Journal of Function Spaces

q-Analysis and Its Applications

Lead Guest Editor: Serkan Araci
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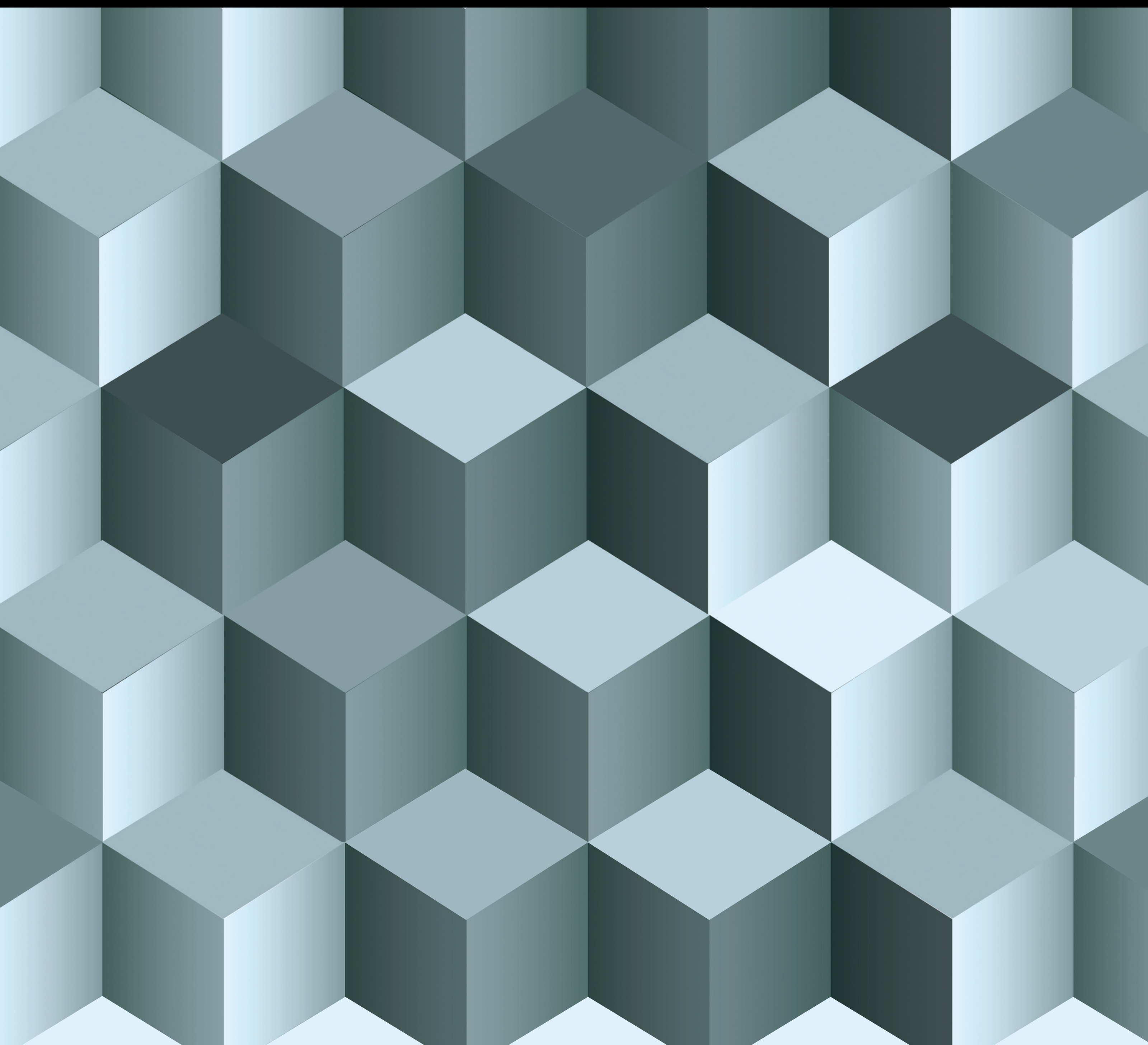

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Research Article

A Note on q-Fubini-Appell Polynomials and Related Properties

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The present article is aimed at introducing and investigating a new class of q-hybrid special polynomials, namely, q-Fubini-Appell polynomials. The generating functions, series representations, and certain other significant relations and identities of this class are established. Some members of q-Fubini-Appell polynomial family are investigated, and some properties of these members are obtained. Further, the class of 3-variable q-Fubini-Appell polynomials is also introduced, and some formulae related to this class are obtained. In addition, the determinant representations for these classes are established.

1. Introduction

The q-calculus subject has gained prominence and numerous popularity during the last three decades or so (see [1–4]). The contemporaneous interest in this subject is due to the fact that q-series has popped in such diverse fields as quantum groups, statistical mechanics, and transcendental number theory. The notations and definitions related to q-calculus used in this article are taken from [2] (see also [5, 6]).

The q-analogues of a number $\ell \in \mathbb{C}$ and the factorial function are, respectively, specified by

$$[\ell]_q = \frac{1-q^\ell}{1-q}, \quad (q \in \mathbb{C} \setminus \{1\}), \tag{1}$$

and

$$[\kappa]_q! = \prod_{l=1}^{\kappa} [l]_q = [1]_q [2]_q [3]_q \cdots [\kappa]_q, \; [0]_q! = 1, \quad \kappa \in \mathbb{N}, q \in \mathbb{C} \setminus \{0, 1\}. \tag{2}$$

The q-binomial coefficient $\begin{bmatrix} \kappa \\ l \end{bmatrix}_q$ is specified by

$$\begin{bmatrix} \kappa \\ l \end{bmatrix}_q = \frac{[\kappa]_q!}{[l]_q! [\kappa - l]_q!}, \quad l = 0, 1, 2, \cdots, \kappa \, ; \kappa \in \mathbb{N}_0. \tag{3}$$

The q-analogue of $(u \oplus v)^\kappa$ is specified as

$$(u \oplus v)_q^\kappa = \sum_{l=0}^{\kappa} \begin{bmatrix} \kappa \\ l \end{bmatrix}_q q^{\binom{\kappa-l}{2}} u^l v^{\kappa-l}. \tag{4}$$

The q-derivative of a function f at a point $\tau \in \mathbb{C} \setminus \{0\}$ is given as

$$D_q f(\tau) = \frac{f(\tau) - f(q\tau)}{\tau - q\tau}, \quad 0 < |q| < 1. \tag{5}$$

The functions

$$e_q(\tau) = \sum_{\kappa=0}^{\infty} \frac{\tau^\kappa}{[\kappa]_q!}, \quad 0 < |q| < 1, |\tau| < |1-q|^{-1}, \tag{6}$$

$$E_q(\tau) = \sum_{\kappa=0}^{\infty} q^{\binom{\kappa}{2}} \frac{\tau^\kappa}{[\kappa]_q!}, \quad 0 < |q| < 1, \tau \in \mathbb{C}, \tag{7}$$

are called q-exponential functions and satisfy the following identities:

$$\begin{gathered} D_q e_q(\tau) = e_q(\tau), \quad D_q E_q(\tau) = E_q(q\tau), \\ e_q(\tau)E_q(-\tau) = E_q(\tau)e_q(-\tau) = 1. \end{gathered} \tag{8}$$

The Fubini polynomials (FP) $\mathscr{F}_\kappa(w)$ [7] (also known as geometric polynomials) are defined as

$$\frac{1}{1 - w(e^\tau - 1)} = \sum_{\kappa=0}^{\infty} \mathscr{F}_\kappa(w) \frac{\tau^\kappa}{\kappa!}, \tag{9}$$

together with the geometric series

$$\frac{1}{1-w} \mathscr{F}_m\left(\frac{w}{1-w}\right) = \sum_{l=0}^{\infty} l^m w^l, \quad |w| < 1. \tag{10}$$

Recently, Duran et al. [8] introduced the q-analogue of the FP $\mathscr{F}_\kappa(w)$, denoted by $\mathscr{F}_{\kappa,q}(w)$ and defined by means of the generating function

$$\frac{1}{1 - w\left(e_q(\tau) - 1\right)} = \sum_{\kappa=0}^{\infty} \mathscr{F}_{\kappa,q}(w) \frac{\tau^\kappa}{[\kappa]_q!}. \tag{11}$$

For $w = 1$, the q-Fubini polynomials (q-FP) $\mathscr{F}_{\kappa,q}(w)$ reduce to the q-Fubini numbers $\mathscr{F}_{\kappa,q}(1) \coloneqq \mathscr{F}_{\kappa,q}$, that is

$$\frac{1}{2 - e_q(\tau)} = \sum_{\kappa=0}^{\infty} \mathscr{F}_{\kappa,q} \frac{\tau^\kappa}{[\kappa]_q!}. \tag{12}$$

Further, we recall the 3-variable q-Fubini polynomials (3Vq-FP) $\mathscr{F}_{\kappa,q}(u, v, w)$ [8] which are given as

$$\frac{1}{1 - w\left(e_q(\tau) - 1\right)} e_q(u\tau) E_q(v\tau) = \sum_{\kappa=0}^{\infty} \mathscr{F}_{\kappa,q}(u, v, w) \frac{\tau^\kappa}{[\kappa]_q!}. \tag{13}$$

Substantial properties of Fubini numbers and polynomials and their q-analogue have been studied and investigated by many researchers (see [7–9] and the references cited therein). Further, these numbers and polynomials have enormous applications in analytic number theory, physics, and the other related areas.

The class of the q-special polynomials such as q-Fubini polynomials, q-Appell polynomials, and certain members belonging to the family of q-Appell polynomials such as q-Bernoulli polynomials and q-Euler polynomials is an expanding field in mathematics [3, 7, 8, 10, 11].

The class of q-Appell polynomial sequences $\{\mathscr{A}_{\kappa,q}(w)\}_{\kappa=0}^{\infty}$ was established and investigated by Al-Salam [1]. These polynomials are defined by means of the generating function

$$\mathscr{A}_q(\tau) e_q(w\tau) = \sum_{\kappa=0}^{\infty} \mathscr{A}_{\kappa,q}(w) \frac{\tau^\kappa}{[\kappa]_q!}, \tag{14}$$

where

$$\mathscr{A}_q(\tau) = \sum_{\kappa=0}^{\infty} \mathscr{A}_{\kappa,q} \frac{\tau^\kappa}{[\kappa]_q!}, \quad \mathscr{A}_q(\tau) \neq 0\,; \mathscr{A}_{0,q} = 1, \tag{15}$$

is an analytic function at $\tau = 0$ and $\mathscr{A}_{\kappa,q} \coloneqq \mathscr{A}_{\kappa,q}(0)$ denotes the q-Appell numbers.

Certain significant members belonging to q-Appell polynomials class are obtained based on suitable selection for the function $\mathscr{A}_q(\tau)$ as

(1) If $\mathscr{A}_q(\tau) = \tau/(e_q(\tau) - 1)$, the q-AP $\mathscr{A}_{\kappa,q}(w)$ reduce to the q-Bernoulli polynomials (q-BP) $\mathfrak{B}_{\kappa,q}(w)$ (see [12, 13]), that is

$$\mathscr{A}_{\kappa,q}(w) \coloneqq \mathfrak{B}_{\kappa,q}(w), \tag{16}$$

where $\mathfrak{B}_{\kappa,q}(w)$ are defined by

$$\frac{\tau}{e_q(\tau) - 1} e_q(w\tau) = \sum_{\kappa=0}^{\infty} \mathfrak{B}_{\kappa,q}(w) \frac{\tau^\kappa}{[\kappa]_q!}, \tag{17}$$

and $\mathfrak{B}_{\kappa,q}$ given by

$$\mathfrak{B}_{\kappa,q} \coloneqq \mathfrak{B}_{\kappa,q}(0), \tag{18}$$

denotes the q-Bernoulli numbers.

(2) If $\mathscr{A}_q(\tau) = 2/(e_q(\tau) + 1)$, the q-AP $\mathscr{A}_{\kappa,q}(w)$ reduce to the q-Euler polynomials (q-EP) $\mathscr{E}_{\kappa,q}(w)$ (see [13, 14]), that is

$$\mathscr{A}_{\kappa,q}(w) \coloneqq \mathscr{E}_{\kappa,q}(w), \tag{19}$$

where $\mathscr{E}_{\kappa,q}(w)$ are defined by

$$\frac{2}{e_q(\tau) + 1} e_q(w\tau) = \sum_{\kappa=0}^{\infty} \mathscr{E}_{\kappa,q}(w) \frac{\tau^\kappa}{[\kappa]_q!}, \tag{20}$$

and $\mathscr{E}_{\kappa,q}$ given by

$$\mathscr{E}_{\kappa,q} := \mathscr{E}_{\kappa,q}(0), \tag{21}$$

denotes the q-Euler numbers.

Also, we recall the family of the numbers denoted by $\mathcal{S}_{2,q}(\kappa, l)$ and defined by

$$\frac{\left(e_q(\tau)-1\right)^l}{[l]_q!} = \sum_{\kappa=l}^{\infty} \mathcal{S}_{2,q}(\kappa, l)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{22}$$

In recent years, many authors have shown their interest to introduce and study new families of q-special polynomials, especially the hybrid type (see [15–17] and the references therein).

The work in this article is summarized as follows: in Section 2, the replacement technique is used to introduce the class of q-Fubini-Appell polynomials by combining the polynomials, q-Fubini polynomials and q-Appell polynomials. In Section 3, the 3-variable q-Fubini-Appell polynomials are introduced which are considered as a generalization of the q-Fubini-Appell polynomials. The generating relations, series representations, and some other useful properties related to these polynomials are established. In Section 4, the determinant representations of these two classes are defined. Further, certain members belonging to these polynomial families are considered, and the corresponding results are also derived.

2. q-Fubini-Appell Polynomials

The q-Fubini-Appell polynomials are established by means of the generating function and series representation. To achieve this, we prove the following results:

Theorem 1. *The q-Fubini-Appell polynomials (q-FAP) ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ are defined by means of the following generating function:*

$$\frac{\mathscr{A}_q(\tau)}{1-w\left(e_q(\tau)-1\right)} = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{23}$$

Proof. Utilizing equation (14), based on expanding the function $e_q(w\tau)$, then replacing the powers of w, i.e., $w^0, w, w^2, \cdots, w^\kappa$ by the corresponding polynomials $\mathscr{F}_{0,q}(w), \mathscr{F}_{1,q}(w), \cdots, \mathscr{F}_{\kappa,q}(w)$ and thereafter summing up the terms in the left-hand side of the resulting equation, we obtain that

$$\mathscr{A}_q(\tau)\sum_{\kappa=0}^{\infty}\mathscr{F}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!} = \sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}\left(\mathscr{F}_{1,q}(w)\right)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{24}$$

Now, denoting the resultant q-FAP in the right hand side of the above equation by ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ and utilizing equation (11) yield the assertion in equation (23).

Remark 2. Taking $w=1$, the q-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ reduce to q-Fubini-Appell numbers (q-FAN) ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}$. Therefore, in view of equation (23), we have

$$\frac{\mathscr{A}_q(\tau)}{2-e_q(\tau)} = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}. \tag{25}$$

Corollary 3. *Taking $\mathscr{A}_q(\tau)=\tau/(e_q(\tau)-1)$ in equation (23), we get the following generating function of the q-Fubini-Bernoulli polynomials (q-FBP) ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w)$.*

$$\frac{\tau}{\left(e_q(\tau)-1\right)\left(1-w\left(e_q(\tau)-1\right)\right)} = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{26}$$

Corollary 4. *Taking $\mathscr{A}_q(\tau)=2/(e_q(\tau)+1)$ in equation (23), we get the following generating function of the q-Fubini-Euler polynomials (q-FEP) ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w)$*

$$\frac{2}{\left(e_q(\tau)+1\right)\left(1-w\left(e_q(\tau)-1\right)\right)} = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{27}$$

Theorem 5. *The following series representation for the q-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ holds true:*

$${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w) = \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathscr{A}_{l,q}\mathscr{F}_{\kappa-l,q}(w). \tag{28}$$

Proof. In view of equations (11) and (15) and equation (23), we have

$$\begin{aligned}\sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!} &= \frac{\mathscr{A}_q(\tau)}{1-w\left(e_q(\tau)-1\right)}\\ &= \left(\sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}\right)\left(\sum_{\kappa=0}^{\infty}\mathscr{F}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}\right)\\ &= \sum_{\kappa=0}^{\infty}\left(\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathscr{A}_{l,q}\mathscr{F}_{\kappa-l,q}(w)\right)\frac{\tau^\kappa}{[\kappa]_q!},\end{aligned} \tag{29}$$

which on comparing the coefficients of $\tau^\kappa/[\kappa]_q!$ yield assertion in equation (28).

Theorem 6. *For $n \in \mathbb{N}_0$, the following series representation for the q-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w)$ holds true:*

$${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(w) = \sum_{l=0}^{\kappa}\sum_{\sigma=0}^{l}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q [\sigma]_q! w^\sigma \mathscr{A}_{\kappa-l,q}\mathcal{S}_{2,q}(l,\sigma). \tag{30}$$

Proof. In view of equations (15), (22), and (23), we can write

$$\begin{aligned}\sum_{\kappa=0}^{\infty} {}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!} &= \frac{\mathcal{A}_q(\tau)}{1-w(e_q(\tau)-1)}\\ &= \sum_{\kappa=0}^{\infty}\mathcal{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}\sum_{\sigma=0}^{\infty} w^\sigma (e_q(\tau)-1)^\sigma\\ &= \sum_{\kappa=0}^{\infty}\mathcal{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}\sum_{\sigma=0}^{\infty} w^\sigma[\sigma]_q!\sum_{l=\sigma}^{\infty}\mathcal{S}_{2,q}(l,\sigma)\frac{\tau^l}{[l]_q!}\\ &= \sum_{\kappa=0}^{\infty}\mathcal{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}\sum_{l=0}^{\infty}\left(\sum_{\sigma=0}^{l} w^\sigma[\sigma]_q!\mathcal{S}_{2,q}(l,\sigma)\right)\frac{\tau^l}{[l]_q!}\\ &= \sum_{\kappa=0}^{\infty}\left(\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathcal{A}_{\kappa-l,q}\sum_{\sigma=0}^{l} w^\sigma[\sigma]_q!\mathcal{S}_{2,q}(l,\sigma)\right)\frac{\tau^\kappa}{[\kappa]_q!},\end{aligned} \tag{31}$$

which on comparing the coefficients of $\tau^\kappa/[\kappa]_q!$ yield assertion in equation (30).

Corollary 7. *Taking $\mathcal{A}_q(\tau)=\tau/(e_q(\tau)-1)$ in equations (28) and (30), we get the following series representations of the q-FBP ${}_{\mathcal{F}}\mathfrak{B}_{\kappa,q}(w)$*

$$\begin{aligned}{}_{\mathcal{F}}\mathfrak{B}_{\kappa,q}(w) &= \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathfrak{B}_{l,q}\mathcal{F}_{\kappa-l,q}(w),\\ {}_{\mathcal{F}}\mathfrak{B}_{\kappa,q}(w) &= \sum_{l=0}^{\kappa}\sum_{\sigma=0}^{l}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q [\sigma]_q! w^\sigma \mathfrak{B}_{\kappa-l,q}\mathcal{S}_{2,q}(l,\sigma), \quad n\in\mathbb{N}_0.\end{aligned} \tag{32}$$

Corollary 8. *Taking $\mathcal{A}_q(\tau)=2/(e_q(\tau)+1)$ in equations (28) and (30), we get the following series representations of the q-FEP ${}_{\mathcal{F}}\mathcal{E}_{\kappa,q}(w)$:*

$$\begin{aligned}{}_{\mathcal{F}}\mathcal{E}_{\kappa,q}(w) &= \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathcal{E}_{l,q}\mathcal{F}_{\kappa-l,q}(w),\\ {}_{\mathcal{F}}\mathcal{E}_{\kappa,q}(w) &= \sum_{l=0}^{\kappa}\sum_{\sigma=0}^{l}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q [\sigma]_q! w^\sigma \mathcal{E}_{\kappa-l,q}\mathcal{S}_{2,q}(l,\sigma), \quad n\in\mathbb{N}_0.\end{aligned} \tag{33}$$

Theorem 9. *The following formula for the q-FAP ${}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(w)$ holds true:*

$$\frac{d}{dw}{}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(w) = \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \left({}_{\mathcal{F}}\mathcal{A}_{\kappa-l,q}(w)\mathcal{F}_{l,q}(1,0,w) - {}_{\mathcal{F}}\mathcal{A}_{l,q}(w)\mathcal{F}_{\kappa-l,q}(w)\right). \tag{34}$$

Proof. Utilizing equation (23), we have

$$\begin{aligned}\frac{d}{dw}\left(\sum_{\kappa=0}^{\infty}{}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(w)\frac{\tau^\kappa}{[\kappa]_q!}\right) &= \frac{d}{dw}\left(\frac{\mathcal{A}_q(\tau)}{1-w(e_q(\tau)-1)}\frac{\mathcal{A}_q(\tau)}{1-w(e_q(\tau)-1)}\right)\\ &= \frac{\mathcal{A}_q(\tau)(e_q(\tau)-1)}{(1-w(e_q(\tau)-1))^2}\\ &= \frac{\mathcal{A}_q(\tau)e_q(\tau)}{(1-w(e_q(\tau)-1))^2} - \frac{\mathcal{A}_q(\tau)}{(1-w(e_q(\tau)-1))^2}\\ &= \sum_{\kappa=0}^{\infty}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathcal{F}}\mathcal{A}_{\kappa-l,q}(w)\mathcal{F}_{l,q}(1,0,w)\frac{\tau^\kappa}{[\kappa]_q!}\\ &\quad - \sum_{\kappa=0}^{\infty}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathcal{F}}\mathcal{A}_{l,q}(w)\mathcal{F}_{\kappa-l,q}(w)\frac{\tau^h}{[h]_q!},\end{aligned} \tag{35}$$

which on equating the coefficients of the like powers of τ yields the assertion in equation (34).

Corollary 10. *Taking $\mathcal{A}_q(\tau)=\tau/(e_q(\tau)-1)$ in equations (34), we get the formula satisfied by the q-FBP ${}_{\mathcal{F}}\mathfrak{B}_{\kappa,q}(w)$ as*

$$\frac{d}{dw}{}_{\mathcal{F}}\mathfrak{B}_{\kappa,q}(w) = \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \left({}_{\mathcal{F}}\mathfrak{B}_{\kappa-l,q}(w)\mathcal{F}_{l,q}(1,0,w) - {}_{\mathcal{F}}\mathfrak{B}_{l,q}(w)\mathcal{F}_{\kappa-l,q}(w)\right). \tag{36}$$

Corollary 11. *Taking $\mathcal{A}_q(\tau)=2/e_q(\tau)+1$ in equations (34), we get the formula satisfied by the q-FEP ${}_{\mathcal{F}}\mathcal{E}_{\kappa,q}(w)$ as*

$$\frac{d}{dw}{}_{\mathcal{F}}\mathcal{E}_{\kappa,q}(w) = \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \left({}_{\mathcal{F}}\mathcal{E}_{\kappa-l,q}(w)\mathcal{F}_{l,q}(1,0,w) - {}_{\mathcal{F}}\mathcal{E}_{l,q}(w)\mathcal{F}_{\kappa-l,q}(w)\right). \tag{37}$$

3. 3-Variable q-Fubini-Appell Polynomials

In this section, the class of 3-variable q-Fubini-Appell polynomials is established, which is a generalization of the class introduced in the previous section. The generating function, series representations, and other formulae for these polynomials are obtained.

Theorem 12. *The 3-variable q-Fubini-Appell polynomials (3Vq-FAP) ${}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(u,v,w)$ are defined by means of the following generating function:*

$$\frac{\mathcal{A}_q(\tau)}{1-w(e_q(\tau)-1)}e_q(u\tau)E_q(v\tau) = \sum_{\kappa=0}^{\infty}{}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!}. \tag{38}$$

Proof. Utilizing equations (13) and (14) and following the same method as in the proof of Theorem 1, we can get the assertion in equation (38).

Remark 13. Setting $w=0$ in equation (38) gives the generating function of the 2-variable q-Appell polynomials (2Vq-AP) $\mathcal{A}_{\kappa,q}(u,v)$ [18], that is

$$\mathscr{A}_q(\tau)e_q(u\tau)E_q(v\tau) = \sum_{\kappa=0}^{\infty} \mathscr{A}_{\kappa,q}(u,v)\frac{\tau^\kappa}{[\kappa]_q!}. \quad (39)$$

Corollary 14. *Taking $\mathscr{A}_q(\tau) = \tau/(e_q(\tau)-1)$ in equation (38), we get the generating function of the 3-variable q-Fubini-Bernoulli polynomials (3Vq-FBP) ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w)$ as*

$$\frac{\tau}{(e_q(\tau)-1)(1-w(e_q(\tau)-1))}e_q(u\tau)E_q(v\tau) = \sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!}. \quad (40)$$

Corollary 15. *Taking $\mathscr{A}_q(\tau) = 2/(e_q(\tau)+1)$ in equation (38), we get the generating function of the 3-variable q-Fubini-Euler polynomials (3Vq-FEP) ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w)$ as*

$$\frac{2}{(e_q(\tau)+1)(1-w(e_q(\tau)-1))}e_q(u\tau)E_q(v\tau) = \sum_{\kappa=0\mathscr{F}}^{\infty} \mathscr{E}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!}. \quad (41)$$

Theorem 16. *The 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ are defined by the series*

$${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) = \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathscr{A}_{l,q}\mathscr{F}_{\kappa-l,q}(u,v,w). \quad (42)$$

Proof. In view of equations (13), (15), and (38), we have

$$\begin{aligned}\sum_{\kappa=0}^{\infty} {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!} &= \frac{\mathscr{A}_q(\tau)}{1-w(e_q(\tau)-1)}e_q(u\tau)E_q(v\tau)\\ &= \left(\sum_{\kappa=0}^{\infty}\mathscr{A}_{\kappa,q}\frac{\tau^\kappa}{[\kappa]_q!}\right)\left(\sum_{\kappa=0}^{\infty}\mathscr{F}_{\kappa,q}(u,v,w)\frac{\tau^\kappa}{[\kappa]_q!}\right)\\ &= \sum_{\kappa=0}^{\infty}\left(\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathscr{A}_{l,q}\mathscr{F}_{\kappa-l,q}(u,v,w)\right)\frac{\tau^\kappa}{[\kappa]_q!},\end{aligned} \quad (43)$$

which on comparing the coefficients of $\tau^\kappa/[\kappa]_q!$ yield assertion in equation (42).

Corollary 17. *Taking $\mathscr{A}_q(\tau) = \tau/(e_q(\tau)-1)$ in equation (42), we get the series representation of the 33Vq-FBP ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w)$ as*

$${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w) = \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathfrak{B}_{l,q}\mathscr{F}_{\kappa-l,q}(u,v,w). \quad (44)$$

Corollary 18. *Taking $\mathscr{A}_q(\tau) = 2/(e_q(\tau)+1)$ in equation (42), we get the series representation of the 33Vq-FEP ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w)$ as*

$${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w) = \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathscr{E}_{l,q}\mathscr{F}_{\kappa-l,q}(u,v,w). \quad (45)$$

Suitably using equations (4), (6), (7), (11), and (23) in generating relation (38) and then making use of the Cauchy product rule in the resultant relations and thereafter comparing the identical powers of τ in both sides of the resultant expressions, we get the formulae given in the following theorem.

Theorem 19. *The 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ satisfy the following formulae*

$$\begin{aligned}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}} \mathscr{A}_{\kappa-l,q}(w)(u\oplus v)_q^l,\\ {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q q^{\binom{\kappa-l}{2}} {}_{\mathscr{F}}\mathscr{A}_{l,q}(u,0,w)v^{\kappa-l},\\ {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q \mathscr{F}_{l,q}(w){}_{\mathscr{F}}\mathscr{A}_{\kappa-l,q}(u,v,0),\\ {}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}} \mathscr{A}_{l,q}(0,v,w)u^{\kappa-l}.\end{aligned} \quad (46)$$

Applying the q-derivatives w.r.t. u and v to generating relation (38), we get the results given in the following theorem.

Theorem 20. *The following identities for the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ hold true:*

$$\begin{aligned}D_{q,u\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= [\kappa]_{q\mathscr{F}}\mathscr{A}_{\kappa-1,q}(u,v,w),\\ D^{\xi}_{q,u\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \frac{[\kappa]_q!}{[\kappa-\xi]_q!}{}_{\mathscr{F}}\mathscr{A}_{\kappa-\xi,q}(u,v,w),\\ D_{q,v\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= [\kappa]_{q\mathscr{F}}\mathscr{A}_{\kappa-1,q}(u,qv,w),\\ D^{\xi}_{q,v\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) &= \frac{[\kappa]_q!}{[\kappa-\xi]_q!}q^{\binom{\xi}{2}}{}_{\mathscr{F}}\mathscr{A}_{\kappa-\xi,q}\left(u,q^{\xi}v,w\right).\end{aligned} \quad (47)$$

Theorem 21. *The following relation for the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ holds true:*

$$\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-l,q}(u,v,w) = \frac{1}{w}\left((w+1){}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w) - \mathscr{A}_{\kappa,q}(u,v)\right). \quad (48)$$

Proof. Consider the identity

$$w\frac{e_q(u\tau)E_q(v\tau)}{1-w(e_q(\tau)-1)}e_q(\tau) = (1+w)\frac{e_q(u\tau)E_q(v\tau)}{1-w(e_q(\tau)-1)} - e_q(u\tau)E_q(v\tau). \quad (49)$$

Now, multiplying both sides of the above identity by $\mathscr{A}_{\kappa,q}(\tau)$ and using equations (6), (38), and (39), we get

$$w\sum_{\kappa=0}^{\infty}\left(\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)\right)\frac{\tau^{\kappa}}{[\kappa]_q!} = \sum_{\kappa=0}^{\infty}\left((1+w)\,{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)-\mathscr{A}_{\kappa,q}(u,v)\right)\frac{\tau^{\kappa}}{[\kappa]_q!}, \tag{50}$$

which on equating the coefficients of the like powers of τ yields the assertion in equation (48).

Now, let us recall the generating function of the 2-variable q-generalized tangent polynomials (2Vq-GTP) $\mathscr{C}_{\kappa,\alpha,q}(u,v)$ [19] given as

$$\frac{2}{e_q(\alpha\tau)+1}e_q(u\tau)E_q(v\tau)=\sum_{\kappa=0}^{\infty}\mathscr{C}_{\kappa,\alpha,q}(u,v)\frac{\tau^{\kappa}}{[\kappa]_q!},\ |\alpha\tau|<\pi,\ \alpha\in\mathbb{R}^{+}, \tag{51}$$

and $\mathscr{C}_{\kappa,\alpha,q}\coloneqq\mathscr{C}_{\kappa,\alpha,q}(0,0)$ denotes the q-generalized tangent numbers (q-GTN).

Theorem 22. *The following relationships between the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ and 2Vq-GTP $\mathscr{C}_{\kappa,\alpha,q}(u,v)$ holds true:*

$${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)=\frac{1}{2}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\cdot\left(\sum_{\sigma=0}^{l}\begin{bmatrix}l\\ \sigma\end{bmatrix}_q\alpha^{\sigma}\mathscr{C}_{\kappa-l,\alpha,q}(u,v)\mathscr{F}_{l-\sigma,q}(w)+{}_{\mathscr{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)\mathscr{C}_{l,\alpha,q}\right). \tag{52}$$

Proof. Utilizing equations (23), (38), and (51), we have

$$\begin{aligned}\sum_{\kappa=0}^{\infty}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)\frac{\tau^{\kappa}}{[\kappa]_q!}&=\frac{\mathscr{A}_q(\tau)}{1-w\left(e_q(\tau)-1\right)}e_q(u\tau)E_q(v\tau)\\&=\frac{\mathscr{A}_q(\tau)}{1-w\left(e_q(\tau)-1\right)}\left(\frac{2}{e_q(\alpha\tau)+1}\right)\left(\frac{e_q(\alpha\tau)+1}{2}\right)e_q(u\tau)E_q(v\tau)\\&=\frac{1}{2}\left[\left(\sum_{\kappa=0}^{\infty}\left(\sum_{\sigma=0}^{\kappa}\begin{bmatrix}\kappa\\ \sigma\end{bmatrix}_q\alpha^{\sigma}\mathscr{F}_{\kappa-\sigma,q}(w)\right)\frac{\tau^{\kappa}}{[\kappa]_q!}\right)\right.\\&\quad\cdot\left(\sum_{\kappa=0}^{\infty}\mathscr{C}_{\kappa,\alpha,q}(u,v)\frac{\tau^{\kappa}}{[\kappa]_q!}\right)+\left(\sum_{\hat{e}=0}^{\infty}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)\frac{\tau^{\kappa}}{[\kappa]_q!}\right)\\&\quad\left.\cdot\left(\sum_{\kappa=0}^{\infty}\mathscr{C}_{\kappa,\alpha,q}\frac{\tau^{\kappa}}{[\kappa]_q!}\right)\right]\\&=\frac{1}{2}\sum_{\kappa=0}^{\infty}\left[\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\sum_{\sigma=0}^{l}\begin{bmatrix}\kappa\\ \sigma\end{bmatrix}_q\alpha^{\sigma}\mathscr{F}_{l-\sigma,q}(w)\mathscr{C}_{\kappa-l,\alpha,q}(u,v)+\sum_{l=0}^{\kappa}\right.\\&\quad\left.\cdot\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-1,q}(u,v,w)\mathscr{C}_{l,\alpha,q}\right]\frac{\tau^{\kappa}}{[\kappa]_q!},\end{aligned} \tag{53}$$

which on comparing the coefficients of $\tau^{\kappa}/[\kappa]_q!$ yield assertion in equation (52).

Since for $\alpha=1$, the 2-variable q-generalized tangent polynomials (2Vq-GTP) $\mathscr{C}_{\kappa,\alpha,q}(u,v)$ reduce to 2-variable q-Euler polynomials $\mathscr{E}_{\kappa,q}(u,v)$ [20]. Therefore, setting $\alpha=1$ in equation (52) gives the following result.

Corollary 23. *The following relationships between the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ and 2Vq-EP $\mathscr{E}_{\kappa,q}(u,v)$ holds true:*

$${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)=\frac{1}{2}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\left(\sum_{\sigma=0}^{l}l\sigma_q\mathscr{E}_{\kappa-l,\alpha,q}(u,v)\mathscr{F}_{l-\sigma,q}(w)+{}_{\mathscr{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)\mathscr{E}_{l,\alpha,q}\right). \tag{54}$$

Let us recall the generating function of the 2-variable q-Euler-Bernoulli polynomials (2Vq-EBP) ${}_{\mathscr{E}}\mathfrak{B}_{\kappa,q}(u,v)$ [16] given by

$$\frac{2\tau}{\left(e_q(\tau)+1\right)\left(e_q(\tau)-1\right)}e_q(u\tau)E_q(v\tau)=\sum_{\kappa=0}^{\infty}{}_{\mathscr{E}}\mathfrak{B}_{\kappa,q}(u,v)\frac{t^{\kappa}}{[\kappa]_q!}. \tag{55}$$

Theorem 24. *The following relationships between the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ and 2Vq-EBP ${}_{\mathscr{E}}\mathfrak{B}_{\kappa,q}(u,v)$ holds true:*

$${}_{\mathscr{F}}\mathscr{A}_{\kappa-1,q}(u,v,w)=\frac{1}{2[k]_q}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\left(\sum_{\sigma=0}^{\kappa}\sum_{h=0}^{l}\begin{bmatrix}\kappa-l\\ \sigma\end{bmatrix}_q\cdot\begin{bmatrix}l\\ h\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-\sigma-l,q}(w)\,{}_{\mathscr{E}}\mathfrak{B}_{l-h,q}(u,v)-{}_{\mathscr{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)\,{}_{\mathscr{E}}\mathfrak{B}_{l,q}\right). \tag{56}$$

Proof. Utilizing equations (6), (38), and (55), we have

$$\begin{aligned}\sum_{\kappa=0}^{\infty}{}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)\frac{\tau^{\kappa}}{[\kappa]_q!}&=\frac{\mathscr{A}_q(\tau)}{1-w\left(e_q(\tau)-1\right)}e_q(u\tau)E_q(v\tau)\\&=\frac{\mathscr{A}_q(\tau)}{1-w\left(e_q(\tau)-1\right)}\left(\frac{2t}{\left(e_q(\tau)+1\right)\left(e_q(\tau)-1\right)}\right)\\&\quad\cdot\left(\frac{\left(e_q(\tau)+1\right)\left(e_q(\tau)-1\right)}{2t}\right)e_q(u\tau)E_q(v\tau)\\&=\frac{1}{2t}\left[\left(\frac{\mathscr{A}_q(\tau)}{1-w\left(e_q(\tau)-1\right)}e_q(\tau)\right)\left(\frac{2te_q(u\tau)E_q(v\tau)}{\left(e_q(\tau)+1\right)\left(e_q(\tau)-1\right)}e_q(\tau)\right)\right.\\&\quad\left.-\left(\frac{\mathscr{A}_q(\tau)e_q(u\tau)E_q(v\tau)}{1-w\left(e_q(\tau)-1\right)}\right)\left(\frac{2t}{\left(e_q(\tau)+1\right)\left(e_q(\tau)-1\right)}\right)\right]\\&=\frac{1}{2t}\left[\left(\sum_{\kappa=0}^{\infty}\sum_{\sigma=0}^{\kappa}\begin{bmatrix}\kappa\\ \sigma\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-\sigma,q}(w)\frac{\tau^{\kappa}}{[\kappa]_q!}\right)\right.\\&\quad\cdot\left(\sum_{l=0}^{\infty}\sum_{h=0}^{l}\begin{bmatrix}l\\ h\end{bmatrix}_q\varepsilon\mathfrak{B}_{l-h,q}(u,v)\frac{\tau^{l}}{[l]_q!}\right)\\&\quad\left.-\sum_{\kappa=0}^{\infty}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)\,{}_{\mathscr{E}}\mathfrak{B}_{l,q}\frac{\tau^{\kappa}}{[\kappa]_q!}\right]\\&=\frac{1}{2t}\sum_{\kappa=0}^{\infty}\left[\sum_{l=0}^{\kappa}\sum_{\sigma=0}^{\kappa}\sum_{h=0}^{l}\begin{bmatrix}\kappa\\ l\end{bmatrix}_q\begin{bmatrix}\kappa-l\\ \sigma\end{bmatrix}_q\right.\\&\quad\cdot\begin{bmatrix}l\\ h\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-l,q}(w)\,{}_{\mathscr{E}}\mathfrak{B}_{l-h,q}(u,v)-\sum_{l=0}^{\kappa}\\&\quad\left.\cdot\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\kappa-l,q}(u,v,w)\,{}_{\mathscr{E}}\mathfrak{B}_{l,q}\right]\frac{\tau^{\kappa}}{[\kappa]_q!},\end{aligned} \tag{57}$$

which on comparing the coefficients of $\tau^\kappa/[\kappa]_q!$ yield assertion in equation (56).

Theorem 25. *The following relationships between the 3Vq-FAP ${}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(u,v,w)$ and 2Vq-AP $\mathcal{A}_{\kappa,q}(u,v)$ holds true:*

$$\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathcal{F}}\mathcal{A}_{\kappa-l,q}\left(u,v,\frac{w}{1-w}\right)=\frac{1}{w}\left({}_{\mathcal{F}}\mathcal{A}_{\kappa,q}\left(u,v,\frac{w}{1-w}\right)-(1-w)\mathcal{A}_{\kappa,q}(u,v)\right).\tag{58}$$

Proof. Replacing w by $w/(1-w)$ in generating relation (38), we have

$$\sum_{\kappa=0}^{\infty}{}_{\mathcal{F}}\mathcal{A}_{\kappa,q}\left(u,v,\frac{w}{1-w}\right)\frac{\tau^\kappa}{[\kappa]_q!}=\frac{\mathcal{A}_q(\tau)}{1-(w/(1-w))\left(e_q(\tau)-1\right)}e_q(u\tau)E_q(v\tau).\tag{59}$$

Rewriting the above equation then using equations (38) and (39), we obtain

$$\sum_{\kappa=0}^{\infty}{}_{\mathcal{F}}\mathcal{A}_{\kappa,q}\left(u,v,\frac{w}{1-w}\right)\frac{\tau^\kappa}{[\kappa]_q!}-w\sum_{\kappa=0}^{\infty}\sum_{l=0}^{\kappa}\begin{bmatrix}\kappa\\ l\end{bmatrix}_{q\mathcal{F}}\mathcal{A}_{\kappa-l,q}\left(u,v,\frac{w}{1-w}\right)\frac{\tau^\kappa}{[\kappa]_q!}=(1-w)\sum_{\kappa=0}^{\infty}\mathcal{A}_{\kappa,q}(u,v)\frac{\tau^\kappa}{[\kappa]_q!}.\tag{60}$$

which on comparing the coefficients of $\tau^\kappa/[\kappa]_q!$ yield assertion in equation (58).

4. Determinant Representations

One of the significant representations of the q-special polynomials is the determinant representation due to its importance for the computational and applied purposes. In 2015, Keleshteri and Mahmudov [18] established the determinant representation of the q-Appell polynomials. In the section, the determinant representations of the q-FAP ${}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(w)$ and the 3Vq-FAP ${}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(u,v,w)$ are introduced.

Definition 26. The determinant representation for the q-FAP ${}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(w)$ of degree κ is given as

$${}_{\mathcal{F}}\mathcal{A}_{0,q}(w)=\frac{1}{\mathcal{B}_{0,q}},\tag{61}$$

$${}_{\mathcal{F}}\mathcal{A}_{\kappa,q}(w)\begin{bmatrix}1 & \mathcal{F}_{1,q} & \mathcal{F}_{2,q}(w) & \cdots & \mathcal{F}_{\kappa-1,q}(w) & \mathcal{F}_{\kappa,q}(w)\\ \mathcal{B}_{0,q} & \mathcal{B}_{1,q} & \mathcal{B}_{2,q} & \cdots & \mathcal{B}_{\kappa-1,q} & \mathcal{B}_{\kappa,q}\\ 0 & \mathcal{B}_{0,q} & \begin{bmatrix}2\\1\end{bmatrix}_q\mathcal{B}_{1,q} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\mathcal{B}_{\kappa-2,q} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\mathcal{B}_{\kappa-1,q}\\ 0 & 0 & \mathcal{B}_{0,q} & \cdots & \begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\mathcal{B}_{\kappa-3,q} & \begin{bmatrix}\kappa\\2\end{bmatrix}_q\mathcal{B}_{\kappa-2,q}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & \mathcal{B}_{0,q} & \begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\mathcal{B}_{1,q}\end{bmatrix},\tag{62}$$

$$\mathcal{B}_{\kappa,q}=-\frac{1}{{}_{\mathcal{F}}\mathcal{A}_{0,q}}\left(\sum_{\nu=1}^{\kappa}\begin{bmatrix}\kappa\\ \nu\end{bmatrix}_{\mathcal{F}}\mathcal{A}_{\nu,q}\mathcal{B}_{\kappa-\nu,q}\right),\quad \mathcal{B}_{0,q}\neq 0,\kappa=1,2,3,\cdots.\tag{63}$$

Setting $\mathcal{B}_{0,q}=1$ and $\mathcal{B}_{\delta,q}=(1/[\delta+1]_q)(\delta=1,2,\cdots,\kappa)$ in equations (61) and (62) gives the determinant representation of the q-FBP ${}_{\mathcal{F}}\mathfrak{B}_{\kappa,q}(w)$ as:

Definition 27. The determinant representation for the q *-FBP* ${}_{\mathcal{F}}\mathfrak{B}_{\kappa,q}(w)$ of degree κ is given as

$${}_{\mathcal{F}}\mathfrak{B}_{0,q}(w)=1,$$

$${}_{\mathcal{F}}\mathfrak{B}_{\kappa,q}(w)=(-1)^\kappa\begin{vmatrix}1 & \mathcal{F}_{1,q}(w) & \mathcal{F}_{2,q}(w) & \cdots & \mathcal{F}_{\kappa-1,q}(w) & \mathcal{F}_{\kappa,q}(w)\\ 1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & \cdots & \frac{1}{[\kappa]_q} & \frac{1}{[\kappa+1]_q}\\ 0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{[2]_q} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\frac{1}{[\kappa-1]_q} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\frac{1}{[\kappa]_q}\\ 0 & 0 & 1 & \cdots & \begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\frac{1}{[\kappa-2]_q} & \begin{bmatrix}\kappa\\2\end{bmatrix}_q\frac{1}{[\kappa-1]_q}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\frac{1}{[2]_q}\end{vmatrix},\quad \kappa=1,2\cdots.\tag{64}$$

Setting $\mathcal{B}_{0,q}=1$ and $\mathcal{B}_{\delta,q}=(1/2)(\delta=1,2,\cdots,\kappa)$ in equations (61) and (62) gives the determinant representation of the q-FEP ${}_{\mathcal{F}}\mathcal{E}_{\kappa,q}(w)$ as:

Definition 28. The determinant representation for the q -FEP ${}_{\mathcal{F}}\mathcal{E}_{\kappa,q}(w)$ of degree κ is given as

$${}_{\mathscr{F}}\varepsilon_{0,q}(w)=1,$$

$${}_{\mathscr{F}}\varepsilon_{\kappa,q}(w)=(-1)^{\kappa}\cdot\begin{vmatrix} 1 & \mathscr{F}_{1,q}(w) & \mathscr{F}_{2,q}(w) & \cdots & \mathscr{F}_{\kappa-1,q}(w) & \mathscr{F}_{\kappa,q}(w)\\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{2} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\frac{1}{2} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\frac{1}{2} \\ 0&0&1&\cdots&\begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\frac{1}{2} & \begin{bmatrix}\kappa\\2\end{bmatrix}_q\frac{1}{2}\\ .&.&.&\cdots&.&.\\ .&.&.&\cdots&.&.\\ 0&0&0&\cdots&1&\begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\frac{1}{2}\end{vmatrix}, \quad \kappa=1,2,3,\cdots. \tag{65}$$

Similarly, the determinant representation of the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$, 3Vq-FBP ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w)$, and 3Vq-FEP ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w)$ are established as:

Definition 29. The determinant representation for the 3Vq-FAP ${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)$ of degree κ is given as

$${}_{\mathscr{F}}\mathscr{A}_{0,q}(u,v,w)=\frac{1}{\mathscr{B}_{0,q}},$$

$${}_{\mathscr{F}}\mathscr{A}_{\kappa,q}(u,v,w)=\frac{(-1)^{\kappa}}{(\mathscr{B}_{0,q})^{\kappa+1}}\cdot\begin{vmatrix} 1 & \mathscr{F}_{1,q}(u,v,w) & \mathscr{F}_{2,q}(u,v,w) & \cdots & \mathscr{F}_{\kappa-1,q}(u,v,w) & \mathscr{F}_{\kappa,q}(u,v,w)\\ \mathscr{B}_{0,q} & \mathscr{B}_{1,q} & \mathscr{B}_{2,q} & \cdots & \mathscr{B}_{\kappa-1,q} & \mathscr{B}_{\kappa,q}\\ 0 & \mathscr{B}_{0,q} & \begin{bmatrix}2\\1\end{bmatrix}_q\mathscr{B}_{1,q} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\mathscr{B}_{\kappa-2,q} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\mathscr{B}_{\kappa-1,q}\\ 0&0&\mathscr{B}_{0,q}&\cdots&\begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\mathscr{B}_{\kappa-3,q}&\begin{bmatrix}\kappa\\2\end{bmatrix}_q\mathscr{B}_{\kappa-2,q}\\ .&.&.&\cdots&.&.\\ .&.&.&\cdots&.&.\\ 0&0&0&\cdots&\mathscr{B}_{0,q}&\begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\mathscr{B}_{1,q}\end{vmatrix},$$

$$\mathscr{B}_{\kappa,q}=-\frac{1}{{}_{\mathscr{F}}\mathscr{A}_{0,q}}\left(\sum_{\nu=1}^{\kappa}\begin{bmatrix}\kappa\\\nu\end{bmatrix}_{q\mathscr{F}}\mathscr{A}_{\nu,q}\mathscr{B}_{\kappa-\nu,q}\right),\ \mathscr{B}_{0,q}\neq 0,\ \kappa=1,2,3,\cdots. \tag{66}$$

Definition 30. The determinant representation for the 3Vq-FBP ${}_{\mathscr{F}}\mathfrak{B}_{\kappa,q}(u,v,w)$ of degree κ is given as

$${}_{\mathscr{F}}\mathfrak{B}_{0,q}(u,v,w)=1,$$

$${}_{\mathscr{F}}\mathfrak{B}_{0,q}(u,v,w)=(-1)^{\kappa}\cdot\begin{vmatrix} 1 & \mathscr{F}_{1,q}(u,v,w) & \mathscr{F}_{2,q}(u,v,w) & \cdots & \mathscr{F}_{\kappa-1,q}(u,v,w) & \mathscr{F}_{\kappa,q}(u,v,w)\\ 1 & & & \cdots & \frac{1}{[\kappa]_q} & \frac{1}{[\kappa+1]_q}\\ 0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{[2]_q} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\frac{1}{[\kappa-1]_q} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\frac{1}{[\kappa]_q}\\ 0&0&1&\cdots&\begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\frac{1}{[\kappa-2]_q}&\begin{bmatrix}\kappa\\2\end{bmatrix}_q\frac{1}{[\kappa-2]_q}\\ .&.&.&\cdots&.&.\\ .&.&.&\cdots&.&.\\ 0&0&0&\cdots&1&\begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\frac{1}{[2]_q}\end{vmatrix},$$
$$\kappa=1,2\cdots. \tag{67}$$

Definition 31. The determinant representation for the 3Vq-FEP ${}_{\mathscr{F}}\mathscr{E}_{\kappa,q}(u,v,w)$ of degree κ is given as

$${}_{\mathscr{F}}\mathscr{E}_{0,q}(u,v,w)=1,$$

$${}_{\mathscr{F}}\varepsilon_{\kappa,q}(u,v,w)=(-1)^{\kappa}\cdot\begin{vmatrix} 1 & \mathscr{F}_{1,q}(u,v,w) & \mathscr{F}_{2,q}(u,v,w) & \cdots & \mathscr{F}_{\kappa-1,q}(u,v,w) & \mathscr{F}_{\kappa,q}(u,v,w)\\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q\frac{1}{2} & \cdots & \begin{bmatrix}\kappa-1\\1\end{bmatrix}_q\frac{1}{2} & \begin{bmatrix}\kappa\\1\end{bmatrix}_q\frac{1}{2} \\ 0&0&1&\cdots&\begin{bmatrix}\kappa-1\\2\end{bmatrix}_q\frac{1}{2} & \begin{bmatrix}\kappa\\2\end{bmatrix}_q\frac{1}{2}\\ .&.&.&\cdots&.&.\\ .&.&.&\cdots&.&.\\ 0&0&0&\cdots&1&\begin{bmatrix}\kappa\\\kappa-1\end{bmatrix}_q\frac{1}{2}\end{vmatrix},$$
$$\kappa=1,2,3\cdots. \tag{68}$$

5. Conclusions

Recently, the Fubini polynomials and their q-analogue have been studied and investigated by many researchers. Motivated by various recent studies related to these type of polynomials (see for example [8, 21, 22]), in this article, we introduced two important families of q-hybrid special polynomials, namely, the q-Fubini-Appell polynomials and 3-variable q-Fubini-Appell polynomials. Certain properties related to these families are derived.

Further investigations along the results obtained in this article, which are associated with many recent generalizations and extensions of the q-Appell polynomial family, especially, the parametric types, may be worthy of consideration in future investigations.

Data Availability

There is no data availability in this manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

A Note on the Poly-Bernoulli Polynomials of the Second Kind

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In this paper, we define the poly-Bernoulli polynomials of the second kind by using the polyexponential function and find some interesting identities of those polynomials. In addition, we define unipoly-Bernoulli polynomials of the second kind and study some properties of those polynomials.

1. Introduction

In the book *Ars Conjectandi*, Bernoulli introduced the Bernoulli number terms of the sum of powers of consecutive integers (see [1, 2]). In [3], Luo and Srivastava defined the Apostol-Bernoulli polynomials and obtained an explicit series representation for their polynomials involving the Gaussian hypergeometric function as well as an explicit series representation involving the Hurwitz function. Frappier defined a generalized Bernoulli polynomials by using the Bessel function of the first kind and found a generalization of a well-known Fourier series representation of Bernoulli polynomials in [4]. In [5], Natalini and Bernardini defined a new class of generalized Bernoulli polynomials and showed that if a differential equation with these polynomials is of order n, then all the considered families of polynomials can be viewed as solutions of differential operators of infinite order. In [6], Kaneko defined the poly-Bernoulli polynomials and found an explicit formula and a duality theorem for those numbers. Khan et al. defined Laguerre-based Hermite-Bernoulli polynomials and derived summation formulas and related bilateral series associated with the newly introduced generating function in [7]. In [8], Jang and Kim defined type 2 degenerate Bernoulli polynomials and showed that these polynomials could be represented linear combinations of the Stirling numbers of the first and the second kinds, Bernoulli polynomials, and those numbers. Moreover, in [9], the degenerate type 2 poly-Bernoulli numbers and polynomials as degenerate versions of such numbers and polynomials were defined, and several explicit expressions and some identities for those numbers and polynomials were derived.

As is well known, *Bernoulli polynomials of order r* are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \tag{1}$$

(see [1, 5, 9, 10]).

In particular, if $r=1$, $B_n(x)=B_n^{(1)}(x)$ are the ordinary Bernoulli polynomials. When $x=0$, $B_n^{(r)}=B_n^{(r)}(0)$ are called the *Bernoulli numbers of order r*. In [1], the relationship between the Bernoulli numbers and zeta functions was studied, and in [2, 8, 10–12], generalized Bernoulli numbers were defined, and the properties of those numbers and polynomials were investigated.

The *Bernoulli polynomials of the second kind* (or the Cauchy polynomials) are defined by the generating function to be

$$\sum_{n=0}^{\infty} b_n(x)\frac{t^n}{n!} = \frac{t}{\log(1+t)}(1+t)^x \tag{2}$$

(see [13–15]).

When $x=0$, $b_n=b_n(0)$ are called the *Bernoulli numbers of the second kind*.

For a nonnegative integer n, the *Stirling numbers of the first kind* are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^i \quad (3)$$

(see [16–18]), where $(x)_0 = x$, $(x)_n = x(x-1)\cdots(x-n+1)$ $(n \geq 0)$. By the direct computation of (3), we derive the following:

$$\frac{1}{n!}(\log(1+t))^n = \sum_{k=n}^{\infty} S_1(k,n)\frac{t^k}{k!} \quad (4)$$

(see [16–19]).

For a given nonnegative integer n, the *Stirling numbers of the second kind* are defined by

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l \quad (5)$$

(see [16–18]).

By (5), we obtain

$$\left(e^t-1\right)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \quad (6)$$

(see [16–19]).

In [17, 19], the authors defined the generalized Stirling numbers of the first and second kinds and generalized binomial coefficients and showed that degenerated special polynomials are represented by linear combinations of those numbers.

The polyexponential function was first studied by Hardy (see [11, 20]), and Kim and Kim defined polyexponential function as an inverse to the polylogarithm function $Li_k(x) = \sum_{n=1}^{\infty}(x^n/n!)$ (see [6, 11, 20, 21]), to be

$$e_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k(n-1)!} \ (k \in \mathbb{Z}) \quad (7)$$

(see [21]).

By (7), we know that $e_1(x) = e^x$.

Recently, some authors applied the polyexponential functions and the polylogarithm functions to degenerate Bernoulli polynomials, type 2 poly-Apostol-Bernoulli polynomials, type 2 degenerate poly-Euler polynomials, and poly-Genocchi polynomials and found many interesting identities about those polynomials (see [11, 12, 20–26]).

In this paper, we define poly-Bernoulli polynomials of the second kind with the polyexponential function and derive some interesting identities between the Stirling numbers of the first kind or the second kind, Bernoulli numbers, Bernoulli numbers of the second kind, and those polynomials. In addition, we define unipoly-Bernoulli polynomials of the second kind and derive some interesting identities of those polynomials.

2. The Poly-Bernoulli Polynomials of the Second Kind

By the definition of the Bernoulli polynomials of the second kind and (7), we define the *poly-Bernoulli polynomials of the second kind* by the generating function to be

$$\sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!} = \frac{e_k(\log(1+t))}{\log(1+t)}(1+t)^x. \quad (8)$$

In particular, if $x=0$, $b_n^{(k)} = b_n^{(k)}(0)$ are called the *poly-Bernoulli numbers of the second kind*. By (8), we know that for each nonnegative integer n,

$$b_n^{(1)}(x) = b_n(x) \quad (9)$$

are the Bernoulli polynomials of the second kind.

Note that

$$\begin{aligned}\sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!} &= \frac{e_k(\log(1+t))}{\log(1+t)}(1+t)^x \\ &= \left(\sum_{l=0}^{\infty} b_l^{(k)}\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\binom{x}{m}t^m\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{x}{m}\binom{n}{m}m!b_{n-m}^{(k)}\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{x}{n-l}\binom{n}{l}(n-l)!b_l^{(k)}\right)\frac{t^n}{n!}.\end{aligned} \quad (10)$$

Hence, by (10), we obtain the following theorem.

Theorem 1. *For each $n \in \mathbb{N} \cup \{0\}$, we have*

$$b_n^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}b_l^{(k)}(x)_{n-l}, \quad (11)$$

where $(x)_k = x(x-1)\cdots(x-k+1)$ is the k-falling factorial.

By replacing t by $e^t - 1$ in (8), we get

$$\begin{aligned}\frac{e_k(t)}{t}e^{xt} &= \left(\frac{1}{t}\sum_{n=1}^{\infty}\frac{t^n}{(n-1)!n^k}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\right) \\ &= \left(\sum_{n=0}^{\infty}\frac{t^n}{n!(n+1)^k}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\frac{x^{n-m}}{(m+1)^k}\right)\frac{t^n}{n!},\end{aligned} \quad (12)$$

and by (6), we have

$$\sum_{n=0}^{\infty} b_n^{(k)} \frac{1}{n!}\left(e^t-1\right)^n = \sum_{n=0}^{\infty} b_n^{(k)}(x) \sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_2(n,m) b_{n-m}^{(k)}(x)\right)\frac{t^n}{n!}. \quad (13)$$

Therefore, by (12) and (13), we obtain the following theorem.

Theorem 2. *For each nonnegative integer n, we have*

$$\sum_{m=0}^{n} S_2(n,m) b_{n-m}^{(k)}(x) = \sum_{m=0}^{n}\binom{n}{m}\frac{x^{n-m}}{(m+1)^k}. \quad (14)$$

In particular, we have

$$\sum_{m=0}^{n} S_2(n,m) b_{n-m}^{(k)} = \frac{1}{(n+1)^k}. \quad (15)$$

From (4) and (8), we derive

$$\begin{aligned}\sum_{n=0}^{\infty} b_n^{(k)}\frac{t^n}{n!} &= \frac{e_k(\log(1+t))}{\log(1+t)} = \frac{1}{\log(1+t)}\sum_{n=1}^{\infty}\frac{(\log(1+t))^n}{(n-1)!n^k}\\ &= \frac{1}{\log(1+t)}\sum_{n=0}^{\infty}\frac{(\log(1+t))^{n+1}}{n!(n+1)^k}\\ &= \frac{1}{\log(1+t)}\sum_{n=0}^{\infty}\frac{1}{(n+1)^{k-1}}\sum_{l=n+1}^{\infty} S_1(l,n+1)\frac{t^l}{l!}\\ &= \frac{t}{\log(1+t)}\sum_{n=0}^{\infty}\frac{1}{(n+1)^{k-1}}\sum_{l=n}^{\infty}\frac{S_1(l+1,n+1)}{l+1}\frac{t^l}{l!}\\ &= \left(\sum_{n=0}^{\infty} b_n\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{1}{(l+1)^{k-1}}\frac{S_1(m+1,l+1)}{m+1}\frac{t^m}{m!}\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\frac{b_{n-m}}{(l+1)^{k-1}}\frac{S_1(m+1,l+1)}{m+1}\right)\frac{t^n}{n!}.\end{aligned} \quad (16)$$

Thus, by (16), we obtain the following theorem.

Theorem 3. *For each $k \in \mathbb{Z}$ and each nonnegative integer n, we have*

$$b_n^{(k)} = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\frac{S_1(m+1,l+1)}{(l+1)^{k-1}(m+1)} b_{n-m}. \quad (17)$$

By (9) and Theorem 3, we get the following corollary.

Corollary 4. *For each $n \in \mathbb{N}\cup\{0\}$, we have*

$$b_n = \sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\frac{S_1(m+1,l+1)}{m+1} b_{n-m}. \quad (18)$$

In Corollary 4, we have

$$\sum_{m=0}^{n}\sum_{l=0}^{m}\binom{n}{m}\frac{S_1(m+1,l+1)}{(m+1)} b_{n-m} = b_n + \sum_{m=1}^{n}\sum_{l=0}^{m}\binom{n}{m}\frac{S_1(m+1,l+1)}{(m+1)} b_{n-m}. \quad (19)$$

Therefore, we obtain the following corollary.

Corollary 5. *For each positive integer n, we have*

$$\sum_{m=1}^{n}\sum_{l=0}^{m}\binom{n}{m}\frac{S_1(m+1,l+1)}{(m+1)} b_{n-m} = 0 \ (n \in \mathbb{N}). \quad (20)$$

In [21], the authors showed that

$$\frac{d}{dx} e_k(x) = \frac{1}{x} e_{k-1}(x) \ (k \geq 2). \quad (21)$$

From (21), we have

$$e_k(x) = \int_0^x \underbrace{\frac{1}{t}\int_0^t \frac{1}{t}\cdots\int_0^t}_{(k-2)-\text{times}} \frac{1}{t}\left(e^t-1\right) dt dt \cdots dt \quad (22)$$

(see [9, 11, 12, 19–21, 23, 25]).

By (22), we can derive the following equations:

$$\begin{aligned}\sum_{n=0}^{\infty} b_n^{(k)}\frac{x^n}{n!} &= \frac{1}{\log(1+x)} e_k(\log(1+t))\\ &= \frac{1}{\log(1+x)}\int_0^x \frac{1}{(1+t)\log(1+t)} e_{k-1}(\log(1+t)) dt\\ &= \frac{1}{\log(1+x)}\int_0^x \frac{1}{(1+t)\log(1+t)}\\ &\quad \underbrace{\int_0^t \frac{1}{(1+t)\log(1+t)}\cdots\int_0^t \frac{t}{(1+t)\log(1+t)}}_{(k-2)-\text{times}} dt dt\cdots dt \ (k\geq 2).\end{aligned} \quad (23)$$

It is well known that

$$\frac{t}{(1+t)\log(1+t)} = \sum_{n=0}^{\infty} B_n^{(n)}\frac{t^n}{n!} \quad (24)$$

(see [9, 12, 23, 25]).

In particular, if we put $k=2$ in (23), then by (23) and (24), we have

$$\begin{aligned}\sum_{n=0}^{\infty} b_n^{(2)} \frac{x^n}{n!} &= \frac{1}{\log(1+x)} e_2(\log(1+t)) \\ &= \frac{1}{\log(1+x)} \int_0^x \frac{1}{(1+t)\log(1+t)} dt \\ &= \frac{1}{\log(1+x)} \sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l!} \int_0^x t^l dt = \frac{x}{\log(1+x)} \sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l+1} \frac{x^l}{l!} \\ &= \left(\sum_{m=0}^{\infty} b_m \frac{x^m}{m!}\right)\left(\sum_{l=0}^{\infty} \frac{B_l^{(l)}}{l+1}\frac{x^l}{l!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\frac{B_l^{(l)}}{l+1} b_{n-l}\right)\frac{x^n}{n!}.\end{aligned} \tag{25}$$

Therefore, by (25), we obtain the following theorem.

Theorem 6. *For a nonnegative integer n, we have*

$$b_n^{(2)} = \sum_{l=0}^{n}\binom{n}{l}\frac{B_l^{(l)}}{l+1} b_{n-l}. \tag{26}$$

3. The Unipoly-Bernoulli Polynomials of the Second Kind

Let p be an arithmetic function which is a real or complex valued function defined on $\mathbb{N}$. In [21], Kim and Kim defined the unipoly function attached to polynomial $p(x)$ by

$$u_k(x \mid p) = \sum_{n=1}^{\infty} \frac{p(n)x^n}{n^k} \ (k \in \mathbb{Z}). \tag{27}$$

In particular, if $p(x)=1$, then

$$u_k(x \mid 1) = \sum_{n=1}^{\infty} \frac{x^n}{n^k} = Li_k(x) \tag{28}$$

is an ordinary polylogarithm function.

Note that by (27), we get

$$\frac{d}{dx} u_k(x \mid p) = \frac{1}{x} u_{k-1}(x \mid p), \tag{29}$$

for $k \geq 2$. In addition, it is well known that

$$u_k(x \mid p) = \int_0^x \frac{1}{t} \underbrace{\int_0^t \frac{1}{t} \cdots \int_0^t}_{(k-2)-\text{times}} \frac{1}{t} u_1(t \mid p) dt dt \cdots dt \tag{30}$$

(see [9, 11, 12, 19, 21, 23, 25]).

In the viewpoint of (8), we define the *unipoly-Bernoulli polynomials of the second kind* as

$$\frac{u_k(\log(1+t) \mid p)}{\log(1+t)} (1+t)^x = \sum_{n=0}^{\infty} b_{n,p}^{(k)}(x) \frac{t^n}{n!}. \tag{31}$$

From (31), we derive the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,p}^{(k)}(x)\frac{t^n}{n!} &= \frac{u_k(\log(1+t) \mid p)}{\log(1+t)}(1+t)^x \\ &= \left(\sum_{n=0}^{\infty} b_{n,p}^{(k)}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(x)_n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}(x)_l b_{n-l,p}^{(k)}\right)\frac{t^n}{n!},\end{aligned} \tag{32}$$

and thus, by (32), we obtain the following theorem.

Theorem 7. *For each nonnegative integer n, we have*

$$b_{n,p}^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l} b_{n-l,p}^{(k)}(x)_l. \tag{33}$$

If we put $p(n) = 1/\Gamma(n)$, then by (31), we get

$$\begin{aligned}\sum_{n=0}^{\infty} b_{n,p}^{(k)}(x)\frac{t^n}{n!} &= \frac{1}{\log(1+t)} u_k\left(\log(1+t) \middle| \frac{1}{\Gamma}\right)(1+t)^x \\ &= \frac{1}{\log(1+t)}\sum_{m=1}^{\infty}\frac{(\log(1+t))^m}{m^k(m-1)!}(1+t)^x \\ &= \frac{e_k(\log(1+t))}{\log(1+t)}(1+t)^x = \sum_{n=0}^{\infty} b_n^{(k)}(x)\frac{t^n}{n!}.\end{aligned} \tag{34}$$

Therefore, by (34), we obtain the following theorem.

Theorem 8. *For a nonnegative integer n, if $p(n) = 1/\Gamma(n)$, then we have*

$$b_{n,p}^{(k)}(x) = b_n^{(k)}(x). \tag{35}$$

In the definition of unipoly-Bernoulli polynomials of the second kind, if $x=0$, then we get

$$\sum_{n=0}^{\infty} b_{n,p}^{(k)} \frac{t^n}{n!} = \frac{1}{\log(1+t)} \sum_{n=1}^{\infty} \frac{p(n)}{n^k} (\log(1+t))^n$$
$$= \frac{1}{\log(1+t)} \sum_{n=0}^{\infty} \frac{p(n+1)}{(n+1)^k} (n+1)! \sum_{l=n+1}^{\infty} S_1(l, n+1) \frac{t^l}{l!}$$
$$= \left(\frac{t}{\log(1+t)}\right) \left(\sum_{n=0}^{\infty} \frac{p(n+1)}{(n+1)^k} (n+1)! \sum_{l=0}^{\infty} S_1 \cdot (n+1+l, n+1) \frac{t^n}{(n+1+l)!}\right)$$
$$= \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!}\right) \left(\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{p(m+1)(m+1)!}{(m+1)^k} S_1 \cdot (n+1, m+1) \frac{t^n}{(n+1)!}\right)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{p(l+1)(l+1)!}{(l+1)^k} \frac{S_1(m+1, l+1)}{m+1} b_{n-m}\right) \frac{t^n}{n!}. \quad (36)$$

Hence, by (34), we obtain the following theorem.

Theorem 9. *For each nonnegative integer n and each integer k , we have*

$$b_{n,p}^{(k)} = \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{p(l+1)(l+1)!}{(l+1)^k} \frac{S_1(m+1, l+1)}{m+1} b_{n-m}. \quad (37)$$

In particular, we have

$$b_{n,1/\Gamma}^{(k)} = \sum_{m=0}^{n} \sum_{l=0}^{m} \binom{n}{m} \frac{S_1(m+1, n+1)}{(l+1)^{k-1}(m+1)} b_{n-m}. \quad (38)$$

By replacing t by $e^t - 1$ in (31), we have

$$\frac{u_k(t \mid p)}{t} e^{xt} = \frac{1}{t} \left(\sum_{n=1}^{\infty} \frac{p(n) t^n}{n^k}\right) e^{xt}$$
$$= \left(\sum_{n=0}^{\infty} \frac{p(n+1) t^n}{(n+1)^k}\right) \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} t^n\right) \quad (39)$$
$$= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \frac{p(l+1)}{(l+1)^k} \frac{n!}{(n-l)!} x^{n-l}\right) \frac{t^n}{n!},$$

and by (6), we get

$$\sum_{n=0}^{\infty} b_{n,p}^{(k)}(x) \frac{1}{n!} (e^t - 1)^n = \sum_{n=0}^{\infty} b_{n,p}^{(k)}(x) \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}$$
$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} S_2(n, m) b_{m,p}^{(k)}(x)\right) \frac{t^n}{n!}. \quad (40)$$

By (39) and (40), we obtain the following theorem.

Theorem 10. *For each nonnegative integer n and each arithmetic function p(n), we have*

$$\sum_{m=0}^{n} S_2(n, m) b_{m,p}^{(k)}(x) = \sum_{l=0}^{n} \frac{p(l+1)}{(l+1)^k} (n)_l x^{n-l}. \quad (41)$$

In particular, we have

$$\sum_{m=0}^{n} S_2(n, m) b_{m,p}^{(k)} = \frac{p(n+1)}{(n+1)^k} n!. \quad (42)$$

4. Conclusion

The polyexponential function was first studied by Hardy. In [21], Kim and Kim modified that function which was again called the polyexponential functions as an inverse to the polylogarithm function. In addition, they defied the unipoly function, attached a arithmetic function p, and found some interesting identities related to Bernoulli numbers, poly-Bernoulli polynomials, and the Stirling numbers of the first kind and second kind. The polyexponential function have been used to define some special polynomials by some researcher and found many interesting identities of those polynomials (see [11, 12, 20–26]).

In this paper, we defined the poly-Bernoulli polynomials of the second kind by using the polyexponential function and found some interesting identities.

In addition, we also define the unipoly-Bernoulli polynomials of the second kind and found some identities which were related to poly-Bernoulli polynomials of the second kind, Bernoulli polynomials, and the Stirling numbers of the first and second kind.

Data Availability

No data was used to support the findings of the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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Research Article

Three-Parameter Logarithm and Entropy

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A three-parameter logarithmic function is derived using the notion of q-analogue and ansatz technique. The derived three-parameter logarithm is shown to be a generalization of the two-parameter logarithmic function of Schwämmle and Tsallis as the latter is the limiting function of the former as the added parameter goes to 1. The inverse of the three-parameter logarithm and other important properties are also proved. A three-parameter entropic function is then defined and is shown to be analytic and hence Lesche-stable, concave, and convex in some ranges of the parameters.

1. Introduction

The concept of entropy provides deep insight into the direction of spontaneous change for many everyday phenomena. For example, a block of ice placed on a hot stove surely melts, while the stove grows cooler. Such a process is called irreversible because no slight change will cause the melted water to turn back into ice while the stove grows hotter [1]. The concept of entropy was first introduced by German physicist Rudolf Clausius as a precise way of expressing the second law of thermodynamics.

The Boltzmann equation for entropy is

$$S = k_B \ln \omega, \tag{1}$$

where k_B is the Boltzmann constant [2] and ω is the number of different ways or microstates in which the energy of the molecules in a system can be arranged on energy levels [3]. The Boltzmann entropy plays a crucial role in the foundation of statistical mechanics and other branches of science [4].

The Boltzmann-Gibbs-Shannon entropy [5, 6] is given by

$$S_{\mathrm{BGS}} \equiv -k \sum_{i=1}^{\omega} p_i \ln p_i = k \sum_{i=1}^{\omega} p_i \ln \frac{1}{p_i}, \tag{2}$$

where

$$\sum_{i=1}^{\omega} p_i = 1. \tag{3}$$

S_{BGS} is a generalization of the Boltzmann entropy because if $p_i = 1/\omega$, for all i,

$$S_{\mathrm{BGS}} = k \ln \omega. \tag{4}$$

Systems presenting long-range interactions and/or long-duration memory have been shown not well described by the Boltzmann-Gibbs statistics. Some examples may be found in gravitational systems, Levy flights, fractals, turbulence physics, and economics. In an attempt to deal with such systems, Tsallis [7] postulated a nonextensive entropy which generalizes Boltzmann-Gibbs entropy through an entropic index q [8]. Another generalization was also suggested by Renyi [9]. Abe [10] proposed how to generate entropy functionals.

Tsallis q-entropy [7] is given by

$$S_q \equiv k \frac{1 - \sum_{\{i=1\}}^{\omega} p_i^q}{q-1} = k \sum_{i=1}^{\omega} p_i \ln_q \frac{1}{p_i}, \tag{5}$$

where $q \in \mathbb{R}$, $\sum_{i=1}^{\omega} p_i = 1$ and

$$\ln_q x \equiv \frac{x^{1-q} - 1}{1 - q}, \ (\ln_1 x = \ln x), \tag{6}$$

which is referred to as q-logarithm. If $p_i = 1/\omega$, for all i, then

$$S_q = k \ln_q \omega. \tag{7}$$

The inverse of the q-logarithm is the q-exponential

$$e_q^x \equiv [1 + (1 - q)x]_+^{1/1-q}, \ \ (e_1^x = e^x), \tag{8}$$

where $[\cdots]_+$ is zero if its argument is nonpositive.

A q-sum and q-product and their calculus studied in [11] were, respectively, defined as follows (these were also mentioned in [5]):

$$\begin{aligned} x \oplus_q y &\equiv x + y + (1 - q)xy, \ (x \oplus_1 y = x + y), \\ x \otimes_q y &\equiv \left(x^{1-q} + y^{1-q} - 1\right)^{\frac{1}{1-q}}, \ (x \otimes_1 y = xy). \end{aligned} \tag{9}$$

The q-logarithm satisfies the following properties:

$$\begin{aligned} \ln_q(xy) &= \ln_q x \oplus_q \ln_q y, \\ \ln_q\left(x \otimes_q y\right) &= \ln_q x + \ln_q y. \end{aligned} \tag{10}$$

Then, a two-parameter logarithm was defined and presented along with a two-parameter entropy in [5]. It was defined as follows:

$$\ln_{q,q'} x = \frac{1}{1 - q'} \left[\exp\left(\frac{1 - q'}{1 - q} \left(x^{1-q} - 1\right) \right) - 1 \right]. \tag{11}$$

The above doubly deformed logarithm satisfies

$$\ln_{q,q'}\left(x \otimes_q y\right) = \ln_{q,q'} x \oplus_{q'} \ln_{q,q'} y. \tag{12}$$

Properties of the two-parameter logarithm and those of the two-parameter entropy were proved in [5]. Probability distribution in the canonical ensemble of the two-parameter entropy was obtained in [12] while applications were discussed in [13].

In Section 2 of the present paper, a three-parameter logarithm $\ln_{q,q',r} x$, where $q, q', r \in \mathbb{R}$, is derived using q-analogues and ansatz technique. In Section 3, the inverse of the three-parameter logarithm is derived and some properties are proved. A three-parameter entropy and its properties are presented in Section 4, and conclusion is given in Section 5.

2. Three-Parameter Logarithm

As $x = e^{\ln x}$, a q-analogue of x will be defined by

$$[x]_q = e^{\ln_q x}, \tag{13}$$

where $\ln_q x$ is defined in (6). Similarly, the q'-analogue of $[x]_q$ is defined by

$$[x]_{q,q}{}' = e^{\ln_{q,q'} x}, \tag{14}$$

where $\ln_{q,q}{}' x$ is as defined in (11), which can be written as

$$\ln_{q,q'} x = \frac{[x]_q^{1-q'} - 1}{1 - q'} = \frac{\left(e^{\ln_q x}\right)^{1-q'} - 1}{1 - q'}. \tag{15}$$

The three-parameter logarithm is then defined as

$$\ln_{q,q}{}'{,r} x = \frac{[x]_{q,q'}^{1-r} - 1}{1 - r} = \frac{\left(e^{\ln_{q,q'} x}\right)^{1-r} - 1}{1 - r}, \tag{16}$$

from which

$$\ln_{q,q}{}'{,r} x \equiv \frac{1}{1 - r} \left\{ e^{\left(1/1-q' \left\{ e^{(1-q')\ \ln_q x} - 1 \right\}\right)^{1-r}} - 1 \right\}. \tag{17}$$

To obtain similar property as that in (12), define $x \otimes_{q,q}{}' y$ as the q'-analogue of $x \otimes_q y$. That is,

$$x \otimes_{q,q}{}' y \equiv \left[x \otimes_q y\right]_q{}' = \left([x]_{q'}^{1-q} + [y]_{q'}^{1-q} - 1\right)^{\frac{1}{1-q}}. \tag{18}$$

Lemma 1. *The following relations hold*

$$\ln_{q,q}{}'\left(x \otimes_{q'} y\right) = \ln_{q,q}{}' x + \ln_{q,q}{}' y, \tag{19}$$

$$\ln_{q,q}{}'{,r}\left(x \otimes_q{}' y\right) = \ln_{q,q}{}'{,r} x \oplus_r \ln_{q,q}{}'{,r} y. \tag{20}$$

Proof. From (16) and (18),

$$\begin{aligned} &\ln_{q,q}{}'\left(x \otimes_q{}' y\right) \\ &= \frac{\left[x \otimes_q{}' y\right]_q^{1-q'} - 1}{1 - q'} = \frac{\left\{\left([x]_q^{1-q'} + [y]_q^{1-q'} - 1\right)^{1/1-q'}\right\}^{1-q'} - 1}{1 - q'} \\ &= \frac{[x]_q^{1-q'} + [y]_q^{1-q'} - 1 - 1}{1 - q'} = \frac{[x]_q^{1-q'} - 1}{1 - q'} + \frac{[y]_q^{1-q'} - 1}{1 - q'} \\ &= \ln_{q,q}{}' x + \ln_{q,q}{}' y. \end{aligned} \tag{21}$$

In similar manner and using (14),

$$\begin{aligned}
&\ln_{q,q',r}\left(x \otimes_{q'} y\right) \\
&= \frac{\left[x \otimes_{q'} y\right]_{q,q'}^{1-r} - 1}{1-r} = \frac{\left\{e^{\ln_{q,q'}\left(x \otimes_{q'} y\right)}\right\}^{1-r} - 1}{1-r} \\
&= \frac{\left(e^{\ln_{q,q'} x + \ln_{q,q'} y}\right)^{1-r} - 1}{1-r} = \frac{\left(e^{\ln_{q,q'} x}\right)^{1-r}\left(e^{\ln_{q,q'} y}\right)^{1-r} - 1}{1-r} \\
&= \frac{\left\{\left(e^{\ln_{q,q'} x}\right)^{1-r} - 1\right\} + \left\{\left(e^{\ln_{q,q'} y}\right)^{1-r} - 1\right\} + \left\{\left(e^{\ln_{q,q'} x}\right)^{1-r} - 1\right\}\left\{\left(e^{\ln_{q,q'} y}\right)^{1-r} - 1\right\}}{1-r}.
\end{aligned} \tag{22}$$

Thus,

$$\begin{aligned}
\ln_{q,q',r}\left(x \otimes_{q'} y\right) &= \frac{\left(e^{\ln_{q,q'} x}\right)^{1-r} - 1}{1-r} + \frac{\left(e^{\ln_{q,q'} y}\right)^{1-r} - 1}{1-r} + (1-r) \\
&\quad \cdot \left[\frac{1}{1-r}\left(e^{\ln_{q,q'} x}\right)^{1-r} - 1\right]\left[\frac{1}{1-r}\left(e^{\ln_{q,q'} y}\right)^{1-r} - 1\right] \\
&= \ln_{q,q',r} x + \ln_{q,q',r} y + (1-r)\left[\ln_{q,q',r} x\right]\left[\ln_{q,q',r} y\right] \\
&= \ln_{q,q',r} x \oplus_r \ln_{q,q',r} y,
\end{aligned} \tag{23}$$

which is the desired relation analogous to (12). ?

