Journal of Mathematics

Fixed Points and Computational Optimization

Lead Guest Editor: Sun Young Cho
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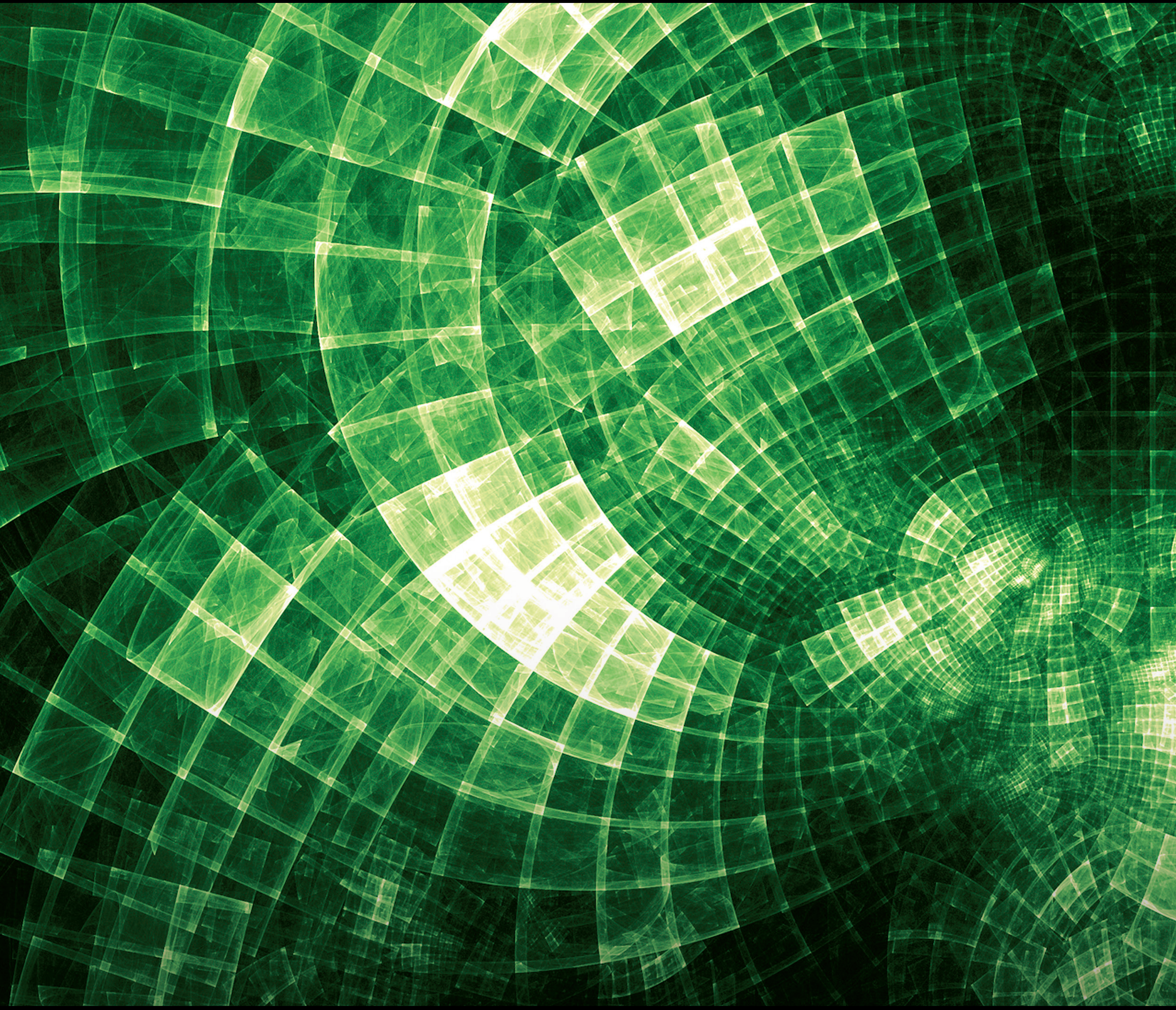

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Research Article

Differential Evolution without the Scale Factor and the Crossover Probability

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Differential evolution has made great achievements in various fields such as computational sciences, engineering optimization, and operations management in the past decades. It is well known that the control parameter setting plays a very important role in terms of the performance improvement of differential evolution. In this paper, a differential evolution without the scale factor and the crossover probability is presented, which eliminates almost all control parameters except for the population size. The proposed algorithm looks upon each individual as a charged particle to decide on the shift of the individual in the direction of the difference based on the attraction-repulsion mechanism in Coulomb's Law. Moreover, Taguchi's parameter design method with the two-level orthogonal array is merged into the crossover operation in order to obtain better individuals in the next generation by means of better combination of factor levels. What is more, a new ratio of the signal-to-noise is proposed for the purpose of fair comparison of the numerical experiment for the tested functions which have an optimal value with 0. Numerical experiments show that the proposed algorithm outperforms the other 5 compared algorithms for the 10 benchmark functions.

1. Introduction

With its efficiency and effectiveness, differential evolution (for short, DE) proposed by Storn and Price has been successfully applied in many different engineering fields [1, 2]. In order to keep improving the performance of DE, various efforts have been devoted over the past decades.

The researchers proposed three discrete DEs for the scheduling problems in the permutation flow shop environment [3]. These approaches focus on converting vectors of the continuous domain into permutation vectors of the discrete domain and self-adjusting the control parameters of these algorithms based on JADE [4] and SADE [5]. The results show that these proposed approaches are promising for scheduling problems.

For the parameter identification of solar cells, the original FSDE in reference [6] was improved, which is the hybridization between free search and DE with opposition-based learning by using a simple greedy strategy instead of a Gaussian noise update in the process of the potential solution generation for the proposed best solution update strategy [7]. Reference [8] also employed a DE with opposition-based learning for estimating optimum hourly energy generation scheduling of a hydro-thermal system.

The authors emphasized the population initialization on increasing the accuracy and convergence speed of DE and designed a new DE variant with a modified initialization scheme by combining the strengths of both chaotic maps and oppositional-based learning strategy in order to generate the initial population with a good quality of mean fitness and diversity of the solutions. Extensive simulation studies on benchmark functions show that the proposed algorithm outperforms its peers [9].

A cultural DE algorithm using a measure of population diversity was proposed as an alternative method for solving the economic load dispatch problems of thermal generators [10]. Based on the cultural algorithm technique using normative and situational knowledge sources, the proposed algorithm is able to balance well the trade-off between the exploration and the exploitation of the search space.

The scale factor F and the crossover probability Cr are two vital parameters in DE, which usually greatly improve the performance. Various strategies for parameter setting have been researched.

The values of $F = 0.5$ and $Cr = 0.9$ were suggested by Storn and Price [1]. The F was set to the normal distribution rand number with expectation 0 and standard deviation 1 for multiobjective optimization in reference [11].

Qin and Suganthan considered F and Cr as the random numbers following normal distribution $F \sim N(0.5, 0.3)$ and $Cr \sim N(Crm, 0.1)$ according to the learning experience, where the parameter Crm is set at 0.5 and updated once every 25 generations [5].

Kim et al. proposed that the scale factor F is calculated by the formula $F = a + b \cdot \text{rand}(0,1)$, where $a + b < 1$ and $a, b > 0$ [12].

Ali and törn empirically obtained an optimal value $Cr = 0.5$ and calculated automatically the scale factor F using the maximum and the minimum for focusing on the exploration at early generation and the exploitation at latter generation [13].

The parameters F and Cr were given, respectively, following α-stable distribution $S_\alpha(0, 0.1, \text{mean}(S_F))$ and $S_\alpha(0, 0.1, \text{mean}(S_{Cr}))$, where S_F and S_{Cr} denote the successfully evolved individuals' F and Cr based on some feedbacks from the optimization process [14].

The scale factor F was set using the Tsallis distribution in economic view for the optimization model in shell-and-tube heat exchangers [15]. F is fist initialized with uniform random values between 0.8 and 1.1, and then is determined by $F = F_{mu} + F_\sigma^2 \cdot P_F$ at each generation, where P_F obeys a q-Gaussian distribution or Tsallis distribution with the means F_{mu} and the variance F_σ^2, the parameter q is linked to the type of distribution that assumes values from 1 to 3.

A self-adaptive scaling factor $F = S \cdot \sqrt{\text{rand}(0,1)^2 \cdot d} - b$ was utilized in reference [16] for maximizing the profit of the distribution company with the several constraints based on the basic idea of the penalty function approach for solving optimal planning of energy storage systems in order to improve the rate of convergence of DE, where S, d, and b are an acceleration factor, a linear decreasing factor, and a deceleration factor, respectively.

Based on the different setups created by a simple orthogonal experimental design method, the paper [17] revealed that DE/best/1/bin with $F = 0.5$ and $Cr = 0.2 + 0.6 * rand(0,1)$ is promising to optimize the vector Jiles–Atherton vector hysteresis model from a workbench containing a rotational single sheet tester. Similarly, the self-adapting parameter strategy was used in reference [18].

Some researchers designed the novel selection operator or employed the classical derivative-free methods in DE or analyzed the search behavior in theory for improving the performance of DE [19–22].

These versions of DE do improve the algorithm performance. However, each of them only is superior to the other in some special aspects. The best setting for the control parameters can be different for different problems. Even though the self-adapting parameter strategies seem to be able to overcome the problem of parameter setting, some new control parameters are used. Several references reported that choosing the proper control parameters for DE is more difficult than expected. How to set reasonably these parameters is a nuisance [2, 23, 24].

A differential evolution without the scale factor and the crossover probability is presented in the paper. The algorithm calculates dynamically the scale factor F using the attraction-repulsion mechanism in Coulomb's Law and executes the crossover operation using Taguchi's parameter design method based on the orthogonal array. The proposed algorithm avoids the parameter settings. Numerical experiments show that the performance of the proposed algorithm is superior to that of the other compared algorithms.

The paper is the extended version which has been further researched based on "almost-parameter-free differential evolution" proposed by Zhang and Liu [24]. There are four different points between them. Firstly, this paper describes in detail the idea and particulars of the proposed algorithm. Secondly, we regard the scale factor F in the mutant equations (13) and (14) in Section 4 as the two different charges for the purpose of a better interpretation of the algorithms' idea and a better numerical experiment results. Thirdly, the vital shortcoming of the original definition of the ratio of the signal to noise (SNR) is analyzed in Section 4 and reveals the fact that it has thought of the optimal value of the tested problem before being solved as 0, then presents a modified definition of SNR for the sake of fairness. Finally, a brief convergence analysis is given under two assumptions.

The main contributions of this paper, which distinguish from the related literatures, are summarized as follows:

(i) Use the electromagnetism-like mechanism to decide on the step length in the direction of the difference for the mutation operation;

(ii) Employ Taguchi's parameter design with a two-level orthogonal array based on a new ratio of the signal to noise that is proposed for the crossover operation;

(iii) Eliminate almost all the control parameters of DE except for the population size.

The remainder of the paper is organized as follows. In Section 2, differential evolution algorithm is briefly introduced. Taguchi's parameter design method is described in the next section. In Section 4, a DE without the parameters is proposed and the convergence in probability is analyzed. In Section 5, the results of numerical experiments are given. Finally, we conclude this paper and consider the further research issues.

2. Differential Evolution

Like other evolutionary algorithms (EAs), DE starts with an initial population individual, followed by the successive operations of mutation, crossover, and selection. However, there are two main differences between them. (i) Mutation is caused not by the small changes of the genes in EAs, but by adding the weighted difference of two randomly selected individuals to a third randomly selected one in DE. The direction information from the current population is used to guide the search process. (ii) New individual is generated by adopting a greedy selection scheme in DE, which is only accepted if it improves on the fitness of the parent individual.

Storn and Price proposed several different mutation strategies [1]:

DE/Rand/1: $V = X_{r1} + F \cdot (X_{r2} - X_{r3})$

DE/Rand/2: $V = X_{r1} + F \cdot (X_{r2} - X_{r3} + X_{r4} - X_{r5})$

DE/Best/2: $V = X_{best} + F \cdot (X_{r2} - X_{r3} + X_{r4} - X_{r5})$

In the above, $r1 \neq r2 \neq r3 \neq r4 \neq r5$, and they are the random numbers distributing uniformly in $[1, NP]$, where NP is denoted by the population size. For the strategy DE/ x/y, x represents the individual being perturbed and y is the number of difference vectors used to disturb x. Take DE/rand/1 as an example, it means that the target individual is randomly selected, and only one difference vector is used.

Although there are several variants of DE, a common variant, which is known as DE/rand/1, or "classic DE," is the most widely used in practice. Hence, this DE is described as follows:

(i) Initialization: like other EAs, classic DE initializes an initial population that distributes uniformly in the feasible domain.

(ii) Mutation: for each parent vector X_i, a mutant vector V_i is generated according to (1) where the random indexes $r1$, $r2$, and $r3$ are mutually distinct integers following uniform distribution in $[1, NP]$ and also are different from the current index i. The scale factor F is used to control the amplification of the differential variation.

$$V_i = X_{r1} + F \cdot (X_{r2} - X_{r3}). \quad (1)$$

(iii) Crossover: the trial individual W_i is generated using the parent and mutant individuals as follows:

$$W_i^j = \begin{cases} V_i^j, & \text{if } r(j) \leq Cr \text{ or } j = \text{rand}n(i), \\ X_i^j, & \text{else.} \end{cases} \quad (2)$$

In the above formula, j is denoted by the j-th component of the individual, $r(j)$ represents a random number with uniform distribution in [0, 1] for each j, the crossover probability Cr is set to a given number in $(0, 1)$, and the integer $randn(i)$ is randomly chosen in $[1, n]$, where n denotes the dimension of the tested problem. The trial individual is a stochastic combination of the parent and mutant individuals. When Cr is equal to 0, at least one of the components of the trial individual will differ from the parent X_i because of the condition $j = randn(i)$.

(iv) Selection: DE implements a very simple selection procedure. The offspring is generated only if the fitness of the offspring is better than that of the parent. Due to the greedy selection scheme, all the individuals of the next generation are as good as or better than their counterparts in the current generation.

$$X_{i+1} = \begin{cases} W_i, & \text{if } f(W_i) < f(X_i), \\ X_i, & \text{otherwise.} \end{cases} \quad (3)$$

The above process ii–iv repeats until the number of function evaluations or the number of the iterations reaches a given constant, namely, the termination criteria are satisfied. Further detailed descriptions about DE can be found in references [1, 23].

3. Taguchi's Parameter Design

Taguchi method [25] is a parameter design approach in the production and process conditions optimization. It can make high-quality products using less development and manufacturing costs. Two major tools used in the Taguchi method are the orthogonal array [26] and the signal-noise-ratio (SNR), which are briefly described as follows.

The orthogonal array is a fractional factorial matrix, which assures a balanced comparison among the factors or its levels. A two-level orthogonal array is a matrix consisting of 1 or 2 arranged in rows and columns. Each row represents the combination of factor levels in each experiment, and each column represents the special level of each factor. Let the element 2 in the orthogonal array be −1, then all column vectors are orthogonal to each other, namely, the dot product is zero. Generally, a two-level orthogonal array is denoted by $L_m(2^{m-1})$, where m, which is equal to 2^k, represents the number of experiments; k is a positive integer; the number 2 shows that each factor has two levels: 1 and 2; $m - 1$ is the number of the factors or columns. The two-level orthogonal arrays are commonly used in practice: $L_4(2^3)$, $L_8(2^7)$, $L_{16}(2^{15})$, and $L_{32}(2^{31})$. For more clearness, the following table (see Table 1) shows the orthogonal array $L_8(2^7)$ with the canonical form.

There are 8 factors in the array $L_8(2^7)$. For each factors, it can choose either 1 or 2. In order to obtain the better or best the combination of factor levels, only 8 experiments are under considered in the two-level orthogonal array $L_8(2^7)$ instead of all combinations of the factors which can reach up to $2^7 = 128$ experiments. The notation E_i represents the i-th experiment or row, and C_j the j-th column vector or factor. For simplicity, the sign $C_{i,j}$ denotes the level of the j-th factor in the i-th experiment. For instance, $C_{3,4} = 1$, $C_{4,3} = 2$, and $C_6 = [1\,2\,2\,1\,1\,2\,2\,1]^T$. If each 2 in array $L_8(2^7)$ is thought of as −1, $C_{i \neq j} \cdot C_j = 0$ for all i and j from 1 to 7.

The conception of the SNR is originally introduced in communication and electronic engineering, which is defined as the ratio of the signal to noise and is used to evaluate the quality of communication. In 1957, Taguchi applied the SNR conception to the design of engineering experiments, hence, Taguchi parameter design method was proposed. This method utilizes the SNR to evaluate quality and applies the orthogonal array to arrange experiments. According to the type of characteristic, the SNR can be classified into smaller-the-better, larger-the-better and nominal-the-best. Given

Table 1: The orthogonal array $L_8(2^7)$ with the canonical form.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
E_1	1	1	1	1	1	1	1
E_2	1	1	1	2	2	2	2
E_3	1	2	2	1	1	2	2
E_4	1	2	2	2	2	1	1
E_5	2	1	2	1	2	1	2
E_6	2	1	2	2	1	2	1
E_7	2	2	1	1	2	2	1
E_8	2	2	1	2	1	1	2

a set of characteristics $y_1, y_2, \ldots, y_n$, then in the case of smaller-the-better characteristic the SNR is as follows:

$$SNR = -10 \cdot \log\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{y_i^2}\right). \tag{4}$$

4. Differential Evolution without F and Cr

After the brief description about classical DE and Taguchi's parameter design, the ideals and the advantage of eliminating the parameters in DE are described, respectively. Finally, the differential evolution without the scale factor and the crossover probability are proposed.

Besides the parameters F and Cr, classic DE has a control parameter NP which are closely related to the problem under consideration. The population size, NP, is typically larger than a threshold value in order to obtain a global optimum and improve the success rate of convergence. However, too large NP may increase the number of function evaluations. Generally, separable and unimodal functions require the smallest population sizes, while parameter-dependent multimodal functions require the largest populations. For simplicity, the parameter NP is set as a constant according to the dimension of the problem under consideration.

The parameter F determines the amplification of the difference. A high (low) value of F makes DE more exploratory (less exploratory). The parameter Cr controls the distribution of coordinate points in the trial individual. A high (low) value of Cr means that the coordinates of the mutant individual dominate the trial individual. Between the two parameters Cr and F, Cr is much more sensitive to the problem's properties and complexity such as the multimodality, while F is more related to the convergence speed.

Finding the optimal values for these parameters is a difficult task as these values are problem specific, especially when one wants to strike a balance between reliability and efficiency. Thus, the performance of DE depends on how these control parameters are selected. However, how to set well these parameters is generally based on trial and error. An optimal parameter setting can be found via the boring preliminary numerical experiments for a special problem, whereas it is not probably optimal for the other problems.

In order to overcome these contradictions, we eliminate the scale factor and the crossover probability with exception of the population size by using the modified attraction-repulsion mechanism and Taguchi method. In the following subsections, how to eliminate these parameters is described in detail.

4.1. Eliminating the Scale Factor F.

According to the attraction-repulsion mechanism in Coulomb's Law, electromagnetism-like (EM) algorithm [27, 28] first calculates the charge of each individual in terms of its objective function value and then determines the resultant force exerted on each particle by all other particles in the population. The charge of each particle determines its power of attraction or repulsion. The particles with better objective function values attract others while those with inferior function values repel.

Like the method of calculating the force, the electromagnetic force exerted on the particle by other particles is obtained by the vector addition following the parallelogram law. For example, the charge of X_1 is less than that of X_2 while is greater than that of X_3 in Figure 1. Thus $EF_{1,2}$ is a repulsive force and $EF_{1,3}$ is an attractive force acting on X_1 by X_2 and X_3, respectively. The resultant force EF_1 exerted on X_1 is $EF_{1,2} + EF_{1,3}$. In a similar way, the resultant forces exerted on X_2, and on X_3 can also be calculated.

The charge Q_i of each X_i is determined by the objective function value of itself relative to that of the current best particle X_{best}:

$$Q_i = \exp\left(-n \cdot \frac{f(X_i) - f(X_{best})}{\sum_{j=1}^{NP}\left(f\left(X_j\right) - f(X_{best})\right)}\right), \tag{5}$$

where n is the dimension of the problem. The force vector $EF_{i,j}$ exerted on X_i by X_j is then determined by

$$EF_{i,j} = \begin{cases} \left(X_j - X_i\right) \cdot \dfrac{Q_i Q_j}{\left\|X_j - X_i\right\|^2}, & \text{if } f\left(X_j\right) \le f(X_i), \\ \left(X_i - X_j\right) \cdot \dfrac{Q_i Q_j}{\left\|X_j - X_i\right\|^2}, & \text{if } f\left(X_j\right) > f(X_i). \end{cases} \tag{6}$$

From (6), the particles with the relatively good objective function values will attract the other particles in the population while the particles with the worse objective function values repel the others. The resultant force vector EF_i exerted on a particle X_i by other $NP - 1$ particles in the population is calculated as follows:

$$EF_i = \sum_{j=1, j\neq i}^{NP} EF_{i,j}. \tag{7}$$

However, each particle has only one particle exerting force on it in a version of EM proposed by Debels et al. [29].

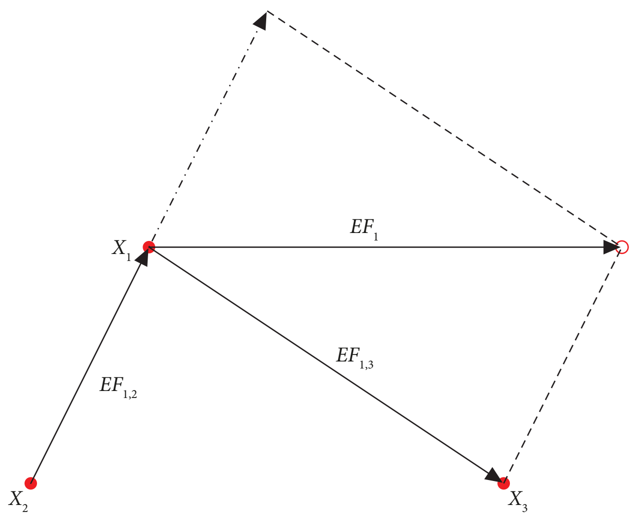


Figure 1: Exertion of forces on X_1 by X_2 and X_3.

In this approach, the charge Q_i^j of X_i is calculated based on the relative difference in the objective function values $f(X_i)$ and $f(X_j)$:

$$Q_i^j = \frac{f(X_i) - f(X_j)}{f(X_{worst}) - f(X_{best})}, \tag{8}$$

where X_{worst} and X_{best} denote, respectively, the worst and the best solutions, X_j is chosen randomly from the population. By the new definition of Q_i^j, obviously, a better(worst) particle X_j gives the higher(lower) Q_i^j value. Moreover, if $f(X_j) \leq f(X_i)$, then Q_i^j is positive, otherwise, Q_i^j is negative. After calculating the charge Q_i^j of X_i, the particle X_i moves to the new particle $X_i + EF_{i,j}$, where

$$EF_{i,j} = Q_i^j \cdot (X_j - X_i). \tag{9}$$

It is obvious that when Q_i^j is positive (negative), X_j attracts(repels) X_i. This modified EM remains the basic ideal of EM, moreover, it is more simple and easier to utilize. Hence, for DE/Rand/1, the mutant individual $V = X_{r1} + F \cdot (X_{r2} - X_{r3})$ can be transformed to

$$\begin{aligned} V &= X_{r1} + F \cdot (X_{r2} - X_{r3}) \\ &= X_{r1} + F \cdot (X_{r2} - X_{r1}) + F \cdot (X_{r1} - X_{r3}) \\ &= X_{r1} + F \cdot (X_{r2} - X_{r1}) + F' \cdot (X_{r3} - X_{r1}), \end{aligned} \tag{10}$$

where $F = -F'$. If we regard the scale factor F and F' in equation (10) as the two different charges Q_i^j as shown in equation (8), viz.

$$F \triangleq Q_{r1}^{r2}, F' \triangleq Q_{r1}^{r3}, \tag{11}$$

then the equation (10) can be interpreted as the motion of the particle X_{r1} in the direction of the resultant force $F_{2,1} + F_{3,1}$. The magnitude of the motion is determined by the scale factors F and F'. Hence, the mutant individual is modified in our algorithm as follows:

$$\begin{aligned} V &= X_{r1} + Q_{r1}^{r2} \cdot (X_{r2} - X_{r1}) + Q_{r1}^{r3} \cdot (X_{r3} - X_{r1}) \\ &= X_{r1} + \left(\frac{f(X_{r1}) - f(X_{r2})}{f(X_{\text{worst}}) - f(X_{\text{best}})} \cdot (X_{r2} - X_{r1}) \right. \\ &\quad \left. + \frac{f(X_{r1}) - f(X_{r3})}{f(X_{\text{worst}}) - f(X_{\text{best}})} \cdot (X_{r3} - X_{r1}) \right). \end{aligned} \tag{12}$$

Similarly, we also have

$$\begin{aligned} V &= X_{r1} + Q_{r2}^{r3} \cdot (X_{r3} - X_{r2}) \\ &= X_{r1} + \frac{f(X_{r2}) - f(X_{r3})}{f(X_{\text{worst}}) - f(X_{\text{best}})} \cdot (X_{r3} - X_{r2}), \end{aligned} \tag{13}$$

or

$$\begin{aligned} V &= X_{r1} + Q_{r3}^{r2} \cdot (X_{r2} - X_{r3}) + Q_{r5}^{r4} \cdot (X_{r4} - X_{r5}) \\ &= X_{r1} + \left(\frac{f(X_{r3}) - f(X_{r2})}{f(X_{\text{worst}}) - f(X_{\text{best}})} \cdot (X_{r2} - X_{r3}) \right. \\ &\quad \left. + \frac{f(X_{r5}) - f(X_{r4})}{f(X_{\text{worst}}) - f(X_{\text{best}})} \cdot (X_{r4} - X_{r5}) \right). \end{aligned} \tag{14}$$

As described , equations (13) and (14) are easy to understand. The idea implied in equation (13) comes from *DE/rand/*1: the individual X_{r1} moves in the direction of $X_{r3} - X_{r2}$. The magnitude of the motion is not controlled artificially in DE/Rand/1, but is determined self-adaptively according to its charge obtained by the particle X_{r2}. The similar interpretation is also done for equation (14).

Besides the self-adaptation of F and the simplicity of calculation, preliminary numerical experiments show that the modified equations (12)–(14) can generally improve the performance of DE, and equation (12) might avoid DE(DE/Ra nd/1) searching wrongly in the direction of "up hill." The detailed description is as follows.

For **s**ix hump camel back function (see $F0$ in Appendix), it is well known that the optimal value is $f^*(\{[-0.08984, 0.71265], [0.08984, -0.71265]\}) = -1.031628$. Given $X_{\text{best}} = [-0.07781, -0.73245]$ and $X_{\text{worst}} = [0.97667, -0.0033774]$, then two cases are given.

Case 1 Let $X_{r1} = [-0.39, -0.91221]$, $X_{r2} = [-0.15301, 0.28698]$, and $X_{r3} = [0.13566, -0.58573]$. Thus, $V = [-0.12891, -0.54275]$ can be obtained by equation (12) (see Figure 2).

Case 2 Let $X_{r1} = [0.15961, 0.48913]$, $X_{r2} = [0.28105, 0.86676]$, and $X_{r3} = [0.94169, -0.23207]$. Then, $V = [-0.4222, 0.97067]$, see Figure 3.

Figures 2 and 3 show the contour of SHCB on $[-1, 1]^2$ with the corresponding function value marked. The stars denote the optimal solutions; the circle denotes the individual X_{r1}; two outer squares 10 represent X_{r2} and X_{r3}, respectively; Two outer real line denote the shift of X_{r1} in direction of the force $EF_{r1,r2}$ and $EF_{r1,r3}$, respectively. The mutant individual V obtained by equation (12) is denoted by the diamond. The inner real line represents the shift of X_{r1} in

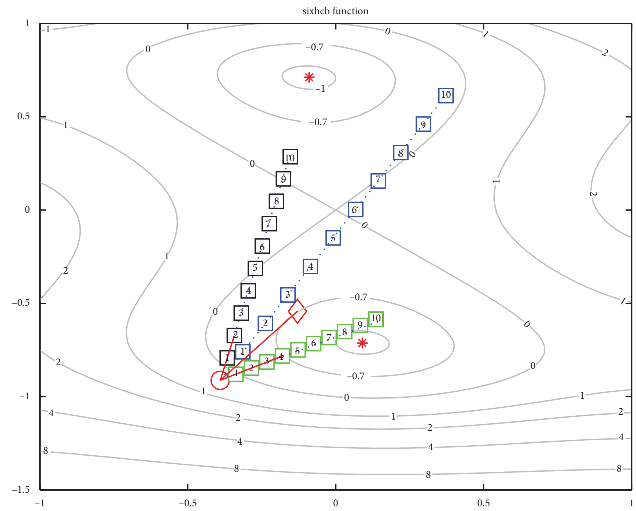


FIGURE 2: Case 1.

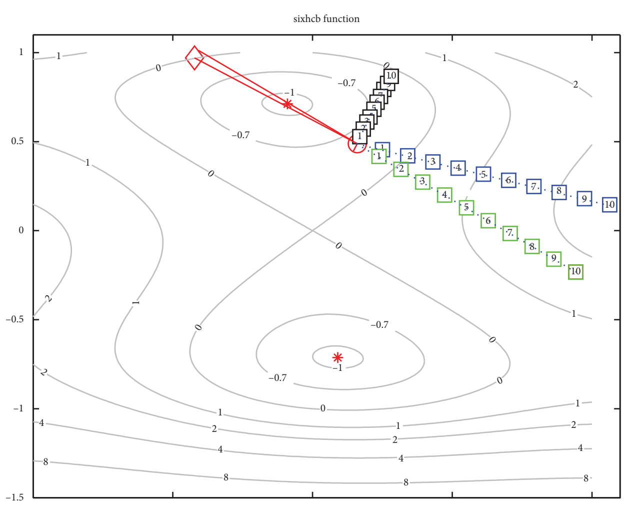


FIGURE 3: Case 2.

direction of the resultant force EF_{r1}. Two bunches of squares locating in outer dashed line denote the motions of the individual X_{r1} in directions of $F \cdot (X_{r2} - X_{r1})$ and $F \cdot (X_{r3} - X_{r1})$ with the different scale factor F, respectively. The scale factor F is chosen orderly from the set $\{0.1, 0.2, \ldots, 0.9, 1\}$, the corresponding results are shown in Figures 2 and 3 by the squares with the different number marked. A bunch of squares between outer squares gives the different mutant individual V (see equation (15)). All squares can be matched by the numbers locating in them.

$$\begin{aligned} V &= X_{r1} + F \cdot [(X_{r2} - X_{r1}) + (X_{r3} - X_{r1})] \\ &= X_{r1} + F \cdot (X_{r2} - X_{r1}) + F \cdot (X_{r3} - X_{r1}). \end{aligned} \tag{15}$$

It is worth noting that equation (15) is different from $V = X_{r1} + F \cdot (X_{r2} - X_{r3} + X_{r4} - X_{r5})$. Five different mutually random individuals are selected in DE/Rand/2 while three individuals in equation (15). However, If $X_{r2} - X_{r1}$ and $X_{r1} - X_{r3}$ are thought of as two new individual, then equation (15) is the same as DE/rand/1 in essence. Thus a comparison is done between equation (12) and equation (15). The two formulas have the similar structure and is easier to distinguish in the figures if some dissimilarities appear in.

From Figure 2, only if $F = \{0.2, 0.3, 0.4\}$, the mutant individual obtained by equation (15) is better, whereas that obtained by equation (12) is closer to the global optimal solution. In Figure 3, it is very clear that equation (12) is superior to equation (15). Though the function value of the individual obtained by equation (15) for $F = 0.2$ is almost same as that of obtained by equation (12), it moves uphill wrong.

4.2. Avoiding the Crossover Probability Cr. Taguchi method can obtain the better combination of the factor level with less cost. In the paper, a two-level orthogonal array $L_m(2^{m-1})$ is used. Since the number of factors (or variables) is $2^k - 1$, where k is an integer greater than 1, the number of experiments m is dependent on the dimension n of the problem. In our paper, m is given as follows:

$$m = \min\left\{2^k \mid k > 1, k \in \mathbb{Z}, 2^k - 1 \geq n\right\}. \tag{16}$$

For instance, if $n = 4$, then $m \geq 3$; if $n = 8$, then $m \geq 4$. In equation (16), the minimal value m subjecting to $m > n$ is chosen for avoiding the possible repeating experiments.

In what follows, a simple algorithm generating the two-level orthogonal array $L_m(2^{m-1})$ is described. The algorithm forms the array by using 2×2 Hadamard matrix H_2.

Definition 1. if any two columns in a matrix H_m consisting of 1 or −1 are orthogonal, then the matrix is called Hadamard matrix [30].

In the above definition, m denotes the order of the Hadamard matrix H_m. There are several operations on Hadamard matrices which preserve the Hadamard property:

(i) Permuting rows (columns)

(ii) Changing the sign of some rows (columns)

(iii) The Kronecker product

If H_m and H_n are known, then $H_{m\cdot n}$ can be obtained by their Kronecker product, namely, by replacing all 1s in H_m by H_n and all -1s by $-H_n$.

Example 1. If $H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, then

$$\begin{aligned} H_2 \otimes H_2 &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \left(\begin{array}{c|c} 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ \hline 1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -1 \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{array} \right) \\ \Rightarrow H_4 &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \end{aligned} \tag{17}$$

$$H_2 \otimes H_4 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \Rightarrow$$

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & E_1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & E_5 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & E_3 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & E_7 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & E_2 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & E_6 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & E_4 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & E_8 \end{pmatrix}$$

where ⊗ denotes the Kronecker product. Hadamard matrix of high order can be similarly generated from that of lower order: $H_2 \otimes H_4 = H_8$, $H_2 \otimes H_8 = H_{16}$, $H_2 \otimes H_{16} = H_{32}$, $H_2 \otimes H_{32} = H_{64}$, etc.

After a Hadamard matrix H_m is obtained, a two-level orthogonal array $L_m(2^{m-1})$ can be given by discarding the all-one column and changing -1 s to 2 s in H_m. However, this obtained array is not generally canonical form. Therefore, the simple exchange of rows can fix it for consistency (see Table 1 and the gray part in H_8).

Recall the notations about the orthogonal array in Section 3: E_i is denoted by the i-th experiment, C_j by the j-th factor and $C_{i,j}$ by the level of the j-th factor in the i-th experiment. The effects of the factors can be defined as follows:

$$E_{C_j,\text{level}} = \sum_{\substack{1 \le i \le m \\ C_{i,j} = \text{level}}} SNR_i. \tag{18}$$

For simplicity, the SNR is calculated as follows:

$$\text{SNR} = \sum_{i=1}^{n} \frac{1}{f_i^2}, \tag{19}$$

where level = {1, 2}, $1 \le i \le m$, and $1 \le j \le m-1$. This conception is used here to evaluate the level of the factor. If $E_{C_j,1} > E_{C_j,2}$, the optimal level of the factor C_j is 1, otherwise, the optimal level is 2. When each $E_{C_j,\text{level}}$ is determined, a new individual (an optimal or near-optimal combination) is generated.

Example 2. An example $\min f(X) = \|X\|_1$ is shown to illustrate this process of Taguchi parameter design method acting on two individuals, where $X \in R^7$. Without loss of generality, let $V = [0, 8, 1, 0, -72, 0, 0]$ and $X = [0, 0, -28, 35, 0, 32, 0]$. This problem has 7 variables (factors), thus according to equation (16):

$m-1=2^3-1=7\geq n$, the orthogonal array $L_8(2^7)$ is chosen(See Table 1). If $C_{i,j}$ is equal to 1 in Table 1, then the corresponding $C_{i,j}$ in Table 2 is the j-th component V^j of the mutant individual V, otherwise, the corresponding $C_{i,j}$ is X^j, see bold in Table 2.

Next, calculate the function value $f(X)$ and the SNR of each combination of the factor level in Table 2, respectively. All results appear in the two most right hand columns. Then the effect of each factor is determined in terms of equation (18) (take C_1 and C_7 as an example).

$$\begin{aligned}
E_{C_1,1} &= \sum_{i=1,2,3,4} SNR_i \\
&= 0.00015 + 0.00017 + 0.00006 + 0.00025 \approx 0.0006, \\
E_{C_1,2} &= \sum_{i=5,6,7,8} SNR_i \\
&= 0.00077 + 0.00003 + 0.00092 + 0.00009 \approx 0.0018, \\
E_{C_7,1} &= \sum_{i=1,4,6,7} SNR_i \\
&= 0.00015 + 0.00025 + 0.00003 + 0.00092 \approx 0.0014, \\
E_{C_7,2} &= \sum_{i=2,3,5,8} SNR_i \\
&= 0.00017 + 0.00006 + 0.00077 + 0.00009 \approx 0.0011.
\end{aligned} \tag{20}$$

Finally, we obtain the new individual or the trial vector W. The optimal level of the factor is decided by its effect. Since $E_{C_1,1} < E_{C_1,2}$, 2 is the optimal level of the factor C_1; $E_{C_2,1} > E_{C_2,2}$, therefore the optimal level of the factor C_2 is 2. The optimal levels of the other factors can be determined in a similar way. The component W^j of the new individual W consists of either V^j or X^j for all j, which is dependent on the optimal level of the factor C_j. If the optimal level is 1, then the corresponding component of the new individual is that of the individual V, otherwise, it is equal to that of the individual X.

Obviously, Taguchi parameter design method executes only 8 experiments instead of all 2^7 combinations of factor levels for obtaining a new individual W with the lower function value 1 (see the last row in Table 2). It is necessary to mention that only the first n columns is used in orthogonal array while the other columns are ignored if $n < m-1$.

In reference [31], the hybrid Taguchi-genetic algorithm (HTGA) is proposed for global numerical optimization with the continuous variables, which uses the systematic reasoning ability of Taguchi parameter design to gain the better genes in the crossover operation. The comparison results between HTGA and OGA/Q [32] show that HTGA can find the optimal or the near-optimal solutions with less function evaluations and better average values. However, this superiority is not very obvious for the tested function with nonzeros optimal values. Let we recall the original definition of SNR in the case of smaller-the-better characteristic, which is described in equation (4), and change it to

$$\mathrm{SNR} = -10\cdot\log\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(y_i-0)^2}\right). \tag{21}$$

In Taguchi method, the item $1/n\sum_{i=1}^{n}1/(y_i-s)^2$ represents the average loss of quality, where s denotes the ideal signal in the case of smaller-the-better characteristic. Therefore, equation (21) shows that HTGA has thought of the optimal value of the tested problem as 0 before this problem is solved. This is unfair and unreasonable. As described above, we found that the superiority of HTGA is not very obvious for those function with nonzero optimal value from Tables IV and V on page 273 and 275 in the reference [31], Hence, SNR is modified as follows:

$$\mathrm{SNR} = \sum_{i=1}^{n}\frac{1}{(f_i-f_t^*)^2}, \tag{22}$$

where f_t^* is defined as the current optimal value after the t-th iteration.

In what follows, the differential evolutions without the scale factor and the crossover probability (for short, DE∖FCr) are proposed. For the sake of clarity, the flowcharts of DE and DE∖FCr(take DE∖FCr2 as an example) are also given in Figure 4, where cross() represents the crossover operation in equation (2) and Taguchi() denotes Taguchi parameter design method in equation (13) (see Algorithm 1).

In Step 2, a termination criterion $|f(X_{\text{worst}})-f(X_{\text{best}})|<\varepsilon$ is given since $f(X_{\text{worst}})-f(X_{\text{best}})$ is the denominator in Eq.(8). When this difference approximates to zero, the numerical stability of the algorithm will lose.

Let $X^{(t)}$ be the population at the t-th generation. Through Step 4-5 in *DE\FCr*, $X^{(t)}$ transforms into the next population $X^{(t+1)}$. Since the limitation of numerical calculation accuracy and $X^{(t+1)}$ relies only on the state of $X^{(t)}$, the population sequence $\{X^{(t)}\}_{t\geq 0}$ generated by DE\FCr can be described as finite-state Markov stochastic process.

Suppose (i) the objection function $f(X)$ has a unique global optimal solution. Let S be the state space of the stochastic process $X^{(t)}$, S^* be the state space of the global optimal solution, and f^* be the global optimal value.

Because of the limitation of state space or search space, the probability that the algorithm can find the optimal solution at the next generation is greater than 0 if it cannot find at the t-th generation, hence, suppose (ii) $P\{X^{(t+1)}=s_j \mid X^{(t)}=s_i\} > \rho > 0$ for $s_i \notin S^*$ and $s_j \in S^*$, where $\rho\in(0,1)$.

Now, we consider the probability $\sum_{s_j\notin S^*}P\{X^{(t+1)}=s_j\}$ that the proposed algorithm can not find the global optimum at the $t+1$ generation.

Step 1: Initialization: population P, population size NP, maximal generation T, current generation $t = 1$, and $\varepsilon = 10^{-100}$, $i = 1$.
Step 2: If $t > T$ or $|f(X_{\text{worst}}) - f(X_{\text{best}})| < \varepsilon$, then output the current optimal value f_t^*.
Step 3: Mutation. For each $X_i \in P_t$ in the population, calculate the mutant individual according to equations (12) or (13) or equation (14). The corresponding algorithm is denoted by DE\FCr1, DE\FCr2 and DE\FCr3, respectively.
Step 4: Crossover. Execute Taguchi parameter design method with the SNR denoted as equation (22) for the individual X_i and the mutant individual V_i, so the trail individual W_i is generated.
Step 5: Selection. If $f(W_i) < f(X_i)$, then $X_i = W_i$ and $i = i + 1$.
Step 6: If $i < = NP$, goto Step 3; otherwise, $t = t + 1$, goto Step 2.

Algorithm 1: (DE\FCr).

$$\begin{aligned}
&\sum_{s_j \notin S^*} P\left\{X^{(t+1)} = s_j\right\} \\
&= \sum_{s_j \notin S^*} \sum_{s_i \in S} P\left\{X^{(t+1)} = s_j, X^{(t)} = s_i\right\} \\
&= \sum_{s_j \notin S^*} \sum_{s_i \in S} P\left\{X^{(t+1)} = s_j \,\middle|\, X^{(t)} = s_i\right\} P\left\{X^{(t)} = s_i\right\} \\
&= \sum_{s_j \notin S^*} \sum_{s_i \notin S^*} P\left\{X^{(t+1)} = s_j \,\middle|\, X^{(t)} = s_i\right\} P\left\{X^{(t)} = s_i\right\} \\
&\quad + \sum_{s_j \notin S^*} \sum_{s_i \in S^*} P\left\{X^{(t+1)} = s_j \,\middle|\, X^{(t)} = s_i\right\} P\left\{X^{(t)} = s_i\right\} \\
&= \sum_{s_j \notin S^*} \sum_{s_i \notin S^*} P\left\{X^{(t+1)} = s_j \,\middle|\, X^{(t)} = s_i\right\} P\left\{X^{(t)} = s_i\right\} \\
&= \sum_{s_i \notin S^*} P\left\{X^{(t)} = s_i\right\} \\
&\quad - \sum_{s_j \in S^*} \sum_{s_i \notin S^*} P\left\{X^{(t+1)} = s_j \,\middle|\, X^{(t)} = s_i\right\} P\left\{X^{(t)} = s_i\right\} \\
&< (1-\rho) \sum_{s_i \notin S^*} P\left\{X^{(t)} = s_i\right\}.
\end{aligned} \tag{23}$$

It is very obvious that the current known optimal solution still can be retained in the next generation from Step 5. Once $DE\backslash FCr$ finds the optimal solution, the $X^{(t+1)}$ will hold the current state S^*. Hence, in the equation (23),

$$\sum_{s_j \notin S^*} \sum_{s_i \in S^*} P\left\{X^{(t+1)} = s_j \,\middle|\, X^{(t)} = s_i\right\} P\left\{X^{(t)} = s_i\right\} = 0. \tag{24}$$

Summarizing the result of equation (23), we have

$$0 \leq \sum_{s \notin S^*} P\left\{X^{(t+1)} = s_j\right\} < (1-\rho) \sum_{s \notin S^*} P\left\{X^{(t)} = s_i\right\}. \tag{25}$$

Because the sequence $\sum_{s \notin S^*} P\{X^{(t)} = s_i\}$ is strictly monotonic decreasing as $t \longrightarrow \infty$, so

$$\lim_{t \longrightarrow \infty} \sum_{s \notin S^*} P\left\{X^{(t)} = s_i\right\} = 0. \tag{26}$$

Therefore,

$$\lim_{t \longrightarrow \infty} P\{f_t^* = f^*\} = 1 - \lim_{t \longrightarrow \infty} \sum_{s \notin S^*} P\left\{X^{(t)} = s_i\right\} = 1. \tag{27}$$

Equation (26) shows that the population sequence generated by $DE\backslash FCr$ can convergence in probability to the global optimum.

5. Numerical Experiments

The proposed Algorithms are executed in Matlab R2017 for the known numerical benchmark functions listed in Appendix with the default parameters $NP = 30$ and $T = 202$. Based on this parameter setting, each $DE\backslash FCr$ needs $30 \times (m + 1)$ function evaluations at each iteration for 30 dimensional tested functions. *DEs* with the four strategies below are compared with our algorithms, respectively.

Strategy 1(DE): DE/Rand/1. $F = 0.5$, $Cr = 0.9$. This is a recommend parameters setting for DE/Rand/1 in most of the references [1–13];

Strategy 2(DEG): $F \sim N(0, 1)$, $C_r = 0.9$ [11];

Strategy 3(DE0.4): $F = 0.4 + 0.4 \cdot rand(0, 1)$, $C_r = 0.9$ [12]

Strategy 4(DEM): $C_r = 0.5$ and F is calculated by the following formula [13]:

$$F = \begin{cases} \max\left\{0.4, 1 - \left|\dfrac{f(X_{\text{worst}})}{f(X_{\text{best}})}\right|\right\}, & \text{if} \left|\dfrac{f(X_{\text{worst}})}{f(X_{\text{best}})}\right| < 1, \\[2ex] \max\left\{0.4, 1 - \left|\dfrac{f(X_{\text{best}})}{f(X_{\text{worst}})}\right|\right\}, & \text{otherwise.} \end{cases} \tag{28}$$

For Strategy 1–4, the population size NP and the maximal generation T are set as 100 and 2000, respectively. All algorithms are performed with 10 independent runs for each tested function with 30 variables. According to these settings, our algorithm has the almost same function evaluations as *DEs* with Strategy 1–4, that is, $100 + 100 \times 2000 = 2000100$ for *DEs* and $30 + 990 \times 202 = 200010$ for $DE\backslash FCr$. Hence, the results listed in Tables 3 and 4 are obtained under the assumption of the not same but different function evaluations. Obviously, the proposed algorithm evaluates 90 function values less than *DEs*. The average values of the

Table 2: The process of Taguchi parameter design method acting on the individuals V and X.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	$f(X)$	SNR
E_1	0	8	1	0	−72	0	0	81	0.00015
E_2	0	8	1	35	0	32	0	76	0.00017
E_3	0	0	−28	0	−72	32	0	132	0.00006
E_4	0	0	−28	35	0	0	0	63	0.00025
E_5	0	8	−28	0	0	0	0	36	0.00077
E_6	0	8	−28	35	−72	32	0	175	0.00003
E_7	0	0	1	0	0	32	0	33	0.00092
E_8	0	0	1	35	−72	0	0	108	0.00009
$E_{C,1}$	0.0006	0.0011	0.0013	0.0019	0.0003	0.0013	0.0014		
$E_{C,2}$	0.0018	0.0013	0.0011	0.0005	0.0021	0.0012	0.0011		
W	0	0	1	0	0	0	0	1	

Table 3: Result comparisons among 7 algorithms for $F1$–$F5$.

Fun	Alg	Best	Worst	Mean	Std	#Elav
	DE	6.445542*E* − 023	4.436277*E* − 022	2.290635*E* − 022	1.108992*E* − 022	200100
	DEG	**1.046763E − 040**	**4.175516E − 039**	**1.128546E − 039**	**1.247818E − 039**	200100
	*DE*0.4	1.348390*E* − 013	1.781579*E* − 012	6.963543*E* − 013	5.084985*E* − 013	200100
*F*1	*DEM*	1.212964*E* − 011	8.731799*E* − 011	3.082813*E* − 011	2.407010*E* − 011	200100
	*DE**FCr*1	3.228810*E* − 022	2.671220*E* − 019	3.358173*E* − 020	8.252692*E* − 020	**200010**
	*DE**FCr*2	2.611086*E* − 024	1.537412*E* + 001	1.546806*E* + 000	4.858513*E* + 000	**200010**
	*DE**FCr*3	1.808723*E* − 025	2.760420*E* − 024	8.956028*E* − 025	9.805783*E* − 025	**200010**
	DE	2.342126*E* − 012	1.023626*E* − 011	5.329959*E* − 012	2.673148*E* − 012	200100
	DEG	**4.440892E − 015**	**7.993606E − 015**	**7.638334E − 015**	**1.123467E − 015**	200100
	*DE*0.4	1.512876*E* − 007	4.988239*E* − 007	3.000771*E* − 007	1.363652*E* − 007	200100
*F*2	*DEM*	4.440892*E* − 015	7.993606*E* − 015	7.283063*E* − 015	1.497956*E* − 015	200100
	*DE**FCr*1	3.135270*E* − 013	2.617373*E* − 011	7.357492*E* − 012	8.539221*E* − 012	**200010**
	*DE**FCr*2	3.996803*E* − 014	1.434325*E* − 008	1.434625*E* − 009	4.535629*E* − 009	**200010**
	*DE**FCr*3	1.509903*E* − 014	1.927347*E* − 013	6.483702*E* − 014	5.207955*E* − 014	**200010**
	DE	4.262079*E* − 003	1.641628*E* − 002	8.883045*E* − 003	3.323616*E* − 003	200100
	DEG	**3.490604E − 003**	**1.002710E − 002**	**5.873780E − 003**	**1.893076E − 003**	200100
	*DE*0.4	7.746132*E* − 003	1.612982*E* − 002	1.284244*E* − 002	2.895560*E* − 003	200100
*F*3	*DEM*	5.340829*E* − 002	8.776692*E* − 002	7.115915*E* − 002	9.972597*E* − 003	200100
	*DE**FCr*1	1.040566*E* − 002	2.893510*E* − 002	1.649259*E* − 002	5.547522*E* − 003	**200010**
	*DE**FCr*2	1.087789*E* − 002	3.538599*E* − 002	2.176316*E* − 002	7.961170*E* − 003	**200010**
	*DE**FCr*3	2.047691*E* − 002	3.617698*E* − 002	2.779340*E* − 002	4.572319*E* − 003	**200010**
	DE	4.869042*E* − 024	1.608507*E* − 022	4.278891*E* − 023	5.275054*E* − 023	200100
	DEG	**1.570545E − 032**	4.146719*E* − 001	4.146719*E* − 002	1.311308*E* − 001	200100
	*DE*0.4	2.196374*E* − 014	1.812240*E* − 013	7.778539*E* − 014	5.733436*E* − 014	200100
*F*4	*DEM*	1.962642*E* + 004	1.637335*E* + 005	8.747525*E* + 004	4.588788*E* + 004	200100
	*DE**FCr*1	5.576314*E* − 024	1.036690*E* − 001	1.036690*E* − 002	3.278302*E* − 002	**200010**
	*DE**FCr*2	1.176714*E* − *E* − 025	3.090863*E* − 001	4.789975*E* − 002	9.877257*E* − 002	**200010**
	*DE**FCr*3	7.336740*E* − 027	**8.506532E − 025**	**1.761171E − 025**	**2.553095E − 025**	**200010**
	DE	3.538508*E* − 023	2.902433*E* − 022	1.590376*E* − 022	8.749484*E* − 023	200100
	DEG	**1.349784E − 032**	1.098737*E* − 002	1.098737*E* − 003	3.474510*E* − 003	200100
	*DE*0.4	1.032194*E* − 013	1.261348*E* − 012	5.065837*E* − 013	4.532350*E* − 013	200100
*F*5	*DEM*	2.220360*E* + 004	1.152671*E* + 005	6.319184*E* + 004	3.036898*E* + 004	200100
	*DE**FCr*1	1.155559*E* − 021	1.691974*E* − 015	1.692275*E* − 016	5.350387*E* − 016	**200010**
	*DE**FCr*2	3.452407*E* − 023	1.247897*E* + 000	2.167329*E* − 001	4.586437*E* − 001	**200010**
	*DE**FCr*3	6.397232*E* − 026	**2.198487E − 024**	**1.088356E − 024**	**6.533660E − 025**	**200010**

The best results in the table are bolded.

obtained results are given in Tables 3 and 4. The number of $f(x)$ evaluations(#EVALU.), the best function value(BEST), the worst function value(WORST), the mean of function values(MEAN) and the standard deviation of the best function values(STD.) are used for the comparisons among these algorithms.

Table 3 summarizes the results obtained by the 7 algorithms for $F1$-$F5$. For $F1$, DEG obtains the best mean of

Table 4: Result comparisons among 7 algorithms for $F6$–$F10$.

Fun	Alg	Best	Worst	Mean	Std	#Elav
F6	DE	**0.000000E + 000**	**0.000000E + 000**	**0.000000E + 000**	**0.000000E + 000**	200100
	DEG	**0.000000E + 000**	1.969000E − 002	5.666035E − 003	6.872102E − 003	200100
	DE0.4	7.412959E − 013	3.983369E − 012	1.929468E − 012	9.211849E − 013	200100
	DEM	8.237855E − 014	2.578935E − 010	2.624131E − 011	8.139491E − 011	200100
	DE\FCr1	**0.000000E + 000**	7.396040E − 003	7.396040E − 004	2.338833E − 003	167650
	DE\FCr2	**0.000000E + 000**	1.477241E − 002	2.216845E − 003	4.986442E − 003	170175
	DE\FCr3	**0.000000E + 000**	**0.000000E + 000**	**0.000000E + 000**	**0.000000E + 000**	**148553**
F7	DE	1.346933E + 002	1.815693E + 002	1.604947E + 002	1.647296E + 001	200100
	DEG	5.969754E + 000	2.089413E + 001	1.492438E + 001	4.115703E + 000	200100
	DE0.4	1.447778E + 002	1.987496E + 002	1.736384E + 002	1.838650E + 001	200100
	DEM	8.895676E + 001	1.067928E + 002	1.005014E + 002	6.041157E + 000	200100
	DE\FCr1	3.979836E + 000	8.954632E + 000	5.870258E + 000	1.654945E + 000	200010
	DE\FCr2	**0.000000E + 000**	2.984877E + 000	6.964713E − 001	1.153657E + 000	152929
	DE\FCr3	**0.000000E + 000**	**0.000000E + 000**	**0.000000E + 000**	**0.000000E + 000**	**173891**
F8	DE	4.734026E + 000	**9.125210E + 000**	**7.317988E + 000**	1.118800E + 000	200100
	DEG	**3.129173E + 000**	7.336176E + 001	2.244019E + 001	2.658720E + 001	200100
	DE0.4	9.489415E + 000	1.273116E + 001	1.136916E + 001	**9.699009E − 001**	200100
	DEM	2.646006E + 001	4.035420E + 002	9.863638E + 001	1.151125E + 002	200100
	DE\FCr1	2.198234E + 001	1.319067E + 002	6.198206E + 001	3.617924E + 001	**200010**
	DE\FCr2	1.491798E + 001	1.011474E + 002	3.406901E + 001	2.894100E + 001	**200010**
	DE\FCr3	1.130096E + 001	7.634987E + 001	2.707867E + 001	1.774072E + 001	**200010**
F9	DE	6.212269E − 011	2.410512E − 010	1.249117E − 010	5.339068E − 011	200100
	DEG	**5.493251E − 023**	**3.940883E − 022**	**1.728275E − 022**	**1.114004E − 022**	200100
	DE0.4	1.482390E − 006	6.510805E − 006	3.673915E − 006	1.530019E − 006	200100
	DEM	6.282339E + 000	7.764649E + 001	6.079609E + 001	2.075941E + 001	200100
	DE\FCr1	8.632854E − 016	4.274906E − 014	9.826062E − 015	1.315808E − 014	**200010**
	DE\FCr2	1.792127E − 016	2.172822E − 001	2.702991E − 002	6.889298E − 002	**200010**
	DE\FCr3	1.435442E − 017	4.834137E − 016	9.552152E − 017	1.422569E − 016	**200010**
F10	DE	**6.521876E − 005**	**4.212407E − 003**	8.509549E − 004	1.555632E − 003	200100
	DEG	7.185030E + 000	2.254554E + 001	1.543480E + 001	5.184946E + 000	200100
	DE0.4	8.266525E − 003	5.784312E − 001	8.974530E − 002	1.759532E − 001	200100
	DEM	1.028353E − 004	2.981224E − 004	**2.063294E − 004**	**5.460973E − 005**	200100
	DE\FCr1	9.550667E + 000	2.092655E + 001	1.313933E + 001	3.863092E + 000	**200010**
	DE\FCr2	9.407884E + 000	2.291669E + 001	1.598827E + 001	4.119512E + 000	**200010**
	DE\FCr3	2.517749E − 002	3.977652E − 001	1.628583E − 001	1.193423E − 001	**200010**

The best results in the table are bolded.

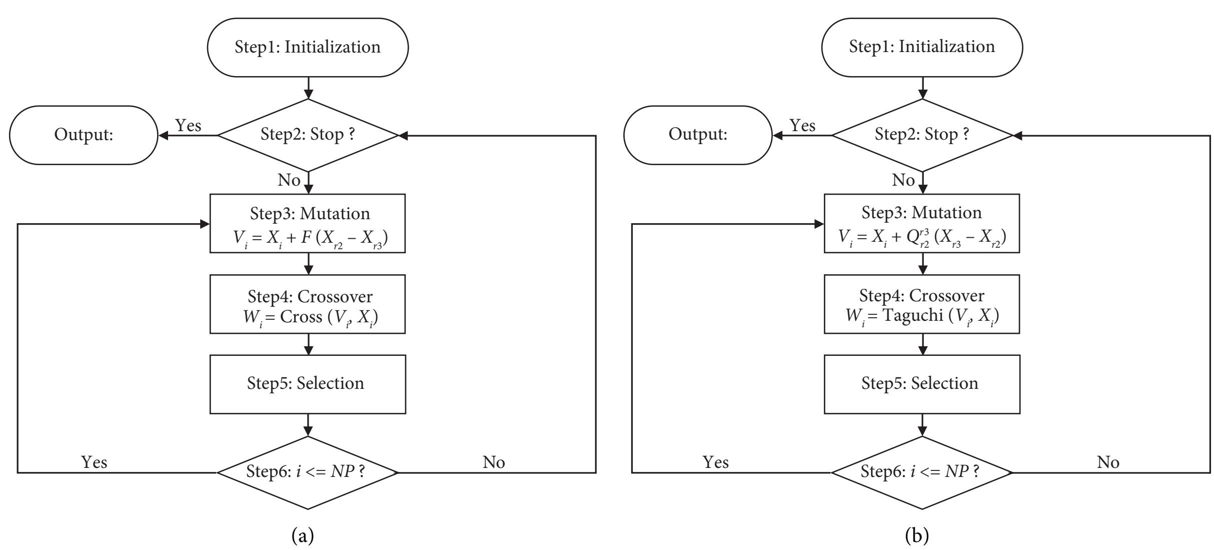
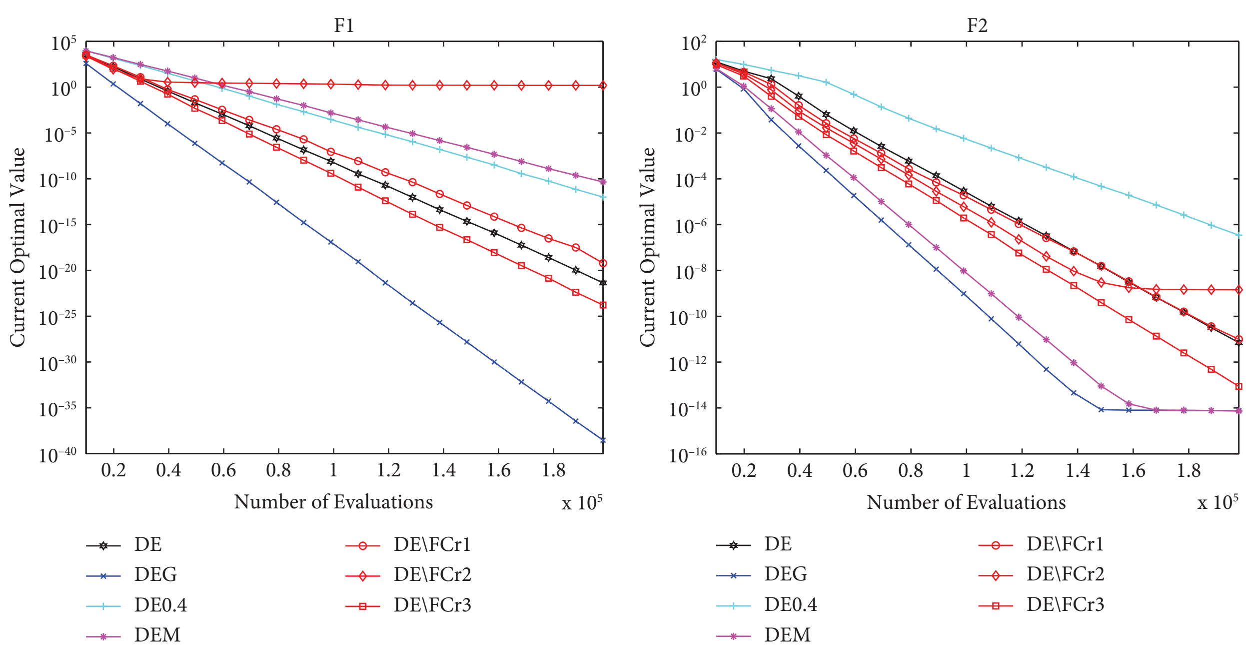


Figure 4: Flowcharts of DE and DE\FCr2. (a) DE. (b) DE\FCr2.

Table 5: Result comparisons between *DE\FCr3* and *DENSO* for *F*1–*F*10.

Fun	Alg	Best	Worst	Mean	Std	#Elav
*F*1	*DE\FCr3*	**1.808723E − 25**	**2.760420E − 24**	**8.956028E − 25**	**9.805783E − 25**	**200010**
	DENSO	8.585195E − 06	1.446810E − 05	1.210160E − 05	1.972164E − 06	200100
*F*2	*DE\FCr3*	**1.509903E − 14**	**1.927347E − 13**	**6.483702E − 14**	**5.207955E − 14**	**200010**
	DENSO	7.522790E − 04	1.429218E − 03	9.793750E − 04	2.043630E − 04	200100
*F*3	*DE\FCr3*	2.047691E − 02	3.617698E − 02	2.779340E − 02	4.572319E − 03	**200010**
	DENSO	**8.027579E − 03**	**2.113126E − 02**	**1.426177E − 02**	**3.372486E − 03**	200100
*F*4	*DE\FCr3*	**7.336740E-27**	**8.506532E-25**	**1.761171E-25**	**2.553095E-25**	**200010**
	DENSO	9.835739E − 07	5.804517E − 06	2.661288E − 06	1.493261E − 06	200100
*F*5	*DE\FCr3*	**6.397232E − 26**	**2.198487E − 24**	**1.088356E − 24**	**6.533660E − 25**	**200010**
	DENSO	1.551095E − 05	8.342158E − 05	3.712426E − 05	2.019380E − 05	200100
*F*6	*DE\FCr3*	**0.000000E + 00**	**0.000000E + 00**	**0.000000E + 00**	**0.000000E + 00**	**148553**
	DENSO	2.194028E − 05	7.591791E − 03	8.146725E − 04	2.381668E − 03	200100
*F*7	*DE\FCr3*	**0.000000E + 00**	**0.000000E + 00**	**0.000000E + 00**	**0.000000E + 00**	**173891**
	DENSO	1.245423E + 01	1.903578E + 01	1.599377E + 01	2.169709E + 00	200100
*F*8	*DE\FCr3*	**1.130096E + 01**	7.634987E + 01	2.707867E + 01	1.774072E + 01	**200010**
	DENSO	2.430580E + 01	**2.530343E + 01**	**2.468209E + 01**	**3.004402E − 01**	200100
*F*9	*DE\FCr3*	**1.435442E + 17**	**4.834137E − 16**	**9.552152E − 17**	**1.422569E − 16**	**200010**
	DENSO	2.975740E + 04	7.035782E + 04	5.295819E + 04	1.148206E + 04	200100
*F*10	*DE\FCr3*	**2.517749E − 02**	**3.977652E − 01**	**1.628583E − 01**	**1.193423E − 01**	**200010**
	DENSO	2.013176E − 01	9.015901E − 01	4.872225E − 01	2.103323E − 01	200100

The best results in the table are bolded.



(a) (b)

Figure 5: Continued.

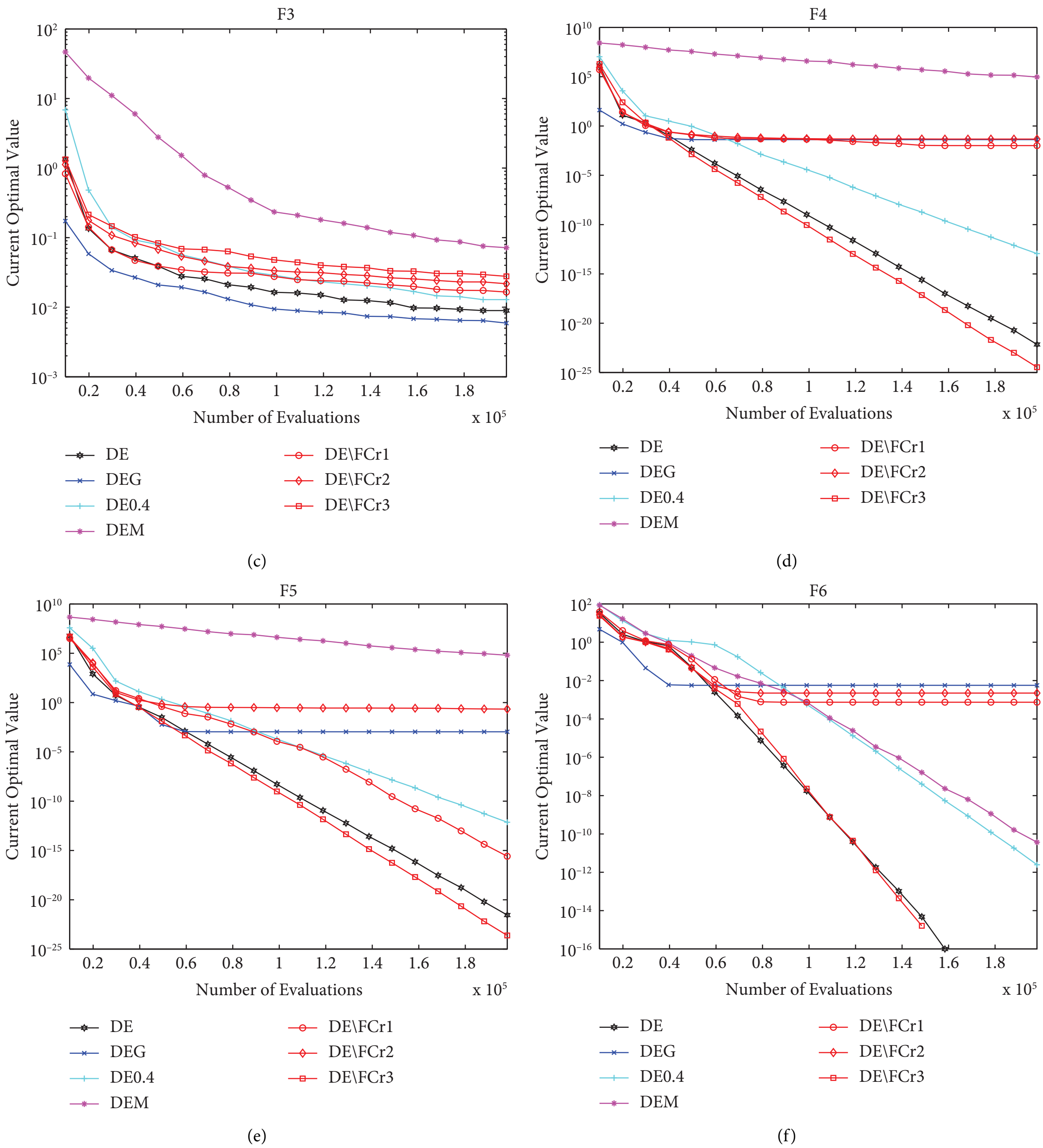


Figure 5: The mean of the current optimal values obtained by 7 algorithms with the number of function evaluations for $F1$–$F6$. (a) $F1$. (b) $F2$. (c) $F3$. (d) $F4$. (e) $F5$. (f) $F6$.

function values and the smallest standard deviation; $DE\backslash FCr1$, $DE\backslash FCr3$ and DE find the better results; The means and standard deviations given by both $DE0.4$ and DEM are worst among all algorithms; After 20 independent runs $DE\backslash FCr2$ finds a better function value, however, it obtains the worst standard deviation because it encounters twice the worst function value 15.37. For $F2$, DEG and DEM find the best function values with the almost same precision $E-15$; The precision given by $DE\backslash FCr3$ is $E-14$; DE and $DE\backslash FCr1$ give the precision of $E-12$; However, the lower precisions provided by $DE\backslash FCr2$ and $DE0.4$ are $E-9$ and $E-7$ respectively. All of algorithms obtain the best function values with the almost same precision for tested function $F3$. For $F4$, the results given by $DE\backslash FCr3$ are best, and those provided by DE are a little bit less promising; $DE0.4$ find the less promising optimum with the precision $E-14$, while the precisions provided by DEG and $DE\backslash FCr2$ reach only $E-2$; DEM fails in finding the optima of $F4$ among 20 independent runs, and traps into the local optima. Both $DE\backslash FCr1$, $DE\backslash FCr3$, DE and $DE0.4$ solve efficiently $F5$

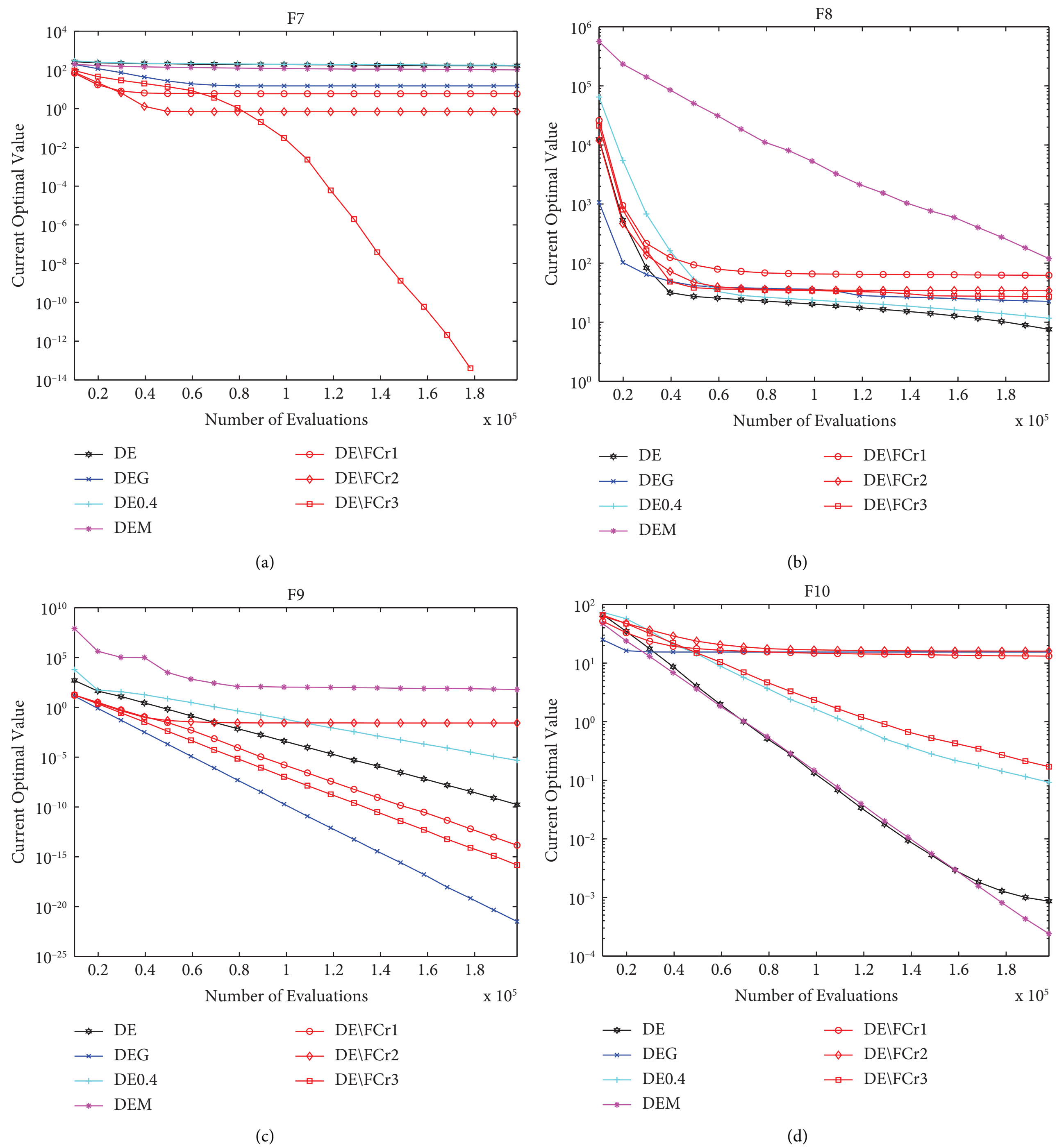


FIGURE 6: The mean of the current optimal values obtained by 7 algorithms with the number of function evaluations for *F*7–*F*10. (a) *F*7. (b) *F*8. (c) *F*9. (d) *F*10.

with the precision $E-24$, $E-22$, $E-16$ and $E-13$ respectively; The results given by DEG and *DE\FCr*2 are worse; Similarly, *DEM* fails to solve *F*5.

In Table 4, Both *DE\FCr*3 and *DE* find the optima of *F*6, however the fewer number of function evaluations 148533 is used by *DE\ FCr*3; The results obtained by *DE*0.4 and *DEM* are sightly worse than those obtained by *DE\FCr*3 and *DE*; However, *DE\FCr*1, *DE\FCr*2 and DEG have the lower precisions of about $E-3$. *DE\FCr*3 finds the optimum of tested function *F*7 with highest precision and smallest STD and the fewer number of function evaluations than the other algorithms; *DE\FCr*1 and *DE\FCr*2 reach the a little bit worse precision, and the anther algorithms fail to find the optimum of *F*7 with 200100 function evaluations. For 30 dimensional Rosenbrock tested function *F*8, none of all algorithms is obviously superior to the other one, namely, all algorithms can not find a satisfactory optima. For *F*9, DEG produces a best results, while $DE\backslash FCr1$ and $DE\backslash FCr3$ provides the sightly worse results; The precisions of the mean function values given by DE and $DE0.4$ reach $E-10$ and $E-6$, respectively; DEM can not find the reasonable result, and only the mean value 60.796 is presented. DE and DEM find the best means of function values of $F10$ with the precision of $E-4$; The precisions given by $DE0.4$ and $DE\backslash FCr3$ are $E-2$ and $E-1$ respectively; DEG, $DE\backslash FCr1$ and $DE\backslash FCr2$ give the almost same results and fail to find a satisfactory optima of $F10$.

In order to show further the efficiency of $DE\backslash FCr$, the means of the current optimal values obtained by 7 algorithms with the almost same number of function evaluations for each tested functions are respectively given in Figures 5 and 6. As mentioned in previous section, $|f(X_{\text{worst}}) - f(X_{\text{best}})| < \varepsilon$ is used for the numerical stability, hence each $DE\backslash FCr$ stops probably before the maximal generation is reached. For convenience to draw the following figures, the current optimal value is recorded repeatedly in succeeding generations if the algorithm stops in advance since we think that the algorithm cannot be improved greatly in succeeding running.

Since the proposed algorithm is not same as the compared algorithms in the number of function evaluations at each iteration, it is inconvenient to draw the evolution curves describing the variations of $MEAN$ with the number of function evaluations in a figure window for the reasonable comparison among all algorithms. Therefore, the current optimal values with the number of function evaluations which is denoted by $t \cdot \text{lcm}\{990, 100\}$ for $t = 1, 2, \ldots, 20$ respectively are drawn in Figures 5 and 6 without considering the number of function evaluations costed by initialization. In fact, the sequence $t \cdot \text{lcm}\{990, 100\}, t = 1, \ldots, 20$ is an arithmetic sequence with the initial term 9900 and the common difference 9900, where lcm denotes the least common multiple. 990 and 100 represent the number of function evaluations of $DE\backslash FCr$ and those of DEs at each iteration, respectively. It needs to be emphasized that each proposed algorithm evaluates 70(=100−30) less than the compared algorithm at each given iteration as above. Consequently, the current optimal value obtained by each compared algorithm under the given number of function evaluations is just recorded at certain generation which is $10 + (t-1) \cdot 10$ for each $DE\backslash FCr$ and $99 + (t-1) \cdot 99$ for DEs. Hence, only according to the recorded current optimal value at each generation can the figures below be given easily.

From each figure(see Figures 5 and 6), DEG outperforms the other algorithms for $F1$, $F2$, and $F9$, whereas $DE\backslash FCr3$ surpasses the other ones for $F4$–F7 and $F10$. For $F1$, $F2$, and $F9$, $DE\backslash FCr3$ is on the top three of 7 algorithms in terms of the performance. However, For $F4$–F7 and $F10$, DEG drops out of the top three almost into the last three. It is worth noting that both $F3$ and $F8$ don't be considered because all algorithms, especially DEG and $DE\backslash FCr3$, obtain the almost same results. We also find that DEG can obtain the optimum with the higher precision at earlier generation than the other algorithms and enters easily into the local optimum at latter generation for $F2$, $F4$–$F6$, and $F10$. $DE\backslash FCr3$ finds the satisfactory results of most of tested functions except $F8$ and also has not the tendency toward the local optimum with the default parameters.

In a summary, $DE\backslash FCr3$ does rather well in terms of the performance, DEG and DE are a little bit less promising, $DE\backslash FCr1$ and $DE0.4$ are even less promising, and $DE\backslash FCr2$ and DEM are worst.

Furthermore, the numerical comparison experiments are done between $DE\backslash FCr3$ and $DENSO$ (see Table 5). $DENSO$ is proposed in reference [19], which employs three other candidate individuals to design a new selection operator for improving the ability to escape the local optimum. In Table 5, $DE\backslash FCr3$ find the optima of the tested functions $F1, F2, F4, F5, F9$ with higher precision. For $F3, F8, F10$, Both algorithms have the almost same precision, however, $DE\backslash FCr3$ reduces 90 function evaluations. Obviously, $DE\backslash FCr3$ give the global minimal value 0 with the fewer $\#ELAV$ for $F6, F7$.

6. Conclusion

For avoiding the settings of the parameters, the differential evolutions without F and Cr are presented. The proposed algorithms use the attraction-repulsion mechanism in Coulomb's Law and Taguchi parameter design method for the purpose of eliminating the scale factor and the crossover probability, respectively. Numerical experiments show that the proposed algorithm $DE\backslash FCr3$, which can balance well between exploration and exploitation, is superior to the compared algorithms with other strategies and can find quickly the optima or the near-optima of the problems. Although a smaller population size 30 is given in the proposed algorithms for all 30 dimensional tested functions, this small population maybe lead to the prematurity of algorithm such as $F8$. However a larger population will expend too many function evaluations because of using the two-level orthogonal array which is related with the dimension of the problems. Obviously, In our algorithms the number of function evaluations of each proposed algorithm at each generation is $(m+1) \cdot NP$. Therefore, as for future work, the following problems are going to be investigated: (i) decrease the function evaluations at each generation and increase the population size without the loss of the algorithmic performance; (ii) analyze the accelerated convergence behavior of the current optimal value f_t^* after the t-th iteration in equation (22).

Appendix

$$
\begin{aligned}
&F0:\ 4x_1^2-2.1x_1^4+\frac{1}{3}x_1^6+x_1x_2-4x_2^2+4x_2^4, x\in[-5,5]^n,\\
&F1:\ \sum_{i=1}^{n}x_i^2, x\in[-100,100]^n,\\
&F2:\ -20\exp\left(-0.2\sqrt{\frac{1}{n}\sum_{i=1}^{n}x_i^2}\right)-\exp\left(\frac{1}{n}\sum_{i=1}^{n}\cos(2\pi x_i)\right)+20+\exp(1), x\in[-32,32]^n,\\
&F3:\ \sum_{i=1}^{n}i\cdot x_i^4+\text{rand}[0,1], x\in[-1.28,1.28]^n,\\
&F4:\ \frac{\pi}{n}\left\{10\sin^2(\pi y_1)+\sum_{i=1}^{n-1}(y_i-1)^2\left[1+10\sin^2(\pi y_{i+1})\right]+(y_n-1)^2\right\}+\sum_{i=1}^{n}u(x_i,10,100,4),\\
&y_i=1+\frac{x_i+1}{4}, i=1,\ldots,n, u(x_i,a,k,m)=\begin{cases}k(x_i-a)^m, x_i>a\\ 0, a\le x_i\le a\\ k(-x_i-a)^m, x_i<-a\end{cases}, x\in[-50,50]^n,\\
&F5:\ \frac{1}{10}\left\{\sin^2(3\pi x_1)+\sum_{i=1}^{n-1}(x_i-1)^2\left[1+\sin^2(3\pi x_{i+1})\right]+(x_n-1)^2\left[1+\sin^2(2\pi x_n)\right]\right\}\\
&\quad+\sum_{i=1}^{n}u(x_i,5,100,4), x\in[-50,50]^n,\\
&F6:\ \sum_{i=1}^{n}\frac{x_i^2}{4000}-\prod_{i=1}^{n}\cos\frac{x_i}{\sqrt{i}}+1, x\in[-600,600]^n,\\
&F7:\ 10n+\sum_{i=1}^{n}\left[x_i^2-10\cos(2\pi x_i)\right], x\in[-5.12,5.12]^n,\\
&F8:\ \sum_{i=1}^{n-1}\left[100\left(x_i^2-x_{i+1}\right)^2+(x_i-1)^2\right], x\in[-5,10]^n,\\
&F9:\ \sum_{i=1}^{n}|x_i|+\prod_{i=1}^{n}|x_i|, x\in[-100,100]^n\\
&F10:\ \max\left\{|x_i|, i=1,2,\cdots,n\right\}, x\in[-100,100]^n.
\end{aligned}
\tag{A.1}
$$

Data Availability

All the data, MATLAB codes, and plots used in the paper can be found at https://github.com/zhang-xiaowei/Code.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Research Article

Existence and Uniqueness for Coupled Systems of Hilfer Type Sequential Fractional Differential Equations Involving Riemann–Stieltjes Integral Multistrip Boundary Conditions

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In this paper, we study a coupled system of Hilfer type sequential fractional differential equations supplemented with Riemann–Stieltjes integral multistrip boundary conditions. The standard tools of the fixed point theory are employed to prove the existence and uniqueness results for the considered problem. Examples are constructed for the illustration of the obtained results.

1. Introduction

Fractional calculus is a generalization of the classical calculus. Fractional differential equations become another necessary tool in solving real-life problems in different research areas such as physics, biology, engineering, and mechanics, see for example the monographs and papers [1–11].

Boundary value problems of fractional differential equations represent an important and interesting branch of applied analysis. Usually, the researchers have given attention in studying fractional differential equations involving Caputo or Riemann–Liouville fractional derivative. But, Caputo or Riemann–Liouville derivative was not considered appropriate in studying some new models in engineering for example. To avoid the difficulties, some new type fractional order derivative operators were introduced in the literature such as Hadamard, Erdeyl-Kober, and Katugampola. Hilfer in [12] introduced a new derivative, which generalizes both Riemann–Liouville and Caputo derivatives. For some applications involving Hilfer fractional derivative, the interested reader is referred to [13–16] and references cited therein.

In [17], Nuchpong et al. investigated a new class of boundary value problems for fractional differential equations for involving sequential Hilfer type fractional derivative and subject to Riemann–Stieltjes integral multistrip boundary conditions of the form

$$\begin{cases} \left({}^{H}D^{\alpha,\beta} + k\,{}^{H}D^{\alpha-1,\beta}\right)u(z) = f(z,u(z)), & z \in [c,d], \\ u(c) = 0, \quad u(d) = \lambda \int_c^d u(s)\mathrm{d}H(s) + \sum_{i-1}^{n} \mu_i \int_{\eta_i}^{\mu_i} u(s)\mathrm{d}s, \end{cases} \tag{1}$$

where ${}^{H}D^{\alpha,\beta}$ denotes the Hilfer fractional derivative operator of order α, $1<\alpha<2$, and parameter β, $0\leq\beta\leq 1$, $f\colon [c,d]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function; $\int_c^d x(s)\mathrm{d}H(s)$ is the Riemann–Stieltjes integral with respect to the function $H\colon [c,d]\longrightarrow\mathbb{R}$, $c\geq 0$, $k,\mu_i\in\mathbb{R}$, $c<\eta_i<\xi_i\leq d$, $i=1,2,\ldots,n$. Existence and uniqueness results are established by using basic tools from fixed point theory.

The study of system of Hilfer type was initiated by Wongcharoen et al. [18], by presenting the following system of fractional differential equations:

$$\begin{cases} {}^{H}D^{\alpha_1,\beta_1}u(z) = f(z,u(z),v(z)), & z\in[c,d],\\ {}^{H}D^{\alpha_2,\beta_2}v(z) = g(z,u(z),v(z)), & z\in[c,d],\\ u(c)=0 \quad u(d)=\displaystyle\sum_{i=1}^{m}\overline{\theta}_i I^{\phi_i}v(\xi_i), & \\ v(c)=0 \quad v(d)=\displaystyle\sum_{j=1}^{n}\overline{\zeta}_j I^{\overline{\phi}_j}u(z_i), & \end{cases} \tag{2}$$

in which ${}^{H}D^{\alpha_1,\beta_1}$ and ${}^{H}D^{\alpha_2,\beta_2}$ indicate the Hilfer fractional derivatives of orders α_1 and α_2, $1<\alpha_1,\alpha_2<2$, and parameters β_1,β_2, $0\leq\beta_1,\beta_2\leq 1$, $f,g\colon [c,d]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions, $c\geq 0$, $\overline{\theta}_i$, $\overline{\zeta}_i\in\mathbb{R}$, and $I^{\phi_i}, I^{\overline{\phi}_j}$ are the Riemann–Liouville fractional integrals of order $\phi_i>0, \overline{\phi}_j>0$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$.

Inspired by the forenamed studies, this article considers the existence and uniqueness of solutions for the following coupled system of Hilfer type fractional differential equations with Riemann–Stieltjes integral multistrip boundary conditions of the form

$$\begin{cases} \left({}^{H}D^{\alpha_1,\beta_1} + \sigma_1 {}^{H}D^{\alpha_1-1,\beta_1}\right)u(z) = f_1(z,u(z),v(z)) & z\in[c,d],\\ \left({}^{H}D^{\alpha_2,\beta_2} + \sigma_2 {}^{H}D^{\alpha_2-1,\beta_2}\right)v(z) = f_2(z,u(z),v(z)) & z\in[c,d],\\ u(c)=0, \quad u(d)=\lambda_1\displaystyle\int_c^d v(s)\mathrm{d}H_1(s) + \sum_{i=1}^{n}\mu_i\int_{\eta_i}^{\xi_i} v(s)\mathrm{d}s, & \\ v(c)=0, \quad v(d)=\lambda_2\displaystyle\int_c^d u(s)\mathrm{d}H_2(s) + \sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r} u(s)\mathrm{d}s, & \end{cases} \tag{3}$$

in which ${}^{H}D^{\alpha_1,\beta_1}$ and ${}^{H}D^{\alpha_2,\beta_2}$ are the Hilfer fractional derivatives of orders $1<\alpha_1,\alpha_2<2$ and parameters β_1,β_2, $0\leq\beta_1,\beta_2\leq 1$, $f_1,f_2\colon [c,d]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ are continuous functions, $\int_c^d(\cdot)\mathrm{d}H_1(s)$, $\int_c^d(\cdot)dH_2(s)$ are the Riemann–Stieltjes integrals with respect to the functions $H_i\colon [c,d]\longrightarrow\mathbb{R}$, $i=1,2$, $c\geq 0$, $\mu_i,\nu_r\in\mathbb{R}$, η_i, ξ_i, ζ_r, $\theta_r\in(c,d)$, $i=1,2,\ldots,n$, $r=1,2,\ldots,p$, λ_1, λ_2, σ_1, $\sigma_2\in\mathbb{R}$.

The remaining of this article has been regulated as follows: In section 2 some concepts, lemmas and theorems are recalled which will be applied throughout this paper. In Section 3, an auxiliary lemma has been proved which concerns a linear variant of system (3) and it is used to convert the coupled system (3) into a fixed point problem. The classical fixed point theorems have been applied in order to obtain the results regarding the existence/uniqueness in Section 4. Thus, the classical Banach fixed point theorem is applied to obtain uniqueness result, while Leray-Schauder alternative and Krasnosel'skiĭ's fixed point theorems are applied to present existence results. Examples are also constructed to illustrate the obtained results.

2. Preliminaries

Now, the following items are reminded which will be applied to fulfil the main results in the next steps.

Throughout the paper, the Banach space of all continuous mappings from $[c,d]$ to $\mathbb{R}$ are denoted by $\mathcal{Y}=C([c,d],\mathbb{R})$ which is equipped with the norm $\|y\|=\sup\{|y(z)|; z\in[c,d]\}$. It is clear that the space

$\mathscr{Y}\times\mathscr{Y}$, equipped with norm defined by $\|(x,y)\| = \|x\| + \|y\|$, is a Banach space.

Also, $AC^n([c,d],\mathbb{R})$ is the n-times absolutely continuous functions defined as

$$AC^n([c,d],\mathbb{R}) = \left\{f\colon [c,d] \longrightarrow \mathbb{R};\, f^{(n-1)} \in AC([c,d],\mathbb{R})\right\}. \tag{4}$$

For a real valued function $g\colon (0,\infty) \longrightarrow \mathbb{R}$, the Riemann–Liouville fractional integral of order $\eta > 0$ is defined by $I^\eta g(t) = \int_0^t ((t-s)^{\eta-1}/\Gamma(\eta)) g(s)\mathrm{d}s$, in which the right-hand side is defined point-wise on $(0,\infty)$, see [2]. Besides, for the function g, the Riemann–Liouville fractional derivative of order $\delta > 0$ is defined by $\{^{RL}\}D^\delta g(t) = (1/\Gamma(n-\delta))(d/\mathrm{d}t)^n \int_0^t (g(s)/(t-s)^{s-n+1})\mathrm{d}s$, in which $n = [\delta]+1$, where $[\delta]$ denotes the integer part of a real number δ, see [2], while the Caputo fractional derivative is defined by $\{^C\}D^s g(t) = (1/\Gamma(n-\delta)) \int_0^t (1/\ \ (t-s)^{\delta-n+1})(d/\mathrm{d}s)^n g(s)\mathrm{d}s$, provided that the right-hand side exists.

Also, the Hilfer fractional derivative of order α and parameter β of a function is defined by

$$^{H}D^{\alpha,\beta}u(t) = I^{\beta(n-\alpha)}D^n I^{(1-\beta)(n-\alpha)}u(t), \tag{5}$$

where $n-1<\alpha<n$, $0\le\beta\le 1$, $t>a$, $D=(d/\mathrm{d}t)$, see [12]. Note that if $\beta = 0$ and $\beta = 1$, the Hilfer derivative is reduced to the Riemann–Liouville and Caputo fractional derivatives, respectively.

The following lemma will be applied to prove a lemma in the next section which presents a pattern of existence of solutions for system (1.3).

Lemma 1 (see [13]). *Let $h \in L(c,d)$, $n_1 - 1 < \overline{\alpha} \le n_1$, $n_1 \in \mathbb{N}$, $0 \le \overline{\beta} \le 1$, $I^{(n_1-\overline{\alpha})(1-\overline{\beta})}h \in AC^k[c,d]$. Then, we have the following relation:*

$$\left(I^{\overline{\alpha}\,H}D^{\overline{\alpha},\overline{\beta}}h\right)(z) = h(z) - \sum_{k=0}^{n_1-1} \frac{(z-c)^{k-(n_1-\overline{\alpha})(1-\overline{\beta})}}{\Gamma\left(k-(n_1-\overline{\alpha})(1-\overline{\beta})+1\right)} \cdot \lim_{z\longrightarrow c^+} \frac{d^k}{dz^k}\left(I^{(1-\beta)(n_1-\overline{\alpha})}h\right)(z). \tag{6}$$

Finally, we collect the fixed point theorems applied to prove the main results in this paper.

Lemma 2 (Banach fixed point theorem, [19]). *Let D be a closed set in X and $T\colon D \longrightarrow D$ satisfies*

$$\begin{gathered} |Tu - Tv| \le \lambda|u-v|, \\ \text{for some}\lambda \in (0,1), \\ \text{for all u, } v \in D. \end{gathered} \tag{7}$$

Then T admits a unique fixed point in D.

Lemma 3 (Leray–Schauder alternative [20]). *Let the set ω be closed bounded convex in X and O an open set contained in ω with $0 \in O$. Then, for the continuous and compact $T\colon \overline{U} \longrightarrow \omega$, either*

(1) $(\mathfrak{a})T$ admits a fixed point in $\overline{U}$, or

(2) $(\mathfrak{a}\mathfrak{a})$ there exists $\mathfrak{u} \in \partial U$ and $\mu \in (0,1)$ with $\mathfrak{u} = \mu T(\mathfrak{u})$.

Lemma 4 (Krasnosel'skiĭ fixed point theorem, [21]). *Let N indicate a closed, bounded, convex, and nonempty subset of a Banach space Y and C, D be operators such that (i) $Cx + Dy \in N$ where $x, y \in N$, (ii) C is compact and continuous, and (iii) D is a contraction mapping. Then, there exists $z \in N$ such that $z = Cz + Dz$.*

3. An Auxiliary Result

Lemma 5. *Let $c \ge 0$, $1 < \alpha_1, \alpha_2 < 2$, $0 \le \beta_1, \beta_2 \le 1$, $\gamma_1 = \alpha_1 + 2\beta_1 - \alpha_1\beta_1$, $\gamma_2 = \alpha_2 + 2\beta_2 - \alpha_2\beta_2$, $h_1, h_2 \in C([c,d],\mathbb{R})$ and $\Theta \ne 0$. Then, the solution of the system*

$$\begin{cases} \left({}^HD^{\alpha_1,\beta_1} + \sigma_1\,{}^HD^{\alpha_1-1,\beta_1}\right)u(z) = h_1(z), & z \in [c,d], \\ \left({}^HD^{\alpha_2,\beta_2} + \sigma_2\,{}^HD^{\alpha_2-1,\beta_2}\right)v(z) = h_2(z), & z \in [c,d], \\ u(c) = 0, \quad u(d) = \lambda_1 \displaystyle\int_c^d v(s)\mathrm{d}H_1(s) + \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} v(s)\mathrm{d}s, \\ v(c) = 0, \quad v(d) = \lambda_2 \displaystyle\int_c^d u(s)\mathrm{d}H_2(s) + \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} u(s)\mathrm{d}s, \end{cases} \tag{8}$$

is given by

$$
\begin{aligned}
u(z) = {} & -\sigma_1 \int_c^z u(s)\mathrm{d}s + I^{\alpha_1} h_1(z) \\
& + \frac{(z-c)^{\gamma_1 - 1}}{\Theta\Gamma(\gamma_1)} \Bigg[G_3 \Bigg(-\sigma_2\lambda_1 \int_c^d \int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) \\
& + \lambda_1 \int_c^d I^{\alpha_2} h_2(s)\mathrm{d}H_1(s) - \sigma_2 \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} \int_c^s v(t)\mathrm{d}t\mathrm{d}s \\
& + \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} I^{\alpha_2} h_2(s)\mathrm{d}s \Bigg) + \sigma_1 \int_c^d u(s)\mathrm{d}s - I^{\alpha_1} h_1(d) \Bigg) \\
& + G_2 \Bigg(-\sigma_1\lambda_2 \int_c^d \int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) + \lambda_2 \int_c^d I^{\alpha_1} h_1(s)\mathrm{d}H_2(s) \\
& - \sigma_1 \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} \int_c^s u(t)\mathrm{d}t\mathrm{d}s + \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} I^{\alpha_1} h_1(s)\mathrm{d}s + \sigma_2 \int_c^d v(s)\mathrm{d}s \\
& - I^{\alpha_2} h_2(d) \Big) \Bigg],
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
v(z) = {} & -\sigma_2 \int_c^z v(s)\mathrm{d}s + I^{\alpha_2} h_2(z) \\
& + \frac{(z-c)^{\gamma_2 - 1}}{\Theta\Gamma(\gamma_2)} \Bigg[G_1 \Bigg(-\sigma_1\lambda_2 \int_c^d \int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) \\
& + \lambda_2 \int_c^d I^{\alpha_1} h_1(s)\mathrm{d}H_2(s) - \sigma_1 \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} \int_c^s u(t)\mathrm{d}t\mathrm{d}s \\
& + \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} I^{\alpha_1} h_1(s)\mathrm{d}s \Bigg) + \sigma_2 \int_c^d v(s)\mathrm{d}s - I^{\alpha_2} h_2(d) \Bigg) \\
& + G_4 \Bigg(-\sigma_2\lambda_1 \int_c^d \int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) + \lambda_1 \int_c^d I^{\alpha_2} h_2(s)\mathrm{d}H_1(s) \\
& - \sigma_2 \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} \int_c^s v(t)\mathrm{d}t\mathrm{d}s + \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} I^{\alpha_2} h_2(s)\mathrm{d}s \Bigg) + \sigma_1 \int_c^d u(s)\mathrm{d}s \\
& - I^{\alpha_1} h_1(d) \Big) \Bigg],
\end{aligned} \tag{10}
$$

where

$$
\begin{aligned}
G_1 &= \frac{(d-c)^{\gamma_1 - 1}}{\Gamma(\gamma_1)}, \\
G_2 &= \lambda_1 \int_c^d \frac{(z-c)^{\gamma_2 - 1}}{\Gamma(\gamma_2)} \mathrm{d}H_1(z) + \sum_{i=1}^n \mu_i \frac{(\xi_i - c)^{\gamma_2} - (\eta_i - c)^{\gamma_2}}{\Gamma(\gamma_2 + 1)}, \\
G_3 &= \frac{(d-c)^{\gamma_2 - 1}}{\Gamma(\gamma_2)}, \\
G_4 &= \lambda_2 \int_c^d \frac{(z-c)^{\gamma_1 - 1}}{\Gamma(\gamma_1)} \mathrm{d}H_2(z) + \sum_{r=1}^p \nu_r \frac{(\theta_r - c)^{\gamma_1} - (\zeta_r - c)^{\gamma_1}}{\Gamma(\gamma_1 + 1)},
\end{aligned} \tag{11}
$$

$$
\Theta = G_1 G_3 - G_2 G_4. \tag{12}
$$

Proof. Let (u, v) be a solution of system (5). By Lemma 2, we have

$$u(z) = k_1\frac{(z-c)^{\gamma_1-2}}{\Gamma(\gamma_1-1)} + k_2\frac{(z-c)^{\gamma_1-1}}{\Gamma(\gamma_1-)} - \sigma_1\int_c^z u(s)\mathrm{d}s + I^{\alpha_1}h_1(z),$$
$$v(z) = d_1\frac{(z-c)^{\gamma_2-2}}{\Gamma(\gamma_2-1)} + d_2\frac{(z-c)^{\gamma_2-1}}{\Gamma(\gamma_2))} - \sigma_2\int_c^z v(s)\mathrm{d}s + I^{\alpha_2}h_2(z), \tag{13}$$

where k_1, k_2, d_1, d_2 are the arbitrary constants, since $(1-\beta_1)(2-\alpha_1) = \gamma_1$ and $(1-\beta_1)(2-\alpha_2) = \gamma_2$. Applying $u(c) = 0$ and $v(c) = 0$, we deduce that $k_1 = d_1 = 0$. Thus, the previous equations become

$$u(z) = k_2\frac{(z-c)^{\gamma_1-1}}{\Gamma(\gamma_1)} - \sigma_1\int_c^z u(s)\mathrm{d}s + I^{\alpha_1}h_1(z), \tag{14}$$

$$v(z) = d_2\frac{(z-c)^{\gamma_2-1}}{\Gamma(\gamma_2)} - \sigma_2\int_c^z v(s)\mathrm{d}s + I^{\alpha_2}h_2(z). \tag{15}$$

Now, applying the boundary conditions $u(d) = \lambda_1\int_c^d v(s)\mathrm{d}H_1(s) + \sum_{i=1}^n \mu_i\int_{\eta_i}^{\xi_i} v(s)\mathrm{d}s$ and $v(d) = \lambda_2\int_c^d u(s)dH_2(s) + \sum_{r=1}^p \int_{\zeta_r}^{\theta_r} u(s)ds$, we get

$$\begin{aligned}
&k_2\frac{(d-c)^{\gamma_1-1}}{\Gamma(\gamma_1)} - d_2\left[\lambda_1\int_c^d \frac{(z-c)^{\gamma_2-1}}{\Gamma(\gamma_2)}\mathrm{d}H_1(z) + \sum_{i=1}^n \mu_i\frac{(\xi_i-c)^{\gamma_2}-(\eta_i-c)^{\gamma_2}}{\Gamma(\gamma_2+1)}\right]\\
&= -\sigma_2\lambda_1\int_c^d\int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) + \lambda_1\int_c^d I^{\alpha_2}h_2(s)\mathrm{d}H_1(s) - \sigma_2\sum_{i=1}^n\mu_i\int_{\eta_i}^{\xi_i}\int_c^s v(t)\mathrm{d}t\mathrm{d}s\\
&\quad+\sum_{i=1}^n\mu_i\int_{\eta_i}^{\xi_i} I^{\alpha_2}h_2(s)\mathrm{d}s + \sigma_1\int_c^d u(s)\mathrm{d}s - I^{\alpha_1}h_1(d),\\
&d_2\frac{(d-c)^{\gamma_2-1}}{\Gamma(\gamma_2)} - k_2\left[\lambda_2\int_c^d \frac{(z-c)^{\gamma_1-1}}{\Gamma(\gamma_1)}\mathrm{d}H_2(z) + \sum_{r=1}^p \nu_r\frac{(\theta_r-c)^{\gamma_1}-(\zeta_r-c)^{\gamma_1}}{\Gamma(\gamma_1+1)}\right]\\
&= -\sigma_1\lambda_2\int_c^d\int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) + \lambda_2\int_c^d I^{\alpha_1}h_1(s)\mathrm{d}H_2(s) - \sigma_1\sum_{r=1}^p\nu_r\int_{\zeta_r}^{\theta_r}\int_c^s u(t)\mathrm{d}t\mathrm{d}s\\
&\quad+\sum_{r=1}^p\nu_r\int_{\zeta_r}^{\theta_r} I^{\alpha_1}h_1(s)\mathrm{d}s + \sigma_2\int_c^d v(s)ds - I^{\alpha_2}h_2(d).
\end{aligned} \tag{16}$$

Consequently, we have the system

$$k_2G_1 - d_2G_2 = \Omega_1,\ d_2G_3 - k_2G_4 = \Omega_2, \tag{17}$$

where

$$\begin{aligned}
\Omega_1 &= -\sigma_2\lambda_1\int_c^d\int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) + \lambda_1\int_c^d I^{\alpha_2}h_2(s)\mathrm{d}H_1(s) - \sigma_2\sum_{i=1}^n\mu_i\int_{\eta_i}^{\xi_i}\int_c^s v(t)\mathrm{d}t\mathrm{d}s\\
&\quad+\sum_{i=1}^n\mu_i\int_{\eta_i}^{\xi_i} I^{\alpha_2}h_2(s)\mathrm{d}s + \sigma_1\int_c^d u(s)\mathrm{d}s - I^{\alpha_1}h_1(d),\\
\Omega_2 &= -\sigma_1\lambda_2\int_c^d\int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) + \lambda_2\int_c^d I^{\alpha_1}h_1(s)\mathrm{d}H_2(s) - \sigma_1\sum_{r=1}^p\nu_r\int_{\zeta_r}^{\theta_r}\int_c^s u(t)\mathrm{d}t\mathrm{d}s\\
&\quad+\sum_{r=1}^p\nu_r\int_{\zeta_r}^{\theta_r} I^{\alpha_1}h_1(s)\mathrm{d}s + \sigma_2\int_c^d v(s)\mathrm{d}s - I^{\alpha_2}h_2\mathrm{d}.
\end{aligned} \tag{18}$$

By solving the above system, we have

$$k_2 = \frac{\Omega_1 G_3 + \Omega_2 G_2}{\Theta}, \qquad d_2 = \frac{G_1 \Omega_2 + G_4 \Omega_1}{\Theta}. \tag{19}$$

Substituting the values of k_2 and d_2 into (10) and (14), respectively, we obtain the solutions (9) and (10). We can obtain the converse by direct computation. The proof is finished. □

4. Existence and Uniqueness Result

Due to Lemma 5, we define an operator $\mathcal{Q}$: $\mathcal{Y} \times \mathcal{Y} \longrightarrow \mathcal{Y} \times \mathcal{Y}$ by

$$\mathcal{Q}(u,v)(z) := \left(\mathcal{Q}_1(u,v)(z), \mathcal{Q}_2(u,v)(z)\right), \tag{20}$$

where

$$\begin{aligned}
&\mathcal{Q}_1(u,v)(z)\\
&= -\sigma_1 \int_c^z u(s)\mathrm{d}s + I^{\alpha_1} f_1(z,u(z),v(z))\\
&\quad + \frac{(z-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)} \Bigg[G_3\Bigg(-\sigma_2\lambda_1 \int_c^d \int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s)\\
&\quad + \lambda_1 \int_c^d I^{\alpha_2} f_2(s,u(s),v(s))\mathrm{d}H_1(s) - \sigma_2 \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} \int_c^s v(t)\mathrm{d}t\mathrm{d}s\\
&\quad + \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} I^{\alpha_2} f_2(s,u(s),v(s))\mathrm{d}s + \sigma_1 \int_c^d u(s)\mathrm{d}s - I^{\alpha_1} f_1(d,u(d),v(d))\\
&\quad + G_2\Bigg(-\sigma_1\lambda_2 \int_c^d \int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) + \lambda_2 \int_c^d I^{\alpha_1} f_1(s,u(s),v(s))\mathrm{d}H_2(s)\\
&\quad - \sigma_1 \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} \int_c^s u(t)\mathrm{d}t\mathrm{d}s + \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} I^{\alpha_1} f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad + \sigma_2 \int_c^d v(s)\mathrm{d}s - I^{\alpha_2} f_2(d,u(d),v(d)),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\mathcal{Q}_2(u,v)(z)\\
&= -\sigma_2 \int_c^z v(s)\mathrm{d}s + I^{\alpha_2} f_2(z,u(z),v(z))\\
&\quad + \frac{(z-c)^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)} G_1 - \sigma_1\lambda_2 \int_c^d \int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s)\\
&\quad + \lambda_2 \int_c^d I^{\alpha_1} f_1(s,(u(s),v(s))\mathrm{d}H_2(s) - \sigma_1 \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} \int_c^s u(t)\mathrm{d}t\mathrm{d}s\\
&\quad + \sum_{r=1}^p \nu_r \int_{\zeta_r}^{\theta_r} I^{\alpha_1} f_1(s,u(s),v(s))\mathrm{d}s + \sigma_2 \int_c^d v(s)\mathrm{d}s - I^{\alpha_2} f_2(d,u(d),v(d))\\
&\quad + G_4\Bigg(-\sigma_2\lambda_1 \int_c^d \int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) + \lambda_1 \int_c^d I^{\alpha_2} f_2(s,u(s),v(s))\mathrm{d}H_1(s)\\
&\quad - \sigma_2 \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} \int_c^s v(t)\mathrm{d}t\mathrm{d}s + \sum_{i=1}^n \mu_i \int_{\eta_i}^{\xi_i} I^{\alpha_2} f_2(s,u(s),v(s))\mathrm{d}s\\
&\quad + \sigma_1 \int_c^d u(s)\mathrm{d}s - I^{\alpha_1} f_1(d,u(d),v(d)) \Bigg) \Bigg].
\end{aligned} \tag{22}$$

For convenience, we set

$$\begin{aligned}
\mathscr{C}_1 &= \frac{1}{\Gamma(\alpha_1+1)} + \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3|\frac{1}{\Gamma(\alpha_1+1)} \\
&\quad + |G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\Bigg(\lambda_2\frac{1}{\Gamma(\alpha_1+1)}\int_c^d (s-c)^{\alpha_1}\mathrm{d}H_2(s) \\
&\quad + \frac{1}{\Gamma(\alpha_1+2)}\sum_{r=1}^{p}|\nu_r|\left[(\theta_r-c)^{\alpha_1+1}-(\zeta_r-c)^{\alpha_1+1}\right]\Bigg), \\
\mathscr{C}_2 &= \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3|\Bigg(\frac{1}{\Gamma(\alpha_2+2)}\sum_{i=1}^{n}|\mu_i|\left[(\xi_i-c)^{\alpha_2+1}-(\eta_i-c)^{\alpha_2+1}\right] \\
&\quad + \frac{\lambda_1}{\Gamma(\alpha_2+1)}\int_c^d (s-c)^{\alpha_2}\mathrm{d}H_1(s)\Bigg) + |G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\frac{1}{\Gamma(\alpha_2+1)}, \\
\mathscr{C}_3 &= |\sigma_1|(d-c) + |G_3|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}(d-c)|\sigma_1| + |G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}(d-c)|\sigma_1||\lambda_2|\int_c^d (s-c)\mathrm{d}H_2(s) \\
&\quad + |G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|\sigma_1|\sum_{r=1}^{p}|\nu_r|\frac{1}{2}\left[(\theta_r-c)^2-(\zeta_r-c)^2\right], \\
\mathscr{C}_4 &= \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3||\sigma_2||\lambda_1|\int_c^d (s-c)\mathrm{d}H_1(s) + \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}|G_3||\sigma_2|\sum_{i=1}^{n}|\mu_i|\frac{1}{2}\left[(\xi_i-c)^2-(\eta_i-c)^2\right] \\
&\quad + |G_2|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}(d-c)|\sigma_2|,
\end{aligned} \tag{23}$$

$$\begin{aligned}
\mathscr{D}_1 &= \frac{1}{\Gamma(\alpha_2+1)} + \frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}|G_1|\frac{1}{\Gamma(\alpha_2+1)} \\
&\quad + |G_4|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}\Bigg(|\lambda_1|\frac{1}{\Gamma(\alpha_2+1)}\int_c^d (s-c)^{\alpha_2}\mathrm{d}H_1(s) + \frac{1}{\Gamma(\alpha_2+2)}\sum_{i=1}^{n}|\mu_i|\left[(\xi_i-c)^{\alpha_2+1}-(\eta_i-c)^{\alpha_2+1}\right]\Bigg), \\
\mathscr{D}_2 &= \frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}|G_1|\Bigg(\frac{1}{\Gamma(\alpha_1+2)}\sum_{r=1}^{p}|\nu_r|\left[(\theta_r-c)^{\alpha_1+1}-(\zeta_r-c)^{\alpha_1+1}\right] + \frac{\lambda_2}{\Gamma(\alpha_1+1)}\int_c^d (s-c)^{\alpha_1}\mathrm{d}H_2(s)\Bigg) \\
&\quad + |G_4|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}\frac{1}{\Gamma(\alpha_1+1)}, \\
\mathscr{D}_3 &= |\sigma_2|(d-c) + |G_1|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}(d-c)|\sigma_2| + |G_4|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}(d-c)|\sigma_2||\lambda_1|\int_c^d (s-c)\mathrm{d}H_1(s) \\
&\quad + |G_4|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}|\sigma_2|\sum_{i=1}^{n}\mu_i\frac{1}{2}\left[(\xi_i-c)^2-(\eta_i-c)^2\right], \\
\mathscr{D}_4 &= \frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}|G_1||\sigma_1||\lambda_2|\int_c^d (s-c)\mathrm{d}H_2(s) + \frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}|G_1||\sigma_1|\sum_{r=1}^{p}|\nu_r|\frac{1}{2}\left[(\theta_r-c)^2-(\zeta_r-c)^2\right] \\
&\quad + |G_4|\frac{(d-c)^{\gamma_2-1}}{|\Theta|\Gamma(\gamma_2)}(d-c)|\sigma_1|.
\end{aligned} \tag{24}$$

Now, Banach's fixed point theorem is applied to present the following uniqueness result.

Theorem 1. *Let* $\Theta \neq 0$ *and* $f_1, f_2\colon [c,d] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ *be two continuous functions such that for all* $z \in [c,d]$ *and* $\overline{u}_i, \overline{v}_i \in \mathbb{R}$, $i = 1, 2$, *we have*

$$\begin{aligned}&\left|f_1(z,\overline{u}_1,\overline{v}_1) - f_1(z,\overline{u}_2,\overline{v}_2)\right| \leq \ell_1\left(\left|\overline{u}_1 - \overline{u}_2\right| + \left|\overline{v}_1 - \overline{v}_2\right|\right),\\ &\left|f_2(z,\overline{u}_1,\overline{v}_1) - f_2(z,\overline{u}_2,\overline{v}_2)\right| \leq \ell_2\left(\left|\overline{u}_1 - \overline{u}_2\right| + \left|\overline{v}_1 - \overline{v}_2\right|\right).\end{aligned} \tag{25}$$

where ℓ_1, ℓ_2 *are positive constants and* $\overline{u}_i, \overline{v}_i \in \mathbb{R}$, $i = 1, 2$. *Then, there exists a unique solution of system (3) on* $[c,d]$ *provided that*

$$\ell_1\left(\mathscr{C}_1 + \mathscr{D}_1\right) + \ell_2\left(\mathscr{C}_2 + \mathscr{D}_2\right) + \mathscr{C}_3 + \mathscr{C}_4 + \mathscr{D}_3 + \mathscr{D}_4 < 1. \tag{26}$$

Proof. It suffices to display that the operator Q has a unique fixed point. For this aim, Banach's theorem will be applied. Put $\sup_{z\in[c,d]}|f_1(z,0,0)| := M < \infty$ and $\sup_{z\in[c,d]}|f_2(z,0,0)| := N < \infty$. Now, we locate $B_r = \{(u,v) \in \mathscr{Y} \times \mathscr{Y};\ \|(u,v)\| \leq r\}$, in which

$$r \geq \frac{M\left(\mathscr{C}_1 + \mathscr{D}_1\right) + N\left(\mathscr{C}_2 + \mathscr{D}_2\right)}{1 - \left[\ell_1\left(\mathscr{C}_1 + \mathscr{D}_1\right) + \ell_2\left(\mathscr{C}_2 + \mathscr{D}_2\right) + \mathscr{C}_3 + \mathscr{C}_4 + \mathscr{D}_3 + \mathscr{D}_4\right]}. \tag{27}$$

First, we indicate that $\mathcal{Q}(B_r) \subseteq B_r$. Assume that $(u,v) \in B_r$ and $z \in [c,d]$. Due to (), we have

$$\left|f_1(z,u(z),v(z))\right| \leq \left|f_1(z,u(z),v(z)) - f_1(z,0,0)\right| + \left|f_1(z,0,0)\right| \leq \ell_1\left(|u(z)| + |v(z)|\right) + M = \ell_1 r + M. \tag{28}$$

Similarly, we have

$$\left|f_2(z,u(z),v(z))\right| \leq \ell_2 r + N. \tag{29}$$

Hence, we infer that

$$\begin{aligned}
&\left|\mathcal{Q}_1(u,v)(z)\right|\\
&\leq r\Bigg[\left|\sigma_1\right|(d-c) + \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\Bigg(\left|\sigma_2\right|\left|\lambda_1\right|\int_c^d (s-c)\mathrm{d}H_1(s)\\
&+\left|\sigma_2\right|\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)\mathrm{d}s + \left|\sigma_1\right|(d-c)\Bigg)\\
&+\left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\Bigg(\left|\sigma_1\right|\left|\lambda_2\right|\int_c^d (s-c)\mathrm{d}H_2(s) + \left|\sigma_1\right|\sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r}(s-c)\mathrm{d}s\\
&+\left|\sigma_2\right|(d-c)\Bigg)\Bigg] + \left(\ell_1 r + M\right)\Bigg[\frac{1}{\Gamma(\alpha_1+1)} + \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\frac{1}{\Gamma(\alpha_1+1)}\\
&+\left|G_2\right|\frac{(d-c)^{\gamma-1}}{|\Theta|\Gamma(\gamma_1)}\Bigg(\left|\lambda_2\right|\frac{1}{\Gamma(\alpha_1+1)}\int_c^d (s-c)^{\alpha_1}\mathrm{d}H_2(s)\\
&+\frac{1}{\Gamma(\alpha_1+1)}\sum_{r=1}^{p}\left|\nu_r\right|\int_{\zeta_r}^{\theta_r}(s-c)^{\alpha_1}\mathrm{d}s\Bigg)\Bigg]\\
&+\left(\ell_2 r + N\right)\Bigg[\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\Bigg(\frac{1}{\Gamma(\alpha_2+1)}\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)^{\alpha_2}\mathrm{d}s\\
&+\frac{\lambda_1}{\Gamma(\alpha_2+1)}\int_c^d (s-c)^{\alpha_2}\mathrm{d}H_1(s)\Bigg) + \left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\frac{1}{\Gamma(\alpha_2+1)}\Bigg]\\
&= \left(\ell_1 r + M\right)\mathscr{C}_1 + \left(\ell_2 r + N\right)\mathscr{C}_2 + r\left(\mathscr{C}_3 + \mathscr{C}_4\right).
\end{aligned} \tag{30}$$

Consequently,

$$\left\|\mathcal{Q}_1(uv)\right\| \leq (\ell_1 r + M)\mathcal{C}_1 + (\ell_2 r + N)\mathcal{C}_2 + r(\mathcal{C}_3 + \mathcal{C}_4). \tag{31}$$

In the same way, we have

$$\left\|\mathcal{Q}_2(u,v)\right\| \leq (\ell_1 r + M)\mathcal{D}_1 + (\ell_2 r + N)\mathcal{D}_2 + r(\mathcal{D}_3 + \mathcal{D}_4). \tag{32}$$

Hence,

$$\begin{aligned}\|\mathcal{Q}(u,v))\| \leq{}& (\ell_1 r + M)(\mathcal{C}_1 + \mathcal{D}_1) + (\ell_2 r + N)(\mathcal{C}_2 + \mathcal{D}_2)\\ &+ r(\mathcal{C}_3 + \mathcal{C}_4 + \mathcal{D}_3 + \mathcal{D}_3) \leq r,\end{aligned} \tag{33}$$

which yields that $\mathcal{Q}(B_r)\subseteq B_r$.

Now, it is proved $\mathcal{Q}$: $\mathcal{Y}\times\mathcal{Y} \longrightarrow \mathcal{Y}\times\mathcal{Y}$ is a contraction. Applying condition (25), for any $(u_1,v_1), (u_2,v_2) \in \mathcal{Y}\times\mathcal{Y}$ and for each $z \in [c,d]$, we have

$$\begin{aligned}
&\left|\mathcal{Q}_1(u_1,v_1)(z) - \mathcal{Q}_1(u_2,v_2)(z)\right|\\
&\leq \ell_1\left(\|u_1-u_2\| + \|v_1-v_2\|\right)\left\{\frac{1}{\Gamma(\alpha_1+1)} + \frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\frac{1}{\Gamma(\alpha_1+1)}\right.\\
&+\left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left(\lambda_2\frac{1}{\Gamma(\alpha_1+1)}\int_c^d (s-c)^{\alpha_1}\mathrm{d}H_2(s)\right.\\
&\left.\left.+\frac{1}{\Gamma(\alpha_1+2)}\sum_{r=1}^{p}\left|\nu_r\right|\int_{\zeta_r}^{\theta_r}(s-c)^{\alpha_1}\mathrm{d}s\right)\right\}\\
&+\ell_2\left(\|u_1-u_2\| + \|v_1-v_2\|\right)\left\{\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\times\right.\\
&\left(\frac{1}{\Gamma(\alpha_2+2)}\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)^{\alpha_2}\mathrm{d}s + \frac{\lambda_1}{\Gamma(\alpha_2+1)}\int_c^d (s-c)^{\alpha_2}\mathrm{d}H_1(s)\right)\\
&\left.+\left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\frac{1}{\Gamma(\alpha_2+1)}\right\} + \|u_1-u_2\|\left\{\left|\sigma_1\right|(d-c)\right.\\
&+\left|G_3\right|\frac{(d-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)}(d-c)\left|\sigma_1\right| + \left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)}(d-c)\times\\
&\left.\left|\sigma_1\right|\left|\lambda_2\right|\int_c^d (s-c)\mathrm{d}H_2(s) + \left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)}\left|\sigma_1\right|\sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r}(s-c)\mathrm{d}s\right\}\\
&+\|v_1-v_2\|\left\{\left|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\right|\left|G_3\right|\left|\sigma_2\right|\left|\lambda_1\right|\int_c^d (s-c)\mathrm{d}H_1(s)\right.\\
&\left.+\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\left|\sigma_2\right|\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)\mathrm{d}s + \left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)}(d-c)\left|\sigma_2\right|\right\}\\
&= (\ell_1\mathcal{C}_1 + \ell_2\mathcal{C}_2)\left(\|u_1-u_2\| + \|v_1-v_2\|\right) + \mathcal{C}_3\|u_1-u_2\| + \mathcal{C}_4\|v_1-v_2\|\\
&\leq ((\ell_1\mathcal{C}_1 + \ell_2\mathcal{C}_2) + \mathcal{C}_3 + \mathcal{C}_4)\left(\|u_1-u_2\| + \|v_1-v_2\|\right),
\end{aligned} \tag{34}$$

and hence

$$\left\|\mathcal{Q}_1(u_1,v_1) - \mathcal{Q}_1(u_2,v_2)\right\| \leq ((\ell_1\mathcal{C}_1 + \ell_2\mathcal{C}_2) + \mathcal{C}_3 + \mathcal{C}_4)\left(\|u_1-u_2\| + \|v_1-v_2\|\right). \tag{35}$$

Furthermore, we deduce that

$$\left\|\mathcal{Q}_2(u_1,v_1)-\mathcal{Q}_2(u_2,v_2)\right\|\le\left(\left(\ell_1\mathcal{D}_1+\ell_2\mathcal{D}_2\right)+\mathcal{D}_3+\mathcal{D}_4\right)\left(\left\|u_1-u_2\right\|+\left\|v_1-v_2\right\|\right). \tag{36}$$

Using (25) and (33), we concluded that

$$\left\|\mathcal{Q}(u_1,v_1)-\mathcal{Q}(u_2,v_2)\right\|\le\left(\ell_1\left(\mathcal{C}_1+\mathcal{D}_1\right)+\ell_2\left(\mathcal{C}_2+\mathcal{D}_2\right)+\mathcal{C}_3+\mathcal{C}_4+\mathcal{D}_3+\mathcal{D}_4\right)\times\left(\left\|u_1-u_2\right\|+\left\|v_1-v_2\right\|\right). \tag{37}$$

As $\ell_1(\mathcal{C}_1+\mathcal{D}_1)+\ell_2(\mathcal{C}_2+\mathcal{D}_2)+\mathcal{C}_3+\mathcal{C}_4+\mathcal{D}_3+\mathcal{D}_4<1$, so the operator $\mathcal{Q}$ is a contraction and by applying Lemma 2, the operator $\mathcal{Q}$ has a unique solution which is the solution of the problem (3). The proof is finished. □

5. Existence Results

Two existence results are proved in this section.

5.1. Existence Result via Leray–Schauder Alternative. The Leray–Schauder alternative (Lemma 3) is used in the proof of our first existence result.

Theorem 2. *Let $\Theta\neq 0$ and f_1, f_2: $[c,d]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be continuous functions. Assume that*

[(H_1)] There exist real constants $u_i, v_i\ge 0$ for $i=1,2$ and $u_0, v_0>0$ such that for all $u,v\in\mathbb{R}$, we have

$$\begin{aligned}\left|f_1(z,u(z),v(z))\right|&\le u_0+u_1|u|+u_2|v|,\\ \left|f_2(z,u(z),v(z))\right|&\le v_0+v_1|u|+v_2|v|.\end{aligned} \tag{38}$$

If $(\mathcal{C}_1+\mathcal{D}_1)u_1+(\mathcal{C}_2+\mathcal{D}_2)v_1+\mathcal{C}_3+\mathcal{D}_3<1$ and $(\mathcal{C}_1+\mathcal{D}_1)u_2+(\mathcal{C}_2+\mathcal{D}_2)v_2+\mathcal{C}_4+\mathcal{D}_4<1$, where $\mathcal{C}_i$, $\mathcal{D}_i$ for $i=1,2,3,4$ are given by (23) and (24), respectively, then system (1.3) admits at least one solution on $[c,d]$.

Proof . The functions f_1, f_2 are continuous on $[c,d]\times\mathbb{R}^2$. Thus, the operator $\mathcal{Q}$ is continuous. Now, we will show that the operator $\mathcal{Q}$: $\mathcal{Y}\times\mathcal{Y}\longrightarrow\mathcal{Y}\times\mathcal{Y}$ is completely continuous. Let $B_r\subset\mathcal{Y}\times\mathcal{Y}$ be a bounded set, where $B_r=\{(u,v)\in\mathcal{Y}\times\mathcal{Y}: \|(u,v)\|\le r\}$. Then, for any $(u,v)\in B_r$, there exist positive real numbers P_1 and P_2 such that $|f_1(z,u(t),v(z))|\le P_1$ and $|f_2(z,u(z),v(z))|\le P_2$.

Thus, for each $(u,v)\in B_r$, we have

$$\begin{aligned}&\left|\mathcal{Q}_1(u,v)(z)\right|\\ &\quad\le\left|\sigma_1\right|\int_c^z|u(s)|\mathrm{d}s+I^{\alpha_1}\left|f_1(z,u(z),v(z))\right|\\ &\quad+\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left[\left|G_3\right|\left(\left|\sigma_2\right\|\lambda_1\right|\int_c^d\left[\int_c^s|v(t)|\mathrm{d}t\right]\mathrm{d}H_1(s)\right.\right.\\ &\quad+\left|\lambda_1\right|\int_c^d\left|f_1(z,u(z),v(z))\right|\mathrm{d}H_1(s)+\left|\sigma_2\right|\sum_{i=1}^n\left|\mu_i\right|\int_{\eta_i}^{\xi_i}\int_a^s|v(t)|\mathrm{d}t\mathrm{d}s\\ &\quad+\sum_{i=1}^n\left|\mu_i\right|\int_{\eta_i}^{\xi_i}I^{\alpha_2}\left|f_2(s,u(s),v(s))\right|\mathrm{d}s\\ &\quad\left.+\left|\sigma_1\right|\int_c^d|u(s)|\mathrm{d}s+I^{\alpha_1}\left|f_1(d,u(d),v(d))\right|\right)\\ &\quad+\left|G_2\right|\left(\left|\sigma_1\right\|\lambda_2\right|\int_c^d\left[\int_c^s|u(t)\mathrm{d}t\right]\mathrm{d}H_2(s)+\left|\sigma_1\right|\sum_{r=1}^p\left|\nu_r\right|\int_{\zeta_r}^{\theta_r}\left[\int_c^s|u(t)\mathrm{d}t\right]\mathrm{d}s\\ &\quad+\left|\lambda_2\right|\int_c^d I^{\alpha_1}\left|f_1(s,u(s),v(s))\right|\mathrm{d}H_2(s)+\left|\sigma_2\right|\int_c^d|v(s)|\mathrm{d}s+I^{\alpha_2}\left|f_2(d,u(d),v(d))\right|\end{aligned}$$

$$
\begin{aligned}
&+\sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r}I^{\alpha_1}\left|f_1(s,u(s),v(s))\right|\mathrm{d}s\Bigg)\Bigg]\\
&\leq r\Bigg[\left|\sigma_1\right|(d-c)+\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\Bigg(\left|\sigma_2\right|\left|\lambda_1\right|\int_c^d(s-c)\mathrm{d}H_1(s)\\
&+\left|\sigma_2\right|\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)\mathrm{d}s+\left|\sigma_1\right|(d-c)\Bigg)\\
&+\left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\Bigg(\left|\sigma_1\right|\left|\lambda_2\right|\int_c^d(s-c)\mathrm{d}H_2(s)+\left|\sigma_1\right|\sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r}(s-c)\mathrm{d}s\\
&+\left|\sigma_2\right|(d-c)\Bigg)\Bigg]+P_1\Bigg[\frac{1}{\Gamma(\alpha_1+1)}+\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\frac{1}{\Gamma(\alpha_1+1)}\\
&+\left|G_2\right|\frac{(d-c)^{\gamma-1}}{|\Theta|\Gamma(\gamma_1)}\Bigg(\left|\lambda_2\right|\frac{1}{\Gamma(\alpha_1+1)}\int_c^d(s-c)^{\alpha_1}\mathrm{d}H_2(s)\\
&+\frac{1}{\Gamma(\alpha_1+1)}\sum_{r=1}^{p}\left|\nu_r\right|\int_{\zeta_r}^{\theta_r}(s-c)^{\alpha_1}\mathrm{d}s\Bigg)\Bigg]\\
&+P_2\Bigg[\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\left|G_3\right|\Bigg(\frac{1}{\Gamma(\alpha_2+1)}\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)^{\alpha_2}\mathrm{d}s\\
&+\frac{\lambda_1}{\Gamma(\alpha_2+1)}\int_c^d(s-c)^{\alpha_2}\mathrm{d}H_1(s)\Bigg)+\left|G_2\right|\frac{(d-c)^{\gamma_1-1}}{|\Theta|\Gamma(\gamma_1)}\frac{1}{\Gamma(\alpha_2+1)}\Bigg]\\
&=P_1\mathscr{C}_1+P_2\mathscr{C}_2+r(\mathscr{C}_3+\mathscr{C}_4),
\end{aligned}
\tag{39}
$$

which yields

$$\left\|\mathcal{Q}_1(u,v)\right\|\leq\mathscr{C}_1P_1+\mathscr{C}_2P_2+r(\mathscr{C}_3+\mathscr{C}_4). \tag{40}$$

Similarly, we obtain that

$$\left\|\mathcal{Q}_2(u,v)\right\|\leq\mathscr{D}_1P_1+\mathscr{D}_2P_2+r(\mathscr{D}_3+\mathscr{D}_4). \tag{41}$$

Hence, from the above inequalities, we get that the operator $\mathcal{Q}$ is uniformly bounded, since

$$
\begin{aligned}
\|\mathcal{Q}(u,v)\|\leq(\mathscr{C}_1+\mathscr{D}_1)P_1+(\mathscr{C}_2+\mathscr{D}_2)P_2\\
+r(\mathscr{C}_3+\mathscr{C}_4+\mathscr{D}_3+\mathscr{D}_4).
\end{aligned}
\tag{42}
$$

Next, we are going to prove that the operator $\mathcal{Q}$ is equicontinuous. Let $\tau_1,\tau_2\in[c,d]$ with $\tau_1<\tau_2$. Then, we have

$$
\begin{aligned}
&\left|\mathcal{Q}_1(u,v)(\tau_2)-\mathcal{Q}_1(u,v)(\tau_1)\right|\\
&\leq\left|\sigma_1\right|r(\tau_2-\tau_1)+P_1\int_c^{\tau_1}\frac{\left[(\tau_2-c)^{\alpha_1-1}-(\tau_2-c)^{\alpha_1-1}\right]}{\Gamma(\alpha_1)}\mathrm{d}s+P_1\int_{\tau_1}^{\tau_2}\frac{(\tau_2-c)^{\alpha_1-1}}{\Gamma(\alpha_1)}\mathrm{d}s\\
&+\frac{\left[(\tau_2-c)^{\gamma_1-1}-(\tau_2-c)^{\gamma_1-1}\right]}{|\Theta|\Gamma(\gamma_1)}\Bigg[r\Bigg(\left|G_3\right|\left|\sigma_2\right|\left|\lambda_1\right|\int_c^d(s-c)\mathrm{d}H_1(s)\\
&+\left|\sigma_2\right|\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)\mathrm{d}s+\left|\sigma_1\right|(d-c)\Bigg)\\
&+\left|G_2r\right|\Bigg(\left|\sigma_1\right|\left|\lambda_2\right|\int_c^d(s-c)\mathrm{d}H_2(s)+\left|\sigma_1\right|\sum_{r=1}^{p}\nu_r\int_{\zeta_r}^{\theta_r}(s-c)\mathrm{d}s
\end{aligned}
$$

$$+\left|\sigma_2\right|(d-c)\Big)+P_1\left[\left|G_3\right|\frac{1}{\Gamma(\alpha_1+1)}+\left|G_2\right|\left(\left|\lambda_2\right|\frac{1}{\Gamma(\alpha_1+1)}\int_c^d(s-c)^{\alpha_1}\mathrm{d}H_2(s)\right.\right.$$
$$\left.\left.+\frac{1}{\Gamma(\alpha_1+1)}\sum_{r=1}^{p}\left|\nu_r\right|\int_{\zeta_r}^{\theta_r}(s-c)^{\alpha_1}\mathrm{d}s\right)\right]$$
$$+P_2\left[\left|G_3\right|\left(\frac{1}{\Gamma(\alpha_2+1)}\sum_{i=1}^{n}\left|\mu_i\right|\int_{\eta_i}^{\xi_i}(s-c)^{\alpha_2}\mathrm{d}s\right.\right.$$
$$\left.\left.+\frac{\left|\lambda_1\right|}{\Gamma(\alpha_2+1)}\int_c^d(s-c)^{\alpha_2}\mathrm{d}H_1(s)\right)+\left|G_2\right|\frac{1}{\Gamma(\alpha_2+1)}\right]\Bigg]. \tag{43}$$

Therefore, we obtain

$$\left|\mathcal{Q}_1(u,v)(\tau_2)-\mathcal{Q}_1(u,v)(\tau_1)\right|\longrightarrow 0,\quad \text{as}\ \tau_1\longrightarrow\tau_2. \tag{44}$$

Analogously, we can get the following inequality:

$$\left|\mathcal{Q}_2(u,v)(\tau_2)-\mathcal{Q}_2(u,v)(\tau_1)\right|\longrightarrow 0,\quad \text{as}\ \tau_1\longrightarrow\tau_2. \tag{45}$$

Hence, the set $\mathcal{Q}\mathcal{B}_r$ is equicontinuous. Accordingly, Arzelá–Ascoli theorem implies that the operator $\mathcal{Q}$ is completely continuous.

Finally, we shall show the boundedness of the set $Z=\{(u,v)\in\mathcal{Y}\times\mathcal{Y}\colon\ (u,v)=\mu\mathcal{Q}(u,v),\ 0\le\mu\le 1\}$. Let any $(u,v)\in Z$, then $(u,v)=\mu\mathcal{Q}(u,v)$. We have, for all $z\in[c,d]$,

$$\begin{aligned} u(z)&=\mu\mathcal{Q}_1(u,v)(z),\\ v(z)&=\mu\mathcal{Q}_2(u,v)(z). \end{aligned} \tag{46}$$

Then, we get

$$\begin{aligned} \|u\|&\le\left(u_0+u_1\|u\|+u_2\|v\|\right)\mathcal{C}_1+\left(v_0+v_1\|u\|+v_2\|v\|\right)\mathcal{C}_2+\|u\|\mathcal{C}_3+\|v\|\mathcal{C}_4,\\ \|v\|&\le\left(u_0+u_1\|u\|+u_2\|v\|\right)\mathcal{D}_1+\left(v_0+v_1\|u\|+v_2\|v\|\right)\mathcal{D}_2+\|u\|\mathcal{D}_3+\|v\|\mathcal{D}_4, \end{aligned} \tag{47}$$

which imply that

$$\begin{aligned} \|u\|+\|v\|\le{}&(\mathcal{C}_1+\mathcal{D}_1)u_0+(\mathcal{C}_2+\mathcal{D}_2)v_0+\left[(\mathcal{C}_1+\mathcal{D}_1)u_1+(\mathcal{C}_2+\mathcal{D}_2)v_1+\mathcal{C}_3+\mathcal{D}_3\right]\|u\|\\ &+\left[(\mathcal{C}_1+\mathcal{D}_1)u_2+(\mathcal{C}_2+\mathcal{D}_2))v_2+\mathcal{C}_4+\mathcal{D}_4\right]\|v\|. \end{aligned} \tag{48}$$

Thus, we obtain

$$\|(u,v)\|\le\frac{(\mathcal{C}_1+\mathcal{D}_1)u_0+(\mathcal{C}_2+\mathcal{D}_2)v_0}{M^*}, \tag{49}$$

where $M^*=\min\{1-(\mathcal{C}_1+\mathcal{D}_1)u_1-(\mathcal{C}_2+\mathcal{D}_2)v_1-(\mathcal{C}_3+\mathcal{D}_3),1-(\mathcal{C}_1+\mathcal{D}_1)u_2-(\mathcal{C}_2+\mathcal{D}_2)v_2-(\mathcal{C}_4+\mathcal{D}_4)\}$, which shows that the set Z is bounded. Therefore, by Leray–Schauder alternative (Lemma 3), the operator $\mathcal{Q}$ has at least one fixed point. Hence, we deduce that problem (3) admits a solution on $[c,d]$, which completes the proof. □

5.2. Existence Result via Krasnosel'skiĭ's Fixed-Point Theorem. Now, Krasnosel'skiĭ's fixed-point theorem (Lemma 4) is applied to prove our second existence result.

Theorem 3. *Assume that $\Theta\ne 0$ and $f_1,f_2\colon[c,d]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are continuous functions satisfying condition (4.8) in Theorem 4.1. Furthermore, suppose that there exist positive constants R_1 and R_2 such that for all $z\in[c,d]$ and $u,v\in\mathbb{R}$, we have*

$$\begin{aligned} \left|f_1(z,u,v)\right|&\le R_1,\\ \left|f_2(z,u,v)\right|&\le R_2. \end{aligned} \tag{50}$$

If $\mathcal{C}_3+\mathcal{C}_4<1$, $\mathcal{D}_3+\mathcal{D}_4<1$ and $\left((d-c)^{\alpha_1}/\Gamma(\alpha_1+1)\right)l_1+\left((d-c)^{\alpha_2}/(d-c)^{\alpha_2}\right)\ell_2<1$, then problem (1.3) admits a solution on $[c,d]$.

Proof. First, we decompose the operator $\mathcal{Q}$ defined by (1) into four operators as

$$\begin{aligned}
\mathcal{S}_1(u,v)(z) &= -\sigma_1\int_c^z u(s)\mathrm{d}s + \frac{(z-c)^{\gamma_1-1}}{\Theta\Gamma(\gamma_1)}\Bigg[G_3\Bigg(-\sigma_2\lambda_1\int_c^d\int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s)\\
&\quad + \lambda_1\int_c^d I^{\alpha_2} f_2(s,u(s),v(s))\mathrm{d}H_1(s) - \sigma_2\sum_{i=1}^n \mu_i\int_{\eta_i}^{\xi_i}\int_c^s v(t)\mathrm{d}t\mathrm{d}s\\
&\quad + \sum_{i=1}^n \mu_i\int_{\eta_i}^{\xi_i} I^{\alpha_2} f_2(s,u(s),v(s))\mathrm{d}s + \sigma_1\int_c^d u(s)\mathrm{d}s - I^{\alpha_1} f_1(d,u(d),v(d))\Bigg)\\
&\quad + G_2\Bigg(-\sigma_1\lambda_2\int_c^d\int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s) + \lambda_2\int_c^d I^{\alpha_1} f_1(s,u(s),v(s))\mathrm{d}H_2(s)\\
&\quad - \sigma_1\sum_{r=1}^p \nu_r\int_{\zeta_r}^{\theta_r}\int_c^s u(t)\mathrm{d}t\mathrm{d}s + \sum_{r=1}^p \nu_r\int_{\zeta_r}^{\theta_r} I^{\alpha_1} f_1(s,u(s),v(s))\mathrm{d}s\\
&\quad + \sigma_2\int_c^d v(s)\mathrm{d}s - I^{\alpha_2} f_2(d,u(d),v(d))\Bigg)\Bigg],\\
\mathcal{S}_2(u,v)(z) &= I^{\alpha_1} f_1(z,u(z),v(z))\, I^{\alpha_1} f_1 u, v(z),\\
\mathcal{S}_3(u,v)(z) &= -\sigma_2\int_c^z v(s)\mathrm{d}s + \frac{(z-c)^{\gamma_2-1}}{\Theta\Gamma(\gamma_2)}\Bigg[G_1\Bigg(-\sigma_1\lambda_2\int_c^d\int_c^s u(t)\mathrm{d}t\mathrm{d}H_2(s)\\
&\quad + \lambda_2\int_c^d I^{\alpha_1} f_1(s,u(s),v(s))\mathrm{d}H_2(s) - \sigma_1\sum_{r=1}^p \nu_r\int_{\zeta_r}^{\theta_r}\int_c^s u(t)\mathrm{d}t\mathrm{d}s\\
&\quad + \sum_{r=1}^p \nu_r\int_{\zeta_r}^{\theta_r} I^{\alpha_1} f_1(s,u(s),v(s))\mathrm{d}s + \sigma_2\int_c^d v(s)\mathrm{d}s - I^{\alpha_2} f_2(d,u(d),v(d))\Bigg)\\
&\quad + G_4\Bigg(-\sigma_2\lambda_1\int_c^d\int_c^s v(t)\mathrm{d}t\mathrm{d}H_1(s) + \lambda_1\int_c^d I^{\alpha_2} f_2(s,u(s),v(s))\mathrm{d}H_1(s)\\
&\quad - \sigma_2\sum_{i=1}^n \mu_i\int_{\eta_i}^{\xi_i}\int_c^s v(t)\mathrm{d}t\mathrm{d}s + \sum_{i=1}^n \mu_i\int_{\eta_i}^{\xi_i} I^{\alpha_2} f_2(s,u(s),v(s))\mathrm{d}s\\
&\quad + \sigma_1\int_c^d u(s)\mathrm{d}s - I^{\alpha_1} f_1(d,u(d),v(d))\Bigg)\Bigg],\\
\mathcal{S}_4(u,v)(z) &= I^{\alpha_2} f_2(z,u(z),v(z))\, I^{\alpha_2} f_2 u, v(z).
\end{aligned} \tag{51}$$

Accordingly, $\mathcal{Q}_1(u,v)(z) = \mathcal{S}_1(u,v)(z) + \mathcal{S}_2(u,v)(z)$ and $\mathcal{Q}_2(u,v)(z) = \mathcal{S}_3(u,v)(z) + \mathcal{S}_4(u,v)(z)$. Let $B_\varepsilon = \{(u,v)\in\mathcal{Y}\times\mathcal{Y};\|(u,v)\|\le\varepsilon\}$ with

$$\varepsilon \ge \max\left\{\frac{\mathcal{C}_1 R_1 + \mathcal{C}_2 R_2}{1-(\mathcal{C}_3+\mathcal{C}_4)}, \frac{\mathcal{D}_1 R_1 + \mathcal{D}_2 R_2}{1-(\mathcal{D}_3+\mathcal{D}_4)}\right\}. \tag{52}$$

First, it is showed that $\mathcal{Q}_1(x,y) + \mathcal{Q}_2(u,v)\in B_\varepsilon$ for all $(x,y),(u,v)\in B_\varepsilon$. According to the proof of Theorem 1, we get

$$\begin{aligned}
&\mathcal{S}_2(u,v)(z) I^{\alpha_1} f_1(z,u(z),v(z)),\\
&\mathcal{S}_4(u,v)(z) I^{\alpha_2} f_2(z,u(z),v(z)).
\end{aligned} \tag{53}$$

Consequently, $\mathcal{Q}_1(x,y) + \mathcal{Q}_2(u,v)\in B_\varepsilon$ and we conclude the condition (i) of Lemma 4. Now, it is indicated that the operator $(\mathcal{S}_2,\mathcal{S}_4)$ is a contraction mapping. For $(x_1,y_1),(x_2,y_2)\in B_\varepsilon$, we infer that

$$\begin{aligned}
\left|\mathcal{S}_2(x_1,y_1)(z) - \mathcal{S}_2(x_2,y_2)(z)\right| &\le I^{\alpha_1}\left|f_1 x_1, y_1 - f_1 x_2, y_2\right|(z)\\
&\le \ell_1\left(\|x_1-x_2\| + \|y_1-y_2\|\right) I^{\alpha_1}(1)(d) \le \ell_1\frac{(d-c)^{\alpha_1}}{\Gamma(\alpha_1+1)}\left(\|x_1-x_2\| + \|y_1-y_2\|\right),
\end{aligned} \tag{54}$$

$$\begin{aligned}
\left|\mathcal{S}_4(x_1,y_1)(z) - \mathcal{S}_4(x_2,y_2)(z)\right|\,| &\le I^{\alpha_2}\left|f_2 x_1, y_1 - f_2 x_2, y_2\right|(z) \le \ell_2\left(\|x_1-x_2\| + \|y_1-y_2\|\right) I^{\alpha_2}(1)(d)\\
&\le \ell_2\frac{(d-c)^{\alpha_2}}{\Gamma(\alpha_2+1)}\left(\|x_1-x_2\| + \|y_1-y_2\|\right).
\end{aligned} \tag{55}$$

As $((d-c)^{\alpha_1}/\Gamma(\alpha_1+1))\ell_1 + (d-c)^{\alpha_2}/\Gamma(\alpha_2+1)\ell_2 < 1$, the operator $(\mathcal{S}_2, \mathcal{S}_4)$ is a contraction and the condition *(iii)* of Lemma 4 is concluded. In the final step, the condition *(ii)* of Lemma 4 is verified for the operator $(\mathcal{S}_1, \mathcal{S}_3)$. As the functions f_1, f_2 are continuous, one can see that the operator $(\mathcal{S}_1, \mathcal{S}_3)$ is continuous. Furthermore, for $(u,v) \in B_\varepsilon$, as in the proof of Theorem 1, we have

$$\begin{aligned} \left|\mathcal{S}_1(u,v)(z)\right| &\le \left(\mathcal{C}_1 - \frac{1}{\Gamma(\alpha_1+1)}\right)R_1 + \mathcal{C}_2 R_2 + \left(\mathcal{C}_3 + \mathcal{C}_4\right)\varepsilon = P^*, \\ \left|\mathcal{S}_3(u,v)(z)\right| &\le \mathcal{D}_1 R_1 + \left(\mathcal{D}_2 - \frac{1}{\Gamma(\alpha_2+1)}\right)R_2 + \left(\mathcal{D}_3 + \mathcal{D}_4\right)\varepsilon = Q^*. \end{aligned} \tag{56}$$

Hence, $\|(\mathcal{S}_1, \mathcal{S}_2)(u,v)\| \le P^* + Q^*$, which implies that $(\mathcal{S}_1, \mathcal{S}_3)B_\varepsilon$ is uniformly bounded. Now, we claim that the set $(\mathcal{S}_1, \mathcal{S}_3)B_\varepsilon$ is equicontinuous. For this aim, let $\tau_1, \tau_2 \in [c,d]$ with $\tau_1 < \tau_2$. For any $(u,v) \in B_\varepsilon$, similar to the proofs of equicontinuous for the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ in 2, we can show that $|\mathcal{S}_1(u,v)(\tau_2) - \mathcal{S}_1(u,v)(\tau_1)|$, $|\mathcal{S}_3(u,v)(\tau_2) - \mathcal{S}_3(u,v)(\tau_1)| \longrightarrow 0$ as $\tau_1 \longrightarrow \tau_2$. Consequently, the set $(\mathcal{S}_1, \mathcal{S}_3)B_\varepsilon$ is equicontinuous and by applying Arzelá-Ascoli theorem, the operator $(\mathcal{S}_1, \mathcal{S}_3)$ will be compact on B_ε. Therefore, by applying Lemma 4, problem (3) has at least one solution on $[c,d]$. This completes the proof. □

Remark 1. In Theorem 2, the functions f_1, f_2 are bounded by linear planes in three-dimension space. While, in Theorem 3, f_1, f_2 are bounded by fixed constants and also satisfied Lipschitz condition in (25).

6. Examples

Now, we present some examples to show the benefits of our results.

Example 1. Consider the following coupled system of Hilfer type sequential fractional differential equations involving Riemann–Stieltjes integral multistrip boundary conditions of the form

$$\begin{aligned} &\left({}^H D^{(5/4)(2/3)} + \frac{1}{20}{}^H D^{(1/4)(2/3)}\right)u(z) = f_1(z,u(z),v(z)), \quad z \in [(1/8),(13/8)], \\ &\left({}^H D^{(7/4)(1/3)} + \frac{1}{15}{}^H D^{(3/4)(1/3)}\right)v(z) = f_2(z,u(z),v(z)), \quad z \in [(1/8),(13/8)], \\ &u\left(\frac{1}{8}\right) = 0, \\ &v\left(\frac{1}{8}\right) = 0, \\ &u\left(\frac{13}{8}\right) = \frac{1}{4}\int_{1/8}^{13/8} v(s)\mathrm{d}\left(e^{-2s}\right) + \frac{2}{7}\int_{1/2}^{5/8} v(s)\mathrm{d}s + \frac{3}{11}\int_{9/8}^{5/4} v(s)\mathrm{d}s, \\ &v\left(\frac{13}{8}\right) = \frac{1}{5}\int_{1/8}^{13/8} u(s)\mathrm{d}\left(e^{-3s}\right) + \frac{4}{13}\int_{1/4}^{3/8} u(s)\mathrm{d}s + \frac{5}{17}\int_{3/4}^{7/8} u(s)\mathrm{d}s + \frac{6}{19}\int_{11/8}^{3/2} u(s)\mathrm{d}s. \end{aligned} \tag{57}$$

Here, $\alpha_1 = 5/4$, $\alpha_2 = 7/4$, $\beta_1 = 2/3$, $\beta_2 = 1/3$, $c = 1/8$, $d = 13/8$, $\lambda_1 = 1/4$, $\lambda_2 = 1/5$, $H_1(t) = e^{-2t}$, $H_2(t) = e^{-3t}$, $n = 2$, $\mu_1 = 2/7$, $\mu_2 = 3/11$, $\eta_1 = 1/2$, $\eta_2 = 9/8$, $\xi_1 = 5/8$, $\xi_2 = 5/4$, $p = 3$, $\nu_1 = 4/13$, $\nu_2 = 5/17$, $\nu_3 = 6/19$, $\zeta_1 = 1/4$, $\zeta_2 = 3/4$, $\zeta_3 = 11/8$, $\theta_1 = 3/8$, $\theta_2 = 7/8$, and $\theta_3 = 3/2$. Then, we can compute that $\gamma_1 = 7/4$, $\gamma_2 = 11/6$, $G_1 \approx 1.474766913$, $G_2 \approx -0.03380798224$, $G_3 \approx 1.490431261$, $G_4 \approx 0.03704876432$, $\Theta \approx 2.199291254$, $\mathcal{C}_1 \approx 1.765659740$, $\mathcal{C}_2 \approx 0.006128241272$, $\mathcal{C}_3 \approx 0.1499796921$, $\mathcal{C}_4 \approx 0.000526941741$, $\mathcal{D}_1 \approx 1.242948896$, $\mathcal{D}_2 \approx 0.06369559511$, $\mathcal{D}_3 \approx 0.1998340850$, and $\mathcal{D}_4 \approx 0.003945721441$.

(i) The Lipschitzian functions f_1, f_2: $[(1/8),(13/8)] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are given by

$$f_1(t,u,v) = \frac{1}{8t+4}\left(\frac{u^2 + 2|u|}{2(1+|u|)}\right) + \frac{\sin|v|}{8t+5} + \frac{1}{2}\cos 2\pi t, \tag{58}$$

$$\begin{aligned} f_2(t,u,v) = {}& \frac{1}{8t+3}\tan^{-1}|u| \\ &+ \frac{e^{(1-8t)}}{8t+7}\left(\frac{4v^2 + 5|v|}{5(1+|v|)}\right) + \frac{1}{4}\log_e t. \end{aligned} \tag{59}$$

From direct computation to (59)-(60), we get

$$\begin{aligned}&\left|f_1(t,u_1,v_1)-f_1(t,u_2,v_2)\right|\le\frac{1}{5}\left|u_1-u_2\right|+\frac{1}{6}\left|v_1-v_2\right|,\\&\left|f_2(t,u_1,v_1)-f_2(t,u_2,v_2)\right|\le\frac{1}{4}\left|u_1-u_2\right|+\frac{1}{8}\left|v_1-v_2\right|,\end{aligned}\tag{60}$$

for $u_1,u_2,v_1,v_2\in\mathbb{R}$. Setting $\ell_1=1/5$ and $\ell_2=1/4$, we obtain $\ell_1(\mathscr{C}_1+\mathscr{D}_1)+\ell_2\ (\mathscr{C}_2+\mathscr{D}_2)+\mathscr{C}_3+\mathscr{C}_4+\mathscr{D}_3+\mathscr{D}_4\approx 0.9734641265<1$. By application of our Theorem 1, the problem of Hilfer type sequential fractional differential system involving Riemann–Stieltjes integral multistrip boundary conditions (58) with (59)-(60) has a unique solution on $[(1/8),(13/8)]$.

(ii) Let the nonlinear functions $f_1,f_2\colon[(1/8),(13/8)]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ be defined by

$$f_1(t,u,v)=\frac{1}{2}\cos^2\pi t+\frac{1}{5}ue^{-v^4}+\frac{|v|^{15}\sin^4u}{3\left(1+v^{14}\right)},\tag{61}$$

$$f_2(t,u,v)=\frac{1}{3}\sin^2\pi t+\frac{u^{18}\cos^{12}v}{4\left(1+|u|^{17}\right)}+\frac{v}{\pi}\tan^{-1}u.\tag{62}$$

We remark that $|f_1(t,u,v)|\le(1/2)+\ (1/5)|u|+(1/3)|v|$ and $|f_2(t,u,v)|\le(1/3)+(1/4)|u|+(1/2)|v|$. Now, we choose the constants as in Theorem 2 by $u_0=1/2$, $u_1=1/5$, $u_2=1/3$, $v_0=1/3$, $v_1=1/4$, and $v_2=1/2$. Then, we can find that $(\mathscr{C}_1+\mathscr{D}_1)u_1+(\mathscr{C}_2+\mathscr{D}_2)v_1+\mathscr{C}_3+\mathscr{D}_3\approx 0.9748101164<1$ and $(\mathscr{C}_1+\mathscr{D}_1)u_2+(\mathscr{C}_2+\mathscr{D}_2)v_2+\mathscr{C}_4+\mathscr{D}_4\approx 0.7915367403<1$. The benefit of Theorem 2 can be used to conclude that the coupled system of Hilfer type sequential fractional differential equations subject to boundary conditions (58) with (62)-(63) has at least one solution on $[(1/8),(13/8)]$.

(iii) Suppose that two Lipschitzian functions $f_1,f_2\colon[(1/8),(13/8)]\times\mathbb{R}^2\longrightarrow\mathbb{R}$ are stated by

$$f_1(t,u,v)=\frac{1}{8t+1}+\frac{1}{8t+2}\left(\frac{|u|}{1+|u|}\right)+\frac{1}{8t+3}\tan^{-1}|v|,\tag{63}$$

$$f_2(t,u,v)=\frac{1}{8t+4}+\frac{1}{3}e^{1-8t}\ \sin|u|+\frac{1}{5}e^{-(1-8t)^2}\left(\frac{|v|}{1+|v|}\right).\tag{64}$$

Actually, we can compute the bounds of the above two functions by $|f_1(t,u,v)|\le(5/6)+(\pi/8)$, $|f_2(t,u,v)|\le(47/60)$, for all $u,v\in\mathbb{R}$. In addition, we can find that $|f_1(t,u_1,v_1)-f_1(t,u_2,v_2)|\le(1/3)|u_1-u_2|+(1/4)|v_1-v_2|$ and $|f_2(t,u_1,v_1)-f_2(t,u_2,v_2)|\le(1/3)\ |u_1-u_2|+(1/5)|v_1-v_2|$, and thus we can set $\ell_1=1/3$ and $\ell_2=1/3$ satisfying condition (4.8) in Theorem 1. Then, we obtain $\mathscr{C}_3+\mathscr{C}_4\approx 0.1505066338<1$, $\mathscr{D}_3+\mathscr{D}_4\approx 0.2037798064<1$, and $((d-c)^{\alpha_1}/\Gamma(\alpha_1+1))\ell_1+((d-c)^{\alpha_1}/\ \Gamma(\alpha_2+1))\ell_2\approx 0.9097463067<1$ that all conditions in Theorem 3 are fulfilled. In this step, we conclude that the problem (58) with (64) has at least one solution on $[(1/8),(13/8)]$. Finally, we observe that the uniqueness result cannot be obtained because $\ell_1(\mathscr{C}_1+\mathscr{D}_1)+\ell_2\ (\mathscr{C}_2+\mathscr{D}_2)+\mathscr{C}_3+\mathscr{C}_4+\mathscr{D}_3+\mathscr{D}_4\approx 1.380430597>1$.

7. Conclusions

In the present research, we studied a coupled system of Hilfer type sequential fractional differential equations supplemented with Riemann–Stieltjes integral multistrip boundary conditions. First, an auxiliary lemma, concerning a linear variant of the considered problem, has been proved which is pivotal to converting the coupled system into a fixed point problem. Then, existence and uniqueness results are established via standard fixed point theorems. Thus, the classical Banach fixed point theorem is applied to obtain a uniqueness result, while Leray–Schauder alternative and Krasnosel'skiĭ's fixed point theorem are applied to present the existence results. Numerical examples are also constructed to illustrate the obtained results. The obtained results are new and enrich the existing literature on coupled systems of Hilfer type sequential fractional differential equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

Further on Inequalities for $(\alpha, h-m)$-Convex Functions via k-Fractional Integral Operators

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The purpose of this article is to demonstrate new generalized k-fractional Hadamard and Fejér–Hadamard integral inequalities for $(\alpha, h-m)$-convex functions. To prove these inequalities, k-fractional integral operators including the generalization of the Mittag–Leffler function are used. The results presented in this article can be considered an important advancement of previously published inequalities.

1. Introduction

Theory of convexity offers an effective and charming area of research and is also a theory that featured prominently and surprisingly in distinct disciplines such as mathematical analysis, optimization, economics, finance, engineering and game theory. Convexity theory is very closely related with the theory of inequalities. Many inequalities well known in the literature are direct applications of the properties of convex functions. The usage of fractional integral operators for getting the generalized types of classic inequalities has become an important method in advanced mathematical studies of inequalities.

One of the convexity theory studies in the literature belongs to Gao et al. [1]. They presented a new type of functions called n-polynomial harmonically exponential type convex, and specified some of their algebraic features. Mehrez and Agarwal [2] established new type of integral inequalities for convex functions and indicated new inequalities for some special and q-special functions. Tariq [3] defined the concept of p–harmonic exponential type convex functions. Also, they investigated some integral inequalities in the form of applications for some means. Another study on convexity theory and inequalities was presented by Butt et al. [4]. They presented the notion of m–polynomial p-harmonic exponential type convex functions and demonstrated various new integral inequalities. Srivastava et al. [5] obtained a new class of the bi-close-to-convex functions described in the open unit disk by using the Borel distribution series of the Mittag–Leffler type. Also, the authors demonstrated the Fekete–Szego type inequalities via the bi-close-to-convex function class.

Fractional calculus, which is the study of integrals and derivatives of fractional order, has expanded significantly over the late nineteenth century. It ranges from chemical, viscoelasticity, and statistical physics to electrical and mechanical engineering. The fundamental working doctrine of fractional analysis is to present new fractional derivative and integral operators, and to analyze the benefits of these operators through the instrument of modeling studies, and collations. Integral operators, which form a significant part of fractional calculus, are now resources of many fields such as inequality theory, engineering, statistics, mathematical biology, and modeling, which take advantage of fractional analysis. Many inequalities have been generalized through the instrument of fractional integral operators and provide construction of new approximations.

One of the fractional calculus studies in the literature belongs to Abdeljawad et al. [6]. They obtained generalized Hermite–Hadamard type inequalities and generalized Simpson type inequalities for (s, m)-convex functions with the help of local fractional integration. Akdemir et al. [7] used generalized fractional integral operators. By using these operators, they proved new and general variants of Chebyshev's inequality. Butt et al. [8] established a general integral identity to acquire new integral inequalities of several Hadamard types. For this purpose, they used a new version of the Atangana–Baleanu integral operator. Khan et al. [9] explored two fractional integral operators related to Fox H-function owing to Saxena and Kumbhat. They proved series expansion of the images of the M-series with the help of these fractional operators. Another study to k-fractional integrals was presented by Qi et al. [10]. They constructed some generalized fractional integral inequalities of the Hermite–Hadamard type via (α, m)-convex functions. Also, they demonstrated that one can get and expand some Riemann–Liouville fractional integral inequalities and classical integral inequalities of Hermite–Hadamard's type. Tunc et al. [11] presented the generalized k-fractional integrals of a function with respect to the another function that generalizes many several types of fractional integrals. Also, they studied trapezoid inequalities for the functions whose derivatives in absolute value are convex. Önalan et al. [12] proved many Hermite–Hadamard type integral inequalities for functions whose absolute values of the second derivatives are s-convex and s-concave using fractional integral operators with the Mittag–Leffler kernel. Zhu et al. [13] explored a weighted integral identity of Simpson-like type. Relying on this identity, they obtained some estimation-type results connected with the weighted Simpson-like type integral inequalities for the first order differentiable functions. Srivastava et al. [14] established the homogeneous q-shift operator and the homogeneous q-difference operator. Based on these operators, they searched generalized Cauchy and Hahn polynomials.

2. Preliminaries

Now let us define some important functions.

Definition 1 (see [15]). A function $\varphi\colon [a, b] \longrightarrow \mathbb{R}$ *is called a convex function, if*

$$\varphi(\eta\iota + (1-\eta)\kappa) \le \eta\varphi(\iota) + (1-\eta)\varphi(\kappa) \tag{1}$$

holds for all $\iota, \kappa \in [a, b]$ and $\eta \in [0, 1]$.

Definition 2 (see [16]). The function $\varphi\colon [0, b] \longrightarrow \mathbb{R}$, $b > 0$, *is called the* (α, m)*-convex function, if*

$$\varphi(\eta\iota + m(1-\eta)\kappa) \le \eta^{\alpha}\varphi(\iota) + m(1-\eta^{\alpha})\varphi(\kappa) \tag{2}$$

holds for all $\iota, \kappa \in [0, b], \eta \in [0, 1]$ and $(\alpha, m) \in [0, 1]^2$.

Definition 3 (see [17]). *A function* $\varphi\colon [0, b] \longrightarrow \mathbb{R}$ *is said to be* (s, m)*-convex, if*

$$\varphi(\eta\iota + m(1-\eta)\kappa) \le \eta^{s}\varphi(\iota) + m(1-\eta^{s})\varphi(\kappa) \tag{3}$$

holds for all $\iota, \kappa \in [0, b]$, $\eta \in [0, 1]$ and $(s, m) \in (0, 1]$.

Definition 4 (see [18]). A function $\varphi\colon [0, b] \longrightarrow \mathbb{R}$ *is said to be* (s, m)*-convex in the second sense, if*

$$\varphi(\eta\iota + m(1-\eta)\kappa) \le \eta^{s}\varphi(\iota) + m(1-\eta)^{s}\varphi(\kappa) \tag{4}$$

holds for all $\iota, \kappa \in [0, b]$, $\eta \in [0, 1]$ and $(s, m) \in (0, 1]^2$.

Definition 5 (see [19]). Let $J \subseteq \mathbb{R}$ *be an interval including* $(0, 1)$ *and let* $h\colon J \longrightarrow \mathbb{R}$ *be a nonnegative function. Then the function* $\varphi\colon [0, b] \longrightarrow \mathbb{R}$ *is called the* $(h - m)$*-convex function, if*

$$\varphi(\eta\iota + m(1-\eta)\kappa) \le h(\eta)\varphi(\iota) + mh(1-\eta)\varphi(\kappa) \tag{5}$$

holds for all $\iota, \kappa \in [0, b]$, $\eta \in [0, 1]$ and $m \in [0, 1]$.

Definition 6 (see [20]). Let $J \subseteq \mathbb{R}$ *be an interval including* $(0, 1)$ *and let* $h\colon J \longrightarrow \mathbb{R}$ *be a nonnegative function. Then the function* $\varphi\colon [0, b] \longrightarrow \mathbb{R}$ *is called the* $(\alpha, h - m)$*-convex function, if*

$$\varphi(\eta\iota + m(1-\eta)\kappa) \le h(\eta^{\alpha})\varphi(\iota) + mh(1-\eta^{\alpha})\varphi(\kappa) \tag{6}$$

holds for all $\iota, \kappa \in [0, b]$, $\eta \in [0, 1]$ and $(\alpha, m) \in [0, 1]^2$.

Remark 1

(i) *By taking* $m = \alpha = 1$ *and* $h(\eta) = \eta$ *in* (6), *we obtain the definition of convex function* (1).

(ii) *By taking* $h(\eta) = \eta$ *in* (6), *we obtain the definition of* (α, m)*-convex function* (2).

(iii) *By taking* $h(\eta) = \eta$ *and* $\alpha = s$ *in* (6), *we obtain the definition of* (s, m)*-convex function* (3).

(iv) *By taking* $h(\eta) = \eta^{s}$ *and* $\alpha = 1$ *in* (6), *we obtain the definition of* (s, m)*-convex function in the second sense* (4).

(v) *By taking* $\alpha = 1$ *in*(6), *we obtain the definition of* (h, m)*-convex function* (5).

(vi) *By taking* $\alpha = m = h(\eta) = 1$ *in* (6), *we obtain the definition of p-function described by Dragomir* et al. *in* [21].

Now let us represent some definitions of fractional integral operators that will form the basis for this article.

Definition 7 (see [22]). Let $\gamma, c, w, \alpha, l, \in \mathbb{C}, \mathfrak{R}(l), \mathfrak{R}(\alpha) > 0$, $\mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0$ *with* $\mu, \delta > 0$, $\tilde{p} \ge 0$, *and* $0 < \nu \le \delta + \mu$. *Let* $\varphi \in L_1[a, b]$, $\iota \in [a, b]$. *In that case, the generalized fractional operators are defined by*

$$\begin{aligned}
\left(F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,a+}\varphi\right)(\iota;\tilde{p}) &= \int_a^{\iota} (\iota-\eta)^{\alpha-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\iota-\eta)^{\mu};\tilde{p}\right)\varphi(\eta)\mathrm{d}\eta,\\
\left(F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,b-}\varphi\right)(\iota;\tilde{p}) &= \int_{\iota}^{b} (\eta-\iota)^{\alpha-1} E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\eta-\iota)^{\mu};\tilde{p}\right)\varphi(\eta)\mathrm{d}\eta,
\end{aligned} \tag{7}$$

where

$$E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}(\eta;\widetilde{p}) = \sum_{n=0}^{\infty}\frac{\beta_{\widetilde{p}}(\gamma+n\nu,c-\gamma)}{\beta(\gamma,c-\gamma)}\frac{(c)_{n\nu}}{\Gamma(\mu n+\alpha)}\frac{\eta^n}{(l)_{n\delta}} \tag{8}$$

is generalized extended Mittag–Leffler function, and $\beta_{\widetilde{p}}$ is the expansion of beta function described as below:

$$\beta_{\widetilde{p}}(\iota,\kappa) = \int_0^1 \eta^{\iota-1}(1-\eta)^{\kappa-1}e^{-(\widetilde{p}/\eta(1-\eta))}\mathrm{d}\eta, \tag{9}$$

where $\mathfrak{R}(\iota), \mathfrak{R}(\kappa), \mathfrak{R}(\widetilde{p}) > 0$.

Definition 8 (see [23]). Let φ, ψ: $[a,b] \longrightarrow \mathbb{R}$ with $0<a<b$, be the functions, φ be positive, $\varphi \in L_1[a,b]$ and ψ be differentiable and strictly increasing. Let (ϕ/ι) be an increasing on $[a,\infty)$, $\gamma, c, w, \alpha, l \in \mathbb{C}, \mathfrak{R}(l), \mathfrak{R}(\alpha) > 0, \mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0$ with $\mu,\delta>0, \widetilde{p}\geq 0$, and $0<\nu\leq\mu+\delta$. In that case, for $\iota \in [a,b]$, the fractional operators are described by

$$\begin{aligned}\left({}_{\psi}F^{\phi,\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,a+}\varphi\right)(\iota;\widetilde{p}) &= \int_a^{\iota}\frac{\phi(\psi(\iota)-\psi(\eta))}{\psi(\iota)-\psi(\eta)}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\psi(\iota)-\psi(\eta))^{\mu};\widetilde{p}\right)\psi\prime(\eta)\varphi(\eta)\mathrm{d}\eta,\\ \left({}_{\psi}F^{\phi,\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,b-}\varphi\right)(\iota;\widetilde{p}) &= \int_{\iota}^{b}\frac{\phi(\psi(\eta)-\psi(\iota))}{\psi(\eta)-\psi(\iota)}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\psi(\eta)-\psi(\iota))^{\mu};\widetilde{p}\right)\psi\prime(\eta)\varphi(\eta)\mathrm{d}\eta.\end{aligned} \tag{10}$$

Definition 9 (see [23]). Let φ, ψ: $[a,b] \longrightarrow \mathbb{R}$ with $0<a<b$, be the functions such that φ be positive and $\varphi \in L_1[a,b]$ and ψ be differentiable and strictly increasing. Let $\gamma, c, w, \alpha, l, \in \mathbb{C}, \mathfrak{R}(l), \mathfrak{R}(\alpha) > 0, \mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0$, $\mu,\delta>0, \widetilde{p}\geq 0$, and $0<\nu\leq\mu+\delta$. In that case, for $\iota \in [a,b]$, the united operators are described by

$$\begin{aligned}\left({}_{\psi}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,a+}\varphi\right)(\iota;\widetilde{p}) &= \int_a^{\iota}(\psi(\iota)-\psi(\eta))^{\alpha-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\psi(\iota)-\psi(\eta))^{\mu};\widetilde{p}\right)\psi'(\eta)\varphi(\eta)\mathrm{d}\eta,\\ \left({}_{\psi}F^{\phi,\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,b-}\varphi\right)(\iota;\widetilde{p}) &= \int_{\iota}^{b}(\psi(\eta)-\psi(\iota))^{\alpha-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\psi(\eta)-\psi(\iota))^{\mu};\widetilde{p}\right)\psi'(\eta)\varphi(\eta)\mathrm{d}\eta.\end{aligned} \tag{11}$$

Recently, Yue et al. defined generalized k-fractional operators including a further extension of Mittag–Leffler function in [24] as noted below:

Definition 10. Let φ, ψ: $[a,b] \longrightarrow \mathbb{R}$ with $0<a<b$; be the functions such that φ be positive and $\varphi \in L_1[a,b]$ and ψ be differentiable and strictly increasing. Let $\gamma, c, w, \alpha, l, \in \mathbb{R}$ and $\alpha>k$, $l,\alpha>0, c>\gamma>0$ with $0<\nu\leq\delta+\mu$, $\widetilde{p}\geq 0$ and $\mu,\delta>0$. In that case, for $\iota \in [a,b]$, the right-left generalized k-fractional operators $({}^{k}_{\psi}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,a+}\varphi)$ and $({}^{k}_{\psi}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,b-}\varphi)$ are defined by

$$\left({}^{k}_{\psi}F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,a+}\varphi\right)(\iota;\widetilde{p}) = \int_a^{\iota}(\psi(\iota)-\psi(\eta))^{(\alpha/k)-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\psi(\iota)-\psi(\eta))^{\mu};\widetilde{p}\right)\psi\prime(\eta)\varphi(\eta)\mathrm{d}\eta, \tag{12}$$

$$\left(g_k F^{\gamma,\delta,\nu,c}_{\mu,\alpha,l,w,b-}\varphi\right)(\iota;\widetilde{p}) = \int_{\iota}^{b}(\psi(\eta)-\psi(\iota))^{(\alpha/k)-1}E^{\gamma,\delta,\nu,c}_{\mu,\alpha,l}\left(w(\psi(\eta)-\psi(\iota))^{\mu};\widetilde{p}\right)\psi\prime(\eta)\varphi(\eta)\mathrm{d}\eta. \tag{13}$$

The following inequality is the admitted Hadamard inequality.

Theorem 1. *Let φ: $[a,b] \longrightarrow \mathbb{R}$ with $a<b$, be a convex function. In that case, the below inequality occurs:*

$$\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \varphi(\iota)d\iota \leq \frac{\varphi(a)+\varphi(b)}{2}. \tag{14}$$

Theorem 2. *Let φ: $[a,b] \longrightarrow \mathbb{R}$ be a convex and ψ: $[a,b] \longrightarrow \mathbb{R}$ be nonnegative and symmetric in respect of $((a+b)/2)$ and integrable. In that case, the below inequality occurs:*

$$\varphi\left(\frac{a+b}{2}\right)\int_a^b \psi(\iota)d\iota \leq \int_a^b \varphi(\iota)\psi(\iota)d\iota \leq \frac{\varphi(a)+\varphi(b)}{2}\int_a^b \psi(\iota)d\iota. \tag{15}$$

This inequality in [25] presented by Fejér is known as a weighted type of Hadamard's inequality.

Many authors have been established several refinements and extensions of the Hadamard and the Fejér–Hadamard inequalities for various fractional integral operators (for details see, [2, 7, 11, 16, 17, 19–21, 26–34] and references therein). This article aims to derive the Hadamard and Fejér–Hadamard inequalities about generalized k-fractional integrals involving Mittag–Leffler functions via $(\alpha, h-m)$-convex functions. In the upcoming section, we will utilize k-fractional integral operators and $(\alpha, h-m)$-convexity to prove the two versions of the Hadamard inequality and the Fejér–Hadamard inequality.

3. The k-Fractional Inequalities of Hadamard and Fejér–Hadamard Type

In this section, we first describe the below generalized k-fractional Hadamard's inequality.

Theorem 3. *Let $h: J \longrightarrow \mathbb{R}$ is nonnegative, nonzero and integrable function and $\varphi, \psi: [a,b] \longrightarrow \mathbb{R}$, $0 \le a < mb$, be the functions such that $\varphi \in L_1[a,b]$ and φ be positive and ψ be differentiable and strictly increasing. If φ is $(\alpha, h-m)$-convex, the below inequalities for k-fractional operators (12) and (13) occur:*

$$
\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}1\right)(m\psi(b);\widetilde{p})\\
&\le h\left(\frac{1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)(m\psi(b);\widetilde{p}) + m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w}m^{\mu},b-}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)\left(\frac{\psi(a)}{m};\widetilde{p}\right)\\
&\le \left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a)) + m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\int_0^1 \eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})h(\eta^{\alpha})\mathrm{d}\eta\\
&+ m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b)) + m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^{2}}\right)\right]\int_0^1 \eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})h(1-\eta^{\alpha})\mathrm{d}\eta,
\end{aligned}
\tag{16}
$$

where $\overline{w} = (w/(m\psi(b)-\psi(a))^{\mu})$ for all $\eta \in [a,b]$.

Proof. Since φ is $(\alpha, h-m)$-convex on $[a,b]$, for all $\iota, \kappa \in [a,b]$, we have

$$
\varphi\left(\frac{\psi(\iota)+m\psi(\kappa)}{2}\right) \le h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(\iota)) + mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(\kappa)). \tag{17}
$$

Setting $\psi(\iota) = \eta\psi(a) + m(1-\eta)\psi(b)$ and $\psi(\kappa) = (\psi(a)/m)(1-\eta) + \eta\psi(b)$ in above inequality, we have

$$
\begin{aligned}
\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right) &\le h\left(\frac{1}{2^{\alpha}}\right)\varphi(\eta\psi(a)+m(1-\eta)\psi(b))\\
&+ mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m}(1-\eta)+\eta g(b)\right).
\end{aligned}
\tag{18}
$$

Multiplying both sides of (18) by $\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})$, then integrating over $[0,1]$, we have

$$
\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\int_0^1 \eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})d\eta\\
&\le h\left(\frac{1}{2^{\alpha}}\right)\int_0^1 \eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\varphi(\eta\psi(a)+m(1-\eta)\psi(b))\mathrm{d}\eta\\
&+ mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1 \eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\varphi\left(\frac{\psi(a)}{m}(1-\eta)+\eta\psi(b)\right)\mathrm{d}\eta.
\end{aligned}
\tag{19}
$$

By specifying $\psi(\iota) = \eta\psi(a) + m(1-\eta)\psi(b)$ and $\psi(\kappa) = (\psi(a)/m)(1-\eta) + \eta\psi(b)$ in (19), we have

$$\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\int_a^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(m\psi(b)-\psi(\iota))^{\mu};\widetilde{p}\right)\psi\prime(\iota)\mathrm{d}\iota\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_a^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(m\psi(b)-\psi(\iota))^{\mu};\widetilde{p}\right)\varphi(\psi(\iota))\psi\prime(\iota)\mathrm{d}\iota\\
&+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{\psi^{-1}(\psi(a)/m)}^{b}\left(\psi(\kappa)-\frac{\psi(a)}{m}\right)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}m^{\mu}\left(\psi(\kappa)-\frac{\psi(a)}{m}\right)^{\mu};\widetilde{p}\right)\varphi(\psi(\kappa))\psi\prime(\kappa)\mathrm{d}\kappa.
\end{aligned}\tag{20}$$

By usage k-fractional operators (12) and (13), the first side of (16) is achieved.

