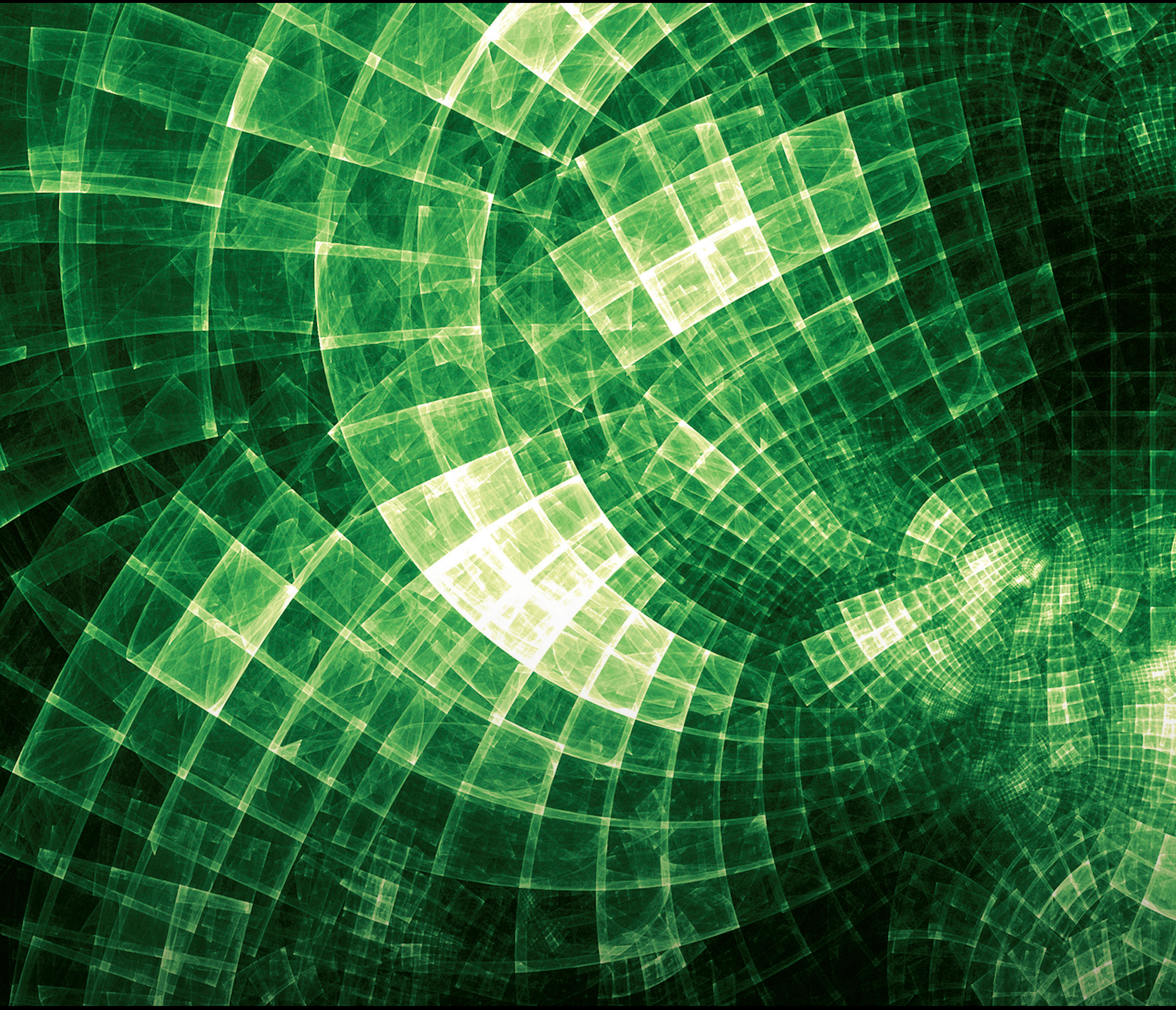
Journal of Mathematics

Soft Computing Algorithms Based on Fuzzy Extensions

Lead Guest Editor: Naeem Jan

Guest Editors: Sami Ullah Khan, Lazim Abdullah, and Kifayat Ullah



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Retraction

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Hindawi
Journal of Mathematics
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Retraction

Retracted: Generalization of Fuzzy Soft BCK/BCI-Algebras

Journal of Mathematics

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Hindawi
Journal of Mathematics
Volume 2023, Article ID 9827948, 1 page
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Retraction

Retracted: Belong and Nonbelong Relations on Double-Framed Soft Sets and Their Applications

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: Pythagorean *m*-Polar Fuzzy Weighted Aggregation Operators and Algorithm for the Investment Strategic Decision Making

Journal of Mathematics

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Hindawi
Journal of Mathematics
Volume 2023, Article ID 9821873, 1 page
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Retraction

Retracted: More on $\mathscr{D}\alpha$-Closed Sets in Topological Spaces

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: On Fuzzy Fixed-Point Results in Complex Valued Extended *b*-Metric Spaces with Application

Journal of Mathematics

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Hindawi
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Retraction

Retracted: General Complex-Valued Overlap Functions

Journal of Mathematics

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Hindawi
Journal of Mathematics
Volume 2023, Article ID 9813989, 1 page
https://doi.org/10.1155/2023/9813989

Retraction

Retracted: TOPSIS Method for Teaching Effect Evaluation of College English with Interval-Valued Intuitionistic Fuzzy Information

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: Hamacher Interactive Hybrid Weighted Averaging Operators under Fermatean Fuzzy Numbers

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: Hybrid Decision-Making Frameworks under Complex Spherical Fuzzy *N*-Soft Sets

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: Control Fuzzy Metric Spaces via Orthogonality with an Application

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: Fire Safety Evaluation for Scenic Spots: An Evidential Best-Worst Method

Journal of Mathematics

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Hindawi
Journal of Mathematics
Volume 2023, Article ID 9795387, 1 page
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Retraction

Retracted: Process Monitoring for Gamma Distributed Product under Neutrosophic Statistics Using Resampling Scheme

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: Certain Notions of Picture Fuzzy Information with Applications

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: New Operators of Cubic Picture Fuzzy Information with Applications

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: Some Complex Intuitionistic Uncertain Linguistic Heronian Mean Operators and Their Application in Multiattribute Group Decision Making

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: Evidence Theory in Picture Fuzzy Set Environment

Journal of Mathematics

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Hindawi
Journal of Mathematics
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Retraction

Retracted: On Multivalued Fuzzy Contractions in Extended *b*-Metric Spaces

Journal of Mathematics

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Hindawi
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Retraction

Retracted: On Three Types of Soft Rough Covering-Based Fuzzy Sets

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Retracted: Some Novel Cosine Similarity Measures Based on Complex Hesitant Fuzzy Sets and Their Applications

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Retraction

Retracted: Types of Complex Fuzzy Relations with Applications in Future Commission Market

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Retraction

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Retraction

Retracted: Some Generalized T-Spherical and Group-Generalized Fuzzy Geometric Aggregation Operators with Application in MADM Problems

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Retraction

Retracted: On Fuzzy Fixed-Point Results in Complex Valued Extended *b*-Metric Spaces with Application

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Research Article

On Fuzzy Fixed-Point Results in Complex Valued Extended *b*-Metric Spaces with Application

Amnah Essa Shammaky,[1] Jamshaid Ahmad ,[2] and Abdelhamied Farrag Sayed[3]

[1]*Department of Mathematics, Faculty of Science, Jazan University, Jazan 45142, Saudi Arabia*
[2]*Department of Mathematics, University of Jeddah, P.O.Box 80327, Jeddah 21589, Saudi Arabia*
[3]*Mathematics Department, Al-Lith University College, Umm Al-Qura University, P.O. Box 112, Al-Lith, Makkah Al-Mukarramah 21961, Saudi Arabia*

Correspondence should be addressed to Jamshaid Ahmad; jamshaid_jasim@yahoo.com

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The aim of this paper is to define fuzzy contraction in the context of complex valued extended b-metric space and prove fuzzy fixed-point results. Our results improve and extend certain recent results in literature. Moreover, we discuss an illustrative example to highlight the realized improvements. As application, we derive fixed-point results for multivalued mappings in the setting of complex valued extended b-metric space.

1. Introduction

In the theory of fixed points, there is vital role of metric spaces which have useful applications in mathematics as well as in computer science, medicine, physics, and biology (see [1–3]). Many mathematicians generalized, improved, and extended the notion of metric spaces to vector-valued metric spaces of Perov [4], b-metric space of Czerwik [5], cone metric spaces of Huang and Zhang [6], and others.

In 2011, Azam et al. [7] introduced the concept of complex valued metric space and obtained some common fixed-point results for rational contraction which consist of a pair of single valued mappings. Later on, many researchers [8–15] worked on this generalized metric space. Ahmad et al. [16] and Azam et al. [17] defined the generalized Housdorff metric function in the setting of complex valued metric space and obtained common fixed-point results for multivalued mappings. In [18], Mukheimer generalized the concept of complex valued metric space to complex valued b-metric space. Recently, Naimatullah et al. [19] introduced the notion of complex valued extended b-metric space as extension of complex valued b-metric space and established some results for rational contractions in this generalized space.

On the contrary, Heilpern [20] introduced the concept of fuzzy mappings in the setting of metric linear spaces and extended Banach Contraction Principle [21]. In 2014, Kutbi et al. [22] established fuzzy fixed-point results in complex valued metric spaces and generalized the results in metric spaces. Owing to the notion of a complex valued metric space, Humaira et al. [23] proved some common fixed-point results under contractive condition for rational expressions.

In this paper, we define the generalized fuzzy contraction in the setting of complex valued extended b-metric space and obtain some fuzzy fixed point results. As application, we derive the main results of Azam et al. [7], Rouzkard and Imdad [9], Ahmad et al. [16], and Kutbi et al. [22] for fuzzy and multivalued mappings in complex valued metric spaces.

2. Preliminaries

In 2011, Azam et al. [7] introduced the complex valued metric space as follows.

Definition 1 (see [7]). Let $\mathbb{C}$ be the set of complex numbers and $\ell_1, \ell_2 \in \mathbb{C}$. A partial order $\precsim$ on $\mathbb{C}$ is defined in this way:

$$\ell_1 \precsim \ell_2 \Leftrightarrow \mathrm{Re}(\ell_1) \le \mathrm{Re}(\ell_2), \mathrm{Im}(\ell_1) \le \mathrm{Im}(\ell_2). \quad (1)$$

It follows that

$$\ell_1 \precsim \ell_2 \quad (2)$$

if one of these assertions is satisfied:

$$\begin{aligned} &\text{(a) } \mathrm{Re}(\ell_1) = Re(\ell_2), \mathrm{Im}(\ell_1) < \mathrm{Im}(\ell_2),\\ &\text{(b) } \mathrm{Re}(\ell_1) < Re(\ell_2), \mathrm{Im}(\ell_1) = \mathrm{Im}(\ell_2),\\ &\text{(c) } \mathrm{Re}(\ell_1) < Re(\ell_2), \mathrm{Im}(\ell_1) < \mathrm{Im}(\ell_2),\\ &\text{(d) } \mathrm{Re}(\ell_1) = Re(\ell_2), \mathrm{Im}(\ell_1) = \mathrm{Im}(\ell_2). \end{aligned} \quad (3)$$

(CV1) $0 \precsim \wp(\ell,\hbar)$, for all $\ell,\hbar \in \mathfrak{Q}$ and $\wp(\ell,\hbar) = 0$ if and only if $\ell = \hbar$

(CV2) $\wp(\ell,\hbar) = \wp(\hbar,\ell)$, for all $\ell,\hbar \in \mathfrak{Q}$

(CV3) $\wp(\ell,\hbar) \precsim \wp(\ell,\nu) + \wp(\nu,\hbar)$, for all $\ell,\hbar,\nu \in \mathfrak{Q}$.

Definition 2 (see [7]). Let $\mathfrak{Q} \neq \varnothing$. A mapping $\wp$: $\mathfrak{Q}\times\mathfrak{Q} \longrightarrow \mathbb{C}$ is said to be a complex valued metric if the following assertions hold.

Then, $(\mathfrak{Q},\wp)$ is called a complex valued metric space (CVMS).

In 2014, Mukheimer [18] introduced the notion of complex valued b-metric space as follows.

(CVB1) $0 \precsim \wp(\ell,\hbar)$, for all $\ell,\hbar \in \mathfrak{Q}$ and $\wp(\ell,\hbar) = 0$ if and only if $\ell = \hbar$

(CVB2) $\wp(\ell,\hbar) = \wp(\hbar,\ell)$, for all $\ell,\hbar \in \mathfrak{Q}$

(CVB3) $\wp(\ell,\hbar) \precsim \pi[\wp(\ell,\nu) + \wp(\nu,\hbar)]$, for all $\ell,\hbar,\nu \in \mathfrak{Q}$

Definition 3 (see [18]). Let $\mathfrak{Q} \neq \varnothing$ and $\pi \ge 1$ be a real number. A mapping $\wp$: $\mathfrak{Q}\times\mathfrak{Q} \longrightarrow \mathbb{C}$ is said to be a complex valued b-metric space if the following assertions hold.

Then, $(\mathfrak{Q},\wp)$ is called a complex valued b- metric space (CVbMS).

Recently, Naimatullah et al. [19] defined the notion of complex valued extended b-metric space in the following way.

(ECVB1) $0 \precsim \wp(\ell,\hbar)$, for all $\ell,\hbar \in \mathfrak{Q}$ and $\wp(\ell,\hbar) = 0$ if and only if $\ell = \hbar$

(ECVB2) $\wp(\ell,\hbar) = \wp(\hbar,\ell)$, for all $\ell,\hbar \in \mathfrak{Q}$

(ECVB3) $\wp(\ell,\hbar) \precsim \varphi(\ell,\hbar)[\wp(\ell,\nu) + \wp(\nu,\hbar)]$, for all $\ell,\hbar,\nu \in \mathfrak{Q}$

Definition 4 (see [19]). Let $\mathfrak{Q} \neq \varnothing$ and φ: $\mathfrak{Q}\times\mathfrak{Q} \longrightarrow [1,\infty)$. A mapping $\wp$: $\mathfrak{Q}\times\mathfrak{Q} \longrightarrow \mathbb{C}$ is called a complex valued extended b-metric if following conditions hold:

Then, $(\mathfrak{Q},\wp)$ is called a complex valued extended b-metric space (CVEbMS).

Lemma 1 (see [19]). *Let $(\mathfrak{Q},\wp)$ be a CVEbMS and let $\{\ell_n\}\subseteq\mathfrak{Q}$. Then, $\{\ell_n\}$ converges to $\ell \Leftrightarrow |\wp(\ell_n,\ell)| \longrightarrow 0$ as $n \longrightarrow \infty$.*

Lemma 2 (see [19]). *Let $(\mathfrak{Q},\wp)$ be a CVEbMS and let $\{\ell_n\}\subseteq\mathfrak{Q}$. Then, $\{\ell_n\}$ is a Cauchy sequence $\Leftrightarrow |\wp(\ell_n,\ell_{n+m})| \longrightarrow 0$ as $n \longrightarrow \infty$, where $m \in \mathbb{N}$.*

Let $(\mathfrak{Q},\wp)$ be a CVEbMS; then, $\mathfrak{CB}(\mathfrak{Q})$ denotes the family of all nonempty, closed, and bounded subsets of $\mathfrak{Q}$.

From now on, we denote $s(\ell_1) = \{\ell_2 \in \mathbb{C}\colon \ell_1 \prec \ell_2\}$ for $\ell_1 \in \mathbb{C}$, and

$$s(\ell_1,\mathfrak{R}_2) = \bigcup_{\ell_2\in\mathfrak{R}_2} s(\wp(\ell_1,\ell_2)) = \bigcup_{\ell_2\in\mathfrak{R}_2} \{\ell \in \mathbb{C}\colon \wp(\ell_1,\ell_2) \prec \ell\}, \quad (4)$$

for $a \in \mathfrak{Q}$ and $\mathfrak{R}_2 \in \mathfrak{CB}(\mathfrak{Q})$.

For $\mathfrak{R}_1,\mathfrak{R}_2 \in \mathfrak{CB}(\mathfrak{Q})$, we denote

$$s(\mathfrak{R}_1,\mathfrak{R}_2) = \left(\bigcap_{\ell_1\in\mathfrak{R}_1} s(\ell_1,\mathfrak{R}_2)\right) \cap \left(\bigcap_{\ell_2\in\mathfrak{R}_2} s(\ell_2,\mathfrak{R}_1)\right). \quad (5)$$

(i) Let $\ell_1,\ell_2 \in \mathbb{C}$. If $\ell_1 \prec \ell_2$, then $s(\ell_2) \subset s(\ell_1)$.

(ii) Let $\ell \in \mathfrak{Q}$ and $\mathfrak{R} \in N(\mathfrak{Q})$. If $\theta \in s(\ell,\mathfrak{R})$, then $\ell \in \mathfrak{R}$.

(iii) Let $\ell \in \mathbb{C}$ and let $\mathfrak{R}_1,\mathfrak{R}_2 \in \mathfrak{CB}(\mathfrak{Q})$ and $\ell_1 \in \mathfrak{R}_1$. If $\ell \in s(\mathfrak{R}_1,\mathfrak{R}_2)$, then $\ell \in s(\ell_1,\mathfrak{R}_2)$, for all $\ell_1 \in \mathfrak{R}_1$, or $\ell \in s(\mathfrak{R}_1,\ell_2)$, for all $\ell_2 \in \mathfrak{R}_2$.

Lemma 3 (see [19]). *Let $(\mathfrak{Q},\wp)$ be a CVEbMS.*

Let $(\mathfrak{Q},\wp)$ be a complex valued extended b-metric space and $\mathfrak{C}(\mathfrak{Q})$ be a collection of nonempty closed subsets of $\mathfrak{Q}$. Let $\mathfrak{I}$: $\mathfrak{Q} \longrightarrow \mathfrak{CB}(\mathfrak{Q})$be a multivalued mapping. For $\ell \in \mathfrak{Q}$ and $\mathfrak{R} \in \mathfrak{CB}(\mathfrak{Q})$, we define

$$W_\ell(\mathfrak{R}) = \{\wp(\ell,\ell_1)\colon \ell_1 \in \mathfrak{R}\}. \quad (6)$$

Thus, for $\ell, y \in \mathfrak{Q}$,

$$W_\ell(\mathfrak{I}y) = \{\wp(\ell,\ell_1)\colon \ell_1 \in \mathfrak{I}y\}. \quad (7)$$

Definition 5 (see [19]). Let $(\mathfrak{Q},\wp)$ be a complex valued metric space. A subset $\mathfrak{R}$ of $\mathfrak{Q}$ is called bounded below if $\exists$ $\ell \in \mathfrak{Q}$, such that $\ell \prec \ell_1$, for all $\ell_1 \in \mathfrak{R}$.

Definition 6 (see [19]). Let $(\mathfrak{Q},\wp)$ be a complex valued metric space. A multivalued mapping $\mathfrak{I}$: $\mathfrak{Q} \longrightarrow 2^{\mathbb{C}}$ is called bounded below if, for each $\ell \in \mathfrak{Q}, \exists\ \ell_x \in \mathbb{C}$,

$$\ell_x \prec u, \quad (8)$$

for all $u \in \mathfrak{I}\ell$.

In 1981, Heilpern [20] utilized the concept of fuzzy set and introduced the notion of fuzzy mappings in metric spaces (MS). A fuzzy set in $\mathfrak{Q}$ is a function with domain $\mathfrak{Q}$ and values in $[0,1]$, and $I^{\mathfrak{Q}}$ is the collection of all fuzzy sets in $\mathfrak{Q}$. If $\mathfrak{R}$ is a fuzzy set and $x \in \mathfrak{Q}$, then the function values

$\mathfrak{R}(\ell)$ is called the grade of membership of ℓ in $\mathfrak{R}$. The α-level set of $\mathfrak{R}$ is denoted by $[\mathfrak{R}]_\alpha$ and is defined as follows:

$$\begin{aligned}[\mathfrak{R}]_\alpha &= \{\ell\colon \mathfrak{R}(\ell) \geq \alpha\} \text{ if } \alpha \in (0,1],\\ [\mathfrak{R}]_0 &= \overline{\{x\colon \mathfrak{R}(\ell) > 0\}}.\end{aligned} \tag{9}$$

Here, $\overline{\mathfrak{R}}$ denotes the closure of the set $\mathfrak{R}$. Let $\mathscr{F}(\mathfrak{Q})$ be the collection of all fuzzy sets in a metric space $\mathfrak{Q}$.

