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# Editorial Board

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*Editorial*

# Recent Advances in Symmetry Analysis and Exact Solutions in Nonlinear Mathematical Physics

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Partial differential equations are used to describe a wide variety of physical phenomena such as fluid dynamics, plasma physics, solid mechanics, and quantum field theory that arise in physics. Many of these equations are nonlinear and, in general, these equations are often very difficult to solve explicitly. Many systematic methods are usually employed to study the nonlinear equations: these include the generalized symmetry method, the Painlevé analysis, the inverse scattering method, the Bäcklund transformation method, the conservation law method, the Cole-Hopf transformation, and the Hirota bilinear method.

Constructing exact solutions, in particular travelling wave solutions, of nonlinear equations plays an important role in soliton theory. Several important direct methods have been developed for obtaining travelling wave solutions to nonlinear partial differential equations such as the inverse scattering method, the tanh-function method, the extended tanh-function method, the $G'/G$ method, the simplest method, and the modified simplest method.

The authors of this special issue had been invited to submit original research articles as well as review articles in the following topics: methods for obtaining solutions of partial differential equations; conservation laws; general methods for obtaining solutions of ordinary differential equations; travelling wave solutions; novel applications in physics.

However, we received 22 papers in these research fields. After a rigorous reviewing process, six articles were finally accepted for publication. These articles contain some new and innovative techniques and ideas that may stimulate further researches in several branches of theory and applications of the transformation groups.

In "The Rational Solutions and Quasi-Periodic Wave Solutions as well as Interactions of $N$-Soliton Solutions for 3 + 1 Dimensional Jimbo-Miwa Equation," H. Yang et al. analyze the rational solutions, quasi-periodic wave solutions obtained by the Hirota method and the theta function for the 3 + 1 dimensional Jimbo-Miwa equation. The knowledge of these solutions allows them to explain the interaction of the $N$-soliton solutions, that is, to show when the soliton solution can be changed into the resonant solution and when there will appear the pursue collision; that is, the soliton with the faster speed will catch up with the soliton with the slower speed (after the collision, the two solitons will continue to spread as the previous speed and the direction).

In "An Efficient Numerical Method for the Solution of the Schrödinger Equation," L. Zhang and T. E. Simos show the original development of a new five-stage symmetric two-step fourteenth-algebraic-order method with vanished phase-lag and its first, second, and third derivatives. More specifically, they firstly introduce the development of the new method and the determination of the local truncation error and successively the local truncation error analysis and the stability. Finally, the interval of periodicity analysis and the efficiency of the new method are considered by applying it to couple of Schrödinger equations.

In the paper titled "Asymptotic Expansion of the Solutions to Time-Space Fractional Kuramoto-Sivashinsky Equations," the attention of authors is devoted to finding the asymptotic expansion of solutions to fractional partial differential equations with initial conditions. The residual power series method is proposed for time-space fractional partial differential equations, where the fractional integral and derivative are described in the sense of Riemann-Liouville integral and Caputo derivative. They apply the method to the linear and nonlinear time-space fractional Kuramoto-Sivashinsky equation with initial value and obtain asymptotic expansion of the solutions, which confirm the accuracy and efficiency of the method.

In "New Periodic Solutions for a Class of Zakharov Equations," C. Sun and S. Ji consider a class of nonlinear Zakharov equations. Inspired by Angluo's idea, by applying the Jacobian elliptic function method, they obtain new periodic solutions for Klein-Gordon Zakharov equations, Zakharov equations, and Zakharov-Rubenchik equations.

In the paper titled "Stability of the Cauchy Additive Functional Equation on Tangle Space and Applications," S. H. Kim introduces real tangles (a generalization of rational tangles) and its operations to enumerating tangles by using the calculus of continued fraction. Moreover, he studies about the analytical structure of tangles, knots, and links by using new operations between real tangles which need not have the topological structure. As for applications of the analytical structure, he proves the generalized Hyers-Ulam stability of the Cauchy additive functional equation $f(x \oplus y) = f(x) \oplus f(y)$ in tangle space which is a set of real tangles with analytic structure and describes the DNA recombination as the action of some enzymes on tangle space.

In their paper titled "The Stochastic Resonance Behaviors of a Generalized Harmonic Oscillator Subject to Multiplicative and Periodically Modulated Noises," S. Zhong et al. study the stochastic resonance (SR) characteristics of a generalized Langevin linear system driven by a multiplicative noise and a periodically modulated noise. They take in consideration a generalized Langevin equation (GLE) driven by an internal noise with long-memory and long-range dependence, such as fractional Gaussian noise and Mittag-Leffler noise. Such a model is appropriate to characterizing the chemical and biological solutions as well as to some nanotechnological devices. An exact analytic expression of the output amplitude is obtained. Based on it, some characteristic features of stochastic resonance phenomenon are revealed. On the other hand, by the use of the exact expression, they obtain the phase diagram for the resonant behaviors of the output amplitude versus noise intensity under different values of system parameters. These results could give the theoretical basis for practical use and control of the SR phenomenon of this mathematical model in future works.

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*Research Article*

# The Stochastic Resonance Behaviors of a Generalized Harmonic Oscillator Subject to Multiplicative and Periodically Modulated Noises

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The stochastic resonance (SR) characteristics of a generalized Langevin linear system driven by a multiplicative noise and a periodically modulated noise are studied (the two noises are correlated). In this paper, we consider a generalized Langevin equation (GLE) driven by an internal noise with long-memory and long-range dependence, such as fractional Gaussian noise (fGn) and Mittag-Leffler noise (M-Ln). Such a model is appropriate to characterize the chemical and biological solutions as well as to some nanotechnological devices. An exact analytic expression of the output amplitude is obtained. Based on it, some characteristic features of stochastic resonance phenomenon are revealed. On the other hand, by the use of the exact expression, we obtain the phase diagram for the resonant behaviors of the output amplitude versus noise intensity under different values of system parameters. These useful results presented in this paper can give the theoretical basis for practical use and control of the SR phenomenon of this mathematical model in future works.

## 1. Introduction

The phenomenon of stochastic resonance (SR) characterized the cooperative effect between weak signal and noise in a nonlinear systems, which was originally perceived for explaining the periodicity of ice ages in early 1980s [1, 2]. In the past three decades, the SR phenomenon attracted great attentions and has been documented in a large number of literatures in biology, physics, chemistry and engineering [3–6], such as SR in magnetic systems [7] and tunnel diode [8] and in cancer growth models [9, 10].

Nowadays, there have been considerable developments in SR, and the original understanding of SR is extended. Firstly, in the initial stage of investigation of SR, the nonlinearity system, noise and periodic signal were thought of as three essential ingredients for the presence of SR. However, in 1996, Berdichevsky and Gitterman [11] found that SR phenomenon can occur in the linear system with multiplicative colored noise [12, 13]. It is because the multiplicative noise breaks the symmetry of the potential of the linear system and this gives rise to the SR phenomenon, which has been explained by Valenti et al. in literatures [14, 15]. Valenti et al. investigated the SR phenomenon in population dynamics, where the multiplicative noise source is Gaussian or Lévy type. Moreover, the appearance of SR in a trapping overdamped monostable system was also investigated in literature [16] recently. Secondly, the conventional definition about SR phenomenon is the signal-to-noise ratio (SNR) versus the noise intensity exhibits a peak [17, 18], whereas the generalized SR [19] implies the nonmonotonic behaviors of a certain function of the output signal (such as the first and second moments, the autocorrelation function) on the system parameters. Thirdly, another interesting extension of SR lies in the fact that output amplitude $A$ attains a maximum value by increasing the driving frequency $\Omega$. Such phenomenon indicates the bona fide SR which was introduced by Gammaitoni et al. appears at some value of $\Omega$ [20, 21].

The SR phenomenon driven by Gaussian noise has been investigated both theoretically and experimentally. However, the Gaussian noise is just an ideal model for actual fluctuations and not always appropriate to describe the real noisy environment. For instance, in the nonequilibrium situation, the stochastic processes describing the interactions of a test particle with the environment exhibit a heavy tailed non-Gaussian distribution. Nowadays, Dybiec et al. investigated the resonant behaviors of a stochastic dynamics system with Lévy stable noise and an isotropic $\alpha$-stable noise with heavy tails and jumps in literatures [22, 23]. Dubkov et al. investigated the Lévy flight superdiffusion as a self-similar Lévy process and derive the fractional Fokker-Planck equation (FFPE) for probability distribution from the Langevin equation with Lévy stable noise [24–26].

Recently, more and more scholars began to pay attention to another important class of non-Gaussian noise, the bounded noise. It should be emphasized is that the bounded noise is a more realistic and versatile mathematical model of stochastic fluctuations in applications, and it is widely applied in the domains of statistical physics, biology, and engineering in the last 20 years. Furthermore, the well-known telegraph noise, such as dichotomous noise (DN) and trichotomous noise that are widely used in the studied of SR phenomena, is a special case of bounded noise. The deepening and development of theoretical studies on bounded noise led to the fact that lots of scholars investigated the effect of bounded noise on the stochastic resonant behaviors in the special model in physics, biology, and engineering [27].

Since Richardson's work in literature [28], a large number of observations related to anomalous diffusion [29–33] have been reported in several scientific fields, for example, brain studies [34, 35], social systems [36], biological cells [37, 38], animal foraging behavior [39], nanoscience [40, 41], and geophysical systems [42]. One of the main aspects of these situations is the correlation functions of the above anomalous diffusion phenomena, which may be related to the non-Markovian characteristic of the stochastic process [43]. The typical characteristic of anomalous diffusion lies in the mean-square displacement (MSD) which satisfies $\langle x(t)^2 \rangle \sim t^{\alpha}$, where the diffusion exponents $0 < \alpha < 1$ and $\alpha > 1$ indicate subdiffusion and superdiffusion, respectively. When $\alpha = 1$, the normal diffusion is recovered [44].

It is well-known that the normal diffusion can be modeled by a Langevin equation, where a Brownian particle subjected to a viscous drag from the surrounding medium is characterized by a friction force, and it also subjected to a stochastic force that arises from the surrounding environment. The friction constant determines how quickly the system exchanges energy with the surrounding environment. For a realistic description of the surrounding environment, it is difficult to choose a universal value of the friction constant. Indeed, in order to depict the real situation more effectively, a different value of the friction constant should be adopted. Hence, a generalization of the Langevin equation is needed, leading to the so-called generalized Langevin equation (GLE) [45]. The GLE is an equation of motion for a non-Markovian stochastic process where the particle has a memory effect to its velocity.

Nowadays, the GLE driven by a fractional Gaussian noise (fGn) [46] with a power-law friction kernel is extensively used for modeling anomalous diffusion processes. For instance, in the study on dynamics of single-molecule when the electron transfer (ET) was used to probe the conformational fluctuations of single-molecule enzyme, the distance between the ET donor and acceptor can be modeled well through a GLE driven by an fGn [47–54]. Besides, Viñales and Despósito have introduced a novel noise whose correlation function is proportional to a Mittag-Leffler function, which is called the Mittag-Leffler noise (M-Ln) [55–57]. The correlation function behaves as a power-law for large times but is nonsingular at the origin due to the inclusion of a characteristic time.

The overwhelming majority of previous studies of SR have related to the case where the external noise and the weak periodic force are introduced additively. However, Dykman et al. [58] studied the case where the signal is multiplied to noise; namely, the noise is modulated by a signal. They found that when an asymmetric bistable system is driven by a signal-modulated noise, stochastic resonance appeared, in contrast to the additive noise, new characteristics emerge, and their results were in agreement with experiments. Furthermore, Cao and Wu [59] studied the SR characteristics of a linear system driven by a signal-modulated noise and an additive noise. It seems that a periodically modulated noise is not uncommon and arises, for example, at the output of any amplifier (optics or radio astronomy) whose amplification factor varies periodically with time.

Due to the synergy of generalized friction kernel of a GLE and the periodically modulated noise, the stochastic resonant behaviors of a GLE can be influenced. In contrast to the case that has been investigated before, new dynamic characteristics emerge. Motivated by the above discussions, we would like to explore the stochastic resonance phenomenon in a generalized harmonic oscillator with multiplicative and periodically modulated noises. Moreover, we consider the GLE is driven by a fractional Gaussian noise and a Mittag-Leffler noise, respectively, in this paper. We focus on the various nonmonotonic behaviors of the output amplitude $A$ with the system parameters and the parameters of the internal driven noise.

The physical motivations of this paper are as follows: (1) in view of the importance of stochastic generalized harmonic oscillator (linear oscillator) with memory in physics, chemistry, and biology and due to the periodically modulated noise arising at the output of the amplifier of the optics device and radio astronomy device, to establish a physical model in which the SR can contain the effects of the two factors, the linearity of the system and the periodical modulation of the noise. (2) The second one is to give a theoretical foundation for the study of SR characteristic features of a generalized harmonic oscillator subject to multiplicative, periodically modulated noises and external periodic force. Our study shows that such a model leads to stochastic resonance phenomenon. Meanwhile, an exact analytic expression of the output amplitude is obtained. Based on it, some characteristic features of SR are revealed.

The paper is organized as follows. Section 2 gives the introductions of the generalized Langevin equation, the fractional Gaussian noise, and the Mittag-Leffler noise. Section 3 gives analytical expression of the output amplitude of the system's steady response. Section 4 presents the simulation results, and gives the discussions. Section 5 gives conclusion.

## 2. System Model

*2.1. The Generalized Langevin Equation.* The generalized Langevin equation (GLE) is an equation of motion for the non-Markovian stochastic process where the particle has a memory effect to its velocity. Anomalous diffusion in physical and biological systems can be formulated in the framework of a GLE that reads as Newton's law for a particle of the unit mass ($m = 1$) [11, 47, 60–66]:

$$\ddot{x}(t) + \gamma \int_0^t K(t-u)\,\dot{x}(u)\,du + \frac{dU(x)}{dx} = \eta(t), \quad (1)$$

where $x(t)$ is the displacement of the Brownian particle at time $t$, $\gamma > 0$ is the friction constant, $K(t)$ represents the memory kernel of the frictional force, and $dU(x)/dx$ is the external force under the potential $U(x)$. The random force $\eta(t)$ is zero-centered and stationary Gaussian that obeys the generalized second fluctuation-dissipation theorem [67]:

$$\langle \eta(t)\,\eta(s)\rangle = C(|t-s|) = k_B T \cdot K(|t-s|), \quad (2)$$

where $k_B$ is the Boltzmann constant and $T$ is the absolute temperature of the environment.

*2.2. The Fractional Gaussian Noise.* Fractional Gaussian noise (fGn) and fractional Brownian motion (fBm) were originally introduced by Mandelbrot and Van Ness [46] for modeling stochastic fractal processes. The fGn $\varepsilon_H(t) = \{\varepsilon_H(t),\ t > 0\}$ with a constant Hurst parameter $1/2 < H < 1$ can be used to more accurately characterize the long-range dependent process [the autocorrelation function $r(k)$ satisfies $\sum_{k=-\infty}^{\infty} r(k) = \infty$] than traditional short-range dependent stochastic process [the autocorrelation function $r(k)$ satisfies $\sum_{k=-\infty}^{\infty} r(k) < \infty$]. The short-range dependent stochastic process is, for example, Markov, Poisson or autoregressive moving average (ARMA) process.

Now consider the one-side normalized fBm which is a Gaussian process $B_H(t) = \{B_H(t),\ t > 0\}$, which shows the properties below [68]:

$$\begin{aligned}
&(1)\quad B_H(0) = 0;\\
&(2)\quad \langle B_H(t)\rangle = 0, \quad t > 0;\\
&(3)\quad \langle B_H(t)\,B_H(s)\rangle = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right),\\
&\qquad\qquad t, s > 0.
\end{aligned} \quad (3)$$

The fGn $\varepsilon_H(t) = \{\varepsilon_H(t),\ t > 0\}$, given by [47–49]

$$\varepsilon_H(t) = \sqrt{2k_B T}\frac{dB_H(t)}{dt}, \quad t > 0, \quad (4)$$

is a stationary Gaussian process with $\langle \varepsilon_H(t)\rangle = 0$. Therefore, according to (3) and (4) and the L'Hospital's rule, the autocorrelation function $C(t)$ of fGn can be derived as

$$\begin{aligned}
C(t) &= \langle \varepsilon_H(0)\,\varepsilon_H(t)\rangle = 2k_B T\\
&\quad \cdot \left\{\lim_{s\to 0^+} \left\langle \frac{[B_H(0+s) - B_H(0)]}{s}\frac{[B_H(t+s) - B_H(t)]}{s}\right\rangle\right\}\\
&= 2k_B T \cdot \left\{\lim_{s\to 0^+} \frac{|t+s|^{2H} + |t-s|^{2H} - 2|t|^{2H}}{2s^2}\right\} = 2k_B T\\
&\quad \cdot \left\{\lim_{s\to 0^+} \frac{2H|t+s|^{2H-1} - 2H|t-s|^{2H-1}}{4s}\right\} = 2k_B T\\
&\quad \cdot \left\{\lim_{s\to 0^+} \frac{2H(2H-1)|t+s|^{2H-2} + 2H(2H-1)|t-s|^{2H-2}}{4}\right\}\\
&= 2k_B T\\
&\quad \cdot \frac{2H(2H-1)|t+0|^{2H-2} + 2H(2H-1)|t-0|^{2H-2}}{4}\\
&= 2k_B T \cdot H(2H-1)\,t^{2H-2}, \quad t > 0.
\end{aligned} \quad (5)$$

*2.3. The Mittag-Leffler Noise.* It is well-known that the physical origin of anomalous diffusion is related to the long-time tail correlations. Thus, in order to model anomalous diffusion process, a lot of different power-law correlation functions are employed in (1) and (2).

Viñales and Despósito have introduced a novel noise whose correlation function is proportional to a Mittag-Leffler function, which is called Mittag-Leffler noise [55–57]. The correlation function of Mittag-Leffler noise behaves as a power-law for large times but is nonsingular at the origin due to the inclusion of a characteristic time. The correlation function of Mittag-Leffler noise is given by

$$C(t) = k_B T \frac{1}{\tau^\alpha} E_\alpha\left[-\left(\frac{|t|}{\tau}\right)^\alpha\right], \quad (6)$$

where $\tau$ is called characteristic memory time and the memory exponent $\alpha$ can be taken as $0 < \alpha < 2$. The $E_\alpha(\cdot)$ denotes the Mittag-Leffler function that is defined through the series

$$E_\alpha(y) = \sum_{j=0}^{\infty} \frac{y^j}{\Gamma(\alpha j + 1)}, \quad (7)$$

which behaves as a stretched exponential for short times and as inverse power-law in the long-time regime when $\alpha \neq 1$. Meanwhile, when $\alpha = 1$, the correlation function equation (4) reduces to the exponential form

$$C(t) = \frac{C_1}{\tau}\exp\left(-\frac{|t|}{\tau}\right), \quad (8)$$

which describes a standard Ornstein-Uhlenbeck process.

*2.4. The System Model.* In this paper, we consider a periodically driven linear system with multiplicative noise and periodically modulated additive noise described by the following generalized Langevin equation:

$$\ddot{x}(t) + \gamma \int_0^t K(t-u)\dot{x}(u)\,du + \left[\omega_0^2 + \xi_1(t)\right]x = A_1 \sin(\Omega t) + A_2 \sin(\Omega t)\xi_2(t) + \eta(t), \tag{9}$$

where $\omega_0$ is the intrinsic frequency of the harmonic oscillator $U(x) = \omega_0^2 x^2/2$. The fluctuations of $\omega_0^2$ in (9) are modeled as a Markovian dichotomous noise $\xi_1(t)$ [69], which consists of jumps between two values $-a$ and $a$, $a > 0$, with stationary probabilities $P_s(-a) = P_s(a) = 1/2$. The statistical properties of $\xi_1(t)$ are

$$\begin{aligned} \langle \xi_1(t)\rangle &= 0, \\ \langle \xi_1(t)\xi_1(s)\rangle &= a^2 \exp(-\nu|t-s|), \end{aligned} \tag{10}$$

where $a^2$ is the noise intensity and $\nu$ is the correlation rate, with $\tau_0 = 1/\nu$ being the correlation time.

$\xi_2(t)$ is a zero mean signal-modulated noise, with coupling strength $D$ with noise $\xi_1(t)$ [70, 71]; that is,

$$\langle \xi_1(t)\xi_2(s)\rangle = D\delta(t-s). \tag{11}$$

In (9), $A_1$ and $\Omega$ are the amplitude and frequency of the external periodic force $A_1\sin(\Omega t)$. Meanwhile, $A_2$ and $\Omega$ are the amplitude and frequency of the periodically modulated additive noise $A_2\sin(\Omega t)\xi_2(t)$, respectively.

In this paper, we assume that the external noise $\xi_1(t), \xi_2(t)$ and the internal noise $\eta(t)$ satisfy $\langle\eta(t)\xi_1(s)\rangle = \langle\eta(t)\xi_2(s)\rangle = 0$ with different origins. In the next section, we will obtain the exact expression of the first moment of the output signal.

## 3. First Moment

*3.1. The Analytical Expression of the Output Amplitude of a GLE.* First of all, we should transfer the stochastic equation (9) to the deterministic equation for the average value $\langle x\rangle$. For this purpose, we use the well-known Shapiro-Loginov [72] procedure which yields, for exponentially correlated noise (10),

$$\left\langle \xi_1 \frac{d^n x}{dt^n}\right\rangle = \left(\frac{d}{dt} + \nu\right)^n \langle \xi_1 x\rangle. \tag{12}$$

Equation (9) depicted the motion of $x(t)$ is bounded by a noisy harmonic force field, by averaging realization of the trajectory of the stochastic equation (9), and, applying the characteristics of the noises $\xi_1(t)$, $\xi_2(t)$, and $\eta(t)$, we obtain the equation of the particle's average displacement $\langle x\rangle$:

$$\frac{d^2\langle x\rangle}{dt^2} + \gamma\int_0^t K(t-u)\frac{d\langle x(u)\rangle}{du}du + \omega_0^2\langle x\rangle + \langle\xi_1 x\rangle = A_1\sin(\Omega t). \tag{13}$$

It can be found that (13) shows the synthetic affections of the particle's average displacement $\langle x\rangle$ and the multiplicative noise coupling term $\langle\xi_1 x\rangle$. In order to deal with the new multiplicative noise coupling term $\langle\xi_1 x\rangle$, we multiply both sides of (9) with $\xi_1(t)$ and then average to construct a closed equations of $\langle x\rangle$ and $\langle\xi_1 x\rangle$:

$$\left\langle \xi_1(t)\frac{d^2x}{dt^2}\right\rangle + \gamma\int_0^t K(t-u)\left\langle \xi_1(t)\frac{dx}{du}\right\rangle du + \omega_0^2\langle\xi_1 x\rangle + a^2\langle x\rangle = A_2 D\sin(\Omega t). \tag{14}$$

Using the Shapiro-Loginov formula (12) and the characteristics of the generalized integration, (14) turns to be

$$\begin{aligned} &\left[\frac{d^2\langle\xi_1 x\rangle}{dt^2} + 2\nu\frac{d\langle\xi_1 x\rangle}{dt} + \nu^2\langle\xi_1 x\rangle\right] + \omega_0^2\langle\xi_1 x\rangle \\ &\quad + a^2\langle x\rangle + \gamma e^{-\nu t}\int_0^t K(t-u) \\ &\quad \cdot\left[\frac{d\langle\xi_1(u)x(u)\rangle}{du} + \nu\langle\xi_1(u)x(u)\rangle\right]e^{\nu u}du \\ &\quad = A_2 D\sin(\Omega t). \end{aligned} \tag{15}$$

To summarize, for the linear generalized Langevin equation (9) to be investigated in this paper, it is a stochastic differential equation driven by an internal noise $\eta(t)$. When we want to obtain the particle's average displacement $\langle x\rangle$ from (9), we average (9) and obtain the traditional classical differential equation (13) for $\langle x\rangle$ and the new multiplicative noise coupling term $\langle\xi_1 x\rangle$. In order to deal with the new coupling term $\langle\xi_1 x\rangle$, we do some mathematical calculations and have another ordinary differential equation (15). Finally, we obtain two linear closed equations (13) and (15) for $x_1 = \langle x\rangle$ and $x_2 = \langle\xi_1 x\rangle$.

In order to solve the closed equations (13) and (15), we use the Laplace transform technique $X_i = \mathrm{LT}[x_i(t)] \triangleq \int_0^{+\infty} x_i(t)e^{-st}dt$, $i = 1, 2$ [73], under the long time limit $t \to \infty$ condition, we obtain the following equations,

$$\begin{aligned} &\left[s^2 + \gamma\widetilde{K}(s)s + \omega_0^2\right]X_1(s) + X_2(s) \\ &\quad = \mathrm{LT}\left[A_1\sin(\Omega t)\right] \\ &a^2X_1(s) + \left[(s+\nu)^2 + \omega_0^2 + \gamma(s+\nu)\widetilde{K}(s+\nu)\right]X_2(s) \\ &\quad = \mathrm{LT}\left[A_2D\sin(\Omega t)\right]. \end{aligned} \tag{16}$$

$\widetilde{K}(s) = \mathrm{LT}[K(t)]$ means performing Laplace transform.

The solutions of (16) can be represented as

$$\begin{aligned} X_1^{(as)}(s) &= \widetilde{H}_{11}(s)\cdot\mathrm{LT}\left[A_1\sin(\Omega t)\right] + \widetilde{H}_{21}(s) \\ &\quad\cdot\mathrm{LT}\left[A_2D\sin(\Omega t)\right], \\ X_2^{(as)}(s) &= \widetilde{H}_{12}(s)\cdot\mathrm{LT}\left[A_1\sin(\Omega t)\right] + \widetilde{H}_{22}(s) \\ &\quad\cdot\mathrm{LT}\left[A_2D\sin(\Omega t)\right], \end{aligned} \tag{17}$$

where $\widetilde{H}_{ki}(s) = \mathrm{LT}[H_{ki}(t)]$, $k, i = 1, 2$, and $\widetilde{H}_{ki}(s)$ can be obtained from (16). Particularly, we have

$$\widetilde{H}_{11}(s) = \frac{1}{\widetilde{H}(s)}\left[\omega_0^2 + (s+\nu)^2 + \gamma(s+\nu)\widetilde{K}(s+\nu)\right],$$
$$\widetilde{H}_{21}(s) = \frac{-1}{\widetilde{H}(s)}, \tag{18}$$

with $\widetilde{H}(s) = [\omega_0^2 + s^2 + \gamma s \cdot \widetilde{K}(s)][\omega_0^2 + (s+\nu)^2 + \gamma(s+\nu)\cdot\widetilde{K}(s+\nu)] - a^2$.

Applying the inverse Laplace transform technique, by the theory of "signals and systems," the product of the Laplace domain functions corresponding to the convolution of the time domain functions, we can obtain the solutions $x_1^{(as)}(t) = \langle x(t)\rangle_{as} \triangleq \lim_{t\to\infty}\langle x(t)\rangle$ and $x_2^{(as)}(t) = \langle \xi_1(t)x(t)\rangle_{as} \triangleq \lim_{t\to\infty}\langle \xi_1(t)x(t)\rangle$ through (17):

$$x_1^{(as)}(t) = H_{11}(t) * [A_1\sin(\Omega t)] + H_{21}(t) * [A_2 D\sin(\Omega t)],$$
$$x_2^{(as)}(t) = H_{12}(t) * [A_1\sin(\Omega t)] + H_{22}(t) * [A_2 D\sin(\Omega t)], \tag{19}$$

where $*$ is the convolution operator.

Equations (13) and (15) with $x_1 = \langle x\rangle$ and $x_2 = \langle \xi_1 x\rangle$ can be regarded as a linear system, and the forces $A_1\sin(\Omega t)$ and $A_2 D\sin(\Omega t)$ can be regarded as the input periodic signals. By the theory of "signals and systems," when we put periodic signals $A_1\sin(\Omega t)$ and $A_2 D\sin(\Omega t)$ into a linear system with system functions $H_{11}(t)$ and $H_{21}(t)$, the output signals are still periodic signals; meanwhile, the frequency of the output signals are the same as the frequency of the input signals. We denote the output signals as $A_{11}\sin(\Omega t + \varphi_{11})$ and $A_{21}\sin(\Omega t + \varphi_{21})$, respectively, which can be shown in Figure 1.

Moreover, the amplitudes $A_{11}$ and $A_{21}$ of the output signals are proportion to the amplitudes $A_1$ and $A_2D$ of the input signals, and the proportion constants are the amplitudes of the frequency response functions $H_{11}(e^{j\Omega})$ and $H_{21}(e^{j\Omega})$; that is, $A_{11} = A_1|H_{11}(e^{j\Omega})|$ and $A_{21} = A_2D|H_{21}(e^{j\Omega})|$.

Thus, from (19), we obtain the following expression of particle's average displacement $x_1^{(as)}(t)$:

$$x_1^{(as)}(t) = H_{11}(t) * [A_1\sin(\Omega t)] + H_{21}(t) * [A_2 D \cdot \sin(\Omega t)] = A_{11}\sin(\Omega t + \varphi_{11}) + A_{21}\sin(\Omega t + \varphi_{21}) = \sqrt{A_{11}^2 + A_{21}^2 + 2A_{11}A_{21}\cos(\varphi_{11}\varphi_{21})} \cdot \sin\left[\Omega t + \frac{\arcsin(A_{11}\sin\varphi_{11} + A_{21}\sin\varphi_{21})}{\sqrt{A_{11}^2 + A_{21}^2 + 2A_{11}A_{21}\cos(\varphi_{11}\varphi_{21})}}\right], \tag{20}$$

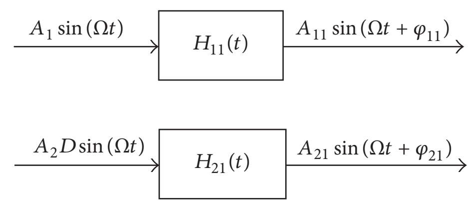



Figure 1: The relationships of the input periodic signals and the output signals by the theory of "signals and systems."

where $A_{11}, A_{21}$ and $\varphi_{11}, \varphi_{21}$ are the amplitude and phase shift of the long-time behaviors of the output signals, respectively.

In addition, we can obtain the expressions of frequency response functions $H_{11}(e^{j\Omega})$ and $H_{21}(e^{j\Omega})$ through (18). Through the derivation, we can suppose that the expressions of $H_{11}(e^{j\Omega})$ and $H_{21}(e^{j\Omega})$ can be simplified as follows:

$$H_{11}(e^{j\Omega}) \triangleq \frac{A_1 + B_1\cdot j}{C_1 + D_1\cdot j},$$
$$H_{21}(e^{j\Omega}) \triangleq \frac{A_2 + B_2\cdot j}{C_2 + D_2\cdot j}, \tag{21}$$

where $A_i, B_i, C_i, D_i$, $i = 1, 2$, are real numbers.

Then, the amplitudes of frequency response functions $H_{11}(e^{j\Omega})$ and $H_{21}(e^{j\Omega})$ are

$$\left|H_{11}(e^{j\Omega})\right| = \sqrt{H_{11}(e^{j\Omega})H_{11}^*(e^{j\Omega})} = \sqrt{\frac{A_1^2 + B_1^2}{C_1^2 + D_1^2}},$$
$$\left|H_{21}(e^{j\Omega})\right| = \sqrt{H_{21}(e^{j\Omega})H_{21}^*(e^{j\Omega})} = \sqrt{\frac{A_2^2 + B_2^2}{C_2^2 + D_2^2}}, \tag{22}$$

where $H_{i1}^*(e^{j\Omega})$, $i = 1, 2$, is the conjugation of $H_{i1}(e^{j\Omega})$, $i = 1, 2$.

Meanwhile, the amplitudes of the output signals are

$$A_{11} = A_1\left|H_{11}(e^{j\Omega})\right| = A_1\sqrt{\frac{A_1^2 + B_1^2}{C_1^2 + D_1^2}},$$
$$A_{22} = A_2D\left|H_{21}(e^{j\Omega})\right| = A_2D\sqrt{\frac{A_2^2 + B_2^2}{C_2^2 + D_2^2}}. \tag{23}$$

In this paper, with the expression of the particle's average displacement $x_1^{(as)}(t)$ in (20), we mainly discuss the resonant behaviors of the output amplitude $A$ which is defined as

$$A \triangleq \sqrt{A_{11}^2 + A_{21}^2 + 2A_{11}A_{21}\cos(\varphi_{11}\varphi_{21})}. \tag{24}$$

It should be emphasized that the following inequality must hold for the sake of the stability of solutions [69]:

$$0 < a^2 \le a_{\mathrm{cr}}^2 = \omega_0^2\left[\omega_0^2 + \nu^2 + \gamma\nu\cdot\widetilde{K}(\nu)\right]. \tag{25}$$

In this paper, we assume the stability condition (25) is satisfied.

*3.2. The Output Amplitude of a GLE with Fractional Gaussian Noise.* When the internal noise $\eta(t)$ in the generalized Langevin equation (9) is fractional Gaussian noise with correlation function (5), from the fluctuation-dissipation theorem (2), we derive the power-law memory kernel $K(t)$ presented by Hurst exponent $H$, $0 < H < 1$:

$$K(t) = \frac{C(t)}{k_B T} = 2H(2H-1)t^{2H-2}. \tag{26}$$

Performing the Laplace transform, we obtain the related $\widetilde{K}(s) = \mathrm{LT}[K(t)]$:

$$\widetilde{K}(s) = \Gamma(2H+1)\cdot s^{1-2H}, \quad 0 < H < 1. \tag{27}$$

From (18), (20), (24), and (27), we get the output amplitude $A$ expressed by (24) with

$$A_{11} = A_1\sqrt{\frac{{f_1}^2 + {f_2}^2}{{f_3}^2 + {f_4}^2}}, \qquad \varphi_{11} = \arctan\left(\frac{f_2 f_3 - f_1 f_4}{f_1 f_3 + f_2 f_4}\right), \tag{28}$$

$$A_{21} = A_2 D\sqrt{\frac{1}{{f_3}^2 + {f_4}^2}}, \qquad \varphi_{21} = \arctan\left(-\frac{f_4}{f_3}\right), \tag{29}$$

where

$$\begin{aligned}
f_1 &= \omega_0^2 + b^2\cos(2\theta) + \gamma b^{2-2H} \\
&\quad\cdot \Gamma(2H+1)\cos[(2-2H)\theta], \\
f_2 &= b^2\sin(2\theta) + \gamma b^{2-2H} \\
&\quad\cdot \Gamma(2H+1)\sin[(2-2H)\theta], \\
f_3 &= M_0 + M_1\cos(2\theta) + M_2\cos[(2-2H)\theta] \\
&\quad + M_3\cos\left[(2-2H)\frac{\pi}{2} + 2\theta\right] \\
&\quad + M_4\cos\left[\left(\frac{\pi}{2}+\theta\right)(2-2H)\right] \\
&\quad + M_5\cos\left[(2-2H)\frac{\pi}{2}\right], \\
f_4 &= M_1\sin(2\theta) + M_2\sin[(2-2H)\theta] \\
&\quad + M_3\sin\left[(2-2H)\frac{\pi}{2} + 2\theta\right] \\
&\quad + M_4\sin\left[\left(\frac{\pi}{2}+\theta\right)(2-2H)\right] \\
&\quad + M_5\sin\left[(2-2H)\frac{\pi}{2}\right], \\
b &= \sqrt{\nu^2 + \Omega^2}, \\
\theta &= \arctan\left(\frac{\Omega}{\nu}\right), \\
M_0 &= \omega_0^4 - a^2 - \Omega^2\omega_0^2, \\
M_1 &= \left(\omega_0^2 - \Omega^2\right)b^2, \\
M_2 &= \gamma\left(\omega_0^2 - \Omega^2\right)b^{2-2H}\Gamma(2H+1), \\
M_3 &= \gamma\Omega^{2-2H}b^2\Gamma(2H+1), \\
M_4 &= \gamma^2\Omega^{2-2H}b^{2-2H}\Gamma^2(2H+1), \\
M_5 &= \gamma\omega_0^2\Omega^{2-2H}\Gamma(2H+1).
\end{aligned} \tag{30}$$

*3.3. The Output Amplitude of a GLE with Mittag-Leffler Noise.* When the internal noise $\eta(t)$ in the generalized Langevin equation (9) is Mittag-Leffler noise with correlation function (6), from the fluctuation-dissipation theorem (2), we derive the Mittag-Leffler memory kernel $K(t)$ presented by memory time $\tau$ and memory exponent $\alpha$:

$$K(t) = \frac{C(t)}{k_B T} = \frac{1}{\tau^\alpha}E_\alpha\left[-\left(\frac{|t|}{\tau}\right)^\alpha\right]. \tag{31}$$

Performing the Laplace transform, we obtain the related $\widetilde{K}(s) = \mathrm{LT}[K(t)]$:

$$\widetilde{K}(s) = \frac{s^{\alpha-1}}{1+(\tau s)^\alpha}, \quad 0 < \alpha < 2,\ \tau > 0. \tag{32}$$

From (18), (20), (24), and (32), we obtain the output amplitude $A$ expressed by (24), with

$$\begin{aligned}
A_{11} &= A_1\sqrt{\frac{{g_1}^2 + {g_2}^2}{{g_3}^2 + {g_4}^2}}, \\
\varphi_{11} &= \arctan\left(\frac{g_2 g_3 - g_1 g_4}{g_1 g_3 + g_2 g_4}\right), \\
A_{21} &= A_2 D\sqrt{\frac{{h_1}^2 + {h_2}^2}{{g_3}^2 + {g_4}^2}}, \\
\varphi_{21} &= \arctan\left(\frac{h_2 g_3 - h_1 g_4}{h_1 g_3 + h_2 g_4}\right),
\end{aligned} \tag{33}$$

where

$$\begin{aligned}
g_1 &= \omega_0^2 + M_7\cos(\alpha\theta) + M_2\cos(2\theta) \\
&\quad + M_2 M_5\cos[(2+\alpha)\theta] \\
&\quad + M_4\omega_0^2\cos\left(\frac{\pi}{2}\alpha\right) \\
&\quad + M_4 M_7\cos\left[\left(\frac{\pi}{2}+\theta\right)\alpha\right]
\end{aligned}$$

$$
\begin{aligned}
&+ M_2M_4\cos\left(2\theta + \frac{\pi}{2}\alpha\right)\\
&+ M_2M_4M_5\cos\left[(2+\alpha)\theta + \frac{\pi}{2}\alpha\right],\\
g_2 &= M_7\sin(\alpha\theta) + M_2\sin(2\theta)\\
&+ M_2M_5\sin[(2+\alpha)\theta] + M_4\omega_0^2\sin\left(\frac{\pi}{2}\alpha\right)\\
&+ M_4M_7\sin\left[\left(\frac{\pi}{2}+\theta\right)\alpha\right]\\
&+ M_2M_4\sin\left(2\theta + \frac{\pi}{2}\alpha\right)\\
&+ M_2M_4M_5\sin\left[(2+\alpha)\theta + \frac{\pi}{2}\alpha\right],\\
h_1 &= -1 - M_4\cos\left(\frac{\pi}{2}\alpha\right) - M_5\cos(\alpha\theta)\\
&- M_4M_5\cos\left[\left(\frac{\pi}{2}+\theta\right)\alpha\right],\\
h_2 &= -M_4\sin\left(\frac{\pi}{2}\alpha\right) - M_5\sin(\alpha\theta)\\
&- M_4M_5\sin\left[\left(\frac{\pi}{2}+\theta\right)\alpha\right],\\
g_3 &= M_8 + M_0M_2\cos(2\theta) + M_9\cos(\alpha\theta)\\
&+ M_0M_2M_5\cos[(2+\alpha)\theta]\\
&+ M_{10}\cos\left(\frac{\pi}{2}\alpha\right) + M_{11}\cos\left(2\theta + \frac{\pi}{2}\alpha\right)\\
&+ M_{12}\cos\left[\left(\frac{\pi}{2}+\theta\right)\alpha\right]\\
&+ M_{13}\cos\left[(2+\alpha)\theta + \frac{\pi}{2}\alpha\right],\\
g_4 &= M_0M_2\sin(2\theta) + M_9\sin(\alpha\theta)\\
&+ M_0M_2M_5\sin[(2+\alpha)\theta]\\
&+ M_{10}\sin\left(\frac{\pi}{2}\alpha\right) + M_{11}\sin\left(2\theta + \frac{\pi}{2}\alpha\right)\\
&+ M_{12}\sin\left[\left(\frac{\pi}{2}+\theta\right)\alpha\right]\\
&+ M_{13}\sin\left[(2+\alpha)\theta + \frac{\pi}{2}\alpha\right],\\
b &= \sqrt{\nu^2 + \Omega^2},\\
\theta &= \arctan\left(\frac{\Omega}{\nu}\right),\\
M_0 &= \omega_0^2 - \Omega^2,\\
M_1 &= \gamma\Omega^\alpha,\\
M_2 &= b^2,\\
M_3 &= \gamma b^\alpha,\\
M_4 &= (\tau\Omega)^\alpha,\\
M_5 &= (\tau b)^\alpha,\\
M_6 &= M_0M_4 + M_1,\\
M_7 &= M_5\omega_0^2 + M_3,\\
M_8 &= \omega_0^2M_0 - a^2,\\
M_9 &= M_0M_7 - M_5a^2,\\
M_{10} &= M_4M_8 + M_1\omega_0^2,\\
M_{11} &= M_2M_6,\\
M_{12} &= M_5\left(M_4M_8 + \omega_0^2M_1\right) + M_3M_6,\\
M_{13} &= M_2M_5M_6.
\end{aligned}
\tag{34}
$$

## 4. Stochastic Resonance Behaviors of a GLE with Multiplicative and Periodically Modulated Noises

In this section, we will perform the numerical simulations on the above analytical expression in (24), with the internal noise $\eta(t)$ modeled as fractional Gaussian noise and Mittag-Leffler noise in Sections 4.1 and 4.2, respectively. It can be seen, from the analytical expression in (24), the behaviors of $A$ are fully determined by the combination of the system parameters $\gamma, \omega_0^2, a^2, \nu, A_1, A_2, D$, and $\Omega$ and the parameters of $\eta(t)$. Based on it, some characteristic features of stochastic resonance behaviors are revealed.

*4.1. The Stochastic Resonance Behaviors of GLE with a Fractional Gaussian Noise.* From (25) and (27), we obtain the stability condition of GLE with fractional Gaussian noise $\eta(t)$:

$$
\begin{aligned}
0 &< a^2 \le a^2_{\text{cr,fGn}}\\
&= \omega_0^2\left[\omega_0^2 + \nu^2 + \gamma\cdot\Gamma(2H+1)\cdot\nu^{2-2H}\right],
\end{aligned}
\tag{35}
$$

with $0 < H < 1$.

