end{aligned} \tag{28}$$

For $v \in [0,1]$, we have

$$\alpha(Tx) > \rho \int_\delta^{1-\delta} H(v,s) ds \times \frac{b}{\rho\eta_1}. \tag{29}$$

So,

$$\alpha(Tx) > \rho \max_{v\in[0,1]} \int_\delta^{1-\delta} H(v,s) ds \times \frac{b}{\rho\eta_1} = b. \tag{30}$$

By (H_4), for $x \in K$, we have

$$\begin{aligned}
\beta(Tx) &= \max_{t\in[0,1]} \left| \int_0^1 \frac{\partial H(t,s)}{\partial t} f_1(s,x,x') ds \right| \\
&< \left| \int_0^1 \left[\frac{\partial G(t,s)}{\partial t} + \frac{1}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right] \right. \\
&\qquad \times f_1(s,x,x') ds \\
&= \left| \int_0^t \left(\frac{1}{2}(1-s)^2 - (t-s) \right) f_1(s,x,x') ds \right. \\
&\qquad + \int_t^1 \frac{1}{2}(1-s)^2 f_1(s,x,x') ds \\
&\qquad + \int_0^1 \left[\frac{1}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right] \\
&\qquad \left. \times f_1(s,x,x') ds \right| \\
&< \left[\int_0^1 \left(\frac{1}{2}(1-s)^2 + (1-s) \right) ds \right. \\
&\qquad \left. + \int_0^1 \left(\frac{1}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right) ds \right] \times \frac{L}{\eta_2} \\
&= \left[\frac{2}{3} + \int_0^1 \left(\frac{1}{1-\sigma} \int_0^1 G(\tau,s) g(\tau) d\tau \right) ds \right] \\
&\qquad \times \frac{L}{\eta_2} = L.
\end{aligned} \tag{31}$$

Theorem 8 implies there is $x \in (\Omega_2 \setminus \overline{\Omega}_1) \cap K$ such that $x = Tx$. So, $x(t)$ is a positive solution for BVP (3) satisfying

$$c < \alpha(x) < b, \qquad \beta(x) < L. \tag{32}$$

Thus, Theorem 10 is completed. □

4. Example

Example 1. Consider the following boundary-value problem

$$\begin{gathered}
x'''(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1, \\
x(0) = 0, \qquad x''(0) = 0, \\
x(1) = \int_0^1 x(t) dt,
\end{gathered} \tag{33}$$

where

$$\begin{aligned}
&f(t,x,y) \\
&= \begin{cases}
\frac{t}{3}x + 2x + |\cos y|, \\
(t,x,y) \in [0,1] \times [0,0.5] \times [-3667, 3667], \\
\frac{109t}{3}(x-0.5) + 25742(x-0.5) + \frac{t}{6} + 1 + |\cos y|, \\
(t,x,y) \in [0,1] \times [0.5,0.6] \times [-3667, 3667], \\
\frac{t}{3}(11-x) + 222(x+11) + |\cos y|, \\
(t,x,y) \in [0,1] \times [0.6,11] \times [-3667, 3667].
\end{cases}
\end{aligned} \tag{34}$$

In this problem, we know that $g(t) = 1$; then we can get $\sigma = \int_0^1 sg(s)ds = 1/2$. Choose $\delta = 1/8 \in (0, 1/2)$; then $\rho = 4\delta^2(1-\delta) = 7/128$.

Furthermore, we obtain

$$\eta_0 = \frac{5}{24}, \qquad \rho\eta_1 = \frac{35}{8192}, \qquad \eta_2 = \frac{3}{4}. \tag{35}$$

If we take $c = 0.5$, $b = 11$, and $L = 3667$, then we get $\rho b \approx 0.601 > 0.6$:

$$f(t,x,y) = \frac{t}{3}x + 2x + |\cos y| \le 2.17 < \frac{c}{\eta_0} \approx 2.4, \tag{36}$$

for $(t,x,y) \in [0,1] \times [0,0.5] \times [-3667,3667]$,

$$\begin{aligned} f(t,x,y) &= \frac{t}{3}(11-x) + 222(x+11) \\ &\quad + |\cos y| \ge 2575.2 > \frac{b}{\rho\eta_1} \approx 2574.1, \end{aligned} \tag{37}$$

for $(t,x,y) \in [\delta, 1-\delta] \times [\delta b, 11] \times [-3667,3667]$,

$$f(t,x,y) \le 4888.8 < \frac{L}{\eta_2} \approx 4889.3, \tag{38}$$

for $(t,x,y) \in [0,1] \times [0,11] \times [-3667,3667]$.

Then all the conditions of Theorem 10 are satisfied. Therefore, by Theorem 10 we know that boundary-value problem (33) has at least one positive solution $x(t)$ satisfying

$$0.5 < \alpha(x) < 11, \qquad \beta(x) < 3667. \tag{39}$$

Acknowledgments

The project is supported by the Natural Science Foundation of China (10971045) and the Natural Science Foundation of Hebei Province (A2013208147).

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 962691, 7 pages
http://dx.doi.org/10.1155/2013/962691

Research Article

Strong Convergence Theorem for Bregman Strongly Nonexpansive Mappings and Equilibrium Problems in Reflexive Banach Spaces

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Received 30 April 2013; Accepted 10 June 2013

Academic Editor: Wei-Shih Du

By using a new hybrid method, a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of Bregman strongly nonexpansive mappings in a reflexive Banach space is proved.

1. Introduction

Throughout this paper, we denote by $\mathbb{R}$ and $\mathbb{R}^+$ the set of all real numbers and all nonnegative real numbers, respectively. We also assume that E is a real reflexive Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, and $\langle\cdot,\cdot\rangle$ is the pairing between E and E^*. Let Θ be a bifunction from $C \times C \to \mathbb{R}$. The equilibrium problem is to find

$$x^* \in C \text{ such that } \Theta(x^*, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The set of such solutions x^* is denoted by $\mathrm{EP}(\Theta)$.

Recall that a mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{2}$$

We denote by $F(T)$ the set of fixed points of T.

Numerous problems in physics, optimization, and economics reduce to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem in a Hilbert spaces; see, for instance, Blum and Oettli [1], Combettes and Hirstoaga [2], and Moudafi [3]. Recently, Tada and Takahashi [4, 5] and S. Takahashi and W. Takahashi [6] obtained weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In particular, Tada and Takahashi [4] established a strong convergence theorem for finding a common element of two sets by using the hybrid method introduced by Nakajo and Takahashi [7]. The authors also proved such a strong convergence theorem in a uniformly convex and uniformly smooth Banach space.

In this paper, motivated by Takahashi et al. [8], we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a Bregman strongly nonexpansive mapping in a real reflexive Banach space by using the shrinking projection method. Using this theorem, we obtain two new strong convergence results for finding a solution of an equilibrium problem and a fixed point of Bregman strongly nonexpansive mappings in a real reflexive Banach space.

2. Preliminaries and Lemmas

In the sequel, we begin by recalling some preliminaries and lemmas which will be used in the proof.

Let E be a real reflexive Banach space with the norm $\|\cdot\|$ and E^* the dual space of E. Throughout this paper, $f : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous, and convex function. We denote by dom f the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$.

Let $x \in \operatorname{int}\operatorname{dom} f$. The subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \le f(y), \forall y \in E\}, \tag{3}$$

where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}. \tag{4}$$

We know that the Young-Fenchel inequality holds:

$$\langle x^*, x\rangle \le f(x) + f^*(x^*), \quad \forall x \in E,\ x^* \in E^*. \tag{5}$$

A function f on E is coercive [9] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\| \to +\infty} f(x) = +\infty. \tag{6}$$

A function f on E is said to be strongly coercive [10] if

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty. \tag{7}$$

For any $x \in \operatorname{int}\operatorname{dom} f$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^\circ(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{8}$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t \to 0^+}((f(x + ty) - f(x))/t)$ exists for any y. In this case, $f^\circ(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \operatorname{int}\operatorname{dom} f$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. It is known that if f is Gâteaux differentiable (resp., Fréchet differentiable) on $\operatorname{int}\operatorname{dom} f$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp., continuous) on $\operatorname{int}\operatorname{dom} f$ (see also [11, 12]). We will need the following result.

Lemma 1 (see [13]). *If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Definition 2 (see [14]). The function f is said to be

(i) essentially smooth, if ∂f is both locally bounded and single valued on its domain,

(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom}\partial f$,

(iii) Legendre if it is both essentially smooth and essentially strictly convex.

Remark 3. Let E be a reflexive Banach space. Then we have the following.

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see [14, Theorem 5.4]).

(ii) $(\partial f)^{-1} = \partial f^*$ (see [12]).

(iii) f is Legendre if and only if f^* is Legendre (see [14, Corollary 5.5]).

(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\operatorname{ran}\nabla f = \operatorname{dom}\nabla f^* = \operatorname{int}\operatorname{dom} f^*$, and $\operatorname{ran}\nabla f^* = \operatorname{dom}\nabla f = \operatorname{int}\operatorname{dom} f$ (see [14, Theorem 5.10]).

Examples of Legendre functions were given in [14, 15]. One important and interesting Legendre function is $(1/p)\|\cdot\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case, the gradient ∇f of f is coincident with the generalized duality mapping of E; that is, $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$ the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that $f : E \to (-\infty, +\infty]$ is Legendre.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$ defined as

$$D_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle \tag{9}$$

is called the Bregman distance with respect to f [16].

Recall that the Bregman projection [17] of $x \in \operatorname{int}\operatorname{dom} f$ onto the nonempty closed and convex set $C \subset \operatorname{dom} f$ is the necessarily unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}. \tag{10}$$

Concerning the Bregman projection, the following are well known.

Lemma 4 (see [18]). *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then*

(a) *$z = P_C^f(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z\rangle \le 0$, for all $y \in C$.*

(b)

$$D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \le D_f(y, x), \quad \forall x \in E,\ y \in C. \tag{11}$$

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The modulus of total convexity of f at $x \in \operatorname{int}\operatorname{dom} f$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x, t) := \inf\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\}. \tag{12}$$

The function f is called totally convex at x if $\nu_f(x, t) > 0$ whenever $t > 0$. The function f is called totally convex if

it is totally convex at any point $x \in \operatorname{int}\operatorname{dom} f$ and is said to be totally convex on bounded sets if $\nu_f(B,t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \operatorname{int}\operatorname{dom} f \times [0,+\infty) \to [0,+\infty]$ defined by

$$\nu_f(B,t) := \inf\left\{\nu_f(x,t) : x \in B \cap \operatorname{dom} f\right\}. \tag{13}$$

The next lemma will be useful in the proof of our main results.

Lemma 5 (see [19]). *If $x \in \operatorname{dom} f$, then the following statements are equivalent.*

(i) *The function f is totally convex at x.*

(ii) *For any sequence $\{y_n\} \subset \operatorname{dom} f$,*

$$\lim_{n\to+\infty} D_f(y_n,x) = 0 \Longrightarrow \lim_{n\to+\infty} \|y_n - x\| = 0. \tag{14}$$

Recall that the function f is called sequentially consistent [18] if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n\to+\infty} D_f(y_n,x_n) = 0 \Longrightarrow \lim_{n\to+\infty} \|y_n - x_n\| = 0. \tag{15}$$

Lemma 6 (see [20]). *The function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 7 (see [21]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n,x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

Lemma 8 (see [21]). *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$, and let C be a nonempty, closed, and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n,x_0) \le D_f(P_C^f x_0, x_0)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $P_C^f x_0$.*

Let C be a convex subset of $\operatorname{int}\operatorname{dom} f$ and let T be a self-mapping of C. A point $p \in C$ is called an asymptotic fixed point of T (see [22, 23]) if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ the set of asymptotic fixed points of T.

Definition 9. A mapping T with a nonempty asymptotic fixed point set $\widehat{F}(T)$ is said to be

(i) Bregman strongly nonexpansive (see [24, 25]) with respect to $\widehat{F}(T)$ if

$$D_f(p,Tx) \le D_f(p,x), \quad \forall x \in C,\ p \in \widehat{F}(T), \tag{16}$$

and if, whenever $\{x_n\} \subset C$ is bounded, $p \in \widehat{F}(T)$ and

$$\lim_{n\to\infty}\left(D_f(p,x_n) - D_f(p,Tx_n)\right) = 0, \tag{17}$$

it follows that

$$\lim_{n\to\infty} D_f(x_n,Tx_n) = 0. \tag{18}$$

(ii) Bregman firmly nonexpansive [26] if, for all $x, y \in C$,

$$\begin{aligned}&\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \\ &\quad \le \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle\end{aligned} \tag{19}$$

or, equivalently,

$$\begin{aligned}&D_f(Tx,Ty) + D_f(Ty,Tx) + D_f(Tx,x) + D_f(Ty,y) \\ &\quad \le D_f(Tx,y) + D_f(Ty,x).\end{aligned} \tag{20}$$

The existence and approximation of Bregman firmly nonexpansive mappings were studied in [26]. It is also known that if T is Bregman firmly nonexpansive and f is Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E, then $F(T) = \widehat{F}(T)$ and $F(T)$ is closed and convex (see [26]). It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) = \widehat{F}(T)$.

Lemma 10 (see [27]). *Let E be a real reflexive Banach space and $f : E \to (-\infty,+\infty]$ a proper lower semicontinuous function; then $f^* : E^* \to (-\infty,+\infty]$ is a proper weak* lower semicontinuous and convex function. Thus, for all $z \in E$, we have*

$$D_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \le \sum_{i=1}^{N} t_i D_f(z,x_i). \tag{21}$$

In order to solve the equilibrium problem, let us assume that a bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions [28]:

(A_1) $\Theta(x,x) = 0$, for all $x \in C$.

(A_2) Θ is monotone; that is, $\Theta(x,y) + \Theta(y,x) \le 0$, for all $x, y \in C$.

(A_3) $\limsup_{t\downarrow 0}\Theta(x + t(z-x), y) \le \Theta(x,y)$ for all $x, z, y \in C$.

(A_4) The function $y \mapsto \Theta(x,y)$ is convex and lower semicontinuous.

The resolvent of a bifunction Θ [29] is the operator $\operatorname{Res}_\Theta^f : E \to 2^C$ defined by

$$\begin{aligned}\operatorname{Res}_\Theta^f(x) = \{&z \in C : \Theta(z,y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \\ &\ge 0,\ \forall y \in C\}.\end{aligned} \tag{22}$$

From Lemma 1 in [24], if $f : E \to (-\infty,+\infty]$ is a strongly coercive and Gâteaux differentiable function and Θ satisfies conditions $(A_1$–$A_4)$, then $\operatorname{dom}(\operatorname{Res}_\Theta^f) = E$. We also know the following lemma which gives us some characterizations of the resolvent $\operatorname{Res}_\Theta^f$.

Lemma 11 (see [24]). *Let E be a real reflexive Banach space and C a nonempty closed convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Legendre function. If the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the conditions (A_1)–(A_4), then the followings hold:*

(i) *Res^f_Θ is single-valued;*

(ii) *Res^f_Θ is a Bregman firmly nonexpansive operator;*

(iii) *$F(Res^f_\Theta) = EP(\Theta)$;*

(iv) *$EP(\Theta)$ is a closed and convex subset of C;*

(v) *for all $x \in E$ and for all $q \in F(Res^f_\Theta)$, we have*

$$D_f\left(q, Res^f_\Theta(x)\right) + D_f\left(Res^f_\Theta(x), x\right) \le D_f(q, x). \quad (23)$$

3. Strong Convergence Theorem

In this section, we proved a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and a fixed point of Bregman strongly nonexpansive mapping in a real reflexive Banach space by using the shrinking projection method.

Theorem 12. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let g be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4) and let T be a Bregman strongly nonexpansive mapping from C into itself such that $F(T) = \widehat{F}(T)$ and $G = F(T) \cap EP(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{aligned} &y_n = \nabla f^*\left(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)\right), \\ &u_n \in C \text{ such that} \\ &g(u_n, y) + \langle \nabla f(u_n) - \nabla f(y_n), y - u_n\rangle \ge 0, \\ &\qquad\qquad \forall y \in C, \\ &C_{n+1} = \left\{z \in C_n : D_f(z, u_n) \le D_f(z, x_n)\right\}, \\ &x_{n+1} = P^f_{C_{n+1}} x \end{aligned} \quad (24)$$

for every $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0,1]$ satisfies $\liminf_{n\to\infty}(1-\alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $P^f_{F(T)\cap EP(g)}x$, where $P^f_{F(T)\cap EP(g)}$ is the Bregman projection of E onto $F(T) \cap EP(g)$.

Proof. We divide the proof of Theorem 12 into five steps.

(I) We first prove that G and C_n both are closed and convex subset of C for all $n \ge 0$. In fact, it follows from Lemma 11 and by Reich and Sabach [26] that $\mathrm{EP}(g)$ and $F(T)$ both are closed and convex. Therefore, G is a closed and convex subset in C. Furthermore, it is obvious that $C_0 = C$ is closed and convex. Suppose that C_n is closed and convex for some $n \ge 1$. Since the inequality $D_f(z, u_n) \le D_f(z, x_n)$ is equivalent to

$$\langle \nabla f(x_n), z - x_n\rangle - \langle \nabla f(u_n), z - u_n\rangle \le f(u_n) - f(x_n). \quad (25)$$

Therefore, we have

$$\begin{aligned} C_{n+1} = \{&z \in C_n : \langle \nabla f(x_n), z - x_n\rangle - \langle \nabla f(u_n), z - u_n\rangle \\ &\le f(u_n) - f(x_n)\}. \end{aligned} \quad (26)$$

This implies that C_{n+1} is closed and convex. The desired *conclusions* are proved. These in turn show that $P^f_{F(T)\cap \mathrm{EP}(g)}x$ and $P^f_{C_n}x$ are well defined.

(II) we prove that $G := F(T) \cap \mathrm{EP}(g) \subset C_n$ for all $n \ge 0$.

Indeed, it is obvious that $G = F(T) \cap \mathrm{EP}(g) \subset C_0 = C$. Suppose that $G \subset C_n$ for some $n \in \mathbb{N}$. Let $u \in G \subset C_n$; since $u_n = \mathrm{Res}^f_g(y_n)$, by Lemma 11 and (21), we have

$$\begin{aligned} D_f(u, u_n) &= D_f\left(u, \mathrm{Res}^f_g y_n\right) \le D_f(u, y_n) \\ &= D_f\left(u, \nabla f^*\left(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(Tx_n)\right)\right) \\ &\le \alpha_n D_f(u, x_n) + (1-\alpha_n) D_f(u, Tx_n) \\ &\le \alpha_n D_f(u, x_n) + (1-\alpha_n) D_f(u, x_n) \\ &= D_f(u, x_n). \end{aligned} \quad (27)$$

Hence, we have $u \in C_{n+1}$. This implies that

$$F(T) \cap \mathrm{EP}(g) \subset C_n, \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (28)$$

So, $\{x_n\}$ is well defined.

(III) We prove that $\{x_n\}$ is a bounded sequence in C.

By the definition of C_n, we have $x_n = P^f_{C_n}x$ for all $n \ge 0$. It follows from Lemma 4(b) that

$$\begin{aligned} D_f(x_n, x) &= D_f\left(P^f_{C_n}x, x\right) \le D_f(u, x) - D_f\left(u, P^f_{C_n}x\right) \\ &\le D_f(u, x), \quad \forall n \ge 0,\ u \in G. \end{aligned} \quad (29)$$

This implies that $\{D_f(x_n, x)\}$ is bounded. By Lemma 7, $\{x_n\}$ is bounded. Since $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, by Lemma 1 ∇f is uniformly continuous and bounded on bounded subsets of E. This implies that $\{\nabla f(x_n)\}$ is bounded.

(IV) Now we proved that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

From $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P^f_{C_n}x$, we have

$$D_f(x_n, x) \le D_f(x_{n+1}, x), \quad \forall n \in \mathbb{N} \cup \{0\}. \quad (30)$$

Thus, $\{D_f(x_n, x)\}$ is nondecreasing. So, the limit of $\{D_f(x_n, x)\}$ exists. Since $D_f(x_{n+1}, x_n) = D_f(x_{n+1}, P^f_{C_n}x) \le D_f(x_{n+1}, x) - D_f(P^f_{C_n}x, x) = D_f(x_{n+1}, x) - D_f(x_n, x)$ for all $n \ge 0$, we

have $\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0$. From $x_{n+1} = P^f_{C_{n+1}} x \in C_{n+1}$, we have

$$D_f(x_{n+1}, u_n) \le D_f(x_{n+1}, x_n), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{31}$$

Therefore, we have

$$\lim_{n\to\infty} D_f(x_{n+1}, u_n) = 0. \tag{32}$$

From Lemma 5, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{33}$$

So, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{34}$$

This means that the sequence $\{u_n\}$ is bounded. Since f is uniformly Fréchet differentiable, it follows from Lemma 1 that ∇f is uniformly continuous. Therefore, we have

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(u_n)\| = 0. \tag{35}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, then f is uniformly continuous on bounded subsets of E (see [30, Theorem 1.8]). It follows that

$$\lim_{n\to\infty} |f(x_n) - f(u_n)| = 0. \tag{36}$$

From the definition of the Bregman distance, we obtain that

$$\begin{aligned} &D_f(u, x_n) - D_f(u, u_n) \\ &\quad = [f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n\rangle] \\ &\qquad - [f(u) - f(u_n) - \langle \nabla f(u_n), u - u_n\rangle] \\ &\quad = (f(u_n) - f(x_n)) + \langle \nabla f(u_n) - \nabla f(x_n), u - u_n\rangle \\ &\qquad + \langle \nabla f(x_n), x_n - u_n\rangle \end{aligned} \tag{37}$$

for any $u \in G$.

It follows from (34)–(37) that

$$\lim_{n\to\infty} \left(D_f(u, x_n) - D_f(u, u_n)\right) = 0. \tag{38}$$

On the other hand, from $u_n = \mathrm{Res}^f_g y_n$ and Lemma 11(v), for any $u \in G$ we have that

$$\begin{aligned} D_f(u_n, y_n) &= D_f\left(\mathrm{Res}^f_g y_n, y_n\right) \\ &\le D_f(u, y_n) - D_f\left(u, \mathrm{Res}^f_g y_n\right) \\ &\le D_f(u, x_n) - D_f\left(u, \mathrm{Res}^f_g y_n\right) \\ &= D_f(u, x_n) - D_f(u, u_n). \end{aligned} \tag{39}$$

So, we have from (38) that

$$\lim_{n\to\infty} D_f(u_n, y_n) = 0. \tag{40}$$

From Lemma 5, we have

$$\lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{41}$$

So, from (34) and (41),we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{42}$$

This means that the sequence $\{y_n\}$ is bounded. Since f is uniformly Fréchet differentiable, it follows from Lemma 1 that

$$\lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(y_n)\| = 0. \tag{43}$$

Since f is uniformly Fréchet differentiable on bounded subsets of E, then f is uniformly continuous on bounded subsets of E (see [30]). It follows that

$$\lim_{n\to\infty} |f(x_n) - f(y_n)| = 0. \tag{44}$$

From the definition of the Bregman distance, we obtain that

$$\begin{aligned} &D_f(u, y_n) - D_f(u, x_n) \\ &\quad = [f(u) - f(y_n) - \langle \nabla f(y_n), u - y_n\rangle] \\ &\qquad - [f(u) - f(x_n) - \langle \nabla f(x_n), u - x_n\rangle] \\ &\quad = (f(x_n) - f(y_n)) - \langle \nabla f(y_n) - \nabla f(x_n), u - y_n\rangle \\ &\qquad + \langle \nabla f(x_n), y_n - x_n\rangle \end{aligned} \tag{45}$$

for any $u \in G$.

It follows from (42) to (45) that

$$\lim_{n\to\infty} \left(D_f(u, y_n) - D_f(u, x_n)\right) = 0. \tag{46}$$

On the other hand, for any $u \in G$ we have

$$\begin{aligned} &D_f(u, y_n) - D_f(u, x_n) \\ &\quad = D_f(u, \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n)\nabla f(Tx_n))) \\ &\qquad - D_f(u, x_n) \\ &\quad \le \alpha_n D_f(u, x_n) + (1 - \alpha_n) D_f(u, Tx_n) - D_f(u, x_n) \\ &\quad = (1 - \alpha_n)\left(D_f(u, Tx_n) - D_f(u, x_n)\right). \end{aligned} \tag{47}$$

This together with (46), (16), and $\lim_{n\to\infty}\alpha_n < 1$ shows that

$$\lim_{k\to\infty} \left(D_f(u, Tx_n) - D_f(u, x_n)\right) = 0. \tag{48}$$

Since T is Bregman strongly nonexpansive, it follows from (48) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{49}$$

(V) Next, we prove that every weak subsequential limit of $\{x_n\}$ belongs to $G = F(T) \cap \mathrm{EP}(g)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. Since T is a Bregman strongly nonexpansive mapping with $F(T) = \widehat{F}(T)$, we have $x^* \in F(T)$.

From $x_{n_k} \rightharpoonup x^*$ and (34), we have $u_{n_k} \rightharpoonup x^*$.

By $u_n = \mathrm{Res}_g^f y_n$, we have

$$g(u_n, y) + \langle \nabla f(u_n) - \nabla f(y_n), y - u_n \rangle \geq 0, \quad \forall y \in C. \tag{50}$$

Replacing n by n_k, we have from (A_2) that

$$\left\langle \nabla f\left(u_{n_k}\right) - \nabla f\left(y_{n_k}\right), y - u_{n_k} \right\rangle \geq -g\left(u_{n_k}, y\right) \geq g\left(y, u_{n_k}\right), \quad \forall y \in C. \tag{51}$$

Since $g(x, \cdot)$ is convex and lower semicontinuous, it is also weakly lower semicontinuous. So, letting $k \to \infty$, we have from (35), (43), and (A_4) that

$$g(y, x^*) \leq 0, \quad \forall y \in C. \tag{52}$$

For $t \in (0, 1]$ and $y \in C$, letting $y_t = ty + (1-t)x^*$, there are $y_t \in C$ and $g(y_t, x^*) \leq 0$. By condition (A_1) and (A_4), we have

$$0 = g(y_t, y_t) \leq tg(y_t, y) + (1-t)\, g(y_t, x^*) \leq tg(y_t, y). \tag{53}$$

Dividing both sides of the above equation by t, we have $g(y_t, y) \geq 0$, for all $y \in C$. Letting $t \downarrow 0$, from condition (A_3), we have

$$g(x^*, y) \geq 0, \quad \forall y \in C. \tag{54}$$

Therefore, $x^* \in \mathrm{EP}(g)$.

(VI) Now, we prove $x_n \to P^f_{F(T)\cap \mathrm{EP}(g)} x$.

Let $w = P^f_{F(T)\cap \mathrm{EP}(g)} x$. From $w \in F(T) \cap \mathrm{EP}(g) \subset C_{n+1}$, we have $D_f(x_{n+1}, x) \leq D_f(w, x)$. Therefore, Lemma 8 implies that $\{x_n\}$ converges strongly to $w = P^f_{F(T)\cap \mathrm{EP}(g)} x$, as claimed. This completes the proof of Theorem 12. □

Corollary 13. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let g be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A_1)–(A_4). Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$, and*

$$\begin{aligned} &u_n \in C \text{ such that} \\ &g(u_n, y) + \langle \nabla f(u_n) - \nabla f(x_n), y - u_n \rangle \geq 0, \quad \forall y \in C, \\ &C_{n+1} = \left\{ z \in C_n : D_f(z, u_n) \leq D_f(z, x_n) \right\}, \\ &x_{n+1} = P^f_{C_{n+1}} x \end{aligned} \tag{55}$$

for every $n \in \mathbb{N} \cup \{0\}$. Then, $\{x_n\}$ converges strongly to $P^f_{EP(g)} x$, where $P^f_{EP(g)}$ is the Bregman projection of E onto $EP(g)$.

Proof. Putting $T = I$ in Theorem 12, we obtain Corollary 13. □

Corollary 14. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. let T be a Bregman strongly nonexpansive mapping from C into itself such that $F(T) = \widehat{F}(T)$ and $G = F(T) \cap EP(g) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 = x \in C$, $C_0 = C$, and*

$$\begin{aligned} u_n &= P^f_C \nabla f^* \left(\alpha_n \nabla f(x_n) + (1-\alpha_n) \nabla f(Tx_n)\right), \\ C_{n+1} &= \left\{ z \in C_n : D_f(z, u_n) \leq D_f(z, x_n) \right\}, \\ x_{n+1} &= P^f_{C_{n+1}} x \end{aligned} \tag{56}$$

for every $n \in \mathbb{N} \cup \{0\}$, where $\{\alpha_n\} \subset [0, 1]$ satisfies $\liminf_{n\to\infty}(1 - \alpha_n) > 0$. Then, $\{x_n\}$ converges strongly to $P^f_{F(T)} x$, where $P^f_{F(T)}$ is the Bregman projection of E onto $F(T)$.

Proof. Putting $g(x, y) = 0$ for all $x, y \in C$ in Theorem 12, we obtain Corollary 14. □

Acknowledgments

This work was supported by Scientific Research Fund of Sichuan Provincial Education Department (11ZB146) and Yunnan University of Finance and Economics.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 352927, 10 pages
http://dx.doi.org/10.1155/2013/352927

Research Article

Some Common Coupled Fixed Point Results for Generalized Contraction in Complex-Valued Metric Spaces

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Received 11 April 2013; Revised 22 May 2013; Accepted 22 May 2013

Academic Editor: Erdal Karapinar

We introduce and study the notion of common coupled fixed points for a pair of mappings in complex valued metric space and demonstrate the existence and uniqueness of the common coupled fixed points in a complete complex-valued metric space in view of diverse contractive conditions. In addition, our investigations are well supported by nontrivial examples.

1. Introduction

Azam et al. [1] introduced the concept of complex-valued metric spaces and obtained sufficient conditions for the existence of common fixed points of a pair of contractive type mappings involving rational expressions. Subsequently, several authors have studied the existence and uniqueness of the fixed points and common fixed points of self-mappings in view of contrasting contractive conditions. Some of these investigations are noted in [2–26].

In [27], Bhaskar and Lakshmikantham introduced the concept of coupled fixed points for a given partially ordered set X. Recently Samet et al. [28, 29] proved that most of the coupled fixed point theorems (on ordered metric spaces) are in fact immediate consequences of well-known fixed point theorems in the literature. In this paper, we deal with the corresponding definition of coupled fixed point for mappings on a complex-valued metric space along with generalized contraction involving rational expressions. Our results extend and improve several fixed point theorems in the literature.

2. Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

$$z_1 \preceq z_2 \quad \text{iff } \operatorname{Re}(z_1) \leq \operatorname{Re}(z_2),\ \operatorname{Im}(z_1) \leq \operatorname{Im}(z_2). \tag{1}$$

Note that $0 \preceq z_1$, z_2 and $z_1 \neq z_2$, $z_1 \preceq z_2$ implies $|z_1| < |z_2|$.

Definition 1. Let X be a nonempty set. Suppose that the self-mapping $d : X \times X \to \mathbb{C}$ satisfies the following:

(1) $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, y) \preceq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is known as a complex valued metric space. A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever, there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A. \tag{2}$$

A point $x \in X$ is a limit point of A whenever, for every $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A \setminus \{x\}) \neq \emptyset. \tag{3}$$

A is called open whenever each element of A is an interior point of A. Moreover, a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F = \{B(x, r) : x \in X, 0 \prec r \in \mathbb{C}\} \tag{4}$$

is a subbasis for a Hausdorff topology τ on X.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x, and x is the limit point of $\{x_n\}$. We denote this by $\lim_{n \to +\infty} x_n = x$, or $x_n \to x$, as $n \to +\infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that, for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then (X, d) is called a complete complex valued metric space. We require the following lemmas.

Lemma 2 (see [1]). *Let (X, d) be a complex valued metric space, and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$ as $n \to +\infty$.*

Lemma 3 (see [1]). *Let (X, d) be a complex valued metric space, and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$ as $n \to +\infty$.*

Definition 4 (*see [27]*). An element $(x, y) \in X \times X$ is called a coupled fixed point of $T : X \times X \to X$ if

$$x = T(x, y), \qquad y = T(y, x). \tag{5}$$

Definition 5. An element $(x, y) \in X \times X$ is called a coupled coincidence point of $S, T : X \times X \to X$ if

$$S(x, y) = T(x, y), \qquad S(y, x) = T(y, x). \tag{6}$$

Example 6. Let $X = \mathbb{R}$ and $S, T : X \times X \to X$ defined as $S(x, y) = x^2 y^2$ and $T(x, y) = (4/3)(x + y)$ for all $x, y \in X$. Then $(0, 0)$, $(1, 2)$, and $(2, 1)$ are coupled coincidence points of S and T.

Example 7. Let $X = \mathbb{R}$ and $S, T : X \times X \to X$ defined as $S(x, y) = x + y + \sin(x + y)$ and $T(x, y) = x + y + xy + \cos(x + y)$ for all $x, y \in X$. Then $(0, \pi/4)$ and $(\pi/4, 0)$ are coupled coincidence points of S and T.

Definition 8. An element $(x, y) \in X \times X$ is called a common coupled fixed point of $S, T : X \times X \to X$ if

$$x = S(x, y) = T(x, y), \qquad y = S(y, x) = T(y, x). \tag{7}$$

Example 9. Let $X = \mathbb{R}$ and $S, T : X \times X \to X$ defined as $S(x, y) = x((x+(y-1)^2)/2)$ and $T(x, y) = x(\sqrt{x^2 + y^2 + 4} - 2)$ for all $x, y \in X$. Then $(0, 0)$, $(1, 2)$, and $(2, 1)$ are common coupled fixed points of S and T.

In the following, we provide common coupled fixed point theorem for a pair of mappings satisfying a rational inequality in complex valued metric spaces.

Theorem 10. *Let (X, d) be a complete complex-valued metric space, and let the mappings $S, T : X \times X \to X$ satisfy*

$$\begin{aligned} d(S(x, y), T(u, v)) \preceq{}& \frac{\alpha(d(x, u) + d(y, v))}{2} \\ &+ (\beta d(x, S(x, y)) d(u, T(u, v)) \\ &\quad + \gamma d(u, S(x, y)) d(x, T(u, v))) \\ &\times (1 + d(x, u) + d(y, v))^{-1} \end{aligned} \tag{8}$$

for all $x, y, u, v \in X$ and α, β, and γ are nonnegative reals with $\alpha + \beta + \gamma < 1$. Then S and T have a unique common coupled fixed point.

Proof. Let x_0 and y_0 be arbitrary points in X. Define $x_{2k+1} = S(x_{2k}, y_{2k})$, $y_{2k+1} = S(y_{2k}, x_{2k})$ and $x_{2k+2} = T(x_{2k+1}, y_{2k+1})$, $y_{2k+2} = T(y_{2k+1}, x_{2k+1})$, for $k = 0, 1, \ldots$

Then,

$$\begin{aligned} &d(x_{2k+1}, x_{2k+2}) \\ &\quad = d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\ &\quad \preceq \frac{\alpha(d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))}{2} \\ &\qquad + (\beta d(x_{2k}, S(x_{2k}, y_{2k})) \\ &\qquad\quad \times d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))) \\ &\qquad \times (1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))^{-1} \\ &\qquad + (\gamma d(x_{2k+1}, S(x_{2k}, y_{2k})) \\ &\qquad\quad \times d(x_{2k}, T(x_{2k+1}, y_{2k+1}))) \\ &\qquad \times (1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))^{-1} \\ &\quad \preceq \frac{\alpha(d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))}{2} \\ &\qquad + \frac{\beta d(x_{2k}, x_{2k+1}) d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})} \\ &\qquad + \frac{\gamma d(x_{2k+1}, x_{2k+1}) d(x_{2k}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})} \end{aligned}$$

$$\leq \frac{\alpha\left(d\left(x_{2k}, x_{2k+1}\right)+d\left(y_{2k}, y_{2k+1}\right)\right)}{2} + \frac{\beta d\left(x_{2k}, x_{2k+1}\right) d\left(x_{2k+1}, x_{2k+2}\right)}{1+d\left(x_{2k}, x_{2k+1}\right)+d\left(y_{2k}, y_{2k+1}\right)}, \tag{9}$$

which implies that

$$\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| \leq \frac{\alpha\left|d\left(x_{2k}, x_{2k+1}\right)+d\left(y_{2k}, y_{2k+1}\right)\right|}{2} + \frac{\beta\left|d\left(x_{2k}, x_{2k+1}\right) d\left(x_{2k+1}, x_{2k+2}\right)\right|}{\left|1+d\left(x_{2k}, x_{2k+1}\right)+d\left(y_{2k}, y_{2k+1}\right)\right|}. \tag{10}$$

Since $|1 + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})| > |d(x_{2k}, x_{2k+1})|$, so we get

$$\begin{aligned}\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| &\leq \frac{\alpha\left|d\left(x_{2k}, x_{2k+1}\right)\right|+\alpha\left|d\left(y_{2k}, y_{2k+1}\right)\right|}{2} \\ &\quad + \beta\left|d\left(x_{2k+1}, x_{2k+2}\right)\right|,\end{aligned} \tag{11}$$

and hence

$$\begin{aligned}\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| &\leq \frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(x_{2k}, x_{2k+1}\right)\right| \\ &\quad + \frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(y_{2k}, y_{2k+1}\right)\right|.\end{aligned} \tag{12}$$

Similarly, one can show that

$$\begin{aligned}\left|d\left(y_{2k+1}, y_{2k+2}\right)\right| &\leq \frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(y_{2k}, y_{2k+1}\right)\right| \\ &\quad + \frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(x_{2k}, x_{2k+1}\right)\right|.\end{aligned} \tag{13}$$

Also,

$$\begin{aligned}&d\left(x_{2k+2}, x_{2k+3}\right) \\ &\quad = d\left(T\left(x_{2k+1}, y_{2k+1}\right), S\left(x_{2k+2}, y_{2k+2}\right)\right) \\ &\quad \leq \frac{\alpha\left(d\left(x_{2k+1}, x_{2k+2}\right)+d\left(y_{2k+1}, y_{2k+2}\right)\right)}{2} \\ &\qquad + \left(\beta d\left(x_{2k+1}, T\left(x_{2k+1}, y_{2k+1}\right)\right)\right. \\ &\qquad\quad \left.\times d\left(x_{2k+2}, S\left(x_{2k+2}, y_{2k+2}\right)\right)\right) \\ &\qquad \times\left(1+d\left(x_{2k+1}, x_{2k+2}\right)+d\left(y_{2k+1}, y_{2k+2}\right)\right)^{-1} \\ &\qquad + \left(\gamma d\left(x_{2k+2}, T\left(x_{2k+1}, y_{2k+1}\right)\right)\right. \\ &\qquad\quad \left.\times d\left(x_{2k+1}, S\left(x_{2k+2}, y_{2k+2}\right)\right)\right) \\ &\qquad \times\left(1+d\left(x_{2k+1}, x_{2k+2}\right)+d\left(y_{2k+1}, y_{2k+2}\right)\right)^{-1} \\ &\quad \leq \frac{\alpha\left(d\left(x_{2k+1}, x_{2k+2}\right)+d\left(y_{2k+1}, y_{2k+2}\right)\right)}{2} \\ &\qquad + \frac{\beta d\left(x_{2k+1}, x_{2k+2}\right) d\left(x_{2k+2}, x_{2k+3}\right)}{1+d\left(x_{2k+1}, x_{2k+2}\right)+d\left(y_{2k+1}, y_{2k+2}\right)} \\ &\qquad + \frac{\gamma d\left(x_{2k+2}, x_{2k+2}\right) d\left(x_{2k+1}, x_{2k+3}\right)}{1+d\left(x_{2k+1}, x_{2k+2}\right)+d\left(y_{2k+1}, y_{2k+2}\right)} \\ &\quad \leq \frac{\alpha\left(d\left(x_{2k+1}, x_{2k+2}\right)+d\left(y_{2k+1}, y_{2k+2}\right)\right)}{2} \\ &\qquad + \frac{\beta d\left(x_{2k+1}, x_{2k+2}\right) d\left(x_{2k+2}, x_{2k+3}\right)}{1+d\left(x_{2k+1}, x_{2k+2}\right)+d\left(y_{2k+1}, y_{2k+2}\right)},\end{aligned} \tag{14}$$

so that

$$\left|d\left(x_{2k+2}, x_{2k+3}\right)\right| \leq \frac{\alpha\left|d\left(x_{2k+1}, x_{2k+2}\right)+d\left(y_{2k+1}, y_{2k+2}\right)\right|}{2} + \frac{\beta\left|d\left(x_{2k+1}, x_{2k+2}\right)\right|\left|d\left(x_{2k+2}, x_{2k+3}\right)\right|}{\left|1+d\left(x_{2k+1}, x_{2k+2}\right)+d\left(y_{2k+1}, y_{2k+2}\right)\right|}. \tag{15}$$

As $|1 + d(x_{2k+1}, x_{2k+2}) + d(y_{2k+1}, y_{2k+2})| > |d(x_{2k+1}, x_{2k+2})|$, therefore

$$\begin{aligned}\left|d\left(x_{2k+2}, x_{2k+3}\right)\right| &\leq \frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| \\ &\quad + \frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(y_{2k+1}, y_{2k+2}\right)\right|.\end{aligned} \tag{16}$$

Similarly, one can show that

$$\begin{aligned}\left|d\left(y_{2k+2}, y_{2k+3}\right)\right| &\leq \frac{\alpha}{1-\beta}\left|d\left(y_{2k+1}, y_{2k+2}\right)\right| \\ &\quad + \frac{1}{2}\left(\frac{\alpha}{1-\beta}\right)\left|d\left(x_{2k+1}, x_{2k+2}\right)\right|.\end{aligned} \tag{17}$$

Adding (12)–(17), we get

$$\begin{aligned}
&\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| + \left|d\left(y_{2k+1}, y_{2k+2}\right)\right| \\
&\quad \leq \frac{\alpha}{1-\beta}\left|d\left(x_{2k}, x_{2k+1}\right)\right| + \frac{\alpha}{1-\beta}\left|d\left(y_{2k}, y_{2k+1}\right)\right| \\
&\left|d\left(x_{2k+2}, x_{2k+3}\right)\right| + \left|d\left(y_{2k+2}, y_{2k+3}\right)\right| \\
&\quad \leq \frac{\alpha}{1-\beta}\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| + \frac{\alpha}{1-\beta}\left|d\left(y_{2k+1}, y_{2k+2}\right)\right|.
\end{aligned} \tag{18}$$

If $h = \alpha/(1-\beta) < 1$, then from (18), we get

$$\begin{aligned}
&\left|d\left(x_n, x_{n+1}\right)\right| + \left|d\left(y_n, y_{n+1}\right)\right| \\
&\quad \leq h\left(\left|d\left(x_{n-1}, x_n\right)\right| + \left|d\left(y_{n-1}, y_n\right)\right|\right) \\
&\quad \leq \cdots \leq h^n\left(\left|d\left(x_0, x_1\right)\right| + \left|d\left(y_0, y_1\right)\right|\right).
\end{aligned} \tag{19}$$

Now if $|d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| = \delta_n$, then

$$\delta_n \leq h\delta_{n-1} \leq \cdots \leq h^n\delta_0. \tag{20}$$

Without loss of generality, we take $m > n$. Since $0 \leq h < 1$, so we get

$$\begin{aligned}
&\left|d\left(x_n, x_m\right)\right| + \left|d\left(y_n, y_m\right)\right| \\
&\quad \leq \left|d\left(x_n, x_{n+1}\right)\right| + \left|d\left(y_n, y_{n+1}\right)\right| + \cdots \\
&\qquad + \left|d\left(x_{m-1}, x_m\right)\right| + \left|d\left(y_{m-1}, y_m\right)\right| \\
&\quad \leq \left[h^n\delta_0 + h^{n+1}\delta_0 + \cdots + h^{m-1}\delta_0\right] \\
&\quad \leq \sum_{i=n}^{m-1} h^i\delta_0 \longrightarrow 0, \quad \text{as } m, n \longrightarrow +\infty.
\end{aligned} \tag{21}$$

This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to +\infty$. We now show that $x = S(x, y)$ and $y = S(y, x)$. We suppose on the contrary that $x \neq S(x, y)$ and $y \neq S(y, x)$ so that $0 \prec d(x, S(x, y)) = l_1$ and $0 \prec d(y, S(y, x)) = l_2$; we would then have

$$\begin{aligned}
l_1 &= d\left(x, S\left(x, y\right)\right) \preceq d\left(x, x_{2k+2}\right) + d\left(x_{2k+2}, S\left(x, y\right)\right) \\
&\preceq d\left(x, x_{2k+2}\right) + d\left(T\left(x_{2k+1}, y_{2k+1}\right), S\left(x, y\right)\right) \\
&\preceq d\left(x, x_{2k+2}\right) + \frac{\alpha\left(d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right)}{2} \\
&\quad + \frac{\beta d\left(x_{2k+1}, T\left(x_{2k+1}, y_{2k+1}\right)\right) d\left(x, S\left(x, y\right)\right)}{1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)} \\
&\quad + \frac{\gamma d\left(x, T\left(x_{2k+1}, y_{2k+1}\right)\right) d\left(x_{2k+1}, S\left(x, y\right)\right)}{1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)} \\
&= d\left(x, x_{2k+2}\right) + \frac{\alpha\left(d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right)}{2} \\
&\quad + \frac{\beta d\left(x_{2k+1}, x_{2k+2}\right) d\left(x, S\left(x, y\right)\right)}{1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)} \\
&\quad + \frac{\gamma d\left(x, x_{2k+2}\right) d\left(x_{2k+1}, S\left(x, y\right)\right)}{1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)}
\end{aligned} \tag{22}$$

so that

$$\begin{aligned}
\left|l_1\right| &\leq \left|d\left(x, x_{2k+2}\right)\right| + \frac{\alpha\left|d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right|}{2} \\
&\quad + \frac{\beta\left|d\left(x_{2k+1}, x_{2k+2}\right)\right|\left|d\left(x, S\left(x, y\right)\right)\right|}{\left|1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right|} \\
&\quad + \frac{\gamma\left|d\left(x, x_{2k+2}\right)\right|\left|d\left(x_{2k+1}, S\left(x, y\right)\right)\right|}{\left|1 + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right|}.
\end{aligned} \tag{23}$$

By taking $k \to +\infty$, we get $|d(x, S(x, y))| = 0$ which is a contradiction so that $x = S(x, y)$. Similarly, one can prove that $y = S(y, x)$. It follows similarly that $x = T(x, y)$ and $y = T(y, x)$. So we have proved that (x, y) is a common coupled fixed point of S and T. We now show that S and T have a unique common coupled fixed point. For this, assume that $(x^*, y^*) \in X$ is a second common coupled fixed point of S and T. Then

$$\begin{aligned}
d\left(x, x^*\right) &= d\left(S\left(x, y\right), T\left(x^*, y^*\right)\right) \\
&\preceq \frac{\alpha\left(d\left(x, x^*\right) + d\left(y, y^*\right)\right)}{2} \\
&\quad + \frac{\beta d\left(x, S\left(x, y\right)\right) d\left(x^*, T\left(x^*, y^*\right)\right)}{1 + d\left(x, x^*\right) + d\left(y, y^*\right)} \\
&\quad + \frac{\gamma d\left(x, T\left(x^*, y^*\right)\right) d\left(x^*, S\left(x, y\right)\right)}{1 + d\left(x, x^*\right) + d\left(y, y^*\right)}
\end{aligned}$$

$$\preceq \frac{\alpha\left(d\left(x,x^*\right)+d\left(y,y^*\right)\right)}{2} + \frac{\beta d\left(x,x\right)d\left(x^*,x^*\right)}{1+d\left(x,x^*\right)+d\left(y,y^*\right)} + \frac{\gamma d\left(x,x^*\right)d\left(x^*,x\right)}{1+d\left(x,x^*\right)+d\left(y,y^*\right)} \tag{24}$$

so that

$$\left|d\left(x,x^*\right)\right| \leq \left|\frac{\alpha\left(d\left(x,x^*\right)+d\left(y,y^*\right)\right)}{2}\right| + \frac{\gamma\left|d\left(x,x^*\right)\right|\left|d\left(x^*,x\right)\right|}{\left|1+d\left(x,x^*\right)+d\left(y,y^*\right)\right|}. \tag{25}$$

Since $|1+d(x,x^*)+d(y,y^*)| > |d(x,x^*)|$, so we get

$$\left|d\left(x,x^*\right)\right| \leq \left|\frac{\alpha\left(d\left(x,x^*\right)+d\left(y,y^*\right)\right)}{2}\right| + \gamma\left|d\left(x,x^*\right)\right| = \left(\frac{\alpha}{2-\alpha-2\gamma}\right)\left|d\left(y,y^*\right)\right|. \tag{26}$$

Similarly, one can easily prove that

$$\left|d\left(y,y^*\right)\right| \leq \left(\frac{\alpha}{2-\alpha-2\gamma}\right)\left|d\left(x,x^*\right)\right|. \tag{27}$$

If we add (26) and (27), we get

$$\left|d\left(x,x^*\right)\right| + \left|d\left(y,y^*\right)\right| \leq \left(\frac{\alpha}{2-\alpha-2\gamma}\right)\left(\left|d\left(x,x^*\right)\right| + \left|d\left(y,y^*\right)\right|\right), \tag{28}$$

which is a contradiction because $\alpha+\beta+\gamma<1$. Thus, we get $x^*=x$ and $y^*=y$, which proves the uniqueness of common coupled fixed point of S and T. □

By setting $S = T$ in Theorem 10, one deduces the following.

Corollary 11. *Let (X,d) be a complete complex-valued metric space, and let the mapping $T: X\times X \to X$ satisfy*

$$\begin{aligned} d\left(T\left(x,y\right),T\left(u,v\right)\right) &\preceq \frac{\alpha\left(d\left(x,u\right)+d\left(y,v\right)\right)}{2} \\ &\quad + \left(\beta d\left(x,T\left(x,y\right)\right)d\left(u,T\left(u,v\right)\right) \right. \\ &\qquad \left. + \gamma d\left(u,T\left(x,y\right)\right)d\left(x,T\left(u,v\right)\right)\right) \\ &\quad \times \left(1+d\left(x,u\right)+d\left(y,v\right)\right)^{-1} \end{aligned} \tag{29}$$

for all x, y, u, $v \in X$, where α, β, and γ are nonnegative reals with $\alpha+\beta+\gamma<1$. Then T has a unique coupled fixed point.

Corollary 12. *Let (X,d) be a complete complex-valued metric space, and let the mapping $T: X\times X \to X$ satisfy*

$$\begin{aligned} d\left(T^n\left(x,y\right),T^n\left(u,v\right)\right) &\preceq \frac{\alpha\left(d\left(x,u\right)+d\left(y,v\right)\right)}{2} \\ &\quad + \left(\beta d\left(x,T^n\left(x,y\right)\right)d\left(u,T^n\left(u,v\right)\right) \right. \\ &\qquad \left. + \gamma d\left(u,T^n\left(x,y\right)\right)d\left(x,T^n\left(u,v\right)\right)\right) \\ &\quad \times \left(1+d\left(x,u\right)+d\left(y,v\right)\right)^{-1} \end{aligned} \tag{30}$$

for all x, y, u, $v \in X$, where α, β, and γ are nonnegative reals with $\alpha+\beta+\gamma<1$. Then, T has a unique coupled fixed point.