Definition 7 (see [20]). Let $\mathfrak{Q}_1$ be a nonempty set and $(\mathfrak{Q}_2, \wp)$ be a MS. A mapping $\mathfrak{I}$ is called fuzzy mapping if $\mathfrak{I}$ is a mapping from $\mathfrak{Q}_1$ into $\mathscr{F}(\mathfrak{Q}_2)$. A fuzzy mapping $\mathfrak{I}$ is a fuzzy subset on $\mathfrak{Q}_1 \times \mathfrak{Q}_2$ with membership function $\mathfrak{I}(x)(y)$. The function $\mathfrak{I}(x)(y)$is the grade of membership of y in $\mathfrak{I}(x)$.

Definition 8 (see [20]). Let $(\mathfrak{Q}, \wp)$ be a MS and $\mathfrak{I}_1, \mathfrak{I}_2\colon \mathfrak{Q} \longrightarrow \mathscr{F}(\mathfrak{Q})$. A point $\ell \in \mathfrak{Q}$ is said to be a fuzzy fixed point of $\mathfrak{I}_2$ if $\ell \in [\mathfrak{I}_2\ell]_\alpha$, for some $\alpha \in [0,1]$. The point $\ell \in \mathfrak{Q}$ is said to be a common fuzzy fixed point of $\mathfrak{I}_1$ and $\mathfrak{I}_2$ if $\ell \in [\mathfrak{I}_1\ell]_\alpha \cap [\mathfrak{I}_2\ell]_\alpha$, for some $\alpha \in [0,1]$.

In 2014, Kutbi et al. [22] used the above notion of fuzzy mappings in complex valued metric space (CVMS) and established the result for these mappings.

In this paper, we establish fuzzy fixed-point results in the setting of complex valued extended b-metric spaces (CVEbMS) and derive the above result of Kutbi et al. [22] for fuzzy mappings and some fixed-point result for multivalued mappings in CVMS.

3. Main Result

Definition 9. Let $(\mathfrak{Q}, \wp)$ be a CVEbMS. The fuzzy mapping $\mathfrak{I}\colon \mathfrak{Q} \longrightarrow \mathscr{F}(\mathfrak{Q})$ is said to have g.l.b. property on $(\mathfrak{Q}, \wp)$ if, for any $\ell \in \mathfrak{Q}$ and any $\alpha \in (0,1]$, greatest lower bound of $W_\ell([\mathfrak{I}\hbar]_\alpha)$ exists in $\mathbb{C}$, $\forall \hbar \in \mathfrak{Q}$. We denote $\wp(\ell, [\mathfrak{I}\hbar]_\alpha)$ by the g.l.b of $W_\ell([\mathfrak{I}\hbar]_\alpha)$. That is,

$$\wp(\ell, [\mathfrak{I}\hbar]_\alpha) = \inf\{\wp(\ell, \nu)\colon \nu \in [\mathfrak{I}\hbar]_\alpha\}. \tag{10}$$

Now, we state our main result in this way.

Theorem 1. *Let $(\mathfrak{Q}, \wp)$ be a complete CVEbMS, $\varphi\colon Q \times Q \longrightarrow [1, \infty)$, and let $\mathfrak{I}_1, \mathfrak{I}_2\colon \mathfrak{Q} \longrightarrow \mathscr{F}(\mathfrak{Q})$ satisfy g.l.b property. Assume that $\exists \alpha \in (0,1]$, such that, for each $\ell \in \mathfrak{Q}$, $[\mathfrak{I}_1\ell]_\alpha, [\mathfrak{I}_2\ell]_\alpha \in CB(\mathfrak{Q})$ and there exist nonnegative real numbers ζ, κ, μ with $\zeta + \kappa + \mu < 1$ and $\lambda(1-\kappa) = \zeta$, where $\lambda \in [0,1)$ such that*

$$\zeta\wp(\ell, \hbar) + \frac{\kappa\wp(\ell, [\mathfrak{I}_1\ell]_\alpha)\wp(\hbar, [\mathfrak{I}_2\hbar]_\alpha) + \mu\wp(\hbar, [\mathfrak{I}_1\ell]_\alpha)\wp(\ell, [\mathfrak{I}_2\hbar]_\alpha)}{1 + \wp(\ell, \hbar)} \in s([\mathfrak{I}_1\ell]_\alpha, [\mathfrak{I}_2\hbar]_\alpha), \tag{11}$$

for all $\ell, \hbar \in \mathfrak{Q}$. If, for each $\ell_0 \in \mathfrak{Q}$, $\lim_{n,m \longrightarrow \infty}\varphi(\ell_n, \ell_m)\lambda < 1$, then $\exists \ell^* \in \mathfrak{Q}$ such that $\ell^* \in [\mathfrak{I}_1\ell^*]_\alpha \cap [\mathfrak{I}_2\ell^*]_\alpha$.

Proof. Let ℓ_0 be an arbitrary point in $\mathfrak{Q}$. By assumption, we can find $\ell_1 \in [\mathfrak{I}_1\ell_0]_\alpha$. So, we have

$$\begin{aligned}&\zeta\wp(\ell_0, \ell_1) + \frac{\kappa\wp(\ell_0, [\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_1, [\mathfrak{I}_2\ell_1]_\alpha)\mu\wp(\ell_1, [\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_0, [\mathfrak{I}_2\ell_1]_\alpha)}{1 + \wp(\ell_0, \ell_1)} \in s([\mathfrak{I}_1\ell_0]_\alpha, [\mathfrak{I}_2\ell_1]_\alpha),\\ &\zeta\wp(\ell_0, \ell_1) + \frac{\kappa\wp(\ell_0, [\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_1, [\mathfrak{I}_2\ell_1]_\alpha) + \mu\wp(\ell_1, [\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_0, [\mathfrak{I}_2\ell_1]_\alpha)}{1 + \wp(\ell_0, \ell_1)} \in \bigcap_{\omega \in [\mathfrak{I}_1\ell_0]_\alpha} s(\omega, [\mathfrak{I}_2\ell_1]_\alpha).\end{aligned} \tag{12}$$

Since $\ell_1 \in [\mathfrak{I}_1\ell_0]_\alpha$, so, we have

$$\zeta\wp(\ell_0, \ell_1) + \frac{\kappa\wp(\ell_0, [\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_1, [\mathfrak{I}_2\ell_1]_\alpha) + \mu\wp(\ell_1, [\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_0, [\mathfrak{I}_2\ell_1]_\alpha)}{1 + \wp(\ell_0, \ell_1)} \in s(\ell_1, [\mathfrak{I}_2\ell_1]_\alpha). \tag{13}$$

By definition,

$$\zeta\wp(\ell_0, \ell_1) + \frac{\kappa\wp(\ell_0, [\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_1, [\mathfrak{I}_2\ell_1]_\alpha) + \mu\wp(\ell_1, [\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_0, [\mathfrak{I}_2\ell_1]_\alpha)}{1 + \wp(\ell_0, \ell_1)} \in \bigcup_{\varrho \in [\mathfrak{I}_2\ell_1]_\alpha} s(\wp(\ell_1, \varrho)). \tag{14}$$

This implies that $\exists \ell_2 \in [\mathfrak{I}_2\ell_1]_\alpha$ such that

$$\zeta\wp(\ell_0,\ell_1) + \frac{\kappa\wp(\ell_0,[\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_1,[\mathfrak{I}_2\ell_1]_\alpha) + \mu\wp(\ell_1,[\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_0,[\mathfrak{I}_2\ell_1]_\alpha)}{1+\wp(\ell_0,\ell_1)} \in s(\wp(\ell_1,\ell_2)). \tag{15}$$

That is,

$$\wp(\ell_1,\ell_2) \preceq \zeta\wp(\ell_0,\ell_1) + \frac{\kappa\wp(\ell_0,[\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_1,[\mathfrak{I}_2\ell_1]_\alpha) + \mu\wp(\ell_1,[\mathfrak{I}_1\ell_0]_\alpha)\wp(\ell_0,[\mathfrak{I}_2\ell_1]_\alpha)}{1+\wp(\ell_0,\ell_1)}. \tag{16}$$

By the meaning of $W_\ell([\mathfrak{I}_2\hbar]_\alpha)$ and $W_\ell([\mathfrak{I}_1\hbar]_\alpha)$ for $\ell,\hbar \in \mathfrak{Q}$, we obtain

$$\begin{aligned}\wp(\ell_1,\ell_2) &\preceq \zeta\wp(\ell_0,\ell_1) + \frac{\kappa\wp(\ell_0,\ell_1)\wp(\ell_1,\ell_2) + \mu\wp(\ell_1,\ell_1)\wp(\ell_0,\ell_2)}{1+\wp(\ell_0,\ell_1)}\\ &= \zeta\wp(\ell_0,\ell_1) + \frac{\kappa\wp(\ell_0,\ell_1)\wp(\ell_1,\ell_2)}{1+\wp(\ell_0,\ell_1)}\\ &= \zeta\wp(\ell_0,\ell_1) + \kappa\wp(\ell_1,\ell_2)\left(\frac{\wp(\ell_0,\ell_1)}{1+\wp(\ell_0,\ell_1)}\right).\end{aligned} \tag{17}$$

This implies

$$\begin{aligned}|\wp(\ell_1,\ell_2)| &\leq \zeta|\wp(\ell_0,\ell_1)| + \kappa|\wp(\ell_1,\ell_2)|\left|\frac{\wp(\ell_0,\ell_1)}{1+\wp(\ell_0,\ell_1)}\right|\\ &\leq \zeta|\wp(\ell_0,\ell_1)| + \kappa|\wp(\ell_1,\ell_2)|,\end{aligned} \tag{18}$$

such that

$$\begin{aligned}|\wp(\ell_1,\ell_2)| &\leq \left(\frac{\zeta}{1-\kappa}\right)|\wp(\ell_0,\ell_1)|\\ &= \lambda|\wp(\ell_0,\ell_1)|.\end{aligned} \tag{19}$$

Similarly, for $\ell_2 \in [\mathfrak{I}_2\ell_1]_\alpha$, we have

$$\begin{aligned}&\zeta\wp(\ell_2,\ell_1) + \frac{\kappa\wp(\ell_1,[\mathfrak{I}_2\ell_1]_\alpha)\wp(\ell_2,[\mathfrak{I}_1\ell_2]_\alpha) + \mu\wp(\ell_2,[\mathfrak{I}_2\ell_1]_\alpha)\wp(\ell_1,[\mathfrak{I}_1\ell_2]_\alpha)}{1+\wp(\ell_2,\ell_1)} \in s([\mathfrak{I}_2\ell_1]_\alpha,[\mathfrak{I}_1\ell_2]_\alpha),\\ &\zeta\wp(\ell_2,\ell_1) + \frac{\kappa\wp(\ell_1,[\mathfrak{I}_2\ell_1]_\alpha)\wp(\ell_2,[\mathfrak{I}_1\ell_2]_\alpha) + \mu\wp(\ell_2,[\mathfrak{I}_2\ell_1]_\alpha)\wp(\ell_1,[\mathfrak{I}_1\ell_2]_\alpha)}{1+\wp(\ell_2,\ell_1)} \in \bigcap_{\omega\in[\mathfrak{I}_2\ell_1]_\alpha} s(\omega,[\mathfrak{I}_1\ell_2]_\alpha).\end{aligned} \tag{20}$$

Since $\ell_2 \in [\mathfrak{I}_2\ell_1]_\alpha$, so, we have

$$\zeta\wp(\ell_2,\ell_1) + \frac{\kappa\wp(\ell_1,[\mathfrak{I}_2\ell_1]_\alpha)\wp(\ell_2,[\mathfrak{I}_1\ell_2]_\alpha) + \mu\wp(\ell_2,[\mathfrak{I}_2\ell_1]_\alpha)\wp(\ell_1,[\mathfrak{I}_1\ell_2]_\alpha)}{1+\wp(\ell_2,\ell_1)} \in s(\ell_2,[\mathfrak{I}_1\ell_2]_\alpha). \tag{21}$$

By definition of "s" function, we have

$$\begin{aligned}&\zeta\wp(\ell_2,\ell_1) + \frac{\kappa\wp(\ell_1,[\mathfrak{I}_2\ell_1]_\alpha)\wp(\ell_2,[\mathfrak{I}_1\ell_2]_\alpha) + \mu\wp(\ell_2,[\mathfrak{I}_2\ell_1]_\alpha)\wp(\ell_1,[\mathfrak{I}_1\ell_2]_\alpha)}{1+\wp(\ell_2,\ell_1)}\\ &\in \bigcup_{\in[\mathfrak{I}_1\ell_2]_\alpha} s(\wp(\ell_2,)).\end{aligned} \tag{22}$$

By definition of "s" function, there exists some $\ell_3 \in [\mathfrak{I}_1\ell_2]_\alpha$, such that

$$\zeta\wp(\ell_2,\ell_1) + \frac{\kappa\wp\big(\ell_{1,}[\mathfrak{I}_2\ell_1]_\alpha\big)\wp(\ell_2,[\mathfrak{I}_1\ell_2]_\alpha) + \mu\wp\big(\ell_{2,}[\mathfrak{I}_2\ell_1]_\alpha\big)\wp(\ell_1,[\mathfrak{I}_1\ell_2]_\alpha)}{1+\wp(\ell_2,\ell_1)} \in s(\wp(\ell_2,\ell_3)). \tag{23}$$

That is,

$$\wp(\ell_2,\ell_3) \preceq \zeta\wp(\ell_2,\ell_1) + \frac{\kappa\wp\big(\ell_{1,}[\mathfrak{I}_2\ell_1]_\alpha\big)\wp(\ell_2,[\mathfrak{I}_1\ell_2]_\alpha) + \mu\wp\big(\ell_{2,}[\mathfrak{I}_2\ell_1]_\alpha\big)\wp(\ell_1,[\mathfrak{I}_1\ell_2]_\alpha)}{1+\wp(\ell_2,\ell_1)}. \tag{24}$$

By the meaning of $W_\ell([\mathfrak{I}_2\hbar]_\alpha)$ and $W_\ell([\mathfrak{I}_1\hbar]_\alpha)$, for $\ell,\hbar \in \mathfrak{Q}$, we obtain

$$\begin{aligned}\wp(\ell_2,\ell_3) &\prec \zeta\wp(\ell_2,\ell_1) + \frac{\kappa\wp(\ell_1,\ell_2)\wp(\ell_2,\ell_3) + \mu\wp\big(\ell_2,\ell_{2,}\big)\wp(\ell_1,\ell_3)}{1+\wp(\ell_2,\ell_1)}\\ &= \zeta\wp(\ell_1,\ell_2) + \kappa\frac{\wp(\ell_1,\ell_2)\wp(\ell_2,\ell_3)}{1+\wp(\ell_1,\ell_2)},\end{aligned} \tag{25}$$

which implies that

$$|\wp(\ell_2,\ell_3)| \le \zeta|\wp(\ell_1,\ell_2)| + \kappa\wp(\ell_2,\ell_3)\frac{|\wp(\ell_1,\ell_2)|}{|1+\wp(\ell_1,\ell_2)|}, \tag{26}$$

which implies

$$\begin{aligned}|\wp(\ell_2,\ell_3)| &\le \left(\frac{\zeta}{1-\kappa}\right)|\wp(\ell_1,\ell_2)|\\ &= \lambda|\wp(\ell_1,\ell_2)|\end{aligned} \tag{27}$$

Inductively, we can construct a sequence $\{\ell_n\}$ in $\mathfrak{Q}$ such that

$$\begin{aligned}&|\wp(\ell_1,\ell_2)| \le \lambda|\wp(\ell_0,\ell_1)|,\\ &|\wp(\ell_2,\ell_3)| \le \lambda^2|\wp(\ell_0,\ell_1)|,\\ &\quad\vdots\\ &\wp(\ell_n,\ell_{n+1}) \le \lambda^n\wp(\ell_0,\ell_1),\end{aligned} \tag{28}$$

for all $n \in \mathbb{N}$. Now, by triangular inequality, for $m > n$, we have

$$\begin{aligned}\wp(\ell_n,\ell_m) \preceq{}& \varphi(\ell_n,\ell_m)\lambda^n\wp(\ell_0,\ell_1)\\ &+ \varphi(\ell_n,\ell_m)\varphi(\ell_{n+1},\ell_m)\lambda^{n+1}\wp(\ell_0,\ell_1)\\ &+\cdots+\\ &\cdot\varphi(\ell_n,\ell_m)\varphi(\ell_{n+1},\ell_m)\cdots\varphi(\ell_{m-2},\ell_m)\varphi(\ell_{m-1},\ell_m)\lambda^{m-1}\wp(\ell_0,\ell_1)\\ &\prec wp(\ell_0,\ell_1)\begin{bmatrix}\varphi(\ell_n,\ell_m)\lambda^n\\ +\varphi(\ell_n,\ell_m)\varphi(\ell_{n+1},\ell_m)\lambda^{n+1}+\cdots+\\ \varphi(\ell_n,\ell_m)\varphi(\ell_{n+1},\ell_m)\cdots\varphi(\ell_{m-2},\ell_m)\varphi(\ell_{m-1},\ell_m)\lambda^{m-1}\end{bmatrix}.\end{aligned} \tag{29}$$

Since $\lim_{n,m\longrightarrow\infty}\varphi(\ell_n,\ell_m)\lambda < 1$, so the series $\sum_{n=1}^{\infty}\lambda^n \prod_{i=1}^{p}\varphi(\ell_i,\ell_m)$ converges by ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^{\infty}\lambda^n\prod_{i=1}^{p}\varphi(\ell_i,\ell_m), \qquad S_n = \sum_{j=1}^{n}\lambda^j\prod_{i=1}^{p}\varphi(\ell_i,\ell_m). \tag{30}$$

Thus, for $m > n$, the above inequality can be written as

$$\wp(\ell_n,\ell_m) \prec \wp(\ell_0,\ell_1)[S_{m-1} - S_n]. \tag{31}$$

Now, by taking $n \longrightarrow \infty$, we obtain

$$|\wp(\ell_n,\ell_m)| \longrightarrow 0. \tag{32}$$

By Lemma 2, we conclude that $\{\ell_n\}$ is a Cauchy sequence. Since $\mathfrak{Q}$ is complete, then there exists an element ℓ^* such that $\ell_n \longrightarrow \varrho^* \in \mathfrak{Q}$ as $n \longrightarrow \infty$. Now, to show $\varrho^* \in \mathfrak{I}_1\varrho^*$ and $\varrho^* \in \mathfrak{I}_2\varrho^*$, from (1), we have

$$\begin{aligned}&\zeta\wp(\ell_{2n},\varrho^*) + \frac{\kappa\wp(\ell_{2n},[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\varrho^*,[\mathfrak{I}_2\varrho^*]_\alpha) + \mu\wp(\varrho^*,[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\ell_{2n},[\mathfrak{I}_2\varrho^*]_\alpha)}{1+\wp(\ell_{2n},\varrho^*)} \in s([\mathfrak{I}_1\ell_{2n}]_\alpha,[\mathfrak{I}_2\varrho^*]_\alpha),\\ &\zeta\wp(\ell_{2n},\varrho^*) + \frac{\kappa\wp(\ell_{2n},[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\varrho^*,[\mathfrak{I}_2\varrho^*]_\alpha) + \mu\wp(\varrho^*,[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\ell_{2n},[\mathfrak{I}_2\varrho^*]_\alpha)}{1+\wp(\ell_{2n},\varrho^*)} \in \bigcap_{\omega\in[\mathfrak{I}_1\ell_{2n}]_\alpha} s(\omega,[\mathfrak{I}_2\varrho^*]_\alpha).\end{aligned} \tag{33}$$

Since $\ell_{2n+1} \in [\mathfrak{I}_1\ell_{2n}]_\alpha$, we have

$$\begin{aligned}&\zeta\wp(\ell_{2n},\varrho^*) + \frac{\kappa\wp(\ell_{2n},[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\varrho^*,[\mathfrak{I}_2\varrho^*]_\alpha) + \mu\wp(\varrho^*,[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\ell_{2n},[\mathfrak{I}_2\varrho^*]_\alpha)}{1+\wp(\ell_{2n},\varrho^*)} \in s(\ell_{2n+1},[\mathfrak{I}_2\varrho^*]_\alpha),\\ &\zeta\wp(\ell_{2n},\varrho^*) + \frac{\kappa\wp(\ell_{2n},[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\varrho^*,[\mathfrak{I}_2\varrho^*]_\alpha) + \mu\wp(\varrho^*,[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\ell_{2n},[\mathfrak{I}_2\varrho^*]_\alpha)}{1+\wp(\ell_{2n},\varrho^*)} \in \bigcup_{\varrho'\in[\mathfrak{I}_2\varrho^*]_\alpha} s\big(\wp\big(\ell_{2n+1},\varrho'\big)\big).\end{aligned} \tag{34}$$