It can be seen from the stability condition (35) that the critical noise intensity $a^2_{\text{cr,fGn}}$ is determined by $\omega_0^2$, $\nu$, $\gamma$, and $H$. Besides, from (24), the behaviors of output amplitude $A$ are fully determined by the combination of the parameters $\gamma, \omega_0^2, a^2, \nu, A_1, A_2, D, \Omega$, and $H$. Thus, in Figure 2, we depict the phase diagram in the $D-\Omega$ plane for the emergence of the stochastic resonance behaviors of $A$ versus $a^2$ at $\omega_0^2 = 1$, $A_1 = 1$, $H = 0.55$, $A_2 = 1$, $\gamma = 0.3$, and $\nu = 0.01$.

In the unshaded region [see the domain (0) of level = 0 which corresponds to Figure 2], the output amplitude $A(a^2)$ varies monotonically as the noise intensity $a^2$ varies, which means the SR phenomenon is impossible. Meanwhile, in the shaded regions, it corresponds to the traditional SR

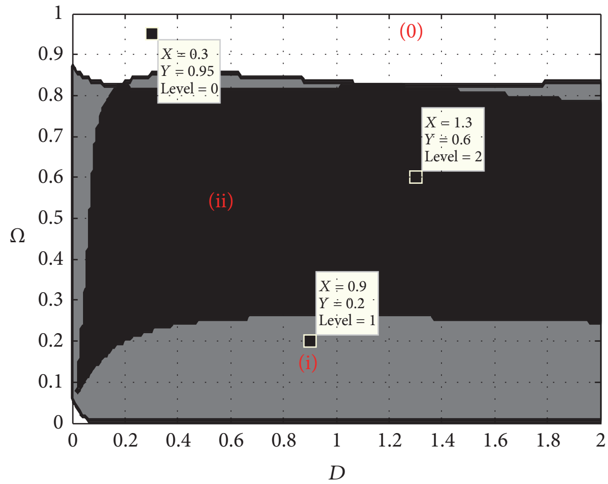
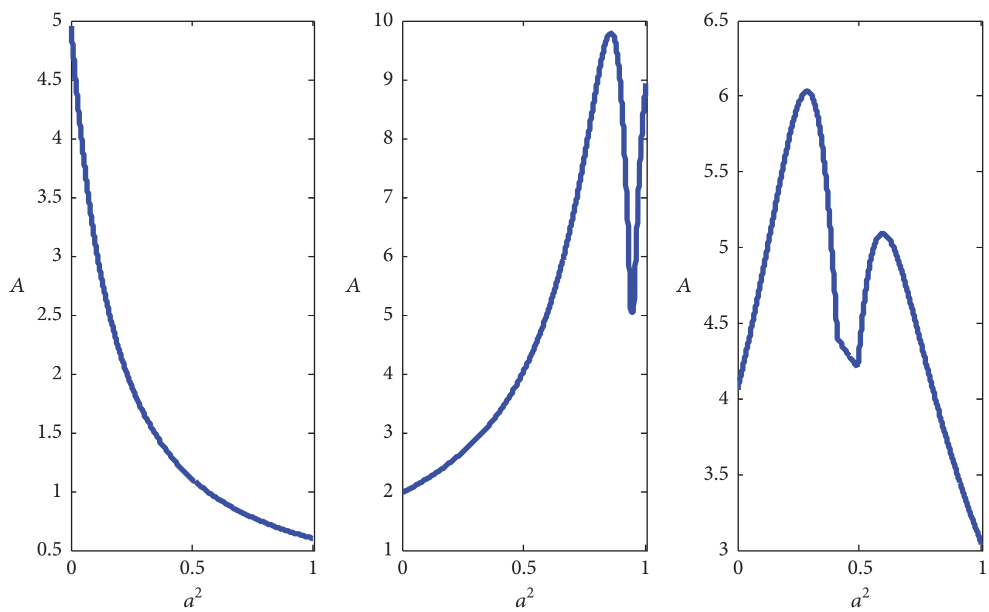



FIGURE 2: The phase diagram for the stochastic resonance behaviors of the output amplitude $A(a^2)$ at $\omega_0^2 = 1$, $A_1 = 1$, $H = 0.55$, $A_2 = 1$, $\gamma = 0.3$, and $\nu = 0.01$, and the values of Level in Figure 2, reflect the number of SR peaks for different combination of $D$ and $\Omega$.




(a) (b) (c)

FIGURE 3: The output amplitude $A$ versus the noise intensity $a^2$ with various $(D, \Omega)$ be chose through Figure 2, with parameters $\omega_0^2 = 1$, $A_1 = 1$, $A_2 = 1$, $\nu = 0.01$, $H = 0.55$, and $\gamma = 0.3$ and (a) $D = 0.3$ and $\Omega = 0.95$; (b) $D = 0.9$ and $\Omega = 0.2$; (c) $D = 1.3$ and $\Omega = 0.6$.

phenomenon taking place, and two phases can be discerned in the resonant domain:

(1) The light shaded region (i) corresponds to the single-peak SR phenomenon [see the domain of level = 1 which corresponds to Figure 2].

(2) The dark shaded region (ii) corresponds to the double-peaks SR phenomenon [see the domain of level = 2 which corresponds to Figure 2].

From Figure 2, we can find that when the driving frequency $\Omega$ is large enough [$\Omega > 0.95$] or small enough [$\Omega < 0.01$], it is impossible to induce the SR phenomenon.

In Figure 3, we plot the curves given by (24) and (28) in which the dependence of the output amplitude $A$ on the noise intensity $a^2$ for different values of the system parameters $(D,\ \Omega)$ can be chosen from Figure 2, to verify the correctness of the results shown in Figure 2. As shown in Figure 3(a), when $D = 0.3$ and $\Omega = 0.95$, which belongs to the unshaded domain in Figure 2, the output amplitude $A(a^2)$ monotonic behavior decreased with the increasing of $a^2$, which means the SR phenomenon does not take place. In Figure 3(b), when $D = 0.9$ and $\Omega = 0.2$, which corresponds to the light grey domain in Figure 2, the curve shows that the output amplitude $A(a^2)$ attains a maximum value at some values of $a^2$; that is, the single-peak SR phenomenon takes place by

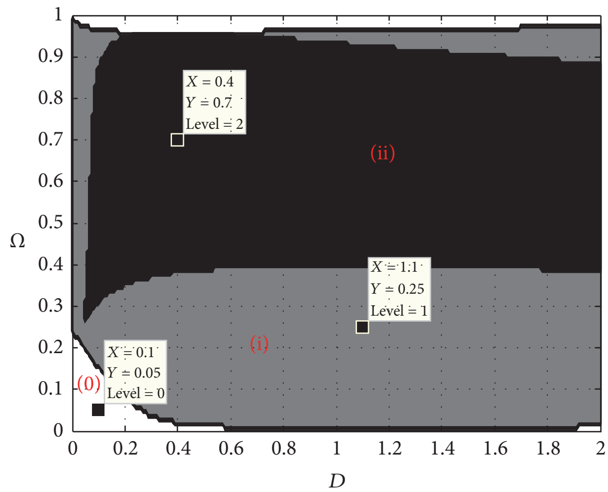



FIGURE 4: The phase diagram for the stochastic resonance behaviors of the output amplitude $A(a^2)$ at $\omega_0^2 = 1, A_1 = 1, \alpha = 0.6, \tau = 0.2, A_2 = 1, \gamma = 0.3$, and $\nu = 0.05$, and the values of level in Figure 3 reflect the number of SR peaks for different combination of $D$ and $\Omega$.

increasing $a^2$. Furthermore, one can see from Figure 3(c) that the double-peaks SR phenomenon happens, for the reason that the parameter combination of $D = 1.3$ and $\Omega = 0.6$ belongs to the dark grey domain in Figure 2. It should be emphasized that the double-peaks SR phenomenon happens because of the presence of two types of noise, external and internal noise sources. The double-peaks SR phenomenon can take place in biological systems, such as neuronal systems [74], in which the internal noise is due to signals coming from all other neurons, and external noise is the environmental noise due to its interaction with the neuronal system.

The main contribution of this section is as follows: with the help of phase diagram for the SR phenomenon, we can effectively control the SR phenomenon of this generalized harmonic system in a certain range and further broaden the application scope of the SR phenomenon in physics, biology, and engineering, such as the detection of weak stimuli by spiking neurons in the presence of certain level of noisy background neural activity [75].

*4.2. The Stochastic Resonance Behaviors of GLE with a Mittag-Leffler Noise.* From (25) and (32), we obtain the stability condition of GLE with a Mittag-Leffler noise $\eta(t)$:

$$0 < a^2 \leq a^2_{\mathrm{cr,MLn}} = \omega_0^2 \left[ \omega_0^2 + \nu^2 + \frac{\gamma \nu^\alpha}{1 + (\tau\nu)^\alpha} \right], \qquad (36)$$

with $0 < \alpha < 2, \tau > 0$.

It is found from the stability condition (36) that the critical noise intensity $a^2_{\mathrm{cr,MLn}}$ is determined by $\omega_0^2, \nu, \gamma, \tau$, and $\alpha$. In Figure 4, the phase diagram in the $D - \Omega$ plane for the emergence of SR phenomenon of $A(a^2)$ at $\omega_0^2 = 1$, $A_1 = 1$, $\alpha = 0.6$, $\tau = 0.2$, $A_2 = 1$, $\gamma = 0.3$, and $\nu = 0.05$ is shown. The same as Figure 2, when parameters $(D, \Omega)$ belong to the unshaded regions (0), the SR phenomenon is impossible; when $(D, \Omega)$ belong to the light grey regions (i), the single-peak SR phenomenon happens; when $(D, \Omega)$ belong to the dark grey regions (ii), the double-peaks SR phenomenon takes place. Moreover, the sufficiently large driving frequency $\Omega$ $[\Omega > 0.98]$ or small enough $\Omega$ $[\Omega < 0.01]$ cannot induce the system to produce SR phenomenon.

In Figure 5, we also show the curves given by (24) and (33) in which the dependence of the output amplitude $A$ on the noise intensity $a^2$ for different values of the system parameters $(D, \Omega)$ can be chosen from Figure 4, to verify the correctness of the results shown in Figure 4.

As shown in Figure 5(a), when $D = 0.1$ and $\Omega = 0.05$, which belongs to the unshaded domain in Figure 4, the output amplitude $A(a^2)$ monotonic increase occurs with the increasing of $a^2$, which means the SR phenomenon does not take place. In Figure 5(b), when $D = 1.1$ and $\Omega = 0.25$, which corresponds to the light grey domain in Figure 4, the curve shows that the output amplitude $A(a^2)$ attains a maximum value at some values of $a^2$; that is, the single-peak SR phenomenon takes place by increasing $a^2$. Furthermore, one can see from Figure 5(c) that the double-peaks SR phenomenon happens, for the reason that the parameter combination of $D = 0.4$ and $\Omega = 0.7$ belongs to the dark grey domain in Figure 4.

## 5. Conclusions

To summarize, in this paper we explore the SR phenomenon in a generalized Langevin equation with multiplicative, periodically modulated noises, and external periodic force. Moreover, the system internal noise is modeled as a fractional Gaussian noise and a Mittag-Leffler noise, respectively. Without loss of generality, the fluctuations of system intrinsic frequency are modeled as a multiplicative dichotomous noise. By the use of the stochastic averaging method and the Laplace transform technique, we obtain the exact expression of the output amplitude $A$ given by (24).

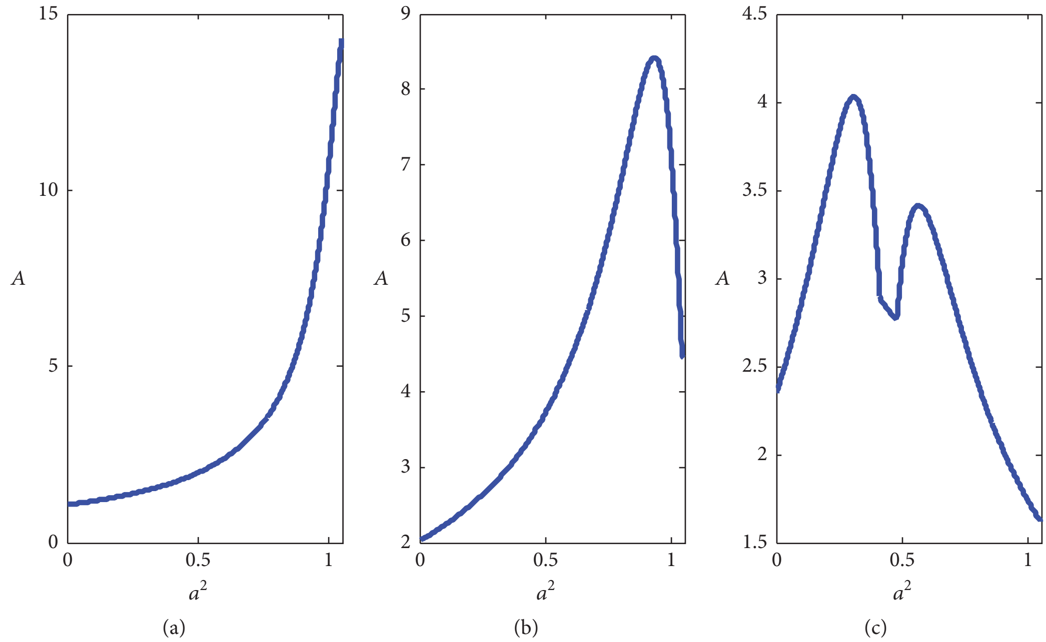



FIGURE 5: The output amplitude $A$ versus the noise intensity $a^2$ with various $(D, \Omega)$ chosen through Figure 4, with parameters $\omega_0^2 = 1$, $A_1 = 1$, $\alpha = 0.6$, $\tau = 0.2$, $A_2 = 1$, $\gamma = 0.3$, and $\nu = 0.05$ and (a) $D = 0.1$ and $\Omega = 0.05$; (b) $D = 1.1$ and $\Omega = 0.25$; (c) $D = 0.4$ and $\Omega = 0.7$.

We focus on the various nonmonotonic behaviors of the output amplitude $A$ with the system parameters $\gamma, \omega_0^2, a^2, \nu, A_1, A_2, D$, and $\Omega$ and the parameters of the internal driven noise. With the exact expression of the output amplitude $A$, we find the conventional SR takes place with the increases of the noise intensity $a^2$ for fractional Gaussian noise and Mittag-Leffler noise, respectively. Moreover, we give the phase diagram in $D-\Omega$ plane for the emergence of SR phenomenon of $A(a^2)$ and find the single-peak and double-peaks SR phenomena.

We believe all the results in this paper not only supply the theoretical investigations of the generalized harmonic oscillator subject to multiplicative, periodically modulated noises and external periodic force but also can suggest some experimental anomalous diffusion results in physical and biological applications in the future [76].

## Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# The Rational Solutions and Quasi-Periodic Wave Solutions as well as Interactions of $N$-Soliton Solutions for 3 + 1 Dimensional Jimbo-Miwa Equation

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The exact rational solutions, quasi-periodic wave solutions, and $N$-soliton solutions of 3 + 1 dimensional Jimbo-Miwa equation are acquired, respectively, by using the Hirota method, whereafter the rational solutions are also called algebraic solitary waves solutions and used to describe the squall lines phenomenon and explained possible formation mechanism of the rainstorm formation which occur in the atmosphere, so the study on the rational solutions of soliton equations has potential application value in the atmosphere field; the soliton fission and fusion are described based on the resonant solution which is a special form of the $N$-soliton solutions. At last, the interactions of the solitons are shown with the aid of $N$-soliton solutions.

## 1. Introduction

Some natural phenomena in the physics and in the biology can be depicted by multitudinous nonlinear partial differential equations. Therefore, the solutions of the partial differential equations become the focus points with which we are concerned [1]. There are various ways to get the solutions, such as the Darboux transformation, Bäcklund transformation [2, 3], Inverse scattering transformation, Homogeneous balance method, and Traveling wave solution [4–6]. However, the methods mentioned above cannot be expressing the periodicity of the partial differential equations.

Unlike the above method, the Hirota method [7, 8] plays a crucial role during obtaining the $N$-soliton solutions by the perturbation and the quasi-periodic wave solutions based on the Riemann theta functions [9]. Hence, it is important to rewrite the partial differential equation into the bilinear forms with the help of the variate transformation. In 2008, Lambert and Springael [10] proposed an explicit way to construct the bilinear forms for the constant coefficient equation.

It is well known that the Hirota method has been widely applied to the 1 + 1 dimensional equations and the 2 + 1 dimensional equations [11–23], but the method is rarely used to the 3 + 1 dimensional equation. As for the 3 + 1 dimensional equation, the quasi-periodic wave solutions happening during the arbitrary two spatial variables $x, y, z$ at one time $t$ or between one spatial variable and the time variable under the other two spatial variables are constants. On the other hand, the rational solutions have attracted more and more attention recently [24–27] because of their graceful structure and potential application value in applied disciplines. The author also applied this kind of rational solutions (also called algebraic solitary wave solutions) to discuss the algebraic Rossby solitary waves and explain the blocking phenomenon which happen in the real atmosphere and ocean [28, 29].

In this paper, we first introduce the well-known 3 + 1 dimensional Jimbo-Miwa equation which has significant efforts in science; it was investigated by Jimbo and Miwa in [30] and its one-soliton solutions were studied by Wazwaz

[31, 32] according to the Tanh-Coth method, and its traveling wave solutions were discussed by Ma and Lee [33] by using rational function transformations. The method provides more systematical and convenient handling of the solution process of nonlinear equations. Lately, we present a brief introduction about the approach and the properties of the Bell-polynomial. Then, the bilinear form of Jimbo-Miwa equation is gained by applying the Bell-polynomial; its rational solutions, quasi-periodic wave solutions, and $N$-soliton solutions are obtained based on the Hirota method and Riemann theta function. Finally, the resonant solution and the interactions of the $N$-soliton solutions are given under the Hirota method.

## 2. The Bell-Polynomial

In order to get the $N$-soliton solutions of the nonlinear evolutions equations (NLEES), we must get the bilinear form of the NLEES; Lambert et al. connected the Bell-polynomial with the Hirota $D$ operator and give rise to an explicit way to construct the bilinear form to the NLEES. Firstly, we are briefly devoted to the notations of the Bell-polynomial.

The definition of the multidimensional Bell-polynomial is as follows:

$$\begin{aligned} & Y_{n_1x_1n_2x_2\cdots n_rx_r}(g) \\ & \quad \equiv Y_{n_1,n_2,\ldots,n_r}\left(g_{l_1x_1,\ldots,l_rx_r}\left(1 \le l_i \le n_i, 0 \le i \le r\right)\right) \\ & \quad = e^{-g}\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}\cdots\partial_{x_r}^{n_r}e^{g}, \end{aligned} \tag{1}$$

where $g$ is a $C^{\infty}$ multivariables' function.

As for a special function $g$ with the variables $x, z$, we give rise to the following several initial value under the definition of the multivariables Bell-polynomial:

$$\begin{aligned} Y_{x,y}(g) &= g_{x,z} + g_x g_z, \\ Y_{x,2z}(g) &= g_{x,2z} + g_x g_{2z} + 2g_{x,z}g_z + g_x g_z^2 \cdots. \end{aligned} \tag{2}$$

Then we provide the redefinition of the binary Bell-polynomial as

$$\begin{aligned} & \tau_{n_1x_1,\ldots,n_rx_r}(v,\omega) \\ & \quad \equiv Y_{n_1,\ldots,n_r}(g)\Big|_{g_{l_1x_1,\ldots,l_rx_r} = \begin{cases} v_{l_1x_1,\ldots,l_rx_r} & \sum_{i=1}^{r} l_i \text{ is odd,} \\ \omega_{l_1x_1,\ldots,l_rx_r} & \sum_{i=1}^{r} l_i \text{ is even,} \end{cases}} \end{aligned} \tag{3}$$

where $v$ and $\omega$ both are the $C^{\infty}$ functions with the variables $x_1, x_2, \ldots, x_r$. We set out some initial expressions depending on (3) as

$$\begin{aligned} \tau_x(v) &= v_x, \\ \tau_{2x}(v,\omega) &= v_x^2 + \omega_{xx}, \\ \tau_{x,y} &= \omega_{x,y} + v_x v_y, \\ \tau_{3x}(v,\omega) &= v_{3x} + 3v_x\omega_{2x} + v_x^3, \\ \tau_{2x,y}(v,\omega) &= v_{2x,y} + 2v_x\omega_{x,y} + v_x^2 v_y + v_y\omega_{2x} \cdots. \end{aligned} \tag{4}$$

The link between the binary Bell-polynomial (3) and the Hirota $D$-operator can be presented through a transformational identity.

$$\begin{aligned} & \tau_{n_1x_1,\ldots,n_rx_r}\left(v = \ln\frac{F}{G}, \omega = \ln FG\right) \\ & \quad = (F\cdot G)^{-1} D_{x_1}^{n_1}\cdots D_{x_r}^{n_r} F\cdot G, \end{aligned} \tag{5}$$

where the Hirota operator is defined by

$$\begin{aligned} & D_{x_1}^{n_1}\cdots D_{x_r}^{n_r} F\cdot G \\ & \quad = \left(\partial_{x_1} - \partial_{x_1'}\right)^{n_1}\cdots\left(\partial_{x_r} - x_r'\right) F(x_1,\ldots,x_r) \\ & \quad \times G\left(x_1',\ldots,x_r'\right)\Big|_{x_1'=x_1,\ldots,x_r'=x_r}. \end{aligned} \tag{6}$$

In particular, when $F = G$, (5) can be read as

$$\begin{aligned} & F^{-2}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F^2 = \tau_{n_1x_1,\ldots,n_rx_r}(v = 0, \omega = 2\ln F) \\ & \quad = \begin{cases} 0, & \sum_{i=1}^{r} n_i \text{ is odd,} \\ P_{n_1x_1,\ldots,n_rx_r}(q), & \sum_{i=1}^{r} n_i \text{ is even.} \end{cases} \end{aligned} \tag{7}$$

In (7), the Bell-polynomial is equal to the $P$-polynomial when $\sum_{i=1}^{r} n_i$ is even, and then we give the first lower order $P$-polynomial:

$$\begin{aligned} P_{2x}(q) &= q_{2x}, \\ P_{4x}(q) &= q_{4x} + 3q_{2x}^2, \\ P_{2x,2y}(q) &= q_{2x,2y} + q_{2x}q_{2y} + 2q_{x,y}^2, \\ P_{x,y}(q) &= q_{x,y}, \\ P_{3x,y}(q) &= q_{3x,y} + 3q_{x,y}q_{2x}\cdots. \end{aligned} \tag{8}$$

As for the NLEES, we can rewrite them as bilinear from with the aid of the $P$-polynomial and show the $N$-soliton solutions and the quasi-periodic wave solutions.

## 3. The Bilinear Form of the 3 + 1 Dimensional Jimbo-Miwa Equation

The 3 + 1 dimensional Jimbo-Miwa equation is

$$\begin{aligned} & u_{xxxy} + 3\left(uu_y\right)_x + 3u_{xx}\partial_x^{-1}u_y + 3u_xu_y + 2u_{yt} - 3u_{xz} \\ & \quad = 0. \end{aligned} \tag{9}$$

Letting $u = q_{xx}$ and substituting it into (9) and integrating twice with respect to $x$ yield

$$q_{xxxy} + 3q_{xx}q_{xy} + 2q_{yt} - 3q_{xz} - \lambda = 0, \tag{10}$$

where $\lambda$ is an arbitrary integral constant. So, in terms of the $P$-polynomial, (10) can be written as

$$P_{3x,y}(q) + 2P_{y,t}(q) - 3P_{x,z}(q) - \lambda = 0. \tag{11}$$

Giving a change of dependent variables

$$q = 2\ln F \Longleftrightarrow u = q_{xx} = 2(\ln F)_{xx}, \tag{12}$$

then we can acquire the bilinear form of (9) as

$$J\left(D_x, D_y, D_z, D_t\right) \equiv \left(D_{3x}D_y + 2D_yD_t - 3D_xD_z\right)F \cdot F - \lambda F^2 = 0, \tag{13}$$

where the definition of the generalized bilinear $D$ operator is

$$\begin{aligned} D_{p,x}^m D_{p,t}^n F \cdot F &= \left(\frac{\partial}{\partial x} + \alpha_p \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} + \alpha_p \frac{\partial}{\partial t'}\right)^n \\ &\cdot F(x,t) F\left(x',t'\right)\Big|_{x'=x,t'=t} = \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} \\ &\cdot \alpha_p^i \alpha_p^j \frac{\partial^{m-i}}{\partial x^{m-i}} \frac{\partial^i}{\partial x'^i} \frac{\partial^{n-j}}{\partial t^{n-j}} \frac{\partial^j}{\partial t'^j} F(x,t) \\ &\cdot F\left(x',t'\right)\Big|_{x'=x,t'=t} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} \binom{m}{i}\binom{n}{j} \alpha_p^i \alpha_p^j \frac{\partial^{m+n-i-j} F(x,t)}{\partial x^{m-i} \partial t^{n-j}} \frac{\partial^{i+j} F(x,t)}{\partial x^i \partial t^j}, \\ &\qquad m, n \geq 0, \end{aligned} \tag{14}$$

where $\alpha_p^s$ is calculated as follows:

$$\begin{aligned} \alpha_p^s &= (-1)^{r_p(s)}, \\ s &= r_p(s) \bmod p, \\ D_{2,t}D_{2,y}F \cdot F &= 2F_{y,t}F - 2F_tF_y, \\ D_{2,x}D_{2,z}F \cdot F &= 2F_{x,z}F - 2F_xF_z, \\ D_{2,x}^3 D_{2,y}F \cdot F &= 2F_{3x,y}F - 2F_{3x}F_y - 6F_{2x,y}F_x \\ &\quad + 6F_{x,y}F_{2x}. \end{aligned} \tag{15}$$

Consequently, let $\lambda = 0$; then the linear combination of (13) constructs the 3 + 1 dimensional Jimbo-Miwa equation as

$$\begin{aligned} &\left(D_x^3 D_y + 2D_yD_t - 3D_xD_z\right)F \cdot F \\ &= 2F_{3x,y}F - 2F_{3x}F_y - 6F_{2x,y}F_x + 6F_{x,y}F_{2x} \\ &\quad + 4F_{y,t}F - 4F_tF_y - 6F_{x,z}F + 6F_xF_z. \end{aligned} \tag{16}$$

## 4. The Rational Solutions of the 3 + 1 Dimensional Jimbo-Miwa Equation

In this section, we use the symbolic computation with Maple and obtain polynomial solutions whose degree of $x$ is less than 3 and degrees of $y$, $z$, and $t$ are less than 2 to the 3 + 1 dimensional Jimbo-Miwa equation:

$$F = \sum_{i=0}^{3}\sum_{s=0}^{2}\sum_{l=0}^{2}\sum_{j=0}^{2} c_{islj} x^i y^s z^l t^j, \tag{17}$$

where $c_{ij}$'s are constants, and we acquire 36 classes of polynomial solutions to (16). Among the 36 classes of solutions, we enumerate 13 classes of solutions and see Appendix A, where the involved constants $c_{islj}$'s are arbitrary provided that the solutions are meaningful. We can confirm that there are 13 distinct classes of rational solutions generated from (12) to the 3 + 1 dimensional Jimbo-Miwa equation (9) by considering the transformation of the coefficient $c_{islj}$; for the detailed expression of rational solutions, see Appendix B.

These above-mentioned solutions are very tedious and difficult to apply in other subjects; here we obtain some reduced form solutions. $u_1$ can be reduced to

$$\begin{aligned} U_1 = &-\left(147t^4x^4 + 168t^3x^4 + 84t^3x^3 + 90t^2x^4\right. \\ &- 42t^4x + 48t^2x^3 + 9t^4 + 24tx^4 - 48t^3x + 18t^2x^2 \\ &+ 12tx^3 + 3x^4 - 60t^2x + 6t^3 - 24tx + 4t^2 - 2t \\ &\left.- 6x + 1\right)\left(7t^2x^3 + 4tx^3 + 3t^2x + 3tx^2 + x^3 + t^2\right. \\ &\left.+ 2tx + t + x + 1\right)^{-2}, \end{aligned} \tag{18}$$

when $c_{i,s,l,j} = 1 + islj$. The picture of the solution (18) is presented in Figure 1.

In the same way, we obtain reduced form solution of $u_5$ as follows:

$$\begin{aligned} U_5 = &\left(-0.3888x^4 - 0.5184x^3y - 0.2592x^2y^2\right. \\ &- 0.2160xy^2 + 0.435456x + 0.577152y \\ &\left.+ 1.0368xy + 0.0828y^2 - 0.5184\right)\left(0.36x^3\right. \\ &\left.+ 0.36x^2y + 0.30xy + 0.2016 - 0.72x + 0.24y\right)^{-2}, \end{aligned} \tag{19}$$

when $c_{i,s,l,j} = (1 + i + s + l + j)/100$ and $t = 0.01$. The picture of the solution (19) is presented in Figure 2.

For the solution $u_{11}$, we have

$$U_{11} = \frac{-(6t+5)^2}{(6xt + 5x + 4)^2}, \tag{20}$$

when $c_{i,s,l,j} = 1 + i + s + l + j$. The picture of the solution (20) is presented in Figure 3.

For the solution $u_{12}$, we get

$$U_{12} = -\frac{1}{(x+y)^2}, \tag{21}$$

when $c_{i,s,l,j} = 1 + is + lj$. The picture of the solution (21) is presented in Figure 4.

*Remark 1.* In fact, these above-mentioned rational solutions are greatly different from those common soliton solutions as the form "sech$^2$." The latter describes that, from the balance between nonlinearity and dispersion, it is possible to have steady waves of a permanent form, which are called classical solitary waves, while, as we know, these solutions which are derived in the paper can be used to describe algebraic solitary waves [28, 29]. During propagation, this kind of solitary

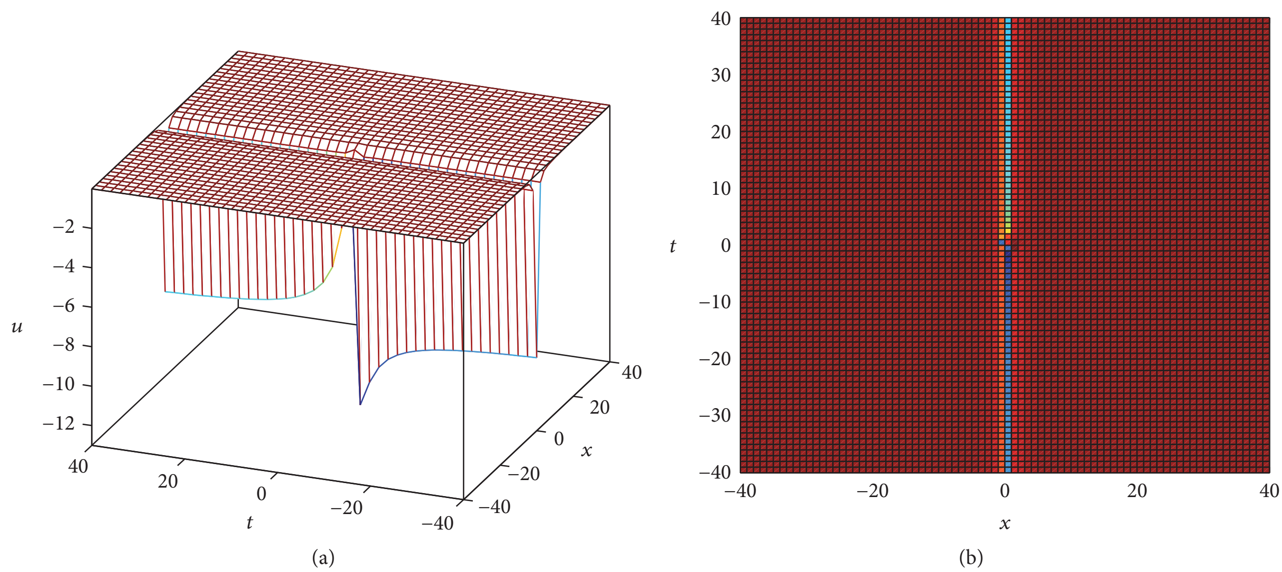



FIGURE 1: Pictures of (18): 3D plot (a) and density plot (b).

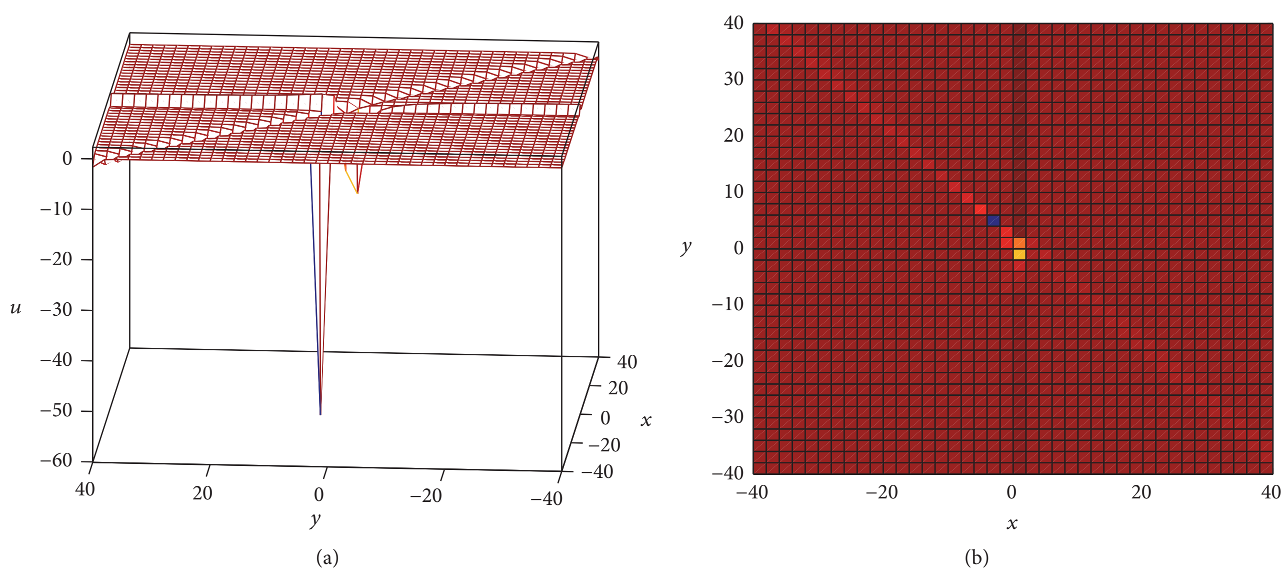



FIGURE 2: Pictures of (19): 3D plot (a) and density plot (b).

waves will have fission and form an interesting phenomenon: solitary waves in a line that the big amplitude solitary wave is at the front and the small amplitude solitary wave is in subsequent, which is deserved in the real atmosphere in the process of thunderstorm and called squall lines. So the rational solutions of soliton equations are used to explain possible formation mechanism of the rainstorm formation. So the study on the rational solutions of soliton equations has potential application value in the atmosphere field.

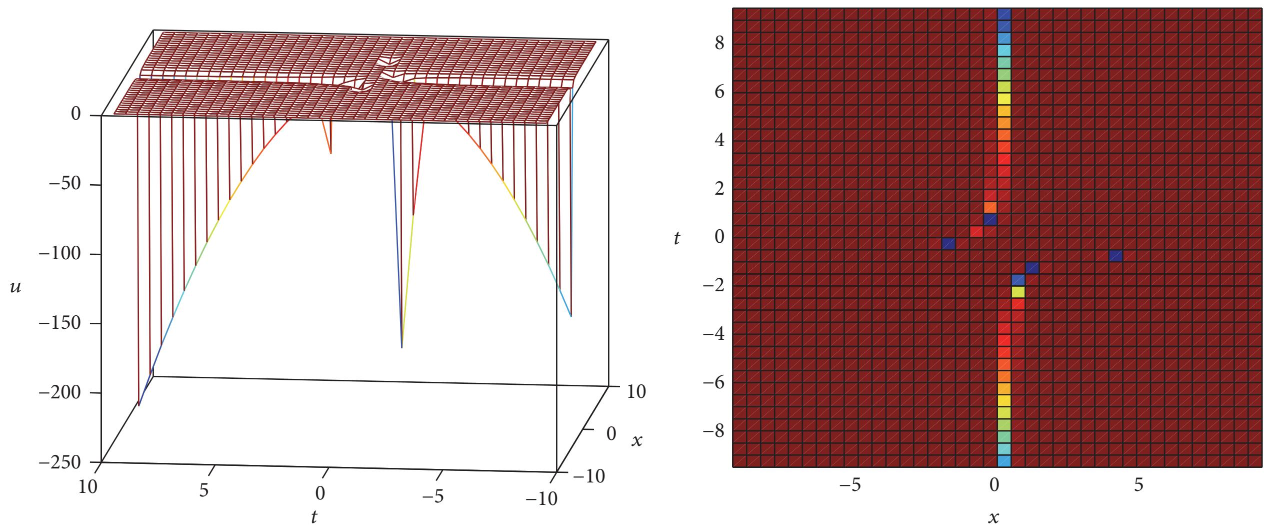
## 5. The Quasi-Periodic Wave Solutions of the 3 + 1 Dimensional Jimbo-Miwa Equation

In this section, we want to get the quasi-periodic wave solution of the 3 + 1 dimensional Jimbo-Miwa equation by applying the Hirota method. Long time ago, the theta functions have been systematically used to construct multiple quasi-periodic solutions [9, 34]. Hence, we let the Riemann function of (9) be

$$F = \sum_{n=-\infty}^{n=\infty} e^{2\pi i n\xi + \pi i n^2 \tau}, \tag{22}$$

where $n \in Z, \tau \in C$, $\operatorname{Im}\tau > 0$, and $\xi = kx + ly + az + wt$; in addition, the parameters $k, l, a, w$ are constant to be determined. Inserting (22) into (13), we have

$$JF \cdot F = J\left(D_x, D_y, D_z, D_t\right) \sum_{n=-\infty}^{n=\infty} e^{2\pi i n\xi + \pi i n^2 \tau} \cdot \sum_{m=-\infty}^{m=\infty} e^{2\pi i m\xi + \pi i m^2 \tau}$$




(a) (b)

Figure 3: Pictures of (20): 3D plot (a) and density plot (b).

$$
\begin{aligned}
&= \sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} J\left(D_x, D_y, D_z, D_t\right) e^{2\pi i n\xi + \pi i n^2 \tau} \\
&\cdot e^{2\pi i m\xi + \pi i m^2 \tau} \\
&= \sum_{n=-\infty}^{n=\infty} \sum_{m=-\infty}^{m=\infty} J\left[2\pi i (n-m)(k,l,a,w)\right] \\
&\cdot e^{2\pi i (n+m)\xi + \pi i (n^2+m^2)\tau} \\
&= \sum_{q=-\infty}^{q=\infty} \left\{ \sum_{n=-\infty}^{n=\infty} J\left[2\pi i (2n-q)(k,l,a,w)\right] \right. \\
&\left. \cdot e^{\pi i (n^2+(q-n)^2)\tau} \right\} e^{2\pi i q\xi} = \sum_{q=-\infty}^{q=\infty} \bar{J}(q) e^{2\pi i q\xi}.
\end{aligned}
\tag{23}
$$

Under the calculation of (23), we can denote that

$$
\begin{aligned}
\bar{J}(q) &= \sum_{n=-\infty}^{n=\infty} J\left[2\pi i (2n-q)(k,l,a,w)\right] e^{\pi i (n^2+(q-n)^2)\tau} \\
&= \sum_{h=-\infty}^{h=\infty} J\left(2\pi i Bk, 2\pi i Bl, 2\pi i Ba, 2\pi i Bw\right) \\
&\cdot e^{\pi i (n^2+(q-h-2)^2)\tau} \cdot e^{2\pi i (q-1)\tau},
\end{aligned}
\tag{24}
$$

where $B = 2h - q + 2$, $q = m + n$, and $h = n - 1$.

From the characters of (24), we can get the following recursion formula:

$$
\bar{J}(q) = \begin{cases} \bar{J}(0) e^{\pi i n q \tau}, & q = 2n, \\ \bar{J}(1) e^{\pi i (2n^2+2n)\tau}, & q = 2n+1. \end{cases}
\tag{25}
$$

If we set $\bar{J}(0) = 0$ and $\bar{J}(1) = 0$, it can satisfy (13); that is,

$$
\begin{aligned}
\bar{J}(0) &= \sum_{n=-\infty}^{n=\infty} \left(256\pi^4 n^4 k^3 l - 32\pi^2 n^2 l w + 48\pi^2 n^2 a k \right. \\
&\left. - \lambda\right) e^{2\pi i n^2 \tau} = 0, \\
\bar{J}(1) &= \sum_{n=-\infty}^{n=\infty} \left(16(2n-1)^4 \pi^4 k^3 l - 8\pi^2 (2n-1)^2 l w \right. \\
&\left. + 12\pi^2 (2n-1)^2 a k - \lambda\right) e^{\pi i (2n^2-2n+1)\tau} = 0.
\end{aligned}
\tag{26}
$$

With the purpose of computational convenience, we can set

$$
\begin{aligned}
q_{11} &= -\sum_{n=-\infty}^{n=\infty} 32\pi^2 n^2 l e^{2\pi i n^2 \tau}, \\
q_{12} &= \sum_{n=-\infty}^{n=\infty} \left(16(2n-1)^4 \pi^4 k^3 l + 12\pi^2 (2n-1)^2 a k\right) \\
&\cdot e^{\pi i (2n^2-2n+1)\tau}, \\
q_{21} &= \sum_{n=-\infty}^{n=\infty} e^{\pi i (2n^2-2n+1)\tau}, \\
q_{22} &= -\sum_{n=-\infty}^{n=\infty} 8\pi^2 (2n-1)^2 l e^{\pi i (2n^2-2n+1)\tau}, \\
q_{31} &= \sum_{n=-\infty}^{n=\infty} e^{2\pi i n^2 \tau}, \\
q_{13} &= \sum_{n=-\infty}^{n=\infty} \left(256\pi^4 n^4 k^3 l + 48\pi^2 n^2 a k\right) e^{2\pi i n^2 \tau}.
\end{aligned}
\tag{27}
$$

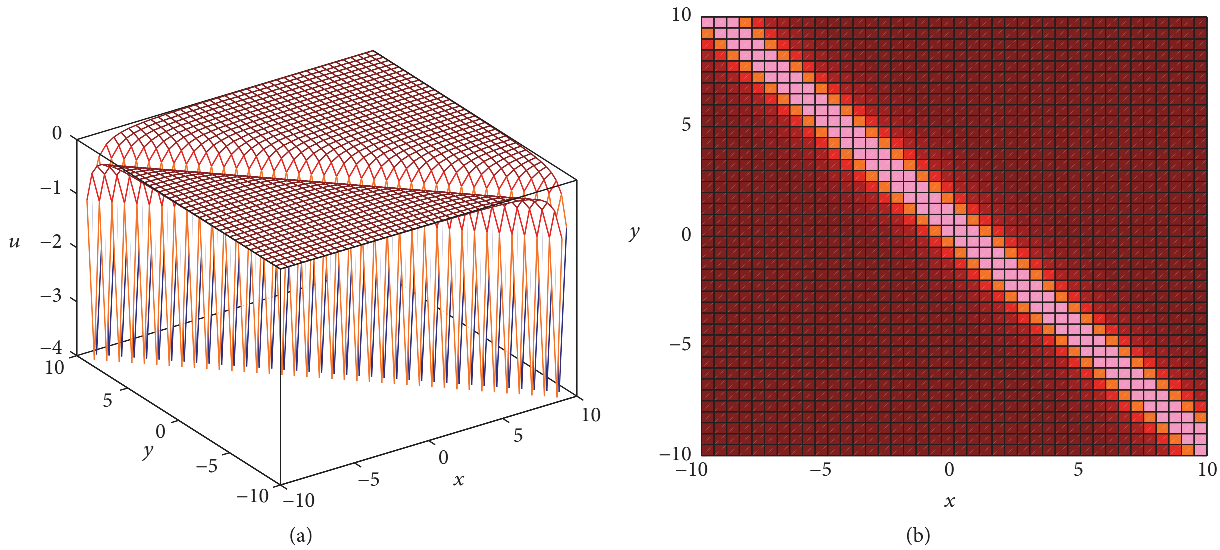



FIGURE 4: Pictures of (21): 3D plot (a) and density plot (b).