Theorem 13. *Let (X,d) be a complete complex-valued metric space, and let the mappings $S,T: X\times X \to X$ satisfy*

$$d\left(S\left(x,y\right),T\left(u,v\right)\right) \preceq \begin{cases} \dfrac{\alpha\left(d\left(x,u\right)+d\left(y,v\right)\right)}{2} + \dfrac{\beta d\left(x,S\left(x,y\right)\right)d\left(u,T\left(u,v\right)\right)}{d\left(x,T\left(u,v\right)\right)+d\left(u,S\left(x,y\right)\right)+d\left(x,u\right)+d\left(y,v\right)}, & \text{if } D\neq 0 \\ 0, & \text{if } D=0 \end{cases} \tag{31}$$

for all $x, y, u, v \in X$, where $D = d(x,T(u,y)) + d(u,S(x,y)) + d(x,u) + d(y,v)$ and α, β are nonnegative reals with $\alpha+\beta<1$. Then S and T have a unique common coupled fixed point.

Proof. Let x_0 and y_0 be arbitrary points in X. Define $x_{2k+1} = S(x_{2k},y_{2k})$, $y_{2k+1} = S(y_{2k},x_{2k})$ and $x_{2k+2} = T(x_{2k+1},y_{2k+1})$, $y_{2k+2} = T(y_{2k+1},x_{2k+1})$, for $k = 0,1,\ldots$.

Now, we assume that

$$\begin{aligned} D_S\left(x_{2k},y_{2k}\right) &= d\left(x_{2k},T\left(x_{2k+1},y_{2k+1}\right)\right) \\ &\quad + d\left(x_{2k+1},S\left(x_{2k},y_{2k}\right)\right) \\ &\quad + d\left(x_{2k},x_{2k+1}\right)+d\left(y_{2k},y_{2k+1}\right) \\ &= d\left(x_{2k},x_{2k+2}\right)+d\left(x_{2k},x_{2k+1}\right) \\ &\quad + d\left(y_{2k},y_{2k+1}\right) \neq 0, \\ D_S\left(y_{2k},x_{2k}\right) &= d\left(y_{2k},T\left(y_{2k+1},x_{2k+1}\right)\right) \\ &\quad + d\left(y_{2k+1},S\left(y_{2k},x_{2k}\right)\right) \\ &\quad + d\left(x_{2k},x_{2k+1}\right)+d\left(y_{2k},y_{2k+1}\right) \\ &= d\left(y_{2k},y_{2k+2}\right)+d\left(x_{2k},x_{2k+1}\right) \\ &\quad + d\left(y_{2k},y_{2k+1}\right) \neq 0. \end{aligned} \tag{32}$$

Then,

$$\begin{aligned} d(x_{2k+1}, x_{2k+2}) &= d(S(x_{2k}, y_{2k}), T(x_{2k+1}, y_{2k+1})) \\ &\preceq \frac{\alpha(d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))}{2} \\ &\quad + \frac{\beta d(x_{2k}, S(x_{2k}, y_{2k}))\, d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))}{D_S(x_{2k}, y_{2k})} \\ &= \frac{\alpha(d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}))}{2} \\ &\quad + (\beta d(x_{2k}, x_{2k+1})\, d(x_{2k+1}, x_{2k+2})) \\ &\quad \times (d(x_{2k}, x_{2k+2}) + d(x_{2k}, x_{2k+1}) \\ &\quad + d(y_{2k}, y_{2k+1}))^{-1} \end{aligned} \tag{33}$$

which implies that

$$\begin{aligned} &|d(x_{2k+1}, x_{2k+2})| \\ &\quad \le \frac{\alpha|d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})|}{2} \\ &\quad + (\beta|d(x_{2k}, x_{2k+1})|\,|d(x_{2k+1}, x_{2k+2})|) \\ &\quad \times (|d(x_{2k}, x_{2k+2}) + d(x_{2k}, x_{2k+1}) \\ &\quad + d(y_{2k}, y_{2k+1})|)^{-1} \\ &\quad \le \frac{\alpha|d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1})|}{2} \\ &\quad + \beta|d(x_{2k}, x_{2k+1})| \end{aligned} \tag{34}$$

as

$$\begin{aligned} &|d(x_{2k+1}, x_{2k+2})| \\ &\quad \le |d(x_{2k+1}, x_{2k}) + d(x_{2k}, x_{2k+2}) + d(y_{2k}, y_{2k+1})|. \end{aligned} \tag{35}$$

Therefore,

$$\begin{aligned} &|d(x_{2k+1}, x_{2k+2})| \\ &\quad \le \frac{(\alpha + 2\beta)}{2}|d(x_{2k}, x_{2k+1})| + \frac{\alpha}{2}|d(y_{2k}, y_{2k+1})|. \end{aligned} \tag{36}$$

Similarly, one can easily prove that

$$\begin{aligned} &|d(y_{2k+1}, y_{2k+2})| \\ &\quad \le \frac{(\alpha + 2\beta)}{2}|d(y_{2k}, y_{2k+1})| + \frac{\alpha}{2}|d(x_{2k}, x_{2k+1})|. \end{aligned} \tag{37}$$

Now, if

$$\begin{aligned} &D_T(x_{2k+1}, y_{2k+1}) \\ &\quad = d(x_{2k+2}, T(x_{2k+1}, y_{2k+1})) \\ &\quad + d(x_{2k+1}, S(x_{2k+2}, y_{2k+2})) \\ &\quad + d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1}) \\ &\quad = d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+1}) \\ &\quad + d(y_{2k+2}, y_{2k+1}) \neq 0, \end{aligned} \tag{38}$$

we get

$$\begin{aligned} &d(x_{2k+2}, x_{2k+3}) \\ &\quad = d(T(x_{2k+1}, y_{2k+1}), S(x_{2k+2}, y_{2k+2})) \\ &\quad \preceq \frac{\alpha(d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1}))}{2} \\ &\quad + (\beta d(x_{2k+2}, S(x_{2k+2}, y_{2k+2})) \\ &\quad \times d(x_{2k+1}, T(x_{2k+1}, y_{2k+1}))) \\ &\quad \times (D_T(x_{2k+1}, y_{2k+1}))^{-1} \\ &\quad = \frac{\alpha(d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1}))}{2} \\ &\quad + (\beta d(x_{2k+2}, x_{2k+3})\, d(x_{2k+1}, x_{2k+2})) \\ &\quad \times (d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+1}) \\ &\quad + d(y_{2k+2}, y_{2k+1}))^{-1}, \end{aligned} \tag{39}$$

which implies that

$$\begin{aligned} &|d(x_{2k+2}, x_{2k+3})| \\ &\quad \le \alpha\left|\frac{\alpha(d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1}))}{2}\right| \\ &\quad + (\beta|d(x_{2k+2}, x_{2k+3})|\,|d(x_{2k+1}, x_{2k+2})|) \\ &\quad \times (|d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+1}) \\ &\quad + d(y_{2k+2}, y_{2k+1})|)^{-1} \\ &\quad \le \alpha\left|\frac{\alpha(d(x_{2k+2}, x_{2k+1}) + d(y_{2k+2}, y_{2k+1}))}{2}\right| \\ &\quad + \beta|d(x_{2k+1}, x_{2k+2})| \end{aligned} \tag{40}$$

as

$$\begin{aligned} &|d(x_{2k+2}, x_{2k+3})| \\ &\quad \le |d(x_{2k+2}, x_{2k+1}) + d(x_{2k+1}, x_{2k+3}) \\ &\quad + d(y_{2k+2}, y_{2k+1})|. \end{aligned} \tag{41}$$

Therefore,

$$\begin{aligned}
&\left|d\left(x_{2k+2}, x_{2k+3}\right)\right| \\
&\quad \leq \frac{\alpha\left|d\left(x_{2k+2}, x_{2k+1}\right)\right|}{2} + \frac{\alpha\left|d\left(y_{2k+2}, y_{2k+1}\right)\right|}{2} \\
&\qquad + \beta\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| \\
&\quad = \frac{(\alpha+2\beta)}{2}\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| \\
&\qquad + \frac{\alpha}{2}\left|d\left(y_{2k+1}, y_{2k+2}\right)\right|.
\end{aligned} \tag{42}$$

Similarly, if $D_T(y_{2k+1}, x_{2k+1}) \neq 0$, one can easily prove that

$$\begin{aligned}
\left|d\left(y_{2k+2}, y_{2k+3}\right)\right| &\leq \frac{(\alpha+2\beta)}{2}\left|d\left(y_{2k+1}, y_{2k+2}\right)\right| \\
&\quad + \frac{\alpha}{2}\left|d\left(x_{2k+1}, x_{2k+2}\right)\right|.
\end{aligned} \tag{43}$$

Adding the inequalities (36)–(43), we get

$$\begin{aligned}
&\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| + \left|d\left(y_{2k+1}, y_{2k+2}\right)\right| \\
&\quad \leq (\alpha+\beta)\left(\left|d\left(x_{2k}, x_{2k+1}\right)\right| + \left|d\left(y_{2k}, y_{2k+1}\right)\right|\right). \\
&\left|d\left(x_{2k+2}, x_{2k+3}\right)\right| + \left|d\left(y_{2k+2}, y_{2k+3}\right)\right| \\
&\quad \leq (\alpha+\beta)\left(\left|d\left(x_{2k+1}, x_{2k+2}\right)\right| + \left|d\left(y_{2k+1}, y_{2k+2}\right)\right|\right).
\end{aligned} \tag{44}$$

If $h = (\alpha+\beta) < 1$, then, from (44), we get

$$\begin{aligned}
&\left|d\left(x_n, x_{n+1}\right)\right| + \left|d\left(y_n, y_{n+1}\right)\right| \\
&\quad \leq h\left(\left|d\left(x_{n-1}, x_n\right)\right| + \left|d\left(y_{n-1}, y_n\right)\right|\right) \\
&\quad \leq \cdots \leq h^n\left(\left|d\left(x_0, x_1\right)\right|\right. \\
&\qquad \left. + \left|d\left(y_0, y_1\right)\right|\right).
\end{aligned} \tag{45}$$

Now if $|d(x_n, x_{n+1})| + |d(y_n, y_{n+1})| = \delta_n$, then

$$\delta_n \leq h\delta_{n-1} \leq \cdots \leq h^n\delta_0. \tag{46}$$

Without loss of generality, we take $m > n$. Since $0 \leq h < 1$, so we get

$$\begin{aligned}
&\left|d\left(x_n, x_m\right)\right| + \left|d\left(y_n, y_m\right)\right| \\
&\quad \leq \left|d\left(x_n, x_{n+1}\right)\right| + \left|d\left(y_n, y_{n+1}\right)\right| + \cdots \\
&\qquad + \left|d\left(x_{m-1}, x_m\right)\right| + \left|d\left(y_{m-1}, y_m\right)\right| \\
&\quad \leq \left[h^n\delta_0 + h^{n+1}\delta_0 + \cdots + h^{m-1}\delta_0\right] \\
&\quad \leq \sum_{i=n}^{m-1} h^i\delta_0 \longrightarrow 0, \quad \text{as } m, n \longrightarrow +\infty.
\end{aligned} \tag{47}$$

This implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete, so there exists $x, y \in X$ such that $x_n \to x$ and $y_n \to y$ as $n \to +\infty$. We now show that $x = S(x, y)$ and $y = S(y, x)$. We suppose on the contrary that $x \neq S(x, y)$ and $y \neq S(y, x)$ so that $0 \prec d(x, S(x, y)) = l_1$ and $0 \prec d(y, S(y, x)) = l_2$; we would then have

$$\begin{aligned}
l_1 &= d\left(x, S\left(x, y\right)\right) \preceq d\left(x, x_{2k+2}\right) + d\left(x_{2k+2}, S\left(x, y\right)\right) \\
&\preceq d\left(x, x_{2k+2}\right) + d\left(T\left(x_{2k+1}, y_{2k+1}\right), S\left(x, y\right)\right) \\
&\preceq d\left(x, x_{2k+2}\right) + \frac{\alpha\left(d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right)}{2} \\
&\qquad + \left(\beta d\left(x_{2k+1}, T\left(x_{2k+1}, y_{2k+1}\right)\right) d\left(x, S\left(x, y\right)\right)\right) \\
&\qquad \times \left(d\left(x_{2k+1}, S\left(x, y\right)\right) + d\left(x, T\left(x_{2k+1}, y_{2k+1}\right)\right)\right. \\
&\qquad\quad \left. + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right)^{-1} \\
&\preceq d\left(x, x_{2k+2}\right) + \frac{\alpha\left(d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right)}{2} \\
&\qquad + \left(\beta l_1 d\left(x_{2k+1}, x_{2k+2}\right)\right) \\
&\qquad \times \left(d\left(x_{2k+1}, S\left(x, y\right)\right) + d\left(x, x_{2k+2}\right)\right. \\
&\qquad\quad \left. + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right)^{-1},
\end{aligned} \tag{48}$$

so that

$$\begin{aligned}
\left|l_1\right| &\leq \left|d\left(x, x_{2k+2}\right)\right| + \frac{\alpha}{2}\left|d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right| \\
&\qquad + \left(\beta\left|l_1\right|\left|d\left(x_{2k+1}, x_{2k+2}\right)\right|\right) \\
&\qquad \times \left(\left|d\left(x_{2k+1}, S\left(x, y\right)\right) + d\left(x, x_{2k+2}\right)\right.\right. \\
&\qquad\quad \left.\left. + d\left(x_{2k+1}, x\right) + d\left(y_{2k+1}, y\right)\right|\right)^{-1}.
\end{aligned} \tag{49}$$

By taking $k \to +\infty$, we get $|d(x, S(x, y))| = 0$ which is a contradiction so that $x = S(x, y)$. Now

$$\begin{aligned}
l_2 &= d\left(y, S\left(y, x\right)\right) \preceq d\left(y, y_{2k+2}\right) + d\left(y_{2k+2}, S\left(y, x\right)\right) \\
&\preceq d\left(y, y_{2k+2}\right) + d\left(T\left(y_{2k+1}, x_{2k+1}\right), S\left(y, x\right)\right) \\
&\preceq d\left(y, y_{2k+2}\right) + \frac{\alpha\left(d\left(y_{2k+1}, y\right) + d\left(x_{2k+1}, x\right)\right)}{2} \\
&\qquad + \left(\beta d\left(y_{2k+1}, T\left(y_{2k+1}, x_{2k+1}\right)\right) d\left(y, S\left(y, x\right)\right)\right) \\
&\qquad \times \left(d\left(y_{2k+1}, S\left(y, x\right)\right) + d\left(y, T\left(y_{2k+1}, x_{2k+1}\right)\right)\right. \\
&\qquad\quad \left. + d\left(y_{2k+1}, y\right) + d\left(x_{2k+1}, x\right)\right)^{-1} \\
&\preceq d\left(y, y_{2k+2}\right) + \frac{\alpha\left(d\left(y_{2k+1}, y\right) + d\left(x_{2k+1}, x\right)\right)}{2} \\
&\qquad + \left(\beta l_2 d\left(y_{2k+1}, y_{2k+2}\right)\right) \\
&\qquad \times \left(d\left(y_{2k+1}, S\left(y, x\right)\right) + d\left(y, y_{2k+2}\right)\right. \\
&\qquad\quad \left. + d\left(y_{2k+1}, y\right) + d\left(x_{2k+1}, x\right)\right)^{-1},
\end{aligned} \tag{50}$$

which implies that

$$\begin{aligned}|l_2| \le{}& |d(y, y_{2k+2})| + \frac{\alpha}{2}|d(y_{2k+1}, y)\, d(x_{2k+1}, x)| \\ &+ (\beta |l_2|\, |d(y_{2k+1}, y_{2k+2})|) \\ &\times (|d(y_{2k+1}, S(y, x)) + d(y, y_{2k+2}) \\ &+ d(y_{2k+1}, y) + d(x_{2k+1}, x)|)^{-1},\end{aligned} \tag{51}$$

Which, on making $k \to +\infty$, gives us $|d(y, S(y, x))| = 0$ which is a contradiction so that $y = S(y, x)$. It follows similarly that $x = T(x, y)$ and $y = T(y, x)$. So we have proved that (x, y) is a common coupled fixed point of S and T. As in Theorem 10, the uniqueness of common coupled fixed point remains a consequence of contraction condition (31).

We have obtained the existence and uniqueness of a unique common coupled fixed point if

$$\begin{aligned}&D_S(x_{2k}, y_{2k}),\ D_S(y_{2k}, x_{2k}), \\ &\quad D_T(x_{2k+1}, y_{2k+1}),\ D_T(y_{2k+1}, x_{2k+1}) \neq 0\end{aligned} \tag{52}$$

for all $k \in \mathbb{N}$. Now, assume that $D_S(x_{2k}, y_{2k}) = 0$ for some $k \in \mathbb{N}$. From

$$d(x_{2k}, x_{2k+2}) + d(x_{2k}, x_{2k+1}) + d(y_{2k}, y_{2k+1}) = 0, \tag{53}$$

we obtain that $x_{2k} = x_{2k+1} = x_{2k+2}$ and $y_{2k} = y_{2k+1}$. If $D_S(y_{2k}, x_{2k}) \neq 0$, using (8), we deduce

$$d(y_{2k+1}, y_{2k+2}) = d(S(y_{2k}, x_{2k}), T(y_{2k+1}, x_{2k+1})) = 0. \tag{54}$$

That is, $y_{2k+1} = y_{2k+2}$ (this equality holds also if $D_S(y_{2k}, x_{2k}) = 0$). The equalities

$$x_{2k} = x_{2k+1} = x_{2k+2}, \qquad y_{2k} = y_{2k+1} = y_{2k+2}, \tag{55}$$

ensure that (x_{2k+1}, y_{2k+1}) is a unique common coupled fixed point of S and T. The same holds if either $D_S(y_{2k}, x_{2k}) = 0$, $D_T(x_{2k+1}, y_{2k+1}) = 0$, or $D_T(y_{2k+1}, x_{2k+1}) = 0$. □

From Theorem 13, if we assume $\alpha = 0$, we obtain the following corollary.

Corollary 14. *Let (X, d) be a complete complex-valued metric space, and let the self-mappings $S, T : X \times X \to X$ satisfy*

$$d(S(x, y), T(u, v)) \preceq \begin{cases} \dfrac{\beta d(x, S(x, y))\, d(u, T(u, v))}{d(x, T(u, v)) + d(u, S(x, y)) + d(x, u) + d(y, v)}, & \text{if } D \neq 0 \\ 0, & \text{if } D \neq 0 \end{cases} \tag{56}$$

for all $x, y, u, v \in X$, where $D = d(x, T(u, y)) + d(u, S(x, y)) + d(x, y) + d(y, v)$ and β is a nonnegative real such that $0 < \beta < 1$. Then S and T have a unique common coupled fixed point.

Corollary 15. *Let (X, d) be a complete complex-valued metric space, and let the mapping $T : X \times X \to X$ satisfy*

$$d(T(x, y), T(u, v)) \preceq \begin{cases} \dfrac{\alpha(d(x, u) + d(y, v))}{2} + \dfrac{\beta d(x, T(x, y))\, d(u, T(u, v))}{d(x, T(u, v)) + d(u, T(x, y)) + d(x, u) + d(y, v)}, & \text{if } D \neq 0 \\ 0, & \text{if } D = 0 \end{cases} \tag{57}$$

for all $x, y, u, v \in X$, where $D = d(x, T(u, y)) + d(u, T(x, y)) + d(x, u) + d(y, v)$ and α, β are nonnegative reals with $\alpha + \beta < 1$. Then T has a unique coupled fixed point.

Corollary 16. *Let (X, d) be a complete complex-valued metric space, and let the mapping $T : X \times X \to X$ satisfy*

$$d(T^n(x, y), T^n(u, v)) \preceq \begin{cases} \dfrac{\alpha(d(x, u) + d(y, v))}{2} + \dfrac{\beta d(x, T^n(x, y))\, d(u, T^n(u, v))}{d(x, T^n(u, v)) + d(u, T^n(x, y)) + d(x, u) + d(y, v)}, & \text{if } D \neq 0 \\ 0, & \text{if } D = 0 \end{cases} \tag{58}$$

for all $x, y, u, v \in X$, where $D = d(x, T^n(u, y)) + d(u, T^n(x, y)) + d(x, u) + d(y, v)$ and α, β are nonnegative reals with $\alpha + \beta < 1$. Then T has a unique coupled fixed point.

Now, we furnished a nontrivial example to support our main result (Theorem 10).

Example 17. Let

$$\begin{aligned}X_1 &= \{z \in \mathbb{C} : \operatorname{Re}(z) \ge 0,\ \operatorname{Im}(z) = 0\}, \\ X_2 &= \{z \in \mathbb{C} : \operatorname{Im}(z) \ge 0,\ \operatorname{Re}(z) = 0\},\end{aligned} \tag{59}$$

and let $X = X_1 \cup X_2$. Consider a complex valued metric $d : X \times X \to \mathbb{C}$ as follows:

$$d(z_1, z_2) = \begin{cases} \dfrac{2}{3}|x_1 - x_2| + \dfrac{i}{2}|x_1 - x_2|, & \text{if } z_1, z_2 \in X_1 \\ \dfrac{1}{2}|y_1 - y_2| + \dfrac{i}{3}|y_1 - y_2|, & \text{if } z_1, z_2 \in X_2 \\ \dfrac{2}{9}(x_1 + y_2) + \dfrac{i}{6}(x_1 + y_2), & \text{if } z_1 \in X_1, z_2 \in X_2 \\ \dfrac{i}{3}(x_2 + y_1) + \dfrac{2i}{9}(x_2 + y_1), & \text{if } z_1 \in X_2, z_2 \in X_1, \end{cases} \tag{60}$$

with $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then (X, d) is a complex valued metric space. Define $S, T : X \times X \to X$ as follows:

$$S(z_1, z_2) = \begin{cases} 0 + \dfrac{x_1 x_2}{4} i & \text{if } z_1, z_2 \in X_1 \\ \dfrac{y_1 y_2}{5} + 0i & \text{if } z_1, z_2 \in X_2 \\ 0 + \dfrac{x_1 y_2}{8} i & \text{if } z_1 \in X_1 \text{ and } z_2 \in X_2 \\ \dfrac{y_1 x_2}{9} + 0i & \text{if } z_1 \in X_2 \text{ and } z_2 \in X_1, \end{cases}$$
$$T(z_1, z_2) = \begin{cases} 0 + \dfrac{x_1 x_2}{6} i & \text{if } z_1, z_2 \in X_1 \\ \dfrac{y_1 y_2}{7} + 0i & \text{if } z_1, z_2 \in X_2 \\ 0 + \dfrac{x_1 y_2}{10} i & \text{if } z_1 \in X_1 \text{ and } z_2 \in X_2 \\ \dfrac{y_1 x_2}{11} + 0i & \text{if } z_1 \in X_2 \text{ and } z_2 \in X_1. \end{cases} \tag{61}$$

By a routine calculation, one can easily verify that the maps S and T satisfy the contraction condition (8) with $\alpha = 3/4$, $\beta = 1/15$, and $\gamma = 2/15$. Notice that the point $(0, 0)$ remains fixed under S and T and is indeed unique common coupled fixed point.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

Acknowledgments

The authors thank the editor and the referees for their valuable comments and suggestions which improved greatly the quality of this paper. Marwan Amin Kutbi gratefully acknowledges the support from the Deanship of Scientific Research (DSR) at King Abdulaziz University (KAU) during this research. Cristina Di Bari is supported by Università degli Studi di Palermo (Local University Project R.S. ex 60%).

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 423040, 8 pages
http://dx.doi.org/10.1155/2013/423040

Research Article

A New Gap Function for Vector Variational Inequalities with an Application

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Received 29 May 2013; Accepted 4 June 2013

Academic Editor: Gue Myung Lee

We consider a vector variational inequality in a finite-dimensional space. A new gap function is proposed, and an equivalent optimization problem for the vector variational inequality is also provided. Under some suitable conditions, we prove that the gap function is directionally differentiable and that any point satisfying the first-order necessary optimality condition for the equivalent optimization problem solves the vector variational inequality. As an application, we use the new gap function to reformulate a stochastic vector variational inequality as a deterministic optimization problem. We solve this optimization problem by employing the sample average approximation method. The convergence of optimal solutions of the approximation problems is also investigated.

1. Introduction

The vector variational inequality (VVI for short), which was first proposed by Giannessi [1], has been widely investigated by many authors (see [2–9] and the references therein). VVI can be used to model a range of vector equilibrium problems in economics, traffic networks, and migration equilibrium problems (see [1]).

One approach for solving a VVI is to transform it into an equivalent optimization problem by using a gap function. A gap function was first introduced to study optimization problems and has become a powerful tool in the study of convex optimization problems. Also a gap function was introduced in the study of scalar variational inequalities. It can reformulate a scalar variational inequality as an equivalent optimization problem, and so some effective solution methods and algorithms for optimization problems can be used to find solutions of variational inequalities. Recently, many authors extended the theory of gap functions to VVI and vector equilibrium problems (see [2, 4, 6–9]). In this paper, we present a new gap function for VVI and reformulate it as an equivalent optimization problem. We also prove that the gap function is directionally differentiable and that any point satisfying the first-order necessary optimality condition for the equivalent optimization problem solves the VVI.

In many practical problems, problem data will involve some uncertain factors. In order to reflect the uncertainties, stochastic vector variational inequalities are needed. Recently, stochastic scalar variational inequalities have received a lot of attention in the literature (see [10–20]). The ERM (expected residual minimization) method was proposed by Chen and Fukushima [11] in the study of stochastic complementarity problems. They formulated a stochastic linear complementarity problem (SLCP) as a minimization problem which minimizes the expectation of a NCP function (also called a residual function) of SLCP and regarded a solution of the minimization problem as a solution of SLCP. This method is the so-called expected residual minimization method. Following the ideas of Chen and Fukushima [11], Zhang and Chen [20] considered stochastic nonlinear complementary problems. Luo and Lin [18, 19] generalized the expected residual minimization method to

solve a stochastic linear and/or nonlinear variational inequality problem. However, in comparison to stochastic scalar variational inequalities, there are very few results in the literature on stochastic vector variational inequalities. In this paper, we consider a deterministic reformulation for the stochastic vector variational inequality (SVVI). Our focus is on the expected residual minimization (ERM) method for the stochastic vector variational inequality. It is well known that VVI is more complicated than a variational inequality, and they model many practical problems. Therefore, it is meaningful and interesting to study stochastic vector variational inequalities.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given. In Section 3, a new gap function for VVI is constructed and some suitable conditions are given to ensure that the new gap function is directionally differentiable and that any point satisfying the first-order necessary condition of optimality for the new optimization problem solves the vector variational inequality. In Section 4, the stochastic VVI is presented and the new gap function is used to reformulate SVVI as a deterministic optimization problem.

2. Preliminaries

In this section, we will introduce some basic notations and preliminary results.

Throughout this paper, denote by x^T the transpose of a vector or matrix x, by $|\cdot|$ the Euclidean norm of a vector or matrix, and by $\langle\cdot,\cdot\rangle$ the inner product in $\mathbb{R}^n$. Let K be a nonempty, closed, and convex set of $\mathbb{R}^n$, $F_i : \mathbb{R}^n \to \mathbb{R}^n$ $(i = 1,\dots,p)$ mappings, and $F := (F_1,\dots,F_p)^T$. The vector variational inequality is to find a vector $x^* \in K$ such that

$$\left(\left\langle F_1(x^*), y - x^*\right\rangle, \dots, \left\langle F_p(x^*), y - x^*\right\rangle\right)^T \notin -\operatorname{int}\mathbb{R}^p_+, \quad \forall y \in K, \tag{1}$$

where $\mathbb{R}^p_+$ is the nonnegative orthant of $\mathbb{R}^p$ and $\operatorname{int}\mathbb{R}^p_+$ denotes the interior of $\mathbb{R}^p_+$. Denote by S the solution set of VVI (1) and by S_ξ the solution set of the following scalar variational inequality (VI_ξ): find a vector $x^* \in K$, such that

$$\left\langle \sum_{j=1}^p \xi_j F_j(x^*), y - x^* \right\rangle \geq 0, \quad \forall y \in K, \tag{2}$$

where $\xi \in B := \{\xi \in \mathbb{R}^p_+ : \sum_{j=1}^p \xi_j = 1\}$.

Definition 1. A function $\varphi_\xi : K \to \mathbb{R}$ is said to be a gap function for VI_ξ (2) if it satisfies the following properties:

(i) $\varphi_\xi(x) \geq 0$, for all $x \in K$;

(ii) $\varphi_\xi(x^*) = 0$ iff x^* solves VI_ξ (2).

Definition 2. A function $\psi : K \to \mathbb{R}$ is said to be a gap function for VVI (1) if it satisfies the following properties:

(i) $\psi(x) \geq 0$, for all $x \in K$;

(ii) $\psi(x^*) = 0$ iff x^* solves VVI (1).

Suppose that G is an $n \times n$ symmetric positive definite matrix. Let

$$\varphi_\xi(x) := \max_{y\in K}\left\{\left\langle \sum_{j=1}^p \xi_j F_j(x), x - y\right\rangle - \frac{1}{2}|x-y|_G^2\right\}, \tag{3}$$

where $|x|_G^2 := \langle x, Gx\rangle$. Note that

$$\sqrt{\lambda_{\min}}\,|x| \leq |x|_G \leq \sqrt{\lambda_{\max}}\,|x|, \tag{4}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalues of G, respectively. It was shown in [21] that the maximum in (3) is attained at

$$H_\xi(x) := \operatorname{Proj}_{K,G}\left(x - G^{-1}\sum_{j=1}^p \xi_j F_j(x)\right), \tag{5}$$

where $\operatorname{Proj}_{K,G}(x)$ is the projection of the point x onto the closed convex set K with respect to the norm $|\cdot|_G$. Thus,

$$\varphi_\xi(x) = \left\langle \sum_{j=1}^p \xi_j F_j(x), x - H_\xi(x)\right\rangle - \frac{1}{2}\left|x - H_\xi(x)\right|_G^2. \tag{6}$$

Lemma 3. *The projection operator $Proj_{K,G}(\cdot)$ is nonexpansive; that is,*

$$\left|Proj_{K,G}(x) - Proj_{K,G}(y)\right|_G \leq |x-y|_G, \quad \forall x, y \in \mathbb{R}^n. \tag{7}$$

Lemma 4. *The function φ_ξ is a gap function for VI_ξ (2) and $x^* \in K$ solves VI_ξ (2) iff it solves the following optimization problem:*

$$\min_{x\in K} \varphi_\xi(x). \tag{8}$$

The gap function φ_ξ is also called the regularized gap function for VI_ξ. When F_j $(j = 1,\dots,p)$ are continuously differentiable, we have the following results.

Lemma 5. *If F_j $(j = 1,\dots,p)$ are continuously differentiable, then φ_ξ is also continuously differentiable in x, and its gradient is given by*

$$\nabla_x\varphi_\xi(x) = \sum_{j=1}^p \xi_j F_j(x) - \left(\sum_{j=1}^p \xi_j \nabla_x F_j(x) - G\right)\left(H_\xi(x) - x\right). \tag{9}$$

Lemma 6. *Assume that F_j are continuously differentiable and that the Jacobian matrixes $\nabla_x F_j(x)$ are positive definite for all $x \in K$ $(j = 1,\dots,p)$. If x is a stationary point of problem (8), that is,*

$$\left\langle \nabla_x\varphi_\xi(x), y - x\right\rangle \geq 0, \quad \forall y \in K, \tag{10}$$

then it solves VI_ξ (2).

3. A New Gap Function for VVI and Its Properties

In this section, based on the regularized gap function φ_ξ for VI_ξ (2), we construct a new gap function for VVI (1) and establish some properties under some mild conditions.

Let

$$\psi(x) := \min_{\xi\in B} \varphi_\xi(x). \tag{11}$$

Before showing that ψ is a gap function for VVI, we first present a useful result.

Lemma 7. *The following assertion is true:*

$$S = \cup_{\xi\in B} S_\xi. \tag{12}$$

Proof. Suppose that $x^* \in \cup_{\xi\in B} S_\xi$. Then, there exists a $\bar{\xi} \in B$ such that $x^* \in S_{\bar{\xi}}$ and

$$\sum_{j=1}^{p} \bar{\xi}_j \left\langle F_j(x^*), y - x^* \right\rangle = \left\langle \sum_{j=1}^{p} \bar{\xi}_j F_j(x^*), y - x^* \right\rangle \geq 0, \quad \forall y \in K. \tag{13}$$

For any fixed $y \in K$, since $\bar{\xi} \in B$, there exists a $j \in \{1, \ldots, p\}$ such that

$$\left\langle F_j(x^*), y - x^* \right\rangle \geq 0, \tag{14}$$

and so

$$\left(\left\langle F_1(x^*), y - x^*\right\rangle, \ldots, \left\langle F_p(x^*), y - x^*\right\rangle\right)^T \notin -\operatorname{int} \mathbb{R}^p_+. \tag{15}$$

Thus, we have

$$\left(\left\langle F_1(x^*), y - x^*\right\rangle, \ldots, \left\langle F_p(x^*), y - x^*\right\rangle\right)^T \notin -\operatorname{int} \mathbb{R}^p_+, \quad \forall y \in K. \tag{16}$$

This implies that $x^* \in S$ and $\cup_{\xi\in B} S_\xi \subset S$.

Conversely, suppose that $x^* \in S$. Then, we have

$$\left(\left\langle F_1(x^*), y - x^*\right\rangle, \ldots, \left\langle F_p(x^*), y - x^*\right\rangle\right)^T \notin -\operatorname{int} \mathbb{R}^p_+, \quad \forall y \in K, \tag{17}$$

and so

$$\left\{\left(\left\langle F_1(x^*), y - x^*\right\rangle, \ldots, \left\langle F_p(x^*), y - x^*\right\rangle\right)^T : \forall y \in K\right\} \cap \left(-\operatorname{int} \mathbb{R}^p_+\right) = \emptyset. \tag{18}$$

Since K is convex, from Theorems 11.1 and 11.3 of [22], it follows that

$$\inf_{y\in K} \left\{ \sum_{j=1}^{p} b_j \left\langle F_j(x^*), y - x^* \right\rangle \right\} \geq \sup_{y\in\left(-\operatorname{int} \mathbb{R}^p_+\right)} \langle b, y\rangle, \tag{19}$$

where $b = (b_1, \ldots, b_p) \neq 0$. Moreover, we have $b > 0$. In fact, if $b_j < 0$ for some $j \in \{1, \ldots, p\}$, then we have

$$\sup_{y\in(-\operatorname{int} \mathbb{R}^p_+)} \langle b, y\rangle = +\infty. \tag{20}$$

On the other hand,

$$\begin{aligned} &\inf_{y\in K} \left\{ \sum_{j=1}^{p} b_j \left\langle F_j(x^*), y - x^* \right\rangle \right\} \\ &\quad \leq \left\{ \sum_{j=1}^{p} b_j \left\langle F_j(x^*), x^* - x^* \right\rangle \right\} = 0 \\ &\quad < \sup_{y\in(-\operatorname{int} \mathbb{R}^p_+)} \langle b, y\rangle, \end{aligned} \tag{21}$$

which is a contradiction. Thus, $b > 0$. This implies that, for any $z \in K$,

$$\begin{aligned} &\sum_{j=1}^{p} b_j \left\langle F_j(x^*), z - x^* \right\rangle \\ &\quad \geq \inf_{y\in K} \left\{ \sum_{j=1}^{p} b_j \left\langle F_j(x^*), y - x^* \right\rangle \right\} \\ &\quad \geq \sup_{y\in\left(-\operatorname{int} \mathbb{R}^p_+\right)} \langle b, y\rangle = 0. \end{aligned} \tag{22}$$

Taking $\xi = b/(\sum_{j=1}^{p} b_j)$, then $\xi \in B$ and

$$\left\langle \sum_{j=1}^{p} \xi_j F_j(x^*), z - x^* \right\rangle \geq 0, \quad \forall z \in K, \tag{23}$$

which implies that $x^* \in S_\xi$ and $S \subset \cup_{\xi\in B} S_\xi$.

This completes the proof. □

Now, we can prove that ψ is a gap function for VVI (1).

Theorem 8. *The function ψ given by* (11) *is a gap function for VVI* (1). *Hence, $x^* \in K$ solves VVI* (1) *iff it solves the following optimization problem:*

$$\min_{x\in K} \psi(x). \tag{24}$$

Proof. Note that for any $\xi \in B$, $\varphi_\xi(x)$ given by (3) is a gap function for VI_ξ (2). It follows from Definition 1 that $\varphi_\xi(x) \geq 0$ for all $x \in K$ and hence $\psi(x) = \min_{\xi\in B} \varphi_\xi(x) \geq 0$ for all $x \in K$.

Assume that $\psi(x^*) = 0$ for some $x^* \in K$. From (6), it is easy to see that $\varphi_\xi(x^*)$ is continuous in ξ. Since B is a closed

and bounded set, there exists a vector $\bar{\xi} \in B$ such that $\psi(x^*) = \varphi_{\bar{\xi}}(x^*)$, which implies that $\varphi_{\bar{\xi}}(x^*) = 0$ and $x^* \in S_{\bar{\xi}}$. It follows from Lemma 7 that x^* solves VVI (1).

Suppose that x^* solves VVI (1). From Lemma 7, it follows that there exists a vector $\bar{\xi} \in B$ such that $x^* \in S_{\bar{\xi}}$ and so $\varphi_{\bar{\xi}}(x^*) = 0$. Since, for all $\xi \in B$, $\varphi_\xi(x^*) \geq 0$, we have $\psi(x^*) = \min_{\xi\in B} \varphi_\xi(x^*) = 0$.

Thus, (11) is a gap function for VVI (1). The last assertion is obvious from the definition of gap function.

This completes the proof. □

Since ψ is constructed based on the regularized gap function for VI_ξ, we wish to call it a regularized gap function for VVI (1). Theorem 8 indicates that in order to get a solution of VVI (1), we only need to solve problem (24). In what follows, we will discuss some properties of the regularized gap function ψ.

Theorem 9. *If F_j $(j = 1, \ldots, p)$ are continuously differentiable, then the regularized gap function ψ is directionally differentiable in any direction $d \in \mathbb{R}^n$, and its directional derivative $\psi'(x; d)$ is given by*

$$\psi'(x; d) = \inf_{\xi\in B(x)} \left\langle \nabla_x \varphi_\xi(x), d \right\rangle, \tag{25}$$

where $B(x) := \{\bar{\xi} \in B : \varphi_{\bar{\xi}}(x) = \min_{\xi\in B} \varphi_\xi(x)\}$.

Proof. It follows since the projection operator is nonexpansive that $\varphi_\xi(x)$ is continuous in (ξ, x). Thus, $B(x)$ is nonempty for any $x \in K$. From Lemma 5, it follows that $\varphi_\xi(x)$ is continuously differentiable in x and that

$$\nabla_x \varphi_\xi(x) = \sum_{j=1}^{p} \xi_j F_j(x) - \left(\sum_{j=1}^{p} \xi_j \nabla_x F_j(x) - G \right) \left(H_\xi(x) - x \right) \tag{26}$$

is continuous in (ξ, x). It follows from Theorem 1 of [23] that ψ is directionally differentiable in any direction $d \in \mathbb{R}^n$, and its directional derivative $\psi'(x; d)$ is given by

$$\psi'(x; d) = \inf_{\xi\in B(x)} \left\langle \nabla_x \varphi_\xi(x), d \right\rangle. \tag{27}$$

This completes the proof. □

By the directional differentiability of ψ shown in Theorem 9, the first-order necessary condition of optimality for problem (24) can be stated as

$$\psi'(x; y - x) \geq 0, \quad \forall y \in K. \tag{28}$$

If one wishes to solve VVI (1) via the optimization problem (24), we need to obtain its global optimal solution. However, since the function ψ is in general nonconvex, it is difficult to find a global optimal solution. Fortunately, we can prove that any point satisfying the first-order condition of optimality becomes a solution of VVI (1). Thus, existing optimization algorithms can be used to find a solution of VVI (1).

Theorem 10. *Assume that F_j are continuously differentiable and that the Jacobian matrixes $\nabla_x F_j(x)$ are positive definite for all $x \in K$ $(j = 1, \ldots, p)$. If $x^* \in K$ satisfies the first-order condition of optimality* (28)*, then x^* solves VVI* (1).

Proof. It follows from Theorem 9 and (28) that, for some $\xi \in B(x^*)$,

$$\left\langle \nabla_x^* \varphi_\xi(x^*), y - x^* \right\rangle \geq 0 \tag{29}$$

holds for any $y \in K$. This implies that x^* is a stationary point of problem (8). It follows from Lemma 6 that x^* solves VI_ξ. From Lemma 7, we see that x^* solves VVI. This completes the proof. □

From Theorems 8 and 10, it is easy to get the following corollary.

Corollary 11. *Assume that the conditions in Theorem 10 are all satisfied. If $x^* \in K$ satisfies the first-order condition of optimality* (28)*, then x^* is a global optimal solution of problem* (24).

4. Stochastic Vector Variational Inequality

In this section, we consider the stochastic vector variational inequality (SVVI). First, we present a deterministic reformulation for SVVI by employing the ERM method and the regularized gap function. Second, we solve this reformulation by the SAA method.

In most important practical applications, the functions F_j $(j = 1, \ldots, p)$ always involve some random factors or uncertainties. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Taking the randomness into account, we get a stochastic vector variational inequality problem (SVVI): find a vector $x^* \in K$ such that

$$\begin{aligned} &\left(\left\langle F_1(x^*, \omega), y - x^* \right\rangle, \ldots, \left\langle F_p(x^*, \omega), y - x^* \right\rangle \right)^T \\ &\qquad \notin -\operatorname{int} \mathbb{R}^p_+, \quad \forall y \in K, \text{ a.s.}, \end{aligned} \tag{30}$$

where $F_j : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ are mappings and a.s. is the abbreviation for "almost surely" under the given probability measure P.

Because of the random element ω, we cannot generally find a vector $x^* \in K$ such that (30) holds almost surely. That is, (30) is not well defined if we think of solving (30) before knowing the realization ω. Therefore, in order to get a reasonable resolution, an appropriate deterministic reformulation for SVVI becomes an important issue in the study of the considered problem. In this section, we will employ the ERM method to solve (30).

Define

$$\psi(x, \omega) := \min_{\xi\in B} \varphi_\xi(x, \omega), \tag{31}$$

where

$$\varphi_\xi(x, \omega) := \max_{y\in K} \left\{ \left\langle \sum_{j=1}^{p} \xi_j F_j(x, \omega), x - y \right\rangle - \frac{1}{2} |x - y|_G^2 \right\}. \tag{32}$$

The maximum in (32) is attained at

$$H_\xi(x,\omega) := \mathrm{Proj}_{K,G}\left(x - G^{-1}\sum_{j=1}^{p}\xi_j F_j(x,\omega)\right), \tag{33}$$

$$\begin{aligned}\varphi_\xi(x,\omega) = &\left\langle \sum_{j=1}^{p}\xi_j F_j(x,\omega), x - H_\xi(x,\omega)\right\rangle \\ &- \frac{1}{2}\left|x - H_\xi(x,\omega)\right|_G^2.\end{aligned} \tag{34}$$

The ERM reformulation is given as follows:

$$\min_{x\in K}\Theta(x) := \mathbb{E}\left[\psi(x,\omega)\right], \tag{35}$$

where $\mathbb{E}$ is the expectation operator.

Note that the objective function $\Theta(x)$ contains the mathematical expectation. In practical applications, it is in general very difficult to calculate $\mathbb{E}[\psi(x,\omega)]$ in a closed form. Thus, we will have to approximate it through discretization. One of the most popular discretization approaches is the sample average approximation method. In general, for an integrable function $\phi : \Omega \to \mathbb{R}$, we approximate the expected value $\mathbb{E}[\phi(\omega)]$ with the sample average $(1/N_k)\sum_{\omega_i\in\Omega_k}\phi(\omega_i)$, where $\omega_1,\ldots,\omega_{N_k}$ are independently and identically distributed random samples of ω and $\Omega_k := \{\omega_1,\ldots,\omega_{N_k}\}$. By the strong law of large numbers, we get the following lemma.

Lemma 12. *If $\phi(\omega)$ is integrable, then*

$$\lim_{k\to\infty}\frac{1}{N_k}\sum_{\omega_i\in\Omega_k}\phi(\omega_i) = \mathbb{E}\left[\phi(\omega)\right] \tag{36}$$

holds with probability one.

Let $\Theta_k(x) := (1/N_k)\sum_{\omega_i\in\Omega_k}\psi(x,\omega_i)$. Applying the above technique, we get the following approximation of (35):

$$\min_{x\in K}\Theta_k(x). \tag{37}$$

In the rest of this section, we focus on the case

$$F_j(x,\omega) := M_j(\omega)x + Q_j(\omega) \tag{38}$$

($j = 1,\ldots,p$), where $M_j : \Omega \to \mathbb{R}^{n\times n}$ and $Q_j : \Omega \to \mathbb{R}^n$ are measurable functions such that

$$\mathbb{E}\left[\left|M_j(\omega)\right|^2\right] < +\infty, \qquad \mathbb{E}\left[\left|Q(\omega)_j\right|^2\right] < +\infty, \qquad j = 1,\ldots,p. \tag{39}$$

This condition implies that

$$\begin{aligned}&\mathbb{E}\left[\sum_{j=1}^{p}M_j(\omega)\right]^2 < +\infty, \\ &\mathbb{E}\left[\left(\sum_{j=1}^{p}\left|M_j(\omega)\right|\right)\left(\sum_{j=1}^{p}\left|Q_j(\omega)\right|\right)\right] < +\infty,\end{aligned} \tag{40}$$

and, for any scalar c,

$$\mathbb{E}\left[\sum_{j=1}^{p}\left(\left|M_j(\omega)\right|c + \left|Q_j(\omega)\right|\right)\right] < +\infty. \tag{41}$$

The following results will be useful in the proof of the convergence result.

Lemma 13. *Let $f,g : \mathbb{R}^p \to \mathbb{R}_+$ be continuous. If $\min_{\xi\in B} f(\xi) < +\infty$ and $\min_{\xi\in B} g(\xi) < +\infty$, then*

$$\left|\min_{\xi\in B} f(\xi) - \min_{\xi\in B} g(\xi)\right| \le \max_{\xi\in B}\left|f(\xi) - g(\xi)\right|. \tag{42}$$

Proof. Without loss of generality, we assume that $\min_{\xi\in B} f(\xi) \le \min_{\xi\in B} g(\xi)$. Let $\bar{\xi}$ minimize f and $\hat{\xi}$ minimize g, respectively. Hence, $f(\bar{\xi}) \le g(\hat{\xi})$ and $g(\hat{\xi}) \le g(\bar{\xi})$. Thus

$$\begin{aligned}\left|\min_{\xi\in B} f(\xi) - \min_{\xi\in B} g(\xi)\right| &= g\left(\hat{\xi}\right) - f\left(\bar{\xi}\right) \\ &\le g\left(\bar{\xi}\right) - f\left(\bar{\xi}\right) \\ &\le \max_{\xi\in B}\left|f(\xi) - g(\xi)\right|.\end{aligned} \tag{43}$$

This completes the proof. □

Lemma 14. *When $F_j(x,\omega) = M_j(\omega)x + Q_j(\omega)$ ($j = 1,\ldots,p$), the function $\varphi_\xi(x,\omega)$ is continuously differentiable in x almost surely, and its gradient is given by*

$$\begin{aligned}\nabla_x\varphi_\xi(x,\omega) = &\sum_{j=1}^{p}\xi_j F_j(x,\omega) \\ &- \left(\sum_{j=1}^{p}\xi_j M_j(\omega) - G\right)\left(H_\xi(x,\omega) - x\right).\end{aligned} \tag{44}$$

Proof. The proof is the same as that of Theorem 3.2 in [21], so we omit it here. □

Lemma 15. *For any $x \in K$, one has*

$$\left|x - H_\xi(x,\omega)\right| \le \frac{2}{\lambda_{\min}}\sum_{j=1}^{p}\left|F_j(x,\omega)\right|. \tag{45}$$

Proof. For any fixed $\omega \in \Omega$, $\varphi_\xi(x,\omega)$ is the gap function of the following scalar variational inequality: find a vector $x^* \in K$, such that

$$\left\langle \sum_{j=1}^{p}\xi_j F_j(x^*,\omega), y - x^*\right\rangle \ge 0, \quad \forall y \in K. \tag{46}$$

Hence, $\varphi_\xi(x,\omega) \geq 0$ for all $x \in K$. From (4) and (34), we have

$$\begin{aligned}
&\frac{1}{2}\left|x - H_\xi(x,\omega)\right|_G^2 \\
&\quad \leq \left\langle \sum_{j=1}^{p} \xi_j F_j(x,\omega), x - H_\xi(x,\omega) \right\rangle \\
&\quad \leq \left|\sum_{j=1}^{p} \xi_j F_j(x,\omega)\right| \left|x - H_\xi(x,\omega)\right| \\
&\quad \leq \frac{1}{\sqrt{\lambda_{\min}}} \sum_{j=1}^{p} \left|F_j(x,\omega)\right| \left|x - H_\xi(x,\omega)\right|_G,
\end{aligned} \tag{47}$$

and so

$$\left|x - H_\xi(x,\omega)\right|_G \leq \frac{2}{\sqrt{\lambda_{\min}}} \sum_{j=1}^{p} \left|F_j(x,\omega)\right|. \tag{48}$$

It follows from (4) that

$$\begin{aligned}
\left|x - H_\xi(x,\omega)\right| &\leq \frac{1}{\sqrt{\lambda_{\min}}} \left|x - H_\xi(x,\omega)\right|_G \\
&\leq \frac{2}{\lambda_{\min}} \sum_{j=1}^{p} \left|F_j(x,\omega)\right|.
\end{aligned} \tag{49}$$

This completes the proof. □

Now, we obtain the convergence of optimal solutions of problem (37) in the following theorem.

Theorem 16. *Let $\{x^k\}$ be a sequence of optimal solutions of problem* (37). *Then, any accumulation point of $\{x^k\}$ is an optimal solution of problem* (35).