This implies that $\exists\varrho_n \in [\mathfrak{I}_2\varrho^*]_\alpha$ such that

$$\zeta\wp(\ell_{2n},\varrho^*) + \frac{\kappa\wp(\ell_{2n},[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\varrho^*,[\mathfrak{I}_2\varrho^*]_\alpha) + \mu\wp(\varrho^*,[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\ell_{2n},[\mathfrak{I}_2\varrho^*]_\alpha)}{1+\wp(\ell_{2n},\varrho^*)} \in s(\wp(\ell_{2n+1},\varrho_n)). \tag{35}$$

That is,

$$\wp(\ell_{2n+1},\varrho_n) \preceq \zeta\wp(\ell_{2n},\varrho^*) + \frac{\kappa\wp(\ell_{2n},[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\varrho^*,[\mathfrak{I}_2\varrho^*]_\alpha) + \mu\wp(\varrho^*,[\mathfrak{I}_1\ell_{2n}]_\alpha)\wp(\ell_{2n},[\mathfrak{I}_2\varrho^*]_\alpha)}{1+\wp(\ell_{2n},\varrho^*)}. \tag{36}$$

The g.l.b property of $\mathfrak{I}_2$ yields

$$\wp(\ell_{2n+1},\varrho_n) \preceq \zeta\wp(\ell_{2n},\varrho^*) + \frac{\kappa\wp(\ell_{2n},\ell_{2n+1})\wp(\varrho^*,\varrho_n) + \mu\wp(\varrho^*,\ell_{2n+1})\wp(\ell_{2n},\varrho_n)}{1+\wp(\ell_{2n},\varrho^*)}. \tag{37}$$

We know that

$$\wp(\varrho^*,\varrho_n) \preceq \theta(\varrho^*,\varrho_n)[\wp(\varrho^*,\ell_{2n+1}) + \wp(\ell_{2n+1},\varrho_n)]. \tag{38}$$

Hence,

$$\begin{aligned}\wp(\varrho^*,\varrho_n) &\preceq \theta(\varrho^*,\varrho_n)\wp(\varrho^*,\ell_{2n+1}) + \zeta\theta(\varrho^*,\varrho_n)\wp(\ell_{2n},\varrho^*)\\ &\quad + \kappa\theta(\varrho^*,\varrho_n)\frac{\wp(\ell_{2n},\ell_{2n+1})\wp(\varrho^*,\varrho_n)}{1+\wp(\ell_{2n},\varrho^*)}\\ &\quad + \mu\theta(\varrho^*,\varrho_n)\frac{\wp(\varrho^*,\ell_{2n+1})\wp(\ell_{2n},\varrho_n)}{1+\wp(\ell_{2n},\varrho^*)}.\end{aligned} \tag{39}$$

It follows that

$$\begin{aligned}\left|\wp\left(\varrho^*,\varrho_n\right)\right| \le \theta\left(\varrho^*,\varrho_n\right)\left|\wp\left(\varrho^*,\ell_{2n+1}\right)\right| + \zeta\theta\left(\varrho^*,\varrho_n\right)\left|\wp\left(\ell_{2n},\varrho^*\right)\right| \\ + \kappa\theta\left(\varrho^*,\varrho_n\right)\frac{\left|\wp\left(\ell_{2n},\ell_{2n+1}\right)\right|\left|\wp\left(\varrho^*,\varrho_n\right)\right|}{\left|1+\wp\left(\ell_{2n},\varrho^*\right)\right|} \\ + \mu\theta\left(\varrho^*,\varrho_n\right)\frac{\left|\wp\left(\varrho^*,\ell_{2n+1}\right)\right|\left|\wp\left(\ell_{2n},\varrho_n\right)\right|}{\left|1+\wp\left(\ell_{2n},\varrho^*\right)\right|}.\end{aligned} \tag{40}$$

Letting $n \longrightarrow \infty$, we get $|\wp(\varrho^*,\varrho_n)| \longrightarrow 0$. By using Lemma 1, we have $\varrho_n \longrightarrow \varrho^*$. Since $[\mathfrak{I}_2\varrho^*]_\alpha$ is closed, so $\varrho^* \in [\mathfrak{I}_2\varrho^*]_\alpha$. Following the similar steps, we can prove that $\varrho^* \in [\mathfrak{I}_1\varrho^*]_\alpha$. Hence, there exists $\varrho^* \in \mathfrak{Q}$ such that $\varrho^* \in [\mathfrak{I}_1\varrho^*]_\alpha \cap [\mathfrak{I}_2\varrho^*]_\alpha$.

By setting $\mu = 0$ in Theorem 1, we get the following Corollary. □

Corollary 1. *Let $(\mathfrak{Q},\wp)$ be a complete CVEbMS, $\varphi\colon \mathfrak{Q}\times\mathfrak{Q} \longrightarrow [1,\infty)$, and let $\mathfrak{I}_1,\mathfrak{I}_2\colon \mathfrak{Q} \longrightarrow \mathcal{F}(\mathfrak{Q})$ satisfy g.l.b property. Assume that $\exists\alpha \in (0,1]$, such that, for each $\ell \in \mathfrak{Q}$, $[\mathfrak{I}_1\ell]_\alpha, [\mathfrak{I}_2\ell]_\alpha \in CB(\mathfrak{Q})$ and there exist nonnegative real numbers ζ,κ with $\zeta+\kappa<1$ and $\lambda(1-\kappa)=\zeta$, where $\lambda \in [0,1)$ such that*

$$\zeta\wp(\ell,\hbar) + \kappa\frac{\wp(\ell,[\mathfrak{I}_1\ell]_\alpha)\wp(\hbar,[\mathfrak{I}_2\hbar]_\alpha)}{1+\wp(\ell,\hbar)} \in s\left([\mathfrak{I}_1\ell]_\alpha,[\mathfrak{I}_2\hbar]_\alpha\right). \tag{41}$$

for all $\ell,\hbar \in \mathfrak{Q}$. If, for each $\ell_0 \in \mathfrak{Q}$, $\lim_{n,m\longrightarrow\infty}\varphi(\ell_n,\ell_m)\lambda < 1$, then $\exists\ell^* \in \mathfrak{Q}$ such that $\ell^* \in [\mathfrak{I}_1\ell^*]_\alpha \cap [\mathfrak{I}_2\ell^*]_\alpha$.

By setting $\mathfrak{I}_1 = \mathfrak{I}_2$ in Theorem 1, we get the following corollary.

Corollary 2. *Let $(\mathfrak{Q},\wp)$ be a complete CVEbMS, $\varphi\colon \mathfrak{Q}\times\mathfrak{Q} \longrightarrow [1,\infty)$, and let $\mathfrak{I}\colon \mathfrak{Q} \longrightarrow \mathcal{F}(\mathfrak{Q})$ satisfy g.l.b property. Assume that $\exists\alpha \in (0,1]$, such that, for each $\ell \in \mathfrak{Q}$, $[\mathfrak{I}\ell] \in CB(\mathfrak{Q})$ and there exist nonnegative real numbers ζ,κ,μ with $\zeta+\kappa+\mu<1$ and $\lambda(1-\kappa)=\zeta$, where $\lambda \in [0,1)$ such that*

$$\zeta\wp(\ell,\hbar) + \frac{\kappa\wp(\ell,[\mathfrak{I}\ell]_\alpha)\wp(\hbar,[\mathfrak{I}\hbar]_\alpha) + \mu\wp(\hbar,[\mathfrak{I}\ell]_\alpha)\wp(\ell,[\mathfrak{I}\hbar]_\alpha)}{1+\wp(\ell,\hbar)} \in s\left([\mathfrak{I}\ell]_\alpha,[\mathfrak{I}\hbar]_\alpha\right), \tag{42}$$

for all $\ell,\hbar \in \mathfrak{Q}$. If, for each $\ell_0 \in \mathfrak{Q}$, $\lim_{n,m\longrightarrow\infty}\varphi(\ell_n,\ell_m)\lambda < 1$, then $\exists\ell^* \in \mathfrak{Q}$ such that $\ell^* \in [\mathfrak{I}\ell^*]_\alpha$.

Remark 1. If we take $\varphi(\ell,\hbar) = 1$ in Theorem 1, then we get main result of Kutbi et al. [22].

4. Application

In this section, we derive some fixed-point results for multivalued mappings as application of our main result.

Theorem 2. *Let $(\mathfrak{Q},\wp)$ be a complete CVEbMS, $\varphi\colon \mathfrak{Q}\times\mathfrak{Q} \longrightarrow [1,\infty)$, and let $\aleph_1,\aleph_2\colon \mathfrak{Q} \longrightarrow CB(\mathfrak{Q})$ satisfy g.l.b property. If there exist nonnegative real numbers ζ,κ,μ with $\zeta+\kappa+\mu<1$ and $\lambda(1-\kappa)=\zeta$, where $\lambda \in [0,1)$ such that*

$$\zeta\wp(\ell,\hbar) + \frac{\kappa\wp(\ell,\aleph_1\ell)\wp(\hbar,\aleph_2\hbar) + \mu\wp(\hbar,\aleph_1\ell)\wp(\ell,\aleph_2\hbar)}{1+\wp(\ell,\hbar)} \in s\left(\aleph_1\ell,\aleph_2\hbar\right), \tag{43}$$

for all $\ell,\hbar \in \mathfrak{Q}$. If, for each $\ell_0 \in \mathfrak{Q}$, $\lim_{n,m\longrightarrow\infty}\varphi(\ell_n,\ell_m)\lambda < 1$, then $\exists\ell^* \in \mathfrak{Q}$ such that $\ell^* \in \aleph_1\ell^* \cap \aleph_2\ell^*$.

Proof. Consider $\mathfrak{I}_1,\mathfrak{I}_2\colon \mathfrak{Q} \longrightarrow \mathcal{F}(\mathfrak{Q})$ defined by

$$\mathfrak{I}_1(\ell)(t) = \begin{cases}\alpha, t \in \aleph_1\ell \\ 0, t \notin \aleph_1\ell\end{cases}, \mathfrak{I}_2(\ell)(t) = \begin{cases}\alpha, t \in \aleph_2\ell \\ 0, t \notin \aleph_2\ell\end{cases}, \tag{44}$$

where $\alpha \in (0,1]$. Then,

$$\begin{aligned}[\mathfrak{I}_1\ell]_\alpha &= \{t\colon \mathfrak{I}_1(\ell)(t) \geq \alpha\} = \aleph_1\ell,\\ [\mathfrak{I}_2\ell]_\alpha &= \aleph_2\ell.\end{aligned} \tag{45}$$

Thus, Theorem 1 can be applied to obtain $\ell^* \in \mathfrak{Q}$ such that

$$\ell^* \in [\mathfrak{I}_1\ell^*]_\alpha \cap [\mathfrak{I}_2\ell^*]_\alpha = \aleph_1\ell^* \cap \aleph_2\ell^*. \tag{46}$$

If we consider just one multivalued mapping, then we get the following result. □

Corollary 3. *Let $(\mathfrak{Q}, \wp)$ be a complete CVEbMS, $\varphi\colon \mathfrak{Q}\times\mathfrak{Q} \longrightarrow [1,\infty)$, and let $\aleph\colon \mathfrak{Q} \longrightarrow CB(\mathfrak{Q})$ satisfy g.l.b property. If there exist nonnegative real numbers ζ, κ, μ with $\zeta + \kappa + \mu < 1$ and $\lambda(1-\kappa) = \zeta$, where $\lambda \in [0,1)$, then*

$$\zeta\wp(\ell,\hbar) + \frac{\kappa\wp(\ell,\aleph\ell)\wp(\hbar,\aleph\hbar) + \mu\wp(\hbar,\aleph\ell)\wp(\ell,\aleph\hbar)}{1+\wp(\ell,\hbar)} \in s(\aleph\ell,\aleph\hbar), \tag{47}$$

for all $\ell,\hbar \in \mathfrak{Q}$. If for each $\ell_0 \in \mathfrak{Q}$, $lim_{n,m\longrightarrow\infty}\varphi(\ell_n,\ell_m)\lambda < 1$, then $\exists\ell^* \in \mathfrak{Q}$ such that $\ell^* \in \aleph\ell^*$.

Corollary 4. *Let $(\mathfrak{Q}, \wp)$ be a complete CVEbMS, $\varphi\colon \mathfrak{Q}\times\mathfrak{Q} \longrightarrow [1,\infty)$, and let $\aleph_1, \aleph_2\colon \mathfrak{Q} \longrightarrow CB(\mathfrak{Q})$ satisfy g.l.b property. If there exist nonnegative real numbers ζ, κ with $\zeta + \kappa < 1$ and $\lambda(1-\kappa) = \zeta$, where $\lambda \in [0,1)$ such that*

$$\zeta\wp(\ell,\hbar) + \kappa\frac{\wp(\ell,\aleph_1\ell)\wp(\hbar,\aleph_2\hbar)}{1+\wp(\ell,\hbar)} \in s(\aleph_1\ell,\aleph_2\hbar), \tag{48}$$

for all $\ell,\hbar \in \mathfrak{Q}$. If, for each $\ell_0 \in \mathfrak{Q}$, $lim_{n,m\longrightarrow\infty}\varphi(\ell_n,\ell_m)\lambda < 1$, then $\exists\ell^* \in \mathfrak{Q}$ such that $\ell^* \in \aleph_1\ell^* \cap \aleph_2\ell^*$.

Proof. Take $\mu = 0$ in Theorem 2. □

Corollary 5. *Let $(\mathfrak{Q}, \wp)$ be a complete CVEbMS, $\varphi\colon \mathfrak{Q}\times\mathfrak{Q} \longrightarrow [1,\infty)$, and let $\aleph_1, \aleph_2\colon \mathfrak{Q} \longrightarrow CB(\mathfrak{Q})$ satisfy g.l.b property. If there exists nonnegative real number $\zeta \in [0,1)$, then*

$$\zeta\wp(\ell,\hbar) \in s(\aleph_1\ell,\aleph_2\hbar), \tag{49}$$

for all $\ell,\hbar \in \mathfrak{Q}$. If, for each $\ell_0 \in \mathfrak{Q}$, $lim_{n,m\longrightarrow\infty}\varphi(\ell_n,\ell_m)\zeta < 1$, then $\exists\ell^* \in \mathfrak{Q}$ such that $\ell^* \in \aleph_1\ell^* \cap \aleph_2\ell^*$.

Proof. Take $\kappa = \mu = 0$ in Theorem 2.

If we take $\varphi(\ell,\hbar) = 1$ in Theorem 2, then we get main result of Ahmad et al. [16] as follows. □

Corollary 6 (see [16]). *Let $(\mathfrak{Q}, \wp)$ be a complete CVMS, and let $\aleph_1, \aleph_2 : \mathfrak{Q} \longrightarrow CB(\mathfrak{Q})$ satisfy g.l.b property. If there exist nonnegative real numbers ζ, κ, μ with $\zeta + \kappa + \mu < 1$ such that*

$$\zeta\wp(\ell,\hbar) + \frac{\kappa\wp(\ell,\aleph_1\ell)\wp(\hbar,\aleph_2\hbar) + \mu\wp(\hbar,\aleph_1\ell)\wp(\ell,\aleph_2\hbar)}{1+\wp(\ell,\hbar)} \in s(\aleph_1\ell,\aleph_2\hbar), \tag{50}$$

for all $\ell,\hbar \in \mathfrak{Q}$, then $\exists\ell^* \in \mathfrak{Q}$ such that $\ell^* \in \aleph_1\ell^* \cap \aleph_2\ell^*$.

The following result is a direct consequence of Theorem 2 if we replace multivalued mappings with single valued mappings.

Theorem 3. *Let $(\mathfrak{Q}, \wp)$ be a complete CVEbMS, $\varphi\colon \mathfrak{Q}\times\mathfrak{Q} \longrightarrow [1,\infty)$, and let $\aleph_1, \aleph_2\colon \mathfrak{Q} \longrightarrow \mathfrak{Q}$. If there exist nonnegative real numbers ζ, κ, μ with $\zeta + \kappa + \mu < 1$ and $\lambda(1-\kappa) = \zeta$, where $\lambda \in [0,1)$ such that*

$$\wp(\aleph_1\ell,\aleph_2\hbar) \prec \zeta\wp(\ell,\hbar) + \frac{\kappa\wp(\ell,\aleph_1\ell)\wp(\hbar,\aleph_2\hbar) + \mu\wp(\hbar,\aleph_1\ell)\wp(\ell,\aleph_2\hbar)}{1+\wp(\ell,\hbar)}, \tag{51}$$

for all $\ell,\hbar \in \mathfrak{Q}$. If, for each $\ell_0 \in \mathfrak{Q}$, $lim_{n,m\longrightarrow\infty}\varphi(\ell_n,\ell_m)\lambda < 1$, then $\exists\ell^* \in \mathfrak{Q}$ such that $\ell^* = \aleph_1\ell^* \cap \aleph_2\ell^*$.

If we take $\varphi(\ell,\hbar) = 1$ in Theorem 3, then we get main result of Rouzkard and Imdad [9] as follows.

Corollary 7 (see [9]). *Let $(\mathfrak{Q}, \wp)$ be a complete CVMS and let $\aleph_1, \aleph_2\colon \mathfrak{Q} \longrightarrow \mathfrak{Q}$. If there exist nonnegative real numbers ζ, κ, μ with $\zeta + \kappa + \mu < 1$ such that*

$$\wp(\aleph_1\ell,\aleph_2\hbar) \prec \zeta\wp(\ell,\hbar) + \frac{\kappa\wp(\ell,\aleph_1\ell)\wp(\hbar,\aleph_2\hbar) + \mu\wp(\hbar,\aleph_1\ell)\wp(\ell,\aleph_2\hbar)}{1+\wp(\ell,\hbar)}, \tag{52}$$

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Retraction

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] G. Muhiuddin and A. M. Alanazi, "(m, n)-Ideals in Semigroups Based on Int-Soft Sets," *Journal of Mathematics*, vol. 2021, Article ID 5546596, 10 pages, 2021.

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Research Article

(m, n)-Ideals in Semigroups Based on Int-Soft Sets

G. Muhiuddin and Abdulaziz M. Alanazi

Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia

Correspondence should be addressed to G. Muhiuddin; chishtygm@gmail.com

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Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, and topological spaces. This provides sufficient motivation to researchers to review various concepts and results from the realm of abstract algebra in the broader framework of fuzzy setting. In this paper, we introduce the notions of int-soft (m, n)-ideals, int-soft $(m, 0)$-ideals, and int-soft $(0, n)$-ideals of semigroups by generalizing the concept of int-soft bi-ideals, int-soft right ideals, and int-soft left ideals in semigroups. In addition, some of the properties of int-soft (m, n)-ideal, int-soft $(m, 0)$-ideal, and int-soft $(0, n)$-ideal are studied. Also, characterizations of various types of semigroups such as (m, n)-regular semigroups, $(m, 0)$-regular semigroups, and $(0, n)$-regular semigroups in terms of their int-soft (m, n)-ideals, int-soft $(m, 0)$-ideals, and int-soft $(0, n)$-ideals are provided.

1. Introduction

Soft set theory of Molodtsov [1] is an important mathematical tool to dealing with uncertainties and fuzzy or vague objects and has huge applications in real-life situations. In soft sets, the problems of uncertainties deal with enough numbers of parameters which make it more accurate than other mathematical tools. Thus, the soft sets are better than the other mathematical tools to describe the uncertainties. Aktaş and Çağman [2] show that the soft sets are more accurate tools to deal the uncertainties by comparing the soft sets to rough and fuzzy sets. The decision-making problem in soft sets had been considered by Maji et al. [3]. In [4], Maji et al. investigated several operations on soft sets. The notions of soft sets introduced in different algebraic structures had been applied and studied by several authors, for example, Aktaş and Çağman [2] for soft groups, Feng et al. [5] for soft semirings, and Naz and Shabir [6, 7] for soft semi-hypergroups.

Song [8] introduced the notions of int-soft semigroups, int-soft left (resp. right) ideals, and int-soft quasi-ideals. Afterthat, Dudek and Jun [9] studied the properties of int-soft left (resp. right) ideals, and characterizations of these int-soft ideal are obtained. Moreover, they introduced the concept of int-soft (generalized) bi-ideals, and characterizations of (int-soft) generalized bi-ideals and int-soft bi-ideals are obtained. Dudek and Jun [9] introduced and characterized the notion of soft interior ideals of semigroups. The concept of union-soft semigroups, union-soft l-ideals, union-soft r-ideals, and union-soft semiprime soft sets have been considered by [10]. In addition, Muhiuddin et al. studied the soft set theory on various aspects (see, for example, [11–21]). For more related concepts, the readers are referred to [22–31].