Then (26) can be written as

$$\begin{aligned} q_{11}w + q_{13} - \lambda q_{31} &= 0, \\ q_{22}w + q_{12} - \lambda q_{21} &= 0. \end{aligned} \tag{28}$$

The parameters $w, \lambda$ can be given by (28) as

$$\begin{aligned} w &= \frac{q_{12}q_{31} - q_{21}q_{13}}{q_{21}q_{11} - q_{31}q_{22}}, \\ \lambda &= \frac{q_{12}q_{11} - q_{22}q_{13}}{q_{21}q_{11} - q_{31}q_{22}}. \end{aligned} \tag{29}$$

Therefore, we can obtain the quasi-periodic wave solution of (9) as

$$u = 2(\ln F)_{xx}, \tag{30}$$

where $F$ satisfies (22); meanwhile, $w$ and $\lambda$ accord with (28). The picture of the quasi-periodic wave solutions of (9) can be shown in Figure 5.

## 6. The $N$-Soliton Solutions of the 3 + 1 Dimensional Jimbo-Miwa Equation

In [35], a multiple Exp-function method was proposed to obtain the multiwave solutions and multisoliton solutions. Here we can give rise to the $N$-soliton solutions of (9) by applying the Hirota direct method and perturbation approach. With the variable transformation $u = 2(\ln F)_{xx}$, (9) has been changed into the bilinear form. We let $\lambda = 0$ and

$$F = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots. \tag{31}$$

Substitute (31) into (13) and compare the power of $\varepsilon$; basing the traditional Hirota method, we can present the $N$-soliton solution as follows:

$$F = \sum_{\mu_i,\mu_j=0,1} \exp\left\{\sum_{i>j}^{N} A_{ij}\mu_i\mu_j + \sum_{j=1}^{N}\mu_j\eta_j\right\}, \tag{32}$$

where

$$\begin{aligned} \eta_j &= k_j x + l_j y + a_j z + w_j t, \\ e^{A_{ij}} &= \frac{2(l_j - l_i)(w_i - w_j) + 3(a_i - a_j)(k_i - k_j) + (k_i - k_j)^3(l_j - l_i)}{2(l_j + l_i)(w_i + w_j) - 3(a_i + a_j)(k_i + k_j) + (k_i + k_j)^3(l_j + l_i)}, \end{aligned} \tag{33}$$

and then we list the one-soliton solution and the two-soliton solution.

*(1) One-Soliton Solution*

$$\begin{aligned} F &= 1 + f_1, \\ f_1 &= e^{\eta}, \\ \eta &= kx + ly + az + wt, \end{aligned} \tag{34}$$

where the coefficients $k, l, a, w$ satisfy $2lw + k^3 l - 3ak = 0$, and then the one-soliton solution is

$$u = 2(\ln F)_{xx} = \frac{k^2}{2}\operatorname{sech}^2\frac{\xi}{2}. \tag{35}$$

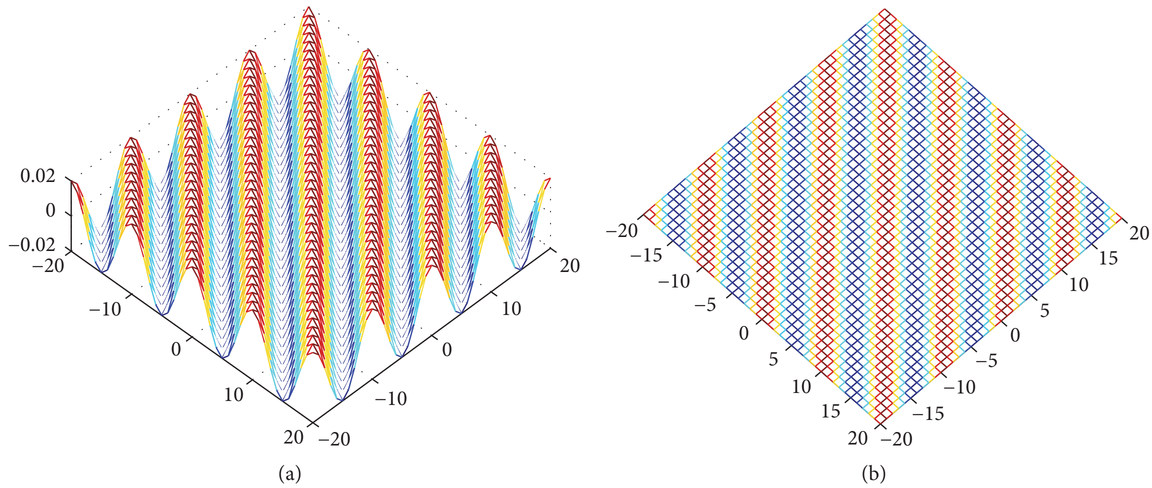



FIGURE 5: One quasi-periodic wave solution to (9) with the parameters $k = 0.1$, $l = 0.1$, $a = 0$, and $\tau = i$. Perspective view of the wave (a); overhead view of the wave (b), with contour plot shown. (The red lines are crests and the blue lines are troughs.)

*(2) Two-Soliton Solution*

$$F = 1 + f_1 + f_2,$$
$$f_1 = e^{\eta_1} + e^{\eta_2},$$
$$f_2 = e^{\eta_1+\eta_2+A_{12}}, \tag{36}$$

where

$$\begin{aligned}\eta_1 &= k_1x + l_1y + a_1z + w_1t,\\ \eta_2 &= k_2x + l_2y + a_2z + w_2t,\\ e^{A_{12}} &= \frac{2(l_2-l_1)(w_1-w_2)+3(a_1-a_2)(k_1-k_2)+(k_1-k_2)^3(l_2-l_1)}{2(l_2+l_1)(w_1+w_2)-3(a_1+a_2)(k_1+k_2)+(k_1+k_2)^3(l_2+l_1)}.\end{aligned} \tag{37}$$

Similarly, the coefficients $k_i, l_i, a_i, w_i$ $(i = 1, 2)$ satisfy $2l_iw_i + k_i^3l_i - 3a_ik_i = 0$, and then the two-soliton solution is

$$u = 2(\ln F)_{xx} = 2[\ln(1 + f_1 + f_2)]_{xx}. \tag{38}$$

As for the two-soliton solution, when the coefficient $e^{A_{ij}} = 0$, the soliton solution can be changed into the resonant solution whose propagation will be shown in Figure 6.

From Figure 6, we can see that, in Figure 6(a), there happened resonance collision to the two-soliton solution and appeared the third soliton; in Figure 6(b) occurs the soliton fusion whose amplitude has been changed after the resonance collision.

When the coefficient $e^{A_{ij}} \neq 0$, there will happen the soliton pursue collision due to the different soliton speed, as we all know that the wave with the faster speed will catch up with the slower speed wave, then collision of the two-soliton can be happening, and the picture of the pursue collision will be shown in Figure 7.

## 7. Conclusion

In this paper, we first introduce a 3 + 1 dimensional Jimbo-Miwa equation and get its bilinear form, rational solutions, quasi-periodic wave solutions, and $N$-soliton solutions based on the Hirota method and the theta function. Afterwards, we analyze the rational solutions and quasi-periodic wave solutions and draw the propagation picture. Finally, we explain the interaction of the $N$-soliton solutions and get a conclusion that when the coefficient $e^{A_{ij}} = 0$, the soliton solution can be turned into the resonant solution, after the resonance collision, and there will appear the soliton fusion phenomenon; when the coefficient $e^{A_{ij}} \neq 0$, there will appear the pursue collision; that is, the soliton with the faster speed will catch up with the soliton with the slower speed, after the collision, and the two-soliton solution will continue spreading in the previous speed and the direction. Although we acquire the solutions of the 3 + 1 dimensional Jimbo-Miwa equation, the integrability [36] of this equation is not discussed; the problem is worthy of exploring.

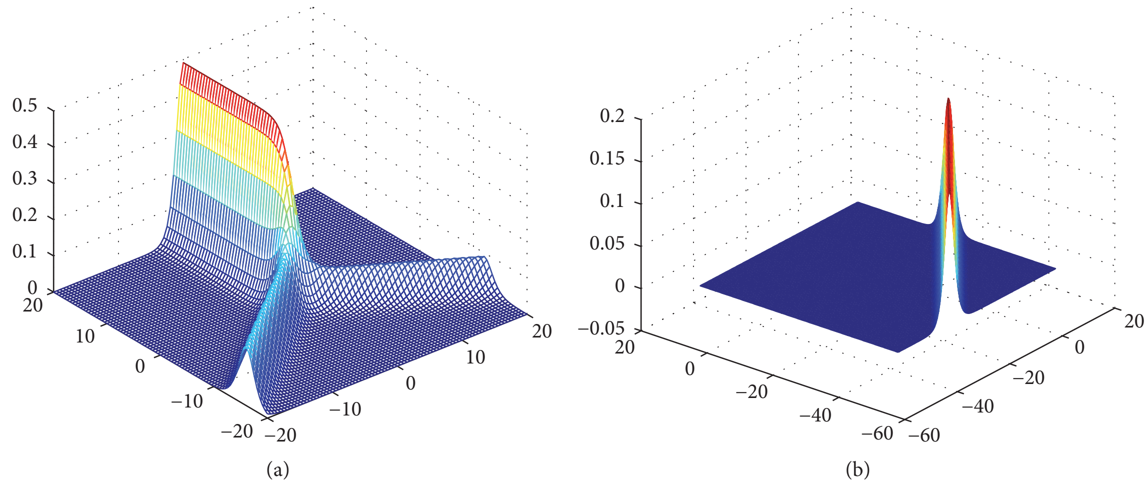



Figure 6: The resonant solution to (9) with the parameters $k_1 = 0.6$, $k_2 = 0.8$, $l_1 = l_2 = 1$, and $a_1 = a_2 = 0$.

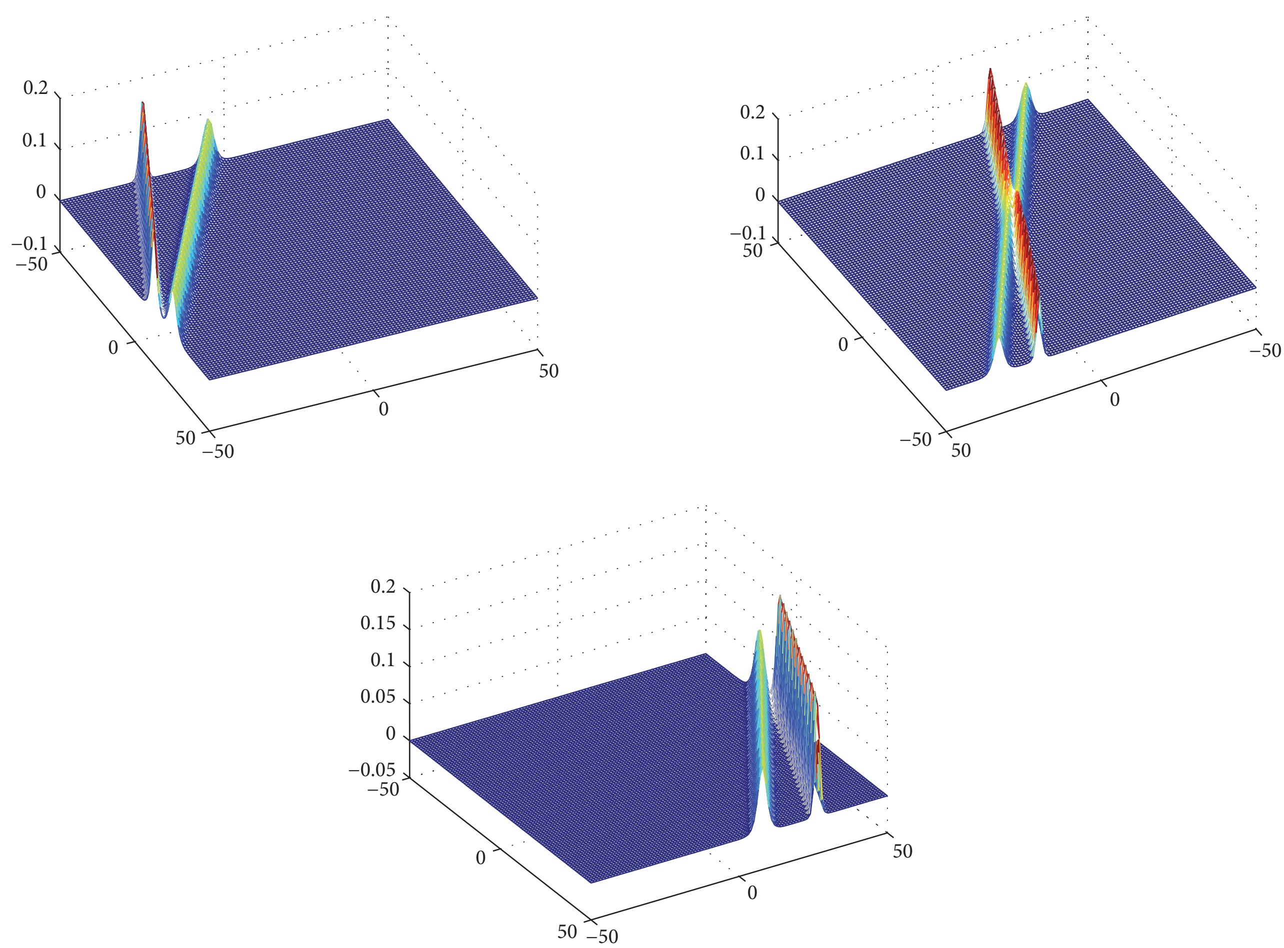



Figure 7: The pursue collision of (35) with $k_1 = 0.6$, $k_2 = 0.8$, $l_1 = 1$, $l_2 = 2$, and $a_1 = a_2 = 0$.

## Appendix

## A. The Expression of $F$

$$F_1 = t^2x^3yzc_{3,1,1,2} + tx^3yzc_{3,1,1,1} + t^2xyzc_{1,1,1,2} + tx^2yzc_{2,1,1,1} + x^3yzc_{3,1,1,0} + txyzc_{1,1,1,1} + tyzc_{0,1,1,1} + xyzc_{1,1,1,0} + yzc_{0,1,1,0} + t^2yzc_{0,1,1,2},$$

$$F_2 = \frac{c_{1,0,1,0}c_{2,1,1,1}zt}{c_{3,1,1,0}} + \frac{\left(c_{1,1,1,0}{c_{2,1,1,1}}^2 + {c_{1,1,1,1}}^2c_{3,1,1,0} + 6c_{2,1,1,1}{c_{3,1,1,0}}^2\right)yzt}{c_{2,1,1,1}c_{3,1,1,0}} + tx^2yzc_{2,1,1,1} + c_{0,2,1,1}y^2zt - \frac{c_{1,0,1,0}c_{1,1,1,1}z}{c_{2,1,1,1}} + c_{1,0,1,0}xz + xyzc_{1,1,1,0} - \frac{c_{1,1,1,1}\left(c_{1,1,1,0}{c_{2,1,1,1}}^2 + {c_{1,1,1,1}}^2c_{3,1,1,0} + 9c_{2,1,1,1}{c_{3,1,1,0}}^2\right)yz}{{c_{2,1,1,1}}^3} + x^3yzc_{3,1,1,0} - \frac{c_{0,2,1,1}c_{1,1,1,1}c_{3,1,1,0}y^2z}{{c_{2,1,1,1}}^2} + \frac{c_{0,2,1,1}c_{3,1,1,0}xy^2z}{c_{2,1,1,1}} + txyzc_{1,1,1,1},$$

$$F_3 = t^2yzc_{0,1,1,2} - \frac{\left({c_{0,1,1,2}}^2c_{2,1,1,1} - c_{0,1,1,2}c_{1,1,1,1}c_{1,1,1,2} - 3c_{1,1,1,2}{c_{2,1,1,1}}^2\right)yzt}{{c_{1,1,1,2}}^2} + tx^2yzc_{2,1,1,1} + \frac{c_{0,1,1,2}c_{1,2,1,1}y^2zt}{c_{1,1,1,2}} + c_{1,2,1,1}xy^2zt + \frac{2c_{0,1,1,2}c_{1,2,1,1}yz^2t}{3c_{2,1,1,1}} + txyzc_{1,1,1,1} + \frac{2c_{1,1,1,2}c_{1,2,1,1}xyz^2t}{3c_{2,1,1,1}} + t^2xyzc_{1,1,1,2},$$

$$F_4 = t^2yzc_{0,1,1,2} + \frac{c_{1,1,1,2}c_{1,2,1,1}y^2zt^2}{c_{2,1,1,1}} - \frac{c_{1,0,1,1}\left(c_{0,1,1,2}c_{2,1,1,1} - c_{1,1,1,1}c_{1,1,1,2}\right)zt}{c_{1,1,1,2}c_{2,1,1,1}} - \frac{\left({c_{0,1,1,2}}^2c_{2,1,1,1} - c_{0,1,1,2}c_{1,1,1,1}c_{1,1,1,2} + 3c_{1,1,1,2}{c_{2,1,1,1}}^2\right)yzt}{{c_{1,1,1,2}}^2} + txyzc_{1,1,1,1} - \frac{c_{1,2,1,1}\left(c_{0,1,1,2}c_{2,1,1,1} - c_{1,1,1,1}c_{1,1,1,2}\right)y^2zt}{c_{1,1,1,2}c_{2,1,1,1}} + c_{1,2,1,1}xy^2zt + \frac{c_{1,0,1,1}c_{1,1,1,2}zt^2}{c_{2,1,1,1}} + t^2xyzc_{1,1,1,2} + c_{1,0,1,1}xzt + tx^2yzc_{2,1,1,1},$$

$$F_5 = -3\frac{c_{0,1,1,1}c_{3,0,1,1}xzt}{c_{2,1,1,1}} + c_{3,0,1,1}x^3zt + tyzc_{0,1,1,1} + txyzc_{1,1,1,1} + tx^2yzc_{2,1,1,1} + 6c_{3,0,1,1}zt^2 + c_{0,0,1,1}zt,$$

$$F_6 = c_{0,0,1,1}zt + tyzc_{0,1,1,1} + tx^2yzc_{2,1,1,1} + c_{0,2,1,1}y^2zt + \frac{{c_{2,1,1,1}}^2xy^2zt}{3c_{3,0,1,1}} + c_{3,0,1,1}x^3zt - 2c_{2,1,1,1}z^2t - 3c_{3,0,1,1}zt^2,$$

$$F_7 = 6c_{3,1,1,1}yzt^2 + \frac{1}{3}\frac{c_{1,0,1,1}c_{2,1,1,1}zt}{c_{3,1,1,1}} + c_{1,0,1,1}xzt + tyzc_{0,1,1,1} + txyzc_{1,1,1,1} + tx^2yzc_{2,1,1,1} + tx^3yzc_{3,1,1,1} + c_{0,2,1,1}y^2zt + 3\frac{c_{0,2,1,1}c_{3,1,1,1}xy^2zt}{c_{2,1,1,1}},$$

$$F_8 = 6c_{3,1,1,1}yzt^2 + txyzc_{1,1,1,1} + tx^2yzc_{2,1,1,1} + tx^3yzc_{3,1,1,1} + \frac{c_{1,2,1,1}c_{2,1,1,1}y^2zt}{3c_{3,1,1,1}} + c_{1,2,1,1}xy^2zt + tyzc_{0,1,1,1},$$

$$F_9 = tx^3yzc_{3,1,1,1} + tx^2yzc_{2,1,1,1} - 3c_{3,1,1,1}yzt^2 + txyzc_{1,1,1,1} + c_{0,2,1,1}y^2zt + tyzc_{0,1,1,1} + c_{0,0,1,1}zt,$$

$$F_{10} = 6c_{3,1,1,1}yzt^2 + \frac{1}{3}\frac{\left(3c_{0,1,1,1}c_{3,0,1,1} + c_{1,0,1,1}c_{2,1,1,1}\right)zt}{c_{3,1,1,1}} + c_{1,0,1,1}xzt + c_{3,0,1,1}x^3zt + txyzc_{1,1,1,1} + tx^2yzc_{2,1,1,1} + tx^3yzc_{3,1,1,1} + 6c_{3,0,1,1}zt^2 + tyzc_{0,1,1,1},$$

$$F_{11} = t^2xyzc_{1,1,1,2} + c_{1,0,0,2}xt^2 + c_{1,1,0,2}xyt^2 + c_{1,2,0,2}xy^2t^2 + c_{1,0,1,2}xzt^2 + \frac{c_{0,0,2,1}c_{1,0,0,2}t}{c_{1,0,2,2}} + c_{1,0,2,2}xz^2t^2 + \frac{c_{1,0,2,1}c_{1,1,2,2}xyz^2t}{c_{1,0,2,2}} + \frac{c_{1,0,2,1}c_{1,2,2,2}xy^2z^2t}{c_{1,0,2,2}} + \frac{c_{1,0,2,1}c_{1,1,1,2}xyzt}{c_{1,0,2,2}} + \frac{c_{1,0,2,1}c_{1,2,1,2}xy^2zt}{c_{1,0,2,2}} + \frac{c_{0,0,2,1}c_{1,1,2,2}yz^2t}{c_{1,0,2,2}} + c_{1,1,2,2}xyz^2t^2 + c_{1,2,2,2}xy^2z^2t^2 + \frac{c_{0,0,2,1}c_{1,2,2,2}y^2z^2t}{c_{1,0,2,2}} + \frac{c_{0,0,2,1}c_{1,2,1,2}y^2zt}{c_{1,0,2,2}} + \frac{c_{1,0,2,1}c_{1,1,0,2}xyt}{c_{1,0,2,2}} + \frac{c_{1,0,2,1}c_{1,2,0,2}xy^2t}{c_{1,0,2,2}} + \frac{c_{0,0,2,1}c_{1,1,1,2}yzt}{c_{1,0,2,2}} + \frac{c_{0,0,2,1}c_{1,2,0,2}y^2t}{c_{1,0,2,2}} + \frac{c_{0,0,2,1}c_{1,1,0,2}yt}{c_{1,0,2,2}} + \frac{c_{0,0,2,1}c_{1,0,1,2}zt}{c_{1,0,2,2}} + \frac{c_{1,0,0,2}c_{1,0,2,1}xt}{c_{1,0,2,2}} + \frac{c_{1,0,1,2}c_{1,0,2,1}xzt}{c_{1,0,2,2}} + c_{1,0,2,1}xz^2t + c_{0,0,2,1}z^2t + c_{1,2,1,2}xy^2zt^2,$$

$$F_{12}=\frac{c_{1,0,2,1}c_{1,1,2,2}xyz^2t}{c_{1,0,2,2}}+c_{1,0,2,2}xz^2t^2+c_{0,1,2,2}yz^2t^2$$
$$+c_{1,1,2,2}xyz^2t^2+c_{1,0,2,1}xz^2t+c_{1,1,0,2}xyt^2+c_{1,0,1,2}xzt^2$$
$$+\frac{c_{0,1,2,2}c_{1,1,2,2}c_{1,0,2,1}y^2z^2t}{c_{1,0,2,2}{}^2}+\frac{c_{0,1,2,2}c_{1,0,1,2}c_{1,0,2,1}yzt}{c_{1,0,2,2}{}^2}$$
$$+\frac{c_{0,1,2,2}c_{1,0,2,1}c_{1,1,1,2}y^2zt}{c_{1,0,2,2}{}^2}+\frac{c_{1,0,2,1}c_{1,1,1,2}xyzt}{c_{1,0,2,2}}$$
$$+\frac{c_{1,0,2,1}c_{1,1,0,2}xyt}{c_{1,0,2,2}}+\frac{c_{1,0,1,2}c_{1,0,2,1}xzt}{c_{1,0,2,2}}+\frac{c_{0,1,2,2}c_{1,1,2,2}y^2z^2t^2}{c_{1,0,2,2}}$$
$$+\frac{c_{0,1,2,2}c_{1,0,2,1}c_{1,1,0,2}y^2t}{c_{1,0,2,2}{}^2}+\frac{c_{0,1,2,2}c_{1,0,0,2}c_{1,0,2,1}yt}{c_{1,0,2,2}{}^2}$$
$$+\frac{c_{0,1,2,2}c_{1,0,0,2}yt^2}{c_{1,0,2,2}}+\frac{c_{0,1,2,2}c_{1,1,0,2}y^2t^2}{c_{1,0,2,2}}+\frac{c_{1,0,0,2}c_{1,0,2,1}xt}{c_{1,0,2,2}}$$
$$+\frac{c_{0,1,2,2}c_{1,0,1,2}yzt^2}{c_{1,0,2,2}}+\frac{c_{0,1,2,2}c_{1,0,2,1}yz^2t}{c_{1,0,2,2}}$$
$$+\frac{c_{0,1,2,2}c_{1,1,1,2}y^2zt^2}{c_{1,0,2,2}}+t^2xyzc_{1,1,1,2}+c_{1,0,0,2}xt^2,$$

$$F_{13}=c_{1,0,1,1}xzt+c_{1,0,0,0}x+c_{1,0,2,1}xz^2t+c_{1,0,0,2}xt^2$$
$$+c_{1,0,1,2}xzt^2+c_{1,0,2,2}xz^2t^2+\frac{c_{0,2,2,2}c_{1,0,0,2}y^2t^2}{c_{1,0,2,2}}$$
$$+\frac{c_{0,2,2,2}c_{1,0,0,1}y^2t}{c_{1,0,2,2}}+\frac{c_{1,0,0,0}c_{1,1,2,2}xy}{c_{1,0,2,2}}+\frac{c_{1,0,0,0}c_{1,2,2,2}xy^2}{c_{1,0,2,2}}$$
$$+\frac{c_{0,2,2,2}c_{1,0,1,0}y^2z}{c_{1,0,2,2}}+c_{1,2,2,2}xy^2z^2t^2+\frac{c_{1,0,1,0}c_{1,2,2,2}xy^2z}{c_{1,0,2,2}}$$
$$+\frac{c_{0,1,2,2}c_{1,0,1,0}yz}{c_{1,0,2,2}}+\frac{c_{0,1,2,2}c_{1,0,1,1}yzt}{c_{1,0,2,2}}+\frac{c_{1,0,1,0}c_{1,1,2,2}xyz}{c_{1,0,2,2}}$$
$$+c_{0,1,2,2}yz^2t^2+\frac{c_{1,0,0,1}c_{1,1,2,2}xyt}{c_{1,0,2,2}}+c_{0,2,2,2}y^2z^2t^2+c_{1,0,0,1}xt$$
$$+\frac{c_{0,1,2,2}c_{1,0,0,0}y}{c_{1,0,2,2}}+\frac{c_{0,2,2,2}c_{1,0,0,0}y^2}{c_{1,0,2,2}}+c_{1,0,1,0}xz$$
$$+\frac{c_{0,1,2,2}c_{1,0,0,2}yt^2}{c_{1,0,2,2}}+\frac{c_{0,1,2,2}c_{1,0,1,2}yzt^2}{c_{1,0,2,2}}+\frac{c_{0,1,2,2}c_{1,0,2,1}yz^2t}{c_{1,0,2,2}}$$
$$+\frac{c_{0,1,2,2}c_{1,0,0,1}yt}{c_{1,0,2,2}}+c_{1,1,2,2}xyz^2t^2+\frac{c_{1,0,1,2}c_{1,2,2,2}xy^2zt^2}{c_{1,0,2,2}}$$
$$+\frac{c_{1,0,1,1}c_{1,1,2,2}xyzt}{c_{1,0,2,2}}+\frac{c_{1,0,1,1}c_{1,2,2,2}xy^2zt}{c_{1,0,2,2}}$$
$$+\frac{c_{1,0,2,1}c_{1,1,2,2}xyz^2t}{c_{1,0,2,2}}+\frac{c_{1,0,2,1}c_{1,2,2,2}xy^2z^2t}{c_{1,0,2,2}}$$
$$+\frac{c_{1,0,0,1}c_{1,2,2,2}xy^2t}{c_{1,0,2,2}}+\frac{c_{0,2,2,2}c_{1,0,2,1}y^2z^2t}{c_{1,0,2,2}}$$
$$+\frac{c_{1,0,1,2}c_{1,1,2,2}xyzt^2}{c_{1,0,2,2}}+\frac{c_{1,0,0,2}c_{1,1,2,2}xyt^2}{c_{1,0,2,2}}$$
$$+\frac{c_{1,0,0,2}c_{1,2,2,2}xy^2t^2}{c_{1,0,2,2}}+\frac{c_{0,2,2,2}c_{1,0,1,2}y^2zt^2}{c_{1,0,2,2}}$$
$$+\frac{c_{0,2,2,2}c_{1,0,1,1}y^2zt}{c_{1,0,2,2}}. \tag{A.1}$$

## B. The Rational Solutions of (3 + 1)-Dimensional Jimbo-Miwa Equation

$$u_1=-\left(3t^4x^4c_{3,1,1,2}{}^2+6t^3x^4c_{3,1,1,1}c_{3,1,1,2}\right.$$
$$+4t^3x^3c_{2,1,1,1}c_{3,1,1,2}+6t^2x^4c_{3,1,1,0}c_{3,1,1,2}$$
$$+3x^4c_{3,1,1,0}{}^2-6t^4xc_{0,1,1,2}c_{3,1,1,2}$$
$$+4t^2x^3c_{2,1,1,1}c_{3,1,1,1}+6tx^4c_{3,1,1,0}c_{3,1,1,1}+t^4c_{1,1,1,2}{}^2$$
$$-6t^3xc_{0,1,1,1}c_{3,1,1,2}-6t^3xc_{0,1,1,2}c_{3,1,1,1}$$
$$+2t^3xc_{1,1,1,2}c_{2,1,1,1}+2t^2x^2c_{2,1,1,1}{}^2$$
$$+4tx^3c_{2,1,1,1}c_{3,1,1,0}+3t^2x^4c_{3,1,1,1}{}^2$$
$$-2t^3c_{0,1,1,2}c_{2,1,1,1}+2t^3c_{1,1,1,1}c_{1,1,1,2}$$
$$-6t^2xc_{0,1,1,0}c_{3,1,1,2}-6t^2xc_{0,1,1,1}c_{3,1,1,1}+t^2c_{1,1,1,1}{}^2$$
$$+2t^2xc_{1,1,1,1}c_{2,1,1,1}-2t^2c_{0,1,1,1}c_{2,1,1,1}$$
$$+2t^2c_{1,1,1,0}c_{1,1,1,2}-6t^2xc_{0,1,1,2}c_{3,1,1,0}+c_{1,1,1,0}{}^2$$
$$-6txc_{0,1,1,0}c_{3,1,1,1}-6txc_{0,1,1,1}c_{3,1,1,0}$$
$$+2txc_{1,1,1,0}c_{2,1,1,1}+2tc_{1,1,1,0}c_{1,1,1,1}-6xc_{0,1,1,0}c_{3,1,1,0}$$
$$\left.-2tc_{0,1,1,0}c_{2,1,1,1}\right)\left(t^2x^3c_{3,1,1,2}+tx^3c_{3,1,1,1}\right.$$
$$+t^2xc_{1,1,1,2}+tx^2c_{2,1,1,1}+x^3c_{3,1,1,0}+t^2c_{0,1,1,2}$$
$$\left.+txc_{1,1,1,1}+tc_{0,1,1,1}+xc_{1,1,1,0}+c_{0,1,1,0}\right)^{-2},$$

$$u_2=-c_{3,1,1,0}c_{2,1,1,1}{}^3\left(2t^2x^2y^2c_{2,1,1,1}{}^5c_{3,1,1,0}\right.$$
$$+4tx^3y^2c_{2,1,1,1}{}^4c_{3,1,1,0}{}^2$$
$$+2t^2xy^2c_{1,1,1,1}c_{2,1,1,1}{}^4c_{3,1,1,0}+3x^4y^2c_{2,1,1,1}{}^3c_{3,1,1,0}{}^3$$
$$-2t^2y^3c_{0,2,1,1}c_{2,1,1,1}{}^4c_{3,1,1,0}$$
$$-4txy^3c_{0,2,1,1}c_{2,1,1,1}{}^3c_{3,1,1,0}{}^2-2t^2y^2c_{1,1,1,0}c_{2,1,1,1}{}^5$$
$$-t^2y^2c_{1,1,1,1}{}^2c_{2,1,1,1}{}^3c_{3,1,1,0}-12t^2y^2c_{2,1,1,1}{}^4c_{3,1,1,0}{}^2$$
$$-4txy^2c_{1,1,1,0}c_{2,1,1,1}{}^4c_{3,1,1,0}$$
$$-6txy^2c_{1,1,1,1}{}^2c_{2,1,1,1}{}^2c_{3,1,1,0}{}^2$$

$$
\begin{aligned}
&- 36txy^2 c_{2,1,1,1}{}^3 c_{3,1,1,0}{}^3 \\
&+ 4ty^3 c_{0,2,1,1} c_{1,1,1,1} c_{2,1,1,1}{}^2 c_{3,1,1,0}{}^2 \\
&+ 6xy^3 c_{0,2,1,1} c_{1,1,1,1} c_{2,1,1,1} c_{3,1,1,0}{}^3 \\
&+ y^4 c_{0,2,1,1}{}^2 c_{2,1,1,1} c_{3,1,1,0}{}^3 - 2t^2 y c_{1,0,1,0} c_{2,1,1,1}{}^5 \\
&- 4txy c_{1,0,1,0} c_{2,1,1,1}{}^4 c_{3,1,1,0} \\
&+ 4ty^2 c_{1,1,1,0} c_{1,1,1,1} c_{2,1,1,1}{}^3 c_{3,1,1,0} \\
&+ 2ty^2 c_{1,1,1,1}{}^3 c_{2,1,1,1} c_{3,1,1,0}{}^2 \\
&+ 18ty^2 c_{1,1,1,1} c_{2,1,1,1}{}^2 c_{3,1,1,0}{}^3 \\
&+ 6xy^2 c_{1,1,1,0} c_{1,1,1,1} c_{2,1,1,1}{}^2 c_{3,1,1,0}{}^2 \\
&+ 6xy^2 c_{1,1,1,1}{}^3 c_{3,1,1,0}{}^3 + 54xy^2 c_{1,1,1,1} c_{2,1,1,1} c_{3,1,1,0}{}^4 \\
&+ 2y^3 c_{0,2,1,1} c_{1,1,1,0} c_{2,1,1,1}{}^2 c_{3,1,1,0}{}^2 \\
&+ 4ty c_{1,0,1,0} c_{1,1,1,1} c_{2,1,1,1}{}^3 c_{3,1,1,0} \\
&+ 6xy c_{1,0,1,0} c_{1,1,1,1} c_{2,1,1,1}{}^2 c_{3,1,1,0}{}^2 \\
&+ c_{1,0,1,0}{}^2 c_{2,1,1,1}{}^3 c_{3,1,1,0} \\
&+ 2y^2 c_{0,2,1,1} c_{1,0,1,0} c_{2,1,1,1}{}^2 c_{3,1,1,0}{}^2 \\
&+ y^2 c_{1,1,1,0}{}^2 c_{2,1,1,1}{}^3 c_{3,1,1,0} \\
&+ 2y c_{1,0,1,0} c_{1,1,1,0} c_{2,1,1,1}{}^3 c_{3,1,1,0} \bigr) \bigl( tx^2 y c_{2,1,1,1}{}^4 c_{3,1,1,0} \\
&+ x^3 y c_{2,1,1,1}{}^3 c_{3,1,1,0}{}^2 + txy c_{1,1,1,1} c_{2,1,1,1}{}^3 c_{3,1,1,0} \\
&+ ty^2 c_{0,2,1,1} c_{2,1,1,1}{}^3 c_{3,1,1,0} + xy^2 c_{0,2,1,1} c_{2,1,1,1}{}^2 c_{3,1,1,0}{}^2 \\
&+ ty c_{1,1,1,0} c_{2,1,1,1}{}^4 + ty c_{1,1,1,1}{}^2 c_{2,1,1,1}{}^2 c_{3,1,1,0} \\
&+ 6ty c_{2,1,1,1}{}^3 c_{3,1,1,0}{}^2 + xy c_{1,1,1,0} c_{2,1,1,1}{}^3 c_{3,1,1,0} \\
&- y^2 c_{0,2,1,1} c_{1,1,1,1} c_{2,1,1,1} c_{3,1,1,0}{}^2 + t c_{1,0,1,0} c_{2,1,1,1}{}^4 \\
&- y c_{1,1,1,1}{}^3 c_{3,1,1,0}{}^2 - y c_{1,1,1,0} c_{1,1,1,1} c_{2,1,1,1}{}^2 c_{3,1,1,0} \\
&- 9y c_{1,1,1,1} c_{2,1,1,1} c_{3,1,1,0}{}^3 - c_{1,0,1,0} c_{1,1,1,1} c_{2,1,1,1}{}^2 c_{3,1,1,0} \\
&+ x c_{1,0,1,0} c_{2,1,1,1}{}^3 c_{3,1,1,0} \bigr)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
u_3 = {}& -c_{1,1,1,2}{}^2 \bigl( 9t^2 c_{1,1,1,2}{}^4 c_{2,1,1,1}{}^2 \\
&+ 12tz c_{1,1,1,2}{}^4 c_{1,2,1,1} c_{2,1,1,1} + 18tx c_{1,1,1,2}{}^3 c_{2,1,1,1}{}^3 \\
&+ 18ty c_{1,1,1,2}{}^3 c_{1,2,1,1} c_{2,1,1,1}{}^2 \\
&+ 18xy c_{1,1,1,2}{}^2 c_{1,2,1,1} c_{2,1,1,1}{}^3 \\
&+ 12xz c_{1,1,1,2}{}^3 c_{1,2,1,1} c_{2,1,1,1}{}^2 + 18x^2 c_{1,1,1,2}{}^2 c_{2,1,1,1}{}^4 \\
&+ 9y^2 c_{1,1,1,2}{}^2 c_{1,2,1,1}{}^2 c_{2,1,1,1}{}^2 \\
&+ 12yz c_{1,1,1,2}{}^3 c_{1,2,1,1}{}^2 c_{2,1,1,1} + 4z^2 c_{1,1,1,2}{}^4 c_{1,2,1,1}{}^2 \\
&- 18t c_{0,1,1,2} c_{1,1,1,2}{}^2 c_{2,1,1,1}{}^3 + 18t c_{1,1,1,1} c_{1,1,1,2}{}^3 c_{2,1,1,1}{}^2 \\
&- 54 c_{1,1,1,2} c_{2,1,1,1}{}^5 + 18x c_{1,1,1,1} c_{1,1,1,2}{}^2 c_{2,1,1,1}{}^3 \\
&- 18y c_{0,1,1,2} c_{1,1,1,2} c_{1,2,1,1} c_{2,1,1,1}{}^3 \\
&+ 18y c_{1,1,1,1} c_{1,1,1,2}{}^2 c_{1,2,1,1} c_{2,1,1,1}{}^2 \\
&- 12z c_{0,1,1,2} c_{1,1,1,2}{}^2 c_{1,2,1,1} c_{2,1,1,1}{}^2 \\
&+ 12z c_{1,1,1,1} c_{1,1,1,2}{}^3 c_{1,2,1,1} c_{2,1,1,1} \\
&+ 9 c_{1,1,1,1}{}^2 c_{1,1,1,2}{}^2 c_{2,1,1,1}{}^2 \\
&- 18 c_{0,1,1,2} c_{1,1,1,1} c_{1,1,1,2} c_{2,1,1,1}{}^3 + 18 c_{0,1,1,2}{}^2 c_{2,1,1,1}{}^4 \bigr) \\
&\cdot \bigl( 3tx c_{1,1,1,2}{}^3 c_{2,1,1,1} + 2xz c_{1,1,1,2}{}^3 c_{1,2,1,1} \\
&+ 3x^2 c_{1,1,1,2}{}^2 c_{2,1,1,1}{}^2 + 3xy c_{1,1,1,2}{}^2 c_{1,2,1,1} c_{2,1,1,1} \\
&+ 3t c_{0,1,1,2} c_{1,1,1,2}{}^2 c_{2,1,1,1} - 3 c_{0,1,1,2}{}^2 c_{2,1,1,1}{}^2 \\
&+ 3x c_{1,1,1,1} c_{1,1,1,2}{}^2 c_{2,1,1,1} \\
&+ 3y c_{0,1,1,2} c_{1,1,1,2} c_{1,2,1,1} c_{2,1,1,1} \\
&+ 3 c_{0,1,1,2} c_{1,1,1,1} c_{1,1,1,2} c_{2,1,1,1} + 2z c_{0,1,1,2} c_{1,1,1,2}{}^2 c_{1,2,1,1} \\
&+ 9 c_{1,1,1,2} c_{2,1,1,1}{}^3 \bigr)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
u_4 = {}& -c_{2,1,1,1}{}^2 c_{1,1,1,2}{}^2 \bigl( 2txy^2 c_{1,1,1,2}{}^3 c_{2,1,1,1} \\
&+ 2x^2 y^2 c_{1,1,1,2}{}^2 c_{2,1,1,1}{}^2 + 2xy^3 c_{1,1,1,2}{}^2 c_{1,2,1,1} c_{2,1,1,1} \\
&+ y^4 c_{1,1,1,2}{}^2 c_{1,2,1,1}{}^2 - 2ty^2 c_{0,1,1,2} c_{1,1,1,2}{}^2 c_{2,1,1,1} \\
&+ 2ty^2 c_{1,1,1,1} c_{1,1,1,2}{}^3 + 2y^2 c_{0,1,1,2}{}^2 c_{2,1,1,1}{}^2 \\
&+ 2xy^2 c_{1,1,1,1} c_{1,1,1,2}{}^2 c_{2,1,1,1} \\
&+ 2y^3 c_{0,1,1,2} c_{1,1,1,2} c_{1,2,1,1} c_{2,1,1,1} \\
&+ 2xy c_{1,0,1,1} c_{1,1,1,2}{}^2 c_{2,1,1,1} + t^2 y^2 c_{1,1,1,2}{}^4 \\
&- 2y^2 c_{0,1,1,2} c_{1,1,1,1} c_{1,1,1,2} c_{2,1,1,1} \\
&+ 2y^2 c_{1,0,1,1} c_{1,1,1,2}{}^2 c_{1,2,1,1} + y^2 c_{1,1,1,1}{}^2 c_{1,1,1,2}{}^2 \\
&+ 6y^2 c_{1,1,1,2} c_{2,1,1,1}{}^3 + 2y c_{0,1,1,2} c_{1,0,1,1} c_{1,1,1,2} c_{2,1,1,1} \\
&+ c_{1,0,1,1}{}^2 c_{1,1,1,2}{}^2 \bigr) \bigl( txy c_{1,1,1,2}{}^3 c_{2,1,1,1} \\
&+ ty^2 c_{1,1,1,2}{}^3 c_{1,2,1,1} + x^2 y c_{1,1,1,2}{}^2 c_{2,1,1,1}{}^2 \\
&+ xy^2 c_{1,1,1,2}{}^2 c_{1,2,1,1} c_{2,1,1,1} + ty c_{0,1,1,2} c_{1,1,1,2}{}^2 c_{2,1,1,1} \\
&+ xy c_{1,1,1,1} c_{1,1,1,2}{}^2 c_{2,1,1,1} \\
&- y^2 c_{0,1,1,2} c_{1,1,1,2} c_{1,2,1,1} c_{2,1,1,1} + t c_{1,0,1,1} c_{1,1,1,2}{}^3
\end{aligned}
$$