To evidence the second side of (16), once again $(\alpha, h-m)$-convexity of φ over $[a,b]$, for $\eta \in [0,1]$, we achieve

$$\begin{aligned}
&h\left(\frac{1}{2^{\alpha}}\right)\varphi(\eta\psi(a)+m(1-\eta)\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left((1-\eta)\frac{\psi(a)}{m}+\eta\psi(b)\right)\\
&\leq h(\eta^{\alpha})\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\\
&+mh(1-\eta^{\alpha})\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^2}\right)\right].
\end{aligned}\tag{21}$$

Multiplying both sides of (21) by $\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})$, next integrating over $[0,1]$, we achieve

$$\begin{aligned}
&h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\varphi(\eta\psi(a)+m(1-\eta)\psi(b))d\eta\\
&+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\varphi\left((1-\eta)\frac{\psi(a)}{m}+\eta\psi(b)\right)d\eta\\
&\leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})h(\eta^{\alpha})d\eta\\
&+m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^2}\right)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})h(1-\eta^{\alpha})d\eta.
\end{aligned}\tag{22}$$

Setting $\psi(\iota) = \eta\psi(a) + m(1-\eta)\psi(b)$ and $\psi(\kappa) = (1-\eta)(\psi(a)/m) + \eta\psi(b)$ in (22), in that case by utilizing k-fractional operators (12) and (13), the second side of (16) is achieved. □

Corollary 1. *By usage (16), anymore k-fractional inequalities are offered as noted below:*

(i) By choosing $\psi = I$ and $\widetilde{p} = w = 0$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}\mathrm{d}\iota\\
&\leq h\left(\frac{1}{2^\alpha}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}\varphi(\iota)\mathrm{d}\iota+m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_{\frac{a}{m}}^{b}\left(\kappa-\frac{a}{m}\right)^{(\tau/k)-1}\varphi(\kappa)\mathrm{d}\kappa\\
&\leq\left[h\left(\frac{1}{2^\alpha}\right)\varphi(a)+m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi(b)\right]\int_0^1\eta^{(\tau/k)-1}h(\eta^\alpha)\mathrm{d}\eta\\
&\quad+m\left[h\left(\frac{1}{2^\alpha}\right)\varphi(b)+m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi\left(\frac{a}{m^2}\right)\right]\int_0^1\eta^{(\tau/k)-1}h(1-\eta^\alpha)\mathrm{d}\eta.
\end{aligned}\tag{23}$$

(ii) By choosing $\psi = I$ and $\widetilde{p} = 0$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(mb-\iota)^\mu\right)\mathrm{d}\iota\\
&\leq h\left(\frac{1}{2^\alpha}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(mb-\iota)^\mu\right)\varphi(\iota)\mathrm{d}\iota\\
&\quad+m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_{(a/m)}^{b}\left(\kappa-\frac{a}{m}\right)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}m^\mu\left(\kappa-\frac{a}{m}\right)^\mu\right)\varphi(\kappa)\mathrm{d}\kappa\\
&\leq\left[h\left(\frac{1}{2^\alpha}\right)\varphi(a)+m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi(b)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^\mu)h(\eta^\alpha)\mathrm{d}\eta\\
&\quad+m\left[h\left(\frac{1}{2^\alpha}\right)\varphi(b)+m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi\left(\frac{a}{m^2}\right)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^\mu)h(1-\eta^\alpha)\mathrm{d}\eta.
\end{aligned}\tag{24}$$

(iii) By setting $m = 1$ and $\psi = I$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{a+b}{2}\right)\int_a^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(b-\iota)^\mu;\widetilde{p}\right)\mathrm{d}\iota\\
&\leq h\left(\frac{1}{2^\alpha}\right)\int_a^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(b-\iota)^\mu;\widetilde{p}\right)\varphi(\iota)\mathrm{d}\iota\\
&\quad+h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_a^{b}(\kappa-a)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(\kappa-a)^\mu;\widetilde{p}\right)\varphi(\kappa)\mathrm{d}\kappa\\
&\leq\left[h\left(\frac{1}{2^\alpha}\right)f(a)+h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi(b)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^\mu;\widetilde{p})h(\eta^\alpha)\mathrm{d}\eta\\
&\quad+\left[h\left(\frac{1}{2^\alpha}\right)\varphi(b)+h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi(a)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^\mu;\widetilde{p})h(1-\eta^\alpha)\mathrm{d}\eta.
\end{aligned}\tag{25}$$

(iv) By choosing $h(\eta) = \eta$ and $\widetilde{p} = w = 0$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\int_a^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}\psi\prime(\iota)\mathrm{d}\iota\\
&\leq\left(\frac{1}{2^\alpha}\right)\int_a^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}\varphi(\psi(\iota))\psi\prime(\iota)\mathrm{d}\iota\\
&\quad+m^{(\tau/k)+1}\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_{\psi^{-1}(\psi(a)/m)}^{b}\left(\psi(\kappa)-\frac{\psi(a)}{m}\right)^{(\tau/k)-1}\varphi(\psi(\kappa))\psi\prime(\kappa)\mathrm{d}\kappa\\
&\leq\left[\left(\frac{1}{2^\alpha}\right)\varphi(\psi(a))+m^{(\tau/k)+1}\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi(\psi(b))\right]\left(\frac{k}{\tau+\alpha k}\right)\\
&\quad+m\left[\left(\frac{1}{2^\alpha}\right)\varphi(\psi(b))+m^{(\tau/k)+1}\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi\left(\frac{\psi(a)}{m^2}\right)\right]\left(\frac{\alpha k^2}{\tau(\tau+\alpha k)}\right).
\end{aligned}\tag{26}$$

(v) By setting $\alpha=1$ and $\psi=I$, we get

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{w}(mb-\iota)^\mu;\widetilde{p})\mathrm{d}\iota\\
&\leq h\left(\frac{1}{2}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{w}(mb-\iota)^\mu;\widetilde{p})\varphi(\iota)\mathrm{d}\iota\\
&\quad+m^{(\tau/k)+1}h\left(\frac{1}{2}\right)\int_{(a/m)}^{b}\left(\kappa-\frac{a}{m}\right)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}m^\mu\left(\kappa-\frac{a}{m}\right)^\mu;\widetilde{p}\right)\varphi(\kappa)\mathrm{d}\kappa\\
&\leq\left[h\left(\frac{1}{2}\right)\varphi(a)+m^{(\tau/k)+1}h\left(\frac{1}{2}\right)\varphi(b)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^\mu;\widetilde{p})h(\eta)\mathrm{d}\eta\\
&\quad+m\left[h\left(\frac{1}{2}\right)\varphi(b)+m^{(\tau/k)+1}h\left(\frac{1}{2}\right)\varphi\left(\frac{a}{m^2}\right)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^\mu;\widetilde{p})h(1-\eta)\mathrm{d}\eta.
\end{aligned}\tag{27}$$

(vi) By setting $\alpha=m=1$, $h(\eta)=\eta$ and $\psi=I$, we get

$$\begin{aligned}
&\varphi\left(\frac{a+b}{2}\right)\int_a^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{w}(b-\iota)^\mu;\widetilde{p})\mathrm{d}\iota\\
&\leq\frac{1}{2}\left[\int_a^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{w}(b-\iota)^\mu;\widetilde{p})\varphi(\iota)\mathrm{d}\iota+\int_a^{b}(\kappa-a)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{w}(\kappa-a)^\mu;\widetilde{p})\varphi(\kappa)\mathrm{d}\kappa\right]\\
&\leq\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\left[\int_0^1\eta^{(\tau/k)}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^\mu;\widetilde{p})\mathrm{d}\eta+\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^\mu;\widetilde{p})(1-\eta)\mathrm{d}\eta\right].
\end{aligned}\tag{28}$$

Remark 2. The above k-fractional inequalities are farther in line with already known conclusions as noted below: (i) By choosing $k=1$ in Corollary 1 *(v), an inequality for extended generalized fractional integrals is acquired. (ii) By choosing $k=1$ and $\widetilde{p}=0$ in* Corollary 1 *(v), Theorem 2.1 of* [28] *is acquired. (iii) By choosing $m=1$, and $h(\eta)=\eta$ in* Corollary 1 *(v), Theorem 2.1 of* [27] *is acquired. (iv) By choosing $\widetilde{p}=w=0$ in* Corollary 1 *(v), Theorem 2.1 of* [20] *is acquired.*

Remark 3. (i) By choosing $k=1$ and $\widetilde{p}=0$ in Remark 1 *(iii), an inequality for extended generalized fractional integrals is acquired. (ii) By choosing $k=1$ and $\widetilde{p}=w=0$ in* Remark 1 *(iii),* Theorem 2 *of* [29] *is acquired. (iii) By choosing $k=1$ in* Remark 1 *(iv), Corollary 2.2 of* [20] *is acquired.*

The below lemma is beneficial to offer the Fejér–Hadamard's inequality for generalized k-fractional integrals.

Lemma 1. *Let $\varphi,\psi\colon [a,b]\longrightarrow\mathbb{R}$ with $0\le a<mb$, be the functions such that $\varphi\in L_1[a,b]$ and φ positive and ψ be differentiable and strictly increasing. If $\varphi(\psi(\iota))=\varphi(\psi(a)+m\psi(b)-\psi(\iota))$, in that case for generalized k-fractional operators (11) and (12), we get*

$$\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)(m\psi(b);\widetilde{p})=\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w}m^{\mu},b-}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)\left(\frac{\psi(a)}{m};\widetilde{p}\right)$$
$$=\frac{1}{2}\left[\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}g\right)(m\psi(b);\widetilde{p})+\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w}m^{\mu},b-}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)\left(\frac{\psi(a)}{m};\widetilde{p}\right)\right], \tag{29}$$

for all $\eta\in[a,b]$.

Proof. By description of generalized k-fractional operators (12) and (13), we get

$$\begin{aligned}\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)(m\psi(b);\widetilde{p})&=\int_a^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(m\psi(b)-\psi(\iota))^{\mu};\widetilde{p}\right)(\varphi^{\circ}\psi)(\iota)\psi\prime(\iota)\mathrm{d}\iota\\&=\int_a^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(m\psi(b)-\psi(\iota))^{\mu};\widetilde{p}\right)\varphi(\psi(\iota))\psi\prime(\iota)\mathrm{d}\iota\\&=\int_a^{g^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(m\psi(b)-\psi(\iota))^{\mu};\widetilde{p}\right)\varphi(\psi(a)+m\psi(b)-\psi(\iota))\psi\prime(\iota)\mathrm{d}\iota.\end{aligned} \tag{30}$$

Setting $\psi(\eta)=\psi(a)+m\psi(b)-\psi(\iota)$ in the above equation and using $\varphi(\psi(\iota))=\varphi(\psi(a)+m\psi(b)-\psi(\iota))$, we have

$$\begin{aligned}\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)(m\psi(b);\widetilde{p})&=\int_{\psi^{-1}(\psi(a)/m)}^{b}(m\psi(\eta)-\psi(a))^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(m\psi(\eta)-\psi(a))^{\mu};\widetilde{p}\right)\varphi(\psi(\eta))\psi\prime(\eta)\mathrm{d}\eta\\&=\int_{\psi^{-1}(\psi(a)/m)}^{b}(m\psi(\eta)-\psi(a))^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(m\psi(\eta)-\psi(a))^{\mu};\widetilde{p}\right)(\varphi^{\circ}\psi)(\eta)\psi\prime(\eta)\mathrm{d}\eta.\end{aligned} \tag{31}$$

This implies

$$\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)(m\psi(b);\widetilde{p})=\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w}m^{\mu},b-}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)\left(\frac{\psi(a)}{m};\widetilde{p}\right). \tag{32}$$

By adding $({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi)(m\psi(b);\widetilde{p})$ on both sides of (32), we have

$$2\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)(m\psi(b);\widetilde{p})=\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w}m^{\mu},b-}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)\left(\frac{\psi(a)}{m};\widetilde{p}\right)+\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}\psi\right)(m\psi(b);\widetilde{p}). \tag{33}$$

From equations (32) and (33), the result can be obtained. □

The first type of Fejér–Hadamard inequality is endued through generalized k-fractional integrals as noted below:

Theorem 4. *Let h: $J \longrightarrow \mathbb{R}$ be nonnegative, nonzero, and integrable function and φ, ψ: $[a, b] \longrightarrow \mathbb{R}$, $0 \leq a < mb$, be the functions such that $\varphi \in L_1[a, b]$ and φ be positive and ψ be differentiable and strictly increasing, r is a nonnegative and integrable function. If φ is $(\alpha, h-m)$-convex and $\varphi(\psi(\iota)) = \varphi(\psi(a) + m\psi(b) - \psi(\iota))$, in that case the below inequalities for generalized k-fractional operators (12) and (13) occur:*

$$\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\left[\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}r^{\circ}\psi\right)(m\psi(b);\widetilde{p})+\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w}m^{\mu},b-}^{\gamma,\delta,\nu,c}r^{\circ}\psi\right)\left(\frac{\psi(a)}{m};\widetilde{p}\right)\right]\\
&\leq 2h\left(\frac{1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}r^{\circ}\psi\right)(m\psi(b);\widetilde{p})\\
&+2m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w}m^{\mu},b-}^{\gamma,\delta,\nu,c}\varphi^{\circ}r^{\circ}\psi\right)\left(\frac{\psi(a)}{m};\widetilde{p}\right)\\
&\leq 2\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\\
&\times\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta\psi(a)+m(1-\eta)\psi(b))h(\eta^{\alpha})d\eta\\
&+2m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^{2}}\right)\right]\\
&\times\int_{0}^{1}t^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta\psi(a)+m(1-\eta)\psi(b))h(1-\eta^{\alpha})d\eta,
\end{aligned}\tag{34}$$

where $\overline{w} = (w/(m\psi(b) - \psi(a))^{\mu})$ for all $\eta \in [a, b]$.

Proof. We demonstrate the claim as follows:

Multiplying both sides of (18) by $\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta\psi(a)+m(1-\eta)\psi(b))$ and then integrating over $[0,1]$, we have

$$\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta\psi(a)+m(1-\eta)\psi(b))\mathrm{d}\eta\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\varphi(\eta\psi(a)+m(1-\eta)\psi(b))r(\eta\psi(a)+m(1-\eta)\psi(b))\mathrm{d}\eta\\
&+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\varphi\left((1-\eta)\frac{\psi(a)}{m}+\eta\psi(b)\right)r(\eta\psi(a)+m(1-\eta)\psi(b))\mathrm{d}\eta.
\end{aligned}\tag{35}$$

By specifying $\psi(\iota) = \eta\psi(a) + m(1-\eta)\psi(b)$ and $\psi(\kappa) = (1-\eta)(\psi(a)/m) + \eta\psi(b)$, that is $\psi(a) + m\psi(b) - \psi(\iota) = (1-\eta)\psi(a) + m\eta\psi(b)$, in (35), then using $\varphi(\psi(\iota)) = \varphi(\psi(a) + m\psi(b) - \psi(\iota))$, we have

$$\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\int_{a}^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(m\psi(b)-\psi(\iota))^{\mu};\widetilde{p}\right)(r^{\circ}\psi)(\iota)\psi\prime(\iota)\mathrm{d}\iota\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_{a}^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(m\psi(b)-\psi(\iota))^{\mu};\widetilde{p}\right)(\varphi^{\circ}\psi)(\iota)(r^{\circ}\psi)(\iota)\psi\prime(\iota)\mathrm{d}\iota\\
&+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{\psi^{-1}(\psi(a)/m)}^{b}(\psi(\iota)-\frac{\psi(a)}{m})^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}m^{\mu}\left(\psi(\iota)-\frac{\psi(a)}{m}\right)^{\mu};\widetilde{p}\right)(\varphi^{\circ}\psi)(\iota)(r^{\circ}\psi)(\iota)\psi\prime(\iota)\mathrm{d}\iota.
\end{aligned}\tag{36}$$

This implies

$$\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}r^{\circ}\psi\right)(m\psi(b);\widetilde{p})\leq h\left(\frac{1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}r^{\circ}\psi\right)(m\psi(b);\widetilde{p}) + m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w}m^{\mu},b-}^{\gamma,\delta,\nu,c}\varphi^{\circ}r^{\circ}\psi\right)\left(\frac{\psi(a)}{m};\widetilde{p}\right). \tag{37}$$

Using Lemma 1 in the above inequality, we have the first side of (34).

To demonstrate second side of (34), multiplying both parts of (21) by $2\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\psi(a)+m(1-\eta)\psi(b))$ and then integrating over $[0,1]$, we have

$$\begin{aligned}
&2h\left(\frac{1}{2^{\alpha}}\right)\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta\psi(a)+m(1-\eta)\psi(b))\varphi(\eta\psi(a)+m(1-\eta)\psi(b))d\eta\\
&\quad+2m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta\psi(a)+m(1-\eta)\psi(b))\varphi\left((1-\eta)\frac{\psi(a)}{m}+\psi(b)\right)\mathrm{d}\eta\\
&\leq 2\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\\
&\quad\times\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta\psi(a)+m(1-\eta)\psi(b))h(\eta^{\alpha})\mathrm{d}\eta\\
&\quad+2m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^{2}}\right)\right]\\
&\quad\times\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta\psi(a)+m(1-\eta)\psi(b))h(1-\eta^{\alpha})\mathrm{d}\eta.
\end{aligned}\tag{38}$$

Setting $\psi(\iota)=\eta\psi(a)+m(1-\eta)\psi(b)$ and $\psi(\kappa)=(1-\eta)(\psi(a)/m)+\eta\psi(b)$, then using $\varphi(\psi(\iota))=\varphi(\psi(a)+m\psi(b)-\psi(\iota))$ in (38), we have

$$\begin{aligned}
&2h\left(\frac{1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w},a+}^{\gamma,\delta,\nu,c}\varphi^{\circ}r^{\circ}\psi\right)(m\psi(b);\widetilde{p})\\
&\quad+2m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{w}m^{\mu},b-}^{\gamma,\delta,\nu,c}\varphi^{\circ}r^{\circ}\psi\right)\left(\frac{\psi(a)}{m};\widetilde{p}\right)\\
&\leq 2\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\\
&\quad\times\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta\psi(a)+m(1-\eta)\psi(b))h(\eta^{\alpha})\mathrm{d}\eta\\
&\quad+2m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^{2}}\right)\right]\\
&\quad\times\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta\psi(a)+m(1-\eta)\psi(b))h(1-\eta^{\alpha})\mathrm{d}\eta.
\end{aligned}\tag{39}$$

By usage Lemma 1 in the above inequality, we have the second side of (34). □

Corollary 2. *By using (34), some more k-fractional inequalities are offered as noted below:*

(i) By choosing $\psi = I$ and $\widetilde{p} = w = 0$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}r(\iota)\mathrm{d}\iota \\
&\leq 2h\left(\frac{1}{2^\alpha}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}(\varphi\circ r)(\iota)\mathrm{d}\iota + 2m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_{(a/m)}^{b}\left(\kappa-\frac{a}{m}\right)^{(\tau/k)-1}(\varphi\circ r)(\kappa)\mathrm{d}\kappa \\
&\leq 2\left[h\left(\frac{1}{2^\alpha}\right)\varphi(a) + m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi(b)\right]\int_0^1\eta^{(\tau/k)-1}r(\eta a+m(1-\eta)b)h(\eta^\alpha)\mathrm{d}\eta \\
&\quad + 2m\left[h\left(\frac{1}{2^\alpha}\right)\varphi(b) + m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi\left(\frac{a}{m^2}\right)\right]\int_0^1\eta^{(\tau/k)-1}r(\eta a+m(1-\eta)b)h(1-\eta^\alpha)\mathrm{d}\eta.
\end{aligned} \tag{40}$$

(ii) By choosing $\widetilde{p} = 0$ and $\psi = I$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(mb-\iota)^\mu\right)r(\iota)\mathrm{d}\iota \\
&\leq 2h\left(\frac{1}{2^\alpha}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(mb-\iota)^\mu\right)(\varphi\circ r)(\iota)\mathrm{d}\iota \\
&\quad + 2m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_{(a/m)}^{b}\left(\kappa-\frac{a}{m}\right)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}m^\mu\left(\kappa-\frac{a}{m}\right)^\mu\right)(\varphi\circ r)(\kappa)\mathrm{d}\kappa \\
&\leq 2\left[h\left(\frac{1}{2^\alpha}\right)\varphi(a) + m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi(b)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^\mu\right)r(\eta a+m(1-\eta)b)h(\eta^\alpha)\mathrm{d}\eta \\
&\quad + 2m\left[h\left(\frac{1}{2^\alpha}\right)\varphi(b) + m^{(\tau/k)+1}h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi\left(\frac{a}{m^2}\right)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^\mu\right)r(\eta a+m(1-\eta)b)h(1-\eta^\alpha)\mathrm{d}\eta.
\end{aligned} \tag{41}$$

(iii) By choosing $m = 1$ and $\psi = I$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{a+b}{2}\right)\int_a^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(b-\iota)^\mu;\widetilde{p}\right)r(\iota)\mathrm{d}\iota \\
&\leq 2h\left(\frac{1}{2^\alpha}\right)\int_a^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(b-\iota)^\mu;\widetilde{p}\right)(\varphi\circ r)(\iota)\mathrm{d}\iota \\
&\quad + 2h\left(\frac{2^\alpha-1}{2^\alpha}\right)\int_a^{b}(\kappa-a)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(\kappa-a)^\mu;\widetilde{p}\right)(\varphi\circ r)(\kappa)\mathrm{d}\kappa \\
&\leq 2\left[h\left(\frac{1}{2^\alpha}\right)\varphi(a) + h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi(b)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^\mu;\widetilde{p}\right)r(\eta a+(1-\eta)b)h(\eta^\alpha)\mathrm{d}\eta \\
&\quad + 2\left[h\left(\frac{1}{2^\alpha}\right)\varphi(b) + h\left(\frac{2^\alpha-1}{2^\alpha}\right)\varphi(a)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^\mu;\widetilde{p}\right)r(\eta a+(1-\eta)b)h(1-\eta^\alpha)\mathrm{d}\eta.
\end{aligned} \tag{42}$$

(iv) *By choosing* $\widetilde{p} = w = 0$ *and* $h(\eta) = \eta$, *we obtain*

$$\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\int_a^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}(r\circ\psi)(\iota)\psi\prime(\iota)\mathrm{d}\iota\\
&\leq\left(\frac{1}{2^{\alpha-1}}\right)\int_a^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}(\varphi\circ r\circ\psi)(\iota)\psi\prime(\iota)\mathrm{d}\iota\\
&+m^{(\tau/k)+1}\left(\frac{2^{\alpha}-1}{2^{\alpha-1}}\right)\int_{\psi^{-1}\left(\frac{\psi(a)}{m}\right)}^{b}(\psi(\kappa)-\frac{\psi(a)}{m})^{(\tau/k)-1}(\varphi\circ r\circ\psi)(\kappa)\psi\prime(\kappa)\mathrm{d}\kappa\\
&\leq 2\left[\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\int_0^1\eta^{(\tau/k)-1}r(\eta\psi(a)+m(1-\eta)\psi(b))(\eta^{\alpha})\mathrm{d}\eta\\
&+2m\left[\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^2}\right)\right]\int_0^1\eta^{(\tau/k)-1}r(\eta\psi(a)+m(1-\eta)\psi(b))(1-\eta^{\alpha})\mathrm{d}\eta.
\end{aligned}\tag{43}$$

(v) *By choosing* $\alpha = 1$ *and* $\psi = I$, *we obtain*

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{w}(mb-\iota)^{\mu};\widetilde{p})r(\iota)\mathrm{d}\iota\\
&\leq 2h\left(\frac{1}{2}\right)\int_a^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{w}(mb-\iota)^{\mu};\widetilde{p})(\varphi\circ r)(\iota)\mathrm{d}\iota\\
&+2m^{(\tau/k)+1}h\left(\frac{1}{2}\right)\int_{(a/m)}^{b}\left(\kappa-\frac{a}{m}\right)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}m^{\mu}\left(\kappa-\frac{a}{m}\right)^{\mu};\widetilde{p}\right)(\varphi\circ r)(\kappa)\mathrm{d}\kappa\\
&\leq 2\left[h\left(\frac{1}{2}\right)\varphi(a)+m^{(\tau/k)+1}h\left(\frac{1}{2}\right)\varphi(b)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta a+m(1-\eta)b)h(\eta)\mathrm{d}\eta\\
&+2m\left[h\left(\frac{1}{2}\right)\varphi(b)+m^{(\tau/k)+1}h\left(\frac{1}{2}\right)\varphi\left(\frac{a}{m^2}\right)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta a+m(1-\eta)b)h(1-\eta)\mathrm{d}\eta.
\end{aligned}\tag{44}$$

(vi) *By choosing* $\alpha = m = 1$, $h(\eta) = \eta$ *and, we obtain*

$$\begin{aligned}
&\varphi\left(\frac{a+b}{2}\right)\int_a^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{w}(b-\iota)^{\mu};\widetilde{p})r(\iota)\mathrm{d}\iota\\
&\leq\left[\int_a^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{w}(b-\iota)^{\mu};\widetilde{p})(\varphi\circ r)(\iota)\mathrm{d}\iota\right.\\
&\left.+\int_a^{b}(\kappa-a)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{w}(\kappa-a)^{\mu};\widetilde{p})(\varphi\circ r)(\kappa)\mathrm{d}\kappa\right]\\
&\leq(\varphi(a)+\varphi(b))\left[\int_0^1\eta^{(\tau/k)}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta a+(1-\eta)b)\mathrm{d}\eta\right.\\
&\left.+\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r(\eta a+(1-\eta)b)(1-\eta)\mathrm{d}\eta\right].
\end{aligned}\tag{45}$$

(vii) By choosing $\alpha = k = 1$ and $\psi = I$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_a^{mb}(mb-\iota)^{\tau-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(mb-\iota)^{\mu};\tilde{p}\right)r(\iota)\mathrm{d}\iota\\
&\leq 2h\left(\frac{1}{2}\right)\int_a^{mb}(mb-\iota)^{\tau-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}(mb-\iota)^{\mu};\tilde{p}\right)(\varphi\circ r)(\iota)\mathrm{d}\iota\\
&\quad+2m^{\tau+1}h\left(\frac{1}{2}\right)\int_{(a/m)}^{b}\left(\kappa-\frac{a}{m}\right)^{\tau-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{w}m^{\mu}\left(\kappa-\frac{a}{m}\right)^{\mu};\tilde{p}\right)(\varphi\circ r)(\kappa)\mathrm{d}\kappa\\
&\leq 2\left[h\left(\frac{1}{2}\right)\varphi(a)+m^{\tau+1}h\left(\frac{1}{2}\right)\varphi(b)\right]\int_0^1\eta^{\tau-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu};\tilde{p}\right)r(\eta a+m(1-\eta)b)h(\eta)\mathrm{d}\eta\\
&\quad+2m\left[h\left(\frac{1}{2}\right)\varphi(b)+m^{\tau+1}h\left(\frac{1}{2}\right)\varphi\left(\frac{a}{m^2}\right)\right]\int_0^1\eta^{\tau-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu};\tilde{p}\right)r(\eta a+m(1-\eta)b)h(1-\eta)\mathrm{d}\eta.
\end{aligned} \tag{46}$$

Remark 4. The above k-fractional inequalities are farther in line with foreknown conclusions as noted below: (i) By choosing $k = 1$ in Corollary 2 *(vi), Theorem 2.2 of* [27] *is acquired. (ii) By choosing $\tilde{p} = 0$ in in* Corollary 2 *(vii), Theorem 2.5 of* [28] *is acquired. (iii) By choosing $k = I$, $\tilde{p} = w = 0$ and $h(\eta) = \eta$ in* Corollary 2 *(v), an inequality for m-convex functions via Riemann–Liouville integrals is acquired. (iv) By choosing $k = 1$ and $\tilde{p} = 0$ in in* Corollary 2 *(vi), an inequality for extended generalized fractional integrals is acquired. (v) By choosing $k = 1$ and $\tilde{p} = w = 0$ in in* Corollary 2 *(vi), Theorem 4* of [26] *is acquired. (vi) By choosing $h(\eta) = \eta$ in in Corollary* 3.2 *(vii), Theorem 3.1* of [27] *is acquired.*

In the subsequent theorem, we offer another type of Hadamard's inequality.

Theorem 5. *Let $h\colon J \longrightarrow \mathbb{R}$ is nonnegative, nonzero and integrable function and $\varphi, \psi\colon [a,b] \longrightarrow \mathbb{R}$, $0 \leq a < mb$, be the functions such that $\varphi \in L_1[a,b]$ and φ be positive and ψ be differentiable and strictly increasing. If φ is $(\alpha, h-m)$-convex, in that case for generalized k-fractional operators (12) and (13), we acquire*

$$\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{\overline{w}},\psi^{-1}(m\psi(b)+\psi(a)/2)+}^{\gamma,\delta,\nu,c}1\right)(m\psi(b);\tilde{p})\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{\overline{w}},\psi^{-1}(m\psi(b)+\psi(a)/2)+}^{\gamma,\delta,\nu,c}\varphi\circ\psi\right)(m\psi(b);\tilde{p})\\
&\quad+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{\overline{w}}m^{\mu},\psi^{-1}(m\psi(b)+\psi(a)/2m)-}^{\gamma,\delta,\nu,c}\varphi\circ\psi\right)\left(\frac{\psi(a)}{m};\tilde{p}\right)\\
&\leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu};\tilde{p}\right)h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)d\eta\\
&\quad+m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^2}\right)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu};\tilde{p}\right)h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta,
\end{aligned} \tag{47}$$

where $\overline{\overline{w}} = (2^{\mu}w/(m\psi(b) - \psi(a))^{\mu})$ for all $\eta \in [a,b]$.

Proof. Setting $\psi(\iota) = (\eta/2)\psi(a) + m(2-\eta/2)\psi(b)$ and $\psi(\kappa) = (2-\eta/2)(\psi(a)/m) + \eta/2\psi(b)$ in (3.2), we have

$$\begin{aligned}
\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)&\leq h\left(\frac{1}{2^{\alpha}}\right)\varphi\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)\\
&\quad+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\left(\frac{2-\eta}{2}\right)\frac{\psi(a)}{m}+\frac{\eta}{2}\psi(b)\right).
\end{aligned} \tag{48}$$

Multiplying both parts of (48) by $\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})$ and then integrating over $[0,1]$, we have

$$\begin{aligned}\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\mathrm{d}\eta\\ \leq h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\varphi\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)\mathrm{d}\eta\\ +mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\varphi\left(\left(\frac{2-\eta}{2}\right)\frac{\psi(a)}{m}+\frac{\eta}{2}\psi(b)\right)\mathrm{d}\eta.\end{aligned}\tag{49}$$

By taking $\psi(\iota)=(\eta/2)\psi(a)+m(2-\eta/2)\psi(b)$ and $\psi(\kappa)=(2-\eta/2)(\psi(a)/m)+\eta/2\psi(b)$ in (49), in that case by usage k-fractional operators (2.12) and (2.13), the first side of (47) is acquired.

To demonstrate the second side of (47), once again $(\alpha,h-m)$-convexity of φ over $[a,b]$, for $\eta\in[0,1]$, we get

$$\begin{aligned}h\left(\frac{1}{2^{\alpha}}\right)\varphi\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\left(\frac{2-\eta}{2}\right)\frac{\psi(a)}{m}+\frac{\eta}{2}\psi(b)\right)\\ \leq h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\\ +mh\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^2}\right)\right].\end{aligned}\tag{50}$$

Multiplying both sides of (50) by $\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})$, then integrating over $[0,1]$, we acquire

$$\begin{aligned}h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\varphi\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)\mathrm{d}\eta\\ +m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})\varphi\left(\left(\frac{2-\eta}{2}\right)\frac{\psi(a)}{m}+\frac{\eta}{2}\psi(b)\right)\mathrm{d}\eta\\ \leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta\\ +m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^2}\right)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta.\end{aligned}\tag{51}$$

Choosing $\psi(\iota)=(\eta/2)\psi(a)+m(2-\eta/2)\psi(b)$ and $\psi(\kappa)=(2-\eta/2)(\psi(a)/m)+\eta/2\psi(b)$ in (51), in that case by usage k-fractional operators (12) and (13), the second side of (47) is acquired. □

Corollary 3. *By using (47), anymore k-fractional inequalities are offered as noted below:*

(i) By choosing $\psi=I$ and $\widetilde{p}=w=0$, we have

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_{a+mb/2}^{mb}(mb-\iota)^{\frac{\tau}{k}-1}d\iota\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_{a+mb/2}^{mb}(mb-\iota)^{\frac{\tau}{k}-1}\varphi(\iota)d\iota+m^{\frac{\tau}{k}+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{a/m}^{a+mb/2}\left(\kappa-\frac{a}{m}\right)^{\frac{\tau}{k}-1}\varphi(\kappa)d\kappa\\
&\leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(a)+m^{\frac{\tau}{k}+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(b)\right]\int_0^1\eta^{\frac{\tau}{k}-1}h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)d\eta\\
&+m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(b)+m^{\frac{\tau}{k}+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{a}{m^{2}}\right)\right]\int_0^1\eta^{\frac{\tau}{k}-1}h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)d\eta\,.
\end{aligned}\tag{52}$$

(ii) By choosing $\tilde{p}=0$ *and* $\psi=I$, *we have*

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_{a+mb/2}^{mb}(mb-\iota)^{\frac{\tau}{k}-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}(mb-\iota)^{\mu}\right)d\iota\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_{a+mb/2}^{mb}(mb-\iota)^{\frac{\tau}{k}-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}(mb-\iota)^{\mu}\right)\varphi(\iota)d\iota\\
&+m^{\frac{\tau}{k}+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{\frac{a}{m}}^{\frac{a+mb}{2}}\left(\kappa-\frac{a}{m}\right)^{\frac{\tau}{k}-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}m^{\mu}\left(\kappa-\frac{a}{m}\right)^{\mu}\right)\varphi(\kappa)d\kappa\\
&\leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(a)+m^{\frac{\tau}{k}+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(b)\right]\int_0^1\eta^{\frac{\tau}{k}-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu}\right)h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)d\eta\\
&+m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(b)+m^{\frac{\tau}{k}+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{a}{m^{2}}\right)\right]\int_0^1\eta^{\frac{\tau}{k}-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu}\right)h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)d\eta\,.
\end{aligned}\tag{53}$$

(iii) By choosing $m=1$ *and* $\psi=I$, *we acquire*

$$\begin{aligned}
&\varphi\left(\frac{a+b}{2}\right)\int_{(a+b/2)}^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}(b-\iota)^{\mu};\tilde{p}\right)\mathrm{d}\iota\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_{(a+b/2)}^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}(b-\iota)^{\mu};\tilde{p}\right)\varphi(\iota)\mathrm{d}\iota\\
&+h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_a^b(\kappa-a)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}(\kappa-a)^{\mu};\tilde{p}\right)\varphi(\kappa)\mathrm{d}\kappa\\
&\leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(a)+h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(b)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu};\tilde{p}\right)h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta\\
&+\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(b)+h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(a)\right]\int_0^1\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu};\tilde{p}\right)h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)d\eta.
\end{aligned}\tag{54}$$

(iv) By choosing $\widetilde{p} = w = 0$ and $h(\eta) = \eta$, we have

$$\begin{aligned}
&\varphi\left(\frac{\psi(a) + m\psi(b)}{2}\right) \int_{\psi^{-1}(\psi(a)+m\psi(b)/2)}^{\psi^{-1}(m\psi(b))} (m\psi(b) - \psi(\iota))^{(\tau/k)-1} \psi\prime(\iota)\mathrm{d}\iota \\
&\leq \left(\frac{1}{2^{\alpha}}\right) \int_{\psi^{-1}(\psi(a)+m\psi(b)/2)}^{\psi^{-1}(m\psi(b))} (m\psi(b) - \psi(\iota))^{(\tau/k)-1} \varphi(\psi(\iota))\psi\prime(\iota)\mathrm{d}\iota \\
&\quad + m^{(\tau/k)+1}\left(\frac{2^{\alpha} - 1}{2^{\alpha}}\right) \int_{\psi^{-1}(\psi(a)/m)}^{\psi^{-1}(\psi(a)+m\psi(b)/2m)} \left(\psi(\kappa) - \frac{\psi(a)}{m}\right)^{(\tau/k)-1} \varphi(\psi(\kappa))\psi\prime(\kappa)\mathrm{d}\kappa \\
&\leq \left[\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a)) + m^{(\tau/k)+1}\left(\frac{2^{\alpha} - 1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\left(\frac{k}{2^{\alpha}(\tau + \alpha k)}\right) \\
&\quad + m\left[\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b)) + m^{(\tau/k)+1}\left(\frac{2^{\alpha} - 1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^2}\right)\right]\left(\frac{k}{\tau} - \frac{k}{2^{\alpha}(\tau + \alpha k)}\right).
\end{aligned} \tag{55}$$

(v) By choosing $\alpha = 1$ and $\psi = I$, we have

$$\begin{aligned}
&\varphi\left(\frac{a + mb}{2}\right) \int_{a+mb/2}^{mb} (mb - \iota)^{(\tau/k)-1} E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{\overline{w}}(mb - \iota)^{\mu}; \widetilde{p})\mathrm{d}\iota \\
&\leq h\left(\frac{1}{2}\right) \int_{a+mb/2}^{mb} (mb - \iota)^{(\tau/k)-1} E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{\overline{w}}(mb - \iota)^{\mu}; \widetilde{p})\varphi(\iota)\mathrm{d}\iota \\
&\quad + m^{(\tau/k)+1} h\left(\frac{1}{2}\right) \int_{(a/m)}^{(a+mb)/2} \left(\kappa - \frac{a}{m}\right)^{(\tau/k)-1} E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}} m^{\mu}\left(\kappa - \frac{a}{m}\right)^{\mu}; \widetilde{p}\right)\varphi(\kappa)\mathrm{d}\kappa \\
&\leq h\left(\frac{1}{2}\right)\left[\varphi(a) + m^{(\tau/k)+1} f(b)\right] \int_0^1 \eta^{(\tau/k)-1} E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu}; \widetilde{p}) h\left(\frac{\eta}{2}\right)\mathrm{d}\eta \\
&\quad + mh\left(\frac{1}{2}\right)\left[\varphi(b) + m^{(\tau/k)+1}\varphi\left(\frac{a}{m^2}\right)\right] \int_0^1 \eta^{(\tau/k)-1} E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu}; \widetilde{p}) h\left(\frac{2 - \eta}{2}\right)\mathrm{d}\eta.
\end{aligned} \tag{56}$$

(vi) By choosing $\alpha = m = 1$, $h(\eta) = \eta$ and $\psi = I$, we have

$$\begin{aligned}
&\varphi\left(\frac{a + b}{2}\right) \int_{(a+b/2)}^{b} (b - \iota)^{(\tau/k)-1} E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}\,(b - \iota)^{\mu}; \widetilde{p}\right)\mathrm{d}\iota \\
&\leq \frac{1}{2}\Big[\int_{(a+b/2)}^{b} (b - \iota)^{(\tau/k)-1} E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{\overline{w}}(b - \iota)^{\mu}; \widetilde{p})\varphi(\iota)\mathrm{d}\iota \\
&\quad + \int_a^{(a+b/2)} (\kappa - a)^{(\tau/k)-1} E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{\overline{w}}(\kappa - a)^{\mu}; \widetilde{p})\varphi(\kappa)\mathrm{d}\kappa\Big] \\
&\leq \left(\frac{\varphi(a) + \varphi(b)}{2}\right) \int_0^1 \eta^{(\tau/k)-1} E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu}; \widetilde{p})\mathrm{d}\eta.
\end{aligned} \tag{57}$$

Remark 5. The above k-fractional inequalities are farther in line with foreknown conclusions as noted below: (i) By choosing $k = 1$ in Corollary 3 *(v), an inequality for extended generalized fractional integrals is acquired. (ii) By choosing $k = 1$ and $\widetilde{p} = 0$ in* Corollary 3 *(v), Theorem 2.2 of* [28] *is acquired.*

The second type of the Fejér–Hadamard's inequality for generalized k-fractional integrals is dedicated as noted below:

Theorem 6. *Let $h\colon J \longrightarrow \mathbb{R}$ is nonnegative, nonzero and integrable function and $\varphi, \psi\colon [a,b] \longrightarrow \mathbb{R}$, $0 \leq a < mb$, be the functions such that $\varphi \in L_1[a,b]$ and φ be positive and ψ be differentiable and strictly increasing, r is a nonnegative and integrable function. If φ is $(\alpha, h-m)$-convex and $\varphi(\psi(\iota)) = \varphi(\psi(a) + m\psi(b) - \psi(\iota))$, in that case the below inequalities for generalized k-fractional operators (12) and (13) occur:*

$$
\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\left({}_{\psi}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\tau,l,\overline{\overline{w}},\psi^{-1}((m\psi(b)+\psi(a))/2)+}r^{\circ}\psi\right)(m\psi(b);\widetilde{p})\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F_{\mu,\tau,l,\overline{\overline{w}}},\psi^{-1}((m\psi(b)+\psi(a))/2)+^{\gamma,\delta,\nu,c}\varphi^{\circ}r^{\circ}\psi\right)(m\psi(b);\widetilde{p})\\
&\quad+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\left({}_{\psi}^{k}F^{\gamma,\delta,\nu,c}_{\mu,\tau,l,\overline{\overline{w}}m^{\mu},\psi^{-1}((m\psi(b)+\psi(a))/2m)-}\varphi^{\circ}r^{\circ}\psi\right)\left(\frac{\psi(a)}{m};\widetilde{p}\right)\\
&\leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\\
&\quad\times\int_0^1\eta^{(\tau/k)-1}E^{\gamma,\delta,\nu,c}_{\mu,\tau,l}(w\eta^{\mu};\widetilde{p})r\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)d\eta\\
&\quad+m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^2}\right)\right]\\
&\quad\times\int_0^1\eta^{(\tau/k)-1}E^{\gamma,\delta,\nu,c}_{\mu,\tau,l}(w\eta^{\mu};\widetilde{p})r\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)d\eta,
\end{aligned}
\tag{58}
$$

where $\overline{\overline{w}} = (2^{\mu}w/(m\psi(b) - \psi(a))^{\mu})$ for all $\eta \in [a^p, b^p]$.

Proof. We demonstrate the claim as follows:

Multiplying (48) by $\eta^{(\tau/k)-1}E^{\gamma,\delta,\nu,c}_{\mu,\tau,l}(w\eta^{\mu};\widetilde{p})r((\eta/2)\psi(a) + m(2-\eta/2)\psi(b))$ and then integrating over $[0,1]$, we have

$$
\begin{aligned}
&\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\int_0^1\eta^{(\tau/k)-1}E^{\gamma,\delta,\nu,c}_{\mu,\tau,l}(w\eta^{\mu};\widetilde{p})r\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)d\eta\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_0^1\eta^{(\tau/k)-1}E^{\gamma,\delta,\nu,c}_{\mu,\tau,l}(w\eta^{\mu};\widetilde{p})\varphi\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)r\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)d\eta\\
&\quad+mh\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1\eta^{(\tau/k)-1}E^{\gamma,\delta,\nu,c}_{\mu,\tau,l}(w\eta^{\mu};\widetilde{p})\varphi\left(\left(\frac{2-\eta}{2}\right)\frac{\psi(a)}{m}+\frac{\eta}{2}\psi(b)\right)r\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)d\eta.
\end{aligned}
\tag{59}
$$

By setting $\psi(\iota) = (\eta/2)\psi(a) + m(2-\eta/2)\psi(b)$ and $\psi(\kappa) = (2-\eta/2)(\psi(a)/m) + \eta/2\psi(b)$, that is, $\psi(a) + m\psi(b) - \psi(\iota) = (2-\eta/2)\psi(a) + m(\eta/2)\psi(b)$, in (59), in that case by usage $\varphi(\psi(\iota)) = \varphi(\psi(a) + m\psi(b) - \psi(\iota))$ and k-fractional integral operators (12) and (13), the first side of (58) is acquired.

To demonstrate the second side of (58), multiplying both parts of (50) by

$$
\eta^{(\tau/k)-1}E^{\gamma,\delta,\nu,c}_{\mu,\tau,l}(w\eta^{\mu};\widetilde{p})r\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right),
\tag{60}
$$

and then integrating over $[0,1]$, we have

$$\begin{aligned}
&h\left(\frac{1}{2^{\alpha}}\right)\int_0^1 \eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu};\widetilde{p}\right)\varphi\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)r\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)\mathrm{d}\eta\\
&\quad+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_0^1 \eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu};\widetilde{p}\right)\varphi\left(\left(\frac{2-\eta}{2}\right)\frac{\psi(a)}{m}+\frac{\eta}{2}\psi(b)\right)r\left(\frac{\eta}{2}g(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)\mathrm{d}\eta\\
&\leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]\\
&\quad\times\int_0^1 \eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu};\widetilde{p}\right)r\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta\\
&\quad+m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^{2}}\right)\right]\\
&\quad\times\int_0^1 \eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(w\eta^{\mu};\widetilde{p}\right)r\left(\frac{\eta}{2}\psi(a)+\left(\frac{2-\eta}{2}\right)\psi(b)\right)h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta.
\end{aligned}\tag{61}$$

Setting $\psi(\iota)=(\eta/2)\psi(a)+m(2-\eta/2)\psi(b)$ and $\psi(\kappa)=(2-\eta/2)(\psi(a)/m)+\eta/2\psi(b)$ in (59), then by using $\varphi(\psi(\iota))=\varphi(\psi(a)+m\psi(b)-\psi(\iota))$ and k-fractional integral operators (12) and (13), the second inequality of (58) is obtained. □

Corollary 4. *By using (58), some more k-fractional inequalities are offered as noted below:*

(i) By choosing $\psi=I$ and $\widetilde{p}=w=0$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_{((a+mb)/2)}^{mb}(mb-\iota)^{(\tau/k)-1}r(\iota)\mathrm{d}\iota\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_{((a+mb)/2)}^{mb}(mb-\iota)^{(\tau/k)-1}\left(\varphi^{\circ}r\right)(\iota)\mathrm{d}\iota\\
&\quad+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{(a/m)}^{((a+mb)/2)}\left(\kappa-\frac{a}{m}\right)^{(\tau/k)-1}\left(\varphi^{\circ}r\right)(\kappa)\mathrm{d}\kappa\\
&\leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(a)+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(b)\right]\\
&\quad\times\int_0^1\eta^{(\tau/k)-1}r\left(\frac{\eta}{2}a+m\left(\frac{2-\eta}{2}\right)b\right)h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta\\
&\quad+m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(b)+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{a}{m^{2}}\right)\right]\\
&\quad\times\int_0^1\eta^{(\tau/k)-1}r\left(\frac{\eta}{2}a+m\left(\frac{2-\eta}{2}\right)b\right)h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta.
\end{aligned}\tag{62}$$

(ii) By choosing $\psi=I$ and $\widetilde{p}=0$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{a+mb}{2}\right)\int_{((a+mb)/2)}^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}(mb-\iota)^{\mu}\right)r(\iota)\mathrm{d}\iota\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_{((a+mb)/2)}^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}(mb-\iota)^{\mu}\right)(\varphi\circ r)(\iota)\mathrm{d}\iota\\
&\quad+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{(a/m)}^{((a+mb)/2)}\left(\kappa-\frac{a}{m}\right)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}m^{\mu}\left(\kappa-\frac{a}{m}\right)^{\mu}\right)(\varphi\circ r)(\kappa)\mathrm{d}\kappa\\
&\leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(a)+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(b)\right]\\
&\quad\times\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu})r\left(\frac{\eta}{2}a+m\left(\frac{2-\eta}{2}\right)b\right)h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta\\
&\quad+m\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(b)+m^{(\tau/k)+1}h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{a}{m^{2}}\right)\right]\\
&\quad\times\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu})r\left(\frac{\eta}{2}a+m\left(\frac{2-\eta}{2}\right)b\right)h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta.
\end{aligned} \tag{63}$$

(iii) By choosing $m=1$ and $\psi=I$, we obtain

$$\begin{aligned}
&\varphi\left(\frac{a+b}{2}\right)\int_{(a+b/2)}^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}(b-\iota)^{\mu};\widetilde{p}\right)r(\iota)\mathrm{d}\iota\\
&\leq h\left(\frac{1}{2^{\alpha}}\right)\int_{(a+b/2)}^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}(b-\iota)^{\mu};\widetilde{p}\right)(\varphi\circ r)(\iota)\mathrm{d}\iota\\
&\quad+h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{a}^{b}(\kappa-a)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}(\kappa-a)^{\mu};\widetilde{p}\right)(\varphi\circ r)(\kappa)\mathrm{d}\kappa\\
&\leq\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(a)+h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(b)\right]\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r\left(\frac{\eta}{2}a+\left(\frac{2-\eta}{2}\right)b\right)h\left(\frac{\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta\\
&\quad+\left[h\left(\frac{1}{2^{\alpha}}\right)\varphi(b)+h\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(a)\right]\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\widetilde{p})r\left(\frac{\eta}{2}a+\left(\frac{2-\eta}{2}\right)b\right)h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta.
\end{aligned} \tag{64}$$

(iv) By choosing $\widetilde{p}=w=0$ and $h(\eta)=\eta$, we obtain

$$\varphi\left(\frac{\psi(a)+m\psi(b)}{2}\right)\int_{\psi^{-1}(\psi(a)+m\psi(b)/2)}^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}r(\psi(\iota))\psi\prime(\iota)\mathrm{d}\iota$$
$$\leq\left(\frac{1}{2^{\alpha}}\right)\int_{\psi^{-1}(\psi(a)+m\psi(b)/2)}^{\psi^{-1}(m\psi(b))}(m\psi(b)-\psi(\iota))^{(\tau/k)-1}(\varphi\circ r\circ\psi)(\iota)\psi\prime(\iota)\mathrm{d}\iota$$
$$+m^{(\tau/k)+1}\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\int_{\psi^{-1}(\psi(a)/2)}^{\psi^{-1}(\psi(a)+m\psi(b)/2m)}(\psi(\kappa)-\frac{\psi(a)}{m})^{(\tau/k)-1}(\varphi\circ r\circ\psi)(\kappa)\psi\prime(\kappa)\mathrm{d}\kappa$$
$$\leq\left[\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(a))+m^{(\tau/k)+1}\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi(\psi(b))\right]$$
$$+m\left[\left(\frac{1}{2^{\alpha}}\right)\varphi(\psi(b))+m^{(\tau/k)+1}\left(\frac{2^{\alpha}-1}{2^{\alpha}}\right)\varphi\left(\frac{\psi(a)}{m^{2}}\right)\right]$$
$$\times\int_{0}^{1}\eta^{(\tau/k)-1}r\left(\frac{\eta}{2}\psi(a)+m\left(\frac{2-\eta}{2}\right)\psi(b)\right)h\left(\frac{2^{\alpha}-\eta^{\alpha}}{2^{\alpha}}\right)\mathrm{d}\eta. \tag{65}$$

(v) By choosing $\alpha=1$ and $\psi=I$, we get

$$\varphi\left(\frac{a+mb}{2}\right)\int_{(a+mb/2)}^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{\overline{w}}(mb-\iota)^{\mu};\tilde{p})r(\iota)\mathrm{d}\iota$$
$$\leq h\left(\frac{1}{2}\right)\int_{(a+mb/2)}^{mb}(mb-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{\overline{w}}(mb-\iota)^{\mu};\tilde{p})(\varphi\circ r)(\iota)\mathrm{d}\iota$$
$$+m^{(\tau/k)+1}h\left(\frac{1}{2}\right)\int_{(a/m)}^{(a+mb/2)}\left(\kappa-\frac{a}{m}\right)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}\left(\overline{\overline{w}}m^{\mu}\left(\kappa-\frac{a}{m}\right)^{\mu};\tilde{p}\right)(\varphi\circ r)(\kappa)\mathrm{d}\kappa$$
$$\leq h\left(\frac{1}{2}\right)\left[\varphi(a)+m^{(\tau/k)+1}\varphi(b)\right]\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\tilde{p})r\left(\frac{\eta}{2}a+m\left(\frac{2-\eta}{2}\right)b\right)h\left(\frac{\eta}{2}\right)\mathrm{d}\eta$$
$$+mh\left(\frac{1}{2}\right)\left[f(b)+m^{(\tau/k)+1}\varphi\left(\frac{a}{m^{2}}\right)\right]\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\tilde{p})r\left(\frac{\eta}{2}a+m\left(\frac{2-\eta}{2}\right)b\right)h\left(\frac{2-\eta}{2}\right)\mathrm{d}\eta. \tag{66}$$

(vi) By choosing $\alpha=m=1, h(\eta)=\eta$ and $\psi=I$, we get

$$\varphi\left(\frac{a+b}{2}\right)\int_{(a+b/2)}^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{\overline{w}}(b-\iota)^{\mu};\tilde{p})r(\iota)d\iota$$
$$\leq\frac{1}{2}[\int_{(a+b/2)}^{b}(b-\iota)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{\overline{w}}(b-\iota)^{\mu};\tilde{p})(\varphi\circ r)(\iota)\mathrm{d}\iota$$
$$+\int_{a}^{(a+b/2)}(\kappa-a)^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(\overline{\overline{w}}(\kappa-a)^{\mu};\tilde{p})(\varphi\circ r)(\kappa)\mathrm{d}\kappa]$$
$$\leq\left(\frac{\varphi(a)+\varphi(b)}{2}\right)\int_{0}^{1}\eta^{(\tau/k)-1}E_{\mu,\tau,l}^{\gamma,\delta,\nu,c}(w\eta^{\mu};\tilde{p})r\left(\frac{\eta}{2}a+\left(\frac{2-\eta}{2}\right)b\right)\mathrm{d}\eta. \tag{67}$$

Remark 6. Those as mentioned above k-fractional inequalities are farther in line with foreknown conclusions as by choosing $k=1$ in Corollary 4 *(v), an inequality for extended generalized fractional integrals is obtained.*

Data Availability

There are no data required for this paper

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All the authors made equal contributions.

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Research Article

Optimal Control Problem Governed by an Evolution Equation and Using Bilinear Regular Feedback

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We solve an optimal control problem governed by an evolution equation using bilinear regular feedback. Using optimization techniques, we show how to approximate the flow of a reaction-diffusion bilinear system by a desired target. For application, we consider the regional flow problem constrained by a bilinear distributed system. The paper ends by an example illustrating the numerical approach of the proposed method.

1. Introduction

Bilinear systems form an important class of dynamic systems for several reasons. Many industrial or natural processes have a bilinear structure. For example, we can cite the transfer of heat by conduction convection, the neutron displacement in a nuclear reactor, and the dynamics of sense organs [1]. Research has shown that bilinear systems are sufficient to approach any nonlinear input-output behavior (see [1, 2]). The control has a double action in the system that allows the adaptation of the model at different levels of input signals. An example is provided by the functioning of sense organ (see [1]).

Optimal control methods continue to provide solutions to many real problems. We cite solutions of smoking models by Mahdy et al. [3] and COVID-19 prediction by Ahmed et al. [4]. Optimal control problems constrained by a distributed bilinear system are initiated by Bradley et al. and Lenhart [5, 6]. In [7], Joshi studies the case of regular velocity terms. Sonawane et al. [8] consider the optimal control for a vibrating string with axial variable. Rao et al. studied plant disease in [9].

Mall et al. propose a uniform method for optimal control problems with control and state constraints (see [10]). Chertovskih et al. in [11] give an indirect method for regular state-constrained optimal control problems in flow fields. Turgut et al. in [12] study an island-based crow search algorithm for solving optimal control problems. Al-Hawasy et al. in [13] consider the optimal control problems for triple elliptic partial differential equations. Bonnet and Frankowska in [14] characterize the necessary optimality conditions for optimal control problems in Wasserstein spaces. Granada and Kovtunenko in [15] consider a shape derivative for optimal control of the nonlinear Brinkman–Forchheimer equation.

For fractional systems, Saidi [16] discusses some results associated to first-order set-valued evolution problems with subdifferentials. Jajarmi and Baleanu [17] consider the fractional optimal control problems with a general derivative operator. Huixian et al. [18] study an averaging result for a class of impulsive fractional neutral stochastic evolution equations. Jafariet al. [19] propose a numerical approach for solving fractional optimal control problems with Mittag–Leffler kernel. Mehandiratta et al. [20] study fractional optimal control problems on a star graph. Heydari et al. [21] propose a numerical solution for an optimal control problems generated by Atangana–Riemann–Liouville fractal-fractional derivative.

The flow problems are one of the most important questions in mathematics. They have applications in several fields such as physics, biology, and engineering. We cite here the problem of controlling the blood flow in a vessel, where we need to calculate the gradient (flow) of the velocity of blood as a rate of change of the blood flow (see [22]).

Recently, many researchers focused on the study of flow problems using optimal control theory. They consider the gradient state of a distributed system and ask if there is an optimal control to reach a desired profile (see [23]). For this approach, one of the most important ideas is called the partial analysis. It has an objective to reach a target on a specific subdomain of the system domain, $\omega \subset D$ (see [24, 25]). For partial work on bilinear distributed systems, Ouzehra et al. [26, 27] study the exact and approximate controllability of reaction-diffusion equation using bilinear control. Zerrik and Ould Sidi [28–31] use partial control problems to orient the dynamics of infinite dimensional systems towards the desired state in a specific area. Zine and Ould Sidi in [32–34] deal with partial control problems in the case of hyperbolic systems. Ould Sidi and Beinane [35, 36] treat the partial flow control problems.

The objective of this paper is to control the flow of equation (1) towards a desired target using the penalization problem 3, and with a more regular spatiotemporal control function. In Section 2, we show the existence of a solution to the studied problems. Next, we give the characterization of its solution considering different types of actions. Section 3 is devoted to the study of the partial flow control problems constrained by bilinear distributed systems with regular optimal control time function. The paper ends by an example illustrating the numerical approach of the proposed method.

2. Flow Problem with Regular Control

Let us consider the system described by

$$\begin{cases} q_m(x,m) = q_{xx}(x,m) - v(x,m)q_x(x,m), & \Gamma, \\ q(x,0) = q_0(x), & D, \\ q = q_x = 0, & \Pi, \end{cases} \tag{1}$$

with a domain $D \subset IR^n$ $(n = 1, 2, 3)$ is open bounded, and its regular boundary is ∂D. Let $M > 0$ and $\Gamma = D \times]0, M[$, $\Pi = \partial D \times]0, M[$, where the space of control is $v \in L^2(0, M, H_0^1(D))$.

Let $q_0(x) \in L^2(D)$ and

$$S = \left\{ \frac{q \in L^2\left(0, M; H_0^1(D)\right)}{q_m \in L^2\left(0, M; H^{-2}(D)\right)} \right\}, \tag{2}$$

represents the state space (see [5]). The system dynamic is $q_{xx} = \Delta q = \sum_{i=1}^n \partial^2 q/\partial x_i^2$, and system (1) has a unique solution q_v in $S \cap L^\infty(0, M; L^2(D))$ (see [37]).

We consider the operator ∇:

$$\begin{aligned} \nabla: H^1(D) &\longrightarrow \left(L^2(D)\right)^n, \\ q &\longrightarrow \nabla q = \left(\frac{\partial q}{\partial x_1}, \ldots, \frac{\partial q}{\partial x_n}\right). \end{aligned} \tag{3}$$

The flow regular optimal control problem of system (1) is

$$\min_{v \in L^2\left(0, M, H_0^1(D)\right)} \Phi_\varepsilon(v), \tag{4}$$

with $\varepsilon > 0$, and Φ_ε is the cost penalty defined by

$$\begin{aligned} \Phi_\varepsilon(v) &= \frac{1}{2}\left\|\nabla q - q^d\right\|^2_{\left(L^2\left(0,M,H_0^1(D)\right)\right)^n} + \frac{\varepsilon}{2}\int_\Gamma \left[v_m^2(x,m) + v_x^2(x,m)\right] \mathrm{d}x\mathrm{d}m \\ &= \frac{1}{2}\sum_{i=1}^n \left\|\frac{\partial q}{\partial x_i} - q_i^d\right\|^2_{L^2\left(0,M,H_0^1(D)\right)} + \frac{\varepsilon}{2}\int_\Gamma \left[v_m^2(x,m) + v_x^2(x,m)\right] \mathrm{d}x\mathrm{d}m, \end{aligned} \tag{5}$$

where the desired flow is $q^d = (q_1^d, \ldots, q_n^d)$.

The main objective is to propose a method to steer the flow of (1) to $q^d(x)$, using the functional (5) and considering a more regular control space $v \in L^2(0, M; H_0^1(D))$. We characterize the solution of (4) through an extension of the Lagrangian method.

2.1. Existence of Solution. In the next theorem, we study the existence of a solution to the flow problem (4).

Theorem 1. *Let us consider q be the solution of the system*

$$\begin{cases} q_m = q_{xx} - v q_x, & \Gamma, \\ q(x,0) = q_0(x), & D, \\ q = q_x = 0, & \Pi. \end{cases} \tag{6}$$

Then, there exists an optimal control v, which is the minimum of (4).

Proof. Let us consider the set $\{\Phi_\varepsilon(v) | v \in L^2(0, M, H_0^1(D))\} \subset IR$, which is a positive nonempty and admits lower bounded. Thus, by choosing a minimizing sequence $(v_n)_n$ which verifies

$$\Phi^* = \lim_{n\longrightarrow+\infty} \Phi(v_n) = \inf_{v\in L^2(0,M,H_0^1(D))} \Phi_\varepsilon(v). \tag{7}$$

Then, the cost $\Phi_\varepsilon(v_n)$ is bounded, and it follows that $\|v_n\|_{L^2(0,M,H_0^1(D))} \leq B$, with B as a positive constant.

We have

$$\begin{aligned} v_n &\rightharpoonup v, \quad L^2(0,M,H_0^1(D)),\\ q^n &\rightharpoonup q, \quad S,\\ q^n_{xx} &\rightharpoonup \chi, \quad S,\\ q^n_x &\rightharpoonup \Lambda, \quad S,\\ q^n_m &\rightharpoonup \Psi, \quad S. \end{aligned} \tag{8}$$

By passing to the limit in the equation $q^n_m(x,m) = q^n_{xx} - v_n q^n_x$, we deduce that $q_m(x,m) = \Psi$, $q \mapsto q_{xx}$, $q_{xx} = \chi$ and $vq_x = \Lambda$. Hence, we obtain

$$q_m = q_{xx} - v(x,m)q_x. \tag{9}$$

From the lower semicontinuity of $\Phi_\varepsilon(v)$:

$$\begin{aligned} \Phi_\varepsilon(v) &= \inf_n \sum_{i=1}^n \frac{1}{2}\int_0^M \int_D \left(\frac{\partial q_n}{\partial x_i} - q_i^d\right)^2 \mathrm{d}x + \frac{\varepsilon}{2}\int_\Gamma \left[v_m^2 + v_x^2\right]_n \mathrm{d}x\mathrm{d}m \\ &\leq \lim_{n\longrightarrow 0} \Phi_\varepsilon(v_n) = \inf_v \Phi_\varepsilon(v). \end{aligned} \tag{10}$$

Therefore, v is a solution of (4). □

2.2. Characterization of Solution. In this section, the aim is to propose a formulation of the solution of our flow problem. Therefore, we should introduce the so-called optimal equation to find the differential of the functional $\Phi_\varepsilon(v)$ in (5). The following lemma mentions the differential of $\Phi_\varepsilon(v)$ with respecting v.

Lemma 2. *A differential of the map*

$$v \in L^2(0,M,H_0^1(D)) \longrightarrow q(v) \in S, \tag{11}$$

is

$$\frac{q(v+\varepsilon l) - q(l)}{\varepsilon} \rightharpoonup \mu, \tag{12}$$

where $\mu = \mu(q,l)$ verifies

$$\begin{cases} \mu_m = \mu_{xx} - v\mu_x - lq_x, & \Gamma,\\ \mu(x,0) = 0, & D,\\ \mu = \mu_x = 0, & \Pi, \end{cases} \tag{13}$$

where $q = q(v)$, $v \in L^2(0,M;H_0^1(D))$, and $d(q(v))l$ is the derivative of $v \longrightarrow q(v)$ with respect v.

Proof. We consider the solution of (13), verifying

$$\|\mu\|_S \leq k_1 \|q\|_{L^\infty(0,M;H_0^1(D))} \|l\|_{L^2(0,M,H_0^1(D))}. \tag{14}$$

Also,

$$\|\mu'\|_S \leq k_2 \|q\|_{L^\infty(0,M;H_0^1(D))} \|l\|_{L^2(0,M,H_0^1(D))}. \tag{15}$$

Thus,

$$\|\mu\|_{\mathscr{C}([0,M];H_0^1(D))} \leq k_3 \|l\|_{L^2(0,M,H_0^1(D))}. \tag{16}$$

Then, we obtain that $l \in L^2(0,M;L^2(D)) \longrightarrow \mu \in \mathscr{C}((0,M);H_0^1(D))$ is bounded (see [5]).

If we put $q_l = q(v+l)$ and $\xi = q_l - q$, then ξ is the state of

$$\begin{cases} \xi_m(x,m) = \xi_{xx} - v(x,m)\xi_x(x,m) - l(x,m)(q_l)_x, & \Gamma,\\ \xi(x,0) = 0, & D,\\ \xi = \xi_x = 0, & \Pi. \end{cases} \tag{17}$$

Thus,

$$\|\xi\|_{L^\infty([0,M];H_0^1(D))} \leq k_4 \|\theta\|_{L^2(0,M,H_0^1(D))}. \tag{18}$$

Let $\gamma = \xi - \mu$ which verifies the system

$$\begin{cases} \gamma_m = \gamma_{xx} + v(x,m)\gamma_x(x,m) + l(x,m)\xi_x, & \Gamma,\\ \gamma(x,0) = 0, & D,\\ \gamma = \gamma_x = 0, & \Pi, \end{cases} \tag{19}$$

$\gamma \in \mathscr{C}(0,M;H_0^1(D))$; consequently,

$$\|\gamma\|_{C([0,M];H_0^1(D))} \leq k \|l\|^2_{L^2(0,M,H_0^1(D))}, \tag{20}$$

and we have

$$\|q(v+l) - q(v) - d(q(v))l\|_{C(0,M;H_0^1(D))} = \|\gamma\|_{C([0,M];H_0^1(D))} \leq k\|l\|^2_{L^2(0,M;H_0^1(D))}, \tag{21}$$

where k_1, k_2, k_3, k_4, and k are a constant positive.

In the following, we define a family of optimal equations.

$$\begin{cases} -\dfrac{\partial p_i}{\partial m} = \dfrac{\partial^2 p_i}{\partial x^2} + \dfrac{\partial (v p_i)}{\partial x} + \left(\dfrac{\partial q}{\partial x_i} - q_i^d\right), & \Gamma, \\ v_x(x,0) = v_x(x,M) = 0, & D, \\ p_i(x,M) = 0, & D, \\ p_i = \dfrac{\partial p_i}{\partial x} = 0, & \Pi. \end{cases} \tag{22}$$

The next lemma characterizes the differential of $\Phi_\varepsilon(v)$. □

Lemma 3. *Let $v \in L^2(0, M, H_0^1(D))$ be the solution of (4), and we obtain*

$$\lim_{\beta \longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l) - \Phi_\varepsilon(v)}{\beta} = \sum_{i=1}^{n} \int_D \int_0^M \frac{\partial \mu(x,m)}{\partial x_i}\left(\frac{\partial q}{\partial x_i} - q_i^d\right) \mathrm{d}m\mathrm{d}x + \varepsilon \int_D \int_0^M [(v_m l_m) + (v_x l_x)] \mathrm{d}m\mathrm{d}x. \tag{23}$$

Proof. The cost $\Phi_\varepsilon(v)$ (5) can be expressed by

$$\Phi_\varepsilon(v) = \frac{1}{2}\sum_{i=1}^{n} \int_D \int_0^M \left(\frac{\partial q}{\partial x_i} - q_i^d\right)^2 \mathrm{d}m\mathrm{d}x + \frac{\varepsilon}{2}\int_D \int_0^M \left[v_m^2 + v_x^2\right] \mathrm{d}m\mathrm{d}x. \tag{24}$$

If we put $q_\beta = q(v+\beta l)$ and $q = q(v)$, using (59), we have

$$\begin{aligned} \lim_{\beta \longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l) - \Phi_\varepsilon(v)}{\beta} &= \lim_{\beta \longrightarrow 0} \sum_{i=1}^{n} \frac{1}{2} \int_D \int_0^M \frac{\left(\left(\partial q_\beta/\partial x_i\right) - q_i^d\right)^2 - \left(\left(\partial q/\partial x_i\right) - q_i^d\right)^2}{\beta} \mathrm{d}m\mathrm{d}x \\ &\quad + \lim_{\beta \longrightarrow 0} \frac{\varepsilon}{2\beta} \int_D \int_0^M \left[(v_m + \beta l_m)^2 - v_m^2 + (v_x + \beta l_x)^2 - v_x^2\right] \mathrm{d}m\mathrm{d}x, \end{aligned} \tag{25}$$

then

$$\begin{aligned} &\lim_{\beta \longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l) - \Phi_\varepsilon(v)}{\beta} \\ &= \lim_{\beta \longrightarrow 0} \sum_{i=1}^{n} \frac{1}{2} \int_D \int_0^M \frac{\left(\left(\partial q_\beta/\partial x_i\right) - \left(\partial q/\partial x_i\right)\right)}{\beta}\left(\frac{\partial q_\beta}{\partial x_i} + \frac{\partial q}{\partial x_i} - 2q_i^d\right) \mathrm{d}m\mathrm{d}x + \lim_{\beta \longrightarrow 0} \varepsilon \int_D \int_0^M [(v_m l_m) + (v_x l_x)](x,m) \mathrm{d}m\mathrm{d}x \\ &= \sum_{i=1}^{n} \int_D \int_0^M \frac{\partial \mu(x,m)}{\partial x_i}\left(\frac{\partial q(x,m)}{\partial x_i} - q_i^d\right) \mathrm{d}m\mathrm{d}x + \int_D \int_0^M \varepsilon[(v_m l_m) + (v_x l_x)](x,m) \mathrm{d}m\mathrm{d}x. \end{aligned} \tag{26}$$

The following theorem proposes a solution of the problem (4). □

Theorem 4. *Let $v \in L^2(0, M; H_0^1(D))$ be a solution of (4); then,*

$$v_{mm} + v_{xx} + \frac{1}{\varepsilon}(Div(p_i))q_x = 0, \quad (27)$$

where $q = q(v)$ is the output of (1), where $p_i = (p_1, \ldots, p_n)$ and $p_i \in C([0, M]; H_0^1(D))$ is the solution of (22).

Proof. Let $l \in L^2(0, M; H_0^1(D))$ and $v + \beta l \in L^2(0, M; H_0^1(D))$ for $\beta > 0$. The extremal of Φ_ε is realized at v; then,

$$0 \leq \lim_{\beta \longrightarrow 0} \frac{\Phi_\varepsilon(v + \beta l) - \Phi_\varepsilon(v)}{\beta}. \quad (28)$$

Lemma 9 gives

$$0 \leq \sum_{i=1}^{n} \int_D \int_0^M \frac{\partial \mu(x, m)}{\partial x_i}\left(\frac{\partial q(x, m)}{\partial x_i} - q_i^d\right) \mathrm{d}m\mathrm{d}x + \int_D \int_0^M \varepsilon[(v_m l_m) + (v_x l_x)](x, m)\mathrm{d}m\mathrm{d}x. \quad (29)$$

Therefore, using (22), we obtain

$$\begin{aligned} 0 \leq &\sum_{i=1}^{n} \int_D \int_0^M \frac{\partial \mu(x, m)}{\partial x_i}\left(-\frac{\partial p_i(x, m)}{\partial m} - \frac{\partial^2 p_i(x, m)}{\partial x^2} - \frac{\partial v p_i(x, m)}{\partial x}\right)\mathrm{d}m\mathrm{d}x \\ &+ \int_D \int_0^M \varepsilon[(v_m l_m) + (v_x l_x)]\mathrm{d}m\mathrm{d}x. \end{aligned} \quad (30)$$

By a simple calculus, we have

$$\begin{aligned} 0 \leq &\sum_{i=1}^{n} \int_D \int_0^M \frac{\partial \mu(x, m)}{\partial x_i}\left(-\frac{\partial p_i(x, m)}{\partial m} - \frac{\partial^2 p_i(x, m)}{\partial x^2} + \frac{\partial v p_i(x, m)}{\partial x}\right) p_i(x, m)\mathrm{d}m\mathrm{d}x \\ &+ \int_D \int_0^M \varepsilon[(v_m l_m) + (v_x l_x)]\mathrm{d}m\mathrm{d}x. \end{aligned} \quad (31)$$

From System(13), we obtain

$$\begin{aligned} 0 \leq &\sum_{i=1}^{n} \int_D \int_0^M \frac{\partial}{\partial x_i} - (l(x, m)q_x) p_i(x, m)\mathrm{d}m\mathrm{d}x \\ &+ \int_D \int_0^M \varepsilon[(v_m l_m) + (v_x l_x)]\mathrm{d}m\mathrm{d}x \\ = &\int_D \int_0^M \left[-l(x, m)q_x\left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} p_i(x, m)\right) + \varepsilon(v_m l_m) + \varepsilon(v_x l_x)\right]\mathrm{d}m\mathrm{d}x. \end{aligned} \quad (32)$$

Moreover, if $l = l(t) \in L^2(0, M; H_0^1 D)$, we deduce

$$-l q_x\left(\sum_{i=1}^{n} \frac{\partial}{\partial x_i} p_i(x, m)\right) - \varepsilon v_{mm} l - \varepsilon v_{xx} l = 0, \quad (33)$$

which allow us to introduce

$$v_{mm} + v_{xx} + \frac{1}{\varepsilon}(Div(p_i))q_x = 0, \quad (34)$$

that the solution v of (4) must satisfy. □

Remark 5. According to equation (2),

(1) If we consider a spatial control function $v = v(x,)$ then the variational formula becomes

$$v_{xx} = -\frac{1}{\varepsilon}(Div(p_i))q_x, \quad (35)$$

(2) If we consider a temporal control function $v = v(m)$, then the variational formula becomes

$$v_{mm} = -\frac{1}{\varepsilon}(Div(p_i))q_x. \tag{36}$$

3. Partial Flow Control Problem

3.1. Problem Statement. We consider the bilinear distributed system (1), with a given $q_0 \in H^1(D)$. System (1) can be rewritten as follows:

$$q(m) = S(m)q_0 + \int_0^m S(m-s)v(s)q(s)\mathrm{d}s, \tag{37}$$

and the solution of (37) are often called the mild solution of (1).

The existence of a unique solution $q_v(x,m)$ in $L^2(0,M;H_0^1(D))$ satisfying (37) can be deduced from [37].

We choose $\omega \in D$, and

$$\begin{aligned} \chi_\omega\colon \left(L^2(D)\right)^n &\longrightarrow \left(L^2(D)\right)^n \\ q &\longrightarrow \chi_\omega q = q|_\omega, \end{aligned} \tag{38}$$

and χ_ω^*; its adjoint is given by

$$\chi_\omega^* q = \begin{cases} q \text{ in } D, \\ 0 \in D\setminus\omega, \end{cases} \tag{39}$$

$$\begin{aligned} \widetilde{\chi}_\omega\colon \left(L^2(D)\right) &\longrightarrow \left(L^2(\omega)\right) \\ q &\longrightarrow \widetilde{\chi}_\omega q = q|_\omega. \end{aligned} \tag{40}$$

Definition 6. Equation (1) is called partial flow controllable on $\omega \subset D$, to $g^d \in (L^2(\omega))^n$ if there exists a control $v \in L^2(0,M,H_0^1(D))$ and $\varepsilon > 0$ such that

$$\left\|\chi_\omega \nabla q_v(M) - g^d\right\|_{(L^2(\omega))^n} \le \varepsilon, \tag{41}$$

where $g^d = (y_1^d, \dots y_n^d)$ is the desired flow in $(L^2(\omega))^n$.

Ouzehra in [26], studies the exact and approximate controllability of distributed bilinear systems. The partial flow control problem of (1) is

$$\min_{v\in L^2(0,M,H_0^1(D))}\Phi_\varepsilon(v), \tag{42}$$

where Φ_ε is presented for $\varepsilon > 0$ by

$$\begin{aligned} \Phi_\varepsilon(v) &= \frac{1}{2}\left\|\chi_\omega \nabla q(M) - g^d\right\|^2_{(L^2(\omega))^n} + \frac{\varepsilon}{2}\int_\Gamma \left[v_m^2(m)\right]\mathrm{d}m \\ &= \frac{1}{2}\sum_{i=1}^n \left\|\widetilde{\chi}_\omega \frac{\partial q(M)}{\partial x_i} - y_i^d\right\|^2_{L^2(\omega)} + \frac{\varepsilon}{2}\int_\Gamma \left[v_m^2(m)\right]\mathrm{d}m. \end{aligned} \tag{43}$$

The objective of the presented problem is to command the flow of (1) to a target state $g^d(x)$, realizing (43), and find $v^* \in L^2(0,M,H_0^1(D))$, verifying

$$\Phi_\varepsilon(v^*) = \min_{v\in L^2(0,M,H_0^1(D))} \Phi_\varepsilon(v). \tag{44}$$

Remark 7. The existence of solutions for the partial flow control problem can be proved in the same way as in the proof of the previous section.

3.2. Characterization of Solution. Now, we are able to formulate the solution of the flow problem (42).

Lemma 8. *A differential of the map*

$$v \in L^2\left(0,M,H_0^1(D)\right) \longrightarrow q(v) \in S, \tag{45}$$

is

$$\frac{q(v+\varepsilon l) - q(l)}{\varepsilon} \longrightarrow \mu, \tag{46}$$

where $\mu = \mu(q,l)$ verifies

$$\begin{cases} \mu_m = \mu_{xx} - v\mu_x - lq_x, & \Gamma, \\ \mu(x,0) = 0, & D, \\ \mu = \mu_x = 0, & \Pi, \end{cases} \tag{47}$$

where $q = q(v)$, $v \in L^2(0,M;H_0^1(D))$, and $d(q(v))l$ is the derivative of $v \longrightarrow q(v)$ with respect v.

Proof. The output of equation (13) satisfies

$$\|\mu\|_S \le k_1\|q\|_{L^\infty(0,M;H_0^1(D))}\|l\|_{L^2(0,M,H_0^1(D))}. \tag{48}$$

Also,

$$\left\|\mu'\right\|_S \le k_2\|q\|_{L^\infty(0,M;H_0^1(D))}\|l\|_{L^2(0,M,H_0^1(D))}. \tag{49}$$

Thus,

$$\|\mu\|_{\mathscr{C}([0,M];H_0^1(D))} \le k_3\|l\|_{L^2(0,M,H_0^1(D))}. \tag{50}$$

Then, we obtain that $l \in L^2(0,M;L^2(D)) \longrightarrow \mu \in \mathscr{C}((0,M);H_0^1(D))$ is bounded (see [5]).

If we put $q_l = q(v+l)$ and $\xi = q_l - q$, then ξ is the state of

$$\begin{cases} \xi_m(x,m) = \xi_{xx} - v(x,m)\xi_x(x,m) - l(x,m)(q_l)_x, & \Gamma, \\ \xi(x,0) = 0, & D, \\ \xi = \xi_x = 0, & \Pi. \end{cases} \tag{51}$$

Thus,

$$\|\xi\|_{L^\infty([0,M]];H_0^1(D))} \le k_4\|l\|_{L^2(0,M,H_0^1(D))}. \tag{52}$$

Let $\gamma = \xi - \mu$ which verifies the system

$$\begin{cases} \gamma_m = \gamma_{xx} + v(x,m)\gamma_x(x,m) + l(x,m)\xi_x, & \Gamma, \\ \gamma(x,0) = 0, & D, \\ \gamma = \gamma_x = 0, & \Pi, \end{cases} \tag{53}$$

$\gamma \in \mathscr{C}(0,M;H_0^1(D))$; consequently,

$$\|\gamma\|_{\mathscr{C}([0,M];H_0^1(D))} \leq k\|l\|^2_{L^2(0,M,H_0^1(D))}, \tag{54}$$

and we have

$$\|q(v+l)-q(v)-d(q(v))l\|_{\mathscr{C}(0,M;H_0^1(D))} = \|\gamma\|_{\mathscr{C}([0,M];H_0^1(D))} \leq k\|l\|^2_{L^2(0,M,H_0^1(D))}. \tag{55}$$

We introduce the family of optimal systems in the case of partial controllability

$$\begin{cases} -(p_i)_m(x,m) = (p_i)_{xx}(x,m) + (v(m)p_i)_x(x,m), & \Gamma, \\ p_i(x,M) = \left(\dfrac{\partial q(M)}{\partial x_i} - \widetilde{\chi}^*_\omega y_i^d\right), & D, \\ p_i(x,m) = (p_i)_x(x,m) = 0, & \Pi, \end{cases} \tag{56}$$

where $\widetilde{\chi}^*_\omega$ is the adjoint of $\widetilde{\chi}_\omega$ defined from $L^2(\omega) \longrightarrow L^2(D)$ by

$$\widetilde{\chi}^*_\omega q(x) = \begin{cases} q(x), & x \in \omega, \\ 0, & x \in D\backslash\omega. \end{cases} \tag{57}$$

The following lemma mentions the differential of $\Phi_\varepsilon(v)$ with respecting v. □

Lemma 9. *If $v \in L^2(0,M)$ is the control realizing (42), μ is the output of (47), and p_i is the solution of (56), we deduce*

$$\lim_{\beta\longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l)-\Phi_\varepsilon(v)}{\beta} = \sum_{i=1}^n \int_\omega \widetilde{\chi}^*_\omega \widetilde{\chi}_\omega \left[\int_0^M \frac{\partial p_i}{\partial m}\frac{\partial \mu(x,m)}{\partial x_i}dm + \int_0^M p_i \frac{\partial}{\partial x_i}\left(\frac{\partial \mu}{\partial m}\right)dm\right]dx + \int_0^M 2\varepsilon l_m v_m dm. \tag{58}$$

Proof. The functional $\Phi_\varepsilon(v)$ given by (43) can take the form

$$\Phi_\varepsilon(v) = \frac{1}{2}\sum_{i=1}^n \int_\omega \left(\widetilde{\chi}_\omega \frac{\partial q}{\partial x_i} - y_i^d\right)^2 dx + \varepsilon \int_0^M v_m^2(m)dm. \tag{59}$$

Let $q_\beta = q(v+\beta l)$ and $q = q(v)$, using $(59)_2$ we have

$$\begin{aligned} \lim_{\beta\longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l)-\Phi_\varepsilon(v)}{\beta} &= \lim_{\beta\longrightarrow 0} \sum_{i=1}^n \frac{1}{2}\int_\omega \frac{\left(\widetilde{\chi}_\omega\left(\partial q_\beta/\partial x_i\right) - y_i^d\right)^2 - \left(\widetilde{\chi}_\omega\left(\partial q/\partial x_i\right) - y_i^d\right)^2}{\beta}dx \\ &+ \lim_{\beta\longrightarrow 0} \frac{\varepsilon}{\beta}\int_0^M \left[\left(v_m+\beta l_m\right)^2 - v_m^2\right]dm. \end{aligned} \tag{60}$$

Consequently,

$$\begin{aligned} \lim_{\beta\longrightarrow 0} \frac{\Phi_\varepsilon(v+\beta l)-\Phi_\varepsilon(v)}{\beta} &= \lim_{\beta\longrightarrow 0} \sum_{i=1}^n \frac{1}{2}\int_\omega \widetilde{\chi}_\omega \frac{\left(\left(\partial q_\beta/\partial x_i\right) - \left(\partial q/\partial x_i\right)\right)}{\beta}\left(\widetilde{\chi}_\omega\left(\partial q_\beta/\partial x_i\right) + \widetilde{\chi}_\omega\left(\partial q/\partial x_i\right) - 2y_i^d\right)dx \\ &+ \lim_{\beta\longrightarrow 0}\int_0^M \left(2\varepsilon l_m v_m + \varepsilon\beta l_m^2\right)dm = \sum_{i=1}^n \int_\omega \widetilde{\chi}_\omega \frac{\partial\mu(x,M)}{\partial x_i}\widetilde{\chi}_\omega\left(\frac{\partial q(x,M)}{\partial x_i} - \widetilde{\chi}^*_\omega y_i^d\right)dx + \int_0^M 2\varepsilon l_m v_m dm \\ &= \sum_{i=1}^n \int_\omega \widetilde{\chi}_\omega \frac{\partial\mu(x,M)}{\partial x_i}\widetilde{\chi}_\omega p_i(x,M)dx + 2\varepsilon\int_0^M l_m v_m dm. \end{aligned} \tag{61}$$

From (56) and (61), we deduce that

$$\lim_{\beta\longrightarrow 0}\frac{\Phi_\varepsilon(v_\varepsilon+\beta l)-\Phi_\varepsilon(v_\varepsilon)}{\beta}=\sum_{i=1}^{n}\int_\omega \tilde{\chi}_\omega^*\tilde{\chi}_\omega\left[\int_0^M\frac{\partial p_i}{\partial m}\frac{\partial\mu(x,m)}{\partial x_i}dm+\int_0^M p_i\frac{\partial}{\partial x_i}\left(\frac{\partial\mu}{\partial m}\right)dm\right]dx+\int_0^M 2\varepsilon l_m v_m dm. \tag{62}$$

Now, we will deduce the solution of (42), exploiting the family of optimal systems. □

Theorem 10. *Let $v\in L^2(0,M;H_0^1)$ be the solution of the partial flow problem, and $q=q(v)$ is its corresponding state of (1), we show that*

$$\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}-2\varepsilon v_{mm}=0, \tag{63}$$

is a solution of problem (42), where $p_i\in C([0,M];H_0^1(D))$ is the unique solution of the adjoint system (56).

Proof. Let $l\in L^2(0,M;H_0^1(D))$ with $v+\beta l\in L^2(0,M;H_0^1(D))$ for $\beta>0$. The functional Φ_ε get its minimum at v, and we deduce

$$0\le\lim_{\beta\longrightarrow 0}\frac{\Phi_\varepsilon(v+\beta l)-\Phi_\varepsilon(v)}{\beta}. \tag{64}$$

Using Lemma 9, replacing $\partial\mu/\partial m$ in system (47), we have

$$\begin{aligned}0&\le\lim_{\beta\longrightarrow 0}\frac{v_\varepsilon(q_\varepsilon+\beta l)-v_\varepsilon(q_\varepsilon)}{\beta}\\&=\sum_{i=1}^{n}\int_\omega\tilde{\chi}_\omega^*\tilde{\chi}_\omega\left[\int_0^M\frac{\partial\mu}{\partial x_i}\frac{\partial p_i}{\partial m}dm+\int_0^M\frac{\partial}{\partial x_i}(\mu_{xx}-v\mu_x-lq_x)p_i dm\right]dx+\int_0^M 2\varepsilon l_m v_m dm\\0&\le\sum_{i=1}^{n}\int_\omega\tilde{\chi}_\omega^*\tilde{\chi}_\omega\left[\int_0^M\frac{\partial\mu}{\partial x_i}\left(\frac{\partial p_i}{\partial m}+\frac{\partial^2 p_i}{\partial x^2}+v(m)\frac{\partial p_i}{\partial x}\right)+l(m)\frac{\partial q_x}{\partial x_i}p_i dm\right]dx+\int_0^M 2\varepsilon l_m v_m dm\\&=\sum_{i=1}^{n}\int_0^M l(m)\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}dm+\int_0^M 2\varepsilon l_m v_m dm\\&=\int_0^M\left[l(m)\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}+2\varepsilon l_m v_m\right]dm.\end{aligned} \tag{65}$$

Consequently, for an arbitrary control $l=l(m)$ we conclude

$$l(m)\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}-2\varepsilon v_{mm}l(m)=0. \tag{66}$$

Then,

$$\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}-2\varepsilon v_{mm}=0. \tag{67}$$

Consequently,

$$v_{mm}=\frac{-1}{2\varepsilon}\sum_{i=1}^{n}\langle\tilde{\chi}_\omega\frac{\partial q_x}{\partial x_i};\tilde{\chi}_\omega p_i\rangle_{L^2(\omega)}. \tag{68}$$

□

4. Example

In this section, we propose the numerical approach to computing the solution of our method (68). We consider the one dimensional bilinear equation

$$\begin{cases}\dfrac{\partial q}{\partial t}+\alpha\dfrac{\partial^2 q}{\partial x^2}=\beta v(x,m)\dfrac{\partial q}{\partial x}, & [0,1]\times[0,1],\\ q(x,0)=q_0(x)=2x, & [0,1],\\ q=0, & \text{at } x=0,1.\end{cases} \tag{69}$$

The operator $-\alpha(\partial^2/\partial x^2)$ admits a set of eigenfunctions $\phi_n(\cdot)$ associated to the eigenvalues λ_n given by

$$\begin{aligned}\phi_n(x)&=\sqrt{2}\sin(n\pi x);\\\lambda_n&=\alpha n^2\pi^2,\\n&\ge 1.\end{aligned} \tag{70}$$

While the operator $-\alpha(\partial^2/\partial x^2)$ of system (69) and the perturbation $\beta v(x,m)\partial q/\partial x$ commute, using Pazy [37], the solution of (69) can be written as

Step 1: Choose
The desired targ y^d et.
The convergence accuracy ζ.
The subregion ω and time M.
Step 2: Until $\|v^{n+1} - v^n\| \leq \zeta$ repeat
Using (71), compute q^n associated to v^n.
Using (72), compute p^n associated to v^n.
Using (74) and (75), compute v^{n+1}.
Step 3: v^n such that $\|v^{n+1} - v^n\| \leq \zeta$ is the minimum of (76).

ALGORITHM 1:Algorithm for calculating the solution of the problem (50).

$$q(x,m) = \sum_{n=1}^{n=N} e^{\alpha n^2 \pi^2 t} < e^{\beta \int_0^m (\partial v/\partial x)(x,m)\mathrm{d}m} q_0, \sqrt{2}\sin(n\pi x) > \sqrt{2}\sin(n\pi x). \tag{71}$$

The one dimensional adjoint system can be written as

$$\begin{cases} -p_m(x,m) = p_{xx}(x,m) + (v(m)p)_x(x,m), & [0,1]\times[0,1], \\ p(x,1) = \left(\dfrac{\partial q(M)}{\partial x} - \widetilde{\chi}_\omega^* y^d\right), & [0,1], \\ p(0,m) = (p)_x(0,m) = 0, & [0,1]. \end{cases} \tag{72}$$

We define the perturbation function

$$f(q,p) = \langle \widetilde{\chi}_\omega \frac{\partial^2 q(m)}{\partial x^2}; \widetilde{\chi}_\omega p(m)\rangle_{L^2(\omega)}. \tag{73}$$

Using (68) and a finite difference schema, the optimal control v can be found by solving

$$\begin{cases} v_{mm}(x,m) = \dfrac{-1}{2\varepsilon} f(q,p), & [0,1]\times[0,1], \\ v(0) = v_m(1) = 0, & [0,1]. \end{cases} \tag{74}$$

By choosing $\varepsilon = 1/n$, we define the following sequence of control $(v^n)_n$ solution of

$$\begin{cases} v_{mm}^{n+1}(x,m) = \dfrac{-n}{2} f(q^n,p^n), & [0,1]\times[0,1], \\ v(0) = v_m(1) = 0, & [0,1], \end{cases} \tag{75}$$

where q^n and p^n are, respectively, the solution of (71) and (72) perturbed by v^n with $v^0 = 0$.

The penalty cost (43) becomes

$$\Phi_n(v^n) = \frac{1}{2}\left\|\chi_\omega \nabla q^n - y^d\right\|^2_{L^2\left(0,1,H_0^1([0,1])\right)} + \frac{1}{2n}\int_0^1\int_0^1 \left[v_m^n(x,tm)\right]^2 \mathrm{d}x\mathrm{d}m. \tag{76}$$

The following convergent Algorithm 1 allows the implementation of our results.

Remark 11.

(1) The distributed bilinear systems (1), are considered with the feedback map $v(x,m)q_x(x,m)$ as multiplication of the control by the velocity of the state system. One can consider another different type of perturbation.
(2) In the case of partial controllability, we use in general temporal control feedback. This type of control is compatible with real applications.
(3) For the simulation point of view, the obtained control formula is easy to calculate numerically. This encourages us to establish numerical approaches and simulations of the proposed problems using Algorithm 1..

5. Conclusion

We consider the flow optimal control problem constrained by a bilinear distributed system. The chosen optimal controls are regular, and the existence of solutions is proved and characterized using optimization techniques. Our method is applied to the partial flow control problem allowing us to control a flow on a specific subdomain of the system domain.

Finally, as an example, we present the numerical approach, which makes it possible to concretize the obtained results.

Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Common Fixed Point Theorems for F-Kannan–Suzuki Type Mappings in TVS-Valued Cone Metric Space with Some Applications

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This research paper generalizes and extends various fixed-point results that demonstrate common fixed-point theorems for F-Kannan–Suzuki type mappings in TVS-valued cone metric spaces. The results are supported using interpretative exemplifications and applications that include nonlinear fractional as well as two-point periodic ordinary differential equations.

1. Introduction

The fixed-point theory is at the foundation of nonlinear analysis, which is a prominent research area of mathematics. Fixed point theory is, in fact, a simple, powerful, and useful tool for nonlinear analysis. It also has fruitful applications in mathematics and in various scientific domains, including physics, chemistry, computer science, etc. As a result, this theory has attracted a large number of researchers who are guiding the theory's growth in various areas.

In 1922, Banach [1] established a fixed point theorem in metric space which states that if $\mathscr{X}$ is a complete metric space and $G\colon \mathscr{X} \longrightarrow \mathscr{X}$ is a contraction map, i.e., $\varrho(G\sigma, G\varsigma) \leq \kappa\varrho(\sigma, \varsigma)$ for all $\sigma, \varsigma \in \mathscr{X}$ and $\kappa \in [0,1)$, then G has a unique fixed point or $G\sigma = \sigma$ has a unique solution. In addition to an acceptable contraction condition, the metrical common fixed-point theorems usually include constraints on commutativity, continuity, completeness, and appropriate containment of ranges of detailed maps. The goal of researchers in this field is to weaken one or more of these conditions. The use of weak conditions of commutativity is to improve common fixed point theorems in analysis. Connell [2] provided an example of a noncomplete metric space X, but every contraction on it has a fixed point. Kannan [3] proposed an alternative contractive condition that was not the same as the Banach contraction condition. Also, Subrahmanyam [4] proved the converse of Banach fixed-point theorem using Kannan mapping. Furthermore, to evaluate a fixed point for a stringent type Kannan contraction, the assumption of continuity of the mapping and the compactness requirement on metric space are necessary.

In 2007, Huang and Zhang [5] generalised the Banach fixed point theorem by introducing the structure of cone metric by substituting real numbers with an ordered Banach space and establishing a convergence criterion for sequences in a cone metric space. In normal cone metric space, Huang and Zhang [5] proved some fixed-point theorems for Kannan type contractive conditions; nevertheless, Rezapour and Hamlbarani [6] neglected this idea in some results by Huang. For normal and nonnormal cones in cone metric spaces, several authors have examined fixed point theorems and common fixed-point theorems for self-mappings. We refer to the reader [7–10] and the references therein. By relaxing the normalcy criteria set by Huang and Zhang [5], Beg et al. [11], investigated common fixed points for a pair of maps on topological vector space (TVS) valued cone metric spaces in 2009. They demonstrated that the class of TVS-valued cone metric spaces is larger than the class of cone metric spaces, used in [12–16] and the references therein. Recently, Hu and Gu [17] proved some fixed point theorems of λ-contractive mappings in Menger PSM-spaces. For a class of contractive mappings, Reich and Alexander [18] generalised fixed points and convergence results. In Hausdroff TVS, Ram and Lai [19] presented the existence results on generalised strong operator equilibrium problems. In TVS-Cone Metric Spaces, Dubey and Mishra [20] demonstrated some fixed-point results of single-valued mapping for -distance. Using some facts about topological vector space, Tas [21] constructed a new notion of a TVS cone S- metric space. Lee [22] introduces chain recurrent set, trapping region, attracting set and repelling set for a flow f on a TVS-cone metric space. By using generalised metric spaces, Ge and Yang [23] proved a common generalisation of TVS-cone metric spaces, partial metric spaces and b-metric spaces, and a unified approach is proposed for some fixed point results. Later, Suzuki [24] and Rida et al. [25] gave a generalisation of the Banach contraction principle that characterises metric completeness.

Wardowski [26] used a new sort of contraction called F-contraction to give an intriguing generalisation of the Banach fixed point theorem. Many scholars have used his method to build new fixed-point theorems since then. The associated results and references can be found in [27–30] and the references therein. Piri and Kumam [28], extended Wardowski's [26] results in 2014 by introducing the notion of F-Suzuki contraction and obtained some intriguing results utilising the Secelean [29] concept. In the complete b-metric spaces, Alsulami et al. [31] demonstrated fixed points of generalised F-Suzuki type Contractions. Budhia et al. [32] proved an extension of almost-F and F-Suzuki contractions with graph and demonstrated some applications to fractional calculus whereas Chandok et al. [33] formulated some fixed point results for the generalised F-Suzuki type contractions in b-metric spaces. Derouiche and Ramou [34] proved new fixed-point results for F-contractions of Suzuki Hardy-Rogers type in b-metric spaces and provided some applications. Beg et al. [11] proposed a fixed point of orthogonal F-Suzuki contraction mapping on 0-complete b-metric spaces with some applications. Mani et al. [35] introduced generalised orthogonal F-contraction and orthogonal F-Suzuki contraction mappings and proved some fixed point theorems for a self-mapping in orthogonal metric space. Vujakovic and Radenovic [36] introduced certain fixed point results for F-contraction of Piri-Kumam-Dung-type mappings in metric spaces.

In 2019, Goswami et al. [27] introduced F-contractive type mappings in b-metric spaces and proved some fixed point results with suitable examples. Recently, Batra et al. [37] noticed in their subsequent analysis that the definition introduced by Goswami et al. [27] is not meaningful in general. Therefore, they provided suitable examples to support their opinion on this definition. Also, due to these reasons, Batra et al. [37] presented F-contraction and Kannan mapping concepts for defining F-Kannan mappings, which is, in a true sense, a generalisation of Kannan mappings.

This paper aims to extend and generalise the results due to Batra et al. [37], Filipovic et al. [38], Morales and Rojas [9], Rahimi et al. [39] and Wangwe and Kumar [40] using a pair of two self-mappings in F-Kannan–Suzuki type mapping in TVS-valued cone metric space, where we consider a map to be sequentially convergent, one to one and continuous. By doing so, we will extend several other results of the same setting in the literature. Finally, we will provide some applications to the nonlinear Riemann–Liouville fractional differential equation and nonlinear Volterra-integral differential equation.

2. Preliminaries

The definitions, lemmas, and theorems will help us prove our main points in the upcoming sections.

In 1968, Kannan [3] developed a new contractive condition and proved the following theorem for self mappings in complete metric spaces as a result of a generalisation of the Banach fixed point theorem.

Theorem 1 (see [3]). *Let $G\colon \mathscr{X} \longrightarrow \mathscr{X}$ be a self mapping on a complete metric space $(\mathscr{X}, \varrho)$ such that*

$$\varrho(G\sigma, G\varsigma) \leq \kappa\{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)\}, \tag{1}$$

for all $\sigma, \varsigma \in \mathscr{X}$ and $0 \leq \kappa \leq (1/2)$. Then, G possesses a unique fixed point $\sigma^* \in \mathscr{X}$ and for any $\sigma \in \mathscr{X}$ the iterate sequence $\{G^n\sigma\}$ converges to σ^*.

Equation (1) is equivalent to

$$\varrho(G\sigma, G\varsigma) \leq \frac{\kappa}{2}\{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)\}, \tag{2}$$

for some $\kappa \in (0, 1)$.

Definition 1 (see [11]). Let $(\mathscr{E}, \tau)$ be always a topological space and $\mathscr{P}$ a subset of $\mathscr{E}$. Then, $\mathscr{P}$ is called a cone if the following hold:

(i) $\mathscr{P}$ is a nonempty, closed and $\mathscr{P} \neq \{0\}$;
(ii) $\lambda\sigma + \mu\varsigma \in \mathscr{P}$ for all $\sigma, \varsigma \in \mathscr{P}$ and nonnegative real number λ, μ;
(iii) $\mathscr{P} \cap (-\mathscr{P}) = \{0\}$.

For given cone $\mathscr{P} \subseteq \mathscr{E}$. If the interior of $\mathscr{P}$ (int$\mathscr{P}$), is nonempty we say that $\mathscr{P}$ is solid. If $\mathscr{P}$ is solid cone, then $\mathscr{P}$ is a component of $\mathscr{P}$, and in this case we use the notation $\sigma \ll \varsigma$ to indicate that $\varsigma - \sigma \in \text{int}\mathscr{P}$. Note that if $\sigma \ll \varsigma$ and $\varsigma \leq \upsilon$, then $\sigma \ll \upsilon$ for all $\sigma, \varsigma, \upsilon \in \text{int}\mathscr{P}$.

The following axioms satisfy TVS-valued cone complete metric space.

Definition 2 (see [11]). Let $\mathscr{X}$ be a nonempty set and the mapping $\varrho\colon \mathscr{X} \times \mathscr{X} \longrightarrow \mathscr{E}$, satisfies the following:

(i) $0 \leq \varrho(\sigma, \varsigma)$, for all $\sigma, \varsigma \in \mathscr{X}$ and $\varrho(\sigma, \varsigma) = 0$ if and only if $\sigma = \varsigma$;
(ii) $\varrho(\sigma, \varsigma) = \varrho(\varsigma, \sigma)$, for all $\sigma, \varsigma \in \mathscr{X}$;
(iii) $\varrho(\sigma, \varsigma) \leq \varrho(\sigma, \upsilon) + \varrho(\upsilon, \varsigma)$, for all $\sigma, \varsigma, \upsilon \in \mathscr{X}$.

Then, ϱ is called a cone metric on $\mathcal{X}$, and $(\mathcal{X}, \varrho)$ is called topological vector space valued cone metric space.

Example 1 (see [12, 9, 41]). Let $\mathscr{E} = (C_{[0,1]}, \mathbb{R}^2)$, $\mathscr{P} = \{(\sigma, \varsigma) \in \mathscr{E} | \sigma, \varsigma \geq 0\} \subset \mathbb{R}^2$, $\mathcal{X} = \mathbb{R}$ and $\varrho\colon \mathcal{X} \times \mathcal{X} \longrightarrow \mathscr{E}$ such that $\varrho(\sigma, \varsigma) = |\sigma - \varsigma| \psi(t)$, where $\psi(t) = e^t$. Then, $(\mathcal{X}, \Sigma)$ is a TVS-valued cone metric space.

The following definition is due to Beg et al. [11] in TVS-valued cone metric space.

Definition 3 (see [8]). Let $(\mathcal{X}, \varrho)$ be a topological vector space valued cone metric space, and let $x \in \mathcal{X}$ and $\{\sigma_n\}_{n\geq 1}$ be a sequence in $\mathcal{X}$. Then,

(i) $\{\sigma_n\}_{n\geq 1}$ converges to $\mathcal{X}$ whenever for every $c \in \mathscr{E}$ with $0 \ll c$ there is a natural number $\mathbb{N}$ such that $\varrho(\sigma_n, \sigma) \ll c$ for all $n \geq \mathbb{N}$. We denote this by

$$\lim_{n\longrightarrow\infty} \sigma_n = \sigma \Leftrightarrow \sigma_n \longrightarrow \sigma. \tag{3}$$

(ii) $\{\sigma_n\}_{n\geq 1}$ is Cauchy sequence whenever for every $c \in \mathscr{E}$ with $0 \ll c$, there is a natural number $\mathbb{N}$ such that $\varrho(\sigma_n, \sigma_m) \ll c$ for all $n, m \geq \mathbb{N}$.

(iii) $(\mathcal{X}, \varrho)$ is called topological vector space valued cone metric space if every Cauchy sequence is convergent.

Definition 4 (see [42]). Let $\mathcal{X}$ be a topological space. If (σ_n) is a sequence of points of $\mathcal{X}$, and if $n_1 < n_2 < \ldots < n_i < \ldots$ is an increasing sequence of positive integers, then the sequence (ς_i) defined by setting $\varsigma_i = \sigma_{n_i}$ is called a subsequence of the sequence (σ_n). The space $\mathcal{X}$ is said to be sequentially compact if every sequence of points of $\mathcal{X}$ has a convergent subsequence.

Definition 5 (see [43]). Let $(\mathcal{X}, d)$ be a metric space. A mapping $G\colon \mathcal{X} \longrightarrow \mathcal{X}$ is said to be sequentially convergent if we have, for every sequence $\{\varsigma_n\}$, if $\{G\varsigma_n\}$ is convergence then $\{\varsigma_n\}$ also is convergence. G is said to be subsequentially convergent if we have, for every sequence $\{\varsigma_n\}$, if $\{G\varsigma_n\}$ is convergence then $\{\varsigma_n\}$ has a convergent subsequence.

The extended version of sequentially convergent mappings in TVS-valued cone metric space is given as follows.

Definition 6 (see [9]). Let $(\mathcal{X}, \varrho)$ be a cone metric space, $\mathscr{P}$ is a solid cone and $G\colon \mathcal{X} \longrightarrow \mathcal{X}$. Then

(i) G is said to be continuous if

$$\lim_{n\longrightarrow\infty} \sigma_n = \sigma \Rightarrow \lim_{n\longrightarrow\infty} G\sigma_n = G\sigma, \tag{4}$$

for all $\sigma_n \in \mathcal{X}$,

(ii) G is said to be sequentially convergent if we have, for every sequence (ς_n), if $G\varsigma_n$ is convergent, then ς_n also is convergent,

(iii) G is said to be subsequentially convergent if we have, for every sequence (ς_n) and $G\varsigma_n$ is convergent, implies ς_n has a convergent subsequence.

In 2011, Filipovic et al. [38] generalised Theorem 3.1 and Theorem 3.5 from [9] by using the sequentially convergent mappings in cone metric space and considered $\mathscr{P}$ to be a solid cone. They proved results on two self mappings as follows.

Definition 7 (see [38]). Let $(\mathcal{X}, \varrho)$ be a cone metric space and $T, f\colon \mathcal{X} \longrightarrow \mathcal{X}$ two mappings. A mapping f is said to be T- Hardy-Rogers contraction if there exists $a_i \geq 0, i = 1, \ldots, 5$ with $\sum_{i=1}^{5} a_i \leq 1$ such that for all $\sigma, \varsigma \in \mathcal{X}$.

$$\begin{aligned}\varrho(Tf\sigma, Tf\varsigma) \leq{}& a_1\varrho(T\sigma, T\varsigma) + a_2\varrho(T\sigma, Tf\sigma) + a_3\varrho(T\varsigma, Tf\varsigma)\\ &+ a_4\varrho(T\sigma, Tf\varsigma) + a_5\varrho(T\varsigma, Tf\sigma).\end{aligned} \tag{5}$$

Theorem 2 (see [38]). *Let $(\mathcal{X}, \varrho)$ be a complete cone metric space and $\mathscr{P}$ a solid cone, in addition let $T\colon \mathcal{X} \longrightarrow \mathcal{X}$ be a one-to-one, continuous mappings and $f\colon \mathcal{X} \longrightarrow \mathcal{X}$ a T-hardy-Rogers contraction. Then,*

(i) For every $\sigma_0 \in \mathcal{X}$ the sequence $Tf^n\sigma_0$ is Cauchy.

(ii) There is $v_{\sigma_0} \in \mathcal{X}$ such that $\lim_{n\longrightarrow\infty} Tf^n\sigma_0 = v_{\sigma_0}$.

(iii) T is sequentially convergent, then $(f^n\sigma_0)$ has a convergent, subsequence.

(iv) There is a unique $u_{\sigma_0} \in \mathcal{X}$ such that $fu_{\sigma_0} = u_{\sigma_0}$.

(v) If T is sequentially convergent, then for each $\sigma_0 \in \mathcal{X}$ the iterate sequence $(f^n\sigma_0)$ converges to u_{σ_0}.

Theorem 3 (see [38]). *Let $(\mathcal{X}, \varrho)$ be a complete cone metric space and $\mathscr{P}$ a solid cone, in addition let $T\colon \mathcal{X} \longrightarrow \mathcal{X}$ be a one-to-one, continuous mappings and $f\colon \mathcal{X} \longrightarrow \mathcal{X}$ such that $F(f) \neq \varnothing$ and that*

$$\varrho\left(Tf\sigma, Tf^2\sigma\right) \preccurlyeq \lambda\varrho(T\sigma, Tf\sigma), \tag{6}$$

holds for some $\lambda \in (0,1)$ and for all $\sigma \in \mathcal{X}, \sigma \neq f\sigma$. Then f has property $\mathscr{P}$.

Remark 1 (see [44]). Let $F(T)$ denote the fixed point set of a map T. A map T has property $\mathscr{P}$ if $F(T) = F(T^n)$ for each $n \in \mathbb{N}$. We shall say that a pair of maps T and f has property Q if $F(T) \cap F(f) = F(T^n) \cap F(f^n)$ for each $n \in \mathbb{N}$.

Secelean [29] proved the following lemma.

Lemma 1 (see [29]). *Let $F\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ be an increasing function and $\{x_n\}$ be a sequence of positive real numbers. Then the following holds:*

(a) If $\lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty$, then $\lim_{n\longrightarrow\infty} \alpha_n = 0$,

(b) If $\inf F = -\infty$ and $\lim_{n\longrightarrow\infty} \alpha_n = 0$, then $\lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty$,

Let $\mathfrak{F}$ be the set of all functions defined as $F\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$, which satisfies the following conditions:

(F1) F is strictly increasing i.e., for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta \Rightarrow F(\alpha) < F(\beta)$

(F2″) there is a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\longrightarrow\infty} F(\alpha_n) = -\infty$ or $\inf F = -\infty$

(F3″) F is continuous on $(0, \infty)$

The following function $F: \mathbb{R}^+ \longrightarrow \mathbb{R}$ belongs to $\mathfrak{F}$:

(i) $F_1(z) = \ln z$

(ii) $F_2(z) = -(1/z)$

(iii) $F_3(z) = -(1/z) + z$

Definition 8 (see [28]). Let $(\mathscr{X}, \varrho)$ be a metric space. A mapping $G: \mathscr{X} \longrightarrow \mathscr{X}$ is said to be an F-Suzuki contraction if there exists $\tau > 0$, such that for all $\sigma, \varsigma \in \mathscr{X}$ with $G\sigma \neq G\varsigma$

$$\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma, \varsigma) \Rightarrow \tau + F(\varrho(G\sigma, G\varsigma)) \leq F(\varrho(\sigma, \varsigma)), \quad (7)$$

where $F \in \mathfrak{F}$.

In 2014, Piri and Kumam [28] established a generalisation of Banach contraction principle, which is as follows:

Theorem 4 (see [28]). *Let $(\mathscr{X}, \varrho)$ be a complete metric space and $G: \mathscr{X} \longrightarrow \mathscr{X}$ be a F-Suzuki contraction. Then G has a unique fixed point $\sigma^* \in \mathscr{X}$ and for every $\sigma_0 \in \mathscr{X}$ a sequence $\{G^n\sigma_0\}_{n\in\mathbb{N}}$ is convergent to σ^*.*

Remark 2 (see [28]). We denote by $\mathfrak{F}$ the set of all functions satisfying F-suzuki type contraction condition due to [28, 29] and let denote by $\mathscr{F}$ the set of all functions satisfying F-contraction condition by Wardowski [26], then

(i) $\mathscr{F} \not\subseteq \mathfrak{F}$

(ii) $\mathfrak{F} \not\subseteq \mathscr{F}$

(iii) $\mathscr{F} \cap \mathfrak{F} \neq \emptyset$

For more details on F-Suzuki contraction mapping, one can see [31–33] and the references therein.

Motivated by Batra et al. [37], we use the following notations: Let $\mathscr{X}$ be a nonempty set and $(\mathscr{X}, \varrho)$ denotes the metric space with metric ϱ. Let the cardinality of a set A is denoted by card$\{A\}$ and Fix G is set of all fixed points of a mapping G.

Batra et al. [37] gave a new generalisation family of contraction called F-Kannan mapping and introduced the following definition:

Definition 9 (see [37]). Let F be a mapping satisfying $(F1) - (F3)$. A mapping $G: \mathscr{X} \longrightarrow \mathscr{X}$ is said to be an F-Kannan mapping if the following holds:

(K1)

$$G\sigma \neq G\varsigma \Rightarrow G\sigma \neq \sigma \text{ or } G\varsigma \neq \varsigma. \quad (8)$$

(K2) $\exists \Upsilon > 0$ such that

$$\Upsilon + F(\varrho(G\sigma, G\varsigma)) \leq F\left[\frac{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)}{2}\right], \quad (9)$$

for all $\sigma, \varsigma \in \mathscr{X}$, with $G\sigma \neq G\varsigma$.

The remark presented below is due to Batra et al. [37].

Remark 3 (see [37]). By properties of F, it follows that every F-Kannan mapping T on a metric space $(\mathscr{X}, \varrho)$, satisfies following condition:

$$\varrho(G\sigma, G\varsigma) \leq \frac{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)}{2}, \quad (10)$$

for every $\sigma, \varsigma \in \mathscr{X}$.

Furthermore, it is concluded that Card$\{\text{Fix}\, G\} \leq 1$. Let G be a self map of a metric space $(\mathscr{X}, \varrho)$. G is said to be a Picard operator (PO) if G has unique fixed point σ^* and $\lim_{n\longrightarrow\infty} G^n\sigma = \sigma^*$ for all $\sigma \in \mathscr{X}$.

Then the family of all functions $F: \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfying the condition $(F1) - (F3)$ is denoted by $\mathscr{F}$.

We recall the following examples from Batra et al. [37] of such functions $F: \mathbb{R}^+ \longrightarrow \mathbb{R}$ which satisfies $(F1) - (F3)$:

Example 2 (see [37]). Let $F_1: \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined as $F_1(z) = \ln(z)$. Then clearly, $(F1) - (F3)$ are satisfied by $F_1(z)$. In fact $(F3)$ holds for every $k \in (0, 1)$

$$\varrho(G\sigma, G\varsigma) \leq e^{-\Upsilon}\left[\frac{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)}{2}\right], \quad (11)$$

for all $\sigma, \varsigma \in \mathscr{X}$ with $G\sigma \neq G\varsigma$.

Thus, if $G: \mathscr{X} \longrightarrow \mathscr{X}$ is a Kannan mapping with constant $\kappa \in (0, 1)$ satisfying

$$\varrho(G\sigma, G\varsigma) \leq \kappa\left[\frac{\varrho(\sigma, G\sigma) + \varrho(\varsigma, G\varsigma)}{2}\right], \quad (12)$$

for every $\sigma, \varsigma \in \mathscr{X}$, then it also satisfies (8) and (11) with $\Upsilon = \ln(1/\kappa)$. In fact, whenever $G\sigma \neq G\varsigma$, then from (12), we get $G\sigma \neq \sigma$ or $G\varsigma \neq \varsigma$.

The following lemma introduced by Batra et al. [37].

Lemma 2 (see [37]). *Let $(\mathscr{X}, \varrho)$ be a metric space and $G: \mathscr{X} \longrightarrow \mathscr{X}$ be a F-Kannan mapping. Then, $\varrho(G^n\sigma, G^{n+1}\sigma) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\sigma \in \mathscr{X}$.*

Batra et al. [37] introduced a F-Kannan mapping using the properties by Subrahmanyam [4] which is an extension of Goswami et al. [27] and Wardowski [26] results. They proved the following result.

Theorem 5 (see [37]). *Let $(\mathscr{X}, \varrho)$ be a complete metric space and suppose $G: \mathscr{X} \longrightarrow \mathscr{X}$ is a F-Kannan mapping, then G is a Picard operator (PO).*

Using the following definitions, we introduce some fundamental properties for a fixed point and common fixed point theorems.

Definition 10 (see [45]). Let (G, f) be a pair of self-mappings on a metric space $(\mathcal{X}, \varrho)$. Then coincidence point of the pair (G, f) is a point $\sigma \in \mathcal{X}$ such that $(G\sigma) = (f\sigma) = \sigma^*$, then σ^* is called coincidence point of the pair (G, f). If $\sigma^* = \sigma$, then σ is said to be a common fixed point of f and G.

Definition 11 (see [46]). Let G, f be self-mappings of a nonempty set $\mathcal{X}$. A point $\sigma \in \mathcal{X}$ is coincidence point of G and f if $t = G\sigma = f\sigma$. The set of coincidence point of G and f is denoted by $C(G, f)$.

Definition 12 (see [46, 47]). Let (T, f) be a pair of self-mappings on a metric space $(\mathcal{X}, \varrho)$. Then, the pair (T, f) is said to be as follows:

(i) Commuting if, for all $\sigma \in \mathcal{X}$, $G(f\sigma) = f(G\sigma)$,

(ii) Weakly commuting if, for all $\varrho(G(f\sigma), f(G\sigma)) \leq \varrho(G\sigma, f\sigma)$,

(iii) Compatible if $\lim_{n\longrightarrow\infty}\varrho(Gf\sigma_n, fG\sigma_n) = 0$, whenever σ_n is a sequence in $\mathcal{X}$ such that $\lim_{n\longrightarrow\infty}G\sigma_n = \lim_{n\longrightarrow\infty}f\sigma_n = t$,

(iv) Weakly compatible if, for all $G(f\sigma) = f(G\sigma)$, for every coincidence point $\sigma \in \mathcal{X}$.

3. Main Results

To prove this section's main results, we commerce by obtaining a more general version of Definition 8 and 9 using a pair of two self mappings in F-Kannan–Suzuki type mapping setting. We denotes $(\mathcal{X}, \varrho)$ as a TVS-valued cone metric space.

Definition 13. Let F be a mapping satisfying $(F1) - (F3)$. A pair of two self mapping $G, f\colon \mathcal{X} \longrightarrow \mathcal{X}$ is said to be an F-Kannan–Suzuki type mapping if the following holds:

(FKS1)

$$Gf\sigma \neq Gf\varsigma \Rightarrow Gf\sigma \neq \sigma \,\text{or}\, Gf\varsigma \neq \varsigma. \tag{13}$$

(FKS2) there exists $\vartheta > 0$ such that

$$\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma, \sigma)$$
$$\Rightarrow \vartheta + F(\varrho(Gf\sigma, Gf\varsigma)) \leq F\left[\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}\right], \tag{14}$$

for all $\sigma, \varsigma \in \mathcal{X}$, with $Gf\sigma \neq Gf\varsigma$ and $F \in \mathfrak{F}$.

Following remark is motivated by the work of Batra et al. [37] given as follows.

Remark 4. By properties of F, it follows that every F-Kannan–Suzuki type mapping G on a TVS-valued cone metric space $(\mathcal{X}, \varrho)$, satisfies the following condition:

$$\varrho(Gf\sigma, Gf\varsigma)) \leq \frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}, \tag{15}$$

for every $\sigma, \varsigma \in \mathcal{X}$.

We give the following examples in the context of a pair of two self mappings:

Example 3. Let $F_1\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined as $F_1(z) = \ln(z)$. Then clearly, $(F1) - (F3)$ are satisfied by $F_1(z)$. In fact $(F3)$ holds for every $\kappa \in (0, 1)$. Moreover, condition (14) takes the form:

$$\varrho(Gf\sigma, Gf\varsigma)) \leq e^{-\vartheta}\left[\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}\right], \tag{16}$$

for all $\sigma, \varsigma \in \mathcal{X}$ with $Gf\sigma \neq Gf\varsigma$.

Thus, if $G, f\colon \mathcal{X} \longrightarrow \mathcal{X}$ is a Kannan mapping with constant $\kappa \in (0, 1)$ satisfying

$$\varrho(Gf\sigma, Gf\varsigma)) \leq \kappa\left[\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}\right]. \tag{17}$$

for every $\sigma, \varsigma \in \mathcal{X}$. Then it also satisfies (16) and (14) with $\vartheta = \ln(1/\kappa)$.

Example 4. Let $F_2\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined as $F_2(z) = -(1/z), z > 0$. Then, $(F1) - (F3)$ are satisfied by $F_2(z)$. Condition (14) takes the form:

$$\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2} \leq \frac{\varrho(Gf\sigma, Gf\varsigma)}{1 - \vartheta\varrho(Gf\sigma, Gf\varsigma)}, \tag{18}$$

for all $\sigma, \varsigma \in \mathcal{X}$ with $Gf\sigma \neq Gf\varsigma$.

Example 5. Let $F_3\colon \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined as $F_3(z) = -(1/z), z > 0$. Then, $(F1) - (F3)$ are satisfied by $F_3(z)$. Condition (14) takes the form:

$$\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2} \leq \frac{\varrho(Gf\sigma, Gf\varsigma)\left([(\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma))/2]^2 - 1\right)}{\varrho(Gf\sigma, Gf\varsigma) + \vartheta\left(\varrho(Gf\sigma, Gf\varsigma)^2 - 1\right)}, \tag{19}$$

for all $\sigma, \varsigma \in \mathcal{X}$ with $Gf\sigma \neq Gf\varsigma$.

We prove the following lemma which is an extension of Lemma 2.

Lemma 3. *Let $(\mathcal{X}, \varrho)$ be a complete TVS-valued cone metric space and $G, f\colon \mathcal{X} \longrightarrow \mathcal{X}$ be an F-Kannan–Suzuki type mapping. Then,*

$$\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right) \longrightarrow 0 \,\text{as}\, n \longrightarrow \infty, \tag{20}$$

for all $\sigma \in \mathcal{X}$.

Proof. Suppose that σ_0 is an arbitrary point in $\mathcal{X}$. If $Gf^n x_0 = Gf^{n+1}\sigma_0$ for some $n \in \mathbb{N}$, then sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ converges in $\mathcal{X}$, and hence the sequence $\varrho(Gf^n\sigma_0, Gf^{n+1}\sigma_0) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\sigma \in \mathcal{X}$.

Assume that $Gf^n\sigma_0 \neq Gf^{n+1}\sigma_0$ for any $n \in \mathbb{N}$. Then, by (14) with $\vartheta > 0$, we get

$$\begin{aligned}&\frac{1}{2}\varrho(\sigma_0, G\sigma_0) < \varrho(\sigma_0, G\sigma_0)\\ &\quad\Rightarrow \vartheta + F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right)\\ &\quad\leq F\left[\frac{\varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)}{2}\right].\end{aligned} \tag{21}$$

By Remark 4, we obtain

$$\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right) \leq \frac{\varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)}{2}. \tag{22}$$

Using (22) in (21), as results yields to

$$\vartheta + F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right) \leq F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right). \tag{23}$$

Letting $n \longrightarrow \infty$ in (23), we get

$$\begin{aligned}\vartheta + 0 &\leq 0,\\ \vartheta &\leq 0,\end{aligned} \tag{24}$$

which is a contradiction. Hence, $\varrho(Gf^n\sigma_0, Gf^{n+1}\sigma_0) \longrightarrow 0$ as $n \longrightarrow \infty$. □

Motivated by Batra et al. [37] and Filipovic et al. [38], we give a proof of an extended version of Theorem 2, 4, and 5 in F-Kannan–Suzuki type mappings with a pair of two self-mappings in complete TVS-valued cone metric space.

Theorem 6. *Let $(\mathcal{X}, \varrho)$ be a complete TVS-valued cone metric space and $\mathcal{P}$ a solid cone, in addition let $G\colon \mathcal{X} \longrightarrow \mathcal{X}$ be a one-to-one, continuous mappings and $f\colon \mathcal{X} \longrightarrow \mathcal{X}$ a G-F-Kannan–Suzuki type contraction. Then,*

(i) For every $\sigma_0 \in \mathcal{X}$ the sequence $Gf^n\sigma_0$ is convergent

(ii) There is $v^* \in X$ such that $\lim_{n\longrightarrow\infty} Gf^n\sigma_0 = v^*$

(iii) G is sequentially convergent, then $(f^n\sigma_0)$ has a convergent, subsequence

(iv) There is a unique $u^* \in X$ such that $fu^* = u^*$

(v) If G is sequentially convergent, then for each $\sigma_0 \in X$ the iterate sequence $(f^n\sigma_0)$ converges to u^*

Proof. By (i), we prove that $\{Gf^n\sigma_0\}$ is a Cauchy sequence. Let $\sigma_0 \in \mathcal{X}$ be arbitrary. If $Gf^n\sigma_0 = Gf^{n+1}\sigma_0$ for some $n \in \mathbb{N}$, then sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ converges in $\mathcal{X}$ and hence the sequence $\varrho(Gf^n\sigma_0, Gf^{n+1}\sigma_0) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\sigma \in \mathcal{X}$. Suppose that $Gf^n\sigma_0 \neq Gf^{n+1}\sigma_0$ for any $n \in \mathbb{N}$. Then, by (14), Lemma 3 with $\vartheta > 0$, we get

$$\begin{aligned}&\frac{1}{2}\varrho(\sigma_n, G\sigma_n) < \varrho(\sigma_n, G\sigma_n) \Rightarrow\\ &\vartheta + F\left(\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)\right) \leq F\left[\frac{\varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)}{2}\right].\end{aligned} \tag{25}$$

From Remark 4, we have

$$\begin{aligned}\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right) &\leq \frac{\varrho\left(Gf^{n-1}\sigma_0, Gf^n x_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right)}{2},\\ 2\varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right) &\leq \varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right) + \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right),\\ \varrho\left(Gf^n\sigma_0, Gf^{n+1}\sigma_0\right) &\leq \varrho\left(Gf^{n-1}\sigma_0, Gf^n\sigma_0\right).\end{aligned} \tag{26}$$

Using (26) in (25), as results yields to

$$\begin{aligned}
&\vartheta + F\left(\varrho\left(Gf^{n}\sigma_0, Gf^{n+1}\sigma_0\right)\right) \leq F\left[\frac{\varrho\left(Gf^{n-1}\sigma_0, Gf^{n}\sigma_0\right) + \varrho\left(Gf^{n}\sigma_0, Gf^{n+1}\sigma_0\right)}{2}\right],\\
&\vartheta + F\left(\varrho\left(Gf^{n}\sigma_0, Gf^{n+1}\sigma_0\right)\right) \leq F\left[\frac{2\varrho\left(Gf^{n-1}\sigma_0, Gf^{n}\sigma_0\right)}{2}\right],\\
&\vartheta + F\left(\varrho\left(Gf^{n}\sigma_0, Gf^{n+1}\sigma_0\right)\right) \leq F\left[\varrho\left(Gf^{n-1}\sigma_0, Gf^{n}\sigma_0\right)\right].
\end{aligned} \tag{27}$$

Letting $n \longrightarrow \infty$ in (27), we get

$$\begin{aligned}
\vartheta + 0 &\leq 0,\\
\vartheta &\leq 0,
\end{aligned} \tag{28}$$

which is a contradiction. Hence, $\varrho(Gf^{n}\sigma_0, Gf^{n+1}\sigma_0) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, $\{Gf^{n}\sigma_0\}$ converges.

Since G is sequentially convergent, using (v), we prove that the iterate of a sequence $f^{n}\sigma_0$ converge to a fixed $u \in \mathcal{X}$. To see this, suppose $\sigma_0 \in \mathcal{X}$ be an arbitrary point in $\mathcal{X}$. Let the sequence $\{\sigma_n\}_{n\geq 1}$ be defined by $\sigma_{n+1} = f\sigma_n = f^{n+1}\sigma_0 = ff^{n}\sigma_0$ and $\sigma_n = f\sigma_{n-1} = f^{n}\sigma_0 = ff^{n-1}\sigma_0$, for $n \geq 1 \in \mathbb{N}$. Thus, we have

$$\begin{aligned}
\varrho(\sigma_n, \sigma_{n+1}) \leq \varrho(f\sigma_{n-1}, f\sigma_n) &= \varrho\left(f^{n}\sigma_0, f^{n+1}\sigma_0\right)\\
&= \varrho\left(ff^{n-1}\sigma_0, ff^{n}\sigma_0\right).
\end{aligned} \tag{29}$$

equivalent to

$$\begin{aligned}
\varrho(G\sigma_n, G\sigma_{n+1}) &\leq \varrho(Gf\sigma_{n-1}, Gf\sigma_n)\\
&= \varrho\left(Gf^{n}\sigma_0, Gf^{n+1}\sigma_0\right)\\
&= \varrho\left(Gff^{n-1}\sigma_0, Tff^{n}\sigma_0\right).
\end{aligned} \tag{30}$$

Let $\sigma = f^{n-1}\sigma_0$ and $\varsigma = f^{n}\sigma_0$, using inequality (14), we obtain

$$\begin{aligned}
&\frac{1}{2}\varrho(\sigma_n, G\sigma_n) < \varrho(\sigma_n, G\sigma_n)\\
&\Rightarrow \vartheta + F\left(\varrho\left(Gff^{n-1}\sigma_0, Gff^{n}\sigma_0\right)\right) \leq F\left[\frac{\varrho\left(Gf^{n-1}\sigma_0, Gff^{n-1}\sigma_0\right) + \varrho\left(Gf^{n}\sigma_0, Gff^{n}\sigma_0\right)}{2}\right],\\
&F\left(\varrho(G\sigma_n, G\sigma_{n+1})\right) \leq F\left[\frac{\varrho(G\sigma_{n-1}, G\sigma_n) + \varrho(G\sigma_n, G\sigma_{n+1})}{2}\right] - \vartheta.
\end{aligned} \tag{31}$$

Using Remark 4 and the increasing property of F, we deduce

$$\begin{aligned}
\varrho\left(Gff^{n-1}\sigma_0, Gff^{n}\sigma_0\right) &\leq \frac{\varrho\left(Gf^{n-1}\sigma_0, Gff^{n-1}\sigma_0\right) + \varrho\left(Gf^{n}\sigma_0, Gff^{n}\sigma_0\right)}{2},\\
\varrho(G\sigma_n, G\sigma_{n+1}) &< \frac{\varrho(G\sigma_{n-1}, G\sigma_n) + \varrho(G\sigma_n, G\sigma_{n+1})}{2},
\end{aligned} \tag{32}$$

and hence,

$$\begin{aligned}
2\varrho(G\sigma_n, G\sigma_{n+1}) - \varrho(G\sigma_n, G\sigma_{n+1}) &< \varrho(G\sigma_{n-1}, G\sigma_n),\\
\varrho(G\sigma_n, G\sigma_{n+1}) &< \varrho(G\sigma_{n-1}, G\sigma_n).
\end{aligned} \tag{33}$$

By $(F1)$, this implies that

$$F\left(\varrho(G\sigma_n, G\sigma_{n+1})\right) < F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right). \tag{34}$$

Consequently, we get

$$\vartheta + F\left(\varrho(G\sigma_n, G\sigma_{n+1})\right) \leq F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right), \tag{35}$$

so

$$F\left(\varrho(G\sigma_n, G\sigma_{n+1})\right) \leq F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right) - \vartheta. \tag{36}$$

By induction and (36), we deduce

$$\begin{aligned}
&F\left(\varrho(G\sigma_{n+1}, G\sigma_{n+2})\right) \leq F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right) - 2\vartheta,\\
&F\left(\varrho(G\sigma_{n+2}, G\sigma_{n+3})\right) \leq F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right) - 3\vartheta,\\
&\Rightarrow F\left(\varrho(G\sigma_n, G\sigma_{n+1})\right) \leq F\left(\varrho(G\sigma_{n-1}, G\sigma_n)\right) - n\vartheta.
\end{aligned} \tag{37}$$

Letting $n \longrightarrow \infty$ in (37), we find that

$$\lim_{n\longrightarrow\infty} F(\varrho(G\sigma_n, G\sigma_{n+1})) = -\infty. \tag{38}$$

Consequently, using Lemma 1 and property $(F2'')$ of F results in

$$\lim_{n\longrightarrow\infty} \varrho(G\sigma_n, G\sigma_{n+1}) = 0. \tag{39}$$

Thus, there exists $n \in \mathbb{N}$ such that

$$\varrho(G\sigma_n, G\sigma_{n+1}) < \varrho\left(G\sigma_n, G^2\sigma_n\right) < c\varrho(\sigma_n, G\sigma_n) < \varrho(\sigma_n, G\sigma_n), \tag{40}$$

which is a contradiction. Hence, we have

$$\lim_{n\longrightarrow\infty} \varrho(\sigma_n, G\sigma_n) = 0. \tag{41}$$

Therefore, we have $\varrho(G\sigma_n, G\sigma_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$. Denote $\alpha_n = \varrho(G\sigma_n, G\sigma_{n+1}) = 0$, for all $n \geq 1$ and $n \in \mathbb{N}$, for F-Kannan–Suzuki type mappings.

By (39), we prove that $\{G\sigma_n\}$ is a Cauchy sequence since $(\mathscr{X}, \varrho)$ is complete. Consider $n, m \in N$ such that $m > n$. Assume on the contrary that there exists $c > 0$ and sequences $\{p(n)\}_{n\geq1}^{\infty}$ and $\{q(n)\}_{n\geq1}^{\infty}$ such that

$$\begin{aligned} p(n) > q(n) > n, \varrho\left(G\sigma_{p(n)}, G\sigma_{q(n)}\right) \\ \geq c, \varrho\left(G\sigma_{p(n)-1}, G\sigma_{q(n)}\right) \leq c, \forall n \in \mathbb{N}. \end{aligned} \tag{42}$$

Using (iii) of Definition 2, we get

$$\begin{aligned} \varrho\left(G\sigma_{p(n)}, G\sigma_{q(n)}\right) &\leq \varrho\left(G\sigma_{p(n)}, G\sigma_{p(n)-1}\right) \\ &\quad + \varrho\left(G\sigma_{p(n)-1}, G\sigma_{q(n)}\right) \\ &\leq \varrho\left(G\sigma_{p(n)}, G\sigma_{p(n)-1}\right) + c. \end{aligned} \tag{43}$$

From (39) and the above inequality, we have

$$\lim_{n\longrightarrow\infty} \varrho\left(G\sigma_{p(n)}, G\sigma_{q(n)}\right) = c. \tag{44}$$

From $(F3'')$, (44), and (14), we get

$$\begin{aligned} &\vartheta + F\left(\varrho\left(G\sigma_{p(n)}, G\sigma_{q(n)}\right)\right) \\ &\quad \leq F\left[\frac{\varrho\left(G\sigma_{p(n)-1}, G\sigma_{p(n)}\right) + \varrho\left(G\sigma_{p(n)}, G\sigma_{q(n)}\right)}{2}\right]. \end{aligned} \tag{45}$$

Equivalently,

$$\begin{aligned} \vartheta + F(c) &\leq F(c), \\ \vartheta &\leq 0, \end{aligned} \tag{46}$$

which is a contradiction. So, $G\sigma_n = G\sigma_m$ for every $m \geq n$ in $\mathscr{X}$. Hence, $\{G\sigma_n\}$ is a Cauchy sequence in $\mathscr{X}$. The completeness of $\mathscr{X}$ ensures the existence of $u^* \in \mathscr{X}$ such that

$$\begin{aligned} \varrho(Gu^*, u^*) &= \lim_{n,m\longrightarrow\infty} \varrho(G\sigma_n, G\sigma_m) = 0 \\ &= \lim_{n\longrightarrow\infty} \varrho(G\sigma_n, u^*) = 0. \end{aligned} \tag{47}$$

By (47) and Definition 6, it follows that $G\sigma_{n+1} \longrightarrow u^*$ as $n \longrightarrow \infty$. By sequential continuity of f and G, we have

$$\begin{aligned} u^* &= \lim_{n\longrightarrow\infty} f^n\sigma_0 = \lim_{n\longrightarrow\infty} \sigma_n = \lim_{n\longrightarrow\infty} \sigma_{n+1} \\ &= \lim_{n\longrightarrow\infty} f\sigma_n = fu^*.u^* \\ &= \lim_{n\longrightarrow\infty} Gf^n\sigma_0 = \lim_{n\longrightarrow\infty} G\sigma_n = \lim_{n\longrightarrow\infty} G\sigma_{n+1} \\ &= \lim_{u\longrightarrow\infty} G^2\sigma_n = Gu^*. \end{aligned} \tag{48}$$

Since $\mathscr{X}$ is a complete metric space, there exists $u^* \in \mathscr{X}$ such that

$$\lim_{n\longrightarrow\infty} G\sigma_n = Gu^* = u^*. \tag{49}$$

Now, we prove that u^* is a fixed point of G. Thus, by (iii) of Definition 2 and $\varrho(u^*, Gu^*) \geq 0$, we have

$$\varrho(u^*, Gu^*) \leq \varrho(u^*, G\sigma_{n+1}) + \varrho(G\sigma_{n+1}, Gu^*). \tag{50}$$

By Remark 4, it implies that

$$\varrho(G\sigma_{n+1}, Gu^*) \leq \frac{\varrho(G\sigma_n, G\sigma_{n+1}) + \varrho(G\sigma_{n+1}, Gu^*)}{2}. \tag{51}$$

Applying (51) in (50), we obtain

$$\varrho(u^*, Gu^*) \leq \varrho(u^*, G\sigma_{n+1}) + \frac{\varrho(G\sigma_n, G\sigma_{n+1}) + \varrho(G\sigma_{n+1}, Gu^*)}{2}. \tag{52}$$

Letting $n \longrightarrow \infty$ and using in above inequality, we get

$$\begin{aligned} \varrho(u^*, Gu^*) &\leq \varrho(u^*, Gu^*) \\ &\quad + \frac{\varrho(u^*, Gu^*) + \varrho(Gu^*, Gu^*)}{2}, \\ \varrho(u^*, Gu^*) &\leq \varrho(u^*, Gu^*) + \frac{\varrho(u^*, Gu^*)}{2}, \\ \varrho(u^*, Gu^*) &\leq \frac{2\varrho(u^*, Gu^*) + \varrho(u^*, Gu^*)}{2}, \\ 2\varrho(u^*, Gu^*) &\leq 2\varrho(u^*, Gu^*) + \varrho(u^*, Gu^*), \\ 2\varrho(u^*, Gu^*) - 2\varrho(u^*, Gu^*) &\leq \varrho(u^*, Gu^*), \\ 0 &\leq \varrho(u^*, Gu^*), \end{aligned} \tag{53}$$

which is a contradiction. Hence, $Gu^* = u^*$.

Next, we prove that u^* is a unique fixed point of G. Assume on contrary that there exists $v^* \in \mathrm{int}(\mathscr{P})$ such that $u^* \neq v^*$ or $Gu^* \neq Gv^*$. Let $G\sigma_n \longrightarrow v^*$ and v^* is a fixed point of G. Using Remark 4 and (14), it follows that $u^* = v^*$ or $Gu^* = Gv^*$ which is a contradiction. Thus, u^* is a unique fixed point of G.

Moreover, G is a subsequentially convergent, $\{f^n\sigma_0\}$ has a convergent subsequence, there exists $\sigma^* \in X$ and $\{f^{n_k}x_0\}_{k=1}^{\infty}$ such that

$$\lim_{k\longrightarrow\infty} f^{n_k}x_0 = v^*. \tag{54}$$

Due to the continuity of G, it implies that

$$\lim_{k\longrightarrow\infty} Gf^{n_k}\sigma_0 = Gv^*. \tag{55}$$

By (49), we conclude that

$$Gv^* = u^*. \tag{56}$$

Using Remark 4 and $\lambda = (1/2)$, we get

$$\begin{aligned}
\varrho\left(Gff^{n_{k-1}}\sigma_0, Tff^{n_k}\sigma_0\right) &\leq \lambda\left(\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) + \varrho\left(Gf^{n_k}\sigma_0, Gff^{n_k}\sigma_0\right),\right.\\
&\leq \lambda\left(\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) + \varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right),\right.\\
&\leq \lambda\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) + \lambda\varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right),\\
(1-\lambda)\varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right) &\leq \lambda\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right),\\
\varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right) &\leq \frac{\lambda}{1-\lambda}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_{k-1}}\sigma_0\right).
\end{aligned} \tag{57}$$

Thus, using (iii) of Definition 2, we have

$$\varrho\left(Gfv^*, Gv^*\right) \leq \varrho\left(Gfv^*, Gf^{n_k+1}\sigma_0\right) + \varrho\left(Gf^{n_k+1}\sigma_0, Gv^*\right). \tag{58}$$

By Remark 4,

$$\begin{aligned}
\varrho\left(Gfv^*, Gf^{n_k+1}\sigma_0\right) &= \varrho\left(Gfv^*, Gff^{n_k}\sigma_0\right)\\
&\leq \lambda\left[\varrho\left(Gv^*, Gfv^*\right) + \varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right)\right].
\end{aligned} \tag{59}$$

Using (57) and (59) in (58), we obtain

$$\begin{aligned}
\varrho\left(Gfv^*, Gv^*\right) &\leq \lambda\left[\varrho\left(Gv^*, Gfv^*\right) + \varrho\left(Gff^{n_{k-1}}\sigma_0, Gff^{n_k}\sigma_0\right)\right] + \varrho\left(Gf^{n_k+1}\sigma_0, Gv^*\right),\\
&\leq \lambda\left[\varrho\left(Gv^*, Gfv^*\right) + \frac{\lambda}{1-\lambda}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_{k-1}}\sigma_0\right)\right] + \varrho\left(Gf^{n_k+1}\sigma_0, Gv^*\right)\\
&\leq \lambda\varrho\left(Gv^*, Gfv^*\right) + \lambda\left(\frac{\lambda}{1-\lambda}\right)^{n_k-1}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_{k-1}}\sigma_0\right) + \varrho\left(Gf^{n_k+1}\sigma_0, Gv^*\right),\\
&\leq \frac{\lambda}{1-\lambda}\left(\frac{\lambda}{1-\lambda}\right)^{n_k-1}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) + \frac{1}{1-\lambda}\varrho\left(Gf^{n_k+1}\sigma_0, Gv^*\right),\\
&\leq \left(\frac{\lambda}{1-\lambda}\right)^{n_k}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) + \frac{1}{1-\lambda}\varrho\left(Gf^{n_k+1}\sigma_0, Gf^{n_k}\sigma_0\right).
\end{aligned} \tag{60}$$

Suppose that

$$\left(\frac{\lambda}{1-\lambda}\right)^{n_k}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) = \frac{c}{2}. \tag{61}$$

$$\frac{\lambda}{1-\lambda}\varrho\left(Gf^{n_k-1}\sigma_0, Gff^{n_k-1}\sigma_0\right) = \frac{c}{2}. \tag{62}$$

Letting $k \longrightarrow \infty$ and using Definition 3, (61), and (62) in (60), we obtain

$$\varrho\left(Gfv^*, Gv^*\right) \leq \frac{c}{2} + \frac{c}{2}, \tag{63}$$

which follows

$$\varrho\left(Gfv^*, Gv^*\right) \leq c. \tag{64}$$

Since G is one to one and continuous, $fv^* = v^*$. So, f has a fixed point. As $Gf^n\sigma_0$ is sequentially convergent, we conclude that $\{Gf^n\sigma_0\}$ converges to the fixed point of f. □

Next, we prove our second main results by extending Theorem 3 using an F-Kannan–Suzuki type mapping in TVS-valued cone metric space.

Theorem 7. *Let $(\mathscr{X}, \varrho)$ be a complete TVS-valued cone metric space and $\mathscr{P}$ a solid cone. In addition, let $G\colon \mathscr{X} \longrightarrow \mathscr{X}$ be a one-to-one, continuous and sequentially mappings and $f\colon \mathscr{X} \longrightarrow \mathscr{X}$ such that $F(f) \neq \varnothing$, $\vartheta > 0$ and that*

$$\begin{aligned}
&\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma, \varsigma)\\
&\Rightarrow \vartheta + F\left(\varrho\left(Gf\sigma, Gf^2\sigma\right)\right) \leq F\left(\varrho\left(G\sigma, Gf\sigma\right)\right),
\end{aligned} \tag{65}$$

holds for some $\lambda \in (0,1)$ and for all $\sigma \in \mathcal{X}, \sigma \neq f\sigma$. Then f has property Q.

Proof. By Remark 1, let $u \in F(G^n) \cap F(f^n)$ for some $n \in \mathbb{N}$. If $u = fu$, that is u is a unique fixed point of G and f. Hence, the proof completed. On contrary, we suppose $u \neq fu$. Let $\sigma = u = f^{n-1}u$ and $\varsigma = fu = ff^{n-1}u$ such that $f^{n-1} \neq ff^{n-1}$ and using (65), we get

$$\begin{aligned} \frac{1}{2}\varrho(u, Gu) &< \varrho(u, fu), \\ \varrho(u, Gu) &< 2\varrho(u, fu), \\ \Rightarrow \vartheta + F\left[\varrho\left(Gff^{n-1}u, Gf^2f^{n-1}u\right)\right] &\leq F\left[\varrho\left(Gf^{n-1}u, Gff^{n-1}u\right)\right], \\ \vartheta + F\left[\varrho\left(Gff^{n-1}u, Gff^nu\right)\right] &\leq F\left[\varrho\left(Gf^{n-1}u, TGf^nu\right)\right]. \end{aligned} \tag{66}$$

Consequently, we get

$$F\left[\varrho\left(Gff^{n-1}u, Gff^nu\right)\right] \leq F\left[\varrho\left(Gf^{n-1}u, Gf^nu\right)\right] - \vartheta. \tag{67}$$

Repeating the same argument several times, we finally obtain

$$F\left[\varrho\left(Gff^{n-1}u, Gff^nu\right)\right] \leq F\left[\varrho\left(Gf^{n-1}u, Gf^nu\right)\right] - n\vartheta. \tag{68}$$

By following similar procedure as the proof of Theorem 6, we can conclude that $\varrho(Gu, Gfu) = c$, i.e., $Gu = Gfu$. Since G is one to one and sequentially convergent, then $u = fu$, which is a contradiction. Hence, $u \in F(G^n) \cap F(f^n)$. □

We give an example where generalised Kannan mapping will not be applicable. However, F- Kannan Suzuki type mapping is applicable.

Example 6. Consider the sequence $\mathcal{X} = \{0, 1\} \cup \{(1/2), (1/3), (1/4), \ldots\}$ and d be an Euclidean metric on $\mathcal{X}$. Then $(\mathcal{X}, \varrho)$ is a TVS-valued cone complete metric space. Let the mapping $f\colon \mathcal{X} \longrightarrow \mathcal{X}$ be determined as follows:

$$\begin{aligned} f(0) &= 0, \\ f(1/i) &= \frac{1}{i+1}, \end{aligned} \tag{69}$$

for $n \geq 2$. Let there exist $\lambda \in [0, (1/2))$, so that, for all $\sigma, \varsigma \in X$ condition (1) is satisfied although is not true for every $\lambda > 0$. That is a contradiction; hence, Kannan's theorem cannot be applicable.

The mapping $G\colon \mathcal{X} \longrightarrow \mathcal{X}$ be determined as

$$\begin{aligned} G(0) &= 0, \\ G(1/i) &= \frac{1}{2^i}. \end{aligned} \tag{70}$$

For all $i \geq 2$, G is continuous, one to one, and subsequentially convergent.

We consider a sequence $\{\sigma_i\}$ in $\mathcal{X}$ and assume that $\mathcal{X}$ is sequentially compact in complete TVS-valued cone metric space. By assumption $\mathcal{X}$ is sequentially compact with $\epsilon = 1$ we can cover the space $\mathcal{X}$ with finitely many balls of radius 1; then one of them contains many $\{\sigma_i\}$ for $i \geq 2$; i.e., there is a ball B_1 of radius 1 so that there is a subsequence of $\{\sigma_i\}$ whose members all belongs to B_1. We denote this subsequence by $\{\sigma_i\}$; thus, all $\{\sigma_i\}$ belongs to B_1.

Similar by sequentially compactness conditions with $\epsilon = (1/2)$, we can find a subsequence $\{\sigma_{i_2}\}$ of $\{\sigma_{i_1}\}$ and a ball B_2 of radius 1/2 so that all $\{\sigma_{i_2}\}$ belongs to B_2. Continuing this way, we obtain for any $k \geq 2$ a subsequence $\{\sigma_{i_k}\}$ of $\{\sigma_{i_k-1}\}$ and a ball B_k of radius 2^{-k} so that all $\{\sigma_{i_k}\}$ belongs to B_k.

Now, let $i, j \in \mathbb{N}$, $j > i$. Then, we show that (f, G) is a F-Kannan–Suzuki type mapping in TVS-valued cone metric space with respect to $F_2(z) = -(1/z)$ and $\vartheta \geq 0$. By using (FKS2) and $F_2(z)$, we have

$$\begin{aligned} &\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma, \varsigma) \\ &\Rightarrow \frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2} \leq \frac{\varrho(Gf\sigma, Gf\varsigma)}{1 - \vartheta\varrho(Gf\sigma, Gf\varsigma)}. \end{aligned} \tag{71}$$

To see this, we now calculate $\varrho(f\sigma, G\varsigma)$ for $\sigma = 1/i, \varsigma = 1/j, i \geq 1$.

$$\begin{aligned} \varrho(G\sigma, Gf\sigma) &= \varrho(T(1/i), Gf(1/i)) \\ &\leq \left|\frac{1}{2^i} - \frac{1}{2^{1/(i+1)}}\right| e^t. \end{aligned} \tag{72}$$

$$\begin{aligned} \varrho(Gf\sigma, Gf\varsigma) &= \varrho(Gf(1/i), Gf(1/j)) \\ &\leq \left|2^{1/(i+1)} - 2^{1/(j+1)}\right| e^t. \end{aligned} \tag{73}$$

$$\begin{aligned} \varrho(G\varsigma, Gf\varsigma) &= \varrho(G(1/j), Gf(1/j)) \\ &\leq \left|2^{1/j} - 2^{1/(j+1)}\right| e^t. \end{aligned} \tag{74}$$

Applying (72), (73), and (74) in (71) becomes

$$\begin{aligned}
&\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma, \varsigma),\\
&\frac{1}{2}\varrho(1/i, G(1/i)) < \varrho(1/i, 1/j),\\
&\varrho(1/i, G(1/j)) < 2\varrho(1/i, 1/j),\\
&\left|\frac{1}{i} - \frac{1}{2^j}\right| e^t < 2\left|\frac{1}{i} - \frac{1}{j}\right| e^t,\\
&\left|\frac{2^i - i}{2^i . i}\right| e^t < 2\left|\frac{j - i}{ij}\right| e^t,\\
&\Rightarrow \frac{\varrho(G(1/i), Gf(1/i)) + \varrho(G(1/j), Gf(1/j))}{2} \le \frac{\varrho(Gf(1/i), Gf(1/j))}{1 - \vartheta\varrho(Gf(1/i), Gf(1/j))},\\
&\Rightarrow \frac{\left|\left(1/2^i\right) - \left(1/2^{1/(i+1)}\right)\right| e^t + \left|2^{1/j} - 2^{1/(j+1)}\right| e^t}{2} \le \frac{\left|2^{1/(i+1)} - 2^{1/(j+1)}\right| e^t}{1 - \vartheta\left|2^{1/(i+1)} - 2^{1/(j+1)}\right| e^t}.
\end{aligned} \tag{75}$$

Thus the inequality (71) and all conditions imposed in Theorem 6 are satisfied. Hence, G and f has unique fixed point that is $v^* = 0$ in $\{\mathscr{P} \subseteq \mathscr{E}\} \in \mathscr{X}$, where $\mathscr{P}$ is a solid cone.

4. Some Applications

Two applications of the theorem stated in the previous part will be presented in this section.

4.1. Existence of a Solution for Nonlinear Riemann–Liouville Type Fractional Differential Equation. As a convolution mapping, the nonlinear fractional differential equation is equally and identically utilized in several applications in the domains of science, engineering, and mathematics.

(i) In image processing: convolutional filtering is used in many essential algorithms in digital image processing, such as edge detection and related procedures. An out-of-focus photograph is created by convolutioning a crisp image with a lens function in optics. This is referred to as bokeh in photography. For example, applying blurring to a picture in image processing software.

(ii) In digital data processing: Savitzky–Golay smoothing filters are used for analyzing spectroscopic data in analytical chemistry. This can boost the signal-to-noise ratio while reducing spectral distortion along with a convolution in statistics that is weighted in moving average.

(iii) In acoustics: reverberation is the convolution of the original sound with echoes from objects surrounding the sound source. Convolution is a technique for mapping the impulse response of a physical room to a digital audio stream in digital signal processing. The imposition of a spectral or rhythmic structure on a sound is known as convolution in electronic music. This envelope or structure is frequently derived from a different sound. Filtering one signal via the other is called convolution of two signals.

(iv) In electrical engineering: the output of a linear time-invariant (LTI) system is obtained by the convolution of one function (the input signal) with a second function (the impulse response). At any one time, the output is the sum of all previous input function values, with the most recent values often having the most influence (expressed as a multiplicative factor). This component is provided by the impulse response function as a function of the time since each input value happened.

(v) In physics: a convolution operation can be found whenever there is a linear system with a "superposition principle." For example, in spectroscopy, line widening owing to the Doppler effect produces a Gaussian spectral line form on its own, whereas collision broadening produces a Lorentzian line shape. The Line form is a convolution of Gaussian and Lorentzian, which is a Voigt function, when both effects are active. The measured fluorescence in time-resolved fluorescence spectroscopy is a sum of exponential decays from each delta pulse, and the excitation signal may be considered as a chain of delta pulses.

(vi) In computational fluid dynamics: the convolution process is used in the large eddy simulation (LES) turbulence model to reduce the range of length scales required in computing, lowering the computational cost.

(vii) In probability theory: the convolution of the distributions of two independent random variables is the probability distribution of the sum of their distributions.

(viii) In kernel density estimation: a distribution is estimated from sample points by convolution with a kernel, such as an isotropic Gaussian.

(ix) In radiotherapy: in the handling of planning systems, the convolution-superposition algorithm is used in the majority of recent computation codes.

The above applications of a convolution show that the fractional derivative as convolution has multiple purposes. It can represent memory, like in the instance of elasticity theory. It may be understood as a filter, with the Caputo and Caputo–Fabrizio types in particular being viewed as a filter of the local derivative with power and exponent functions (one can see in [48]).

The purpose of this section is to provide an application of Theorem 6 to find a common solution of a nonlinear fractional differential equation, where we can apply F-Kannan–Suzuki type mappings in complete TVS-valued cone metric spaces.

Here, we investigates the Riemann–Liouville derivative fractional integral of order $\alpha > 0$. This form of fractional derivative for a continuous function g: $[0,\infty) \longrightarrow \mathbb{R}$ denoted by $D_a^\alpha f$, is given by

$$\begin{aligned}(D_{0+}^{\alpha})g(t) &= \left(\frac{d}{dt}\right)^{n-1}(I_{0+}^{\alpha})g(t)\\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_0^t (t-s)^{n-\alpha-1}g(s)\mathrm{d}s,\end{aligned} \tag{76}$$

where $[\alpha]$ denotes the integer part of the real number α and $n = [\alpha] + 1$, provided that the right hand side is pointwise defined on $(0,\infty)$. (see [49–54]). Also, the Riemann–Liouville fractional integral of order α is given by

$$(I_0^\alpha)g(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}g(s)\mathrm{d}s, \tag{77}$$

for $\alpha > 0$. The notation $[\alpha]$ stands for largest integer not greater than α. If $\alpha = m \in \mathbb{N}$, then $(D_{0+}^m)g(t) = g^m(t)$, for $t > 0$ and if $\alpha = 0$, then $(D_{0+}^0)g(t) = g(t)$ for $t > 0$.

The following nonlinear fractional differential equation with integral boundary valued conditions is inspired by Kilbas et al. [55], Cabada and Hamdi [56], and Cabada and Wang [50].

$$\begin{cases} D_{0+}^{\alpha}\sigma(t) + g(t,\sigma(t)) = 0, & 0 < t < 1,\\ \sigma(0) = \sigma'(0), & \\ \sigma'(1) = \lambda\int_0^1 \sigma(s)\mathrm{d}s, & 0 < \lambda < 1,\end{cases} \tag{78}$$

where D_{0+}^α denotes the Riemann–Liouville fractional derivative of order α and g: $[0,1] \longrightarrow \mathcal{X}$ is a continuous function.

We recall the following lemmas from Bai, and Lü [57].

Lemma 4. *Let $\alpha > 0$. If we assume $\sigma \in C(0,1) \cap L(0,1)$, then the fractional differential equation:*

$$D_{0+}^{\alpha}\sigma(t) = 0, \tag{79}$$

has

$$\sigma(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{80}$$

$C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, as unique solution.

Since $D_{0+}^\alpha I_{0+}^\alpha \sigma(t) = \sigma$ for all $\sigma \in C(0,1) \cap L(0,1)$. From Lemma 4, we deduce the following lemma.

Lemma 5. *Assume that $\sigma \in C(0,1) \cap L(0,1)$, with fractional derivative of order $\alpha > 0$ that belongs to $\sigma \in C(0,1) \cap L(0,1)$. Then,*

$$I_{0+}^{\alpha} D_{0+}^{\alpha}\sigma(t) = \sigma(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}, \tag{81}$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, as unique solution.

The unique solution of (78) is given by

$$\sigma(t) = \int_a^t G(t,s)g(s,u(s))\mathrm{d}s. \tag{82}$$

Recall that the Green function related to the problem (78) is given by

$$G_f(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \le s \le t \le 1,\\[2ex] \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}, & 0 \le t \le s \le 1.\end{cases} \tag{83}$$

Consider the space $\mathcal{X} = (C[0,1], \mathbb{R}^n)$, $\mathcal{E} = C[0,1]$ be endowed with the ordering $\sigma \le \varsigma$ if $\sigma(t) \le \varsigma(t)$ for all $t \in C[0,1]$ and define $\mathcal{P} \in \mathcal{E}$ by $\mathcal{P} = \{(\sigma,\varsigma) \in \mathcal{E}$: $\sigma(t), \varsigma(t) \ge 0\} \subset \mathbb{R}^2$, $\mathcal{X} = \mathbb{R}$.

This space defines the metric ϱ: $\mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{E}$ such that

$$\varrho(\sigma,\varsigma) = \sup_{t\in[0,1]}\{|\sigma(t) - \varsigma(t)|\}\psi(t), \tag{84}$$

$\forall \sigma, \varsigma \in \mathcal{X}$ and $\psi(t) = e^t$. Then, $(\mathcal{X}, \varrho)$ is a TVS-valued cone metric space. A function $\sigma \in C([0,1], \mathcal{X})$ is a unique solution of the fractional differential integral equation (82) if and only if $\sigma = u^*$ is a solution of the nonlinear fractional differential equation (78).

Now, we prove the following theorem.

Theorem 8. *Suppose the following condition hold:*

(i) $G_f(t,s) \in C([0,1]\times[0,1], X) \geq 0$ *for all* $t, s \in [0,1]$

(ii) $\int_0^1 G_f(t,s) \leq \gamma(s)$ *for all* $t, s \in [0,1]$

(iii) $g \in C([0,1]\times\mathcal{X}, \mathcal{X})$ *is sequentially continuous*

(iv) *there exists a continuous function* $g\colon [0,1]\times\mathbb{R} \longrightarrow \mathbb{R}_+$, *such that*

$$|g(t,\sigma(s)) - g(t,\varsigma(s))| \leq e^{-\vartheta}\gamma(s)|\sigma(s) - \varsigma(s)|, \tag{85}$$

for all $t \in [0,1]$ *and* $\vartheta > 0$, *such that*

$$\gamma(s) = \frac{t^{\alpha-1}[\alpha\lambda + \alpha(\alpha+1)] - (\alpha+1)[\alpha t^\alpha + \lambda t^\alpha]}{\alpha(\alpha+1)(\alpha-\lambda)\Gamma(\alpha)}. \tag{86}$$

Then, the fractional differential Equation 4.1 has a common solution as a fixed point $\sigma^* \in C([0,1], \mathcal{X})$.

Proof. Let us define a map $G, f\colon \mathcal{P} \longrightarrow \mathcal{E}$ by

$$Gf\sigma(t) = \int_0^1 G_f(t,s) g(s,\sigma(s))\mathrm{d}s, \tag{87}$$

for $t \in [0,1]$, then $Gf^n\sigma_0$ is sequentially continuous. This implies that $f \in Gf^n\sigma_0$ and $f^n\sigma_0$ possess a fixed point $u^* \in Gf$. To prove the existence of fixed point of Gf, we prove that Gf is sequentially and contraction. To show Gf is sequentially continuous, let $Gf\sigma \neq Gf\varsigma$, for all $\sigma, \varsigma \in [0,1]$. By condition (iv), we have

$$\begin{aligned}
|Gf\sigma - Gf\varsigma| &= \left|\int_0^1 G_f(t,s)g(s,\sigma(s))\mathrm{d}s - \int_0^1 G_f(t,s)g(s,\varsigma(s))\mathrm{d}s\right|,\\
&\leq \int_0^1 G_f(t,s)|g(s,\sigma(s)) - g(s,\varsigma(s))|\mathrm{d}s,\\
&\leq \left[\int_0^t \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s) - (\alpha-\lambda)(t-s)^{\alpha-1}}{(\alpha-\lambda)\Gamma(\alpha)}\mathrm{d}s + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(\alpha-\lambda)\Gamma(\alpha)}\mathrm{d}s\right] e^{-\vartheta}|\sigma(s) - \varsigma(s)|e^t,\\
&\leq \left[\frac{t^{\alpha-1}[\alpha\lambda + \alpha(\alpha+1)] - (\alpha+1)[\alpha t^\alpha + \lambda t^\alpha]}{\alpha(\alpha+1)(\alpha-\lambda)\Gamma(\alpha)}\right] e^{-\vartheta}|\sigma(s) - \varsigma(s)|.
\end{aligned} \tag{88}$$

This implies that

$$|Gf\sigma, Gf\varsigma| \leq e^{-\vartheta}\gamma(s)|\sigma - \varsigma|e^t. \tag{89}$$

Since $\gamma(s) < 1$, we have

$$|Gf\sigma, Gf\varsigma| \leq e^{-\vartheta}|\sigma - \varsigma|e^t. \tag{90}$$

Thus, for each $\sigma, \varsigma \in \mathcal{X}$, we have

$$\varrho(Gf\sigma, Gf\varsigma) \leq e^{-\vartheta}\mathbb{M}(\sigma,\varsigma). \tag{91}$$

Taking logarithms on both sides of (91) using $F_1(z) = \ln(z)$ and the property of F, we get

$$\ln(\varrho(Gf\sigma, Gf\varsigma)) \leq \ln\left(e^{-\vartheta}\mathbb{M}(\sigma,\varsigma)\right) \tag{92}$$

equivalent to

$$\vartheta + F(\varrho(Gf\sigma, Gf\varsigma)) \leq F(\mathbb{M}(\sigma,\varsigma)), \tag{93}$$

where

$$\mathbb{M}(\sigma,\varsigma) = \frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}. \tag{94}$$

Using (94) in (93) and applying F-Kannan–Suzuki type conditions leads to

$$\begin{aligned}
&\frac{1}{2}\varrho(\sigma, G\sigma) < \varrho(\sigma,\varsigma)\\
&\Rightarrow \vartheta + F(\varrho(Gf\sigma, Gf\varsigma)) \leq F\left[\frac{\varrho(G\sigma, Gf\sigma) + \varrho(G\varsigma, Gf\varsigma)}{2}\right].
\end{aligned} \tag{95}$$

For $\gamma \in [0,1)$, $\vartheta > 0$ satisfies F-Kannan–Suzuki type mapping. Therefore, Gf is a contraction mapping on X. Since all the conditions of Theorem 8 are satisfied. Therefore, there exists $u^* \in C([0,1])$ a common fixed point of G and f, that is, u^* is a solution to fractional nonlinear differential equation (78). □

4.2. The Existence of Coincidence Solution for Nonlinear Volterra-Integral Equations. This section investigates the coincidence solution for nonlinear Volterra-integral equations in the setting of TVS-valued cone metric spaces. Nieto [58] initiated the study of the existing solution of an ordinary differential equation. Since then, several authors utilized his ideas to find the solution of ordinary differential equations. In detail, one can see the literature in [55, 59–62] and the references therein.

Integral equation methods help solve many problems of the applied fields like mathematical economics and optimal control theory because this problem is often reduced to integral equations.

Integral equations appear in several forms. However, in this section, we are interested with the integral equation, namely, Volterra integral-differential equation which is of the form

$$u^n(t,\sigma) = f(t,\sigma) + \int_a^\sigma K(\sigma,t,u(t))\mathrm{d}t, \quad \text{where } u^n = \frac{d^n u}{d\sigma^n}. \tag{96}$$

The following integral equation inspired by [12, 63–66].

$$\begin{aligned} u(\sigma,\varsigma) = l(\sigma,\varsigma) &+ \int_0^\sigma g(\sigma,\varsigma,\varepsilon,u(\varepsilon,\varsigma))\mathrm{d}\varepsilon \\ &+ \int_0^\sigma\int_0^\varsigma h(\sigma,\varsigma,\nu,\tau,u(\nu,\tau))\mathrm{d}\varepsilon\mathrm{d}\nu, \end{aligned} \tag{97}$$

where l, g, h are given functions and u is unknown function to be found.

Let $C(G,f)$ be the class of continuous functions from the set G to the set f. We denote $\mathscr{E} = \mathbb{R}^+ \times \mathbb{R}^+$, $\mathscr{E}_1 = \{l(\sigma,\varsigma,s)\colon 0 \le s \le \sigma \le \infty, \varsigma \in \mathbb{R}^+\}$ and $\mathscr{E}_2 = \{l(\sigma,\varsigma,s,t)\colon 0 \le s \le \sigma \le \infty, 0 \le t \le \varsigma \le \infty\}$. We denote that $l \in C(\mathscr{E},\mathbb{R})$, $g \in C(\mathscr{E}_1 \times \mathbb{R},\mathbb{R})$ and $h \in C(\mathscr{E}_2 \times \mathbb{R},\mathbb{R})$

Denote $\mathscr{X}$ be the space of functions $z \in C(\mathbb{R}^+ \times \mathbb{R}^+,\mathbb{R})$ satisfying

$$|z(\sigma,t)| = O\left(e^{\lambda(\sigma+\varsigma)}\right), \tag{98}$$

where λ is a positive constant, that is,

$$|z(\sigma,\varsigma)| \le M_0\left(e^{\lambda(\sigma+\varsigma)}\right), \tag{99}$$

for constant $M_0 > 0$. Let $(\mathscr{X}, \|.\|)$ be a Banach space. Define a norm in the space X by

$$|z|_{\mathscr{X}} = \sup_{(\sigma,\varsigma)\in\mathscr{E}} \left[|z(\sigma,\varsigma)|e^{(-\lambda(\sigma+\varsigma))}\right]. \tag{100}$$

Define the mapping $G, f\colon \mathscr{X} \times \mathscr{X} \longrightarrow [0,\infty)$ by

$$\begin{aligned} Gf^n u(\sigma,\varsigma) = l(\sigma,\varsigma) &+ \int_0^\sigma g(\sigma,\varsigma,\varepsilon,u(\varepsilon,\varsigma))\mathrm{d}\varepsilon \\ &+ \int_0^\sigma\int_0^\varsigma h(\sigma,\varsigma,\nu,\tau,u(\nu,\tau))\mathrm{d}\varepsilon\mathrm{d}\nu, \end{aligned} \tag{101}$$

and

$$\begin{aligned} Gf^n v(\sigma,\varsigma) = l(\sigma,\varsigma) &+ \int_0^\sigma g(\sigma,\varsigma,\varepsilon,v(\varepsilon,y))\mathrm{d}\varepsilon \\ &+ \int_0^\sigma\int_0^\varsigma h\sigma,\varsigma,\nu,\tau,v(\nu,\tau))\mathrm{d}\varepsilon\mathrm{d}\nu, \end{aligned} \tag{102}$$

for $u, v \in \mathscr{X}$. The coincidence fixed point of $Gf^n u$ and $Gf^n v$ is also a solution of the Volterra integral-differential equation (97).

Now we prove the results by establishing the existence solution of a coincidence fixed point for a pair of self mappings:

Theorem 9. *Suppose the following conditions holds:*

(i) *For the continuous functions $g, h \in \mathscr{X}$, we have*

$$\begin{aligned} |g(\sigma,\varsigma,\varepsilon,u(\varepsilon,\varsigma)) - g(\sigma,\varsigma,\varepsilon,v(\varepsilon,\varsigma))| &\le \gamma_1(\sigma,\varsigma,\varepsilon)|u-v|, \\ |h(\sigma,\varsigma,\nu,\tau,u(\nu,\tau)) - h(\sigma,\varsigma,\nu,\tau,v(\nu,\tau))| &\le \gamma_2(\sigma,\varsigma,\nu,\tau)|u-v|, \end{aligned} \tag{103}$$

where $\gamma_1 \in C(\mathscr{E}_1, [0,\infty))$ and $\gamma_2 \in C(\mathscr{E}_2, [0,\infty))$.

(ii) *There exists a nonnegative constant δ such that $\delta < 1$ and*

$$\int_0^\sigma \gamma_1(\sigma,\varsigma,\varepsilon)e^{\lambda(\varepsilon+\varsigma)}\mathrm{d}\varepsilon + \int_0^\sigma\int_0^\varsigma \gamma_2(\sigma,\varsigma,\nu,\tau)e^{\lambda(\nu+\tau)}\mathrm{d}\tau\mathrm{d}\nu \le \delta e^{\lambda(\sigma+\varsigma)-\vartheta}, \tag{104}$$

for all $\sigma,\varsigma,\varepsilon,\nu,\tau \in \mathscr{E}_1 \cup \mathscr{E}_2$.

Then, the nonlinear Volterra-integral equation (97) has a unique solution in $\mathscr{E}_1 \cup \mathscr{E}_2$ which is the coincidence fixed point of equations (101) and (102).

Proof. Let $G, f\colon \mathscr{X} \longrightarrow \mathscr{X}$ be two operators such that $Gf^v \in \mathscr{X}$ and $Gf^n v \in \mathscr{X}$. Now we verify that the two operators are contractive maps in $\mathscr{X}$. Let $u, v \in \mathscr{X}$. On contrary we claim that G and f are not contractive maps in $\mathscr{X}$. From equations (101) and (102), using condition (i) and (ii) of Theorem 9, we have

$$\begin{aligned}
\left|Gf^n u - Gf^n v\right|_{\mathcal{X}} &= l(\sigma,\varsigma) + \int_0^\sigma g(\sigma,\varsigma,\varepsilon,u(\varepsilon,\varsigma))\mathrm{d}\varepsilon + \int_0^\sigma\int_0^\varsigma h(\sigma,\varsigma,\nu,\tau,u(\nu,\tau))\mathrm{d}\tau\mathrm{d}\nu \\
&\quad - l(\sigma,\varsigma) - \int_0^\sigma g(\sigma,\varsigma,\varepsilon,v(\varepsilon,\varsigma))\mathrm{d}\varepsilon - \int_0^\sigma\int_0^\varsigma h(\sigma,\varsigma,\nu,\tau,v(\nu,\tau))\mathrm{d}\tau\mathrm{d}\nu, \\
&\le \int_0^\sigma |g(\sigma,\varsigma,\varepsilon,u(\varepsilon,\varsigma)) - g(\sigma,\varsigma,\varepsilon,v(\varepsilon,\varsigma))|\mathrm{d}\varepsilon \\
&\quad + \int_0^\sigma\int_0^\varsigma |h(\sigma,\varsigma,\nu,\tau,u(\nu,\tau)) - h(\sigma,\varsigma,\nu,\tau,v(\nu,\tau))|\mathrm{d}\tau\mathrm{d}\nu, \\
&\le \left[\int_0^\sigma \gamma_1(\sigma,\varsigma,\varepsilon)e^{\lambda(\sigma+\varsigma)}\mathrm{d}\varepsilon + \int_0^\sigma\int_0^\varsigma \gamma_2(\sigma,\varsigma,\nu,\tau)e^{\lambda(\nu+\tau)}\mathrm{d}\tau\mathrm{d}\nu\right]|u-v|_{\mathcal{X}}, \\
&\le \delta e^{\lambda(\sigma+\varsigma)-\vartheta}|u-v|_{\mathcal{X}}, \\
&\le \delta e^{\lambda(\sigma+\varsigma)-\vartheta}|u-v|_{\mathcal{X}}, \\
\left|Gf^n u - Gf^n v\right|_{\mathcal{X}} &\le \delta e^{-\vartheta}|u-v|_{\mathcal{X}} e^{\lambda(\sigma+\varsigma)}, \\
\varrho(Gfu,Gfv) &\le \delta e^{-\vartheta} M(u,v),
\end{aligned} \tag{105}$$

which is a contradiction. Hence u is a common fixed of G and f, also a solution to integral (97).

From (105), since $\delta < 1$ and using FKS2 of Definition 13, where

$$M(u,v) = \frac{\varrho(Gu,Gfu) + \varrho(Gv,Gfv)}{2}, \tag{106}$$

we have

$$\varrho(Gfu,Gfv) \le e^{-\vartheta} M(u,v). \tag{107}$$

Using $F_1(z) = \ln z$ by taking natural logarithms in both sides of (107), we get

$$\vartheta + \varrho(Gfu,Gfv) \le M(u,v). \tag{108}$$

By (106), we obtain a F-Kannan–Suzuki contraction as defined in Definition 13. Thus, all conditions imposed in Theorem 6 and Theorem 9 are satisfied. Hence, u^* is a common fixed point of G and f in X. □

5. Conclusion

The novelty of this study to fixed point theory is the fixed point result given in Theorem 6. This theorem provides the common fixed points conditions for a pair of two self mappings in TVS-valued cone metric spaces. This paper extended and generalised the results due to Batra et al. [37], Filipovic et al. [38], Morales and Rojas [9], Rahimi et al. [39], and Wangwe and Kumar [40] using a pair of two self-mappings in F-Kannan–Suzuki type mapping in TVS-valued cone metric space, where we consider a map to be sequentially convergent, one to one and continuous. By doing so, we extended several other results of the same setting in the literature. These results have some applications in many areas of applied mathematics, especially in nonlinear Riemann–Liouville fractional differential equation and nonlinear Volterra-integral differential equation.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. In addition, all authors read and approved the final manuscript.

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Research Article

Two New Weak Convergence Algorithms for Solving Bilevel Pseudomonotone Equilibrium Problem in Hilbert Space

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In this paper, we introduce two new subgradient extragradient algorithms to find the solution of a bilevel equilibrium problem in which the pseudomonotone and Lipschitz-type continuous bifunctions are involved in a real Hilbert space. The first method needs the prior knowledge of the Lipschitz constants of the bifunctions while the second method uses a self-adaptive process to deal with the unknown knowledge of the Lipschitz constant of the bifunctions. The weak convergence of the proposed algorithms is proved under some simple conditions on the input parameters. Our algorithms are very different from the existing related results in the literature. Finally, some numerical experiments are presented to illustrate the performance of the proposed algorithms and to compare them with other related methods.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let g: $H \times H \longrightarrow \mathbb{R}$ be a bifunction with $g(x, x) = 0$ for all $x \in C$. The equilibrium problem (EP for short) is associated with g and C to find $z \in C$ such that

$$g(z, y) \geq 0, \quad \forall y \in C. \tag{1}$$

The solution set of (1) is denoted by $EP(g, C)$.

If $g(x, y) = \langle G(x), y - x \rangle$ for all $x, y \in H$, where G is a mapping from H into itself, then the problem (1) becomes the following variational inequality problem (VIP for short):

$$\text{find } x^* \in C \text{ such that } \langle G(x^*), y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{2}$$

The solution set of (2) is denoted by $VI(G, C)$.

The EP (1) has a simple form and is very general in the sense that it includes, as special cases, the variational inequality problem, fixed point problem, complementarity problem, optimization problem as well as the Nash equilibrium problem; see,for example [1,2]. Many methods have been proposed for approximating a solution of the EP (1). Mastroeni [3] used the auxiliary problem principle which was first introduced for solving the optimization problems to solve EP (1) and presented the iteration algorithm in the form

$$x_0 \in C, x_{n+1} = \operatorname{argmin}\left\{\lambda g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}, \tag{3}$$

where the stepsize $\lambda > 0$. For obtaining the convergence of this algorithm, the bifunction g is required to be strongly monotone and Lipschitz-type continuous. To avoid the hypothesis of the strong monotonicity, Quoc et al. [4] first proposed the extragradient method (or the proximal-like methods) in which two strongly convex problems are solved at each iteration. The extragradient method is as follows: $x_0 \in C$ and

$$\begin{cases} y_n = \operatorname{argmin}\left\{\lambda g(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}, \\ x_{n+1} = \operatorname{argmin}\left\{\lambda g(y_n, y) + \frac{1}{2}\|y - x_n\|^2 : y \in C\right\}. \end{cases} \tag{4}$$

In 2018, Hieu [5] presented a new extragradient method for solving the EP (1.1) as follows: $x_0, y_0 \in C$ and

$$\begin{cases} x_{n+1} = \text{argmin}\left\{\lambda_n f(y_n, y) + \frac{1}{2}\|y - x_n\|^2: y \in C\right\}, \\ y_{n+1} = \text{argmin}\left\{\lambda_{n+1} f(y_n, y) + \frac{1}{2}\|y - x_{n+1}\|^2: y \in C\right\}, n \geq 0, \end{cases} \tag{5}$$

where $\{\lambda_n\} \subset (0, \infty)$ is a nonincreasing sequence and f is a strongly pseudomonotone and Lipschitz-type continuous mapping.

In 2011, Censor et al. [6] proposed a new method, which is called the subgradient extragradient method, for solving the VIP (2). In 2016, Hieu [7] extended this method to the EP (1.1). In 2019, inspired by [5,7], Liu and Kong [8] introduced the following subgradient extragradient method for solving the EP (1): $x_0, y_0 \in C$ and

$$\begin{cases} x_1 = \text{argmin}\left\{\lambda f(y_0, y) + \frac{1}{2}\|y - x_0\|^2: y \in C\right\}, \\ y_1 = \text{argmin}\left\{\lambda f(y_0, y) + \frac{1}{2}\|y - x_1\|^2: y \in C\right\}, \\ x_{n+1} = \text{argmin}\left\{\lambda f(y_0, y) + \frac{1}{2}\|y - x_n\|^2: y \in H_n\right\}, \\ y_{n+1} = \text{argmin}\left\{\lambda f(y_0, y) + \frac{1}{2}\|y - x_{n+1}\|^2: y \in C\right\}, n \geq 1, \end{cases} \tag{6}$$

where $H_n = \{z \in H: \langle x_n - \lambda w_{n-1} - y_n, z - y_n\rangle \leq 0\}$ and $w_{n-1} \in \partial_2 f(y_{n-1}, y_n)$, and f is a pseudomonotone and Lipschitz-type continuous mapping.

The advantage of equations (5) and (6) is that only one value of f at y_n is computed at each iteration. On the recent methods for solving the EP (1), we refer the readers to [9–15].

In this paper, our interest is the bilevel equilibrium problem (BEP for short) which consists of the following:

$$\text{find } \overline{x} \in EP(g, C) \text{ such that } f(\overline{x}, y) \geq 0, \quad \forall y \in EP(g, C), \tag{7}$$

where $f: H \times H \longrightarrow \mathbb{R}$ with $f(x, x) = 0$ for all $x \in H$. The BEPs are the special cases of mathematical programs with equilibrium constraints and also are the generalization of variational inequality over equilibrium constraints, hierarchical minimization problems, and complementarity problems. The methods for solving BEPs have been studied extensively by many authors. Moudafi [16] introduced a proximal method and proved the weak convergence to a solution of the BEP (7). Dinh and Muu [17] proposed a penalty and gap function method for solving the BEP (7). Quy [18] introduced an algorithm by combining the proximal method with the Halpern method for solving bilevel monotone equilibrium and fixed point problem. Yuying et al. [19] presented an extragradient method as follows:

$$\begin{cases} y_n = \text{argmin}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2: y \in C\right\}, \\ z_n = \text{argmin}\left\{\lambda_n g(y_n, y) + \frac{1}{2}\|y - x_n\|^2: y \in C\right\}, \\ x_{n+1} = \eta_n x_n + (1 - \eta_n) z_n - \alpha_n \mu w_n, w_n \in \partial_2 f(z_n, z_n), \end{cases} \tag{8}$$

where $\{\alpha_n\} \subset (0, 1)$, $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}]$ with $\underline{\lambda} > 0$, and $\{\eta_n\} \subset [0, 1 - \alpha_n]$. Anh and An [20] proposed the following subgradient extragradient method for solving the BEP (7):

$$\begin{cases} y_n = \text{argmin}\left\{\lambda_n g(x_n, y) + \frac{1}{2}\|y - x_n\|^2: y \in C\right\}, \\ z_n = \text{argmin}\left\{\lambda_n g(y_n, y) + \frac{1}{2}\|z - x_n\|^2: z \in H_n\right\}, \\ x_{n+1} = \text{argmin}\left\{\beta_n f(z_n, y) + \frac{1}{2}\|t - z_n\|^2: z \in C\right\}, \end{cases} \tag{9}$$

where $H_n = \langle v \in H: \langle x_n - \lambda_n w_n - y_n, v - y_n\rangle \leq 0\rangle$ with $w_n \in \partial_2 g(x_n, y_n)$, $\{\lambda_n\}$ and $\{\beta_n\}$ are two nonnegative sequences.

Observe that in the works mentioned above, the bifunction g is monotone or pseudomonotone while f is strongly monotone, and then, the algorithms have a strong convergence. In this paper, inspired by [8,20], we propose two new subgradient extragradient methods for solving the BEP (7) where both the bifunction f and g are pseudomonotone. The first method needs the prior knowledge of the Lipschitz constants of the bifunctions while the second method uses a self-adaptive process to deal with the unknown knowledge of the Lipschitz constant of the bifunctions. The weak convergence of the proposed algorithms is proved under some sufficient assumptions. Finally, some numerical experiments are presented to illustrate the performance of the proposed algorithms and to compare them with other related methods.

2. Preliminaries

Let H be a real Hilbert space, $\mathbb{R}$ be the set of all real numbers, and $\mathbb{N}$ be the set of all positive integers. We list some well-known definitions and properties which will be used in our following analysis.

Definition 1. A mapping $F: H \longrightarrow H$ is said to be

(i) monotone if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in H; \tag{10}$$

(ii) pseudomonotone if

$$\langle F(y), x - y\rangle \geq 0 \Rightarrow \langle F(x), x - y\rangle \geq 0, \quad \forall x, y \in H; \tag{11}$$

(iii) L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|F(x) - F(y)\| \leq \|x - y\|, \quad \forall x, y \in H. \tag{12}$$

Definition 2. A bifunction $f\colon H \times H \longrightarrow \mathbb{R}$ is said to be

(i) pseudomonotone on C if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \quad \forall x, y \in C. \tag{13}$$

(ii) Lipschitz-type continuous on C if there exists the constants $c_1 > 0$ and $c_2 > 0$ such that

$$f(x, z) \leq f(x, y) + f(y, z) + c_1\|x - y\|^2 + c_2\|y - z\|^2, \quad \forall x, y, z \in C. \tag{14}$$

Remark 1. If F is L-Lipschitz continuous on H, then for each $x, y \in H$, $f(x, y) = \langle F(x), y - x\rangle$ is Lipschitz-type continuous with the constants $c_1 = c_2 = (L/2)$; see [21] for details.

Let C be a nonempty closed and convex subset of H. For each $x \in H$, there exists a unique point in C, denoted by $P_C x$, such that

$$P_C x = \arg\min\{\|y - x\|: y \in C\}, \tag{15}$$

P_C is said to be the metric projection from H onto C. The following lemma characterizes the property of P_C.

Lemma 1. *Let $P_C\colon H \longrightarrow C$ be the metric projection. Then,*

(i) $z = P_C x$ if and only if

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C. \tag{16}$$

(ii) for all $y \in C$ and $x \in H$,

$$\left\|y - P_C x\right\|^2 + \left\|P_C x - x\right\|^2 \leq \|x - y\|^2. \tag{17}$$

Remark 2. For any given $\overline{x} \in H$ and $v \in H$ with $v \neq 0$, let $T = \{x \in H: \langle v, x - \overline{x}\rangle \leq 0\}$. Then, for all $y \in H$, the projection $\Pi_T(y)$ is defined by

$$\prod_T(y) = y - \max\left\{0, \frac{\langle v, y - \overline{x}\rangle}{\|v\|^2}\right\}v. \tag{18}$$

The formula (18) gives us an explicit manner to compute the projection of any point onto a half-space; see [22] for details.

Definition 3.

(1) The normal cone N_C of C at $x \in C$ is defined by

$$N_C(x) = \{w \in H: \langle w, y - x\rangle \leq 0, \forall y \in C\}. \tag{19}$$

(2) The subdifferentiable of a convex function $g\colon C \longrightarrow \mathbb{R}$ at $x \in H$ is defined by

$$\partial g(x) = \{w \in H: g(y) - g(x) \geq \langle w, y - x\rangle, \forall y \in C\}. \tag{20}$$

Lemma 2 (see [23]). *Let $g\colon C \longrightarrow \mathbb{R}$ be a convex subdifferentiable and lower semicontinuous function on C. Then, x^* is a solution to the following convex problem:*

$$\min\{g(x): x \in C\}, \tag{21}$$

if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(x^*)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.

For a proper, convex, and lower semicontinuous function: $h\colon C \longrightarrow (-\infty, +\infty]$ and $\lambda > 0$, the proximal mapping of h with λ is defined by

$$\operatorname{prog}_{\lambda h}(x) = \operatorname{argmin}\left\{\lambda h(y) + \frac{1}{2}\|x - y\|^2: y \in C\right\}, x \in C. \tag{22}$$

Lemma 3 (see [24, 25]). *For all $x, y \in C$ and $\lambda > 0$, the following inequality holds:*

$$\lambda\left(h(y) - h\left(\operatorname{prog}_{\lambda h}(x)\right)\right) \geq \langle x - \operatorname{prog}_{\lambda h}(x), y - \operatorname{prog}_{\lambda h}(x)\rangle. \tag{23}$$

Remark 3. From Lemma 3, we note that if $x = \operatorname{prog}_{\lambda h}(x)$, then

$$x \in \operatorname{argmin}\{h(y): y \in C\} = \left\{x \in C: h(x) = \min_{y \in C} h(y)\right\}. \tag{24}$$

Lemma 4 (see [26]). *Let $\{a_n\}$ and $\{c_n\}$ be two sequences of nonnegative real numbers satisfying the condition*

$$a_{n+1} \leq a_n + c_n, \quad \forall n \in \mathbb{N}. \tag{25}$$

If $\sum_n c_n < \infty$, then $\lim_{n \longrightarrow \infty} a_n$ exists.