Proof. Let x^* be an accumulation point of $\{x^k\}$. Without loss of generality, we assume that x^k itself converges to x^* as k tends to infinity. It is obvious that $x^* \in K$. At first, we will show that

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^k, \omega_i\right) = \mathbb{E}\left[\psi\left(x^*, \omega\right)\right]. \tag{50}$$

From Lemma 12, it suffices to show that

$$\lim_{k\to\infty} \left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^k, \omega_i\right) - \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^*, \omega_i\right) \right| = 0. \tag{51}$$

It follows from Lemma 13 and the mean-value theorem that

$$\begin{aligned}
&\left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^k, \omega_i\right) - \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^*, \omega_i\right) \right| \\
&\quad \leq \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \left|\psi\left(x^k, \omega_i\right) - \psi\left(x^*, \omega_i\right)\right| \\
&\quad = \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \left|\min_{\xi \in B} \varphi_\xi\left(x^k, \omega_i\right) - \min_{\xi \in B} \varphi_\xi\left(x^*, \omega_i\right)\right| \\
&\quad \leq \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \max_{\xi \in B} \left|\varphi_\xi\left(x^k, \omega_i\right) - \varphi_\xi\left(x^*, \omega_i\right)\right| \\
&\quad \leq \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \max_{\xi \in B} \left|\nabla_x \varphi_\xi\left(y^{k,i}, \omega_i\right)\right| \left|x^k - x^*\right|,
\end{aligned} \tag{52}$$

where $y^{k,i} = \lambda_{k,i} x^k + (1 - \lambda_{k,i}) x^*$ with $\lambda_{k,i} \in [0,1]$. Because $\lim_{k\to+\infty} x^k = x^*$, there exists a constant C such that $|x^k| \leq C$ for each k. By the definition of $y^{k,i}$, we have $|y^{k,i}| \leq C$. Hence, for any $\xi \in B$ and $\omega_i \in \Omega_k$,

$$\begin{aligned}
&\left|\nabla_x \varphi_\xi\left(y^{k,i}, \omega_i\right)\right| \\
&\quad = \left| \sum_{j=1}^{p} \xi_j F_j\left(y^{k,i}, \omega_i\right) \right. \\
&\qquad - \left(\sum_{j=1}^{p} \xi_j \nabla_x F_j\left(y^{k,i}, \omega_i\right) - G \right) \\
&\qquad \left. \times \left(H_\xi\left(y^{k,i}, \omega_i\right) - y^{k,i}\right) \right| \\
&\quad \leq \sum_{j=1}^{p} \left|\xi_j\right| \left|F_j\left(y^{k,i}, \omega_i\right)\right| \\
&\qquad + \left(\sum_{j=1}^{p} \left|\xi_j\right| \left|\nabla_x F_j\left(y^{k,i}, \omega_i\right)\right| + |G| \right) \\
&\qquad \times \left|H_\xi\left(y^{k,i}, \omega_i\right) - y^{k,i}\right| \\
&\quad \leq \sum_{j=1}^{p} \left|F_j\left(y^{k,i}, \omega_i\right)\right| \\
&\qquad + \frac{2}{\lambda_{\min}} \left(\sum_{j=1}^{p} \left|\nabla_x F_j\left(y^{k,i}, \omega_i\right)\right| + |G| \right) \\
&\qquad \times \sum_{j=1}^{p} \left|F_j\left(y^{k,i}, \omega_i\right)\right|
\end{aligned}$$

$$\leq \left(\frac{2}{\lambda_{\min}} \sum_{j=1}^{p} \left| M_j(\omega_i) \right| + \frac{2}{\lambda_{\min}} |G| + 1 \right) \times \sum_{j=1}^{p} \left(\left| M_j(\omega_i) \right| C + \left| Q_j(\omega_i) \right| \right)$$
$$= \frac{2C}{\lambda_{\min}} \left(\sum_{j=1}^{p} \left| M_j(\omega_i) \right| \right)^2 + \sum_{j=1}^{p} \left(\left| M_j(\omega_i) \right| C + \left| Q_j(\omega_i) \right| \right) \left(\frac{2}{\lambda_{\min}} |G| + 1 \right) + \frac{2}{\lambda_{\min}} \left(\sum_{j=1}^{p} \left| M_j(\omega_i) \right| \right) \left(\sum_{j=1}^{p} \left| Q_j(\omega_i) \right| \right), \tag{53}$$

where the second inequality is from Lemma 15. Thus,

$$\begin{aligned} &\left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^k, \omega_i\right) - \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^*, \omega_i\right) \right| \\ &\leq \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \max_{\xi \in B} \left| \nabla_x \varphi_\xi\left(y^{k,i}, \omega_i\right) \right| \left| x^k - x^* \right| \\ &\leq \frac{2C}{\lambda_{\min}} \left| x^k - x^* \right| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \left(\sum_{j=1}^{p} \left| M_j(\omega_i) \right| \right)^2 \\ &\quad + \left| x^k - x^* \right| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \sum_{j=1}^{p} \left(\left| M_j(\omega_i) \right| C + \left| Q_j(\omega_i) \right| \right) \\ &\qquad \times \left(\frac{2}{\lambda_{\min}} |G| + 1 \right) \\ &\quad + \frac{2}{\lambda_{\min}} \left| x^k - x^* \right| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \left(\sum_{j=1}^{p} \left| M_j(\omega_i) \right| \right) \\ &\qquad \times \left(\sum_{j=1}^{p} \left| Q_j(\omega_i) \right| \right). \end{aligned} \tag{54}$$

From (40) and (41), each term in the last inequality above converges to zero, and so

$$\lim_{k \to \infty} \left| \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^k, \omega_i\right) - \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^*, \omega_i\right) \right| = 0. \tag{55}$$

Since

$$\lim_{k \to \infty} \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^*, \omega_i\right) = \mathbb{E}\left[\psi\left(x^*, \omega\right)\right], \tag{56}$$

(50) is true.

Now, we are in the position to show that x^* is a solution of problem (35).

Since x^k solves problem (37) for each k, we have that, for any $x \in K$,

$$\frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x^k, \omega_i\right) \leq \frac{1}{N_k} \sum_{\omega_i \in \Omega_k} \psi\left(x, \omega_i\right). \tag{57}$$

Letting $k \to \infty$ above, we get from Lemma 12 and (50) that

$$\mathbb{E}\left[\psi\left(x^*, \omega\right)\right] \leq \mathbb{E}\left[\psi\left(x, \omega\right)\right], \tag{58}$$

which means that x^* is an optimal solution of problem (35). This completes the proof. □

Acknowledgments

This work was supported by the National Natural Science Foundation of China (11171237, 11101069, and 71101099), by the Construction Foundation of Southwest University for Nationalities for the subject of applied economics (2011XWD-S0202), and by Sichuan University (SKQY201330).

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 831491, 7 pages
http://dx.doi.org/10.1155/2013/831491

Research Article

On Cyclic Generalized Weakly C-Contractions on Partial Metric Spaces

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Received 23 April 2013; Accepted 29 May 2013

Academic Editor: Wei-Shih Du

We give new results of a cyclic generalized weakly C-contraction in partial metric space. The results of this paper extend, generalize, and improve some fixed point theorems in the literature.

1. Introduction and Preliminaries

The notion of partial metric space [1], represented by the abbreviation PMS, departs from the usual metric spaces due to removing the assumption of self-distance. In other words, in PMS self-distance needs not to be zero. This interesting distance function is defined by Matthews [1], as a generalization metric to study in computer science, in particular, to get a more efficient programs in computer science. In the remarkable publication of Matthews [1], a characterization of the Banach Contraction Principle was given in the context of PMS. Due to its wide application potential [2–6], PMS and its topological properties are considered by many authors [7–25]. Very recently, Haghi et al. [26] proved that some obtained results in the context of PMS can be deduced from earlier results in the setting of usual metric space.

In the sequel, $\mathbb{R}^+$, $\mathbb{N}^*$ will represent the set of all real nonnegative numbers and the set of all positive natural numbers, respectively. Moreover, we use the abbreviations MS, CMS, PMS, and CMPS for metric space, complete metric space, partial metric space, and complete partial metric space, respectively. Let Λ be the collection of function $\varphi : [0,1) \to [0,1)$ which is nondecreasing, continuous together with the property $\varphi(t) > 0$ for $t \in (0,1)$ and $\varphi(0) = 0$. The following definition introduced by Chatterjea [27] to generalize the Banach contraction principle.

Definition 1. Suppose that (X,d) is an MS. A mapping $T : X \to X$ is said to be a C-contraction if there exists $\alpha \in (0,1/2)$ such that the following inequality holds:

$$d(Tx,Ty) \le \alpha\left(d(x,Ty) + d(y,Tx)\right) \quad \forall x,y \in X. \tag{1}$$

Moreover, Chatterjea [27] reported that every C-contraction $T : X \to X$ has a unique fixed point, where (X,d) is a complete metric space. Recently, Choudhury [28] introduced a generalization of C-contraction inspired by the notion of weak ϕ-contraction (see, e.g., [29, 30]).

Definition 2. Suppose that (X,d) is an MS. A self-mapping T on X is called a weakly C-contractive if

$$\begin{aligned} d(Tx,Ty) \le{}& \frac{1}{2}\left[d(x,Ty) + d(y,Tx)\right] \\ &- \varphi\left(d(x,Ty), d(y,Tx)\right), \end{aligned} \tag{2}$$

for all $x,y \in X$, where the mapping $\varphi : [0,+\infty)^2 \to [0,+\infty)$ is continuous and has the following property:

$$\varphi(x,y) = 0 \quad \text{iff } x = y = 0. \tag{3}$$

The notion of weakly C-contractive can be also called a weak C-contraction. In [28], the author proves that on

the setting of CMS, every weak C-contraction possesses a unique fixed point.

On other hand, in 2003, Kirk et al. [31] introduce cyclic contraction and give a characterization of the celebrated fixed-point theorem of Banach (known also as the Banach contraction mapping principle) in the set-up cyclic contraction. The authors [31] introduced the notion of cyclic representation in the following way.

Definition 3 (see [31]). Suppose that (X, d) is an MS and T is a self-mapping on X. Let m be a natural number and let X_i, $i = 1, \ldots, m$ be nonempty sets. Then, $X = \bigcup_{i=1}^{m} X_i$ is called a cyclic representation of X with respect to T if

$$X_1 \subset X_2, \ldots, T(X_{m-1}) \subset X_m, \quad T(X_m) \subset X_1, \tag{4}$$

where $X_{m+1} = X_1$.

Kirk et al. [31] prove that a self-mapping T, on a cyclic representation of X, possesses a fixed point if

$$d(Tx, Ty) \leq \varphi(d(x, y)), \quad \forall x \in X_i,\ y \in X_{i+1}, \quad i \in \{1, 2, \ldots, m\}, \tag{5}$$

where (X, d) is a CMS and $\varphi : [0, 1) \to [0, 1)$ is a function, upper semicontinuous from the right and $0 \leq \varphi(t) < t$ for $t > 0$.

Recently, Păcurar and Rus [32] generalize the result of Kirk et al. [31] via the notion of cyclic ϕ-contraction. Following the paper of Păcurar and Rus [32], the notion of cyclic weak-ϕ-contraction was introduced by Karapınar [33]. Let Λ be the collection of function $\varphi : [0, 1) \to [0, 1)$ which is nondecreasing, continuous together with the property $\varphi(t) > 0$ for $t \in (0, 1)$ and $\varphi(0) = 0$.

Definition 4 (see [33]). Suppose that (X, d) is an MS and T is a self-mapping on X. Let m be a natural number and let X_i, $i = 1, \ldots, m$, be nonempty closed sets. Assume that $X = \bigcup_{i=1}^{m} X_i$ is a cyclic representation of X with respect to T. A mapping $T : X \to X$ is said to be a cyclic weaker φ-contraction if there exists $\varphi \in \Lambda$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \tag{6}$$

for any $x \in X_i$, $y \in X_{i+1}$, $i = 1, 2, \ldots, m$, where $X_{m+1} = X_1$.

The author [33] shows that a self-mapping T, on a cyclic representation of X, possesses a fixed point if T is a cyclic weaker φ-contraction on a CMS (X, d).

In the last decade, the existence and uniqueness of a fixed point of various cyclic contractions in the context of PMS have been investigated and improved by several authors, see, for example, [7, 11].

In this paper, we derive some fixed-point result on certain cyclic contractions in the setup of complete PMS. Presented results of the paper extend, improve, and generalize some recent results on the topic in the literature. Among them, we list a few of them as follows: [7, 11, 13, 17, 28, 34].

For the sake of completeness, we call up some basic definitions and essential results in PMS. For more details, see, for example, [1, 7, 8, 17, 22].

Definition 5. Let X be a nonempty set. A function $p : X \times X \to [0, \infty)$ is called partial metric if the following conditions hold:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$,

for all $x, y, z \in X$. A pair (X, p) is called partial metric space.

It is evident that if $p(x, y) = 0$, then $x = y$ due to assumptions (p_1) and (p_2). However, if $x = y$, then $p(x, y)$ need not be 0. It is also known that a PMS generates a topology which is T_0. We say that a sequence $\{x_n\}$ is convergent to a point $x \in X$ in (X, p) if $\lim_{n\to\infty} p(x, x_n) = p(x, x)$, denoted as $x_n \to x$ $(n \to \infty)$ or $\lim_{n\to\infty} x_n = x$, with respect to the corresponding topology. We underline the simple fact that a limit of a sequence in a PMS need not be unique. Notice also that the function $p(\cdot, \cdot)$ need not be continuous; that is, $x_n \to x$ and $y_n \to y$ need not yield $p(x_n, y_n) \to p(x, y)$.

There is strong correlation between partial metric and metric. For example, a mapping $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{7}$$

forms a metric on X, where p is a partial metric. It is called the corresponding metric of partial metric.

Example 6. Let $X = [0, \infty)$. The pair (X, p) is an elementary example of a PMS, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Notice that the corresponding metric is

$$d_p(x, y) = 2\max\{x, y\} - x - y = |x - y|. \tag{8}$$

Example 7. The mapping $p(x, y) = \rho(x, y) + c$ forms a partial metric on X. Note also that the corresponding metric is $d_p(x, y) = 2d(x, y)$, where (X, ρ) is a metric space and $c \geq 0$ is arbitrary.

For the further nontrivial examples of PMS, they can be found in [1–6].

Definition 8. Let (X, p) be a PMS. Then

(1) a sequence $\{x_n\}$ is called a Cauchy if the limit of $p(x_n, x_m)$ as $n, m \to \infty$ exists (and is finite). If every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$, then the space (X, p) is called complete,

(2) let $\{x_n\}$ be a sequence in (X, p). If $\lim_{n,m\to\infty} p(x_n, x_m) = 0$, then the sequence $\{x_n\}$ is called 0-Cauchy. Analogously, if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x, x) = 0$, then the space (X, p) is called 0-complete and denoted by 0-CPMS [22].

This lemma can be found in some recent publication on the topic, see, for example, [2–6].

Lemma 9. *Let (X, p) be a PMS. Then*

(a) *a sequence $\{x_n\}$ is Cauchy in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p),*

(b) *a PMS (X, p) is complete if and only if the metric space (X, d_p) is complete. Furthermore, $\lim_{n\to+\infty} d_p(x_n, x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to+\infty} p(x_n, x) = \lim_{n,m\to+\infty} p(x_n, x_m). \quad (9)$$

Lemma 10. *Let (X, p) be a PMS.*

(a) *If $p(x_n, z) \to p(z, z) = 0$ as $n \to \infty$, then $p(x_n, y) \to p(z, y)$ as $n \to \infty$ for each $y \in X$ [8, 9, 18].*

(b) *If (X, p) is complete, then it is 0-complete [22].*

The converse assertion of (b) does not hold; for the counter examples, see [22]. Note that every closed subset of a 0-CPMS is 0-complete.

Let Φ be the class of functions $\varphi : [0,\infty)^3 \to [0,\infty)$ which is lower semicontinuous and satisfying $\varphi(x, y, z) = 0 \Leftrightarrow x = y = z = 0$.

In what follows we introduce the notion of a cyclic generalized weakly C-contraction in PMS.

Definition 11. Assume that (X, p) is a PMS and m is a natural number. Suppose that $X_1, X_2, \ldots, X_m$ are closed nonempty subsets of (X, d_p) and $Y = \bigcup_{i=1}^m X_i$ is a cyclic representation of Y with respect to T; a mapping $T : Y \to Y$ is said to be a cyclic generalized weakly C-contraction if

$$\begin{aligned} p(Tx, Ty) \le \frac{1}{4}\big[&p(x, Tx) + p(y, Ty) + p(x, Ty) \\ &+ p(y, Tx)\big] \\ &- \varphi\Big(p(x, Tx), p(y, Ty) \\ &\frac{1}{2}\big[p(x, Ty) + p(y, Tx)\big]\Big), \end{aligned} \quad (10)$$

for any $x \in X_i$, $y \in X_{i+1}$, $i = 1, 2, \ldots, m$, where $X_{m+1} = X_1$ and $\varphi \in \Phi$.

In this paper, we establish a fixed point theorem for cyclic generalized weakly C-contractions in the frame of CMPS.

2. Main Results

We present the fundamental result of this paper as follows.

Theorem 12. *Assume that (X, p) is a 0-CPMS and $T : Y \to Y$ is a cyclic generalized weakly C-contraction. Then, the mapping T has a unique fixed point $z \in \bigcap_{i=1}^n X_i$, and $p(z, z) = 0$.*

Proof. Take $x_0 \in Y$; that is, there is some i_0 with $x_0 \in X_{i_0}$. Since $T(X_{i_0}) \subseteq X_{i_0+1}$ implies that $Tx_0 \in X_{i_0+1}$, we find $x_1 \in X_{i_0+1}$ such that $Tx_0 = x_1$. By using the same argument, we construct the sequence $x_{n+1} = Tx_n$, where $x_n \in X_{i_n}$. Consequently, for $n \ge 0$, there exists $i_n \in \{1, 2, \ldots, m\}$ such that $x_n \in X_{i_n}$ and $x_{n+1} \in X_{i_n+1}$. We suppose that $x_n \ne x_{n+1}$ for all n. Indeed, if $x_{n_0} = x_{n_0+1}$ for some n_0, then we conclude that $Tx_{n_0} = x_{n_0}$; that is, x_{n_0} is the desired fixed point of T. Consequently, the proof is completed.

Due to (10), we derive that

$$p(x_n, x_{n+1}) = p(Tx_{n-1}, Tx_n) \quad (11)$$

$$\begin{aligned} \le \frac{1}{4}\big[&p(x_{n-1}, Tx_{n-1}) + p(x_n, Tx_n) + p(x_{n-1}, Tx_n) \\ &+ p(x_n, Tx_{n-1})\big] \\ &- \varphi\Big(p(x_{n-1}, Tx_{n-1}), p(x_n, Tx_n), \\ &\frac{1}{2}\big[p(x_{n-1}, Tx_n) + p(x_n, Tx_{n-1})\big]\Big) \end{aligned} \quad (12)$$

$$\begin{aligned} = \frac{1}{4}\big[&p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + p(x_{n-1}, x_{n+1}) \\ &+ p(x_n, x_n)\big] \\ &- \varphi\Big(p(x_{n-1}, x_n), p(x_n, x_{n+1}), \frac{1}{2}\big[p(x_{n-1}, x_{n+1}) \\ &+ p(x_n, x_n)\big]\Big) \end{aligned} \quad (13)$$

$$\begin{aligned} \le \frac{1}{4}\big[&p(x_{n-1}, x_n) + p(x_n, x_{n+1}) + p(x_{n-1}, x_{n+1}) \\ &+ p(x_n, x_n)\big] \end{aligned} \quad (14)$$

$$\le \frac{1}{2}\big[p(x_{n-1}, x_n) + p(x_n, x_{n+1})\big] \text{ by } (p_4), \quad (15)$$

for all $n \ge 1$. As a result, we find that

$$p(x_n, x_{n+1}) \le p(x_{n-1}, x_n), \quad (16)$$

for all $n \ge 1$. We set $t_n = p(x_n, x_{n-1})$. On the occasion of the facts above, $\{t_n\}$ is a nonincreasing sequence of nonnegative real numbers. Consequently, there exists $L \ge 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = L. \quad (17)$$

We will prove that $L = 0$. Suppose, to the contrary, that $L > 0$. From (14) and (15) we derive that

$$\begin{aligned} p(x_n, x_{n+1}) &\le \frac{1}{4}\big[p(x_{n-1}, x_n) + p(x_n, x_{n+1}) \\ &\quad + p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\big] \\ &\le \frac{1}{2}\big[p(x_{n-1}, x_n) + p(x_n, x_{n+1})\big], \end{aligned} \quad (18)$$

for any $n \in \mathbb{N}^*$. Letting $n \to \infty$ in (18), we have

$$\begin{aligned} L &= \lim_{n\to\infty} p(x_{n+1}, x_n) \\ &\le \lim_{n\to\infty} \frac{1}{4}\big[2L + p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\big] \le L. \end{aligned} \quad (19)$$

This yields that

$$\lim_{n\to\infty} p(x_n, x_n) + p(x_{n-1}, x_{n+1}) = 2L. \tag{20}$$

On the other hand, by (13) we have

$$\begin{aligned} t_{n+1} &\le \frac{1}{4}\left[t_n + t_{n+1} + p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\right] \\ &\quad - \varphi\left(t_n, t_{n+1}, \frac{1}{2}\left[p(x_{n-1}, x_{n+1}) + p(x_n, x_n)\right]\right) \\ &\le \frac{1}{2}\left[t_n + t_{n+1}\right]. \end{aligned} \tag{21}$$

Letting $n \to \infty$ in inequality (21), we get that

$$L \le L - \varphi(L, L, L) \le L. \tag{22}$$

Since $\varphi(x, y, z) = 0 \Leftrightarrow x = y = z = 0$, we get $L = 0$.

Due to (p_2), we have $0 \le p(x_n, x_n) \le p(x_n, x_{n+1})$. Hence, $\lim_{n\to\infty} p(x_n, x_n) = 0$. Then, by (20) we conclude that $\lim_{n\to\infty} p(x_{n-1}, x_{n+1}) = 0$.

Hence, we have

$$\begin{aligned} \lim_{n\to\infty} p(x_n, x_{n+1}) &= \lim_{n\to\infty} p(x_n, x_n) \\ &= \lim_{n\to\infty} p(x_{n-1}, x_{n+1}) = 0. \end{aligned} \tag{23}$$

We assert that the sequence $\{x_n\}$ is Cauchy. To reach this goal, the standard techniques in the literature will be used (see, e.g., [17]). For the sake of completeness, we explicitly prove that $\{x_n\}$ is Cauchy. First assert that

(K) for each $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that if $r, q \ge n$ with $r - q \equiv 1(m)$, then $p(x_r, x_q) < \varepsilon$.

Suppose, to the contrary, that there is $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ if $r_n > q_n \ge n$ with $r_n - q_n \equiv 1(m)$, then

$$p(x_{q_n}, x_{r_n}) \ge \varepsilon. \tag{24}$$

We examine the case $n > 2m$. So, taking $q_n \ge n$ into account, we can choose r_n with $r_n > q_n$ in a way that it is the smallest integer satisfying $r_n - q_n \equiv 1(m)$ and $p(x_{q_n}, x_{r_n}) \ge \varepsilon$. Hence, $p(x_{q_n}, x_{r_n - m}) \le \varepsilon$, by using the triangular inequality

$$\begin{aligned} \varepsilon &\le p(x_{q_n}, x_{r_n}) \\ &\le p(x_{q_n}, x_{r_n - m}) + \sum_{i=1}^{m} p(x_{r_n - i}, x_{r_{n-i+1}}) - \sum_{i=1}^{m} p(x_{r_n - i}, x_{r_n - i}) \\ &< \varepsilon + \sum_{i=1}^{m} p(x_{r_n - i}, x_{r_{n-i+1}}). \end{aligned} \tag{25}$$

Letting $n \to \infty$ in (25) and keeping in mind $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we obtain

$$\lim_{n\to\infty} p(x_{q_n}, x_{r_n}) = \varepsilon. \tag{26}$$

Again, by (p_4)

$$\begin{aligned} \varepsilon &\le p(x_{q_n}, x_{r_n}) \\ &\le p(x_{q_n}, x_{q_n+1}) + p(x_{q_n+1}, x_{r_n+1}) + p(x_{r_n+1}, x_{r_n}) \\ &\quad - p(x_{q_n+1}, x_{q_n+1}) - p(x_{r_n+1}, x_{r_n+1}) \\ &\le p(x_{q_n}, x_{q_n+1}) + p(x_{q_n+1}, x_{q_n}) + p(x_{q_n}, x_{r_n}) \\ &\quad + p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{r_n}) \\ &\quad - p(x_{q_n}, x_{q_n}) - p(x_{r_n}, x_{r_n}) \\ &\le 2p(x_{q_n}, x_{q_n+1}) + p(x_{q_n}, x_{r_n}) + 2p(x_{r_n}, x_{r_n+1}). \end{aligned} \tag{27}$$

Taking (23) and (26) into account, we get

$$\lim_{n\to\infty} p(x_{q_n+1}, x_{r_n+1}) = \varepsilon, \tag{28}$$

as $n \to \infty$ in (26).

By (p_4) we have the following inequalities:

$$\begin{aligned} p(x_{q_n}, x_{r_n+1}) &\le p(x_{q_n}, x_{r_n}) + p(x_{r_n}, x_{r_n+1}) \\ &\quad - p(x_{r_n}, x_{r_n}) \\ &\le p(x_{q_n}, x_{r_n}) + p(x_{r_n}, x_{r_n+1}), \\ p(x_{q_n}, x_{r_n}) &\le p(x_{q_n}, x_{r_n+1}) + p(x_{r_n}, x_{r_n+1}) \\ &\quad - p(x_{r_n+1}, x_{r_n+1}) \\ &\le p(x_{q_n}, x_{r_n+1}) + p(x_{r_n}, x_{r_n+1}). \end{aligned} \tag{29}$$

Letting $n \to \infty$ in (29) we derived that

$$\lim_{n\to\infty} p(x_{q_n}, x_{r_n+1}) = \varepsilon. \tag{30}$$

Again by (p_4) we have

$$\begin{aligned} p(x_{r_n}, x_{q_n+1}) &\le p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{q_n+1}) \\ &\quad - p(x_{r_n+1}, x_{r_n+1}) \\ &\le p(x_{r_n}, x_{r_n+1}) + p(x_{r_n+1}, x_{q_n+1}), \\ p(x_{r_n+1}, x_{q_n+1}) &\le p(x_{r_n+1}, x_{r_n}) + p(x_{r_n}, x_{q_n+1}) - p(x_{r_n}, x_{r_n}) \\ &\le p(x_{r_n+1}, x_{r_n}) + p(x_{r_n}, x_{q_n+1}). \end{aligned} \tag{31}$$

Letting $n \to \infty$ in (31) we derived that

$$p(x_{r_n}, x_{q_n+1}) = \varepsilon. \tag{32}$$

Since x_{q_n} and x_{r_n} lie in distinct adjacently labeled sets X_i and X_{i+1} for certain $1 \le i \le m$, keeping in mind that T is a cyclic generalized weakly C-contraction, we have

$$\begin{aligned} p\left(x_{q_n+1}, x_{r_n+1}\right) &= p\left(Tx_{q_n}, Tx_{r_n}\right) \\ &\le \frac{1}{4}\left[p\left(x_{q_n}, Tx_{q_n}\right) + p\left(x_{r_n}, Tx_{r_n}\right)\right. \\ &\quad \left. + p\left(x_{q_n}, Tx_{r_n}\right) + p\left(x_{r_n}, Tx_{q_n}\right)\right] \\ &\quad - \varphi\left(p\left(x_{q_n}, Tx_{q_n}\right), p\left(x_{r_n}, Tx_{r_n}\right),\right. \\ &\quad \left.\frac{1}{2}\left[p\left(x_{q_n}, Tx_{r_n}\right) + p\left(x_{r_n}, Tx_{q_n}\right)\right]\right) \\ &\le \frac{1}{4}\left[p\left(x_{q_n}, x_{q_n+1}\right) + p\left(x_{r_n}, x_{r_n+1}\right)\right. \\ &\quad \left. + p\left(x_{q_n}, x_{r_n+1}\right) + p\left(x_{r_n}, x_{q_n+1}\right)\right] \\ &\quad - \varphi\left(p\left(x_{q_n}, x_{q_n+1}\right), p\left(x_{r_n}, x_{r_n+1}\right),\right. \\ &\quad \left.\frac{1}{2}\left[p\left(x_{q_n}, x_{r_n+1}\right) + p\left(x_{r_n}, x_{q_n+1}\right)\right]\right). \end{aligned} \tag{33}$$

Taking into account (23), (26), (28), (30), (32), and the lower semicontinuity of φ, letting $n \to \infty$ in the inequality above, we find that

$$\varepsilon \le \frac{1}{4}[0 + 0 + \varepsilon + \varepsilon] - \varphi(0, 0, \varepsilon) \le \frac{1}{2}\varepsilon, \tag{34}$$

which is a contradiction. Hence, (K) holds.

We are ready to show that the sequence $\{x_n\}$ is Cauchy. Fix $\varepsilon > 0$. Due to the assumptions, one can find $n_0 \in \mathbb{N}$ such that if $r, q \ge n_0$ with $r - q \equiv 1(m)$,

$$p\left(x_r, x_q\right) \le \frac{\varepsilon}{2}. \tag{35}$$

Since $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we also find $n_1 \in \mathbb{N}$ such that

$$p\left(x_n, x_{n+1}\right) \le \frac{\varepsilon}{2m}, \tag{36}$$

for any $n \ge n_1$. Assume that $r, s \ge \max\{n_0, n_1\}$ and $s > r$. Consequently, there is a $k \in \{1, 2, \ldots, m\}$ with $s - r \equiv k(m)$. Therefore, $s - r + \alpha \equiv 1(m)$ for $\alpha = m - k + 1$. Thus, we obtain for $j \in \{1, \ldots, m\}$, $s + j - r \equiv 1(m)$

$$\begin{aligned} p\left(x_r, x_s\right) &\le p\left(x_r, x_{s+j}\right) + p\left(x_{s+j}, x_{s+j-1}\right) + \cdots + p\left(x_{s+1}, x_s\right) \\ &\quad - \left[p\left(x_{s+j}, x_{s+j}\right) + \cdots + p\left(x_{s+1}, x_{s+1}\right)\right] \\ &\le p\left(x_r, x_{s+j}\right) + p\left(x_{s+j}, x_{s+j-1}\right) + \cdots \\ &\quad + p\left(x_{s+1}, x_s\right). \end{aligned} \tag{37}$$

By (35) and (36) together with the last inequality, we find that

$$p\left(x_r, x_s\right) \le \frac{\varepsilon}{2} + j \times \frac{\varepsilon}{2m} \le \frac{\varepsilon}{2} + m \times \frac{\varepsilon}{2m} = \varepsilon, \tag{38}$$

which yields that the sequence $\{x_n\}$ is Cauchy. Regarding that ε is arbitrary, we conclude that $\{x_n\}$ is a 0-Cauchy sequence.

Taking into account that Y is closed in (X, p), we observe that (Y, p) is also 0-complete. Thus, there exists $x \in Y = \bigcup_{i=1}^{m} X_i$ such that $\lim_{n\to\infty} x_n = x$ in (Y, p); equivalently

$$p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = 0. \tag{39}$$

Now, we assert that x is a fixed point of T. First, we observed that the sequence (x_n) has infinite terms in each X_i for $i \in \{1, 2, \ldots, m\}$, since $\lim_{n\to\infty} x_n = x$ and as $Y = \bigcup_{i=1}^{m} X_i$ is a cyclic representation of Y with respect to T. Assume that $x \in X_i$ and $Tx \in X_{i+1}$. We consider a subsequence x_{n_k} of (x_n) with $x_{n_k} \in X_{i-1}$. Notice that such subsequence exists due to the above-mentioned comment. By applying the contractive condition, we find

$$\begin{aligned} p\left(x_{n_{k+1}}, Tx\right) &= p\left(Tx_{n_k}, Tx\right) \\ &\le \frac{1}{4}\left[p\left(x_{n_k}, Tx_{n_k}\right) + p(x, Tx)\right. \\ &\quad \left. + p\left(x_{n_k}, Tx\right) + p\left(x, Tx_{n_k}\right)\right] \\ &\quad - \varphi\left(p\left(x_{n_k}, Tx_{n_k}\right), p(x, Tx),\right. \\ &\quad \left.\frac{1}{2}\left[p\left(x_{n_k}, Tx\right) + p\left(x, Tx_{n_k}\right)\right]\right) \\ &= \frac{1}{4}\left[p\left(x_{n_k}, x_{n_k+1}\right) + p(x, Tx)\right. \\ &\quad \left. + p\left(x_{n_k}, Tx\right) + p\left(x, x_{n_k+1}\right)\right] \\ &\quad - \varphi\left(p\left(x_{n_k}, x_{n_k+1}\right), p(x, Tx),\right. \\ &\quad \left.\frac{1}{2}\left[p\left(x_{n_k}, Tx\right) + p\left(x, x_{n_k+1}\right)\right]\right). \end{aligned} \tag{40}$$

Letting $n \to \infty$ and by using $x_{n_k} \to x$, together with the lower semicontinuity of φ, we get

$$\begin{aligned} p(x, Tx) &\le \frac{1}{2}p(x, Tx) - \varphi\left(0, p(x, Tx), \frac{1}{2}p(x, Tx)\right) \\ &\le \frac{1}{2}p(x, Tx). \end{aligned} \tag{41}$$

So $p(x, Tx) = 0$ which yields that $Tx = x$. We will prove the uniqueness of x to complete the proof. Suppose, on the contrary, that $y, z \in X$ are distinct fixed points of T. We observe that $y, z \in \bigcap_{i=1}^{m} X_i$, since T is cyclic mapping and

$y, z \in X$ are fixed points of T. Due to mentioned contractive condition, we derive that

$$\begin{aligned} p(y,z) &= p(Ty,Tx) \\ &\le \frac{1}{4}\left[p(y,Ty)+p(z,Tz)+p(y,Tz)+p(z,Ty)\right] \\ &\quad -\varphi\Big(p(y,Ty), p(z,Tz), \\ &\qquad \frac{1}{2}\left[p(y,Tz)+p(z,Ty)\right]\Big), \end{aligned} \tag{42}$$

that is,

$$\begin{aligned} p(y,z) &\le \frac{1}{2}p(y,z) - \varphi\left(0,0,\frac{1}{2}\left[p(y,z)+p(z,y)\right]\right) \\ &\le \frac{1}{2}p(y,z). \end{aligned} \tag{43}$$

This gives us $p(y,z) = 0$; that is, $y = z$. □

Corollary 13. *Suppose that (X,p) is a 0-CPMS, $m \in \mathbb{N}$, $X_1, X_2, \ldots, X_m$ are nonempty closed subsets of X. Let $T : Y \to Y$ be and let $Y = \bigcup_{i=1}^{m} X_i$. $Y = \bigcup_{i=1}^{m} X_i$ be a cyclic representation of X with respect to T.*

If there exists $\beta \in [0, 1/4)$ such that

$$\begin{aligned} p(Tx,Ty) \le \beta\big[&p(x,Tx)+p(y,Ty) \\ &+p(x,Ty)+p(y,Tx)\big], \end{aligned} \tag{44}$$

for any $x \in X_i$, $y \in X_{i+1}$, $i = 1, 2, \ldots, m$, where $X_{m+1} = X_1$, then, T has a fixed point $z \in \bigcap_{i=1}^{n} X_i$ and $p(z,z) = 0$.

Proof. Let $\beta \in [0, 1/4)$. Hence, it suffices to take the function $\varphi : [0,+\infty)^3 \to [0,+\infty)$ defined by $\varphi(a,b,c) = (1/4-\beta)(a+b+2c)$. It is evident that φ satisfies the following conditions:

(1) $\varphi(a,b,c) = 0$ if and only if $a = b = c = 0$, and

(2) φ is lower semi-continuous.

The results follow when we apply Theorem 12. □

Theorem 14. *Suppose that (X,d) is a 0-CPMS. If the mapping $T : X \to X$ satisfies*

$$\begin{aligned} p(Tx,Ty) \le \frac{1}{4}\big[&p(x,Tx)+p(y,Ty) \\ &+p(x,Ty)+p(y,Tx)\big] \\ -\varphi\Big(&p(x,Tx), p(x,Ty), \\ &\frac{1}{2}\left[p(x,Ty)+p(y,Tx)\right]\Big), \end{aligned} \tag{45}$$

for any $x, y \in X$, where $\phi \in \Phi$, then, it has a unique fixed point $z \in X$ with $p(z,z) = 0$.

Proof. It is sufficient to take $X_i = X$ for $i = 1, \ldots, m$ in Theorem 12. □

Remark 15. Let us remark that if in Definition 11 we consider the following condition

$$\begin{aligned} p(Tx,Ty) \le \max\Big\{&p(x,x), p(y,y), \\ &\frac{1}{4}\big[p(x,Tx)+p(y,Ty)+p(x,Ty) \\ &+p(y,Tx)\big]\Big\} \\ -\varphi\Big(&p(x,Tx), p(x,Ty), \\ &\frac{1}{2}\left[p(x,Ty)+p(y,Tx)\right]\Big), \end{aligned} \tag{46}$$

instead of (10), then by following the lines in the proof of Theorem 12, we obtain the same conclusions in our results.

Example 16. Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$. It is clear that (X,p) is a 0-complete partial metric space. Fix $m \in \mathbb{N}$ and define $Y = \cup_{i=1}^{m} X_i$, where $X_i = [0,(1/2^i)]$ for $i = 1, 2, \ldots, m$. Let $T : X \to X$ and let $\varphi : [0,\infty)^3 \to [0,\infty)^3$ be defined as

$$\varphi(p,q,r) = \begin{cases} 0 & \text{if } p = q = r = 0, \\ \dfrac{(p+q+r)}{36} & \text{otherwise,} \end{cases} \tag{47}$$

respectively. Then, all the conditions of Theorem 12 are satisfied. Hence, T has a unique fixed point, namely, 0.

Remark 17. Notice that we get the same results if we replace 0-CPMS with CPMS.

Acknowledgment

Vladimir Rakocević is supported by Grant no. 174025 of the Ministry of Science, Technology, and Development, Serbia.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 101439, 6 pages
http://dx.doi.org/10.1155/2013/101439

Research Article

Existence and Convergence Theorems of Best Proximity Points

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Received 22 February 2013; Accepted 16 April 2013

Academic Editor: Gue Myung Lee

The aim of this paper is to prove some best proximity point theorems for new classes of cyclic mappings, called pointwise cyclic orbital contractions and asymptotic pointwise cyclic orbital contractions. We also prove a convergence theorem of best proximity point for relatively nonexpansive mappings in uniformly convex Banach spaces.

1. Introduction and Preliminaries

Let (X, d) be a metric space, and let A, B be subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. In 2003, Kirk et al. [1] proved the following generalization of Banach contraction principle.

Theorem 1 (see [1]). *Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that T is a cyclic mapping such that*

$$d(Tx, Ty) \leq \alpha d(x, y), \tag{1}$$

for some $\alpha \in (0, 1)$ and for all $x \in A$, $y \in B$. Then T has a unique fixed point in $A \cap B$.

In [2] Eldred and Veeramani introduced the class of *cyclic contractions* as follows.

Definition 2 (see [2]). Let A and B be nonempty subsets of a metric space X. A mapping $T : A \cup B \to A \cup B$ is said to be a cyclic contraction if T is cyclic and

$$d(Tx, Ty) \leq \alpha d(x, y) + (1 - \alpha) \operatorname{dist}(A, B), \tag{2}$$

for some $\alpha \in (0, 1)$ and for all $x \in A$, $y \in B$.

Let T be a cyclic mapping. A point $x \in A \cup B$ is said to be a *best proximity point* for T provided that $d(x, Tx) = \operatorname{dist}(A, B)$, where

$$\operatorname{dist}(A, B) := \inf \{d(x, y) : x \in A, y \in B\}. \tag{3}$$

Note that if $\operatorname{dist}(A, B) = 0$, then the best proximity point is nothing but a fixed point of T.

The next theorem ensures existence, uniqueness, and convergence of best proximity point for cyclic contractions in uniformly convex Banach spaces.

Theorem 3 (see [2]). *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a cyclic contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$ and $\|x - Tx\| = \operatorname{dist}(A, B)$.*

Recently, Suzuki et al. in [3] introduced the notion of property UC which is a kind of geometric property for subsets of a metric space X.

Definition 4 (see [3]). Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy property UC if the following holds.

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_n d(x_n, y_n) = \operatorname{dist}(A, B)$ and $\lim_n d(z_n, y_n) = \operatorname{dist}(A, B)$, then we have $\lim_n d(x_n, z_n) = 0$.

We mention that if A and B are nonempty subsets of a uniformly convex Banach space X such that A is convex, then (A, B) satisfies the property UC. Other examples of pairs having the property UC can be found in [3]. Here, we state the following two lemmas of [3].

Lemma 5 (see [3]). *Let A and B be nonempty subsets of a metric space (X, d). Assume that (A, B) satisfies the property UC. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either of the following holds:*

$$\begin{gathered}\lim_{m\to\infty}\sup_{n\geq m} d(x_m, y_n) = d(A, B)\\ \text{or}\quad \lim_{n\to\infty}\sup_{m\geq n} d(x_m, y_n) = d(A, B).\end{gathered} \tag{4}$$

Then $\{x_n\}$ is a Cauchy sequence.

Lemma 6 (see [3]). *Let (X, d) be a metric space and let A and B be nonempty subsets of X such that (A, B) satisfies the property UC. Let $T : A\cup B \to A\cup B$ be a cyclic map such that*

$$\begin{gathered}d\left(T^2x, Tx\right) \leq d(x, Tx) \quad \forall x \in A,\\ d\left(T^2x, Tx\right) < d(x, Tx) \quad \forall x \in A\\ \text{with } \operatorname{dist}(A, B) < d(x, Tx).\end{gathered} \tag{5}$$

For a point $z \in A$, the following are equivalent:

(i) *z is a best proximity point of T;*

(ii) *z is a fixed point of T^2.*

Throughout this paper, (A, B) stands for a nonempty pair in a metric space (X, d). When we say that a pair (A, B) satisfies a specific property, we mean that both A and B satisfy the mentioned property. Also, we define $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C$ and $B \subseteq D$. Moreover, we use the following notations:

$$\begin{gathered}\delta_x(A) = \sup\{d(x, y) : y \in A\} \quad \forall x \in X,\\ \delta(A, B) = \sup\{d(x, y) : x \in A, y \in B\},\\ \operatorname{diam}(A) = \delta(A, A).\end{gathered} \tag{6}$$

For a cyclic mapping $T : A\cup B \to A\cup B$ and $x \in A\cup B$, we define the orbit setting at x by

$$\mathcal{O}_{T^2}x := \left\{x, T^2x, T^4x, \ldots, T^{2n}x, \ldots\right\}, \tag{7}$$

where $T^{2n}x = T(T^{2n-1}x)$ for $n \geq 1$ and $T^0x = x$. We set

$$\mathcal{O}_{T^2}(x, y) := \mathcal{O}_{T^2}(x) \cup \mathcal{O}_{T^2}(y), \tag{8}$$

for all $x, y \in A\cup B$. Note that if $(x, y) \in A\times B$, then $\mathcal{O}_{T^2}x \subseteq A$ and $\mathcal{O}_{T^2}y \subseteq B$. Also, the set of all best proximity points of the mapping T in A will be denoted by B.P.P$(T) \cap A$.

We mention that a mapping $T : A \cup B \to A \cup B$ is said to be *relatively nonexpansive* provided that T is cyclic and satisfies the condition $\|Tx - Ty\| \leq \|x - y\|$ for each $(x, y) \in A \times B$. Note that a relatively nonexpansive mapping need not be a continuous mapping. Also every nonexpansive self-map can be considered as a relatively nonexpansive mapping.

In 2005 Eldred et al. in [4] introduced a geometric concept called *proximal normal structure*. Using this notion they proved that if (A, B) is a nonempty weakly compact convex pair in a Banach space X and $T : A \cup B \to A \cup B$ is a relatively nonexpansive mapping, then there exists $(x, y) \in A \times B$ such that $\| x - Tx \| = \| Ty - y \| = \operatorname{dist}(A, B)$. For more details on this subject, we refer the reader to [5–10].

2. Pointwise Cyclic Orbital Contractions

In [11], the notion of *pointwise cyclic contractions* was introduced as follows.

Definition 7 (see [11]). Let (A, B) be a pair of subsets of a metric space (X, d). Let $T : A \cup B \to A \cup B$ be a cyclic mapping. T is said to be a pointwise cyclic contraction if for each $(x, y) \in A \times B$ there exist $0 \leq \alpha(x) < 1, 0 \leq \alpha(y) < 1$ such that

$$\begin{gathered}d(Tx, Ty) \leq \alpha(x)\, d(x, y) + (1 - \alpha(x)) \operatorname{dist}(A, B)\\ \forall y \in B,\\ d(Tx, Ty) \leq \alpha(y)\, d(x, y) + (1 - \alpha(y)) \operatorname{dist}(A, B)\\ \forall x \in A.\end{gathered} \tag{9}$$

The following result was proved in [11].

Theorem 8 (see [11]). *Let (A, B) be a nonempty weakly compact convex pair in a Banach space X and suppose that T is a pointwise cyclic contraction mapping. Then there exists $(x, y) \in A \times B$ such that $\|x - Tx\| = \|y - Ty\| = \operatorname{dist}(A, B)$.*

In this section, we introduce a new class of cyclic mappings, called *pointwise cyclic orbital contractions*, which contains the pointwise cyclic contractions as a subclass. For such mappings, we study the existence of best proximity points in Banach spaces.

Definition 9. Let (A, B) be a pair of subsets of a metric space (X, d). A cyclic mapping $T : A \cup B \to A \cup B$ is said to be a pointwise cyclic orbital contraction if there exists $\alpha : A \cup B \to [0, 1)$ such that for each $(x, y) \in A \times B$

$$\begin{gathered}d(Tx, Ty) \leq \alpha(x)\, \delta_x(\mathcal{O}_{T^2}y) + (1 - \alpha(x)) \operatorname{dist}(A, B)\\ \forall y \in B,\end{gathered} \tag{10}$$

$$\begin{gathered}d(Tx, Ty) \leq \alpha(y)\, \delta_y(\mathcal{O}_{T^2}x) + (1 - \alpha(y)) \operatorname{dist}(A, B)\\ \forall x \in A.\end{gathered} \tag{11}$$

It is clear that the class of pointwise cyclic orbital contractions contains the class of pointwise cyclic contractions as a subclass. The following example shows that the converse need

not be true. Moreover, it is interesting to note that a pointwise cyclic orbital contraction may not be relatively nonexpansive.

Example 10. Let $X := \mathbb{R}$ with the usual metric. For $A = B = [0, 1/2]$, define $T : A \cup B \to A \cup B$ by

$$Tx = \begin{cases} \frac{1}{8}x & \text{if } 0 \le x \le \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} < x \le \frac{1}{2}. \end{cases} \tag{12}$$

Then T is pointwise cyclic orbital contraction with $\alpha(x) = 7/8$ for all $x \in A$.

Proof. If either $0 \le x, y \le 1/4$ or $1/4 < x, y \le 1/2$, then it is easy to see that relations (10) and (11) hold. Suppose that $0 \le x \le 1/4$ and $1/4 < y \le 1/2$. Thus,

$$\begin{gathered} d(Tx, Ty) = \frac{1}{8}x, \\ \delta_x(\mathcal{O}_{T^2} y) = \sup_{n\ge 0} \left| x - T^{2n} y \right| = \max\{x, y - x\}. \end{gathered} \tag{13}$$

Hence,

$$d(Tx, Ty) = \frac{1}{8}x \le \frac{7}{8}\max\{x, y - x\} = \alpha(x)\,\delta_x(\mathcal{O}_{T^2} y), \tag{14}$$

that is, (10) holds. Also, by the fact that $\delta_y(\mathcal{O}_{T^2} x) = \sup_{n\ge 0}|y - T^{2n}x| = y$ then

$$d(Tx, Ty) = \frac{1}{8}x \le \frac{7}{8}y = \alpha(y)\,\delta_x(\mathcal{O}_{T^2} x), \tag{15}$$

which implies that (10) and (11) hold. Thus, T is a pointwise cyclic orbital contraction. Now, we show that T is not pointwise cyclic contraction. Indeed, if there exists a function $\alpha : A \cup B \to [0, 1)$ such that $d(Tx, Ty) \le \alpha(x)d(x, y)$ for all $(x, y) \in A \times B$, then for $x = 1/4$ and $y = 26/100$ we must have

$$\frac{1}{8} \times \frac{25}{100} = d(Tx, Ty) \le \alpha(x)\,d(x, y) = \alpha\left(\frac{25}{100}\right) \times \frac{1}{100}, \tag{16}$$

and hence $25/8 \le \alpha(1/4)$, which is a contradiction. Therefore, T is not pointwise cyclic contraction. Moreover, we note that since T is not continuous, T is not relatively nonexpansive. □

Let us state our main result of this section.

Theorem 11. *Let (A, B) be a nonempty weakly compact convex pair in a Banach space X. If $T : A \cup B \to A \cup B$ is a pointwise cyclic orbital contraction, then the set of best proximity points of T is nonempty.*

Proof. Let Σ denote the collection of all nonempty weakly compact convex pairs (E, F) which are subsets of (A, B) and such that T is cyclic on $E \cup F$. Then Σ is nonempty, since $(A, B) \in \Sigma$. Σ is partially ordered by reverse inclusion; that is, $(A, B) \le (C, D) \Leftrightarrow (C, D) \subseteq (A, B)$. It is easy to check that every increasing chain in Σ is bounded above. Hence by Zorn's lemma we can get a minimal element say $(K_1, K_2) \in \Sigma$. We have

$$\left(\overline{co}(T(K_2)), \overline{co}(T(K_1))\right) \subseteq (K_1, K_2). \tag{17}$$

Moreover

$$T\left(\overline{co}(T(K_2))\right) \subseteq T(K_1) \subseteq \overline{co}(T(K_1)), \tag{18}$$

and also

$$T\left(\overline{co}(T(K_1))\right) \subseteq \overline{co}(T(K_2)). \tag{19}$$

Now, by the minimality of (K_1, K_2), we have $\overline{co}(T(K_2)) = K_1$, $\overline{co}(T(K_1)) = K_2$. Suppose that $a \in K_1$. Then for each $y \in K_2$ we have

$$\begin{aligned} \|Ta - Ty\| &\le \alpha(a)\,\delta_a(\mathcal{O}_{T^2} y) + (1 - \alpha(a))\,\mathrm{dist}(A, B) \\ &\le \alpha(a)\,\delta_a(K_2) + (1 - \alpha(a))\,\mathrm{dist}(A, B), \end{aligned} \tag{20}$$

which implies that $T(K_2) \subseteq \mathcal{B}(Ta; \alpha(a)\delta_a(K_2) + (1 - \alpha(a))\,\mathrm{dist}(A, B))$. Hence,

$$\begin{aligned} K_1 &= \overline{co}(T(K_2)) \\ &\subseteq \mathcal{B}\left(Ta; \alpha(a)\,\delta_a(K_2) + (1 - \alpha(a))\,\mathrm{dist}(A, B)\right). \end{aligned} \tag{21}$$

Thus, for each $x \in K_1$ we must have

$$\|x - Ta\| \le \alpha(a)\,\delta_a(K_2) + (1 - \alpha(a))\,\mathrm{dist}(A, B), \tag{22}$$

which ensures that

$$\delta_{Ta}(K_1) \le \alpha(a)\,\delta_a(K_2) + (1 - \alpha(a))\,\mathrm{dist}(A, B). \tag{23}$$

Similarly, we can see that if $b \in K_2$, then

$$\delta_{Tb}(K_2) \le \alpha(b)\,\delta_b(K_1) + (1 - \alpha(b))\,\mathrm{dist}(A, B). \tag{24}$$

Assume that (p, q) is a fixed element in $K_1 \times K_2$. Let $\delta_p(K_2) \le \delta_q(K_1)$. Set $r := \delta_p(K_2)$ and

$$\begin{gathered} E := \left\{y \in K_2 : \delta_y(K_1) \le r\right\}, \\ F := \left\{x \in K_1 : \delta_x(K_2) \le r\right\}. \end{gathered} \tag{25}$$

Obviously, $p \in F$. Also, from (23) $Tp \in E$ and then (E, F) is a nonempty pair. Besides, it is easy to see that

$$\begin{gathered} E := \bigcap_{a\in K_1} \mathcal{B}(a; r) \cap K_2, \\ F := \bigcap_{b\in K_2} \mathcal{B}(b; r) \cap K_1. \end{gathered} \tag{26}$$

Now, let $y \in E$. Then $y \in K_2$ and by (24), $\delta_{Ty}(K_2) \le \delta_y(K_1) \le r$ which implies that $Ty \in F$. Hence, $T(E) \subseteq F$. Similarly, by relation (23) we conclude that $T(F) \subseteq E$. That is,

T is cyclic on $E \cup F$. By the minimality of (K_1, K_2) we must have $F = K_1$ and $E = K_2$. Therefore,

$$\delta_x(K_2) \le r, \qquad \delta_y(K_1) \le r, \tag{27}$$

for each $(x, y) \in A \times B$. Then for all $(x, y) \in A \times B$ we have

$$\delta_x(K_2) \le \delta_q(K_1), \qquad \delta_y(K_1) \le \delta_p(K_2). \tag{28}$$

Particularly, $\delta_p(K_2) \le \delta_q(K_1) \le \delta_p(K_2)$. Thus,

$$\delta_p(K_2) = \delta_q(K_1). \tag{29}$$

Similar argument implies that if $\delta_q(K_1) \le \delta_p(K_2)$, then relation (29) is to be achieved. Therefore, (29) holds for all $(p, q) \in K_1 \times K_2$. To complete the proof of the theorem, we consider the following cases.