$$
\begin{aligned}
&- yc_{0,1,1,2}{}^{2}c_{2,1,1,1}{}^{2} + y^{2}c_{1,1,1,1}c_{1,1,1,2}{}^{2}c_{1,2,1,1} \\
&+ xc_{1,0,1,1}c_{1,1,1,2}{}^{2}c_{2,1,1,1} - 3yc_{1,1,1,2}c_{2,1,1,1}{}^{3} \\
&+ c_{1,0,1,1}c_{1,1,1,1}c_{1,1,1,2}{}^{2} - c_{0,1,1,2}c_{1,0,1,1}c_{1,1,1,2}c_{2,1,1,1} \\
&+ yc_{0,1,1,2}c_{1,1,1,1}c_{1,1,1,2}c_{2,1,1,1}\Big)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
u_{5} = {}&\Big(-3x^{4}c_{2,1,1,1}{}^{2}c_{3,0,1,1}{}^{2} - 4x^{3}yc_{2,1,1,1}{}^{3}c_{3,0,1,1} \\
&- 2xy^{2}c_{1,1,1,1}c_{2,1,1,1}{}^{3} + 36txc_{2,1,1,1}{}^{2}c_{3,0,1,1}{}^{2} \\
&- 2x^{2}y^{2}c_{2,1,1,1}{}^{4} + 12tyc_{2,1,1,1}{}^{3}c_{3,0,1,1} \\
&+ 12xyc_{0,1,1,1}c_{2,1,1,1}{}^{2}c_{3,0,1,1} + 2y^{2}c_{0,1,1,1}c_{2,1,1,1}{}^{3} \\
&- y^{2}c_{1,1,1,1}{}^{2}c_{2,1,1,1}{}^{2} + 6xc_{0,0,1,1}c_{2,1,1,1}{}^{2}c_{3,0,1,1} \\
&+ 2yc_{0,0,1,1}c_{2,1,1,1}{}^{3} + 6tc_{2,1,1,1}c_{3,0,1,1} \\
&- 9c_{0,1,1,1}{}^{2}c_{3,0,1,1}{}^{2}\Big)\Big(x^{3}c_{2,1,1,1}c_{3,0,1,1} + x^{2}yc_{2,1,1,1}{}^{2} \\
&+ 6yc_{0,1,1,1}c_{1,1,1,1}c_{2,1,1,1}c_{3,0,1,1} - 3xc_{0,1,1,1}c_{3,0,1,1} \\
&+ yc_{0,1,1,1}c_{2,1,1,1} + c_{0,0,1,1}c_{2,1,1,1} + xyc_{1,1,1,1}c_{2,1,1,1}\Big)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
u_{6} = {}&-\Big(27x^{4}c_{3,0,1,1}{}^{4} + 36x^{3}yc_{2,1,1,1}c_{3,0,1,1}{}^{3} \\
&+ 18x^{2}y^{2}c_{2,1,1,1}{}^{2}c_{3,0,1,1}{}^{2} - 18y^{3}c_{0,2,1,1}c_{2,1,1,1}c_{3,0,1,1}{}^{2} \\
&+ 6xy^{3}c_{2,1,1,1}{}^{3}c_{3,0,1,1} + y^{4}c_{2,1,1,1}{}^{4} \\
&- 54xy^{2}c_{0,2,1,1}c_{3,0,1,1}{}^{3} + 162txc_{3,0,1,1}{}^{4} \\
&+ 54tyc_{2,1,1,1}c_{3,0,1,1}{}^{3} - 54xyc_{0,1,1,1}c_{3,0,1,1}{}^{3} \\
&+ 108xzc_{2,1,1,1}c_{3,0,1,1}{}^{3} - 18y^{2}c_{0,1,1,1}c_{2,1,1,1}c_{3,0,1,1}{}^{2} \\
&+ 36yzc_{2,1,1,1}{}^{2}c_{3,0,1,1}{}^{2} - 54xc_{0,0,1,1}c_{3,0,1,1}{}^{3} \\
&- 18yc_{0,0,1,1}c_{2,1,1,1}c_{3,0,1,1}{}^{2}\Big)\Big(-3x^{3}c_{3,0,1,1}{}^{2} \\
&- xy^{2}c_{2,1,1,1}{}^{2} + 6zc_{2,1,1,1}c_{3,0,1,1} - 3y^{2}c_{0,2,1,1}c_{3,0,1,1} \\
&+ 9tc_{3,0,1,1}{}^{2} - 3yc_{0,1,1,1}c_{3,0,1,1} - 3c_{0,0,1,1}c_{3,0,1,1} \\
&- 3x^{2}yc_{2,1,1,1}c_{3,0,1,1}\Big)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
u_{7} = {}&3c_{3,1,1,1}\Big(-9x^{4}y^{2}c_{2,1,1,1}{}^{2}c_{3,1,1,1}{}^{3} \\
&- 12x^{3}y^{2}c_{2,1,1,1}{}^{3}c_{3,1,1,1}{}^{2} \\
&+ 18xy^{2}c_{0,1,1,1}c_{2,1,1,1}{}^{2}c_{3,1,1,1}{}^{2} \\
&+ 108txy^{2}c_{2,1,1,1}{}^{2}c_{3,1,1,1}{}^{3} - 6x^{2}y^{2}c_{2,1,1,1}{}^{4}c_{3,1,1,1} \\
&- 27y^{4}c_{0,2,1,1}{}^{2}c_{3,1,1,1}{}^{3} + 36ty^{2}c_{2,1,1,1}{}^{3}c_{3,1,1,1}{}^{2} \\
&- 6xy^{2}c_{1,1,1,1}c_{2,1,1,1}{}^{3}c_{3,1,1,1} \\
&- 18y^{3}c_{0,2,1,1}c_{1,1,1,1}c_{2,1,1,1}c_{3,1,1,1}{}^{2} \\
&+ 6y^{3}c_{0,2,1,1}c_{2,1,1,1}{}^{3}c_{3,1,1,1} + 6y^{2}c_{0,1,1,1}c_{2,1,1,1}{}^{3}c_{3,1,1,1} \\
&- 18y^{2}c_{0,2,1,1}c_{1,0,1,1}c_{2,1,1,1}c_{3,1,1,1}{}^{2} \\
&- 3y^{2}c_{1,1,1,1}{}^{2}c_{2,1,1,1}{}^{2}c_{3,1,1,1} \\
&- 6yc_{1,0,1,1}c_{1,1,1,1}c_{2,1,1,1}{}^{2}c_{3,1,1,1} + 2yc_{1,0,1,1}c_{2,1,1,1}{}^{4} \\
&- 3c_{1,0,1,1}{}^{2}c_{2,1,1,1}{}^{2}c_{3,1,1,1}\Big)\Big(c_{1,0,1,1}c_{2,1,1,1}{}^{2} \\
&+ 3xc_{1,0,1,1}c_{2,1,1,1}c_{3,1,1,1} + 9xy^{2}c_{0,2,1,1}c_{3,1,1,1}{}^{2} \\
&+ 18tyc_{2,1,1,1}c_{3,1,1,1}{}^{2} + 3xyc_{1,1,1,1}c_{2,1,1,1}c_{3,1,1,1} \\
&+ 3x^{2}yc_{2,1,1,1}{}^{2}c_{3,1,1,1} + 3y^{2}c_{0,2,1,1}c_{2,1,1,1}c_{3,1,1,1} \\
&+ 3yc_{0,1,1,1}c_{2,1,1,1}c_{3,1,1,1} + 3x^{3}yc_{2,1,1,1}c_{3,1,1,1}{}^{2}\Big)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
u_{8} = {}&3c_{3,1,1,1}\Big(-12x^{3}c_{2,1,1,1}c_{3,1,1,1}{}^{2} - 6x^{2}c_{2,1,1,1}{}^{2}c_{3,1,1,1} \\
&- 3y^{2}c_{1,2,1,1}{}^{2}c_{3,1,1,1} - 6xc_{1,1,1,1}c_{2,1,1,1}c_{3,1,1,1} \\
&+ 108txc_{3,1,1,1}{}^{3} - 9x^{4}c_{3,1,1,1}{}^{3} + 36tc_{2,1,1,1}c_{3,1,1,1}{}^{2} \\
&+ 18xc_{0,1,1,1}c_{3,1,1,1}{}^{2} - 6yc_{1,1,1,1}c_{1,2,1,1}c_{3,1,1,1} \\
&+ 2yc_{1,2,1,1}c_{2,1,1,1}{}^{2} + 6c_{0,1,1,1}c_{2,1,1,1}c_{3,1,1,1} \\
&- 3c_{1,1,1,1}{}^{2}c_{3,1,1,1}\Big)\Big(3x^{3}c_{3,1,1,1}{}^{2} + 3x^{2}c_{2,1,1,1}c_{3,1,1,1} \\
&+ 3xyc_{1,2,1,1}c_{3,1,1,1} + 18tc_{3,1,1,1}{}^{2} + 3xc_{1,1,1,1}c_{3,1,1,1} \\
&+ yc_{1,2,1,1}c_{2,1,1,1} + 3c_{0,1,1,1}c_{3,1,1,1}\Big)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
u_{9} = {}&-y\Big(3x^{4}yc_{3,1,1,1}{}^{2} + 4x^{3}yc_{2,1,1,1}c_{3,1,1,1} \\
&+ 18txyc_{3,1,1,1}{}^{2} - 6xy^{2}c_{0,2,1,1}c_{3,1,1,1} \\
&+ 6tyc_{2,1,1,1}c_{3,1,1,1} + 2x^{2}yc_{2,1,1,1}{}^{2} - 6xyc_{0,1,1,1}c_{3,1,1,1} \\
&+ 2xyc_{1,1,1,1}c_{2,1,1,1} - 2y^{2}c_{0,2,1,1}c_{2,1,1,1} \\
&- 6xc_{0,0,1,1}c_{3,1,1,1} - 2yc_{0,1,1,1}c_{2,1,1,1} + yc_{1,1,1,1}{}^{2} \\
&- 2c_{0,0,1,1}c_{2,1,1,1}\Big)\Big(-x^{3}yc_{3,1,1,1} - x^{2}yc_{2,1,1,1} \\
&+ 3tyc_{3,1,1,1} - xyc_{1,1,1,1} - y^{2}c_{0,2,1,1} - yc_{0,1,1,1} \\
&- c_{0,0,1,1}\Big)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
u_{10} = {}&3c_{3,1,1,1}\Big(-9x^{4}y^{2}c_{3,1,1,1}{}^{3} - 18x^{4}yc_{3,0,1,1}c_{3,1,1,1}{}^{2} \\
&- 12x^{3}y^{2}c_{2,1,1,1}c_{3,1,1,1}{}^{2} + 108txy^{2}c_{3,1,1,1}{}^{3} \\
&- 9x^{4}c_{3,0,1,1}{}^{2}c_{3,1,1,1} - 6x^{2}y^{2}c_{2,1,1,1}{}^{2}c_{3,1,1,1} \\
&+ 216txyc_{3,0,1,1}c_{3,1,1,1}{}^{2} + 6y^{2}c_{0,1,1,1}c_{2,1,1,1}c_{3,1,1,1} \\
&+ 36ty^{2}c_{2,1,1,1}c_{3,1,1,1}{}^{2} + 18xy^{2}c_{0,1,1,1}c_{3,1,1,1}{}^{2} \\
&- 6xy^{2}c_{1,1,1,1}c_{2,1,1,1}c_{3,1,1,1} - 3c_{1,0,1,1}{}^{2}c_{3,1,1,1}
\end{aligned}
$$

$$
\begin{aligned}
&+36tyc_{2,1,1,1}c_{3,0,1,1}c_{3,1,1,1}+36xyc_{0,1,1,1}c_{3,0,1,1}c_{3,1,1,1}\\
&-12x^3yc_{2,1,1,1}c_{3,0,1,1}c_{3,1,1,1}-3y^2c_{1,1,1,1}{}^2c_{3,1,1,1}\\
&+18xc_{0,1,1,1}c_{3,0,1,1}{}^2+6xc_{1,0,1,1}c_{2,1,1,1}c_{3,0,1,1}\\
&+6yc_{0,1,1,1}c_{2,1,1,1}c_{3,0,1,1}-6yc_{1,0,1,1}c_{1,1,1,1}c_{3,1,1,1}\\
&+2yc_{1,0,1,1}c_{2,1,1,1}{}^2+108txc_{3,0,1,1}{}^2c_{3,1,1,1}\Big)\\
&\cdot\Big(3x^3yc_{3,1,1,1}{}^2+3x^2yc_{2,1,1,1}c_{3,1,1,1}+18tyc_{3,1,1,1}{}^2\\
&+3xyc_{1,1,1,1}c_{3,1,1,1}+18tc_{3,0,1,1}c_{3,1,1,1}\\
&+3xc_{1,0,1,1}c_{3,1,1,1}+3yc_{0,1,1,1}c_{3,1,1,1}+3c_{0,1,1,1}c_{3,0,1,1}\\
&+3x^3c_{3,0,1,1}c_{3,1,1,1}+c_{1,0,1,1}c_{2,1,1,1}\Big)^{-2},\\
u_{11}&=-\left(tc_{1,0,2,2}+c_{1,0,2,1}\right)^2\left(txc_{1,0,2,2}+xc_{1,0,2,1}\right.\\
&\left.+c_{0,0,2,1}\right)^{-2},\\
u_{12}&=-c_{1,0,2,2}{}^2\left(xc_{1,0,2,2}+yc_{0,1,2,2}\right)^{-2},\\
u_{13}&=-\left(y^2c_{1,2,2,2}+yc_{1,1,2,2}+c_{1,0,2,2}\right)^2\left(xy^2c_{1,2,2,2}\right.\\
&\left.+xyc_{1,1,2,2}+y^2c_{0,2,2,2}+xc_{1,0,2,2}+yc_{0,1,2,2}\right)^{-2}.
\end{aligned}
\tag{B.1}
$$

## Competing Interests

The authors declare that they have no competing interests.

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*Research Article*

# Asymptotic Expansion of the Solutions to Time-Space Fractional Kuramoto-Sivashinsky Equations

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This paper is devoted to finding the asymptotic expansion of solutions to fractional partial differential equations with initial conditions. A new method, the residual power series method, is proposed for time-space fractional partial differential equations, where the fractional integral and derivative are described in the sense of Riemann-Liouville integral and Caputo derivative. We apply the method to the linear and nonlinear time-space fractional Kuramoto-Sivashinsky equation with initial value and obtain asymptotic expansion of the solutions, which demonstrates the accuracy and efficiency of the method.

## 1. Introduction

The Kuramoto-Sivashinsky (KS) equation in one space dimension,

$$D_t u(x,t) + D_x^4 u(x,t) + D_x^2 u(x,t) + u(x,t) D_x u(x,t) = 0, \quad (1)$$

has attracted a great deal of interest as a model for complex spatiotemporal dynamics in spatially extended systems and as a paradigm for finite-dimensional dynamics in a partial differential equation. $D_x^2 u$ term in (1) is responsible for an instability at large scales; dissipative term $D_x^4 u$ provides damping at small scales; and the nonlinear term $uD_x u$ stabilizes by transferring energy between large and small scales. The KS equation dates back to the mid-1970s and was first introduced by Kuramoto [1] in the study of phase turbulence in the Belousov-Zhabotinsky reaction-diffusion systems. An extension of this equation to two or more spatial dimensions was given by Sivashinsky [2, 3] in modelling small thermal diffusive instabilities in laminar flame fronts and in small perturbations from a reference Poiseuille flow of a flame layer on an inclined plane. In one space dimension it is also used as model for the problem of Bénard convection in an elongated box, and it may be used to describe long waves on the interface between two viscous fluids and unstable drift wave in plasmas. As a dynamical system, KS equation is known for its chaotic solutions and complicated behavior due to the terms that appear. Because of this fact, KS equation was studied extensively as a paradigm of finite dynamics in a partial differential equation. Its multimodal, oscillatory, and chaotic solutions have been investigated [4–8]; its nonintegrability was established via Painlevé analysis [9] and due to its bifurcation behavior a connection to low finite-dimensional dynamical systems is established [10, 11]. The KS equation is nonintegrable; therefore the exact solution of this equation is not obtainable and only numerical schemes have been proposed [12, 13].

Partial differential equations (PDEs) which arise in real-world physical problems are often too complicated to be solved exactly and even if an exact solution is obtainable, the required calculations may be too complicated to be practical or difficult to interpret the outcome. Very recently, some practical approximate analytical solutions are proposed to solve KS equation, such as Chebyshev spectral collocation scheme [14], lattice Boltzmann technique [15], local discontinuous Galerkin method [16], tanh function method [17], variational iteration method [18], perturbation methods [19], classical and nonclassical symmetries method for the KS equation dispersive effects [20], Riccati expansion method [21], and Lie symmetry method [22], and see also [23–27]. In the last few decades, fractional-order models are found to be more adequate than integer-order models for some real-world problems. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, and so forth involves derivatives of fractional order. In particular, for the construction of the approximate solutions of the fractional PDEs, various methods are proposed: finite difference method [28], the finite element method [29], the differential transformation method [30], the fractional subequation method [31], the fractional complex transform method [32], the modified simple equation method [33], the variational iteration method [34], the Lagrange characteristic method [35], the iteration method [36], and so on. For the time-fractional KS equation, in [37] the authors constructed the analytical exact solutions via fractional complex transform, and they obtained new types of exact analytical solutions. In [38], Rezazadeh and Ziabary found travelling wave solutions by the general time-space KS equation by a subequation method.

In this work, we apply residual power series (RPS) method to construct the asymptotic expansion of the solution to the more general linear KS equation

$$D_t^{\alpha} u(x,t) + \beta D_x^{4\tau} u(x,t) + \gamma D_x^{2\sigma} u(x,t) + \delta D_x^{\eta} u(x,t) = 0 \tag{2}$$

and the nonlinear KS equation

$$D_t^{\alpha} u(x,t) + \beta D_x^{4\tau} u(x,t) + \gamma D_x^{2\sigma} u(x,t) + \delta u(x,t) D_x^{\eta} u(x,t) = 0 \tag{3}$$

with initial value

$$u(x,0) = a_0(x) \in C^{\infty}(\mathbb{R}), \tag{4}$$

where $0 < \alpha, \eta \le 1$, $3/4 < \tau \le 1$, $1/2 < \sigma \le 1$, $\beta$, $\gamma$, and $\delta$ are any arbitrary constants, and $(x,t) \in \mathbb{R} \times \mathbb{R}$. The general response expression contains different parameters describing the order of the fractional derivative that can be varied to obtain various responses. The fractional power series solutions can be obtained by the RPS method. Particularly, if we take special parameters $\alpha = 1$, $\tau = 1$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, and $\delta = 1$ (here the equation is integer-order) and special initial condition $a_0(x) = x$, the exact solution of linear KS equation is

$$u(x,t) = x - t, \tag{5}$$

and the exact solution of nonlinear KS equation also can be obtained:

$$u(x,t) = \frac{x}{1+t}. \tag{6}$$

Here the solution of nonlinear KS equation we obtained is different from Porshokouhi and Ghanbari's work in [39] for the integer-order KS equation. They take the travel wave initial condition

$$a_0(x) = c + \frac{5}{19}\sqrt{\frac{11}{19}}\left(11\tanh^3\left(k\left(x-x_0\right)\right) - 9\tanh\left(k\left(x-x_0\right)\right)\right) \tag{7}$$

and get the exact traveling wave solution

$$u(x,t) = c + \frac{5}{19}\sqrt{\frac{11}{19}}\left(11\tanh^3\left(k\left(x-ct-x_0\right)\right) - 9\tanh\left(k\left(x-ct-x_0\right)\right)\right) \tag{8}$$

of (2) with the same special parameters $\alpha = 1$, $\tau = 1/4$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, and $\delta = 1$. Their skills mainly depend on the variational iteration method and obtain the different numerical examples with different parameters. To the best of information of the authors, no previous research work has been done using proposed technique for solving time-space fractional KS equation. Our method can be applied to the time-space linear and nonlinear fractional KS equations. The main advantage of the RPS method is that it can be applied directly for all types of differential equation, because it depends on the recursive differentiation of time-fractional derivative and uses given initial conditions to calculate coefficients of the multiple fractional power series solution with minimal calculations. Another important advantage is that this method does not require linearization, perturbation, or discretization of the variables; it is not affected by computational round-off errors and does not require large computer memory and extensive time.

The rest of this paper is organised as follows. In Section 2, some necessary concepts on the theory of fractional calculus are presented. The main steps of the PRS method are proposed in Section 3. Section 4 is the application of RPS method to construct analytical solution of linear and nonlinear time-space fractional KS equation with initial value. The paper is concluded with some general remarks in Section 5.

## 2. Some Concepts on the Theory of Fractional Calculus

There are several definitions of the fractional integral and fractional derivative, which are not necessarily equivalent to each other (see [40–42]). Riemann-Liouville integral and Caputo derivative are the two most used forms which have been introduced in [43–45]. In this section, we give some notions we need in this paper.

*Definition 1* (see [40, 41, 43]). A real function $f(t)$, $t > 0$, is said to be in the space $C_{\mu}$, $\mu \in \mathbb{R}$, if there exists a real number

$p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, +\infty)$, and it is said to be in the space $C_\mu^n$ if $f^{(n)}(t) \in C_\mu$, $n \in \mathbb{N}$.

*Definition 2* (see [40, 41, 43]). The Riemann-Liouville fractional integral operator of order $\alpha \geqslant 0$ of a function $f \in C_\mu$, $\mu \geqslant -1$, is defined as

$$
{}_{t_0}I_t^{\alpha} f(t) := \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)\, d\tau, & \alpha > 0,\ t > \tau > t_0, \\ f(t), & \alpha = 0, \end{cases} \tag{9}
$$

where the symbol ${}_{t_0}I_t^{\alpha}$ represents the $\alpha$th Riemann-Liouville fractional integral of $f$ of $t$ between the limits $t_0$ and $t$.

*Property 1* (see [40, 41, 43]). Here the properties of the operator ${}_{t_0}I_t^{\alpha}$ are given: for $f \in C_\mu$, $\mu \geqslant -1$, $\alpha, \beta, C \in \mathbb{R}$, and $\gamma \geqslant -1$,

$$
\begin{aligned}
&\text{(Pro1)} \quad {}_{t_0}I_t^{\alpha}\, {}_{t_0}I_t^{\beta} f(t) = {}_{t_0}I_t^{\alpha+\beta} f(t) = {}_{t_0}I_t^{\beta}\, {}_{t_0}I_t^{\alpha} f(t), \\
&\text{(Pro2.)} \quad {}_{t_0}I_t^{\alpha} C = \frac{C}{\Gamma(\alpha+1)} (t-t_0)^{\alpha}, \\
&\text{(Pro3.)} \quad {}_{t_0}I_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}\Big|_{t_0}^{t}.
\end{aligned} \tag{10}
$$

*Definition 3* (see [40, 41, 43]). The Caputo fractional derivative of order $\alpha > 0$ of $f \in C_{-1}^n$, $n \in \mathbb{N}$, is defined as

$$
{}_{t_0}D_t^{\alpha} f(t) := \begin{cases} {}_{t_0}I_t^{n-\alpha} f^{(n)}(t), & n-1 < \alpha < n,\ t > t_0, \\ \dfrac{d^n f(t)}{dt^n}\Big|_{t=t_0}, & \alpha = n, \end{cases} \tag{11}
$$

where the symbol ${}_{t_0}D_t^{\alpha} f(t)$ represents the $\alpha$th Caputo fractional derivative of $f$ with respect to $t$ at $t_0$.

*Property 2* (see [40, 41, 43]). Here the properties of the operator ${}_{t_0}D_t^{\alpha}$ are given: for $\gamma > -1$, $t > s \geqslant 0$, and $C \in \mathbb{R}$,

$$
\begin{aligned}
&\text{(Pro1.)} \quad {}_{t_0}D_t^{\alpha}\, {}_{t_0}D_t^{\beta} f(t) = {}_{t_0}D_t^{\alpha+\beta} f(t) \\
&\qquad\qquad = {}_{t_0}D_t^{\beta}\, {}_{t_0}D_t^{\alpha} f(t), \\
&\text{(Pro2.)} \quad {}_{t_0}D_t^{\alpha} C = 0, \\
&\text{(Pro3.)} \quad {}_{t_0}D_t^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}\Big|_{t=t_0}.
\end{aligned} \tag{12}
$$

*Remark 4* (see [41, 46]). The Caputo time-fractional derivative operator of order $\alpha$ of function $u(x,t)$ with respect to $t$ at $t_0$ is defined as

$$
{}_{t_0}D_t^{\alpha} u(x,t) := \begin{cases} {}_{t_0}I_t^{n-\alpha} \dfrac{\partial^n u(x,t)}{\partial t^n}, & n-1 < \alpha < n,\ t > t_0, \\ \dfrac{d^n f(t)}{dt^n}\Big|_{t=t_0}, & \alpha = n \in \mathbb{N}, \end{cases} \tag{13}
$$

where $x \in \mathbb{R}$ and $t > 0$.

*Remark 5* (see [41, 46]). The Caputo space-fractional derivative operator of order $\beta$ of function $u(x,t)$ with respect to $x$ at $x_0$ is defined as

$$
{}_{x_0}D_x^{\beta} u(x,t) := \begin{cases} {}_{x_0}I_x^{m-\beta} \dfrac{\partial^m u(x,t)}{\partial t^m}, & m-1 < \alpha < m,\ x > x_0, \\ \dfrac{\partial^m u(x,t)}{\partial x^m}\Big|_{x=x_0}, & \alpha = m \in \mathbb{N}, \end{cases} \tag{14}
$$

where $x \in \mathbb{R}$ and $t > 0$.

*Definition 6* (see [41, 46]). A power series representation of the form

$$
\sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha} := c_0 + c_1 (t-t_0)^{\alpha} + c_2 (t-t_0)^{2\alpha} + \cdots, \tag{15}
$$

where $m-1 < \alpha \leq m$ and $t \geqslant t_0$, is called a fractional power series (FPS) about $t_0$, where $t$ is a variable and $c_n$ are constants called the coefficients of the series.

**Theorem 7.** *Suppose that $f$ has a FPS representation at $t_0$ of the form*

$$
f(t) = \sum_{n=0}^{\infty} c_n (t-t_0)^{n\alpha}, \quad 0 \leqslant m-1 < \alpha \leqslant m,\ t_0 \leqslant t < t_0 + R, \tag{16}
$$

*and R is the radius of convergence of the FPS. If $f(t) \in C[t_0, t_0 + R)$ and ${}_{t_0}D_t^{n\alpha} f(t) \in C(t_0, t_0 + R)$ for $n = 0, 1, 2, \ldots$, then the coefficients $c_n$ will take the form of*

$$
c_n = \frac{{}_{t_0}D_t^{n\alpha} f(t)}{\Gamma(n\alpha+1)}, \tag{17}
$$

*where ${}_{t_0}D_t^{n\alpha} = {}_{t_0}D_t^{\alpha} \cdot {}_{t_0}D_t^{\alpha} \cdot \cdots \cdot {}_{t_0}D_t^{\alpha}$ (n-times).*

This result is similar to [46, Theorem 2.2] and [47, Theorem 3.4]. It is convenient to give the details for the following applications; thus we write the process of the proof in the form of function with one variable.

*Proof.* First of all, notice that if we put $t = t_0$ into (16), it yields

$$
c_0 = f(t_0) = \frac{{}_{t_0}D_t^{0\alpha} f(t)}{\Gamma(0\alpha+1)}. \tag{18}
$$

Applying the operator ${}_{t_0}D_t^{\alpha}$ one time on (16) leads to

$$
c_1 = \frac{{}_{t_0}D_t^{\alpha} f(t)}{\Gamma(\alpha+1)}. \tag{19}
$$

Again, by applying the operator ${}_{t_0}D_t^{\alpha}$ two times on (16), one can obtain

$$
c_2 = \frac{{}_{t_0}D_t^{2\alpha} f(t)}{\Gamma(2\alpha+1)}. \tag{20}
$$

By now, the pattern is clearly found, if we continue applying recursively the operator ${}_{t_0}D_t^{\alpha}$ $n$-times on (16); then it is easy to discover the following form for $c_n$:

$$c_n = \frac{{}_{t_0}D_t^{n\alpha} f(t)}{\Gamma(n\alpha + 1)}, \quad n = 0, 1, 2, \ldots; \tag{21}$$

that is, it has the same form as (17), which completes the proof. □

Following the similar result as [46, Theorem 2.3] or [48, Remark 8], we can obtain the following corollary.

**Corollary 8.** *If $f(t) = u(x,t)$, then we have*

$$u(x,t) = \sum_{n=0}^{\infty} c_n(x)(t - t_0)^{n\alpha}, \tag{22}$$
$$0 \leqslant m - 1 < \alpha \leqslant m, \ x \in \mathbb{R}, \ t_0 \leqslant t < t_0 + R,$$

*and $R$ is the radius of convergence of the FPS; if $D_t^{n\alpha} \in C(\mathbb{R} \times (t_0, t_0 + R))$, $n = 0, 1, 2, \ldots$, then the coefficients are given by $c_n(x) = {}_{t_0}D_t^{n\alpha} u(x,t)/\Gamma(n\alpha + 1)$.*

## 3. Algorithm of RPS Method

Let us consider the higher-order time-space fractional differential equation with initial values as follows:

$$\begin{aligned} &D_t^{m\alpha} u(x,t) + G(x,t) = 0, \\ &u(x,0) = a_0(x), \ D_t^{\alpha} u(x,t)\big|_{t=0} \\ &\quad = a_1(x), \ldots, D_t^{(m-1)\alpha} u(x,t)\big|_{t=0} = a_{m-1}(x), \end{aligned} \tag{23}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$, and

$$\begin{aligned} G(x,t) = F\big(&u, D_t^{\alpha} u, D_t^{2\alpha} u, \ldots, D_t^{(m-1)\alpha} u, D_x^{\beta_1} u, D_x^{\beta_2} u, \\ &\ldots, D_x^{\beta_l} u\big). \end{aligned} \tag{24}$$

$u(x,\cdot)$ and $G(x,\cdot)$ are analytical function about $t$, $(m-1)/m < \alpha \leqslant 1$, $m \in \mathbb{N}$, $p - 1 < \beta_p \leqslant p$, and $p = 1, 2, \ldots, l$; $a_q(x) \in C^{\infty}(\mathbb{R})$, $q = 0, 1, 2, \ldots, m-1$.

Assume that $u(x,t)$ is analytical about $t$; the solution of the system can be written in the form of

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t), \tag{25}$$

where $u_n(x,t)$ are terms of approximations and are given as

$$u_n(x,t) = C_n(x) t^{n\alpha}, \tag{26}$$
$$x \in \mathbb{R}, \ |t| < R, \ n = 0, 1, 2, \ldots,$$

where $R$ is the radius of convergence of above series. Obviously, when $i = 0, 1, 2, \ldots, m-1$, using the terms $D_t^{i\alpha} u(x,t)$ which satisfy the initial condition, we can get

$$\begin{aligned} &a_i(x) = D_t^{i\alpha} u(x,t)\big|_{t=0} = C_i(x)\Gamma(i\alpha + 1) \Longrightarrow \\ &u_i(x,t) = C_i(x) t^{i\alpha} = \frac{D_t^{i\alpha} u(x,t)\big|_{t=0}}{\Gamma(i\alpha + 1)} t^{i\alpha} \\ &\quad = \frac{a_i(x)}{\Gamma(i\alpha + 1)} t^{i\alpha}, \quad i = 0, 1, \ldots, m-1. \end{aligned} \tag{27}$$

So we have the initial guess approximation of $u(x,t)$ in the following form:

$$\begin{aligned} u^{\text{initial}}(x,t) &:= u_0(x,t) + u_1(x,t) + \cdots + u_{m-1}(x,t) \\ &= a_0(x) + \frac{a_1(x)}{\Gamma(\alpha + 1)} t^{\alpha} + \cdots \\ &\quad + \frac{a_{m-1}(x)}{\Gamma((m-1)\alpha + 1)} t^{(m-1)\alpha}. \end{aligned} \tag{28}$$

On the other aspect as well, if we choose $u^{\text{initial}}(x,t)$ as initial guess approximation $u_i(x,t)$ for $i = m, m+1, m+2, \ldots$, the approximate solutions of $u(x,t)$ of (23) by the $k$th truncated series are

$$u^k(x,t) := u^{\text{initial}}(x,t) + \sum_{i=m}^{k} C_i(x) t^{i\alpha}, \tag{29}$$
$$k = m, m+1, m+2, \ldots.$$

Before applying RPS method for solving (23), we first give some notations:

$$\operatorname{Res}(x,t) := D_t^{m\alpha} u(x,t) + G(x,t). \tag{30}$$

Substituting the $k$th truncated series $u^k(x,t)$ into (23), we can obtain the following definition for $k$th residual function:

$$\operatorname{Res}^k(x,t) := D_t^{m\alpha} u^k(x,t) + G^k(x,t), \tag{31}$$

where

$$\begin{aligned} G^k(x,t) = F\big(&u^k, D_t^{\alpha} u^k, D_t^{2\alpha} u^k, \ldots, D_t^{(m-1)\alpha} u^k, D_x^{\beta_1} u^k, \\ &D_x^{\beta_2} u^k, \ldots, D_x^{\beta_{1n}} u^k, D_{x_1}^{\beta_l} u^k\big). \end{aligned} \tag{32}$$

Then, we have following facts:

$$\begin{aligned} &(1) \quad \lim_{k\to\infty} u^k(x,t) = u(x,t); \\ &(2) \quad \operatorname{Res}(x,t) = 0; \\ &(3) \quad \operatorname{Res}^{\infty}(x,t) = \lim_{k\to+\infty} \operatorname{Res}^k(x,t) = \operatorname{Res}(x,t) = 0, \\ &\qquad x \in \mathbb{R}, \ |t| < R. \end{aligned} \tag{33}$$

These show that the residual function $\mathrm{Res}^{\infty}(x,t)$ is infinitely many times differentiable at $t=0$. On the other hand, we can show that

$$\begin{aligned} 0 &= D_t^{(k-m)\alpha}\ \mathrm{Res}^{\infty}(x,t)\big|_{t=0} \\ &= C_k(x)\,\Gamma(k\alpha+1) + D_t^{(k-m)\alpha}G^k(x,t)\big|_{t=0} \Longrightarrow \\ C_k(x) &= -\frac{D_t^{(k-m)\alpha}G^k(x,t)\big|_{t=0}}{\Gamma(k\alpha+1)}, \\ &\qquad k = m, m+1, m+2, \ldots. \end{aligned} \tag{34}$$

Since $-D_t^{(k-m)\alpha}G^k(x,t)|_{t=0}$ is not dependent on $t$, denoting it by $f_k(x)$, by Theorem 7, (34) can be written as

$$\begin{aligned} C_k(x) &= \frac{f_k(x)}{\Gamma(k\alpha+1)}, \\ f_k(x) &= -D_t^{(k-m)\alpha}\ G^k(x,t)\big|_{t=0}, \\ &\qquad k = m, m+1, m+2, \ldots. \end{aligned} \tag{35}$$

In fact, the relation of (35) is a fundamental rule in the RPS method and its applications. So the fractional power series solution of (23) is

$$\begin{aligned} u(x,t) &= u^{\text{initial}}(x,t) + \sum_{i=m}^{\infty} C_i(x)\,t^{i\alpha} \\ &= \sum_{i=0}^{m-1}\frac{a_i(x)}{\Gamma(i\alpha+1)}t^{i\alpha} + \sum_{i=m}^{\infty}\frac{f_i(x)}{\Gamma(i\alpha+1)}t^{i\alpha}, \end{aligned} \tag{36}$$

where $a_i(x)$ $(i = 0,1,2,\ldots,m-1)$ are given by the initial conditions and $f_i(x)$ $(i = m, m+1, m+2, \ldots)$ have been constructed by RPS method in the form in (35).

## 4. Application of RPS Method to Time-Space Fractional KS Equation

In this section, we apply the RPS method to the linear and nonlinear time-space fractional KS equation with the initial conditions. The fractional power series solutions can be obtained by the recursive equation (35) with time-fractional derivative, while it will use the given initial conditions. And the fractional power series solutions we consider in the following examples are all in the convergence radius of the series.

*4.1. Linear Time-Space Fractional KS Equation.* Consider the linear time-space fractional KS equation

$$\begin{aligned} &D_t^{\alpha}u(x,t) + \beta D_x^{4\tau}u(x,t) + \gamma D_x^{2\sigma}u(x,t) + \delta D_x^{\eta}u(x,t) \\ &\quad = 0. \end{aligned} \tag{37}$$

In (37), if $u(x,t)$ is analytical about $t$, then $u(x,t)$ can be written as the fractional power series:

$$u(x,t) = \sum_{n=0}^{\infty}\frac{f_n(x)}{\Gamma(n\alpha+1)}t^{n\alpha}. \tag{38}$$

If we can take the initial value,

$$u(x,0) = a_0(x) \in C^{\infty}(\mathbb{R}). \tag{39}$$

According to the initial condition, it is obvious that $f_0(x) = a_0(x)$. Before getting the coefficients $f_n(x)$ $(n = 1,2,\ldots)$, we first present some notations as in Section 3:

$$\begin{aligned} G(x,t) &= \beta D_x^{4\tau}u(x,t) + \gamma D_x^{2\sigma}u(x,t) \\ &\quad + \delta D_x^{\eta}u(x,t); \\ \mathrm{Res}(x,t) &= D_t^{\alpha}u(x,t) + G(x,t); \\ u^k(x,t) &= \sum_{j=0}^{k}\frac{f_j(x)}{\Gamma(j\alpha+1)}t^{j\alpha} \\ &= a_0(x) + \sum_{j=1}^{k}\frac{f_j(x)}{\Gamma(j\alpha+1)}t^{j\alpha}; \\ G^k(x,t) &= \beta D_x^{4\tau}u^k(x,t) + \gamma D_x^{2\sigma}u^k(x,t) \\ &\quad + \delta D_x^{\eta}u^k(x,t); \\ \mathrm{Res}^k(x,t) &= D_t^{\alpha}u_k(x,t) + G^k(x,t). \end{aligned} \tag{40}$$

It follows from the facts in (33) that the residual function $\mathrm{Res}^{\infty}(x,t)$ is infinitely many times differentiable at $t=0$. On the other hand, it follows from Definition 3 and (40) that

$$\begin{aligned} 0 &= D_t^{(k-1)\alpha}\ \mathrm{Res}^{\infty}(x,t)\big|_{t=0} = D_t^{(k-1)\alpha}\ \mathrm{Res}^{k}(x,t)\big|_{t=0} \\ &= D_t^{(k-1)\alpha}\left(D_t^{\alpha}u^k(x,t) + G^k(x,t)\right)\big|_{t=0} \\ &= D_t^{k\alpha}u^k(x,t)\big|_{t=0} + D_t^{(k-1)\alpha}G^k(x,t)\big|_{t=0} \\ &= f_k(x) + D_t^{(k-1)\alpha}G^k(x,t)\big|_{t=0}, \end{aligned} \tag{41}$$

which gives

$$f_k(x) = -D_t^{(k-1)\alpha}\ G^k(x,t)\big|_{t=0}, \quad (k = 1,2,3,\ldots). \tag{42}$$

Equation (42) gives the iterative formula for the coefficients $f_n(x)$ $(n = 1,2,\ldots)$, so we can obtain

$$\begin{aligned} f_0(x) &= a_0(x), \\ f_1(x) &= -\left(\beta D_x^{4\tau}a_0(x) + \gamma D_x^{2\sigma}a_0(x) + \delta D_x^{\eta}a_0(x)\right) \\ &\triangleq a_1(x), \\ f_2(x) &= -\left(\beta D_x^{4\tau}a_1(x) + \gamma D_x^{2\sigma}a_1(x) + \delta D_x^{\eta}a_1(x)\right) \\ &\triangleq a_2(x); \\ f_3(x) &= -\left(\beta D_x^{4\tau}a_2(x) + \gamma D_x^{2\sigma}a_2(x) + \delta D_x^{\eta}a_2(x)\right) \\ &\triangleq a_3(x). \end{aligned} \tag{43}$$

For general $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
& f_k(x) \\
& \quad = -\left(\beta D_x^{4\tau} a_{k-1}(x) + \gamma D_x^{2\sigma} a_{k-1}(x) + \delta D_x^{\eta} a_{k-1}(x)\right) \\
& \quad \triangleq a_k(x).
\end{aligned} \tag{44}
$$

So the $k$th approximate solution of (37) with initial value (39) is

$$
u^k(x,t) = \sum_{i=0}^{k} \frac{f_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha} = \sum_{i=0}^{k} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \quad k = 1,2,3,\ldots, \tag{45}
$$

where $a_i(x)$ $(i = 0,1,2,\ldots,k)$ are given by (39) and (44).