The results of this paper are arranged as follows. Section 2 summarises some concepts and properties related to semigroups, soft sets, and int-soft ideals that are required to establish our key results, while Section 3 presents the principle of int-soft (m, n)-ideals. We prove that the int-soft bi-ideals are int-soft (m, n)-ideals for each positive integer m, n, but the converse is not necessarily valid. Then, we prove that the A subset of the S semigroup is (m, n)-ideal of S if and only if $(\widehat{\chi A}, S)$ over U is an int-soft (m, n)-ideal over U. Also, we prove that a soft set $(\widetilde{\mathscr{K}}, S)$ over U is an int-soft (m, n)-ideal over U if and only if $(\widetilde{\mathscr{K}}^{m} \circ \widehat{\chi_S} \circ \widetilde{\mathscr{K}}^{n}, S) \subseteq (\widetilde{\mathscr{K}}, S)$. Moreover, we characterize (m, n) regular semigroups in terms of int-soft (m, n)-ideals over U. In this respect, we prove that a semigroup S is (m, n)-regular if and only if $(\widetilde{\mathscr{K}}, S) = (\widetilde{\mathscr{K}}^{m} \circ \widehat{\chi_S} \circ \widetilde{\mathscr{K}}^{n}, S)$ for each int-soft (m, n)-ideal

$(\widehat{\mathscr{K}}, S)$ over U. In Section 4, first, we present the idea of int-soft $(m, 0)$-ideal and $(0, n)$-ideal over U. After that, we obtain some analogues' results to the previous section. Furthermore, we prove that a semigroup S is (m, n)-regular if and only if $(\widehat{\mathscr{K}} \cap \widehat{\mathscr{G}}, S) = (\widehat{\mathscr{K}}^m \circ \widehat{\mathscr{G}} \cap \widehat{\mathscr{K}} \circ \widehat{\mathscr{G}}^n, S)$ for each int-soft $(m, 0)$-ideal $(\widehat{\mathscr{K}}, S)$ and for each int-soft $(0, n)$-ideal $(\widehat{\mathscr{G}}, S)$ over U. At the end of this section, we provide the existence theorem for int-soft (m, n)-ideal over U and for the minimality of int-soft (m, n)-ideal over U. We also provide a conclusion in Section 5 that contains the direction for certain potential work.

2. Preliminaries

Let S be a semigroup. For $(\varnothing \neq)\Omega, \mho \subseteq S$, $\Omega\mho$ is defined as $\Omega\mho = \{\upsilon\hbar \mid \upsilon \in \Omega, \hbar \in \mho\}$. A subset $(\varnothing \neq)\Omega$ of S is called a sub-semigroup of S if $\upsilon\hbar \in \Omega\, \forall \upsilon, \hbar \in \Omega$. A subset $(\varnothing \neq)\Omega$ of S is called a left (resp. right) ideal of S if $S\Omega \subseteq \Omega$ (resp. $\Omega S \subseteq \Omega$) and is called an ideal of S if Ω is both left and right ideal of S. A sub-semigroup $\mho$ of S is called a bi-ideal of S if $\mho S \mho \subseteq \mho$.

Let U be a universal set and let E be a set of parameters. Let $\mathscr{P}(U)$ denote the power set of U and let $\Omega \subseteq E$. A pair $(\widehat{\mathscr{K}}, \Omega)$ is called a soft set (over U) [32] if $F\colon \Omega \longrightarrow \mathscr{P}(U)$ is a mapping. We denote the set of all soft sets over U with parameter set S by $\mathscr{S}_S(U)$.

Let $(\widehat{\mathscr{K}}, \Omega)$ and $(\widehat{\mathscr{G}}, \mho)$ be soft sets over U. Then, $(G, \mho)$ is called a soft subset of $(\widehat{\mathscr{K}}, \Omega)$ if $\mho \subseteq \Omega$ and $\widehat{\mathscr{G}}(\upsilon) \subseteq \widehat{\mathscr{K}}(\upsilon)$, $\forall \upsilon \in \mho$.

Let $(\widehat{\mathscr{K}}, \Omega)$ and $(\widehat{\mathscr{G}}, \Omega)$ be two soft sets. Then, for each $\upsilon \in \Omega$, the union and intersection are defined as

$$\begin{aligned} (\widehat{\mathscr{K}} \Cup \widehat{\mathscr{G}})(\upsilon) &= \widehat{\mathscr{K}}(\upsilon) \cup \widehat{\mathscr{G}}(\upsilon), \\ (\widehat{\mathscr{K}} \Cap \widehat{\mathscr{G}})(\upsilon) &= \widehat{\mathscr{K}}(\upsilon) \cap \widehat{\mathscr{G}}(\upsilon). \end{aligned} \tag{1}$$

For any two soft sets $(\widehat{\mathscr{K}}, \Omega)$ and $(\widehat{\mathscr{G}}, \Omega)$ of S, the int-soft product $\widehat{\mathscr{K}} \circ \widehat{\mathscr{G}}$ is defined as

$$(\widehat{\mathscr{K}} \circ \widehat{\mathscr{G}})(\upsilon) = \begin{cases} \bigcup_{\upsilon = \hbar\kappa} \left\{\widehat{\mathscr{K}}(\hbar) \cap \widehat{\mathscr{G}}(\kappa)\right\}, & \text{if there exist } \hbar,\ \kappa \in S \text{ such that } \upsilon = \hbar\kappa, \\ \varnothing, & \text{otherwise.} \end{cases} \tag{2}$$

A soft set $(\widehat{\mathscr{K}}, S)$ over U is called an int-soft right (resp. Left) ideal over U if $\widehat{\mathscr{K}}(\upsilon\kappa) \supseteq \widehat{\mathscr{K}}(\upsilon)$ (resp. $\widehat{\mathscr{K}}(\upsilon\kappa) \supseteq \widehat{\mathscr{K}}(\kappa)$) for all $\upsilon, \kappa \in S$. It is called an int-soft ideal over U if it is both int-soft left and int-soft right ideal over U. An int-soft sub-semigroup $(\widehat{\mathscr{K}}, S)$ over U is called an int-soft bi-ideal over U if $\widehat{\mathscr{K}}(\upsilon\kappa\hbar) \supseteq \widehat{\mathscr{K}}(\upsilon) \cap \widehat{\mathscr{K}}(\hbar)$ for all $\upsilon, \kappa, \hbar \in S$. The set of all int-soft left (resp. Right) ideals and int-soft bi-ideals over U will be denoted by $\mathscr{I}_L(U)$ (resp. $\mathscr{I}_R(U)$) and $\mathscr{I}_B(U)$.

More concepts related to our study in different aspects have been studied in [33–39].

For $(\varnothing \neq)\Omega \subseteq S$, the characteristic soft set over U is denoted by $(\widehat{\chi_\Omega}, S)$ and defined as

$$\widehat{\chi_\Omega}(\upsilon) = \begin{cases} U, & \text{if } \upsilon \in \Omega, \\ \varnothing, & \text{if } \upsilon \notin \Omega. \end{cases} \tag{3}$$

Let $(\varnothing \neq)\Omega, \mho \subseteq S$. Then, we have (1) $\chi_\Omega \circ \chi_\mho = \chi_{\Omega\mho}$ and (2) $\chi_\Omega \cap \chi_\mho = \chi_{\Omega\cap\mho}$.

The concept of (m, n)-ideals of semigroups was introduced by Lajos [40] as follows. Let S be a semigroup and m, n be nonnegative integers. Then, a sub-semigroup Ω of S is said to be an (m, n)-ideal of S if $\Omega^m S \Omega^n \subseteq \Omega$. After that, the concept of (m, n)-ideals in various algebraic structures such as ordered semigroups, LA-semigroups, and fuzzy semigroups had been studied by, for instance, Akram et al. [41], Bussaban and Changphas [42], Changphas [43], Mahboob et al. [44], and many others.

We denote by $[\upsilon]_{(m,n)}$ the principal (m, n)-ideal, $[\upsilon]_{(m,0)}$ the principal $(m, 0)$-ideal, and $[\upsilon]_{(0,n)}$ the principal $(0, n)$-ideal generated by an element υ of S, respectively. They were given by Krgovic [45] as follows:

$$\begin{aligned} [\upsilon]_{(m,n)} &= \bigcup_{i=1}^{m+n} \upsilon^i \cup \upsilon^m S \upsilon^n, \\ [\upsilon]_{(m,0)} &= \bigcup_{i=1}^{m} \upsilon^i \cup \upsilon^m S, \\ [\upsilon]_{(0,n)} &= \bigcup_{i=1}^{n} \upsilon^i \cup S \upsilon^n. \end{aligned} \tag{4}$$

In whatever follows, $\mathscr{M}_{(m,n)}$, $\mathscr{M}_{(m,0)}$, and $\mathscr{M}_{(0,n)}$ denote the set of all (m, n)-ideals, $(m, 0)$-ideals, and $(0, n)$-ideals of S.

3. Int-Soft (m, n)-Ideals

Definition 1. An int-soft sub-semigroup $(\widehat{\mathscr{K}}, S)$ over U is called an int-soft (m, n)-ideal over U if

$$\widehat{\mathscr{K}}(\hbar_1\hbar_2, \ldots, \hbar_m\kappa\upsilon_1\upsilon_2, \ldots, \upsilon_m) \supseteq \widehat{\mathscr{K}}(\hbar_1) \cap \widehat{\mathscr{K}}(\hbar_2) \cap \cdots \widehat{\mathscr{K}}(\hbar_m) \cap \widehat{\mathscr{K}}(\upsilon_1) \cap \widehat{\mathscr{K}}(\upsilon_2) \cap \cdots \cap \widehat{\mathscr{K}}(\upsilon_n), \tag{5}$$

for all $\hbar_1, \hbar_2, \ldots, \hbar_n, \kappa, \upsilon_1, \upsilon_2, \ldots, \upsilon_m \in S$.

The set of all int-soft (m,n)-ideals over U will be denoted by $\mathscr{I}_{(m,n)}(U)$.

Example 1. Let $S=\{0,\upsilon,\hbar\}$. Define the binary operation $'\cdot'$ on S as follows.

$\cdot$	0	υ	$\hbar$
0	0	0	0
υ	0	0	0
$\hbar$	0	0	υ

Then, $(S,\cdot)$ is a semigroup. Define $(\widehat{\mathscr{K}},S)\in\mathscr{S}_S(U)$ as

$$\widehat{\mathscr{K}}(\kappa)=\begin{cases}U_1, & \text{if } \kappa\in\{0,\upsilon\},\\ U_2, & \text{if } \kappa=\hbar,\end{cases} \tag{6}$$

where $U_1,U_2\subseteq U$ such that $U_2\subseteq U_1$. It is straightforward to verify that $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)$.

Lemma 1. *In S, $(\widehat{\mathscr{K}},S)\in\mathscr{I}_B(U)\Rightarrow(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)$.*

Proof (straightforward). □

Remark 1. In general, in a semigroup S, $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)\nRightarrow(\widehat{\mathscr{K}},S)\in\mathscr{I}_B(U)$.

Example 2. Let $S=\{0,\upsilon,\hbar,\kappa\}$. Define the binary operation $'\cdot'$ on S as follows.

$\cdot$	0	υ	$\hbar$	κ
0	0	0	0	0
υ	υ	υ	υ	υ
$\hbar$	$\hbar$	$\hbar$	$\hbar$	$\hbar$
κ	0	0	υ	0

Then, S is a semigroup. Define $(\widehat{\mathscr{K}},S)\in\mathscr{S}_S(U)$ as

$$\widehat{\mathscr{K}}(\omega)=\begin{cases}U, & \text{if } \omega\in\{0,\kappa\},\\ \varnothing, & \text{if } \omega\in\{\upsilon,\hbar\}.\end{cases} \tag{7}$$

Then, $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)$, $\forall m,n\geq 2$, but $\widehat{\mathscr{K}}\notin\mathscr{I}_B(U)$ because $\varnothing=\widehat{\mathscr{K}}(\upsilon)=\widehat{\mathscr{K}}(\kappa\hbar 0)\nsupseteq\widehat{\mathscr{K}}(\kappa)\cap\widehat{\mathscr{K}}(0)=U$.

Theorem 1. *Let $(\widehat{\mathscr{K}},S),(\widehat{\mathscr{F}},S)\in\mathscr{I}_{(m,n)}(U)$. Then, $(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}},S)\in\mathscr{I}_{(m,n)}(U)$.*

Proof. Let $\upsilon,\hbar\in S$. We have

$$(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}})(\upsilon\hbar)=\widehat{\mathscr{K}}(\upsilon\hbar)\cap\widehat{\mathscr{F}}(\upsilon\hbar),=\widehat{\mathscr{K}}(\upsilon)\cap\widehat{\mathscr{K}}(\hbar)\cap\widehat{\mathscr{F}}(\upsilon)\cap\widehat{\mathscr{F}}(\hbar)=(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}})(\upsilon)\cap(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}})(\hbar). \tag{8}$$

Let $\upsilon_1,\upsilon_2,\ldots,\upsilon_m,\kappa,\hbar_1,\hbar_2,\ldots,\hbar_n\in S$. Now, we have

$$\begin{aligned}&(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}})(\upsilon_1\upsilon_2,\ldots,\upsilon_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n)\supseteq\widehat{\mathscr{K}}(\upsilon_1\upsilon_2,\ldots,\upsilon_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n)\cap\widehat{\mathscr{F}}(\upsilon_1\upsilon_2,\ldots,\upsilon_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n),\\ &\supseteq\widehat{\mathscr{K}}(\upsilon_1)\cap\widehat{\mathscr{K}}(\upsilon_2)\cap\cdots\cap\widehat{\mathscr{K}}(\upsilon_m)\cap\widehat{\mathscr{K}}(\hbar_1)\cap\widehat{\mathscr{K}}(\hbar_2)\cap\cdots\cap\widehat{\mathscr{K}}(\hbar_n)\cap\widehat{\mathscr{F}}(\upsilon_1)\cap\widehat{\mathscr{F}}(\upsilon_2)\\ &\cap\cdots\cap\widehat{\mathscr{F}}(\upsilon_m)\cap\widehat{\mathscr{F}}(\hbar_1)\cap\widehat{\mathscr{F}}(\hbar_2)\cap\cdots\cap\widehat{\mathscr{F}}(\hbar_n)\\ &\supseteq(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}})(\upsilon_1)\cap(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}})(\upsilon_2)\cap\cdots\cap(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}})(\upsilon_m)\cap(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}})(\hbar_1)\cap(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}})(\hbar_2)\cap\cdots\cap(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}})(\hbar_n).\end{aligned} \tag{9}$$

Therefore, $(\widehat{\mathscr{K}}\,\widetilde{\cap}\,\widehat{\mathscr{F}},S)\in\mathscr{I}_{(m,n)}(U)$. □

Theorem 2. *Let $(\varnothing\neq)\Omega\subseteq S$. Then, $(\varnothing\neq)\Omega\in\mathscr{M}_{(m,n)}\Leftrightarrow(\widehat{\chi_\Omega},S)\in\mathscr{I}_{(m,n)}(U)$.*

Proof. $(\Rightarrow)$ Let $\upsilon_1,\upsilon_2,\ldots,\upsilon_m,\kappa,\hbar_1,\hbar_2,\ldots,\hbar_n\in S$. Below are the cases we have:

Case 1. If $x_k\notin\Omega$ for some $k\in\{1,2,\ldots,m\}$, then

$$\widehat{\chi_\Omega}(\upsilon_1\upsilon_2,\ldots,\upsilon_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n)\supseteq\widehat{\chi_\Omega}(\upsilon_1)\cap\widehat{\chi_\Omega}(\upsilon_2)\cap\cdots\cap\widehat{\chi_\Omega}(\upsilon_m)\cap\widehat{\chi_\Omega}(\hbar_1)\cap\widehat{\chi_\Omega}(\hbar_2)\cap\cdots\cap\widehat{\chi_\Omega}(\hbar_n)\}. \tag{10}$$

Case 2. If $y_l\notin\Omega$ for some $l\in\{1,2,\ldots,n\}$, then

$$\widehat{\chi_\Omega}(\upsilon_1\upsilon_2,\ldots,\upsilon_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n)\supseteq\widehat{\chi_\Omega}(\upsilon_1)\cap\widehat{\chi_\Omega}(\upsilon_2)\cap\cdots\cap\widehat{\chi_\Omega}(\upsilon_m)\cap\widehat{\chi_\Omega}(\hbar_1)\cap\widehat{\chi_\Omega}(\hbar_2)\cap\cdots\cap\widehat{\chi_\Omega}(\hbar_n). \tag{11}$$

When $x_k \notin \Omega$ and $y_l \notin \Omega$ for $k \in \{1,2,\ldots,m\}$ and $l \in \{1,2,\ldots,n\}$ are used in previous cases.

Case 3. If $x_k, y_l \in \Omega$, $\forall k \in \{1,2,\ldots,m\}$ and $l \in \{1,2,\ldots,n\}$, then $v_1v_2,\ldots,v_m z\hbar_1\hbar_2,\ldots,\hbar_n \in \Omega^m S\Omega^n \subseteq \Omega$. Therefore,

$$\begin{aligned}&\widehat{\chi_\Omega}(v_1v_2,\ldots,v_m c\hbar_1\hbar_2,\ldots,\hbar_n) = 1,\\ &\supseteq \widehat{\chi_\Omega}(v_1)\cap\widehat{\chi_\Omega}(v_2)\cap\cdots\cap\widehat{\chi_\Omega}(v_m)\cap\widehat{\chi_\Omega}(\hbar_1)\cap\widehat{\chi_\Omega}(\hbar_2)\cap\cdots\cap\widehat{\chi_\Omega}(\hbar_n)\}.\end{aligned} \tag{12}$$

Hence, $(\widehat{\chi_\Omega}, S) \in \mathcal{I}_{(m,n)}(U)$.

($\Leftarrow$) Let $v_1,v_2,\ldots,v_m,\hbar_1,\hbar_2,\ldots,\hbar_n \in \Omega$ and $\kappa \in S$. Then, $\widehat{\chi_\Omega}(v_1v_2,\ldots,v_m c\hbar_1\hbar_2,\ldots,\hbar_n) \supseteq \widehat{\chi_\Omega}(v_1)\cap\widehat{\chi}_\Omega(v_2)\cap\cdots\cap\widehat{\chi_\Omega}(v_m)\cap\widehat{\chi_\Omega}(\hbar_1)\cap\widehat{\chi_\Omega}(\hbar_2)\cap\cdots\cap\widehat{\chi_\Omega}(\hbar_n) = 1$ implies $\widehat{\chi_\Omega}(v_1v_2,\ldots,v_m c\hbar_1\hbar_2,\ldots,\hbar_n) = 1$. Therefore, $v_1v_2,\ldots,v_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n \in \Omega$. Thus, $\Omega^m S\Omega^n \subseteq \Omega$, as required. □

Theorem 3. *Let* $(\widehat{\mathcal{K}}, S) \in \mathcal{S}_S(U)$. *Then,* $(\widehat{\mathcal{K}}, S) \in \mathcal{I}_{(m,n)}(U) \Leftrightarrow (\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S \circ \widehat{\mathcal{K}}^n, S) \subseteq (\widehat{\mathcal{K}}, S)$.