3. Main Results

In this section, let $\mathbb{N}$ denotes the set of all positive integers, $N_0 = \mathbb{N} \cup \{0\}$, H be a real Hilbert space, and C be a nonempty closed convex subset of H. The notation "$\rightharpoonup$" denotes the weak converge. Let $f, g\colon H \times H \longrightarrow \mathbb{R}$ be two bifunctions satisfying the following conditions:

(A1) f and g are pseudomonotone on H

(A2) for each $y \in H$, $\limsup_{n \longrightarrow \infty} f(x_n, y) \leq f(x, y)$ and $\limsup_{n \longrightarrow \infty} g(x_n, y) \leq g(x, y)$ for every sequence $x_n \rightharpoonup x$

(A3) $f(x, \cdot)$ and $g(x, \cdot)$ are convex, lower semicontinuous, and subdifferentiable of H for each $x \in H$

(A4) g and f are Lipschitz-type continuous on H with the constants c_1, c_2 and d_1, d_2, respectively; that is, for all $x, y, z \in H$,

$$g(x,z) \le g(x,y) + g(y,z) + c_1\|x-y\|^2 + c_2\|y-z\|^2,$$
$$f(x,z) \le f(x,y) + f(y,z) + d_1\|x-y\|^2 + d_2\|y-z\|^2. \tag{26}$$

In this section, the solution set of the BEP (7) is denoted by Ω; that is, $\Omega = \{x \in E(g,C): f(\overline{x},y) \ge 0, \forall y \in E\ P(g,C)\}$, and assume that $\Omega \ne \varnothing$.

Now, we introduce the first algorithm for finding a point $\overline{x} \in \Omega$.

Remark 4. By using the notation "prog" in Section 2, u_n, t_n, x_{n+1}, and y_{n+1} may be rewritten as

$$\begin{cases} u_n = \text{prog}_{\beta g(x_n,\cdot)}(x_n), \\ t_n = \text{prog}_{\beta g(u_n,\cdot)}(x_n), \end{cases}$$
$$\begin{cases} x_{n+1} = \text{prog}_{\lambda f(y_n,\cdot)}(x_n), \\ y_{n+1} = \text{prog}_{\lambda f(y_n,\cdot)}(x_{n+1}). \end{cases} \tag{27}$$

Note that since $f(x,\cdot)$ and $g(x,\cdot)$ are convex and lower semicontinuous on H for each $x \in H$, for any given $\beta > 0$, $u \in H$, and the closed convex subset $D \subset H$, from [27], Proposition 12.15, and Definition 12.23, it follows that both

$$\text{argmin}\left\{\beta g(u,y) + \frac{1}{2}\|x-y\|^2 : y \in D\right\},$$
$$\text{argmin}\left\{\beta f(u,t) + \frac{1}{2}\|x-t\|^2 : t \in D\right\}, \tag{28}$$

are a singleton. Hence, u_n, t_n, x_{n+1}, and y_{n+1} in Algorithm 1 are obtained uniquely at each step.

Remark 5. From (A1)–(A4), it follows that (i) $EP(g,C)$ and $EP(f,C)$ are closed and convex; see [4]; (ii) $g(x,x) = 0$ and $f(x,x) = 0$ for all $x \in C$; see [28].

The following remark shows that the stop criterion in Step 3 is meaning.

Remark 6. Suppose that $x_{n+1} = y_n = x_n = u_n$ for some $n \in \mathbb{N}$. By $u_n = x_n$, the definition of u_n, and Lemma 3, we get

$$\beta(g(u_n,y) - g(x_n,u_n)) = \beta(g(x_n,y) - g(x_n,x_n)) \ge 0, \quad \forall y \in C, \tag{29}$$

which with $\beta > 0$ Remark 5 implies that $x_n \in EP(g,C)$. Similarly, by $x_{n+1} = y_n = x_n$, the definition of x_{n+1}, and Lemma 3, we can prove that $x_n \in EP(f,H_n)$. By the proof of Lemma 4, we see $EP(g,C) \subset H_n$ for all $n \in \mathbb{N}$. So $f(x_n,y) \ge 0$ for all $y \in EP(g,C)$. It follows that $x_n \in \Omega$.

Lemma 5. *Assume that $\beta \in (0, \min\{(1/2c_1), (1/2c_2)\})$. Then, $C \subset C_k$, $EP(g,C) \subset T_n \subset H_{n+1}$ for each $n \in \mathbb{N}_0$.*

Proof. We first show that $C \subset C_k$ for each $n \in \mathbb{N}_0$. By Lemma 1 and the definition of u_k, we have

$$0 \in \partial_2\left\{\beta g(x_n,y) + \frac{1}{2}\|x_n-y\|^2\right\}(u_n) + N_C(u_n). \tag{30}$$

Thus, for $w_n \in \partial_2 g(x_n,u_n)$, there exists $\overline{w}_n \in N_C(u_n)$ such that

$$\beta w_n + u_n - x_n + \overline{w}_n = 0. \tag{31}$$

So,

$$\langle x_n - u_n, y - u_n\rangle = \beta\langle w_n, y-u_n\rangle + \langle \overline{w}_n, y-u_n\rangle, \quad \forall y \in C. \tag{32}$$

Since $\overline{w}_n \in N_C(u_n)$, we have $\langle \overline{w}_n, y-u_n\rangle \le 0$ for all $y \in C$. Hence, $\beta\langle w_n, y-u_n\rangle \ge \langle x_n-u_n, y-u_n\rangle$ for all $y \in C$, which implies that $\langle x_n - \beta w_n - u_n, y - u_n\rangle \le 0$ for all $y \in C$. This shows that $C \subset C_n$ for each $\mathbb{N}_0$.

Next, we show that $EP(g,C) \subset T_n$ for each $n \in \mathbb{N}_0$. By Lemma 3 and the definition of t_n in Remark 4, we have

$$\beta(g(u_n,y) - g(u_n,t_n)) \ge \langle x_n - t_n, y - t_n\rangle, \quad \forall y \in C_n, \forall n \in \mathbb{N}_0. \tag{33}$$

Note, we have proved that $C \subset C_n$ for each $n \in \mathbb{N}_0$. So substituting any $x' \in EP(g,C) \subset C$ into (33), we obtain

$$\beta(g(u_n,x') - g(u_n,t_n)) \ge \langle x_n - t_n, x' - t_n\rangle, \forall n \in \mathbb{N}_0. \tag{34}$$

Since $u_n \in C$ and g is pseudomonotone on H, we have $g(u_n,x') \le 0$. Then, (34) implies that

$$\langle x_n - t_n, t_n - x'\rangle \ge \beta g(u_n,t_n), \forall n \in \mathbb{N}_0. \tag{35}$$

Now applying (A4) to g, we have

$$g(u_n,t_n) \ge g(x_n,t_n) - g(x_n,u_n) - c_1\|u_n - x_n\|^2 - c_2\|t_n - u_n\|^2, \quad \forall n \in \mathbb{N}_0. \tag{36}$$

Combining (35) and (36), we get

$$\langle x_n - t_n, t_n - x'\rangle \ge \beta\left(g(x_n,t_n) - g(x_n,u_n) - c_1\|u_n - x_n\|^2 - c_2\|t_n - u_n\|^2\right), \quad \forall n \in \mathbb{N}_0. \tag{37}$$

On the other hand, by the definition of $w_n \in \partial_2 g(x_n,u_n)$, we have

$$\beta(g(x_n,y) - g(x_n,u_n)) \ge \langle \beta w_n, y - u_n\rangle, \quad \forall y \in H, \forall n \in \mathbb{N}_0. \tag{38}$$

Since $t_n \in C_n$, we have

Initialization: Choose $x_0, y_0, y_{-1} \in C$ and the parameters $\beta > 0$ and $\lambda > 0$. Put $n = 0$.

Step 1. For given x_n, solve the strongly convex problems: $\begin{cases} u_n = \operatorname{argmin}\{\beta g(x_n, y) + 1/2\|x_n - y\|^2\colon y \in C\}, \\ t_n = \operatorname{argmin}\{\beta g(u_n, t) + 1/2\|x_n - y\|^2\colon t \in C_n\}, \end{cases}$

where $C_n = \{v \in H\colon \langle x_n - \beta w_n - u_n, v - u_n\rangle \le 0\}$ with $w_n \in \partial_2 g(x_n, u_n)$.

Step 2. Solve the strongly convex problems: $\begin{cases} x_{n+1} = \arg\min\{\lambda f(y_n, y) + 1/2\|x_n - y\|^2\colon y \in H_n\}, \\ y_{n+1} = \arg\min\{\lambda f(y_n, y) + 1/2\|x_{n+1} - y\|^2\colon y \in T_n\}, \end{cases}$

where $H_n = \{z \in H\colon \langle x_n - \lambda v_n - y_n, z - y_n\rangle \le 0\}$ with $v_n \in \partial_2 f(y_{n-1}, y_n), T_n = \{z \in H\colon \|z - t_n\| \le \|z - x_n\|\}$.

Step 3. If $x_{n+1} = y_n = x_n = u_n$, then the algorithm stops, $x_n \in \Omega$; otherwise, set $n = n + 1$ and return to Step 1.

Algorithm 1: (Extragradient-like method without prior constants).

$$\langle x_n - u_n, t_n - u_n\rangle \le \beta\langle w_n, t_n - u_n\rangle, \tag{39}$$

which with (38) implies that

$$\beta(g(x_n, t_n) - g(x_n, u_n)) \ge \langle x_n - u_n, t_n - u_n\rangle, \forall n \in \mathbb{N}_0. \tag{40}$$

From (37) and (40) and

$$2\langle t_n - x_n, x' - t_n\rangle = \|x_n - x'\|^2 - \|t_n - x_n\|^2 - \|t_n - x'\|^2, \tag{41}$$

it follows that

$$\begin{aligned} &\|x_n - x'\|^2 - \|t_n - x_n\|^2 - \|t_n - x'\|^2 \\ &\quad \ge 2\langle x_n - u_n, t_n - u_n\rangle - 2\beta c_1\|x_n - u_n\|^2 \\ &\qquad - 2\beta c_2\|u_n - t_n\|^2, \forall n \in \mathbb{N}_0. \end{aligned} \tag{42}$$

Hence,

$$\begin{aligned} \|t_n - x'\|^2 &\le \|x_n - x'\|^2 - \|t_n - x_n\|^2 \\ &\quad - 2\langle x_n - u_n, t_n - u_n\rangle + 2\beta\Big(c_1\|x_n - u_n\|^2 \\ &\quad + c_2\|u_n - t_n\|^2\Big) \\ &= \|x_n - x'\|^2 - \|t_n - u_n\|^2 - \|u_n - x_n\|^2 \\ &\quad + 2\beta\Big(c_1\|x_n - u_n\|^2 + c_2\|u_n - t_n\|^2\Big) \\ &= \|x_n - x'\|^2 - (1 - 2\beta c_1)\|x_n - u_n\|^2 \\ &\quad + (1 - 2\beta c_2)\|u_n - t_n\|^2, \quad \forall n \in \mathbb{N}_0. \end{aligned} \tag{43}$$

In particular, from $1 - 2\beta c_1 > 0$ and $1 - 2\beta c_2 > 0$, it follows that

$$\|t_n - x'\|^2 \le \|x_n - x'\|^2, \quad \forall n \in \mathbb{N}_0, \tag{44}$$

which implies that $x' \in T_n$. Since $x' \in EP(g, C)$ is arbitrary, it follows that $EP(g, C) \subset T_n$ for each $n \in \mathbb{N}_0$.

Finally, we prove that $T_n \subset H_{n+1}$ for each $n \in \mathbb{N}_0$. By the definition of y_{n+1} in Remark 4 and Lemma 3, we have

$$0 \in \lambda\partial_2 f(y_n, y_{n+1}) + y_{n+1} - x_{n+1} + N_{T_n}(y_{n+1}), \forall n \in \mathbb{N}_0. \tag{45}$$

Thus, for $v_n \in \partial_2 f(y_{n-1}, y_n)$, there exists $w_n \in N_{T_n}(y_{n+1})$ such that

$$\lambda v_n + y_{n+1} - x_{n+1} + w_n = 0, \quad \forall n \in \mathbb{N}_0. \tag{46}$$

It follows that

$$\begin{aligned} \langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle &= \lambda\langle v_n, y - y_{n+1}\rangle \\ &\quad + \langle w_n, y - y_{n+1}\rangle, \\ &\quad \forall y \in T_n, \forall n \in \mathbb{N}_0. \end{aligned} \tag{47}$$

Since $w_n \in N_{T_n}(y_{n+1})$, we have $\langle w_n, y - y_{n+1}\rangle \le 0$ for all $y \in T_n$. Hence, $\lambda\langle v_n, y - y_{n+1}\rangle \ge \langle x_{n+1} - y_{n+1}, y - y_{n+1}\rangle$ for all $y \in T_n$ and $n \in \mathbb{N}_0$, which with the definition of H_{n+1} implies that $T_n \subset H_{n+1}$ for each $n \in \mathbb{N}_0$. This completes the proof. □

Lemma 6. *Assume that $\beta \in (0, \min\{(1/2c_1), (1/2c_2)\})$ and $\lambda \in (0, (1/2d_2 + 4d_1))$. Let $\{x_n\}$ be the sequence generated by Algorithm 1. For all $x^* \in \Omega$, the limit of $\{\|x^* - x_n\|^2\}$ exists, and*

$$\begin{aligned} \lambda f(y_n, y) &\ge \lambda\Big[\langle x_n - y_n, x_{n+1} - y_n\rangle - c_1\|y_{n-1} - y_n\|^2 - c_2\|y_n - x_{n+1}\|^2\Big] \\ &\quad + \langle x_n - x_{n+1}, y - x_{n+1}\rangle, \forall y \in EP(g, C), \forall n \in \mathbb{N}. \end{aligned} \tag{48}$$

Proof. Since $\{\|x^* - x_n\|^2\}$, from the definition of H_n, it follows that

$$\langle x_n - \lambda v_n - y_n, x_{n+1} - y_n\rangle \le 0, \quad \forall n \in \mathbb{N}, \tag{49}$$

that is,

$$\lambda\langle v_n, x_{n+1} - y_n\rangle \geq \langle x_n - y_n, x_{n+1} - y_n\rangle, \quad \forall n \in \mathbb{N}. \tag{50}$$

By $v_n \in \partial_2 f(y_{n-1}, y_n)$ and the definition of subdifferential, we have

$$f(y_{n-1}, y) - f(y_{n-1}, y_n) \geq \langle v_n, y - y_n\rangle, \quad \forall y \in H, \forall n \in \mathbb{N}. \tag{51}$$

Replacing y in (51) with x_{n+1}, we get

$$f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) \geq \langle v_n, x_{n+1} - y_n\rangle, \quad \forall n \in \mathbb{N}. \tag{52}$$

Combining (50) and (52), we have

$$\lambda\left(f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)\right) \geq \langle x_n - y_n, x_{n+1} - y_n\rangle, \quad \forall n \in \mathbb{N}. \tag{53}$$

By Lemma 3 and the definition of x_{n+1}, we have

$$\lambda\left(f(y_n, y) - f(y_n, x_{n+1})\right) \geq \langle x_n - x_{n+1}, y - x_{n+1}\rangle, \quad \forall y \in H_n, \forall n \in \mathbb{N}. \tag{54}$$

Substituting $y = x^* \in \Omega$ into (54), we obtain

$$\lambda\left(f(y_n, x^*) - f(y_n, x_{n+1})\right) \geq \langle x_n - x_{n+1}, x^* - x_{n+1}\rangle, \quad \forall n \in \mathbb{N}. \tag{55}$$

Note that (A1) implies that $f(y_n, x^*) \leq 0$, which with (55) leads to

$$\lambda f(y_n, x_{n+1}) \leq \langle x_n - x_{n+1}, x_{n+1} - x^*\rangle, \quad \forall n \in \mathbb{N}. \tag{56}$$

On the other hand, by the Lipschitz-type continuity of f, we have

$$f(y_n, x_{n+1}) \geq f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - d_1\|y_n - y_{n-1}\|^2 - d_2\|x_{n+1} - y_n\|^2, \forall n \in \mathbb{N}. \tag{57}$$

By (56) and (57), we obtain

$$\langle x_n - x_{n+1}, x_{n+1} - x^*\rangle \geq \lambda\left(f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - d_1\|y_n - y_{n-1}\|^2 - d_2\|x_{n+1} - y_n\|^2\right), \tag{58}$$

which with (53) implies that

$$\begin{aligned}\langle x_n - x_{n+1}, x_{n+1} - x^*\rangle &\geq \langle x_n - y_n, x_{n+1} - y_n\rangle - \lambda\left(d_1\|y_n - y_{n-1}\|^2 + d_2\|x_{n+1} - y_n\|^2\right)\\ &= \frac{1}{2}\left(\|y_n - x_n\|^2 + \|x_{n+1} - y_n\|^2 - \|x_{n+1} - x_n\|^2\right) - \lambda\left(d_1\|y_n - y_{n-1}\|^2 + d_2\|x_{n+1} - y_n\|^2\right), \quad \forall n \in \mathbb{N}.\end{aligned} \tag{59}$$

Since

$$\langle x_n - x_{n+1}, x_{n+1} - x^*\rangle = \frac{1}{2}\left(\|x_n - x^*\|^2 - \|x_{n+1} - x_n\|^2 - \|x_{n+1} - x^*\|^2\right), \tag{60}$$

by (59) we get

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2\|^2 - \|y_n - x_n\|^2 - \|x_{n+1} - y_n\|^2 + 2\lambda d_1\|y_n - y_{n-1}\|^2 + 2\lambda d_2\|x_{n+1} - y_n\|^2\\ &\leq \|x_n - x^*\|^2\|^2 - \|y_n - x_n\|^2 - \|x_{n+1} - y_n\|^2 + 2\lambda d_1\left(\|y_n - x_n\| + \|x_n - y_{n-1}\|\right)^2 + 2\lambda d_2\|x_{n+1} - y_n\|^2\\ &\leq \|x_n - x^*\|^2\|^2 - \|y_n - x_n\|^2 - \|x_{n+1} - y_n\|^2 + 4\lambda d_1\left(\|y_n - x_n\|^2 + \|x_n - y_{n-1}\|^2\right) + 2\lambda d_2\|x_{n+1} - y_n\|^2\\ &= \|x_n - x^*\|^2\|^2 - (1 - 4\lambda d_1)\|y_n - x_n\|^2 - (1 - 2\lambda d_2)\|x_{n+1} - y_n\|^2 + 4\lambda d_1\|x_n - y_{n-1}\|^2, \forall n \in \mathbb{N}.\end{aligned} \tag{61}$$

Fix $N \in \mathbb{N}$. For all $m \in \mathbb{N}$ with $m > N$, by (61) we have

$$\|x^* - x_{m+1}\|^2 \le \|x^* - x_N\|^2 - (1 - (4\lambda d_1 + 2\lambda d_2)) \sum_{n=N}^{m} \|x_{n+1} - y_n\|^2 - (1 - 4\lambda c_1) \sum_{n=N}^{m} \|y_n - x_n\|^2 + 4\lambda c_1 \|x_N - y_{N-1}\|^2. \quad (62)$$

Hence,

$$(1 - 4\lambda d_1 - 2\lambda d_2) \sum_{n=N}^{m} \|x_{n+1} - y_n\|^2 + (1 - 4\lambda d_1) \sum_{n=N}^{m} \|y_n - x_n\|^2 < \|x^* - x_N\|^2 < \infty, \quad \forall m > N, \quad (63)$$

which with $4\lambda d_1 + 2\lambda d_2 < 1$ leads to

$$\sum_{n=1}^{\infty} \|x_{n+1} - y_n\|^2 < \infty, \qquad \sum_{n=1}^{\infty} \|y_n - x_n\|^2 < \infty. \quad (64)$$

It follows that

$$\lim_{n \longrightarrow \infty} \|x_n - y_n\|^2 = 0, \qquad \lim_{n \longrightarrow \infty} \|x_{n+1} - y_n\|^2 = 0. \quad (65)$$

By (65) and the triangle inequality of norm, we obtain

$$\|x_n - x_{n+1}\| \le \|x_n - y_n\| + \|y_n - x_{n+1}\| \longrightarrow 0, \text{ as } n \longrightarrow \infty, \quad (66)$$

$$\|y_n - y_{n+1}\| \le \|y_n - x_n\| + \|x_n - x_{n+1}\| + \|x_{n+1} - y_{n+1}\| \longrightarrow 0, \text{ as } n \longrightarrow \infty, \quad (67)$$

and

$$\|y_{n+1} - x_n\| \le \|y_{n+1} - y_n\| + \|y_n - x_n\| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \quad (68)$$

Note that (61) implies

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 + 4\lambda d_1 \|y_{n-1} - x_n\|^2, \quad \forall n \in \mathbb{N}. \quad (69)$$

From (62) and (69) and Lemma 3, it follows that the limit of $\{\|x_n - x^*\|^2\}$ exists.

Finally, by (54), (56), and (52), we get

$$\begin{aligned} \lambda f(y_n, y) &\ge \lambda f(y_n, x_{n+1}) + \langle x_n - x_{n+1}, y - x_{n+1} \rangle \\ &\ge \lambda \left[f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - d_1 \|y_{n-1} - y_n\|^2 - d_2 \|y_n - x_{n+1}\|^2 \right] + \langle x_n - x_{n+1}, y - x_{n+1} \rangle \\ &\ge \lambda \left[\langle x_n - y_n, x_{n+1} - y_n \rangle - d_1 \|y_{n-1} - y_n\|^2 - d_2 \|y_n - x_{n+1}\|^2 \right] + \langle x_n - x_{n+1}, y - x_{n+1} \rangle, \quad \forall y \in H_n, \forall n \in \mathbb{N}. \end{aligned} \quad (70)$$

Note that Lemma 4 has shown that $EP(g, C) \subset T_n \subset H_{n+1}$ for each $n \in \mathbb{N}_0$. So by (70), we have

$$\lambda f(y_n, y) \ge \lambda \left[\langle x_n - y_n, x_{n+1} - y_n \rangle - d_1 \|y_{n-1} - y_n\|^2 - d_2 \|y_n - x_{n+1}\|^2 \right] + \langle x_n - x_{n+1}, y - x_{n+1} \rangle, \quad \forall y \in EP(g, C), \forall n \in \mathbb{N}. \quad (71)$$

This completes the proof. □

Theorem 1. *If the parameters β and λ satisfy the conditions:*

$$\beta \in \left(0, \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}\right) \text{ and } \lambda \in \left(0, \frac{1}{2d_2 + 4d_1}\right), \quad (72)$$

then the sequence $\{x_n\}$ generated by Algorithm 1 converges weakly to the point $\overline{x} = \lim_{n\longrightarrow\infty} P_\Omega x_n$.

Proof. Since $y_{n+1} \in T_n$ for each $n \in \mathbb{N}_0$, by (66) we have

$$\|t_n - y_{n+1}\| \le \|y_{n+1} - x_n\| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \quad (73)$$

Furthermore, by (68) and (73), we get

$$\|t_n - x_n\| \le \|t_n - y_{n+1}\| + \|y_{n+1} - x_n\| \longrightarrow 0, \text{ as } n \longrightarrow \infty. \quad (74)$$

From Lemma 5, it follows that $\{x_n\}$ is bounded. This fact with (74) implies that $\{t_n\}$ is also bounded.

Take $x' \in EP(g, C)$ and put $M = \sup_{n\in\mathbb{N}} (\|x_n - x'\| + \|t_n - x'\|)$. By (43) and (74), we have

$$\begin{aligned}\left(1 - 2\beta c_1\|x_n - u_n\|^2 + (1 - 2\beta c_2)\|u_n - t_n\|^2 \le \|x_n - x'\|^2 - \|t_n - x'\|^2 \right.\\ \le \|x_n - t_n\|\left(\|x_n - x'\| + \|t_n - x'\|\right)\\ \le M\|x_n - t_n\| \longrightarrow 0, \text{ as } n \longrightarrow \infty,\end{aligned} \quad (75)$$

which with $1 - 2\beta c_1 > 0$ and $1 - 2\beta c_2 > 0$ implies

$$\begin{aligned}\|x_n - u_n\|^2 &\longrightarrow 0,\\ \|u_n - t_n\|^2 &\longrightarrow 0, \text{ as } n \longrightarrow \infty.\end{aligned} \quad (76)$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ weakly converging to $\overline{x} \in H$. By (73), we can conclude that $\{u_{n_k}\}$ also weakly converges to $\overline{x}$. Since C is closed and $\{u_n\} \subset C$ for all $n \in \mathbb{N}$, it follows that $\overline{x} \in C$. We show that $\overline{x} \in EP(g, C)$. In fact, by (33), (36), and (40), we get

$$\beta g(u_{n_k}, y) \ge \beta\left(\langle x_{n_k} - u_{n_k}, t_{n_k} - u_{n_k}\rangle - c_1\|u_{n_k} - x_{n_k}\|^2 - c_2\|t_{n_k} - u_{n_k}\|^2\right) + \langle x_{n_k} - t_{n_k}, y - t_{n_k}\rangle, \quad \forall y \in C, \forall k \in \mathbb{N}. \quad (77)$$

Letting $k \longrightarrow \infty$ in (77), by (74), (76), and (A2), we get

$$\beta g(\overline{x}, ty) \ge \limsup_{k\longrightarrow\infty} g(u_{n_k}, y) \ge 0, \forall y \in C, \quad (78)$$

which with $\beta > 0$ implies that $\overline{x} \in EP(g, C)$.

Next, we prove that $\overline{x} \in \Omega$. To end this, we need to show that

$$f(\overline{x}, y) \ge 0, \forall y \in EP(g, C). \quad (79)$$

In fact, by (73), we have

$$\begin{aligned}\lambda f(y_{n_k}, y) \ge \lambda\left[\langle x_{n_k} - y_{n_k}, x_{n_k+1} - y_{n_k}\rangle - d_1\|y_{n_k-1} - y_{n_k}\|^2 - d_2\|y_{n_k} - x_{n_k+1}\|^2\right]\\ + \langle x_{n_k} - x_{n_k+1}, y - x_{n_k+1}\rangle, \forall y \in EP(g, C), \forall k \in \mathbb{N}_0.\end{aligned} \quad (80)$$

Letting $k \longrightarrow \infty$ in (80), by (65)–(67) and (A2), we have

$$\lambda f(\overline{x}, y) \ge \limsup_{k\longrightarrow\infty} f(y_{n_k}, y) \ge 0, \forall y \in EP(g, C), \quad (81)$$

which with $\lambda > 0$ implies that $f(\overline{x}, y) \ge 0$ for all $y \in EP(g, C)$. So, $\overline{x} \in \Omega$.

Now, we prove that the whole sequence $\{x_n\}$ converges weakly to the point $\overline{x}$. Indeed, assume that there exists a different subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to $x^\dagger$ with $\overline{x} \ne x^\dagger$. By arguing similarly as above, it follows that $x^\dagger \in \Omega$. Note that in the proof of Lemma 5, we have shown that the limits of $\{\|x_n - x^\dagger\|\}$ and $\{\|x_n - \overline{x}\|\}$ exist. So by Opial's theorem [29], we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|x_n - \overline{x}\| &= \liminf_{k\longrightarrow\infty}\|x_{n_k} - \overline{x}\| < \liminf_{k\longrightarrow\infty}\|x_{n_k} - x^\dagger\|\\ &= \lim_{n\longrightarrow\infty}\|x_n - x^\dagger\| = \liminf_{i\longrightarrow\infty}\|x_{n_i} - x^\dagger\|\\ &< \liminf_{i\longrightarrow\infty}\|x_{n_i} - \overline{x}\| = \lim_{n\longrightarrow\infty}\|x_n - \overline{x}\|.\end{aligned} \quad (82)$$

It is a contradiction. Hence, $\overline{x} = x^\dagger$. Therefore, the whole sequence $\{x_n\}$ converges weakly to the point $\overline{x}$.

Finally, we prove $\overline{x} = \lim_{n\longrightarrow\infty} P_\Omega x_n$. Let $w_n = P_\Omega x_n$ for all $n \geq 1$. It is easy to see that $\{w_n\}$ is bounded from the boundedness of $\{x_n\}$. We show that $\{w_n\}$ is a Cauchy sequence. By Lemma 1 and the definition of w_{n+1}, we have

$$\|w_{n+1} - x_{n+1}\|^2 \leq \|w_n - x_{n+1}\|^2, \forall n \in \mathbb{N}_0. \tag{83}$$

Since $w_n \in \Omega$, replacing x^* in (69) with w_n, we get

$$\|w_n - x_{n+1}\|^2 \leq \|x_n - w_n\|^2 + 4\lambda d_1 \|y_{n-1} - x_n\|^2, \forall n \in \mathbb{N}. \tag{84}$$

From (83) and (84), it follows that

$$\|w_{n+1} - x_{n+1}\|^2 \leq \|x_n - w_n\|^2 + 4\lambda d_1 \|y_{n-1} - x_n\|^2, \forall n \in \mathbb{N}. \tag{85}$$

From (64), (85), and Lemma 3, it follows that the limit of $\{\|w_n - x_n\|^2\}$ exists. For all $m, n \in \mathbb{N}$ with $m > n$, since $w_n \in \Omega$, by (69), we deduce

$$\begin{aligned}\|x_m - w_n\|^2 &\leq \|x_{m-1} - w_n\|^2 + 4\lambda d_1 \|y_{m-2} - x_{m-1}\|^2 \\ &\leq \cdots \leq \|x_n - w_n\|^2 + 4\lambda d_1 \sum_{k=n}^{m-1} \|y_{k-1} - x_k\|^2.\end{aligned} \tag{86}$$

From $w_m = P_\Omega x_m$ and $w_n \in \Omega$, by Lemma 1 and (86), we have

$$\begin{aligned}\|w_n - w_m\|^2 &\leq \|w_n - x_m\|^2 - \|w_m - x_m\|^2 \\ &\leq \|x_n - w_n\|^2 + 4\lambda d_1 \sum_{k=n}^{m-1} \|y_{k-1} - x_k\|^2 \\ &\quad - \|w_m - x_m\|^2, \forall m > n.\end{aligned} \tag{87}$$

Since $\lim_{n\longrightarrow\infty} \|x_n - w_n\|^2$ exists, letting $m, n \longrightarrow \infty$ in (87), by (64), we get $\lim_{n,m\longrightarrow\infty} \|w_n - w_m\|^2 = 0$. Consequently, $\{w_n\}$ is a Cauchy sequence. Since Ω is closed, $\{w_n\}$ converges strongly to some $x' \in \Omega$. Now, we prove that $\overline{x} = x'$. In fact, it follows from Lemma 1, $w_n = P_\Omega x_n$ and $\overline{x} \in \Omega$ that $\langle \overline{x} - w_n, w_n - x_n \rangle \geq 0$. Since $w_n \longrightarrow x'$ and $x_n \rightharpoonup \overline{x}$, we have $\langle \overline{x} - x', x' - \overline{x} \rangle \geq 0$. This shows that $\overline{x} = x' = \lim_{n\longrightarrow\infty} P_\Omega x_n$. This completes the proof. □

In Algorithm 1, c_1 and c_2 need to be known as the input parameters. The following algorithm is a modification in which c_1 and c_2 do not need to be known. □

Remark 7. By (A4), we have

$$\begin{aligned} g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) &\leq c_1 \|u_n - x_n\|^2 + c_2 \|t_n - u_n\|^2 \\ &\leq c\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right),\end{aligned} \tag{88}$$

where $c = \max\{c_1, c_2\}$. If $g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) \leq 0$, then

$$\begin{aligned}\beta_{n+1} &= \min\left\{\beta_n, \frac{\mu\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right)}{g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n)}\right\} \\ &\geq \min\left\{\beta_n, \frac{\mu\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right)}{c\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2\right)}\right\} = \min\left\{\beta_n, \frac{\mu}{c}\right\} \geq \cdots \geq \min\left\{\beta_0, \frac{\mu}{c}\right\},\end{aligned} \tag{89}$$

Note that from the definition of β_{n+1}, it follows that $\beta_{n+1} \geq \min\{\beta_0, (\mu/c)\}$ still holds even if $g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) \leq 0$. Since $\{\beta_n\}$ is nonincreasing and bounded from below by $\min\{\beta_0, (\mu/c)\}$, there exists $\beta > 0$ such that

$$\lim_{n\longrightarrow\infty} \beta_n = \beta. \tag{90}$$

Similarly, we can conclude that there exists $\lambda > 0$ such that

$$\lim_{n\longrightarrow\infty} \lambda_n = \lambda > 0. \tag{91}$$

Theorem 2. *The sequence $\{x_n\}$ generated by Algorithm 2 converges weakly to the point $\overline{x} = \lim_{n\longrightarrow\infty} P_\Omega x_n$.*

Proof. Repeating the proof of (37) and (40), we can get, for all $n \in \mathbb{N}$,

$$\lambda_n (f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n)) \geq \langle x_n - y_n, x_{n+1} - y_n \rangle, \tag{92}$$

and

$$\lambda_n f(y_n, x_{n+1}) \leq \langle x_n - x_{n+1}, x_{n+1} - x^* \rangle. \tag{93}$$

Initialization: Choose $x_0, y_{-1}, y_0 \in C$, the parameters $\beta_0, \lambda_0 > 0$, and $\mu \in (0, 1/4)$. Put $n = 0$.

Step 1. For given x_n, solve the strongly convex problems: $\begin{cases} u_n = \text{argmin}\{\beta_n g(x_n, y) + 1/2\|x_n - y\|^2 \colon y \in C\}, \\ t_n = \text{argmin}\{\beta_n g(u_n, t) + 1/2\|x_n - y\|^2 \colon t \in C_k\}, \end{cases}$

where $C_n = \{v \in H \colon \langle x_n - \beta_n w_n - u_n, v - u_n\rangle \le 0\}$ with $w_n \in \partial_2 g(x_n, u_n)$.

Step 2. Solve the strongly convex problems: $\begin{cases} x_{n+1} = \arg\min\{\lambda_n f(y_n, y) + 1/2\|x_n - y\|^2 \colon y \in H_n\}, \\ y_{n+1} = \arg\min\{\lambda_{n+1} f(y_n, y) + 1/2\|x_{n+1} - y\|^2 \colon y \in T_n\}, \end{cases}$

where $H_n = \{z \in H \colon \langle x_n - \lambda_n v_n - y_n, z - y_n\rangle \le 0\}$ with $v_n \in \partial_2 f(y_{n-1}, y_n)$, $T_n = \{z \in H \colon \|z - t_n\| \le \|z - x_n\|\}$,

$$\lambda_{n+1} = \begin{cases} \lambda_n, & f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1}) \le 0, \\ \min\{\lambda_n, \mu(\|y_n - y_{n-1}\|^2 + \|y_n - x_{n+1}\|^2)/f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1})\}, & \text{otherwise.} \end{cases}$$

Step 3. Modify β_{n+1} by the following formula:

$$\beta_{n+1} = \begin{cases} \beta_n, & g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) \le 0, \\ \min\{\beta_n, \mu(\|u_n - x_n\|^2 + \|t_n - u_n\|^2)/g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n)\}, & \text{otherwise.} \end{cases}$$

Step 4. If $x_{n+1} = y_{n+1} = x_n = y_n$, then the algorithm stops, $x_n \in \Omega$; otherwise, set $n = n + 1$ and return to Step 1.

Algorithm 2: (Extragradient-like method without prior constants).

By (92) and (93), we have

$$\begin{aligned} \lambda_n(f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1})) &\ge \langle x_n - y_n, x_{n+1} - y_n\rangle - \langle x_n - x_{n+1}, x_{n+1} - x^*\rangle \\ &= \frac{1}{2}\left[\left(\|y_n - x_n\|^2 + \|x_{n+1} - y_n\|^2 - \|x_{n+1} - x_n\|^2\right) - \left(\|x_n - x^*\|^2 - \|x_{n+1} - x_n\|^2 - \|x_{n+1} - x^*\|^2\right)\right], \forall n \in \mathbb{N}. \end{aligned} \tag{94}$$

By the definition of λ_{n+1}, in the case when $f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1}) > 0$, we have

$$f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1}) \le \frac{\mu\left(\|y_n - y_{n-1}\|^2 + \|y_n - x_{n+1}\|^2\right)}{\lambda_{n+1}}, \quad \forall n \in \mathbb{N}. \tag{95}$$

It is emphasized here that (95) still holds even if

$$f(y_{n-1}, x_{n+1}) - f(y_{n-1}, y_n) - f(y_n, x_{n+1}) \le 0. \tag{96}$$

So, combining (94) with (95), we obtain

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\le \|x_n - x^*\|^2 - \left(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2\right) + \frac{2\mu\lambda_n}{\lambda_{n+1}}\left(\|y_n - y_{n-1}\|^2 + \|y_n - x_{n+1}\|^2\right) \\ &\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \left(1 - \frac{2\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1} - y_n\|^2 + \frac{4\mu\lambda_n}{\lambda_{n+1}}\left(\|y_n - x_n\|^2 + \|x_n - y_{n-1}\|^2\right) \\ &= \|x_n - x^*\|^2 - \left(1 - \frac{4\mu\lambda_n}{\lambda_{n+1}}\right)\|x_n - y_n\|^2 - \left(1 - \frac{2\mu\lambda_n}{\lambda_{n+1}}\right)\|x_{n+1} - y_n\|^2 + \frac{4\mu\lambda_n}{\lambda_{n+1}}\|x_n - y_{n-1}\|^2, \forall n \in \mathbb{N}. \end{aligned} \tag{97}$$

Since $\lambda_n \longrightarrow \lambda > 0$, it follows that $\lim_{n\longrightarrow\infty}(4\mu\lambda_n/\lambda_{n+1}) = 4\mu < 1$. Thus, for a fixed number $\epsilon \in (4\mu, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{4\mu\lambda_n}{\lambda_{n+1}} < \epsilon, \quad \forall n \geq n_0. \tag{98}$$

By (97) and (98), we have

$$\left\|x_{n+1} - x^*\right\|^2 \leq \left\|x_n - x^*\right\|^2 - (1-\epsilon)\left\|x_n - y_n\right\|^2 - \left(1 - \frac{\epsilon}{2}\right)\left\|x_{n+1} - y_n\right\|^2 + \epsilon\left\|x_n - y_{n-1}\right\|^2, \quad \forall n \geq n_0. \tag{99}$$

which is a similar result with (61).

On the other hand, for all $x' \in EP(g, C)$, repeating the proof of (35) and (38), we have

$$\langle x_n - u_n, t_n - u_n\rangle + \langle x_n - t_n, x' - t_n\rangle \leq \beta_n\left[g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n)\right], \quad \forall n \in \mathbb{N}_0. \tag{100}$$

By the definition of β_{n+1}, if $g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) > 0$, then

$$g(x_n, t_n) - g(x_n, u_n) - g(u_n, t_n) \leq \frac{\mu\left(\left\|u_n - x_n\right\|^2 + \left\|t_n - u_n\right\|^2\right)}{\beta_{n+1}}, \quad \forall n \in \mathbb{N}_0. \tag{101}$$

Note that the definition of β_{n+1} implies that (101) still holds even if $g(x_n, t_n) g(x_n, u_n) - g(u_n, t_n) \leq 0$. So by (100) and (101), we obtain

$$\langle x_n - u_n, t_n - u_n\rangle + \langle x_n - t_n, x' - t_n\rangle \leq \frac{\mu\beta_n\left(\left\|u_n - x_n\right\|^2 + \left\|t_n - u_n\right\|^2\right)}{\beta_{n+1}}, \forall n \in \mathbb{N}_0. \tag{102}$$

Now, by (102) and $2\langle t_n - x_n, t_n - x'\rangle = \|x_n - x'\|^2 - \|t_n - x_n\|^2 - \|t_n - x'\|^2$, we have

$$\begin{aligned}
\left\|t_n - x'\right\|^2 &\leq \left\|x_n - x'\right\|^2 - \left\|t_n - x_n\right\|^2 - 2\langle x_n - u_n, t_n - u_n\rangle + \frac{2\mu\beta_n\left(\left\|u_n - x_n\right\|^2 + \left\|t_n - u_n\right\|^2\right)}{\beta_{n+1}} \\
&= \left\|x_n - x'\right\|^2 - \left\|t_n - u_n\right\|^2 - \left\|u_n - x_n\right\|^2 + \frac{2\mu\beta_n\left(\left\|u_n - x_n\right\|^2 + \left\|t_n - u_n\right\|^2\right)}{\beta_{n+1}} \\
&= \left\|x_n - x'\right\|^2 - \left(1 - \frac{2\mu\beta_n}{\beta_{n+1}}\right)\left\|x_n - u_n\right\|^2 + \left(1 - \frac{2\mu\beta_n}{\beta_{n+1}}\right)\left\|u_n - t_n\right\|^2, \forall n \in \mathbb{N}_0.
\end{aligned} \tag{103}$$

Since $\beta_n \longrightarrow \beta > 0$, it follows that $\lim_{n\longrightarrow\infty}(2\mu\beta_n/\beta_{n+1}) = 2\mu < 1$. Thus, for a fixed number $\tau \in (2\mu, 1)$, there exists $m_0 \in \mathbb{N}$ such that

$$\frac{2\mu\beta_n}{\beta_{n+1}} < \tau, \ \forall n \geq m_0. \tag{104}$$

By (103) and (104), we have

$$\begin{aligned}\|t_n - x'\|^2 &\leq \|x_n - x'\|^2 - (1-\tau)\|x_n - u_n\|^2 \\ &\quad + (1-\tau)\|u_n - t_n\|^2 \\ &\leq \|x_n - x'\|^2, \ \forall n \geq m_0.\end{aligned} \tag{105}$$

Finally, by arguing similarly to the proof of Lemma 5 and Theorem 1, we can obtain the desired conclusion. This completes the proof.

As an application of the results above, we consider the following bilevel variational inequality problem (BVIP for short):

$$\text{find } \bar{x} \in VI(G,C) \text{ such that } \langle F(\bar{x}), y - \bar{x}\rangle \geq 0, \quad \forall y \in VI(G,C), \tag{106}$$

where F and G be the mappings from H into itself. We denote the solution set of (106) by Γ, that is,

$$\Gamma = \{z \in VI(G,C): \langle F(z), y - z\rangle \geq 0, \ \forall y \in VI(G,C)\}. \tag{107}$$

□

Corollary 1. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $F, G: H \longrightarrow H$ be the pseudomonotone and Lipschitz continuous mappings with the Lipschitz constants L_1 and L_2 satisfy the following conditions:*

$$(B) \limsup_{n\longrightarrow\infty}\langle F(x_n), y - x_n\rangle \leq \langle F(\hat{x}), y - \bar{x}\rangle \text{ and } \limsup_{n\longrightarrow\infty}\langle G(x_n), y - x_n\rangle \leq \langle G(\hat{x}), y - \bar{x}\rangle \tag{108}$$

for every sequence $\{x_n\}$ converging weakly to $\hat{x}$.

Assume that $\Gamma \neq \emptyset$. Take the parameters $\beta \in (0, (1/L_1)), \lambda \in (0, (1/3L_2))$, the initial points $x_0, y_0, y_{-1} \in H$ and generate the sequence $\{x_n\}$ in the following manner:

$$\begin{cases} u_n = P_C(x_n - \beta G(x_n)), \\ t_n = P_{C_n}(x_n - \beta G(u_n)), \\ x_{n+1} = P_{H_n}(x_n - \lambda F(y_n)), \\ y_{n+1} = P_{T_n}(x_{n+1} - \lambda F(y_n)), n \geq 0, \end{cases} \tag{109}$$

where $C_n = \{y \in H: \langle x_n - \beta G(x_n) - u_n, y - u_n\rangle \leq 0\}$, $H_n = \{y \in H: \langle x_n - \beta F(y_{n-1}) - y_n, y - y_n\rangle \leq 0\}$, and T_n is defined as in Algorithm 1. Then, the sequence $\{x_n\}$ generated by (109) converges weakly to the point $\bar{x} = \lim_{n\longrightarrow\infty} P_\Gamma x_n$.

Proof. Let $g(x,y) = \langle G(x), y - x\rangle$ and $f(x,y) = \langle F(x), y - x\rangle$ for all $x, y \in H$. Since F is pseudomonotone on H, it follows that $f(x,y) = \langle F(x), y - x\rangle \geq 0 \Rightarrow f(y,x) = \langle F(y), x - y\rangle \leq 0$. So f is pseudomonotone on H. It is obvious that f satisfies the condition (A3). In addition, if $x_n \rightharpoonup \hat{x}$, by (B), we have

$$\begin{aligned}\limsup_{n\longrightarrow\infty} f(x_n, y) &= \limsup_{n\longrightarrow\infty}\langle F(x_n), y - x_n\rangle \\ &\leq \langle F(\hat{x}), y - \hat{x}\rangle = f(\hat{x}, y).\end{aligned} \tag{110}$$

So f satisfies the condition (A2). Finally, since F is L_1-Lipschitz continuous, f is Lipschitz-type continuous with the constant $d_1 = d_2 = (L_1/2)$; see Remark 1. Thus, f satisfies the conditions (A1)–(A4). Similarly, g also satisfies the conditions (A1)–(A4). In particular, g satisfies (A4) with $c_1 = c_2 = (L_2/2)$. So, the conditions on β and λ in Lemma 5 become the ones in Corollary 1. On the other hand, by Algorithm 1,

$$u_n = \operatorname{argmin}\left\{\beta g(x_n, y) + \frac{1}{2}\|x_n - y\|^2: \ y \in C\right\} \tag{111}$$

is equivalent to $u_n = P_C(x_n - \beta G(x_n))$. Similarly, t_n, x_{n+1}, and y_{n+1} in Algorithm 1 are equivalent to t_n, x_{n+1}, and y_{n+1} in Corollary 1, respectively. By Theorem 1, the desired conclusion can be obtained. This completes the proof.

Since the proof process of the following corollary is similar to the one of Corollary 3.1, we give the following corollary and omit the proof process. □

Corollary 2. *Let H be a real Hilbert space and C be a nonempty closed and convex subset of H. Let $F, G: H \longrightarrow H$ be the pseudomonotone and Lipschitz continuous mappings with the Lipschitz constants L_1 and L_2 satisfying the condition (B) in Corollary 3.1. Assume that $\Gamma \neq \emptyset$. The parameters $\beta_0 > 0, \lambda_0 > 0$, $\mu \in (\beta_0, (1/4))$, the initial points $x_0, y_0, y_{-1} \in H$ are taken, and the sequence $\{x_n\}$ is generated by the following manner:*

$$\begin{cases} u_n = P_C(x_n - \beta_n G(x_n)), \\ t_n = P_{C_n}(x_n - \beta_n G(u_n)), \\ x_{n+1} = P_{H_n}(x_n - \lambda_n F(y_n)), \\ y_{n+1} = P_{T_n}(x_{n+1} - \lambda_{n+1} F(y_n)), n \geq 0, \end{cases} \tag{112}$$

where C_n, H_n, and T_n are defined as in Corollary 1,

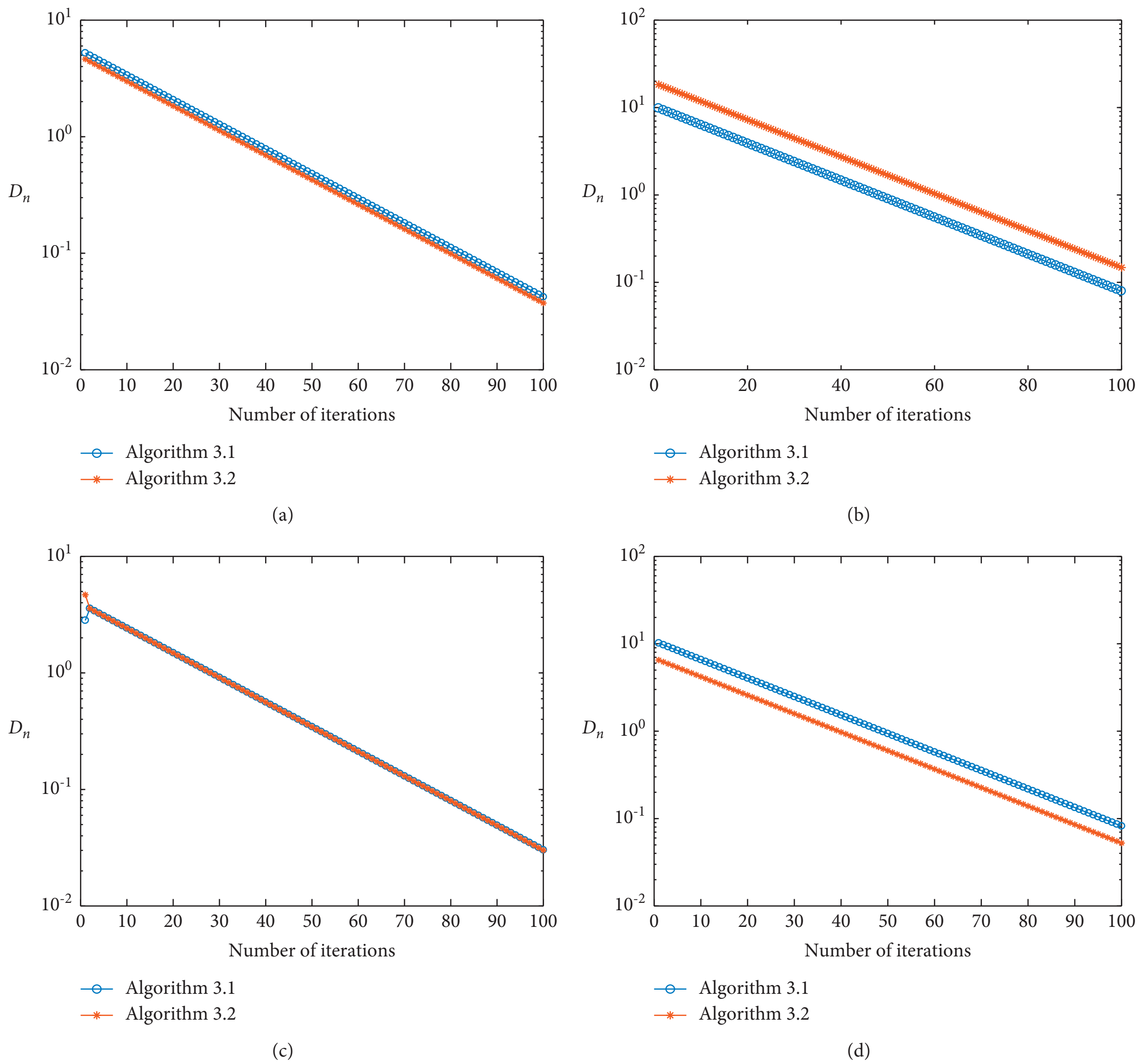

FIGURE 1: Experiment in different $\mathbb{R}^m$.

$$\lambda_{n+1} = \begin{cases} \lambda_n, & \langle F(y_{n-1}) - F(y_n), x_{n+1} - y_n \rangle \leq 0, \\ \min\left\{ \lambda_n, \dfrac{\mu\left(\|y_n - y_{n-1}\|^2 + \|y_n - x_{n+1}\|^2 \right)}{\langle F(y_{n-1}) - F(y_n), x_{n+1} - y_n \rangle} \right\}, & \text{otherwise}, \end{cases} \tag{113}$$

and β_{n+1} *is modified by*

$$\beta_{n+1} = \begin{cases} \beta_n, & \langle G(x_n) - G(u_n), t_n - u_n \rangle \leq 0, \\ \min\left\{ \lambda_n, \dfrac{\mu\left(\|u_n - x_n\|^2 + \|t_n - u_n\|^2 \right)}{\langle G(x_n) - G(u_n), t_n - u_n \rangle} \right\}, & \text{otherwise}. \end{cases} \tag{114}$$

Then, the sequence $\{x_n\}$ generated by (112) converges weakly to the point $\overline{x} = \lim_{n\longrightarrow\infty} P_\Gamma x_n$.

Remark 8. Since C_n, H_n, and T_n are half-spaces, from Remark 2, it follows that t_n, x_{n+1}, and y_{n+1} in Corollary 1 and 2 can be computed explicitly.

4. Numerical Examples

In this section, we give two examples to illustrate the convergence of Algorithm 1 and 2. The programs are written in Matlab 2016b, and the examples are computed on a PC Intel(R) Core (TM) i5-4260U CPU, 2.00 GHz, Ram 4.00 GB.

We first give the following example to illustrate the effectiveness of Algorithm 1 and 2.

Example 1. Let $H = \mathbb{R}^m$ and $C = \{x \in \mathbb{R}^m: x_1 \geq 0, x_i \geq 1, \forall i \in \{2, \ldots, m\}\}$. Let $g: H \times H \longrightarrow \mathbb{R}$ be defined by

$$g(x, y) = \sum_{i=2}^{m} (y_i - x_i)\|x\|, \quad \forall x = (x_1, \ldots, x_m),\ y = (y_1, \ldots, y_m) \in H. \tag{115}$$

It is known that g satisfies the conditions (A1)–(A4). In particular, g is Lipschitz-type continuous with the constants $c_1 = c_2 = 2$; see [30] for details. Let $f: H \times H \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = \langle Ax, y - x\rangle, \quad \forall x, y \in H, \tag{116}$$

where $Ax = ((x_1/2), (x_1 - 1/2), \ldots, (x_m - 1/2))$. It is easy to see that f satisfies the conditions (A1)–(A4). In particular, f is Lipschitz-type continuous with the constants $d_1 = d_2 = (1/4)$. The solution set of the bilevel equilibrium problem (7) in this example is found as $\Omega = \{(0, 1, \ldots, 1)\}$.

We choose the initial points x_0, y_0, y_{-1} randomly from the interval (0,5) for Algorithm 1 and 2, the input parameters $\lambda = \beta = 0.1$ for Algorithm 1, and $\lambda_0 = \beta_0 = \mu = 0.2$ for Algorithm 2. The maximum iteration of 100 as the stop criterion is used for Algorithm 1 and 2. The numerical results with the different dimensions m are shown in Figure 1. In this figures, the x-axis represents the number of iterations while the y-axis is for the value of D_n generated by Algorithm 1 and 2, where

$$D_n = \left\| x_n - (0, 1, \ldots, 1) \right\|. \tag{117}$$

From the computed results, we see the effectiveness of Algorithm 1 and 2.

The next example was ever used in [20]. Here, we use this example to illustrate the convergence of Algorithm 1 and 2 and compare the computed results with Algorithm 2.1 in [20].

Example 2. Let $C = \{x \in \mathbb{R}^5: -1 \leq x_i \leq 1, \forall i = 1, \ldots, 5\}$ and $f: \mathbb{R}^5 \times \mathbb{R}^5$ be defined by

$$f(x, y) = \langle F(x) + Qy + q, y - x\rangle, \forall x, y \in \mathbb{R}^5, \tag{118}$$

where $Q = AA^T + B + D$ with

$$A = \begin{pmatrix} -2 & 1 & 0 & 1 & -1 \\ 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 1 & 3 & 1 & 0 \\ 2 & 0 & 1 & -1 & 3 \end{pmatrix},$$
$$B = \begin{pmatrix} 0 & 1 & 2 & 1 & -1 \\ -1 & 3 & 2 & 0 & 2 \\ -2 & -2 & 1 & 1 & -3 \\ -1 & 0 & -1 & 1 & 0 \\ 1 & -2 & 3 & 0 & 2 \end{pmatrix}, \tag{119}$$
$$D = \begin{pmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 22 \end{pmatrix},$$
$$F(x) = (\xi x_1 + \xi x_2 + \sin(x_1), -\xi x_1 + \xi x_2 + \sin(x_2), (\xi - 1)x_3, (\xi - 1)x_4, (\xi - 1)x_5),$$

and q is a vector in $\mathbb{R}^5$.

Let $f: \mathbb{R}^5 \times \mathbb{R}^5$ be defined by

$$g(x, y) = \langle Px + \overline{P}y + p, y - x\rangle, \forall x, y \in \mathbb{R}^5, \tag{120}$$

where p is a vector in $\mathbb{R}^5$ and $P = 2\overline{P} + I$ with

Table 1: Computed results for Algorithm 3.1 with the different parameters.

Test prob.	ξ	β	λ	No. iter.	CPU-times (s)
1	65	$(1/3c_1)$	$(1/7d_1)$	8	1.534 5
2	69	$(1/3c_1)$	$(1/7d_1)$	17	1.711 3
3	55	$(1/5c_1)$	$(1/7d_1)$	22	2.023 4
4	55	$(1/5c_1)$	$(1/7d_1)$	13	1.743 1
5	70	$(1/5c_1)$	$(1/10d_1)$	15	1.711 3
6	80	$(1/5c_1)$	$(1/10d_1)$	24	1.988 5
7	95	$(1/8c_1)$	$(1/15d_1)$	35	2.233 1
8	45	$(1/8c_1)$	$(1/15d_1)$	21	1.993 5
9	100	$(1/10c_1)$	$(1/10d_1)$	19	2.100 3
10	150	$(1/10c_1)$	$(1/10d_1)$	22	2.002 3

Table 2: Computed results for Algorithm 3.2 with the different parameters.

Test prob.	ξ	β_0	λ_0	μ	No. iteration	CPU times (s)
1	65	$(1/3c_1)$	$(1/7d_1)$	0.2	12	4.324 3
2	69	$(1/3c_1)$	$(1/7d_1)$	0.2	23	4.001 2
3	55	$(1/5c_1)$	$(1/7d_1)$	0.15	22	3.023 4
4	55	$(1/5c_1)$	$(1/7d_1)$	0.15	32	5.111 3
5	70	$(1/5c_1)$	$(1/10d_1)$	0.1	25	4.875 0
6	80	$(1/5c_1)$	$(1/10d_1)$	0.1	29	5.114 6
7	95	$(1/8c_1)$	$(1/15d_1)$	0.1	33	6.143 7
8	45	$(1/8c_1)$	$(1/15d_1)$	0.12	31	5.887 5
9	100	$(1/10c_1)$	$(1/10d_1)$	0.09	29	4.997 3
10	150	$(1/10c_1)$	$(1/10d_1)$	0.09	31	5.880 3

Table 3: Computed results for Algorithm 2.1 in [20] with the different parameters.

Test prob.	ξ	β_n	λ_n	No. iteration	CPU times (s)
1	65	$(2\eta/2d_1^2(n^2+2))$	$(1/2c_1+100n)$	8	1.500 2
2	69	$(2\eta/2d_1^2(2n^2+12))$	$(1/2c_1+200n)$	7	0.887 2
3	55	$(2\eta/2d_1^2(2n^2+20))$	$(1/2c_1+500n)$	12	1.687
4	55	$(2\eta/2d_1^2(2n^2+15))$	$(1/2c_1+600n)$	13	1.400 3
5	70	$(2\eta/2d_1^2(2n^2+20))$	$(1/2c_1+1000n)$	5	1.711 3
6	80	$(2\eta/2d_1^2(2n^2+30))$	$(1/2c_1+1000n)$	5	1.422 2
7	95	$(2\eta/2d_1^2(n^2+100))$	$(1/2c_1+500n)$	7	0.983 1
8	45	$(2\eta/2d_1^2(n^2+150))$	$(1/2c_1+500n)$	9	0.800 5
9	100	$(2\eta/2d_1^2(n^2+200))$	$(1/2c_1+1000n)$	7	0.965 9
10	150	$(2\eta/2d_1^2(2n^2+1))$	$(1/2c_1+200n)$	14	2.110 3

$$\overline{P} = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 1 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 0 & 1 & 0 & 5.5 \end{pmatrix}. \tag{121}$$

It is known that f and g satisfy all the conditions required in [20] and Section 3 of this paper. In particular, f is Lipschitz-type continuous with the constants $d_1 = d_2 = (1/2)(\sqrt{2(2\xi^2+2\xi+1)} + \|Q\|)$, where $\|Q\| = 58.9677$ and g is Lipschitz-type continuous with the constants $c_1 = c_2 = (1/2)\|\overline{P}+I\|$; see [20].

We choose the initial point $x_0 = y_0 = y_{-1} = (1,1,1,1,1)$ for Algorithm 1 and 2 and $x_0 = (1,0,0,1,1)$ for Algorithm 2.1 in [20]. The stop criterion is $D_n \le 10^{-3}$, where

$$D_n = \max\left\{\left\|x_n - y_n\right\|, \left\|x_{n+1} - y_n\right\|\right\} \tag{122}$$

for the three algorithms. The computed results are presented in Tables 1–3 for Algorithm 1, 2, and Algorithm 2.1 in [20], respectively. In Table 3, $\eta = \xi - 1 - \|Q\|$.

From the computed results, we see that Algorithm 2 needs more CPU times and iterations over Algorithm 1 and Algorithm 2.1 in [20]. The course may be that Algorithm 2 involves a self-adaptive process of computing the values of β_{n+1} and λ_{n+1}.

5. Conclusion

We have proposed two iterative algorithms for finding the solution of a bilevel equilibrium problem in a real Hilbert space. The sequence generated by our algorithms converges weakly to the solution. Furthermore, we reported some numerical results to support our algorithms. How to obtain the strong convergence of Algorithm 1 and 2 without the additional assumptions is our future investigation.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Stability Analysis of a Ratio-Dependent Predator-Prey Model

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In this study, a ratio-dependent predator-prey model is investigated. The local stability and global stability of the nonnegative boundary equilibrium and positive equilibrium of the model are discussed, respectively. Sufficient condition is obtained for the existence of Hopf bifurcation at the positive equilibrium.

1. Introduction

Recently, the predator-prey models have been studied by many authors [1–8]. In general, a predator-prey model has the following forms:

$$\begin{cases} \dot{x} = xf(x) - p(x)y, \\ \dot{y} = kp(x)y - yg(y), \end{cases} \tag{1}$$

where $x(t)$ and $y(t)$ are the densities of the prey and predator population at time t, respectively. The function $f(x)$ represents the growth of the prey population rate, $g(y)$ represents the growth rate of predator population, and $p(x)$ represents the functional response function of predator population to prey population. In [1], Xu et al. used the function $p(x) = x^2/(x^2 + my^2)$ as the functional response function of predator population to prey population. The time delay due to the gestation of the predator is discussed in [1].

It is noted that in model (1), each individual's prey admits the same risk to be attacked by predators and each individual predator admits the same ability to feed on prey. This assumption seems not to be realistic for many animals. In natural world, there are many species whose individuals pass through an immature stage. Stage structure is a natural phenomenon and represents, for example, the division of a population into immature and mature individuals. In the last two decades, stage-structured models have received great attention [3–7, 9].

Based on above discussion, we study the following predator-prey model:

$$\begin{cases} \dot{x}_1(t) = rx_2(t) - (d_1 + r_1)x_1(t) - \dfrac{a_1x_1^2(t)y_2(t)}{x_1^2(t) + my_2^2(t)}, \\ \dot{x}_2(t) = r_1x_1(t) - \mathrm{d}_2x_2(t) - ax_2^2(t), \\ \dot{y}_1(t) = \dfrac{a_2x_1^2(t-\tau)y_2(t-\tau)}{x_1^2(t-\tau) + my_2^2(t-\tau)} - (r_2 + d_3)y_1(t), \\ \dot{y}_2(t) = r_2y_1(t) - d_4y_2(t), \end{cases} \tag{2}$$

where $x_1(t)$ and $x_2(t)$ are the densities of the immature and mature prey at time t and $y_1(t)$ and $y_2(t)$ are the densities of the immature and mature predators at time t. In model (2), all parameters are positive constants. $\tau \geq 0$ is the time delay due to the gestation of the predator. $x^2/(x^2 + my^2)$ is the ratio-dependent functional response.

Model (2) is of the following initial conditions:

$$\begin{aligned}&x_1(\theta)=\phi_1(\theta)\geq 0,\\&x_2(\theta)=\phi_2(\theta)\geq 0,\\&y_1(\theta)=\varphi_1(\theta)\geq 0,\\&y_2(\theta)=\varphi_2(\theta)\geq 0,\quad \theta\in[-\tau,0),\\&\phi_1(0)>0,\\&\phi_2(0)>0,\\&\varphi_1(0)>0,\\&\varphi_2(0)>0,\phi_1(\theta),\phi_2(\theta),\varphi_1(\theta),\varphi_2(\theta))\in C\left([-\tau,0],R^4_{+0}\right).\end{aligned}\tag{3}$$

The organization of this study is as follows. In Section 2, we discuss the local stability of the nonnegative boundary equilibrium and the positive equilibrium of models (2) and (3). The existence of a Hopf bifurcation for models (2) and (3) at the positive equilibrium is also established. Sufficient conditions are derived for the global stability of the nonnegative boundary equilibrium and positive equilibrium of models (2) and (3) in Section 3, respectively.

2. Local Stability and Hopf Bifurcation

In this section, by analyzing the corresponding characteristic equations, we study the local stability of each of nonnegative equilibria and the existence of a Hopf bifurcation at the positive equilibrium of models (2) and (3).

If $rr_1>d_2(r_1+d_1)$, model (2) has a nonnegative boundary equilibrium $E_1(x_1',x_2',0,0)$, where

$$\begin{aligned}x_1'&=\frac{r[rr_1-d_2(r_1+d_1)]}{a(r_1+d_1)^2},\\x_2'&=\frac{rr_1-d_2(r_1+d_1)}{a(r_1+d_1)}.\end{aligned}\tag{4}$$

If $(H_1)a_2r_2>d_4(r_2+d_3),rr_1-d_2(r_1+d_1)/a_1d_2>d_4(r_2+d_3)/a_2r_2h$, model (2) has a positive equilibrium $E^+(x_1^+,x_2^+,y_1^+,y_2^+)$, where

$$\begin{aligned}x_1^+&=\frac{r}{r_1+d_1+a_1h/1+mh^2}x_2^+,\\x_2^+&=\frac{1}{a}\left(\frac{rr_1}{r_1+d_1+a_1h/1+mh^2}-d_2\right),\\y_1^+&=\frac{d_4}{r_2}hx_1^*+\\y_2^+&=hx_1^+,\\h&=\sqrt{\frac{a_2r_2-d_4(r_2+d_3)}{md_4(r_2+d_3)}}.\end{aligned}\tag{5}$$

The characteristic equation of model (2) at $E_1(x_1',x_2',0,0)$ takes the following form:

$$\begin{aligned}&\left[\lambda^2+(r_1+d_1+d_2+2ax_2')\lambda+rr_1-d_2(r_1+d_1)\right]\\&\cdot\left[\lambda^2+g_1\lambda+g_0+h_0e^{-\lambda\tau}\right]=0,\end{aligned}\tag{6}$$

where $g_1=r_2+d_3+d_4,g_0=d_4(r_2+d_3),h_0=-a_2r_2$. When $rr_1>d_2(r_1+d_1)$, all roots of equation,

$$\lambda^2+(r_1+d_1+d_2+2ax_2^+)\lambda+rr_1-d_2(r_1+d_1)=0,\tag{7}$$

are negative. Now, we consider the roots of the following equation. $\lambda^2+g_1\lambda+g_0+h_0e^{-\lambda\tau}=0$. By calculating, we obtain

$$\begin{aligned}g_1^2-2g_0&=d_4^2+(r_2+d_3)^2>0,g_0^2-h_0^2\\&=d_4^2(r_2+d_3)^2-(a_2r_2)^2.\end{aligned}\tag{8}$$

When $d_4(r_2+d_3)>a_2r_2$, we get $g_0^2-h_0^2>0$. Therefore, E_1 is locally stable for all $\tau>0$. When $d_4(r_2+d_3)<a_2r_2$, we get $g_0^2-h_0^2<0$. Thus, E_1 is unstable.

The characteristic equation of model (2) at E^+ is of the form

$$\begin{aligned}&\lambda^4+P_3\lambda^3+P_2\lambda^2+P_1\lambda+P_0\\&+\left(Q_2\lambda^2+Q_1\lambda+Q_0\right)e^{-\lambda\tau}=0,\end{aligned}\tag{9}$$

where $P_3=r_1+d_1+a_1\alpha+d_2+2ax_2^++r_2+d_3+d_4,P_2=d_4(r_2+d_3)+(r_2+d_3+d_4)\quad(r_1+d_1+a_1\alpha+d_2+2ax_2^+)+(r_1+d_1+a_1\alpha)(d_2+2ax_2^+)-rr_1,\quad P_1=d_4(r_2+d_3)(r_1+d_1+a_1\alpha+d_2+2ax_2^+)+(r_2+d_3+d_4)[(r_1+d_1+a_1\alpha)(d_2+2ax_2^+)-rr_1],P_0=d_4(r_2+d_3)[(r_1+d_1+a_1\alpha)(d_2+2ax_2^+)-rr_1],Q_2=-a_2r_2\beta,Q_1=-a_2r_2\beta(r_1+d_1+d_2+2ax_2^+),\quad Q_0=-a_2r_2\beta[(r_1+d_1)(d_2+2ax_2^+)-rr_1],\ \alpha=2mx_1^+(y_2^+)^3/[(x_1^+)^2m(y_2^+)^2]^2,\beta=(x_1^+)^4/[(x_1^+)^2+m(y_2^+)^2]^2$.

Let $\tau=0$; then, (9) has the following form:

$$\lambda^4+P_3\lambda^3+(P_2+Q_2)\lambda^2+(P_1+Q_1)\lambda+P_0+Q_0=0.\tag{10}$$

Note that $P_3>0$. When

$$\begin{aligned}&[(H_2)]P_3(P_2+Q_2)>(P_1+Q_1),(P_1+Q_1)\\&\cdot[P_3(P_2+Q_2)-(P_1+Q_1)]>P_3^2(P_0+Q_0)>0,\end{aligned}\tag{11}$$

then positive equilibrium E^+ is locally asymptotically stable.

Let (H_1) and (H_2) hold. If $i\omega\,(\omega>0)$ is a solution of (9), by calculation, we can obtain

$$\omega^8+f_3\omega^6+f_2\omega^4+f_1\omega^2+f_0=0,\tag{12}$$

where

$$\begin{aligned}
f_3 &= P_3^2 - 2P_2 = d_4^2 + (r_2+d_3)^2 + (r_1+d_1+a_1\alpha)^2 + (d_2+2ax_2^+)^2 + 2rr_1 > 0,\\
f_2 &= P_2^2 + 2P_0 - 2P_1P_3 - Q_2^2 = \left[d_4^2(r_2+d_3)^2 - (a_2r_2\beta)^2\right] + \left[(r_1+d_1+a_1\alpha)(d_2+2ax_2^+) - rr_1\right]^2\\
&\quad + \left[d_4^2 + (r_2+d_3)^2\right]\left[(r_1+d_1+a_1\alpha)^2 + (d_2+2ax_2^+)^2 + 2rr_1\right] > 0,\\
f_1 &= P_1^2 - 2P_0P_2 + 2Q_0Q_2 - Q_1^2 = \left[d_4^2(r_2+d_3)^2 - (a_2r_2\beta)^2\right]\left[(r_1+d_1)^2 + (d_2+2ax_2^+)^2 + 2rr_1\right]\\
&\quad + \left[d_4^2 + (r_2+d_3)^2\right]\left[(r_1+d_1+a_1\alpha)(d_2+2ax_2^+) - rr_1\right]^2\\
&\quad + d_4^2(r_2+d_3)^2\left[2a_1\alpha(r_1+d_1) + (a_1\alpha)^2\right] > 0,\\
f_0 &= P_0^2 - Q_0^2 = (P_0+Q_0)(P_0-Q_0),
\end{aligned} \tag{13}$$

when $P_0 > Q_0$, E^+ is locally asymptotically stable for all $\tau > 0$. When $P_0 < Q_0$, ω_0 is the positive root of (12); in this case, (9) has a pair of roots $\pm i\omega_0$. By (12), we obtain

$$\tau_k = \frac{2k\pi}{\omega_0} + \frac{1}{\omega_0}\arccos\frac{\left(Q_2\omega_0^2 - Q_0\right)\left(\omega_0^4 - P_2\omega_0^2 + P_0\right) + Q_1\omega_0\left(P_3\omega_0^3 - P_1\omega_0\right)}{\left(Q_1\omega_0\right)^2 + \left(Q_2\omega_0^2 - Q_0\right)^2}, \quad k = 0, 1, 2, \ldots,. \tag{14}$$

Therefore, E^+ remains stable for $\tau < \tau_0$.

Differentiating (9) with respect to τ, we obtain that

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{4\lambda^3 + 3P_3\lambda^2 + 2P_2\lambda + P_1}{-\lambda\left(\lambda^4 + P_3\lambda^3 + P_2\lambda^2 + P_1\lambda + P_0\right)} + \frac{2Q_2\lambda + Q_1}{\lambda\left(Q_2\lambda^2 + Q_1\lambda + Q_0\right)} - \frac{\tau}{\lambda}. \tag{15}$$

Hence, we get

$$\begin{aligned}
\operatorname{sgn}\left\{\frac{d(\mathrm{Re}\lambda)}{d\tau}\right\}_{\lambda=i\omega_0} &= \operatorname{sgn}\left\{\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=i\omega_0}\\
&= \operatorname{sgn}\left\{\frac{\left(3P_3\omega_0^2 - P_1\right)\left(P_3\omega_0^2 - P_1\right) + 2\left(2\omega_0^2 - P_2\right)\left(\omega_0^4 - P_2\omega_0^2 + P_0\right)}{\omega_0^2\left(P_1 - P_3\omega_0^2\right)^2 + \left(\omega_0^4 - P_2\omega_0^2 + P_0\right)^2}\right.\\
&\quad \left. + \frac{2Q_2\left(Q_0 - Q_2\omega_0^2\right) - Q_1^2}{\left(Q_3\omega_0^3 - Q_1\omega_0\right)^2 + \left(Q_2\omega_0^2 - q_0\right)^2}\right\}\\
&= \operatorname{sgn}\left\{\frac{4\omega_0^6 + 3f_3\omega_0^4 + 2f_2\omega_0^2 + f_1}{\left(Q_1\omega_0\right)^2 + \left(Q_2\omega_0^2 - Q_0\right)^2}\right\} > 0.
\end{aligned} \tag{16}$$

Therefore, as $\tau = \tau_0$, $\omega = \omega_0$, there is Hopf bifurcation. From above discussion, we have the following results.

Theorem 1. *For model (2) with (3), we have the following:*

(i) Let $rr_1 > d_2(r_1 + d_1)$; if $a_2r_2 < d_4(r_2 + d_3)$, then E_1 is locally asymptotically stable; if $a_2r_2 > d_4(r_2 + d_3)$, then E_1 is unstable.

(ii) Assume (H_1) and (H_2) hold; if $P_0 > Q_0$, then E^+ is locally asymptotically stable for all $\tau \geq 0$; if $P_0 < Q_0$,

then there exists a $\tau_0 > 0$, s.t., E^+ is locally asymptotically stable if $0 < \tau < \tau_0$ and unstable if $\tau > \tau_0$. When $\tau = \tau_0$, models (2) and (3) undergo Hopf bifurcation at E^+.

3. Global Stability

In this section, by using an iteration technique, we discuss the global stability of the nonnegative equilibria E_1 and E^+ of models (2) and (3), respectively.

Theorem 2. *Let*

$$[(H_3)] r r_1 > d_2 (r_1 + d_1) + \frac{a_1 d_2}{2\sqrt{m}}, \quad a_2 r_2 < d_4 (r_2 + d_3), \tag{17}$$

hold; then, the nonnegative boundary equilibrium E_1 of model (2) is globally stable.

Proof. It follows from the positive solution of model (2), and we can obtain

$$\begin{aligned} \dot{x}_1(t) &\leqslant r x_2(t) - (d_1 + r_1) x_1(t), \\ \dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \end{aligned} \tag{18}$$

By Lemma 2.2 of [5] and comparison, we have

$$\begin{aligned} \limsup_{t\longrightarrow+\infty} x_1(t) &\le \frac{r[r r_1 - d_2(r_1 + d_1)]}{a(r_1 + d_1)^2}, \\ \limsup_{t\longrightarrow+\infty} x_2(t) &\le \frac{r r_1 - d_2(r_1 + d_1)}{a(r_1 + d_1)}. \end{aligned} \tag{19}$$

Therefore, there is a positive number t_1, for sufficiently small positive number ε, such that as $t > t_1$, $x_1(t) \le x_1' + \varepsilon$. Hence, for $t > t_1 + \tau$, we derive that

$$\begin{aligned} \dot{y}_1(t) &\le \frac{a_2 (x_1' + \varepsilon)^2 y_2(t-\tau)}{(x_1' + \varepsilon)^2 + m y_2^2(t-\tau)} - (r_2 + d_3) y_1(t), \\ u\dot{y}_2(t) &= r_2 y_1(t) - d_4 y_2(t). \end{aligned} \tag{20}$$

By Lemma 2.2 of [5] and comparison, we can obtain

$$\begin{aligned} \lim_{t\longrightarrow+\infty} y_1(t) &= 0, \\ \lim_{t\longrightarrow+\infty} y_2(t) &= 0. \end{aligned} \tag{21}$$

Therefore, there is a positive number $t_2.t_1$, such that if $t > t_2$, $y_2(t) < \varepsilon$.

For $t > t_2$, we derive from model (2) that

$$\begin{aligned} \dot{x}_1(t) &\ge r x_2(t) - (r_1 + d_1) x_1(t) - \frac{a_1}{2\sqrt{m}} x_1(t) \\ \dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \end{aligned} \tag{22}$$

By Lemma 2.2 of [5] and comparison, we have

$$\begin{aligned} \lim\inf_{t\longrightarrow+\infty} x_1(t) &\ge \frac{r}{a(r_1 + d_1 + a_1/2\sqrt{m})} \left[\frac{r r_1}{r_1 + d_1 + a_1/2\sqrt{m}} - d_2 \right] := \underline{x}_1, \\ \lim\inf_{t\longrightarrow+\infty} x_1(t) &\ge \frac{1}{a} \left[\frac{r r_1}{r_1 + d_1 + a_1/2\sqrt{m}} - d_2 \right]. \end{aligned} \tag{23}$$

By model (2), it follows that

$$\begin{aligned} \dot{x}_1(t) &\ge r x_2(t) - (r_1 + d_1) x_1(t) - \frac{a_1 \varepsilon}{\underline{x}_1} x_1(t), \\ \dot{x}_2(t) &= r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \end{aligned} \tag{24}$$

By Lemma 2.4 of [3] and comparison, we obtain that

$$\begin{aligned} \liminf_{t\longrightarrow+\infty} x_1(t) &\ge \frac{r[r r_1 - d_2(r_1 + d_1)]}{a(r_1 + d_1)^2}, \\ \liminf_{t\longrightarrow+\infty} x_2(t) &\ge \frac{r r_1 - d_2(r_1 + d_1)}{a(r_1 + d_1)}, \end{aligned} \tag{25}$$

which together with (19) and (21) yields

$$\lim_{t\longrightarrow+\infty} (x_1(t), x_2(t), y_1(t), y_2(t)) = (x_1', x_2', 0, 0). \tag{26}$$

Hence, the equilibrium $E_1(x_1', x_2', 0, 0)$ of model (2) is globally stable. □

Theorem 3. *Assume (H_1), (H_2), and $P_0 > Q_0$ hold; if*

$$\begin{aligned} [(H_4)] \frac{r r_1 - d_2(r_1 + d_1)}{a_1 d_2} &> \frac{1}{2\sqrt{m}}, a_2 r_2 (r_1 + d_1) \\ &< a_1 d_4 (r_2 + d_3) h, \end{aligned} \tag{27}$$

then the positive equilibrium $E^+(x_1^+, x_2^+, y_1^+, y_2^+)$ of model (2) is global stability.

Proof. Let

$$U_{x_i} = \limsup_{t\longrightarrow+\infty} x_i(t),$$
$$L_{x_i} = \liminf_{t\longrightarrow+\infty} x_i(t),$$
$$U_{y_i} = \limsup_{t\longrightarrow+\infty} y_i(t),$$
$$L_{y_i} = \liminf_{t\longrightarrow+\infty} y_i(t), \quad (i=1,2). \tag{28}$$

By the first two equations of model (2), we can obtain that

$$\dot{x}_1(t) \le r x_2(t) - (d_1 + r_1) x_1(t),$$
$$\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \tag{29}$$

By Lemma 2.2 of [5] and comparison, we have

$$U_{x_1} = \limsup_{t\longrightarrow+\infty} x_1(t) \le \frac{r[rr_1 - d_2(r_1+d_1)]}{a(r_1+d_1)^2} := M_1^{x_1},$$
$$U_{x_2} = \limsup_{t\longrightarrow+\infty} x_2(t) \le \frac{rr_1 - d_2(r_1+d_1)}{a(r_1+d_1)} := M_1^{x_2}. \tag{30}$$

So, for sufficiently small positive number ε, there exists a positive number t_1, such that if $t > t_1$, then $x_1(t) \le M_1^{x_1} + \varepsilon$.

For $t > t_1 + \tau$, by the last two equations of model (2), we get

$$\dot{y}_1(t) \le \frac{a_2 (M_1^{x_1}+\varepsilon)^2 y_2(t-\tau)}{(M_1^{x_1}+\varepsilon)^2 + m y_2^2(t-\tau)} - (r_2+d_3) y_1(t). \dot{x}_2(t) = r_2 x_1(t) - d_4 x_2(t). \tag{31}$$

By Lemma 2.2 of [5] and comparison, we obtain

$$U_{y_1} = \limsup_{t\longrightarrow+\infty} y_1(t) \le \frac{d_4}{r_2} h M_1^{x_1} := M_1^{y_1},$$
$$U_{y_2} = \limsup_{t\longrightarrow+\infty} y_2(t) = h M_1^{x_1} := M_1^{y_2}. \tag{32}$$

Hence, $U_{y_1} \le M_1^{y_1}$, $U_{y_2} \le M_1^{y_2}$, in which

$$M_1^{y_1} = \frac{a_2 r_2 - d_4(r_2+d_3)}{m r_2 (r_2+d_3)} M_1^{x_1},$$
$$M_1^{y_2} = \frac{a_2 r_2 - d_4(r_2+d_3)}{m d_4 (r_2+d_3)} M_1^{x_1}. \tag{33}$$

Therefore, for sufficiently small positive number ε, there is $t_2 \ge t_1 + \tau$, such that if $t > t_2$, $y_2(t) \le M_1^{y_2} + \varepsilon$.

For $t > t_2$, by the first two equations of model (2), we have

$$\dot{x}_1(t) \ge r x_2(t) - (r_1+d_1) x_1(t) - \frac{a_1}{2\sqrt{m}} x_1(t),$$
$$\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \tag{34}$$

By Lemma 2.4 of [3] and comparison, we derive that

$$L_{x_1} = \liminf_{t\longrightarrow\infty} x_1(t) \ge \frac{r[rr_1 - d_2(r_1+d_1+a_1/2\sqrt{m})]}{a(r_1+d_1+a_1/2\sqrt{m})^2} := N_1^{x_1},$$
$$L_{x_2} = \liminf_{t\longrightarrow\infty} x_2(t) \ge \frac{rr_1 - d_2(r_1+d_1+a_1/2\sqrt{m})}{a(r_1+d_1+a_1/2\sqrt{m})} := N_1^{x_2}. \tag{35}$$

Hence, for sufficiently small positive number ε, there is $t_3 \ge t_2$, such that if $t > t_3$, $x_1(t) \ge N_1^{x_1} - \varepsilon$.

For $t > t_3 + \tau$, it follows from the last two equations of model (2) that

$$\dot{y}_1(t) \ge \frac{a_2 (N_1^{x_1}-\varepsilon)^2 y_2(t-\tau)}{(N_1^{x_1}-\varepsilon)^2 + m y_2^2(t-\tau)} - (d_3+r_2) y_1(t). \dot{x}_2(t) = r_2 x_1(t) - d_4 x_2(t). \tag{36}$$

By Lemma 2.4 of [3] and comparison, we can obtain

$$L_{y_1} = \liminf_{t\longrightarrow+\infty} y_1(t) \le \frac{d_4}{r_2} h N_1^{x_1} := N_1^{y_1},$$
$$L_{y_2} = \limsup_{t\longrightarrow+\infty} y_2(t) = h N_1^{x_1} := N_1^{y_2}. \tag{37}$$

Therefore, for sufficiently small positive number ε, there is a positive number $t_4 \ge t_3 + \tau$, such that if $t > t_4$, $y_2(t) \ge N_1^{y_2} - \varepsilon$. In this case, by the first two equations of model (2), we have

$$\dot{x}_1(t) \le r x_2(t) - (d_1+r_1) x_1(t) - \frac{a_1 (N_1^{x_1}-\varepsilon)(N_1^{y_2}-\varepsilon)}{(M_1^{x_1}+\varepsilon)^2 + m (M_1^{y_2}+\varepsilon)^2} x_1(t),$$
$$\dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \tag{38}$$

For sufficiently small positive number ε, if (H_4) holds, by Lemma 2.2 of [5] and a comparison argument, we can obtain

$$U_{x_1} = \limsup_{t\longrightarrow+\infty} x_1(t) \le \frac{r\left[rr_1 - d_2\left(r_1+d_1+a_1 N_1^{x_1} N_1^{y_2}/(M_1^{x_1})^2 + m (M_1^{y_2})^2\right)\right]}{a\left(r_1+d_1+a_1 N_1^{x_1} N_1^{y_2}/(M_1^{x_1})^2 + m(M_1^{y_2})^2\right)^2} := M_2^{x_1},$$
$$U_{x_2} = \limsup_{t\longrightarrow+\infty} x_2(t) \le \frac{rr_1 - d_2\left(r_1+d_1+a_1 N_1^{x_1} N_1^{y_2}/(M_1^{x_1})^2 + m(M_1^{y_2})^2\right)}{a\left(r_1+d_1+a_1 N_1^{x_1} N_1^{y_2}/(M_1^{x_1})^2 + m(M_1^{y_2})^2\right)} := M_2^{x_2}. \tag{39}$$

Therefore, for sufficiently small positive number ε, there is $t_5 \geq t_4$, such that if $t > t_5$, $x_1(t) \leq M_2^{x_1} + \varepsilon$.

From the last two equations of model (2), we obtain that for $t > t_5 + \tau$,

$$\dot{y}_1(t) \leq \frac{a_2 (M_2^{x_1} + \varepsilon)^2 y_2(t-\tau)}{(M_2^{x_1} + \varepsilon)^2 + m y_2^2(t-\tau)} - (d_3 + r_2) y_1(t), \quad \dot{x}_2(t) = r_2 x_1(t) - d_4 x_2(t). \tag{40}$$

By Lemma 2.2 of [5] and comparison, if $a_2 r_2 > d_4(r_2 + d_3)$ holds, we have

$$U_{y_1} = \limsup_{t\longrightarrow+\infty} y_1(t) \leq \frac{d_4}{r_2} M_2^{x_1} := M_2^{y_1}, \quad U_{y_2} = \limsup_{t\longrightarrow+\infty} y_2(t) \leq h M_2^{x_1} := M_2^{y_2}. \tag{41}$$

Hence, for $\varepsilon > 0$ sufficiently small, there is a $T_6 \geq T_5 + \tau$, such that if $t > T_6$, $y_2(t) \leq M_2^{y_2} + \varepsilon$.

Again, for sufficiently small positive number ε and $t > t_6$, by the first two equations of model (2), we have

$$\dot{x}_1(t) \geq r x_2(t) - (d_1 + r_1) x_1(t) - \frac{a_1 (M_2^{x_1} + \varepsilon)(M_2^{y_2} + \varepsilon)}{(N_1^{x_1} - \varepsilon)^2 + m (N_1^{y_2} - \varepsilon)^2} x_1(t), \quad \dot{x}_2(t) = r_1 x_1(t) - d_2 x_2(t) - a x_2^2(t). \tag{42}$$

By Lemma 2.4 of [3] and comparison, if (H_4) holds, we can obtain

$$L_{x_1} = \liminf_{t\longrightarrow+\infty} x_1(t) \geq \frac{r\left[r r_1 - d_2\left(r_1 + d_1 + a_1 M_2^{x_1} M_2^{y_2} / (N_1^{x_1})^2 + m (N_1^{y_2})^2\right)\right]}{a\left(r_1 + d_1 + a_1 M_2^{x_1} M_2^{y_2} / (N_1^{x_1})^2 + m (N_1^{y_2})^2\right)^2} := N_2^{x_1},$$
$$L_{x_2} = \liminf_{t\longrightarrow+\infty} x_2(t) \geq \frac{r r_1 - d_2\left(r_1 + d_1 + a_1 M_2^{x_1} M_2^{y_2} / (N_1^{x_1})^2 + m (N_1^{y_2})^2\right)}{a\left(r_1 + d_1 + a_1 M_2^{x_1} M_2^{y_2} / (N_1^{x_1})^2 + m (N_1^{y_2})^2\right)} := N_2^{x_2}. \tag{43}$$

So, there is a positive number $t_7 \geq t_6$, for $t > t_7$, $x_1(t) \geq N_2^{x_1} - \varepsilon$.

For sufficiently small positive number ε and $t >_7 + \tau$, from the last two equations of model (2), we can derive

$$\dot{y}_1(t) \geq \frac{a_2 (N_2^{x_1} - \varepsilon)^2 y_2(t-\tau)}{(N_2^{x_1} - \varepsilon)^2 + m y_2^2(t-\tau)} - (d_3 + r_2) y_1(t), \quad \dot{x}_2(t) = r_2 x_1(t) - d_4 x_2(t). \tag{44}$$

By Lemma 2.4 of [3] and comparison, if $a_2 r_2 > d_4(d_3 + r_2)$, we have

$$U_{y_1} = \limsup_{t\longrightarrow+\infty} y_1(t) \geq \frac{d_4}{r_2} N_2^{x_1} := N_2^{y_1}, \quad U_{y_2} = \limsup_{t\longrightarrow+\infty} y_2(t) \geq h N_2^{x_1} := N_2^{y_2}. \tag{45}$$

Repeat the above process; for $n \geq 2$, we can obtain eight sequences:

$$M_n^{x_1}, M_n^{x_2}, M_n^{y_1}, M_n^{y_2}, N_n^{x_1}, N_n^{x_2}, N_n^{y_1}, N_n^{y_2} \ (n = 1, 2,), \tag{46}$$

in which

$$M_n^{x_1} = \frac{r}{r_1 + d_1 + a_1 N_{n-1}^{x_1} N_{n-1}^{y_2} / (M_{n-1}^{x_1})^2 + m (M_{n-1}^{y_2})^2} M_n^{x_2},$$
$$M_n^{x_2} = \frac{r r_1 - d_2\left(r_1 + d_1 + a_1 N_{n-1}^{x_1} N_{n-1}^{y_2} / (M_{n-1}^{x_1})^2 + m (M_{n-1}^{y_2})^2\right)}{a\left(r_1 + d_1 + a_1 N_{n-1}^{x_1} N_{n-1}^{y_2} / (M_{n-1}^{x_1})^2 + m (M_{n-1}^{y_2})^2\right)},$$
$$M_n^{y_1} = \frac{d_4}{r_2} h M_n^{x_1},$$
$$M_n^{y_2} = h M_n^{x_1},$$
$$N_n^{x_1} = \frac{r}{r_1 + d_1 + a_1 N_{n-1}^{x_1} N_{n-1}^{y_2} / (M_{n-1}^{x_1})^2 + m (M_{n-1}^{y_2})^2} N_n^{x_2},$$
$$N_n^{x_2} = \frac{r r_1 - d_2\left(r_1 + d_1 + a_1 M_n^{x_1} M_n^{y_2} / (N_{n-1}^{x_1})^2 + m (N_{n-1}^{y_2})^2\right)}{a\left(r_1 + d_1 + a_1 M_n^{x_1} M_n^{y_2} / (N_{n-1}^{x_1})^2 + m (N_{n-1}^{y_2})^2\right)},$$
$$N_n^{y_1} = \frac{d_4}{r_2} h N_n^{x_1},$$
$$N_n^{y_2} = h N_n^{x_1}. \tag{47}$$

It is noted that

$$N_n^{x_i} \leq L_{x_i} \leq U_{x_i} \leq M_n^{x_i}, N_n^{y_i} \leq L_{y_i} \leq U_{y_i} \leq M_n^{y_i}, \ (i = 1, 2). \tag{48}$$

Direct calculation, we have $M_n^{x_i}$ and $M_n^{y_i}$ as nonincreasing, and $N_n^{x_i}$ and $N_n^{y_i}$ as nondecreasing. Therefore, the limits of sequences in $M_n^{x_i}$, $M_n^{y_i}$, $N_n^{x_i}$, and $N_n^{y_i}$ exist. Let

$$\text{lim} t \longrightarrow +\infty M_n^{x_i} = \overline{x}_i,$$
$$\text{lim} t \longrightarrow +\infty N_n^{x_i} = \underline{x}_i,$$
$$\text{lim} t \longrightarrow +\infty M_n^{y_i} = \overline{y}_i,$$
$$\text{lim} t \longrightarrow +\infty N_n^{y_i} = \underline{y}_i, \ (i = 1, 2). \tag{49}$$

We have

$$\overline{x}_1 = \frac{r}{r_1 + d_1 + a_1 \underline{x}_1 \underline{y}_2 / (\overline{x}_1)^2 + m(\overline{y}_2)^2} \overline{x}_2,$$

$$\overline{x}_2 = \frac{rr_1 - d_2\left(r_1 + d_1 + a_1 \underline{x}_1 \underline{y}_2 / (\overline{x}_1)^2 + m(\overline{y}_2)^2\right)}{a\left(r_1 + d_1 + a_1 \underline{x}_1 \underline{y}_2 / (\overline{x}_1)^2 + m(\overline{y}_2)^2\right)},$$

$$\overline{y}_1 = \frac{d_4}{r_2} h \overline{x}_1,$$

$$\overline{y}_2 = h \overline{x}_1,$$

$$\underline{x}_1 = \frac{r}{r_1 + d_1 + a_1 \overline{x}_1 \overline{y}_2 / (\underline{x}_1)^2 + m(\underline{y}_2)^2} \underline{x}_2,$$

$$\underline{x}_2 = \frac{rr_1 - d_2\left(r_1 + d_1 + a_1 \overline{x}_1 \overline{y}_2 / (\underline{x}_1)^2 + m(\underline{y}_2)^2\right)}{a\left(r_1 + d_1 + a_1 \overline{x}_1 \overline{y}_2 / (\underline{x}_1)^2 + m(\underline{y}_2)^2\right)}, \tag{50}$$

$$\underline{y}_1 = \frac{d_4}{r_2} h \underline{x}_1,$$

$$\underline{y}_2 = h \underline{x}_1.$$

Now, we prove that $\overline{x}_i = \underline{x}_i, \overline{y}_i = \underline{y}_i$, $(i = 1, 2)$. By (50), we can obtain

$$\begin{aligned} a\left[(r_1 + d_1)(1 + mh^2)(\overline{x}_1)^2 + a_1 h (\underline{x}_1)^2\right] &= r(1 + mh^2)\left[rr_1 - d_2(r_1 + d_1)\right](\overline{x}_1)^3 \\ &\quad - rd_2 a_1 h(1 + mh^2)(\overline{x}_1)(\underline{x}_1)^2, \\ a\left[(r_1 + d_1)(1 + mh^2)(\underline{x}_1)^2 + a_1 h (\overline{x}_1)^2\right] &= r(1 + mh^2)\left[rr_1 - d_2(r_1 + d_1)\right](\underline{x}_1)^3 \\ &\quad - rd_2 a_1 h(1 + mh^2)(\underline{x}_1)(\overline{x}_1)^2. \end{aligned} \tag{51}$$

From above two equations, we have

$$\begin{aligned} &a\left[(r_1 + d_1)^2(1 + mh^2)^2 - (a_1 h)^2\right]\left[(\underline{x}_1)^2 + (\overline{x}_1)^2\right](\overline{x}_1 + \underline{x}_1)(\overline{x}_1 - \underline{x}_1) \\ &= \left[(1 + mh^2)(rr_1 - d_2(r_1 + d_1))\left((\overline{x}_1)^2 + \overline{x}_1 \underline{x}_1 + (\underline{x}_1)^2\right) + rd_2 a_1 h(1 + mh^2)\overline{x}_1 \underline{x}_1\right](\overline{x}_1 - \underline{x}_1). \end{aligned} \tag{52}$$

If $\overline{x}_1 \neq \underline{x}_1$, then we obtain

$$\begin{aligned} &a\left[(r_1 + d_1)^2(1 + mh^2)^2 - (a_1 h)^2\right]\left[(\underline{x}_1)^2 + (\overline{x}_1)^2\right](\overline{x}_1 + \underline{x}_1) \\ &= (1 + mh^2)\left[rr_1 - d_2(r_1 + d_1)\right]\left[(\overline{x}_1)^2 + \overline{x}_1 \underline{x}_1 + (\underline{x}_1)^2\right] + rd_2 a_1 h(1 + mh^2)\overline{x}_1 \underline{x}_1 \end{aligned}. \tag{53}$$

Since $rr_1 > d_2(r_1 + d_1), \overline{x}_1 > 0, \underline{x}_1 > 0$, therefore, $(r_1 + d_1)(1 + mh^2) > a_1 h$. This is a contradiction. So, $\overline{x}_1 = \underline{x}_1$. By (50), we have $\overline{x}_2 = \underline{x}_2 \overline{y}_1 = \underline{y}_1$ and $\overline{y}_2 = \underline{y}_2$. Therefore, the positive equilibrium E^+ is globally stable.

4. Discussion

In this study, we have studied a ratio-dependent predator-prey model with stage structure for the prey and predator. A time delay due to the gestation of the predator is considered. By using the eigenvalue theory, we have obtained the sufficient conditions for the local stability of the nonnegative equilibria of model (2). The existence of Hopf bifurcation is given. By the iteration technique and comparison arguments, sufficient conditions have been established for the global stability of the nonnegative equilibria. From Theorem 2, we know that if (H_3) holds, the predator population will go to extinction. By Theorem 3, we learn that if (H_1) and (H_4) hold, then both the predator and prey species of model (2) are permanent [10, 11].

Data Availability

The [DATA TYPE] data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On Existence of Fixed Points for Multivalued Generalized w_b-Contractive Mappings and Applications

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In this study, we present some new results on the existence of fixed points for multivalued generalized w_b-contractive mappings in the frame work of metric type spaces. Consequently, presented results unify and generalize several known metric fixed-point results. In support of our main results, examples are provided to show that the results are genuine generalization of the known corresponding results of metric fixed-point theory.

1. Introduction

The concept of a metric space plays a vital role in the development of metric fixed-point theory and nonlinear functional analysis and also in various scientific branches. In the literature, this notion of metric space has been extended in several directions by reducing or modifying the metric axioms. Czerwik [1, 2] introduced and studied the concept of b-metric space (metric type space), where the triangle inequality replaced with the weaker condition. In fact, the basic idea of b-metric was given by Bakhtin [3]. It has been observed that the class of metric type spaces is effectively larger than the class of metric spaces [1]. In literature, a number of metric fixed-point results for single-valued and multivalued mappings have been shown; see, for example, [4–11] and references therein.

Using a concept of the Hausdorff–Pompieu metric, Nadler [12] introduced a notion of multivalued contraction mappings and proved splendid result in metric fixed-point theory for such mappings, known as Nadler contraction principle. Due to its importance, matric fixed-point theory of multivalued contractions has been further developed in various directions by a number of authors. A real generalization of the Nadler contraction principle is obtained by Mizoguchi and Takahashi [13]. Without using the idea of the Hausdorff–Pompieu metric, a number of authors obtained interesting fixed-point results and improved various results of metric fixed-point theory including the results of Nadler and Mizoguchi–Takahashi and others. See, for example, [14–17] and references therein.

In [18] Kada et al. introduced a concept of generalized distance, namely, w-distance on metric spaces, and improved some classical results by replacing the involved metric by a generalized distance. Based on this set up, a number of authors studied fixed-point results of mappings with respect to w-distance. Suzuki and Takahashi [19] introduced notions of single-valued and multivalued weakly contractive mappings and studied the existence of fixed points for such mappings. Consequently, they generalized the Banach contraction principle and Nadler contraction principle. For further fixed-point results with applications, see, for example, [20–24] and references therein. In [25], Hussain et al. defined w-distance on metric-type spaces called wt-distance (here, we call it w_b-distance), and they proved fixed-point and common fixed-point results for single-valued mappings with respect to w_b-distance. A number of articles with applications on this topic can also be found in [26–29] and references therein.

2. Preliminaries

Now, we recall some notations, concepts, and facts which are useful for our results.

Let (S, d) be a metric space. Let 2^S denote a collection of nonempty subsets of S, $Cl(S)$ denote a collection of nonempty closed subsets of S, $CB(S)$ denote a collection of nonempty closed bounded subsets of S, and $K(S)$ denote a collection of all nonempty compact subsets of S. An element $u \in S$ is called a *fixed point* of a multivalued mapping $J: S \longrightarrow 2^S$ if $u \in J(u)$. We denote $Fix(J) = \{u \in S: u \in J(u)\}$. A sequence $\{u_n\}$ in S is called an *orbit* of J at $u_0 \in S$ if $u_n \in J(u_{n-1})$, for all $n \geq 1$. A map $f: S \longrightarrow \mathbb{R}$ is called *J-orbitally lower semicontinuous* at $z \in S$ if, for any orbit $\{u_n\}$ of J at $u_0 \in S$ with $u_n \longrightarrow z$ implies $f(z) \leq \liminf_{n\longrightarrow\infty} f(u_n)$. For a constant $c \in (0, 1)$, we say a function $\xi_c: [0, \infty) \longrightarrow [0, c)$ is a strong MT-function if $\limsup_{r\longrightarrow s^+} \xi_c(r) < c$, for all $s \in [0, \infty)$. In case $c = 1$, the function ξ_1 is denoted by ξ, known as MT-function. It has been observed that a function ξ is MT-function if and only if, for any strictly decreasing sequence $\{u_n\}$ in $[0, \infty)$, we have $0 \leq \sup_n \xi(u_n) < 1$. For more characterizations of MT-function, see [30].

Using the concept of Hausdorff–Pompieu metric, Nadler [12] proved a multivalued version of the well-known Banach contraction principle.

Theorem 1 (see [12]). *Let (S, d) be a complete metric space and let $J: S \longrightarrow CB(S)$ be a multivalued contraction mapping (that is, for a fixed constant $h \in (0, 1)$ and for each $u, v \in S$, $H(J(u), J(v)) \leq h\, d(u, v)$, where H is the Hausdorff–Pompieu metric on $CB(S)$). Then, $Fix(J) \neq \varnothing$.*

This result known as Nadler contraction principle, which has been generalized in various directions. The first real generalization of Theorem 1 is obtained by Mizoguchi and Takahashi [13].

Theorem 2 (see [13]). *Let (S, d) be a complete metric space and let $J: S \longrightarrow CB(S)$ be a multivalued mapping. Assume that there exists MT-function ξ such that, for each $u, v \in S$,*

$$H(J(u), J(v)) \leq \xi(d(u, v))d(u, v). \quad (1)$$

Then, $Fix(J) \neq \varnothing$.

On the contrary, without using the concept of the Hausdorff–Pompieu metric, Feng and Liu [16] generalized Theorem 1 as follows.

Theorem 3 (see [16]). *Let (S, d) be a complete metric space and let $J: S \longrightarrow Cl(S)$ be a multivalued mapping. Suppose that a real-valued function g on S, $g(u) = d(u, J(u))$, is lower semicontinuous. Then, $Fix(J) \neq \varnothing$ provided there exist constants $c, h \in (0, 1), h < c$, such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} c d(u, v) &\leq g(u), \\ g(v) &\leq h d(u, v). \end{aligned} \quad (2)$$

Later, Klim and Wardowski [17] generalized Theorem 3 as follows.

Theorem 4 (see [17]). *Let (S, d) be a complete metric space and let $J: S \longrightarrow Cl(S)$ be a multivalued mapping such that a real-valued function g on S, $g(u) = d(u, J(u))$, is lower semicontinuous. Then, $Fix(J) \neq \varnothing$ provided that there exists a strong MT-function ξ_c such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} c d(u, v) &\leq g(u), \\ g(v) &\leq \xi_c(d(u, v))d(u, v). \end{aligned} \quad (3)$$

Using MT-functions, Klim and Wardowski [17] also proved fixed-point result for compact valued mappings of metric spaces as follows.

Theorem 5 (see [17]). *Let (S, d) be a complete metric space and let $J: S \longrightarrow K(S)$ be a multivalued mapping such that a real-valued function g on S, $g(u) = d(u, J(u))$, is lower semicontinuous. Then, $Fix(J) \neq \varnothing$ provided that there exists MT-function ξ such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} d(u, v) &= g(u), \\ g(v) &\leq \xi(d(u, v))d(u, v). \end{aligned} \quad (4)$$

It is worth mentioning that Theorem 4 generalizes Theorem 1 and Theorem 3 but does not generalize Theorem 2 because the strong-MT-function ξ_c in Theorem 4 is stronger than the MT-function ξ used in Theorem 2 as $c < 1$.

However, some more general interesting fixed-point results in this direction obtained by Ćirić [14, 15] unify and generalize the corresponding abovementioned results.

In [18], Kada et al. introduced the concept of w-distance as follows.

Let (S, d) be a metric space. A function $p: S \times S \longrightarrow [0, \infty)$ is called w-distance on S if it satisfies the following, for each $u, v, t \in S$:

(1) $p(u, t) \leq p(u, v) + p(v, t)$

(2) A function $p(u, \cdot): S \longrightarrow [0, \infty)$ is lower semicontinuous

(3) For any $\epsilon > 0$, there exists $\delta > 0$ such that $p(t, u) \leq \delta$ and $p(t, v) \leq \delta$ imply $d(u, v) \leq \epsilon$

Using the concept of w-distances, they improved a number of known important results of metric fixed-point theory. Note that, in general, for $u, v \in S$, $p(u, v) \neq p(v, u)$ and not either of the implications $p(u, v) = 0 \Leftrightarrow u = v$ necessarily hold. Clearly, the metric d is a w-distance on S. Let $(W, \|.\|)$ be a normed space. Then, the functions $p_1, p_2: W \times W \longrightarrow [0, \infty)$ defined by $p_1(u, v) = \|v\|$ and $p_2(u, v) = \|u\| + \|v\|$, for all $u, v \in W$, are w-distances [18]. For more examples and properties of the w-distance, see [18, 19, 24]. Using the concept of w-distance, Suzuki and Takahashi [19] introduced single-valued and multivalued weakly contractive mappings and then improved the Banach contraction principle and Nadler contraction principle. For further general fixed-point results in this direction, see [20, 21, 23, 24] and references therein.

Czerwik [1, 2] introduced a concept of b-metric space as follows.

Let S be a nonempty set. Let $\Delta: S \times S \longrightarrow [0, \infty)$ be a function which satisfies the following, for all $u, v, t \in S$:

(1) $\Delta(u, v) = 0$ if and only if $u = v$

(2) $\Delta(u, v) = \Delta(v, u)$

(3) $\Delta(u, v) \leq b[\Delta(u, t) + \Delta(t, v)]$, for some $b \geq 1$

Then, Δ is called a b-metric on S and (S, Δ) is known as b-metric space (also known a metric-type space [8, 25]). In the sequel, we also call it metric-type space. Obviously, for $b = 1$, we obtain a standard metric on S. In fact, the class of metric-type spaces is effectively larger than the class of metric spaces. Indeed, let $S = \mathbb{R}$ be endowed with a mapping $\Delta: S \times S \longrightarrow \mathbb{R}^+$ defined by $\Delta(u, v) = (u - v)^2$, for each $u, v \in S$. Then, (S, Δ) is a metric-type space with $b = 2$, but it is not a metric space [3]. For more examples of metric-type spaces, see [1, 31]. It is worth to point out that, unlike the case of standard metric, the b-metric Δ is not necessarily continuous due to the modified triangle inequality. In general, Δ is not continuous in each variable [5]. However, a metric-type space can be endowed with a topology induced by its convergence [5] and almost all the concepts and results which are valid for metric spaces can be extended to the framework of metric type spaces. In fact, for metric-type spaces, the notions of convergence, Cauchy sequence, and completeness and continuity can be defined similarly as in metric spaces, see [4, 8, 25]. Let us recall few such useful notions and facts in the framework of metric-type spaces.

Let (S, Δ) be a metric-type space and let $\{u_n\}$ be a sequence in S. Then,

(1) $\{u_n\}$ converges in S if there exists $u \in S$ such that $\lim_{n \longrightarrow \infty} \Delta(u_n, u) = 0$

(2) $\{u_n\}$ is a Cauchy sequence in S if $\lim_{n,m \longrightarrow \infty} \Delta(u_n, u_m) = 0$

(3) (S, Δ) is complete if every Cauchy sequence in S is convergent in S

(4) A real-valued function f on S is b-lower semicontinuous at a point $u \in S$ if $f(u) \leq \liminf_{n \longrightarrow \infty} b f(u_n)$ whenever $u_n \longrightarrow u$

Recently, fixed-point theory for metric-type spaces studied and developed by many authors, for example, see [5, 7, 28, 32] and references therein.

Motivated by the work of Kada et al. [18] and Hussain et al. [25] introduced wt-distance (here, we call it w_b-distance) in the setting of metric-type space as follows.

Let (S, Δ) be a metric-type space with constant $b \geq 1$. Then, a function $p_b: S \times S \longrightarrow [0, \infty)$ is called a w_b-distance on S if, for any $u, v, t \in S$, the following hold:

(1) $p_b(u, t) \leq b[p_b(u, v) + p_b(v, t)]$

(2) $p_b(u, \cdot): S \longrightarrow [0, \infty)$ is b-lower semicontinuous (i.e., if, for any sequence $\{v_n\}$ in S, $v_n \longrightarrow v \in S$, then $p_b(u, v) \leq \liminf_{n \longrightarrow \infty} b p_b(u, v_n)$)

(3) For any $\epsilon > 0$, there exists $\delta > 0$ such that $p_b(t, u) \leq \delta$ and $p_b(t, v) \leq \delta$ imply $\Delta(u, v) \leq \epsilon$

Remark 1. Note that, for $b = 1$, each w_b-distance is a w-distance. In general, w_b-distance is not symmetric, see [25]. In fact, the class of w_b-distances is much larger than the class of w-distance, see [28]. Every b-metric is w_b-distance [25], but the converse is not true, see [28].

Example 1 (see [25]). Let $S = \mathbb{R}$ and $\Delta(u, v) = (u - v)^2$; then,

(1) The function $p_b: S \times S \longrightarrow [0, \infty)$ defined by $p_b(u, v) = |u|^2 + |v|^2$ for every $u, v \in S$ is a w_b-distance on S

(2) The function $p_b: S \times S \longrightarrow [0, \infty)$ defined by $p_b(u, v) = |v|^2$ for every $u, v \in S$ is a w_b-distance on S

The following result is an analogue of Lemma 1 of [18], stated and used in [25, 26].

Lemma 1 (see [25]). *Let (S, Δ) be a metric-type space with constant $b \geq 1$ and let p_b be a w_b-distance on S. Let $\{u_n\}$ and $\{v_n\}$ be sequences in S; let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, \infty)$ converging to zero. Then, the following hold, for each $u, v, t \in S$.*

(i) If $p_b(u_n, v) \leq \alpha_n$ and $p_b(u_n, t) \leq \beta_n$, for any $n \in \mathbb{N}$, then $v = t$. In particular, if $p_b(u, v) = 0$ and $p_b(u, t) = 0$, then $v = t$.

(ii) If $p_b(u_n, v_n) \leq \alpha_n$ and $p_b(u_n, t) \leq \beta_n$, for any $n \in \mathbb{N}$, then $\Delta(v_n, t) \longrightarrow 0$.

(iii) If $p_b(u_n, u_m) \leq \alpha_n$, for any $n, m \in \mathbb{N}$ with $m > n$, then $\{u_n\}$ is a Cauchy sequence.

(iv) If $p_b(v, u_n) \leq \alpha_n$, for any $n \in \mathbb{N}$, then $\{u_n\}$ is a Cauchy sequence.

Lemma 2 (see [33]). *Let A be a closed subset of a metric-type space (S, Δ) and p_b be a w_b-distance on S. Suppose that there exists $z \in S$ such that $p_b(z, z) = 0$. Then, $p_b(z, A) = 0$ $\Leftrightarrow z \in A$, where $p_b(z, A) = \inf\{p_b(z, w): w \in A\}$.*

Recently, some interesting results appeared in metric fixed-point theory with respect to w_b-distance on metric-type spaces; for example, see [25–28] and references therein.

Let $A \in (0, +\infty]$. Consider a real-valued function ψ on $[0, A)$ satisfying the following conditions:

(1) $\psi(0) = 0$ and $\psi(r) > 0$, for each $r \in (0, A)$.

(2) ψ is nondecreasing on $[0, A)$.

(3) ψ is subadditive, that is,

$$\psi(r_1 + r_2) \leq \psi(r_1) + \psi(r_2), \quad \text{for all } r_1, r_2 \in (0, A). \tag{5}$$

Examples and properties of such functions can be found in [34].

We define

$$\Omega[0, A) = \{\psi: \psi \text{ satisfies } (1) - (3)\}. \tag{6}$$

The real-valued function ψ plays an important role in metric fixed-point theory; for example, see [22, 34, 35] and references therein. Among others, Latif and Abdou [22] proved some interesting fixed-point results for multivalued mapping with respect to w-distance. For example, the following results unify and extend a number of known metric fixed-point results.

Theorem 6 (see [22]). *Let (S, d) be a complete metric space with a w-distance p. Let $J: S \longrightarrow Cl(S)$ be a multivalued mapping such that a real-valued function g on S, $g(u) = p(u, J(u))$, is lower semicontinuous. Assume that there exist $c \in (0,1)$ and $\eta \in \Omega[0, A)$ such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} &c\eta(p(u,v)) \leq \eta(g(u)),\\ &\eta(g(v)) \leq \xi_c(p(u,v))\eta(p(u,v)), \end{aligned} \tag{7}$$

where ξ_c is a strong MT-function. Then, there exists $w_0 \in S$ such that $g(w_0) = 0$. Moreover, if $p(w_0, w_0) = 0$, then $w_0 \in J(w_0)$.

Theorem 7 (see [22]). *Let (S, d) be a complete metric space with a w-distance p. Let $J: S \longrightarrow Cl(S)$ be a multivalued mapping such that a real-valued function g on S, $g(u) = p(u, J(u))$, is lower semicontinuous. Assume that there exists $\eta \in \Omega[0, A)$ such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} &\eta(p(u,v)) = \eta(g(u)),\\ &\eta(g(v)) \leq \xi(p(u,v))\eta(p(u,v)), \end{aligned} \tag{8}$$

where ξ is MT-function. Then, there exists $w_0 \in S$ such that $g(w_0) = 0$. Moreover, if $p(w_0, w_0) = 0$, then $w_0 \in J(w_0)$.

The aim of this paper is to present some more general results on the existence of fixed points related to multivalued generalized w_b-contractive mappings defined on metric-type spaces. In particular, such mappings involve the function $\psi \circ p_b$, where $\psi \in \Omega[0, A)$ and the function p_b is a w_b-distance on a metric-type space. Consequently, our results unify and generalize the corresponding several known metric fixed-point results.

3. Results

Now, we present our first main result on the existence of fixed points for multivalued mapping with respect to w_b-distance, which improve and generalize a number of known fixed-point results including Theorem 6.

Throughout this section, (S, Δ) is a complete metric-type space and p_b is a w_b-distance on S.

Theorem 8. *Let $J: S \longrightarrow Cl(S)$ be a multivalued mapping. Assume that there exist a strong MT-function ξ_c and a function $\psi \in \Omega[0, A)$ such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned} &c\psi(p_b(u,v)) \leq \psi(q(u)),\\ &\psi(q(v)) \leq \xi_c(p_b(u,v))\psi(p_b(u,v)), \end{aligned} \tag{9}$$

where q is a real-valued function on S defined by $q(u) = p_b(u, J(u))$. Then,

(1) For any $u_0 \in S$, there exist an orbit $\{u_n\}$ of J at u_0 and $z_0 \in S$ such that $\lim_{n\longrightarrow\infty} u_n = z_0$.

(2) $p_b(z_0, J(z_0)) = 0$ if and only if the function q is J-orbitally b-lower semicontinuous at z_0. Moreover, if $p_b(z_0, z_0) = 0$, then $z_0 \in Fix(J)$.

Proof. Let u_0 be an arbitrary but fixed element of S. Then, there exists $u_1 \in J(u_0)$ such that

$$c\psi(p_b(u_0,u_1)) \leq \psi(q(u_0)), \tag{10}$$

$$\psi(q(u_1)) \leq \xi_c(p_b(u_0,u_1))\psi(p_b(u_0,u_1)), \tag{11}$$

$$\xi_c(p_b(u_0,u_1)) < c. \tag{12}$$

Thus, we have

$$\psi(q(u_0)) - \psi(q(u_1)) \geq [c - \xi_c(p_b(u_0,u_1))]\psi(p_b(u_0,u_1)) > 0. \tag{13}$$

Similarly, for $u_1 \in S$, there exists $u_2 \in J(u_1)$ such that

$$\begin{aligned} &c\psi(p_b(u_1,u_2)) \leq \psi(q(u_1)),\\ &\psi(q(u_2)) \leq \xi_c(p_b(u_1,u_2))\psi(p_b(u_1,u_2)),\\ &\xi_c(p_b(u_1,u_2)) < c, \end{aligned} \tag{14}$$

$$\begin{aligned} \psi(q(u_1)) - \psi(q(u_2)) \geq [c - \xi_c(p_b(u_1,u_2))]\\ \psi(p_b(u_1,u_2)) > 0. \end{aligned} \tag{15}$$

Continuing this process, we obtain an orbit $\{u_n\}$ of J at $u_0 \in S$ such that $u_{n+1} \in J(u_n)$ satisfying

$$c\psi(p_b(u_n,u_{n+1})) \leq \psi(q(u_n)), \tag{16}$$

$$\psi(q(u_{n+1})) \leq \xi_c(p_b(u_n,u_{n+1}))\psi(p_b(u_n,u_{n+1})), \tag{17}$$

$$\xi_c(p_b(u_n,u_{n+1})) < c, \tag{18}$$

$$\begin{aligned} \psi(q(u_n)) - \psi(q(u_{n+1})) \geq [c - \xi_c(p_b(u_n,u_{n+1}))]\\ \cdot \psi(p_b(u_n,u_{n+1})) > 0, \end{aligned} \tag{19}$$

which imply that

$$\psi(q(u_{n+1})) < \psi(q(u_n)), \quad n \in \mathbb{N} \cup \{0\}. \tag{20}$$

While from (11), (12), and (14), it follows that

$$\begin{aligned} \psi(p_b(u_1,u_2)) \leq \frac{1}{c}\psi(q(u_1)) \leq \frac{1}{c}\xi_c(p_b(u_0,u_1))\psi(p_b(u_0,u_1))\\ < \psi(p_b(u_0,u_1)). \end{aligned} \tag{21}$$

Thus, for each $n \in \mathbb{N}$, we obtain

$$\psi(p_b(u_n, u_{n+1})) < \psi(p_b(u_{n-1}, u_n)). \tag{22}$$

By (20) and (22), we note that the sequences $\{\psi(q(u_n))\}$ and $\{\psi(p_b(u_n, u_{n+1}))\}$ are decreasing. Now, since ψ is nondecreasing, it follows that $\{q(u_n)\}$ and $\{p_b(u_n, u_{n+1})\}$ are decreasing sequences and are bounded from below, thus convergent. Now, by the definition of the function ξ_c, there exists $\delta \in [0, c)$ such that

$$\limsup_{n\longrightarrow\infty} \xi_c(p_b(u_n, u_{n+1})) = \delta. \tag{23}$$

Thus, for any $c_0 \in (\delta, c)$ with $c_0 c^{-1} \in (0, b^{-1})$, there exists $n_0 \in \mathbb{N}$ such that

$$\xi_c(p_b(u_n, u_{n+1})) < c_0, \quad \text{for all } n > n_0, \tag{24}$$

and thus, for all $n > n_0$, we have

$$\xi_c(p_b(u_n, u_{n+1})) \times \cdots \times \xi_c\left(p_b\left(u_{n_0+1}, u_{n_0+2}\right)\right) < c_0^{n-n_0}. \tag{25}$$

Note that, for all $n > n_0$, we have

$$\begin{aligned}
\psi(q(u_{n+1})) &\le \xi_c(p_b(u_n, u_{n+1}))\psi(p_b(u_n, u_{n+1})) \\
&\le \frac{1}{c}\xi_c(p_b(u_n, u_{n+1}))\psi(q(u_n)) \\
&\le \frac{1}{c^2}\xi_c(p_b(u_n, u_{n+1}))\xi_c(p_b(u_{n-1}, u_n))\psi(q(u_{n-1})) \\
&\vdots \\
&\le \frac{1}{c^n}\left[\xi_c(p_b(u_n, u_{n+1})) \times \cdots \times \xi_c(p_b(u_1, u_2))\right]\psi(q(u_1)) \\
&= \frac{\xi_c(p_b(u_n, u_{n+1})) \times \cdots \times \xi_c\left(p_b\left(u_{n_0+1}, u_{n_0+2}\right)\right)}{c^{n-n_0}} \times \frac{\xi_c\left(p_b\left(u_{n_0}, u_{n_0+1}\right)\right) \times \cdots \times \xi_c(p_b(u_1, u_2))\psi(q(u_1))}{c^{n_0}},
\end{aligned} \tag{26}$$

and thus,

$$\psi(q(u_{n+1})) < \left(\frac{c_0}{c}\right)^{n-n_0} \frac{\xi_c\left(p_b\left(u_{n_0}, u_{n_0+1}\right)\right) \times \cdots \times \xi_c(p_b(u_1, u_2))\psi(q(u_1))}{c^{n_0}}. \tag{27}$$

Put $\lambda = c_0 c^{-1}$ and since $\lambda < 1$, we have $\lim_{n\longrightarrow\infty}\lambda^{n-n_0} = 0$, and hence, the decreasing sequence $\{\psi(q(u_n))\}$ converges to 0. Thus, we have

$$\lim_{n\longrightarrow\infty} q(u_n) = 0. \tag{28}$$

Now, we show that $\{u_n\}$ is a Cauchy sequence. From (16), (17), and (24), we note that, for all $n > n_0$,

$$\begin{aligned}
\psi(p_b(u_n, u_{n+1})) &\le \frac{1}{c}\psi(q(u_n)) \\
&\le \frac{1}{c}\xi_c(p_b(u_{n-1}, u_n))\psi(p_b(u_{n-1}, u_n)) \\
&< \lambda\psi(p_b(u_{n-1}, u_n)) \\
&< \lambda^2\psi(p_b(u_{n-2}, u_{n-1})) \\
&\vdots \\
&< \lambda^n\psi(p_b(u_0, u_1)).
\end{aligned} \tag{29}$$

Thus, we have

$$\psi(p_b(u_n, u_{n+1})) \le \lambda^n\psi(p_b(u_0, u_1)), \quad n \in \mathbb{N} \cup \{0\}. \tag{30}$$

Now, for any $n, m \in \mathbb{N}, m > n$,

$$\begin{aligned}
\psi(p_b(u_n, u_m)) &\le b(\psi(p_b(u_n, u_{n+1})) + \psi(p_b(u_{n+1}, u_m))) \\
&\le b\psi(p_b(u_n, u_{n+1})) + b(b(\psi(p_b(u_{n+1}, u_{n+2})) + \psi(p_b(u_{n+2}, u_m)))) \\
&= b\psi(p_b(u_n, u_{n+1})) + b^2\psi(p_b(u_{n+1}, u_{n+2})) + b^2\psi(p_b(u_{n+2}, u_m)) \\
&\le b\psi(p_b(u_n, u_{n+1})) + b^2\psi(p_b(u_{n+1}, u_{n+2})) + b^2(b(\psi(p_b(u_{n+2}, u_{n+3})) + \psi(p_b(u_{n+3}, u_m)))) \\
&= b\psi(p_b(u_n, u_{n+1})) + b^2\psi(p_b(u_{n+1}, u_{n+2})) + b^3(\psi(p_b(u_{n+2}, u_{n+3})) + \psi(p_b(u_{n+3}, u_m))) \\
&\vdots \\
&\le b\psi(p_b(u_n, u_{n+1})) + b^2\psi(p_b(u_{n+1}, u_{n+2})) + \ldots + b^{m-n-1}(\psi(p_b(u_{m-2}, u_{m-1})) + \psi(p_b(u_{m-1}, u_m))) \\
&\le b\lambda^n\psi(p_b(u_0, u_1)) + b^2\lambda^{n+1}\psi(p_b(u_0, u_1)) + \ldots + b^{m-n-1}\lambda^{m-2}\psi(p_b(u_0, u_1)) + b^{m-n-1}\lambda^{m-1}\psi(p_b(u_0, u_1)) \\
&= b\lambda^n\left(1 + b\lambda + (b\lambda)^2 + \cdots + (b\lambda)^{m-n-2} + b^{m-n-2}\lambda^{m-n-1}\right)\psi(p_b(u_0, u_1)).
\end{aligned} \tag{31}$$

Since $\lambda < b^{-1}$, thus, for $m, n \in \mathbb{N}$ with $m > n > n_0$, we obtain

$$\psi(p_b(u_n, u_m)) \le \frac{b\lambda^n}{1 - b\lambda}\psi(p_b(u_0, u_1)). \tag{32}$$

Thus, since $(b\lambda^n/(1 - b\lambda)) \longrightarrow 0$ as $n \longrightarrow +\infty$, we have $\lim_{n,m\longrightarrow+\infty}\psi(p_b(u_n, u_m)) = 0$, and thus,

$$\lim_{n,m\longrightarrow+\infty} p_b(u_n, u_m) = 0. \tag{33}$$

By Lemma 1 (iii), $\{u_n\}$ is Cauchy sequence in S. Since S is complete, $\{u_n\}$ converges to some point $z_0 \in S$. Note that the sequence $\{u_n\}$ is an orbit of J at $u_0 \in S$ with $u_n \longrightarrow z_0$. Now, suppose that the function q is J-orbitally b-lower semicontinuous at z_0; then, using (28), we have

$$0 \le q(z_0) \le \liminf_{n\longrightarrow\infty} bq(u_n) = 0, \tag{34}$$

and hence, $q(z_0) = p_b(z_0, J(z_0)) = 0$. Conversely, if $p_b(z_0, J(z_0)) = q(z_0) = 0$, then, clearly, the function q is J-orbitally b-lower semicontinuity at z_0 because $q(z_0) = 0 \le \liminf_{n\longrightarrow\infty} bq(u_n)$. Furthermore, if $p_b(z_0, z_0) = 0$, then, since $J(z_0)$ is closed, it follows from Lemma 2 that $z_0 \in J(z_0)$.

If we consider in Theorem 8, a constant mapping $\xi_c(s) = \tau$ and $s \in (0, \infty)$, where $\tau \in (0, c)$; then, we have the following result. □

Corollary 1. *Let $J: S \longrightarrow Cl(S)$ be a multivalued mapping satisfying that, for any constants $c \in (0, 1)$ and for each $u \in S$, there is $v \in L_c^u$ such that*

$$\psi(q(v)) \le \tau\psi(p_b(u, v)), \tag{35}$$

where $L_c^u = \{v \in J(u): c\psi(p_b(u, v)) \le \psi(q(u))\}$ and a real-valued function q on S defined by $q(u) = p_b(u, J(u))$ is b-lower semicontinuous. Then, there exists $z_0 \in S$ such that $q(z_0) = 0$. Furthermore, if $p_b(z_0, z_0) = 0$, then $z_0 \in Fix(J)$.

Remark 2.

(1) Theorem 8 generalizes the fixed-point result (Theorem 2.2 of [36]). Indeed, if we take $\psi(s) = s$, for all $s \in [0, A)$ and $b = 1$ (i.e., w_b-distance as a w-distance) in Theorem 8, then we get Theorem 2.2 of [36]. Consequently, Theorem 8 also extends Theorem 4, which contains Theorem 3.

(2) Theorem 8 generalizes the fixed-point result (Theorem 2.1 of [22]) for w_b-distance in the frame work of metric-type spaces.

(3) Theorem 8 contains Theorem 2.1 of [35] as a special case.

(4) Corollary 1 contains the fixed-point results (Corollary 2.2 of [22] and Theorem 3.3 of [37]).

Now, without using b-lower semicontinuity of the function q, we present a fixed-point result for multivalued mappings which extends the fixed-point result of Theorem 2.4 of [36] and reduces to Theorem 2.4 of [22].

Theorem 9. *Suppose that all the hypotheses of Theorem 8 (except the b-lower semicontinuity of the function q) hold. Assume that*

$$\inf\{\psi(p_b(u, z)) + \psi(q(u)): u \in S\} > 0, \tag{36}$$

for every $z \in S$ with $z \notin J(z)$. Then, $Fix(J) \neq \emptyset$.

Proof. Following the proof of Theorem 8, there exists an orbit $\{u_n\}$ of J at $u_0 \in S$, which turns as a Cauchy sequence in a complete space S. Then, there exists $z_0 \in S$ such that the sequence $\{u_n\}$ converges to z_0. Thus, by the b-lower semicontinuity of the function $p_b(u_n, \cdot)$, and from the proof of Theorem 8, it follows that, for all $n > n_0$,

$$\psi(p_b(u_n, z_0)) \le \liminf_{m\longrightarrow\infty} b\psi(p_b(u_n, u_m)) \le \frac{b^2\lambda^n}{1 - b\lambda}\psi(p_b(u_0, u_1)), \tag{37}$$

where $\lambda = c_0/c < 1$. Also, note that

$$q(u_n) = p_b(u_n, J(u_n)) \leq p_b(u_n, u_{n+1}), \tag{38}$$

for all n, and since the function ψ is nondecreasing, we have

$$\psi(q(u_n)) \leq \psi(p_b(u_n, u_{n+1})), \tag{39}$$

and thus,

$$\psi(q(u_n)) < \lambda^n \psi(p_b(u_0, u_1)). \tag{40}$$

Assume that $z_0 \notin J(z_0)$. Then, we have

$$\begin{aligned}
0 &< \inf\{\psi(p_b(u, z_0)) + \psi(q(u)) : u \in S\} \\
&\leq \inf\{\psi(p_b(u_n, z_0)) + \psi(q(u_n)) : n > n_0\} \\
&< \inf\left\{\frac{b^2\lambda^n}{1 - b\lambda}\psi(p_b(u_0, u_1)) + \lambda^n \psi(p_b(u_0, u_1)) : n > n_0\right\} \\
&= \left\{\frac{b^2 - b\lambda + 1}{1 - b\lambda}\psi(p_b(u_0, u_1))\right\}\inf\{\lambda^n : n > n_0\} = 0,
\end{aligned} \tag{41}$$

which is impossible, and hence, $z_0 \in Fix(J)$.

Using MT-functions (instead of strong MT-functions), we present a fixed-point result for multivalued w_b-contraction mappings which extends Theorem 2.5 of [22] and thus contains a number of known metric fixed-point results. □

Theorem 10. *Let $J: S \longrightarrow Cl(S)$ be a multivalued mapping. Assume that there exist an MT-function ξ and a function $\psi \in \Omega[0, A)$ such that, for any $u \in S$, there is $v \in J(u)$ satisfying*

$$\begin{aligned}
&\psi(p_b(u, v)) = \psi(q(u)), \\
&\psi(q(v)) \leq \xi(p_b(u, v))\psi(p_b(u, v)),
\end{aligned} \tag{42}$$

where q is a real-valued function on S defined by $q(u) = p_b(u, J(u))$. Then,

(1) For any $u_0 \in S$, there exist an orbit $\{u_n\}$ of J at u_0 and $z_0 \in S$ such that $lim_{n\longrightarrow\infty} u_n = z_0$.

(2) $p_b(z_0, J(z_0)) = 0$ if and only if the function q is J-orbitally b-lower semicontinuous at z_0. Moreover, if $p_b(z_0, z_0) = 0$, then $z_0 \in Fix(J)$.

Proof. Let u_0 be an arbitrary but fixed element of S. Then, there is $u_1 \in J(u_0)$ such that

$$\begin{aligned}
&\psi(p_b(u_0, u_1)) = \psi(q(u_0)), \\
&\psi(q(u_1)) \leq \xi(p_b(u_0, u_1))\psi(p_b(u_0, u_1)), \\
&\xi(p_b(u_0, u_1)) < 1.
\end{aligned} \tag{43}$$

Following the proof of Theorem 8, there exists a Cauchy sequence $\{u_n\}$ in the complete space S such that $u_{n+1} \in J(u_n)$ (that is, $\{u_n\}$ is an orbit of J at u_0) satisfying

$$\begin{aligned}
&\psi(p_b(u_n, u_{n+1})) = \psi(q(u_n)), \\
&\psi(q(u_{n+1})) \leq \xi(p_b(u_n, u_{n+1}))\psi(p_b(u_n, u_{n+1})), \\
&\xi(p_b(u_n, u_{n+1})) < 1.
\end{aligned} \tag{44}$$

Consequently, there exists $z_0 \in S$ such that the sequence $\{u_n\}$ converges to z_0. Now, the rest of the proof follows as of Theorem 8. □

Remark 3.

(1) For $b = 1$, Theorem 10 reduces to Theorem 2.5 of [22].

(2) If we take $b = 1$ and $\psi(s) = s$, for each $s \in [0, A)$ in Theorem 10, the we obtain Theorem 2.5 of [38].

(3) It turns out that Theorem 10 also generalizes Theorem 7 of [14] and Theorem 2.4 of [35].

Using the same technique as in the proof of Theorem 9, we get the following fixed-point result (in the absence of the b-lower semicontinuity of the function q), which contains fixed-point result (Theorem 2.7 of [22]) as a special case.

Theorem 11. *Suppose that all the hypotheses of Theorem 10 except the b-lower semicontinuity of the function q hold. Assume that*

$$\inf\{\psi(p_b(u, z)) + \psi(q(u)) : u \in S\} > 0, \tag{45}$$

for every $z \in S$ with $z \notin J(z)$. Then, $Fix(J) \neq \varnothing$.

4. Example

Now, we present examples which show that our main results, namely, Theorems 8 and 10 are genuine generalizations of Theorem 2.1 of [22] and Theorem 2.5 of [22], respectively.

Example 2. Let $S = [0, 1]$. Define $\Delta(u, v) = (u - v)^2$, for all $u, v \in S$. Then, S is a metric-type space with $b = 2$. Define a w_b-distance function on S by $p_b(u, v) = v^2$, for all $u, v \in S$. Let $J: S \longrightarrow Cl(S)$ be defined by

$$J(u) = \begin{cases} \left\{\frac{1}{2}u^2\right\}; & u \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ \left\{0, \frac{17}{96}, \frac{1}{4}\right\}; & u = \frac{15}{32}. \end{cases} \tag{46}$$

Let $A \in [1, \infty)$ and let $c = 1/2$. Define a function $\psi: [0, A) \longrightarrow \mathbb{R}$ by $\psi(s) = s^{1/2}$. Clearly, $\psi \in \Omega[0, A)$. Define $\xi_c: [0, \infty) \longrightarrow [0, c)$ as follows:

$$\xi_c(s) = \begin{cases} \frac{3}{4}s^{1/2}; & s \in \left[0, \frac{1}{2}\right), \\ \frac{3}{8}; & s \in \left[\frac{1}{2}, \infty\right). \end{cases} \tag{47}$$

Clearly, ξ_c is a strong MT-function. Also, note that

$$q(u) = p_b(u, J(u)) = \begin{cases} \frac{1}{4}u^4; & u \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 1\right], \\ 0; & u = \frac{15}{32}. \end{cases} \quad (48)$$

Now, for each $u \in [0, 15/32)t \cup n(q15/32, 1]$, we have $J(u) = \{(1/2)u^2\}$. Take $v = (1/2)u^2 \in J(u)$; then, we have

$$p_b(u, v) = q(u) = \frac{1}{4}u^4. \quad (49)$$

Thus, for $u \in [0, 1], u \neq 15/32$, we have

$$c\psi(p_b(u, v)) \leq \psi(q(u)), \quad (50)$$

$$\psi(q(v)) = \psi\left(p_b\left(\frac{1}{2}u^2, \frac{1}{2}\left(\frac{1}{2}u^2\right)^2\right)\right) = \psi\left(\frac{1}{64}u^8\right) = \frac{1}{8}u^4 \leq \frac{3}{16}u^4 = \xi_c(p_b(u, v))\psi(p_b(u, v)). \quad (51)$$

Now, let $u = 15/32$; then, we have $J(u) = \{0, 17/96, 1/4\}$. Clearly, there exists $v = 0 \in J(u)$ such that

$$c\psi(p_b(u, v)) = 0 = \psi(q(u)), \quad (52)$$

$$\psi(q(v)) = \psi(p_b(0, 0)) = \xi_c(p_b(u, v))\psi(p_b(u, v)). \quad (53)$$

Note that, for each $u \in [0, 1]$, all the conditions of Theorem 8 are satisfied, and hence, it follows that $Fix(J) \neq \emptyset$. Note that $Fix(J) = \{0\}$.

Clearly, p_b is not a metric d, even not a w-distance p on S, and thus, J does not satisfy the hypotheses of Theorem 2.1 of [22]. Note that the mapping J also does not satisfy the hypotheses of Theorem 2.5 of [22].

Example 3. Let $S = \{0\} \cup \{(1/n): n \in \mathbb{N}\}$. Denote $\Lambda = \{0\} \cup \{(1/2n): n \in \mathbb{N}\}$. Clearly, $\Lambda \subseteq S$. Let $\Delta: S \times S \longrightarrow [0, \infty)$ be defined by

$$\Delta(u, v) = \begin{cases} 0; & \text{if } u = v, \\ 2; & \text{if } u \neq v \in \{0, 1\}, \\ |u - v|; & \text{if } u \neq v \in \Lambda, \\ 4; & \text{otherwise}. \end{cases} \quad (54)$$

Then, S is a metric-type space with $b = 8/3$ (see [28]). Define a w_b-distance $p_b: S \times S \longrightarrow [0, \infty)$ by

$$p_b(u, v) = \begin{cases} 0; & \text{if } u = v, \\ 2; & \text{if } u \neq v \in \{0, 1\}, \\ \max\{3(u - v), 2(v - u)\}; & \text{if } u \neq v \in \Lambda, \\ 4; & \text{otherwise}. \end{cases} \quad (55)$$

Let $J: S \longrightarrow Cl(S)$ be defined by

$$J(u) = \begin{cases} \left\{\frac{1}{11}u\right\}; & \text{if } u \in \Lambda, \\ \left\{0, \frac{1}{3}\right\}; & \text{otherwise}. \end{cases} \quad (56)$$

Let $A \in [1, \infty)$. Define a function $\psi: [0, A) \longrightarrow \mathbb{R}$ by $\psi(s) = s^{1/2}$. Clearly, $\psi \in \Omega[0, A)$. Define $\xi: [0, \infty) \longrightarrow [0, 1)$ as follows:

$$\xi(s) = \begin{cases} \frac{1}{4}s; & \text{if } s \in \Lambda, \\ \frac{1}{2}; & \text{otherwise}. \end{cases} \quad (57)$$

Clearly, ξ is MT-function. We need to examine the following cases:

Case I: suppose $u \in \Lambda \setminus \{0\}$; we have $J(u) = \{(1/11)u\}$ and so

$$q(u) = p_b(u, J(u)) = \max\left\{3\left(u - \frac{1}{11}u\right), 2\left(\frac{1}{11}u - u\right)\right\} = \frac{30}{11}u. \quad (58)$$

Take $v = (1/11)u \in J(u)$; then, clearly, $v \in \Lambda$, and we have

$$p_b(u, v) = q(u) = \frac{30}{11}u. \quad (59)$$

Thus, for $u \in \Lambda \setminus \{0\}$, we have

$$\psi(p_b(u, v)) = \psi(q(u)), \quad (60)$$

$$\psi(q(v)) = \psi\left(p_b\left(\frac{1}{11}u, \frac{1}{(11)^2}u\right)\right) = \psi\left(\frac{30}{(11)^2}u\right) = \frac{\sqrt{30}}{11}u^{1/2} \leq \frac{1}{2}\sqrt{\frac{30}{11}}u^{1/2} = \xi(p_b(u, v))\psi(p_b(u, v)). \quad (61)$$

Case II: suppose $u = 0$; then, we have $J(u) = \{(1/11)u\} = \{0\}$. Clearly, there exists $v = 0 \in J(u)$ such that

$$\psi(p_b(u, v)) = 0 = \psi(q(u)), \quad (62)$$

$$\psi(q(v)) = \psi(p_b(0, 0)) = 0 = \xi(p_b(u, v))\psi(p_b(u, v)). \quad (63)$$

Case III: suppose $u = 1$; then, we have $J(u) = \{0, 1/3\}$. Clearly, there exists $v = 0 \in J(u)$ such that

$$\psi(p_b(u,v)) = \sqrt{2} = \psi(q(u)), \tag{64}$$

$$\begin{aligned}\psi(q(v)) &= \psi(p_b(0,0)) = 0 \leq \frac{1}{\sqrt{2}} \\ &= \xi(p_b(u,v))\psi(p_b(u,v)).\end{aligned} \tag{65}$$

Case IV: suppose $u = (1/3)$; then, we have $J(u) = \{0, 1/3\}$. Clearly, there exists $v = (1/3) \in J(u)$ such that

$$\psi(p_b(u,v)) = 0 = \psi(q(u)), \tag{66}$$

$$\psi(q(v)) = \psi\left(p_b\left(\frac{1}{3},\frac{1}{3}\right)\right) = 0 = \xi(p_b(u,v))\psi(p_b(u,v)). \tag{67}$$

Case V: suppose $u \in S \setminus (\Lambda \cup \{1, (1/3)\})$; then, we have $J(u) = \{0, (1/3)\}$. Clearly, there exists $v = (1/3) \in J(u)$ such that

$$\psi(p_b(u,v)) = 2 = \psi(q(u)), \tag{68}$$

$$\psi(q(v)) = \psi\left(p_b\left(\frac{1}{3},\frac{1}{3}\right)\right) = 0 \leq 1 = \xi(p_b(u,v))\psi(p_b(u,v)). \tag{69}$$

Note that, for each $u \in S$, all the conditions of Theorem 10 are satisfied, and hence, it follows that $Fix(J) \neq \emptyset$. Note that $Fix(J) = \{0\}$.

Not that the w_b-distance p_b is not a metric d, even not a w-distance p on S, and thus, J do not satisfy the hypotheses of Theorem 2.5 of [22].

5. Conclusion

Among others, Feng-Liu [16], Klim and Wardowski [17], and Ćirić [14] studied the existence of fixed points for multivalued contractive-type mappings without using the Hausdorff–Pompieu metric, and consequently, they generalized some classical known fixed-point results including Theorems 1 and 2. In this study, we established some general fixed-point results for multivalued generalized contractive mappings on metric-type spaces (instead of normal metric spaces) with respect to w_b-distances. Presented results generalize and improve a number of known fixed-point results, including the corresponding fixed-point results which are stated in Section 2. In support of our main fixed-point theorems, examples are also provided. Note that the family of metric-type spaces is effectively larger than one of metric spaces, and hence, our theorems are more general, different from the classical results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

PID Controller Parameter Optimized by Reformative Artificial Bee Colony Algorithm

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The PID parameters determine the PID controller performance. A reformative artificial bee colony (RABC) algorithm is proposed for the PID parameter optimization problem. The algorithm balances the exploitation capability and exploration capability of the ABC algorithm by introducing a global optimal solution and improving the food source probability. The proposed algorithm is validated by simulation with six benchmark functions, and the results show that the RABC algorithm has higher search accuracy and faster search speed than other variants of the artificial bee colony algorithm. The RABC algorithm-optimized PID controller has better control with minimum overshoot and fast response, as verified by comparison with PSO-PID, DE-PID, and GA-PID methods in three typical systems.

1. Introduction

The PID controller has been the most widely used and mature controller in the industrial production process [1]. Despite the emergence of various new controllers in the control field, PID controllers are still in the dominant position with their simple structure, easy implementation, and robustness. The PID controllers are widely used in chemical, electric power, metallurgy, and other industrial control sites. In the industrial control process, more than 95% of the loops have a PID structure [2]. The suitability of controller parameters has an important impact on the quality of the controller. With the development of modern industry, the traditional PID controller can no longer meet the requirements of control systems with high order, time lag, and nonlinearity. The traditional method of PID parameter optimization can no longer fully adapt to the exploitation of modern industry, so it is very important to study a new and efficient PID parameter optimization technique for engineering practice [3].

For the optimization of the parameters of the PID controller, the researchers used various optimization techniques. Feng et al. [4] proposed an improved genetic algorithm (IGA), to search for the PID controller parameters, for the robotic excavator. Özdemir et al. [5] proposed a new metaheuristic optimization algorithm, optical inspired optimization (OIO) algorithm to optimize PID controllers. The method has better performance in terms of maximum overshoot and stabilization time. Chen et al. [6] proposed a fuzzy PID controller optimized by an improved ant colony algorithm. The improvements of nonlinear incremental evaporation rate and pheromone incremental update were proposed in the IACO algorithm to improve the quality of the solution. Hekimo glu et al. [7] proposed atomic search optimization (ASO) algorithm and chaotic ASO (ChASO) to determine the optimal parameters of fractional-order proportional + integral + derivative (FOPID) controller. Bingul et al. [8] proposed a new time-domain performance criterion for the rectification design of the proportional-integral-derivative (PID) controller in an automatic voltage regulator (AVR) using the cuckoo search algorithm. This performance criterion is chosen to minimize the maximum overshoot, rise time, stabilization time, and steady-state error of the terminal voltage. Ekinci et al. [9] used an improved kidney-inspired algorithm (IKA) and a new objective function. The main objective of the method is to optimize the transient response of the AVR system to obtain the optimal values of the three gains (K_p, K_i, and K_d) of the PID controller by minimizing the maximum overshoot, stabilization time, rise time, and peak time values of the terminal voltage and eliminating the steady-state error. Batiha et al. [10] implemented two optimization algorithms, particle swarm optimization (PSO) and bacteria foraging optimization (BFO) algorithms, for the purpose of tuning the fractional-order PID-controller. Huang and Chuang [11] proposed an artificial bee colony optimization (ABC) algorithm incorporating fuzzy theory to optimize the PID controller by introducing fractional-order proportional-integral-derivative (FOPID) control strategy. Panoeiro et al. [12] optimized PID controller parameters by bionic optimization technique. Optimization of PID parameters based on the Ziegler–Nichols (ZN) rule was proposed in [13] as a fractional-order PID controller optimization method based on radial basis function (RBF) neural network. A fractional-order fuzzy proportional integral differential (FOFPID) controller was proposed by Sharma et al. [14]. The cuckoo search algorithm (CSA) optimization technique was used to optimize all the controller parameters. Chang et al. proposed a new adaptive genetic algorithm for PID controller design, and they found that the fractional-order PID controller significantly reduced the overshoot and stabilization time compared to the optimized conventional PID controller [15]. Bingul [16] used a differential evolution (DE) algorithm for tuning the PID controller for unstable and time-lagged integral processes. The results show that the PID-tuned DE has faster stabilization time, less or no overshoot, and higher robustness. Cao and Cao [17] demonstrated parameter optimization of a fractional-order controller based on improved PSO. The improved particle swarm algorithm has a faster search speed and better solution than the genetic algorithm. Maiti et al. [18] used PSO to design fractional-order PID controllers. They significantly reduced the overshoot percentage, rise, and adjustment time using the FOPID controller compared to the PID controller. Alfi and Modares [19] used a novel adaptive PSO (APSO) algorithm to find the optimal system parameters for unstable nonlinear systems and optimal parameters for PID controllers. Some scholars applied the particle swarm algorithm (PSO) [20–22] to optimize the PID controller parameters to improve the search speed. Improving the PID structure is too tedious. The exploitation capability and exploration capability of intelligent algorithms in parameter optimization of PID controllers need further improvement.

Artificial bee colony algorithm is a swarm intelligence optimization algorithm, proposed by Karaboga, to simulate the process of honey bee foraging [23]. The algorithm is easy to implement control with few parameters and has good optimization performance. Therefore, the contributions of this paper are as follows:

(1) The RABC algorithm is proposed to improve the exploitation capability and exploration capability of the ABC algorithm.

(2) The PID controller is based on the RABC algorithm (RABC-PID). The effectiveness of this controller is verified by comparing it with the controllers optimized by the other three methods.

The rest of this paper is organized as follows. Section 2 introduces the basic artificial swarm algorithm, the modified artificial swarm algorithm, and the superiority of the modified artificial swarm algorithm verified by six benchmark functions. The principle of optimized PID controller parameters and the whole optimization process of the modified artificial swarm algorithm is introduced. Section 3 establishes three typical models, and Section 4 conducts experimental simulations to validate the superiority of the RABC algorithm by comparing it with three other intelligent methods. Section 5 concludes the whole paper.

2. Proposed Method

2.1. Basic Artificial Bee Colony Algorithm. The basic artificial bee colony algorithm divides the population into three types: employed bees, following bees, and scout bees, and sets a search phase for each type of artificial bee, i.e., employed bee phase, following bee phase, and scout bee phase. Initially, the algorithm uses random initialization to generate the initial population, which is shown in the following:

$$X_i^j = X_{\min}^j + \mathrm{rand}(0,1)\left(X_{\max}^j - X_{\min}^j\right), \tag{1}$$

where $i = 1, \ldots, \mathrm{SN}$, $j = 1, \ldots, D$. SN denotes the population size, D denotes the problem dimension, rand (0, 1) is a random number between 0 and 1, and $X_{\max}^j$ and $X_{\min}^j$ denote the upper and lower bounds of the jth dimension of an individual, respectively.

In the employed bee phase, the employed bee searches for food sources by performing a random search of the feasible domain by equation (2) and passes the food source multiplication to the following bee waiting in the hive to search for the food source.

$$V_i^j = X_i^j + \varphi_i^j\left(X_i^j - X_k^j\right), \tag{2}$$

where $k \in (1, \ldots, \mathrm{SN})$ is a randomly selected indicator for different i, which means that there is only one randomly selected solution in generating the new candidate solution; $j \in (1, \ldots, D)$ is a randomly selected indicator, which means that only one dimension has changed between the new candidate solution and the old one. φ_i^j is a random number uniformly distributed on $[-1, 1]$.

In the following bee stage, according to the food source information passed back to the hive by the employed bee, the following bee selects the food source using roulette according to the probability calculated in equation (3) below, and the nectar collection process still uses equation (2) to update the food source randomly.

$$P_i = \frac{\mathrm{fit}_i}{\sum_{j=1}^{\mathrm{SN}} \mathrm{fit}_j}, \tag{3}$$

where fit_i denotes the adaptation value of the ith food source.

During the scout bee phase, scout bees discard all food sources that have already been mined for honey, exceeding the limit for all food sources. An employed bee on the abandoned food source will then transform into a scout bee, randomly searching for new food sources according to equation (1). Throughout the search process of the algorithm, the food source corresponds to the candidate solution of the optimization problem, and the quality of the food source represents the merit of the candidate solution.

2.2. Reformative Artificial Bee Colony Algorithm. It is well known that the exploration ability [24] and the exploitation ability of an algorithm are two conflicting aspects that affect the performance of an algorithm. In other words, the enhanced exploration ability of the algorithm will inevitably affect the exploitation ability of the algorithm, which may reduce the convergence speed of the algorithm, while the enhanced exploitation ability of the algorithm will also inevitably affect the exploration ability of the algorithm, which may lead the algorithm to fall into local optimum. The artificial bee colony algorithm can be seen to have stronger exploration ability and weaker exploitation ability due to its random search feature. In this paper, a reformative artificial bee colony algorithm (RABC) is proposed. The optimal solution idea is proposed for the problem of the weak exploitation ability of the artificial bee colony algorithm. The algorithm selects the global optimal bee location and its food source location when updating the location of the food source. When the colony is updated, the bees are allowed to refer to the global bee with the best food source and move to the food source with better quality, and the global optimal is updated as the colony is constantly updated, which allows the bees to use the optimal food source as a reference when acquiring food source information and improves the exploration capability of the algorithm to some extent. Equation (2) for searching food sources is changed to the following equation:

$$V_i^j = X_{i,\text{best}}^j + \varphi_i^j\left(X_{i,\text{best}}^j - X_i^j\right), \tag{4}$$

where $X_{i,\text{best}}^j$ is the global current optimal food source.

The following probability of following bees is determined by the proportion of the current food source's fitness among all food sources. When some better food sources do not differ much from the optimal food source, it will lead to a lower probability of following the optimal food source by the following bees. For some bad food sources, due to the random selectivity of the following bees, they have the opportunity to follow them instead, which leads to a slower speed of finding the optimal food source and shows a slow convergence speed in the algorithm. Because of this, this paper proposes to take the current optimal food source of the colony as a reference, so that the following bees are more inclined to choose the food source with high quality and improve the speed of the colony to find the optimal food source. Thus, the probability P_i of following bees following the hiring bees is changed to the following equation:

$$P_i = \frac{0.8\text{fit}_i}{\max \text{fit}_i + 0.2}, \tag{5}$$

where $\max\ \text{fit}_i$ is the fitness value of the highest solution.

Both employed and following bees use equation (4) to update the food source location, which allows a portion of the bee's information to be exchanged with the globally optimal bee, ensuring that the bee is not disturbed by the locally optimal bee, but also that the bee moves to a better food source led by the globally optimal bee. Using equation (5) to judge the quality of food sources, following bees are more likely to follow honey-harvesting bees that have high-quality food sources and thus exploit high-quality food sources.

2.3. Benchmark Function Simulation Verification. To check the optimization performance of the RABC algorithm, the same ABC, GABC [25], and GBABC [26] algorithms are tested for comparison experiments on six benchmark functions. The basic characteristics of the test functions are given in Table 1. The test dimension of the test function is $D = 50$. In the experiments, the population number SN is 100, the limit is 50, the maximum cycle number MaxCycle is set to 5000, and the algorithm is run 30 times independently. The experimental results are shown in Table 2.

Table 2 gives the experimental results of the six benchmark functions, including the mean and standard deviation. From the table, we can see that the RABC algorithm has the smallest standard deviation among the six benchmark functions, which proves that the RABC algorithm has the best stability. The mean value is closest to the optimal value, which proves that the RABC algorithm has the highest search accuracy.

Figure 1 gives the benchmark function convergence curve graph. From the figure, it can be seen that the RABC algorithm outperforms both the basic artificial bee colony algorithm and the other two improved artificial bee colony algorithms in terms of convergence speed performance and search accuracy. The experimental results indicate that RABC has better optimization performance.

2.4. PID Controller Based on RABC (RABC-PID). The RABC algorithm is used to optimize the PID parameters, which is essentially a parameter optimization problem based on a certain objective function, i.e., finding the optimal values in the parameter space of K_p, K_i, and K_d variables to optimize the control performance of the system. The control block diagram is shown in Figure 2.

The RABC algorithm optimizes the PID parameters by taking the error on the system as the evaluation function of the RABC, i.e., the fitness function input, calculating the value of the fitness function, and then adjusting the three PID parameters according to the fitness of the function to make the control performance of the system optimal.

The selection of the objective function is an important process. To achieve the optimal comprehensive performance of the whole system, we need to use some indicators that can reflect the comprehensive performance. At this stage, the common comprehensive performance evaluation criteria are mainly based on the connection between the deviation of the system $e\ (t) = r\ (t) - y\ (t)$ and time t. There are four main comprehensive performance evaluation criteria: integral of squared error (ISE), integral of time-weighted squared error

Table 1: Benchmark function definition domain and optimal value.

Function	Name	Definition domain	Optimal value
F1	Beale	(−4.5, 4.5)	0
F2	Camel3	(−5, 5)	0
F3	Hump	(−5, 5)	0
F4	Rastrigin	(−5.12, 5.12)	0
F5	Goldstein	(−2.2)	3
F6	Easom	(−100, 100)	−1

Table 2: Benchmark function experimental results.

Function		ABC	GABC	GBABC	RABC
F1	Mean	0.00218509	0.000172577	0.000144142	**6.49866e − 05**
	Std	0.00129199	0.000147622	9.31112e − 05	**2.46584e − 05**
F2	Mean	0.00338069	0.0024648	5.02474e − 05	**9.78549e − 06**
	Std	0.00235156	0.0025263	1.80147e − 05	**1.96744e − 06**
F3	Mean	0.00130278	0.00134831	5.77273e − 05	**1.17531e − 05**
	Std	0.00040916	0.00025792	6.32839e − 05	**4.90377e − 06**
F4	Mean	0.0287649	0.0123041	4.1950e − 10	**1.02564e − 11**
	Std	0.0374303	0.0081333	4.3844e − 10	**8.64572e − 11**
F5	Mean	3.2728	3.03062	3.0111	**3.00295**
	Std	0.337852	0.00717374	0.00522465	**0.00379628**
F6	Mean	−0.0548194	−0.351209	−0.41055	**−0.771196**
	Std	0.3775246	0.352833	0.0580604	**0.00293816**

(ITSE), integral of absolute error (IAE), integrated time absolute error (ITAE) [27].

Integrated time absolute error (ITAE), which integrates the speed, stability, and accuracy of the system, is widely used to optimize the comprehensive performance index of the PID controller. ITAE can be expressed as

$$\text{ITAE} = \int_0^{\infty} t|e(t)|\mathrm{d}t. \tag{6}$$

To prevent the control from being too large, the weighted integrated time absolute error (WITAE) is proposed. The ITAE performance index is used as the minimum adaptation function for parameter selection, and the squared term of the control input is added to the objective function. WITAE can be expressed as

$$\text{WITAE} = \int_0^{t} \left[J_1 t|e(t)| + J_2 t u(t)^2\right]\mathrm{d}t, \tag{7}$$

where J_1, J_2 are the weights and WITAE is the adaptation value. Once the overshoot is generated and the penalty function is applied [28], equation (8) is used as the performance evaluation criteria.

$$\text{WITAE} = \int_0^{t} \left[J_1 t|e(t)| + J_2 t u(t)^2 + J_3|e(t)|\right]\mathrm{d}t, \tag{8}$$

where J_3 is the weight value, $J_3 \gg J_1$. Usually, $J_1 = 0.999$, $J_2 = 0.001$, and $J_3 = 100$. The fitness function is expressed as

$$\text{fit}_i = \begin{cases} \dfrac{1}{1+\text{WITAE}}, & \text{WITAE} \geq 0, \\ 1 + |\text{WITAE}|, & \text{WITAE} < 0. \end{cases} \tag{9}$$

The higher the value of fit_i is, the higher the probability that the food source will be selected for search. The RABC algorithm optimizes the PID controller parameters flow as shown in Figure 3.

3. Three Typical Models

3.1. DC Motor Modeling. The motor circuit structure is shown in Figure 4. The air-gap flux is constant, and the control of the motor is realized by adjusting the voltage of the armature circuit, to realize the regulation of the motor speed. For such a motor object, some of its parameter characteristics such as torque constant and viscous friction coefficient are not available, so it can be regarded as a black box in modeling, and the modeling can be completed by using input and output data and using system identification.

First, determine the object model structure, and the air-gap flux is known, so its electromagnetic torque T_i is proportional to the armature current i_a:

$$T_i = K_i i_a, \tag{10}$$

where K_i is the torque constant of the motor.

The armature circuit voltage balance equation can be expressed as

$$u_a = R_a i_a + L_a \frac{\mathrm{d}i_a}{\mathrm{d}t} + e_a, \tag{11}$$

where R_a and L_a are the resistance and inductance of the armature circuit, respectively.

The relationship between the motor's counter-electromotive force e_a and its angular velocity ω is as follows:

$$e_a = K_e \omega, \tag{12}$$

where K_e is the counter-electromotive force constant. The equation of torque balance on the motor shaft can be expressed as

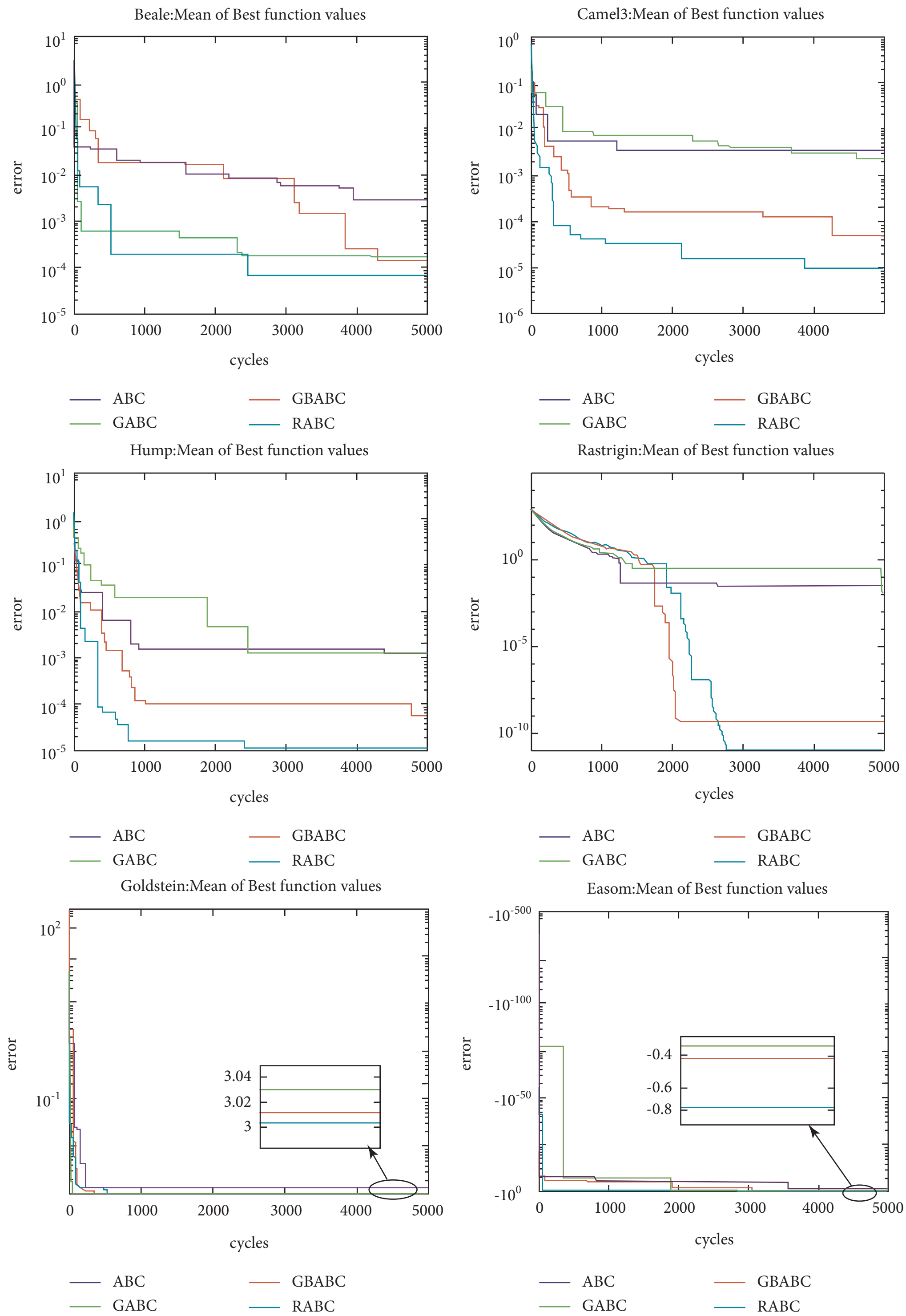

FIGURE 1: Convergence curve of benchmark function.

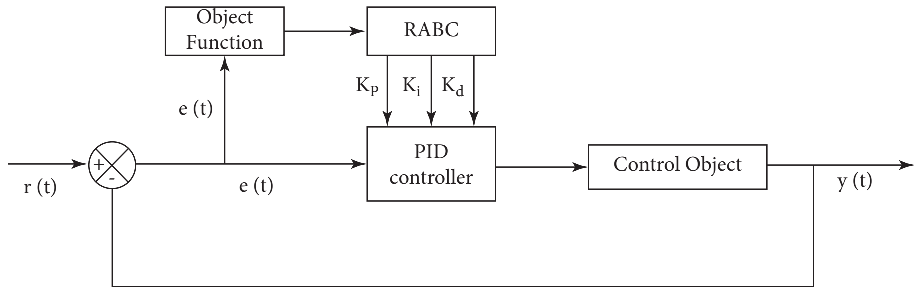

Figure 2: PID parameter tuning block diagram of reformative artificial bee colony algorithm.

$$T_i = J\frac{d\omega}{dt} + b\omega + T_L, \tag{13}$$

where J is the equivalent rotational inertia, b is the equivalent viscous friction coefficient, and T_L is the load torque. With u_a as input voltage and ω as output speed, the following differential equation is obtained by eliminating i_a, e_a, and T_i:

$$\begin{aligned} & L_a J\frac{d^2\omega}{dt} + (R_a J + bL_a)\frac{d\omega}{dt} + (K_e K_t + R_a b)\omega \\ & \quad + \frac{R_a T_L + L_a T}{K_t} = K_t u_a. \end{aligned} \tag{14}$$

The term of $((R_a T_L + L_a T)/K_t)$ can be ignored without participating in the calculation. Therefore, differential equation (14) can be transformed into the following transfer function (15) according to the Laplace transformation.

$$G(s) = \frac{K_t}{L_a J s^2 + (R_a J + bL_a)s + (K_e K_t + R_a b)}. \tag{15}$$

At this point, the armature-controlled DC motor can be viewed as an oscillating link. Usually, the inductance L_a in the armature circuit is small, and if its effect is neglected, transfer function (15) can be approximated as a first-order transfer function, which can be expressed as follows:

$$G(s) = \frac{K}{Ts + 1}, \tag{16}$$

where $K = (K_t/(K_e K_t + R_a b))$ is the gain constant of the motor and $T = (R_a J/(K_e K_t + R_a b))$ is the time constant of the motor.

3.2. Mathematical Model of Double Capacity Water Tank (DCWT). The double-volume tank is a typical second-order time-lag system, which is schematically shown in Figure 5.

In Figure 5, A_1 and A_2 represent the bottom area of the tank, q_1, q_2, and q_3 represent the water flow, R_1 and R_2 represent the resistance of the valves V_1 and V_2, which is called liquid resistance of valve resistance, and $\Delta q = (\Delta h/R)$. According to the material balance on tank 1, there is the following equation:

$$\Delta q_1 - \Delta q_2 = A_1\frac{d\Delta h_1}{dt}, \tag{17}$$

$$\Delta q_2 = \frac{\Delta h_1}{R_2}. \tag{18}$$

Therefore, differential equations (17) and (18) can be transformed into the following transfer functions (19) and (20) according to the Laplace transformation.

$$\Delta Q_1(S) - \Delta Q_2(S) = A_1 S \Delta H_1(S), \tag{19}$$

$$\Delta Q_2(S) = \frac{\Delta H_1(S)}{R_2}. \tag{20}$$

Similarly, the differential equation and transfer function of tank 2 can be obtained:

$$\begin{aligned} \Delta q_2 - \Delta q_3 &= A_2\frac{d\Delta h_2}{dt}, \\ \Delta q_2 &= \frac{\Delta h_2}{R_3}, \\ \Delta Q_2(S) - \Delta Q_3(S) &= A_2 S \Delta H_2(S), \\ \Delta Q_3(S) &= \frac{\Delta H_2(S)}{R_2}. \end{aligned} \tag{21}$$

The transfer function of DCWT can be expressed as follows:

$$\begin{aligned} W_o(S) &= \frac{\Delta H_2(S)}{\Delta Q_1(S)} = \frac{R_2}{(A_1 R_2 S + 1)(A_2 R_3 S + 1)} \\ &= \frac{K}{(T_1 S + 1)(T_2 S + 1)}, \end{aligned} \tag{22}$$

where $T_1 = A_1 R_2$ is the time constant of tank 1, $T_2 = A_2 R_3$ is the time constant of tank 2, and K is the amplification factor of the dual-capacity object. If the system also has a pure delay, transfer function (22) can be changed as follows:

$$W_o(S) = \frac{\Delta H_2(S)}{\Delta Q_1(S)} = \frac{K}{(T_1 S + 1)(T_2 S + 1)} e^{-\tau s}. \tag{23}$$

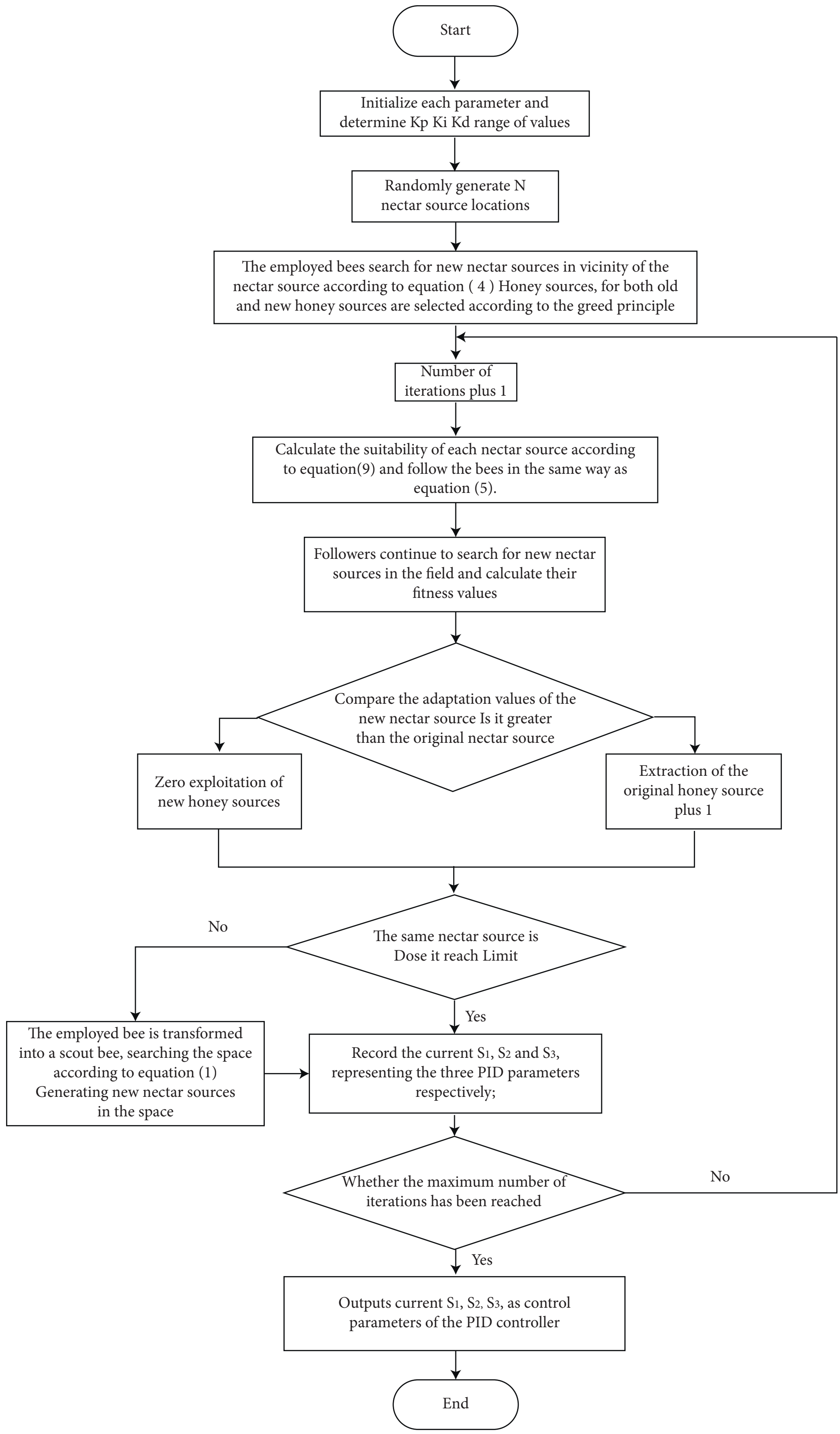

Figure 3: The flowchart of the reformative artificial bee colony algorithm for optimizing PID controller parameters.

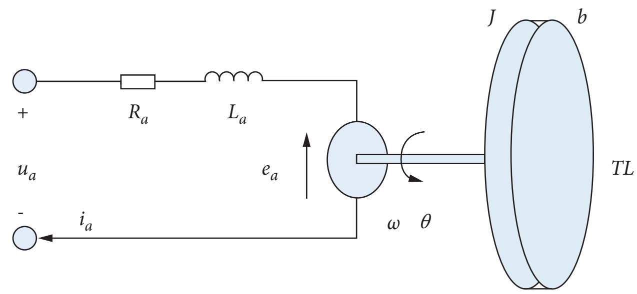

FIGURE 4: Motor circuit structure diagram.

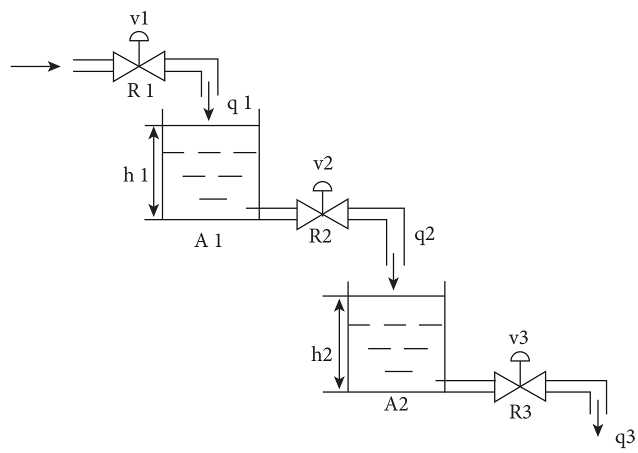

FIGURE 5: Schematic diagram of dual-capacity water tank.

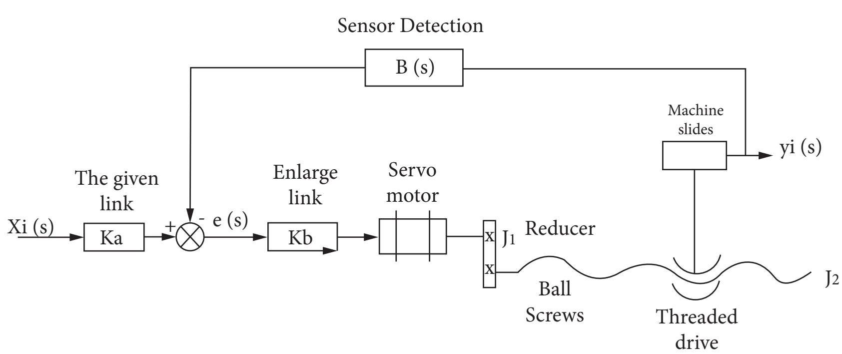

FIGURE 6: Schematic diagram of the position control system of the machine tool sliding table.

3.3. Position Control System for the Sliding Table (PCSST). The schematic diagram of the PCSST is shown in Figure 6. In the figure, $x_i(S)$ is the given input signal; K_a is the given link; $y_i(S)$ is the output signal; and $B(s)$ is the sensor detection link. $e(s)$ is the error signal, which is converted to a voltage signal to drive the servo motor after the amplification link K_b. The modeling process of the PCSST is explained as follows.

3.3.1. Position Closed-Loop Model. The schematic diagram of the electro-hydraulic servo slide table position control system in this study is shown in Figure 6. There are photoelectric sensors installed on both sides of the x-axis direction of the machine slide, which detect the slide motion position in real time and use the position closed-loop control to continuously correct the position error $e(s)$ to achieve zero-error position closed-loop control. Therefore, the closed-loop position control model is established as follows:

$$\begin{aligned} U_a(S) &= K_b\left[p_{wc}(s) - p_{wf}(s)\right], \\ p_{wf}(s) &= K_w x_h(s), \end{aligned} \tag{24}$$

where $U_a(S)$ is the position loop voltage; K_b is the position amplification factor; $p_{wc}(s)$ represent the position loop

Table 3: Transfer function.

Model	Coefficient setting	Transfer function
DC	$K = 5$ $T = 6$	$G(s) = (5/(6s+1))$
DCWT	$\tau = 5$ $K = 10$ $T_1 = 3$ $T_2 = 5$	$G(S) = (10/(3s+1)(5s+1))e^{-5\tau}$
PCSST	$a_3 = 5,\ a_2 = 3$ $a_1 = 2.45,\ a_0 = 18$ $K_aK_bK_wK_Ti_{pl} = 16$	$G(s) = (16/(5s^3 + 3s^2 + 2.45s + 18))$

Table 4: The optimal parameters of the PID controllers of the three models.

		K_p	K_i	K_d
RABC	DC	15.8565	3.08603	−0.874396
	DCWT	8.456	5.61285	0.573094
	PCSST	5.9645	3.72379	0.86088
PSO	DC	12.5	2.9477	−1.0005
	DCWT	6.872	3.063	0.13548
	PCSST	5.2364	2.0749	0.18095
DE	DC	14.56	3.04633	−0.93457
	DCWT	5.465	2.871	0.15037
	PCSST	5.22386	1.959	0.18219
GA	DC	13.5	1.462	−1.02
	DCWT	4.2827	2.84296	0.13236
	PCSST	4.0379	1.6632	0.14135

initial pulses; $p_{wf}(s)$ represent the position loop sensor feedback pulses; K_w is the position gain factor; and $x_h(s)$ is the slide table movement.

3.3.2. Servomotor Model. The servomotor drive model can be expressed as follows:

$$\begin{bmatrix} i \\ \ddot{\theta} \end{bmatrix} = \begin{pmatrix} -\dfrac{r}{l} & -\dfrac{p_k\lambda_f}{l} \\ \dfrac{3p_n\lambda_f}{2J_1} & 0 \end{pmatrix} \begin{bmatrix} i \\ \omega \end{bmatrix} + \begin{bmatrix} \dfrac{u}{l} \\ -\dfrac{T}{l} \end{bmatrix}, \quad (25)$$

where J_1 is the servomotor inertia; λ_f is the motor magnetic field coefficient; l is the inductance coefficient; p_k is the pole logarithm; i is the current value; θ is the rotation angle; and T is the servomotor torque, and it can be expressed as follows:

$$T = K_T i = \frac{3}{2}P_n\lambda_f i, \quad (26)$$

where K_T is the torque coefficient.

3.3.3. Working Slide Drive Model. The machine slide completes the movement along the X-axis direction under the ball screw thread drive, and the slide position movement model can be simplified to a ball screw linear motion model as follows:

$$x_l(s) = i_{pt}\theta_l = \frac{H_h i}{2\pi}\theta_m, \quad (27)$$

where $x_i(s)$ is the displacement of the table; i is the ratio of the ball screw; H_h is the total ball screw travel; and θ_m is the turning angle of the screw.

3.3.4. Closed-Loop Total Transfer Model of Table Position. From the above three models, closed-loop series connection to form the total transfer function of the electro-hydraulic position closed-loop control system of the working slide of the combined machine tool can be obtained.

$$G(s) = \frac{x(s)}{y(s)} = \frac{K_aK_bK_wK_Ti_{pl}}{a_3s^3 + a_2s^2 + a_1s^1 + a_0}. \quad (28)$$

4. Simulation Validation

The RABC balances the exploitation capability of the basic artificial bee colony algorithm with the exploration capability. In this paper, we verify the superiority of the RABC algorithm to optimize PID parameters by four intelligent algorithms.

4.1. Performance Indicators of the Control System. The performance indicators of control systems are divided into transient performance indicators and steady-state performance indicators. Transient performance refers to the transient behavior of the output of a control system during

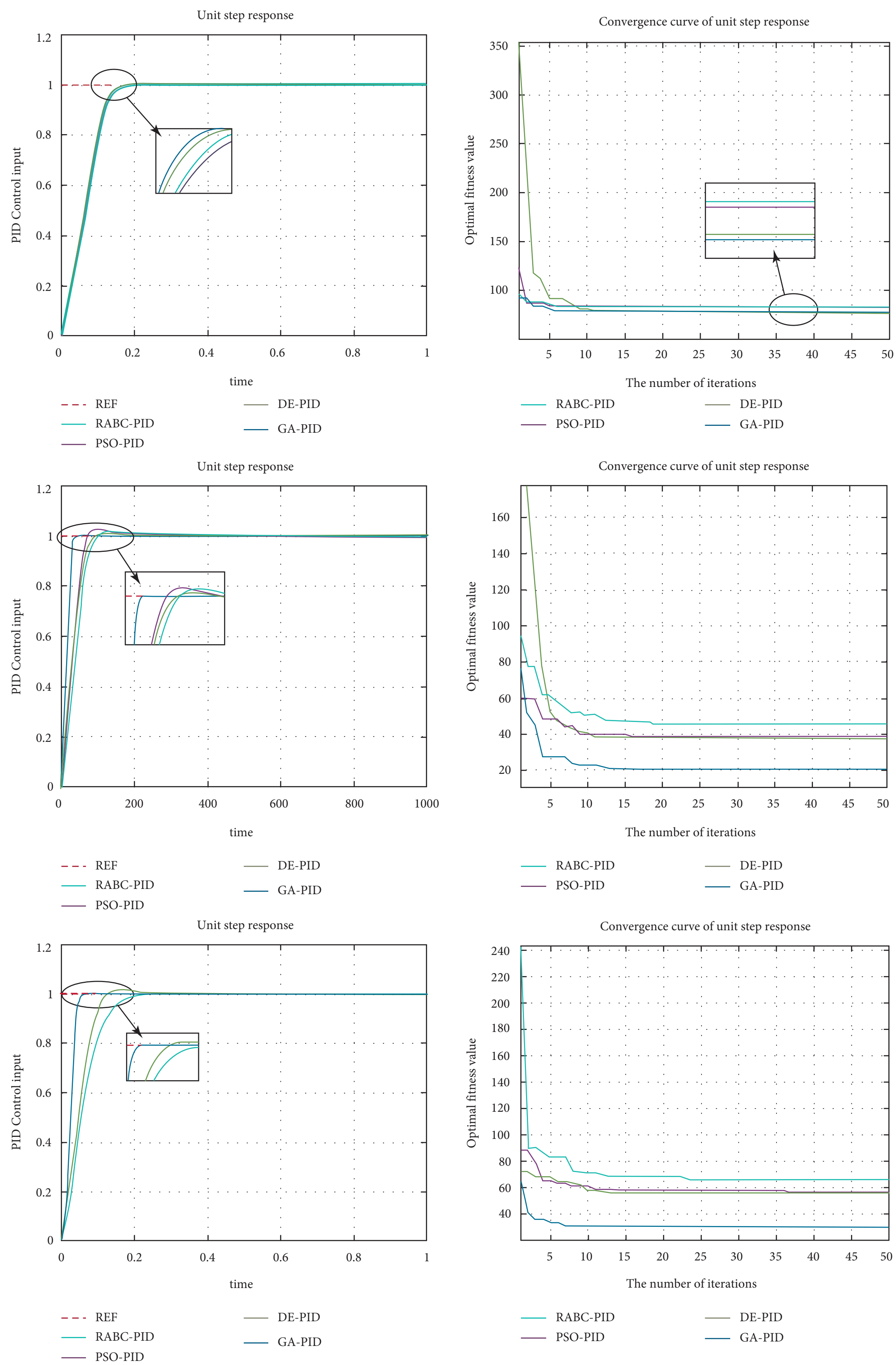

Figure 7: Unit step response curve and convergence curve.