Specifically, if taking $\alpha = 1$, $\tau = 1$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, and $\delta = 1$ (the equation becomes integer order) and special initial value $a_0(x) = x$, we can obtain

$$
\begin{aligned}
& a_1(x) = -1, \\
& a_k(x) = 0, \quad k = 2,3,\ldots.
\end{aligned} \tag{46}
$$

So, we can obtain the $k$th approximate power series solution of integer-order equation (37):

$$
u^k(x,t) = \sum_{i=0}^{k} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha} = x - t. \tag{47}
$$

Thus, the exact solution is

$$
u(x,t) = \lim_{k\to\infty} u^k(x,t) = x - t, \tag{48}
$$

when $\alpha = 1$, $\tau = 1$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, and $a_0(x) = x$.

*4.2. Nonlinear Time-Space Fractional KS Equation.* Let us rewrite the nonlinear time-space fractional KS equation:

$$
\begin{aligned}
& D_t^{\alpha} u(x,t) + \beta D_x^{4\tau} u(x,t) + \gamma D_x^{2\sigma} u(x,t) \\
& \quad + \delta u(x,t) D_x^{\eta} u(x,t) = 0.
\end{aligned} \tag{49}
$$

If $u(x,t)$ is analytical about $t$ of (49), then $u(x,t)$ can be written as the fractional power series:

$$
u(x,t) = \sum_{n=0}^{\infty} \frac{f_n(x)}{\Gamma(n\alpha+1)} t^{n\alpha}. \tag{50}
$$

Here, the initial value satisfies

$$
u(x,0) = a_0(x) \in C^{\infty}(\mathbb{R}). \tag{51}
$$

According to the initial condition, it is obvious that $f_0(x) = a_0(x)$.

Before getting the coefficients $f_n(x)$ $(n = 1,2,\ldots)$, we also present some symbols:

$$
\begin{aligned}
G(x,t) &= \beta D_x^{4\tau} u(x,t) + \gamma D_x^{2\sigma} u(x,t) \\
&\quad + \delta u(x,t) D_x^{\eta} u(x,t); \\
\operatorname{Res}(x,t) &= D_t^{\alpha} u(x,t) + G(x,t); \\
u^k(x,t) &= \sum_{j=0}^{k} \frac{f_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha} \\
&= a_0(x) + \sum_{j=1}^{k} \frac{f_j(x)}{\Gamma(j\alpha+1)} t^{j\alpha}; \\
G^k(x,t) &= \beta D_x^{4\tau} u^k(x,t) + \gamma D_x^{2\sigma} u^k(x,t) \\
&\quad + \delta u^k(x,t) D_x^{\eta} u^k(x,t); \\
\operatorname{Res}^k(x,t) &= D_t^{\alpha} u_k(x,t) + G^k(x,t).
\end{aligned} \tag{52}
$$

It also follows from the facts in (33) that the residual function $\operatorname{Res}^{\infty}(x,t)$ is infinitely many times differentiable at $t = 0$.

On the other hand, it follows from Definition 3 and the notations above that

$$
\begin{aligned}
0 &= D_t^{(k-1)\alpha} \operatorname{Res}^{\infty}(x,t)\big|_{t=0} = D_t^{(k-1)\alpha} \operatorname{Res}^k(x,t)\big|_{t=0} \\
&= D_t^{(k-1)\alpha} \left(D_t^{\alpha} u^k(x,t) + G^k(x,t)\right)\big|_{t=0} \\
&= D_t^{k\alpha} u^k(x,t)\big|_{t=0} + D_t^{(k-1)\alpha} G^k(x,t)\big|_{t=0} \\
&= f_k(x) + D_t^{(k-1)\alpha} G^k(x,t)\big|_{t=0},
\end{aligned} \tag{53}
$$

which gives

$$
f_k(x) = -D_t^{(k-1)\alpha} G^k(x,t)\big|_{t=0}, \quad (k = 1,2,3,\ldots). \tag{54}
$$

Equation (54) gives the iterative formula for the coefficients $f_n(x)$ $(n = 1,2,\ldots)$, so we can obtain

$$
\begin{aligned}
& f_0(x) = a_0(x), \\
& f_1(x) = -\left(\beta D_x^{4\tau} a_0(x) + \gamma D_x^{2\sigma} a_0(x)\right. \\
& \qquad \left. + \delta a_0(x) D_x^{\eta} a_0(x)\right) \triangleq a_1(x), \\
& f_2(x) = -\left(\beta D_x^{4\tau} a_1(x) + \gamma D_x^{2\sigma} a_1(x)\right. \\
& \qquad \left. + \delta a_0(x) D_x^{\eta} a_1(x) + \delta a_1(x) D_x^{\eta} a_0(x)\right) \triangleq a_2(x); \\
& f_3(x) = -\left(\beta D_x^{4\tau} a_2(x) + \gamma D_x^{2\sigma} a_2(x)\right. \\
& \qquad + \delta a_0(x) D_x^{\eta} a_2(x) + \delta \frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2} a_1(x) D_x^{\eta} a_1(x) \\
& \qquad \left. + \delta a_2(x) D_x^{\eta} a_0(x)\right) \triangleq a_3(x).
\end{aligned} \tag{55}
$$

For general $k \in \mathbb{Z}$, we have

$$f_k(x) = -\left(\beta D_x^{4\tau} a_{k-1}(x) + \gamma D_x^{2\sigma} a_{k-1}(x) + \delta \sum_{j=0}^{k-1} \frac{\Gamma((k-1)\alpha+1)}{\Gamma(j\alpha+1)\Gamma((k-1-j)\alpha+1)} a_j(x) \cdot D_x^{\eta} a_{k-1-j}(x)\right) \triangleq a_k(x). \quad (56)$$

So the $k$th approximate solution of (51) with initial value (51) is

$$u^k(x,t) = \sum_{i=0}^{k} \frac{f_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha} = \sum_{i=0}^{k} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha}, \quad k = 1, 2, 3, \ldots, \quad (57)$$

where $a_i(x)$ $(i = 0, 1, 2, \ldots, k)$ are given by (51) and (56).

In particular, if taking $\alpha = 1$, $\tau = 1$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, $\delta = 1$, and $a_0(x) = x$, we can obtain

$$\begin{aligned}
a_1(x) &= -x, \\
a_2(x) &= 2!x, \\
a_3(x) &= -3!x, \\
&\vdots \\
a_{k-1}(x) &= (-1)^{k-1}(k-1)!x, \\
a_k(x) &= D_x^4 a_{k-1}(x) + D_x^2 a_{k-1}(x) + \sum_{j=0}^{k-1} \frac{\Gamma((k-1)\alpha+1)}{\Gamma(j\alpha+1)\Gamma((k-1-j)\alpha+1)} a_j(x) \\
&\quad \cdot D_x a_{k-1-j}(x) = \sum_{j=0}^{k-1} \frac{\Gamma((k-1)+1)}{\Gamma(j+1)\Gamma((k-1-j)+1)} \\
&\quad \cdot (-1)^j j!x(-1)^{k-1-j}(k-1-j)! \\
&= \sum_{j=0}^{k-1} \frac{(k-1)!}{j!(k-1-j)!}(-1)^{k-1} j!x(k-1-j)! \\
&= (-1)^k (k-1)! \cdot kx = (-1)^k k!x, \quad k = 1, 2, 3, \ldots.
\end{aligned} \quad (58)$$

So the $k$th approximate FPS solution of (49) with initial value (51) is

$$u^k(x,t) = \sum_{i=0}^{k} \frac{a_i(x)}{\Gamma(i\alpha+1)} t^{i\alpha} = \sum_{i=0}^{k} \frac{(-1)^i i!x}{i!} t^i = x\sum_{i=0}^{k}(-t)^i = x\frac{1-(-t)^k}{1+t}. \quad (59)$$

When $t \in (-1, 1)$, it shows

$$u(x,t) = \lim_{k\to\infty} u^k(x,t) = \lim_{k\to\infty} x\frac{1-(-t)^k}{1+t} = \frac{x}{1+t}, \quad (60)$$

which is the exact solution for (49) with special parameters $\alpha = 1$, $\tau = 1$, $\sigma = 1$, $\eta = 1$, $\beta = 1$, $\gamma = 1$, and $\delta = 1$ and special initial value $a_0(x) = x$.

## 5. Concluding Remarks

In this paper, we have used a new method: the residual power series method for the general fractional differential equations. The asymptotic expansion of the solutions can be obtained successfully with respect to initial conditions which are infinitely differentiable by the RPS method. We apply RPS method to linear and nonlinear Kuramoto-Sivashinsky equation with infinitely differential initial conditions and obtain the asymptotic expansion of the solutions. Particularly, if taking special parameters and special initial value, the analytical solutions are obtained. These applications show that this method is efficient and does not require linearization or perturbation; it is not affected by computational round-off errors and does not require large computer memory and extensive time.

## Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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*Research Article*

# An Efficient Numerical Method for the Solution of the Schrödinger Equation

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The development of a new five-stage symmetric two-step fourteenth-algebraic order method with vanished phase-lag and its first, second, and third derivatives is presented in this paper for the first time in the literature. More specifically we will study (1) the development of the new method, (2) the determination of the local truncation error (LTE) of the new method, (3) the local truncation error analysis which will be based on test equation which is the radial time independent Schrödinger equation, (4) the stability and the interval of periodicity analysis of the new developed method which will be based on a scalar test equation with frequency different than the frequency of the scalar test equation used for the phase-lag analysis, and (5) the efficiency of the new obtained method based on its application to the coupled Schrödinger equations.

## 1. Introduction

The approximate solution of the close-coupling differential equations of the Schrödinger type is studied in this paper. The above-mentioned problem has the following form:

$$\left[\frac{d^2}{dx^2}+k_i^2-\frac{l_i\left(l_i+1\right)}{x^2}-V_{ii}\right]y_{ij}=\sum_{m=1}^{N}V_{im}y_{mj}, \tag{1}$$

where $1 \leq i \leq N$ and $m \neq i$ and the boundary conditions are as follows:

$$y_{ij}=0 \quad \text{at } x=0 \tag{2}$$

$$y_{ij}\sim k_i x j_{l_i}\left(k_i x\right)\delta_{ij}+\left(\frac{k_i}{k_j}\right)^{1/2}K_{ij}k_i x n_{li}\left(k_i x\right), \tag{3}$$

where $j_l(x)$ and $n_l(x)$ are the spherical Bessel and Neumann functions. We will examine the case in which all channels are open (see [1]).

Defining a matrix $K'$ and diagonal matrices $M$, $N$ by (see [1])

$$\begin{aligned} K'_{ij}&=\left(\frac{k_i}{k_j}\right)^{1/2}K_{ij},\\ M_{ij}&=k_i x j_{l_i}\left(k_i x\right)\delta_{ij},\\ N_{ij}&=k_i x n_{l_i}\left(k_i x\right)\delta_{ij}\end{aligned} \tag{4}$$

we obtain a new form of the asymptotic condition (3):

$$\mathbf{y}\sim\mathbf{M}+\mathbf{N}\mathbf{K}'. \tag{5}$$

In several scientific areas (e.g., quantum chemistry, theoretical physics, material science, atomic physics, and molecular physics) there exists a real problem which is the rotational excitation of a diatomic molecule by neutral particle impact. The mathematical model of this problem can be expressed with close-coupling differential equations arising from the Schrödinger equation. Denoting, as in [1], the entrance channel by the quantum numbers $(j, l)$, the exit channels by $(j', l')$, and the total angular momentum by $J = j + l = j' + l'$, we find that

$$\left[\frac{d^2}{dx^2} + k_{j'j}^2 - \frac{l'\left(l'+1\right)}{x^2}\right] y_{j'l'}^{Jjl}(x) = \frac{2\mu}{\hbar^2}\sum_{j''}\sum_{l''}\left\langle j'l'; J\left|V\right| j''l''; J\right\rangle y_{j''l''}^{Jjl}(x), \tag{6}$$

where

$$k_{j'j} = \frac{2\mu}{\hbar^2}\left[E + \frac{\hbar^2}{2I}\left\{j(j+1) - j'\left(j'+1\right)\right\}\right]. \tag{7}$$

$E$ is the kinetic energy of the incident particle in the center-of-mass system, $I$ is the moment of inertia of the rotator, and $\mu$ is the reduced mass of the system.

The above-described problem will be solved numerically via finite difference method of the form of special multistep method.

The multistep finite difference method has the general form

$$\sum_{i=-m}^{m} c_i y_{n+i} = h^2 \sum_{i=-m}^{m} b_i f\left(x_{n+i}, y_{n+i}\right), \tag{8}$$

where $\{x_i\}_{i=-m}^{m}$ are distinct points within the integration area and $h$ given by $h = |x_{i+1} - x_i|$, $i = 1 - m(1)m - 1$ is *the step size or step length of the integration*.

*Remark 1.* A method (8) is called symmetric multistep method or symmetric $2m$-step method if $c_{-i} = c_i$ and $b_{-i} = b_i$, $i = 0(1)m$.

If we apply the symmetric $2m$-step method $(i = -m(1)m)$, to the scalar test equation

$$y'' = -\phi^2 y, \tag{9}$$

we obtain the difference equation

$$A_m(v)\, y_{n+m} + \cdots + A_1(v)\, y_{n+1} + A_0(v)\, y_n + A_1(v)\, y_{n-1} + \cdots + A_m(v)\, y_{n-m} = 0 \tag{10}$$

and the associated characteristic equation

$$A_m(v)\,\lambda^m + \cdots + A_1(v)\,\lambda + A_0(v) + A_1(v)\,\lambda^{-1} + \cdots + A_m(v)\,\lambda^{-m} = 0, \tag{11}$$

where $v = \phi h$, $h$ is the step length, and $A_j(v)$ $j = 0(1)k$ are polynomials of $v$.

We give some definitions.

*Definition 2* (see [2]). For a symmetric $2m$-step method with characteristic equation given by (11) one will say that it has an interval of periodicity $(0, v_0^2)$ if, for all $v \in (0, v_0^2)$, the roots $\lambda_i$, $i = 1(1)2m$, of (11) satisfy

$$\begin{aligned} \lambda_1 &= e^{i\theta(v)}, \\ \lambda_2 &= e^{-i\theta(v)}, \\ \left|\lambda_i\right| &\leq 1, \quad i = 3(1)2m, \end{aligned} \tag{12}$$

where $\theta(v)$ is a real function of $v$.

*Definition 3* (see [2]). One calls P-stable method a multistep method if its interval of periodicity is equal to $(0, \infty)$.

*Definition 4.* One calls singularly P-stable method a multistep method if its interval of periodicity is equal to $(0, \infty) - S$ (where $S$ is a set of distinct points).

*Definition 5* (see [3, 4]). For a symmetric $2m$-step method with the characteristic equation given by (11), one defines as the phase-lag the leading term in the expansion of

$$t = v - \theta(v). \tag{13}$$

Then, if the quantity $t = O(v^{q+1})$ as $v \to \infty$, the order of the phase-lag is $q$.

*Definition 6* (see [5]). One calls symmetric $2m$-step method *phase-fitted* if its phase-lag is equal to zero.

**Theorem 7** (see [3]). *The symmetric $2m$-step method with characteristic equation given by (11) has phase-lag order $q$ and phase-lag constant $c$ given by*

$$-cv^{q+2} + O\left(v^{q+4}\right) = \frac{2A_m(v)\cos(mv) + \cdots + 2A_j(v)\cos(jv) + \cdots + A_0(v)}{2m^2 A_m(v) + \cdots + 2j^2 A_j(v) + \cdots + 2A_1(v)}. \tag{14}$$

## 2. The New Five-Stage Fourteenth-Algebraic Order P-Stable Two-Step Method with Vanished Phase-Lag and Its First, Second, and Third Derivatives

We consider the following family of five-stage symmetric two-step methods:

$$\begin{aligned} \hat{y}_n &= y_n - a_0 h^2\left(f_{n+1} - 2f_n + f_{n-1}\right), \\ \check{y}_n &= y_n - a_1 h^2\left(f_{n+1} - 2\hat{f}_n + f_{n-1}\right) - 2a_2 h^2 \hat{f}_n, \\ \hat{y}_{n+1/2} &= \frac{1}{2}\left(y_n + y_{n+1}\right) - h^2\left[a_3 \check{f}_n + \left(\frac{1}{8} - a_3\right) f_{n+1}\right], \\ \hat{y}_{n-1/2} &= \frac{1}{2}\left(y_n + y_{n-1}\right) - h^2\left[a_3 \check{f}_n + \left(\frac{1}{8} - a_3\right) f_{n-1}\right], \\ y_{n+1} + a_4 y_n + y_{n-1} &= h^2\left[b_1\left(f_{n+1} + f_{n-1}\right) + b_0 f_n + b_2\left(\hat{f}_{n+1/2} + \hat{f}_{n-1/2}\right)\right], \end{aligned} \tag{15}$$

where $a_0 = 45469/862066800$, $a_1 = -2793/26878564$, $a_2 = -86919/13439282$, $a_3 = 6719641/52720800$, $f_{n+i} = y''(x_{n+i}, y_{n+i})$, $i = -1(1/2)1$, $\widehat{f}_n = y''(x_n, \widehat{y}_n)$, $\check{f}_n = y''(x_n, \check{y}_n)$, and $a_4, b_j$, $j = 0(1)2$, are free parameters.

Application of the above-mentioned method (15) to the scalar test equation (9) leads to the difference equation (10) and the characteristic equation (11) with $m = 1$ and

$$\begin{aligned} A_1(v) &= 1 + v^2\left(b_1 + \frac{1}{2}b_2\right) - v^4 b_2\left(a_3 - \frac{1}{8}\right) \\ &\quad + 2v^6 b_2 a_1 a_3 + 4v^8 b_2 a_0 a_3 (a_2 - a_1), \\ A_0(v) &= a_4 + v^2(b_0 + b_2) + 2v^4 b_2 a_3 \\ &\quad + 4v^6 b_2 a_3 (a_2 - a_1) \\ &\quad - 8v^8 b_2 a_0 a_3 (a_2 - a_1). \end{aligned} \tag{16}$$

If we request the new method (15) to have vanished phase-lag and its first, second, and third derivatives, then the following system of equations is obtained:

$$\begin{aligned} \text{Phase-Lag (PL)} &= \frac{T_0}{T_{\text{denom}}} = 0, \\ \text{First Derivative of the Phase-Lag} &= \frac{T_1}{T_{\text{denom}}^2} = 0, \\ \text{Second Derivative of the Phase-Lag} &= \frac{T_2}{T_{\text{denom}}^3} = 0, \\ \text{Third Derivative of the Phase-Lag} &= \frac{T_3}{T_{\text{denom}}^4} = 0, \end{aligned} \tag{17}$$

where $T_j$, $j = 0(1)3$, are given in Appendix A.

Now solving the system of (17) we determine the other coefficients of the new obtained three-stage two-step method:

$$\begin{aligned} b_0 &= \frac{1}{136407}\frac{T_4}{T_{\text{denom1}}}, \\ b_1 &= \frac{1}{136407}\frac{T_5}{T_{\text{denom1}}}, \\ b_2 &= \frac{1}{45469}\frac{T_6}{T_{\text{denom1}}}, \\ a_4 &= \frac{T_7}{T_{\text{denom2}}}, \end{aligned} \tag{18}$$

where the formulae $T_j$, $j = 4(1)7$, $T_{\text{denom1}}$, and $T_{\text{denom2}}$ are given in Appendix B.

Additionally to the above formulae for the coefficients $a_4$, $b_j$, $j = 0(1)2$, we give also the following Taylor series expansions of these coefficients, for the case of heavy cancelations for some values of $|v|$ in the formulae given by (18):

$$\begin{aligned} b_0 &= \frac{73205}{63882} + \frac{268231049v^{10}}{262060331977892448} \\ &\quad - \frac{473946402833v^{12}}{22453329243865824944640\,0} \\ &\quad - \frac{8411469616088504 47v^{14}}{26876635104907392458734080000} \\ &\quad - \frac{347112521555505 20483v^{16}}{2384065040345705340659547832 3200} \\ &\quad - \frac{4501755747002920707034 19v^{18}}{70169739320300080471881097683456 00000} \\ &\quad + \cdots, \\ b_1 &= \frac{51911}{383292} + \frac{268231049v^{10}}{1572361991867354688} \\ &\quad - \frac{61738277963993v^{12}}{134719975463194949667840 0} \\ &\quad - \frac{559868876269811659v^{14}}{1612598106294443547524044 80000} \\ &\quad - \frac{602760578937670659833v^{16}}{35760975605185580109893217484 80000} \\ &\quad - \frac{5264385774221384920342 51v^{18}}{842036871843600965662573172201472 00000} \\ &\quad + \cdots, \\ b_2 &= -\frac{19970}{95823} - \frac{268231049v^{10}}{393090497966838672} \\ &\quad + \frac{15790029293123v^{12}}{33679993865798737416960 0} \\ &\quad + \frac{181541151070183v^{14}}{403149526573610886881011 20} \\ &\quad + \frac{21664254981643052111v^{16}}{894024390129639502747330437 12000} \\ &\quad + \frac{836495539466931839 7937v^{18}}{842036871843600965662573172201472 000} \\ &\quad + \cdots, \\ a_4 &= -2 + \frac{134317v^{16}}{15752663892725760} \\ &\quad + \frac{1170023347v^{18}}{4217775757277322240000} + \cdots. \end{aligned} \tag{19}$$

The behavior of the coefficients is given in Figure 1.

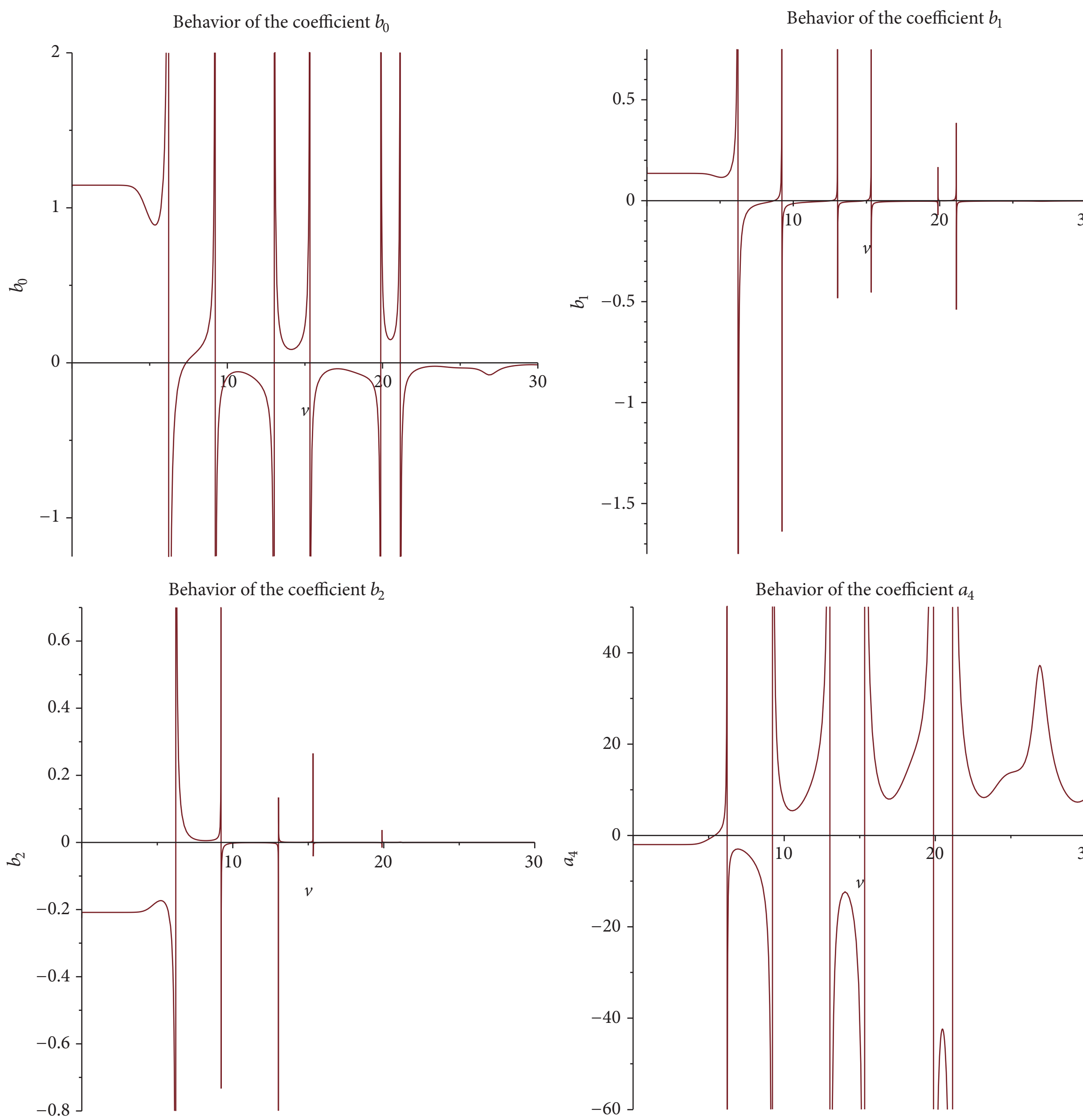


FIGURE 1: Behavior of the coefficients of the new obtained method given by (18) for several values of $v = \phi h$.

The local truncation error of the new developed five-stage two-step method (15) with the coefficients given by (18)-(19), which is indicated as NM2S5S3DV, is given by

$$\mathrm{LTE}_{\mathrm{NM2S5S3DV}} = \frac{134317}{551343236245401600} h^{16} \left( y_n^{(16)} + 56\phi^{10} y_n^{(6)} + 140\phi^{12} y_n^{(4)} + 120\phi^{14} y_n^{(2)} + 35\phi^{16} y_n \right) + O\left(h^{18}\right). \quad (20)$$

## 3. Analysis of the Method

*3.1. Error Analysis.* The test equation used for the local truncation error (LTE) analysis is given by

$$y''(x) = \left(V(x) - V_c + G\right) y(x), \quad (21)$$

where $V(x)$ is a potential function, $V_c$ is a constant value approximation of the potential for the specific $x$, and $G = V_c - E$ and $E$ is the energy. This is the radial time independent Schrödinger equation.

We will study the following methods.

*3.1.1. Classical Method (i.e., Method (15) with Constant Coefficients)*

$$\mathrm{LTE}_{\mathrm{CL}} = \frac{134317}{551343236245401600} h^{16} y_n^{(16)} + O\left(h^{18}\right). \quad (22)$$

*3.1.2. The Five-Stage Two-Step Method with Vanished Phase-Lag and Its First, Second, and Third Derivatives Developed in Section 3*

$$\begin{aligned}\mathrm{LTE}_{\mathrm{NM2S5S3DV}} &= \frac{134317}{551343236245401600} h^{16} \left( y_n^{(16)} \right. \\ &\quad \left. + 56\phi^{10} y_n^{(6)} + 140\phi^{12} y_n^{(4)} + 120\phi^{14} y_n^{(2)} + 35\phi^{16} y_n \right) \\ &\quad + O\left(h^{18}\right).\end{aligned} \tag{23}$$

If we substitute the higher order derivatives requested in the LTE formulae, which are obtained using the test problem (21), into the LTE expressions, we produce new formulae of LTE which have the general form

$$\mathrm{LTE} = h^p \sum_{i=0}^{m} a_i G^i, \tag{24}$$

where $a_i$ are constant numbers (classical methods) or formulae of $\phi$ (fitted methods) and $p$ is the algebraic order of the specific method.

Two cases of the parameter $G$ are studied:

(i) *The Energy and the Potential Are Closed to Each Other.* Consequently, $G = V_c - E \approx 0 \Rightarrow G^i = 0,\ i = 1, 2, \ldots$.

Therefore, the local truncation error for the classical method (constant coefficients) and the local truncation error for the five-stage two-step method with eliminated phase-lag and its first, second, and third derivatives developed in Section 3 are the same since the formulae of the LTE are free from $G$ (i.e., LTE $= h^p a_0$ in (24)) and the free from $G$ terms (i.e., the terms of the formulae which do not have the quantity $G$) in the local truncation errors in this case are the same and are given in Appendix C. From the above it is easy to see that, for these values of $G$, the methods are of comparable accuracy.

(ii) *The Energy and the Potential Have Big Difference.* Consequently, $G \gg 0$ or $G \ll 0$ and $|G|$ is a large number. For this case the most accurate method is the method with the minimum power of $G$ in the formulae of LTE (i.e., the method with the highest accuracy is the method with minimum $i$ in (24)).

We give now the asymptotic expansions of the local truncation errors.

*3.1.3. Classical Method*

$$\begin{aligned}\mathrm{LTE}_{\mathrm{CL}} &= \frac{134317}{551343236245401600} h^{16} \left( y(x) G^8 + \cdots \right) \\ &\quad + O\left(h^{18}\right).\end{aligned} \tag{25}$$

*3.1.4. The Four-Stage Two-Step P-Stable Method with Vanished Phase-Lag and Its First, Second, and Third Derivatives Developed in Section 3*

$$\begin{aligned}\mathrm{LTE}_{\mathrm{NM2S5S3DV}} &= \frac{134317}{9845414932953600} \\ &\quad \cdot h^{16} \left[ \left( 20 g(x) y(x) \frac{d^2}{dx^2} g(x) \right. \right. \\ &\quad + 15 \left( \frac{d}{dx} g(x) \right)^2 y(x) \\ &\quad + 10 \left( \frac{d^3}{dx^3} g(x) \right) \frac{d}{dx} y(x) \\ &\quad \left. \left. + 31 \left( \frac{d^4}{dx^4} g(x) \right) y(x) \right) G^5 + \cdots \right] + O\left(h^{18}\right).\end{aligned} \tag{26}$$

The above leads us to the following theorem.

**Theorem 8.** *(i) Classical method (i.e., method (15) with constant coefficients): for this method the error increases as the eighth power of G.*

*(ii) Fourteenth-algebraic order five-stage two-step method with vanished phase-lag and its first, second, and third derivatives developed in Section 3: for this method the error increases as the fifth power of G.*

*Therefore, for large values of $|G| = |V_c - E|$, the new developed fourteenth-algebraic order five-stage two-step method with vanished phase-lag and its first, second, and third derivatives developed in Section 3 is the most accurate method for the numerical solution of the radial Schrödinger equation.*

*3.2. Stability and Interval of Periodicity Analysis.* Let us define the scalar test equation for the stability and interval of periodicity analysis:

$$y'' = -\omega^2 y. \tag{27}$$

*Remark 9.* Comparing the test equations (9) and (27), we have that the frequency $\phi$ is not equal to the frequency $\omega$; that is, $\omega \neq \phi$.

The difference equation which is produced after application of the new method (15) with the coefficients given by (18)-(19) to the scalar test equation (27) is given by

$$A_1(s, v)\left(y_{n+1} + y_{n-1}\right) + A_0(s, v)\, y_n = 0, \tag{28}$$

where

$$\begin{aligned}A_1(s, v) &= 1 + \frac{1}{8}\left(8b_1 + 4b_2\right) s^2 + \frac{1}{8} b_2 \left(-8a_3 + 1\right) s^4 \\ &\quad + 4a_3 b_2 \left( v^2 a_0 a_2 + \frac{1}{2} a_1 \right) s^6 \\ &\quad - 4a_0 a_1 a_3 b_2 s^8, \\ A_0(s, v) &= a_4 + \left(b_0 + b_2\right) s^2 + 4a_3 b_2 \left( v^2 a_2 + \frac{1}{2} \right) s^4\end{aligned}$$

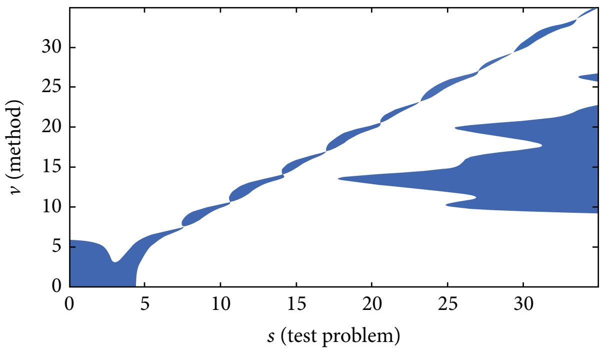


FIGURE 2: $s$-$v$ plane of the new obtained five-stage symmetric two-step fourteenth-algebraic order method with vanished phase-lag and its first, second, and third derivatives.

$$- 8a_3b_2\left(v^2a_0a_2+\frac{1}{2}a_1\right)s^6 + 8a_0a_1a_3b_2s^8, \tag{29}$$

$s=\omega h$, and $v=\phi h$.

Taking into account the coefficients $a_i$, $i = 0,1$, and $b_i$, $i=0(1)2$, and their substitution into formulae (29), the new stability polynomials are produced:

$$A_1(s,v)=\frac{T_8}{T_{\text{denom3}}}, \quad A_0(s,v)=\frac{T_9}{T_{\text{denom3}}}, \tag{30}$$

where the formulae $T_k$, $k=8,9$, and $T_{\text{denom3}}$ are given in Appendix D.

$s$-$v$ plane of the new produced method is shown in Figure 2.

*Remark 10.* Observing $s$-$v$ region we can define two areas:

(i) The shadowed area which defines the area where the method is stable.

(ii) The white area which defines the area where the method is unstable.

For problems of the form of the coupled equations arising from the Schrödinger equation and related problems (in quantum chemistry, material science, theoretical physics, atomic physics, astronomy, astrophysics, physical chemistry, and chemical physics), the area of the region which plays critical role in the stability of the numerical methods is the surroundings of the first diagonal of the $s$-$v$ plane, where $s=v$. Studying this case, we found that the interval of periodicity is equal to $(0,24)$.

The above development leads to the following theorem.

**Theorem 11.** *The five-stage symmetric two-step method developed in Section 3*

(i) *is of fourteenth algebraic order,*

(ii) *has vanished the phase-lag and its first, second, and third derivatives,*

(iii) *has an interval of periodicity equal to* $(0,24)$ *(when* $s=v$*).*

## 4. Numerical Results

We will apply the new produced method to the approximate solution for coupled differential equations of the Schrödinger type.

*4.1. Error Estimation.* For the numerical solution of the previously referred problem we will use variable step methods. An algorithm is called variable step when it is based on the change of the step size of the integration using a local truncation error estimation (LTEE) procedure. In the past decades several methods of constant or variable steps have been developed for the approximation of coupled differential equations of the Schrödinger type and related problems (see, e.g., [1–55]).

In our numerical tests we use local estimation procedure which is based on an embedded pair and on the fact that we have better approximation for the problems with oscillatory and/or periodical solutions having maximal algebraic order of the methods.

The local truncation error in $y_{n+1}^L$ is estimated by

$$\text{LTE}=\left|y_{n+1}^H-y_{n+1}^L\right|, \tag{31}$$

where $y_{n+1}^L$ gives the lower algebraic order solution which is obtained using the twelfth-algebraic order method developed in [54] and $y_{n+1}^H$ gives the higher order solution which is obtained using the five-stage symmetric two-step method of fourteenth algebraic order with vanished phase-lag and its first, second, and third derivatives developed in Section 3.

In our numerical tests the changes of the step sizes are reduced on duplication of step sizes. We use the following procedure:

(i) If LTE $<$ acc then the step size is duplicated; that is, $h_{n+1}=2h_n$.

(ii) If acc $\le$ LTE $\le$ 100 acc then the step size remains stable; that is, $h_{n+1}=h_n$.

(iii) If 100 acc $<$ LTE then the step size is halved and the step is repeated; that is, $h_{n+1}=(1/2)h_n$.

In the above, $h_n$ is the step length used for the $n$th step of the integration and acc is the requested accuracy of the local truncation error (LTE).

*Remark 12.* The local extrapolation technique is also used; that is, while for a local truncation error estimation less than acc we use the lower algebraic order solution $y_{n+1}^L$ it is the higher algebraic order solution $y_{n+1}^H$ which is accepted at each point as integration.

*4.2. Coupled Differential Equations.* We will present the numerical solution of the coupled Schrödinger equations

(1) using the boundary conditions (2) and (3). Mathematical models of this form are observed in many real problems in quantum chemistry, material science, theoretical physics, atomic physics, physical chemistry and chemical physics, and so forth. The methodology fully described in [1] will be used for the approximate solution for this problem.

Equation (1) contains the potential $V$ which can be presented as (see for details [1])

$$V\left(x, \widehat{\mathbf{k}}_{j'j} \widehat{\mathbf{k}}_{jj}\right) = V_0(x) P_0\left(\widehat{\mathbf{k}}_{j'j} \widehat{\mathbf{k}}_{jj}\right) + V_2(x) P_2\left(\widehat{\mathbf{k}}_{j'j} \widehat{\mathbf{k}}_{jj}\right). \tag{32}$$

Based on (32), the coupling matrix element is given by

$$\left\langle j'l'; J \left| V \right| j''l''; J \right\rangle = \delta_{j'j''} \delta_{l'l''} V_0(x) + f_2\left(j'l', j''l''; J\right) V_2(x), \tag{33}$$

where $f_2$ coefficients are obtained from formulas given by Bernstein et al. [56] and $\widehat{\mathbf{k}}_{j'j}$ is a unit vector parallel to the wave vector $\mathbf{k}_{j'j}$ and $P_i$, $i = 0, 2$, are Legendre polynomials (see for details [57]). The above leads to the new form of boundary conditions:

$$y_{j'l'}^{Jjl}(x) = 0 \quad \text{at } x = 0 \tag{34}$$

$$y_{j'l'}^{Jjl}(x) \sim \delta_{jj'} \delta_{ll'} \exp\left[-i\left(k_{jj} x - \frac{1}{2} l\pi\right)\right], \\ -\left(\frac{k_i}{k_j}\right)^{1/2} S^J\left(jl; j'l'\right) \exp\left[i\left(k_{j'j} x - \frac{1}{2} l'\pi\right)\right], \tag{35}$$

where

$$\mathbf{S} = (\mathbf{I} + \mathbf{iK})(\mathbf{I} - \mathbf{iK})^{-1}. \tag{36}$$

We will use the variable step method described in Section 4.1 in order to compute the cross sections for rotational excitation of molecular hydrogen by impact of various heavy particles.

We use the following parameters in our example:

$$\begin{gathered} \frac{2\mu}{\hbar^2} = 1000.0, \\ \frac{\mu}{I} = 2.351, \\ E = 1.1, \\ V_0(x) = \frac{1}{x^{12}} - 2\frac{1}{x^6}, \\ V_2(x) = 0.2283 V_0(x). \end{gathered} \tag{37}$$

For our test we take $J = 6$ and we will consider excitation of the rotator from $j = 0$ state to levels up to $j' = 2, 4$, and 6 which is equivalent to sets of *four, nine, and sixteen coupled differential equations*, respectively. Using the methodology fully described by Bernstein [57] and Allison [1], we consider the potential infinite for $x < x_0$. The wave functions then tend to zero and then the boundary conditions (34) are given by

$$y_{j'l'}^{Jjl}(x_0) = 0. \tag{38}$$

Table 1: Coupled differential equations. Real time of computation (in seconds) (RTC) and maximum absolute error (MErr) to calculate $|S|^2$ for the variable step methods, Method I–Method X. acc $= 10^{-6}$. We note that $h_{max}$ is the maximum step size. $N$ indicates the number of equations of the set of coupled differential equations.

| Method | $N$ | $h_{max}$ | RTC | MErr |
|---|---|---|---|---|
| Method I | 4 | 0.014 | 3.25 | $1.2 \times 10^{-3}$ |
| | 9 | 0.014 | 23.51 | $5.7 \times 10^{-2}$ |
| | 16 | 0.014 | 99.15 | $6.8 \times 10^{-1}$ |
| Method II | 4 | 0.056 | 1.55 | $8.9 \times 10^{-4}$ |
| | 9 | 0.056 | 8.43 | $7.4 \times 10^{-3}$ |
| | 16 | 0.056 | 43.32 | $8.6 \times 10^{-2}$ |
| Method III | 4 | 0.007 | 45.15 | $9.0 \times 10^{0}$ |
| | 9 | | | |
| | 16 | | | |
| Method IV | 4 | 0.112 | 0.39 | $1.1 \times 10^{-5}$ |
| | 9 | 0.112 | 3.48 | $2.8 \times 10^{-4}$ |
| | 16 | 0.112 | 19.31 | $1.3 \times 10^{-3}$ |
| Method V | 4 | 0.448 | 0.14 | $3.4 \times 10^{-7}$ |
| | 9 | 0.448 | 1.37 | $5.8 \times 10^{-7}$ |
| | 16 | 0.448 | 9.58 | $8.2 \times 10^{-7}$ |
| Method VI | 4 | 0.448 | 0.09 | $2.9 \times 10^{-7}$ |
| | 9 | 0.448 | 1.10 | $4.5 \times 10^{-7}$ |
| | 16 | 0.448 | 8.57 | $7.4 \times 10^{-7}$ |
| Method VII | 4 | 0.448 | 0.06 | $1.3 \times 10^{-7}$ |
| | 9 | 0.448 | 1.04 | $1.7 \times 10^{-7}$ |
| | 16 | 0.448 | 7.58 | $2.9 \times 10^{-7}$ |
| Method VIII | 4 | 0.448 | 0.04 | $8.8 \times 10^{-8}$ |
| | 9 | 0.448 | 1.02 | $9.2 \times 10^{-8}$ |
| | 16 | 0.448 | 7.48 | $8.9 \times 10^{-8}$ |
| Method IX | 4 | 0.448 | 0.04 | $9.7 \times 10^{-8}$ |
| | 9 | 0.448 | 1.01 | $1.2 \times 10^{-7}$ |
| | 16 | 0.448 | 7.15 | $2.3 \times 10^{-7}$ |
| Method X | 4 | 0.896 | 0.02 | $7.0 \times 10^{-8}$ |
| | 9 | 0.896 | 0.45 | $5.7 \times 10^{-8}$ |
| | 16 | 0.896 | 6.11 | $6.5 \times 10^{-8}$ |

For comparison purposes the following variable step methods are used:

(i) *Method I*. The iterative Numerov method of Allison [1].