One can also derive (17) using ansatz. To do this, let $x = y$ in (20). Then,

$$\ln_{q,q',r}\left(x \otimes_{q'} x\right) = \ln_{q,q',r} x \oplus_r \ln_{q,q',r} x. \tag{24}$$

Lemma 2. *If* $\ln_{q,q',r} x = G(\ln_{q,q'} x) = G(z)$, *then*

$$G(2z) = 2G(z) + (1-r)[G(z)]^2. \tag{25}$$

Moreover, when $z = \ln_{q,q}{}' x$, *the ansatz*

$$G(z) = \frac{1}{1-r}\left(b^z - 1\right), \tag{26}$$

satisfies equation (25).

Proof. Note that from (21)

$$\begin{aligned}
\ln_{q,q',r}\left(x \otimes_{q'} x\right) &= G\left(\ln_{q,q'}\left(x \otimes_{q'} x\right)\right) \\
&= G\left(\ln_{q,q'} x + \ln_{q,q'} x\right) \\
&= G\left(2 \ln_{q,q'} x\right) = G(2z).
\end{aligned} \tag{27}$$

Thus, from (23) and (20),

$$\begin{aligned}
G\left(2 \ln_{q,q'} x\right) &= \ln_{q,q',r} x \oplus_r \ln_{q,q',r} x = \ln_{q,q',r} x + \ln_{q,q',r} x \\
&\quad + (1-r)\left(\ln_{q,q',r} x\right)^2 = 2G\left(\ln_{q,q'} x\right) + (1-r) \\
&\quad \cdot \left[G\left(\ln_{q,q'} x\right)\right]^2 G(2z) = 2G(z) + (1-r)[G(z)]^2.
\end{aligned} \tag{28}$$

Then, the ansatz in (26) will give

$$\begin{aligned}
&2G(z) + (1-r)[G(z)]^2 \\
&= 2 \cdot \frac{1}{1-r}\left(b^z - 1\right) + (1-r)\left[\frac{1}{1-r}\left(b^z - 1\right)\right]^2 \\
&= \frac{2}{1-r}\left(b^z - 1\right) + \frac{\left(b^z - 1\right)^2}{1-r} = \frac{2b^z - 2 + b^{2z} - 2b^z + 1}{1-r} \\
&= \frac{2b^z - 2 + b^{2z} - 2b^z + 1}{1-r} = \frac{b^{2z} - 1}{1-r} = G(2z),
\end{aligned} \tag{29}$$

which means that (26) solves equation (25). ?

Lemma 2. implies that

$$G(z) = G\left(\ln_{q,q'} x\right) = \ln_{q,q',r} x = \frac{1}{1-r}\left(b^{\ln_{q,q'} x} - 1\right). \tag{30}$$

Using the property that $d/dx \ln_{q,q',r} x|_{x=1} = 1$,which is a natural property of a logarithmic function, it is determined that $b = e^{1-r}$. Consequently,

$$\ln_{q,q',r} x = \frac{1}{1-r}\left(e^{(1-r)\ \ln_{q,q'} x} - 1\right). \tag{31}$$

Explicitly,

$$\ln_{q,q',r} x = \frac{1}{1-r}\left(e^{1-r/1-q'\ \exp\left(\left[\left(1-q'\right)/(1-q)\right]\left(x^{1-q}-1\right)\right)} - 1\right), \tag{32}$$

which is the same as that in (17). The preceding equation can be written as

$$\ln_{q,q',r} x = \ln_r e^{\ln_{q,q'} x}. \tag{33}$$

It can be easily verified that

$$\lim_{r \to 1} \ln_{q,q',r} x = \ln_{q,q'} x. \tag{34}$$

Graphs of $\ln_{q,q',r} x$ for $q = q' = r$ are shown in Figure 1 while graphs of $\ln_{q,q',r} x$ with one fixed parameter are shown in Figure 2.

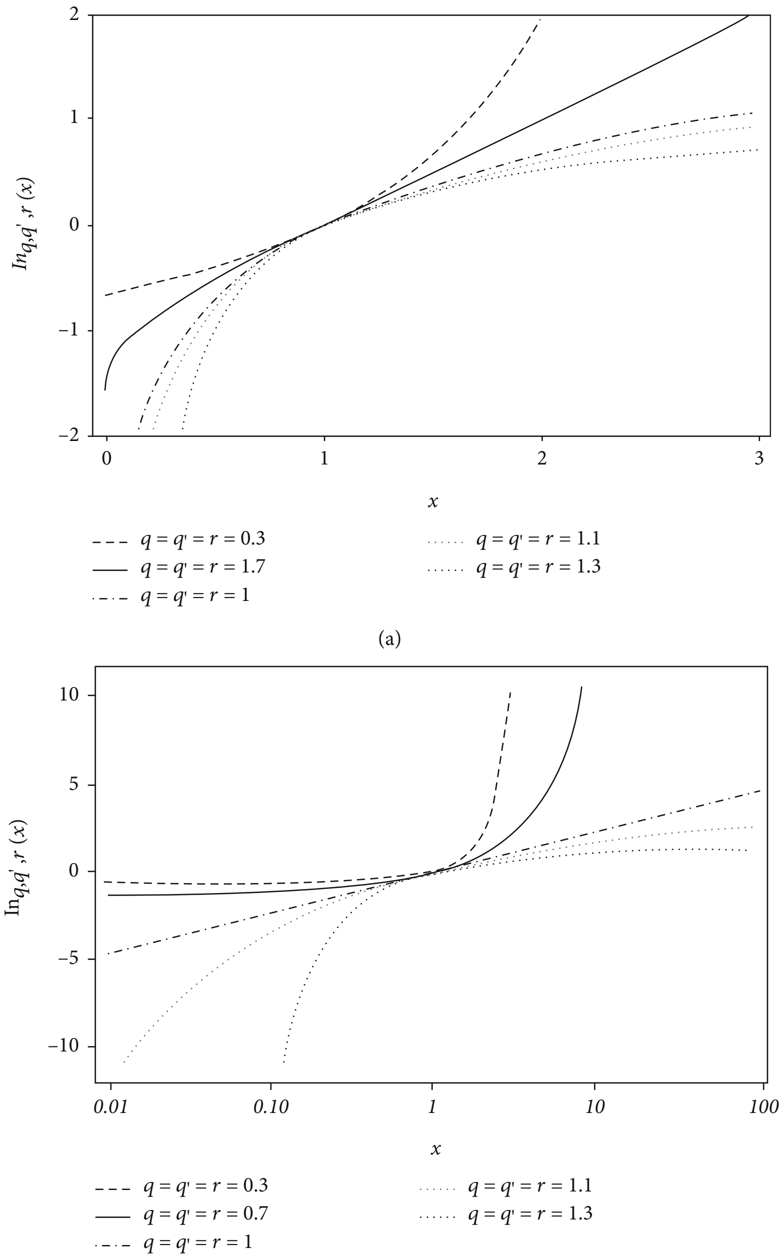


Figure 1: Illustration of the three-parameter logarithm in equation (32), setting $q = q' = r$ in (a) linear scales and (b) semilogarithmic scales.

3. Properties

In this section, the inverse of the three-parameter logarithmic function will be derived. It is also verified that the derivative of this logarithm at $x = 1$ is 1 and that the value of the function at $x = 1$ is zero. Moreover, it is shown that the following equality holds

$$\ln_{q,q',r}\frac{1}{x} = -\ln_{2-q,2-q',2-r}x. \tag{35}$$

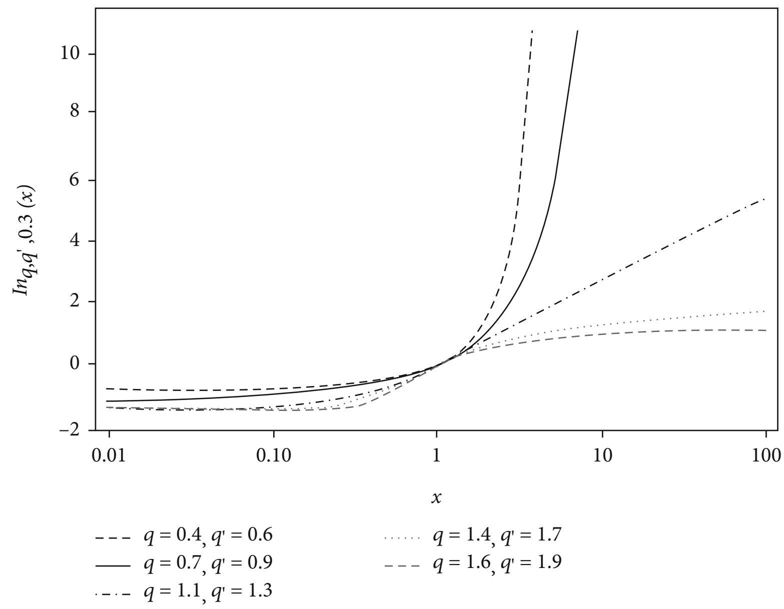
It follows from (16) that the three-parameter logarithmic function is an increasing function of x. Thus, a unique inverse function exists.

Theorem 3. *The inverse of the three-logarithmic function is given by*

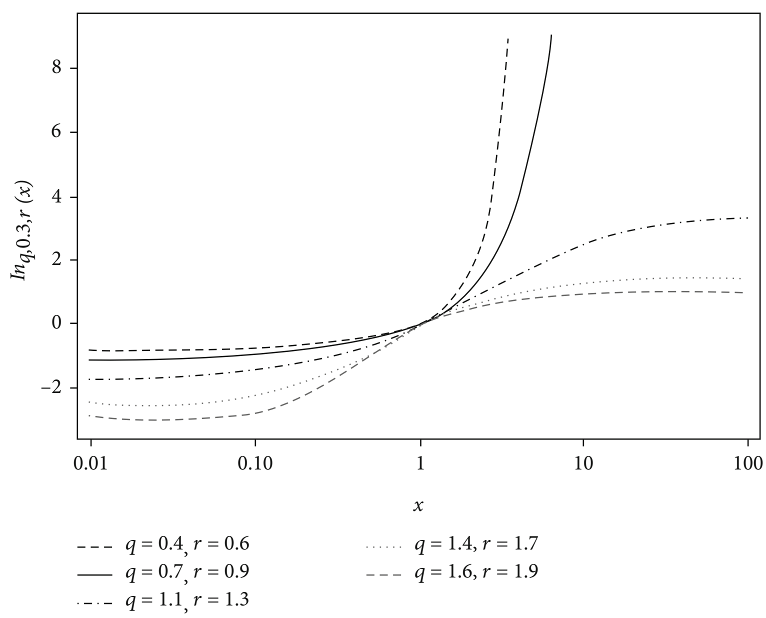
$$e^{y}_{q,q',r} = \exp_q\left\{\ln e_{q'}^{\ln e_r^{y}}\right\}. \tag{36}$$

Proof. To find the inverse function, let $y = \ln_{q,q',r}(x)$ and solve for x. That is,

$$y = \frac{1}{1-r}\left\{\exp\left(\frac{1-r}{1-q'}\ \exp\left(\frac{1-q'}{1-q}\left(x^{1-q}-1\right)\right)-1\right)-1\right\}, \tag{37}$$

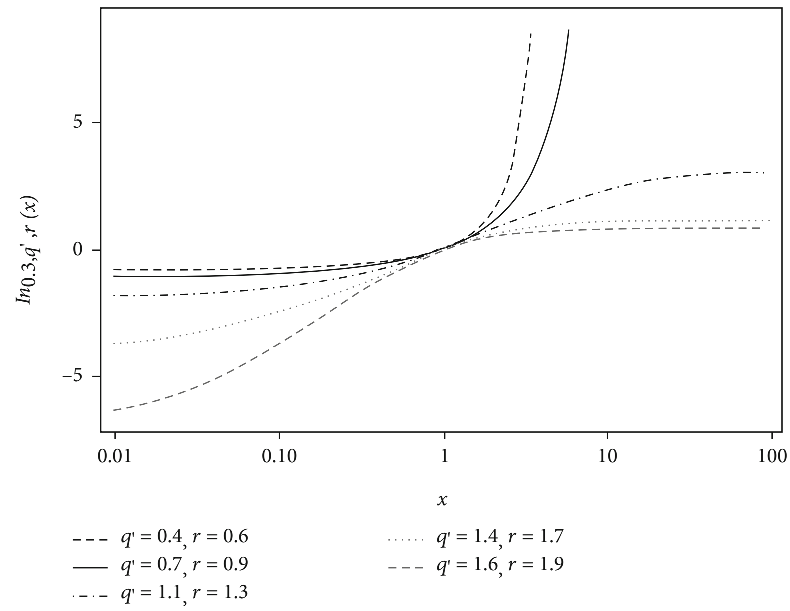


(a)



(b)

Figure 2: Continued.



(c)

FIGURE 2: (a) Illustration of the three-parameter logarithm for fixed value of r. (b) Illustration of the three-parameter logarithm for fixed value of q'. (c) Illustration of the three-parameter logarithm for fixed value of q.

from which

$$x=\left\{1+\frac{1-q}{1-q'}\ln\left[1+\frac{1-q'}{1-r}\ln\{1+(1-r)y\}\right]\right\}^{1/1-q}. \tag{38}$$

Thus, the inverse function is given by

$$\begin{aligned} e^{y}_{q,q',r} = \exp_{q,q',r}y &= \left\{1+\frac{1-q}{1-q'}\ln\left[1 + \frac{1-q'}{1-r}\ln\{1+(1-r)y\}\right]\right\}^{1/1-q} \\ &= \left\{1+\frac{1-q}{1-q'}\ln\left[1+\left(1-q'\right)\ln\{1+(1-r)y\}^{1/1-r}\right]\right\}^{1/1-q} \\ &= \left\{1+\frac{1-q}{1-q'}\ln\left[1+\left(1-q'\right)\ln e_r^y\right]\right\}^{1/1-q} \\ &= \left\{1+(1-q)\ln\left[1+\left(1-q'\right)\ln e_r^y\right]^{1/1-q'}\right\}^{1/1-q} \\ &= \left\{1+(1-q)\ln e_{q'}^{\ln e_r^y}\right\}^{1/1-q} = e_q^{\ln e_{q'}^{\ln e_r^y}} = \exp_q\left\{\ln e_q^{\ln e_r^y}\right\}, \end{aligned} \tag{39}$$

where the q-exponential e_q^x is defined in (8).

Theorem 4. *The three-parameter logarithm satisfies the following properties:*

(1) $(d/dx)\ln_{q,q',r}x|_{x=1}=1$,

(2) $\ln_{q,q',r}1=0$,

(3) The slope of $\ln_{q,q',r}x$ *is positive for all* $x>0$

(4) $\ln_{q,q',r}(1/x)=-\ln_{2-q,2-q',2-r}x$.

Proof. To find the derivative, use (17) to obtain

$$\frac{d}{dx}\ln_{q,q',r}x = x^{-q}\exp\left\{\frac{1-r}{1-q'}\left(e^{\left(1-q'\right)\ln_q 1}-1\right)-1\right\}=0. \tag{40}$$

From (40), the slope of $\ln_{q,q',r}x$ is positive for all $x>0$. This is also observed in Figures 1 and 2.

To prove part (4) of the theorem, let $q\longrightarrow 2-q$, $q'\longrightarrow 2-q'$, and $r\longrightarrow 2-r$. From [5],

$$\ln_{q,q'}\frac{1}{x}=-\ln_{2-q,2-q'}x, \tag{41}$$

then

$$\begin{aligned} \ln_{q,q'}(1/x) &= \frac{\left(e^{\ln_{\{q,q'\}}(1/x)}\right)^{1-r}-1}{1-r} = \frac{\left(e^{-\ln_{2-q,2-q'}x}\right)^{1-r}-1}{1-r} \\ &= \frac{\left(e^{\ln_{2-q,2-q'}x}\right)^{r-1}-1}{-(r-1)} = \frac{-\left\{\left(e^{\ln_{2-q,2-q'}x}\right)^{1-(2-r)}-1\right\}}{1-(2-r)} \\ &= -\ln_{2-q,2-q',2-r}x.? \end{aligned} \tag{42}$$

4. Three-Parameter Entropy

A three-parameter generalization of the Boltzmann-Gibbs-Shannon entropy is constructed here, and its properties are proved. Based on the three-parameter logarithm the entropic function is defined as follows:

$$S_{q,q',r} \equiv k \sum_{i=1}^{\omega} p_i \ln_{q,q',r} \frac{1}{p_i}. \tag{43}$$

If $p_i = 1/\omega, \forall i$,

$$S_{q,q',r} = k \ln_{q,q',r} \omega, \tag{44}$$

where ω is the number of states.

4.1. Lesche-Stability (or Experimental Robustness). The functional form of $\ln_{q,q',r} x$ given in the previous section is analytic in x as $\ln_{q,q',r} x$ is analytic in x. Consequently, $S_{q,q',r}$ is Lesche-stable.

4.2. Expansibility. An entropic function S satisfies this condition if a zero probability ($p_i = 0$) state does not contribute to the entropy. That is, $S(p_1, p_2, \cdots, p_w, 0) = S(p_1, p_2, \cdots, p_w)$ for any distribution $\{p_i\}$. Observe that in the limit $p_i = 0$, $\ln_{q,q',r} 1/p_i$ is finite if one of q, q', r is greater than 1. Consequently,

$$S_{q,q',r}(p_1, p_2, \cdots, p_w, 0) = S_{q,q',r}(p_1, p_2, \cdots, p_w) \tag{45}$$

provided that one of q, q', r is greater than 1.

4.3. Concavity. Concavity of the entropic function $S_{q,q',r}$ is assured if

$$\frac{d^2}{dp_i^2}\left(p_i \ln_{q,q',r} \frac{1}{p_i}\right) < 0. \tag{46}$$

Theorem 5. *The three-parameter entropic function $S_{q,q',r}$ is concave provided $q + q' + r > 2$.*

Proof. By manual calculation (which is a bit tedious),

$$\begin{aligned}\frac{d^2}{dp_i^2}\left(p_i \ln_{q,q',r} \frac{1}{p_i}\right) &= \exp\left\{\frac{1-r}{1-q'}\left(e^{(1-q')\ln_q 1/p_i} - 1\right)\right\} \\ &\quad \cdot e^{(1-q')\ln_q 1/p_i} \times \left\{-q p_i^{q-2} + \left(1-q'\right) p_i^{2q-3} \right. \\ &\quad \left. + (1-r) p_i^{2q-3} e^{(1-q')\ln_q 1/p_i}\right\}.\end{aligned} \tag{47}$$

In the limit $p_i \longrightarrow 1$, the second derivative given in (47) is less than zero if $q + q' + r > 2$. Thus, concavity of $S_{q,q',r}$ is guaranteed if $q + q' + r > 2$. In the limit $p_i \longrightarrow 0$, concavity is guaranteed if $r > 1$. If $r < 1$, concavity holds if $q > 1$.

4.4. Convexity. A twice-differentiable function of a single variable is convex if and only if its second derivative is nonnegative on its entire domain. The analysis on the convexity of $S_{q,q',r}$ is analogous to that of its concavity. In the limit $p_i \longrightarrow 1$, convexity is guaranteed if $q + q' + r \le 2$. In the limit $p_i \longrightarrow 0$, convexity is assured if $q, r < 1$. Thus, we have the following theorem.

Theorem 6. *The three-parameter entropic function $S_{q,q}{}'{,r}$ is convex provided $q + q' + r \le 2$.*

Concavity of $S_{q,q',r}$ is illustrated in Figure 3(a) while convexity is illustrated in Figure 3(b).

4.5. Composability. An entropic function S is said to be composable if for events A and B,

$$S(A + B) = \Phi\,(S(A), S(B), \text{indices}), \tag{48}$$

where Φ is some single-valued function [5]. The Boltzmann-Gibbs-Shannon entropy satisfies

$$S_{\mathrm{BGS}}(A + B) = S_{\mathrm{BGS}}(A) + S_{\mathrm{BGS}}(B). \tag{49}$$

Hence, it is composable and additive. The one-parameter entropy S_q,for $q \neq 1$ is also composable as it satisfies

$$\frac{S_q^{A+B}}{k} = \frac{S_q^A}{k} \oplus_q \frac{S_q^B}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\frac{S_q(A)}{k}\frac{S_q(B)}{k}. \tag{50}$$

The two-parameter entropy $S_{q,q'}$ [5] satisfies, in the microcanonical ensemble (i.e., equal probabilities), that

$$Y\left(S^{A+B}\right) = Y\left(S^A\right) + Y\left(S^B\right) + \frac{1-q'}{1-q} Y\left(S^A\right) Y\left(S^B\right), \tag{51}$$

where

$$Y(S) \equiv 1 + \frac{1-q}{1-q'} \ln\left[1 + \left(1-q'\right)\frac{S}{k}\right]. \tag{52}$$

However, this does not hold true for arbitrary distributions $\{p_i\}$, which means $S_{q,q}{}'$ is not composable in general. For the 3-parameter entropy $S_{q,q',r}$, a similar property as that of (51) is obtained as shown in the following theorem.

Theorem 7. *The three-parameter entropy $S_{q,q',r}$ satisfies*

$$U\left(S^{A+B}\right) = U\left(S^A\right) + U\left(S^B\right) + \frac{1-q'}{1-q} U\left(S^A\right) U\left(S^B\right), \tag{53}$$

where

$$U(S) = \ln\left[1 + \frac{1-q'}{1-r} \ln\left[1 + (1-r)\frac{S}{k}\right]\right]. \tag{54}$$

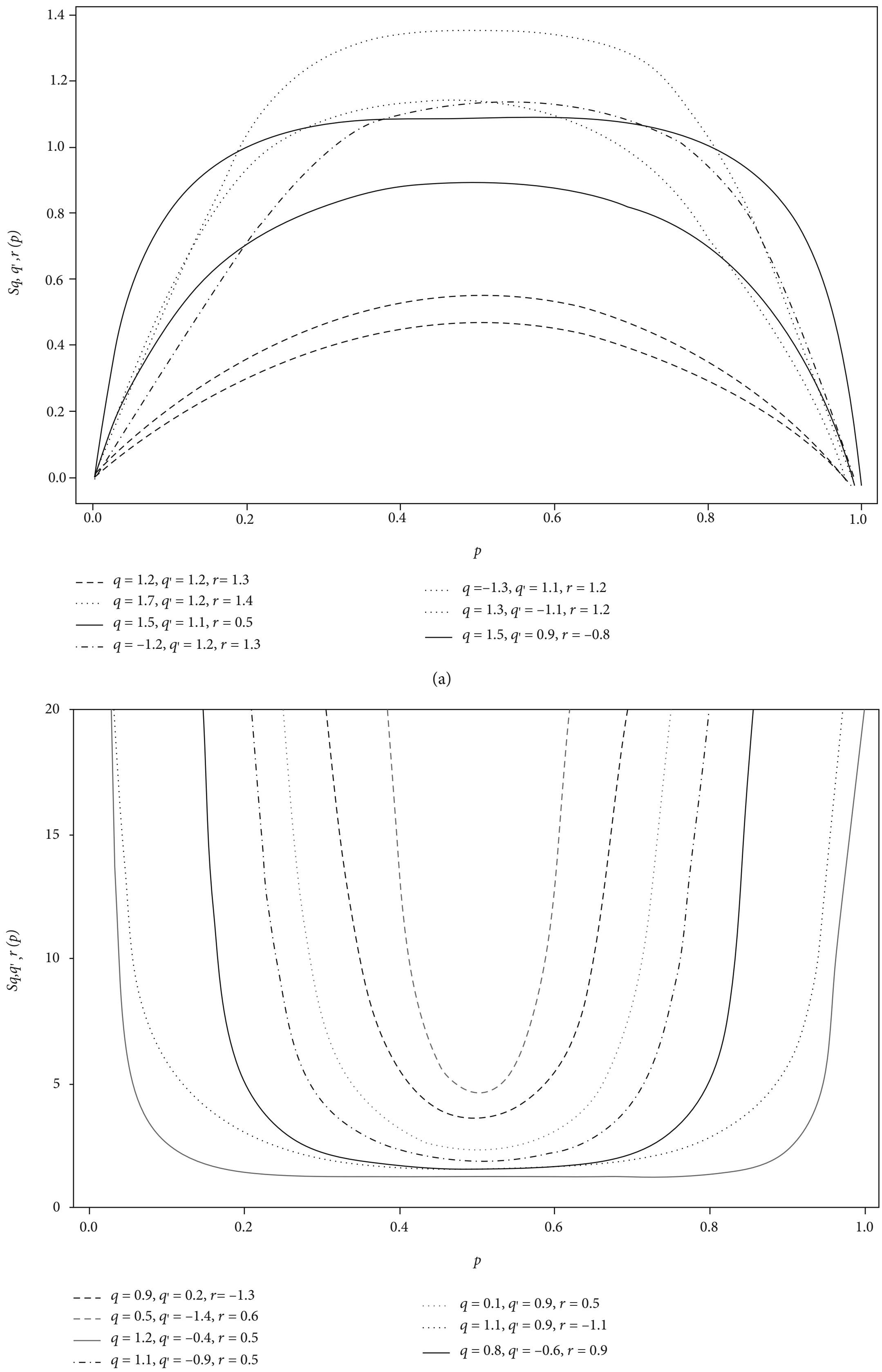


Figure 3: Illustration of the three-parameter entropic function: (a) concavity and (b) convexity.

Proof. Note that

$$\ln_{q,q}{}' (W_A W_B) = \frac{1}{1-q'}\left[e^{(1-q')\ \ln_q (W_A W_B)} - 1\right] = \frac{S^{A+B}_{q,q'}}{k}, \tag{55}$$

from which

$$\begin{aligned}\frac{S^A_{q,q',r}}{k} &= \ln_{q,q',r} W_A = \frac{1}{1-r}\left[e^{(1-r)\ \ln_{q,q'} W_A} - 1\right] \\ &= \frac{1}{1-r}\left[e^{(1-r) S^A_{q,q'}/k} - 1\right].\end{aligned} \tag{56}$$

Similarly,

$$\begin{aligned}\frac{S^B_{q,q',r}}{k} &= \ln_{q,q}{}' r W_B = \frac{1}{1-r}\left[e^{(1-r)S^B_{q,q'}/k} - 1\right], \\ \frac{S^{A+B}_{q,q',r}}{k} &= \ln_{\{q,q',r\}} W_A W_B = \frac{1}{1-r}\left[e^{(1-r)S^{A+B}_{q,q'}/k} - 1\right] \\ &= \frac{1}{1-r} e^{(1-r)S^{A+b}_{q,q'}/k} - \frac{1}{1-r}.\end{aligned} \tag{57}$$

From (57),

$$\ln\left[(1-r)\frac{S^{A+B}_{q,q',r}}{k} + 1\right] = (1-r)\frac{S^{A+B}_{q,q'}}{k}. \tag{58}$$

Using the following result in [5],

$$\frac{S^{A+B}_{q,q'}}{k} = \frac{1}{1-q'}\left\{e^{1-q'/1-q\ \ln\left[1+(1-q')S^A_{q,q'}/k\right]\cdot\ln\left[1+(1-q')S^B_{q,q'}/k\right]}\left[1+(1-q')S^A_{q,q'}/k\right]\left[1+(1-q')S^B_{q,q'}/k\right] - 1\right\}. \tag{59}$$

Equation (58) becomes

$$\begin{aligned}\ln\left[1+(1-r)\frac{S^{A+B}_{q,q',r}}{k}\right] = \frac{1-r}{1-q'}&\left\{e^{[(1-q')/(1-q)]\ \ln\left[1+[(1-q')/(1-r)]\ \ln\left[1+(1-r)S^A_{q,q',r}/k\right]\right]\cdot\ln\left[1+[(1-q')/(1-r)]\ \ln\left[1+(1-r)S^B_{q,q',r}/k\right]\right]}\right. \\ &\left.\times\left[1+\frac{1-q'}{1-r}\ \ln\left[1+(1-r)\frac{S^A_{q,q',r}}{k}\right]\right]\times\left[1+\frac{1-q'}{1-r}\ \ln\left[1+(1-r)\frac{S^B_{q,q',r}}{k}\right]\right] - 1\right\}.\end{aligned} \tag{60}$$

Now, with

$$U(S) = \ln\left[1+\frac{1-q'}{1-r}\ \ln\left[1+(1-r)\frac{S}{k}\right]\right], \tag{61}$$

we have

$$\begin{aligned}1 &+ \frac{1-q'}{1-r}\ \ln\left[1+(1-r)\frac{S^{A+B}_{q,q',r}}{k}\right] \\ &= e^{[(1-q')/(1-q)]U(S^A)\cdot U(S^B)} \\ &\quad\times\left[1+\frac{1-q'}{1-r}\ \ln\left[1+(1-r)\frac{S^A_{q,q',r}}{k}\right]\right] \\ &\quad\times\left[1+\frac{1-q'}{1-r}\ \ln\left[1+(1-r)\frac{S^B_{q,q',r}}{k}\right]\right].\end{aligned} \tag{62}$$

Consequently,

$$\begin{aligned}\ln&\left[1+\frac{1-q'}{1-r}\ \ln\left[1+(1-r)\frac{S^{A+B}_{q,q',r}}{k}\right]\right] \\ &= \frac{1-q'}{1-q}U(S^A)\cdot U(S^B) \\ &\quad+\ln\left[1+\frac{1-q'}{1-r}\ \ln\left[1+(1-r)\frac{S^A_{q,q',r}}{k}\right]\right] \\ &\quad+\ln\left[1+\frac{1-q'}{1-r}\ \ln\left[1+(1-r)\frac{S^B_{q,q',r}}{k}\right]\right],\end{aligned} \tag{63}$$

which can be written as

$$U(S^{A+B}) = U(S^A) + U(S^B) + \frac{1-q'}{1-q}U(S^A)U(S^B). \tag{64}$$

In view of the noncomposability of the 2-parameter entropy, $S_{q,q',r}$ is also noncomposable.

5. Conclusion

It is shown that the two-parameter logarithm of Schwämmle and Tsallis [5] can be generalized to three-parameter logarithm using q-analogues. Consequently, a three-parameter entropic function is defined, and its properties are proved. It will be interesting to study the applicability of the three-parameter entropy to adiabatic ensembles [13] and other ensembles [14] and how these applications relate to generalized Lambert W function.

Data Availability

The computer programs and articles used to generate the graphs and support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author's declare that they have no conflicts of interest.

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Research Article

Inequalities of Hardy Type via Superquadratic Functions with General Kernels and Measures for Several Variables on Time Scales

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We use the properties of superquadratic functions to produce various improvements and popularizations on time scales of the Hardy form inequalities and their converses. Also, we include various examples and interpretations of the disparities in the literature that exist. In particular, our findings can be seen as refinements of some recent results closely linked to the time-scale inequalities of the classical Hardy, Pólya-Knopp, and Hardy-Hilbert. Some continuous inequalities are derived from the main results as special cases. The essential results will be proved by making use of some algebraic inequalities such as the Minkowski inequality, the refined Jensen inequality, and the Bernoulli inequality on time scales.

1. Introduction

In [1], Hardy claimed this fundamental inequality and proved it:

$$\int_0^\infty \left(\frac{1}{\theta}\int_0^\theta g(\eta)d\eta\right)^q d\theta \le \left(\frac{q}{q-1}\right)^q \int_0^\infty g^q(\theta)d\theta, \qquad (1)$$

where $1 < q < \infty$, $g \ge 0$, and $(q/(q-1))^q$ are sharp. They have emerged in the literature since the discovery of (1) numerous papers concerned with new arguments, generalizations, and extensions. One of the most common generalizations for (1) is the disparity of Pólya-Knopp's inequality (see [2]), which is

$$\int_0^\infty \exp\left(\frac{1}{\theta}\int_0^\theta \ln g(\eta)d\eta\right)d\theta \le e\int_0^\infty g(\theta)d\theta. \qquad (2)$$

In [3], Kaijser et al. signalized that both (1) and (2) are special states of the Hardy-Knopp's inequality:

$$\int_0^\infty \Theta\left(\frac{1}{\theta}\int_0^\theta g(\eta)d\eta\right)\frac{d\theta}{\theta} \le \int_0^\infty \Theta(g(\theta))d\theta\frac{d\theta}{\theta}, \qquad (3)$$

where $\Theta \in C((0,\infty),\mathbb{R})$ is a convex function.

In [4], Cizmeija et al. proved that if $\zeta : (0,\alpha) \longrightarrow \mathbb{R} \ge 0$, Θ is a convex on (β,γ) where $-\infty \le \beta \le \gamma \le \infty$, $g : (0,\alpha) \longrightarrow \mathbb{R}$ with $g(\theta) \in (\beta,\gamma)$, $\forall\theta \in (0,\alpha)$ as an integrable function and υ is defined by

$$\upsilon(\eta) \coloneqq \eta\int_\eta^\alpha \frac{\zeta(\theta)}{\theta^2}d\theta, \quad \text{for } \eta \in (0,\alpha), \qquad (4)$$

then the integral inequality

$$\int_0^\infty \zeta(\theta)\Theta\left(\frac{1}{\theta}\int_0^\theta g(\eta)d\eta\right)\frac{d\theta}{\theta} \le \int_0^\infty \upsilon(\theta)\Theta(g(\theta))\frac{d\theta}{\theta}, \quad (5)$$

is valid.

In [5], Kaijser et al. applied the inequality of Jensen for convex functions and the theorem of Fubini to establish an invitingly popularization (1). Particularly, it was proved that if $\zeta : (0, \alpha) \longrightarrow \mathbb{R} \ge 0$ and $l : (0, \alpha) \times (0, \alpha) \longrightarrow \mathbb{R} \ge 0, 0 < \alpha \le \infty$ such that

$$L(\theta) \coloneqq \int_0^\theta l(\theta, \eta)d\eta > 0, \theta \in (0, \alpha), \quad (6)$$

and $\Theta \in C(I, \mathbb{R}), I \subseteq \mathbb{R}$ is a convex function, $g : (0, \alpha) \longrightarrow \mathbb{R}$ such that $g(\theta) \in I, \forall\theta \in (0, \alpha)$ be integrable function, and υ is defined by

$$\upsilon(\eta) \coloneqq \eta\int_\eta^\alpha \xi(\theta)\frac{l(\theta, \eta)}{L(\theta)}\frac{d\theta}{\theta} < \infty, \quad \eta \in (0, \alpha), \quad (7)$$

then the integral inequality

$$\int_0^\infty \xi(\theta)\Theta(A_l g(\theta))\frac{d\theta}{\theta} \le \int_0^\infty \upsilon(\theta)\Theta(g(\theta))\frac{d\theta}{\theta}, \quad (8)$$

is valid, where $A_l g$ is defined by

$$A_l g(\theta) \coloneqq \frac{1}{L(\theta)}\int_0^\theta l(\theta, \eta)g(\eta)d\eta, \theta \in (0, \alpha). \quad (9)$$

As a popularization of (8), Krulic et al. [6] have demonstrated that if $(\Omega_1, \sum_1, \mu_1)$ and $(\Omega_2, \sum_2, \mu_2)$ are two measure spaces with positive σ finite measures $\zeta : \Omega_1 \longrightarrow \mathbb{R} \ge 0$ and $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \ge 0$ such that

$$L(\theta) \coloneqq \int_{\Omega_2} l(\theta, \eta)d\mu_2(\eta) > 0, \quad \theta \in \Omega_1, \quad (10)$$

and Θ is a convex function on an interval $I \subseteq \mathbb{R}$, $g : \Omega_2 \longrightarrow \mathbb{R} \ge\ > 0$ with $g(\Omega_2) \subseteq I$ be measurable function and υ is defined by

$$\upsilon(\eta) \coloneqq \left(\int_{\Omega_1} \xi(\theta)\left(\frac{l(\theta, \eta)}{L(\theta)}\right)^{q/p} d\mu_1(\theta)\right)^{q/p} < \infty, \quad \eta \in \Omega_2, \quad (11)$$

then the integral inequality

$$\left(\int_{\Omega_1} \xi(\theta)\Theta^{q/p}(A_l g(\theta))d\mu_1(\theta)\right)^{1/q} \le \left(\int_{\Omega_2} \upsilon(\eta)\Theta(g(\eta))d\mu_2(\eta)\right)^{1/q}, \quad (12)$$

is valid, where $0 < p \le q < \infty$ and $A_l g : \Omega_1 \longrightarrow \mathbb{R}$ are defined by

$$A_l g(\theta) \coloneqq \frac{1}{L(\theta)}\int_{\Omega_2} l(\theta, \eta)g(\eta)d\mu_2(\eta), \quad \theta \in \Omega_1. \quad (13)$$

Observe that inequality (12) is a generalization of Hardy inequality (1). Namely, let $\Omega_1 = \Omega_2 = \mathbb{R}_+ = (0, \infty)$, $d\mu_1(\theta) = d\theta, d\mu_2(\eta) = d\eta$ and $u(\theta) = 1/\theta$, and if $1 < p = q < \infty$ and $\Theta : [0, \infty) \longrightarrow \mathbb{R}$ are defined by $\Theta(\theta) = \theta^p$, then (1) is followed directly from (12), which can be rewritten with $g(\eta^{p/(p-1)})\eta^{1/(p-1)}$ instead of $g(\eta)$ and

$$l(\theta, \eta) \coloneqq \frac{1}{\theta}\chi_{0<\eta\le\theta<\infty}(\theta, \eta). \quad (14)$$

In the same setting, except with $g(\eta)\eta^{1/p}$ instead of $g(\eta)$ and with

$$l(\theta, \eta) \coloneqq \left(\frac{\theta}{\eta}\right)^{1/q}(\theta + \eta)^{-1}, \quad (15)$$

relation (12) becomes the Hardy-Hilbert integral inequality (see [7]).

$$\int_0^\infty \left(\int_0^\infty \frac{g(\eta)}{\theta + \eta}d\eta\right)^p d\theta \le \left(\frac{\pi}{\sin(\pi/p)}\right)^p \int_0^\infty g^p(\theta)d\theta. \quad (16)$$

In [8], Abramovich et al. considered a superquadratic function Θ instead of a convex function Θ and obtained the following refinement of inequality (12) in the particular case $p = q$, as

$$\int_{\Omega_1} \zeta(\theta)\Theta(A_l g(\theta))d\mu_1(\theta) + \int_{\Omega_1}\int_{\Omega_2} \zeta(\theta)\frac{l(\theta, \eta)}{L(\theta)}\Theta(|g(\eta) - A_l g(\theta)|)d\mu_1(\theta)d\mu_2(\eta) \le \int_{\Omega_2} \upsilon(\eta)\Theta(g(\eta))d\mu_2(\eta). \quad (17)$$

In [9], Aleksandra et al. proved that, if $\lambda \le 1, (\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two measure spaces with positive σ-finite measures, $\zeta : \Omega_1 \longrightarrow \mathbb{R} \ge 0, l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \ge 0$ such that $L : \Omega_1 \longrightarrow \mathbb{R}$ is defined as in (10), $\Theta \in C(I, \mathbb{R})$, $I \subseteq \mathbb{R}$ is a convex function, $g : \Omega_2 \longrightarrow \mathbb{R} \ge 0$ such that $g(\Omega_2) \subseteq I$ be measurable function and is defined by

$$\upsilon(\eta) \coloneqq \left(\int_{\Omega_1} \zeta(\theta)\left(\frac{l(\theta, \eta)}{L(\theta)}\right)^\lambda d\mu_1(\theta)\right)^{1/\lambda} < \infty, \quad \eta \in \Omega_2, \quad (18)$$

then the integral inequality

$$\int_{\Omega_1} \zeta(\theta)\Theta^{\lambda}(A_l g(\theta))d\mu_1(\theta) + \lambda \int_{\Omega_1}\int_{\Omega_2} \zeta(\theta)\frac{l(\theta,\eta)}{L(\theta)} \cdot \Theta^{\lambda-1}(|g(\eta) - A_l g(\theta)|)d\mu_1(\theta)d\mu_2(\eta) \leq \left(\int_{\Omega_2} v(\eta)\Theta(g(\eta))d\mu_2(\eta)\right)^{\lambda}, \quad (19)$$

is valid, where $A_l g : \Omega_1 \longrightarrow \mathbb{R}$ is defined by (13).

In the past few years, several researchers have suggested the study of dynamic time-scale inequalities. In [10], the authors showed a number of Hardy-type inequalities with a general kernel on time scale. Namely, they have determined that if $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces, $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that

$$L(\theta) \coloneqq \int_{\Omega_2} l(\theta,\eta)\Delta\eta < \infty, \theta \in \Omega_1, \quad (20)$$

and $\zeta : \Omega_1 \longrightarrow \mathbb{R}_+ \geq 0$ such that

$$v(\eta) \coloneqq \int_{\Omega_1} \frac{l(\theta,\eta)\zeta(\theta)}{L(\theta)}\Delta\theta < \infty, \quad \eta \in \Omega_2, \quad (21)$$

then the integral inequality

$$\int_{\Omega_1} \zeta(\theta)\Theta\left(\frac{1}{L(\theta)}\int_{\Lambda} l(\theta,\eta)g(\eta)\Delta\eta\right)\Delta\theta \leq \int_{\Omega_2} v(\eta)\Theta(g(\eta))\Delta\eta, \quad (22)$$

is available for all $\Delta\mu_2$-integrable $g : \Omega_2 \longrightarrow \mathbb{R}$ such that $g(\Omega_2) \subset I$ and $\Theta \in C(I, \mathbb{R}), I \subset \mathbb{R}$ are a convex function.

Moreover, Donchev et al. [11] improved the inequality (22) by replacing the function $g(\eta)$ by an m-tuple of functions $\boldsymbol{g}(\eta) = (g_1(\eta), g_2(\eta), \cdots, g_m(\eta))$ such that $g_1(\eta), g_2(\eta), \cdots, g_m(\eta)$ are $\Delta\mu_2$-integrable on Ω_2 in the following way. If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces, $U \subset \mathbb{R}^m$ a convex set and $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R}_+$ such that

$$L(\theta) \coloneqq \int_{\Omega_2} l(\theta,\eta)\Delta\eta < \infty, \quad \theta \in \Omega_1, \quad (23)$$

and $\zeta : \Omega_1 \longrightarrow \mathbb{R}$ such that

$$v(\eta) \coloneqq \int_{\Omega_1} \frac{l(\theta,\eta)\zeta(\theta)}{L(\theta)}\Delta\theta < \infty, \quad \eta \in \Omega_2, \quad (24)$$

then for every a convex function Θ, the integral inequality

$$\int_{\Omega_1} \zeta(\theta)\Theta\left(\frac{1}{L(\theta)}\int_{\Omega_2} l(\theta,\eta)\boldsymbol{g}(\eta)\Delta\eta\right)\Delta\theta \leq \int_{\Omega_2} v(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta, \quad (25)$$

is available for all $\Delta\mu_2$-integrable functions $\boldsymbol{g} : \Omega_2 \longrightarrow \mathbb{R}^m$ such that $\boldsymbol{g}(\Omega_2) \subset U \subset \mathbb{R}^m$.

In [12], the authors have specified the time scale version of (17). That is, they proved it if $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces with positive σ-finite measures, $\zeta : \Omega_1 \longrightarrow \mathbb{R} \geq 0$ and $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that $l(\theta, .)$ is a $\Delta\mu_2$-integrable function for $\theta \in \Omega_2$, and $L : \Omega_1 \longrightarrow \mathbb{R}$ is defined as

$$L(\theta) \coloneqq \int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta) > 0, \quad \theta \in \Omega_1, \quad (26)$$

$$v(\eta) \coloneqq \int_{\Omega_1} \zeta(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Delta\mu_1(\theta) < \infty, \quad \eta \in \Omega_2. \quad (27)$$

If $\Theta : [\alpha, \infty) \longrightarrow \mathbb{R} \geq 0, (\alpha \geq 0)$ and a superquadratic function, then

$$\int_{\Omega_1} \zeta(\theta)\Theta(A_l g(\theta))\Delta\mu_1(\theta) + \int_{\Omega_1}\int_{\Omega_2} \zeta(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Theta \cdot (|g(\eta) - A_l g(\theta)|)\Delta\mu_1(\theta)\Delta\mu_2(\eta) \leq \int_{\Omega_2} v(\eta)\Theta(g(\eta))\Delta\mu_2(\eta), \quad (28)$$

is available for all $\Delta\mu_2$-integrable function $g : \Omega_2 \longrightarrow \mathbb{R} \geq 0$, and $A_l g$ is defined by

$$(A_l g)(\theta) \coloneqq \frac{1}{L(\theta)}\int_{\Omega_2} l(\theta,\eta)g(\eta)\Delta\mu_2(\eta), \quad \theta \in \Omega_1. \quad (29)$$

In [13], Saker et al. obtained the following refined Jensen's inequality for superquadratic

$$\Theta\left(\frac{\int_{\Omega_2} l(\theta,\eta)g(\eta)\Delta\mu_2(\eta)}{\int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta)}\right) \leq \int_{\Omega_2} \frac{l(\theta,\eta)}{\int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta)}[\Theta(g(\eta)) - \Theta(|g(\eta) - A_l g(\theta)|)]\Delta\mu_2(\eta), \quad (30)$$

and in the same paper, he employed the above result to derive the following inequality of Hardy type:

$$\int_{\Omega_1} \zeta(\theta)\Theta^{\lambda}(A_l g(\theta))\Delta\mu_1(\theta) + \lambda\int_{\Omega_1}\int_{\Omega_2} \zeta(\theta) \cdot \frac{l(\theta,\eta)}{L(\theta)}\Theta^{\lambda-1}(A_l g(\theta))\Theta(|g(\eta) - A_l g(\theta)|)\Delta\mu_1(\eta) \leq \left(\int_{\Omega_2} v(\eta)\Theta(g(\eta))\Delta\mu_2(\eta)\right)^{\lambda}, \quad (31)$$

where

$$v(\eta) := \left(\int_{\Omega_1} \zeta(\theta) \left(\frac{l(\theta,\eta)}{L(\theta)} \right)^{\lambda} \Delta\mu_1(\theta) \right)^{1/\lambda} < \infty, \quad \mu \in \Omega_2, \tag{32}$$

$\lambda \geq 1$, $\zeta : \Omega_1 \longrightarrow \mathbb{R} \geq 0$, and $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that $l(\theta, .)$ is a $\Delta\mu_2$-integrable function for $\theta \in \Omega_2$ and $L : \Omega_1 \longrightarrow \mathbb{R}$ is defined by (26), $\Theta : [0,\infty) \longrightarrow \mathbb{R} \geq 0$ is a superquadratic function, and $A_l g$ is defined by (29).

Another development of Hardy-type inequality (28) has been made by Bibi [14] and Fabelurin [15] as follows. If $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces, $\zeta : \Omega_1 \longrightarrow \mathbb{R} \geq 0$ and $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that $l(\theta, .)$ are a $\Delta\mu_2$-integrable function for $\theta \in \Omega_2$, $L : \Omega_1 \longrightarrow \mathbb{R}$ is defined by (26) and $\Theta \in C(K_m, \mathbb{R})$ is a superquadratic function, then

$$\begin{aligned} &\int_{\Omega_1} \xi(\theta)\Theta((A_l\boldsymbol{g})(\theta))\Delta\mu_1(\theta) + \int_{\Omega_1}\int_{\Omega_2} \xi(\theta) \\ &\cdot \frac{l(\theta,\eta)}{L(\theta)} \Theta(|\boldsymbol{g}(\eta) - (A_l\boldsymbol{g})(\theta)|)\Delta\mu_2(\eta) \\ &\leq \int_{\Omega_2} v(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta), \end{aligned} \tag{33}$$

is available for all $\Delta\mu_2$-integrable functions $\boldsymbol{g} : \Omega_2 \longrightarrow \mathbb{R}^m$ such that $\boldsymbol{g}(\Omega_2) \subset K_m$, where $A_l\boldsymbol{g} : \Omega_1 \longrightarrow \mathbb{R}$ is defined by

$$(A_l\boldsymbol{g})(\theta) := \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta,\eta)\boldsymbol{g}(\eta)\Delta\mu_2(\eta), \quad \theta \in \Omega_1. \tag{34}$$

For developing of dynamic inequalities on time scale calculus, we refer the reader to the articles [16–26].

Motivated by the above results, our major aim in this paper is to deduce few nouveau general Hardy-type inequalities for multivariate superquadratic functions that involve more general kernels on arbitrary time scales.

The paper is governed as follows: We remember some basic notions, definitions, and results of multivariate superquadratic functions on time scales in Preliminaries. In Inequalities with General Kernel, we obtain the extensions to the general kernel of Hardy-type inequality. In Inequalities with Specific Time Scales, we extend the latest results from Inequalities with General Kernel to several specific time scales. In Inequalities with Specific Time Scales, we discuss several particular cases of Hardy-type inequality by choosing such special kernels. In Inequalities with Specific Kernels, we derive enhanced forms of certain well-known Hardy-Hilbert-type inequalities.

2. Preliminaries

In this section, we will present some fundamental concepts and effects to integrals of time scales and for multivariate superquadratic functions which will be useful to deduce our major results. Let $\mathbb{R}^m$ be the Euclidean space, $\boldsymbol{\theta} := (\theta_1, \theta_2, \cdots, \theta_m) \in \mathbb{R}^m$, $\boldsymbol{\eta} := (\eta_1, \eta_2, \cdots, \eta_m) \in \mathbb{R}^m$, and $\boldsymbol{g}(t) := (g_1(t), g_2(t), \cdots, g_m(t))$ be the function defined on $\boldsymbol{\theta} \subset \mathbb{R}^m$. Throughout this supplement, we utilize the following notations:

$$\begin{aligned} &\boldsymbol{\theta}.\boldsymbol{\eta} := (\theta_1\eta_1, \theta_2\eta_2, \cdots, \theta_m\eta_m), \\ &|\boldsymbol{\theta}| := (|\theta_1|, |\theta_2|, \cdots, |\theta_m|) \text{ and} \\ &\langle \boldsymbol{\theta}, \boldsymbol{\eta} \rangle := \sum_{i=1}^{m} \theta_i\eta_i. \end{aligned} \tag{35}$$

Also, $\boldsymbol{\theta} \leq \boldsymbol{\eta}(\theta < \eta)$ means that $\theta_i \leq \eta_i(\theta_i < \eta_i)$, $\forall 1 \leq i \leq m$, and $\mathbf{0} := (0, 0, \cdots, 0)$ is the null vector. The subsets K_m and K_m^+ in $\mathbb{R}^m$ are defined by

$$\begin{aligned} K_m &:= [0,\infty)^m := \{\boldsymbol{\theta} \in \mathbb{R}^m : \mathbf{0} \leq \boldsymbol{\theta}\}, \\ K_m^+ &:= [0,\infty)^m := \{\boldsymbol{\theta} \in \mathbb{R}^m : \mathbf{0} < \boldsymbol{\theta}\}. \end{aligned} \tag{36}$$

Now, we arraign the definition and few essential properties of superquadratic functions that premised in [27].

Definition 1. A function $\Theta : K_m \longrightarrow \mathbb{R}$ is named a superquadratic function if $\forall \boldsymbol{\theta} \in K_m$, $\exists c(\theta) \in \mathbb{R}^m$ such that

$$\Theta(\boldsymbol{\eta}) - \Theta(\boldsymbol{\theta}) - \Theta(|\boldsymbol{\eta} - \boldsymbol{\theta}|) \geq \langle c(\boldsymbol{\theta}), \boldsymbol{\eta} - \boldsymbol{\theta} \rangle, \forall \boldsymbol{\eta} \in K_m. \tag{37}$$

If $-\Theta$ is a superquadratic, then Θ is a subquadratic, and the reverse inequality of (37) is available.

In the following, we recall a couple of beneficial examples of a superquadratic function.

Example 1. By [2], Example 1, the power function $\Theta : [0,\infty) \longrightarrow \mathbb{R}$, defined by $\Theta(\theta) := \theta^p$, is called a superquadratic if $p \geq 2$ and a subquadratic if $1 < p \leq 2$ (it is also readily seen that if $0 < p \leq 1$ then θ^p is a subquadratic function). Since the sum of superquadratic functions is also superquadratic, then

$$\Theta(\boldsymbol{\theta}) := \sum_{i=1}^{m} \theta_i^p, \tag{38}$$

is a superquadratic on K_m for each $p \geq 2$.

Example 2 ([2], Examples 4, 5, and 6,). By utilizing the same argument as in Example 1, the functions $\Theta_1, \Theta_2, \Theta_3 : K_m \longrightarrow \mathbb{R}$ defined as

$$\Theta_1(\boldsymbol{\theta}) \coloneqq \sum_{i=1}^{m}(\theta_i \cosh \theta_i - \sinh \theta_i),$$
$$\Theta_2(\boldsymbol{\theta}) \coloneqq \ln\left(1+\sum_{i=1}^{m}\theta_i\right) - \sum_{i=1}^{m}\theta_i, \tag{39}$$
$$\Theta_3(\boldsymbol{\theta}) \coloneqq \begin{cases} \sum_{i=1,i\neq j}^{m} \theta_i^2 \ln \theta_i, & \text{if } \theta_i > 0, \theta_j = 0, \\ 0, & \text{if } \boldsymbol{\theta} = 0, \end{cases}$$

are superquadratic.

The following lemma shows that nonnegative superquadratic functions are indeed convex functions.

Lemma 2. *Suppose that Θ is a superquadratic with $\mathbf{c}(\theta) \coloneqq (c_1(\theta), c_2(\theta), \cdots, c_n(\theta))$ as in Definition 1. Then*

(i) $\Theta(0) \leq 0$ and $c_i(0) \leq 0 \forall 1 \leq i \leq m$

(ii) If $\Theta(0) \coloneqq 0$ and $\nabla\Theta(0) \coloneqq 0$, then $c_i(\theta) \coloneqq \partial_i g(\theta)$, whenever $\partial_i g(\theta)$ exists for some index $1 \leq i \leq m$ at $\boldsymbol{\theta} \in K_m$

(iii) If $\Theta \geq 0$, then Θ is convex and $\Theta(0) \coloneqq 0$ and $\nabla\Theta(0) \coloneqq 0$.

In the following, we recall the inequality of Minkowski and the inequality of Jensen for superquadratic functions on time scales which are utilized in the proof of the essential results. The following definitions and theorems are referred from [28, 29]. Let $\mathbb{T}_i$, $1 \leq i \leq m$ be time scales, and

$$\begin{aligned} \Lambda^m &\coloneqq \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_m \\ &\coloneqq \{t = (t_1, t_2, \cdots, t_m) \colon t_i \in \mathbb{T}_i, 1 \leq i \leq m\}, \end{aligned} \tag{40}$$

is called an m-dimensional time scale. Consider E to be Δ-measurable subplot of Λ^m and $g : E \longrightarrow \mathbb{R}$ a Δ-measurable function; then, the corresponding Δ-integral named Lebesgue Δ-integral is denoted by

$$\begin{aligned} &\int_E g(t_1, t_2, \cdots, t_m)\Delta_1 t_1 \cdots \Delta_m t_m, \\ &\int_E g(t)\Delta t, \int_E g d\mu_\Delta \text{or} \int_E g(t) d\mu_\Delta(t), \end{aligned} \tag{41}$$

where μ_Δ is a σ-additive Lebesgue Δ-measure on Λ^m. Also, if $\boldsymbol{g}(t) \coloneqq (g_1(t), g_2(t), \cdots, g_m(t))$ is an m-tuple of functions such that $g_1, g_2, \cdots, g_m$ are Lebesgue Δ-integrable on E, then $\int_E \boldsymbol{g} d\mu_\Delta$ denotes the m-tuple:

$$\left(\int_E g_1 d\mu_\Delta, \cdots, \int_E g_m d\mu_\Delta\right), \tag{42}$$

i.e., Δ-integral acts on each component of g.

Lemma 3. *Assume $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces, and suppose that $u \geq 0, v \geq 0$ and $g \geq 0$ on Ω_1, Ω_2 and $\Omega_1 \times \Omega_2$, respectively. If $q \geq 1$, then*

$$\begin{aligned} &\left(\int_{\Omega_1}\left(\int_{\Omega_2} g(\theta,\eta)v(\eta)d\mu_2(\eta)\right)^q u(\theta)d\mu_1(\theta)\right)^{1/q} \\ &\leq \int_{\Omega_2}\left(\int_{\Omega_1} g^q(\theta,\eta)u(\theta)d\mu_1(\theta)\right)v(\eta)d\mu_2(\eta), \end{aligned} \tag{43}$$

is available provided all integrals in (43) exist. If $0 < q < 1$ and

$$\int_{\Omega_1}\left(\int_{\Omega_2} gv d\mu_2\right)^q u d\mu_1 > 0, \quad \int_{\Omega_2} gv d\mu_2 > 0, \tag{44}$$

is available, then (43) is reversed. For $q < 0$, in addition with (44), if

$$\int_{\Omega_1} g^q u d\mu_1 > 0, \tag{45}$$

is available, then the sign of (43) is reversed.

Theorem 4 ([14], Theorem 3.1). *Assume $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two finite-dimensional time-scale measure spaces. Let $\Theta \in C(K_m, \mathbb{R}) \geq 0$ be continuous and superquadratic, $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that $l(\theta, .)$ is $\Delta\mu_2$-integrable for $\theta \in \Omega_2$. Then, the inequality*

$$\Theta\left(\frac{\int_{\Omega_2} l(\theta,\eta)\boldsymbol{g}(\eta)\Delta\mu_2(\eta)}{\int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta)}\right) \leq \frac{\int_{\Omega_2} l(\theta,\eta)\left(\Theta(\boldsymbol{g}(\eta)) - \Theta\left(\left|\boldsymbol{g}(\eta) - 1/\int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta)\int_{\Omega_2} l(\theta,\eta)\boldsymbol{g}(\eta)\Delta\mu_2(\eta)\right|\right)\right)\Delta\mu_2(\eta)}{\int_{\Omega_2} l(\theta,\eta)\Delta\mu_2(\eta)}, \tag{46}$$

holds for all functions $\boldsymbol{g}$ such that $\boldsymbol{g}(E) \subset K_m$. If Θ is a subquadratic, then (46) is reversed.

3. Inequalities with General Kernel

In this section, we get the Hardy inequality for several variables via multivariate superquadratic functions. Before presenting the results, we labeled the following hypothesis.

(A1) $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ are two time-scale measure spaces with positive σ-finite measures

(A2) $l : \Omega_1 \times \Omega_2 \longrightarrow \mathbb{R} \geq 0$ such that

$$L(\theta) := \int_{\Omega_2} l(\theta, \eta) \Delta\mu_2(\eta) < \infty, \quad \theta \in \Omega_1. \tag{47}$$

(A3) $\xi : \Omega_1 \longrightarrow \mathbb{R}$ is $\Delta\mu_1$-integrable, and the function ω is defined by

$$\omega(\eta) := \left(\int_{\Omega_1} \xi(\theta) \left(\frac{l(\theta, \eta)}{L(\theta)} \right)^{\lambda} \Delta\mu_1(\theta) \right)^{1/\lambda} < \infty, \quad \eta \in \Omega_2, \tag{48}$$

where $\lambda \geq 1$.

Theorem 5. *Assume (A1)–(A3) are satisfied. If $\Theta \in C(K_m, \mathbb{R}) \geq 0$ and is superquadratic, then*

$$\begin{aligned} &\int_{\Omega_1} \xi(\theta) \Theta^{\lambda}((A_l \boldsymbol{g})(\theta)) \Delta\mu_1(\theta) + \lambda \int_{\Omega_1} \int_{\Omega_2} \xi(\theta) \frac{l(\theta, \eta)}{L(\theta)} \\ &\quad \cdot \Theta^{\lambda-1}((A_k \boldsymbol{g})(\theta)) \Theta(|\boldsymbol{g}(\eta) - A_1 \boldsymbol{g}(\theta)|) \Delta\mu_2(\eta) \\ &\leq \left(\int_{\Omega_2} \omega(\eta) \Theta(\boldsymbol{g}(\eta)) \Delta\mu_2(\eta) \right)^{\lambda}, \end{aligned} \tag{49}$$

is available for $\boldsymbol{g} : \Omega_2 \longrightarrow \mathbb{R}^m$ that is a nonnegative $\Delta\mu_2$ -integrable function such that $\boldsymbol{g}(\Omega_2) \subset K_m$ and $A_l \boldsymbol{g} : \Omega_1 \longrightarrow \mathbb{R}$ defined by

$$(A_l \boldsymbol{g})(\theta) := \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \boldsymbol{g}(\eta) \Delta\mu_2(\eta), \quad \theta \in \Omega_1. \tag{50}$$

If Θ is subquadratic and $0 < \lambda < 1$, then (49) is reversed.

Proof. We begin with an explicit identity

$$\Theta((A_l \boldsymbol{g})(\theta)) := \Theta\left(\frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \boldsymbol{g}(\eta) \Delta\mu_2(\eta) \right). \tag{51}$$

By applying the refined Jensen inequality (46) on (51), we find

$$\begin{aligned} &\Theta((A_l \boldsymbol{g})(\theta)) + \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(|\boldsymbol{g}(\eta) - A_l \boldsymbol{g}(\theta)|) \Delta\mu_2(\eta) \\ &\quad \leq \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(\boldsymbol{g}(\eta)) \Delta\mu_2(\eta). \end{aligned} \tag{52}$$

Then, since $\lambda \geq 1$ and $\Theta \geq 0$, we get

$$\begin{aligned} &\left(\Theta((A_l \boldsymbol{g})(\theta)) + \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(|\boldsymbol{g}(\eta) - (A_l \boldsymbol{g})(\theta)|) \Delta\mu_2(\eta) \right)^{\lambda} \\ &\quad \leq \left(\frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(\boldsymbol{g}(\eta)) \Delta\mu_2(\eta) \right)^{\lambda}. \end{aligned} \tag{53}$$

Furthermost, by utilizing the famous inequality of Bernoulli, it ensues that the L. H. S. of (53) became

$$\begin{aligned} &\left(\Theta((A_l \boldsymbol{g})(\theta)) + \frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(|\boldsymbol{g}(\eta) - (A_l \boldsymbol{g})(\theta)|) \Delta\mu_2(\eta) \right)^{\lambda} \\ &\quad \leq \Theta^{\lambda}((A_l \boldsymbol{g})(\theta)) + \lambda \frac{\Theta^{\lambda-1}((A_l \boldsymbol{g})(\theta))}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta \\ &\qquad \cdot (|\boldsymbol{g}(\eta) - (A_l \boldsymbol{g})(\theta)|) \Delta\mu_2(\eta), \end{aligned} \tag{54}$$

that is, we get

$$\begin{aligned} &\Theta^{\lambda}((A_l \boldsymbol{g})(\theta)) + \lambda \frac{\Theta^{\lambda-1}((A_l \boldsymbol{g})(\theta))}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta \\ &\quad \cdot (|\boldsymbol{g}(\eta) - (A_l \boldsymbol{g})(\theta)|) \Delta\mu_2(\eta) \\ &\quad \leq \left(\frac{1}{L(\theta)} \int_{\Omega_2} l(\theta, \eta) \Theta(\boldsymbol{g}(\eta)) \Delta\mu_2(\eta) \right)^{\lambda}. \end{aligned} \tag{55}$$

Multiplying (55) by $\xi(\theta)$ and integrating it over Ω_1 with respect to $\Delta\mu_1(\theta)$, we have

$$\begin{aligned} &\int_{\Omega_1} \xi(\theta) \Theta^{\lambda}((A_l \boldsymbol{g})(\theta)) \Delta\mu_1(\theta) + \lambda \int_{\Omega_1} \xi(\theta) \\ &\quad \cdot \left(\frac{1}{L(\theta)} \int_{\Omega_1} l(\theta, \eta) \Theta(|\boldsymbol{g}(\eta) - (A_l \boldsymbol{g})(\theta)|) \Delta\mu_2(\eta) \right) \Delta\mu_1(\theta) \\ &\leq \int_{\Omega_1} \xi(\theta) \left(\frac{1}{L(\theta)} \int_{\Omega_1} l(\theta, \eta) \Theta(\boldsymbol{g}(\eta)) \Delta\mu_2(\eta) \right)^{\lambda} \Delta\mu_1(\theta). \end{aligned} \tag{56}$$

Applying the inequality of Minkowski on the R. H. S. of (56), we get

$$\int_{\Omega_1}\xi(\theta)\Theta\left(\frac{1}{L(\theta)}\int_{\Omega 2}l(\theta,\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{\lambda}\Delta\mu_1(\theta)$$
$$\leq\left(\int_{\Omega_2}\Theta(\boldsymbol{g}(\eta))\left(\int_{\Omega_2}\xi(\theta)\left(\frac{l(\theta,\eta)}{L(\theta)}\right)^{\lambda}\Delta\mu_1(\theta)\right)^{1/\lambda}\Delta\mu_2(\eta)\right)^{\lambda}. \tag{57}$$

Finally, substituting (57) into (56) and utilizing the definition (48) of the weight function ω, we get

$$\int_{\Omega_1}\xi(\theta)\Theta^{\lambda}((A_l\boldsymbol{g})(\theta))\Delta\mu_1(\theta)+\lambda\int_{\Omega_1}\int_{\Omega_2}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Theta^{\lambda-1}$$
$$\cdot((A_l\boldsymbol{g})(\theta))\Theta(|\boldsymbol{g}(\eta)-(A_l\boldsymbol{g})(\theta)|)\Delta\mu_1(\theta)\Delta\mu_2(\eta)$$
$$\leq\left(\int_{\Omega_2}\omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{\lambda}, \tag{58}$$

which is (49). If Θ is subquadratic and $0<\lambda<1$, the corresponding results can be obtained similarly.

Remark 6. If $\lambda=1$ and $m=1$ in Theorem 5, then (49) reduces to (28) premised in Introduction.

Remark 7. For the Lebesgue scale measures $\Delta\mu_1(\theta)=\Delta\theta$, $\Delta\mu_2(\eta)$ and $m=1$, Theorem 5 coincides with Theorem 2.1.1 in [30].

Remark 8. As a special case of Theorem 5 when $\mathbb{T}=\mathbb{R}$ and $m=1$, we have the inequality (19).

Corollary 9. *Given that ξ and $(A_l\boldsymbol{g})(\theta)$ are as in Theorem 5 and $\omega\geq 0$, then, since $\Theta\geq 0$ and superquadratic, the second term on the L. H. S. of (49) is nonnegative and the integral inequality*

$$\int_{\Omega_1}\xi(\theta)\Theta^{\lambda}((A_l\boldsymbol{g})(\theta))\Delta\mu_1(\theta)\leq\left(\int_{\Omega 2}\omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{\lambda}, \tag{59}$$

is valid.

Remark 10. By taking $\lambda=1$ in Corollary 9, inequality (59) reduces to (25).

Remark 11. For the Lebesgue scale measures $\Delta\mu_1(\theta)=\Delta\theta$, $\Delta\mu_2(\eta)=\Delta\eta$ and $m=1$, Corollary 9 coincides with Corollary 2.1.2 in [30].

Remark 12. Rewrite (49) with $\lambda=qp^{-1}\geq 1$ such that $0<p\leq q<\infty$ or $-\infty<p\leq q<0$; then

$$\int_{\Omega_1}\xi(\theta)\Theta^{q/p}((A_l\boldsymbol{g})(\theta))\Delta\mu_1(\theta)+\frac{q}{p}\int_{\Omega_1}\int_{\Omega_2}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Theta^{q/p-1}$$
$$\cdot((A_l\boldsymbol{g})(\theta))\Theta(|\boldsymbol{g}(\eta)-(A_l\boldsymbol{g})(\theta)|)\Delta\mu_1(\theta)\Delta\mu_2(\eta) \tag{60}$$
$$\leq\left(\int_{\Omega_2}\omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{q/p}.$$

Remark 13. For $m=1$, inequality (60) coincides with inequality (3.13) in ([28], Remark 3.5).

Remark 14. In Remark 12, since $\Theta\geq 0$, then the second term on the L. H. S. of (60) is nonnegative. Hence, (60) reduces to

$$\int_{\Omega_1}\xi(\theta)\Theta^{q/p}((A_l\boldsymbol{g})(\theta))\Delta\mu_1(\theta)\leq\left(\int_{\Omega_2}\omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{q/p}, \tag{61}$$

which is a refinement of the Hardy-type inequality in ([27], Remark 2.1.4) and [6].

In the following, we labeled some specific superquadratic functions starting with power functions.

Theorem 15. *Assume (A1)–(A3) are satisfied. If $\boldsymbol{g}_i:\Omega_2\longrightarrow\mathbb{R}(1\leq i\leq m)$ are $\Delta\mu_2$-integrable functions such that $\boldsymbol{g}_i(\Omega_2)\subset[0,\infty)$, then the inequality*

$$\int_{\Omega_1}\xi(\theta)\left(\sum_{i=1}^{m}(A_l\boldsymbol{g}_i)^p(\theta)\right)^{r}\Delta\mu_1(\theta)+\lambda\int_{\Omega_1}\int_{\Omega_2}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}$$
$$\cdot\left(\sum_{i=1}^{m}(A_k\boldsymbol{g}_i)^p(\theta)\right)^{\lambda-1}\left(\sum_{i=1}^{m}\left|\boldsymbol{g}_i(\eta)-(A_l\boldsymbol{g}_i)(\theta)^p\right|\right)$$
$$\cdot\Delta\mu_1(\theta)\Delta\mu_2(\eta)\leq\left(\int_{\Omega_2}\omega(\eta)\left(\sum_{i=1}^{m}(\boldsymbol{g}_i(\eta))^p\right)\Delta\mu_2(\eta)\right)^{\lambda}, \tag{62}$$

is valid, where $p\geq 2$ and

$$(A_lg_i)(\theta):=\frac{1}{L(\theta)}\int_{\Omega_2}l(\theta,\eta)g_i(\eta)\Delta\mu_2(\eta),\theta\in\Omega_1. \tag{63}$$

If $0<\lambda<1$ and $1<p\leq 2$, then (62) is reversed.

Proof. We get the result from Theorem 5 by putting

$$\Theta(\theta):=\sum_{i=1}^{m}\theta_i^p, \tag{64}$$

in (49).

Remark 16. For $m = 1$, Theorem 15 reduces to Corollary 3.1 in [13]. In particular, for $p = 1$ and $\lambda = 1$, Theorem 15 reduces to Remark 3.11 in [13].

Remark 17. For the Lebesgue scale measures $\Delta\mu_1(\theta) = \Delta\theta$, $\Delta\mu_2(\eta) = \Delta\eta$ and $m = 1$. Theorem 15 coincides with Corollary 2.1.5 in [30].

Theorem 18. *Assume (A1)–(A3) are satisfied. If $g_i : \Omega_2 \longrightarrow \mathbb{R}(1 \le i \le m)$ are $\Delta\mu_2$-integrable functions such that $g_i(\Omega_2) \subset [0, \infty)$, then the inequality*

$$\begin{aligned}&\int_{\Omega_1} \xi(\theta)\left(\sum_{i=1}^{m}(\exp\,(A_i g_i)(\theta) - (A_i g_i)(\theta) - 1)\right)^{\lambda} \Delta\mu_1(\theta) + I\\ &\quad\le \left(\int_{\Omega_2} \omega(\eta)\left(\sum_{i=1}^{m}(g_i(\eta) - \log g_i(\eta) - 1)\right)\Delta\mu_2(\eta)\right)^{\lambda},\end{aligned} \tag{65}$$

is valid, where

$$\begin{aligned}I :=& \lambda\int_{\Omega_1}\int_{\Omega_2} \xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\left(\sum_{i=1}^{m}(\exp\,|\log g_i(\eta)\right.\\ &\left. - (A_l g_i)(\theta)| - |\log g_i(\eta) - (A_l g_i)(\theta)| - 1)\right)\\ &\times\left(\sum_{i=1}^{m}(\exp\,(A_l g_i)(\theta) - (A_l g_i)(\theta) - 1)\right)^{\lambda-1} \Delta\mu_1(\theta)\Delta\mu_2(\eta),\end{aligned} \tag{66}$$

and

$$(A_l g_i)(\theta) := \frac{1}{L(\theta)}\int_{\Omega_2} l(\theta,\eta)\,\log g_i(\eta)\Delta\mu_2(\eta), \theta \in \Omega_1. \tag{67}$$

If $0 < \lambda < 1$, then (65) is reversed.

Proof. We get the result from Theorem 5 by putting

$$\Theta(\theta) := \sum_{i=1}^{m}(\exp\,(\theta_i) - \theta_i - 1), \tag{68}$$

in (49) and with $\log g(\eta)$ instead of $g(\eta)$.

Remark 19. By taking $m = 1$ in Theorem 18, inequality (65) reduces to inequality 3.16 in [28], Corollary 3.2.

Remark 20. For $m = 1$ and $\lambda = 1$, the relation (65) that is regarded as a generalization and a refinement of the Pólya-Knopp's inequality which coincided with Remark 3.12 in [13].

Theorem 21. *Assume (A1)–(A3) are satisfied. If $g_i : \Omega_2 \longrightarrow \mathbb{R}(1 \le i \le m)$ are $\Delta\mu_2$-integrable functions such that $g_i(\Omega_2) \subset [0, \infty)$, then the inequality*

$$\begin{aligned}&\int_{\Omega_1}\sum_{i=1}^{m}[(A_i g_i)(\theta)\,\cosh\,(A_i g_i)(\theta) - \sinh\,(A_i g_i)(\theta)]^{\lambda}\\ &\quad\cdot\xi(\theta)\Delta\mu_1(\theta) + \lambda\int_{\Omega_1}\int_{\Omega 2}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\\ &\quad\cdot\left(\sum_{i=1}^{m}[(A_i g_i)(\theta)\,\cosh\,(A_i g_i)(\theta) - \sinh\,(A_i g_i)(\theta)]\right)^{\lambda-1}\\ &\quad\times\sum_{i=1}^{m}[|g_i(\eta) - (A_i g_i)(\theta)|\,\cosh\,(|g_i(\eta) - (A_i g_i)(\theta)|)\\ &\quad - \sinh\,(|g_i(\eta) - (A_i g_i)(\theta)|)]\Delta\mu_1(\theta)\Delta\mu_2(\eta)\\ &\le\left(\int_{\Omega_2}\omega(\eta)\sum_{i=1}^{m}([g_i(\eta) - \cosh\,(g_i(\eta)) - \sinh\,(g_i(\eta))]\Delta\mu_2(\eta))\right)^{\lambda},\end{aligned} \tag{69}$$

is valid, where $A_l g_i$ is defined as in (63). If $0 < \lambda < 1$, then (69) is reversed.

Proof. We get the result from Theorem 5 by putting

$$\Theta(\theta) := \sum_{i=1}^{m}(\theta_i\,\cosh\,\theta_i - \sinh\,\theta_i), \tag{70}$$

in (49).

Remark 22. For $\lambda = 1$, Theorem 21 reduces to Theorem 2.5 in [14]. In particular, for $m = 1$ and $\lambda = 1$, Theorem 21 coincides with Corollary 2.6 in [14].

Theorem 23. *Assume (A1)–(A3) are satisfied. If$g_i : \Omega_2 \longrightarrow \mathbb{R}(1 \le i \le m)$ are $\Delta\mu_2$-integrable functions such that $g_i(\Omega_2) \subset [0, \infty)$, then the inequality.*

$$\begin{aligned}&\int_{\Omega_1}\xi(\theta)\left(\ln\left(1 + \sum_{i=1}^{m}(A_i g_i)(\theta)\right) - \sum_{i=1}^{m}(A_i g_i)(\theta)\right)^{\lambda}\Delta\mu_1(\theta)\\ &\quad + \lambda\int_{\Omega_1}\int_{\Omega_1}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\left(\ln\left(1 + \sum_{i=1}^{m}(A_i g_i)(\theta)\right)\right.\\ &\quad\left. - \sum_{i=1}^{m}(A_i g_i)(\theta)\right)^{\lambda-1}\times\left(\ln\left(1 + \sum_{i=1}^{m}|g_i(\eta)A_i g_i(\theta)|\right)\right.\\ &\quad\left. - \sum_{i=1}^{m}|g_i(\eta)A_i g_i(\theta)|\right)^{\lambda}\Delta\mu_1(\theta)\Delta\mu_2(\eta)\\ &\le\left(\int_{\Omega_2}\omega(\eta)\left(\ln\left(1 + \sum_{i=1}^{m}g_i(\eta)\right) - \sum_{i=1}^{m}g_i(\eta)\right)\Delta\mu_2(\eta)\right)^{\lambda},\end{aligned} \tag{71}$$

is valid, where $A_l g_i$ is defined as in (63). If $0 < \lambda < 1$, then (71) is reversed.

Proof. We get the result from Theorem 5 by putting

$$\Theta(\theta) \coloneqq \sum_{i=1}^{m} \theta_i^2 \ln \theta_i, \quad (72)$$

in (49) with the assumption 0 ln 0 = 0.

Remark 24. For $\lambda = 1$, Theorem 23 reduces to Theorem 2.7 in [14]. In particular, for $m = 1$ and $\lambda = 1$, Theorem 23 coincides with Corollary 2.8 in [14].

Theorem 25. *Assume (A1)–(A3) are satisfied. If $g_i : \Omega_2 \longrightarrow \mathbb{R}(1 \le i \le m)$ are $\Delta\mu_2$-integrable functions such that $g_i(\Omega_2) \subset [0, \infty)$, then the inequality*

$$\begin{aligned}
&\int_{\Omega_1} \xi(\theta)\left(\ln\left(1+\sum_{i=1}^{m}(A_i g_i)(\theta)\right)-\sum_{i=1}^{m}(A_i g_i)(\theta)\right)^{\lambda}\Delta\mu_1(\theta)\\
&+\lambda\int_{\Omega_1}\int_{\Omega_1}\xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\left(\ln\left(1+\sum_{i=1}^{m}(A_i g_i)(\theta)\right)\right.\\
&\left.-\sum_{i=1}^{m}(A_i g_i)(\theta)\right)^{\lambda-1}\times\left(\ln\left(1+\sum_{i=1}^{m}|g_i(\eta)A_i g_i(\theta)|\right)\right.\\
&\left.-\sum_{i=1}^{m}|g_i(\eta)A_i g_i(\theta)|\right)^{\lambda}\Delta\mu_1(\theta)\Delta\mu_2(\eta)\\
&\le\left(\int_{\Omega_2}\omega(\eta)\left(\ln\left(1+\sum_{i=1}^{m}g_i(\eta)\right)-\sum_{i=1}^{m}g_i(\eta)\right)\Delta\mu_2(\eta)\right)^{\lambda},
\end{aligned} \quad (73)$$

is valid, where $A_l g_i$ is defined as in (63). If $0 < \lambda < 1$, then (73) is reversed.

Proof. We get the result from Theorem 5 by taking

$$\Theta(\theta) \coloneqq \ln\left(1 + \sum_{i=1}^{m} \theta_i\right) - \sum_{i=1}^{m} \theta_i, \quad (74)$$

in (49).

Remark 26. For $\lambda = 1$, Theorem 25 reduces to Theorem 2.9 in [14]. In particular, for $m = 1$ and $\lambda = 1$, Theorem 25 coincides with Corollary 2.10 in [14].

Now, to wrap up this section, we consider yet another implementation of Theorem 5 rigged with finite measure spaces.