Case 1. If $\delta_p(K_2) = \operatorname{dist}(A, B)$, then we have

$$\|p - Tp\| \le \delta_p(K_2) = \operatorname{dist}(A, B), \tag{30}$$

that is, p is a best proximity point of T.

Case 2. If $\delta_p(K_2) > \operatorname{dist}(A, B)$, it now follows from (23) and (29) that

$$\begin{aligned} \delta_p(K_2) &= \delta_{Tp}(K_1) \\ &\le \alpha(p)\,\delta_p(K_2) + (1 - \alpha(p))\operatorname{dist}(A, B) \\ &< \delta_p(K_2), \end{aligned} \tag{31}$$

which is a contradiction. Hence, each point of K_1 is a best proximity point of T and so $K_1 \subseteq \text{B.P.P}(T) \cap A$. Similarly, we can see that $K_2 \subseteq \text{B.P.P}(T) \cap B$. Thus, for each $(x, y) \in K_1 \times K_2$ we must have

$$\|x - Tx\| = \|Ty - y\| = \operatorname{dist}(A, B). \tag{32}$$

□

3. Asymptotic Pointwise Cyclic Orbital Contractions

Definition 12. Let (A, B) be a pair of subsets of a metric space (X, d). A cyclic mapping $T : A \cup B \to A \cup B$ is said to be an asymptotic pointwise cyclic orbital contraction if for each $(x, y) \in A \times B$,

$$\begin{aligned} d\left(T^{2n}x, T^{2n}y\right) &\le \alpha_n(x)\operatorname{diam}\mathcal{O}_{T^2}(x, y) \\ &\quad + (1 - \alpha_n(x))\operatorname{dist}(A, B) \quad \forall y \in B, \\ d\left(T^{2n}x, T^{2n}y\right) &\le \alpha_n(y)\operatorname{diam}\mathcal{O}_{T^2}(x, y) \\ &\quad + (1 - \alpha_n(y))\operatorname{dist}(A, B) \quad \forall x \in A, \end{aligned} \tag{33}$$

where for each $n \in \mathbb{N}$, $\alpha_n : A \cup B \to \mathbb{R}^+$ and $\limsup_{n\to\infty}\alpha_n(x) \le \eta$ for some $0 < \eta < 1$ and for all $x \in A \cup B$.

The following theorem establishes existence and convergence of a best proximity point for asymptotic pointwise cyclic orbital contractions in metric spaces with the property UC.

Theorem 13. *Let (A, B) be a nonempty closed pair in a complete metric space (X, d) such that (A, B) satisfies the property UC. Assume that $T : A \cup B \to A \cup B$ is an asymptotic pointwise cyclic orbital contraction such that T is continuous on A. If there exists $x \in A$ such that the orbit of T at x is bounded, then T has a best proximity point in A. Moreover, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point of T.*

Proof. Let $x \in A$. We note that the sequence $\{\operatorname{diam}[\mathcal{O}_{T^2}(T^{2n}x, T^{2n+1}x)]\}$ is decreasing and bounded below by $\operatorname{dist}(A, B)$. Let $\operatorname{diam}[\mathcal{O}_{T^2}(T^{2n}x, T^{2n+1}x)] \to r_x \ge \operatorname{dist}(A, B)$. We claim that $r_x = \operatorname{dist}(A, B)$. For all $k_1, k_2 \in \mathbb{N}$ with $k_1 \le k_2$ we have

$$\begin{aligned} &d\left(T^{2(n+k_1)}x, T^{2(n+k_2)}(Tx)\right) \\ &\quad \le \alpha_{n+k_1}(x)\operatorname{diam}\left[\mathcal{O}_{T^2}(x, Tx)\right] \\ &\qquad + \left(1 - \alpha_{n+k_1}(x)\right)\operatorname{dist}(A, B). \end{aligned} \tag{34}$$

Taking the supremum with respect to k_1 and k_2 and then letting $n \to \infty$ we obtain

$$r_x \le \eta\operatorname{diam}\left[\mathcal{O}_{T^2}(x, Tx)\right] + (1 - \eta)\operatorname{dist}(A, B). \tag{35}$$

Besides, for each $m \in \mathbb{N}$ we have

$$\begin{aligned} r_x &= \lim_{n\to\infty}\operatorname{diam}\left[\mathcal{O}_{T^2}\left(T^{2n}\left(T^{2m}x\right), T^{2n}\left(T^{2m}(Tx)\right)\right)\right] \\ &\le \eta\operatorname{diam}\mathcal{O}_{T^2}\left(T^{2m}x, T^{2m}(Tx)\right) + (1 - \eta)\operatorname{dist}(A, B). \end{aligned} \tag{36}$$

Now, if $m \to \infty$ we obtain

$$r_x \le \eta r_x + (1 - \eta)\operatorname{dist}(A, B), \tag{37}$$

and hence $r_x = \operatorname{dist}(A, B)$. We now conclude that

$$\lim_{n\to\infty}\sup_{m\ge n} d\left(T^{2n}x, T^{2m+1}x\right) = \operatorname{dist}(A, B). \tag{38}$$

Since (A, B) has the property UC, by Lemma 5 $\{T^{2n}x\}$ is a Cauchy sequence. Suppose that $x_{2n} \to p$. Continuity of T on A implies that $x_{2n+1} \to Tp$. Thus, $d(p, Tp) = \operatorname{dist}(A, B)$. That is, p is a best proximity point of the mapping T in A. □

The next corollary is a direct result of Theorem 13.

Corollary 14 (compare to Theorem 3). *Let (A, B) be a nonempty closed pair in a uniformly convex Banach space X such that A is convex. Assume that $T : A \cup B \to A \cup B$ is an asymptotic pointwise cyclic orbital contraction such that T is continuous on A. If there exists $x \in A$ such that the orbit of T at x is bounded, then T has a best proximity point in A. Moreover, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges to the best proximity point of T.*

4. A Convergence Theorem

In this section, we give a convergence theorem of best proximity point for cyclic mappings which is derived from Ishikawa's convergence theorem ([12]). We begin with the following proposition which is an inequality characterization of uniformly convex Banach spaces.

Proposition 15 (see [13]). *Let X be a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing, continuous and convex function $\varphi : [0,1) \to [0,1)$ such that $\varphi(0) = 0$ and*

$$\|\lambda x+(1-\lambda)y\|^2 \le \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\varphi(\|x-y\|), \tag{39}$$

for all $\lambda \in [0,1]$ and all $x, y \in X$ such that $\|x\| \le r$ and $\|y\| \le r$.

Definition 16. Let (A, B) be a nonempty pair of subsets of a normed linear space X. Suppose that $T : A \cup B \to A \cup B$ is a cyclic mapping on $A \cup B$. We say that T is hemicompactness on A provided that each sequence $\{x_n\}$ in A with $\|x_n - T^2x_n\| \to 0$ has a convergent subsequence.

It is clear that if A is compact set, then each cyclic mapping defined on $A \cup B$ is hemicompactness, where B is a nonempty subset of X.

Theorem 17. *Let (A, B) be a nonempty, bounded, closed, and convex pair in a uniformly convex Banach space X. Assume that $T : A \cup B \to A \cup B$ is a cyclic relatively nonexpansive mapping such that T is hemicompactness on A and T^2 is continuous and satisfies the condition*

$$\|T^2x - Tx\| < \|Tx - x\|, \tag{40}$$

for all $x \in A \cup B$ with $\|x - Tx\| > \operatorname{dist}(A, B)$. Define a sequence $\{x_n\}$ in A by $x_1 \in A$ and

$$x_{n+1} = \alpha x_n + (1-\alpha)T^2x_n, \tag{41}$$

for $n \in \mathbb{N}$, where α is a real number belonging to $(0,1)$. Then $\{x_n\}$ converges strongly to a best proximity point of T in A.

Proof. Since (A, B) is a bounded, closed, and convex pair in a uniformly convex Banach space X, the relatively nonexpansive mapping T has a best proximity point in B ([4]). Also, we note that both of the (A, B) and (B, A) have the property UC. So, by Lemma 6 a point $p \in B$ is a best proximity point of the mapping T if and only if p is a fixed point of the mapping $T^2|_B$. We now have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&\quad = \|\alpha x_n + (1-\alpha)T^2x_n - T^2p\| \\
&\quad = \|\alpha x_n + (1-\alpha)T^2x_n - \alpha T^2p - (1-\alpha)T^2p\| \\
&\quad \le \alpha\|x_n - p\| + (1-\alpha)\|T^2x_n - T^2p\| \\
&\quad \le \alpha\|x_n - p\| + (1-\alpha)\|x_n - p\| = \|x_n - p\|.
\end{aligned} \tag{42}$$

Therefore, $\{\|x_n - p\|\}$ is a decreasing sequence and hence $\{\|x_n - p\|\}$ is convergent. So $\{x_n\}$ is bounded. From the uniform convexity of a Banach space X and by Proposition 15, there exists a strictly increasing, continuous and convex function $\varphi : [0,1) \to [0,1)$ such that $\varphi(0) = 0$ and

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \|\alpha(x_n - p) + (1-\alpha)(T^2x_n - p)\|^2 \\
&\quad \le \alpha\|x_n - p\|^2 + (1-\alpha)\|T^2x_n - T^2p\|^2 \\
&\qquad - \alpha(1-\alpha)\varphi(\|x_n - T^2x_n\|) \\
&\quad \le \|x_n - p\|^2 - \alpha(1-\alpha)\varphi(\|x_n - T^2x_n\|).
\end{aligned} \tag{43}$$

Thus

$$\alpha(1-\alpha)\varphi(\|x_n - T^2x_n\|) \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2, \tag{44}$$

which implies that $\varphi(\|x_n - T^2x_n\|) \to 0$. Since φ is strictly increasing and continuous at 0, it follows that $\|x_n - T^2x_n\| \to 0$.

$$\|x_n - T^2x_n\| \longrightarrow 0. \tag{45}$$

On the other hand, since T^2 is hemicompactness on A, there exists a subsequence $\{x_{n_j}\}$ of the sequence $\{x_n\}$ such that $x_{n_j} \to q \in A$. By the continuity of the mapping T^2 on A, we have $T^2x_{n_j} \to T^2q$. Since $\|x_{n_j} - T^2x_{n_j}\| \to 0$, we obtain $q = T^2q$. Hence $q \in A$ is a fixed point of the mapping T^2 in A and again by Lemma 6, q is a best proximity point of T in A and $x_n \to q \in A$ strongly. □

Acknowledgement

The research of N. Shahzad was partially supported by Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 145190, 9 pages
http://dx.doi.org/10.1155/2013/145190

Research Article

Common Fixed Point Theorems of New Contractive Conditions in Fuzzy Metric Spaces

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Received 26 February 2013; Accepted 8 May 2013

Academic Editor: Tamaki Tanaka

Some new limit contractive conditions in fuzzy metric spaces are introduced, by using property $(E.A)$, some common fixed point theorems for four maps are proved in GV-fuzzy metric spaces. As an application of our results, some new contractive conditions are presented, and some common fixed point theorems are proved under these contractive conditions. The contractive conditions presented in this paper contain or generalize many contractive conditions that appeared in the literatures. Some examples are given to illustrate that our results are real generalizations for the results in the references and to show that our limit contractive conditions are important for the existence of fixed point.

1. Introduction

The theory of fuzzy sets was first introduced by Zadeh [1], after many authors introduced the notion of fuzzy metric spaces in different ways (see [2–5]). In particular, Kramosil and Michálek [4] generalized the concept of probabilistic metric space given by Menger [6] to the fuzzy framework. Later on, George and Veeramani [2] modified the concept of fuzzy metric space introduced by Kramosil and Michálek and defined the Hausdorff and first countable topology on the modified fuzzy metric space. Actually, this topology can also be constructed on each fuzzy metric space in the sense of Kramosil and Michálek and it is metrizbale [2, 7]. Other recent contributions to the study of fuzzy metric spaces in the sense of [2] may be found in [8, 9]. Since then, many authors have proved fixed point and common fixed point theorems in fuzzy metric spaces in the sense of [2]. Especially, we want to emphasize that some common fixed point theorems for φ-type contraction maps in fuzzy metric spaces have been recently obtained in [10–16].

Quite recently, Miheţ [13] proved some existence theorems of common fixed point for two self-mappings f, g of a fuzzy metric space $(X, M, *)$ under the following contractive condition:

$$\begin{aligned} M(fx, fy, t) &\geq \varphi(\min\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t), \\ &\qquad M(fx, gy, t), M(fy, gx, t)\}) \end{aligned} \tag{1}$$

for all $x, y \in X$ and $t > 0$, where $\varphi : [0,1] \to [0,1]$ is continuous and nondecreasing on $[0,1]$, and $\varphi(x) > x$ for all $x \in [0,1]$.

C. Vetro and P. Vetro [15] proved some existence theorems of common fixed point for two self-mappings f, g of a fuzzy metric space $(X, M, *)$ under the following contractive condition:

$$\begin{aligned} &\frac{1}{M(fx, fy, t)} - 1 \\ &\leq r\left(\frac{1}{\min\{M(gx, gy, t), M(fx, gx, t), M(fy, gy, t)\}} - 1\right) \end{aligned} \tag{2}$$

for all $x, y \in X$ and $t > 0$, where $r : [0, +\infty) \to [0, +\infty)$ with $r(\tau) < \tau$ for every $\tau > 0$, an upper semicontinuous function.

Gopal et al. [10] proved some existence theorems of common fixed point for four self-mappings A, B, S and T of a fuzzy metric space $(X, M, *)$ under the following contractive condition:

$$\frac{1}{M(Ax, By, t)} - 1 \le r\left(\frac{1}{\min\{M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t)\}} - 1\right) \tag{3}$$

for all $x, y \in X$ and $t > 0$, where $r : [0, +\infty) \to [0, +\infty)$ with $r(\tau) < \tau$ for every $\tau > 0$, an upper semicontinuous function.

Imdad and Ali [11], Vijayaraju and Sajath [16] proved some existence theorems of common fixed point for four self-mappings A, B, S, and T of a fuzzy metric space $(X, M, *)$ under the following contractive condition:

$$M(Ax, By, t) \ge \phi(\min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t)\}) \tag{4}$$

for all $x, y \in X$ and $t > 0$, where $\phi : [0, 1] \to [0, 1]$ with $\phi(s) > s$ whenever $0 < s < 1$ is a continuous or increasing and left-continuous function.

Imdad et al. [12] proved some existence theorems of common fixed point for four self-mappings f, g, S, and T of a fuzzy metric space $(X, M, *)$ under the following contractive condition:

$$M(fx, gy, t) \ge \varphi(\min\{M(Sx, Ty, t), M(fx, Sx, t), M(fy, Ty, t), M(fx, Ty, t), M(fy, Sx, t)\}) \tag{5}$$

for all $x, y \in X$ and $t > 0$, where $\varphi : [0, 1] \to [0, 1]$ is continuous and nondecreasing on $[0, 1]$, and $\varphi(x) > x$ for all $x \in [0, 1]$.

Shen et al. [14] proved an existence theorem of fixed point for self-mapping T of a fuzzy metric space $(X, M, *)$ under the following contractive condition:

$$\varphi(M(Tx, Ty, t)) \le k(t) \cdot \varphi(M(x, y, t)) \tag{6}$$

for all $x, y \in X$ and $t > 0$, where $\varphi : [0, 1] \to [0, 1]$ satisfies the following properties:

(P1) φ is strictly decreasing and left continuous;

(P2) $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

Furthermore, let k be a function from $(0, \infty)$ into $(0, 1)$.

The purpose of this paper is to present limit contractive conditions to unify all of these φ-type nonlinear contractive conditions. Then, by using property $(E.A)$, some common fixed point theorems for four maps are proved in GV-fuzzy metric spaces. As an application of our limit contraction condition, we present some new φ-type integral contractive conditions and some common fixed point theorems for four maps in GV-fuzzy metric spaces under these contractive conditions. Our results generalize the corresponding results in [10–16]. Some examples are given to illustrate that our results are real generalizations for the results in the references and show that our limit contractive conditions are important for the existence of fixed point.

For the reader's convenience, we recall some terminologies from the theory of fuzzy metric spaces, which will be used in what follows.

Definition 1 (see [4]). A *continuous* t-norm in the sense of Kramosil and Michálek is a binary operation $*$ on $[0, 1]$ satisfying the following conditions:

(1) $*$ is associative and commutative,

(2) $a * 1 = a$ for all $a \in [0, 1]$,

(3) $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$,

(4) the mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ is continuous.

Three typical examples of continuous t-norm are $a * b = \max\{a + b - 1, 0\}$, $a * b = ab$, and $a * b = \min\{a, b\}$.

Definition 2 (see [2]). A *fuzzy metric space* in the sense of George and Veeramani is a triple $(X, M, *)$, where X is a nonempty set, M is a fuzzy set on $X^2 \times (0, \infty)$, and $*$ is a continuous t-norm such that the following conditions are satisfied for all $x, y, z \in X$ and $t, s > 0$

(GV-1) $M(x, y, t) > 0$;

(GV-2) $M(x, y, t) = 1$ if and only if $x = y$;

(GV-3) $M(x, y, t) = M(y, x, t)$;

(GV-4) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$;

(GV-5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

In what follows, fuzzy metric spaces in the sense of George and Veeramani will be called *GV-fuzzy metric spaces*.

Lemma 3 (see [17]). *Let $(X, M, *)$ be a GV-fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t for all $x, y \in X$.*

Definition 4 (see [13]). Let $(X, M, *)$ be a GV-fuzzy metric space. Then one has the following:

(1) a sequence $\{x_n\}$ in X is said to be *convergent* to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(2) a sequence $\{x_n\}$ in X is said to be *Cauchy sequence* if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p \in \mathbb{N}$.

(3) a fuzzy metric space is called *complete* if every Cauchy sequence converges in X.

Lemma 5 (see [5]). *Let $(X, M, *)$ be a GV-fuzzy metric space. Then M is a continuous function on $X^2 \times (0, \infty)$.*

Definition 6 (see [18]). Let $(X, M, *)$ be a GV-fuzzy metric space. Then two self-mappings A and S of $(X, M, *)$ satisfy *property* $(E.A)$ if there exists a sequence $\{x_n\}$ in X and z in X such that $\{Ax_n\}$ and $\{Sx_n\}$ converge to z that is, for any $t > 0$,

$$\lim_{n\to\infty} M(Ax_n, z, t) = \lim_{n\to\infty} M(Sx_n, z, t) = 1. \tag{7}$$

Definition 7 (see [18]). Let $(X, M, *)$ be a GV-fuzzy metric space. Then two pairs of self-mappings A, S and B, T of $(X, M, *)$ are said to share *common property* $(E.A)$ if there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that, for any $t > 0$,

$$\begin{aligned}\lim_{n\to\infty} M(Ax_n, z, t) &= \lim_{n\to\infty} M(Sx_n, z, t) \\ &= \lim_{n\to\infty} M(By_n, z, t) = \lim_{n\to\infty} M(Ty_n, z, t) = 1\end{aligned} \tag{8}$$

for some $z \in X$.

Definition 8 (see [19]). Let $(X, M, *)$ be a GV-fuzzy metric space. Then two self-mappings A and S of $(X, M, *)$ are said to be *weak compatible* if they commute at their coincidence point; that is,

$$Ax = Sx \quad \text{implies } SAx = ASx. \tag{9}$$

2. Main Results

Let $(X, M, *)$ be a GV-fuzzy metric space, and let A, B, S, and T be self-mappings of $(X, M, *)$. For any $x, y \in X$ and $t > 0$, we define

$$\begin{aligned}\operatorname{Min}(x, y, t) &= \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t), \\ &\qquad M(Sx, By, t), M(Ty, Ax, t)\}.\end{aligned} \tag{10}$$

Consider that $(C1)$ $0 < \underline{\lim}_{n\to\infty} \operatorname{Min}(x_n, y_n, t) = L(t) < 1$ implies that $\overline{\lim}_{n\to\infty} M(Ax_n, By_n, t) > L(t)$ for all $t > 0$ and any sequence $\{x_n\}$ and $\{y_n\}$ in X.

Theorem 9. *Let $(X, M, *)$ be a GV-fuzzy metric space, and A, B, S, and T be self-mappings of $(X, M, *)$ such that one has the following:*

(1) *$(C1)$ holds;*

(2) *A, S are weakly compatible and B, T are weakly compatible;*

(3) *A, S satisfy property $(E.A)$ or B, T satisfy property $(E.A)$;*

(4) *$AX \subset TX$ and $BX \subset SX$;*

(5) *one of the range of the mappings A, B, S, or T is a closed subspace of X.*

Then A, B, S, and T have a unique common fixed point.

Proof. Suppose that B, T satisfy the property $(E.A)$. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} M(Bx_n, z, t) = \lim_{n\to\infty} M(Tx_n, z, t) = 1 \tag{11}$$

for all $t > 0$ and some $z \in X$.

Since $BX \subset SX$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$. Hence $\lim_{n\to\infty} M(Sy_n, z, t) = 1$ for all $t > 0$.

Suppose that SX is a closed subspace of X. Then $z = Su$ for some $u \in X$. Subsequently, we have that

$$\begin{aligned}\lim_{n\to\infty} M(Bx_n, Su, t) &= \lim_{n\to\infty} M(Tx_n, Su, t) \\ &= \lim_{n\to\infty} M(Sy_n, Su, t) = 1\end{aligned} \tag{12}$$

for all $t > 0$. Then by using Lemma 5 we have that

$$\begin{aligned}\lim_{n\to\infty} M(Tx_n, Bx_n, t) &= 1, \\ \lim_{n\to\infty} M(Tx_n, Au, t) &= M(Su, Au, t).\end{aligned} \tag{13}$$

Thus, we have that

$$\begin{aligned}\underline{\lim}_{n\to\infty} \operatorname{Min}(u, x_n, t) &= \underline{\lim}_{n\to\infty} \min\{M(Su, Tx_n, t), M(Su, Au, t), \\ &\qquad M(Tx_n, Bx_n, t), M(Su, Bx_n, t), \\ &\qquad M(Tx_n, Au, t)\} \\ &= M(Su, Au, t) = \lim_{n\to\infty} M(Bx_n, Au, t)\end{aligned} \tag{14}$$

for any $t > 0$; that is,

$$\underline{\lim}_{n\to\infty} \operatorname{Min}(u, x_n, t) = \overline{\lim}_{n\to\infty} M(Au, Bx_n, t) = M(Su, Au, t). \tag{15}$$

If $Su \neq Au$, then there exists $t > 0$ such that $0 < M(Su, Au, t) < 1$. This and $(C1)$ imply that

$$\overline{\lim}_{n\to\infty} M(Au, Bx_n, t) > \underline{\lim}_{n\to\infty} \operatorname{Min}(u, x_n, t). \tag{16}$$

This is a contradiction. Thus, we have that $Su = Au$. The weak compatibility of A and S implies that $ASu = SAu$, and then $AAu = ASu = SAu = SSu$.

On the other hand, since $AX \subset TX$, there exists $v \in X$ such that $Au = Tv$. Since for any $t > 0$

$$\begin{aligned}\operatorname{Min}(u, v, t) &= \min\{M(Su, Tv, t), M(Su, Au, t), M(Tv, Bv, t), \\ &\qquad M(Su, Bv, t), M(Tv, Au, t)\} = M(Au, Bv, t),\end{aligned} \tag{17}$$

by $(C1)$, we get $M(Au, Bv, t) = 1$, which implies that $Au = Bv$; that is, $Au = Su = Tv = Bv$. The weak compatibility of B and T implies that $BTv = TBv$ and $TTv = TBv = BTv = BBv$.

Let us show that Au is a common fixed point of A, B, T, and S. Since

$$\begin{aligned}
&\operatorname{Min}(Au, v, t)\\
&\quad = \min\{M(SAu, Tv, t), M(SAu, AAu, t),\\
&\qquad M(Tv, Bv, t), M(SAu, Bv, t), M(Tv, AAu, t)\}\\
&\quad = \min\{M(AAu, Bv, t), 1, 1, M(AAu, Bv, t),\\
&\qquad M(Tv, AAu, t)\}\\
&\quad = M(AAu, Bv, t),
\end{aligned} \tag{18}$$

by $(C1)$, we get that $M(AAu, Bv, t) = 1$, which implies that $AAu = Bv$. Therefore, $AAu = SAu = Bv = Au$, and Au is a common fixed point of A and S. Similarly, we can prove that Bv is a common fixed point of B and T. Noting that $Au = Bv$, we conclude that Au is a common fixed point A, B, T, and S. The proof is similar when TX is assumed to be a closed subspace of X. The cases in which AX or BX is a closed subspace of X are similar to the cases in which TX or SX, respectively, is closed since $AX \subset TX$ and $BX \subset SX$. If $Au = Bu = Tu = Su = u$ and $Av = Bv = Tv = Su = v$, then

$$\begin{aligned}
&\operatorname{Min}(u, v, t)\\
&\quad = \min\{M(Su, Tv, t), M(Su, Au, t), M(Tv, Bv, t),\\
&\qquad M(Su, Bv, t), M(Tv, Au, t)\}\\
&\quad = \min\{M(Au, Tv, t), 1, 1, M(Su, Bv, t),\\
&\qquad M(Tv, Au, t)\}\\
&\quad = M(Au, Bv, t).
\end{aligned} \tag{19}$$

Therefore, by $(C1)$, we have $u = v$; that is, the common fixed point is unique. This completes the proof. □

To introduce some integral contractive conditions, let $h : [0, 1] \to \mathbb{R}_+$ be nonnegative, Lebesgue integrable, and satisfy

$$\int_0^{\varepsilon} h(t)\,dt > 0 \tag{20}$$

for each $0 < \varepsilon \le 1$. We denote $d = \int_0^1 h(t)dt$.

Theorem 10. *Let $(X, M, *)$ be a GV-fuzzy metric space, and A, B, S, and T be self-mappings of $(X, M, *)$. If one of the following conditions is satisfied*

$(C2)$ *There exists a function $\omega : [0, d] \to [0, d]$ such that for any $0 < s < d$, $\omega(s) < s$, $\overline{\lim}_{u \to s}\omega(u) < s$ and for any $x, y \in X$, $t \in (0, \infty)$, $0 < \operatorname{Min}(x, y, t) < 1$ implies that*

$$\omega\left(\int_0^{M(Ax,By,t)} h(s)\,ds\right) \ge \int_0^{\operatorname{Min}(x,y,t)} h(s)\,ds. \tag{21}$$

$(C3)$ *There exists a function $\phi : [0, d] \to [0, d]$ such that, for any $0 < s < d$, $\phi(s) > s$, $\underline{\lim}_{t \to s}\phi(t) > s$, and for any $x, y \in X$, $t \in (0, \infty)$, $0 < \operatorname{Min}(x, y, t) < 1$ implies that*

$$\int_0^{M(Ax,By,t)} h(s)\,ds \ge \phi\left(n\int_0^{\operatorname{Min}(x,y,t)} h(s)\,ds\right). \tag{22}$$

Then $(C1)$ holds.

Proof. $(C2) \Rightarrow (C1)$. Assume that $(C2)$ holds. If $\{x_n\}$ and $\{y_n\}$ in X and $0 < \underline{\lim}_{n\to\infty} \operatorname{Min}(x_n, y_n, t) = L(t) < 1$, then

$$\begin{aligned}
&\overline{\lim_{n\to\infty}}\,\omega\left(\int_0^{M(Ax_n,By_n,t)} h(\tau)\,d\tau\right)\\
&\quad \ge \overline{\lim_{n\to\infty}} \int_0^{\operatorname{Min}(x_n,y_n,t)} h(\tau)\,d\tau\\
&\quad \ge \int_0^{\underline{\lim}_{n\to\infty}\operatorname{Min}(x_n,y_n,t)} h(\tau)\,d\tau = \int_0^{L(t)} h(\tau)\,d\tau.
\end{aligned} \tag{23}$$

If $\overline{\lim}_{n\to\infty} M(Ax_n, By_n, t) = 1$, then $\overline{\lim}_{n\to\infty} M(Ax_n, By_n, t) > L(t)$. If $\overline{\lim}_{n\to\infty} M(Ax_n, By_n, t) = r(t) < 1$, then there exists a subsequence $\{M(Ax_{n_i}, By_{n_i}, t)\}$ such that

$$\lim_{i\to\infty} M\left(Ax_{n_i}, By_{n_i}, t\right) = \overline{\lim_{n\to\infty}} M\left(Ax_n, By_n, t\right) = r(t). \tag{24}$$

Thus,

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \int_0^{M(Ax_n,By_n,t)} h(\tau)\,d\tau\\
&\quad = \lim_{i\to\infty} \int_0^{M(Ax_{n_i},By_{n_i},t)} h(\tau)\,d\tau\\
&\quad = \int_0^{r(t)} h(\tau)\,d\tau > \overline{\lim_{u\to\int_0^{r(t)} h(\tau)d\tau}}\,\omega(u)\\
&\quad \ge \overline{\lim_{i\to\infty}}\,\omega\left(\int_0^{\operatorname{Min}(x_{n_i},y_{n_i},t)} h(\tau)\,d\tau\right) \ge \int_0^{L(t)} h(\tau)\,d\tau.
\end{aligned} \tag{25}$$

This implies that

$$\overline{\lim_{n\to\infty}} M\left(Ax_n, By_n, t\right) = r(t) > L(t). \tag{26}$$

Thus, $(C1)$ holds.

$(C3) \Rightarrow (C1)$. Assume that $(C3)$ holds. If $\{x_n\}$ and $\{y_n\}$ in X and $0 < \underline{\lim}_{n\to\infty} \operatorname{Min}(x_n, y_n, t) = L(t) < 1$, then there exists a subsequence $\{\operatorname{Min}(x_{n_i}, y_{n_i}, t)\}$ such that $\lim_{i\to\infty} \operatorname{Min}(x_{n_i}, y_{n_i}, t) = L(t)$. This implies that

$$\begin{aligned}
&\underline{\lim_{i\to\infty}}\,\phi\left(\int_0^{\operatorname{Min}(x_{n_i},y_{n_i},t)} h(\tau)\,d\tau\right)\\
&\quad \ge \underline{\lim_{t\to\int_0^{L(t)} h(\tau)d\tau}}\,\phi(t) > \int_0^{L(t)} h(\tau)\,d\tau\\
&\quad = \lim_{i\to\infty} \int_0^{\operatorname{Min}(x_{n_i},y_{n_i},t)} h(\tau)\,d\tau.
\end{aligned} \tag{27}$$

It follows from

$$\int_0^{M(Ax,By,t)} h(\tau)\,d\tau \geq \phi\left(\int_0^{\mathrm{Min}(x,y,t)} h(\tau)\,d\tau\right) \tag{28}$$

that we can get

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} \int_0^{M(Ax_n,By_n,t)} h(\tau)\,d\tau \\
&\quad = \int_0^{\overline{\lim}_{n\to\infty} M(Ax_n,By_n,t)} h(\tau)\,d\tau \\
&\quad \geq \overline{\lim_{n\to\infty}} \phi\left(\int_0^{\mathrm{Min}(x_n,y_n,t)} h(\tau)\,d\tau\right) \\
&\quad \geq \overline{\lim_{i\to\infty}} \phi\left(\int_0^{\mathrm{Min}(x_{n_i},y_{n_i},t)} h(\tau)\,d\tau\right) \\
&\quad \geq \underline{\lim_{i\to\infty}} \phi\left(\int_0^{\mathrm{Min}(x_{n_i},y_{n_i},t)} h(\tau)\,d\tau\right) > \int_0^{L(t)} h(\tau)\,d\tau.
\end{aligned} \tag{29}$$

This implies that $\overline{\lim}_{n\to\infty} M(Ax_n, By_n, t) > L(t)$; that is, $(C1)$ holds. □

It follows from Theorems 9 and 10 that we have the following fixed point theorems for integral type contractive mappings.

Theorem 11. *Let $(X, M, *)$ be a GV-fuzzy metric space, and let A, B, S, and T be self-mappings of $(X, M, *)$ such that one has the following*

(1) *one of $(C2)$-$(C3)$ holds;*

(2) *A, S are weakly compatible and B, T are weakly compatible;*

(3) *A, S satisfy the property $(E.A)$ or B, T satisfy the property $(E.A)$;*

(4) *$AX \subset TX$ and $BX \subset SX$;*

(5) *one of the range of the mappings A, B, S, or T is a closed subspace of X.*

Then A, B, S, and T have a unique common fixed point.

In $(C2)$ and $(C3)$, by taking $h(t) \equiv 1$, we have the following contractive conditions.

$(C4)$ There exists a function $\omega : [0,1] \to [0,1]$ such that for any $0 < s < 1$, $\omega(s) < s$, $\overline{\lim}_{u\to s}\omega(u) < s$ and for any $x, y \in X$, $t \in (0,\infty)$, $0 < \mathrm{Min}\,(x,y,t) < 1$ implies that

$$\omega\left(M\left(x,y,t\right)\right) \geq \mathrm{Min}\,\left(x,y,t\right). \tag{30}$$

$(C5)$ There exists a function $\phi : [0,1] \to [0,1]$ such that, for any $0 < s < 1$, $\phi(s) > s$, $\underline{\lim}_{t\to s}\phi(t) > s$, and for any $x, y \in X$, $t \in (0,\infty)$, $0 < \mathrm{Min}\,(x,y,t) < 1$ implies that

$$M\left(Ax,By,t\right) \geq \phi\left(\mathrm{Min}\,\left(x,y,t\right)\right). \tag{31}$$

Corollary 12. *Let $(X, M, *)$ be a GV-fuzzy metric space, and let A, B, S, and T be self-mappings of $(X, M, *)$ such that one has the following:*

(1) *one of $(C4)$-$(C5)$ holds;*

(2) *A, S are weakly compatible and B, T are weakly compatible;*

(3) *A, S satisfy property $(E.A)$ or B, T satisfy the property $(E.A)$;*

(4) *$AX \subset TX$ and $BX \subset SX$;*

(5) *one of the range of the mappings A, B, S, or T is a closed subspace of X.*

Then A, B, S and T have a unique common fixed point.

Remark 13. As a special case of $(C4)$, we can take function $\omega : [0,1] \to [0,1]$ as one of the following:

(1) ω is nonincreasing, for any $0 < s < 1$, $\omega(s) < s$, $\overline{\lim}_{u\to s^-}\omega(u) < s$;

(2) ω is an upper semicontinuous function such that, for any $0 < s < 1$, $\omega(s) < s$;

(3) ω is non-increasing and left-upper semicontinuous such that $\omega(s) < s$ for any $0 < s < 1$.

As a special case of $(C5)$, we can take function $\phi : [0,1] \to [0,1]$ as one of the follows:

(1) ϕ is a nondecreasing and left-continuous function such that, for any $0 < s < 1$, $\phi(s) > s$;

(2) ϕ is a a lower semi-continuous function such that for any $0 < s < 1$, $\phi(s) > s$;

(3) $\phi(s) = s + \psi(s)$, where $\psi : [0,1] \to [0,1]$ is a continuous function with for any $0 < s < 1$, $\psi(s) > 0$.

From Corollary 12, we have the following corollaries.

Corollary 14. *Let $(X, M, *)$ be a GV-fuzzy metric space, and let A, B, S, and T be self-mappings of $(X, M, *)$ such that one has the following:*

(1) *there exists an upper semicontinuous function $r : [0,+\infty) \to [0,+\infty)$ with $r(s) < s$ for any $s > 0$ such that for all $x, y \in X$ and $t > 0$*

$$\frac{1}{M\left(Ax,By,t\right)} - 1 \leq r\left(\frac{1}{\mathrm{Min}\,\left(x,y,t\right)} - 1\right); \tag{32}$$

(2) *A, S are weakly compatible and B, T are weakly compatible;*

(3) *A, S satisfy property $(E.A)$ or B, T satisfy property $(E.A)$;*

(4) *$AX \subset TX$ and $BX \subset SX$;*

(5) *one of the range of the mappings A, B, S or T is a closed subspace of X.*

Then A, B, S, and T have a unique common fixed point.

Proof. Define function $\phi : [0,1] \to [0,1]$ by

$$\phi(t) = \begin{cases} 0 & \text{if } t = 0; \\ \dfrac{1}{1 + r((1/t) - 1)} & \text{if } t \in (0,1]. \end{cases} \tag{33}$$

Then, for any $s > 0$,

$$\begin{aligned} \underline{\lim}_{t \to s} \phi(t) &= \frac{1}{1 + \overline{\lim}_{t \to s} r((1/t) - 1)} \\ &= \frac{1}{1 + r((1/s) - 1)} > \frac{1}{1 + (1/s) - 1} = s, \end{aligned} \tag{34}$$

and (32) can be rewritten as: for any $x, y \in X$, $t \in (0, \infty)$, $0 < \mathrm{Min}(x, y, t) < 1$ implies that

$$M(Ax, By, t) \geq \phi(\mathrm{Min}(x, y, t)). \tag{35}$$

That is, $(C5)$ holds. Then the conclusion can be deduced from Corollary 12. This completes the proof. □

Corollary 15. *Let $(X, M, *)$ be a GV-fuzzy metric space, and A, B, S, and T be self-mappings of $(X, M, *)$ such that one has the following:*

(1) *there exists a strictly decreasing and left continuous function $\varphi : [0,1] \to [0,1]$ with $\varphi(\lambda) = 0$ if and only if $\lambda = 1$ and function $k : (0, \infty) \to (0,1)$ such that for all $x, y \in X$ $(x \neq y)$ and $t > 0$*

$$\varphi(M(Ax, By, t)) \leq k(t) \cdot \varphi(\mathrm{Min}(x, y, t)); \tag{36}$$

(2) *A, S are weakly compatible and B, T are weakly compatible;*

(3) *A, S satisfy property $(E.A)$ or B, T satisfy property $(E.A)$;*

(4) *$AX \subset TX$ and $BX \subset SX$;*

(5) *one of the range of the mappings A, B, S, or T is a closed subspace of X.*

Then, A, B, S and T have a unique common fixed point.

Proof. Define function $\phi : [0,1] \times (0, +\infty) \to [0,1]$ by

$$\phi(s, t) = \varphi^{-1}(k(t)\varphi(s)). \tag{37}$$

Since φ is strictly decreasing and left continuous, we have that φ^{-1} is strictly decreasing and right continuous and $\phi(s,t)$ is increasing in s. Then we have that

$$\underline{\lim}_{u \to s^-} \phi(u, t) = \varphi^{-1}(k(t)\varphi(s)). \tag{38}$$

Also we have that $\phi(u,t) \geq \phi(s,t)$ for $u > s$; this shows that

$$\underline{\lim}_{u \to s^+} \phi(u, t) \geq \varphi^{-1}(k(t)\varphi(s)). \tag{39}$$

That is, we can get that

$$\underline{\lim}_{u \to s} \phi(u, t) \geq \phi(s, t) = \varphi^{-1}(k(t)\varphi(s)) > \varphi^{-1}(\varphi(s)) = s, \tag{40}$$

and (36) can be rewritten as follows: for any $x, y \in X$, $t \in (0, \infty)$, $0 < \mathrm{Min}(x, y, t) < 1$ implies that

$$M(Ax, By, t) \geq \phi(\mathrm{Min}(x, y, t), t). \tag{41}$$

If $\{x_n\}$ and $\{y_n\}$ in X and $0 < \underline{\lim}_{n \to \infty} \mathrm{Min}(x_n, y_n, t) = L(t) < 1$, then there exists a subsequence $\{\mathrm{Min}(x_{n_i}, y_{n_i}, t)\}$ such that $\lim_{i \to \infty} \mathrm{Min}(x_{n_i}, y_{n_i}, t) = L(t)$. This shows that

$$\begin{aligned} &\underline{\lim}_{i \to \infty} \phi\left(\mathrm{Min}\left(x_{n_i}, y_{n_i}, t\right), t\right) \\ &\quad \geq \underline{\lim}_{s \to L(t)} \phi(s, t) > L(t) \\ &\quad = \lim_{i \to \infty} \mathrm{Min}\left(x_{n_i}, y_{n_i}, t\right). \end{aligned} \tag{42}$$

It follows from $M(Ax, By, t) \geq \phi(\mathrm{Min}(x, y, t), t)$ that we can get

$$\begin{aligned} &\overline{\lim}_{n \to \infty} M(Ax_n, By_n, t) \\ &\quad \geq \overline{\lim}_{n \to \infty} \phi(\mathrm{Min}(x_n, y_n, t), t) \\ &\quad \geq \overline{\lim}_{i \to \infty} \phi\left(\mathrm{Min}\left(x_{n_i}, y_{n_i}, t\right), t\right) \\ &\quad \geq \underline{\lim}_{i \to \infty} \phi\left(\mathrm{Min}\left(x_{n_i}, y_{n_i}, t\right), t\right) > L(t). \end{aligned} \tag{43}$$

Thus, we have that $(C1)$ holds. The conclusion can be deduced from Theorem 9. This completes the proof. □

Remark 16. The main result Theorems 3.1 and 2.1 in [13] are the special cases of our Theorem 9 and Corollary 14 for $A = B = f$ and $S = T = g$. Especially, it follows from Corollary 12 that the condition "φ is nondecreasing" is not needed. Therefore, our results improve and generalize the results in [13].

Let $(X, M, *)$ be a GV-fuzzy metric space, and let A, B, S, and T be self-mappings of $(X, M, *)$. For any $x, y \in X$ and $t > 0$, we define

$$\begin{aligned} &\widetilde{\mathrm{Min}}(x, y, t) \\ &\quad = \min\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t)\}. \end{aligned} \tag{44}$$

Consider that $(\widetilde{C}1)$ $0 < \underline{\lim}_{n \to \infty} \widetilde{\mathrm{Min}}(x_n, y_n, t) = \widetilde{L}(t) < 1$ implies $\overline{\lim}_{n \to \infty} M(Ax_n, By_n, t) > \widetilde{L}(t)$ for all $t > 0$ and any sequence $\{x_n\}$ and $\{y_n\}$ in X.

By (44), we can write the conditions $(\widetilde{C}2)$–$(\widetilde{C}5)$ which correspond to the conditions $(C2)$–$(C5)$ in Theorem 9, Theorem 10 and Corollary 12 by replacing $\mathrm{Min}(x, y, t)$ with $\widetilde{\mathrm{Min}}(x, y, t)$. It is similar to the proof of Theorem 10, we can know that one of $(\widetilde{C}4)$-$(\widetilde{C}5)$ can imply $(\widetilde{C}1)$ holds.

Corollary 17. *Let $(X, M, *)$ be a GV-fuzzy metric space, and A, B, S and T be self-mappings of $(X, M, *)$ such that one has the following:*

(1) *$(\widetilde{C}1)$ or one of $(\widetilde{C}4)$-$(\widetilde{C}5)$ holds;*

(2) *A, S are weakly compatible and B, T are weakly compatible;*

(3) *A, S satisfy property $(E.A)$ or B, T satisfy property $(E.A)$;*

(4) *$AX \subset TX$ and $BX \subset SX$;*

(5) *one of the range of the mappings A, B, S, or T is a closed subspace of X.*

Then A, B, S, and T have a unique common fixed point.

Proof. It is clear that we only need to prove Corollary 17 for $(\widetilde{C}1)$. Assume that sequence $\{x_n\}$ and $\{y_n\}$ in X, $0 < \underline{\lim}_{n\to\infty} \operatorname{Min}(x_n, y_n, t) = L(t) < 1$. Then

$$\lim_{n\to\infty} \widetilde{\operatorname{Min}}(x_n, y_n, t) = \widetilde{L}(t) \geq \underline{\lim}_{n\to\infty} \operatorname{Min}(x_n, y_n, t) = L(t). \tag{45}$$

If $\underline{\lim}_{n\to\infty} \widetilde{\operatorname{Min}}(x_n, y_n, t) = \widetilde{L}(t) < 1$, then by $(\widetilde{C}1)$ we have that

$$\overline{\lim_{n\to\infty}} M(Ax_n, By_n, t) > \widetilde{L}(t) \geq L(t). \tag{46}$$

If $\underline{\lim}_{n\to\infty} \widetilde{\operatorname{Min}}(x_n, y_n, t) = \widetilde{L}(t) = 1$, then, for any n, either $\widetilde{\operatorname{Min}}(x_n, y_n, t) < 1$ or $\widetilde{\operatorname{Min}}(x_n, y_n, t) = 1$. If $\widetilde{\operatorname{Min}}(x_n, y_n, t) < 1$, then by $(\widetilde{C}1)$ we have that $M(Ax_n, By_n, t) > \widetilde{\operatorname{Min}}(x_n, y_n, t)$. If $\widetilde{\operatorname{Min}}(x_n, y_n, t) = 1$, then by (44) we have that $Ax_n = By_n$. By (GV-2) we get that $M(Ax_n, By_n, t) = 1 = \widetilde{\operatorname{Min}}(x_n, y_n, t)$. Thus, $(\widetilde{C}1)$ implies that $M(Ax_n, By_n, t) \geq \widetilde{\operatorname{Min}}(x_n, y_n, t)$ for any n. This implies that

$$\begin{aligned} &\overline{\lim_{n\to\infty}} M(Ax_n, By_n, t) \\ &\quad \geq \underline{\lim}_{n\to\infty} \widetilde{\operatorname{Min}}(x_n, y_n, t) = \widetilde{L}(t) = 1 > L(t). \end{aligned} \tag{47}$$

Therefore, $(\widetilde{C}1)$ implies that $(C1)$ holds. Then by Theorem 9 we know that A, B, S, and T have a unique common fixed point. This completes the proof. □

Remark 18. In Theorem 9 and Corollaries 12–17, conditions (3), (4), and (5) can be replaced by the following conditions:

$(3')$ A, S and B, T share the common property $(E.A)$;

$(4')$ the range, of the mappings S and T are closed subspaces of X.

The proof can be got by properly modifying the proof of Theorem 9, Corollaries 12–17. Thus, from Theorem 9, Corollaries 12–17 and Remark 13, we can see that our results generalize and improve the results in [10–16].

Example 19. Let $X = [0, 1]$, $M(x, y, t) = t/(t + |x - y|)$ for every $x, y \in X$ and $t > 0$ and $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$. Then $(X, M, *)$ is a fuzzy metric space. Define $A = B$ and $T = S : X \to X$ by $Ax = \sin(x/(1 + x))$ and $Tx = \sin x$ for all $x \in X$. Then we have the following:

(1) A and S satisfy the property $(E.A)$ for the sequence $x_n = 1/n, n = 1, 2, \ldots$,

(2) A and S are weakly compatible,

(3) $AX = [0, \sin(1/2)] \subset TX = [0, \sin 1]$, and AX, TX are closed;

(4) $(C1)$ holds. In fact, if $\{x_n\}$ and $\{y_n\}$ in X and $0 < \underline{\lim}_{n\to\infty} \operatorname{Min}(x_n, y_n, t) = L(t) < 1$, then there exists a subsequence $\{\operatorname{Min}(x_{n_i}, y_{n_i}, t)\}$ such that $\lim_{i\to\infty} \operatorname{Min}(x_{n_i}, y_{n_i}, t) = L(t)$. Since $\{x_n\}$ and $\{y_n\}$ in X, $\{x_n\}$ and $\{y_n\}$ have convergent subsequence. With out of generality, assume that $\lim_{i\to\infty} x_{n_i} = x_0$, and $\lim_{i\to\infty} y_{n_i} = y_0$. Then we have that

$$\begin{aligned} &\lim_{i\to\infty} M(Ax_{n_i}, By_{n_i}, t) \\ &\quad = \frac{t}{t + |\sin(x_0/(1 + x_0)) - \sin(y_0/(1 + y_0))|}, \\ &\lim_{i\to\infty} \operatorname{Min}(x_{n_i}, y_{n_i}, t) \\ &\quad = \min\left\{\frac{t}{t + |\sin x_0 - \sin y_0|}, \right. \\ &\qquad \frac{t}{t + |\sin x_0 - \sin(x_0/(1 + x_0))|}, \\ &\qquad \frac{t}{t + |\sin y_0 - \sin(y_0/(1 + y_0))|}, \\ &\qquad \frac{t}{t + |\sin x_0 - \sin(y_0/(1 + y_0))|}, \\ &\qquad \left.\frac{t}{t + |\sin y_0 - \sin(x_0/(1 + x_0))|}\right\} = L(t). \end{aligned} \tag{48}$$

If $0 \leq x_0 < y_0$, then

$$\begin{aligned} L(t) = \min\Bigg\{&\frac{t}{t + \sin y_0 - \sin x_0}, \\ &\frac{t}{t + \sin x_0 - \sin(x_0/(1 + x_0))}, \\ &\frac{t}{t + \sin y_0 - \sin(y_0/(1 + y_0))}, \\ &\frac{t}{t + |\sin x_0 - \sin(y_0/(1 + y_0))|}, \\ &\frac{t}{t + \sin y_0 - \sin(x_0/(1 + x_0))}\Bigg\}. \end{aligned} \tag{49}$$

It is clear that

$$\begin{aligned} \sin y_0 - \sin \frac{x_0}{1+x_0} &> \sin y_0 - \sin x_0, \\ \sin y_0 - \sin \frac{x_0}{1+x_0} &> \sin x_0 - \sin \frac{x_0}{1+x_0}, \\ \sin y_0 - \sin \frac{x_0}{1+x_0} &> \sin y_0 - \sin \frac{y_0}{1+y_0}. \end{aligned} \tag{50}$$

It follows from

$$\sin y_0 + \sin x_0 > \sin \frac{x_0}{1+x_0} + \sin \frac{y_0}{1+y_0} \tag{51}$$

that

$$\sin y_0 - \sin \frac{x_0}{1+x_0} > \sin \frac{y_0}{1+y_0} - \sin x_0. \tag{52}$$

On the other hand,

$$\begin{aligned} \sin y_0 - \sin \frac{x_0}{1+x_0} &> \sin x_0 - \sin \frac{x_0}{1+x_0} \\ &> \sin x_0 - \sin \frac{y_0}{1+y_0}. \end{aligned} \tag{53}$$

Thus, we have that

$$\sin y_0 - \sin \frac{x_0}{1+x_0} > \left| \sin x_0 - \sin \frac{y_0}{1+y_0} \right|. \tag{54}$$

It follows from those inequalities that

$$L(t) = \frac{t}{t + \sin y_0 - \sin\left(x_0/\left(1+x_0\right)\right)}. \tag{55}$$

Similarly, if $0 \le y_0 < x_0$, then we can get that

$$L(t) = \frac{t}{t + \sin x_0 - \sin\left(y_0/\left(1+y_0\right)\right)}, \tag{56}$$

and if $x_0 = y_0 > 0$, then we have that

$$L(t) = \frac{t}{t + \sin x_0 - \sin\left(x_0/\left(1+x_0\right)\right)}. \tag{57}$$

Thus, we have that

$$L(t) = \begin{cases} \dfrac{t}{t + \sin y_0 - \sin\left(x_0/\left(1+x_0\right)\right)} & \text{if } 0 \le x_0 < y_0; \\ \dfrac{t}{t + \sin x_0 - \sin\left(y_0/\left(1+y_0\right)\right)} & \text{if } 0 \le y_0 < x_0; \\ \dfrac{t}{t + \sin x_0 - \sin\left(x_0/\left(1+x_0\right)\right)} & \text{if } x_0 = y_0 > 0. \end{cases} \tag{58}$$

It is clear that

$$\begin{aligned} &\frac{t}{t + \sin\left(y_0/\left(1+y_0\right)\right) - \sin\left(x_0/\left(1+x_0\right)\right)} \\ &\quad > \frac{t}{t + \sin y_0 - \sin\left(x_0/\left(1+x_0\right)\right)} \\ &\qquad \text{if } 0 \le x_0 < y_0; \end{aligned} \tag{59}$$

$$\begin{aligned} &\frac{t}{t + \sin\left(x_0/\left(1+x_0\right)\right) - \sin\left(y_0/\left(1+y_0\right)\right)} \\ &\quad > \frac{t}{t + \sin x_0 - \sin\left(y_0/\left(1+y_0\right)\right)} \\ &\qquad \text{if } 0 \le y_0 < x_0; \end{aligned} \tag{60}$$

$$1 > \frac{t}{t + \sin x_0 - \sin\left(x_0/\left(1+x_0\right)\right)} \quad \text{if } x_0 = y_0 > 0. \tag{61}$$

This shows that

$$\begin{aligned} &\overline{\lim_{n\to\infty}} M\left(Ax_n, By_n, t\right) \\ &\quad \ge \lim_{i\to\infty} M\left(Ax_{n_i}, By_{n_i}, t\right) > L(t) \\ &\quad = \lim_{i\to\infty} \operatorname{Min}\left(x_{n_i}, y_{n_i}, t\right) = \underline{\lim_{n\to\infty}} \operatorname{Min}\left(x_n, y_n, t\right). \end{aligned} \tag{62}$$

Thus, $(C1)$ holds.