Proof. ($\Rightarrow$) Let $a \in S$. If $(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S \circ \widehat{\mathcal{K}}^n)(a) = \varnothing$, then $(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S \circ \widehat{\mathcal{K}}^n, S) \subseteq (\widehat{\mathcal{K}}, S)$. In the other case, when $(f^m \circ S \circ f^n)(a) \neq \varnothing$, then there exist elements $r, s \in S$ such that $a = rs$, $(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S)(r) \neq \varnothing$ and $\widehat{\mathcal{K}}^n(s) \neq \varnothing$. As $(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S)(s) \neq \varnothing$, there exist $u_1, v_1 \in S$ such that $x = u_1v_1$, $\widehat{\mathcal{K}}^m(u_1) \neq \varnothing$ and $\widehat{\chi}_S(v_1) = U$. It is easy to show that there exist $u_2, v_2, \ldots, u_m, v_m \in S$ such that, for any $l \in \{2,\ldots,m\}$, we have $u_{l-1} = u_lv_l$, $\widehat{\mathcal{K}}(u_l) \neq 0$ and $\widehat{\mathcal{K}}^{m-l+1}(v_l) \neq \varnothing$. As $\widehat{\mathcal{K}}^n(y) \neq \varnothing$, there exist $u_1', v_1' \in S$ such that $y = u_1'v_1'$, $\widehat{\mathcal{K}}(u_1') \neq \varnothing$ and $\widehat{\mathcal{K}}^{n-1}(v_1') \neq \varnothing$. Similarly, there exist $u_2', v_2', \ldots, u_{n-1}', v_{n-1}' \in S$ such that, for $l \in \{2,\ldots,n-1\}$, we have $u_{l-1} = u_l'v_l', \widehat{\mathcal{K}}(u_l') \neq \varnothing$ and $\widehat{\mathcal{K}}^{n-l}(v_l') \neq \varnothing$. Now, we have

$$\begin{aligned}
\left(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S \circ \widehat{\mathcal{K}}^n\right)(a) &= \bigcup_{a=rs}\left\{\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S\right)(r) \cap \widehat{\mathcal{K}}^n(s)\},\\
&= \bigcup_{a=rs}\left\{\bigcup_{r=u_1v_1}\left\{\widehat{\mathcal{K}}^m(u_1)\cap\widehat{\chi}_S(v_1)\right\}\cap\bigcup_{s=u_1'v_1'}\left\{\widehat{\mathcal{K}}(u_1')\cap\widehat{\mathcal{K}}^{n-1}(v_1')\right\}\right\}\\
&= \bigcup_{a=rs}\bigcup_{r=u_1v_1}\bigcup_{s=u_1'v_1'}\left\{\widehat{\mathcal{K}}^m(u_1)\cap S(v_1)\cap\widehat{\mathcal{K}}(u_1')\cap\widehat{\mathcal{K}}^{n-1}(v_1')\right\}\\
&= \bigcup_{a=rs}\bigcup_{r=u_1v_1}\bigcup_{s=u_1'v_1'}\left\{\widehat{\mathcal{K}}^m(u_1)\cap\widehat{\mathcal{K}}(u_1')\cap\widehat{\mathcal{K}}^{n-1}(v_1')\right\}\\
&= \bigcup_{a=rs}\bigcup_{x=u_1v_1}\bigcup_{y=u_1'v_1'}\left\{\bigcup_{u_1=u_2v_2}\left\{\widehat{\mathcal{K}}(u_2)\cap\widehat{\mathcal{K}}^{m-1}(v_2)\right\}\cap\bigcup_{u_1'=u_2'v_2'}\left\{\widehat{\mathcal{K}}(u_2')\cap\widehat{\mathcal{K}}^{n-2}(v_2')\right\}\cap\widehat{\mathcal{K}}(v_1')\right\}\\
&= \bigcup_{a=xy}\bigcup_{x=u_1v_1}\bigcup_{y=u_1'v_1'}\bigcup_{u_1=u_2v_2}\bigcup_{u_1'=u_2'v_2'}\left\{\widehat{\mathcal{K}}(u_2)\cap\widehat{\mathcal{K}}^{m-1}(v_2)\cap\widehat{\mathcal{K}}(u_2')\cap\widehat{\mathcal{K}}^{n-2}(v_2')\cap\widehat{\mathcal{K}}(v_1')\right\}\\
&= \bigcup_{a=xy}\bigcup_{x=u_1v_1}\bigcup_{y=u_1'v_1'}\bigcup_{u_1=u_2v_2}\bigcup_{u_1'=u_2'v_2'}\cdots\bigcup_{u_{m-1}=u_m,v_m}\bigcup_{u_{n-2}'=u_{n-1}'v_{n-1}'}\\
&\quad\left\{\widehat{\mathcal{K}}(u_2)\cap\widehat{\mathcal{K}}(u_3)\cap\cdots\cap\widehat{\mathcal{K}}(u_m)\cap\widehat{\mathcal{K}}(v_m)\cap\widehat{\mathcal{K}}(u_{n-1}')\cap\widehat{\mathcal{K}}(v_{n-1}')\cap\cdots\cap\widehat{\mathcal{K}}(v_2')\cap\widehat{\mathcal{K}}(v_1')\right\}\\
&\subseteq \bigcup_{a=xy}\left\{\widehat{\mathcal{K}}(u_2u_3,\ldots,u_mv_mv_1u_1'u_2',\ldots,u_{n-1}'v_{n-1}')\right\}\\
&= \bigcup_{a=xy}\left\{\widehat{\mathcal{K}}(xy)\right\}, \quad (\text{since } x = u_2u_3,\ldots,u_mv_mv_1 \text{ and } y = u_1'u_2',\ldots,u_{n-1}'v_{n-1}')\\
&= \widehat{\mathcal{K}}(a).
\end{aligned} \tag{13}$$

($\Leftarrow$) For any $v_1,v_2,\ldots,v_m,\kappa,\hbar_1,\hbar_2,\ldots,\hbar_n \in S$, let $a = v_1v_2,\ldots,v_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n$. Since $(\widehat{\mathcal{K}}^m \circ \widehat{\chi}_S \circ \widehat{\mathcal{K}}^n, S) \subseteq (\widehat{\mathcal{K}}, S)$, we have

$$\begin{aligned}
\widehat{\mathscr{K}}\left(v_1v_2,\ldots,v_m\kappa\hbar_1\hbar_2,\ldots,\hbar_n\right) = \widehat{\mathscr{K}}(a),&\\
&\supseteq \left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathscr{K}}^n\right)(a)\\
&= \bigcup_{a=pq}\left\{\left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\right)(p)\cap\widehat{\mathscr{K}}^n(q)\right\}\\
&\supseteq \left\{\left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\right)\left(v_1v_2,\ldots,v_m\kappa\right)\cap\widehat{\mathscr{K}}^n\left(\hbar_1\hbar_2,\ldots,\hbar_n\right)\right\}\\
&\supseteq \left\{\bigcup_{v_1v_2,\ldots,v_m\kappa=uv}\left\{\widehat{\mathscr{K}}^m(u)\cap\widehat{\chi}_S(v)\right\}\cap\bigcup_{\hbar_1\hbar_2,\ldots,\hbar_n=u'v'}\left\{\widehat{\mathscr{K}}(u')\cap\widehat{\mathscr{K}}^{n-1}(v')\right\}\right\}\\
&\supseteq \left\{\left\{\widehat{\mathscr{K}}^m\left(v_1v_2,\ldots,v_m\right)\cap\widehat{\chi}_S(\kappa)\right\}\cap\left\{\widehat{\mathscr{K}}(\hbar_1)\cap\widehat{\mathscr{K}}^{n-1}\left(\hbar_2,\ldots,\hbar_{n-1}\hbar_n\right)\right\}\right\}\\
&\supseteq \left\{\widehat{\mathscr{K}}^m\left(v_1v_2,\ldots,v_m\right)\cap\widehat{\mathscr{K}}^{n-1}\left(\hbar_1\hbar_2,\ldots,\hbar_{n-1}\right)\cap\widehat{\mathscr{K}}(\hbar_n)\right\}\\
&\vdots\\
&\supseteq \left\{\widehat{\mathscr{K}}(v_1)\cap\widehat{\mathscr{K}}(v_2)\cap\cdots\cap\widehat{\mathscr{K}}(v_m)\cap\widehat{\mathscr{K}}(\hbar_1)\cap\widehat{\mathscr{K}}(\hbar_2)\cap\cdots\cap\widehat{\mathscr{K}}(\hbar_n)\right\}.
\end{aligned}\tag{14}$$

Hence, $(\widehat{\mathscr{K}}, S)\in\mathscr{I}_{(m,n)}(U)$. □

Definition 2. A semigroup S is called the (m,n)-regular if, $\forall\, a\in S\exists x\in S$ such that $a=a^m x a^n$.

Lemma 2. *If S is (m,n)-regular, $(\widehat{\mathscr{K}}, S)\in\mathscr{I}_{(m,n)}(U)\nRightarrow(\widehat{\mathscr{K}}, S)\in\mathscr{I}_B(U)$.*

Proof. Suppose that $(\widehat{\mathscr{K}}, S)\in\mathscr{I}_{(m,n)}(U)$ and $v,\kappa,\hbar\in S$. Since S is (m,n)-regular, $v\kappa\hbar = v^m p v^n\kappa\hbar^m q\hbar^n$ for some $p,q\in S$. Therefore,

$$\begin{aligned}
\widehat{\mathscr{K}}(v\kappa\hbar) &= \widehat{\mathscr{K}}\left(v^m p v^n\kappa\hbar^m q\hbar^n\right),\\
&= \widehat{\mathscr{K}}\left(v^m\left(p v^n\kappa\hbar^m q\right)\hbar^n\right)\\
&\supseteq\left\{\widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}(\hbar)\right\},
\end{aligned}\tag{15}$$

as required. □

Lemma 3. *Let $(\widehat{\mathscr{K}}, S)\in\mathscr{S}_S(U)$. Then, $\widehat{\mathscr{K}}(v)\subseteq\widehat{\mathscr{K}}^l(v^l)$, $\forall l\in\mathbb{Z}^+$ and $v\in S$.*

Proof. Let $v\in S$. As $v^l = vv^{l-1}$, we have

$$\begin{aligned}
\widehat{\mathscr{K}}^l\left(v^l\right) &= \bigcup_{v^l=\hbar\kappa}\left\{\widehat{\mathscr{K}}(\hbar)\cap\widehat{\mathscr{K}}^{l-1}(\kappa)\right\},\\
&\supseteq \widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}^{l-1}\left(v^{l-1}\right)\\
&= \widehat{\mathscr{K}}(v)\cap\bigcup_{v^{l-1}=\hbar'\kappa'}\left\{\widehat{\mathscr{K}}(\hbar')\cap\widehat{\mathscr{K}}^{l-2}(\kappa')\right\}\\
&\supseteq \widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}^{l-2}\left(v^{l-2}\right)\\
&\vdots\\
&\supseteq \widehat{\mathscr{K}}(v)\cap\cdots\cap\widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}(v)\\
&= \widehat{\mathscr{K}}(v).
\end{aligned}\tag{16}$$

□

Theorem 4. *S is (m,n)-regular $\Leftrightarrow(\widehat{\mathscr{K}}, S)\subseteq(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathscr{K}}^n, S)$, $\forall(\widehat{\mathscr{K}}, S)\in\mathscr{S}_S(U)$.*

Proof. ($\Rightarrow$) Let $v\in S$. Then, $v=v^m x v^n$ for some $x\in S$. We have

$$\begin{aligned}
\left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\circ\widehat{\mathscr{K}}^n\right)(v) &= \bigcup_{v=rs}\left\{\left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\right)(r)\cap\widehat{\mathscr{K}}^n(s)\right\},\\
&\supseteq\left(\widehat{\mathscr{K}}^m\circ\widehat{\chi}_S\right)\left(v^m x\right)\cap\widehat{\mathscr{K}}^n\left(v^n\right)\\
&= \bigcup_{v^m x=pq}\left\{\widehat{\mathscr{K}}^m(p)\cap\widehat{\chi}_S(q)\right\}\cap\widehat{\mathscr{K}}^n\left(v^n\right)\\
&\supseteq \widehat{\mathscr{K}}^m\left(v^m\right)\cap\widehat{\chi}_S(x)\cap\widehat{\mathscr{K}}^n\left(v^n\right)\\
&= \widehat{\mathscr{K}}^m\left(v^m\right)\cap\widehat{\mathscr{K}}^n\left(v^n\right)\supseteq\widehat{\mathscr{K}}(v)\cap\widehat{\mathscr{K}}(v),\quad\text{by Lemma 3}\\
&= \widehat{\mathscr{K}}(v).
\end{aligned}\tag{17}$$

Therefore, $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}^{m} \circ \chi_S \circ \widehat{\mathscr{K}}^{n}, S)$.

($\Leftarrow$) Let $v \in S$. Since $(\widehat{\chi_v}, S) \in \mathscr{S}_S(U)$, so by Theorem 2, $(\widehat{\chi_v}, S) \subseteq (\widehat{\chi_v}^{m} \circ \widehat{\chi_S} \circ \widehat{\chi_v}^{n}, S)$. Therefore, $\widehat{\chi_v}(x) \subseteq \widehat{\chi_v}^{m} \circ \widehat{\chi_S} \circ \widehat{\chi_v}^{n}(x) = \chi_{v^m S v^n}(x)$. It follows that $v \in v^m S v^n$, and so, S is (m, n)-regular. □

Theorem 5. *S is (m, n)-regular $\Leftrightarrow (\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}}^{m} \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^{n}, S) \forall (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$.*

Proof. ($\Rightarrow$) Suppose that S is (m, n)-regular and $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$. Then, by Theorems 3 and 4, $(\widehat{\mathscr{K}}^{m} \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^{n}, S) \subseteq (\widehat{\mathscr{K}}, S)$ and $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}^{m} \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^{n}, S)$. Hence, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}}^{m} \circ \chi_S \circ \widehat{\mathscr{K}}^{n}, S)$.

($\Leftarrow$) Suppose that $\omega \in S$. As $[\omega]_{(m,n)} \in \mathscr{M}_{(m,n)}$, by Theorem 2, $(\chi_{[\omega]_{(m,n)}}, S) \in \mathscr{I}_{(m,n)}(U)$. Thus, by hypothesis, we have

$$\chi_{[\omega]_{(m,n)}} = \chi^{m}_{[\omega]_{(m,n)}} \circ \widehat{\chi_S} \circ \chi^{n}_{[\omega]_{(m,n)}} = \chi_{([\omega]_{(m,n)})^{m} S ([\omega]_{(m,n)})^{n}}. \tag{18}$$

Therefore, $[\omega]_{(m,n)} = ([\omega]_{(m,n)})^{m} S ([\omega]_{(m,n)})^{n}$. By Lemma 1 in [4], $[\omega]_{(m,n)} = \omega^m S \omega^n$. Thus, $\omega \in \omega^m S \omega^n$, as required. □

Lemma 4. *If $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$ and $(\widehat{\mathscr{F}}, S)$ is an int-soft sub-semigroup over U, such that*

$$\left(\widehat{\mathscr{K}}^{m} \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^{n}, S\right) \subseteq (\widehat{\mathscr{F}}, S) \subseteq (\widehat{\mathscr{K}}, S), \tag{19}$$

then $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$.

Proof. As $(\widehat{\mathscr{F}}, S)$ is an int-soft sub-semigroup over U, by Theorem 3, it is sufficient to show that $(\widehat{\mathscr{F}}^{m} \circ \widehat{\chi_S} \circ \widehat{\mathscr{F}}^{n}, S) \subseteq (\widehat{\mathscr{F}}, S)$. Now,

$$\left(\widehat{\mathscr{F}}^{m} \circ \widehat{\chi_S} \circ \widehat{\mathscr{F}}^{n}\right)(v) \subseteq \left(\widehat{\mathscr{F}}^{m} \circ \widehat{\chi_S} \circ \widehat{\mathscr{F}}^{n}\right)(v) \subseteq \widehat{\mathscr{F}}(v). \tag{20}$$

Hence, $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$. □

Lemma 5. *Let $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$ and $(\widehat{\mathscr{F}}, S) \in \mathscr{S}_S(U)$. If $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S) \subseteq (\widehat{\mathscr{K}}, S)$ or $(\widehat{\mathscr{F}} \circ \widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}, S)$, then*

(1) $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$

(2) $(\widehat{\mathscr{F}} \circ \widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$

Proof. When $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S) \subseteq (\widehat{\mathscr{K}}, S)$, then we have

$$\begin{aligned}((\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}) \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}))(v) &\subseteq (\widehat{\mathscr{K}} \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}))(v) \\ &= (\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})(v) \\ &\subseteq (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})(v).\end{aligned} \tag{21}$$

It follows that $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S)$ is an int-soft sub-semigroup over U. Also, we have

$$\begin{aligned}\left((\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})^{m} \circ \widehat{\chi_S} \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})^{n}\right)(v) &= \left((\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})^{m} \circ \widehat{\chi_S} \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})^{n-1} \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})\right)(v), \\ &\subseteq \left(\widehat{\mathscr{K}}^{m} \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^{n-1} \circ (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})\right)(v) \\ &\subseteq \left(\widehat{\mathscr{K}}^{m} \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^{n} \circ \widehat{\mathscr{F}}\right)(v) \subseteq (\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}})(v).\end{aligned} \tag{22}$$

Thus, $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$. Similarly, when $(\widehat{\mathscr{F}} \circ \widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}, S)$, then $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,n)}(U)$. Similar to (1), it can be verified. □

4. Int-Soft $(m, 0)$-Ideals and Int-Soft $(0, n)$-Ideals

Definition 3. An int-soft sub-semigroup $(\widehat{\mathscr{K}}, S)$ over S is called an int-soft $(m, 0)$-ideal over U if

$$\widehat{\mathscr{K}}(v_1 v_2, \ldots, v_m \kappa) \supseteq \widehat{\mathscr{K}}(v_1) \cap \widehat{\mathscr{K}}(v_2) \cap \cdots \cap \widehat{\mathscr{K}}(v_m), \tag{23}$$

for all $v_1, v_2, \ldots, v_m, \kappa \in S$.

An int-soft $(0, n)$-ideal can be described dually.

Whatever follows, we denote the set of all int-soft $(m, 0)$-ideals and $(0, n)$-ideals over U by $\mathscr{I}_{(m,0)}(U)$ and $\mathscr{I}_{(0,n)}(U)$.

Example 3. Let $S = \{0, v, \hbar, \kappa\}$. Define the binary operation $' \cdot '$ on S as follows.

$\cdot$	0	v	$\hbar$	κ
0	0	0	0	0
v	0	0	κ	0
$\hbar$	0	0	0	0
κ	0	0	0	0

Then, S is a semigroup. Define $(\widehat{\mathscr{K}}, S), (\widehat{\mathscr{F}}, S) \in \mathscr{S}_S(U)$ as

$$\begin{aligned}\widehat{\mathscr{K}}(\omega) &= \begin{cases} U, & \text{if } \omega \in \{0, \hbar\}, \\ \varnothing, & \text{if } \omega \in \{v, \kappa\}, \end{cases} \\ \widehat{\mathscr{F}}(\omega) &= \begin{cases} V, & \text{if } \omega \in \{0, v\}, \\ \varnothing, & \text{if } \omega \in \{\hbar, \kappa\}. \end{cases}\end{aligned} \tag{24}$$

It is straightforward to verify that $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(0,n)}(U)$.

Lemma 6. *In S, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_R(U)$ (resp. $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_L(U)$) $\Rightarrow (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$ (resp. $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(0,n)}(U)$).*

Proof (straightforward). □

Remark 2. In general, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$ (resp. $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(0,n)}(U)) \nRightarrow (\widehat{\mathscr{K}}, S) \in \mathscr{I}_R(U)$ (resp. $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_L(U)$).

Example 4. In Example 3, $(\widehat{\mathscr{K}}, S) \in \mathcal{S}_S(U) \Rightarrow (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U), \mathscr{I}_{(0,n)}(U)\ \forall m, n \geq 2$, but $(\widehat{\mathscr{K}}, S) \notin \mathscr{I}_R(U), \mathscr{I}_L(U)$.

Definition 4. A semigroup S is called the $(m,0)$-regular (resp. $(0,n)$-regular) if $\forall v \in S \exists \hbar \in S$ such that $v = v^m \hbar$ (resp. $v = \hbar v^n$).

Lemma 7. *The following assertions hold:*

(1) *In $(m,0)$-regular S, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U) \Rightarrow (\widehat{\mathscr{K}}, S) \in \mathscr{I}_R(U)$*

(2) *In $(0,n)$-regular S, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(0,n)}(U) \Rightarrow (\widehat{\mathscr{K}}, S) \in \mathscr{I}_L(U)$*

Proof. Let $v, \hbar \in S$. Since S is $(m,0)$-regular, so $\exists \kappa \in S$ such that $v\hbar = v^m \kappa \hbar$. Therefore, we have

$$\widehat{\mathscr{K}}(v\hbar) = \widehat{\mathscr{K}}(v^m \kappa \hbar) = \widehat{\mathscr{K}}(v^m(\kappa\hbar)) \supseteq \widehat{\mathscr{K}}(v). \tag{25}$$

Hence, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_R(U)$. (2). Similarly, this can be proved. □

Lemma 8. *Let $(\varnothing \neq)\Omega \subseteq S$. Then, $\Omega \in \mathscr{M}_{(m,0)}$ (resp. $\Omega \in\in \mathscr{M}_{(0,n)}$) $\Leftrightarrow$ the $(\widehat{\chi_\Omega}, S) \in \mathscr{I}_{(m,0)}(U)$ (resp. $(\widehat{\chi_\Omega}, S) \in \mathscr{I}_{(0,n)}(U)$).*

Proof. $(\Rightarrow)$ Let $v_1, v_2, \ldots, v_m, \kappa \in S$. If $x_k \notin \Omega$, for some $k \in \{1, 2, \ldots, m\}$, then $\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m \kappa) \supseteq \widehat{\chi_\Omega}(v_1) \cap \widehat{\chi_\Omega}(v_2) \cap \cdots \cap \widehat{\chi_\Omega}(v_m)$. If $x_k \in \Omega$ for each $k \in \{1, 2, \ldots, m\}$, then $v_1 v_2, \ldots, v_m \kappa \in \Omega^m S \subseteq \Omega$. Therefore,

$$\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m c) = 1 \supseteq \widehat{\chi_\Omega}(v_1) \cap \widehat{\chi_\Omega}(v_2) \cap \cdots \cap \widehat{\chi_\Omega}(v_m). \tag{26}$$

Hence, $(\widehat{\chi_\Omega}, S) \in \mathscr{I}_{(m,0)}(U)$.