Table 5: Performance index.

		t_r	t_p	σ %	t_s	WITAE
RABC	DC	**0.114**	**0.176**	**0.038882**	**0.134**	**77.28**
	DCWT	**28**	**10**	**0**	**32**	**20.48**
	PCSST	**0.043**	**0.074**	**0.089273**	**0.052**	**30.28**
PSO	DC	0.124	0.241	0.049552	0.157	82.04
	DCWT	60	100	3.1078	123	38.64
	PCSST	0.095	0.103	1.9173	0.117	55.95
DE	DC	0.116	0.183	0.075293	0.138	78.05
	DCWT	65	115	1.1421	82	37.38
	PCSST	0.096	0.164	1.7873	0.118	56.65
GA	DC	0.127	0.293	0.082143	0.166	82.86
	DCWT	68	116	1.5401	84	45.93
	PCSST	0.126	0.305	0.15743	0.177	66.34

the transition process, which is the so-called transition process in which the system is transferred from one steady state to another under the action of an external input signal. In control systems, the unit step response of the system is generally used to define the indicators of the transient performance of the system, which are usually rise time t_r, peak time t_p, peak time t_p, and overshoot σ %.

4.2. Simulation Verification. After simulation verification in Matlab, three typical mathematical model transfer functions are shown in Table 3.

To verify the optimization performance of the RABC algorithm, the comparison with three intelligent algorithms PSO, DE, and GA is verified. After the simulation verification in Matlab, the optimal parameter setting of each algorithm is obtained. The parameters of each algorithm were set as follows: PSO algorithm $C_1 = C_2 = 2$, initial value of W is 0.9, $W_{\max} = 0.9$, $W_{\min} = 0.4$, $V_{\max} = 1$, $V_{\min} = -1$, population size SN = 50, and iterations = 50. DE algorithm crossover probability Pcr = 0.8, scaling factor $F = 0.85$, population size SN = 50, and iterations = 50. GA algorithm crossover probability Pc = 0.7, variation probability Pm = 0.3, population size SN = 50, iterations = 50. RABC algorithm population size SN is set to 50, limit number of iterations is 50, and iteration number is set to 50.

The simulation time for DC and PCSST is set to 1 s. The simulation time for DCWT is set to 1000 s, since DCWT is a time-delay system. The experimental results are shown in Table 4.

From the unit step response curve in Figure 7, it can be seen that the RABC algorithm has no overshoot amount in the dual-capacity tank system, and the rise time, regulation time, and peak time are the shortest. The RABC algorithm has almost no overshoot in the other two systems, and the rise time, regulation time, and peak time are also the shortest. This indicates that the RABC algorithm optimizes the PID controller parameters best. From the unit step response convergence curve in Figure 7, it can be seen that the RABC algorithm has the fastest convergence speed and the highest convergence accuracy among the three systems. This also proves that the RABC algorithm optimizes the PID controller parameters best.

From Table 5, it can be clearly seen that the adaptation values obtained by the RABC algorithm in optimizing the PID controller parameters are smaller than those of the other three optimization algorithms for the DC motor, dual-capacity water tank, and machine tool control systems. This also fully illustrates that the RABC algorithm is more accurate than the other three intelligent algorithms. From Table 5, it can be also seen that the PID controller of the dual-capacity water tank system optimized by the RABC algorithm has a fast response, no overshoot, and short regulation time and can enter the steady-state zone quickly, which reflects a better control effect. The PID controller of the other two systems optimized by the RABC algorithm has almost no overshoot, and the rise time, regulation time, and peak time are also the shortest. In summary, RABC algorithm-optimized PID controller has a better control effect.

A good controller has some robustness to cope with changes in parameters. For DC model, the parameter T is set to 10, 15, and 20, respectively. For DCWT model, the parameter T_1 is set to 5, 10, and 15, respectively. For PCSST model, the parameter a_3 is set to 15, 20, and 25, respectively. The simulation results are presented in Figures 8–10 and Tables 6–8. It can be seen from Figures 8–10 that the method still yields better results in the case of variation of model parameters. From Tables 6–8, we can see that the RABC-PID method still outperforms the other three methods in terms of overshoot, rise time, regulation time, and peak time when the parameters are changed through simulation experiments, which also proves that the RABC-PID has stronger robustness.

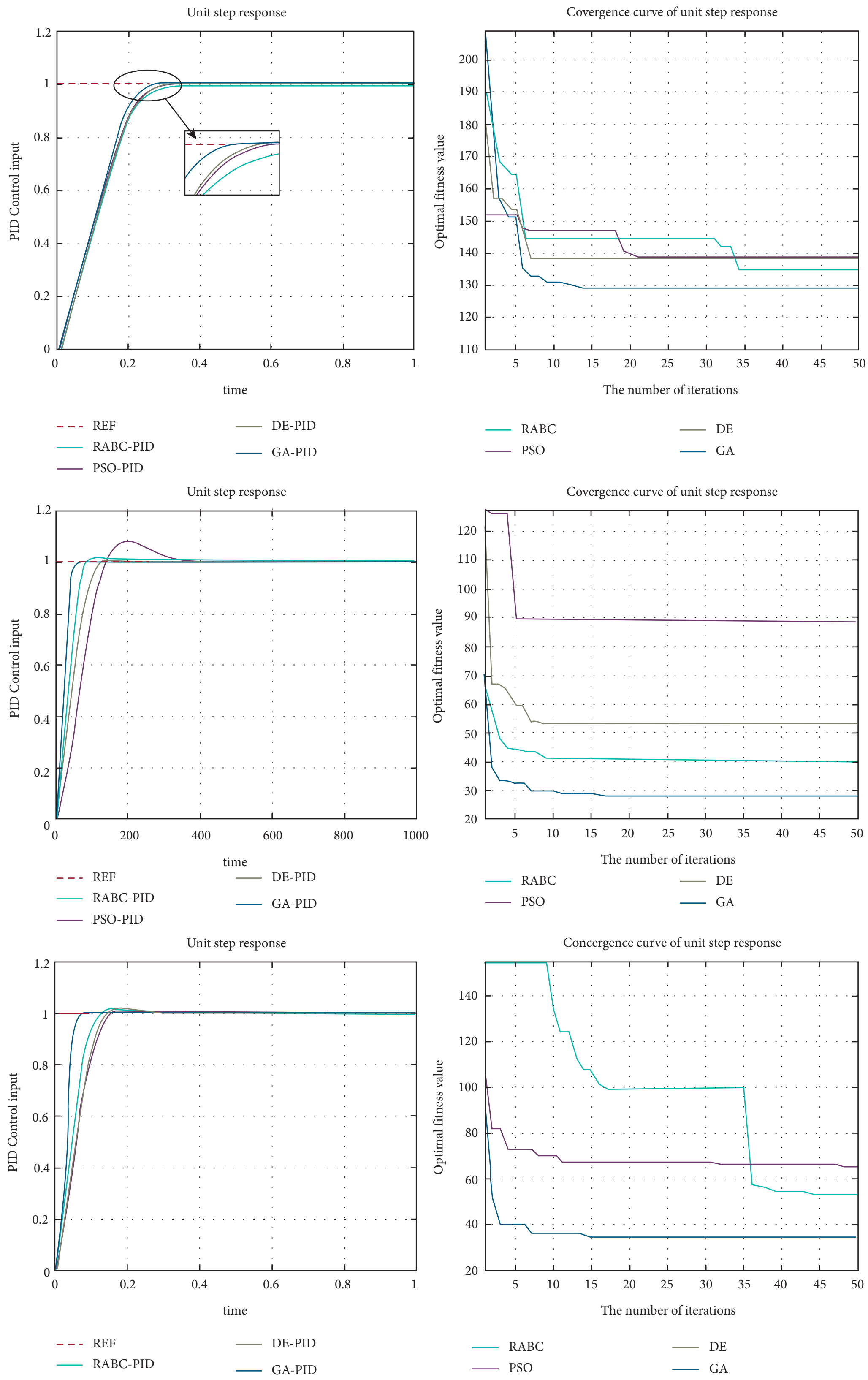

FIGURE 8: Unit step response curve and convergence curve when the parameters $T = 10$, $T_1 = 5$, and $a_3 = 15$.

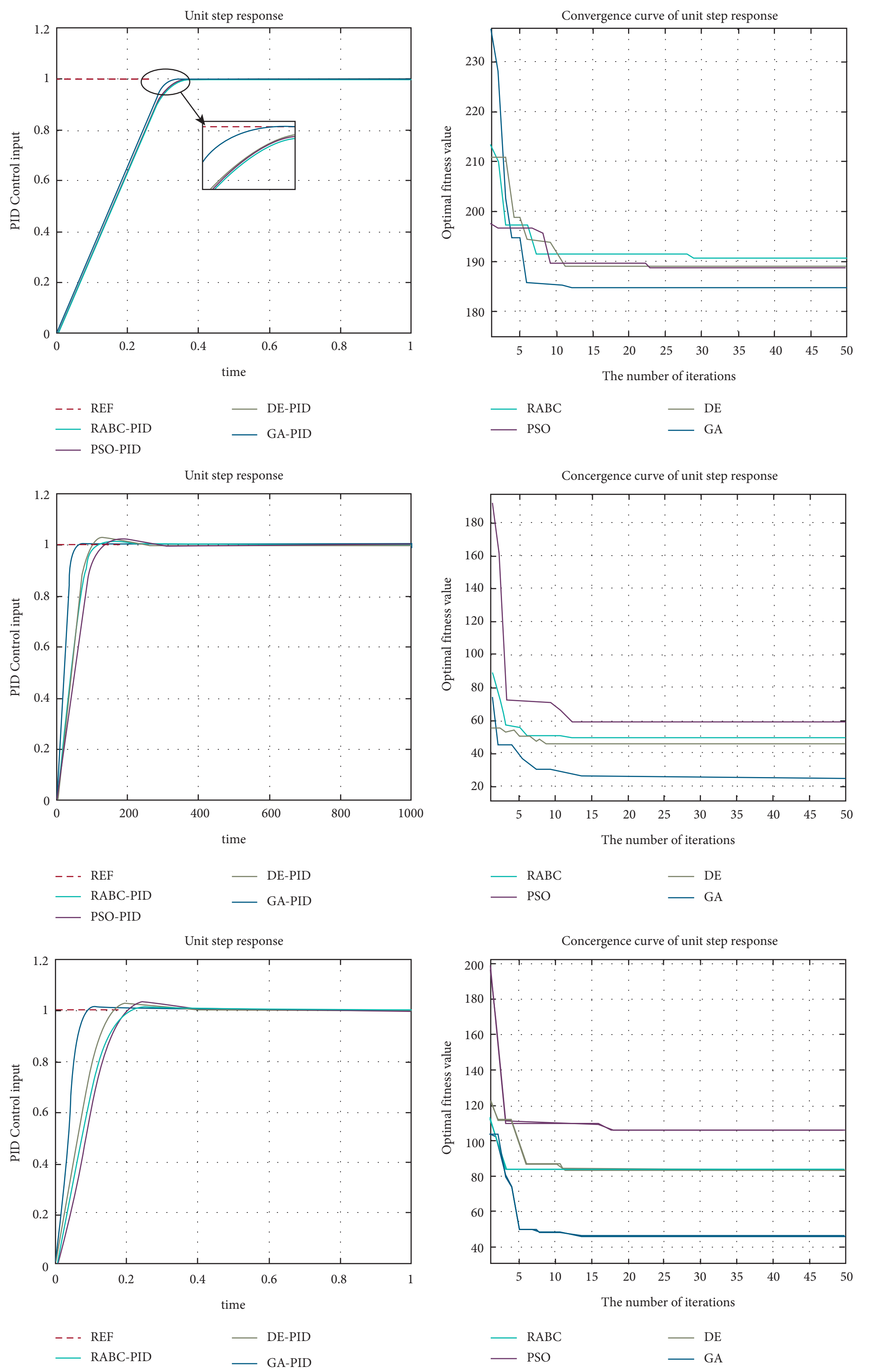

Figure 9: Unit step response curve and convergence curve when the parameters $T = 15$, $T_1 = 10$, and $a_3 = 20$.

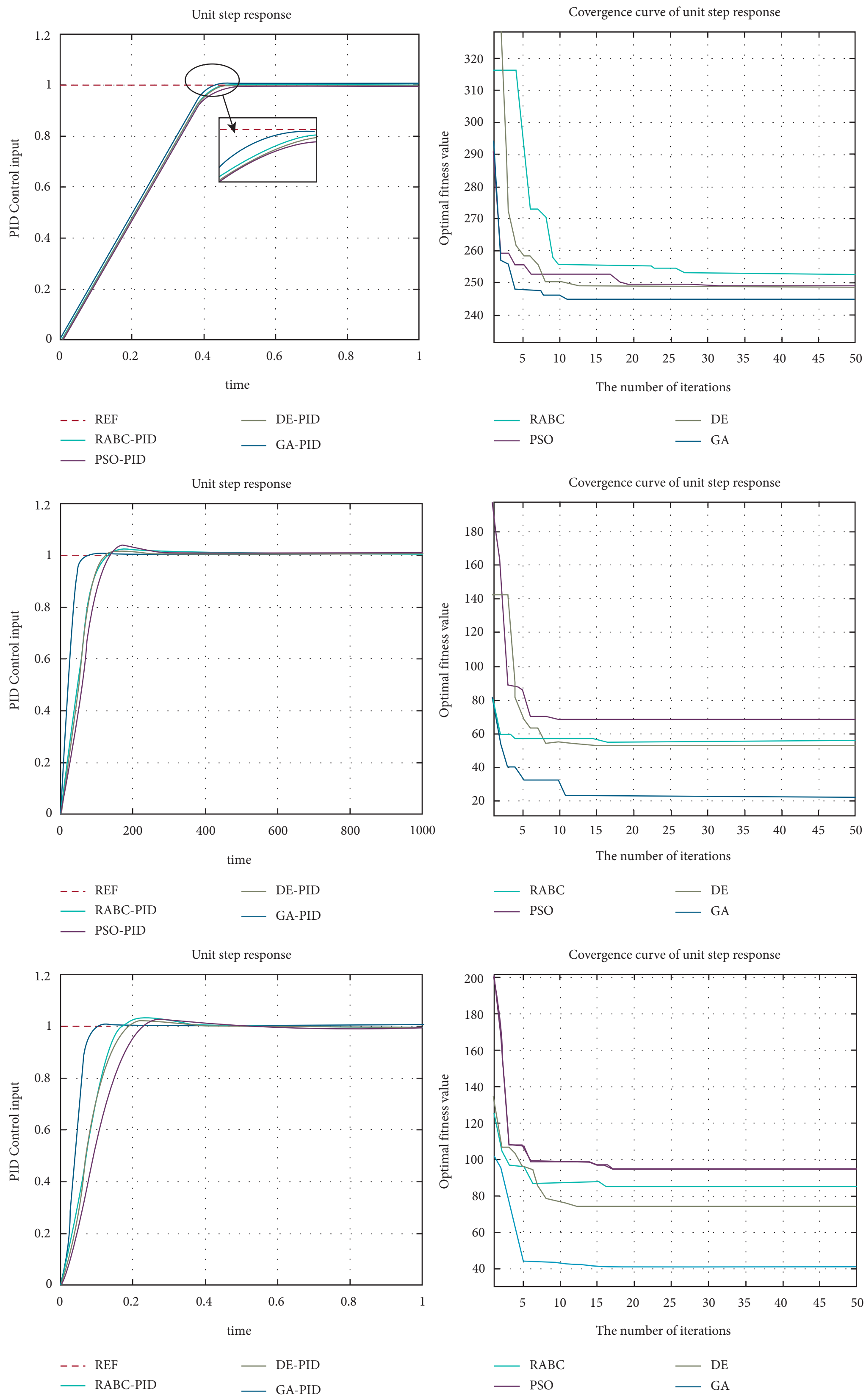

FIGURE 10: Unit step response curve and convergence curve when the parameters $T = 20$, $T_1 = 15$, and $a_3 = 25$.

Table 6: Performance index when the parameters $T=10$, $T_1=5$, and $a_3=15$.

		t_r	t_p	σ %	t_s	WITAE
RABC	DC	**0.193**	**0.302**	**0.15831**	**0.232**	**128.9**
	DCWT	**40**	**69**	**0.073972**	**48**	**28.45**
	PCSST	**0.049**	**0.085**	**0.12111**	**0.06**	**34.49**
PSO	DC	0.211	0.357	0.29852	0.263	138.5
	DCWT	119	201	7.5223	303	40.03
	PCSST	0.114	0.119	1.3367	0.143	52.98
DE	DC	0.209	0.345	0.42965	0.259	138.4
	DCWT	91	156	1.6836	112	52.83
	PCSST	0.11	0.187	2.051	0.196	64.6
GA	DC	0.211	0.711	0.16068	0.289	134.9
	DCWT	66	112	1.813	80	89.15
	PCSST	0.09	0.156	1.4905	0.112	65.28

Table 7: Performance Index when the parameters $T=15$, $T_1=10$, and $a_3=20$.

		t_r	t_p	σ %	t_s	WITAE
RABC	DC	**0.277**	**0.381**	**0.004674**	**0.307**	**184.8**
	DCWT	**40**	**71**	**0.248.2**	**50**	**25.63**
	PCSST	**0.064**	**0.111**	**0.54352**	**0.081**	**46.17**
PSO	DC	0.283	0.438	0.080586	0.336	188.9
	DCWT	98	169	1.9875	121	58.68
	PCSST	0.157	0.266	2.8769	0.323	105.6
DE	DC	0.283	0.425	0.12588	0.333	188.8
	DCWT	79	137	1.4829	98	45.6
	MTCS	0.125	0.213	2.2948	0.239	84.63
GA	DC	0.285	0.422	0.28155	0.336	190.6
	DCWT	83	142	2.02663	146	50.28
	PCSST	0.152	0.271	1.0545	0.193	83.17

Table 8: Performance index when the parameters $T=20$, $T_1=15$, and $a_3=25$.

		t_r	t_p	σ %	t_s	WITAE
RABC	DC	**0.368**	**0.496**	**0.008282**	**0.408**	**245**
	DCWT	**46**	**81**	**0.33053**	**57**	**29.48**
	PCSST	**0.071**	**0.123**	**0.64848**	**0.089**	**41.43**
PSO	DC	0.374	0.611	0.013246	0.443	249.1
	DCWT	110	186	3.024	228	68.17
	PCSST	0.117	0.3	2.767	0.36	94.74
DE	DC	0.373	0.545	0.10481	0.434	248.8
	DCWT	90	156	1.6576	112	52.67
	PCSST	0.139	0.236	2.4837	0.273	74.49
GA	DC	0.373	0.502	0.89717	0.423	253
	DCWT	94	162	1.9269	116	55.42
	PCSST	0.136	0.228	3.3499	0.288	85.31

5. Conclusions

In this paper, we propose an RABC algorithm to optimize the PID controller, to address the defects of the traditional ABC algorithm that the following bees follow the employed bees with too much randomness and the bee colony is not easy to approach the optimal food source, resulting in slow convergence and low accuracy, to increase the probability of following the employed bees that have a higher quality food source, and to introduce the global optimal bees to lead the other bees to move to a better food source. The RABC algorithm can quickly obtain the optimized value of the theory and improve the convergence speed and convergence accuracy. The PID controller optimized by the RABC algorithm has the characteristics of no overshoot and fast response and has a better control effect.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Forcing Strong Convergence of a Mann-Based Iteration for Nonexpansive and Monotone Operators in a Hilbert Space

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Mann iteration is weakly convergent in infinite dimensional spaces. We, in this paper, use the nearest point projection to force the strong convergence of a Mann-based iteration for nonexpansive and monotone operators. A strong convergence theorem of common elements is obtained in an infinite dimensional Hilbert space. No compact conditions are needed.

1. Introduction: Preliminaries

In the real world, there are a lot of nonlinear phenomena, which can be modelled into variational inequalities and variational inclusions, such as signal processing, image recovery, and machine learning; see, e.g., [1–7] and the references therein. Fixed point methods are powerful and popular for dealing various nonlinear operator equations and inequalities in abstract spaces, in particular, for variational inequalities and variational inclusions. Recently, various efficient fixed point methods have been introduced and investigated; see, e.g., [8–13] and the references therein. Let T be a nonlinear operator on a Hilbert space H, which is endowed with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. The fixed point set of T is presented by $\mathrm{Fix}(T)$. Recall that T is said to be contractive iff there is a real number $a \in (0,1)$ such that

$$\|Tx - Ty\| \leq a\|x - y\|, \quad \forall x, y \in H. \tag{1}$$

Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H. \tag{2}$$

Recall that T is said to be firmly nonexpansive iff

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \quad \forall x, y \in H. \tag{3}$$

It is clear that the class of firmly nonexpansive mappings is a special class of nonexpansive mappings. One knows the projection operator (see below) is firmly nonexpansive. The class of nonexpansive operators is significant in various nonlinear equations and mathematical programming computation. It also has wide real applications in applied and industrial fields. For various iterative methods, Mann iteration is popular for dealing with fixed points of nonexpansive operators. It reads

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n x_n, \tag{4}$$

where $\{\alpha_n\}$ is a real number sequence in the interval $(0,1)$. However, the Mann iteration is weakly convergent only in infinite dimensional spaces; see, e.g., [14] and the references therein. To force the strong convergence without possible compact assumptions, various regularized methods have been investigated in Hilbert spaces and Banach spaces recently; see, e.g., [15–19] and the references therein. One of the efficient regularized methods is the Halpern iteration, which reads

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n x, \tag{5}$$

where $\{\alpha_n\}$ is a real number sequence in the interval $(0,1)$ and x is a fixed anchor. With some conditions on $\{\alpha_n\}$, it was proved that $\{x_n\}$ converges to x, which is a special fixed point of T, that is, the nearest point in $\mathrm{Fix}(T)$ to x. Halpern [20] pointed out that conditions (c1) $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$ and (c2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary if the Halpern iteration scheme converges in norm. In view of (c2), the Halpern iteration may not be a fast iteration. Recently, a number of researchers investigated the problem of removing (c2) with the aid of projections; see, e.g., [21–24] and the references therein. In 2000, Moudafi [25] further proposed the viscosity approximation iteration, which reads as follows:

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Sx_n, \quad (6)$$

where S is a contraction. This approximation method, which improves the property of the class of nonexpansive mappings, is popular from the viewpoint of variational inequalities. Indeed, the fixed point also solves a monotone variational inequality with S. Another popular regularized method is the hybrid projection method, which was considered by Nakajo and Takahashi [18] for fixed points of nonexpansive mappings first. Indeed, they studied the following algorithm:

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)Tx_n + \alpha_n x_n, \\ Q_n = \{x \in C\colon \langle x_n - x, x_n - x_0\rangle \le 0\}, \\ C_n = \{x \in C\colon \|x - y_n\| \le \|x - x_n\|\}, \\ x_{n+1} = \mathrm{Proj}_{Q_n \cap C_n} x_0, \end{cases} \quad (7)$$

where C is a closed, convex, and nonempty subset of H and $\mathrm{Proj}_{Q_n\cap C_n}$ is the nearest point projection onto the intersection set. They obtained a strong convergence theorem for nonexpansive mappings in a real Hilbert spaces without compact assumption on T. For more general nonlinear mappings though the projection-based method, we refer to [26–30] and the references therein.

Let C be a convex and closed subset of a real Hilbert space H. From now on, Proj_C is borrowed to denote the nearest projection onto subset C, i.e., $\mathrm{Proj}_C(x) \coloneqq \arg\min\{\|x-y\|, y \in C\}$. Let A be a nonlinear mapping on H. Recall that A is said to be

(1) *Strongly monotone* iff there exists a positive constant ξ such that $\langle Ax - Ay, x - y\rangle \ge \xi\|x-y\|^2$, $\forall x, y \in H$

(2) *Monotone* iff $\langle Ax - Ay, x - y\rangle \ge 0$, $\forall x, y \in H$

(3) *Cocoercive* iff there exists a positive constant $\forall x, y \in H$ such that $\langle Ax - Ay, x - y\rangle \ge \xi\|Ax - Ay\|^2$, $\forall x, y \in H$

Let $B\colon H \rightrightarrows H$ be a multivalued nonlinear mapping. Next, we turn our attention to the class of multivalued mappings. B is said to be a monotone mapping if and only if for all $x, y \in H$, $f \in By$, and $e \in Bx \Rightarrow \langle e - f, x - x\rangle > 0$. The symbol $B^{-1}(0)$ is used to stand for the set of zero points of B. Mapping B is said to be a maximally monotone mapping iff the graph of B, $\mathrm{Graph}(B)$, is not contained in the graph of any other monotone mapping properly. Let $J_\beta^B = (I\ d + \beta B)^{-1}$, where $I\ d$ is the identity mapping and β is a constant. This operator is called the resolvent of B. Its domain is denoted by $\mathrm{Dom}(B)$ in this paper. It is clear $B^{-1}(0) = \mathrm{Fix}(J_\beta^B)$.

Consider the following variational inclusion problem, which finds a point $x \in C$ such that $x \in (B+A)^{-1}(0)$, where B is a multivalued maximally monotone mapping and A is a ξ-cocoercive mapping. For the inclusion problem, splitting methods (FB, PR, and DR) are popular for zero points of the sum of the monotone mappings. Splitting methods were considered by many authors for image recovery, signal processing, and machine learning. The FB-type splitting method means an iterative method for which each iteration involves only with the individual operators not the sum. In this paper, with the condition that the solution set is nonempty, we consider finding a $\theta \in C$ such that $\theta \in F(T) \cap (B+A)^{-1}(0)$, where T is a nonexpansive mapping with a nonempty fixed point set, B is a multivalued maximally monotone mapping, and A is a ξ-cocoercive mapping. We establish a strong convergence with the aid of hybrid projection and FB splitting in a Hilbert space. Our strong convergence theorem requires less restriction on parameter sequences and the operators.

To show our main findings, we also need the following necessary tools.

The nearest point projection operator Proj_C has the following property:

$$\|\mathrm{Proj}_C y - \mathrm{Proj}_C x\|^2 \le \langle y - x, \mathrm{Proj}_C y - \mathrm{Proj}_C(x)\rangle, \quad \forall x, y \in H. \quad (8)$$

Lemma 1 (see [31]). *Let H be a Hilbert space, and let C be a convex, closed, and nonempty subset of H. Let T be a nonexpansive mapping on C. Then, $Fix(T)$ is convex and closed.*

Remark 1. Let H be a Hilbert space, and let C be a convex, closed, and nonempty subset of H. Let $A\colon C \longrightarrow H$ be a ξ-cocoercive mapping, and let $B\colon H \rightrightarrows H$ be a multivalued maximally monotone operator. Then, $\mathrm{Fix}(J_\beta^B(I\ d - \beta A)) = (B+A)^{-1}(0)$, where β is some constant and $I\ d$ is the identity mapping. Besides, the resolvent is firmly nonexpansive. From Lemma 1, we have that $(B+A)^{-1}(0)$ is convex and closed.

Lemma 2 (see [31]). *Let H be a Hilbert space, and let C be a convex, closed, and nonempty subset of H. Let T be a nonexpansive mapping on C. Then, $I\ d - T$ is demiclosed (let $\{x_n\}$ be a sequence weakly converging to x, and let $Tx_n - x_n \longrightarrow \infty$ be $n \longrightarrow \infty$. Then, x is a fixed point of T).*

2. Main Results

Theorem 1. *Assume that H is a Hilbert space and C is a convex and closed subset in space H. Assume that A is a single-valued ξ-cocoercive mapping from set C to space H and B is a set-valued maximally monotone mapping from H to H. Assume that T is a nonexpansive mapping from C to C, and $CSS(B, A, T) = (B+A)^{-1}(0) \cap Fix(T)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are positive real number sequences. Let $\{x_n\}$ be a sequence in set C generated in the following iterative process:*

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)Tx_n + \alpha_n J_{\beta_n}^B (x_n - \beta_n A x_n), \\ Q_n = \{x \in C: \langle x_n - x, x_n - x_0 \rangle \leq 0\}, \\ C_n = \{x \in C: \|x - y_n\| \leq \|x - x_n\|\}, \\ x_{n+1} = \mathrm{Proj}_{Q_n \cap C_n} x_0, \end{cases} \quad (9)$$

where $J_{\beta_n}^B$ is the resolvent mapping $(Id + \beta_n B)^{-1}$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (i) $1 > \alpha_n \geq \alpha > 0$ with α being a fixed real number and (ii) $0 < \beta \leq \beta_n \leq \beta' < 2\xi$ with β and β' being two fixed real numbers. Then, the sequence $\{x_n\}$ converges strongly to $\mathrm{Proj}_{CSS(A,B,T)} x_0$.

Proof. From Lemma 1, we have that $\mathrm{Fix}(T)$ is convex and closed. From Remark 1, we have that $(B+A)^{-1}(0)$ is convex and closed. Hence, $CSS(B,A,T)$ is convex and closed. This shows that the metric (nearest point) projection onto the set is well-defined.

Note that $\|x - y_n\|^2 \leq \|x - x_n\|^2$ is equivalent to $2\langle x, x_n - y_n\rangle \leq \|x_n\|^2 - \|y_n\|^2$. Let x and x' be the points in C_n. Then,

$$\begin{aligned} 2r\langle x, x_n - y_n\rangle &\leq r\left(\|x_n\|^2 - \|y_n\|^2\right), \\ 2(1-r)\langle x', x_n - y_n\rangle &\leq (1-r)\left(\|x_n\|^2 - \|y_n\|^2\right), \end{aligned} \quad (10)$$

where r is a real number in $(0,1)$. Adding the two inequalities above, we have

$$2\langle rx + (1-r)x', x_n - y_n\rangle \leq \|x_n\|^2 - \|y_n\|^2, \quad (11)$$

that is,

$$\|rx + (1-r)x' - y_n\| \leq \|rx + (1-r)x' - x_n\|. \quad (12)$$

It shows that $rx + (1-r)x' \in C_n$. C_n is convex. The closedness of C_n is obvious. The definition of ξ-cocoercive mappings send us to the situation $\mathrm{Id} - \beta_n A$ is a nonexpansive mapping for each n. Indeed, for any $w, v \in C$,

$$\begin{aligned} &\|(Id - \beta_n A)w - (Id - \beta_n A)v\|^2 \\ &= \beta_n^2 \|Aw - Av\|^2 - 2\beta_n \langle Aw - Av, w - v\rangle + \|w - v\|^2 \\ &\leq \beta_n(\beta_n - 2\xi)\|Aw - Av\|^2 + \|w - v\|^2. \end{aligned} \quad (13)$$

This indicates $\mathrm{Id} - \beta_n A$ is a mapping of nonexpansive. Observe that $CSS(B,A,T) \subset C_n$. Indeed, from the nonexpansivity of the resolvent, we have

$$\begin{aligned} \|y_n - p\| &\leq (1-\alpha_n)\|Tx_n - p\| + \alpha_n \left\|J_{\beta_n}^B (x_n - \beta_n A x_n) - p\right\| \\ &= (1-\alpha_n)\|Tx_n - Tp\| + \alpha_n \left\|J_{\beta_n}^B (x_n - \beta_n A x_n) - J_{\beta_n}^B (p - \beta_n A p)\right\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|(\mathrm{Id} - \beta_n A)x_n - (\mathrm{Id} - \beta_n A)p\| \\ &\leq \|x_n - p\|, \quad \forall p \in CSS(A,B,T). \end{aligned} \quad (14)$$

So, we complete the proof $CSS(A,B,T) \subset C_n$.

On the contrary, it is obvious that Q_n is convex and closed. Next, one shows that $CSS(B,A,T) \subset Q_n \cap C_n$. Borrowing $C_0 = C$, we have $CSS(B,A,T) \subset Q_0 \cap C_0$. Let x_m be a given vector, and $CSS(B,A,T) \subset Q_m \cap C_m$ for some positive integer m. There is a vector $x_{m+1} \in Q_m \cap C_m$ with $x_{m+1} = \mathrm{Proj}_{Q_m \cap C_m} x_0$. There holds $\langle x_0 - x_{m+1}, x_{m+1} - j\rangle \geq 0$ for all $j \in Q_m \cap C_m$. Borrowing $CSS(B,A,T) \subset Q_m \cap C_m$, we get $CSS(B,A,T) \subset Q_{m+1}$. Thus, $CSS(B,A,T) \subset Q_{m+1} \cap C_{m+1}$. Hence, $CSS(B,A,T) \subset Q_n \cap C_n$ for all n.

One next observes that x_n is a bounded sequence. As we have showed that $CSS(B,A,T)$ is convex and closed set in C, a unique vector $\mu \in CSS(A,B,T)$ with $\mu = \mathrm{Proj}_{CSS(A,B,T)} x_0$ is guaranteed. We have the construction of x_{n+1}, that is, $\mathrm{Proj}_{Q_n \cap C_n} x_0 = x_{n+1}$. So,

$$\|x_0 - x_{n+1}\| \leq \|x_0 - v\|, \quad (15)$$

for each $v \in Q_n \cap C_n$. By $\mu \in CSS(A,B,T) \subset Q_n \cap C_n$, we obtain

$$\|x_0 - x_{n+1}\| \leq \|x_0 - \mu\|, \quad (16)$$

that infers x_n is a bounded sequence. Our next step shows $\|x_{n+1} - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Because $x_n = \mathrm{Proj}_{Q_n} x_0$ and $x_{n+1} \in Q_n \cap C_n \subset Q_n$, one infers that

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|. \quad (17)$$

Borrowing the conclusion (x_n is a bounded sequence), one infers that the limit of $\{\|x_0 - x_n\|\}$ exists. We may suppose that $\lim_{n\longrightarrow\infty} \|x_0 - x_n\| = d > 0$. Observe

$$\begin{aligned} &\|x_{n+1} - x_0\|^2 - \|x_0 - x_n\|^2 \\ &\geq \|x_{n+1} - x_0\|^2 - \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle \\ &= \|x_n - x_{n+1}\|^2 \geq 0, \end{aligned} \quad (18)$$

thanks to $\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0$ ($x_{n+1} \in Q_n$ and the property of the metric projection). By the limit of the limit of $\{\|x_0 - x_n\|\}$, one infers $\lim_{n\longrightarrow\infty} \|x_{n+1} - x_n\|^2 = 0$.

Note that x_{n+1} is in C_n. So,

$$\|x_{n+1} - y_n\| \leq \|x_{n+1} - x_n\|. \quad (19)$$

That indicates that $x_{n+1} - y_n \longrightarrow 0$ as $n \longrightarrow \infty$. Furthermore, $x_n - y_n \longrightarrow 0$ as $n \longrightarrow \infty$. Let $z_n = J^B_{\beta_n}(x_n - \beta_n A x_n)$. For any $p \in \mathrm{CSS}(B, A, T)$, ξ-cocoercive and resolvent operators send us to

$$\begin{aligned}
&\|p - z_n\|^2 \\
&\leq \left\| J^B_{\beta_n}(p - \beta_n A p) - J^B_{\beta_n}(x_n - \beta_n A x_n)\right\|^2 \\
&\leq \|(p - \beta_n A p) - (x_n - \beta_n A x_n)\|^2 \\
&\leq \|p - x_n\|^2 - (2\xi - \beta_n)\beta_n \|Ap - Ax_n\|^2.
\end{aligned} \tag{20}$$

So,

$$\begin{aligned}
&\|p - y_n\|^2 \\
&\leq (1 - \alpha_n)\|Tx_n - p\|^2 + \alpha_n \|z_n - p\|^2 \\
&\leq (1 - \alpha_n)\|Tx_n - Tp\|^2 + \alpha_n\left(\|p - x_n\|^2 - (2\xi - \beta_n)\beta_n\|Ap - Ax_n\|^2\right) \\
&\leq \|p - x_n\|^2 - \alpha_n(2\xi - \beta_n)\beta_n\|Ap - Ax_n\|^2.
\end{aligned} \tag{21}$$

That is,

$$\alpha_n(2\xi - \beta_n)\beta_n\|Ap - Ax_n\|^2 \leq \|y_n - x_n\|\left(\|p - x_n\| + \|p - y_n\|\right). \tag{22}$$

By the fact that $\|y_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we have $Ax_n - Ap \longrightarrow 0$ as $n \longrightarrow \infty$. By the firm nonexpansivitity of the resolvent operator, we also have

$$\begin{aligned}
&\|p - z_n^2\| \\
&\leq \langle p - z_n, (p - \beta_n A p) - (x_n - \beta_n A x_n)\rangle \\
&= \frac{1}{2}\left(\|p - x_n\|^2 + \|p - z_n\|^2 - \|x_n - z_n - \beta_n(Ap - Ax_n)\|^2\right) \\
&\leq \frac{1}{2}\left(\|p - x_n\|^2 + \|p - z_n\|^2 - \|x_n - z_n\|^2 - \beta_n^2\|Ap - Ax_n\|^2 + 2\beta_n\|x_n - z_n\|\|Ap - Ax_n\|\right) \\
&\leq \frac{1}{2}\left(\|p - x_n\|^2 + \|p - z_n\|^2 - \|x_n - z_n\|^2 + 2\beta_n\|x_n - z_n\|\|Ap - Ax_n\|\right),
\end{aligned} \tag{23}$$

which holds that

$$\begin{aligned}
\|p - z_n\|^2 &\leq \|p - x_n\|^2 - \|x_n - z_n\|^2 + 2\beta_n\|x_n - z_n\|\|Ap - Ax_n\|, \\
\|p - y_n\|^2 &\leq (1 - \alpha_n)\|Tx_n - Tp\|^2 + \alpha_n\left\|J^B_{\beta_n}(x_n - \beta_n A x_n) - p\right\|^2 \\
&\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|z_n - p\|^2 \\
&\leq \|x_n - p\|^2 - \alpha_n\|x_n - z_n\|^2 + 2\beta_n\alpha_n\|x_n - z_n\|\|Ap - Ax_n\|.
\end{aligned} \tag{24}$$

So, $\alpha_n\|x_n - z_n\|^2 \le \|x_n - y_n\|C + 2\beta_n\alpha_n\|x_n - z_n\|\|Ap - Ax_n\|$, where C is some constant. By the requirement on the control parameter and the result that $Ap - Ax_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $x_n - z_n \longrightarrow \infty$ as $n \longrightarrow \infty$. With a simple calculation, we have $x_n - Tx_n \longrightarrow \infty$ as $n \longrightarrow \infty$. We have the fact that $(\mathrm{Id} - \beta_n A)x_n \in (\mathrm{Id} + \beta_n B)z_n$. It holds

$$\frac{x_n - z_n}{\beta_n} - Ax_n \in Bz_n. \tag{25}$$

By the assumption that B is maximally monotone,

$$\langle \frac{x_n - z_n}{\beta_n} - Ax_n - u, z_n - v\rangle \ge 0, \tag{26}$$

for any $u \in Bv$. By the result that $\{x_n\}$ is a bounded sequence, there is a subsequence $\{x_{n_m}\}$ converges to θ weakly. The ξ-cocoercive mappings yield $Ax_{n_m} \longrightarrow A\theta$. It holds $\langle -A\theta - u, \theta - v\rangle \ge 0$. It shows $0 \in (B + A)(\theta)$. Note that $\mathrm{Id} - T$ is demiclosed (Lemma 2). One asserts $\theta \in \mathrm{Fix}(T)$. One next shows that $\theta = \mathrm{Proj}_{\mathrm{CSS}(B,A,T)}x_0$ and x_n converges to it strongly. Set $\overline{x} = \mathrm{Proj}_{\mathrm{CSS}(B,A,T)}x_0$. Since the functional $\|\cdot\|$ is weakly lower semicontinuous, one has

$$\|\overline{x} - x_0\| \le \|\theta - x_0\| \le \liminf_{m\longrightarrow\infty}\|x_0 - x_{n_m}\| \le \limsup_{m\longrightarrow\infty}\|x_0 - x_{n_m}\| \le \|\overline{x} - x_0\|. \tag{27}$$

One gets $\|\overline{x} - x_0\| = \lim_{m\longrightarrow\infty}\|x_0 - x_{n_m}\| = \|\theta - x_0\|$. Since the framework is a Hilbert space, one gets $x_n \longrightarrow \theta$ as $n \longrightarrow \infty$. This finishes this theorem. □

Let

$$\partial f(x) = \{z \in H\colon f(x) + \langle y - x, z\rangle \le f(y), \forall y \in H\}, \quad \forall, x \in H, \tag{28}$$

where $f\colon H \longrightarrow (-\infty, \infty]$ is a proper, convex, and lower semicontinuous function. Rockfellar [32] proved that ∂f is a multivalued maximally monotone operator. Let C be a closed, convex, and nonempty subset of H and i_C be the indicator function of C, that is,

$$i_C x = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases} \tag{29}$$

Furthermore, we define the normal cone $N_C(v)$ of C at v as follows:

$$N_C v = \{z \in H\colon \langle z, y - v\rangle \le 0, \forall y \in H\}, \tag{30}$$

for any $v \in C$. Then, $i_C\colon H \longrightarrow (-\infty, \infty]$ is proper, convex, and lower semicontinuous on H. ∂i_C is a maximally monotone operator. Let $\mathrm{Res}_\lambda x = (\mathrm{Id} + \lambda\, \partial i_C)^{-1}x$. So, $\partial i_C x = N_C x$ and $x \in C$; we obtain

$$v = J_\lambda^{\partial i_C} x \Longleftrightarrow v = Proj_C x, \tag{31}$$

where $Proj_C^{\partial i_C}$ is the metric projection onto C. This yields $x \in (A + \partial i_C)^{-1}(0) \Longleftrightarrow x \in \mathrm{VI}(A, C)$, where $\mathrm{VI}(A, C)$ denotes the classical variational inequality, that is, find a point $x \in C$ such that $\langle Ax, y - x\rangle \ge 0$ for all $y \in C$.

Corollary 1. *Assume that H is a Hilbert space and C is a convex and closed subset in space H. Assume that A is a single-valued ξ-cocoercive mapping from set C to space H. Assume that T is a nonexpansive mapping from C to C and $VI(A, C) \cap Fix(T)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are positive real number sequences. Let $\{x_n\}$ be a sequence in set C generated in the following iterative process:*

$$\begin{cases} x_0 \in C, \\ y_n = (1 - \alpha_n)Tx_n + \alpha_n \mathrm{Proj}_C(x_n - \beta_n Ax_n), \\ Q_n = \{x \in C\colon \langle x_n - x, x_n - x_0\rangle \le 0\}, \\ C_n = \{x \in C\colon \|x - y_n\| \le \|x - x_n\|\}, \\ x_{n+1} = \mathrm{Proj}_{Q_n \cap C_n} x_0. \end{cases} \tag{32}$$

Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (i) $1 > \alpha_n \ge \alpha > 0$ with α being a fixed real number and (ii) $0 < \beta \le \beta_n \le \beta' < 2\xi$ with β and β' being two fixed real numbers. Then, the sequence $\{x_n\}$ converges strongly to $\mathrm{Proj}_{\mathrm{VI}(A,C)\cap\mathrm{Fix}(T)}x_0$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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Research Article

An Iterative Algorithm for Solving Fixed Point Problems and Quasimonotone Variational Inequalities

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In this paper, we survey a common problem of the fixed point problem and the quasimonotone variational inequality problem in Hilbert spaces. We suggest an iterative algorithm for finding a common element of the solution of a quasimonotone variational inequality and the fixed point of a pseudocontractive operator. Convergence theorems are shown under some mild conditions. Several corollaries are also obtained.

1. Introduction

Let H be a real Hilbert space with an inner product $\langle\cdot,\cdot\rangle$ and an induced norm $\|\cdot\|$. Let C be a nonempty closed and convex subset of H. Let $f\colon C\longrightarrow H$ be a nonlinear operator. In this paper, our work is closely related to a classical variational inequality of finding a point $x^{\dagger}\in C$ such that

$$\left\langle f\left(x^{\dagger}\right), x-x^{\dagger}\right\rangle \geq 0,\quad \forall x\in C. \tag{1}$$

We use $\mathrm{Sol}(C,f)$ to denote the solution set of (1).

It is well known that variational inequality problems provide a general mathematical framework for a large number of problems arising in optimization [1–8]. For example, constrained optimization problems such as LP and NLP are special cases of variational inequalities, and systems of equations and complementarity problems can be cast as variational inequalities. Thus, variational inequality problems have many applications, including those in transportation networks [9], signal processing [10, 11], regression analysis [12], equilibrium problems [13, 14], fixed point problems [15–19], and complementarity problems [1, 20]. There are numerous iterative algorithms for solving variational inequalities and related problems, (see for examples [21–31]).

Let $\varphi\colon C\longrightarrow\mathbb{R}$ be a convex function. Letting $f(x)=\nabla\varphi(x)$, the variational inequality (1) is equivalent to the following minimization problem:

$$\min_{x\in C}\varphi(x), \tag{2}$$

which implies that we can use the following projection-gradient algorithm [32–35] to solve variational inequality (1), i.e., an iterative sequence $\{u_n\}$ generated by the recursive form:

$$u_{n+1}=\mathrm{proj}_C\left[u_n-\varsigma_n f\left(u_n\right)\right], \tag{3}$$

where $\varsigma_n>0$ is the step size, and $\mathrm{proj}_C\colon H\longrightarrow C$ is the metric projection.

The sequence $\{u_n\}$ generated by the projection-gradient algorithm is the convergent provided. f is strongly (pseudo) monotone (see [25, 36]), or f is inverse strongly monotone (see [10, 35]). However, if f is plain monotone, then the sequence $\{u_n\}$ generated by (3) does not necessarily converge. To overcome this flaw, many iterative methods have been proposed, such as the proximal point method [37, 38], Korpelevich's extragradient method [39–41] and its variant forms [42–44], the subgradient extragradient method [45, 46], and Tseng's method [47]. Especially, Bot et al. [48] suggested the following Tseng-type forward-backward-forward algorithm:

$$\begin{cases} v_n = P_C(u_n - \lambda f(u_n)), \\ u_{n+1} = \mu_k(v_n + \lambda(f(u_n) - f(v_n)) + (1-\mu_k)u_n, \quad \forall n \geq 0. \end{cases} \tag{4}$$

Bot et al. [48] proved that the sequence $\{u_n\}$ generated by (4) converges weakly to an element in Sol (C, f) provided f is pseudomonotone and sequentially weakly continuous.

Let $\mathrm{Sol}^d(C, f)$ be the solution set of the dual variational inequality of (1), that is,

$$\mathrm{Sol}^d(C, f) \coloneqq \{u \in C | \langle f(x), x - u\rangle \geq 0, \quad \forall x \in C\}. \tag{5}$$

where $\mathrm{Sol}^d(C, f)$ is the closed convex. If C is convex and f is continuous, then $\mathrm{Sol}^d(C, f) \subset \mathrm{Sol}(C, f)$.

To show the convergence of the sequence $\{u_n\}$, a common condition $\mathrm{Sol}(C, f) \subset \mathrm{Sol}^d(C, f)$ has been added, that is,

$$\langle f(x), x - u\rangle \geq 0, \quad \forall u \in \mathrm{Sol}(C, f) \text{ and } x \in C, \tag{6}$$

which is a direct consequence of the pseudomonotonicity of f. But this conclusion (that is, $\mathrm{Sol}(C, f) \subset \mathrm{Sol}^d(C, f)$) is false, if f is quasimonotone.

The main purpose of this paper is to introduce a self-adaptive forward-backward-forward algorithm to solve quasimonotone variational inequalities (1) and the fixed point problem of pseudocontractive operators. The algorithm is designed such that the step-sizes are dynamically chosen and its convergence is guaranteed without prior knowledge of the Lipschitz constant of f. We prove that the proposed algorithm converges weakly to a common element of the solution of a quasimonotone variational inequality and the fixed point of a pseudocontractive operator under some additional conditions.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T: C \longrightarrow C$ be a nonlinear operator. $\mathrm{Fix}(T)$ is used to denote the set of fixed points of T, i.e., $\mathrm{Fix}(T)\colon = \{x \in C | x = Tx\}$. " $\rightharpoonup$ " and " $\longrightarrow$ " is used to denote weak convergence and strong convergence, respectively. Let $\{u_n\}$ be a sequence in H. $\omega_w(u_n)$ is used to denote the set of all weak cluster points of $\{u_n\}$, i.e., $\omega_w(u_n) = \{u^\dagger \colon \exists \{u_{n_i}\} \subset \{u_n\} \text{ such that } u_{n_i} \rightharpoonup u^\dagger \ (i \longrightarrow \infty)\}$.

Let $f: C \longrightarrow H$ be a nonlinear operator. We recall that f is said to be

(i) pseudomonotone if

$$\langle f(x^\dagger), x - x^\dagger\rangle \geq 0 \text{ implies } \langle f(x), x - x^\dagger\rangle \geq 0, \quad \forall x, x^\dagger \in C \tag{7}$$

(ii) quasimonotone if

$$\langle f(x^\dagger), x - x^\dagger\rangle > 0 \text{ implies} \langle f(x), x - x^\dagger\rangle \geq 0, \quad \forall x, x^\dagger \in C \tag{8}$$

(iii) L-Lipschitz continuous if there exists some constant $L > 0$ such that

$$\|f(x) - f(x^\dagger)\| \leq L\|x - x^\dagger\|, \text{for all } x, x^\dagger \in C \tag{9}$$

(iv) sequently weakly continuous if $u_n \rightharpoonup \tilde{x}$ implies that $f(u_n) \rightharpoonup f(\tilde{x})$.

We recall that an operator $T: C \longrightarrow C$ is said to be pseudocontractive if

$$\|T(x) - T(x^\dagger)\|^2 \leq \|x - x^\dagger\|^2 + \|(I-T)x - (I-T)x^\dagger\|^2, \tag{10}$$

for all $x, x^\dagger \in C$.

For fixed $x \in H$, there exists a unique $x^\dagger \in C$ satisfying $\|x - x^\dagger\| = \inf\{\|x - \tilde{x}\| \colon \tilde{x} \in C\}$. $x^\dagger$ is denoted by $\mathrm{proj}_C[x]$. The projection proj_C has the following basic property: for given $x \in H$,

$$\langle x - \mathrm{proj}_C[x], y - \mathrm{proj}_C[x]\rangle \leq 0, \quad \forall y \in C. \tag{11}$$

Applying this characteristic inequality, we have the following equivalence relation:

$$x^\dagger \in \mathrm{Sol}(f, C) \Leftrightarrow x^\dagger = \mathrm{proj}_C[x^\dagger - \varsigma f(x^\dagger)], \quad \forall \varsigma > 0. \tag{12}$$

In a Hilbert space H, we have

$$\|\zeta u + (1-\zeta)u^\dagger\|^2 = \zeta\|u\|^2 + (1-\zeta)\|u^\dagger\|^2 - \zeta(1-\zeta)\|u - u^\dagger\|^2, \tag{13}$$

$\forall u, u^\dagger \in H$ and $\forall \zeta \in [0, 1]$.

Lemma 1 (see [44]). *Let C be a nonempty, convex, and closed subset of a Hilbert space H. We assume that $T: C \longrightarrow C$ is an L-Lipschitz pseudocontractive operator. Then, for all $\tilde{u} \in C$ and $u^\dagger \in Fix(T)$, we have*

$$\|u^\dagger - T[(1-\varpi)\tilde{u} + \varpi T(\tilde{u})]\|^2 \leq \|\tilde{u} - u^\dagger\|^2 + (1-\varpi)\|\tilde{u} - T[(1-\varpi)\tilde{u} + \varpi T(\tilde{u})]\|^2, \tag{14}$$

where $0 < \varpi < 1/\sqrt{1 + L^2} + 1$.

Lemma 2 (see [14]). *Let C be a nonempty, convex, and closed subset of a Hilbert space H. Let $T: C \longrightarrow C$ be a continuous pseudocontractive operator. Then,*

(i) $Fix(T) \subset C$ is closed and convex

(ii) T is a demiclosedness, i.e., $u_n \rightharpoonup \tilde{z}$ and $T(u_n) \longrightarrow z^\dagger$ imply that $T(\tilde{z}) = z^\dagger$

3. Main Results

In this section, we introduce our main results. Let C be a nonempty closed convex subset of a real Hilbert space H. We assume that the following conditions are satisfied:

(C1): the operator $f: H \longrightarrow H$ is quasimonotone; κ-Lipschitz continuous and satisfies the following property (P):

$$\left.\begin{array}{l} H \ni x_n \rightharpoonup x^\dagger \in H \text{ as } n \longrightarrow \infty \\ \liminf_{n\longrightarrow+\infty}\|f(x_n)\| = 0 \end{array}\right\} \text{imply that } f\left(x^\dagger\right) = 0 \tag{15}$$

(C2): the operator $T: H \longrightarrow H$ is pseudocontractive and L-Lipschitz continuous

(C3): $\Gamma: = \mathrm{Sol}^d(C, f) \cap \mathrm{Fix}(T) \neq \varnothing$ and $\{x \in C: f(x) = 0\} \setminus \mathrm{Sol}^d(C, f)$ is a finite set

Remark 1. If the operator f is sequentially weakly continuous, then f satisfies the property (P).

Next, we present an iterative algorithm for finding a common point in Γ. Let $\{\zeta_n\}$, $\{\alpha_n\}$, and $\{\varpi_n\}$ be three sequences in $(0,1)$. Let $\beta \in (0,1)$ and $\varsigma_0 > 0$ be two constants.

Algorithm 1. Initialization: let $u_0 \in H$ be an initial guess. We set $n = 0$.

Step 1. Let the n-th iterate u_n be given. We compute

$$\left\{\begin{array}{l} \hat{v}_n = (1-\varpi_n)u_n + \varpi_n T(u_n) \\ v_n = (1-\alpha_n)u_n + \alpha_n T(\hat{v}_n) \end{array}\right. . \tag{16}$$

Step 2. Let the n-th step size ς_n be known. We compute

$$w_n = \mathrm{proj}_C[v_n - \varsigma_n f(v_n)], \tag{17}$$

and

$$u_{n+1} = (1-\zeta_n)v_n + \zeta_n w_n + \zeta_n\varsigma_n[f(v_n) - f(w_n)]. \tag{18}$$

Step 3. We update the $n+1$-th step size by the following form:

$$\varsigma_{n+1} = \left\{\begin{array}{ll} \min\left\{\varsigma_n, \dfrac{\beta\|w_n - v_n\|}{\|f(w_n) - f(v_n)\|}\right\}, & \text{if } f(w_n) \neq f(v_n), \\ \varsigma_n, & \textit{else}. \end{array}\right. \tag{19}$$

We set $n: = n+1$ and return to step 1.

Based on Algorithm 1, we have the following remark.

Remark 2. (i) By (17), if at some step $w_n = v_n = \mathrm{proj}_C[v_n - \varsigma_n f(v_n)]$, then $v_n \in \mathrm{Sol}(C, f)$. (ii) By (19), $\varsigma_{n+1} \leq \varsigma_n$ and $\varsigma_n \geq \min\{\varsigma_0, \beta/\kappa\}$ for all n, so $\lim_{n\longrightarrow\infty}\varsigma_n = \varsigma^\dagger$ exists, and $\varsigma^\dagger \geq \min\{\varsigma_0, \beta/\kappa\} > 0$.

Next, we prove the convergence of Algorithm 1.

Theorem 1. *Suppose that* $0 < \underline{\alpha} < \alpha_n < \overline{\alpha} < \varpi_n < \overline{\varpi} < 1/\sqrt{1+L^2} + 1\,(\forall n \geq 0)$ *and* $0 < \liminf_{n\longrightarrow\infty}\zeta_n \leq \limsup_{n\longrightarrow\infty}\zeta_n < 1$. *Then, the sequence* $\{u_n\}$ *generated by Algorithm 1 converges weakly to some point in* Γ.

Proof. Let $\tilde{x} \in \Gamma$. Since $\tilde{x} \in \mathrm{Sol}^d(C, f) \subset C$, from (11) and (17), we have

$$\langle w_n - v_n + \varsigma_n f(v_n), w_n - \tilde{x}\rangle \leq 0, \tag{20}$$

which yields that

$$\langle w_n - v_n, w_n - \tilde{x}\rangle \leq \varsigma_n\langle f(v_n), \tilde{x} - w_n\rangle. \tag{21}$$

Noting that $w_n \in C$ and $\tilde{x} \in \mathrm{Sol}^d(C, f)$, we have

$$\langle f(w_n), \tilde{x} - w_n\rangle \leq 0. \tag{22}$$

Combining (21) and (23), we obtain

$$\langle w_n - v_n, w_n - \tilde{x}\rangle + \varsigma_n\langle f(v_n) - f(w_n), w_n - \tilde{x}\rangle \leq 0. \tag{23}$$

In Hilbert space H, we have $\langle x-y, x-z\rangle = 1/2(\|x-y\|^2 + \|x-z\|^2 - \|y-z\|^2)$ for all $x, y, z \in H$. Setting $x = w_n$, $y = v_n$, and $z = \tilde{x}$, we deduce $\langle w_n - v_n, w_n - \tilde{x}\rangle = 1/2(\|w_n - v_n\|^2 + \|w_n - \tilde{x}\|^2 - \|v_n - \tilde{x}\|^2)$. This together with (1) implies that

$$\begin{aligned} &\frac{1}{2}\left(\|w_n - v_n\|^2 + \|w_n - \tilde{x}\|^2 - \|v_n - \tilde{x}\|^2\right) \\ &\quad + \varsigma_n\langle f(v_n) - f(w_n), w_n - \tilde{x}\rangle \leq 0, \end{aligned} \tag{24}$$

and it follows that

$$\begin{aligned} \|w_n - \tilde{x}\|^2 \leq{}& \|v_n - \tilde{x}\|^2 - 2\varsigma_n\langle f(v_n) - f(w_n), w_n - \tilde{x}\rangle \\ & - \|w_n - v_n\|^2. \end{aligned} \tag{25}$$

Based on (18), we have

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &= \|(1-\zeta_n)(v_n-\tilde{x})+\zeta_n(w_n-\tilde{x})+\zeta_n\varsigma_n[f(v_n)-f(w_n)]\|^2\\ &= \|(1-\zeta_n)(v_n-\tilde{x})+\zeta_n(w_n-\tilde{x})\|^2+\zeta_n^2\varsigma_n^2\|f(v_n)-f(w_n)\|^2\\ &\quad+2\zeta_n(1-\zeta_n)\varsigma_n\langle v_n-\tilde{x}, f(v_n)-f(w_n)\rangle\\ &\quad+2\zeta_n^2\varsigma_n\langle w_n-\tilde{x}, f(v_n)-f(w_n)\rangle.\end{aligned} \tag{26}$$

Using (13) and from (26), we deduce

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &= (1-\zeta_n)\|v_n-\tilde{x}\|+\zeta_n\|w_n-\tilde{x}\|^2-\zeta_n(1-\zeta_n)\|v_n-w_n\|^2\\ &\quad+\zeta_n^2\varsigma_n^2\|f(v_n)-f(w_n)\|^2+2\zeta_n^2\varsigma_n\langle w_n-\tilde{x}, f(v_n)-f(w_n)\rangle\\ &\quad+2\zeta_n(1-\zeta_n)\varsigma_n\langle v_n-\tilde{x}, f(v_n)-f(w_n)\rangle.\end{aligned} \tag{27}$$

According to (25) and (27), we obtain

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &\le \|v_n-\tilde{x}\|-\zeta_n(2-\zeta_n)\|v_n-w_n\|^2+\zeta_n^2\varsigma_n^2\|f(v_n)-f(w_n)\|^2\\ &\quad+2\zeta_n(1-\zeta_n)\varsigma_n\langle v_n-w_n, f(v_n)-f(w_n)\rangle\\ &\le \|v_n-\tilde{x}\|-\zeta_n(2-\zeta_n)\|v_n-w_n\|^2+\zeta_n^2\varsigma_n^2\|f(v_n)-f(w_n)\|^2\\ &\quad+2\zeta_n(1-\zeta_n)\varsigma_n\|v_n-w_n\|\|f(v_n)-f(w_n)\|.\end{aligned} \tag{28}$$

Thanks to (19), $\|f(w_n)-f(v_n)\|\le\beta\|w_n-v_n\|/\varsigma_{n+1}$. This together with (28) implies that

$$\begin{aligned}\|u_{n+1}-\tilde{x}\|^2 &\le \|v_n-\tilde{x}\|-\zeta_n(2-\zeta_n)\|v_n-w_n\|^2+\zeta_n^2\beta^2\frac{\varsigma_n^2}{\varsigma_{n+1}^2}\|w_n-v_n\|^2\\ &\quad+2\zeta_n(1-\zeta_n)\beta\frac{\varsigma_n}{\varsigma_{n+1}}\|v_n-w_n\|^2\\ &= \|v_n-\tilde{x}\|^2-\zeta_n\left[2-\zeta_n-\zeta_n\beta^2\frac{\varsigma_n^2}{\varsigma_{n+1}^2}-2(1-\zeta_n)\beta\frac{\varsigma_n}{\varsigma_{n+1}}\right]\|v_n-w_n\|^2.\end{aligned} \tag{29}$$

It is noted that $0<\liminf_{n\longrightarrow\infty}\zeta_n\le\limsup_{n\longrightarrow\infty}\zeta_n<1$ and $\lim_{n\longrightarrow\infty}\varsigma_n/\varsigma_{n+1}=1$. Then, we have $\liminf_{n\longrightarrow\infty}\zeta_n[2-\zeta_n-\zeta_n\beta^2\varsigma_n^2/\varsigma_{n+1}^2-2(1-\zeta_n)\beta\varsigma_n/\varsigma_{n+1}]>0$. So, there exists a positive constant θ and a positive integer $\mathcal{N}$ such that when $n\ge\mathcal{N}$,\

$$\zeta_n\left[2-\zeta_n-\zeta_n\beta^2\frac{\varsigma_n^2}{\varsigma_{n+1}^2}-2(1-\zeta_n)\beta\frac{\varsigma_n}{\varsigma_{n+1}}\right]\ge\theta. \tag{30}$$

In combination with (29), we get

$$\|u_{n+1}-\tilde{x}\|^2\le\|v_n-\tilde{x}\|-\theta\|v_n-w_n\|^2, n\ge\mathcal{N}. \tag{31}$$

By (13) and (16), we obtain

$$\begin{aligned}\|v_n-\tilde{x}\|^2 &= \|(1-\alpha_n)(u_n-\tilde{x})+\alpha_n(T(\hat{v}_n)-\tilde{x})\|^2\\ &= (1-\alpha_n)\|u_n-\tilde{x}\|^2+\alpha_n\|T(\hat{v}_n)-\tilde{x}\|^2\\ &\quad-\alpha_n(1-\alpha_n)\|u_n-T(\hat{v}_n)\|^2.\end{aligned} \tag{32}$$

Using Lemma 1, we have

$$\begin{aligned}\|T(\hat{v}_n)-\tilde{x}\|^2 &= \|T[(1-\varpi_n)u_n+\varpi_nT(u_n)]-\tilde{x}\|^2\\ &\le \|u_n-\tilde{x}\|^2+(1-\varpi_n)\|u_n-T(\hat{v}_n)\|^2.\end{aligned} \tag{33}$$

Substituting (33) into (32), we get

$$\|v_n-\tilde{x}\|^2\le\|u_n-\tilde{x}\|^2+(\alpha_n-\varpi_n)\alpha_n\|u_n-T(\hat{v}_n)\|^2, \tag{34}$$

which results, together with (31), that

$$\|u_{n+1}-\tilde{x}\|^2 \le \|u_n-\tilde{x}\|^2 - (\varpi_n-\alpha_n)\alpha_n\|u_n - T(\hat{v}_n)\|^2 - \theta\|v_n-w_n\|^2, n \ge \mathcal{N}, \tag{35}$$

which implies that

$$(\varpi_n-\alpha_n)\alpha_n\|u_n - T(\hat{v}_n)\|^2 + \theta\|v_n-w_n\|^2 \le \|u_n-\tilde{x}\|^2 - \|u_{n+1}-\tilde{x}\|^2, n \ge \mathcal{N}. \tag{36}$$

By assumption, $\liminf_{n\longrightarrow\infty}(\varpi_n-\alpha_n)\alpha_n > 0$. From (35), we conclude that $\|u_{n+1}-\tilde{x}\| \le \|u_n-\tilde{x}\|, n \ge \mathcal{N}$. Therefore, $\lim_{n\longrightarrow\infty}\|u_n-\tilde{x}\|$ exists, and the sequence $\{u_n\}$ is bounded.

In combination with (36), we derive

$$\lim_{n\longrightarrow\infty}\|u_n - T(\hat{v}_n)\| = 0, \tag{37}$$

$$\lim_{n\longrightarrow\infty}\|v_n - w_n\| = 0. \tag{38}$$

By (16), $v_n - u_n = \alpha_n(T(\hat{v}_n) - u_n)$, it follows from (37) that

$$\lim_{n\longrightarrow\infty}\|v_n - u_n\| = 0. \tag{39}$$

From (38) and the Lipschitz continuity of f, we have

$$\lim_{n\longrightarrow\infty}\|f(v_n) - f(w_n)\| = 0. \tag{40}$$

According to the boundedness of the sequence $\{u_n\}$, we conclude that the sequence $\{v_n\}$ is bounded by (34) and the sequence $\{w_n\}$ is bounded because of $\|w_n\| \le \|v_n\| + \varsigma_n\|f(v_n)\|$ by (17).

Since T is L-Lipschitz continuous, we have

$$\begin{aligned}\|u_n - T(u_n)\| &\le \|u_n - T(\hat{v}_n)\| + \|T(\hat{v}_n) - T(u_n)\|\\ &\le \|u_n - T(\hat{v}_n)\| + L\varpi_n\|u_n - T(u_n)\|.\end{aligned} \tag{41}$$

It follows that

$$\|u_n - T(u_n)\| \le \frac{1}{1-L\varpi_n}\|u_n - T(\hat{v}_n)\|. \tag{42}$$

This together with (37) implies that

$$\lim_{n\longrightarrow\infty}\|u_n - T(u_n)\| = 0. \tag{43}$$

By virtue of (18), (38), and (40), we have

$$\lim_{n\longrightarrow\infty}\|u_{n+1} - v_n\| = 0. \tag{44}$$

Next, we show that $\omega_w(u_n) \subset \Gamma$. Selecting any $x^* \in \omega_w(u_n)$ and letting $\{u_{n_i}\}$ to be a subsequence of $\{u_n\}$ such that $u_{n_i} \rightharpoonup x^*$ as $i \longrightarrow \infty$, from (38) and (39), we have $v_{n_i} \rightharpoonup x^*$ and $w_{n_i} \rightharpoonup x^*$. Taking into account (43) and Lemma 2, we obtain that $x^* \in \mathrm{Fix}(T)$. Next, we show that $x^* \in \mathrm{Sol}(C, f)$. Based on (11) and $w_{n_i} = \mathrm{proj}_C[v_{n_i} - \varsigma_{n_i}f(v_{n_i})]$, we receive

$$\left\langle w_{n_i} - v_{n_i} + \varsigma_{n_i}f(v_{n_i}), w_{n_i} - x^\dagger\right\rangle \le 0, \quad \forall x^\dagger \in C, \tag{45}$$

which yields

$$\begin{aligned}&\frac{1}{\varsigma_{n_i}}\left\langle v_{n_i} - w_{n_i}, u - w_{n_i}\right\rangle + \left\langle f(v_{n_i}), w_{n_i} - v_{n_i}\right\rangle\\ &\quad\le \left\langle f(v_{n_i}), x^\dagger - v_{n_i}\right\rangle, \quad \forall x^\dagger \in C.\end{aligned} \tag{46}$$

Owing to (39), $\lim_{i\longrightarrow\infty}\|v_{n_i} - w_{n_i}\| = 0$. It follows from (46) that

$$\liminf_{i\longrightarrow\infty}\left\langle f(v_{n_i}), x^\dagger - v_{n_i}\right\rangle \ge 0, \quad \forall x^\dagger \in C. \tag{47}$$

There are two possible cases: $\liminf_{i\longrightarrow+\infty}\|f(v_{n_i})\| = 0$ and $\liminf_{i\longrightarrow+\infty}\|f(v_{n_i})\| > 0$.

If $\liminf_{i\longrightarrow+\infty}\|f(v_{n_i})\| = 0$, by $v_{n_i} \rightharpoonup x^*$ and f satisfying (16), we obtain that $f(x^*) = 0$. If $\liminf_{i\longrightarrow+\infty}\|f(v_{n_i})\| > 0$, then there exists an integer $\mathcal{I} > 0$ satisfying $f(v_{n_i}) \ne 0$ for all $i \ge \mathcal{I}$. By (47), we achieve

$$\liminf_{i\longrightarrow+\infty}\left\langle \frac{f(v_{n_i})}{\|f(v_{n_i})\|}, x^\dagger - v_{n_i}\right\rangle \ge 0, \quad \forall x^\dagger \in C. \tag{48}$$

Let $\{\xi_j\}$ be a positive strictly decreasing sequence such that $\xi_j \longrightarrow 0$ as $j \longrightarrow +\infty$. By virtue of (48), there exists a strictly increasing subsequence $\{n_{i_j}\}$ satisfying $n_{i_j} \ge \mathcal{I}$ and $\forall j \ge 0$,

$$\left\langle \frac{f(v_{n_{i_j}})}{\|f(v_{n_{i_j}})\|}, x^\dagger - v_{n_{i_j}}\right\rangle + \xi_j > 0, \quad \forall x^\dagger \in C, \tag{49}$$

which results that

$$\left\langle f(v_{n_{i_j}}), x^\dagger - v_{n_{i_j}}\right\rangle + \xi_j\|f(v_{n_{i_j}})\| > 0, \quad \forall x^\dagger \in C, \forall j \ge 0. \tag{50}$$

We set $\tilde{v}_j = f(v_{n_{i_j}})/\|f(v_{n_{i_j}})\|^2$ for all $j \ge 0$. Then, $\left\langle f(v_{n_{i_j}}), \tilde{v}_j\right\rangle = 1$ for each $j \ge 0$. Owing to (50), we have

$$\left\langle f(v_{n_{i_j}}), x^\dagger + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j - v_{n_{i_j}}\right\rangle > 0, \quad \forall x^\dagger \in C, \forall j \ge 0. \tag{51}$$

Since f is quasimonotone on H, by (51), we get

$$\begin{aligned}&\left\langle f\left(x^\dagger + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j\right), x^\dagger + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j - v_{n_{i_j}}\right\rangle \ge 0,\\ &\qquad \forall x^\dagger \in C, \forall j \ge 0.\end{aligned} \tag{52}$$

Since $\lim_{j\longrightarrow+\infty}\xi_j\|f(v_{n_{i_j}})\|\|\tilde{v}_j\| = \lim_{j\longrightarrow+\infty}\xi_j = 0$ and f is Lipschitz continuous, $\lim_{j\longrightarrow\infty}f(x + \xi_j\|f(v_{n_{i_j}})\|\tilde{v}_j) = f(x)$. Letting $j \longrightarrow +\infty$ in (52), we deduce

$$\left\langle f\left(x^{\dagger}\right), x^{\dagger} - x^{*}\right\rangle \geq 0, \quad \forall x^{\dagger} \in C, \tag{53}$$

which means $x^* \in \mathrm{Sol}^d(C, f)$.

Next, we show that x^* is the unique weak cluster point of $\{u_n\}$ in $\mathrm{Sol}^d(C, f)$. Let $\overline{x} \in \mathrm{Sol}^d(C, f)$ be another weak cluster point of $\{u_n\}$. Then, there exists a sequence $\{u_n\}$ of $\{u_n\}$ satisfying $u_{n_j} \rightharpoonup \overline{x}$ as $j \longrightarrow +\infty$. We note that for all $k \geq 0$,

$$2\langle u_n, x^* - \overline{x}\rangle = \left\|u_n - \overline{x}\right\|^2 - \left\|u_n - x^*\right\|^2 + \left\|x^*\right\|^2 - \|\overline{x}\|^2. \tag{54}$$

We note that $\lim_{n\longrightarrow+\infty}\|u_n - x^*\|$ and $\lim_{n\longrightarrow+\infty}\|u_n - \overline{x}\|$ exist. From (54), $\lim_{n\longrightarrow+\infty}\langle u_n, x^* - \overline{x}\rangle$ exists. Hence,

$$\lim_{i\longrightarrow+\infty}\left\langle u_{n_i}, x^* - \overline{x}\right\rangle = \lim_{j\longrightarrow+\infty}\left\langle u_{n_j}, x^* - \overline{x}\right\rangle. \tag{55}$$

Since $u_{n_i} \rightharpoonup x^*$ and $u_{n_j} \rightharpoonup \overline{x}$, from (55), we have

$$\langle x^*, x^* - \overline{x}\rangle = \langle \overline{x}, x^* - \overline{x}\rangle, \tag{56}$$

which implies that $\|x^* - \overline{x}\|^2 = 0$, and hence $x^* = \overline{x}$. Therefore, $\{u_n\}$ has the unique weak cluster point in $\mathrm{Sol}^d(C, f)$. By the condition (C3), $\{x \in C, f(x) = 0\} \setminus \mathrm{Sol}^d(C, f)$ is a finite set. Therefore, $\{u_n\}$ has finite weak cluster points in $\mathrm{Sol}(C, f)$ denoted by $q_1, q_2, \ldots, q_m$. We set $N_0 = \{1, 2, \ldots, m\}$ and $\nu = \min\left\{\|q_j - q_k\|/3, j, k \in N_0, j \neq k\right\}$. Let q_j, $j \in N_0$ be any weak cluster point in $\mathrm{Sol}(C, f)$ and $\left\{u_{n_i}^j\right\}$ be a subsequence of $\{u_n\}$ satisfying $u_{n_i}^j \rightharpoonup q_j$ as $i \longrightarrow +\infty$. Then, we have

$$\lim_{i\longrightarrow+\infty}\left\langle u_{n_i}^j, \frac{q_j - q_k}{\left\|q_j - q_k\right\|}\right\rangle = \left\langle q_j, \frac{q_j - q_k}{\left\|q_j - q_k\right\|}\right\rangle, \quad \forall k \in N_0 \text{ and } k \neq j. \tag{57}$$

By the definition of ν, we have $\forall k \neq j$,

$$\begin{aligned}\left\langle q_j, \frac{q_j - q_k}{\left\|q_j - q_k\right\|}\right\rangle &= \frac{\left\|q_j - q_k\right\|}{2} + \frac{\left\|q_j\right\|^2 - \left\|q_k\right\|^2}{2\left\|q_j - q_k\right\|} \\ &> \nu + \frac{\left\|q_j\right\|^2 - \left\|q_k\right\|^2}{2\left\|q_j - q_k\right\|}.\end{aligned} \tag{58}$$

In the light of (57) and (58), there exists an integer int_i^j such that when $i \geq \mathrm{int}_i^j$,

$$u_{n_i}^j \in \left\{x\colon \left\langle x, \frac{q_j - q_k}{\left\|q_j - q_k\right\|}\right\rangle > \nu + \frac{\left\|q_j\right\|^2 - \left\|q_k\right\|^2}{2\left\|q_j - q_k\right\|}\right\}, \quad k \in N_0, k \neq j. \tag{59}$$

We write

$$Sb_j = \bigcap_{k=1,k\neq j}^{m}\left\{x\colon \left\langle x, \frac{q_j - q_k}{\left\|q_j - q_k\right\|}\right\rangle > \nu + \frac{\left\|q_j\right\|^2 - \left\|q_k\right\|^2}{2\left\|q_j - q_k\right\|}\right\}. \tag{60}$$

Taking into account (59) and (60), we have $u_{n_i}^j \in Sb_j$ when $i \geq \max\left\{\mathrm{int}_i^j, j \in N_0\right\}$.

Now, we show that $u_n \in \cup_{j=1}^m Sb_j$ for a large enough n. If not, there exists a subsequence $\left\{u_{n_l}\right\}$ of $\{u_n\}$ such that $u_{n_l} \notin \cup_{j=1}^m Sb_j$. By the boundedness of $\left\{u_{n_l}\right\}$, there exists a subsequence of $\left\{u_{n_l}\right\}$ convergent weakly to x^*. Without the loss of generality, we still denote the subsequence as $\left\{u_{n_l}\right\}$. According to assumptions, $u_{n_l} \notin \cup_{j=1}^m Sb_j$, so $u_{n_l} \notin Sb_j$ for any $j \in N_0$. Therefore, there exists a subsequence $\left\{u_{n_{l_s}}\right\}$ of $\left\{u_{n_l}\right\}$ such that when $\forall s \geq 0$,

$$u_{n_{l_s}} \notin \left\{x\colon \left\langle x, \frac{q_j - q_k}{\left\|q_j - q_k\right\|}\right\rangle > \nu + \frac{\left\|q_j\right\|^2 - \left\|q_k\right\|^2}{2\left\|q_j - q_k\right\|}\right\}, \quad k \in N_0, k \neq j. \tag{61}$$

Thus,

$$x^* \notin \left\{x\colon \left\langle x, \frac{q_j - q_k}{\left\|q_j - q_k\right\|}\right\rangle > \nu + \frac{\left\|q_j\right\|^2 - \left\|q_k\right\|^2}{2\left\|q_j - q_k\right\|}\right\}, \quad k \in N_0, k \neq j, \tag{62}$$

which implies that $x^* \neq q_j$ and $j \in N_0$. This is impossible. So, for a large enough positive integer N_1, $u_n \in \cup_{j=1}^m Sb_j$ when $n \geq N_1$.

Next, we show that $\{u_n\}$ has the unique weak cluster point in $\mathrm{Sol}(C, f)$. First, there exists a positive integer $N_2 \geq N_1$ such that $\|u_{n+1} - u_n\| < \nu$ for all $n \geq N_2$. We assume that $\{u_n\}$ has at least two weak cluster points in $\mathrm{Sol}(C, f)$. Then, there exists $\widehat{n} \geq N_2$ such that $u_{\widehat{n}} \in Sb_j$ and $u_{\widehat{n}+1} \in Sb_k$, where $j, k \in N_0$ and $m \geq 2$, that is,

$$u_{\widehat{n}} \in Sb_j = \bigcap_{k=1,k\neq j}^{m}\left\{x\colon \left\langle x, \frac{q_j - q_k}{\left\|q_j - q_k\right\|}\right\rangle > \nu + \frac{\left\|q_j\right\|^2 - \left\|q_k\right\|^2}{2\left\|q_j - q_k\right\|}\right\}, \tag{63}$$

and

$$u_{\widehat{n}+1} \in Sb_k = \bigcap_{j=1,j\neq k}^{m}\left\{x\colon \left\langle x, \frac{q_k - q_j}{\left\|q_k - q_j\right\|}\right\rangle > \nu + \frac{\left\|q_k\right\|^2 - \left\|q_j\right\|^2}{2\left\|q_k - q_j\right\|}\right\}. \tag{64}$$

Therefore,

$$\left\langle u_{\widehat{n}}, \frac{q_j - q_k}{\left\|q_j - q_k\right\|}\right\rangle > \nu + \frac{\left\|q_j\right\|^2 - \left\|q_k\right\|^2}{2\left\|q_j - q_k\right\|}, \tag{65}$$

and

$$\left\langle u_{\widehat{n+1}}, \frac{q_k - q_j}{\|q_k - q_j\|} \right\rangle > \nu + \frac{\|q_k\|^2 - \|q_j\|^2}{2\|q_k - q_j\|}. \tag{66}$$

Combining (65) and (66), we achieve

$$\left\langle u_{\widehat{n}} - u_{\widehat{n+1}}, \frac{q_j - q_k}{\|q_j - q_k\|} \right\rangle > 2\nu. \tag{67}$$

At the same time, we have

$$\|u_{\widehat{n+1}} - u_{\widehat{n}}\| < \nu. \tag{68}$$

Based on (67) and (68), we deduce

$$2\nu < \left\langle u_{\widehat{n}} - u_{\widehat{n+1}}, \frac{q_j - q_k}{\|q_j - q_k\|} g \right\rangle \le \|u_{\widehat{n}} - u_{\widehat{n+1}}\| < \nu. \tag{69}$$

This leads to a contradiction. Then, $\{u_n\}$ has the unique weak cluster point in $\mathrm{Sol}(C, f)$. So, the sequence $\{u_n\}$ has the unique weak cluster point $x^* \in \Gamma$. Therefore, the sequence $\{u_n\}$ converges weakly to $x^* \in \Gamma$. This completes the proof. □

Based on Algorithm 1 and Theorem 1, we can obtain the following algorithms and the corresponding corollaries.

Algorithm 2. Initialization: let $u_0 \in H$ be an initial guess. We set $n = 0$.

Step 1. Let the n-th iterate u_n and the n-th step size ς_n be given. We compute

$$w_n = \mathrm{proj}_C[u_n - \varsigma_n f(u_n)], \tag{70}$$

and

$$u_{n+1} = (1 - \zeta_n)u_n + \zeta_n w_n + \zeta_n \varsigma_n [f(u_n) - f(w_n)]. \tag{71}$$

Step 2. We update the $n+1$-th step size by the following form:

$$\varsigma_{n+1} = \begin{cases} \min\left\{\varsigma_n, \dfrac{\beta\|w_n - u_n\|}{\|f(w_n) - f(u_n)\|}\right\}, & \text{if } f(w_n) \ne f(u_n), \\ \varsigma_n, & \text{else.} \end{cases} \tag{72}$$

We set $n: = n + 1$ and return to step 1.

Corollary 1. *We assume that the operator $f\colon H \longrightarrow H$ is quasimonotone, κ-Lipschitz continuous and satisfies the property (P). Suppose that $Sol^d(C, f) \ne \emptyset$, $\{x \in C\colon f(x) = 0\} \setminus Sol^d(C, f)$ is a finite set and $0 < \liminf_{n\longrightarrow\infty}\zeta_n \le \limsup_{n\longrightarrow\infty}\zeta_n < 1$. Then, the sequence $\{u_n\}$ generated by Algorithm 2 converges weakly to some point in $Sol(C, f)$.*

Algorithm 3. Initialization: let $u_0 \in C$ and $\varsigma_0 > 0$. We set $n = 0$.

Step 1. For known u_n, we compute

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T[(1 - \varpi_n)u_n + \varpi_n T(u_n)]. \tag{73}$$

Step 2. We set $n: = n + 1$ and return to step 1.

Corollary 2. *We assume that $T\colon C \longrightarrow C$ is a pseudocontractive and L-Lipschitz continuous operator. We suppose that $Fix(T) \ne \emptyset$ and $0 < \underline{\alpha} < \alpha_n < \overline{\alpha} < \varpi_n < \overline{\varpi} < 1/\sqrt{1 + L^2} + 1\, (\forall n \ge 0)$. Then, the sequence $\{u_n\}$ generated by Algorithm 3 converges weakly to some point in $Fix(T)$.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Best Proximity Points of Multivalued Hardy-Roger's Type (Cyclic) Contractive Mappings of *b*-Metric Spaces

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In this article, we introduce a new type of generalized multivalued Hardy and Roger's type proximal contractive and proximal cyclic contractive mappings of b-metric spaces and develop some results for the existence of best proximity point(s). Moreover, we obtain some results for the existence and uniqueness of best proximity points for single-valued mappings. Examples are given to explain the main results.

1. Introduction

The metric fixed point theory plays a very fundamental role in many fields of mathematics especially in nonlinear analysis and some related disciplines. The fundamental tool of this theory is the Banach contraction principle (shortly BCP) [1] which states that if a self-mapping $T\colon P \longrightarrow P$ of a complete metric space (P, ϱ) with metric ϱ satisfies

$$\varrho(Tp_1, Tp_2) \leq k\varrho(p_1, p_2), \tag{1}$$

for all $p_1, p_2 \in P$, and for some $k \in [0, 1)$, T has a unique fixed point, that is, there exists a point $p \in P$, such that

$$\varrho(p, Tp) = 0. \tag{2}$$

A mapping that satisfies (1) is known as Banach contraction. After this remarkable result, many mathematicians contributed for the development of fixed-point theory by producing many results with different generalized contractive mappings in complete metric spaces, for details one can see [2–8] and the references therein. One of the important generalizations of BCP was presented by Edelstein [9] in 1962. Later on, many mathematicians generalized Edelstein's result, for instance Meir and Keeler [10] in 1969 and Reich [11] in 1971. Reich's result has been further generalized by Hardy and Roger [12] in 1973 as follows.

Theorem 1. *Let (P, ϱ) be a metric space and $T\colon P \longrightarrow P$ a self-mapping satisfying the following conditions for all $p_1, p_2 \in P$:*

(1)

$$\varrho(Tp_1, Tp_2) \leq \alpha\varrho(p_1, Tp_1) + \beta\varrho(p_2, Tp_2) + \gamma\varrho(p_1, Tp_2) + \delta\varrho(p_2, Tp_1) + \tau\varrho(p_1, p_2), \tag{3}$$

where $\alpha, \beta, \gamma, \delta, \tau$ are nonnegative reals.
Set $\Omega = \alpha + \beta + \gamma + \delta + \tau$. Then,

(a) If P is complete and $\Omega < 1$, then T has a unique fixed point

(b) If (1) is modified as

$(1')$ for all $p_1 \neq p_2$ implies

$$\varrho(Tp_1, Tp_2) < \alpha\varrho(p_1, Tp_1) + \beta\varrho(p_2, Tp_2) + \gamma\varrho(p_1, Tp_2) + \delta\varrho(p_2, Tp_1) + \tau\varrho(p_1, p_2), \tag{4}$$

and P is compact, T is continuous, and $\Omega = 1$*; then, T has a unique fixed point.*

Nadler [13] in 1969 generalized the BCP in the context of multivalued mappings of complete metric spaces. Later on, Nadler's result has been generalized by Prolla [14] in 1983.

Meanwhile, the metric space has been generalized to b-metric space; by then, the fixed point theory has been further generalized for single-valued and multivalued mappings in the context of b-metric space, for instance, Bakhtin [15] in 1989 and Czerwik [16] in 1993.

For nonself mapping, $T: R \longrightarrow S$ (R and S are two nonempty sets), such that $R \cap T(R) = \varnothing$ (empty set); then, it is not possible to find the fixed point of T. The best way to deal with such situation is to explore a point r in R, such that

$$\varrho(r, \mathrm{Tr}) = \varrho(R, S), \tag{5}$$

where

$$\varrho(R, S) = \inf_{r \in R, s \in S} \varrho(r, s), \tag{6}$$

and if such a point in R exists, it is called the best proximity point of T. If $R = S$, then the best proximity point becomes a fixed point. So, best proximity point theory is the proper generalization of fixed-point theory. Fan's result [17] in 1969 was probably the first attempt in this direction.

Later on, many mathematicians extended Fan's result and developed some best proximity point results. For more details, one can see [18]. Best proximity point theory has been further developed by using different proximal contractions, for more details, one can see references [19–23].

Kirk [24] in 2003 introduced cyclic contraction and developed some fixed points results. Later on in 2006, Eldered and Veeramani [25] developed some best proximity point results for cyclic contractions.

Basha in 2019 [21] introduced proximal contractive and proximal cyclic contractive mappings and developed some results for the existence and uniqueness of best proximity point.

Recently, in 2021, Hiranmoy et al. [26] introduced proximal Kannan-type and proximal cyclic Kannan-type contractive mappings in metric spaces (compare with [21]) and developed some best proximity point results.

Motivated by the contractive mappings of Hiranmoy, we introduce the notion of multivalued Hardy and Roger's type proximal and cyclic proximal contractive mappings and develop some results for the existence of best proximity points in b-metric space. Furthermore, we give some examples to explain the results.

2. Preliminaries

Throughout this article, $\mathbb{R}, \mathbb{R}^+, \mathbb{N}, \mathbb{N}_1$, and $\wp(P)$ denote the set of reals, nonnegative reals, positive integers, nonnegative integers, and collection of nonempty subsets of P, respectively.

Definition 2. Let P be a nonempty set and $b \geq 1$ a real number. The mapping $\varrho_b: P \times P \longrightarrow [0, \infty)$ is a b-metric and (P, ϱ_b) is called b-metric space if ϱ_b satisfies the following axioms:

(b_1) $\varrho_b(p_1, p_2) = 0$ if and only if $p_1 = p_2$

(b_2) $\varrho_b(p_1, p_2) = \varrho_b(p_2, p_1)$

(b_3) $\varrho_b(p_1, p_2) \leq b[\varrho_b(p_1, p_3) + \varrho_b(p_3, p_2)]$, for all $p_1, p_3, p_2 \in P$.

Throughout this paper, ϱ and ϱ_b denote metric and b-metric, respectively. Now, suppose that R and S are two nonempty subsets of (P, ϱ_b). Define

$$\begin{aligned}\varrho_b(R, S) &= \inf\{\varrho_b(r, s): r \in R, s \in S\},\\ R_0 &= \{r \in R: \varrho_b(r, s) = \varrho_b(R, S) \text{ for some } s \in S\},\\ S_0 &= \{s \in S: \varrho_b(r, s) = \varrho_b(R, S) \text{ for some } r \in R\}.\end{aligned} \tag{7}$$

Definition 3. A b-metric space (P, ϱ_b) is boundedly compact if every bounded sequence in P has a convergent subsequence (compare with [27]).

Definition 4 (see [26]). Let R and S be two nonempty subsets of (P, ϱ). A mapping $T: R \longrightarrow S$ is said to be a proximal Kannan-type contractive mapping if

$$\begin{aligned}&\left.\begin{aligned}\varrho(r_1, \mathrm{Tr}_3) &= \varrho(R, S)\\ \varrho(r_2, \mathrm{Tr}_4) &= \varrho(R, S)\\ r_3 &\neq r_4\end{aligned}\right\} \text{implies } \varrho(r_1, r_2) < \frac{1}{2}\left(\varrho(r_1, r_3) + \varrho(r_2, r_4)\right),\\ &\left.\begin{aligned}\varrho(r_1, \mathrm{Tr}_3) &= \varrho(R, S)\\ \varrho(r_2, \mathrm{Tr}_4) &= \varrho(R, S)\\ r_3 &= r_4\end{aligned}\right\} \text{implies } \varrho(r_1, r_2) \leq \frac{1}{2}\left(\varrho(r_1, r_3) + \varrho(r_2, r_4)\right),\end{aligned} \tag{8}$$

hold for all $r_1, r_2, r_3, r_4 \in R$.

Definition 5. Let R and S be two nonempty subsets of (P, ϱ_b). Then, a mapping $T\colon R\cup S \longrightarrow R\cup S$ is said to be cyclic if $T(R) \subset S$ and $T(S) \subset R$ (compare with [26]).

Definition 6 (see [26]). *Let R and S be two nonempty subsets of (P, ϱ). A cyclic mapping $T\colon R\cup S \longrightarrow R\cup S$ is said to be a proximal cyclic Kannan-type contractive mapping if*

$$\left.\begin{array}{l}\varrho(r_1, \mathrm{Tr}_3) = \varrho(R,S)\\ \varrho(r_2, \mathrm{Tr}_4) = \varrho(R,S)\\ \varrho(r_3, r_4) > \varrho(R,S)\end{array}\right\}\text{implies } \varrho(r_1, r_2) < \frac{1}{2}\left(\varrho(r_1, r_3) + \varrho(r_2, r_4)\right),$$
$$\left.\begin{array}{l}\varrho(r_1, \mathrm{Tr}_3) = \varrho(R,S)\\ \varrho(r_2, \mathrm{Tr}_4) = \varrho(R,S)\\ \varrho(r_3, r_4) = \varrho(R,S)\end{array}\right\}\text{implies } \varrho(r_1, r_2) = \varrho(R,S), \tag{9}$$

that hold for all $r_1, r_2, r_3, r_4 \in R$.

In the following, we introduce a compact weak proximal pair in b-metric space.

Definition 7. Let R and S be two nonempty subsets of (P, ϱ_b). The pair (R, S) is said to be a compact weak proximal pair if for bounded sequences (r_n) in R and (s_n) in S with $\varrho_b(r_n, s_n) \longrightarrow \varrho_b(R, S)$ as $n \longrightarrow \infty$, the sequences (r_n) and (s_n) have convergent subsequences in R and S, respectively (compare with [26]).

Remark 8. Note that if $R = S$ in above definition, then (R, R) is a compact weak proximal pair if and only if R is boundedly compact.

Now, we present a lemma in the context of b-metric space (analogous to [[26], Lemma 2.2]) that will be used in the sequel to prove our main results.

Lemma 9. *Let R and S be two nonempty subsets of (P, ϱ_b), such that at least one of R and S is bounded, and (R, S) is a compact weak proximal pair. Then, $R_0 \neq \varnothing$, and hence, so is S_0.*

Proof. As

$$\varrho_b(R, S) = \inf\{\varrho_b(r, s)\colon r \in R, s \in S\}, \tag{10}$$

so for each $n \in N$, there exists $r_n \in R$ and $s_n \in S$, such that

$$\varrho_b(R, S) \leq \varrho_b(r_n, s_n) < \varrho_b(R, S) + \frac{1}{n}. \tag{11}$$

Therefore, the sequence $(\varrho_b(r_n, s_n))$ converges to $\varrho_b(R, S)$. Now, we assume that R is bounded. So, there exists a positive real number K, such that $\varrho_b(r_n, r_m) \leq K$ for all $n, m \in \mathbb{N}$, so we have

$$\begin{aligned}\varrho_b(s_n, s_m) &\leq b\left(\varrho_b(s_n, r_n) + \varrho_b(r_n, s_m)\right)\\ &\leq b\left[\varrho_b(s_n, r_n) + b\left(\varrho_b(r_n, r_m) + \varrho_b(r_m, s_m)\right)\right],\end{aligned} \tag{12}$$

which implies

$$\varrho_b(s_n, s_m) < b\left[\varrho_b(R, S) + 1 + b\left(K + \varrho_b(R, S) + 1\right).\right. \tag{13}$$

Therefore, (r_n) and (s_n) are bounded sequences. So by compact weak proximality of the pair (R, S), there exist subsequences (r_{n_k}) of (r_n) and (s_{n_k}) of (s_n), such that (r_{n_k}) converges to $r_\star \in R$ and (s_{n_k}) converges to $s_\star \in S$. Therefore,

$$\varrho_b\left(r_{n_k}, s_{n_k}\right) \longrightarrow \varrho_b(r_\star, s_\star) \text{ as } k \longrightarrow \infty. \tag{14}$$

Thus, we have

$$\varrho_b(r_\star, s_\star) = \varrho_b(R, S). \tag{15}$$

So, $r_\star \in R_0$ and $s_\star \in S_0$. Hence, $R_0 \neq \varnothing$ and $S_0 \neq \varnothing$. Similarly, if S is bounded, then $R_0 \neq \varnothing$ and $S_0 \neq \varnothing$. □

Theorem 10 (see [26]). *Let R and S be two nonempty subsets of (P, ϱ), such that at least one of R and S is bounded, and (R, S) is a compact weak proximal pair. Let $T\colon R \longrightarrow S$ be a proximal Kannan-type contractive mapping and assume that*

(i) $T(R_0) \subset S_0$

(ii) If (r_n) and (s_n) are two bounded sequences in R and S, respectively, such that $(\varrho(r_n, s_n))$ converges to $\varrho(R, S)$, then $\lim_{n\longrightarrow\infty} \varrho(r_n, r_{n+1}) = 0$.

Then, T has a unique best proximity point in R.

Theorem 11 (see [26]). *Let R and S be two nonempty subsets of (P, ϱ), such that at least one of R and S is bounded, and (R, S) is a compact weak proximal pair. Let $T\colon R\cup S \longrightarrow R\cup S$ be a proximal cyclic Kannan-type contractive mapping and assume that the following conditions hold:*

(i) $T(R_0) \subset S_0$ and $T(S_0) \subset R_0$

(ii) If (r_n) and (s_n) are two bounded sequences in R and S, respectively, such that $(\varrho(r_n, s_n))$ converges to $\varrho(R, S)$, then $\lim_{n\longrightarrow\infty} \varrho(r_n, r_{n+1}) = 0$.

Then, the following conditions hold:

(a) There exist $r \in R$ and $s \in S$, such that $\varrho(r, \mathrm{Tr}) = \varrho(R, S)$ and $\varrho(s, Ts) = \varrho(R, S)$

(b) If $r \in R$ and $s \in S$, such that $\varrho(r, \mathrm{Tr}) = \varrho(R, S)$ and $\varrho(s, Ts) = \varrho(R, S)$, then $\varrho(r, s) = \varrho(R, S)$.

Now, we introduce the notions of a new type of generalized multivalued Hardy and Roger's proximal contractive and proximal cyclic contractive mappings.

Definition 12. Let R and S be two nonempty subsets of (P, ϱ_b). A mapping $T\colon R \longrightarrow \wp(S)$ is said to be a new type of generalized multivalued Hardy and Roger's proximal contractive mapping if

$$\left.\begin{array}{l}\varrho_b(r_1, \mathrm{Tr}_3) = \varrho_b(R, S)\\ \varrho_b(r_2, \mathrm{Tr}_4) = \varrho_b(R, S)\\ r_3 \neq r_4\end{array}\right\},$$

implies,

$$\left.\begin{array}{l}\varrho_b(r_1, r_2) < \alpha\varrho_b(r_1, r_3) + \beta\varrho_b(r_2, r_4) + \dfrac{\gamma}{b^2}\varrho_b(r_3, r_4)\\ + \dfrac{\delta}{b}\varrho_b(r_1, r_4) + \dfrac{\tau}{b}\varrho_b(r_2, r_3)\end{array}\right\},$$

$$\left.\begin{array}{l}\varrho_b(r_1, \mathrm{Tr}_3) = \varrho_b(R, S)\\ \varrho_b(r_2, \mathrm{Tr}_4) = \varrho_b(R, S)\\ r_3 = r_4\end{array}\right\},$$

implies

$$\begin{aligned}\varrho_b(r_1, r_2) &\leq \alpha\varrho_b(r_1, r_3) + \beta\varrho_b(r_2, r_3)\\ &\quad + \frac{\delta}{b}\varrho_b(r_1, r_4) + \frac{\tau}{b}\varrho_b(r_2, r_3),\end{aligned} \tag{16}$$

which hold for all $r_1, r_2, r_3, r_4 \in R$, where

$$\alpha + \beta + \gamma + 2\tau = 1, \beta \neq 1, \gamma + \delta + \tau < 1 \tag{17}$$

$,\alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}^+$.

Remark 13. If in the Definition 12, we replace $T\colon R \longrightarrow \wp(S)$ by $T\colon R \longrightarrow S$, then T is said to be a new type of generalized Hardy and Roger's proximal contractive mapping.

Definition 14. Let R and S be two nonempty subsets of (P, ϱ_b). A multivalued cyclic mapping $T\colon R \cup S \longrightarrow \wp(R) \cup \wp(S)$ is said to be a new type of generalized multivalued Hardy and Roger's proximal cyclic contractive mapping if

$$\left.\begin{array}{l}\varrho_b(r_1, \mathrm{Tr}_3) = \varrho_b(R, S)\\ \varrho_b(r_2, \mathrm{Tr}_4) = \varrho_b(R, S)\\ \varrho_b(r_3, r_4) > \varrho_b(R, S)\end{array}\right\},$$

implies,

$$\left.\begin{array}{l}\varrho_b(r_1, r_2) < \alpha\varrho_b(r_1, r_3) + \beta\varrho_b(r_2, r_4) + \dfrac{\gamma}{b^2}\varrho_b(r_3, r_4)\\ + \dfrac{\delta}{b}\varrho_b(r_1, r_4) + \dfrac{\tau}{b}\varrho_b(r_2, r_3)\end{array}\right\},$$

$$\left.\begin{array}{l}\varrho_b(r_1, \mathrm{Tr}_3) = \varrho_b(R, S)\\ \varrho_b(r_2, \mathrm{Tr}_4) = \varrho_b(R, S)\\ \varrho_b(r_3, r_4) = \varrho_b(R, S)\end{array}\right\} \text{implies} \varrho_b(r_1, r_2) = \varrho_b(R, S), \tag{18}$$

which hold for all $r_1, r_2, r_3, r_4 \in R$, where

$$\alpha + \beta + \gamma + 2\tau = 1, \alpha \neq 1, \beta \neq 1, \gamma + \delta + \tau < 1, \alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}^+. \tag{19}$$

Remark 15. If in the Definition 14, we replace $T\colon R \cup S \longrightarrow \wp(R) \cup \wp(S)$ by $T\colon R \cup S \longrightarrow R \cup S$, then T is said to be a new type of generalized Hardy and Roger's proximal cyclic contractive mapping.

3. Best Proximity Points Results for a New Type of Multivalued Hardy and Roger's Proximal Contractive Mappings in b-metric Space

The following is our main result of this section.

Theorem 16. *Let R and S be two nonempty subsets of (P, ϱ_b), such that at least one of R and S is bounded and (R, S) is a compact weak proximal pair. Let $T\colon R \longrightarrow \wp(S)$ be a new type of generalized multivalued Hardy and Roger's proximal contractive mapping. Further assume that*

(i) For each $r \in R_0$, $\mathrm{Tr} \subset S_0$

(ii) If (r_n) and (s_n) are two bounded sequences in R and S, respectively, such that $(\varrho_b(r_n, s_n))$ converges to $\varrho_b(R, S)$, then $\lim_{n\longrightarrow\infty}\varrho_b(r_n, r_{n+1}) = 0$.

Then, T has a best proximity point.

Proof. Lemma 9 implies $R_0 \neq \varnothing$. Let $r_0 \in R_0$; then, $\mathrm{Tr}_0 \subset S_0$. We can pick an element $s_1 \in \mathrm{Tr}_0 \subset S_0$, so that there exists $r_1 \in R$, such that

$$\varrho_b(r_1, s_1) = \varrho_b(R, S). \quad (20)$$

Continuing this way, we can construct sequences (r_n) in R and (s_n) in Tr_{n-1}, such that

$$\varrho_b(r_n, s_n) = \varrho_b(R, S), \quad (21)$$

for all $n \in \mathbb{N}$. Therefore,

$$\varrho_b(R, S) \leq \varrho_b(r_n, Tr_{n-1}) \leq \varrho_b(r_n, s_n) = \varrho_b(R, S), \quad (22)$$

that is,

$$\varrho_b(r_n, Tr_{n-1}) = \varrho_b(R, S). \quad (23)$$

If $r_n = r_{n-1}$ for some $n \in \mathbb{N}$, then r_{n-1} is the best proximity point of T, and the proof is completed. So, we may assume that $r_n \neq r_{n-1}$ for all $n \in \mathbb{N}$. Now, we show that (r_n) and (s_n) are bounded sequences. As we have

$$\begin{gathered} \varrho_b(r_n, Tr_{n-1}) = \varrho_b(R, S), \\ \varrho_b(r_{n+1}, \mathrm{Tr}_n) = \varrho_b(R, S), \\ r_n \neq r_{n-1}, \end{gathered} \quad (24)$$

so by the given condition, we obtain

$$\begin{aligned} \varrho_b(r_n, r_{n+1}) &< \alpha\varrho_b(r_n, r_{n-1}) + \beta\varrho_b(r_{n+1}, r_n) + \frac{\gamma}{b^2}\varrho_b(r_{n-1}, r_n), \\ &\quad + \frac{\delta}{b}\varrho_b(r_n, r_n) + \frac{\tau}{b}\varrho_b(r_{n+1}, r_{n-1}) \\ &\leq \alpha\varrho_b(r_n, r_{n-1}) + \beta\varrho_b(r_{n+1}, r_n) \\ &\quad + \gamma\varrho_b(r_{n-1}, r_n) + \tau(\varrho_b(r_{n-1}, r_n) + \varrho_b(r_n, r_{n+1})), \end{aligned} \quad (25)$$

which implies

$$(1 - (\beta + \tau))\varrho_b(r_n, r_{n+1}) < (\alpha + \gamma + \tau)\varrho_b(r_{n-1}, r_n). \quad (26)$$

If $1 - (\beta + \tau) = 0$, then $\beta + \tau = 1$, so (17) implies $\alpha = \gamma = \tau = 0$, and so $\beta = 1$, a contradiction. Therefore, $1 - (\beta + \tau) \neq 0$ and $1 - (\beta + \tau) = \alpha + \gamma + \tau$. Hence, we get

$$\varrho_b(r_n, r_{n+1}) < \varrho_b(r_{n-1}, r_n), \quad (27)$$

that is,

$$\varrho_b(r_n, r_{n+1}) < \varrho_b(r_{n-1}, r_n) < \cdots < \varrho_b(r_0, r_1) = K\,(\text{say}). \quad (28)$$

As

$$\begin{gathered} \varrho_b(r_n, Tr_{n-1}) = \varrho_b(R, S), \\ \varrho_b(r_m, Tr_{m-1}) = \varrho_b(R, S), \end{gathered} \quad (29)$$

so, if $r_{n-1} = r_{m-1}$, then we have

$$\begin{aligned} \varrho_b(r_n, r_m) &\leq b(\varrho_b(r_n, r_{m-1}) + \varrho_b(r_{m-1}, r_m)), \\ &\leq b[b(\varrho_b(r_n, r_{n-1}) + \varrho_b(r_{n-1}, r_{m-1})) \\ &\quad + \varrho_b(r_{m-1}, r_m)], \\ &\leq (b^2 + b)K. \end{aligned} \quad (30)$$

If $r_{n-1} \neq r_{m-1}$, then

$$\begin{aligned} \varrho_b(r_n, r_m) &< \alpha\varrho_b(r_n, r_{n-1}) + \beta\varrho_b(r_m, r_{m-1}) + \frac{\gamma}{b^2}\varrho_b(r_{n-1}, r_{m-1}), \\ &\quad + \frac{\delta}{b}\varrho_b(r_n, r_{m-1}) + \frac{\tau}{b}\varrho_b(r_m, r_{n-1}) \\ &\leq (\alpha + \beta)K + \frac{\gamma}{b^2}[b(\varrho_b(r_{n-1}, r_n) + b(\varrho_b(r_n, r_m) + \varrho_b(r_m, r_{m-1})))] \\ &\quad + \delta(\varrho_b(r_n, r_m) + \varrho_b(r_m, r_{m-1})) + \tau(\varrho_b(r_m, r_n) + \varrho_b(r_n, r_{n-1})), \end{aligned} \quad (31)$$

which implies

$$(1 - (\gamma + \delta + \tau)\varrho_b(r_n, r_m) < (\alpha + \beta + 2\gamma + \delta + \tau)K), \quad (32)$$

and by (2), $\gamma + \delta + \tau < 1$. Therefore,

$$\varrho_b(r_n, r_m) < \frac{(\alpha + \beta + \gamma + \delta + \tau)}{1 - (\gamma + \delta + \tau)}K. \quad (33)$$

Hence, (r_n) is a bounded sequence. Furthermore, for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned} \varrho_b(s_n, s_m) &\leq b(\varrho_b(s_n, r_n) + \varrho_b(r_n, s_m)) \\ &\leq b\varrho_b(R, S) + b^2(\varrho_b(r_n, r_m) + \varrho_b(r_m, s_m)) \\ &\leq (b + b^2)\varrho_b(R, S) + b^2\frac{(\alpha + \beta + \gamma + \delta + \tau)}{1 - (\gamma + \delta + \tau)}K. \end{aligned} \quad (34)$$

Therefore, (s_n) is also a bounded sequence. From (69), it is clear that $(\varrho_b(r_n, r_{n+1}))$ is a decreasing sequence of nonnegative real numbers and hence convergent. Using hypothesis (ii), $(\varrho_b(r_n, r_{n+1}))$ converges to 0. Now, by compact weak proximality of the pair (R, S), there exist two subsequences (r_{n_k}) of (r_n) and (s_{n_k}) of (s_n), such that (r_{n_k}) converges to some $r_\star \in R$ and (s_{n_k}) converges to some $s_\star \in S$. Consequently,

$$\varrho_b(r_{n_k}, s_{n_k}) \longrightarrow \varrho_b(r_\star, s_\star) \text{as } k \longrightarrow \infty. \tag{35}$$

Consequently,

$$\varrho_b(r_\star, s_\star) = \varrho_b(R, S). \tag{36}$$

Thus, $r_\star \in R_0$, which implies $Tr_\star \subset S_0$. For each $\nu \in Tr_\star$, there exists $v \in R$, such that $\varrho_b(v, \nu) = \varrho_b(R, S)$. Now,

$$\varrho_b(R, S) \leq \varrho_b(v, Tr_\star) \leq \varrho_b(v, \nu) = \varrho_b(R, S), \tag{37}$$

which implies

$$\varrho_b(v, Tr_\star) = \varrho_b(R, S). \tag{38}$$

Moreover, we have

$$\varrho_b(r_{n_k+1}, r_\star) \leq b(\varrho_b(r_{n_k+1}, r_{n_k}) + \varrho_b(r_{n_k}, r_\star)). \tag{39}$$

Letting $k \longrightarrow \infty$, we get

$$\lim_{k \longrightarrow \infty} r_{n_k+1} = r_\star, \tag{40}$$

then using the facts

$$\begin{aligned} \varrho_b(r_{n_k+1}, Tr_{n_k}) &= \varrho_b(R, S), \\ \varrho_b(v, Tr_\star) &= \varrho_b(R, S), \end{aligned} \tag{41}$$

we get

$$\begin{aligned} \varrho_b(r_{n_k+1}, v) &\leq \alpha\varrho_b(r_{n_k+1}, r_{n_k}) + \beta\varrho_b(v, r_\star) + \frac{\gamma}{b^2}\varrho_b(r_{n_k}, r_\star) \\ &\quad + \frac{\delta}{b}\varrho_b(r_{n_k+1}, r_\star) + \frac{\tau}{b}\varrho_b(v, r_{n_k}) \\ &\leq \alpha\varrho_b(r_{n_k+1}, r_{n_k}) + \beta\varrho_b(v, r_\star) + \gamma\varrho_b(r_{n_k}, r_\star) \\ &\quad + \delta\varrho_b(r_{n_k+1}, r_\star) + \tau\varrho_b(v, r_{n_k}). \end{aligned} \tag{42}$$

Letting $k \longrightarrow \infty$, we get

$$[1 - (\beta + \tau)]\varrho_b(v, r_\star) \leq 0. \tag{43}$$

It implies $v = r_\star$. Thus, we have $\varrho_b(r_\star, Tr_\star) = \varrho_b(R, S)$, that is, $r_\star$ is a best proximity point of T. This completes the proof.

Now, we give an example to explain our claim. □

Example 17. Let $P = \mathbb{R}$, $R = [1, 2]$, and $S = [1/4, 1/2]$. Consider

$$\varrho_b(p_1, p_2) = |p_1 - p_2|^2, \tag{44}$$

for all $p_1, p_2 \in \mathbb{R}$. Then, ϱ_b is a b-metric on P with $b = 2$. It implies $\varrho_b(R, S) = \{1/4\}$, $R_0 = \{1\}$, and $S_0 = \{1/2\}$; now, define a mapping $T: R \longrightarrow \wp(S)$ as follows:

$$\text{Tr} = \left\{\frac{1}{2^n}: 1 \leq n \leq r\right\}. \tag{45}$$

It implies for each $r \in R_0$, $\text{Tr} \subset S_0$. Now, we check T is a new type of generalized multivalued Hardy and Roger's proximal contractive mapping. Let

$$r_1, r_2, r_3, r_4 \in R. \tag{46}$$

Then, we discuss two possible cases.

Case 1: if

$$r_1, r_2, r_3, r_4 > 1, \tag{47}$$

then

$$\begin{aligned} \varrho_b(r_1, \text{Tr}_3) &\neq d(R, S), \\ \varrho_b(r_2, \text{Tr}_4) &\neq d(R, S). \end{aligned} \tag{48}$$

Case 2: if

$$r_1 = r_2 = r_3 = r_4 = 1, \tag{49}$$

then

$$\begin{aligned} \varrho_b(r_1, \text{Tr}_3) &= d(R, S), \\ \varrho_b(r_2, \text{Tr}_4) &= d(R, S), \\ r_3 &= r_4. \end{aligned} \tag{50}$$

It implies

$$\begin{aligned} \varrho_b(r_1, r_2) = 0 &\leq \alpha\varrho_b(r_1, r_3) + \beta\varrho_b(r_2, r_4) + \frac{\delta}{b}\varrho_b(r_1, r_4) \\ &\quad + \frac{\tau}{b}\varrho_b(r_2, r_3), \end{aligned} \tag{51}$$

where

$$\alpha + \beta + \gamma + 2\tau = 1, \alpha \neq 1, \beta \neq 1, \gamma + \delta + \tau < 1, \tag{52}$$

$\alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}^+$. So, all axioms of Theorem 16 are satisfied. Hence, T has the best proximity points set $\{1\}$.

Theorem 18. *Let R and S be two nonempty subsets of (P, ϱ_b), such that at least one of R and S is bounded, and (R, S) is a compact weak proximal pair. Let $T: R \longrightarrow S$ be a new type of generalized Hardy and Roger's proximal contractive mapping and assume that*

(i) For each $r \in R_0$, $\text{Tr} \in S_0$

(ii) If (r_n) and (s_n) are two bounded sequences in R and S, respectively, such that $(\varrho_b(r_n, s_n))$ converges to $\varrho_b(R, S)$, then $\lim_{n \longrightarrow \infty} \varrho_b(r_n, r_{n+1}) = 0$

Then, T has a unique best proximity point.

Proof. Existence of best proximity point follows from Theorem 16. Now, to prove the uniqueness, consider r_1 and r_2 be two distinct best proximity points of T. Then, we have

$$\begin{aligned}\varrho_b(r_1, \mathrm{Tr}_1) &= \varrho_b(R,S),\\ \varrho_b(r_2, \mathrm{Tr}_2) &= \varrho_b(R,S),\\ r_1 &\neq r_2.\end{aligned} \tag{53}$$

It implies

$$\begin{aligned}\varrho_b(r_1,r_2) &< \alpha\varrho_b(r_1,r_1) + \beta\varrho_b(r_2,r_2) + \frac{\gamma}{b^2}\varrho_b(r_1,r_2) + \frac{\delta}{b}\varrho_b(r_1,r_2) + \frac{\tau}{b}\varrho_b(r_1,r_2),\\ &\leq \alpha\varrho_b(r_1,r_1) + \beta\varrho_b(r_2,r_2) + \gamma\varrho_b(r_1,r_2) + \delta\varrho_b(r_1,r_2) + \tau\varrho_b(r_1,r_2),\end{aligned} \tag{54}$$

so

$$\varrho_b(r_1,r_2) < (\gamma+\delta+\tau)\varrho_b(r_1,r_2) < \varrho_b(r_1,r_2), \tag{55}$$

a contradiction as $\gamma+\delta+\tau<1$. Hence, T has a unique best proximity point. □

Corollary 19. *If we take in Theorem 18* $b=1$ *and* $\alpha=\beta=1/2, \gamma=\delta=\tau=0$, *then we get Theorem 10.*

4. Best Proximity Points Results for a New Type of Multivalued Hardy and Roger's Proximal Cyclic Contractive Mappings in b-metric Space

In this section, we consider new type of multivalued Hardy and Roger's proximal cyclic contractive mapping for the existence of best proximity points.

Theorem 20. *Let R and S be two nonempty subsets of* (P, ϱ_b), *such that at least one of R and S is a bounded subset of P and* (R,S) *is a compact weak proximal pair. Let* $T\colon R\cup S \longrightarrow \wp(R)\cup\wp(S)$ *be a new type of generalized multivalued Hardy and Roger's proximal cyclic contractive mapping and assume that*

(i) For each $r\in R_0$, $Ts\subset S_0$, *and for each* $s\in S_0$, $Ts\subset R_0$, *and*

(ii) If (r_n) *and* (s_n) *are two bounded sequences in R and S, respectively, such that* $(\varrho_b(r_n,s_n))$ *converges to* $\varrho_b(R,S)$, *then* $\lim_{n\longrightarrow\infty}\varrho_b(r_n,r_{n+1})=0$.

Then, there exist $r\in R$ *and* $s\in S$, *such that* $\varrho_b(r,\mathrm{Tr}) = \varrho_b(R,S)$ *and* $\varrho_b(s,Ts)=\varrho_b(R,S)$, *and furthermore,* $\varrho_b(r,s)=\varrho_b(R,S)$.

Proof. Since (R,S) is a compact weakly proximal pair and at least one of R and S is bounded, so by Lemma 9, it follows that $R_0\neq 0$ and $S_0\neq 0$. Let $r_0\in R_0$ and $s_0\in S_0$ imply $\mathrm{Tr}_0\subset S_0$ and $Ts_0\subset R_0$, so there exists $r_1\in R$ and $s_1\in S$, such that

$$\varrho_b(r_1,\nu_1) = \varrho_b(R,S), \tag{56}$$

$\nu_1\in Tr_0$. Continuing this way, we construct sequences (r_n) in R, (s_n) in S, (ν_n) in Tr_{n-1}, and (υ_n) in Ts_{n-1}, such that

$$\varrho_b(r_n,\nu_n) = \varrho_b(R,S), \tag{57}$$

and

$$\varrho_b(s_n,\upsilon_n) = \varrho_b(R,S). \tag{58}$$

It implies

$$\varrho_b(R,S) \leq \varrho_b(r_n, Tr_{n-1}) \leq \varrho_b(r_n,\nu_n) = \varrho_b(R,S), \tag{59}$$

and

$$\varrho_b(R,S) \leq \varrho_b(s_n, Ts_{n-1}) \leq \varrho_b(s_n,\upsilon_n) = \varrho_b(R,S), \tag{60}$$

for all $n\in\mathbb{N}$. Thus, we have

$$\begin{aligned}\varrho_b(r_n, Tr_{n-1}) &= \varrho_b(R,S),\\ \varrho_b(s_n, Ts_{n-1}) &= \varrho_b(R,S).\end{aligned} \tag{61}$$

First, we assume that R is bounded. Then, there exists a positive real number K, such that $\varrho_b(r_n,r_m)\leq K$ for all $n,m\in\mathbb{N}$. Therefore, for all $n,m\in\mathbb{N}$, we have

$$\begin{aligned}\varrho_b(\nu_n,\nu_m) &\leq b(\varrho_b(\nu_n,r_n) + \varrho_b(r_n,\nu_m)),\\ &\leq b(\varrho_b(R,S) + b(\varrho_b(r_n,r_m) + \varrho_b(r_m,\nu_m))),\\ &\leq b(\varrho_b(R,S) + bK + b\varrho_b(R,S)),\end{aligned} \tag{62}$$

implies

$$\varrho_b(\nu_n,\nu_m) \leq (b+b^2)\varrho_b(R,S) + b^2K. \tag{63}$$

Therefore, (ν_n) is bounded. Also, T is cyclic, so for each $s\in S$, $Ts\subset R$, and so $\upsilon_n\in R$, for all $n\in\mathbb{N}$. Therefore, there exists a positive real number K_1, such that $\varrho_b(\upsilon_n,\upsilon_m)\leq K_1$. It implies (υ_n) is bounded, so

$$\begin{aligned}\varrho_b(s_n,s_m) &\leq b(\varrho_b(s_n,\upsilon_n) + \varrho_b(\upsilon_n,s_m)),\\ &\leq b(\varrho_b(R,S) + b(\varrho_b(\upsilon_n,\upsilon_m) + \varrho_b(\upsilon_m,s_m))),\\ &\leq (b+b^2)\varrho_b(R,S) + b^2K_1.\end{aligned} \tag{64}$$

It implies (s_n) is bounded. Thus, (r_n), (s_n), (ν_n), and (v_n) all are bounded sequences. On a similar line, we can prove (r_n), (s_n), (ν_n), and (v_n) are bounded whenever S is bounded. Since (R,S) is a compact weak pair, therefore, there exist subsequences (r_{n_k}), (s_{n_k}), (ν_{n_k}), and (v_{n_k}) of (r_n), (s_n), (ν_n), and (v_n), respectively, such that $r_{n_k} \longrightarrow r_\star \in R$, $s_{n_k} \longrightarrow s_\star \in S$, $\nu_{n_k} \longrightarrow \nu_\star \in S$, and $v_{n_k} \longrightarrow v_\star \in R$, as $k \longrightarrow \infty$. First, we show that $\varrho_b(r_\star, Tr_\star) = \varrho_b(R,S)$. As we have

$$\varrho_b(r_n, Tr_{n-1}) = \varrho_b(R,S), \tag{65}$$

and

$$\varrho_b(r_{n+1}, \mathrm{Tr}_n) = \varrho_b(R,S), \tag{66}$$

so if $\varrho_b(r_{n-1}, r_n) > \varrho_b(R,S)$, then we have

$$\begin{aligned}\varrho_b(r_n, r_{n+1}) &< \alpha\varrho_b(r_n, r_{n-1}) + \beta\varrho_b(r_{n+1}, r_n) + \frac{\gamma}{b^2}\varrho_b(r_{n-1}, r_n)\\ &\quad + \frac{\delta}{b}\varrho_b(r_n, r_n) + \frac{\tau}{b}\varrho_b(r_{n+1}, r_{n-1}),\\ &\le \alpha\varrho_b(r_n, r_{n-1}) + \beta\varrho_b(r_{n+1}, r_n) + \gamma\varrho_b(r_{n-1}, r_n) + \tau(\varrho_b(r_{n-1}, r_n) + \varrho_b(r_n, r_{n+1}))\\ &\le (\alpha+\gamma)\varrho_b(r_{n-1}, r_n) + \beta\varrho_b(r_n, r_{n+1}) + \tau(\varrho_b(r_{n-1}, r_n) + \varrho_b(r_n, r_{n+1})).\end{aligned} \tag{67}$$

So,

$$(1-(\beta+\tau))\varrho_b(r_n, r_{n+1}) < (\alpha+\gamma+\tau)\varrho_b(r_{n-1}, r_n). \tag{68}$$

If $1-(\beta+\tau)=0$, then $\beta+\tau=1$, so (19) implies $\alpha=\gamma=\delta=\tau=0$, and $\beta=1$ is a contradiction. Therefore,

$$\varrho_b(r_n, r_{n+1}) < \varrho_b(r_{n-1}, r_n). \tag{69}$$

If

$$\varrho_b(r_{n-1}, r_n) = \varrho_b(R,S), \tag{70}$$

then

$$\varrho_b(r_n, r_{n+1}) = \varrho_b(r_{n-1}, r_n). \tag{71}$$

Therefore,

$$\varrho_b(r_n, r_{n+1}) \le \varrho_b(r_{n-1}, r_n), \tag{72}$$

for all $n \in \mathbb{N}$, and hence, the sequence $(\varrho_b(r_n, r_{n+1}))$ is a convergent sequence of real numbers. By hypothesis (ii), it follows that $(\varrho_b(r_n, r_{n+1}))$ converges to 0. Now,

$$\begin{aligned}\varrho_b\left(r_{n_k+1}, r_\star\right) &\le b\left(\varrho_b\left(r_{n_k+1}, r_{n_k}\right) + \varrho_b\left(r_{n_k}, r_\star\right)\right)\\ &\longrightarrow 0 \text{ as } k \longrightarrow \infty\end{aligned} \tag{73}$$

.

Therefore, $\lim_{k\longrightarrow\infty} r_{n_k+1} = r_\star$. Again, we have $\varrho_b(r_{n_k}, \nu_{n_k}) \longrightarrow \varrho_b(r_\star, \nu_\star)$ as $k \longrightarrow \infty$, and hence, we get

$$\varrho_b(r_\star, \nu_\star) = \varrho_b(R,S). \tag{74}$$

So, $r_\star \in R_0$ implies $Tr_\star \subset S_0$. Thus, there exists $v \in R$, such that

$$\varrho_b(v, \nu) = \varrho_b(R,S), \tag{75}$$

where $\nu \in Tr_\star$. It implies

$$\varrho_b(v, Tr_\star) = \varrho_b(R,S). \tag{76}$$

If

$$\varrho_b\left(r_{n_k}, r_\star\right) = \varrho_b(R,S), \tag{77}$$

only for finitely many k, then we can exclude those r_{n_k} from (r_{n_k}) and then assume

$$\varrho_b\left(r_{n_k}, r_\star\right) > \varrho_b(R,S) \tag{78}$$

for all k. If

$$\varrho_b\left(r_{n_k}, r_\star\right) = \varrho_b(R,S), \tag{79}$$

for infinitely many k, then we can extract a subsequence $(r_{n_{k_l}})$ from (r_{n_k}), such that

$$\varrho_b\left(r_{n_{k_l}}, r_\star\right) = \varrho_b(R,S), \tag{80}$$

for all l. This gives

$$\lim_{l\longrightarrow\infty} \varrho_b\left(r_{n_{k_l}}, r_\star\right) = \varrho_b(R,S) \text{ implies } \varrho_b(R,S) = 0. \tag{81}$$

From the relations

$$\left.\begin{aligned}\varrho_b(v, Tr_\star) &= \varrho_b(R,S),\\ \varrho_b\left(r_{n_{k_l}+1}, Tr_{n_{k_l}}\right) &= \varrho_b(R,S),\end{aligned}\right\} \tag{82}$$

and

$$\varrho_b\left(r_{n_{k_l}}, r_\star\right) = \varrho_b(R,S), \tag{83}$$

we get

$$\varrho_b\left(r_{n_{k_l+1}}, v\right) = \varrho_b(R,S) = 0, \tag{84}$$

for all l. Taking limit as $l \longrightarrow \infty$, we get

$$\varrho_b(r_\star, v) = 0, \quad (85)$$

so $r_\star = v$. Therefore, we have

$$\varrho_b(r_\star, Tr_\star) = \varrho_b(R, S). \quad (86)$$

Next, we assume that

$$\varrho_b\left(r_{n_{k_l}}, r_\star\right) > \varrho_b(R, S), \quad (87)$$

for all k; then, from relations

$$\left.\begin{aligned} &\varrho_b(v, Tr_\star) = \varrho_b(R, S),\\ &\varrho_b\left(r_{n_{k_l}+1}, Tr_{n_{k_l}}\right) = \varrho_b(R, S), \end{aligned}\right\} \quad (88)$$

we get

$$\varrho_b\left(r_{n_{k_l}+1}, v\right) < \alpha\varrho_b(v, r_\star) + \beta\varrho_b\left(r_{n_{k_l}+1}, r_{n_{k_l}}\right) + \frac{\gamma}{b^2}\varrho_b\left(r_\star, r_{n_{k_l}}\right) + \frac{\delta}{b}\varrho_b\left(r_{n_{k_l}+1}, r_\star\right) + \frac{\tau}{b}\varrho_b\left(v, r_{n_{k_l}}\right). \quad (89)$$

Taking limit as $k \longrightarrow \infty$ in above, we get

$$\begin{aligned} &\varrho_b(v, r_\star) \le \alpha\varrho_b(v, r_\star) + \frac{\tau}{b}\varrho_b(v, r_\star),\\ &\varrho_b(v, r_\star) \le \alpha\varrho_b(v, r_\star) + \tau\varrho_b(v, r_\star). \end{aligned} \quad (90)$$

It implies

$$(1 - (\alpha + \tau))\varrho_b(v, r_\star) \le 0. \quad (91)$$

If $\alpha + \tau = 1$, then (19) implies $\beta = \gamma = \tau = 0$ which implies $\alpha = 1$, a contradiction, so

$$\varrho_b(v, r_\star) = 0 \text{ implies } v = r_\star. \quad (92)$$

Hence,

$$\varrho_b(r_\star, Tr_\star) = \varrho_b(R, S). \quad (93)$$

Similarly, we can prove

$$\varrho_b(s_\star, Ts_\star) = \varrho_b(R, S). \quad (94)$$

Now, let $r \in R, s \in S$, such that

$$\varrho_b(r, \mathrm{Tr}) = \varrho_b(R, S), \quad (95)$$

and

$$\varrho_b(s, Ts) = \varrho_b(R, S). \quad (96)$$

If $\varrho_b(r, s) > \varrho_b(R, S)$, then

$$\varrho_b(r, s) < \alpha\varrho_b(r, r) + \beta\varrho_b(s, s) + \frac{\gamma}{b^2}\varrho_b(r, s) + \frac{\delta}{b}\varrho_b(r, s) + \frac{\tau}{b}\varrho_b(r, s), \quad (97)$$

so

$$\varrho_b(r, s) < \gamma\varrho_b(r, s) + \delta\varrho_b(r, s) + \tau\varrho_b(r, s). \quad (98)$$

It implies

$$(1 - (\gamma + \delta + \tau))\varrho_b(r, s) < 0, \quad (99)$$

which further implies

$$\varrho_b(r, s) < 0, \quad (100)$$

a contradiction. So, $\varrho_b(r, s) = \varrho_b(R, S)$. This completes the proof. □

Theorem 21. *Let R and S be two nonempty subsets of (P, ϱ_b), such that at least one of R and S is a bounded subset of P and (R, S) is a compact weak proximal pair. Let $T\colon R \cup S \longrightarrow R \cup S$ be a new type of generalized Hardy and Roger's proximal cyclic contractive mapping and assume that*

(i) For each $r \in R_0$, $\mathrm{Tr} \in S_0$, and for each $s \in S_0$, $Ts \in R_0$,

(ii) If (r_n) and (s_n) are two bounded sequences in R and S, respectively, such that $(\varrho_b(x_n, y_n))$ converges to $\varrho_b(R, S)$, then $\lim_{n\longrightarrow\infty}\varrho_b(r_n, r_{n+1}) = 0$.

Then, there exist $r \in R$ and $s \in S$, such that $\varrho_b(r, \mathrm{Tr}) = \varrho_b(R, S)$ and $\varrho_b(s, Ts) = \varrho_b(R, S)$. Furthermore, $\varrho_b(r, s) = \varrho_b(R, S)$.

Proof. Following Theorem 20, we can get the required result. □

Corollary 22. *If we take $b = 1$, $\alpha = \beta = 1/2$, and $\gamma = \delta = \tau = 0$ in Theorem 21, we get Theorem 11.*

5. Some Fixed Points Results

In this section, we derive some fixed points results from our main results.

Theorem 23. *Let (P, ϱ_b) be a boundedly compact b-metric space; then,*

(i) A mapping $T\colon P \longrightarrow \wp(P)$, such that for $q_1, q_2 \in P$, there exist $p_1 \in Tq_1$ and $p_2 \in Tq_2$, such that

$$\varrho_b(p_1, p_2) < \alpha\varrho_b(p_1, q_1) + \beta\varrho_b(p_2, q_2) + \frac{\gamma}{b^2}\varrho_b(q_1, q_2) + \frac{\delta}{b}\varrho_b(p_1, q_2) + \frac{\tau}{b}\varrho_b(p_2, q_1), \text{s.} \quad (101)$$

For $q_1 \neq q_2$
And

$$\left.\begin{aligned} &\varrho_b(p_1, p_2) \le \alpha\varrho_b(p_1, q_1) + \beta\varrho_b(p_2, q_2) +\\ &\frac{\delta}{b}\varrho_b(p_1, q_2) + \frac{\tau}{b}\varrho_b(p_2, q_1) \end{aligned}\right\}, \quad (102)$$

For $q_1 = q_2$, *where*

$$\alpha + \beta + \gamma + 2\tau = 1, \beta \neq 1, \gamma + \delta + \tau < 1, \tag{103}$$

$\alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}^+$.

(ii) If (p_n) *is a bounded sequence in* P, *then* $\lim_{n\longrightarrow\infty}(p_n, p_{n+1}) = 0$.

Then, $Fix(T)$ *(set of fixed points of* T*) is nonempty*

Theorem 24. *Let* (P, ϱ_b) *be a boundedly compact b-metric space; then,*

(i) A mapping $T\colon P \longrightarrow P$, *such that for all* $q_1, q_2 \in P$ *with* $q_1 \neq q_2$,

$$\varrho_b(Tq_1, Tq_2) < \alpha\varrho_b(Tq_1, q_1) + \beta\varrho_b(Tq_2, q_2) + \frac{\gamma}{b^2}\varrho_b(q_1, q_2) + \frac{\delta}{b}\varrho_b(Tq_1, q_2) + \frac{\tau}{b}\varrho_b(Tq_2, q_1), \tag{104}$$

where

$$\alpha + \beta + \gamma + 2\tau = 1, \beta \neq 1, \gamma + \delta + \tau < 1, \tag{105}$$

$\alpha, \beta, \gamma, \delta, \tau \in \mathbb{R}^+$.

(ii) If (p_n) *is a bounded sequence in* P, *then* $\lim_{n\longrightarrow\infty}(p_n, p_{n+1}) = 0$

Then, $Fix(T)$ *is singleton.*

6. Conclusion

We presented a new type of generalized multivalued Hardy and Roger's proximal contractive and proximal cyclic contractive mappings in b-metric spaces and developed results for the existence of best proximity points. Furthermore, we have derived results for the existence and uniqueness of best proximity points for new type of generalized Hardy and Roger's proximal contractive and proximal cyclic contractive mappings. Our results are the generalization of the results already existing in literature.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Integral Equations Approach in Complex-Valued Generalized *b*-Metric Spaces

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In this paper, we study a rational type common fixed-point theorem (CFP theorem) in complex-valued generalized b-metric spaces (G_b-metric spaces) by using three self-mappings under the generalized contraction conditions. We find CFP and prove its uniqueness. To justify our result, we provide an illustrative example. Furthermore, we present a supportive application of the three Urysohn type integral equations (UTIEs) for the validity of our result. The UTIEs are

$$\begin{aligned}\gamma_1(q) &= \int_{k_1}^{k_2} Q_1(q,r,\gamma_1(r))dr + \hbar_1(q),\\ \gamma_2(q) &= \int_{k_1}^{k_2} Q_2(q,r,\gamma_2(r))dr + \hbar_2(q),\\ \gamma_3(q) &= \int_{k_1}^{k_2} Q_3(q,r,\gamma_3(r))dr + \hbar_3(q),\end{aligned} \tag{1}$$

where $q \in [k_1,k_2], \gamma_1,\gamma_2,\gamma_3,\hbar_1,\hbar_2,\hbar_3 \in \Upsilon$, where $\Upsilon = C([k_1,k_2],\mathbb{R}^n)$ is the set of all real-valued continuous functions defined on $[k_1,k_2]$ and Q_1,Q_2,Q_3: $[k_1, k_2]\times[k_1,k_2]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

1. Introduction

In 1922, Banach [1] proved a “Banach contraction principle” which is stated as “a single-valued contractive type mapping in a complete metric space has a unique FP.” Later on, this principle generalized in many directions, and some researchers contributed to the theory of FP. In [2], Nazam et al. presented the idea of weakly increasing F-contractions and established some results in ordered partial-metric space with an application. Hu and Gu [3] presented a new idea of Menger probabilistic S-metric space by using the concepts of probabilistic and S-metric spaces. They investigated its topological characteristics and established some FP theorems with illustrative examples.

Bakhtin [4] introduced the idea of b-metric space. After that, Czerwik [5] established some FP results by using b-metric spaces. In [6], Boriceanu et al. extended the concept of fractal operator theory by introducing it in b-metric spaces and presented some generalized CFP results. While, Akkouchi [7] used the concept of an implicit relation in the said spaces and established CFP results for contractive type maps. Došenović at el. [8] discussed, complemented, improved, generalized, and enriched some FP results of $(\beta-\psi_1-\psi_2)$ contraction in ordered b-metric spaces. They made a different approach of taking Picard’s sequence which help shorten the proof comparing other previous studies in literature. They also complimented and enriched CFP results for $\beta_{q,\phi}$ s, ψ contraction maps.

Delfani et al. [9] contributed in ordered b-metric spaces by establishing FP theorems. In these results, they introduced the generalizations of $F-ts$ and $(\psi, \phi, F-ts)$ contractions. They also provided a suitable example to verify their FP result. In [10], Karapinar et al. contributed in b-metric spaces by generalizing it to prove FP results in view of (α, ψ)-Meir-Keeler type contractions. These results improved and unified some previous results. Abdeljawad et al. [11] extended the b-metric spaces to double controlled metric spaces by improving control functions $\alpha(\gamma_1, \gamma_2)$ and $\mu(\gamma_1, \gamma_2)$ on the right side of b-triangle, that is, $d(\gamma_1, \gamma_2) \leq \alpha(\gamma_1, \gamma_3) d(\gamma_1, \gamma_3) + \mu(\gamma_3, \gamma_2) d(\gamma_3, \gamma_2), \forall \gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. They also provided some examples in which two functions are not comparable.

Mustafa and Sims [12] introduced the generalized idea of metric space and established some FP theorems by using Dhage's theory. Later on, Mustafa et al. [13] proved some modified contractive-type FP results in this space. Abbas and Rhoades [14] started to investigate CFP results in the said spaces. In [15], Chugh et al. established the property P in G-metric space and proved some results. In this context, Mohanta et al. [16] contributed by establishing a CFP result which improved and supplemented some of the existing results. Mustafa introduced a mapping pair and obtained many CFP results for different contraction conditions. He supported these results through examples.

In 2014, Aghajani et al. [17] presented the idea of G_b-metric space and proved a weakly compatible CFP theorem. Aydi [18] improved, complemented, unified, and generalized some well-known existing results in said spaces and established some coupled and tripled coincidence point results. Gupta in [19] extended some existing results from literature and obtained various FP results in G_b-metric spaces. In [20], Makran et al. proved general CFP theorem by using multivalued maps and established its application. Mebawondu et al. [21] proved FP results for a different contraction type maps which involve C-class, α_s^δ-admissible, Suzuki-type maps in G_b-metric spaces.

Ege [22] gave the idea of complex-valued G_b-metric space and proved "Banach contraction principle and Kannan's contraction theorems for FP." Ege [23] used the idea of α-series to prove CFP results in said spaces. He also introduced $\alpha-\psi$ contraction maps to prove CFP results. Kumar and Vashistha [24] introduced the idea of coupled FP for mapping in the said space. They proved coupled FP results and supported it by providing an example satisfying their main result. Recently, Mehmood et al. [25] proved some CFP theorems by using compatible single-valued contractive type mappings on complex-valued b-metric spaces with an application.

This article presents a contraction theorem in complex-valued G_b-metric spaces by using three self-maps to establish a generalized CFP-theorem. This result improves, extends, and modifies some of the existing results (e.g, [22, 26]). We present an example to support our work. We also present an application of the three UTIEs for the existence of a common solution to support our work. This study is organized as follows: Section 2 consists of preliminary concepts. In Section 3, we present a CFP theorem with an illustrative example. While in Section 4, we present an application to support our main work. Finally, in Section 5, we discuss the conclusion of our work.

2. Preliminaries

Let the complex numbers be denoted by $\mathbb{C}$ and $z_i, z_{ii} \in \mathbb{C}$. Now, we define $\leq$ as $z_i \leq z_{ii}$, iff $R_e(z_i) \leq R_e(z_{ii})$ and $I_m(z_i) \leq I_m(z_{ii})$, where the real and imaginary parts of the complex number are denoted by R_e and I_m, respectively. Accordingly, $z_i \leq z_{ii}$, if any one of the following conditions holds:

(1) $R_e(z_i) = R_e(z_{ii})$ and $I_m(z_i) = I_m(z_{ii})$,
(2) $R_e(z_i) < R_e(z_{ii})$ and $I_m(z_i) = I_m(z_{ii})$,
(3) $R_e(z_i) = R_e(z_{ii})$ and $I_m(z_i) < I_m(z_{ii})$,
(4) $R_e(z_i) < R_e(z_{ii})$ and $I_m(z_i) < I_m(z_{ii})$.

Particularly, we can write $z_i \leq z_{ii}$ if $z_i \neq z_{ii}$ and one of (2), (3), and (4) is satisfied.

Remark 1 (see [27]). The following given properties can be held and verified if

(1) $\beta_1, \beta_2 \in \mathbb{R}$ and $\beta_1 \leq \beta_2 \Rightarrow \beta_1 y \leq \beta_2 y \ \forall y \in \mathbb{C}$,
(2) $0 \leq z_i \leq z_{ii} \Rightarrow |z_i| < |z_{ii}|$,
(3) $z_i \leq z_{ii}$ and $z_{ii} < z_{iii} \Rightarrow z_i < z_{iii}$.

Definition 1 (see [17]). Let $\Upsilon \neq \varnothing$ set and $b \geq 1$ be a given real number. A mapping $G: \Upsilon \times \Upsilon \times \Upsilon \longrightarrow \mathbb{R}^+$ is called a G_b-metric if G holds the following axioms:

(1) $G(\gamma_1, \gamma_2, \gamma_3) = 0$ if $\gamma_1 = \gamma_2 = \gamma_3$
(2) $0 < G(\gamma_1, \gamma_1, \gamma_2)$ for all $\gamma_1, \gamma_2 \in \Upsilon$ with $\gamma_1 \neq \gamma_2$
(3) $G(\gamma_1, \gamma_1, \gamma_2) \leq G(\gamma_1, \gamma_2, \gamma_3)$ for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$ with $\gamma_2 \neq \gamma_3$
(4) $G(\gamma_1, \gamma_2, \gamma_3) = G(p\{\gamma_1, \gamma_2, \gamma_3\})$, where p is a permutation of $\gamma_1, \gamma_2, \gamma_3$
(5) $G(\gamma_1, \gamma_2, \gamma_3) \leq b[G(\gamma_1, a, a) + G(a, \gamma_2, \gamma_3)]$ for all $\gamma_1, \gamma_2, \gamma_3, a \in \Upsilon$

Then, a pair (Υ, G) is called a G_b-metric space.

Note that each G-metric space is a G_b-metric space with $b = 1$.

Definition 2 (see [22]). Let $\Upsilon \neq \varnothing$ set and $b \geq 1$ be a given real number. A mapping $G: \Upsilon \times \Upsilon \times \Upsilon \longrightarrow \mathbb{C}$ is called a complex-valued G_b-metric if G holds the following axioms:

(1) $G(\gamma_1, \gamma_2, \gamma_3) = 0$ if $\gamma_1 = \gamma_2 = \gamma_3$
(2) $0 < G(\gamma_1, \gamma_1, \gamma_2)$ for all $\gamma_1, \gamma_2 \in \Upsilon$ with $\gamma_1 \neq \gamma_2$
(3) $G(\gamma_1, \gamma_1, \gamma_2) \leq G(\gamma_1, \gamma_2, \gamma_3)$ for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$ with $\gamma_2 \neq \gamma_3$

(4) $G(\gamma_1,\gamma_2,\gamma_3) = G(p\{\gamma_1,\gamma_2,\gamma_3\})$, where p is a permutation of $\gamma_1,\gamma_2,\gamma_3$

(5) $G(\gamma_1,\gamma_2,\gamma_3) \le b[G(\gamma_1,a,a) + G(a,\gamma_2,\gamma_3)]$ for all $\gamma_1,\gamma_2,\gamma_3,a \in \Upsilon$

Then, a pair (Υ, G) is called a complex-valued G_b-metric space.

Example 1. Let $\Upsilon = 0,\infty)$ and a mapping $G\colon \Upsilon\times\Upsilon\times\Upsilon \longrightarrow \mathbb{C}$ be defined as

$$G(\gamma_1,\gamma_2,\gamma_3) = \left(\frac{3}{4}|\gamma_1-\gamma_2| + \frac{3}{4}|\gamma_2-\gamma_3| + \frac{3}{4}|\gamma_3-\gamma_1|\right)^2 + i\left(\frac{3}{4}|\gamma_1-\gamma_2| + \frac{3}{4}|\gamma_2-\gamma_3| + \frac{3}{4}|\gamma_3-\gamma_1|\right)^2, \tag{2}$$

for all $\gamma_1,\gamma_2,\gamma_3 \in \Upsilon$. Then, G is a complex-valued G_b-metric with $b=2$, but it is not G-metric. To see this, let $\gamma_1 = 3, \gamma_2 = 5, \gamma_3 = 7, a = 7/2$, and we get $G(3,5,7) = 576/16 + 576/16i$, $G(3,7/2,7/2) = 9/16 + 9/16i$, and $G(7/2,5,7) = 441/16 + 441/16i$; thus,$G(3,5,7) = 576/16 + 576/16i \nleq 450/16 + 450/16i = G(3,7/2,7/2) + G(7/2,5,7)$.

Clearly, property (5) of definition (2.3) is satisfied with $b=2$; hence,

$G(\gamma_1,\gamma_2,\gamma_3) = (3/4|\gamma_1-\gamma_2| + 3/4|\gamma_2-\gamma_3| + 3/4|\gamma_3-\gamma_1|)^2 + i(3/4|\gamma_1-\gamma_2| + 3/4|\gamma_2-\gamma_3| + 3/4|\gamma_3-\gamma_1|)^2$ is G_b-metric.

Proposition 1 (see [22]). *Let (Υ, G) be a complex-valued G_b-metric space. Then, $\forall \gamma_1,\gamma_2,\gamma_3 \in \Upsilon$.*

(1) $G(\gamma_1,\gamma_2,\gamma_3) \le b(G(\gamma_1,\gamma_1,\gamma_2) + G(\gamma_1,\gamma_1,\gamma_3))$

(2) $G(\gamma_1,\gamma_2,\gamma_2) \le 2bG(\gamma_1,\gamma_1,\gamma_2)$

Definition 3 (see [22]). Let (Υ, G) be a complex-valued G_b-metric space and $\{\gamma_j\}$ be a sequence in Υ.

(1) $\{\gamma_j\}$ is complex-valued G_b-convergent to γ if for every $0 < a \in \mathbb{C}$, $\exists k \in \mathbb{N}$, such that $G(\gamma,\gamma_j,\gamma_m) < a, \forall j,m \ge k$.

(2) A sequence $\{\gamma_j\}$ is called complex-valued G_b-Cauchy if for every $0 < a \in \mathbb{C}$, $\exists k \in \mathbb{N}$, such that $G(\gamma_j,\gamma_m,\gamma_l) < a, \forall j,m,l \ge k$.

(3) If every complex-valued G_b-Cauchy sequence is complex-valued G_b-convergent in (Υ, G), then a pair (Υ, G) is called complex-valued G_b-complete.

Proposition 2 (see [22]). *Let (Υ, G) be a complex-valued G_b-metric space and $\{\gamma_j\}$ be a sequence in Υ. Then, $\{\gamma_j\}$ is complex-valued G_b-convergent to γ if and only if $|G(\gamma,\gamma_j,\gamma_m)| \longrightarrow 0$ as $n,m \longrightarrow \infty$.*

Theorem 1 (see [22]). *Let (Υ, G) be a complex-valued G_b-metric space; then, for a sequence $\{\gamma_j\}$ in Υ and a point $\gamma \in \Upsilon$, the following are equivalent:*

(1) $\{\gamma_j\}$ is complex-valued G_b-convergent to γ

(2) $|G(\gamma_j,\gamma_j,\gamma)| \longrightarrow 0$ as $j \longrightarrow \infty$

(3) $|G(\gamma_j,\gamma,\gamma)| \longrightarrow 0$ as $j \longrightarrow \infty$

(4) $|G(\gamma_m,\gamma_j,\gamma)| \longrightarrow 0$ as $j \longrightarrow \infty$

Proposition 3 (see [22]). *Let (Υ, G) be a complex-valued G_b-metric space and $\{\gamma_j\}$ be a sequence in Υ. Then, $\{\gamma_j\}$ is a complex-valued G_b-convergent to γ if and only if $|G(\gamma,\gamma_j,\gamma_m)| \longrightarrow 0$ as $j,m \longrightarrow \infty$.Proof: Assume that $\{\gamma_j\}$ is complex-valued G_b-convergent to γ and let*

$$a = \frac{\varepsilon}{\sqrt{2}} + i\frac{\varepsilon}{\sqrt{2}}, \forall \varepsilon > 0. \tag{3}$$

Then, we have $0 < a \in \mathbb{C}$, and there is a natural number k, such that $G(\gamma,\gamma_j,\gamma_m) < a$ for all $n,m \ge k$. Thus, $|G(\gamma,\gamma_j,\gamma_m)| < |a| = \varepsilon$ for all $j,m \ge k$, and so, $|G(\gamma,\gamma_j,\gamma_m)| \longrightarrow 0$ as $j,m \longrightarrow \infty$.

Suppose that $|G(\gamma,\gamma_j,\gamma_m)| \longrightarrow 0$ as $j,m \longrightarrow \infty$. For a given $a \in \mathbb{C}$ with $0 < a$, there exists a real number $\delta > 0$, such that for $z \in \mathbb{C}$,

$$|z| < \delta \Rightarrow z < a. \tag{4}$$

Considering δ, we have a natural number k, such that $|G(\gamma,\gamma_j,\gamma_m)| < \delta$ for all $j,m \ge k$. This means that $G(\gamma,\gamma_j,\gamma_m) < a$ for all $j,m \ge k$, i.e., $\{\gamma_j\}$ is complex-valued G_b-convergent to γ.

3. Main Result

Theorem 2. *Let (Υ, G) be a complete complex-valued G_b-metric space with coefficient $b \ge 1$ and $F_1,F_2,F_3\colon \Upsilon \longrightarrow \Upsilon$ be mappings satisfying*

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \le \beta_1 G(\gamma_1,\gamma_2,\gamma_3) + \beta_2 W(\gamma_1,\gamma_2,\gamma_3), \tag{5}$$

where

$$W(\gamma_1,\gamma_2,\gamma_3) = \max\left\{\begin{array}{c} G(\gamma_1,\gamma_2,F_2\gamma_2), G(F_1\gamma_1,F_1\gamma_1,\gamma_2), G(F_1\gamma_1,\gamma_2,\gamma_2), \\ G(F_2\gamma_2,F_2\gamma_2,\gamma_3), G(F_2\gamma_2,\gamma_3,\gamma_3), \\ \dfrac{G(\gamma_1,F_1\gamma_1,F_1\gamma_1)\cdot G(\gamma_2,F_2\gamma_2,F_2\gamma_2)}{1+G(\gamma_1,\gamma_2,\gamma_2)}, \\ \dfrac{G(\gamma_2,F_2\gamma_2,F_2\gamma_2)\cdot G(\gamma_3,F_3\gamma_3,F_3\gamma_3)}{1+G(F_1\gamma_1,\gamma_3,\gamma_3)}, \\ \dfrac{G(\gamma_3,F_3\gamma_3,F_3\gamma_3)\cdot G(\gamma_1,F_1\gamma_1,F_1\gamma_1)}{1+G(F_2\gamma_2,F_3\gamma_3,F_3\gamma_3)} \end{array}\right\}, \tag{6}$$

for all $\gamma_1,\gamma_2,\gamma_3 \in \Upsilon$, $\beta_1,\beta_2 \in 0, 1/3)$, such that $(\beta_1+\beta_2) < 1/3$, $b\beta_2 < 1/3$, and $(\beta_1 + 2b\beta_2) < 2/3$. Then, F_1, F_2, and F_3 have a unique CFP in Υ.

Proof. Fix $\gamma_0 \in \Upsilon$, and $\{\gamma_j\}$ be a sequence in Υ, such that

$$\begin{aligned} \gamma_{3n+1} &= F_1\gamma_{3j}, \\ \gamma_{3j+2} &= F_2\gamma_{3j+1}, \\ \gamma_{3j+3} &= F_3\gamma_{3j+2} \forall n \geq 0. \end{aligned} \tag{7}$$

Now, by using (5),

$$\begin{aligned} G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3}) &= G(F_1\gamma_{3j},F_2\gamma_{3j+1},F_3\gamma_{3j+2}) \\ &\leq \beta_1 G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}) \\ &\quad + \beta_2 W(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}). \end{aligned} \tag{8}$$

This implies that

$$\begin{aligned} \left|G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3})\right| &\leq \beta_1 \left|G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2})\right| \\ &\quad + \beta_2 \left|W(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2})\right| \end{aligned} \tag{9}$$

where

$$\begin{aligned} W(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}) &= \max\left\{\begin{array}{c} G(\gamma_{3j},\gamma_{3j+1},F_2\gamma_{3j+1}), G(F_1\gamma_{3j},F_1\gamma_{3j},\gamma_{3j+1}), G(F_1\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}), \\ G(F_2\gamma_{3j+1},F_2\gamma_{3j+1},\gamma_{3j+2}), G(F_2\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}), \\ \dfrac{G(\gamma_{3j},F_1\gamma_{3j},F_1\gamma_{3j})\cdot G(\gamma_{3j+1},F_2\gamma_{3j+1},F_2\gamma_{3j+1})}{1+G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})}, \\ \dfrac{G(\gamma_{3j+1},F_2\gamma_{3j+1},F_2\gamma_{3j+1})\cdot G(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2})}{1+G(F_1\gamma_{3j},\gamma_{3j+2},\gamma_{3j+2})}, \\ \dfrac{G(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2})\cdot G(\gamma_{3j},F_1\gamma_{3j},F_1\gamma_{3j})}{1+G(F_2\gamma_{3j+1},F_3\gamma_{3j+2},F_3\gamma_{3j+2})} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}), G(\gamma_{3j+1},\gamma_{3j+1},\gamma_{3j+1}), G(\gamma_{3j+1},\gamma_{3j+1},\gamma_{3j+1}), \\ G(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}), G(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}), \\ \dfrac{G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})\cdot G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2})}{1+G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})}, \\ \dfrac{G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2})\cdot G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})}{1+G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2})}, \\ \dfrac{G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})\cdot G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})}{1+G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})} \end{array}\right\}. \end{aligned} \tag{10}$$

This implies that

$$\begin{aligned}
&\left|W\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|\\
&\leq \max\left\{\begin{array}{c}
\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\left|G\left(\gamma_{3j+1},\gamma_{3j+1},\gamma_{3j+1}\right)\right|,\left|G\left(\gamma_{3j+1},\gamma_{3j+1},\gamma_{3j+1}\right)\right|,\\
\left|G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right)\right|,\left|G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right)\right|,\\
\dfrac{\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|\cdot\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|}{\left|1+G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|},\\
\dfrac{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|\cdot\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}{\left|1+G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|},\\
\dfrac{\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|\cdot\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|}{\left|1+G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}
\end{array}\right\}\\
&\leq \max\left\{\begin{array}{c}
\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|,\\
\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|,\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|
\end{array}\right\}.
\end{aligned} \tag{11}$$

By using Definition 2.3 (3) and after simplification, we get that

$$\left|W\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|\leq \max\left\{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3}\right)\right|,\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|\right\}. \tag{12}$$

Now, there are two possibilities:

(i) If $|G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3})|$ is a maximum term in $\{|G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3})|,|G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2})|\}$, then $|W(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2})| = |G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3})|$; so, after simplification, (3.3) can be written as

$$\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq a_1\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\text{where}\quad a_1=\beta_1/\left(1-\beta_2\right). \tag{13}$$

(ii) If $|G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2})|$ is a maximum term in $\{|G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3})|,|G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2})|\}$, then $|W(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2})| = |G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2})|$; so, after simplification, (3.3) can be written as

$$\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq a_2\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\text{where}\quad a_2=\left(\beta_1+\beta_2\right). \tag{14}$$

Let $a{:}\ = \max\{a_1,a_2\}<1/3$; then, from (13) and (14), for all $n\geq 0$, we have

$$\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq a\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|. \tag{15}$$

Similarly,

$$\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|\leq a\left|G\left(\gamma_{3j-1},\gamma_{3j},\gamma_{3j+1}\right)\right|. \tag{16}$$

Now, from (16) and (15) and by induction, we have that

$$\begin{aligned}\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+3}\right)\right| &\leq a\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|\\ &\leq a^2\left|G\left(\gamma_{3j-1},\gamma_{3j},\gamma_{3j+1}\right)\right.\\ &\leq \cdots \leq a^{3j+1}\left|G\left(\gamma_0,\gamma_1,\gamma_2\right)\right|.\end{aligned} \tag{17}$$

Let $m,n\in\mathbb{N}$ and $>n$, and we have that

$$\begin{aligned}\left|G\left(\gamma_j,\gamma_m,\gamma_m\right)\right| &\leq b\left|G\left(\gamma_j,\gamma_{n+1},\gamma_{n+1}\right)\right|+b\left|G\left(\gamma_{n+1},\gamma_m,\gamma_m\right)\right|\\ &\leq b\left|G\left(\gamma_j,\gamma_{n+1},\gamma_{n+1}\right)\right|+b^2\left|G\left(\gamma_{n+1},\gamma_{n+2},\gamma_{n+2}\right)\right|+\cdots+b^{m-n}\left|G\left(\gamma_{m-1},\gamma_m,\gamma_m\right)\right|\\ &\leq ba^n\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|+b^2a^{n+1}\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|+\cdots+b^{m-n}a^{m-1}\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &\leq\left[ba^n+b^2a^{n+1}+\cdots+b^{m-n}a^{m-1}\right]\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &=\left[ba^n+b^2a^{n+1}+\cdots+b^{m-n}a^{m-1}\right]\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &=ba^n\left[1+ba+b^2a^2\ldots+b^{m-(n+1)}a^{m-(n+1)}\right]\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &=ba^n\sum_{t=0}^{m-(n+1)}b^ta^t\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &\leq ba^n\sum_{t=0}^{\infty}b^ta^t\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\\ &=\frac{ba^n}{1-ba}\left|G\left(\gamma_0,\gamma_1,\gamma_1\right)\right|\longrightarrow 0, as\, n\longrightarrow\infty.\end{aligned} \tag{18}$$

By Proposition 2.5 (1), we have $|G(\gamma_j,\gamma_m,\gamma_l)|\leq b\left\{|G(\gamma_j,\gamma_m,\gamma_m)|+|G(\gamma_l,\gamma_m,\gamma_m)|\right\}$ for $n,m,l\in\mathbb{N}$. If we take limit as $n,m,l\longrightarrow\infty$, we obtain $|G(\gamma_j,\gamma_m,\gamma_l)|\longrightarrow 0$. It implies $\left\{\gamma_j\right\}$ is a G_b-Cauchy sequence. Since, Υ is complete complex-valued G_b-metric space, $\exists\xi\in\Upsilon$, such that, $\gamma_j\longrightarrow\xi$, as $n\longrightarrow\infty$ or $\lim_{n\longrightarrow\infty}\gamma_j=\xi$. We have to show that $F_1\xi=\xi$; by contrary case, let $F_1\xi\neq\xi$. Now by (5),

$$\begin{aligned}G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right) &= G\left(F_1\xi,F_2\gamma_{3j+1},F_3\gamma_{3j+2}\right)\\ &\leq\beta_1G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\\ &\quad+\beta_2W\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right).\end{aligned} \tag{19}$$

This implies that

$$\left|G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq\beta_1\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|+\beta_2\left|W\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|, \tag{20}$$

where

$$W\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)=\max\left\{\begin{array}{c}G\left(\xi,\gamma_{3j+1},F_2\gamma_{3j+1}\right),G\left(F_1\xi,F_1\xi,\gamma_{3j+1}\right),G\left(F_1\xi,\gamma_{3j+1},\gamma_{3j+1}\right),\\ G\left(F_2\gamma_{3j+1},F_2\gamma_{3j+1},\gamma_{3j+2}\right),G\left(F_2\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right),\\ \dfrac{G(\xi,F_1\xi,F_1\xi)\cdot G\left(\gamma_{3j+1},F_2\gamma_{3j+1},F_2\gamma_{3j+1}\right)}{1+G\left(\xi,\gamma_{3j+1},\gamma_{3j+1}\right)},\\ \dfrac{G\left(\gamma_{3j+1},F_2\gamma_{3j+1},F_2\gamma_{3j+1}\right)\cdot G\left(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)}{1+G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+2}\right)},\\ \dfrac{G\left(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)\cdot G(\xi,F_1\xi,F_1\xi)}{1+G\left(F_2\gamma_{3j+1},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)}\end{array}\right\}$$
$$=\max\left\{\begin{array}{c}G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right),G\left(F_1\xi,F_1\xi,\gamma_{3j+1}\right),G\left(F_1\xi,\gamma_{3j+1},\gamma_{3j+1}\right),\\ G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right),G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right),\\ \dfrac{G(\xi,F_1\xi,F_1\xi)\cdot G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)}{1+G\left(\xi,\gamma_{3j+1},\gamma_{3j+1}\right)},\\ \dfrac{G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\cdot G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)}{1+G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+2}\right)},\\ \dfrac{G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\cdot G(\xi,F_1\xi,F_1\xi)}{1+G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)}\end{array}\right\}. \tag{21}$$

This implies that

$$\left|W\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|\leq\max\left\{\begin{array}{c}\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\left|G\left(F_1\xi,F_1\xi,\gamma_{3j+1}\right)\right|,\left|G\left(F_1\xi,\gamma_{3j+1},\gamma_{3j+1}\right)\right|,\\ \left|G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right)\right|,\left|G\left(\gamma_{3j+2},\gamma_{3j+2},\gamma_{3j+2}\right)\right|,\\ \dfrac{\left|G(\xi,F_1\xi,F_1\xi)\right|\cdot\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|}{\left|1+G\left(\xi,\gamma_{3j+1},\gamma_{3j+1}\right)\right|},\\ \dfrac{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|\cdot\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}{\left|1+G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+2}\right)\right|},\\ \dfrac{\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|\cdot\left|G\xi,F_1\xi,F_1\xi\right|}{\left|1+G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}\end{array}\right\}. \tag{22}$$

After simplification, we get that

$$\left|W\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|\leq\max\left\{\begin{array}{c}\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\left|G\left(F_1\xi,F_1\xi,\gamma_{3j+1}\right)\right|,\left|G\left(F_1\xi,\gamma_{3j+1},\gamma_{3j+1}\right)\right|,\\ \dfrac{\left|G\left(\xi,F_1\xi,F_1\xi\right)\right|\cdot\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|}{\left|1+G\left(\xi,\gamma_{3j+1},\gamma_{3j+1}\right)\right|},\\ \dfrac{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|\cdot\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}{\left|1+G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+2}\right)\right|},\\ \dfrac{\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|\cdot\left|G\left(\xi,F_1\xi,F_1\xi\right)\right|}{\left|1+G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}\end{array}\right\}. \tag{23}$$

Now, there are six possibilities:

(i) f $|G(\xi,\gamma_{3j+1},\gamma_{3j+2})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq\beta_1\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|+\beta_2\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|. \tag{24}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G\left(F_1\xi,\xi,\xi\right)\right|\leq\beta_1|G(\xi,\xi,\xi)|+\beta_2|G(\xi,\xi,\xi)|. \tag{25}$$

This implies that $|G(F_1\xi,\xi,\xi)|=0$; thus,

$$F_1\xi=\xi. \tag{26}$$

(ii) If $|G(F_1\xi,F_1\xi,\gamma_{3j+1})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq\beta_1\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|+\beta_2\left|G\left(F_1\xi,F_1\xi,\gamma_{3j+1}\right)\right|. \tag{27}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\begin{aligned}\left|G\left(F_1\xi,\xi,\xi\right)\right|&\leq\beta_1|G(\xi,\xi,\xi)|+\beta_2\left|G\left(F_1\xi,F_1\xi,\xi\right)\right|\\&\leq 2b\beta_2\left|G\left(F_1\xi,\xi,\xi\right)\right|\because(\text{Definition 2.5 (2)}).\end{aligned} \tag{28}$$

This implies that $(1-2b\beta_2)|G(F_1\xi,\xi,\xi)|\leq 0$ is a contradiction, where $(1-2b\beta_2)\neq 0\Rightarrow|G(F_1\xi,\xi,\xi)|=0$; thus,

$$F_1\xi=\xi. \tag{29}$$

(iii) If $|G(F_1\xi,\gamma_{3j+1},\gamma_{3j+1})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq\beta_1\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|+\beta_2\left|G\left(F_1\xi,\gamma_{3j+1},\gamma_{3j+1}\right)\right|. \tag{30}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G\left(F_1\xi,\xi,\xi\right)\right|\leq\beta_1|G(\xi,\xi,\xi)|+\beta_2\left|G\left(F_1\xi,\xi,\xi\right)\right|. \tag{31}$$

This implies that $(1-\beta_2)|G(F_1\xi,\xi,\xi)|\leq 0$ is a contradiction, where $(1-\beta_2)\neq 0\Rightarrow|G(F_1\xi,\xi,\xi)|=0$; thus,

$$F_1\xi=\xi. \tag{32}$$

(iv) If $|G(\xi,F_1\xi,F_1\xi)|\cdot|G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2})|/|1+G(\xi,\gamma_{3j+1},\gamma_{3j+1})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq\beta_1\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|+\beta_2\frac{\left|G\left(\xi,F_1\xi,F_1\xi\right)\right|\cdot\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|}{\left|1+G\left(\xi,\gamma_{3j+1},\gamma_{3j+1}\right)\right|}. \tag{33}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$|G(F_1\xi,\xi,\xi)|\leq\beta_1|G(\xi,\xi,\xi)|+\beta_2\frac{|G(\xi,F_1\xi,F_1\xi)|\cdot|G(\xi,\xi,\xi)|}{|1+G(\xi,\xi,\xi)|}. \tag{34}$$

This implies that $|G(F_1\xi,\xi,\xi)|=0$; thus,

$$F_1\xi=\xi. \tag{35}$$

(v) If $|G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2})|\cdot|G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})|/|1+G(F_1\xi,\gamma_{3j+2},\gamma_{3j+2})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq\beta_1\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|+\beta_2\frac{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|\cdot\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}{\left|1+G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+2}\right)\right|} \tag{36}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$|G(F_1\xi,\xi,\xi)|\leq\beta_1|G(\xi,\xi,\xi)|+\beta_2\frac{|G(\xi,\xi,\xi)|\cdot|G(\xi,\xi,\xi)|}{|1+G(F_1\xi,\xi,\xi)|}. \tag{37}$$

This implies that $|G(F_1\xi,\xi,\xi)|=0$; thus,

$$F_1\xi=\xi. \tag{38}$$

(vi) If $|G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})|\cdot|G(\xi,F_1\xi,F_1\xi)|/|1+G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})|$ is a maximum term in (23), then (21) can be written as

$$\left|G\left(F_1\xi,\gamma_{3j+2},\gamma_{3j+3}\right)\right|\leq\beta_1\left|G\left(\xi,\gamma_{3j+1},\gamma_{3j+2}\right)\right|+\beta_2\frac{\left|G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|\cdot\left|G(\xi,F_1\xi,F_1\xi)\right|}{\left|1+G\left(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3}\right)\right|}. \tag{39}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$|G(F_1\xi,\xi,\xi)|\leq\beta_1|G(\xi,\xi,\xi)|+\beta_2\frac{|G(\xi,\xi,\xi)|\cdot|G(\xi,F_1\xi,F_1\xi)|}{|1+G(\xi,\xi,\xi)|}. \tag{40}$$

This implies that $|G(F_1\xi,\xi,\xi)|=0$; thus,

$$F_1\xi=\xi. \tag{41}$$

From (26)–(41), we get that ξ is a FP of F_1, that is,

$$F_1\xi=\xi. \tag{42}$$

Next, we have to show that $F_2\xi=\xi$; by contrary case, let $F_2\xi\neq\xi$. Now by (5),

$$\begin{aligned}G\left(\gamma_{3j+1},F_2\xi,\gamma_{3j+3}\right)&=G\left(F_1\gamma_{3j},F_2\xi,F_3\gamma_{3j+2}\right)\\&\leq\beta_1G\left(\gamma_{3j},\xi,\gamma_{3j+2}\right)+\beta_2W\left(\gamma_{3j},\xi,\gamma_{3j+2}\right).\end{aligned} \tag{43}$$

This implies that

$$\left|G\left(\gamma_{3j+1},F_2\xi,\gamma_{3j+3}\right)\right|\leq\beta_1\left|G\left(\gamma_{3j},\xi,\gamma_{3j+2}\right)\right|+\beta_2\left|W\left(\gamma_{3j},\xi,\gamma_{3j+2}\right)\right|, \tag{44}$$

where

$$W\left(\gamma_{3j},\xi,\gamma_{3j+2}\right)=\max\left\{\begin{array}{c}G\left(\gamma_{3j},\xi,F_2\xi\right),G\left(F_1\gamma_{3j},F_1\gamma_{3j},\xi\right),G\left(F_1\gamma_{3j},\xi,\xi\right),\\ G\left(F_2\xi,F_2\xi,\gamma_{3j+2}\right),G\left(F_2\xi,\gamma_{3j+2},\gamma_{3j+2}\right),\\ \dfrac{G\left(\gamma_{3j},F_1\gamma_{3j},F_1\gamma_{3j}\right)\cdot G(\xi,F_2\xi,F_2\xi)}{1+G\left(\gamma_{3j},\xi,\xi\right)},\\ \dfrac{G(\xi,F_2\xi,F_2\xi)\cdot G\left(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)}{1+G\left(F_1\gamma_{3j},\gamma_{3j+2},\gamma_{3j+2}\right)},\\ \dfrac{G\left(\gamma_{3j+2},F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)\cdot G\left(\gamma_{3j},F_1\gamma_{3j},F_1\gamma_{3j}\right)}{1+G\left(F_2\xi,F_3\gamma_{3j+2},F_3\gamma_{3j+2}\right)}\end{array}\right\}$$

$$= \max\left\{\begin{array}{c} G(\gamma_{3j},\xi,F_2\xi), G(\gamma_{3j+1},\gamma_{3j+1},\xi), G(\gamma_{3j+1},\xi,\xi), \\ G(F_2\xi,F_2\xi,\gamma_{3j+2}), G(F_2\xi,\gamma_{3j+2},\gamma_{3j+2}), \\ \dfrac{G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})\cdot G(\xi,F_2\xi,F_2\xi)}{1+G(\gamma_{3j},\xi,\xi)}, \\ \dfrac{G(\xi,F_2\xi,F_2\xi)\cdot G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})}{1+G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2})}, \\ \dfrac{G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})\cdot G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})}{1+G(F_2\xi,\gamma_{3j+3},\gamma_{3j+3})} \end{array}\right\}. \tag{45}$$

This implies that

$$\left|W(\gamma_{3j},\xi,\gamma_{3j+2})\right| = \max\left\{\begin{array}{c} \left|G(\gamma_{3j},\xi,F_2\xi)\right|, \left|G(\gamma_{3j+1},\gamma_{3j+1},\xi)\right|, \left|G(\gamma_{3j+1},\xi,\xi)\right|, \\ \left|G(F_2\xi,F_2\xi,\gamma_{3j+2})\right|, \left|G(F_2\xi,\gamma_{3j+2},\gamma_{3j+2})\right|, \\ \dfrac{\left|G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})\right|\cdot\left|G(\xi,F_2\xi,F_2\xi)\right|}{\left|1+G(\gamma_{3j},\xi,\xi)\right|}, \\ \dfrac{\left|G(\xi,F_2\xi,F_2\xi)\right|\cdot\left|G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})\right|}{\left|1+G(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2})\right|}, \\ \dfrac{\left|G(\gamma_{3j+2},\gamma_{3j+3},\gamma_{3j+3})\right|\cdot\left|G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1})\right|}{\left|1+G(F_2\xi,\gamma_{3j+3},\gamma_{3j+3})\right|}. \end{array}\right\} \tag{46}$$

Now, there are eight possibilities:

(i) If $|G(\gamma_{3j},\xi,F_2\xi)|$ is a maximum term in (46), then after simplification, (44) can be written as

$$\left|G(\gamma_{3j+1},F_2\xi,\gamma_{3j+3})\right| \le \beta_1\left|G(\gamma_{3j},\xi,\gamma_{3j+2})\right| + \beta_2\left|G(\gamma_{3j},\xi,F_2\xi)\right|. \tag{47}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi,F_2\xi,\xi)\right| \le \beta_1|G(\xi,\xi,\xi)| + \beta_2\left|G(\xi,\xi,F_2\xi)\right|. \tag{48}$$

This implies that $(1-\beta_2)|(G\xi,F_2\xi,\xi)| \le 0$ is a contradiction, where $(1-\beta_2)\ne 0 \Rightarrow |G(\xi,F_2\xi,\xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{49}$$

(ii) If $|G(\gamma_{3j+1},\gamma_{3j+1},\xi)|$ is a maximum term in (46), then (44) can be written as

$$\left|G(\gamma_{3j+1},F_2\xi,\gamma_{3j+3})\right| \le \beta_1\left|G(\gamma_{3j},\xi,\gamma_{3j+2})\right| + \beta_2\left|G(\gamma_{3j+1},\gamma_{3j+1},\xi)\right|. \tag{50}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi,F_2\xi,\xi)\right| \le \beta_1|G(\xi,\xi,\xi)| + \beta_2|G(\xi,\xi,\xi)|. \tag{51}$$

This implies that $|G(\xi,F_2\xi,\xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{52}$$

(iii) If $|G(\gamma_{3j+1},\xi,\xi)|$ is a maximum term in (46), then (44) can be written as

$$\left|G(\gamma_{3j+1},F_2\xi,\gamma_{3j+3})\right| \le \beta_1\left|G(\gamma_{3j},\xi,\gamma_{3j+2})\right| + \beta_2\left|G(\gamma_{3j+1},\xi,\xi)\right|. \tag{53}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi,F_2\xi,\xi)\right| \le \beta_1|G(\xi,\xi,\xi)| + \beta_2|G(\xi,\xi,\xi)|. \tag{54}$$

This implies that $|G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{55}$$

(iv) If $|G(F_2\xi, F_2\xi, \gamma_{3j+2})|$ is a maximum term in (46), then (44) can be written as

$$\left|G\left(\gamma_{3j+1}, F_2\xi, \gamma_{3j+3}\right)\right| \leq \beta_1 \left|G\left(\gamma_{3j}, \xi, \gamma_{3j+2}\right)\right| + \beta_2 \left|G\left(F_2\xi, F_2\xi, \gamma_{3j+2}\right)\right|. \tag{56}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\begin{aligned}\left|G(\xi, F_2\xi, \xi)\right| &\leq \beta_1 |G(\xi, \xi, \xi)| + \beta_2 \left|G(F_2\xi, F_2\xi, \xi)\right| \\ &\leq 2b\beta_2 \left|G(\xi, F_2\xi, \xi)\right| \because (\text{Proposition 2.5 (2)}).\end{aligned} \tag{57}$$

This implies that $(1 - 2b\beta_2)|G(\xi, F_2\xi, \xi)| \leq 0$ is a contradiction, where $(1 - 2b\beta_2) \neq 0 \Rightarrow |G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{58}$$

(v) If $|G(F_2\xi, \gamma_{3j+2}, \gamma_{3j+2})|$ is a maximum term in (46), then (44) can be written as

$$\left|G\left(\gamma_{3j+1}, F_2\xi, \gamma_{3j+3}\right)\right| \leq \beta_1 \left|G\left(\gamma_{3j}, \xi, \gamma_{3j+2}\right)\right| + \beta_2 \left|G\left(F_2\xi, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|. \tag{59}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi, F_2\xi, \xi)\right| \leq \beta_1 |G(\xi, \xi, \xi)| + \beta_2 \left|G(F_2\xi, \xi, \xi)\right|. \tag{60}$$

This implies that $(1 - \beta_2)|G(\xi, F_2\xi, \xi)| \leq 0$ is a contradiction, where $(1 - \beta_2) \neq 0 \Rightarrow |G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{61}$$

(vi) If $|G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1})| \cdot |G(\xi, F_2\xi, F_2\xi)|/|1+G(\gamma_{3j}, \xi, \xi)|$ is a maximum term in (46), then (44) can be written as

$$\left|G\left(\gamma_{3j+1}, F_2\xi, \gamma_{3j+3}\right)\right| \leq \beta_1 \left|G\left(\gamma_{3j}, \xi, \gamma_{3j+2}\right)\right| + \beta_2 \frac{\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right| \cdot \left|G(\xi, F_2\xi, F_2\xi)\right|}{\left|1 + G\left(\gamma_{3j}, \xi, \xi\right)\right|}. \tag{62}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi, F_2\xi, \xi)\right| \leq \beta_1 |G(\xi, \xi, \xi)| + \beta_2 \frac{|G(\xi, \xi, \xi)| \cdot \left|G(\xi, F_2\xi, F_2\xi)\right|}{|1 + G(\xi, \xi, \xi)|}. \tag{63}$$

This implies that $|G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{64}$$

(vii) If $|G(\xi, F_2\xi, F_2\xi)| \cdot |G(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3})|/|1+G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2})|$ is a maximum term in (46), then (44) can be written as

$$\left|G\left(\gamma_{3j+1}, F_2\xi, \gamma_{3j+3}\right)\right| \leq \beta_1 \left|G\left(\gamma_{3j}, \xi, \gamma_{3j+2}\right)\right| + \beta_2 \frac{\left|G(\xi, F_2\xi, F_2\xi)\right| \cdot \left|G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right|}{\left|1 + G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|}. \tag{65}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$|G(\xi, F_2\xi, \xi)| \leq \beta_1 |G(\xi, \xi, \xi)| + \beta_2 \frac{|G(\xi, F_2\xi, F_2\xi)| \cdot |G(\xi, \xi, \xi)|}{|1 + G(\xi, \xi, \xi)|}. \tag{66}$$

This implies that $|G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{67}$$

(viii) If $|G(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3})| \cdot |G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1})| /|1 + G(F_2\xi, \gamma_{3j+3}, \gamma_{3j+3})|$ is a maximum term in (46), then (44) can be written as

$$\left|G\left(\gamma_{3j+1}, F_2\xi, \gamma_{3j+3}\right)\right| \leq \beta_1 \left|G\left(\gamma_{3j}, \xi, \gamma_{3j+2}\right)\right| + \beta_2 \frac{\left|G\left(\gamma_{3j+2}, \gamma_{3j+3}, \gamma_{3j+3}\right)\right| \cdot \left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|}{\left|1 + G\left(F_2\xi,_{3j+3}, \gamma_{3j+3}\right)\right|}. \tag{68}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi, F_2\xi, \xi)\right| \leq \beta_1|G(\xi, \xi, \xi)| + \beta_2\frac{|G(\xi, \xi, \xi)| \cdot |G(\xi, \xi, \xi)|}{\left|1 + G(F_2\xi, \xi, \xi)\right|}. \tag{69}$$

This implies that $|G(\xi, F_2\xi, \xi)| = 0$; thus,

$$F_2\xi = \xi. \tag{70}$$

From (49) to (70), we find that ξ is a FP of F_2, that is,

$$F_2\xi = \xi. \tag{71}$$

Now, we have to show that $F_3\xi = \xi$; by contrary case, let $F_3\xi \neq \xi$. Now, by (5),

$$\begin{aligned} G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right) &= G\left(F_1\gamma_{3j}, F_2\gamma_{3j+1}, F_3\xi\right) \\ &\leq \beta_1 G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right) + \beta_2 W\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right). \end{aligned} \tag{72}$$

This implies that

$$\begin{aligned} \left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right)\right| \leq \beta_1&\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right| \\ &+ \beta_2\left|W\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right|, \end{aligned} \tag{73}$$

where

$$\begin{aligned} W\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right) &= \max\left\{\begin{array}{c} G\left(\gamma_{3j}, \gamma_{3j+1}, F_2\gamma_{3j+1}\right), G\left(F_1\gamma_{3j}, F_1\gamma_{3j}, \gamma_{3j+1}\right), G\left(F_1\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right), \\ G\left(F_2\gamma_{3j+1}, F_2\gamma_{3j+1}, \xi\right), G\left(F_2\gamma_{3j+1}, \xi, \xi\right), \\ \dfrac{G\left(\gamma_{3j}, F_1\gamma_{3j}, F_1\gamma_{3j}\right) \cdot G\left(\gamma_{3j+1}, F_2\gamma_{3j+1}, F_2\gamma_{3j+1}\right)}{1 + G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)}, \\ \dfrac{G\left(\gamma_{3j+1}, F_2\gamma_{3j+1}, F_2\gamma_{3j+1}\right) \cdot G(\xi, F_3\xi, F_3\xi)}{1 + G\left(F_1\gamma_{3j}, \xi, \xi\right)}, \\ \dfrac{G(\xi, F_3\xi, F_3\xi) \cdot G\left(\gamma_{3j}, F_1\gamma_{3j}, F_1\gamma_{3j}\right)}{1 + G\left(F_2\gamma_{3j+1}, F_3\xi, F_3\xi\right)} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+2}\right), G\left(\gamma_{3j+1}, \gamma_{3j+1}, \gamma_{3j+1}\right), G\left(\gamma_{3j+1}, \gamma_{3j+1}, \gamma_{3j+1}\right), \\ G\left(\gamma_{3j+2}, \gamma_{3j+2}, \xi\right), G\left(\gamma_{3j+2}, \xi, \xi\right), \\ \dfrac{G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right) \cdot G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)}{1 + G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)}, \\ \dfrac{G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right) \cdot G(\xi, F_3\xi, F_3\xi)}{1 + G\left(\gamma_{3j+1}, \xi, \xi\right)}, \\ \dfrac{G(\xi, F_3\xi, F_3\xi) \cdot G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)}{1 + G\left(\gamma_{3j+2}, F_3\xi, F_3\xi\right)} \end{array}\right\}. \end{aligned} \tag{74}$$

This implies that

$$
\begin{aligned}
&\left|W\left(\gamma_{3j},\gamma_{3j+1},\xi\right)\right| \\
&= \max\left\{\begin{array}{c}
\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\left|G\left(\gamma_{3j+1},\gamma_{3j+1},\gamma_{3j+1}\right)\right|,\left|G\left(\gamma_{3j+1},\gamma_{3j+1},\gamma_{3j+1}\right)\right|,\\
\left|G\left(\gamma_{3j+2},\gamma_{3j+2},\xi\right)\right|,\left|G\left(\gamma_{3j+2},\xi,\xi\right)\right|,\\
\dfrac{\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|\cdot\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|}{\left|1+G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|},\\
\dfrac{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|\cdot\left|G\left(\xi,F_3\xi,F_3\xi\right)\right|}{\left|1+G\left(\gamma_{3j+1},\xi,\xi\right)\right|},\\
\dfrac{\left|G\left(\xi,F_3\xi,F_3\xi\right)\right|\cdot\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|}{\left|1+G\left(\gamma_{3j+2},F_3\xi,F_3\xi\right)\right|}
\end{array}\right\}
\end{aligned}
\tag{75}
$$

After simplification, we get that

$$
\begin{aligned}
&\left|W\left(\gamma_{3j},\gamma_{3j+1},\xi\right)\right| \\
&= \max\left\{\begin{array}{c}
\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|,\left|G\left(\gamma_{3j+2},\gamma_{3j+2},\xi\right)\right|,\left|G\left(\gamma_{3j+2},\xi,\xi\right)\right|,\\
\dfrac{\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|\cdot\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|}{\left|1+G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|},\\
\dfrac{\left|G\left(\gamma_{3j+1},\gamma_{3j+2},\gamma_{3j+2}\right)\right|\cdot\left|G\left(\xi,F_3\xi,F_3\xi\right)\right|}{\left|1+G\left(\gamma_{3j+1},\xi,\xi\right)\right|},\\
\dfrac{\left|G\left(\xi,F_3\xi,F_3\xi\right)\right|\cdot\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+1}\right)\right|}{\left|1+G\left(\gamma_{3j+2},F_3\xi,F_3\xi\right)\right|}
\end{array}\right\}.
\end{aligned}
\tag{76}
$$

Now, there are six possibilities:

(i) If $|G(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2})|$ is a maximum term in (76), then (73) can be written as

$$
\begin{aligned}
\left|G\left(\gamma_{3j+1},\gamma_{3j+2},F_3\xi\right)\right| \le{}& \beta_1\left|G\left(\gamma_{3j},\gamma_{3j+1},\xi\right)\right| \\
&+\beta_2\left|G\left(\gamma_{3j},\gamma_{3j+1},\gamma_{3j+2}\right)\right|.
\end{aligned}
\tag{77}
$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$
\left|G\left(\xi,\xi,F_3\xi\right)\right| \le \beta_1|G(\xi,\xi,\xi)|+\beta_2|G(\xi,\xi,\xi)|. \tag{78}
$$

This implies that $|G(\xi,\xi,F_3\xi)|=0$; thus,

$$
F_3\xi=\xi. \tag{79}
$$

(ii) If $|G(\gamma_{3j+2},\gamma_{3j+2},\xi)|$ is a maximum term in (76), then (73) can be written as

$$
\begin{aligned}
\left|G\left(\gamma_{3j+1},\gamma_{3j+2},F_3\xi\right)\right| \le{}& \beta_1\left|G\left(\gamma_{3j},\gamma_{3j+1}\xi\right)\right| \\
&+\beta_2\left|G\left(\gamma_{3j+2},\gamma_{3j+2},\xi\right)\right|.
\end{aligned}
\tag{80}
$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$
\left|G\left(\xi,\xi,F_3\xi\right)\right| \le \beta_1|G(\xi,\xi,\xi)|+\beta_2|G(\xi,\xi,\xi)|. \tag{81}
$$

This implies that $|G(\xi,\xi,F_3\xi)|=0$; thus,

$$
F_3\xi=\xi. \tag{82}
$$

(iii) If $|G(\gamma_{3j+2}, \xi, \xi)|$ is a maximum term in (76), then (73) can be written as

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right)\right| \leq \beta_1\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right| + \beta_2\left|G\left(\gamma_{3j+2}, \xi, \xi\right)\right|, \tag{83}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi, \xi, F_3\xi)\right| \leq \beta_1|G(\xi, \xi, \xi)| + \beta_2|G(\xi, \xi, \xi)|. \tag{84}$$

This implies that $|G(\xi, \xi, F_3\xi)| = 0$; thus,

$$F_3\xi = \xi. \tag{85}$$

(iv) If $|G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1})| \cdot |G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2})| / |1 + G(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1})|$ is a maximum term in (76), then (73) can be written as

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right)\right| \leq \beta_1\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right| + \beta_2\frac{\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right| \cdot \left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right|}{\left|1 + G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|}. \tag{86}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi, \xi, F_3\xi)\right| \leq \beta_1|G(\xi, \xi, \xi)| + \beta_2\frac{|G(\xi, \xi, \xi)| \cdot |G(\xi, \xi, \xi)|}{|1 + G(\xi, \xi, \xi)|}. \tag{87}$$

This implies that $|G(\xi, \xi, F_3\xi)| = 0$; thus,

$$F_3\xi = \xi. \tag{88}$$

(v) If $|G(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2})| / |1 + G(\gamma_{3j+1}, \xi, \xi)|$ is a maximum term in (76), then (73) can be written as

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right)\right| \leq \beta_1\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right| + \beta_2\frac{\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, \gamma_{3j+2}\right)\right| \cdot \left|G(\xi, F_3\xi, F_3\xi)\right|}{\left|1 + G\left(\gamma_{3j+1}, \xi, \xi\right)\right|}. \tag{89}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi, \xi, F_3\xi)\right| \leq \beta_1|G(\xi, \xi, \xi)| + \beta_2\frac{|G(\xi, \xi, \xi)| \cdot \left|G(\xi, F_3\xi, F_3\xi)\right|}{|1 + G(\xi, \xi, \xi)|}. \tag{90}$$

This implies that $|G(\xi, \xi, F_3\xi)| = 0$; thus,

$$F_3\xi = \xi. \tag{91}$$

(vi) If $|G(\xi, F_3\xi, F_3\xi)| / |1 + G(\gamma_{3j+2}, F_3\xi, F_3\xi)|$ is a maximum term in (76), then (73) can be written as

$$\left|G\left(\gamma_{3j+1}, \gamma_{3j+2}, F_3\xi\right)\right| \leq \beta_1\left|G\left(\gamma_{3j}, \gamma_{3j+1}, \xi\right)\right| + \beta_2\frac{\left|G(\xi, F_3\xi, F_3\xi)\right| \cdot \left|G\left(\gamma_{3j}, \gamma_{3j+1}, \gamma_{3j+1}\right)\right|}{\left|1 + G\left(\gamma_{3j+2}, F_3\xi, F_3\xi\right)\right|}. \tag{92}$$

Applying $\lim_{n\longrightarrow\infty}$ on both sides, we get

$$\left|G(\xi, \xi, F_3\xi)\right| \leq \beta_1|G(\xi, \xi, \xi)| + \beta_2\frac{\left|G(\xi, F_3\xi, F_3\xi)\right| \cdot |G(\xi, \xi, \xi)|}{\left|1 + G(\xi, F_3\xi, F_3\xi)\right|}. \tag{93}$$

This implies that $|G(\xi, \xi, F_3\xi)| = 0$; thus,

$$F_3\xi = \xi. \tag{94}$$

From (79)–(94), we find that ξ is a FP of F_3, that is,

$$F_3\xi = \xi. \tag{95}$$

Hence proved that ξ is a CFP of F_1, F_2 and F_3, that is,

$$F_1\xi = F_2\xi = F_3\xi = \xi. \tag{96}$$

Uniqueness: assume that $\xi^* \in \Upsilon$ is another CFP of the mappings F_1, F_2, and F_3, such that

$$\begin{gathered} F_1\xi = F_2\xi = F_3\xi = \xi, \\ F_1\xi^* = F_2\xi^* = F_3\xi^* = \xi^*. \end{gathered} \tag{97}$$

Then, from (5), we have that

$$\begin{aligned} G(\xi, \xi^*, \xi^*) &= G(F_1\xi, F_2\xi^*, F_3\xi^*) \\ &\leq \beta_1 G(\xi, \xi^*, \xi^*) + \beta_2 W(\xi, \xi^*, \xi^*). \end{aligned} \tag{98}$$

This implies that

$$\left|G\left(\xi,\xi^*,\xi^*\right)\right| \le \beta_1\left|G\left(\xi,\xi^*,\xi^*\right)\right| + \beta_2\left|W\left(\xi,\xi^*,\xi^*\right)\right|, \tag{99}$$

where

$$\begin{aligned} W\left(\xi,\xi^*,\xi^*\right) &= \max\left\{\begin{array}{c} G\left(\xi,\xi^*,F_2\xi^*\right), G\left(F_1\xi,F_1\xi,\xi^*\right), G\left(F_1\xi,\xi^*,\xi^*\right), \\ G\left(F_2\xi^*,F_2\xi^*,\xi^*\right), G\left(F_2\xi^*,\xi^*,\xi^*\right), \\ \dfrac{G\left(\xi,F_1\xi,F_1\xi\right)\cdot G\left(\xi^*,F_2\xi^*,F_2\xi^*\right)}{\left(1+G\left(\xi,\xi^*,\xi^*\right)\right)}, \\ \dfrac{G\left(\xi^*,F_2\xi^*,F_2\xi^*\right)\cdot G\left(\xi^*,F_3\xi^*,F_3\xi^*\right)}{\left(1+G\left(F_1\xi,\xi^*,\xi^*\right)\right)}, \\ \dfrac{G\left(\xi^*,F_3\xi^*,F_3\xi^*\right)\cdot G\left(\xi,F_1\xi,F_1\xi\right)}{\left(1+G\left(F_2\xi^*,F_3\xi^*,F_3\xi^*\right)\right)} \end{array}\right\} \\ &= \max\left\{\begin{array}{c} G\left(\xi,\xi^*,\xi^*\right), G\left(\xi,\xi,\xi^*\right), G\left(\xi,\xi^*,\xi^*\right), \\ G\left(\xi^*,\xi^*,\xi^*\right), G\left(\xi^*,\xi^*,\xi^*\right), \\ \dfrac{G\left(\xi,\xi,\xi\right)\cdot G\left(\xi^*,\xi^*,\xi^*\right)}{\left(1+G\left(\xi,\xi^*,\xi^*\right)\right)}, \\ \dfrac{G\left(\xi^*,\xi^*,\xi^*\right)\cdot G\left(\xi^*,\xi^*,\xi^*\right)}{\left(1+G\left(\xi,\xi^*,\xi^*\right)\right)}, \\ \dfrac{G\left(\xi^*,\xi^*,\xi^*\right)\cdot G\left(\xi,\xi,\xi\right)}{\left(1+G\left(\xi^*,\xi^*,\xi^*\right)\right)} \end{array}\right\}. \end{aligned} \tag{100}$$

This implies that

$$\left|W\left(\xi,\xi^*,\xi^*\right)\right| = \max\left\{\begin{array}{c} \left|G\left(\xi,\xi^*,\xi^*\right)\right|, \left|G\left(\xi,\xi,\xi^*\right)\right|, \left|G\left(\xi,\xi^*,\xi^*\right)\right|, \\ \left|G\left(\xi^*,\xi^*,\xi^*\right)\right|, \left|G\left(\xi^*,\xi^*,\xi^*\right)\right|, \\ \dfrac{\left|G\left(\xi,\xi,\xi\right)\right|\cdot\left|G\left(\xi^*,\xi^*,\xi^*\right)\right|}{\left|1+G\left(\xi,\xi^*,\xi^*\right)\right|}, \\ \dfrac{\left|G\left(\xi^*,\xi^*,\xi^*\right)\right|\cdot\left|G\left(\xi^*,\xi^*,\xi^*\right)\right|}{\left|1+G\left(\xi,\xi^*,\xi^*\right)\right|}, \\ \dfrac{\left|\left(\xi^*,\xi^*,\xi^*\right)\right|\cdot\left|G\left(\xi,\xi,\xi\right)\right|}{\left|1+G\left(\xi^*,\xi^*,\xi^*\right)\right|} \end{array}\right\}. \tag{101}$$

Now, by using Proposition 2.5 (2) and after simplifying, we get that

$$\left|W\left(\xi,\xi^{*},\xi^{*}\right)\right| \leq \max\left|G\left(\xi,\xi^{*},\xi^{*}\right)\right|, 2b\left|G\left(\xi,\xi^{*},\xi^{*}\right)\right|. \tag{102}$$

Clearly, $2b|G(\xi,\xi^{*},\xi^{*})|$ is a maximum term in (102), so (99) can be written as

$$\left|G\left(\xi,\xi^{*},\xi^{*}\right)\right| \leq \beta_1\left|G\left(\xi,\xi^{*},\xi^{*}\right)\right| + 2b\beta_2\left|G\left(\xi,\xi^{*},\xi^{*}\right)\right|. \tag{103}$$

This implies that $(1-\beta_1-2b\beta_2)|G(\xi,\xi^{*},\xi^{*})| \leq 0$ is a contradiction, where $(1-\beta_1-2b\beta_2) \neq 0 \Rightarrow |G(\xi,\xi^{*},\xi^{*})| = 0 \Rightarrow \xi = \xi^{*}$. Hence proved that F_1, F_2, and F_3 have a unique CFP in Υ.