Corollary 27. *Let the supposition of Theorem 5 be satisfied and denote $\int_{\Omega_1} \Delta\mu_1(\theta) = |\Omega_1|$ and $\int_{\Omega_2} \Delta\mu_2(\theta) = |\Omega_2|$ such that $|\Omega_1|, |\Omega_2| < \infty$: setting $l(\theta, \eta)$ and $\xi(\theta) = 1$. Then, $L(\theta) = \int_{\Omega_2} \Delta\mu_2(\theta) = |\Omega_2|$ and*

$$\begin{aligned}
\omega(\eta) &\coloneqq \left(\int_{\Omega_1}\left(\frac{1}{|\Omega_2|}\right)^{\lambda}\Delta\mu_1(\theta)\right)^{1/\lambda}\\
&= \left(\frac{1}{|\Omega_2|^{\lambda}}\int_{\Omega_1}\Delta\mu_1(\theta)\right)^{1/\lambda} = \frac{|\Omega_1|^{1/\lambda}}{|\Omega_2|}.
\end{aligned} \quad (75)$$

Hence, the following inequality

$$\begin{aligned}
&\int_{\Omega_1}\Theta\left(\frac{1}{|\Omega_2|}\int_{\Omega_2}\boldsymbol{g}(\eta)\Delta\mu_2(\eta)\right)^{\lambda}\Delta\mu_1(\theta)\\
&+\frac{\lambda}{|\Omega_2|}\int_{\Omega_1}\int_{\Omega_2}\Theta\left(\frac{1}{|\Omega_2|}\int_{\Omega_2}\boldsymbol{g}(\eta)\Delta\mu_2(\eta)\right)^{\lambda-1}\\
&\times\Theta\left(\left|\boldsymbol{g}(\eta)-\frac{1}{|\Omega_2|}\int_{\Omega_2}\boldsymbol{g}(\eta)\Delta\mu_2(\eta)\right|\right)\Delta\mu_1(\theta)\Delta\mu_2(\eta)\\
&\le\frac{|\Omega_1|}{|\Omega_2|}\left(\int_{\Omega_2}\Theta(\boldsymbol{g}(\eta))\Delta\mu_2(\eta)\right)^{\lambda},
\end{aligned} \quad (76)$$

is valid. If Θ is subquadratic and $0 < \lambda < 1$, then (76) is reversed.

Remark 28. By taking $m = 1$ in Corollary 27, inequality (76) reduces to inequality 3.19 in [28], Corollary 3.2.

Remark 29. For the Lebesgue scale measures $\Delta\mu_1(\theta) = \Delta\theta$, $\Delta\mu_2(\eta) = \Delta\eta$ and $m = 1$, Corollary 27 coincides with Corollary 2.1.6 in [30].

Remark 30. For $\mathbb{T} = \mathbb{R}$, $m = 1$, and $\lambda = 1$, Corollary 27 reduces to Corollary 3.3 in [8].

4. Inequalities with Specific Time Scales

In this section, by selecting few different time scales, we get some consequential inequalities. More precisely, assume $0 \le \alpha < \beta \le \infty$ are points in $\mathbb{T}$ and $S_1 \coloneqq \{(\theta, \eta) \in \mathbb{T} : 0 \le \alpha < \eta \le \theta < \beta\}$. Applying Theorem 5 to $\Omega_2 = \Omega_2 = [\alpha, \beta)_{\mathbb{T}}$, $\Delta\mu_1(\theta) = \Delta\theta$, and $\Delta\mu_2(\eta) = \Delta\eta$, we get the following conclusion.

Theorem 31. *Assume $0 \le \alpha < \beta \le \infty$ and $l : [\alpha, \beta)_{\mathbb{T}} \times [\alpha, \beta)_{\mathbb{T}} \longrightarrow \mathbb{R} \ge 0$ such as $L(\theta) \coloneqq \int_{\alpha}^{\theta} k(\theta, \eta)\Delta\eta < \infty, \theta \in [\alpha, \beta)_{\mathbb{T}}$*

Suppose that $\xi(\theta)$: $[\alpha, \beta)_{\mathbb{T}} \longrightarrow \mathbb{R}$ and

$$\omega(\eta) \coloneqq \left(\int_{\eta}^{\beta} \xi(\theta)\left(\frac{l(\theta,\eta)}{L(\theta)}\right)^{\lambda}\Delta\theta\right)^{1/\lambda} < \infty, \eta \in [\alpha, \beta)_{\mathbb{T}}, \quad (77)$$

where $\lambda \geq 1$. *If* $\Theta \in C(K_m, \mathbb{R}) \geq 0$ *and is superquadratic, then*

$$\int_\alpha^\beta \xi(\theta)\Theta^\lambda((A_l\boldsymbol{g})(\theta))\Delta\theta + \lambda\int_\alpha^\beta\int_\alpha^\theta \xi(\theta)\frac{l(\theta,\eta)}{L(\theta)}\Theta^{\lambda-1} \cdot ((A_l\boldsymbol{g})(\theta))\Theta(|\boldsymbol{g}(\eta) - (A_l\boldsymbol{g})(\theta)|)\Delta\theta\Delta\eta \leq \left(\int_\alpha^\beta \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta\right)^\lambda, \tag{78}$$

is available for all nonnegative integrable functions $\boldsymbol{g} : [\alpha,\beta)_{\mathbb{T}} \longrightarrow \mathbb{R}^m$ *and for* $A_l\boldsymbol{g} : [\alpha,\beta)_{\mathbb{T}} \longrightarrow \mathbb{R}$ *defined as*

$$(A_l\boldsymbol{g})(\theta) \coloneqq \frac{1}{L(\theta)}\int_\alpha^\theta l(\theta,\eta)\boldsymbol{g}(\eta)\Delta\eta,\ \theta \in [\alpha,\beta)_{\mathbb{T}}. \tag{79}$$

If $0 < \lambda < 1$ *and* Θ *are subquadratic, then (78) is reversed.*

Remark 32. By taking $m = 1$ and replacing $\xi(\theta)$, $\omega(\eta)$, and $l(\theta,\eta)$, respectively, $\xi(\theta)/(\theta-\alpha)$, $\omega(\eta)/(\eta-\alpha)$, and $l_{\chi_{S_1}}(\theta,\eta)$ where χ_{S_1} denotes the characteristic function over S_1 in Theorem 31, inequality (78) reduces to inequality 4.1 in [28], Theorem 4.1.

On the other hand, for $0 \leq \alpha < \beta \leq \infty$, consider the set

$$S_2 \coloneqq \{(\theta,\eta) \in \mathbb{T} : \beta < \theta \leq \eta < \infty\}. \tag{80}$$

Then, putting $\Omega_1 = \Omega_2 = [\beta,\infty)_{\mathbb{T}}$ where $\mathbb{T}$ is a time scale, $\Delta\mu_1(\theta) = \Delta\theta$ and $\Delta\mu_2(\eta) = \Delta\eta$. We obtain a dual form of Theorem 31 as follows.

Theorem 33. *Suppose that* $0 \leq \beta < \infty\tilde{\xi}(\theta) : [\beta,\infty)_{\mathbb{T}} \longrightarrow \mathbb{R} \geq 0$ *and* $\tilde{l} : [\beta,\infty)_{\mathbb{T}} \times [\beta,\infty)_{\mathbb{T}} \times [\beta,\infty)_{\mathbb{T}} \longrightarrow \mathbb{R} \geq 0$ *such that*

$$\tilde{L}(\theta) \coloneqq \int_\theta^\infty \tilde{l}(\theta,\eta)\Delta\eta < \infty, \quad \theta \in [\beta,\infty)_{\mathbb{T}}, \quad \tilde{\omega}(\eta) \coloneqq \left(\int_\beta^\eta \tilde{\xi}(\theta)\left(\frac{\tilde{l}(\theta,\eta)}{L(\theta)}\right)^\lambda \Delta\theta\right)^{1/\lambda} < \infty, \eta \in [\beta,\infty)_{\mathbb{T}}, \tag{81}$$

where $\lambda \geq 1$. *If* $\Theta \in C(K_m\mathbb{R}) \geq 0$ *and superquadratic, then*

$$\int_\beta^\infty \tilde{\xi}(\theta)\Theta^\lambda((A_l\boldsymbol{g})(\theta))\Delta\theta + \lambda\int_\beta^\infty\int_\theta^\infty \tilde{\xi}(\theta)\frac{\tilde{l}(\theta,\eta)}{L(\theta)}\Theta^{\lambda-1} \cdot \left(\left(\tilde{A}_l\boldsymbol{g}\right)(\theta)\right)\Theta(|\boldsymbol{g}(\eta) - (A_l\boldsymbol{g})(\theta)|)\Delta\theta\Delta\eta \leq \left(\int_\beta^\infty \tilde{\omega}(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta\right), \tag{82}$$

is available for all nonnegative $\Delta\eta$ *-integrable functions* $\boldsymbol{g} : [\beta,\infty)_{\mathbb{T}} \longrightarrow \mathbb{R}^m$ *and for the operator* $\tilde{A}_l g : [\beta,\infty)_{\mathbb{T}} \longrightarrow \mathbb{R}$ *defined by*

$$\left(\tilde{A}_l\boldsymbol{g}\right)(\theta) \coloneqq \frac{1}{\tilde{L}(\theta)}\int_\theta^\infty \tilde{l}(\theta,\eta)\boldsymbol{g}(\eta)\Delta\eta, \quad \theta \in [\beta,\infty)_{\mathbb{T}}. \tag{83}$$

If Θ *is subquadratic and* $0 < \lambda < 1$, *then (82) is reversed.*

Remark 34. By taking $m = 1$ and replacing $\tilde{\xi}(\theta), \tilde{\omega}(\eta)$, and $\tilde{l}(\theta,\eta)$, respectively, by $\tilde{\xi}(\theta)/(\theta-\alpha), \tilde{\omega}(\eta)/(\eta-\alpha)$, and $\tilde{l}_{\chi_{S_2}}(\theta,\eta)$ where χ_{S_2} denotes the characteristic function over S_2 in Theorem 33; inequality (82) reduces to inequality 4.7 in [28], Theorem 4.2.

5. Inequalities with Specific Kernels

In this section, we find some consequential inequalities of the Hardy type by selecting specific kernels and weight functions.

Corollary 35. *Suppose that the assumptions of Theorem 31 are satisfied only with*

$$l(\theta,\eta) \coloneqq 0, \quad \text{if } \alpha \leq \eta \leq \sigma(\theta) \leq \beta. \tag{84}$$

Define

$$L(\theta) \coloneqq \int_\alpha^{\sigma(\theta)} l(\theta,\eta)\Delta\eta > 0, \quad \theta \in [\alpha,\beta)_{\mathbb{T}}. \tag{85}$$

If $\Theta \in C(K_m, \mathbb{R}) \geq 0$ *and is superquadratic, then (78) is available for all nonnegative* $\Delta\eta$*-integrable functions* $g : [\alpha,\beta)_{\mathbb{T}} \longrightarrow \mathbb{R}^m$ *defined as*

$$(A_l\boldsymbol{g})(\theta) \coloneqq \frac{1}{L(\theta)}\int_\alpha^\sigma l(\theta,\eta)\,\boldsymbol{g}(\eta)\Delta\eta, \quad \theta \in [\alpha,\beta)_{\mathbb{T}}. \tag{86}$$

If Θ *is subquadratic and* $0 < \lambda < 1$, *then (78) is reversed.*

Corollary 36. *Assume that the assumptions of Theorem 31 is satisfied only with*

$$l(\theta,\eta) \coloneqq 0, \text{if } \alpha \leq \sigma(\theta) \leq \eta \leq \beta. \tag{87}$$

Define

$$L(\theta) \coloneqq \int_{\sigma(\theta)}^\beta l(\theta,\eta)\Delta\eta > 0, \quad \theta \in [\alpha,\beta)_{\mathbb{T}}. \tag{88}$$

If $\Theta \in C(K_m, \mathbb{R}) \geq 0$ *and is superquadratic, then (78) is available for all nonnegative integrable functions* $\boldsymbol{g} : [\alpha,\beta)_{\mathbb{T}} \longrightarrow \mathbb{R}^m$

$$(A_l\boldsymbol{g})(\theta) \coloneqq \frac{1}{L(\theta)}\int_{\sigma(\theta)}^\beta l(\theta,\eta)g(\eta)\Delta\eta, \quad \theta \in [\alpha,\beta)_{\mathbb{T}}. \tag{89}$$

If Θ *is subquadratic and* $0 < \lambda < 1$, *then (78) is reversed.*

Corollary 37. *Assume that the assumptions of Theorem 31 is satisfied only with* $l : [\alpha, \beta)_{\mathbb{T}} \times [\alpha, \beta)_{\mathbb{T}} \longrightarrow \mathbb{R}$ *defined as*

$$l(\theta, \eta) := \begin{cases} 1, & \text{if } 0 \le \alpha \le \eta < \sigma(\theta) \le \beta, \\ 0, & \text{otherwise,} \end{cases} \quad (90)$$

and $\xi(\theta)$: $[\alpha, \beta)_{\mathbb{T}} \longrightarrow \mathbb{R}$; *then* $L(\theta) := \int_\alpha^{\sigma(\theta)} l(\theta, \eta)\Delta\eta = \sigma(\theta) - \alpha, \theta \in [\alpha, \beta)_{\mathbb{T}}$, *and* $A_l \boldsymbol{g}(\theta)$ *in this case is the classical Hardy and denoted by*

$$(H\boldsymbol{g})(\theta) := \frac{1}{\sigma(\theta) - \alpha} \int_\alpha^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta, \quad \theta \in [\alpha, \beta)_{\mathbb{T}}. \quad (91)$$

If we let

$$\omega(\eta) := \left(\int_\eta^\beta \xi(\theta) \left(\frac{1}{\sigma(\theta) - \alpha} \right)^\lambda \Delta\theta \right) < \infty, \quad \eta \in [\alpha, \beta)_{\mathbb{T}}, \quad (92)$$

where $\lambda \ge 1$, *then (78) became*

$$\begin{aligned} &\int_\alpha^\beta \xi(\theta)\Theta^\lambda \left(\frac{1}{\sigma(\theta) - \alpha} \int_\alpha^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \Delta\theta \\ &\quad + \lambda \int_\alpha^\beta \int_\eta^\beta \Theta^{\lambda-1} \left(\frac{1}{\sigma(\theta) - \alpha} \int_\alpha^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \Theta \\ &\quad \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\sigma(\theta) - \alpha} \int_\alpha^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{\xi(\theta)}{\sigma(\theta) - \alpha} \Delta\theta\Delta\eta \\ &\le \left(\int_\alpha^\beta \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta \right). \end{aligned} \quad (93)$$

If Θ *is subquadratic and* $0 < \lambda < 1$, *then (93) is reversed.*

Remark 38. For $m = 1$ and replacing $\xi(\theta), \omega(\eta)$ by $\xi(\theta)/(\theta - \alpha)$ and $\omega(\eta)/(\eta - \alpha)$ in (93), Corollary 37 coincides with Example 4.1 in [13].

Remark 39. By taking $\mathbb{T} = \mathbb{R}, \alpha = 0$, and replacing $\xi(\theta), \omega(\eta)$ by $\xi(\theta)/\theta$ and$\omega(\eta)/\eta$ in (93), we have

$$\begin{aligned} &\int_0^\beta \xi(\theta)\Theta^\lambda \left(\theta^{-1} \int_0^\theta \boldsymbol{g}(\eta)d\eta \right) \frac{d\theta}{\theta} \\ &\quad + \lambda \int_0^\beta \int_\eta^\beta \Theta^{\lambda-1} \left(\frac{1}{\theta} \int_0^\theta \boldsymbol{g}(\eta)d\eta \right) \Theta \\ &\quad \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\theta} \int_0^\theta \boldsymbol{g}(\eta)d\eta \right| \right) \frac{\xi(\theta)}{\theta^2} d\theta d\eta \\ &\le \left(\int_0^\beta \omega(\eta)\Theta(\boldsymbol{g}(\eta)) \frac{d\eta}{\eta} \right)^\lambda, \end{aligned} \quad (94)$$

where

$$\omega(\eta) := \eta \left(\int_\eta^\beta \xi(\theta) \left(\frac{1}{\theta} \right)^\lambda \frac{d\theta}{\theta} \right)^{1/\lambda}, \eta \in [0, \beta). \quad (95)$$

If Θ is subquadratic and $0 < \lambda < 1$, then (94) is reversed, which is a refinement of 4.6 in [28], Remark 4.2.

Corollary 40. *In Corollary 37, if* $\alpha = 0$ *and* $\xi(\theta) = 1/\theta$, *then (93) reduces to*

$$\begin{aligned} &\int_0^\beta \Theta^\lambda \left(\frac{1}{\sigma(\theta)} \int_0^\theta \boldsymbol{g}(\eta)d\eta \right) \frac{d\theta}{\theta} \\ &\quad + \lambda \int_0^\beta \int_\eta^\beta \Theta^{\lambda-1} \left(\frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \Theta \\ &\quad \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\theta\sigma(\theta)} \Delta\theta\Delta\eta \\ &\le \left(\int_0^\beta \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta \right)^\lambda, \end{aligned} \quad (96)$$

where

$$\omega(\eta) := \left(\int_\eta^\beta \left(\frac{1}{\sigma(\theta)} \right)^\lambda \frac{\Delta\theta}{\theta} \right)^{1/\lambda} < \infty, \eta \in [\alpha, \beta)_{\mathbb{T}}. \quad (97)$$

Furthermore, if $\beta = \infty$, *then (96) becomes*

$$\begin{aligned} &\int_0^\infty \Theta^\lambda \left(\frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \frac{\Delta\theta}{\theta} \\ &\quad + \lambda \int_0^\infty \int_\eta^\infty \Theta^{\lambda-1} \left(\frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \Theta \\ &\quad \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\theta\sigma(\theta)} \Delta\theta\Delta\eta \\ &\le \left(\int_0^\infty \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta \right)^\lambda, \end{aligned} \quad (98)$$

where

$$\omega(\eta) := \left(\int_\eta^\infty \left(\frac{1}{\sigma(\theta)} \right)^\lambda \frac{\Delta\theta}{\theta} \right)^{1/\lambda} < \infty, \eta \in [\alpha, \infty)_{\mathbb{T}}. \quad (99)$$

Remark 41. For $\lambda = 1$, inequality (96) reduces to

$$\begin{aligned} &\int_0^\beta \Theta \left(\frac{1}{\sigma(\theta)} \int_0^\theta \boldsymbol{g}(\eta)\Delta\eta \right) \frac{\Delta\theta}{\theta} + \int_0^\beta \int_\eta^\beta \Theta \\ &\quad \cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\sigma(\theta)} \int_0^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\theta\sigma(\theta)} \Delta\theta\Delta\eta \\ &\le \int_0^\beta \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta, \end{aligned} \quad (100)$$

where

$$\omega(\eta) := \int_{\eta}^{\beta} \left(\frac{\Delta\theta}{\theta\sigma(\theta)} \right) = \left(\frac{1}{\eta} - \frac{1}{\beta} \right), \eta \in [\alpha, \beta)_{\mathbb{T}}, \quad (101)$$

while inequality (98) reduces to

$$\begin{aligned} &\int_{0}^{\infty} \Theta\left(\frac{1}{\sigma(\theta)} \int_{0}^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right) \frac{\Delta\theta}{\theta} + \int_{0}^{\infty}\int_{\eta}^{\infty} \Theta \\ &\cdot \left(\left| \boldsymbol{g}(\eta) - \frac{1}{\sigma(\theta)} \int_{0}^{\sigma(\theta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\theta\sigma(\theta)} \Delta\theta\Delta\eta \quad (102) \\ &\leq \int_{0}^{\infty} \Theta(\boldsymbol{g}(\eta)) \frac{\Delta\eta}{\eta}. \end{aligned}$$

Example 3. Considering Theorem 33 with $l : [\beta, \infty)_{\mathbb{T}} \times [\beta, \infty)_{\mathbb{T}} \longrightarrow \mathbb{R}$ defined by

$$l(\theta, \eta) := \begin{cases} {}^{1/\eta\sigma(\eta)} \text{if } \eta \geq \theta \\ 0, \text{otherwise,} \end{cases} \quad (103)$$

and $\xi(\theta)$: $[\beta, \infty)_{\mathbb{T}} \longrightarrow \mathbb{R} \geq 0$, then

$$\begin{aligned} L(\theta) &:= \int_{\theta}^{\infty} l(\theta, \eta)\Delta\eta = \int_{\theta}^{\infty} \frac{1}{\eta\sigma(\eta)} = -\int_{\theta}^{\infty} \left(\frac{1}{\eta} \right)^{\Delta} \Delta\eta \\ &= \frac{1}{\theta}, \theta \in [\beta, \infty)_{\mathbb{T}}. \end{aligned} \quad (104)$$

The operator $A_l g(\theta)$ is defined as

$$(A_l\boldsymbol{g})(\theta) := \theta\int_{\theta}^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta, \theta \in [\beta, \infty)_{\mathbb{T}}, \quad (105)$$

and if we let

$$\omega(\eta) := \left(\int_{\eta}^{\beta} \theta^{-1} \left(\frac{\theta}{\eta\sigma(\eta)} \right)^{\lambda} \Delta\theta \right)^{1/\lambda} < \infty, \eta \in [\beta, \infty)_{\mathbb{T}}, \quad (106)$$

where $\lambda \geq 1$, then (82) became

$$\begin{aligned} &\int_{0}^{\infty} \Theta^{\lambda} \left(\theta\int_{\theta}^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta \right) \frac{\Delta\theta}{\theta} \\ &+ \lambda\int_{\beta}^{\infty}\int_{\theta}^{\infty} \Theta^{\lambda-1} \left(\theta\int_{\theta}^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta \right) \Theta \\ &\cdot \left(\left| \boldsymbol{g}(\eta) - \theta\int_{0}^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\eta\sigma(\eta)} \Delta\theta\Delta\eta \quad (107) \\ &\leq \left(\int_{\beta}^{\infty} \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta \right)^{\lambda}. \end{aligned}$$

If Θ is subquadratic and $0 < \lambda < 1$, then (107) is reversed.

Remark 42. For $\lambda = 1$, inequality (107) reduces to

$$\begin{aligned} &\int_{\beta}^{\infty} \Theta\left(\theta\int_{\beta}^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta \right) \frac{\Delta\theta}{\theta} + \lambda\int_{\beta}^{\infty}\int_{\theta}^{\infty} \Theta \\ &\cdot \left(\left| \boldsymbol{g}(\eta) - \theta\int_{0}^{\infty} \frac{1}{\eta\sigma(\eta)} \boldsymbol{g}(\eta)\Delta\eta \right| \right) \frac{1}{\eta\sigma(\eta)} \Delta\theta\Delta\eta \quad (108) \\ &\leq \int_{\beta}^{\infty} \omega(\eta)\Theta(\boldsymbol{g}(\eta))\Delta\eta, \end{aligned}$$

where

$$\omega(\eta) := \frac{1}{\eta\sigma(\eta)} \int_{\eta}^{\beta} \Delta\theta = \frac{1}{\eta\sigma(\eta)} (\beta - \eta). \quad (109)$$

6. Some Particular Cases

In this section, we obtain a popularization and a refinement of the classical inequality of the Hardy-Hilbert type (16) for numerous variables on time scales. It is clarified in the result below.

Theorem 43. *Assume that the assumptions of Theorem 31 are satisfied only with $\Omega_1 = \Omega_2 = [0, \infty)_{\mathbb{T}}$, $p > 1$, $\lambda > 0$ and replace $\Delta\mu_1(\theta)$ and $\Delta\mu_2(\eta)$ by the Lebesgue scale measure $\Delta\theta$ and $\Delta\eta$. Furthermore, define*

$$\begin{aligned} L_1(\theta) &:= \int_{0}^{\infty} \frac{(\theta/\eta)^{-1/p}}{\theta + \eta} \Delta\eta \text{ and } L_2(\eta) \\ &:= \left(\int_{0}^{\infty} \left(\frac{\theta/\eta^{1-(1/p)}}{\theta + \eta} \right)^{\lambda} \Delta\theta \right)^{1/\lambda}. \end{aligned} \quad (110)$$

If $\lambda \geq 1$ and $p \geq 2$, then

$$\begin{aligned} &\int_{0}^{\infty} (L_1(\theta))^{\lambda(1-p)} \left(\int_{0}^{\infty} \frac{\boldsymbol{g}(\eta)}{\theta + \eta} \Delta\eta \right)^{\lambda p} \Delta\theta \\ &+ \lambda\int_{0}^{\infty}\int_{0}^{\infty} \eta^{-1/p} L_1^{(\lambda-1)(1-p)}(\theta) \left(\int_{0}^{\infty} \frac{\boldsymbol{g}(\eta)}{\theta + \eta} \Delta\eta \right)^{p(\lambda-1)} \\ &\times \left| \boldsymbol{g}(\eta)\eta 1/p - \frac{1}{L_1(\theta)} \int_{0}^{\infty} \frac{\boldsymbol{g}(\eta)}{\theta + \eta} \Delta\eta \right|^{p} \frac{\theta_p^{1}{}^{-1}}{\theta + \eta} \Delta\theta\Delta\eta \\ &\leq \left(\int_{0}^{\infty} L_2(\eta)\boldsymbol{g}(\eta)\Delta\eta \right)^{\lambda}, \end{aligned} \quad (111)$$

is available for all nonnegative integrable $\Delta\eta$-integrable functions $\boldsymbol{g} : [\alpha, \beta)_{\mathbb{T}} \longrightarrow \mathbb{R}^m$. If $0 < \lambda < 1$, then (111) is reversed.

Proof. Utilizing $\xi(\theta) := (L_1(\theta)/\theta)^{\lambda}$ and

$$l(\theta, \eta) := \begin{cases} \dfrac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta + \eta}, \text{if } \theta \neq 0, \eta \neq 0, \theta + \eta \neq 0 \\ 0, \text{otherwise,} \end{cases} \quad (112)$$

in Theorem 15, we obtain

$$\begin{aligned} L(\theta) &:= \int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}\Delta\eta = L_1(\theta), \\ \omega(\eta) &:= \left(\int_0^\infty \xi(\theta)\left(\frac{l(\theta,\eta)}{L(\theta)}\right)^\lambda \Delta\theta\right) := \left(\int_0^\infty \left(\frac{L_1(\theta)}{\theta}\right)^\lambda \left(\frac{l(\theta,\eta)}{L(\theta)}\right)^\lambda \Delta\theta\right)^{\frac{1}{\lambda}} \\ &:= \left(\int_0^\infty \left(\frac{l(\theta,\eta)}{\theta}\right)^\lambda \Delta\theta\right)^{\frac{1}{\lambda}} := \left(\left(\eta^{-1}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{1-1/p}}{\theta+\eta}\right)^\lambda \Delta\theta\right)^{\frac{1}{\lambda}} \\ &:= \frac{1}{\eta}\left(\left(\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{1-1/p}}{\theta+\eta}\right)^\lambda \Delta\theta\right)^{\frac{1}{\lambda}} \frac{L_2(\eta)}{\eta}, \end{aligned} \tag{113}$$

and the operator $(A_l\boldsymbol{g})(\theta)$ in this case is defined as

$$(A_l\boldsymbol{g})(\theta) := \frac{1}{L_1(\theta)}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}\boldsymbol{g}(\eta)\Delta\eta. \tag{114}$$

Utilizing $(A_l\boldsymbol{g})(\theta)$ in (62), we obtain

$$\begin{aligned} &\int_0^\infty \left(\frac{L_1(\theta)}{\theta}\right)^\lambda \left(\frac{1}{L_1(\theta)}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}\boldsymbol{g}(\eta)\Delta\eta\right)^{\lambda p}\Delta\theta \\ &+\lambda\int_0^\infty\int_0^\infty \left(\frac{L_1(\theta)}{\theta}\right)^\lambda \left(\frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{(\theta+\eta)L_1(\theta)}\right)\left(\frac{1}{L_1(\theta)}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}\boldsymbol{g}(\eta)\Delta\eta\right)^{p(\lambda-1)} \\ &\times\left|\boldsymbol{g}(\eta)-\frac{1}{L_1(\theta)}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}\boldsymbol{g}(\eta)\Delta\eta\right|^p \Delta\theta\Delta\eta \\ &\le \left(\int_0^\infty \frac{K_2(\eta)}{\eta}\boldsymbol{g}^p(\eta)\Delta\eta\right)^\lambda. \end{aligned} \tag{115}$$

Hence,

$$\begin{aligned} &\int_0^\infty (L_1(\theta))^{\lambda(1-p)}\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)\eta^{-1/p}}{\theta+\eta}\Delta\eta\right)^{\lambda p}\Delta\theta \\ &+\lambda\int_0^\infty\int_0^\infty L_1^{(\lambda-1)(1-p)}(\theta)\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)\eta^{-\frac{1}{p}}}{\theta+\eta}\Delta\eta\right)^{p(\lambda-1)} \\ &\times\left|\boldsymbol{g}(\eta)-\frac{1}{L_1(\theta)}\left(\frac{1}{\theta}\right)\int_0^\infty \frac{\boldsymbol{g}(\eta)\eta^{-p^{-1}}}{\theta+\eta}\Delta\eta\right|^p \left(\frac{\eta^{-p^{-1}}}{\theta+\eta}\right)\left(\theta^{-1}\right)^{1-p^{-1}}\Delta\theta\Delta\eta \\ &\le \left(\int_0^\infty \frac{K_2(\eta)}{\eta}\boldsymbol{g}^p(\eta)\Delta\eta\right)^\lambda. \end{aligned} \tag{116}$$

Finally, replacing $g(\eta)$ by $g(\eta)\eta^{1/p}$ in (116), we get (111). The cases $0 < \lambda < 1$ and $1 < p \le 2$ are proved in the same way.

Remark 44. For $m = 1$, Theorem 43 reduces to Theorem 5.1 in [13]. In particular, for $\lambda = 1$, Theorem 43 is a refinement of Theorem 5.5 in [10].

Remark 45. By taking $\mathbb{T} = \mathbb{R}$, $\lambda = 1$, and $p \ge 2$, in Theorem 43 and utilizing the known fact that

$$\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{-1/p}}{\theta+\eta}d\eta = \int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{1-1/p}}{\theta+\eta}d\theta = \frac{\pi}{\sin\left(\frac{\pi}{p}\right)}, \tag{117}$$

then (111) becomes

$$\begin{aligned} &\int_0^\infty \left(\int_0^\infty \frac{\boldsymbol{g}(\eta)}{\theta+\eta}d\eta\right)^p d\theta + \left(\frac{\pi}{\sin(\pi p^{-1})}\right)^{p-1}\int_0^\infty\int_0^\infty \eta^{-p^{-1}} \\ &\cdot\left|\boldsymbol{g}(\eta)\eta^{p^{-1}} - \frac{\sin(\pi p^{-1})}{\pi}\theta^{p^{-1}}\int_0^\infty \frac{\boldsymbol{g}(\eta)}{\theta+\eta}d\eta\right|^p \frac{\theta^{p-1_{-1}}}{\theta+\eta}d\theta d\eta \\ &\le \left(\frac{\pi}{\sin(\pi p^{-1})}\right)^p \int_0^\infty \boldsymbol{g}^p(\eta)d\eta, \end{aligned} \tag{118}$$

which is a refinement of (16). For $m = 1$, (118) has been established in [3], Corollary 3.2.

In the following theorem, we introduce a generalized form of (111) on time scales.

Theorem 46. *Suppose that $\lambda > 0$, $p > 1$ and $s, \delta \in \mathbb{R}$. Furthermore, assume*

$$\begin{aligned} &\left[\left(\int_0^\infty \frac{\theta^\delta(\eta/\theta)^{(s-2/p)+1}}{(\theta+\eta)^s}\right)^\lambda \Delta\theta\right]^{1/\lambda} \text{ and } L_1(\theta) \\ &:= \int_0^\infty \frac{\left(\eta\theta^{-1}\right)^{s-2/p}}{(\theta+\eta)^s}\Delta\eta, \end{aligned} \tag{119}$$

where $\lambda \ge 1$ and $p \ge 2$; then

$$\begin{aligned} &\int_0^\infty (L_1(\theta))^{\lambda(1-p)}\theta^{\lambda(\delta-s+1)}\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)}{(\theta+\eta)^s}\Delta\eta\right)^{p\lambda} \\ &+\lambda\int_0^\infty\int_0^\infty \eta_p^{2-s}L_1^{(\lambda-1)(1-p)}(\theta)\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)}{(\theta+\eta)^s}\Delta\eta\right)^{p(\lambda-1)} \\ &\times\left|\boldsymbol{g}(\eta)\eta_p^{2-s} - \frac{1}{L_1(\theta)}\theta_p^{2-s}\int_0^\infty \frac{\boldsymbol{g}(\eta)}{(\theta+\eta)^s}\Delta\eta\right|^p \\ &\cdot\frac{\theta^{p\lambda+(s-2)(1+p\lambda-p)/p}}{(\theta+\eta)^s}\Delta\theta\Delta\eta \le \left(\int_0^\infty L_2(\eta)\boldsymbol{g}^p(\eta)\Delta\eta\right)^\lambda, \end{aligned} \tag{120}$$

is available for all nonnegative integrable functions $g : [\alpha,\beta)_{\mathbb{T}} \longrightarrow \mathbb{R}^m$. If $0 < \lambda < 1$ and $1 < p \le 2$, then (120) is reversed.

Proof. Rewrite (62) in Theorem 15 with $\Omega_1 = \Omega_2 = [0,\infty)_{\mathbb{T}}$, $\Delta\mu_1(\theta) = \Delta\theta$, and $\Delta\mu_2(\eta) = \Delta\eta$. Let us define $\xi(\theta) := \left(L_1(\theta)\theta^{\delta-1}\right)^\lambda$ and

$$l(\theta,\eta) := \begin{cases} \dfrac{\left(\frac{\eta}{\theta}\right)^{s-2/p}}{(\theta+\eta)^s}, \text{if } \theta \neq 0, \eta \neq 0, \theta+\eta \neq 0 \\ 0, \text{otherwise.} \end{cases} \tag{121}$$

We have

$$\begin{aligned} L(\theta) &:= \int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{s-2/p}}{(\theta+\eta)^s} \Delta\eta = L_1(\theta), \\ \omega(\eta) &:= \int_0^\infty \xi(\theta) \left(\frac{l(\theta,\eta)}{L(\theta)} \Delta\theta\right)^{\lambda-1} := \left(\int_0^\infty \frac{L_1(\theta)\theta^\delta}{\theta} \frac{l(\theta,\eta)}{L_1(\theta)} \Delta\theta\right)^{\lambda-1} \\ &:= \left(\int_0^\infty \frac{L_1(\theta)\theta^\delta}{\theta} \frac{l(\theta,\eta)}{L_1(\theta)} \Delta\theta\right)^{\lambda-1} := \left(\frac{1}{\eta}\int_0^\infty \frac{\theta^\delta \left(\frac{\eta}{\theta}\right)^{\frac{s-2}{P}}}{\theta(\theta+\eta)^s} g(\eta)\Delta\theta\right)^{\lambda-1} \\ &:= \frac{1}{\eta}\left(\left(\int_0^\infty \frac{\theta^\delta \left(\frac{\eta}{\theta}\right)^{\frac{s-2}{P}+1}}{(\theta+\eta)^s}\right)^{\lambda} \Delta\theta\right)^{\lambda-1} := \frac{L_2(\eta)}{\eta}, \end{aligned} \tag{122}$$

and the operator $(A_l g)(\theta)$ in this case is defined as

$$(A_l \boldsymbol{g})(\theta) := \frac{1}{L_1(\theta)} \int_\theta^\infty \frac{\left(\frac{\eta}{\theta}\right)^{s-2/p}}{(\theta+\eta)^s} \boldsymbol{g}(\eta)\Delta\eta. \tag{123}$$

Now, substituting L, ω and $(A_l g)(\theta)$ in (62), we get

$$\begin{aligned} &\int_0^\infty \left(L_1(\theta)\theta^{\delta-1}\right)^\lambda \left(\frac{1}{L_1(\theta)^s}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{s-2/p}}{(\theta+\eta)^s}\boldsymbol{g}(\eta)\Delta\theta\right)^{p\lambda} \Delta\theta \\ &\quad + \lambda\int_0^\infty\int_0^\infty \left(L_1(\theta)\theta^{\delta-1}\right)^\lambda \left(\frac{\left(\frac{\eta}{\theta}\right)^{s-2/p}}{(\theta+\eta)^s L_1(\theta)}\right)\left(\frac{1}{L_1(\theta)}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{s-2/p}}{(\theta+\eta)^s}\boldsymbol{g}(\eta)\Delta\theta\right)^{p(\lambda-1)} \\ &\quad \times \left|\boldsymbol{g}(\eta) - \frac{1}{L_1(\theta)}\int_0^\infty \frac{\left(\frac{\eta}{\theta}\right)^{s-2/p}}{(\theta+\eta)^s}\boldsymbol{g}(\eta)\Delta\eta\right|^p \Delta\theta\Delta\eta \\ &\leq \left(\int_0^\infty \frac{L_2(\eta)}{\eta}\boldsymbol{g}^p(\eta)\Delta\eta\right)^\lambda. \end{aligned} \tag{124}$$

Hence,

$$\begin{aligned} &\int_0^\infty (L_1(\theta))^{\lambda(1-p)}\theta^{\delta\lambda}\left(\frac{1}{\theta}\right)^{\lambda(s-1)}\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)^{s-2/p}}{(\theta+\eta)^s}\Delta\eta\right)^{p\lambda} \\ &\quad + \lambda\int_0^\infty\int_0^\infty {L_1}^{(\lambda-1)(1-p)}(\theta)\left(\int_0^\infty \frac{\boldsymbol{g}(\eta)^{s-2/p}}{(\theta+\eta)^s}\Delta\eta\right)^{p(\lambda-1)} \\ &\quad \times \left|\boldsymbol{g}(\eta) - \frac{1}{L_1(\theta)}\theta_p^{s-2}\int_0^\infty \frac{(\eta)^{s-2/p}\boldsymbol{g}(\eta)}{(\theta+\eta)^s}\Delta\eta\right|^p \left(\frac{(\eta)_p^{s-2}}{(\theta+\eta)^s}\right) \\ &\quad \cdot \left(\frac{1}{\theta}\right)\frac{\theta^{p\lambda+(s-2)(1+p\lambda-p)/p}}{(\theta+\eta)^s}\Delta\theta\Delta\eta \leq \left(\int_0^\infty L_2(\eta)\boldsymbol{g}^p(\eta)\Delta\eta\right)^\lambda. \end{aligned} \tag{125}$$

Finally, considering (125) with $g(\eta)\eta^{(2-s/p)}$ instead of $g(\eta)$, we obtain (120). The cases $0 < \lambda < 1$ and $1 < p \leq 2$ are proved in the same way.

Remark 47. For $m = 1$, Theorem 46 coincides with Theorem 5.2 in [13].

Remark 48. Clearly, for $p > 1, \delta = 0$, and $s = 1$, Theorem 46 reduces to Theorem 43.

7. Conclusion and Future Work

The study of dynamic inequalities on time scales has a lot of scope. This research article is devoted to some general Hardy-type dynamic inequalities and their converses on time scales. Inequalities are considered in rather general forms and contain several special integral inequalities. In particular, our findings can be seen as refinements of some recent results closely linked to the time-scale inequalities of the classical Hardy, Pólya-Knopp, and Hardy-Hilbert. We use some algebraic inequalities such as the Minkowski inequality, the refined Jensen inequality and the Bernoulli inequality on time scales to prove the essential results in this paper. The performance of the superquadratic method for functions is reliable and effective to obtain new dynamic inequalities on time scales. This method has more advantages: it is direct and concise. Thus, the proposed method can be extended to some forms for Hardy's and related dynamic inequalities in mathematical and physical sciences. Our computed outcomes can be very useful as a starting point to get some continuous inequalities, especially from the obtained dynamic inequalities. In the future, we will get some discrete inequalities from the main results. Also, we will suppose that $g(t) := (g_1(t), \cdots, g(t))$ is an m-tuple of functions and $t = (t_1, t_2, \cdots, t_n)$ is n-tuple of variables to get the general forms of Hardy's and related inequalities on time scales. Similarly, in the future, we can present such inequalities by using Riemann-Liouville-type fractional integrals and fractional derivatives on time scales. It will also be very interesting to present such inequalities on quantum calculus.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally. All the authors read and approved the final manuscript.

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Research Article

Approximation Properties of λ-Gamma Operators Based on q-Integers

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In the present paper, we will introduce λ-Gamma operators based on q-integers. First, the auxiliary results about the moments are presented, and the central moments of these operators are also estimated. Then, we discuss some local approximation properties of these operators by means of modulus of continuity and Peetre $\mathscr{K}$-functional. And the rate of convergence and weighted approximation for these operators are researched. Furthermore, we investigate the Voronovskaja type theorems including the quantitative q-Voronovskaja type theorem and q-Grüss-Voronovskaja theorem.

1. Introduction

Gamma operators are very important positive linear operators and have been widely used in probability theory and computational mathematics. For $f \in C(\mathbb{R}^+)$, $n = 1, 2, 3, \cdots$ where $\mathbb{R}^+ = (0,\infty)$ and $C(\mathbb{R}^+)$ be the space of all continuous functions f on the interval $\mathbb{R}^+$, the Gamma operators were introduced in [1] by

$$G_n(f;x) = \frac{1}{n!}\int_0^\infty e^{-t}t^n f\left(\frac{nx}{t}\right)\mathrm{d}t, \quad x \in \mathbb{R}^+. \tag{1}$$

We can learn some properties of Gamma operators and their modified operators in [2–7]. In [8], Qi et al. defined new Gamma operators as follows:

$$G_{n,\lambda}(f;x) = \frac{1}{n!}\int_0^\infty e^{-t}t^n \left(\frac{n}{t}\right)^\lambda f\left(\frac{nx}{t}\right)\mathrm{d}t, \quad x \in \mathbb{R}^+. \tag{2}$$

where $f \in C(\mathbb{R}^+)$, $\lambda \in \mathbb{N} = \{0, 1, 2, \cdots\}$. Obviously, if $f^{(\lambda)} \in C(\mathbb{R}^+)$, then $(G_n(f;x))^{(\lambda)} = G_{n,\lambda}(f^{(\lambda)};x)$. Meantime, $G_{n,\lambda}(1;x) = (n^\lambda(n-\lambda)!/n!) \neq 1$ (while $\lambda \neq 0$). In order to preserve the constant, we defined λ-Gamma operators as follows:

Definition 1. For $f \in C(\mathbb{R}^+)$, $\lambda \in \mathbb{N}$, $n = \lambda, \lambda+1, \cdots$, the λ-Gamma operators are defined by

$$\mathscr{G}_{n,\lambda}(f;x) = \frac{1}{(n-\lambda)!}\int_0^\infty e^{-t}t^{n-\lambda} f\left(\frac{nx}{t}\right)\mathrm{d}t, \quad x \in \mathbb{R}^+. \tag{3}$$

Let us recall some useful concepts and notations from q-calculus, which can be founded in [9–11]. For nonnegative integer i, the q-integer $[i]_q$ and q-factorial $[i]_q!$ are defined by

$$[i]_q = 1 + q + \cdots + q^{i-1} = \begin{cases} \dfrac{1-q^i}{1-q}, & q \neq 1, \\ i, & q = 1, \end{cases}$$
$$[i]_q! = \begin{cases} [1]_q[2]_q\cdots[i]_q, & i \geq 1, \\ 1, & i = 0. \end{cases} \tag{4}$$

Further, q-power basis can be defined by

$$(x+y)_q^i=\begin{cases}(x+y)(x+qy)\cdots\left(x+q^{i-1}y\right), & i=1,2,\cdots,\\ 1, & i=0,\end{cases}$$
$$(x-y)_q^i=\begin{cases}(x-y)(x-qy)\cdots\left(x-q^{i-1}y\right), & i=1,2,\cdots,\\ 1, & i=0.\end{cases} \tag{5}$$

The q-derivative D_qf of a function f can be defined by

$$\left(D_qf\right)(x)=\frac{f(x)-f(qx)}{(1-q)x}, \quad \text{if} \quad x\neq 0, \tag{6}$$

and $(D_qf)(0)=f'(0)$ provided $f'(0)$ exists. High-order q-derivatives can be defined by $D_q^0f=f$, $D_q^i=D_q(D_q^{i-1}f)$, $i=1,2,\cdots$. The formula for the q-derivative of a product is $D_q(f(x)g(x))=D_q(f(x))g(x)+D_q(g(x))f(qx)$. We easily know that if a function f is continuous on an interval which does not include 0, then f is continuous q-differentiable.

The q-improper integral of function f can be defined by

$$\int_0^{\infty/1-q}f(t)\mathrm{d}_qt=\sum_{i=-\infty}^{\infty}f\left(\frac{q^i}{1-q}\right)q^i, \quad q\in(0,1). \tag{7}$$

The q-analogue of the classical exponential function e^x is

$$E_q(x)=\sum_{i=0}^{\infty}q^{((i(i-1))/2)}\frac{x^i}{[i]_q!}=(1+(1-q)x)_q^{\infty}, \quad q\in(0,1). \tag{8}$$

The q-Gamma function is defined by

$$\Gamma_q(s)=\int_0^{(\infty/1-q)}x^{(s-1)}E_q(-qx)\mathrm{d}_qx, \quad s\in\mathbb{R}^+, \tag{9}$$

and satisfies the functional relation: $\Gamma_q(s+1)=[s]_q\Gamma_q(s)$, $\Gamma_q(1)=1$. Moreover, for any nonnegative integer $i>0$, the relation holds: $\Gamma_q(i+1)=[i]_q!$.

Now, we construct the q-analogue of λ-Gamma operators using q-Gamma function as follows.

Definition 2. For $f:\mathbb{R}^+\to\mathbb{R}$, $q\in(0,1)$, $\lambda\in\mathbb{N}$, $n=\lambda,\lambda+1,\cdots$, the q-analogue of λ-Gamma operators (3) are defined as

$$\mathscr{G}_{n,\lambda}^q(f;x)=\frac{1}{[n-\lambda]_q!}\int_0^{\infty/1-q}f\left(\frac{[n]_qx}{t}\right)E_q(-qt)t^{n-\lambda}\mathrm{d}_qt, \quad x\in\mathbb{R}^+. \tag{10}$$

The paper is organized as follows: In Section 1, we introduce the history of Gamma operators, recall some basic notations about the q-calculus, and construct λ-Gamma operators based on q-integers with q-Gamma function. In Section 2, we obtain the auxiliary results about the moment computation formula. The second- and fourth-order central moments computation formula and other quantitative properties are also presented. In Section 3, we discuss local approximation about the operators by means of modulus of continuity and Peetre $\mathscr{K}$-functional. In Section 4 and Section 5, the rate of convergence and weighted approximation for these operators are researched. In the last section, we firstly prove quantitative q-Voronovskaja type theorems in terms of weighted modulus of continuity, and then the q-Grüss-Voronovskaja theorem in the quantitative mean is also presented (for the quantitative q-Voronovskaja type theorem0 and the q-Grüss-Voronovskaja theorem for the other operators, see also [12, 13]).

2. Auxiliary Results

In this section, we will give some lemmas and corollaries, which are necessary to obtain the approximation properties of the operators $\mathscr{G}_{n,\lambda}^q(f;x)$.

Lemma 3. *For $q\in(0,1)$, $\lambda\in\mathbb{N}$, $i\in\mathbb{N}$, $n=\lambda+i,\lambda+i+1,\cdots$, the following formula holds:*

$$\mathscr{G}_{n,\lambda}^q\left(t^i;x\right)=\frac{[n]_q^i[n-\lambda-i]_q!}{[n-\lambda]_q!}x^i. \tag{11}$$

Proof. According to the properties of q-Gamma function, we have

$$\begin{aligned}\mathscr{G}_{n,\lambda}^q\left(t^i;x\right)&=\frac{1}{[n-\lambda]_q!}\int_0^{\infty/1-q}\left(\frac{[n]_qx}{t}\right)^iE_q(-qt)t^{n-\lambda}\mathrm{d}_qt\\&=\frac{[n]_q^ix^i}{[n-\lambda]_q!}\int_0^{\infty/1-q}t^{n-\lambda-i}E_q(-qt)\mathrm{d}_qt\\&=\frac{[n]_q^ix^i\Gamma_q(n-\lambda-i+1)}{[n-\lambda]_q!}\\&=\frac{[n]_q^i[n-\lambda-i]_q!}{[n-\lambda]_q!}x^i.\end{aligned} \tag{12}$$

Lemma 3 is proved.

Corollary 4. *By the lemma given above and some elementary calculations, we can get the results*

$$A(x)=\mathscr{G}_{n,\lambda}^q(t-x;x)=\frac{q^{n-\lambda}[\lambda]_q}{[n-\lambda]_q}x \quad \text{for} \quad n\geq\lambda+1,$$

$$\begin{aligned}B(x)&=\mathscr{G}_{n,\lambda}^q\left((t-x)^2;x\right)\\&=\left(\frac{q^{n-\lambda-1}[\lambda+1]_q}{[n-\lambda-1]_q}-\frac{q^{n-\lambda}[\lambda]_q}{[n-\lambda]_q}+\frac{q^{2n-2\lambda-1}[\lambda]_q[\lambda+1]_q}{[n-\lambda-1]_q[n-\lambda]_q}\right)x^2 \quad \text{for} \quad n\\&\geq\lambda+2.\end{aligned} \tag{13}$$

Lemma 5. *Let $q=(q_n)$ be a sequence satisfying $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a \in [0,1]$. Then, for each $x \in \mathbb{R}^+$, $A_n(x) := \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)$, $B_n(x) := \mathscr{G}_{n,\lambda}^{q_n}((t-x)^2;x)$, we can obtain*

$$\lim_{n\to\infty}[n]_{q_n}A_n(x)=\lambda ax, \quad \lim_{n\to\infty}[n]_{q_n}B_n(x)=ax^2, \tag{14}$$

$$\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)^3;x\right)=O\left(\frac{1}{[n]_{q_n}^2}\right), \tag{15}$$

$$\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)^4;x\right)=O\left(\frac{1}{[n]_{q_n}^2}\right), \tag{16}$$

$$\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)^6;x\right)=O\left(\frac{1}{[n]_{q_n}^3}\right). \tag{17}$$

Proof. By Lemma 3, we can easily get (14). Without loss of generality, we only prove equation (15). Equation (16) and equation (17) can be proved in some way. Set $\mathscr{G}_{n,\lambda}^{q}(t^i;x)=C(i)x^i$, $i=1,2,3$, and $\mathscr{G}_{n,\lambda}^{q_n}((t-x)^3;x)=C(4)x^3$. Using $([n]_{q_n})/([n-\lambda-i]_{q_n})=1+((q_n^{n-\lambda-i}[\lambda+i]_{q_n})/([n-\lambda-i]_{q_n}))$, $i=0,1,2,\cdots,n-\lambda$, we can easily get

$$C(1)=1+\frac{q_n^{n-\lambda}[\lambda]_{q_n}}{[n-\lambda]_{q_n}},$$

$$C(2)=1+\frac{q_n^{n-\lambda}[\lambda]_{q_n}}{[n-\lambda]_{q_n}}+\frac{q_n^{n-\lambda-1}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}}+\frac{q_n^{2n-2\lambda-1}[\lambda]_{q_n}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}[n-\lambda]_{q_n}},$$

$$\begin{aligned}C(3)=1&+\frac{q_n^{n-\lambda}[\lambda]_{q_n}}{[n-\lambda]_{q_n}}+\frac{q_n^{n-\lambda-1}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}}+\frac{q_n^{n-\lambda-2}[\lambda+2]_{q_n}}{[n-\lambda-2]_{q_n}}\\&+\frac{q_n^{2n-2\lambda-3}[\lambda+2]_{q_n}[\lambda+1]_{q_n}}{[n-\lambda-2]_{q_n}[n-\lambda-1]_{q_n}}+\frac{q_n^{2n-2\lambda-2}[\lambda+2]_{q_n}[\lambda]_{q_n}}{[n-\lambda-2]_{q_n}[n-\lambda]_{q_n}}\\&+\frac{q_n^{2n-2\lambda-1}[\lambda]_{q_n}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}[n-\lambda]_{q_n}}+o\left(\frac{1}{[n]_{q_n}^2}\right).\end{aligned} \tag{18}$$

Combining

$$\begin{aligned}C(4)&=C(3)-3C(2)+3C(1)-1\\&=\frac{q_n^{n-\lambda}[\lambda]_{q_n}}{[n-\lambda]_{q_n}}-\frac{2q_n^{n-\lambda-1}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}}+\frac{q_n^{n-\lambda-2}[\lambda+2]_{q_n}}{[n-\lambda-2]_{q_n}}+\frac{q_n^{2n-2\lambda-3}[\lambda+2]_{q_n}[\lambda+1]_{q_n}}{[n-\lambda-2]_{q_n}[n-\lambda-1]_{q_n}}\\&\quad+\frac{q_n^{2n-2\lambda-2}[\lambda+2]_{q_n}[\lambda]_{q_n}}{[n-\lambda-2]_{q_n}[n-\lambda]_{q_n}}-\frac{2q_n^{2n-2\lambda-1}[\lambda]_{q_n}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}[n-\lambda]_{q_n}}\\&\quad+o\left(\frac{1}{[n]_{q_n}^2}\right):=\frac{q_n^{n-\lambda}[\lambda]_{q_n}}{[n-\lambda]_{q_n}}-\frac{2q_n^{n-\lambda-1}[\lambda+1]_{q_n}}{[n-\lambda-1]_{q_n}}\\&\quad+\frac{q_n^{n-\lambda-2}[\lambda+2]_{q_n}}{[n-\lambda-2]_{q_n}}+I=\frac{q_n^{n-\lambda-2}}{\prod_{i=0}^{2}[n-\lambda-i]_{q_n}}\\&\quad\cdot\left([\lambda+2]_{q_n}[n-\lambda]_{q_n}[n-\lambda-1]_{q_n}+q_n^2[\lambda]_{q_n}[n-\lambda-2]_{q_n}[n-\lambda-1]_{q_n}\right.\\&\quad\left.-2q_n[\lambda+1]_{q_n}[n-\lambda-2]_{q_n}[n-\lambda]_{q_n}\right)+I\\&=\frac{q_n^{n-\lambda-2}}{\prod_{i=0}^{2}[n-\lambda-i]_{q_n}}\left([n-\lambda]_{q_n}\left([\lambda+2]_{q_n}[n-\lambda-1]_{q_n}-q_n[\lambda+1]_{q_n}[n-\lambda-2]_{q_n}\right)-q_n[n-\lambda-2]_{q_n}\right.\\&\quad\left.\cdot\left([\lambda+1]_{q_n}[n-\lambda]_{q_n}-q_n[\lambda]_{q_n}[n-\lambda-1]_{q_n}\right)\right)+I\\&=\frac{q_n^{n-\lambda-2}}{\prod_{i=0}^{2}[n-\lambda-i]_{q_n}}\left([n-\lambda]_{q_n}[n]_{q_n}-q_n[n-\lambda-2]_{q_n}[n]_{q_n}\right)+I\\&=\frac{q_n^{n-\lambda-2}[n]_{q_n}\left(1+q_n^{n-\lambda-1}\right)}{\prod_{i=0}^{2}[n-\lambda-i]_{q_n}}+I,\end{aligned}$$

$$\lim_{n\to\infty}[n]_{q_n}^2 I=(3\lambda+2)a^2, \tag{19}$$

we have $\lim_{n\to\infty}[n]_{q_n}^2\mathscr{G}_{n,\lambda}^{q_n}((t-x)^3;x)=3(\lambda+1)a^2x^3+ax^3$. This means that equation (15) is obtained. Thus, the proof of Lemma 5 is accomplished.

Lemma 6. *Let $q=(q_n)$ be a sequence satisfying $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a \in [0,1]$. Then, for each $x \in \mathbb{R}^+$, the following relations*

$$\mathscr{G}_{n,\lambda}^{q_n}\left(\left|(t-x)_{q_n}^2\right|;x\right)\le O\left(\frac{1}{[n]_{q_n}}\right), \tag{20}$$

$$\mathscr{G}_{n,\lambda}^{q_n}\left(\left|(t-x)_{q_n}^2\right|(t-x)^4;x\right)\le O\left(\frac{1}{[n]_{q_n}^3}\right), \tag{21}$$

hold.

Proof. By the definition of q-power basis, we have $(t-x)_{q_n}^2=(t-x)(t-q_nx)=(t-x)^2+(1-q_n^n)x(t-x)=(t-x)^2+x((1-q_n^n)/([n]_{q_n}))(t-x)$. Thus, we can write $|(t-x)_{q_n}^2|\le(t-x)^2+x((1-q_n^n)/([n]_{q_n}))|t-x|$. Using the monotonicity of the operators $\mathscr{G}_{n,\lambda}^{q_n}$ and the Cauchy-Schwarz inequality, we can get

$$\begin{aligned}\mathcal{G}_{n,\lambda}^{q_n}\left(\left|(t-x)_{q_n}^2\right|;x\right) &\le \mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right) + x\frac{1-q_n^n}{[n]_{q_n}}\mathcal{G}_{n,\lambda}^{q_n}(|t-x|;x) \le \mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right) \\ &+ x\frac{1-q_n^n}{[n]_{q_n}}\sqrt{\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right)} \le O\left(\frac{1}{[n]_{q_n}}\right) + O\left(\frac{1}{[n]_{q_n}^{3/2}}\right) = O\left(\frac{1}{[n]_{q_n}}\right).\end{aligned} \tag{22}$$

The inequality (21) can be get in the same way. Using the monotonicity of the operators $\mathcal{G}_{n,\lambda}^{q_n}$, (16)and (17), Cauchy-Schwarz inequality, respectively, we can obtain

$$\begin{aligned}\mathcal{G}_{n,\lambda}^{q_n}\left(\left|(t-x)_{q_n}^2\right|(t-x)^4;x\right) &\le \mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^6;x\right) + \frac{x}{[n]_{q_n}}\mathcal{G}_{n,\lambda}^{q_n}\left(|t-x|^5;x\right) \le \mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^6;x\right) \\ &+ \frac{x}{[n]_{q_n}}\sqrt{\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^4;x\right)\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^6;x\right)} \le O\left(\frac{1}{[n]_{q_n}^3}\right) + O\left(\frac{1}{[n]_{q_n}^{7/2}}\right) = O\left(\frac{1}{[n]_{q_n}^3}\right).\end{aligned} \tag{23}$$

Thus, we complete the proof.

3. Local Approximation

Let $C_B(\mathbb{R}^+)$ be the space of all real-valued continuous bounded functions f on $\mathbb{R}^+$, endowed with the norm $\|f\| = \sup_{x\in\mathbb{R}^+} |f(x)|$. Moreover, the Peetre's $\mathcal{K}$-functional is defined by

$$\mathcal{K}_2(f;\delta) = \inf_{h\in C_B^2(\mathbb{R}^+)}\left\{\|f-h\| + \delta\|h''\|\right\}, \tag{24}$$

where $C_B^2(\mathbb{R}^+) \coloneqq \{h \in C_B(\mathbb{R}^+): h', h'' \in C_B(\mathbb{R}^+)\}$. By ([14], p. 177, Theorem 2.4), there exists an absolute constant $C > 0$ such that

$$\mathcal{K}_2(f;\delta) \le C\omega_2\left(f;\sqrt{\delta}\right), \tag{25}$$

where $\delta > 0$ and the second-order modulus of smoothness is defined by

$$\omega_2\left(f;\sqrt{\delta}\right) = \sup_{0<t\le\delta}\sup_{x\in\mathbb{R}^+}|f(x+2t) - 2f(x+t) + f(x)|, \quad f \in C_B(\mathbb{R}^+). \tag{26}$$

The usual modulus of smoothness is defined by

$$\omega(f;\delta) = \sup_{0<t\le\delta}\sup_{x\in\mathbb{R}^+}|f(x+t) - f(x)|, \quad f \in C_B(\mathbb{R}^+). \tag{27}$$

Theorem 7. *Let $f \in C_B(\mathbb{R}^+)$, $q \in (0,1)$, $\lambda = 1, 2, \cdots$. Then for all $x \in \mathbb{R}^+$ and $n \ge \lambda + 1$, there exists an absolute $C_1 = 4C$ such that*

$$\left|\mathcal{G}_{n,\lambda}^{q}(f;x) - f(x)\right| \le C_1\omega_2\left(f;\sqrt{A^2(x) + B(x)}\right) + \omega(f;|A(x)|). \tag{28}$$

Proof. Using Definition 2, we easily obtain $|\mathcal{G}_{n,\lambda}^{q}(f;x)| \le \|f\|$ for all $f \in C_B(\mathbb{R}^+)$. Next, we define new operators by

$$\mathcal{P}_{n,\lambda}^{q}(f;x) = \mathcal{G}_{n,\lambda}^{q}(f;x) + f(x) - f(A(x)+x), \quad x \in \mathbb{R}^+. \tag{29}$$

We can get $\mathcal{P}_{n,\lambda}^{q}(t-x;x) = \mathcal{G}_{n,\lambda}^{q}(t-x;x) - A(x) = 0$ and $|\mathcal{P}_{n,\lambda}^{q}(f;x)| \le 3\|f\|$ for all $f \in C_B(\mathbb{R}^+)$. For $x, t \in \mathbb{R}^+$ and $h \in C_B^2(\mathbb{R}^+)$, using Taylor's expansion, we can write

$$h(t) = h(x) + h'(x)(t-x) + \int_x^t h''(u)(t-u)\mathrm{d}u. \tag{30}$$

Hence,

$$\begin{aligned}\left|\mathcal{P}_{n,\lambda}^{q}(h;x) - h(x)\right| &= \left|h'(x)\mathcal{P}_{n,\lambda}^{q}(t-x;x) + \mathcal{P}_{n,\lambda}^{q}\left(\int_x^t h''(u)(t-u)\mathrm{d}u;x\right)\right| \\ &\le \left|\mathcal{P}_{n,\lambda}^{q}\left(\int_x^t h''(u)(t-u)\mathrm{d}u;x\right)\right| \\ &\le \left|\mathcal{G}_{n,\lambda}^{q}\left(\int_x^t h''(u)(t-u)\mathrm{d}u;x\right) - \int_x^{A(x)+x} h''(u)(A(x)+x-u)\mathrm{d}u\right| \le \mathcal{G}_{n,\lambda}^{q} \\ &\cdot \left(\int_x^t |h''(u)|(t-u)\mathrm{d}u;x\right) + \left|\int_x^{A(x)+x} |h''(u)|(A(x)+x-u)\mathrm{d}u\right| \\ &\le \left(B(x) + A^2(x)\right)\|h''\|.\end{aligned} \tag{31}$$

Further, for all $h \in C_B^2(\mathbb{R}^+)$, we can write

$$\begin{aligned}\left|\mathscr{G}_{n,\lambda}^{q}(f;x)-f(x)\right| &= \left|\mathscr{P}_{n,\lambda}^{q}(f;x)+f(A(x)+x)-2f(x)\right| \\ &\le \left|\mathscr{P}_{n,\lambda}^{q}(f-h;x)-(f-h)(x)\right| \\ &\quad + \left|\mathscr{P}_{n,\lambda}^{q}(h;x)-h(x)\right| + |f(A(x)+x) \\ &\quad - f(x)| \le 4\|f-h\| + \left(A^2(x)+B(x)\right)\|h''\| \\ &\quad + \omega(f;|A(x)|).\end{aligned} \tag{32}$$

Taking infimum over all h and using (25), we can get the desired conclusion.

Corollary 8. *Let $f \in C_B(\mathbb{R}^+)$, $q \in (0,1)$. Then for all $x \in \mathbb{R}^+$ and $n \ge 1$, there exists an absolute $C_1 = 4C$ such that*

$$\left|\mathscr{G}_{n,0}^{q}(f;x)-f(x)\right| \le C_1\omega_2\left(f;\sqrt{B(x)}\right). \tag{33}$$

Corollary 9. *Let $f \in C_B(\mathbb{R}^+)$, $q=(q_n)$ be a sequence satisfying $q_n \in (0,1)$, $q_n \longrightarrow 1$ and $q_n^n \longrightarrow a \in [0,1]$ as $n \longrightarrow \infty$, the limit*

$$\lim_{n\to\infty} \mathscr{G}_{n,\lambda}^{q_n}(f;x) = f(x) \tag{34}$$

holds for all $x \in \mathbb{R}^+$.

4. Rate of Convergence

As is known, if f is not uniformly continuous on $\mathbb{R}^+$, we cannot get $\omega(f;\delta)\to 0$ as $\delta \to 0$. To research the rate of convergence of the operators $\mathscr{G}_{n,\lambda}^{q_n}$ on $\mathbb{R}^+$, we recall the weighted modulus of continuity $\Omega(f;\delta)$(see [15] or [16]). First, we shall consider the following three classes of functions:

$$B_2(\mathbb{R}^+) := \left\{f:\mathbb{R}^+\to\mathbb{R}; |f(x)| \le C_f\left(1+x^2\right)\right\}, \tag{35}$$

where C_f is a positive constant which depends only on f,

$$\begin{aligned} C_2(\mathbb{R}^+) &:= \{f \in B_2(\mathbb{R}^+): f \quad \text{is continuous}\}, \\ C_2^0(\mathbb{R}^+) &:= \left\{f \in B_2(\mathbb{R}^+): \lim_{x\to\infty}\frac{f(x)}{1+x^2} \quad \text{is finite}\right\}.\end{aligned} \tag{36}$$

The space $C_2^0(\mathbb{R}^+)$ is a linear normed space endowed with the norm $\|f\|_2 = \sup_{x\in\mathbb{R}^+}((|f(x)|)/(1+x^2))$. For any $f \in C_2(\mathbb{R}^+)$, $\Omega(f;\delta)$ is defined by

$$\Omega(f;\delta) = \sup_{0\le t<\delta, x\in\mathbb{R}^+} \frac{|f(x+t)-f(x)|}{(1+t^2)(1+x^2)}, \tag{37}$$

if $f \in C_2^0(\mathbb{R}^+)$, then $\Omega(f;\delta)$ has the following properties:

(i) $\lim_{\delta\to 0^+}\Omega(f;\delta) = 0$

(ii) $\Omega(f;\rho\delta) \le 2(1+\rho)(1+\delta^2)\Omega(f;\delta)$, $\rho \in \mathbb{R}^+$

In [17–19], the following inequality was introduced and used

$$\begin{aligned}|f(t)-f(x)| &\le \left(1+(t-x)^2\right)\left(1+x^2\right)\Omega(f;|t-x|) \\ &\le 2\left(1+\frac{|t-x|}{\delta}\right)\left(1+\delta^2\right)\left(1+(t-x)^2\right)\left(1+x^2\right)\Omega(f;\delta) \\ &\le \begin{cases} 4\left(1+\delta^2\right)^2\left(1+x^2\right)\Omega(f;\delta), & |t-x|<\delta, \\ 4\left(1+\delta^2\right)^2\left(1+x^2\right)\dfrac{(t-x)^4}{\delta^4}\Omega(f;\delta), & |t-x|\ge\delta.\end{cases}\end{aligned} \tag{38}$$

Meanwhile, we introduce the modulus of continuity of $f \in C(0,\mathbf{a}](\mathbf{a}>0)$ by $\omega_{\mathbf{a}}(f;\delta) = \sup_{|t-x|\le\delta x, t\in(0,\mathbf{a}]} \sup |f(t)-f(x)|$.

The following is a theorem of the rate of convergence for the operators $\mathscr{G}_{n,\lambda}^{q}$:

Theorem 9. *Let $f \in C_2(\mathbb{R}^+)$, $\lambda \in \mathbb{N}$, $n = \lambda+1, \lambda+2, \cdots$, $\mathbf{a} \in \mathbb{R}^+$, we have*

$$\|\mathscr{G}_{n,\lambda}^{q}(f;x)-f\|_{C(0,\mathbf{a}]} \le 4C_f\left(1+\mathbf{a}^2\right)B(\mathbf{a}) + 2\omega_{\mathbf{a}+1}\left(f,\sqrt{B(\mathbf{a})}\right). \tag{39}$$

Proof. For any $x \in (0,\mathbf{a}]$, $t \in (\mathbf{a}+1,\infty)$, we can easily obtain $1 \le (t-\mathbf{a})^2 \le (t-x)^2$, therefore

$$\begin{aligned}|f(t)-f(x)| &\le C_f\left(2+x^2+t^2\right) \\ &\le C_f\left(2+3x^2+2(t-x)^2\right) \\ &\le 4C_f\left(1+\mathbf{a}^2\right)(t-x)^2.\end{aligned} \tag{40}$$

If $t \in (0,\mathbf{a}+1)$, for any $\delta \in \mathbb{R}^+$, we can obtain

$$|f(t)-f(x)| \le \omega_{\mathbf{a}+1}(f;|t-x|) \le \left(1+\frac{|t-x|}{\delta}\right)\omega_{\mathbf{a}+1}(f;\delta). \tag{41}$$

Combining (39) with (40), we can get

$$|f(t)-f(x)| \le 4C_f\left(1+\mathbf{a}^2\right)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{\mathbf{a}+1}(f;\delta). \tag{42}$$

By Cauchy-Schwarz's inequality and Corollary 4, for all $x \in (0,\mathbf{a}]$, we have

$$
\begin{aligned}
|\mathcal{G}_{n,\lambda}^{q}(f;x)-f(x)| &\le \mathcal{G}_{n,\lambda}^{q}(|f(t)-f(x)|;x)\\
&\le 4C_f\left(1+\mathbf{a}^2\right)\mathcal{G}_{n,\lambda}^{q}\left((t-x)^2;x\right)+\mathcal{G}_{n,\lambda}^{q}\left(\left(1+\frac{|t-x|}{\delta}\right);x\right)\omega_{\mathbf{a}+1}(f,\delta)\\
&\le 4C_f\left(1+\mathbf{a}^2\right)\mathcal{G}_{n,\lambda}^{q}\left((t-x)^2;x\right)+\omega_{\mathbf{a}+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\mathcal{G}_{n,\lambda}^{q}\left((t-x)^2;x\right)}\right)\\
&\le 4C_f\left(1+\mathbf{a}^2\right)B(x)+\omega_{\mathbf{a}+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{B(x)}\right)\\
&\le 4C_f\left(1+\mathbf{a}^2\right)B(\mathbf{a})+\omega_{\mathbf{a}+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{B(\mathbf{a})}\right).
\end{aligned}
\tag{43}
$$

By choosing $\delta=\sqrt{B(\mathbf{a})}$ and taking supremum over all $x\in(0,\mathbf{a}]$, we can get the desired results.

Theorem 10. *$q=(q_n)$ be a sequence satisfying $q_n\in(0,1)$, $q_n\to 1$, and $q_n^n\to a$ as $n\to\infty$ and $f\in C_2^0(\mathbb{R}^+)$; then, there exists a positive integer $N\in\mathbb{N}_+$ such that for all $n>N$ and $\upsilon>0$, the inequality*

$$
\sup_{x\in\mathbb{R}^+}\frac{\left|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{\upsilon}}\le 64\Omega\left(f;\frac{1}{\sqrt{[n]_{q_n}}}\right),
\tag{44}
$$

holds.

Proof. Using (14) and (16), there exists a positive integer $N\in\mathbb{N}_+$ such that for all $n>N$,

$$
\begin{aligned}
&\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right)\le\frac{9}{4[n]_{q_n}},\\
&\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^4;x\right)\le 1.
\end{aligned}
\tag{45}
$$

By Cauchy-Schwarz's inequality, we can get

$$
\mathcal{G}_{n,\lambda}^{q_n}(|t-x|;x)\le\sqrt{\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right)}\le\frac{3}{2\sqrt{[n]_{q_n}}},
\tag{46}
$$

$$
\begin{aligned}
\mathcal{G}_{n,\lambda}^{q_n}\left(|t-x|^3;x\right)&\le\sqrt{\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^2;x\right)}\sqrt{\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)^4;x\right)}\\
&\le\frac{3}{2\sqrt{[n]_{q_n}}}.
\end{aligned}
\tag{47}
$$

Since $\mathcal{G}_{n,\lambda}^{q_n}$ is linear and positive, using (38), (46), and (47), for any $\delta\in(0,1)$, we can obtain

$$
\begin{aligned}
\left|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)\right|&\le 16\left(1+x^2\right)\Omega(f;\delta)\\
&\quad\cdot\left(1+\frac{\mathcal{G}_{n,\lambda}^{q_n}\left(|t-x|+|t-x|^3;x\right)}{\delta}\right)\\
&\le 16\left(1+x^2\right)\left(1+\frac{3}{\delta\sqrt{[n]_{q_n}}}\right)\Omega(f;\delta).
\end{aligned}
\tag{48}
$$

Taking $\delta=1/\sqrt{[n]_{q_n}}$, we complete the proof.

5. Weighted Approximation

In this section, we will discuss the weighted approximation theorems for the operators $\mathcal{G}_{n,\lambda}^{q_n}$.

Theorem 11. *Let $q=(q_n)$ be a sequence satisfying $q_n\in(0,1)$, $q_n\to 1$, and $q_n^n\to a\in[0,1]$ as $n\to\infty$ and $f\in C_2^0(\mathbb{R}^+)$, we have*

$$
\lim_{n\to\infty}\|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f\|_2=0.
\tag{49}
$$

Proof. Using Korovkin's theorem (see [20]), it is sufficient to verify the following three conditions:

$$
\lim_{n\to\infty}\|\mathcal{G}_{n}^{p_n,q_n}\left(t^k\right)-x^k\|_2=0,\ k=0,1,2.
\tag{50}
$$

Since $\mathcal{G}_{n}^{p_n,q_n}(1;x)=1$, (51) holds for $k=1$. By Lemma 3 and $\lim_{n\to\infty}([n]_{q_n}/[n-\lambda]_{q_n})=\lim_{n\to\infty}([n]_{q_n}/[n-\lambda-1]_{q_n})=1$, we can easily obtain

$$\begin{aligned}\|\mathcal{G}_{n,\lambda}^{q_n}(t;x)-x\|_2&=\sup_{x\in\mathbb{R}^+}\frac{1}{1+x^2}\left|\mathcal{G}_{n,\lambda}^{q_n}(t;x)-x\right|\\&=\sup_{x\in\mathbb{R}^+}\frac{x}{1+x^2}\left|\frac{[n]_{q_n}}{[n-\lambda]_{q_n}}-1\right|\\&\le\left|\frac{[n]_{q_n}}{[n-\lambda]_{q_n}}-1\right|\to 0, n\to\infty.\end{aligned}$$

$$\begin{aligned}\|\mathcal{G}_{n,\lambda}^{q_n}(t^2;x)-x^2\|_2&=\sup_{x\in\mathbb{R}^+}\frac{1}{1+x^2}\left|\mathcal{G}_{n,\lambda}^{q_n}(t^2;x)-x^2\right|\\&=\sup_{x\in\mathbb{R}^+}\frac{x^2}{1+x^2}\left|\frac{[n]_{q_n}^2}{[n-\lambda]_{q_n}[n-\lambda-1]_{q_n}}-1\right|\\&\le\left|\frac{[n]_{q_n}^2}{[n-\lambda]_{q_n}[n-\lambda-1]_{q_n}}-1\right|\to 0, n\to\infty.\end{aligned} \tag{51}$$

We can draw the final conclusion through all the estimates above.

Theorem 12. *Let $q=(q_n)$ be a sequence satisfying $q_n\in(0,1)$, $q_n\to 1$ and $q_n^n\to 1$ as $n\to\infty$ and $f\in C_2^0(\mathbb{R}^+)$. For any $f\in C_2^0(\mathbb{R}^+)$ and $v>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}^+}\frac{|\mathcal{G}_n^{p_n,q_n}(f;x)-f(x)|}{(1+x^2)^{1+v}}=0. \tag{52}$$

Proof. Let $x_0\in\mathbb{R}^+$ be arbitrary but fixed. Then,

$$\begin{aligned}\sup_{x\in\mathbb{R}^+}\frac{|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)|}{(1+x^2)^{1+v}}&\le\sup_{x\in(0,x_0)}\frac{|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)|}{(1+x^2)^{1+v}}\\&\quad+\sup_{x\in[x_0,\infty)}\frac{|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)|}{(1+x^2)^{1+v}}\\&\le\|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f\|_{C(0,x_0]}\\&\quad+\|f\|_2\sup_{x\in[x_0,\infty)}\frac{|\mathcal{G}_{n,\lambda}^{q_n}((1+t^2);x)|}{(1+x^2)^{1+v}}\\&\quad+\sup_{x\in[x_0,\infty)}\frac{|f(x)|}{(1+x^2)^{1+v}}.\end{aligned} \tag{53}$$

Since $|f(x)|\le\|f\|_2(1+x^2)$, we have $\sup_{x\in[x_0,\infty)}((|f(x)|)/(1+x^2)^{1+v})\le((\|f\|_2)/(1+x_0^2)^v)$. Let $\varepsilon>0$ be arbitrary, we can choose x_0 to be so large that

$$\frac{\|f\|_2}{(1+x_0^2)^v}<\varepsilon. \tag{54}$$

In view of Corollary 9, while $x\in[x_0,\infty)$, we obtain

$$\begin{aligned}\|f\|_2\lim_{n\to\infty}\frac{|\mathcal{G}_{n,\lambda}^{q_n}((1+t^2);x)|}{(1+x^2)^{1+v}}&=\|f\|_2\frac{(1+x^2)}{(1+x^2)^{1+v}}=\frac{\|f\|_2}{(1+x^2)^v}\\&\le\frac{\|f\|_2}{(1+x_0^2)^v}<\varepsilon.\end{aligned} \tag{55}$$

Using Theorem 9, we can see that the first term of the inequality (53) implies that

$$\|\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f\|_{C(0,x_0]}<\varepsilon,\quad\text{as}\quad n\to\infty. \tag{56}$$

Combining (53)–(56), we get the desired result.

6. Voronovskaja Type Theorems

As is known, Voronovskaja type theorems of many positive operators are widely researched and discussed (see [21–28]). In this section, we will discuss the quantitative q-Voronovskaja theorem and q-Grüss-Voronovskaja theorem.

6.1. Quantitative q-Voronovskaja Theorem. In this subsection, we will obtain the Quantitative q-Voronovskaja theorem and Voronovskaja type asymptotic formula for the operators $\mathcal{G}_{n,\lambda}^{q_n}$.

Theorem 13. *Let $q=(q_n)$ be a sequence satisfying $q_n\in(0,1)$, $q_n\to 1$ and $q_n^n\to a\in[0,1]$ as $n\to\infty$ and $f\in C_2^0(\mathbb{R}^+)$ satisfy $D_{q_n}^2 f\in C_2^0(\mathbb{R}^+)$. Then, the inequality*

$$\begin{aligned}&\left|[n]_{q_n}\left(\mathcal{G}_{n,\lambda}^{q_n}(f;x)-f(x)-D_{q_n}f(x)A_n(x)\right)\right.\\&\left.\quad-\frac{[n]_{q_n}B_n(x)+(1-q_n^n)A_n(x)x}{[2]_{q_n}!}D_{q_n}^2f(x)\right|\\&\le O(1)\Omega\left(D_{q_n}^2f;\frac{1}{\sqrt{[n]_{q_n}}}\right),\end{aligned} \tag{57}$$

holds for any $x\in\mathbb{R}^+$.