$(\Leftarrow)$ Let $v_1, v_2, \ldots, v_m \in \Omega$ and $\kappa \in S$. Then, $\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m c) \supseteq \widehat{\chi_\Omega}(v_1) \cap \widehat{\chi_\Omega}(v_2) \cap \cdots \cap \widehat{\chi_\Omega}(v_m) = 1$ implies $\widehat{\chi_\Omega}(v_1 v_2, \ldots, v_m c) = 1$. Therefore, $v_1 v_2, \ldots, v_m c \in \Omega$. Thus, $\Omega^m S \subseteq \Omega$, as required. □

Theorem 6. *Let $(\widehat{\mathscr{K}}, S)$ be any int-soft sub-semigroup over U. Then, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$ (resp. $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(0,n)}(U)$) $\Leftrightarrow (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S) \subseteq (\widehat{\mathscr{K}}, S)$ (resp. $(\widehat{\chi_S} \circ \widehat{\mathscr{K}}^n, S) \subseteq (\widehat{\mathscr{K}}, S)$).*

Proof. It is similar to the proof of Theorem 3. □

Lemma 9. *Let S be (m,n)-regular, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$, and $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(0,n)}(U)$. Then, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}}, S)$ and $(\widehat{\mathscr{F}}, S) = (\widehat{\mathscr{F}} \circ \widehat{\mathscr{F}}, S)$.*

Proof. Let $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$. Then, $(\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}, S)$. We have

$$\begin{aligned}\widehat{\mathscr{K}}(x) &\subseteq \left(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^n\right)(x) = \left(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^{n-1} \circ \widehat{\mathscr{K}}\right)(x),\\ &\subseteq \left(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^{n-1} \circ \widehat{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^n\right)(x)\\ &\subseteq \left(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S} \circ \widehat{\mathscr{K}}^m \circ \widehat{\chi_S}\right)(x)\\ &\subseteq (\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}})(x),\end{aligned} \tag{27}$$

so we obtain $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}}, S)$. Hence, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}} \circ \widehat{\mathscr{K}}, S)$. □

Theorem 7. *In S, the following assertions are true:*

(1) *S is $(m,0)$-regular $\Leftrightarrow (\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S)$, $\forall (\widehat{\mathscr{K}}, S) \in \mathcal{S}_S(U)$*

(2) *S is $(0,n)$-regular $\Leftrightarrow (\widehat{\mathscr{K}}, S) \subseteq (\widehat{\chi_S} \circ \widehat{\mathscr{K}}^n, S)$, $\forall (\widehat{\mathscr{K}}, S) \in \mathcal{S}_S(U)$*

Proof. $(\Rightarrow)$ Let $v \in S$. Then, $\exists \hbar \in S$ such that $v = v^m \hbar$. Now, we have

$$\begin{aligned}\left(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}\right)(v) &= \bigcup_{v=\kappa s}\left\{\left(\widehat{\mathscr{K}}^m\right)(\kappa) \cap \widehat{\chi_S}(s)\right\},\\ &\supseteq \widehat{\mathscr{K}}^m(v^m) \cap \widehat{\chi_S}(\hbar)\\ &= \widehat{\mathscr{K}}^m(v^m)\\ &\supseteq \widehat{\mathscr{K}}(v).\end{aligned} \tag{28}$$

Therefore, $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S)$.

$(\Leftarrow)$ Take any $v \in S$. Since $(\widehat{\chi_v}, S) \in \mathcal{S}_S(U)$, $(\widehat{\chi_v}, S) \subseteq (\widehat{\chi_v}^m \circ S, S)$. Therefore, $\widehat{\chi_v}(\hbar) \subseteq \widehat{\chi_v}^m \circ S(\hbar) = \widehat{\chi_{v^m S}}(\hbar)$. It follows that $v \in v^m S$ and so, S is $(m,0)$-regular. Similar to (1), (2) can be verified. □

Theorem 8. *The following assertions are true in S:*

(1) *S is $(m,0)$-regular $\Leftrightarrow (\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S)$, $\forall (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$*

(2) *S is $(0,n)$-regular $\Leftrightarrow (\widehat{\mathscr{K}}, S) = (\widehat{\chi_S} \circ \widehat{\mathscr{K}}^n, S)$, $\forall (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(0,n)}(U)$*

Proof. (1) $(\Rightarrow)$ Suppose that S is $(m,0)$-regular and $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$. Then, by Theorems 7 and 6, we have $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S)$ and $(\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S) \subseteq (\widehat{\mathscr{K}}, S)$. Hence, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}}^m \circ \widehat{\chi_S}, S)$.

$(\Leftarrow)$ Take $R \in\in \mathscr{M}_{(m,0)}$. By Lemma 8, $(\chi_R, S) \in \mathscr{I}_{(m,0)}(U)$. By hypothesis $(\chi_R, S) = (\chi_R^m \circ \widehat{\chi_S}, S)$. So, $\chi_R(\hbar) = \chi_R^m \circ \widehat{\chi_S}(\hbar) = \chi_{R^m S}(\hbar)$, and it follows that $R^m S = R$. Therefore, by Theorem 1 in [45], S is (m,n)-regular. Similar to (1), (2) can be verified. □

Theorem 9. *S is (m,n)-regular $\Leftrightarrow (\widehat{\mathscr{K}} \cap \widehat{\mathscr{F}}, S) = (\widehat{\mathscr{K}}^m \circ \widehat{\mathscr{F}}^n, S)$, $\forall (\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$, and $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(0,n)}(U)$.*

Proof. $(\Rightarrow)$ Suppose that $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(0,n)}(U)$. As S is (m,n)-regular, we have

$$(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)\subseteq\left(\left(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}}\right)^{m}\circ\widehat{\chi}_{S}\circ\left(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}}\right)^{n},S\right)\subseteq\left(\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_{S}\circ\widehat{\mathscr{F}}^{n},S\right). \tag{29}$$

By Theorem 8 and Lemma 9, we have $(\widehat{\chi}_S\circ\widehat{\mathscr{F}}^{n},S)=(\widehat{\mathscr{F}},S)$ and $(\widehat{\mathscr{F}},S)=(\widehat{\mathscr{F}}^{n},S)$. Therefore, $(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)\subseteq(\widehat{\mathscr{K}}^{m}\circ\widehat{\mathscr{F}}^{n},S)$. Also, $(\widehat{\mathscr{K}}^{m}\circ\widehat{\mathscr{F}}^{n},S)\subseteq(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)$. Therefore, $(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)=(\widehat{\mathscr{K}}^{m}\circ\widehat{\mathscr{F}}^{n},S)$.

($\Leftarrow$) Take $R\in\mathscr{M}_{(m,0)}$ and $L\in\mathscr{M}_{(0,n)}$. By Lemma 2, $(\widehat{\chi}_R,S)\in\mathscr{I}_{(m,0)}(U)$ and $(\widehat{\chi}_L,S)\in\mathscr{I}_{(0,n)}(U)$. By hypothesis, we have

$$\widehat{\chi}_{R\cap L}=\widehat{\chi}_R\wedge\widehat{\chi}_L=\widehat{\chi}_R{}^{m}\circ\widehat{\chi}_L{}^{n}=\widehat{\chi}_{R^mL^n}, \tag{30}$$

it follows that $R\cap L=R^mL^n$. Thus, by Theorem 12 in [44], S is (m,n)-regular. □

Corollary 1. *If S is (m,n)-regular, then $(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)=(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}},S)$, $\forall(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}},S)\in\mathscr{I}_{(0,n)}(U)$.*

Theorem 10. *S is (m,n)-regular $\Leftrightarrow(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)=(\widehat{\mathscr{K}}^{m}\circ\widehat{\mathscr{F}}\cap\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}}^{n},S)\,\forall\,(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}},S)\in\mathscr{I}_{(0,n)}(U)$.*

Proof. ($\Rightarrow$) Suppose that $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}},S)\in\mathscr{I}_{(0,n)}(U)$. As S is (m,n)-regular, we have

$$(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)\subseteq\left(\left(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}}\right)^{m}\circ\widehat{\chi}_{S}\circ\left(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}}\right)^{n},S\right)\subseteq\left(\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_{S}\circ\widehat{\mathscr{F}}^{n},S\right)\subseteq\left(\widehat{\mathscr{K}}^{m}\circ\widehat{\mathscr{F}},S\right), \tag{31}$$

and so, $(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)\subseteq(\widehat{\mathscr{K}}^{m}\circ\widehat{\mathscr{F}},S)$. Similarly, $(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)\subseteq(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}}^{n},S)$. Thus, $(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)\subseteq(\widehat{\mathscr{K}}^{m}\circ\widehat{\mathscr{F}}\cap\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}}^{n},S)$. Since $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}},S)\in\mathscr{I}_{(0,n)}(U)$, the reverse inclusion holds. Hence, $(\widehat{\mathscr{K}}\cap\widehat{\mathscr{F}},S)=(\widehat{\mathscr{K}}^{m}\circ\widehat{\mathscr{F}}\cap\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}}^{n},S)$.

($\Leftarrow$) Take $R\in\mathscr{M}_{(m,0)}$ and $L\in\mathscr{M}_{(0,n)}$. By Lemma 8, $(\widehat{\chi}_R,S)\in\mathscr{I}_{(m,0)}(U)$ and $(\widehat{\chi}_L,S)\in\mathscr{I}_{(0,n)}(U)$. Observe that, by hypothesis, we have

$$\widehat{\chi}_{R\cap L}=\widehat{\chi}_R\cap\widehat{\chi}_L=\widehat{\chi}_R{}^{m}\circ\widehat{\chi}_L\cap\widehat{\chi}_R\circ\widehat{\chi}_L{}^{n}=\widehat{\chi}_{R^mL\cap RL^n}, \tag{32}$$

and it follows that $R\cap L=R^mL\cap RL^n$. Therefore, by Theorem 3 in [45], S is (m,n)-regular. □

Lemma 10. *For $(\widehat{\mathscr{K}},S)\in\mathscr{S}_S(U)$, $(\widehat{\mathscr{K}}\cup\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S,S)\in\mathscr{I}_{(m,0)}(U)$ (resp. $(\widehat{\mathscr{K}}\cup\widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n},S)\in\mathscr{I}_{(0,n)}(U)$).*

Proof (straightforward). □

Lemma 11. *In (m,n)-regular semigroup S, for each $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)$, there exist $(\widehat{\mathscr{G}},S)\in\mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}},S)\in\mathscr{I}_{(0,n)}(U)$ such that $(\widehat{\mathscr{K}},S)=(\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}},S)$.*

Proof. Suppose that $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,n)}(U)$. Then, $(\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n},S)\subseteq(\widehat{\mathscr{K}},S)$. As S is (m,n)-regular, $(\widehat{\mathscr{K}},S)\subseteq(\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n},S)$. Therefore, $(\widehat{\mathscr{K}},S)=(\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n},S)$. Let $(\widehat{\mathscr{G}},S)=(\widehat{\mathscr{K}}\cup\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S,S)$ and $(\widehat{\mathscr{F}},S)=(\widehat{\mathscr{K}}\cup\widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n},S)$. By Lemma 9, $(\widehat{\mathscr{G}},S)\in\mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}},S)\in\mathscr{I}_{(0,n)}(U)$. Since S is (m,n)-regular, $(\widehat{\mathscr{G}},S)=(\widehat{\mathscr{K}}\cup\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S,S)=(\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S,S)$ and $(\widehat{\mathscr{F}},S)=(\widehat{\mathscr{K}}\cup\widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n},S)=(\widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n},S)$, so

$$(\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}},S)=\left(\widehat{\mathscr{K}}^{m}\circ S\circ S\circ\widehat{\mathscr{K}}^{n},S\right)=\left(\widehat{\mathscr{K}}^{m}\circ S\circ\widehat{\mathscr{K}}^{n},S\right)=(\widehat{\mathscr{K}},S), \tag{33}$$

as required. □

Lemma 12. *In (m,n)-regular semigroup S, $\forall(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}},S)\in\mathscr{S}_S(U)$, $(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}},S)\in\mathscr{I}_{(m,n)}(U)$.*

Proof. Let $(\widehat{\mathscr{K}},S)\in\mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}},S)\in\mathscr{S}_S(U)$. Now,

$$\begin{aligned}\left(\left(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}}\right)^{m}\circ\widehat{\chi}_S\circ\left(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}}\right)^{n}\right)(v)&=\left(\underbrace{(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\circ(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\circ\cdots\circ(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})}_{m-\text{times}}\circ\widehat{\chi}_S\circ\underbrace{(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\circ(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\circ\cdots\circ(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})}_{n-\text{times}}\right)(v),\\&=\left((\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\circ\underbrace{(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\circ(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\circ\cdots\circ(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})}_{m-1-\text{times}}\circ\widehat{\chi}_S\circ\underbrace{(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\circ(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\circ\cdots\circ(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})}_{n-1-\text{times}}\circ(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\right)(v)\\&\subseteq\left((\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\circ\widehat{\chi}_S\circ\widehat{\chi}_S\circ\widehat{\chi}_S\circ(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})\right)(v)\subseteq\left(\widehat{\mathscr{K}}\circ\widehat{\chi}_S\circ\widehat{\mathscr{F}}\right)(v)\\&\subseteq\left(\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S\circ\widehat{\mathscr{F}}\right)(v)\quad\text{(by Lemma 9)}\\&\subseteq(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}})(v).\end{aligned} \tag{34}$$

Therefore, $(\widehat{\mathscr{K}}\circ\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(\mathrm{m,n})}(\mathrm{U})$.

By Lemmas 11 and 12, we have the following. ☐

Theorem 11. *Let S be a (m,n)-regular and $(\widehat{\mathscr{K}}, S) \in \mathscr{S}_S(U)$. Then, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U) \Leftrightarrow$ there exist $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(m,0)}(U)$ and $(\widehat{\mathscr{F}}, S) \in \mathscr{I}_{(0,n)}(U)$ such that $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{F}}\circ\widehat{\mathscr{F}}, S)$.*

Definition 5. An int-soft (m,n)-ideal $(\widehat{\mathscr{K}}, S)$ over U is called minimal if, for all int-soft (m,n)-ideal $(\widehat{\mathscr{K}}', S)$ over U, $(\widehat{\mathscr{K}}', S) \subseteq (\widehat{\mathscr{K}}, S)$ implies $(\widehat{\mathscr{K}}', S) = (\widehat{\mathscr{K}}, S)$.

Dually, a minimum int-soft $(m,0)$-ideal and minimal int-soft $(0,n)$-ideal over U can be described.

Theorem 12. *In (m,n)-regular semigroup S, a soft set $(\widehat{\mathscr{K}}, tS)$ over U is a minimal int-soft (m,n)-ideal over $U \Leftrightarrow$ there exist a minimal int-soft $(m,0)$-ideal $(\widehat{\mathscr{G}}, S)$ and a minimal int-soft $(0,n)$-ideal $(\widehat{\mathscr{F}}, S)$ over U such that $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}, S)$.*

Proof. $(\Rightarrow)$ Let $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)(U)}$ be minimal. By Lemma 11, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S)\circ(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S)$. We show that $(\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S) \in \mathscr{I}_{(\mathrm{m},0)(\mathrm{U})}$ is minimal. To show this, let $(\widehat{\mathscr{K}}', S) \in \mathscr{I}_{(m,0)}(U)$ such that $(\widehat{\mathscr{K}}', S) \subseteq (\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S)$. Since S is (m,n)-regular, so, by Corollary 1, $(\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S) \cap (\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S) = (\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S)\circ(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S)$. Again, by Corollary 1, $(\widehat{\mathscr{K}}', S)\circ(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S) = (\widehat{\mathscr{K}}', S)\cap(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S) \subseteq (\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S)\cap(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S) = (\widehat{\mathscr{K}}, S)$. By Lemma 12, $(\widehat{\mathscr{K}}', S)\circ(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S) \in \mathscr{I}_{(m,n)}(U)$. Since $(\widehat{\mathscr{K}}', S)\circ(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S) \subseteq (\widehat{\mathscr{K}}, S)$, by minimality of the int-soft (m,n)-ideal $(\widehat{\mathscr{K}}, S)$ over U, we have $(\widehat{\mathscr{K}}', S)\circ(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S) = (\widehat{\mathscr{K}}, S)$. Therefore, $(\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S)\cap(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S) = (\widehat{\mathscr{K}}', S)\cap(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S)$. As $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S)\cap(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S)$, we have $(\widehat{\mathscr{K}}, S) \subseteq (\widehat{\mathscr{K}}', S)$. So, $(\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S) \subseteq (\widehat{\mathscr{K}}', S)$. Hence, $(\widehat{\mathscr{K}}', S) = (\widehat{\mathscr{K}}, S)$. Thus, $(\widehat{\mathscr{K}} \cup \widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S) \in \mathscr{I}_{(\mathrm{m},0)(\mathrm{U})}$ is minimal. Similarly, $(\widehat{\mathscr{K}} \cup \widehat{\chi}_S\circ\widehat{\mathscr{K}}^{n}, S) \in \mathscr{I}_{(0,\mathrm{n})(\mathrm{U})}$ is minimal.

$(\Leftarrow)$ Assume that $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}, S)$ for some minimal int-soft $(m,0)$-ideal $(\widehat{\mathscr{G}}, S)$ and minimal int-soft $(0,n)$-ideal $(\widehat{\mathscr{F}}, S)$ over U. By Lemma 11, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)}(U)$. To show that $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)(U)}$ is minimal, let $(\widehat{\mathscr{W}}, S) \in \mathscr{I}_{(m,n)}(U)$ such that $(\widehat{\mathscr{W}}, S) \subseteq (\widehat{\mathscr{K}}, S)$. Then, $(\widehat{\mathscr{W}}^{m}\circ\widehat{\chi}_S, S) \subseteq (\widehat{\mathscr{K}}^{m}\circ\widehat{\chi}_S, S) \subseteq ((\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}})^{m}\circ\widehat{\chi}_S, S) = ((\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}})\circ(\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}})\circ\cdots\circ(\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}})\circ\widehat{\chi}_S) \subseteq ((\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}})\circ(\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}})\circ\cdots\circ(\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}})\circ\widehat{\chi}_S \subseteq \widehat{\mathscr{G}}\circ\widehat{\chi}_S \subseteq (\widehat{\mathscr{G}}^{m}\circ\widehat{\chi}_S\circ\widehat{\mathscr{G}}^{n})\circ\widehat{\chi}_S \subseteq \widehat{\mathscr{G}}^{m}\circ\widehat{\chi}_S \subseteq \widehat{\mathscr{G}}$.

As $(\widehat{\mathscr{W}}^{m}\circ\widehat{\chi}_S, S) \in \mathscr{I}_{(\mathrm{m},0)}(\mathrm{U})$ and $(\widehat{\mathscr{G}}, S) \in \mathscr{I}_{(m,0)(U)}$ is minimal, $(\widehat{\mathscr{W}}^{m}\circ\widehat{\chi}_S, S) = (\widehat{\mathscr{G}}, S)$. Similarly, $(\widehat{\chi}_S\circ\widehat{\mathscr{W}}^{n}, S) = (\widehat{\mathscr{F}}, S)$. Now, $(\widehat{\mathscr{K}}, S) = (\widehat{\mathscr{G}}\circ\widehat{\mathscr{F}}, S) = ((\widehat{\mathscr{W}}^{m}\circ\widehat{\chi}_S)\circ(\widehat{\chi}_S\circ\widehat{\mathscr{W}}^{n}), S) \subseteq (\widehat{\mathscr{W}}^{m}\circ\widehat{\chi}_S\circ\widehat{\mathscr{W}}^{n}, S) \subseteq (\widehat{\mathscr{W}}, S)$. Hence, $(\widehat{\mathscr{K}}, S) \in \mathscr{I}_{(m,n)(U)}$ is minimal. ☐

Corollary 2. *There is at least one minimal int-soft (m,n)-ideal over U in (m,n)-regular semigroup $S \Leftrightarrow S$ has at least one minimal int-soft $(m,0)$-ideal and one minimal int-soft $(0,n)$-ideal over U.*

5. Conclusion

The main purpose of this article is to present in semigroups the ideas of int-soft (m,n)-ideals, int-soft $(m,0)$-ideals, and int-soft $(0,n)$-ideals. If we take $m = 1 = n$ in the int-soft (m,n)-ideals, int-soft $(m,0)$-ideals, and int-soft $(0,n)$-ideals in particular, then we get the int-soft bi-ideals, int-soft right ideals, and int-soft left ideals. The ideas proposed in this paper can also be seen to be more general than int-soft bi-ideals, int-soft right ideals, and int-soft left ideals. Also, if we place $m = 1 = n$ in the results of this paper, then the results of [8] are deduced as corollaries, which is the main application of the results of this paper.