Corollary 1. *Let (Υ, G) be a complete complex-valued G_b-metric space with coefficient $b \geq 1$ and $F_1, F_2, F_3\colon \Upsilon \longrightarrow \Upsilon$ be mappings satisfying*

$$G\left(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3\right) \leq \beta_1 G\left(\gamma_1,\gamma_2,\gamma_3\right) + \beta_2 W\left(\gamma_1,\gamma_2,\gamma_3\right), \tag{104}$$

where

$$W\left(\gamma_1,\gamma_2,\gamma_3\right) = \max\left\{\begin{array}{c} G\left(\gamma_1,\gamma_2,F_2\gamma_2\right), G\left(F_1\gamma_1,F_1\gamma_1,\gamma_2\right), G\left(F_1\gamma_1,\gamma_2,\gamma_2\right), \\ G\left(F_2\gamma_2,F_2\gamma_2,\gamma_3\right), G\left(F_2\gamma_2,\gamma_3,\gamma_3\right), \\ \dfrac{G\left(\gamma_1,F_1\gamma_1,F_1\gamma_1\right)\cdot G\left(\gamma_2,F_2\gamma_2,F_2\gamma_2\right)}{1+G\left(\gamma_1,\gamma_2,\gamma_2\right)}, \end{array}\right\}, \tag{105}$$

for all $\gamma_1,\gamma_2,\gamma_3 \in \Upsilon$, $\beta_1,\beta_2 \in 0, 1/3)$, such that $(\beta_1+\beta_2) < 1/3$, $b\beta_2 < 1/3$, and $(\beta_1+2b\beta_2) < 2/3$; then, F_1, F_2, and F_3 have a unique CFP in Υ.

Corollary 2. *Let (Υ, G) be a complete complex-valued G_b-metric space with coefficient $b \geq 1$ and $F_1, F_2, F_3\colon \Upsilon \longrightarrow \Upsilon$ be mappings satisfying*

$$G\left(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3\right) \leq \beta_1 G\left(\gamma_1,\gamma_2,\gamma_3\right) + \beta_2 W\left(\gamma_1,\gamma_2,\gamma_3\right), \tag{106}$$

where

$$W\left(\gamma_1,\gamma_2,\gamma_3\right) = \max\left\{\begin{array}{c} G\left(\gamma_1,\gamma_2,F_2\gamma_2\right), G\left(F_1\gamma_1,F_1\gamma_1,\gamma_2\right), G\left(F_1\gamma_1,\gamma_2,\gamma_2\right), \\ G\left(F_2\gamma_2,F_2\gamma_2,\gamma_3\right), G\left(F_2\gamma_2,\gamma_3,\gamma_3\right), \\ \dfrac{G\left(\gamma_2,F_2\gamma_2,F_2\gamma_2\right)\cdot G\left(\gamma_3,F_3\gamma_3,F_3\gamma_3\right)}{1+G\left(F_1\gamma_1,\gamma_3,\gamma_3\right)}, \end{array}\right\}, \tag{107}$$

for all $\gamma_1,\gamma_2,\gamma_3 \in \Upsilon$, $\beta_1,\beta_2 \in 0, 1/3)$, such that $(\beta_1+\beta_2) < 1/3$, $b\beta_2 < 1/3$, and $(\beta_1+2b\beta_2) < 2/3$; then, F_1, F_2, and F_3 have a unique CFP in Υ.

Corollary 3. *Let (Υ, G) be a complete complex-valued G_b-metric space with coefficient $b \geq 1$ and $F_1, F_2, F_3\colon \Upsilon \longrightarrow \Upsilon$ be mappings satisfying*

$$G\left(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3\right) \leq \beta_1 G\left(\gamma_1,\gamma_2,\gamma_3\right) + \beta_2 W\left(\gamma_1,\gamma_2,\gamma_3\right), \tag{108}$$

where

$$W\left(\gamma_1,\gamma_2,\gamma_3\right) = \max\left\{\begin{array}{c} G\left(\gamma_1,\gamma_2,F_2\gamma_2\right), G\left(F_1\gamma_1,F_1\gamma_1,\gamma_2\right), G\left(F_1\gamma_1,\gamma_2,\gamma_2\right), \\ G\left(F_2\gamma_2,F_2\gamma_2,\gamma_3\right), G\left(F_2\gamma_2,\gamma_3,\gamma_3\right), \\ \dfrac{G\left(\gamma_3,F_3\gamma_3,F_3\gamma_3\right)\cdot G\left(\gamma_1,F_1\gamma_1,F_1\gamma_1\right)}{1+G\left(F_2\gamma_2,F_3\gamma_3,F_3\gamma_3\right)} \end{array}\right\}, \tag{109}$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$, $\beta_1, \beta_2 \in 0, 1/3)$, *such that* $(\beta_1 + \beta_2) < 1/3$, $b\beta_2 < 1/3$, *and* $(\beta_1 + 2b\beta_2) < 2/3$; *then,* F_1, F_2, *and* F_3 *have a unique CFP in* Υ.

Remark 2. If we put $\beta_2 = 0$ and $F_1 = F_2 = F_3$ in Theorem 2, we get (Theorem 3.7 [22]).

Example 2. Let (Υ, G) be a complex-valued G_b-metric space, where $\Upsilon = [0, 1]$ and $G: \Upsilon \times \Upsilon \times \Upsilon \longrightarrow \mathbb{C}$, with $G(\gamma_1, \gamma_2, \gamma_3) = (3/4|\gamma_1 - \gamma_2| + 3/4|\gamma_2 - \gamma_3| + 3/4|\gamma_3 - \gamma_1|)^2 + i\,(3/4|\gamma_1 - \gamma_2| + 3/4|\gamma_2 - \gamma_3| + 3/4|\gamma_3 - \gamma_1|)^2$, for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Now, we define $F_1, F_2, F_3: \Upsilon \longrightarrow \Upsilon$ as

$$F_1\gamma = F_2\gamma = F_3\gamma = \ln\left(1 + \frac{\gamma}{5 + \gamma}\right). \tag{110}$$

Notice that

$$|G(\gamma_1, \gamma_2, \gamma_3)|, |W(\gamma_1, \gamma_2, \gamma_3)| \geq 0. \tag{111}$$

In all regards, it is enough to show that $G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3)$, for all $\gamma_1, \gamma_2, \gamma_3 \in [0, 1]$ and $\beta_1, \beta_2 \in 0, 1/3)$, with $\beta_1 + \beta_2 < 1/3$, $b\beta_2 < 1/3$, and $\beta_1 + 2b\beta_2 < 2/3$.

$$\begin{aligned}
G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) &= \left(\frac{3}{4}|F_1\gamma_1 - F_2\gamma_2| + \frac{3}{4}|F_2\gamma_2 - F_3\gamma_3| + \frac{3}{4}|F_3\gamma_3 - F_1\gamma_1|\right)^2 \\
&\quad + i\left(\frac{3}{4}|F_1\gamma_1 - F_2\gamma_2| + \frac{3}{4}|F_2\gamma_2 - F_3\gamma_3| + \frac{3}{4}|F_3\gamma_3 - F_1\gamma_1|\right)^2 \\
&= \left(\begin{array}{c}\frac{3}{4}\left|\ln\left(1 + \frac{\gamma_1}{5 + \gamma_1}\right) - \ln\left(1 + \frac{\gamma_2}{5 + \gamma_2}\right)\right| + \frac{3}{4}\left|\ln\left(1 + \frac{\gamma_2}{5 + \gamma_2}\right) - \ln\left(1 + \frac{\gamma_3}{5 + \gamma_3}\right)\right| \\ + \frac{3}{4}\left|\ln\left(1 + \frac{\gamma_3}{5 + \gamma_3}\right) - \ln\left(1 + \frac{\gamma_1}{5 + \gamma_1}\right)\right|\end{array}\right)^2 \\
&\quad + i\left(\begin{array}{c}\frac{3}{4}\left|\ln\left(1 + \frac{\gamma_1}{5 + \gamma_1}\right) - \ln\left(1 + \frac{\gamma_2}{5 + \gamma_2}\right)\right| + \frac{3}{4}\left|\ln\left(1 + \frac{\gamma_2}{5 + \gamma_2}\right) - \ln\left(1 + \frac{\gamma_3}{5 + \gamma_3}\right)\right| \\ + \frac{3}{4}\left|\ln\left(1 + \frac{\gamma_3}{5 + \gamma_3}\right) - \ln\left(1 + \frac{\gamma_1}{5 + \gamma_1}\right)\right|\end{array}\right)^2 \\
&\leq \left(\frac{3}{4}\left|\frac{\gamma_1}{5 + \gamma_1} - \frac{\gamma_2}{5 + \gamma_2}\right| + \frac{3}{4}\left|\frac{\gamma_2}{5 + \gamma_2} - \frac{\gamma_3}{5 + \gamma_3}\right| + \frac{3}{4}\left|\frac{\gamma_3}{5 + \gamma_3} - \frac{\gamma_1}{5 + \gamma_1}\right|\right)^2 \\
&\quad + i\left(\frac{3}{4}\left|\frac{\gamma_1}{5 + \gamma_1} - \frac{\gamma_2}{5 + \gamma_2}\right| + \frac{3}{4}\left|\frac{\gamma_2}{5 + \gamma_2} - \frac{\gamma_3}{5 + \gamma_3}\right| + \frac{3}{4}\left|\frac{\gamma_3}{5 + \gamma_3} - \frac{\gamma_1}{5 + \gamma_1}\right|\right)^2 \\
&\leq \left(\frac{3}{4}\left|\frac{5\gamma_1 - 5\gamma_2}{25}\right| + \frac{3}{4}\left|\frac{5\gamma_2 - 5\gamma_3}{25}\right| + \frac{3}{4}\left|\frac{5\gamma_3 - 5\gamma_1}{25}\right|\right)^2 \\
&\quad + i\left(\frac{3}{4}\left|\frac{5\gamma_1 - 5\gamma_2}{25}\right| + \frac{3}{4}\left|\frac{5\gamma_2 - 5\gamma_3}{25}\right| + \frac{3}{4}\left|\frac{5\gamma_3 - 5\gamma_1}{25}\right|\right)^2 \\
&= \frac{1}{25}\left[\begin{array}{c}\left(\frac{3}{4}|\gamma_1 - \gamma_2| + \frac{3}{4}|\gamma_2 - \gamma_3| + \frac{3}{4}|\gamma_3 - \gamma_1|\right)^2 \\ + i\left(\frac{3}{4}|\gamma_1 - \gamma_2| + \frac{3}{4}|\gamma_2 - \gamma_3| + \frac{3}{4}|\gamma_3 - \gamma_1|\right)^2\end{array}\right].
\end{aligned} \tag{112}$$

$$G(\gamma_1, \gamma_2, \gamma_3) = \left[\begin{array}{c}\left(\frac{3}{4}|\gamma_1 - \gamma_2| + \frac{3}{4}|\gamma_2 - \gamma_3| + \frac{3}{4}|\gamma_3 - \gamma_1|\right)^2 \\ + i\left(\frac{3}{4}|\gamma_1 - \gamma_2| + \frac{3}{4}|\gamma_2 - \gamma_3| + \frac{3}{4}|\gamma_3 - \gamma_1|\right)^2\end{array}\right]. \tag{113}$$

For $\gamma_1, \gamma_2, \gamma_3 \in [0, 1]$, we discuss different cases with $\beta_1 = 1/10, \beta_2 = 1/20$, and $b = 2$. Hence,

$$\beta_1 + \beta_2 = \frac{1}{10} + \frac{1}{20} < \frac{1}{3}, b\beta_2 = 2\left(\frac{1}{20}\right) < \frac{1}{3}, \quad (114)$$

$$\beta_1 + 2b\beta_2 = \frac{1}{10} + 2(2)\frac{1}{20} < \frac{2}{3}. \quad (115)$$

Case 1: let $\gamma_1 = 0, \gamma_2 = 0, \gamma_3 = 0$; then from (112) and (113), directly we get that $G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3)$. Hence, (3.1) is satisfied with $\beta_1 = 1/10, \beta_2 = 1/20$, and $b = 2$.

Case 2: let $\gamma_1 = 0, \gamma_2 = 0, \gamma_3 = 1/2$; then from (112) and (113), we find $G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3)$ is satisfy with $\beta_1 = 1/10$, i.e.,

$$\frac{1}{25}\begin{bmatrix} \left(\frac{3}{4}|0-0| + \frac{3}{4}\left|0-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-0\right|\right)^2 \\ +i\left(\frac{3}{4}|0-0| + \frac{3}{4}\left|0-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-0\right|\right)^2 \end{bmatrix} \leq \beta_1 \begin{bmatrix} \left(\frac{3}{4}|0-0| + \frac{3}{4}\left|0-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-0\right|\right)^2 \\ +i\left(\frac{3}{4}|0-0| + \frac{3}{4}\left|0-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-0\right|\right)^2 \end{bmatrix}$$
$$\Rightarrow \frac{9}{400} + i\frac{9}{400} \leq \frac{9}{160} + i\frac{9}{160}. \quad (116)$$

Hence, (3.1) is satisfied with $\beta_1 = 1/10, \beta_2 = 1/20$, and $b = 2$.

Case 3: let $\gamma_1 = 1/2, \gamma_2 = 1/3, \gamma_3 = 1/4$; then from (112) and (113), we find $G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3)$ is satisfy with $\beta_1 = 1/10$, i.e.,

$$\frac{1}{25}\begin{bmatrix} \left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{3}\right| + \frac{3}{4}\left|\frac{1}{3}-\frac{1}{4}\right| + \frac{3}{4}\left|\frac{1}{4}-\frac{1}{2}\right|\right)^2 \\ +i\left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{3}\right| + \frac{3}{4}\left|\frac{1}{3}-\frac{1}{4}\right| + \frac{3}{4}\left|\frac{1}{4}-\frac{1}{2}\right|\right)^2 \end{bmatrix} \leq \beta_1 \begin{bmatrix} \left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{3}\right| + \frac{3}{4}\left|\frac{1}{3}-\frac{1}{4}\right| + \frac{3}{4}\left|\frac{1}{4}-\frac{1}{2}\right|\right)^2 \\ +i\left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{3}\right| + \frac{3}{4}\left|\frac{1}{3}-\frac{1}{4}\right| + \frac{3}{4}\left|\frac{1}{4}-\frac{1}{2}\right|\right)^2 \end{bmatrix}$$
$$\Rightarrow \frac{9}{1600} + i\frac{9}{1600} \leq \frac{9}{640} + i\frac{9}{640}. \quad (117)$$

Hence, (3.1) is satisfied with $\beta_1 = 1/10, \beta_2 = 1/20$, and $b = 2$.

Case 4: let $\gamma_1 = 1/2, \gamma_2 = 1/2, \gamma_3 = 1$; then from (112) and (113), we find $G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \beta_1 G(\gamma_1, \gamma_2, \gamma_3)$ is satisfy with $\beta_1 = 1/10$, i.e.,

$$\frac{1}{25}\begin{bmatrix} \left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-1\right| + \frac{3}{4}\left|1-\frac{1}{2}\right|\right)^2 \\ +i\left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-1\right| + \frac{3}{4}\left|1-\frac{1}{2}\right|\right)^2 \end{bmatrix} \leq \beta_1 \begin{bmatrix} \left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-1\right| + \frac{3}{4}\left|1-\frac{1}{2}\right|\right)^2 \\ +i\left(\frac{3}{4}\left|\frac{1}{2}-\frac{1}{2}\right| + \frac{3}{4}\left|\frac{1}{2}-1\right| + \frac{3}{4}\left|1-\frac{1}{2}\right|\right)^2 \end{bmatrix}$$
$$\Rightarrow \frac{9}{400} + i\frac{9}{400} \leq \frac{9}{160} + i\frac{9}{160}. \quad (118)$$

Hence, (3.1) is satisfied with $\beta_1 = 1/10, \beta_2 = 1/20$, and $b = 2$. Thus, all the conditions of Theorem 2 are satisfied with noticing that the point $0 \in \Upsilon$, which remains fixed under mappings F_1, F_2, and F_3, is indeed unique.

4. Applications

In this section, we present an application of the three UTIEs to support our main work. Let $\Upsilon = C([k_1, k_2], \mathbb{R}^n)$ be the set of all real-valued continuous functions defined on $[k_1, k_2]$. Now, we state and prove a result based on the three UTIEs to uplift our work.

Theorem 3. *Let $\Upsilon = C([k_1, k_2], \mathbb{R}^n)$, where $[k_1, k_2] \subseteq \mathbb{R}$ and $G: \Upsilon \times \Upsilon \times \Upsilon \longrightarrow \mathbb{C}$ are defined as*

$$G(\gamma_1, \gamma_2, \gamma_3) = \begin{pmatrix} \|\gamma_1(q) - \gamma_2(q)\| \\ +\gamma_2(q) - \gamma_3(q)\| \\ +\gamma_3(q) - \gamma_1(q)\| \end{pmatrix}^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \quad (119)$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$ and $q \in [k_1, k_2]$. Consider the UTIEs are

$$\begin{aligned} \gamma_1(q) &= \int_{k_1}^{k_2} Q_1(q, r, \gamma_1(r)) dr + \hbar_1(q), \\ \gamma_2(q) &= \int_{k_1}^{k_2} Q_2(q, r, \gamma_2(r)) dr + \hbar_2(q), \\ \gamma_3(q) &= \int_{k_1}^{k_2} Q_3(q, r, \gamma_3(r)) dr + \hbar_3(q), \end{aligned} \quad (120)$$

where $r \in [k_1, k_2]$. Let Q_1, Q_2, Q_3: $[k_1, k_2] \times [k_1, k_2] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are such that $D_{\gamma_1}, E_{\gamma_2}, F_{\gamma_3} \in \Upsilon$; for every $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$, we have that

$$\begin{aligned} D_{\gamma_1}(q) &= \int_{k_1}^{k_2} Q_1(q, r, \gamma_1(r)) dr, \\ E_{\gamma_2}(q) &= \int_{k_1}^{k_2} Q_2(q, r, \gamma_2(r)) dr, \\ F_{\gamma_3}(q) &= \int_{k_1}^{k_2} Q_3(q, r, \gamma_3(r)) dr. \end{aligned} \quad (121)$$

If there exists $\mu \in (0, 1)$, such that for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$,

$$\begin{pmatrix} \|D_{\gamma_1}(q) - E_{\gamma_2}(q) + \hbar_1(q) - \hbar_2(q)\| \\ +E_{\gamma_2}(q) - F_{\gamma_3}(q) + \hbar_2(q) - \hbar_3(q)\| \\ +F_{\gamma_3}(q) - D_{\gamma_1}(q) + \hbar_3(q) - \hbar_1(q)\| \end{pmatrix}^2 \sqrt{1 + k_1^2} e^{i \cot k_1} \leq \mu M(\gamma_1, \gamma_2, \gamma_3), \quad (122)$$

where

$$M(\gamma_1, \gamma_2, \gamma_3) = \max\{A_1(\gamma_1, \gamma_2, \gamma_3)(q), A_2(\gamma_1, \gamma_2, \gamma_3)(q)\}, \quad (123)$$

with

$$A_1(\gamma_1, \gamma_2, \gamma_3)(q) = \begin{pmatrix} \|\gamma_1(q) - \gamma_2(q)\| \\ +\gamma_2(q) - \gamma_3(q)\| \\ +\gamma_3(q) - \gamma_1(q)\| \end{pmatrix}^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \quad (124)$$

$$A_2(\gamma_1, \gamma_2, \gamma_3)(q) = \max \left\{ \begin{array}{c} a_1(\gamma_1, \gamma_2, \gamma_3)(q), a_2(\gamma_1, \gamma_2, \gamma_3)(q), \\ a_3(\gamma_1, \gamma_2, \gamma_3)(q), a_4(\gamma_1, \gamma_2, \gamma_3)(q), \\ a_5(\gamma_1, \gamma_2, \gamma_3)(q), a_6(\gamma_1, \gamma_2, \gamma_3)(q), \\ a_7(\gamma_1, \gamma_2, \gamma_3)(q), a_8(\gamma_1, \gamma_2, \gamma_3)(q) \end{array} \right\}, \quad (125)$$

where

$$a_1(\gamma_1,\gamma_2,\gamma_3)(q) = \begin{pmatrix} \gamma_1(q) - \gamma_2(q) \\ +E_{\gamma_2}(q) + \hbar_2(q) - \gamma_2(q) \\ +E_{\gamma_2}(q) + \hbar_2(q) - \gamma_1(q) \end{pmatrix}^2 \sqrt{1+k_1^2}e^{i\cot k_1},$$
$$a_2(\gamma_1,\gamma_2,\gamma_3)(q) = \left(2D_{\gamma_1}(q) + \hbar_1(q) - \gamma_2(q)\right)^2 \sqrt{1+k_1^2}e^{i\cot k_1},$$
$$a_3(\gamma_1,\gamma_2,\gamma_3)(q) = \left(2D_{\gamma_1}(q) + \hbar_1(q) - \gamma_2(q)\right)^2 \sqrt{1+k_1^2}e^{i\cot k_1},$$
$$a_4(\gamma_1,\gamma_2,\gamma_3)(q) = \left(2E_{\gamma_2}(q) + \hbar_2(q) - \gamma_3(q)\right)^2 \sqrt{1+k_1^2}e^{i\cot k_1},$$
$$a_5(\gamma_1,\gamma_2,\gamma_3)(q) = \left(2E_{\gamma_2}(q) + \hbar_2(q) - \gamma_3(q)\right)^2 \sqrt{1+k_1^2}e^{i\cot k_1},$$
$$a_6(\gamma_1,\gamma_2,\gamma_3)(q) = \frac{\begin{pmatrix} 4D_{\gamma_1}(q) + \hbar_1(q) - \gamma_1(q) \\ \cdot E_{\gamma_2}(q) + \hbar_2(q) - \gamma_2(q) \end{pmatrix}^2 \left(\sqrt{1+k_1^2}\,e^{i\cot k_1}\right)^2}{1 + \left(2\gamma_1(q) - \gamma_2(q)\right)^2 \sqrt{1+k_1^2}e^{i\cot k_1}},$$
$$a_7(\gamma_1,\gamma_2,\gamma_3)(q) = \frac{\begin{pmatrix} 4E_{\gamma_2}(q) + \hbar_2(q) - \gamma_2(q) \\ \cdot F_{\gamma_3}(q) + \hbar_3(q) - \gamma_3(q) \end{pmatrix}^2 \left(\sqrt{1+k_1^2}\,e^{i\cot k_1}\right)^2}{1 + \left(2D_{\gamma_1}(q) + \hbar_1(q) - \gamma_3(q)\right)^2 \sqrt{1+k_1^2}e^{i\cot k_1}},$$
$$a_8(\gamma_1,\gamma_2,\gamma_3)(q) = \frac{\begin{pmatrix} 4F_{\gamma_3}(q) + \hbar_3(q) - \gamma_3(q) \\ \cdot D_{\gamma_1}(q) + \hbar_1(q) - \gamma_1(q) \end{pmatrix}^2 \left(\sqrt{1+k_1^2}\,e^{i\cot k_1}\right)^2}{1 + \left(2E_{\gamma_2}(q) + \hbar_2(q) - F_{\gamma_3}(q) - \hbar_3(q)\right)^2 \sqrt{1+k_1^2}e^{i\cot k_1}}. \tag{126}$$

Then, the three UTIEs, i.e., (120) have a unique common solution.

Proof 1. Define $F_1, F_2, F_3 \colon \Upsilon \longrightarrow \Upsilon$ as

$$\begin{aligned} F_1\gamma_1 &= F_1\gamma_1(q) = D_{\gamma_1}(q) + \hbar_1(q) = D_{\gamma_1} + \hbar_1, \gamma_1(q) = \gamma_1, \\ F_2\gamma_2 &= F_2\gamma_2(q) = E_{\gamma_2}(q) + \hbar_2(q) = E_{\gamma_2} + \hbar_2, \gamma_2(q) = \gamma_2, \\ F_3\gamma_3 &= F_3\gamma_3(q) = F_{\gamma_3}(q) + \hbar_3(q) = F_{\gamma_3} + \hbar_3, \gamma_3(q) = \gamma_3. \end{aligned} \tag{127}$$

Then, we have the following main two cases:

(1) If $A_1(\gamma_1,\gamma_2,\gamma_3)(q)$ is the maximum term in $\{A_1(\gamma_1,\gamma_2,\gamma_3)(q), A_2(\gamma_1,\gamma_2,\gamma_3)(q)\}$, then from (122), (123), and (127), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \le \mu \begin{pmatrix} \|\gamma_1 - \gamma_2\| \\ +\gamma_2 - \gamma_3\| \\ +\gamma_3 - \gamma_1\| \end{pmatrix}^2 \sqrt{1+k_1^2}e^{i\cot k_1}, \tag{128}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy the hypothesis of Theorem 2 with $\mu = \beta_1$ and $\beta_2 = 0$ in (31). Then, the given three UTIEs i.e., (4.1) have a unique common solution in Υ.

(2) If $A_2(\gamma_1,\gamma_2,\gamma_3)(q)$ is the maximum term in $\{A_1(\gamma_1,\gamma_2,\gamma_3)(q), A_2(\gamma_1,\gamma_2,\gamma_3)(q)\}$, then from (123), we have that

$$M(\gamma_1,\gamma_2,\gamma_3) = A_2(\gamma_1,\gamma_2,\gamma_3)(q). \tag{129}$$

Then, there are furthermore eight subcases arising:

(i) If $a_1(\gamma_1,\gamma_2,\gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1,\gamma_2,\gamma_3)(q) = a_1(\gamma_1,\gamma_2,\gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \le \mu \begin{pmatrix} \|\gamma_1 - \gamma_2\| \\ +E_{\gamma_2} + \hbar_2 - \gamma_2\| \\ +E_{\gamma_2} + \hbar_2 - \gamma_1\| \end{pmatrix}^2 \sqrt{1+k_1^2}e^{i\cot k_1}, \tag{130}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy the hypothesis of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (31). Then, the given three UTIEs, i.e., (4.1) have a unique common solution in Υ.

(ii) If $a_2(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_2(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu\left(2D_{\gamma_1} + \hbar_1 - \gamma_2\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \tag{131}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy the hypothesis of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(iii) If $a_3(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_3(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu\left(2D_{\gamma_1} + \hbar_1 - \gamma_2\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \tag{132}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy the hypothesis of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(iv) If $a_4(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_4(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu\left(2E_{\gamma_2} + \hbar_2 - \gamma_3\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \tag{133}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy the hypothesis of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(v) If $a_5(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_5(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu\left(2E_{\gamma_2} + \hbar_2 - \gamma_3\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \tag{134}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy all the conditions of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(vi) If $a_6(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_6(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu \frac{\begin{pmatrix} 4D_{\gamma_1} + \hbar_1 - \gamma_1\| \\ \cdot E_{\gamma_2} + \hbar_2 - \gamma_2\| \end{pmatrix}^2 \left(\sqrt{1 + k_1^2}\, e^{i \cot k_1}\right)^2}{1 + \left(2\gamma_1 - \gamma_2\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}}, \tag{135}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy all the conditions of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(vii) If $a_7(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_7(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu \frac{\begin{pmatrix} 4E_{\gamma_2} + \hbar_2 - \gamma_2\| \\ \cdot F_{\gamma_3} + \hbar_3 - \gamma_3\| \end{pmatrix}^2 \left(\sqrt{1 + k_1^2}\, e^{i \cot k_1}\right)^2}{1 + \left(2D_{\gamma_1} + \hbar_1 - \gamma_3\|\right)^2 \sqrt{1 + k_1^2} e^{i \cot k_1}}, \tag{136}$$

For all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy all the conditions of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ.

(viii) If $a_8(\gamma_1, \gamma_2, \gamma_3)(q)$ is the maximum term in (125), then $A_2(\gamma_1, \gamma_2, \gamma_3)(q) = a_8(\gamma_1, \gamma_2, \gamma_3)(q)$; now, from (122), (127), and (129), we have that

$$G(F_1\gamma_1, F_2\gamma_2, F_3\gamma_3) \leq \mu \frac{\begin{pmatrix} 4F_{\gamma_3} + \hbar_3 - \gamma_3\| \\ \cdot D_{\gamma_1} + \hbar_1 - \gamma_1\| \end{pmatrix}^2 \left(\sqrt{1+k_1^2}\, e^{i\cot k_1}\right)^2}{1 + \left(2E_{\gamma_2} + \hbar_2 - F_{\gamma_3} - \hbar_3\|\right)^2 \sqrt{1+k_1^2} e^{i\cot k_1}}, \tag{137}$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Upsilon$. Thus, the maps F_1, F_2, and F_3 satisfy all the conditions of Theorem 2 with $\mu = \beta_2$ and $\beta_1 = 0$ in (5). Then, the given three UTIEs, i.e., (120) have a unique common solution in Υ. □

5. Conclusions

We have established a generalized CFP-theorem in complex-valued G_b-metric spaces for three self-mappings. In this result, we have used a generalized rational contraction condition and proved a unique CFP-theorem. To justify our result, we presented an illustrative example in the said space by using three self-maps. Also, we present an application of integral equations to get the existing result for a common solution to support our work. By using this concept, one can prove different contractive-type FP and CFP results for many self-mappings in complex-valued G_b-metric spaces with different types of integral operators.

Data Availability

No datasets were generated or analyzed to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Nonexpansive Mappings on New Premodular Special Space of Sequences

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For different premodular, which is a generalization of modular, defined by weighted Orlicz sequence space and its prequasi operator ideal, we have examined the existence of a fixed point for both Kannan contraction and nonexpansive mappings acting on these spaces. Some numerous numerical experiments and practical applications are presented to support our results.

1. Introduction

The spaces of all, bounded, r-absolutely summable, and null sequences of real numbers will be denoted throughout the article by $\mathbf{R}^{\mathscr{L}^+}$, ℓ_∞, ℓ_r, and c_0, respectively, where $\mathscr{L}^+$ is the set of nonnegative integers.

Definition 1. [1, 2] *An Orlicz function is a function $M\colon [0,\infty) \longrightarrow [0,\infty)$, which is continuous and strictly increasing with $M(0)=0$, $M(v)>0$ for $v>0$, and $M(v)\longrightarrow\infty$, as $v\longrightarrow\infty$.*

Definition 2. An Orlicz function M is said to satisfy Δ_2-condition for every values of $v\geq 0$, if there is $k>0$, such that $M(2v)\leq kM(v)$. The Δ_2-condition is equivalent to $M(lv)\leq klM(v)$ for every values of $l>1$ and v.

Lindentrauss and Tzafriri [3] utilized the idea of a convex Orlicz function to define Orlicz sequence space:

$$\ell_M=\left\{v\in\mathbf{R}^{\mathscr{L}^+}\colon \rho(\omega v)<\infty \text{for some } \omega>0\right\}, \text{where } \rho(v) = \sum_{y=0}^{\infty} M\left(\left|v_y\right|\right). \tag{1}$$

$(\ell_M, \|.\|)$ is a Banach space with the Luxemburg norm:

$$\|v\| = \inf\left\{\omega>0\colon \rho\left(\frac{v}{\omega}\right)\leq 1\right\}. \tag{2}$$

Every Orlicz sequence space contains a subspace that is isomorphic to c_0 or ℓ_r, for some $1\leq r<\infty$ ([4], Theorem 4.a.9). The space of all bounded linear operators from a Banach space $\mathfrak{X}$ into a Banach space $\mathfrak{Y}$ will be denoted by $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ and if $\mathfrak{X}=\mathfrak{Y}$, we write $\mathscr{B}(\mathfrak{X})$. $e_x=\{0,0,\ldots,1,0,0,\ldots\}$, while 1 lies in the x^{th} place, with $x\in\mathscr{L}^+$.

Definition 3. [5] *An s-number function is a mapping from $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ into $[0,\infty)^{\mathscr{L}^+}$ which transforms every map $H\in\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ to $(s_x(H))_{x=0}^{\infty}$ satisfying the next conditions:*

(i) $\|H\| = s_0(H)\geq s_1(H)\geq s_2(H)\geq\cdots\geq 0$, *for every* $H\in\mathscr{B}(\mathfrak{X},\mathfrak{Y})$,
(ii) $s_{y+x-1}(H_1+H_2)\leq s_y(H_1)+s_x(H_2)$, *for every* $H_1,H_2\in\mathscr{B}(\mathfrak{X},\mathfrak{Y})$, *and* $y,x\in\mathscr{L}^+$,
(iii) *ideal property:* $s_x(UTH)\leq\|U\|s_x(T)\|H\|$, *for every* $H\in\mathscr{B}(\mathfrak{X}_0,\mathfrak{X})$, $T\in\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ *and* $U\in\mathscr{B}(\mathfrak{Y},\mathfrak{Y}_0)$, *where* $\mathfrak{X}_0$ *and* $\mathfrak{Y}_0$ *are any two Banach spaces,*

(iv) *for* $H \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ *and* $\omega \in \mathbf{R}$, *we have* $s_x(\omega H) = |\omega| s_x(H)$,

(v) *rank property: If* $\operatorname{rank}(H) \leq x$, *then* $s_x(H) = 0$, *for all* $H \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$,

(vi) *norming property:* $s_{l \geq x}(I_x) = 0$ *or* $s_{l < x}(I_x) = 1$, *where* I_x *explains the unit map on the x-dimensional Hilbert space* ℓ_2^x.

The xth approximation number, $\alpha_x(W)$, is defined as

$$\alpha_x(H) = \inf\{\|H - Y\|: Y \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}) \text{ and } \operatorname{rank}(Y) \leq x\}. \tag{3}$$

Notations 1. The sets S_W, $S_W(\mathfrak{X}, \mathfrak{Y})$, S_W^{app}, and $S_W^{\text{app}}(\mathfrak{X}, \mathfrak{Y})$ (cf. [6]) are defined as follows:

$$\begin{aligned} S_W &:= \{S_W(\mathfrak{X}, \mathfrak{Y})\}, \text{ where } S_W(\mathfrak{X}, \mathfrak{Y}) \\ &:= \{H \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}):\ ((s_x(H))_{x=0}^{\infty} \in W\}. \text{Also} \\ S_W^{\text{app}} &:= \{S_W^{\text{app}}(\mathfrak{X}, \mathfrak{Y})\}, \text{ where } S_W^{\text{app}}(\mathfrak{X}, \mathfrak{Y}) \\ &:= \{H \in \mathscr{B}(\mathfrak{X}, \mathfrak{Y}):\ ((\alpha_x(H))_{x=0}^{\infty} \in W\}. \end{aligned} \tag{4}$$

Fixed point theory, Banach space geometry, normal series theory, ideal transformations, and approximation theory are all examples of ideal operator theorems and summability. Faried and Bakery [6] established the concept of a prequasi operator ideal that encapsulates the quasi operator ideal. Bakery and Abou Elmatty investigated the sufficient (but not necessary) conditions on $\ell(\gamma, r)$ that allowed $S_{\ell(\gamma,r)}$ to build a simple Banach prequasi operator ideal in [7]. For varied weights and powers, the prequasi operator ideal $S_{\ell(\gamma,r)}^{\text{app}}$ was once rigorously contained and small prequasi operator ideal. Several mathematicians were able to investigate many extensions for contraction maps defined on the space or on the space itself thanks to the Banach fixed point theorem [8]. Kannan [9] investigated an example of a class of operators that perform the same fixed point actions as contractions but are not continuous. Kannan operators in modular vector spaces have only been described by Ghoncheh [10]. He demonstrated the existence of a Kannan mapping fixed point in complete modular spaces with Fatou property. For more details on Kannan's fixed point theorems and modular vector spaces (see [11–14]). Bakery and Mohamed [15] introduced the concept of the prequasi norm on $\ell^{((r_a))}$ with variable exponent in $(0, 1]$. They looked at the Fatou property of different prequasi norms on $\ell^{((r_a))}$, as well as the sufficient requirements on $\ell^{((r_a))}$ with the definite prequasi norm to construct prequasi Banach and closed space. They also demonstrated the existence of a fixed point of Kannan prequasi norm contraction maps on $\ell^{((r_a))}$ and the prequasi Banach operator ideal constructed by $\ell^{((r_a))}$ and s-numbers. Recently, Reich and Zaslavski [16] showed the existence of a unique fixed point for nonlinear contractive self-mappings of a nonbounded closed subset of a Banach space. They extended this conclusion to contractive mappings, which map into a Banach space a closed subset of the space. For nonexpansive mappings defined by an intersection of a finite number of closed bounded and convex nonempty subsets in Banach spaces, Dehici and Redjel [17] obtained certain fixed point results. According to Bendahmane and Bendoukha [18], a (p, q)-metric space is a generalization of the metric and S-metric spaces. They equipped them a Hausdorff topology and specified several fundamental features. Several well-known findings from fixed point theory are generalized to these new spaces. The paper is structured as follows: we present conditions on the weighted Orlicz sequence space $(\ell_M(\lambda))_\mu$, under definite prequasi norm of μ to construct prequasi Banach and closed sequence space in Section 3. The Fatou property of $\ell_M(\lambda)$ has been investigated for various prequasi norms. In Section 4, the existence of fixed point for Kannan μ-contraction mapping acting on $(\ell_M(\lambda))_\mu$ equipped with different prequasi norms are presented. Several numerical experiments are shown to demonstrate our results. In Section 5, the conditions for which the space $(\ell_M(\lambda))_\mu$ satisfies the property (R) and has the μ-normal structure property are presented. The existence of a fixed point of Kannan prequasi norm nonexpansive mapping on $(\ell_M(\lambda))_\mu$ has been given. In Section 6, we explain the existence of a fixed point of Kannan prequasi norm contraction mapping in the prequasi Banach operator ideal $S_{(\ell_M(\lambda))_\mu}$. In Section 7, we give some applications to the existence of solutions of summable equations.

2. Definitions and Preliminaries

Here and after, the space of all functions $\mu: Y \longrightarrow [0, \infty)$ is $[0, \infty)^Y$, θ is the zero vector of Y, $[x/2]$ is the integral part of $x/2$, F is the space of finite sequences, and $\mathscr{B}$ is the class of each bounded linear mapping between any two Banach spaces. Nakano [19] introduced the concept of modular vector spaces.

Definition 4. Let Y be a vector space. A function $\mu \in [0, \infty]^Y$ *is called modular if the following conditions hold:*

(i) *If* $\beta \in Y$, $\beta = \theta \Leftrightarrow \mu(\beta) = 0$ *and* $\mu(\beta) \geq 0$,

(ii) *if* $\beta \in Y$ *and* $|\omega| = 1$, *then* $\mu(\omega\beta) = \mu(\beta)$,

(iii) *assume* $\beta, \eta \in Y$ *and* $\omega \in [0, 1]$, *then* $\mu(\omega\beta + (1 - \omega)\eta) \leq \mu(\beta) + \mu(\eta)$.

The concept of premodular vector spaces, which is more general than modular vector spaces.

Definition 5. [6] The linear space of sequences Y is said to be a special space of sequences (sss), if:

(1) $\{e_x\}_{x \in \mathscr{Z}^+} \subseteq Y$,

(2) *Y is solid, i.e., for* $\beta = (\beta_x) \in \mathbf{R}^{\mathscr{Z}^+}$, $\eta = (\eta_x) \in Y$ *and* $|\beta_x| \leq |\eta_x|$, *for all* $x \in \mathscr{Z}^+$, *then* $\beta \in Y$,

(3) *If* $(\beta_x)_{x=0}^{\infty} \in Y$, *then* $(\beta_{[x/2]})_{x=0}^{\infty} \in Y$.

Definition 6. [6] A subclass Y_μ of Y is called a premodular (sss), if we have $\mu \in [0, \infty)^{Y^\mu}$ that satisfies the following conditions:

(i) *When* $\beta \in Y$, $\beta = \theta \Leftrightarrow \mu(\beta) = 0$,

(ii) *For every* $\beta \in Y$ *and* $\omega \in \mathbf{R}$, *then there is* $B \geq 1$ *with* $\mu(\omega\beta) \leq B|\omega|\mu(\beta)$,

(iii) $\mu(\beta + \eta) \leq J(\mu(\beta) + \mu(\eta))$, *for all* $\beta, \eta \in Y$, *holds for some* $J \geq 1$,

(iv) *If* $x \in \mathscr{Z}^+$ *and* $|\beta_x| \leq |\eta_x|$, *then* $\mu((\beta_x)) \leq \mu((\eta_x))$,

(v) *For some* $J_0 \geq 1$, *we have* $\mu((\beta_x)) \leq \mu((\beta_{[x/2]})) \leq J_0\mu((\beta_x))$,

(vi) $\overline{F} = Y_\mu$,

(vii) *There exists* $\varsigma > 0$ *such that* $\mu(\omega, 0, 0, 0, \ldots) \geq \varsigma|\omega|\mu(1, 0, 0, 0, \ldots)$, *for all* $\omega \in \mathbf{R}$.

Example 1. The function $\mu(\beta) = (\sum_{x\in\mathscr{Z}^+} \sqrt[5]{|\beta_x|})^5$ *is a premodular (not a modular) on the vector space* $\ell_{1/5}$. *As for every* $\beta, \eta \in \ell_{1/5}$, *one has*

$$\mu\left(\frac{\beta + \eta}{2}\right) = \left(\sum_{x \in \mathscr{Z}^+} \sqrt[5]{\left|\frac{\beta_x + \eta_x}{2}\right|}\right)^5 \leq 8(\mu(\beta) + \mu(\eta)). \qquad (5)$$

Definition 7. [15] Suppose Y is a (sss). The function $\mu \in [0, \infty)^Y$ is said to be prequasi norm on Y, if it holds the settings (i), (ii), and (iii) of Definition 6.

Theorem 1. *[15] Let* Y *be a premodular (sss), then it is prequasi normed (sss).*

Theorem 2. *[15]* Y *is a prequasi normed (sss), when it is quasi-normed (sss).*

Definition 8. [20]

(i) *The prequasi norm* μ *on* X_μ *is said to be* μ*-convex, when* $\mu(\omega\beta + (1-\omega)\eta) \leq \omega\mu(\beta) + (1-\omega)\mu(\eta)$, *for all* $\omega \in [0, 1]$ *and* $\beta, \eta \in X_\mu$.

(ii) $\{\beta_x\}_{x\in\mathscr{Z}^+} \subseteq (X)_\mu$ *is* μ*-convergent to* $\beta \in (X)_\mu$, *if and only if,* $\lim_{x\longrightarrow\infty} \mu(\beta_x - \beta) = 0$. *If the* μ*-limit exists, hence it is unique.*

(iii) $\{\beta_x\}_{x\in\mathscr{Z}^+} \subseteq (X)_\mu$ *is* μ*-Cauchy, if* $\lim_{x,h\longrightarrow\infty} \mu(\beta_x - \beta_h) = 0$.

(iv) $\Phi \subset (X)_\mu$ *is* μ*-closed, if for every* μ*-converging* $\{\beta_x\}_{x\in\mathscr{Z}^+} \subset \Phi$ *to* β, *then* $\beta \in \Phi$.

(v) $\Phi \subset (X)_\mu$ *is* μ*-bounded, if* $v_\mu(\Phi) = \sup\{\mu(\beta - \eta): \beta, \eta \in \Phi\} < \infty$.

(vi) *The* μ*-ball of radius* $r \geq 0$ *and center* β, *for every* $\beta \in (X)_\mu$, *is defined as*

$$\mathscr{B}_\mu(\beta, r) = \{\eta \in (X)_\mu: \mu(\beta - \eta) \leq r\}. \qquad (6)$$

(vii) *A prequasi norm* μ *on* X *satisfies the Fatou property, if for every sequence* $\{\eta^x\} \subseteq (X)_\mu$ *with* $\lim_{x\longrightarrow\infty} \mu(\eta^x - \eta) = 0$ *and any* $\beta \in (X)_\mu$, *we have* $\mu(\beta - \eta) \leq \sup_m \inf_{x\geq m} \mu(\beta - \eta^x)$.

Recall that the μ-balls are μ-closed under the Fatou property.

Definition 9. [21] A subclass $\mathscr{G}$ of $\mathscr{B}$ is called an operator ideal, if every vector $\mathscr{G}(\mathfrak{X}, \mathfrak{Y}) = \mathscr{G} \cap \mathscr{B}(\mathfrak{X}, \mathfrak{Y})$ holds the following conditions:

(i) $I_\gamma \in \mathscr{G}$, *where* γ *indicates Banach space of one dimension.*

(ii) *The space* $\mathscr{G}(\mathfrak{X}, \mathfrak{Y})$ *is linear over* $\mathbf{R}$.

(iii) *If* $H \in \mathscr{B}(\mathfrak{X}_0, \mathfrak{X})$, $T \in \mathscr{G}(\mathfrak{X}, \mathfrak{Y})$, *and* $V \in \mathscr{B}(\mathfrak{Y}, \mathfrak{Y}_0)$, *then* $VTH \in \mathscr{G}(\mathfrak{X}_0, \mathfrak{Y}_0)$, *where* $\mathfrak{X}_0$ *and* $\mathfrak{Y}_0$ *are normed spaces.*

Recall that the quasi operator ideals are a special case of the prequasi operator ideals.

Definition 10. [6] A function $\Upsilon \in [0, \infty)^{\mathscr{G}}$ is said to be a prequasi norm on the ideal $\mathscr{G}$ if the following conditions verify:

(1) *Suppose* $H \in \mathscr{G}(\mathfrak{X}, \mathfrak{Y})$, $\Upsilon(H) \geq 0$ *and* $\Upsilon(H) = 0$, *if and only if,* $H = 0$,

(2) *there exists* $D \geq 1$ *such that* $\Upsilon(\omega H) \leq D|\omega|\Upsilon(H)$, *for every* $H \in \mathscr{G}(\mathfrak{X}, \mathfrak{Y})$ *and* $\omega \in \mathbf{R}$,

(3) *we have* $J \geq 1$ *so that* $\Upsilon(H_1 + H_2) \leq J[\Upsilon(H_1) + \Upsilon(H_2)]$, *for all* $H_1, H_2 \in \mathscr{G}(\mathfrak{X}, \mathfrak{Y})$,

(4) *we get* $\omega \geq 1$ *so that if* $H \in \mathscr{B}(\mathfrak{X}_0, \mathfrak{X})$, $T \in \mathscr{G}(\mathfrak{X}, \mathfrak{Y})$, *and* $V \in \mathscr{B}(\mathfrak{Y}, \mathfrak{Y}_0)$, *then* $\Upsilon(VTH) \leq \omega\|V\|\Upsilon(T)\|H\|$.

Theorem 3. *[15] The function* $\Upsilon(H) = \mu(s_x(H))_{x=0}^\infty$ *is a prequasi norm on* S_{Y_μ}, *when* Y_μ *is a premodular (sss).*

Theorem 4. *[6] If* Υ *is a quasi norm on the ideal* $\mathscr{G}$, *then* Υ *is a prequasi norm on the ideal* $\mathscr{G}$.

Lemma 1. *[22, 23] Assume* $M: (0, \infty) \longrightarrow [0, \infty)$ *is a continuous function and strictly increasing with* $\lim_{x\longrightarrow 0} M(x) = 0$, *and if the functions* $M(x)$ *and* $\ln(M(e^x))$ *are convex on* $[0, \infty)$, *then*

$$M^{-1}\left(\sum_{x=0}^\infty \lambda_x M(\beta_x + \eta_x)\right) \leq M^{-1}\left(\sum_{x=0}^\infty \lambda_x M(\beta_x)\right) + M^{-1}\left(\sum_{x=0}^\infty \lambda_x M(\eta_x)\right). \qquad (7)$$

$\lambda_x, \beta_x, \eta_x \in [0, \infty)$, *for all* $x \in \mathscr{Z}^+$ *and* $\sum_{x=0}^\infty \lambda_x = 1$.

3. Main Results

3.1. Properties of Different Prequasi Norms. In this section, we have studied some topological structures and the Fatou property of the weighted Orlicz sequence space, $\ell_M(\lambda)$, for various prequasi norms.

Lemma 2. *If* M *is a concave Orlicz function, then* $M(x + y) \leq M(x) + M(y)$, *for all* $x, y \in [0, \infty)$.

Proof. It is easy so omitted. □

Theorem 5. *$(\ell_M(\lambda))_\mu$, where $\mu(\beta)=\sum_{y=0}^{\infty}\lambda_y M(|\beta_y|)$, for each $\beta\in\ell_M(\lambda)$, is a premodular (sss), if M is a concave Orlicz function or convex Orlicz function satisfying Δ_2-condition.*

Proof. Suppose M is a convex Orlicz function satisfying Δ_2-condition. First, we must demonstrate that $\ell_M(\lambda)$ is a (sss):

(1)

(i) Let $\beta, t\in\ell_M(\lambda)$. As M is a strictly increasing and convex function satisfying Δ_2-condition, we get

$$\begin{aligned}\mu(\beta+\eta)&=\sum_{y=0}^{\infty}\lambda_y M\left(\left|\beta_y+\eta_y\right|\right)\\&\leq\frac{k}{2}\left[\sum_{y=0}^{\infty}\lambda_y M\left(\left|\beta_y\right|\right)+\sum_{y=0}^{\infty}\lambda_y M\left(\left|\eta_y\right|\right)\right]\\&=\frac{k}{2}(\mu(\beta)+\mu(\eta))<\infty,\end{aligned}\tag{8}$$

this implies $\beta+\eta\in\ell_M(\lambda)$.

(ii) Suppose $\omega\in\mathbf{R}$ and $\beta\in\ell_M(\lambda)$. Since M satisfies Δ_2-condition, we have

$$\begin{aligned}\mu(\omega\beta)&=\sum_{y=0}^{\infty}\lambda_y M\left(\left|\omega\beta_y\right|\right)\\&\leq k|\omega|\sum_{y=0}^{\infty}\lambda_y M\left(\left|\beta_y\right|\right)\leq D|\omega|\mu(\beta)<\infty.\end{aligned}\tag{9}$$

So $\omega\beta\in\ell_M(\lambda)$. Therefore, from conditions 1 (i) and (ii), one has $\ell_M(\lambda)$ is linear. We have $e_y\in\ell_M(\lambda)$, for every $y\in\mathscr{Z}^+$, as

(2) Let $|\beta_y|\leq|\eta_y|$, for every $y\in\mathscr{Z}^+$ and $\eta\in\ell_M(\lambda)$. Since M is a nondecreasing function, then

$$\begin{aligned}\mu(\beta)&=\sum_{y=0}^{\infty}\lambda_y M\left(\left|\beta_y\right|\right)\\&\leq\sum_{y=0}^{\infty}\lambda_y M\left(\left|\eta_y\right|\right)=\mu(\eta)<\infty,\end{aligned}\tag{10}$$

one has $\beta\in\ell_M(\lambda)$.

(3) Assume $(\beta_y)\in\ell_M(\lambda)$, we get

$$\begin{aligned}\mu\left(\left(\beta_{[y/2]}\right)\right)&=\sum_{y=0}^{\infty}\lambda_y M\left(\left|\beta_{[y/2]}\right|\right)\\&\leq 2\sum_{y=0}^{\infty}\lambda_y M\left(\left|\beta_y\right|\right)=2\mu\left(\left(\beta_y\right)\right)<\infty,\end{aligned}\tag{11}$$

then $(\beta_{[y/2]})\in\ell_M(\lambda)$. Second, to prove that the functional μ on $\ell_M(\lambda)$ is a premodular:

(i) Obviously, $\mu(\beta)\geq 0$ and $\mu(\beta)=0\Leftrightarrow\beta=\theta$.

(ii) There are $D=\max\{1,k\}\geq 1$ with $\mu(\omega\beta)\leq D|\omega|\mu(\beta)$, for every $\beta\in\ell_M(\lambda)$ and $\omega\in\mathbf{R}$.

(iii) There exists $J=\max\{1,k/2\}\geq 1$ with $\mu(\beta+\eta)\leq J(\mu(\beta)+\mu(\eta))$, for every $\beta,\eta\in\ell_M(\lambda)$.

(iv) Follows the proof part (2).

(v) Follows from the proof part (3) that $J_0=2\geq 1$.

(vi) Obviously, $\overline{F}=\ell_M(\lambda)$.

(vii) There exists $0<\varsigma\leq M_0(|\omega|)/|\omega|M_0(1)$, for $\omega\neq 0$ or $\varsigma>0$, for $\omega=0$ so that $\mu(\omega,0,0,0,\ldots)\geq\varsigma|\omega|\mu(1,0,0,0,\ldots)$.

If M is a concave Orlicz function. By applying Lemma 2 and the parallel proof follows. □

Theorem 6. *If M is a concave Orlicz function or convex Orlicz function satisfying Δ_2-condition, then $(\ell_M(\lambda))_\mu$ is a prequasi Banach (sss), where $\mu(\beta)=\sum_{y=0}^{\infty}\lambda_y M(|\beta_y|)$, for each $\beta\in\ell_M(\lambda)$.*

Proof. Suppose M is a convex Orlicz function satisfying Δ_2-condition. By using Theorem 5, the space $(\ell_M(\lambda))_\mu$ is a premodular (sss). From Theorem 1, the space $(\ell_M(\lambda))_\mu$ is a prequasi normed (sss). To prove that $(\ell_M(\lambda))_\mu$ is a prequasi Banach (sss), let $\beta^r=(\beta_y^r)_{y=0}^{\infty}$ be a Cauchy sequence in $(\ell_M(\lambda))_\mu$. Therefore, for all $\epsilon\in(0,1)$, we have that for every $r,t\geq r_0$, we get

$$\mu\left(\beta^r-\beta^t\right)=\sum_{y=0}^{\infty}\lambda_y M\left(\left|\beta_y^r-\beta_y^t\right|\right)<\epsilon.\tag{12}$$

Hence, for $r,t\geq r_0$ and $y\in\mathscr{Z}^+$, one has $|\beta_y^r-\beta_y^t|<\epsilon$ Then (β_y^t) is a Cauchy sequence in $\mathbf{R}$, for fixed $y\in\mathscr{Z}^+$. This gives $\lim_{t\longrightarrow\infty}\beta_y^t=\beta_y^0$, for constant $y\in\mathscr{Z}^+$. Therefore, $\mu(\beta^r-\beta^0)<\epsilon$, for all $r\geq r_0$. To investigate that $\beta^0\in\ell_M(\lambda)$, one has $\mu(\beta^0)=\mu(\beta^0-\beta^r+\beta^r)\leq J(\mu(\beta^r-\beta^0)+\mu(\beta^r))<\infty$, so $\beta^0\in\ell_M(\lambda)$. This implies that $(\ell_M(\lambda))_\mu$ is a prequasi Banach (sss). If M is a concave Orlicz function. By applying Lemma 2 and the parallel proof follows. □

Theorem 7. *If M is a concave Orlicz function or convex Orlicz function satisfying Δ_2-condition, then $(\ell_M(\lambda))_\mu$ is a prequasi closed (sss), where $\mu(\beta)=\sum_{y=0}^{\infty}\lambda_y M(|\beta_y|)$, for every $\beta\in\ell_M(\lambda)$.*

Proof. Let M be a convex Orlicz function satisfying Δ_2-condition. According to Theorem 5, the space $(\ell_M(\lambda))_\mu$ is a premodular (sss). From Theorem 1, the space $(\ell_M(\lambda))_\mu$ is a prequasi normed (sss). To prove that $(\ell_M(\lambda))_\mu$ is a prequasi closed (sss), suppose $\beta^r=(\beta_y^r)_{y=0}^{\infty}\in(\ell_M(\lambda))_\mu$ and $\lim_{r\longrightarrow\infty}\mu(\beta^r-\beta^0)=0$, hence for all $\epsilon\in(0,1)$, one has $r_0\in\mathscr{Z}^+$ so that for every $r\geq r_0$, we have

$$\mu\left(\beta^r-\beta^0\right)=\sum_{y=0}^{\infty}\lambda_y M\left(\left|\beta_y^r-\beta_y^0\right|\right)<\epsilon.\tag{13}$$

Therefore, for $r\geq r_0$ and $y\in\mathscr{Z}^+$, one has $|\beta_y^r-\beta_y^0|<\epsilon$. Hence, (β_y^r) is a convergent sequence in $\mathbf{R}$, for constant $y\in\mathscr{Z}^+$. So, $\lim_{r\longrightarrow\infty}\beta_y^r=\beta_y^0$, for constant $y\in\mathscr{Z}^+$. Finally to show that $\beta^0\in\ell_M(\lambda)$, one has

$$\mu(\beta^0) = \mu(\beta^0 - \beta^r + \beta^r) \leq J(\mu(\beta^r - \beta^0) + \mu(\beta^r)) < \infty. \tag{14}$$

Hence, HTML translation failed. This implies that $(\ell_M(\lambda))_\mu$ is a prequasi closed (sss). If M is a concave Orlicz function, by applying Lemma 2 and the parallel proof follows. □

Theorem 8. *If M is a convex Orlicz function satisfying Δ_2-condition and $\ln(M(e^x))$ is convex, then the function $\mu(\beta) = M^{-1}(\sum_{y=0}^{\infty} \lambda_y M(|\beta_y|))$ verifies the Fatou property, for all $\beta \in \ell_M(\lambda)$.*

Proof. Assume that $\{\eta^b\} \subseteq (\ell_M(\lambda))_\mu$ such that $\lim_{b\longrightarrow\infty} \mu(\eta^b - \eta) = 0$. As the space $(\ell_M(\lambda))_\mu$ is a prequasi closed space, one has $t \in (\ell_M(\lambda))_\mu$. Hence, for every $\beta \in (\ell_M(\lambda))_\mu$, from Lemma 1, we have

$$\begin{aligned}\mu(\beta - \eta) &= M^{-1}\left(\sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y|)\right) \leq M^{-1}\left(\sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y^b|)\right) + M^{-1}\left(\sum_{y=0}^{\infty} \lambda_y M(|\eta_y^b - \eta_y|)\right) \\ &\leq \sup_j \inf_{b\geq j} \mu(\beta - \eta^b).\end{aligned} \tag{15}$$

Hence, μ satisfies the Fatou property. □

Theorem 9. *If M is a concave Orlicz function, then the function $\mu(\beta) = \sum_{y=0}^{\infty} \lambda_y M(|\beta_y|)$ holds the Fatou property, for all $\beta \in \ell_M(\lambda)$.*

Proof. Suppose $\{\eta^b\} \subseteq (\ell_M(\lambda))_\mu$ so that $\lim_{b\longrightarrow\infty} \mu(\eta^b - \eta) = 0$. As the space $(\ell_M(\lambda))_\mu$ is a prequasi closed space; hence, $\eta \in (\ell_M(\lambda))_\mu$. As M is continuous, concave and $M(0) = 0$. Therefore, for every $\beta \in (\ell_M(\lambda))_\mu$, one has

$$\begin{aligned}\mu(\beta - \eta) &= \sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y|) \leq \sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y^b|) \\ &\quad + \sum_{y=0}^{\infty} \lambda_y M(|\eta_y^b - \eta_y|) \leq \sup_j \inf_{b\geq j} \mu(\beta - \eta^b).\end{aligned} \tag{16}$$

Hence, μ satisfies the Fatou property. □

Theorem 10. *The function $\mu(\beta) = \sum_{y=0}^{\infty} \lambda_y M(|\beta_y|)$ does not satisfy the Fatou property, for all $\beta \in \ell_M(\lambda)$, if M is a strictly convex Orlicz function satisfying Δ_2-condition.*

Proof. Since M is a strictly convex Orlicz function satisfying Δ_2-condition, then there exists $k > 2$ such that $2M(u) < M(2u) < kM(u)$, for all $u \geq 0$. Let the conditions be fulfilled and $\{\eta^b\} \subseteq (\ell_M(\lambda))_\mu$ with $\lim_{b\longrightarrow\infty} \mu(\eta^b - \eta) = 0$. As the space $(\ell_M(\lambda))_\mu$ is a prequasi closed space; hence, $\eta \in (\ell_M(\lambda))_\mu$. Since M is continuous, then for any $\beta \in (\ell_M(\lambda))_\mu$, we have

$$\mu(\beta - \eta) = \sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y|) \leq \frac{k}{2}\left[\sum_{y=0}^{\infty} \lambda_y M(|\beta_y - \eta_y^b|) + \sum_{y=0}^{\infty} \lambda_y M(|\eta_y^b - \eta_y|)\right] \leq \frac{k}{2} \sup_j \inf_{b\geq j} \mu(\beta - \eta^b). \tag{17}$$

Therefore, μ does not hold the Fatou property. □

Example 2. *For every $\beta \in \ell_M(\lambda)$, the function $\mu(\beta) = \ln(1 + \sum_{y=0}^{\infty} \lambda_y (e^{|\beta_y|} - 1))$ is a prequasi norm, not quasi, and not a norm.*

Example 3. *For all $\beta \in \ell_M(\lambda)$, the function $\mu(\beta) = (\sum_{y=0}^{\infty} \lambda_y \sqrt{|\beta_y|})^2$ is a prequasi norm, quasi norm, and not a norm.*

Example 4. *The function $\mu(\beta) = \inf\{\kappa > 0: \sum_{y=0}^{\infty} \lambda_y M(|\beta_y|/\kappa) \leq 1\}$ is a prequasi norm, a quasi* norm, *and a* norm *on $\ell_M(\lambda)$.*

4. Kannan μ-Contraction Operator

We now define Kannan μ-Lipschitzian mapping acting on $(\ell_M(\lambda))_\mu$. The sufficient conditions for a fixed point of Kannan contraction mapping on $(\ell_M(\lambda))_\mu$ under various prequasi norms are investigated.

Definition 11. An operator H: $(\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$ is called a Kannan μ-Lipschitzian, if there exists $\nu \geq 0$, so that

$$\mu(H\beta - H\eta) \leq \nu(\mu(H\beta - \beta) + \mu(H\eta - \eta)), \tag{18}$$

for every $\beta, \eta \in (\ell_M(\lambda))_\mu$.

(1) The operator H is said to be Kannan μ-contraction, when $\nu \in [0, 1/2)$.

(2) The operator H is said to be Kannan μ-nonexpansive, whenever $\nu = 1/2$.

A vector $\beta \in (\ell_M(\lambda))_\mu$ is called a fixed point of H, when $H(\beta) = \beta$.

Theorem 11. *If M is a convex Orlicz function satisfying Δ_2-condition and $\ln(M(e^x))$ is convex, and $H: (\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$ is Kannan μ-contraction mapping, where $\mu(\beta) = M^{-1}(\sum_{y=0}^{\infty}\lambda_y M(|\beta_y|))$, for all $\beta \in \ell_M(\lambda)$; hence, H has a unique fixed point.*

Proof. Assume that $\beta \in \ell_M(\lambda)$, one has $H^t\beta \in \ell_M(\lambda)$. Since H is a Kannan μ-contraction mapping, we have

$$\mu\left(H^{t+1}\beta - H^t\beta\right) \le \nu\left(\mu\left(H^{t+1}\beta - H^t\beta\right) + \mu\left(H^t\beta - H^{t-1}\beta\right)\right) \Rightarrow$$
$$\mu\left(H^{t+1}\beta - H^t\beta\right) \le \frac{\nu}{1-\nu}\mu\left(H^t\beta - H^{t-1}\beta\right) \le \left(\frac{\nu}{1-\nu}\right)^2\mu\left(H^{t-1}\beta - H^{t-2}\beta\right) \le \cdots \le \left(\frac{\nu}{1-\nu}\right)^t\mu(H\beta - \beta). \tag{19}$$

Therefore, for every $t, \nu \in \mathscr{X}^+$ with $\nu > t$, then we get

$$\mu\left(H^t\beta - H^\nu\beta\right) \le \nu\left(\mu\left(H^t\beta - H^{t-1}\beta\right) + \mu\left(H^\nu\beta - H^{\nu-1}\beta\right)\right) \le \nu\left(\left(\frac{\nu}{1-\nu}\right)^{t-1} + \left(\frac{\nu}{1-\nu}\right)^{\nu-1}\right)\mu(H\beta - \beta). \tag{20}$$

So, $\{H^t\beta\}$ is a Cauchy sequence in $(\ell_M(\lambda))_\mu$. As the space $(\ell_M(\lambda))_\mu$ is prequasi Banach space. Therefore, there is $\eta \in (\ell_M(\lambda))_\mu$ such that $\lim_{t\longrightarrow\infty} H^t\beta = \eta$. To prove that $H\eta = \eta$. As μ holds the Fatou property, we obtain

$$\mu(H\eta - \eta) \le \sup_p \inf_{t\ge p}\mu\left(H^{t+1}\beta - H^t\beta\right) \le \sup_p \inf_{t\ge p}\left(\frac{\nu}{1-\nu}\right)^t\mu(H\beta - \beta) = 0, \tag{21}$$

hence $H\eta = \eta$. Hence, η is a fixed point of H. To prove the uniqueness of the fixed point. For different fixed points $\zeta, \eta \in (\ell_M(\lambda))_\mu$ of H. We have that

$$\mu(\zeta - \eta) \le \mu(H\zeta - H\eta) \le \nu(\mu(H\zeta - \zeta) + \mu(H\eta - \eta)) = 0. \tag{22}$$

Therefore, $\zeta = \eta$. □

Corollary 1. *Let M be a convex Orlicz function satisfying Δ_2-condition and $\ln(M(e^x))$ be convex, and $H: (\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$ be Kannan μ-contraction mapping, with $\mu(\beta) = M^{-1}(\sum_{y=0}^{\infty}\lambda_y M(|\beta_y|))$, for every $\beta \in \ell_M(\lambda)$, then H has a unique fixed point ζ such that $\mu(H^t\beta - \zeta) \le \nu(\nu/1-\nu)^{t-1}\mu(H\beta - \beta)$.*

Proof. From Theorem 11, there is a unique fixed point ζ of H. Hence, one has

$$\mu\left(H^t\beta - \zeta\right) = \mu\left(H^t\beta - H\zeta\right) \le \nu\left(\mu\left(H^t\beta - H^{t-1}\beta\right) + \mu(H\zeta - \zeta)\right) = \nu\left(\frac{\nu}{1-\nu}\right)^{t-1}\mu(H\beta - \beta). \tag{23}$$

□

Theorem 12. *Suppose M is a concave Orlicz function, and $H: (\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$ is Kannan μ-contraction mapping, where $\mu(\beta) = \sum_{y=0}^{\infty}\lambda_y M(|\beta_y|)$, for all $\beta \in \ell_M(\lambda)$; hence, H has a unique fixed point.*

Proof. It is easy so omitted. □

Definition 13. Assume $(\ell_M(\lambda))_\mu$ is a pr-quasi normed (sss), $H: (\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$ and $\zeta \in (\ell_M(\lambda))_\mu$. The operator H is called μ-sequentially continuous at ζ, if and only if, when $\lim_{y\longrightarrow\infty}\mu(\beta_y - \zeta) = 0$, then $\lim_{y\longrightarrow\infty}\mu(H\beta_y - H\zeta) = 0$.

Theorem 14. *Let M be a strictly convex Orlicz function satisfying Δ_2-condition, and $H: (\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$, where $\mu(\beta) = \sum_{y=0}^{\infty}\lambda_y M(|\beta_y|)$, for every $\beta \in \ell_M(\lambda)$. The element $\eta \in (\ell_M(\lambda))_\mu$ is the unique fixed point of H, if the next conditions are satisfied:*

(i) H is Kannan μ-contraction mapping,

(ii) H is μ-sequentially continuous at a point $\eta \in (\ell_M(\lambda))_\mu$,

(iii) There exists $\beta \in (\ell_M(\lambda))_\mu$ such that the sequence of iterates $\{H^t\beta\}$ has a subsequence $\{H^{t_p}\beta\}$ converging to η.

Proof. Since M is a strictly convex Orlicz function satisfying Δ_2-condition, then there exists $k > 2$ such that $2M(u) < M(2u) < kM(u)$, for all $u \ge 0$. Let the conditions be verified. If η is not a fixed point of H, then $H\eta \ne \eta$. By the conditions (ii) and (iii), we have

$$\lim_{t_p\longrightarrow\infty}\mu\left(H^{t_p}\beta - \eta\right) = 0, \qquad \lim_{t_p\longrightarrow\infty}\mu\left(H^{t_p+1}\beta - H\eta\right) = 0. \tag{24}$$

As the operator H is Kannan μ-contraction, one can see

$$\begin{aligned}0 < \mu(H\eta - \eta) &= \mu\left(\left(H\eta - H^{t_p+1}\beta\right) + \left(H^{t_p}\beta - \eta\right) + \left(H^{t_p+1}\beta - H^{t_p}\beta\right)\right)\\ &\leq \frac{k^2}{4}\mu\left(H^{t_p+1}\beta - H\eta\right) + \frac{k^2}{4}\mu\left(H^{t_p}\beta - \eta\right) + \frac{k}{2}\nu\left(\frac{\nu}{1-\nu}\right)^{t_p-1}\mu(H\beta - \beta).\end{aligned} \tag{25}$$

Since $t_p \longrightarrow \infty$, this gives a contradiction. Hence, η is a fixed point of H. To prove that the uniqueness of the fixed point η. For different fixed points $\eta, \zeta \in (\ell_M(\lambda))_\mu$ of H. Therefore, one has

$$\mu(\eta - \zeta) \leq \mu(H\eta - H\zeta) \leq \nu(\mu(H\eta - \eta) + \mu(H\zeta - \zeta)) = 0. \tag{26}$$

So, $\eta = \zeta$. □

Example 15. Assume H: $(\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$, *where* $M(t) = \sqrt[3]{t} + \sqrt[4]{t}$ *and* $\mu(\beta) = \sum_{y=0}^{\infty}\lambda_y M(|\beta_y|)$, *for all* $\beta \in \ell_M(\lambda)$ *and*

$$H(\beta) = \begin{cases} \dfrac{\beta}{18}, & \mu(\beta) \in [0,1), \\ \\ \dfrac{\beta}{20}, & \mu(\beta) \in [1,\infty). \end{cases} \tag{27}$$

As for each $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [0,1)$, one has

$$\begin{aligned}\mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{18} - \frac{\beta_2}{18}\right) \leq \frac{1}{\sqrt[4]{17}}\left(\mu\left(\frac{17\beta_1}{18}\right) + \mu\left(\frac{17\beta_2}{18}\right)\right)\\ &= \frac{1}{\sqrt[4]{17}}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).\end{aligned} \tag{28}$$

For all $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [1,\infty)$, one has

$$\begin{aligned}\mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{20} - \frac{\beta_2}{20}\right) \leq \frac{1}{\sqrt[4]{19}}\left(\mu\left(\frac{19\beta_1}{20}\right) + \mu\left(\frac{19\beta_2}{20}\right)\right)\\ &= \frac{1}{\sqrt[4]{19}}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).\end{aligned} \tag{29}$$

For all $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1) \in [0,1)$ and $\mu(\beta_2) \in [1,\infty)$, we obtain

$$\begin{aligned}\mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{18} - \frac{\beta_2}{20}\right) \leq \frac{1}{\sqrt[4]{17}}\mu\left(\frac{17\beta_1}{18}\right) + \frac{1}{\sqrt[4]{19}}\mu\left(\frac{19\beta_2}{20}\right)\\ &\leq \frac{1}{\sqrt[4]{17}}\left(\mu\left(\frac{17\beta_1}{18}\right) + \mu\left(\frac{19\beta_2}{20}\right)\right)\\ &= \frac{1}{\sqrt[4]{17}}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).\end{aligned} \tag{30}$$

Hence, the operator H is Kannan μ-contraction. As μ verifies the Fatou property. From Theorem 11, the operator H has a unique fixed point $\theta \in (\ell_M(\lambda))_\mu$.

Assume $\{\beta^{(y)}\} \subseteq (\ell_M(\lambda))_\mu$ is such that $\lim_{y\longrightarrow\infty}\mu(\beta^{(y)} - \beta^{(0)}) = 0$, where $\beta^{(0)} \in (\ell_M(\lambda))_\mu$ with $\mu(\beta^{(0)}) = 1$.

As the prequasi norm μ is continuous, one can see

$$\begin{aligned}\lim_{t_p\longrightarrow\infty}\mu\left(H\beta^{(y)} - H\beta^{(0)}\right) &= \lim_{t_p\longrightarrow\infty}\mu\left(\frac{\beta^{(y)}}{18} - \frac{\beta^{(0)}}{20}\right)\\ &= \mu\left(\frac{\beta^{(0)}}{180}\right) > 0.\end{aligned} \tag{31}$$

Therefore, H is not μ-sequentially continuous at $\beta^{(0)}$. Hence, the operator H is not continuous at $\beta^{(0)}$.

Let $\mu(\beta) = [\sum_{y=0}^{\infty}\lambda_y M(|\beta_y|)]^4$, for all $\beta \in \ell_M(\lambda)$.

As for all $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [0,1)$, one has

$$\begin{aligned}\mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{18} - \frac{\beta_2}{18}\right) \leq \frac{8}{17}\left(\mu\left(\frac{17\beta_1}{18}\right) + \mu\left(\frac{17\beta_2}{18}\right)\right)\\ &= \frac{8}{17}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).\end{aligned} \tag{32}$$

For all $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [1,\infty)$, one has

$$\begin{aligned}\mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{20} - \frac{\beta_2}{20}\right) \leq \frac{8}{19}\left(\mu\left(\frac{19\beta_1}{20}\right) + \mu\left(\frac{19\beta_2}{20}\right)\right)\\ &= \frac{8}{19}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).\end{aligned} \tag{33}$$

For all $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1) \in [0,1)$ and $\mu(\beta_2) \in [1,\infty)$, we get

$$\begin{aligned}\mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{18} - \frac{\beta_2}{20}\right) \leq \frac{8}{17}\mu\left(\frac{17\beta_1}{18}\right) + \frac{8}{19}\mu\left(\frac{19\beta_2}{20}\right)\\ &\leq \frac{8}{17}\left(\mu\left(\frac{17\beta_1}{18}\right) + \mu\left(\frac{19\beta_2}{20}\right)\right)\\ &= \frac{8}{17}(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)).\end{aligned} \tag{34}$$

So, the operator H is Kannan μ-contraction and $H^t(\beta) = \begin{cases} \beta/18^t & \mu(\beta) \in [0,1) \\ \beta/20^t & \mu(\beta) \in [1,\infty) \end{cases}$

Clearly, H is μ-sequentially continuous at $\theta \in (\ell_M(\lambda))_\mu$ and $\{H^t\beta\}$ contains a subsequence $\{H^{t_p}\beta\}$ converging to θ. From Theorem 14, then $\theta \in (\ell_M(\lambda))_\mu$ is the unique fixed point of H.

Example 5. Assume $H: (\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$, where $M(t) = t^2$ and $\mu(\beta) = \sqrt{\sum_{y=0}^{\infty} \lambda_y M(|\beta_y|)}$, for all $\beta \in \ell_M(\lambda)$ and

$$H(\beta) = \begin{cases} \dfrac{\beta}{4}, & \mu(\beta) \in [0,1), \\ \dfrac{\beta}{5}, & \mu(\beta) \in [1,\infty). \end{cases} \tag{35}$$

As for each $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [0,1)$, one has

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{4} - \frac{\beta_2}{4}\right) \le \frac{1}{3}\left(\mu\left(\frac{3\beta_1}{4}\right) + \mu\left(\frac{3\beta_2}{4}\right)\right) \\ &= \frac{1}{3}\left(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)\right). \end{aligned} \tag{36}$$

For all $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [1,\infty)$, one has

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{5} - \frac{\beta_2}{5}\right) \le \frac{1}{4}\left(\mu\left(\frac{4\beta_1}{5}\right) + \mu\left(\frac{4\beta_2}{5}\right)\right) \\ &= \frac{1}{4}\left(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)\right). \end{aligned} \tag{37}$$

For all $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1) \in [0,1)$ and $\mu(\beta_2) \in [1,\infty)$, we get

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{4} - \frac{\beta_2}{5}\right) \le \frac{1}{3}\mu\left(\frac{3\beta_1}{4}\right) + \frac{1}{4}\mu\left(\frac{4\beta_2}{5}\right) \\ &\le \frac{1}{3}\left(\mu\left(\frac{3\beta_1}{4}\right) + \mu\left(\frac{4\beta_2}{5}\right)\right) \\ &= \frac{1}{3}\left(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)\right). \end{aligned} \tag{38}$$

Hence, the operator H is Kannan μ-contraction. As μ satisfies the Fatou property. From Theorem 11, the operator H has one fixed point $\theta \in (\ell_M(\lambda))_\mu$.

Suppose $\{\beta^{(y)}\} \subseteq (\ell_M(\lambda))_\mu$ is so that $\lim_{y\longrightarrow\infty} \mu(\beta^{(y)} - \beta^{(0)}) = 0$, where.

$\beta^{(0)} \in (\ell_M(\lambda))_\mu$ with $\mu(\beta^{(0)}) = 1$. As the prequasi norm μ is continuous, one can see

$$\begin{aligned} \lim_{y\longrightarrow\infty} \mu\left(H\beta^{(y)} - H\beta^{(0)}\right) &= \lim_{y\longrightarrow\infty} \mu\left(\frac{\beta^{(y)}}{4} - \frac{\beta^{(0)}}{5}\right) \\ &= \mu\left(\frac{\beta^{(0)}}{20}\right) > 0. \end{aligned} \tag{39}$$

Therefore, H is not μ-sequentially continuous at $\beta^{(0)}$. Hence, the map H is not continuous at $\beta^{(0)}$.

Let $\mu(\beta) = \sum_{y=0}^{\infty} \lambda_y M(|\beta_y|)$, for every $\beta \in \ell_M(\lambda)$.

As for each $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [0,1)$, one has

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{4} - \frac{\beta_2}{4}\right) \le \frac{2}{9}\left(\mu\left(\frac{3\beta_1}{4}\right) + \mu\left(\frac{3\beta_2}{4}\right)\right) \\ &= \frac{2}{9}\left(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)\right). \end{aligned} \tag{40}$$

For all $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1), \mu(\beta_2) \in [1,\infty)$, one has

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{5} - \frac{\beta_2}{5}\right) \le \frac{1}{8}\left(\mu\left(\frac{4\beta_1}{5}\right) + \mu\left(\frac{4\beta_2}{5}\right)\right) \\ &= \frac{1}{8}\left(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)\right). \end{aligned} \tag{41}$$

For all $\beta_1, \beta_2 \in (\ell_M(\lambda))_\mu$ with $\mu(\beta_1) \in [0,1)$ and $\mu(\beta_2) \in [1,\infty)$, we obtain

$$\begin{aligned} \mu(H\beta_1 - H\beta_2) &= \mu\left(\frac{\beta_1}{4} - \frac{\beta_2}{5}\right) \le \frac{2}{9}\mu\left(\frac{3\beta_1}{4}\right) + \frac{1}{8}\mu\left(\frac{4\beta_2}{5}\right) \\ &\le \frac{2}{9}\left(\mu\left(\frac{3\beta_1}{4}\right) + \mu\left(\frac{4\beta_2}{5}\right)\right) \\ &= \frac{2}{9}\left(\mu(H\beta_1 - \beta_1) + \mu(H\beta_2 - \beta_2)\right). \end{aligned} \tag{42}$$

So, the operator H is Kannan μ-contraction and. $H^t(\beta) = \begin{cases} \beta/4^t & \mu(\beta) \in [0,1) \\ \beta/5^t & \mu(\beta) \in [1,\infty) \end{cases}$

Obviously, H is μ-sequentially continuous at $\theta \in (\ell_M(\lambda))_\mu$ and $\{H^t\beta\}$ has a subsequence $\{H^{t_p}\beta\}$ converging to θ. From Theorem 14, then $\theta \in (\ell_M(\lambda))_\mu$ is the unique fixed point of H.

Example 16. Suppose $H: (\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$, where $M(t) = \sqrt[3]{t} + \sqrt[4]{t}$ and $\mu(\beta) = \left(\sum_{y=0}^{\infty} \lambda_y M(|\beta_y|)\right)^4$, for every $\beta \in \ell_M(\lambda)$ and

$$H(\beta) = \begin{cases} \dfrac{1}{18}(e_0 + \beta), & \beta_0 \in \left(-\infty, \dfrac{1}{17}\right), \\ \dfrac{1}{17}e_0, & \beta_0 = \dfrac{1}{17}, \\ \dfrac{1}{18}e_0, & \beta_0 \in \left(\dfrac{1}{17}, \infty\right). \end{cases} \tag{43}$$

As for each $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (-\infty, 1/17)$, one has

$$\begin{aligned} \mu(H\beta - H\eta) &= \mu\left(\frac{1}{18}(\beta_0 - \eta_0, \beta_1 - \eta_1, \beta_2 - \eta_2, \ldots)\right) \\ &\le \frac{8}{17}\left(\mu\left(\frac{17\beta}{18}\right) + \mu\left(\frac{17t}{18}\right)\right) \\ &\le \frac{8}{17}\left(\mu(H\beta - \beta) + \mu(H\eta - \eta)\right). \end{aligned} \tag{44}$$

For every $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (1/17, \infty)$, then for all $\epsilon > 0$ one has

$$\mu(H\beta - H\eta) = 0 \leq \epsilon(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \quad (45)$$

For every $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/17)$ and $\eta_0 \in (1/17, \infty)$, we get

$$\begin{aligned}\mu(H\beta - H\eta) &= \mu\left(\frac{\beta}{18}\right) \leq \frac{1}{17}\mu\left(\frac{17\beta}{18}\right) = \frac{1}{17}\mu(H\beta - \beta)\\ &\leq \frac{1}{17}(\mu(H\beta - \beta) + \mu(H\eta - \eta)).\end{aligned} \quad (46)$$

Hence, the operator H is Kannan μ-contraction. Evidently, H is μ-sequentially continuous at $1/17e_0 \in (\ell_M(\lambda))_\mu$, and we have $\beta \in (\ell_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/17)$ under $\{H^t\beta\} = \{\sum_{n=1}^{t} 1/18^n e_0 + 1/18^t\beta\}$ contains a subsequence $\{H^{t_p}\beta\} = \left\{\sum_{n=1}^{t_p} 1/18^n e_0 + 1/18^{t_p}\beta\right\}$ converging to $1/17e_0$. From Theorem 14, the map H has a unique fixed point $1/17e_0 \in (\ell_M(\lambda))_\mu$. Observe that H is not continuous at $1/17e_0 \in (\ell_M(\lambda))_\mu$.

If $\mu(\beta) = \sum_{y\in\mathscr{I}^+}\lambda_y M(|\beta_y|)$, for every $\beta \in \ell_M(\lambda)$. As for all $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (-\infty, 1/17)$, one has

$$\begin{aligned}\mu(H\beta - H\eta) &= \mu\left(\frac{1}{18}(\beta_0 - \eta_0, \beta_1 - \eta_1, \beta_2 - \eta_2, \ldots)\right)\\ &\leq \frac{1}{\sqrt[4]{17}}\left(\mu\left(\frac{17\beta}{18}\right) + \mu\left(\frac{17t}{18}\right)\right)\\ &\leq \frac{1}{\sqrt[4]{17}}(\mu(H\beta - \beta) + \mu(H\eta - \eta)).\end{aligned} \quad (47)$$

For each $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (1/17, \infty)$, then for all $\epsilon > 0$ we get

$$\mu(H\beta - H\eta) = 0 \leq \epsilon(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \quad (48)$$

For every $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/17)$ and $\eta_0 \in (1/17, \infty)$, this gives

$$\begin{aligned}\mu(H\beta - H\eta) &= \mu\left(\frac{\beta}{18}\right) \leq \frac{1}{\sqrt[4]{17}}\mu\left(\frac{17\beta}{18}\right) = \frac{1}{\sqrt[4]{17}}\mu(H\beta - \beta)\\ &\leq \frac{1}{\sqrt[4]{17}}(\mu(H\beta - \beta) + \mu(H\eta - \eta)).\end{aligned} \quad (49)$$

So, the operator H is Kannan μ-contraction. As μ satisfies the Fatou property. From Theorem 11, the operator H holds one fixed point $1/17e_0 \in (\ell_M(\lambda))_\mu$.

Example 6. Assume $H: (\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$, where $M(t) = t^2 + 2t$ and $\mu(\beta) = \sum_{y=0}^{\infty}\lambda_y M(|\beta_y|)$, for every $\beta \in \ell_M(\lambda)$ and

$$H(\beta) = \begin{cases} \frac{1}{6}(e_1 + \beta), & \beta_0 \in \left(-\infty, \frac{1}{5}\right),\\ \frac{1}{5}e_1, & \beta_0 = \frac{1}{5},\\ \frac{1}{6}e_1, & \beta_0 \in \left(\frac{1}{5}, \infty\right).\end{cases} \quad (50)$$

As for each $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (-\infty, 1/5)$, one has

$$\begin{aligned}\mu(H\beta - H\eta) &= \mu\left(\frac{1}{6}(\beta_0 - \eta_0, \beta_1 - \eta_1, \beta_2 - \eta_2, \ldots)\right)\\ &\leq \frac{2}{5}\left(\mu\left(\frac{5\beta}{6}\right) + \mu\left(\frac{5t}{6}\right)\right)\\ &\leq \frac{2}{5}(\mu(H\beta - \beta) + \mu(H\nu - \eta)).\end{aligned} \quad (51)$$

Suppose $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (1/5, \infty)$, then for any $\epsilon > 0$ we obtain

$$\mu(H\beta - H\eta) = 0 \leq \epsilon(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \quad (52)$$

Assume $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/5)$ and $\eta_0 \in (1/5, \infty)$, one can see

$$\begin{aligned}\mu(H\beta - H\eta) &= \mu\left(\frac{\beta}{6}\right) \leq \frac{1}{5}\mu\left(\frac{5\beta}{6}\right) = \frac{1}{5}\mu(H\beta - \beta)\\ &\leq \frac{1}{5}(\mu(H\beta - \beta) + \mu(H\eta - \eta)).\end{aligned} \quad (53)$$

Hence, the operator H is Kannan μ-contraction. Clearly, H is μ-sequentially continuous at $1/5e_1 \in (\ell_M(\lambda))_\mu$ and there exists $\beta \in (\ell_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/5)$ under $\{H^t\beta\} = \left\{\sum_{n=1}^{n=1} 1/6^n e_1 + 1/6^t\beta\right\}$ contains a subsequence $\{H^{t_p}\beta\} = \left\{\sum_{n=1}^{t_p} 1/6^n e_1 + 1/6^{t_p}\beta\right\}$ converging to $1/5e_1$. From Theorem 14, the operator H holds a unique fixed point $1/5e_1 \in (\ell_M(\lambda))_\mu$. Observe that H is not continuous at $1/5e_1 \in (\ell_M(\lambda))_\mu$.

If $M(t) = t^2$ and $\mu(\beta) = \sqrt{\sum_{y=0}^{\infty}\lambda_y M(|\beta_y|)}$, for every $\beta \in \ell_M(\lambda)$.

Since for all $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (-\infty, 1/5)$, one has

$$\begin{aligned}\mu(H\beta - H\eta) &= \mu\left(\frac{1}{6}(\beta_0 - \eta_0, \beta_1 - \eta_1, \beta_2 - \eta_2, \ldots)\right)\\ &\leq \frac{1}{5}\left(\mu\left(\frac{5\beta}{6}\right) + \mu\left(\frac{5\eta}{6}\right)\right)\\ &\leq \frac{1}{5}(\mu(H\beta - \beta) + \mu(H\eta - \eta)).\end{aligned} \quad (54)$$

If $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0, \eta_0 \in (1/5, \infty)$, then for all $\epsilon > 0$ one has

$$\mu(H\beta - H\eta) = 0 \le \epsilon(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \quad (55)$$

Assume $\beta, \eta \in (\ell_M(\lambda))_\mu$ with $\beta_0 \in (-\infty, 1/5)$ and $\eta_0 \in (1/5, \infty)$, we get

$$\mu(H\beta - H\eta) = \mu\left(\frac{\beta}{6}\right) \le \frac{1}{5}\mu\left(\frac{5\beta}{6}\right) = \frac{1}{5}\mu(H\beta - \beta) \le \frac{1}{5}(\mu(H\beta - \beta) + \mu(H\eta - \eta)). \quad (56)$$

So, the operator H is Kannan μ-contraction. As μ satisfies the Fatou property. From Theorem 11, the operator H contains one fixed point $1/5e_1 \in (\ell_M(\lambda))_\mu$.

5. Kannan Nonexpansive Operator

We have presented in this section the uniform convexity of the space $(\ell_M(\lambda))_\mu$, where

$$\ell_M(\lambda) = \left\{u \in \mathbf{R}^{\mathscr{I}^+} : \varrho(\omega u) < \infty, \text{for some } \omega > 0\right\}, \quad (57)$$

and $\varrho(u) = \sum_{y=0}^{\infty} \lambda_y M(|u_y|)$, under the Luxemburg norm

$$\mu(u) = \inf\left\{\omega > 0 \colon \varrho\left(\frac{u}{\omega}\right) \le 1\right\}. \quad (58)$$

Definition 12.

(1) *The continuous function M is called strictly convex (SC), if*

$$M\left(\frac{v+t}{2}\right) < \frac{M(v) + M(t)}{2}, \quad (59)$$

for all $v, t \in [0, \infty)$ and $v \ne t$.

(2) [24] The following statements are equivalent:

(i) M is a uniformly convex function on $[0, \infty)$.

(ii) For any $\epsilon > 0$ and $u_0 > 0$, there exists a number $\delta \in (0, 1)$ such that for all u, v, and $|u - v| \ge \epsilon \max\{|u|, |v|\} \ge \epsilon u_0$ imply

$$M\left(\frac{u+v}{2}\right) \le \frac{1-\delta}{2}(M(u) + M(v)), \quad (60)$$

if $u \ge u_0$.

(iii) For any $u_0 > 0$ and $a \in (0, 1)$, there exists a number $\delta \in (0, 1)$ such that

if $u \ge u_0$.

(3) [25] A normed space (X, μ) is said to be strictly convex if for any $u, v \in X$ and $b > 0$ satisfying $\mu(u) \le b$, $\mu(v) \le b$, and $\mu(u - v) > 0$ imply $\mu(u + v/2) < b$.

(4) [26] A normed space (X, μ) is said to be uniformly convex if for any $b > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $u, v \in X$ satisfying $\mu(u) \le b$, $\mu(v) \le b$ and $\mu(u - v) \ge \epsilon$ imply $\mu(u + v/2) \le b - \delta$.

Theorem 17. *If* $\lim_{n\longrightarrow\infty} \mu(x_n) = b$, $\lim_{n\longrightarrow\infty} \mu(y_n) = b$ *and* $\lim_{n\longrightarrow\infty} \mu(x_n + y_n/2) = b$ *imply* $\lim_{n\longrightarrow\infty} \mu(x_n - y_n) = 0$, *for all* $\{x_n\}, \{y_n\} \subset \ell_M(\lambda)$ *and* $b > 0$, *then* $\ell_M(\lambda)$ *is uniformly convex, where M is a convex Orlicz function satisfying Δ_2-condition.*

Proof. Let the conditions be satisfied and $\ell_M(\lambda)$ is not uniformly convex, then there exists $\epsilon_0 > 0$ and $\{x_n\}, \{y_n\} \subset \ell_M(\lambda)$ such that $\mu(x_n) \le b$, $\mu(y_n) \le b$, $\mu(x_n - y_n) \ge \epsilon_0$ we get $\mu(x_n + y_n/2) > b - 1/n$, for some $b > 0$. To prove that $\lim_{n\longrightarrow\infty} \mu(x_n) = b$, let $\lim_{n\longrightarrow\infty} \mu(x_n) = b_1 < b$ and $\lim_{n\longrightarrow\infty} \mu(y_n) = b$. Since M is satisfying Δ_2-condition, we have $\lim_{n\longrightarrow\infty} \varrho(x_n/b_1) = 1$ and $\lim_{n\longrightarrow\infty} \varrho(x_n/b) = 1$. Hence,

$$\lim_{n\longrightarrow\infty} \varrho\left(\frac{x_n + y_n}{2b}\right) \le \frac{b_1}{2b} \lim_{n\longrightarrow\infty} \varrho\left(\frac{x_n}{b_1}\right) + \frac{1}{2} \lim_{n\longrightarrow\infty} \varrho\left(\frac{y_n}{b}\right)\Bigg) < 1. \quad (61)$$

This is equivalent to $\overline{\lim}_{n\longrightarrow\infty} \mu(x_n + y_n/2) < b$. This contradicts $\lim_{n\longrightarrow\infty} \mu(x_n + y_n/2) > b$, so $\lim_{n\longrightarrow\infty} \varrho(x_n) = b$. Similarly, we can prove that $\lim_{n\longrightarrow\infty} \varrho(y_n) b$. Also since

$$1 < \underline{\lim}_{n\longrightarrow\infty} \varrho\left(\frac{x_n + y_n}{2b}\right) \le \lim_{n\longrightarrow\infty} \varrho\left(\frac{x_n + y_n}{2b}\right) \le \overline{\lim}_{n\longrightarrow\infty} \varrho\left(\frac{x_n + y_n}{2b}\right) \le \frac{1}{2}\left(\overline{\lim}_{n\longrightarrow\infty} \varrho\left(\frac{x_n}{b}\right) + \overline{\lim}_{n\longrightarrow\infty} \varrho\left(\frac{y_n}{b}\right)\right) = 1. \quad (62)$$

Then $\lim_{n\longrightarrow\infty} \varrho(x_n + y_n/2b) = 1$. This implies $\lim_{n\longrightarrow\infty} \mu(x_n + y_n/2) = b$. But $\lim_{n\longrightarrow\infty} \mu(x_n - y_n) \ge \epsilon_0 > 0$, this gives a contradiction. □

Theorem 18. *The space $\ell_M(\lambda)$ is uniformly convex, if M is a uniformly convex Orlicz function satisfying Δ_2-condition.*

Proof. Assume the settings are satisfied, $\lim_{n\longrightarrow\infty} \mu(x_n) = b$, $\lim_{n\longrightarrow\infty} \mu(y_n) = b$, and $\lim_{n\longrightarrow\infty} \mu(x_n + y_n/2) = b$, we will prove that $\lim_{n\longrightarrow\infty} \mu(x_n - y_n) = 0$. For any $\epsilon \in (0, 1/2)$, let us choose $u_0 > 0$ such that $M(2u_0) < \epsilon$. Since M is uniformly convex, then there exists $\delta \in (0, 1)$ such that $|u - v| \ge \epsilon \max\{|u|, |v|\} \ge \epsilon u_0$ imply

$$M\left(\frac{u+v}{2}\right) \leq \frac{1-\delta}{2}\left(M(u)+M(v)\right). \quad (63)$$

For each $n \in \mathscr{Z}^+$, put

$$G_n = \left\{ i \in \mathscr{Z}^+ : \left|\frac{x_n(i)}{b}\right|, \left|\frac{y_n(i)}{b}\right| < u_0 \right\},$$

$$E_n = \left\{ i \in \mathscr{Z}^+ : \left|\frac{x_n(i)-y_n(i)}{b}\right| < \epsilon \max\left\{\left|\frac{x_n(i)}{b}\right|, \left|\frac{y_n(i)}{b}\right|\right\} < \epsilon u_0 \right\},$$

$$F_n = \left\{ i \in \mathscr{Z}^+ : \left|\frac{x_n(i)-y_n(i)}{b}\right| \geq \epsilon \max\left\{\left|\frac{x_n(i)}{b}\right|, \left|\frac{y_n(i)}{b}\right|\right\} \geq \epsilon u_0 \right\}. \quad (64)$$

Then we deduce

$$\sum_{i\in G_n} M\left(\left|\frac{x_n(i)-y_n(i)}{b}\right|\right) \leq M(2u_0), \quad (65)$$

and thus

$$\sum_{i\in E_n} M\left(\left|\frac{x_n(i)-y_n(i)}{b}\right|\right) \leq 2\epsilon \sum_{i\in\mathscr{Z}^+} M\left(\frac{|x_n(i)|+|y_n(i)|}{2b}\right) \leq \epsilon\left(\sum_{i\in\mathscr{Z}^+} M\left(\left|\frac{x_n(i)}{b}\right|\right) + \sum_{i\in\mathscr{Z}^+} M\left(\left|\frac{y_n(i)}{b}\right|\right)\right) \leq 2\epsilon. \quad (66)$$

Hence, we get

$$\begin{aligned} 0 \longleftarrow & \frac{\mu(x_n)+\mu(y_n)}{2} - \mu\left(\frac{x_n+y_n}{2}\right) \\ &= \frac{\sum_{i\in\mathscr{Z}^+} M(|x_n(i)/b|) + \sum_{i\in\mathscr{Z}^+} M(y_n(i)/b)}{2} - \sum_{i\in\mathscr{Z}^+} M\left(\frac{|x_n(i)+y_n(i)|}{2b}\right) \\ &\geq \frac{\sum_{i\in F_n} M(|x_n(i)/b|) + \sum_{i\in F_n} M(y_n(i)/b)}{2} - \sum_{i\in F_n} M\left(\frac{|x_n(i)+y_n(i)|}{2b}\right) \\ &\geq \frac{\sum_{i\in F_n} M(|x_n(i)/b|) + \sum_{i\in F_n} M(y_n(i)/b)}{2} - \frac{1-\delta}{2}\left[\sum_{i\in F_n} M(|x_n(i)/b|) + \sum_{i\in F_n} M(y_n(i)/b)\right] \\ &= \frac{\delta}{2}\left[\sum_{i\in F_n} M(|x_n(i)/b|) + \sum_{i\in F_n} M(y_n(i)/b)\right]. \end{aligned} \quad (67)$$

Since u_0 and ϵ are arbitrary, then $\lim_{n\longrightarrow\infty} \varrho(x_n - y_n/2b) = 0$. As M verifies Δ_2-condition. Therefore, $\lim_{n\longrightarrow\infty} \mu(x_n - y_n) = 0$. From Theorem 17, the proof follows.

Here, we discuss the property (R) and the μ-normal structure property of the space $(\ell_M(\lambda))_\mu$. □

Definition 13. The space $(Y)_\mu$ holds the property (R), if for all decreasing sequence $\{\Phi_x\}_{x\in\mathscr{Z}^+}$ of μ-closed and μ-convex nonempty subsets of $(Y)_\mu$ so that $\sup_{x\in\mathscr{Z}^+} d_\mu(\beta, \Phi_x) < \infty$, for some $\beta \in (Y)_\mu$; hence, we have $\cap_{x\in\mathscr{Z}^+}\Phi_x \neq \varnothing$.

Definition 14. The space $(Y)_\mu$ holds the μ-normal structure property if for all nonempty μ-bounded, μ-convex, and μ-closed subset Φ of $(Y)_\mu$ not decreased to one point, there exists $\beta \in \Phi$ with

$$\sup_{\eta\in\Phi} \mu(\beta-\eta) < v_\mu(\Phi) \coloneqq \sup\{\mu(\beta-\eta): \beta, \eta \in \Phi\} < \infty. \quad (68)$$

Theorem 19. *If M is a uniformly convex Orlicz function satisfying Δ_2-condition, then*

(1) Assume Φ is a nonempty μ-closed and μ-convex subset of $(\ell_M(\lambda))_\mu$. For $\beta \in (\ell_M(\lambda))_\mu$ with

$$d_\mu(\beta, \Phi) = \inf\{\mu(\beta-\eta): \eta \in \Phi\} < \infty. \quad (69)$$

Therefore, we have one $\phi \in \Phi$ with $d_\mu(\beta,\Phi) = \mu(\beta-\phi)$.

(2) $(\ell_M(\lambda))_\mu$ satisfies the property (R).

Proof. For (1), assume $\beta \notin \Phi$ as Φ is μ-closed. So, one has $D \coloneqq d_\mu(\beta,\Phi) > 0$. Therefore, there is $\eta_t \in \Phi$ so that $\lim_{t\longrightarrow\infty} \mu(\beta-\eta_t) = D$. To prove that $\{\eta_t\}$ is a μ-Cauchy. For any two subsequences $\{\eta_{t_a}\}$ and $\{\eta_{t_b}\} \subset \{\eta_t\}$, we have $\mu(\eta_{t_a} + \eta_{t_b}/2 - \beta) \geq D$, as $\lim_{a\longrightarrow\infty} \mu(\beta-\eta_{t_a}) = D$ and $\lim_{b\longrightarrow\infty} \mu(\beta-\eta_{t_b}) = D$. Moreover,

$$\mu\left(\frac{\eta_{t_a}+\eta_{t_b}}{2} - \beta\right) = \mu\left(\frac{\eta_{t_a}-\beta}{2} + \frac{\eta_{t_b}-\beta}{2}\right) < \frac{1}{2}(D+D) = D. \quad (70)$$

Therefore, we have $\lim_{a,b\longrightarrow\infty}\mu(\eta_{t_a}+\eta_{t_b}/2-\beta)=D$. Since the space $(\ell_M(\lambda))_\mu$ is uniformly convex, we get

$$\lim_{a,b\longrightarrow\infty}\mu\left(\eta_{t_a}-\beta-\left(\eta_{t_b}-\beta\right)\right)=\lim_{a,b\longrightarrow\infty}\mu\left(\eta_{t_a}-\eta_{t_b}\right)=0. \tag{71}$$

Thus, $\{\eta_t\}$ is a μ-Cauchy in Φ. Since Φ is closed and the space $(\ell_M(\lambda))_\mu$ is complete, then there exists $\phi\in\Phi$ with $\mu(\beta-\phi)=d_\mu(\beta,\Phi)$. Since the space $(\ell_M(\lambda))_\mu$ is uniformly convex, then it is (SC), which implies the uniqueness of ϕ. To show (2), for some $t_0\in\mathscr{X}^+$, suppose $\beta\notin\Phi_{t_0}$. Since $(d_\mu(\beta,\Phi_t))_{t\in\mathscr{X}^+}\in\ell_\infty$ is increasing. Set $\lim_{t\longrightarrow\infty}d_\mu(\beta,\Phi_t)=D$, when $D>0$. Otherwise, $\beta\in\Phi_t$, for each $t\in\mathscr{X}^+$. From (1), we have a unique $\eta_t\in\Phi_t$ with $d_\mu(\beta,\Phi_t)=\mu(\beta-\eta_t)$, for all $t\in\mathscr{X}^+$. A consistent proof will show that $\{\eta_t/2\}\mu$-converges to some $\eta\in(\ell_M(\lambda))_\mu$. Since $\{\Phi_t\}$ are μ-convex, decreasing and μ-closed, we get $2\eta\in\cap_{t\in\mathscr{X}^+}\Phi_t$. □

Theorem 20. *If M is a uniformly convex Orlicz function satisfying Δ_2-condition, then $(\ell_M(\lambda))_\mu$ has the μ-normal structure property.*

Proof. Let the conditions are satisfied. Theorem 18 gives that $(\ell_M(\lambda))_\mu$ is uniformly convex. Assume Φ is a μ-bounded, μ-convex, and μ-closed subset of $(\ell_M(\lambda))_\mu$ not decreased to one point. Hence, $v_\mu(\Phi)>0$. Set $D=v_\mu(\Phi)$. Let $\beta,\eta\in\Phi$ with $\beta\neq\eta$. Hence, $\mu(\beta-\eta/2)>0$. For every $\phi\in\Phi$, one has $\mu(\beta-\phi)\le D$ and $\mu(\eta-\phi)\le D$. As Φ is μ-convex, then $\beta+\eta/2\in\Phi$. Hence,

$$\mu\left(\frac{\beta+\eta}{2}-\phi\right)=\mu\left(\frac{(\beta-\phi)+(\eta-\phi)}{2}\right)<D, \tag{72}$$

for every $\phi\in\Phi$. So

$$\sup_{\phi\in\Phi}\mu\left(\frac{\beta+\eta}{2}-\phi\right)<D=v_\mu(\Phi). \tag{73}$$

□

Lemma 3. *Let the space $(\ell_M(\lambda))_\mu$ verify the (R) property and the μ-quasi-normal property. Assume Φ is a nonempty μ-bounded, μ-convex, and μ-closed subset of $(\ell_M(\lambda))_\mu$. Suppose $H:\Phi\longrightarrow\Phi$ is a Kannan μ-nonexpansive mapping. For $x>0$. If $W_x=\{\beta\in\Phi:\ \mu(\beta-H(\beta))\le x\}\neq\varnothing$. Set*

$$\Phi_x=\cap\left\{\mathscr{B}_\mu(t,v):\ H(W_x)\subset\mathscr{B}_\mu(t,v)\right\}\cap\Phi. \tag{74}$$

Then Φ_x is a nonempty, μ-convex, μ-closed subset of Φ with $H(\Phi_x)\subset\Phi_x\subset W_x$ and $v_\mu(\Phi_x)\le x$.

Proof. As $H(W_x)\subset\Phi_x$, this gives $\Phi_x\neq\varnothing$. Since the μ-balls are μ-convex, and μ-closed, then Φ_x is a μ-closed and μ-convex subset of Φ. To prove that $\Phi_x\subset W_x$. Assume $\beta\in\Phi_x$. If $\mu(\beta-H(\beta))=0$, we have $\beta\in W_x$. Otherwise, suppose $\mu(\beta-H(\beta))>0$. Set

$$t=\sup\{\mu(H(\zeta)-H(\beta)):\ \zeta\in W_x\}. \tag{75}$$

From the definition of t, then $H(W_x)\subset\mathscr{B}_\mu(H(\beta),t)$. Hence, $\Phi_x\subset\mathscr{B}_\mu(H(\beta),t)$, which implies $\mu(\beta-H(\beta))\le t$. Assume $d>0$. Hence, there is $\zeta\in W_x$ so that $t-d\le\mu(H(\zeta)-H(\beta))$. Then

$$\begin{aligned}\mu(\beta-H(\beta))-d\le t-d&\le\mu(H(\zeta)-H(\beta))\\&\le\frac{1}{2}(\mu(\beta-H(\beta))+\mu(\zeta-H(\zeta)))\\&\le\frac{1}{2}(\mu(\beta-H(\beta))+x).\end{aligned} \tag{76}$$

Since d is arbitrarily positive, we have $\mu(\beta-H(\beta))\le x$, then we have $\beta\in W_x$. For $H(W_x)\subset\Phi_x$, we get $H(\Phi_x)\subset H(W_x)\subset\Phi_x$, this indicates Φ_x is H-invariant. Consequent to prove that $v_\mu(\Phi_x)\le x$. As

$$\mu(H(\beta)-H(\eta))\le\frac{1}{2}(\mu(\beta-H(\beta))+\mu(\eta-H(\eta))), \tag{77}$$

For every $\beta,\eta\in W_x$. Let $\beta\in W_x$. So $H(W_x)\subset\mathscr{B}_\mu(H(\beta),x)$. From the definition of Φ_x, one has $\Phi_x\subset\mathscr{B}_\mu(H(\beta),x)$. Hence, $H(\beta)\in\cap_{\eta\in\Phi_x}\mathscr{B}_\mu(\eta,x)$. Therefore, we have $\mu(\eta-\zeta)\le x$, for every $\eta,\zeta\in\Phi_x$, which implies $v_\mu(\Phi_x)\le x$. This finishes the proof.

In this part, we give enough settings on $(\ell_M(\lambda))_\mu$ so that the Kannan μ– nonexpansive mapping defined on it contains a fixed point. □

Theorem 21. *Let $(\ell_M(\lambda))_\mu$ hold the μ-quasinormal property and the (R) property. Assume Φ is a nonempty, μ-convex, μ-closed, and μ-bounded subset of $(\ell_M(\lambda))_\mu$. If $H:\Phi\longrightarrow\Phi$ is a Kannan μ-nonexpansive mapping, then H has a fixed point.*

Proof. Let $x_t=x_0+1/t$, for all $t\ge1$, where $x_0=\inf\{\mu(\beta-H(\beta)):\ \beta\in\Phi\}$. We have for each $t\ge1$ that $W_{x_t}=\{\beta\in\Phi:\ \mu(\beta-H(\beta))\le x_t\}\neq\varnothing$. Suppose Φ_{x_t} explained as in Lemma 3. Clearly, $\{\Phi_{x_t}\}$ is a decreasing sequence of nonempty μ-bounded, μ-closed, and μ-convex subsets of Φ. The property (R) gives that $\Phi_\infty=\cap_{t\ge1}\Phi_{x_t}\neq\varnothing$. Let $\beta\in\Phi_\infty$, we have $\mu(\beta-H(\beta))\le x_t$, for every $t\ge1$. If $t\longrightarrow\infty$, one has $\mu(\beta-H(\beta))\le x_0$, which implies $\mu(\beta-H(\beta))=x_0$. Hence, $W_{x_0}\neq\varnothing$. So $x_0=0$. Otherwise, $x_0>0$ which investigates that H has no fixed point. Assume Φ_{x_0} as defined in Lemma 3. Since H has no fixed point and Φ_{x_0} is H-invariant, hence Φ_{x_0} holds more than one point, which gives, $v_\mu(\Phi_{x_0})>0$. By the μ-quasinormal property, one has $\beta\in\Phi_{x_0}$ with

$$\mu(\beta-\eta)<v_\mu\left(\Phi_{x_0}\right)\le x_0, \tag{78}$$

for every $\eta\in\Phi_{x_0}$. By Lemma 3, we have $\Phi_{x_0}\subset W_{x_0}$. By definition of Φ_{x_0}, then $H(\beta)\in W_{x_0}\subset\Phi_{x_0}$. Obviously, one has

$$\mu(\beta-H(\beta))<v_\mu\left(\Phi_{x_0}\right)\le x_0, \tag{79}$$

which contradicts the definition of x_0. So $x_0=0$ this implies that any point in W_{x_0} is a fixed point of H, i.e., H has a fixed point in Φ.

According to Theorem 19, Theorem 20, and Theorem 21, we obtain the next corollary: □

Corollary 2. *If M is a uniformly convex Orlicz function satisfying Δ_2-condition. Assume Φ is a nonempty, μ-convex, μ-closed, and μ-bounded subset of $(\ell_M(\lambda))_\mu$. Suppose $H\colon \Phi \longrightarrow \Phi$ is a Kannan μ-nonexpansive operator. Then H holds a fixed point.*

Example 7. Let $H\colon \Phi \longrightarrow \Phi$ with $H(\beta) = \begin{cases} \beta/4, & \mu(\beta) \in [0,1), \\ \beta/5, & \mu(\beta) \in [1,\infty), \end{cases}$

where $\Phi = \{\beta \in (\ell_M(\lambda))_\mu\colon \beta_0 = \beta_1 = 0\}$, where $\varrho(\beta) = \sum_{x=0}^{\infty} \lambda_x |\beta_x|^2$, for every $\beta \in (\ell_M(\lambda))_\mu$. As Example 5, the operator H is Kannan μ-contraction mapping. So it is Kannan μ-nonexpansive operator. Clearly, Φ is a nonempty, μ-convex, μ-closed and μ-bounded subset of $(\ell_M(\lambda))_\mu$. By Corollary 2, the operator H has a fixed point in Φ.

6. Kannan $\Upsilon-$ Contraction Mapping on $S_{(\ell_M(\lambda))_\mu}$

For any two Banach spaces $\mathfrak{X}$ and $\mathfrak{Y}$, we examine in this section the existence of a fixed point of Kannan $\Upsilon-$ contraction mapping on $S_{(\ell_M(\lambda))_\mu}$, where $\Upsilon(Q) = M^{-1}(\sum_{y=0}^{\infty} \lambda_y M(|s_y(Q)|))$, for all $Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X},\mathfrak{Y})$.

Theorem 22. *If M is a convex Orlicz function satisfying Δ_2-condition and $\ln(M(e^x))$ is convex, then $(S_{(\ell_M(\lambda))_\mu}, \Upsilon)$ is a prequasi Banach operator ideal, where $\Upsilon(Q) = \mu((s_y(Q))_{y=0}^{\infty})$.*

Proof. As Theorem 5, the space $(\ell_M(\lambda))_\mu$ is a premodular (sss). Therefore, from Theorem 3, one has $\Upsilon(Q) = \mu((s_y(Q))_{y=0}^{\infty})$ is a prequasi norm on $S_{(\ell_M(\lambda))_\mu}$. Suppose $Q_r \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X},\mathfrak{Y})$ is a Cauchy sequence. As $\mathscr{B}(\mathfrak{X},\mathfrak{Y}) \supseteq S_{(\ell_M(\lambda))_\mu}(\mathfrak{X},\mathfrak{Y})$, one obtains

$$\Upsilon(Q_r - Q_t) = \mu\Big(\big(s_y(Q_r - Q_t)\big)_{y=0}^{\infty}\Big) \geq \mu(s_0(Q_r - Q_t), 0, 0, 0, \ldots,) = M^{-1}\big(\lambda_0 \|Q_r - Q_t\|\big). \tag{80}$$

Hence $(Q_r)_{r\in\mathscr{Z}^+}$ is a Cauchy sequence in $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$. Since $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$ is a Banach space, so there is $Q \in \mathscr{B}(\mathfrak{X},\mathfrak{Y})$ with $\lim_{r\longrightarrow\infty}\|Q_r - Q\| = 0$. Since $(s_y(Q_r))_{y=0}^{\infty} \in (\ell_M(\lambda))_\mu$, for every $r \in \mathscr{Z}^+$. We have

$$\begin{aligned}\Upsilon(Q) &= \mu\Big(\big(s_y(Q)\big)_{y=0}^{\infty}\Big) = \mu\Big(\big(s_y(Q - Q_r + Q_r)\big)_{y=0}^{\infty}\Big) \leq \mu\Big(\big(s_{[y/2]}(Q - Q_r)\big)_{y=0}^{\infty}\Big) + \mu\Big(\big(s_{[y/2]}(Q_r)_{y=0}^{\infty}\big)\Big) \\ &\leq \mu\Big(\big(\|Q_r - Q\|\big)_{y=0}^{\infty}\Big) + 2\mu\Big(\big(s_y(Q_r)_{y=0}^{\infty}\big)\Big) < \epsilon.\end{aligned} \tag{81}$$

Therefore, $(s_y(Q))_{y=0}^{\infty} \in (\ell_M(\lambda))_\mu$, this implies $Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X},\mathfrak{Y})$. □

Theorem 23. *If M is a convex Orlicz function satisfying Δ_2-condition and $\ln(M(e^x))$ is convex, then $(S_{(\ell_M(\lambda))_\mu}, \Upsilon)$ is a prequasi closed operator ideal, where $\Upsilon(Q) = \mu((s_y(Q))_{y=0}^{\infty})$.*

Proof. As Theorem 5, the space $(\ell_M(\lambda))_\mu$ is a premodular (sss). Therefore, from Theorem 3, one has $\Upsilon(Q) = \mu((s_y(Q))_{y=0}^{\infty})$ is a prequasi norm on $S_{(\ell_M(\lambda))_\mu}$. Assume $Q_r \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X},\mathfrak{Y})$, for every $r \in \mathscr{Z}^+$ and $\lim_{r\longrightarrow\infty}\Upsilon(Q_r - Q) = 0$. Hence, there is $\varsigma > 0$ and since $\mathscr{B}(\mathfrak{X},\mathfrak{Y}) \supseteq S_{(\ell_M(\lambda))_\mu}(\mathfrak{X},\mathfrak{Y})$, we get

$$\begin{aligned}\Upsilon(Q_r - Q) &= \mu\Big(\big(s_y(Q_r - Q)\big)_{y=0}^{\infty}\Big) \geq \mu(s_0(Q_r - Q), 0, 0, 0, \ldots) \\ &= M^{-1}\big(\lambda_0 \|Q_r - Q\|\big).\end{aligned} \tag{82}$$

Hence $(Q_r)_{r\in\mathscr{Z}^+}$ is convergent in $\mathscr{B}(\mathfrak{X},\mathfrak{Y})$. i.e., $\lim_{r\longrightarrow\infty}\|Q_r - Q\| = 0$ and while $(s_y(Q_r))_{y=0}^{\infty} \in (\ell_M(\lambda))_\mu$, for all $r \in \mathscr{Z}^+$ and $(\ell_M(\lambda))_\mu$ is a premodular (sss). Hence, we have

$$\begin{aligned}\Upsilon(Q) &= \mu\Big(\big(s_y(Q)\big)_{y=0}^{\infty}\Big) = \mu\Big(\big(s_y(Q - Q_r + Q_r)\big)_{y=0}^{\infty}\Big) \\ &\leq \mu\Big(\big(s_{[y/2]}(Q - Q_r)\big)_{y=0}^{\infty}\Big) + \mu\Big(\big(s_{[y/2]}(Q_r)_{y=0}^{\infty}\big)\Big) \\ &\leq \mu\Big(\big(\|Q_r - Q\|\big)_{y=0}^{\infty}\Big) + 2\mu\Big(\big(s_y(Q_r)_{y=0}^{\infty}\big)\Big) < \epsilon,\end{aligned} \tag{83}$$

we have $(s_y(Q))_{y=0}^{\infty} \in (\ell_M(\lambda))_\mu$, then $Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X},\mathfrak{Y})$. □

Definition 15. A prequasi norm Υ on the ideal $S_{(\ell_M(\lambda))_\mu}$, where $\Upsilon(Q) = \mu((s_y(Q))_{y=0}^{\infty})$, satisfies the Fatou property if for any sequence $\{Q_y\}_{y\in\mathscr{Z}^+} \subseteq S_{(\ell_M(\lambda))_\Upsilon}(\mathfrak{X},\mathfrak{Y})$ with $\lim_{y\longrightarrow\infty}\Upsilon(Q_y - Q) = 0$ and any $V \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X},\mathfrak{Y})$, then $\Upsilon(V - Q) \leq \sup_y \inf_{j\geq y} \mu(V - Q_j)$.

Theorem 25. *The prequasinorm $\Upsilon(Q) = M^{-1}(\sum_{y=0}^{\infty} \lambda_y M(s_y(Q)))$, for all $Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X},\mathfrak{Y})$ does not satisfy the Fatou property, if M is a convex Orlicz function satisfying Δ_2-condition and $\ln(M(e^x))$ is convex.*

Proof. Assume the settings are satisfied and $\{Q_t\}_{t\in\mathscr{Z}^+} \subseteq S_{(\ell_M(\lambda))_\mu}(\mathfrak{X},\mathfrak{Y})$ with $\lim_{t\longrightarrow\infty}\Upsilon(Q_t - Q) = 0$. Since the space $S_{(\ell_M(\lambda))_\mu}$ is a prequasi closed ideal, then,

$Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$. Hence, for any $V \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$, we have

$$\begin{aligned}\Upsilon(V-Q) &= M^{-1}\left(\sum_{y=0}^{\infty}\lambda_y M\left(s_y(V-Q)\right)\right) \le M^{-1}\left(\sum_{y=0}^{\infty}\lambda_y M\left(s_{[y/2]}\left(V-Q_j\right)\right)\right) + M^{-1}\left(\sum_{y=0}^{\infty}\lambda_y M\left(s_{[[y/2]]}\left(Q_j-Q\right)\right)\right) \\ &\le 2\sup_t \inf_{j\ge t} M^{-1}\left(\sum_{y=0}^{\infty}\lambda_y M\left(s_y\left(V-Q_j\right)\right)\right).\end{aligned} \quad (84)$$

Hence, Υ does not satisfy the Fatou property. □

Definition 16. An operator $P: S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y}) \longrightarrow S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ is called a Kannan Υ-Lipschitzian, if there exists $\nu \ge 0$, so that for every $Q, T \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$, we have

$$\Upsilon(PQ-PT) \le \nu(\Upsilon(PQ-Q) + \Upsilon(PT-T)). \quad (85)$$

(1) If $\nu \in [0, 1/2)$, the operator P is said to be Kannan Υ-contraction.

(2) If $\nu = 1/2$, the operator P is said to be Kannan Υ-nonexpansive.

Definition 17. An operator $P: S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y}) \longrightarrow S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ is said to be Υ-sequentially continuous at V, if and only if, when $\lim_{t\longrightarrow\infty}\Upsilon(Q_t - V) = 0$, then $\lim_{t\longrightarrow\infty}\Upsilon(PQ_t - PV) = 0$.

Theorem 26. *If M is a convex Orlicz function satisfying Δ_2-condition and $\ln(M(e^x))$ is conve, and $P: S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y}) \longrightarrow S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$. The point $T \in S_{(\ell_M(\lambda))_\Upsilon}(\mathfrak{X}, \mathfrak{Y})$ is the unique fixed point of P, when the following conditions are satisfied:*

(i) P is Kannan Υ-contraction mapping,

(ii) P is Υ-sequentially continuous at a point $T \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$,

(iii) There exists $V \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ so that the sequence of iterates $\{P^t V\}$ has a subsequence $\{P^{t_j} V\}$ converging to T.

Proof. Let the conditions be verified. If T is not a fixed point of P, then $PT \ne T$. From the conditions (ii) and (iii), we have

$$\begin{aligned}\lim_{t_j\longrightarrow\infty}\Upsilon\left(P^{t_j}V - T\right) &= 0,\\ \lim_{t_j\longrightarrow\infty}\Upsilon\left(P^{t_j+1}V - PT\right) &= 0.\end{aligned} \quad (86)$$

Since P is Kannan Υ-contraction mapping, one can see

$$\begin{aligned}0 < \Upsilon(PT-T) &= \Upsilon\left(\left(PT - P^{t_j+1}V\right) + \left(P^{t_j}V - T\right) + \left(P^{t_j+1}V - P^{t_j}V\right)\right)\\ &\le 2\Upsilon\left(P^{t_j+1}V - PT\right) + 4\Upsilon\left(P^{t_j}V - T\right) + 4\nu\left(\frac{\nu}{1-\nu}\right)^{t_j-1}\Upsilon(PV-V).\end{aligned} \quad (87)$$

Since $t_j \longrightarrow \infty$, we have a contradiction. Hence, T is a fixed point of P. To prove the uniqueness of the fixed point T. Let we have two different fixed points $T, U \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ of P. Therefore, one has

$$\begin{aligned}\Upsilon(T-U) &\le \Upsilon(PT-PU)\\ &\le \nu(\Upsilon(PT-T) + \Upsilon(PU-U)) = 0.\end{aligned} \quad (88)$$

Hence, $T = U$. □

Example 8. Assume $P: S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y}) \longrightarrow S_{(\ell_M(\lambda))_\Upsilon}(\mathfrak{X}, \mathfrak{Y})$, where $\Upsilon(Q) = \sum_{y=0}^{\infty}\lambda_y\sqrt{s_y(Q)}$, for every $Q \in S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y})$ and

$$P(Q) = \begin{cases}\dfrac{Q}{26}, & \Upsilon(Q) \in [0,1),\\[2ex] \dfrac{Q}{37}, & \Upsilon(Q) \in [1,\infty).\end{cases} \quad (89)$$

As for every $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $\Upsilon(Q_1), \Upsilon(Q_2) \in [0,1)$, one has

$$\begin{aligned}\Upsilon(PQ_1 - PQ_2) &= \Upsilon\left(\frac{Q_1}{26} - \frac{Q_2}{26}\right) \le \frac{2}{5}\left(\Upsilon\left(\frac{25Q_1}{26}\right) + \Upsilon\left(\frac{25Q_2}{26}\right)\right)\\ &= \frac{2}{5}(\Upsilon(PQ_1 - Q_1) + \Upsilon(PQ_2 - Q_2)).\end{aligned} \quad (90)$$

For each $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $\Upsilon(Q_1), \Upsilon(Q_2) \in [1, \infty)$, we get

$$\Upsilon(PQ_1 - PQ_2) = \Upsilon\left(\frac{Q_1}{37} - \frac{Q_2}{37}\right) \le \frac{1}{3}\left(\Upsilon\left(\frac{36Q_1}{37}\right) + \Upsilon\left(\frac{36Q_2}{37}\right)\right) = \frac{1}{3}\left(\Upsilon(PQ_1 - Q_1) + \Upsilon(PQ_2 - Q_2)\right). \tag{91}$$

For each $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $\Upsilon(Q_1) \in [0, 1)$ and $\Upsilon(Q_2) \in [1, \infty)$, one can see

$$\begin{aligned}\Upsilon(PQ_1 - PQ_2) &= \Upsilon\left(\frac{Q_1}{26} - \frac{Q_2}{37}\right) \le \frac{2}{5}\Upsilon\left(\frac{25Q_1}{26}\right) + \frac{1}{3}\Upsilon\left(\frac{36Q_2}{37}\right) \le \frac{2}{5}\left(\Upsilon\left(\frac{25Q_1}{26}\right) + \Upsilon\left(\frac{36Q_2}{37}\right)\right)\\ &= \frac{2}{5}\left(\Upsilon(PQ_1 - Q_1) + \Upsilon(PQ_2 - Q_2)\right).\end{aligned} \tag{92}$$

So, the operator Q is Kannan Υ-contraction and $P^t(Q) = \begin{cases} Q/26^t, & \Upsilon(Q) \in [0, 1), \\ Q/37^t, & \Upsilon(Q) \in [1, \infty).\end{cases}$

Clearly, P is Υ-sequentially continuous at the zero operator $\Theta \in S_{(\ell_M(\lambda))_\mu}$ and $\{P^t Q\}$ has a subsequence $\{P^{t_j} Q\}$ converging to Θ. From Theorem 27, the zero operator $\Theta \in S_{(\ell_M(\lambda))_\mu}$ is the unique fixed point of P. Suppose with $\lim_{t\longrightarrow\infty} \Upsilon(Q^{(t)} - Q^{(0)}) = 0 - b \pm \sqrt{b^2 - 4ac}/2a$, where $Q^{(0)} \in S_{(\ell_M(\lambda))_\mu}$ with $\Upsilon(Q^{(0)}) = 1$. From the continuously of the prequasi norm Υ, one has

$$\begin{aligned}\lim_{t\longrightarrow\infty} \Upsilon\left(PQ^{(t)} - PQ^{(0)}\right) &= \lim_{t\longrightarrow\infty} \Upsilon\left(\frac{Q^{(t)}}{26} - \frac{Q^{(0)}}{37}\right)\\ &= \Upsilon\left(\frac{11Q^{(0)}}{962}\right) > 0.\end{aligned} \tag{93}$$

So P is not Υ-sequentially continuous at $Q^{(0)}$. This implies the operator P is not continuous at $Q^{(0)}$.

Example 9. Suppose $P\colon S_{(\ell_M(\lambda))_\mu}(\mathfrak{X}, \mathfrak{Y}) \longrightarrow S_{(\ell_M(\lambda))_\Upsilon}(\mathfrak{X}, \mathfrak{Y})$, *where* $\Upsilon(Q) = \sqrt{\sum_{y=0}^{\infty} \lambda_y (s_y(Q))^2}$, *for every* $Q \in S_{(\ell_M(\lambda))_\Upsilon}(\mathfrak{X}, \mathfrak{Y})$ *and*

$$P(Q) = \begin{cases} \dfrac{Q}{5}, & \Upsilon(Q) \in [0, 1), \\[2ex] \dfrac{Q}{6}, & \Upsilon(Q) \in [1, \infty).\end{cases} \tag{94}$$

As for each $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $\Upsilon(Q_1), \Upsilon(Q_2) \in [0, 1)$, one can see

$$\begin{aligned}\Upsilon(PQ_1 - PQ_2) &= \Upsilon\left(\frac{Q_1}{5} - \frac{Q_2}{5}\right) \le \frac{\sqrt{2}}{4}\left(\Upsilon\left(\frac{4Q_1}{5}\right) + \Upsilon\left(\frac{4Q_2}{5}\right)\right)\\ &= \frac{\sqrt{2}}{4}\left(\Upsilon(PQ_1 - Q_1) + \Upsilon(PQ_2 - Q_2)\right).\end{aligned} \tag{95}$$

For every $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $\Upsilon(Q_1), \Upsilon(Q_2) \in [1, \infty)$, this implies

$$\begin{aligned}\Upsilon(PQ_1 - PQ_2) &= \Upsilon\left(\frac{Q_1}{6} - \frac{Q_2}{6}\right) \le \frac{\sqrt{2}}{5}\left(\Upsilon\left(\frac{5Q_1}{6}\right) + \Upsilon\left(\frac{5Q_2}{6}\right)\right)\\ &= \frac{\sqrt{2}}{5}\left(\Upsilon(PQ_1 - Q_1) + \Upsilon(PQ_2 - Q_2)\right).\end{aligned} \tag{96}$$

For each $Q_1, Q_2 \in S_{(\ell_M(\lambda))_\mu}$ with $\Upsilon(Q_1) \in [0, 1)$ and $\Upsilon(Q_2) \in [1, \infty)$, one obtains

$$\begin{aligned}\Upsilon(PQ_1 - PQ_2) &= \Upsilon\left(\frac{Q_1}{5} - \frac{Q_2}{6}\right) \le \frac{\sqrt{2}}{4}\Upsilon\left(\frac{4Q_1}{5}\right) + \frac{\sqrt{2}}{5}\Upsilon\left(\frac{5Q_2}{6}\right)\\ &\le \frac{\sqrt{2}}{4}\left(\Upsilon\left(\frac{4Q_1}{5}\right) + \Upsilon\left(\frac{5Q_2}{6}\right)\right)\\ &= \frac{\sqrt{2}}{4}\left(\Upsilon(PQ_1 - Q_1) + \Upsilon(PQ_2 - Q_2)\right).\end{aligned} \tag{97}$$

So, the operator Q is Kannan Υ-contraction and. $P^t(Q) = \begin{cases} Q/5^t, & \Upsilon(Q) \in [0, 1), \\ Q/6^t, & \Upsilon(Q) \in [1, \infty).\end{cases}$

Evidently, P is Υ-sequentially continuous at the zero operator $\Theta \in S_{(\ell_M(\lambda))_\mu}$ and $\{P^t Q\}$ has a subsequence $\{P^{t_j} Q\}$ converging to Θ. From Theorem 27, the zero operator $\Theta \in S_{(\ell_M(\lambda))_\mu}$ is the unique fixed point of P. Suppose

$\{Q^{(t)}\} \subseteq S_{(\ell_M(\lambda))_\mu}$ with $\lim_{t\longrightarrow\infty} \Upsilon(Q^{(t)} - Q^{(0)}) = 0$, where $Q^{(0)} \in S_{(\ell_M(\lambda))_\mu}$ with $\Upsilon(Q^{(0)}) = 1$. From the continuously of the prequasi norm Υ, one has

$$\begin{aligned}\lim_{t\longrightarrow\infty} \Upsilon\left(PQ^{(t)} - PQ^{(0)}\right) &= \lim_{t\longrightarrow\infty} \Upsilon\left(\frac{Q^{(t)}}{5} - \frac{Q^{(0)}}{6}\right)\\ &= \Upsilon\left(\frac{Q^{(0)}}{30}\right) > 0.\end{aligned} \tag{98}$$

So P is not Υ-sequentially continuous at $Q^{(0)}$. Hence, the operator P is not continuous at $Q^{(0)}$.

7. Applications on Summable Equations

We investigate here a solution to (101), which studied by many authors (see [27–29]), in $(\ell_M(\lambda))_\mu$.

$$\beta_x = r_x + \sum_{y=0}^{\infty} D(x,y)h\left(y,\beta_y\right). \tag{99}$$

Suppose H: $(\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$ constructed by

$$H(\beta_x)_{x\in\mathscr{Z}^+} = \left(r_x + \sum_{y=0}^{\infty} D(x,y)h\left(y,\beta_y\right)\right)_{x\in\mathscr{Z}^+}. \tag{100}$$

Theorem 27. *If M is a convex Orlicz function satisfying Δ_2-condition and $\ln(M(e^x))$ is convex, D: $\mathscr{Z}^{+2} \longrightarrow \mathbf{R}$, h: $\mathscr{Z}^+ \times \mathbf{R} \longrightarrow \mathbf{R}$, r: $\mathscr{Z}^+ \longrightarrow \mathbf{R}$, and for all $x \in \mathscr{Z}^+$, there exists $\nu \in [0, 1/2)$, with*

$$\begin{aligned}&M\left(\left|\sum_{y\in\mathscr{Z}^+} D(x,y)\left(h\left(y,\beta_y\right) - h\left(y,\eta_y\right)\right)\right|\right)\\ &\leq M(\nu)\left[M\left(\left|r_x - \beta_x + \sum_{y=0}^{\infty} D(x,y)f\left(y,\beta_y\right)\right|\right) + M\left(\left|r_x - \eta_x + \sum_{y=0}^{\infty} D(x,y)f\left(y,\eta_y\right)\right|\right)\right],\end{aligned} \tag{101}$$

then equation (101) hold a solution in $(\ell_M(\lambda))_\mu$, where $\mu(\beta) = M^{-1}(\sum_{x=0}^{\infty} M(|\beta_x|))$, for every $\beta \in \ell_M(\lambda)$.

Proof. Suppose the setups are verified. We have

$$\begin{aligned}\mu(H\beta - H\eta) = M^{-1}\left(\sum_{x\in\mathscr{Z}^+} \lambda_x M\left(\left|H\beta_x - H\eta_x\right|\right)\right) &= M^{-1}\left(\sum_{x\in\mathscr{Z}^+} \lambda_x M\left(\left|\sum_{m\in\mathscr{Z}^+} D(x,y)\left[h\left(y,\beta_y\right) - h\left(y,\eta_y\right)\right]\right|\right)\right)\\ &\leq \nu M^{-1}\left(\sum_{x\in\mathscr{Z}^+} \lambda_x M\left(\left|r_x - \beta_x + \sum_{m=0}^{\infty} D(x,y)h\left(y,\beta_y\right)\right|\right)\right)\\ &\quad + \nu M^{-1}\left(\sum_{x\in\mathscr{Z}^+} \lambda_x M\left(\left|r_x - \eta_x + \sum_{m=0}^{\infty} D(x,y)h\left(y,\eta_y\right)\right|\right)\right)\\ &= \nu(\mu(H\beta - \beta) + \mu(H\eta - \eta)).\end{aligned} \tag{102}$$

In view of Theorem 11, there exists a unique solution of equation (101) in $(\ell_M(\lambda))_\mu$. □

Example 10. For the space $(\ell_M(\lambda))_\mu$, where $\mu(\beta) = \sqrt[4]{\sum_{x\in\mathscr{Z}^+}|\beta_x|^4}$, for all $\beta \in \ell_M(\lambda)$. Assume the summable equations are defined as

$$\beta_x = e^{-(3x+6)} + \sum_{y=0}^{\infty} (-1)^x \left(\frac{\beta_x}{x^2 + y^2 + 1}\right)^{\nu} \sin y, \tag{103}$$

where $\nu > 2$ and let H: $(\ell_M(\lambda))_\mu \longrightarrow (\ell_M(\lambda))_\mu$ is defined by

$$H(\beta_x)_{x\in\mathscr{Z}^+} = \left(e^{-(3x+6)} + \sum_{y=0}^{\infty} (-1)^x \left(\frac{\beta_x}{x^2 + y^2 + 1}\right)^{\nu} \sin y\right)_{x\in\mathscr{Z}^+}. \tag{104}$$

We have

$$\left|\sum_{y=0}^{\infty}(-1)^x\left(\frac{\beta_x}{x^2+y^2+1}\right)^{\nu}(\sin y-\sin y)\right|^4 \leq \frac{1}{81}\left[\left|e^{-(3x+6)}-\beta_x+\sum_{y=0}^{\infty}(-1)^x\left(\frac{\beta_x}{x^2+y^2+1}\right)^{\nu}\sin y\right|^4+\left|e^{-(3x+6)}-\eta_x+\sum_{y=0}^{\infty}(-1)^x\left(\frac{\eta_x}{x^2+y^2+1}\right)^{\nu}\sin y\right|^4\right]. \tag{105}$$

By Theorem 27, the summable equations (105) have one solution in $(\ell_M(\lambda))_\mu$.

Theorem 30. *If M is a concave Orlicz function, $D\colon \mathscr{Z}^{+2}\longrightarrow \mathbf{R}$, $h\colon \mathscr{Z}^{+}\times\mathbf{R}\longrightarrow\mathbf{R}$, $r\colon\mathscr{Z}^{+}\longrightarrow\mathbf{R}$, and for every $x\in\mathscr{Z}^{+}$, there exists $\nu\in[0,1/2)$, with*

$$M\left(\left|\sum_{y\in\mathscr{Z}^+}D(x,y)\left(h(y,\beta_y)-h(y,\eta_y)\right)\right|\right) \leq \nu\left[M\left(\left|r_x-\beta_x+\sum_{y=0}^{\infty}D(x,y)f(y,\beta_y)\right|\right)+M\left(\left|r_x-\eta_x+\sum_{y=0}^{\infty}D(x,y)f(y,\eta_y)\right|\right)\right], \tag{106}$$

then equation (101) contains one solution in $(\ell_M(\lambda))_\mu$, where $\mu(\beta)=\sum_{x=0}^{\infty}M(|\beta_x|)$, for each $\beta\in\ell_M(\lambda)$.

Proof. Suppose the setups are verified. One has

$$\begin{aligned}\mu(H\beta-H\eta)&=\sum_{x\in\mathscr{Z}^+}\lambda_x M\left(\left|H\beta_x-H\eta_x\right|\right)=\sum_{x\in\mathscr{Z}^+}\lambda_x M\left(\left|\sum_{y\in\mathscr{Z}^+}D(x,y)\left(h(y,\beta_y)-h(y,\eta_y)\right)\right|\right)\\ &\leq\nu\left[\sum_{x\in\mathscr{Z}^+}\lambda_x M\left(\left|r_x-\beta_x+\sum_{y=0}^{\infty}D(x,y)h(y,\beta_y)\right|\right)+\sum_{x\in\mathscr{Z}^+}\lambda_x M\left(\left|r_x-\eta_x+\sum_{y=0}^{\infty}D(x,y)h(y,\eta_y)\right|\right)\right]\\ &=\nu(\mu(H\beta-\beta)+\mu(H\eta-\eta)).\end{aligned} \tag{107}$$

In view of Theorem 12, there exists a unique solution of equation (101) in $(\ell_M(\lambda))_\mu$. ☐

Example 11. For the space $(\ell_M(\lambda))_\mu$, where $\mu(\beta)=\sum_{x\in\mathscr{Z}^+}\sqrt[3]{|\beta_x|}$, for every $\beta\in\ell_M(\lambda)$. Assume the summable equations

$$\beta_x=e^{-(3x+6)}+\sum_{y=0}^{\infty}(-1)^{x+y}\left(\frac{e^{|\beta_x|}}{x^2+y^2+1}\right)^{\nu}, \tag{108}$$

where $\nu>2$ and let $H\colon(\ell_M(\lambda))_\mu\longrightarrow(\ell_M(\lambda))_\mu$ is defined by

$$H(\beta_x)_{x\in\mathscr{Z}^+}=\left(e^{-(3x+6)}+\sum_{y=0}^{\infty}(-1)^{x+y}\left(\frac{e^{|\beta_x|}}{x^2+y^2+1}\right)^{\nu}\right)_{x\in\mathscr{Z}^+}. \tag{109}$$

It is easy to see that

$$\left|\sum_{y=0}^{\infty}(-1)^x\left(\frac{e^{|\beta_x|}}{x^2+y^2+1}\right)^{\nu}\left((-1)^y-(-1)^y\right)\right|^{1/3} \leq\frac{1}{3}\left[\left|e^{-(3x+6)}-\beta_x+\sum_{y=0}^{\infty}(-1)^{x+y}\left(\frac{e^{|\beta_x|}}{x^2+y^2+1}\right)^{\nu}\right|^{1/3}+\left|e^{-(3x+6)}-\eta_x+\sum_{y=0}^{\infty}(-1)^{x+y}\left(\frac{e^{|\beta_x|}}{x^2+y^2+1}\right)^{\nu}\right|^{1/3}\right]. \tag{110}$$

By Theorem 30, the summable equation (105) has an unique solution in $(\ell_M(\lambda))_\mu$.

Example 12. Given the sequence space $(\ell_M(\lambda))_\mu$, *where* $\mu(\beta) = \sqrt{\sum_{x\in\mathcal{Z}^+}|\beta_x|^2}$, *for all* $\beta \in \ell_M(\lambda)$. *Consider the summable* equations (110), *with* $x \geq 2$ *and* $v > 2$ *and let* $H: \Phi \longrightarrow \Phi$, *where* $\Phi = \left\{\beta \in (\ell_M(\lambda))_\mu: \beta_0 = \beta_1 = 0\right\}$, *defined by*

$$H(\beta_x)_{x\geq 2} = \left(e^{-(3x+6)} + \sum_{y=0}^{\infty}(-1)^{x+y}\left(\frac{e^{|\beta_x|}}{x^2+y^2+1}\right)^v\right)_{x\geq 2}. \tag{111}$$

Obviously, Φ is a nonempty, μ-convex, μ-closed, and μ-bounded subset of $(\ell_M(\lambda))_\mu$. It is easy to see that

$$\begin{aligned}&\left|\sum_{y=0}^{\infty}(-1)^x\left(\frac{e^{|\beta_x|}}{x^2+y^2+1}\right)^v\left((-1)^y-(-1)^y\right)\right|^2\\ &\leq\frac{1}{9}\left[\left|e^{-(3x+6)}-\beta_x+\sum_{y=0}^{\infty}(-1)^{x+y}\left(\frac{e^{|\beta_x|}}{x^2+y^2+1}\right)^v\right|^2+\left|e^{-(3x+6)}-\eta_x+\sum_{y=0}^{\infty}(-1)^{x+y}\left(\frac{e^{|\eta_x|}}{x^2+y^2+1}\right)^v\right|^2\right].\end{aligned} \tag{112}$$

By Theorem 27 and Corollary 2, the summable equation (110) have a solution in Φ.

8. Conclusion

We explored the presence of a fixed point for both Kannan contraction and nonexpansive mappings working on various premodular, which is a generalization of modular, defined by weighted Orlicz sequence space and its pre-quasi operator ideal. Numerous numerical experiments and practical applications are used to substantiate our findings.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed equally to the writing of this paper. All the authors read and approved the final manuscript.

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Research Article

On Some Coincidence Best Proximity Point Results

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In this paper, we discuss some (coincidence) best proximity point results for generalized proximal contractions and $\lambda-\mu$-proximal Geraghty contractions in controlled metric type spaces. To clarify our study, various examples are given and some conclusions are drawn.

1. Introduction and Preliminaries

To solve the equation $\check{T}p = p$ ($\check{T}$ is a mapping defined on a subset of a metric space, a simplified linear space, or a topological vector space), fixed point theory is an important tool. A nonself-mapping $\check{T}: J \longrightarrow K$ may not have a fixed point. From this perspective, the best approximation theorem and the best proximity point are relevant. A classical best approximation theorem was due to Fan [1], i.e., if J is a nonempty compact convex subset of a Hausdorff locally convex topological vector space X with a seminorm p and $\check{T}: J \longrightarrow X$ is a continuous mapping, then there is an element p in J satisfying the condition that $\Psi(p, \check{T}p) = \Psi(\check{T}p, J)$. Many subsequent extensions and variations of Fan's theorem have occurred, including references [2, 3].

However, even though the best approximation theorems provide an approximate solution to the equation $\check{T}p = p$, they do not provide an ideal approximate solution. Moreover, the theorem of the best proximity point specifies adequate criteria for the presence of an element p to reduce the error $\Psi(p, \check{T}p)$. For a nonself-mapping $\check{T}: J \longrightarrow K$, $\Psi(p, \check{T}p)$ is at least $\Psi(J, K)$ for all p in J, then the best proximity point theorem establishes a globally optimal solution of error $\Psi(p, \check{T}p)$ by constraining an approximate solution p of the equation $\check{T}p = p$ to the condition that $\Psi(p, \check{T}p) = \Psi(J, K)$. Such an ideal approximate solution $\check{T}p = p$ is the best proximity point of the nonself-mapping $\check{T}: J \longrightarrow K$. For sure, the best proximity point hypotheses are a logical augmentation of fixed point hypotheses, on the grounds that the best proximity point is a fixed point in the light of self-mappings.

The best proximity point hypotheses have been demonstrated in [4]. Anuradha and Veeramani have tested the proximal pointwise contractions for the presence of a best proximity point [2]. Generally, several best proximity point theorems were analyzed for multiple variants of contractions in [5–14]. A best proximity point theorem for contraction mappings was presented in [15]. Some interesting common best proximity theorems have been discussed in [7, 15].

Nadler [16] was the first who generalized the Banach contraction principle for multivaluated mappings. Later, several works appeared in this direction. For more details, see [17–20]. The best proximity point hypotheses for different sorts of multivalued mappings have likewise been obtained in [21, 22].

Recently, the authors in [23] introduced a controlled metric type space in which the function of extended b-metric spaces was substituted by a function $\alpha(p, q)$ depending on the parameters of the left-hand side of the triangular inequality. The primary goal of this article is to include the best proximity point theorems for generalized and modified proximal contractions in the context of complete controlled type metric spaces, thus providing an optimal approximate solution to the equation $\check{T}p = p$. It is acknowledged that the previous best proximity point theorems include the well-known Banach contraction principle and some of its generalizations.

First, we state the following useful definitions in the sequel.

Definition 1 (see [6]). Let (X, Ψ) be a metric space having a pair of nonempty subsets (J, K) such that J_0 is nonempty. The pair (J, K) has the P-property if and only if

$$\left.\begin{array}{l}\Psi(p_1, q_1) = \Psi(J, K)\\ \Psi(p_2, q_2) = \Psi(J, K)\end{array}\right\}, \quad \text{implies}\, \Psi(p_1, p_2) = \Psi(q_1, q_2), \tag{1}$$

where $p_1, p_2 \in J_0$ and $q_1, q_2 \in K_0$.

Definition 2 (see [24]). Let (X, Ψ) be a metric space having a pair of nonempty subsets J and K. Let $\check{T}\colon J \longrightarrow K$ and $\mu\colon J \times J \longrightarrow [0, \infty)$. The mapping $\check{T}$ is said to be μ-proximal admissible if

$$\left.\begin{array}{l}\mu(p_1, p_2) \geq 1\\ \Psi(u_1, \check{T}p_1) = \Psi(J, K)\\ \Psi(u_2, \check{T}p_2) = \Psi(J, K)\end{array}\right\}, \quad \text{implies}\, \mu(u_1, u_2) \geq 1, \tag{2}$$

for all $p_1, p_2, u_1, u_2 \in J$.

Definition 3 (see [25]). Let $\mathscr{B}(X)$ represent the closed and bounded subsets of X. Let H be the Pompeiu–Hausdroff metric induced by metric Ψ defined by

$$H(J, K) = \max\left\{\sup_{a\in J} \mathscr{D}(a, K), \sup_{b\in K} \mathscr{D}(b, J)\right\}, \tag{3}$$

for $J, K \subseteq \mathscr{B}(X)$, where

$$\mathscr{D}(a, K) = \inf\{\Psi(a, b)\colon b \in K\}. \tag{4}$$

Definition 4 (see [23]). Let X be a nonempty set, and consider $\alpha\colon X \times X \longrightarrow [1, \infty)$ as a function. Let $\Psi\colon X \times X \longrightarrow [0, \infty)$ satisfying

(1) $\Psi(p_1, p_2) = 0$ if and only if $p_1 = p_2$
(2) $\Psi(p_1, p_2) = \Psi(p_2, p_1)$
(3) $\Psi(p_1, p_2) \leq \alpha(p_1, p_3)\Psi(p_1, p_3) + \alpha(p_3, p_2)\Psi(p_3, p_2)$, for all $p_1, p_2, p_3 \in X$, then (X, Ψ) is called a controlled metric type space

From now on, (X, Ψ) is a controlled metric type space.

Definition 5 (see [23]). A sequence $\{p_n\}$ in a controlled metric type space (X, Ψ) converges to some p in X if for each positive ε, there is some positive N_ε such that $\Psi(p_n, p) < \varepsilon$ for each $n \geq N_\varepsilon$. It can be written as

$$\lim_{n\longrightarrow\infty} p_n = p. \tag{5}$$

Definition 6 (see [23]). The sequence $\{p_n\}$ in a controlled metric type space (X, Ψ) is said to be a Cauchy sequence, if for every $\varepsilon > 0$, $\Psi(p_n, p_m) < \varepsilon$ for all $m, n \geq N_\varepsilon$, where $N_\varepsilon \in \mathbb{N}$.

Definition 7 (see [23]). A controlled metric type space (X, Ψ) is said to be complete if every Cauchy sequence is convergent in X.

Definition 8 (see [23]). Let $p \in X$ and $\varepsilon > 0$.

(1) The open ball $K(p, \varepsilon)$ is defined as follows:

$$K(p, \varepsilon) = \{q \in X, \Psi(p, q) < \varepsilon\}. \tag{6}$$

(2) The mapping $\check{T}\colon X \longrightarrow X$ is said continuous at $p \in X$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\check{T}(K(p, \delta)) \subseteq K(\check{T}p, \varepsilon). \tag{7}$$

Clearly, if $\check{T}$ is continuous at p in the controlled metric type space (X, Ψ), then $p_n \longrightarrow p$ implies that $\check{T}p_n \longrightarrow \check{T}p$ as $n \longrightarrow \infty$.

Definition 9 (see [26]). Define the function $\mathscr{H}\colon \mathscr{L}(X) \times \mathscr{L}(X) \longrightarrow [0, \infty]$ by

$$\mathscr{H}(J, K) = \begin{cases} \max\left\{\sup_{a\in J} \mathscr{D}(a, K), \sup_{b\in K} \mathscr{D}(b, J)\right\}, & \text{if the maximum exists,}\\ \infty, & \text{otherwise,}\end{cases} \tag{8}$$

for $J, K \subseteq \mathscr{L}(X)$ (it represents the set of closed subsets of X), where

$$\mathscr{D}(a, K) = \inf\{\Psi(a, b)\colon b \in K\}, \quad \text{for } K \subset X. \tag{9}$$

Let J and K be two nonempty subsets of X. Define

$$\begin{aligned} J_0 &= \{p \in J\colon \Psi(p, q) = \Psi(J, K) \text{ for some } q \in K\}, \\ K_0 &= \{q \in K\colon \Psi(p, q) = \Psi(J, K) \text{ for some } p \in J\}, \end{aligned} \tag{10}$$

where

$$\Psi(J, K) = \inf\{\Psi(p, q)\colon p \in J, q \in K\} \text{ (distance of a set } J \text{ to a set } K), \tag{11}$$

and we will denote

$$\mathscr{D}^*(p, q) = \Psi(p, q) - \Psi(J, K), \quad \text{for all } p \in J, q \in K. \tag{12}$$

Theorem 1 (see [26]). *The function $\mathscr{H}\colon \mathscr{L}(X) \times \mathscr{L}(X) \longrightarrow [0, \infty]$ is a generalized Pompeiu–Hausdroff controlled metric space on $\mathscr{L}(X)$.*

Remark 1 (see [26]). Let $(\mathscr{L}(X), H)$ be a generalized Pompeiu–Hausdroff-controlled metric type space. Then, the following assertions hold (for all bounded and closed subsets $J, K, C,$ and D of X):

(1) $H(C, D) = 0$ is equivalent to $C = D$

(2) $H(C, D) = H(D, C)$

(3) $H(J, C) \leq \max\sup_{a \in J} \alpha(a, b), \alpha(b, J) H(J, K) + \max \alpha(b, C), \sup_{c \in C} \alpha(c, b), H(K, C)$

Theorem 2 (see [26]). *If (X, Ψ) is a complete controlled metric space with $\lim_{n,m \longrightarrow \infty} \alpha(p_n, p_m) k < 1$, for all $p_n, p_m \in X$, where $k \geq 1$, then $(\mathscr{L}(X), \mathscr{H})$ is complete.*

2. Coincidence Best Proximity Points for Generalized Proximal Contractions

In this section, we will discuss some best proximity point theorems using the multivalued concept on a controlled metric space (X, Ψ).

From now and onward, J and K are nonempty subsets of a controlled metric type space (X, Ψ) (until otherwise stated). Define $\alpha\colon X \times X \longrightarrow [1, \infty)$ by $\alpha_*(p, J) = \inf\{\alpha(p, a), \text{for all } a \in J\}$ and $\alpha_*(J, K) = \inf\{\alpha(a, b), \text{for all } a \in J \text{ and } b \in K\}$, where $\alpha\colon X \times X \longrightarrow [1, \infty)$ and J and K are nonempty subsets of X.

Definition 10 (see [26]). A mapping $\breve{T}\colon X \longrightarrow \mathscr{B}(X)$ is continuous in a controlled metric type space (X, Ψ) at $p \in X$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\breve{T}(K(p, \delta)) \subseteq K(\breve{T}p, \varepsilon), \tag{13}$$

where $K(p, \varepsilon)$ is given as

$$K(p, \varepsilon) = \{q \in X, \Psi(p, q) < \varepsilon\}. \tag{14}$$

Clearly, if $\breve{T}$ is continuous at p, then $p_n \longrightarrow p$ implies that $\breve{T}p_n \longrightarrow \breve{T}p$ as $n \longrightarrow \infty$.

We introduce the following.

Definition 11. Let (X, Ψ) be a controlled metric type space having two nonempty subsets J and K. Let $\breve{T}\colon J \longrightarrow K$ be a mapping. A point $p \in J$ is said to be a best proximity point of the mapping $\breve{T}$ if

$$\Psi(p, \breve{T}p) = \Psi(J, K). \tag{15}$$

Definition 12. Let (X, Ψ) be a controlled metric type space having two nonempty subsets J and K. A nonempty set J is said to be approximately compact with respect to K if every sequence $\{p_n\}$ in J satisfying the condition that $\mathscr{D}(q, p_n) \longrightarrow \mathscr{D}(q, J)$ for some q in K has a convergent subsequence.

Definition 13. Given $\breve{T}\colon J \longrightarrow \mathscr{B}(K)$ and $\breve{g}\colon J \longrightarrow J$. A pair of mappings $(\breve{g}, \breve{T})$ is said to be a β-generalized proximal contraction if there exists a real number $\beta \in [0, 1)$ such that

$$\left.\begin{aligned} \mathscr{D}(\breve{g}u_1, \breve{T}p_1) &= \Psi(J, K) \\ \mathscr{D}(\breve{g}u_2, \breve{T}p_2) &= \Psi(J, K) \end{aligned}\right\}, \quad \text{implies } \mathscr{H}(\breve{T}u_1, \breve{T}u_2) \leq \beta \mathscr{H}(\breve{T}p_1, \breve{T}p_2), \tag{16}$$

for all u_1, u_2, p_1, and p_2 in J.

Definition 14. A mapping $\breve{T}: J \longrightarrow \mathscr{B}(K)$ is said to be a $\beta_{\breve{T}}$-generalized proximal contraction if there exists $\beta \in [0,1)$ such that

$$\left.\begin{array}{l}\mathscr{D}\left(u_1, \breve{T}p_1\right) = \Psi(J,K)\\ \mathscr{D}\left(u_2, \breve{T}p_2\right) = \Psi(J,K)\end{array}\right\}, \quad \text{implies } \mathscr{H}\left(\breve{T}u_1, \breve{T}u_2\right) \leq \beta \mathscr{H}\left(\breve{T}p_1, \breve{T}p_2\right), \tag{17}$$

for all u_1, u_2, p_1, and p_2 in J.

Note that, if we take $\breve{g} = I_J$ (the identity mapping on J), then every β-generalized proximal contraction will reduce to a $\beta_{\breve{T}}$-generalized proximal contraction.

Definition 15. Let (X, Ψ) be a controlled metric type space having two nonempty subsets J and K. Let $\breve{T}: J \longrightarrow K$ and $\breve{g}: J \longrightarrow J$ be mappings. A point $p \in J$ is said to be a coincidence best proximity point of the pair of mappings $(\breve{g}, \breve{T})$ if

$$\Psi(\breve{g}p, \breve{T}p) = \Psi(J,K). \tag{18}$$

Remark 2. If we take $\breve{g} = I_J$ (the identity mapping over J), then every coincidence best proximity point becomes a best proximity point of the mapping $\breve{T}$.

If $J \cap K \neq \varnothing$ or $\Psi(J,K) = 0$, then every best proximity point will reduce to a fixed point of the mapping $\breve{T}$.

Our first main result is stated as follows:

Theorem 3. *Let (X, Ψ) be a controlled metric type space having two nonempty subsets J and K. Let $\breve{T}: J \longrightarrow \mathscr{B}(K)$ and $\breve{g}: J \longrightarrow J$ be one-to-one and continuous mappings. Assume that K is a closed subset and J is approximately compact with respect to K with $\breve{T}(J_0) \subseteq K_0$ and $J_0 \subseteq \breve{g}(J_0)$. Further, assume that the pair $(\breve{g}, \breve{T})$ is a β-generalized proximal contraction such that*

$$\sup_{m\geq 1} \lim_{i\longrightarrow\infty} \max\left\{\sup_{q_i \in \breve{T}p_i} \alpha(q_i, q_{i+1}), \alpha\left(q_{i+1}, \breve{T}p_i\right)\right\} \max\left\{\sup_{q_i \in \breve{T}p_i} \alpha(q_i, q_m), \alpha\left(q_m, \breve{T}p_i\right)\right\} < \frac{1}{k}, \tag{19}$$

and $\lim_{n\longrightarrow\infty} \alpha_*(\breve{g}p_n, \breve{T}p_{n-1}) = 1$, where $k \in (0,1)$. Then, there exists a coincidence best proximity point of the pair $(\breve{g}, \breve{T})$.

Proof. Let p_0 be an arbitrary element in J_0. Since $\breve{T}(J_0)$ is contained in K_0 and J_0 is contained in $\breve{g}(J_0)$, there exists an element p_1 in J_0 such that

$$\mathscr{D}\left(\breve{g}p_1, \breve{T}p_0\right) = \Psi(J,K). \tag{20}$$

Again, since $\breve{T}p_1$ is an element of $\breve{T}(J_0)$ which is contained in K_0 and J_0 is contained in $\breve{g}(J_0)$, it follows that there is an element p_2 in J_0 such that

$$\mathscr{D}\left(\breve{g}p_2, \breve{T}p_1\right) = \Psi(J,K). \tag{21}$$

This process can be continued by selecting p_n in J_0 satisfying the condition as follows:

$$\mathscr{D}\left(\breve{g}p_n, \breve{T}p_{n-1}\right) = \Psi(J,K). \tag{22}$$

Having selected $\{p_n\}$ satisfying the condition, there exists an element p_{n+1} in J_0 satisfying

$$\mathscr{D}\left(\breve{g}p_{n+1}, \breve{T}p_n\right) = \Psi(J,K), \tag{23}$$

for every integer $n \geq 0$.

Since the pair $(\breve{g}, \breve{T})$ is a β-generalized proximal contraction, by using equations (22) and (23), we obtain

$$\mathscr{H}\left(\breve{T}p_{n+1}, \breve{T}p_n\right) \leq \beta \mathscr{H}\left(\breve{T}p_n, \breve{T}p_{n-1}\right), \quad \text{for each } n \geq 1. \tag{24}$$

We deduce that

$$\mathscr{H}\left(\breve{T}p_{n+1}, \breve{T}p_n\right) \leq \beta^n \mathscr{H}\left(\breve{T}p_1, \breve{T}p_0\right), \quad \text{for each } n \geq 0. \tag{25}$$

Now, we have to prove that $\left\{\breve{T}p_n\right\}$ is a Cauchy sequence, for all natural numbers $n, m \in \mathbb{N}$ with $n < m$,

$$
\begin{aligned}
\mathscr{H}\left(\check{T}p_n,\check{T}p_m\right) \le{}& \max\left\{\sup_{q_n\in\check{T}p_n}\alpha\left(q_n,q_{n+1}\right),\alpha\left(q_{n+1},\check{T}p_n\right)\right\}\mathscr{H}\left(\check{T}p_n,\check{T}p_{n+1}\right)\\
&+\max\left\{\sup_{q_{n+1}\in\check{T}p_{n+1}}\alpha\left(q_{n+1},q_m\right),\alpha\left(q_m,\check{T}p_{n+1}\right)\right\}\mathscr{H}\left(\check{T}p_{n+1},\check{T}p_m\right)\\
&\le\max\left\{\sup_{q_n\in\check{T}p_n}\alpha\left(q_n,q_{n+1}\right),\alpha\left(q_{n+1},\check{T}p_n\right)\right\}\mathscr{H}\left(\check{T}p_n,\check{T}p_{n+1}\right)\\
&+\max\left\{\sup_{q_{n+1}\in\check{T}p_{n+1}}\alpha\left(q_{n+1},q_{n+2}\right),\alpha\left(q_{n+2},\check{T}p_{n+1}\right)\right\}\max\left\{\sup_{q_{n+1}\in\check{T}p_{n+1}}\alpha\left(q_{n+1},q_m\right),\alpha\left(q_m,\check{T}p_{n+1}\right)\right\}\mathscr{H}\left(\check{T}p_{n+1},\check{T}p_{n+2}\right)\\
&+\max\left\{s\sup_{q_{n+1}\in\check{T}p_{n+1}}\alpha\left(q_{n+1},q_m\right),\alpha\left(q_m,\check{T}p_{n+1}\right)\right\}\max\left\{\sup_{q_{n+2}\in\check{T}p_{n+2}}\alpha\left(q_{n+2},q_m\right),\alpha\left(q_m,\check{T}p_{n+2}\right)\right\}\mathscr{H}\left(\check{T}p_{n+2},\check{T}p_m\right)\\
\le{}&\max\left\{\sup_{q_n\in\check{T}p_n}\alpha\left(q_n,q_{n+1}\right),\alpha\left(q_{n+1},\check{T}p_n\right)\right\}\mathscr{H}\left(\check{T}p_n,\check{T}p_{n+1}\right)\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\max\left\{\sup_{q_j\in\check{T}p_j}\alpha\left(q_j,q_m\right),\alpha\left(q_m,\check{T}p_j\right)\right\}\right)\max\left\{\sup_{q_i\in\check{T}p_i}\alpha\left(q_i,q_{i+1}\right),\alpha\left(q_{i+1},\check{T}p_i\right)\right\}\mathscr{H}\left(\check{T}p_i,\check{T}p_{i+1}\right)\\
&+\prod_{k=n+1}^{m-1}\max\left\{\sup_{q_{n_k}\in\check{T}p_k}\alpha\left(q_k,q_m\right),\alpha\left(q_m,\check{T}p_k\right)\right\}\mathscr{H}\left(\check{T}p_{m-1},\check{T}p_m\right)\\
\le{}&\max\left\{\sup_{q_n\in\check{T}p_n}\alpha\left(q_n,q_{n+1}\right),\alpha\left(q_{n+1},\check{T}p_n\right)\right\}\beta^n\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right)\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\max\left\{\sup_{q_j\in\check{T}p_j}\alpha\left(q_j,q_m\right),\alpha\left(q_m,\check{T}p_j\right)\right\}\right)\max\left\{\sup_{q_i\in\check{T}p_i}\alpha\left(q_i,q_{i+1}\right),\alpha\left(q_{i+1},\check{T}p_i\right)\right\}\beta^i\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right)\\
&+\prod_{k=n+1}^{m-1}\max\left\{\sup_{q_{n_k}\in\check{T}p_k}\alpha\left(q_k,q_m\right),\alpha\left(q_m,\check{T}p_k\right)\right\}\beta^{m-1}\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right)\\
\le{}&\max\left\{\sup_{q_n\in\check{T}p_n}\alpha\left(q_n,q_{n+1}\right),\alpha\left(q_{n+1},\check{T}p_n\right)\right\}\beta^n\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right)\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\max\left\{\sup_{q_j\in\check{T}p_j}\alpha\left(q_j,q_m\right),\alpha\left(q_m,\check{T}p_j\right)\right\}\right)\max\left\{\sup_{q_i\in\check{T}p_i}\alpha\left(q_i,q_{i+1}\right),\alpha\left(q_{i+1},\check{T}p_i\right)\right\}\beta^i\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right)\\
&+\prod_{k=n+1}^{m-1}\max\left\{\sup_{q_{n_k}\in\check{T}p_k}\alpha\left(q_k,q_m\right),\alpha\left(q_m,\check{T}p_k\right)\right\}\max\left\{\sup_{q_{m-1}\in\check{T}p_{m-1}}\alpha\left(q_{m-1},q_m\right),\alpha\left(q_m,\check{T}p_{m-1}\right)\right\}\beta^{m-1}\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right)\\
={}&\max\left\{\sup_{q_n\in\check{T}p_n}\alpha\left(q_n,q_{n+1}\right),\alpha\left(q_{n+1},\check{T}p_n\right)\right\}\beta^n\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\max\left\{\sup_{q_j\in\check{T}p_j}\alpha\left(q_j,q_m\right),\alpha\left(q_m,\check{T}p_j\right)\right\}\right)\max\left\{\sup_{q_i\in\check{T}p_i}\alpha\left(q_i,q_{i+1}\right),\alpha\left(q_{i+1},\check{T}p_i\right)\right\}\beta^i\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right)\\
&\le\max\left\{\sup_{q_n\in\check{T}p_n}\alpha\left(q_n,q_{n+1}\right),\alpha\left(q_{n+1},\check{T}p_n\right)\right\}\beta^n\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right)\\
&+\sum_{i=n+1}^{m-2}\left(\prod_{j=0}^{i}\max\left\{\sup_{q_j\in\check{T}p_j}\alpha\left(q_j,q_m\right),\alpha\left(q_m,\check{T}p_j\right)\right\}\right)\max\left\{\sup_{q_i\in\check{T}p_i}\alpha\left(q_i,q_{i+1}\right),\alpha\left(q_{i+1},\check{T}p_i\right)\right\}\beta^i\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right).
\end{aligned}
\tag{26}
$$

Assume that

$$
S_{m-2}=\sum_{i=n+1}^{m-2}\left(\prod_{j=0}^{i}\max\left\{\sup_{q_j\in\check{T}p_j}\alpha\left(q_j,q_m\right),\alpha\left(q_m,\check{T}p_j\right)\right\}\right)\max\left\{\sup_{q_i\in\check{T}p_i}\alpha\left(q_i,q_{i+1}\right),\alpha\left(q_{i+1},\check{T}p_i\right)\right\}\beta^i.
\tag{27}
$$

Then, we obtain

$$
\mathscr{H}\left(\check{T}p_n,\check{T}p_m\right)\le\mathscr{H}\left(\check{T}p_0,\check{T}p_1\right)\left[\beta^n\max\left\{\sup_{q_n\in\check{T}p_n}\alpha\left(q_n,q_{n+1}\right),\alpha\left(q_{n+1},\check{T}p_n\right)\right\}+\left(S_{m-1}-S_n\right)\right].
\tag{28}
$$

Using the ratio test, we have

$$
a_i=\prod_{j=0}^{i}\max\left\{\sup_{q_j\in\check{T}p_j}\alpha\left(q_j,q_m\right),\alpha\left(q_m,\check{T}p_j\right)\right\}\max\left\{\sup_{q_i\in\check{T}p_i}\alpha\left(q_i,q_{i+1}\right),\alpha\left(q_{i+1},\check{T}p_i\right)\right\}\beta^i,
\tag{29}
$$

where $(a_{i+1}/a_i)<(1/k)$. Taking limit as $n,m\longrightarrow\infty$, we obtain

$$
\lim_{n\longrightarrow\infty}\mathscr{H}\left(\check{T}p_n,\check{T}p_m\right)=0.
\tag{30}
$$

That is, $\{\check{T}p_n\}$ is a Cauchy sequence in the complete generalized Pompeiu–Hausdroff controlled metric type space $(\mathscr{B}(X),\mathscr{H})$; hence, it converges to some q in K (as the set K is closed). Therefore,

$$
\begin{aligned}
\Psi(q,J)\le\Psi\left(q,\breve{g}p_n\right)&\le\alpha_*\left(q,\check{T}p_{n-1}\right)\mathscr{D}\left(q,\check{T}p_{n-1}\right)+\alpha_*\left(\check{T}p_{n-1},\breve{g}p_n\right)\mathscr{D}\left(\check{T}p_{n-1},\breve{g}p_n\right)\\
&=\alpha_*\left(q,\check{T}p_{n-1}\right)\mathscr{D}\left(q,\check{T}p_{n-1}\right)+\alpha_*\left(\check{T}p_{n-1},\breve{g}p_n\right)\Psi(J,K).
\end{aligned}
\tag{31}
$$

Taking $\lim_{n\longrightarrow\infty}$ on both sides of the above inequality, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\Psi(q,\breve{g}p_n) &\leq \lim_{n\longrightarrow\infty}\left[\alpha_*(q,\check{T}p_{n-1})\mathscr{D}(q,\check{T}p_{n-1})+\alpha_*(\check{T}p_{n-1},\breve{g}p_n)\Psi(J,K)\right]\\ &\leq \Psi(J,K)\\ &\leq \Psi(q,J).\end{aligned} \tag{32}$$

Therefore, $\Psi(q,\breve{g}p_n)\longrightarrow\Psi(q,J)$. In view of the fact that J is approximately compact with respect to K, $\{\breve{g}p_n\}$ has a subsequence $\{\breve{g}p_{n_k}\}$ converging to some $z=\breve{g}p\in J$ for some $p\in J_0$. Thus,

$$\Psi(z,q)=\lim_{k\longrightarrow\infty}\mathscr{D}(\breve{g}p_{n_k},\check{T}p_{n_{k-1}})=\Psi(J,K). \tag{33}$$

Therefore, z is a member of J_0. Since J_0 is contained in $\breve{g}(J_0)$ and $z=\breve{g}p$ for some p in J_0, $\breve{g}p_{n_k}\longrightarrow\breve{g}p$ and $\breve{g}$ is a one-to-one continuous mapping, so $p_{n_k}\longrightarrow p$. Since $\check{T}$ is continuous, it can be concluded that $\check{T}p_{n_k}\longrightarrow\check{T}p$. This implies that

$$\mathscr{D}(\breve{g}p,\check{T}p)=\lim_{k\longrightarrow\infty}\mathscr{D}(\breve{g}p_{n_k},\check{T}p_{n_{k-1}})=\Psi(J,K). \tag{34}$$

That is, p is a coincidence best proximity point of the pair $(\breve{g},\check{T})$.