Proof. Using the q-Taylor expansion formula (58), we have

$$\begin{aligned}f(t)&=f(x)+D_{q_n}f(x)(t-x)+\frac{D_{q_n}^2f(\xi)}{[2]_{q_n}!}(t-x)_{q_n}^2\\&=f(x)+D_{q_n}f(x)(t-x)+\frac{D_{q_n}^2f(x)}{[2]_{q_n}!}(t-x)_{q_n}^2+R_2(t,x;q_n),\end{aligned} \tag{58}$$

where ξ is a number between t and x and

$$R_2(t,x;q_n)=\frac{D_{q_n}^2f(\xi)-D_{q_n}^2f(x)}{[2]_{q_n}!}(t-x)_{q_n}^2. \tag{59}$$

Applying the operators $\mathcal{G}_{n,\lambda}^{q_n}$ to both sides of (58) and using $(t-x)_{q_n}^2 = (t-x)^2 + ((1-q_n^n)/([n]_{q_n}))(t-x)x$, we have

$$\begin{aligned}&\left|\mathcal{G}_{n,\lambda}^{q_n}(f;x) - f(x) - D_{q_n}f(x)A_n(x) - \frac{D_{q_n}^2 f(x)}{[2]_{q_n}!}\mathcal{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)\right| \\ &= \Bigg|\mathcal{G}_{n,\lambda}^{q_n}(f;x) - f(x) - D_{q_n}f(x)A_n(x) \\ &\quad - \frac{B_n(x) + \left(((1-q_n^n)A_n(x))/\left([n]_{q_n}\right)\right)x}{[2]_{q_n}!}D_{q_n}^2 f(x)\Bigg| \\ &\le \mathcal{G}_{n,\lambda}^{q_n}(|R_2(t,x;q_n)|;x).\end{aligned} \tag{60}$$

Multiplying the above inequality by $[n]_{q_n}$, we have

$$\begin{aligned}&\Bigg|[n]_{q_n}\left(\mathcal{G}_{n,\lambda}^{q_n}(f;x) - f(x) - D_{q_n}f(x)A_n(x)\right) \\ &\quad - \frac{[n]_{q_n}B_n(x) + (1-q_n^n)A_n(x)x}{[2]_{q_n}!}D_{q_n}^2 f(x)\Bigg| \\ &\le [n]_{q_n}\mathcal{G}_{n,\lambda}^{q_n}(|R_2(t,x;q_n)|;x).\end{aligned} \tag{61}$$

Furthermore,

$$\begin{aligned}\left|\frac{D_{q_n}^2 f(\xi) - D_{q_n}^2 f(x)}{[2]_{q_n}!}\right| &\le \frac{1}{[2]_{q_n}!}\Omega\left(D_{q_n}^2 f;|\xi-x|\left(1+(\xi-x)^2\right)\left(1+x^2\right)\right) \\ &\le \frac{1}{[2]_{q_n}!}\Omega\left(D_{q_n}^2 f;|t-x|\left(1+(t-x)^2\right)\left(1+x^2\right)\right) \\ &\le \frac{2}{[2]_{q_n}!}\left(1+\frac{|t-x|}{\delta}\right)\left(1+(t-x)^2\right) \\ &\quad\cdot\left(1+\delta^2\right)\left(1+x^2\right)\Omega\left(D_{q_n}^2 f;\delta\right) \\ &\le 16\left(1+x^2\right)\left(1+\frac{(t-x)^4}{\delta^4}\right)\Omega\left(D_{q_n}^2 f;\delta\right),\end{aligned} \tag{62}$$

for all $\delta \in (0,1)$. Hence,

$$|R_2(t,x;q_n)| \le 16\left(1+x^2\right)\left(|(t-x)_{q_n}^2| + \frac{|(t-x)_{q_n}^2|(t-x)^4}{\delta^4}\right)\Omega\left(D_{q_n}^2 f;\delta\right). \tag{63}$$

Using (20), (21), for any $x \in \mathbb{R}^+$, we can write

$$\begin{aligned}\mathcal{G}_{n,\lambda}^{q_n}(|R_2(t,x;q_n)|;x) &\le 16\left(1+x^2\right)\Bigg(\mathcal{G}_{n,\lambda}^{q_n}\left(|(t-x)_{q_n}^2|;x\right) \\ &\quad + \frac{\mathcal{G}_{n,\lambda}^{q_n}\left(|(t-x)_{q_n}^2|(t-x)^4;x\right)}{\delta^4}\Bigg)\Omega\left(D_{q_n}^2 f;\delta\right) \\ &\le \left(O\left(\frac{1}{[n]_{q_n}}\right) + \frac{1}{\delta^4}O\left(\frac{1}{[n]_{q_n}^3}\right)\right)\Omega\left(D_{q_n}^2 f;\delta\right).\end{aligned} \tag{64}$$

If we choose $\delta = 1/\sqrt{[n]_{q_n}}$, we can easily get

$$[n]_{q_n}\mathcal{G}_{n,\lambda}^{q_n}(|R_2(t,x;q_n)|;x) \le O(1)\Omega\left(D_{q_n}^2 f;\frac{1}{\sqrt{[n]_{q_n}}}\right), \tag{65}$$

which completes the proof of Theorem 13.

Corollary 14. *Let (q_n) be a sequence satisfying $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a \in [0,1]$ as $n \to \infty$ and $f \in C_2^0(\mathbb{R}^+)$ satisfy $f'' \in C_2^0(\mathbb{R}^+)$. Then, we can obtain*

$$\lim_{n\to\infty}[n]_{q_n}\left(\mathcal{G}_{n,\lambda}^{q_n}(f;x) - f(x)\right) = \lambda a x f'(x) + \frac{1}{2}a x^2 f''(x). \tag{66}$$

6.2. q-Grüss-Voronovskaja Theorem. In this subsection, we will obtain the q-Grüss-Voronovskaja theorem and its quantitative version for the operators $\mathcal{G}_{n,\lambda}^{q_n}$.

Theorem 15. *Let $q = (q_n)$ be a sequence satisfying $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a \in [0,1]$ as $n \to \infty$ and $f, g \in C_2^0(\mathbb{R}^+)$ satisfy $D_{q_n}^2 f, D_{q_n}^2 g, D_{q_n}^2(fg) \in C_2^0(\mathbb{R}^+)$. Then, the following inequality*

$$\begin{aligned}&[n]_{q_n}\Bigg|\mathcal{G}_{n,\lambda}^{q_n}(fg;x) - \mathcal{G}_{n,\lambda}^{q_n}(f;x)\mathcal{G}_{n,\lambda}^{q_n}(g;x) \\ &\quad - \frac{D_{q_n}(f(q_n x))\left(D_{q_n}(g(x)) + D_{q_n}(g(q_n x))\right)}{[2]_{q_n}!}\mathcal{G}_{n,\lambda}^{q_n} \\ &\quad\cdot\left((t-x)_{q_n}^2;x\right)\Bigg| \le O(1)\Omega\left(D_{q_n}^2(fg);\frac{1}{\sqrt{[n]_{q_n}}}\right) \\ &\quad + O(1)\left(\|f\|_2 + O\left(\frac{1}{[n]_{q_n}}\right)\left(\|D_{q_n}f\|_2 + \|D_{q_n}^2 f\|_2\right)\right)\Omega \\ &\quad\cdot\left(D_{q_n}^2 g;\frac{1}{\sqrt{[n]_{q_n}}}\right) + O(1)\left(\|g\|_2 + O\left(\frac{1}{[n]_{q_n}}\right)\right. \\ &\quad\cdot\left.\left(\|D_{q_n}g\|_2 + \|D_{q_n}^2 g\|_2\right)\right)\Omega\left(D_{q_n}^2 f;\frac{1}{\sqrt{[n]_{q_n}}}\right) \\ &\quad + O\left(\frac{1}{[n]_{q_n}}\right)\left(\|D_{q_n}f\|_2 + \|D_{q_n}^2 f\|_2\right)\left(\|D_{q_n}g\|_2 + \|D_{q_n}^2 g\|_2\right) \\ &\quad + O\left(\frac{1}{[n]_{q_n}}\right)\Omega\left(D_{q_n}^2 f;\frac{1}{\sqrt{[n]_{q_n}}}\right)\Omega\left(D_{q_n}^2 g;\frac{1}{\sqrt{[n]_{q_n}}}\right),\end{aligned} \tag{67}$$

holds for any $x \in \mathbb{R}^+$.

Proof. Using the equalities

$$\begin{aligned}D_{q_n}(f(x)g(x)) &= D_{q_n}(f(x))g(x) + f(q_n x)D_{q_n}(g(x)), \\ D_{q_n}^2(f(x)g(x)) &= D_{q_n}^2(f(x))g(x) + D_{q_n}(f(q_n x))D_{q_n}(g(x)) \\ &\quad + f(q_n x)D_{q_n}^2(g(x)) + D_{q_n}(f(q_n x))D_{q_n}(g(q_n x)),\end{aligned} \tag{68}$$

by simple computations, for $x \in \mathbb{R}^+$ and $n = \lambda + 1, \cdots$, we can obtain

$$\begin{aligned}
&\mathscr{G}_{n,\lambda}^{q_n}(fg;x) - \mathscr{G}_{n,\lambda}^{q_n}(f;x)\mathscr{G}_{n,\lambda}^{q_n}(g;x) = \mathscr{G}_{n,\lambda}^{q_n}(fg;x) \\
&\quad - f(x)g(x) - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(f(x)g(x)) \\
&\quad - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(f(x)g(x)) \\
&\quad - g(x)\left(\mathscr{G}_{n,\lambda}^{q_n}(f;x) - f(x) - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(f(x)) - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(f(x))\right) \\
&\quad - \mathscr{G}_{n,\lambda}^{q_n}(f;x)\left(\mathscr{G}_{n,\lambda}^{q_n}(g;x) - g(x) - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(g(x)) - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(g(x))\right) \\
&\quad + \frac{D_{q_n}(f(q_nx))\left(D_{q_n}(g(x)) + D_{q_n}(g(q_nx))\right)}{[2]_{q_n}!}\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right) \\
&\quad - D_{q_n}^2(g(x))\left(\mathscr{G}_{n,\lambda}^{q_n}(f;x) - f(x)\right)\frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!} \\
&\quad + D_{q_n}^2(g(x))D_{q_n}(f(x))\frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}(q_n-1)x \\
&\quad - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)\left(\mathscr{G}_{n,\lambda}^{q_n}(f;x) - f(q_nx)\right)D_{q_n}(g(x)).
\end{aligned} \tag{69}$$

Hence, we can write

$$\begin{aligned}
&\mathscr{G}_{n,\lambda}^{q_n}(fg;x) - \mathscr{G}_{n,\lambda}^{q_n}(f;x)\mathscr{G}_{n,\lambda}^{q_n}(g;x) \\
&\quad - \frac{D_{q_n}(f(q_nx))\left(D_{q_n}(g(x)) + D_{q_n}(g(q_nx))\right)}{[2]_{q_n}!}\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right) \\
&\quad = \mathscr{G}_{n,\lambda}^{q_n}(fg;x) - f(x)g(x) \\
&\quad - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(f(x)g(x)) \\
&\quad - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(f(x)g(x)) \\
&\quad - g(x)\left(\mathscr{G}_{n,\lambda}^{q_n}(f;x) - f(x) - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(f(x)) - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(f(x))\right) \\
&\quad - f(x)\left(\mathscr{G}_{n,\lambda}^{q_n}(g;x) - g(x) - \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)D_{q_n}(g(x)) - \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}D_{q_n}^2(g(x))\right) + (f(x) \\
&\quad - \mathscr{G}_{n,\lambda}^{q_n}(f;x))\left(\mathscr{G}_{n,\lambda}^{q_n}(g;x) - g(x)\right) \\
&\quad + \mathscr{G}_{n,\lambda}^{q_n}(t-x;x)(q_n-1)xD_{q_n}(f(x))D_{q_n}(g(x)) \\
&\quad + \frac{\mathscr{G}_{n,\lambda}^{q_n}\left((t-x)_{q_n}^2;x\right)}{[2]_{q_n}!}(q_n-1)xD_{q_n}f(x)D_{q_n}^2g(x) := I_1 \\
&\quad + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned} \tag{70}$$

By Theorem 13, for any fixed $x \in \mathbb{R}^+$, we can easily have the following estimates

$$[n]_{q_n}|I_1| \le O(1)\Omega\left(D_{q_n}^2(fg);\frac{1}{\sqrt{[n]_{q_n}}}\right), \tag{71}$$

$$\begin{aligned}
[n]_{q_n}|I_2| &\le |g(x)|O(1)\Omega\left(D_{q_n}^2(f);\frac{1}{\sqrt{[n]_{q_n}}}\right) \\
&\le \|g\|_2 O(1)\Omega\left(D_{q_n}^2(f);\frac{1}{\sqrt{[n]_{q_n}}}\right),
\end{aligned} \tag{72}$$

$$\begin{aligned}
[n]_{q_n}|I_3| &\le |f(x)|O(1)\Omega\left(D_{q_n}^2(g);\frac{1}{\sqrt{[n]_{q_n}}}\right) \\
&\le \|f\|_2 O(1)\Omega\left(D_{q_n}^2(g);\frac{1}{\sqrt{[n]_{q_n}}}\right).
\end{aligned} \tag{73}$$

Using (14), (20), and $|q_n - 1| = (|q_n^n - 1|/[n]_{q_n}) \le O(1/[n]_{q_n})$, we have

$$\begin{aligned}
[n]_{q_n}|I_5| &\le O\left(\frac{1}{[n]_{q_n}}\right)\|D_{q_n}f\|_2\|D_{q_n}g\|_2, \\
[n]_{q_n}|I_6| &\le O\left(\frac{1}{[n]_{q_n}}\right)\|D_{q_n}f\|_2\|D_{q_n}^2g\|_2.
\end{aligned} \tag{74}$$

Using (14), (20), and Theorem 13, we can get

$$\begin{aligned}
\left|\mathscr{G}_{n,\lambda}^{q_n}(f;x) - f(x)\right| &\le O\left(\frac{1}{[n]_{q_n}}\right)\left(\|D_{q_n}f\|_2 + \|D_{q_n}^2f\|_2\right) \\
&\quad + O\left(\frac{1}{[n]_{q_n}}\right)\Omega\left(D_{q_n}^2(f);\frac{1}{\sqrt{[n]_{q_n}}}\right),
\end{aligned} \tag{75}$$

hence, we can know

$$\begin{aligned}
[n]_{q_n}|I_4| &\le O\left(\frac{1}{[n]_{q_n}}\right)\left(\left(\|D_{q_n}f\|_2 + \|D_{q_n}^2f\|_2\right)\left(\|D_{q_n}g\|_2 + \|D_{q_n}^2g\|_2\right)\right) \\
&\quad + O\left(\frac{1}{[n]_{q_n}}\right)\Omega\left(D_{q_n}^2(f);\frac{1}{\sqrt{[n]_{q_n}}}\right)\left(\|D_{q_n}g\|_2 + \|D_{q_n}^2g\|_2\right) \\
&\quad + O\left(\frac{1}{[n]_{q_n}}\right)\Omega\left(D_{q_n}^2(g);\frac{1}{\sqrt{[n]_{q_n}}}\right)\left(\|D_{q_n}f\|_2 + \|D_{q_n}^2f\|_2\right) \\
&\quad + O\left(\frac{1}{[n]_{q_n}}\right)\Omega\left(D_{q_n}^2(f);\frac{1}{\sqrt{[n]_{q_n}}}\right)\Omega\left(D_{q_n}^2(g);\frac{1}{\sqrt{[n]_{q_n}}}\right).
\end{aligned} \tag{76}$$

Combining (71)–(76), we complete the proof of Theorem 15.

Corollary 16. *Let $q=(q_n)$ be a sequence satisfying $q_n \in (0,1)$, $q_n \to 1$ and $q_n^n \to a \in [0,1]$ as $n\to\infty$ and $f, g \in C_2^0(\mathbb{R}^+)$ satisfy $f'', g'', (fg)'' \in C_2^0(\mathbb{R}^+)$. Then, the following limit equality*

$$\lim_{n\to\infty}[n]_{q_n}\left(\mathscr{G}_{n,\lambda}^{q_n}(fg;x)-\mathscr{G}_{n,\lambda}^{q_n}(f;x)\mathscr{G}_{n,\lambda}^{q_n}(g;x)\right)=af'(x)g'(x)x^2, \tag{77}$$

holds for any $x \in \mathbb{R}^+$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors read and approved the final manuscript.

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Research Article

On Janowski Analytic (p, q)-Starlike Functions in Symmetric Circular Domain

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The main object of the present paper is to apply the concepts of (p, q)-derivative by establishing a new subclass of analytic functions connected with symmetric circular domain. Further, we investigate necessary and sufficient conditions for functions belonging to this class. Convex combination, weighted mean, arithmetic mean, growth theorem, and convolution property are also determined.

1. Introduction and Definitions

Quantum calculus or q-calculus is a generalization of classical calculus without the notation of limits. The theory of q-calculus is established by Jackson, for details see [1, 2]. Due to its numerous applications in various branches of applied sciences and mathematics, for example, physics, operator theory, numerical analysis, and differential equations, attracted researchers to this field. A detailed study on applications of q-calculus in operator theory may be found in [3]. The geometric interpretation of q-calculus has been recognized through studies on quantum groups. Starlikeness and convexity are two major properties of analytic functions. Ismail et al. [4] investigated the generalized starlike function $\mathcal{S}^*$, and certain subclasses close-to-convex functions of q-Mittag-Leffler functions were studied by Srivastava and Bansal [5], also the reader is referred to [6–12] for more details.

The foundation of quantum calculus is on one parameter, while the postquantum calculus or simply (p, q)-calculus is the generalization of q-calculus based on two parameters. By setting $p = 1$ in (p, q)-calculus, the q-calculus is obtained. The (p, q)-integer was considered by Chakrabarti and Jagannathan [13], also see the work [14–18]. The idea of q-starlike is extended to (p, q)-stalikeness by Raza et al. [19]. Before we define our new class in this field, we give some basics for a better understanding of the work to follow.

Let $\mathcal{A}$ represent the family of function f that are analytic in the open unit disc $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ having the series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, (z \in \mathfrak{D}). \quad (1)$$

A function $f(z)$ of the form (1) is subordinate to function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, symbolically represented $f(z) \prec g(z)$, if there occur a Schwarz function $w(z)$ with limitation that $w(0) = 0$, and $|w(z)| \le 1$, then $f(z) = g(w(z))$. While the convolution of these functions can be defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, (z \in \mathfrak{D}). \quad (2)$$

For $0 < q < 1$, the q-derivative of a function f is defined by

$$\partial_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, (z \neq 0, q \neq 1), \quad (3)$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + \sum_{l=1}^{n-1} q^l, \quad [0, q] = 0, \quad (4)$$

see [13] for details.

Also for $0 < p < q < 1$, the (p, q)-derivative of a function f is defined in [2] as

$$\partial_{p,q} f(z) = \frac{f(pz) - f(qz)}{z(p-q)}, (z \neq 0, p \neq q). \quad (5)$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, 3, \cdots\}$ and $z \in \mathfrak{D}, \partial_{p,q}(\sum_{n=1}^{\infty} a_n z^n) = \sum_{n=1}^{\infty} [n]_{p,q} a_n z^{n-1}$,

where

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}. \quad (6)$$

We note that $\partial_{1,q} f(z) = \partial_q f(z)$ (for more on this topic one should read [20–22]).

Sakaguchi [23], in year 1956, established the class of starlike functions with respect to symmetrical points denoted by $\mathcal{S}_s^*$ of holomorphic univalent functions in $\mathfrak{D}$ if the below condition is satisfies

$$\operatorname{Re} \frac{2zf'(z)}{f(z) - f(-z)} > 0, (z \in \mathfrak{A}). \quad (7)$$

Motivated by the work of [19, 23, 24], we now define $\mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$ given below.

Definition 1. Let $-\mathscr{D} \leq \mathscr{C} < \mathscr{D} \leq 1, 0 < p < q \leq 1$ and $-1 \leq m < l \leq 1$, then the function $f \in \mathscr{A}$ is in the class $\mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$ if it satisfies

$$\frac{(l-m)z\partial_{p,q} f(z)}{f(lz) - f(mz)} \prec \frac{1 + \mathscr{C}z}{1 + \mathscr{D}z}, (z \in \mathfrak{D}), \quad (8)$$

where the symbol "$\prec$" indicates the well-known subordination.

We note that $\mathcal{S}_{1,q}^*(l, m, \mathscr{C}, \mathscr{D}) = \mathcal{S}_q^*(l, m, \mathscr{C}, \mathscr{D})$, where

$$\mathcal{S}_q^*(l, m, \mathscr{C}, \mathscr{D}) = \left\{ f \in \mathscr{A} : \frac{(l-m)z\partial_q f(z)}{f(lz) - f(mz)} \prec \frac{1 + \mathscr{C}z}{1 + \mathscr{D}z}, (z \in \mathfrak{D}) \right\}, \quad (9)$$

and

$$\lim_{q \to 1^-} \mathcal{S}_q^*(1, -1, \mathscr{C}, \mathscr{D}) = \mathcal{S}^*(\mathscr{C}, \mathscr{D}) = \left\{ f \in \mathscr{A} : \frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + \mathscr{C}z}{1 + \mathscr{D}z}, (z \in \mathfrak{D}) \right\}. \quad (10)$$

Equivalently, a function $f \in \mathscr{A}$ is in the $\mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$ if and only if

$$\left| \frac{(l-m)z\partial_{p,q} f(z)/f(lz) - f(mz) - 1}{\mathscr{D}((l-m)z\partial_{p,q} f(z)/f(lz) - f(mz)) - \mathscr{C}} \right| < 1, (z \in \mathfrak{D}). \quad (11)$$

In our main results, in the next section, we evaluate the criteria for functions belonging to this newly defined class. After that, the convex combination property for this class will be discussed. Then utilizing these results, the weighted mean and arithmetic mean properties will be investigated. Further, convolution type results will be discussed in the form of two theorems. At the end of this article, a conclusion and future work will be presented.

2. Main Results

Theorem 2. *Let $f \in \mathscr{A}$ be of the form (1). Then the function $f \in \mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$, if and only if the following inequality holds*

$$\sum_{n=2}^{\infty} \left\{ [n]_{p,q}(1 + \mathscr{D}) - (\mathscr{C} + 1) \frac{l^n - m^n}{l - m} \right\} |a_n| < (\mathscr{D} - \mathscr{C}). \quad (12)$$

Proof. Let us suppose that the first inequality (12) holds. Then to show that $f \in \mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$, we only need to prove the inequality (11). For this consider

$$\begin{aligned} &\left| \frac{(l-m)z\partial_{p,q} f(z)/f(lz) - f(mz) - 1}{D((l-m)z\partial_{p,q} f(z)/f(lz) - f(mz)) - C} \right| \\ &= \left| \frac{\sum_{n=2}^{\infty} \left[[n]_{p,q} - l^n - m^n/l - m \right] \alpha_n z^n}{(D-C)z - \sum_{n=2}^{\infty} \left[D[n]_{p,q} - C(l^n - m^n/l - m) \alpha_n z^n \right]} \right| \\ &\leq \frac{\sum_{n=2}^{\infty} \left[[n]_{p,q} - l^n - m^n/l - m \right] |\alpha_n|}{(D-C) - \sum_{n=2}^{\infty} \left[D[n]_{p,q} - C(l^n - m^n/l - m) \right] |\alpha_n|} < 1, \end{aligned} \quad (13)$$

where we used and this completes the direct part. Conversely, let $f \in \mathcal{S}_{p,q}^*(l, m, \mathscr{C}, \mathscr{D})$ be of from (1). Then from (11), we have for $z \in \mathfrak{D}$,

$$\left|\frac{(l-m)z\partial_{p,q}f(z)/f(lz)-f(mz)-1}{D\left((l-m)z\partial_{p,q}f(z)/f(lz)-f(mz)\right)-C}\right| = \left|\frac{\sum_{n=2}^{\infty}\left[[n]_{p,q}-l^n-m^n/l-m\right]|\alpha_n z^n|}{(D-C)z-\sum_{n=2}^{\infty}\left[D[n]_{p,q}-C(l^n-m^n/l-m)\right]|\alpha_n z^n|}\right|<1 \quad (14)$$

Since $|\mathrm{Re}z|<|z|<1$, we have

$$\mathrm{Re}\left\{\frac{\sum_{n=2}^{\infty}\left[[n]_{p,q}-l^n-m^n/l-m\right]|\alpha_n z^n|}{(D-C)z-\sum_{n=2}^{\infty}\left[D[n]_{p,q}-C(l^n-m^n/l-m)\right]|\alpha_n z^n|}\right\}<1 \quad (15)$$

Now we choose values of z on the real axis such that $(l-m)z\partial_{p,q}f(z)/f(lz)-f(mz)$ is real. Upon clearing the denominator in (15) and letting $z\longrightarrow 1^-$ through real values, we obtain the required inequality (12).

Theorem 3. *Let $f_i\in\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$ and having power series representations*

$$f_i(z)=z+\sum_{k=1}^{\infty}a_{k,i}z^k, \text{for } i=1,2,3,\cdots,t. \quad (16)$$

Then $\Phi\in\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$, where

$$\Phi(z)=\sum_{i=1}^{t}\omega_i f_i(z) \text{ with } \sum_{i=1}^{t}\omega_i=1. \quad (17)$$

Proof. By Theorem 2, one can write

$$\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}|a_{n,i}|<1. \quad (18)$$

Therefore

$$\begin{aligned}\Phi(z)&=\sum_{i=1}^{t}\omega_i\left(z+\sum_{n=2}^{\infty}a_{n,i}z^n\right)\\&=z+\sum_{i=1}^{t}\sum_{n=2}^{\infty}\omega_i a_{n,i}z^n\\&=z+\sum_{n=2}^{\infty}\left(\sum_{i=1}^{t}\omega_i a_{n,i}\right)z^n;\end{aligned} \quad (19)$$

however,

$$\begin{aligned}&\sum_{n=2}^{\infty}\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\left(\left|\sum_{i=1}^{t}\omega_i a_{n,i}\right|\right)\\&=\sum_{i=1}^{t}\omega_i\left[\sum_{n=2}^{\infty},\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})},|a_{n,i}|\right]\le 1,\end{aligned} \quad (20)$$

then $\Phi\in\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$. Hence, the proof is completed.

Theorem 4. *If $f_1,f_2\in\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$, then their weighted mean ψ_k is also in $\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$, where ψ_k is defined by*

$$\psi_k(z)=\left\{\frac{(1-k)f_1(z)+(1+k)f_2(z)}{2}\right\}. \quad (21)$$

Proof. From (21), one can easily write

$$\psi_k(z)=z+\sum_{n=2}^{\infty}\left\{\frac{(1-k)a_n+(1+k)b_n}{2}\right\}z^n. \quad (22)$$

To prove that $\psi_k\in\mathcal{S}^*_{p,q}(l,m,\mathscr{C},\mathscr{D})$, it is enough to show that

$$\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}\cdot\left\{\frac{(1-k)a_n+(1+k)b_n}{2}\right\}<1. \quad (23)$$

For this, consider

$$\begin{aligned}&\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}\cdot\left\{\frac{(1-k)a_n+(1+k)b_n}{2}\right\}\\&=\frac{(1-j)}{2}\cdot\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}|a_n|\\&\quad+\frac{(1+j)}{2}\cdot\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D})-(\mathscr{C}+1)(l^n-m^n/l-m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}|b_n|\\&<\frac{(1-k)}{2}+\frac{(1+k)}{2}=1,\end{aligned} \quad (24)$$

where we have used inequality (12). Which completes the proof.

Theorem 5. *Let $f_i \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$, with $i = 1, 2, \cdots, j$. Then, their arithmetic mean φ of f_i*

$$\varphi(z) = \frac{1}{j}\sum_{i=1}^{j} f_i(z), \tag{25}$$

is also in the class $\mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$.

Proof. From (25), we can write

$$\varphi(z) = \frac{1}{j}\sum_{i=1}^{j}\left(z + \sum_{n=2}^{\infty} a_{n,i} z^n\right) = z + \sum_{n=2}^{\infty}\left(\frac{1}{j}\sum_{i=1}^{j} a_{n,i}\right) z^n. \tag{26}$$

Since $f_i \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$ for every $i = 1, 2, \cdots, j$, using (12), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D}) - (\mathscr{C}+1)(l^n - m^n/l - m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}\cdot\left|\frac{1}{j}\sum_{i=1}^{j} a_{n,i}\right| \\
&= \frac{1}{j}\sum_{i=1}^{j}\left(\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D}) - (\mathscr{C}+1)(l^n - m^n/l - m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}\cdot\left|a_{n,i}\right|\right) \\
&\le \frac{1}{j}\sum_{i=1}^{j}(1) = 1,
\end{aligned} \tag{27}$$

which complete the proof.

Theorem 6. *Let $f \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$. Then for $|z| = r, 0 < r < 1$,*

$$r - \delta_{p,q}(l, m, \mathscr{C}, \mathscr{D}) r^2 < |f(z)| < r + \delta_{p,q}(l, m, \mathscr{C}, \mathscr{D}) r^2, \tag{28}$$

where

$$\delta_{p,q}(l, m, \mathscr{C}, \mathscr{D}) = \frac{(\mathscr{C}-\mathscr{D})}{[2]_{p,q}(1-\mathscr{D}) + (\mathscr{C}+1)(l+m)}. \tag{29}$$

$$r - \gamma_{p,q}(l, m, \mathscr{C}, \mathscr{D}) r^2 < \left|\partial_{p,q} f(z)\right| < r + \gamma_{p,q}(l, m, \mathscr{C}, \mathscr{D}) r^2, \tag{30}$$

where

$$\gamma_{p,q}(l, m, \mathscr{C}, \mathscr{D}) = \frac{(\mathscr{C}-\mathscr{D})}{(1-\mathscr{D}) + (\mathscr{C}+1)(l+m)}. \tag{31}$$

Proof. To prove (28), consider

$$|f(z)| \le r + \sum_{n=2}^{\infty} |a_n||r|^n, \tag{32}$$

as $0 < r < 1$ so $r^n < r^2$ hence

$$|f(z)| < r + r^2\sum_{n=2}^{\infty} |a_n| \le r + \frac{(\mathscr{C}-\mathscr{D})}{[2]_{p,q}(1-\mathscr{D}) + (\mathscr{C}+1)(l+m)} r^2. \tag{33}$$

Similarly,

$$\begin{aligned}
|f(z)| &\ge r - \sum_{n=2}^{\infty} |a_n||r|^n > r - r^2\sum_{n=2}^{\infty} |a_n| \\
&\ge r - \frac{(\mathscr{C}-\mathscr{D})}{[2]_{p,q}(1-\mathscr{D}) + (\mathscr{C}+1)(l+m)} r^2.
\end{aligned} \tag{34}$$

Hence complete the proof of (28). Similarly, we can prove (30).

Theorem 7. *Let $f_i \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$, such that*

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n, \ i = 1, 2, \tag{35}$$

with condition $|a_{n,2}| \le 1$, then $f_1 * f_2 \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$.

Proof. Since form (35), we have

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} z^n, \ i = 1, 2. \tag{36}$$

Then convolution is defined as

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \tag{37}$$

Since $f_2 \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$, with limitation that $|a_{n,2}| \le 1$. Therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D}) - (\mathscr{C}+1)(l^n - m^n/l - m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}|a_{n,1}||a_{n,2}| \\
&\le \sum_{n=2}^{\infty}\left\{\frac{\left[[n]_{p,q}(1+\mathscr{D}) - (\mathscr{C}+1)(l^n - m^n/l - m)\right]}{(\mathscr{D}-\mathscr{C})}\right\}|a_{n,1}| < 1.
\end{aligned} \tag{38}$$

Hence $f_1 * f_2 \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$.

Theorem 8. *Let $f(z) \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$. Then*

$$\frac{1}{z}\left[f(z) * \left(\frac{(1+\mathscr{D}e^{i\theta})z}{(1-pz)(1-qz)} - \frac{(1+\mathscr{C}e^{i\theta})z}{(1-lz)(1-mz)}\right)\right] \ne 0. \tag{39}$$

Proof. Let $f(z) \in \mathcal{S}^*_{p,q}(l, m, \mathscr{C}, \mathscr{D})$. Then by definition of subordination, there exists a Schwarz function $w(z)$, such that $w(0) = 0$ and $|w(z)| < 1$,

$$\frac{(l-m)z\partial_{p,q}f(z)}{f(lz)-f(mz)} = \frac{1+\mathscr{C}w(z)}{1+\mathscr{D}w(z)}, \tag{40}$$

equivalently,

$$\frac{(l-m)z\partial_{p,q}f(z)}{f(lz)-f(mz)} \neq \frac{1+\mathscr{C}e^{i\theta}}{1+\mathscr{D}e^{i\theta}}, \tag{41}$$

$$z\partial_{p,q}f(z)\left(1+\mathscr{D}e^{i\theta}\right) - \frac{f(lz)-f(mz)}{l-m}\left(1+\mathscr{C}e^{i\theta}\right) \neq 0, \tag{42}$$

using the relations

$$\begin{aligned} z\partial_{p,q}f(z) &= f(z) * \frac{z}{(1-pz)(1-qz)}, \\ \frac{f(lz)-f(mz)}{l-m} &= f(z) * \left[\frac{z}{(1-lz)(1-mz)}\right], \end{aligned} \tag{43}$$

now (42), becomes

$$\frac{1}{z}\left[f(z) * \left(\frac{\left(1+\mathscr{D}e^{i\theta}\right)z}{(1-pz)(1-qz)} - \frac{\left(1+\mathscr{C}e^{i\theta}\right)z}{(1-lz)(1-mz)}\right)\right] \neq 0, \tag{44}$$

which completes the proof.

3. Conclusions

Utilizing the concepts of postquantum calculus, we defined a new subclass of analytic functions associated with symmetric circular domain. For this class, we investigated some useful results such as necessary and sufficient problem, convex combination, weight mean, arithmetic mean, distortion bounds, and convolution property. There are some problems open for researchers such as radii problems, extreme point theorem, analytic criteria, and integral mean of inequality. Moreover, this concept is new and can be extended to meromorphic functions and harmonic functions.

Data Availability

Data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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Research Article

On Some Integral Inequalities in Quantum Calculus

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The objective of this paper is to establish q-analogue of some well-known inequalities in analysis, namely, Poincaré-type inequalities, Sobolev-type inequalities, and Lyapunov-type inequalities. Our obtained results may serve as a useful source of inspiration for future works in quantum calculus.

1. Introduction and Preliminaries

Mathematical inequalities play a crucial role in the development of various branches of mathematics as well as other disciplines of science. In particular, integral inequalities involving the function and its gradient provide important tools in the proof of regularity of solutions to differential and partial differential equations, stability, boundedness, and approximations. One of these categories of inequalities is the Poincaré-type inequality. Namely, if Ω is a bounded (or bounded at least in one direction) domain of $\mathbb{R}^N$, then, there exists a constant $C = C(\Omega) > 0$ such that for all $u \in H_0^1(\Omega)$,

$$\int_\Omega |u(x)|^2\, dx \leq C \int_\Omega |\nabla u(x)|^2 dx. \quad (1)$$

For a smooth bounded domain Ω, the best constant C satisfying the above inequality is equal to $\lambda(\Omega)^{-1}$, where $\lambda(\Omega)$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$, and Δ is the Laplacian operator (see, e.g., [1–5]). Due to the importance of Poincaré inequality in the qualitative analysis of partial differential equations and also in numerical analysis, numerous contributions dealing with generalizations and extensions of this inequality appeared in the literature (see, e.g., [6–17] and the references therein). Another important inequality involving the function and its gradient is the Sobolev inequality (see [18, 19]). Namely, if u is a smooth function of compact support in $\mathbb{R}^2$, then

$$\int_{\mathbb{R}^2} u^4(x)\, dx \leq \frac{\kappa}{2} \left(\int_{\mathbb{R}^2} u^2(x)\, dx \right) \left(\int_{\mathbb{R}^2} |\nabla u(x)|^2\, dx \right), \quad (2)$$

where $\kappa > 0$ is a dimensionless constant and ∇u denotes the gradient of u. For further results related to Sobolev-type inequalities and their applications, see, for example, [20–26].

Lyapunov's inequality is one of the important results in analysis. It was shown that this inequality is very useful in the study of spectral properties of differential equations, namely, stability of solutions, eigenvalues, and disconjugacy criteria. More precisely, consider the second order differential equation

$$-\vartheta''(t) = f(t)\vartheta(t),\ m_1 < t < m_2, \quad (3)$$

under the Dirichlet boundary conditions

$$\vartheta(m_i) = 0, \quad i = 1, 2, \quad (4)$$

where $f \in C([m_1, m_2])$. Obviously, the trivial function $\vartheta \equiv 0$ is a solution to (3)–(4). Lyapunov's inequality provides a necessary criterion for the existence of a nontrivial solution. Namely, if $\vartheta \in C^1([m_1, m_2])$ is a nontrivial solution to (3)–(4), then (see Lyapunov [27] and Borg

[28])

$$\int_{m_1}^{m_2} |f(t)|\, dt > \frac{4}{m_1 - m_2}. \tag{5}$$

Since the appearance of the above result, numerous contributions related to Lyapunov-type inequalities have been published (see, e.g., [18, 29–32] and the references therein).

On the other hand, because of its usefulness in several areas of physics (thermostatistics, conformal quantum mechanics, nuclear and high energy physics, black holes, etc.), the theory of quantum calculus received a considerable attention by many researchers from various disciplines (see, e.g., [33–35]).

In this paper, motivated by the abovementioned contributions, our goal is to derive q-analogs of some Poincaré-type inequalities, Sobolev-type inequalities, and Lyapunov-type inequalities. Notice that only the one dimensional case is considered in this work.

We recall below some notions and properties related to q-calculus (see, e.g., [36–51] and the references therein).

We first fix $q \in (0,1)$. Let $\mathbb{N}$ be the set of positive natural numbers, i.e., $\mathbb{N} = \{1, 2, 3, \cdots\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Definition 1. The q -derivative of a function $\vartheta \in C^1([0,T])$ ($T > 0$) is defined by

$$D_q\vartheta(t) = \begin{cases} \dfrac{\vartheta(t) - \vartheta(qt)}{(1-q)t} & \text{if } 0 < t \le T, \\ \vartheta'(0) & \text{if } t = 0. \end{cases} \tag{6}$$

Remark 2. Using L'Hospital's rule, one obtains

$$\lim_{t \to 0^+} D_q\vartheta(t) = \vartheta'(0), \tag{7}$$

which shows that $D_q\vartheta \in C([0,T])$ for all $\vartheta \in C^1([0,T])$.

Remark 3. It can be easily seen that

$$\lim_{q \to 1^-} D_q\vartheta(t) = \vartheta'(t), \quad 0 \le t \le T. \tag{8}$$

Lemma 4 (see [45]). *Let $\vartheta, \rho \in C^1([0,T])$. Then*

$$D_q(\vartheta\rho)(t) = \vartheta(qt)D_q\rho(t) + \rho(t)D_q\vartheta(t). \tag{9}$$

Definition 5. The q-integral of a function $\vartheta \in C([0,T])$ is defined by

$$\int_0^t \vartheta(\xi)d_q\xi = (1-q)t\sum_{\sigma=0}^{\infty} q^\sigma \vartheta(q^\sigma t), \quad 0 \le t \le T, \tag{10}$$

and

$$\int_s^t \vartheta(\xi)d_q\xi = \int_0^t \vartheta(\xi)d_q\xi - \int_0^s \vartheta(\xi)d_q\xi, \quad 0 < s \le t \le T. \tag{11}$$

Remark 6. Obviously, if $\vartheta \in C([0,T])$, then

$$\int_0^t \vartheta(\xi)d_q\xi < \infty, \int_s^t \vartheta(\xi)d_q\xi < \infty, \quad 0 < s \le t \le T. \tag{12}$$

Lemma 7 (see [39]). *Let $\vartheta, \rho \in C([0,T])$, $0 \le t \le T$, $p > 1$ and $p' = p/p-1$. Then*

(i) $\left|\int_0^t \vartheta(\xi)d_q\xi\right| \le \int_0^t |\vartheta(\xi)|d_q\xi$

(ii) For all $\sigma \in \mathbb{N}_0$, $\vartheta(q^\sigma t) \le \rho(q^\sigma t)\int_0^t \vartheta(\xi)\, d_q\xi \le \int_0^t \rho(\xi)\, d_q\xi$

(iii) $\int_0^T |\vartheta(\xi)||\rho(\xi)|d_q\xi \le \left(\int_0^T |\vartheta(\xi)|^p\, d_q\xi\right)^{1/p} \left(\int_0^T |\rho(\xi)|^{p'}\, d_q\xi\right)^{1/p'}$.

Lemma 8 (see [45]). *Let $\vartheta \in C^1([0,T])$. Then*

(i) $\int_s^t D_q\vartheta(\xi)d_q\xi = \vartheta(t) - \vartheta(s), 0 \le s \le t \le T$

(ii) $D_q\int_0^t \vartheta(\xi)d_q\xi = \vartheta(t), 0 < t \le T$

Remark 9. Notice that in general, for $0 < s < t \le T$,

$$\left|\int_s^t \vartheta(\xi)d_q\xi\right| \le \int_s^t |\vartheta(\xi)|d_q\xi. \tag{13}$$

Namely, following [40], consider the function $\vartheta : [0,1] \to \mathbb{R}$ defined by

$$\vartheta(\xi) = \begin{cases} \dfrac{1}{1-q}(4q^{-n}\xi - 1 - 3q) & \text{if } q^{n+1} \le \xi \le \dfrac{q^n(1+q)}{2}, n \in \mathbb{N}_0, \\ \dfrac{4}{1-q}(-\xi q^{-n} + 1) - 1 & \text{if } \dfrac{q^n(1+q)}{2} \le \xi \le q^n, n \in \mathbb{N}_0, \\ 0 & \text{if } \xi = 0. \end{cases} \tag{14}$$

Then, one has

$$\vartheta(q^n) = -1 \quad \text{and} \quad \vartheta\left(\frac{q^n(1+q)}{2}\right) = 1, \quad \text{for all } n \in \mathbb{N}_0. \tag{15}$$

Therefore, an elementary calculation shows that

$$\int_{1+q/2}^{1} \vartheta(\xi) d_q\xi = -\frac{3+q}{2} \quad \text{and} \quad \int_{1+q/2}^{1} |\vartheta(\xi)| d_q\xi = \frac{1-q}{2}. \tag{16}$$

Hence, one has

$$\left| \int_{1+q/2}^{1} \vartheta(\xi)\, d_q\xi \right| > \int_{1+q/2}^{1} |\vartheta(\xi)| d_q\xi. \tag{17}$$

We have the following integration by parts rule.

Lemma 10 (see [45]). *Let $\vartheta_i \in C^1([0,T])$, $i=1,2$. Then*

$$\int_0^T \vartheta_1(\xi)\left(D_q\vartheta_2\right)(\xi)\, d_q\xi = \left[\vartheta_1(\xi)\vartheta_2(\xi)\right]_{\xi=0}^{T} - \int_0^T \vartheta_2(q\xi)\left(D_q\vartheta_1\right)(\xi)\, d_q\xi. \tag{18}$$

Let us introduce the set

$$\Lambda_q = \{q^n : n \in \mathbb{N}\} \cup \{0\}. \tag{19}$$

Let $T \in \Lambda_q$, $T>0$, i.e.,

$$T = q^k, \text{for some } k \in \mathbb{N}, \tag{20}$$

and $I_q = [0,T] \cap \Lambda_q$, i.e.,

$$I_q = \left\{ q^{i+k} : i \in \mathbb{N}_0 \right\} \cup \{0\}. \tag{21}$$

Let $s, t \in I_q$ be such that $0<s<t$, i.e., $t=q^{i+k}$ for some $i \in \mathbb{N}_0$ and $s = q^{i+k+j}$ for some $j \in \mathbb{N}$. In this case, for $\vartheta \in C([0, T])$, by Definition 5, one has

$$\begin{aligned} \int_s^t \vartheta(\xi)\, d_q\xi &= \int_0^t \vartheta(\xi)\, d_q\xi - \int_0^s \vartheta(\xi)\, d_q\xi = (1-q)t\sum_{\sigma=0}^{\infty} q^\sigma \vartheta(q^\sigma t) \\ &- (1-q)s\sum_{\sigma=0}^{\infty} q^\sigma\vartheta(q^\sigma s) = (1-q)\left(\sum_{\sigma=0}^{\infty} tq^\sigma\vartheta(q^\sigma t) \right. \\ &\left. - \sum_{\sigma=0}^{\infty} sq^\sigma \vartheta(q^\sigma s)\right) = (1-q)\left(\sum_{\sigma=0}^{\infty} q^{\sigma+i+k}\vartheta\left(q^{\sigma+i+k}\right) \right. \\ &\left. - \sum_{\sigma=0}^{\infty} q^{\sigma+i+k+j}\vartheta\left(q^{\sigma+i+k+j}\right)\right) = (1-q) \\ &\left(\sum_{n=i+k}^{\infty} q^n\vartheta(q^n) - \sum_{n=i+k+j}^{\infty} q^n\vartheta(q^n)\right) \\ &= (1-q)\sum_{n=i+k}^{i+k+j-1} q^n \vartheta(q^n), \end{aligned} \tag{22}$$

which is a finite sum. Hence, one deduces the following property.

Lemma 11. *Let $\vartheta, \rho \in C([0,T])$, where $T \in \Lambda_q$, $T>0$. Let $s, t \in I_q$ be such that $0<s<t$. Then*

$$\left| \int_s^t \vartheta(\xi)\, d_q\xi \right| \le \int_s^t |\vartheta(\xi)|\, d_q\xi. \tag{23}$$

2. Poincaré and Sobolev Type Inequalities

Let $q \in (0,1)$ be fixed.

Theorem 12. *Let $p>1$ and $T \in \Lambda_q$, $T>0$. Let $\vartheta \in C^1([0,T])$ be such that*

$$\vartheta(0) = \vartheta(T) = 0. \tag{24}$$

Then

$$\int_0^T |\vartheta(\xi)|^p d_q\xi \le \left(\frac{T}{2}\right)^p \int_0^T \left|D_q\vartheta(\xi)\right|^p d_q\xi. \tag{25}$$

Proof. Let $t = q^\sigma T$, where $\sigma \in \mathbb{N}$. Notice that since $T \in \Lambda_q$, then $t \in I_q$. By property (i) of Lemma 8, one has

$$\vartheta(t) - \vartheta(0) = \int_0^t D_q\vartheta(\xi) d_q\xi. \tag{26}$$

Since $\vartheta(0)=0$, it holds that

$$\vartheta(t) = \int_0^t D_q\vartheta(\xi) d_q\xi. \tag{27}$$

Next, by property (i) of Lemma 7, one obtains

$$|\vartheta(t)| \le \int_0^t \left|D_q\vartheta(\xi)\right| d_q\xi. \tag{28}$$

Again, using property (i) of Lemma 8, and the fact that $\vartheta(T)=0$, one obtains

$$-\vartheta(t) = \int_t^T D_q\vartheta(\xi) d_q\xi. \tag{29}$$

Hence, by Lemma 11, one deduces that

$$|\vartheta(t)| \le \int_t^T \left|D_q\vartheta(\xi)\right| d_q\xi. \tag{30}$$

Combining (28) with (30), it holds that

$$|\vartheta(t)| \le \frac{1}{2}\int_0^T \left|D_q\vartheta(\xi)\right| d_q\xi. \tag{31}$$

On the other hand, by Hölder's inequality (see property

(iii) of Lemma 7), one has

$$\int_0^T |D_q\vartheta(\xi)| d_q\xi \le \left(\int_0^T |D_q\vartheta(\xi)|^p d_q\xi\right)^{1/p} \left(\int_0^T 1 d_q\xi\right)^{p-1/p}. \tag{32}$$

Notice that

$$\int_0^T 1 d_q\xi = (1-q)T\sum_{n=0}^{\infty} q^n = T. \tag{33}$$

Therefore,

$$\int_0^T |D_q\vartheta(\xi)| d_q\xi \le T^{p-1/p}\left(\int_0^T |D_q\vartheta(\xi)|^p d_q\xi\right)^{1/p}. \tag{34}$$

Combining (31) with (34), one deduces that

$$|\vartheta(t)| \le \frac{T^{p-1/p}}{2}\left(\int_0^T |D_q\vartheta(\xi)|^p d_q\xi\right)^{1/p}, \tag{35}$$

which yields

$$|\vartheta(t)|^p \le \frac{T^{p-1}}{2^p}\int_0^T |D_q\vartheta(\xi)|^p d_q\xi, \tag{36}$$

i.e.,

$$|\vartheta(q^\sigma T)|^p \le \frac{T^{p-1}}{2^p}\int_0^T |D_q\vartheta(\xi)|^p d_q\xi, \quad \sigma\in\mathbb{N}. \tag{37}$$

Notice that since $\vartheta(T)=0$, the above inequality is also true for $\sigma=0$. Hence, by property (ii) of Lemma 7, one deduces that

$$\int_0^T |\vartheta(\xi)|^p d_q\xi \le \frac{T^{p-1}}{2^p}\int_0^T |D_q\vartheta(\xi)|^p d_q\xi \int_0^T 1\, d_q\xi. \tag{38}$$

Finally, (25) follows from (33) and (38).

Remark 13. Inequality (25) is the one dimensional q-analog of the Poincaré-type inequality derived by Pachpatte [11].

Theorem 14. *Let $p_1, p_2 > 1$ and $T\in\Lambda_q$, $T>0$. Let $\vartheta_1, \vartheta_2 \in C^1([0,T])$ be such that*

$$\vartheta_i(0)=\vartheta_i(T)=0, i=1,2. \tag{39}$$

Then

$$\int_0^T |\vartheta_1(\xi)|^{p_1}|\vartheta_2(\xi)|^{p_2} d_q\xi \le \frac{1}{2}\left(\frac{T}{2}\right)^{p_1+p_2} \int_0^T \left(|D_q\vartheta_1(\xi)|^{2p_1} + |D_q\vartheta_2(\xi)|^{2p_2}\right) d_q\xi. \tag{40}$$

Proof. From (37) and (70), one has

$$\left|\vartheta_1(q^\sigma T)\,\right|^{p_1} \le \frac{T^{p_1-1}}{2^{p_1}}\int_0^T \left|D_q\vartheta_1(\xi)\right|^{p_1} d_q\xi, \sigma\in\mathbb{N}_0, \tag{41}$$

$$|\vartheta_2(q^\sigma T)|^{p_2} \le \frac{T^{p_2-1}}{2^{p_2}}\int_0^T |D_q\vartheta_2(\xi)|^{p_2} d_q\xi, \quad \sigma\in\mathbb{N}_0. \tag{42}$$

Multiplying (41) by (42), one obtains

$$|\vartheta_1(q^\sigma T)|^{p_1}|\vartheta_2(q^\sigma T)|^{p_2} \le \frac{T^{p_1+p_2-2}}{2^{p_1+p_2}}\left(\int_0^T |D_q\vartheta_1(\xi)|^{p_1} d_q\xi\right) \cdot \left(\int_0^T |D_q\vartheta_2(\xi)|^{p_2} d_q\xi\right).$$

Next, using the inequality $2AB \le A^2+B^2$, $A, B\in\mathbb{R}$, one deduces that

$$|\vartheta_1(q^\sigma T)|^{p_1}|\vartheta_2(q^\sigma T)|^{p_2} \le \frac{T^{p_1+p_2-2}}{2^{p_1+p_2+1}}\left[\left(\int_0^T |D_q\vartheta_1(\xi)|^{p_1} d_q\xi\right)^2 + \left(\int_0^T |D_q\vartheta_2(\xi)|^{p_2} d_q\xi\right)^2\right]. \tag{44}$$

On the other hand, by Hölder's inequality (see property (iii) of Lemma 7), for $i=1,2$, one has

$$\left(\int_0^T |D_q\vartheta_i(\xi)|^{p_i} d_q\xi\right)^2 \le T\int_0^T |D_q\vartheta_i(\xi)|^{2p_i} d_q\xi. \tag{45}$$

Hence, combining (44) with (45), it holds that

$$|\vartheta_1(q^\sigma T)|^{p_1}|\vartheta_2(q^\sigma T)|^{p_2} \le \frac{T^{p_1+p_2-1}}{2^{p_1+p_2+1}}\int_0^T \cdot \left(|D_q\vartheta_1(\xi)|^{2p_1} + |D_q\vartheta_2(\xi)|^{2p_2}\right) d_q\xi. \tag{46}$$

Since the above inequality holds for all $\sigma\in\mathbb{N}_0$, by property (ii) of Lemma 7, one deduces that

$$\int_0^T |\vartheta_1(\xi)|^{p_1}|\vartheta_2(\xi)|^{p_2} d_q\xi \le \frac{T^{p_1+p_2-1}}{2^{p_1+p_2+1}}\int_0^T \cdot \left(|D_q\vartheta_1(\xi)|^{2p_1} + |D_q\vartheta_2(\xi)\,|\right) d_q\xi \int_0^T 1\, d_q\xi. \tag{47}$$

Finally, (40) follows from (33) and (47).

Remark 15. Inequality (40) is the one dimensional q-analog of the Poincaré-type inequality derived by Pachpatte [10].

Theorem 16. *Let $p>1$, $m>(p/2(p-1))$, $N\in\mathbb{N}$ and $T\in\Lambda_q$, $T>0$. Let $\vartheta_i\in C^1([0,T])$, $i=1,2,\cdots,N$ be such that*

$$\vartheta_i(0)=\vartheta_i(T)=0. \tag{48}$$

Then

$$\left[\int_0^T\left(\sum_{i=1}^N|\vartheta_i(\xi)|^2\right)^{p/p-1}d_q\xi\right]^{2m(p-1)/p} \cdot \leq \frac{1}{N}\left(\frac{N}{4}\right)^{2m}T^{(6m-1)p-2m/p}\sum_{i=1}^N\int_0^T\left|D_q\vartheta_i(\xi)\right|^{4m}d_q\xi. \quad (49)$$

Proof. Let $t=q^\sigma T$, where $\sigma\in\mathbb{N}$. From (31), one has

$$|\vartheta_i(t)|\leq\frac{1}{2}\int_0^T|D_q\vartheta_i(\xi)|\,d_q\xi, i=1,2,\cdots,N. \quad (50)$$

On the other hand, by Hölder's inequality (see property (iii) of Lemma 7) and (33), one has

$$\left(\int_0^T|D_q\vartheta_i(\xi)|\,d_q\xi\right)^2\leq T\int_0^T\left|D_q\vartheta_i(\xi)\right|^2d_q\xi. \quad (51)$$

Hence, by (50), one deduces that

$$|\vartheta_i(t)|^2\leq\frac{T}{4}\int_0^T\left|D_q\vartheta_i(\xi)\right|^2d_q\xi,\quad i=1,2,\cdots,N, \quad (52)$$

which yields

$$\left(\sum_{i=1}^N|\vartheta_i(t)|^2\right)^{p/p-1}\leq\left(\frac{T}{4}\right)^{p/p-1}\left(\sum_{i=1}^N\int_0^T\left|D_q\vartheta_i(\xi)\right|^2d_q\xi\right)^{p/p-1}. \quad (53)$$

Next, using the discrete version of Hölder's inequality, one obtains

$$\begin{aligned}&\left(\sum_{i=1}^N|\vartheta_i(t)|^2\right)^{p/p-1}\\ &\leq\left(\frac{T}{4}\right)^{p/p-1}\left(\sum_{i=1}^N1\right)^{1/p-1}\sum_{i=1}^N\left(\int_0^T\left|D_q\vartheta_i(\xi)\right|^2d_q\xi\right)^{p/p-1}\\ &=N^{1/p-1}\left(\frac{T}{4}\right)^{p/p-1}\sum_{i=1}^N\left(\int_0^T\left|D_q\vartheta_i(\xi)\right|^2d_q\xi\right)^{p/p-1}.\end{aligned} \quad (54)$$

On the other hand, by Hölder's inequality (see property (iii) of Lemma 7) and (33), one has

$$\begin{aligned}&\int_0^T\left|D_q\vartheta_i(\xi)\right|^2d_q\xi\\ &\leq\left(\int_0^T1\,d_q\xi\right)^{1/p}\left(\int_0^T\left|D_q\vartheta_i(\xi)\right|^{2p/p-1}d_q\xi\right)^{p-1/p}\\ &=T^{1/p}\left(\int_0^T\left|D_q\vartheta_i(\xi)\right|^{2p/p-1}d_q\xi\right)^{p-1/p}.\end{aligned} \quad (55)$$

Therefore, by (54), one deduces that

$$\left(\sum_{i=1}^N|\vartheta_i(t)|^2\right)^{p/p-1}\leq N^{1/p-1}\left(\frac{T}{4}\right)^{p/p-1}T^{1/p-1}\sum_{i=1}^N\int_0^T\left|D_q\vartheta_i(t)\right|^{2p/p-1}d_q\xi. \quad (56)$$

Since the above inequality is true for all $\sigma\in\mathbb{N}_0$ (recall that $t=q^\sigma T$), by property (ii) of Lemma 7, and using (33), one deduces that

$$\int_0^T\left(\sum_{i=1}^N|\vartheta_i(\xi)|^2\right)^{p/p-1}d_q\xi\leq N^{1/p-1}\left(\frac{T^2}{4}\right)^{p/p-1}\sum_{i=1}^N\int_0^T\left|D_q\vartheta_i(t)\right|^{2p/p-1}d_q\xi, \quad (57)$$

which yields

$$\left[\int_0^T\left(\sum_{i=1}^N|\vartheta_i(\xi)|^2\right)^{p/p-1}d_q\xi\right]^{2m(p-1)/p} \cdot \leq N^{2m/p}\left(\frac{T^2}{4}\right)^{2m}\left(\sum_{i=1}^N\int_0^T\left|D_q\vartheta_i(t)\right|^{2p/p-1}d_q\xi\right)^{2m(p-1)/p}. \quad (58)$$

Next, using Hölder's inequality with exponents $2m(p-1)/2m(p-1)-p$ and $2m(p-1)/p$ (notice that $2m(p-1)/p>1$ by assumption), one obtains

$$\begin{aligned}&\int_0^T\left|D_q\vartheta_i(t)\right|^{2p/p-1}d_q\xi\\ &\leq\left(\int_0^T1\,d_q\xi\right)^{2m(p-1)-p/2m(p-1)}\left(\int_0^T\left|D_q\vartheta_i(t)\right|^{4m}d_q\xi\right)^{p/2m(p-1)}\\ &=T^{2m(p-1)-p/2m(p-1)}\left(\int_0^T\left|D_q\vartheta_i(t)\right|^{4m}d_q\xi\right)^{p/2m(p-1)},\end{aligned} \quad (59)$$

which yields

$$\sum_{i=1}^N\int_0^T\left|D_q\vartheta_i(t)\right|^{2p/p-1}d_q\xi\leq T^{2m(p-1)-p/2m(p-1)}\sum_{i=1}^N\left(\int_0^T\left|D_q\vartheta_i(t)\right|^{4m}d_q\xi\right)^{p/2m(p-1)}. \quad (60)$$

Furthermore, the discrete Hölder's inequality shows that

$$\sum_{i=1}^N\left(\int_0^T\left|D_q\vartheta_i(t)\right|^{4m}d_q\xi\right)^{p/2m(p-1)}\leq N^{2m(p-1)-p/2m(p-1)} \cdot\left(\sum_{i=1}^N\int_0^T\left|D_q\vartheta_i(t)\right|^{4m}d_q\xi\right)^{p/2m(p-1)}. \quad (61)$$

Hence, by (60), one deduces that

$$\sum_{i=1}^{N}\int_0^T \left|D_q\vartheta_i(t)\right|^{2p/p-1} d_q\xi \le T^{2m(p-1)-p/2m(p-1)} N^{2m(p-1)-p/2m(p-1)} \cdot \left(\sum_{i=1}^{N}\int_0^T \left|D_q\vartheta_i(t)\right|^{4m} d_q\xi\right)^{p/2m(p-1)}. \quad (62)$$

Finally, combining (58) with (62), (49) follows.

Remark 17. Inequality (49) is the one dimensional q-analog of the Poincaré-type inequality derived by Pachpatte [12].

Theorem 18. *Let $T \in \Lambda_q$, $T > 0$. Let $\vartheta \in C^1([0,T])$ be such that*

$$\vartheta(0) = \vartheta(T) = 0. \quad (63)$$

Then

$$\int_0^T \vartheta^2(\xi)\, d_q\xi \le \frac{T}{2}\left(\int_0^T (|\vartheta(q\xi)| + |\vartheta(\xi)|)^2\, d_q\xi\right)^{1/2}\left(\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right)^{1/2}. \quad (64)$$

Proof. Let $t = q^\sigma T$, where $\sigma \in \mathbb{N}$. By Lemma 4, property (i) of Lemma 8, and using the boundary conditions, one has

$$\vartheta^2(t) = \int_0^t (\vartheta(q\xi) + \vartheta(\xi)) D_q\vartheta(\xi)\, d_q\xi \quad (65)$$

and

$$-\vartheta^2(t) = \int_t^T (\vartheta(q\xi) + \vartheta(\xi)) D_q\vartheta(\xi)\, d_q\xi. \quad (66)$$

Combining (65) with (66), it holds that

$$\vartheta^2(t) \le \frac{1}{2}\int_0^T (|\vartheta(q\xi)| + |\vartheta(\xi)|)\left|D_q\vartheta(\xi)\right| d_q\xi. \quad (67)$$

Using Hölder's inequality, one obtains

$$\vartheta^2(t) \le \frac{1}{2}\left(\int_0^T (|\vartheta(q\xi)| + |\vartheta(\xi)|)^2 d_q\xi\right)^{1/2}\left(\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right)^{1/2}. \quad (68)$$

Since the above inequality holds for all $\sigma \in \mathbb{N}_0$, using property (ii) of Lemma 7, integrating over $(0,T)$, and using (33), (64) follows.

Remark 19. Inequality (64) is the one dimensional q-analog of the Sobolev-type inequality derived by Pachpatte [11].

3. Lyapunov-Type Inequalities

We fix $q \in (0,1)$ and $T \in \Lambda_q$, $T > 0$. Consider the second order q-difference equation

$$-D_q\left(D_q\vartheta\right)(t/q) + a(t)D_q\vartheta(t) = f(t)\varphi(\vartheta(t)), \quad 0 < t < T, \quad (69)$$

under the boundary conditions

$$\vartheta(0) = \vartheta(T) = 0, \quad (70)$$

where $a, f \in C([0,T])$ and $\varphi : \mathbb{R} \to \mathbb{R}$. We suppose that there exists a constant $L_\varphi > 0$ such that

$$|\varphi(x)| \le L_\varphi |x|, \quad x \in \mathbb{R}. \quad (71)$$

Obviously, from (71), one has $\varphi(0) = 0$. Hence, $\vartheta \equiv 0$ is a trivial solution to (69) and (70). The following theorem provides a necessary condition for the existence of a nontrivial solution to (69) and (70) satisfying $\vartheta(t) \ne 0$, $0 < t < T$.

Theorem 20. *Suppose that $\vartheta \in C^1([0,T])$ is a solution to (69) and (70) satisfying*

$$\vartheta(t) \ne 0, \quad 0 < t < T. \quad (72)$$

Then

$$1 \le L_\varphi \int_0^T \sqrt{\frac{\xi(T-\xi)}{4}}|f(\xi)| d_q\xi + \left(\int_0^T \sqrt{\frac{\xi(T-\xi)}{4}}|a(\xi)|^2 d_q\xi\right)^{1/2}. \quad (73)$$

Proof. Let $s = q^\sigma T$, where $\sigma \in \mathbb{N}$. Since $\vartheta(0) = 0$, using property (i) of Lemma 8, one has

$$\vartheta(s) = \int_0^s D_q\vartheta(\xi) d_q\xi. \quad (74)$$

By Hölder's inequality (see property (iii) of Lemma 7) and (33), one obtains

$$|\vartheta(s)| \le \sqrt{s}\left(\int_0^s \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right)^{1/2}, \quad (75)$$

which yields

$$|\vartheta(s)|^2 \le s\int_0^s \left|D_q\vartheta(\xi)\right|^2 d_q\xi. \quad (76)$$

Similarly, since $\vartheta(T) = 0$, one has

$$-\vartheta(s) = \int_s^T D_q\vartheta(\xi) d_q\xi, \quad (77)$$

which implies that (see Lemma 11)

$$|\vartheta(s)| \le \int_s^T \left|D_q\vartheta(\xi)\right| d_q\xi. \quad (78)$$

Since $s, T \in I_q$, then $\int_s^T |D_q\vartheta(\xi)| d_q\xi$ is a finite sum (see (22)). Hence, we can apply Hölder's inequality to get

$$|\vartheta(s)|^2 \le (T-s)\int_s^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi. \quad (79)$$

Multiplying (76) by (79), one obtains

$$|\vartheta(s)|^4 \le s(T-s)\left(\int_0^s \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right)\left(\int_s^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right), \quad (80)$$

i.e.,

$$|\vartheta(s)|^2 \le \sqrt{s(T-s)}\left(\int_0^s |D_q\vartheta(\xi)|^2 d_q\xi\right)^{1/2}\left(\int_s^T |D_q\vartheta(\xi)|^2 d_q\xi\right)^{1/2}. \quad (81)$$

Using the inequality $2AB \le A^2 + B^2$, $A, B \in \mathbb{R}$, it holds that

$$|\vartheta(s)|^2 \le \frac{\sqrt{s(T-s)}}{2}\left[\int_0^s \left|D_q\vartheta(\xi)\right|^2 d_q\xi + \int_s^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right], \quad (82)$$

i.e., (recall that $s = q^\sigma T$ and $\vartheta(T) = 0$)

$$|\vartheta(q^\sigma T)|^2 \le \frac{\sqrt{q^\sigma T(T - q^\sigma T)}}{2}\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi, \quad \sigma \in \mathbb{N}_0. \quad (83)$$

Consider now the function

$$w(t) = D_q\vartheta(t/q), \quad 0 < t < T. \quad (84)$$

By (69), one has

$$-D_q w(t) + a(t)D_q\vartheta(t)(t) = f(t)\varphi(\vartheta(t)), \quad 0 < t < T. \quad (85)$$

Multiplying (85) by $\vartheta(t)$ and integrating over $(0, T)$, one obtains

$$-\int_0^T \vartheta(\xi)D_q w(\xi)d_q\xi + \int_0^T a(\xi)\vartheta(\xi)D_q\vartheta(\xi)d_q\xi = \int_0^T f(\xi)\varphi(\vartheta(\xi))\vartheta(\xi)d_q\xi. \quad (86)$$

On the other hand, using the integration by parts rule (see Lemma 10) and the boundary conditions (70), one has

$$-\int_0^T \vartheta(\xi)D_q w(\xi)d_q\xi = \int_0^T w(q\xi)D_q\vartheta(\xi)d_q\xi. \quad (87)$$

Hence, by (86) and the definition of w, one deduces that

$$\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi = \int_0^T f(\xi)\varphi(\vartheta(\xi))\vartheta(\xi)d_q\xi - \int_0^T \boxtimes a(\xi)\vartheta(\xi)D_q\vartheta(\xi)d_q\xi. \quad (88)$$

Next, using (71), one obtains

$$\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi \le L_\varphi\int_0^T |f(\xi)|\vartheta^2(\xi)d_q\xi + \int_0^T |a(\xi)||\vartheta(\xi)|\left|D_q\vartheta(\xi)\right|d_q\xi. \quad (89)$$

Furthermore, by (83) and property (ii) of Lemma 7, one deduces that

$$\begin{aligned}\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi \le{}& \frac{L_\varphi}{2}\left(\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi\right)\\ &\cdot\left(\int_0^T |f(\xi)|\sqrt{\xi(T-\xi)}d_q\xi\right)\\ &+ \frac{1}{\sqrt{2}}\left(\int_0^T |a(\xi)|[\xi(T-\xi)]^{1/4}|D_q\vartheta(\xi)|d_q\xi\right)\\ &\cdot\left(\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi\right)^{1/2}.\end{aligned} \quad (90)$$

Therefore, by Hölder's inequality, it holds that

$$\begin{aligned}\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi \le{}& \frac{L_\varphi}{2}\left(\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right)\\ &\cdot\left(\int_0^T |f(\xi)|\sqrt{\xi(T-\xi)}d_q\xi\right)\\ &+ \frac{1}{\sqrt{2}}\left(\int_0^T |a(\xi)|^2\sqrt{\xi(T-\xi)}d_q\xi\right)^{1/2}\\ &\cdot\left(\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi\right).\end{aligned} \quad (91)$$

Next, we claim that

$$\int_0^T \left|D_q\vartheta(\xi)\right|^2 d_q\xi \ne 0. \quad (92)$$

Indeed, suppose that $\int_0^T |D_q\vartheta(\xi)|^2 d_q\xi = 0$. By Definition 5, one obtains

$$(1-q)T\sum_{\tau=0}^{\infty} q^\tau \left|D_q\vartheta(q^\tau T)\right|\left|D_q\vartheta(q^\tau T)\right|^2 = 0, \quad (93)$$

which yields

$$D_q\vartheta(q^{\tau}T)=0, \tau\in\mathbb{N}_0. \quad (94)$$

In particular, for $\tau=1$, one has

$$D_q\vartheta(T)=0, \quad (95)$$

i.e.,

$$\vartheta(T)-\vartheta(qT)=0. \quad (96)$$

Since $\vartheta(T)=0$, one deduces that $\vartheta(qT)=0$, which contradicts (72). This proves (92). Now, dividing (91) by $\int_0^T |D_q\vartheta(\xi)|^2\, d_q\xi>0$, it holds that

$$1\le \frac{L_\varphi}{2}\int_0^T |f(\xi)|\sqrt{\xi(T-\xi)}\, d_q\xi+\frac{1}{\sqrt{2}}\left(\int_0^T |a(\xi)|^2\sqrt{\xi(T-\xi)}\, d_q\xi\right)^{1/2}, \quad (97)$$

which yields (73).

Using the inequality

$$\xi(T-\xi)\le\frac{T^2}{4},\quad 0<\xi<T, \quad (98)$$

one deduces from Theorem 20 the following result.

Corollary 21. *Suppose that $\vartheta\in C^1([0,T])$ is a solution to (69) and (70) satisfying (72). Then*

$$1\le\frac{L_\varphi T}{4}\int_0^T |f(\xi)|\, d_q\xi+\frac{\sqrt{T}}{2}\left(\int_0^T |a(\xi)|^2\, d_q\xi\right)^{1/2}. \quad (99)$$

Consider now the second order q-difference equation

$$-D_q\left(D_q\vartheta\right)(t/q)=f(t)\varphi(\vartheta(t)),\quad 0<t<T, \quad (100)$$

under the boundary conditions (70), where $f\in C([0,T])$ and $\varphi:\mathbb{R}\to\mathbb{R}$ satisfies (71). Notice that (100) is a special case of (69) with $a\equiv 0$. Hence, by Theorem 20 and Corollary 21, one deduces the following results.

Corollary 3.2. *Suppose that $\vartheta\in C^1([0,T])$ is a solution to (100) and (70) satisfying (72). Then*

$$\int_0^T \sqrt{\xi(T-\xi)}|f(\xi)|\, d_q\xi\ge\frac{2}{L_\varphi}. \quad (101)$$

Corollary 22. *Suppose that $\vartheta\in C^1([0,T])$ is a solution to (100) and (70) satisfying (72). Then*

$$\int_0^T |f(\xi)|\, d_q\xi\ge\frac{4}{L_\varphi T}. \quad (102)$$

Remark 23. Inequality (102) with $\varphi(x)=x$ ($L_\varphi=1$) is the q-analogue of Lyapunov inequality (5) with $m_1=0$ *and* $m_2=T$.

4. Conclusion

Integral inequalities involving the function and its gradient are very useful in the study of existence, uniqueness, and qualitative properties of solutions to ordinary and partial differential equations. Motivated by the importance of q-calculus in applications, integral inequalities involving the function and its q-derivative are obtained. Namely, we derived the q-analogue of some Poincaré-type inequalities and Sobolev-type inequalities. We also established the q-analogue of some Lyapunov-type inequalities. We hope that our results will serve as a useful inspiration for future works in the context of q-calculus.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to this work.

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Research Article

Nonlinear Fractional q-Difference Equation with Fractional Hadamard and Quantum Integral Nonlocal Conditions

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In this paper, we establish existence and uniqueness results for a boundary value problem consisting by a nonlinear fractional q-difference equation subject to a new type of boundary condition, combining the fractional Hadamard and quantum integrals. Our analysis is based on Banach's fixed point theorem, a fixed point theorem for nonlinear contractions, Krasnosel'ski i's fixed point theorem, and Leray-Schauder nonlinear alternative. Examples are given to illustrate our results.

1. Introduction

The aim of this paper is to investigate the existence and uniqueness of solutions for a nonlinear fractional q-difference equation subject to fractional Hadamard and quantum integral condition of the form:

$$\begin{cases} D_q^{\alpha}x(t) = f(t, x(t)), 1 < \alpha \le 2, t \in (0, T), \\ x(0) = 0, \sum_{i=1}^{n} \gamma_i I_{p_i}^{\mu_i} x(\xi_i) = \sum_{j=1}^{m} \beta_j J^{\sigma_j} x\left(\eta_j\right), \end{cases} \quad (1)$$

where D_q^{α} is the fractional q-derivative of order α, with a quantum number $q \in (0, 1)$, $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a nonlinear continuous function, $I_{p_i}^{\mu_i}$ denotes the fractional quantum integral of order $\mu_i > 0$, with quantum number $0 < p_i < 1$, J^{σ_j} is the Hadamard fractional integral of order $\sigma_j > 0$, γ_i and β_j are given constants, and $\xi_i, \eta_j \in (0, T)$ are fixed points, for $i = 1, \cdots, n$ and $j = 1, \cdots, m$.

The subject of fractional differential equations has recently evolved into an interesting subject for many researchers due to its multiple applications in economics, engineering, physics, chemistry, signal analysis, etc. Various types of fractional derivative and integral operator were studied: Riemann-Liouville, conformable fractional integral operators, Caputo, Hadamard, Erdelyi-Kober, Grünwald-Letnikov, Marchaud, and Riesz are just a few to name. The Hadamard-type fractional derivative differs from the preceding ones in the sense that the kernel of the integral and derivative contain logarithmic function of arbitrary exponent. Details and properties of Hadamard fractional derivatives and integrals can be found in Kilbas et al. [1]. Recently, there were some results on Hadamard-type fractional differential equations, see [2–11] and references cited therein.

Nonlinear fractional q-difference equations appear in the mathematical modeling of many phenomena in engineering and science and have attracted much attention by many researchers, see for example [12–21] and references therein.

In the present paper, the novelty lies in the fact that we combine in boundary conditions both Hadamard and quantum integrals. To the best of our knowledge, this type of boundary condition appears for the first time in the literature. It is important to notice that we are combining in our work, fractional calculus, and quantum calculus. The key tool for this combination is the Property 2.25 of [1].

Some special cases of the second condition of (1) can be seen by reducing $m = n = 1$ as

$$\frac{\gamma_1}{\Gamma_{p_1}(\mu_1)}\int_{0^+}^{\xi_1}(\xi_1 - p_1 s)_{p_1}^{(\mu_1-1)}x(s)\,d_{p_1}s = \frac{\beta_1}{\Gamma(\sigma_1)}\int_{0^+}^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1-1}\frac{x(s)}{s}ds, \tag{2}$$

which is mixed quantum and Hadamard calculus. If $p_1 = 1$, then we have

$$\frac{\gamma_1}{\Gamma(\mu_1)}\int_{0^+}^{\xi_1}(\xi_1 - s)^{\mu_1-1}x(s)ds = \frac{\beta_1}{\Gamma(\sigma_1)}\int_{0^+}^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\sigma_1-1}\frac{x(s)}{s}ds, \tag{3}$$

which is also mixed Riemann-Liouville and Hadamard fractional integral condition. If $\mu_1 = \sigma_1 = 1$, we have integral condition of the form:

$$\gamma_1\int_{0^+}^{\xi_1}x(s)ds = \beta_1\int_{0^+}^{\eta_1}\frac{x(s)}{s}ds, \tag{4}$$

which is a variety used in physical boundary value problems.

We establish existence and uniqueness results by using standard fixed point theorems. We prove two existence and uniqueness results with the help of the Banach contraction mapping principle and a fixed point theorem on nonlinear contractions due to Boyd and Wong. Moreover, we prove two existence results, one via Leray-Schauder nonlinear alternative and another one via Krasnosel'ski i's fixed point theorem.

The paper is organized as follows: in Section 2, we recall some preliminary facts that we need in the sequel. In Section 3, we prove our main results. Some examples to illustrate our results are presented in Section 4.

2. Preliminaries

To present the preliminary, we suggest the basic quantum calculus in the book of Kac and Cheung [22], fractional quantum calculus in [23–25], and the Hadamard fractional calculus in [1]. Let a fixed constant $q \in (0,1)$ be a quantum number. The q-number is defined by

$$[a]_q = \frac{1-q^a}{1-q}, a \in \mathbb{R}. \tag{5}$$

For example, $[3]_q = 1 + q + q^2$. The q-power function for any $a, b \in \mathbb{R}$, $a \neq 0$, is defined as

$$(a-b)_q^{(\gamma)} = a^\gamma \prod_{i=0}^{\infty}\frac{1-(b/a)q^i}{1-(b/a)q^{\gamma+i}}. \tag{6}$$

If $\gamma = k \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$, then $(a-b)_q^{(k)} = \prod_{i=0}^{k-1}(a - bq^i)$ and $(a-b)_q^{(0)} := 1$. For example, $(a-b)_q^{(3)} = (a-b)(a-qb)(a-q^2b)$. The notation of q-power function is appeared in kernels of fractional q-calculus as Definitions 1 and 2. Now, the q-gamma function $\Gamma_q(t)$ is defined by

$$\Gamma_q(t) = \frac{(1-q)_q^{(t-1)}}{(1-q)^{t-1}}, \quad \text{for} \quad t \in \mathbb{R} \setminus \{0, -1, -2, \cdots\}. \tag{7}$$

Now, we observe that $\Gamma_q(t+1) = [t]_q\Gamma_q(t)$. Next, we discuss about the q-derivative of a function $f : [0, \infty) \longrightarrow \mathbb{R}$ which is defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \neq 0, \text{and } D_q f(0) = \lim_{t\to 0} D_q f(t). \tag{8}$$

If $f'(t)$ exists, then $\lim_{q\to 1} D_q f(t) = f'(t)$. The q-integral formula can be presented as

$$(I_q f)(t) = \int_0^t f(s)d_q s = t(1-q)\sum_{n=0}^{\infty} q^n f(tq^n), t \in [0, \infty). \tag{9}$$

The higher order of q-derivative and q-integral operators is

$$\left(D_q^k h\right)(t) = D_q\left(D_q^{k-1}f\right)(t) \text{ and } \left(I_q^k h\right)(t) = I_q\left(I_q^{k-1}f\right)(t), k \in \mathbb{N}, \tag{10}$$

with $(D_q^0 f)(t) = f(t)$ and $(I_q^0 f)(t) = f(t)$. Next, the fundamental theorem of calculus for operators D_q and I_q can be stated as formulas

$$\left(D_q I_q f\right)(t) = f(t), \tag{11}$$

and if f is continuous at the point $t = 0$, then

$$\left(I_q D_q f\right)(t) = f(t) - f(0). \tag{12}$$

Let us give the definitions of fractional quantum calculus of the Riemann-Liouville type fractional derivative and also integral operators.

Definition 1 [24]. Let a constant $\alpha \geq 0$ and f be the function on $[0, \infty)$. The Riemann-Liouville fractional q -integral of f order α is defined by

$$\left(I_q^\alpha f\right)(t) = \frac{1}{\Gamma_q(\alpha)} \int_{0^+}^t (t - qs)_q^{(\alpha-1)} f(s)\, d_q s, \quad \alpha > 0, \quad t \in (0,\infty), \tag{13}$$

and $(I_q^0 f)(t) = f(t)$.

Definition 2 [24]. The Riemann-Liouville fractional q -derivative of order $\alpha \geq 0$ of a function $f : [0,\infty) \longrightarrow \mathbb{R}$ is given by

$$\left(D_q^\alpha f\right)(t) = \left(D_q^n I_q^{n-\alpha} f\right)(t) = \frac{1}{\Gamma_q(n-\alpha)} D_q^n \int_{0^+}^t \cdot (t - qs)_q^{(n-\alpha-1)} f(s)\, d_q s, \quad \alpha > 0, \tag{14}$$

and $(D_q^0 f)(t) = f(t)$, where n is the smallest integer greater than or equal to α.

Now, for $t, s > 0$, the q-beta function is presented by

$$B_q(t,s) = \int_{0^+}^1 u^{(t-1)} (1 - qu)_q^{(s-1)} d_q u, \tag{15}$$

which is related to the q-gamma function by

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}. \tag{16}$$

The fundamental formulas for fractional quantum calculus are in the following lemma.

Lemma 3 [24, 26]. *Let $\alpha, \beta \geq 0$, n be a positive integer and f be a function defined in $[0, \infty)$. Then, the following formulas hold*

$$\begin{aligned}
&\left(I_q^\beta I_q^\alpha f\right)(t) = \left(I_q^{\alpha+\beta} f\right)(t),\\
&\left(D_q^\alpha I_q^\alpha f\right)(t) = f(t),\\
&\left(I_q^\beta D_q^n f\right)(t) = \left(D_q^n I_q^\beta f\right)(t) - \sum_{k=0}^{n-1} \frac{t^{\beta-n+k}}{\Gamma_q(\beta+k-n+1)} \left(D_q^k f\right)(0).
\end{aligned} \tag{17}$$

The fractional q-integration of the two deferent quantum numbers is given by lemma.

Lemma 4 [27]. *Let constants $\alpha, \beta > 0$ and $0 < p, q < 1$ be quantum numbers. Then, for $\eta \in \mathbb{R}_+$, we have*

$$I_p^\alpha I_q^\beta (1)(\eta) = \frac{\Gamma_p(\beta+1)}{\Gamma_p(\alpha+\beta+1)\Gamma_q(\beta+1)} \eta^{\alpha+\beta}. \tag{18}$$

The Hadamard fractional calculus is the subject of fractional derivative and integral which have logarithm kernels inside the singular integral formulas as in the definitions.

Definition 5 [1]. The Hadamard derivative of fractional order α for a function $f : [0,\infty) \longrightarrow \mathbb{R}$ is defined as

$$^H D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_{0^+}^t \cdot \left(\log \frac{t}{s}\right)^{n-\alpha-1} \frac{f(s)}{s} ds, n = [\alpha] + 1, \tag{19}$$

where the notation $[\alpha]$ denotes the integer part of the real number α, $\log(\cdot) = \log_e(\cdot)$, and Γ is the usual Gamma function.