In the future work, one can further study these concepts to various algebraic structures such as semi-hypergroups, semi-hyperrings, rings, LA-semigroups, BL-algebras, MTL-algebras, R0-algebras, MV-algebras, EQ-algebras, and lattice implication algebras.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Generalization of Fuzzy Soft BCK/BCI-Algebras

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Generalization of Fuzzy Soft BCK/BCI-Algebras

N. Alam,[1] G. Muhiuddin ,[2] S. Obeidat,[1] H. N. Zaidi ,[1] A. Altaleb,[1] and J. M. Aqib[3]

[1]*Department of Basic Sciences, Deanship of Preparatory Year, University of Háil, Háil 2440, Saudi Arabia*
[2]*Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia*
[3]*Department of Software Engineering, College of Computer Science, University of Háil, Háil 2440, Saudi Arabia*

Correspondence should be addressed to G. Muhiuddin; chishtygm@gmail.com

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In this paper, the notions of $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebras and $(\in, \in\vee q)$-fuzzy soft sub-BCK/BCI-algebras are introduced, and related properties are investigated. Furthermore, relations between fuzzy soft BCK/BCI-algebras and $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebras are displayed. Moreover, conditions for an $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebra to be a fuzzy soft BCK/BCI-algebra are provided. Also, the union, the extended intersection, and the "AND"-operation of two $(\in, \in\vee q)$-fuzzy soft (sub-)BCK/BCI-algebras are discussed, and a characterization of an $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebra is established.

1. Introduction

The uncertainty which appeared in economics, engineering, environmental science, medical science, social science, and so on is too complicated to be captured within a traditional mathematical framework. In order to overcome this situation, a number of approaches including fuzzy set theory [1, 2], probability theory, rough set theory [3, 4], vague set theory [5], and the interval mathematics [6] have been developed. The concept of soft set was introduced by Molodtsov [7] as a new mathematical method to deal with uncertainties free from the errors being occurred in the existing theories. Later, Maji et al. [8, 9] defined fuzzy soft sets and also described how soft set theory is applied to the problem of decision making. Study on the soft set theory is currently moving forward quickly. In [10], Jun et al. discussed the intersection-soft filters in R_0-algberas. Roh and Jun [11] studied positive implicative ideals of BCK-algebras based on intersectional soft sets. Roy and Mayi [12] gave results on applying fuzzy soft sets to the problem of decision making. Aygünoğlu and Aygün [13] proposed and investigated the notion of a fuzzy soft group. Furthermore, Jun et al. [14] applied the theory of fuzzy soft sets to BCK/BCI-algebras and introduced the notion of fuzzy soft BCK/BCI-algebras (briefly, FSB-algebras) and related notions. Moreover, Muhiuddin et al. studied and applied the soft set theory to the different algebraic structures on various aspects (see, e.g., [15–23]). Also, some related concepts based on the present work are studied in [24–33].

In this paper, we define the notions of $(\in, \in\vee q)$-FSB-algebras and $(\in, \in\vee q)$-fuzzy soft sub-BCK/BCI-algebras. Further, we investigate related properties and consider relations between fuzzy soft BCK/BCI-algebras and $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebras. Moreover, we prove that every FSB-algebra over X is an $(\in, \in\vee q)$-FSB-algebra over X and also show by an example that the converse of the aforesaid statement is not true in general. In fact, we provide a condition for an $(\in, \in\vee q)$-FSB-algebra to be a FSB-algebra. In addition, we discuss the union, the extended intersection, and the "AND"-operation of two $(\in, \in\vee q)$-FSB-algebras. Finally, we establish a characterization of an $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebra. The paper is organized as follows. Section 2 summarizes some definitions and properties related to BCK/BCI-algebras, fuzzy sets, soft sets, and fuzzy soft sets which are needed to develop our main results. In Section 3, the notions of FSB-algebras are studied and the concepts of θ-identity and θ-absolute FSB-algebras are introduced. Section 4 is devoted to the study of $(\in, \in\vee q)$-FSB-algebra. The paper ends with a conclusion and a list of references.

2. Preliminaries

A BCK/BCI-algebra is the most important class of logical algebras which was introduced by K. Iséki.

By a BCI-algebra, we mean a system $(\tilde{\mathscr{X}};\ *, 0)$, where $\tilde{\mathscr{X}}$ be a nonempty set with a constant 0 and a binary operation $*$ if

(i) $(\forall \varpi, \varrho, \vartheta \in \tilde{\mathscr{X}})(((\varpi * \varrho) * (\varpi * \vartheta)) * (\vartheta * \varrho) = 0)$

(ii) $(\forall \varpi, \varrho \in \tilde{\mathscr{X}})((\varpi * (\varpi * \varrho)) * \varrho = 0)$

(iii) $(\forall \varpi \in \tilde{\mathscr{X}})(\varpi * \varpi = 0)$

(iv) $(\forall \varpi, \varrho \in \tilde{\mathscr{X}})(\varpi * \varrho = 0, \varrho * \varpi = 0 \Rightarrow \varpi = \varrho)$

If a BCI-algebra $\tilde{\mathscr{X}}$ satisfies

(v) $(\forall \varpi \in \tilde{\mathscr{X}})(0 * \varpi = 0)$,

then $\tilde{\mathscr{X}}$ is called a BCK-algebra. Any BCK-algebra $\tilde{\mathscr{X}}$ satisfies

(a1) $(\forall \varpi \in \tilde{\mathscr{X}})(\varpi * 0 = \varpi)$,

(a2) $(\forall \varpi, \varrho, \vartheta \in \tilde{\mathscr{X}})(\varpi \leq \varrho \Longrightarrow \varpi * \vartheta \leq \varrho * \vartheta, \vartheta * \varrho \leq \vartheta * \varpi)$,

(a3) $(\forall \varpi, \varrho, \vartheta \in \tilde{\mathscr{X}})((\varpi * \varrho) * \vartheta = (\varpi * \vartheta) * \varrho)$,

(a4) $(\forall \varpi, \varrho, \vartheta \in \tilde{\mathscr{X}})((\varpi * \vartheta) * (\varrho * \vartheta) \leq \varpi * \varrho)$

where $\varpi \leq \varrho$ if and only if $\varpi * \varrho = 0$.

The following conditions are satisfied in any BCI-algebra $\tilde{\mathscr{X}}$:

(a5) $(\forall \varpi, \varrho, \vartheta \in \tilde{\mathscr{X}})(0 * (0 * ((\varpi * \vartheta) * (\varrho * \vartheta))) = (0 * \varrho) * (0 * \varpi))$.

(a6) $(\forall \varpi, \varrho \in \tilde{\mathscr{X}})(0 * (0 * (\varpi * \varrho)) = (0 * \varrho) * (0 * \varpi))$.

In a BCK/BCI-algebra $\tilde{\mathscr{X}}$, a nonempty subset T of $\tilde{\mathscr{X}}$ is called a BCK/BCI-subalgebra of $\tilde{\mathscr{X}}$ if $\varpi * \varrho \in T\, \forall \varpi, \varrho \in T$.

In a BCK/BCI-algebra $\tilde{\mathscr{X}}$, a fuzzy set μ in $\tilde{\mathscr{X}}$ is called a fuzzy BCK/BCI-algebra if it satisfies

$$(\forall \varpi, \varrho \in \tilde{\mathscr{X}})(\mu(\varpi * \varrho) \geq \min\{\mu(\varpi), \mu(\varrho)\}). \tag{1}$$

In a set $\tilde{\mathscr{X}}$, a fuzzy set μ in $\tilde{\mathscr{X}}$ of the form

$$\mu(\vartheta) \coloneqq \begin{cases} t \in (0, 1], & \text{if } \vartheta = \varpi, \\ 0, & \text{if } \vartheta \neq \varpi, \end{cases} \tag{2}$$

is called a fuzzy point with support ϖ and value t and is denoted by ϖ_t.

For a fuzzy set μ in a set $\tilde{\mathscr{X}}$ and a fuzzy point ϖ_t, Pu and Liu [34] presented the symbol $\varpi_t \alpha \mu$, where $\alpha \in \{\in, q, \in\vee q, \in\wedge q\}$. If $\varpi_t \in \mu$ (resp. $\varpi_t\, q\mu$), then we mean $\mu(\varpi) \geq t$ (resp. $\mu(\varpi) + t > 1$), and in this case, ϖ_t is said to belong to (resp. be quasi-coincident with) a fuzzy set μ. If $\varpi_t \in\vee q\mu$ (resp. $\varpi_t \in\wedge q\mu$), then we mean $\varpi_t \in \mu$ or $\varpi_t q\mu$ (resp. $\varpi_t \in \mu$ and $\varpi_t q\mu$).

For an initial universe set U and a set of parameters E, let $P(U)$ denote the power set of U and $\Omega \subset E$. Molodtsov [7] defined the soft set as follows.

Definition 1 (see [7]). A pair (ζ, Ω) is called a soft set over U, where ζ is a function given by

$$\zeta\colon \Omega \longrightarrow P(U). \tag{3}$$

The set $\zeta(\varepsilon)$ for $\varepsilon \in \Omega$ may be considered as the set of ε-approximate elements of the soft set (ζ, Ω). Clearly, a soft set is not a set. We refer the reader to [7] for illustration where several examples are presented.

Let $\mathscr{F}(U)$ denote the set of all fuzzy sets in U.

Definition 2 (see [9]). A pair $(\tilde{\zeta}, \Omega)$ is called a fuzzy soft set over U where $\tilde{\zeta}$ is a mapping given by

$$\tilde{\zeta}\colon \Omega \longrightarrow \mathscr{F}(U). \tag{4}$$

For all $\varpi \in \Omega$, $\tilde{\zeta}[\varpi] \in \mathscr{F}(U)$ and it is called fuzzy value set of parameter ϖ. If $\tilde{\zeta}[\varpi]$, for all $\varpi \in \Omega$, is a crisp subset of U, then $(\tilde{\zeta}, \Omega)$ is degenerated to be the standard soft set. Thus, fuzzy soft sets are a generalization of standard soft sets.

We will use $\mathscr{FS}(U)$ to denote the set of all fuzzy soft sets over U.

Definition 3 (see [9]). Let $(\tilde{\zeta}, \Omega), (\tilde{\eta}, \Omega) \in \mathscr{FS}(U)$. The union of $(\tilde{\zeta}, \Omega)$ and $(\tilde{\eta}, \mho)$ is defined to be the fuzzy soft set $(\tilde{\xi}, \Upsilon)$ satisfying the following conditions:

(i) $\Upsilon = \Omega \cup \mho$,

(ii) for all $\theta \in \Upsilon$,

$$\tilde{\xi}[\theta] = \begin{cases} \tilde{\zeta}[\theta], & \text{if } \theta \in \Omega \backslash \mho, \\ \tilde{\eta}[\theta], & \text{if } \theta \in \mho \backslash \Omega, \\ \tilde{\zeta}[\theta] \cup \tilde{\eta}[\theta], & \text{if } \theta \in \Omega \cap \mho. \end{cases} \tag{5}$$

In this case, we write $(\tilde{\zeta}, \Omega) \widetilde{\cup} (\tilde{\eta}, \mho) = (\tilde{\xi}, \Upsilon)$.

Definition 4 (see [9]). If $(\tilde{\zeta}, \Omega), (\tilde{\eta}, \mho) \in \mathscr{FS}(U)$, then "$(\tilde{\zeta}, \Omega)$ AND $(\tilde{\eta}, \mho)$" denoted by $(\tilde{\zeta}, \Omega) \tilde{\wedge} (\tilde{\eta}, \mho)$ is defined by

$$(\tilde{\zeta}, \Omega) \tilde{\wedge} (\tilde{\eta}, \mho) = (\tilde{\xi}, \Omega \times \mho), \tag{6}$$

where $\tilde{\xi}[\alpha, \beta] = \tilde{\zeta}[\alpha] \cap \tilde{\eta}[\beta] \forall (\alpha, \beta) \in \Omega \times \mho$.

Definition 5 (see [35]). For two soft sets $(\tilde{\zeta}, \Omega)$ and $(\tilde{\eta}, \mho)$, the extended intersection is the soft set $(\tilde{\xi}, \Upsilon)$ where $\Upsilon = \Omega \cup \mho$, and for every $\theta \in \Upsilon$,

$$\tilde{\xi}[\theta] = \begin{cases} \tilde{\zeta}[\theta], & \text{if } \theta \in \Omega \backslash \mho, \\ \tilde{\eta}[\theta], & \text{if } \theta \in \mho \backslash \Omega, \\ \tilde{\zeta}[\theta] \cap \tilde{\eta}[\theta], & \text{if } \theta \in \Omega \cap \mho. \end{cases} \tag{7}$$

We write $(\tilde{\zeta}, \Omega) \widetilde{\cap}_\theta (\tilde{\eta}, \mho) = (\tilde{\xi}, \Upsilon)$.

Definition 6 (see [35]). Let $(\tilde{\zeta}, \Omega), (\tilde{\eta}, \mho) \in \mathscr{FS}(U)$ such that $\Omega \cap \mho \neq \varnothing$. The restricted intersection of $(\tilde{\zeta}, \Omega)$ and $(\tilde{\eta}, \mho)$ is denoted by $(\tilde{\zeta}, \Omega) \widetilde{\cap}_r (\tilde{\eta}, \mho)$ and is defined as $(\tilde{\zeta}, \Omega) \widetilde{\cap}_r (\tilde{\eta}, \mho) = (\tilde{\xi}, \Upsilon)$, where $\Upsilon = \Omega \cap \mho$ and for all $c \in \Upsilon$, $\tilde{\xi}[c] = \tilde{\zeta}[c] \cap \tilde{\eta}[c]$.

3. $(\in, \in\vee q)$-Fuzzy Soft BCK/BCI-Algebras

Definition 7 (see [36]). A fuzzy set μ in $\widetilde{\mathcal{X}}$ is said to be an $(\in, \in\vee q)$-fuzzy subalgebra of $\widetilde{\mathcal{X}}$ if

$$(\forall \hbar, \kappa \in \widetilde{\mathcal{X}})(\forall \omega_1, \omega_2 \in (0,1])\left(\hbar_{\omega_1}, \kappa_{\omega_2} \in \mu \Longrightarrow (\hbar * \kappa)_{\min\{\omega_1,\omega_2\}} \in\vee q\mu\right). \tag{8}$$

Definition 8. Let $(\widetilde{\zeta}, \Omega) \in \mathcal{F}\mathcal{S}(\widetilde{\mathcal{X}})$ where $\Omega \subseteq E$. If there exists a parameter $\upsilon \in \Omega$ such that $\widetilde{\zeta}[\upsilon]$ is an $(\in, \in\vee q)$-fuzzy subalgebra of $\widetilde{\mathcal{X}}$, we say that $(\widetilde{\zeta}, \Omega)$ is an $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebra over $\widetilde{\mathcal{X}}$ based on a parameter υ. If $(\widetilde{\zeta}, \Omega)$ is an $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebra over $\widetilde{\mathcal{X}}$ based on all parameters, we say that $(\widetilde{\zeta}, \Omega)$ is an $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebra over $\widetilde{\mathcal{X}}$.

The notion $\mathcal{F}\mathcal{S}_{\mathcal{BCK}/\mathcal{BCI}}\mathcal{A}(\widetilde{\mathcal{X}})$ will be used for the set of all $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebras.

Example 1. Let $\widetilde{\mathcal{X}} = \{0, \imath, \mathcal{J}, \ell\}$ be a BCI-algebra with the following table.

$*$	0	$\imath$	$\mathcal{J}$	ℓ
0	0	$\imath$	$\mathcal{J}$	ℓ
$\imath$	$\imath$	0	ℓ	$\mathcal{J}$
$\mathcal{J}$	$\mathcal{J}$	ℓ	0	$\imath$
ℓ	ℓ	$\mathcal{J}$	$\imath$	0

Let $\Omega = \{e_1, e_2, e_3\}$ and let $(\widetilde{\zeta}, \Omega) \in \mathcal{F}\mathcal{S}(\widetilde{\mathcal{X}})$. Then, $\widetilde{\zeta}[e_1]$, $\widetilde{\zeta}[e_2]$, and $\widetilde{\zeta}[e_3]$ are fuzzy sets in $\widetilde{\mathcal{X}}$. We define them as follows:

$\widetilde{\zeta}$	0	$\imath$	$\mathcal{J}$	ℓ
e_1	0.6	0.7	0.3	0.3
e_2	0.8	0.2	0.4	0.2
e_3	0.6	0.3	0.3	0.7

Then, $(\widetilde{\zeta}, \Omega)$ is an $(\in, \in\vee q)$-fuzzy soft BCI-algebra over $\widetilde{\mathcal{X}}$.

Proposition 1. *If $(\widetilde{\zeta}, \Omega) \in \mathcal{F}\mathcal{S}_{\mathcal{BCK}/\mathcal{BCI}}\mathcal{A}(\widetilde{\mathcal{X}})$, then*

$$(\forall \hbar \in \widetilde{\mathcal{X}})\left(\widetilde{\zeta}[\upsilon](0) \geq \min\left\{\widetilde{\zeta}[\upsilon](\hbar), 0.5\right\}\right), \tag{9}$$

where υ is any parameter in Ω.

Proof. For $\hbar \in \widetilde{\mathcal{X}}$ and $\upsilon \in \Omega$, we have

$$\begin{aligned}\widetilde{\zeta}[\upsilon](0) &= \widetilde{F}[\upsilon](\hbar * \hbar) \geq \min\left\{\widetilde{\zeta}[\upsilon](\hbar), \widetilde{F}[\upsilon](\hbar), 0.5\right\} \\ &= \min\left\{\widetilde{\zeta}[\upsilon](\hbar), 0.5\right\}.\end{aligned} \tag{10}$$

Hence, $\widetilde{\zeta}[\upsilon](0) \geq \min\{\widetilde{F}[\upsilon](\hbar), 0.5\}$ for all $\hbar \in \widetilde{\mathcal{X}}$ and any parameter υ in Ω. □

Theorem 1. *Let $(\widetilde{\zeta}, \Omega) \in \mathcal{F}\mathcal{S}_{\mathcal{BCK}/\mathcal{BCI}}\mathcal{A}(\widetilde{\mathcal{X}})$. If $\rho \subseteq \Omega$, then $(\widetilde{\zeta}|_\rho, \mho) \in \mathcal{F}\mathcal{S}_{\mathcal{BCK}/\mathcal{BCI}}\mathcal{A}(\widetilde{\mathcal{X}})$.*

Proof (straightforward) □

The following example shows that there exists $(\widetilde{\zeta}, \Omega) \in \mathcal{F}\mathcal{S}(\widetilde{\mathcal{X}})$ such that

(i) $(\widetilde{\zeta}, \Omega)$ is not an $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebra over $\widetilde{\mathcal{X}}$

(ii) There exists a subset ρ of Ω such that $(\widetilde{\zeta}|_\rho, \mho)$ is an $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebra over $\widetilde{\mathcal{X}}$

Example 2. Consider a BCK-algebra $\widetilde{\mathcal{X}} = \{\{0, \imath, \mathcal{J}, \kappa, \ell\}$ with the following table.