(5) By Theorem 9, A and T have common fixed point.

Example 20. Let $X = [0,1]$, $M(x,y,t) = t/(t + |x-y|)$ for every $x, y \in X$ and $t > 0$ and $a * b = \min\{a,b\}$ for all $a, b \in [0,1]$. Then $(X, M, *)$ is a fuzzy metric space. Define $A = B$ and $T = S : X \to X$ by

$$Ax = \begin{cases} \dfrac{1}{2} & \text{if } x = 0; \\ 0 & \text{if } x = 1; \\ \dfrac{x}{2} & \text{if } x \in (0,1); \end{cases} \tag{63}$$

and $Tx = x$ for all $x \in X$. Then we have the following:

(1) A and S satisfy the property $(E.A)$ for the sequence $x_n = 1/n$, $n = 1, 2, \ldots$;

(2) A and S are weakly compatible;

(3) $AX = [0, 1/2] \subset TX = [0,1]$, and AX, TX are closed;

(4) $(C1)$ does not hold. In fact, for $x_n = 1/n$ and $y_n = 0$, $n = 1, 2, \ldots$, we have that

$$\begin{aligned}
&\operatorname{Min}(x_n, y_n, t) \\
&\quad = \min\left\{M\left(\frac{1}{n}, 0, t\right), M\left(\frac{1}{n}, \frac{1}{2n}, t\right), M\left(0, \frac{1}{2}, t\right),\right. \\
&\qquad \left. M\left(\frac{1}{n}, \frac{1}{2}, t\right), M\left(0, \frac{1}{n}, t\right)\right\} \\
&\quad = \min\left\{\frac{t}{t + (1/n)}, \frac{t}{t + (1/2n)}, \frac{t}{t + (1/2)},\right. \\
&\qquad \left. \frac{t}{t + (1/2) - (1/n)}, \frac{t}{t + (1/n)}\right\} \longrightarrow \frac{t}{t + (1/2)} \\
&\qquad\qquad (n \longrightarrow \infty),
\end{aligned}$$

$$\begin{aligned}
&\overline{\lim_{n\to\infty}} M(Ax_n, By_n, t) \\
&\quad = \overline{\lim_{n\to\infty}} M\left(\frac{1}{2n}, \frac{1}{2}, t\right) \\
&\quad = \overline{\lim_{n\to\infty}} \frac{t}{t + (1/2) - (1/2n)} = \frac{t}{t + (1/2)}.
\end{aligned} \tag{64}$$

Thus, $(C1)$ does not hold.

(5) A and T have no common fixed point.

Example 19 does not satisfy the φ-type contractive conditions used in [10–16]. Example 20 shows that if $(C1)$ does not hold, then A and T may have no common fixed point. Thus, $(C1)$ is important for the existence of common fixed point.

Acknowledgments

The authors would like to thank the referees for useful comments and suggestions. This research was supported by the NNSF of China (fund no. 11171286).

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 302438, 7 pages
http://dx.doi.org/10.1155/2013/302438

Research Article

Fixed Point Theory of Weak Contractions in Partially Ordered Metric Spaces

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Received 12 March 2013; Accepted 20 April 2013

Academic Editor: Erdal Karapınar

We prove two new fixed point theorems in the framework of partially ordered metric spaces. Our results generalize and improve many recent fixed point theorems in the literature.

1. Introduction and Preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers. Let (X, d) be a metric space, D a subset of X, and $f : D \to X$ a map. We say f is contractive if there exists $\alpha \in [0, 1)$ such that, for all $x, y \in D$,

$$d(fx, fy) \leq \alpha \cdot d(x, y). \tag{1}$$

The well-known Banach's fixed point theorem asserts that if $D = X$, f is contractive and (X, d) is complete, then f has a unique fixed point in X. In nonlinear analysis, the study of fixed points of given mappings satisfying certain contractive conditions in various abstract spaces has been investigated deeply. The Banach contraction principle [1] is one of the initial and crucial results in this direction. Also, this principle has many generalizations. For instance, Alber and Guerre-Delabriere in [2] suggested a generalization of the Banach contraction mapping principle by introducing the concept of weak contraction in Hilbert spaces. In [2], the authors also proved that the result of Eslamian and Abkar [3] is equivalent to the result of Dutta and Choudhury [4]. Later, weakly contractive mappings and mappings satisfying other weak contractive inequalities have been discussed in several works, some of which are noted in [4–16].

In 2008, Dutta and Choudhury proved the following theorem.

Theorem 1 (see [4]). *Let (X, d) be a complete metric space, and let $f : X \to X$ be such that*

$$\psi(d(fx, fy)) \leq \psi(d(x, y)) - \phi(d(x, y)), \quad \text{for each } x, y \in X, \tag{2}$$

where $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous and nondecreasing, and $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. Then f has a fixed point in X.

Recently, Eslamian and Abkar [3] proved the following theorem.

Theorem 2 (see [3]). *Let (X, d) be a complete metric space, and let $f : X \to X$ be such that*

$$\psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x, y)), \quad \text{for each } x, y \in X, \tag{3}$$

where $\psi, \alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, and

$$\begin{gathered} \psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall t > 0, \\ \psi(t) = 0 \quad \text{if and only if } t = 0, \quad \alpha(0) = \beta(0) = 0. \end{gathered} \tag{4}$$

Then f has a fixed point in X.

In the recent, fixed point theory has developed rapidly in partially ordered metric spaces (e.g., [17–22]).

In 2012, Choudhury and Kundu [23] proved the following fixed point theorem as a generalization of Theorem 2.

Theorem 3 (see [23]). *Let $(X, \sqsubseteq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space and let $f : X \to X$ be a nondecreasing mapping such that*

$$\psi(d(fx, fy)) \leq \alpha(d(x, y)) - \beta(d(x, y)), \quad \text{for each } x, y \in X \text{ such that } x \sqsubseteq y, \tag{5}$$

where $\psi, \alpha, \beta : \mathbb{R}^+ \to \mathbb{R}^+$ are such that ψ is continuous and nondecreasing, α is continuous, β is lower semicontinuous, and

$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \forall t > 0, \qquad \psi(t) = 0 \quad \text{if and only if } t = 0, \quad \alpha(0) = \beta(0) = 0. \tag{6}$$

Also, if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{7}$$

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then f and g have a coincidence point in X.

In this paper, we prove two new fixed point theorems in the framework of partially ordered metric spaces. Our results generalize and improve many recent fixed point theorems in the literature.

2. Fixed Point Results (I)

We start with the following definition.

Definition 4. Let $(X, \sqsubseteq)$ be a partially ordered set and $f : X \to X$. Then f is said to be monotone nondecreasing if, for $x, y \in X$,

$$x \sqsubseteq y \Longrightarrow fx \sqsubseteq fy. \tag{8}$$

Let $(X, \sqsubseteq)$ be a partially ordered set. $x, y \in X$ are said to be comparable if either $x \sqsubseteq y$ or $y \sqsubseteq x$ holds.

In the section, we denote by Ψ the class of functions $\psi : \mathbb{R}^{+3} \to \mathbb{R}^+$ satisfying the following conditions:

(ψ_1) ψ is an increasing, continuous function in each coordinate;

(ψ_2) for all $t \in \mathbb{R}^+$, $\psi(t, t, t) \leq t$, $\psi(0, 0, t) \leq t$ and $\psi(t, 0, 0) \leq t$.

Next, we denote by Φ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(ϕ_1) ϕ is a continuous, nondecreasing function;

(ϕ_2) $\phi(t) > 0$ for $t > 0$ and $\phi(0) = 0$;

(ϕ_3) ϕ is subadditive; that is, $\phi(t_1 + t_2) \leq \phi(t_1) + \phi(t_2)$ for all $t_1, t_2 > 0$.

And we denote the following sets of functions:

$$\Theta = \{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ \quad \text{such that } \varphi \text{ is continuous}\}, \qquad \Xi = \{\xi : \mathbb{R}^+ \to \mathbb{R}^+ \quad \text{such that } \xi \text{ is lower continuous}\}. \tag{9}$$

Let X be a nonempty set, and let $(X, \sqsubseteq)$ be a partially ordered set endowed with a metric d. Then, the triple $(X, \sqsubseteq, d)$ is called a partially ordered complete metric space.

We now state the main fixed point theorem for $(\varphi, \psi, \phi, \xi)$-contractions in partially ordered metric spaces, as follows.

Theorem 5. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be monotone nondecreasing, and*

$$\varphi(d(fx, fy)) \leq \psi(\phi(d(x, y)), \phi(d(x, fx)), \phi(d(y, fy))) - \xi(\max\{d(x, y), d(x, fx), d(y, fy)\}), \tag{10}$$

for all comparable $x, y \in X$, where $\varphi \in \Theta$, $\psi \in \Psi$, $\phi \in \Phi$, and $\xi \in \Xi$, and

$$\varphi(t) - \phi(t) + \xi(t) > 0 \quad \forall t > 0, \qquad \varphi(t) = 0 \quad \text{if and only if } t = 0, \quad \phi(0) = \xi(0) = 0. \tag{11}$$

Suppose that either

(a) *f is continuous or*

(b) *if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{12}$$

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then f has a fixed point in X.

Proof. Since f is nondecreasing, by induction, we construct the sequence $\{x_n\}$ recursively as

$$x_n = f^n x_0 = fx_{n-1} \quad \forall n \in \mathbb{N}. \tag{13}$$

Thus, we also conclude that

$$x_0 \sqsubset x_1 = fx_0 \sqsubseteq x_2 = fx_1 \sqsubseteq \cdots \sqsubseteq x_n = fx_{n-1} \sqsubseteq \cdots. \tag{14}$$

If any two consecutive terms in (14) are equal, then the f has a fixed point, and hence the proof is completed. So we may assume that

$$d(x_{n-1}, x_n) \neq 0, \quad \forall n \in \mathbb{N}. \tag{15}$$

Now, we claim that $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$. If not, we assume that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$ for some $n \in \mathbb{N}$;

substituting $x = x_n$ and $y = x_{n+1}$ in (10) and using the definition of the function ψ, we have

$$\begin{aligned}
&\psi(\phi(d(x_n, x_{n+1})), \phi(d(x_n, fx_n)), \phi(d(x_{n+1}, fx_{n+1})))\\
&\quad = \psi(\phi(d(x_n, x_{n+1})), \phi(d(x_n, x_{n+1})), \phi(d(x_{n+1}, x_{n+2})))\\
&\quad \le \phi(d(x_{n+1}, x_{n+2})),\\
&\xi(\max\{d(x_n, x_{n+1}), d(x_n, fx_n), d(x_{n+1}, fx_{n+1})\})\\
&\quad = \xi(\max\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\})\\
&\quad = \xi(d(x_{n+1}, x_{n+2})),
\end{aligned} \tag{16}$$

and hence

$$\begin{aligned}
&\varphi(d(x_{n+1}, x_{n+2}))\\
&\quad = \varphi(d(fx_n, fx_{n+1}))\\
&\quad \le \phi(d(x_{n+1}, x_{n+2})) - \xi(d(x_{n+1}, x_{n+2})).
\end{aligned} \tag{17}$$

Since $\varphi(t) - \phi(t) + \xi(t) > 0$ for all $t > 0$, we have that $d(x_{n+1}, x_{n+2}) = 0$, which contradicts (15). Therefore, we conclude that

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n) \quad \forall n \in \mathbb{N}. \tag{18}$$

From the previous argument, we also have that for each $n \in \mathbb{N}$

$$\varphi(d(x_n, x_{n+1})) \le \phi(d(x_{n-1}, x_n)) - \xi(d(x_{n-1}, x_n)). \tag{19}$$

It follows in (18) that the sequence $\{d(x_n, x_{n+1})\}$ is monotone decreasing; it must converge to some $\eta \ge 0$. Taking limit as $n \to \infty$ in (19) and using the continuities of φ and ϕ and the lower semicontinuous of ξ, we get

$$\varphi(\eta) \le \phi(\eta) - \xi(\eta), \tag{20}$$

which implies that $\eta = 0$. So we conclude that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{21}$$

□

We next claim that $\{x_n\}$ is a Cauchy sequence; that is, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \ge n$, then $d(x_p, x_q) < \varepsilon$.

Suppose, on the contrary, that there exists $\epsilon > 0$ such that, for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \ge n$ satisfying

$$d(x_{q_n}, x_{p_n}) \ge \epsilon. \tag{22}$$

Further, corresponding to $q_n \ge n$, we can choose p_n in such a way that it the smallest integer with $p_n > q_n \ge n$ and $d(x_{q_n}, x_{p_n}) \ge \epsilon$. Therefore $d(x_{q_n}, x_{p_n-1}) < \epsilon$. Now we have that for all $n \in \mathbb{N}$

$$\begin{aligned}
\epsilon &\le d(x_{p_n}, x_{q_n})\\
&\le d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n})\\
&< d(x_{p_n}, x_{p_n-1}) + \epsilon.
\end{aligned} \tag{23}$$

By letting $n \to \infty$. we get that

$$\lim_{n\to\infty} d(x_{p_n}, x_{q_n}) = \epsilon. \tag{24}$$

On the other hand, we have

$$\begin{aligned}
&d(x_{p_n}, x_{q_n})\\
&\quad \le d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n-1}) + d(x_{q_n-1}, x_{q_n}),\\
&d(x_{p_n-1}, x_{q_n-1})\\
&\quad \le d(x_{p_n-1}, x_{p_n}) + d(x_{p_n}, x_{q_n}) + d(x_{q_n}, x_{q_n-1}).
\end{aligned} \tag{25}$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d(x_{p_n-1}, x_{q_n-1}) = \epsilon. \tag{26}$$

By (14), we have that the elements x_{p_n} and x_{q_n} are comparable. Substituting $x = x_{p_n-1}$ and $y = x_{q_n-1}$ in (10), we have that, for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\psi(\phi(d(x_{p_n-1}, x_{q_n-1})), \phi(d(x_{p_n-1}, fx_{p_n-1})),\\
&\quad \phi(d(x_{q_n-1}, fx_{q_n-1})))\\
&\qquad \le \psi(\phi(d(x_{p_n-1}, x_{q_n-1})), \phi(d(x_{p_n-1}, x_{p_n})),\\
&\qquad\quad \phi(d(x_{q_n-1}, x_{q_n}))),\\
&M(x_{p_n-1}, x_{q_n-1})\\
&\quad = \max\{d(x_{p_n-1}, x_{q_n-1}), d(x_{p_n-1}, fx_{p_n-1}),\\
&\qquad d(x_{q_n-1}, fx_{q_n-1})\}\\
&\quad = \max\{d(x_{p_n-1}, x_{q_n-1}), d(x_{p_n-1}, x_{p_n}),\\
&\qquad d(x_{q_n-1}, x_{q_n})\}.
\end{aligned} \tag{27}$$

By the previous argument and using inequality (10), we can conclude that

$$\begin{aligned}
\varphi(\epsilon) &\le \psi(\phi(\epsilon), 0, 0) - \xi(\epsilon)\\
&\le \phi(\epsilon) - \xi(\epsilon),
\end{aligned} \tag{28}$$

which implies that $\epsilon = 0$, a contradiction. Therefore, the sequence $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} x_n = \nu. \tag{29}$$

Suppose that (a) holds. Then

$$\nu = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f\nu. \tag{30}$$

Thus, ν is a fixed point in X.

Suppose that (b) holds; that is, $x_n \sqsubseteq \nu$ for all $n \in \mathbb{N}$. Substituting $x = x_n$ and $y = \nu$ in (10), we have that

$$\begin{aligned}\varphi(d(x_{n+1}, f\nu)) &= \varphi(d(fx_n, f\nu)) \\ &\le \psi(\phi(d(x_n,\nu)), \phi(d(x_n, fx_n)), \phi(d(\nu, f\nu))) \\ &\quad - \xi(\max\{d(x_n,\nu), d(x_n, fx_n), d(\nu, f\nu)\}).\end{aligned} \tag{31}$$

Taking limit as $n \to \infty$ in equality (31), we have

$$\begin{aligned}\varphi(d(\nu, f\nu)) &\le \psi(\phi(0), \phi(o), \phi(d(\nu, f\nu))) - \xi(d(\nu, f\nu)) \\ &\le \phi(d(\nu, f\nu)) - \xi(d(\nu, f\nu)),\end{aligned} \tag{32}$$

which implies that $d(\nu, f\nu) = 0$; that is $\nu = f\nu$. So we complete the proof.

If we let

$$\begin{aligned}&\psi(\phi(d(x,y)), \phi(d(x,fx)), \phi(d(y,fy))) \\ &\quad = \max\{\phi(d(x,y)), \phi(d(x,fx)), \phi(d(y,fy))\},\end{aligned} \tag{33}$$

it is easy to get the following theorem.

Theorem 6. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space. Let $f : X \to X$ be monotone nondecreasing, and*

$$\begin{aligned}\varphi(d(fx, fy)) &\le \max\{\phi(d(x,y)), \phi(d(x,fx)), \phi(d(y,fy))\} \\ &\quad - \xi(\max\{d(x,y), d(x,fx), d(y,fy)\}),\end{aligned} \tag{34}$$

for all comparable $x, y \in X$, where $\varphi \in \Theta$, $\phi \in \Phi$, and $\xi \in \Xi$, and

$$\begin{gathered}\varphi(t) - \phi(t) + \xi(t) > 0 \quad \forall t > 0, \\ \varphi(t) = 0 \quad \text{if and only if } t = 0, \quad \phi(0) = \xi(0) = 0.\end{gathered} \tag{35}$$

Suppose that either

(a) *f is continuous or*

(b) *if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{36}$$

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then f has a fixed point in X.

3. Fixed Point Results (II)

In the section, we denote by Ψ the class of functions $\psi : \mathbb{R}^{+3} \to \mathbb{R}^+$ satisfying the following conditions:

(ψ_1) ψ is an increasing and continuous function in each coordinate;

(ψ_2) for $t \in \mathbb{R}^+$, $\phi(t,t,t) \le t$, $\phi(t,0,0) \le t$, and $\phi(0,0,t) \le t$.

Next, we denote by Θ the class of functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(φ_1) φ is continuous and nondecreasing;

(φ_2) for $t > 0$, $\varphi(t) > 0$ and $\varphi(0) = 0$.

And we denote by Φ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions.

(ϕ_1) ϕ is continuous;

(ϕ_2) for $t > 0$, $\phi(t) > 0$ and $\phi(0) = 0$.

We now state the main fixed point theorem for the (ψ, φ, ϕ)-contractions in partially ordered metric spaces, as follows.

Theorem 7. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f : X \to X$ be monotone nondecreasing, and*

$$\begin{aligned}\varphi(d(fx, fy)) &\le \psi(\varphi(d(x,y)), \varphi(d(x,fx)), \varphi(d(y,fy))) \\ &\quad - \phi(M(x,y)) + L \cdot m(x,y),\end{aligned} \tag{37}$$

for all comparable $x, y \in X$ and $\psi \in \Psi$, $\varphi \in \Theta$, $\phi \in \Phi$, where $L > 0$ and

$$\begin{aligned}M(x,y) &= \max\{d(x,y), d(x,fx), d(y,fy)\}, \\ m(x,y) &= \min\{d(x,y), d(x,fx), d(y,fy), d(x,fy), \\ &\qquad d(y,fx)\}.\end{aligned} \tag{38}$$

Suppose that either

(a) *f is continuous or*

(b) *if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{39}$$

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then f has a fixed point in X.

Proof. If $fx_0 = x_0$, then the proof is finished. Suppose that $x_0 \sqsubset fx_0$. Since f is nondecreasing, by induction, we construct the sequence $\{x_n\}$ recursively as

$$x_n = f^n x_0 = fx_{n-1} \quad \forall n \in \mathbb{N}. \tag{40}$$

Thus, we also conclude that

$$x_0 \sqsubset x_1 = fx_0 \sqsubseteq x_2 = fx_1 \sqsubseteq \cdots \sqsubseteq x_n = fx_{n-1} \sqsubseteq \cdots. \tag{41}$$

We now claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{42}$$

Put $x = x_{n-1}$ and $y = x_n$ in (37). Note that

$$\begin{aligned} m(x_{n-1}, x_n) &= \min\{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), \\ &\quad d(x_n, fx_n), d(x_{n-1}, fx_n), d(x_n, fx_{n-1})\} \\ &= \min\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ &\quad d(x_{n-1}, x_{n+1}), d(x_n, x_n)\} \\ &= 0. \end{aligned} \tag{43}$$

So, we obtain that

$$\begin{aligned} &\varphi(d(x_n, x_{n+1})) \\ &= \varphi(d(fx_{n-1}, fx_n)) \\ &\le \psi(\varphi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, fx_{n-1})), \\ &\quad \varphi(d(x_n, fx_n))) - \phi(M(x_{n-1}, x_n)) \\ &\le \psi(\varphi(d(x_{n-1}, x_n)), \varphi(d(x_{n-1}, x_n)), \varphi(d(x_n, x_{n+1}))) \\ &\quad - \phi(M(x_{n-1}, x_n)), \end{aligned} \tag{44}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_n)\} \\ &= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned} \tag{45}$$

We now claim that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \forall n \in \mathbb{N}. \tag{46}$$

If not, we assume that $d(x_{n-1}, x_n) \le d(x_n, x_{n+1})$; then $\varphi(d(x_{n-1}, x_n)) \le \varphi(d(x_n, x_{n+1}))$, since φ is non-decreasing. Using inequality (44) and the conditions of the function ψ, we have that, for each $n \in \mathbb{N}$,

$$\varphi(d(x_n, x_{n+1})) \le \varphi(d(x_n, x_{n+1})) - \phi(d(x_n, x_{n+1})), \tag{47}$$

which implies that $\phi(d(x_n, x_{n+1})) = 0$, and hence $d(x_n, x_{n+1}) = 0$. This contradicts our initial assumption.

From the previous argument, we have that, for each $n \in \mathbb{N}$,

$$\begin{aligned} &\varphi(d(x_n, x_{n+1})) \le \varphi(d(x_{n-1}, x_n)) - \phi(d(x_{n-1}, x_n)), \\ &d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \end{aligned} \tag{48}$$

And since the sequence $\{d(x_n, x_{n+1})\}$ is decreasing, it must converge to some $\eta \ge 0$. Taking limit as $n \to \infty$ in (48) and by the continuity of φ and ϕ, we get

$$\varphi(\eta) \le \varphi(\eta) - \phi(\eta), \tag{49}$$

and so we conclude that $\phi(\eta) = 0$ and $\eta = 0$.

We next claim that $\{x_n\}$ is Cauchy; that is, for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that if $p, q \ge n$, then $d(x_p, x_q) < \varepsilon$.

Suppose, on the contrary, that there exists $\epsilon > 0$ such that, for any $n \in \mathbb{N}$, there are $p_n, q_n \in \mathbb{N}$ with $p_n > q_n \ge n$ satisfying

$$d(x_{q_n}, x_{p_n}) \ge \epsilon. \tag{50}$$

Further, corresponding to $q_n \ge n$, we can choose p_n in such a way that it the smallest integer with $p_n > q_n \ge n$ and $d(x_{q_n}, x_{p_n}) \ge \epsilon$. Therefore $d(x_{q_n}, x_{p_n-1}) < \epsilon$. By the rectangular inequality, we have

$$\begin{aligned} \epsilon &\le d(x_{p_n}, x_{q_n}) \\ &\le d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n}) \\ &< d(x_{p_n}, x_{p_n-1}) + \epsilon. \end{aligned} \tag{51}$$

Letting $n \to \infty$, then we get

$$\lim_{n\to\infty} d(x_{p_n}, x_{q_n}) = \epsilon. \tag{52}$$

On the other hand, we have

$$\begin{aligned} &d(x_{p_n}, x_{q_n}) \\ &\quad \le d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n-1}) + d(x_{q_n-1}, x_{q_n}), \\ &d(x_{p_n-1}, x_{q_n-1}) \\ &\quad \le d(x_{p_n-1}, x_{p_n}) + d(x_{p_n}, x_{q_n}) + d(x_{q_n}, x_{q_n-1}). \end{aligned} \tag{53}$$

By letting $n \to \infty$, we get that

$$\lim_{n\to\infty} d(x_{p_n-1}, x_{q_n-1}) = \epsilon. \tag{54}$$

Using inequalities (37), (52), and (54) and putting $x = x_{p_n-1}$ and $y = x_{q_n-1}$, we have that

$$\begin{aligned} &\varphi(d(x_{p_n}, x_{q_n})) \\ &= \varphi(d(fx_{p_n-1}, fx_{q_n-1})) \\ &\le \psi(\varphi(d(x_{p_n-1}, x_{q_n-1})), \varphi(d(x_{p_n-1}, fx_{p_n-1})), \\ &\quad \varphi(d(x_{q_n-1}, fx_{q_n-1}))) \\ &\quad - \phi(M(x_{p_n-1}, x_{q_n-1})) + L \cdot m(x_{p_n-1}, x_{q_n-1}) \\ &= \psi(\varphi(d(x_{p_n-1}, x_{q_n-1})), \varphi(d(x_{p_n-1}, x_{p_n})), \\ &\quad \varphi(d(x_{q_n-1}, x_{q_n}))) \\ &\quad - \phi(M(x_{p_n-1}, x_{q_n-1})) + L \cdot m(x_{p_n-1}, x_{q_n-1}), \end{aligned} \tag{55}$$

where

$$\begin{aligned} M\left(x_{p_n-1}, x_{q_n-1}\right) &= \max\left\{d\left(x_{p_n-1}, x_{q_n-1}\right), d\left(x_{p_n-1}, x_{p_n}\right), d\left(x_{q_n-1}, x_{q_n}\right)\right\}, \\ m\left(x_{p_n-1}, x_{q_n-1}\right) &= \min\left\{d\left(x_{p_n-1}, x_{q_n-1}\right), d\left(x_{p_n-1}, x_{p_n}\right), \right. \\ &\qquad \left. d\left(x_{q_n-1}, x_{q_n}\right), d\left(x_{p_n-1}, x_{q_n}\right), d\left(x_{q_n-1}, x_{p_n}\right)\right\}. \end{aligned} \tag{56}$$

Letting $n \to \infty$, then we obtain that

$$\begin{aligned} &\lim_{n\to\infty} M\left(x_{p_n-1}, x_{q_n-1}\right) = \epsilon, \\ &\lim_{n\to\infty} m\left(x_{p_n-1}, x_{q_n-1}\right) = 0, \\ &\varphi(\epsilon) \le \psi(\varphi(\epsilon), 0, 0) - \phi(\epsilon) \le \varphi(\epsilon) - \phi(\epsilon). \end{aligned} \tag{57}$$

This implies that $\phi(\epsilon) = 0$, and hence $\epsilon = 0$. So we get a contraction. Therefore $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $\nu \in X$ such that

$$\lim_{n\to\infty} x_n = \nu. \tag{58}$$

Suppose that (a) holds. Then

$$\nu = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f\nu. \tag{59}$$

Thus, ν is a fixed point in X

Suppose that (b) holds; that is, $x_n \sqsubseteq \nu$ for all $n \in \mathbb{N}$. Substituting $x = x_n$ and $y = \nu$ in (37), we have that

$$\begin{aligned} &\varphi(d(x_{n+1}, f\nu)) \\ &\quad = \varphi(d(fx_n, f\nu)) \\ &\quad \le \psi(\varphi(d(x_n, \nu)), \varphi(d(x_n, fx_n)), \varphi(d(\nu, f\nu))) \\ &\qquad - \phi(M(x_n, \nu)) + L \cdot m(x_n, \nu), \end{aligned} \tag{60}$$

where

$$\begin{aligned} M(x_n, \nu) &= \max\{d(x_n, \nu), d(x_n, fx_n), d(\nu, f\nu)\}, \\ m(x_n, \nu) &= \min\{d(x_n, \nu), d(x_n, fx_n), d(\nu, f\nu), d(x_n, f\nu) \\ &\qquad d(\nu, fx_n)\}. \end{aligned} \tag{61}$$

Letting $n \to \infty$, then we obtain that

$$\begin{aligned} &M(x_n, \nu) \longrightarrow d(\nu, f\nu), \qquad m(x_n, \nu) \longrightarrow 0, \\ &\varphi(d(\nu, f\nu)) \le \psi(\varphi(0), \varphi(0), \varphi(d(\nu, f\nu))) - \phi(d(\nu, f\nu)) \\ &\qquad \le \varphi(d(\nu, f\nu)) - \phi(d(\nu, f\nu)), \end{aligned} \tag{62}$$

which implies that $d(\nu, f\nu) = 0$; that is, $\nu = f\nu$. So we complete the proof. □

If we let

$$\begin{aligned} &\psi(\varphi(d(x, y)), \varphi(d(x, fx)), \varphi(d(y, fy))) \\ &\quad = \max\{\varphi(d(x, y)), \varphi(d(x, fx)), \varphi(d(y, fy))\}, \end{aligned} \tag{63}$$

it is easy to get the following theorem.

Theorem 8. *Let $(X, \sqsubseteq, d)$ be a partially ordered complete metric space, and let $f : X \to X$ be monotone nondecreasing, and*

$$\begin{aligned} &\varphi(d(fx, fy)) \\ &\quad \le \max\{\varphi(d(x, y)), \varphi(d(x, fx)), \varphi(d(y, fy))\} \\ &\qquad - \phi(M(x, y)) + L \cdot m(x, y), \end{aligned} \tag{64}$$

for all comparable $x, y \in X$ and $\varphi \in \Theta$, $\phi \in \Phi$, where $L > 0$ and

$$\begin{aligned} M(x, y) &= \max\{d(x, y), d(x, fx), d(y, fy)\}, \\ m(x, y) &= \min\{d(x, y), d(x, fx), d(y, fy), d(x, fy), \\ &\qquad d(y, fx)\}. \end{aligned} \tag{65}$$

Suppose that either

(a) *f is continuous or*

(b) *if any nondecreasing sequence $\{x_n\}$ in X converges to ν, then one assumes that*

$$x_n \sqsubseteq \nu \quad \forall n \in \mathbb{N}. \tag{66}$$

If there exists $x_0 \in X$ with $x_0 \sqsubseteq fx_0$, then f has a fixed point in X.

Acknowledgment

The authors would like to thank the referee(s) for the many useful comments and suggestions for the improvement of the paper.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 541079, 8 pages
http://dx.doi.org/10.1155/2013/541079

Research Article

Strong Convergence Theorems for Solutions of Equations of Hammerstein Type

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Received 11 March 2013; Accepted 18 April 2013

Academic Editor: Erdal Karapınar

We consider an auxiliary operator, defined in a real Hilbert space in terms of K and F, that is, monotone and Lipschitz mappings (resp., monotone and bounded mappings). We use an explicit iterative process that converges strongly to a solution of equation of Hammerstein type. Furthermore, our results improve related results in the literature.

1. Introduction

Let H be a real Hilbert space. A mapping $A : D(A) \subseteq H \to H$ is said to be monotone if $\langle Ax - Ay, x - y\rangle \geq 0$ for every $x, y \in D(A)$. A is called maximal monotone if it is monotone and the $R(I + rA) = H$, the range of $(I + rA)$, for each $r > 0$, where I is the identity mapping on H. A is said to satisfy the range condition if $\mathrm{cl}(D(A)) \subseteq R(I + rA)$ for each $r > 0$. For monotone mappings, there are many related equations of evolution. Several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's function, can be put in operator form as

$$u + KFu = 0, \tag{1}$$

where K and F are monotone mappings. In fact, (1) comes from the following integral equation of Hammerstein type [1]:

$$u(x) + \int_{\Omega} k(x, y) f(y, u(y))\, dy = h(x), \tag{2}$$

where dy is a σ-finite measure on the measure space Ω; the real kernel is defined by $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general, nonlinear, and h is a given function on Ω. If we now define an operator K by

$$Kv(x) = \int_{\Omega} k(x, y) v(y)\, dy, \quad x \in \Omega, \tag{3}$$

and the so-called superposition or Nemytskii operator by $Fu(y) := f(y, u(y))$, then (2) can be put in (1) (without loss of generality, we may assume that $h \equiv 0$).

Note that equations of Hammerstein type play a crucial role in the theory of optimal control systems and in automation and network theory, and several existence and uniqueness theorems have been proved for equations of the Hammerstein type. For details, one can refer to [2–7].

In 2005, Chidume and Zegeye [8] constructed an iterative process as follows:

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n (Fu_n - v_n) - \lambda_n \theta_n (u_n - w), \\ v_{n+1} &= v_n - \lambda_n (Kv_n + u_n) - \lambda_n \theta_n (v_n - w), \quad n \in \mathbb{N}, \end{aligned} \tag{4}$$

where H is a real Hilbert space, F and K: $H \to H$ are bounded monotone mappings satisfying the range condition, $w \in H$, and $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ are sequences in $(0, 1)$. Chidume and Zegeye [8] show that this sequence converges strongly to the solution of (1) under suitable conditions.

In 2011, Chidume and Ofoedu [9] introduced a coupled explicit iterative process as follows:

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n \alpha_n (Fu_n - v_n) - \lambda_n \theta_n (u_n - u_1), \\ v_{n+1} &= v_n - \lambda_n \alpha_n (Kv_n + u_n) - \lambda_n \theta_n (v_n - v_1), \quad n \in \mathbb{N}, \end{aligned} \tag{5}$$

where E is a uniformly smooth real Banach space, F and K: $E \to E$ are bounded and monotone mappings, and $\{\lambda_n\}_{n\in\mathbb{N}}$,

$\{\theta_n\}_{n\in\mathbb{N}}$, and $\{\alpha_n\}_{n\in\mathbb{N}}$ are sequences in $(0,1)$. Chidume and Ofoedu [9] gave a strong convergence theorem for approximation of the solution of (1) under suitable conditions.

In 2012, Chidume and Djitté [10] consider the following iterative process:

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n (Fu_n - v_n) - \lambda_n\theta_n (u_n - u_1), \\ v_{n+1} &= v_n - \lambda_n (Kv_n + u_n) - \lambda_n\theta_n (v_n - v_1), \quad n \in \mathbb{N}, \end{aligned} \tag{6}$$

where H is a real Hilbert space, K and F are bounded and maximal monotone mappings, $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ are sequences in $(0,1)$ and Chidume and Djitté [10] show that this iterative process converges to an approximate solution of nonlinear equations of Hammerstein type under suitable conditions.

Motivated by the previous works, in this paper, we consider an auxiliary operator, defined in a real Hilbert space in terms of K and F, that is monotone and Lipschitz mapping, or monotone and bounded mappings. We use an explicit iterative process that converges strongly to a solution of equation of Hammerstein type. Furthermore, our results improve related results in the literature.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively.

Lemma 1. *Let H be a real Hilbert space. One has $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$ for all $x, y \in H$.*

Lemma 2 (see [11]). *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}_{n\in\mathbb{N}}$ a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\{u_n\}_{n\in\mathbb{N}}$ a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} u_n < \infty$, $\{t_n\}_{n\in\mathbb{N}}$ a sequence of real numbers with $\limsup t_n \le 0$. Suppose that $a_{n+1} \le (1-\alpha_n)a_n + \alpha_n t_n + u_n$ for each $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} a_n = 0$.*

Let ℓ^∞ be the Banach space of bounded sequences with the supremum norm. A linear functional μ on ℓ^∞ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1,1,1,\ldots)$. For $x = (x_1, x_2, x_3, \ldots)$, the value $\mu(x)$ is also denoted by $\mu_n(x_n)$. A mean μ on ℓ^∞ is called a Banach limit if it satisfies $\mu_n(x_n) = \mu_n(x_{n+1})$. If μ is a Banach limit on ℓ^∞, then for $x = (x_1, x_2, x_3, \ldots) \in \ell^\infty$,

$$\liminf_{n\to\infty} x_n \le \mu_n(x_n) \le \limsup_{n\to\infty} x_n. \tag{7}$$

In particular, if $x = (x_1, x_2, x_3, \ldots) \in \ell^\infty$ and $\lim_{n\to\infty} x_n = a \in \mathbb{R}$, then we have $\mu(x) = \mu_n(x_n) = a$. For details, we can refer to [12].

Lemma 3 (see [13]). *Let α be a real number and $(x_0, x_1, \ldots) \in \ell^\infty$ such that $\mu_n x_n \le \alpha$ for all Banach limit μ on ℓ^∞. If $\limsup_{n\to\infty}(x_{n+1} - x_n) \le 0$, then, $\limsup_{n\to\infty} x_n \le \alpha$.*

Lemma 4 (see [14]). *Let C be a nonempty closed convex subset of a Hilbert space H, let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in H, and let μ be a Banach limit on ℓ^∞. Let $g : C \to \mathbb{R}$ be defined by $g(z) = \mu_n\|x_n - z\|^2$ for each $z \in C$. Then there exists a unique $z_0 \in C$ such that $g(z_0) = \min\{g(z) : z \in C\}$.*

Lemma 5 (see [15]). *Let H be a Hilbert space, let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in H, and let μ be a mean on ℓ^∞. Then, there exists a unique point $z_0 \in H$ such that $\mu_n\langle x_n, y\rangle = \langle z_0, y\rangle$ for each $y \in H$. Indeed, $z_0 \in \overline{\mathrm{co}}\{x_n : n \in \mathbb{N}\}$.*

Let H be a real Hilbert space. Let $W := H \times H$ with norm

$$\|z\| := \left(\|u\|^2 + \|v\|^2\right)^{1/2}, \quad \text{where } z = (u,v) \in W. \tag{8}$$

Hence, W is a real Hilbert space with inner product $\langle w_1, w_2\rangle = \langle u_1, u_2\rangle + \langle v_1, v_2\rangle$ for all $w_1 = (u_l, v_1)$, $w_2 = (u_2, v_2) \in W$ [8].

Lemma 6. *Let H be a real Hilbert space, and let $W := H \times H$. Let $F, K : H \to H$ be two mappings, and let $A : W \to W$ be defined by*

$$Aw := (Fu - v, Kv + u) \quad \text{for each } w = (u,v) \in W. \tag{9}$$

(i) *If F and K are monotone mappings, then A is a monotone mapping [8, Lemma 3.1].*

(ii) *If F and K are bounded mappings, then A is a bounded mapping [8, Lemma 3.1].*

(iii) *If F and K are Lipschitz mappings with Lipschitz constants L_1 and L_2, respectively, then A is a Lipschitz mapping. Indeed, the Lipschitz constant of A is $2(L+1)$, where $L := \max\{L_1, L_2\}$ [16, Remark 13.6].*

3. Main Results (I)

Let H be a real Hilbert space. Let $F, K : H \to H$ be two mappings, and let $A : H \times H \to H \times H$ be defined by $Aw = (Fu - v, Kv + u)$ for each $w = (u,v) \in H \times H$. Then, we observe that $\overline{u} \in H$ is a solution of $u + KFu = 0$ if and only if $\overline{w} = (\overline{u}, \overline{v})$ is a solution of $Aw = 0$ in $H \times H$ for $\overline{v} = F\overline{u}$.

Theorem 7. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ be sequences in $(0,1]$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary u_1, $v_1 \in H$ by*

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n\alpha_n (Fu_n - v_n) - \lambda_n\theta_n (u_n - u_1), \\ v_{n+1} &= v_n - \lambda_n\alpha_n (Kv_n + u_n) - \lambda_n\theta_n (v_n - v_1), \quad n \in \mathbb{N}. \end{aligned} \tag{10}$$

Suppose that one of the following conditions holds:

$$\begin{aligned}&\text{(i)}\ \lim_{n\to\infty}\left(\frac{\alpha_n^2}{\theta_n}\right)=\lim_{n\to\infty}\lambda_n\alpha_n=0;\\&\text{(ii)}\ \lim_{n\to\infty}\left(\frac{\lambda_n}{\theta_n}\right)=0;\\&\text{(iii)}\ \lim_{n\to\infty}\left(\frac{\alpha_n}{\theta_n}\right)=0.\end{aligned}\tag{11}$$

Then, the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, $\{Fu_n\}_{n\in\mathbb{N}}$, *and* $\{Kv_n\}_{n\in\mathbb{N}}$ *are bounded.*

Proof. Since F and K are Lipschitz mappings, we may assume that the Lipschitz constants of F and K are L_1 and L_2, respectively. Let

$$L=\max\{L_1,L_2\},\qquad r_0:=\frac{1}{32(L+1)^2}.\tag{12}$$

Let $W:=H\times H$ with the norm $\|w\|:=(\|u\|^2+\|v\|^2)^{1/2}$ for each $w=(u,v)\in H\times H$. Take any $\overline{u}\in H$ such that $\overline{u}$ is solution of $u+KFu=0$, and let $\overline{u}$ be fixed. Let $\overline{v}=F\overline{u}$ and $\overline{w}=(\overline{u},\overline{v})$. We observe that $\overline{u}=-K\overline{v}$. For each $n\in\mathbb{N}$, let $w_n:=(u_n,v_n)$.

For each $n\in\mathbb{N}$, it follows from Lemma 1 that

$$\begin{aligned}&\|u_{n+1}-\overline{u}\|^2\\&\quad=\|(1-\lambda_n\theta_n)(u_n-\overline{u})+\lambda_n(\theta_nu_1-\alpha_nFu_n+\alpha_nv_n-\theta_n\overline{u})\|^2\\&\quad\le(1-\lambda_n\theta_n)^2\|u_n-\overline{u}\|^2\\&\qquad+2\lambda_n\langle\theta_nu_1-\alpha_nFu_n+\alpha_nv_n-\theta_n\overline{u},u_{n+1}-\overline{u}\rangle\\&\quad\le(1-\lambda_n\theta_n)\|u_n-\overline{u}\|^2\\&\qquad+2\lambda_n\langle\theta_nu_1-\alpha_nFu_n+\alpha_nv_n-\theta_n\overline{u},u_{n+1}-\overline{u}\rangle.\end{aligned}\tag{13}$$

Similarly, we have

$$\begin{aligned}&\|v_{n+1}-\overline{v}\|^2\\&\quad\le(1-\lambda_n\theta_n)\|v_n-\overline{v}\|^2\\&\qquad+2\lambda_n\langle\theta_nv_1-\alpha_nKv_n-\alpha_nu_n-\theta_n\overline{u},v_{n+1}-\overline{v}\rangle.\end{aligned}\tag{14}$$

For each $n\in\mathbb{N}$, by (13) and (14), we know that

$$\begin{aligned}&\|w_{n+1}-\overline{w}\|^2\\&\quad\le(1-\lambda_n\theta_n)\|w_n-\overline{w}\|^2\\&\qquad+2\lambda_n\langle\theta_nw_1-\alpha_nAw_n-\theta_n\overline{w},w_{n+1}-\overline{w}\rangle\\&\quad=(1-\lambda_n\theta_n)\|w_n-\overline{w}\|^2+2\lambda_n\theta_n\langle w_1-\overline{w},w_{n+1}-\overline{w}\rangle\\&\qquad-2\lambda_n\alpha_n\langle Aw_n,w_{n+1}-\overline{w}\rangle.\end{aligned}\tag{15}$$

For each $n\in\mathbb{N}$, since A is monotone and $A\overline{w}=0$, we know that

$$\begin{aligned}&\langle Aw_n,w_{n+1}-\overline{w}\rangle\\&\quad=\langle Aw_n,w_n-\lambda_n\alpha_nAw_n-\lambda_n\theta_n(w_n-w_1)-\overline{w}\rangle\\&\quad=\langle Aw_n,w_n-\overline{w}\rangle+\langle Aw_n,-\lambda_n\alpha_nAw_n-\lambda_n\theta_n(w_n-w_1)\rangle\\&\quad\ge\langle Aw_n,-\lambda_n\alpha_nAw_n-\lambda_n\theta_n(w_n-w_1)\rangle.\end{aligned}\tag{16}$$

Hence, for each $n\in\mathbb{N}$, it follows from (15) and (16) that

$$\begin{aligned}&\|w_{n+1}-\overline{w}\|^2\\&\quad\le(1-\lambda_n\theta_n)\|w_n-\overline{w}\|^2+2\lambda_n\theta_n\langle w_1-\overline{w},w_{n+1}-\overline{w}\rangle\\&\qquad-2\lambda_n\alpha_n\langle Aw_n,w_{n+1}-\overline{w}\rangle\\&\quad\le(1-\lambda_n\theta_n)\|w_n-\overline{w}\|^2+2\lambda_n\theta_n\langle w_1-\overline{w},w_{n+1}-\overline{w}\rangle\\&\qquad+2\lambda_n\alpha_n\langle Aw_n,\lambda_n\alpha_nAw_n+\lambda_n\theta_nw_n-\lambda_n\theta_nw_1\rangle\\&\quad\le(1-\lambda_n\theta_n)\|w_n-\overline{w}\|^2+2\lambda_n\theta_n\langle w_1-\overline{w},w_{n+1}-\overline{w}\rangle\\&\qquad+2\lambda_n\alpha_n\|Aw_n\|\cdot(\lambda_n\alpha_n\|Aw_n\|+\lambda_n\theta_n\|w_n-w_1\|)\\&\quad\le(1-\lambda_n\theta_n)\|w_n-\overline{w}\|^2+2\lambda_n\theta_n\langle w_1-\overline{w},w_{n+1}-\overline{w}\rangle\\&\qquad+4\lambda_n\alpha_n(L+1)\|w_n-\overline{w}\|\\&\qquad\cdot(2\lambda_n\alpha_n(L+1)\|w_n-\overline{w}\|+\lambda_n\theta_n\|w_n-w_1\|)\\&\quad=(1-\lambda_n\theta_n)\|w_n-\overline{w}\|^2+2\lambda_n\theta_n\langle w_1-\overline{w},w_{n+1}-\overline{w}\rangle\\&\qquad+8\lambda_n^2\alpha_n^2(L+1)^2\|w_n-\overline{w}\|^2\\&\qquad+4\lambda_n^2\alpha_n\theta_n(L+1)\|w_n-\overline{w}\|\cdot\|w_n-w_1\|.\end{aligned}\tag{17}$$

For conditions (i)–(iii), we only need to consider one case since the proof is similar. Now, we assume that $\lim_{n\to\infty}(\lambda_n/\theta_n)=0$. Then, there exists $n_0\in\mathbb{N}$ such that $\lambda_n/\theta_n<r_0$ for each $n\ge n_0$. Choose $r>0$ such that $w_1\in B(\overline{w},r/4)$ and $w_{n_0}\in B(\overline{w},r/4)$. Let $B:=\overline{B(\overline{w},r)}$.

Now, we want to show that $w_n\in B$ for each $n\ge n_0$. Clearly, $w_{n_0}\in B(\overline{w},r)$. Suppose that $w_n\in B$ for some $n\ge n_0$. Then, $w_{n+1}\in B$. Indeed, if not, then we have

$$\|w_n-\overline{w}\|\le r<\|w_{n+1}-\overline{w}\|.\tag{18}$$

Hence, by (17) and (18), we get

$$\begin{aligned}\|w_{n+1}-\overline{w}\|^2\le{}&(1-\lambda_n\theta_n)\|w_{n+1}-\overline{w}\|^2\\&+2\lambda_n\theta_n\|w_1-\overline{w}\|\cdot\|w_{n+1}-\overline{w}\|\\&+8\lambda_n^2\alpha_n^2(L+1)^2\|w_n-\overline{w}\|\cdot\|w_{n+1}-\overline{w}\|\\&+4\lambda_n^2\alpha_n\theta_n(L+1)\|w_{n+1}-\overline{w}\|\cdot\|w_n-w_1\|.\end{aligned}\tag{19}$$

By (19), we have

$$\begin{aligned}
&\lambda_n\theta_n \|w_{n+1}-\overline{w}\| \\
&\quad\le 2\lambda_n\theta_n \|w_1-\overline{w}\| + 8\lambda_n^2\alpha_n^2(L+1)^2 \|w_n-\overline{w}\| \\
&\qquad + 4\lambda_n^2\alpha_n\theta_n (L+1) \|w_n-w_1\| \\
&\quad\le 2\lambda_n\theta_n\cdot\frac{r}{4} + 8\lambda_n^2\alpha_n^2(L+1)^2 r + 4\lambda_n^2\alpha_n\theta_n (L+1) \\
&\qquad \cdot (\|w_n-\overline{w}\| + \|w_1-\overline{w}\|) \\
&\quad\le 2\lambda_n\theta_n\cdot\frac{r}{4} + 8\lambda_n^2\alpha_n^2(L+1)^2 r + 4\lambda_n^2\alpha_n\theta_n (L+1)\cdot\frac{5r}{4}.
\end{aligned} \tag{20}$$

This implies that

$$\begin{aligned}
r &< \|w_{n+1}-\overline{w}\| \\
&\le \frac{1}{2}r + 8\lambda_n\frac{\alpha_n^2}{\theta_n}(L+1)^2 r + \frac{5\lambda_n\alpha_n\theta_n}{\theta_n}(L+1) r \\
&\le \frac{1}{2}r + \frac{8\lambda_n}{\theta_n}(L+1)^2 r + \frac{8\lambda_n}{\theta_n}(L+1) r \\
&\le r.
\end{aligned} \tag{21}$$

This leads to a contradiction. So, $w_{n+1}\in B$. Hence, by mathematical induction, we know that $\{w_n\}_{n\ge n_0}\subseteq B$. Therefore, $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are bounded sequences. Furthermore, $\{Fu_n\}_{n\in\mathbb{N}}$ and $\{Kv_n\}_{n\in\mathbb{N}}$ are bounded sequences since F and K are Lipschitz mappings. For conditions (ii) and (iii), the proof is similar. Therefore, the proof is completed. □

Remark 8. (i) Theorem 7 improves the conditions of [17, Theorem 3.1] if the space E in [17] is reduced to a real Hilbert space. Indeed, [17, Theorem 3.1] assumes that $\lim_{n\to\infty}\lambda_n = \lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}(\lambda_n/\theta_n) = \lim_{n\to\infty}(\alpha_n/\theta_n) = 0$.

(ii) Furthermore, we know that it is impossible to assume that $\alpha_n = \theta_n = 1$ in [17, Theorem 3.1]. However, we can choose $\alpha_n = \theta_n = 1$ in our result. Indeed, if $\alpha_n = \theta_n = 1$ and $\lambda_n = \beta_n$, then we have the following result as a special case of Theorem 7.

Corollary 9. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{aligned}
u_{n+1} &= u_n - \beta_n(Fu_n - v_n) - \beta_n(u_n - u_1), \\
v_{n+1} &= v_n - \beta_n(Kv_n + u_n) - \beta_n(v_n - v_1), \quad n\in\mathbb{N}.
\end{aligned} \tag{22}$$

If $\lim_{n\to\infty}\beta_n = 0$, then the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, $\{Fu_n\}_{n\in\mathbb{N}}$, and $\{Kv_n\}_{n\in\mathbb{N}}$ are bounded.