Definition 6 [1]. The Hadamard fractional integral of order α for a function $f : [0,\infty) \longrightarrow \mathbb{R}$ is defined by

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_{0^+}^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s} ds, \alpha > 0, \tag{20}$$

provided the integral in right hand side exists.

The key tool for combining the two type of fractional calculus in our work is the following lemma.

Lemma 7 ([1], Property 2.25). *Let $\alpha > 0$ and $\beta > 0$. The following formulas hold*

$$J^\alpha t^\beta = \beta^{-\alpha} t^\beta \text{ and } ^H D^\alpha t^\beta = \beta^\alpha t^\beta. \tag{21}$$

To accomplish our main purpose, we will use the fixed point theory for considering an operator equation $x = \mathbb{Q}x$. For finding the operator $\mathbb{Q}$, let us see the following lemma.

Lemma 8. *Suppose that the points ξ_i, $\eta_j \in [0, T]$ and the constant*

$$\Omega = \sum_{i=1}^n \gamma_i \frac{\Gamma_{p_i}(\alpha)\xi_i^{\alpha+\mu_i-1}}{\Gamma_{p_i}(\alpha+\mu_i)} - \sum_{j=1}^m \beta_j (\alpha-1)^{-\sigma_j} \eta_j^{\alpha-1} \neq 0, \tag{22}$$

where α, μ_i, p_i, γ_i, σ_j, β_j, $i = 1, \cdots, n$, and $j = 1, \cdots, m$ are defined in problem (1). Then, the linear fractional q-difference equation

$$D_q^\alpha x(t) = h(t), 0 < t < T, \tag{23}$$

where $h : [0, T] \longrightarrow \mathbb{R}$, *and subject to mixed fractional integrals of Hadamard and quantum boundary conditions*

$$x(0) = 0, \sum_{i=1}^{n} \gamma_i I_{p_i}^{\mu_i} x(\xi_i) = \sum_{j=1}^{m} \beta_j J^{\sigma_j} x\left(\eta_j\right) \quad (24)$$

is equivalent to the linear integral equation

$$x(t) = \frac{1}{\Omega}\left[\sum_{j=1}^{m} t^{\alpha-1}\beta_j J^{\sigma_j} I_q^{\alpha} h\left(\eta_j\right) - \sum_{i=1}^{n} t^{\alpha-1}\gamma_i I_{p_i}^{\mu_i} I_q^{\alpha} h(\xi_i)\right] + I_q^{\alpha} h(t). \quad (25)$$

Proof. Since $\alpha \in (1, 2]$, then (23) can be written as

$$D_q^2 I_q^{2-\alpha} x(t) = h(t), 0 < t < T. \quad (26)$$

Applying the fractional q-integral of order α and using Lemma 3, we obtain

$$\begin{aligned} I_q^{\alpha} D_q^2 I_q^{2-\alpha} x(t) &= D_q^2 I_q^{\alpha} I_q^{2-\alpha} x(t) - k_1 t^{\alpha-1} - k_2 t^{\alpha-2} \\ &= x(t) - k_1 t^{\alpha-1} - k_2 t^{\alpha-2} = I_q^{\alpha} h(t), \end{aligned} \quad (27)$$

which yields

$$x(t) = k_1 t^{\alpha-1} + k_2 t^{\alpha-2} + I_q^{\alpha} h(t), \quad (28)$$

where $k_1, k_2 \in \mathbb{R}$. The first boundary condition of (24) implies that $k_2 = 0$. Then, (28) is reduced to

$$x(t) = k_1 t^{\alpha-1} + I_q^{\alpha} h(t). \quad (29)$$

Now, we apply the fractional quantum integral of Riemann-Liouville of order μ_i with quantum number p_i to (29) as

$$I_{p_i}^{\mu_i} x(t) = k_1 \frac{\Gamma_{p_i}(\alpha) t^{\alpha+\mu_i-1}}{\Gamma_{p_i}(\alpha+\mu_i)} + I_{p_i}^{\mu_i} I_q^{\alpha} h(t). \quad (30)$$

Using Lemma 7 for taking the Hadamard fractional integral of order σ_j to (29), we get

$$J^{\sigma_j} x(t) = k_1 (\alpha-1)^{-\sigma_j} t^{\alpha-1} + J^{\sigma_j} I_q^{\alpha} h(t). \quad (31)$$

From the second boundary condition of (24) and above two equations, it follows that

$$\begin{aligned} k_1 \sum_{i=1}^{n} \gamma_i \frac{\Gamma_{p_i}(\alpha)\xi_i^{\alpha+\mu_i-1}}{\Gamma_{p_i}(\alpha+\mu_i)} &+ \sum_{i=1}^{n} \gamma_i I_{p_i}^{\mu_i} I_q^{\alpha} h(\xi_i) \\ &= k_1 \sum_{j=1}^{m} \beta_j (\alpha-1)^{-\sigma_j} \eta_j^{\alpha-1} + \sum_{j=1}^{m} \beta_j J^{\sigma_j} I_q^{\alpha} h\left(\eta_j\right), \end{aligned} \quad (32)$$

and consequently

$$k_1 = \frac{1}{\Omega}\left[\sum_{j=1}^{m} \beta_j J^{\sigma_j} I_q^{\alpha} h\left(\eta_j\right) - \sum_{i=1}^{n} \gamma_i I_{p_i}^{\mu_i} I_q^{\alpha} h(\xi_i)\right], \quad (33)$$

where the nonzero constant Ω is defined by (22). Substituting the constant k_1 in (29), then, we obtain (25), which is the solution of BVP (23) and (24). The converse can be obtained by a direct computation. The proof is completed.

3. Main Results

At first, we denote by $\mathscr{C} = C([0, T], \mathbb{R})$ the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$ endowed with the sup norm as $\|x\| = \sup\{|x(t)|, t \in [0, T]\}$. In view of Lemma 8 and replacing the function h by $f(t, x(t))$, we define the operator $\mathscr{Q} : \mathscr{C} \longrightarrow \mathscr{C}$ by

$$\mathscr{Q}x(t) = \frac{t^{\alpha-1}}{\Omega}\left[\sum_{j=1}^{m} \beta_j J^{\sigma_j} I_q^{\alpha} f_x\left(\eta_j\right) - \sum_{i=1}^{n} \gamma_i I_{p_i}^{\mu_i} I_q^{\alpha} f_x(\xi_i)\right] + I_q^{\alpha} f_x(t), \quad (34)$$

where $I_q^{\alpha} f_x(v)$ is denoted by

$$\begin{aligned} I_q^{\alpha} f_x(v) &= \frac{1}{\Gamma_q(\alpha)} \int_{0^+}^{v} (v - qs)_q^{(\alpha-1)} f(s, x(s))\, d_q s \\ &:= g(v), \quad v \in \left\{t, \xi_i, \eta_j\right\}, \end{aligned} \quad (35)$$

while $J^{\sigma_j} I_q^{\alpha} f_x(\eta_j)$ and $I_{p_i}^{\mu_i} I_q^{\alpha} f_x(\xi_i)$ are the Hadamard and quantum fractional integrals of a function g as

$$\begin{aligned} J^{\sigma_j} I_q^{\alpha} f_x\left(\eta_j\right) &= \frac{1}{\Gamma(\sigma_j)} \int_{0^+}^{\eta_j} \left(\log \frac{\eta_j}{s}\right)^{\sigma_j-1} \frac{g(s)}{s} ds, \\ I_{p_i}^{\mu_i} I_q^{\alpha} f_x(\xi_i) &= \frac{1}{\Gamma_{p_i}(\mu_i)} \int_{0^+}^{\xi_i} (\xi_i - p_i s)_{p_i}^{(\mu_i-1)} g(s)\, d_{p_i} s, \end{aligned} \quad (36)$$

respectively. Now, we are going to prove the main results which are the existence criteria of solution for nonlocal mixed fractional integrals boundary value problem (1). The first, an existence and uniqueness result for (1), is given by using Banach's fixed point theorem.

Theorem 9. *Let* $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ *be a nonlinear continuous function satisfying the assumption.*

(H_1) There exists a positive constant L such that $|f(t, x) - f(t, y)| \leq L|x - y|$, for each $t \in [0, T]$ and $x, y \in \mathbb{R}$.

If

$$L\Phi < 1, \quad (37)$$

where Φ is given by

$$\Phi = \frac{T^{\alpha}}{\Gamma_q(\alpha+1)}\left[\frac{1}{T|\Omega|}\sum_{j=1}^{m}\left|\beta_j\right|\alpha^{-\sigma_j}\eta_j^{\alpha} + \frac{1}{T|\Omega|}\sum_{i=1}^{n}|\gamma_i|\frac{\Gamma_{p_i}(\alpha+1)}{\Gamma_{p_i}(\alpha+\mu_i+1)}\xi_i^{\alpha+\mu_i}+1\right], \tag{38}$$

then the boundary value problem (1) has a unique solution on $[0, T]$.

Proof. The result allows from the operator equation $x = \mathcal{Q}x$, where the operator $\mathcal{Q}$ is defined by (34). The Banach fixed point theorem is used to show that $\mathcal{Q}$ has a fixed point which is the unique solution of problem (1). Since the function f is continuous, then, we can set $\sup\{|f(t,0)|, t\in[0,T]\} = M < \infty$. After that, we define the radius r satisfying

$$r \geq \frac{\Phi M}{1-\Phi L} \tag{39}$$

of a ball $B_r = \{x \in C : \|x\| \leq r\}$. For any $x \in B_r$, we see that

$$\begin{aligned}|\mathcal{Q}x(t)| &\leq \sup_{t\in[0,T]}\frac{t^{\alpha-1}}{|\Omega|}\sum_{j=1}^{m}\left|\beta_j\right|J^{\sigma_j}I_q^{\alpha}|f_x|\left(\eta_j\right) + \frac{t^{\alpha-1}}{|\Omega|}\sum_{i=1}^{n}|\gamma_i|I_{p_i}^{\mu_i}I_q^{\alpha}|f_x|(\xi_i)+I_q^{\alpha}|f_x|(t)\Bigg]\\
&\leq \frac{T^{\alpha-1}}{|\Omega|}\sum_{j=1}^{m}\left|\beta_j\right|J^{\sigma_j}I_q^{\alpha}(|f_x-f_0|+|f_0|)\left(\eta_j\right) + I_q^{\alpha}(|f_x-f_0|+|f_0|)(T)+\frac{T^{\alpha-1}}{|\Omega|}\\
&\quad\cdot\sum_{i=1}^{n}|\gamma_i|I_{p_i}^{\mu_i}I_q^{\alpha}(|f_x-f_0|+|f_0|)(\xi_i)\\
&\leq (Lr+M)\left[\frac{T^{\alpha-1}}{|\Omega|}\sum_{j=1}^{m}\left|\beta_j\right|J^{\sigma_j}I_q^{\alpha}(1)\left(\eta_j\right) + \frac{T^{\alpha-1}}{|\Omega|}\sum_{i=1}^{n}|\gamma_i|I_{p_i}^{\mu_i}I_q^{\alpha}(1)(\xi_i)+I_q^{\alpha}(1)(T)\right],\end{aligned} \tag{40}$$

in which we used the following fact:

$$\begin{aligned}|f_x-f_0|+|f_0| &= |f(\nu,x(\nu))-f(\nu,0)|+|f(\nu,0)|\\ &\leq L|x|+M\leq Lr+M,\end{aligned} \tag{41}$$

where $\nu\in\{T,\xi_i,\eta_j\}$. By applying Lemmas 4 and 2.3, we have

$$\begin{aligned}I_{p_i}^{\mu_i}I_q^{\alpha}(1)(\xi_i) &= \frac{\Gamma_{p_i}(\alpha+1)}{\Gamma_{p_i}(\alpha+\mu_i+1)\Gamma_q(\alpha+1)}\xi_i^{\alpha+\mu_i},\\
J^{\sigma_j}I_q^{\alpha}(1)\left(\eta_j\right) &= \frac{1}{\Gamma_q(\alpha+1)}\left(J^{\sigma_j}t^{\alpha}\right)\left(\eta_j\right) = \frac{1}{\Gamma_q(\alpha+1)}\alpha^{-\sigma_j}\eta_j^{\alpha}.\end{aligned} \tag{42}$$

Then, we obtain

$$\begin{aligned}|\mathcal{Q}x(t)| &\leq \frac{(Lr+M)T^{\alpha}}{\Gamma_q(\alpha+1)}\left[\frac{1}{T|\Omega|}\sum_{j=1}^{m}|\beta_j|\alpha^{-\sigma_j}\eta_j^{\alpha} + \frac{1}{T|\Omega|}\sum_{i=1}^{n}|\gamma_i|\frac{\Gamma_{p_i}(\alpha+1)}{\Gamma_{p_i}(\alpha+\mu_i+1)}\xi_i^{\alpha+\mu_i}+1\right]\\ &= (Lr+M)\Phi\leq r.\end{aligned} \tag{43}$$

From this, we conclude that $\|\mathcal{Q}x\|\leq r$ which yields $\mathcal{Q}B_r\subset B_r$.

Next, we will prove that the operator $\mathcal{Q}$ is a contraction. Let $x, y\in\mathcal{C}$, and for each $t\in[0,T]$, then, we have

$$\begin{aligned}|\mathcal{Q}x(t)-\mathcal{Q}y(t)| &\leq \frac{T^{\alpha-1}}{|\Omega|}\sum_{j=1}^{m}|\beta_j|J^{\sigma_j}I_q^{\alpha}|f_x-f_y|\left(\eta_j\right)+\frac{T^{\alpha-1}}{|\Omega|}\\
&\quad\cdot\sum_{i=1}^{n}|\gamma_i|I_{p_i}^{\mu_i}I_q^{\alpha}|f_x-f_y|(\xi_i)+I_q^{\alpha}|f_x-f_y|(t)\\
&\leq \left(\frac{T^{\alpha-1}}{|\Omega|}\sum_{j=1}^{m}|\beta_j|J^{\sigma_j}I_q^{\alpha}(1)\left(\eta_j\right)+\frac{T^{\alpha-1}}{|\Omega|}\cdot\sum_{i=1}^{n}|\gamma_i|I_{p_i}^{\mu_i}I_q^{\alpha}(1)(\xi_i)+I_q^{\alpha}(1)(T)\right)\\
&\quad\cdot L\|x-y\| = L\Phi\|x-y\|.\end{aligned} \tag{44}$$

Hence, we get the result that $\|\mathcal{Q}x-\mathcal{Q}y\|\leq L\Phi\|x-y\|$. As $L\Phi<1$, from (37), the operator $\mathcal{Q}$ is a contraction. Applying the well known Banach fixed point theorem, it follows that $\mathcal{Q}$ has a fixed point which is the unique solution of the boundary value problem (1). This completes the proof.

Next, the nonlinear contraction theorem will be used to prove a second existence and uniqueness result.

Definition 10. Let E be a Banach space and let $\mathcal{A}: E\longrightarrow E$ be a mapping. The operator $\mathcal{A}$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ such that $\Psi(0)=0$ and $\Psi(t)<t$ for all $t>0$ with the property:

$$\|\mathcal{A}x-\mathcal{A}y\|\leq\Psi(\|x-y\|), \forall x,y\in E. \tag{45}$$

Lemma 11 (see [28]). *Let E be a Banach space and let $\mathcal{A}: E\longrightarrow E$ be a nonlinear contraction. Then, $\mathcal{A}$ has a unique fixed point in E.*

Theorem 12. *Suppose that a continuous function $f:[0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ satisfies the condition:*

$(H_2)|f(t,x)-f(t,y)| \le h(t)(|x-y|/H^* + |x-y|)$, $t \in [0, T]$, $x, y \in \mathbb{R}$, *where the function* $h : [0,T] \longrightarrow \mathbb{R}^+$ *is continuous, and a positive constant* H^* *is defined by*

$$H^* = \frac{T^{\alpha-1}}{|\Omega|}\sum_{j=1}^{m}\left|\beta_j\right| J^{\sigma_j} I_q^{\alpha} h\left(\eta_j\right) + \frac{T^{\alpha-1}}{|\Omega|}\sum_{i=1}^{n}|\gamma_i| I_{p_i}^{\mu_i} I_q^{\alpha} h(\xi_i) + I_q^{\alpha} h(T). \tag{46}$$

Then, the mixed fractional Hadamard and quantum integrals nonlocal problem (1) has a unique solution on $[0,T]$.

Proof. Let us consider the operator $\mathcal{Q} : \mathscr{C} \longrightarrow \mathscr{C}$ defined in (34) and define a continuous nondecreasing function $\Psi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by

$$\Psi(\lambda) = \frac{H^* \lambda}{H^* + \lambda}, \quad \forall \lambda \ge 0. \tag{47}$$

Then, we see that the function Ψ satisfies $\Psi(0) = 0$ and $\Psi(\lambda) < \lambda$ for all $\lambda > 0$.

Next, for any $x, y \in \mathscr{C}$ and for each $t \in [0,T]$, we obtain

$$\begin{aligned}
|Qx(t) - Qy(t)| &\le \frac{t^{\alpha-1}}{|\Omega|}\sum_{j=1}^{m}\left|\beta_j\right| J^{\sigma_j} I_q^{\alpha}\left|f_x - f_y\right|\left(\eta_j\right) \\
&\quad + \frac{t^{\alpha-1}}{|\Omega|}\sum_{i=1}^{n}|\gamma_i| I_{p_i}^{\mu_i} I_q^{\alpha}\left|f_x - f_y\right|(\xi_i) + I_q^{\alpha}\left|f_x - f_y\right|(t) \\
&\le \frac{T^{\alpha-1}}{|\Omega|}\sum_{j=1}^{m}\left|\beta_j\right| J^{\sigma_j} I_q^{\alpha}\left(h\frac{|x-y|}{H^* + |x-y|}\right)\left(\eta_j\right) \\
&\quad + \frac{T^{\alpha-1}}{|\Omega|}\sum_{i=1}^{n}|\gamma_i| I_{p_i}^{\mu_i} I_q^{\alpha}\left(h\frac{|x-y|}{H^* + |x-y|}\right)(\xi_i) \\
&\quad + I_q^{\alpha}\left(h\frac{|x-y|}{H^* + |x-y|}\right)(T) \le \frac{\Psi(\|x-y\|)}{H^*} \\
&\quad \cdot \left[\frac{T^{\alpha-1}}{|\Omega|}\sum_{j=1}^{m}\left|\beta_j\right| J^{\sigma_j} I_q^{\alpha} h\left(\eta_j\right) + \frac{T^{\alpha-1}}{|\Omega|}\sum_{i=1}^{n}\right. \\
&\quad \left. \cdot |\gamma_i| I_{p_i}^{\mu_i} I_q^{\alpha} h(\xi_i) + I_q^{\alpha} h(T)\right] = \Psi(\|x-y\|),
\end{aligned} \tag{48}$$

which implies that $\|Qx - Qy\| \le \Psi(\|x-y\|)$ and also satisfies Definition 10. Therefore, $\mathcal{Q}$ is a nonlinear contraction. Thus, by applying Lemma 11, the operator $\mathcal{Q}$ has a unique fixed point which is the unique solution of the boundary value problem (1). The proof is finished.

Next, the first existence result will be obtained by applying the following theorem.

Theorem 13 (Nonlinear alternative for single valued maps) [29]. *Let E be a Banach space, C a closed, convex subset of E, U be an open subset of C, and* $0 \in U$. *Suppose that* $\mathscr{A} : \bar{U} \longrightarrow C$ *is a continuous, compact (that is,* $\mathscr{A}(\bar{U})$ *is a relatively compact subset of C) map. Then, either*

(i) $\mathscr{A}$ *has a fixed point in* $\bar{U}$, *or*

(ii) *There is a* $x \in \partial U$ *(the boundary of U in C) and* $\lambda \in (0,1)$ *with* $x = \lambda \mathscr{A}(x)$.

Theorem 14. *Suppose that* $f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is a nonlinear continuous function which satisfies the following conditions:*

(H_3) *there exists a continuous nondecreasing function* $\psi : [0,\infty) \longrightarrow (0,\infty)$ *and also a function* $p \in C([0,T], \mathbb{R}^+)$ *such that*

$$|f(t,x)| \le p(t)\psi(|x|) \text{for each} (t,x) \in [0,T] \times \mathbb{R} \tag{49}$$

(H_4) *there exists a positive constant N such that*

$$\frac{N}{\psi(N)\|p\|\Phi} > 1, \tag{50}$$

where Φ *defined by (38). Then, the problem (1) has at least one solution on* $[0,T]$.

Proof. For a positive number ρ, we let $B_\rho = \{x \in \mathscr{C} : \|x\| \le \rho\}$ be a bounded ball in $\mathscr{C}$. Now, we will prove that the set $\mathcal{Q}B_\rho$ is uniformly bounded. For $t \in [0,T]$, we can compute that

$$\begin{aligned}
|Qx(t)| &\le \frac{T^{\alpha-1}}{|\Omega|}\left[\sum_{j=1}^{m}\left|\beta_j\right| J^{\sigma_j} I_q^{\alpha}|f_x|\left(\eta_j\right) + \sum_{i=1}^{n}|\gamma_i| I_{p_i}^{\mu_i} I_q^{\alpha} \mid f_x \mid (\xi_i)\right] + I_q^{\alpha}|f_x|(T) \\
&\le \|p\|\psi(\|x\|)\frac{T^{\alpha}}{\Gamma_q(\alpha+1)}\left[\frac{1}{T|\Omega|}\sum_{j=1}^{m}\left|\beta_j\right|\alpha^{-\sigma_j}\eta_j + \frac{1}{T|\Omega|}\sum_{i=1}^{n}|\gamma_i|\frac{\Gamma_{p_i}(\alpha+1)}{\Gamma_{p_i}(\alpha+\mu_i+1)}\xi_i^{\alpha+\mu_i} + 1\right] \\
&\le \|p\|\psi(\rho)\Phi,
\end{aligned} \tag{51}$$

which can be deduced that

$$\|\mathcal{Q}x\| \le \|p\|\psi(\rho)\Phi. \tag{52}$$

Then, the set $\mathcal{Q}B_\rho$ is uniformly bounded. Next, we will show that the set $\mathcal{Q}B_\rho$ is equicontinuous set of $\mathscr{C}$.

For any two points $\tau_1, \tau_2 \in [0, T]$ with $\tau_1 < \tau_2$ and $x \in B_\rho$, we have

$$\begin{aligned}
&|Qx(\tau_2) - Qx(\tau_1)| \\
&\le \frac{\left|\tau_2^{\alpha-1} - \tau_1^{\alpha-1}\right|}{|\Omega|}\left[\sum_{j=1}^{m}\left|\beta_j\right| J^{\sigma_j} I_q^{\alpha}|f_x|\left(\eta_j\right) + \sum_{i=1}^{n}|\gamma_i| I_{p_i}^{\mu_i} I_q^{\alpha}|f_x|(\xi_i)\right] + \left|\frac{1}{\Gamma_q(\alpha)}\int_0^{\tau_1}\left[(t_2 - qs)_q^{(\alpha-1)} - (t_1 - s)_q^{(\alpha-1)}\right] f(s, x(s))\, d_q s + \frac{1}{\Gamma_q(\alpha)} \cdot \int_{\tau_1}^{\tau_2}(t_2 - qs)_q^{(\alpha-1)} f(s, x(s))\, d_q s\right| \\
&\le \frac{\|p\|\psi(\rho)\left|\tau_2^{\alpha-1} - \tau_1^{\alpha-1}\right|}{|\Omega|}\sum_{j=1}^{m}|\beta_j|\frac{\alpha^{-\sigma_j}\eta_j^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{\|p\|\psi(\rho)\left|\tau_2^{\alpha-1} - \tau_1^{\alpha-1}\right|}{|\Omega|}\sum_{i=1}^{n}|\gamma_i| \cdot \frac{\Gamma_{p_i}(\alpha+1)}{\Gamma_{p_i}(\alpha+\mu_i+1)\Gamma_q(\alpha+1)}\xi_i^{\alpha+\mu_i} \\
&\quad \cdot + \frac{\|p\|\psi(\rho)}{\Gamma_q(\alpha+1)}\left[2(\tau_2 - \tau_1)^{(\alpha)} + \left|\tau_2^{(\alpha)} - \tau_1^{(\alpha)}\right|\right].
\end{aligned} \tag{53}$$

As $\tau_2 - \tau_1 \longrightarrow 0$, the right hand side of the above inequality converses to zero, independently of $x \in B_\rho$. Then, the set $\mathcal{Q}B_\rho$ is equicontinuous. Thus, we conclude that the set $\mathcal{Q}B_\rho$ is relatively compact. Therefore, by the Arzel a'-Ascoli theorem, the operator $\mathcal{Q} : \mathcal{C} \longrightarrow \mathcal{C}$ is completely continuous.

Finally, we show that the operator $\mathcal{Q}$ cannot be fulfilled the condition (ii) in Theorem 13. Then, we have to claim that there exists an open set $U \subset B_\rho$ with $x \ne \lambda \mathcal{Q}x$ for $\lambda \in (0, 1)$ and $x \in \partial U$. Then, for each $t \in [0, T]$, we apply the computation in the first step, that is

$$|x(t)| \le \|p\|\psi(\|x\|)\Phi \tag{54}$$

which yields inequality

$$\frac{\|x\|}{\|p\|\psi(\|x\|)\Phi} \le 1. \tag{55}$$

The condition (H_4) implies that there exists a constant N such that $\|x\| \ne N$. Now, we define the set

$$U = \left\{x \in B_\rho : \|x\| < N\right\}. \tag{56}$$

From the previous results, we obtain that the operator $\mathcal{Q} \sim \bar{U} \longrightarrow \mathcal{C}$ is continuous and completely continuous. Then, there is no $x \in \partial U$ such that $x = \lambda Qx$ for some $\lambda \in (0, 1)$. By applying the nonlinear alternative of the Leray-Schauder type, we get that the operator $\mathcal{Q}$ has a fixed point $x \in \bar{U}$ which is a solution of the nonlinear fractional q-difference equation with fractional Hadamard and quantum integral nonlocal conditions. This finishes the proof.

The next existence result is based on Krasnosel'ski i's fixed point theorem which can be used to relax the condition in Theorem 9.

Theorem 15 (Krasnosel'ski i's fixed point theorem) [30]. *Let C be a closed, bounded, convex, and nonempty subset of a Banach space E. Let $\mathcal{A}, \mathcal{B}$ be the operators such that (a) $\mathcal{A}x + \mathcal{B}y \in C$ whenever $x, y \in C$; (b) $\mathcal{A}$ is compact and continuous; (c) $\mathcal{B}$ is a contraction mapping. Then, there exists $z \in C$ such that $z = \mathcal{A}z + \mathcal{B}z$.*

Theorem 16. *Assume that a continuous function $f : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is satisfied condition (H_1) in Theorem 9 and is bounded as the following condition:*

(i) $(H_5)|f(t, x)| \le \kappa(t), \quad \forall (t, x) \in [0, T] \times \mathbb{R}$, and $\kappa \in C([0, T], \mathbb{R}^+)$.

If inequality

$$\frac{LT^\alpha}{\Gamma_q(\alpha+1)} < 1 \tag{57}$$

holds, then the nonlocal problem (1) has at least one solution on $[0, T]$.

Proof. Now, we define $\sup\{|\kappa(t)| : t \in [0, T]\} = \|\kappa\|$ and choose a positive constant $\bar{r}$ such that

$$\bar{r} \ge \|\kappa\|\Phi, \tag{58}$$

where Φ is defined by (38), to be a radius of the ball $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \le \bar{r}\}$. Furthermore, we set the operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on $B_{\bar{r}}$ as $\mathcal{A}$ and $\mathcal{B}$ in Theorem 15, respectively, by

$$\begin{aligned}
\mathcal{Q}_1 x(t) &= \frac{t^{\alpha-1}}{\Omega}\left[\sum_{j=1}^{m}\beta_j J^{\sigma_j} I_q^{\alpha} f_x\left(\eta_j\right) - \sum_{i=1}^{n}\gamma_i I_{p_i}^{\mu_i} I_q^{\alpha} f_x(\xi_i)\right], t \in [0, T], \\
\mathcal{Q}_2 x(t) &= I_q^{\alpha} f_x(t) t \in [0, T].
\end{aligned} \tag{59}$$

The combination of two operators shows $\mathcal{Q}_1 + \mathcal{Q}_2 = \mathcal{Q}$. We have

$$\|\mathcal{Q}_1 x + \mathcal{Q}_2 y\| \leq \|\kappa\| \left[\frac{T^{\alpha-1}}{|\Omega|} \sum_{j=1}^{m} \left|\beta_j\right| \frac{\alpha^{-\sigma_j}\eta_j^{\alpha}}{\Gamma_q(\alpha+1)} + \frac{T^{\alpha-1}}{|\Omega|} \sum_{i=1}^{n} |\gamma_i| \frac{\Gamma_{p_i}(\alpha+1)}{\Gamma_{p_i}(\alpha+\mu_i+1)\Gamma_q(\alpha+1)} \xi_i^{\alpha+\mu_i} + \frac{T^{\alpha}}{\Gamma_q(\alpha+1)} \right] = \|\kappa\| \Phi \leq \bar{r}. \tag{60}$$

Therefore, we have $\mathcal{Q}_1 x + \mathcal{Q}_2 y \in B_{\bar{r}}$, and thus condition (a) of Theorem 15 is satisfied. Since the function f is fulfilled by condition (H_1) in Theorem 9, then the operator $\mathcal{Q}_2$ is a contraction mapping with inequality (57).

Finally, we will show that the operator $\mathcal{Q}_1$ should satisfy condition (b) in Theorem 15. Using the continuity of f, we can show that the operator $\mathcal{Q}_1$ is continuous. The uniformly boundedness of the set $\mathcal{Q}_1 B_{\bar{r}}$ can be shown by

$$\|\mathcal{Q}_1 x\| \leq \frac{\|\kappa\| T^{\alpha-1}}{|\Omega| \, \Gamma_q(\alpha+1)} \left[\sum_{j=1}^{m} \left|\beta_j\right| \alpha^{-\sigma_j}\eta_j^{\alpha} + \sum_{i=1}^{n} |\gamma_i| \frac{\Gamma_{p_i}(\alpha+1)}{\Gamma_{p_i}(\alpha+\mu_i+1)} \xi_i^{\alpha+\mu_i} \right]. \tag{61}$$

To prove $\mathcal{Q}_1 B_{\bar{r}}$ is equicontinuous set, we let two points $t_1, t_2 \in [0, T]$, $t_2 < t_1$. For any $x \in B_{\bar{r}}$, we have

$$\begin{aligned} &|\mathcal{Q}_1 x(t_1) - \mathcal{Q}_1 x(t_2)| \\ &\leq \|\kappa\| \frac{\left|t_1^{\alpha-1} - t_2^{\alpha-1}\right|}{|\Omega|} \sum_{j=1}^{m} \left|\beta_j\right| \frac{\alpha^{-\sigma_j}\eta_j^{\alpha}}{\Gamma_q(\alpha+1)} \\ &\quad + \|\kappa\| \frac{\left|t_1^{\alpha-1} - t_2^{\alpha-1}\right|}{|\Omega|} \sum_{i=1}^{n} |\gamma_i| \frac{\Gamma_{p_i}(\alpha+1)}{\Gamma_{p_i}(\alpha+\mu_i+1)\Gamma_q(\alpha+1)} \xi_i^{\alpha+\mu_i}, \end{aligned} \tag{62}$$

which converses to zero independently of x as $|t_1 - t_2| \longrightarrow 0$. So, $\mathcal{Q}_1 \mathrm{B}_{\bar{r}}$ is an equicontinuous set. Therefore, $\mathcal{Q}_1 B_{\bar{r}}$ is a relative compact and by the Arzelá-Ascoli theorem, $\mathcal{Q}_1$ is compact on $B_{\bar{r}}$. Thus, the assumptions $(a), (b)$, and (c) of Krasnosel'ski i's fixed point theorem are satisfied. Then, the nonlinear fractional q-difference equation with fractional Hadamard and quantum integral nonlocal conditions (1) has at least one solution on $[0, T]$. The proof is completed.

Remark 17. The interchanging of operators $\mathcal{Q}_1$ and $\mathcal{Q}_2$ gives another result by replacing inequality (57) by the following condition:

$$\frac{L T^{\alpha-1}}{|\Omega| \Gamma_q(\alpha+1)} \left[\sum_{j=1}^{m} \left|\beta_j\right| \alpha^{-\sigma_j}\eta_j^{\alpha} + \sum_{i=1}^{n} |\gamma_i| \frac{\Gamma_{p_i}(\alpha+1)}{\Gamma_{p_i}(\alpha+\mu_i+1)} \xi_i^{\alpha+\mu_i} \right] < 1. \tag{63}$$

4. Examples

Example 18. Consider the nonlinear fractional q -difference equation with fractional Hadamard and quantum integral nonlocal conditions of the form:

$$\begin{cases} D_{1/2}^{3/2} x(t) = f(t, x(t)), \, t \in (0, 2), \\ x(0) = 0, \, \dfrac{3}{8} I_{1/6}^{1/2} x\left(\dfrac{1}{4}\right) + \dfrac{2}{5} I_{1/3}^{3/2} x\left(\dfrac{1}{2}\right) + \dfrac{1}{9} I_{1/2}^{5/2} x\left(\dfrac{3}{2}\right) \\ = \dfrac{1}{3} J^{1/3} x\left(\dfrac{1}{5}\right) + \dfrac{4}{9} J^{2/3} x\left(\dfrac{3}{5}\right) + \dfrac{7}{12} J^{4/3} x\left(\dfrac{7}{5}\right) + \dfrac{8}{15} J^{5/3} x\left(\dfrac{9}{5}\right). \end{cases} \tag{64}$$

Here, $\alpha = 3/2$, $q = 1/2$, $T = 2$, $\gamma_1 = 3/8$, $\gamma_2 = 2/5$, $\gamma_3 = 1/9$, $\mu_1 = 1/2$, $\mu_2 = 3/2$, $\mu_3 = 5/2$, $p_1 = 1/6$, $p_2 = 1/3$, $p_3 = 1/2$, $\xi_1 = 1/4$, $\xi_2 = 1/2$, $\xi_3 = 3/2$, $n = 3$, $\beta_1 = 1/3$, $\beta_2 = 4/9$, $\beta_3 = 7/12$, $\beta_4 = 8/15$, $\sigma_1 = 1/3$, $\sigma_2 = 2/3$, $\sigma_3 = 4/3$, $\sigma_4 = 5/3$, $\eta_1 = 1/5$, $\eta_2 = 3/5$, $\eta_3 = 7/5$, $\eta_4 = 9/5$, $m = 4$. Then, we can compute constants as $|\Omega| \approx 2.51852$ and $\Phi \approx 3.27524$.

(i) Let the nonlinear function f be defined by

$$f(t, x) = \frac{e^{-\cos^2 t}}{(t+2)^3} \left(\frac{x^2 + 2\,|\,x\,|}{1 + |x\,|} \right) + \frac{t^2}{4} + 1. \tag{65}$$

Then, by direct computation, we get $|f(t, x) - f(t, y)| \leq (1/4)|x - y|$, which satisfies condition (H_1) in Theorem 9 with $L = 1/4$. Therefore, we have

$$L\Phi \approx 0.81881 < 1. \tag{66}$$

By the conclusion of Theorem 9, the boundary value problem (64) with (65) has a unique solution on $[0, 2]$.

(ii) Consider now the function f by

$$f(t, x) = \frac{1}{(t+2)^3} \left(\frac{x^{18}}{x^{16} + 1} + 1 \right). \tag{67}$$

Then, we can see that

$$|f(t, x)| = \left| \frac{1}{(t+2)^3} \left(\frac{x^{18}}{x^{16} + 1} + 1 \right) \right| \leq \frac{1}{(t+2)^3} \left(x^2 + 1 \right). \tag{68}$$

Setting $p(t) = 1/(t+2)^3$ and $\psi(x) = x^2 + 1$, we have $\|p\| = 1/8$, and there exists a constant $N \in (0.52019, 1.92238)$ satisfying inequality in (H_4). Hence, all assumptions in Theorem 14 are completed. Thus, the problem (64) with (67) has at least one solution on $[0, 2]$.

(iii) If the function f is

$$f(t, x) = \frac{\sin^2 t}{m} \left(\frac{|x|}{|x| + 1} \right) + \frac{1}{4}, \quad m \in \mathbb{R}^+, \tag{69}$$

then, we have $|f(t, x) - f(t, y)| \leq (1/m)|x - y|$ with $L = 1/m$. If $m \leq \Phi \approx 3.27524$, then Theorem 9 cannot be used to apply for the problem (64) with (69). For example, if $m = 2$, then

$L\Phi \approx 1.63762 > 1$. But the inequality in Remark 17 is satisfied as

$$\frac{1}{2} \cdot \frac{T^{\alpha-1}}{|\Omega|\Gamma_q(\alpha+1)} \left[\sum_{j=1}^{m} \left|\beta_j\right| \alpha^{-\sigma_j} \eta_j^{\alpha} + \sum_{i=1}^{n} |\gamma_i| \frac{\Gamma_{p_i}(\alpha+1)}{\Gamma_{p_i}(\alpha+\mu_i+1)} \xi_i^{\alpha+\mu_i} \right] = 0.44979 < 1. \tag{70}$$

Hence, by applying Theorem 16 and Remark 17, the problem (64) with (69) has at least one solution on $[0, 2]$.

5. Conclusion

We investigated the existence and uniqueness of solutions for a nonlocal boundary value problem involving a q-difference equation, supplemented with a new type of boundary condition, including both Hadamard fractional and quantum integrals. In our first two results, we establish the existence and uniqueness of solutions by using Banach's fixed point theorem and a fixed point theorem for nonlinear contractions due to Boyd and Wong. Then, we used the Leray-Schauder nonlinear alternative and Krasnosel'ski i's fixed point theorem to derive two existence results. Examples are also presented to illustrate our results. It is worthwhile to point out that the results presented in this paper are new and significantly contribute to the existing literature on the topic.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Research Article

A Novel Picture Fuzzy n-Banach Space with Some New Contractive Conditions and Their Fixed Point Results

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A picture fuzzy n-normed linear space (N_{PF}), a mixture of a picture fuzzy set and an n-normed linear space, is a proficient concept to cope with uncertain and unpredictable real-life problems. The purpose of this manuscript is to present some novel contractive conditions based on N_{PF}. By using these contractive conditions, we explore some fixed point theorems in a picture fuzzy n-Banach space (B_{PF}). The discussed modified results are more general than those in the existing literature which are based on an intuitionistic fuzzy n-Banach space (B_{IF}) and a fuzzy n-Banach space. To express the reliability and effectiveness of the main results, we present several examples to support our main theorems.

1. Introduction

In various real-life problems, for a suitable mapping, the existence of a solution and existence of a fixed point (FP) are equivalent. Thus, the existence of a FP is a proficient technique to cope with awkward and difficult problems in real-life issues. Various scholars have utilized such results in the environment of many fields [1, 2]. The extensive useful techniques capable with both algebraic and topological properties are those of a normed linear space (NLS), but the continuous maps are more proficient in the sense of NLS. Moreover, in a metric space, every contractive map is uniformly continuous. One of the fundamental applications of Banach's contraction principle is the "Picard's theorem," which is the basic theorem for the existence and uniqueness of solution to the ordinary differential equations. Various scholars have utilized this application in the environment of a partial differential equation [3], in the Gauss-Seidel method for evaluating systems of linear equations [4], in the proof of the inverse function theorem [5], etc.

The theory of a fuzzy set (FS) was investigated by Zadeh [6], characterized by only positive grades restricted to $[0, 1]$. FS has achieved more success due to its ability to cope with complications and troubles. However, in some practice cases, the concept of FS cannot cope with complications and uncertainty because of lack of knowledge of the problem. Therefore, Atanassov [7] investigated the intuitionistic FS (IFS) containing both positive and negative grades, whose sum is bounded to $[0, 1]$. IFS is regarded as a more improved way to cope with complex and awkward information. Further, Cường [8] investigated the picture FS (PFS) including positive, abstinence, and negative grades, whose sum is bounded to $[0, 1]$. PFS is regarded as a more improved way to deal with even more complex information. For more related works, we may refer to References [9–16].

Keeping the advantages of the PFS, the objective of this manuscript is summarized in the following ways:

(1) To present some novel contractive conditions, we used N_{PF} as a basis

(2) By using these contractive conditions, some fixed point theorems are explored for a picture fuzzy n-Banach space (B_{PF}). These results are more modified and more general than the existing results which are based on an intuitionistic fuzzy n-Banach space (B_{IF}) and a fuzzy n-Banach space

(3) To express the reliability and effectiveness of the explored approaches, we explain examples in support of the main results

The rest of this manuscript is summarized in the following ways: In Section 2, we review some basic notions like N_{IF} and their related properties used in the presented work. In Section 3, we describe the notion of N_{PF} and their fundamental properties. In Section 4, we present some novel contractive conditions based on N_{PF}. By using these contractive conditions, we instigate some fixed point theorems for a picture fuzzy n-Banach space (B_{PF}). Finally, the conclusion of this manuscript is discussed in Section 5.

2. Preliminaries

The purpose of this section is to review some existing notions, like N_{IF} and their related properties. Throughout this section, the symbols R^{+}_{Rn}, R_{Rn}, N_{Nn}, $\mathcal{X}_{UNI}$, M_m, N_n, $*_{ct}$, and $\circ_{ctc}$ represent the positive real numbers, real numbers, natural numbers, universal set, supporting grade, supporting against, continuous t-norm, and continuous t-conorm, respectively.

Definition 1. [9]. A N_{IF} is stated by $(\mathcal{X}_{UNI}, M_m, A_a, N_n, *_{ct}, \circ_{ctc})$, where M_m, A_a, N_n is defined on $(x_1, x_2, \cdots, x_k, p) \in \mathcal{X}^k_{UNI} \times (0,\infty)$, where the following conditions hold:

(i) $M_m(x_1, x_2, \cdots, x_k, p) + A_a(x_1, x_2, \cdots, x_k, p) + N_n(x_1, x_2, \cdots, x_k, p) \le 1$

(ii) $M_m(x_1, x_2, \cdots, x_k, p) > 0$

(iii) $M_m(x_1, x_2, \cdots, x_k, p) = 1$ iff $x_1, x_2, \cdots, x_k$ are linearly dependent

(iv) $M_m(x_1, x_2, \cdots, x_k, p)$ is invariant under any permutation of $x_1, x_2, \cdots, x_k$

(v) $M_m(x_1, x_2, \cdots, \alpha x_k, p) = M_m(x_1, x_2, \cdots, x_k, p/|\alpha|)$ if $\alpha \neq 0 \in \mathbb{R}_{Rn}$

(vi) $M_m(x_1, x_2, \cdots, x_k + x'_k, p + q) \ge \min(M_m(x_1, x_2, \cdots, x_k, p), M_m(x_1, x_2, \cdots, x'_k, q))$

(vii) $M_m(x_1, x_2, \cdots, x_k + x_k, .)$ is a nondecreasing function of $\mathbb{R}^+_{Rn}$ and $\log_{p\to\infty} M_m(x_1, x_2, \cdots, x_k + x_k, p) = 1$

(viii) $N_n(x_1, x_2, \cdots, x_k, p) < 1$

(ix) $N_n(x_1, x_2, \cdots, x_k, p) = 0$ iff $x_1, x_2, \cdots, x_k$ are linearly dependent

(x) $N_n(x_1, x_2, \cdots, x_k, p)$ is invariant under any permutation of $x_1, x_2, \cdots, x_k$

(xi) $N_n(x_1, x_2, \cdots, \alpha x_k, p) = N_n(x_1, x_2, \cdots, x_k, p/|\alpha|)$ if $\alpha \neq 0 \in \mathbb{R}_{Rn}$

(xii) $N_n(x_1, x_2, \cdots, x_k + x'_k, p + q) \ge \max(N_n(x_1, x_2, \cdots, x_k, p), N_n(x_1, x_2, \cdots, x'_k, q))$

(xiii) $N_n(x_1, x_2, \cdots, x_k + x_k, .)$ is a nonincreasing function of $\mathbb{R}^+_{Rn}$ and $\log_{p\to\infty} N_n(x_1, x_2, \cdots, x_k + x_k, p) = 0$

(xiv) Further, $M_n(x_1, x_2, \cdots, x_k, p) > 0$ and $N_n(x_1, x_2, \cdots, x_k, p) < 1$ imply $x = 0, \forall p > 0$

(xv) For $p \neq 0, M_n(x_1, x_2, \cdots, x_k, .)$ and $N_n(x_1, x_2, \cdots, x_k, .)$ are continuous functions of $\mathbb{R}^+_{Rn}$ and are strictly increasing and strictly decreasing, respectively, on the subset $\{p : 0 < M_m(x_1, x_2, \cdots, x_k, p), N_n(x_1, x_2, \cdots, x_k, p) < 1\}$ of $\mathbb{R}^+_{Rn}$

Moreover, we explain some important theories based on convergent and Cauchy convergent sequences.

Definition 2. [9]. Consider $N_{IF}(\mathcal{X}_{UNI}, M_m, N_n, *_{ct}, \circ_{ctc})$; then, the sequence $x = \{x_{\mathfrak{f}}\}$ in $\mathcal{X}_{UNI}$ is convergent to $\boldsymbol{g} \in \mathcal{X}_{UNI}$ based on the intuitionistic fuzzy n-norm $(M_m, N_n)^k$ if for every $\epsilon, p > 0$ and $\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1} \in \mathcal{X}_{UNI}$, there exists $\tilde{\omega}_0 \in \mathbb{N}_{Nn}$ such that

$$\begin{aligned} M_n(\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1}, x_{\tilde{\omega}} - \boldsymbol{g}, p) > 1 - \epsilon, \\ N_n(\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1}, x_{\tilde{\omega}} - \boldsymbol{g}, p) < \epsilon, \end{aligned} \tag{1}$$

for all $\tilde{\omega} \ge \tilde{\omega}_0$ and it is represented by $(M_m, N_n)^k - \lim x_{\tilde{\omega}} = \boldsymbol{g}$.

Definition 3. [9]. Let $N_{IF}(\mathcal{X}_{UNI}, M_m, N_n, *_{ct}, \circ_{ctc})$; then, the sequence $x = \{x_{\mathfrak{f}}\}$ in $\mathcal{X}_{UNI}$ is Cauchy convergent based on the intuitionistic fuzzy n-norm $(M_m, N_n)^k$ if for every ϵ, $p > 0$ and $\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1} \in \mathcal{X}_{UNI}$, there exists $\tilde{\omega}_0 \in \mathbb{N}_{Nn}$ such that

$$\begin{aligned} M_n(\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1}, x_{\tilde{\omega}} - x_{\Upsilon}, p) > 1 - \epsilon, \\ N_n(\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1}, x_{\tilde{\omega}} - x_{\Upsilon}, p) < \epsilon, \end{aligned} \tag{2}$$

for all $\tilde{\omega}, \Upsilon \ge \tilde{\omega}_0$ and it is represented by $(M_m, N_n)^k - \lim x_{\tilde{\omega}} = x_{\Upsilon}$.

3. Picture Fuzzy n-Normed Linear Space

The purpose of this section is to explore some new approaches like N_{PF} and their related properties, which are extensively efficient for the proof of our main work in the next section. Throughout this section, the symbols $\mathcal{X}_{UNI}$, M_m, A_a, N_n, $*_{ct}$, and $\circ_{ctc}$ represented the universal set, supporting grade, abstinence grade, supporting against, continuous t-norm, and continuous t-conorm, respectively.

Definition 4. A N_{PF} is stated as $(\mathcal{X}_{UNI}, M_m, A_a, N_n, *_{ct}, \circ_{ctc})$, where M_m, A_a, N_n is defined on $(x_1, x_2, \cdots, x_k, p) \in \mathcal{X}^k_{UNI} \times (0,\infty)$, where the following conditions hold:

(i) $M_m(x_1, x_2, \cdots, x_k, p) + A_a(x_1, x_2, \cdots, x_k, p) + N_n(x_1, x_2, \cdots, x_k, p) \leq 1$

(ii) $M_m(x_1, x_2, \cdots, x_k, p) > 0$

(iii) $M_m(x_1, x_2, \cdots, x_k, p) = 1$ iff $x_1, x_2, \cdots, x_k$ are linearly dependent

(iv) $M_m(x_1, x_2, \cdots, x_k, p)$ is invariant under any permutation of $x_1, x_2, \cdots, x_k$

(v) $M_m(x_1, x_2, \cdots, \alpha x_k, p) = M_m(x_1, x_2, \cdots, x_k, p/|\alpha|)$ if $\alpha \neq 0 \in \mathbb{R}_{Rn}$

(vi) $M_m(x_1, x_2, \cdots, x_k + x_k', p + q) \geq \min(M_m(x_1, x_2, \cdots, x_k, p), M_m(x_1, x_2, \cdots, x_k', q))$

(vii) $M_m(x_1, x_2, \cdots, x_k + x_k, .)$ is a nondecreasing function of $\mathbb{R}^+_{Rn}$ and $\log_{p\to\infty} M_m(x_1, x_2, \cdots, x_k + x_k, p) = 1$

(viii) $A_a(x_1, x_2, \cdots, x_k, p) < 1$

(ix) $A_a(x_1, x_2, \cdots, x_k, p) = 0$ iff $x_1, x_2, \cdots, x_k$ are linearly dependent

(x) $A_a(x_1, x_2, \cdots, x_k, p)$ is invariant under any permutation of $x_1, x_2, \cdots, x_k$

(xi) $A_a(x_1, x_2, \cdots, \alpha x_k, p) = A_a(x_1, x_2, \cdots, x_k, p/|\alpha|)$ if $\alpha \neq 0 \in \mathbb{R}_{Rn}$

(xii) $A_a(x_1, x_2, \cdots, x_k + x_k', p + q) \geq \max(A_a(x_1, x_2, \cdots, x_k, p), A_a(x_1, x_2, \cdots, x_k', q))$

(xiii) $A_a(x_1, x_2, \cdots, x_k + x_k, .)$ is a nonincreasing function of $\mathbb{R}^+_{Rn}$ and $\log_{p\to\infty} A_a(x_1, x_2, \cdots, x_k + x_k, p) = 0$

(xiv) $N_n(x_1, x_2, \cdots, x_k, p) < 1$

(xv) $N_n(x_1, x_2, \cdots, x_k, p) = 0$ iff $x_1, x_2, \cdots, x_k$ are linearly dependent

(xvi) $N_n(x_1, x_2, \cdots, x_k, p)$ is invariant under any permutation of $x_1, x_2, \cdots, x_k$

(xvii) $N_n(x_1, x_2, \cdots, \alpha x_k, p) = N_n(x_1, x_2, \cdots, x_k, p/|\alpha|)$ if $\alpha \neq 0 \in \mathbb{R}_{Rn}$

(xviii) $N_n(x_1, x_2, \cdots, x_k + x_k', p + q) \geq \max(N_n(x_1, x_2, \cdots, x_k, p), N_n(x_1, x_2, \cdots, x_k', q))$

(xix) $(x_1, x_2, \cdots, x_k + x_k, .)$ is a nonincreasing function of $\mathbb{R}^+_{Rn}$ and $\log_{p\to\infty} N_n(x_1, x_2, \cdots, x_k + x_k, p) = 0$

(xx) Further, $M_n(x_1, x_2, \cdots, x_k, p) > 0, A_n(x_1, x_2, \cdots, x_k, p) < 1$ and $N_n(x_1, x_2, \cdots, x_k, p) < 1$; then, $x = 0$, $\forall p > 0$

(xxi) For $p \neq 0$, $M_n(x_1, x_2, \cdots, x_k, .)$, $A_n(x_1, x_2, \cdots, x_k, .)$, and $N_n(x_1, x_2, \cdots, x_k, .)$ are continuous functions of $\mathbb{R}^+_{Rn}$ and also strictly increasing and strictly decreasing, respectively, on the subset $\{p : 0 < M_m(x_1, x_2, \cdots, x_k, p), A_a(x_1, x_2, \cdots, x_k, p), N_n(x_1, x_2, \cdots, x_k, p) < 1\}$ of $\mathbb{R}^+_{Rn}$

Moreover, we explain some important theories based on convergent and Cauchy convergent sequences.

Definition 5. For a $N_{PF}(\mathcal{X}_{UNI}, M_m, A_a, N_n, *_{ct}, \circ_{ctc})$, the sequence $x = \{x_\mathfrak{f}\}$ in $\mathcal{X}_{UNI}$ is convergent to $\mathscr{g} \in \mathcal{X}_{UNI}$ based on the picture fuzzy n-norm $(M_m, A_a, N_n)^k$ if for every $\epsilon, p > 0$ and $\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1} \in \mathcal{X}_{UNI}$, there exists $\tilde{\omega}_0 \in \mathbb{N}_{Nn}$ such that

$$\begin{aligned} &M_n(\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1}, x_{\tilde{\omega}} - \mathscr{g}, p) > 1 - \epsilon, \\ &A_n(\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1}, x_{\tilde{\omega}} - \mathscr{g}, p) < \epsilon, \\ &N_n(\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1}, x_{\tilde{\omega}} - \mathscr{g}, p) < \epsilon, \end{aligned} \tag{3}$$

for all $\tilde{\omega} \geq \tilde{\omega}_0$ and it is represented by $(M_m, A_a, N_n)^k - \lim x_{\tilde{\omega}} = \mathscr{g}$.

Definition 6. For a $N_{PF}(\mathcal{X}_{UNI}, M_m, A_a, N_n, *_{ct}, \circ_{ctc})$, the sequence $x = \{x_\mathfrak{f}\}$ in $\mathcal{X}_{UNI}$ is Cauchy convergent based on the picture fuzzy n-norm $(M_m, A_a, N_n)^k$ if for every $\epsilon, p > 0$ and $\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1} \in \mathcal{X}_{UNI}$, there exists $\tilde{\omega}_0 \in \mathbb{N}_{Nn}$ such that

$$\begin{aligned} &M_n(\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1}, x_{\tilde{\omega}} - x_{\Upsilon}, p) > 1 - \epsilon, \\ &A_n(\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1}, x_{\tilde{\omega}} - x_{\Upsilon}, p) < \epsilon, \\ &N_n(\ddot{y}_1, \ddot{y}_2, \cdots, \ddot{y}_{k-1}, x_{\tilde{\omega}} - x_{\Upsilon}, p) < \epsilon, \end{aligned} \tag{4}$$

for all $\tilde{\omega}, \Upsilon \geq \tilde{\omega}_0$ and it is represented by $(M_m, A_a, N_n)^k - \lim x_{\tilde{\omega}} = x_{\Upsilon}$.

Remark 7. The following assumptions are important for our main results.

(1) Suppose S_{m-1} is the set of functions $\Psi_{m-1} : [0,+\infty) \longrightarrow [0,+\infty)$ such that

(i) Ψ_{m-1} is continuous and nondecreasing

(ii) $\Psi_{m-1}(p) = 0 \Leftrightarrow p = 0$

(2) Suppose S_{a-2}, S_{n-3} is the set of functions $\Psi_{a-2}, \Psi_{n-3} : [0,+\infty) \longrightarrow [0,+\infty)$ such that

(i) Ψ_{a-2}, Ψ_{n-3} is continuous and nonincreasing

(ii) $\Psi_{a-2}(p), \Psi_{n-3}(p) = 0 \Leftrightarrow p = 0$

(3) Suppose $\overline{\mathcal{T}}_{m-1}$ is the set of functions $\Theta_1 : [0,+\infty) \longrightarrow [0,+\infty)$ such that

(i) Θ_1 is continuous and strictly increasing

(ii) $\Theta_1(p) = 0 \Leftrightarrow p = 0$

(4) Suppose $\overline{T}_{a-2}, \overline{T}_{n-3}$ is the set of functions with $\Theta_2, \Theta_3 : [0,+\infty) \longrightarrow [0,+\infty)$ such that

(i) Θ_2, Θ_3 is continuous and strictly decreasing

(ii) $\Theta_2(p), \Theta_3(p) = 0 \Leftrightarrow p = 0$

4. Contractive Mappings Based on the Picture Fuzzy n-Banach Space

Based on the definitions introduced in Section 3, we describe some contractive mappings using the B_{PF} named as picture fuzzy n-normed contractive mapping (N_{CM}) and verify it with the help of numerical examples.

Definition 8. For a $N_{PF}(\mathcal{X}_{UNI}, M_m, A_a, N_n, *_{ct}, \circ_{ctc})$, the mapping $T : \mathcal{X}_{UNI} \longrightarrow \mathcal{X}_{UNI}$ is called N_{CM}, if

$$\left.\begin{array}{l} M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) \le M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p) \\ A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) \ge A_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p) \\ N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) \ge N_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p) \end{array}\right\}, \tag{5}$$

for all $x_1, x_2, \cdots, x_{k-1} \in \mathcal{X}_{UNI}, x, \ddot{y} \in \mathcal{X}_{UNI}, p > 0$.

Further, based on equation (5) and using Remark 7, we explore the following results, which are very helpful for future work.

Theorem 9. *For a $N_{PF}(\mathcal{X}_{UNI}, M_m, A_a, N_n, *_{ct}, \circ_{ctc})$, we define $N_{CM}, T : \mathcal{X}_{UNI} \longrightarrow \mathcal{X}_{UNI}$ such that*

$$\left.\begin{array}{l} M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) \ge \alpha \Longrightarrow M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p - \Psi_{m-1}(p)) \ge \alpha \\ A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) < 1 - \alpha \Longrightarrow A_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p - \Psi_{a-2}(p)) < 1 - \alpha \\ N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) < 1 - \alpha \Longrightarrow N_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p - \Psi_{n-3}(p)) < 1 - \alpha \end{array}\right\}, \tag{6}$$

where $\Psi_{m-1} \in S_{m-1}$, $\Psi_{a-2} \in S_{a-2}$, and $\Psi_{n-3} \in S_{n-3}$, for all $x_1, x_2, \cdots, x_{k-1} \in \mathcal{X}_{UNI}, x, \ddot{y} \in \mathcal{X}_{UNI}, p > 0$ with $\alpha \in (0, 1]$. Then, T possesses a unique fixed point in $\mathcal{X}_{UNI}$.

Proof. Let $x_0 \in \mathcal{X}_{UNI}$ with $x_{k+1} = T(x_k) \forall k \in \mathbb{N}_{Nn}$. By using Remark 7 and inequality (6), we get

$$\begin{aligned} & M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) \\ & \quad \le M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p - \Psi_{m-1}(p)), \\ & A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) \\ & \quad \ge A_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p - \Psi_{a-2}(p)), \\ & N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) \\ & \quad \ge N_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p - \Psi_{n-3}(p)). \end{aligned} \tag{7}$$

Further, we write the above equations as

$$\begin{aligned} & M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) \\ & \quad \le M_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, p - \Psi_{m-1}(p)) \\ & \quad \le M_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, p), \\ & A_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) \\ & \quad \ge A_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, p - \Psi_{a-2}(p)) \\ & \quad \ge A_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, p), \\ & N_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) \\ & \quad \ge N_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, p - \Psi_{n-3}(p)) \\ & \quad \le N_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, p). \end{aligned} \tag{8}$$

It is clear from the above analysis that $\{M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p)\}$ is a bounded nondecreasing sequence while $\{A_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p)\}$ and $\{N_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p)\}$ are bounded nonincreasing sequences. Then, the limit of these equations exists. Hence,

$$\begin{aligned} & M_n(x_2, x_3, \cdots, x_{k-1}, x_1 - x_0, p + \Psi_{m-1}(p)) \\ & \quad \le M_n(x_2, x_3, \cdots, x_k, x_2 - x_1, p + \Psi_{m-1}(p) - \Psi_{m-1}(p + \Psi_{m-1}(p))) \\ & \quad \le M_n(x_2, x_3, \cdots, x_k, x_2 - x_1, p), \\ & A_n(x_2, x_3, \cdots, x_{k-1}, x_1 - x_0, p + \Psi_{a-1}(p)) \\ & \quad \ge A_n(x_2, x_3, \cdots, x_k, x_2 - x_1, p), \\ & N_n(x_2, x_3, \cdots, x_{k-1}, x_1 - x_0, p + \Psi_{n-1}(p)) \\ & \quad \ge N_n(x_2, x_3, \cdots, x_k, x_2 - x_1, p). \end{aligned} \tag{9}$$

By using the induction on k, we have

$$\begin{aligned} & M_n(x_2, x_3, \cdots, x_{k-1}, x_1 - x_0, p + \Psi_{m-1}(p)) \\ & \quad \le M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p), \\ & A_n(x_2, x_3, \cdots, x_{k-1}, x_1 - x_0, p + \Psi_{a-1}(p)) \\ & \quad \ge A_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p), \\ & N_n(x_2, x_3, \cdots, x_{k-1}, x_1 - x_0, p + \Psi_{n-1}(p)) \\ & \quad \ge N_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p). \end{aligned} \tag{10}$$

As $k \longrightarrow \infty$, we have

$$\lim_{k\to\infty} M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) = 1,$$
$$\lim_{k\to\infty} A_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) = 0, \quad (11)$$
$$\lim_{k\to\infty} N_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) = 0.$$

Supposing $p, \epsilon > 0$, we have

$$M_n\left(x_1, x_2, \cdots, x_{k-1}, x_{K+1} - x_K, \frac{p}{2}\right) \geq 1 - \epsilon,$$
$$M_n\left(x_1, x_2, \cdots, x_{k-1}, x_{K+1} - x_K, \Psi_{m-1}\left(\frac{p}{2}\right)\right) \geq 1 - \epsilon. \quad (12)$$

Similarly, from abstinence and falsity grades, we have

$$A_n\left(x_1, x_2, \cdots, x_{k-1}, x_{K+1} - x_K, \Psi_{a-2}\left(\frac{p}{2}\right)\right) < \epsilon,$$
$$N_n\left(x_1, x_2, \cdots, x_{k-1}, x_{K+1} - x_K, \Psi_{n-3}\left(\frac{p}{2}\right)\right) < \epsilon. \quad (13)$$

By using the above analysis, we write, if $M_n(x_1, x_2, \cdots, x_{k-1}, x - x_K, (p/2)) \geq 1 - \epsilon$ and $A_n(x_1, x_2, \cdots, x_{k-1}, x - x_K, p/2), N_n(x_1, x_2, \cdots, x_{k-1}, x - x_K, p/2) < \epsilon$, then

$$M_n\left(x_1, x_2, \cdots, x_{k-1}, T(x) - x_K, \frac{p}{2}\right)$$
$$\geq \min\left(M_n\left(x_1, x_2, \cdots, x_{k-1}, T(x) - T(x_K), \frac{p}{2} - \Psi_{m-1}\left(\frac{p}{2}\right)\right)\right),$$
$$M_n\left(x_1, x_2, \cdots, x_{k-1}, T(x) - x_K, \Psi_{m-1}\left(\frac{p}{2}\right)\right)$$
$$\geq \min\left(M_n\left(x_1, x_2, \cdots, x_{k-1}, x - x_K, \frac{p}{2}\right)\right),$$
$$M_n\left(x_1, x_2, \cdots, x_{k-1}, x_{K+1} - x_K, \Psi_{m-1}\left(\frac{p}{2}\right)\right) \geq 1 - \epsilon. \quad (14)$$

Similarly, solving the grades of abstinence and falsity, we have

$$A_n\left(x_1, x_2, \cdots, x_{k-1}, x_{K+1} - x_K, \Psi_{a-2}\left(\frac{p}{2}\right)\right) < \epsilon,$$
$$N_n\left(x_1, x_2, \cdots, x_{k-1}, x_{K+1} - x_K, \Psi_{n-3}\left(\frac{p}{2}\right)\right) < \epsilon. \quad (15)$$

Therefore,

$$M_n\left(x_1, x_2, \cdots, x_{k-1}, x_k - x_K, \frac{p}{2}\right) \geq 1 - \epsilon. \quad (16)$$

Similarly, dealing with the grades of abstinence and falsity, we have

$$A_n\left(x_1, x_2, \cdots, x_{k-1}, x_k - x_K, \frac{p}{2}\right) < \epsilon,$$
$$N_n\left(x_1, x_2, \cdots, x_{k-1}, x_k - x_K, \frac{p}{2}\right) < \epsilon. \quad (17)$$

Then, for all $k \geq \mathbb{N}_{\mathrm{Nn}}$,

$$M_n(x_1, x_2, \cdots, x_{k-1}, x_k - x_\Upsilon, p)$$
$$\geq \min\{M_n(x_1, x_2, \cdots, x_{k-1}, x_k - x_K, p), M_n \cdot (x_1, x_2, \cdots, x_{k-1}, x_\Upsilon - x_K, p)\} \geq 1 - \epsilon,$$
$$M_n(x_1, x_2, \cdots, x_{k-1}, x_\Upsilon - x_K, p) \geq 1 - \epsilon. \quad (18)$$

Also, we find

$$A_n(x_1, x_2, \cdots, x_{k-1}, x_k - x_\Upsilon, p) < \epsilon,$$
$$N_n(x_1, x_2, \cdots, x_{k-1}, x_k - x_\Upsilon, p) < \epsilon. \quad (19)$$

Since ϵ is arbitrary and the sequence $\{x_{\mathfrak{f}}\}$ is Cauchy, hence they are convergent. Therefore, $\lim\{x_{\mathfrak{f}}\} = x$.

Suppose $p, \epsilon > 0$; then, there exists $k_0 \in \mathbb{N}_{\mathrm{Nn}}$ such that

$$M_n\left(x_1, x_2, \cdots, x_{k-1}, x_k - x, \frac{p}{2}\right) \geq 1 - \epsilon,$$
$$M_n\left(x_1, x_2, \cdots, x_{k-1}, x - x_k, \Psi_{m-1}\left(\frac{p}{2}\right)\right) \geq 1 - \epsilon. \quad (20)$$

Moreover, doing the same process to abstinence and falsity grades, we obtain

$$A_n\left(x_1, x_2, \cdots, x_{k-1}, x_k - x, \Psi_{a-2}\left(\frac{p}{2}\right)\right) < \epsilon,$$
$$N_n\left(x_1, x_2, \cdots, x_{k-1}, x_k - x, \Psi_{n-3}\left(\frac{p}{2}\right)\right) < \epsilon, \quad (21)$$

for all $k \geq k_0$. Hence,

$$M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p)$$
$$\geq \min(M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x_{k+1}, p - \Psi_{m-1}(p))),$$

$$M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x, \Psi_{m-1}(p))$$
$$\geq \min(M_n(x_1, x_2, \cdots, x_{k-1}, x - x_k, p)),$$

$$M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x, \Psi_{m-1}(p)) \geq 1 - \epsilon,$$

$$A_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) < \epsilon,$$

$$N_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) < \epsilon, \quad (22)$$

for all $k \geq k_0$. Therefore,

$$M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) = 1,$$
$$A_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) = 0, \quad (23)$$
$$N_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) = 0,$$

for all $p > 0$. Hence, $T(x) = x$; that is, T has a fixed point in $\mathcal{X}_{\mathrm{UNI}}$. Next, we prove its uniqueness. For this, we suppose $\acute{y}$ is another fixed point of T in $\mathcal{X}_{\mathrm{UNI}}$; then,

$$\begin{aligned}
&M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p + k\Psi_{m-1}(p)) \\
&\quad \geq \min\left(M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p)\right) \\
&\quad = M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p), \\
&A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p + k\Psi_{a-2}(p)) \\
&\quad = M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p), \\
&N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p + k\Psi_{n-3}(p)) \\
&\quad = M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p),
\end{aligned} \tag{24}$$

for all $k \in \mathbb{N}_{\mathrm{Nn}}$ and $k \longrightarrow \infty$; then,

$$\begin{aligned}
M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &= 1, \\
A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &= 0, \\
N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &= 0,
\end{aligned} \tag{25}$$

for all $p > 0$. Hence, $x = \ddot{y}$. Thus, T has a unique fixed point in X_{UNI}.

Example 10. For a Banach space $(X_{\mathrm{UNI}}, \|.\|)$, we define a mapping $T : X_{\mathrm{UNI}} \longrightarrow X_{\mathrm{UNI}}$ such that for all $x, \ddot{y} \in X_{\mathrm{UNI}}$,

$$\begin{aligned}
\|T(x) - T(\ddot{y})\| &\leq \|x - \ddot{y}\| - \Psi_{m-1}\|x - \ddot{y}\|, \\
\|T(x) - T(\ddot{y})\| &\geq \|x - \ddot{y}\| - \Psi_{a-2}\|x - \ddot{y}\|, \\
\|T(x) - T(\ddot{y})\| &\geq \|x - \ddot{y}\| - \Psi_{n-3}\|x - \ddot{y}\|.
\end{aligned} \tag{26}$$

We know that $\Psi_{m-1} \in S_{m-1}$, $\Psi_{a-2} \in S_{a-2}$, and $\Psi_{n-3} \in S_{n-3}$. We consider that $\Psi_{m-1}(\beta p) \leq \beta\Psi_{m-1}(p)$, $\Psi_{a-2}(\beta p) \geq \beta \Psi_{a-2}(p)$, and $\Psi_{n-3}(\beta p) \geq \beta\Psi_{n-3}(p)$, where $p > 0$ and $\beta \in [0, 1]$. Now, we describe the picture fuzzy n-norm M_n, A_n, and N_n:

$$\begin{aligned}
&M_n(x_1, x_2, \cdots, x_{k-1}, x_k, p) \\
&\quad = \begin{cases} \dfrac{p}{\|x_1, x_2, \cdots, x_{k-1}, x_k\|}, & 0 < p \leq \|x_1, x_2, \cdots, x_{k-1}, x_k\|, \\ 1, & \|x_1, x_2, \cdots, x_{k-1}, x_k\| < p, \\ 0, & p \leq 0, \end{cases} \\
&A_n(x_1, x_2, \cdots, x_{k-1}, x_k, p) \\
&\quad = \begin{cases} 1 - \dfrac{p}{\|x_1, x_2, \cdots, x_{k-1}, x_k\|}, & 0 < p \leq \|x_1, x_2, \cdots, x_{k-1}, x_k\|, \\ 0, & \|x_1, x_2, \cdots, x_{k-1}, x_k\| < p, \\ 1, & p \leq 0, \end{cases} \\
&N_n(x_1, x_2, \cdots, x_{k-1}, x_k, p) \\
&\quad = \begin{cases} 1 - \dfrac{p}{\|x_1, x_2, \cdots, x_{k-1}, x_k\|}, & 0 < p \leq \|x_1, x_2, \cdots, x_{k-1}, x_k\|, \\ 0, & \|x_1, x_2, \cdots, x_{k-1}, x_k\| < p, \\ 1, & p \leq 0. \end{cases}
\end{aligned} \tag{27}$$

We consider that

$$\begin{aligned}
M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &\geq \alpha, \\
A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &< 1 - \alpha, \\
N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &< 1 - \alpha.
\end{aligned} \tag{28}$$

The first three parts are discussed for the truth grade. We have the following cases:

Case 1. Suppose $0 < p \leq \|x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p\|$; then,

$$\begin{aligned}
M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &= \frac{p}{\|x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p\|} \geq \alpha, \\
p &\geq \alpha\|x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p\|.
\end{aligned} \tag{29}$$

Further, we write

$$\begin{aligned}
&\alpha\|x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y})\| \\
&\quad \leq \alpha\|x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}\| - \alpha\Psi_{m-1}\|x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}\| \\
&\quad \leq \alpha\|x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}\| - \Psi_{m-1}(\alpha\|x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}\|) \\
&\quad \leq p - \Psi_{m-1}(p).
\end{aligned} \tag{30}$$

Therefore, we get

$$\begin{aligned}
&M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p - \Psi_{m-1}(p)) \\
&\quad = \frac{p - \Psi_{m-1}(p)}{\|x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y})\|} \geq \alpha.
\end{aligned} \tag{31}$$

Case 2. Suppose $\|x_1, x_2, \cdots, x_{k-1}, x_k\| < p$; then,

$$\begin{aligned}
\|x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y})\| &\leq \|x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}\| \\
&\quad - \Psi_{m-1}\|x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}\| \\
&\leq p - \Psi_{m-1}(p).
\end{aligned} \tag{32}$$

Therefore, we get

$$M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p - \Psi_{m-1}(p)) = 1 \geq \alpha. \tag{33}$$

Case 3. Suppose $p \leq 0$ and $M_n(x_1, x_2, \cdots, x_{k-1}, x_k, p) = 0$; then,

$$M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), p - \Psi_{m-1}(p)) = 1 \geq \alpha. \tag{34}$$

Similarly, we can prove these conditions for abstinence and falsity grades. Hence, the solution is completed. Further, we instigate more results based on B_{PF} to show the proficiency of the discussed results.

Theorem 11. *For a $B_{PF}(\mathcal{X}_{UNI}, M_m, A_a, N_n, *_{ct}, \circ_{ctc})$, the grade of truth, abstinence, and falsity satisfies the conditions of Definition 4. Now, we define the decreasing mapping Γ_1 and increasing mappings Γ_2 and Γ_3, such that $\Gamma_1 : (0,+\infty) \longrightarrow [0,1)$ and $\Gamma_2 : (0,+\infty) \longrightarrow [0,1), \Gamma_3 : (0,+\infty) \longrightarrow [0,1)$ with $T : \mathcal{X}_{UNI} \longrightarrow \mathcal{X}_{UNI}$, such that*

$$\left.\begin{aligned} &M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) \geq \alpha \Longrightarrow M_n\left(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), \Theta_1^{-1}(\Gamma_1(P)\Theta_1(P))\right) \geq \alpha \\ &A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) < 1-\alpha \Longrightarrow A_n\left(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), \Theta_2^{-1}(\Gamma_2(P)\Theta_2(P))\right) < 1-\alpha \\ &N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) < 1-\alpha \Longrightarrow N_n\left(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), \Theta_3^{-1}(\Gamma_3(P)\Theta_3(P))\right) < 1-\alpha \end{aligned}\right\}, \tag{35}$$

where $\Theta_1 \in \overline{\tau}_{m-1}$, $\Theta_2 \in \overline{\tau}_{a-2}$, and $\Theta_3 \in \overline{\tau}_{n-3}$, for all $x_1, x_2, \cdots, x_{k-1} \in \mathcal{X}_{UNI}, x, \ddot{y} \in \mathcal{X}_{UNI}, p > 0$ with $\alpha \in (0,1]$. Then, T has a unique fixed point in $\mathbb{X}_{UNI}$.

Proof. Let $x_0 \in \mathcal{X}_{\mathrm{UNI}}$ with $x_{k+1} = T(x_k) \forall k \in \mathbb{N}_{\mathrm{Nn}}$. By using Remark 7 and inequality (35), we get

$$\begin{aligned} M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &\leq M_n(x_1, x_2, \cdots, x_{k-1}, T(x) \\ &\quad - T(\ddot{y}), \Theta_1^{-1}(\Gamma_1(P)\Theta_1(P))), \\ A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &\geq A_n(x_1, x_2, \cdots, x_{k-1}, T(x) \\ &\quad - T(\ddot{y}), \Theta_2^{-1}(\Gamma_2(P)\Theta_2(P))), \\ N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &\geq N_n(x_1, x_2, \cdots, x_{k-1}, T(x) \\ &\quad - T(\ddot{y}), \Theta_3^{-1}(\Gamma_3(P)\Theta_3(P))). \end{aligned} \tag{36}$$

Further, we write the above equations as

$$\begin{aligned} &M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) \\ &\quad \leq M_n\left(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, \Theta_1^{-1}(\Gamma_1(P)\Theta_1(P))\right) \\ &\quad \leq M_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, p), \\ &A_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) \\ &\quad \geq A_n\left(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, \Theta_2^{-1}(\Gamma_2(P)\Theta_2(P))\right) \\ &\quad \geq A_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, p), \\ &N_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) \\ &\quad \geq N_n\left(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, \Theta_3^{-1}(\Gamma_3(P)\Theta_3(P))\right) \\ &\quad \leq N_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, p). \end{aligned} \tag{37}$$

It is clear from the above analysis that $\{M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p)\}$ is a bounded nondecreasing sequence and $\{A_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p)\}$ and $\{N_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p)\}$ are the bounded nonincreasing sequences. Then, the limit of these equations exists. We suppose that

$$\begin{aligned} &\lim_{k\to\infty} M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) < \beta_1 < 1, \\ &\lim_{k\to\infty} A_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) > \beta_2 > 1, \\ &\lim_{k\to\infty} N_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) > \beta_3 > 1. \end{aligned} \tag{38}$$

Therefore, we have

$$\begin{aligned} &M_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, q) \geq M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, q), \\ &A_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, q) \leq A_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, q), \\ &N_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, q) \leq N_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, q), \\ &0 < p \leq \|x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}\|_{\beta_1} \leq \|x_2, x_3, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_1}, \\ &0 < p \leq \|x_2, x_3, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_2} \leq \|x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}\|_{\beta_2}, \\ &0 < p \leq \|x_2, x_3, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_3} \leq \|x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}\|_{\beta_3}. \end{aligned} \tag{39}$$

Then, the limit of these equations also exists. We have

$$\begin{aligned} &\lim_{k\to\infty} \|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_1} = b_1, \\ &\lim_{k\to\infty} \|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_2} = b_2, \\ &\lim_{k\to\infty} \|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_3} = b_3. \end{aligned} \tag{40}$$

If $M_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, q) \geq \beta_1$, $A_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, q) < \beta_2$, and $N_n(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, q) < \beta_3$, then

$$\begin{aligned} &M_n\left(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, \Theta_1^{-1}(\Gamma_1(P)\Theta_1(P))\right) \\ &\quad \geq M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, q) \geq \beta_1, \\ &A_n\left(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, \Theta_2^{-1}(\Gamma_2(P)\Theta_2(P))\right) \\ &\quad \leq A_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, q) < \beta_2, \\ &M_n\left(x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}, \Theta_3^{-1}(\Gamma_3(P)\Theta_3(P))\right) \\ &\quad \leq M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, q) < \beta_3. \end{aligned} \tag{41}$$

Therefore,

$$\begin{aligned} &\|x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}\|_{\beta_1} \leq \Theta_1^{-1}(\Gamma_1(P)\Theta_1(P)), \\ &\Theta_1\|x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}\|_{\beta_1} \leq (\Gamma_1(P)\Theta_1(P)) \\ &\quad \leq \left(\Gamma_1\left(\|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_1}\right)\Theta_1(P)\right) \\ &\quad \leq (\Gamma_1(b_1)\Theta_1(P)). \end{aligned} \tag{42}$$

Similarly, we can find that

$$\begin{aligned}\Theta_2\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_2} &\geq \Theta_2^{-1}(\Gamma_2(P)\Theta_2(P))\\ &\geq (\Gamma_2(b_2)\Theta_2(P)),\\ \Theta_3\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_3} &\geq \Theta_3^{-1}(\Gamma_3(P)\Theta_3(P))\\ &\geq (\Gamma_3(b_3)\Theta_3(P)).\end{aligned} \tag{43}$$

And it is clear that $p\longrightarrow \|x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_1}$, $\|x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_2}$, $\|x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_3}$; then,

$$\begin{aligned}&\Theta_1\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_1}\\ &\quad\leq \left(\Gamma_1(b_1)\Theta_1\left(\|x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_1}\right)\right).\end{aligned} \tag{44}$$

Again,

$$\begin{aligned}&\Theta_2\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_2}\\ &\quad\geq \left(\Gamma_2(b_2)\Theta_2\left(\|x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_2}\right)\right),\\ &\Theta_3\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_3}\\ &\quad\geq \left(\Gamma_3(b_3)\Theta_3\left(\|x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_3}\right)\right).\end{aligned} \tag{45}$$

Thus, we get

$$\begin{aligned}&\Theta_1(b_1)\leq \Gamma_1(b_1)\Theta_1(b_1)\Longrightarrow \Gamma_1(b_1)\geq 1,\\ &\Theta_2(b_2)\geq \Gamma_2(b_2)\Theta_2(b_2)\Longrightarrow \Gamma_2(b_2)\leq 1,\\ &\Theta_3(b_3)\geq \Gamma_3(b_3)\Theta_3(b_3)\Longrightarrow \Gamma_3(b_3)\leq 1,\end{aligned} \tag{46}$$

which is a contradiction; hence,

$$\begin{aligned}&\lim_{k\to\infty} M_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)=1,\\ &\lim_{k\to\infty} A_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)=0,\\ &\lim_{k\to\infty} N_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)=0.\end{aligned} \tag{47}$$

Suppose p, $\epsilon>0$. We have

$$M_n\left(x_1,x_2,\cdots,x_{k-1},x_{K+1}-x_K,\frac{p}{2}-\Theta_1^{-1}\left(\Gamma_1\left(\frac{P}{2}\right)\Theta_1\left(\frac{P}{2}\right)\right)\right)\geq 1-\epsilon. \tag{48}$$

Similarly, for abstinence and falsity grades, we have

$$\begin{aligned}&A_n\left(x_1,x_2,\cdots,x_{k-1},x_{K+1}-x_K,\frac{p}{2}-\Theta_2^{-1}\left(\Gamma_2\left(\frac{P}{2}\right)\Theta_2\left(\frac{P}{2}\right)\right)\right)<\epsilon,\\ &N_n\left(x_1,x_2,\cdots,x_{k-1},x_{K+1}-x_K,\frac{p}{2}-\Theta_3^{-1}\left(\Gamma_3\left(\frac{P}{2}\right)\Theta_3\left(\frac{P}{2}\right)\right)\right)<\epsilon.\end{aligned} \tag{49}$$

By using the above analysis, we get, if $M_n(x_1,x_2,\cdots,x_{k-1},x-x_K,p/2)\geq 1-\epsilon$ and $A_n(x_1,x_2,\cdots,x_{k-1},x-x_K,p/2)$, $N_n(x_1,x_2,\cdots,x_{k-1},x-x_K,p/2)<\epsilon$, then

$$\begin{aligned}&M_n\left(x_1,x_2,\cdots,x_{k-1},T(x)-x_K,\frac{p}{2}\right)\\ &\quad\geq \min\left(M_n\left(x_1,x_2,\cdots,x_{k-1},T(x)-T(x_K),\frac{p}{2}\right.\right.\\ &\quad\left.\left.-\Theta_1^{-1}\left(\Gamma_1\left(\frac{P}{2}\right)\Theta_1\left(\frac{P}{2}\right)\right)\right)\right),\\ &M_n\left(x_1,x_2,\cdots,x_{k-1},T(x)-x_K,\Theta_1^{-1}\left(\Gamma_1\left(\frac{P}{2}\right)\Theta_1\left(\frac{P}{2}\right)\right)\right)\\ &\quad\geq \min\left(M_n\left(x_1,x_2,\cdots,x_{k-1},x-x_K,\frac{p}{2}\right)\right),\\ &M_n\left(x_1,x_2,\cdots,x_{k-1},x_{K+1}-x_K,\Theta_1^{-1}\left(\Gamma_1\left(\frac{P}{2}\right)\Theta_1\left(\frac{P}{2}\right)\right)\right)\geq 1-\epsilon.\end{aligned} \tag{50}$$

Similarly, resolving the grades of abstinence and falsity, we have

$$\begin{aligned}&A_n\left(x_1,x_2,\cdots,x_{k-1},x_{K+1}-x_K,\Theta_2^{-1}\left(\Gamma_2\left(\frac{P}{2}\right)\Theta_2\left(\frac{P}{2}\right)\right)\right)<\epsilon,\\ &N_n\left(x_1,x_2,\cdots,x_{k-1},x_{K+1}-x_K,\Theta_3^{-1}\left(\Gamma_3\left(\frac{P}{2}\right)\Theta_3\left(\frac{P}{2}\right)\right)\right)<\epsilon.\end{aligned} \tag{51}$$

Therefore,

$$M_n\left(x_1,x_2,\cdots,x_{k-1},x_k-x_K,\frac{p}{2}\right)\geq 1-\epsilon. \tag{52}$$

Also, we note

$$\begin{aligned}&A_n\left(x_1,x_2,\cdots,x_{k-1},x_k-x_K,\frac{p}{2}\right)<\epsilon,\\ &N_n\left(x_1,x_2,\cdots,x_{k-1},x_k-x_K,\frac{p}{2}\right)<\epsilon.\end{aligned} \tag{53}$$

Then, for all $k\geq \mathbb{N}_{\mathrm{Nn}}$,

$$\begin{aligned}&M_n(x_1,x_2,\cdots,x_{k-1},x_k-x_Y,p)\\ &\quad\geq \min\{M_n(x_1,x_2,\cdots,x_{k-1},x_k-x_K,p),M_n\\ &\quad\cdot(x_1,x_2,\cdots,x_{k-1},x_Y-x_K,p)\}\geq 1-\epsilon,\\ &M_n(x_1,x_2,\cdots,x_{k-1},x_Y-x_K,p)\geq 1-\epsilon.\end{aligned} \tag{54}$$

Further, we find

$$\begin{aligned}&A_n(x_1,x_2,\cdots,x_{k-1},x_k-x_Y,p)<\epsilon,\\ &N_n(x_1,x_2,\cdots,x_{k-1},x_k-x_Y,p)<\epsilon.\end{aligned} \tag{55}$$

Since ϵ is arbitrary and the sequence $\{x_{\mathfrak{f}}\}$ is Cauchy, hence they are convergent. Therefore, $\lim\{x_{\mathfrak{f}}\}=x$.