To prove the uniqueness of the coincidence best proximity point of the pair of mappings $(\breve{g},\check{T})$, suppose that there is another coincidence best proximity point $q\neq p$ of the pair $(\breve{g},\check{T})$. We have

$$\begin{aligned}\mathscr{D}(\breve{g}p,\check{T}p)&=\Psi(J,K),\\ \mathscr{D}(\breve{g}q,\check{T}q)&=\Psi(J,K).\end{aligned} \tag{35}$$

As the mapping $\check{T}$ is one-to-one on the set J and $p\neq q$, one has $\mathscr{H}(\check{T}p,\check{T}q)>0$. Since the pair $(\breve{g},\check{T})$ is a β-generalized proximal contraction, one can write

$$(0<)\mathscr{H}(\check{T}p,\check{T}q)\leq\beta\mathscr{H}(\check{T}p,\check{T}q)<\mathscr{H}(\check{T}p,\check{T}q). \tag{36}$$

It is a contradiction. □

Corollary 1. *Let $\check{T}: J\longrightarrow\mathscr{B}(K)$ and $\alpha_*: X\times X\longrightarrow[1,\infty)$ be mappings, where K is a closed subset and J is approximately compact with respect to K with $\check{T}(J_0)\subseteq K_0$. Suppose that $\check{T}$ is a continuous and $\beta_{\check{T}}$-generalized proximal contraction such that*

$$\begin{aligned}&\sup_{m\geq1}\lim_{i\longrightarrow\infty}\alpha_*(p_i,p_{i+1})\alpha_*(p_i,p_m)<\frac{1}{k},\\ &\lim_{n\longrightarrow\infty}\alpha_*(p_n,\check{T}p_{n-1})=1,\quad \text{where } k\in(0,1),\end{aligned} \tag{37}$$

then there exists a unique best proximity point of $\check{T}$.

Proof. If we take identity mapping $\breve{g}=I_J$ ($\breve{g}$ is identity on J), the remaining proof is same as in Theorem 3 □

Definition 16. Let $\check{T}: J\longrightarrow K$ and $\breve{g}: J\longrightarrow J$. A pair of mappings $(\breve{g},\check{T})$ is said to be a β-modified proximal contraction if there exists $\beta\in[0,1)$ such that

$$\left.\begin{aligned}\Psi(\breve{g}u_1,\check{T}p_1)&=\Psi(J,K)\\ \Psi(\breve{g}u_2,\check{T}p_2)&=\Psi(J,K)\end{aligned}\right\},\quad \text{implies } \Psi(\check{T}u_1,\check{T}u_2)\leq\beta\Psi(\check{T}p_1,\check{T}p_2), \tag{38}$$

for all u_1,u_2,p_1, and p_2 in J.

Definition 17. A mapping $\check{T}: J\longrightarrow K$ is said to be a $\beta_{\check{T}}$-modified proximal contraction if there exists $\beta\in[0,1)$ such that

$$\left.\begin{aligned}\Psi(u_1,\check{T}p_1)&=\Psi(J,K)\\ \Psi(u_2,\check{T}p_2)&=\Psi(J,K)\end{aligned}\right\},\quad \text{implies } \Psi(\check{T}u_1,\check{T}u_2)\leq\beta\Psi(\check{T}p_1,\check{T}p_2), \tag{39}$$

for all u_1,u_2,p_1, and p_2 in J.

Note that if we take $\breve{g}=I_J$ (the identity mapping on J), then every β-modified proximal contraction is a $\beta_{\check{T}}$-modified proximal contraction.

Theorem 4. *Let $\check{T}: J\longrightarrow K$ and $\breve{g}: J\longrightarrow J$ be two continuous and one-to-one mappings, where K is a closed subset and J is approximately compact with respect to K with $\check{T}(J_0)\subseteq K_0$ and $J_0\subseteq\breve{g}(J_0)$. If the pair $(\breve{g},\check{T})$ is a β-modified proximal contraction and*

$$\begin{aligned}&\sup_{m\geq1}\lim_{i\longrightarrow\infty}\alpha(p_i,p_{i+1})\alpha(p_i,p_m)<\frac{1}{k},\\ &\lim_{n\longrightarrow\infty}\alpha(\breve{g}p_n,\check{T}p_{n-1})=1,\quad \text{where } k\in(0,1),\end{aligned} \tag{40}$$

then there exists a unique coincidence best proximity point of the pair $(\breve{g},\check{T})$.

Proof. Let p_0 be an arbitrary element in J_0. Since $\check{T}(J_0)$ is contained in K_0 and J_0 is contained in $\breve{g}(J_0)$, there exists an element p_1 in J_0 such that

$$\Psi(\breve{g}p_1,\check{T}p_0)=\Psi(J,K). \tag{41}$$

Since $\check{T}p_1$ is an element of $\check{T}(J_0)$ which is contained in K_0 and J_0 is contained in $\breve{g}(J_0)$, it follows that there exists an element p_2 in J_0 such that

$$\Psi(\breve{g}p_2,\check{T}p_1)=\Psi(J,K). \tag{42}$$

By continuing this process, we can construct a sequence $\{p_n\}$ in J_0, satisfying the condition as follows:

$$\Psi(\breve{g}p_n,\check{T}p_{n-1})=\Psi(J,K). \tag{43}$$

Having chosen $\{p_n\}$ in J_0, there exists an element p_{n+1} in J_0, such that

$$\Psi(\breve{g}p_{n+1},\check{T}p_n)=\Psi(J,K), \tag{44}$$

for every positive integer n. Since the pair $(\breve{g},\breve{T})$ is a β-modified proximal contraction from equations (43) and (44), we obtain

$$\Psi\left(\breve{T}p_{n+1},\breve{T}p_n\right)\le\beta\Psi\left(\breve{T}p_n,\breve{T}p_{n-1}\right). \tag{45}$$

Recursively, we have

$$\Psi\left(\breve{T}p_{n+1},\breve{T}p_n\right)\le\beta^n\Psi\left(\breve{T}p_1,\breve{T}p_0\right). \tag{46}$$

Now, we have to prove that $\{\breve{T}p_n\}$ is a Cauchy sequence. For all natural numbers $n,m\in\mathbb{N}$ with $n<m$, we have

$$\begin{aligned}
\Psi\left(\breve{T}p_n,\breve{T}p_m\right)&\le\alpha\left(\breve{T}p_n,\breve{T}p_{n+1}\right)\Psi\left(\breve{T}p_n,\breve{T}p_{n+1}\right)+\alpha\left(\breve{T}p_{n+1},\breve{T}p_m\right)\Psi\left(\breve{T}p_{n+1},\breve{T}p_m\right)\\
&\le\alpha\left(\breve{T}p_n,\breve{T}p_{n+1}\right)\Psi\left(\breve{T}p_n,\breve{T}p_{n+1}\right)+\alpha\left(\breve{T}p_{n+1},\breve{T}p_m\right)\alpha\left(\breve{T}p_{n+1},\breve{T}p_{n+2}\right)\Psi\left(\breve{T}p_{n+1},\breve{T}p_{n+2}\right)\\
&\quad+\alpha\left(\breve{T}p_{n+1},\breve{T}p_m\right)\alpha\left(\breve{T}p_{n+2},\breve{T}p_m\right)\Psi\left(\breve{T}p_{n+2},\breve{T}p_m\right)\\
&\le\alpha\left(\breve{T}p_n,\breve{T}p_{n+1}\right)\Psi\left(\breve{T}p_n,\breve{T}p_{n+1}\right)+\alpha\left(\breve{T}p_{n+1},\breve{T}p_m\right)\alpha\left(\breve{T}p_{n+1},\breve{T}p_{n+2}\right)\Psi\left(\breve{T}p_{n+1},\breve{T}p_{n+2}\right)\\
&\quad+\alpha\left(\breve{T}p_{n+1},\breve{T}p_m\right)\alpha\left(\breve{T}p_{n+2},\breve{T}p_m\right)\alpha\left(\breve{T}p_{n+2},\breve{T}p_{n+3}\right)\Psi\left(\breve{T}p_{n+2},\breve{T}p_{n+3}\right)+\alpha\left(\breve{T}p_{n+1},\breve{T}p_m\right)\\
&\quad\alpha\left(\breve{T}p_{n+2},\breve{T}p_m\right)\alpha\left(\breve{T}p_{n+3},\breve{T}p_m\right)\Psi\left(\breve{T}p_{n+3},\breve{T}p_m\right)\\
&\le\alpha\left(\breve{T}p_n,\breve{T}p_{n+1}\right)\Psi\left(\breve{T}p_n,\breve{T}p_{n+1}\right)+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha\left(\breve{T}p_j,\breve{T}p_m\right)\right)\alpha\left(\breve{T}p_i,\breve{T}p_{i+1}\right)\Psi\left(\breve{T}p_i,\breve{T}p_{i+1}\right)\\
&\quad+\prod_{k=n+1}^{m-1}\alpha\left(\breve{T}p_k,\breve{T}p_m\right)\Psi\left(\breve{T}p_{m-1},\breve{T}p_m\right)\\
&\le\alpha\left(\breve{T}p_n,\breve{T}p_{n+1}\right)\beta^n\Psi\left(\breve{T}p_0,\breve{T}p_1\right)+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha\left(\breve{T}p_j,\breve{T}p_m\right)\right)\alpha\left(\breve{T}p_i,\breve{T}p_{i+1}\right)\beta^i\Psi\left(\breve{T}p_0,\breve{T}p_1\right)\\
&\quad+\prod_{k=n+1}^{m-1}\alpha\left(\breve{T}p_k,\breve{T}p_m\right)\beta^{m-1}\Psi\left(\breve{T}p_0,\breve{T}p_1\right)\\
&\le\alpha\left(\breve{T}p_i,\breve{T}p_{i+1}\right)\beta^n\Psi\left(\breve{T}p_0,\breve{T}p_1\right)+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha\left(\breve{T}p_j,\breve{T}p_m\right)\right)\alpha\left(\breve{T}p_i,\breve{T}p_{i+1}\right)\beta^i\Psi\left(\breve{T}p_0,\breve{T}p_1\right)\\
&\quad+\prod_{k=n+1}^{m-1}\alpha\left(\breve{T}p_k,\breve{T}p_m\right)\alpha\left(\breve{T}p_{m-1},\breve{T}p_m\right)\beta^{m-1}\Psi\left(\breve{T}p_0,\breve{T}p_1\right)\\
&=\alpha\left(\breve{T}p_n,\breve{T}p_{n+1}\right)\beta^n\Psi\left(\breve{T}p_0,\breve{T}p_1\right)+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\alpha\left(\breve{T}p_j,\breve{T}p_m\right)\right)\alpha\left(\breve{T}p_i,\breve{T}p_{i+1}\right)\beta^i\Psi\left(\breve{T}p_0,\breve{T}p_1\right)\\
&\le\alpha\left(\breve{T}p_n,\breve{T}p_{n+1}\right)\beta^n\Psi\left(\breve{T}p_0,\breve{T}p_1\right)+\sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\alpha\left(\breve{T}p_j,\breve{T}p_m\right)\right)\alpha\left(\breve{T}p_i,\breve{T}p_{i+1}\right)\beta^i\Psi\left(\breve{T}p_0,\breve{T}p_1\right).
\end{aligned} \tag{47}$$

Assume that

$$S_l=\sum_{i=0}^{l}\left(\prod_{j=0}^{i}\alpha\left(\breve{T}p_j,\breve{T}p_m\right)\right)\alpha\left(\breve{T}p_i,\breve{T}p_{i+1}\right)\beta^i. \tag{48}$$

It follows that

$$\Psi\left(\breve{T}p_n,\breve{T}p_m\right)\le\Psi\left(\breve{T}p_0,\breve{T}p_1\right)\left[\beta^n\alpha\left(\breve{T}p_n,\breve{T}p_{n+1}\right)+\left(S_{m-1}-S_n\right)\right]. \tag{49}$$

Using the ratio test, we have

$$a_i=\prod_{j=0}^{i}\alpha\left(\breve{T}p_j,\breve{T}p_m\right)\alpha\left(\breve{T}p_i,\breve{T}p_{i+1}\right)\beta^i,\quad\text{where }\frac{a_{i+1}}{a_i}<\frac{1}{k}. \tag{50}$$

By applying limit $m,n\longrightarrow\infty$ in inequality (49), we get

$$\lim_{n\longrightarrow\infty}\Psi\left(\breve{T}p_n,\breve{T}p_m\right)=0, \tag{51}$$

which shows that $\{\breve{T}p_n\}$ is a Cauchy sequence; hence, it is convergent to some q in K (as the set K is closed). Therefore,

$$\begin{aligned}\Psi(q,J) &\leq \Psi\left(q,\breve{g}p_n\right) \leq \alpha\left(q,\check{T}p_{n-1}\right)\Psi\left(q,\check{T}p_{n-1}\right) + \alpha\left(\check{T}p_{n-1},\breve{g}p_n\right)\Psi\left(\check{T}p_{n-1},\breve{g}p_n\right)\\ &= \alpha\left(q,\check{T}p_{n-1}\right)\Psi\left(q,\check{T}p_{n-1}\right) + \alpha\left(\check{T}p_{n-1},\breve{g}p_n\right)\Psi(J,K).\end{aligned} \tag{52}$$

Taking limit $n \longrightarrow \infty$ on both sides of the above inequality, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\Psi\left(q,\breve{g}p_n\right) &\leq \lim_{n\longrightarrow\infty}\left[\alpha\left(q,\check{T}p_{n-1}\right)\Psi\left(q,\check{T}p_{n-1}\right) + \alpha\left(\check{T}p_{n-1},\breve{g}p_n\right)\Psi(J,K)\right]\\ &\leq \Psi(J,K)\\ &\leq \Psi(q,J).\end{aligned} \tag{53}$$

Therefore, $\Psi(q,\breve{g}p_n) \longrightarrow \Psi(q,J)$. In view of the fact that J is approximately compact with respect to K, $\{\breve{g}p_n\}$ has a subsequence $\{\breve{g}p_{n_k}\}$ converging to some $z = \breve{g}p \in J$ for some $p \in J_0$. It follows that

$$\Psi(z,q) = \lim_{k\longrightarrow\infty}\Psi\left(\breve{g}p_{n_k},\check{T}p_{n_{k-1}}\right) = \Psi(J,K). \tag{54}$$

Therefore, z is an element of J_0. Since J_0 is contained in $\breve{g}(J_0)$, we have $z = \breve{g}p$ for some p in J_0. As $\breve{g}p_{n_k} \longrightarrow \breve{g}p$ and $\breve{g}$ is a one-to-one continuous mapping, $p_{n_k} \longrightarrow p$. Since $\check{T}$ is continuous, it can be concluded that $\check{T}p_{n_k} \longrightarrow \check{T}p$. Hence,

$$\Psi(\breve{g}p,\check{T}p) = \lim_{k\longrightarrow\infty}\Psi\left(\breve{g}p_{n_k},\check{T}p_{n_{k-1}}\right) = \Psi(J,K). \tag{55}$$

To prove the uniqueness, suppose that q is another coincidence best proximity point of the pair $(\breve{g},\check{T})$ such that $p \neq q$. Then,

$$\begin{aligned}\Psi(\breve{g}p,\check{T}p) &= \Psi(J,K),\\ \Psi(\breve{g}q,\check{T}q) &= \Psi(J,K).\end{aligned} \tag{56}$$

Since the pair $(\breve{g},\check{T})$ is a β-modified proximal contraction, we have

$$0 < \Psi(\check{T}p,\check{T}q) \leq \beta\Psi(\check{T}p,\check{T}q) < \Psi(\check{T}p,\check{T}q), \tag{57}$$

which is a contradiction (as $\check{T}$ is one-to-one mapping on J). Hence, the pair $(\breve{g},\check{T})$ has a unique coincidence best proximity point. □

Corollary 2. *Let $\check{T}$: $J \longrightarrow K$ be a given continuous mapping, where K is a closed subset and J is approximately compact with respect to K with $\check{T}(J_0)\subseteq K_0$. If $\check{T}$ is a $\beta_{\check{T}}$-modified proximal contraction and suppose that*

$$\begin{aligned}&\sup_{m\geq 1}\lim_{i\longrightarrow\infty}\alpha\left(p_i,p_{i+1}\right)\alpha\left(p_i,p_m\right) < \frac{1}{k},\\ &\lim_{n\longrightarrow\infty}\alpha\left(p_n,\check{T}p_{n-1}\right) = 1, \quad \text{where } k \in (0,1),\end{aligned} \tag{58}$$

then there exists a unique best proximity point of $\check{T}$.

Proof. If we take $\breve{g} = I_J$ (the identity mapping over the set J), the remaining proof is same as Theorem 4. □

Example 1. Let $X = \{0,1,2,3,4,5\}$. Consider the function Ψ given as $\Psi(p,p) = 0$ and $\Psi(p,q) = \Psi(q,p)$, where

Ψ	0	1	2	3	4	5
0	0	1/14	1/13	1/15	1/12	1/11
1	1/14	0	2/3	3/4	1/15	4/5
2	1/13	2/3	0	1/9	1/8	1/15
3	1/15	3/4	1/9	0	7/8	8/9
4	1/12	1/15	1/8	7/8	0	1/4
5	1/11	4/5	1/15	8/9	1/4	0

Take α: $X \times X \longrightarrow [1,\infty)$ to be symmetric which is defined as $\alpha(p,q) = 19p + 21q$. It is easy to see that (X,Ψ) is a *controlled metric type space*. Take $J = \{0,1,2\}$ and $K = \{3,4,5\}$. Obviously, $\Psi(J,K) = (1/15)$, $J_0 = J$, and $K_0 = K$. Now, consider $\check{T}$: $J \longrightarrow K$ as follows:

$$\check{T}p = \begin{cases} 3, & \text{if } p = \{0,1\},\\ 5, & \text{if } p = 2.\end{cases} \tag{59}$$

Clearly, $\check{T}(J_0)\subseteq K_0$. Define $\breve{g}$: $J \longrightarrow J$ by

$$\breve{g}p = \begin{cases} 0, & \text{if } p = 0,\\ 1, & \text{if } p = 2,\\ 2, & \text{if } p = 1.\end{cases} \tag{60}$$

We get $J_0\subseteq\breve{g}(J_0)$. Now, we have to show that the pair $(\breve{g},\check{T})$ satisfies

$$\begin{aligned}\Psi(\breve{g}0,\check{T}1) &= \Psi(0,3) = \Psi(J,K),\\ \Psi(\breve{g}1,\check{T}2) &= \Psi(2,5) = \Psi(J,K),\end{aligned} \tag{61}$$

where $u_1 = 0, u_2 = 1, p_1 = 1$, and $p_2 = 2$. Since the pair $(\breve{g},\check{T})$ is a *β-modified proximal contraction:*

$$\Psi(\check{T}0,\check{T}1) \leq \beta\Psi(\check{T}1,\check{T}2), \tag{62}$$

for every $\beta \in [0,1)$, the pair $(\breve{g},\check{T})$ is a *β-modified proximal contraction*. Hence, 0 is the unique *coincidence best proximity point* of $\check{T}$ and $\breve{g}$.

Definition 18. Let $\check{T}$: $J \longrightarrow K$ and $\breve{g}$: $J \longrightarrow J$. A pair of mappings $(\breve{g},\check{T})$ is said to be a β– proximal contraction if there exists $\beta \in [0,1)$ such that

$$\left.\begin{aligned}\Psi\left(\breve{g}u_1,\check{T}p_1\right) &= \Psi(J,K)\\ \Psi\left(\breve{g}u_2,\check{T}p_2\right) &= \Psi(J,K)\end{aligned}\right\}, \quad \text{implies } \Psi\left(\breve{g}u_1,\breve{g}u_2\right) \leq \beta\Psi\left(p_1,p_2\right), \tag{63}$$

for all u_1, u_2, p_1, and p_2 in J.

Definition 19. A mapping $\check{T}$: $J \longrightarrow K$ is said to be a $\beta_{\check{T}}$-proximal contraction if there exists $\beta \in [0,1)$ such that

$$\left.\begin{aligned}\Psi\left(u_1,\check{T}p_1\right) &= \Psi(J,K)\\ \Psi\left(u_2,\check{T}p_2\right) &= \Psi(J,K)\end{aligned}\right\}, \quad \text{implies } \Psi\left(u_1,u_2\right) \leq \beta\Psi\left(p_1,p_2\right), \tag{64}$$

for all u_1, u_2, p_1, and p_2 in J.

Note that, if we take $\breve{g} = I_J$, then every β-proximal contraction is a $\beta_{\check{T}}$-proximal contraction.

Theorem 5. *Let $\check{T}: J \longrightarrow K$ and $\breve{g}: J \longrightarrow J$ be continuous mappings, where K is a closed subset and J is approximately compact with respect to K with $\check{T}(J_0) \subseteq K_0$ and $J_0 \subseteq \breve{g}(J_0)$. Suppose that $\breve{g}$ is an expansive mapping and the pair $(\breve{g}, \check{T})$ is a β-proximal contraction such that*

$$\begin{aligned} &\sup_{m\geq 1} \lim_{i\longrightarrow\infty} \alpha(p_i, p_{i+1})\alpha(p_i, p_m) < \frac{1}{k}, \\ &\lim_{n\longrightarrow\infty} \alpha\left(\breve{g}p_n, \check{T}p_{n-1}\right) = 1, \quad \text{where } k \in (0,1), \end{aligned} \tag{65}$$

then there exists a unique coincidence best proximity point of the pair $(\breve{g}, \check{T})$.

Proof. Let p_0 be an arbitrary element in J_0. Since $\check{T}(J_0)$ is contained in K_0 and J_0 is contained in $\breve{g}(J_0)$, there exists an element p_1 in J_0 such that

$$\Psi\left(\breve{g}p_1, \check{T}p_0\right) = \Psi(J, K). \tag{66}$$

Again, since $\check{T}p_1$ is an element of $\check{T}(J_0)$ which is contained in K_0 and J_0 is contained in $\breve{g}(J_0)$, it follows that there is an element p_2 in J_0, such that

$$\Psi\left(\breve{g}p_2, \check{T}p_1\right) = \Psi(J, K). \tag{67}$$

This process can be continued by selecting p_n in J_0 so that

$$\Psi\left(\breve{g}p_{n+1}, \check{T}p_n\right) = \Psi(J, K). \tag{68}$$

Since the pair $(\breve{g}, \check{T})$ is a β-proximal contraction, we have

$$\Psi\left(\breve{g}p_{n+1}, \breve{g}p_n\right) \leq \beta\Psi(p_n, p_{n-1}). \tag{69}$$

As $\breve{g}$ is an expansive mapping, one writes

$$\Psi(p_{n+1}, p_n) \leq \Psi\left(\breve{g}p_{n+1}, \breve{g}p_n\right) \leq \beta^n\Psi(p_1, p_0), \tag{70}$$

so we have

$$\Psi(p_{n+1}, p_n) \leq \beta^n\Psi(p_1, p_0). \tag{71}$$

We claim that $\{p_n\}$ is a Cauchy sequence. For all natural numbers $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned} \Psi(p_n, p_m) \leq\ & \alpha(p_n, p_{n+1})\Psi(p_n, p_{n+1}) + \alpha(p_{n+1}, p_m)\Psi(p_{n+1}, p_m) \\ \leq\ & \alpha(p_n, p_{n+1})\Psi(p_n, p_{n+1}) + \alpha(p_{n+1}, p_m)\alpha(p_{n+1}, p_{n+2})\Psi(p_{n+1}, p_{n+2}) \\ & + \alpha(p_{n+1}, p_m)\alpha(p_{n+2}, p_m)\Psi(p_{n+2}, p_m) \\ \leq\ & \alpha(p_n, p_{n+1})\Psi(p_n, p_{n+1}) + \alpha(p_{n+1}, p_m)\alpha(p_{n+1}, p_{n+2})\Psi(p_{n+1}, p_{n+2}) \\ & + \alpha(p_{n+1}, p_m)\alpha(p_{n+2}, p_m)\alpha(p_{n+2}, p_{n+3})\Psi(p_{n+2}, p_{n+3}) + \alpha(p_{n+1}, p_m) \\ & \alpha(p_{n+2}, p_m)\alpha(p_{n+3}, p_m)\Psi(p_{n+3}, p_m) \\ \leq\ & \alpha(p_n, p_{n+1})\Psi(p_n, p_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha\left(p_j, p_m\right)\right)\alpha(p_i, p_{i+1})\Psi(p_i, p_{i+1}) \\ & + \prod_{k=n+1}^{m-1}\alpha(p_k, p_m)\Psi(p_{m-1}, p_m) \\ \leq\ & \alpha\left(\check{T}p_n, \check{T}p_{n+1}\right)\beta^n\Psi(p_0, p_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha\left(p_j, p_m\right)\right)\alpha(p_i, p_{i+1})\beta^i\Psi(p_0, p_1) \\ & + \prod_{k=n+1}^{m-1}\alpha(p_k, p_m)\beta^{m-1}\Psi(p_0, p_1) \\ \leq\ & \alpha(p_n, p_{n+1})\beta^n\Psi(p_0, p_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha\left(p_j, p_m\right)\right)\alpha(p_i, p_{i+1})\beta^i\Psi(p_0, p_1) \\ & + \prod_{k=n+1}^{m-1}\alpha(p_k, p_m)\alpha(p_{m-1}, p_m)\beta^{m-1}\Psi(p_0, p_1) \\ =\ & \alpha(p_n, p_{n+1})\beta^n\Psi(p_0, p_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\alpha\left(p_j, p_m\right)\right)\alpha(p_i, p_{i+1})\beta^i\Psi(p_0, p_1) \\ \leq\ & \alpha(p_n, p_{n+1})\beta^n\Psi(p_0, p_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\alpha\left(p_j, p_m\right)\right)\alpha(p_i, p_{i+1})\beta^i\Psi(p_0, p_1). \end{aligned} \tag{72}$$

Assume that

$$S_l = \sum_{i=0}^{l} \left(\prod_{j=0}^{i} \alpha\left(p_j, p_m\right) \right) \alpha\left(p_i, p_{i+1}\right) \beta^i. \tag{73}$$

Then, we obtain

$$\Psi\left(p_n, p_m\right) \le \Psi\left(p_0, p_1\right)\left[\beta^n \alpha\left(p_n, p_{n+1}\right) + \left(S_{m-1} - S_n\right)\right]. \tag{74}$$

Using the ratio test, we have

$$a_i = \prod_{j=0}^{i} \alpha\left(p_j, p_m\right) \alpha\left(p_i, p_{i+1}\right) \beta^i, \quad \text{where } \frac{a_{i+1}}{a_i} < \frac{1}{k}. \tag{75}$$

By taking limit as $n, m \longrightarrow \infty$, (74) becomes

$$\lim_{n \longrightarrow \infty} \Psi\left(p_n, p_m\right) = 0. \tag{76}$$

Therefore, $\{p_n\}$ is a Cauchy sequence in the complete controlled metric type space (X, Ψ); hence, it is convergent to some p in J (as set J is closed). Since $\breve{g}$ and $\check{T}$ are continuous, we have

$$\Psi(\breve{g}p, \check{T}p) = \lim_{n \longrightarrow \infty} \Psi\left(\breve{g}p_{n+1}, \check{T}p_n\right) = \Psi(J, K). \tag{77}$$

Hence, p is the unique coincidence best proximity point of the pair $(\breve{g}, \check{T})$. To prove the uniqueness, suppose that q is another coincidence best proximity point of the pair $(\breve{g}, \check{T})$ such that $p \neq q$. Then,

$$\begin{aligned} \Psi(\breve{g}p, \check{T}p) &= \Psi(J, K), \\ \Psi(\breve{g}q, \check{T}q) &= \Psi(J, K). \end{aligned} \tag{78}$$

Since the pair $(\breve{g}, \check{T})$ is a β-modified proximal contraction, we have

$$\Psi(p, q) \le \Psi(\breve{g}p, \breve{g}q) \le \beta \Psi(p, q) < \Psi(p, q), \tag{79}$$

which is a contradiction. Hence, the pair $(\breve{g}, \check{T})$ has a unique coincidence best proximity point. ☐

Corollary 3. *Let $\check{T}: J \longrightarrow K$ be a continuous mapping, where K is closed subset and J is approximately compact with respect to K with $\check{T}(J_0) \subseteq K_0$. If $\check{T}$ is a $\beta_{\check{T}}$-proximal contraction and suppose that*

$$\begin{aligned} &\sup_{m \ge 1} \lim_{i \longrightarrow \infty} \alpha\left(p_i, p_{i+1}\right) \alpha\left(p_i, p_m\right) < \frac{1}{k}, \\ &\lim_{n \longrightarrow \infty} \alpha\left(p_n, \check{T}p_{n-1}\right) = 1, \quad \text{where } k \in (0, 1), \end{aligned} \tag{80}$$

then there exists a best proximity point of $\check{T}$.

Proof. If we take identity mapping $\breve{g} = I_J$, the remaining proof is same as Theorem 5. ☐

3. Coincidence Best Proximity Points for Geraghty Type Proximal Contractive Mappings

First, we need to define a generalized Geraghty type proximal contractive mapping.

From now and onward, F is a class of all nondecreasing functions $\lambda: [0, \infty) \longrightarrow [0, 1)$ such that for any bounded sequence $\{t_n\}$ of positive real numbers, $\lambda\{t_n\} \longrightarrow 1$ implies $t_n \longrightarrow 0$.

Definition 20. Let (X, Ψ) be a controlled metric type space having a pair of nonempty subsets (J, K) such that J_0 is nonempty. Then, a pair (J, K) has the P-property if and only if

$$\left.\begin{aligned} \Psi\left(p_1, q_1\right) &= \Psi(J, K) \\ \Psi\left(p_2, q_2\right) &= \Psi(J, K) \end{aligned}\right\} \quad \text{implies } \Psi\left(p_1, p_2\right) = \Psi\left(q_1, q_2\right). \tag{81}$$

Definition 21. Let $\check{T}: J \longrightarrow \mathscr{B}(K)$, $\breve{g}: J \longrightarrow J$ are mappings. A pair $(\breve{g}, \check{T})$ is said to be a $\lambda - \mu$-proximal Geraghty contraction if $\mu: J \times J \longrightarrow [0, \infty)$ is such that

$$\left.\begin{aligned} &\mu(p, q) \ge 1 \\ &\mathscr{D}(\breve{g}u, \check{T}p) = \Psi(J, K) \\ &\mathscr{D}(\breve{g}v, \check{T}q) = \Psi(J, K) \end{aligned}\right\} \quad \text{implies that } \mu(p, q)\mathscr{H}(\check{T}p, \check{T}q) \le \lambda(M(u, v, p, q))M(u, v, p, q), \tag{82}$$

where

$$\begin{aligned} M(u, v, p, q) = \max\Bigg\{ &\Psi(\breve{g}p, \breve{g}q), \frac{\mathscr{D}(\breve{g}p, \check{T}p) - \alpha_*(\breve{g}q, \check{T}p)\Psi(J, K)}{\alpha_*(\breve{g}p, \breve{g}q)}, \\ &\mathscr{D}^*(\breve{g}u, \check{T}p), \frac{\mathscr{D}(\breve{g}u, \check{T}q) - \alpha_*(\breve{g}v, \check{T}q)\Psi(J, K)}{\alpha_*(\breve{g}u, \breve{g}v)} \Bigg\}, \end{aligned} \tag{83}$$

for all $u, v, p, q \in J$, where $\lambda \in F$.

Definition 22. A mapping $\check{T}: J \longrightarrow \mathscr{B}(K)$ is said to be a $(\lambda-\mu)_{\check{T}}$-proximal Geraghty contraction if $\mu: J\times J \longrightarrow [0,\infty)$ is such that

$$\left.\begin{array}{l}\mu(p,q)\geq 1\\ \mathscr{D}(u,\check{T}p)=\Psi(J,K)\\ \mathscr{D}(v,\check{T}q)=\Psi(J,K)\end{array}\right\},\quad \text{implies that}\, \mu(p,q)\mathscr{H}(\check{T}p,\check{T}q)\leq\lambda(M(u,v,p,q))M(u,v,p,q), \tag{84}$$

where

$$M(u,v,p,q)=\max\left\{\Psi(p,q),\frac{\Psi(p,\check{T}p)-\alpha_*(q,\check{T}p)\Psi(J,K)}{\alpha_*(p,q)},\right.$$
$$\left.\Psi^*(u,\check{T}p),\frac{\Psi(u,\check{T}q)-\alpha_*(v,\check{T}q)\Psi(J,K)}{\alpha_*(u,v)}\right\}, \tag{85}$$

for all $u,v,p,q\in J$, where $\lambda\in F$.

If we take $\breve{g}=I_J$ (the identity mapping over J), then every $\lambda-\mu$-proximal Geraghty contraction will reduce to a $\lambda-\mu$-generalized proximal Geraghty contraction.

Theorem 6. *Let $\check{T}: J\longrightarrow\mathscr{B}(K)$, $\breve{g}: J\longrightarrow J$, and $\mu: J\times J\longrightarrow[0,+\infty)$ be mappings, where J is a closed subset and the pair (J,K) satisfies the P-property with $\check{T}(J_0)\subseteq K_0$ and $J_0\subseteq\breve{g}(J_0)$. If a pair of continuous mappings $(\breve{g},\check{T})$ is a $\lambda-\mu$-proximal Geraghty contraction, where $\check{T}$ is μ-proximal admissible, then there exist elements $p_0,p_1\in J_0$ such that $\mathscr{D}(\breve{g}p_1,\check{T}p_0)=\Psi(J,K)$ and $\mu(p_0,p_1)\geq 1$. If $\{p_n\}$ is a sequence in J such that $\mu(p_n,p_{n+1})\geq 1$ and suppose that*

$$\sup_{m\geq 1}\lim_{i\longrightarrow\infty}\alpha_*(p_i,p_{i+1})\alpha_*(p_i,p_m)<\frac{1}{k},\quad \text{where}\, k\in(0,1), \tag{86}$$

then the pair $(\breve{g},\check{T})$ has a unique coincidence best proximity point $p^*\in J$.

Proof. From the given condition, there exist $p_0,p_1\in J_0$ such that $\mathscr{D}(\breve{g}p_1,\check{T}p_0)=\Psi(J,K)$ and $\mu(p_0,p_1)\geq 1$. As $\check{T}(J_0)\subseteq K_0$, there exists $p_2\in J_0$ such that $\mathscr{D}(\breve{g}p_2,\check{T}p_1)=\Psi(J,K)$. As $\check{T}$ is μ-proximal admissible, $\mu(p_0,p_1)\geq 1$,

$$\begin{aligned}\mathscr{D}(\breve{g}p_1,\check{T}p_0)&=\Psi(J,K),\\ \mathscr{D}(\breve{g}p_2,\check{T}p_1)&=\Psi(J,K),\end{aligned} \tag{87}$$

using the P-property $\Psi(\breve{g}p_1,\breve{g}p_2)=\mathscr{H}(\check{T}p_0,\check{T}p_1)$. Since the pair $(\breve{g},\check{T})$ is a $\lambda-\mu$-proximal Geraghty contraction with $\mu(p_1,p_2)\geq 1$, we have

$$\Psi(\breve{g}p_1,\breve{g}p_2)\leq\lambda(M(p_0,p_1,p_1,p_2))M(p_0,p_1,p_1,p_2), \tag{88}$$

where

$$\begin{aligned}M(p_0,p_1,p_1,p_2)&\leq\max\left\{\Psi(\breve{g}p_0,\breve{g}p_1),\frac{\mathscr{D}(\breve{g}p_0,\check{T}p_0)-\alpha_*(\breve{g}p_1,\check{T}p_0)\Psi(J,K)}{\alpha_*(\breve{g}p_0,\breve{g}p_1)},\right.\\
&\qquad\left.\mathscr{D}^*(\breve{g}p_1,\check{T}p_0),\frac{\mathscr{D}(\breve{g}p_1,\check{T}p_1)-\alpha_*(\breve{g}p_2,\check{T}p_1)\Psi(J,K)}{\alpha_*(\breve{g}p_1,\breve{g}p_2)}\right\}\\
&\leq\max\left\{\Psi(\breve{g}p_0,\breve{g}p_1),\frac{\alpha_*(\breve{g}p_0,\breve{g}p_1)\Psi(\breve{g}p_0,\breve{g}p_1)+\alpha_*(\breve{g}p_1,\check{T}p_0)\mathscr{D}(\breve{g}p_1,\check{T}p_0)}{\alpha_*(\breve{g}p_0,\breve{g}p_1)}\right.\\
&\qquad-\frac{\alpha_*(\breve{g}p_1,\check{T}p_0)\Psi(J,K)}{\alpha_*(\breve{g}p_0,\breve{g}p_1)},\mathscr{D}(\breve{g}p_1,\check{T}p_0)-\Psi(J,K),\frac{\alpha_*(\breve{g}p_1,\breve{g}p_2)\Psi(\breve{g}p_1,\breve{g}p_2)}{\alpha_*(\breve{g}p_1,\breve{g}p_2)}\\
&\qquad\left.+\frac{\alpha_*(\breve{g}p_2,\check{T}p_1)\mathscr{D}(\breve{g}p_2,\check{T}p_1)-\alpha_*(\breve{g}p_2,\check{T}p_1)\Psi(J,K)}{\alpha_*(\breve{g}p_1,\breve{g}p_2)}\right\}\\
&\leq\max\left\{\Psi(\breve{g}p_0,\breve{g}p_1),\Psi(\breve{g}p_0,\breve{g}p_1),0,\Psi(\breve{g}p_1,\breve{g}p_2)\right\},\end{aligned} \tag{89}$$

and we have

$$M(p_0, p_1, p_1, p_2) \leq \max\{\Psi(\breve{g}p_0, \breve{g}p_1), \Psi(\breve{g}p_1, \breve{g}p_2)\}. \quad (90)$$

If

$$\max\{\Psi(\breve{g}p_0, \breve{g}p_1), \Psi(\breve{g}p_1, \breve{g}p_2)\} = \Psi(\breve{g}p_1, \breve{g}p_2), \quad (91)$$

then inequality (88) becomes

$$\Psi(\breve{g}p_1, \breve{g}p_2) \leq \lambda(\Psi(\breve{g}p_1, \breve{g}p_2))\Psi(\breve{g}p_1, \breve{g}p_2), \quad (92)$$

which is a contradiction. So, we can conclude that

$$\Psi(\breve{g}p_1, \breve{g}p_2) \leq \lambda(\Psi(\breve{g}p_0, \breve{g}p_1))\Psi(\breve{g}p_0, \breve{g}p_1). \quad (93)$$

Further, by the fact that $\check{T}(J_0) \subseteq K_0$, there exists $p_3 \in J_0$ such that $\mathscr{D}(\breve{g}p_3, \check{T}p_2) = \Psi(J, K)$. As $\check{T}$ is μ-proximal admissible mapping, where $\mu(p_2, p_3) \geq 1$,

$$\begin{aligned} \mathscr{D}(\breve{g}p_2, \check{T}p_1) &= \Psi(J, K), \\ \mathscr{D}(\breve{g}p_3, \check{T}p_2) &= \Psi(J, K), \end{aligned} \quad (94)$$

using the P-Property, we have $\Psi(\breve{g}p_2, \breve{g}p_3) = \mathscr{H}(\check{T}p_1, \check{T}p_2)$. Since the pair $(\breve{g}, \check{T})$ is a $\lambda - \mu$-proximal Geraghty mapping with $\mu(p_2, p_3) \geq 1$, one writes

$$\Psi(\breve{g}p_2, \breve{g}p_3) \leq \lambda(M(p_2, p_3, p_1, p_2))M(p_2, p_3, p_1, p_2), \quad (95)$$

where

$$\begin{aligned} M(p_2, p_3, p_1, p_2) \leq \max\Bigg\{ & \Psi(\breve{g}p_1, \breve{g}p_2), \frac{\mathscr{D}(\breve{g}p_1, \check{T}p_1) - \alpha_*(\breve{g}p_2, \check{T}p_1)\Psi(J, K)}{\alpha_*(\breve{g}p_1, \breve{g}p_2)}, \\ & \mathscr{D}^*(\breve{g}p_2, \check{T}p_1), \frac{\mathscr{D}(\breve{g}p_2, \check{T}p_2) - \alpha_*(\breve{g}p_3, \check{T}p_2)\Psi(J, K)}{\alpha_*(\breve{g}p_2, \breve{g}p_3)} \Bigg\} \\ \leq \max\Bigg\{ & \Psi(\breve{g}p_1, \breve{g}p_2), \frac{\alpha_*(\breve{g}p_1, \breve{g}p_2)\Psi(\breve{g}p_1, \breve{g}p_2) + \alpha_*(\breve{g}p_2, \check{T}p_1)\mathscr{D}(\breve{g}p_2, \check{T}p_1)}{\alpha_*(\breve{g}p_1, \breve{g}p_2)} \\ & - \frac{\alpha_*(\breve{g}p_2, \check{T}p_1)\Psi(J, K)}{\alpha_*(\breve{g}p_1, \breve{g}p_2)}, \mathscr{D}(\breve{g}p_2, \check{T}p_1) - \Psi(J, K), \frac{\alpha_*(\breve{g}p_2, \breve{g}p_3)\Psi(\breve{g}p_2, \breve{g}p_3)}{\alpha_*(\breve{g}p_2, \breve{g}p_3)} \\ & + \frac{\alpha_*(\breve{g}p_3, \check{T}p_2)\mathscr{D}(\breve{g}p_3, \check{T}p_2) - \alpha_*(\breve{g}p_3, \check{T}p_2)\Psi(J, K)}{\alpha_*(\breve{g}p_2, \breve{g}p_3)} \Bigg\} \\ \leq \max\{ & \Psi(\breve{g}p_1, \breve{g}p_2), \Psi(\breve{g}p_1, \breve{g}p_2), 0, \Psi(\breve{g}p_2, \breve{g}p_3)\}, \end{aligned} \quad (96)$$

and we have

$$M(p_2, p_3, p_1, p_2) \leq \max\{\Psi(\breve{g}p_1, \breve{g}p_2), \Psi(\breve{g}p_2, \breve{g}p_3)\}. \quad (97)$$

If

$$\max\{\Psi(\breve{g}p_1, \breve{g}p_2), \Psi(\breve{g}p_2, \breve{g}p_3)\} = \Psi(\breve{g}p_2, \breve{g}p_3), \quad (98)$$

then inequality (95) becomes

$$\Psi(\breve{g}p_2, \breve{g}p_3) \leq \lambda(\Psi(\breve{g}p_2, \breve{g}p_3))\Psi(\breve{g}p_2, \breve{g}p_3), \quad (99)$$

which is a contradiction. Thus,

$$\Psi(\breve{g}p_2, \breve{g}p_3) \leq \lambda(\Psi(\breve{g}p_1, \breve{g}p_2))\Psi(\breve{g}p_1, \breve{g}p_2). \quad (100)$$

Similarly, we can construct a sequence $\{p_n\} \subseteq J_0$, where $\mu(p_n, p_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \mathscr{D}(\breve{g}p_n, \check{T}p_{n-1}) &= \Psi(J, K), \\ \mathscr{D}(\breve{g}p_{n+1}, \check{T}p_n) &= \Psi(J, K), \end{aligned} \quad (101)$$

using the P-property $\Psi(\breve{g}p_n, \breve{g}p_{n+1}) = \mathscr{H}(\check{T}p_n, \check{T}p_{n-1})$. Since the pair $(\breve{g}, \check{T})$ is a $\lambda - \mu$-proximal Geraghty contraction with $\mu(p_n, p_{n+1}) \geq 1$, we get

$$\Psi(\breve{g}p_n, \breve{g}p_{n+1}) \leq \lambda(M(p_n, p_{n+1}, p_{n-1}, p_n))M(p_n, p_{n+1}, p_{n-1}, p_n), \quad (102)$$

where

$$M(p_n, p_{n+1}, p_{n-1}, p_n) \leq \max\left\{\Psi(\breve{g}p_{n-1}, \breve{g}p_n), \frac{\mathscr{D}(\breve{g}p_{n-1}, \check{T}p_{n-1}) - \alpha_*(\breve{g}p_n, \check{T}p_{n-1})\Psi(J,K)}{\alpha_*(\breve{g}p_{n-1}, \breve{g}p_n)},\right.$$
$$\left.\mathscr{D}^*(\breve{g}p_n, \check{T}p_{n-1}), \frac{\mathscr{D}(\breve{g}p_n, \check{T}p_n) - \alpha_*(\breve{g}p_{n+1}, \check{T}p_n)\Psi(J,K)}{\alpha_*(\breve{g}p_n, \breve{g}p_{n+1})}\right\}$$
$$\leq \max\left\{\Psi(\breve{g}p_{n-1}, \breve{g}p_n), \frac{\alpha_*(\breve{g}p_{n-1}, \breve{g}p_n)\Psi(\breve{g}p_{n-1}, \breve{g}p_n) + \alpha_*(\breve{g}p_n, \check{T}p_{n-1})\mathscr{D}(\breve{g}p_n, \check{T}p_{n-1})}{\alpha_*(\breve{g}p_{n-1}, \breve{g}p_n)}\right.$$
$$-\frac{\alpha_*(\breve{g}p_n, \check{T}p_{n-1})\Psi(J,K)}{\alpha_*(\breve{g}p_{n-1}, \breve{g}p_n)}\mathscr{D}(\breve{g}p_n, \check{T}p_{n-1}) - \Psi(J,K)$$
$$\left.\frac{\alpha_*(\breve{g}p_n, \breve{g}p_{n+1})\Psi(\breve{g}p_n, \breve{g}p_{n+1}) + \alpha_*((gp_{n+1}, \check{T}p_n)\mathscr{D}(\breve{g}p_{n+1}, \check{T}p_n)}{\alpha_*(\breve{g}p_n, \breve{g}p_{n+1})} - \frac{\alpha_*(\breve{g}p_{n+1}, \check{T}p_n)\Psi(J,K)}{\alpha_*(\breve{g}p_n, \breve{g}p_{n+1})}\right\}$$
$$\leq \max\{\Psi(\breve{g}p_{n-1}, \breve{g}p_n), \Psi(\breve{g}p_{n-1}, \breve{g}p_n), 0, \Psi(\breve{g}p_n, \breve{g}p_{n+1})\}. \tag{103}$$

After simplification, we have

$$M(p_n, p_{n+1}, p_{n-1}, p_n) \leq \max\{\Psi(\breve{g}p_{n-1}, \breve{g}p_n), \Psi(\breve{g}p_n, \breve{g}p_{n+1})\}. \tag{104}$$

If

$$\max\{\Psi(\breve{g}p_{n-1}, \breve{g}p_n), \Psi(\breve{g}p_n, \breve{g}p_{n+1})\} = \Psi(\breve{g}p_n, \breve{g}p_{n+1}), \tag{105}$$

then inequality (102) becomes

$$\Psi(\breve{g}p_n, \breve{g}p_{n+1}) \leq \lambda(\Psi(\breve{g}p_n, \breve{g}p_{n+1}))\Psi(\breve{g}p_n, \breve{g}p_{n+1}), \tag{106}$$

which is a contradiction. So, we conclude that

$$\Psi(\breve{g}p_n, \breve{g}p_{n+1}) \leq \lambda(\Psi(\breve{g}p_{n-1}, \breve{g}p_n))\Psi(\breve{g}p_{n-1}, \breve{g}p_n). \tag{107}$$

Further,

$$\Psi(\breve{g}p_n, \breve{g}p_{n+1}) \leq \lambda(\Psi(\breve{g}p_{n-1}, \breve{g}p_n))\Psi(\breve{g}p_{n-1}, \breve{g}p_n) \leq \Psi(\breve{g}p_{n-1}, \breve{g}p_n), \tag{108}$$

which shows that $\{\Psi(\breve{g}p_n, \breve{g}p_{n+1})\}$ is a decreasing sequence. Since $\lambda \in F$, from (108), we have

$$\lambda(\Psi(\breve{g}p_n, \breve{g}p_{n+1})) \leq \lambda(\Psi(\breve{g}p_{n-1}, \breve{g}p_n)),$$
$$\lambda(\Psi(\breve{g}p_{n-1}, \breve{g}p_n)) \leq \lambda(\Psi(\breve{g}p_{n-2}, \breve{g}p_{n-1})). \tag{109}$$

Continuing on the same lines, we can write

$$\lambda(\Psi(\breve{g}p_{n-1}, \breve{g}p_n)) \leq \lambda(\Psi(\breve{g}p_{n-2}, \breve{g}p_{n-1})) \leq \cdots \leq \lambda(\Psi(\breve{g}p_0, \breve{g}p_1)). \tag{110}$$

Using inequality (107),

$$\Psi(\breve{g}p_{n-1}, \breve{g}p_n) \leq \lambda(\Psi(\breve{g}p_{n-2}, \breve{g}p_{n-1}))\Psi(\breve{g}p_{n-2}, \breve{g}p_{n-1}). \tag{111}$$

From inequalities (107) and (111), we have

$$\Psi(\breve{g}p_n, \breve{g}p_{n+1}) \leq \lambda(\Psi(\breve{g}p_{n-1}, \breve{g}p_n))\Psi(\breve{g}p_{n-1}, \breve{g}p_n)$$
$$\leq \lambda(\Psi(\breve{g}p_{n-1}, \breve{g}p_n))\lambda(\Psi(\breve{g}p_{n-2}, \breve{g}p_{n-1}))$$
$$\cdot \Psi(\breve{g}p_{n-2}, \breve{g}p_{n-1}). \tag{112}$$

Following on similar lines, we have

$$\Psi(\breve{g}p_n, \breve{g}p_{n+1}) \leq \lambda(\Psi(\breve{g}p_{n-1}, \breve{g}p_n))\lambda(\Psi(\breve{g}p_{n-2}, \breve{g}p_{n-1})), \ldots, \lambda(\Psi(\breve{g}p_0, \breve{g}p_1))\Psi(\breve{g}p_0, \breve{g}p_1)$$
$$= \lambda^n(\Psi(\breve{g}p_0, \breve{g}p_1))\Psi(\breve{g}p_0, \breve{g}p_1). \tag{113}$$

We deduce

$$\Psi(\breve{g}p_n, \breve{g}p_{n+1}) \leq \lambda^n(\Psi(\breve{g}p_0, \breve{g}p_1))\Psi(\breve{g}p_0, \breve{g}p_1). \tag{114}$$

From (108), suppose that $\Psi(\breve{g}p_{n-1}, \breve{g}p_n) > 0$, so we can conclude

$$\frac{\Psi(\breve{g}p_n, \breve{g}p_{n+1})}{\Psi(\breve{g}p_{n-1}, \breve{g}p_n)} \leq \lambda(\Psi(\breve{g}p_{n-1}, \breve{g}p_n)) \leq 1, \quad \text{for all } n \geq 1. \tag{115}$$

Let $l = \lim_{n \longrightarrow +\infty} \Psi(\breve{g}p_{n-1}, \breve{g}p_n)$. Using equation (108) and letting $n \longrightarrow +\infty$, we obtain that

$$\frac{l}{l} = 1 \le \lim_{n\longrightarrow+\infty} \lambda\left(\Psi\left(\breve{g}p_{n-1}, \breve{g}p_n\right)\right) \le 1. \tag{116}$$

Thus, $\lim_{n\longrightarrow+\infty}\Psi\left(\breve{g}p_{n-1}, \breve{g}p_n\right) = 1$. Using the definition of λ, we conclude that

$$\lim_{n\longrightarrow+\infty} \Psi\left(\breve{g}p_{n-1}, \breve{g}p_n\right) = 0. \tag{117}$$

Now, we have to show that $\left\{\breve{g}p_n\right\}$ is a Cauchy sequence. For all natural numbers $n, m \in \mathbb{N}$ with $n < m$, we have

$$\begin{aligned}
\Psi\left(\breve{g}p_n, \breve{g}p_m\right) &\le \alpha_*\left(\breve{g}p_n, \breve{g}p_{n+1}\right)\Psi\left(\breve{g}p_n, \breve{g}p_{n+1}\right) + \alpha_*\left(\breve{g}p_{n+1}, \breve{g}p_m\right)\Psi\left(\breve{g}p_{n+1}, \breve{g}p_m\right)\\
&\le \alpha_*\left(\breve{g}p_n, \breve{g}p_{n+1}\right)\Psi\left(\breve{g}p_n, \breve{g}p_{n+1}\right) + \alpha_*\left(\breve{g}p_{n+1}, \breve{g}p_m\right)\alpha_*\left(\breve{g}p_{n+1}, \breve{g}p_{n+2}\right)\Psi\left(\breve{g}p_{n+1}, \breve{g}p_{n+2}\right)\\
&\quad + \alpha_*\left(\breve{g}p_{n+1}, \breve{g}p_m\right)\alpha_*\left(\breve{g}p_{n+2}, \breve{g}p_m\right)\Psi\left(\breve{g}p_{n+2}, \breve{g}p_m\right)\\
&\le \alpha_*\left(\breve{g}p_n, \breve{g}p_{n+1}\right)\Psi\left(\breve{g}p_n, \breve{g}p_{n+1}\right) + \alpha_*\left(\breve{g}p_{n+1}, \breve{g}p_m\right)\alpha_*\left(\breve{g}p_{n+1}, \breve{g}p_{n+2}\right)\Psi\left(\breve{g}p_{n+1}, \breve{g}p_{n+2}\right)\\
&\quad + \alpha_*\left(\breve{g}p_{n+1}, \breve{g}p_m\right)\alpha_*\left(\left(gp_{n+2}, \breve{g}p_m\right)\alpha_*\left(\breve{g}p_{n+2}, \breve{g}p_{n+3}\right)\Psi\left(\breve{g}p_{n+2}, \breve{g}p_{n+3}\right) + \alpha_*\left(\breve{g}p_{n+1}, \breve{g}p_m\right)\right.\\
&\quad \alpha_*\left(\breve{g}p_{n+2}, \breve{g}p_m\right)\alpha_*\left(\breve{g}p_{n+3}, \breve{g}p_m\right)\Psi\left(\breve{g}p_{n+3}, \breve{g}p_m\right)\\
&\le \alpha_*\left(\breve{g}p_n, \breve{g}p_{n+1}\right)\Psi\left(\breve{g}p_n, \breve{g}p_{n+1}\right)\\
&\quad + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} \alpha_*\left(\breve{g}p_j, \breve{g}p_m\right)\right)\alpha_*\left(\breve{g}p_i, \breve{g}p_{i+1}\right)\Psi\left(\breve{g}p_i, \breve{g}p_{i+1}\right) + \prod_{k=n+1}^{m-1} \alpha_*\left(\breve{g}p_k, \breve{g}p_m\right)\Psi\left(\breve{g}p_{m-1}, \breve{g}p_m\right)\\
&\le \alpha_*\left(\breve{g}p_n, \breve{g}p_{n+1}\right)\lambda^n\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\Psi\left(\breve{g}p_0, \breve{g}p_1\right) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} \alpha_*\left(\breve{g}p_j, \breve{g}p_m\right)\right)\alpha_*\left(\breve{g}p_i, \breve{g}p_{i+1}\right)\\
&\quad \lambda^i\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\Psi\left(\breve{g}p_0, \breve{g}p_1\right) + \prod_{k=n+1}^{m-1} \alpha_*\left(\breve{g}p_k, \breve{g}p_m\right)\lambda^{m-1}\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\\
&\le \alpha_*\left(\breve{g}p_n, \breve{g}p_{n+1}\right)\lambda^n\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\\
&\quad + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i} \alpha_*\left(\breve{g}p_j, \breve{g}p_m\right)\right)\alpha_*\left(\breve{g}p_i, \breve{g}p_{i+1}\right)\lambda\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\\
&\quad + \prod_{k=n+1}^{m-1} \alpha_*\left(\breve{g}p_k, \breve{g}p_m\right)\alpha_*\left(\breve{g}p_{m-1}, \breve{g}p_m\right)\\
&\quad \lambda^{m-1}\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\Psi\left(\breve{g}p_0, \breve{g}p_1\right) = \alpha_*\left(\breve{g}p_n, \breve{g}p_{n+1}\right)\lambda^n\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\\
&\quad + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i} \alpha_*\left(\breve{g}p_j, \breve{g}p_m\right)\right)\alpha_*\left(\breve{g}p_i, \breve{g}p_{i+1}\right)\lambda^i\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\\
&\le \alpha_*\left(\breve{g}p_n, \breve{g}p_{n+1}\right)\lambda^n\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\\
&\quad + \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i} \alpha_*\left(\breve{g}p_j, \breve{g}p_m\right)\right)\alpha_*\left(\breve{g}p_i, \breve{g}p_{i+1}\right)\\
&\quad \lambda^i\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\Psi\left(\breve{g}p_0, \breve{g}p_1\right).
\end{aligned} \tag{118}$$

Assume that

$$S_l = \sum_{i=0}^{l}\left(\prod_{j=0}^{i} \alpha_*\left(\breve{g}p_j, \breve{g}p_m\right)\right)\alpha_*\left(\breve{g}p_i, \breve{g}p_{i+1}\right)\lambda^i\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right). \tag{119}$$

Then, we obtain

$$\begin{aligned}
\Psi\left(\breve{g}p_n, \breve{g}p_m\right) \le \Psi\left(\breve{g}p_0, \breve{g}p_1\right)&\left[\lambda^n\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right)\alpha_*\left(\breve{g}p_n, \breve{g}p_{n+1}\right)\right.\\
&\left. + \left(S_{m-1} - S_n\right)\right].
\end{aligned} \tag{120}$$

Using the ratio test, we have

$$a_i = \prod_{j=0}^{i} \alpha_*\left(\breve{g}p_j, \breve{g}p_m\right)\alpha_*\left(\breve{g}p_i, \breve{g}p_{i+1}\right)\lambda^i\left(\Psi\left(\breve{g}p_0, \breve{g}p_1\right)\right),$$
$$\text{where } \frac{a_{i+1}}{a_i} < \frac{1}{k}. \tag{121}$$

Taking limit as $n, m \longrightarrow \infty$, inequality (120) becomes

$$\lim_{n,m\longrightarrow\infty} \Psi\left(\breve{g}p_n, \breve{g}p_m\right) = 0. \tag{122}$$

This implies that $\left\{\breve{g}p_n\right\}$ is a Cauchy sequence in the complete controlled metric type space (X, Ψ); hence, it is convergent and suppose that it converges to some p^* in $J_0 \subseteq J$ (as set J is closed), which assures that the sequence $\{p_n\} \subseteq J_0$ since $p_n \longrightarrow p^*$. As $(\breve{g}, \check{T})$ is a pair of continuous mappings, one writes

$$\mathscr{D}\left(\breve{g}p^*, \check{T}p^*\right) = \Psi(J, K). \tag{123}$$

Therefore, p^* is a coincidence best proximity point of the pair $(\breve{g}, \check{T})$.

For uniqueness, suppose that there are two distinct coincidence best proximity points of $(\breve{g}, \check{T})$ such that $p^* \neq q^*$. Thus, $s = \Psi(p^*, q^*) > 0$. Since $\Psi(\breve{g}p^*, \check{T}p^*) = \Psi(\breve{g}q^*, \check{T}q^*) = \Psi(J, K)$, using the P-property, we conclude that $s = \mathscr{H}(\check{T}p^*, \check{T}q^*)$. Since the pair $(\breve{g}, \check{T})$ is a $\lambda - \mu$-proximal Geraghty contraction, we obtain $s \leq \lambda(s)s$. Thus, $\lambda(s) \geq 1$. Since $\lambda(s) \geq 1$, we conclude that $\lambda(s) = 1$ and therefore $s = 0$, which is contradiction. □

Example 2. Let $X = \{0, 1, 2, 3, 4, 5\}$ be endowed with the function Ψ given as $\Psi(p, q) = \Psi(q, p)$ and $\Psi(p, p) = 0$, where

Ψ	0	1	2	3	4	5
0	0	1/2	1/3	1/10	1/5	1/6
1	1/2	0	1/4	2/3	1/10	3/4
2	1/3	1/4	0	6/7	7/8	1/10
3	1/10	2/3	6/7	0	1/2	1/3
4	1/5	1/10	7/8	1/2	0	1/4
5	1/6	3/4	1/10	1/3	1/4	0

Take α: $X \times X \longrightarrow [1, \infty)$ to be symmetric and defined as $\alpha(p, q) = 16p + 18q$. It is easy to see that (X, Ψ) is controlled type metric space. Suppose $J = \{0, 1, 2\}$ and $K = \{3, 4, 5\}$. After a simple calculation, $\Psi(J, K) = (1/10)$, the P-property is satisfied, $J_0 = J$, and $K_0 = K$. Consider

$$\check{T}p = \begin{cases} 3, & \text{if } p = 2, \\ \{3, 4\}, & \text{if } p = \{0, 1\}, \end{cases}$$
$$\breve{g}p = \begin{cases} 0, & \text{if } p = 0, \\ 1, & \text{if } p = 2, \\ 2, & \text{if } p = 1. \end{cases} \tag{124}$$

Clearly, $\check{T}(J_0) \subseteq K_0$ and $J_0 \subseteq \breve{g}(J_0)$. Now, we have to show that the pair $(\breve{g}, \check{T})$ is a $\lambda - \mu$-proximal Geraghty contraction:

$$\mu(p, q)\mathscr{H}(\check{T}p, \check{T}q) \leq \lambda(M(u, v, p, q))M(u, v, p, q), \tag{125}$$

for all $u, v, p, q \in J$ and for the function μ: $J \times J \longrightarrow [0, +\infty)$ is defined by

$$\mu(p, q) = \Psi(p, q) + 1. \tag{126}$$

Hence,

$$\begin{aligned} &\mathscr{D}(\breve{g}0, \check{T}2) = \mathscr{D}(1, 3) = \Psi(J, K), \\ &\mathscr{D}(\breve{g}2, \check{T}1) = \mathscr{D}(1, \{3, 4\}) = \Psi(J, K). \end{aligned} \tag{127}$$

After simple calculations, $\mathscr{H}(\check{T}p, \check{T}q) = \mathscr{H}(3, \{3, 4\}) = 0$, $\mu(p, q) = \Psi(3, \{3, 4\}) + 1 = 1$, and

$$\begin{aligned} M(0, 2, 2, 1) &= \max\left\{\Psi(\breve{g}2, \breve{g}1), \frac{\mathscr{D}(\breve{g}2, \check{T}2) - \alpha_*(\breve{g}1, \check{T}2)(1/10)}{\alpha_*(\breve{g}2, \breve{g}1)}\right. \\ &\qquad \left.\mathscr{D}^*(\breve{g}0, \check{T}2), \frac{\mathscr{D}(\breve{g}0, \check{T}1) - \alpha_*(\breve{g}2, \check{T}1)(1/10)}{\alpha_*(\breve{g}0, \breve{g}2)}\right\} \\ &= \max\left\{\Psi(1, 2), \frac{\mathscr{D}(1, 3) - \alpha_*(2, 3)(1/10)}{\alpha_*(1, 2)}\right. \\ &\qquad \left.\mathscr{D}^*(0, 3), \frac{\mathscr{D}(0, \{3, 4\}) - \alpha_*(1, \{3, 4\})(1/10)}{\alpha_*(0, 1)}\right\} \\ &= \max\left\{\frac{1}{4}, \frac{-238}{1560}, 0, \frac{-69}{180}\right\} = \frac{1}{4}. \end{aligned} \tag{128}$$

Now, we have to show that the pair $(\breve{g}, \check{T})$ is a $\lambda-\mu$-proximal Geraghty contraction:

$$\begin{aligned}(1)(0) &\leq \lambda(M(u,v,p,q))\left(\frac{1}{4}\right)\\ &\leq \lambda(M(u,v,p,q))\left(\frac{1}{4}\right),\end{aligned} \tag{129}$$

and for every $\lambda\colon [0,\infty) \longrightarrow [0,1]$, the pair $(\breve{g}, \check{T})$ is a $\lambda-\mu$-proximal Geraghty contraction. Hence, 0 is the unique coincidence point of the pair of mappings $(\breve{g}, \check{T})$.

Corollary 4. *Let $\check{T}\colon J \longrightarrow \mathscr{B}(K)$ be a continuous mapping, where J is a closed subset and the pair (J,K) satisfies the P-property with $\check{T}(J_0)\subseteq K_0$. If $\check{T}$ is a $(\lambda-\mu)_{\check{T}}$-proximal Geraghty contraction, where $\check{T}$ is μ-proximal admissible, then there exist elements $p_0, p_1 \in J_0$ such that $\mathscr{D}(p_1, \check{T}p_0) = \Psi(J,K)$ and $\mu(p_0,p_1)\geq 1$. If $\{p_n\}$ is a sequence in J such that $\mu(p_n,p_{n+1})\geq 1$ and suppose that*

$$\sup_{m\geq 1}\lim_{i\longrightarrow\infty}\alpha_*(p_i,p_{i+1})\alpha_*(p_i,p_m) < \frac{1}{k}, \quad \text{where } k\in(0,1), \tag{130}$$

then $\check{T}$ has a unique best proximity point $p^* \in J$.

Proof. If we take $\breve{g} = I_J$ (an identity mapping over J), the remaining proof is same as Theorem 6. □

Definition 23. Let $\check{T}\colon J \longrightarrow K$, $\breve{g}\colon J \longrightarrow J$, and $\mu\colon J\times J \longrightarrow [0,+\infty)$ be mappings. A pair of mappings $(\breve{g}, \check{T})$ is said to be a $\lambda-\mu$-modified proximal Geraghty contraction if

$$\left.\begin{aligned}&\mu(p,q)\geq 1\\ &\Psi(\breve{g}u,\check{T}p) = \Psi(J,K)\\ &\Psi(\breve{g}v,\check{T}q) = \Psi(J,K)\end{aligned}\right\}, \quad \text{implies } \mu(p,q)\Psi(\check{T}p,\check{T}q)\leq \lambda(M(u,v,p,q))M(u,v,p,q), \tag{131}$$

where

$$\begin{aligned}M(u,v,p,q) = \max\Bigg\{&\Psi(\breve{g}p,\breve{g}q), \frac{\Psi(\breve{g}p,\check{T}p)-\alpha(\breve{g}q,\check{T}p)\Psi(J,K)}{\alpha(\breve{g}p,\breve{g}q)},\\ &\Psi^*(\breve{g}u,\check{T}p), \frac{\Psi(\breve{g}u,\check{T}q)-\alpha(\breve{g}v,\check{T}q)\Psi(J,K)}{\alpha(\breve{g}u,\breve{g}v)}\Bigg\},\end{aligned} \tag{132}$$

for all $u,v,p,q\in J$, where $\lambda\in F$.

Definition 24. Let $\check{T}\colon J\longrightarrow K$ and $\mu\colon J\times J\longrightarrow[0,+\infty)$ be mappings. A mapping $\check{T}$ is said to be a *$(\lambda-\mu)_{\check{T}}$-modified proximal Geraghty contraction* if

$$\left.\begin{aligned}&\mu(p,q)\geq 1\\ &\Psi(u,\check{T}p) = \Psi(J,K)\\ &\Psi(v,\check{T}q) = \Psi(J,K)\end{aligned}\right\}, \quad \text{implies } \mu(p,q)\Psi(\check{T}p,\check{T}q)\leq \lambda(M(u,v,p,q))M(u,v,p,q), \tag{133}$$

where

$$\begin{aligned}M(u,v,p,q) = \max\Bigg\{&\Psi(p,q), \frac{\Psi(p,\check{T}p)-\alpha(q,\check{T}p)\Psi(J,K)}{\alpha(p,q)},\\ &\Psi^*(u,\check{T}p), \frac{\Psi(u,\check{T}q)-\alpha(v,\check{T}q)\Psi(J,K)}{\alpha(u,v)}\Bigg\},\end{aligned} \tag{134}$$

for all $u, v, p, q \in J$, where $\lambda \in F$.

Note that, if we take $\breve{g} = I_J$ (an identity mapping over J), then every $\lambda - \mu$-modified proximal Geraghty contraction will reduce to a $(\lambda - \mu)_{\breve{T}}$-modified proximal Geraghty contraction.

Theorem 7. *Let $\breve{T}: J \longrightarrow K$ and $\breve{g}: J \longrightarrow J$ be continuous mappings, where J is closed subset and the pair (J, K) satisfies the P-property with $\breve{T}(J_0) \subseteq K_0$ and $J_0 \subseteq \breve{g}(J_0)$. If the pair of mappings $(\breve{g}, \breve{T})$ is a $\lambda - \mu$-modified proximal Geraghty contraction, where $\breve{T}$ is a μ-proximal admissible, then there exist elements $p_0, p_1 \in J_0$ such that $\Psi(\breve{g}p_1, \breve{T}p_0) = \Psi(J, K)$ and $\mu(p_0, p_1) \geq 1$. If $\{p_n\}$ is a sequence in J such that $\mu(p_n, p_{n+1}) \geq 1$ and suppose that*

$$\sup_{m\geq 1} \lim_{i\longrightarrow\infty} \alpha(p_i, p_{i+1})\alpha(p_i, p_m) < \frac{1}{k}, \quad \text{where } k \in (0,1), \tag{135}$$

then the pair $(\breve{g}, \breve{T})$ has a unique coincidence best proximity point $p^* \in J$.

Proof. It is a simple consequence of Theorem 6. □

Corollary 5. *Let $\breve{T}: J \longrightarrow K$ be a continuous mapping, where J is closed subset and the pair (J, K) satisfies the P-property with $\breve{T}(J_0) \subseteq K_0$. If $\breve{T}$ is a $(\lambda - \mu)_{\breve{T}}$-modified proximal Geraghty contraction, where $\breve{T}$ is a μ-proximal admissible, then there exist elements $p_0, p_1 \in J_0$ such that $\Psi(\breve{g}p_1, \breve{T}p_0) = \Psi(J, K)$ and $\mu(p_0, p_1) \geq 1$. If $\{p_n\}$ is a sequence in J such that $\mu(p_n, p_{n+1}) \geq 1$ and suppose that*

$$\sup_{m\geq 1} \lim_{i\longrightarrow\infty} \alpha(p_i, p_{i+1})\alpha(p_i, p_m) < \frac{1}{k}, \quad \text{where } k \in (0,1), \tag{136}$$

then the pair $(\breve{g}, \breve{T})$ has a unique best proximity point $p^* \in J$.

Proof. If we take $\breve{g} = I_J$, the remaining proof is same as Theorem 7. □

4. Conclusion

In our paper, we ensured the existence of some best proximity point results via the multivalued concept in controlled metric spaces. To our knowledge, we are the first who worked on best proximity points for the class of multivalued mappings in this setting. We open the door for new perspectives when dealing with new generalized multivalued (proximal) contractions.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest concerning the publication of this article.

Authors' Contributions

All authors contributed equally and significantly for writing this article. All authors read and approved the final manuscript.

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Research Article

Multiple-Sets Split Common Fixed-Point Problems for Demicontractive Mappings

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In this paper, we are concerned with the multiple-sets split common fixed-point problems whenever the involved mappings are demicontractive. We first study several properties of demicontractive mappings and particularly their connection with directed mappings. By making use of these properties, we propose some new iterative methods for solving multiple-sets split common fixed-point problems, as well as multiple-sets spit feasibility problems. Under mild conditions, we establish their weak convergence of the proposed methods.

1. Introduction

The split common fixed-point problem (SCFP) requires finding an element in a fixed-point set such that its image under a linear transformation belongs to another fixed-point set. Formally, it consists in finding $x \in H_1$ such that

$$x \in F(U), Ax \in F(T), \tag{1}$$

where $A: H_1 \longrightarrow H_2$ is a bounded linear mapping from a Hilbert space H_1 into another Hilbert space H_2, and $F(U)$ and $F(T)$ are respectively the fixed-point sets of nonlinear mappings $U: H_1 \longrightarrow H_1$ and $T: H_2 \longrightarrow H_2$. Specially, if U and T are both metric projections, then problem (1) is reduced to the well-known split feasibility problem (SFP) [1]. Actually, the SFP can be formulated as finding $x \in H_1$ such that

$$x \in C, Ax \in Q, \tag{2}$$

where $C \subseteq H_1$ and $Q \subseteq H_2$ are nonempty closed convex sets, and mapping A is as above. These two problems recently have been extensively investigated since they play an important role in various areas including signal processing and image reconstruction [2–6].

We assume throughout the paper that problem (1) is consistent, which means that its solution set is nonempty. Censor and Segal [7] studied problem (1) when U and T are directed mappings. In this situation, they proposed the following method:

$$x_{n+1} = U\left[x_n - \tau_n A^* (I - T)Ax_n\right], \tag{3}$$

where A^* is the conjugate of A, I stands for the identity mapping, and τ_n is a properly chosen stepsize. It is shown that if τ_n is chosen in $(0, 2/\|A\|^2)$, then (7) converges weakly to a solution of (1). Subsequently, this result was extended to more general cases (see, e.g., [8–17]). Since the choice of the stepsize is related to $\|A\|$, thus to implement (7), one has to compute (or at least estimate) the norm $\|A\|$, which is generally not easy in practice. A way avoiding this is to adopt variable stepsize which ultimately has no relation with $\|A\|$ [9, 10, 18]. In this connection, Wang and Cui [10] proposed the following stepsize:

$$\tau_n = \begin{cases} \dfrac{\left\|(I-T)Ax_n\right\|^2}{\left\|A^*(I-T)Ax_n\right\|^2}, & \left\|(I-T)Ax_n\right\| \neq 0; \\ 0, & \left\|(I-T)Ax_n\right\| = 0. \end{cases} \tag{4}$$

On the other hand, Wang [19] proposed a new method:

$$x_{n+1} = x_n - \tau_n\left[(I-U)x_n + A^*(I-T)Ax_n\right], \tag{5}$$

where $\{\tau_n\} \subset (0, \infty)$ is chosen such that

$$\tau_n = \frac{\|(I-U)x_n\|^2 + \|(I-T)Ax_n\|^2}{\|(I-U)x_n + A^*(I-T)Ax_n\|^2}. \quad (6)$$

It is clear that the selection of stepsizes (8) and (6) does not rely on the norm $\|A\|$, which in turn improves the performance of the original algorithm. Assume that U and T are both directed such that $I-T$ and $I-U$ are demiclosed at 0. It is shown that the sequence $\{x_n\}$ generated by (7) and (8) or (5) and (6) converges weakly to a solution of problem (1).

Now, let us consider the multiple-sets split common fixed-point problem (MSCFP) that is more general than the SCFP. Formally, it consists in finding $x \in H_1$ such that

$$x \in \bigcap_{i=1}^{t} F(U_i), Ax \in \bigcap_{j=1}^{s} F(T_j), \quad (7)$$

where t and s are two positive integers, $A\colon H_1 \longrightarrow H_2$ is a bounded linear mapping from a Hilbert space H_1 into another Hilbert space H_2, and $F(U_i)$ and $F(T_j)$ are respectively the fixed-point sets of nonlinear mappings $U_i\colon H_1 \longrightarrow H_1$, $i = 1, 2, \ldots, t$ and $T_j\colon H_2 \longrightarrow H_2$, $j = 1, 2, \ldots, s$. Specially, if these nonlinear mappings are all metric projections, problem (7) is reduced to the well-known MSFP [20]. Actually, it can be formulated as the problem of finding $x \in H_1$ such that

$$x \in \bigcap_{i=1}^{t} C_i, Ax \in \bigcap_{j=1}^{s} Q_j, \quad (8)$$

where t and s are two positive integers, $A\colon H_1 \longrightarrow H_2$ is as above, and $\{C_i\}_{i=1}^{t} \subset H_1$ and $\{Q_j\}_{j=1}^{s} \subset H_2$ are two classes of nonempty convex closed subsets.

Inspired by the works mentioned above, we are aimed to introduce and analyze iterative methods for solving the MSCFP in Hilbert spaces. We first study several properties of demicontractive mappings and especially find its connection with the directed mapping. By making use of these properties, we propose a new iterative algorithm for solving the MSCFP, as well as MSFP. Under mild conditions, we obtain the weak convergence of the proposed algorithm. Our results extend the related works from the case of two-sets to the case of multiple-sets.

2. Preliminary

Throughout the paper, assume that H, H_1, H_2 are real Hilbert spaces, and $F(T)$ denotes its fixed-point set of a mapping T. The following formula plays an important role in the subsequent analysis.

Lemma 1 (see [21]). *Let $s, t \in \mathbb{R}$ and $x, y \in H$. It then follows that*

$$\|tx + sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - ts\|x-y\|^2. \quad (9)$$

We next recall the definition of several important classes of nonlinear mappings.

Definition 1 (see [21]). Let T be a mapping from H into H.

(i) T is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in H. \quad (10)$$

(ii) T is firmly nonexpansive if

$$\begin{aligned}\|Tx - Ty\|^2 &\le \|x-y\|^2 \\ &\quad - \|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in H.\end{aligned} \quad (11)$$

(iii) T is k-strictly pseudocontractive $(k<1)$ if

$$\begin{aligned}\|Tx - Ty\|^2 &\le \|x-y\|^2 \\ &\quad + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in H.\end{aligned} \quad (12)$$

Definition 2 (see [21]). Let $T\colon H \longrightarrow H$ be a mapping with $F(T) \neq \varnothing$.

(i) T is quasinonexpansive if

$$\|Tx - y\| \le \|x - y\|, \quad \forall (x, y) \in H \times F(T). \quad (13)$$

(ii) T is directed if

$$\|Tx - y\|^2 \le \|x-y\|^2 - \|(I-T)x\|^2, \quad \forall (x, y) \in H \times F(T). \quad (14)$$

(iii) T is k-demicontractive $(k<1)$ if

$$\begin{aligned}\|Tx - y\|^2 &\le \|x-y\|^2 + k\|((I-T))x\|^2, \\ &\quad \forall (x, y) \in H \times F(T).\end{aligned} \quad (15)$$

It is clear that a directed mapping is −1-demicontractive, while a quasinonexpansive mapping is 0-demicontractive. It is also clear that a firmly nonexpansive mapping is −1-strictly pseudocontractive, while a nonexpansive mapping is 0-strictly pseudocontractive.

It is well known that a mapping T is firmly nonexpansive if and only if $2T - I$ is nonexpansive (cf. [21]). Analogously, we can easily get the following lemma, which presents a characteristic of directed mappings by using quasinonexpansive mappings.

Lemma 2 *A mapping T is directed if and only if $2T - I$ is quasinonexpansive.*

We now study properties of demicontractive mappings.

Lemma 3 (see [22]). *Let $T\colon H \longrightarrow H$ be k-demicontractive $(k<1)$ with $F(T) \neq \varnothing$. Then, the following hold.*

(i) $\langle Tx - z, (I-T)x\rangle \geq 0, \quad \forall z \in F(T), x \in H$;

(ii) $\langle x - z, (I-T)x\rangle \geq \|(I-T)x\|^2, \quad \forall z \in F(T), x \in H$.

Lemma 4. *For each $i = 1, 2, \ldots, t$, assume that $T_i\colon H \longrightarrow H$ is k_i-demicontractive with $k_i < 1$. Let $T = 1/2\sum_{i=1}^t \omega_i((1+k_i)I + (1-k_i)T_i)$, where $0 < \omega_i < 1, \sum_{i=1}^t \omega_i = 1$. If $\cap_{i=1}^t F(T_i)$ is nonempty, then*

$$F(T) = \bigcap_{i=1}^{t} F(T_i). \tag{16}$$

Proof. We first show $\cap_{i=1}^t F(T_i) \subseteq F(T)$. Pick $x \in \cap_{i=1}^t F(T_i)$. It then follows that

$$\begin{aligned} Tx &= \frac{1}{2}\sum_{i=1}^{t} \omega_i ((1+k_i)x + (1-k_i)T_i x) \\ &= \frac{1}{2}\sum_{i=1}^{t} \omega_i ((1+k_i)x + (1-k_i)x) \\ &= \frac{1}{2}\sum_{i=1}^{t} \omega_i 2x = x. \end{aligned} \tag{17}$$

Since x is chosen arbitrarily, we have $\cap_{i=1}^t F(T_i) \subseteq F(T)$.

It suffices to show that $F(T) \subseteq \cap_{i=1}^t F(T_i)$. Fix $z \in \cap_{i=1}^t F(T_i)$ and choose any $x \in F(T)$. Since $Tx = x$ and T_i is k_i-demicontractive, we have

$$\begin{aligned} 0 &= 4\langle Tx - x, x - z\rangle \\ &= 2\sum_{i=1}^{t} \omega_i (1-k_i)\langle T_i x - x, x - z\rangle \\ &\geq \sum_{i=1}^{t} \omega_i (1-k_i)^2 \|T_i x - x\|^2. \end{aligned} \tag{18}$$

Thus, $\sum_{i=1}^t \omega_i (1-k_i)^2 \|x - T_i x\|^2 = 0$. Since $\omega_i(1-k_i) > 0$, we have $\|x - T_i x\| = 0$ for all $i = 1, 2 \ldots t$. Moreover, since x is chosen arbitrarily, we get $F(T) \subseteq \cap_{i=1}^t F(T_i)$. Hence, the proof is complete. □

Lemma 5. *For each $i = 1, 2 \ldots t$, assume that $T_i\colon H \longrightarrow H$ is k_i-demicontractive with $k_i < 1$. Let $T = 1/2\sum_{i=1}^t \omega_i((1+k_i)I + (1-k_i)T_i)$, where $0 < \omega_i < 1, \sum_{i=1}^t \omega_i = 1$. If $\cap_{i=1}^t F(T_i)$ is nonempty, then T is directed. Moreover, if for each $i = 1, 2 \ldots t$, $I - T_i$ is demiclosed at 0, then $I - T$ is also demiclosed at 0.*

Proof. By Lemma 4, we have $F(T) = \cap_{i=1}^t F(T_i) \neq \varnothing$. By Lemma 2, it suffices to show that $2T - I = \sum_{i=1}^t \omega_i (k_i I + (1-k_i)T_i)$ is quasinonexpansive. To this end, fix any $(x, z) \in H \times F(T)$. By Lemma 1 and the property of demicontractions that

$$\begin{aligned} \|(k_i x + (1-k_i)T_i x) - z\|^2 &= \|k_i(x-z) + (1-k_i)(T_i x - z)\|^2 \\ &= k_i\|x-z\|^2 + (1-k_i)\|T_i x - z\|^2 - k_i(1-k_i)\|(I-T_i)x\|^2 \\ &\leq k_i\|x-z\|^2 + (1-k_i)\left(\|x-z_i\|^2 + k_i\|(I-T_i)x\|^2\right) - k_i(1-k_i)\|(I-T_i)x\|^2 \\ &= \|x-z\|^2, \end{aligned} \tag{19}$$

hence $\|(k_i x + (1-k_i)T_i x) - z\| \leq \|x - z\|$ for all $i = 1, 2 \ldots t$. It then follows that

$$\begin{aligned} \|(2T-I)x - z\| &= \left\|\sum_{i=1}^{t} \omega_i (k_i x + (1-k_i)T_i x) - z\right\| \\ &\leq \sum_{i=1}^{t} \omega_i \|(k_i x + (1-k_i)T_i x) - z\| \\ &\leq \sum_{i=1}^{t} \omega_i \|x - z\| \\ &= \|x - z\|. \end{aligned} \tag{20}$$

Thus, $2T - I$ is quasinonexpansive, which implies T is directed.

Let us now prove the second assertion. By Lemma 4, we have $F(T) = \cap_{i=1}^t F(T_i) \neq \varnothing$. Let $\{x_n\} \subset H$ be such that $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Fix $z \in F(T)$. Since T_i is k_i-demicontractive, we have

$$\begin{aligned} 4\langle Tx_n - x_n, x_n - z\rangle &= 2\sum_{i=1}^{t} \omega_i (1-k_i)\langle T_i x_n - x_n, x_n - z\rangle \\ &\geq \sum_{i=1}^{t} \omega_i (1-k_i)^2 \|T_i x_n - x_n\|^2. \end{aligned} \tag{21}$$

Since $\omega_i(1-k_i) > 0$, we have $\lim_n \|x_n - T_i x_n\| = 0$, which, by our hypothesis, implies $\lim_n \|x - T_i x\| = 0$ for all $i = 1, 2 \ldots t$, that is, $x \in \cap_{i=1}^t F(T_i)$. By Lemma 4, the proof is complete. □

Finally, we end this section by recalling two weak convergence theorems of iterative methods for approximating a solution of the two-sets SCFP (1).

Theorem 1 (see [10], Theorem 3.1). *(Assume that U and T are both directed such that $I - U$ and $I - T$ are both demiclosed at 0. Then, the sequence $\{x_n\}$, generated by (7) and (8), converges weakly to a solution of problem (1).*

Theorem 2 (see [19], Theorem 3.4). *Assume that U and T are both directed such that $I-U$ and $I-T$ are both demiclosed at 0. Then, the sequence $\{x_n\}$, generated by (5) and (6), converges weakly to a solution of problem (1).*

3. The Case for Demicontractive Mappings

In this section, we are concerned with the multiple-sets split common feasibility problem and we assume that (7) is consistent, which means that its solution set is nonempty. First, motivated by (7) and (8), we propose the first algorithm for solving problem (7).

Algorithm 1. Let x_0 be arbitrary and choose $\{\alpha_i\}_{i=1}^t \subset (0,1)$ with $\sum_{i=1}^t \alpha_i = 1$, $\{\beta_j\}_{j=1}^s \subset (0,1)$ with $\sum_{j=1}^s \beta_j = 1$. Given x_n, update the next iteration via

$$\begin{cases} y_n = x_n - \tau_n \sum_{j=1}^{s} \beta_j (1-l_j) A^* (I-T_j) A x_n \\ x_{n+1} = \dfrac{1}{2} \sum_{i=1}^{t} \alpha_i ((1+k_i) y_n + (1-k_i) U_i y_n), \end{cases} \tag{22}$$

where $\tau_n = 0$ if $\|\sum_{j=1}^s \beta_j (1-l_j)(I-T_j)Ax_n\| = 0$; otherwise,

$$\tau_n = \frac{\left\|\sum_{j=1}^s \beta_j (1-l_j)(I-T_j)Ax_n\right\|^2}{\left\|\sum_{j=1}^s \beta_j (1-l_j) A^*(I-T_j)Ax_n\right\|^2}. \tag{23}$$

Theorem 3. *Assume that U_i and T_j are respectively k_i and l_j-demicontractive such that $I-U_i$ and $I-T_j$ are demiclosed at 0 for $i = 1,2,\ldots,t$ and $j = 1,2,\ldots,s$. Then, the sequence $\{x_n\}$, generated by Algorithm 1, converges weakly to a solution of (7).*

Proof. Let $U = 1/2\sum_{i=1}^t \alpha_i ((1+k_i)I + (1-k_i)U_i)$ and $T = 1/2\sum_{j=1}^s \beta_j ((1+l_j)I + (1-l_j)T_j)$. Thus, we can rewrite Algorithm 1 as

$$x_{n+1} = U(x_n - \tau_n A^*(I-T)Ax_n), \tag{24}$$

where $\tau_n = 0$ if $\|(I-T)Ax_n\| = 0$; otherwise,

$$\tau_n = \frac{\|(I-T)Ax_n\|^2}{\|A^*(I-T)Ax_n\|^2}. \tag{25}$$

By Lemma 5, U and T are both directed such as $I-T$ and $I-U$ are demiclosed at 0. It then follows from Theorem 1 that $\{x_n\}$ weakly converges to a point x that satisfies $x \in F(U)$ and $Ax \in F(T)$. Moreover, by Lemma 4, we conclude that $x \in \cap_i F(U_i)$ and $Ax \in \cap_j F(T_j)$, that is, x is a solution of problem (7). □

Motivated by (5) and (6), we propose the second algorithm for solving problem (7).

Algorithm 2. Let x_0 be arbitrary and choose $\{\alpha_i\}_{i=1}^t \subset (0,1)$ with $\sum_{i=1}^t \alpha_i = 1$, $\{\beta_j\}_{j=1}^s \subset (0,1)$ with $\sum_{j=1}^s \beta_j = 1$. Given x_n, if

$$\left\|\sum_{i=1}^{t} \alpha_i (1-k_i)(I-U_i)x_n + \sum_{j=1}^{s} \beta_j (1-l_j) A^*(I-T_j)Ax_n\right\| = 0, \tag{26}$$

then stop; otherwise, update the next iteration via

$$x_{n+1} = x_n - \tau_n \left[\sum_{i=1}^{t} (1-k_i)(I-U_i)x_n + \sum_{j=1}^{s} \beta_j (1-l_j) A^*(I-T_j)Ax_n\right], \tag{27}$$

where

$$\tau_n = \frac{\left\|\sum_{i=1}^t \alpha_i (I-U_i)(1-k_i)x_n\right\|^2 + \left\|\sum_{j=1}^s \beta_j (1-l_j) A^*(I-T_j)Ax_n\right\|^2}{2\left\|\sum_{i=1}^t (1-k_i)(I-U_i)x_n + \sum_{j=1}^s \beta_j (1-l_j)Ax_n\right\|^2}. \tag{28}$$

Theorem 4. *Assume that U_i and T_j are respectively k_i and l_j-demicontractive such that $I-U_i$ and $I-T_j$ are demiclosed at 0 for $i = 1,2,\ldots,t$ and $j = 1,2,\ldots,s$. Then, the sequence $\{x_n\}$, generated by Algorithm 2, converges weakly to a solution of (7).*

Proof. Let $U = 1/2\sum_{i=1}^t \alpha_i ((1+k_i)I + (1-k_i)U_i)$ and $T = 1/2\sum_{j=1}^s \beta_j ((1+l_j)I + (1-l_j)T_j)$. Thus, we can rewrite Algorithm 2 as $x_{n+1} = x_n - \tau_n[(I-U)x_n + A^*(I-T)Ax_n]$, where

$$\tau_n = \frac{\|(I-U)x_n\|^2 + \|(I-T)Ax_n\|^2}{\|(I-U)x_n + A^*(I-T)Ax_n\|^2}. \tag{29}$$

By Lemma 5, U and T are both directed such as $I-T$ and $I-U$ are demiclosed at 0. It then follows from Theorem 2 that $\{x_n\}$ weakly converges to a point x that satisfies $x \in F(U)$ and $Ax \in F(T)$. Moreover, by Lemma 4, we conclude that $x \in \cap_i F(U_i)$ and $Ax \in \cap_j F(T_j)$, that is, x is a solution of problem (7). □

4. Multiple-Sets Split Feasibility Problem

In this section, we apply the previous result to approximate a solution of the multiple-sets split feasibility problem (MSFP). Also, we assume that problem (8) is consistent, which means that its solution set is nonempty. By applying Algorithm 1, we obtain the first algorithm for solving (8).

Algorithm 3. Let x_0 be arbitrary and choose $\{\alpha_i\}_{i=1}^t \subset (0,1)$ with $\sum_{i=1}^t \alpha_i = 1$, $\{\beta_j\}_{j=1}^s \subset (0,1)$ with $\sum_{j=1}^s \beta_j = 1$. Given x_n, update the next iteration via

$$x_{n+1} = \sum_{i=1}^t \alpha_i P_{C_i}\left[x_n - \tau_n A^* \sum_{j=1}^s \beta_j \left(I - P_{Q_j}\right) A x_n\right], \quad (30)$$

where $\tau_n = 0$ if $\|\sum_{j=1}^s \beta_j (1 - l_j)(I - T_j) A x_n\| = 0$; otherwise,

$$\tau_n = \frac{\left\|\sum_{j=1}^s \beta_j \left(I - P_{Q_j}\right) A x_n\right\|^2}{\left\|\sum_{j=1}^s \beta_j A^* \left(I - P_{Q_j}\right) A x_n\right\|^2}. \quad (31)$$

Theorem 5. *The sequence $\{x_n\}$, generated by Algorithm 3, converges weakly to a solution of (2).*

Proof. It suffices to notice that both P_{C_i} and P_{Q_j} are -1-demicontractive, which implies $k_i = l_j = -1$ for all $i = 1, \ldots, t$, $j = 1, \ldots, s$. Applying Theorem 3 yields the desired assertion. □

Next, we propose the second algorithm for solving (8) by applying Algorithm 2.

Algorithm 4. Let x_0 be arbitrary and choose $\{\alpha_i\}_{i=1}^t \subset (0,1)$ with $\sum_{i=1}^t \alpha_i = 1$, $\{\beta_j\}_{j=1}^s \subset (0,1)$ with $\sum_{j=1}^s \beta_j = 1$. Given x_n, if

$$\left\|\sum_{i=1}^t \alpha_i \left(I - P_{C_i}\right) x_n + \sum_{j=1}^s \beta_j A^* \left(I - P_{Q_j}\right) A x_n\right\| = 0, \quad (32)$$

then stop; otherwise, update the next iteration via

$$x_{n+1} = x_n - \tau_n \left[\sum_{i=1}^t \alpha_i \left(I - P_{C_i}\right) x_n + \sum_{j=1}^s \beta_j A^* \left(I - P_{Q_j}\right) A x_n\right], \quad (33)$$

where

$$\tau_n = \frac{\left\|\sum_{i=1}^t \alpha_i \left(I - P_{C_i}\right) x_n\right\|^2 + \left\|\sum_{j=1}^s \beta_j \left(I - P_{Q_j}\right) A x_n\right\|^2}{\left\|\sum_{i=1}^t \alpha_i \left(I - P_{C_i}\right) x_n + \sum_{j=1}^s \beta_j A^* \left(I - P_{Q_j}\right) A x_n\right\|^2}. \quad (34)$$

Theorem 6. *The sequence $\{x_n\}$, generated by Algorithm 4, converges weakly to a solution of (8).*

Proof. It suffices to notice that both P_{C_i} and P_{Q_j} are -1-demicontractive, which implies $k_i = l_j = -1$ for all $i = 1, \ldots, t$, $j = 1, \ldots, s$. Applying Theorem 4 yields the desired assertion. □

5. Conclusion

In this paper, we consider the MSCFP whenever the involved mappings are demicontractive. We obtained several properties of demicontractive mappings and particularly their connection with directed mappings. These properties enable us to propose some new iterative methods for solving MSCFP, as well as MSFP. Under mild conditions, we establish their weak convergence of the proposed methods. Our results extend the existing works from the case of two-sets to the case of multiple-sets.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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Research Article

On Fixed Point Findings for Diverse Contractions in b Dislocated-Multiplicative Metric Spaces

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In our present research study, we present the idea of b dislocated-multiplicative metric space (abbrev. bd-multiplicative metric space) that is generalization of b-multiplicative metric space and dislocated-multiplicative metric space. Furthermore, we prove some of the fixed point theorems in bd-multiplicative metric spaces. Also, we get common fixed point findings for fuzzy mappings in these spaces. Our findings are improved and more generalized form of several findings (see, e.g., [5, 6]).

1. Introduction

In 2008, the idea of multiplicative calculus was defined by Bashirov et al. [1] and then the conception of multiplicative metric spaces (multiplicative distance) was introduced by Çevikel and Özavşasar [2]. Czerwik [3] presented concepts of b-metric space that is the popularization of metric space. Dosenovic et al. in section Future Work in [4] presented the idea of b-multiplicative metric spaces. After that, Ali et al. in [5] studied fixed point theorems for single-valued and multivalued mappings on b-multiplicative metric spaces.

Furthermore, several authors obtained some fixed point findings for mappings satisfying different contractive conditions (see, e.g., [6–8]). The idea of fuzzy mappings was initially studied by Weiss [9] and Butnariu [10]. Then, the concept of fuzzy mappings was studied by Heilpern [11]. Many of the fixed point theorems for fuzzy contraction mappings in the metric linear space were proved (e.g., [12–15]), which are the fuzzy extension for the Banach contraction principle. The concept of b-multiplicative metric spaces, as one of the useful generalizations of multiplicative metric spaces, was first used by Dosenovic et al. in [4],and Ali et al. in [5] study fixed point theorems for single-valued and multivalued mappings on b-multiplicative metric spaces.

Our findings in b-multiplicative metric space, d-multiplicative metric space, and multiplicative metric space can be obtained as corollaries of our findings.

In this part, we list some of the concepts which we will use in our major findings.

The definition of b-multiplicative metric space is given as follows.

Definition 1 (see [4, 5]). Suppose that X is a nonempty set and $s \geq 1$ is a given real number. A function $d\colon X \times X \longrightarrow [1, \infty)$ is considered as a b-multiplicative metric if it satisfies the following conditions: $\forall \eta, \xi, z \in X$,

(i) $d(\eta, \xi) \geq 1$
(ii) $d(\eta, \xi) = 1$ iff $\eta = \xi$
(iii) $d(\eta, \xi) = d(\xi, \eta)$
(iv) $d(\eta, \xi) \leq d(\eta, z)^s \cdot d(z, \xi)^s$

Example 1 (see [5]). Let $X = [0, \infty)$. Define a function $d\colon X \times X \longrightarrow [1, \infty)$, $d(\eta, \xi) = a^{(\eta - \xi)^2}$, where $a > 1$ is any fixed real number. Then, (X, d) is a b-multiplicative metric with $s = 2$.

In [11, 16], an element in any fuzzy set has a degree of belonging, a membership function may be used in order to introduce the value of degree of belonging for any element to a set, and the value of degree of belonging takes real values on the whole closed interval $[0, 1]$. The membership function is

$$\mu_A \colon X \longrightarrow [0, 1]. \tag{1}$$

Suppose (X, d) is a metric linear space. In X, a fuzzy set is a function $A\colon X \longrightarrow [0, 1]$. Thus, it is an element of I^X, where $I = [0, 1]$. If A is a fuzzy set and $\eta \in X$, then the function value $A(\eta)$ is considered as the grade of membership of η in A.

I^X denotes to the collection of all fuzzy sets in X. The α-level set of A is defined by

$$A_\alpha = \{\eta\colon A(\eta) \geq \alpha\} \text{ with } \alpha \in (0,1] \text{ and } A_0 = \overline{\{\eta\colon A(\eta) > 0\}}, \tag{2}$$

whenever $\overline{\{\}}$ is the closure of set (nonfuzzy) {}.

Definition 2 (see [17]). A fuzzy set A in X is an approximate quantity if its α-level set is a nonempty compact subset (nonfuzzy) of X for each $\alpha \in [0, 1]$.

The set of all an approximate quantities denoted by $W^*(X)$ is a subcollection of $\mathfrak{F}(X)$.

Ozavsar and Cevikel [2] prove that every multiplicative contraction in a complete multiplicative metric space has a unique fixed point.

Definition 3 (see [2]). Assume that (X, d) is a multiplicative metric space. A mapping $g\colon X \longrightarrow X$ is called multiplicative contraction if

$$\exists \lambda \in [0, 1)\colon d(g\eta_1, g\eta_2) \leq d(\eta_1, \eta_2)^\lambda \ \forall \eta_1, \eta_2 \in X. \tag{3}$$

Theorem 1 (see [2]). *Assume that (X, d) is a multiplicative metric space. A mapping $g\colon X \longrightarrow X$ is called multiplicative contraction. Then, g has a unique fixed point.*

Theorem 2 (see [4]). *Suppose that (X, d) is a complete multiplicative metric space and a continuous function $g\colon X \longrightarrow X$, $\lambda \in [0, 1)$ such that*

$$d(g\eta, g\xi) \leq \{\max\{d(\eta, g\eta), d(\xi, g\xi)\}\}^\lambda. \tag{4}$$

Then, g has a unique fixed point.

In 2015, Kang et al. [18] introduced the concept of compatible mappings as follows.

Definition 4. Let (X, d) be a multiplicative metric space. The mappings $f, F\colon X \longrightarrow X$; then, (f, F) is called compatible if and only if $ft = Ft$ for some t in X implying $fFt = Fft$.

Many authors studied many fixed point theorems for compatible mappings in multiplicative metric space and employed it to prove a common fixed point theorem (see [4, 18]).

In this paper, we introduce the new notion of bd-multiplicative metric space. We prove fixed point theorems for single mappings and a common fixed point for fuzzy mappings in bd-multiplicative metric space. As illustrative application, we state some of our theorems on Cartesian product in these spaces.

2. Fixed Point Theorems in bd-Multiplicative Metric Spaces

In this part, we present the conception of bd-multiplicative metric space. Also, we introduce some of the fixed point theories and show our main findings with the help of some examples in this space.

Definition 5. Suppose that $X \neq \phi$ and $s \geq 1$ is a given real number. A function $d\colon X \times X \longrightarrow \mathbb{R}^+$ is called bd-multiplicative metric space if it satisfies the following conditions: $\forall \eta, \xi, z \in X$,

(i) $d(\eta, \xi) \geq 1$
(ii) $d(\eta, \xi) = 1$ implies $\eta = \xi$
(iii) $d(\eta, \xi) = d(\xi, \eta)$
(iv) $d(\eta, \xi) \leq [d(\eta, z) \cdot d(z, \xi)]^s$

Example 2. Let $X = \mathbb{R}^+ \cup \{0\}$. Define $d\colon X \times X \longrightarrow [1, \infty)$ as

$$d(\eta, \xi) = a^{(\eta+\xi)^2}, \ \forall \eta, \xi \in X, \ a \geq 1. \tag{5}$$

Then, (X, d) is bd-multiplicative metric space with $s = 2$.

Example 3. Let $X = [1, \infty)$. Define $d\colon X \times X \longrightarrow [1, \infty)$ as

$$d(\eta, \xi) = a^{((\eta-1)+(\xi-1))^2}, \ \forall \eta, \xi \in X, \ a \geq 1. \tag{6}$$

Then, (X, d) is bd-multiplicative metric space with $s = 2$.

Definition 6. Let (X, d) be an bd-multiplicative metric space. We say that $\{\eta_n\}$ converges to η if and only if

$$d(\eta_n, \eta) \longrightarrow_{bd} 1, \text{ as } n \longrightarrow \infty. \tag{7}$$

Definition 7. Let (X, d) be an bd-multiplicative metric space. We say that $\{\eta_n\}$ is bd-multiplicative Cauchy if and only if

$$d(\eta_n, \eta_m) \longrightarrow_{bd} 1, \text{ as } n, m \longrightarrow \infty. \tag{8}$$

Definition 8. An bd-multiplicative metric space (X, d) is complete if every multiplicative Cauchy sequence in X is convergent.

Now, we state the following lemma without proof.

Lemma 1. *Suppose that (X, d) is bd-multiplicative metric space. Then, any subsequence of convergent sequence in X is convergent.*

The following theorem is the generalization of Theorem 3.2 in [2].

Theorem 3. *Suppose that (X, d) is a complete bd-multiplicative metric space and a continuous function $g\colon D \longrightarrow X$; $D \subseteq X$ satisfies*

$$d(g\eta, g\xi) \leq d(\eta, \xi)^k, \tag{9}$$

where η, $\xi \in D$ and $k \in [0, 1/s)$. Then, g has a unique fixed point.

Proof. Let η_0 be an arbitrary point in X; then by hypothesis, there exists η_1 such that $\eta_1 = g\eta_0$. In a similar way, one can obtain a sequence $\{\eta_n\} \subseteq X$ such that

$$\begin{aligned}
\eta_n &= g\eta_{n-1} = g^n \eta_0, \\
d(\eta_n, \eta_{n+m}) &\leq d(\eta_n, \eta_{n+1})^{s^n} \cdot d(\eta_{n+1}, \eta_{n+2})^{s^{n+1}} \ldots d \\
&\quad \cdot (\eta_{n+m-1}, \eta_{n+m})^{s^{n+m-1}} \\
&= d(g^n \eta_0, g^n \eta_1)^{s^n} \cdot d\left(g^{n+1} \eta_0, g^{n+1} \eta_1\right)^{s^{n+1}} \ldots d \\
&\quad \cdot \left(g^{n+m-1} \eta_0, g^{n+m-1} \eta_1\right)^{s^{n+m-1}} \\
&\leq d(\eta_0, \eta_1)^{(ks)^n} \cdot d(\eta_0, \eta_1)^{(ks)^{n+1}} \ldots d \\
&\quad \cdot (\eta_0, \eta_1)^{(ks)^{n+m-1}} \\
&\leq d(\eta_0, \eta_1)^{(ks)^n/1-ks}.
\end{aligned} \tag{10}$$

Then,

$$d(\eta_n, \eta_{n+m}) \leq d(\eta_0, \eta_1)^{(ks)^n/1-ks}. \tag{11}$$

As $n \longrightarrow \infty$ in (11) and $k < 1/s \Rightarrow ks < 1$, then $\{\eta_n\}$ is a multiplicative Cauchy sequence.

Since (X, d) is complete, then $\{\eta_n\}$ is convergent such that $\lim_{n\longrightarrow\infty} \eta_n = \eta^*$. However,

$$\begin{aligned}
\eta^* &= \lim_{n\longrightarrow\infty} \eta_{n+1} \\
&= \lim_{n\longrightarrow\infty} g\eta_n \\
&= g\lim_{n\longrightarrow\infty} \eta_n \\
&= g\eta^*.
\end{aligned} \tag{12}$$

Therefore, η^* is a fixed point of g. Suppose that $g\eta^* = \eta^*$, $g\overline{\eta} = \overline{\eta}$, and $\overline{\eta} \neq \overline{\eta}$.

$$d(\eta^*, \overline{\eta}) = d(g\eta^*, g\overline{\eta}) \leq d(\eta^*, \overline{\eta})^k \leq d(\eta^*, \overline{\eta}). \tag{13}$$

This is a contradiction with assumption; then, $\eta^* = \overline{\eta}$. Then, g has a unique fixed point. □

Example 4. Suppose that $X = [1, \infty)$, (X, d) is bd-multiplicative metric space and $d(\eta, \xi) = a^{((\eta-1)+(\xi-1))^2}$, where $a = 2$ with $s = 2$. Define $g\colon X \longrightarrow X$ such that $g\eta = (\eta + 1)/2$:

$$\begin{aligned}
d(g\eta, g\xi) &= 2^{((\eta-1)/2+(\xi-1)/2)^2} \\
&= 2^{1/4((\eta-1)+(\xi-1))^2} \\
&= d(\eta, \xi)^{1/4}.
\end{aligned} \tag{14}$$

Then, (1) holds such that $k = 1/4$. Therefore, g has a unique fixed point $1 \in X$.

Corollary 1. *Suppose that (X, d) is a complete multiplicative metric space and a continuous function $g\colon X \longrightarrow X$ satisfies*

$$d(g\eta, g\xi) \leq d(\eta, \xi)^k, \tag{15}$$

where $\eta, \xi \in D$ and $k \in [0, 1)$. Then, g has a unique fixed point.

Corollary 2. *Suppose that (X, d) is a complete b-multiplicative metric space and a continuous function $g\colon X \longrightarrow X$ satisfies*

$$d(g\eta, g\xi) \leq d(\eta, \xi)^k, \tag{16}$$

where $\eta, \xi \in X$ and $k \in [0, 1/s)$. Then, g has a unique fixed point.

The following theorem is the generalization of Theorem 2.32 in [4].

Theorem 4. *Suppose that (X, d) is a complete bd-multiplicative metric space and a continuous function $g\colon D \longrightarrow X$, $k \in [0, 1/s)$, such that*

$$d(g\eta, g\xi) \leq \{\max\{d(\eta, \xi), d(\eta, g\eta), d(\xi, g\xi)\}\}^k. \tag{17}$$

Then, g has a unique fixed point.

Proof. Let η_0 be an arbitrary point in X; then by hypothesis, there exists η_1 such that $\eta_1 = g\eta_0$.

In a similar way, one can obtain $\eta_2 \in X$ such that $\eta_2 = g\eta_1$.

$$\begin{aligned}
d(\eta_1, \eta_2) &= d(g\eta_0, g\eta_1) \\
&\leq \max\{d(\eta_0, \eta_1), d(\eta_0, g\eta_0), d(\eta_1, g\eta_1)\}^k \\
&= \max\{d(\eta_0, \eta_1), d(\eta_0, \eta_1), d(\eta_1, \eta_2)\}^k \\
&= \max\{d(\eta_0, \eta_1), d(\eta_1, \eta_2)\}^k \\
&= d(\eta_0, \eta_1)^k.
\end{aligned} \tag{18}$$

Otherwise, we have a contradiction, that is, $d(\eta_1, \eta_2) \leq d(\eta_1, \eta_2)^k$.

$$\begin{aligned}
d(\eta_2, \eta_3) &= d(g\eta_1, g\eta_2) \\
&\leq \max\{d(\eta_1, \eta_2),\ d(\eta_1, g\eta_1),\ d(\eta_2, g\eta_2)\}^k \\
&= \max\{d(\eta_1, \eta_2),\ d(\eta_1, \eta_2),\ d(\eta_2, \eta_3)\}^k \\
&= \max\{d(\eta_1, \eta_2), d(\eta_2, \eta_3)\}^k \\
&= d(\eta_0, \eta_1)^{k^2}.
\end{aligned} \tag{19}$$

Continuing in this way, we produce a sequence $\{\eta_n\}$ in X such that $\{\eta_n\} = g(\eta_{n-1})$ and

$$\begin{aligned} d(\eta_n,\eta_{n+1}) &= d(g\eta_{n-1}, g(\eta_n)) \\ &\leq \max\{d(\eta_{n-1},\eta_n), d(\eta_{n-1}, g\eta_{n-1}), d(\eta_n, g\eta_n)\}^k \\ &= \max\{d(\eta_{n-1},\eta_n), d(\eta_{n-1},\eta_n), d(\eta_n,\eta_{n+1})\}^k \\ &= \max\{d(\eta_{n-1},\eta_n), d(\eta_n,\eta_{n+1})\}^k \\ &= \left\{d(\eta_0,\eta_1)^{k^{n-1}}\right\}^k \\ &= d(\eta_0,\eta_1)^{k^n}, \end{aligned} \tag{20}$$

for each $n \in \mathbb{N}$. It follows by induction that $d(\eta_n,\eta_{n+1}) \leq d(\eta_0,\eta_1)^{k^n}$. However,

$$\begin{aligned} d(\eta_n,\eta_{n+m}) &\leq d(\eta_n,\eta_{n+1})^{s^n} \cdot d(\eta_{n+1},\eta_{n+2})^{s^{n+1}} \\ &\quad \ldots d(\eta_{n+m-1},\eta_{n+m})^{s^{n+m-1}} \\ &= d(g^n\eta_0, g^n\eta_1)^{s^n} \cdot d\left(g^{n+1}\eta_0, g^{n+1}\eta_1\right)^{s^{n+1}} \\ &\quad \ldots d\left(g^{n+m-1}\eta_0, g^{n+m-1}\eta_1\right)^{s^{n+m-1}} \\ &\leq d(\eta_0,\eta_1)^{(ks)^n} \cdot d(\eta_0,\eta_1)^{(ks)^{n+1}} \\ &\quad \ldots d(\eta_0,\eta_1)^{(ks)^{n+m-1}} \\ &\leq d(\eta_0,\eta_1)^{(ks)^n\left[1+(ks)+(ks)^2+\ldots\right]} \\ &\leq d(\eta_0,\eta_1)^{(ks)^n/1-ks}. \end{aligned} \tag{21}$$

As $k \in (0,1/s)$, $m,n \longrightarrow \infty$, $ks<1$, and (X,d) is complete, then $\{\eta_n\}$ is a multiplicative Cauchy sequence in X and there exists $\eta^* \in X$ such that $d(\eta_n,\eta^*) \longrightarrow_{bd} 1$.

From Lemma 1, $\eta^* = \lim_{n\longrightarrow\infty}\eta_{n+1} = \lim_{n\longrightarrow\infty} g\eta_n = g\lim_{n\longrightarrow\infty}\eta_n = g\eta^*$.

Then, η^* is fixed point of g, and $g\eta^* = \eta^*$. Suppose that g has another fixed point $\overline{\eta}$ such that $g\overline{\eta} = \overline{\eta}$ and $\overline{\eta} \neq \overline{\eta}$.

$$\begin{aligned} d(\eta^*,\overline{\eta}) &= d(g\eta^*, g\overline{\eta}) \\ &\leq \max\{d(\eta^*,\overline{\eta}), d(\eta^*, g\eta^*), d(\overline{\eta}, g\overline{\eta})\}^k \\ &= \max\{d(\eta^*,\overline{\eta}), d(\eta^*,\eta^*), d(\overline{\eta},\overline{\eta})\}^k \\ &\leq \max\{d(\eta^*,\overline{\eta}), d(\eta^*,\eta_n)^s \cdot d(\eta_n,\eta^*)^s, d(\overline{\eta},\eta_n)^s \\ &\quad \cdot d(\eta_n,\overline{\eta})^s\}^k \\ &= \max\{d(\eta^*,\overline{\eta}), 1\}^k \\ &= d(\eta^*,\overline{\eta})^k \\ &\leq d(\eta^*,\overline{\eta}). \end{aligned} \tag{22}$$

This is a contradiction with assumption; then, $\eta^* = \overline{\eta}$. Then, g has a unique fixed point. □

Corollary 3. *Suppose that (X,d) is a complete multiplicative metric space and a continuous function g: $X \longrightarrow X$ satisfies*

$$d(g\eta, g\xi) \leq \{\max\{d(\eta,\xi), d(\eta, g\eta), d(\xi, g\xi)\}\}^k, \tag{23}$$

where η, $\xi \in X$ and $k \in [0,1)$. Then, g has a unique fixed point.

Corollary 4. *Suppose that (X,d) is a complete b-multiplicative metric space and a continuous function g: $X \longrightarrow X$ satisfies*

$$d(g\eta, g\xi) \leq \{\max\{d(\eta,\xi), d(\eta, g\eta), d(\xi, g\xi)\}\}^k, \tag{24}$$

where η, $\xi \in X$ and $k \in [0,1/s)$. Then, g has a unique fixed point.

3. Common Fixed Point Theorems for Fuzzy Mappings in *bd*-Multiplicative Metric Spaces

Definition 9. Suppose that X is an arbitrary set and Y is *bd*-multiplicative-metric space. A mapping F is stated according to be a fuzzy mapping iff F is a function from the set X into $W^*(Y)$, i.e., $F(\eta) \in W^*(Y)$, for each $\eta \in X$.

Definition 10. Suppose that (X,d) is a *bd*-multiplicative metric space. The functions g: $Y \subseteq X \longrightarrow X$ and G: $Y \longrightarrow W^*(Y)$. A hybrid pair (g,G) is called D-compatible iff $\{gt\} \subset Gt$ for some $t \in Y$ implies $gGt \subset Ggt$.

Definition 11. Suppose that (X,d) is a *bd*-multiplicative metric space. Two maps G and g are said to be occasionally coincidentally idempotent if $g^2\eta = g\eta$ for some $C(g,G)$, where $C(g,G)$ refers to the set of all coincidence points of two mappings g and G, i.e.,

$$C(g,G) = \{\eta\colon g\eta = G\eta\}. \tag{25}$$

Now, we state the following lemma without proof.

Lemma 2. *Suppose that (X,d) is a bd-multiplicative metric space and $M \subseteq W^*(X)$. Then,*

$$\overline{M} = \{\eta \in X\colon d(\eta, M) = 1\}. \tag{26}$$

Corollary 5. *Suppose that (X,d) is a bd-multiplicative metric space and $M \subseteq W^*(X)$ and $d(\eta,M) = 1$ if and only if $\eta \in \overline{M} = M$.*

Lemma 3. *Suppose that (X,d) is a bd-multiplicative metric space, G: $X \longrightarrow W^*(X)$ is a fuzzy map, and $\eta_0 \in X$. Then, there exists $\eta_1 \in X$ such that $\{\eta_1\} \subseteq G(\eta_0)$.*

Theorem 5. *Suppose that (X,d) is a complete bd-multiplicative metric space and two continuous mappings g,f: $X \longrightarrow X$ satisfy*

$$d(f\eta, f\xi) \leq d(\eta,\xi)^k \text{ and } d(g\eta, g\xi) \leq d(\eta,\xi)^k, \tag{27}$$

where $\eta, \xi \in X$ and $k \in [0, 1/s)$ and two fuzzy mappings $G, F\colon X \longrightarrow W^*(X)$, such that

(i) $\{GX\}_\alpha \subset f(X)$, $\{FX\}_\alpha \subset g(X)$

(ii) *The pairs (G, g) and (F, f) are D-compatible and occasionally idempotent mappings*

Then, there exists $\eta^* \in X$ such that $\eta^* = f\eta^* = g\eta^*$ and $\eta^* \in \{F\eta^*\}_\alpha \cap \{G\eta^*\}_\alpha$.

Proof. Suppose η_0 is an arbitrary point in X. Then, there is $\{\xi_1\} = \{g\eta_1\} \subset \{F\eta_0\}_\alpha$, from Lemma 3.2; then, there exists $\{\xi_2\} = \{f\eta_2\} \subset \{G\eta_1\}_\alpha$, where

$$\{\xi_{2n+1}\} = \{g\eta_{2n+1}\} \subset \{F\eta_{2n}\}_\alpha, \tag{28}$$

$$\{\xi_{2n+2}\} = \{f\eta_{2n+2}\} \subset \{G\eta_{2n+1}\}_\alpha. \tag{29}$$

From (11) in Theorem 3,

$$d(\xi_n, \xi_m) \leq d(\xi_0, \xi_1)^{(ks)^n/1-ks}. \tag{30}$$

As $n \longrightarrow \infty$, $k < 1/s \Rightarrow ks < 1$ that implies $\{\xi_n\}$ is a multiplicative Cauchy sequence.

Since (X, d) is complete, then $\{\eta_n\}$ is convergent such that

$$\lim_{n\longrightarrow\infty} d(\xi_n, \xi_m) = 1. \tag{31}$$

Next, we prove that $\eta^* \in \{Fz\}_\alpha$.

$$\begin{aligned} d(fz, f\{Fz\}_\alpha) &= d\left(f^2 z, f\{Fz\}_\alpha\right) \\ &= d(f\eta^*, f\{Fz\}_\alpha) \\ &\leq d(\eta^*, \{Fz\}_\alpha)^k \\ &\leq d(\eta^*, fz)^{ks} \cdot d(fz, \{Fz\}_\alpha)^{ks} \\ &= d(\eta^*, fz)^{ks} \cdot d(\eta^*, \{Fz\}_\alpha)^{ks} \\ &= d(\eta^*, \eta^*)^{ks} \cdot d(\eta^*, \{Fz\}_\alpha)^{ks}. \\ &\Downarrow \\ d(\eta^*, \{Fz\}_\alpha)^{k-ks} &\leq d(\eta^*, \eta^*)^{ks} \\ &\leq d(\eta^*, \xi_{2n+2})^{ks^2} \cdot d(\xi_{2n+2}, \eta^*)^{ks^2} \\ &= d(\xi_{2n+2}, \eta^*)^{ks^2} \cdot d(\xi_{2n+2}, \eta^*)^{ks^2}. \end{aligned} \tag{32}$$

From Lemma 1, ξ_{2n+2} is a convergent sequence, i.e., $d(\xi_{2n+2}, \eta^*) \longrightarrow 1$ as $n \longrightarrow \infty$,i.e.,

$$\begin{aligned} d(\eta^*, \{Fz\}_\alpha)^{k-ks} &\leq 1. \\ d(\eta^*, \{Fz\}_\alpha) &\leq 1. \end{aligned} \tag{33}$$

Then, we have $d(\eta^*, \{Fz\}_\alpha) = 1$ and Corollary 5 illustrates that $\eta^* \in \overline{\{Fz\}_\alpha} = \{Fz\}_\alpha$.

Since $\eta^* = fz \in \{Fz\}_\alpha \subset g(X)$, there exists $w \in X$ suchthat $\eta^* = gw$.

Similar to the previous steps, we can prove that $\eta^* = gw \in \{Gw\}_\alpha$.

As two pairs (G, g) and (F, f) are D-compatible,

$$\begin{aligned} \{\eta^*\} = \{fz\} \subset \{Fz\}_\alpha \text{ and } \{\eta^*\} \\ = \{gw\} \subset \{Gw\}_\alpha. \end{aligned} \tag{34}$$

Moreover,

$$\{f\eta^*\} = \{ffz\} \subset \{fFz\}_\alpha \subset \{Ffz\}_\alpha = \{F\eta^*\}_\alpha. \tag{35}$$

$$\begin{aligned} \{g\eta^*\} = \{ggw\} \subset \{gGw\}_\alpha \subset \{Ggw\}_\alpha \\ = \{G\eta^*\}_\alpha. \end{aligned} \tag{36}$$

Now, we show that $\eta^* = f\eta^*$ and $\eta^* = g\eta^*$. Since $\{\xi_n\}$ and $\{\eta_n\}$ are convergent sequences, from Lemma 1,

$$\begin{aligned} \eta^* &= \lim_{n\longrightarrow\infty} \xi_{2n+2} \\ &= \lim_{n\longrightarrow\infty} f\eta_{2n+2} \\ &= f \lim_{n\longrightarrow\infty} \eta_{2n+2} \\ &= f\eta^*. \end{aligned} \tag{37}$$

$$\begin{aligned} \eta^* &= \lim_{n\longrightarrow\infty} \xi_{2n+1} \\ &= \lim_{n\longrightarrow\infty} g\eta_{2n+1} \\ &= g \lim_{n\longrightarrow\infty} \eta_{2n+1} \\ &= g\eta^*. \end{aligned} \tag{38}$$

Then, $\eta^* = f\eta^* = g\eta^*$ and $\eta^* \in \{F\eta^*\}_\alpha \cap \{G\eta^*\}_\alpha$. □

Example 5. Suppose that $X = [0, 1]$, (X, d) is a bd-multiplicative metric space defined by $d(\eta, \xi) = a^{(\eta+\xi)^2}$, and $a > 1$. Define maps $g, f\colon X \longrightarrow X$ as $g\eta = \eta^2$, $f\eta = \eta^3 \,\forall\, \eta$, and $\xi \in X$. Also, define two fuzzy mappings $G, F\colon X \longrightarrow W^*(X)$ as

$$(F\eta)(\xi) = \begin{cases} 0 & if \quad 0 \leq \xi < 1/5 \\ 1/3 & if \quad 1/5 \leq \xi \leq \eta^3 \\ 2/3 & if \quad \eta^3 < \xi < 4/5 \\ 1 & if \quad 4/5 \leq \xi \leq 1 \end{cases}, \tag{39}$$

and

$$(G\eta)(\xi) = \begin{cases} 0 & if \quad 0 \leq \xi < 1/4 \\ 1/6 & if \quad 1/4 \leq \xi \leq \eta^2 \\ 1/9 & if \quad \eta^2 < \xi < 6/5 \\ 1 & if \quad 6/5 \leq \xi \leq 1 \end{cases}. \tag{40}$$

Now, for $\alpha = 1/3$, $f\{F\eta\}_{1/3} = [(1/125), \eta^9] \subset [(1/5), \eta^3] = \{Ff\eta\}_{1/3}$ and for $\alpha = 1/6$, $g\{G\eta\}_{1/6} = [(1/16), \eta^4] \subset [(1/4), \eta^4] = \{Gg\eta\}_{1/6}$; i.e., (g, G) and (f, F) are D-compatible. Finally, $f1 = ff1 \in [(1/5), 1] = Ff1$ and $gg1 = g1 \in [1/4, 1] = Gg1$; i.e., (f, F) and (g, G) are occasionally coincidentally idempotent. Then, $1 = f1 = g1 \in [1/5, 1] \cap [1/4, 1] = \{F1\}_{1/3} \cap \{G1\}_{1/6}$ is a common fixed point.

Example 6. Suppose that $X = [2/5, 3/2]$, (X, d) is a bd-multiplicative metric space defined by $d(\eta, \xi) = 2^{((\eta - (2/5)) + (\xi - (2/5)))^2}$. Define maps $g, f\colon X \longrightarrow X$ as $g\eta = 1/2(\eta + 2/5)$, $f\eta = (\eta + 2)/6 \,\forall\, \eta$, and $\xi \in X$. Also, define two fuzzy mappings $G, F\colon X \longrightarrow W^*(X)$ as

$$(F\eta)(\xi) = \begin{cases} 0 & if \quad 0 \leq \xi < 1/5 \\ 1/3 & if \quad 1/5 \leq \xi < (\eta + 2)/6, \\ 2/3 & if \quad (\eta + 2)/6 \leq \xi \leq 1 \end{cases} \tag{41}$$

and

$$(G\eta)(\xi) = \begin{cases} 0 & \text{if } 0 \leq \xi < \dfrac{1}{4} \\ \dfrac{1}{6} & \text{if } \dfrac{1}{4} \leq \xi < \dfrac{1}{2}\left(\eta + \dfrac{2}{5}\right). \\ \dfrac{1}{4} & \text{if } \dfrac{1}{2}\left(\eta + \dfrac{2}{5}\right) \leq \xi \leq 1 \end{cases} \tag{42}$$

Now, for $\alpha = 2/3$,

$$f\{F\eta\}_{2/3} = [(\eta + 14)/36, 1/2] \subset [(\eta + 14)/36, 1] = \{Ff\eta\}_{2/3}, \tag{43}$$

and for $\alpha = 1/4$,

$$\begin{aligned} g\{G\eta\}_{1/4} &= [(1/4\eta + 1/10), 7/20] \subset [(1/4\eta + 1/10), 1] \\ &= \{Gg\eta\}_{1/4}. \end{aligned} \tag{44}$$

Hence, (g, G) and (f, F) are D-compatible. Finally,

$$\begin{aligned} f2/5 &= ff2/5 \in [4/10, 1] \\ &= Ff2/5 \end{aligned} \tag{45}$$

and

$$\begin{aligned} gg2/5 &= g2/5 \in [2/10, 1] \\ &= Gg2/5. \end{aligned} \tag{46}$$

Therefore, (f, F) and (g, G) are occasionally coincidentally idempotent. Furthermore,

$$\begin{aligned} d(f\eta, f\xi) &= 2^{(((\eta+2/6)-(2/5))+((\xi+2/6)-(2/5)))^2} \\ &= 2^{\frac{1}{36}((\eta - (2/5)) + (\xi - (2/5)))^2} \\ &= d(\eta, \xi)^{1/36}. \end{aligned} \tag{47}$$

Then, $2/5 = f2/5 = g2/5 \in [4/10, 1] \cap [2/10, 1] = \{F2/5\}_{2/3} \cap \{G2/5\}_{1/4}$ is a common fixed point.

We concluded the following corollary when we set $f = g$ in Theorem 5.

Corollary 6. *Let $Y \subset X$, and suppose continuous mapping $g\colon Y \longrightarrow X$ satisfies $d(g\eta, g\xi) \leq d(\eta, \xi)^k$, where $\eta, \xi \in Y$, $k \in [0, 1/s)$ and fuzzy mapping $G\colon Y \longrightarrow W^*(X)$ such that*

(i) $\{GY\}_\alpha \subset g(Y)$

(ii) The pair (G, g) is D-compatible and occasionally idempotent mappings

Then, there exists $\eta^* \in Y$ such that $\eta^* = g\eta^* \in \{G\eta^*\}_\alpha$.

Theorem 6. *Let $Y \subset X$ and suppose two continuous mappings $g, \quad f\colon Y \longrightarrow X$, satisfy*

$$d(f\eta, f\xi) \leq d(\eta, \xi)^k \text{and}\, d(g\eta, g\xi) \leq d(\eta, \xi)^k, \tag{48}$$

where $\eta, \quad \xi \in Y$, $k \in [0, 1/s)$, and $\{F_n\}_\alpha\colon Y \longrightarrow W^*(X)$ such that $\forall\, \eta \in Y$,

(i) $\{F_l Y\}_\alpha \subset f(Y)$ and $\{F_k Y\}_\alpha \subset g(Y)$

(ii) The pairs (F_l, f) and (F_k, g) are D-compatible and occasionally idempotent mappings

Then, there exists $\eta^* \in Y$ such that $\eta^* = f\eta^* = g\eta^*$ and $\eta^* \in \cap_{n=0}^{\infty}\{F_n\eta^*\}_\alpha$.

Proof. The proof of this theorem is completed, when putting $F_l = G$ and $F_k = F$ in Theorem 5. □

Remark 1. If $F_l = G$ and $F_k = F$, then Theorem 6 implies Theorem 5.

Theorem 7. *Suppose that (X, d) is a complete bd-multiplicative metric space and two continuous mappings $g, \quad f\colon X \longrightarrow X$ satisfy*

$$\begin{aligned} d(f\eta, f\xi) &\leq \{\max\{d(\eta, \xi), d(\eta, f\eta), d(\xi, f\xi)\}\}^k, \\ d(g\eta, g\xi) &\leq \{\max\{d(\eta, \xi), d(\eta, g\eta), d(\xi, g\xi)\}\}^k, \end{aligned} \tag{49}$$

where $\eta, \quad \xi \in X$, $k \in [0, 1/s)$, and two fuzzy mappings $F, \quad G\colon X \longrightarrow W^*(X)$, such that

(i) $\{GX\}_\alpha \subset f(X)$, $\{FX\}_\alpha \subset g(X)$

(ii) The pairs (F, f) and (G, g) are D-compatible and occasionally idempotent mappings

Then, there exists $\eta^* \in X$: $\eta^* = f\eta^* = g\eta^*$ and $\eta^* \in \{F\eta^*\}_\alpha \cap \{G\eta^*\}_\alpha$.

Proof. Let η_0 be an arbitrary point in X; then, there exists $\xi_1 = g\eta_1 \in \{F\eta_0\}_\alpha$ and from Lemma 3, there exists $\{\xi_2\} = \{f\eta_2\} \subset \{G\eta_1\}_\alpha$ such that

$$\begin{aligned} \{\xi_{2n+1}\} &= \{g\eta_{2n+1}\} \subset \{F\eta_{2n}\}_\alpha, \\ \{\xi_{2n+2}\} &= \{f\eta_{2n+2}\} \subset \{G\eta_{2n+1}\}_\alpha. \end{aligned} \tag{50}$$

Since $\xi_1 = g\eta_1 \in \{F\eta_0\}$, there exists $\xi_2 = f\eta_2 \subset \{G\eta_1\}_\alpha$, and from Theorem 4,

$$d(\xi_n, \xi_m) \leq d(\xi_0, \xi_1)^{(ks)^n/1-ks}. \tag{51}$$

As $n \longrightarrow \infty$, $k < 1/s \Rightarrow ks < 1$ that implies $\{\xi_n\}$ is a multiplicative Cauchy sequence. Since (X, d) is complete, then $\{\eta_n\}$ is convergent such that

$$\lim_{n\longrightarrow\infty} d(\xi_n, \xi_m) = 1. \tag{52}$$

As $\{\xi_{2n+2}\}$ is a Cauchy sequence in $f(X)$ and $f(X)$ is joint orbitally complete, then there exists $\eta^* \in X$ such that

$$\lim_{n\longrightarrow\infty} \xi_{2n+2} = \eta^*, \quad \eta^* = fz\ \forall z \in X. \tag{53}$$

We prove that $\eta^* \in \{Fz\}_\alpha$ and

$$\begin{aligned} d(\eta^*, \{Fz\}_\alpha) &= d(fz, \{Fz\}_\alpha) \le d(fz, \eta^*)^s \cdot d(\eta^*, \{Fz\}_\alpha)^s \\ &= d(ffz, fz)^s \cdot d(\eta^*, \{Fz\}_\alpha)^s \\ &= d\left(f^2 z, f^2 z\right)^s \cdot d(\eta^*, \{Fz\}_\alpha)^s \\ &\le \left\{\max\left\{d(fz, fz), d\left(fz, f^2 z\right), d\left(fz, f^2 z\right)\right\}\right\}^{ks} \\ &\quad \cdot d(\eta^*, \{Fz\}_\alpha)^s \\ &= \{\max\{d(fz, fz), d(fz, fz) \cdot d(fz, fz)\}\}^{ks} \\ &\quad \cdot d(\eta^*, \{Fz\}_\alpha)^s \\ &= \{\max\{d(\eta^*, \eta^*), d(\eta^*, \eta^*), d(\eta^*, \eta^*)\}\}^{ks} \\ &\quad \cdot d(\eta^*, \{Fz\}_\alpha)^s \\ &= d(\eta^*, \eta^*)^{ks} \cdot d(\eta^*, \{Fz\}_\alpha)^s. \\ d(\eta^*, \{Fz\}_\alpha)^{1-s} &\le d(\eta^*, \eta^*)^{ks} \\ &\le \left(d(\eta^*, \xi_{2n+2})^s \cdot d(\xi_{2n+2}, \eta^*)^s\right)^{ks}. \end{aligned} \tag{54}$$

As $n \longrightarrow \infty$, then $d(\eta^*, \xi_{2n+2}) \longrightarrow_{bd} 1$. However, ξ_{2n+2} is a convergent sequence; i.e.,

$$\begin{aligned} d(\eta^*, \{Fz\}_\alpha)^{1-s} &\le 1, \\ d(\eta^*, \{Fz\}_\alpha) &\le 1. \end{aligned} \tag{55}$$

Then, we have $d(\eta^*, \{Fz\}_\alpha) = 1$ and Corollary 5 illustrates that $\eta^* \in \overline{\{Fz\}_\alpha} = \{Fz\}_\alpha$.

Since $\{\eta^*\} = \{fz\} \subset \{Fz\}_\alpha \subset g(X)$, there exists $w \in X$ such that $\eta^* = gw$.

Similar to the previous steps, we can prove that $\eta^* = gw \in \{Gw\}_\alpha$.

As two pairs (G, g) and (F, f) are D-compatible,

$$\begin{aligned} \{\eta^*\} &= \{fz\} \subset Fz \quad \text{and}\ \{\eta^*\} \\ &= \{gw\} \subset \{Gw\}_\alpha, \end{aligned} \tag{56}$$

and therefore

$$\begin{aligned} \{f\eta^*\} &= \{ffz\} \subset \{fFz\}_\alpha \subset \{Ffz\}_\alpha \\ &= \{F\eta^*\}_\alpha, \end{aligned} \tag{57}$$

$$\begin{aligned} \{g\eta^*\} &= \{ggw\} \subset \{gGw\}_\alpha \subset \{Ggw\}_\alpha \\ &= \{G\eta^*\}_\alpha. \end{aligned} \tag{58}$$

Now, we show that $\eta^* = f\eta^*$ and $\eta^* = g\eta^*$. Since $\{\xi_n\}$ and $\{\eta_n\}$ are convergent sequences, then

$$\begin{aligned} \eta^* &= \lim_{n\longrightarrow\infty} \xi_{2n+2} \\ &= \lim_{n\longrightarrow\infty} f\eta_{2n+2} \\ &= f \lim_{n\longrightarrow\infty} \eta_{2n+2} \\ &= f\eta^*, \end{aligned} \tag{59}$$

$$\begin{aligned} \eta^* &= \lim_{n\longrightarrow\infty} \xi_{2n+1} \\ &= \lim_{n\longrightarrow\infty} g\eta_{2n+1} \\ &= g \lim_{n\longrightarrow\infty} \eta_{2n+1} \\ &= g\eta^*. \end{aligned} \tag{60}$$

Then, f, g, F, and G have a common fixed point. □

Theorem 8. *Suppose that (X, d) is a complete bd-multiplicative metric space, $Y \subset X$, and two continuous mappings g, $f: Y \longrightarrow X$ satisfy*

$$\begin{aligned} d(f\eta, f\xi) &\le \{\max\{d(\eta, \xi), d(\eta, f\eta), d(\xi, f\xi)\}\}^k, \\ d(g\eta, g\xi) &\le \{\max\{d(\eta, \xi), d(\eta, g\eta), d(\xi, g\xi)\}\}^k, \end{aligned} \tag{61}$$

where η, $\xi \in Y$, $k \in [0, 1/s)$, and $\{F_n\}: Y \longrightarrow W^*(X)$, such that

(i) $\{F_l Y\}_\alpha \subset f(Y)$, $\{F_k Y\}_\alpha \subset g(Y)$, $k = 2n+1$, $l = 2n+2$, and $n \in \mathbb{N}$

(ii) The pairs (F_k, f) and (F_l, g) are D-compatible and occasionally idempotent mappings

Then, there exist $\eta^* \in Y$ such that $\eta^* = f\eta^* = g\eta^*$ and $\eta^* \in \cap_{n=0}^{\infty} \{F_n \eta^*\}_\alpha$.

4. Applications

In this section, we give some applications on our main results. We state some of our theorems on Cartesian product without proof.

Theorem 9. *Suppose that (X, d) is a complete bd-multiplicative metric space. The map $g: D^2 = D \times D \longrightarrow D^2$, $D \subseteq X$ satisfies*

$$d(g(a, c), g(b, d)) \le d((a, c), (b, d))^k, \tag{62}$$

where (a, c), $(b, d) \in D^2$ and $k \in [0, 1/s)$. Then, g has a unique fixed point.

The next example illustrates the previous theory.

Example 7. Suppose that $X = [1, \infty)$ and $D = \{(1, \eta): \eta \in X\}$. Define $d: D^2 \longrightarrow [1, \infty)$ as

$$d((a, c), (b, d)) = a^{((c/b) - 1 + (d/a) - 1)^2}, \tag{63}$$

$\forall\ (a, c)$, $(b, d) \in D$, $a \ge 1$. Then, (X, d) is a complete bd-multiplicative metric space with $s = 2$.

Let $g: D \longrightarrow D$ be a function defined by $g((1,\eta)) = (1, (\eta+1)/2)$. Then, condition (3) holds. However,

$$\begin{aligned} d(g(1,\eta), g(1,\xi)) &= d(1, (1+\eta)/2), (1, (1+\xi)/2) \\ &= a^{((\eta-1)/2+\xi-1/2)^2} \\ &= a^{1/4(\eta-1+\xi-1)^2} \\ &= d((1,\eta),(1,\xi))^{1/4}, \end{aligned} \tag{64}$$

with $k = 1/4$. It is obvious that $(1,1) \in D^2$ is a unique fixed point of a map g.

Theorem 10. *Suppose that (X,d) is a complete bd-multiplicative metric space and $g: D^2 \longrightarrow D^2$, $k \in [0,1/s)$, such that*

$$\begin{aligned} d(g(a,c), g(b,d)) \le \{\max\{d((a,c),(b,d)), d((a,c), g(a,c)), \\ \cdot\, d((b,d), g(b,d))\}\}^k. \end{aligned} \tag{65}$$

Then, g has a unique fixed point.

Theorem 12. *Suppose that (X,d) is a complete bd-multiplicative metric space and two continuous mappings g, $f: D^2 \longrightarrow D^2$ satisfy*

$$\begin{aligned} d(f(a,c), \quad f(b,d)) \le d((a,c),(b,d))^k, \\ d(g(a,c), \quad g(b,d)) \le d((a,c),(b,d))^k, \end{aligned} \tag{66}$$

where $(a,c), (b,d) \in D^2$, $k \in [0,1/s)$, and two fuzzy mappings G, $F: D^2 \longrightarrow W^*(X)$, such that

(i) $\{GD^2\}_\alpha \subset f(D^2)$ and $\{FD^2\}_\alpha \subset g(D^2)$

(ii) The pairs (F,f) and (G,g) are D-compatible and occasionally idempotent mappings

Then f, g, F, and G have a common fixed point.

Theorem 13. *Suppose that (X,d) is a complete bd-multiplicative metric space, $D \subset X$, and two mappings g, $f: D^2 \longrightarrow D^2$ satisfy*

$$\begin{aligned} d(f(a,c), f(b,d)) \le d((a,c),(b,d))^k, \\ d(g(a,c), g(b,d)) \le d((a,c),(b,d))^k, \end{aligned} \tag{67}$$

where $(a,c), (b,d) \in D^2$, $k \in [0,1/s)$, and $\{F_n\}: D^2 \longrightarrow W^*(X)$ such that

(i) $\{F_l D^2\}_\alpha \subset f(D^2)$, $\{F_k D^2\}_\alpha \subset g(D^2)$, $k = 2n+1$, $l = 2n+2$, and $n \in \mathbb{N}$

(ii) The pairs (F_k, f) and (F_l, g) are D-compatible and occasionally idempotent mappings

Then, there exists $\eta^* \in D \times D$ such that $\eta^* = f\eta^* = g\eta^*$ and $\eta^* \in \cap_{n=0}^{\infty}\{F_n \eta^*\}_\alpha$.

Theorem 14. *Suppose that (X,d) is a complete bd-multiplicative metric space and two mappings g, $f: D^2 \longrightarrow D^2$, $D \subset X$, satisfy*

$$\begin{aligned} d(f(a,c), f(b,d)) \le \{\max\{d((a,c),(b,d)), d((a,c), f(a,c)), d((b,d), f(b,d))\}\}^k, \\ d(g(a,c), g(b,d)) \le \{\max\{d((a,c),(b,d)), d((a,c), g(a,c)), d((b,d), g(b,d))\}\}^k, \end{aligned} \tag{68}$$

where $(a,c), (b,d) \in D^2$, $k \in [0,1/s)$, $\{F_n\}_\alpha: D^2 \longrightarrow W^*(X)$, such that

(i) $\{F_l D^2\}_\alpha \subset f(D^2)$, $\{F_k D^2\}_\alpha \subset g(D^2)$, $k = 2n+1$, $l = 2n+2$, and $n \in \mathbb{N}$

(ii) The pairs (F_k, f) and (F_l, g) are D-compatible and occasionally idempotent mappings

Then, there exists $\eta^* \in D^2$ such that $\eta^* = f\eta^* = g\eta^*$ and $\eta^* \in \cap_{n=0}^{\infty}\{F_n \eta^*\}_\alpha$.

5. Conclusion

In this paper, we introduced the concept of *bd*-multiplicative metric spaces. We studied some of the fixed point theorems in these spaces. Also, we obtain common fixed point theorems for fuzzy mappings in complete *bd*-multiplicative metric spaces. Finally, we get some of applications on our main findings. We hope that our presented idea herein will be a source of motivation for other researchers to extend and improve these findings for their applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Double Controlled Partial Metric Type Spaces and Convergence Results

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In this paper, we firstly propose the notion of double controlled partial metric type spaces, which is a generalization of controlled metric type spaces, partial metric spaces, and double controlled metric type spaces. Secondly, our aim is to study the existence of fixed points for Kannan type contractions in the context of double controlled partial metric type spaces. The proposed results enrich, theorize, and sharpen a multitude of pioneer results in the context of metric fixed point theory. Additionally, we provide numerical examples to illustrate the essence of our obtained theoretical results.

1. Introduction and Preliminaries

The study of fixed points of given mappings satisfying certain contractive conditions in various abstract spaces has been at the middle of vigorous research activity. Banach contraction mapping principle has attracted the eye of the many authors to generalize, extend, and improve the metric fixed point theory. For this purpose, the authors considered the extension of metric fixed point theory to different abstract spaces such as symmetric spaces, quasimetric spaces, fuzzy metric spaces, partial metric spaces, probabilistic metric spaces, and spaces with graph.

The notion of b-metric spaces was first presented by Bakhtin [1] and Czerwik [2]. Many writers have since obtained a number of fixed point solutions in b-metric spaces for single and multivalued operators. We reference Kamran et al. [3] (see also [4, 5]), who presented extended b-metric spaces by manipulating the triangle inequality rather than utilizing control functions, as one of the generalizations concerning b-metric spaces. Following that, in 2018, Abdeljawad et al. [6, 7] established the concepts of controlled metric type spaces and double controlled metric type spaces, respectively. Souayah and Mrad [8] proposed a more broad idea of controlled partial metric type spaces in 2019. It is useful to establish the extensions of the contraction principle from metric spaces to b-metric spaces, and therefore the controlled metric type of spaces is useful to prove the existence and uniqueness of theorems for many forms of integral and differential equations. Some interesting applications can be found in the recent papers [4, 9–15]. It is always interesting to find novel applications dealing with engineering science and technology using fixed point technique.

On the other hand, the notion of partial metric space was given by Matthews [16, 17] in 1992, which is the generalization of the usual metric space in which $d(x, x)$ is not zero. After that, many researchers worked on the partial metric type spaces to discover the existence of fixed point and their uniqueness. In 2019, Gu and Shatanawi [18] expounded some coupled fixed point theorems in the context of partial metric spaces for hybrid pairs of mappings satisfying a symmetric type contraction. In 2020, Nguyen and Tram [19] demonstrated various fixed point results involving involution mappings. Recently, in 2021, Javaid et al. [20]

propounded fixed point results in the setting orthogonal partial metric spaces with application. Researchers can refer to [14, 21–23] for further information on fixed points in partial type metric spaces.

Taking into consideration the facts mentioned above, in this article, we introduce the concept of double controlled partial metric type space, which is an extension of the controlled metric type spaces, double controlled metric type spaces, and controlled partial metric type spaces. We also look into the existence and uniqueness of fixed point results, which are Kannan contractions' extensions.

Let us begin by reviewing the definition of double controlled metric space as follows.

Definition 1 (see [6]). Let X be a nonempty set and consider the functions α, μ: $X \times X \longrightarrow [1, \infty)$.

Let d: $X \times X \longrightarrow [0, \infty)$ satisfy

(1) $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$,

(2) $d(x_1, x_2) = d(x_2, x_1)$,

(3) $d(x_1, x_2) \leq \alpha(x_1, x_3) d(x_1, x_3) + \mu(x_3, x_2) d(x_3, x_2)$, for all $x_1, x_2, x_3 \in X$, then (X, d) is called a double controlled metric type space.

2. Double Controlled Partial Metric Type Spaces

The following is the formal definition of the double controlled partial metric type space which generalizes the notation of controlled metric type spaces, double controlled metric type spaces, and partial metric spaces.

Definition 2. Let X be a nonempty set consider α, μ: $X \times X \longrightarrow [1, \infty)$ be a function.

Let d: $X \times X \longrightarrow [0, \infty)$ satisfy

(1) $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$,

(2) $d(x_1, x_2) = d(x_2, x_1)$,

(3) $d(x_1, x_1) \leq d(x_1, x_2)$,

(4) $d(x_1, x_2) \leq \alpha(x_1, x_3) d(x_1, x_3) + \mu(x_3, x_2) d(x_3, x_2)$, for all x_1, x_2, $x_3 \in X$, then (X, d) is called a double controlled partial metric type space.

Note that double controlled partial metric type space is more extensive than the double controlled metric type space.

Example 1. A double controlled partial metric type space is not necessarily a double controlled metric type space.

Let $X = \{0, 1, 2, 3, 4\}$ and take d: $X \times X \longrightarrow [0, \infty)$. Consider α, μ: $X \times X \longrightarrow [1, \infty)$, where

$$\begin{aligned} \alpha(x, y) &= d(x, y) + 5, \\ \mu(x, y) &= d(x, y) + 7. \end{aligned} \tag{1}$$

Let the metric d be defined by the following (Table 1).

It is easy to verify that $(p1)$ and $(p2)$ are true.

We prove condition (3) with different cases, that is, $d(x_1, x_1) \leq d(x_1, x_2)$, for all $x_1, x_2 \in X$ and $x_1 \neq x_2$.

Table 1: Metric d defined in Example 1.

d	0	1	2	3	4
0	1/27	1/4	1/5	1/6	1/7
1	1/4	1/28	2/7	2/9	2/11
2	1/5	2/7	1/29	3/11	3/13
3	1/6	2/9	3/11	1/28	4/13
4	1/7	2/11	3/13	4/13	1/27

Case (i): let $d(x_1, x_1) = d(0, 0) = (1/27)$, $d(0, 0) \leq d(x_1, x_2)$, satisfied for all $x_1, x_2 \in X$ and $x_1 \neq x_2$.

Case (ii): let $d(x_1, x_1) = d(1, 1) = (1/28)$, $d(1, 1) \leq d(x_1, x_2)$, satisfied for all $x_1, x_2 \in X$ and $x_1 \neq x_2$

Case (iii): let $d(x_1, x_1) = d(2, 2) = (1/29)$, $d(2, 2) \leq d(x_1, x_2)$, satisfied for all l $x_1, x_2 \in X$ and $x_1 \neq x_2$.

Case (iv): let $d(x_1, x_1) = d(3, 3) = (1/28)$, $d(3, 3) \leq d(x_1, x_2)$, satisfied for all $x_1, x_2 \in X$ and $x_1 \neq x_2$.

Case (v): let $d(x_1, x_1) = d(4, 4) = (1/27)$, $d(4, 4) \leq d(x_1, x_2)$, satisfied for all $x_1, x_2 \in X$ and $x_1 \neq x_2$.

Now, we will prove the property $(p4)$.

Case (i): to satisfy $d(0, 0)$, we have

$$\begin{aligned} &d(0, 0) \leq \alpha(0, 0) d(0, 0) + \mu(0, 0) d(0, 0) \\ &0.0370 \leq 0.4513, \\ &d(0, 0) \leq \alpha(0, 1) d(0, 1) + \mu(1, 0) d(1, 0) \\ &0.0370 \leq 3.1250, \\ &d(0, 0) \leq \alpha(0, 2) d(0, 2) + \mu(2, 0) d(2, 0) \\ &0.0370 \leq 2.48, \\ &d(0, 0) \leq \alpha(0, 3) d(0, 3) + \mu(3, 0) d(3, 0) \\ &0.0370 \leq 2.0555, \\ &d(0, 0) \leq \alpha(0, 4) d(0, 4) + \mu(4, 0) d(4, 0) \\ &0.0370 \leq 1.7551. \end{aligned} \tag{2}$$

Case (ii): now, we have to satisfy $d(0, 1) = d(1, 0)$:

$$\begin{aligned} &d(0, 1) \leq \alpha(0, 0) d(0, 0) + \mu(0, 1) d(0, 1) \\ &0.25 \leq 1.9990, \\ &d(0, 1) \leq \alpha(0, 1) d(0, 1) + \mu(1, 1) d(1, 1) \\ &0.25 \leq 1.5637, \\ &d(0, 1) \leq \alpha(0, 2) d(0, 2) + \mu(2, 1) d(2, 1) \\ &0.25 \leq 3.1216, \\ &d(0, 1) \leq \alpha(0, 3) d(0, 3) + \mu(3, 1) d(3, 1) \\ &0.25 \leq 2.4660, \\ &d(0, 1) \leq \alpha(0, 4) d(0, 4) + \mu(4, 1) d(4, 1) \\ &0.25 \leq 2.0404. \end{aligned} \tag{3}$$

Case (iii): to prove $d(0, 2) = d(2, 0)$, we have

$$\begin{aligned}
&d(0,2) \le \alpha(0,0)d(0,0) + \mu(0,2)d(0,2)\\
&0.2 \le 1.6265,\\
&d(0,2) \le \alpha(0,1)d(0,1) + \mu(1,2)d(1,2)\\
&0.2 \le 3.3941,\\
&d(0,2) \le \alpha(0,2)d(0,2) + \mu(2,2)d(2,2)\\
&0.2 \le 1.2825,\\
&d(0,2) \le \alpha(0,3)d(0,3) + \mu(3,2)d(3,2)\\
&0.2 \le 2.8445,\\
&d(0,2) \le \alpha(0,4)d(0,4) + \mu(4,2)d(4,2)\\
&0.2 \le 2.4033.
\end{aligned} \tag{4}$$

Case (iv): in order to show $d(0,3) = d(3,0)$, we proceed as follows:

$$\begin{aligned}
&d(0,3) \le \alpha(0,0)d(0,0) + \mu(0,3)d(0,3)\\
&0.1666 \le 1.3810,\\
&d(0,3) \le \alpha(0,1)d(0,1) + \mu(1,3)d(1,3)\\
&0.1666 \le 2.9174,\\
&d(0,3) \le \alpha(0,2)d(0,2) + \mu(2,3)d(2,3)\\
&0.1666 \le 3.0234,\\
&d(0,3) \le \alpha(0,3)d(0,3) + \mu(3,3)d(3,3)\\
&0.1666 \le 1.1123,\\
&d(0,3) \le \alpha(0,4)d(0,4) + \mu(4,3)d(4,3)\\
&0.1666 \le 2.9832.
\end{aligned} \tag{5}$$

Case (v): now, we have to satisfy $d(0,4) = d(4,0)$:

$$\begin{aligned}
&d(0,4) \le \alpha(0,0)d(0,0) + \mu(0,4)d(0,4)\\
&0.1428 \le 1.2069,\\
&d(0,4) \le \alpha(0,1)d(0,1) + \mu(1,4)d(1,4)\\
&0.1428 \le 2.6182,\\
&d(0,4) \le \alpha(0,2)d(0,2) + \mu(2,4)d(2,4)\\
&0.1428 \le 2.7086,\\
&d(0,4) \le \alpha(0,3)d(0,3) + \mu(3,4)d(3,4)\\
&0.1428 \le 3.1096,\\
&d(0,4) \le \alpha(0,4)d(0,4) + \mu(4,4)d(4,4)\\
&0.1428 \le 0.9953.
\end{aligned} \tag{6}$$

Case (vi): for the case $d(1,1)$, we have

$$\begin{aligned}
&d(1,1) \le \alpha(1,0)d(1,0) + \mu(0,1)d(0,1)\\
&0.03571 \le 3.125,\\
&d(1,1) \le \alpha(1,1)d(1,1) + \mu(1,1)d(1,1)\\
&0.03571 \le 0.4311,\\
&d(1,1) \le \alpha(1,2)d(1,2) + \mu(2,1)d(2,1)\\
&0.03571 \le 3.5918,\\
&d(1,1) \le \alpha(1,3)d(1,3) + \mu(3,1)d(3,1)\\
&0.03571 \le 2.7654,\\
&d(1,1) \le \alpha(1,4)d(1,4) + \mu(1,4)d(4,1)\\
&0.03571 \le 2.2479.
\end{aligned} \tag{7}$$

Case (vii): to satisfy $d(1,2) = d(2,1)$, we have

$$\begin{aligned}
&d(1,2) \le \alpha(1,0)d(1,0) + \mu(0,2)d(0,2)\\
&0.2857 \le 2.7525,\\
&d(1,2) \le \alpha(1,1)d(1,1) + \mu(1,2)d(1,2)\\
&0.2857 \le 2.2614,\\
&d(1,2) \le \alpha(1,2)d(1,2) + \mu(2,2)d(2,2)\\
&0.2857 \le 1.7527,\\
&d(1,2) \le \alpha(1,3)d(1,3) + \mu(3,2)d(3,2)\\
&0.2857 \le 3.1439,\\
&d(1,2) \le \alpha(1,4)d(1,4) + \mu(4,2)d(4,2)\\
&0.2857 \le 2.6107.
\end{aligned} \tag{8}$$

Case (viii): now, we have to satisfy $d(1,3) = d(3,1)$:

$$\begin{aligned}
&d(1,3) \le \alpha(1,0)d(1,0) + \mu(0,3)d(0,3)\\
&0.2222 \le 2.5069,\\
&d(1,3) \le \alpha(1,1)d(1,1) + \mu(1,3)d(1,3)\\
&0.2222 \le 1.7847,\\
&d(1,3) \le \alpha(1,2)d(1,2) + \mu(2,3)d(2,3)\\
&0.2222 \le 3.4936,\\
&d(1,3) \le \alpha(1,3)d(1,3) + \mu(3,3)d(3,3)\\
&0.2222 \le 1.4117,\\
&d(1,3) \le \alpha(1,4)d(1,4) + \mu(4,3)d(4,3)\\
&0.2222 \le 3.19066.
\end{aligned} \tag{9}$$

Case (ix): for the case $d(1,4) = d(4,1)$, consider the following:

$$\begin{aligned}
&d(1,4) \le \alpha(1,0)d(1,0) + \mu(0,4)d(0,4)\\
&0.1818 \le 2.3329,\\
&d(1,4) \le \alpha(1,1)d(1,1) + \mu(1,4)d(1,4)\\
&0.1818 \le 1.4856,\\
&d(1,4) \le \alpha(1,2)d(1,2) + \mu(2,4)d(2,4)\\
&0.1818 \le 3.1788,\\
&d(1,4) \le \alpha(1,3)d(1,3) + \mu(3,4)d(3,4)\\
&0.1818 \le 3.4090,\\
&d(1,4) \le \alpha(1,4)d(1,4) + \mu(4,4)d(4,4)\\
&0.1818 \le 1.2027.
\end{aligned} \tag{10}$$

Case (x): for the case $d(2,2)$, we have

$$\begin{aligned}
&d(2,2) \le \alpha(2,0)d(2,0) + \mu(0,2)d(0,2)\\
&0.0344 \le 2.48,\\
&d(2,2) \le \alpha(2,1)d(2,1) + \mu(1,2)d(1,2)\\
&0.0344 \le 3.5918,\\
&d(2,2) \le \alpha(2,2)d(2,2) + \mu(2,2)d(2,2)\\
&0.0344 \le 0.4161,\\
&d(2,2) \le \alpha(2,3)d(2,3) + \mu(2,3)d(2,3)\\
&0.0344 \le 3.4214,\\
&d(2,2) \le \alpha(2,4)d(2,4) + \mu(4,2)d(4,2)\\
&0.0344 \le 2.8757.
\end{aligned} \tag{11}$$

Case (xi): to satisfy $d(2,3) = d(3,2)$, we proceed as follows:

$$\begin{aligned}
&d(2,3) \le \alpha(2,0)d(2,0) + \mu(0,3)d(0,3)\\
&0.2727 \le 2.2344,\\
&d(2,3) \le \alpha(2,1)d(2,1) + \mu(1,3)d(1,3)\\
&0.2727 \le 3.1151,\\
&d(2,3) \le \alpha(2,2)d(2,2) + \mu(2,3)d(2,3)\\
&0.2727 \le 2.1570,\\
&d(2,3) \le \alpha(2,3)d(2,3) + \mu(3,3)d(3,3)\\
&0.2727 \le 1.6892,\\
&d(2,3) \le \alpha(2,4)d(2,4) + \mu(4,3)d(4,3)\\
&0.2727 \le 3.4556.
\end{aligned} \tag{12}$$

Case (xii): next, we have to satisfy $d(2,4) = d(4,2)$:

$$\begin{aligned}
&d(2,4) \le \alpha(2,0)d(2,0) + \mu(0,4)d(0,4)\\
&0.2307 \le 2.0604,\\
&d(2,4) \le \alpha(2,1)d(2,1) + \mu(1,4)d(1,4)\\
&0.2307 \le 2.8159,\\
&d(2,4) \le \alpha(2,2)d(2,2) + \mu(2,4)d(2,4)\\
&0.2307 \le 1.8422,\\
&d(2,4) \le \alpha(2,3)d(2,3) + \mu(3,4)d(3,4)\\
&0.2307 \le 3.6865,\\
&d(2,4) \le \alpha(2,4)d(2,4) + \mu(4,4)d(4,4)\\
&0.2307 \le 1.4677.
\end{aligned} \tag{13}$$

Case(xiii): now, for the case $d(3,3)$, we consider

$$\begin{aligned}
&d(3,3) \le \alpha(3,0)d(3,0) + \mu(0,3)d(0,3)\\
&0.03571 \le 2.0555,\\
&d(3,3) \le \alpha(3,1)d(3,1) + \mu(1,3)d(1,3)\\
&0.03571 \le 2.7654,\\
&d(3,3) \le \alpha(3,2)d(3,2) + \mu(2,3)d(2,3)\\
&0.03571 \le 3.4214,\\
&d(3,3) \le \alpha(3,3)d(3,3) + \mu(3,3)d(3,3)\\
&0.03571 \le 0.4311,\\
&d(3,3) \le \alpha(3,4)d(3,4) + \mu(4,3)d(4,3)\\
&0.03571 \le 3.8816.
\end{aligned} \tag{14}$$

Case (xiv): now, we have to satisfy $d(3,4) = d(4,3)$:

$$\begin{aligned}
&d(3,4) \le \alpha(3,0)d(3,0) + \mu(0,4)d(0,4)\\
&0.3076 \le 1.8855,\\
&d(3,4) \le \alpha(3,1)d(3,1) + \mu(1,4)d(1,4)\\
&0.3076 \le 2.4662,\\
&d(3,4) \le \alpha(3,2)d(3,2) + \mu(2,4)d(2,4)\\
&0.3076 \le 3.1066,\\
&d(3,4) \le \alpha(3,3)d(3,3) + \mu(3,4)d(3,4)\\
&0.3076 \le 2.4283,\\
&d(3,4) \le \alpha(3,4)d(3,4) + \mu(4,4)d(4,4)\\
&0.3076 \le 1.8937.
\end{aligned} \tag{15}$$

Case (xv): lastly, for the case $d(4,4)$, we have

$$\begin{aligned}
&d(4,4) \le \alpha(4,0)d(4,0) + \mu(0,4)d(0,4)\\
&0.0370 \le 1.7551,\\
&d(4,4) \le \alpha(4,1)d(4,1) + \mu(1,4)d(1,4)\\
&0.0370 \le 2.2479,\\
&d(4,4) \le \alpha(4,2)d(4,2) + \mu(2,4)d(2,4)\\
&0.0370 \le 2.8757,\\
&d(4,4) \le \alpha(4,3)d(4,3) + \mu(3,4)d(3,4)\\
&0.0370 \le 3.8816,\\
&d(4,4) \le \alpha(4,4)d(4,4) + \mu(4,4)d(4,4)\\
&0.0370 \le 0.4471.
\end{aligned} \tag{16}$$

Therefore, (X,d) is a double controlled partial metric type space but is not a double controlled metric type space since $d(x,x)$ is not equal to zero all the time.

We define Cauchy and convergent sequence in double controlled partial metric type spaces as follows.

Definition 3. Let (X,d) be a double controlled partial metric type space; the sequence $\{x_n\}_{n\ge 0}$ converges to some x in X, if $\lim_{n,m\longrightarrow\infty} d(x_n,x) = d(x,x)$; in this case, we write $\lim_{n\longrightarrow\infty} x_n = x$.

Definition 4. The sequence $\{x_n\}$ in a double controlled partial metric type space (X,d) is said to be Cauchy sequence, if $\lim_{n,m\longrightarrow\infty} d(x_n,x_m)$ exists and is finite.

Definition 5. A double controlled partial metric type space (X, d) is said to be complete if every Cauchy sequence x in X converges to a point $x \in X$, that is, $d(x, x) = \lim_{n,m \longrightarrow \infty} d(x_n, x_m)$.

Definition 6. Let (X, d) be a double controlled partial metric type space. Let $x \in X$ and $\varepsilon > 0$.

(i) The open ball $B_p(x, \varepsilon)$ is

$$B_p(x, \varepsilon) = \{y \in X, d(x, y) < d(x, x) + \varepsilon\}. \tag{17}$$

(ii) The mapping $T\colon X \longrightarrow X$ is said to be continuous at $x \in X$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$T\big(B_p(x, \delta)\big) \subseteq B_p(Tx, \varepsilon). \tag{18}$$

Therefore, if T is continuous at x in the double controlled partial metric type space (X, d), then $x_n \longrightarrow x$ implies that $Tx_n \longrightarrow Tx$ as $n \longrightarrow \infty$

3. Some Novel Results

This section is devoted to discuss some fixed point results in double controlled partial metric type space (X, d). The main result of this article is given by the following theorem.

Theorem 1. *Let (X, d) be a complete double controlled partial metric type space by the functions $\alpha, \mu\colon X \times X \longrightarrow [1, \infty)$. Suppose that $f\colon X \longrightarrow X$ satisfies*

$$d(fx, fy) \leq \beta[d(x, fx) + (y, fy)], \tag{19}$$

for all $x, y \in X$, where $\beta \in (0, (1/2))$. For $x_0 \in X$, take $x_n = f^n x_0$, assuming that

$$\sup_{m \geq 1} \lim_{i \longrightarrow \infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})} \mu(x_i, x_m) < \frac{1}{k}, \quad \text{where } k \in (0, 1). \tag{20}$$

Furthermore, assume that for every $x \in X$, $\lim_{n \longrightarrow \infty} \alpha(x, x_n)$, $\lim_{n \longrightarrow \infty} \alpha(x_n, x)$, $\lim_{n \longrightarrow \infty} \mu(x, x_n)$, and $\lim_{n \longrightarrow \infty} \mu(x_n, x)$ exist and are finite. Then, the sequence $\{x_n\}$ converges to some $u \in X$; moreover, if α and μ satisfy the following assumptions,

$$\lim_{n \longrightarrow \infty} \frac{\alpha(u, x_{n+1})}{1 - \beta\mu(x_{n+1}, fu)} \leq 0, \tag{21}$$

then f has a unique fixed point.

Proof. Consider $x_n = f^n x_0$, let $x_1 \in X$ be arbitrary, and let $x_2 = fx_1$ and let $x_3 = fx_2$ be chosen.

By using (19), we get

$$\begin{aligned} d(x_2, x_3) &= d(fx_1, fx_2) \leq \beta[d(x_1, fx_1) + d(x_2, fx_2)] \\ &= \beta[d(x_1, x_2) + d(x_2, x_3)]. \end{aligned} \tag{22}$$

Then,

$$d(x_2, x_3) \leq \frac{\beta}{1 - \beta} d(x_1, x_2), \quad \text{where } \frac{\beta}{1 - \beta} = \eta \in [0, 1). \tag{23}$$

By repeating the same procedure in inequality (23), we obtain

$$d(x_n, x_{n+1}) \leq \eta^{n-1} d(x_1, x_2). \tag{24}$$

Now, we have to show that $\{x_n\}$ is Cauchy sequence. Since (X, d) is a double controlled partial metric type space, for all natural numbers $n, m \in N$ with $n < m$, we acquire

$$\begin{aligned} d(x_n, x_m) &\leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) + \mu(x_{n+1}, x_m) d(x_{n+1}, x_m) \\ &\leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_{n+2}) \mu(x_{n+1}, x_m) d(x_{n+1}, x_{n+2}) \\ &\quad + \mu(x_{n+1}, x_m) \mu(x_{n+2}, x_m) d(x_{n+2}, x_m) \\ &\leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_{n+2}) \mu(x_{n+1}, x_m) d(x_{n+1}, x_{n+2}) \\ &\quad + \alpha(x_{n+2}, x_{n+3}) \mu(x_{n+1}, x_m) \mu(x_{n+2}, x_m) d(x_{n+2}, x_{n+3}) \\ &\quad + \mu(x_{n+1}, x_m) \mu(x_{n+2}, x_m) \mu(x_{n+3}, x_m) d(x_{n+3}, x_m) \\ &\leq \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \mu(x_j, x_m) \right) \alpha(x_i, x_{i+1}) d(x_i, x_{i+1}) \\ &\quad + \prod_{k=n+1}^{m-1} \mu(x_k, x_m) d(x_{m-1}, x_m) \end{aligned}$$

$$\begin{aligned}
&\leq \alpha(x_n,x_{n+1})\eta^n d(x_0,x_1)+\sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(x_j,x_m)\right)\alpha(x_i,x_{i+1})\eta^i d(x_0,x_1)\\
&\quad+\prod_{k=n+1}^{m-1}\mu(x_k,x_m)\eta^{m-1}d(x_0,x_1)\\
&\quad+\prod_{k=n+1}^{m-1}\mu(x_k,x_m)\alpha(x_{m-1},x_m)\eta^{m-1}d(x_0,x_1)\\
&=\alpha(x_n,x_{n+1})\eta^n d(x_0,x_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(x_j,x_m)\right)\alpha(x_i,x_{i+1})\eta^i d(x_0,x_1)\\
&\leq \alpha(x_n,x_{n+1})\eta^n d(x_0,x_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(x_j,x_m)\right)\alpha(x_i,x_{i+1})\eta^i d(x_0,x_1)\\
&\leq \alpha(x_n,x_{n+1})\eta^n d(x_0,x_1)+\sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(x_j,x_m)\right)\alpha(x_i,x_{i+1})\eta^i d(x_0,x_1).
\end{aligned} \tag{25}$$

Assume that

$$S_p=\sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\mu(x_j,x_m)\right)\alpha(x_i,x_{i+1})\eta^i d(x_0,x_1). \tag{26}$$

Then, we obtain

$$d(x_n,x_m)\leq d(x_0,x_1)\left[\eta^n\alpha(x_n,x_{n+1})+(S_{m-1}-S_n)\right]. \tag{27}$$

Using ratio test, we have

$$a_i=\left(\prod_{j=0}^{i}\mu(x_j,x_m)\right)\alpha(x_i,x_{i+1})\eta^i d(x_0,x_1),\quad \text{where } \frac{a_{i+1}}{a_i}<\frac{1}{\eta}. \tag{28}$$

Taking $\lim_{n,m\longrightarrow\infty}$, (27) becomes

$$\lim_{n,m\longrightarrow\infty} d(x_n,x_m)=0. \tag{29}$$

This implies that $\{x_n\}$ is a Cauchy sequence in a complete double controlled metric type space (X,d), so $\{x_n\}$ converges to some $u\in X$. Now, we have to prove that u is a fixed point of T, so we need to verify that

$$d(u,fu)=d(u,u)=d(fu,fu). \tag{30}$$

From the $(p3)$, we have

$$\begin{aligned}
d(u,u)&\leq d(u,fu),\\
d(fu,fu)&\leq d(u,fu).
\end{aligned} \tag{31}$$

Hence, for proving $fu=u$, it is sufficient to prove that $d(u,u)\geq d(u,fu)$ and $d(fu,fu)\geq d(u,fu)$. The triangular inequality yields that

$$\begin{aligned}
d(u,fu)&\leq \alpha(u,x_{n+1})d(u,x_{n+1})+\mu(x_{n+1},fu)d(x_{n+1},fu)\\
&\leq \alpha(u,x_{n+1})d(u,x_{n+1})+\mu(x_{n+1},fu)d(fx_n,fu)\\
&\leq \alpha(u,x_{n+1})d(u,x_{n+1})+\beta\mu(x_{n+1},fu)d(x_n,fx_n)\\
&\quad+\beta\mu(x_{n+1},fu)d(u,fu).
\end{aligned} \tag{32}$$

Taking limit as $n\longrightarrow\infty$, we obtain

$$d(u,fu)\leq \lim_{n\longrightarrow\infty}\frac{\alpha(u,x_{n+1})}{1-\beta\mu(x_{n+1},fu)d(u,fu)}d(u,u). \tag{33}$$

Utilizing condition (21), we get

$$d(u,fu)\leq d(u,u). \tag{34}$$

On the other hand,

$$\begin{aligned}
d(u,fu)&\leq \alpha(u,fu)d(u,fu)+\mu(fu,fu)d(fu,fu)\\
&\leq \alpha(u,fu)d(u,fu)\\
&\quad+\mu(fu,fu)\beta[d(u,fu)+d(u,fu)]\\
&\leq \alpha(u,fu)d(u,fu)+\beta\mu(fu,fu)d(u,fu)\\
&\quad+\beta\mu(fu,fu)d(u,fu)\\
&\leq \frac{\alpha(u,fu)}{1-\beta\mu(fu,fu)}d(fu,fu).
\end{aligned} \tag{35}$$

Hence, we get

$$d(u,fu)\leq d(u,u). \tag{36}$$

From (31)–(36), we obtain

$$u=fu. \tag{37}$$

Uniqueness: assume that there are two fixed points u and v of T, then

$$\begin{aligned}
d(u,v)=d(fu,fv)&\leq\beta[d(u,fu)+d(v,fv)]\\
&=\beta[d(u,u)+d(v,v)].
\end{aligned} \tag{38}$$

Furthermore, we have

$$d(u,u) = d(fu, fu) \leq 2\beta d(u, fu) = 2\beta d(u,u), \quad (39)$$

where $\beta > 1$, then $d(u,u) = 0$, similarly

$$d(v,v) = d(fv, fv) \leq 2\beta d(v, fv) = 2\beta d(v,v). \quad (40)$$

Then, $d(v,v) = 0$. Since $d(u,u) = d(v,v) = 0$, then $d(u,v) = 0$. Therefore, $d(u,u) = d(v,v) = d(u,v)$, which gives $u = v$ and T has a unique fixed point. □

Definition 7. Let (X,d) be complete double controlled partial type metric space; a mapping $T\colon X \longrightarrow X$ is sequentially convergent. For every sequence $\{x_n\}$, if $\{fx_n\}$ is convergent, then $\{x_n\}$ also converges. Also, f is said to be subsequentially convergent. For every sequence $\{x_n\}$, if $\{fx_n\}$ is convergent, then $\{x_n\}$ has a convergent subsequence.

Theorem 2. *Let (X,d) be a complete double controlled partial metric type space and $f, g\colon X \longrightarrow X$ be mapping such that f is continuous, one-to-one, and subsequentially convergent*

$$d(fgx, fgy) \leq \beta[d(fx, fgx) + (fy, fgy)]. \quad (41)$$

For all $x, y \in X$, where $\beta \in (0, (1/2))$. For $x_0 \in X$, take $x_n = g^n x_0$, assuming that

$$\sup_{m\geq 1} \lim_{i\longrightarrow\infty} \frac{\alpha(fx_{i+1}, fx_{i+2})}{\alpha(fx_i, fx_{i+1})} \mu(fx_i, fx_m) < \frac{1}{k}, \quad \text{where } k \in (0,1). \quad (42)$$

Furthermore, assume that for every $x \in X$, $\lim_{n\longrightarrow\infty} \alpha(x, x_n)$, $\lim_{n\longrightarrow\infty}\alpha(x_n, x)$, $\lim_{n\longrightarrow\infty}\mu(x, x_n)$, and $\lim_{n\longrightarrow\infty}\mu(x_n, x)$ exist and are finite. Then, g has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X and consider the sequence $\{x_n\}$ defined in the hypothesis of the theorem. From (41), we obtain

$$\begin{aligned} d(fx_n, fx_{n+1}) &= d(fgx_{n-1}, fx_n) \\ &\leq \beta[d(fx_{n-1}, fgx_{n-1}) + d(fx_n, fgx_n)] \\ &= \beta[d(fx_{n-1}, fgx_{n-1}) + d(fx_n, fx_{n+1})] \\ &= \frac{\beta}{1-\beta} d(fx_{n-1}, fgx_{n-1}). \end{aligned} \quad (43)$$

By induction, we get

$$d(fx_n, fx_{n+1}) \leq \left(\frac{\beta}{1-\beta}\right)^n d(fgx_{n-1}, fx_n) = \eta^n d(fgx_0, fx_1), \quad (44)$$

where

$$\frac{\beta}{1-\beta} = \eta \in [0,1). \quad (45)$$

Now, we have to show that $\{fx_n\}$ is a Cauchy sequence. Since (X,d) is double controlled partial metric type space for all natural numbers $n, m \in N$ with $n < m$, we get

$$\begin{aligned} d(fx_n, fx_m) &\leq \alpha(fx_n, fx_{n+1})d(fx_n, fx_{n+1}) + \mu(fx_{n+1}, fx_m)d(fx_{n+1}, fx_m) \\ &\leq \alpha(fx_n, fx_{n+1})d(fx_n, fx_{n+1}) + \alpha(fx_{n+1}, fx_{n+2})\mu(fx_{n+1}, fx_m)d(fx_{n+1}, fx_{n+2}) \\ &\quad + \mu(fx_{n+1}, fx_m)\mu(fx_{n+2}, fx_m)d(fx_{n+2}, fx_m) \\ &\leq \alpha(fx_n, fx_{n+1})d(fx_n, fx_{n+1}) + \alpha(fx_{n+1}, fx_{n+2})\mu(fx_{n+1}, fx_m)d(fx_{n+1}, fx_{n+2}) \\ &\quad + \alpha(fx_{n+2}, fx_{n+3})\mu(fx_{n+1}, fx_m)\mu(fx_{n+2}, fx_m)d(fx_{n+2}, fx_{n+3}) \\ &\quad + \mu(fx_{n+1}, fx_m)\mu(fx_{n+2}, fx_m)\mu(fx_{n+3}, fx_m)d(fx_{n+3}, fx_m) \\ &\leq \alpha(fx_n, fx_{n+1})d(fx_n, fx_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu\big(fx_j, fx_m\big)\right)\alpha(fx_i, fx_{i+1})d(fx_i, fx_{i+1}) \\ &\quad + \prod_{k=n+1}^{m-1}\mu(fx_k, fx_m)d(fx_{m-1}, fx_m) \\ &\leq \alpha(fx_n, fx_{n+1})\eta^n d(fx_0, fx_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu\big(fx_j, fx_m\big)\right)\alpha(fx_i, fx_{i+1})\eta^i d(fx_0, fx_1) \\ &\quad + \prod_{k=n+1}^{m-1}\mu(fx_k, fx_m)\eta^{m-1} d(fx_0, fx_1) \end{aligned}$$

$$\leq \alpha(fx_n, fx_{n+1})\eta^n d(fx_0, fx_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\mu(fx_j, fx_m)\right)\alpha(fx_i, fx_{i+1})\eta^i d(fx_0, fx_1)$$
$$+ \prod_{k=n+1}^{m-1}\mu(fx_k, fx_m)\alpha(fx_{m-1}, fx_m)\eta^{m-1}d(fx_0, fx_1)$$
$$= \alpha(fx_n, fx_{n+1})\eta^n d(fx_0, fx_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(fx_j, fx_m)\right)\alpha(fx_i, fx_{i+1})\eta^i d(fx_0, fx_1)$$
$$\leq \alpha(fx_n, fx_{n+1})\eta^n d(fx_0, fx_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(fx_j, fx_m)\right)\alpha(fx_i, fx_{i+1})\eta^i d(fx_0, fx_1)$$
$$\leq \alpha(fx_n, fx_{n+1})\eta^n d(fx_0, fx_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\mu(fx_j, fx_m)\right)\alpha(fx_i, fx_{i+1})\eta^i d(fx_0, fx_1). \quad (46)$$

Assume that

$$S_p = \sum_{i=n+1}^{m-1}\left(\prod_{j=0}^{i}\mu(fx_j, fx_m)\right)\alpha(fx_i, fx_{i+1})\eta^i d(fx_0, fx_1). \quad (47)$$

Then, we obtain

$$d(fx_n, fx_m) \leq d(fx_0, fx_1)\left[\eta^n\alpha(fx_n, fx_{n+1}) + (S_{m-1} - S_n)\right]. \quad (48)$$

Using ratio test, we have

$$a_i = \left(\prod_{j=0}^{i}\mu(fx_j, fx_m)\right)\alpha(fx_i, fx_{i+1})\eta^i d(fx_0, fx_1), \quad \text{where } \frac{a_{i+1}}{a_i} < \frac{1}{\eta}. \quad (49)$$

Taking $\lim_{n,m\longrightarrow\infty}$ inequality, (48) reduces to

$$\lim_{n,m\longrightarrow\infty} d(fx_n, fx_m) = 0. \quad (50)$$

This amounts to say that $\{fx_n\}$ is a Cauchy sequence in a complete double controlled partial metric type space (X, d), hence there exists $v \in X$ such that

$$\lim_{n\longrightarrow\infty} fx_n = v. \quad (51)$$

Since f is convergent, the sequence $\{x_n\}$ has a convergent subsequence denoted by $\{x_{n_k}\}_{k=1}^{\infty}$ such that

$$\lim_{k\longrightarrow\infty} x_{n_k} = u. \quad (52)$$

Using the continuity of f, we obtain

$$\lim_{k\longrightarrow\infty} fx_{n_k} = fu. \quad (53)$$

From (51) and (53), we conclude that $fu = v$. Making use of triangular inequality, we get

$$\begin{aligned}
d(fgu, fu) &\leq \alpha(fgu, fg^{n_k}x_0)d(fgu, fg^{n_k}x_0) + \mu(fg^{n_k}x_0, fu)d(fg^{n_k}x_0, fu)\\
&\leq \alpha(fgu, fg^{n_k}x_0)\beta\left[d(fu, fgu) + d\left(fg^{n_k-1}x_0, fg^{n_k}x_0\right)\right]\\
&\quad + \mu(fg^{n_k}x_0, fu)d(fg^{n_k}x_0, fu)\\
&\leq \beta\alpha(fgu, fg^{n_k}x_0)d(fu, fgu) + \beta\alpha(fgu, fg^{n_k}x_0)\, d\left(fg^{n_k-1}x_0, fg^{n_k}x_0\right)\\
&\quad + \mu(fg^{n_k}x_0, fu)d(fg^{n_k}x_0, fu)\\
&\leq \frac{\beta\alpha\left(fgu, fx_{n_k}\right)}{1 - \beta\alpha\left(fgu, fx_{n_k}\right)}d\left(fx_{n_k-1}, fx_{n_k}\right) + \frac{\beta\mu\left(fu, fx_{n_k}\right)}{1 - \beta\alpha\left(fgu, fx_{n_k}\right)}d\left(fx_{n_k}, fu\right)\\
&\leq \frac{\beta\alpha\left(fgu, fx_{n_k}\right)}{1 - \beta\alpha\left(fgu, fx_{n_k}\right)}\left(\frac{\beta}{1-\beta}\right)^{n_k-1}d(fx_0, fx_1) + \frac{\beta\mu\left(fu, fx_{n_k}\right)}{1 - \beta\alpha\left(fgu, fx_{n_k}\right)}d\left(fx_{n_k}, fu\right).
\end{aligned} \quad (54)$$

Proceeding the $\lim_{k\longrightarrow\infty}$, we obtain

$$d(fgu, fu) \le \text{constant} \times d(fu, fu), \tag{55}$$

which proves that $d(fu, fu) = 0$. From the triangular inequality, we have

$$d(fu, fu) \le \alpha(fu,u)d(fu,u) + \mu(u, fu)d(u, fu). \tag{56}$$

Suppose that $\alpha(fu,u) \le \mu(u, fu)$, then

$$d(fu, fu) \le 2\alpha(fu,u)d(fu,u). \tag{57}$$

On the other hand,

$$\begin{aligned} d(u, fu) &\le \alpha(u,u)d(u,u) + \mu(u, fu)d(u, fu) \\ &\le \frac{\alpha(u,u)}{1-\mu(u, fu)} d(u,u). \end{aligned} \tag{58}$$

Note that if $\mu\colon X\times X \longrightarrow [1,\infty)$, then $1-\mu(u, fu) \le 0$ and we get $d(u, fu) = 0$. Thus, from (57), we obtain

$$d(fu, fu) = 0. \tag{59}$$

From (55) and (57), we deduce that $d(fgu, fu) = 0$. To check the property $(p1)$, i.e.,

$$d(fgu, fu) = d(fu, fu) = d(fgu, fgu) = 0. \tag{60}$$

It is easy to see that

$$\begin{aligned} d(fgu, fgu) &\le \beta[d(fu, fgu) + d(fu, fgu)] \\ &= 2\beta d(fu, fgu) = 0. \end{aligned} \tag{61}$$

Thus, $fgu = fu$, since f is one-to-one, $gu = u$. Therefore, u is a fixed point of g.