(ii) *Method II*. The variable step method of Raptis and Cash [47].

(iii) *Method III*. The embedded Runge-Kutta Dormand and Prince method 5(4) [49].

(iv) *Method IV*. The embedded Runge-Kutta method ERK4(2) developed in Simos [41].

(v) *Method V*. The embedded symmetric two-step method developed in [50].

(vi) *Method VI*. The embedded symmetric two-step method developed in [52].

(vii) *Method VII*. The embedded symmetric two-step method developed in [52].

(viii) *Method VIII*. The embedded symmetric two-step method developed in [53].

(ix) *Method IX*. The embedded symmetric two-step method developed in [54].

(x) *Method X*. The developed embedded symmetric two-step method developed in this paper.

The results are presented in Table 1. More specifically, we present (1) the requested real time of computation by the variable step methods mentioned above in order to calculate the square of the modulus of **S** matrix for sets of 4, 9, and 6 coupled differential equations and (2) the maximum error in the computation of the square of the modulus of **S** matrix.

## 5. Conclusions

In this paper we introduce, for the first time in the literature, a new five-stage symmetric two-step fourteenth-algebraic order family of methods and we produced a method of the family with vanished phase-lag and its first, second, and third derivatives. In this paper,

(1) we introduced the new family of methods,

(2) we developed the new method of the new family with vanished phase-lag and its first, second, and third derivatives,

(3) we analyzed the local truncation error,

(4) we analyzed the interval of periodicity and the stability of the new developed method,

(5) finally, we analyzed the efficiency of the new obtained method on the numerical solution for coupled Schrödinger equations.

From the analysis presented above, we conclude that the new developed method is much more efficient than the known ones for the numerical solution for the coupled Schrödinger equations.

All computations were carried out on IBM PC-AT compatible 80486 using double precision arithmetic with 16 significant digits' accuracy (IEEE standard).

## Appendix

### A. Formulae of $T_j$, $j = 0(1)3$, and $T_{\text{denom}}$

$$
\begin{aligned}
T_0 = {} & \big(45469 v^8 b_2 + 7038360 b_2 v^6 + 652886640 b_2 v^4 - 265712832000 \\
& + (-265712832000 b_1 - 132856416000 b_2)\, v^2\big) \cos(v) - 45469 v^8 b_2 \\
& + 431033400 b_2 v^6 - 33866990640 b_2 v^4 + (-132856416000 b_0 - 132856416000 b_2)\, v^2 \\
& - 132856416000 a_4, \\
T_1 = {} & -2067429961 \left( -\frac{265712832000}{45469} + v^8 b_2 + \frac{7038360 b_2 v^6}{45469} + \frac{652886640 b_2 v^4}{45469} \right. \\
& \left. + \left( -\frac{265712832000 b_1}{45469} - \frac{132856416000 b_2}{45469} \right) v^2 \right)^2 \sin(v) - 39837369710880 \\
& \cdot \left( v^{12} {b_2}^2 - \frac{4393400 v^{10} {b_2}^2}{28973} - \frac{26360400 v^8 b_2}{28973} \left( b_0 + 2 b_1 + \frac{153226717033 b_2}{4994087615} \right) \right. \\
& - \frac{17669903328000 b_2 v^6}{188196191} \left( b_0 + \frac{45469 a_4}{3519180} - \frac{16290 b_1}{133} - \frac{8012 b_2}{133} + \frac{45469}{1759590} \right) \\
& - \frac{819540618336000 b_2 v^4}{188196191} \left( b_0 + \frac{8379 a_4}{259082} + \frac{13439282 b_1}{129541} + \frac{6849182 b_2}{129541} - \frac{513135}{129541} \right)
\end{aligned}
$$

$$-\frac{1639081236672000v^2b_2}{188196191}\left(a_4+\frac{13439282}{129541}\right)$$

$$+\left(\frac{166768965158400000a_4}{188196191}-\frac{333537930316800000}{188196191}\right)b_2$$

$$+\frac{333537930316800000a_4b_1}{188196191}-\frac{333537930316800000b_0}{188196191}\Bigg)v,$$

$$T_2=-94003972896709\left(-\frac{265712832000}{45469}+v^8b_2+\frac{7038360b_2v^6}{45469}+\frac{652886640b_2v^4}{45469}\right.$$

$$\left.+\left(-\frac{265712832000b_1}{45469}-\frac{132856416000b_2}{45469}\right)v^2\right)^3\cos(v)+5434096090152008160v^{20}{b_2}^3$$

$$-16537464242256682368000v^{18}{b_2}^3-115361960678781505920000{b_2}^2$$

$$\cdot\left(b_0+2b_1+\frac{227930737228b_2}{4994087615}\right)v^{16}-2295953942267723189760000{b_2}^2$$

$$\cdot\left(b_0+\frac{45469a_4}{5278770}-\frac{16290b_1}{133}-\frac{7009130426558198b_2}{125536380378255}+\frac{45469}{2639385}\right)v^{14}$$

$$-19473400902538405380096000{b_2}^2\left(b_0+\frac{6730730601a_4}{290822990464}+\frac{379717697333b_1}{18176436904}\right.$$

$$\left.+\frac{127130652893979671b_2}{15971271578806720}-\frac{356864310411}{145411495232}\right)v^{12}$$

$$-48153927914830003875840000000b_2\left(\frac{541122054664{b_2}^2}{24970438075}\right.$$

$$+\left(\frac{72734780969b_0}{99881752300}+\frac{12770448547a_4}{1198581027600}+\frac{2191635189987b_1}{49940876150}+\frac{63965810599}{599290513800}\right)$$

$$\left.\cdot b_2+b_1\left(b_0+2b_1\right)\right)v^{10}-298158903937735957315584000000000b_2$$

$$\cdot\left(-\frac{1773970067669{b_2}^2}{73624597200}+\left(\frac{90405467881b_0}{147249194400}+\frac{583935539227a_4}{6184466164800}\right.\right.$$

$$\left.-\frac{8073535624019b_1}{73624597200}+\frac{141363876421}{114527151200}\right)b_2-\frac{16290{b_1}^2}{133}+\left(b_0-\frac{45469a_4}{14076720}+\frac{591097}{7038360}\right)b_1$$

$$\left.+\frac{45469b_0}{1005480}\right)v^8-460959572442991450610073600000000b_2\left(\frac{3424591{b_2}^2}{129541}\right.$$

$$+\left(\frac{1}{2}b_0-\frac{12668968199a_4}{682950515280}+\frac{13568823b_1}{129541}-\frac{2869992664019}{341475257640}\right)b_2+\frac{13439282{b_1}^2}{129541}$$

$$\left.+\left(b_0-\frac{11172a_4}{129541}-\frac{2907765}{129541}\right)b_1+\frac{69825b_0}{259082}+\frac{45469a_4}{23317380}+\frac{45469}{11658690}\right)v^6$$

$$-55315148693158974073208832000000b_2\left(\left(\left(-\frac{3}{8}a_4+\frac{7237805}{518164}\right)b_2+\left(-\frac{3}{4}a_4+\frac{6719641}{259082}\right)b_1\right.\right.$$

$$\left.+b_0+\frac{13965a_4}{1036328}-\frac{855225}{518164}\right)v^4+((-140701539050488631027957760000000000a_4$$

$$+281403078100977262055915520000000000)\,b_2{}^2+((-562806156201954524111831040000000000a_4$$

$$+562806156201954524111831040000000000)\,b_1+281403078100977262055915520000000000b_0$$

$$-276575743465794870366044160000000a_4-28693459297029315946323640320000000)\,b_2$$

$$+562806156201954524111831040000000000b_1\,(-a_4b_1+b_0))\,v^2$$

$$+(93801026033659087351971840000000000a_4-187602052067318174703943680000000000)\,b_2$$

$$+187602052067318174703943680000000000a_4b_1-187602052067318174703943680000000000b_0,$$

$$T_3=4274266643640461521\left(-\frac{265712832000}{45469}+v^8b_2+\frac{7038360b_2v^6}{45469}+\frac{652886640b_2v^4}{45469}\right.$$

$$\left.+\left(-\frac{265712832000b_1}{45469}-\frac{132856416000b_2}{45469}\right)v^2\right)^4\sin(v)-9883316604924866361081 60$$

$$\cdot\left(v^{26}b_2{}^4-\frac{703331165780v^{24}b_2{}^4}{1317373337}-\frac{17573600b_2{}^3v^{22}}{4139}\left(b_0+2b_1+\frac{52341891953b_2}{998817523}\right)\right.$$

$$-\frac{229708743264000v^{20}b_2{}^3}{188196191}\left(b_0+\frac{45469a_4}{6099912}-\frac{16290b_1}{133}-\frac{20280808515977707b_2}{362660654426070}+\frac{45469}{3049956}\right)$$

$$-\frac{121875152985790934400 0v^{18}b_2{}^3}{8557092608579}\left(b_0+\frac{8254669605a_4}{385284849578}-\frac{663166829662b_1}{192642424789}\right.$$

$$\left.-\frac{2052773129691092548765b_2}{192414629552462777647}-\frac{394861571517}{192642424789}\right)$$

$$-\frac{11673827561088000000b_2{}^2v^{16}}{188196191}\left(\frac{259818415586440661023 03b_2{}^2}{997636444251855529000}\right.$$

$$+\left(\frac{14436736503248b_0}{20643288160585}+\frac{672095448881a_4}{83900671932000}+\frac{613721035812897 84b_1}{1135380848832175}-\frac{6194383101859}{41950335966000}\right)b_2$$

$$\left.+b_1\,(b_0+2b_1)\right)-\frac{845121609800838881280000000b_2{}^2v^{14}}{8557092608579}\left(-\frac{200253323882994581b_2{}^2}{15064365645390600}\right.$$

$$+\left(\frac{17737276339168999b_0}{30128731290781200}+\frac{4562567732722819a_4}{632703357106405200}-\frac{56067458474270819b_1}{792861349757400}+\frac{170062002887713063}{316351678553202600}\right)b_2$$

$$-\frac{105220{b_1}^2}{1197}+\left(b_0+\frac{45469a_4}{12669048}+\frac{591097}{15836310}\right)b_1+\frac{45469b_0}{3016440}\right)$$

$$-\frac{24444125578632065770546790400000000{b_2}^2v^{12}}{389082443819478551}\left(-\frac{18787681229658547{b_2}^2}{7985635789403360}\right.$$

$$+\left(\frac{8969884197456053b_0}{15971271578806720}+\frac{3422548232166571a_4}{383310517891361280}-\frac{136160476648092927b_1}{7985635789403360}-\frac{280627602486327149}{191655258945680640}\right)b_2$$

$$-\frac{222153379571{b_1}^2}{9088218452}+\left(b_0+\frac{380984751a_4}{72705747616}-\frac{154401947847}{36352873808}\right)b_1$$

$$+\frac{1650933921b_0}{36352873808}+\frac{2067429961a_4}{8724689713920}+\frac{2067429961}{4362344856960}\right)-\frac{4431265402194779258880000000000b_2v^{10}}{8557092608579}$$

$$\cdot\left(\frac{4028587739494373897{b_2}^3}{438820490554820000}+\left(\frac{4349124035232111147b_0}{877640981109640000}+\frac{2926099586132929a_4}{1755281962219280000}\right.\right.$$

$$+\frac{15313712383120673847b_1}{438820490554820000}-\frac{144221396158735377}{877640981109640000}\right){b_2}^2+\left(\frac{15500862502{b_1}^2}{454007965}\right.$$

$$+\left(\frac{72734780969b_0}{49940876150}-\frac{7528278019a_4}{1997635046000}-\frac{539180954813}{499408761500}\right)b_1$$

$$+\frac{98566938253b_0}{1997635046000}+\frac{49343a_4}{87868000}-\frac{100387}{2196700}\right)b_2+{b_1}^2\left(b_0+2b_1\right)\right)$$

$$-\frac{623776823123832930890612736000000000v^8b_2}{389082443819478551}\left(-\frac{665172562469{b_2}^3}{73624597200}\right.$$

$$+\left(\frac{53593169281b_0}{147249194400}-\frac{2554367727834151a_4}{1811388909895488000}-\frac{547986492173b_1}{8180510800}+\frac{267706836641709133}{143004387623328000}\right){b_2}^2$$

$$+\left(-\frac{5855940613619{b_1}^2}{36812298600}+\left(\frac{90405467881b_0}{73624597200}+\frac{2390206957a_4}{309223308240}+\frac{1205462023201}{386529135300}\right)b_1\right.$$

$$+\frac{28440383893b_0}{220873791600}+\frac{12770448547a_4}{7421359397760}-\frac{109828438403}{3710679698880}\right)b_2+\left(-\frac{16290{b_1}^2}{133}+\left(b_0+\frac{45469a_4}{1407672}+\frac{45469}{185220}\right)b_1+\frac{45469b_0}{502740}\right)b_1\right)$$

$$-\frac{1421741999377084654844751052800000000b_2v^6}{389082443819478551}\left(\left(\frac{23281985157a_4}{167808706810}-\frac{2683972235033}{83904353405}\right){b_2}^2\right.$$

$$+\left(\left(\frac{96906582357a_4}{83904353405}-\frac{14553382412876}{83904353405}\right)b_1+b_0+\frac{58393539227a_4}{1761991421505}+\frac{439542930952}{251713060215}\right)b_2$$

$$+\left(\frac{29449838880a_4}{16780870681}-\frac{3607051694400}{16780870681}\right){b_1}^2+\left(-\frac{3995270092a_4}{352398284301}+\frac{51938511196}{352398284301}\right)b_1+\frac{9988175230b_0}{117466094767}\right)$$

$$+\frac{231449118350975844469748269056000000000b_2v^4}{389082443819478551}\left(\left(-\frac{1}{4}a_4+\frac{1}{2}\right){b_2}^2\right.$$

$$+\left(\left(-a_4+1\right)b_1+\frac{1}{2}b_0+\frac{12668968199a_4}{2731802061120}+\frac{430638152579}{1365901030560}\right)b_2$$

$$-a_4{b_1}^2+\left(b_0+\frac{2793a_4}{129541}+\frac{513135}{259082}\right)b_1-\frac{19551b_0}{518164}-\frac{45469a_4}{186539040}-\frac{45469}{93269520}\right)$$

$$\begin{aligned}
&+\Bigg(\bigg(\frac{29436101984019845312296648704000000000000a_4}{389082443819478551}-\frac{5887220396803969062459329740800000000000}{389082443819478551}\bigg)b_2^{\,3}\\
&+\Bigg(\bigg(\frac{1766166119041190718737798922240000000000000a_4}{389082443819478551}-\frac{235488815872158762498373189632000000000000}{389082443819478551}\bigg)b_1\\
&+\frac{66150270245146553335607721984000000000}{9489815702914111}-\frac{5887220396803969062459329740800000000000b_0}{389082443819478551}\\
&+\frac{173586838763231883352311201792000000000a_4}{389082443819478551}\bigg)b_2^{\,2}\\
&+\Bigg(\bigg(\frac{3532332238082381437475597844480000000000000a_4}{389082443819478551}-\frac{235488815872158762498373189632000000000000}{389082443819478551}\bigg)b_1^{\,2}\\
&+\bigg(\frac{6002944955979456984694203875328000000000}{389082443819478551}-\frac{235488815872158762498373189632000000000000b_0}{389082443819478551}\\
&+\frac{347173677526463766704622403584000000000a_4}{389082443819478551}\bigg)b_1+\frac{38200467852207663324090531840000000000}{389082443819478551}\\
&-\frac{31188841156191646544530636800000000000a_4}{389082443819478551}-\frac{289311397938719805587185336320000000000b_0}{389082443819478551}\bigg)b_2\\
&-\frac{235488815872158762498373189632000000000000b_1^{\,2}\left(-a_4b_1+b_0\right)}{389082443819478551}\Bigg)v^2\\
&+\bigg(\frac{11774440793607938124918659481600000000000}{389082443819478551}-\frac{5887220396803969062459329740800000000000a_4}{389082443819478551}\bigg)b_2^{\,2}\\
&+\Bigg(\bigg(\frac{235488815872158762498373189632000000000000}{389082443819478551}-\frac{235488815872158762498373189632000000000000a_4}{389082443819478551}\bigg)b_1\\
&-\frac{6002944955979456984694203875328000000000}{389082443819478551}+\frac{11774440793607938124918659481600000000000b_0}{389082443819478551}\\
&-\frac{578622795877439611174370672640000000000a_4}{389082443819478551}\bigg)b_2+\frac{235488815872158762498373189632000000000000b_1\left(-a_4b_1+b_0\right)}{389082443819478551}\Bigg)v,\\
&T_{\text{denom}}=45469v^8b_2+7038360b_2v^6+652886640b_2v^4-265712832000+\left(-265712832000b_1-132856416000b_2\right)v^2.
\end{aligned}\tag{A.1}$$

## B. Formulae for the Coefficients of the Produced Method $T_j,\ j=4(1)7,\ T_{\text{denom1}}$, and $T_{\text{denom2}}$

$$\begin{aligned}
T_4={}&\left(15277584v^6+844603200v^4\right.\\
&\left.+15669279360v^2\right)(\cos(v))^3+\Big(\big(-2182512v^7\\
&-153643056v^5-4378489920v^3\\
&+15669279360v\big)\sin(v)+363752v^8\\
&-2593839192v^6+148398964560v^4\\
&+672116719680v^2+797138496000\Big)(\cos(v))^2\\
&+\Big(\big(-4365024v^7+20674325616v^5\\
&-490147842240v^3-812807775360v\big)\sin(v)\\
&-30555168v^6-1689206400v^4
\end{aligned}$$

$$
\begin{aligned}
&- 31338558720v^2\Big)\cos(v) + \Big(13095072v^7 \\
&+ 1013523840v^5 + 31338558720v^3\Big)\sin(v) \\
&+ 727504v^8 - 5149484424v^6 + 206280915120v^4 \\
&+ 937829551680v^2 - 797138496000, \\
T_5 &= \Big(-181876v^8 - 17295684v^6 - 1200197880v^4 \\
&+ 128939096160v^2 + 398569248000\Big)(\cos(v))^2 \\
&+ \Big(\Big(-2182512v^7 - 176559432v^5 - 5645394720v^3 \\
&- 7834639680v\Big)\sin(v) - 7638792v^6 \\
&+ 25862004000v^4 - 406403887680v^2\Big)\cos(v) \\
&+ \Big(-1091256v^7 + 5180039592v^5 \\
&- 161329966560v^3 + 406403887680v\Big)\sin(v) \\
&- 363752v^8 - 38410764v^6 - 2294820360v^4 \\
&+ 277464791520v^2 - 398569248000, \\
T_6 &= \Big(-88570944000v^2 - 265712832000\Big)(\cos(v))^2 \\
&- 177141888000v^2 + 265712832000, \\
T_{\text{denom1}} &= v^3\left(v\left(v^6 + \frac{1191127v^4}{45469} + \frac{6478080v^2}{45469}\right.\right. \\
&\left. - \frac{4570206480}{45469}\right)(\cos(v))^2 + \left(\left(16v^6\right.\right. \\
&\left. + \frac{50850600v^4}{45469} + \frac{1487650080v^2}{45469} - \frac{2611546560}{45469}\right) \\
&\cdot \sin(v) + 8v^5 + \frac{1724133600v^3}{45469} \\
&\left. - \frac{135467962560v}{45469}\right)\cos(v) + \left(8v^6\right. \\
&- \frac{1718677320v^4}{45469} + \frac{29638785120v^2}{45469} \\
&\left. + \frac{135467962560}{45469}\right)\sin(v) + 2v^7 + \frac{12521841v^5}{45469} \\
&\left. + \frac{674562000v^3}{45469} + \frac{7181753040v}{45469}\right), \\
T_7 &= \Big(-727504v^5 + 56306880v^3 + 5223093120v\Big) \\
&\cdot (\cos(v))^3 + \Big(\Big(727504v^6 + 67219440v^4 \\
&+ 2247792960v^2 + 5223093120\Big)\sin(v) - 90938v^7 \\
&+ 581713426v^5 - 48439997760v^3 \\
&+ 474137868960v\Big)(\cos(v))^2 + \Big(\Big(1455008v^6 \\
&- 6907446960v^4 + 211658354880v^2 \\
&- 270935925120\Big)\sin(v) + 1455008v^5 \\
&- 112613760v^3 - 10446186240v\Big)\cos(v) \\
&+ \Big(-4365024v^6 - 337841280v^4 \\
&- 10446186240v^2\Big)\sin(v) - 181876v^7 \\
&+ 1141692670v^5 - 22742250720v^3 \\
&- 203201943840v, \\
T_{\text{denom2}} &= \Big(45469v^7 + 1191127v^5 + 6478080v^3 \\
&- 4570206480v\Big)(\cos(v))^2 + \Big(\Big(727504v^6 \\
&+ 50850600v^4 + 1487650080v^2 - 2611546560\Big) \\
&\cdot \sin(v) + 363752v^5 + 1724133600v^3 \\
&- 135467962560v\Big)\cos(v) + \Big(363752v^6 \\
&- 1718677320v^4 + 29638785120v^2 \\
&+ 135467962560\Big)\sin(v) + 90938v^7 + 12521841v^5 \\
&+ 674562000v^3 + 7181753040v.
\end{aligned} \tag{B.1}
$$

## C. Formulae of the Asymptotic Form of the Local Truncation Errors $\mathrm{LTE}_{\mathrm{CL}} = \mathrm{LTE}_{\mathrm{NM2S5S3DV}} = a_0$

Formulae of the asymptotic form of the local truncation errors are as follows:

$$
\begin{aligned}
&\mathrm{LTE}_{\mathrm{CL}} \\
&= \mathrm{LTE}_{\mathrm{NM2S5S3DV}} = a_0 \\
&= \frac{134317\,(g(x))^2\,((d/dx)\,y(x))\left((d^4/dx^4)\,g(x)\right)\left(d^3/dx^3\right)g(x)}{3390294398400} + \frac{14371919 g(x)\,((d/dx)\,y(x))\left((d^3/dx^3)\,g(x)\right)^2 (d/dx)\,g(x)}{153834608327400} \\
&+ \frac{222563269\,(g(x))^2\,y(x)\left((d^7/dx^7)\,g(x)\right)(d/dx)\,g(x)}{22972634843558400} + \frac{3238785821\,(g(x))^2\,y(x)\left((d^6/dx^6)\,g(x)\right)\left(d^2/dx^2\right)g(x)}{137835809061350400} \\
&+ \frac{42041221\,(g(x))^4\,y(x)\left((d^2/dx^2)\,g(x)\right)^2}{13127219910604800} + \frac{134317\left((d^3/dx^3)\,g(x)\right)((d/dx)\,y(x))\left(d^8/dx^8\right)g(x)}{36063790963200} \\
&+ \frac{166687397 g(x)\,y(x)\left((d^5/dx^5)\,g(x)\right)((d/dx)\,g(x))\left(d^2/dx^2\right)g(x)}{1378358090613504} + \frac{757413563\,(g(x))^2\,y(x)\left((d^3/dx^3)\,g(x)\right)((d/dx)\,g(x))\left(d^2/dx^2\right)g(x)}{5301377271590400} \\
&+ \frac{5917066801 g(x)\,y(x)\left((d^4/dx^4)\,g(x)\right)\left((d^3/dx^3)\,g(x)\right)(d/dx)\,g(x)}{34458952265337600} + \frac{5506997\,(g(x))^3\,((d/dx)\,y(x))\left((d^4/dx^4)\,g(x)\right)(d/dx)\,g(x)}{615338433309600} \\
&+ \frac{447409927\,(g(x))^2\,y(x)\left((d^5/dx^5)\,g(x)\right)\left(d^3/dx^3\right)g(x)}{11486317421779200} + \frac{11954213\,(g(x))^2\,y(x)\left(d^{10}/dx^{10}\right)g(x)}{45945269687116800} \\
&+ \frac{6312899 g(x)\,((d/dx)\,y(x))\left(d^{11}/dx^{11}\right)g(x)}{68917904530675200} + \frac{134317\,(g(x))^6\,y(x)\left(d^2/dx^2\right)g(x)}{2187869985100800} \\
&+ \frac{7118801\,(g(x))^3\,((d/dx)\,y(x))\left((d^3/dx^3)\,g(x)\right)\left(d^2/dx^2\right)g(x)}{492270746647680} + \frac{189252653\,((d/dx)\,g(x))^2\,y(x)\left((d^3/dx^3)\,g(x)\right)^2}{2812975695129600} \\
&+ \frac{2552023\,(g(x))^5\,y(x)\,((d/dx)\,g(x))^2}{9845414932953600} + \frac{5506997\,(g(x))^2\,y(x)\,((d/dx)\,g(x))^4}{984541493295360} \\
&+ \frac{68367353\,((d/dx)\,g(x))^2\,((d/dx)\,y(x))\left(d^7/dx^7\right)g(x)}{6265264048243200} + \frac{6312899\,((d/dx)\,g(x))^2\,y(x)\left(d^8/dx^8\right)g(x)}{1790075442355200} \\
&+ \frac{134317\,(g(x))^3\,((d/dx)\,y(x))\,((d/dx)\,g(x))^3}{70324392378240} + \frac{99260263\,((d/dx)\,g(x))^4\,y(x)\left(d^2/dx^2\right)g(x)}{4922707466476800} \\
&+ \frac{4969729\,(g(x))^4\,((d/dx)\,y(x))\left(d^5/dx^5\right)g(x)}{6563609955302400} + \frac{134317\,(g(x))^5\,y(x)\left(d^4/dx^4\right)g(x)}{307669216654800} \\
&+ \frac{3089291\left((d^2/dx^2)\,g(x)\right)y(x)\left((d^4/dx^4)\,g(x)\right)^2}{108191372889600} + \frac{2011665709 g(x)\,y(x)\left((d^3/dx^3)\,g(x)\right)((d/dx)\,g(x))^3}{34458952265337600} \\
&+ \frac{2283389 g(x)\,y(x)\left((d^2/dx^2)\,g(x)\right)\left((d^3/dx^3)\,g(x)\right)^2}{18031895481600} + \frac{686494187 g(x)\,y(x)\left((d^2/dx^2)\,g(x)\right)^2((d/dx)\,g(x))^2}{6265264048243200} \\
&+ \frac{8999239\left((d^2/dx^2)\,g(x)\right)^2((d/dx)\,y(x))\left(d^5/dx^5\right)g(x)}{168297691161600} + \frac{134317\,(g(x))^4\,((d/dx)\,y(x))\,((d/dx)\,g(x))\left(d^2/dx^2\right)g(x)}{70324392378240} \\
&+ \frac{13834651\,(g(x))^4\,y(x)\,((d/dx)\,g(x))\left(d^3/dx^3\right)g(x)}{2812975695129600} + \frac{75620471\left((d^2/dx^2)\,g(x)\right)y(x)\left((d^3/dx^3)\,g(x)\right)\left(d^5/dx^5\right)g(x)}{1514679220454400} \\
&+ \frac{59502431\,((d/dx)\,g(x))\,((d/dx)\,y(x))\left((d^5/dx^5)\,g(x)\right)\left(d^3/dx^3\right)g(x)}{703243923782400} + \frac{6312899\,((d/dx)\,g(x))\,y(x)\left((d^5/dx^5)\,g(x)\right)\left(d^4/dx^4\right)g(x)}{162734131123200} \\
&+ \frac{429411449\,((d/dx)\,g(x))^2\,((d/dx)\,y(x))\left((d^3/dx^3)\,g(x)\right)\left(d^2/dx^2\right)g(x)}{2461353733238400} + \frac{1112010443\,((d/dx)\,g(x))^2\,y(x)\left((d^4/dx^4)\,g(x)\right)\left(d^2/dx^2\right)g(x)}{9845414932953600} \\
&+ \frac{134317\,(g(x))^6\,((d/dx)\,y(x))\,(d/dx)\,g(x)}{9845414932953600} + \frac{29952691 g(x)\,y(x)\left((d^6/dx^6)\,g(x)\right)\left(d^4/dx^4\right)g(x)}{1566316012060800}
\end{aligned}
$$

$$+\frac{134317\left((d/dx)\,g(x)\right)\left((d/dx)\,y(x)\right)\left(d^{10}/dx^{10}\right)g(x)}{255725063193600}+\frac{134317\left(\left(d^{3}/dx^{3}\right)g(x)\right)y(x)\left(d^{9}/dx^{9}\right)g(x)}{233027572377600}+\frac{134317\left(g(x)\right)^{8}y(x)}{551343236245401600}$$

$$+\frac{134317\left((d/dx)\,g(x)\right)^{5}(d/dx)\,y(x)}{44751886058880}+\frac{134317\left(\left(d^{14}/dx^{14}\right)g(x)\right)y(x)}{551343236245401600}+\frac{134317\left(\left(d^{13}/dx^{13}\right)g(x)\right)(d/dx)\,y(x)}{39381659731814400}$$

$$+\frac{134317\left(\left(d^{3}/dx^{3}\right)g(x)\right)^{3}(d/dx)\,y(x)}{4507973870400}+\frac{134317\left(\left(d^{2}/dx^{2}\right)g(x)\right)^{4}y(x)}{8975876861952}+\frac{134317\left(\left(d^{6}/dx^{6}\right)g(x)\right)^{2}y(x)}{183597481267200}$$

$$+\frac{155673403g(x)\,y(x)\left(\left(d^{6}/dx^{6}\right)g(x)\right)\left((d/dx)\,g(x)\right)^{2}}{5301377271590400}+\frac{671585\left(\left(d^{2}/dx^{2}\right)g(x)\right)^{2}y(x)\left(d^{6}/dx^{6}\right)g(x)}{40391445878784}+\frac{134317\left((d/dx)\,g(x)\right)y(x)\left(d^{11}/dx^{11}\right)g(x)}{1458579990067200}$$

$$+\frac{671585\left((d/dx)\,g(x)\right)^{3}\left((d/dx)\,y(x)\right)\left(d^{4}/dx^{4}\right)g(x)}{17900754423552}+\frac{20013233\left((d/dx)\,g(x)\right)^{3}y(x)\left(d^{5}/dx^{5}\right)g(x)}{895037721177600}+\frac{3089291g(x)\,y(x)\left(d^{12}/dx^{12}\right)g(x)}{137835809061350400}$$

$$+\frac{123437323\left(g(x)\right)^{3}y(x)\left(\left(d^{3}/dx^{3}\right)g(x)\right)^{2}}{8614738066334400}+\frac{1581851309\left(g(x)\right)^{2}y(x)\left(\left(d^{4}/dx^{4}\right)g(x)\right)^{2}}{68917904530675200}+\frac{134317\left(g(x)\right)^{5}\left((d/dx)\,y(x)\right)\left(d^{3}/dx^{3}\right)g(x)}{703243923782400}$$

$$+\frac{134317\left(\left(d^{2}/dx^{2}\right)g(x)\right)y(x)\left(d^{10}/dx^{10}\right)g(x)}{504893073484800}+\frac{2552023\left(\left(d^{2}/dx^{2}\right)g(x)\right)\left((d/dx)\,y(x)\right)\left(d^{9}/dx^{9}\right)g(x)}{1514679220454400}$$

$$+\frac{134317\left(\left(d^{5}/dx^{5}\right)g(x)\right)\left((d/dx)\,y(x)\right)\left(d^{6}/dx^{6}\right)g(x)}{17212263868800}+\frac{136869023g(x)\,y(x)\left(\left(d^{5}/dx^{5}\right)g(x)\right)^{2}}{12530528096486400}$$

$$+\frac{134317\left((d/dx)\,g(x)\right)\left((d/dx)\,y(x)\right)\left(\left(d^{4}/dx^{4}\right)g(x)\right)^{2}}{2712235518720}+\frac{59502431\left(g(x)\right)^{3}y(x)\left(d^{8}/dx^{8}\right)g(x)}{68917904530675200}+\frac{2283389\left(g(x)\right)^{3}\left((d/dx)\,y(x)\right)\left(d^{7}/dx^{7}\right)g(x)}{2153684516583600}$$

$$+\frac{2552023\left(\left(d^{5}/dx^{5}\right)g(x)\right)y(x)\left(d^{7}/dx^{7}\right)g(x)}{1927773553305600}+\frac{53861117g(x)\left((d/dx)\,y(x)\right)\left(\left(d^{4}/dx^{4}\right)g(x)\right)\left((d/dx)\,g(x)\right)\left(d^{2}/dx^{2}\right)g(x)}{351621961891200}$$

$$+\frac{279782311\left(g(x)\right)^{4}y(x)\left(d^{6}/dx^{6}\right)g(x)}{275671618122700800}+\frac{134317\left(\left(d^{4}/dx^{4}\right)g(x)\right)y(x)\left(d^{8}/dx^{8}\right)g(x)}{137698110950400}$$

$$+\frac{4163827\left(g(x)\right)^{2}\left((d/dx)\,y(x)\right)\left(d^{9}/dx^{9}\right)g(x)}{7657544947852800}+\frac{165344227g(x)\,y(x)\left(\left(d^{2}/dx^{2}\right)g(x)\right)^{2}\left(d^{4}/dx^{4}\right)g(x)}{1566316012060800}$$

$$+\frac{426187841\left(g(x)\right)^{3}y(x)\left(\left(d^{5}/dx^{5}\right)g(x)\right)(d/dx)\,g(x)}{34458952265337600}+\frac{134317\left((d/dx)\,g(x)\right)\left((d/dx)\,y(x)\right)\left(\left(d^{2}/dx^{2}\right)g(x)\right)^{3}}{1823224987584}$$

$$+\frac{4160200441\left(g(x)\right)^{2}y(x)\left(\left(d^{2}/dx^{2}\right)g(x)\right)^{3}}{137835809061350400}+\frac{134317\left(\left(d^{3}/dx^{3}\right)g(x)\right)^{2}y(x)\left(d^{4}/dx^{4}\right)g(x)}{4161206649600}$$

$$+\frac{134317\left(\left(d^{4}/dx^{4}\right)g(x)\right)\left((d/dx)\,y(x)\right)\left(d^{7}/dx^{7}\right)g(x)}{21906517651200}+\frac{159702913\left(g(x)\right)^{2}\left((d/dx)\,y(x)\right)\left(\left(d^{3}/dx^{3}\right)g(x)\right)\left((d/dx)\,g(x)\right)^{2}}{4922707466476800}$$

$$+\frac{1578090433\left(g(x)\right)^{2}y(x)\left(\left(d^{4}/dx^{4}\right)g(x)\right)\left((d/dx)\,g(x)\right)^{2}}{34458952265337600}+\frac{13566017\left(g(x)\right)^{2}\left((d/dx)\,y(x)\right)\left(\left(d^{6}/dx^{6}\right)g(x)\right)(d/dx)\,g(x)}{1093934992550400}$$

$$+\frac{89589439\left(g(x)\right)^{2}\left((d/dx)\,y(x)\right)\left(\left(d^{5}/dx^{5}\right)g(x)\right)\left(d^{2}/dx^{2}\right)g(x)}{3281804977651200}+\frac{432635057g(x)\,y(x)\left(\left(d^{8}/dx^{8}\right)g(x)\right)\left(d^{2}/dx^{2}\right)g(x)}{68917904530675200}$$

$$+\frac{876955693g(x)\,y(x)\left(\left(d^{7}/dx^{7}\right)g(x)\right)\left(d^{3}/dx^{3}\right)g(x)}{68917904530675200}+\frac{84754027\left(g(x)\right)^{3}y(x)\left((d/dx)\,g(x)\right)^{2}\left(d^{2}/dx^{2}\right)g(x)}{4922707466476800}$$

$$+\frac{6312899g(x)\,y(x)\left(\left(d^{9}/dx^{9}\right)g(x)\right)(d/dx)\,g(x)}{2871579355444800}+\frac{589248679g(x)\left((d/dx)\,y(x)\right)\left(\left(d^{2}/dx^{2}\right)g(x)\right)^{2}\left(d^{3}/dx^{3}\right)g(x)}{4922707466476800}$$

$$+\frac{60308333g(x)\left((d/dx)\,y(x)\right)\left(\left(d^{8}/dx^{8}\right)g(x)\right)(d/dx)\,g(x)}{11486317421779200}+\frac{248352133g(x)\left((d/dx)\,y(x)\right)\left(\left(d^{7}/dx^{7}\right)g(x)\right)\left(d^{2}/dx^{2}\right)g(x)}{17229476132668800}$$

$$+\frac{66218281g(x)\left((d/dx)\,y(x)\right)\left(\left(d^{6}/dx^{6}\right)g(x)\right)\left(d^{3}/dx^{3}\right)g(x)}{2461353733238400}+\frac{50906143g(x)\left((d/dx)\,y(x)\right)\left(\left(d^{2}/dx^{2}\right)g(x)\right)\left((d/dx)\,g(x)\right)^{3}}{1230676866619200}$$

$$+\frac{37743077\,(g(x))^2\,((d/dx)\,y(x))\left((d^2/dx^2)\,g(x)\right)^2(d/dx)\,g(x)}{895037721177600}+\frac{22430939g(x)\,((d/dx)\,y(x))\left((d^5/dx^5)\,g(x)\right)((d/dx)\,g(x))^2}{546967496275200}$$
$$+\frac{41235319\,((d/dx)\,g(x))\,y(x)\left((d^2/dx^2)\,g(x)\right)^2\left(d^3/dx^3\right)g(x)}{262544398212096}+\frac{170716907\,((d/dx)\,g(x))\,((d/dx)\,y(x))\left((d^6/dx^6)\,g(x)\right)\left(d^2/dx^2\right)g(x)}{3281804977651200}$$
$$+\frac{30758593\,((d/dx)\,g(x))\,y(x)\left((d^7/dx^7)\,g(x)\right)\left(d^2/dx^2\right)g(x)}{1837810787484672}+\frac{64337843\,((d/dx)\,g(x))\,y(x)\left((d^6/dx^6)\,g(x)\right)\left(d^3/dx^3\right)g(x)}{2187869985100800}$$
$$+\frac{28340887\left((d^2/dx^2)\,g(x)\right)((d/dx)\,y(x))\left((d^3/dx^3)\,g(x)\right)\left(d^4/dx^4\right)g(x)}{189334902556800}$$
$$+\frac{80455883\,(g(x))^3\,y(x)\left((d^4/dx^4)\,g(x)\right)\left(d^2/dx^2\right)g(x)}{3445895226533760}+\frac{1477487g(x)\,((d/dx)\,y(x))\left((d^5/dx^5)\,g(x)\right)\left(d^4/dx^4\right)g(x)}{40683532780800}. \tag{C.1}$$

## D. Formulae for the Stability Polynomials $T_k,\ k=8,9$, and $T_{\text{denom3}}$

$$\begin{aligned}
T_8 = {} & 159532850484000v^2s^2 \\
& + 26588808414000s^4v^2 \\
& + 285508033854s^6v^2 \\
& - 30236885s^8v^2 + 1881961910s^6v^4 \\
& - 7406900100s^2v^8 \\
& - 766584691740s^2v^6 \\
& - 47158235397000s^2v^4 \\
& + 17776560240\sin(v)\,v^9 \\
& + 17776560240\cos(v)\,v^8 \\
& - 83991760628400\sin(v)\,v^7 \\
& + 84258409032000\cos(v)\,v^6 \\
& + 1448447428814400\sin(v)\,v^5 \\
& - 6620319330307200\cos(v)\,v^4 \\
& + 6620319330307200\sin(v)\,v^3 \\
& + 33124136824800v^6 \\
& + 239299275726000v^4 \\
& + 641047557915v^8 \\
& - 17776560240\sin(v)\,s^2v^7 \\
& - 124435921680\cos(v)\,s^2v^6 \\
& + 84382844953680\sin(v)\,s^2v^5 \\
& + 421292045160000\cos(v)\,s^2v^4 \\
& - 2628065155262400\sin(v)\,s^2v^3 \\
& - 6620319330307200\cos(v)\,s^2v^2 \\
& + 6620319330307200\sin(v)\,s^2v \\
& - 15953285048400s^4 \\
& - 171982326600s^6 \\
& + 5555175075v^{10} \\
& + 18142131s^8 \\
& + 1111035015v^{10}\cos(2v) \\
& - 18142131s^8\cos(2v) \\
& + 29105188245v^8\cos(2v) \\
& + 171982326600s^6\cos(2v) \\
& + 158291884800v^6\cos(2v) \\
& + 15953285048400s^4\cos(2v) \\
& - 111672995338800v^4\cos(2v) \\
& + 17776560240v^9\sin(2v) \\
& + 1242534411000v^7\sin(2v) \\
& + 36350729704800v^5\sin(2v) \\
& - 63813140193600v^3\sin(2v) \\
& - 17776560240s^2v^7\sin(2v) \\
& - 1438076573640s^2v^5\sin(2v)
\end{aligned}$$

$$
\begin{aligned}
&- 45981739994400 s^2 v^3 \sin(2v) \\
&- 63813140193600 s^2 v \sin(2v) \\
&- 6047377 s^8 v^2 \cos(2v) \\
&+ 376392382 s^6 v^4 \cos(2v) \\
&- 1481380020 s^2 v^8 \cos(2v) \\
&+ 58456619346 s^6 v^2 \cos(2v) \\
&- 140873346180 s^2 v^6 \cos(2v) \\
&+ 5317761682800 s^4 v^2 \cos(2v) \\
&- 9775611732600 s^2 v^4 \cos(2v) \\
&- 31906570096800 v^2 s^2 \cos(2v),
\end{aligned}
$$

$$
\begin{aligned}
T_9 = {} & 9930478995460800 v^2 s^2 \\
&- 2779874954539200 s^4 v^2 \\
&- 571016067708 s^6 v^2 \\
&+ 60473770 s^8 v^2 \\
&- 3763923820 s^6 v^4 \\
&+ 14813800200 s^2 v^8 \\
&- 105011921485800 s^2 v^6 \\
&+ 4569025673646000 s^2 v^4 \\
&- 204430442760 \sin(v) v^9 \\
&+ 44441400600 \cos(v) v^8 \\
&- 15689049845400 \sin(v) v^7 \\
&- 3439646532000 \cos(v) v^6 \\
&- 483042711060000 \sin(v) v^5 \\
&- 319065700968000 \cos(v) v^4 \\
&+ 63813140193600 \sin(v) v^3 \\
&- 2295045137952000 v^6 \\
&+ 1655079832576800 v^4 \\
&+ 70008688347210 v^8 \\
&+ 204430442760 \sin(v) s^2 v^7 \\
&- 311089804200 \cos(v) s^2 v^6 \\
&+ 15884592008040 \sin(v) s^2 v^5 \\
&- 17198232660000 \cos(v) s^2 v^4 \\
&+ 492673721349600 \sin(v) s^2 v^3 \\
&- 319065700968000 \cos(v) s^2 v^2 \\
&+ 63813140193600 \sin(v) s^2 v \\
&+ 1655079832576800 s^4 \\
&+ 343964653200 s^6 \\
&- 11110350150 v^{10} \\
&- 36284262 s^8 \\
&- 2222070030 v^{10} \cos(2v) \\
&+ 36284262 s^8 \cos(2v) \\
&+ 14214167564310 v^8 \cos(2v) \\
&- 343964653200 s^6 \cos(2v) \\
&- 1183631345265600 v^6 \cos(2v) \\
&- 1655079832576800 s^4 \cos(2v) \\
&+ 11585558828037600 v^4 \cos(2v) \\
&+ 35553120480 v^9 \sin(2v) \\
&- 168783466467600 v^7 \sin(2v) \\
&+ 5171871901492800 v^5 \sin(2v) \\
&- 6620319330307200 v^3 \sin(2v) \\
&- 35553120480 s^2 v^7 \sin(2v) \\
&+ 168392382142320 s^2 v^5 \sin(2v) \\
&- 3992254175044800 s^2 v^3 \sin(2v) \\
&- 6620319330307200 s^2 v \sin(2v) \\
&+ 12094754 s^8 v^2 \cos(2v) \\
&- 752784764 s^6 v^4 \cos(2v) \\
&+ 2962760040 s^2 v^8 \cos(2v) \\
&- 116913238692 s^6 v^2 \cos(2v) \\
&- 21126820218840 s^2 v^6 \cos(2v) \\
&- 530284710614400 s^4 v^2 \cos(2v) \\
&+ 1208709566341200 s^2 v^4 \cos(2v) \\
&+ 3310159665153600 v^2 s^2 \cos(2v)
\end{aligned}
$$

$$
\begin{aligned}
&+ 35680944852000 s^4 v^4 \\
&+ 3439646532000 s^2 v^4 \cos(3v) \\
&+ 63813140193600 v^2 s^2 \cos(3v) \\
&+ 63813140193600 v^3 \sin(3v) \\
&+ 821253508200 v^7 \sin(3v) \\
&+ 27462410488800 v^5 \sin(3v) \\
&+ 8888280120 v^9 \sin(3v) \\
&+ 63813140193600 v^4 \cos(3v) \\
&- 8888280120 v^8 \cos(3v) \\
&+ 687929306400 v^6 \cos(3v) \\
&- 8888280120 s^2 v^7 \sin(3v) \\
&+ 62217960840 s^2 v^6 \cos(3v) \\
&+ 7136188970400 s^4 v^4 \cos(2v) \\
&- 625711345560 s^2 v^5 \sin(3v) \\
&- 17831400199200 s^2 v^3 \sin(3v) \\
&+ 63813140193600 s^2 v \sin(3v),
\end{aligned}
$$

$$
\begin{aligned}
T_{\text{denom3}}
&= 1111035015 v^{10} \cos(2v) \\
&+ 17776560240 \sin(v) v^9 \\
&+ 17776560240 v^9 \sin(2v) \\
&+ 5555175075 v^{10} \\
&+ 17776560240 \cos(v) v^8 \\
&+ 29105188245 v^8 \cos(2v) \\
&- 83991760628400 \sin(v) v^7 \\
&+ 1242534411000 v^7 \sin(2v) \\
&+ 641047557915 v^8 \\
&+ 84258409032000 \cos(v) v^6 \\
&+ 158291884800 v^6 \cos(2v) \\
&+ 1448447428814400 \sin(v) v^5 \\
&+ 36350729704800 v^5 \sin(2v) \\
&+ 33124136824800 v^6 \\
&- 6620319330307200 \cos(v) v^4 \\
&- 111672995338800 v^4 \cos(2v) \\
&+ 6620319330307200 \sin(v) v^3 \\
&- 63813140193600 v^3 \sin(2v) \\
&+ 239299275726000 v^4.
\end{aligned}
\tag{D.1}
$$

## Additional Points

Theodore E. Simos is highly cited researcher, active member of the European Academy of Sciences and Arts, active member of the European Academy of Sciences, and corresponding member of European Academy of Arts, Sciences, and Humanities.