In fact, following the same argument as the proof of Theorem 7, we can get the following result.

Theorem 10. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ be sequences in $(0,1]$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{aligned}
u_{n+1} &= u_n - \alpha_n(Fu_n - v_n) - \theta_n(u_n - u_1), \\
v_{n+1} &= v_n - \alpha_n(Kv_n + u_n) - \theta_n(v_n - v_1), \quad n\in\mathbb{N}.
\end{aligned} \tag{23}$$

Suppose that one of the following conditions holds:

$$\begin{aligned}
&\text{(i)} \lim_{n\to\infty}\left(\frac{\alpha_n^2}{\theta_n}\right) = \lim_{n\to\infty}\alpha_n = 0; \\
&\text{(ii)} \lim_{n\to\infty}\left(\frac{\alpha_n}{\theta_n}\right) = 0.
\end{aligned} \tag{24}$$

Then, the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, $\{Fu_n\}_{n\in\mathbb{N}}$, and $\{Kv_n\}_{n\in\mathbb{N}}$ are bounded.

Remark 11. Corollary 9 is also a special case of Theorem 10.

Theorem 12. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ be sequences in $(0,1]$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary u_1, $v_1 \in H$ by*

$$\begin{aligned}
u_{n+1} &= u_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), \\
v_{n+1} &= v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad n\in\mathbb{N}.
\end{aligned} \tag{25}$$

Assume that

(i) $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$; $\lim_{n\to\infty}\lambda_n\alpha_n = \lim_{n\to\infty}\lambda_n\theta_n = 0$;

(ii) *one of the following conditions holds:*

$$\begin{aligned}
&\text{(a)} \lim_{n\to\infty}\left(\frac{\alpha_n^2}{\theta_n}\right) = 0; \\
&\text{(b)} \lim_{n\to\infty}\left(\frac{\lambda_n}{\theta_n}\right) = 0; \\
&\text{(c)} \lim_{n\to\infty}\left(\frac{\alpha_n}{\theta_n}\right) = 0;
\end{aligned} \tag{26}$$

(iii) *one of the following conditions holds:*

$$\begin{aligned}
&\text{(d)} \sum_{n=1}^{\infty}\lambda_n^2\alpha_n^2 < \infty; \\
&\text{(e)} \lim_{n\to\infty}\left(\frac{\lambda_n\alpha_n^2}{\theta_n}\right) = 0.
\end{aligned} \tag{27}$$

Then, there exists a subset $K_{\min}$ of $H\times H$ such that if $(\overline{u},\overline{v})\in K_{\min}$ with $\overline{v} = F\overline{u}$, then the sequence $\{u_n\}$ converges strongly to $\overline{u}$.

Proof. Let B and n_0 be the same as the proof of Theorem 7. Let μ be a Banach limit on ℓ^∞. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with $x_1 = w_1$ and $x_n = w_{n_0+n-2}$ for each $n \geq 2$. Clearly, $\{x_n\}_{n\in\mathbb{N}} \subseteq B$. By Lemma 4, there is a unique $x \in B$ such that

$$\mu_n \|x_n - x\|^2 = \min_{y\in B} \mu_n \|x_n - y\|^2. \tag{28}$$

Let $K_{\min} = \{x\}$, and we assume that $\overline{w} = (\overline{u}, \overline{v}) \in K_{\min}$. Let $t \in [0,1]$. Then, $tw_1 + (1-t)\overline{w} \in B$. Hence, for each $t \in (0,1)$, it follows from Lemma 1 that

$$\begin{aligned}\mu_n \|x_n - \overline{w}\|^2 &\leq \mu_n \|x_n - tw_1 - (1-t)\overline{w}\|^2 \\ &\leq \mu_n \|x_n - \overline{w}\|^2 \\ &\quad - 2t\mu_n \langle w_1 - \overline{w}, x_n - tw_1 - (1-t)\overline{w}\rangle .\end{aligned} \tag{29}$$

By Lemma 5, there exists $z_0 \in H$ such that $\mu_n\langle x_n, y\rangle = \langle z_0, y\rangle$ for each $y \in H$. By (29), for each $t \in (0,1)$,

$$\begin{aligned}&\langle w_1 - \overline{w}, z_0 - tw_1 - (1-t)\overline{w}\rangle \\ &\quad = \mu_n \langle w_1 - \overline{w}, x_n - tw_1 - (1-t)\overline{w}\rangle \leq 0.\end{aligned} \tag{30}$$

In (30), letting $t \to 0$, we get

$$\mu_n \langle w_1 - \overline{w}, x_n - \overline{w}\rangle = \langle w_1 - \overline{w}, z_0 - \overline{w}\rangle \leq 0. \tag{31}$$

Clearly,

$$\limsup_{n\to\infty} \left(\langle w_1 - \overline{w}, x_{n+1} - \overline{w}\rangle - \langle w_1 - \overline{w}, x_n - \overline{w}\rangle\right) \leq 0. \tag{32}$$

By (31), (32), and Lemma 3, we know that $\limsup_{n\to\infty}\langle w_1 - \overline{w}, x_n - \overline{w}\rangle \leq 0$.

Hence,

$$\limsup_{n\to\infty} \langle w_1 - \overline{w}, w_n - \overline{w}\rangle \leq 0. \tag{33}$$

By (17), (33), and Lemma 2, we know that $\lim_{n\to\infty}\|w_n - \overline{w}\| = 0$. Therefore, $\{u_n\}$ converges strongly to $\overline{u}$. □

Remark 13. (i) The conclusion of Theorem 12 is still true if $\sum_{n=1}^\infty \lambda_n\theta_n = \infty$, $\lim_{n\to\infty}(\alpha_n^2/\theta_n) = 0$, and $\lim_{n\to\infty}\lambda_n\alpha_n = \lim_{n\to\infty}\lambda_n\theta_n = 0$.

(ii) The conclusion of Theorem 12 is still true if $\sum_{n=1}^\infty \lambda_n\theta_n = \infty$, $\lim_{n\to\infty}(\alpha_n/\theta_n) = 0$, and $\lim_{n\to\infty}\lambda_n = 0$.

Remark 14. Following the same argument as in Remark 8, we know that Theorem 12 improves the conditions of [17, Theorem 3.2] if the space E is reduced to a real Hilbert space.

In Theorem 12, if $\alpha_n = \theta_n = 1$ and $\lambda_n = \beta_n$ for each $n \in \mathbb{N}$, then we have the following result.

Corollary 15. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz and monotone mappings. Suppose that $u + KFu = 0$ has a solution $\overline{u}$ in H. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{aligned}u_{n+1} &= u_n - \beta_n(Fu_n - v_n) - \beta_n(u_n - u_1), \\ v_{n+1} &= v_n - \beta_n(Kv_n + u_n) - \beta_n(v_n - v_1), \quad n \in \mathbb{N}.\end{aligned} \tag{34}$$

Assume that $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^\infty \beta_n = \infty$. Then there exists a subset $K_{\min}$ of $H \times H$ such that if $(\overline{u}, \overline{v}) \in K_{\min}$ with $\overline{v} = F\overline{u}$, then the sequence $\{u_n\}$ converges strongly to $\overline{u}$.

In Theorem 12, if $\alpha_n = 1$ for each $n \in \mathbb{N}$, then we have the following result.

Corollary 16. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$. Let $\{\theta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1]$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{aligned}u_{n+1} &= u_n - \lambda_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1), \\ v_{n+1} &= v_n - \lambda_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad n \in \mathbb{N}.\end{aligned} \tag{35}$$

Assume that $\sum_{n=1}^\infty \lambda_n\theta_n = \infty$, $\lim_{n\to\infty}\lambda_n\theta_n = 0$, and $\lim_{n\to\infty}(\lambda_n/\theta_n) = 0$. Then, there exists a subset $K_{\min}$ of $H \times H$ such that if $(\overline{u}, \overline{v}) \in K_{\min}$ with $\overline{v} = F\overline{u}$, then the sequence $\{u_n\}$ converges strongly to $\overline{u}$.

In fact, following the same argument as the proof of Theorem 12, we get the following result.

Theorem 17. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ be sequences in $(0,1]$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{aligned}u_{n+1} &= u_n - \alpha_n(Fu_n - v_n) - \theta_n(u_n - u_1), \\ v_{n+1} &= v_n - \alpha_n(Kv_n + u_n) - \theta_n(v_n - v_1), \quad n \in \mathbb{N}.\end{aligned} \tag{36}$$

Assume that $\sum_{n=1}^\infty \theta_n = \infty$, $\lim_{n\to\infty}\theta_n = 0$, and $\lim_{n\to\infty}(\alpha_n^2/\theta_n) = 0$. Then, there exists a subset $K_{\min}$ of $H \times H$ such that if $(\overline{u}, \overline{v}) \in K_{\min}$ with $\overline{v} = F\overline{u}$, then the sequence $\{u_n\}$ converges strongly to $\overline{u}$.

Remark 18. Corollary 15 is also a special case of Theorem 17.

4. Main Results (II)

In this section, we consider that F and K are bounded mappings.

Theorem 19. *Let H be a real Hilbert space. Let $F, K : H \to H$ be bounded and monotone mappings. Suppose that $u+KFu = 0$ has a solution in H. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ be sequences in $(0,1]$. Let $\{u_n\}_{n\in\mathbb{N}}$ and*

$\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary u_1, $v_1 \in H$ by

$$u_{n+1} = u_n - \lambda_n\alpha_n(Fu_n - v_n) - \lambda_n\theta_n(u_n - u_1),$$
$$v_{n+1} = v_n - \lambda_n\alpha_n(Kv_n + u_n) - \lambda_n\theta_n(v_n - v_1), \quad n \in \mathbb{N}. \tag{37}$$

Suppose that one of the following conditions holds:

$$\begin{aligned}
&\text{(i)} \lim_{n\to\infty}\left(\frac{\alpha_n^2}{\theta_n}\right) = \lim_{n\to\infty}\lambda_n\alpha_n = 0;\\
&\text{(ii)} \lim_{n\to\infty}\left(\frac{\lambda_n}{\theta_n}\right) = 0;\\
&\text{(iii)} \lim_{n\to\infty}\left(\frac{\lambda_n}{\theta_n}\right) = \lim_{n\to\infty}\lambda_n\alpha_n = 0;\\
&\text{(iv)} \lim_{n\to\infty}\left(\frac{\alpha_n}{\theta_n}\right) = 0.
\end{aligned} \tag{38}$$

Then, the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, $\{Fu_n\}_{n\in\mathbb{N}}$, and $\{Kv_n\}_{n\in\mathbb{N}}$ are bounded.

Proof. Since F and K are bounded, we know that A is bounded. Hence, $||A|| < \infty$. From the proof of Theorem 7, we know that

$$\begin{aligned}
&\|w_{n+1} - \overline{w}\|^2\\
&\quad\le (1-\lambda_n\theta_n)\|w_n - \overline{w}\|^2 + 2\lambda_n\theta_n\langle w_1 - \overline{w}, w_{n+1} - \overline{w}\rangle\\
&\qquad + 2\lambda_n\alpha_n\|Aw_n\|\cdot(\lambda_n\alpha_n\|Aw_n\| + \lambda_n\theta_n\|w_n - w_1\|)\\
&\quad\le (1-\lambda_n\theta_n)\|w_n - \overline{w}\|^2 + 2\lambda_n\theta_n\langle w_1 - \overline{w}, w_{n+1} - \overline{w}\rangle\\
&\qquad + 2\lambda_n^2\alpha_n^2\|A\|^2\cdot\|w_n\|^2 + 2\lambda_n^2\alpha_n\theta_n\|A\|\cdot\|w_n\|\cdot\|w_n - w_1\|
\end{aligned} \tag{39}$$

for each $n \in \mathbb{N}$.

Choose $r_0 > 0$ such that $\max\{||w_1||, ||\overline{w}||\} \le r_0/8$. For conditions (i)–(iv), we only need to consider one case since the proof is similar. Now, we assume that $\lim_{n\to\infty}(\lambda_n/\theta_n) = 0$. Then, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\lambda_n}{\theta_n} < \max\left\{\frac{8}{81\|A\|^2}, \frac{1}{10\|A\|}\right\} \tag{40}$$

for each $n \ge n_0$. Choose $r > 1$ such that $r_0 < r$ and $||w_{n_0}|| \le r/8$. Let $B := B(\overline{w}, r)$. Clearly, $||w_{n_0} - \overline{w}|| \le r/4$.

Now, we want to show that $w_n \in B$ for each $n \ge n_0$. Clearly, $w_{n_0} \in B(\overline{w}, r)$. Suppose that $w_n \in B$ for some $n \ge n_0$. Then, $w_{n+1} \in B$. Indeed, if not, then we have

$$\|w_n - \overline{w}\| \le r < \|w_{n+1} - \overline{w}\|. \tag{41}$$

Clearly, $||w_n|| \le ||w_n - \overline{w}|| + ||\overline{w}|| \le 9r/8$. Hence, by (39) and (41), we get

$$\begin{aligned}
&\|w_{n+1} - \overline{w}\|^2\\
&\quad\le (1-\lambda_n\theta_n)\|w_n - \overline{w}\|^2 + 2\lambda_n\theta_n\|w_1 - \overline{w}\|\cdot\|w_{n+1} - \overline{w}\|\\
&\qquad + 2\lambda_n^2\alpha_n^2\|A\|^2\cdot\|w_n\|^2 + 2\lambda_n^2\alpha_n\theta_n\|A\|\cdot\|w_n\|\cdot\|w_n - w_1\|\\
&\quad\le (1-\lambda_n\theta_n)r^2 + 2\lambda_n\theta_n\frac{r}{4}\cdot\|w_{n+1} - \overline{w}\|\\
&\qquad + 2\lambda_n^2\alpha_n^2\|A\|^2\cdot\frac{81}{64}r^2 + 2\lambda_n^2\alpha_n\theta_n\|A\|\cdot\frac{10}{8}r^2\\
&\quad\le (1-\lambda_n\theta_n)\|w_{n+1} - \overline{w}\|^2 + \lambda_n\theta_n\frac{r}{2}\cdot\|w_{n+1} - \overline{w}\|\\
&\qquad + \lambda_n^2\alpha_n^2\|A\|^2\cdot\frac{81}{32}r^2 + \lambda_n^2\alpha_n\theta_n\|A\|\cdot\frac{5}{2}r^2.
\end{aligned} \tag{42}$$

This implies that

$$\begin{aligned}
\|w_{n+1} - \overline{w}\|^2 \le{}& \frac{r}{2}\cdot\|w_{n+1} - \overline{w}\| + \frac{81}{32}\cdot\frac{\lambda_n\alpha_n^2}{\theta_n}\|A\|^2\cdot r^2\\
&+ \frac{5}{2}\cdot\frac{\lambda_n\alpha_n\theta_n}{\theta_n}\|A\|\cdot r^2.
\end{aligned} \tag{43}$$

Furthermore, we get

$$\begin{aligned}
r < \|w_{n+1} - \overline{w}\| &\le \frac{r}{2} + \frac{81}{32}\frac{\lambda_n\alpha_n^2}{\theta_n}\|A\|^2\cdot r + \frac{5}{2}\cdot\frac{\lambda_n\alpha_n\theta_n}{\theta_n}\|A\|\cdot r\\
&\le \frac{r}{2} + \frac{81}{32}\frac{\lambda_n}{\theta_n}\|A\|^2\cdot r + \frac{5}{2}\cdot\frac{\lambda_n}{\theta_n}\|A\|\cdot r\\
&\le r.
\end{aligned} \tag{44}$$

This leads to a contradiction. So, $w_{n+1} \in B$. Hence, by mathematical induction, we know that $\{w_n\}_{n\ge n_0} \subseteq B$. Therefore, $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ are bounded sequences. Furthermore, $\{Fu_n\}_{n\in\mathbb{N}}$ and $\{Kv_n\}_{n\in\mathbb{N}}$ are bounded sequences since F and K are bounded mappings. □

Remark 20. (i) Theorem 19 improves the conditions of [9, Theorem 3.1] if the space E is reduced to a real Hilbert space. Indeed, [9, Theorem 3.1] assumes that $\lim_{n\to\infty}(\lambda_n/\theta_n) = \lim_{n\to\infty}(\alpha_n/\theta_n) = 0$.

(ii) Furthermore, we know that it is impossible to assume that $\alpha_n = \theta_n = 1$ in [9, Theorem 3.1]. However, we can choose $\alpha_n = \theta_n = 1$ in our result. Indeed, if $\alpha_n = \theta_n = 1$ and $\lambda_n = \beta_n$, then we have the following result as special case of Theorem 19.

Corollary 21. *Let H be a real Hilbert space. Let $F, K : H \to H$ be bounded and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, 1)$.*

Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by

$$\begin{aligned} u_{n+1} &= u_n - \beta_n (Fu_n - v_n) - \beta_n (u_n - u_1), \\ v_{n+1} &= v_n - \beta_n (Kv_n + u_n) - \beta_n (v_n - v_1), \quad n \in \mathbb{N}. \end{aligned} \tag{45}$$

If $\lim_{n\to\infty}\beta_n = 0$, then the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, $\{Fu_n\}_{n\in\mathbb{N}}$, and $\{Kv_n\}_{n\in\mathbb{N}}$ are bounded.

Following the same argument as the proof of Theorem 19, we get the following result. Note that Corollary 21 is also a special case of the following result.

Theorem 22. *Let H be a real Hilbert space. Let $F, K : H \to H$ be bounded and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ be sequences in $(0, 1]$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{aligned} u_{n+1} &= u_n - \alpha_n (Fu_n - v_n) - \theta_n (u_n - u_1), \\ v_{n+1} &= v_n - \alpha_n (Kv_n + u_n) - \theta_n (v_n - v_1), \quad n \in \mathbb{N}. \end{aligned} \tag{46}$$

Suppose that one of the following conditions holds:

$$\begin{aligned} &\text{(i)} \lim_{n\to\infty}\left(\frac{\alpha_n^2}{\theta_n}\right) = \lim_{n\to\infty}\alpha_n = 0; \\ &\text{(ii)} \lim_{n\to\infty}\left(\frac{\alpha_n}{\theta_n}\right) = 0. \end{aligned} \tag{47}$$

Then, the sequences $\{u_n\}_{n\in\mathbb{N}}$, $\{v_n\}_{n\in\mathbb{N}}$, $\{Fu_n\}_{n\in\mathbb{N}}$, and $\{Kv_n\}_{n\in\mathbb{N}}$ are bounded.

Following the similar argument as the proof of Theorem 12, we get the following result.

Theorem 23. *Let H be a real Hilbert space. Let $F, K : H \to H$ be bounded and monotone mappings. Suppose that $u+KFu = 0$ has a solution in H. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, 1)$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ be sequences in $(0, 1]$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary u_1, $v_1 \in H$ by*

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n\alpha_n (Fu_n - v_n) - \lambda_n\theta_n (u_n - u_1), \\ v_{n+1} &= v_n - \lambda_n\alpha_n (Kv_n + u_n) - \lambda_n\theta_n (v_n - v_1), \quad n \in \mathbb{N}. \end{aligned} \tag{48}$$

Assume that

(i) $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$; $\lim_{n\to\infty}\lambda_n\alpha_n = \lim_{n\to\infty}\lambda_n\theta_n = 0$;

(ii) *one of the following conditions holds:*

$$\begin{aligned} &\text{(a)} \lim_{n\to\infty}\left(\frac{\alpha_n^2}{\theta_n}\right) = 0; \\ &\text{(b)} \lim_{n\to\infty}\left(\frac{\lambda_n}{\theta_n}\right) = 0; \\ &\text{(c)} \lim_{n\to\infty}\left(\frac{\alpha_n}{\theta_n}\right) = 0; \end{aligned} \tag{49}$$

(iii) *one of the following conditions holds:*

$$\begin{aligned} &\text{(d)} \sum_{n=1}^{\infty}\lambda_n^2\alpha_n^2 < \infty; \qquad \sum_{n=1}^{\infty}\lambda_n^2\alpha_n\theta_n < \infty; \\ &\text{(e)} \sum_{n=1}^{\infty}\lambda_n^2\alpha_n < \infty; \\ &\text{(f)} \lim_{n\to\infty}\left(\frac{\lambda_n\alpha_n^2}{\theta_n}\right) = 0. \end{aligned} \tag{50}$$

Then, there exists a subset $K_{\min}$ of $H \times H$ such that if $(\overline{u}, \overline{v}) \in K_{\min}$ with $\overline{v} = F\overline{u}$, then the sequence $\{u_n\}$ converges strongly to $\overline{u}$.

Remark 24. (i) The conclusion of Theorem 23 is still true if $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$, $\lim_{n\to\infty}(\alpha_n^2/\theta_n) = 0$, and $\lim_{n\to\infty}\lambda_n\alpha_n = \lim_{n\to\infty}\lambda_n\theta_n = 0$. Furthermore, the conclusion of Theorem 23 is still true if $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$, $\lim_{n\to\infty}(\alpha_n/\theta_n) = 0$, and $\lim_{n\to\infty}\lambda_n = 0$.

(ii) Theorem 23 improves the conditions of [9, Theorem 3.2] if the space E is reduced to a real Hilbert space. Indeed, [9, Theorem 3.2] assumes that $\lim_{n\to\infty}(\lambda_n/\theta_n) = \lim_{n\to\infty}(\alpha_n/\theta_n) = 0$.

The following is a special case of Theorem 23.

Corollary 25. *Let H be a real Hilbert space. Let $F, K : H \to H$ be bounded and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, 1)$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary u_1, $v_1 \in H$ by*

$$\begin{aligned} u_{n+1} &= u_n - \beta_n (Fu_n - v_n) - \beta_n (u_n - u_1), \\ v_{n+1} &= v_n - \beta_n (Kv_n + u_n) - \beta_n (v_n - v_1), \quad n \in \mathbb{N}. \end{aligned} \tag{51}$$

Assume that $\sum_{n=1}^{\infty}\beta_n = \infty$ and $\lim_{n\to\infty}\beta_n = 0$. Then, there exists a subset $K_{\min}$ of $H \times H$ such that if $(\overline{u}, \overline{v}) \in K_{\min}$ with $\overline{v} = F\overline{u}$, then the sequence $\{u_n\}$ converges strongly to $\overline{u}$.

The following is also a special case of Theorem 23.

Corollary 26. *Let H be a real Hilbert space. Let $F, K : H \to H$ be bounded and monotone mappings. Suppose that $u+KFu = 0$ has a solution in H. Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, 1)$. Let $\{\theta_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, 1]$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary u_1, $v_1 \in H$ by*

$$\begin{aligned} u_{n+1} &= u_n - \lambda_n\alpha_n (Fu_n - v_n) - \lambda_n\theta_n (u_n - u_1), \\ v_{n+1} &= v_n - \lambda_n\alpha_n (Kv_n + u_n) - \lambda_n\theta_n (v_n - v_1), \quad n \in \mathbb{N}. \end{aligned} \tag{52}$$

Assume that $\sum_{n=1}^{\infty}\lambda_n\theta_n = \infty$, $\lim_{n\to\infty}\lambda_n\theta_n = 0$, and $\lim_{n\to\infty}(\lambda_n/\theta_n) = 0$. Then, there exists a subset $K_{\min}$ of $H \times H$ such that if $(\overline{u}, \overline{v}) \in K_{\min}$ with $\overline{v} = F\overline{u}$, then the sequence $\{u_n\}$ converges strongly to $\overline{u}$.

Furthermore, we get the following result. Note that Corollary 25 is also a special case of the following result.

Theorem 27. *Let H be a real Hilbert space. Let $F, K : H \to H$ be bounded and monotone mappings. Suppose that $u + KFu = 0$ has a solution in H. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\theta_n\}_{n\in\mathbb{N}}$ be sequences in $(0, 1]$. Let $\{u_n\}_{n\in\mathbb{N}}$ and $\{v_n\}_{n\in\mathbb{N}}$ be sequences in H defined iteratively from arbitrary $u_1, v_1 \in H$ by*

$$\begin{aligned} u_{n+1} &= u_n - \alpha_n (Fu_n - v_n) - \theta_n (u_n - u_1), \\ v_{n+1} &= v_n - \alpha_n (Kv_n + u_n) - \theta_n (v_n - v_1), \quad n \in \mathbb{N}. \end{aligned} \tag{53}$$

Assume that $\sum_{n=1}^{\infty} \theta_n = \infty$, $\lim_{n\to\infty}\theta_n = 0$, and $\lim_{n\to\infty}(\alpha_n^2/\theta_n) = 0$. Then, there exists a subset $K_{\min}$ of $H \times H$ such that if $(\overline{u}, \overline{v}) \in K_{\min}$ with $\overline{v} = F\overline{u}$, then the sequence $\{u_n\}$ converges strongly to $\overline{u}$.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 534127, 10 pages
http://dx.doi.org/10.1155/2013/534127

Research Article

Best Proximity Points for Generalized α-ψ-Proximal Contractive Type Mappings

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Received 19 February 2013; Accepted 7 April 2013

Academic Editor: Lai-Jiu Lin

We introduce a new class of non-self-contractive mappings. For such mappings, we study the existence and uniqueness of best proximity points. Several applications and interesting consequences of our obtained results are derived.

1. Introduction and Preliminaries

Let A and B be two nonempty subsets of a metric space (X, d). An element $x \in A$ is said to be a fixed point of a given map $T : A \to B$ if $Tx = x$. Clearly, $T(A) \cap A \neq \emptyset$ is a necessary (but not sufficient) condition for the existence of a fixed point of T. If $T(A) \cap A = \emptyset$, then $d(x, Tx) > 0$ for all $x \in A$ that is, the set of fixed points of T is empty. In a such situation, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point analysis has been developed in this direction.

An element $a \in A$ is called a best proximity point of T if

$$d(a, Ta) = d(A, B), \tag{1}$$

where

$$d(A, B) = \inf \{d(x, y) : x \in A, \ y \in B\}. \tag{2}$$

Because of the fact that $d(x, Tx) \geq d(A, B)$ for all $x \in A$, the global minimum of the mapping $x \mapsto d(x, Tx)$ is attained at a best proximity point. Clearly, if the underlying mapping is a self-mapping, then it can be observed that a best proximity point is essentially a fixed point. The goal of best proximity point theory is to furnish sufficient conditions that assure the existence of such points. For more details on this approach, we refer the reader to [1–12] and references therein.

Recently, Samet et al. [13] introduced a new class of contractive mappings called α-ψ-contractive type mappings. Let (X, d) be a metric space.

Definition 1. A self-mapping $T : X \to X$ is said to be an α-ψ-contraction, where $\alpha : X \times X \to [0, \infty)$ and ψ is a (c)-comparison function, if

$$\alpha(x, y) d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in X. \tag{3}$$

Definition 2. A self-mapping $T : X \to X$ is said to be α-admissible, where $\alpha : X \times X \to [0, \infty)$, if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Longrightarrow \alpha(Tx, Ty) \geq 1. \tag{4}$$

The main results obtained in [13] are the following fixed point theorems.

Theorem 3. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*

(iii) *T is continuous.*

Then, T has a fixed point; that is, there exists $x \in X$ such that $Tx = x$.

Theorem 4. *Let (X, d) be a complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$;

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in X$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* n.

Then, T *has a fixed point.*

Theorem 5. *In addition to the hypotheses of Theorem 3 (resp., Theorem 4), suppose that for all* $(x, y) \in X \times X$, *there exists* $z \in X$ *such that* $\alpha(x, z) \geq 1$ *and* $\alpha(y, z) \geq 1$. *Then we have a unique fixed point.*

It was shown in [13, 14] that various types of contractive mappings belong to the class of α-ψ-contractive type mappings (classical contractive mappings, contractive mappings on ordered metric spaces, cyclic contractive mappings, etc.). For other works in this direction, we refer the reader to [15, 16].

In a very recent paper, Jleli and Samet [17] established some best proximity point results for α-ψ-contractive type mappings. Before presenting the main results obtained in [17], we need to fix some notations and recall some definitions.

Let A and B, two nonempty subsets of a metric space (X, d). We will use the following notations:

$$d(A, B) := \inf\{d(a, b) : a \in A,\ b \in B\},$$
$$A_0 := \{a \in A : d(a, b) = d(A, B) \text{ for some } b \in B\}, \quad (5)$$
$$B_0 := \{b \in B : d(a, b) = d(A, B) \text{ for some } a \in A\}.$$

Definition 6. An element $x^* \in A$ is said to be a best proximity point of the non-self-mapping $T : A \to B$ if it satisfies the condition that

$$d(x^*, Tx^*) = d(A, B). \quad (6)$$

The following concept was introduced in [11].

Definition 7. Let (A, B) be a pair of nonempty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then, the pair (A, B) is said to have the P-property if and only if

$$\left.\begin{array}{l} d(x_1, y_1) = d(A, B) \\ d(x_2, y_2) = d(A, B) \end{array}\right\} \Longrightarrow d(x_1, x_2) = d(y_1, y_2), \quad (7)$$

where $x_1, x_2 \in A$ and $y_1, y_2 \in B$.

The following concepts were introduced in [17].

Definition 8. Let $T : A \to B$ and $\alpha : A \times A \to [0, \infty)$. We say that T is α-proximal admissible if

$$\left.\begin{array}{c} \alpha(x_1, x_2) \geq 1 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \Longrightarrow \alpha(u_1, u_2) \geq 1, \quad (8)$$

for all $x_1, x_2, u_1, u_2 \in A$.

Clearly, if $A = B$, T is α-proximal admissible implies that T is α-admissible.

Definition 9. A non-self-mapping $T : A \to B$ is said to be an α-ψ-proximal contraction, where $\alpha : A \times A \to [0, \infty)$ and ψ is a (c)-comparison function, if

$$\alpha(x, y)\, d(Tx, Ty) \leq \psi(d(x, y)), \quad \forall x, y \in A. \quad (9)$$

The main results obtained in [17] are the following.

Theorem 10. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X, d) *such that* A_0 *is nonempty. Let* $\alpha : A \times A \to [0, \infty)$ *and* ψ *be a (c)-comparison function. Suppose that* $T : A \to B$ *is a non-self-mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$, *and* (A, B) *satisfies the P-property;*

(ii) T *is* α*-proximal admissible;*

(iii) *there exist elements* x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq 1; \quad (10)$$

(iv) T *is a continuous* α-ψ*-proximal contraction.*

Then, there exists an element $x^* \in A_0$ *such that*

$$d(x^*, Tx^*) = d(A, B). \quad (11)$$

Theorem 11. *Let* A *and* B *be nonempty closed subsets of a complete metric space* (X, d) *such that* A_0 *is nonempty. Let* $\alpha : A \times A \to [0, \infty)$ *and* ψ *be a (c)-comparison function. Suppose that* $T : A \to B$ *is a non-self-mapping satisfying the following conditions:*

(i) $T(A_0) \subseteq B_0$, *and* (A, B) *satisfies the P-property;*

(ii) T *is* α*-proximal admissible;*

(iii) *there exist elements* x_0 *and* x_1 *in* A_0 *such that*

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq 1; \quad (12)$$

(iv) T *is an* α-ψ*-proximal contraction;*

(v) *if* $\{x_n\}$ *is a sequence in* A *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to x \in A$ *as* $n \to \infty$, *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x) \geq 1$ *for all* k.

Then, there exists an element $x^* \in A_0$ *such that*

$$d(x^*, Tx^*) = d(A, B). \quad (13)$$

Theorem 12. *In addition to the hypotheses of Theorem 10 (resp., Theorem 11), suppose that for all* $(x, y) \in \alpha^{-1}([0, 1[)$, *there exists* $z \in A_0$ *such that* $\alpha(x, z) \geq 1$ *and* $\alpha(y, z) \geq 1$. *Then,* T *has a unique best proximity point.*

In this paper, we extend and generalize the above results by introducing a new family of non-self-contractive mappings that will be called the class of generalized α-ψ-proximal contractive type mappings. For such mappings, we discuss the existence and uniqueness of best proximity points. Various applications and interesting consequences are derived from our main results.

2. Main Results

All the notations presented in the previous section will be used through this paper.

We denote by Ψ the set of nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that

$$\sum_{n=1}^{\infty} \psi^n(t) < \infty, \quad \forall t > 0, \tag{14}$$

where ψ^n is the nth iterate of ψ. These functions are known in the literature as (c)-comparison functions. It is easily proved that if ψ is a (c)-comparison function, then $\psi(t) < t$ for all $t > 0$.

We introduce the following concept.

Definition 13. A non-self-mapping $T : A \to B$ is said to be a generalized α-ψ-proximal contraction, where $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$, if

$$\alpha(x,y)\, d(Tx,Ty) \le \psi(M(x,y)), \quad \forall x,y \in A, \tag{15}$$

where

$$\begin{aligned} M(x,y) &= \max\Big\{d(x,y), \frac{1}{2}[d(x,Tx) + d(y,Ty)] - d(A,B), \\ &\qquad \frac{1}{2}[d(y,Tx) + d(x,Ty)] - d(A,B)\Big\}. \end{aligned} \tag{16}$$

Our first main result is the following best proximity point theorem.

Theorem 14. *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A,B) satisfies the P-property;*

(ii) *T is α-proximal admissible;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1,Tx_0) = d(A,B), \qquad \alpha(x_0,x_1) \ge 1; \tag{17}$$

(iv) *T is a continuous generalized α-ψ-proximal contraction.*

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*,Tx^*) = d(A,B). \tag{18}$$

Proof. From condition (iii), there exist elements x_0 and x_1 in A_0 such that

$$d(x_1,Tx_0) = d(A,B), \qquad \alpha(x_0,x_1) \ge 1. \tag{19}$$

Since $T(A_0) \subseteq B_0$, there exists $x_2 \in A_0$ such that

$$d(x_2,Tx_1) = d(A,B). \tag{20}$$

Now, we have

$$\begin{aligned} &\alpha(x_0,x_1) \ge 1, \\ &d(x_1,Tx_0) = d(A,B), \\ &d(x_2,Tx_1) = d(A,B). \end{aligned} \tag{21}$$

Since T is α-proximal admissible, this implies that $\alpha(x_1,x_2) \ge 1$. Thus, we have

$$d(x_2,Tx_1) = d(A,B), \qquad \alpha(x_1,x_2) \ge 1. \tag{22}$$

Again, since $T(A_0) \subseteq B_0$, there exists $x_3 \in A_0$ such that

$$d(x_3,Tx_2) = d(A,B). \tag{23}$$

Now, we have

$$\begin{aligned} &\alpha(x_1,x_2) \ge 1, \\ &d(x_2,Tx_1) = d(A,B), \\ &d(x_3,Tx_2) = d(A,B). \end{aligned} \tag{24}$$

Since T is α-proximal admissible, this implies that $\alpha(x_2,x_3) \ge 1$. Thus, we have

$$d(x_3,Tx_2) = d(A,B), \qquad \alpha(x_2,x_3) \ge 1. \tag{25}$$

Continuing this process, by induction, we can construct a sequence $\{x_n\} \subset A_0$ such that

$$\begin{aligned} &d(x_{n+1},Tx_n) = d(A,B), \\ &\alpha(x_n,x_{n+1}) \ge 1, \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{26}$$

Since (A,B) satisfies the P-property, we conclude from (26) that

$$d(x_n,x_{n+1}) = d(Tx_{n-1},Tx_n), \quad \forall n \in \mathbb{N}. \tag{27}$$

From condition (iv), that is, T is a generalized α-ψ-proximal contraction, for all $n \in \mathbb{N}$, we have

$$\alpha(x_{n-1},x_n)\, d(Tx_{n-1},Tx_n) \le \psi(M(x_{n-1},x_n)). \tag{28}$$

On the other hand, using (26) and (27), we have

$$\begin{aligned}
&M(x_{n-1},x_n)\\
&\quad=\max\Big\{d(x_{n-1},x_n),\\
&\qquad\frac{1}{2}\left[d(x_{n-1},Tx_{n-1})+d(x_n,Tx_n)\right]-d(A,B),\\
&\qquad\frac{1}{2}\left[d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)\right]-d(A,B)\Big\}\\
&\quad\le\max\Big\{d(x_{n-1},x_n),\\
&\qquad\frac{1}{2}\left[d(x_{n-1},x_n)+d(x_n,Tx_{n-1})\right.\\
&\qquad\quad\left.+d(x_n,x_{n+1})+d(x_{n+1},Tx_n)\right]-d(A,B),\\
&\qquad\frac{1}{2}\left[d(A,B)+d(x_{n-1},x_n)\right.\\
&\qquad\quad\left.+d(x_n,Tx_{n-1})+d(Tx_{n-1},Tx_n)\right]-d(A,B)\Big\}\\
&\quad=\max\Big\{d(x_{n-1},x_n),\\
&\qquad\frac{1}{2}\left[d(x_{n-1},x_n)+d(A,B)\right.\\
&\qquad\quad\left.+d(x_n,x_{n+1})+d(A,B)\right]-d(A,B),\\
&\qquad\frac{1}{2}\left[d(A,B)+d(x_{n-1},x_n)\right.\\
&\qquad\quad\left.+d(A,B)+d(x_n,x_{n+1})\right]-d(A,B)\Big\}\\
&\quad=\max\Big\{d(x_{n-1},x_n),\\
&\qquad\frac{1}{2}\left[d(x_{n-1},x_n)+d(x_n,x_{n+1})\right],\\
&\qquad\frac{1}{2}\left[d(x_{n-1},x_n)+d(x_n,x_{n+1})\right]\Big\}\\
&\quad=\max\Big\{d(x_{n-1},x_n),\frac{1}{2}\left[d(x_{n-1},x_n)+d(x_n,x_{n+1})\right]\Big\}\\
&\quad\le\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}.
\end{aligned}\tag{29}$$

Thus, we proved that

$$\begin{aligned}
&M(x_{n-1},x_n)\\
&\quad\le\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\},\quad\forall n\in\mathbb{N}.
\end{aligned}\tag{30}$$

Using the above inequality, (26), (27), and (28), and taking in consideration that ψ is a nondecreasing function, we get that

$$\begin{aligned}
&d(x_{n+1},x_n)\\
&\quad\le\psi(\max\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\}),\quad\forall n\in\mathbb{N}.
\end{aligned}\tag{31}$$

If for some $N\in\mathbb{N}\cup\{0\}$, we have $x_{N+1}=x_N$, from (26), we get that $d(x_N,Tx_N)=d(A,B)$; that is, x_N is a best proximity point. So, we can suppose that

$$d(x_{n+1},x_n)>0,\quad\forall n\in\mathbb{N}\cup\{0\}.\tag{32}$$

Suppose that $\max\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\}=d(x_n,x_{n+1})$. Using (32) and since $\psi(t)<t$ for all $t>0$, we have

$$d(x_{n+1},x_n)\le\psi(d(x_n,x_{n+1}))<d(x_n,x_{n+1}),\tag{33}$$

which is a contradiction. Thus, we have

$$\begin{aligned}
&\max\{d(x_n,x_{n-1}),d(x_n,x_{n+1})\}\\
&\quad=d(x_n,x_{n-1}),\quad\forall n\in\mathbb{N}.
\end{aligned}\tag{34}$$

Now, from (31), we get that

$$d(x_{n+1},x_n)\le\psi(d(x_n,x_{n-1})),\quad\forall n\in\mathbb{N}.\tag{35}$$

Using the monotony of ψ, by induction, it follows from (35) that

$$d(x_{n+1},x_n)\le\psi^n(d(x_1,x_0)),\quad\forall n\in\mathbb{N}\cup\{0\}.\tag{36}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence in the metric space (X,d). Let $\varepsilon>0$ be fixed. Since $\sum_{n=1}^{\infty}\psi^n(d(x_1,x_0))<\infty$, there exists some positive integer $h=h(\varepsilon)$ such that

$$\sum_{k\ge h}\psi^k(d(x_1,x_0))<\varepsilon.\tag{37}$$

Let $m>n>h$, using the triangular inequality, (36) and (37), we obtain

$$\begin{aligned}
d(x_n,x_m)&\le\sum_{k=n}^{m-1}d(x_k,x_{k+1})\\
&\le\sum_{k=n}^{m-1}\psi^k(d(x_1,x_0))\\
&\le\sum_{k\ge h}\psi^k(d(x_1,x_0))\\
&<\varepsilon.
\end{aligned}\tag{38}$$

Thus, $\{x_n\}$ is a Cauchy sequence in the metric space (X,d). Since (X,d) is complete and A is closed, there exists some $x^*\in A$ such that $x_n\to x^*$ as $n\to\infty$. On the other hand, T is a continuous mapping. Then, we have $Tx_n\to Tx^*$ as $n\to\infty$. The continuity of the metric function d implies that $d(A,B)=d(x_{n+1},Tx_n)\to d(x^*,Tx^*)$ as $n\to\infty$. Therefore, $d(x^*,Tx^*)=d(A,B)$. This completes the proof of the theorem. □

In the next result, we remove the continuity hypothesis, assuming the following condition in A:

(H) If $\{x_n\}$ is a sequence in A such that $\alpha(x_n,x_{n+1})\ge1$ for all n and $x_n\to x\in A$ as $n\to\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)},x)\ge1$ for all k.

Theorem 15. *Let A and B be nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $\alpha : A \times A \to [0,\infty)$ and $\psi \in \Psi$. Suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A,B) satisfies the P-property;*

(ii) *T is α-proximal admissible;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A,B), \qquad \alpha(x_0, x_1) \geq 1; \tag{39}$$

(iv) *(H) holds, and T is a generalized α-ψ-proximal contraction.*

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A,B). \tag{40}$$

Proof. Following the proof of Theorem 14, there exists a Cauchy sequence $\{x_n\} \subset A$ such that (26) holds, and $x_n \to x^* \in A$ as $n \to \infty$. From the condition (H), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq 1$ for all k. Since T is a generalized α-ψ-proximal contraction, we get that

$$\begin{aligned} &d(Tx_{n(k)}, Tx^*) \\ &\quad \leq \alpha(x_{n(k)}, x^*)\, d(Tx_{n(k)}, Tx^*) \\ &\quad \leq \psi(M(x_{n(k)}, x^*)), \quad \forall k, \end{aligned} \tag{41}$$

where

$$\begin{aligned} &M(x_{n(k)}, x^*) \\ &\quad = \max\Bigg\{d(x_{n(k)}, x^*), \\ &\qquad \frac{d(x_{n(k)}, Tx_{n(k)}) + d(x^*, Tx^*)}{2} - d(A,B), \\ &\qquad \frac{d(x_{n(k)}, Tx^*) + d(x^*, Tx_{n(k)})}{2} - d(A,B)\Bigg\}. \end{aligned} \tag{42}$$

On the other hand, from (26), for all k, we have

$$\begin{aligned} &d(x^*, Tx^*) \\ &\quad \leq d(x^*, x_{n(k)+1}) + d(x_{n(k)+1}, Tx_{n(k)}) \\ &\qquad + d(Tx_{n(k)}, Tx^*) \\ &\quad = d(x^*, x_{n(k)+1}) + d(A,B) \\ &\qquad + d(Tx_{n(k)}, Tx^*). \end{aligned} \tag{43}$$

Thus, we have

$$\begin{aligned} &d(Tx_{n(k)}, Tx^*) \\ &\quad \geq d(x^*, Tx^*) - d(x^*, x_{n(k)+1}) - d(A,B), \quad \forall k. \end{aligned} \tag{44}$$

Combining (41) with (44), we get that

$$\begin{aligned} &d(x^*, Tx^*) - d(x^*, x_{n(k)+1}) - d(A,B) \\ &\quad \leq \psi(M(x_{n(k)}, x^*)), \quad \forall k. \end{aligned} \tag{45}$$

From (26), for all k, we have

$$\begin{aligned} &M(x_{n(k)}, x^*) \\ &= \max\Bigg\{d(x_{n(k)}, x^*), \\ &\qquad \frac{d(x_{n(k)}, Tx_{n(k)}) + d(x^*, Tx^*)}{2} - d(A,B), \\ &\qquad \frac{d(x_{n(k)}, Tx^*) + d(x^*, Tx_{n(k)})}{2} - d(A,B)\Bigg\} \\ &\leq \max\Bigg\{d(x_{n(k)}, x^*), \\ &\qquad \frac{d(x_{n(k)}, x_{n(k)+1}) + d(x_{n(k)+1}, Tx_{n(k)}) + d(x^*, Tx^*)}{2} \\ &\qquad - d(A,B), \\ &\qquad (d(x_{n(k)}, x^*) + d(x^*, Tx^*) + d(x^*, x_{n(k)+1}) \\ &\qquad + d(x_{n(k)+1}, Tx_{n(k)})) \times (2)^{-1} - d(A,B)\Bigg\} \\ &= \max\Bigg\{d(x_{n(k)}, x^*), \\ &\qquad \frac{d(x_{n(k)}, x_{n(k)+1}) + d(A,B) + d(x^*, Tx^*)}{2} \\ &\qquad - d(A,B), \\ &\qquad \frac{d(x_{n(k)}, x^*) + d(x^*, Tx^*) + d(x^*, x_{n(k)+1}) + d(A,B)}{2} \\ &\qquad - d(A,B)\Bigg\} \\ &:= \zeta(x_{n(k)}, x^*). \end{aligned} \tag{46}$$

Since ψ is a nondecreasing function, we get from (45) that

$$\begin{aligned} &d(x^*, Tx^*) - d(x^*, x_{n(k)+1}) - d(A,B) \\ &\quad \leq \psi(\zeta(x_{n(k)}, x^*)), \quad \forall k. \end{aligned} \tag{47}$$

Suppose that $d(x^*, Tx^*) - d(A, B) > 0$. In this case, we have

$$\lim_{k\to\infty} \zeta(x_{n(k)}, x^*) = \max\left\{0, \frac{d(x^*, Tx^*) - d(A, B)}{2}, \frac{d(x^*, Tx^*) - d(A, B)}{2}\right\}, \tag{48}$$

that is,

$$\lim_{k\to\infty} \zeta(x_{n(k)}, x^*) = \frac{d(x^*, Tx^*) - d(A, B)}{2}. \tag{49}$$

Since

$$\frac{d(x^*, Tx^*) - d(A, B)}{2} > 0, \tag{50}$$

for k large enough, we have $\zeta(x_{n(k)}, x^*) > 0$. On the other hand, we have $\psi(t) < t$ for all $t > 0$. Then, from (47), we get that

$$d(x^*, Tx^*) - d(x^*, x_{n(k)+1}) - d(A, B) < \zeta(x_{n(k)}, x^*), \quad \text{for } k \text{ large enough}. \tag{51}$$

Using (49) and letting $k \to \infty$ in the above inequality, we obtain that

$$d(x^*, Tx^*) - d(A, B) \le \frac{d(x^*, Tx^*) - d(A, B)}{2}, \tag{52}$$

which is a contradiction. Thus, we deduce that x^* is a best proximity point of T; that is, $d(x^*, Tx^*) = d(A, B)$. □

The next result gives us a sufficient condition that assures the uniqueness of the best proximity point. We need the following definition.

Definition 16. Let $T : A \to B$ be a non-self-mapping and $\alpha : A \times A \to [0, \infty)$. We say that T is (α, d) regular if for all $(x, y) \in \alpha^{-1}([0, 1[)$, there exists $z \in A_0$ such that

$$\alpha(x, z) \ge 1, \qquad \alpha(y, z) \ge 1. \tag{53}$$

Theorem 17. *In addition to the hypotheses of Theorem 14 (resp., Theorem 15), suppose that T is (α, d) regular. Then, T has a unique best proximity point.*

Proof. From the proof of Theorem 14 (resp., Theorem 15), we know that the set of best proximity points of T is nonempty ($x^* \in A_0$ is a best proximity point). Suppose that $y^* \in A_0$ is another best proximity point of T, that is,

$$d(Tx^*, x^*) = d(Ty^*, y^*) = d(A, B). \tag{54}$$

Using the P-property and (54), we get that

$$d(Tx^*, Ty^*) = d(x^*, y^*). \tag{55}$$

We distinguish two cases.

Case 1. If $\alpha(x^*, y^*) \ge 1$.

Since T is a generalized α-ψ-proximal contraction, using (55), we obtain that

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\le \alpha(x^*, y^*)\, d(Tx^*, Ty^*) \\ &\le \psi(M(x^*, y^*)), \end{aligned} \tag{56}$$

where from (54) and (55), we have

$$\begin{aligned} M(x^*, y^*) &= \max\Big\{d(x^*, y^*), \frac{1}{2}[d(x^*, Tx^*) + d(y^*, Ty^*)] - d(A, B), \frac{1}{2}[d(y^*, Tx^*) + d(x^*, Ty^*)] - d(A, B)\Big\} \\ &= \max\Big\{d(x^*, y^*), 0, \frac{1}{2}[d(y^*, Tx^*) + d(x^*, Ty^*)] - d(A, B)\Big\} \\ &\le \max\Big\{d(x^*, y^*), \frac{1}{2}[d(y^*, Ty^*) + d(Ty^*, Tx^*) + d(x^*, Tx^*) + d(Tx^*, Ty^*)] - d(A, B)\Big\} \\ &= \max\{d(x^*, y^*), d(x^*, y^*)\} = d(x^*, y^*). \end{aligned} \tag{57}$$

This equality with (56) imply that

$$d(x^*, y^*) \le \psi(d(x^*, y^*)). \tag{58}$$

Since $\psi(t) < t$ for all $t > 0$, the above inequality holds only if $d(x^*, y^*) = 0$, that is, $x^* = y^*$.

Case 2. If $\alpha(x^*, y^*) < 1$.