Suppose $p, \epsilon > 0$; then, there exists $k_0 \in \mathbb{N}_{\mathrm{Nn}}$ such that

$$M_n\left(x_1, x_2, \cdots, x_{k-1}, x - x_k, p - \Theta_1^{-1}(\Gamma_1(p)\Theta_1(p))\right) \geq 1 - \epsilon. \quad (56)$$

Similarly, observing for abstinence and falsity grades, we have

$$\begin{aligned} &A_n\left(x_1, x_2, \cdots, x_{k-1}, x_k - x, p - \Theta_2^{-1}(\Gamma_2(p)\Theta_2(p))\right) < \epsilon, \\ &N_n\left(x_1, x_2, \cdots, x_{k-1}, x_k - x, p - \Theta_3^{-1}(\Gamma_3(p)\Theta_3(p))\right) < \epsilon, \end{aligned} \quad (57)$$

for all $k \geq k_0$. Hence,

$$\begin{aligned} &M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) \\ &\quad \geq \min\left(M_n\left(x_1, x_2, \cdots, x_{k-1}, T(x) - x_{k+1}, p - \Theta_1^{-1}(\Gamma_1(p)\Theta_1(p))\right)\right), \\ &M_n\left(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x, \Theta_1^{-1}(\Gamma_1(p)\Theta_1(p))\right) \\ &\quad \geq \min\left(M_n(x_1, x_2, \cdots, x_{k-1}, x - x_k, p)\right), \\ &M_n\left(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x, \Theta_1^{-1}(\Gamma_1(p)\Theta_1(p))\right) \geq 1 - \epsilon, \\ &A_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) < \epsilon, \\ &N_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) < \epsilon, \end{aligned} \quad (58)$$

for all $k \geq k_0$. Therefore,

$$\begin{aligned} &M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) = 1, \\ &A_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) = 0, \\ &N_n(x_1, x_2, \cdots, x_{k-1}, T(x) - x, p) = 0, \end{aligned} \quad (59)$$

for all $p > 0$. Hence, $T(x) = x$; that is, T has a fixed point in $\mathcal{X}_{\mathrm{UNI}}$. Next, we prove the uniqueness of the fixed point. For this, we suppose $\ddot{y}$ is another fixed point T in $\mathcal{X}_{\mathrm{UNI}}$; then,

$$\begin{aligned} M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &\geq \min\left(M_n(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), \Theta_1^{-1}(\Gamma_1(p)\Theta_1(p)))\right) \\ &= M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p), \\ A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &= M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p), \\ N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &= M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p). \end{aligned} \quad (60)$$

Hence, $p = \Theta_1^{-1}(\Gamma_1(p)\Theta_1(p)) \Longrightarrow \Theta_1(p) = \Gamma_1(p)\Theta_1(p)$,

$$\begin{aligned} &p = \Theta_2^{-1}(\Gamma_2(p)\Theta_2(p)) \Longrightarrow \Theta_2(p) = \Gamma_2(p)\Theta_2(p), \\ &p = \Theta_3^{-1}(\Gamma_3(p)\Theta_3(p)) \Longrightarrow \Theta_3(p) = \Gamma_3(p)\Theta_3(p). \end{aligned} \quad (61)$$

Therefore, $\Gamma_1(p) = 1$, $\Gamma_2(p) = 1$, and $\Gamma_3(p) = 1$. It is a contradiction; thus, for all $k \in \mathbb{N}_{\mathrm{Nn}}$ and $k \longrightarrow \infty$, we obtain

$$\begin{aligned} &M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) = 1, \\ &A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) = 0, \\ &N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) = 0, \end{aligned} \quad (62)$$

for all $p > 0$. Hence, $x = \ddot{y}$. Hence, T has a unique fixed point in $\mathcal{X}_{\mathrm{UNI}}$.

Example 12. For a Banach space $(\mathcal{X}_{\mathrm{UNI}}, \|.\|)$, we define the decreasing mapping Γ_1 and increasing mappings Γ_2 and Γ_3, such that $\Gamma_1 : (0,+\infty) \longrightarrow [0, 1)$ and $\Gamma_2 : (0,+\infty) \longrightarrow [0, 1)$, $\Gamma_3 : (0,+\infty) \longrightarrow [0, 1)$, and $T : \mathcal{X}_{\mathrm{UNI}} \longrightarrow \mathcal{X}_{\mathrm{UNI}}$ are such that for all $x, \ddot{y} \in \mathcal{X}_{\mathrm{UNI}}$,

$$\begin{aligned} &\Theta_1\|T(x) - T(\ddot{y})\| \leq \Gamma_1\|x - \ddot{y}\| - \Theta_1\|x - \ddot{y}\|, \\ &\Theta_2\|T(x) - T(\ddot{y})\| \geq \Gamma_2\|x - \ddot{y}\| - \Theta_2\|x - \ddot{y}\|, \\ &\Theta_3\|T(x) - T(\ddot{y})\| \geq \Gamma_3\|x - \ddot{y}\| - \Theta_3\|x - \ddot{y}\|, \end{aligned} \quad (63)$$

where $\Theta_1 \in \overline{\mathcal{T}}_{m-1}$, $\Theta_2 \in \overline{\mathcal{T}}_{a-2}$, and $\Theta_3 \in \overline{\mathcal{T}}_{n-3}$. Suppose that $\Gamma_1 \Theta_1$ is nondecreasing and $\Gamma_2\Theta_2$, $\Gamma_3\Theta_3$ are nonincreasing functions with

$$\begin{aligned} &\beta_1\left(\Theta_1^{-1}(\Gamma_1(p)\Theta_1(p))\right) \leq \Theta_1^{-1}(\Gamma_1(\beta_1 p)\Theta_1(\beta_1 p)), \\ &\beta_2\left(\Theta_2^{-1}(\Gamma_2(p)\Theta_2(p))\right) \geq \Theta_2^{-1}(\Gamma_2(\beta_2 p)\Theta_2(\beta_2 p)), \\ &\beta_3\left(\Theta_3^{-1}(\Gamma_3(p)\Theta_3(p))\right) \geq \Theta_3^{-1}(\Gamma_3(\beta_3 p)\Theta_3(\beta_3 p)), \end{aligned} \quad (64)$$

for all $p \in [0,+\infty)$, $\beta_1, \beta_2, \beta_3 \in [0, 1]$. Further, define picture fuzzy n-norm M_n, A_n, N_n as in Example 10. Consider that

$$\begin{aligned} &M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) \geq \alpha, \\ &A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) < 1 - \alpha, \\ &N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) < 1 - \alpha. \end{aligned} \quad (65)$$

By using the three cases of Example 10 and using Theorem 11, we explore that the function T has a unique fixed point in X_{UNI}. Hence, the solution is completed. Further, we have utilized more results based on B_{PF} to show the proficiency of the proven approaches.

Theorem 13. *Let $B_{PF}(\mathcal{X}_{UNI}, M_m, A_a, N_n, *_{ct}, \circ_{ctc})$. Let the grade of truth, abstinence, and falsity satisfy the conditions of Definition 4. Now, we define the mapping $T : \mathcal{X}_{UNI} \longrightarrow \mathcal{X}_{UNI}$, such that*

$$M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) \geq \alpha \Longrightarrow M_n\left(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), \Theta_1^{-1}\left(\Theta_1(P) - \Theta_1'(P)\right)\right) \geq \alpha$$

$$A_n(x_1,x_2,\cdots,x_{k-1},x-\acute{y},p)<1-\alpha \Longrightarrow A_n(x_1,x_2,\cdots,x_{k-1},T(x)-T(\acute{y}),\Theta_2^{-1}\left(\Theta_2(P)-\Theta_2'(P)\right)\right)<1-\alpha$$
$$N_n(x_1,x_2,\cdots,x_{k-1},x-\acute{y},p)<1-\alpha \Longrightarrow N_n(x_1,x_2,\cdots,x_{k-1},T(x)-T(\acute{y}),\Theta_3^{-1}\left(\Theta_3(P)-\Theta_3'(P)\right)\right)<1-\alpha\}, \tag{66}$$

where $(\Theta_1(P),\Theta_1'(P))\in\overline{\tau}_{m-1}$, $(\Theta_2(P),\Theta_2'(P))\in\overline{\tau}_{a-2}$, *and* $(\Theta_3(P)-\Theta_3'(P))\in\overline{\tau}_{n-3}$, *for all* $x_1,x_2,\cdots,x_{k-1}\in X_{UNI}, x,\acute{y}\in X_{UNI}, p>0$ *with* $\alpha\in(0,1]$ *and* $(\Theta_1(P)\geq\Theta_1'(P)),(\Theta_2(P)\geq\Theta_2'(P)),(\Theta_3(P)\geq\Theta_3'(P))$. *Then, T has a unique fixed point in* X_{UNI}.

Proof. Let $x_0\in X_{\mathrm{UNI}}$ with $x_{k+1}=T(x_k)\forall k\in\mathbb{N}_{\mathrm{Nn}}$. By using Remark 7 and equation (66), we get

$$\begin{aligned}
&M_n(x_1,x_2,\cdots,x_{k-1},x-\acute{y},p)\\
&\quad\leq M_n\left(x_1,x_2,\cdots,x_{k-1},T(x)-T(\acute{y}),\Theta_1^{-1}\left(\Theta_1(P)-\Theta_1'(P)\right)\right),\\
&A_n(x_1,x_2,\cdots,x_{k-1},x-\acute{y},p)\\
&\quad\geq A_n\left(x_1,x_2,\cdots,x_{k-1},T(x)-T(\acute{y}),\Theta_2^{-1}\left(\Theta_2(P)-\Theta_2'(P)\right)\right),\\
&N_n(x_1,x_2,\cdots,x_{k-1},x-\acute{y},p)\\
&\quad\geq N_n\left(x_1,x_2,\cdots,x_{k-1},T(x)-T(\acute{y}),\Theta_3^{-1}\left(\Theta_3(P)-\Theta_3'(P)\right)\right).
\end{aligned}\tag{67}$$

Further, from the above equations, we obtain

$$\begin{aligned}
&M_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)\\
&\quad\leq M_n\left(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},\Theta_1^{-1}\left(\Theta_1(P)-\Theta_1'(P)\right)\right)\\
&\quad\leq M_n(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},p),\\
&A_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)\\
&\quad\geq A_n\left(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},\Theta_2^{-1}\left(\Theta_2(P)-\Theta_2'(P)\right)\right)\\
&\quad\geq A_n(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},p),\\
&N_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)\\
&\quad\geq N_n\left(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},\Theta_3^{-1}\left(\Theta_3(P)-\Theta_3'(P)\right)\right)\\
&\quad\leq N_n(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},p).
\end{aligned}\tag{68}$$

It is clear from the above analysis that $\{M_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)\}$ is a bounded nondecreasing sequence and the sequences $\{A_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)\}$ and $\{N_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)\}$ are bounded and nonincreasing. Then, the limit of these equations exists. We suppose that

$$\begin{aligned}
&\lim_{k\to\infty}M_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)<\beta_1<1,\\
&\lim_{k\to\infty}A_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)>\beta_2>1,\\
&\lim_{k\to\infty}N_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,p)>\beta_3>1.
\end{aligned}\tag{69}$$

Therefore, we have

$$\begin{aligned}
&M_n(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},q)\geq M_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,q),\\
&A_n(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},q)\leq A_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,q),\\
&N_n(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},q)\leq N_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,q),\\
&0<p\leq\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_1}\leq\|x_2,x_3,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_1},\\
&0<p\leq\|x_2,x_3,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_2}\leq\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_2},\\
&0<p\leq\|x_2,x_3,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_3}\leq\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_3}.
\end{aligned}\tag{70}$$

Then, the limit of these equations also exists. We have

$$\begin{aligned}
&\lim_{k\to\infty}\|x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_1}=b_1,\\
&\lim_{k\to\infty}\|x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_2}=b_2,\\
&\lim_{k\to\infty}\|x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k\|_{\beta_3}=b_3.
\end{aligned}\tag{71}$$

If $M_n(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},q)\geq\beta_1$, $A_n(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},q)<\beta_2$, and $N_n(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},q)<\beta_3$, then

$$\begin{aligned}
&M_n\left(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},\Theta_1^{-1}\left(\Theta_1(P)-\Theta_1'(P)\right)\right)\\
&\quad\geq M_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,q)\geq\beta_1,\\
&A_n\left(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},\Theta_2^{-1}\left(\Theta_2(P)-\Theta_2'(P)\right)\right)\\
&\quad\leq A_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,q)<\beta_2,\\
&M_n\left(x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1},\Theta_3^{-1}\left(\Theta_3(P)-\Theta_3'(P)\right)\right)\\
&\quad\leq M_n(x_1,x_2,\cdots,x_{k-1},x_{k+1}-x_k,q)<\beta_3.
\end{aligned}\tag{72}$$

Therefore,

$$\begin{aligned}
&\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_1}\leq\Theta_1^{-1}\left(\Theta_1(P)-\Theta_1'(P)\right),\\
&\Theta_1\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_1}\leq\left(\Theta_1(P)-\Theta_1'(P)\right).
\end{aligned}\tag{73}$$

Similarly, we can find

$$\begin{aligned}
&\Theta_2\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_2}\geq\left(\Theta_2(P)-\Theta_2'(P)\right),\\
&\Theta_3\|x_2,x_3,\cdots,x_k,x_{k+2}-x_{k+1}\|_{\beta_3}\geq\left(\Theta_3(P)-\Theta_3'(P)\right).
\end{aligned}\tag{74}$$

Clearly, $p \longrightarrow \|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_1}, \|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_2}, \|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_3}$; hence,

$$\Theta_1 \|x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}\|_{\beta_1} \le \left(\Theta_1\left(\|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_1}\right) - \Theta_1'\left(\|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_1}\right)\right). \tag{75}$$

Also, we write

$$\begin{aligned}\Theta_2 \|x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}\|_{\beta_2} &\ge \left(\Theta_2\left(\|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_1}\right) - \Theta_2'\left(\|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_1}\right)\right),\\ \Theta_3 \|x_2, x_3, \cdots, x_k, x_{k+2} - x_{k+1}\|_{\beta_3} &\ge \left(\Theta_3\left(\|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_1}\right) - \Theta_3'\left(\|x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k\|_{\beta_1}\right)\right).\end{aligned} \tag{76}$$

Thus, we get

$$0 \le \Theta_1(p) \le \Theta_1(b_1) \le \Theta_1(b_1) - \Theta_1'(b_1) \le \Theta_1(b_1). \tag{77}$$

Similarly,

$$\begin{aligned}\Theta_2(p) &\ge \Theta_2(b_2) \ge \Theta_2(b_2) - \Theta_2'(b_2) \ge \Theta_2(b_2),\\ \Theta_3(p) &\ge \Theta_3(b_3) \ge \Theta_3(b_3) - \Theta_3'(b_3) \ge \Theta_3(b_3).\end{aligned} \tag{78}$$

It is a contradiction; hence,

$$\begin{aligned}&\lim_{k\to\infty} M_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) = 1,\\ &\lim_{k\to\infty} A_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) = 0,\\ &\lim_{k\to\infty} N_n(x_1, x_2, \cdots, x_{k-1}, x_{k+1} - x_k, p) = 0.\end{aligned} \tag{79}$$

The rest of the proof to express for all $p > 0$ the equation $T(x) = x$ can be obtained using the similar technique of Theorem 9 and Theorem 11; that is, T has a fixed point in $\mathcal{X}_{\mathrm{UNI}}$. Next, we prove the uniqueness of the fixed point. For this, we suppose $\ddot{y}$ is another fixed point T in $\mathcal{X}_{\mathrm{UNI}}$; then,

$$\begin{aligned}&M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p)\\ &\quad \ge \min\left(M_n\left(x_1, x_2, \cdots, x_{k-1}, T(x) - T(\ddot{y}), \Theta_1^{-1}\left(\Theta_1(P) - \Theta_1'(P)\right)\right)\right)\\ &\quad = M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p),\\ &A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) = M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p),\\ &N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) = M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p).\end{aligned} \tag{80}$$

Hence, $p = \Theta_1^{-1}(\Theta_1(P) - \Theta_1'(P)) \Longrightarrow \Theta_1(p) = \Theta_1(P) - \Theta_1'(P)$,

$$\begin{aligned}p &= \Theta_2^{-1}\left(\Theta_2(P) - \Theta_2'(P)\right) \Longrightarrow \Theta_2(p) = \Theta_2(P) - \Theta_2'(P),\\ p &= \Theta_3^{-1}\left(\Theta_3(P) - \Theta_3'(P)\right) \Longrightarrow \Theta_3(p) = \Theta_3(P) - \Theta_3'(P).\end{aligned} \tag{81}$$

Therefore, $\Theta_1'(p) = 1$, $\Theta_2'(p) = 1$, and $\Theta_3'(p) = 1$. It is a contradiction; thus, as $k \longrightarrow \infty$, we get

$$\begin{aligned}M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &= 1,\\ A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &= 0,\\ N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &= 0,\end{aligned} \tag{82}$$

for all $p > 0$. Hence, $x = \ddot{y}$. Thus, T has a unique fixed point in $\mathcal{X}_{\mathrm{UNI}}$.

Example 14. For a Banach space $(\mathcal{X}_{\mathrm{UNI}}, \|.\|)$, we define the mapping $T : \mathcal{X}_{\mathrm{UNI}} \longrightarrow \mathcal{X}_{\mathrm{UNI}}$ such that for all $x, \ddot{y} \in \mathcal{X}_{\mathrm{UNI}}$,

$$\begin{aligned}\Theta_1\|T(x) - T(\ddot{y})\| &\le \Theta_1\|x - \ddot{y}\| - \Theta_1'\|x - \ddot{y}\|,\\ \Theta_2\|T(x) - T(\ddot{y})\| &\ge \Theta_2\|x - \ddot{y}\| - \Theta_2'\|x - \ddot{y}\|,\\ \Theta_3\|T(x) - T(\ddot{y})\| &\ge \Theta_3\|x - \ddot{y}\| - \Theta_3'\|x - \ddot{y}\|,\end{aligned} \tag{83}$$

where $\Theta_1, \Theta_1' \in \overline{\mathcal{T}}_{m-1}$, $\Theta_2, \Theta_2' \in \overline{\mathcal{T}}_{a-2}$, and $\Theta_3, \Theta_3' \in \overline{\mathcal{T}}_{n-3}$. Suppose $\Theta_1 - \Theta_1'$ is nondecreasing and $\Theta_2 - \Theta_2', \Theta_3 - \Theta_3'$ are nonincreasing functions with

$$\begin{aligned}\beta_1\left(\Theta_1^{-1}\left(\Theta_1 - \Theta_1'\right)\right) &\le \Theta_1^{-1}\left(\Theta_1(\beta_1 p) - \Theta_1'(\beta_1 p)\right),\\ \beta_2\left(\Theta_2^{-1}(\Gamma_2(p)\Theta_2(p))\right) &\ge \Theta_2^{-1}\left(\Theta_2(\beta_2 p) - \Theta_2'(\beta_2 p)\right),\\ \beta_3\left(\Theta_3^{-1}(\Gamma_3(p)\Theta_3(p))\right) &\ge \Theta_3^{-1}\left(\Theta_3(\beta_3 p) - \Theta_3'(\beta_3 p)\right),\end{aligned} \tag{84}$$

for all $p \in [0, +\infty)$ and $\beta_1, \beta_2, \beta_3 \in [0, 1]$. Further, define picture fuzzy n-norm M_n, A_n, N_n as in Example 10. Consider that

$$\begin{aligned}M_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &\ge \alpha,\\ A_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &< 1 - \alpha,\\ N_n(x_1, x_2, \cdots, x_{k-1}, x - \ddot{y}, p) &< 1 - \alpha.\end{aligned} \tag{85}$$

We explore that the function T has a unique fixed point in X_{UNI}. Hence, the solution is completed.

5. Conclusion

A picture fuzzy set is more proficient and more capable than an intuitionistic fuzzy set and fuzzy to cope with uncertain and unpredictable information in realistic issues. Keeping the advantages of the picture fuzzy set and a n-norm linear space, the manuscript made the following advancements in the existing literature:

(1) The novel picture fuzzy n-norm linear space and its basic properties are explored

(2) Some novel contractive conditions based on N_{PF} are presented. By using these contractive conditions, we have explored some fixed point theorems for a picture fuzzy n-Banach space (B_{PF}). It was observed that these results are more modified and more general than the existing ones in the literature, which are based on intuitionistic fuzzy n-Banach spaces (B_{IF}) and fuzzy n-Banach spaces

(3) The reliability and effectiveness of the obtained main theorems are expressed, and several examples are presented afterwards

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

We declare that we do not have any commercial or associative interests that represent conflicts of interest in connection with this manuscript. There are no professional or other personal interests that can inappropriately influence our submitted work.

Authors' Contributions

All authors contributed equally to the writing of this article. All authors read and approved the final manuscript.

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Research Article

Certain Subclasses of β-Uniformly q-Starlike and β-Uniformly q-Convex Functions

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In this paper, the authors introduced certain subclasses β-uniformly q-starlike and β-uniformly q-convex functions of order α involving the q-derivative operator defined in the open unit disc. Coefficient bounds were also investigated.

1. Introduction

The q-analysis is a generalization of the ordinary analysis. The application of the q-calculus was first introduced by Jackson [1–3]. In geometric function theory, the q-hypergeometric functions were first used by Srivastava [4]. The q-calculus provides valuable tools that have been used to define several subclasses of the normalized analytic function in the open unit disk $\mathbb{U}$. Ismail et al. [5] were the first to study a certain class $\mathcal{S}^*$ of starlike functions by using the q-derivative operator. Recently, new subclasses of analytic functions associated with q-derivative operators are introduced and discussed, see for example [4, 6–18]. Motivated by the importance of q-analysis, in this paper, we introduce the classes of β-uniformly q-starlike and β-uniformly q-convex functions defined by the q-derivative operator in the open unit disc, as a generalization of β-uniformly starlike and β-uniformly convex functions.

First, we recall some basic notations and definitions from q-calculus, which are used in this paper. The q-derivative of the function f is defined as follows [1–3]:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}, \quad (z \neq 0, 0 < q < 1). \tag{1}$$

From equation (1), it is clear that if f and g are the two functions, then

$$D_q(f(z) + g(z)) = D_q f(z) + D_q g(z), \tag{2}$$

$$Dq(cf(z)) = cD_q f(z), \tag{3}$$

where c is a constant. We note that $D_q f(z) \longrightarrow f'(z)$ as $q \longrightarrow 1^-$, where f' is the ordinary derivative of the function f.

In particular, using equation (1), the q-derivative of the function $h(z) = z^n$ is as follows:

$$D_q h(z) = [n]_q z^{n-1}, \tag{4}$$

where $[n]_q$ denotes the q-number and is given as follows:

$$[n]_{q=}\frac{1-q^n}{1-q}, \quad (0<q<1). \tag{5}$$

Since we note that $[n]_q \longrightarrow n$ as $q \longrightarrow 1^-$, therefore, in view of equation (4), $D_q h(z) \longrightarrow h'(z)$ as $q \longrightarrow 1^-$, where $h'(z)$ denotes the ordinary derivative of the function $h(z)$ with respect to z.

In this paper, we consider the classes $\mathscr{A}$ and $\mathscr{T}$ of the functions, analytic in the open unit disc $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$, of the following forms, respectively:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n, \tag{6}$$

$$f(z)=z-\sum_{n=2}^{\infty}a_n z^n, \quad (a_n\geq 0). \tag{7}$$

Also, using equations (2), (3), (4), and (6), we get the following q-derivatives of the function f:

$$D_q f(z)=1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}, \quad (0<q<1), \tag{8}$$

$$D_q\left(zD_q f(z)\right)=1+\sum_{n=2}^{\infty}[n]_q^2 a_n z^{n-1}, \quad (0<q<1), \tag{9}$$

where $[n]_q$ is given by equation (5).

The classes of starlike functions of order $\alpha(0\leq\alpha<1)$ and convex functions of order $\alpha(0\leq\alpha<1)$, denoted by $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$, respectively, are defined as follows [19]:

$$\mathscr{S}^*(\alpha)=\left\{f\in\mathscr{A}:\Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha\right\}, \tag{10}$$

$$\mathscr{K}(\alpha)=\left\{f\in\mathscr{A}:\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha\right\}. \tag{11}$$

It is clear that $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ are the subclasses of the class $\mathscr{A}$.

The classes of β-uniformly starlike functions of order α and β-uniformly convex functions of order α, denoted by $\mathscr{SD}(\alpha,\beta)$ and $\mathscr{KD}(\alpha,\beta)$, respectively, are defined as follows [20]:

$$\mathscr{SD}(\alpha,\beta)=\left\{f\in\mathscr{A}:\Re\left(\frac{zf'(z)}{f(z)}-\alpha\right)>\beta\left|\frac{zf'(z)}{f(z)}-1\right|\right\}, \tag{12}$$

$$\mathscr{KD}(\alpha,\beta)=\left\{f\in\mathscr{A}:\Re\left(\frac{zf''(z)}{f'(z)}-\alpha\right)>\beta\left|\frac{zf''(z)}{f'(z)}-1\right|\right\}, \tag{13}$$

where $z\in\mathbb{U}$, $0\leq\alpha<1$, and $\beta\geq 0$.

The class of q-starlike functions of order μ, denoted by $\mathscr{S}_q^*(\mu)$, is defined as follows [13]:

$$\mathscr{S}_q^*(\mu)=\left\{f\in\mathscr{A}:\Re\left(\frac{zD_q f(z)}{f(z)}\right)>\mu, \quad (z\in\mathbb{U};0\leq\mu<1)\right\}. \tag{14}$$

Also, the class of q-convex functions of order μ, denoted by $C_q(\mu)$, is defined as [13]:

$$C_q(\mu)=\left\{f\in\mathscr{A}:\Re\left(\frac{D^q\left(zD_q f(z)\right)}{D_q f(z)}\right)>\mu, \quad (z\in\mathbb{U};0\leq\mu<1)\right\}. \tag{15}$$

The analytic function g is said to be subordinate to the analytic function f in $\mathbb{U}$ [21], represented as follows:

$$g(z)\prec f(z) \text{ or } g\prec f, \tag{16}$$

if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with

$$\begin{aligned} w(0)&=0,\\ |w(z)|&<1, \end{aligned} \tag{17}$$

such that

$$g(z)=f(w(z)), \quad (z\in\mathbb{U}). \tag{18}$$

In the next section, we introduce the classes of β-uniformly q-starlike and β-uniformly q-convex functions of order α, denoted by $\mathcal{S}_q(\alpha,\beta)$ and $\mathscr{UCV}_q(\alpha,\beta)$, respectively. Also, we obtain the coefficient bounds of the functions belonging to these classes.

2. Coefficient Bounds

Since the q-derivative is a generalized form of the ordinary derivative, therefore, in view of definitions of $\mathscr{SD}(\alpha,\beta)$ and $\mathscr{KD}(\alpha,\beta)$, we define the classes of β-uniformly q-starlike and β-uniformly q-convex functions of order α, denoted by $S_q(\alpha,\beta)$ and $\mathscr{UCV}_q(\alpha,\beta)$, respectively, by replacing the ordinary derivative with the q-derivative in equations (12) and (13).

We provide the respective definitions of the classes $S_q(\alpha,\beta)$ and $\mathscr{UCV}_q(\alpha,\beta)$.

Definition 1. The function $f \in \mathscr{A}$ is said to be β-uniformly q-starlike of order α, if it satisfies the following inequality:

$$\Re\left(\frac{zD_q(f(z))}{f(z)} - \alpha\right) > \beta\left|\frac{zD_q(f(z))}{f(z)}\right| - 1, \tag{19}$$

where $0 < q < 1$, $\beta \geq 0$, $0 \leq \alpha < 1$, and $z \in \mathbb{U}$.

Definition 2. The function $f \in \mathscr{A}$ is said to be β-uniformly q-convex of order α, if it satisfies the following inequality:

$$\Re\left(\frac{D_q\left(zD_{qf(z)}\right)}{D_qf(z)} - \alpha\right) > \beta\left|\frac{D_q\left(zD_{qf(z)}\right)}{D_qf(z)} - 1\right|, \tag{20}$$

where $0 < q < 1$, $\beta \geq 0$, $0 \leq \alpha < 1$, and $z \in \mathbb{U}$.

Further, we define the classes $\mathscr{T}\mathscr{S}_q(\alpha, \beta)$ and $\mathscr{U}\mathscr{C}\mathscr{T}_q(\alpha, \beta)$ containing functions with negative coefficients and satisfying inequalities (19) and (20), respectively, as follows:

$$\begin{aligned} \mathscr{T}\mathscr{S}_q(\alpha, \beta) &= \mathscr{S}_q(\alpha, \beta) \cap \mathrm{T}, \\ \mathscr{U}\mathscr{C}\mathscr{T}_q(\alpha, \beta) &= \mathscr{U}\mathscr{C}\mathscr{V}_q(\alpha, \beta) \cap \mathrm{T}. \end{aligned} \tag{21}$$

Remark 3. We note that

(i) $\lim_{q\to 1^-}\mathscr{S}_q(\alpha, \beta) = \mathscr{S}\mathscr{D}(\alpha, \beta)$ and $\lim_{q\to 1^-}\mathscr{U}\mathscr{C}\mathscr{V}_q(\alpha, \beta) = \mathscr{K}\mathscr{D}(\alpha, \beta)$

(ii) $\mathscr{S}_q(\alpha, \beta) = \mathscr{S}_q^*\mathrm{D}(\alpha)$ and $\mathscr{U}\mathscr{C}\mathscr{V}_q(\alpha, 0) = \mathscr{C}_q(\alpha)$ (see [8]).

(iii) $\lim_{q\to 1^-}\mathscr{S}_q(\alpha, 0) = \mathscr{S}^*(\alpha)$ and $\lim_{q\to 1^-}\mathscr{U}\mathscr{C}\mathscr{V}_q(\alpha, 0) = \mathscr{K}(\alpha)$

Now, the relation between the subclasses $\mathscr{S}_q^*(\mu)$ and $\mathscr{S}_q(\alpha, \beta)$is given by the following result.

Theorem 4. *Let $f \in \mathscr{S}_q(\alpha, \beta)$, then $f \in \mathscr{S}_q^*((\alpha+\beta)/(1+\beta))$, where $\beta \geq 0$, $0 \leq \alpha < 1$, and $0 < q < 1$.*

Proof. If $f \in \mathscr{S}_q(\alpha, \beta)$, then in view of Definition 1 and using the fact that $-\Re < (z) \leq |z|$, we get

$$\Re\left(\frac{zD_q(f(z))}{f(z)} - \alpha\right) > \beta\left|\frac{zD_q(f(z))}{f(z)} - 1\right| \geq \beta\Re\left(\frac{zD_q(f(z))}{f(z)} - 1\right), \tag{22}$$

which implies that

$$\Re\left(\frac{zD_q(f(z))}{f(z)}\right) - \alpha > \beta\Re\left(\frac{zD_q(f(z))}{f(z)}\right) + \beta, \tag{23}$$

then

$$\Re\left(\frac{zD_q(f(z))}{f(z)}\right) > \frac{\alpha+\beta}{1+\beta}. \tag{24}$$

Since $\beta \geq 0$ and $0 \leq \alpha < 1$, then $0 \leq (\alpha+\beta)/(1+\beta) < 1$. Hence, in view of equation (14), we obtain $f \in \mathscr{S}_q^*((\alpha+\beta)/(1+\beta))$.

Also, the relation between the subclasses $C_q(\alpha)$ and $\mathscr{U}\mathscr{C}\mathscr{V}_q(\alpha, \beta)$is given by the following result.

Theorem 5. *Let $f \in \mathscr{U}\mathscr{C}\mathscr{V}_q(\alpha, \beta)$, then $f \in \mathscr{C}_q((\alpha+\beta)/(1+\beta))$, where $\beta \geq 0$, $0 \leq \alpha < 1$, and $0 < q < 1$.*

Proof. If $f \in \mathscr{U}\mathscr{C}\mathscr{V}_q(\alpha, \beta)$, then in view of Definition 2 and using the fact that $-\Re < (z) \leq |z|$, we get

$$\begin{aligned} \Re\left(\frac{D_q(zD_q(z))}{D_qf(z)} - \alpha\right) &> \beta\left|\frac{D_q(zD_qf(z))}{D_qf(z)} - 1\right| \\ &\geq -\beta\Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)} - 1\right), \end{aligned} \tag{25}$$

which implies that

$$\Re\left(\frac{D_q(zD_q(z))}{D_qf(z)}\right) - \alpha > -\beta\Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right) + \beta, \tag{26}$$

then

$$\Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right) > \frac{\alpha+\beta}{1+\beta}, \tag{27}$$

since $\beta \geq 0$ and $0 \leq \alpha < 1$, then $0 \leq (\alpha+\beta)/(1+\beta) < 1$. Hence, in view of equation (15), we obtain $f \in \mathscr{C}_q((\alpha+\beta)/(1+\beta))$.

Next, the coefficient bound of the class $\mathscr{S}_q(\alpha, \beta)$ is given by the following result.

Theorem 6. *A function $f \in \mathscr{A}$ belongs to the class $\mathscr{S}_q(\alpha, \beta)$ if*

$$\sum_{n=2}^{\infty}\left([n]_q(1+\beta) - (\alpha+\beta)\right)|a_n| \leq 1 - \alpha, \tag{28}$$

where $0 < q < 1$, $\beta \geq 0$, $0 \leq \alpha < 1$, and $[n]_q$ denotes the q-number.

Proof. Now, using the fact that $-\Re(z) \leq |z|$, we have

$$\begin{aligned} &\beta\left|\frac{zD_q(f(z))}{f(z)} - 1\right| - \Re\left(\frac{zD_q(f(z))}{f(z)} - 1\right) \\ &\leq (1+\beta)\left|\frac{zD_q(f(z))}{f(z)} - 1\right|. \end{aligned} \tag{29}$$

Using equations (6) and (8) in the right hand side of inequality (29), we get

$$(1+\beta)\left|\frac{zD_q(f(z))}{f(z)}-1\right| = (1+\beta)\left|\frac{\sum_{n=2}^{\infty}\left([n]_q-1\right)a_n z^{n-1}}{1+\sum_{n=2}^{\infty}a_n z^{n-1}}\right|. \tag{30}$$

Since $|z|<1$, therefore, from the above inequality, we get

$$(1+\beta)\left|\frac{zD_q(f(z))}{f(z)}-1\right| < \frac{(1+\beta)\sum_{n=2}^{\infty}\left([n]_q-1|a_n|\right)}{1-\sum_{n=2}^{\infty}|a_n|}. \tag{31}$$

Combining inequalities (29) and (31), we get

$$\beta\left|\frac{zD_q(f(z))}{f(z)}-1\right| - \Re\left(\frac{zD_q(f(z))}{f(z)}-1\right) < \frac{(1+\beta)\sum_{n=2}^{\infty}\left([n]_q-1\right)|a_n|}{1-\sum_{n=2}^{\infty}|a_n|}. \tag{32}$$

If $(/)((1+\beta)\sum_{n=2}^{\infty}([n]_q-1)|a_n|)/(1-\sum_{n=2}^{\infty}|a_n|)<1-\alpha$, which is equivalent to inequality (28), then from inequality (32) we get

$$\beta\left|\frac{zD_q(f(z))}{f(z)}-1\right| - \Re\left(\frac{zD_q(f(z))}{f(z)}-1\right) \le 1-a, \tag{33}$$

which is equivalent to inequality (19). Thus, in view of Definition 1, the function $f\in\mathcal{S}_q(\alpha,\beta)$.

Also, we obtain the coefficient bound for $f\in\mathcal{T}\mathcal{S}_q(\alpha,\beta)$ in the following result.

Theorem 7. *The function $f\in\mathcal{T}$ belongs to the class $\mathcal{T}\mathcal{S}_q(\alpha,\beta)$, if and only if*

$$\sum_{n=2}^{\infty}\left([n]_q(1+\beta)-(\alpha+\beta)\right)a_n \le 1-\alpha, \tag{34}$$

where $0<q<1$, $\beta\ge 0$, $0\le\alpha<1$, and $[n]_q$ denotes the q -number.

Proof. Since $\mathcal{T}$ is a subclass of class $\mathcal{A}$, therefore in view of Theorem 6, the sufficient condition of our result holds. Now, we need to prove only the necessary condition. Let $f\in\mathcal{T}\mathcal{S}_q(\alpha,\beta)$ and taking z real, then from inequality (19), we have

$$\frac{zD_q(f(z))}{f(z)}-\alpha > \beta\left|\frac{zD_q(f(z))}{f(z)}-1\right|. \tag{35}$$

Now, using equations (7) and (8) in inequality (35), we get

$$\frac{1-\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}}{1-\sum_{n=2}^{\infty}a_n z^{n-1}}-\alpha > \frac{\sum_{n=2}^{\infty}\beta\left([n]_q a_n z^{n-1}\right)}{1-\sum_{n=2}^{\infty}a_n z^{n-1}}, \tag{36}$$

then, letting $z\to 1$ along the real axis, inequality (36), gives the condition (34).

The coefficient bound of the class $UCV_q(\alpha,\beta)$ is given by the following result.

Theorem 8. *A function $f\in\mathcal{A}$ belongs to the class $\mathcal{UCV}_q(\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[n]_q\left([n]_q(1+\beta)-(\alpha+\beta)\right)|a_n| \le 1-\alpha, \tag{37}$$

where $0<q<1$, $\beta\ge 0$, $0\le\alpha<1$, and $[n]_q$ denotes the q-number.

Proof. Now, using the fact that $-\Re<(z)\le|z|$, we have

$$\beta\left|\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right| - \Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right) \le (1+\beta)\left|\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right|. \tag{38}$$

Using equations (8) and (9) in the right hand side of inequality (38), we get

$$(1+\beta)\left|\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right| = (1+\beta)\left|\frac{1+\sum_{n=2}^{\infty}[n]_q^2 a_n z^{n-1}}{1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}}-1\right|. \tag{39}$$

Since $|z|<1$, therefore, from the above inequality, we get

$$(1+\beta)\left|\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right| < \frac{(1+\beta)\sum_{n=2}^{\infty}\left([n]_q^2-[n]_q\right)|a_n|}{1-\sum_{n=2}^{\infty}[n]_q|a_n|}. \tag{40}$$

Combining inequalities (38) and (40), we get

$$\beta\left|\frac{D_q(D_qf(z))}{D_qf(z)}-1\right| - \Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right) < \frac{(1+\beta)\sum_{n=2}^{\infty}\left([n]_q^2-[n]_q\right)|a_n|}{1-\sum_{n=2}^{\infty}[n]_q|a_n|}. \tag{41}$$

If $((1+\beta)\sum_{n=2}^{\infty}([n]_q^2-[n]_q)|a_n|)/(1-\sum_{n=2}^{\infty}[n]_q|a_n|)<1-\alpha$, which is equivalent to inequality (37), then from inequality (41), we get

$$\beta\left|\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right|-\Re\left(\frac{(D_q(D_qfz))}{D_qf(z)}-1\right)\le 1-\alpha, \quad (42)$$

which is equivalent to inequality (20). Thus, in view of Definition 2, the function $f\in\mathcal{UCV}_q(\alpha,\beta)$.

The coefficient bound for $f\in\mathcal{UCT}_q(\alpha,\beta)$ is given by the following result.

Theorem 9. *The function $f\in\mathcal{T}$ belongs to the class $\mathcal{UCT}_q(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty}[n]_q\Big([n]_q(1+\beta)-(\alpha+\beta)\Big)a_n\le 1-\alpha, \quad (43)$$

where $0<q<1$, $\beta\ge 0$, $0\le\alpha<1$, and $[n]_q$ denotes the q-number.

Proof. Since $\mathcal{T}$ is a subclass of class $\mathcal{A}$, therefore, in view of Theorem 8, the sufficient condition holds. Now, we need to prove only the necessary condition. Let f belong to the class $\mathcal{UCT}_q(\alpha,\beta)$ and taking z real, then from inequality (20), we have

$$\frac{D_q(zD_qf(z))}{D_qf(z)}-\alpha>\beta\left|\frac{D_q(zD_qf(z))}{D_qf(z)}-1\right|. \quad (44)$$

Now, using equations (8) and (9) in inequality (44), we get

$$\frac{1-\sum_{n=2}^{\infty}[n]_q^2a_nz^{n-1}}{1-\sum_{n=2}^{\infty}[n]_qa_nz^{n-1}}-\alpha>\frac{\sum_{n=2}^{\infty}\beta\Big([n]_q^2-[n]_qa_nz^{n-1}\Big)}{1-\sum_{n=2}^{\infty}[n]_qa_nz^{n-1}}, \quad (45)$$

then letting $z\to 1$ along real axis, inequality (45) gives condition (43).

We note that, $q\to 1^-$ in Theorems 6 and 8, we get the coefficient bounds for the functions belonging to the classes $\mathcal{SD}(\alpha,\beta)$ and $\mathcal{KD}(\alpha,\beta)$ in [20], respectively.

In the next section, we obtain the extreme points for the functions belonging to the classes $\mathcal{TS}_q(\alpha,\beta)$ and $\mathcal{UCT}_q(\alpha,\beta)$.

3. Extreme Points

The extreme points of $f\in\mathcal{TS}_q(\alpha,\beta)$ are given by the following result.

Theorem 10. *Let $\{f_n(z)\}_{n\in\mathbb{N}}$ be sequences of functions such that*

$$f_1(z)=z,$$
$$f_n(z)=z-\frac{1-a}{[n]_q(1+\beta)-(a+\beta)}z^n,$$
$$(n\ge 2,\ 0<q<1,\ \beta\ge 0,\ 0\le\alpha<1), \quad (46)$$

where $[n]_q$ denotes the q-number. Then f belongs to $\mathcal{TS}_q(\alpha,\beta)$ if and only if f can be expressed as the form

$$f(z)=\sum_{n=1}^{\infty}\lambda_nf_n(z), \quad (47)$$

where $\lambda_n\ge 0(n\ge 1)$ and $\sum_{n=1}^{\infty}\lambda_n=1$.

Proof. Let $f\in\mathcal{TS}_q(\alpha,\beta)$, then in view of Theorem 7, inequality (34) holds. Since $a_n\ge 0(n\ge 1)$ and $0\le\alpha<1$, therefore from inequality (34), we have

$$\Big([n]_q(1+\beta)-(\alpha+\beta)\Big)a_n\le 1-\alpha, \quad n\ge 2. \quad (48)$$

Thus, if we take

$$\lambda_n\frac{[n]_q(1+\beta)-(\alpha+\beta)}{1-\alpha}a_n, \quad n\ge 2, \quad (49)$$

since $\lambda_1\ge 0$, then, $\lambda_n\ge 0(n\ge 1)$.

Substituting$a_n$from equation (49) with$a_n$from equation (7), we get:

$$f(z)=z-\sum_{n=2}^{\infty}\frac{1-\alpha}{[n]_q(1+\beta)-(\alpha+\beta)}\lambda_nz^n. \quad (50)$$

Since $\sum_{n=1}^{\infty}\lambda_n=1$, therefore, we have

$$\begin{aligned}f(z)&=\lambda_1z+\sum_{n=2}^{\infty}\lambda_nz-\sum_{n=2}^{\infty}\frac{1-\alpha}{[n]_q(1+\beta)-(\alpha+\beta)}\lambda_nz^n\\&=\lambda_1z+\sum_{n=2}^{\infty}\lambda_n\left(z-\frac{1-\alpha}{[n]_q(1+\beta)-(\alpha+\beta)}z^n\right),\end{aligned} \quad (51)$$

since $f_1(z)=z$ and $f_n(z)$ is given by equation (46). Therefore, from equation (51), we get the assertion (47). Conversely, let f be expressible in the form (47), which on using equation (46), gives

$$f(z)=z-\sum_{n=2}^{\infty}\frac{1-\alpha}{[n]_q(1+\beta)-(\alpha+\beta)}\lambda_nz^n, \quad (52)$$

which can be expressed as follows:

$$f(z) = z - \sum_{n=2}^{\infty} \eta_n z^n, \tag{53}$$

where

$$\eta_n = \frac{1-\alpha}{[n]_q(1+\beta)-(\alpha+\beta)} \lambda_n, \quad n \geq 2. \tag{54}$$

Now, to prove that the function f, given by equation (53), belongs to the class $\mathcal{TS}_q(\alpha, \beta)$, we need to show that the coefficients $\eta_n (n \geq 2)$ satisfy the inequality (34).

Since $\lambda_1 \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$, therefore from equation (54), we have

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{[n]_q(1+\beta)-(\alpha+\beta)}{1-\alpha} \eta_n \\ &\quad = \sum_{n=2}^{\infty} \frac{[n]_q(1+\beta)-(\alpha+\beta)}{1-\alpha} \frac{1-\alpha}{[n]_q(1+\beta)-(\alpha+\beta)} \lambda_n \\ &\quad = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1. \end{aligned} \tag{55}$$

Thus, we get

$$\sum_{n=2}^{\infty} \left([n]_q(1+\beta) - (\alpha+\beta) \eta_n \leq 1 - \alpha \right). \tag{56}$$

Therefore, in view of Theorem 7 and the above inequality, we proved that the function f, given by equation (53), belongs to the class $\mathcal{TS}_q(\alpha, \beta)$.

Also, the extreme points of $f \in \mathcal{UCT}_q(\alpha, \beta)$ are given by the following result.

Theorem 11. *Let $\{f_n(z)\}_{n \in \mathbb{N}}$ be a sequence of functions such that*

$$\begin{aligned} &f_1(z) = z, \\ &f_n = z - \frac{1-\alpha}{[n]_q \left([n]_q(1+\beta) - (\alpha+\beta) \right)} z^n, \end{aligned} \tag{57}$$

where $n \geq 2$, $0 < q < 1$, $\beta \geq 0$, and $0 \leq \alpha < 1$. Then, f belongs to $\mathcal{UCT}_q(\alpha, \beta)$ if and only if f can be expressed in the form given by equation (47) in terms of functions $f_n (n \geq 2)$, given by equation (57), and $\lambda_n \geq 0 (n \geq 1)$, $\sum_{n=1}^{\infty} \lambda_n = 1$.

Proof. Let $f \in \mathcal{UCT}_q(\alpha, \beta)$, then from inequality (43), we have

$$\left([n]_q \left([n]_q(1+\beta) - (\alpha+\beta) \right) \right) a_n \leq 1 - \alpha \quad (n \geq 2). \tag{58}$$

If we set

$$\lambda_n = \frac{[n]_q \left([n]_q(1+\beta) - (\alpha+\beta) \right)}{1-\alpha} a_n \quad (n \geq 2), \tag{59}$$

since $\lambda_1 = 1$, then $\lambda_n \geq 0 \, (n \geq 1)$. Then, substituting a_n from equation (59) with a_n equation (7), we get

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1-\alpha}{[n]_q \left([n]_q(1+\beta) - (\alpha+\beta) \right)} \lambda_n z^n. \tag{60}$$

Since $\sum_{n=1}^{\infty} \lambda_n = 1$, therefore, we have

$$f(z) = \lambda_1 z + \sum_{n=2}^{\infty} \left(\frac{1-\alpha}{[n]_q \left([n]_q(1+\beta) - (\alpha+\beta) \right)} z^n \right), \tag{61}$$

since $f_1(z) = z$ and $f_n(z)$ is given by equation (57). Therefore, from equation (61), we get assertion (47).

Conversely, let f be expressible in the form (47), which on using equation (60), gives

$$f(z) = z - \sum_{n=2}^{\infty} \frac{1-\alpha}{[n]_q \left([n]_q(1+\beta) - (\alpha+\beta) \right)} \lambda_n z^n, \tag{62}$$

which can be expressed as

$$f(z) = z - \sum_{n=2}^{\infty} \eta_n z^n, \tag{63}$$

where

$$\eta_n = \frac{1-\alpha}{[n]_q \left([n]_q(1+\beta) - (\alpha+\beta) \right)} \lambda_n, \quad n \geq 2. \tag{64}$$

Now, to prove that function f is given by equation (63) and belongs to the class $\mathcal{UCT}_q(\alpha, \beta)$, we need to show that the coefficient $\eta_n (n \geq 2)$ satisfies inequality (43). Since $\lambda_1 \geq 0$ and $\sum_{n=1}^{\infty} \lambda_n = 1$,, therefore from equation (64), we have

$$\sum_{n=2}^{\infty} \frac{[n]_q \left([n]_q(1+\beta) - (\alpha+\beta) \right)}{1-\alpha} \eta_n = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1. \tag{65}$$

Thus, we get

$$\sum_{n=2}^{\infty} [n]_q \left([n]_q(1+\beta) - (\alpha+\beta) \right) \eta_n < 1 - \alpha. \tag{66}$$

Therefore, in view of Theorem 9 and the above inequality, we proved that function f, given by equation (63), belongs to the class $\in \mathcal{UCT}_q(\alpha, \beta)$.

4. Partial Sums

The sequence of partial sums of the function $f(z) \in \mathcal{A}$, is defined as [22].

$$f_k(z) = z + \sum_{n=2}^{k} a_n z^n \quad (k \in \mathbb{N}\,;\, z \in \mathbb{U}). \tag{67}$$

Now, we find the bounds of the real part of the ratio of the complex valued function $f \in \mathcal{A}$ to its partial sums f_k $(k \in \mathbb{N})$, for the function to be in the class $\mathcal{S}_q(\alpha, \beta)$ in the following result.

Theorem 12. *Let $f(z) \in \mathcal{A}$ in the form (6) and suppose that*

$$\sum_{n=2}^{\infty} c_n |a_n| \leq 1, \tag{68}$$

where

$$c_n \frac{[n]_q(1+\beta) + (\alpha+\beta)}{1-\alpha} \quad (n \geq 2\,;\, 0 < q < 1,\ \beta \geq 0,\ 0 \leq \alpha < 1), \tag{69}$$

then $f(z) \in \mathcal{S}_q(\alpha, \beta)$. Further, the following inequalities hold:

$$\operatorname{Re}\left(\frac{f(z)}{f_k(z)}\right) \geq 1 - \frac{1}{c_{k+1}}, \tag{70}$$

$$\operatorname{Re}\left(\frac{f_k(z)}{f(z)}\right) \geq \frac{c_{k+1}}{1 + c_{k+1}}, \tag{71}$$

where

$$c_n \geq \begin{cases} 1, & \text{if } n = 2, 3, \cdots, k, \\ c_{k+1}, & \text{if } n = k+1, k+2, \cdots, \end{cases} \tag{72}$$

Proof. Since $\{[n]q\}_{n\geq 2}$ is increasing and $\beta \geq 0$, $\alpha < 1$, therefore, in view of equation (69), $\{cn\}_{n\geq 2}$ is an increasing sequence. Then, $c_{n+1} \geq c_n$, $\forall n$ and

$$c_n \geq c_2 \frac{[2]_q(1+\beta) - (\alpha+\beta)}{1+\alpha} \geq \frac{[1]_q(1+\beta) - (\alpha+\beta)}{1-\alpha}. \tag{73}$$

Since $[1]_q = 1$, therefore, we have

$$c_n \geq 1, \quad \forall n. \tag{74}$$

Thus, for the particular value k of n, condition (72) holds. In view of the first inequality of condition (72), we have

$$\sum_{n=2}^{k} |a_n| + c_{k+1} \sum_{n=_{k+1}}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} c_n |a_n|, \tag{75}$$

which in view of inequality (68), gives

$$\sum_{n=2}^{k} |a_n| + c_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq 1, \tag{76}$$

or, equivalently

$$c_{k+1} \sum_{n=k+1}^{\infty} |a_n| \leq 1 - \sum_{n=2}^{k} |a_n|. \tag{77}$$

Now, for some fixed positive integer k, we define

$$h_1(z) := 1 + \frac{c_{k+1}(f(z) - f_k(z))}{f_k(z)}. \tag{78}$$

Now, using equations (6) and (67), equation (78) gives

$$h_1(z) = 1 + \frac{c_{k+1}\left(\sum_{n=k+1}^{\infty} a_n z^{n-1}\right)}{1 + \sum_{n=2}^{k} a_n z^{n-1}}. \tag{79}$$

From equation (79), we have

$$\begin{aligned} \left|\frac{h_1(z) - 1}{h_1(z) + 1}\right| &= \left|\frac{c_{k+1}\sum_{n=k+1}^{\infty} a_n z^{n-1}}{2 + 2\sum_{n=2}^{k} a_n z^{n-1} + c_{k+1}\sum_{n=k+1}^{\infty} a_n z^{n-1}}\right| \\ &\leq \frac{c_{k+1}\sum_{n=k+1}^{\infty} |a_n|}{2 - 2\sum_{n=2}^{k} |a_n| - c_{k+1}\sum_{n=k+1}^{\infty} |a_n|}. \end{aligned} \tag{80}$$

In view of inequality (77), the above inequality gives $|(h_1(z) - 1)/(h_1(z) + 1)| \leq 1$, which implies

$$\operatorname{Re}(h_1(z)) \geq 0. \tag{81}$$

Since each $c_n \in \mathbb{R}$, therefore, using equation (79) in inequality (81), we get assertion (70).

Again, since $\{c_n\}_{n\geq 2}$ is an increasing function and $c_n \geq 1$, $\forall_n \geq 2$, therefore, we have

$$\sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} c_n |a_n|, \tag{82}$$

which in view of inequality (68), gives

$$\sum_{n=2}^{\infty} |a_n| \leq 1. \tag{83}$$

Now, we define the function $h_2(z)$ as follows:

$$h_2(z) = (1 + c_{k+1})\left(\frac{f_k(z)}{f(z)}\right) - c_{k+1}. \tag{84}$$

Using equations (6) and (67) in equation (84), we get

$$h_2(z) = 1 - \frac{(c_{k+1} + 1)\sum_{n=k+1}^{\infty} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} a_n z^{n-1}}. \tag{85}$$

From equation (85), we have

$$\begin{aligned}\left|\frac{h_2(z)-1}{h_2(z)+1}\right| &= \left|\frac{-(c_{k+1}+1)\sum_{n=k+1}^{\infty}a_n z^{n-1}}{2+2\sum_{n=2}^{\infty}a_n z^{n-1}-(c_{k+1}+1)\sum_{n=k+1}^{\infty}a_n z^{n-1}}\right| \\ &= \left|\frac{-(c_{k+1}+1)\sum_{n=k+1}^{\infty}a_n z^{n-1}}{2+2\sum_{n=2}^{k}a_n z^{n-1}+(1-c_{k+1})\sum_{n=k+1}^{\infty}a_n z^{n-1}}\right| \\ &\le \frac{(c_{k+1}+1)\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{k}|a_n|-(1-c_{k+1})\sum_{n=k+1}^{\infty}|a_n|} \\ &\le \frac{(c_{k+1}+1)\sum_{n=k+1}^{\infty}|a_n|}{2-2\sum_{n=2}^{\infty}|a_n|+(c_{k+1}+1)\sum_{n=k+1}^{\infty}|a_n|},\end{aligned} \tag{86}$$

using inequality (83) in inequality (86), we get $|(h_2(z)-1)/(h_2(z)+1)| \le 1$, which implies

$$\operatorname{Re}\,(h_2(z)) \ge 0. \tag{87}$$

Therefore, using equation (84) in inequality (87), we get assertion (71).

Now, we find the bounds of the real part of the ratio of the complex valued function $f \in \mathscr{A}$ to its partial sums $f_k (k \in \mathbb{N})$, for the function to be in the class $\mathscr{UCV}_q(\alpha, \beta)$ in the following result.

Theorem 13. *Let $f(z) \in \mathscr{A}$ be in the form given by equation (6) and*

$$\sum_{n=2}^{\infty} s_n |a_n| \le 1, \tag{88}$$

where

$$s_n \frac{[n]_q\left([n]_q(1+\beta)-(\alpha+\beta)\right)}{1-\alpha} \quad (n \ge 2\,;\, \beta \ge 0, 0 \le \alpha < 1, 0 < q < 1). \tag{89}$$

Then, $f(z) \in \mathscr{UCV}_q(\alpha, \beta)$. Further, the following inequalities hold:

$$\operatorname{Re}\left(\frac{f(z)}{f_k(z)}\right) > 1 - \frac{1}{s_{k+1}}, \tag{90}$$

$$\operatorname{Re}\left(\frac{f_k(z)}{f(z)}\right) > \frac{s_{k+1}}{1+s_{k+1}}, \tag{91}$$

where

$$s_n \ge \begin{cases} 1, & \text{if } n = 2, 3, \cdots k, \\ s_{k+1}, & \text{if } n = k+1, k+2, \cdots. \end{cases} \tag{92}$$

Proof. Using Theorem 6 and following the same steps involved in the proof of Theorem 12, we get assertion (90) and (91).

In the next section, we discuss the integral means inequality for the functions belonging to the classes $\mathscr{TS}_q(\alpha, \beta)$ and $\mathscr{UCT}_q(\alpha, \beta)$.

5. Integral Means Inequality

Silverman [23] has been using the subordination principle to show that the integral $\int_0^{2\pi} |f(re^{1\theta})|^{\sigma} d\theta$ $(\sigma > 0, 0 < r < 1)$ attains its maximum value in class $\mathscr{T}$, when $f_2(z) = z - (z^2/2)$. Then, he applied that principle to solve the integral means inequality $\int_0^{2\pi} |f(re^{i\theta})|^{\sigma} d\theta \le \int_0^{2\pi} |f_2(re^{i\theta})|^{\sigma} d\theta$. Also, he found the integral means inequality for the classes $\mathscr{S}^*(\alpha)$ and $\mathscr{K}(\alpha)$ with negative coefficients.

First, we need to mention the following lemma [24].

Lemma 14. *If f and g are two analytic functions in $\mathbb{U}$ in the form $\mathscr{T}$ and $f \prec g$, then*

$$\int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^{\sigma} d\theta \le \int_0^{2\pi} \left|g\left(re^{i\theta}\right)\right|^{\sigma} d\theta, \tag{93}$$

where $\sigma > 0$, $0 < r < 1$, and $z = re^{i\theta}$.

Now, we establish the integral means inequality for the functions belonging to the class $\mathscr{TS}_q(\alpha, \beta)$.

Theorem 15. *Let f be of the form given by equation (7) that belongs to the class $\mathscr{TS}_q(\alpha, \beta)$ and $f_2(z)$ be defined as follows:*

$$f_2(z) = 1 \frac{1-\alpha}{[2]_q(1+\beta)-(\alpha+\beta)} z, \tag{94}$$

then, for $z = re^{i\theta} (0 < r < 1)$, we have

$$\int_0^{2\pi} |f(z)|^{\sigma} d\theta \le \int_0^{2\pi} |f_2(z)|^{\sigma} d\theta \quad (\sigma > 0). \tag{95}$$

Proof. We define the function $w_1(z)$ as follows:

$$w_1(z) = \sum_{n=2}^{\infty} \frac{[2]_q(1+\beta)-(\alpha+\beta)}{1-\alpha} a_n z^{n-1}. \tag{96}$$

From the above equation, we have

$$w_1(0) = 0. \tag{97}$$

Again, from equation (96), we have

$$\begin{aligned}|w_1(z)| &= \sum_{n=2}^{\infty} \frac{[2]_q(1+\beta)-(\alpha+\beta)}{1-\alpha} a_n z^{n-1} \\ &\le \sum_{n=2}^{\infty} \frac{[2]_q(1+\beta)-(\alpha+\beta)}{1-\alpha} a_n |z|^{n-1},\end{aligned} \tag{98}$$

since $z = re^{i\theta} (0 < r < 1)$ implies $|z| = |r| < 1$, and using inequality (37), therefore, from the above inequality, we have

$$|w_1| \leq \sum_{n=2}^{\infty} \frac{[n]_q(1+\beta)-(\alpha+\beta)}{1-\alpha} a_n \leq 1. \tag{99}$$

From equation (96), we have

$$1-\sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\alpha}{[2]_q(1+\beta)-(\alpha+\beta)} w_1(z). \tag{100}$$

Since w_1 is analytic in $\mathbb{U}$, therefore in view of equations (18), (96), (97), and (100); inequality (99); and the subordination principle, we have

$$1-\sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\alpha}{[2]_q(1+\beta)-(\alpha+\beta)} z. \tag{101}$$

Since, the function on the both sides of the above relation are analytic in $\mathbb{U}$, therefore, in view of Lemma 14 and equation (94), we get assertion (95).

Next, we establish the integral means inequality for the functions belonging to the class $\mathscr{UCT}_q(\alpha, \beta)$ with the positive coefficients.

Theorem 16. *Let f belong to the class $\mathscr{UCT}_q(\alpha, \beta)$ and $f_3(z)$ is defined by*

$$f_3(z) = 1 - \frac{1-\alpha}{[2]_q\left([2]_q(1+\beta)-(\alpha+\beta)\right)} z, \tag{102}$$

then, for $z = re^{i\theta} (0 < r < 1)$, we have

$$\int_0^{2\pi} |f(z)|^\sigma d\theta \leq \int_0^{2\pi} |f_3(z)|^\sigma d\theta \quad (\sigma > 0). \tag{103}$$

Proof. We define the function $w_2(z)$ as follows:

$$w_2(z) = \sum_{n=2}^{\infty} \frac{[2]_q\left([2]_q(1+\beta)-(\alpha+\beta)\right)}{1-\alpha} a_n z^{n-1}. \tag{104}$$

From the above equation, we have

$$w_2(0) = 0. \tag{105}$$

Again, from equation (104), we have

$$\begin{aligned} |w_2(z)| &= \left| \sum_{n=2}^{\infty} \frac{[2]_q\left([2]_q(1+\beta)-(\alpha+\beta)\right)}{1-\alpha} a_n z^{n-1} \right| \\ &\leq \sum_{n=2}^{\infty} \frac{[2]_q\left([2]_q(1+\beta)-(\alpha+\beta)\right)}{1-\alpha} a_n |z|^{n-1}, \end{aligned} \tag{106}$$

since $z = re^{i\theta}$, then $|z| = |r| < 1$ and using inequality (103), therefore, from the above inequality, we have

$$|w_2(z)| \leq \sum_{n=2}^{\infty} \frac{[n]_q\left([n]_q(1+\beta)-(\alpha+\beta)\right)}{1-\alpha} a_n < 1. \tag{107}$$

From equation (104), we have

$$1-\sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1-\alpha}{[2]_q\left([2]_q(1+\beta)-(\alpha+\beta)\right)} w_2(z). \tag{108}$$

Since w_2 is analytic in $\mathbb{U}$, therefore, in view of equations (18), (104), (105), (108); inequality (107); and the subordination principle, we have

$$1-\sum_{n=2}^{\infty} a_n z^{n-1} \prec - \frac{1-\alpha}{[2]_q\left([2]_q(1+\beta)-(\alpha+\beta)\right)} z. \tag{109}$$

Since, the function on the both sides of the above relation are analytic in $\mathbb{U}$, therefore, in view of Lemma 14 and equation (102), we get assertion (103).

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors equally contributed to this manuscript and approved the final version.

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Research Article

New Estimates of $q_1 q_2$-Ostrowski-Type Inequalities within a Class of n-Polynomial Prevexity of Functions

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In this article, we develop a novel framework to study for a new class of preinvex functions depending on arbitrary nonnegative function, which is called n-polynomial preinvex functions. We use the n-polynomial preinvex functions to develop $q_1 q_2$-analogues of the Ostrowski-type integral inequalities on coordinates. Different features and properties of excitement for quantum calculus have been examined through a systematic way. We are discussing about the suggestions and different results of the quantum inequalities of the Ostrowski-type by inferring a new identity for $q_1 q_2$-differentiable function. However, the problem has been proven to utilize the obtained identity, we give $q_1 q_2$-analogues of the Ostrowski-type integrals inequalities which are connected with the n-polynomial preinvex functions on coordinates. Our results are the generalizations of the results in earlier papers.

1. Introduction

Calculus is an imperative study of the derivatives and integrals. The classical derivative was convoluted with the strength regulation kind kernel, and eventually, this gave upward thrust to new calculus referred to as the quantum calculus. In mathematics, quantum calculus (named q-calculus) is the study of calculus without limits. The interest in this subject has exploded, and the q-calculus has in the last twenty years served as a bridge between mathematics and physics. The q -calculus has numerous applications in various fields of mathematics, for example, dynamical systems, number theory, combinatorics, special functions, fractals, and also for scientific problems in some applied areas such as computer science, quantum mechanics, and quantum physics. Jackson [1] defined the q-analogue of derivative and integral operator as well as provided some of their applications. It is imperative to mention that quantum integral inequalities are more practical and informative than their classical counterparts. It has been mainly due to the fact that quantum integral inequalities can describe the hereditary properties of the processes and phenomena under investigation. Historically, the subject of quantum calculus can be traced back to Euler and Jacobi, but in recent decades, it has experienced a rapid development. As a result, new generalizations of the classical concepts of quantum calculus have been initiated and reviewed in many literature. Tariboon and Ntouyas [2, 3] proposed the quantum calculus concepts on finite intervals and obtained several q-analogues of classical mathematical objects, which inspired many other researchers to study the subject in depth, and as a consequence, numerous novel results concerning quantum analogues of classical mathematical results have been launched. Noor et al. [4] obtained new q-analogues of inequality utilizing the first-order q-differentiable convex function.

Inequality plays an irreplaceable role in the development of mathematics. Very recently, many new inequalities such as the Hermite-Hadamard-type inequality [5–9], Petrović-type inequality [10], Pólya-Szegö and Ćebyšev-type inequalities [11], Ostrowski-type inequality [12], reverse Minkowski inequality [13], Jensen-type inequality [14–16], Bessel function inequality [17], trigonometric and hyperbolic functions inequalities [18], fractional integral inequality [19–22], complete and generalized elliptic integrals inequalities [23–28], generalized convex function inequality [29–31], and mean values inequality [32–34] have been discovered by many researchers. While the concept of classical convexity has been brought into a streamline by mathematical inequalities [35–50]. In fact, convex function and its connection with mathematical inequalities have wide applications in the estimation of some parameters in scientific observations and calculations [51–65]. In recent years, the classical concept of convexity has been extended and generalized in different directions, one of the important generalization of convexity is the invexity, which was studied by Hanson [66]; this work has greatly expanded the role of invexity in optimization. In [67, 68], the authors introduced a class of functions, which is called preinvexity as a generalization of convex functions.

Now, we recall the classical and well-known Hermite-Hadamard inequality [69], which can be stated as

$$\Phi\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(\gamma)d\gamma \leq \frac{\Phi(\xi_1)+\phi(\xi_2)}{2}, \quad (1)$$

for all $\xi_1, \xi_2 \in I$ if $\Phi : I \longrightarrow \mathbb{R}$ is a convex function.

Ostrowski [70] established an integral inequality for continuous and differentiable function as follows.

Theorem 1 (See [70]). *Let $\Phi : [\xi 1, \xi 2] \longrightarrow \mathbb{R}$ be continuous and differentiable on (ξ_1, ξ_2) such that $|\Phi'(\gamma)| \leq M$ for all $\gamma \in (\xi_1, \xi_2)$. Then, one has*

$$\left|\Phi(\varrho) - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(\gamma)d\gamma\right| \leq \left[\frac{1}{4} + \frac{(\varrho-\xi_1+\xi_2/2)^2}{(\xi_2-\xi_1)^2}\right](\xi_2-\xi_1)M, \quad (2)$$

for all $\varrho \in [\xi_1, \xi_2]$ with the best possible constant 1/4.

The inequality (2) can be described in an identical kind as

$$\left|\Phi(\varrho) - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(\gamma)d\gamma\right| \leq \frac{M}{\xi_2-\xi_1}\left[\frac{(\varrho-\xi_1)^2+(\varrho)^2}{2}\right]. \quad (3)$$

Latif et al. [71] generalized the Ostrowski inequality (2) to the coordinated convex function by establishing an identity as follows.

Theorem 2 (See [71]). *Let $\xi_1 < \xi_2, \xi_3 < \xi_4$ and $\Phi : [\xi_1, \xi_2] \times [\xi_3, \xi_4] \longrightarrow \mathbb{R}$ be continuous and differentiable on $(\xi_1, \xi_2) \times (\xi_3, \xi_4)$ such that $\partial^2\Phi/\partial_z\partial_w \in L([\xi_1, \xi_2] \times [\xi_3, \xi_4])$.*

Then the identity

$$\begin{aligned}
&\Phi(\varrho,\rho) + \frac{1}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_{\xi_1}^{\xi_2}\int_{\xi_3}^{\xi_4}\Phi(u,v)dudv - y \\
&= \frac{(\varrho-\xi_1)^2(\rho-\xi_3)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z\partial w}\Phi \\
&\cdot(z\varrho+(1-z)\xi_1, w\rho+(1-w)\xi_3)dzdw \\
&- \frac{(\varrho-\xi_1)^2(\xi_4-\rho)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z\partial w}\Phi \\
&\cdot(z\varrho+(1-z)\xi_1, w\rho+(1-w)\xi_4)dzdw \\
&- \frac{(\xi_2-\varrho)^2(\rho-\xi_3)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z\partial w}\Phi \\
&\cdot(z\varrho+(1-z)\xi_2, w\rho+(1-w)\xi_3)dzdw \\
&+ \frac{(\xi_2-\varrho)^2(\xi_4-\rho)^2}{(\xi_2-\xi_1)(\xi_4-\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2}{\partial z\partial w}\Phi \\
&\cdot(z\varrho+(1-z)\xi_2, w\rho+(1-w)\xi_4)dzdw,
\end{aligned} \quad (4)$$

holds for all $(\varrho, \rho) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$, where

$$y = \frac{1}{\xi_4-\xi_3}\int_{\xi_3}^{\xi_4}\Phi(\varrho,v)dv + \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(u,\rho)du. \quad (5)$$

Noor et al. [4] presented the Ostrowski-type inequality for quantum calculus.

Theorem 3 (See [4]). *Let $q \in (0,1), \xi_1 < \xi_2$ and $\Phi[\xi_1, \xi_2] \longrightarrow \mathbb{R}$ be continuous such that ${}_{\xi_1}D_q\Phi$ is integrable on (ξ_1, ξ_2). Then*

$$\begin{aligned}
&\Phi(\varrho) - \frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi(u)\xi_1 d_q u = \frac{q(\varrho-\xi_1)^2}{\xi_2-\xi_1}\int_0^1 z_{\xi_1}D_q\Phi \\
&\cdot((1-z)\xi_1+z\varrho)_0 d_q z + \frac{q(\xi_2-\varrho)^2}{\xi_2-\xi_1}\int_0^1 z_{\xi 1}D_q\Phi \\
&\cdot((1-z)\xi_2+z\varrho)0d_q z.
\end{aligned} \quad (6)$$

The following quantum integral version of the Hermite-Hadamard-type inequality for the coordinated convex function was proved by Alp and Sarıkaya [72].

Theorem 4 (See [72]). *Let $q_1, q_2 \in (0,1)$, $\xi_1 < \xi_2$, $\xi_3 < \xi_4$ and $\Phi : \mathcal{N} = [\xi_1, \xi_2] \times [\xi_3, \xi_4] \longrightarrow \mathbb{R}$ be a coordinated convex* *function on $\mathcal{N}$. Then one has*

$$\begin{aligned}
&\Phi\left(\frac{q_1\xi_1+\xi_2}{1+q_1}, \frac{q_2\xi_3+\xi_4}{1+q_2}\right) \\
&\leq \frac{1}{2}\left[\frac{1}{\xi_2-\xi_1}\int_{\xi_1}^{\xi_2}\Phi\left(z, \frac{q_2\xi_3+\xi_4}{1+q_2}\right)_{\xi_1}d_{q1}z \right.\\
&\quad \left.+\frac{1}{\xi_3-\xi_4}\int_{\xi_3}^{\xi_4}\Phi\left(\frac{q_1\xi_1+\xi_2}{1+q_1}, w\right)_{\xi_3}d_{q_2}w\right] \\
&\leq \frac{1}{(\xi_1-\xi_2)(\xi_3-\xi_4)}\int_{\xi_1}^{\xi_2}\int_{\xi_3}^{\xi_4}\Phi(z,w)_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w \\
&\leq \frac{1}{4}\left[\frac{1}{(1+q_1)(\xi_2-\xi_1)}\left(q_2\int_{\xi_1}^{\xi_2}\Phi(z,\xi_3)_{\xi_1}d_{q_1}z+\int_{\xi_1}^{\xi_2}\Phi(z,\xi_4)_{\xi_1}d_{q_1}z\right)\right.\\
&\quad \left.+\frac{1}{(1+q_2)(\xi_3-\xi_4)}\left(q_1\int_{\xi_3}^{\xi_3}\Phi(\xi_1,w)_{\xi_3}d_{q_2}w+\int_{\xi_3}^{\xi_4}\Phi(\xi_2,w)_{\xi_3}d_{q_2}w\right)\right] \\
&\leq \frac{q_1q_2\Phi(\xi_1,\xi_3)+q_2\Phi(\xi_2,\xi_3)+q_1\Phi(\xi_1,\xi_4)+\Phi(\xi_2,\xi_4)}{(1+q_1)(1+q_2)}.
\end{aligned} \tag{7}$$

Kalsoom et al. [73] found the quantum integral inequality for two parameters function on the finite rectangle.

Next, we present the definitions of q_1q_2-derivative and integral, and their two known results.

Definition 5. Let $q_1, q_2 \in (0,1)$, $\xi_1 < \xi_2$, $\xi_3 < \xi_4$ and $\Phi : [\xi_1, \xi_2] \times [\xi_3, \xi_4] \longrightarrow \mathbb{R}$ be a continuous function. Then, the partially q_1-derivative, q_2-derivative, and q_1q_2-derivative at $(z, w) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$ for the function Φ are defined by

$$\begin{aligned}
\frac{{}_{\xi_1}\partial_{q_1}\Phi(z,w)}{{}_{\xi_1}\partial_{q_1}z} &= \frac{\Phi(z,w)-\Phi(q_1z+(1-q_1)\xi_1,w)}{(1-q_1)(z-\xi_1)}\,(z\neq\xi_1), \\
\frac{{}_{\xi_3}\partial_{q_2}\Phi(z,w)}{{}_{\xi_3}\partial_{q_2}w} &= \frac{\Phi(z,w)-\Phi(z,q_2w+(1-q_2)\xi_3)}{(1-q_2)(w-\xi_3)}\,(w\neq\xi_3), \\
\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Phi(z,w)}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w} &= \frac{1}{(1-q_1)(1-q_2)(z-\xi_1)(w-\xi_3)} \\
&\quad\times[\Phi(q_1z+(1-q_1)\xi_1, q_2w+(1-q_2)\xi_3) \\
&\quad-\Phi(q_1z+(1-q_1)\xi_1,w) \\
&\quad-\Phi(z,q_2w+(1-q_2)\xi_3)+\Phi(z,w)] \\
&\quad(z\neq\xi_1, w\neq\xi_3),
\end{aligned} \tag{8}$$

respectively. The function Φ is said to be partially q_1-, q_2-, and q_1q_2-differentiable on $[\xi_1, \xi_2] \times [\xi_3, \xi_4]$ if ${}_{\xi_1}\partial_{q1}\Phi(z,w)/{}_{\xi_1}\partial_{q1}z$, ${}_{\xi_3}\partial_{q2}\Phi(z,w)/{}_{\xi_3}\partial_{q2}w$, and ${}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Phi(z,w)/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w$ exist for all $(z,w) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$.