## Competing Interests

The authors declare that they have no competing interests.

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*Research Article*

# Stability of the Cauchy Additive Functional Equation on Tangle Space and Applications

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We introduce real tangle and its operations, as a generalization of rational tangle and its operations, to enumerating tangles by using the calculus of continued fraction and moreover we study the analytical structure of tangles, knots, and links by using new operations between real tangles which need not have the topological structure. As applications of the analytical structure, we prove the generalized Hyers-Ulam stability of the Cauchy additive functional equation $f(x \oplus y) = f(x) \oplus f(y)$ in tangle space which is a set of real tangles with analytic structure and describe the DNA recombination as the action of some enzymes on tangle space.

## 1. Introduction

In 1970, Conway introduced rational tangles and algebraic tangles for enumerating knots and links by using Conway notation. The rational tangles are defined as the family of tangles that can be transformed into the trivial tangle by sequence of twisting of the endpoints. Given a tangle, two operations, called the numerator and denominator, by connecting the endpoints of the tangle produce knots or 2-component links. To enumerating and classifying knots, the theory of general tangles has been introduced in [1].

Moreover the rational tangles are classified by their fractions by means of the fact that two rational tangles are isotopic if and only if they have the same fraction [1]. This implies the known result that the rational tangles correspond to the rational numbers one to one. It is clear that every rational number can be written as continued fractions with all numerators equal to 1 and that every real number $r$ corresponds to a unique continued fraction, which is finite if $r$ is rational and infinite if $r$ is irrational. Thus the continued fractions give the relationship between the analytical structure and topological structure under a certain restricted operator. See [2], for example. There are some operations that can be performed on tangles as the sum, multiplication, rotation, mirror image, and inverted image.

Topologically, the sum and multiplication on tangles are defined as connecting two endpoints of one tangle to two endpoints of another. However they are not commutative and do not preserve the class of rational tangles. Furthermore the sum and multiplication of two rational tangles are a rational tangle if and only if one of two is an integer tangle [3]. Thus the set of rational tangles is not a group because it was discovered that not all rational tangles form a closed set under the sum and multiplication. Considering a braid of rational tangles, a series of strands that are always descending, the set of braids is a group under braid multiplication.

In 1940, Ulam introduced the stability problem of functional equations during talk before a Mathematical Colloquium at the University of Wisconsin [4]:

Given a group $G_1$, a metric group $(G_2, d)$ and a positive number $\epsilon$, does there exist a number $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$?

Analytically, the stability problem of functional equations originated from a question of Ulam concerning the stability of group homomorphisms. The functional equation

$$f(x + y) = f(x) + f(y) \tag{1}$$

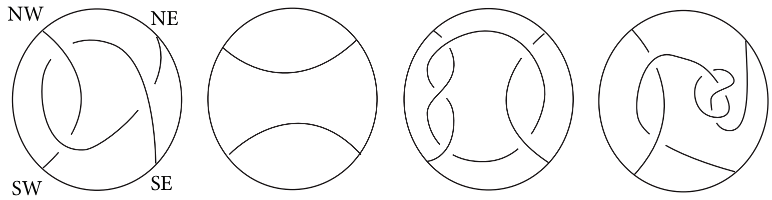


Figure 1: Rational, trivial, prime, locally knotted tangles.

is called the Cauchy additive functional equation. In particular, every solution of the Cauchy additive functional equation is said to be an additive mapping. In [5], Hyers gave the first affirmative partial answer to the question of Ulam for Banach spaces. In [6], Hyers' theorem was generalized by Aoki for additive mappings and by Rassias for linear mappings by considering an unbounded Cauchy difference in [7]. In [8], a generalization of the Rassias theorem was obtained by Găvruţa by replacing the unbounded Cauchy difference by a general control function in Rassias' approach. There are many interesting stability problems of several functional equations that have been extensively investigated by a number of authors. See [9–14].

In recent years, new applications of tangles to the field of molecular biology have been developed. In particular, knot theory gives a nice way to model DNA recombination. The relationship between topology and DNA began in the 1950s with the discovery of the helical Crick-Watson structure of duplex DNA. The mathematical model is the tangle model for site-specific recombination, which was first introduced by Sumners [15]. This model uses knot theory to study enzyme mechanisms. Therefore rational tangles are of fundamental importance for the classification of knots and the study of DNA recombination. In this paper, we introduce new tangles called real tangles to apply the stability problem and DNA recombination on tangles.

In Section 2, we introduce real tangles and operations between tangles which can be performed to make up tangle space and having analytical structure. Moreover we show that the operations together with two real tangles will always generate a real tangle. In Section 3, we prove the Hyers-Ulam stability of the Cauchy additive functional equation in tangle space and study the DNA recombination on real tangles, as applications of knots or links.

## 2. Continued Fractions and Tangle Space

A rational tangle is a proper embedding of two unoriented arcs (strings) $t_1$ and $t_2$ in 3-ball $B^3$ so that the endpoints of the arcs go to a specific set of 4 points on the equator of $B^3$, usually labeled NW, NE, SW, SE. This is equivalent to saying that rational tangles are defined as the family of tangles that can be transformed into the trivial tangle by a sequence of twisting of the endpoints. Note that there are tangles that cannot be obtained in this fashion: they are the prime tangles and locally knotted tangles. For example, see Figure 1.

Geometrically, we have the following operations between rational tangles: the integer (the horizontal) tangles, denoted by $R(n)$, consist in $n$ horizontal twists, $n \in Z$, the mirror image of $R(n)$, denoted by $-R(n)$ or $R(-n)$, is obtained from $R(n)$ by switching all the crossing, and the rotation of $R(n)$, denoted by $R^R(n)$, is obtained by rotation $R(n)$ counterclockwise by $90^\circ$. Moreover the inverse (the vertical) tangle of $R(n)$, denoted by $R(1/n)$, is defined by $-R^R(n)$ or $R^R(-n)$ as the composition of the rotation and mirror of $R(n)$. For example, $R(1/3) = -R^R(3)$, and $R(1/-3) = -R^R(-3)$. For the trivial tangle $R(0)$ we define $R(\infty) = R^R(0)$ or $R(1/0)$.

Generally, every rational tangle can be represented by the continued fractions $C(a_1, a_2, \ldots, a_n)$ as following Conway notation:

$$C(a_1, a_2, \ldots, a_n) = a_1 + \frac{1}{a_2 + \cdots + (1/(a_{(n-1)} + (1/a_n)))} \tag{2}$$

for $a_1 \in Z$, $a_2, \ldots, a_n \in Z - \{0\}$, and $n$ even or odd and we denote it by $R(a_1, a_2, \ldots, a_n)$. See Figure 2.

By Conway [1], rational tangles are classified by fractions by fact of the following: two rational tangles are isotopic if and only if they have the same fraction. For example, $C(2, -2, 3)$ and $C(1, 2, 2)$ represent the same tangles up to isotopy because they have a fraction $7/5$. Therefore the rational tangles $R(a_1, a_2, \ldots, a_n)$ with the exception of $\{R(0), R(\pm 1), R(\infty)\}$ are said to be in canonical form if $|a_n| > 1$, $a_i \neq 0$ for $2 \leq i \leq n$. Note that all nonzero entries have the same sign and every rational tangle has a unique canonical form. The canonical form of the example above is $R(1, 2, 2)$ and the following corollary, which is a direct result of Conway's theorem [1], will give us a means of classifying rational tangles by way of fractions.

**Corollary 1.** *There is a one-to-one correspondence between canonical rational tangles and rational numbers $\beta/\alpha \in Q \cup \{\infty\}$, where $\alpha \in N \cup \{0\}$, $\beta \in Z$, $\gcd(\alpha, \beta) = 1$, $R(\infty) = R(1/0)$.*

Now we define infinite tangles by infinite continued fractions of irrational numbers that the chain of fractions never ends as the following:

$$C(a_1, a_2, a_3, \ldots) = a_1 + \frac{1}{a_2 + (1/(a_3 + (1/(a_4 + \cdots))))}, \tag{3}$$

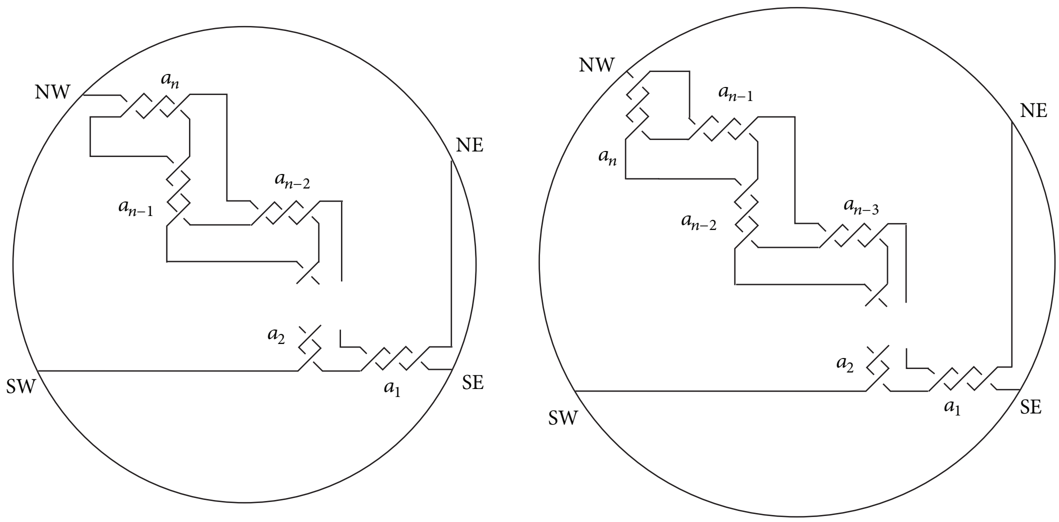


Figure 2: Rational tangles $R(a_1, a_2, \ldots, a_n)$ for $n$ odd or even.

where $a_1$ is allowed to be 0, but all subsequent terms $a_i$ must be positive; that is, $a_1, a_2, \ldots \in Z$, $a_i > 0$ for $i \geq 2$. Note that let $C_i = C(a_1, a_2, \ldots, a_i)$ for $i \geq 1$ and then the limit

$$C(a_1, a_2, a_3, \ldots) = \lim_{i\to\infty} C_i \tag{4}$$

is a unique irrational number and that let $R_i = R(a_1, a_2, \ldots, a_i)$ for $i \geq 1$ be canonical rational tangles and then the infinite tangles, denoted by $R(a_1, a_2, a_3, \ldots)$, are defined by the limit of canonical rational tangles $R_i$ as

$$R(a_1, a_2, a_3, \ldots) = \lim_{i\to\infty} R_i. \tag{5}$$

*Example 2.* Let $C_1 = C(1), C_2 = (1,1), \ldots, C_i = C(1,1,1,\ldots,1)$, and then the limit has an irrational number

$$\lim_{i\to\infty} C_i = C(1,1,1,\ldots) = \frac{1+\sqrt{2}}{2}. \tag{6}$$

**Corollary 3.** *There is a one-to-one correspondence between infinite tangles and irrational numbers.*

Note that $\alpha \in R - Q$ is quadratic irrational if and only if it is of the form

$$\alpha = \frac{a+\sqrt{b}}{c}, \tag{7}$$

where $a, b, c \in Z$, $b > 0$, $c \neq 0$, and $b$ is not the square of a rational number. Thus an irrational number is called quadratic irrational if it is a solution of a quadratic equation $Ax^2 + Bx + D = 0$, where $A, B, D \in Z$ and $A \neq 0$. Moreover $\alpha$ is eventually periodic of the form

$$C\left(a_1, a_2, \ldots, a_N, \overline{a_{N+1}, \ldots, a_{N+p}}\right), \tag{8}$$

where the bar indicates the periodic part with $p$ terms. Thus an infinite tangle is said to be periodic if it has eventually periodic of the form

$$R\left(a_1, a_2, \ldots, a_N, \overline{a_{N+1}, \ldots, a_{N+p}}\right). \tag{9}$$

See Figure 5 for $N = 1$ and $p = 2$.

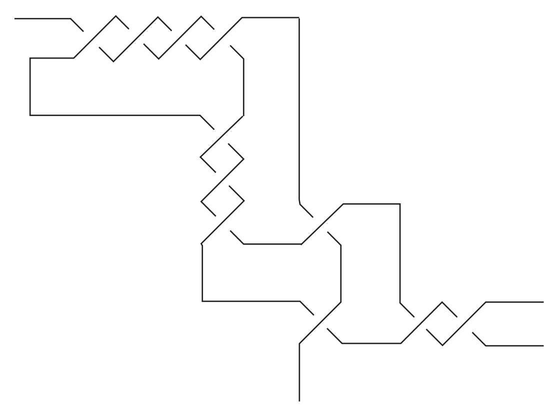

Figure 3: Canonical rational tangle $R(2, 1, 1, 3, 4)$.

**Corollary 4.** *There is a one-to-one correspondence between infinite periodic tangles and quadratic irrational numbers.*

Finally, tangles are said to be real if it is rational tangles or infinite tangles, and so the real tangles are finite if it is rational tangles and infinite if it is infinite tangles. Thus continued fractions of finite real tangles are rational and continued fractions of infinite real tangles are irrational. Moreover infinite real tangle is periodic if it is infinite periodic tangles, and so continued fractions of infinite periodic real tangles are quadratic irrational numbers. Let $r$ be a real number that corresponds to finite or infinite continued fractions. Then, by corollaries, the fact that $R(r)$ has a unique real tangle can be proved. The following is examples of the corollaries above.

*Example 5.* (1) Let $77/30 \in Q$ be a rational number. Then $R(77/30) = R(2,1,1,3,4)$ is a canonical rational tangle of $77/30$. See Figure 3.

(2) Let $e = 2.718281\cdots \in R - Q$ be an irrational number used as the base of the natural logarithm function. Then

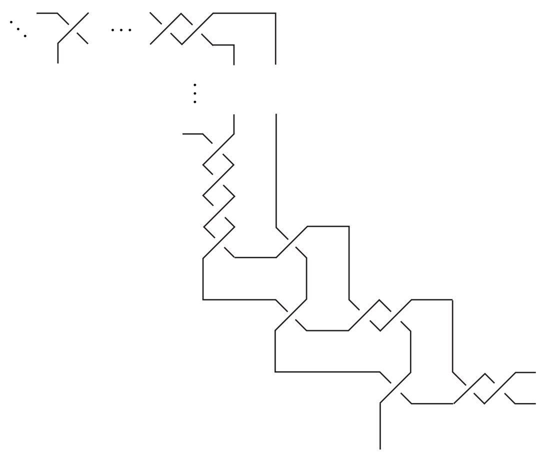

Figure 4: Infinite tangle $R(2, 1, 2, 1, 1, 4, \ldots)$.

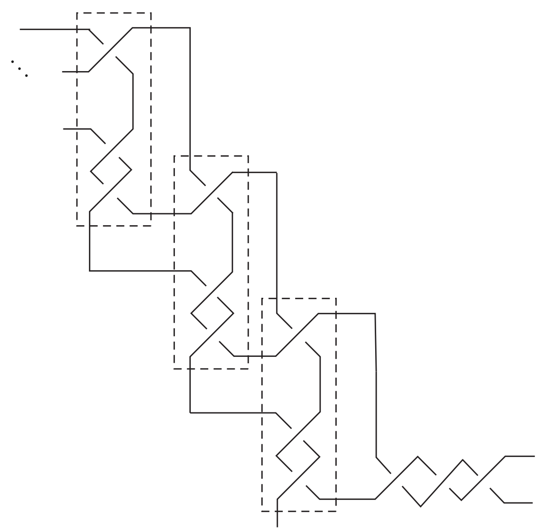

Figure 5: Infinite periodic tangle $R(3, \overline{2, 1})$.

$R(e) = R(2, 1, 2, 1, 1, 4, 1, 1, \ldots)$ is an infinite tangle of $e$. See Figure 4.

(3) Considering a quadratic irrational number $(5+\sqrt{3})/2$, $R((5+\sqrt{3})/2) = R(3, 2, 1, 2, 1, \ldots) = R(3, \overline{2, 1})$ is an infinite periodic tangle of quadratic irrational number $(5+\sqrt{3})/2$. See Figure 5. In Figure 5, the boxes mean periodic parts as $R(\overline{2, 1})$.

Now we introduce the operations on real tangles with analytical structure, which need not have the topological structure. However, on rational tangles, our operations are applicable to geometrical results obtained from topological structure. Our operations need to discuss the generalized Hyers-Ulam stability of the Cauchy additive functional equation $f(x+y) = f(x)+f(y)$ and DNA recombinations in next section.

Let functions $p : R \times R \to R$, defined by $p(r_1, r_2) = r_1 + r_2$, and $m : R \times R \to R$, defined by $m(r_1, r_2) = r_1 \times r_2$ be two binary operators on $R$, and $\phi$ a map from the set of real tangles $T$ to the set of real number $R$ in which tangles $R(a_1, a_2, \ldots, a_n)$ or $R(a_1, a_2, a_3, \ldots)$ are corresponding to the rational numbers or irrational numbers, respectively, one to one. Then for $\phi : T \to R$ and each tangles $t_1, t_2 \in T$, we define a map $\phi^* : T \times T \to R \times R$ by $\phi^*(t_1, t_2) = (\phi(t_1), \phi(t_2))$ and two binary operators $\oplus$ and $\otimes$ on a nonempty set $T$ by

$$\begin{aligned} \oplus &: T \times T \longrightarrow T, \\ \otimes &: T \times T \longrightarrow T, \end{aligned} \tag{10}$$

where

$$\begin{aligned} \oplus(t_1, t_2) &= \phi^{-1} p \phi^* (t_1, t_2), \\ \otimes(t_1, t_2) &= \phi^{-1} m \phi^* (t_1, t_2). \end{aligned} \tag{11}$$

For convenience, we write $t_1 \oplus t_2$ and $t_1 \otimes t_2$ by $\oplus(t_1, t_2)$ and $\otimes(t_1, t_2)$, respectively.

**Lemma 6.** *Let $\phi : T \to R$ be a map from the set of real tangles to the set of real numbers at which real tangles $R(a_1, a_2, \ldots, a_n)$ or $R(a_1, a_2, \ldots)$ are corresponding to the real number $r$, where $r$ has continued fraction $C(a_1, a_2, \ldots, a_n)$ or $C(a_1, a_2, \ldots)$. Then $\phi$ satisfies the following properties: for all $t, t_1, t_2 \in T$,*

$$\begin{aligned} &(1) \quad \phi(-t) = -\phi(t) \\ &(2) \quad \phi\left(-t^R\right) = \frac{1}{\phi(t)} \\ &(3) \quad t \oplus (-t) = R(0) \\ &(4) \quad t \otimes \left(-t^R\right) = R(1) \\ &(5) \quad \phi(t_1 \oplus t_2) = \phi(t_1) + \phi(t_2) \\ &(6) \quad \phi(t_1 \otimes t_2) = \phi(t_1) \times \phi(t_2). \end{aligned} \tag{12}$$

*Proof.* Let $t$ be a real tangle in $T$ and $\phi(t) = r \in R$, continued fraction corresponding to $t$.

(1) Since $-t$ is obtained from $t$ by switching all the crossing, $\phi(-t) = -r$ and so $\phi(-t) = -\phi(t)$.

(2) Since $t^R$ is obtained by rotation $t$ counterclockwise by $90^\circ$, $\phi(t^R) = -1/r$ and so $\phi(-t^R) = 1/r$ by (1). Thus $\phi(-t^R) = 1/\phi(t)$.

(3) $t \oplus (-t) = \phi^{-1} p \phi^*(t, -t) = \phi^{-1} p(\phi(t), \phi(-t)) = \phi^{-1}(\phi(t) + \phi(-t)) = \phi^{-1}(0) = R(0)$ by (1).

(4) $t \otimes (-t^R) = \phi^{-1} m \phi^*(t, -t^R) = \phi^{-1} m(\phi(t), \phi(-t^R)) = \phi^{-1}(\phi(t) \times \phi(-t^R)) = \phi^{-1}(1) = R(1)$ by (2).

(5) $\phi(t_1 \oplus t_2) = \phi(\phi^{-1} p \phi^*(t_1, t_2)) = p(\phi(t_1), \phi(t_2)) = \phi(t_1) + \phi(t_2)$.

(6) $\phi(t_1 \otimes t_2) = \phi(\phi^{-1} m \phi^*(t_1, t_2)) = m(\phi(t_1), \phi(t_2)) = \phi(t_1) \times \phi(t_2)$. ☐

In the following, we show that operators $\oplus$ and $\otimes$ together with two real tangles will always generate a real tangle.

**Theorem 7.** *Let $T$ be the set of real tangles and $\oplus$ the binary operation on $T$. Then $(T, \oplus)$ is a group.*

*Proof.* To show associative of $\oplus$, let $t_1, t_2, t_3 \in T$.

Then

$$(t_1 \oplus t_2) \oplus t_3 = \phi^{-1} p\phi^* \left(\phi^{-1} p\phi^* (t_1, t_2), t_3\right)$$
$$= \phi^{-1} p\phi^* \left(\phi^{-1} (\phi(t_1) + \phi(t_2)), t_3\right)$$
$$= \phi^{-1} ((\phi(t_1) + \phi(t_2)) + \phi(t_3))$$
$$= \phi^{-1} (\phi(t_1) + (\phi(t_2) + \phi(t_3))) \quad (13)$$
$$= \phi^{-1} p\phi^* \left(t_1, \phi^{-1} (\phi(t_2) + \phi(t_3))\right)$$
$$= \phi^{-1} p\phi^* \left(t_1, \phi^{-1} p\phi^* (t_2, t_3)\right)$$
$$= t_1 \oplus (t_2 \oplus t_3).$$

For the remainder, the identity element is the trivial tangle $R(0) \in T$ and the inverse of $t \in T$ is $-t$. See Lemma 6. Thus the set $T$ forms a group with respect to $\oplus$. □

In particular, for $t_1, t_2 \in T$, we write $t_1 \ominus t_2$ for $t_1 \oplus (-t_2)$.

**Theorem 8.** *Let $T$ be the set of real tangles and $\otimes$ the binary operation on $T$. Then $(T, \otimes)$ is a group.*

*Proof.* To show associative of $\otimes$, let $t_1, t_2, t_3 \in T$. Then

$$(t_1 \otimes t_2) \otimes t_3 = \phi^{-1} m\phi^* \left(\phi^{-1} m\phi^* (t_1, t_2), t_3\right)$$
$$= \phi^{-1} m\phi^* \left(\phi^{-1} (\phi(t_1) \times \phi(t_2)), t_3\right)$$
$$= \phi^{-1} ((\phi(t_1) \times \phi(t_2)) \times \phi(t_3))$$
$$= \phi^{-1} (\phi(t_1) \times (\phi(t_2) \times \phi(t_3))) \quad (14)$$
$$= \phi^{-1} m\phi^* \left(t_1, \phi^{-1} (\phi(t_2) \times \phi(t_3))\right)$$
$$= \phi^{-1} m\phi^* \left(t_1, \phi^{-1} m\phi^* (t_2, t_3)\right)$$
$$= t_1 \otimes (t_2 \otimes t_3).$$

For the remainder, the identity element is the integer tangle $R(1) \in T$ and the inverse of $t \in T$ is $-t^R$ and denoted as $t^{-1}$, where $t^R$ means rotation counterclockwise by 90°. See Lemma 6. Thus the set $T$ forms a group with respect to $\otimes$. □

**Corollary 9.** *$(T, \oplus)$ and $(T, \otimes)$ are abelian groups.*

For the symbolization, we write $nt$ for $t \oplus t \oplus \cdots \oplus t$($n$ summands) and write $t^n$ for $t \otimes t \otimes \cdots \otimes t$($n$ products). Note that, by distributive law between two operations, we have

$$(t_1 \oplus t_2) \otimes (t_3 \oplus t_4) \Longleftrightarrow$$
$$(t_1 \otimes t_3) \oplus (t_1 \otimes t_4) \oplus (t_2 \otimes t_3) \oplus (t_2 \otimes t_4). \quad (15)$$

For $t_1, t_2, t \in T$ and $r \in R$, let $\oplus : T \times T \to T$ and $\cdot : R \times T \to T$ be addition and scalar multiplication operators, respectively, defined by $\oplus(t_1, t_2) = t_1 \oplus t_2$ and $\cdot(r, t) = \phi^{-1} m(r, \phi(t)) \in T$, denoted by $r \cdot t$, where $(\cdot, \cdot) : R \times T \to R \times R$ is a map. Then the set $T$ of the real tangles with addition and scalar multiplication operators satisfy the following result.

**Theorem 10.** *$(T, \oplus, \cdot)$ is a vector space.*

*Proof.* By Theorem 7, $(T, \oplus)$ is a group. Moreover, we have the following properties:

$$(1) \quad r \cdot (t_1 \oplus t_2) = \phi^{-1} m(r, \phi(t_1 \oplus t_2))$$
$$= \phi^{-1} m\left(r, \phi\phi^{-1} p\phi^* (t_1, t_2)\right)$$
$$= \phi^{-1} m(r, \phi(t_1) + \phi(t_2))$$
$$= \phi^{-1} (r \times (\phi(t_1) + \phi(t_2)))$$
$$= \phi^{-1} (r \times \phi(t_1) + r \times \phi(t_2)) \quad (16)$$
$$= \phi^{-1} p\left(\phi\phi^{-1} m(r, \phi(t_1)), \phi\phi^{-1} m(r, \phi(t_2))\right)$$
$$= \phi^{-1} p(\phi(r \cdot t_1), \phi(r \cdot t_2))$$
$$= \phi^{-1} p\phi^* (r \cdot t_1, r \cdot t_2)$$
$$= (r \cdot t_1) \oplus (r \cdot t_2).$$

Similarly,

$$(2) \quad (r_1 + r_2) \cdot t = (r_1 \cdot t) \oplus (r_2 \cdot t),$$
$$(3) \quad r_1 \cdot (r_2 \cdot t) = (r_1 r_2) \cdot t, \quad (17)$$
$$(4) \quad 1 \cdot t = t$$

hold. Thus $(T, \oplus, \cdot)$ is a vector space over $R$. □

In particular, we write $r(t_1 \oplus t_2)$ for $r \cdot (t_1 \oplus t_2)$; that is, the operator $\cdot$ is often omitted.

*Remark 11.* We have the following relations, which are proved from the above facts

$$(1) \quad -(t_1 \oplus t_2) = -t_1 \oplus (-t_2),$$
$$(2) \quad -(t_1 \ominus t_2) = (-t_1 \oplus t_2), \quad (18)$$
$$(3) \quad t_1 \ominus t_2 = -t_2 \oplus t_1.$$

If we define a function $D : T \times T \to R^+$ as $D(t_1, t_2) = |\phi(t_1) - \phi(t_2)|$ for each $t_1, t_1 \in T$, then $(T, D)$ is a metric space from the following theorem.

**Theorem 12.** *$(T, D)$ is a metric space.*

*Proof.* Let $t_1, t_2, t_3 \in T$. Then

$$(1) \quad D(t_1, t_2) = 0 \Longleftrightarrow$$
$$|\phi(t_1) - \phi(t_2)| = 0 \Longleftrightarrow$$
$$\phi(t_1) = \phi(t_2) \Longleftrightarrow$$
$$t_1 = t_2,$$
$$(2) \quad D(t_1, t_2) = |\phi(t_1) - \phi(t_2)| = |\phi(t_2) - \phi(t_1)|$$
$$= D(t_2, t_1),$$

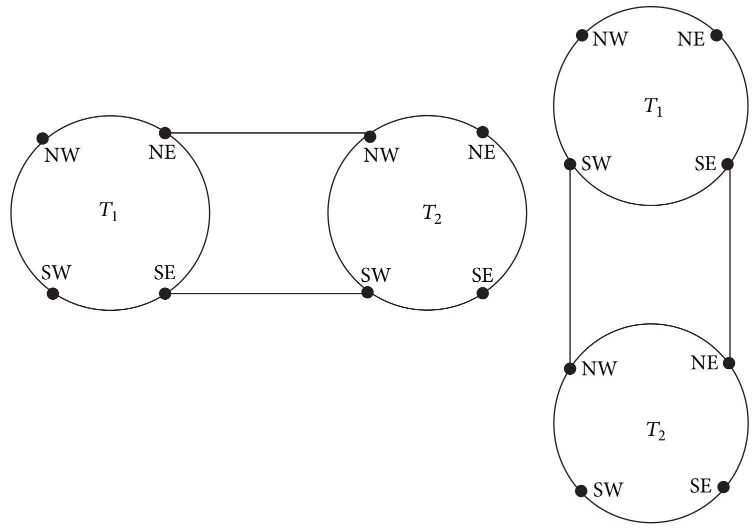


Figure 6: Addition and multiplication of tangles.

$$\begin{aligned}(3)\quad D(t_1,t_2) &= |\phi(t_1)-\phi(t_2)| \\ &= |\phi(t_1)-\phi(t_3)+\phi(t_3)-\phi(t_2)| \\ &\le |\phi(t_1)-\phi(t_3)| + |\phi(t_3)-\phi(t_2)| \\ &\le D(t_1,t_3)+D(t_3,t_2).\end{aligned} \tag{19}$$

Thus $(T, D)$ is a metric space. □

Define the norm of $t$ by $\|t\| = D(R(0), t)$. Then we have that $\|t\| = |\phi(t)|$ and $D(t_1, t_2) = |\phi(t_1) - \phi(t_2)| = |\phi(t_1) + \phi(-t_2)| = |\phi(t_1 \oplus (-t_2))| = \|t_1 \oplus (-t_2)\| = \|t_1 \ominus t_2\|$.

*Remark 13.* We have the following relations:

$$\begin{aligned}(1)\quad \|t_1\otimes t_2\| &= D(R(0), t_1\otimes t_2) = |0-\phi(t_1\otimes t_2)| \\ &= |\phi(t_1)\times\phi(t_2)| \\ &= |\phi(t_1)|\times|\phi(t_2)| \\ &= \|t_1\|\times\|t_2\|, \\ (2)\quad \|t_1\oplus t_2\| &= D(R(0), t_1\oplus t_2) = |0-\phi(t_1\oplus t_2)| \\ &= |\phi(t_1)+\phi(t_2)| \\ &\le |\phi(t_1)|+|\phi(t_2)| \\ &= \|t_1\|+\|t_2\|.\end{aligned} \tag{20}$$

Define an inequality $\prec$ (resp. $\preccurlyeq$) of $t_1, t_2$ on $T$ as $t_1 \prec t_2$ (resp. $t_1 \preccurlyeq t_2$) $\Leftrightarrow \phi(t_1) < \phi(t_2)$ (resp. $\phi(t_1) \le \phi(t_2)$).

*Remark 14.* (1) For each $\epsilon > 0$, $\|t\| < \epsilon$ means that $-t_\epsilon \prec t \prec t_\epsilon$ for some $\phi(t_\epsilon) = \epsilon$.

(2) For each $\epsilon > 0$, we have that

$$\begin{aligned}\|t_1\ominus t_2\| < \epsilon &\Longleftrightarrow \\ -t_\epsilon \prec t_1\ominus t_2 \prec t_\epsilon &\Longleftrightarrow \\ -t_\epsilon\oplus t_2 \prec t_1 \prec t_\epsilon\oplus t_2 &\Longleftrightarrow \\ t_2\ominus t_\epsilon \prec t_1 \prec t_2\oplus t_\epsilon,&\end{aligned} \tag{21}$$

for some $\phi(t_\epsilon) = \epsilon$.

In order to determine a group (generally, a vector space) from the set of rational tangles (generally, real tangles), two binary operators $\oplus$ and $\otimes$ are necessary. For other operators, restricted on rational tangles, addition (denote by #) and multiplication (denote by $*$) of horizontal and vertical rational tangles are considered in [1]. In detail, the multiplication of two rational tangles is defined as connecting the top two ends of one tangle to the bottom two endpoints of another, and the addition of two rational tangles is defined as connecting the two leftmost endpoints of one tangle with the two rightmost points of the other as shown in Figure 6.

However the addition of two rational tangles is not necessarily rational, but it can be algebraic tangle [1]. For example, it can be easily seen that the sum of $R(1/2)$ and $R(1/2)$ is not a rational tangle. As the results, in [3], the multiplication (resp., addition) of two rational tangles will be rational tangle if one of two is a vertical (resp., horizontal) tangle. Note that, as a special case of rational tangles, the set of braids is a group under the multiplication. Therefore two operators $\oplus$ and $\otimes$ on rational tangles are a generalization of operators # and $*$ introduced in [1]. For two operators $\oplus$ and $\otimes$ on real tangles, we do not know yet whether it has a topological or geometrical structure.

In Section 3, we will study some applications for two operators $\oplus$ and $\otimes$ on real tangles. In this paper, the set $(T, D)$ of the real tangles with a metric $D$ is called the tangle space.

## 3. Some Applications on Tangle Space

*3.1. Tangle Space and Stability.* Let $T$ be the tangle space and $f : T \to T$ a mapping. Then we prove the generalized Hyers-Ulam stability of the Cauchy additive functional equation as follows.

**Theorem 15.** *Let $f : T \to T$ be a mapping such that*

$$\|f(x \oplus y) \ominus (f(x) \oplus f(y))\| < \epsilon \tag{22}$$

*for all $x, y \in T$ and for some $\epsilon > 0$. Then there exists a unique additive mapping $Q : T \to T$ such that $\|f(x) \ominus Q(x)\| < \epsilon$ for all $x \in T$.*

*Proof.* Suppose that $f : T \to T$ is a mapping such that

$$\|f(x \oplus y) \ominus (f(x) \oplus f(y))\| < \epsilon \tag{23}$$

for all $x, y \in T$ and for some $\epsilon > 0$. Then we have

$$\begin{gathered}\|f(x \oplus y) \ominus (f(x) \oplus f(y))\| < \epsilon \Longrightarrow \\ f(x) \oplus f(y) \ominus a_\epsilon \prec f(x \oplus y) \prec f(x) \oplus f(y) \oplus a_\epsilon,\end{gathered} \tag{I}$$

for some $\phi(a_\epsilon) = \epsilon$.

(1) Putting $x = y$ in (I),

$$2f(x) \ominus a_\epsilon \prec f(2x) \prec 2f(x) \oplus a_\epsilon \Longrightarrow \tag{II}$$

$$\begin{gathered}f(x) \ominus \frac{a_\epsilon}{2} \prec \frac{f(2x)}{2} \prec f(x) \oplus \frac{a_\epsilon}{2} \Longrightarrow \\ \left\|\frac{f(2x)}{2} \ominus f(x)\right\| < \frac{\epsilon}{2}.\end{gathered} \tag{III}$$

(2) Putting $x = 2x$ in (II),

$$\begin{aligned}&2f(2x) \ominus a_\epsilon \prec f(4x) \prec 2f(2x) \oplus a_\epsilon \Longrightarrow \\ &\frac{f(2x)}{2} \ominus \frac{a_\epsilon}{4} \prec \frac{f(4x)}{4} \prec \frac{f(2x)}{2} \oplus \frac{a_\epsilon}{4} \Longrightarrow \\ &f(x) \ominus \frac{a_\epsilon}{2} \ominus \frac{a_\epsilon}{4} \prec \frac{f(4x)}{4} \\ &\quad \prec f(x) \oplus \frac{a_\epsilon}{2} \oplus \frac{a_\epsilon}{4} \ (\because \text{by (III)}) \Longrightarrow \\ &f(x) \ominus \frac{3}{4}a_\epsilon \prec \frac{f(4x)}{4} \prec f(x) \oplus \frac{3}{4}a_\epsilon \Longrightarrow \\ &\left\|\frac{f(4x)}{4} \ominus f(x)\right\| < \frac{3}{4}\epsilon.\end{aligned} \tag{IV}$$

(3) Putting $x = 4x$ in (II),

$$\begin{aligned}&2f(4x) \ominus a_\epsilon \prec f(8x) \prec 2f(4x) \oplus a_\epsilon \Longrightarrow \\ &\frac{f(4x)}{4} \ominus \frac{a_\epsilon}{8} \prec \frac{f(8x)}{8} \prec \frac{f(4x)}{4} \oplus \frac{a_\epsilon}{8} \Longrightarrow \\ &f(x) \ominus \frac{a_\epsilon}{2} \ominus \frac{a_\epsilon}{4} \ominus \frac{a_\epsilon}{8} \prec \frac{f(8x)}{8} \\ &\quad \prec f(x) \oplus \frac{a_\epsilon}{2} \oplus \frac{a_\epsilon}{4} \\ &\quad \oplus \frac{a_\epsilon}{8} \ (\because \text{by (IV)}) \Longrightarrow \\ &f(x) \ominus \frac{7}{8}a_\epsilon \prec \frac{f(8x)}{8} \prec f(x) \oplus \frac{7}{8}a_\epsilon \Longrightarrow \\ &\left\|\frac{f(8x)}{8} \ominus f(x)\right\| < \frac{7}{8}\epsilon.\end{aligned} \tag{24}$$

Putting recursively $x = (2^{n-1})x$ in (II), we have

$$\left\|\frac{f(2^n x)}{2^n} \ominus f(x)\right\| < \frac{2^n - 1}{2^n}\epsilon < \epsilon, \tag{V}$$

where $n \geq 1$.

Define $Q : T \to T$ by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{25}$$

for all $x \in T$; that is, $\lim_{n\to\infty}\|f(2^n x)/2^n \ominus Q(x)\| = 0$.