Uniqueness: let u, v be two fixed points of g, then $gu = u$ and $gv = v$. From the condition $(p3)$, we have

$$d(fv, fv) \le d(fu, fv), \tag{62}$$

$$d(fu, fu) \le d(fu, fv). \tag{63}$$

On the other hand, using the triangular inequality, we get

$$\begin{aligned} d(fu, fv) &= d(u,v) \\ &\le a(u,u)d(u,u) + \mu(u,v)d(u,v) \\ &\le \frac{a(u, u)}{1-\mu(u, v)} d(u, u). \end{aligned} \tag{64}$$

Since $\mu\colon X\times X \longrightarrow [1,\infty)$, then $1-\mu(u,v) \le 0$ and we get $d(fu, fv) = 0$. Therefore, from (62) and (63), we obtain that

$$d(fu, fv) = d(fu, fu) = d(fv, fv) = 0. \tag{65}$$

Utilizing the property $(p1)$ of the double controlled partial metric type space, we obtain $fu = fv$. Hence, f is one-to-one so that $u = v$. Finally, by replacing $\{n_k\}$ with $\{n\}$, we conclude that $\{x_n\}$ converges to u as $n \longrightarrow \infty$. Thus, the sequence $\{x_n\}$ converges to the unique fixed point g. □

Corollary 1 (Banach contraction). *Let (X, d) be a complete double controlled partial metric type space by the functions $\alpha, \mu\colon X\times X \longrightarrow [1,\infty)$. Suppose that $f\colon X \longrightarrow X$ satisfies*

$$d(fx, fy) \le \beta d(x, y), \tag{66}$$

for all $x, y \in X$, where $\beta \in (0, (1/2))$. For $x_0 \in X$, take $x_n = f^n x_0$, assuming that

$$\sup_{m\ge1} \lim_{i\longrightarrow\infty} \frac{\alpha(x_{i+1}, x_{i+2})}{\alpha(x_i, x_{i+1})}\mu(x_i, x_m) < \frac{1}{k}, \quad \text{where } k \in (0,1). \tag{67}$$

Furthermore, assume that for every $x \in X$, $\lim_{n\longrightarrow\infty} \alpha(x, x_n)$, $\lim_{n\longrightarrow\infty}\alpha(x_n, x)$, $\lim_{n\longrightarrow\infty}\mu(x, x_n)$, and $\lim_{n\longrightarrow\infty}\mu(x_n, x)$ exist and are finite. Then, the sequence $\{x_n\}$ converges to some $u \in X$; moreover, if α and μ satisfy the following assumptions,

$$\lim_{n\longrightarrow\infty} \frac{\alpha(u, x_{n+1})}{1-\beta\mu(x_{n+1}, fu)} \le 0, \tag{68}$$

then f has a unique fixed point.

Remark 1. Results presented in this manuscript generalize, enrich, and theorize the prominent results due to Kannan [24] and Bojor [25] in the framework of double controlled partial metric type spaces.

Example 2. Let $X = \{0,1,2\}$; consider the function d given as follows: (Table 2)

Given $\alpha, \mu\colon X\times X \longrightarrow [1,\infty)$ is defined as

$$\begin{aligned} \alpha(x, y) &= d(x, y) + 5, \\ \mu(x, y) &= d(x, y) + 7. \end{aligned} \tag{69}$$

It is easy to verify that given d equipped with X is double controlled partial metric type space but not double controlled metric type space because $d(x,x) \ne 0$ for all $x \in X$. Now, we define a mapping $f\colon X \longrightarrow X$ by the following:

$$f(x) = \begin{cases} 1, & \text{when } x = \{1,2\}, \\ 2, & \text{when } x = 0. \end{cases} \tag{70}$$

Choose $f0 = 2$ and $f2 = 1$, then by using (19), we acquire

$$\begin{aligned} d(f0,\ f2) &\le \beta[d(0, f0) + d(2, f2)] \\ d(2,1) &\le \beta[d(0,2) + d(2,1)] \\ \frac{1}{5} &\le \beta\left(\frac{2}{7} + \frac{1}{5}\right) \\ \frac{1}{5} &\le \beta\left(\frac{17}{35}\right). \end{aligned} \tag{71}$$

Since $\beta \in (0, (1/2))$, we choose $\beta = (8/17)$; taking $x_0 = 0$ and $k = (1/8)$, it is clear that condition (20) is satisfied as follows:

Table 2: Metric d defined in Example 2.

d	0	1	2
0	1/27	1/4	2/7
1	1/4	1/28	1/5
2	2/7	1/5	1/29

$$\sup_{m\geq 1}\lim_{i\longrightarrow\infty}\frac{\alpha(x_{i+1},x_{i+2})}{\alpha(x_i,x_{i+1})}\mu(x_i,x_m)=\frac{1768}{245}<8=\frac{1}{k}. \tag{72}$$

Since inequality (20) is satisfied for every $x\in X$, additionally, for each $x\in X$, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\alpha(x,x_n)&=\max(0,x)<\infty,\\ \lim_{n\longrightarrow\infty}\alpha(x_n,x)&=\max(x,0)<\infty,\\ \lim_{n\longrightarrow\infty}\mu(x,x_n)&=\max(0,x)<\infty,\\ \lim_{n\longrightarrow\infty}\mu(x_n,x)&=\max(x,0)<\infty.\end{aligned} \tag{73}$$

Therefore, all the hypotheses of Theorem 1 are contended and 1 is the unique fixed point of f.

4. Conclusions

We launched a new concept of double controlled partial metric type spaces which expands the ideas of certain variants of metric spaces, viz., controlled metric type spaces, double controlled metric type spaces, and partial metric spaces. The introduced results sum up and broaden some previous writing, and some illustrative examples are investigated to show the potency of our work.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Yosida Complementarity Problem with Yosida Variational Inequality Problem and Yosida Proximal Operator Equation Involving XOR-Operation

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Due to the importance of Yosida approximation operator, we generalized the variational inequality problem and its equivalent problems by using Yosida approximation operator. The aim of this work is to introduce and study a Yosida complementarity problem, a Yosida variational inequality problem, and a Yosida proximal operator equation involving XOR-operation. We prove an existence result together with convergence analysis for Yosida proximal operator equation involving XOR-operation. For this purpose, we establish an algorithm based on fixed point formulation. Our approach is based on a proximal operator technique involving a subdifferential operator. As an application of our main result, we provide a numerical example using the MATLAB program R2018a. Comparing different iterations, a computational table is assembled and some graphs are plotted to show the convergence of iterative sequences for different initial values.

1. Introduction

Stampacchia [1] and Ficchera [2] originated the study of variational inequalities, separately. Variational inequalities are mathematical models for many problems occurring in physics, engineering sciences, transportation planning, financial problems, and in many industrial strategies, etc. (see, for example, [3–11]). In 1968, Cottle and Dantzig [12] proposed linear complementarity problem which appear continually in computational mechanics. It is interesting to note that finding the solution of linear complementarity problem is associated with minimizing some quadratic function. However, in 1964, Cottle [13] in his Ph. D thesis introduced nonlinear complementarity problem which is closely related to Hartman and Stampacchia variational inequality problem. The proximal operator technique is useful to establish equivalence between variational inequalities and proximal operator equations. The proximal operator equation approach is used to solve variational inequalities and related optimization problems.

XOR is a logical operation and represents the inequality function, that is, the output is true if the inputs are not alike; otherwise, the output is false. An easy way to remember XOR is "must have one or the other but not both." It is important to note that XOR does not leak information about the original plain text. The inner XOR is the encryption and the outer XOR is the decryption, that is, the exact XOR function can be used for both encryption and decryption. Consider a string of binary digits 10101 and XOR the string 10111 with it to get 00010. That is, the original string is encoded and the second string becomes key; if we XOR our key with our encoded string, we get our original string back. XOR allows to easily encrypt and decrypt a string; the other logical operations do not.

The possible strategy of solving stochastic notion of multivalued differential equation in finite dimensional space is based on Yosida approximation approach. The existence of multivalued stochastic differential equation in finite dimensional space with a time-independent, deterministic maximal monotone operator through Yosida approximation approach was first discussed by Petterson [14]. Yosida approximation operators are used to solve wave equations, heat equations, etc. For more details and recent past developments about complementarity problems, variational inequalities, proximal operator equations, Yosida approximation operator, and related topics, we refer to [15–28] and references therein.

Motivated by all the above discussed concepts, in this paper, we consider and study a Yosida complementarity problem, a Yosida variational inequality problem, and a Yosida proximal operator equation involving XOR-operation. Some equivalence results are proved. To obtain the solution of Yosida proximal operator equation involving XOR-operation, we define an algorithm based on fixed point formulation. Convergence criteria are also discussed. In support of our main result, an example is provided using MATLAB program R2018a. A comparison of different iterations is assembled in the form of a computational table, and the convergence of the iterative sequences is shown by some graphs for different initial values.

2. Preliminaries and Basic Results

We suppose that H is a real ordered positive Hilbert Space with its norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, $\mathscr{C}\subseteq H$ is a closed convex pointed cone, d is the metric induced by the norm $\|\cdot\|$, $\widetilde{C}B(H)$ is the family of nonempty, closed, and bounded subsets of H, and $D(\cdot,\cdot)$ is the Hausdorff metric on $\widetilde{C}B(H)$.

The following definitions, concepts, and results are required for the presentation of this paper.

Definition 1. A convex cone is a subset of a vector space over an ordered field that is closed under linear combinations with positive coefficients.

Definition 2. Two elements x and y of a set X are said to be comparable with respect to a binary operation $\leq$, if at least one of $x\leq y$ or $y\leq x$ is true. Comparable elements x and y are denoted by $x\propto y$.

Definition 3. A partial order is any binary relation which is reflexive, antisymmetric, and transitive.

Definition 4. Suppose $\text{lub}\{x,y\}$ and $\text{glb}\{x,y\}$ for the set $\{x,y\}$ exist; then, XOR and XNOR operations denoted by $\oplus$ and $\odot$ are defined as follows:

(i) $x\oplus y=(x-y)\vee(y-x)$

(ii) $x\odot y=(x-y)\wedge(y-x)$, where $x\vee y=\text{lub}\{x,y\}$, $x\wedge y=\text{glb}\{x,y\}$, lub means the least upper bound, and glb means the greatest lower bound

Proposition 1 (see [29]). *Let $\oplus$ be an XOR-operation and $\odot$ be an XNOR operation. Then, the following axioms are true:*

(i) $x\odot x=0, x\odot y=y\odot x=-(x\oplus y)=-(y\oplus x)$

(ii) $x\oplus x=0, x\oplus y=y\oplus x, 0\leq x\oplus y$

(iii) $x\oplus 0=x$

(iv) $0\leq x\oplus y$, *if* $x\propto y$

(v) If $x\propto y$, *then* $x\oplus y=0$ *if and only if* $x=y$

(vi) $\|0\oplus 0\|=\|0\|$

(vii) $\|x\oplus y\|\leq\|x-y\|$

(viii) If $x\propto y$, *then* $\|x\oplus y\|=\|x-y\|$

Definition 5. Let $N\colon H\times H\times H\longrightarrow H$ be a single-valued mapping and $A\colon\mathscr{C}\longrightarrow\widetilde{C}B(H)$ be a multivalued mapping. Then

(i) N is said to be Lipschitz continuous in the first argument if there exists a constant $\lambda_{N_1}>0$ such that

$$\|N(u_1,\cdot,\cdot)-N(u_2,\cdot,\cdot)\|\leq\lambda_{N_1}\|u_1-u_2\|,\quad\forall x_1,x_2\in\mathscr{C},u_1\in A(x_1)\text{ and }u_2\in A(x_2),\tag{1}$$

(ii) N is said to be Lipschitz continuous in the second argument if there exists a constant $\lambda_{N_2}>0$ such that

$$\|N(\cdot,u_1,\cdot)-N(\cdot,u_2,\cdot)\|\leq\lambda_{N_2}\|u_1-u_2\|,\quad\forall x_1,x_2\in\mathscr{C},u_1\in A(x_1)\text{ and }u_2\in A(x_2).\tag{2}$$

Similarly, we can define the Lipschitz continuity of N in the third argument.

Definition 6. A multivalued mapping $A: \mathscr{C} \longrightarrow \widetilde{C}B(H)$ is said to be D-Lipschitz continuous if for any $x, y \in \mathscr{C}$, there exists a constant $\lambda_{D_A} > 0$ such that

$$D(A(x), A(y)) \leq \lambda_{D_A} \|x - y\|. \tag{3}$$

Definition 7 (see [30]). Let $\psi: H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex functional. A vector $w \in H$ is called subgradient of ψ at $x \in do\ m\psi$, if

$$\langle w, y - x \rangle \leq \psi(y) - \psi(x), \quad \forall y \in H. \tag{4}$$

The set of all subgradients of ψ at x is denoted by $\partial\psi(x)$. The mapping $\partial\psi : H \longrightarrow 2^H$ defined by

$$\partial\psi(x) = \{w \in H: \langle w, y - x \rangle \leq \psi(y) - \psi(x), \forall y \in H\} \tag{5}$$

is called subdifferential of ψ.

Definition 8. Let $P: \mathscr{C} \longrightarrow \mathscr{C}$ be a mapping and $\psi: \mathscr{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex functional. The proximal operator $J_\rho^{\partial\psi}: \mathscr{C} \longrightarrow \mathscr{C}$ is defined by

$$J_\rho^{\partial\psi}(x) = [P + \rho\partial\psi]^{-1}(x), \quad \forall x \in \mathscr{C}, \tag{6}$$

where $\rho > 0$ is a constant.

Definition 9. The Yosida approximation operator of ψ is defined by

$$Y_\rho^{\partial\psi}(x) = \frac{1}{\rho}\left[I - J_\rho^{\partial\psi}\right](x), \quad \forall x \in \mathscr{C}, \tag{7}$$

where $\rho > 0$ is a constant.

Furthermore, we prove some propositions related to proximal operator and Yosida approximation operator.

Proposition 2. *Let $P: \mathscr{C} \longrightarrow \mathscr{C}$ and $\psi: \mathscr{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be linear mappings, then the proximal operator $J_\rho^{\partial\psi}$ is linear. That is*

$$\alpha J_\rho^{\partial\psi}(x) = J_\rho^{\partial\psi}(\alpha x), \tag{8}$$

provided

$$\left(J_\rho^{\partial\psi}\right)^{-1}\left(J_\rho^{\partial\psi}(x)\right) = \left(\left(J_\rho^{\partial\psi}\right)^{-1}\left(J_\rho^{\partial\psi}\right)\right)(x) = x, \quad \forall x \in \mathscr{C} \text{ and } \alpha > 0. \tag{9}$$

Proof. Using the definition of $J_\rho^{\partial\psi}$, linearity of P and ψ, and Theorem 1.48 and Theorem 1.49 of [31], we have

$$\begin{aligned}
\alpha J_\rho^{\partial\psi}(x) &= J_\rho^{\partial\psi}\left(J_\rho^{\partial\psi}\right)^{-1}\left(\alpha J_\rho^{\partial\psi}(x)\right) \\
&= J_\rho^{\partial\psi}[P + \rho\partial\psi]\left(\alpha J_\rho^{\partial\psi}(x)\right) \\
&= J_\rho^{\partial\psi}\left[P\left(\alpha J_\rho^{\partial\psi}(x)\right) + \rho\partial\psi\left(\alpha J_\rho^{\partial\psi}(x)\right)\right] \\
&= J_\rho^{\partial\psi}\left[P\left(\alpha J_\rho^{\partial\psi}(x)\right) + \rho\partial\psi\left(\alpha J_\rho^{\partial\psi}(x)\right)\right] \\
&= J_\rho^{\partial\psi}\left[P\left(\alpha J_\rho^{\partial\psi}(x)\right) + \rho\partial\left(\alpha\psi\left(J_\rho^{\partial\psi}(x)\right)\right)\right] \\
&= J_\rho^{\partial\psi}\left[\alpha P\left(J_\rho^{\partial\psi}(x)\right) + \rho\alpha\, \partial\psi\left(J_\rho^{\partial\psi}(x)\right)\right] \\
&= J_\rho^{\partial\psi}\left[\alpha[P + \rho\partial\psi]\left(J_\rho^{\partial\psi}(x)\right)\right] \\
&= J_\rho^{\partial\psi}\left[\alpha\left(J_\rho^{\partial\psi}\right)^{-1} J_\rho^{\partial\psi}(x)\right] \\
&= J_\rho^{\partial\psi}\left[\alpha\left(\left(J_\rho^{\partial\psi}\right)^{-1}\left(J_\rho^{\partial\psi}\right)\right)(x)\right] \\
&= J_\rho^{\partial\psi}[\alpha x].
\end{aligned} \tag{10}$$

□

Proposition 3. *The Yosida approximation operator $Y_\rho^{\partial\psi}: \mathscr{C} \longrightarrow \mathscr{C}$ is linear, that is,*

$$Y_\rho^{\partial\psi}(\alpha x) = \alpha Y_\rho^{\partial\psi}(x), \quad \forall x \in \mathscr{C}. \tag{11}$$

Proof. Using the definition of $Y_\rho^{\partial\psi}$ and Proposition 2, we have

$$\begin{aligned}
Y_\rho^{\partial\psi}(\alpha x) &= \frac{1}{\rho}\left[I - J_\rho^{\partial\psi}\right](\alpha x) \\
&= \frac{1}{\rho}\left[\alpha x - J_\rho^{\partial\psi}(\alpha x)\right] \\
&= \frac{1}{\rho}\left[\alpha x - \alpha J_\rho^{\partial\psi}(x)\right] \\
&= \frac{\alpha}{\rho}\left[I - J_\rho^{\partial\psi}\right](x) \\
&= \alpha Y_\rho^{\partial\psi}(x), \quad \forall x \in \mathscr{C}.
\end{aligned} \tag{12}$$

□

Proposition 4. *The proximal operator $J_\rho^{\partial\psi}: \mathscr{C} \longrightarrow \mathscr{C}$ is Lipschitz continuous, provided $P: \mathscr{C} \longrightarrow \mathscr{C}$ is strongly monotone with respect to $J_\rho^{\partial\psi}$ with constant $\mu > 0$, ψ is strongly convex with modulus $\lambda > 0$, and $J_\rho^{\partial\psi}$ is strongly monotone with constant $\sigma > 0$, where $\sigma = 2\lambda$.*

Proof. Let $x, y \in \mathscr{C}$, then

$$J_\rho^{\partial\psi}(x) = [P + \rho\partial\psi]^{-1}(x), \tag{13}$$

$$J_\rho^{\partial\psi}(y) = [P + \rho\partial\psi]^{-1}(y). \tag{14}$$

Thus,

$$\frac{1}{\rho}\left[x - P\left(J_\rho^{\partial\psi}(x)\right)\right] \in \partial\psi\left(J_\rho^{\partial\psi}(x)\right), \tag{15}$$

$$\frac{1}{\rho}\left[y - P\left(J_\rho^{\partial\psi}(y)\right)\right] \in \partial\psi\left(J_\rho^{\partial\psi}(y)\right). \tag{16}$$

As ψ is strongly convex with modulus $\lambda > 0$, then the proximal operator $J_\rho^{\partial\psi}$ is strongly monotone with constant $\sigma > 0$, where $\sigma = 2\lambda$ (see [31]). Therefore,

$$\begin{aligned}\sigma\left\|J_\rho^{\partial\psi}(x) - J_\rho^{\partial\psi}(y)\right\|^2 &\leq \left\langle\frac{1}{\rho}\left(x - P\left(J_\rho^{\partial\psi}(x)\right)\right) - \frac{1}{\rho}\left(y - P\left(J_\rho^{\partial\psi}(y)\right)\right), x - y\right\rangle \\ &= \frac{1}{\rho}\left\langle x - y - \left(P\left(J_\rho^{\partial\psi}(x)\right) - P\left(J_\rho^{\partial\psi}(y)\right)\right), x - y\right\rangle \\ &= \frac{1}{\rho}\left[\langle x - y, x - y\rangle - \left\langle P\left(J_\rho^{\partial\psi}(x)\right) - P\left(J_\rho^{\partial\psi}(y)\right), x - y\right\rangle\right].\end{aligned} \tag{17}$$

Since P is strongly monotone with respect to $J_\rho^{\partial\psi}$ with constant $\mu > 0$, we have

$$\sigma\left\|J_\rho^{\partial\psi}(x) - J_\rho^{\partial\psi}(y)\right\|^2 \leq \frac{1}{\rho}\left[\|x - y\|^2 - \mu\left\|J_\rho^{\partial\psi}(x) - J_\rho^{\partial\psi}(y)\right\|^2\right], \tag{18}$$

which implies that

$$\left\|J_\rho^{\partial\psi}(x) - J_\rho^{\partial\psi}(y)\right\| \leq \theta\|x - y\|, \quad \text{where } \theta = \frac{1}{\sqrt{\sigma\rho + \mu}}. \tag{19}$$

That is, $J_\rho^{\partial\psi}$ is Lipschitz continuous. □

Proposition 5. *The Yosida approximation operator is strongly monotone if all the conditions of Proposition 4 hold.*

Proof. Using the Lipschitz continuity of proximal operator $J_\rho^{\partial\psi}$, we have

$$\begin{aligned}\left\langle Y_\rho^{\partial\psi}(x) - Y_\rho^{\partial\psi}(y), x - y\right\rangle &= \left\langle\frac{1}{\rho}\left[I - J_\rho^{\partial\psi}\right](x) - \frac{1}{\rho}\left[I - J_\rho^{\partial\psi}\right](y), x - y\right\rangle \\ &= \frac{1}{\rho}\left[\langle x - y, x - y\rangle - \left\langle J_\rho^{\partial\psi}(x) - J_\rho^{\partial\psi}(y), x - y\right\rangle\right] \\ &\geq \frac{1}{\rho}\left[\|x - y\|^2 - \left\|J_\rho^{\partial\psi}(x) - J_\rho^{\partial\psi}(y)\right\|\|x - y\|\right] \\ &\geq \frac{1}{\rho}\left[\|x - y\|^2 - \theta\|x - y\|^2\right] \\ &= \left(\frac{1 - \theta}{\rho}\right)\|x - y\|^2 \\ &= \delta_y\|x - y\|^2, \quad \text{where } \delta_y = \left(\frac{1 - \theta}{\rho}\right).\end{aligned} \tag{20}$$

□

3. Description of the Problems and Equivalence Lemmas

Let H be a real ordered positive Hilbert space and $\mathscr{C} \subseteq H$ be a closed convex pointed cone. Let $A, B, C\colon \mathscr{C} \longrightarrow \widetilde{C}B(H)$ be the multivalued mappings and $N\colon H \times H \times H \longrightarrow H$ be a single-valued mapping. Suppose $\psi\colon \mathscr{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a proper, convex functional and $Y_\rho^{\partial\psi}\colon \mathscr{C} \longrightarrow \mathscr{C}$ is the Yosida approximation operator. We consider the following Yosida complementarity problem involving XOR-operation.

Find $x \in \mathscr{C}, u \in A(x), v \in B(x), w \in C(x)$ such that

$$\begin{aligned} &\langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle \oplus \psi\left(Y_\rho^{\partial\psi}(x)\right) = 0,\\ &\langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle \oplus \psi\left(Y_\rho^{\partial\psi}(y)\right) \geq 0, \quad \forall y \in \mathscr{C}. \end{aligned} \tag{21}$$

From problem (21), one can easily obtain the complementarity problems studied by Huang et al. [32], Yin and Xu [33], Flores-Bazán and López [34], Isac [35, 36] and Farajzadeh and Harandi [37], etc.

In connection with Yosida complementarity problem involving XOR-operation (21), we mention the following Yosida variational inequality problem involving XOR-operation.

Find $x \in \mathscr{C}, u \in A(x), v \in B(x), w \in C(x)$ such that

$$\begin{aligned} &\langle N(u,v,w), Y_\rho^{\partial\psi}(y) - Y_\rho^{\partial\psi}(x)\rangle \oplus \left(\psi\left(Y_\rho^{\partial\psi}(y)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right)\right)\\ &\geq 0, \quad \forall y \in \mathscr{C}. \end{aligned} \tag{22}$$

In acquaintance with Yosida variational inequality problem involving XOR-operation (22), we mention the following Yosida proximal operator equation involving XOR-operation.

Find $x, z \in \mathscr{C}, u \in A(x), v \in B(x), w \in C(x)$ such that

$$N(u,v,w) \oplus \rho^{-1} R_\rho^{\partial\psi}(z) = 0, \tag{23}$$

where $\rho > 0$ is a constant, $R_\rho^{\partial\psi} = [I - P(J_\rho^{\partial\psi})], J_\rho^{\partial\psi}$ is the proximal operator, $P: \mathscr{C} \longrightarrow \mathscr{C}$ is a mapping, and $z = P(Y_\rho^{\partial\psi}(x)) + \rho N(u,v,w)$.

The equivalence between problem (21) and problem (22) and the equivalence between problem (22) and problem (23) are given as follows.

Lemma 1. *Let $A, B, C: \mathscr{C} \longrightarrow \widetilde{C}B(H)$ be the multivalued mappings and $N: H \times H \times H \longrightarrow H$ be a single-valued mapping. Suppose $\psi: \mathscr{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a linear, proper functional. Let $Y_\rho^{\partial\psi}: \mathscr{C} \longrightarrow \mathscr{C}$ be the Yosida approximation operator. If $\langle N(u,v,w), Y_\rho^{\partial\psi}\rangle \propto \psi(Y_\rho^{\partial\psi}(x))$, for all $x \in \mathscr{C}, u \in A(x), v \in B(x), w \in C(x)$, then the Yosida complementarity problem involving XOR-operation (21) and the Yosida variational inequality problem involving XOR-operation (22) are equivalent.*

Proof. Let the Yosida complementarity problem involving XOR-operation (21) holds. We have

$$\begin{aligned} &\langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle \oplus \psi\left(Y_\rho^{\partial\psi}(x)\right) = 0,\\ &\langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle \oplus \psi\left(Y_\rho^{\partial\psi}(y)\right) \geq 0, \quad \forall x, y \in \mathscr{C}, u \in A(x), v \in B(x), w \in C(x). \end{aligned} \tag{24}$$

Since $\langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle \propto \psi(Y_\rho^{\partial\psi}(x))$, using (v) of Proposition 1, we have

$$\langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle = \psi\left(Y_\rho^{\partial\psi}(x)\right). \tag{25}$$

Also, $\langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle \oplus \psi(Y_\rho^{\partial\psi}(y)) \geq 0$, we have

$$\langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle \oplus \left(\psi\left(Y_\rho^{\partial\psi}(y)\right) \oplus \psi\left(Y_\rho^{\partial\psi}(y)\right) \geq \psi\left(Y_\rho^{\partial\psi}(y)\right). \tag{26}$$

By (ii) and (iii) of Proposition 1, we have

$$\langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle \geq \psi\left(Y_\rho^{\partial\psi}(y)\right). \tag{27}$$

Using the properties of inner product, we can write

$$\begin{aligned} \langle N(u,v,w), Y_\rho^{\partial\psi}(y) - Y_\rho^{\partial\psi}(x)\rangle &= \langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle\\ &\quad - \langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle. \end{aligned} \tag{28}$$

Applying (25) and (27), (28) becomes

$$\begin{aligned} &\langle N(u,v,w), Y_\rho^{\partial\psi}(y) - Y_\rho^{\partial\psi}(x)\rangle \geq \psi\left(Y_\rho^{\partial\psi}(y)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right),\\ &\langle N(u,v,w), Y_\rho^{\partial\psi}(y) - Y_\rho^{\partial\psi}(x)\rangle \oplus \left(\psi\left(Y_\rho^{\partial\psi}(y)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right)\right) \geq 0, \end{aligned} \tag{29}$$

which is the Yosida variational inequality problem involving XOR-operation (22).

On the other hand, let the Yosida variational inequality problem (22) holds. That is, $x \in \mathscr{C}, u \in A(x), v \in B(x), w \in C(x)$ such that

$$\begin{aligned} &\langle N(u,v,w), Y_\rho^{\partial\psi}(y) - Y_\rho^{\partial\psi}(x)\rangle \oplus \left(\psi\left(Y_\rho^{\partial\psi}(y)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right)\right)\\ &\geq 0, \quad \forall y \in \mathscr{C}. \end{aligned} \tag{30}$$

As $\mathscr{C}$ is a closed convex pointed cone, $y = 2x \in \mathscr{C}$ as well as $y = (1/2)x \in \mathscr{C}$. Putting $y = 2x$ and $y = (1/2)x$ and using linearity of ψ and Proposition 3, we have

$$\begin{aligned}
&\langle N(u,v,w), Y_\rho^{\partial\psi}(2x) - Y_\rho^{\partial\psi}(x)\rangle \oplus \left(\psi\left(Y_\rho^{\partial\psi}(2x)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right)\right) \geq 0\\
&\langle N(u,v,w), 2Y_\rho^{\partial\psi}(x) - Y_\rho^{\partial\psi}(x)\rangle \oplus \left(\psi\left(2Y_\rho^{\partial\psi}(x)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right)\right) \geq 0\\
&\langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle \oplus \left(2\psi\left(Y_\rho^{\partial\psi}(x)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right)\right) \geq 0\\
&\langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle \oplus \psi\left(Y_\rho^{\partial\psi}(x)\right) \geq 0,\\
&\langle N(u,v,w), Y_\rho^{\partial\psi}\left(\frac{1}{2}x\right) - Y_\rho^{\partial\psi}(x)\rangle \oplus \left(\psi\left(Y_\rho^{\partial\psi}\left(\frac{1}{2}x\right)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right)\right) \geq 0\\
&\langle N(u,v,w), \frac{-1}{2}Y_\rho^{\partial\psi}(x)\rangle \oplus \left(\frac{-1}{2}\psi\left(Y_\rho^{\partial\psi}(x)\right)\right) \geq 0\\
&\langle N(u,v,w), \frac{-1}{2}Y_\rho^{\partial\psi}(x)\rangle \geq \frac{-1}{2}\left(\psi\left(Y_\rho^{\partial\psi}(x)\right)\right)\\
&\langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle \leq \psi\left(Y_\rho^{\partial\psi}(x)\right).
\end{aligned} \tag{31}$$

Thus, we have

$$\langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle \oplus \psi\left(Y_\rho^{\partial\psi}(x)\right) \leq 0. \tag{32}$$

Adding (31) and (32), we have

$$\langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle \oplus \psi\left(Y_\rho^{\partial\psi}(x)\right) = 0. \tag{33}$$

Since

$$\langle N(u,v,w), Y_\rho^{\partial\psi}(y) - Y_\rho^{\partial\psi}(x)\rangle \oplus \left(\psi\left(Y_\rho^{\partial\psi}(y)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right)\right) \geq 0, \tag{34}$$

we have

$$\begin{aligned}
&\langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle - \langle N(u,v,w), Y_\rho^{\partial\psi}(x)\rangle\\
&\geq \psi\left(Y_\rho^{\partial\psi}(y)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right).
\end{aligned} \tag{35}$$

Using (25), from the above inequality, we have

$$\langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle - \psi\left(Y_\rho^{\partial\psi}(x)\right) \geq \psi\left(Y_\rho^{\partial\psi}(y)\right) - \psi\left(Y_\rho^{\partial\psi}(x)\right), \tag{36}$$

it follows that

$$\langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle \geq \psi\left(Y_\rho^{\partial\psi}(y)\right). \tag{37}$$

Using (ii) of Proposition 1, we have

$$\begin{aligned}
&\langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle \oplus \psi\left(Y_\rho^{\partial\psi}(y)\right)\\
&\geq \psi\left(Y_\rho^{\partial\psi}(y)\right) \oplus \psi\left(Y_\rho^{\partial\psi}(y)\right),
\end{aligned} \tag{38}$$

$$\langle N(u,v,w), Y_\rho^{\partial\psi}(y)\rangle \oplus \psi\left(Y_\rho^{\partial\psi}(y)\right) \geq 0. \tag{39}$$

Combination of (33) and (39) is the required Yosida complementarity problem involving XOR-operation (21). □

The following Lemma guarantees the equivalence between the Yosida variational inequality problem involving XOR-operation (22) and a fixed point equation.

Lemma 2. *Let $P: \mathscr{C} \longrightarrow \mathscr{C}$ be a mapping, then the Yosida variational inequality problem involving XOR-operation (22) has a solution $x \in \mathscr{C}, u \in A(x), v \in B(x), w \in C(x)$, if and only if it satisfies the equation:*

$$Y_\rho^{\partial\psi}(x) = J_\rho^{\partial\psi}\left[P\left(Y_\rho^{\partial\psi}(x)\right) + \rho N(u,v,w)\right], \tag{40}$$

where $\rho > 0$ is a constant.

Proof. Let $x \in \mathscr{C}, u \in A(x), v \in B(x), w \in C(x)$ satisfy equation (40), that is,

$$Y_\rho^{\partial\psi}(x) = J_\rho^{\partial\psi}\left[P\left(Y_\rho^{\partial\psi}(x)\right) + \rho N(u,v,w)\right]. \tag{41}$$

Using the definition of the proximal operator $J_\rho^{\partial\psi}$ and from the above equation, we have

$$\begin{aligned} Y_\rho^{\partial\psi}(x) &= [P+\rho\partial\psi]^{-1}\left[P\left(Y_\rho^{\partial\psi}(x)\right)+\rho N(u,v,w)\right] \\ P\left(Y_\rho^{\partial\psi}(x)\right)+\rho\partial\psi\left(Y_\rho^{\partial\psi}(x)\right) &= P\left(Y_\rho^{\partial\psi}(x)\right)+\rho N(u,v,w) \quad \text{(that is,)} \\ N(u,v,w) &\in \partial\psi\left(Y_\rho^{\partial\psi}(x)\right). \quad \text{(which gives us)} \end{aligned} \tag{42}$$

Applying the definition of subdifferential operator, the above inclusion holds if and only if

$$\psi\left(Y_\rho^{\partial\psi}(y)\right)-\psi\left(Y_\rho^{\partial\psi}(x)\right)\geq\langle N(u,v,w),\psi\left(Y_\rho^{\partial\psi}(y)\right)-\psi\left(Y_\rho^{\partial\psi}(x)\right)\rangle. \tag{43}$$

Using (ii) of Proposition 1, we have

$$\begin{aligned} &\langle N(u,v,w),\psi\left(Y_\rho^{\partial\psi}(y)\right)-\psi\left(Y_\rho^{\partial\psi}(x)\right)\rangle\oplus\left(\psi\left(Y_\rho^{\partial\psi}(y)\right)-\psi\left(Y_\rho^{\partial\psi}(x)\right)\right) \\ &\geq\langle N(u,v,w),\psi\left(Y_\rho^{\partial\psi}(y)\right)-\psi\left(Y_\rho^{\partial\psi}(x)\right)\rangle\oplus\langle N(u,v,w),\psi\left(Y_\rho^{\partial\psi}(y)\right)-\psi\left(Y_\rho^{\partial\psi}(x)\right)\rangle. \end{aligned} \tag{44}$$

It follows that

$$\begin{aligned} &\langle N(u,v,w),Y_\rho^{\partial\psi}(y)-Y_\rho^{\partial\psi}(x)\rangle\oplus\left(\psi\left(Y_\rho^{\partial\psi}(y)\right)-\psi\left(Y_\rho^{\partial\psi}(x)\right)\right) \\ &\geq 0,\quad \forall y\in\mathscr{C}, \end{aligned} \tag{45}$$

which is the required Yosida variational inequality problem involving XOR-operation (22). □

The Lemma mentioned below ensures the equivalence between the Yosida variational inequality problem involving XOR-operation (22) and the Yosida proximal operator equation involving XOR-operation (23).

Lemma 3. *Suppose $N(u,v,w)\propto R_\rho^{\partial\psi}(z)$ and $P\colon\mathscr{C}\longrightarrow\mathscr{C}$ is a one-one mapping. Then $x\in\mathscr{C},u\in A(x),v\in B(x),w\in C(x)$ is the solution of the Yosida variational inequality problem involving XOR-operation (22) if and only if $x,z\in\mathscr{C},u\in A(x),v\in B(x),w\in C(x)$ satisfy the Yosida proximal operator equation involving XOR-operation (23), where $R_\rho^{\partial\psi}=[I-P(J_\rho^{\partial\psi})]$, in which $J_\rho^{\partial\psi}$ is the proximal operator and $P(J_\rho^{\partial\psi}(z))=P(J_\rho^{\partial\psi})(z)$.*

Proof. Let $x\in\mathscr{C},u\in A(x),v\in B(x),w\in C(x)$ be the solution of the Yosida variational inequality problem involving XOR-operation (22). Then by Lemma 2, it satisfies the equation:

$$Y_\rho^{\partial\psi}(x)=J_\rho^{\partial\psi}\left[P\left(Y_\rho^{\partial\psi}(x)\right)+\rho N(u,v,w)\right]. \tag{46}$$

Let $z=P(Y_\rho^{\partial\psi}(x))+\rho N(u,v,w)$, then

$$\begin{aligned} Y_\rho^{\partial\psi}(x) &= J_\rho^{\partial\psi}(z), \\ \text{As } z &= P\left(J_\rho^{\partial\psi}(z)\right)+\rho N(u,v,w) \\ z-P\left(J_\rho^{\partial\psi}(z)\right) &= \rho N(u,v,w) \\ \left(I-P\left(J_\rho^{\partial\psi}\right)\right)(z) &= \rho N(u,v,w) \\ R_\rho^{\partial\psi}(z) &= \rho N(u,v,w) \\ \rho^{-1}R_\rho^{\partial\psi}(z) &= N(u,v,w),\ \text{where } P\left(J_\rho^{\partial\psi}(z)\right)=P\left(J_\rho^{\partial\psi}\right)(z). \end{aligned} \tag{47}$$

Using (ii) of Proposition 1, we have

$$N(u,v,w)\oplus\rho^{-1}R_\rho^{\partial\psi}(z)=N(u,v,w)\oplus N(u,v,w)=0. \tag{48}$$

Thus, we have

$$N(u,v,w)\oplus\rho^{-1}R_\rho^{\partial\psi}(z)=0, \tag{49}$$

which is the required Yosida proximal operator equation involving XOR-operation (23).

Conversely, let $x,z\in\mathscr{C},u\in A(x),v\in B(x),w\in C(x)$ be the solution of Yosida proximal operator equation involving XOR-operation (23).

That is, we have

$$N(u,v,w)\oplus\rho^{-1}R_\rho^{\partial\psi}(z)=0. \tag{50}$$

Using (v) of Proposition 1, definition of $R_\rho^{\partial\psi}$ and comparability of $N(u,v,w)$ with $R_\rho^{\partial\psi}(z)$, we obtain

$$\begin{aligned} \rho N(u,v,w) &= R_\rho^{\partial\psi}(z)=\left[I-P\left(J_\rho^{\partial\psi}\right)\right](z) \\ &= z-P\left(J_\rho^{\partial\psi}\right)(z) \\ &= P\left(Y_\rho^{\partial\psi}(x)\right)+\rho N(u,v,w)-P\left(J_\rho^{\partial\psi}\left(\left(P\left(Y_\rho^{\partial\psi}(x)\right)+\rho N(u,v,w)\right)\right)\right). \end{aligned} \tag{51}$$

From above, we have

$$P\left(Y_\rho^{\partial\psi}(x)\right) = P\left(J_\rho^{\partial\psi}\left(P\left(Y_\rho^{\partial\psi}(x)\right) + \rho N(u,v,w)\right)\right). \tag{52}$$

Since P is a one-one mapping, we obtain

$$Y_\rho^{\partial\psi}(x) = J_\rho^{\partial\psi}\left[P\left(Y_\rho^{\partial\psi}(x)\right) + \rho N(u,v,w)\right]. \tag{53}$$

Applying Lemma 2, we conclude that $x \in \mathscr{C}, u \in A(x), v \in B(x), w \in C(x)$ is the solution of Yosida variational inequality problem involving XOR-operation (22). □

4. Algorithm and Existence Results

Invoking Lemmas 2 and 3, we suggest the following algorithm for solving Yosida proximal operator equation involving XOR-operation (23).

Algorithm 1. For any $x_0, z_0 \in \mathscr{C}, u_0 \in A(x_0), v_0 \in B(x_0), w_0 \in C(x_0)$, we let

$$z_1 = P\left(Y_\rho^{\partial\psi}(x_0)\right) + \rho N(u_0, v_0, w_0). \tag{54}$$

Take any $x_1 \in \mathscr{C}$ such that

$$Y_\rho^{\partial\psi}(x_1) = J_\rho^{\partial\psi}(z_1). \tag{55}$$

Since $u_0 \in A(x_0), v_0 \in B(x_0), w_0 \in C(x_0)$, by Nadler's theorem [38], there exist $u_1 \in A(x_1), v_0 \in B(x_1), w_0 \in C(x_1)$, using (viii) of Proposition 1 and comparability of $u_0, u_1; v_0, v_1$ and w_0, w_1, we have

$$\begin{aligned}
\|u_0 \oplus u_1\| &= \|u_0 - u_1\| \leq (1+1)D\left(A\left(Y_\rho^{\partial\psi}(x_0)\right), A\left(Y_\rho^{\partial\psi}(x_1)\right)\right),\\
\|v_0 \oplus v_1\| &= \|v_0 - v_1\| \leq (1+1)D\left(B\left(Y_\rho^{\partial\psi}(x_0)\right), B\left(Y_\rho^{\partial\psi}(x_1)\right)\right),\\
\|w_0 \oplus w_1\| &= \|w_0 - w_1\| \leq (1+1)D\left(C\left(Y_\rho^{\partial\psi}(x_0)\right), C\left(Y_\rho^{\partial\psi}(x_1)\right)\right),
\end{aligned} \tag{56}$$

where $D(\cdot,\cdot)$ is the Hausdorff metric on $\widetilde{C}B(H)$.

Let $z_2 = P(Y_\rho^{\partial\psi}(x_1)) + \rho N(u_1, v_1, w_1)$ and take any $x_2 \in \mathscr{C}$ such that

$$Y_\rho^{\partial\psi}(x_2) = J_\rho^{\partial\psi}(z_2). \tag{57}$$

Continuing the above procedure, we compute the sequences $\{x_n\}, \{u_n\}, \{v_n\}$ and $\{z_n\}$ by the schemes given below:

$$\begin{aligned}
&Y_\rho^{\partial\psi}(x_n) = J_\rho^{\partial\psi}(z_n),\\
&u_n \in A(x_n), u_{n+1} \in A(x_{n+1}) \text{ such that } u_n \propto u_{n+1},\\
&\|u_n \oplus u_{n+1}\| = \|u_n - u_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right)D\left(A\left(Y_\rho^{\partial\psi}(x_n)\right), A\left(Y_\rho^{\partial\psi}(x_{n+1})\right)\right),\\
&v_n \in B(x_n), v_{n+1} \in B(x_{n+1}) \text{ such that } v_n \propto v_{n+1},\\
&\|v_n \oplus v_{n+1}\| = \|v_n - v_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right)D\left(B\left(Y_\rho^{\partial\psi}(x_n)\right), B\left(Y_\rho^{\partial\psi}(x_{n+1})\right)\right),\\
&w_n \in C(x_n), w_{n+1} \in C(x_{n+1}) \text{ such that } w_n \propto w_{n+1},\\
&\|w_n \oplus w_{n+1}\| = \|w_n - w_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right)D\left(C\left(Y_\rho^{\partial\psi}(x_n)\right), C\left(Y_\rho^{\partial\psi}(x_{n+1})\right)\right),\\
&z_{n+1} = P\left(Y_\rho^{\partial\psi}(x_n)\right) + \rho N(u_n, v_n, w_n),
\end{aligned} \tag{58}$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, 3, \cdots$.

Theorem 1. *Let H be a real ordered positive Hilbert Space and $\mathscr{C} \subseteq H$ be a closed convex pointed cone. Let $A, B, C\colon \mathscr{C} \longrightarrow \widetilde{C}B(H)$ be the D-Lipschitz continuous mappings with constants $\lambda_{D_A}, \lambda_{D_B}$, and λ_{D_C}, respectively. Let $N\colon H \times H \times H \longrightarrow H$ be a single-valued mapping such that N is Lipschitz continuous in first, second, and third arguments with constants $\lambda_{N_1}, \lambda_{N_2}$, and λ_{N_3}, respectively. Let $Y_\rho^{\partial\psi}\colon \mathscr{C} \longrightarrow \mathscr{C}$ be the Yosida approximation operator such that $Y_\rho^{\partial\psi}$ is strongly monotone with constant δ_y and $J_\rho^{\partial\psi}\colon \mathscr{C} \longrightarrow \mathscr{C}$ be the proximal operator such that $J_\rho^{\partial\psi}$ is Lipschitz continuous with constant θ. Suppose $P\colon \mathscr{C} \longrightarrow \mathscr{C}$ be a Lipschitz continuous mapping with constant λ_P, strongly monotone with respect to $J_\rho^{\partial\psi}$ with constant μ and $\psi\colon H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a strongly convex, subdifferentiable, proper functional satisfying $Y_\rho^{\partial\psi}(x) \in do\, m(\partial\psi)$. Suppose that $z_{n+1} \propto z_n$, for $n = 0, 1, 2, \cdots$ and if the following condition is satisfied:*

$$\begin{aligned}
&\theta < 2\delta_y, \xi(\theta) < \frac{1 - \lambda_P\theta}{\rho\theta}, \quad \text{where } \xi(\theta)\\
&= \lambda_{N_1}\lambda_{D_A} + \lambda_{N_2}\lambda_{D_B} + \lambda_{N_3}\lambda_{D_C} \text{ and } \theta = \frac{1}{\sqrt{\sigma\rho + \mu}},
\end{aligned} \tag{59}$$

then there exists $x, z \in \mathscr{C}, u \in A(x), v \in B(x), w \in C(x)$ satisfying the Yosida proximal operator equation involving XOR-operation (23) and the sequences $\{x_n\}, \{z_n\}, \{u_n\}, \{v_n\}$

and $\{w_n\}$ generated by Algorithm 1 converge strongly to $x, z, u, v,$ and w, respectively.

Proof. Using (x) of Algorithm 1 and (ii) of Proposition 1, we have

$$\begin{aligned} 0 &\le z_{n+1} \oplus z_n \\ &= \left[P\left(Y_\rho^{\partial\psi}(x_n)\right) + \rho N(u_n, v_n, w_n)\right] \oplus \left[P\left(Y_\rho^{\partial\psi}(x_n)\right) + \rho N(u_{n-1}, v_{n-1}, w_{n-1})\right] \\ &= \left[P\left(Y_\rho^{\partial\psi}(x_n)\right) \oplus P\left(Y_\rho^{\partial\psi}(x_{n-1})\right)\right] + \rho\left[N(u_n, v_n, w_n) \oplus N(u_{n-1}, v_{n-1}, w_{n-1})\right]. \end{aligned} \tag{60}$$

It follows from (60) that

$$\begin{aligned} \|z_{n+1} \oplus z_n\| \le{}& \left\|P\left(Y_\rho^{\partial\psi}(x_n)\right) \oplus P\left(Y_\rho^{\partial\psi}(x_{n-1})\right)\right\| \\ &+ \rho\|N(u_n, v_n, w_n) \oplus N(u_{n-1}, v_{n-1}, w_{n-1})\|. \end{aligned} \tag{61}$$

Since $z_{n+1} \propto z_n$ and using (vii) and (viii) of Proposition 1, from (61), we obtain

$$\begin{aligned} \|z_{n+1} - z_n\| \le{}& \left\|P\left(Y_\rho^{\partial\psi}(x_n)\right) - P\left(Y_\rho^{\partial\psi}(x_{n-1})\right)\right\| \\ &+ \rho\|N(u_n, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})\|. \end{aligned} \tag{62}$$

Since N is Lipschitz continuous in all the three arguments with constants $\lambda_{N_1}, \lambda_{N_2}$, and λ_{N_3}, respectively, and A, B, C are D-Lipschitz continuous mappings with constants $\lambda_{D_A}, \lambda_{D_B}, \lambda_{D_C}$, respectively, and using (vii), (viii), (ix) of Algorithm 1, we have

$$\begin{aligned} &\|N(u_n, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})\| = \|N(u_n, v_n, w_n) - N(u_{n-1}, v_n, w_n) + N(u_{n-1}, v_n, w_n) \\ &\quad - N(u_{n-1}, v_{n-1}, w_n) + N(u_{n-1}, v_{n-1}, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})\| \\ &\le \lambda_{N_1}\|u_n - u_{n-1}\| + \lambda_{N_2}\|v_n - v_{n-1}\|\lambda_{N_3}\|w_n - w_{n-1}\| \\ &\le \lambda_{N_1}\left[\left(1 + \frac{1}{n}\right)D\left(A\left(Y_\rho^{\partial\psi}(x_n)\right), A\left(Y_\rho^{\partial\psi}(x_{n-1})\right)\right)\right] \\ &\quad + \lambda_{N_2}\left[\left(1 + \frac{1}{n}\right)D\left(B\left(Y_\rho^{\partial\psi}(x_n)\right), B\left(Y_\rho^{\partial\psi}(x_{n-1})\right)\right)\right] \\ &\quad + \lambda_{N_3}\left[\left(1 + \frac{1}{n}\right)D\left(C\left(Y_\rho^{\partial\psi}(x_n)\right), C\left(Y_\rho^{\partial\psi}(x_{n-1})\right)\right)\right] \\ &\le \left[\lambda_{N_1}\left(1 + \frac{1}{n}\right)\lambda_{D_A}\left\|Y_\rho^{\partial\psi}(x_n) - Y_\rho^{\partial\psi}(x_{n-1})\right\|\right] \\ &\quad + \left[\lambda_{N_2}\left(1 + \frac{1}{n}\right)\lambda_{D_B}\left\|Y_\rho^{\partial\psi}(x_n) - Y_\rho^{\partial\psi}(x_{n-1})\right\|\right] \\ &\quad + \left[\lambda_{N_3}\left(1 + \frac{1}{n}\right)\lambda_{D_C}\left\|Y_\rho^{\partial\psi}(x_n) - Y_\rho^{\partial\psi}(x_{n-1})\right\|\right] \\ &= \left(\lambda_{N_1}\lambda_{D_A} + \lambda_{N_2}\lambda_{D_B} + \lambda_{N_3}\lambda_{D_C}\right)\left(1 + \frac{1}{n}\right)\left\|Y_\rho^{\partial\psi}(x_n) - Y_\rho^{\partial\psi}(x_{n-1})\right\|. \end{aligned} \tag{63}$$

Using strong monotonicity of the Yosida approximation operator $Y_\rho^{\partial\psi}$ with constant δ_Y and Lipschitz continuity of the proximal operator $J_\rho^{\partial\psi}$ with constant θ, we have

$$\begin{aligned} \|x_n - x_{n-1}\|^2 &= \left\|J_\rho^{\partial\psi}(z_n) - J_\rho^{\partial\psi}(z_{n-1}) - \left[Y_\rho^{\partial\psi}(x_n) - x_n - \left(Y_\rho^{\partial\psi}(x_{n-1}) - x_{n-1}\right)\right]\right\|^2 \\ &\le \left\|J_\rho^{\partial\psi}(z_n) - J_\rho^{\partial\psi}(z_{n-1})\right\|^2 - 2\langle Y_\rho^{\partial\psi}(x_n) - Y_\rho^{\partial\psi}(x_{n-1}), x_n - x_{n-1}\rangle + \|x_n - x_{n-1}\|^2 \\ &\le \theta^2\|z_n - z_{n-1}\|^2 - 2\delta_y\|x_n - x_{n-1}\|^2 + \|x_n - x_{n-1}\|^2. \end{aligned} \tag{64}$$

It follows that

$$2\delta_y\left\|x_n - x_{n-1}\right\|^2 \le \theta^2\left\|z_n - z_{n-1}\right\|^2,$$
$$\left\|x_n - x_{n-1}\right\| \le \frac{\theta^2}{\sqrt{2\delta_y}}\left\|z_n - z_{n-1}\right\|, \tag{65}$$
$$\left\|x_n - x_{n-1}\right\| \le \xi(y)\left\|z_n - z_{n-1}\right\|,$$

where $\xi(y) = \theta^2/\sqrt{2\delta_y}$.

Combining (62) and (63), using Lipschitz continuity of $P, J_\rho^{\partial\psi}$ and (vi) of Algorithm 1, we have

$$\begin{aligned}\left\|z_{n+1} - z_n\right\| &\le \lambda_P\left\|\left(Y_\rho^{\partial\psi}(x_n)\right) - \left(Y_\rho^{\partial\psi}(x_{n-1})\right)\right\| \\ &\quad + \rho\left(\lambda_{N_1}\lambda_{D_A} + \lambda_{N_2}\lambda_{D_B} + \lambda_{N_3}\lambda_{D_C}\right)\left(1 + \frac{1}{n}\right)\left\|Y_\rho^{\partial\psi}(x_n) - Y_\rho^{\partial\psi}(x_{n-1})\right\|,\end{aligned} \tag{66}$$

$$\begin{aligned}\left\|z_{n+1} - z_n\right\| &\le \left[\lambda_P + \rho\xi_n(\theta)\right]\left\|Y_\rho^{\partial\psi}(x_n) - Y_\rho^{\partial\psi}(x_{n-1})\right\| \\ &\le \left[\lambda_P + \rho\xi_n(\theta)\right]\left\|J_\rho^{\partial\psi}(z_n) - J_\rho^{\partial\psi}(z_{n-1})\right\| \\ &= \left[\lambda_P + \rho\xi_n(\theta)\right]\theta\left\|z_n - z_{n-1}\right\| \\ &= \left[\lambda_P\theta + \rho\xi_n(\theta)\theta\right]\left\|z_n - z_{n-1}\right\| \\ &= S_n(\theta)\left\|z_n - z_{n-1}\right\|,\end{aligned} \tag{67}$$

where $S_n(\theta) = \lambda_P\theta + \rho\xi_n(\theta)\theta$, $\theta = (1/\sqrt{\sigma\rho + \mu})$ and $\xi_n(\theta) = (\lambda_{N_1}\lambda_{D_A} + \lambda_{N_2}\lambda_{D_B} + \lambda_{N_3}\lambda_{D_C})(1 + (1/n))$.

Letting $S(\theta) = \lambda_P\theta + \rho\xi(\theta)\theta$, where $\xi(\theta) = (\lambda_{N_1}\lambda_{D_A} + \lambda_{N_2}\lambda_{D_B} + \lambda_{N_3}\lambda_{D_C})$, it follows that $S_n(\theta) \longrightarrow S(\theta)$ as $n \longrightarrow \infty$. From (59), we have $\xi(y) < 1$ and $S(\theta) < 1$. Consequently, we conclude from (65) and (67) that $\{x_n\}$ and $\{z_n\}$ both are Cauchy sequences. Since H is complete and $\mathscr{C} \subseteq H$ is a closed convex subset of H and thus $\mathscr{C}$ is also complete, we may assume that $x_n \longrightarrow x \in \mathscr{C}$ and $z_n \longrightarrow z \in \mathscr{C}$. From (vii), (viii), and (ix) of Algorithm 1, it follows that $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are also Cauchy sequences such that $u_n \longrightarrow u, v_n \longrightarrow v$ and $w_n \longrightarrow w$, as $n \longrightarrow \infty$.

It can be shown easily by using the techniques of [28] that $u \in A(x), v \in B(x)$, and $w \in C(x)$. By Lemma 3, we conclude that $x, z \in \mathscr{C}, u \in A(x), v \in B(x)$, and $w \in C(x)$ is the solution of Yosida proximal operator equation involving XOR-operation (23). □

We provide the following numerical example using MATLAB program R2018a along with a computational table and a convergence graphs for different initial values in support of Algorithm 1 and Theorem 1.

Example 1. Suppose $\mathscr{C} = H = [0, \infty)$. Let $P: \mathscr{C} \longrightarrow \mathscr{C}$ and $\psi: \mathscr{C} \longrightarrow \mathbb{R} \cup \{+\infty\}$ be the mappings such that for $x \in \mathscr{C}$,

$$P(x) = \frac{x}{2},$$
$$\psi(x) = x^2, \tag{68}$$

Then $\partial\psi(x) = \{2x\}$, the sub differential of ψ.

Since $\psi''(x) = 2 > 0$. Hence, ψ is strongly convex with modulus $\lambda = 2$.

For $\rho = 1$, the proximal operator $J_\rho^{\partial\psi}$ is given by

$$J_\rho^{\partial\psi} = [P + \rho\partial\psi]^{-1}(x) = \frac{2x}{5}, \text{where } [P + \rho\partial\psi](x) = \frac{5x}{2}. \tag{69}$$

It is simple to see that P is Lipschitz continuous with constant $\lambda_p = (11/10)$, strongly monotone with respect to $J_\rho^{\partial\psi}$ with constant $\mu = 1/3$, and $J_\rho^{\partial\psi}$ is Lipschitz continuous with constant $\theta = \sqrt{2}/3$.

In view of proximal operator calculated above, the Yosida approximation operator is given by

$$Y_\rho^{\partial\psi}(x) = \frac{1}{\rho}\left[I - J_\rho^{\partial\psi}\right](x) = \frac{3x}{5}. \tag{70}$$

Also,

$$\begin{aligned}\langle Y_\rho^{\partial\psi}(x) - Y_\rho^{\partial\psi}(y), x - y\rangle &= \langle\frac{3x}{5} - \frac{3y}{5}, x - y\rangle \\ &= \langle\frac{3}{5}(x - y), x - y\rangle \\ &= \frac{3}{5}\langle x - y, x - y\rangle = \frac{3}{5}\|x - y\|^2 \\ &\ge \frac{2}{5}\|x - y\|^2.\end{aligned} \tag{71}$$

Hence, $Y_\rho^{\partial\psi}$ is strongly monotone with constant $\delta_y = (2/5)$.

Let us consider the mappings $N\colon H\times H\times H \longrightarrow H$ and $A,B,C\colon \mathscr{C}\longrightarrow \widetilde{C}B(H)$ such that

$$\begin{aligned} A(x) &= \left\{\frac{x}{7}\right\},\\ B(x) &= \left\{\frac{x}{5}\right\},\\ C(x) &= \left\{\frac{x}{6}\right\},\\ N(u,v,w) &= \frac{u}{2}+\frac{v}{2}+\frac{w}{2}, \end{aligned} \tag{72}$$

where $x\in\mathscr{C}, u\in A(x), v\in B(x)$, and $w\in C(x)$.

$$\begin{aligned} \text{Clearly, } D(A(x),A(y)) &= \max\left\{\sup_{x\in A(x)} d(x,F(y)), \sup_{y\in F(y)} d(F(x),y)\right\}\\ &\le \max\left\{\left\|\frac{x}{7}-\frac{y}{7}\right\|, \left\|\frac{y}{7}-\frac{x}{7}\right\|\right\}\\ &= \frac{1}{7}\max\{\|x-y\|,\|y-x\|\}\\ &\le \frac{1}{5}\|x-y\|, \end{aligned} \tag{73}$$

that is, $D(A(x),A(y))\le (1/5)\|x-y\|$.

Thus, A is D-Lipschitz continuous with constant $\lambda_{D_A}=(1/5)$. Similarly, we can obtain that B and C are D-Lipschitz continuous with constants $\lambda_{D_B}=(1/3)$ and $\lambda_{D_C}=(1/3)$, respectively.

N is Lipschitz continuous in all the three arguments with constants $\lambda_{N_1}=\lambda_{N_2}=\lambda_{N_3}=1$.

$$\begin{aligned} \text{Then, } N(u,v,w) &= \frac{x}{14}+\frac{x}{10}+\frac{x}{12}\\ &= \frac{107x}{420},\\ \text{Since } z &= P\left(Y_\rho^{\partial\psi}(x)\right)+\rho N(u,v,w)\\ &= \frac{Y_\rho^{\partial\psi}(x)}{2}+N(u,v,w)\\ &= \frac{1}{2}\cdot\frac{3x}{5}+N(u,v,w),\\ z &= \frac{3x}{10}+N(u,v,w),\\ J_\rho^{\partial\psi}(z) &= \frac{6x}{50}+\frac{2N(u,v,w)}{5}. \end{aligned} \tag{74}$$

Hence,

$$R_\rho^{\partial\psi}(z) = \left[I-J_\rho^{\partial\psi}\right](z) = \frac{3}{5}\left[\frac{3x}{10}+N(u,v,w)\right]. \tag{75}$$

Below we show that condition (59) is satisfied.

For $\lambda_P=11/10, \rho=1, \delta_y=2/5, \mu=1/3, \lambda=2, \sigma=2\lambda=4, \theta=\sqrt{2}/3, \lambda_{N_1}=\lambda_{N_2}=\lambda_{N_3}=1, \lambda_{D_A}=1/5, \lambda_{D_B}=1/3, \lambda_{D_C}=1/3, \xi(\theta)=\lambda_{N_1}\lambda_{D_A}+\lambda_{N_2}\lambda_{D_B}+\lambda_{N_3}\lambda_{D_C}=0.86$ and $(1-\lambda_P\theta)/\rho\theta=1.021$. Hence, $\theta<2\delta_y$ and $\xi(\theta)<(1-\lambda_P\theta)/\rho\theta$. That is, condition (59) is satisfied.

For, $x=0$, the Yosida proximal operator equation involving XOR-operator (23) is fulfilled.

$$\begin{aligned} &\text{That is, } N(u,v,w)\oplus\rho^{-1}R_\rho^{\partial\psi}(z)\\ &= N(u,v,w)\oplus\frac{3}{5}\left[\frac{3x}{10}+N(u,v,w)\right]=0. \end{aligned} \tag{76}$$

Furthermore, we obtain the sequences $\{x_n\}$ and $\{z_n\}$ generated by iterative Algorithm 1 as

$$\begin{aligned} z_{n+1} &= P\left(Y_\rho^{\partial\psi}(x_n)\right)+\rho N(u_n,v_n,w_n)\\ &= \frac{Y_\rho^{\partial\psi}(x_n)}{2}+\frac{107x_n}{420}\\ &= \frac{1}{2}\cdot\frac{3x_n}{5}+\frac{107x_n}{420}\\ &= \frac{3x_n}{10}+\frac{107x_n}{420},\\ &\text{also, } Y^{\partial\psi}{}_\rho(x_n) = J_\rho^{\partial\psi}(z_n)\\ \frac{3x_n}{5} &= \frac{2z_n}{5}\\ x_n &= \frac{2z_n}{5}. \end{aligned} \tag{77}$$

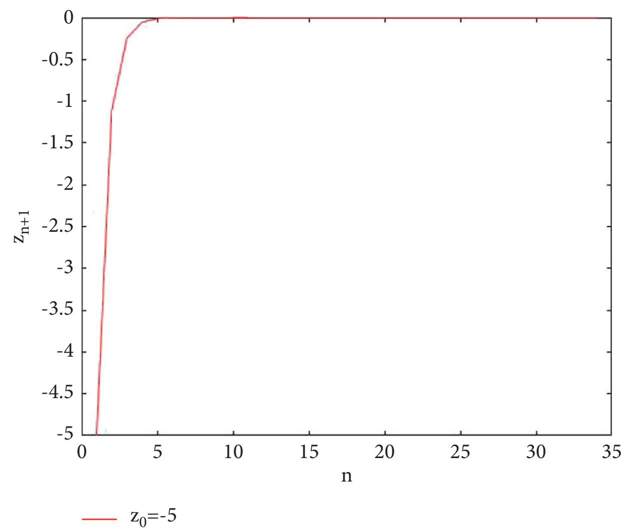

Figure 1: Case I: for the initial value $z_0 = -5$.

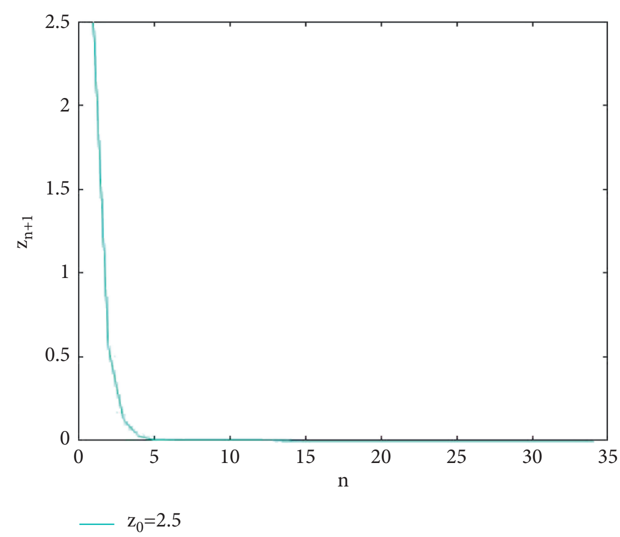

Figure 2: Case II: for the initial value $z_0 = 2.5$.

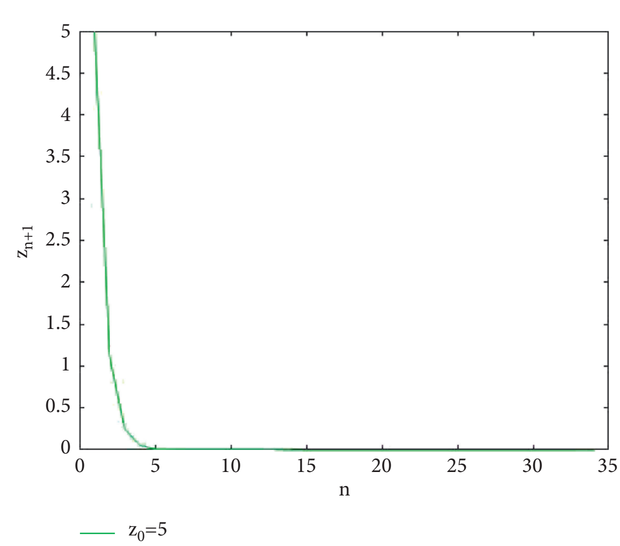

Figure 3: Case III: for the initial value $z_0 = 5$.

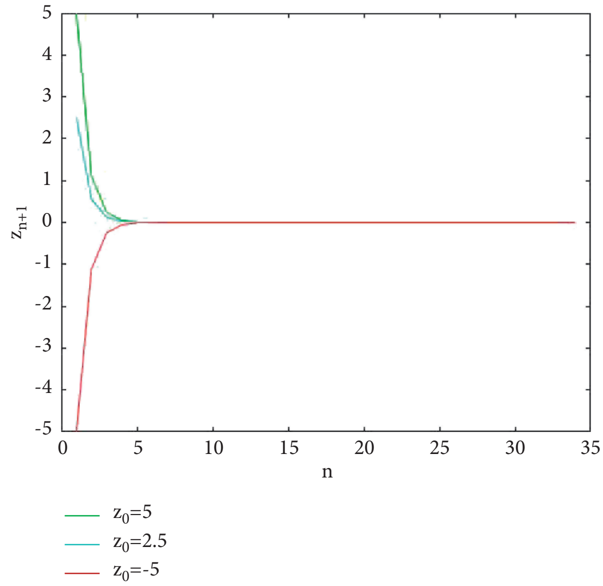

FIGURE 4: By combining all the above graphs, we get the following conjoining graph of convergence.

TABLE 1: The values of z_n with initial values $z_0 = -5$, $z_0 = 2.5$, and $z_0 = 5$.

No. of iterations	For $z_0 = -5$ z_n	For $z_0 = 2.5$ z_n	For $z_0 = 5$ z_n
$n = 1$	-5	2.5	5
$n = 2$	-1.10952380952381	0.554761904761905	1.10952380952381
$n = 3$	-0.246208616780045	0.123104308390023	0.246208616780045
$n = 4$	-0.0546348644854767	0.0273174322427384	0.0546348644854767
$n = 5$	-0.0121237365953486	0.00606186829767432	0.0121237365953486
$n = 10$	$-6.52333379900596e-06$	$3.26166689950298e-06$	$6.52333379900596e-06$
$n = 15$	$-3.50996440070956e-09$	$1.75498220035478e-09$	$3.50996440070956e-09$
$n = 20$	$-1.88858189291582e-12$	$9.44290946457911e-13$	$1.88858189291582e-12$
$n = 25$	$-1.01617599469911e-15$	$5.08087997349556e-16$	$1.01617599469911e-15$
$n = 27$	$-5.00382572119910e-17$	$2.50191286059955e-17$	$5.00382572119910e-17$
$n = 29$	$-2.46397001884969e-18$	$1.23198500942484e-18$	$2.46397001884969e-18$
$n = 30$	$-5.46766680373311e-19$	$2.73383340186656e-19$	$5.46766680373311e-19$

From (77) and (78), we have

$$z_{n+1} = \left(\frac{3}{10} + \frac{107}{420}\right)\frac{2z_n}{5}, \qquad z_{n+1} = \frac{233}{1050}z_n. \tag{78}$$

Clearly, the sequence $\{z_n\}$ converges to 0, and consequently, the sequence $\{x_n\}$ also converges to 0.

It is shown in Figures 1–3 that, for initial values $z_0 = -5, 2.5$, and 5, the sequence $\{z_n\}$ converges to 0. A consolidated graph using Figures 1–3 is provided in Figure 4. In Table 1, comparing different initial values of $\{z_n\}$ and for different iterations, it is obtained that the sequence $\{z_n\}$ converges to 0.

5. Conclusion

In this work, we introduce and study three new problems, that is, a Yosida complementarity problem, a Yosida variational inequality problem, and a Yosida proximal operator equation involving XOR-operation. It is shown that Yosida complementarity problem involving XOR-operation is equivalent to a Yosida variational inequality problem involving XOR-operation and Yosida variational inequality problem involving XOR-operation is equivalent to a Yosida proximal operator equation involving XOR-operation. An algorithm is established to obtain the solution of Yosida proximal operator equation involving XOR-operation. Finally, an existence and convergence result is proved. A numerical example is given in support of our main result.

It is still an open and interesting problem that how to establish equivalence between Yosida complementarity problem involving XOR-operation and Yosida proximal operator equation problem involving XOR-operation.

Data Availability

No data were used to support this study.

Disclosure

A variant form of Yosida variational inequality and Yosida proximal operator equation involving XOR-operation was considered in "Some Problems Concerning Generalized Variational Inequalities", Ph. D Thesis, (2009), AMU Aligarh [39]. In this variant form, neither the concept of Yosida approximation operator nor the concept of XOR-operation was used. Moreover, no complementarity problem was considered in the abovementioned thesis.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On the Hadamard Well-Posedness of Generalized Mixed Variational Inequalities in Banach Spaces

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We introduce a new concept of Hadamard well-posedness of a generalized mixed variational inequality in a Banach space. The relations between the Levitin–Polyak well-posedness and Hadamard well-posedness for a generalized mixed variational inequality are studied. The characterizations of Hadamard well-posedness for a generalized mixed variational inequality are established.

1. Introduction

In [1], Tykhonov first introduced the well-posedness of a minimization problem, which means that it has a unique minimizer and every minimizing sequence converges to the unique minimizer. There are two concepts of well-posedness which are Tykhonov well-posedness [1] and Hadamard well-posedness [2].

Recently, variational inequality (VI) has been extensively studied due to the facts that it has many potential applications and that it is closely related to a differentiable minimization problem. Well-posedness for a variational inequality has been then extensively investigated. See, e.g., [3–11] and the references therein.

In 2013, Li and Xia [12] introduced the concept of Hadamard well-posedness of a general mixed variational inequality in Banach spaces. Under some suitable conditions, relations between Levitin–Polyak well-posedness and Hadamard well-posedness of a general mixed variational inequality were presented. They also established some characterizations of Hadamard well-posedness for a general mixed variational inequality. Very recently, some scholars still focused on the study of the well-posedness of various classes of variational inequalities, see e.g., generalized variational-hemivariational inequalities with perturbations in [13], completely generalized mixed variational inequalities in [14], noncompact generalized mixed variational inequalities in [15], generalized variational inequality with generalized mixed variational inequality constraint in [16], systems of generalized mixed quasivariational inclusion problems in [17], systems of time-dependent hemivariational inequalities in [18], and generalized hemivariational inequalities in [19].

Motivated and inspired by the research work going on this field, we introduce a new concept of Hadamard well-posedness for a generalized mixed variational inequality in a Banach space. Under some suitable conditions, the relations between the Levitin–Polyak well-posedness and Hadamard well-posedness for a generalized mixed variational inequality are studied. We also establish some characterizations of Hadamard well-posedness for a generalized mixed variational inequality. Finally, we prove that under suitable conditions, the Hadamard well-posedness of a generalized mixed variational inequality is equivalent to the existence and uniqueness of its solutions. Our results improve, extend, and develop the earlier and recent ones announced by some others, e.g., Ceng and Yao [7] and Li and Xia [12, 20].

2. Preliminaries

Let X be a real reflexive Banach space with its dual X^* and K be a nonempty, closed subset of X. We use the same notations in [12]. For more details about these notations and relevant definitions, please consult relevant reference; see, e.g., [12] (following [21]). Let X' be the collection of all affine functions defined on X. It is obvious that $X^* \subset X'$. Let U be the collection of all nonempty set-valued mappings $F: X \longrightarrow 2^{X^*}$, and $\tau(X)$ be the collection of all mappings $P: X \longrightarrow 2^{X'}$ such that for any $x \in X$, there exist $F \in U$ and $\lambda \in \mathbf{R}$ such that

$$\langle P(x), x - y\rangle = \langle F(x), x - y\rangle + \lambda, \quad \forall y \in K. \tag{1}$$

For any $P_1, P_2 \in \tau(X)$, it follows that there exist $F_i \in U$ and $\lambda_i \in \mathbf{R}$, $i = 1, 2$ such that

$$\begin{aligned} \langle P_1(x), x - y\rangle &= \langle F_1(x), x - y\rangle + \lambda_1, \quad \forall y \in K, \\ \langle P_2(x), x - y\rangle &= \langle F_2(x), x - y\rangle + \lambda_2, \quad \forall y \in K. \end{aligned} \tag{2}$$

We define

$$d(P_1, P_2) = \begin{cases} |\lambda_1 - \lambda_2|, & F_1 = F_2, \\ 1 + |\lambda_1 - \lambda_2|, & F_1 \neq F_2. \end{cases} \tag{3}$$

It can be routinely checked that $(\tau(X), d)$ is a metric space. In particular, if U is the collection of all single-valued mappings $F: X \longrightarrow X^*$ and $\tau(X)$ is the collection of all single-valued mappings $P: X \longrightarrow X'$ such that for any $x \in X$, there exist $F \in U$ and $\lambda \in \mathbf{R}$ such that

$$\langle P(x), x - y\rangle = \langle F(x), x - y\rangle + \lambda, \quad \forall y \in K, \tag{4}$$

then the above metric space $(\tau(X), d)$ reduces to the metric space $(\tau(X), d)$ defined in [[12], p. 1619]. In this case, it is clear that the metric space $(\tau(X), d)$ is a special case of metric space (Γ, d) defined in [[21], p. 377].

Let $C(X)$ be the collection of all nonempty closed subsets of X endowed with the usual Hausdorff metric $\mathcal{H}(\cdot,\cdot)$, that is, for every $A_1, A_2 \in C(X)$,

$$\mathcal{H}(A_1, A_2) = \max\{e(A_1, A_2), e(A_2, A_1)\}, \tag{5}$$

where $e(A_1, A_2) = \sup_{a\in A_1} d'(a, A_2)$ with $d'(a, A_2) = \inf_{b\in A_2}\|a - b\|$. Let $\{A_n\}$ be a sequence of nonempty closed subsets of X. We say that A_n converges to A in the Hausdorff metric iff $\mathcal{H}(A_n, A) \longrightarrow 0$ as $n \longrightarrow \infty$.

Let $B(X)$ be the family of all real-valued functions on X; we define

$$d_1(\phi_1, \phi_2) = \sup_{x\in X}|\phi_1(x) - \phi_2(x)|, \tag{6}$$

where $\phi_1, \phi_2 \in B(X)$; it can be routinely checked that $(B(X), d_1)$ is a metric space.

Let M be the collection of all (P, ϕ, K) such that

(i) $P \in \tau(X)$;
(ii) $\phi \in B(X)$;
(iii) $K \in C(X)$.

Then, for any $(P_1, \phi_1, K_1), (P_2, \phi_2, K_2) \in M$, we define

$$\rho((P_1, \phi_1, K_1), (P_2, \phi_2, K_2)) = d(P_1, P_2) + d_1(\phi_1, \phi_2) + \mathcal{H}(K_1, K_2). \tag{7}$$

Clearly, (M, ρ) is a metric space.

Let $F: X \longrightarrow 2^{X^*}$ be a set-valued mapping, and $\phi: X \longrightarrow \mathbf{R}\cup\{+\infty\}$ be a proper, convex, and lower semicontinuous functional. Consider the following generalized mixed variational inequality associated with (F, ϕ, K):

$$\begin{aligned} &\text{GMVI}(F, \phi, K)\text{: find } x \in K \text{ such that for some } u \in F(x), \\ &\langle u, x - y\rangle + \phi(x) - \phi(y) \leq 0, \quad \forall y \in K. \end{aligned} \tag{8}$$

We denote by $S(F, \phi, K)$ the solution set of $\text{GMVI}(F, \phi, K)$. In what follows, we first introduce new concept of Hadamard well-posedness for $\text{GMVI}(F, \phi, K)$. It is worth mentioning that some similar ideas have also been presented in [22, 23] very recently.

Definition 1. A generalized mixed variational inequality $\text{GMVI}(F, \phi, K)$ is called Hadamard well-posed if it has a unique solution $x^* \in K$, and if for every sequence of triples $\{(P_n, \phi_n, K_n)\} \subset M$ converging to (F, ϕ, K) and every sequence $\{x_n\}$ such that $x_n \in S(P_n, \phi_n, K_n)$ for each $n \in \mathbf{N}$, it follows that $x_n \longrightarrow x^*$, where $K \cup \{x_n\} \subseteq K_n$ for all $n \in \mathbf{N}$.

Definition 2 (see [20]). A sequence $\{x_n\} \subset X$ is called a LP approximating sequence for $\text{GMVI}(F, \phi, K)$, if there exist $w_n \in X$ with $w_n \longrightarrow 0$ and $0 < \epsilon_n \longrightarrow 0$ such that $x_n + w_n \in K$ for all $n \in \mathbf{N}$, and there exists $u_n \in F(x_n)$ such that

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \leq \epsilon_n, \quad \forall y \in K, n \in N. \tag{9}$$

Definition 3 (see [20]). We say that $\text{GMVI}(F, \phi, K)$ is LP well-posed if $\text{GMVI}(F, \phi, K)$ has a unique solution and every LP approximating sequence for $\text{GMVI}(F, \phi, K)$ converges strongly to the unique solution.

The product space $C(X) \times B(X)$ is equipped by a product metric, that is, $\mathcal{H}(A_1, A_2) + d_1(f_1, f_2)$, where $A_1, A_2 \in C(X)$ and $f_1, f_2 \in B(X)$. Let further $BC^0(X)$ be the family of all real-valued continuous functions on X; it is easy to check that $(BC^0(X), d_1)$ is a metric space and we write $Q = C(X) \times BC^0(X)$. Now, we can easily get the following lemma.

Lemma 1 (see [12]). *Let the pair $(A, f) \in Q$, then the function $(A, f) \longmapsto \inf(A, f)$ is upper semicontinuous.*

We consider the following gap function for GMVI(F,ϕ,K):

$$g(x)=\sup_{y\in K}\inf_{u\in F(x)}\{\langle u,x-y\rangle+\phi(x)-\phi(y)\},\quad \forall x\in X. \tag{10}$$

Lemma 2. *The following statements hold:*

(i) $g(x)\geq 0$, $\forall x\in K$;

(ii) $\overline{x}\in K$ *solves GMVI*$(F,\phi,K)\Longleftrightarrow g(\overline{x})=0$.

Proof. For each $x\in K$, we have

$$\begin{aligned} g(x)&=\sup_{y\in K}\inf_{u\in F(x)}\{\langle u,x-y\rangle+\phi(x)-\phi(y)\}\\ &\geq \inf_{u\in F(x)}\{\langle u,x-x\rangle+\phi(x)-\phi(x)\}\\ &=0. \end{aligned} \tag{11}$$

Observe that

$$\begin{aligned} &\overline{x}\text{ solves GMVI}(F,\phi,K)\\ &\Longleftrightarrow \exists\overline{u}\in F(\overline{x})\text{ s.t. }\langle\overline{u},\overline{x}-y\rangle+\phi(\overline{x})-\phi(y)\leq 0,\quad\forall y\in K\\ &\Longleftrightarrow \sup_{y\in K}\{\langle\overline{u},\overline{x}-y\rangle+\phi(\overline{x})-\phi(y)\}\leq 0\\ &\Longleftrightarrow 0\leq g(\overline{x})\leq\sup_{y\in K}\{\langle\overline{u},\overline{x}-y\rangle+\phi(\overline{x})-\phi(y)\}\leq 0\\ &\Longleftrightarrow g(\overline{x})=0. \end{aligned} \tag{12}$$

This completes the proof.

We also consider the following optimization problem:

$$(\text{OP}):\ \inf_{x\in K}g(x), \tag{13}$$

with $g(x)$ defined by (10). Its optimal solution set will be denoted by $\arg\min(K,g)$ and the optimal value will be denoted by $\inf(K,g)$, respectively.

The following Definitions 4–6 can be found in [12]. However, Definition 7 is conventional. □

Definition 4. A sequence $\{x_n\}\subset X$ is called an LP minimizing sequence for (OP) if there exists $u_n\in F(x_n)$ such that

$$\begin{aligned} g_n(x_n)&\longrightarrow\inf(K,g),\\ d'(x_n,K)&\longrightarrow 0, \end{aligned} \tag{14}$$

where $g_n(x_n)=\sup_{y\in K\cup\{x_n\}}\{\langle u_n,x_n-y\rangle+\phi(x_n)-\phi(y)\}$, $\forall n\in\mathbf{N}$.

Definition 5. We say that (OP) is LP well-posed if and only if (OP) has a unique solution and every LP minimizing sequence for (OP) converges strongly to the unique solution.

Definition 6. A nonempty-valued function $f\colon X\longrightarrow\mathbf{R}\cup\{+\infty\}$ is said to be uniformly continuous, if for any $\epsilon>0$, there exists $\delta>0$ such that for all $x,y\in X$ with $\|x-y\|<\delta$, one has $|f(x)-f(y)|<\varepsilon$.

Definition 7. A nonempty set-valued mapping $F\colon X\longrightarrow 2^{X^*}$ is said to be monotone, if for all $x,y\in X$, $u\in F(x)$ and $v\in F(y)$,

$$\langle u-v,x-y\rangle\geq 0. \tag{15}$$

Definition 8 (see [7]). Let $\mathscr{H}(\cdot,\cdot)$ be the Hausdorff metric on the collection $\text{CB}(X)$ of all nonempty, closed, and bounded subsets of X, which is defined by $\mathscr{H}(A,B)=\max\{e(A,B),e(B,A)\}$ for A and B in $\text{CB}(X)$. A nonempty set-valued mapping $F\colon X\longrightarrow 2^{X^*}$ is said to be

(i) $\mathscr{H}$-hemicontinuous, if for any $x,y\in X$, the function $t\longmapsto\mathscr{H}(F(x+t(y-x)),F(x))$ from $[0,1]$ into $\mathbf{R}^+=[0,+\infty)$ is continuous at 0^+;

(ii) $\mathscr{H}$-uniformly continuous, if for any $\epsilon>0$, there exists $\delta>0$ such that for all $x,y\in X$ with $\|x-y\|<\delta$, one has $\mathscr{H}(F(x),F(y))<\epsilon$.

The following proposition is a special case of Lemma 2.2 in [7].

Proposition 1 (see [20]). *Let K be a nonempty, closed, and convex subset of X, $F\colon X\longrightarrow 2^{X^*}$ be a nonempty compact-valued mapping which is $\mathscr{H}$-hemicontinuous and monotone, and $\phi\colon X\longrightarrow\mathbf{R}\cup\{+\infty\}$ be a proper and convex functional. Then, for a given $x\in K$, the following statements are equivalent:*

(i) there exists $u\in F(x)$ such that $\langle u,x-y\rangle+\phi(x)-\phi(y)\leq 0$, $\forall y\in K$;

(ii) $\langle v,x-y\rangle+\phi(x)-\phi(y)\leq 0$, $\forall y\in K, v\in F(y)$.

We can also prove the following lemma easily.

Lemma 3. *Let K be a nonempty, closed subset of X, $F\colon X\longrightarrow 2^{X^*}$ be a nonempty set-valued mapping, and $\phi\colon X\longrightarrow\mathbf{R}\cup\{+\infty\}$ be a functional on X. Then, the following statements are equivalent:*

(i) GMVI(F,ϕ,K) is LP well-posed;

(ii) (OP) is LP well-posed with $g(x)$ defined by (10).

Proof. We first claim that (i) ⇒ (ii). Indeed, suppose that GMVI(F,ϕ,K) is LP well-posed and $x^*\in K$ is the unique solution of GMVI(F,ϕ,K). By Lemma 2, $x^*\in K$ is the unique solution of (OP). Then, we get $\inf(K,g)=0$. Let $\{x_n\}\subset X$ be a LP minimizing sequence for (OP). Then, there exists $u_n\in F(x_n)$ such that

$$\begin{aligned} g_n(x_n)&\longrightarrow\inf(K,g)=0,\\ d'(x_n,K)&\longrightarrow 0, \end{aligned} \tag{16}$$

where $g_n(x_n)=\sup_{y\in K\cup\{x_n\}}\{\langle u_n,x_n-y\rangle+\phi(x_n)-\phi(y)\}$, $\forall n\in\mathbf{N}$. So, we deduce that

$$\sup_{y\in K}\{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\} \le \sup_{y\in K\cup\{x_n\}}\{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\} = g_n(x_n), \quad (17)$$

which immediately yields

$$\limsup_{n\longrightarrow\infty}\sup_{y\in K}\{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\} \le 0. \quad (18)$$

Thus, there exist $0 < \epsilon_n \longrightarrow 0$ and $u_n \in F(x_n)$ such that

$$\sup_{y\in K}\{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\} \le \epsilon_n; \quad (19)$$

that is,

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \epsilon_n, \quad \forall y \in K. \quad (20)$$

Furthermore, from $d'(x_n, K) \longrightarrow 0$ it follows that there exists $\overline{x}_n \in K$ such that $\|\overline{x}_n - x_n\| \longrightarrow 0$. Putting $w_n = \overline{x}_n - x_n$, we get $x_n + w_n = \overline{x}_n \in K$ with $w_n \longrightarrow 0$. Therefore, $\{x_n\}$ is a LP approximating sequence for $\mathrm{GMVI}(F, \phi, K)$, and hence $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. This means that (OP) is LP well-posed.

We show that (ii) ⇒ (i). Indeed, suppose that (OP) is LP well-posed and $x^* \in K$ is the unique solution of (OP). By Lemma 2, $x^* \in K$ is the unique solution of $\mathrm{GMVI}(F, \phi, K)$. Then, we get $\inf(K, g) = 0$. Let $\{x_n\} \subset X$ be a LP approximating sequence for $\mathrm{GMVI}(F, \phi, K)$. Then, there exist $w_n \in X$ with $w_n \longrightarrow 0$ and $0 < \epsilon_n \longrightarrow 0$ such that $x_n + w_n \in K$ for all $n \in \mathbf{N}$, and there exists $u_n \in F(x_n)$ such that

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \epsilon_n, \quad \forall y \in K, n \in \mathbf{N}, \quad (21)$$

that is,

$$\sup_{y\in K}\{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\} \le \epsilon_n, \quad \forall n \in \mathbf{N}. \quad (22)$$

Putting $\overline{x}_n = x_n + w_n$ for all $n \in \mathbf{N}$, we get $\overline{x}_n \in K$ for all $n \in \mathbf{N}$. Then, it is easy to see that $d'(x_n, K) \le \|x_n - \overline{x}_n\| = \|w_n\| \longrightarrow 0$. Observe that for all $n \in \mathbf{N}$,

$$\begin{aligned} 0 &\le g_n(x_n) \\ &= \sup_{y\in K\cup\{x_n\}}\{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\} \\ &= \max\left\{\sup_{y\in K}\{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\}, 0\right\} \\ &\le \epsilon_n \longrightarrow 0. \end{aligned} \quad (23)$$

Consequently, $\{x_n\}$ is a LP minimizing sequence for (OP), and hence $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. This means that $\mathrm{GMVI}(F, \phi, K)$ is LP well-posed. □

3. Well-Posedness

In this section, we investigate the relations between Levitin–Polyak well-posedness and Hadamard well-posedness of a generalized mixed variational inequality.

Theorem 1. *Let K be a nonempty, closed subset of X and $F: X \longrightarrow 2^{X^*}$ be a nonempty set-valued mapping. Let $\phi: X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be a functional. Then, $\mathrm{GMVI}(F, \phi, K)$ is LP well-posed whenever $\mathrm{GMVI}(F, \phi, K)$ is Hadamard well-posed.*

Proof. Suppose that $\mathrm{GMVI}(F, \phi, K)$ is Hadamard well-posed and $x^* \in K$ is the unique solution of $\mathrm{GMVI}(F, \phi, K)$. Let $\{x_n\} \subset X$ be an LP approximating sequence for $\mathrm{GMVI}(F, \phi, K)$. Then, there exist $w_n \in X$ with $w_n \longrightarrow 0$ and $0 < \epsilon_n \longrightarrow 0$ such that $x_n + w_n \in K$ for all $n \in \mathbf{N}$, and there exists $u_n \in F(x_n)$ such that

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \epsilon_n, \quad \forall y \in K, \quad (24)$$

that is,

$$\sup_{y\in K}\{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\} \le \epsilon_n, \quad \forall n \in N. \quad (25)$$

So, it follows from $x_n + w_n \in K$ that there exists $\overline{x}_n \in K$ such that $x_n + w_n = \overline{x}_n$ for all $n \in \mathbf{N}$. Thus, we get $d'(x_n, K) \le \|x_n - \overline{x}_n\| = \|w_n\| \longrightarrow 0$. For each $n \in \mathbf{N}, x \in X$, we construct a sequence $\{(F_n, \phi_n, K_n)\}$ as follows:

$$\langle F_n(x), x - y\rangle = \langle F(x), x - y\rangle - \epsilon_n, \quad \forall y \in K, \quad (26)$$

$$\phi_n(x) = \phi(x) - \epsilon_n, \quad (27)$$

and $K_n = K \cup \{x_n\}$.

It is obvious that $x_n \in K_n, F_n \in \tau(X), \phi_n \in B(X)$, and $K_n \in C(X)$. It follows from (24)–(27) that

$$\begin{aligned} &\langle F_n(x_n), x_n - y\rangle + \phi_n(x_n) - \phi_n(y) \\ &= \langle F_n(x_n), x_n - y\rangle - \epsilon_n + \phi(x_n) - \epsilon_n - [\phi_n(y) - \epsilon_n] \\ &= \langle F_n(x_n), x_n - y\rangle + \phi(x_n) - \phi_n(y) - \epsilon_n, \quad \forall y \in K. \end{aligned} \quad (28)$$

Since $K_n = K \cup \{x_n\}$, it follows from (24) that

$$\begin{aligned} 0 &\le \mathcal{G}_n(x_n) \\ &= \sup_{y\in K_n}\inf\{\langle F_n(x_n), x_n - y\rangle + \phi_n(x_n) - \phi_n(y)\} \\ &= \sup_{y\in K_n}\inf\{\langle F_n(x_n), x_n - y\rangle + \phi(x_n) - \phi_n(y) - \epsilon_n\} \\ &= \sup_{y\in K_n}\inf_{u\in F(x_n)}\{\langle u, x_n - y\rangle + \phi(x_n) - \phi_n(y) - \epsilon_n\} \\ &\le \sup_{y\in K_n}\{\langle u, x_n - y\rangle + \phi(x_n) - \phi_n(y) - \epsilon_n\} \le 0. \end{aligned} \quad (29)$$

That is, $\mathcal{G}_n(x_n) = 0$ for all $n \in \mathbf{N}$. So, it follows from Lemma 2 that $x_n \in S(F_n, \phi_n, K_n)$ for all $n \in \mathbf{N}$. From (3) and (26), we have $d(F_n, F) = |\epsilon_n| \longrightarrow 0$. Again from (6) and (27), we have $d_1(\phi_n, \phi) \longrightarrow 0$. We also obtain that $\mathcal{H}(K_n, K) = d'(x_n, K) \longrightarrow 0$. Thus, we have $\rho((F_n, \phi_n, K_n), (F, \phi, K)) \longrightarrow 0$. Since $\mathrm{GMVI}(F, \phi, K)$ is Hadamard well-posed, we know that $\{x_n\}$ converges strongly to the unique solution x^*

of GMVI(F, ϕ, K). Thus, GMVI(F, ϕ, K) is LP well-posed. The proof is complete.

Next, we have the following result which can be regarded as the reverse of Theorem 1 under the uniform continuity of the function ϕ. □

Theorem 2. *Let K be a nonempty, closed subset of X and $F: X \longrightarrow 2^{X^*}$ be a nonempty set-valued mapping. Let $\phi: X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be uniformly continuous on X. Then, GMVI(F, ϕ, K) is Hadamard well-posed whenever GMVI(F, ϕ, K) is LP well-posed.*

Proof. Suppose that GMVI(F, ϕ, K) is LP well-posed and $x^* \in K$ is the unique solution of GMVI(F, ϕ, K). Let

$$g(x) = \sup_{y \in K} \inf\{\langle F(x), x - y\rangle + \phi(x) - \phi(y)\}, \quad \forall x \in X. \tag{30}$$

Since GMVI(F, ϕ, K) has the unique solution $x^* \in K$, by Lemma 2, we know that (OP) has the unique solution $x^* \in K$. That is, $\inf(K, g) = 0$ and $\arg\min(K, g) = \{x^*\}$. Let $\{(F_n, \phi_n, K_n)\} \in \tau(X) \times B(X) \times C(X)$, (F_n, ϕ_n, K_n) converges to (F, ϕ, K) and $x_n \in S(F_n, \phi_n, K_n)$, where $K \cup \{x_n\} \subseteq K_n$. So, it follows from $x_n \in S(F_n, \phi_n, K_n)$ that there exists $\tilde{u}_n \in F_n(x_n)$ such that

$$\langle \tilde{u}_n, x_n - y\rangle + \phi_n(x_n) - \phi_n(y) \leq 0, \quad \forall y \in K_n, \tag{31}$$

which immediately yields

$$\langle \tilde{u}_n, x_n - y\rangle + \phi_n(x_n) - \phi_n(y) \leq 0, \quad \forall y \in K. \tag{32}$$

For any $x \in X$, let

$$\mathcal{G}_n(x) = \sup_{y \in K_n} \inf\{\langle F_n(x), x - y\rangle + \phi_n(x) - \phi_n(y)\}, \quad \forall x \in X. \tag{33}$$

From Lemma 1, it is easy to see that

(i) $\mathcal{G}_n(x) \geq 0$, $\forall x \in K_n$;

(ii) for any $x \in K_n$, $\mathcal{G}_n(x) = 0 \Longleftrightarrow x \in S(F_n, \phi_n, K_n)$.

It follows from (ii) and $x_n \in S(F_n, \phi_n, K_n)$ that $\mathcal{G}_n(x_n) = \inf(K_n, \mathcal{G}) = 0$. On the other hand, note that $\rho((F_n, \phi_n, K_n), (F, \phi, K)) \longrightarrow 0$. Then, we deduce that $d(F_n, F) \longrightarrow 0$, $d_1(\phi_n, \phi) \longrightarrow 0$ and $\mathcal{H}(K_n, K) \longrightarrow 0$. Since $d(F_n, F) \longrightarrow 0$, it follows from (3) that there exists $0 < \epsilon_n \longrightarrow 0$ such that for any $x \in X$,

$$\langle F_n(x), x - y\rangle = \langle F(x), x - y\rangle - \epsilon_n, \quad \forall y \in K. \tag{34}$$

In particular, we have

$$\langle F_n(x_n), x_n - y\rangle = \langle F(x_n), x_n - y\rangle - \epsilon_n, \quad \forall y \in K. \tag{35}$$

From $\tilde{u}_n \in F_n(x_n)$ it follows that there exists $u_n \in F(x_n)$ such that

$$\langle \tilde{u}_n, x_n - y\rangle = \langle u_n, x_n - y\rangle - \epsilon_n, \quad \forall y \in K. \tag{36}$$

This together with (32) leads to

$$\langle u_n, x_n - y\rangle + \phi_n(x_n) - \phi_n(y) \leq \epsilon_n, \quad \forall y \in K, \tag{37}$$

which can be rewritten as follows:

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \leq \epsilon_n + \phi(x_n) - \phi_n(x_n) - (\phi(y) - \phi_n(y)), \quad \forall y \in K. \tag{38}$$

Next, we claim that

$$\lim_{n \longrightarrow \infty} \left|\phi(x_n) - \phi_n(x_n) - (\phi(y) - \phi_n(y))\right| = 0, \quad \text{uniformly for } y \in X. \tag{39}$$

As a matter of fact, for any $\delta > 0$, since $d_1(\phi_n, \phi) = \sup_{x \in X}|\phi_n(x) - \phi(x)| \longrightarrow 0$, there exists an integer $N \geq 1$ such that for all $n \geq N$,

$$\sup_{x \in X}\left|\phi_n(x) - \phi(x)\right| \leq \delta. \tag{40}$$

It follows that for any $x \in X$,

$$\phi(x) - \delta \leq \phi_n(x) \leq \phi(x) + \delta, \quad n \geq N. \tag{41}$$

So, for any $x, y \in X$, we have

$$\begin{aligned} \phi_n(x) - \phi_n(y) &\leq \phi(x) + \delta - (\phi(y) - \delta) \\ &= \phi(x) - \phi(y) + 2\delta, \quad n \geq N. \end{aligned} \tag{42}$$

Meantime, we also have

$$\begin{aligned} \phi(x) - \phi(y) - 2\delta &= \phi(x) - \delta - (\phi(y) + \delta) \\ &\leq \phi_n(x) - \phi_n(y), \quad n \geq N. \end{aligned} \tag{43}$$

Then, for any $x, y \in X$,

$$\left|\phi_n(x) - \phi_n(y) - (\phi(x) - \phi(y))\right| \leq 2\delta, \quad n \geq N. \tag{44}$$

In particular, for any $y \in X$, we get

$$\left|\phi_n(x_n) - \phi_n(y) - (\phi(x_n) - \phi(y))\right| \leq 2\delta, \quad n \geq N. \tag{45}$$

This means that (39) holds.

Finally, from (38) and (39) and $0 < \epsilon_n \longrightarrow 0$, we have

$$
\begin{aligned}
0 \le g_n(x_n) &= \sup_{y\in K\cup\{x_n\}} \{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\} \\
&\le \max\left\{\sup_{y\in K}\{\epsilon_n + \phi(x_n) - \phi_n(x_n) - (\phi(y) - \phi_n(y))\}, 0\right\} \\
&\le \epsilon_n + \sup_{y\in K}\left|\phi(x_n) - \phi_n(x_n) - (\phi(y) - \phi_n(y))\right| \longrightarrow 0.
\end{aligned}
\tag{46}
$$

Since $K_n \longrightarrow K$ in the Hausdorff metric and $x_n \in K_n$, we have $d'(x_n, K) \longrightarrow 0$. Thus, $\{x_n\}$ is an LP minimizing sequence for (OP). Since $\mathrm{GMVI}(F, \phi, K)$ is LP well-posed, according to Lemma 3, we know that (OP) is LP well-posed. Therefore, $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. So, it follows that $\mathrm{GMVI}(F, \phi, K)$ is Hadamard well-posed. The proof is complete. □

Remark 1. Theorems 1 and 2 improve, extend, and develop Theorems 3.1 and 3.2 in [12] to a great extent because the generalized mixed variational inequality considered in Theorems 1 and 2 is more general than the general mixed variational inequality considered in [[12], Theorems 3.1 and 3.2].

4. Metric Characterization and Conditions for Hadamard Well-Posedness

In this section, we derive the metric characterization of Hadamard well-posedness for a generalized mixed variational inequality and prove that under suitable conditions, the Hadamard well-posedness of a generalized mixed variational inequality is equivalent to the existence and uniqueness of its solutions.

To characterize the Hadamard well-posedness for a generalized mixed variational inequality $\mathrm{GMVI}(F, \phi, K)$, we define

$$
\Omega(\epsilon) = \{x \in X\colon d'(x, K) \le \epsilon, \text{and there exists } u \in F(x) \text{ such that } \forall y \in K, \langle u, x - y\rangle + \phi(x) - \phi(y) \le \epsilon\}, \quad \forall \epsilon \ge 0. \tag{47}
$$

Theorem 3. *Let $(F, \phi, K) \in \tau(X) \times B(X) \times C(X)$, K be convex, $F\colon X \longrightarrow 2^{X^*}$ be a nonempty compact-valued mapping which is $\mathscr{H}$-hemicontinuous and monotone, and $\phi\colon X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be proper, convex, and uniformly continuous on X. Then, $\mathrm{GMVI}(F, \phi, K)$ is Hadamard well-posed if and only if*

$$
\Omega(\epsilon) \ne \emptyset, \quad \forall \epsilon > 0 \text{ and } \operatorname{diam}(\Omega(\epsilon)) \longrightarrow 0 \text{ as } \epsilon \longrightarrow 0. \tag{48}
$$

Proof. Assume that $\mathrm{GMVI}(F, \phi, K)$ is Hadamard well-posed. Then, $\mathrm{GMVI}(F, \phi, K)$ has a unique solution which lies in $\Omega(\epsilon)$ for all $\epsilon > 0$. Put $x_0 \in S(F, \phi, K)$. Obviously, $x_0 \in \Omega(\epsilon)$ for all $\epsilon > 0$. If $\operatorname{diam}(\Omega(\epsilon)) \not\rightarrow 0$ as $\epsilon \longrightarrow 0$, then for some $\delta > 0, 0 < \epsilon_n \longrightarrow 0$ such that for n sufficiently large,

$$
\operatorname{diam}(\Omega(\epsilon_n)) > \delta > 0. \tag{49}
$$

Thus, we can find points $x_n \in \Omega(\epsilon_n)$ such that

$$
\|x_n - x_0\| > \frac{\delta}{2}. \tag{50}
$$

Since $x_n \in \Omega(\epsilon_n)$, we have

$$
d'(x_n, K) \le \epsilon_n, \tag{51}
$$

and there exists $u_n \in F(x_n)$ such that

$$
\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \epsilon_n, \quad \forall y \in K. \tag{52}
$$

Now, we construct a sequence $\{(F_n, \phi_n, K_n)\}$ as follows:

$$
\begin{aligned}
\langle F_n(x), x - y\rangle &= \langle F_n(x), x - y\rangle - \epsilon_n, \quad \forall y \in K, \\
\phi_n(x) &= \phi(x) - \epsilon_n, \\
K_n &= K \bigcup \{x_n\}.
\end{aligned}
\tag{53}
$$

It is obvious that $x_n \in K_n, F_n \tau(X), \phi_n \in B(X)$, and $K_n \in C(X)$. By the similar argument to that in the proof of Theorem 1, we have $x_n \in S(F_n, \phi_n, K_n)$. Observe that $d(F_n, F) = |\epsilon_n| \longrightarrow 0, d_1(\phi_n, \phi) \longrightarrow 0$, and $\mathscr{H}(K_n, K) = d'(x_n, K) \longrightarrow 0$. Thus, we have $\rho((F_n, \phi_n, K_n), (F, \phi, K)) \longrightarrow 0$. Since $\mathrm{GMVI}(F, \phi, K)$ is Hadamard well-posed, one has $x_n \longrightarrow x_0$, a contradiction to (50).

Conversely, suppose that condition (48) holds. Let $(F_n, \phi_n, K_n) \longrightarrow (F, \phi, K)$, and $x_n \in S(F_n, \phi_n, K_n)$, where $K \cup \{x_n\} \subseteq K_n$, $n = 1, 2, \ldots$. So, it follows from $x_n \in S(F_n, \phi_n, K_n)$ that there exists $\tilde{u}_n \in F_n(x_n)$ such that

$$
\langle \tilde{u}_n, x_n - y\rangle + \phi_n(x_n) - \phi_n(y) \le 0, \quad \forall y \in K_n, \tag{54}
$$

which immediately yields

$$
\langle \tilde{u}_n, x_n - y\rangle + \phi_n(x_n) - \phi_n(y) \le 0, \quad \forall y \in K. \tag{55}
$$

Furthermore, note that $\rho((F_n, \phi_n, K_n), (F, \phi, K)) \longrightarrow 0$. Then, we deduce that $d(F_n, F) \longrightarrow 0, d_1(\phi_n, \phi) \longrightarrow 0$ and $\mathscr{H}(K_n, K) \longrightarrow 0$. Since $d(F_n, F) \longrightarrow 0$, it follows from (3) that there exists $0 < \epsilon_n' \longrightarrow 0$ such that for any $x \in X$,

$$
\langle F_n(x), x - y\rangle = \langle F(x), x - y\rangle - \epsilon_n', \quad \forall y \in K. \tag{56}
$$

In particular, we have

$$
\langle F_n(x_n), x_n - y\rangle = \langle F(x_n), x_n - y\rangle - \epsilon_n', \quad \forall y \in K. \tag{57}
$$

From $\tilde{u}_n \in F_n(x_n)$, it follows that there exists $u_n \in F(x_n)$ such that

$$\langle \tilde{u}_n, x_n - y\rangle = \langle u_n, x_n - y\rangle - \epsilon_n', \quad \forall y \in K. \tag{58}$$

This together with (55) leads to

$$\langle u_n, x_n - y\rangle + \phi_n(x_n) - \phi_n(y) \le \epsilon_n', \quad \forall y \in K, \tag{59}$$

which can be rewritten as follows:

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \epsilon_n' + \phi(x_n) - \phi_n(x_n) - (\phi(y) - \phi_n(y)), \quad \forall y \in K. \tag{60}$$

Repeating the same argument as that of (39) in the proof of Theorem 2, we get

$$\lim_{n\longrightarrow\infty} |\phi(x_n) - \phi_n(x_n) - (\phi(y) - \phi_n(y))| = 0, \quad \text{uniformly for } y \in X. \tag{61}$$

Taking into account that $K_n \longrightarrow K$ in the Hausdorff metric and $x_n \in K_n$, we have $d'(x_n, K) \longrightarrow 0$. Thus, there exists $0 < \epsilon_n^* \longrightarrow 0$ such that $d'(x_n, K) \le \epsilon_n^*$ and

$$\sup_{y\in X} |\phi(x_n) - \phi_n(x_n) - (\phi(y) - \phi_n(y))| \le \epsilon_n^*, \quad \forall n \in N. \tag{62}$$

Set $\epsilon_n = \epsilon_n' + \epsilon_n^*$. Then, it follows from (60) that

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \epsilon_n, \quad \forall y \in K. \tag{63}$$

This means that $x_n \in \Omega(\epsilon_n)$ for all $n \in \mathbf{N}$. From (48), we know that $\{x_n\}$ is a Cauchy sequence and so it converges strongly a point $\overline{x} \in K$. Since F is monotone and ϕ is lower semicontinuous, it follows from (63) that for any $y \in K, v \in F(y)$,

$$\begin{aligned} &\langle v, \overline{x} - y\rangle + \phi(\overline{x}) - \phi(y) \\ &\quad \cdot \liminf_{n\longrightarrow\infty} \{\langle v, x_n - y\rangle + \phi(x_n) - \phi(y)\} \\ &\le \liminf_{n\longrightarrow\infty} \{\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y)\} \\ &\le \liminf_{n\longrightarrow\infty} \epsilon_n = 0. \end{aligned} \tag{64}$$

So, from Proposition 1, it is easy to see that $\overline{x}$ solves $\text{GMVI}(F, \phi, K)$.

To complete the proof, we need only to prove that $\text{GMVI}(F, \phi, K)$ has a unique solution. Assume by contradiction that $\text{GMVI}(F, \phi, K)$ has two distinct solutions x_1 and x_2 in K. Then, it is easy to see that $x_1, x_2 \in \Omega(\epsilon)$ for all $\epsilon > 0$ and

$$0 < \|x_1 - x_2\| \le \text{diam}(\Omega(\epsilon)) \longrightarrow 0, \tag{65}$$

a contradiction to (48). The proof is complete.

Next, we prove that the Hadamard well-posedness of a generalized mixed variational inequality is equivalent to the existence and uniqueness of its solutions under suitable conditions. □

Theorem 4. *Let $(F, \phi, K) \in \tau(X) \times B(X) \times C(X)$, K be convex, $F: X \longrightarrow 2^{X^*}$ be a nonempty compact-valued mapping which is $\mathscr{H}$-hemicontinuous and monotone, and $\phi: X \longrightarrow \mathbf{R} \cup \{+\infty\}$ be proper, convex, and uniformly continuous on X. Then, $\text{GMVI}(F, \phi, K)$ is Hadamard well-posed if and only if it has a unique solution.*

Proof. The necessity is obvious. For the sufficiency, suppose that $\text{GMVI}(F, \phi, K)$ has a unique solution x^*. If $\text{GMVI}(F, \phi, K)$ is not Hadamard well-posed, then there exists $\{(F_n, \phi_n, K_n)\} \subset M$ converging to (F, ϕ, K) with $x_n \in S(F_n, \phi_n, K_n)$ such that $\{x_n\}$ do not converge to x^*, where $K \cup \{x_n\} \subseteq K_n$, $n = 1, 2, \ldots$. So, it follows from $x_n \in S(F_n, \phi_n, K_n)$ that there exists $\tilde{u}_n \in F_n(x_n)$ such that

$$\langle \tilde{u}_n, x_n - y\rangle + \phi_n(x_n) - \phi_n(y) \le 0, \quad \forall y \in K_n, \tag{66}$$

which immediately yields

$$\langle \tilde{u}_n, x_n - y\rangle + \phi_n(x_n) - \phi_n(y) \le 0, \quad \forall y \in K. \tag{67}$$

Furthermore, note that $\rho((F_n, \phi_n, K_n), (F, \phi, K)) \longrightarrow 0$. Then, we deduce that $d(F_n, F) \longrightarrow 0, d_1(\phi_n, \phi) \longrightarrow 0$ and $\mathscr{H}(K_n, K) \longrightarrow 0$. Since $d(F_n, F) \longrightarrow 0$, it follows from (3) that there exists $0 < \epsilon_n' \longrightarrow 0$ such that for any $x \in X$,

$$\langle F_n(x), x - y\rangle = \langle F(x), x - y\rangle - \epsilon_n', \quad \forall y \in K. \tag{68}$$

In particular, we have

$$\langle F_n(x_n), x_n - y\rangle = \langle F(x_n), x_n - y\rangle - \epsilon_n', \quad \forall y \in K. \tag{69}$$

From $\tilde{u}_n \in F_n(x_n)$, it follows that there exists $u_n \in F(x_n)$ such that

$$\langle \tilde{u}_n, x_n - y\rangle = \langle u_n, x_n - y\rangle - \epsilon_n', \quad \forall y \in K. \tag{70}$$

This together with (67) leads to

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \epsilon_n' + \phi(x_n) - \phi_n(x_n) - (\phi(y) - \phi_n(y)), \quad \forall y \in K. \tag{71}$$

Repeating the same argument as that of (63) in the proof of Theorem 3, we obtain that there exists $0 < \epsilon_n \longrightarrow 0$ such that $d'(x_n, K) \le \epsilon_n$ and

$$\langle u_n, x_n - y\rangle + \phi(x_n) - \phi(y) \le \epsilon_n, \quad \forall y \in K. \tag{72}$$

From $d'(x_n, K) \le \epsilon_n < \epsilon_n + (1/n)$, it follows that there exists $\overline{x}_n \in K$ such that $\|\overline{x}_n - x_n\| < \epsilon_n + (1/n) \longrightarrow 0$. Putting $w_n = \overline{x}_n - x_n$, we get $\overline{x}_n = w_n + x_n$ with $w_n \longrightarrow 0$.

We claim that $\{x_n\}$ is bounded. As a matter of fact, if $\{x_n\}$ is unbounded, then $\{\overline{x}_n\}$ is an unbounded sequence in K. Without loss of generality, we may assume that $\|\overline{x}_n\| \longrightarrow +\infty$. Let

$$t_n = \frac{1}{\|\overline{x}_n - x^*\|}, \quad z_n = x^* + t_n(\overline{x}_n - x^*). \tag{73}$$

Without loss of generality, we may assume that $t_n \in (0,1]$ and $z_n \rightharpoonup z (\ne x^*)$. Then, we have for each $y \in K, v \in F(y)$,

$$\begin{aligned}\langle v, z - y\rangle &= \langle v, z - z_n\rangle + \langle v, z_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, \overline{x}_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, x_n + w_n - x^*\rangle + \langle v, x^* - y\rangle \\ &= \langle v, z - z_n\rangle + t_n\langle v, x_n - y\rangle + (1 - t_n)\langle v, x^* - y\rangle \\ &\quad + t_n\langle v, w_n\rangle.\end{aligned} \tag{74}$$

Since x^* is the unique solution of $\mathrm{GMVI}(F, \phi, K)$, there exists $u^* \in F(x^*)$ such that

$$\langle u^*, x^* - y\rangle + \phi(x^*) - \phi(y) \le 0, \quad \forall y \in K. \tag{75}$$

Since F is monotone, we have

$$\begin{aligned}\langle v, x^* - y\rangle &\le \langle u^*, x^* - y\rangle, \\ \langle v, x_n - y\rangle &\le \langle u_n, x_n - y\rangle.\end{aligned} \tag{76}$$

It follows from (72)–(76) and the convexity of ϕ that for all $v \in F(y)$,

$$\begin{aligned}&\langle v, z - y\rangle \\ &\le \langle v, z - z_n\rangle + t_n\phi(y) - t_n\phi(x_n) + t_n\epsilon_n + (1 - t_n)(\phi(y) - \phi(x^*)) + t_n\langle v, w_n\rangle \\ &= \langle v, z - z_n\rangle + \phi(y) - [t_n\phi(x_n) + (1 - t_n)\phi(x^*)] + t_n\epsilon_n + t_n\langle v, w_n\rangle \\ &= \langle v, z - z_n\rangle + \phi(y) - [t_n\phi(\overline{x}_n) + (1 - t_n)\phi(x^*) + t_n\phi(x_n) - t_n\phi(\overline{x}_n)] + t_n\epsilon_n + t_n\langle v, w_n\rangle \\ &\le \langle v, z - z_n\rangle + \phi(y) - \phi(z_n) - t_n[\phi(x_n) - \phi(\overline{x}_n)] + t_n\epsilon_n + t_n\langle v, w_n\rangle, \quad \forall y \in K.\end{aligned} \tag{77}$$

Since ϕ is uniformly continuous, we have

$$\begin{aligned}&\langle v, z - y\rangle \\ &\le \liminf_{n\longrightarrow\infty}\{\langle v, z - y\rangle + \phi(y) - \phi(z_n) - t_n[\phi(x_n) - \phi(\overline{x}_n)] \\ &\quad + t_n\epsilon_n + t_n\langle v, w_n\rangle\} \\ &\le \phi(y) - \phi(z), \quad \forall y \in K.\end{aligned} \tag{78}$$

This together with Proposition 1 implies that z solves $\mathrm{GMVI}(F, \phi, K)$, a contradiction. Thus, $\{x_n\}$ is bounded.

Next, we claim that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \longrightarrow \overline{x}$ as $k \longrightarrow \infty$. Clearly, $\overline{x} \in K$. It follows from (72) that

$$\langle u_{n_k}, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y) \le \epsilon_{n_k}, \quad \forall y \in K. \tag{79}$$

Since F is monotone and ϕ is uniformly continuous, we have

$$\begin{aligned}&\langle v, \overline{x} - y\rangle + \phi(\overline{x}) - \phi(y) \\ &= \liminf_{k\longrightarrow\infty}\{\langle v, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\} \\ &\le \liminf_{k\longrightarrow\infty}\{\langle u_{n_k}, x_{n_k} - y\rangle + \phi(x_{n_k}) - \phi(y)\} \\ &\le \liminf_{k\longrightarrow\infty}\epsilon_{n_k} = 0, \quad \forall y \in K,\ v \in F(y).\end{aligned} \tag{80}$$

This together with Proposition 1 implies that $\overline{x}$ solves $\mathrm{GMVI}(F, \phi, K)$. Since $\mathrm{GMVI}(F, \phi, K)$ has a unique solution x^*, we have $\overline{x} = x^*$. Thus, $x_n \longrightarrow x^*$, which reaches a contradiction. So, $\mathrm{GMVI}(F, \phi, K)$ is Hadamard well-posed. The proof is complete. □

5. Concluding Remarks

Theorems 3 and 4 improve, extend, and develop Theorems 4.1 and 4.2 in [12] to a great extent because the generalized mixed variational inequality considered in Theorems 3 and 4 is more general than the general mixed variational inequality considered in ([12], Theorems 4.1 and 4.2). In addition, Theorems 3 and 4 also improve, extend, and develop Theorems 3.1 and 6.1 in [7] and Theorems 3.1 and 6.1 in [20] to a great extent because Levitin–Polyak well-posedness of a generalized mixed variational inequality is replaced by Hadamard well-posedness of a generalized mixed variational inequality.

Data Availability

All data generated or analyzed during this study are included in this published article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Research Article

Generalized Interpolative Contractions and an Application

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In this study, we introduce a property (P) and the generalized interpolative contractions of types I, II, III, and IV. We investigate certain conditions for the existence of fixed points of generalized interpolative contractions. We derive several new results from the main theorems. As an application, we resolve the Urysohn integral equation.

1. Introduction

Fixed-point theory is an outstanding example of a central principle with multiple implementations. In diverse areas, such as differential equations and artificial intelligence, it has always been a significant theoretical method. Furthermore, the development of accurate and efficient techniques for computing fixed points has significantly increased the concept's utility for applications, making fixed-point methods a major tool in the arsenal of the applied mathematician. The key element in the metric fixed-point theory is the Banach contraction principle (BCP). It states that *every contraction, in the complete metric space, admits a unique fixed point*. This principle has been generalized by many ways (see [1]). Recently, Gordji et al. [2] presented a new generalization of the BCP by defining the notion of orthogonal sets and hence orthogonal metric spaces. They presented an example supporting the fact that their main theorem is a real generalization of the BCP. Baghani et al. [3] extended the work of [2] to F-contractions. Chandok et al. [4] extended the results given in [3] to multivalued F-contractions.

On the contrary, Karapinar [5] introduced interpolative contractions and presented a method to obtain fixed points of such contractions. Karapinar et al. [6–9], in subsequent papers, investigated Rus–Reich–Ćirić-type interpolative contractions, Hardy–Rogers-type interpolative contractions, Rus–Reich–Ćirić-type ω-interpolative contractions, and Boyd–Wong- and Matkowski-type interpolative contractions to ensure the existence of fixed points in variant (generalized) metric spaces. Gautam et al. [10] presented some fixed-point results for Chatterjea and cyclic Chatterjea interpolative contractions in complete quasi-partial b-metric spaces. Debnath et al. [11] proved some fixed-point theorems for Rus–Reich–Ćirić- and Hardy–Rogers-type interpolative contractions in b-metric spaces.

Boyd–Wong [12] generalized the well-known Banach contraction principle (BCP) [13] by introducing a control function $\Psi: [0,\infty) \longrightarrow [0,\infty)$, verifying the below conditions for each $\mathcal{J} > 0$:

(1) $\Psi(\mathcal{J}) < \mathcal{J}$
(2) $\lim_{\ell \longrightarrow \mathcal{J}+} \Psi(\ell) < \mathcal{J}$

The related result of Boyd–Wong [12] is as follows.

Theorem 1. *Let S: $X \longrightarrow X$ be a self-mapping on a complete metric space (X, d) so that*

$$d(S\ell, S\mathcal{J}) \leq \Psi(d(\ell, \mathcal{J})), \quad \text{for all } \ell, \mathcal{J} \in X, \tag{1}$$

where $\Psi: [0,\infty) \longrightarrow [0,\infty)$ verifies (1)-(2). Then, S has a unique fixed point in X (say, ρ) and the sequence $(S^n\ell)$ is convergent to ρ, for each $\ell \in X$.

It is noted that Theorem 1 is an improvement of main results of Rakotch [14] and Browder [15]. The Boyd–Wong idea has been generalized by Matkowski [16], Samet et al. [17], Karapinar et al. [18], Pasicki [19], and Proinov [20], respectively. Recently, Nazam et al. [21] introduced several conditions on the newly introduced functions $\Psi, \phi\colon (0,\infty) \longrightarrow \mathbb{R}$ to generalize and improve the results in [12, 16–20].

The Banach contraction principle (BCP) and its generalization (GBCP) have been extensively applied to show the existence of solutions to various mathematical models. For instant, in [22–27], authors have applied GBCP to show the existence of solution to a matrix equation:

$$X = \mathbb{D} + \left(\sum_{i=1}^{m} \mathbb{W}_i^* X \mathbb{W}_i + \sum_{i=1}^{m} \mathbb{G}_i^* X \mathbb{G}_i \right), \tag{2}$$

where $\mathbb{D} \in \mathrm{P}^{(m)}$ (set of $m \times m$ positive definite matrices) and $\mathbb{W}_i$ and $\mathbb{G}_i$ are arbitrary $m \times m$ matrices for each i and are entries of block matrices given by

$$\begin{aligned} \mathbb{W} &= \begin{bmatrix} \mathbb{W}_1 \\ \mathbb{W}_2 \\ \mathbb{W}_3 \\ \vdots \\ \mathbb{W}_m \end{bmatrix}, \\ \mathbb{G} &= \begin{bmatrix} \mathbb{G}_1 \\ \mathbb{G}_2 \\ \mathbb{G}_3 \\ \vdots \\ \mathbb{G}_m \end{bmatrix}. \end{aligned} \tag{3}$$

Consider the system of fractional differential equations:

$$\begin{aligned} {}^{C}D^{\beta} f(\nu) &= K_1(\nu, f(\nu)), \\ {}^{C}D^{\beta} g(\nu) &= K_2(\nu, g(\nu)), \end{aligned} \tag{4}$$

under boundary conditions,

$$\begin{aligned} f(0) &= 0, \\ If(1) &= f'(0), \\ g(0) &= 0, \\ Ig(1) &= g'(0), \end{aligned} \tag{5}$$

where ${}^{C}D^{\beta}$ denotes CFD of order β defined by

$$\begin{aligned} {}^{C}D^{\beta} f(\nu) &= \frac{1}{\Gamma(n-\beta)} \int_0^{\nu} (\nu-\eta)^{n-\beta-1} f^n(\eta) \mathrm{d}\eta, \\ {}^{C}D^{\beta} g(\nu) &= \frac{1}{\Gamma(n-\beta)} \int_0^{\nu} (\nu-\eta)^{n-\beta-1} g^n(\eta) \mathrm{d}\eta. \end{aligned} \tag{6}$$

The existence of solutions of the above system has been shown in [21] by using GBCP. In [28], authors have employed the GBCP for the existence of solutions to a system of integral equations:

$$\begin{aligned} u(t) &= f(t) + \int_a^t K(t, x, S(u)(t)) \mathrm{d}x, \\ w(t) &= f(t) + \int_a^t J(t, x, T(w)(t)) \mathrm{d}x, \end{aligned} \tag{7}$$

for all $u, w \in C([a,b])$, $x, t \in [a,b]$, and $a > 0$, where $f\colon M \longrightarrow \mathbb{R}$ is a continuous function, $K, J\colon [a,b] \times [a,b] \times M \longrightarrow \mathbb{R}$ are lower semicontinuous operators, and $S, T\colon C([a,b]) \longrightarrow C([a,b])$.

In this paper, motivated by the interpolation notion of contractions and the applications of GBCP, we investigate different conditions on the functions Ψ, Φ to show the existence of fixed points of generalized interpolative contractions (a new GBCP) of type I, II, III, and IV and hence, we apply GBCP of type I to resolve the Urysohn integral equation.

2. Preliminaries

Before stating our main results, we need to define some basic notions for better understanding of readers.

Definition 1 (see [2]). Let $\perp$ be a binary relation defined on a nonempty set $\mathscr{A}$ (i.e., $\perp \subset \mathscr{A} \times \mathscr{A}$) verifying the property (O). Then, $(\mathscr{A}, \perp)$ is called an orthogonal set (in short, O-set):

$$\text{(O): there is } a \in \mathscr{A} \text{ such that either } a \perp \mathscr{J} \text{ or } \mathscr{J} \perp a, \quad \forall \mathscr{J} \in \mathscr{A}. \tag{8}$$

Example 1. Let $\mathscr{A}$ be the set of integers. Consider $a \perp \theta$ if and only if $a \equiv 1 \pmod{\theta}$. Then, $(\mathscr{A}, \perp)$ is an O-set. Indeed, $1 \perp \theta$ for each θ.

Definition 2 (see [2]). A sequence $\{\hbar_n\colon n \text{ is a positive integer}\}$ is said to be an O-sequence if either $\hbar_n \perp \hbar_{n+1}$ or $\hbar_{n+1} \perp \hbar_n$, for all n.

Definition 3 (see [2]). The O-set $(\mathscr{A}, \perp)$ endowed with a metric d is called an O-metric space (in short, OMS) denoted by $(\mathscr{A}, \perp, d)$.

Definition 4 (see [2]). The O-sequence $\{\hbar_n\} \subset \mathscr{A}$ is said to be O-Cauchy if $\lim_{n,m\longrightarrow\infty} d(\hbar_n, \hbar_m) = 0$. If each O-Cauchy sequence converges in $\mathscr{A}$, then $\mathscr{A}$ is called O-complete.

Remark 1. Each complete metric space is O-complete, but the converse is not true in general (see [2], for details).

Lemma 1. *Let $(X, \perp, d)$ be an OMS and $\{\imath_n\} \subset X$ be an O-sequence, verifying $\lim_{n\longrightarrow\infty} d(\imath_n, \imath_{n+1}) = 0$. If the sequence $\{\imath_n\}$ is not Cauchy, then there are $\{\imath_{n_k}\}$, $\{\imath_{m_k}\}$, and $\xi > 0$ such that*

$$\lim_{k\longrightarrow\infty} \mathrm{d}\left(l_{n_k+1}, l_{m_k+1}\right) = \xi+, \tag{9}$$

$$\lim_{k\longrightarrow\infty} \mathrm{d}\left(l_{n_k}, l_{m_k}\right) = \mathrm{d}\left(l_{n_k+1}, l_{m_k}\right) = \mathrm{d}\left(l_{n_k}, l_{m_k+1}\right) = \xi. \tag{10}$$

The proof of this lemma has the same arguments that are given in [20]. We omit details.

Definition 5. Let $T: \mathscr{A} \longrightarrow \mathscr{A}$ be a self-mapping. An element $v \in \mathscr{A}$ is said to be a fixed point of T if $v = Tv$.

Definition 6 (see [3]). Let $(\mathscr{A}, \perp, d)$ be an OMS and $\perp \subset \mathscr{A} \times \mathscr{A}$ be a binary relation. $\mathscr{A}$ is called $\perp$-regular if, for each sequence $\{\imath_n\} \subset \mathscr{A}$ so that $\imath_n \perp \imath_{n+1}$ for each $n \geq 0$ and $\imath_n \longrightarrow \imath$ as $n \longrightarrow \infty$, we have either $\imath_n \perp \imath$, or $\imath \perp \imath_n$, for all $n \geq 0$.

Definition 7 (see [2]). A mapping $T: \mathscr{A} \longrightarrow \mathscr{A}$ is said to be asymptotically regular at a point v of X if

$$\lim_{\imath \longrightarrow \infty} \mathrm{d}\left(T^{\imath} v, T^{\imath+1} v\right) = 0. \tag{11}$$

If T is asymptotically regular at each point in $\mathscr{A}$, then it is named as an asymptotically regular mapping.

3. $(\Psi, \Phi)_{\perp}$-Interpolative Contractions and Related Fixed-Point Results

In this section, we initiate the notion of $(\Psi, \Phi)_{\perp}$-interpolative contractions. We consider various conditions on control functions Ψ, Φ to ensure the existence of fixed points of $(\Psi, \Phi)_{\perp}$-interpolative contractions. In the following, we develop the strategy towards main results.

Let $\Lambda = \{(a, v) \in \mathscr{A} \times \mathscr{A}: a \perp v\}$.

Definition 8. A mapping $f: \mathscr{A} \times \mathscr{A} \longrightarrow [1, \infty)$ is said to be strictly $\perp$-admissible if $f(a, \theta) > 1$, for all $a, \theta \in \mathscr{A}$, with $a \perp \theta$ and $f(a, \theta) = 1$ otherwise.

Example 2. Let $\mathscr{A} = [0, 1)$, and we define the relation $\perp \subset \mathscr{A} \times \mathscr{A}$ by

$$a \perp \theta \text{ if } a\theta \in \{a, \theta\} \subset \mathscr{A}. \tag{12}$$

Then, $\mathscr{A}$ is O-set. Define $f: \mathscr{A} \times \mathscr{A} \longrightarrow [1, \infty)$ by

$$f(a, \theta) = \begin{cases} a + \dfrac{2}{1+\theta}, & \text{if } a \perp \theta, \\ 1, & \text{otherwise.} \end{cases} \tag{13}$$

Then, f is $\perp$-admissible.

Definition 9. Let $T: \mathscr{A} \longrightarrow \mathscr{A}$ and $\perp \subset \mathscr{A} \times \mathscr{A}$ be a binary relation. Such T is called $\perp$-preserving if, for each $q \in \mathscr{A}$ and $p = T(q)$ such that $q \perp p$ or $p \perp q$, there is $\omega = T(p)$ such that $p \perp \omega$ or $\omega \perp p$.

Example 3. Let $\mathscr{A} = [0, 1)$, and we define the relation $\perp \subset \mathscr{A} \times \mathscr{A}$ by

$$a \perp \theta \text{ if } a\theta \in \{a, \theta\} \subset \mathscr{A}. \tag{14}$$

Then, $\mathscr{A}$ is an O-set. We define $S: \mathscr{A} \longrightarrow \mathscr{A}$ by

$$S(a) = \begin{cases} \dfrac{a+1}{7}, & \text{if } a \in \mathbb{Q} \cap \mathscr{A}, \\ 0, & \text{if } a \in \mathbb{Q}^c \cap \mathscr{A}. \end{cases} \tag{15}$$

Then, S is $\perp$-preserving. Indeed, for $a = 0$, there is $\theta = S(0) = 1/7$ such that either $a \perp \theta$ or $\theta \perp a$, and then, there is $\ell = S(\theta)$ such that either $\ell \perp \theta$ or $\theta \perp \ell$.

Let $(\mathscr{A}, d)$ be a metric space. For a mapping $S: \mathscr{A} \longrightarrow \mathscr{A}$ and positive real numbers a, b, c, we define the mappings $\check{F}_1, \check{F}_2, \check{F}_3, \check{F}_4: \mathscr{A} \times \mathscr{A} \longrightarrow [0, \infty)$ by

$$\begin{aligned}
&\check{F}_1(\ell, \mathscr{J}) = d(\ell, \mathscr{J}) [d(\ell, S\ell)]^{1/(a-b)(a-c)} [d(\mathscr{J}, S\mathscr{J})]^{1/(a-b)(a-c)} \\
&[d(\ell, S\ell) + d(\mathscr{J}, S\mathscr{J})]^{1/(b-a)(b-c)} [d(\ell, S\mathscr{J}) + d(\mathscr{J}, S\ell)]^{1/(c-a)(c-b)}, \\
&\check{F}_2(\ell, \mathscr{J}) = d(\ell, \mathscr{J}) [d(\ell, S\ell)]^{a/(a-b)(a-c)} [d(\mathscr{J}, S\mathscr{J})]^{a/(a-b)(a-c)} \\
&[d(\ell, S\ell) + d(\mathscr{J}, S\mathscr{J})]^{b/(b-a)(b-c)} [d(\ell, S\mathscr{J}) + d(\mathscr{J}, S\ell)]^{c/(c-a)(c-b)}, \\
&\check{F}_3(\ell, \mathscr{J}) = \max \left\{ \begin{array}{c} d(\ell, \mathscr{J}), [d(\ell, S\ell)]^{a^2/(a-b)(a-c)} [d(\mathscr{J}, S\mathscr{J})]^{a^2/(a-b)(a-c)} \\ [d(\ell, S\ell) + d(\mathscr{J}, S\mathscr{J})]^{b^2/(b-a)(b-c)} \\ [d(\ell, S\mathscr{J}) + d(\mathscr{J}, S\ell)]^{c^2/(c-a)(c-b)} \end{array} \right\}, \\
&\check{F}_4(\ell, \mathscr{J}) = d(\ell, \mathscr{J})^{a^3/(a-b)(a-c)} d(\mathscr{J}, S\mathscr{J})^{a^3/(a-b)(a-c)} \\
&[d(\ell, S\ell) + d(\mathscr{J}, S\mathscr{J})]^{b^3/(b-a)(b-c)} [d(\ell, S\mathscr{J}) + d(\mathscr{J}, S\ell)]^{c^3/(c-a)(c-b)}.
\end{aligned} \tag{16}$$

It is important to note that, despite $a, b, c > 0$, some exponents are negative; for example, if $a > b$, $a > c$, and $b > c$, then $1/(b-a)(b-c) < 0$. If any one of a, b, c goes to ∞, then $\check{F}_1(\ell, \mathscr{J}) = d(\ell, \mathscr{J})$. Moreover, we have the following interesting facts about the exponents that can be proved by using basic algebraic tools:

$$\begin{aligned}
&\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-a)(c-b)}=0,\\
&\frac{a}{(a-b)(a-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}=0,\\
&\frac{a^2}{(a-b)(a-c)}+\frac{b^2}{(b-a)(b-c)}+\frac{c^2}{(c-a)(c-b)}=1,\\
&\frac{a^3}{(a-b)(a-c)}+\frac{b^3}{(b-a)(b-c)}+\frac{c^3}{(c-a)(c-b)}=a+b+c.
\end{aligned} \tag{17}$$

The following observations are essential for the proofs of main theorems.

Observation 1. The following inequality holds for all $a, b \geq 2$ and $r \geq 1$:

$$(a+b)^r \leq (ab)^r. \tag{18}$$

Proof. We note that the equality holds for $a = b = 2$. We can assume that $a \geq b$; then, $a = \eta b$, $\eta \geq 1$. Let $b = t$ so that $a = \eta t$, $t \geq 2$. Define the function $f\colon [2,\infty) \longrightarrow (-\infty,\infty)$ by

$$f(t) = \left(\eta t^2\right)^r - (\eta t + t)^r, \quad \forall t \in [2,\infty). \tag{19}$$

This implies that

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(f(t)) = \frac{rt^{r-1}}{(\eta+1)^r}\left[2t^r\left(\frac{\eta}{\eta+1}\right)^r - 1\right]. \tag{20}$$

Since $2t^r(\eta/\eta+1)^r > 1$ (otherwise $t < 1$), we have $f'(t) > 0$. This implies that $f(t) \geq 0$; hence, $(\eta t^2)^r - (\eta t + t)^r \geq 0$, that is, $(a+b)^r \leq (ab)^r$. □

Observation 2. Let $K \geq 2$. For any nonempty set $\mathscr{A}$, we define the mapping $d\colon \mathscr{A} \times \mathscr{A} \longrightarrow [0,\infty)$ by

$$d(u,v) = \begin{cases} K, & \text{if } u \neq v, \\ 0, & \text{if } u = v. \end{cases} \tag{21}$$

Then, the pair $(\mathscr{A}, d)$ is a metric space.

Definition 10. Let $(\mathscr{A}, d)$ be a metric space. A mapping $T\colon \mathscr{A} \longrightarrow \mathscr{A}$ is said to have property P if, for any real number r, it satisfies the following inequality:

$$\begin{aligned}
&\left(d(\ell, T(\ell)) + d\left(T(\ell), T^2(\ell)\right)\right)^r \\
&\quad \leq \left(d(\ell, T(\ell)) d\left(T(\ell), T^2(\ell)\right)\right)^r, \quad \forall \ell \in \mathscr{A}.
\end{aligned} \tag{22}$$

Example 4. Let $\mathscr{A} = [1,\infty)$ and consider the metric d defined by $d(u,v) = |u - v|$ for all $u, v \in \mathscr{A}$. The mapping $T\colon \mathscr{A} \longrightarrow \mathscr{A}$ defined by $T(\ell) = K\ell$, for all $\ell \in \mathscr{A}$ and $K \geq (5/2)$, satisfies the property P. Indeed,

$$\begin{aligned}
&\left(d(\ell, T(\ell)) + d\left(T(\ell), T^2(\ell)\right)\right)^r \\
&\quad = [(K-1)|\ell| + (K-1)|K\ell|]^r \\
&\quad \leq [(K-1)(K+1)|\ell|]^r \leq \left[(K-1)^2 K|\ell|^2\right]^r \\
&\quad = \left(d(\ell, T(\ell)) d\left(T(\ell), T^2(\ell)\right)\right)^r.
\end{aligned} \tag{23}$$

Example 5. Every identity mapping satisfies the property P. The constant mapping does not satisfy the property P. The mapping $T\colon \mathscr{A} \longrightarrow \mathscr{A}$ is defined by $T(\ell) = 0$ for all $\ell \in \mathscr{A}$ which satisfies the property P only for $\ell = 0$.

Example 6. Let $\mathscr{A} = (-\infty,\infty)$. The mapping $T\colon \mathscr{A} \longrightarrow \mathscr{A}$ defined by $T(\ell) = 2 - 3\ell$ for all $\ell \in \mathscr{A}$ satisfies the property P. In fact, the mapping $T\colon \mathscr{A} \longrightarrow \mathscr{A}$ defined by $T(\ell) = a - b\ell$, for all $\ell \in \mathscr{A}$, for $b > a$, satisfies the property P.

Example 7. Let $\mathscr{A} = [2.5,\infty)$. The mapping $T\colon \mathscr{A} \longrightarrow \mathscr{A}$ defined by $T(\ell) = 2 - 3\ell$ for all $\ell \in \mathscr{A}$ satisfies the property P.

Example 8. Let $\mathscr{A} = [1,\infty)$. The mapping $T\colon \mathscr{A} \longrightarrow \mathscr{A}$ defined by $T(\ell) = 1/\ell^2$ for all $\ell \in \mathscr{A}$ satisfies the property P.

Remark 2. The proof of Theorem 2 depends largely on the use of either "Observations 1 and 2" or "Property P."

We proceed with the property P.

Definition 11. Let $(\mathscr{A}, \perp, d)$ be an OMS. A mapping $S\colon \mathscr{A} \longrightarrow \mathscr{A}$ is said to be a $(\Psi,\Phi)_\perp$-interpolative fractional contraction of types I, II, III, and IV, for $i = 1,2,3,4$, respectively, if there exist a strictly $\perp$-admissible mapping f and $a, b, c \in (0,\infty]$, for $i = 1$, and $a, b, c \in (0,\infty)$, for $i = 2,3,4$, such that

$$\Psi(f(\ell,\mathscr{J}) d(S\ell, S\mathscr{J})) \leq \Phi\left(\check{F}_i(\ell,\mathscr{J})\right), \tag{24}$$

for all $\ell, \mathscr{J} \in \Lambda$ and $d(S\ell, S\mathscr{J}) > 0$.

If either $a = \infty$ or $b = \infty$ or $c = \infty$ in $(\Psi,\Phi)_\perp$-interpolative fractional contraction of type I, we receive the recently announced (ψ,ϕ)-contraction by Proinov [20] which provided $\ell, \mathscr{J} \notin \Lambda$.

We also note that, for $\Phi(\ell) = \Psi(\ell) - \tau$ and $\Psi(\ell) = \ln(\ell)$, for all $\ell \in (0,\infty)$, $\tau > 0$, contraction (24) ($i = 1$) can be written as follows:

$$\begin{aligned}
\tau + \ln(f(\ell,\mathscr{J}) d(S\ell, S\mathscr{J})) &\leq \ln(d(\ell,\mathscr{J})) + \frac{1}{(a-b)(a-c)}\ln(d(\ell, S\ell)) \\
&+ \frac{1}{(a-b)(a-c)}\ln(d(\mathscr{J}, S\mathscr{J})) + \frac{1}{(a-b)(a-c)}\ln[d(\ell, S\ell) + d(\mathscr{J}, S\mathscr{J})] \\
&+ \frac{1}{(a-b)(a-c)}\ln[d(\ell, S\mathscr{J}) + d(\mathscr{J}, S\ell)],
\end{aligned} \tag{25}$$

and then, we have

$$\tau + \Psi(f(\ell, \mathcal{J}) d(S\ell, S\mathcal{J})) \leq \Psi(d(\ell, \mathcal{J})) + \frac{1}{(a-b)(a-c)}\Psi(d(\ell, S\ell)) + \frac{1}{(a-b)(a-c)}\Psi(d(\mathcal{J}, S\mathcal{J})) + \frac{1}{(a-b)(a-c)}\Psi[d(\ell, S\ell) + d(\mathcal{J}, S\mathcal{J})] + \frac{1}{(a-b)(a-c)}\Psi[d(\ell, S\mathcal{J}) + d(\mathcal{J}, S\ell)]. \tag{26}$$

This represents a general version of the contraction introduced by Wardowski [29], and if either $a = \infty$ or $b = \infty$ or $c = \infty$ and $\ell, \mathcal{J} \notin \Lambda$, then type I represents an F-contraction [29].

Remark 3. It is very important to note that the set of self-mappings satisfying property P and contraction (24) is not empty. For example, the mappings $S(\ell) = 2 - 3\ell$, for all $\ell \in (\infty, \infty)$, and $S(\ell) = 2\ell - 1$, for all $\ell \in [2.5, \infty)$, satisfy both the property P and contraction (24) with $\Phi(\ell) = \Psi(\ell) - \tau$ and $\Psi(\ell) = \ln(\ell)$, for all $\ell \in (0, \infty)$, where $\tau > 0$.

In the next result, we give a set of conditions that guarantee the existence of a fixed point of a self-mapping S.

Theorem 2. *Let $(\mathcal{A}, \perp, d)$ be an $\perp$-regular O-complete metric space (in short, OCMS). Let $S: \mathcal{A} \longrightarrow \mathcal{A}$ be an $\perp$-preserving mapping verifying (24) for $i = 1$ and property P. Suppose the relation $\perp$ is transitive and the functions $\Psi, \Phi: (0, \infty) \longrightarrow (-\infty, \infty)$ are so that*

(i) For each $\hbar_0 \in \mathcal{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_1 \perp \hbar_0$ or $\hbar_0 \perp \hbar_1$

(ii) Ψ, Φ are nondecreasing and $\Phi(\mathcal{J}) < \Psi(\mathcal{J})$, for all $\mathcal{J} > 0$

(iii) $\lim sup_{\mathcal{J} \longrightarrow \delta+} \Phi(\mathcal{J}) < \Psi(\delta+)$, for all $\delta > 0$

(iv) $\lim sup_{a \longrightarrow 0} \Phi(a) \leq \liminf_{a \longrightarrow \xi+} \Psi(a)$

Then, S admits a fixed point in $\mathcal{A}$.

Proof. Step 1: simplification of $\check{F}_1(\hbar_{n-1}, \hbar_n)$:

$$\begin{aligned}
\check{F}_1(\hbar_{n-1}, \hbar_n) &= d(\hbar_{n-1}, \hbar_n) d(\hbar_{n-1}, S\hbar_{n-1})^{1/(a-b)(a-c)} d(\hbar_n, S\hbar_n)^{1/(a-b)(a-c)} \\
&\quad [d(\hbar_{n-1}, S\hbar_{n-1}) + d(\hbar_n, S\hbar_n)]^{1/(b-a)(b-c)} \\
&\quad [d(\hbar_{n-1}, S\hbar_n) + d(\hbar_n, S\hbar_{n-1})]^{1/(c-a)(c-b)} \\
&\leq d(\hbar_{n-1}, \hbar_n) d(\hbar_{n-1}, \hbar_n)^{1/(a-b)(a-c)} d(\hbar_n, \hbar_{n+1})^{1/(a-b)(a-c)} \\
&\quad [d(\hbar_{n-1}, \hbar_n) + d(\hbar_n, \hbar_{n+1})]^{1/(b-a)(b-c)} \\
&\quad [d(\hbar_{n-1}, \hbar_{n+1}) + d(\hbar_n, \hbar_n)]^{1/(c-a)(c-b)} \\
&\leq d(\hbar_{n-1}, \hbar_n) d(\hbar_{n-1}, \hbar_n)^{1/(a-b)(a-c)} d(\hbar_n, \hbar_{n+1})^{1/(a-b)(a-c)} \\
&\quad [d(\hbar_{n-1}, \hbar_n) + d(\hbar_n, \hbar_{n+1})]^{1/(b-a)(b-c)} [d(\hbar_{n-1}, \hbar_n) + d(\hbar_n, \hbar_{n+1})]^{1/(c-a)(c-b)} \\
&= d(\hbar_{n-1}, \hbar_n) d(\hbar_{n-1}, \hbar_n)^{1/(a-b)(a-c)} d(\hbar_n, \hbar_{n+1})^{1/(a-b)(a-c)} \\
&\quad [d(\hbar_{n-1}, \hbar_n) + d(\hbar_n, \hbar_{n+1})]^{1/(b-a)(b-c) + 1/(c-a)(c-b)} \\
&\leq d(\hbar_{n-1}, \hbar_n) d(\hbar_{n-1}, \hbar_n)^{1/(a-b)(a-c)} d(\hbar_n, \hbar_{n+1})^{1/(a-b)(a-c)} \\
&\quad [d(\hbar_{n-1}, \hbar_n) d(\hbar_n, \hbar_{n+1})]^{1/(b-a)(b-c) + 1/(c-a)(c-b)} \text{ by property } P \\
&= d(\hbar_{n-1}, \hbar_n)^{1 + 1/(a-b)(a-c) + 1/(b-a)(b-c) + 1/(c-a)(c-b)} \\
&\quad d(\hbar_n, \hbar_{n+1})^{1/(a-b)(a-c) + 1/(b-a)(b-c) + 1/(c-a)(c-b)} \\
&= d(\hbar_{n-1}, \hbar_n),
\end{aligned} \tag{27}$$

Step 2: by (i), for an arbitrary $\hbar_0 \in \mathscr{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. It is assumed that S is an $\perp$-preserving mapping, so there is $\hbar_2 = S(\hbar_1)$ such that $\hbar_1 \perp \hbar_2$ or $\hbar_2 \perp \hbar_1$, and then, there is $\hbar_3 = S(\hbar_2)$ such that $\hbar_2 \perp \hbar_3$ or $\hbar_3 \perp \hbar_2$. In general, there is $\hbar_{n+1} = S(\hbar_n)$ such that $\hbar_n \perp \hbar_{n+1}$ or $\hbar_{n+1} \perp \hbar_n$ for all. Hence, $f(\hbar_n, \hbar_{n+1}) > 1$, for all $n \geq 0$. Note that if $\hbar_n = S(\hbar_n)$, then $\hbar_n$ is a fixed point of S, for all $n \geq 0$. We assume that $\hbar_n \neq S(\hbar_n)$, for all $n \geq 0$. Thus, $d(S\hbar_{n-1}, S\hbar_n) > 0$, for each $n \neq 1$ (otherwise, $\hbar_n = S\hbar_n$, for some n). Let $h_n = d(\hbar_n, \hbar_{n+1})$, for all $n \geq 0$. By the first part of (ii) and (24) ($i = 1$), we have

$$\Psi(h_n) < \Psi(f(\hbar_{n-1}, \hbar_n) d(S(\hbar_{n-1}), S(\hbar_n))) \leq \Phi(\check{F}_1(\hbar_{n-1}, \hbar_n)) \leq \Phi(h_{n-1}). \tag{28}$$

In view of second part of (ii), we write

$$\Psi(h_n) \leq \Phi(h_{n-1}) < \Psi(h_{n-1}). \tag{29}$$

Since Ψ is nondecreasing, one gets $h_n < h_{n-1}$, for each $n \geq 1$. This shows that the sequence $\{h_n\}$ is decreasing, so there is $L \geq 0$ such that $\lim_{n \longrightarrow \infty} h_n = L+$. If $L > 0$, by (29), one obtains

$$\Psi(L+) = \lim_{n \longrightarrow \infty} \Psi(h_n) \leq \lim_{n \longrightarrow \infty} \sup \Phi(h_{n-1}) \leq \lim_{a \longrightarrow L+} \sup \Phi(a). \tag{30}$$

This contradicts (iii), so $L = 0$, i.e., S is an asymptotically regular mapping.

Step 3: we claim that $\{\hbar_n\}$ is a Cauchy sequence. If not, then, by Lemma 1, there are $\{\hbar_{n_k}\}$ and $\{\hbar_{m_k}\}$ of $\{\hbar_n\}$ and $\xi > 0$ such that (9) and (10) hold. By (9), we infer that $d(\hbar_{n_k+1}, \hbar_{m_k+1}) > \xi$. Since $\hbar_n \perp \hbar_{n+1}$, for all $n \geq 0$, by transitivity of $\perp$, we have $\hbar_{n_k} \perp \hbar_{m_k}$ and hence, $f(\hbar_{n_k}, \hbar_{m_k}) > 1$ for all $k \geq 1$. Letting $\ell = \hbar_{n_k}$ and $J = \hbar_{m_k}$ in (24) $(i = 1)$, we have, for each $k \geq 1$,

$$\Psi\left(d\left(\hbar_{n_k+1}, \hbar_{m_k+1}\right)\right) \leq \Psi\left(f\left(\hbar_{n_k}, \hbar_{m_k}\right) d\left(S\hbar_{n_k}, S\hbar_{m_k}\right)\right) \leq \Phi\left(\check{F}_1\left(\hbar_{n_k}, \hbar_{m_k}\right)\right). \tag{31}$$

We note that

$$\begin{aligned}
&\check{F}_1\left(\hbar_{n_k}, \hbar_{m_k}\right) = d\left(\hbar_{n_k}, \hbar_{m_k}\right) d\left(\hbar_{n_k}, S\hbar_{n_k}\right)^{1/(a-b)(a-c)} d\left(\hbar_{m_k}, S\hbar_{m_k}\right)^{1/(a-b)(a-c)} \\
&\left[d\left(\hbar_{n_k}, S\hbar_{n_k}\right) + d\left(\hbar_{m_k}, S\hbar_{m_k}\right)\right]^{1/(b-a)(b-c)} \left[d\left(\hbar_{n_k}, S\hbar_{m_k}\right) + d\left(\hbar_{m_k}, S\hbar_{n_k}\right)\right]^{1/(c-a)(c-b)} \\
&\quad \leq d\left(\hbar_{n_k}, \hbar_{m_k}\right) d\left(\hbar_{n_k}, \hbar_{n_k+1}\right)^{1/(a-b)(a-c)} d\left(\hbar_{m_k}, \hbar_{m_k+1}\right)^{1/(a-b)(a-c)} \\
&\left[d\left(\hbar_{n_k}, \hbar_{n_k+1}\right) + d\left(\hbar_{m_k}, \hbar_{m_k+1}\right)\right]^{1/(b-a)(b-c)} \\
&\left[d\left(\hbar_{n_k}, \hbar_{m_k+1}\right) + d\left(\hbar_{m_k}, \hbar_{n_k+1}\right)\right]^{1/(c-a)(c-b)} = B_k.
\end{aligned} \tag{32}$$

If $\hbar_k = d(\hbar_{n_k+1}, \hbar_{m_k+1})$, we have

$$\Psi(\hbar_k) \leq \Phi(B_k), \text{ for all } k \geq 1. \tag{33}$$

By (9), we have $\lim_{k \longrightarrow \infty} \hbar_k = \xi+$, and (33) implies

$$\liminf_{a \longrightarrow \xi+} \Psi(a) \leq \liminf_{k \longrightarrow \infty} \Psi(\hbar_k) \leq \limsup_{k \longrightarrow \infty} \Phi(B_k) \leq \limsup_{a \longrightarrow 0} \Phi(a). \tag{34}$$

It is a contradiction to (iv), so $\{\hbar_n\}$ is a Cauchy sequence in the OCMS $(\mathscr{A}, \perp, d)$; hence, there is $a^* \in \mathscr{A}$ so that $\hbar_n \longrightarrow a^*$ as $n \longrightarrow \infty$, and the $\perp$-regularity of $(\mathscr{A}, \perp, d)$ yields that $\hbar_n \perp a^*$ or $a^* \perp \hbar_n$. Thus, $f(\hbar_n, a^*) > 1$. We claim that $d(a^*, S(a^*)) = 0$. Assume that $d(\hbar_{n+1}, S(a^*)) > 0$ for infinitely many values of n. By (24) $(i = 1)$,

$$\Psi(d(\hbar_{n+1}, S(a^*))) \leq \Psi(f(\hbar_n, a^*) d(S(\hbar_n), S(a^*))) \leq \Phi(\check{F}_1(\hbar_n, a^*)). \tag{35}$$

By the first part of (ii), we get $d(\hbar_{n+1}, S(a^*)) < \check{F}_1(\hbar_n, a^*)$. Applying limit $n \longrightarrow \infty$, we obtain $d(a^*, S(a^*)) \leq 0$. This implies that $d(a^*, S(a^*)) = 0$; hence, $a^* = S(a^*)$. □

Next result gives an idea on conditions ensuring the existence of fixed points of S verifying (24) $(i = 1)$.

Theorem 3. *Let $(\mathscr{A}, \perp, d)$ be an $\perp$-regular OCMS. Let $S: \mathscr{A} \longrightarrow \mathscr{A}$ be an $\perp$-preserving mapping verifying (24) $(i = 1)$ and property P. Assume the relation $\perp$ is transitive and the functions $\Psi, \Phi: (0, \infty) \longrightarrow (-\infty, \infty)$ are such that*

(i) For each $\hbar_0 \in \mathscr{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$

(ii) $\Phi(\mathscr{J}) < \Psi(\mathscr{J})$, for all $\mathscr{J} > 0$

(iii) $\inf_{a > \xi > 0} \Psi(a) > -\infty$

(iv) If $\{\Psi(\hbar_n)\}$ and $\{\Phi(\hbar_n)\}$ are converging to the same limit and $\{\Psi(\hbar_n)\}$ is strictly decreasing, then $\lim_{n \longrightarrow \infty} \hbar_n = 0$

(v) $\limsup_{a \longrightarrow 0} \Phi(a) < \liminf_{a \longrightarrow \xi+} \Psi(a)$, for all $\xi > 0$

(vi) $\limsup_{a \longrightarrow 0} \Phi(a) < \liminf_{a \longrightarrow \xi} \Psi(a)$, for all $\xi > 0$

Then, S possesses a fixed point in $\mathscr{A}$.

Proof. Note that we need (i)-(iv) to show that S is an asymptotically regular. Condition (v) is needed to establish that $\{\hbar_n\}$ is Cauchy and (vi) is useful to ensure that the mapping S has a fixed point.

By (i), for an arbitrary $\hbar_0 \in \mathscr{A}$, there is $\hbar_1 = S(\hbar_0)$ so that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. Since S is $\perp$-preserving, there is $\hbar_2 = S(\hbar_1)$ so that $\hbar_1 \perp \hbar_2$ or $\hbar_2 \perp \hbar_1$, and then, $\hbar_3 = S(\hbar_2)$ so that $\hbar_2 \perp \hbar_3$

or $\hbar_3 \perp \hbar_2$. In general, there is $\hbar_{n+1} = S(\hbar_n)$ in order that $\hbar_n \perp \hbar_{n+1}$ or $\hbar_{n+1} \perp \hbar_n$, for all $n \geq 0$. Hence, $f(\hbar_n, \hbar_{n+1}) > 1$. Note that if $\hbar_n = S(\hbar_n)$, then $\hbar_n$ is a fixed point of S. Suppose that $\hbar_n \neq S(\hbar_n)$, for all $n \geq 0$. Thus, $d(S\hbar_{n-1}, S\hbar_n) > 0$ (otherwise $\hbar_n = S\hbar_n$). Since $f(\hbar_n, \hbar_{n+1}) > 1$, by (ii) and (24) $(i = 1)$, we write

$$\begin{aligned}\Psi(d(\hbar_n, \hbar_{n+1})) &\leq \Psi(f(\hbar_{n-1}, \hbar_n) d(S(\hbar_{n-1}), S(\hbar_n)))\\ &\leq \Phi(\check{F}_1(\hbar_{n-1}, \hbar_n))\\ &\leq \Psi(d(\hbar_{n-1}, \hbar_n)).\end{aligned} \tag{36}$$

Inequality (36) shows that $\{\Psi(d(\hbar_{n-1}, \hbar_n))\}$ is strictly decreasing. If it is not bounded below, in view of (iii), we get $\inf_{d(\hbar_{n-1}, \hbar_n) > \xi} \Psi(d(\hbar_{n-1}, \hbar_n)) > -\infty$. This implies that

$$\lim \inf_{d(\hbar_{n-1}, \hbar_n) \longrightarrow \xi+} \Psi(d(\hbar_{n-1}, \hbar_n)) > -\infty. \tag{37}$$

Thus, $\lim_{n \longrightarrow \infty} d(\hbar_{n-1}, \hbar_n) = 0$; otherwise, we have

$$\lim \inf_{d(\hbar_{n-1}, \hbar_n) \longrightarrow \xi+} \Psi(d(\hbar_{n-1}, \hbar_n)) = -\infty, \tag{38}$$

(i.e., a contradiction to (iii)). If it is bounded below, then $\{\Psi(d(\hbar_{n-1}, \hbar_n))\}$ is a convergent sequence, and by (36), $\{\Phi(d(\hbar_{n-1}, \hbar_n))\}$ also converges and both have the same limit. Thus, by (iv), one gets $\lim_{n \longrightarrow \infty} d(\hbar_{n-1}, \hbar_n) = 0$. Hence, S is asymptotically regular.

Now, we claim that $\{\hbar_n\}$ is a Cauchy sequence. If $\{\hbar_n\}$ is not a Cauchy sequence, so, by Lemma 1, there exist $\{\hbar_{n_k}\}$ and $\{\hbar_{m_k}\}$ and $\xi > 0$ such that (9) and (10) hold. By (9), we infer that $d(\hbar_{n_k+1}, \hbar_{m_k+1}) > \xi$. Since $\hbar_n \perp \hbar_{n+1}$, for all $n \geq 0$, so, by transitivity of $\perp$, we have $\hbar_{n_k} \perp \hbar_{m_k}$, and hence, $f(\hbar_{n_k}, \hbar_{m_k}) > 1$ for all $k \geq 1$. Letting $x = \hbar_{n_k}$ and $y = \hbar_{m_k}$ in (24), one writes, for all $k \geq 1$,

$$\begin{aligned}\Psi\left(d\left(\hbar_{n_k+1}, \hbar_{m_k+1}\right)\right) &\leq \Psi\left(f\left(\hbar_{n_k}, \hbar_{m_k}\right) d\left(S\hbar_{n_k}, S\hbar_{m_k}\right)\right)\\ &\leq \Phi\left(\left(\check{F}_1\left(\hbar_{n_k}, \hbar_{m_k}\right)\right)\right).\end{aligned} \tag{39}$$

We note that

$$\begin{aligned}&\check{F}_1\left(\hbar_{n_k}, \hbar_{m_k}\right) = d\left(\hbar_{n_k}, \hbar_{m_k}\right) d\left(\hbar_{n_k}, S\hbar_{n_k}\right)^{1/(a-b)(a-c)} d\left(\hbar_{m_k}, S\hbar_{m_k}\right)^{1/(a-b)(a-c)}\\ &\left[d\left(\hbar_{n_k}, S\hbar_{n_k}\right) + d\left(\hbar_{m_k}, S\hbar_{m_k}\right)\right]^{1/(b-a)(b-c)} \left[d\left(\hbar_{n_k}, S\hbar_{m_k}\right) + d\left(\hbar_{m_k}, S\hbar_{n_k}\right)\right]^{1/(c-a)(c-b)}\\ &\quad \leq d\left(\hbar_{n_k}, \hbar_{m_k}\right) d\left(\hbar_{n_k}, \hbar_{n_k+1}\right)^{1/(a-b)(a-c)} d\left(\hbar_{m_k}, \hbar_{m_k+1}\right)^{1/(a-b)(a-c)}\\ &\left[d\left(\hbar_{n_k}, \hbar_{n_k+1}\right) + d\left(\hbar_{m_k}, \hbar_{m_k+1}\right)\right]^{1/(b-a)(b-c)}\\ &\left[d\left(\hbar_{n_k}, \hbar_{m_k+1}\right) + d\left(\hbar_{m_k}, \hbar_{n_k+1}\right)\right]^{1/(c-a)(c-b)} = B_k.\end{aligned} \tag{40}$$

If $\hbar_k = d(\hbar_{n_k+1}, \hbar_{m_k+1})$, we have

$$\Psi(\hbar_k) \leq \Phi(B_k), \text{ for all } k \geq 1. \tag{41}$$

By (9), we have $\lim_{k \longrightarrow \infty} \hbar_k = \xi+$ and (41) implies

$$\lim \inf_{a \longrightarrow \xi+} \Psi(a) \leq \liminf_{k \longrightarrow \infty} \Psi(\hbar_k) \leq \limsup_{k \longrightarrow \infty} \Phi(B_k) \leq \limsup_{a \longrightarrow 0} \Phi(a). \tag{42}$$

It contradicts (v), so $\{\hbar_n\}$ is a Cauchy sequence in the OCMS $\mathscr{A}$. Hence, there is $a^* \in \mathscr{A}$ in order that $\hbar_n \longrightarrow a^*$ as $n \longrightarrow \infty$.

To show that $Sa^* = a^*$, we have two cases:

Case 1: if $d(\hbar_{n+1}, Sa^*) = 0$, for some $n \geq 0$, then, since

$$d(a^*, Sa^*) \leq d(a^*, \hbar_{n+1}) + d(\hbar_{n+1}, Sa^*) = d(a^*, \hbar_{n+1}), \tag{43}$$

taking limit $n \longrightarrow \infty$ on both sides, we have $d(a^*, Sa^*) \leq 0$. This implies $d(a^*, S(a^*)) = 0$; thus, $a^* = S(a^*)$.

Case 2: if, for all $n \geq 0$, $d(\hbar_{n+1}, Sa^*) > 0$, then by $\perp$-regularity of $\mathscr{A}$, we find $\hbar_n \perp a^*$ or $a^* \perp \hbar_n$, so $f(\hbar_n, a^*) > 1$. By (24) $(i = 1)$, one writes

$$\begin{aligned}\Psi(d(\hbar_{n+1}, Sa^*)) &\leq \Psi(f(\hbar_n, a^*) d(S\hbar_n, Sa^*))\\ &\leq \Phi(\check{F}_1(\hbar_n, a^*)) \text{ for all } n \geq 0.\end{aligned} \tag{44}$$

By taking $H_n = d(\hbar_{n+1}, Sa^*)$ and $b_n = \check{F}_1(\hbar_n, a^*)$, one writes

$$\Psi(H_n) \leq \Phi(b_n) \text{ for all } n \geq 0. \tag{45}$$

Take $\xi = d(a^*, Sa^*)$. Note that $H_n \longrightarrow \xi$ and $b_n \longrightarrow 0$ as $n \longrightarrow \infty$. Applying limits on (45), we have

$$\begin{aligned}\lim \inf_{a \longrightarrow \xi} \Psi(a) &\leq \lim \inf_{n \longrightarrow \infty} \Psi(H_n) \leq \lim \sup_{n \longrightarrow \infty} \Phi(b_n)\\ &\leq \lim \inf_{a \longrightarrow 0} \Phi(a).\end{aligned} \tag{46}$$

This contradicts (vi) if $\xi > 0$. Thus, we have $d(a^*, Sa^*) = 0$, i.e., $a^* = Sa^*$, that is, a^* is a fixed point of S. □

Remark 4. Observe that

$$\begin{aligned}
\check{F}_2(\hbar_{n-1},\hbar_n) &= d(\hbar_{n-1},\hbar_n)d(\hbar_{n-1},S\hbar_{n-1})^{a/(a-b)(a-c)}d(\hbar_n,S\hbar_n)^{a/(a-b)(a-c)}\\
&\quad [d(\hbar_{n-1},S\hbar_{n-1})+d(\hbar_n,S\hbar_n)]^{b/(b-a)(b-c)}\\
&\quad [d(\hbar_{n-1},S\hbar_n)+d(\hbar_n,S\hbar_{n-1})]^{c/(c-a)(c-b)}\\
&\le d(\hbar_{n-1},\hbar_n)d(\hbar_{n-1},\hbar_n)^{a/(a-b)(a-c)}d(\hbar_n,\hbar_{n+1})^{a/(a-b)(a-c)}\\
&\quad [d(\hbar_{n-1},\hbar_n)+d(\hbar_n,\hbar_{n+1})]^{b/(b-a)(b-c)}\\
&\quad [d(\hbar_{n-1},\hbar_{n+1})+d(\hbar_n,\hbar_n)]^{c/(c-a)(c-b)}\\
&\le d(\hbar_{n-1},\hbar_n)d(\hbar_{n-1},\hbar_n)^{a/(a-b)(a-c)}d(\hbar_n,\hbar_{n+1})^{a/(a-b)(a-c)}\\
&\quad [d(\hbar_{n-1},\hbar_n)+d(\hbar_n,\hbar_{n+1})]^{b/(b-a)(b-c)}[d(\hbar_{n-1},\hbar_n)+d(\hbar_n,\hbar_{n+1})]^{c/(c-a)(c-b)}\\
&= d(\hbar_{n-1},\hbar_n)d(\hbar_{n-1},\hbar_n)^{a/(a-b)(a-c)}d(\hbar_n,\hbar_{n+1})^{a/(a-b)(a-c)}\\
&\quad [d(\hbar_{n-1},\hbar_n)+d(\hbar_n,\hbar_{n+1})]^{b/(b-a)(b-c)+c/(c-a)(c-b)}\\
&\le d(\hbar_{n-1},\hbar_n)d(\hbar_{n-1},\hbar_n)^{a/(a-b)(a-c)}d(\hbar_n,\hbar_{n+1})^{a/(a-b)(a-c)}\\
&\quad [d(\hbar_{n-1},\hbar_n)d(\hbar_n,\hbar_{n+1})]^{b/(b-a)(b-c)+c/(c-a)(c-b)} \text{ by property } P\\
&= d(\hbar_{n-1},\hbar_n)^{1+a/(a-b)(a-c)+b/(b-a)(b-c)+c/(c-a)(c-b)}\\
&\quad d(\hbar_n,\hbar_{n+1})^{a/(a-b)(a-c)+b/(b-a)(b-c)+c/(c-a)(c-b)}\\
&= d(\hbar_{n-1},\hbar_n),\\
\check{F}_3(\hbar_{n-1},\hbar_n) &= \max\left\{\begin{array}{c} d(\hbar_{n-1},\hbar_n), d(\hbar_{n-1},S\hbar_{n-1})^{a^2/(a-b)(a-c)}d(\hbar_n,S\hbar_n)^{a^2/(a-b)(a-c)}\\ [d(\hbar_{n-1},S\hbar_{n-1})+d(\hbar_n,S\hbar_n)]^{b^2/(b-a)(b-c)}\\ [d(\hbar_{n-1},S\hbar_n)+d(\hbar_n,S\hbar_{n-1})]^{c^2/(c-a)(c-b)}\end{array}\right\}\\
&= \max\left\{\begin{array}{c} d(\hbar_{n-1},\hbar_n), d(\hbar_{n-1},\hbar_n)^{a^2/(a-b)(a-c)}d(\hbar_n,\hbar_{n+1})^{a^2/(a-b)(a-c)}\\ [d(\hbar_{n-1},\hbar_n)+d(\hbar_n,\hbar_{n+1})]^{b^2/(b-a)(b-c)}\\ [d(\hbar_{n-1},\hbar_{n+1})+d(\hbar_n,\hbar_n)]^{c^2/(c-a)(c-b)}\end{array}\right\}\\
&\le \max\left\{\begin{array}{c} d(\hbar_{n-1},\hbar_n), d(\hbar_{n-1},\hbar_n)^{a^2/(a-b)(a-c)}d(\hbar_n,\hbar_{n+1})^{a^2/(a-b)(a-c)}\\ [d(\hbar_{n-1},\hbar_n)+d(\hbar_n,\hbar_{n+1})]^{b^2/(b-a)(b-c)}\\ [d(\hbar_{n-1},\hbar_n)+d(\hbar_n,\hbar_{n+1})]^{c^2/(c-a)(c-b)}\end{array}\right\}\\
&\le \max\left\{\begin{array}{c} d(\hbar_{n-1},\hbar_n), d(\hbar_{n-1},\hbar_n)^{a^2/(a-b)(a-c)}d(\hbar_n,\hbar_{n+1})^{a^2/(a-b)(a-c)}\\ [d(\hbar_{n-1},\hbar_n)d(\hbar_n,\hbar_{n+1})]^{b^2/(b-a)(b-c)}\\ [d(\hbar_{n-1},\hbar_n)d(\hbar_n,\hbar_{n+1})]^{c^2/(c-a)(c-b)} \text{ by property } P\end{array}\right\}\\
&= \max\left\{\begin{array}{c} d(\hbar_{n-1},\hbar_n),\\ d(\hbar_{n-1},\hbar_n)^{a^2/(a-b)(a-c)+b^2/(b-a)(b-c)+c^2/(c-a)(c-b)}\\ d(\hbar_n,\hbar_{n+1})^{a^2/(a-b)(a-c)+b^2/(b-a)(b-c)+c^2/(c-a)(c-b)}\end{array}\right\}\\
&= \max\{d(\hbar_{n-1},\hbar_n), d(\hbar_n,\hbar_{n+1})\},
\end{aligned}$$

$$\begin{aligned}
\check{F}_4(\hbar_{n-1},\hbar_n) &= d(\hbar_{n-1},\hbar_n)^{a^3/(a-b)(a-c)} d(\hbar_n, S\hbar_n)^{a^3/(a-b)(a-c)} \\
&\quad [d(\hbar_{n-1}, S\hbar_{n-1}) + d(\hbar_n, S\hbar_n)]^{b^3/(b-a)(b-c)} [d(\hbar_{n-1}, S\hbar_n) + d(\hbar_n, S\hbar_{n-1})]^{c^3/(c-a)(c-b)} \\
&= d(\hbar_{n-1},\hbar_n)^{a^3/(a-b)(a-c)} d(\hbar_n, \hbar_{n+1})^{a^3/(a-b)(a-c)} \\
&\quad [d(\hbar_{n-1}, \hbar_n) + d(\hbar_n, \hbar_{n+1})]^{b^3/(b-a)(b-c)} [d(\hbar_{n-1}, \hbar_{n+1}) + d(\hbar_n, \hbar_n)]^{c^3/(c-a)(c-b)} \\
&\leq d(\hbar_{n-1},\hbar_n)^{a^3/(a-b)(a-c)} d(\hbar_n, \hbar_{n+1})^{a^3/(a-b)(a-c)} \\
&\quad [d(\hbar_{n-1}, \hbar_n) d(\hbar_n, \hbar_{n+1})]^{b^3/(b-a)(b-c)} [d(\hbar_{n-1}, \hbar_n) d(\hbar_n, \hbar_{n+1})]^{c^3/(c-a)(c-b)} \\
&= [d(\hbar_{n-1}, \hbar_n) d(\hbar_n, \hbar_{n+1})]^{a^3/(a-b)(a-c)+b^3/(b-a)(b-c)+c^3/(c-a)(c-b)} \\
&= [d(\hbar_{n-1}, \hbar_n) d(\hbar_n, \hbar_{n+1})]^{(a+b+c)} \\
&\leq \max\{d(\hbar_{n-1}, \hbar_n), d(\hbar_n, \hbar_{n+1})\}.
\end{aligned} \tag{47}$$

The next two results address the $(\Psi,\Phi)_\perp$-interpolative fractional contractions of types II and III.

Theorem 4. *Let $(\mathscr{A},\perp,d)$ be an $\perp$-regular OCMS. Let $S\colon \mathscr{A}\longrightarrow\mathscr{A}$ be an $\perp$-preserving mapping verifying (24) for $i=2,3$ and property P. Suppose the relation $\perp$ is transitive, and the functions $\Psi,\Phi\colon (0,\infty)\longrightarrow(-\infty,\infty)$ are so that*

(i) For each $\hbar_0\in\mathscr{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_1\perp\hbar_0$ or $\hbar_0\perp\hbar_1$

(ii) Ψ,Φ are nondecreasing and $\Phi(\mathscr{J})<\Psi(\mathscr{J})$, for all $\mathscr{J}>0$

(iii) $\limsup_{J\longrightarrow\delta+}\Phi(\mathscr{J})<\Psi(\delta+)$, for all $\delta>0$

(iv) $\limsup_{a\longrightarrow 0}\Phi(a)\leq\liminf_{a\longrightarrow\xi+}\Psi(a)$

Then, S has a fixed point in $\mathscr{A}$.

Proof. Keeping in view the simplifications for $\check{F}_2(\hbar_{n-1},\hbar_n)$ and $\check{F}_3(\hbar_{n-1},\hbar_n)$ given in Remark 4 with the fact that $d(\hbar_{n-1},\hbar_n)>d(\hbar_n,\hbar_{n+1})$ and following the proof of Theorem 2, we assert that S admits a fixed point in $\mathscr{A}$. If $d(\hbar_{n-1},\hbar_n)<d(\hbar_n,\hbar_{n+1})$, then we have a contradiction to the definition of function Ψ. □

Theorem 5. *Let $(\mathscr{A},\perp,d)$ be an $\perp$-regular OCMS. Let $S\colon \mathscr{A}\longrightarrow\mathscr{A}$ be an $\perp$-preserving mapping verifying (24) ($i=2,3$) and property P. Assume the relation $\perp$ is transitive, and the functions $\Psi,\Phi\colon (0,\infty)\longrightarrow(-\infty,\infty)$ are so that*

(i) For each $\hbar_0\in\mathscr{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_0\perp\hbar_1$ or $\hbar_1\perp\hbar_0$

(ii) $\Phi(\mathscr{J})<\Psi(\mathscr{J})$, for all $\mathscr{J}>0$

(iii) $\inf_{a>\xi>0}\Psi(a)>-\infty$

(iv) If $\{\Psi(\hbar_n)\}$ and $\{\Phi(\hbar_n)\}$ are converging to the same limit and $\{\Psi(\hbar_n)\}$ is strictly decreasing, then $\lim_{n\longrightarrow\infty}\hbar_n=0$

(v) $\limsup_{a\longrightarrow 0}\Phi(a)<\liminf_{a\longrightarrow\xi+}\Psi(a)$, for all $\xi>0$

(vi) $\limsup_{a\longrightarrow 0}\Phi(a)<\liminf_{a\longrightarrow\xi}\Psi(a)$, for all $\xi>0$

Then, S possesses a fixed point in $\mathscr{A}$.

Proof. Keeping in view the simplifications for $\check{F}_2(\hbar_{n-1},\hbar_n)$ and $\check{F}_3(\hbar_{n-1},\hbar_n)$ given in Remark 4 with the fact that $d(\hbar_{n-1},\hbar_n)>d(\hbar_n,\hbar_{n+1})$ and following the proof of Theorem 2, we assert that S admits a fixed point in $\mathscr{A}$. If $d(\hbar_{n-1},\hbar_n)<d(\hbar_n,\hbar_{n+1})$, then we have a contradiction to the definition of function Ψ. □

The next two results address the $(\Psi,\Phi)_\perp$-interpolative fractional contraction of type IV.

Theorem 6. *Let $(\mathscr{A},\perp,d)$ be an $\perp$-regular OCMS. Let $S\colon \mathscr{A}\longrightarrow\mathscr{A}$ be an $\perp$-preserving mapping verifying (24) for $i=4$ with $a+b+c<0.5$ and property P. Suppose the relation $\perp$ is transitive and the functions $\Psi,\Phi\colon (0,\infty)\longrightarrow(-\infty,\infty)$ are so that*

(i) For each $\hbar_0\in\mathscr{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_1\perp\hbar_0$ or $\hbar_0\perp\hbar_1$

(ii) Ψ,Φ are nondecreasing and $\Phi(\mathscr{J})<\Psi(\mathscr{J})$, for all $\mathscr{J}>0$

(iii) $\limsup_{\mathscr{J}\longrightarrow\delta+}\Phi(\mathscr{J})<\Psi(\delta+)$, for all $\delta>0$

(iv) $\limsup_{a\longrightarrow 0}\Phi(a)\leq\liminf_{a\longrightarrow\xi+}\Psi(a)$

Then, S has a fixed point in $\mathscr{A}$.

Proof. Keeping in view the simplifications for $\check{F}_4(\hbar_{n-1},\hbar_n)$ given in Remark 4 and following the proof of Theorem 4, we assert that S admits a fixed point in $\mathscr{A}$. □

Theorem 7. *Let $(\mathscr{A},\perp,d)$ be an $\perp$-regular OCMS. Let $S\colon \mathscr{A}\longrightarrow\mathscr{A}$ be an $\perp$-preserving mapping verifying (24) ($i=4$) with $a+b+c<0.5$ and property P. Assume the relation $\perp$ is transitive and the functions $\Psi,\Phi\colon (0,\infty)\longrightarrow(-\infty,\infty)$ are so that*

(i) For each $\hbar_0\in\mathscr{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_0\perp\hbar_1$ or $\hbar_1\perp\hbar_0$

(ii) $\Phi(\mathscr{J})<\Psi(\mathscr{J})$, for all $\mathscr{J}>0$

(iii) $\inf_{a>\xi>0}\Psi(a)>-\infty$

(iv) If $\{\Psi(\hbar_n)\}$ and $\{\Phi(\hbar_n)\}$ are converging to the same limit and $\{\Psi(\hbar_n)\}$ is strictly decreasing, then $\lim_{n\longrightarrow\infty}\hbar_n=0$

(v) $\limsup_{a\longrightarrow 0}\Phi(a)<\liminf_{a\longrightarrow\xi+}\Psi(a)$, for all $\xi>0$

(vi) $\limsup_{a\longrightarrow 0}\Phi(a)<\liminf_{a\longrightarrow\xi}\Psi(a)$, for all $\xi>0$

Then, S possesses a fixed point in $\mathscr{A}$.

Proof. Keeping in view the simplifications for $\check{F}_4(\hbar_{n-1}, \hbar_n)$ given in Remark 4 and following the proof of Theorem 5, we assert that S admits a fixed point in $\mathscr{A}$. □

4. The Generality of the Main Results

Let us define $\Psi(\mathscr{J}) = \mathscr{J}$, for all $\mathscr{J} > 0$, in any one of Theorems 2 and 3, we receive a general version of the interpolative Boyd–Wong fixed-point theorem proved in [9], and defining $\Phi(\mathscr{J}) = \Psi(\mathscr{J}) - \tau$ in Theorem 2, we receive the following result (interpolative fractional version of Wardowski fixed-point theorem with only monotonicity condition on Ψ).

Corollary 1. *Let $(\mathscr{A}, d)$ be a complete metric space. Let $S\colon \mathscr{A} \longrightarrow \mathscr{A}$ be a mapping so that*

$$\begin{aligned}&\Psi(d(S\ell, S\mathscr{J})) \leq \Psi(\check{F}_i(\ell, \mathscr{J})) - \tau \quad \forall \ell, \mathscr{J} \in \mathscr{A},\\ &i = 1, 2, 3, 4 \text{ provided } d(S\ell, S\mathscr{J}) > 0,\end{aligned} \tag{48}$$

where $\Psi\colon (0, \infty) \longrightarrow \mathbb{R}$ is nondecreasing and $\tau > 0$. Then, there is a fixed point of S in $\mathscr{A}$.

If we define $\Phi(\mathscr{J}) = \Psi(\mathscr{J}) - \tau(\mathscr{J})$ in Theorem 2, we get an interpolative fractional version of fixed-point theorem presented in [4].

Corollary 2. *Let $(\mathscr{A}, d)$ be a complete metric space. Let $S\colon \mathscr{A} \longrightarrow \mathscr{A}$ be a mapping so that*

$$\begin{aligned}&\tau(d(\ell, \mathscr{J})) + \Psi(d(S\ell, S\mathscr{J})) \leq \Psi(\check{F}_i(\ell, \mathscr{J})) \quad \forall \ell, \mathscr{J} \in \mathscr{A},\\ &i = 1, 2, 3, 4 \text{ provided } d(S\ell, S\mathscr{J}) > 0,\end{aligned} \tag{49}$$

where $\Psi\colon (0, \infty) \longrightarrow \mathbb{R}$ is nondecreasing and $\liminf_{a \longrightarrow t+} \tau(a) > 0, \forall t \geq 0$. Then, S has a fixed point in $\mathscr{A}$.

We receive the following interpolative fractional version of Moradi theorem [30] if we take $\Phi(\mathscr{J}) = h(\Psi(\mathscr{J}))$ in Theorem 2.

Corollary 3. *Let $(\mathscr{A}, \perp, d)$ be an $\perp$-regular OCMS. Let $S\colon \mathscr{A} \longrightarrow \mathscr{A}$ be an $\perp$-preserving mapping so that*

$$\begin{aligned}&\Psi(f(\ell, \mathscr{J}) d(S\ell, S\mathscr{J})) \leq h(\Psi(\check{F}_i(\ell, \mathscr{J}))) \quad \forall \ell, \mathscr{J} \in \Lambda,\\ &i = 1, 2, 3, 4 \text{ provided } d(S\ell, S\mathscr{J}) > 0,\end{aligned} \tag{50}$$

where

(i) $h\colon I \longrightarrow [0, \infty)$ is an upper semicontinuous function with $h(\mathscr{J}) < \mathscr{J}$, for all $\mathscr{J} \in I \subset \mathbb{R}$

(ii) $\Psi\colon (0, \infty) \longrightarrow I$ is nondecreasing

Assume that, for each $\hbar_0 \in \mathscr{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. Then, S has a unique fixed point in $\mathscr{A}$.