Definition 6. Let $q_1, q_2 \in (0,1)$, $\xi_1 < \xi_2$, $\xi_3 < \xi_4$ and $\Phi : [\xi_1, \xi_2] \times [\xi_3, \xi_4] \longrightarrow \mathbb{R}$ be a continuous function. Then the q_1q_2-integral of the function Φ on $[\xi_1, \xi_2] \times [\xi_3, \xi_4]$ is defined by

$$\begin{aligned}
&\int_{\xi_3}^{t}\int_{\xi_1}^{s}\Phi(z,w)_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w = (1-q_1)(1-q_2)(s-\xi_1)(t-\xi_3) \\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\Phi(q_1^ns+(1-q_1^n)\xi_1, q_2^mt+(1-q_2^m)\xi_3) \\
&\quad\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q_1^nq_2^m\Phi(q_1^ns+(1-q_1^n)\xi_1, q_2^mt+(1-q_2^m)\xi_3),
\end{aligned} \tag{9}$$

for $(s,t) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$.

Theorem 7. *Let $\xi_1 < \xi_2$, $\xi_3 < \xi_4$ and $\Phi : [\xi_1, \xi_2] \times [\xi_3, \xi_4] \longrightarrow \mathbb{R}$ be a continuous function. Then, we have the identities*

$$\begin{aligned}
&\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}s_{\xi_3}\partial_{q_2}t}\int_{\xi_4}^{t}\int_{\xi_1}^{s}\Phi(z,w)_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w = \Phi(s,t), \\
&\int_{\xi_3}^{t}\int_{\xi_1}^{s}\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Phi(z,w)}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\,{}_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w = \Phi(s,t), \\
&\int_{t_1}^{t}\int_{s_1}^{s}\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Phi(z,w)}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\,{}_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w = \Phi(s,t)-\Phi(s,t_1) \\
&\quad -\Phi(s_1,t)+\Phi(s_1,t_1),
\end{aligned} \tag{10}$$

for $(s_1, t_1) \in (\xi_1, s) \times (\xi_4, t)$.

Theorem 8. *Let $a \in \mathbb{R}$, $\xi_1 < \xi_2$, $<\xi_3 < \xi_4$ and $\Phi_1, \Phi_2 : [\xi_1, \xi_2] \times [\xi_3, \xi_4] \longrightarrow \mathcal{R}$ be continuous functions. Then, the identities*

$$\begin{aligned}
&\int_{\xi_3}^{t}\int_{\xi_1}^{s}[\Phi_1(z,w)+\Phi_2(z,w)]_{\xi_1}d_{q_1}z_{\xi_4}d_{q_2}w \\
&\quad=\int_{\xi_3}^{t}\int_{\xi_1}^{s}\Phi_1(z,w)_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w+\int_{\xi_3}^{t}\int_{\xi_1}^{s}\Phi_2(z,w)_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w, \\
&\int_{\xi_3}^{t}\int_{\xi_1}^{s}a\Phi(z,w)_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w = a\int_{\xi_3}^{t}\int_{\xi_1}^{s}\Phi(z,w)_{\xi_1}d_{q_1}z_{\xi_3}d_{q_2}w,
\end{aligned} \tag{11}$$

holds for $(s,t) \in [\xi_1, \xi_2] \times [\xi_3, \xi_4]$.

Very recently, Toplu et al. [74] improved the Hermite-Hadamard inequality (1) by investigating the n-polynomial convexity. The main purpose of the article is to introduce the notion of n-polynomial preinvex function, provide a new generalized quantum integral identity, establish new quantum analogues of Ostrowski-type inequalities for the n-polynomial preinvex function on coordinates, and generalize and unify the previous known results.

2. Discussions and Main Results

In the beginning of this section, we introduce the definition of n-polynomial prevexity.

Definition 9 (See [75]). Let $\eta(\cdot,\cdot)$: $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a continuous bi-function. Then, $\Omega_\eta \subset \mathbb{R}^n$ is said to be invex if

$$\xi_1 + \gamma\eta(\xi_2, \xi_1) \in \Omega_\eta, \tag{12}$$

for all $\xi_1, \xi_2 \in \Omega_\eta$ and $\gamma \in [0, 1]$.

Definition 10 (See [67]). The function $\Phi : \Omega_\eta \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be preinvex if

$$\Phi(\xi_1 + \gamma\eta(\xi_2, \xi_1)) \leq (1-\gamma)\Phi(\xi_1) + \gamma\Phi(\xi_2), \tag{13}$$

for all $\xi_1, \xi_2 \in \Omega_\eta$ and $\gamma \in [0, 1]$.

Definition 11. Let $n \in \mathbb{N}$. Then, the nonnegative function $\Phi : \Omega_\eta \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is said to be n-polynomial preinvex if

$$\begin{aligned}\Phi(\xi_1 + \gamma\eta(\xi_2, \xi_1)) \leq& \frac{1}{n}\sum_{p=1}^{n}\left[1-(1-\gamma)^p\right]\Phi(\xi_1)\\ &+ \frac{1}{n}\sum_{p=1}^{n}[1-(1-\gamma^p)]\Phi(\xi_2),\end{aligned} \tag{14}$$

for all $\xi_1, \xi_2 \in \Omega_\eta$ and $\gamma \in [0, 1]$.

Note that if $n = 1$, then the definition of n-polynomial preinvex function reduce to the definition of preinvex function.

If we take $n = 2$, then we have 2-polynomial preinvex function inequality

$$\Phi(\xi_1 + \gamma\eta(\xi_2, \xi_1)) \leq \frac{3\gamma-\gamma^2}{2}\Phi(\xi_1) + \frac{1-\gamma-\gamma^2}{2}\Phi(\xi_2). \tag{15}$$

Proposition 12. *Let $\xi_2 > 0$ and $\Phi_\alpha : [\xi_1, \xi_1 + \eta(\xi_2, \xi_1)] \longrightarrow \mathbb{R}$ be an arbitrary family of n -polynomial preinvex functions and $\Phi(\xi) = \sup_\alpha \Phi_\alpha(\xi)$. If $J_\eta = \{u \in [\xi_1, \xi_1 + \eta(\xi_2, \xi_1)]: \Phi(u) < 1\}$ is nonempty, then J_η is an interval and Φ is an n-polynomial preinvex function on J_η.*

Proof. Let $\gamma \in [0, 1]$ and $\xi_1, \xi_1 + \eta(\xi_2, \xi_1) \in J_\eta$. Then, we have

$$\begin{aligned}\Phi(\xi_1 + \gamma\eta(\xi_2, \xi_1)) &= \sup_\alpha \Phi_\alpha(\xi_1 + \gamma\eta(\xi_2, \xi_1))\\ &\leq \sup_\alpha\left[\frac{1}{n}\sum_{p=1}^{n}\left[1-(1-\gamma)^p\right]\Phi_\alpha(\xi_1) + \frac{1}{n}\sum_{p=1}^{n}[1-\gamma]^p\Phi_\alpha(\xi_2)\right]\\ &\leq \frac{1}{n}\sum_{p=1}^{n}\left[1-(1-\gamma)^p\right]\sup_\alpha\Phi_\alpha(\xi_1) + \frac{1}{n}\sum_{p=1}^{n}(1-\gamma)^p\sup_\alpha\Phi_\alpha(\xi_2)\\ &\leq \frac{1}{n}\sum_{p=1}^{n}\left[1-(1-\gamma)^p\right]\Phi(\xi_1) + \frac{1}{n}\sum_{p=1}^{n}(1-\gamma)^p\Phi(\xi_2) < \infty.\end{aligned} \tag{16}$$

This completes the proof.

In order to establish new quantum analogues of the Ostrowski-type inequalities on coordinates for the n-polynomial preinvex function, we need a key lemma, which we present in this section.

Lemma 13. *Let $q_1, q_2 \in (0, 1), \xi_1 < \xi_2, \xi_3 < \xi_4, \mathcal{N} = [\xi_1, \xi_2] \times [\xi_3, \xi_4], \mathcal{N}^\circ$, be the interior of $\mathcal{N}$, and $\Phi : \mathcal{N} \longrightarrow \mathbb{R}$ be mixed partial q_1q_2-differentiable on $\mathcal{N}^\circ$ such that $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2} / {}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w)\Phi|$ is continuous and integrable on $[\xi_1, \xi_1 + \eta_1(\xi_2, \xi_1)] \times [\xi_3, \xi_3 + \eta_2(\xi_4, \xi_3)] \subset \mathcal{N}^\circ$ for $\eta_1(\xi_2, \xi_1), \eta_2(\xi_4, \xi_3) > 0$. Then, we have the identity*

$$\begin{aligned}\Omega_{q_1q_2}(\xi_1, \xi_2, \xi_3, \xi_4)(\Phi) =& -\frac{q_1q_2[\eta_1(\varrho, \xi_1)]^2[\eta_2(\rho, \xi_3)]^2}{\eta_1(\xi_2, \xi_1)\eta_2(\xi_4, \xi_3)}\\ &\times\int_0^1\int_0^1 zw\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_1 + z\eta_1(\varrho, \xi_1), \xi_3 + w\eta_2(\rho, \xi_3))_0 d_{q_1}z_0 d_{q_2}w\\ &- \frac{q_1q_2[\eta_1(\varrho, \xi_1)]^2[\eta_2(\xi_4, \rho)]^2}{\eta_1(\xi_2, \xi_1)\eta_2(\xi_4, \xi_3)}\\ &\times\int_0^1\int_0^1 zw\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_1 + z\eta_1(\varrho, \xi_1), \xi_4 + w\eta_2(\rho, \xi_4))_0 d_{q_1}z_0 d_{q_2}w\\ &- \frac{q_1q_2[\eta_1(\xi_2, \varrho)]^2[\eta_2(\rho, \xi_3)]^2}{\eta_1(\xi_2, \xi_1)\eta_2(\xi_4, \xi_3)}\\ &\times\int_0^1\int_0^1 zw\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_2 + z\eta_1(\varrho, \xi_2), \xi_3 + w\eta_2(\rho, \xi_3))_0 d_{q_1}z_0 d_{q_2}w\\ &- \frac{q_1q_2[\eta_1(\xi_2, \varrho)]^2[\eta_2(\xi_4, \rho)]^2}{\eta_1(\xi_2, \xi_1)\eta_2(\xi_4, \xi_3)}\\ &\times\int_0^1\int_0^1 zw\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1} z_{\xi_3}\partial_{q_2} w}\Phi(\xi_2 + z\eta_1(\varrho, \xi_2), \xi_4 + w\eta_2(\rho, \xi_4))_0 d_{q_1}z_0 d_{q_2}w,\end{aligned} \tag{17}$$

where

$$\begin{aligned}
\Omega_{q_1q_2}(\xi_1,\xi_2,\xi_3,\xi_4)(\Phi) &= \Phi(\varrho,\rho) \\
&+ \frac{1}{\eta_2(\xi_4,\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)} \Phi(\varrho,\nu)_0 d_{q_2}\nu + \int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)} \Phi(\varrho,\nu)_0 d_{q_2}\nu\right] \\
&+ \frac{1}{\eta_1(\xi_2,\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)} \Phi(u,\rho)_0 d_{q_1}u + \int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)} \Phi(u,\rho)_0 d_{q_1}\nu\right] - \mathcal{Q}, \\
\mathcal{Q} &= \frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)} \Phi(u,\nu)_0 d_{q_1}u_0 d_{q_2}\nu \\
&+ \frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)} \Phi(u,\nu)_0 d_{q_1}u_0 d_{q_2}\nu \\
&+ \frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)} \Phi(u,\nu)_0 d_{q_1}u_0 d_{q_2}\nu \\
&+ \frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)} \Phi(u,\nu)_0 d_{q_1}u_0 d_{q_2}\nu.
\end{aligned} \tag{18}$$

Proof. Considering

$$\begin{aligned}
&-\int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))_0 d_{q_1}z_0 d_{q_2}w \\
&-\int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))_0 d_{q_1}z_0 d_{q_2}w \\
&-\int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi 3+w\eta_2(\rho,\xi_3))_0 d_{q_1}z_0 d_{q_2}w \\
&-\int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))_0 d_{q_1}z_0 d_{q_2}w,
\end{aligned} \tag{19}$$

it follows from the definitions of partial q_1q_2-derivative and q_1q_2-integral that

$$\begin{aligned}
&\int_0^1\int_0^1 zw \frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))_0 d_{q_1}z_0 d_{q_2}w \\
&= \frac{1}{(1-q_1)(1-q_2)\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \times \Bigg[\int_0^1\int_0^1 \Phi(\xi_1+zq_1\eta_1(\varrho,\xi_1),\xi_3 \\
&+ wq_2\eta_2(\rho,\xi_2))_0 d_{q_1}z_0 d_{q_2}w - \int_0^1\int_0^1 \Phi(\xi_1+zq_1\eta_1(\varrho,\xi_1),\xi_3 \\
&+ w\eta_2(\rho,\xi_3))_0 d_{q_1}z_0 d_{q_2}w - \int_0^1\int_0^1 \Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+wq_2\eta_2(\rho,\xi_3))_0 d_{q_1}z_0 d_{q_2}w \\
&+ \int_0^1\int_0^1 \Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))_0 d_{q_1}z_0 d_{q_2}w\Bigg] = \frac{1}{\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \\
&\times \Bigg[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi\Big(\xi_1+q_1^{n+1}\eta_1(\varrho,\xi_1),\xi_3+q_2^{m+1}\eta_2(\rho,\xi_3) \\
&- \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n+1}\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3))\Big) \\
&- \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^{m+1}\eta_2(\rho,\xi_3)) \\
&+ \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^{n+1}\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3))\Bigg] \\
&= \frac{1}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)) \\
&- \frac{1}{q_1\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)) \\
&- \frac{1}{q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)) \\
&+ \frac{1}{\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)).
\end{aligned} \tag{20}$$

Note that

$$\begin{aligned}
&\frac{1}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)) \\
&= -\frac{\Phi(\varrho,\rho)}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} - \frac{1}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=0}^{\infty} q^n \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\rho) \\
&- \frac{1}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{m=0}^{\infty} q_2^m(\varrho,\xi_3+q_2^m\eta_2(\rho,\xi_3)) - \frac{1}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \\
&\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)),
\end{aligned} \tag{21}$$

$$\begin{aligned}
&-\frac{1}{q_1\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)) \\
&= \frac{1}{q_1\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{m=0}^{\infty} q_2^m \Phi(\varrho,\xi_3+q_2^m\eta_2(\rho,\xi_3)) - \frac{1}{q_1\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)} \\
&\times \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q_1^n q_2^m \Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)),
\end{aligned} \tag{22}$$

$$-\frac{1}{q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}q_1^nq_2^m\Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3))$$
$$=\frac{1}{q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\sum_{n=0}^{\infty}q_1^n\Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\rho)-\frac{1}{q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}\times\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)). \tag{23}$$

From (20)–(23) we get

$$\int_0^1\int_0^1 zw\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))_0d_{q_1}z_0d_{q_2}w$$
$$=-\frac{\Phi(\varrho,\rho)}{q_2q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}-\frac{(1-q_2)\eta_2(\rho,\xi_3)}{q_1q_2\eta_1(\varrho,\xi_1)[\eta_2(\rho,\xi_3)]^2}\sum_{m=0}^{\infty}q_2^m(\varrho,\xi_3+q_2^m\eta_2(\rho,\xi_3))-\frac{(1-q_1)\eta_1(\varrho,\xi_1)}{q_1q_2[\eta_1(\varrho,\xi_1)]^2\eta_2(\rho,\xi_3)}\sum_{n=0}^{\infty}q_2^n\Phi(\xi_1+q_1^m\eta_1(\varrho,\xi_1),\rho)$$
$$+\frac{(1-q_1)(1-q_2)\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_1^nq_2^m\Phi(\xi_1+q_1^n\eta_1(\varrho,\xi_1),\xi_3+q_2^m\eta_2(\rho,\xi_3)), \tag{24}$$

$$\int_0^1\int_0^1 zw\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))_0d_{q_1}z_0d_{q_2}w$$
$$=-\frac{\Phi(\varrho,\rho)}{q_1q_2\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}-\frac{\Phi(\varrho,\rho)}{q_1q_2\eta_1(\varrho,\xi_1)[\eta_2(\rho,\xi_3)]^2}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,\nu)_0d_{q_2}\nu-\frac{1}{q_1q_2[\eta_1(\varrho,\xi_1)]^2\eta_2(\rho,\xi_3)}\int_{\xi_1}^{\xi_3+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)_0d_{q_1}\nu$$
$$+\frac{1}{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}\int_{\xi_1}^{\xi_2+\eta_1(\varrho,\xi_2)}\int_{\xi_3}^{\xi_3+\eta_2(\varrho,\xi_3)}\Phi(u,\nu)_0d_{q_1}u_0d_{q_2}\nu. \tag{25}$$

Multiplying both sides of equality (25) by

$$\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}, \tag{26}$$

leads to

$$\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_1(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{\xi_1\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\times\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))_0d_{q_1}z_0d_{q_2}w$$
$$=-\frac{\eta_1(\varrho,\xi_1)\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Phi(\varrho,\rho)-\frac{\eta_1(\varrho,\xi_1)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,\nu)_0d_{q_2}\nu$$
$$-\frac{\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)_0d_{q_1}u+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(u,\nu)_0d_{q_1}u_0d_{q_2}\nu. \tag{27}$$

Similarly, we have

$$\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\times\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+z\eta_2(\rho,\xi_4))_0d_{q_1}z_0d_{q_2}w$$
$$=-\frac{\eta_1(\varrho,\xi_1)\eta_2(\xi_4,\rho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Phi(\varrho,\rho)-\frac{\eta_1(\varrho,\xi_1)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(\varrho,\nu)_0d_{q_2}\nu$$
$$+\frac{\eta_2(\xi_4,\rho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)_0d_{q_1}u+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(u,\nu)_0d_{q_1}u_0d_{q_2}\nu,$$
$$\times\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\times\Phi(\xi_2+z\eta_1(\varrho,\xi_2)+w\eta_2(\rho,\xi_3))_0d_{q_1}z_0d_{q_2}w$$
$$=-\frac{\eta_1(\xi_2,\varrho)\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Phi(\varrho,\rho)-\frac{\eta_1(\xi_2,\varrho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,\nu)_0d_{q_2}\nu$$
$$-\frac{\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\rho,\xi_2)}\Phi(u,\rho)_0d_{q_1}u+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(u,\nu)_0d_{q_1}u_0d_{q_2}\nu, \tag{28}$$

$$\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\times\Phi(\xi_2+z\eta_1(\varrho,\xi_2)+w\eta_2(\rho,\xi_3))_0d_{q_1}z_0d_{q_2}w$$
$$=-\frac{\eta_1(\xi_2,\varrho)\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Phi(\varrho,\rho)-\frac{\eta_1(\xi_2,\varrho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)_0d_{q_2}v \quad (29)$$
$$-\frac{\eta_2(\rho,\xi_3)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\rho,\xi_2)}\Phi(u,\rho)_0d_{q_1}u+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(u,v)_0d_{q_1}u_0d_{q_2}v,$$

$$\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_0^1\int_0^1 zw\frac{\partial^2_{q_1,q_2}}{_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\times\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))_0d_{q_1}z_0d_{q_2}w$$
$$=-\frac{\eta_1(\xi_2,\varrho)\eta_2(\xi_4,\rho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\Phi(\varrho,\rho)-\frac{\eta_1(\xi_2,\varrho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(\varrho,v)_0d_{q_2}v \quad (30)$$
$$-\frac{\eta_2(\xi_4,\rho)}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\Phi(u,\rho)_0d_{q_1}u+\frac{1}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(u,v)_0d_{q_1}u_0dq_2v.$$

Therefore, Lemma 13 follows from (19), (27), (28), (29), and (30).

Theorem 14. *Let* $q_1,q_2\in(0,1)$, $\xi_1<\xi_2$, $\xi_3<\xi_4,\Phi:\mathcal{N}\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$ *be mixed partial* q_1q_2 *differentiable on* N° *(the interior of* $\mathcal{N}$*) such that its mixed partial* q_1q_2*-derivatives is continuous and integrable on* $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]\subset\mathcal{N}^\circ$ *for* $\eta_1(\xi_2,\xi_1),\eta_2(\xi_4,\xi_3)>0$. *If* $|(\xi_1,\xi_3\partial^2_{q_1,q_2}/\xi_1\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi|$ *is an n-polynomial preinvex function on the coordinates on* $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]$ *and* $|(\xi_1,\xi_3\partial^2_{q_1,q_2}/\xi_1\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi(\varrho,\rho)|\le M$ *for* $\varrho,\rho\in\mathcal{N}$, *then we have*

$$\left|\Phi(\varrho,\rho)+\frac{1}{\eta_2(\xi_4,\xi_1)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)_0d_{q_2}v+\int_{\xi_4}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)_0d_{q_2}v\right]\right.$$
$$\left.+\frac{1}{\eta_1(\xi_2,\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)_0d_{q_1}u+\int_{\xi_2}^{\xi_2,\eta_1(\varrho,\xi_3)}\Phi(u,\rho)_0d_{q_1}u\right]-Q\right|$$
$$\le q_1q_2M\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right)\left[\frac{[\eta_1(\varrho,\xi_1)]^2+[\eta_1(\xi_2,\varrho)]^2}{\eta_1(\xi_2,\xi_1)}\right]$$
$$\times\left[\frac{[\eta_2(\rho,\xi_3)]^2+[\eta_2(\xi_4,\rho)]^2}{\eta_2(\xi_4,\xi_3)}\right], \quad (31)$$

where Q is defined in Lemma 13.

Proof. It follows from (19) and the properties of the n-polynomial preinvexity function of the function $|(\partial^2_{q_1,q_2}/\xi_1\partial_{q_1}z_{\xi_3}\partial_{q2}w)\Phi|$ on coordinates that

$$\left|\Phi(\varrho,\rho)+\frac{1}{\eta_2(\xi_4\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,\rho)_0d_{q_2}v+\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(\varrho,\rho)_0d_{q_2}v\right]+\frac{1}{\eta_1(\xi_2\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)_0d_{q1}u+\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\Phi(u,\rho)_0d_{q1}u\right]-Q\right|$$
$$\le\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2\xi_1)\eta_2(\xi_4\xi_3)}\int_0^1\int_0^1 zw\left|\frac{_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|_0d_{q_1}z_0d_{q_2}w$$
$$+\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2\xi_1)\eta_2(\xi_4\xi_3)}\int_0^1\int_0^1 zw\left|\frac{_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|_0d_{q_1}z_0d_{q_2}w \quad (32)$$
$$+\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2\xi_1)\eta_2(\xi_4\xi_3)}\int_0^1\int_0^1 zw\left|\frac{_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\right|_0d_{q_1}z_0d_{q_2}w$$
$$+\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2\xi_1)\eta_2(\xi_4\xi_3)}\int_0^1\int_0^1 zw\left|\frac{_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\right|_0d_{q_1}z_0d_{q_2}w.$$

Considering the first integral

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w \le \int_0^1 w\left\{\int_0^1 z\left[\frac{1}{n}\sum_{p=1}^n\left[1-(1-z)^p\right]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\varrho,\xi_3+w\eta_2(\rho,\xi_3))\right| + \frac{1}{n}\sum_{p=1}^n[1-z^p]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1,\xi_3+w\eta_2(\rho,\xi_3))\right|\right]{}_0d_{q_1}z\right\}\cdot{}_0d_{q_2}w, \tag{33}$$

in view of the Definition 6 for $k = 1, 2$, we get

$$\mathscr{A}_{qk}=\frac{1}{n}\sum_{p=1}^n\int_0^1 z\left[1-(1-z)^p\right]{}_0d_{qk}z=\frac{1}{1+q_k}-\frac{(1-q_k)}{n}\sum_{p=1}^n\sum_{e=0}^\infty q_k^{2e}(1-q_k^e)^p,$$
$$\mathscr{B}_{qk}=\frac{1}{n}\sum_{p=1}^n\int_0^1 z[1-z^p]{}_0d_{qk}z=\frac{1}{1+q_k}-\frac{1}{n}\sum_{p=1}^n\frac{1-q_k}{1-q_k^{p+2}}. \tag{34}$$

From (33) and computing the q_1-integral, we get

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w \le \int_0^1 w\left[\begin{array}{c}\mathscr{A}_{q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\varrho,\xi_3+w\eta_2(\rho,\xi_3))\right| \\ +\mathscr{B}_{q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q2}w}\Phi(\xi_1,\xi_3+w\eta_2(\rho,\xi_3))\right|\end{array}\right]{}_0d_{q2}w. \tag{35}$$

Computing the q_2-integral and utilizing the fact $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z\partial_{q_2}w)\Phi(\varrho,\rho)|\le M$ for $\varrho,\rho\in\mathscr{N}$, inequality (35) leads to the conclusion that

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w \le M\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right). \tag{36}$$

Analogously, we also have

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|{}_0d_{q_1}z_0d_{q_2}w \le M\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right), \tag{37}$$

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\right|{}_0d_{q_1}z_0d_{q_2}w \le M\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right), \tag{38}$$

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\right|{}_0d_{q_1}z_0d_{q_2}w \le M\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right). \tag{39}$$

Making use of the inequalities (36), (37), (38), and (39) and the fact that

$$[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2+[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_4,\rho)]^2 + [\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2+[\eta_1(\xi_2,\varrho)]^2[\eta_2(\xi_4,\rho)]^2 = \left[[\eta_1(\varrho,\xi_1)]^2+[\eta_1(\xi_2,\varrho)]^2\right]\left[[\eta_2(\rho,\xi_3)]^2+[\eta_2(\xi_4,\rho)]^2\right], \tag{40}$$

we get the desired inequality (31).

Theorem 15. *Let $\gamma_1,\gamma_2>1$ with $(1/\gamma_2)+(1/\gamma_1)=1$, $q_1,q_2\in(0,1)$, and $\Phi:\mathscr{N}\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$ be mixed partial q_1q_2-differentiable on N° such that its mixed partial q_1q_2-derivative is continuous and integrable on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]\subset\mathscr{N}^\circ$ for $\eta_1(\xi_2,\xi_1),\eta_2(\xi_4,\xi_3)>0$. If $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi|^{\gamma_2}$ is an n-polynomial preinvex function on the coordinates on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]$ and $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi(\varrho,\rho)|\le M$ for $\varrho,\rho,\in\mathscr{N}$, then we have*

$$\left|\Phi(\varrho,\rho)+\frac{1}{\eta_2(\xi_4,\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,\rho){}_0d_{q_2}v+\int_{\xi_4}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,\rho){}_0d_{q_2}v\right]\frac{1}{\eta_1(\xi_2,\xi_1)}\cdot\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho){}_0d_{q_1}u+\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\Phi(u,\rho){}_0d_{q_1}u\right]-\mathscr{Q}\right| \le \frac{q_1q_2M\sqrt[\gamma_2]{\left(\mathscr{C}_{q_1}+\mathscr{D}_{q_1}\right)\left(\mathscr{C}_{q_2}+\mathscr{D}_{q_2}\right)}}{\sqrt[\gamma_1]{[\gamma_1+1]_{q_1}[\gamma_1+1]_{q_2}}}\cdot\left[\frac{[\eta_1(\varrho,\xi_1)]^2+[\eta_1(\xi_2,\varrho)]^2}{\eta_1(\xi_2,\xi_1)}\right]\times\left[\frac{[\eta_2(\rho,\xi_3)]^2+[\eta_2(\xi_4,\rho)]^2}{\eta_2(\xi_4,\xi_3)}\right], \tag{41}$$

where $\mathcal{Q}$ is defined in Lemma 13.

Proof. It follows from (19), the Hölder inequality and the property of n-polynomial preinvexity of the function $|\partial^2_{q_1,q_2}\Phi/\xi_1\partial_{q_1}z_{\xi_3}\partial_{q2}w|^{\gamma_2}$ on coordinates that

$$\begin{aligned}
&\Bigg|\Phi(\varrho,\rho)+\frac{1}{\eta_2(\xi_4,\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)\,{}_0d_{q_2}v\right.\\
&\quad\left.+\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(\varrho,v)\,{}_0d_{q_2}v\right]\\
&\quad+\frac{1}{\eta_1(\xi_2,\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)\,{}_0d_{q_1}u\right.\\
&\quad\left.+\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\Phi(u,\rho)\,{}_0d_{q_1}u\right]-\mathcal{Q}\Bigg|\\
&\le\left(\int_0^1\int_0^1 z^{\gamma_1}w^{\gamma_1}\,{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{1/\gamma_1}\\
&\quad\times\left[\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\right.\\
&\quad\cdot\left(\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{1/\gamma_2}\\
&\quad+\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_3,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\\
&\quad\cdot\left(\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma_2}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{1/\gamma_2}\\
&\quad+\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\\
&\quad\cdot\left(\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{1/\gamma_2}\\
&\quad+\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\\
&\quad\left.\cdot\left(\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma_2}{}_0d_{q_1}z\,{}_0d_{q_2}w\right)^{\frac{1}{\gamma_2}}\right],
\end{aligned}\tag{42}$$

for all $\varrho,\rho\in N$.

From the n-polynomial preinvexity of the function $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi|^{\gamma_2}$, we get

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&\le\int_0^1\left\{\int_0^1\left[\frac{1}{n}\sum_{p=1}^{n}\left[1-(1-z)^p\right]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\varrho,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right.\right.\\
&\quad\left.\left.+\frac{1}{n}\sum_{p=1}^{n}[1-z^p]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right]{}_0d_{q_1}z\right\}{}_0d_{q_2}w.
\end{aligned}\tag{43}$$

Computing the q_1-integral on the right-hand side of (43), we have

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&\le\int_0^1\left[\frac{1}{n}\sum_{p=1}^{n}\left[1-(1-z)^p\right]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\varrho,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right.\\
&\quad\left.+\frac{1}{n}\sum_{p=1}^{n}[1-z^p]\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right]{}_0d_{q_1}z.
\end{aligned}\tag{44}$$

In view of the Definition 6 for $k=1,2$, we obtain

$$\begin{aligned}
\mathscr{C}_{q_k}&=\frac{1}{n}\sum_{p=1}^{n}\int_0^1\left[1-(1-z)^p\right]{}_0d_{q_k}z=1-\frac{(1-q_k)}{n}\sum_{p=1}^{n}\sum_{e=0}^{\infty}q_k^e(1-q_k^e)^p,\\
\mathscr{D}_{q_k}&=\frac{1}{n}\sum_{p=1}^{n}\int_0^1[1-z^p]{}_0d_{q_k}z=1-\frac{1}{n}\sum_{p=1}^{n}\frac{1-q_k}{1-q_k^{p+2}}.
\end{aligned}\tag{45}$$

Therefore, we get

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&\le\int_0^1\left[\mathscr{C}_{q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\varrho,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right.\\
&\quad\left.+\mathscr{D}_{q_1}\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1,\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}\right]{}_0d_{q_2}w.
\end{aligned}\tag{46}$$

Similarly, computing the q_2-integral and utilizing the fact $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z\partial_{q_2}w)\Phi(\varrho,\rho)|\le M$ for $\varrho,\rho\in\mathcal{N}$ on the right-hand side of (46), one has

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z\,{}_0d_{q_2}w\\
&\le M^{\gamma_2}\left(\mathscr{C}_{q_1}+\mathscr{D}_{q_1}\right)\left(\mathscr{C}_{q_2}+\mathscr{D}_{q_2}\right).
\end{aligned}\tag{47}$$

Analogously, we also can get

$$\begin{aligned}
&\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma_2}{}_0d_{q_1}z\,{}_0d_{q_2}w,\\
&\le M^{\gamma_2}\left(\mathscr{C}_{q_1}+\mathscr{D}_{q_1}\right)\left(\mathscr{C}_{q_2}+\mathscr{D}_{q_2}\right),
\end{aligned}\tag{48}$$

$$\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w \le M^{\gamma_2}\left(\mathscr{C}_{q_1}+\mathscr{D}_{q_1}\right)\left(\mathscr{C}_{q_2}+\mathscr{D}_{q_2}\right), \tag{49}$$

$$\int_0^1\int_0^1\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma_2}{}_0d_{q_1}z_0d_{q_2}w \le M^{\gamma_2}\left(\mathscr{C}_{q_1}+\mathscr{D}_{q_1}\right)\left(\mathscr{C}_{q_2}+\mathscr{D}_{q_2}\right). \tag{50}$$

Therefore, the desired inequality (41) follows from (47), (48), (49), and (50) and the fact that

$$\int_0^1\int_0^1 z^{\gamma_1}w^{\gamma_1}{}_0d_{q_1}z_0d_{q_2}w=\frac{1}{[\gamma_1+1]_{q_1}[\gamma_1+1]_{q_2}}, \tag{51}$$

where $[\gamma_1+1]_{q1}$ and $[\gamma_1+1]_{q2}$ are the q_1- and q_2-analogues of γ_1+1 and γ_2+1, respectively.

Theorem 16. *Let $q_1,q_2\in(0,1)$, $\gamma>1$ and $\Phi:\mathscr{N}\subseteq\mathbb{R}^2\longrightarrow\mathbb{R}$ be mixed partial q_1q_2-differentiable over $\mathscr{N}^\circ$ such that its partial q_1q_2-derivative is continuous and integrable on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]\subset\mathscr{N}^\circ$ for $\eta_1(\xi_2,\xi_1),\eta_2(\xi_4,\xi_3)>0$. If $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi|$ is an n-polynomial preinvex function on the coordinates on $[\xi_1,\xi_1+\eta_1(\xi_2,\xi_1)]\times[\xi_3,\xi_3+\eta_2(\xi_4,\xi_3)]$ and $|({}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w)\Phi(\varrho,\rho)|\le M$ for $\varrho,\rho\in\mathscr{N}$, then we have the inequality*

$$\left|\Phi(\varrho,\rho)+\frac{1}{\eta_2(\xi_4,\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)_0d_{q_2}v+\int_{\xi_4}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,v)_0d_{q_2}v\right]+\frac{1}{\eta_1(\xi_2,\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)_0d_{q_1}u+\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\Phi(u,\rho)_0d_{q_1}u\right]-\mathscr{Q}\right| \le \frac{q_1q_2M\sqrt[\gamma]{\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right)}}{[(1+q_1)(1+q_2)]^{1-1/\gamma}}\times\left[\frac{[\eta_1(\varrho,\xi_1)]^2+[\eta_1(\xi_2,\varrho)]^2}{\eta_1(\xi_2,\xi_1)}\right]\left[\frac{[\eta_2(\rho,\xi_3)]^2+[\eta_2(\xi_4,\rho)]^2}{\eta_2(\xi_4,\xi_3)}\right], \tag{52}$$

where $\mathscr{Q}$ is defined in Lemma 13.

for all $\varrho,\rho\in N$.

Proof. It follows from (19), the power mean inequality and the property of n-polynomial preinvexity of the function $|{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}\Phi/{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w|$ on coordinates that

$$\begin{aligned}
&\left|\Phi(\varrho,\rho)+\frac{1}{\eta_2(\xi_4,\xi_3)}\left[\int_{\xi_3}^{\xi_3+\eta_2(\rho,\xi_3)}\Phi(\varrho,\rho)_0d_{q_2}v+\int_{\xi_4}^{\xi_4+\eta_2(\rho,\xi_4)}\Phi(\varrho,v)_0d_{q_2}v\right]+\frac{1}{\eta_1(\xi_2,\xi_1)}\left[\int_{\xi_1}^{\xi_1+\eta_1(\varrho,\xi_1)}\Phi(u,\rho)_0d_{q_1}u+\int_{\xi_2}^{\xi_2+\eta_1(\varrho,\xi_2)}\Phi(u,\rho)_0d_{q_1}u\right]-\mathscr{Q}\right|\\
&\le\left(\int_0^1\int_0^1 zw_0d_{q_1}z_0d_{q_2}w\right)^{1-(1/\gamma)}\times\left[\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\cdot\left(\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma}{}_0d_{q_1}z_0d_{q_2}w\right)^{1/\gamma}\right]\\
&+\left[\frac{q_1q_2[\eta_1(\varrho,\xi_1)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\left(\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma}{}_0d_{q_1}z_0d_{q_2}w\right)^{1/\gamma}\right]\\
&+\left[\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\rho,\xi_3)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\left(\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma}{}_0d_{q_1}z_0d_{q_2}w\right)^{1/\gamma}\right]\\
&+\left[\frac{q_1q_2[\eta_1(\xi_2,\varrho)]^2[\eta_2(\xi_4,\rho)]^2}{\eta_1(\xi_2,\xi_1)\eta_2(\xi_4,\xi_3)}\left(\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma}{}_0d_{q_1}z_0d_{q_2}w\right)^{1/\gamma}\right],
\end{aligned} \tag{53}$$

By similar argument as in Theorem 14, we can prove that

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma}{}_0d_{q_1}z\,{}_0d_{q_2}w \leq M^{\gamma}\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right), \tag{54}$$

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_1+z\eta_1(\varrho,\xi_1),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma}{}_0d_{q_1}z\,{}_0d_{q_2}w \leq M^{\gamma}\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right), \tag{55}$$

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_3+w\eta_2(\rho,\xi_3))\right|^{\gamma}{}_0d_{q_1}z\,{}_0d_{q_2}w \leq M^{\gamma}\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right), \tag{56}$$

$$\int_0^1\int_0^1 zw\left|\frac{{}_{\xi_1,\xi_3}\partial^2_{q_1,q_2}}{{}_{\xi_1}\partial_{q_1}z\,{}_{\xi_3}\partial_{q_2}w}\Phi(\xi_2+z\eta_1(\varrho,\xi_2),\xi_4+w\eta_2(\rho,\xi_4))\right|^{\gamma}{}_0d_{q_1}z\,{}_0d_{q_2}w \leq M^{\gamma}\left(\mathscr{A}_{q_1}+\mathscr{B}_{q_1}\right)\left(\mathscr{A}_{q_2}+\mathscr{B}_{q_2}\right). \tag{57}$$

Now by making use of the inequalities (54), (55), (56), and (57) and the fact that

$$\int_0^1\int_0^1 zw\,{}_0d_{q_1}z\,{}_0d_{q_2}w = \frac{1}{(1+q_1)(1+q_2)}, \tag{58}$$

we get the desired inequality (52).

3. Conclusion

In the article, we have introduced a new class of preinvex functions which is named n-polynomial preinvex functions and discovered a new quantum integral identity involving second-order mixed partial differentiable function. By using the obtained quantum integral identity as an auxiliary result, we have established several q_1q_2-Ostrowski-type inequalities for the class of n-polynomial preinvex functions on coordinates, which have improved and unified many previously known results. Our given ideas and approaches may lead to a lot of follow-up research for the interested readers.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

H. Kalsoom introduced the Definition 11, carried out the proof of Proposition 12 and Lemma 13, and drafted the manuscript. M. Idrees carried out the proof of Theorem 14. D. Baleanu carried out the proof of Theorem 15. Y.-M. Chu provided the main idea of the manuscript, carried out the proof of Theorem 16, completed the final revision and submitted the article. All authors read and approved the final manuscript.

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Research Article

On New Unified Bounds for a Family of Functions via Fractional q-Calculus Theory

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The present article deals with the new estimates in q-calculus and fractional q-calculus on a time scale $\mathbb{T}_{t_0} = \{0\} \cup \{t : t = t_0 q^n, n$ is a nonnegative integer$\}$, where $t_0 \in \mathbb{R}$ and $0 < q < 1$. The role of fractional time scale q-calculus can be found as one of the prominent techniques to generate some variants for a class of positive functions $n \, (n \in \mathbb{N})$.Finally, our work will provide foundation and motivation for further investigation on time-fractional q-calculus systems that have an intriguing application in quantum theory and special relativity theory.

1. Introduction

Fractional differential equations were executed to demonstrate tremendous innovations for different issues in the physical sciences [1–15]. Since most frameworks involve recollections, the scientists are agreeing with the nonlocality of the fractional operators make it progressively functional in demonstrating the classical derivatives. Recently, nonlocal fractional derivatives without the singular kernel have been exhibited and contemplated [16, 17]. However, there are no solid numerical defenses of the new sorts of fractional derivatives; their applications were demonstrated by numerous analysts [18, 19]. Furthermore, presently we have the utilization of fractional calculus in fields like science, material science, and building and among different zones. It is a stunner of the fractional calculus that we have such a large number of valuable meanings of differential and integral operators, for instance, Saigo, conformable, Riemann-Liouville, Katugampola, Hadamard, Erdélyi-Kober, Liouville, local, and Weyl types. These operators are having their significance and applications in picture handling, science, hydrodynamics, and viscoelastic. For a detailed depiction of the origination of fractional calculus, advancement, and applications, we refer the interested readers to the notable books and research articles [20–22].

Hilger [23] began the theories of time scales in his doctoral dissertation and combined discrete and continuous analysis [24, 25]. At that time, this theory has received a lot of attention. In the book written by Bohner and Peterson [26] on the issues of time scale, a brief summary is given and several time calculations are performed. Over the past decade, many analysts working in special applications have proved a reasonable number of dynamic inequalities on a time scale [27, 28]. Several researchers have created various results relating to fractional calculus on time scales to obtain the corresponding dynamic variants [29].

In the eighteenth century (1707–1783), Euler initiated calculus with no limits refer to as quantum calculus. Jackson began a deliberate investigation of q-calculus and presented the q-definite integrals. Additionally, he was the first to create q-calculus in an efficient manner. Few selected branches of pure and applied mathematics, such as combinatorics, Gauss hypergeometric functions, orthogonal polynomials, dynamic, and quantum theory, have been enhanced by the exploration work of different researchers.

Motivated, by what we mentioned above, we extend the idea of fractional q-calculus type operators with a time scale to arbitrary positive order, provide several bounds for a family of $n \in \mathbb{N}$, and finally prove several variants for time-fractional q-calculus theory. These new results have utilities in the monotonicity for this nabla continuous fractional operator with singular and nonsingular kernel and compare them to the discrete classical ones. The time-fractional q-calculus under consideration in this paper have kernels different from classical nabla fractional differences with kernels depending on the rising factorial powers, and we believe that they bring new kernels with new memories, which may be of different interest for applications. The idea is quite new and seems to have opened new doors of investigation towards various scientific fields of research including engineering, fluid dynamics, meteorology, analysis, and aerodynamics.

Inequalities have wild applications in pure and applied mathematics [30–33]. Very recently, many new inequalities such as Hermite-Hadamard type inequality [34–38], Petrović type inequality [39], Pólya-Szegö type inequality [40], Ostrowski type inequality [41], reverse Minkowski inequality [42], Jensen type inequality [43, 44], Bessel function inequality [45], trigonometric and hyperbolic functions inequalities [46], fractional integral inequality [47–51], complete and generalized elliptic integrals inequalities [52–57], generalized convex function inequality [58–60], and mean values inequality [61–63] have been discovered by many researchers.

Variants regarding fractional integral operators are the use of noteworthy significant strategies amongst researchers and accumulate fertile functional applications in various areas of science [64, 65]. We state some of them, that is, the variants of Minkowski, Hardy, Opial, Hermite-Hadamard, Grüss, Lyenger, Ostrowski, C ebyšev, and Pólya-Szegö, and others. Such applications of fractional integral operators compelled us to show the generalization by using a family of n positive functions involving time-fractional q-calculus integrals operators.

Owing to the above phenomena, the key aim of this research is to demonstrate the notations and primary definitions of our noteworthy time-fractional q-calculus operator. Also, we present the results concerning for a class of family of $n (n \in \mathbb{N})$ continuous positive decreasing functions on $[\varsigma_1, \varsigma_2]$ by employing a time-fractional q-calculus operator. Finally, it is emphasized that combining these two approaches, q-fractional calculus and time scale analysis, could be the most efficient way of incorporating inequalities into both times and q-components for quantum theory and special relativity theory.

2. Preliminaries

Let us recall some necessary definitions and preliminary results that are used for further discussion. For more details, we may refer to [33].

Definition 1 (See [33]). The particular time scale $\mathbb{T}_{t_0}$ is defined by

$$\mathbb{T}_{t_0} = \{t : t = t_0 q^n, n \text{ is a nonnegative integer}\} \cup \{0\}, 0 < q < 1. \tag{1}$$

If there is no confusion concerning t_0, we will denote $\mathbb{T}_{t_0}$ by $\mathbb{T}$.

Definition 2. The q-factorial function is defined in the following way

$$\begin{aligned} (\zeta - \varphi)^{\underline{(n)}} &= (\zeta - \varphi)(\zeta - q\varphi)(\zeta - q^{\varphi}) \cdots (\zeta - q^n \varphi), n \in \mathbb{N}, \\ (\zeta - \varphi)^{\underline{(n)}} &= \zeta^n \prod_{\kappa=0}^{\infty} \frac{1 - (\varphi/\zeta) q^\kappa}{1 - (\varphi/\zeta) q^{n+\kappa}}, \quad n \notin \mathbb{N}. \end{aligned} \tag{2}$$

Definition 3. The q-derivative of the q-factorial function with respect to ζ is defined by

$$\nabla_q (\zeta - \varphi)^{\underline{(n)}} = \frac{1 - q^n}{1 - q} (\zeta - \varphi)^{\underline{(n-1)}}, \tag{3}$$

and the q-derivative of the q-factorial function with respect to s is defined by

$$\nabla_q (\zeta - \varphi)^{\underline{(n)}} = -\frac{1 - q^n}{1 - q} (\zeta - q\varphi)^{\underline{(n-1)}}. \tag{4}$$

Definition 4. The q-exponential function is defined as

$$e_q(\zeta) = \prod_{\kappa=0}^{\infty} (1 - q^\kappa \zeta), \quad e_q(0) = 1. \tag{5}$$

Definition 5. The q-Gamma function is defined by

$$\Gamma_q(\beta) = \frac{1}{1 - q} \int_0^1 \left(\frac{\zeta}{1 - q} \right)^{\beta - 1} e_q(q\zeta) \nabla \zeta, \beta \in \mathbb{R}^+. \tag{6}$$

Remark 6. We observe that

$$\Gamma_q(\beta + 1) = [\beta]_q \Gamma_q(\beta), \quad \beta \in \mathbb{R}^+, \tag{7}$$

and $[\beta]_q = 1 - q^\beta / 1 - q$.

Definition 7. The fractional q-integral is defined as

$$\nabla_q^{-\beta}\Psi(\zeta)=\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi(\varphi)\nabla\varphi. \quad (8)$$

Remark 8. Let $\Psi(\zeta)=1$. Then Definition 7 gives

$$\nabla_q^{-\beta}(1)=\frac{1}{\Gamma_q(\beta)}\frac{q-1}{q^{\beta}-1}\zeta^{\underline{\beta}}=\frac{1}{\Gamma_q(\beta+1)}\zeta^{\underline{\beta}}. \quad (9)$$

3. Main Results

Now we demonstrate the left fractional q integral operator on an arbitrary time scale $\mathbb{T}$ to derive the generalization of some classical inequalities.

Theorem 9. *Let $\alpha>0$, $\eta\geq\delta>0$, $\beta\in\mathbb{C}$ with $\Re(\beta)>0$, and Ψ be a continuous positive decreasing function defined on $\mathbb{T}_{t_0}$. Then, one has*

$$\frac{\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]}{\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\delta}(\zeta)]}\geq\frac{\nabla_{\varsigma_1^+,q}^{-\beta}[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)]}{\nabla_{\varsigma_1^+,q}^{-\beta}[(\zeta-\varsigma_1)^{\alpha}\Psi^{\delta}(\zeta)]}. \quad (10)$$

Proof. Using the hypothesis given in Theorem 9, we have

$$((\omega-\varsigma_1)^{\alpha}-(\varphi-\varsigma_1)^{\alpha})\left(\Psi^{\eta-\delta}(\varphi)-\Psi^{\eta-\delta}(\omega)\right)\geq 0, \quad (11)$$

where $\alpha>0$, $\eta\geq\delta>0$, and $\varphi,\omega\in[\varsigma_1,\zeta]$.

It follows from (11) that

$$(\omega-\varsigma_1)^{\alpha}\Psi^{\eta-\delta}(\varphi)-(\varphi-\varsigma_1)^{\alpha}\Psi^{\eta-\delta}(\omega)-(\omega-\varsigma_1)^{\alpha}\Psi^{\eta-\delta}(\omega)+(\varphi-\varsigma_1)^{\alpha}\Psi^{\eta-\delta}(\varphi)\geq 0. \quad (12)$$

Multiplying (12) by $1/\Gamma_q(\beta)(\zeta-q\varphi)^{\beta-1/}\Psi^{\delta}(\varphi)$,$\varphi\in(\varsigma_1,\zeta)$, we have

$$\begin{aligned}&\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\left[(\omega-\varsigma_1)^{\alpha}\Psi^{\eta-\delta}(\varphi)-(\varphi-\varsigma_1)^{\alpha}\Psi^{\eta-\delta}(\omega)-(\omega-\varsigma_1)^{\alpha}\Psi^{\eta-\delta}(\omega)+(\varphi-\varsigma_1)^{\alpha}\Psi^{\eta-\delta}(\varphi)\right]\Psi^{\delta}(\varphi)\\&=(\omega-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\delta}(\varphi)\Psi^{\eta-\delta}(\varphi)\\&-(\varphi-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\delta}(\varphi)\Psi^{\eta-\delta}(\omega)\\&-(\omega-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\delta}(\varphi)\Psi^{\eta-\delta}(\omega)\\&+(\varphi-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\delta}(\varphi)\Psi^{\eta-\delta}(\varphi)\geq 0.\end{aligned} \quad (13)$$

Integrating on both sides of (13) for φ over (ς_1,ζ), we have

$$\begin{aligned}&(\omega-\varsigma_1)^{\alpha}\int_{\varsigma_1}^{\zeta}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\delta}(\varphi)\Psi^{\eta-\delta}(\varphi)\nabla\varphi-(\varphi-\varsigma_1)^{\alpha}\\&\cdot\int_{\varsigma_1}^{\zeta}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\delta}(\varphi)\Psi^{\eta-\delta}(\omega)\nabla\varphi-(\omega-\varsigma_1)^{\alpha}\\&\cdot\int_{\varsigma_1}^{\zeta}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\delta}(\varphi)\Psi^{\eta-\delta}(\omega)\nabla\varphi+(\varphi-\varsigma_1)^{\alpha}\\&\cdot\int_{\varsigma_1}^{\zeta}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\delta}(\varphi)\Psi^{\eta-\delta}(\varphi)\nabla\varphi\geq 0,\end{aligned} \quad (14)$$

that is

$$\begin{aligned}&(\omega-\varsigma_1)^{\alpha}\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]\right)+\Psi^{\eta-\delta}(\omega)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\delta}(\zeta)\right]\right)\\&-(\omega-\varsigma_1)^{\alpha}\Psi^{\eta-\delta}(\omega)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\right]\right)-\left(\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)\right]\right)\geq 0.\end{aligned} \quad (15)$$

Multiplying (15) by $1/\Gamma_q(\beta)(\zeta-q\omega)^{\beta-1/}\Psi^{\delta}(\omega),\omega\in(\varsigma_1,\zeta)$, and integrating for ω over (ς_1,ζ) shows

$$\begin{aligned}&\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\delta}(\zeta)\right]\right)\\&-\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)\right]\right)\left(\left[\Psi^{\delta}(\zeta)\right]\right)\geq 0.\end{aligned} \quad (16)$$

Dividing the above inequality by $(\nabla_{\varsigma_1^+,q}^{-\beta}[(\zeta-\varsigma_1)^{\alpha}\Psi^{\delta}(\zeta)])$ $(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\delta}(\zeta)])$, we get the desired inequality (10).

Theorem 10. *Let $\alpha>0$, and $\eta\geq\delta>0$, $\beta,\lambda\in\mathbb{C}$ with $\Re(\beta)>0$ and $\Re(\lambda)>0$, and Ψ be a continuous positive decreasing function defined on $\mathbb{T}_{t_0}$. Then the time-fractional q-integral satisfies the inequality*

$$\frac{\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[(\zeta-\varsigma_1)^{\alpha}\Psi^{\delta}(\zeta)]\right)+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\Psi^{\eta}(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}[(\zeta-\varsigma_1)^{\alpha}\Psi^{\delta}(\zeta)]\right)}{\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\Psi^{\delta}(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)]\right)+\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\delta}(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)]\right)}\geq 1. \quad (17)$$

Proof. Multiplying both sides of (15) by $1/\Gamma_q(\lambda)(\zeta-q\omega)^{\lambda-1/}$ $\Psi^{\delta}(\omega)$, $\omega\in(\varsigma_1,\zeta)$ and integrating for ω over (ς_1,ζ) shows

$$\begin{aligned}&\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\delta}(\zeta)\right]\right)+\nabla_{\varsigma_1^+,q}^{-\lambda}[\Psi^{\eta}(\zeta)]\Big)\\&\cdot\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\delta}(\zeta)\right]\right)-\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)\\&\cdot\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)\right]\right)-\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\right]\right)\\&\cdot\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)\right]\right)\geq 0.\end{aligned} \quad (18)$$

Dividing (18) by

$$\begin{aligned}&\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)\right]\right)\\&-\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\right]\right)\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^{\alpha}\Psi^{\eta}(\zeta)\right]\right),\end{aligned}\tag{19}$$

we get the desired inequality (17).

Theorem 11. *Let $\alpha>0$, $\eta\geq\delta>0$, $\beta\in\mathbb{C}$ with $\Re(\beta)>0$, Ψ be a continuous positive decreasing function defined on $\mathbb{T}_{t_0}$, and $\hbar$ be a continuous positive increasing function on $\mathbb{T}_{t_0}$. Then the time-fractional q -integral satisfies the inequality*

$$\frac{\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\hbar^{\alpha}(\zeta)\right]\right)}{\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\right]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)\hbar^{\alpha}(\zeta)]\right)}\geq 1.\tag{20}$$

Proof. Using the hypothesis given in Theorem 11, we have

$$(\hbar^{\alpha}(\omega)-\hbar^{\alpha}(\varphi))\left(\Psi^{\eta-\delta}(\varphi)-\Psi^{\eta-\delta}(\omega)\right)\geq 0,\tag{21}$$

where $\alpha>0$, $\eta\geq\delta>0$, and $\varphi,\omega\in[\varsigma_1,\zeta]$. From (21), we have

$$\begin{aligned}&\hbar^{\alpha}(\omega)\Psi^{\eta-\delta}(\varphi)-\hbar^{\alpha}(\varphi)\Psi^{\eta-\delta}(\omega)\\&+\hbar^{\alpha}(\omega)\Psi^{\eta-\delta}(\omega)-\hbar^{\alpha}(\varphi)\Psi^{\eta-\delta}(\varphi)\geq 0.\end{aligned}\tag{22}$$

Taking product of (22) by $1/\Gamma_q(\beta)(\zeta-q\varphi)^{\beta-1/}\Psi^{\delta}(\varphi),\varphi\in(\varsigma_1,\zeta)$, we get

$$\begin{aligned}&\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\delta}(\varphi)\left[\hbar^{\alpha}(\omega)\Psi^{\eta-\delta}(\varphi)-\hbar^{\alpha}(\varphi)\Psi^{\eta-\delta}(\omega)\right.\\&\left.+\hbar^{\alpha}(\omega)\Psi^{\eta-\delta}(\omega)-\hbar^{\alpha}(\varphi)\Psi^{\eta-\delta}(\varphi)\right]\\&=\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta}(\varphi)-\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}\\&\cdot(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta-\delta}(\omega)\Psi^{\delta}(\varphi)+\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta-\delta}\\&\cdot(\omega)\Psi^{\delta}(\varphi)-\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta}(\varphi)\geq 0.\end{aligned}\tag{23}$$

Integrating (23) for φ over (ς_1,ζ), we obtain

$$\begin{aligned}&\frac{\hbar^{\alpha}(\omega)}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta}(\varphi)\nabla\varphi-\frac{\hbar^{\alpha}(\varphi)}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta-\delta}\\&\cdot(\omega)\Psi^{\delta}(\varphi)\nabla\varphi+\frac{\hbar^{\alpha}(\omega)}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta-\delta}(\omega)\Psi^{\delta}(\varphi)\nabla\varphi\\&-\frac{\hbar^{\alpha}(\varphi)}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\Psi^{\eta}(\varphi)\nabla\varphi\geq 0.\end{aligned}\tag{24}$$

It follows that

$$\begin{aligned}&\hbar^{\alpha}(\omega)\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]\right)+\Psi^{\eta-\delta}(\omega)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\\&-\hbar^{\alpha}(\omega)\Psi^{\eta-\delta}(\omega)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\right]\right)-\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\geq 0.\end{aligned}\tag{25}$$

Again, taking the product (15) by $1/\Gamma_q(\beta)(\zeta-q\omega)^{\beta-1/}\Psi^{\delta}(\omega),\omega\in(\varsigma_1,\zeta)$, and integrating for ω over (ς_1,ζ) gives

$$\begin{aligned}&\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\hbar^{\alpha}(\zeta)\right]\right)\\&-\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\Psi^{\delta}(\zeta)\right]\right)\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)\hbar^{\alpha}(\zeta)]\right)\geq 0,\end{aligned}\tag{26}$$

which completes the proof of the desired inequality (20).

Theorem 12. *Let $\alpha>0$, $\eta\geq\delta>0$, $\Re(\lambda),\Re(\beta)>0$ with $\Re(\lambda),\Re(\beta)>0,\Psi$ be a continuous positive decreasing function defined on $\mathbb{T}_{t_0}$ and $\hbar$ be a continuous positive increasing function on $\mathbb{T}_{t_0}$. Then, one has*

$$\frac{\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\Psi^{\delta}(\zeta)\right]\right)+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\Psi^{\eta}(\zeta)]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\theta}(\zeta)\Psi^{\delta}(\zeta)\right]\right)}{\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)}\geq 1.\tag{27}$$

Proof. Multiplying both sides of (25) by $1/\Gamma_q(\beta)(\zeta-q\omega)^{\beta-1/}\Psi^{\delta}(\omega),\omega\in(\varsigma_1,\zeta)$, and integrating for ω over (ς_1,ζ) leads to the conclusion that

$$\begin{aligned}&\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]\,\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\\&+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\Psi^{\eta}(\zeta)]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\theta}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\\&-\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)\\&-\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)]\,\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)\geq 0.\end{aligned}\tag{28}$$

It follows that

$$\begin{aligned}&\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\Psi^{\eta}(\zeta)]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\\&+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\Psi^{\eta}(\zeta)]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\theta}(\zeta)\Psi^{\delta}(\zeta)\right]\right)\\&\geq\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)\\&+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right).\end{aligned}\tag{29}$$

Dividing above inequality by

$$\left(\nabla_{\varsigma_1^+,q}^{-\beta}[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)]\,\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right)+\left(\nabla_{\varsigma_1^+,q}^{-\lambda}[\hbar^{\alpha}(\zeta)\Psi^{\eta}(\zeta)]\,\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\Psi^{\delta}(\zeta)\right]\right),\tag{30}$$

we get the desired inequality (27).

Now, we demonstrate the fractional q-integral to derive some inequalities for a class of n-decreasing positive functions.

Theorem 13. *Let $\alpha > 0$, $\eta \geq \delta_\kappa > 0$ for any fixed $\kappa \in \{1, 2, 3, \cdots, n\}$, $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$, and $\{\Psi_j, j = 1, 2, 3, \cdots, n\}$ be a sequence of continuous positive decreasing functions defined on $\mathbb{T}_{t_0}$. Then, the time-fractional q -integral satisfies the inequality*

$$\frac{\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]}{\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]} \geq \frac{\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]}{\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]}. \tag{31}$$

Proof. Since $\{\Psi_j, j = 1, 2, 3, \cdots, n\}$ is a sequence of continuous positive decreasing functions on $[\varsigma_1, \zeta]$, we have

$$\left((\omega-\varsigma_1)^{\alpha}-(\varphi-\varsigma_1)^{\alpha}\right)\left(\Psi_\kappa^{\eta-\delta\kappa}(\varphi)-\Psi_\kappa^{\eta-\delta\kappa}(\omega)\right)\geq 0, \tag{32}$$

for any fixed $\kappa \in \{1, 2, 3, \cdots, n\}$, $\alpha > 0$, $\eta \geq \delta_\kappa > 0$ and $\varphi, \omega \in [\varsigma_1, \zeta]$.

It follows from (32) that

$$\begin{aligned}&(\omega-\varsigma_1)^{\alpha}\Psi_\kappa^{\eta-\delta\kappa}(\varphi)+(\varphi-\varsigma_1)^{\alpha}\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\&\quad\geq(\omega-\varsigma_1)^{\alpha}\Psi_\kappa^{\eta-\delta\kappa}(\omega)+(\varphi-\varsigma_1)^{\alpha}\Psi_\kappa^{\eta-\delta\kappa}(\varphi).\end{aligned} \tag{33}$$

Taking the product of (22) by $1/\Gamma_q(\beta)(\zeta - q\varphi)^{\underline{\beta-1}/}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)$, $\varphi \in (\varsigma_1, \zeta)$, and integrating for φ over (ς_1, ζ), we have

$$\begin{aligned}&\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\Big[(\omega-\varsigma_1)^{\alpha}\Psi_\kappa^{\eta-\delta\kappa}(\varphi)+(\varphi-\varsigma_1)^{\alpha}\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\&\quad-(\omega-\varsigma_1)^{\alpha}\Psi_\kappa^{\eta-\delta\kappa}(\omega)-(\varphi-\varsigma_1)^{\alpha}\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\Big]\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\\&\quad=(\omega-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\\&\quad+(\varphi-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\&\quad-(\omega-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\&\quad-(\varphi-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\geq 0.\end{aligned} \tag{34}$$

Integrating (34) for φ over (ς_1, ζ), we get

$$\begin{aligned}&(\omega-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\nabla\varphi\\&\quad+(\varphi-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla\varphi\\&\quad-(\omega-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla\varphi\\&\quad-(\varphi-\varsigma_1)^{\alpha}\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\underline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\nabla\varphi\geq 0.\end{aligned} \tag{35}$$

It follows from (35) that

$$\begin{aligned}&(\omega-\varsigma_1)^{\alpha}\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]+\Psi_\kappa^{\eta-\delta\kappa}(\omega)\,\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&\quad\geq(\omega-\varsigma_1)^{\alpha}\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&\quad+\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right].\end{aligned} \tag{36}$$

Again, taking the product of (36) by $1/\Gamma_q(\beta)(\zeta - q\omega)^{\underline{\beta-1}/}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\omega)$, $\omega \in (\varsigma_1, \zeta)$, and integrating for ω over (ς_1, ζ), we obtain

$$\begin{aligned}&\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&\quad\geq\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right],\end{aligned} \tag{37}$$

which gives the desired inequality (31).

Theorem 14. *Let $\alpha > 0$, $\eta \geq \delta_\kappa > 0$ for any fixed $\kappa \in \{1, 2, 3, \cdots, n\}$, $\beta, \lambda \in \mathbb{C}$ with $\Re(\beta) > 0$, $\Re(\lambda) > 0$, and $\{\Psi_j, j = 1, 2, 3, \cdots, n\}$ be a sequence of continuous positive decreasing functions defined on $\mathbb{T}_{t_0}$. Then, we have the inequality*

$$\begin{aligned}&\Bigg(\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla^{-\lambda}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&\quad+\nabla^{-\lambda}_{\varsigma_1^+,q}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\Bigg)/\\&\quad\Bigg(\nabla^{-\beta}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla^{-\lambda}_{\varsigma_1^+,q}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\&\quad+\nabla^{-\lambda}_{\varsigma_1^+,q}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla^{-\beta}_{\varsigma_1^+,q}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\Bigg)\geq 1.\end{aligned} \tag{38}$$

Proof. Taking product on both sides of (36) by $1/\Gamma_q(\lambda)(\zeta - q\theta)^{\lambda-1}/\prod_{j=1}^{n}\Psi_j^{\delta_j}(\omega), \omega \in (\varsigma_1, \zeta)$, and integrating for ω over (ς_1, ζ), we get

$$\begin{aligned}
&\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\
&\quad+\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\
&\quad\geq\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\
&\quad+\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right].
\end{aligned}\tag{39}$$

Dividing the above inequality by

$$\begin{aligned}
&\nabla_{\varsigma_1^+,q}^{-\beta}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\
&\quad+\nabla_{\varsigma_1^+,q}^{-\lambda}\left[(\zeta-\varsigma_1)^{\alpha}\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right],
\end{aligned}\tag{40}$$

gives the desired inequality (38).

Theorem 15. *Let $\alpha > 0$, $\eta \geq \delta_\kappa > 0$ for any fixed $\kappa \in \{1, 2, 3, \cdots, n\}$, $\beta \in \mathbb{C}$ with $\Re(\beta) > 0$, and $\hbar$ and $\Psi_j (j = 1, 2, 3, \cdots, n)$ be the continuous positive decreasing functions defined on $\mathbb{T}_{t_0}$. Then, the time-fractional q -integral satisfies the inequality*

$$\frac{\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]}{\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]}\geq 1. \tag{41}$$

Proof. It follows from the given hypothesis that

$$\left(\hbar^{\alpha}(\omega)-\hbar^{\alpha}(\varphi)\right)\left(\Psi_\kappa^{\eta-\delta\kappa}(\varphi)-\Psi_\kappa^{\eta-\delta\kappa}(\omega)\right)\geq 0, \tag{42}$$

for any fixed $\kappa \in \{1, 2, 3, \cdots, n\}, \alpha > 0$, $\eta \geq \delta_\kappa > 0$, and $\varphi, \omega \in [\varsigma_1, \zeta]$.

Inequality (42) leads to

$$\begin{aligned}
&\hbar^{\alpha}(\omega)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)+\hbar^{\alpha}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)-\hbar^{\alpha}(\omega)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\
&\quad-\hbar^{\alpha}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\geq 0.
\end{aligned}\tag{43}$$

Taking the product on both sides of (43) by $1/\Gamma_q(\beta)(\zeta - q\varphi)^{\beta-1}/\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi), \varphi \in (\varsigma_1, \zeta)$, and integrating for φ over (ς_1, ζ), we obtain

$$\begin{aligned}
&\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\\
&\quad+\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\
&\quad-\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\\
&\quad-\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\geq 0.
\end{aligned}\tag{44}$$

Integrating (44) for φ over (ς_1, ζ), we have

$$\begin{aligned}
&\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\nabla\varphi\\
&\quad+\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla\varphi\\
&\quad-\hbar^{\alpha}(\omega)\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla\varphi\\
&\quad-\hbar^{\alpha}(\varphi)\frac{1}{\Gamma_q(\beta)}\int_{\varsigma_1}^{\zeta}(\zeta-q\varphi)^{\overline{\beta-1}}\prod_{j=1}^{n}\Psi_j^{\delta_j}(\varphi)\Psi_\kappa^{\eta-\delta\kappa}(\varphi)\nabla\varphi\geq 0.
\end{aligned}\tag{45}$$

From (43), we clearly see that

$$\begin{aligned}
&\hbar^{\alpha}(\omega)\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]+\Psi_\kappa^{\eta-\delta\kappa}(\zeta)\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\
&\quad-\hbar^{\alpha}(\omega)\Psi_\kappa^{\eta-\delta\kappa}(\omega)\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]-\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\geq 0.
\end{aligned}\tag{46}$$

Again, taking the product on both sides of (46) by $1/\Gamma_q(\beta)(\zeta - q\theta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\omega), \omega \in (\varsigma_1, \zeta)$, and integrating for ω over (ς_1, ζ), we have

$$\begin{aligned}
&\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\\
&\quad-\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta j}(\zeta)\right]\geq 0,
\end{aligned}\tag{47}$$

which completes the proof of the desired inequality (41).

Theorem 16. *Let $\alpha > 0, \eta \geq \delta_\kappa > 0$ for any fixed $\kappa \in \{1, 2, 3, \cdots, n\}$, $\beta, \lambda \in \mathbb{C}$, with $\Re(\beta) > 0, \Re(\lambda) > 0$, $\{\Psi_j, j = 1, 2, 3, \cdots, n\}$ be a sequence of continuous positive decreasing functions defined on $\mathbb{T}_{t_0}$ and $\hbar$ be a continuous positive increasing functions defined on $\mathbb{T}_{t_0}$. Then*

$$\left(\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\right.$$
$$\left.+\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\right)/$$
$$\left(\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\right.$$
$$\left.+\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\right)\geq 1. \tag{48}$$

Proof. Multiplying both sides of (46) by $1/\Gamma_q(\lambda)(\zeta-q\omega)^{\lambda-1}/\prod_{j=1}^{n}\Psi_j^{\delta_j}(\omega), \omega\in(\varsigma_1,\zeta)$, and integrating for ω over (ς_1,ζ), we have

$$\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]$$
$$+\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]$$
$$-\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]$$
$$-\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\geq 0. \tag{49}$$

It follows that

$$\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]$$
$$+\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]$$
$$\geq\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]$$
$$+\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]. \tag{50}$$

Dividing both sides of the above inequality by

$$\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]$$
$$+\nabla_{\varsigma_1^+,q}^{-\lambda}\left[\hbar^{\alpha}(\zeta)\prod_{j\neq\kappa}^{n}\Psi_j^{\delta_j}\Psi_\kappa^{\eta}(\zeta)\right]\nabla_{\varsigma_1^+,q}^{-\beta}\left[\prod_{j=1}^{n}\Psi_j^{\delta_j}(\zeta)\right], \tag{51}$$

gives desired inequality (48).

4. Conclusion

In this note, we have derived certain variants by using the time-fractional q-calculus operator related to a class of n positive continuous, and decreasing functions on the interval $[\varsigma_1,\varsigma_2]$ are elaborated. In [66], Liu et al. investigated thought-provoking integral inequalities for continuous functions on $[\varsigma_1,\varsigma_2]$. Recently, Dahmani [67] has presented the more generalizations of the work of [66] by utilizing the Riemann-Liouville fractional integral operators. If we take into account $\mathbb{T}=\mathbb{R}$ and $q=1$, then our findings are the special cases of the results proposed by Dahmani [67]. The established relationship highlighted the importance of selecting appropriate combinations and validated q-fractional time scale approaches for special relativity theory and quantum mechanics. From the existence and uniqueness viewpoint, it is found that the q-fractional order controls potentially provide the tools to better represent measured that cannot be fit to the classical model.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

L. Xu provided the main ideas of the article and carried out the proof of Theorem 9. Y.-M. Chu drafted the manuscript and carried out the proof of Theorem 10. S. Rashid carried out the proof of Theorem 11 and Theorem 12, completed the final revision, and submitted the article. A. A.El-Deeb carried out the proof of Theorems 13 and 14. K. S. Nisar carried out the proof of Theorems 15 and 16. All authors read and approved the final manuscript.

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Research Article

On the (p, q)-Humbert Functions from the View Point of the Generating Function Method

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The main object of the present paper is to construct new (p, q)-analogy definitions of various families of (p, q)-Humbert functions using the generating function method as a starting point. This study shows a class of several results of (p, q)-Humbert functions with the help of the generating functions such as explicit representations and recurrence relations, especially differential recurrence relations, and prove some of their significant properties of these functions.

1. Introduction

In the last quarter of 20th century, q-calculus appeared as a connection between mathematics and physics. We have also a generalization of q-calculus with one more parameter, we can say it is a two-parameter quantum calculus. Generally, it is called (p, q)-calculus. The theory of (p, q)-calculus or post quantum calculus has recently been applied in many areas of mathematics, physics and engineering, such as biology, mechanics, economics, electrochemistry, probability theory, approximation theory, statistics, number theory, quantum theory, theory of relativity, and statistical mechanics, etc. For more details on this topic (p, q)-calculus, see, for example, [1–6]. Burban and Klimyk [3], Duran et al. [7–10], Jagannathan [11], Jagannathan and Srinivasa [12], Sahai and Yadav [13] have earlier investigated some properties of the two parameter quantum calculus. Sadjang [14–16] introduced the two ((p, q)-analogues of the Laplace transform, two (p, q)-Taylor formulas for polynomials, (p, q)-Appell polynomials and developed some their properties. Mursaleen et al. [17, 18] investigated the (p, q)-analogues of Bernstein operators and approximation properties of (p, q)-Bernstein operators that are a generalization of q-Bernstein operators. Khan and Lobiya [19] have nicely discussed a lot of applications in different approximation theory areas, such as per Weirstarass approximation theorems, basic hypergeometric functions, orthogonal polynomials and can be used in differential equations as well as computer-aided geometric designs. Recently, Pasricha and Varma presented and introduced the Humbert function $J_{m,n}(x)$ in [20, 21]. In [22], Srivastava and Shehata have earlier studied the q-Humbert functions. The motivation of these generalizations q-Humbert functions is to provide appropriate application areas of mathematical, physical and engineering such as numerical analysis, approximation theory and computer-aided geometric design (see the recent papers [1, 6, 19, 23] and the references therein).

The main purpose of this paper is to obtain explicit formulas for the various families of (p, q)-Humbert functions for $0 < |q| < |p| \leq 1$ for p, q in $\mathbb{C}$. We mainly use the (p, q)-calculus in the theory of special functions. This work is organized as follows. More precisely, we define the numerous (known or new) (p, q)-Humbert functions and discuss some significant properties such as explicit representations, recurrence relations and some new generating functions in Section 2. In Section 3, especially recurrence relations and some interesting differential recurrence relations for the (p, q)-Humbert functions are discussed. In Section 4, the conclusion and perspectives are given to illustrate the main results.

1.1. Basic Definitions and Miscellaneous Results. To convenience of the reader, we provide a summary of the mathematical notations and some basic definitions of (p, q)-calculus where $0 < |q| < |p| \leq 1$ for $p, q \in \mathbb{C}$, operations and notations we need to be used in this work. We use the following standard notations: $\mathbb{N} = 1, 2, 3, \cdots$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \cdots\}$. The symbols $\mathbb{N}$ and $\mathbb{C}$ denote the sets of natural numbers and complex numbers, respectively.

The q-number $[\alpha]_q$ and q-factorial $[n]_q!$ are defined as follows: (see [22])

$$[\alpha]_q = \frac{1-q^\alpha}{1-q}, 0 < |q| < 1 ; q \in \mathbb{C} - \{1\} ; \alpha \in \mathbb{C} \quad (1)$$

$$[n]_q! = \prod_{k=1}^{n} [k]_q = [1]_q [2]_q \cdots [n]_q = \prod_{k=1}^{n} \frac{1-q^k}{1-q}, q \neq 1 ; n \in \mathbb{N}, \quad (2)$$

$$q! = 1, 0 < |q| < 1 ; q \in \mathbb{C} - \{1\}. \quad (3)$$

In [22], the q-Humbert functions is defined by

$$J_{m,n}^{(1)}(x \mid q) = \sum_{k=0}^{\infty} \frac{(-1)^k}{[k]_{p,q}! [m+k]_{p,q}! [n+k]_{p,q}!} \left(\frac{x}{3}\right)^{m+n+3k}. \quad (4)$$

The (p, q)-number (bibasic number or twin-basic number) is denoted by $[\alpha]_{p,q}$ and is defined by the following notation [15]

$$[\alpha]_{p,q} = \frac{p^\alpha - q^\alpha}{p-q}, 0 < |q| < |p| \leq 1 ; p, q, \alpha \in \mathbb{C}. \quad (5)$$

For $p, q, \alpha \in \mathbb{C}$ and $0 < |q| < |p| \leq 1$ for $p, q, \alpha \in \mathbb{C}$, the (p, q)-number and (p, q)-factorial are given as follows: (see [11, 12, 15])

$$[n]_{p,q} = \begin{cases} \dfrac{p^n - q^n}{p-q}, & n \in \mathbb{N} ; \\ 0, & n = 0 \end{cases}$$
$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}! = [n]_{p,q} [n-1]_{p,q} \cdots [2]_{p,q} [1]_{p,q}, n \geq 1 \text{ and } [0]_{p,q}! = 0. \quad (6)$$

The (p, q)-number $[n]_{p,q}$ is a natural generalization of the q-number in (3) such that

$$\lim_{p \to 1} [n]_{p,q} = [n]_q. \quad (7)$$

The (p, q)-number satisfies the following addition properties

$$\begin{aligned} [n]_{p,q} &= p^{-k} [n+k]_{p,q} - q^n p^{-k} [k]_{p,q} \\ &= q^{-k} [n+k]_{p,q} - p^n q^{-k} [k]_{p,q}, n, k \in \mathbb{N}. \end{aligned} \quad (8)$$

The (p, q)-factorial is denoted by $[n]_{p,q}!$ and is defined by (see [6, 11, 12])

$$[n]_{p,q}! = \begin{cases} \displaystyle\prod_{k=1}^{n} [k]_{p,q} = \frac{((p,q) ; (p,q))_n}{(p-q)^n}, & n \geq 1 ; \\ 1, & n = 0, \end{cases} \quad (9)$$

where

$$((a,b) ; (p,q))_n = \begin{cases} \displaystyle\prod_{r=0}^{n-1} (ap^r - bq^r), & n > 0 ; \\ 1, & n = 0 ; \\ \dfrac{1}{\prod_{r=0}^{-n} (ap^{-r} - bq^{-r})}, & n < 0. \end{cases} \quad (10)$$

As in the q-case, there are many definitions of the (p, q)-exponential function. The following two (p, q)-analogues of exponential function will be frequently used throughout this paper:

The (p, q)-exponential function is defined by (see [12, 16])

$$e_{p,q}(x) = \sum_{k=0}^{\infty} \frac{p^{\binom{k}{2}} x^k}{[k]_{p,q}!}. \quad (11)$$

The (p, q)-complementary exponential function is defined by

$$E_{p,q}(x) = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}} x^k}{[k]_{p,q}!}. \quad (12)$$

It is easy to see that (see [15, 16])

$$\begin{aligned} & e_{p,q}(x) E_{p,q}(-x) = 1, \\ & e_{\frac{1}{p},\frac{1}{q}}(x) = E_{p,q}(x), E_{\frac{1}{p},\frac{1}{q}}(x) = e_{p,q}(x). \end{aligned} \quad (13)$$

Let f be a function defined on a subset of real or complex plane. The (p, q)-derivative operator of the function f is defined as follows (see [15, 24, 25])

$$D_{p,q} f(x) = \frac{f(px) - f(qx)}{(p-q)x}, x \neq 0, \quad (14)$$

and $(D_{p,q} f)(0) = f'(0)$, provided that f is differentiable at 0, which satisfies the following relations (see [14, 16])

$$D_{p,q} e_{p,q}(\mu x) = a e_{p,q}(\mu p x), \quad (15)$$

$$D_{p,q} E_{p,q}(\mu x) = a E_{p,q}(\mu q x), \mu \in \mathbb{C}. \quad (16)$$

The (p,q)-derivative operator satisfy the following product rules as follows: (see [11, 12, 14, 15])

$$D_{p,q}[f_1(x)f_2(x)] = f_2(px)D_{p,q}\{f_1(x)\} + f_1(qx)D_{p,q}\{f_2(x)\} \tag{17}$$

$$\begin{aligned} D_{p,q}[f_1(x)f_2(x)f_3(x)] = {} & f_3(px)f_2(px)D_{p,q}\{f_1(x)\} \\ & + f_3(px)f_1(qx)D_{p,q}\{f_2(x)\} \\ & + f_1(qx)f_2(qx)D_{p,q}\{f_3(x)\}. \end{aligned} \tag{18}$$

Our purpose is to generalize the class of Bessel functions, by using the same approach exposed above and is to define our main problem on the generalized (p,q)-Humbert functions. In particular, we will present some particular cases of functions which are belonging to the family of (p,q)-Humbert functions which are introduced as the third (p,q)-Humbert functions.

2. Definitions of New (p,q)-Analogue of the (p,q)-Humbert Functions and Some Basic Properties

Here we apply the notion of (p,q)-analogue of the generating function to obtain explicit formulas for generalized (p,q)-Humbert functions and give some interesting significant properties for these functions.

Definition 1. Let us define the product of symmetric (p,q)-exponential functions as the generating function of the (p,q)-Humbert functions of the first kind as follows:

$$\begin{aligned} F_1(x;u,t\mid p,q) &= e_{p,q}\left(\frac{xu}{3}\right)e_{p,q}\left(\frac{xt}{3}\right)e_{p,q}\left(-\frac{x}{3ut}\right) \\ &= \sum_{m,n=-\infty}^{\infty} J^{(1)}_{m,n}(x\mid p,q)u^m t^n. \end{aligned} \tag{19}$$

Remark 2. Note that in eq. (19), if we put $p=1$, then (p,q)-Humbert functions reduces to the q-Humbert functions defined in [22].

Remark 3. When $q \longrightarrow p = 1$, the (p, q)-Humbert functions reduce to the classical Humbert functions defined in [20, 21].