By hypothesis, there exists $z_0 \in A_0$ such that $\alpha(x^*, z_0) \ge 1$ and $\alpha(y^*, z_0) \ge 1$. Since $T(A_0) \subseteq B_0$, there exists $z_1 \in A_0$ such that

$$d(z_1, Tz_0) = d(A, B). \tag{59}$$

Now, we have

$$\begin{aligned} &\alpha(x^*, z_0) \ge 1, \\ &d(x^*, Tx^*) = d(A, B), \\ &d(z_1, Tz_0) = d(A, B). \end{aligned} \tag{60}$$

Since T is α-proximal admissible, we get that $\alpha(x^*, z_1) \ge 1$. Thus, we have

$$d(z_1, Tz_0) = d(A, B), \qquad \alpha(x^*, z_1) \ge 1. \tag{61}$$

Continuing this process, by induction, we can construct a sequence $\{z_n\}$ in A_0 such that

$$d(z_{n+1}, Tz_n) = d(A, B), \quad \alpha(x^*, z_n) \geq 1, \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{62}$$

Using the P-property and (62), we get that

$$d(z_{n+1}, x^*) = d(Tz_n, Tx^*), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{63}$$

Since T is a generalized α-ψ-proximal contraction, we have

$$\begin{aligned} &\alpha(z_{n+1}, x^*) d(Tz_n, Tx^*) \\ &\quad \leq \psi(M(z_n, x^*)), \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{64}$$

Combining the above inequality with (63), we get that

$$\begin{aligned} &\alpha(z_{n+1}, x^*) d(z_{n+1}, x^*) \\ &\quad \leq \psi(M(z_n, x^*)), \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{65}$$

This implies from (62) that

$$d(z_{n+1}, x^*) \leq \psi(M(z_n, x^*)), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{66}$$

On the other hand, from (63), for all $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned} &M(z_n, x^*) \\ &\quad = \max\Big\{d(z_n, x^*), \\ &\qquad \frac{1}{2}[d(z_n, Tz_n) + d(x^*, Tx^*)] - d(A, B), \\ &\qquad \frac{1}{2}[d(x^*, Tz_n) + d(z_n, Tx^*)] - d(A, B)\Big\} \\ &\quad = \max\Big\{d(z_n, x^*), \\ &\qquad \frac{1}{2}[d(z_n, x^*) + d(x^*, Tx^*) \\ &\qquad\quad + d(Tx^*, Tz_n) + d(A, B)] - d(A, B), \\ &\qquad \frac{1}{2}[d(x^*, Tx^*) + d(Tx^*, Tz_n) \\ &\qquad\quad + d(z_n, x^*) + d(x^*, Tx^*)] - d(A, B)\Big\} \\ &\quad = \max\Big\{d(z_n, x^*), \frac{1}{2}[d(z_n, x^*) + d(z_{n+1}, x^*)]\Big\} \\ &\quad \leq \max\{d(z_n, x^*), d(z_{n+1}, x^*)\}. \end{aligned} \tag{67}$$

Combining the above inequality with (71), we get that

$$\begin{aligned} &d(z_{n+1}, x^*) \\ &\quad \leq \psi(\max\{d(z_n, x^*), d(z_{n+1}, x^*)\}), \\ &\qquad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{68}$$

Suppose that for some N, we have $z_N = x^*$. From (63), we get that $z_n = x^*$ for all $n \geq N$. This implies that $z_n \to x^*$ as $n \to \infty$. Now, suppose that $d(z_n, x^*) > 0$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\psi(t) < t$ for all $t > 0$, the inequality (68) holds only if

$$\begin{aligned} &\max\{d(z_n, x^*), d(z_{n+1}, x^*)\} \\ &\quad = d(z_n, x^*), \quad \forall n \in \mathbb{N} \cup \{0\}. \end{aligned} \tag{69}$$

Now, we have

$$d(z_{n+1}, x^*) \leq \psi(d(z_n, x^*)), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{70}$$

By induction, we then derive

$$d(z_n, x^*) \leq \psi^n(d(z_0, x^*)), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{71}$$

Letting $n \to \infty$ in (71), we obtain that $z_n \to x^*$ as $n \to \infty$. So, in all cases, we have $z_n \to x^*$ as $n \to \infty$. Similarly, we can prove that $z_n \to y^*$ as $n \to \infty$. By uniqueness of the limit, we obtain that $x^* = y^*$. □

3. Applications

3.1. Standard Best Proximity Point Results. We have the following best proximity point result.

Corollary 18. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\psi \in \Psi$ and suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *$d(Tx, Ty) \leq \psi(M(x, y))$, for all $x, y \in A$.*

Then, there exists a unique element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B). \tag{72}$$

Proof. Consider the mapping $\alpha : A \times A \to [0, \infty)$ defined by:

$$\alpha(x, y) = 1, \quad \forall x, y \in A. \tag{73}$$

From the definition of α, clearly T is α-proximal admissible and also it is an α-ψ-proximal contraction. On the other hand, for any $x \in A_0$, since $T(A_0) \subseteq B_0$, there exists $y \in A_0$ such that $d(Tx, y) = d(A, B)$. Moreover, from condition (ii), T is a continuous mapping. Now, all the hypotheses of Theorem 14 are satisfied and the existence of the best proximity point follows from Theorem 14. The uniqueness is an immediate consequence of the definition of α and Theorem 17. □

Taking in Corollary 18 $\psi(t) = kt$, where $k \in (0, 1)$, we obtain the following best proximity point result.

Corollary 19. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Suppose*

that $T : A \to B$ is a non-self-mapping satisfying the following conditions:

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *there exists $k \in (0, 1)$ such that $d(Tx, Ty) \leq kM(x, y)$, for all $x, y \in A$.*

Then, there exists a unique element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B). \tag{74}$$

3.2. Best Proximity Points on a Metric Space Endowed with an Arbitrary Binary Relation. Before presenting our results, we need a few preliminaries.

Let (X, d) be a metric space and $\mathscr{R}$ be a binary relation over X. Denote

$$\mathscr{S} = \mathscr{R} \cup \mathscr{R}^{-1}; \tag{75}$$

this is the symmetric relation attached to R. Clearly,

$$x, y \in X, \quad x\mathscr{S}y \Longleftrightarrow x\mathscr{R}y \text{ or } y\mathscr{R}x. \tag{76}$$

Definition 20. We say that $T : A \to B$ is a proximal comparative mapping if

$$\left.\begin{array}{c} x_1\mathscr{S}x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \Longrightarrow u_1\mathscr{S}u_2, \tag{77}$$

for all $x_1, x_2, u_1, u_2 \in A$.

We have the following best proximity point result.

Corollary 21. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\mathscr{R}$ be a binary relation over X. Suppose that $T : A \to B$ is a continuous non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *T is a proximal comparative mapping;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0\mathscr{S}x_1; \tag{78}$$

(iv) *there exists $\psi \in \Psi$ such that*

$$x, y \in A, \quad x\mathscr{S}y \Longrightarrow d(Tx, Ty) \leq \psi(M(x, y)). \tag{79}$$

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B). \tag{80}$$

Proof. Define the mapping $\alpha : A \times A \to [0, \infty)$ by:

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x\mathscr{S}y, \\ 0 & \text{otherwise.} \end{cases} \tag{81}$$

Suppose that

$$\begin{gathered} \alpha(x_1, x_2) \geq 1, \\ d(u_1, Tx_1) = d(A, B), \\ d(u_2, Tx_2) = d(A, B), \end{gathered} \tag{82}$$

for some $x_1, x_2, u_1, u_2 \in A$. By the definition of α, we get that

$$\begin{gathered} x_1\mathscr{S}x_2, \\ d(u_1, Tx_1) = d(A, B), \\ d(u_2, Tx_2) = d(A, B). \end{gathered} \tag{83}$$

Condition (ii) implies that $u_1\mathscr{S}u_2$, which gives us from the definition of α that $\alpha(u_1, u_2) \geq 1$. Thus, we proved that T is α-proximal admissible. Condition (iii) implies that

$$d(x_1, Tx_0) = d(A, B), \qquad \alpha(x_0, x_1) \geq 1. \tag{84}$$

Finally, condition (iv) implies that

$$\alpha(x, y)\, d(Tx, Ty) \leq \psi(M(x, y)), \quad \forall x, y \in A, \tag{85}$$

that is, T is a generalized α-ψ-proximal contraction. Now, all the hypotheses of Theorem 14 are satisfied, and the desired result follows immediately from this theorem. □

In order to remove the continuity assumption, we need the following condition:

($\mathscr{H}$) if the sequence $\{x_n\}$ in X and the point $x \in X$ are such that $x_n\mathscr{S}x_{n+1}$ for all n and $\lim_{n\to\infty} d(x_n, x) = 0$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)}\mathscr{S}x$ for all k.

Corollary 22. *Let A and B be nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $\mathscr{R}$ be a binary relation over X. Suppose that $T : A \to B$ is a non-self-mapping satisfying the following conditions:*

(i) *$T(A_0) \subseteq B_0$, and (A, B) satisfies the P-property;*

(ii) *T is a proximal comparative mapping;*

(iii) *there exist elements x_0 and x_1 in A_0 such that*

$$d(x_1, Tx_0) = d(A, B), \quad x_0\mathscr{S}x_1; \tag{86}$$

(iv) *there exists $\psi \in \Psi$ such that*

$$x, y \in A, \quad x\mathscr{S}y \Longrightarrow d(Tx, Ty) \leq \psi(M(x, y)), \tag{87}$$

(v) *($\mathscr{H}$) holds.*

Then, there exists an element $x^* \in A_0$ such that

$$d(x^*, Tx^*) = d(A, B). \tag{88}$$

Proof. The result follows from Theorem 15 by considering the mapping α given by (81), and by observing that, condition ($\mathscr{H}$) implies condition (H). □

Corollary 23. *In addition to the hypotheses of Corollary 21 (resp., Corollary 22), suppose that the following condition holds: for all $(x, y) \in A \times A$ with $(x, y) \notin \mathcal{S}$, there exists $z \in A_0$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$. Then, T has a unique best proximity point.*

Proof. The result follows from Theorem 17 by considering the mapping α given by (81). □

3.3. Related Fixed Point Theorems

3.3.1. Fixed Points for Generalized α-ψ Contractive Type Mappings. The concept of generalized α-ψ contractive type mappings was introduced recently in [14].

Definition 24. Let A be a nonempty subset of a metric space (X, d) and $T : A \to A$ be a self-mapping. We say that T is a generalized α-ψ contractive mapping if there exist two functions $\alpha : A \times A \to [0, \infty)$ and $\psi \in \Psi$ such that for all $x, y \in A$, we have

$$\begin{aligned} &\alpha(x, y)\, d(Tx, Ty) \\ &\quad \le \psi\left(\max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \right.\right. \\ &\qquad \left.\left. \frac{d(y, Tx) + d(x, Ty)}{2}\right\}\right). \end{aligned} \tag{89}$$

Taking $A = B$ in Theorems 14–17, we obtain the following fixed point results established in [14].

Corollary 25. *Let A be a nonempty closed subset of a complete metric space (X, d). Let $T : A \to A$ be a generalized α-ψ contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in A$ such that $\alpha(x_0, Tx_0) \ge 1$;*

(iii) *T is continuous.*

Then, T has a fixed point.

Corollary 26. *Let A be a nonempty closed subset of a complete metric space (X, d). Let $T : A \to A$ be a generalized α-ψ contractive mapping satisfying the following conditions:*

(i) *T is α-admissible;*

(ii) *there exists $x_0 \in A$ such that $\alpha(x_0, Tx_0) \ge 1$;*

(iii) *condition (H) holds.*

Then, T has a fixed point.

Corollary 27. *In addition to the hypotheses of Corollary 25 (resp., Corollary 26), suppose that for all $(x, y) \in \alpha^{-1}([0, 1[)$, there exists $z \in A$ such that*

$$\alpha(x, z) \ge 1, \qquad \alpha(y, z) \ge 1. \tag{90}$$

Then, T has a unique fixed point.

3.3.2. Fixed Points on a Metric Space Endowed with an Arbitrary Binary Relation. We recall the following concept introduced in [18].

Let A be a nonempty closed subset of a complete metric space (X, d). Suppose that X is endowed with an arbitrary binary relation $\mathcal{R}$. We denote by $\mathcal{S}$ the symmetric relation attached to $\mathcal{R}$. Let $T : A \to A$ be a given mapping.

Definition 28. We say that $T : A \to A$ is a comparative mapping if T maps comparable elements into comparable elements, that is,

$$x, y \in A, \quad x\mathcal{S}y \Longrightarrow Tx\mathcal{S}Ty. \tag{91}$$

We have the following fixed point theorem.

Corollary 29. *Assume that $T : A \to A$ is a continuous comparative map, and*

$$\begin{aligned} &x, y \in A, \\ &x\mathcal{S}y \Longrightarrow \psi\left(\max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \right.\right. \\ &\qquad \left.\left. \frac{d(y, Tx) + d(x, Ty)}{2}\right\}\right), \end{aligned} \tag{92}$$

where $\psi \in \Psi$. Suppose also that there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$. Then, T has a fixed point.

Proof. It follows from Corollary 21 by taking $A = B$ and remarking that if $A = B$, a comparative map is a proximal comparative map. □

Remark that a self-mapping $T : A \to A$ satisfying the property (92) is not necessarily continuous (see Example 2.2 in [18]).

Similarly, Taking $A = B$ in Corollary 22, we obtain the following fixed point result.

Corollary 30. *Assume that $T : A \to A$ is a comparative map satisfying (92) for some $\psi \in \Psi$. Suppose also that there exists $x_0 \in X$ such that $x_0\mathcal{S}Tx_0$. If $(\mathcal{H})$ holds, then T has a fixed point.*

The uniqueness of the fixed point follows from Corollary 23 by taking $A = B$.

Corollary 31. *In addition to the hypotheses of Corollary 29 (resp., Corollary 30), suppose that the following condition holds: for all $(x, y) \in A \times A$ with $(x, y) \notin \mathcal{S}$, there exists $z \in A$ such that $x\mathcal{S}z$ and $y\mathcal{S}z$. Then, T has a unique fixed point.*

Acknowledgment

This work is supported by the Research Center, College of Science, King Saud University.

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Hindawi Publishing Corporation
Journal of Applied Mathematics
Volume 2013, Article ID 230392, 17 pages
http://dx.doi.org/10.1155/2013/230392

Research Article

A New Computational Technique for Common Solutions between Systems of Generalized Mixed Equilibrium and Fixed Point Problems

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Received 18 February 2013; Accepted 1 April 2013

Academic Editor: Wei-Shih Du

We introduce a new iterative algorithm for finding a common element of a fixed point problem of amenable semigroups of nonexpansive mappings, the set solutions of a system of a general system of generalized equilibria in a real Hilbert space. Then, we prove the strong convergence of the proposed iterative algorithm to a common element of the above three sets under some suitable conditions. As applications, at the end of the paper, we apply our results to find the minimum-norm solutions which solve some quadratic minimization problems. The results obtained in this paper extend and improve many recent ones announced by many others.

1. Introduction

Throughout this paper, we denoted by $\mathbb{R}$ the set of all real numbers. We always assume that H is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$ and C is a nonempty, closed, and convex subset of H. P_C denotes the metric projection of H onto C. A mapping $T : C \to C$ is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \le L\|x - y\|, \quad \forall x, y \in C. \tag{1}$$

If $0 < L < 1$, then T is a contraction, and if $L = 1$, then T is a nonexpansive mapping. We denote by $\mathrm{Fix}(T)$ the set of all fixed points set of the mapping T; that is, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$.

A mapping $F : C \to H$ is said to be *monotone* if

$$\langle Fx - Fy, x - y\rangle \ge 0, \quad \forall x, y \in C. \tag{2}$$

A mapping $F : C \to H$ is said to be *strongly monotone* if there exists $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \ge \eta\|x - y\|^2, \quad \forall x, y \in C. \tag{3}$$

Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $\Theta : C \times C \to \mathbb{R}$ an equilibrium bifunction, and $\Psi : C \to H$ a nonlinear mapping. The *generalized mixed equilibrium problem* is to find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) + \langle \Psi x^*, y - x^*\rangle \ge 0, \quad \forall y \in C, \tag{4}$$

which was introduced and studied by Peng and Yao [1]. The set of solutions of problem (4) is denoted by $\mathrm{GMEP}(\Theta, \varphi, \Psi)$. As special cases of problem (4), we have the following results.

(1) If $\Psi = 0$, then problem (4) reduces to *mixed equilibrium problem*. Find $x^* \in C$ such that

$$\Theta(x^*, y) + \varphi(y) - \varphi(x^*) \ge 0, \quad \forall y \in C, \tag{5}$$

which was considered by Ceng and Yao [2]. The set of solutions of problem (5) is denoted by $\mathrm{MEP}(\Theta)$.

(2) If $\varphi = 0$, then problem (4) reduces to *generalized equilibrium problem*. Find $x^* \in C$ such that

$$\Theta(x^*, y) + \langle \Psi x^*, y - x^*\rangle \ge 0, \quad \forall y \in C, \tag{6}$$

which was considered by S. Takahashi and W. Takahashi [3]. The set of solutions of problem (6) is denoted by GEP(Θ, Ψ).

(3) If $\Psi = \varphi = 0$, then problem (4) reduces to *equilibrium problem*. Find $x^* \in C$ such that

$$\Theta(x^*, y) \geq 0, \quad \forall y \in C. \tag{7}$$

The set of solutions of problem (7) is denoted by EP(Θ).

(4) If $\Theta = \varphi = 0$, then problem (4) reduces to *classical variational inequality problem*. Find $x^* \in C$ such that

$$\langle \Psi x^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{8}$$

The set of solutions of problem (8) is denoted by VI(C, Ψ). It is known that $x^* \in C$ is a solution of the problem (8) if and only if x^* is a fixed point of the mapping $P_C(I - \lambda\Psi)$, where $\lambda > 0$ is a constant and I is the identity mapping.

The problem (4) is very general in the sense that it includes several problems, namely, fixed point problems, optimization problems, saddle point problems, complementarity problems, variational inequality problems, minimax problems, Nash equilibrium problems in noncooperative games, and others as special cases. Numerous problems in physics, optimization, and economics reduce to find a solution of problem (4) (see, e.g., [4–9]). Several iterative methods to solve the fixed point problems, variational inequality problems, and equilibrium problems are proposed in the literature (see, e.g., [1–3, 10–18]) and the references therein.

Let $A_1, A_2 : C \to H$ be two mappings. Ceng and Yao [12] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &G_2(x^*, x) + \langle A_2 y^*, x - x^* \rangle + \frac{1}{\lambda_2}\langle x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &G_1(y^*, y) + \langle A_1 x^*, y - y^* \rangle + \frac{1}{\lambda_1}\langle y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in C, \end{aligned} \tag{9}$$

which is called *a general system of generalized equilibria*, where $\lambda_1 > 0$ and $\lambda_2 > 0$ are two constants. In particular, if $G_1 = G_2 = G$ and $A_1 = A_2 = A$, then problem (9) reduces to the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &G(x^*, x) + \langle A y^*, x - x^* \rangle + \frac{1}{\lambda_2}\langle x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &G(y^*, y) + \langle A x^*, y - y^* \rangle + \frac{1}{\lambda_1}\langle y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in C, \end{aligned} \tag{10}$$

which is called *a new system of generalized equilibria*, where $\lambda_1 > 0$ and $\lambda_2 > 0$ are two constants.

If $G_1 = G_2 = \Theta$, $A_1 = A_2 = A$, and $x^* = y^*$, then problem (9) reduces to problem (7).

If $G_1 = G_2 = 0$, then problem (9) reduces to *a general system of variational inequalities*. Find $(x^*, y^*) \in C \times C$ such that

$$\begin{aligned} &\langle \lambda_2 A_2 y^* + x^* - y^*, x - x^* \rangle \geq 0, \quad \forall x \in C, \\ &\langle \lambda_1 A_1 x^* + y^* - x^*, y - y^* \rangle \geq 0, \quad \forall y \in C, \end{aligned} \tag{11}$$

where $\lambda_1 > 0$ and $\lambda_2 > 0$ are two constants, which is introduced and considered by Ceng et al. [19].

In 2010, Ceng and Yao [12] proposed the following relaxed extragradient-like method for finding a common solution of generalized mixed equilibrium problems, a system of generalized equilibria (9), and a fixed point problem of a k-strictly pseudocontractive self-mapping S on C as follows:

$$\begin{aligned} z_n &= S_{r_n}^{(\Theta,\varphi)}(x_n - r_n \Psi x_n), \\ y_n &= S_{\lambda_1}^{G_1}\left[S_{\lambda_2}^{G_2}(z_n - \lambda_2 A_2 z_n) - \lambda_1 A_1 S_{\lambda_2}^{G_2}(z_n - \lambda_2 A_2 z_n)\right], \\ x_{n+1} &= \alpha_n u + \beta_n x_n + \gamma_n y_n + \delta_n S y_n, \quad \forall n \geq 0, \end{aligned} \tag{12}$$

where $\Psi, A_1, A_2 : C \to H$ are α-inverse strongly monotone, $\tilde{\alpha}_1$-inverse strongly monotone, and $\tilde{\alpha}_2$-inverse strongly monotone, respectively. They proved strong convergence of the related extragradient-like algorithm (12) under some appropriate conditions $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_n\} \subset [0, 1]$ satisfying $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$, for all $n \geq 0$, to $\hat{x} = P_\Omega \hat{x}$, where $\Omega = \mathrm{Fix}(S) \cap \mathrm{GMEP}(\Theta, \varphi, \Psi) \cap \mathrm{Fix}(K)$, with the mapping $K : C \to C$ defined by

$$Kx = S_{\lambda_1}^{G_1}\left[S_{\lambda_2}^{G_2}(x - \lambda_2 A_2 x) - \lambda_1 A_1 S_{\lambda_2}^{G_2}(x - \lambda_2 A_2 x)\right], \quad \forall x \in C. \tag{13}$$

Very recently, Ceng et al. [11] introduced an iterative method for finding fixed points of a nonexpansive mapping T on a nonempty, closed, and convex subset C in a real Hilbert space H as follows:

$$x_{n+1} = P_C\left[\alpha_n \gamma V x_n + (I - \alpha_n \mu F) T x_n\right], \quad \forall n \geq 0, \tag{14}$$

where P_C is a metric projection from H onto C, V is an L-Lipschitzian mapping with a constant $L \geq 0$, and F is a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa, \eta > 0$ and $0 < \mu < 2\eta/\kappa^2$. Then, they proved that the sequences generated by (14) converge strongly to a unique solution of variational inequality as follows:

$$\langle (\mu F - \gamma V) x^*, x^* - x \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{15}$$

In this paper, motivated and inspired by the previous facts, we first introduce the following problem of finding $(x_1^*, x_2^*, \ldots, x_M^*) \in C \times C \times \cdots \times C$ such that

$$\begin{aligned}
&G_M(x_1^*, x_1) + \langle A_M x_M^*, x_1 - x_1^* \rangle \\
&\quad + \frac{1}{\lambda_M} \langle x_1^* - x_M^*, x_1 - x_1^* \rangle \geq 0, \quad \forall x_1 \in C, \\
&G_{M-1}(x_M^*, x_M) + \langle A_{M-1} x_{M-1}^*, x_M - x_M^* \rangle \\
&\quad + \frac{1}{\lambda_{M-1}} \langle x_M^* - x_{M-1}^*, x_M - x_M^* \rangle \geq 0, \quad \forall x_M \in C, \\
&\qquad \vdots \\
&G_2(x_3^*, x_3) + \langle A_2 x_2^*, x_3 - x_3^* \rangle + \frac{1}{\lambda_2} \langle x_3^* - x_2^*, x_3 - x_3^* \rangle \geq 0, \\
&\qquad \forall x_3 \in C, \\
&G_1(x_2^*, x_2) + \langle A_1 x_1^*, x_2 - x_2^* \rangle + \frac{1}{\lambda_1} \langle x_2^* - x_1^*, x_2 - x_2^* \rangle \geq 0, \\
&\qquad \forall x_2 \in C,
\end{aligned} \tag{16}$$

which is called a *more general system of generalized equilibria in Hilbert spaces*, where $\lambda_i > 0$ for all $i \in \{1, 2, \ldots, M\}$. In particular, if $M = 2$, $x_1^* = x^*$, $x_2^* = y^*$, $x_1 = x$, and $x_2 = y$, then problem (16) reduces to problem (9). Finally, by combining the relaxed extragradient-like algorithm (12) with the general iterative algorithm (14), we introduce a new iterative method for finding a common element of a fixed point problem of a nonexpansive semigroup, the set solutions of a general system of generalized equilibria in a real Hilbert space. We prove the strong convergence of the proposed iterative algorithm to a common element of the previous three sets under some suitable conditions. Furthermore, we apply our results to finding the minimum-norm solutions which solve some quadratic minimization problem. The main result extends various results existing in the current literature.

2. Preliminaries

Let S be a semigroup. We denote by ℓ^∞ the Banach space of all bounded real-valued functionals on S with supremum norm. For each $s \in S$, we define the left and right translation operators $l(s)$ and $r(s)$ on $\ell^\infty(S)$ by

$$(l(s) f)(t) = f(st), \qquad (r(s) f)(t) = f(ts), \tag{17}$$

for each $t \in S$ and $f \in \ell^\infty(S)$, respectively. Let X be a subspace of $\ell^\infty(S)$ containing 1. An element μ in the dual space X^* of X is said to be a *mean* on X if $\|\mu\| = \mu(1) = 1$. It is well known that μ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s), \tag{18}$$

for each $f \in X$. We often write $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$.

Let X be a translation invariant subspace of $\ell^\infty(S)$ (i.e., $l(s)X \subset X$ and $r(s)X \subset X$ for each $s \in S$) containing 1. Then, a mean μ on X is said to be *left invariant* (resp., right invariant) if $\mu(l(s)f) = \mu(f)$ (resp., $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant [20–22]. S is said to be *left* (resp., *right*) *amenable* if X has a left (resp., right) invariant mean. S is a amenable if S is left and right amenable. In this case, $\ell^\infty(S)$ also has an invariant mean. As is well known, $\ell^\infty(S)$ is amenable when S is commutative semigroup; see [23]. A net $\{\mu_\alpha\}$ of means on X is said to be *left regular* if

$$\lim_\alpha \|l_s^* \mu_\alpha - \mu_\alpha\| = 0, \tag{19}$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Let C be a nonempty, closed, and convex subset of H. A family $\mathcal{S} = \{T(s) : s \in S\}$ is called a *nonexpansive semigroup* on C if for each $s \in S$, the mapping $T(s) : C \to C$ is nonexpansive and $T(st) = T(ts)$ for each $s, t \in S$. We denote by $\mathrm{Fix}(\mathcal{S})$ the set of common fixed point of $\mathcal{S}$; that is,

$$\mathrm{Fix}(\mathcal{S}) = \bigcap_{s \in S} \mathrm{Fix}(T(s)) = \bigcap_{s \in S} \{x \in C : T(s)x = x\}. \tag{20}$$

Throughout this paper, the open ball of radius r centered at 0 is denoted by B_r, and for a subset D of H by $\overline{\mathrm{co}}\, D$, we denote the closed convex hull of D. For $\epsilon > 0$ and a mapping $T : D \to H$, the set of ϵ-approximate fixed point of T is denoted by $F_\epsilon(T, D)$; that is, $F_\epsilon(T, D) = \{x \in D : \|x - Tx\| \leq \epsilon\}$.

In order to prove our main results, we need the following lemmas.

Lemma 1 (see [23–25]). *Let f be a function of a semigroup S into a Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact, and let X be a subspace of $\ell^\infty(S)$ containing all the functions $t \mapsto \langle f(t), x^* \rangle$ with $x^* \in E^*$. Then, for any $\mu \in X^*$, there exists a unique element f_μ in E such that*

$$\langle f_\mu, x^* \rangle = \mu_t \langle f(t), x^* \rangle, \tag{21}$$

for all $x^* \in E^*$. Moreover, if μ is a mean on X, then

$$\int f(t)\, d\mu(t) \in \overline{\mathrm{co}}\{f(t) : t \in S\}. \tag{22}$$

One can write f_μ by $\int f(t) d\mu(t)$.

Lemma 2 (see [23–25]). *Let C be a closed and convex subset of a Hilbert space H, $\mathcal{S} = \{T(s) : s \in S\}$ a nonexpansive semigroup from C into C such that $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$, and X a subspace of ℓ^∞ containing 1, the mapping $t \mapsto \langle T(t)x, y \rangle$ an element of X for each $x \in C$ and $y \in H$, and μ a mean on X.*

If one writes $T(\mu)x$ instead of $\int T_t x d\mu(t)$, then the following hold:

(i) *$T(\mu)$ is nonexpansive mapping from C into C;*

(ii) *$T(\mu)x = x$ for each $x \in \mathrm{Fix}(\mathcal{S})$;*

(iii) *$T(\mu)x \in \overline{\mathrm{co}}\{T_t x : t \in S\}$ for each $x \in C$;*

(iv) *if μ is left invariant, then $T(\mu)$ is a nonexpansive retraction from C onto* $\mathrm{Fix}(\mathcal{S})$.

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let C be a nonempty, closed, and convex subset of H. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Also, a mapping $I : C \to C$ denotes the identity mapping. For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x - y\|, \quad \forall y \in C. \tag{23}$$

Such a projection P_C is called the *metric projection* of H onto C. We know that P_C is a firmly nonexpansive mapping of H onto C; that is,

$$\langle x - y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{24}$$

It is known that

$$z = P_C x \Longleftrightarrow \langle x - z, y - z\rangle \le 0, \quad \forall x \in H,\ y \in C. \tag{25}$$

In a real Hilbert space H, it is well known that

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle, \tag{26}$$

$$\begin{aligned}\|\lambda x + (1-\lambda) y\|^2 &= \lambda\|x\|^2 + (1-\lambda)\|y\|^2 \\ &\quad - \lambda(1-\lambda)\|x - y\|^2,\end{aligned} \tag{27}$$

for all $x, y \in H$ and $\lambda \in [0,1]$.

If $A : C \to H$ is α-inverse strongly monotone, then it is obvious that A is $1/\alpha$-Lipschitz continuous. We also have that, for all $x, y \in C$ and $\lambda > 0$,

$$\begin{aligned}&\|(I - \lambda A)x - (I - \lambda)y\|^2 \\ &\quad = \|(x - y) - \lambda(Ax - Ay)\|^2 \\ &\quad = \|x - y\|^2 - 2\lambda\langle Ax - Ay, x - y\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\quad \le \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2.\end{aligned} \tag{28}$$

In particular, if $\lambda < 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

For solving the equilibrium problem, let us assume that the bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;

(A2) Θ is monotone, that is, $\Theta(x, y) + \Theta(y, x) \le 0$ for each $x, y \in C$;

(A3) Θ is upper semicontinuous, that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} \Theta(tz + (1-t)x, y) \le \Theta(x, y); \tag{29}$$

(A4) $\Theta(x, \cdot)$ is convex and weakly lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for all $z \in C \setminus D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0; \tag{30}$$

(B2) C is a bounded set.

Lemma 3 (see [1]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions $(A1) - (A4)$, and let $\varphi : C \to \mathbb{R}$ be a lower semicontinuous and convex function. For $r > 0$ and $x \in H$, define a mapping $S_r^{(\Theta,\varphi)} : H \to C$ as follows:*

$$\begin{aligned}&S_r^{(\Theta,\varphi)}(x) \\ &\quad = \Big\{y \in C : \Theta(y, z) + \varphi(z) - \varphi(y) \\ &\qquad + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall z \in C\Big\}.\end{aligned} \tag{31}$$

Assume that either $(B1)$ or $(B2)$ holds. Then, the following hold:

(i) *$S_r^{(\Theta,\varphi)} \ne \emptyset$ for all $x \in H$ and $S_r^{(\Theta,\varphi)}$ is single valued;*

(ii) *$S_r^{(\Theta,\varphi)}$ is firmly nonexpansive, that is, for all $x, y \in H$,*

$$\left\|S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y\right\|^2 \le \left\langle S_r^{(\Theta,\varphi)}x - S_r^{(\Theta,\varphi)}y, x - y\right\rangle; \tag{32}$$

(iii) $\mathrm{Fix}(S_r^{(\Theta,\varphi)}) = \mathrm{MEP}(\Theta, \varphi)$;

(iv) $\mathrm{MEP}(\Theta, \varphi)$ *is closed and convex.*

Remark 4. If $\varphi = 0$, then $S_r^{(\Theta,\varphi)}$ is rewritten as S_r^{Θ} (see [12, Lemma 2.1] for more details).

Lemma 5 (see [26]). *Let $\{x_n\}$ and $\{l_n\}$ be bounded sequences in a Banach space X, and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)l_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty}\|l_n - x_n\| = 0$.*

Lemma 6 (Demiclosedness Principle [27]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \ne \emptyset$. If $\{x_n\}$ is a sequence in C that converges weakly to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$; in particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

Lemma 7 (see [28]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \sigma_n)a_n + \delta_n, \tag{33}$$

where $\{\sigma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=0}^{\infty}\sigma_n = \infty$;

(ii) $\limsup_{n\to\infty}(\delta_n/\sigma_n) \le 0$ *or* $\sum_{n=0}^{\infty}|\delta_n| < \infty$.

Then, $\lim_{n\to\infty}a_n = 0$.

The following lemma can be found in [29, 30]. For the sake of the completeness, we include its proof in a real Hilbert space version.

Lemma 8. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $F : C \to X$ be a κ-Lipschitzian and η-strongly monotone operator. Let $0 < \mu < 2\eta/\kappa^2$ and $\tau = \mu(\eta - \mu\kappa^2/2)$. Then, for each $t \in (0, \min\{1, 1/2\tau\})$, the mapping $S : C \to H$ defined by $S := I - t\mu F$ is a contraction with constant $1 - t\tau$.*

Proof. Since $0 < \mu < 2\eta/\kappa^2$ and $t \in (0, \min\{1, 1/2\tau\})$, this implies that $1 - t\tau \in (0, 1)$. For all $x, y \in C$, we have

$$\begin{aligned}
\|Sx - Sy\|^2 &= \|(I - t\mu F)x - (I - t\mu F)y\|^2 \\
&= \|(x - y)x - t\mu(Fx - Fy)\|^2 \\
&= \|x - y\|^2 - 2t\mu\langle Fx - Fy, x - y\rangle \\
&\quad + t^2\mu^2\|Fx - Fy\|^2 \\
&\le \|x - y\|^2 - 2t\mu\eta\|x - y\|^2 \\
&\quad + t^2\mu^2\kappa^2\|x - y\|^2 \\
&\le \left[1 - t\mu\left(2\eta - \mu\kappa^2\right)\right]\|x - y\|^2 \\
&= \left[1 - 2t\mu\left(\eta - \frac{\mu\kappa^2}{2}\right)\right]\|x - y\|^2 \\
&\le \left[1 - t\mu\left(\eta - \frac{\mu\kappa^2}{2}\right)\right]^2\|x - y\|^2 \\
&= (1 - t\tau)^2\|x - y\|^2.
\end{aligned} \tag{34}$$

It follows that

$$\|Sx - Sy\| \le (1 - t\tau)\|x - y\|. \tag{35}$$

Hence, we have that $S := I - t\mu F$ is a contraction with constant $1 - t\tau$. This completes the proof. □

Lemma 9. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A_i : C \to H$ $(i = 1, 2, \ldots, M)$ be a finite family of α_i-inverse strongly monotone operator. Let $K : C \to C$ be a mapping defined by*

$$\begin{aligned}
Kx := {} & S_{\lambda_M}^{G_M}(I - \lambda_M A_M) S_{\lambda_{M-1}}^{G_{M-1}} \\
& \times (I - \lambda_{M-1}A_{M-1}) \cdots S_{\lambda_1}^{G_1}(I - \lambda_1 A_1)x, \quad \forall x \in C.
\end{aligned} \tag{36}$$

If $0 < \lambda_i < 2\alpha_i$ for all $i = 1, 2, \ldots, M$, then $K : C \to C$ is nonexpansive.

Proof. Put $Q^i = S_{\lambda_i}^{G_i}(I - \lambda_i A_i)S_{\lambda_{i-1}}^{G_{i-1}}(I - \lambda_{i-1}A_{i-1}) \cdots S_{\lambda_1}^{G_1}(I - \lambda_1 A_1)$ for $i = 1, 2, \ldots, M$ and $Q^0 = I$. Then, $K = Q^M$. For all $x, y \in C$, it follows from (28) that

$$\begin{aligned}
&\|Kx - Ky\| \\
&\quad = \|Q^M x - Q^M y\| \\
&\quad = \|S_{\lambda_M}^{G_M}(I - \lambda_M A_M)Q^{M-1}x - S_{\lambda_M}^{G_M}(I - \lambda_M A_M)Q^{M-1}y\| \\
&\quad \le \|(I - \lambda_M A_M)Q^{M-1}x - (I - \lambda_M A_M)Q^{M-1}y\| \\
&\quad \le \|Q^{M-1}x - Q^{M-1}y\| \\
&\quad \vdots \\
&\quad \le \|Q^0 x - Q^0 y\| \\
&\quad = \|x - y\|,
\end{aligned} \tag{37}$$

which implies that K is nonexpansive. This completes the proof. □

Lemma 10. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A_i : C \to H$ $(i = 1, 2, \ldots, M)$ be a nonlinear mapping. For given $(x_1^*, x_2^*, \ldots, x_M^*) \in C \times C \times \cdots \times C$, where $x^* = x_1^*$, $x_i^* = S_{\lambda_{i-1}}^{G_{i-1}}(I - \lambda_{i-1}A_{i-1})x_{i-1}^*$, for $i = 2, 3, \ldots, M$, and $x_1^* = S_{\lambda_M}^{G_M}(I - \lambda_M A_M)x_M^*$. Then, $(x_1^*, x_2^*, \ldots, x_M^*)$ is a solution of the problem (16) if and only if x^* is a fixed point of the mapping K defined as in Lemma 9.*

Proof. Let $(x_1^*, x_2^*, \ldots, x_M^*) \in C \times C \times \cdots \times C$ be a solution of the problem (16). Then, we have

$$\begin{aligned}
&G_M(x_1^*, x_1) + \langle A_M x_M^*, x_1 - x_1^*\rangle \\
&\quad + \frac{1}{\lambda_M}\langle x_1^* - x_M^*, x_1 - x_1^*\rangle \ge 0, \quad \forall x_1 \in C, \\
&G_{M-1}(x_M^*, x_M) + \langle A_{M-1}x_{M-1}^*, x_M - x_M^*\rangle \\
&\quad + \frac{1}{\lambda_{M-1}}\langle x_M^* - x_{M-1}^*, x_M - x_M^*\rangle \ge 0, \quad \forall x_M \in C, \\
&\qquad \vdots \\
&G_2(x_3^*, x_3) + \langle A_2 x_2^*, x_3 - x_3^*\rangle \\
&\quad + \frac{1}{\lambda_2}\langle x_3^* - x_2^*, x_3 - x_3^*\rangle \ge 0, \quad \forall x_3 \in C, \\
&G_1(x_2^*, x_2) + \langle A_1 x_1^*, x_2 - x_2^*\rangle \\
&\quad + \frac{1}{\lambda_1}\langle x_2^* - x_1^*, x_2 - x_2^*\rangle \ge 0, \quad \forall x_2 \in C, \\
&\qquad \Updownarrow
\end{aligned}$$

$$x_1^* = S_{\lambda_M}^{G_M}(I-\lambda_M A_M)x_M^*$$
$$x_2^* = S_{\lambda_1}^{G_1}(I-\lambda_1 A_1)x_1^*,$$
$$\vdots$$
$$x_{M-1}^* = S_{\lambda_{M-2}}^{G_{M-2}}(I-\lambda_{M-2}A_{M-2})x_{M-2}^*,$$
$$x_M^* = S_{\lambda_{M-1}}^{G_{M-1}}(I-\lambda_{M-1}A_{M-1})x_{M-1}^*,$$
$$\Updownarrow$$
$$x^* = S_{\lambda_M}^{G_M}(I-\lambda_M A_M)S_{\lambda_{M-1}}^{G_{M-1}} \times (I-\lambda_{M-1}A_{M-1})\cdots S_{\lambda_1}^{G_1}(I-\lambda_1 A_1)x^* = Kx^*. \tag{38}$$

This completes the proof. □

3. Main Results

Theorem 11. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\Theta_k : C\times C \to \mathbb{R}$ $(k = 1,2,\ldots,N)$ a finite family of bifunctions which satisfy (A1)–(A4), $\varphi_k : C \to \mathbb{R}$ $(k = 1,2,\ldots,N)$ a finite family of lower semicontinuous and convex functions, and $\Psi_k : C \to H$ $(k = 1,2,\ldots,N)$ a finite family of a μ_k-inverse strongly monotone mapping and $A_k : C \to H$ $(k = 1,2,\ldots,M)$ a finite family of an α_k-inverse strongly monotone mapping. Let S be a semigroup, and let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$. Let X be a left invariant subspace of $\ell^\infty(S)$ such that $1 \in X$ and the function $t \to \langle T(t)x, y\rangle$ is an element of X for $x \in C$ and $y \in H$. Let $\{\mu_n\}$ be a left regular sequence of means on X such that $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\| = 0$. Let $F : C \to H$ be a κ-Lipschitzian and η-strongly monotone operator with constants $\kappa,\eta > 0$, and let $V : C \to H$ be an L-Lipschitzian mapping with a constant $L \geq 0$. Let $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \mu\kappa^2/2)$. Assume that $\mathcal{F} := \bigcap_{k=1}^N \mathrm{GMEP}(\Theta_k,\varphi_k,\Psi_k) \cap (K) \cap \mathrm{Fix}(\mathcal{S}) \neq \emptyset$, where K is defined as in Lemma 9. For given $x_1 \in C$, let $\{x_n\}$ be a sequence defined by*

$$\begin{aligned}
u_n &= S_{r_{N,n}}^{(\Theta_N,\varphi_N)}(I-r_{N,n}\Psi_N)S_{r_{N-1,n}}^{(\Theta_{N-1},\varphi_{N-1})} \times (I-r_{N-1,n}\Psi_{N-1})\cdots S_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I-r_{1,n}\Psi_1)x_n,\\
y_n &= S_{\lambda_M}^{G_M}(I-\lambda_M A_M)S_{\lambda_{M-1}}^{G_{M-1}} \times (I-\lambda_{M-1}A_{M-1})\cdots S_{\lambda_1}^{G_1}(I-\lambda_1 A_1)u_n,\\
x_{n+1} &= \beta_n x_n + (1-\beta_n)P_C[\alpha_n\gamma V x_n + (I-\alpha_n\mu F)T(\mu_n)y_n], \quad \forall n \geq 1,
\end{aligned} \tag{39}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$, and $\{r_{k,n}\}_{k=1}^N$ is a sequence such that $\{r_{k,n}\}_{k=1}^N \subset [a_k,b_k] \subset (0,2\gamma_k)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;

(C2) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$;

(C3) $\liminf_{n\to\infty} r_{k,n} > 0$ and $\lim_{n\to\infty}(r_{k,n}/r_{k,n+1}) = 1$ for all $k \in \{1,2,\ldots,N\}$.

Then, the sequence $\{x_n\}$ defined by (39) converges strongly to $\hat{x} \in \mathcal{F}$ as $n \to \infty$, where $\hat{x}$ solves uniquely the variational inequality

$$\langle(\mu F - \gamma V)\hat{x}, \hat{x} - v\rangle \leq 0, \quad \forall v \in \mathcal{F}. \tag{40}$$

Equivalently, one has $\hat{x} = P_{\mathcal{F}}(I-\mu F+\gamma V)\hat{x}$.

Proof. Note that from condition (C1), we may assume, without loss of generality, that $\alpha_n \leq \min\{1, 1/2\tau\}$ for all $n \in \mathbb{N}$. First, we show that $\{x_n\}$ is bounded. Set

$$\begin{aligned}
G_n^k &:= S_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I-r_{k,n}\Psi_k)S_{r_{k-1,n}}^{(\Theta_{k-1},\varphi_{k-1})} \times (I-r_{k-1,n}\Psi_{k-1})\cdots S_{r_{1,n}}^{(\Theta_1,\varphi_1)}(I-r_{1,n}\Psi_1),\\
&\qquad \forall k \in \{1,2,\ldots,N\},\ n \in \mathbb{N},\\
Q^i &:= S_{\lambda_i}^{G_i}(I-\lambda_i A_i)S_{\lambda_{i-1}}^{G_{i-1}}(I-\lambda_{i-1}A_{i-1})\cdots S_{\lambda_1}^{G_1}(I-\lambda_1 A_1),\\
&\qquad \forall i \in \{1,2,\ldots,M\},
\end{aligned} \tag{41}$$

$G_n^0 = Q^0 = I$. Then, we have $u_n = G_n^N x_n$ and $y_n = Q^M u_n$. From Lemmas 3 and 9, we have that G_n^N and Q^M are nonexpansive. Take $x^* \in \mathcal{F}$; we have

$$\|u_n - x^*\| = \|G_n^N x_n - G_n^N x^*\| \leq \|x_n - x^*\|. \tag{42}$$

By Lemma 10, we have $x^* = Q^M x^*$. It follows from (42) that

$$\begin{aligned}
\|y_n - x^*\| &= \|Q^M u_n - Q^M x^*\|\\
&\leq \|u_n - x^*\|\\
&\leq \|x_n - x^*\|.
\end{aligned} \tag{43}$$

Set

$$z_n := P_C[\alpha_n\gamma V x_n + (I-\alpha_n\mu F)T(\mu_n)y_n], \quad \forall n \in \mathbb{N}. \tag{44}$$

Then, we can rewrite (39) as $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$. From Lemma 8 and (43), we have

$$\begin{aligned}
&\|z_n - x^*\|\\
&= \|P_C[\alpha_n\gamma V x_n + (I-\alpha\mu F)T(\mu_n)y_n] - P_C x^*\|\\
&\leq \|\alpha_n(\gamma V x_n - \mu F x^*) + (I-\alpha_n\mu F)(T(\mu_n)y_n - x^*)\|\\
&\leq \alpha_n\|\gamma V x_n - \mu F x^*\| + (1-\alpha_n\tau)\|T(\mu_n)y_n - x^*\|\\
&\leq \alpha_n\gamma\|V x_n - V x^*\| + \alpha_n\|\gamma V x^* - \mu F x^*\|\\
&\quad + (1-\alpha_n\tau)\|y_n - x^*\|\\
&\leq (1-\alpha_n(\tau-\gamma L))\|x_n - x^*\| + \alpha_n\|\gamma V x^* - \mu F x^*\|.
\end{aligned} \tag{45}$$

It follows from (45) that

$$\begin{aligned}
&\|x_{n+1}-x^*\| \\
&\quad = \|\beta_n(x_n-x^*)+(1-\beta_n)(z_n-x^*)\| \\
&\quad \le \beta_n\|x_n-x^*\|+(1-\beta_n)\|z_n-x^*\| \\
&\quad \le \beta_n\|x_n-x^*\|+(1-\beta_n) \\
&\qquad \times [(1-\alpha_n(\tau-\gamma L))\|x_n-x^*\|+\alpha_n\|\gamma Vx^*-\mu Fx^*\|] \\
&\quad = (1-\alpha_n(1-\beta_n)(\tau-\gamma L))\|x_n-x^*\| \\
&\qquad + \alpha_n(1-\beta_n)(\tau-\gamma L)\frac{\|\gamma Vx^*-\mu Fx^*\|}{\tau-\gamma L}.
\end{aligned} \tag{46}$$

By induction, we have

$$\|x_n-x^*\| \le \max\left\{\|x_1-x^*\|, \frac{\|\gamma Vx^*-\mu Fx^*\|}{\tau-\gamma L}\right\}, \quad \forall n \ge 1. \tag{47}$$

Hence, $\{x_n\}$ is bounded, and so are $\{Vx_n\}$ and $\{FT(\mu_n)y_n\}$.