Defining $h(\mathscr{J}) = \mathscr{J}^\delta$ and $\delta \in (0, 1)$ in Corollary 3, we have the next result.

Corollary 4. *Let $(\mathscr{A}, \perp, d)$ be an $\perp$-regular and OCMS. Let $S\colon \mathscr{A} \longrightarrow \mathscr{A}$ be an $\perp$-preserving mapping so that*

$$\begin{aligned}&\Psi(f(\ell, \mathscr{J}) d(S\ell, S\mathscr{J})) \leq (\Psi(\check{F}_i(\ell, \mathscr{J})))^r \quad \forall \ell, \mathscr{J} \in \Lambda,\\ &i = 1, 2, 3, 4 \text{ provided } d(S\ell, S\mathscr{J}) > 0,\end{aligned} \tag{51}$$

where $\Psi\colon (0, \infty) \longrightarrow (0, 1)$ is nondecreasing. Assume that, for each $\hbar_0 \in \mathscr{A}$, there is $\hbar_1 = S(\hbar_0)$ such that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. Then, S has a fixed point in $\mathscr{A}$.

Observe that Corollary 4 is an improvement of Jleli–Samet fixed-point theorem [31] and the results of Li and Jiang [32] and Ahmad et al. [33].

An improvement of Skof fixed-point theorem [34] may be stated by putting $\Phi(\mathscr{J}) = \lambda \Psi(\mathscr{J})$ in Theorem 2, for $i = 1$, with either $a = \infty$ or $b = \infty$ or $c = \infty$.

Corollary 5. *Let $(\mathscr{A}, \perp, d)$ be an $\perp$-regular OCMS. Let $S\colon \mathscr{A} \longrightarrow \mathscr{A}$ be an $\perp$-preserving mapping so that*

$$\begin{aligned}&\Psi(f(\ell, \mathscr{J}) d(S\ell, S\mathscr{J})) \leq \lambda \Psi(\check{F}_1(\ell, \mathscr{J})) \quad \forall \ell,\\ &\mathscr{J} \in \Lambda, \text{ provided } d(S\ell, S\mathscr{J}) > 0,\end{aligned} \tag{52}$$

where $\Psi\colon (0, \infty) \longrightarrow (0, \infty)$ is nondecreasing and $\lambda \in (0, 1)$. Assume that, for each $\hbar_0 \in \mathscr{A}$, there is $\hbar_1 = S(\hbar_0)$ so that $\hbar_0 \perp \hbar_1$ or $\hbar_1 \perp \hbar_0$. Then, S has a unique fixed point in $\mathscr{A}$.

5. The Existence of the Solution to Urysohn Integral Equation (UIE)

In this section, we will apply Theorem 2 for the existence of the unique solution to UIE:

$$\ell(\hbar) = f(\hbar) + \int_{IR} K_1(\hbar, s, \ell(s)) \mathrm{d}s. \tag{53}$$

This integral equation encapsulates both Volterra integral equation (VIE) and Fredholm integral equation (FIE), depending on the region of integration (IR). If IR $= (a, x)$, where a is fixed, then UIE is VIE, and for IR $= (a, b)$, where a, b are fixed, UIE is FIE. In the literature, one can find many approaches to find a unique solution to UIE (see [35–39] and references therein). We are interested to use a fixed-point technique for this purpose. The fixed-point technique is simple and elegant to show the existence of a unique solution to further mathematical models.

Let IR be a set of finite measure and $\mathscr{L}^2_{\mathrm{IR}} = \left\{\ell \mid \int_{\mathrm{IR}} |\ell(s)|^2 \mathrm{d}s < \infty\right\}$.

Define the norm $\|.\|\colon \mathscr{L}^2_{\mathrm{IR}} \longrightarrow [0, \infty)$ by

$$\|\ell\|_2 = \sqrt{\int_{\mathrm{IR}} |\ell(s)|^2 \mathrm{d}s}, \text{ for all } \ell, \mathscr{J} \in \mathscr{L}^2_{\mathrm{IR}}. \tag{54}$$

An equivalent norm can be defined as follows:

$$\|\ell\|_{2,\nu} = \sqrt{\sup\left\{e^{-\nu \int_{\mathrm{IR}} \alpha(s) \mathrm{d}s} \int_{\mathrm{IR}} |\ell(s)|^2 \mathrm{d}s\right\}}, \text{ for all } \ell \in \mathscr{L}^2_{\mathrm{IR}}, \nu > 1. \tag{55}$$

Then, $(\mathscr{L}^2_{\mathrm{IR}}, \|.\|_{2,\nu})$ is a Banach space. Let $\mathscr{A} = \{\ell \in \mathscr{L}^2_{\mathrm{IR}}\colon \ell(s) > 0 \text{ for almost every } s\}$. The metric d_ν

associated to norm $\|.\|_{2,\nu}$ is given by $d_\nu(\ell,\mathcal{J}) = \|\ell - \mathcal{J}\|_{2,\nu}$, for all $\ell,\mathcal{J} \in \mathcal{A}$. Define an orthogonal relation $\perp$ on $\mathcal{A}$ by

$$a\perp v \text{ if and only if } a(s)v(s) \geq v(s), \text{ for all } a, v \in \mathcal{A}. \tag{56}$$

Then, $(\mathcal{A},\perp,d)$ is an OCMS (see Theorem 4.1 in [3]). Let L: $\mathcal{A}\times\mathcal{A} \longrightarrow (1,\infty)$ be defined by

$$L(\delta,t) = e^{\|\delta+t\|_{\mathcal{L}^2}} \text{ for all } \delta,t \in \mathcal{A} \text{ with } \delta\perp t. \tag{57}$$

Then, L is a strictly $\perp$-admissible mapping. Put $M = \inf\{L(\delta,t), \forall\delta,t \in \mathcal{A} \text{ with } \delta\perp t\}$. Let

(A1) The kernel K_1: $\mathrm{IR}\times\mathrm{IR}\times\mathbb{R} \longrightarrow \mathbb{R}$ satisfies Carathéodory conditions and

$$\begin{aligned}&\left|K_1(\hbar,s,\ell(s))\right| \leq w(\hbar,s) + e(\hbar,s)|\ell(s)|; w,\\ &e \in \mathcal{L}^2(\mathrm{IR}\times\mathrm{IR}), e(\hbar,s) > 0.\end{aligned} \tag{58}$$

(A2) The function f: $\mathrm{IR} \longrightarrow [1,\infty)$ is continuous and bounded on IR.

(A3) There exists a positive constant C such that

$$\sup_{\hbar\in\mathrm{IR}} \int_{IR} \left|K_1(\hbar,s)\right| ds \leq C. \tag{59}$$

(A4) For any $\ell_0 \in \mathcal{L}^2_{\mathrm{IR}}$, there is $\ell_1 = R(\ell_0)$ such that $\ell_1\perp\ell_0$ or $\ell_0\perp\ell_1$.

(A5) There exists a nonnegative and measurable function q: $\mathrm{IR}\times\mathrm{IR} \longrightarrow \mathbb{R}$ such that

$$\alpha(\hbar) := \int_{\mathrm{IR}} q^2(\hbar,s)ds \leq \frac{1}{\nu M^2} \tag{60}$$

and integrable over IR with

$$\left|K_1(\hbar,s,\ell(s)) - K_1(\hbar,s,\mathcal{J}(s))\right| \leq q(\hbar,s)|\ell(s) - \mathcal{J}(s)|, \tag{61}$$

for all $\hbar, s \in \mathrm{IR}$ and $\ell,\mathcal{J} \in \mathcal{A}$ with $\ell\perp\mathcal{J}$.

Theorem 8. *Suppose that the mappings f and K_1 mentioned above satisfy conditions (A1)–(A5); then, the UIE (53) has a unique solution.*

Proof. Define the mapping R: $\mathcal{A} \longrightarrow \mathcal{A}$, in accordance with the abovementioned notations, by

$$(R\ell)(\hbar) = f(\hbar) + \int_{\mathrm{IR}} K_1(\hbar,s,\ell(s))ds. \tag{62}$$

The operator R is $\perp$-preserving: let $\ell, J \in \mathcal{A}$ with $\ell\perp J$; then, $\ell(s)J(s) \geq J(s)$. Since, for almost every $\hbar \in \mathrm{IR}$,

$$(R\ell)(\hbar) = f(\hbar) + \int_{\mathrm{IR}} K_1(\hbar,s,\ell(s))ds \geq 1, \tag{63}$$

this implies that $(R\ell)(\hbar)(RJ)(\hbar) \geq (RJ)(\hbar)$. Thus, $(R\ell)\perp(RJ)$.

Self-operator: conditions (A1) and (A3) imply that R is continuous and compact mapping from $\mathcal{A}$ to $\mathcal{A}$ (see Lemma 3 in [35]).

By (A4), for any $\ell_0 \in \mathcal{A}$, there is $\ell_1 = R(\ell_0)$ such that $\ell_1\perp\ell_0$ or $\ell_0\perp\ell_1$, and using the fact that R is $\perp$-preserving, we have $\ell_n = R^n(\ell_0)$ with $\ell_n\perp\ell_{n+1}$ or $\ell_{n+1}\perp\ell_n$, for all $n \geq 0$. We will check the contractive condition (24) of Theorem 2 in the next lines. By (A5) and Holder inequality, we have

$$\begin{aligned}|(R\ell)(\hbar) - (R\mathcal{J})(\hbar)|^2 &= \left|\int_{IR} K_1(\hbar,s,\ell(s))ds - \int_{IR} K_1(\hbar,s,\mathcal{J}(s))ds\right|^2\\ &\leq \left(\int_{IR}\left|K_1(\hbar,s,\ell(s)) - K_1(\hbar,s,\mathcal{J}(s))\right|ds\right)^2\\ &\leq \left(\int_{IR} q(\hbar,s)|\ell(s) - \mathcal{J}(s)|ds\right)^2\\ &\leq \int_{IR} q^2(\hbar,s)ds \cdot \int_{IR}|\ell(s) - \mathcal{J}(s)|^2 ds\\ &= \alpha(\hbar)\int_{IR}|\ell(s) - \mathcal{J}(s)|^2 ds.\end{aligned} \tag{64}$$

This implies, by integrating with respect to $\hbar$,

$$\begin{aligned}\int_{IR}|(R\ell)(\hbar) - (R\mathcal{J})(\hbar)|^2 d\hbar &\leq \int_{IR}\left(\alpha(\hbar)\int_{IR}|\ell(s) - \mathcal{J}(s)|^2 ds\right)d\hbar\\ &= \int_{IR}\left(\alpha(\hbar)e^{\nu\int_{\mathrm{IR}}\alpha(s)ds} \cdot e^{-\nu\int_{\mathrm{IR}}\alpha(s)ds}\int_{IR}|\ell(s) - \mathcal{J}(s)|^2 ds\right)d\hbar\\ &\leq \|\ell - \mathcal{J}\|^2_{2,\nu}\int_{IR}\alpha(\hbar)e^{\nu\int_{\mathrm{IR}}\alpha(s)ds} d\hbar\\ &\leq \frac{1}{\nu M^2}\|\ell - \mathcal{J}\|^2_{2,\nu} e^{\nu\int_{\mathrm{IR}}\alpha(s)ds}.\end{aligned} \tag{65}$$

Thus, we have

$$M^2 e^{-\nu \int_{\mathrm{IR}} \alpha(s)\mathrm{d}s} \int_{IR} |(R\ell)(\hbar) - (R\mathscr{J})(\hbar)|^2 d\hbar \le \frac{1}{\nu} \|\ell - \mathscr{J}\|_{2,\nu}^2. \tag{66}$$

This implies that

$$M^2 \|(R\ell) - (R\mathscr{J})\|_{2,\nu}^2 \le \frac{1}{\nu} \|\ell - \mathscr{J}\|_{2,\nu}^2. \tag{67}$$

That is,

$$L(\ell, \mathscr{J}) d_\nu((R\ell), (R\mathscr{J})) \le \sqrt{\frac{1}{\nu}} d_\nu(\ell, \mathscr{J}). \tag{68}$$

Taking ln on both sides and defining $\Psi(t) = \ln(t)$ with $\Phi(t) = \Psi(t) - \tau$, $\tau > 0$, we have

$$\begin{aligned} &\Psi(L(\ell, \mathscr{J}) d_\nu((R\ell), (R\mathscr{J}))) \le \Phi(\check{F}_1(\ell, \mathscr{J})), \\ &\tau = -\ln\left(\sqrt{\frac{1}{\nu}}\right), a = \infty. \end{aligned} \tag{69}$$

The defined Ψ and Φ satisfy remaining conditions of Theorem 2. Hence, by Theorem 2, the operator R has a unique fixed point. This means that the UIE (53) has a unique solution. □

6. Conclusion

The $(\Psi, \Phi)_\perp$ interpolative contractions are broad enough to include well-known contractions. The presented theorems provide a general criterion for the existence of a unique fixed point of $(\Psi, \Phi)_\perp$ interpolative contraction mappings. Fixed-point methodology is used to investigate the presence of a solution to a UIE.

Data Availability

Data sharing is not applicable to this article as no dataset was generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors equally conceived the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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Research Article

A Hybrid Model of Extreme Learning Machine Based on Bat and Cuckoo Search Algorithm for Regression and Multiclass Classification

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Extreme learning machine (ELM), as a new simple feedforward neural network learning algorithm, has been extensively used in practical applications because of its good generalization performance and fast learning speed. However, the standard ELM requires more hidden nodes in the application due to the random assignment of hidden layer parameters, which in turn has disadvantages such as poorly hidden layer sparsity, low adjustment ability, and complex network structure. In this paper, we propose a hybrid ELM algorithm based on the bat and cuckoo search algorithm to optimize the input weight and threshold of the ELM algorithm. We test the numerical experimental performance of function approximation and classification problems under a few benchmark datasets; simulation results show that the proposed algorithm can obtain significantly better prediction accuracy compared to similar algorithms.

1. Introduction

In recent years, artificial intelligence algorithms have drawn extensive attention from scientific research. As an important part of artificial intelligence, machine learning has been widely used in data mining [1], speech recognition [2], feature selection [3, 4], learning incentivization strategy [5], natural language processing [6], and the nonlinear function approximation and benchmark problem [7]. As a branch of machine learning, neural networks have been successfully applied in many tasks of learning from data. However, most of the traditional neural networks use the gradient learning algorithm for network training, which makes the network make problems such as low training efficiency, slow speed, and easy to fall into local optimal.

Extreme Learning Machine (ELM) is a new method of training artificial neural networks and includes supervised training methods, which is a kind of neural network structure put forward by Huang et al. using single hidden layer feedforward networks (SLFN) [8–10]. Huang et al. [11] argue that the existing neural networks have some defects in learning speed; the main reason for the low rate of learning is that all the parameters on the network are determined repeatedly by a training method. In the ELM learning algorithm, the weight feedback and threshold are generated randomly. Then, the output of the hidden layer matrix is used to calculate the final output weight. Computing the final weights was obtained using Moore–Penrose (MP) generalized inverse. Compared with other neural networks based on the gradient learning algorithm, the ELM learning algorithm has great advantages in learning speed, and it is capable of producing good generalization performance and greatly reduces the computational complexity of complex application problems [12, 13]. Meanwhile, these good performances have been widely promoted in various practical application fields such as biomedicine [14–16], fault diagnosis [17, 18], and indoor positioning systems [19, 20]. However, since the input parameters are generated randomly and the ELM requires a large number of hidden neurons, the amplitude of the output weight will be large when the output matrix of the hidden layer is ill, which will cause the trained model to fall into the local minimum and show the phenomenon of overfitting [21]. In [22, 23], an ELM based on different regularization was proposed to effectively overcome the overfitting phenomenon. The accuracy and effectiveness of the ELM algorithm largely rest with the internal parameters of the model. So as to choose the suitable model parameters, many researchers use a bionic optimization algorithm to optimize the input weights and thresholds.

In the literature [24], the improved ELM algorithm was proposed, which used a differential evolution algorithm to choose the input weights and then used MP generalized inverse analysis to determine the output weights. This improvement enables it to obtain better generalization performance in a compact network. In the literature [25], the coral reefs optimization (CRO – ELM) has been used for carrying out evolution in ELM weights to enhance the performance of these machines. A new evolutionary algorithm, particle swarm optimization (PSO – ELM), is introduced to optimize the input weight and hidden bias of ELM [26, 27] so that the network has better generalization performance in the benchmark classification experiment and is more suitable for some prediction problems. A real-coded genetic algorithm (RCGA – ELM) was proposed [28] to select the number of hidden neurons and the input weights, such that the generalization performance of the classifier is a maximum. But it needed to adjust many parameters in genetic operators artificially. The cuckoo search algorithm (CS – ELM) was proposed [29–33], which was used to pretrain the ELM ensuring optimal solutions and to further improve the accuracy and stability of CS – ELM. References [34, 35] proposed ICS model, which combines the improved cuckoo search algorithm with ELM. Both CS – ELM and ICS – ELM select the input weights and biases before calculating the output weights, and they ensure the full column rank of the hidden layer output matrix.

Bat algorithm (BA) [36, 37] and cuckoo search algorithm (CS) [38, 39] are two new heuristic swarm intelligence optimization algorithms. Bat algorithm has the advantages of a simple model, fast convergence rate, strong global optimization, and so on and has been widely used in engineering optimization, model identification, and other problems. The cuckoo search algorithm has the characteristics of simple and efficient, few parameters, easy to implement, and excellent random search path and has been successfully applied to medical image optimization [40], multiobjective optimization [41], image processing [42], and other practical problems. Literature [43] shows that bat algorithm and cuckoo search algorithm have great advantages over genetic algorithm and particle swarm optimization in the new metaheuristic environment. In this paper, we combine the BACS hybrid algorithm with traditional ELM and propose an optimization algorithm of ELM based on BACS. The basic thought of the BACS – ELM algorithm is to use the BACS algorithm to train the input weight and threshold value randomly generated by ELM to find the optimal parameter and then determine the output weights by using MP generalized inverse so as to improve the convergence speed and stability of the network model. The main contributions are as follows:

(1) Based on the idea of a group intelligence optimization algorithm, this paper introduces how to train ELM by BACS hybrid algorithm. By using this method, the input weights and thresholds of the ELM network can be reasonably optimized to solve the randomness problem of hidden layer parameters so that the network parameters can reach the optimum.
(2) By improving the traditional ELM network by BACS hybrid algorithm, the local and global optimization problems are effectively balanced, and the generalization performance of the network is improved.
(3) Nonlinear function fitting and classification problems present that the BACS – ELM algorithm can acquire better approximation effect and generalization performance than other algorithms.

The rest of the paper is arranged as follows: Section 2 introduces the traditional ELM network model and algorithm. Section 3 introduces the principles and implementation steps of the bat algorithm and cuckoo search algorithm. The hybrid algorithm of Extreme Learning Machine based on the bat cuckoo algorithm is described in Section 4. Some numerical experiments are discussed in Section 5. Section 6 offers some conclusions for this paper.

2. The Preliminary of ELM

In this section, we begin with the introduction of standard ELM, the network model of ELM is shown in Figure 1, and its network model can be divided into three layers, which are the input layer, hidden layer, and output layer. All of these works provide fundamental theoretical support for the new method proposed next. $(x_j, o_j) \in R^n \times R^m$ represents P arbitrary various samples, where $x_j = (x_{j1}, x_{j2}, \ldots, x_{jn})^T \in R^n$ and $o_j = (o_{j1}, o_{j2}, \ldots, o_{jm})^T \in R^m$; the traditional SLFN with L hidden nodes can be mathematically modeled as

$$h_L(\mathbf{x}_j) = \sum_{i=1}^{L} \beta_i G(\mathbf{w}_i, b_i, \mathbf{x}_j) = \mathbf{t}_j, \quad j = 1, 2, \ldots, P, \tag{1}$$

where $G(\mathbf{w}_i, b_i, \mathbf{x}_j)$ is an activation function, which can take various kinds forms, such as the sigmoid function:

$$G(\mathbf{w}, b, \mathbf{x}) = \frac{1}{1 + \exp\left(-\left(\mathbf{w}^T \mathbf{x}^T + b\right)\right)} \tag{2}$$

or Gaussian function:

$$G(\mathbf{w}, b, \mathbf{x}) = \exp\left(-b\|\mathbf{w} - \mathbf{x}\|^2\right). \tag{3}$$

The above SLFN can approximate these P samples in the training process of gradual iteration. When the learning error is reduced to zero, $\sum_{j=1}^{P} \|\mathbf{t}_j - \mathbf{o}_j\| = 0$, the learning capacity of the ELM is optimal, and then there exist $(\mathbf{w}_i, b_i)$ and β_i such that

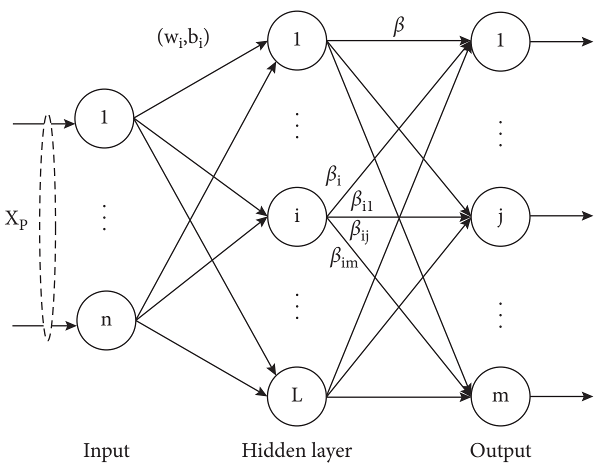

Figure 1: The structure of the ELM model.

$$\sum_{i=1}^{L} \beta_i G\left(\mathbf{w}_i, b_i, \mathbf{x}_j\right) = \mathbf{o}_j, \tag{4}$$

where $\mathbf{w}_i = (w_{i1}, w_{i2}, \ldots, w_{in})^T \in \mathbf{R}^n$ is the input weight, which links the i-th hidden node as presented in Figure 1, $b_i \in \mathbf{R}$ is the threshold of the i-th hidden node and is generated randomly, $\beta_i = (\beta_{i1}, \beta_{i2}, \ldots, \beta_{im})^T \in \mathbf{R}^m$ is the output weight of the i-th hidden node, and $\mathbf{t}_j$ represents the actual output of input $\mathbf{x}_j$ in the network.

The above P equations can be rewritten as the following matrix form:

$$\mathbf{H}\beta = \mathbf{O}, \tag{5}$$

where

$$\begin{aligned} &\mathbf{H}\left(\mathbf{w}_1, \ldots, \mathbf{w}_L, b_1, \ldots, b_L, \mathbf{x}_1, \ldots, \mathbf{x}_P\right) \\ &= \begin{bmatrix} G\left(\mathbf{w}_1, b_1, \mathbf{x}_1\right) & \cdots & G\left(\mathbf{w}_L, b_L, \mathbf{x}_1\right) \\ \vdots & \ddots & \vdots \\ G\left(\mathbf{w}_1, b_1, \mathbf{x}_P\right) & \cdots & G\left(\mathbf{w}_L, b_L, \mathbf{x}_P\right) \end{bmatrix}_{P\times L}, \\ &\beta = \begin{bmatrix} \beta_1^T \\ \vdots \\ \beta_L^T \end{bmatrix}_{L\times m}, \\ &\mathbf{O} = \begin{bmatrix} \mathbf{o}_1^T \\ \vdots \\ \mathbf{o}_P^T \end{bmatrix}_{P\times m}, \end{aligned} \tag{6}$$

where H is called the output matrix of the hidden layer and β represents the final output matrix. The basic principle of ELM is to obtain the output weight β through formula $\mathbf{H}\beta = \mathbf{O}$.

In practical training, the number of nodes L in the hidden layer is usually less than the number of training samples P. Therefore, on the premise that the activation function is differentiable, input weights and thresholds randomly selected before training should remain unchanged during training. In this way, the output weight of the network can be obtained by solving the least squares of the following linear system:

$$\min_{\beta} \|\mathbf{H}\beta - \mathbf{O}\|, \tag{7}$$

and the explicit solution is

$$\widehat{\beta} = \mathbf{H}^{\dagger}\mathbf{O}, \tag{8}$$

where $\mathbf{H}^{\dagger}$ represents the MP generalized inverse of $\mathbf{H}$ [44]. Therefore, ELM can be described as follows (Algorithm 1).

3. Algorithm Description

3.1. Bat Algorithm. Bat algorithm (BA) is a swarm intelligence optimization algorithm that simulates the predation behavior of bats. Because of its simple model, fast convergence speed, and strong global optimization, it has been widely used in data mining, wireless sensors, and power systems. However, there are also some problems in practical applications, such as easy premature convergence and low optimization accuracy.

The bat algorithm determines the optimal bat in the current search space by adjusting the frequency, wavelength, and loudness and then obtains the optimal solution to the optimization problem. For this algorithm, in order to simulate this predation behavior, the following assumptions are proposed in the process:

(1) All bat individuals can use echolocation to perceive the distance and distinguish the difference between the target and the obstacle in a special way

(2) The bat flies randomly at position x_i at speed v_i, finds the target with frequency $f_{\min}$, variable wavelength λ, and loudness A_0, and automatically adjusts the wavelength (or frequency) and pulse emission rate $r \in [0, 1]$ through the distance from the target and so on

(3) Assume that the loudness changes from the maximum value A_0 to the minimum value $A_{\min}$

Assuming that, in the search space with dimension d, the number of iterations is t, the update formulas for the frequency, velocity, and position of the bat individual i in the t-th generation are as follows:

$$f_i = f_{\min} + (f_{\max} - f_{\min})\beta, \tag{9}$$

$$v_i^t = v_i^{t-1} + \left(x_i^{t-1} - x^*\right) f_i, \tag{10}$$

$$x_i^t = x_i^{t-1} + v_i^t, \tag{11}$$

where f_i represents the frequency of the i-th bat and its adjustment range is $[f_{\min}, f_{\max}]$, β is a random number that obeys a uniform distribution in [0, 1], and x^* represents the current optimal solution.

For the current local search domain, a random number rand_1 is generated. If $\text{rand}_1 > \text{rand}_i$, the current new solution is generated by the random disturbance of the optimal solution. The update formula is as follows:

Input: given a training set $(\mathbf{x}_j, \mathbf{o}_j) \in \mathbf{R}^n \times \mathbf{R}^m$, activation function is $G(\mathbf{w}_i, b_i, \mathbf{x}_j)$, and the hidden nodes number is L.
Output β.
Step 1: setting learning parameters for hidden nodes $\mathbf{w}_i$ and $\mathbf{b}_i$, $1 \le i \le L$.
Step 2: calculate the output matrix $\mathbf{H}$ based on (5).
Step 3: calculate the output weight $\beta = \mathbf{H}^{\dagger}\mathbf{O}$.

Algorithm 1: ELM algorithm.

$$x_{\text{new}} = x_{\text{old}} + \varepsilon A^t, \tag{12}$$

where ε is a random number in $[-1, 1]$ and A^t represents the average loudness of the bat population.

When the bat is constantly approaching the target, its loudness A will drop to a fixed value, and at this time, r will continue to increase. Randomly generate a number rand_2; if $\text{rand}_2 < A_i$ and the new fitness value $f(x_{\text{new}}) > f(x_{\text{old}})$, the new solution generated by (12) is accepted; that is $x_i^{t+1} = x_{new}$. The update formula for the loudness A_i and pulse rate r_i of the first bat is as follows:

$$A_i^{t+1} = \alpha A_i^t, \tag{13}$$

$$r_i^{t+1} = r_i^0 [1 - \exp(-\sigma t)], \tag{14}$$

where α represents the loudness attenuation coefficient and $0 < \alpha < 1$. σ represents the pulse frequency enhancement coefficient and $\sigma > 0$.

3.2. Cuckoo Search Algorithm. The cuckoo search algorithm (CS) is simplification and simulation of the cuckoo nest finding and spawning behavior. The special habit of cuckoos is parasitic brooding; that is, other host birds hatch and brood on their behalf. In order to make this phenomenon difficult to detect, the bird will first find a bird with similar characteristics to its own egg as the host during the breeding period. After being recognized by the host bird, the egg is removed or the host rebuilds the nest. In order to simulate its reproductive behavior, the following assumptions are proposed in the process:

(1) Each cuckoo lays only one egg at a time and randomly selects the nest to hatch
(2) The best bird's nest is retained to the next generation
(3) The number of available bird nests n remains unchanged; there is a probability (p_a) that the host bird finds foreign eggs, $p_a \in [0, 1]$

For the cuckoo search algorithm, randomly initialize n bird nest positions in the d-dimensional search space and leave the best position to the next generation. The new position is generated by Levy flight. Then the cuckoo's nest search path and position update formula are as follows:

$$x_i^{t+1} = x_i^t + \alpha \oplus \text{Levy}(\lambda), \tag{15}$$

where x_i^t represents the position of the i-th bird nest in the t-th generation, α represents the step-length control factor and $\alpha > 0$, $\oplus$ is the point-to-point multiplication, $\text{Levy}(\lambda)$ is the random search path, and $\text{Levy} \sim u = t^{-\lambda}\,(1 < \lambda \le 3)$.

After the position is updated, compare the random numbers r and p_a, and $0 \le r \le 1$; if $r > p_a$, then use the random walk method to change the position so as to retain a set of better values and obtain the current optimal bird nest position and optimal solution through iteration. The update formula is as follows:

$$x_i^{t+1} = x_i^t + \tau\left(x_m^t - x_k^t\right), \tag{16}$$

where τ represents the uniformly distributed scaling factor within [0, 1] and both x_m^t and x_k^t represent the random solution in the t-th generation.

3.3. Bat Cuckoo Hybrid Algorithm. Although the bat algorithm has low convergence accuracy, its global search ability is strong; in order to improve the quality of the cuckoo population, the bat algorithm is integrated into the cuckoo algorithm for optimization, and a bat cuckoo hybrid algorithm (BACS) is proposed. For this algorithm, the nest position obtained by the cuckoo algorithm is not directly used as the initial position, but the bat algorithm is used to continue to optimize the optimal value after the position is updated, which greatly accelerates the global search ability of the algorithm. Therefore, the integration of the two algorithms effectively balances the problem of local and global optimization. Based on this, the specific steps of the bat cuckoo hybrid algorithm are shown in Table 1.

4. Hybrid Algorithm of Extreme Learning Machine Based on Bat Cuckoo Algorithm

Extreme Learning Machine (ELM) selects hidden layer parameters randomly and does not need to update iteratively during training, and the output weight can be determined by the least square solution, which greatly accelerates the learning process. Although ELM overcomes the shortcomings of the traditional gradient descent algorithm, the number of hidden nodes still needs to be set in advance, which may lead to many redundant nodes. Therefore, ELM requires more random hidden nodes in some applications than traditional neural network algorithms. However, this will lead to a decrease in the sparsity and regulation ability of the hidden layer, the complexity of the network structure, and the extension of the training time and finally affect the generalization ability and robustness of the network.

Table 1: Steps of the bat cuckoo hybrid algorithm.

Step 1: initialize the basic parameters and set the loop termination criteria
Step 2: initialize the location of the bird nest, calculate the fitness value of each bird nest, and obtain the optimal position and optimal value
Step 3: record the optimal position of the previous generation, update according to formula (15) to obtain a new set of positions, calculate the fitness value, and compare it with the value of the previous generation to determine the current better position
Step 4: compare the random number r with p_a; if $r > p_a$, update the position randomly; otherwise, it will not change
Step 5: use the new position as the initial point of the bat algorithm and use equations (9)–(14) to update the position of the bird nest
Step 6: record the position of step 5 and calculate the fitness value to determine the current optimal position and optimal value
Step 7: if the termination conditions are met, continue to the next step; otherwise, go to step 3
Step 8: output the global optimal position, and the algorithm ends

BACS algorithm has the characteristics of strong search accuracy, fast convergence speed, and not easy to fall into local best and effectively balances local and global search. Using this optimization ability, the hidden layer parameters of ELM are selected appropriately to solve the problem that the hidden layer parameters need to be optimized due to randomness. Therefore, this paper considers the use of the BACS algorithm to optimize ELM so as to propose a hybrid algorithm of Extreme Learning Machine based on the bat cuckoo algorithm (BACS – ELM). We first use the BACS algorithm to train the input weights and thresholds randomly generated by ELM. The population is taken as the initially hidden layer parameter of ELM, and the fitness function of the BACS algorithm is used to conduct iterative optimization. The position of the individual of the population is constantly adjusted to find the optimal hidden layer parameter until the maximum number of iterations or search accuracy is reached. At the end of the iteration, the optimal individual position is obtained, and the optimized results are used as the input weights and thresholds of ELM to train the network so as to improve the convergence speed and stability of the network model. To prevent the problem of output saturation caused by excessive input value, we use the following formula to normalize the data:

$$y = \frac{x - x_{\min}}{x_{\max} - x_{\min}}, \tag{17}$$

where x is the original data and $x_{\max}$ and $x_{\min}$ represent the maximum and minimum values of the original data, respectively.

Next, the input weights and thresholds of ELM were represented by the cuckoo individuals using real coding rules. On the basis of Section 2, the number of neurons in the input layer and hidden layer is fixed as n and L , respectively. Therefore, the calculation formula of the coding length of the cuckoo individual is

$$D = (n + 1) * L. \tag{18}$$

Individual position of cuckoo can be expressed as

$$X = (x_1, x_2, x_3, \ldots, x_{L\times n+L}). \tag{19}$$

The input weights w_i and thresholds b_i of ELM are mapped to the individual position of the cuckoo, the population is randomly initialized, and the obtained random individuals are assigned to the input weights and thresholds of ELM one by one and placed in the ELM network. Here, the assignments of input weights and thresholds are, respectively, expressed as follows:

$$\begin{aligned} w_i &= (x_1, x_2, x_3, \ldots, x_{L\times n}), \\ b_i &= (x_{L\times n+1}, x_{L\times n+2}, \ldots, x_{L\times n+L}). \end{aligned} \tag{20}$$

In the training sample process of ELM, in order to evaluate the prediction performance more objectively, we used the root mean square error as the evaluation index of model prediction, so the fitness function was designed as

$$\text{RMSE} = \sqrt{\frac{1}{P}\sum_{j=1}^{P}\left(t_j - o_j\right)^2}, \tag{21}$$

where P is the total number of samples, $T = (t_1, t_2, \ldots, t_P)$ represents the actual output value of samples, and $O = (o_1, o_2, \ldots, o_P)$ represents the expected output value of samples. Table 2 shows the specific implementation steps of the BACS – ELM algorithm.

5. Experimental Results

In order to verify the performance of the proposed algorithm, a function fitting and several classification problems are tested in this section, and the validity of BACS – ELM is tested by comparing it with the *ELM*, *BA – ELM*, and *CS – ELM* algorithms.

5.1. Function Fitting. In order to declare the performance of the proposed algorithm more intuitively and effectively, we take into account adopting ELM, BA – ELM, CS – ELM, and BACS – ELM to approximate the Sinc function and then compare the function approximation capabilities. The expression for the Sinc function is defined as follows:

$$f x = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases} \tag{22}$$

The training set and test set of 5000 samples were selected, respectively, and the input variables x_i obey the uniform distribution in the interval $[-10, 10]$. In order to increase the authenticity and improve the generalization performance of the algorithm, random noise was added to the training samples, whereas the testing data remained noise-free. For different optimization methods, the initial parameter settings are presented in Table 3, and the maximum iteration number is set $I = 100$. The activation function is the RBF function, and the fitness function is

Table 2: Steps of the BACS – ELM learning algorithm.

Step 1: initialize the basic parameters and set the loop termination criteria
Step 2: initialize the cuckoo individual, code the input weights and thresholds of ELM into the individual, and each individual represents an ELM network structure
Step 3: normalize the training data and random initial individual position and calculate the fitness value in line with equation (21)
Step 4: record the optimal position, obtain a group of new positions according to equation (15), calculate the fitness value, and determine the current optimal position
Step 5: compare the random numbers r with P_a; if $r > P_a$, update the position randomly; otherwise, it will not change
Step 6: take the new position as the starting point of BA, and randomly generate rand_1; if $\text{rand}_1 > r_i$, update the current optimal position; otherwise, go to step 7
Step 7: randomly generate rand_1; if $\text{rand}_1 < A_i \&\& f(x_{\text{new}}) > f(x_{\text{old}})$, replace x_{new} with the current position x_i^{t+1} or do not update x_{new}
Step 8: calculate the fitness value of each individual, and determine the current optimal position and optimal value
Step 9: if the termination condition is met, proceed to the next step; otherwise, go to step 4
Step 10: the individual cuckoo is decoded into the input weights and thresholds of ELM; obtain the optimal ELM network structure according to these parameters

Table 3: The population parameter setting of three optimization methods.

Optimization algorithm	Parameter setting
BA – ELM	Bat population size $N = 20$, loudness $A_0 = 1$, and pulse emissivity $r = 0.5$ Loudness attenuation coefficient $\alpha = 0.9$ and pulse frequency interval $[f_{\min}, f(\max)] = [0, 2]$
CS – ELM	Initial nest size $N = 20$, discovery probability $P_a = 0.25$, and step size control factor $\alpha = 0.5$ Scaling factor $\gamma = 0.5$
BACS – ELM	Bat population size $N = 20$, loudness $A_0 = 1$, and pulse emissivity $r = 0.5$ Loudness attenuation coefficient $\alpha = 0.9$ and pulse frequency interval $[f_{\min}, f(\max)] = [0, 2]$ Discovery probability $P_a = 0.25$, step size control factor $\alpha = 0.5$, and scaling factor $\gamma = 0.5$

RMSE. In order to compare the results of each algorithm more objectively, each experiment was run 20 times and then took the mean value.

The selection of the number of hidden nodes will have a direct influence on the performance of the model. Therefore, the experiment on BACS – ELM was carried out by adjusting the number of hidden nodes, and the test results obtained are shown in Table 4. The results show that the function has the best fitting effect when the number of hidden nodes is 12, and the mean square error of training and testing tends to be stable with the increase of nodes. To ensure the performance of the algorithm and reduce the complexity of the model, the architecture of the optimized ELM network can be determined as 1-12-1.

Then, based on the selection of the above parameter values, simulation experiments were carried out on the ELM, BA – ELM, CS – ELM, and BACS – ELM algorithms. It can be seen from Figure 2 that the approximation effect of the BACS – ELM algorithm is better than that of other algorithms. Moreover, the performance comparison of each algorithm is shown in Table 5. According to the displayed results, the test RMSE value of the BACS – ELM algorithm is the smallest, which means that the algorithm has higher accuracy and better stability. As can be seen from the training time in the table, due to the randomness of hidden layer parameters of ELM, it has a very fast learning speed, but the fitting effect is not ideal.

The results also show that the three optimization methods are all effective. But there is little difference in training and testing time between the BA – ELM, CS – ELM, and BACS – ELM algorithms and the advantages of learning efficiency are not embodied. Nevertheless, the ELM model based on the BACS algorithm greatly improves the convergence accuracy of function fitting, so the computational efficiency is also within the acceptable range.

5.2. Classification Problems. In this section, in order to more accurately appraise the effectiveness of the BASC – ELM algorithm, the performance of the algorithm will be compared on multiple classification problems. The relevant information of the dataset is given in Table 6. The initial parameter setting of each group was consistent with the above. The maximum iterations number $I = 100$ and the activation function was the Sigmoid function. Each group of experiments was run 20 times to take the average value.

Figure 3 shows the comparison of the classification accuracy of the algorithm in different datasets with the change of the number of nodes. Figure 3(a) is based on the variation trend of breast cancer; it can be seen from the figure that ELM needs the most nodes to achieve relatively high accuracy, while other algorithms all achieve the highest accuracy when the node is 20, and further speaking, BASC – ELM is slightly better. Figure 3(b) is based on the changing trend of heart failure. It can be seen from the figure that the four algorithms all show a similar curve trend when the number of hidden nodes increases and they all have the best accuracy when the node is 20, but at this time, BASC – ELM has the highest value of 84.23%. Figure 3(c) is based on the variation trend of Iris. BASC – ELM has the best accuracy when the node is 10, which is 5 fewer nodes than other algorithms when they get the maximum value. Figure 3(d) is based on the changing trend of the vertebral column. It can be seen from the graph that BASC – ELM only needs the

Table 4: The influence of hidden node number on BACS – ELM algorithm.

Number of hidden nodes	2	4	6	8	10	12	14	16
Training RMSE	0.1370	0.1183	0.1109	0.1094	0.1082	0.1081	0.1087	0.1081
Testing RMSE	0.0902	0.0516	0.0279	0.0098	0.0095	0.0084	0.0087	0.0085

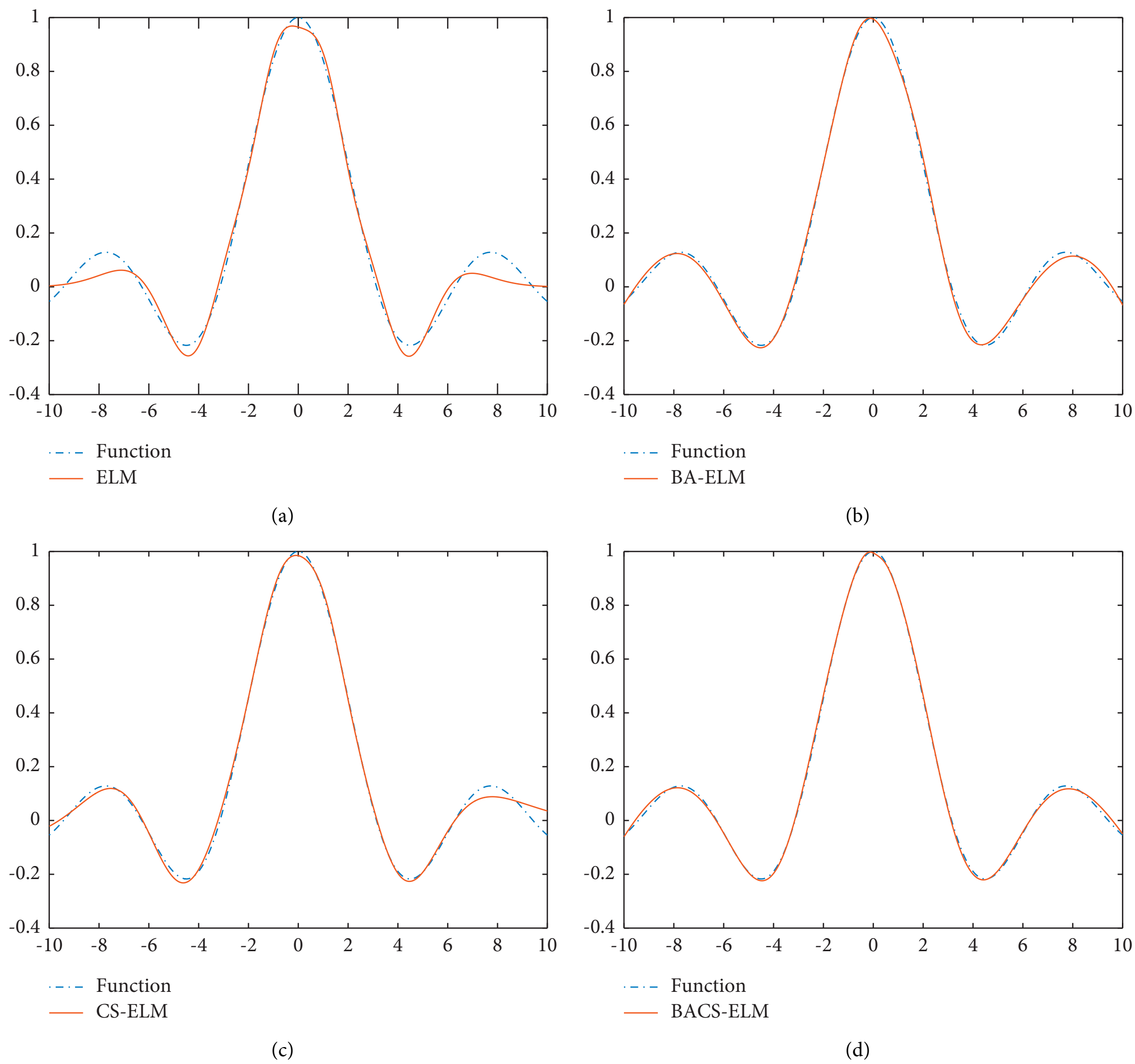

Figure 2: Comparison of the fitting effects of the four algorithms: (a) ELM, (b) BA – ELM, (c) CS – ELM, and (d) BACS – ELM.

Table 5: The performance comparison of four algorithms on Sinc function.

Algorithms	Training time (s)	Testing time (s)	Training RMSE	Testing RMSE
ELM	0.0056	0.0018	0.1351	0.0453
BA – ELM	40.6553	0.0019	0.1095	0.0119
CS – ELM	35.9939	0.0022	0.1107	0.0148
BASC – ELM	42.7807	0.0026	0.1081	0.0084

minimum number of nodes to obtain the best results, and the accuracy value fluctuates little, which indicates that the algorithm can achieve better stability.

Next, in order to better explain the accuracy of the BACS – ELM algorithm in classification experiments, Figure 4 presents the fitness curves of the BA – ELM, CS – ELM, and BACS – ELM algorithms under four classification problems, respectively. To maintain the consistency of the experimental environment, the number of hidden nodes for each problem was set as 20, 20, 15, and 30,

Table 6: The detailed description of the classification dataset.

Dataset	Training samples	Testing samples	Attribute	Classes
Breast cancer	80	36	9	2
Heart failure	209	90	12	2
Iris	105	45	4	3
Vertebral column	208	102	6	3

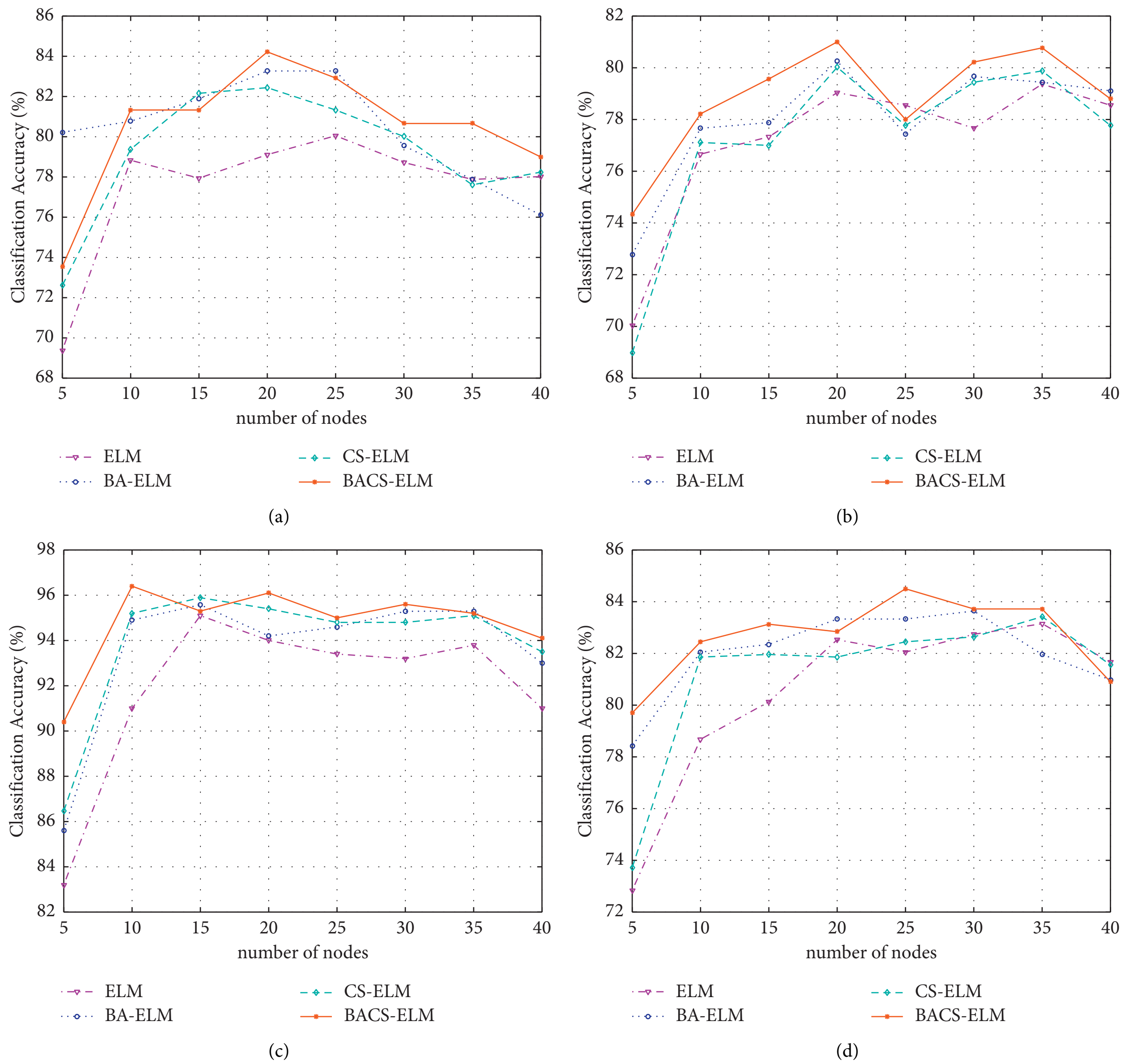

Figure 3: The graph of the classification accuracy over different datasets with the change of the number of nodes: (a) breast cancer; (b) heart failure; (c) iris; (d) vertebral column.

respectively, while other parameters were unchanged. As can be seen from Figures 4(a)–4(d), for different datasets, compared with the BA – ELM and CS – ELM algorithms, the BACS – ELM algorithm can obtain the best fitness function value in the case of the least number of iterations. This is because when the BACS algorithm optimizes the input weights and thresholds of ELM, it has a strong local optimization ability at the initial stage of search and makes full use of the global optimization ability of the BA algorithm. The combination of the two greatly improves the convergence accuracy.

Based on the above analysis, the performance results of the four algorithms on the number of hidden nodes, training time, training, and test accuracy are also given in the

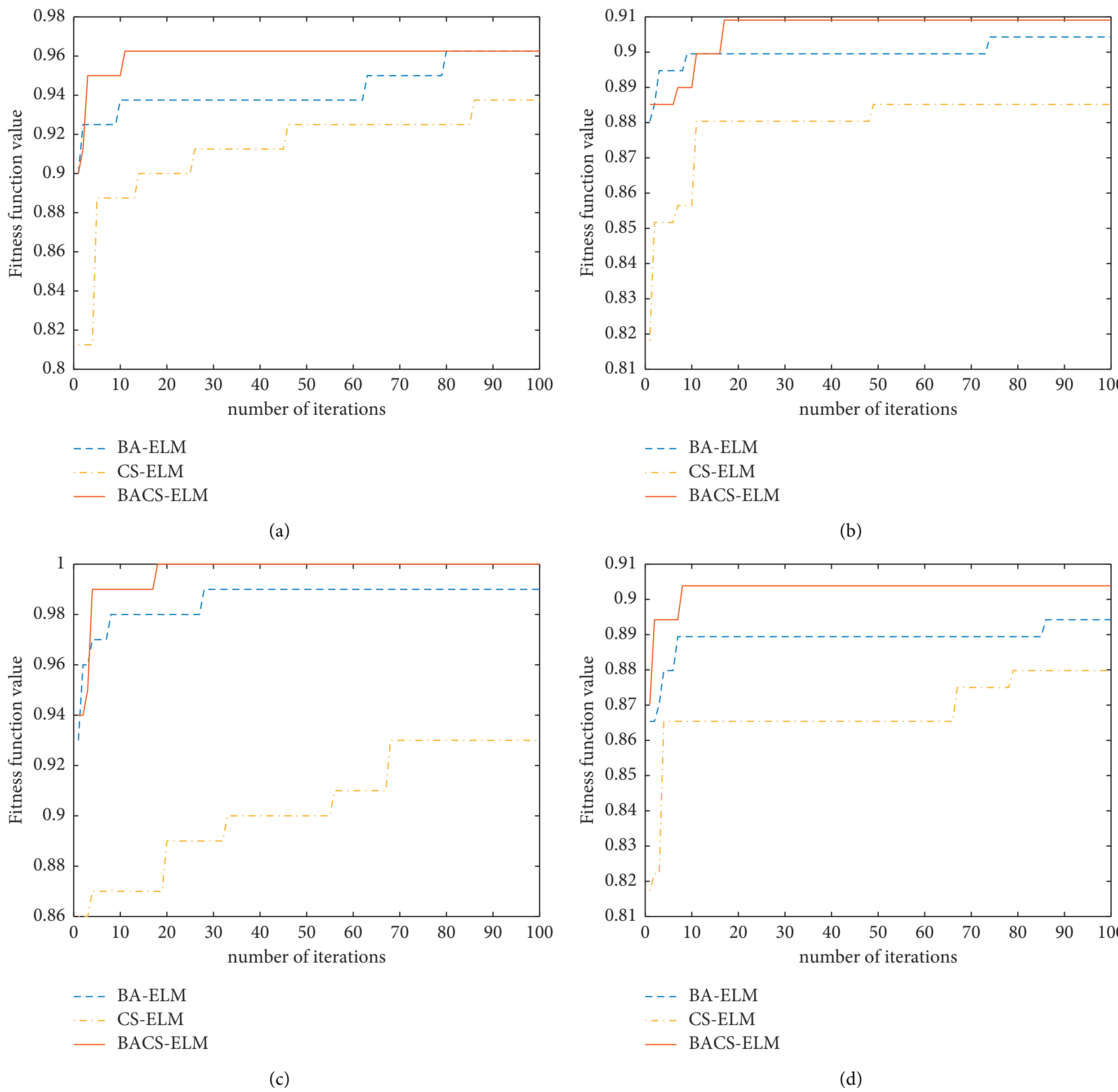

Figure 4: Comparison of fitness function curves on different datasets: (a) breast cancer; (b) heart failure; (c) iris; (d) vertebral column.

experiment. It can be clearly seen from Table 7 that the BACS – ELM algorithm can achieve the best test accuracy under the minimum number of hidden nodes in all the four datasets, which indicates that the algorithm can effectively optimize the parameters of the hidden layer of the ELM model by using BACS algorithm and then obtain a more appropriate and simplified network structure. At the same time, the best generalization performance and classification ability are obtained. In terms of computing time or efficiency, hidden layer parameters of ELM do not need to be iteratively tuned, so the learning speed is very fast, but the success rate of its classification is very low. In Table 7, we did not list the test time data because the values of the four algorithms for different datasets in the experimental results are very low, and the size is similar; that is to say, the impact of the data on the overall experiment results cannot be regarded as an evaluation item. Compared with the other two optimization methods, although the BACS – ELM algorithm is slightly worse in learning efficiency, it shows great advantages in classification accuracy.

Table 7: Performance comparison for classification problems.

Datasets	Algorithm	Training accuracy (%)	Testing accuracy (%)	Number of hidden nodes	Train time (s)
Breast cancer	ELML	84.05	80.05	25	0.0046
	BA – ELM	87.65	83.27	20	7.8443
	CS – ELM	86.00	82.44	20	7.2638
	BACS – ELM	89.27	84.23	20	9.2870
Heart failure	ELML	82.26	79.04	20	0.0069
	BA – ELM	85.64	80.25	20	9.0520
	CS – ELM	84.35	80.02	20	8.1702
	BACS – ELM	86.29	81.00	20	10.236
Iris	ELML	96.79	95.10	15	0.0062
	BA – ELM	97.90	95.58	15	5.2547
	CS – ELM	97.00	95.89	15	5.8250
	BACS – ELM	98.40	96.40	10	6.7234
Vertebral column	ELML	85.33	82.74	35	0.0078
	BA – ELM	86.87	83.64	30	10.568
	CS – ELM	86.82	83.42	35	11.534
	BACS – ELM	88.07	84.50	25	13.939

6. Conclusions

In this paper, we propose a hybrid Extreme Learning Machine algorithm based on the bat and cuckoo search algorithm to optimize the input weight and threshold of the traditional ELM algorithm, thus improving the disadvantages of traditional ELM, such as poor sparsity of hidden layer, low adjustment ability, and complex network structure. Meanwhile, the BACS algorithm has the characteristics of strong searching accuracy, fast convergence speed, and not easy to fall into the local optimal, which effectively balances the local and global optimization problems. Therefore, the proposed BACS-ELM algorithm can effectively solve the optimization problem due to the randomness of hidden layer parameters and improve the generalization performance of the network.

Experimental results show that the BACS-ELM algorithm is superior to other algorithms in function fitting and classification. In the future, we consider extending the BACS-ELM algorithm to practical application problems and solving a wider class of even tougher optimization problems.

Data Availability

All data included in this study are available upon request by contact with the corresponding author.

Disclosure

This manuscript is the authors' original work and has not been published nor has it been submitted simultaneously elsewhere.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Analysis of Subgradient Extragradient Iterative Schemes for Variational Inequalities

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In this paper, we investigate the monotone variational inequality in Hilbert spaces. Based on Censor's subgradient extragradient method, we propose two modified subgradient extragradient algorithms with self-adaptive and inertial techniques for finding the solution of the monotone variational inequality in real Hilbert spaces. Strong convergence analysis of the proposed algorithms have been obtained under some mild conditions.

1. Introduction

Let H be a real Hilbert space and $S \in H$ be a nonempty closed convex subset. Let $f: H \longrightarrow H$ be an operator. In this work, we investigate the following variational inequality problem (VIPs):

$$\text{find a point } u^{\ddagger} \in S, \quad \text{s.t.} \langle f(u^{\ddagger}), x - u^{\ddagger} \rangle \geq 0, \ \forall x \in S. \tag{1}$$

Denote by $\mathrm{Sol}(S, f)$ the solution set of (1). The VIPs is an important tool to study various problems in the domain of mechanics, optimization, transportation, fixed point, economics equilibrium, contract problems in elasticity, and other branches of mathematics, see [1–17]. Therefore, VIPs have received much attention by many scholars, see [18–30]. There are a variety of methods to solve the VIPs, such as regularization method and projection method [31–39]. In this work, we focus on projection method.

As we all know that the gradient projection algorithm is the simplest and oldest method ([40, 41]), the method is defined as follows:

$$u^{k+1} = P_S(u^k - \gamma f(u^k)), \tag{2}$$

where $P_{\mathscr{S}}: H \longrightarrow S$ is the metric projection and γ is some positive number.

In order to obtain a convergent result, this algorithm requires that the operator f is strongly monotone. In order to avoid the strong monotonicity hypothesis, Korpelevich [42] proposed the extragradient algorithm which is stated as follows:

$$\begin{cases} x^k = P_S(u^k - \gamma f(u^k)), \\ u^{k+1} = P_S(u^k - \gamma f(x^k)), \end{cases} \tag{3}$$

where $\gamma \in (0, (1/L))$ and operator f is monotone and Lipschitz continuous in H.

Note that the algorithm (3) calculates two projections on S in each iteration. If the set S is more complicated, there will be a lot of calculations. In order to overcome this difficulty, Censor et al. [43] constructed a half space with subdifferentiation and proposed subgradient extragradient method which is defined by

$$\begin{cases} x^k = P_S\left(u^k - \gamma f\left(u^k\right)\right), \\ T^k = \left\{z \in H | \langle u^k - \gamma f\left(u^k\right) - x^k, z - x^k \rangle \le 0\right\}, \\ u^{k+1} = P_{T^k}\left(u^k - \gamma f\left(x^k\right)\right). \end{cases} \tag{4}$$

Recently, Dong et al. [44] improved the algorithm (4) with self-adaptive stepsize which generates a sequence $\{u^k\}$ by the following form:

$$\begin{cases} x^k = P_{\mathcal{S}}\left(u^k - \gamma^k f\left(u^k\right)\right), \\ \gamma^k \left\| f\left(u^k\right) - f\left(x^k\right) \right\| \le \sigma \left\| u^k - x^k \right\|, \quad \forall \sigma \in (0,1), \\ T^k = \left\{z \in H \langle u^k - \gamma^k f\left(u^k\right) - x^k, z - x^k \rangle \le 0\right\}, \\ u^{k+1} = P_{T^k}\left(u^k - \tau \zeta^k \gamma^k f\left(x^k\right)\right), \end{cases} \tag{5}$$

where $\zeta^k = (\langle u^k - x^k, \kappa(u^k, x^k)\rangle / \|\kappa(u^k, x^k)\|^2)$ and $\kappa(u^k, x^k) = (u^k - x^k) - \gamma^k (f(u^k) - f(x^k))$.

Weak convergence of Algorithm (5) has been obtained. Motivated and inspired by the above work, in this paper, we continue to investigate iterative algorithms for solving the monotone variational inequality in Hilbert spaces. We construct two modified subgradient extragradient algorithms for finding the solution of the monotone variational inequality. Our algorithms combine self-adaptive technique and inertial method. Under some mild conditions, we prove that the proposed algorithms converge strongly to a solution of the monotone variational inequality.

The organizational structure of this paper is as follows. In Section 2, we present some definitions and preliminary results, which will be used in further analysis of the proposed algorithms. In Section 3, we proposed two modified subgradient extragradient algorithms and prove strong convergence theorems.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Use "$\rightharpoonup$" and "$\longrightarrow$" to denote weak and strong convergence, respectively. Let $\{x^k\}$ be a sequence in H. We use $\omega_w(x^k)$ to denote the set of all weak cluster points of $\{x^k\}$, i.e.,

$$\omega_w\left(x^k\right) = \left\{x^\dagger \colon \exists \left\{x^{k_i}\right\} \subset \left\{x^k\right\} \text{ such that } x^{k_i} \rightharpoonup x^\dagger \text{ as } i \longrightarrow \infty\right\}. \tag{6}$$

For $\forall u, v \in H$, and $\lambda \in R$, the following results hold

$$\|u + v\|^2 \le \|u\|^2 + 2\langle v, u + v\rangle, \tag{7}$$

$$\|\lambda u + (1-\lambda)v\|^2 = \lambda\|u\|^2 + (1-\lambda)\|v\|^2 - \lambda(1-\lambda)\|u - v\|^2. \tag{8}$$

Definition 1. Let $f \colon H \longrightarrow H$ be an operator. Recall that the operator f is said to be

(i) Monotone if

$$\langle f(u) - f(v), u - v\rangle \ge 0, \quad \forall u, v \in H. \tag{9}$$

(ii) Strongly monotone if there exists $\gamma > 0$ s.t.

$$\langle f(u) - f(v), u - v\rangle \ge \gamma\|u - v\|^2, \quad \forall u, v \in H. \tag{10}$$

(iii) L-Lipschitz continuous if there exists $L > 0$ s.t.

$$\|f(u) - f(v)\| \le L\|u - v\|, \quad \forall u, v \in H. \tag{11}$$

If $L < 1$, f is said to be L-contractive.

Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique point $P_C(x) \in C$ such that

$$\left\|x - P_C(x)\right\| \le \|y - x\|, \quad \forall y \in C. \tag{12}$$

It is well known that P_C satisfies [45]

$$\langle x - P_C(x), y - P_C(x)\rangle \le 0, \tag{13}$$

$$\left\|x - P_C(x)\right\|^2 + \left\|y - P_C(x)\right\|^2 \le \|x - y\|^2, \tag{14}$$

for all $x \in H$ and $y \in C$.

Lemma 1 (see [46]). *Let $\{b^k\}$ be a real number sequence. Suppose that there exists a subsequence $\{b^{k_m}\}$ of $\{b^k\}$ such that $b^{k_m} < b^{k_m+1}$ for all $m \in \mathbb{N}$. Define the sequence $\{\gamma(k)\}$ as follows:*

$$\gamma(k) = \max\left\{n \in \mathbb{N} | \overline{k}_0 \le n \le k, b^n \le b^{n+1}\right\}, \tag{15}$$

for each $k \ge \overline{k}_0$. Then, the following inequality holds:

$$0 \le b^k \le b^{\gamma(k)+1}, \tag{16}$$

for each $k \ge \overline{k}_0$. Further, for all $k \ge \overline{k}_0$, the sequence $\{\gamma(k)\}$ is nondecreasing and

$$\lim_{k \longrightarrow \infty} \gamma(k) = +\infty. \tag{17}$$

Lemma 2 (see [33]). *Suppose that the sequence $\left\{\delta^k\right\}$ of real numbers is nonnegative and there exists $k_0 \in \mathbb{N}$ such that*

$$\delta^{k+1} \le \left(1 - \gamma^k\right)\delta^k + \gamma^k l^k, \tag{18}$$

for each $k \ge k_0$, where the sequences $\{\gamma^k\}$ and $\{l^k\}$ satisfy the following conditions:

$$\begin{gathered} \left\{\gamma^k\right\} \subset (0,1), \\ \lim_{k \longrightarrow \infty} \gamma^k = 0, \\ \sum_{k=1}^{\infty} \gamma^k = \infty, \\ \limsup_{k \longrightarrow \infty} l^k \le 0. \end{gathered} \tag{19}$$

Then, $\lim_{k \longrightarrow \infty} \delta^k = 0$.

3. Main Result

In this section, we present our main results.

Let S be a nonempty closed convex subset of a real Hilbert space H. Suppose that the following three conditions are satisfied:

(C1): the set $\mathrm{Sol}(S, f)$ is not empty;

(C2): the operate f is monotone;

(C3): the operate f is L-Lipschitz continuous.

Let $\sigma, \rho \in (0,1)$, $\tau \in (0,2)$, and $\gamma^0 > 0$ be four constants. Let $\{\theta^k\}, \{\varepsilon^k\} \subset (0,1)$, and $\{\lambda^k\} \subset [a,b] \subset (0,1)$ be three sequences, satisfying

$$\begin{aligned} \sum_{n=1}^{\infty} \theta^k &= +\infty, \\ \lim_{k\longrightarrow\infty} \theta^k &= 0, \\ \varepsilon^k &= o(\theta^k). \end{aligned} \tag{20}$$

Next, we introduce an iterative algorithm for solving (1).

Lemma 3. *If $x^k = v^k$ or $\kappa(v^k, x^k) = 0$ in Algorithm 1, then $x^k \in Sol(S, f)$.*

Proof. Since f is L-Lipschitz continuous, we obtain

$$\begin{aligned} \|\kappa(v^k, x^k)\| &= \|v^k - x^k - \gamma^k(f(v^k) - f(x^k))\| \\ &\geq \|v^k - x^k\| - \gamma^k\|f(v^k) - f(x^k)\| \\ &\geq \|v^k - x^k\| - \gamma^k L\|v^k - x^k\| = (1 - \gamma^k L)\|v^k - x^k\|, \\ \|\kappa(v^k, x^k)\| &= \|v^k - x^k - \gamma^k(f(v^k) - f(x^k))\| \\ &\leq \|v^k - x^k\| + \gamma^k\|f(v^k) - f(x^k)\| \\ &\leq \|v^k - x^k\| + \gamma^k L\|v^k - x^k\| = (1 + \gamma^k L)\|v^k - x^k\|. \end{aligned} \tag{21}$$

It follows that

$$(1 - \gamma^k L)\|v^k - x^k\| \leq \|\kappa(v^k, x^k)\| \leq (1 + \gamma^k L)\|v^k - x^k\|. \tag{22}$$

Consequently, $v^k = x^k \Leftrightarrow \kappa(v^k, x^k) = 0$. Furthermore, if $v^k = x^k$ or $\kappa(v^k, x^k) = 0$, we have

$$x^k = P_S(x^k - \gamma^k f(x^k)). \tag{23}$$

Combining (13) and (23), we get

$$\langle x^k - \gamma^k f(x^k) - x^k, x^k - z\rangle \geq 0, \quad \forall z \in S, \tag{24}$$

which implies that

$$\langle f(x^k), z - x^k\rangle \geq 0, \quad \forall z \in S. \tag{25}$$

This completes the proof. □

Lemma 4. *The sequence $\{\gamma^k\}_{k=0}^{\infty}$ generated by Algorithm 1 is monotonically decreasing, and $\gamma^k \leq \min\{\gamma^0, (\sigma/L)\}$ for each $k \geq 0$.*

Proof. Obviously, by the definition of $\{\gamma^{k+1}\}$, we have $\{\gamma^k\}$ is a monotonically decreasing sequence. Then, $\gamma^k \geq \gamma^0, \forall n > 0$. Since f is Lipschitz continuous, we have

$$\|f(u^k) - f(x^k)\| \leq L\|u^k - x^k\|. \tag{26}$$

In the case of $f(u^k) \neq f(x^k)$, we have

$$\frac{\sigma\|u^k - x^k\|}{\|f(u^k) - f(x^k)\|} \geq \frac{\sigma}{L}. \tag{27}$$

Obviously, the lower bound of $\{\gamma^k\}$ is $\min\{\gamma^0, (\sigma/L)\}$. This completes the proof. □

Lemma 5. *Let $\{\zeta^k\}$ be the sequence generated by Algorithm 1. Then, we have*

$$\zeta^k \geq \frac{1-\sigma}{1+\sigma^2}. \tag{28}$$

Proof. Combining Lemma 4 and Cauchy–Schwartz inequality, we have

$$\begin{aligned} \langle v^k - x^k, \kappa(v^k, x^k)\rangle &= \langle v^k - x^k, v^k - x^k - \gamma^k(f(v^k) - f(x^k))\rangle \\ &= \|v^k - x^k\|^2 - \gamma^k\langle v^k - x^k, f(v^k) - f(x^k)\rangle \\ &\geq \|v^k - x^k\|^2 - \gamma^k\|v^k - x^k\|\|f(v^k) - f(x^k)\| \\ &\geq \|v^k - x^k\|^2 - \gamma^k L\|v^k - x^k\|^2 \\ &= (1 - \gamma^k L)\|v^k - x^k\|^2 \\ &\geq (1-\sigma)\|v^k - x^k\|^2. \end{aligned} \tag{29}$$

Since f is monotone and Lipschitz continuous, then we obtain

$$\begin{aligned} \|\kappa(v^k, x^k)\|^2 &= \|v^k - x^k - \gamma^k(f(v^k) - f(x^k))\|^2 \\ &= \|v^k - x^k\|^2 + (\gamma^k)^2\|f(v^k) - f(x^k)\|^2 \\ &\quad - 2\gamma^k\langle v^k - x^k, f(v^k) - f(x^k)\rangle \\ &\leq \|v^k - x^k\|^2 + (\gamma^k)^2 L^2\|v^k - x^k\|^2 \\ &= \left(1 + (\gamma^k)^2 L^2\right)\|v^k - x^k\|^2 \\ &\leq (1+\sigma^2)\|v^k - x^k\|^2. \end{aligned} \tag{30}$$

From (29) and (30), we have

$$\zeta^k = \frac{\langle v^k - x^k, \kappa(v^k, x^k)\rangle}{\|\kappa(v^k, x^k)\|^2} \geq \frac{1-\sigma}{1+\sigma^2}. \tag{31}$$

This completes the proof. □

Lemma 6. *Let $u^\ddagger \in Sol(S, f)$. Then,*

$$\left\|t^k - u^\ddagger\right\|^2 \leq \left\|v^k - u^\ddagger\right\|^2 - \left\|\left(v^k - t^k\right) - \tau\zeta^k \kappa\left(v^k, x^k\right)\right\|^2 - \tau(2-\tau)\left(\zeta^k\right)^2\left\|\kappa\left(v^k, x^k\right)\right\|^2. \quad (32)$$

Proof. From (13) and Algorithm 1, we have

$$\begin{aligned}\left\|t^k - u^\ddagger\right\|^2 &\leq \left\|P_{T^k}\left(v^k - \tau\zeta^k\gamma^k f\left(x^k\right)\right) - P_{T^k}u^\ddagger\right\|^2 \\ &\leq \langle t^k - u^\ddagger, v^k - \tau\zeta^k\gamma^k f\left(x^k\right) - u^\ddagger\rangle \\ &= \frac{1}{2}\left\|t^k - u^\ddagger\right\|^2 + \frac{1}{2}\left\|v^k - \tau\zeta^k\gamma^k f\left(x^k\right) - u^\ddagger\right\|^2 \\ &\quad - \frac{1}{2}\left\|t^k - v^k + \tau\zeta^k\gamma^k f\left(x^k\right)\right\|^2 \\ &= \frac{1}{2}\left\|t^k - u^\ddagger\right\|^2 + \frac{1}{2}\left\|v^k - u^\ddagger\right\|^2 \\ &\quad + \frac{1}{2}\tau^2\left(\zeta^k\right)^2\left(\gamma^k\right)^2\left\|f\left(x^k\right)\right\|^2 \\ &\quad - \langle v^k - u^\ddagger, \tau\zeta^k\gamma^k f\left(x^k\right)\rangle - \frac{1}{2}\left\|t^k - v^k\right\|^2 \\ &\quad - \frac{1}{2}\tau^2\left(\zeta^k\right)^2\left(\gamma^k\right)^2\left\|f\left(x^k\right)\right\|^2 - \langle t^k - v^k, \tau\zeta^k\gamma^k f\left(x^k\right)\rangle \\ &= \frac{1}{2}\left\|t^k - u^\ddagger\right\|^2 + \frac{1}{2}\left\|v^k - u^\ddagger\right\|^2 - \frac{1}{2}\left\|t^k - v^k\right\|^2 \\ &\quad - \langle t^k - u^\ddagger, \tau\zeta^k\gamma^k f\left(x^k\right)\rangle.\end{aligned} \quad (33)$$

It follows that

$$2\left\|t^k - u^\ddagger\right\|^2 \leq \left\|t^k - u^\ddagger\right\|^2 + \left\|v^k - u^\ddagger\right\|^2 - \left\|t^k - v^k\right\|^2 - 2\tau\zeta^k\gamma^k\langle t^k - u^\ddagger, f\left(x^k\right)\rangle, \quad (34)$$

or equivalently

$$\left\|t^k - u^\ddagger\right\|^2 \leq \left\|v^k - u^\ddagger\right\|^2 - \left\|t^k - v^k\right\|^2 - 2\tau\zeta^k\gamma^k\langle t^k - u^\ddagger, f\left(x^k\right)\rangle. \quad (35)$$

We deduce from $x^k \in C$ and $u^\ddagger \in \mathrm{Sol}(S, f)$ that $\langle f(u^\ddagger), x^k - u^\ddagger\rangle \geq 0$. It follows from the monotonicity of operator f that $\langle f(x^k) - f(u^\ddagger), x^k - u^\ddagger\rangle \geq 0$. Then, $\langle f(x^k), x^k - u^\ddagger\rangle \geq 0$. It equates that $\langle f(x^k), t^k - u^\ddagger\rangle \geq \langle f(x^k), t^k - x^k\rangle$. Thus,

$$-2\tau\zeta^k\gamma^k\langle f\left(x^k\right), t^k - u^\ddagger\rangle \leq -2\tau\zeta^k\gamma^k\langle f\left(x^k\right), t^k - x^k\rangle. \quad (36)$$

On the other hand, combining $t^k \in T^k$ and Algorithm 1, we obtain

$$\langle\kappa\left(v^k, x^k\right), t^k - x^k\rangle \leq \gamma^k\langle f\left(x^k\right), t^k - x^k\rangle. \quad (37)$$

This implies that

$$-2\tau\zeta^k\gamma^k\langle f\left(x^k\right), t^k - u^\ddagger\rangle \leq -2\tau\zeta^k\langle\kappa\left(v^k, x^k\right), t^k - x^k\rangle. \quad (38)$$

Hence, we obtain

$$\begin{aligned}-2\tau\zeta^k\gamma^k\langle f\left(x^k\right), t^k - u^\ddagger\rangle &\leq -2\tau\zeta^k\langle\kappa\left(v^k, x^k\right), t^k - x^k\rangle \\ &\leq -2\tau\zeta^k\langle\kappa\left(v^k, x^k\right), v^k - x^k\rangle \\ &\quad + 2\tau\zeta^k\langle\kappa\left(v^k, x^k\right), v^k - t^k\rangle.\end{aligned} \quad (39)$$

Now, we calculate $-2\tau\zeta^k\langle\kappa(v^k, x^k), v^k - x^k\rangle$ and $2\tau\zeta^k\langle\kappa(v^k, x^k), v^k - t^k\rangle$ separately. From the definition of $\{\zeta^k\}$, we get

$$-2\tau\zeta^k\langle\kappa\left(v^k, x^k\right), v^k - x^k\rangle = -2\tau\left(\zeta^k\right)^2\left\|\kappa\left(v^k, x^k\right)\right\|^2. \quad (40)$$

Meanwhile,

$$2\tau\zeta^k\langle\kappa\left(v^k, x^k\right), v^k - t^k\rangle = \left\|v^k - t^k\right\|^2 + \tau^2\left(\zeta^k\right)^2\left\|\kappa\left(v^k, x^k\right)\right\|^2 - \left\|v^k - t^k - \tau\zeta^k\kappa\left(v^k, x^k\right)\right\|^2. \quad (41)$$

This implies that

$$\begin{aligned}-2\tau\zeta^k\gamma^k\langle f\left(x^k\right), t^k - u^\ddagger\rangle &\leq -2\tau\left(\zeta^k\right)^2\left\|\kappa\left(v^k, x^k\right)\right\|^2 \\ &\quad + \left\|v^k - t^k\right\|^2 + \tau^2\left(\zeta^k\right)^2\left\|\kappa\left(v^k, x^k\right)\right\|^2 \\ - \left\|v^k - t^k - \tau\zeta^k\kappa\left(v^k, x^k\right)\right\|^2 &= \left\|v^k - t^k\right\|^2 - \left\|v^k - t^k - \tau\zeta^k\kappa\left(v^k, x^k\right)\right\|^2 \\ &\quad - \tau(2-\tau)\left(\zeta^k\right)^2\left\|\kappa\left(v^k, x^k\right)\right\|^2.\end{aligned} \quad (42)$$

So, we get

$$\left\|t^k - u^\ddagger\right\|^2 \leq \left\|v^k - u^\ddagger\right\|^2 - \left\|\left(v^k - t^k\right) - \tau\zeta^k\kappa\left(v^k, x^k\right)\right\|^2 - \tau(2-\tau)\left(\zeta^k\right)^2\left\|\kappa\left(v^k, x^k\right)\right\|^2. \quad (43)$$

This completes the proof. □

Theorem 1. *The sequence $\{u^k\}$ generated by Algorithm 1 converges strongly to $u^\ddagger \in Sol(S, f)$.*

Proof. We divide the proof into four claims.

Claim 1. We prove the boundedness of the sequences $\{u^k\}$ and $\{t^k\}$. Indeed, from Algorithm 1 and Lemma 6, we get

$$\begin{aligned}\left\|u^{k+1} - u^\ddagger\right\| &= \left\|\left(1 - \theta^k - \lambda^k\right)v^k + \lambda^k t^k - u^\ddagger\right\| \\ &= \left\|\left(1 - \theta^k - \lambda^k\right)\left(v^k - u^\ddagger\right) + \lambda^k\left(t^k - u^\ddagger\right) - \theta^k u^\ddagger\right\| \\ &\leq \left\|\left(1 - \theta^k - \lambda^k\right)\left(v^k - u^\ddagger\right) + \lambda^k\left(t^k - u^\ddagger\right)\right\| + \theta^k\left\|u^\ddagger\right\| \\ &\leq \left(1 - \theta^k - \lambda^k\right)\left\|v^k - u^\ddagger\right\| + \lambda^k\left\|t^k - u^\ddagger\right\| + \theta^k\left\|u^\ddagger\right\|\end{aligned}$$

Initialization. Choose $u^0, u^1 \in H$ arbitrarily.
Step 1. Choose ρ^k s.t. $0 \le \rho^k \le \bar{\rho}^k$, where $\bar{\rho}^k = \begin{cases} \min\{\rho, (\varepsilon^k/\|u^k - u^{k-1}\|)\}, & \text{if } u^k \ne u^{k-1} \\ \rho, & \text{otherwise} \end{cases}$
Calculate $v^k = u^k + \rho^k (u^k - u^{k-1})$
Step 2. Calculate $x^k = P_S(v^k - \gamma^k f(v^k))$,
where, $\gamma^{k+1} = \begin{cases} \min\{\gamma^k, (\sigma\|u^k - x^k\|/\|f(u^k) - f(x^k)\|)\}, & \text{if } f(u^k) \ne f(x^k) \\ \gamma^k, & \text{otherwise} \end{cases}$
Step 3. Construct the half space T^k as follows $T^k = \{z \in H | \langle v^k - \gamma^k f(v^k) - x^k, z - x^k \rangle \le 0\}$
Calculate $t^k = P_{T^k}(v^k - \tau\zeta^k\gamma^k f(x^k))$
where $\zeta^k = (\langle v^k - x^k, \kappa(v^k, x^k)\rangle / \|\kappa(v^k, x^k)\|^2)$
and $\kappa(v^k, x^k) = v^k - x^k - \gamma^k (f(v^k) - f(x^k))$
Step 4. Compute $u^{k+1} = (1 - \theta^k - \lambda^k)v^k + \lambda^k t^k$
If $x^k = v^k$, then stop and $x^k \in \mathrm{Sol}(S, f)$. Otherwise, set $k: = k + 1$ and return to step 1.

Algorithm 1: Strong convergence algorithm with contractive technique.

$$\begin{aligned} &\le (1 - \theta^k)\|v^k - u^\ddagger\| - \lambda^k\|v^k - u^\ddagger\| + \lambda^k\|v^k - u^\ddagger\| + \theta^k\|u^\ddagger\| \\ &= (1 - \theta^k)\|v^k - u^\ddagger\| + \theta^k\|u^\ddagger\|. \end{aligned} \tag{44}$$

Combining Algorithm 1 and (44), we obtain

$$\begin{aligned} \|u^{k+1} - u^\ddagger\| &= (1 - \theta^k)\|v^k - u^\ddagger\| + \theta^k\|u^\ddagger\| \\ &= (1 - \theta^k)\|u^k + \rho^k(u^k - u^{k-1}) - u^\ddagger\| + \theta^k\|u^\ddagger\| \\ &\le (1 - \theta^k)\|u^k - u^\ddagger\| + \rho^k(1 - \theta^k) \\ &\quad \|u^k - u^{k-1}\| + \theta^k\|u^\ddagger\| \\ &= (1 - \theta^k)\|u^k - u^\ddagger\| + \theta^k(\varsigma^k + \|u^\ddagger\|), \end{aligned} \tag{45}$$

where

$$\varsigma^k = (1 - \theta^k)\frac{\rho^k}{\theta^k}\|u^k - u^{k-1}\|. \tag{46}$$

Taking into account $\varepsilon^k = o(\theta^k)$ and the definition of ρ^k, we get

$$\lim_{k \longrightarrow \infty} \varsigma^k = 0. \tag{47}$$

Then, the sequence $\{\varsigma^k\}$ is bounded. Let $M = \sup_{k \ge 1}(\varsigma^k + \|u^\ddagger\|)$. We obtain from (45) that

$$\begin{aligned} \|u^{k+1} - u^\ddagger\| &\le (1 - \theta^k)\|u^k - u^\ddagger\| + \theta^k M \\ &\le \max\{\|u^k - u^\ddagger\|, M\}. \end{aligned} \tag{48}$$

For $\forall k \ge k_0$, we have

$$\|u^{k+1} - u^\ddagger\| \le \max\{\|u^k - u^\ddagger\|, M\}. \tag{49}$$

It follows that the sequence $\{u^k\}$ is bounded. Therefore, the sequence $\{t^k\}$ is bounded.

Claim 2. We prove that the following holds:

$$\begin{aligned} \|u^{k+1} - u^\ddagger\|^2 \le (1 - \theta^k)^2\|v^k - u^\ddagger\|^2 - 2\lambda^k\theta^k \\ \langle v^k - t^k, u^{k+1} - u^\ddagger\rangle + 2\theta^k\langle u^\ddagger, u^{k+1} - u^\ddagger\rangle. \end{aligned} \tag{50}$$

Set $z^k = (1 - \lambda^k)v^k + \lambda^k t^k$. Then, $v^k - z^k = \lambda^k(v^k - t^k)$. Therefore,

$$\begin{aligned} u^{k+1} &= (1 - \theta^k - \lambda^k)v^k + \lambda^k t^k = z^k - \theta^k v^k \\ &= (1 - \theta^k)z^k - \theta^k(v^k - z^k) \\ &= (1 - \theta^k)z^k - \theta^k\lambda^k(v^k - t^k). \end{aligned} \tag{51}$$

From Lemma 6, we have

$$\|t^k - u^\ddagger\| \le \|v^k - u^\ddagger\|, \tag{52}$$

which implies that

$$\|z^k - u^\ddagger\|^2 \le (1 - \lambda^k)\|v^k - u^\ddagger\|^2 + \lambda^k\|t^k - u^\ddagger\|^2 \le \|v^k - u^\ddagger\|^2. \tag{53}$$

By (7), (51), and (53), we get

$$\begin{aligned} \|u^{k+1} - u^\ddagger\|^2 &= \|(1 - \theta^k)z^k - \theta^k\lambda^k(v^k - t^k) - u^\ddagger\|^2 \\ &= \|(1 - \theta^k)(z^k - u^\ddagger) - \theta^k\lambda^k(v^k - t^k) - \theta^k u^\ddagger\|^2 \\ &\le (1 - \theta^k)^2\|z^k - u^\ddagger\|^2 - 2\theta^k\lambda^k\langle v^k - t^k, u^{k+1} - u^\ddagger\rangle - 2\theta^k\langle u^\ddagger, u^{k+1} - u^\ddagger\rangle \\ &\le (1 - \theta^k)^2\|v^k - u^\ddagger\|^2 - 2\theta^k\lambda^k\langle v^k - t^k, u^{k+1} - u^\ddagger\rangle - 2\theta^k\langle u^\ddagger, u^{k+1} - u^\ddagger\rangle \\ &= (1 - \theta^k)^2\|v^k - u^\ddagger\|^2 - 2\theta^k\lambda^k\langle v^k - t^k, u^{k+1} - u^\ddagger\rangle + 2\theta^k\langle - u^\ddagger, u^{k+1} - u^\ddagger\rangle. \end{aligned} \tag{54}$$

Claim 3. By (8) and Algorithm 1, we obtain

$$\begin{aligned}\left\|v^k-u^\ddagger\right\|^2 &= \left\|u^k+\rho^k\left(u^k-u^{k-1}\right)-u^\ddagger\right\|^2\\ &= \left\|\left(1+\rho^k\right)\left(u^k-u^\ddagger\right)-\rho^k\left(u^{k-1}-u^\ddagger\right)\right\|^2\\ &= \left(1+\rho^k\right)\left\|u^k-u^\ddagger\right\|^2-\rho^k\left\|u^{k-1}-u^\ddagger\right\|^2+\rho^k\left(1+\rho^k\right)\left\|u^k-u^{k-1}\right\|^2\\ &\le \left(1+\rho^k\right)\left\|u^k-u^\ddagger\right\|^2-\rho^k\left\|u^{k-1}-u^\ddagger\right\|^2+2\rho^k\left\|u^k-u^{k-1}\right\|^2\\ &= \left\|u^k-u^\ddagger\right\|^2+\rho^k\left(\left\|u^k-u^\ddagger\right\|^2-\left\|u^{k-1}-u^\ddagger\right\|^2\right)+2\rho^k\left\|u^k-u^{k-1}\right\|^2.\end{aligned} \tag{55}$$

Using Lemma (8) and (52), we get

$$\begin{aligned}\left\|u^{k+1}-u^\ddagger\right\|^2 &= \left\|\left(1-\theta^k-\lambda^k\right)\left(v^k-u^\ddagger\right)+\lambda^k\left(t^k-u^\ddagger\right)+\theta^k\left(-u^\ddagger\right)\right\|^2\\ &\le \left(1-\theta^k-\lambda^k\right)\left\|v^k-u^\ddagger\right\|^2+\lambda^k\left\|t^k-u^\ddagger\right\|^2+\theta^k\left\|u^\ddagger\right\|^2\\ &\le \left(1-\theta^k-\lambda^k\right)\left\|v^k-u^\ddagger\right\|^2+\theta^k\left\|u^\ddagger\right\|^2\\ &\quad+\lambda^k\left(\left\|v^k-u^\ddagger\right\|^2-\left\|v^k-t^k-\tau\zeta^k\kappa\left(v^k,x^k\right)\right\|^2\right.\\ &\quad\left.-\tau(2-\tau)\left(\zeta^k\right)^2\left\|\kappa\left(v^k,x^k\right)\right\|^2\right)\\ &\le \left(1-\theta^k\right)\left\|v^k-u^\ddagger\right\|^2+\theta^k\left\|u^\ddagger\right\|^2-\lambda^k\left\|v^k-t^k-\tau\zeta^k\kappa\left(v^k,x^k\right)\right\|^2\\ &\quad-\lambda^k\tau(2-\tau)\left(\zeta^k\right)^2\left\|\kappa\left(v^k,x^k\right)\right\|^2\Big).\end{aligned} \tag{56}$$

From (55) and (56), we get

$$\begin{aligned}\left\|u^{k+1}-u^\ddagger\right\|^2 &\le \left(1-\theta^k\right)\left\|u^k+\rho^k\left(u^k-u^{k-1}\right)-u^\ddagger\right\|^2+\theta^k\left\|u^\ddagger\right\|^2\\ &\quad-\lambda^k\left\|v^k-t^k-\tau\zeta^k\kappa\left(v^k,x^k\right)\right\|^2-\lambda^k\tau(2-\tau)\left(\zeta^k\right)^2\left\|\kappa\left(v^k,x^k\right)\right\|^2\\ &\le \left(1-\theta^k\right)\left\|u^k-u^\ddagger\right\|^2+\rho^k\left(1-\theta^k\right)\left(\left\|u^k-u^\ddagger\right\|^2-\left\|u^{k-1}-u^\ddagger\right\|^2\right)\\ &\quad+2\rho^k\left(1-\theta^k\right)\left\|u^k-u^{k-1}\right\|^2+\theta^k\left\|u^\ddagger\right\|^2\\ &\quad-\lambda^k\left\|v^k-t^k-\tau\zeta^k\kappa\left(v^k,x^k\right)\right\|^2-\lambda^k\tau(2-\tau)\left(\zeta^k\right)^2\left\|\kappa\left(v^k,x^k\right)\right\|^2\\ &\le \left\|u^k-u^\ddagger\right\|^2+\rho^k\left(1-\theta^k\right)\left(\left\|u^k-u^\ddagger\right\|^2-\left\|u^{k-1}-u^\ddagger\right\|^2\right)\\ &\quad+2\rho^k\left(1-\theta^k\right)\left\|u^k-u^{k-1}\right\|^2+\theta^k\left\|u^\ddagger\right\|^2\\ &\quad-\lambda^k\left\|v^k-t^k-\tau\zeta^k\kappa\left(v^k,x^k\right)\right\|^2-\lambda^k\tau(2-\tau)\left(\zeta^k\right)^2\left\|\kappa\left(v^k,x^k\right)\right\|^2.\end{aligned} \tag{57}$$

Claim 4. Next, we will consider two different cases to prove the strong convergence of the sequence $\left\{\|u^k-u^\ddagger\|^2\right\}$.

Case 1. There exists an $N\in\mathbb{N}$ s.t. $\|u^{k+1}-u^\ddagger\|^2\le\|u^k-u^\ddagger\|^2, \forall k\ge N$. Obviously, the limit of the sequence $\left\{\|u^k-u^\ddagger\|^2\right\}$ exists which implies that

$\lim_{k\longrightarrow\infty}\|u^{k+1}-u^k\|=0$. In (57), taking the limit as $k\longrightarrow\infty$, we deduce

$$\lim_{k\longrightarrow\infty}\left\|\kappa\left(v^k,x^k\right)\right\|=0, \tag{58}$$

$$\lim_{k\longrightarrow\infty}\left\|v^k-t^k-\tau\zeta^k\kappa\left(v^k,x^k\right)\right\|^2=0. \tag{59}$$

On the other hand, we have

$$\left\|v^k-t^k\right\|\le\left\|v^k-t^k-\tau\zeta^k\kappa\left(v^k,x^k\right)\right\|+\tau\zeta^k\left\|\kappa\left(v^k,x^k\right)\right\|. \tag{60}$$

So, we have $\lim_{k\longrightarrow\infty}\|v^k-t^k\|=0$.
Combining Lemma 3 and (58), we obtain

$$\lim_{k\longrightarrow\infty}\left\|v^k-x^k\right\|=0. \tag{61}$$

Now, we show that $w_w(u^k)\subset \mathrm{Sol}(S,f)$. Choose $p^\ddagger\in w_w(u^k)$. It implies that there exists a subsequence $\{u^{n_k}\}$ of $\{u^k\}$ which converges weakly to $p^\ddagger$. Therefore, $v^{n_k}\rightharpoonup p^\ddagger$. Due to $\lim_{k\longrightarrow\infty}\|v^k-x^k\|=0$, we obtain $x^{n_k}\rightharpoonup p^\ddagger\in\mathcal{S}$. By Algorithm 1, we have

$$\langle x^k-v^k+\gamma^k f\left(v^k\right),u-x^k\rangle\ge 0,\quad \forall u\in C. \tag{62}$$

Since f is monotone, we have

$$\begin{aligned}0&\le\langle x^k-v^k,u-x^k\rangle+\gamma^k\langle f\left(v^k\right),u-x^k\rangle\\&=\langle x^k-v^k,u-x^k\rangle+\gamma^k\langle f\left(v^k\right),u-v^k\rangle\\&\quad+\gamma^k\langle f\left(v^k\right),v^k-x^k\rangle\\&\le\langle x^k-v^k,u-x^k\rangle+\gamma^k\langle f(u),u-v^k\rangle\\&\quad+\gamma^k\langle f\left(v^k\right),v^k-x^k\rangle.\end{aligned} \tag{63}$$

Taking the limit in (63) as $k\longrightarrow\infty$, we get

$$\langle f(u),u-p^\ddagger\rangle\ge 0,\quad\forall u\in C, \tag{64}$$

which implies that $w_w(u^k)\in\mathrm{Sol}(S,f)$.
Set $b^k=\|u^k-u^\ddagger\|^2$ for all $k\ge 0$. By (65) for $q=u^\ddagger$, we obtain

$$\begin{aligned}b^{k+1}&\le\left(1-\theta^k\right)\left\|v^k-q\right\|^2+\theta^k\Big[-2\lambda^k\left\|v^k-t^k\right\|\left\|u^{k+1}-q\right\|\\&\quad+2\langle -q,u^{k+1}-q\rangle\Big].\end{aligned} \tag{65}$$

We deduce from Algorithm 1 that

$$\begin{aligned}\left\|v^k-q\right\|^2&\le\left(\left\|u^k-q\right\|+\rho^k\left\|u^k-u^{k-1}\right\|\right)^2\\&=\left\|u^k-q\right\|^2+\left(\rho^k\right)^2\left\|u^k-u^{k-1}\right\|^2\\&\quad+2\rho^k\left\|u^k-q\right\|\left\|u^k-u^{k-1}\right\|\\&\le\left\|u^k-q\right\|^2+\rho^k\left\|u^k-u^{k-1}\right\|^2\\&\quad+2\rho^k\left\|u^k-q\right\|\left\|u^k-u^{k-1}\right\|\\&\le b^k+3K\rho^k\left\|u^k-u^{k-1}\right\|,\end{aligned} \tag{66}$$

where

$$K=\sup_{k\ge1}\left\{\left\|u^k-u^{k-1}\right\|,\left\|u^k-q\right\|\right\}. \tag{67}$$

By virtue of (65) and (66), we have

$$b^{k+1}\le\left(1-\theta^k\right)b^k+\delta^k, \tag{68}$$

where

$$\begin{aligned}\delta^k&=\theta^k\Bigg[3K\left(1-\theta^k\right)\frac{\rho^k}{\theta^k}\left\|u^k-u^{k-1}\right\|-2\lambda^k\left\|v^k-t^k\right\|\\&\quad\left\|u^{k+1}-q\right\|+2\langle -q,u^{k+1}-q\rangle\Bigg].\end{aligned} \tag{69}$$

So, we get

$$\limsup_{k\ge1}\langle -q,u^{k+1}-q\rangle=\sup_{u^\ddagger\in w_w}\left(u^k\right)\langle -q,u^\ddagger-q\rangle\le 0. \tag{70}$$

From (70), we deduce that $q\in P_{\mathrm{Sol}(S,f)}(0)$. Combining the property of projection, $\lim_{k\longrightarrow\infty}\|v^k-t^k\|^2=0$ and $\lim_{k\longrightarrow\infty}(\rho^k/\theta^k)\|u^k-u^{k-1}\|=0$, we have $\limsup_{k\ge1}\delta^k\le 0$. By Lemma 2, we obtain $b^k=\|u^k-u^\ddagger\|^2\longrightarrow 0\,(k\longrightarrow\infty)$. Therefore, the sequence $\{u^k\}$ converges strongly to $u^\ddagger$.
Case 2. There exists a subsequence $\{b^{k_i}\}\subset\{b^k\}_{k\ge\widetilde{k}_0}$ s.t. $b^{\widetilde{k}_i}\le b^{\widetilde{k}_i+1}$ for $\forall i\ge 0$. From Lemma 2, we can deduce

$$\begin{aligned}b^{\gamma(k)}&\le b^{\gamma(k)+1},\\b^k&\le b^{\gamma(k)+1},\end{aligned} \tag{71}$$

for each $k\ge\widetilde{k}_0$, where $\gamma(k)=\max\left\{n\in\mathbb{N}|\widetilde{k}_0\le n\le k,b^n\le b^{n+1}\right\}$. Further, the sequence $\{\gamma(k)\}_{k\ge\widetilde{k}_0}$ is nondecreasing (i.e., $\lim_{k\longrightarrow\infty}\gamma(k)=\infty$). Let $b^k=\|u^k-u^\ddagger\|^2$. By (71) and Claim 3 for $q=u^\ddagger$, we obtain

$$\begin{aligned}&\lambda^{\gamma(k)}\Big[\left\|v^{\gamma(k)}-t^{\gamma(k)}-\gamma\zeta^{\gamma(k)}d\left(v^{\gamma(k)},x^{\gamma(k)}\right)\right\|^2\\&\quad+\tau(2-\tau)\left(\zeta^{\gamma(k)}\right)^2\left\|d\left(v^{\gamma(k)},x^{\gamma(k)}\right)\right\|^2\Big]\\&\le\rho^{\gamma(k)}\left(1-\theta^{\gamma(k)}\right)\left(b^{\gamma(k)}-b^{\gamma(n)-1}\right)\\&\quad+2\rho^{\gamma(k)}\left(1-\theta^{\gamma(k)}\right)\left\|u^{\gamma(k)}-u^{\gamma(n)-1}\right\|^2+\theta^{\gamma(k)}\|q\|^2.\end{aligned} \tag{72}$$

We deduce from the definition of b^k that

$$
\begin{aligned}
b^{\gamma(k)} - b^{\gamma(k)-1} &= \left\|u^{\gamma(k)} - q\right\|^2 - \left\|u^{\gamma(k)-1} - q\right\|^2 \\
&= \left(\left\|u^{\gamma(k)} - q\right\| - \left\|u^{\gamma(k)-1} - q\right\|\right) \\
&\quad \left(\left\|u^{\gamma(k)} - q\right\| + \left\|u^{\gamma(k)-1} - q\right\|\right) \\
&\leq \left\|u^{\gamma(k)} - u^{\gamma(k)-1}\right\|\left(\left\|u^{\gamma(k)} - q\right\| + \left\|u^{\gamma(k)-1} - q\right\|\right).
\end{aligned} \tag{73}
$$

Combining (72) and (73), we have

$$
\begin{aligned}
&\lambda^{\gamma(k)}\Big[\left\|v^{\gamma(k)} - t^{\gamma(k)} - \gamma\zeta^{\gamma(k)} d\left(v^{\gamma(k)}, u^{\gamma(k)}\right)\right\|^2 \\
&\quad + \tau(2-\tau)\left(\zeta^{\gamma(k)}\right)^2\left\|d\left(v^{\gamma(k)}, u^{\gamma(k)}\right)\right\|^2\Big] \\
&\leq \rho^{\gamma(k)}\left(1 - \theta^{\gamma(k)}\right)\left[\left\|u^{\gamma(k)} - u^{\gamma(k)-1}\right\|\left(\left\|u^{\gamma(k)} - q\right\| - \left\|u^{\gamma(k)-1} - q\right\|\right)\right] \\
&\quad + 2\rho^{\gamma(k)}\left(1 - \theta^{\gamma(k)}\right)\left\|u^{\gamma(k)} - u^{\gamma(k)-1}\right\|^2 + \theta^{\gamma(k)}\|q\|^2.
\end{aligned} \tag{74}
$$

Similarly, we have $\rho^{\gamma(k)}(1 - \theta^{\gamma(k)})\|u^{\gamma(k)} - u^{\gamma(k)-1}\| \longrightarrow 0$. It follows that

$$
\begin{aligned}
&w_w\left(u^{\gamma(k)}\right) \subset \mathrm{Sol}(S, f), \\
&\lim_{k\longrightarrow\infty}\left\|t^{\gamma(k)} - u^{\gamma(k)}\right\|^2 = \lim_{k\longrightarrow\infty}\left\|t^{\gamma(k)} - v^{\gamma(k)}\right\|^2 = 0,
\end{aligned} \tag{75}
$$

and

$$
\begin{aligned}
b^{\gamma(k)+1} &\leq \left(1 - \theta^{\gamma(k)}\right)b^{\gamma(k)} + \theta^{\gamma(k)} \\
&\quad \left[3K\left(1 - \theta^{\gamma(k)}\right)\frac{\rho^{\gamma(k)}}{\theta^{\gamma(k)}}\left\|u^{\gamma(k)} - u^{\gamma(k)-1}\right\|\right. \\
&\quad \left. - 2\lambda^{\gamma(k)}\left\|v^{\gamma(k)} - t^{\gamma(k)}\right\|\left\|u^{\gamma(k)+1} - q\right\| \quad + 2\langle -q, u^{\gamma(k)+1} - q\rangle.\right.
\end{aligned} \tag{76}
$$

Since $b^{\gamma(k)} \leq b^{\gamma(k)+1}$ and $\theta^{\gamma(k)} > 0$, from (76), we have

$$
\begin{aligned}
b^{\gamma(k)} &\leq 3K\left(1 - \theta^{\gamma(k)}\right)\frac{\rho^{\gamma(k)}}{\theta^{\gamma(k)}}\left\|u^{\gamma(k)} - u^{\gamma(k)-1}\right\| \\
&\quad - 2\lambda^{\gamma(k)}\left\|v^{\gamma(k)} - t^{\gamma(k)}\right\|\left\|u^{\gamma(k)+1} - q\right\| + 2\langle -q, u^{\gamma(k)+1} - q\rangle.
\end{aligned} \tag{77}
$$

Since $q \in P_{\mathrm{Sol}(S,f)}(0)$ and $w_w(u^{\gamma(k)}) \subset \mathrm{Sol}(S, f)$, we have $\limsup_{k\longrightarrow\infty}\langle -q, u^{\gamma(k)+1} - q\rangle = \sup_{k\longrightarrow\infty}\langle -q, u^{\ddagger} - q\rangle \leq 0$. By (75), (77), and $(\rho^{\gamma(k)}/\theta^{\gamma(k)})\|u^{\gamma(k)} - u^{\gamma(k)-1}\| \longrightarrow 0$, we get

$$
\limsup_{k\longrightarrow\infty} b^{\gamma(k)} \leq 2\sup_{q\in w_w}\left(u^{\gamma(k)}\right)\langle -q, u^{\ddagger} - q\rangle \leq 0. \tag{78}
$$

It follows from (76) that

$$
\begin{aligned}
&\limsup_{k\longrightarrow\infty} b^{\gamma(k)+1} \leq 0 \\
&\text{or} \lim_{k\longrightarrow\infty} b^{\gamma(k)+1} = 0.
\end{aligned} \tag{79}
$$

Hence, $\lim_{k\longrightarrow\infty} b^k = 0$. Therefore, the sequence $\{u^k\}$ converges strongly to $u^{\ddagger}$. This completes the proof.

Suppose that g: H $\longrightarrow$ H is a ρ-contractive operator. Next, we propose an iterative algorithm with viscosity item. □

Theorem 2. *The sequence $\{u^k\}$ generated by Algorithm 1 converges strongly to $u^{\ddagger} = P_{Sol(S,f)}g(u^{\ddagger})$.*

Proof. We divide the proof into 4 claims.

Claim 1. We prove the boundedness of the sequences $\{g(v^k)\}, \{x^k\}$ and $\{t^k\}$. From Algorithm 1, we get

$$
\begin{aligned}
\left\|v^k - u^{\ddagger}\right\| &= \left\|u^k - \rho^k\left(u^k - u^{k-1}\right) - u^{\ddagger}\right\| \\
&\leq \left\|u^k - u^{\ddagger}\right\| + \rho^k\left\|u^k - u^{k-1}\right\| \\
&= \left\|u^k - u^{\ddagger}\right\| + \theta^k\frac{\rho^k}{\theta^k}\left\|u^k - u^{k-1}\right\|.
\end{aligned} \tag{80}
$$

From Algorithm 1, we obtain $(\rho^k/\theta^k)\|u^k - u^{k-1}\| \longrightarrow 0$, $(k \longrightarrow \infty)$. Then, $\exists M_1 > 0$ s.t.

$$
\frac{\rho^k}{\theta^k}\left\|u^k - u^{k-1}\right\| \leq M_1, \quad \forall k > 0. \tag{81}
$$

By Algorithm 1 and (81), we have

$$
\begin{aligned}
\left\|u^{k+1} - u^{\ddagger}\right\| &= \left\|\theta^k g\left(v^k\right) + \left(1 - \theta^k\right)t^k - u^{\ddagger}\right\| \\
&= \left\|\theta^k\left(g\left(v^k\right) - u^{\ddagger}\right) + \left(1 - \theta^k\right)\left(t^k - u^{\ddagger}\right)\right\| \\
&\leq \theta^k\left\|g\left(v^k\right) - u^{\ddagger}\right\| + \left(1 - \theta^k\right)\left\|t^k - u^{\ddagger}\right\| \\
&\leq \theta^k\left\|g\left(v^k\right) - g\left(u^{\ddagger}\right)\right\| + \theta^k\left\|g\left(u^{\ddagger}\right) - u^{\ddagger}\right\| + \left(1 - \theta^k\right)\left\|t^k - u^{\ddagger}\right\| \\
&\leq \theta^k\rho\left\|v^k - u^{\ddagger}\right\| + \theta^k\left\|g\left(u^{\ddagger}\right) - u^{\ddagger}\right\| + \left(1 - \theta^k\right)\left\|v^k - u^{\ddagger}\right\|
\end{aligned}
$$

Initialization. Choose $u^0, u^1 \in \mathrm{H}$ arbitrarily.
Step 1. Choose ρ^k s.t. $0 \le \rho^k \le \overline{\rho}^k$, where $\overline{\rho}^k = \begin{cases} \min\{\rho, (\varepsilon^k/\|u^k - u^{k-1}\|)\} & \text{if } u^k \ne u^{k-1} \\ \rho & \text{otherwise} \end{cases}$
Calculate $v^k = u^k + \rho^k (u^k - u^{k-1})$
Step 2. Calculate $x^k = P_S(v^k - \gamma^k f(v^k))$,
where $\gamma^{k+1} = \begin{cases} \min\{\gamma^k, (\sigma\|u^k - x^k\|/\|fu^k - fx^k\|)\} & \text{iff } u^k \ne fx^k \\ \gamma^k & \text{otherwise} \end{cases}$
Step 3. Construct the half space T^k as follows $T^k = \{z \in \mathrm{H} | \langle v^k - \gamma^k f v^k - x^k, z - x^k \rangle \le 0\}$.
Calculate $t^k = P_{T^k}(v^k - \tau\zeta^k\gamma^k f(x^k))$,
where $\zeta^k = (\langle v^k - x^k, \kappa(v^k, x^k)\rangle / \|\kappa(v^k, x^k)\|^2)$
and $\kappa(v^k, x^k) = (v^k, x^k) - \gamma^k (f(v^k) - f(x^k))$.
Step 4. Compute $u^{k+1} = \theta^k g(v^k) + (1 - \theta^k) t^k$.
If $x^k = v^k$, then stop and $x^k \in \mathrm{Sol}(S, f)$. Otherwise, set $k := k + 1$ and return to step 1.

Algorithm 2: Strong convergence algorithm with viscosity term.

$$
\begin{aligned}
&\le \left(1 - (1-\rho)\theta^k\right)\left\|v^k - u^\ddagger\right\| + \theta^k\left\|g\left(u^\ddagger\right) - u^\ddagger\right\| \\
&\le \left(1 - (1-\rho)\theta^k\right)\left\|u^k + \rho^k\left(u^k - u^{k-1}\right) - u^\ddagger\right\| + \theta^k\left\|g\left(u^\ddagger\right) - u^\ddagger\right\| \\
&\le \left(1 - (1-\rho)\theta^k\right)\left\|u^k - u^\ddagger\right\| + \left(1 - (1-\rho)\theta^k\right)\rho^k\left\|u^k - u^{k-1}\right\| + \theta^k\left\|g\left(u^\ddagger\right) - u^\ddagger\right\| \\
&\le \left(1 - (1-\rho)\theta^k\right)\left\|u^k - u^\ddagger\right\| + \left(1 - (1-\rho)\theta^k\right)\theta^k\frac{\rho^k}{\theta^k}\left\|u^k - u^{k-1}\right\| + \theta^k\left\|g\left(u^\ddagger\right) - u^\ddagger\right\|.
\end{aligned}
\tag{82}
$$

From (81) and (82), we have

$$
\begin{aligned}
\left\|u^{k+1} - u^\ddagger\right\| &\le \left(1 - (1-\rho)\theta^k\right)\left\|u^k - u^\ddagger\right\| + \left(1 - (1-\rho)\theta^k\right)\theta^k\frac{\rho^k}{\theta^k}\left\|u^k - u^{k-1}\right\| + \theta^k\left\|g\left(u^\ddagger\right) - u^\ddagger\right\| \\
&\le \left(1 - (1-\rho)\theta^k\right)\left\|u^k - u^\ddagger\right\| + \left(1 - (1-\rho)\theta^k\right)\theta^k M_1 + \theta^k\left\|g\left(u^\ddagger\right) - u^\ddagger\right\| \le \left(1 - (1-\rho)\theta^k\right)\left\|u^k - u^\ddagger\right\| \\
&\quad + (1-\rho)\theta^k\frac{\left(1 - (1-\rho)\theta^k\right)M_1 + \left\|g\left(u^\ddagger\right) - u^\ddagger\right\|}{1-\rho} \\
&\le \left(1 - (1-\rho)\theta^k\right)\left\|u^k - u^\ddagger\right\| + (1-\rho)\theta^k\frac{M_1 + \left\|g\left(u^\ddagger\right) - u^\ddagger\right\|}{1-\rho} \\
&\le \max\left\{\left\|u^k - u^\ddagger\right\|, \frac{M_1 + \left\|g\left(u^\ddagger\right) - u^\ddagger\right\|}{1-\rho}\right\} \le \cdots \le \max\left\{\left\|u^k - u^\ddagger\right\|, \frac{M_1 + \left\|g\left(u^\ddagger\right) - u^\ddagger\right\|}{1-\rho}\right\}.
\end{aligned}
\tag{83}
$$

It is obvious that the sequence $\{u^k\}$ is bounded. Furthermore, the sequences $\{g(v^k)\}, \{x^k\}$ and $\{t^k\}$ are bounded.

Claim 2. From (7) and Algorithm 1, we have

$$
\begin{aligned}
\left\|u^{k+1} - u^\ddagger\right\|^2 &= \left\|\theta^k g\left(v^k\right) + \left(1 - \theta^k\right)t^k - u^\ddagger\right\|^2 \\
&= \left\|\theta^k\left(g\left(v^k\right) - g\left(u^\ddagger\right)\right) + \left(1 - \theta^k\right)\left(t^k - u^\ddagger\right) + \theta^k\left(g\left(u^\ddagger\right) - u^\ddagger\right)\right\|^2 \\
&\le \left\|\theta^k\left(g\left(v^k\right) - g\left(u^\ddagger\right)\right) + \left(1 - \theta^k\right)\left(t^k - u^\ddagger\right)\right\|^2
\end{aligned}
$$

$$+ 2\theta^k \langle g(u^\ddagger) - u^\ddagger, u^{k+1} - u^\ddagger \rangle \leq \theta^k \|g(v^k) - g(u^\ddagger)\|^2 + (1-\theta^k)\|t^k - u^\ddagger\|^2$$
$$+ 2\theta^k \langle g(u^\ddagger) - u^\ddagger, u^{k+1} - u^\ddagger \rangle \leq \theta^k \rho \|v^k - u^\ddagger\|^2 + (1-\theta^k)\|v^k - u^\ddagger\|^2 + 2\theta^k \langle g(u^\ddagger) - u^\ddagger, u^{k+1} - u^\ddagger \rangle$$
$$= \left(1 - (1-\rho)\theta^k\right)\|v^k - u^\ddagger\|^2 + 2\theta^k \langle g(u^\ddagger) - u^\ddagger, u^{k+1} - u^\ddagger \rangle. \tag{84}$$

Claim 3. By (8) and (55), we obtain

$$\begin{aligned}
\|u^{k+1} - u^\ddagger\|^2 &= \|\theta^k (g(v^k) - u^\ddagger) + (1-\theta^k)(t^k - u^\ddagger)\|^2 \\
&\leq \theta^k \|g(v^k) - u^\ddagger\|^2 + (1-\theta^k)\|t^k - u^\ddagger\|^2 - \theta^k(1-\theta^k)\|g(v^k) - t^k\|^2 \\
&\leq \theta^k \|g(v^k) - u^\ddagger\|^2 + (1-\theta^k)\|t^k - u^\ddagger\|^2 \leq \theta^k \|g(v^k) - u^\ddagger\|^2 + (1-\theta^k)\|v^k - u^\ddagger\|^2 \\
&\quad - (1-\theta^k)\|v^k - t^k - \tau\zeta^k \kappa(v^k, x^k)\|^2 - (1-\theta^k)\tau(2-\tau)(\zeta^k)^2 \|\kappa(v^k, x^k)\|^2.
\end{aligned} \tag{85}$$

From (85) and (55), we obtain

$$\begin{aligned}
\|u^{k+1} - u^\ddagger\|^2 &\leq \theta^k \|g(v^k) - u^\ddagger\|^2 + (1-\theta^k)\|u^k - u^\ddagger\|^2 \\
&\quad + (1-\theta^k)\rho^k \left(\|u^k - u^\ddagger\|^2 - \|u^{k-1} - u^\ddagger\|^2\right) \\
&\quad + 2(1-\theta^k)\rho^k \|u^k - u^{k-1}\|^2 - (1-\theta^k)\|(v^k - t^k) - \tau\zeta^k \kappa(v^k, x^k)\|^2 \\
&\quad - (1-\theta^k)\tau(2-\tau)(\zeta^k)^2 \|\kappa(v^k, x^k)\|^2 \\
&\leq \theta^k \|g(v^k) - u^\ddagger\|^2 + \|u^k - u^\ddagger\|^2 + (1-\theta^k)\rho^k \left(\|u^k - u^\ddagger\|^2 - \|u^{k-1} - u^\ddagger\|^2\right) \\
&\quad + 2(1-\theta^k)\rho^k \|u^k - u^{k-1}\|^2 - (1-\theta^k)\|v^k - t^k - \tau\zeta^k \kappa(v^k, x^k)\|^2 - (1-\theta^k)\tau(2-\tau)(\zeta^k)^2 \|\kappa(v^k, x^k)\|^2.
\end{aligned} \tag{86}$$

This implies that

$$\begin{aligned}
&(1-\theta^k)\|v^k - t^k - \tau\zeta^k \kappa(v^k, x^k)\|^2 + (1-\theta^k)\tau(2-\tau)(\zeta^k)^2 \|\kappa(v^k, x^k)\|^2 \\
&\quad - (1-\theta^k)\rho^k \left(\|u^k - u^\ddagger\|^2 - \|u^{k-1} - u^\ddagger\|^2\right) - 2(1-\theta^k)\rho^k \|u^k - u^{k-1}\|^2 \\
&\leq \|u^k - u^\ddagger\|^2 - \|u^{k+1} - u^\ddagger\|^2 + \theta^k \|g(v^k) - u^\ddagger\|^2.
\end{aligned} \tag{87}$$

Claim 4. According to Claim 3, we can see that there are two possible cases.

Case 1. There exists an $N \in \mathbb{N}$, s.t. $\|u^{k+1} - u^\ddagger\|^2 \leq \|u^k - u^\ddagger\|^2$ for $\forall k > N$. It follows that $\lim_{k\longrightarrow\infty}\|u^k - u^\ddagger\|$ exists. From (86) and $\lim_{k\longrightarrow\infty}\theta^k = 0$, we have

$$\begin{aligned}
\lim_{k\longrightarrow\infty} \|\kappa(v^k, x^k)\| &= 0, \\
\lim_{k\longrightarrow\infty} \|v^k - t^k - \tau\zeta^k \kappa(v^k, x^k)\| &= 0.
\end{aligned} \tag{88}$$

Note that

$$\left\|v^k - t^k\right\|^2 \leq \left\|v^k - t^k - \tau\zeta^k\kappa\left(v^k, x^k\right)\right\| + \left\|\tau\zeta^k\kappa\left(v^k, x^k\right)\right\|. \tag{89}$$

So,

$$\lim_{k\longrightarrow\infty}\left\|v^k - t^k\right\| = 0. \tag{90}$$

Similarly, we can obtain

$$w_w\left(u^k\right) \subset \mathrm{Sol}(S, f). \tag{91}$$

Set $b^k = \|u^k - q\|^2$ for all $k \geq 0$. By (84) for $q = u^\ddagger$, we get

$$\begin{aligned} b^{k+1} &\leq \left(1 - (1-\rho)\theta^k\right)\left\|v^k - u^\ddagger\right\|^2 \\ &\quad + 2\theta^k\langle g\left(u^\ddagger\right) - u^\ddagger, u^{k+1} - u^\ddagger\rangle. \end{aligned} \tag{92}$$

It follows from (66) and (92) that

$$b^{k+1} \leq \left(1 - (1-\rho)\theta^k\right)b^k + \delta^k, \tag{93}$$

where

$$\begin{aligned} \delta^k &= 3K\left(1 - (1-\rho)\theta^k\right)\rho^k\left\|u^k - u^{k-1}\right\| \\ &\quad + 2\theta^k\langle g(q) - q, u^{k+1} - q\rangle. \end{aligned} \tag{94}$$

Then,

$$\begin{aligned} &\limsup_{k\geq 1}\langle g(q) - q, u^{k+1} - q\rangle = \limsup_{u^\ddagger\in w_w} \\ &\left(u^k\right)\langle g(q) - q, u^\ddagger - q\rangle \leq 0. \end{aligned} \tag{95}$$

Hence, we deduce $q \in P_{\mathrm{Sol}(S,f)}(0)$. Combining the property of projection, $\lim_{k\longrightarrow\infty}\|v^k - t^k\|^2 = 0$ and $\lim_{k\longrightarrow\infty}(\rho^k/\theta^k)\|u^k - u^{k-1}\| = 0$, we have $\limsup_{k\geq 1}\delta^k \leq 0$. By Lemma 2, we obtain $b^k = \|u^k - u^\ddagger\|^2 \longrightarrow 0 (k \longrightarrow \infty)$. Therefore, the sequence $\{u^k\}$ converges strongly to $u^\ddagger$.

Case 2. There exists a subsequence $\{b^{k_i}\} \subset \{b^k\}_{k\geq\widetilde{k}_0}$, s.t. $b^{k_i} \leq b^{k_i+1}$ for each $i \geq 0$. From Lemma 1, we deduce that

$$\begin{aligned} b^{\gamma(k)} &\leq b^{\gamma(k)+1}, \\ b^k &\leq b^{\gamma(k)+1}, \end{aligned} \tag{96}$$

for all $k \geq \widetilde{k}_0$, where $\gamma(k) = \max\left\{n \in \mathbb{N}|\widetilde{k}_0 \leq n \leq k, b^n \leq b^{n+1}\right\}$. Therefore, the sequence $\{\gamma(k)\}_{k\geq\widetilde{k}_0}$ is nondecreasing (i.e., $\lim_{k\longrightarrow\infty}\gamma(k) = \infty$). By (96) and Claim 3 for $q = u^\ddagger$, we obtain

$$\begin{aligned} &\left(1 - \theta^{\gamma(k)}\right)\left\|\left(v^{\gamma(k)} - t^{\gamma(k)}\right) - \tau\zeta^{(\gamma(k))}\kappa\left(v^{\gamma(k)}, x^{\gamma(k)}\right)\right\|^2 \\ &\quad + \left(1 - \theta^{\gamma(k)}\right)\tau(2-\tau)\left(\zeta^{\gamma(k)}\right)^2\left\|\kappa\left(v^{\gamma(k)}, x^{\gamma(k)}\right)\right\|^2 \\ &\leq \left\|u^{\gamma(k)} - q\right\|^2 - \left\|u^{\gamma(n)+1} - u^\ddagger\right\|^2 + \theta^{\gamma(k)}\left\|g\left(v^{\gamma(k)}\right) - q\right\|^2 \\ &\quad + \left(1 - \theta^{\gamma(k)}\right)\rho^{\gamma(k)}\left(b^{\gamma(k)} - b^{\gamma(k)-1}\right) + 2\left(1 - \theta^{\gamma(k)}\right)\rho^{\gamma(k)} \\ &\quad \left\|u^{\gamma(k)} - u^{\gamma(k)-1}\right\|^2. \end{aligned} \tag{97}$$

From (73) and (97), we get

$$\begin{aligned} &\left(1 - \theta^{\gamma(k)}\right)\left\|\left(v^{\gamma(k)} - t^{\gamma(k)}\right) - \tau\zeta^{\gamma(k)}\kappa\left(v^{\gamma(k)}, x^{\gamma(k)}\right)\right\|^2 \\ &\quad + \left(1 - \theta^{\gamma(k)}\right)\tau(2-\tau)\left(\zeta^{\gamma(k)}\right)^2\left\|\kappa\left(v^{\gamma(k)}, x^{\gamma(k)}\right)\right\|^2 \\ &\leq \left\|u^{\gamma(k)} - u^\ddagger\right\|^2 - \left\|u^{\gamma(k)+1} - u^\ddagger\right\|^2 + \theta^{\gamma(k)}\left\|g\left(v^{\gamma(k)}\right) - u^\ddagger\right\|^2 \\ &\quad + 2\left(1 - \theta^{\gamma(k)}\right)\rho^{\gamma(k)}\left\|u^{\gamma(k)} - u^{\gamma(n)-1}\right\|^2 + \left(1 - \theta^{\gamma(k)}\right)\rho^{\gamma(k)} \\ &\quad \left\|u^{\gamma(k)} - u^{\gamma(n)-1}\right\|\left(\left\|u^{\gamma(k)} - q\right\| + \left\|u^{\gamma(n)-1} - q\right\|\right). \end{aligned} \tag{98}$$

Using Claim 1 and (98), we have $\lim_{k\longrightarrow\infty}(1 - \theta^{\gamma(k)})\rho^{\gamma(k)}\|u^{\gamma(k)} - u^{\gamma(k)-1}\| = 0$. Therefore,

$$\begin{aligned} &w_w\left(u^{\gamma(k)}\right) \subset \mathrm{Sol}(S, f), \\ &\lim_{k\longrightarrow\infty}\left\|t^{\gamma(k)} - u^{\gamma(k)}\right\|^2 = \lim_{k\longrightarrow\infty}\left\|t^{\gamma(k)} - v^{\gamma(k)}\right\|^2 = 0, \\ &b^{\gamma(k)+1} \leq \left(1 - (1-\rho)\theta^{\gamma(k)}\right)b^{\gamma(k)} + 3K \\ &\quad \left(1 - (1-\rho)\theta^{\gamma(k)}\right)\rho^{\gamma(k)}\left\|u^{\gamma(k)} - u^{\gamma(k)-1}\right\| \\ &\quad + 2\theta^{\gamma(k)}\langle g(q) - q, u^{\gamma(k)+1} - q\rangle. \end{aligned} \tag{99}$$

Since $b^{\gamma(k)} \leq b^{\gamma(k)+1}$ and $b^{\gamma(k)} \geq 0$, we receive

$$\begin{aligned} b^{\gamma(k)} &\leq 3K\left(1 - (1-\rho)\theta^{\gamma(k)}\right)\rho^{\gamma(k)}\left\|u^{\gamma(k)} - u^{\gamma(k)-1}\right\| \\ &\quad + 2\theta^{\gamma(k)}\langle g(q) - q, u^{\gamma(k)+1} - q\rangle. \end{aligned} \tag{100}$$

Note that $q \in P_{\mathrm{Sol}(S,f)}(0)$ and $w_w(u^{\gamma(k)}) \subset \mathrm{Sol}(S, f)$. By the property of projection, we have

$$\begin{aligned} &\limsup_{k\longrightarrow\infty}\langle g(q) - q, u^{\gamma(k)+1} - q\rangle = \sup_{u^\ddagger\in w_w}\left(u^{\gamma(k)}\right) \\ &\langle g(q) - q, u^\ddagger - q\rangle \leq 0. \end{aligned} \tag{101}$$

Since $(1 - \theta^{\gamma(k)})\rho^{\gamma(k)}\|u^{\gamma(k)} - u^{\gamma(n)-1}\| \longrightarrow 0$, we deduce

$$\limsup_{k\longrightarrow\infty} a_{\gamma(k)} \leq 2\sup_{u^\ddagger\in w_w}\left(x_{\gamma(k)}\right)\langle g(q) - q, u^\ddagger - q\rangle \leq 0. \tag{102}$$

So, $\limsup_{k\longrightarrow\infty}b^{\gamma(k)+1} \leq 0$ or $\lim_{k\longrightarrow\infty}b^{\gamma(k)+1} = 0$. Hence, $\lim_{k\longrightarrow\infty}b^k = 0$ which implies that the sequence $\{u^k\}$ converges strongly to $u^\ddagger$. This completes the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Research Article

On Credit Risk Contagion of Supply Chain Finance under COVID-19

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Existing research on credit risk contagion of supply chain finance pays more attention to the influence of network internal structure on the process of risk contagion. The spread of COVID-19 has had a huge impact on the supply chain, with a large number of enterprises experiencing difficulties in operation, resulting in increased credit risks in supply chain finance. Under the impact of the epidemic, this paper explores the transmission speed and steady state of credit risk when the supply chain finance network is affected by external impact so that we can have a more complete understanding of the ability of supply chain finance to resist risks. The simulation results show that external shocks of different degrees will increase the number of initial infected enterprises and lead to the increase in credit risk contagion speed but have no significant impact on network steady state; the speed of credit risk contagion is positively correlated with network complexity but not significantly affected by network size; core enterprises infected will increase the rate of credit risk contagion. The intensity of policy intervention has obvious curative effect on the risk caused by external shock. When the supply chain financial network is affected by external shocks, the intensity, time, and pertinence of policy response can effectively prevent the credit risk contagion.

1. Introduction

Since the outbreak in 2020, COVID-19 has spread to 191 countries and regions, with more than 200 million confirmed cases and 4.4 million deaths. As a result of the epidemic, the unemployment rate has gradually increased, enterprises have suspended production, and economic indicators have declined significantly. Optimization methods, in particular, fixed-point methods, are efficient and powerful for solving various real problems in traffic and transportation, medical imaging, modern management, financial engineering (see, e.g., [1–5]). For the supply chain, the continuous decline of market demand and production stagnation of upstream and downstream enterprises will lead to serious operational difficulties for enterprises within the supply chain. Repeated outbreaks caused by mutations in the virus have cast a shadow over the recovery of global supply chain. Against the backdrop of the pandemic, supply chain finance is also facing severe impact.

Supply chain finance integrates logistics, capital flow, and information flow and forms a corresponding network structure. With the continuous advancement of financial globalization, network complexity is strengthened [6], while making the network more efficient, it also makes it more vulnerable to default and bankruptcy [7]. The credit default of one participant will cause losses of other members and eventually lead to credit risk contagion throughout the supply chain network [8–11]. The spread of COVID-19 has significantly increased the network credit risk of supply chain finance. The supply chain finance plays a role in resisting the external impact and effectively offsetting losses caused by the COVID-19 outbreak; therefore, it is important to study the influence of external impact on supply chain financial credit risk infection and corresponding policies.

The rest of this paper is organized as follows. Section 2 is literature review, which mainly summarizes the existing research on credit risk of supply chain finance. Section 3 is the construction of complex network of supply chain finance, including model selection and model construction. Section 4 is the simulation results and discussion, the design of the relevant simulation experiments, and the results of in-depth analysis. Section 5 is the conclusion.

2. Literature Review

Supply chain finance is different from traditional financing methods. By integrating the financing process of all participants and optimizing the capital flow in the supply chain, supply chain finance improves the overall capital management level of the supply chain [12–14]. The existing research on credit risk of supply chain finance mainly focuses on risk identification and evaluation [8], and the research method is mainly based around the construction of the risk evaluation index system. Mou et al. [9] believed that core enterprises are crucial to the evaluation of credit risks in the supply chain, and the FAHP method is used to quantitatively measure and evaluate the credit risks of core enterprises. Zhu et al. [15] believed that SMEs are in a vulnerable position in the supply chain, but their credit risk status is of great significance to supply chain finance. With the help of RS-multiboosting machine learning model, they built a prediction model of credit risk for SMEs in supply chain finance, which has a good effect. Wang et al. [16] believed that the development of smart city and smart finance makes the financial risks of SMEs more complex and uses the improved PROMETHEE method to build a more accurate credit risk evaluation model of SMEs. However, enterprises in the supply chain do not operate in isolation, and the credit risk within the supply chain is bound to infect the whole supply chain, causing serious impact on the network.

On the topic of credit risk contagion, existing research mainly focuses on the influence of internal structure of supply chain on the process of credit risk contagion. Zhao et al. [17] established a scale-free complex network model of supply chain finance and found that network structure had a significant impact on credit risk contagion. Xie et al. [18] made a game analysis of the two-tier supply chain with financing constraints by constructing two-tier financing channels in the supply chain and found that financing structure has a significant impact on credit risk contagion, while the contagion effect under dual-channel financing mechanism is weak. With the help of the SIRS infectious disease model, Wang et al. [11] deeply discussed the influence of multiple factors on supply chain risk contagion, including enterprise risk preference, operational robustness and flexibility, completeness of market information, especially network topology. However, in addition to internal factors, external shocks also have an impact on the process of credit risk contagion. Therefore, this paper mainly studies the impact of external shocks (COVID-19) on the credit risk contagion process of supply chain finance.

3. Construction of Supply Chain Finance Network

3.1. Selection of Network Model and Infection Model. From the perspective of the development history of complex network, complex network can be divided into regular network, random graph, small world network, and scale-free network. A regular network is one in which any two nodes are directly connected by an edge; random graph refers to the network formed by connecting nodes randomly according to probability. Both small world network and random network conform to Poisson distribution, which results that the degree of a large number of nodes in the network is concentrated near k, and there are no points with a relatively high degree. This kind of network is also called uniform network. In the structure of supply chain finance network, newly joined SMEs tend to be connected with core enterprises with high degree, which makes core enterprises have a very high degree, while SMEs have a relatively low degree. This characteristic of supply chain finance network is exactly in line with the characteristics of scale-free network. Therefore, BA scale-free network is selected as the basis to construct supply chain finance network.

The premise of virus transmission is a network environment conducive to transmission, and only mutual contact and relationship can complete the transmission process, which is consistent with the transmission of credit risk. Therefore, the epidemic model has been widely used to study the transmission of credit risk. The SIS virus infection model is a classical model of transmission dynamics in complex networks and has been applied in a large number of studies on credit risk [17, 19]. The SIS virus infection model has the advantage that it has no limitation on the network scale and no special requirements on the network transmission direction and can accurately reflect the dynamic process of credit risk infection in the network. Therefore, the SIS virus infection model is very suitable for this paper as the basic model of credit risk infection.

3.2. Construction of Complex Network of Supply Chain Finance. Based on the discussion of model selection in the previous section and the special requirements of the network model in related studies [17, 19], this paper proposes the following hypotheses for the supply chain finance network:

(i) *Hypothesis 1.* Supply chain finance network is an undirected network. Generally, risk contagion is transmitted from upstream to downstream, but for supply chain finance network, it is mainly manifested as upstream and downstream cross infection, and there is no clear direction. Therefore, this article sets the network as an undirected network.

(ii) *Hypothesis 2.* Nodes in the network are only infected by neighboring nodes. In the supply chain finance network, the connection between two nodes is the channel of risk contagion, and nodes without direct contact have little influence on each other, which is not considered in this paper.

(iii) *Hypothesis 3.* Credit risk in supply chain finance network transmits with a certain probability. Even if it is connected to the infected node, it does not necessarily mean that it will be affected accordingly. The probability of infection is related to the strength of the connection between the two nodes. In this paper, to simplify the infection model, this probability is set as a constant.

(iv) *Hypothesis 4*. The structure and scale of supply chain finance network will not change in the process of credit risk contagion. In the real world, the network is always in a dynamic process, some nodes will join, and also some nodes will disconnect. In this paper, the network model is simplified, and the changes of network structure and scale in the process of infection are not considered.

Assume that the probability of credit risk infection in the network is λ, $0 < \lambda < 1$; at time t, the density of infected individuals on the network is $\rho(t)$. As t approaches infinity, the network reaches a steady state ρ. Considering that the scale-free network is a nonuniform network, the infected density of the node with the degree of k at the time of t is $\rho_k(t)$. Each node has a certain recovery capability, and the recovery coefficient is set as γ. Through the comprehensive analysis of the above contents and the characteristics of the SIS model, we can get the following equation at time t:

$$\frac{\partial \rho_k(t)}{\partial t} = -\gamma \rho_k(t) + \lambda k\left[1-\rho_k(t)\right]\Theta(\rho(t)). \tag{1}$$

The first term on the right is the annihilation term, and the density of infected nodes decreases with the speed γ. The second item is the generation term, and it is proportional to the probability of infection λ, density of susceptible nodes $[1-\rho_k(t)]$, the degree of node k, and $\Theta(\rho(t))$. $\Theta(\rho(t))$ represents the probability that any given edge is connected to an infected node, namely, $\Theta(\rho(t)) = (1/\langle k\rangle)\sum_k kP(k)\rho_k(t)$. We assume the infected density of the node with degree k at steady state ρ_k.

Using the steady-state condition $(\partial \rho_k(t)/\partial t) = 0$, we have the following:

$$\rho_k = \frac{\lambda k \Theta(\rho)}{\gamma + \lambda k \Theta(\rho)}. \tag{2}$$

According to equation (2), we find that at steady state, the density of infected nodes with degree k is positively correlated to the infection probability λ and the degree of the node k but is negatively correlated to the enterprise's resilience γ.

With equation (2) and $\Theta(\rho(t))$, we have the following:

$$\Theta = \frac{1}{\langle k\rangle}\sum_k kP(k)\frac{\lambda k\Theta}{\gamma + \lambda k\Theta}. \tag{3}$$

If the equation has a nontrivial solution $\Theta \neq 0$, the following conditions must be met:

$$\frac{d}{d\Theta}\left(\frac{1}{\langle k\rangle}\sum_k kP(k)\frac{\lambda k\Theta}{\gamma + \lambda k\Theta}\right)\Bigg|_{\Theta=0} \geq 1. \tag{4}$$

Namely,

$$\sum_k \frac{\lambda k^2 P(k)}{\gamma\langle k\rangle} = \frac{\lambda\langle k^2\rangle}{\langle k\rangle} \geq 1. \tag{5}$$

The critical value λ_c of BA scale-free network is as follows:

$$\lambda_c = \frac{\gamma\langle k\rangle}{\langle k^2\rangle}. \tag{6}$$

In BA scale-free network,

$$\begin{aligned}\langle k\rangle &= \int_m^\infty kP(k)dk = 2m,\\ \langle k^2\rangle &= \sum_k k^2 P(k),\\ P(k) &= 2m^2k^{-3}.\end{aligned} \tag{7}$$

Assume that the maximum degree in the supply chain finance network is k_l; as the network goes to infinity, k_l tends to mN^2, so $\langle k^2\rangle \approx 2m^2\ln(k_l/m)$. The critical value λ_c can be represented as

$$\lambda_c = \frac{\gamma\langle k\rangle}{\langle k^2\rangle} = \frac{\gamma}{2m\ln N}. \tag{8}$$

According to equation (8), we find that the critical value λ_c is negatively correlated with the network structure m and the network size N but is positively correlated to the enterprise's resilience γ.

4. Simulation Results and Discussion under Epidemic Impact

4.1. Simulation Algorithm Design

(1) BA scale-free network is constructed. According to the existing research [17, 18, 20], this paper sets the basic parameters of the network as $N = 1000$ and $m = 5$.

(2) According to the scope of the epidemic, the initial infected nodes were randomly selected from the network.

(3) The number of newly infected nodes are determined. Assume that the status of the node i is $S(i)$; when the node is infected, $S(i) = 1$; if not infected, $S(i) = 0$; External impact on node i can be expressed as

$$\beta(i) = 1-(1-\lambda)^{\alpha_i}, \tag{9}$$

where $\alpha_i = \sum_{j=1}^N a_{ij}S_j, i \neq j$ indicates the number of infected nodes in adjacent nodes of the node i, a_{ij} indicates the connection status between node i and node j, $a_{ij} = 1$ indicates that the two nodes are directly connected; otherwise, they are not adjacent nodes; $\beta(i)$ denotes the probability that node i is infected by at least one neighboring infected node, which can be used to represent its external impact.

The risk threshold C_i of node i can be expressed as

$$C_i = (\delta + c_i)\frac{k_i}{2m}. \tag{10}$$

The current assets owned by enterprises have a certain resistance to credit risk, so the resistance of enterprises is expressed as δ, which is set to 0.25. To represent the differences of enterprises, we introduce random number c_i

$(0 < c_i < 1)$. If $C_i < \beta(i)$, node i is infected. Otherwise, node i is not infected.

4.2. Analysis of Simulation Results under Epidemic Impact

4.2.1. Determine the Risk of Contagion λ. First, we need to determine the probability of risk contagion λ matching the credit risk contagion network, which is related to λ_c in equation (8); when $\lambda > \lambda_c$, credit risk becomes contagious; otherwise, credit risk will not be contagious. According to equation (8), we set the initial network state to $N = 1000$, $m = 5$, and $\gamma = 0.2$ and simulated the influence of network parameters γ, m, and N on λ. The simulation results are shown in Figure 1.

According to the above simulation results, we found that with the increase in self-healing rate γ, λ_c increased continuously. With different γ, λ_c is between 0.2 and 0.3. For network complexity m, when $m = 1$, the density of infected nodes in the steady state is 0, indicating that there is no risk infection within the network. With the increase in network complexity, λ_c gradually decreases, while λ_c has a maximum value of about 0.3. There is a negative correlation between the network size and λ_c; when $N = 200$, λ_c is about 0.21, and when $N = 1000$, λ_c is about 0.25. Therefore, in order to ensure that the simulation results are not affected by risk contagion probability, λ is set as 0.3 in the subsequent simulation experiments of this paper.

4.2.2. Analysis of Simulation Results of Enterprise Self-Healing Mode under Epidemic Impact. We first consider that in the case of no policy intervention, the enterprise has a certain self-healing rate γ. That is, the enterprise has a certain probability to recover to a healthy state. Compared with the credit default of an internal enterprise, the outbreak of COVID-19 will have a more serious impact, mainly reflected in the following: a large number of enterprises are faced with shutdown and production, demand decline, labor shortage, and other problems, which leads to a sudden increase in the number of initially infected enterprises. Under the epidemic, the probability of default of core enterprises increases. As core enterprises are connected with a large number of enterprises, default will inevitably have a serious impact on the entire network. Therefore, in the adjustment of the initial network state, we increased the initial number of infected nodes m_0 to represent the impact caused by the epidemic impact. The initial number of infected nodes indicates the degree to which different industries are affected by the epidemic. Finally, the influence of the number of core enterprises infected on credit risk contagion is discussed. First, we consider the impact of the initial number of infected nodes on the supply chain finance network, and the simulation results are shown in Figure 2.

According to the simulation results, we find that the initial number of infected nodes m_0 is basically positively correlated with the contagion speed of credit risk. However, for the steady state, the difference between Figures 2(a) and 2(b) is not obvious because the risk contagion threshold of SMEs C_i is relatively low. Without policy intervention and increase in the initial number of infected nodes, most nodes will be infected. Now, we will discuss the impact of core enterprises infection on credit risk contagion. The simulation results are shown in Figure 3.

According to the simulation results, it is found that when the initial number of infected nodes is small ($m_0 = 10$) and $c = 0$, there is a difference between the credit risk contagion rate and the infection rate of core enterprises. With the increase in external shock intensity (initial infection number), the difference of credit risk contagion speed gradually decreases. The main reason for this phenomenon is that the default of core enterprises will affect a large number of small- and medium-sized enterprises, which makes the contagion speed increase rapidly. The increase in initial infection nodes will also increase the transmission speed of credit risks, thus narrowing the gap between $c = 0$ and other cases.

Considering the impact of network nature on credit risk contagion, we will next discuss the resistance of network structure to epidemic impact. There are differences between different types of supply chains, which are related to the attributes of each enterprise and the connections between enterprises. Under the impact of the epidemic, it is worth exploring whether different self-healing rates, network complexity, and network size will affect the transmission of credit risk. Here, set the basic network parameters to $\lambda = 0.3$, $c = 0$, $N = 1000$, and $m = 5$. First, we discuss the self-healing ability of enterprises and its impact on credit risk contagion. The simulation results are shown in Figure 4.

According to the simulation results, we find that the enterprise self-healing rate γ has a significant impact on steady state and credit risk contagion speed. When $m_0 = 5$ and γ greater than 0.4, the steady state is 0, and there is no credit risk contagion. With the continuous decrease in γ, the contagion rate of credit risk gradually increases. When $\gamma = 0.3$, the network basically reaches a steady state at 36 steps. When $\gamma = 0.1$, it takes only 20 steps to basically reach steady state. When $m_0 = 5$ and $\gamma = 0.3$, the steady state is basically maintained at 80%. When $\gamma = 0.1$, the final steady state increases to 90%. This conclusion is basically consistent in the face of different degrees of initial shock: even when the initial number of infected nodes reaches 200, the final steady-state difference between $\gamma = 0.1$ and $\gamma = 0.5$ remains at 10%. It shows that the enterprise self-healing rate has a regulating effect on the final steady state of the network. In order to ensure that the enterprise self-healing rate would not affect the authenticity of other simulation results, the basic value of the enterprise self-healing rate was set as 0.2.

The impact of network complexity on credit risk contagion under different external shocks is shown in Figure 5. According to the simulation results, we find that network complexity has a significant impact on the rate of credit risk contagion. When $m_0 = 5$, there is no credit risk contagion in the network where m is less than 2. With the increase in m, the contagion rate of credit risk keeps increasing. When $m = 6$, the network has basically reached steady state in 6 steps. With the increase in external impact, the law is basically consistent: even if $m_0 = 200$, the number of infected nodes at the initial stage of $m = 1$ increases rapidly; however, after 4 steps, the number of infected nodes began to decline

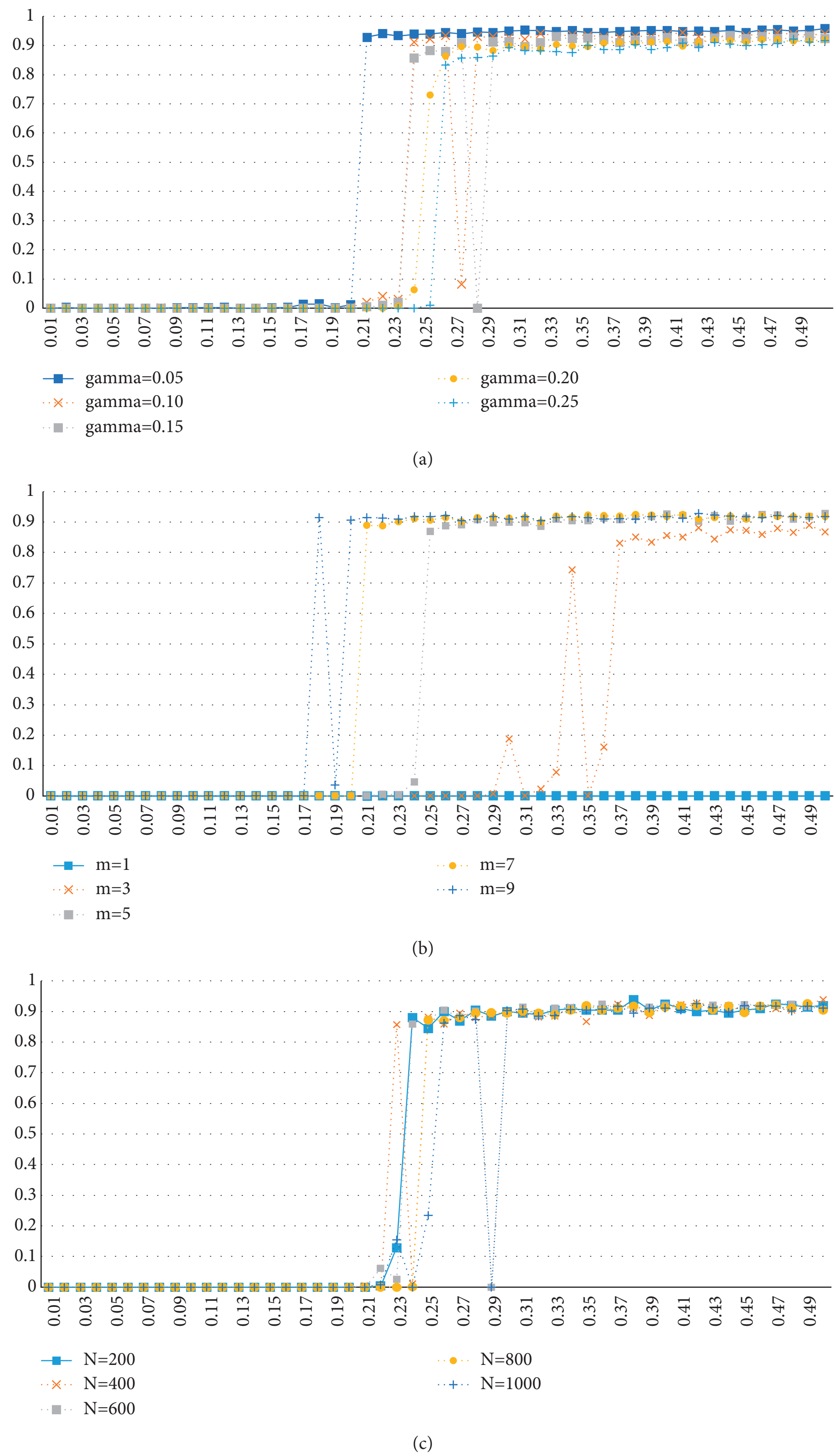


Figure 1: The influence of network parameters on risk contagion probability λ. (a) γ ranges from 0.05 to 0.25; step length is 0.05. (b) m ranges from 1 to 9; step length is 2. (c) N ranges from 200 to 1000; step length is 200.

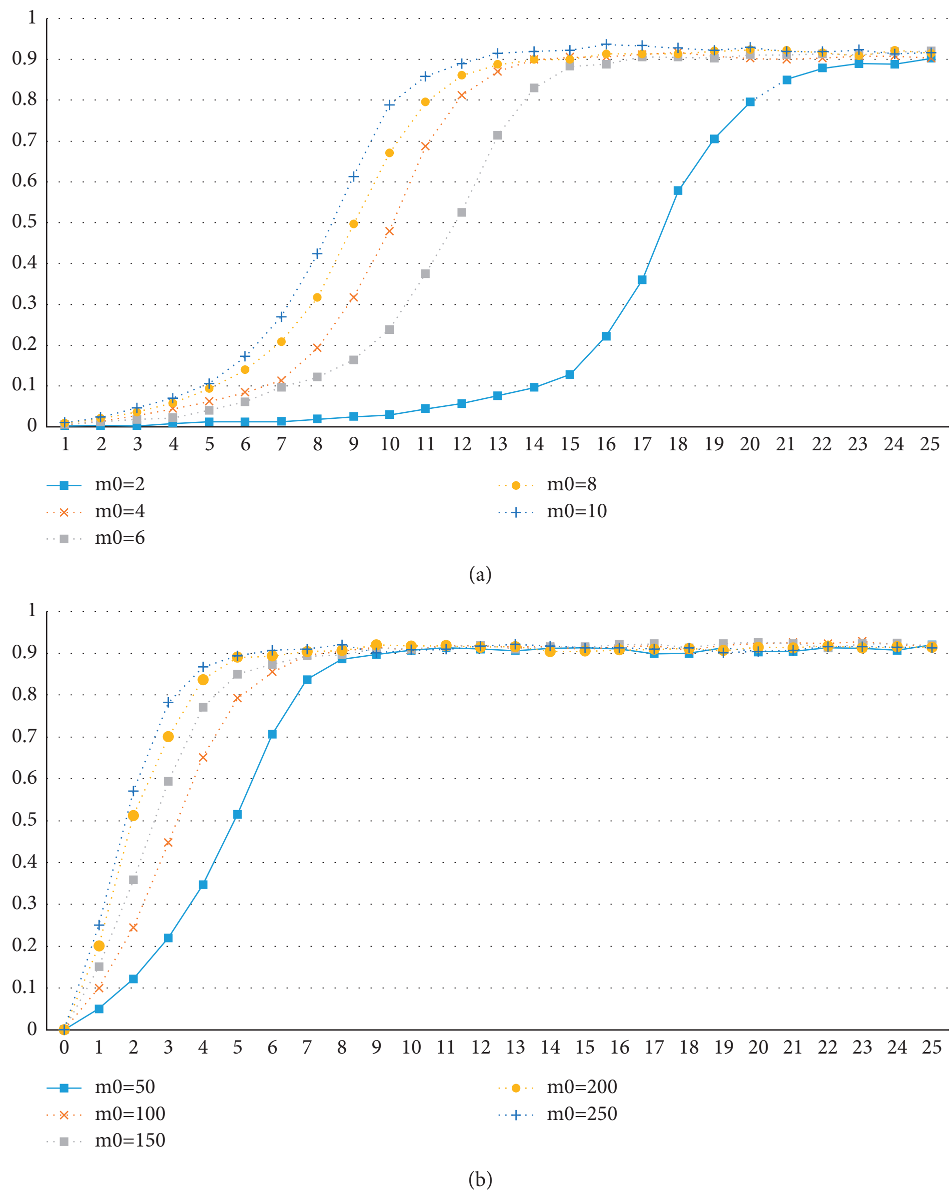
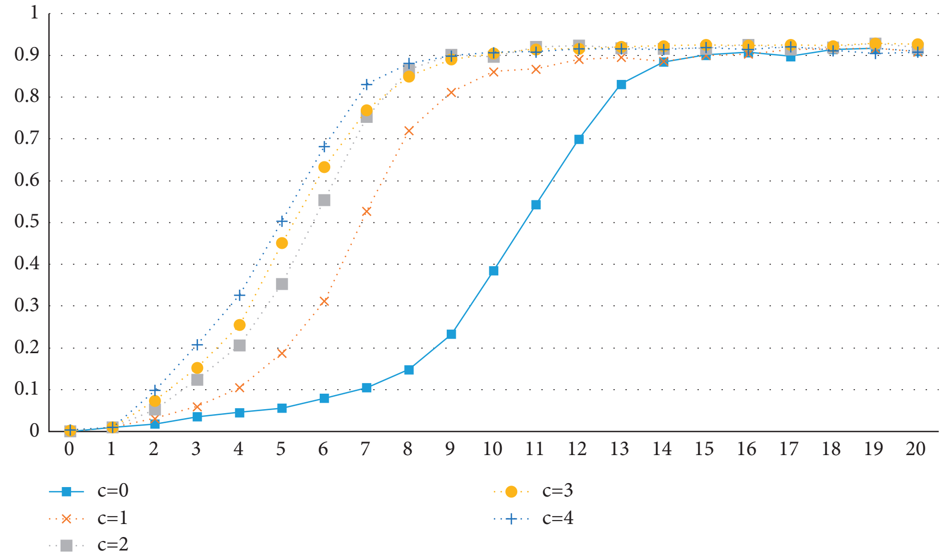


FIGURE 2: Effect of initial number of infected nodes m_0 on ρ_k. (a) m_0 ranges from 2 to 10; step length is 2. (b) m_0 ranges from 50 to 250; step length is 50.

and finally stopped infection. When $m = 2$, the contagion rate of credit is also low, but it still reaches steady state after 50 steps. This experimental result shows that network complexity is positively correlated with the transmission speed of credit risk. Although network complexity improves the efficiency of supply chain finance to some extent, when confronted with external shocks, more connections between enterprises become the channel of credit risk transmission, thus accelerating the transmission of credit risk.

Next, we discuss the impact of network size on credit risk contagion, and the simulation results are shown in Figure 6. We find that network size has no significant influence on the final steady state: at different scales, network size does not have a significant influence on the steady state, and this conclusion is still valid under the circumstance of increasing external shocks. When the initial shock is small, the network size has a certain impact on the transmission speed of credit risk. When the scale is large, the transmission speed of credit risk is relatively slow. However, with the increasing external shock, the curve highly overlaps, and the impact of network size can be almost ignored.

4.2.3. Analysis of Simulation Results of Policy Intervention Mode under Epidemic Impact. In order to better study the influence of policy intervention on credit risk contagion, the enterprise recovery time t_d is introduced into the network to represent the time step required for an enterprise to recover from an infected state to a healthy state under policy intervention. The smaller t_d is, the stronger the policy



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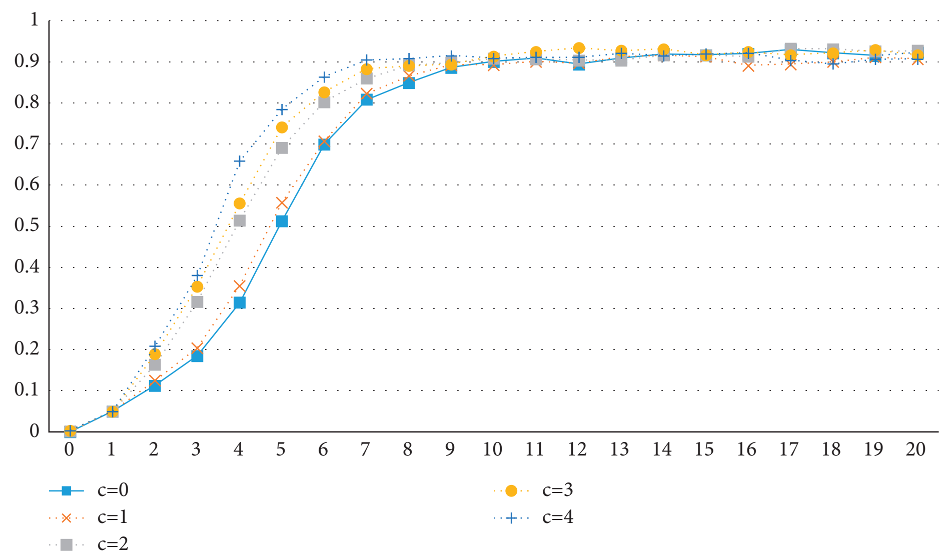

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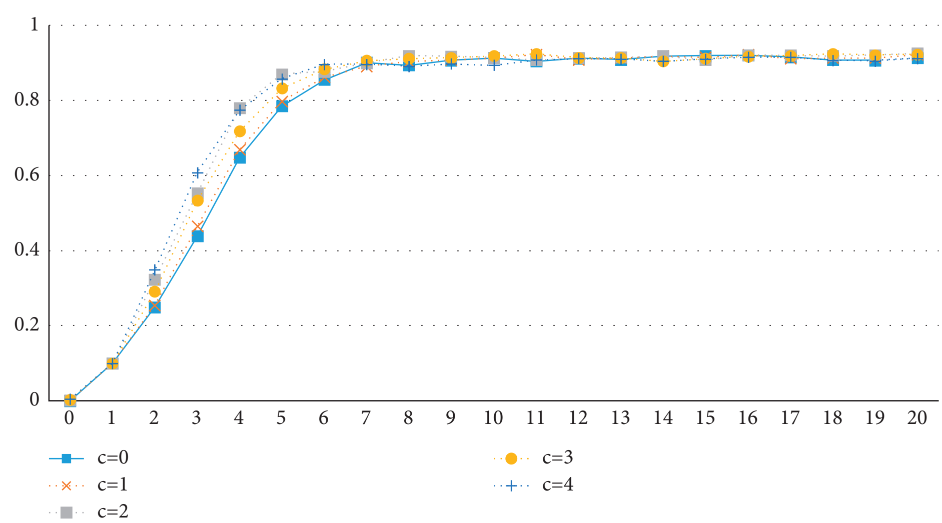
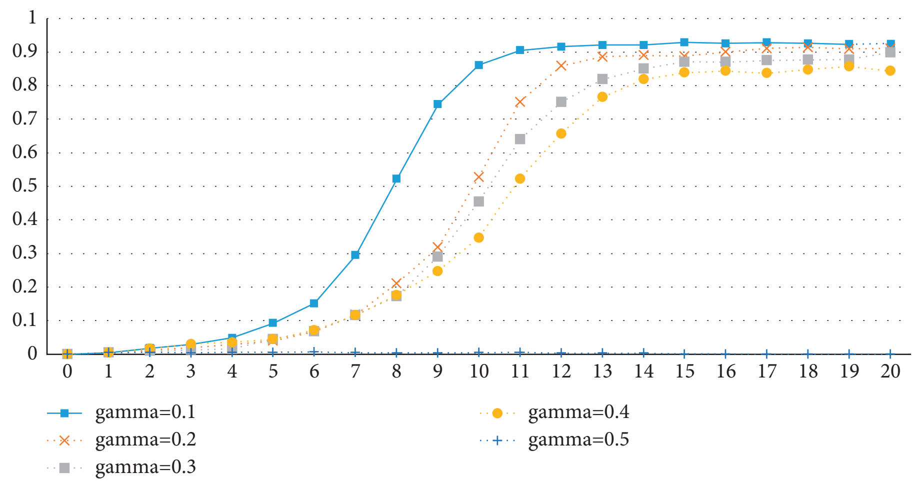


(c)

FIGURE 3: Effect of infected core enterprises c on ρ_k. (a) $m_0 = 10$; c ranges from 0 to 4; step length is 1. (b) $m_0 = 50$; c ranges from 0 to 4; step length is 1. (c) $m_0 = 100$; c ranges from 0 to 4; step length is 1.



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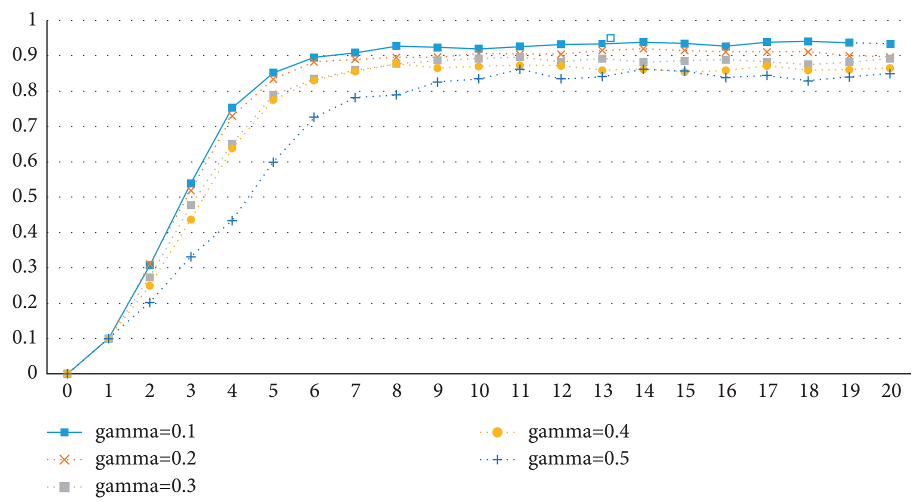

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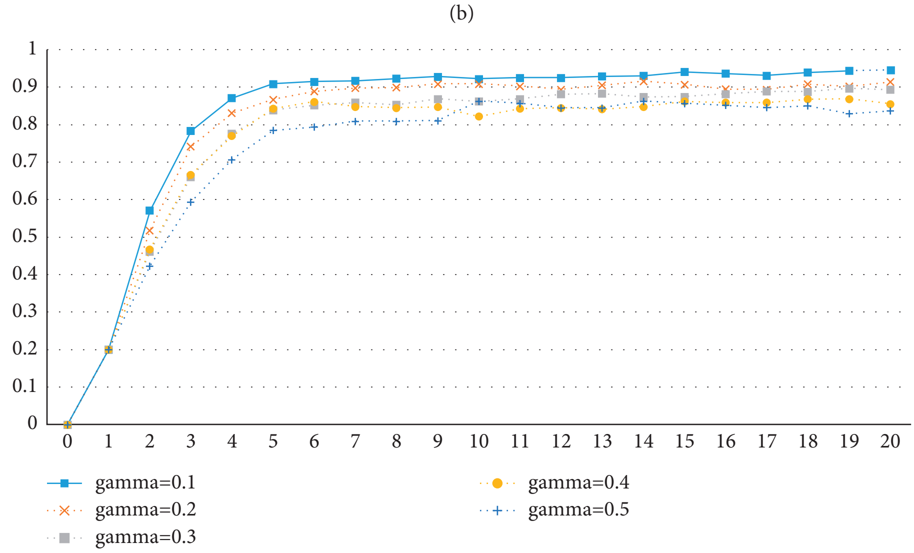


(c)

Figure 4: The impact of enterprise self-healing rate γ on ρ_k. (a) $m_0 = 5$; γ ranges from 0.1 to 0.5; step length is 0.1. (b) $m_0 = 100$; γ ranges from 0.1 to 0.5; step length is 0.1. (c) $m_0 = 200$; γ ranges from 0.1 to 0.5; step length is 0.1.

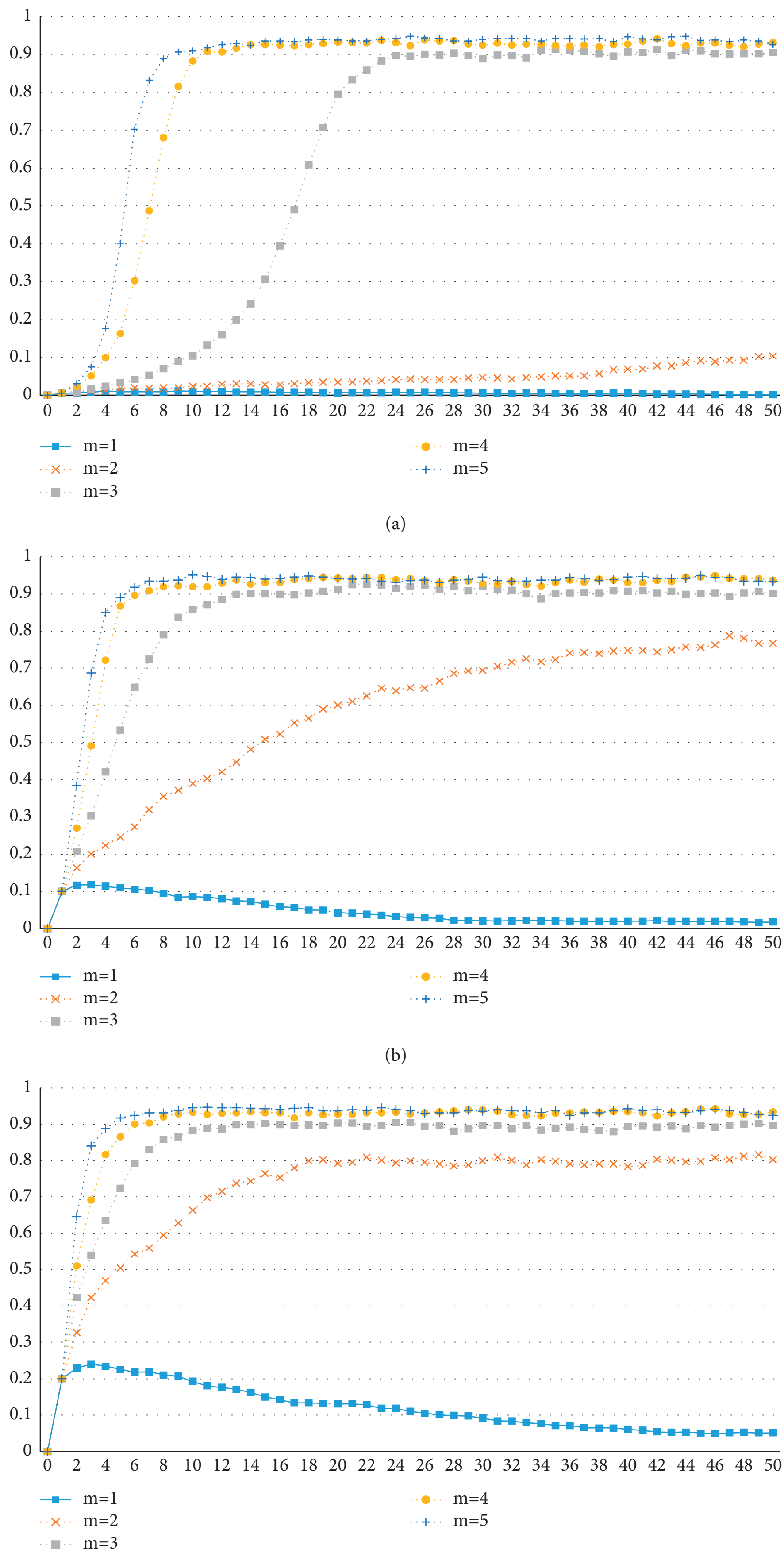
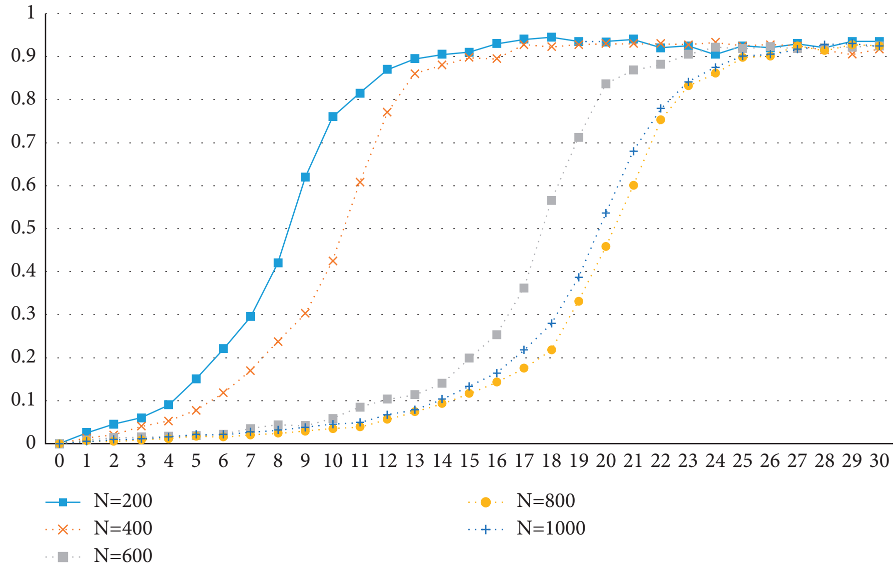


FIGURE 5: Impact of network complexity m on ρ_k. (a) $m_0 = 5$; m ranges from 1 to 5; step length is 1. (b) $m_0 = 100$; m ranges from 1 to 5; step length is 1. (c) $m_0 = 200$; m ranges from 1 to 6; step length is 1.



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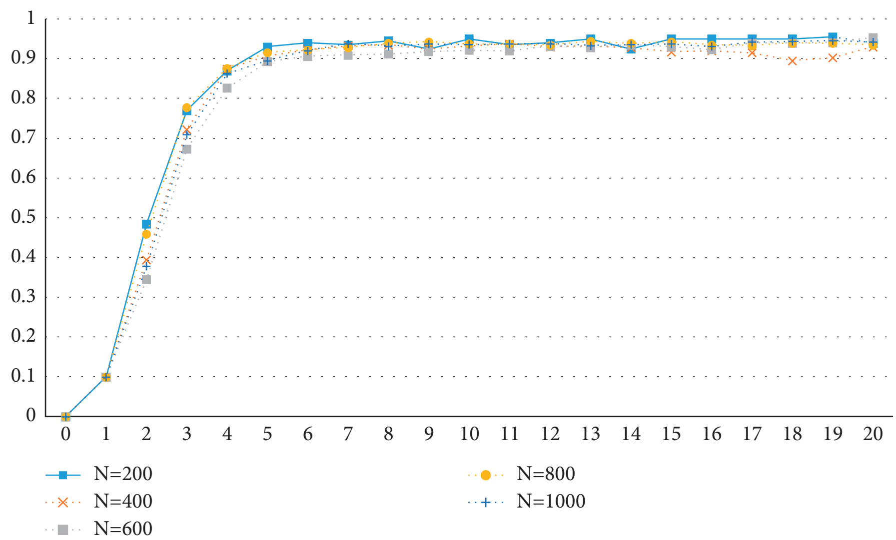

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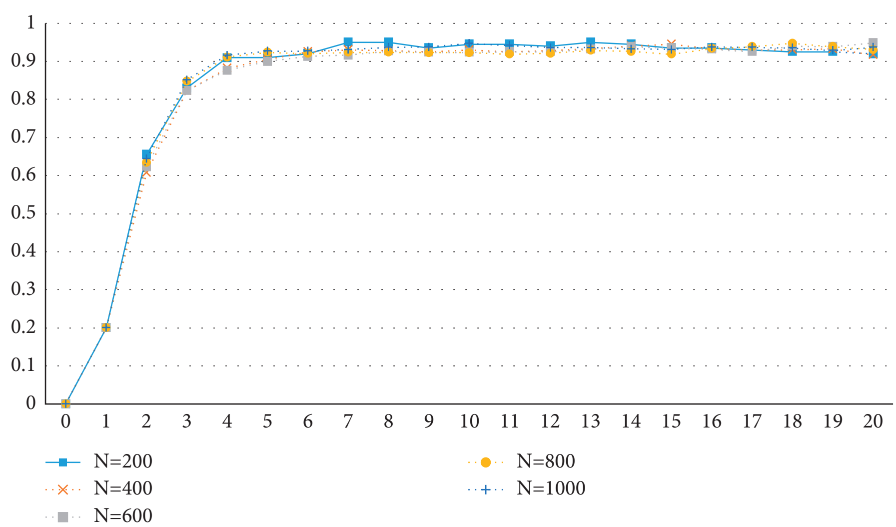


(c)

FIGURE 6: Impact of network size N on ρ_k. (a) $m_0 = 5$; n ranges from 100 to 1000; step length is 200. (b) $m_0 = 100$; n ranges from 100 to 1000; step length is 200. (c) $m_0 = 200$; n ranges from 100 to 1000; step length is 200.

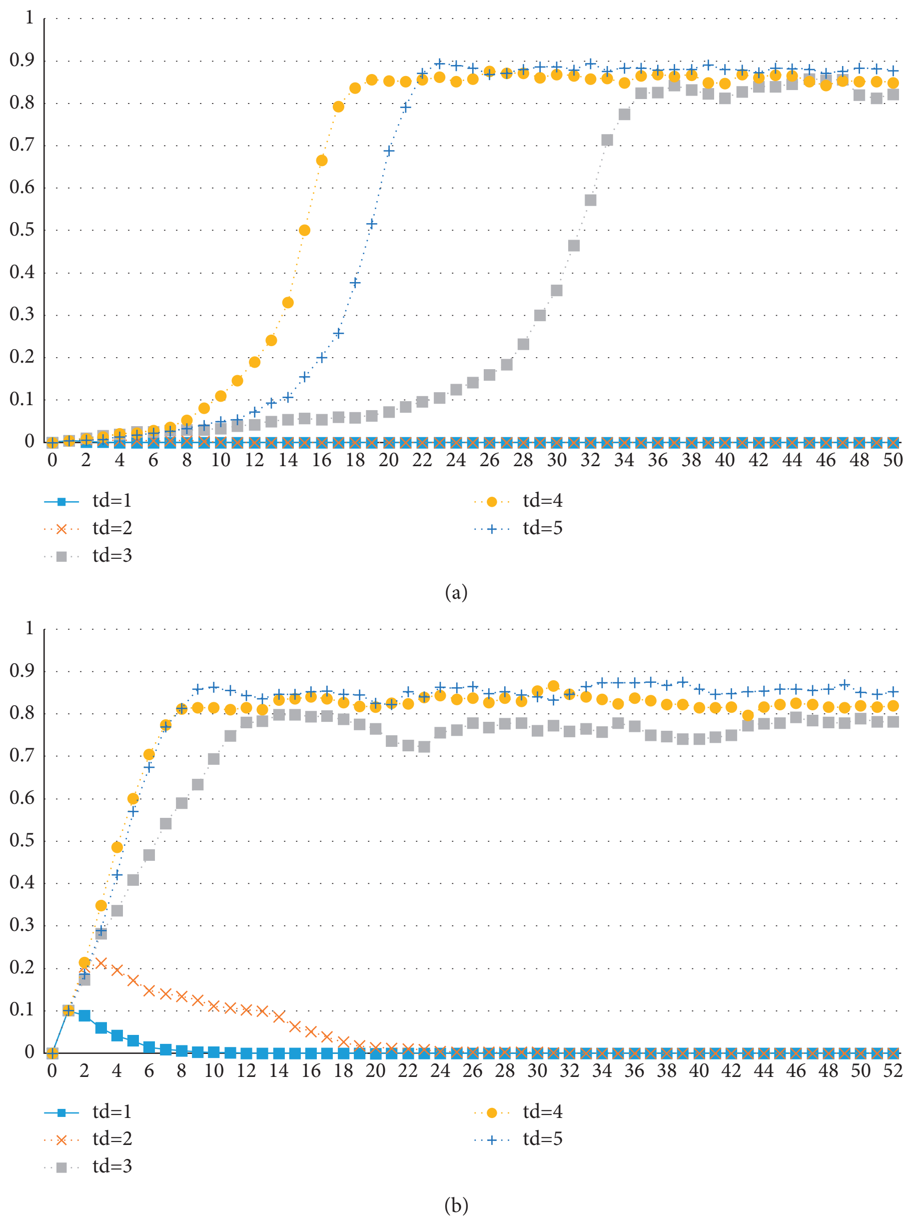
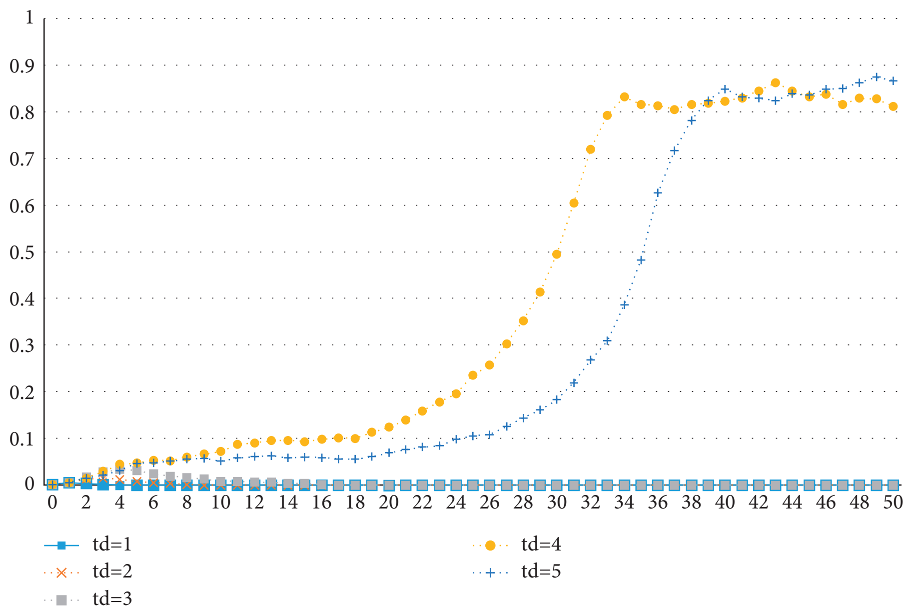


Figure 7: Impact of policy intervention t_d on ρ_k $(c = 0)$. (a) $m_0 = 5$; t_d ranges from 1 to 5; step length is 1. (b) $m_0 = 100$; t_d ranges from 1 to 5; step length is 1.

intervention is; otherwise, the weaker it is. We defined $t_d \leq 2$ as a strong intervention, $t_d = 3$ as a moderate intervention, and $t_d \geq 4$ as a weak intervention. First of all, we consider the suppression effect of policy intensity on epidemic impact when there is no initial infection of core enterprises $(c = 0)$, and the simulation results are shown in Figure 7.

According to the simulation results, we find that in the face of different degrees of external shocks, policy intervention has a significant restraining effect on the rate of credit risk contagion: when the external shocks are small $(m_0 = 5)$, $t_d \leq 2$ can effectively suppress the credit risk contagion. Even if the policy intervention is weak $(t_d \geq 3)$, the contagion rate of credit risk is still low in the initial stage. When the initial impact is enhanced, the inhibitory effect of strong intervention on the epidemic impact is still relatively obvious, but the inhibitory effect of $t_d \geq 3$ on the transmission rate of credit risk is weakened. When $m_0 = 100$, $t_d = 3$ could not completely inhibit the transmission of credit risk. There are two possible explanations for the above conclusions: first, the impact of the epidemic on different industries is different and the intervention intensity is different; second, different levels of intervention are required at different stages of the epidemic. Therefore, the intensity of policy intervention in different industries should be different, and the effect of intervention in the early stage of the outbreak is better, and the cost of intervention is lower. Next, we consider the case of $c = 1$, and the simulation results are shown in Figure 8.

The infection of core enterprises has a great impact on the network. In the case of $m_0 = 5$, $t_d < 3$ can also restrain the



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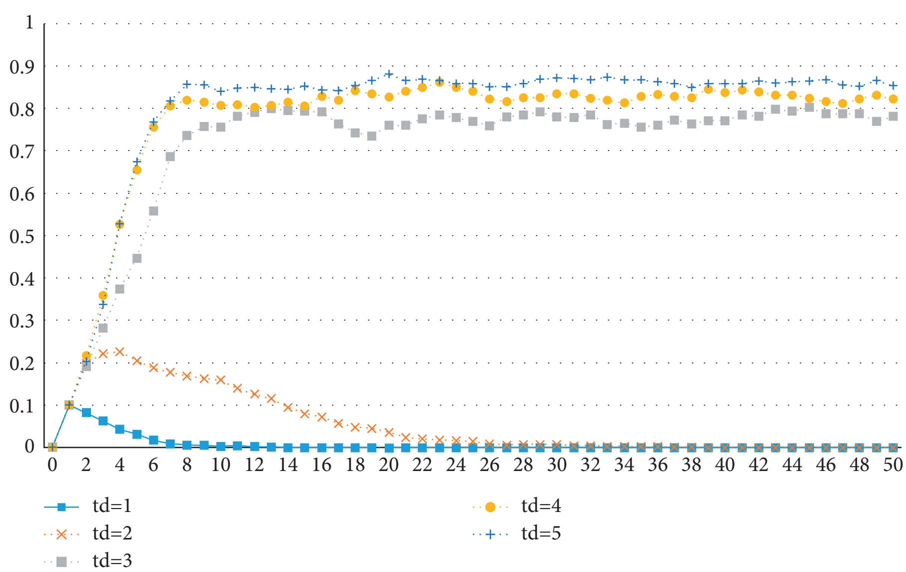

(b)

FIGURE 8: Impact of policy intervention t_d on $\rho_k(c = 1)$. (a) $m_0 = 5$; t_d ranges from 1 to 5; step length is 1. (b) $m_0 = 100$; t_d ranges from 1 to 5; step length is 1.

credit risk contagion. However, with the strengthening of external shocks ($m_0 = 100$), $t_d = 3$ is unable to restrain the credit risk contagion, and the credit risk contagion speed increases greatly when $t_d \geq 4$. Due to the importance of the health status of core enterprises to the network risk transmission, attention should be paid in the process of the outbreak.

5. Conclusion

This paper conducts a simulation experiment on the credit risk contagion of supply chain finance network under the COVID-19 pandemic. First, considering the mode of self-healing without policy intervention, we find that the increase in the initial number of infected nodes will significantly increase the contagion rate of credit risk, but the impact on the steady state is relatively limited. The self-healing rate of enterprises can significantly reduce the speed of credit risk contagion and the density of infected nodes in the steady state. As the complexity of network increases the channels of credit risk transmission, the speed of risk transmission increases. However, the network size has no significant influence on the infection rate and the steady state. However, when the external shock is large enough, the steady state of the network is above 80%, which indicates that in the absence of policy intervention, the self-healing ability of enterprises alone cannot resist the impact of external shock on the network, resulting in the failure of the supply chain finance network to play its due function.

Second, under the external shock, policy intervention is essential to ensure the normal operation of the supply chain and reduces the possibility of credit risk contagion. In the case of small external impact, even moderate intensity intervention can achieve the effect of infection suppression. However, as external shocks grow, so too should the scale of intervention. This shows that the impact of the epidemic on different industries is different, and policy intervention measures should also be targeted. In the early stage of the epidemic, timely and targeted government intervention will achieve obvious results.

Finally, due to the importance of its position, the speed of the credit risk contagion will be significantly increased once credit default occurs in core enterprises of the supply chain. Therefore, whilst focusing on key industries and preventing the supply chain network from being severely impacted, the health of core enterprises should be focused on.

This paper has some limitations and could be improved in the follow-up research. First of all, this paper ignores the particularity of different supply chain financial networks and selects the same contagion probability. Subsequent studies can further explore the characteristics of specific networks. In addition, this paper assumes that the scale of the network remains unchanged and does not consider the entry and exit mechanism of nodes. In order to be closer to the reality, corresponding discussions can be made in subsequent studies.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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