From (19) and using (11), we have

$$\begin{aligned} F_1(x;u,t\mid p,q) &= e_{p,q}\left(\frac{xu}{3}\right)e_{p,q}\left(\frac{xt}{3}\right)e_{p,q}\left(-\frac{x}{3ut}\right) \\ &= \sum_{r=0}^{\infty}\frac{p^{r(r-1)/2}}{[r]_{p,q}!}\left(\frac{xu}{3}\right)^r \sum_{i=0}^{\infty}\frac{p^{i(i-1)/2}}{[i]_{p,q}!} \\ &\quad \cdot \left(\frac{xt}{3}\right)^i \sum_{k=0}^{\infty}\frac{p^{k(k-1)/2}}{[k]_{p,q}!}\left(-\frac{x}{3ut}\right)^k \\ &= \sum_{i,r,k=0}^{\infty}\frac{(-1)^k p^{\binom{r}{2}+\binom{i}{2}+\binom{k}{2}}}{[k]_{p,q}![i]_{p,q}![r]_{p,q}!} \\ &\quad \cdot \left(\frac{x}{3}\right)^{k+i+r} u^{r-k}t^{i-k}. \end{aligned} \tag{20}$$

Replace r by $m+k$ and i by $n+k$ to get

$$\begin{aligned} &\sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}u^m t^n \\ &= \sum_{m,n=-\infty}^{\infty} J^{(1)}_{m,n}(x\mid p,q)u^m t^n. \end{aligned} \tag{21}$$

Explicitly, we get the explicit expression of (p,q)-Humbert functions $J^{(1)}_{m,n}(x\mid p,q)$ as the following power series

$$J^{(1)}_{m,n}(x\mid p,q) = \sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}. \tag{22}$$

By (9), the series expansions of the (p,q)-Humbert functions $J^{(1)}_{m,n}(x\mid p,q)$ are given as

$$J^{(1)}_{m,n}(x\mid p,q) = \frac{1}{((p,q);(p,q))_n((p,q);(p,q))_n} \times \sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}, \tag{23}$$

or

$$J^{(1)}_{m,n}(x\mid p,q)=\frac{1}{\Gamma_{p,q}(m+1)\Gamma_{p,q}(n+1)}\left(\frac{x}{3(p-q)}\right)^{m+n}\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{3k}, \tag{24}$$

equivalently, we have

$$J^{(1)}_{m,n}(x\mid p,q)=\frac{((p^{m+1},q^{m+1});(p,q))_\infty((p^{n+1},q^{n+1});(p,q))_\infty}{((p,q);(p,q))_\infty((p,q);(p,q))_\infty}\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}+\binom{n+k}{2}+\binom{k}{2}}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}. \tag{25}$$

Lemma 4. *Let n and m are integers, then the function $J^{(1)}_{m,n}(x\mid p,q)$ satisfies the following relations*

$$J^{(1)}_{-m,n}(x\mid p,q)=(-1)^m J^{(1)}_{m,n+m}(x\mid p,q) \tag{26}$$

$$J^{(1)}_{m,-n}(x\mid p,q)=(-1)^n J^{(1)}_{m+n,m}(x\mid p,q). \tag{27}$$

Proof. From the definition of (p,q)-Humbert functions $J^{(1)}_{m,n}(x\mid p,q)$, we have

$$\begin{aligned}J^{(1)}_{-m,n}(x\mid p,q)&=\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{-m+k}{2}+\binom{n+k}{2}+\binom{k}{2}}}{[k]_{p,q}![-m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{-m+n+3k}\\&=\sum_{k=m}^{\infty}\frac{(-1)^k p^{\binom{-m+k}{2}+\binom{n+k}{2}+\binom{k}{2}}}{[k]_{p,q}![-m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{-m+n+3k}.\end{aligned} \tag{28}$$

Replacing $s=k-m$, we obtain

$$\begin{aligned}J^{(1)}_{-m,n}(x\mid p,q)=\sum_{s=0}^{\infty}&\frac{(-1)^{s+m} p^{\binom{s}{2}+\binom{n+m+s}{2}+\binom{m+s}{2}}}{[s]_{p,q}![s+m]_{p,q}![n+m+s]_{p,q}!}\\&\cdot\left(\frac{x}{3}\right)^{2m+n+3s}=(-1)^m J^{(1)}_{m,n+m}(x\mid p,q).\end{aligned} \tag{29}$$

The equation (27) can be proved in a like manner.

Lemma 5. *The function $J^{(1)}_{m,n}(x\mid p,q)$ satisfies the following properties*

$$\begin{aligned}J^{(1)}_{-m,-n}(x\mid p,q)&=(-1)^m J^{(1)}_{m,m-n}(x\mid p,q)\\&=(-1)^n J^{(1)}_{n-m,n}(x\mid p,q),\end{aligned} \tag{30}$$

where n and m are integers.

Proof. From the definition of (p,q)-Humbert functions $J^{(1)}_{m,n}(x\mid p,q)$, we have

$$\begin{aligned}J^{(1)}_{-m,-n}(x\mid p,q)&=\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{-m+k}{2}+\binom{-n+k}{2}+\binom{k}{2}}}{[k]_{p,q}![-m+k]_{p,q}![-n+k]_{p,q}!}\left(\frac{x}{3}\right)^{-m-n+3k}\\&=\sum_{k=\max\{n,m\}}^{\infty}\frac{(-1)^k p^{\binom{-m+k}{2}+\binom{-n+k}{2}+\binom{k}{2}}}{[k]_{p,q}![-m+k]_{p,q}![-n+k]_{p,q}!}\left(\frac{x}{3}\right)^{-m-n+3k}.\end{aligned} \tag{31}$$

Upon setting $s=k-m$ in the Eq. (31), we get

$$\begin{aligned}J^{(1)}_{-m,-n}(x\mid p,q)&=\sum_{s=0}^{\infty}\frac{(-1)^{s+m} p^{\binom{s}{2}+\binom{m-n+s}{2}+\binom{m+s}{2}}}{[s]_{p,q}![m+s]_{p,q}![m-n+s]_{p,q}!}\left(\frac{x}{3}\right)^{2m-n+3s}\\&=(-1)^m J^{(1)}_{m,m-n}(x\mid p,q).\end{aligned} \tag{32}$$

Upon setting $s=k-n$ in the Eq. (31), we get

$$J_{-m,-n}^{(1)}(x\mid p,q)=\sum_{s=0}^{\infty}\frac{(-1)^{s+n}p^{\binom{n-m+s}{2}+\binom{n+s}{2}+\binom{s}{2}}}{[s]_{p,q}![n-m+s]_{p,q}![n+s]_{p,q}!}\left(\frac{x}{3}\right)^{2n-m+3s} = (-1)^n J_{n-m,n}^{(1)}(x\mid p,q). \tag{33}$$

Now, we define that the generating function of (p,q)-Humbert functions of the second kind.

Definition 6. The generating function $F_2(x;u,t\mid p,q)$ *of* (p,q)-Humbert functions of the second kind is defined by

$$F_2(x;u,t\mid p,q)=E_{p,q}\left(\frac{xu}{3}\right)E_{p,q}\left(\frac{xt}{3}\right)E_{p,q}\left(-\frac{qx}{3ut}\right) = \sum_{m,n=-\infty}^{\infty} q^{\binom{n}{2}+\binom{m}{2}} J_{m,n}^{(2)}(x\mid p,q)u^m t^n. \tag{34}$$

From the generating function of the (p,q)-Humbert functions $J_{m,n}^{(2)}(x\mid p,q)$, we have

$$\begin{aligned}F_2(x;u,t\mid p,q)&=E_{p,q}\left(\frac{xu}{3}\right)E_{p,q}\left(\frac{xt}{3}\right)E_{p,q}\left(-\frac{qx}{3ut}\right)\\&=\sum_{r=0}^{\infty}\frac{q^{r(r-1)/2}}{[r]_{p,q}!}\left(\frac{xu}{3}\right)^r\sum_{i=0}^{\infty}\frac{q^{i(i-1)/2}}{[i]_{p,q}!}\cdot\left(\frac{xt}{3}\right)^i\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{[k]_{p,q}!}\left(-\frac{qx}{3ut}\right)^k\\&=\sum_{i,r,k=0}^{\infty}\frac{(-1)^k q^{r(r-1)+i(i-1)+k(k+1)/2}}{[k]_{p,q}![i]_{p,q}![r]_{p,q}!}\cdot\left(\frac{x}{3}\right)^{k+i+r}u^{r-k}t^{i-k}.\end{aligned} \tag{35}$$

Now, substituting r by $m+k$ and i by $n+k$ in the last equation, we obtain the following equality

$$\sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{(m+k)(m+k-1)+(n+k)(n+k-1)+k(k+1)/2}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}u^m t^n = \sum_{m,n=-\infty}^{\infty} q^{m(m-1)+n(n-1)/2} J_{m,n}^{(2)}(x\mid p,q)u^m t^n. \tag{36}$$

Explicitly, we get the explicit expression of (p,q)-Humbert functions $J_{m,n}^{(2)}(x\mid p,q)$ as the following power series

$$J_{m,n}^{(2)}(x\mid p,q)=\sum_{k=0}^{\infty}\frac{(-1)^k q^{1/2k(3k-1+2(m+n))}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k} \tag{37}$$

equivalently, we have

$$\begin{aligned}J_{m,n}^{(2)}(x\mid p,q)={}&\frac{1}{((p,q);(p,q))_n((p,q);(p,q))_n}\\&\times\sum_{k=0}^{\infty}\frac{(-1)^k q^{1/2k(3k-1+2(m+n))}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}\\={}&\frac{1}{\Gamma_{p,q}(m+1)\Gamma_{p,q}(n+1)}\left(\frac{x}{3(p-q)}\right)^{m+n}\\&\times\sum_{k=0}^{\infty}\frac{(-1)^k q^{1/2k(3k-1+2(m+n))}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{3k}\\={}&\frac{((p^{m+1},q^{m+1});(p,q))_\infty((p^{n+1},q^{n+1});(p,q))_\infty}{((p,q);(p,q))_\infty((p,q);(p,q))_\infty}\\&\times\sum_{k=0}^{\infty}\frac{(-1)^k q^{1/2\,k(3k-1+2(m+n))}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}.\end{aligned} \tag{38}$$

Lemma 7. *The connection between generating functions of the (p,q) -Humbert functions $J^{(1)}_{m,n}(x \mid p,q)$ and $J^{(2)}_{m,n}(x \mid p,q)$ is given by*

$$J^{(1)}_{m,n}\left(q^{1/3}x \mid \frac{1}{p},\frac{1}{q}\right) = q^{1/3(n+m)+\binom{n}{2}+\binom{m}{2}} J^{(2)}_{m,n}(x \mid p,q). \tag{39}$$

Proof. If we set that

$$x = q^{1/3}x,\ t = q^{-1/3}t,\ u = q^{-1/3}u \tag{40}$$

in (19) and using $e_{1/p,1/q}(x) = E_{p,q}(x)$, we get

$$\begin{aligned} F_1\left(q^{1/3}x\,;q^{-1/3}u,q^{-1/3}t \mid \frac{1}{p},\frac{1}{q}\right) &= E_{p,q}\left(\frac{xu}{3}\right)E_{p,q}\left(\frac{xt}{3}\right)E_{p,q}\left(-\frac{qx}{3ut}\right) \\ &= \sum_{m,n=-\infty}^{\infty} J^{(1)}_{m,n}\left(q^{1/3}x \mid \frac{1}{p},\frac{1}{q}\right)q^{-1/3(m+n)}u^m t^n \end{aligned} \tag{41}$$

and (34), we obtain (39).

Definition 8. The generating function $F_3(x\,;u,t \mid p,q)$ of the (p,q) -Humbert functions of the third kind $J^{(3)}_{m,n}(x \mid p,q)$ is given by

$$\begin{aligned} F_3(x\,;u,t \mid p,q) &= e_{p,q}\left(\frac{xu}{3}\right)e_{p,q}\left(\frac{xt}{3}\right)E_{p,q}\left(-\frac{qx}{3ut}\right) \\ &= \sum_{m,n=-\infty}^{\infty} J^{(3)}_{m,n}(x \mid p,q)u^m t^n. \end{aligned} \tag{42}$$

Using (42), (11) and (12), we have

$$\begin{aligned} &F_3(x\,;u,t \mid p,q) \\ &= \sum_{r=0}^{\infty}\frac{p^{r(r-1)/2}}{[r]_{p,q}!}\left(\frac{xu}{3}\right)^r \sum_{i=0}^{\infty}\frac{p^{i(i-1)/2}}{[i]_{p,q}!}\left(\frac{xt}{3}\right)^i \sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{[k]_{p,q}!}\left(-\frac{qx}{3ut}\right)^k \\ &= \sum_{i,r,k=0}^{\infty}\frac{(-1)^k p^{\binom{r}{2}+\binom{i}{2}} q^{\binom{k}{2}} q^k}{[k]_{p,q}![i]_{p,q}![r]_{p,q}!}\left(\frac{x}{3}\right)^{k+i+r}u^{r-k}t^{i-k}. \end{aligned} \tag{43}$$

Substituting r by $m+k$ and i by $n+k$ in the last equation, we obtain the following equality

$$\begin{aligned} &\sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}+\binom{n+k}{2}} q^{\binom{k+1}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}u^m t^n \\ &= \sum_{m,n=-\infty}^{\infty} J^{(3)}_{m,n}(x \mid p,q)u^m t^n. \end{aligned} \tag{44}$$

Explicitly, we get the explicit expression of (p,q)-Humbert functions $J^{(3)}_{m,n}(x \mid p,q)$ of the third kind as the following power series

$$J^{(3)}_{m,n}(x \mid p,q) = \sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}+\binom{n+k}{2}} q^{\binom{k+1}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}, \tag{45}$$

or, equivalently, we get

$$\begin{aligned} J^{(3)}_{m,n}(x \mid p,q) &= \frac{1}{((p,q)\,;(p,q))_n((p,q)\,;(p,q))_n} \\ &\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}+\binom{n+k}{2}} q^{\binom{k+1}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k} \\ &= \frac{1}{\Gamma_{p,q}(m+1)\Gamma_{p,q}(n+1)}\left(\frac{x}{3(p-q)}\right)^{m+n} \\ &\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}+\binom{n+k}{2}} q^{\binom{k+1}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{3k} \\ &= \frac{((p^{m+1},q^{m+1})\,;(p,q))_\infty((p^{n+1},q^{n+1})\,;(p,q))_\infty}{((p,q)\,;(p,q))_\infty((p,q)\,;(p,q))_\infty} \\ &\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}+\binom{n+k}{2}} q^{\binom{k+1}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}. \end{aligned} \tag{46}$$

Definition 9. A fourth generating function $F_4(x\,;u,t\,|\,p,q)$ of the (p,q) -Humbert functions $J^{(4)}_{m,n}(x\,|\,p,q)$ of the fourth kind is defined by

$$F_4(x\,;u,t\,|\,p,q)=E_{p,q}\left(\frac{qxu}{3}\right)E_{p,q}\left(\frac{qxt}{3}\right)e_{p,q}\left(-\frac{x}{3ut}\right)=\sum_{m,n=-\infty}^{\infty}J^{(4)}_{m,n}(x\,|\,p,q)u^mt^n. \tag{47}$$

Using (47), (11) and (12), we have

$$\begin{aligned}F_4(x\,;u,t\,|\,p,q)&=\sum_{r=0}^{\infty}\frac{q^{r(r-1)/2}}{[r]_{p,q}!}\left(\frac{qxu}{3}\right)^r\sum_{i=0}^{\infty}\frac{q^{i(i-1)/2}}{[i]_{p,q}!}\left(\frac{qxt}{3}\right)^i\cdot\sum_{k=0}^{\infty}\frac{p^{k(k-1)/2}}{[k]_{p,q}!}\left(-\frac{x}{3ut}\right)^k\\&=\sum_{i,r,k=0}^{\infty}\frac{(-1)^kq^{\binom{r}{2}+\binom{i}{2}}p^{\binom{k}{2}}q^{r+i}}{[k]_{p,q}![i]_{p,q}![r]_{p,q}!}\cdot\left(\frac{x}{3}\right)^{k+i+r}u^{r-k}t^{i-k}.\end{aligned} \tag{48}$$

Replace r by $m+k$ and i by $n+k$ to get

$$\sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{m+k+1}{2}+\binom{n+k+1}{2}}p^{\binom{k}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}u^mt^n=\sum_{m,n=-\infty}^{\infty}J^{(4)}_{m,n}(x\,|\,p,q)u^mt^n. \tag{49}$$

Explicitly, we obtain the explicit expressions of (p,q) -Humbert functions $J^{(4)}_{m,n}(x\,|\,p,q)$ as

$$\begin{aligned}J^{(4)}_{m,n}(x\,|\,p,q)&=\sum_{k=0}^{\infty}\frac{(-1)^kp^{\binom{k}{2}}q^{\binom{m+k+1}{2}+\binom{n+k+1}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}\\&=\frac{1}{((p,q)\,;(p,q))_n((p,q)\,;(p,q))_n}\\&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^kp^{\binom{k}{2}}q^{\binom{m+k+1}{2}+\binom{n+k+1}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}\\&=\frac{1}{\Gamma_{p,q}(m+1)\Gamma_{p,q}(n+1)}\left(\frac{x}{3(p-q)}\right)^{m+n}\\&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^kp^{\binom{k}{2}}q^{\binom{m+k+1}{2}+\binom{n+k+1}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{3k}\\&=\frac{((p^{m+1},q^{m+1})\,;(p,q))_\infty((p^{n+1},q^{n+1})\,;(p,q))_\infty}{((p,q)\,;(p,q))_\infty((p,q)\,;(p,q))_\infty}\\&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^kp^{\binom{k}{2}}q^{\binom{m+k+1}{2}+\binom{n+k+1}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}.\end{aligned} \tag{50}$$

Definition 10. The generating function $F_5(x\,;u,t\,|\,p,q)$ of the (p,q) -Humbert functions $J^{(5)}_{m,n}(x\,|\,p,q)$ of the fifth kind is defined as

$$F_5(x\,;u,t\,|\,p,q)=e_{p,q}\left(\frac{xu}{3}\right)E_{p,q}\left(\frac{qxt}{3}\right)E_{p,q}\left(-\frac{qx}{3ut}\right)=\sum_{m,n=-\infty}^{\infty}q^{\frac{1}{2}n(n-1)}J^{(5)}_{m,n}(x\,|\,p,q)u^m t^n. \quad (51)$$

Using (11) (12) and (51), we have

$$F_5(x\,;u,t\,|\,p,q)=\sum_{r=0}^{\infty}\frac{p^{r(r-1)/2}}{[r]_{p,q}!}\left(\frac{xu}{3}\right)^r\sum_{i=0}^{\infty}\frac{q^{i(i-1)/2}}{[i]_{p,q}!}\cdot\left(\frac{qxt}{3}\right)^i\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{[k]_{p,q}!}\left(-\frac{qx}{3ut}\right)^k$$

$$=\sum_{i,r,k=0}^{\infty}\frac{(-1)^k p^{\binom{r}{2}}q^{\binom{i}{2}+\binom{k}{2}}q^{i+k}}{[k]_{p,q}![i]_{p,q}![r]_{p,q}!}\left(\frac{x}{3}\right)^{k+i+r}u^{r-k}t^{i-k}. \quad (52)$$

Upon setting $r=m+k$ and $i-n+k$ in the above equation, we get

$$\sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}}q^{\binom{n+k+1}{2}+\binom{k+1}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}u^m t^n=\sum_{m,n=-\infty}^{\infty}q^{\binom{n}{2}}J^{(5)}_{m,n}(x\,|\,p,q)u^m t^n. \quad (53)$$

Explicitly, we obtain the explicit representations of (p,q) -Humbert functions $J^{(5)}_{m,n}(x\,|\,p,q)$ of the fifth kind as the following power series

$$\begin{aligned}J^{(5)}_{m,n}(x\,|\,p,q)&=\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}}q^{\binom{n+k+1}{2}+\binom{k+1}{2}-\binom{n}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}\\
&=\frac{1}{((p,q)\,;(p,q))_n((p,q)\,;(p,q))_n}\\
&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}}q^{\binom{n+k+1}{2}+\binom{k+1}{2}-\binom{n}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}\\
&=\frac{1}{\Gamma_{p,q}(m+1)\Gamma_{p,q}(n+1)}\left(\frac{x}{3(p-q)}\right)^{m+n}\\
&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}}q^{\binom{n+k+1}{2}+\binom{k+1}{2}-\binom{n}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{3k}\\
&=\frac{((p^{m+1},q^{m+1})\,;(p,q))_\infty((p^{n+1},q^{n+1})\,;(p,q))_\infty}{((p,q)\,;(p,q))_\infty((p,q)\,;(p,q))_\infty}\\
&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{m+k}{2}}q^{\binom{n+k+1}{2}+\binom{k+1}{2}-\binom{n}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}.\end{aligned} \quad (54)$$

Definition 11. *The generating function $F_6(x;u,t\mid p,q)$ of the (p,q)-Humbert functions $J^{(6)}_{m,n}(x\mid p,q)$ of the sixth kind is defined by*

$$F_6(x;u,t\mid p,q)=E_{p,q}\left(\frac{qxu}{3}\right)e_{p,q}\left(\frac{xt}{3}\right)E_{p,q}\left(-\frac{qx}{3ut}\right)=\sum_{m,n=-\infty}^{\infty}q^{\frac{1}{2}m(m-1)}J^{(6)}_{m,n}(x\mid p,q)u^m t^n. \tag{55}$$

From (55), (11) and (12), we have

$$\begin{aligned}F_6(x;u,t\mid p,q)&=\sum_{r=0}^{\infty}\frac{q^{r(r-1)/2}}{[r]_{p,q}!}\left(\frac{qxu}{3}\right)^r\sum_{i=0}^{\infty}\frac{p^{i(i-1)/2}}{[i]_{p,q}!}\cdot\left(\frac{xt}{3}\right)^i\sum_{k=0}^{\infty}\frac{q^{k(k-1)/2}}{[k]_{p,q}!}\left(-\frac{qx}{3ut}\right)^k\\&=\sum_{i,r,k=0}^{\infty}\frac{(-1)^k p^{\binom{i}{2}}q^{\binom{r}{2}+\binom{k}{2}}q^{r+k}}{[k]_{p,q}![i]_{p,q}![r]_{p,q}!}\left(\frac{x}{3}\right)^{k+i+r}u^{r-k}t^{i-k}.\end{aligned} \tag{56}$$

Substituting r by $m+k$ and i by $n+k$ in the last equation, we get the following equality

$$\sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{n+k}{2}}q^{\binom{m+k+1}{2}+\binom{k+1}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}u^m t^n=\sum_{m,n=-\infty}^{\infty}q^{\binom{m}{2}}J^{(6)}_{m,n}(x\mid p,q)u^m t^n. \tag{57}$$

Explicitly, we obtain the explicit representations of (p,q)-Humbert functions $J^{(6)}_{m,n}(x\mid p,q)$ of the sixth kind as the following power series

$$\begin{aligned}J^{(6)}_{m,n}(x\mid p,q)&=\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{n+k}{2}}q^{\binom{m+k+1}{2}+\binom{k+1}{2}-\binom{m}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}\\&=\frac{1}{((p,q);(p,q))_n((p,q);(p,q))_n}\\&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{n+k}{2}}q^{\binom{m+k+1}{2}+\binom{k+1}{2}-\binom{m}{2}}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}\\&=\frac{1}{\Gamma_{p,q}(m+1)\Gamma_{p,q}(n+1)}\left(\frac{x}{3(p-q)}\right)^{m+n}\\&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{n+k}{2}}q^{\binom{m+k+1}{2}+\binom{k+1}{2}-\binom{m}{2}}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{3k}\\&=\frac{((p^{m+1},q^{m+1});(p,q))_\infty((p^{n+1},q^{n+1});(p,q))_\infty}{((p,q);(p,q))_\infty((p,q);(p,q))_\infty}\\&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{n+k}{2}}q^{\binom{m+k+1}{2}+\binom{k+1}{2}-\binom{m}{2}}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}.\end{aligned} \tag{58}$$

Definition 12. The generating function $F_7(x\,;u,t\mid p,q)$ of the (p,q) -Humbert functions $J_{m,n}^{(7)}(x\mid p,q)$ of the seventh kind is defined by

$$F_7(x\,;u,t\mid p,q)=e_{p,q}\left(\frac{xu}{3}\right)E_{p,q}\left(\frac{qxt}{3}\right)e_{p,q}\left(-\frac{x}{3ut}\right)=\sum_{m,n=-\infty}^{\infty}J_{m,n}^{(7)}(x\mid p,q)u^mt^n. \tag{59}$$

From (11), (12) and ((59), we have

$$F_7(x\,;u,t\mid p,q)=\sum_{r=0}^{\infty}\frac{p^{r(r-1)/2}}{[r]_{p,q}!}\left(\frac{xu}{3}\right)^r\sum_{i=0}^{\infty}\frac{q^{i(i-1)/2}}{[i]_{p,q}!}\cdot\left(\frac{qxt}{3}\right)^i\sum_{k=0}^{\infty}\frac{p^{k(k-1)/2}}{[k]_{p,q}!}\left(-\frac{x}{3ut}\right)^k$$

$$=\sum_{i,r,k=0}^{\infty}\frac{(-1)^kq^{\binom{i}{2}}p^{\binom{r}{2}+\binom{k}{2}}q^i}{[k]_{p,q}![i]_{p,q}![r]_{p,q}!}\left(\frac{x}{3}\right)^{k+i+r}u^{r-k}t^{i-k}. \tag{60}$$

Replacing r by $m+k$ and i by $n+k$ in the above equation, we obtain the following equality

$$\sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{n+k+1}{2}}p^{\binom{m+k}{2}+\binom{k}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}u^mt^n=\sum_{m,n=-\infty}^{\infty}J_{m,n}^{(7)}(x\mid p,q)u^mt^n. \tag{61}$$

Explicitly, we get the explicit expressions of (p,q)-Humbert functions $J_{m,n}^{(7)}(x\mid p,q)$ of the seventh kind as the following power series

$$J_{m,n}^{(7)}(x\mid p,q)=\sum_{k=0}^{\infty}\frac{(-1)^kq^{\binom{n+k+1}{2}}p^{\binom{m+k}{2}+\binom{k}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}$$

$$\begin{aligned}J_{m,n}^{(7)}(x\mid p,q)&=\frac{1}{((p,q)\,;(p,q))_n((p,q)\,;(p,q))_n}\\&\times\sum_{k=0}^{\infty}\frac{(-1)^kp^{\binom{m+k}{2}+\binom{k}{2}}q^{\binom{n+k+1}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}\\&=\frac{1}{\Gamma_{p,q}(m+1)\Gamma_{p,q}(n+1)}\left(\frac{x}{3(p-q)}\right)^{m+n}\\&\times\sum_{k=0}^{\infty}\frac{(-1)^kp^{\binom{m+k}{2}+\binom{k}{2}}q^{\binom{n+k+1}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{3k}\\&=\frac{((p^{m+1},q^{m+1})\,;(p,q))_\infty((p^{n+1},q^{n+1})\,;(p,q))_\infty}{((p,q)\,;(p,q))_\infty((p,q)\,;(p,q))_\infty}\\&\times\sum_{k=0}^{\infty}\frac{(-1)^kp^{\binom{m+k}{2}+\binom{k}{2}}q^{\binom{n+k+1}{2}}}{((p,q)\,;(p,q))_k((p^{m+1},q^{m+1})\,;(p,q))_k((p^{n+1},q^{n+1})\,;(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}.\end{aligned} \tag{62}$$

Definition 13. The generating function $F_8(x;u,t\mid p,q)$ of the (p,q) -Humbert functions $J_{m,n}^{(8)}(x\mid p,q)$ of the eighth kind is defined by

$$F_8(x;u,t\mid p,q)=E_{p,q}\left(\frac{qxu}{3}\right)e_{p,q}\left(\frac{xt}{3}\right)e_{p,q}\left(-\frac{x}{3ut}\right)=\sum_{m,n=-\infty}^{\infty}J_{m,n}^{(8)}(x\mid p,q)u^m t^n. \tag{63}$$

From (11), (12) and (63), we have

$$F_8(x;u,t\mid p,q)=\sum_{r=0}^{\infty}\frac{q^{r(r-1)/2}}{[r]_{p,q}!}\left(\frac{qxu}{3}\right)^r\sum_{i=0}^{\infty}\frac{p^{i(i-1)/2}}{[i]_{p,q}!}\cdot\left(\frac{xt}{3}\right)^i\sum_{k=0}^{\infty}\frac{p^{k(k-1)/2}}{[k]_{p,q}!}\left(-\frac{x}{3ut}\right)^k$$

$$=\sum_{i,r,k=0}^{\infty}\frac{(-1)^k q^{\binom{r}{2}}p^{\binom{i}{2}+\binom{k}{2}}q^r}{[k]_{p,q}![i]_{p,q}![r]_{p,q}!}\left(\frac{x}{3}\right)^{k+i+r}u^{r-k}t^{i-k}. \tag{64}$$

Substituting r by $m+k$ and i by $n+k$ in the above equation, we get the following equality

$$\sum_{m,n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{m+k+1}{2}}p^{\binom{n+k}{2}+\binom{k}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}u^m t^n=\sum_{m,n=-\infty}^{\infty}J_{m,n}^{(8)}(x\mid p,q)u^m t^n. \tag{65}$$

Explicitly, we obtain the explicit expressions of (p,q)-Humbert functions $J_{m,n}^{(6)}(x\mid p,q)$ of the eighth kind as the following power series

$$J_{m,n}^{(8)}(x\mid p,q)=\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{m+k+1}{2}}p^{\binom{n+k}{2}+\binom{k}{2}}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}.$$

$$\begin{aligned}J_{m,n}^{(8)}(x\mid p,q)&=\frac{1}{((p,q);(p,q))_n((p,q);(p,q))_n}\\&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{n+k}{2}+\binom{k}{2}}q^{\binom{m+k+1}{2}}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}\\&=\frac{1}{\Gamma_{p,q}(m+1)\Gamma_{p,q}(n+1)}\left(\frac{x}{3(p-q)}\right)^{m+n}\\&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{n+k}{2}+\binom{k}{2}}q^{\binom{m+k+1}{2}}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{3k}\\&=\frac{((p^{m+1},q^{m+1});(p,q))_\infty((p^{n+1},q^{n+1});(p,q))_\infty}{((p,q);(p,q))_\infty((p,q);(p,q))_\infty}\\&\quad\times\sum_{k=0}^{\infty}\frac{(-1)^k p^{\binom{n+k}{2}+\binom{k}{2}}q^{\binom{m+k+1}{2}}}{((p,q);(p,q))_k((p^{m+1},q^{m+1});(p,q))_k((p^{n+1},q^{n+1});(p,q))_k}\left(\frac{(p-q)x}{3}\right)^{m+n+3k}.\end{aligned} \tag{66}$$

Furthermore, we show the relations between generating functions for the (p,q)-Humbert functions.

Theorem 14. *The connections between generating functions of the (p,q) -Humbert functions of all kinds are given by*

$$F_4(x;u,t\mid p,q)=F_3\left(qx;u,t\mid \frac{1}{p},\frac{1}{q}\right),$$
$$F_7(x;u,t\mid p,q)=F_6\left(qx;u,t\mid \frac{1}{p},\frac{1}{q}\right) \tag{67}$$

and

$$F_8(x;u,t\mid p,q)=F_5\left(qx;u,t\mid \frac{1}{p},\frac{1}{q}\right). \tag{68}$$

Further examples can be discussed, but are omitted for the sake of conciseness.

Theorem 15. *(Multiplication theorem) The links between the generating functions for the (p,q) -Humbert functions of all kinds*

$$\begin{aligned}
F_1(x;u,t\mid p,q)&=F_3(x;u,t\mid p,q)e_{p,q}\left(\frac{qx}{3ut}\right)e_{p,q}\left(-\frac{x}{3ut}\right),\\
F_3(x;u,t\mid p,q)&=F_1(x;u,t\mid p,q)E_{p,q}\left(-\frac{qx}{3ut}\right)E_{p,q}\left(\frac{x}{3ut}\right),\\
F_3(x;u,t\mid p,q)&=F_6(x;u,t\mid p,q)e_{p,q}\left(\frac{xu}{3}\right)e_{p,q}\left(-\frac{qxu}{3}\right),\\
F_6(x;u,t\mid p,q)&=F_3(x;u,t\mid p,q)E_{p,q}\left(-\frac{xu}{3}\right)E_{p,q}\left(\frac{qxu}{3}\right),\\
F_5(x;u,t\mid p,q)&=F_7(x;u,t\mid p,q)E_{p,q}\left(-\frac{qx}{3ut}\right)E_{p,q}\left(\frac{x}{3ut}\right),\\
F_7(x;u,t\mid p,q)&=F_5(x;u,t\mid p,q)e_{p,q}\left(\frac{qx}{3ut}\right)e_{p,q}\left(-\frac{x}{3ut}\right),\\
F_6(x;u,t\mid p,q)&=F_8(x;u,t\mid p,q)E_{p,q}\left(-\frac{qx}{3ut}\right)E_{p,q}\left(\frac{x}{3ut}\right),\\
F_8(x;u,t\mid p,q)&=F_6(x;u,t\mid p,q)e_{p,q}\left(\frac{qx}{3ut}\right)e_{p,q}\left(-\frac{x}{3ut}\right),\\
F_7(x;u,t\mid p,q)&=F_1(x;u,t\mid p,q)E_{p,q}\left(\frac{-xt}{3}\right)E_{p,q}\left(\frac{qxt}{3}\right),\\
F_1(x;u,t\mid p,q)&=F_7(x;u,t\mid p,q)e_{p,q}\left(\frac{xt}{3}\right)e_{p,q}\left(-\frac{qxt}{3}\right)\\
F_1(x;u,t\mid p,q)&=F_8(x;u,t\mid p,q)e_{p,q}\left(-\frac{qxu}{3}\right)E_{p,q}\left(\frac{xu}{3}\right),
\end{aligned} \tag{69}$$

$$F_8(x;u,t\mid p,q)=F_1(x;u,t\mid p,q)E_{p,q}\left(\frac{qxu}{3}\right)e_{p,q}\left(-\frac{xu}{3}\right). \tag{70}$$

3. The Recurrence Relations

In this section, we show the significant interesting recurrence relations for the (p,q)-Humbert functions of the first kind so far introduced can be established with respect to x on their generating functions in different ways.

Theorem 16. *The (p,q) -Humbert functions $J^{(1)}_{m,n}(x\mid p,q)$ satisfy the recurrence relations*

$$\begin{aligned}
&J^{(1)}_{m-1,n}(px\mid p,q)+p^{n-2m-1/3}q^{2m-n+1/3}J^{(1)}_{m,n-1}(p^{2/3}q^{1/3}x\mid p,q)\\
&\quad -p^{-m+n+2/3}q^{m+n+2/3}J^{(1)}_{m+1,n+1}(p^{1/3}q^{2/3}x\mid p,q)\\
&=3D_{p,q}\left\{J^{(1)}_{m,n}(x\mid p,q)\right\},
\end{aligned} \tag{71}$$

$$\begin{aligned}
&p^{m-2n-1/3}q^{2n-m+1/3}J^{(1)}_{m-1,n}(p^{2/3}q^{1/3}x\mid p,q)+J^{(1)}_{m,n-1}(px\mid p,q)\\
&\quad -p^{-m+n+2/3}q^{m+n+2/3}J^{(1)}_{m+1,n+1}(p^{1/3}q^{2/3}x\mid p,q)\\
&=3D_{p,q}\left\{J^{(1)}_{m,n}(x\mid p,q)\right\},
\end{aligned} \tag{72}$$

$$\begin{aligned}
&J^{(1)}_{m-1,n}(px\mid p,q)+p^{2n-m-2/3}q^{m-2n+2/3}J^{(1)}_{m,n-1}(p^{1/3}q^{2/3}x\mid p,q)\\
&\quad -p^{n-2m-1/3}q^{2m-n+1/3}J^{(1)}_{m+1,n+1}(p^{2/3}q^{1/3}x\mid p,q)\\
&=3D_{p,q}\left\{J^{(1)}_{m,n}(x\mid p,q)\right\},
\end{aligned} \tag{73}$$

$$\begin{aligned}
&p^{2n-m-2/3}q^{n-2m+1/3}J^{(1)}_{m-1,n}(p^{1/3}q^{2/3}x\mid p,q)+J^{(1)}_{m,n-1}(px\mid p,q)\\
&\quad -p^{m-2n-1/3}q^{2n-m+1/3}J^{(1)}_{m+1,n+1}(p^{2/3}q^{1/3}x\mid p,q)\\
&=3D_{p,q}\left\{J^{(1)}_{m,n}(x\mid p,q)\right\},
\end{aligned} \tag{74}$$

$$\begin{aligned}
&p^{m+n-1/3}q^{1-m-n/3}J^{(1)}_{m-1,n}(p^{2/3}q^{1/3}x\mid p,q)\\
&\quad +p^{2n-m-2/3}q^{m-2n+2/3}J^{(1)}_{m,n-1}\\
&\quad \cdot(p^{1/3}q^{2/3}x\mid p,q)-J^{(1)}_{m+1,n+1}(px\mid p,q)\\
&=3D_{p,q}\left\{J^{(1)}_{m,n}(x\mid p,q)\right\}
\end{aligned} \tag{75}$$

$$\begin{aligned}
&p^{2m-n-2/3}q^{n-2m+2/3}J^{(1)}_{m-1,n}(p^{1/3}q^{2/3}x\mid p,q)\\
&\quad +p^{m+n-1/3}q^{1-m-n/3}J^{(1)}_{m,n-1}\\
&\quad \cdot(p^{2/3}q^{1/3}x\mid p,q)-J^{(1)}_{m+1,n+1}(px\mid p,q)\\
&=3D_{p,q}\left\{J^{(1)}_{m,n}(x\mid p,q)\right\}.
\end{aligned} \tag{76}$$

Proof. By applying the (p,q)-derivative operator on both sides of Eq. (19), using (15) and (18), we get

$$\frac{1}{3}\left[ue_{p,q}\left(\frac{pxu}{3}\right)e_{p,q}\left(\frac{pxt}{3}\right)e_{p,q}\left(-\frac{px}{3ut}\right) + te_{p,q}\left(\frac{qxu}{3}\right)e_{p,q}\left(\frac{pxt}{3}\right)e_{p,q}\left(-\frac{px}{3ut}\right) - \frac{1}{ut}e_{p,q}\left(\frac{qxu}{3}\right)e_{p,q}\left(\frac{qxt}{3}\right)e_{p,q}\left(-\frac{px}{3ut}\right)\right] = \sum_{m,n=-\infty}^{\infty} D_{p,q}J_{m,n}^{(1)}(x\mid p,q)u^m t^n. \tag{77}$$

Taking $x=p^{2/3}q^{1/3}x$, $u=p^{-2/3}q^{2/3}u$ and $t=p^{1/3}q^{-1/3}t$ in Eq. (19), then we get the result

$$p^{1/3}q^{-1/3}te_{p,q}\left(\frac{qxu}{3}\right)e_{p,q}\left(\frac{pxt}{3}\right)e_{p,q}\left(-\frac{px}{3ut}\right) = \sum_{m,n=-\infty}^{\infty} p^{n-2m/3}q^{2m-n/3}J_{m,n-1}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right)u^m t^n. \tag{78}$$

Using the generating function (19), and taking $x=p^{1/3}q^{2/3}x$, $u=p^{-1/3}q^{1/3}u$ and $t=p^{-1/3}q^{1/3}t$, we have

$$\frac{1}{p^{-2/3}q^{2/3}ut}e_{p,q}\left(\frac{qxu}{3}\right)e_{p,q}\left(\frac{qxt}{3}\right)e_{p,q}\left(-\frac{px}{3ut}\right) = \sum_{m,n=-\infty}^{\infty} p^{-(m+n/3)}q^{m+n/3}J_{m+1,n+1}^{(1)}\left(p^{1/3}q^{2/3}x\mid p,q\right)u^m t^n. \tag{79}$$

Using Eqs. (77), (78) and (79), we give the following relation

$$\frac{1}{3}\left[\sum_{m,n=-\infty}^{\infty} J_{m,n}^{(1)}(px\mid p,q)u^{m+1}t^n + \sum_{m,n=-\infty}^{\infty} p^{2m-n-1/3}q^{2m-n+1/3}J_{m,n-1}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right)u^m t^n - \sum_{m,n=-\infty}^{\infty} p^{2-m-n/3}q^{m+n+2/3}J_{m+1,n+1}^{(1)}\left(p^{1/3}q^{2/3}x\mid p,q\right)u^m t^n\right] = \sum_{m,n=-\infty}^{\infty} D_{p,q}J_{m,n}^{(1)}(x\mid p,q)u^m t^n. \tag{80}$$

Thus, we obtain the recurrence relation (71). Similarly, the other equations of this theorem can be proved.

Theorem 17. *The (p,q) -Humbert functions $J_{m,n}^{(1)}(x\mid p,q)$ have the following recurrence relations*

$$J_{m-1,n}^{(1)}(px\mid p,q) + p^{n-2m-1/3}q^{2m-n+1/3}J_{m,n-1}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right) = p^{m-2n-1/3}q^{2n-m+1/3}J_{m-1,n}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right) + J_{m,n-1}^{(1)}(px\mid p,q), \tag{81}$$

$$J_{m-1,n}^{(1)}(px\mid p,q) + p^{2n-m-2/3}q^{m-2n+2/3}J_{m,n-1}^{(1)}\left(p^{1/3}q^{2/3}x\mid p,q\right) - p^{n-2m-1/3}q^{2m-n+1/3}J_{m+1,n+1}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right) = p^{2m-n-2/3}q^{n-2m+2/3}J_{m-1,n}^{(1)}\left(p^{1/3}q^{2/3}x\mid p,q\right) + J_{m,n-1}^{(1)}(px\mid p,q) - p^{m-2n-1/3}q^{2n-m+1/3}J_{m+1,n+1}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right), \tag{82}$$

$$p^{m+n-1/3}q^{1-m-n/3}J_{m-1,n}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right) + p^{2n-m-2/3}q^{m-2n+2/3}J_{m,n-1}^{(1)}\left(p^{1/3}q^{2/3}x\mid p,q\right) = p^{2m-n-2/3}q^{n-2m+2/3}J_{m-1,n}^{(1)}\left(p^{1/3}q^{2/3}x\mid p,q\right) + p^{m+n-1/3}q^{1-m-n/3}J_{m,n-1}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right), \tag{83}$$

$$p^{n-2m-1/3}q^{2m-n+1/3}J_{m,n-1}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right) - p^{-(m+n+2/3)}q^{m+n+2/3}J_{m+1,n+1}^{(1)}\left(p^{1/3}q^{2/3}x\mid p,q\right) = p^{2n-m-2/3}q^{m-2n+2/3}J_{m,n-1}^{(1)}\left(p^{1/3}q^{2/3}x\mid p,q\right) - p^{n-2m-1/3}q^{2m-n+1/3}J_{m+1,n+1}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right) \tag{84}$$

$$p^{m-2n-1/3}q^{2n-m+1/3}J_{m-1,n}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right) - p^{-(m+n+2/3)}q^{m+n+2/3}J_{m+1,n+1}^{(1)}\left(p^{1/3}q^{2/3}x\mid p,q\right) = p^{2m-n-2/3}q^{n-2m+2/3}J_{m-1,n}^{(1)}\left(p^{1/3}q^{2/3}x\mid p,q\right) - p^{m-2n-1/3}q^{2n-m+1/3}J_{m+1,n+1}^{(1)}\left(p^{2/3}q^{1/3}x\mid p,q\right). \tag{85}$$

Proof. By using (71) and (72), we obtain (81). In similar way, the Eqs. (82), (83), (84) and (85) can be proven.

Theorem 18. *The (p,q) -Humbert functions $J_{m,n}^{(1)}(x\mid p,q)$ satisfy the following recurrence relations*

$$3\frac{[n]_{p,q}}{x}J_{m,n}^{(1)}(x\mid p,q) = p^{m+n-1/3}J_{m,n-1}^{(1)}\left(p^{-(1/3)}x\mid p,q\right) + p^{m+n-1/3}q^n J_{m+1,n+1}^{(1)}\left(p^{-(1/3)}x\mid p,q\right), \tag{86}$$

$$3\frac{[m]_{p,q}}{x}J_{m,n}^{(1)}(x\mid p,q) = p^{m+n-1/3}J_{m-1,n}^{(1)}\left(p^{-(1/3)}x\mid p,q\right) + p^{m+n-1/3}q^m J_{m+1,n+1}^{(1)}\left(p^{-(1/3)}x\mid p,q\right), \tag{87}$$

$$3\frac{[n]_{p,q}}{x}J^{(1)}_{m,n}(x\mid p,q)=q^{m+n-1/3}J^{(1)}_{m,n-1}\left(q^{-(1/3)}x\mid p,q\right)+q^{m+n-1/3}p^{n}J^{(1)}_{m+1,n+1}\left(q^{-(1/3)}x\mid p,q\right)\tag{88}$$

$$3\frac{[m]_{p,q}}{x}J^{(1)}_{m,n}(x\mid p,q)=q^{m+n-1/3}J^{(1)}_{m-1,n}\left(q^{-(1/3)}x\mid p,q\right)+q^{m+n-1/3}p^{m}J^{(1)}_{m+1,n+1}\left(q^{-(1/3)}x\mid p,q\right).\tag{89}$$

Proof. Multiplying both sides of Eq. (22) by $[n]_{p,q}$ and noting that

$$[n]_{p,q}=p^{-k}[n+k]_{p,q}-p^{-k}q^{n}[k]_{p,q},\tag{90}$$

and, we get

$$\begin{aligned}
[n]_{p,q}J^{(1)}_{m,n}(x\mid p,q)
&=\sum_{k=0}^{\infty}\frac{(-1)^k[n]_{p,q}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}\\
&=\sum_{k=0}^{\infty}\frac{(-1)^k\left(p^{-k}[n+k]_{p,q}-p^{-k}q^{n}[k]_{p,q}\right)}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}\\
&=\sum_{k=0}^{\infty}\frac{(-1)^k p^{-k}[n+k]_{p,q}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}
-\sum_{k=0}^{\infty}\frac{(-1)^k p^{-k}q^{n}[k]_{p,q}}{[k]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}\\
&=\sum_{k=0}^{\infty}\frac{(-1)^k p^{-k}}{[k]_{p,q}![m+k]_{p,q}![n+k-1]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}
-\sum_{k=0}^{\infty}\frac{(-1)^k p^{-k}q^{n}}{[k-1]_{p,q}![m+k]_{p,q}![n+k]_{p,q}!}\left(\frac{x}{3}\right)^{m+n+3k}\\
&=p^{m+n-1/3}\frac{x}{3}\sum_{k=0}^{\infty}\frac{(-1)^k}{[k]_{p,q}![m+k]_{p,q}![n+k-1]_{p,q}!}
\cdot\left(\frac{p^{-1/3}x}{3}\right)^{m+n-1+3k}+p^{m+n-1/3}q^{n}\frac{x}{3}\sum_{k=0}^{\infty}\\
&\quad\cdot\frac{(-1)^k}{[k]_{p,q}![m+k+1]_{p,q}![n+k+1]_{p,q}!}\left(\frac{p^{-1/3}x}{3}\right)^{m+n+3k+2}.
\end{aligned}\tag{91}$$

Using (22) and (91), we obtain (86). Similarly, we can prove (87), (88) and (89).

Theorem 19. *The (p,q) -Humbert functions $J^{(1)}_{m,n}(x\mid p,q)$ have the following recurrence relations*

$$q^{-(m+n-1/3)}J^{(1)}_{m,n-1}\left(q^{1/3}x\mid p,q\right)=p^{-(m+n-1/3)}J^{(1)}_{m,n-1}\left(p^{1/3}x\mid p,q\right)+(p-q)\frac{x}{3}J^{(1)}_{m+1,n}(x\mid p,q),\tag{92}$$

$$q^{-(m+n-1/3)}J^{(1)}_{m-1,n}\left(q^{1/3}x\mid p,q\right)=p^{-(m+n-1/3)}J^{(1)}_{m-1,n}\left(p^{1/3}x\mid p,q\right)+(p-q)\frac{x}{3}J^{(1)}_{m,n+1}(x\mid p,q),\tag{93}$$

$$q^{-(m+n-2/3)}J^{(1)}_{m-1,n-1}\left(q^{1/3}x\mid p,q\right)=p^{-(m+n-2/3)}J^{(1)}_{m-1,n-1}\left(p^{1/3}x\mid p,q\right)+(p-q)\frac{x}{3}J^{(1)}_{m,n}(x\mid p,q),\tag{94}$$

$$p^{-(m+n-1/3)}J^{(1)}_{m,n-1}\left(p^{1/3}x\mid p,q\right)=q^{-(m+n-1/3)}J^{(1)}_{m,n-1}\left(q^{1/3}x\mid p,q\right)-(p-q)\frac{x}{3}J^{(1)}_{m+1,n}(x\mid p,q),\tag{95}$$

$$p^{-(m+n-1/3)}J^{(1)}_{m-1,n}\left(p^{1/3}x\mid p,q\right)=q^{-(m+n-1/3)}J^{(1)}_{m-1,n}\left(q^{1/3}x\mid p,q\right)-(p-q)\frac{x}{3}J^{(1)}_{m,n+1}(x\mid p,q)\tag{96}$$

$$p^{-(m+n-2/3)}J^{(1)}_{m-1,n-1}\left(p^{1/3}x\mid p,q\right)=q^{-(m+n-2/3)}J^{(1)}_{m-1,n-1}\left(q^{1/3}x\mid p,q\right)-(p-q)\frac{x}{3}J^{(1)}_{m,n}(x\mid p,q).\tag{97}$$

Proof. By (22), we consider

$$\begin{aligned}
q^{-(m+n-1/3)}J^{(1)}_{m,n-1}\left(q^{1/3}x\mid p,q\right)
&=q^{-(m+n-1/3)}\sum_{k=0}^{\infty}\frac{(-1)^k}{[k]_{p,q}![m+k]_{p,q}![n+k-1]_{p,q}!}\cdot\left(\frac{xq^{1/3}}{3}\right)^{m+n-1+3k}\\
&=\sum_{k=0}^{\infty}\frac{(-1)^k}{[k]_{p,q}![m+k]_{p,q}![n+k-1]_{p,q}!}\left(\frac{x}{3}\right)^{m+n-1+3k}q^{k}.
\end{aligned}\tag{98}$$

Using the following identity

$$q^{k}=p^{k}-(p-q)[k]_{p,q},\tag{99}$$

we get

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{[k]_{p,q}![m+k]_{p,q}![n+k-1]_{p,q}!}\left(\frac{x}{3}\right)^{m+n-1+3k}q^k$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{[k]_{p,q}![m+k]_{p,q}![n+k-1]_{p,q}!}\left(\frac{x}{3}\right)^{m+n-1+3k}\cdot\left(p^k-(p-q)[k]_{p,q}\right)$$
$$=\sum_{k=0}^{\infty}\frac{(-1)^k p^k}{[k]_{p,q}![m+k]_{p,q}![n+k-1]_{p,q}!}\left(\frac{x}{3}\right)^{m+n-1+3k}-(p-q)\sum_{k=0}^{\infty}\frac{(-1)^k[k]_{p,q}}{[k]_{p,q}![m+k]_{p,q}![n+k-1]_{p,q}!}\left(\frac{x}{3}\right)^{m+n-1+3k}$$
$$=p^{-(m+n-1/3)}J^{(1)}_{m,n-1}\left(p^{1/3}x\mid p,q\right)+(p-q)\frac{x}{3}J^{(1)}_{m+1,n}(x\mid p,q). \tag{100}$$

Thus, the Eq. (92) is proved. In the same way, equations (93), (94), (95), (96) and (97) can be proved.

Similar recurrence relations can be achieved by using the generating function; in fact, by differentiating with respect to u and v, separately, we have:

Theorem 20. *The (p,q) -Humbert functions satisfy the following properties:*

$$\frac{x}{3}\left[p^{m-2n-2/3}J^{(1)}_{m,n}\left(p^{2/3}x\mid p,q\right)+p^{-(m+n+1/3)}q^{2m-n-1/3}J^{(1)}_{m+2,n+1}\left(q^{1/3}x\mid p,q\right)\right]=[m+1]_{p,q}J^{(1)}_{m+1,n}(x\mid p,q), \tag{101}$$

$$\frac{x}{3}\left[p^{m-2n-2/3}J^{(1)}_{m+2,n+1}\left(p^{2/3}x\mid p,q\right)+p^{2m-n-1/3}q^{-(m+n+1/3)}J^{(1)}_{m,n}\left(p^{1/3}q^{1/3}x\mid p,q\right)\right]=[m+1]_{p,q}J^{(1)}_{m+1,n}(x\mid p,q), \tag{102}$$

$$\frac{x}{3}\left[p^{n-2m-2/3}J^{(1)}_{m,n}\left(p^{2/3}x\mid p,q\right)+p^{-(n+m+1/3)}q^{2n-m-1/3}J^{(1)}_{m+1,n+2}\left(p^{1/3}q^{1/3}x\mid p,q\right)\right]=[n+1]_{p,q}J^{(1)}_{m,n+1}(x\mid p,q) \tag{103}$$

$$\frac{x}{3}\left[p^{n-2m-2/3}J^{(1)}_{m+1,n+2}\left(p^{2/3}x\mid p,q\right)+p^{2n-m-1/3}q^{-(m+n+1/3)}J^{(1)}_{m,n}\left(p^{1/3}q^{1/3}x\mid p,q\right)\right]=[n+1]_{p,q}J^{(1)}_{m,n+1}(x\mid p,q). \tag{104}$$

Proof. Differentiating with respect to u in (19), using (15) and (17), we get

$$\frac{x}{3}\left[e_{p,q}\left(\frac{pxu}{3}\right)e_{p,q}\left(\frac{xt}{3}\right)e_{p,q}\left(-\frac{px}{3ut}\right)+\frac{1}{u^2t}e_{p,q}\left(\frac{qxu}{3}\right)e_{p,q}\left(\frac{xt}{3}\right)e_{p,q}\left(-\frac{px}{3ut}\right)\right]=\sum_{m,n=-\infty}^{\infty}[m]_{p,q}J^{(1)}_{m,n}(x\mid p,q)u^{m-1}t^n. \tag{105}$$

Taking $x=p^{2/3}x$, $u=p^{1/3}u$ and $t=p^{-2/3}t$, we have

$$\frac{p^{2/3}x}{3}e_{p,q}\left(\frac{pxu}{3}\right)e_{p,q}\left(\frac{xt}{3}\right)e_{p,q}\left(-\frac{px}{3ut}\right)=\sum_{m,n=-\infty}^{\infty}p^{m-2n/3}J^{(1)}_{m,n}\left(p^{2/3}x\mid p,q\right)u^mt^n. \tag{106}$$

Setting $x=p^{1/3}q^{1/3}x$, $u=p^{-1/3}q^{2/3}u$ and $t=p^{-1/3}q^{-1/3}t$, we get

$$\frac{p^{4/3}x}{3q^{2/3}u^2t}e_{p,q}\left(\frac{qxu}{3}\right)e_{p,q}\left(\frac{xt}{3}\right)e_{p,q}\left(-\frac{px}{3ut}\right)=\sum_{m,n=-\infty}^{\infty}p^{-(m+n/3)}q^{2m-n/3}J^{(1)}_{m+2,n+1}\left(p^{1/3}q^{1/3}x\mid p,q\right)u^mt^n. \tag{107}$$

Using (105) and by means of the results (106) and (107), we arrive at the following equality:

$$\frac{x}{3}\left[\sum_{m,n=-\infty}^{\infty}p^{m-2n-2/3}J^{(1)}_{m,n}\left(p^{2/3}x\mid p,q\right)u^mt^n+\sum_{m,n=-\infty}^{\infty}p^{-(m+n+1/3)}q^{2m-n-1/3}J^{(1)}_{m+2,n+1}\left(q^{1/3}x\mid p,q\right)u^mt^n\right]\cdot\sum_{m,n=-\infty}^{\infty}[m+1]_{p,q}J^{(1)}_{m+1,n}(x\mid p,q)u^mt^n. \tag{108}$$

Thus, we obtain the result (101). Proceeding on parallel lines as mentioned above, the relations (102), (103) and (104) are immediate consequences of the definitions (19), (15) and (17).

4. Conclusion and Perspectives

The (p,q)-Humbert functions or the twin-basic Humbert functions have various applications in the field of mathematical physics and engineering sciences and so on. There are some results that have been noticed in this study. We have seen some particular cases of (p,q)-Humbert functions of the first kind that can be introduced belonging to the family of (p,q)-Humbert functions. The (p,q)-Humbert functions of the first kind allow us to describe many aspects of computational analysis. It is also interesting to explore how these classes of (p,q)-Humbert functions of the first kind can be described in terms of (p,q)-Humbert functions of the different types. Many properties of these new transforms have been proved and should be a starting point of many other works. For this, the researchers recommended to study these other seven families of (p, q)-Humbert functions from these extensions as a parallel study of this work. Further work will be carried out in the next future in other fields of interest.

Data Availability

No data were used to support this paper.

Conflicts of Interest

The author of this paper declare that they have no conflicts of interest.

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Research Article

On Transformation Involving Basic Analogue of Multivariable *H*-Function

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In this article, fractional order q-integrals and q-derivatives involving a basic analogue of multivariable H-function have been obtained. We give an application concerning the basic analogue of multivariable H-function and q-extension of the Leibniz rule for the fractional q-derivative for a product of two basic functions. We also give the corollary concerning basic analogue of multivariable Meijer's G-function as a particular case of the main result.

1. Introduction and Preliminaries

The q-calculus is not of recent appearance, it was introduced in the twenties of last century. In 1910, Jackson [1] introduced and developed q-calculus systematically. The fractional q-calculus is the expansion of ordinary fractional calculus in the q-theory. Recently, there was a significant work done by many authors in the area of q-calculus due to lots of applications in mathematics, statistics, and physics.

Since special functions play significant roles in mathematical physics, it is persuaded to think that some deformation of the ordinary special functions based on the q-calculus can also play comparable roles in this area of research. Further, many authors have derived images of various q-special functions under fractional q-calculus operators; see, for example, [2–7], and may more. The q-fractional integrals and derivatives was firstly studied by Al-Salam [8] (see also, [9]). Many researchers have used these operators to evaluate fractional q-calculus formulas for various special function, general class of q-polynomials, basic analogue of Fox's H-function, fractional q-calculus formulas for various special function, and etc. One may refer to the recent work [2–7, 10–14] on fractional q-calculus. Throughout this article, let $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$, and N be the sets of integers, complex numbers, real numbers, positive real numbers, and positive integers, respectively, and let. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

The objective of this article is to establish fractional q-integral and q-derivative of Riemann-Liouville type involving a basic analogue of multivariable H-function. We also give an application of q-Leibniz formula.

In the q-calculus theory, for a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, we have a q-real number $[a]q$

and q-shifted factorial (q-analogue of the Pochhammer symbol) as given by

$$[a]_q \frac{1-q^a}{1-q}, (a;q)_n = \prod_{i=1}^{n-1}\left(1-aq^i\right)\left(a \in \mathbb{R}, n \in \mathbb{N} \cup \{\infty\}\right). \tag{1}$$

The q-Factorial function is defined by

$$(a;q)_n = \begin{cases} 1, \text{ if } n = 0 \\ \prod_{k=0}^{n=1} \left(1 - aq^k\right), \text{ if } n \in \mathbb{N}. \end{cases} \quad (2)$$

Its extension is

$$(a;q)_n = \frac{(a;q)\infty}{(aq^n;q)\infty} (n \in \mathbb{Z}), \quad (3)$$

which can be elaborated to $n = \alpha \in \mathbb{C}$, given by

$$(a;q)_\alpha = \frac{(a;q)\infty}{(aq^n;q)\infty} (\alpha \in \mathbb{C}; |q| < 1), \quad (4)$$

where the principal value of q^α is taken.

In terms of the q-gamma function, (2) can be written as

$$(a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)} (a \in \mathbb{R} \setminus \{0,-1,-2,\}), \quad (5)$$

where the q-gamma function [15] is given by

$$\Gamma_q(a) = \frac{(q;q)_\infty (1-q)^{1-a}}{(q^a;q)_\infty} = \frac{(q;q)_{a-1}}{(1-q)^{a-1}} (|q| < 1; a \in \mathbb{R} \setminus \{0,-1,-2,\}), \quad (6)$$

obviously,

$$\Gamma_q(a+1) = [a]_q \Gamma_q(a) q! (|q| < 1). \quad (7)$$

The q-analogue of the familiar Riemann-Liouville fractional integral operator of a function $f(x)$is defined by (see Al-Salam [8])

$$I_q^\mu\{f(x)\} = \frac{1}{\Gamma_q(\mu)} \int_0^x (x - tq)_{\mu-1} f(t) d_q t \, (\Re(\mu) > 0, |q| < 1), \quad (8)$$

also q-analogue of the power function is defined as

$$(x-y)_v = x^v \left(\frac{y}{x};q\right)_v = x^v \prod_{n=0}^{\infty} \left[\frac{1-(y/x)q^n}{1-(y/x)q^{n+v}}\right], x \neq 0. \quad (9)$$

The basic integral is given by (see Gasper and Rahman [15])

$$\int_0^x f(t) d_q t = x(1-q) \sum_{k=0}^{\infty} q^k f\left(xq^k\right). \quad (10)$$

The equation (8) in conjunction with (10) yield the following series representation of the Riemann-Liouville fractional integral operator

$$I_q^\mu f(x) = \frac{x^\mu (1-q)}{\Gamma_q(\mu)} \sum_{k=0}^{\infty} q^k \left[1 - q^{k+1}\right]_{\mu-1} f\left(xq^k\right). \quad (11)$$

In particular, for $f(x) = x^{\lambda-1}$, we have [4]

$$I_q^\mu\left(x^{\lambda-1}\right) = \frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda+\mu)} x\lambda + \mu - 1 \, (\Re(\lambda+\mu) > 0). \quad (12)$$

2. Basic Analogue of Multivariable H-Function

In this section, we introduce the basic analogue of multivariable H-function [16, 17], given by the following manner:

$$H(z_1, z_r; q) = H^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q':p_1q_1;\cdots;p_r,q_r} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} ; q \left| \begin{matrix} \left(a_j; \alpha_j^{(r)} \cdots, \alpha_j^{(r)}\right)_{1,p} : \left(c_j', \gamma'\right)_{1,p_1}, \cdots, \left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,p_r} \\ \left(b_j; \beta_j', \cdots, \beta_j^{(r)}\right)_{1,q'} : \left(d_j', \delta'\right)_{1,q_1}, \cdots, \left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,q_1} \end{matrix} \right. \right) \quad (13)$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L1} \cdots \int_{L_r} \pi^r \phi(s_1, \cdots, s_r; q) \prod_{i=1}^{r} \theta_i(s_i; q) x_1^{s_1} \cdots x_r^{s_r} d_q s_1 \cdots d_q s_r,$$

where $\omega = \sqrt{-1}$, and

$$\phi(s_1,\cdots,s_r;q) = \frac{\prod_{j=1}^{n} G\left(q^{1-\alpha_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i}\right)}{\prod_{j=n+1}^{p} G\left(q^{a_j-\sum_i^r \alpha_j^{(i)}s_i}\right)\prod_{j=1}^{q'} G\left(q^{1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}s_i}\right)},$$
$$\theta_i(s_1;q) = \frac{\prod_{j=1}^{m_i} G\left(q^{d_j^{(i)}-\delta_j^{(i)}s_i}\right)\prod_{j=1}^{n_i} G\left(q^{1-c_j^{(i)}+\gamma_j^{(i)}s_i}\right)}{\prod_{j=m_i+1}^{q_i} G\left(q^{1-d_j^{(i)}+\delta_j^{(i)}s_i}\right)\prod_{j=n_i+1}^{p_i} G\left(q^{c_j^{(i)}+\gamma_j^{(i)}s_i}\right)G(q^{1-s_i})\sin\pi s_i}, \quad (14)$$

here, $i = 1, \cdots, r$ and

$$G(q^a) = \left[\prod_{n=0}^{\infty}(1-q^{a+n})\right]^{-1} = \frac{1}{(q^a;q)_\infty}. \quad (15)$$

The integers $n, p, q, m_i, n_i, p_i, q_i$ are constrained by the inequalities $0 \le n \le p$, $0 \le q'$, $1 \le m_i \le q_i$ and $0 \le n_i \le p_i$, $i = 1, \cdots, r$. The poles of integrand are assumed to be simple. The quantities$a_j(j=1,\cdots,p)$; $c_j^{(i)}(j=1,\cdots,p_i)$; $b_j(j=1,\cdots,q')$; $d_j^{(i)}$ $(j=1,\cdots,q_i, i=1,\cdots,r)$ are complex numbers and the following quantities $\alpha_j^{(i)}(j=1,\cdots,p)$; $\gamma_j^{(i)}(j=1,\cdots,p_i)$; $\beta_j^{(i)}(j=1,\cdots,q')$; $\delta_j^{(i)}(j=1,\cdots,q_i, i=1,\cdots,r)$ are positive real numbers.

The contour L_i in the complex s_i-plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $\omega\infty$ with indentations, if necessary to make certain that all the poles of $G(q^{d_j^{(i)}+\delta_j^{(i)}s_i})$ $(j=1,\cdots,m_i)$ are separated from those of, $G(q^{1-c_j^{(i)}+\gamma_j^{(i)}s_i})$ $(i=1,\cdots,n_i)G(q^{1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}s_i})(j=1,\cdots,n)$. For large values of $|s_i|$, the integrals converge if $\Re(s\log(z_i) - \log\ \sin\pi s_i) < 0$ $(i=1,\cdots,r)$.

If the quantities $\alpha_j^{(i)}(j=1,\cdots,p)$; $\gamma_j^{(i)}(j=1,\cdots,p_i)$; $\beta_j^{(i)}$ $(j=1,\cdots,q')$; $\delta_j^{(i)}(j=1,\cdots,q_i) = 1$ for $i = 1, \cdots, r$, then the basic analogue of multivariable H-function reduces in basic analogue of multivariable Meijer's G-function defined by Khadia and Goyal [18], we obtain

$$G(z_1,\cdots,z_r;q) = G^{0,n:m_1,n_1;\cdots;m_r,n_r}_{p,q':p_1q_1;\cdots;p_r,q_r}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} ;q \left| \begin{matrix} (a_j)_p : \left(c_j'\right)_{p_1}, \cdots, \left(c_j^{(r)}\right)_{p_r} \\ (b_j)_{q'} : \left(d_j'\right)_{q_1}, \cdots, \left(d_j^{(r)}\right)_{q_1} \end{matrix} \right.\right)$$
$$= \frac{1}{(2\pi\omega)^r}\int_{L1}\cdots\int_{L_r}\pi^r v(s_1,\cdots,s_r;q)\prod_{i=1}^{r} v_i(s_i;q) z_1^{s_1}\cdots z_r^{s_r} d_q s_1 \cdots d_q s_r, \quad (16)$$

where

$$v(s_1,\cdots,s_r;q) = \frac{\prod_{j=1}^{n} G\left(q^{1-\alpha_j+\sum_{i=1}^{r}s_i}\right)}{\prod_{j=n+1}^{p} G\left(q^{a_j-\sum_{i=1}^{r}s_i}\right)\prod_{j=1}^{q'} G\left(q^{1-b_j+\sum_{i=1}^{r}s_i}\right)},$$
$$v(s_1;q) = \frac{\prod_{j=1}^{m_i} G\left(q^{d_j^{(i)}-s_i}\right)\prod_{j=1}^{n_i} G\left(q^{1-c_j^{(i)}+s_i}\right)}{\prod_{j=m_i+1}^{q_i} G\left(q^{1-d_j^{(i)}+s_i}\right)\prod_{j=n_i+1}^{p_i} G\left(q^{1-c_j^{(i)}-s_i}\right)G(q^{1-s_i})\sin\pi s_i}, \quad (17)$$

where $i = 1, \cdots, r$; the integers $n, p, q, m_i, n_i, p_i, q_i$ are constrained by the inequalities $0 \le n \le p$, $0 \le q'$, $1 \le m_i \le q_i$ and $0 \le n_i \le p_i$, $i = 1, \cdots, r$. The poles of integrand are assumed to be simple. The quantities $\alpha_j^{(i)}(j=1,\cdots,p)$; $c_j^{(i)}(j=1,\cdots,p_i)$; $b_j(j=1,\cdots,q')$; $d_j^{(i)}(j=1,\cdots,q_i;i=1,\cdots,r)$ are complex numbers.

The contour L_i in the complex s_i-plane is of the Mellin-Barnes type which runs from $-\omega\infty$ to $\omega\infty$ with indentations, if necessary to ensure that all the poles of $G(q^{d_j^{(i)}+\delta_j^{(i)}s_i})(j=1,\cdots,m_i)$ are separated from those of $G(q^{1-c_j^{(i)}+s_i})(i=1,\cdots,n_i)G(q^{1-a_j+\sum_{i=1}^{r}s_i})(j=1,\cdots,n)$. For large values of $|s_i|$, the integrals converge if $\mathfrak{R}(s\log(z_i)-\log\sin\pi s_i)<0(i=1,\cdots,r)$.

3. Main Results

In this section, we establish two fractional q-integral formulas about the basic analogue of multivariable H-function.

Let

$$U=m_1,n_1;\cdots;m_r,n_r;V=p_1,q_1;\cdots;p_r,q_r;$$
$$A=\left(a_j;\alpha_j',\cdots,\alpha_j^{(r)}\right)_{1,p};B=\left(c_j',\gamma'\right)_{1,p_1},\cdots,\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r};$$
$$C=\left(b_j;\beta_j',\cdots,\beta_j^{(r)}\right)_{1,q'};D=\left(d_j',\delta'\right)_{1,q_1},\cdots,\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r}. \tag{18}$$

Theorem 1. *Let $\mathfrak{R}(\mu)>0$, $|q|<1$, the Riemann-Liouville fractional q-integral of a product of two basic functions exists, and we have*

$$I_q^{\mu}\left\{x^{\lambda-1}H_{p,q':V}^{0,n:U}\left(\begin{matrix}z_1x^{P1}\\ \cdot\\ \cdot\\ \cdot\\ z_1x^{\mathrm{Pr}}\end{matrix};q\left|\begin{matrix}A:B\\ C:D\end{matrix}\right.\right)\right\}$$
$$=(1-q)^{\mu}x^{\lambda+\mu-1}H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{matrix}z_1x^{P1}\\ \cdot\\ \cdot\\ \cdot\\ z_1x^{\mathrm{Pr}}\end{matrix};q\left|\begin{matrix}(1-\lambda;p_1,\cdots,p_r)A:B\\ C(1-\lambda-\mu;p_1,\cdots,p_r):D\end{matrix}\right.\right), \tag{19}$$

where $p_i\in\mathbb{N}$, $\mathfrak{R}(s\ \log(z_i)-\log\ \sin\ \pi s_i)<0$ for $i=1,\cdots,r$.

Proof. To prove the result (19), we consider the left hand side of equation (19) (say I) and take the definitions (8) and (13) into account, we have

$$\mathcal{I}=\frac{1}{\Gamma_q(\alpha)}\int_0^x(x-yq)_{\alpha-1}\frac{y^{\lambda-1}}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\phi(s_1,\cdots,s_r;q)\cdot\prod_{i=1}^{r}\theta_i(s_i;q)\times z_1^{s_r}\cdots z_r^{s_r}y^{\rho_1s_1+\cdots\rho_rs_r}d_qs_1\cdots d_qs_rd_qy. \tag{20}$$

Interchanging the order of integrations which is permissible under the given conditions, we obtain

$$\mathcal{I}=\frac{1}{\Gamma_q(\alpha)(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\phi(s_1,\cdots,s_r;q)\prod_{i=1}^{r}\theta_i(s_i;q)z_1^{s_r}\cdots z_r^{s_r}\times\int_0^x(x-yq)_{\alpha-1}\left\{y^{\rho_1s_1+\cdots\rho_rs_r+\lambda-1}\right\}\mathrm{d}_qy\mathrm{d}_qs_1\cdots\mathrm{d}_qs_r. \tag{21}$$

The above equation writes

$$\mathcal{I}=\frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\phi(s,t;q)\theta_1(s;q)\cdots\theta_r(s;q)z_1^{s_1}\cdots z_r^{\rho_r}\times I_q^{\mu}\left\{x^{\rho_1s_1+\cdots+\rho_rs_r+\lambda-1}\right\}\mathrm{d}_qs_1\cdots\mathrm{d}_qs_r. \tag{22}$$

Now using the result (12), then the equation (22) reduces as

$$\mathcal{I}=\frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r}\pi^r\phi(s,t;q)\theta_1(s;q)\cdots\theta_r(s;q)z_1^{s_1}\cdots z_r^{\rho_r}\times\frac{\Gamma_q(\rho_1s_1+\cdots\rho_rs_r+\lambda)}{\Gamma_q(\rho_1s_1+\cdots\rho_rs_r+\lambda+\mu)}x^{\rho_1s_1+\cdots\rho_rs_r+\lambda+\mu-1}. \tag{23}$$

Next, interpreting the q-Mellin-Barnes multiple integrals contour in terms of the basic analogue of multivariable H-function, then we get the desired result (19).

If we replace μ by $-\mu$ in Theorem 1, and use the fractional q- derivative operator defined as

$$I_q^{-\mu}\{f(x)\}=D_{x,q}^{\mu}f(x)=\frac{1}{\Gamma_q(-\mu)}\int_0^x(x-tq)_{-\mu-1}f(t)\mathrm{d}_qt(\mathfrak{R}(\mu)<0), \tag{24}$$

and power function formula

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}\right\}=\frac{\Gamma_q(\lambda)}{\Gamma_q(\lambda-\mu)}x^{\lambda-\mu-1}(\lambda\neq-1,-2,\cdots), \tag{25}$$

then we have the following result:

Theorem 2. *Let $\Re(\mu) > 0$, $|q| < 1$ the Riemann Liouville fractional q-derivative of a product of two basic functions exists, and given by*

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}H_{p,q':V}^{0,n:U}\left(\begin{matrix} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{\rho_r}\end{matrix}\; ;q\left|\begin{matrix} A:B \\ C:D\end{matrix}\right.\right)\right\} = (1-q)^{-\mu}x^{\lambda-\mu-1}H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{matrix} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{\rho_r}\end{matrix}\; ;q\left|\begin{matrix} (1-\lambda;\rho_1,\cdots,\rho_r)A:B \\ C(1-\lambda+\mu;\rho_1,\cdots,\rho_r): D\end{matrix}\right.\right), \tag{26}$$

where $\rho_i \in \mathbb{N}$, $\Re(s\log(z_i) - \log\sin\pi s_i) < 0$ for $i = 1, \cdots, r$.

Proof. The proof of result asserted by Theorem 2 runs parallel to that of Theorem 1.

The details are, therefore, being omitted.

4. Leibniz's Application

In this section, we give an application concerning the basic analogue of multivariable H-function and q-extension of the Leibniz rule for the fractional q-derivative for a product of two basic functions.

We have the q-extension of the Leibniz rule for the fractional q-derivatives for a product of two basic functions in terms of a series involving the fractional q-derivatives of the function, in the following manner [9]:

Lemma 3.

$$D_{x,q}^{\alpha}\{W(x)Y(x)\} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}[q^{-\mu};q]_n}{(q;q)_n}D_{x,q}^{\mu-n}\{W(xq^n)\}D_{x,q}^{n}\{Y(x)\}, \tag{27}$$

where $W(x)$ and $Y(x)$ are two regular functions.

Theorem 4. *Let $\Re(\mu) < 0$,, then the Riemann-Liouville fractional q-derivative of a product of two basic function exists and given by*

$$H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{matrix} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{\rho_r}\end{matrix}\; ;q\left|\begin{matrix} (1-\lambda;\rho_1,\cdots,\rho_r), A:B \\ C, (1-\lambda+\mu;\rho_1,\cdots,\rho_r): D\end{matrix}\right.\right) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n\lambda+(n(n-1)/2)}[q^{-\mu};q]_n}{(q;q)_n(q^{\lambda};q)_{n-\mu}}H_{p+1,q'+1:V}^{0,n+1:U} \cdot \left(\begin{matrix} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{\rho_r}\end{matrix}\; ;q\left|\begin{matrix} (0;\rho_1,\cdots,\rho_r)A:B \\ C(n;\rho_1,\cdots,\rho_r): D\end{matrix}\right.\right), \tag{28}$$

where $p_i \in \mathbb{N}$, $\Re(s\ \log(z_i) - \log\ \sin\ \pi s_i) < 0$ for $i = 1, \cdots, r$.

Proof. For applying q-Leibniz rule, we let

$$W(x) = x^{\lambda-1} \text{ and } Y(x)H_{p,q':V}^{0,n:U}\left(\begin{matrix} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{\rho_r}\end{matrix}\; ;q\left|\begin{matrix} A:B \\ C:D\end{matrix}\right.\right). \tag{29}$$

By using the lemma 3, we have

$$D_{x,q}^{\mu}\left\{x^{\lambda-1}H_{p,q':V}^{0,n:U}\left(\begin{matrix} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{\rho_r}\end{matrix}\; ;q\left|\begin{matrix} A:B \\ C:D\end{matrix}\right.\right)\right\} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2}[q^{-\mu};q]_n}{(q;q)_n}D_{x,q}^{\mu-n}\left\{(xq^n)^{\lambda-1}\right\}D_{x,q}^{n} \cdot\{H(z_1x^{\rho_1},\cdots,z_rx^{\rho_r};q)\}. \tag{30}$$

Next, by setting $\lambda = 1$ and using the Theorem 2, we arrive at

$$D_{x,q}^{n}\{H(z_1x^{\rho_1},\cdots,z_rx^{\rho_r};q)\} = (1-q)^{-\mu}x^{-\mu}\times H_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{matrix} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{\rho_r}\end{matrix};q\left|\begin{matrix}(0;\rho_1,\cdots,\rho_r)A:B \\ C(n;\rho_1,\cdots,\rho_r):D\end{matrix}\right.\right), \tag{31}$$

where $\rho_i \in \mathbb{N}, \Re(s\log(z_i)-\log\sin\pi s_i)<0(i=1,\cdots,r)$.

Now, by using (25) and (31) we obtain the desired result (28) after several algebraic manipulations.

5. Particular Case

In this section, the basic analogue of multivariable H-function reduces in basic analogue of multivariable Meijer's G-function [18].

Let

$$\begin{aligned} &U=m_1,n_1;\cdots;m_r,n_r;V=p_1,q_1;\cdots;p_r,q_r;\\ &A_1=(a_j)_{1,p};B_1=(c_j)_{1,p_1},\cdots,\left(c_j^{(r)}\right)_{1,p_r};C_1(b_j)_{1,q'};\\ &D_1=(d_j)_{1q_1},\cdots,\left(d_j^{(r)}\right)_{1,q_r}. \end{aligned} \tag{32}$$

Corollary 5.

$$D_{x,q}^{\mu}\left\{t^{\lambda-1}G_{p+1,q'+1:V}^{0,n+1:U}\left(\begin{matrix} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{\rho_r}\end{matrix};q\left|\begin{matrix}A_1:B_1 \\ C_1:D_1\end{matrix}\right.\right)\right\} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n\lambda+(n(n-1)/2)}[q^{-\mu};q]_n}{(q;q)_n(q^{\lambda};q)_{n-\mu}}G_{p+1,q'+1:V}^{0,n+1:U} \cdot \left(\begin{matrix} z_1x^{\rho_1} \\ \cdot \\ \cdot \\ \cdot \\ z_rx^{\rho_r}\end{matrix};q\left|\begin{matrix}(0;\rho_1,\cdots,\rho_r)A_1:B_1 \\ C_1,(n;\rho_1,\cdots,\rho_r):D_1\end{matrix}\right.\right), \tag{33}$$

where $p_i \in \mathbb{N}, \Re(s\log(z_i)-\log\sin\pi s_i)<0$ *for* $i=1,\cdots,r$.

Remark 6. If the basic analogue of multivariable H-function reduces in basic analogue of Srivastava-Daout function [19], *then we obtain the results given by Purohit* et al. [20].

Remark 7. If the basic analogue of multivariable H-function reduces in basic analogue of H-function of two variables defined by Saxena et al. [21], *we obtain the result due to Yadav et al.* [7]. *Further, if the basic analogue of multivariable H-function reduces in basic analogue of H-function of one variable defined by Saxena et al.* [22], *then we can easily obtain the similar result.*

6. Conclusion

In the present article, we have proposed the fractional order q-integrals and q-derivatives involving a basic analogue of multivariable H-function. The significance of our derived results lies in their diverse generality. By specializing the various parameters as well as variables in the basic analogue of multivariable H-function, we can obtain a large number of results involving a remarkably wide range of useful basic functions (or product of such basic functions) of one and several variables. Hence, the derived formulas in this article are most general in character and may reaffirm to be useful in several interesting cases appearing in literature.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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