Now putting $x = 2^n x$ and $y = 2^n y$ in $\|f(x \oplus y) \ominus f(x) \ominus f(y)\| < \epsilon$, we have

$$\begin{aligned}&\|f(2^n x \oplus 2^n y) \ominus f(2^n x) \ominus f(2^n y)\| < \epsilon \Longrightarrow \\ &\left\|\frac{f(2^n(x \oplus y))}{2^n} \ominus \frac{f(2^n x)}{2^n} \ominus \frac{f(2^n y)}{2^n}\right\| < \frac{\epsilon}{2^n} \Longrightarrow \\ &\left\|\lim_{n\to\infty}\frac{f(2^n(x \oplus y))}{2^n} \ominus \lim_{n\to\infty}\frac{f(2^n x)}{2^n} \ominus \lim_{n\to\infty}\frac{f(2^n y)}{2^n}\right\| \\ &\quad < \lim_{n\to\infty}\frac{\epsilon}{2^n} = 0.\end{aligned} \tag{26}$$

Thus $\|Q(x \oplus y) \ominus Q(x) \ominus Q(y)\| = 0$; that is, $Q(x \oplus y) = Q(x) \oplus Q(y)$. Moreover, from (V),

$$\left\|\lim_{n\to\infty}\frac{f(2^n x)}{2^n} \ominus \lim_{n\to\infty} f(x)\right\| < \lim_{n\to\infty}\left(\frac{2^n - 1}{2^n}\right)\epsilon = \epsilon. \tag{27}$$

Thus we have $\|Q(x) \ominus f(x)\| < \epsilon$. Therefore this means that $Q$ is an additive mapping such that $\|Q(x) \ominus f(x)\| < \epsilon$.

To prove the uniqueness of the additive mapping $Q$, assume that there is another additive mapping $Q' : T \to T$ such that

$$\begin{gathered}Q'(x \oplus y) = Q'(x) \oplus Q'(y), \\ \|f(x) \ominus Q'(x)\| < \epsilon.\end{gathered} \tag{28}$$

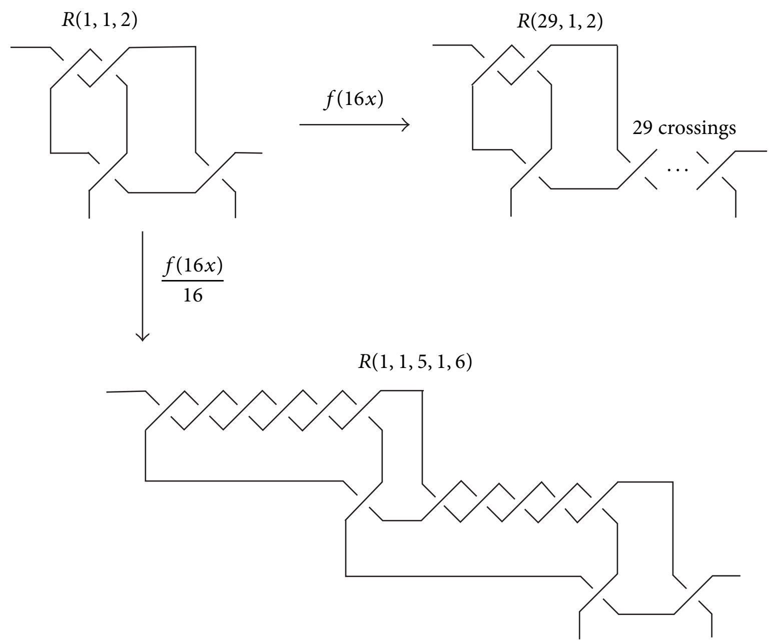


Figure 7

From the fact $Q'(2^n x) = 2^n Q'(x)$, we have $Q'(x) = Q'(2^n x)/2^n$. Since $\|f(x) \ominus Q'(x)\| < \epsilon$, we have that

$$\begin{aligned}
& Q'(x) \ominus a_\epsilon \prec f(x) \prec Q'(x) \oplus a_\epsilon \Longrightarrow \\
& Q'(2^n x) \ominus a_\epsilon \prec f(2^n x) \prec Q'(2^n x) \oplus a_\epsilon \Longrightarrow \\
& \frac{Q'(2^n x)}{2^n} \ominus \frac{a_\epsilon}{2^n} \prec \frac{f(2^n x)}{2^n} \prec \frac{Q'(2^n x)}{2^n} \oplus \frac{a_\epsilon}{2^n} \Longrightarrow \\
& Q'(x) \ominus \frac{a_\epsilon}{2^n} \prec \frac{f(2^n x)}{2^n} \prec Q'(x) \oplus \frac{a_\epsilon}{2^n} \Longrightarrow \\
& \left\| \frac{f(2^n x)}{2^n} \ominus Q'(x) \right\| < \frac{\epsilon}{2^n} \Longrightarrow \\
& \left\| \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \ominus \lim_{n\to\infty} Q'(x) \right\| < \lim_{n\to\infty} \frac{\epsilon}{2^n} = 0,
\end{aligned} \tag{29}$$

where $\epsilon > 0$ and $\phi(a_\epsilon) = \epsilon$. Thus we have $\|Q(x) \ominus Q'(x)\| = 0$; that is, $Q(x) = Q'(x)$. This completes the proof. □

For example, let $f : T \to T$ be a mapping defined by $f(x) = r_1 \cdot x \oplus r_2$, where $r_1, r_2 \in R$, $x \in T$. In fact, $r_2$ means $r_2 \cdot R(1)$. Then $f$ is not additive mapping. However a mapping $f : T \to T$ defined by $f(x) = r \cdot x$, $r \in R$ and $x \in T$ is additive. In tangle space $T$, let $x = R(1,1,2)$, $f(x) = x \oplus 3$, and $n = 4$; then $f(16x) = R(29,1,2)$ and $f(16x)/16 = R(1,1,5,1,6)$, where $16x = R(26,1,2)$. See Figure 7.

Generally, for each $n$, the real tangles are as the following:

$$\frac{f(2^n x)}{2^n} = R\left(\frac{14 + \sum_{k=1}^{n} 52^{k-1}}{2^n 3}\right) \tag{30}$$

and so the additive mapping $Q(x)$ is real tangle as the following:

$$Q(x) = \lim_{n\to\infty} R\left(\frac{14 + \sum_{k=1}^{n} 52^{k-1}}{2^n 3}\right). \tag{31}$$

*3.2. Tangle Space and Rational Knot or Link.* Suppose that $T' \subset T$ is the set of rational tangles. Given $t \in T'$, the numerator closure $N(t)$ is formed by connecting the NW and NE endpoints and the SW and SE endpoints, and the denominator closure $D(t)$ is formed by connecting the NW and SW endpoints and connecting the NE and SE endpoints. We note that two operations $N(t)$ and $D(t)$ by connecting the endpoints of $t$ produce knots or 2-component links, called rational knot or link if $t$ is a rational tangle, and that every 2-bridge knot is a rational knot because it can be obtained as the numerator or denominator closure of a rational tangle. See Figure 8.

Let $T' \subset T$ be the set of rational tangles and $K$ the set of rational knots or links. Then for given $t \in T'$, it allows defining a function $N : T' \to K$ in order that $N(t)$ is the numerator closure. The following theorem discusses equivalence of rational knots or links obtained by taking the numerator closure of rational tangles. We call this theorem the tangle classification theorem

**Theorem 16** (see [16]). *Let $R(p/q)$ and $R(p'/q')$ be the rational tangles with reduced fractions $p/q$ and $p'/q'$, respectively. Then $N(R(p/q))$ and $N(R(p'/q'))$ are topologically equivalent if and only if $p = p'$ and $q^{\pm} \equiv q' \bmod p$.*

For example, $N(R(0,3,2)) = N(R(2/7)) = N(R(2/1)) = N(R(2))$ because $7 \equiv 1 \bmod 2$.

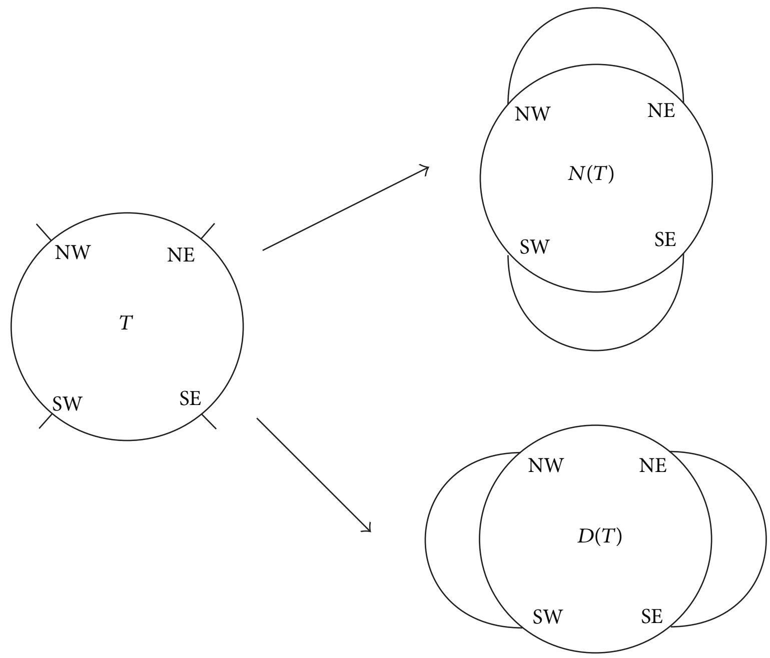


Figure 8: The numerator closure and denominator closure of tangle.

**Corollary 17.** *If two rational tangles are isotopic, then their each numerator's closures are topological equivalent.*

*Proof.* Let $R(p/q)$ and $R(p'/q')$ be the rational tangles with reduced fractions $p/q$ and $p'/q'$, respectively. Then $p = p'$ and $q = q'$ because $R(p/q)$ and $R(p'/q')$ are isotopic.

Thus, by Theorem 16, $N(R(p/q))$ and $N(R(p'/q'))$ are topological equivalent. □

However there is a counterexample for the converse of Corollary 17 as follows.

*Example 18.* Let $R(1,2,2)$ and $R(2,3)$ be two rational tangles with fractions 7/5 and 7/3, respectively. By Theorem 16, $N(R(7/5))$ and $N(R(7/3))$ are topological equivalent, but two tangles $R(7/5)$ and $R(7/3)$ are not isotopic.

Define the numerator closure of the sum of two rational tangles as the following:

$$N\left(R\left(\frac{y_1}{x_1}\right)\oplus R\left(\frac{y_2}{x_2}\right)\right)=N\left(R\left(\frac{x_2y_1+x_1y_2}{x_1x_2}\right)\right), \quad (32)$$

where $\gcd(x_1,y_1) = \gcd(x_2,y_2) = 1$. Note that the rational knot or link

$$N\left(R\left(\frac{x_2y_1+x_1y_2}{x_1x_2}\right)\right) \quad (33)$$

is denoted by $b(x_2y_1+x_1y_2, x_1x_2)$, called the 2-bridge knot or link, and that $b(x_2y_1+x_1y_2, x_1x_2)$ is to be the 2-bridge knot if $x_2y_1+x_1y_2$ is odd number and the 2-bridge link if not.

A tangle equation is an equation of the form $N(A\oplus B) = K$, where $A, B \in T'$ and $K \in K$. Solving equations of this type will be useful in the tangle model and gaining a better understanding of certain enzyme mechanisms [15].

*Example 19.* Considering rational tangles $R(2)$ and $R(23/17)$, then $R(2)\oplus R(23/17)$ is the rational tangle $R(3,2,1,5)$ because of $R(23/17) = R(1,2,1,5)$. Thus a tangle equation $N(R(2)\oplus R(23/17)) = N(R(3,2,1,5))$ is representing the 2-bridge knot $b(57,17)$ from the computation of the numerator closure above.

If one of the tangles in the equation is unknown and the other tangle and the knot $K$ are known, then there is one tangle as the solution of equation, but it is not unique. In fact, let $A$ be known rational tangle and $K$ rational knot or link. Then there are two different rational tangles as the solution of the equation $N(X\oplus A) = K$ which is the topological equivalent under numerator operation in Theorem 16.

*Example 20.* Let $A = R(1/3)$, 2-bridge knot $K = b(5,2)$ known, and $X = R(x)$ unknown. Then $X = R(13/6)$ is a solution of the equation $N(X\oplus A) = K$. However $b(5,2)$ and $b(5,3)$ are topological equivalent from Theorem 16. Thus $X = R(4/3)$ is the other solution of the equation $N(X\oplus A) = K$ if $K = b(5,3)$.

From Theorem 16 and Corollary 17, we obtain the following corollary by the method as in Example 20.

**Corollary 21.** *Let $A$ and $K$ be known rational tangle in $T'$ and rational knot or link in $K$, respectively. Then there exist two solutions $X \in T'$ of the equation $N(X\oplus A) = K$.*

*3.3. Tangle Space and DNA.* Suppose that tangles $S$, $T$, and $R$ below are rational. As discussed in the introduction of Section 1, DNA must be topologically manipulated by enzymes in order for vital life processes to occur. The actions of some enzymes can be described as site-specific

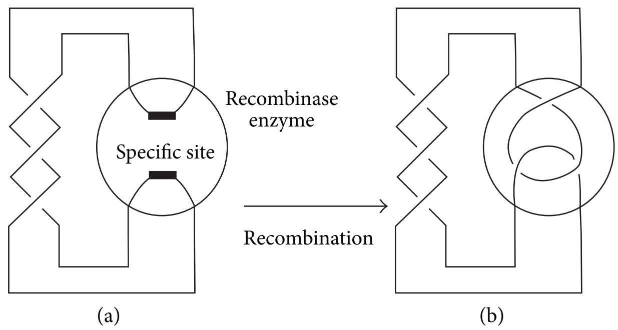


FIGURE 9: Example of a site-specific recombination.

recombination. Site-specific recombination is a process by which a piece of DNA is moved to another position on the molecule or to import a foreign piece of a DNA molecule into it. Recombination is used for gene rearrangement, gene regulation, copy number control, and gene therapy. This process is mediated by an enzyme called a recombinase. A small segment of the genetic sequence of the DNA that is recognized by the recombinase is called a recombination site or a specific site. See Figure 9. Note that the tangle in Figure 9 is where the enzyme acts.

The DNA molecule and the enzyme itself are called the synaptic complexes. Before recombination the DNA molecule is called the substract, that is, it is unchanged by the enzyme. After recombination the DNA molecule is called the product. In Figure 9, (a) is the substract and (b) is the product. This is the result which replaces a tangle (or enzyme) with a new tangle, called the recombination tangle. Thus the following tangle equations hold:

$$\begin{aligned} N(S \oplus T) &= \text{the substract}, \\ N(S \oplus R) &= \text{the product}, \end{aligned} \tag{34}$$

where the product is a result that the enzyme replaces a tangle $T$ with a tangle $R$. Generally it will repeat the tangle replacement a number of times. If it is possible to observe the substract and the product; then the ideal situation would be to determinate tangles $S$, $T$, and $R$ from the tangle equations. However it is a hard question in general to solve the tangle equations because there are only two equations but three unknowns. As above, the tangle model has been used to mathematically show the enzyme mechanism of recombination. See [17] for similar examples.

*Example 22.* Let the knot types of the substrate and the product yielding equations in the recombination variables $S$, $T$, and $R$ be as follows:

$$\begin{aligned} N(S \oplus T) &= \text{the unknot } b(1,1), \\ N(S \oplus R) &= \text{the trefoil knot } b(3,1). \end{aligned} \tag{35}$$

Then solutions of the equations are either $(S,T,R) = (R(-1/2), R(0), R(2))$ or $(S,T,R) = (R(1/2), R(0), R(-2))$.

In our study of tangle space with operator $\oplus$, it is still unknown how to construct a link or knot associated with a given real tangle and analyze DNA molecules by real tangles.

## Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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*Research Article*

# New Periodic Solutions for a Class of Zakharov Equations

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Through applying the Jacobian elliptic function method, we obtain the periodic solution for a series of nonlinear Zakharov equations, which contain Klein-Gordon Zakharov equations, Zakharov equations, and Zakharov-Rubenchik equations.

## 1. Introduction

For most of nonlinear evolution equations, we have many methods to obtain their exactly solutions, such as hyperbolic function expansion method [1], the transformation method [2], the trial function method [3], the automated method [4], and the extended tanh-function method [5]. But these methods can only obtain solitary wave solution and cannot be used to deduce periodic solutions. The Jacobian function method provides a way to find periodic solutions for some nonlinear evolution equations. In particular, in the research of plasma physics theory, quantum mechanics, fluid mechanics, and optical fiber communication, we frequently meet kinds of Zakharov equations.

Hence, in this paper, inspired by Angulo Pava's work [6], we are concerned to obtain exact periodic solutions of a series of Zakharov equations,

$$\begin{aligned} u_{tt} - u_{xx} + u + nu &= 0, \\ n_{tt} - n_{xx} &= \left(|u|^2\right)_{xx}, \\ u_{tt} - u_{xx} - |v|_{xx} &= 0, \\ iv_t + \alpha v_{xx} - uv &= 0, \end{aligned} \tag{1}$$

and Zakharov-Rubenchik equations,

$$\begin{aligned} iB_t + \omega B_{xx} - k\left(u - \frac{1}{2}\lambda\rho + q|B|^2 B\right) &= 0, \\ \theta\rho_t + \left(u - \lambda\rho\right)_x &= -k|B|_x^2, \\ \theta u_t + \left(\beta\rho - \lambda u\right)_x &= \frac{1}{2}k\lambda|B|_x^2. \end{aligned} \tag{2}$$

## 2. The Periodic Solution for Klein-Gordon Zakharov Equations

The Klein-Gordon Zakharov equations are used to show the interaction between langmuir wave and ion wave in the plasma, which has the following form:

$$\begin{aligned} u_{tt} - u_{xx} + u + nu &= 0, \\ n_{tt} - n_{xx} &= \left(|u|^2\right)_{xx}, \end{aligned} \tag{3}$$

where $u(x,t)$ denotes the biggest moment scale component produced by electron in electric field. $n(x,t)$ denotes the speed of deviations between the ions at any position and that at equilibrium position.

Now, we suppose that it possesses solitary wave solutions of the following form:

$$\begin{aligned} u(x,t) &= e^{i\theta}\varphi(\xi), \\ n(x,t) &= n(\xi), \\ \xi &= x - ct, \end{aligned} \tag{4}$$

where $c$ is a traveling wave speed and $c^2 < 1$ is a constant.

By substituting (4) into (3), we can obtain

$$c^2\varphi'' - \varphi'' + \varphi + \varphi n = 0, \tag{5}$$

$$\left(c^2 - 1\right)n'' = \left(\varphi^2\right)''. \tag{6}$$

By (6), we have

$$\left(c^2-1\right)n-\varphi^2=c_1\xi+m, \tag{7}$$

where $c_1$, $m$ are integration constants.

In what follows, we are concerned with the periodic solution of (3); thus we need to require $c_1=0$.

Therefore, (7) implies

$$n=\frac{\varphi^2+m}{c^2-1}. \tag{8}$$

Moreover, through (8) and (5), we have that

$$\left(c^2-1\right)\varphi''+\varphi+\frac{\varphi^3+\varphi m}{c^2-1}=0. \tag{9}$$

Multiplying (9) by $\varphi'$ and integrating once, we obtain

$$\frac{c^2-1}{2}\left(\varphi'\right)^2+\frac{\varphi^2}{2}+\frac{\varphi^4+2m\varphi^2}{c^2-1}\frac{1}{4}=h, \tag{10}$$

where $h$ is a nonzero integration constant. Furthermore, $h, m$, and $c$ satisfy the following condition:

$$\begin{gathered}\left(1-c^2-m\right)^2-4h\left(1-c^2\right)\geq 0,\\ \left(1-c^2-m\right)-\sqrt{\left(1-c^2-m\right)^2-4\left(1-c^2\right)h}\geq 0,\\ 0<h<\frac{\left(1-c^2-m\right)^2}{4\left(1-c^2\right)},\end{gathered} \tag{11}$$

so that

$$\left(\varphi'\right)^2=\frac{1}{2}\frac{1}{\left(c^2-1\right)^2}\left(\eta_1^2-\varphi^2\right)\left(\varphi^2-\eta_2^2\right), \tag{12}$$

where $-\eta_1$, $\eta_1$, $-\eta_2$, and $\eta_2$ are the zeros of the polynomial $F(\lambda)=-((c^2-1+m)/(c^2-1)^2)\lambda^2-(1/2)(\lambda^4/(c^2-1)^2)+2h/(c^2-1)$. Without losing generality, we assume that $\eta_1>\eta_2>0$. Therefore, we can deduce that $\eta_1\leq\varphi\leq\eta_2$; $\eta_1$ and $\eta_2$ satisfy

$$\begin{gathered}\eta_1^2+\eta_2^2=2\left(1-c^2\right)-2m,\\ \eta_1^2\eta_2^2=4h\left(1-c^2\right).\end{gathered} \tag{13}$$

Let $\phi=\varphi/\eta_1$, $k^2=(\eta_1^2-\eta_2^2)/\eta_1^2$. Hence, (12) can be written as

$$\left(\phi'\right)^2=\frac{1}{2}\frac{1}{\left(c^2-1\right)^2}\eta_1^2\left(1-\phi^2\right)\left(\phi^2-1+k^2\right). \tag{14}$$

Moreover, we define a new variable $\psi$, which satisfies $\psi'\geq 0$, $\psi(0)=0$, by the relation $\phi^2=1-k^2\sin^2\psi$; through a tedious computation, we obtain that

$$\left(\psi'\right)^2=\frac{1}{2}\frac{1}{\left(c^2-1\right)^2}\eta_1^2\left(1-k^2\sin^2\psi\right). \tag{15}$$

Then we obtain

$$\int_0^{\psi(\xi)}\frac{d\tau}{\sqrt{1-k^2\sin^2\tau}}=\frac{\sqrt{2}}{2}\frac{1}{1-c^2}\eta_1\xi. \tag{16}$$

According to the definition of the Jacobian elliptic function $y=\mathrm{sn}(u;k)$, we can obtain that $\sin\psi=\mathrm{sn}(\eta_1\sqrt{\beta}\xi,k)$. Here, $\beta=\sqrt{2}/2(1-c^2)$. So

$$\begin{aligned}\phi(\xi)&=\sqrt{1-k^2\mathrm{sn}^2\left(\eta_1\frac{\sqrt{2}}{2}\frac{1}{1-c^2}\xi,k\right)}\\ &=\mathrm{dn}\left(\eta_1\frac{\sqrt{2}}{2}\frac{1}{1-c^2}\xi,k\right).\end{aligned} \tag{17}$$

By returning to initial variable, we obtain that

$$\varphi(\xi)=\eta_1\mathrm{dn}\left(\eta_1\frac{\sqrt{2}}{2}\frac{1}{1-c^2}\xi,k\right) \tag{18}$$

is a dnoidal solution of (10).

Furthermore, dn has fundamental period $2K(k)$; that is, $\mathrm{dn}(u+2k;k)=\mathrm{dn}(u;k)$, and $K(k)$ is the complete elliptic integral of first kind. So the dnoidal wave solution $\varphi$ has fundamental period, $T$, given by

$$T=\frac{2\sqrt{2}K(k)\left(1-c^2\right)}{\eta_1}. \tag{19}$$

So, by applying the method of the Jacobian elliptic function and inspired by Angulo Pava's ideas, we obtain that (3) has the periodic traveling solution of the following form:

$$\begin{aligned}u(x,t)&=e^{i\theta}\varphi(\xi)=e^{i\theta}\eta_1\mathrm{dn}\left(\eta_1\frac{\sqrt{2}}{2}\frac{1}{1-c^2}\xi,k\right),\\ n(x,t)&\\ &=\frac{\left(\eta_1\mathrm{dn}\left(\eta_1\left(\sqrt{2}/2\right)\left(1/\left(1-c^2\right)\right)\xi,k\right)\right)^2+m}{c^2-1}.\end{aligned} \tag{20}$$

Moreover, $T$ and $k^2$ can be also rewritten as the following form:

$$\begin{aligned}T\left(\eta_2\right)&=\frac{2\sqrt{2}K\left(k\left(\eta_2\right)\right)\left(1-c^2\right)}{\sqrt{2\left(1-c^2\right)-2m-\eta_2^2}},\\ k^2&=\frac{2\left(1-c^2\right)-2m-2\eta_2^2}{2\left(1-c^2\right)-2m-\eta_2^2}.\end{aligned} \tag{21}$$

Furthermore, if $\eta_2\to 0$ then $k(\eta_2)\to 1$. Hence $K(k(\eta_2))\to+\infty$, so that $T(\eta_2)\to+\infty$. If $\eta_2\to\sqrt{1-c^2-m}$, then $k(\eta_2)\to 0$. Hence $K(k(\eta_2))\to\pi/2$, so that $T(\eta_2)\to\sqrt{2}\pi(1-c^2)/\sqrt{1-c^2-m}$.

Next we will show that, for an arbitrary but fixed $L$, $\sqrt{1-(L^2+L\sqrt{L^2-8\pi m})/4\pi}<|c_0|<\sqrt{1+(L^2+L\sqrt{L^2-8\pi m})/4\pi}$. We can obtain that there is a unique $\eta_{2,0} = \eta_{2,0}(c_0) \in (0, \sqrt{1-c_0^2-m})$ such that $T(\eta_2) = T(\eta_2(c)) = L$ is a fundamental period for the dnoidal wave solution (18).

**Theorem 1.** *Let $L > 0$ be arbitrary but fixed. Consider $\sqrt{1-(L^2+L\sqrt{L^2-8\pi m})/4\pi} < |c_0| < \sqrt{1+(L^2+L\sqrt{L^2-8\pi m})/4\pi}$ and unique $\eta_{2,0} = \eta_{2,0}(c_0) \in (0, \sqrt{1-c_0^2-m})$, such that $T(\eta_2) = L$. Then, there exist an interval $U(c_0)$, an interval $I(\eta_{2,0})$, and a unique smooth function $F : U(c_0) \to I(\eta_{2,0})$ such that $F(c) = \eta_2$ and*

$$L = \frac{2\sqrt{2}K(k(\eta_2))(1-c^2)}{\sqrt{2(1-c^2)-2m-\eta_2^2}}, \tag{22}$$

*where $c \in U(c_0)$, $\eta_2 \in U(\eta_{2,0})$.*

*Proof.* Based on the ideas establish in Angulo Pava's work [6], we will give a brief proof. Now, we consider the open set

$$\begin{aligned}\Omega = \Bigg\{ &(\eta_2, c) : \eta_2 \in \left(0, \sqrt{1-c_0^2-m}\right), \; c \\ &\in \left(-\sqrt{1+\frac{L^2+L\sqrt{L^2-8\pi m}}{4\pi}},\right. \\ &\left.\sqrt{1+\frac{L^2+L\sqrt{L^2-8\pi m}}{4\pi}}\right) \cup \left(-\infty,\right. \\ &\left.-\sqrt{1-\frac{L^2+L\sqrt{L^2-8\pi m}}{4\pi}}\right) \\ &\cup \left(\sqrt{1-\frac{L^2+L\sqrt{L^2-8\pi m}}{4\pi}}, +\infty\right)\Bigg\} \in R^2.\end{aligned} \tag{23}$$

We define $\Phi : \Omega \to R$ by

$$\Phi(\eta_2, c) = \frac{2\sqrt{2}K(k(\eta_2))(1-c^2)}{\sqrt{2(1-c^2)-2m-\eta_2^2}}. \tag{24}$$

Here, $k^2 = (2(1-c^2)-2m-2\eta_2^2)/(2(1-c^2)-2m-\eta_2^2)$. Hypothesise $\Phi(\eta_{2,0}, c_0) = L$. In what follows, we will prove that $\partial\Phi/\partial\eta_2 < 0$.

From (24), we can obtain that

$$\begin{aligned}\frac{\partial\Phi}{\partial\eta_2} &= \frac{2\sqrt{2}\eta_2 K(k(\eta_2))(1-c^2)}{(2(1-c^2)-2m-\eta_2^2)^{3/2}} \\ &+ \frac{2\sqrt{2}(1-c^2)}{\sqrt{2(1-c^2)-2m-\eta_2^2}} \frac{dK}{dk} \frac{\partial k}{\partial\eta_2}.\end{aligned} \tag{25}$$

From $k^2$, we can deduce $k(\eta_2, c)$ is a strictly decreasing function of $\eta_2$.

According to Jacobian elliptic function theory [7], we have

$$\frac{dK}{dk} = \frac{E-(1-k^2)K}{k(1-k^2)}, \tag{26}$$

$$\frac{dE}{dk} = \frac{E-K}{k}. \tag{27}$$

Here, $E$ is the complete elliptic integral of second kind.

Next, we adopt the reduction to absurdity to prove $\partial\Phi/\partial\eta_2 < 0$. Now, we assume that $\partial\Phi/\partial\eta_2 \geq 0$. So, we have the following inequality:

$$\begin{aligned}&k\left(2(1-c^2-m)-\eta_2^2\right)K(k(\eta_2)) \\ &\quad \geq 2(1-c^2-m)\frac{dK}{dk}.\end{aligned} \tag{28}$$

Indeed, substituting (26) into (25) and using the method of enlarging and reducing, we obtain that

$$\begin{aligned}&(1-k^2)\left[4(1-c^2-m)-2\eta_2^2\right]K(k) \\ &\quad \geq 2(1-c^2-m)E(k).\end{aligned} \tag{29}$$

From $k^2$, we can easily deduce that

$$\begin{aligned}1-k^2 &= \frac{\eta_2^2}{2(1-c^2)-2m-\eta_2^2}, \\ 2-k^2 &= \frac{2(1-c^2)-2m}{2(1-c^2)-2m-\eta_2^2}.\end{aligned} \tag{30}$$

Hence,

$$2(1-k^2)K(k) \geq (2-k^2)E(k). \tag{31}$$

Let $\gamma^2(\eta_2, c) = 1-k^2$, $\gamma' = -k(\partial k/\partial\eta_2)(1/\gamma) > 0$. So, $\gamma$ is an increasing function of $\eta_2 \in (0, \sqrt{1-c^2-m})$. Moreover, $\gamma(0) = 1$, $\gamma(\sqrt{1-c^2-m}) = 1$.

Define

$$f(\gamma) = (1+\gamma^2)E\left(\sqrt{1-\gamma^2}\right) - 2\gamma^2 K\left(\sqrt{1-\gamma^2}\right). \tag{32}$$

Due to $\partial\Phi/\partial\eta_2 \geq 0$, $f(\gamma) \leq 0$.

However, by (27) and a simple computation,

$$f'(\gamma) = 3\gamma\left(E\left(\sqrt{1-\gamma^2}\right) - K\left(\sqrt{1-\gamma^2}\right)\right) < 0, \tag{33}$$

so $f(\gamma)$ is a decreasing function. Furthermore, $f(1) = 0$, so for $\gamma \in (0,1)$, $f(\gamma) > 0$.

It is in conflict with our assumption. So we obtain our affirmation that $\partial\Phi/\partial\eta_2 < 0$. Hence, by the implicit function theorem, there exists a unique smooth function $F$, defined in a neighborhood, $U(c_0)$ of $c_0$, so that $\Phi(\eta_2(c), c) = L$ for $c \in U(c_0)$. Hence, we obtain (22). □

## 3. The Periodic Solution for Zakharov Equations

Now, we consider the following Zakharov equations:

$$\begin{aligned} &u_{tt} - u_{xx} - (|v|)^2_{xx} = 0, \\ &iv_t - \alpha v_{xx} - uv = 0, \end{aligned} \tag{34}$$

which describe the high frequency moment of plasma, where $u(x,t)$ denotes the ion number density variation, $v(x,t)$ denotes the electric field intensity of slowly varying amplitude, and $\alpha \in R$.

We seek the solitary solutions of (34) in the form

$$\begin{aligned} u &= u(\xi), \\ v &= \varphi(\xi) e^{i\xi}, \\ \xi &= x - ct, \end{aligned} \tag{35}$$

where $u$ and $\varphi$ are real functions, $c$ is a traveling speed, and $c^2 \neq 1$.

Substituting (35) into (34), we can obtain that

$$\left(c^2 - 1\right) u'' - \left(\varphi^2\right)'' = 0, \tag{36}$$

$$-\alpha\varphi'' - (c + 2\alpha) i\varphi' + (c + \alpha)\varphi - u\varphi = 0. \tag{37}$$

By (36), we can deduce

$$\left(c^2 - 1\right) u - \varphi^2 = c_2\xi + r. \tag{38}$$

Here, $c_2$, $r$ are integration constants.

Next, we consider periodic solutions of (34). So, we need to require $c_2 = 0$. Therefore, by (38)

$$u = \frac{\varphi^2 + r}{c^2 - 1}. \tag{39}$$

Furthermore, by (37), we can obtain that

$$c = -2\alpha. \tag{40}$$

So, by (39) and (40), (37) can be rewritten:

$$\varphi'' + \varphi\left(1 + \frac{2r}{c - c^3}\right) + \frac{2\varphi^3}{c - c^3} = 0. \tag{41}$$

Multiplying (41) by $\varphi'$ and integrating once, we obtain

$$\frac{1}{2}\left(\varphi'\right)^2 + \frac{1}{2}\varphi^2\left(1 + \frac{2r}{c - c^3}\right) + \frac{1}{2}\varphi^4 = h, \tag{42}$$

where $h$ is a nonzero integration constant.

Hence,

$$\left(\varphi'\right)^2 = \left(a^2 + \varphi^2\right)\left(b^2 - \varphi^2\right). \tag{43}$$

Here, $-ai$, $ai$, $-b$, and $b$ are polynomial roots of $F(t) = -t^4 - t^2(a^2 - b^2) + 2h$, $b > 0$, and

$$\begin{aligned} a^2 - b^2 &= 1 + \frac{2r}{c - c^3}, \\ a^2 b^2 &= 2h. \end{aligned} \tag{44}$$

Let $\chi = \varphi/b$ (suppose $\chi(0) = 0$) and $k^2 = b^2/(a^2 + b^2)$. Hence, (43) becomes

$$\left(\chi'\right)^2 = b^2\left(1 - \chi^2\right)\left(\frac{a^2}{b^2} + \chi^2\right). \tag{45}$$

Now, we define $\chi^2 = 1 - \sin^2\psi$, so we get that

$$\left(\psi'\right)^2 = \left(a^2 + b^2\right)\left(1 - k^2\sin^2\psi\right). \tag{46}$$

Let $\tau = \sqrt{a^2 + b^2}$. According to the definition of the Jacobian elliptic function sn, we can obtain $\sin\psi(\xi) = \mathrm{sn}(\tau\xi, k)$. So that, $\chi^2 = 1 - \mathrm{sn}^2(\tau\xi, k)$.

So, we obtain the solution of (41):

$$\varphi(\xi) = b\,\mathrm{cn}(\tau\xi, k). \tag{47}$$

According to Jacobian elliptic functions theory, cn has period $4K(k)$, and we can obtain that the cnoidal wave solution has period $T$, which is given by

$$T = \frac{4K(k)}{\tau}. \tag{48}$$

So, we can obtain the periodic solutions of (34):

$$\begin{aligned} u(x,t) &= u(\xi) = \frac{(b\,\mathrm{cn}(\tau\xi, k))^2 + r}{c^2 - 1}, \\ v(x,t) &= v(\xi) = be^{i\xi}\mathrm{cn}(\tau\xi, k). \end{aligned} \tag{49}$$

Furthermore, it follows that for $k^2 = b^2/(a^2 + b^2)$, we have $k^2 \in (0, 1/2)$.

If $\tau \to 0$, then $k \to 0$, and $K(k) \to \pi/2$, $T(\tau) \to +\infty$. Furthermore, if $\tau \to +\infty$, $T(\tau) \to 0$.

Next, we will prove that for an arbitrary but fixed $L > 0$, there exists a unique $\alpha = \alpha(c) \in (0, +\infty)$ such that $T(\alpha(c)) = L$ is a fundamental period of the cnoidal wave solution (47). So, we have the following theorem.

**Theorem 2.** *Let $L > 0$ but fixed. Consider $c_0 > 0$ and the unique $\tau_0 = \tau_0(c_0) \in (0, +\infty)$ such that $T(\tau_0) = L$. Then, there exists an internal $A(c_0)$ around $c_0$, an internal $B(b_0)$ around $b_0$, and a unique smooth function $\Upsilon$ such that $\Upsilon(c_0) = \alpha_0$ and*

$$L = \frac{4K(k)}{\tau}, \tag{50}$$

*where $c \in A(c_0)$, $\tau = \Upsilon(c)$.*

*Proof.* The proof is similar to that of Theorem 1. For details see Theorem 1 (see also Angulo Pava [6, 8, 9]). □

## 4. The Periodic Solution for Zakharov-Rubenchik Equations

Now, we consider the Zakharov-Rubenchik equations:

$$\begin{aligned} &iB_t + \omega B_{xx} - \ell\left(u - \frac{1}{2}\lambda\rho + q|B|^2 B\right) = 0, \\ &\theta\rho_t + \left(u - \lambda\rho\right)_x = -\ell|B|^2_x, \\ &\theta u_t + \left(\beta\rho - \lambda u\right)_x = \frac{1}{2}\ell\lambda|B|^2_x, \end{aligned} \tag{51}$$

which describe the dynamics of small amplitude Alfvén waves propagating in a plasma. Here, $B(x,t)$ denotes the magnetic field, $u(x,t)$ the fluid speed, and $\rho(x,t)$ the density of mass. Moreover, $\omega$, $\ell$, $\lambda$, $\theta$, $\beta$, $q$ are real constants.

Motivated by Oliveira [10], we look for the solitary waves of (51), as follows:

$$\begin{aligned} B(x,t) &= e^{i\xi} A(\xi), \\ u(x,t) &= aA^2(\xi), \\ \rho(x,t) &= bA^2(\xi), \\ \xi &= x - ct, \end{aligned} \tag{52}$$

where $a$, $b$, and $A$ are real functions and $c$ denotes traveling speed.

Putting (52) into (51), from the second and third equations of (51), we can deduce

$$\begin{aligned} a = a(c) &= \frac{\ell\left[(\lambda/2)(\lambda + c\theta) - \beta\right]}{\beta - (c\theta + \lambda)^2}, \\ b = b(c) &= \frac{\ell(-c\theta - \lambda/2)}{\beta - (c\theta + \lambda)^2}. \end{aligned} \tag{53}$$

Moreover, from the first equation of (51), we can obtain that

$$\begin{aligned} &(c-\omega)A + i(2\omega - c)A' + \omega A'' \\ &\quad - \ell\left(a - \frac{1}{2}\lambda b + q\right)A^3 = 0. \end{aligned} \tag{54}$$

So, by (54), it implies

$$c = 2\omega, \tag{55}$$

$$A'' + A - \frac{2\ell\left(a - (\lambda/2)b + q\right)}{c}A^3 = 0. \tag{56}$$

Let $b_1 = -\ell(a - (\lambda/2)b + q)/c > 0$ and multiplying (56) by $A'$ and integrating once, we obtain

$$\left[A'\right]^2 = b_1\left(A^2 + a_1^2\right)\left(b_2^2 - A^2\right), \tag{57}$$

where $-a_1 i$, $a_1 i$, $-b_2$, $b_2$ are the polynomial roots of $F(r) = -b_1 r^4 - r^2 + h$. Moreover,

$$\begin{aligned} b_2^2 - a_1^2 &= -\frac{1}{b_1}, \\ b_2^2 a_1^2 &= \frac{h}{b_1}. \end{aligned} \tag{58}$$

Let $\chi = A/b_2$ (suppose $\chi(0) = 0$) and $k^2 = b_2^2/(a_1^2 + b_2^2)$. Hence, (57) becomes

$$\left(\chi'\right)^2 = b_1 b_2^2\left(1 - \chi^2\right)\left(\frac{a_1^2}{b_2^2} + \chi^2\right). \tag{59}$$

Now, we define $\chi^2 = 1 - \sin^2\psi$, so we get that

$$\left(\psi'\right)^2 = b_1\left(a_1^2 + b_2^2\right)\left(1 - k^2\sin^2\psi\right). \tag{60}$$

Let $\alpha = \sqrt{b_1(a_1^2 + b_2^2)}$. According to the definition of the Jacobin elliptic function sn, we can obtain $\sin\psi(\xi) = \mathrm{sn}(\alpha\xi, k)$. So, $\chi^2 = 1 - \mathrm{sn}^2(\alpha\xi, k)$.

So, we obtain the solution of (56):

$$A(\xi) = b_2\,\mathrm{cn}(\alpha\xi, k). \tag{61}$$

Since cn has fundamental period $4K(k)$, we can obtain that solution (61) has fundamental period, $T$, which is

$$T = \frac{4K(k)}{\sqrt{b_1\left(a_1^2 + b_2^2\right)}}. \tag{62}$$

Hence, (51) have the periodic solutions of the following form:

$$\begin{aligned} B(\xi) &= e^{i\xi} b_2\,\mathrm{cn}(\alpha\xi, k), \\ u(\xi) &= ab_2^2\,\mathrm{cn}^2(\alpha\xi, k), \\ \rho(\xi) &= bb_2^2\,\mathrm{cn}^2(\alpha\xi, k). \end{aligned} \tag{63}$$

Moreover, from (58) and the definition of $k^2$, it follows that

$$\begin{aligned} k^2 &= \frac{a_1^2 - 1/b_1}{2a_1^2 - 1/b_1}, \\ T(a_1) &= \frac{4K(k)}{\sqrt{2a_1^2 b_1 - 1}}. \end{aligned} \tag{64}$$

Moreover, if $a_1 \to 0$, $k^2 \to 1$, $T(a_1) \to +\infty$. If $a_1 \to +\infty$, $k^2 \to 1/2$, $T(a_1) \to 0$.

Next, we will show that for an arbitrary but fixed $L > 0$, there exists $a_1 = a_1(c) \in (0, +\infty)$ such that $T(a_1(c)) = L$ is a fundamental period of the cnoidal wave solution (61). Motivated by Angulo Pava's result [6], we have the following theorem.

**Theorem 3.** *For $L > 0$ arbitrary and fixed, consider $c_0 > 0$ and the unique $a_1 = a_1(c_0) \in (0, +\infty)$. Then, there exist an interval $C(c_0)$ around $c_0$, an interval $D(a_1)$ around $a_1$, and a unique smooth function $H : C(c_0) \to D(a_1)$ such that $H(c_0) = a_1$ and*

$$L = \frac{4K(k)}{\sqrt{2a_1^2 b_1 - 1}}, \tag{65}$$

*where $c \in C(c_0)$, $a_1 = H(c)$.*

*Proof.* The idea and method are similar to those of Theorem 1. For details, please see Theorem 1 (see also Angulo Pava [6, 8, 9]). □

## 5. Conclusion

Inspired by Angulo Pava's idea, by applying Jacobian elliptic function method, we have obtained new period wave solution for Klein-Gordon Zakharov equations, Zakharov equations, and Zakharov-Rubenchik equations. In particular, the solutions of (18), (47), and (58) were not found in the previous work. The method can help to look for periodic solution for a class of nonlinear equations.

## Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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