Next, we show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{48}$$

Observe that

$$\lim_{n\to\infty}\|T(\mu_{n+1})y_n - T(\mu_n)y_n\| = 0. \tag{49}$$

Indeed,

$$\begin{aligned}
&\|T(\mu_{n+1})y_n - T(\mu_n)y_n\| \\
&\quad = \sup_{\|z\|=1}|\langle T(\mu_{n+1})y_n - T(\mu_n)y_n, z\rangle| \\
&\quad = \sup_{\|z\|=1}|(\mu_{n+1})_s\langle T(s)y_n, z\rangle - (\mu_n)_s\langle T(s)y_n, z\rangle| \\
&\quad \le \|\mu_{n+1}-\mu_n\|\sup_{s\in S}\|T(s)y_n\|.
\end{aligned} \tag{50}$$

Since $\{y_n\}$ is bounded and $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\| = 0$, then (49) holds. We observe that

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|Q^M u_{n+1} - Q^M u_n\| \\
&\le \|u_{n+1}-u_n\|.
\end{aligned} \tag{51}$$

Let $\{\omega_n\}$ be a bounded sequence in C. Now, we show that

$$\lim_{n\to\infty}\|G_n^N\omega_n - G_{n+1}^N\omega_n\| = 0. \tag{52}$$

For the previous purpose, put $D_n^k = S_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}\Psi_k)$, and we first show that

$$\lim_{n\to\infty}\|D_{n+1}^k\omega_n - D_n^k\omega_n\| = 0, \quad \forall k \in \{1,2,\ldots,N\}. \tag{53}$$

In fact, since $D_n^k\omega_n \in \mathrm{GMEP}(\Theta_k,\varphi_k,\Psi_k)$ and $D_{n+1}^k\omega_n \in \mathrm{GMEP}(\Theta_k,\varphi_k,\Psi_k)$, we have

$$\begin{aligned}
&\Theta_k(D_n^k\omega_n, y) + \varphi_k(y) - \varphi_k(D_n^k\omega_n) \\
&\quad + \langle\Psi_k\omega_n, y - D_n^k\omega_n\rangle \\
&\quad + \frac{1}{r_{k,n}}\langle y - D_n^k\omega_n, D_n^k\omega_n - \omega_n\rangle \ge 0, \\
&\qquad \forall y \in C,
\end{aligned} \tag{54}$$

$$\begin{aligned}
&\Theta_k(D_{n+1}^k\omega_n, y) + \varphi_k(y) - \varphi_k(D_{n+1}^k\omega_n) \\
&\quad + \langle\Psi_k\omega_n, y - D_{n+1}^k\omega_n\rangle \\
&\quad + \frac{1}{r_{k,n+1}}\langle y - D_{n+1}^k\omega_n, D_{n+1}^k\omega_n - \omega_n\rangle \ge 0, \\
&\qquad \forall y \in C.
\end{aligned} \tag{55}$$

Substituting $y = D_{n+1}^k\omega_n$ in (54) and $y = D_n^k\omega_n$ in (55), then add these two inequalities, and using (A2), we obtain

$$\begin{aligned}
&\left\langle D_{n+1}^k\omega_n - D_n^k\omega_n, \frac{1}{r_{k,n}}(D_n^k\omega_n - \omega_n)\right. \\
&\qquad \left. - \frac{1}{r_{k,n+1}}(D_{n+1}^k\omega_n - \omega_n)\right\rangle \ge 0.
\end{aligned} \tag{56}$$

Hence,

$$\begin{aligned}
&\left\langle D_{n+1}^k\omega_n - D_n^k\omega_n, D_n^k\omega_n - D_{n+1}^k\omega_n + D_{n+1}^k\omega_n\right. \\
&\qquad \left. -\omega_n - \frac{r_{k,n}}{r_{n,k+1}}(D_{n+1}^k\omega_n - \omega_n)\right\rangle \ge 0;
\end{aligned} \tag{57}$$

we derive from (57) that

$$\begin{aligned}
&\|D_{n+1}^k\omega_n - D_n^k\omega_n\|^2 \\
&\quad \le \left\langle D_{n+1}^k\omega_n - D_n^k\omega_n, \left(1 - \frac{r_{k,n}}{r_{k,n+1}}\right)(D_{n+1}^k\omega_n - \omega_n)\right\rangle \\
&\quad \le \|D_{n+1}^k\omega_n - D_n^k\omega_n\|\left|1 - \frac{r_{k,n}}{r_{k,n+1}}\right|\|D_{n+1}^k\omega_n - \omega_n\|,
\end{aligned} \tag{58}$$

which implies that

$$\|D_{n+1}^k\omega_n - D_n^k\omega_n\| \le \left|1 - \frac{r_{k,n}}{r_{k,n+1}}\right|\|D_{n+1}^k\omega_n - \omega_n\|. \tag{59}$$

Noticing that condition $(C3)$ implies that (53) holds, from the definition of G_n^N and the nonexpansiveness of D_n^k, we have

$$\begin{aligned}
&\left\|G_n^N\omega_n - G_{n+1}^N\omega_n\right\| \\
&\quad = \left\|D_n^N G_n^{N-1}\omega_n - D_{n+1}^N G_{n+1}^{N-1}\omega_n\right\| \\
&\quad \le \left\|D_n^N G_n^{N-1}\omega_n - D_{n+1}^N G_n^{N-1}\omega_n\right\| \\
&\qquad + \left\|D_{n+1}^N G_n^{N-1}\omega_n - D_{n+1}^N G_{n+1}^{N-1}\omega_n\right\| \\
&\quad \le \left\|D_n^N G_n^{N-1}\omega_n - D_{n+1}^N G_n^{N-1}\omega_n\right\| \\
&\qquad + \left\|G_n^{N-1}\omega_n - G_{n+1}^{N-1}\omega_n\right\| \\
&\quad \le \left\|D_n^N G_n^{N-1}\omega_n - D_{n+1}^N G_n^{N-1}\omega_n\right\| \\
&\qquad + \left\|D_n^{N-1} G_n^{N-2}\omega_n - D_{n+1}^{N-1} G_{n+1}^{N-2}\omega_n\right\| \\
&\qquad + \left\|G_n^{N-2}\omega_n - G_{n+1}^{N-2}\omega_n\right\| \\
&\quad \le \left\|D_n^N G_n^{N-1}\omega_n - D_{n+1}^N G_n^{N-1}\omega_n\right\| \\
&\qquad + \left\|D_n^{N-1} G_n^{N-2}\omega_n - D_{n+1}^{N-1} G_{n+1}^{N-2}\omega_n\right\| + \cdots \\
&\qquad + \left\|D_n^2\omega_n - D_{n+1}^2\omega_n\right\| + \left\|D_n^1\omega_n - D_{n+1}^1\omega_n\right\|,
\end{aligned} \tag{60}$$

for which (52) follows by (53). Since $u_n = G_n^N x_n$ and $u_{n+1} = G_{n+1}^N x_{n+1}$, we have

$$\begin{aligned}
&\left\|u_n - u_{n+1}\right\| \\
&\quad = \left\|G_n^N x_n - G_{n+1}^N x_{n+1}\right\| \\
&\quad \le \left\|G_n^N x_n - G_{n+1}^N x_n\right\| + \left\|G_{n+1}^N x_n - G_{n+1}^N x_{n+1}\right\| \\
&\quad \le \left\|G_n^N x_n - G_{n+1}^N x_n\right\| + \left\|x_n - x_{n+1}\right\|.
\end{aligned} \tag{61}$$

Put a constant $M_1 > 0$ such that

$$\begin{aligned}
M_1 = \sup_{n\ge 1}\{&\gamma\left\|Vx_{n+1}\right\| + \mu\left\|FT(t_{n+1})y_{n+1}\right\|, \\
&\gamma\left\|Vx_n\right\| + \mu\left\|FT(t_n)y_n\right\|\}.
\end{aligned} \tag{62}$$

From definition of $\{z_n\}$, we note that

$$\begin{aligned}
&\left\|z_{n+1} - z_n\right\| \\
&\quad = \left\|P_C\left[\alpha_{n+1}\gamma Vx_{n+1} + (I - \alpha_{n+1}\mu F)T(\mu_{n+1})y_{n+1}\right]\right. \\
&\qquad \left. - P_C\left[\alpha_n\gamma Vx_n + (I - \alpha_n\mu F)T(\mu_n)y_n\right]\right\| \\
&\quad \le \alpha_{n+1}\left\|\gamma Vx_{n+1} - \mu FT(\mu_{n+1})y_{n+1}\right\| \\
&\qquad + \alpha_n\left\|\gamma Vx_{n+1} - \mu FT(\mu_n)y_n\right\| \\
&\qquad + \left\|T(\mu_{n+1})y_{n+1} - T(\mu_n)y_n\right\| \\
&\quad \le \alpha_{n+1}\left\|\gamma Vx_{n+1} - \mu FT(\mu_{n+1})y_{n+1}\right\| \\
&\qquad + \alpha_n\left\|\gamma Vx_{n+1} - \mu FT(\mu_n)y_n\right\| \\
&\qquad + \left\|T(\mu_{n+1})y_{n+1} - T(\mu_{n+1})y_n\right\| \\
&\qquad + \left\|T(\mu_{n+1})y_n - T(\mu_n)y_n\right\| \\
&\quad \le (\alpha_{n+1} + \alpha_n)M_1 + \left\|y_{n+1} - y_n\right\| \\
&\qquad + \left\|T(\mu_{n+1})y_n - T(\mu_n)y_n\right\|.
\end{aligned} \tag{63}$$

It follows from (51), (61), and (63) that

$$\begin{aligned}
\left\|z_{n+1} - z_n\right\| \le\ & (\alpha_{n+1} + \alpha_n)M_1 + \left\|G_n^N x_n - G_{n+1}^N x_n\right\| \\
& + \left\|x_{n+1} - x_n\right\| + \left\|T(\mu_{n+1})y_n - T(\mu_n)y_n\right\|.
\end{aligned} \tag{64}$$

From condition $(C1)$, (49), and (52), we have

$$\limsup_{n\to\infty}\left(\left\|z_{n+1} - z_n\right\| - \left\|x_{n+1} - x_n\right\|\right) \le 0. \tag{65}$$

Hence, by Lemma 5, we obtain

$$\lim_{n\to\infty}\left\|z_n - x_n\right\| = 0. \tag{66}$$

Consequently,

$$\lim_{n\to\infty}\left\|x_{n+1} - x_n\right\| = \lim_{n\to\infty}(1 - \beta_n)\left\|z_n - x_n\right\| = 0. \tag{67}$$

From condition $(C1)$, we have

$$\begin{aligned}
&\left\|z_n - T(\mu_n)y_n\right\| \\
&\quad = \left\|P_C\left[\alpha_n\gamma Vx_n + (I - \alpha_n\mu F)T(\mu_n)y_n\right] - P_C T(\mu_n)y_n\right\| \\
&\quad \le \alpha_n\left\|\gamma Vx_n - \mu FT(\mu_n)y_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{68}$$

From (66) and (68), we have

$$\begin{aligned}
\left\|x_n - T(\mu_n)y_n\right\| \le\ & \left\|x_n - z_n\right\| \\
& + \left\|z_n - T(\mu_n)y_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{69}$$

Set $z_n = P_C v_n$, where $v_n = \alpha_n\gamma Vx_n + (I - \alpha_n\mu F)T(\mu_n)y_n$. From (25), we have

$$\begin{aligned}
\left\|z_n - x^*\right\| =\ & \left\langle v_n - x^*, z_n - x^*\right\rangle \\
& + \left\langle P_C v_n - v_n, P_C v_n - x^*\right\rangle \\
\le\ & \left\langle v_n - x^*, z_n - x^*\right\rangle \\
=\ & \alpha_n\left\langle \gamma Vx_n - \mu Fx^*, z_n - x^*\right\rangle \\
& + \left\langle (I - \alpha_n\mu F)(T(\mu_n)y_n - x^*), z_n - x^*\right\rangle
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n\tau)\|y_n - x^*\|\|z_n - x^*\| \\
&\quad + \alpha_n \langle \gamma V x_n - \mu F x^*, z_n - x^* \rangle \\
&\leq \frac{(1-\alpha_n\tau)}{2}\left(\|y_n - x^*\|^2 + \|z_n - x^*\|^2\right) \\
&\quad + \alpha_n \langle \gamma V x_n - \mu F x^*, z_n - x^* \rangle \\
&\leq \frac{(1-\alpha_n\tau)}{2}\|y_n - x^*\|^2 + \frac{1}{2}\|z_n - x^*\|^2 \\
&\quad + \alpha_n \langle \gamma V x_n - \mu F x^*, z_n - x^* \rangle .
\end{aligned} \tag{70}$$

It follows that

$$\begin{aligned}
\|z_n - x^*\|^2 &\leq \|y_n - x^*\|^2 + 2\alpha_n \langle \gamma V x_n - \mu F x^*, z_n - x^* \rangle \\
&\leq \|y_n - x^*\|^2 + 2\alpha_n \|\gamma V x_n - \mu F x^*\| \|z_n - x^*\| .
\end{aligned} \tag{71}$$

By the convexity of $\|\cdot\|^2$ and (71), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad = \|\beta_n (x_n - x^*) + (1-\beta_n)(z_n - x^*)\|^2 \\
&\quad \leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\|z_n - x^*\|^2 \\
&\quad \leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n) \\
&\qquad \times \left\{\|y_n - x^*\|^2 + 2\alpha_n \|\gamma V x_n - \mu F x^*\| \|z_n - x^*\|\right\} \\
&\quad \leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + 2\alpha_n (1-\beta_n) M_2,
\end{aligned} \tag{72}$$

where $M_2 > 0$ is an appropriate constant such that $M_2 = \sup_{n\geq 1}\{\|\gamma V x_n - \mu F x^*\| \|z_n - x^*\|\}$.

Next, we show that

$$\lim_{n\to\infty} \left\|G_n^{k-1} x_n - G_n^k x_n\right\| = 0, \quad \forall k \in \{1, 2, \ldots, N\} . \tag{73}$$

From (28), we have

$$\begin{aligned}
&\left\|G_n^k x_n - x^*\right\|^2 \\
&\quad = \left\|S_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n}\Psi_k) G_n^{k-1} x_n - S_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n}\Psi_k) x^*\right\|^2 \\
&\quad \leq \left\|(I - r_{k,n}\Psi_k) G_n^{k-1} - (I - r_{k,n}\Psi_k) x^*\right\|^2 \\
&\quad \leq \left\|G_n^{k-1} x_n - x^*\right\|^2 + r_{k,n}(r_{k,n} - 2\mu_k)\left\|\Psi_k G_n^{k-1} x_n - \Psi_k x^*\right\|^2 \\
&\quad \leq \|x_n - x^*\|^2 + r_{k,n}(r_{k,n} - 2\mu_k)\left\|\Psi_k G_n^{k-1} x_n - \Psi_k x^*\right\|^2 .
\end{aligned} \tag{74}$$

From (42), for all $k \in \{1, 2, \ldots, N\}$, we note that

$$\begin{aligned}
\|y_n - x^*\|^2 &= \left\|Q^M u_n - x^*\right\| \leq \|u_n - x^*\|^2 \\
&= \left\|G_n^N x_n - x^*\right\|^2 \leq \left\|G_n^k x_n - x^*\right\|^2 .
\end{aligned} \tag{75}$$

From (72) and (75), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\left\|G_n^N x_n - x^*\right\|^2 \\
&\qquad + 2\alpha_n (1-\beta_n) M_2 \\
&\quad \leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\left\|G_n^k x_n - x^*\right\|^2 \\
&\qquad + 2\alpha_n (1-\beta_n) M_2 .
\end{aligned} \tag{76}$$

Substituting (74) into (72), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \leq \beta_n \|x_n - x^*\|^2 + (1-\beta_n) \\
&\qquad \times \left\{\|x_n - x^*\|^2 + r_{k,n}(r_{k,n} - 2\mu_k)\left\|\Psi_k G_n^{k-1} x_n - \Psi_k x^*\right\|^2\right\} \\
&\qquad + 2\alpha_n (1-\beta_n) M_2 \\
&\quad = \|x_n - x^*\|^2 + (1-\beta_n) r_{k,n}(r_{k,n} - 2\mu_k) \\
&\qquad \times \left\|\Psi_k G_n^{k-1} x_n - \Psi_k x^*\right\|^2 \\
&\qquad + 2\alpha_n (1-\beta_n) M_2 ,
\end{aligned} \tag{77}$$

which in turn implies that

$$\begin{aligned}
&(1-\beta_n) r_{k,n}(2\mu_k - r_{k,n})\left\|\Psi_k G_n^{k-1} x_n - \Psi_k x^*\right\|^2 \\
&\quad \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n (1-\beta_n) M_2 \\
&\quad \leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| \\
&\qquad + 2\alpha_n (1-\beta_n) M_2 .
\end{aligned} \tag{78}$$

Since $\liminf_{n\to\infty}(1-\beta_n) > 0$, $0 < r_{k,n} < 2\mu_k$, for all $k \in \{1, 2, \ldots, N\}$, from (C1) and (67), we obtain that

$$\lim_{n\to\infty} \left\|\Psi_k G_n^{k-1} x_n - \Psi_k x^*\right\| = 0, \quad \forall k \in \{1, 2, \ldots, N\} . \tag{79}$$

On the other hand, from Lemma 3 and (26), we have

$$\begin{aligned}
&\left\|G_n^k x_n - x^*\right\|^2 \\
&\quad = \left\|S_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n}\Psi_k) G_n^{k-1} x_n - S_{r_{k,n}}^{(\Theta_k,\varphi_k)} (I - r_{k,n}\Psi_k) x^*\right\|^2 \\
&\quad \leq \left\langle (I - r_{k,n}\Psi_k) G_n^{k-1} x_n - (I - r_{k,n}\Psi_k) x^*, G_n^k x_n - x^* \right\rangle \\
&\quad = \frac{1}{2}\Big(\left\|(I - r_{k,n}\Psi_k) G_n^{k-1} x_n - (I - r_{k,n}\Psi_k) x^*\right\|^2 \\
&\qquad + \left\|G_n^k x_n - x^*\right\|^2 \\
&\qquad - \Big\|(I - r_{k,n}\Psi_k) G_n^{k-1} x_n - (I - r_{k,n}\Psi_k) x^* \\
&\qquad - \left(G_n^k x_n - x^*\right)\Big\|^2\Big)
\end{aligned}$$

$$\leq \frac{1}{2}\left(\left\|G_n^{k-1}x_n - x^*\right\|^2 + \left\|G_n^k - x^*\right\|^2 - \left\|G_n^{k-1}x_n - G_n^k x_n - r_{k,n}\left(\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right)\right\|^2\right), \tag{80}$$

which in turn implies that

$$\begin{aligned}
&\left\|G_n^k x_n - x^*\right\|^2 \\
&\quad\leq \left\|G_n^{k-1}x_n - x^*\right\|^2 \\
&\qquad - \left\|G_n^{k-1}x_n - G_n^k - r_{k,n}(\Psi_k G_n^{k-1} - \Psi_k x^*)\right\|^2 \\
&\quad= \left\|G_n^{k-1}x_n - x^*\right\|^2 - \left\|G_n^{k-1}x_n - G_n^k x_n\right\| \\
&\qquad - r_{k,n}^2\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\|^2 \\
&\qquad + 2r_{k,n}\left\langle G_n^{k-1}x_n - G_n^k x_n, \Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\rangle \\
&\quad\leq \left\|G_n^{k-1}x_n - x^*\right\|^2 - \left\|G_n^{k-1}x_n - G_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n}\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\| \\
&\quad\leq \left\|x_n - x^*\right\|^2 - \left\|G_n^{k-1}x_n - G_n^k x_n\right\|^2 \\
&\qquad + 2r_{k,n}\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\|.
\end{aligned} \tag{81}$$

Substituting (81) into (76), we have

$$\begin{aligned}
&\left\|x_{n+1} - x^*\right\|^2 \\
&\quad\leq \beta_n\left\|x_n - x^*\right\|^2 + (1-\beta_n) \\
&\qquad\times\left\{\left\|x_n - x^*\right\|^2 - \left\|G_n^{k-1}x_n - G_n^k x_n\right\|^2\right. \\
&\qquad\quad\left. +2r_{k,n}\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\|\right\} \\
&\qquad + 2\alpha_n(1-\beta_n)M_2 \\
&\quad= \left\|x_n - x^*\right\|^2 - (1-\beta_n)\left\|G_n^{k-1}x_n - G_n^k x_n\right\| \\
&\qquad + 2r_{k,n}(1-\beta_n)\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\| \\
&\qquad + 2\alpha_n(1-\beta_n)M_2,
\end{aligned} \tag{82}$$

which in turn implies that

$$\begin{aligned}
&(1-\beta_n)\left\|G_n^{k-1}x_n - G_n^k x_n\right\| \\
&\quad\leq \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 \\
&\qquad + 2r_{k,n}(1-\beta_n)\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\| \\
&\qquad + 2\alpha_n(1-\beta_n)M_2 \\
&\quad\leq \left(\left\|x_n - x^*\right\| + \left\|x_{n+1} - x^*\right\|\right)\left\|x_{n+1} - x_n\right\| \\
&\qquad + 2r_{k,n}(1-\beta_n)\left\|G_n^{k-1}x_n - G_n^k x_n\right\|\left\|\Psi_k G_n^{k-1}x_n - \Psi_k x^*\right\| \\
&\qquad + 2\alpha_n(1-\beta_n)M_2.
\end{aligned} \tag{83}$$

Since $\liminf_{n\to\infty}(1-\beta_n) > 0$, from $(C1)$, (67), and (79), we obtain that (73) holds. Consequently,

$$\begin{aligned}
\left\|x_n - u_n\right\| &= \left\|G_n^0 - G_n^N x_n\right\| \\
&\leq \left\|G_n^0 x_n - G_n^1 x_n\right\| + \left\|G_n^1 x_n - G_n^2 x_n\right\| + \cdots \\
&\quad + \left\|G_n^{N-1}x_n - G_n^N x_n\right\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned} \tag{84}$$

Next, we show that

$$\lim_{n\to\infty}\left\|A_i Q^{i-1}u_n - A_i Q^{i-1}x^*\right\| = 0, \quad \forall i \in \{1, 2, \ldots, M\}. \tag{85}$$

From (28), we have

$$\begin{aligned}
&\left\|Q^M u_n - Q^M x^*\right\|^2 \\
&\quad= \left\|S_{\lambda_M}^{G_M}(I - \lambda_M A_M)Q^{M-1}u_n - S_{\lambda_M}^{G_M}(I - \lambda_M A_M)Q^{M-1}x^*\right\|^2 \\
&\quad\leq \left\|(I - \lambda_M A_M)Q^{M-1}u_n - (I - \lambda_M A_M)Q^{M-1}x^*\right\|^2 \\
&\quad\leq \left\|Q^{M-1}u_n - Q^{M-1}x^*\right\|^2 + \lambda_M(\lambda_M - 2\alpha_M) \\
&\qquad\times\left\|A_M Q^{M-1}u_n - A_M Q^{M-1}x^*\right\|^2.
\end{aligned} \tag{86}$$

By induction, we have

$$\begin{aligned}
&\left\|Q^M u_n - Q^M x^*\right\|^2 \\
&\quad\leq \left\|u_n - x^*\right\| + \sum_{i=1}^{M}\lambda_i(\lambda_i - 2\alpha_i)\left\|A_i Q^{i-1}u_n - A_i Q^{i-1}x^*\right\|^2 \\
&\quad\leq \left\|x_n - x^*\right\| + \sum_{i=1}^{M}\lambda_i(\lambda_i - 2\alpha_i)\left\|A_i Q^{i-1}u_n - A_i Q^{i-1}x^*\right\|^2.
\end{aligned} \tag{87}$$

From (72) and (75), we have

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 \leq \beta_n\left\|x_n - x^*\right\|^2 + (1-\beta_n)\left\|Q^M u_n - Q^M x^*\right\|^2 \\
+ 2\alpha_n(1-\beta_n)M_2.
\end{aligned} \tag{88}$$

Substituting (87) into (88), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad\le \beta_n\|x_n-x^*\|^2+(1-\beta_n) \\
&\qquad\times\Big\{\|x_n-x^*\|^2 \\
&\qquad\quad+\sum_{i=1}^{M}\lambda_i(\lambda_i-2\alpha_i)\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|^2\Big\} \\
&\qquad+2\alpha_n(1-\beta_n)M_2 \\
&\quad=\|x_n-x^*\|^2 \\
&\qquad+(1-\beta_n)\sum_{i=1}^{M}\lambda_i(\lambda_i-2\alpha_i)\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|^2 \\
&\qquad+2\alpha_n(1-\beta_n)M_2,
\end{aligned} \tag{89}$$

which in turn implies that

$$\begin{aligned}
&(1-\beta_n)\sum_{i=1}^{M}\lambda_i(2\alpha_i-\lambda_i)\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|^2 \\
&\quad\le\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\alpha_n(1-\beta_n)M_2 \\
&\quad\le(\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\| \\
&\qquad+2\alpha_n(1-\beta_n)M_2.
\end{aligned} \tag{90}$$

Since $\liminf_{n\to\infty}(1-\beta_n)>0$, from $(C1)$ and (67), we obtain that (85) holds.

On the other hand, from (24) and (26), we have

$$\begin{aligned}
&\left\|Q^Mu_n-Q^Mx^*\right\|^2 \\
&\quad=\left\|S_{\lambda_M}^{G_M}(I-\lambda_MA_M)Q^{M-1}u_n-S_{\lambda_M}^{G_M}(I-\lambda_MA_M)Q^{M-1}x^*\right\|^2 \\
&\quad\le\left\langle(I-\lambda_MA_M)Q^{M-1}u_n-(I-\lambda_MA_M)Q^{M-1}x^*,\right. \\
&\qquad\left.Q^Mu_n-Q^Mx^*\right\rangle \\
&\quad=\frac{1}{2}\Big(\left\|(I-\lambda_MA_M)Q^{M-1}u_n-(I-\lambda_MA_M)Q^{M-1}x^*\right\|^2 \\
&\qquad+\left\|Q^Mu_n-Q^Mx^*\right\|^2 \\
&\qquad-\Big\|(I-\lambda_MA_M)Q^{M-1}u_n \\
&\qquad\quad-(I-\lambda_MA_M)x^*-(Q^Mu_n-Q^Mx^*)\Big\|^2\Big) \\
&\quad\le\frac{1}{2}\Big(\left\|Q^{M-1}u_n-Q^{M-1}x^*\right\|^2+\left\|Q^Mu_n-Q^Mx^*\right\|^2 \\
&\qquad-\Big\|Q^{M-1}u_n-Q^Mu_n+Q^Mx^*-Q^{M-1}x^* \\
&\qquad\quad-\lambda_M\left(A_MQ^{M-1}u_n-A_MQ^{M-1}x^*\right)\Big\|^2\Big),
\end{aligned} \tag{91}$$

which in turn implies that

$$\begin{aligned}
&\left\|Q^Mu_n-Q^Mx^*\right\|^2 \\
&\quad\le\left\|Q^{M-1}u_n-Q^{M-1}x^*\right\|^2 \\
&\qquad-\Big\|Q^{M-1}u_n-Q^Mu_n+Q^Mx^*-Q^{M-1}x^* \\
&\qquad\quad-\lambda_M(A_MQ^{M-1}u_n-A_MQ^{M-1}x^*)\Big\|^2 \\
&\quad=\left\|Q^{M-1}u_n-Q^{M-1}x^*\right\|^2 \\
&\qquad-\left\|Q^{M-1}u_n-Q^Mu_n-Q^Mx^*-Q^{M-1}x^*\right\|^2 \\
&\qquad-\lambda_M^2\left\|A_MQ^{M-1}u_n-A_MQ^{M-1}x^*\right\|^2 \\
&\qquad+2\lambda_M\Big\langle Q^{M-1}u_n-Q^Mu_n+Q^Mx^*-Q^{M-1}x^*, \\
&\qquad\quad A_MQ^{M-1}u_n-A_MQ^{M-1}x^*\Big\rangle \\
&\quad\le\left\|Q^{M-1}u_n-Q^{M-1}x^*\right\|^2 \\
&\qquad-\left\|Q^{M-1}u_n-Q^Mu_n+Q^Mx^*-Q^{M-1}x^*\right\|^2 \\
&\qquad+2\lambda_M\left\|Q^{M-1}u_n-Q^Mu_n+Q^Mx^*-Q^{M-1}x^*\right\| \\
&\qquad\times\left\|A_MQ^{M-1}u_n-A_MQ^{M-1}x^*\right\|.
\end{aligned} \tag{92}$$

By induction, we have

$$\begin{aligned}
&\left\|Q^Mu_n-Q^Mx^*\right\|^2 \\
&\quad\le\|u_n-x^*\| \\
&\qquad-\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|^2 \\
&\qquad+\sum_{i=1}^{N}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\| \\
&\qquad\quad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\| \\
&\quad\le\|x_n-x^*\|-\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|^2 \\
&\qquad+\sum_{i=1}^{N}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\| \\
&\qquad\quad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|.
\end{aligned} \tag{93}$$

Substituting (93) into (88), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 \\
&\quad\le \beta_n\|x_n-x^*\|^2+(1-\beta_n) \\
&\qquad\times\Bigg\{\|x_n-x^*\|-\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|^2 \\
&\qquad\quad+\sum_{i=1}^{M}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\| \\
&\qquad\qquad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\|\Bigg\} \\
&\qquad+2\alpha_n(1-\beta_n)M_2 \\
&\quad\le \|x_n-x^*\|^2 \\
&\qquad-(1-\beta_n)\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|^2 \\
&\qquad+(1-\beta_n) \\
&\qquad\times\sum_{i=1}^{M}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\| \\
&\qquad\quad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\| \\
&\qquad+2\alpha_n(1-\beta_n)M_2,
\end{aligned} \tag{94}$$

which in turn implies that

$$\begin{aligned}
&(1-\beta_n)\sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|^2 \\
&\quad\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\qquad+(1-\beta_n)\sum_{i=1}^{M}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\| \\
&\qquad\quad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\| \\
&\qquad+2\alpha_n(1-\beta_n)M_2 \\
&\quad\le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\| \\
&\qquad+(1-\beta_n)\sum_{i=1}^{M}2\lambda_i\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\| \\
&\qquad\quad\times\left\|A_iQ^{i-1}u_n-A_iQ^{i-1}x^*\right\| \\
&\qquad+2\alpha_n(1-\beta_n)M_2.
\end{aligned} \tag{95}$$

Since $\liminf_{n\to\infty}(1-\beta_n)>0$, from $(C1)$, (67), and (85), we obtain that

$$\lim_{n\to\infty}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\|=0, \quad \forall i\in\{1,2,\ldots,M\}. \tag{96}$$

Consequently,

$$\begin{aligned}
\|u_n-y_n\| &= \left\|Q^0u_n-Q^Mu_n\right\| \\
&\le \sum_{i=1}^{M}\left\|Q^{i-1}u_n-Q^iu_n+Q^ix^*-Q^{i-1}x^*\right\| \\
&\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{97}$$

It follows from (84) and (97) that

$$\begin{aligned}
\|x_n-y_n\| &\le \|x_n-u_n\|+\|u_n-y_n\| \\
&\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned} \tag{98}$$

Next, we show that

$$\lim_{n\to\infty}\|x_n-T(t)x_n\|=0, \quad \forall t\in S. \tag{99}$$

Put

$$M^*=\max\left\{\|x_1-x^*\|,\frac{1}{\tau-\gamma L}\|\gamma Vx^*-\mu Fx^*\|\right\}. \tag{100}$$

Set $D=\{y\in C:\|y-x^*\|\le M^*\}$. We remark that D is nonempty, bounded, closed, and convex set, and $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are in D. We will show that

$$\limsup_{n\to\infty}\sup_{y\in D}\|T(\mu_n)y-T(t)T(\mu_n)y\|=0, \quad \forall t\in S. \tag{101}$$

To complete our proof, we follow the proof line as in [31] (see also [23, 32, 33]). Let $\epsilon>0$. By [34, Theorem 1.2], there exists $\delta>0$ such that

$$\overline{\text{co}}\,F_\delta(T(t);D)+B_\delta\subset F_\epsilon(T(t);D), \quad \forall t\in S. \tag{102}$$

Also by [34, Corollary 1.1], there exists a natural number N such that

$$\left\|\frac{1}{N+1}\sum_{i=0}^{N}T(t^is)y-T(t)\left(\frac{1}{N+1}\sum_{i=0}^{N}T(t^i)y\right)\right\|\le\delta, \tag{103}$$

for all $t, s \in S$ and $y \in D$. Let $t \in S$. Since $\{\mu_n\}$ is strongly left regular, there exists $n_0 \in \mathbb{N}$ such that $\|\mu_n - l^*_{t^i}\mu_n\| \le \delta/(M^* + \|w\|)$ for all $n \ge n_0$ and $i = 1, 2, \ldots, N$. Then, we have

$$\begin{aligned}
&\sup_{y\in D}\left\|T(\mu_n)y - \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)\, y\, d\mu_n(s)\right\| \\
&\quad = \sup_{y\in D}\sup_{\|z\|=1}\left|\langle T(\mu_n)y, z\rangle \right. \\
&\qquad\qquad \left. - \left\langle \int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)\, y\, d\mu_n(s), z\right\rangle\right| \\
&\quad = \sup_{y\in D}\sup_{\|z\|=1}\left|\frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T(s)y, z\rangle \right. \\
&\qquad\qquad \left. - \frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T(t^i)y, z\rangle\right| \\
&\quad \le \frac{1}{N+1} \\
&\qquad \times \sum_{i=0}^{N}\sup_{y\in D}\sup_{\|z\|=1}\left|(\mu_n)_s\langle T(s)y, z\rangle - (l^*_{t^i}\mu_n)_s\langle T(s)y, z\rangle\right| \\
&\quad \le \max_{i=1,2,3,\ldots,N}\|\mu_n - l^*_{t^i}\mu_n\|(M^* + \|w\|) \le \delta, \quad \forall n \ge n_0.
\end{aligned} \tag{104}$$

On the other hand, by Lemma 2, we have

$$\begin{aligned}
&\int \frac{1}{N+1}\sum_{i=0}^{N} T(t^i s)\, y\, d\mu_n(s) \\
&\quad \in \overline{\mathrm{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T(t^i)T(s)y : s \in S\right\}.
\end{aligned} \tag{105}$$

Combining (103)–(105), we have

$$\begin{aligned}
T(\mu_n)y &= \frac{1}{N+1}\sum_{i=1}^{N} T(t^i s)\, y\, d\mu_n(s) \\
&\quad + \left(T(\mu_n)y - \int \frac{1}{N+1}\sum_{i=1}^{N} T(t^i s)\, y\, d\mu_n(s)\right) \\
&\in \overline{\mathrm{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T(t^i)(T(s)y) : s \in S\right\} + B_\delta \\
&\subset \overline{\mathrm{co}}\, F_\delta(T(t); D) + B_\delta,
\end{aligned} \tag{106}$$

for all $y \in D$ and $n \ge n_0$. Therefore,

$$\limsup_{n\to\infty}\sup_{y\in D}\|T(\mu_n)y - T(t)T(\mu_n)y\| \le \epsilon. \tag{107}$$

Since $\epsilon > 0$ is arbitrary, we obtain that (101) holds. Let $t \in S$ and $\epsilon > 0$. Then, there exists $\delta > 0$ satisfying (102). From (101) and condition $(C2)$, there exists $a, b \in (0, 1)$ such that $0 < a \le \beta_n \le b < 1$ and $T(\mu_n)y \in F_\delta(T(t); D)$ for all $y \in D$. From (69), there exists $k_0 \in \mathbb{N}$ such that $\|x_n - T(\mu_n)y_n\| < \delta/b$ for all $n > k_0$. Then, from (102) and (106), we have

$$\begin{aligned}
x_{n+1} &= \beta_n x_n + (1-\beta_n)T(\mu_n)y_n \\
&= T(\mu_n)y_n + \beta_n(x_n - T(\mu_n)y_n) \\
&\in F_\delta(T(t); D) + B_\delta \subset F_\epsilon(T(t); D),
\end{aligned} \tag{108}$$

for all $n > k_0$. Hence, $\limsup_{n\to\infty}\|x_n - T(t)x_n\| \le \epsilon$. Since $\epsilon > 0$ is arbitrary, we obtain that (99) holds.

Next, we show that

$$\limsup_{n\to\infty}\langle \gamma V\hat{x} - \mu F\hat{x}, z_n - \hat{x}\rangle \le 0, \tag{109}$$

where $\hat{x} = P_{\mathcal{F}}(I - \mu F + \gamma V)\hat{x}$. To show this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle \gamma V\hat{x} - \mu F\hat{x}, x_n - \hat{x}\rangle = \lim_{i\to\infty}\left\langle \gamma V\hat{x} - \mu F\hat{x}, x_{n_i} - \hat{x}\right\rangle. \tag{110}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. Now, we show that $v \in \mathcal{F}$.

(i) We first show that $v \in \mathrm{Fix}(\mathcal{S})$. From (99), we have $\|x_n - T(t)x_n\| \to 0$ as $n \to \infty$ for all $t \in S$. Then, from Demiclosedness Principle 2.6, we get $v \in \mathrm{Fix}(\mathcal{S})$.

(ii) We show that $v \in \mathrm{Fix}(K)$, where K is defined as in Lemma 9. Then, from (97), we have

$$\begin{aligned}
\|y_n - Ky_n\| = \|Ku_n - Ky_n\| &\le \|u_n - y_n\| \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{111}$$

and from (98), we also have $\|x_n - Kx_n\| \to 0$. By Demiclosedness Principle 2.6, we get $v \in \mathrm{Fix}(K)$.

(iii) We show that $v \in \bigcap_{k=1}^{N}\mathrm{GMEP}(\Theta_k, \varphi_k, \Psi_k)$. Note that $G_n^k x_n = S_{r_{k,n}}^{(\Theta_k,\varphi_k)}(I - r_{k,n}\Psi_k)G_n^{k-1}x_n$, for all $k \in \{1, 2, \ldots, N\}$. Then, we have

$$\begin{aligned}
&\Theta_k(G_n^k x_n, y) + \varphi_k(y) - \varphi_k(G_n^k) \\
&\quad + \left\langle \Psi_k G_n^{k-1}x_n, y - G_n^k x_n\right\rangle \\
&\quad + \frac{1}{r_{k,n}}\left\langle y - G_n^k x_n, G_n^k x_n - G_n^{k-1}x_n\right\rangle \ge 0.
\end{aligned} \tag{112}$$

Replacing n by n_i in the last inequality and using $(A2)$, we have

$$\begin{aligned}
&\varphi_k(y) - \varphi_k(G_{n_i}^k x_{n_i}) + \left\langle \Psi_k G_{n_i}^{k-1}x_{n_i}, y - G_{n_i}^k x_{n_i}\right\rangle \\
&\quad + \frac{1}{r_{k,n}}\left\langle y - G_{n_i}^k x_{n_i}, G_{n_i}^k x_{n_i} - G_{n_i}^{k-1}x_{n_i}\right\rangle \ge \Theta_k(y, G_{n_i}^k x_{n_i}).
\end{aligned} \tag{113}$$

Let $y_t = ty + (1-t)v$ for all $t \in (0,1]$ and $y \in C$. This implies that $y_t \in C$. Then, we have

$$\begin{aligned}
&\left\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k y_t \right\rangle \\
&\quad \geq \varphi_k\left(G_{n_i}^k x_{n_i}\right) - \varphi_k\left(y_t\right) + \left\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k y_t \right\rangle \\
&\qquad - \left\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k G_n^{k-1} x_{n_i} \right\rangle \\
&\qquad - \left\langle y_t - G_{n_i}^k x_{n_i}, \frac{G_{n_i}^k x_{n_i} - G_{n_i}^{k-1} x_{n_i}}{r_{k,n_i}} \right\rangle + \Theta_k\left(y_t, G_{n_i}^k x_{n_i}\right) \\
&\quad = \varphi_k\left(G_{n_i}^k x_{n_i}\right) - \varphi_k\left(y_t\right) + \left\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k y_t - \Psi_k G_{n_i}^k x_{n_i} \right\rangle \\
&\qquad + \left\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k G_{n_i}^k x_{n_i} - \Psi_k G_{n_i}^{k-1} x_{n_i} \right\rangle \\
&\qquad - \left\langle y_t - G_{n_i}^k x_{n_i}, \frac{G_{n_i}^k x_{n_i} - G_{n_i}^{k-1} x_{n_i}}{r_{k,n_i}} \right\rangle + \Theta_k\left(y_t, G_{n_i}^k x_{n_i}\right).
\end{aligned} \tag{114}$$

From (73), we have $\|\Psi_k G_{n_i}^k x_{n_i} - \Psi_k G_{n_i}^{k-1} x_{n_i}\| \to 0$ as $i \to \infty$. Furthermore, by the monotonicity of Ψ_k, we obtain $\langle y_t - G_{n_i}^k x_{n_i}, \Psi_k y_t - \Psi_k G_{n_i}^k x_{n_i} \rangle \geq 0$. Then, from $(A4)$, we obtain

$$\left\langle y_t - v, \Psi_k y_t \right\rangle \geq \varphi_k(v) - \varphi_k\left(y_t\right) + \Theta_k\left(y_t, v\right). \tag{115}$$

Using $(A1)$, $(A4)$, and (115), we also obtain

$$\begin{aligned}
0 &= \Theta_k\left(y_t, y_t\right) + \varphi_k\left(y_t\right) - \varphi_k\left(y_t\right) \\
&\leq t\Theta_k\left(y_t, y\right) + (1-t)\,\Theta_k\left(y_t, v\right) + (1-t)\,\varphi_k(v) - \varphi_k\left(y_t\right) \\
&\leq t\left[\Theta_k\left(y_t, y\right) + \varphi_k(y) - \varphi_k\left(y_t\right)\right] + (1-t)\left\langle y_t - v, \Psi_k y_t \right\rangle \\
&= t\left[\Theta_k\left(y_t, y\right) + \varphi_k(y) - \varphi_k\left(y_t\right)\right] + (1-t)\,t\left\langle y - v, \Psi_k y_t \right\rangle,
\end{aligned} \tag{116}$$

and, hence,

$$0 \leq \Theta_k\left(y_t, y\right) + \varphi_k(y) - \varphi_k\left(y_t\right) + (1-t)\left\langle y - v, \Psi_k y_t \right\rangle. \tag{117}$$

Letting $t \to 0$ and using $(A3)$, we have, for each $y \in C$,

$$0 \leq \Theta_k(v, y) + \varphi_k(y) - \varphi_k(v) + \left\langle y - v, \Psi_k v \right\rangle. \tag{118}$$

This implies that $v \in \mathrm{GMEP}(\Theta_k, \varphi_k, \Psi_k)$. Hence, $v \in \bigcap_{k=1}^N \mathrm{GMEP}(\Theta_k, \varphi_k, \Psi_k)$. Therefore,

$$v \in \mathcal{F} := \bigcap_{k=1}^{N} \mathrm{GMEP}\left(\Theta_k, \varphi_k, \Psi_k\right) \cap \mathrm{Fix}(K) \cap \mathrm{Fix}(\mathcal{S}). \tag{119}$$

From (66) and (110), we obtain

$$\begin{aligned}
\limsup_{n\to\infty} \left\langle \gamma V\hat{x} - \mu F\hat{x}, z_n - \hat{x} \right\rangle &= \limsup_{n\to\infty} \left\langle \gamma V\hat{x} - \mu F\hat{x}, x_n - \hat{x} \right\rangle \\
&= \lim_{i\to\infty} \left\langle \gamma V\hat{x} - \mu F\hat{x}, x_{n_i} - \hat{x} \right\rangle \\
&= \left\langle \gamma V\hat{x} - \mu F\hat{x}, v - \hat{x} \right\rangle \leq 0.
\end{aligned} \tag{120}$$

Finally, we show that $x_n \to \hat{x}$ as $n \to \infty$. Notice that $z_n = P_C v_n$, where $v_n = \alpha_n \gamma V x_n + (I - \alpha_n \mu F) T(\mu_n) y_n$. Then, from (25), we have

$$\begin{aligned}
\left\|z_n - \hat{x}\right\|^2 &= \left\langle v_n - \hat{x}, z_n - \hat{x} \right\rangle + \left\langle P_C v_n - v_n, P_C v_n - \hat{x} \right\rangle \\
&\leq \left\langle v_n - \hat{x}, z_n - \hat{x} \right\rangle \\
&= \alpha_n \gamma \left\langle V x_n - V\hat{x}, z_n - \hat{x} \right\rangle \\
&\quad + \alpha_n \left\langle \gamma V\hat{x} - \mu F\hat{x}, z_n - \hat{x} \right\rangle \\
&\quad + \left\langle (I - \alpha_n \mu F)\left(T\left(\mu_n\right) y_n - \hat{x}\right), z_n - \hat{x} \right\rangle \\
&\leq \left(1 - \alpha_n\left(1 - \beta_n\right)(\tau - \gamma L)\right)\left\|x_n - \hat{x}\right\|^2 \\
&\quad + \alpha_n \left\langle \gamma V\hat{x} - \mu F\hat{x}, z_n - \hat{x} \right\rangle.
\end{aligned} \tag{121}$$

It follows from (121) that

$$\begin{aligned}
\left\|x_{n+1} - \hat{x}\right\|^2 &\leq \beta_n \left\|x_n - \hat{x}\right\|^2 + \left(1 - \beta_n\right)\left\|z_n - \hat{x}\right\|^2 \\
&\leq \beta_n \left\|x_n - \hat{x}\right\|^2 + \left(1 - \beta_n\right) \\
&\quad \times \left\{\left(1 - \alpha_n\left(1 - \beta_n\right)(\tau - \gamma L)\right)\left\|x_n - \hat{x}\right\|^2 \right. \\
&\qquad \left. + \alpha_n \left\langle \gamma V\hat{x} - \mu F\hat{x}, z_n - \hat{x} \right\rangle \right\} \\
&\leq \left(1 - \alpha_n\left(1 - \beta_n\right)(\tau - \gamma L)\right)\left\|x_n - \hat{x}\right\|^2 \\
&\quad + \alpha_n\left(1 - \beta_n\right)\left\langle \gamma V\hat{x} - \mu F\hat{x}, z_n - \hat{x} \right\rangle.
\end{aligned} \tag{122}$$

Put $\sigma_n := \alpha_n(1 - \beta_n)(\tau - \gamma L)$ and $\delta_n := \alpha_n(1 - \beta_n)\langle \gamma V\hat{x} - \mu F\hat{x}, z_n - \hat{x} \rangle$. Then, (122) reduces to formula

$$\left\|x_{n+1} - \hat{x}\right\|^2 \leq \left(1 - \sigma_n\right)\left\|x_n - \hat{x}\right\|^2 + \delta_n. \tag{123}$$

It is easily seen that $\sum_{n=1}^{\infty} \sigma_n = \infty$, and (using (120))

$$\limsup_{n\to\infty} \frac{\delta_n}{\sigma_n} = \frac{1}{\tau - \gamma L} \limsup_{n\to\infty} \left\langle \gamma V\hat{x} - \mu F\hat{x}, z_n - \hat{x} \right\rangle \leq 0. \tag{124}$$

Hence, by Lemma 7, we conclude that $x_n \to \hat{x}$ as $n \to \infty$. This completes the proof. □

Using the results proved in [35] (see also [32]), we obtain the following results.

Corollary 12. *Let C, H, Θ_k, φ_k, Ψ_k, A_k, F, and V be the same as in Theorem 11. Let S and T be nonexpansive mappings on C with $ST = TS$. Assume that $\mathcal{F} := \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \cap \bigcap_{n=1}^{\infty} \mathrm{GMEP}(\Theta_k, \varphi_k, \Psi_k) \cap \mathrm{Fix}(K) \neq \emptyset$, where K is defined as in*

Lemma 9. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_{k,n}\}_{k=1}^N$ be sequences satisfying (C1)–(C3). Then, the sequence $\{x_n\}$ defined by

$$\begin{aligned} u_n &= S_{r_{N,n}}^{(\Theta_N,\varphi_N)}\left(I - r_{N,n}\Psi_N\right) S_{r_{N-1,n}}^{(\Theta_{N-1},\varphi_{N-1})} \\ &\quad \times \left(I - r_{N-1,n}\Psi_{N-1}\right)\cdots S_{r_{1,n}}^{(\Theta_1,\varphi_1)}\left(I - r_{1,n}\Psi_1\right) x_n, \\ y_n &= S_{\lambda_M}^{G_M}\left(I - \lambda_M A_M\right) S_{\lambda_{M-1}}^{G_{M-1}} \\ &\quad \times \left(I - \lambda_{M-1}A_{M-1}\right)\cdots S_{\lambda_1}^{G_1}\left(I - \lambda_1 A_1\right) u_n, \\ x_{n+1} &= \beta_n x_n + \left(1 - \beta_n\right) \\ &\quad \times P_C\left[\alpha_n \gamma V x_n + \left(I - \alpha_n \mu F\right)\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} S^i T^j y_n\right], \\ &\qquad \forall n \geq 1, \end{aligned} \tag{125}$$

converges strongly to $\hat{x} \in \mathcal{F}$, where $\hat{x}$ solves uniquely the variational inequality (40).

Corollary 13. *Let C, H, Θ_k, φ_k, Ψ_k, A_k, F, and V be the same as in Theorem 11. Let $\mathcal{S} = \{T(t) : t > 0\}$ be a strongly continuous nonexpansive semigroup on C. Assume that $\Omega := \mathrm{Fix}(\mathcal{S}) \cap \bigcap_{n=1}^{\infty} \mathrm{GMEP}(\Theta_k, \varphi_k, \Psi_k) \cap \mathrm{Fix}(K) \neq \emptyset$, where K is defined as in Lemma 9. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_{k,n}\}_{k=1}^N$ be sequences satisfying (C1)–(C3). Then, the sequence $\{x_n\}$ defined by*

$$\begin{aligned} u_n &= S_{r_{N,n}}^{(\Theta_N,\varphi_N)}\left(I - r_{N,n}\Psi_N\right) S_{r_{N-1,n}}^{(\Theta_{N-1},\varphi_{N-1})} \\ &\quad \times \left(I - r_{N-1,n}\Psi_{N-1}\right)\cdots S_{r_{1,n}}^{(\Theta_1,\varphi_1)}\left(I - r_{1,n}\Psi_1\right) x_n, \\ y_n &= S_{\lambda_M}^{G_M}\left(I - \lambda_M A_M\right) S_{\lambda_{M-1}}^{G_{M-1}} \\ &\quad \times \left(I - \lambda_{M-1}A_{M-1}\right)\cdots S_{\lambda_1}^{G_1}\left(I - \lambda_1 A_1\right) u_n, \\ x_{n+1} &= \beta_n x_n + \left(1 - \beta_n\right) \\ &\quad \times P_C\left[\alpha_n \gamma V x_n + \left(I - \alpha_n \mu F\right)\frac{1}{t_n}\int_0^{t_n} T(s)\, y_n ds\right], \\ &\qquad \forall n \geq 1, \end{aligned} \tag{126}$$

where $\{t_n\}$ is an increasing sequence in $(0,\infty)$ with $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$, converges strongly to $\hat{x} \in \mathcal{F}$, where $\hat{x}$ solves uniquely the variational inequality (40).

Corollary 14. *Let C, H, Θ_k, φ_k, Ψ_k, A_k, F, and V be the same as in Theorem 11. Let $\mathcal{S} = \{T(t) : t > 0\}$ be a strongly continuous nonexpansive semigroup on C. Assume that $\Omega := \mathrm{Fix}(\mathcal{S}) \cap \bigcap_{n=1}^{\infty} \mathrm{GMEP}(\Theta_k, \varphi_k, \Psi_k) \cap \mathrm{Fix}(K) \neq \emptyset$, where K is defined as in Lemma 9. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_{k,n}\}_{k=1}^N$ be sequences satisfying (C1)–(C3). Then, the sequence $\{x_n\}$ defined by*

$$\begin{aligned} u_n &= S_{r_{N,n}}^{(\Theta_N,\varphi_N)}\left(I - r_{N,n}\Psi_N\right) S_{r_{N-1,n}}^{(\Theta_{N-1},\varphi_{N-1})} \\ &\quad \times \left(I - r_{N-1,n}\Psi_{N-1}\right)\cdots S_{r_{1,n}}^{(\Theta_1,\varphi_1)}\left(I - r_{1,n}\Psi_1\right) x_n, \\ y_n &= S_{\lambda_M}^{G_M}\left(I - \lambda_M A_M\right) S_{\lambda_{M-1}}^{G_{M-1}} \\ &\quad \times \left(I - \lambda_{M-1}A_{M-1}\right)\cdots S_{\lambda_1}^{G_1}\left(I - \lambda_1 A_1\right) u_n, \\ x_{n+1} &= \beta_n x_n + \left(1 - \beta_n\right) P_C \\ &\quad \times \left[\alpha_n \gamma V x_n + \left(I - \alpha_n \mu F\right)\right. \\ &\qquad \left.\times \left(a_n \int_0^{\infty} \exp\left(-a_n s\right) T(s)\, y_n\right) ds\right], \quad \forall n \geq 1, \end{aligned} \tag{127}$$

where $\{a_n\}$ is a decreasing sequence in $(0,\infty)$ with $\lim_{n\to\infty} a_n = 0$, converges strongly to $\hat{x} \in \Omega$, where $\hat{x}$ solves uniquely the variational inequality (40).

4. Some Applications

In this section, as applications, we will apply Theorem 11 to find minimum-norm solutions $\hat{x} = P_\Omega(0)$ of some variational inequalities. Namely, find a point $\hat{x}$ which solves uniquely the following quadratic minimization problem:

$$\|\hat{x}\|^2 = \min_{x\in\Omega}\|x\|^2. \tag{128}$$

Minimum-norm solutions have been applied widely in several branches of pure and applied sciences, for example, defining the pseudoinverse of a bounded linear operator, signal processing, and many other problems in a convex polyhedron and a hyperplane (see [36, 37]).

Recently, some iterative methods have been studied to find the minimum-norm fixed point of nonexpansive mappings and their generalizations (see, e.g. [38–49] and the references therein).

Using Theorem 11 and Corollaries 12, 13, and 14, we immediately have the following results, respectively.

Theorem 15. *Let C and H be the same as in Theorem 11. Let $\mathcal{S} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on C such that $\mathcal{F} := \mathrm{Fix}(\mathcal{S}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences satisfying (C1)–(C3). Then, the sequence $\{x_n\}$ defined by*

$$x_{n+1} = \beta_n x_n + \left(1 - \beta_n\right) P_C\left[\left(1 - \alpha_n\right) T\left(\mu_n\right) x_n\right], \quad \forall n \geq 1, \tag{129}$$

converges strongly to $\hat{x} \in \mathcal{F}$, where $\hat{x} = P_{\mathcal{F}}(0)$ is the minimum-norm fixed point of $\mathcal{F}$, where $\hat{x}$ solves uniquely the quadratic minimization problem (128).

Theorem 16. *Let C and H be the same as in Corollary 12. Let S and T be nonexpansive mappings on C with $ST = TS$ such*

that $\mathcal{F} := \mathrm{Fix}(S) \cap \mathrm{Fix}(T) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences satisfying* (C1)–(C3). *Then, the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) P_C \times \left[(1-\alpha_n) \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} S^i T^j x_n \right], \quad \forall n \geq 1, \tag{130}$$

converges strongly to $\hat{x} \in \mathcal{F}$, *where* $\hat{x} = P_{\mathcal{F}}(0)$ *is the minimum-norm fixed point of* $\mathcal{F}$, *where* $\hat{x}$ *solves uniquely the quadratic minimization problem* (128).

Theorem 17. *Let C and H be the same as in Corollary 13. Let* $\mathcal{S} = \{T(t) : t > 0\}$ *be a strongly continuous nonexpansive semigroup on C such that* $\mathcal{F} := \mathrm{Fix}(\mathcal{S}) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences satisfying* (C1)–(C3). *Then, the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) P_C \left[(1-\alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s) x_n ds \right], \quad \forall n \geq 1, \tag{131}$$

where $\{t_n\}$ *is an increasing sequence in* $(0,\infty)$ *with* $\lim_{n\to\infty}(t_n/t_{n+1}) = 1$, *converges strongly to* $\hat{x} \in \mathcal{F}$, *where* $\hat{x} = P_{\mathcal{F}}(0)$ *is the minimum-norm fixed point of* $\mathcal{F}$, *where* $\hat{x}$ *solves uniquely the quadratic minimization problem* (128).

Theorem 18. *Let C and H be the same as in Corollary 14. Let* $\mathcal{S} = \{T(t) : t > 0\}$ *be a nonexpansive semigroup on C such that* $\mathcal{F} := \mathrm{Fix}(\mathcal{S}) \neq \emptyset$. *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be sequences satisfying* (C1)–(C3). *Then, the sequence* $\{x_n\}$ *defined by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n) P_C \times \left[(1-\alpha_n) \left(a_n \int_0^{\infty} \exp(-a_n s) T(s) x_n ds \right) \right], \quad \forall n \geq 1, \tag{132}$$

where $\{a_n\}$ *is a decreasing sequence in* $(0,\infty)$ *with* $\lim_{n\to\infty} a_n = 0$, *converges strongly to* $\hat{x} \in \mathcal{F}$, *where* $\hat{x} = P_{\mathcal{F}}(0)$ *is the minimum-norm fixed point of* $\mathcal{F}$, *where* $\hat{x}$ *solves uniquely the quadratic minimization problem* (128).

Acknowledgment

The authors were supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (Grant no. NRU56000508).

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