$*$	0	$\imath$	$\mathcal{J}$	κ	ℓ
0	0	0	0	0	0
$\imath$	$\imath$	0	$\imath$	0	0
$\mathcal{J}$	$\mathcal{J}$	$\mathcal{J}$	0	$\mathcal{J}$	0
κ	κ	κ	κ	0	0
ℓ	ℓ	ℓ	κ	$\mathcal{J}$	0

Let $\Omega = \{e_1, e_2, e_3, e_4, e_5\}$ and let $(\widetilde{\zeta}, \Omega) \in \mathcal{F}\mathcal{S}(\widetilde{\mathcal{X}})$. Then, $\widetilde{\zeta}[e_1]$, $\widetilde{\zeta}[e_2]$, $\widetilde{\zeta}[e_3]$, $\widetilde{\zeta}[e_4]$, and $\widetilde{\zeta}[e_5]$ are fuzzy sets in $\widetilde{\mathcal{X}}$. We define them as follows:

$\widetilde{\zeta}$	0	$\imath$	$\mathcal{J}$	κ	ℓ
e_1	0.7	0.6	0.2	0.4	0.2
e_2	0.6	0.3	0.8	0.2	0.4
e_3	0.9	0.4	0.9	0.3	0.3
e_4	0.8	0.1	0.1	0.3	0.8
e_5	0.6	0.4	0.4	0.7	0.4

Then, $(\widetilde{\zeta}, \Omega)$ is not an $(\in, \in\vee q)$-fuzzy soft BCK-algebra over $\widetilde{\mathcal{X}}$ since it is not an $(\in, \in\vee q)$-fuzzy soft BCK-algebra over $\widetilde{\mathcal{X}}$ based on two parameters e_2 and e_4. However, if we take $\rho = \{e_1, e_3, e_5\}$, then $(\widetilde{\zeta}|_\rho, \mho)$ is described as follows:

$\widetilde{\zeta}|_\rho$	0	$\imath$	$\mathcal{J}$	κ	ℓ
e_1	0.7	0.6	0.2	0.4	0.2
e_3	0.9	0.4	0.9	0.3	0.3
e_5	0.6	0.4	0.4	0.7	0.4

and it is an $(\in, \in\vee q)$-fuzzy soft BCK-algebra over $\widetilde{\mathcal{X}}$.

Theorem 2. *Every fuzzy soft BCK/BCI-algebra over $\widetilde{\mathcal{X}}$ is an $(\in, \in\vee q)$-fuzzy soft BCK/BCI-algebra over $\widetilde{\mathcal{X}}$.*

Proof (straightforward) □

The converse of Theorem 2 is not true as follows.

Example 3. Consider $(\widetilde{\zeta}, \Omega) \in \mathcal{F}\mathcal{S}_{\mathcal{BCI}}\mathcal{A}(\widetilde{\mathcal{X}})$ in Example 1. We know that $(\widetilde{\zeta}, \Omega)$ is not a fuzzy soft BCI-algebra over $\widetilde{\mathcal{X}}$

since $(\tilde{\zeta},\Omega)$ is not a fuzzy soft BCI-algebra over $\tilde{\mathscr{X}}$ based on the parameter e_1 as $\tilde{\zeta}[e_1](0)=0.6<0.7=\tilde{\zeta}[e_1](a)$.

Lemma 1 (see [36]). *A fuzzy set μ in $\tilde{\mathscr{X}}$ is an $(\in,\in\vee q)$-fuzzy subalgebra of*

$$\tilde{\mathscr{X}}\Leftrightarrow(\forall\hbar,\kappa\in\tilde{\mathscr{X}})(\mu(\hbar*\kappa)\geq\min\{\mu(\hbar),\mu(\kappa),0.5\}). \quad (11)$$

According to the Lemma 1, the following theorem is straightforward.

Theorem 3. *A fuzzy soft set $(\tilde{\zeta},\Omega)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$ if and only if*

$$(\forall\hbar,\kappa\in\tilde{\mathscr{X}})(\forall u\in\Omega)\left(\tilde{\zeta}[u](\hbar*\kappa)\geq\min\left\{\tilde{\zeta}[u](\hbar),\tilde{\zeta}[u](\kappa),0.5\right\}\right). \quad (12)$$

Theorem 4. *If $(\tilde{\zeta},\Omega)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$ such that*

$$(\forall v\in\Omega)(\forall\hbar\in\tilde{\mathscr{X}})(\tilde{\zeta}[v](\hbar)<0.5), \quad (13)$$

then $(\tilde{\zeta},\Omega)$ is a fuzzy soft BCK/BCI-algebra over $\tilde{\mathscr{X}}$.

Proof. Let $\hbar,\kappa\in\tilde{\mathscr{X}}$ and $v\in\Omega$. Since $(\tilde{\zeta},\Omega)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$, it follows from Theorem 3 and (13) that

$$\begin{aligned}\tilde{\zeta}[v](\hbar*\kappa)&\geq\min\left\{\tilde{\zeta}[v](\hbar),\tilde{\zeta}[v](\kappa),0.5\right\}\\&=\min\left\{\tilde{\zeta}[v](\hbar),\tilde{\zeta}[v](\kappa)\right\}.\end{aligned} \quad (14)$$

Therefore, $(\tilde{\zeta},\Omega)$ is a fuzzy soft BCK/BCI-algebra over $\tilde{\mathscr{X}}$. □

Theorem 5. *If $(\tilde{\zeta},\Omega),(\tilde{\eta},\mho)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$, then the extended intersection of $(\tilde{\zeta},\Omega)$ and $(\tilde{\eta},\mho)$ is an $(\in,\in\vee q)$-fuzzy soft BCK/BCI-algebra over $\tilde{\mathscr{X}}$.*

Proof. Let $(\tilde{\zeta},\Omega)\tilde{\cap}_e(\tilde{\eta},\mho)=(\tilde{\xi},\Upsilon)$ be the extended intersection of $(\tilde{\zeta},\Omega)$ and $(\tilde{\eta},\mho)$. Then, $\Upsilon=\Omega\cup\mho$. For any $v\in\Upsilon$, if $v\in\Omega\setminus\mho$ (resp. $v\in\mho\setminus\Omega$), then $\tilde{\xi}[v]=\tilde{\zeta}[v]\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$ (resp. $\tilde{\xi}[v]=\tilde{\eta}[v]\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$). If $\Omega\cap\mho\neq\varnothing$, then $\tilde{\xi}[v]=\tilde{\zeta}[v]\cap\tilde{\eta}[v]\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$ for all $v\in\Omega\cap\mho$ since the intersection of two $(\in,\in\vee q)$-fuzzy BCK/BCI-algebras is an $(\in,\in\vee q)$-fuzzy BCK/BCI-algebra. Therefore, $(\tilde{\xi},\Upsilon)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$. □

Corollary 1. *The restricted intersection of two $(\in,\in\vee q)$-fuzzy soft BCK/BCI-algebras is an $(\in,\in\vee q)$-fuzzy soft BCK/BCI-algebra.*

Theorem 6. *Let $(\tilde{\zeta},\Omega),(\tilde{\eta},\mho)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$. If $\Omega\cap\mho=\varnothing$, then the union $(\tilde{\zeta},\Omega)\tilde{\cup}(\tilde{\eta},\mho)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$.*

Proof. By Definition 3, we can write $(\tilde{\zeta},\Omega)\tilde{\cup}(\tilde{\eta},\mho)=(\tilde{\xi},\Upsilon)$, where $\Upsilon=\Omega\cup\mho$ and for all $e\in\Upsilon$,

$$\tilde{\xi}[e]=\begin{cases}\tilde{\zeta}[e], & \text{if } e\in\Omega\setminus\mho,\\ \tilde{\eta}[e], & \text{if } e\in\mho\setminus\Omega,\\ \tilde{\zeta}[e]\cup\tilde{\eta}[e], & \text{if } e\in\Omega\cap\mho.\end{cases} \quad (15)$$

Since $v\in\Omega\setminus\mho$ or $v\in\mho\setminus\Omega$ for all $v\in\Upsilon$. If $v\in\Omega\setminus\mho$, then $\tilde{\xi}[v]=\tilde{\zeta}[v]\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$ because $(\tilde{\zeta},\Omega)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$. If $v\in\mho\setminus\Omega$, then $\tilde{\xi}[v]=\tilde{\eta}[v]\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$ because $(\tilde{\eta},\mho)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$. Hence, $(\tilde{\xi},\Upsilon)=(\tilde{\zeta},\Omega)\tilde{\cup}(\tilde{\eta},\mho)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$. □

The following illustration shows that Theorem 6 is not valid if $\Omega\cap\mho\neq\varnothing$.

Example 4. Let $\tilde{\mathscr{X}}=\{0,1,\imath,\mathscr{J},\kappa\}$ be a BCI-algebra with the following table.

$*$	0	1	$\imath$	$\mathscr{J}$	κ
0	0	0	$\imath$	$\mathscr{J}$	κ
1	1	0	$\imath$	$\mathscr{J}$	κ
$\imath$	$\imath$	$\imath$	0	κ	$\mathscr{J}$
$\mathscr{J}$	$\mathscr{J}$	$\mathscr{J}$	κ	0	$\imath$
κ	κ	κ	$\mathscr{J}$	$\imath$	0

Consider sets of parameters as follows:

$$\begin{aligned}\Omega&=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\},\\ \mho&=\{\alpha_3,\alpha_4,\beta_5\}.\end{aligned} \quad (16)$$

Then, Ω and $\mho$ are not disjoint sets of parameters. Let $(\tilde{\zeta},\Omega)$ be a fuzzy soft set over $\tilde{\mathscr{X}}$. Then, $\tilde{\zeta}[\alpha_1]$, $\tilde{\zeta}[\alpha_2]$, $\tilde{\zeta}[\alpha_3]$, and $\tilde{\zeta}[\alpha_4]$ are fuzzy sets in $\tilde{\mathscr{X}}$. We define them as follows:

$\tilde{\zeta}$	0	1	$\imath$	$\mathscr{J}$	κ
α_1	0.7	0.6	0.3	0.3	0.3
α_2	0.6	0.5	0.4	0.2	0.2
α_3	0.8	0.5	0.1	0.3	0.1
α_4	0.5	0.5	0.2	0.2	0.4

Then, $(\tilde{\zeta},\Omega)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$. Let $(\tilde{\eta},\mho)\in\mathscr{FS}(\tilde{\mathscr{X}})$. Then, $\tilde{\eta}[\alpha_3]$, $\tilde{\eta}[\alpha_4]$, and $\tilde{\eta}[\beta_5]$ are fuzzy sets in $\tilde{\mathscr{X}}$. We define them as follows:

$\tilde{\eta}$	0	1	$\imath$	$\mathscr{J}$	κ
α_3	0.8	0.6	0.3	0.1	0.1
α_4	0.7	0.6	0.3	0.3	0.5
β_5	0.9	0.5	0.2	0.4	0.2

Then, $(\tilde{\eta},\mho)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$, and the union

$$(\tilde{\zeta},\Omega)\tilde{\cup}(\tilde{\eta},\mho)=(\tilde{\xi},\Upsilon), \quad (17)$$

of $(\tilde{\zeta},\Omega)$ and $(\tilde{\eta},\mho)$ is described as follows:

$\tilde{\xi}$	0	1	$\imath$	$\mathscr{J}$	κ
α_1	0.7	0.6	0.3	0.3	0.3
α_2	0.6	0.5	0.4	0.2	0.2
α_3	0.8	0.6	0.3	0.3	0.1
α_4	0.7	0.6	0.3	0.3	0.5
β_5	0.9	0.5	0.2	0.4	0.2

For a parameter $\alpha_3 \in \Omega \cap \tilde{\mho}$, we have

$$\begin{aligned}
\tilde{\xi}[\alpha_3](\mathscr{J} * \imath) &= \left(\tilde{\zeta}[\alpha_3] \cup \tilde{\eta}[\alpha_3]\right)(\mathscr{J} * \imath) \\
&= \left(\tilde{\zeta}[\alpha_3] \cup \tilde{\eta}[\alpha_3]\right)(\kappa) \\
&= \max\left\{\tilde{\zeta}[\alpha_3](\kappa), \tilde{\eta}[\alpha_3](\kappa)\right\} \\
&= \max\{0.1, 0.1\} = 0.1, \\
&\quad \min\left\{\tilde{\xi}[\alpha_3](\mathscr{J}), \tilde{\xi}[\alpha_3](\imath), 0.5\right\} \\
&= \min\left\{\left(\tilde{\zeta}[\alpha_3] \cup \tilde{\eta}[\alpha_3]\right)(\mathscr{J}), \left(\tilde{\zeta}[\alpha_3] \cup \tilde{\eta}[\alpha_3]\right)(\imath), 0.5\right\} \\
&= \min\left\{\max\left\{\tilde{\zeta}[\alpha_3](\mathscr{J}), \tilde{\eta}[\alpha_3](\mathscr{J})\right\}, \max\left\{\tilde{\zeta}[\alpha_3](\mathscr{J}), \tilde{\eta}[\alpha_3](\mathscr{J})\right\}, 0.5\right\} \\
&= \min\{\max\{0.3, 0.1\}, \max\{0.1, 0.3\}, 0.5\} \\
&= 0.3.
\end{aligned} \tag{18}$$

Thus, from Theorem 3, $(\tilde{\xi}, \Upsilon) \notin \mathscr{F}\mathscr{S}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\widetilde{\mathscr{X}})$ based on the parameter α_3 and so that $(\tilde{\xi}, \Upsilon) \notin \mathscr{F}\mathscr{S}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\widetilde{\mathscr{X}})$.

Theorem 7. *If* $(\tilde{\zeta}, \Omega), (\tilde{\eta}, \tilde{\mho}) \in \mathscr{F}\mathscr{S}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\widetilde{\mathscr{X}})$, *then* $(\tilde{\zeta}, \Omega)\tilde{\wedge}(\tilde{\eta}, \tilde{\mho}) \in \mathscr{F}\mathscr{S}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\widetilde{\mathscr{X}})$.

Proof. By Definition 4, we have

$$(\tilde{\zeta}, \Omega)\tilde{\wedge}(\tilde{\eta}, \tilde{\mho}) = (\tilde{\xi}, \Omega \times \tilde{\mho}), \tag{19}$$

where $\tilde{\xi}[u, v] = \tilde{\zeta}[u] \cap \tilde{\eta}[v], \forall (u, v) \in \Omega \times \rho$. For any $\hbar, \kappa \in \widetilde{\mathscr{X}}$, we have

$$\begin{aligned}
\tilde{\xi}[u, v](\hbar * \kappa) = (\tilde{\zeta}[u] \cap \tilde{\eta}[v])(\hbar * \kappa) &= \min\left\{\tilde{\zeta}[u](\hbar * \kappa), \tilde{\eta}[v](\hbar * \kappa)\right\} \\
&\geq \min\left\{\min\left\{\tilde{\zeta}[u](\hbar), \tilde{\zeta}[u](\kappa), 0.5\right\}, \min\{\tilde{\eta}[v](\hbar), \tilde{\eta}[v](\kappa), 0.5\}\right\} \\
&= \min\left\{\min\left\{\tilde{\zeta}[u](\hbar), \tilde{\eta}[v](\hbar)\right\}, \min\left\{\tilde{\zeta}[u](\kappa), \tilde{\eta}[v](\kappa)\right\}, 0.5\right\} \\
&= \min\left\{(\tilde{\zeta}[u] \cap \tilde{\eta}[v])(\hbar), (\tilde{\zeta}[u] \cap \tilde{\eta}[v])(\kappa), 0.5\right\} \\
&= \min\left\{\tilde{\xi}[u, v](\hbar), \tilde{\xi}[u, v](\kappa), 0.5\right\}.
\end{aligned} \tag{20}$$

Hence, $(\tilde{\xi}, \Omega \times \tilde{\mho}) = (\tilde{\zeta}, \Omega)\tilde{\wedge}(\tilde{\eta}, \tilde{\mho}) \in \mathscr{F}\mathscr{S}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\widetilde{\mathscr{X}})$ based on (u, v) by using Theorem 3. Since (u, v) is arbitrary,

$$(\tilde{\xi}, \Omega \times \tilde{\mho}) = (\tilde{\zeta}, \Omega)\tilde{\wedge}(\tilde{\eta}, \tilde{\mho}) \in \mathscr{F}\mathscr{S}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\widetilde{\mathscr{X}}). \tag{21}$$

□

Definition 9. Let $(\tilde{\zeta}, \Omega), (\tilde{\eta}, \tilde{\mho}) \in \mathscr{F}\mathscr{S}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\widetilde{\mathscr{X}})$. We say that $(\tilde{\zeta}, \Omega)$ is an $(\in, \in\vee q)$-fuzzy soft sub-BCK/BCI-algebra of $(\tilde{\eta}, \tilde{\mho})$ if

(1) $\Omega \subseteq \rho$,

(2) $\tilde{\zeta}[u]$ is an $(\in, \in\vee q)$-fuzzy sub-BCK/BCI-algebra of $\tilde{\eta}[u]$ for all $u \in \Omega$; that is, $\tilde{\zeta}[u]$ is an $(\in, \in\vee q)$-fuzzy BCK/BCI-algebra satisfying the following condition:

$$(\forall \hbar \in \widetilde{\mathscr{X}})(\tilde{\zeta}[u](\hbar) \leq \tilde{\eta}[u](\hbar)). \tag{22}$$

Example 5. Let $(\tilde{\zeta}, \Omega) \in \mathscr{F}\mathscr{S}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\widetilde{\mathscr{X}})$ in Example 1. For a subset $\rho = \{e_1, e_3\}$ of Ω, let $(\tilde{\eta}, \tilde{\mho})$ be fuzzy soft set over $\widetilde{\mathscr{X}}$ which is defined as follows:

$\tilde{\eta}$	0	$\imath$	$\mathscr{J}$	κ
e_1	0.56	0.67	0.23	0.23
e_3	0.56	0.23	0.23	0.67

Then, $(\tilde{\eta},\tilde{\mho})$ is an $(\in,\in\vee q)$-fuzzy soft sub-BCI-algebra of $(\tilde{\zeta},\Omega)$.

Example 6. Let $\tilde{\mathscr{X}}=\{0,1,2,3,4\}$ be a BCK-algebra with the following Cayley table.

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	0	2
3	3	2	1	0	3
4	4	4	4	4	0

Let $\rho=\{e_1,e_2,e_3,e_4,e_5\}$ be a set of parameters and let $(\tilde{\eta},\tilde{\mho})\in\mathscr{FS}(\tilde{\mathscr{X}})$ which is defined as follows:

$\tilde{\eta}$	0	1	2	3	4
e_1	0.6	0.2	0.8	0.2	0.4
e_2	0.7	0.7	0.3	0.3	0.5
e_3	0.8	0.1	0.3	0.1	0.4
e_4	0.6	0.6	0.3	0.3	0.6
e_5	0.9	0.3	0.4	0.3	0.2

Then, $(\tilde{\eta},\tilde{\mho})\in\mathscr{FS}_{\mathscr{BCK}}\mathscr{A}(\tilde{\mathscr{X}})$. For a subset $\Omega=\{e_1,e_3,e_4\}$ of ρ, let $(\tilde{\zeta},\Omega)\in\mathscr{FS}(\tilde{\mathscr{X}})$ defined by

$\tilde{\zeta}$	0	1	2	3	4
e_1	0.56	0.2	0.78	0.2	0.34
e_3	0.78	0.1	0.23	0.1	0.34
e_4	0.56	0.56	0.3	0.3	0.56

Then, $(\tilde{\zeta},\Omega)$ is an $(\in,\in\vee q)$-fuzzy soft sub-BCK-algebra of $(\tilde{\eta},\tilde{\mho})$.

Theorem 8. *Let $(\tilde{\zeta},\Omega),(\tilde{\eta},\Omega)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$. If $\tilde{\zeta}[u]\subseteq\tilde{\eta}[u]$ for all $u\in\Omega$, then $(\tilde{\zeta},\Omega)$ is an $(\in,\in\vee q)$-fuzzy soft sub-BCK/BCI-algebra of $(\tilde{\eta},\Omega)$.*

Proof (straightforward) □

Theorem 9. *Let $(\tilde{\xi},\Upsilon)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$. If $(\tilde{\zeta},\Omega)$ and $(\tilde{\eta},\tilde{\mho})$ are $(\in,\in\vee q)$-fuzzy soft sub-BCK/BCI-algebras of $(\tilde{\xi},\Upsilon)$, then so is the extended intersection of $(\tilde{\zeta},\Omega)$ and $(\tilde{\eta},\tilde{\mho})$.*

Proof. The proof is followed from Theorem 5 and Definition 9. □

Theorem 10. *Let $(\tilde{\xi},\Upsilon)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$. If $(\tilde{\zeta},\Omega)$ and $(\tilde{\eta},\tilde{\mho})$ are $(\in,\in\vee q)$-fuzzy soft sub-BCK/BCI-algebras of $(\tilde{\xi},\Upsilon)$, then so is the union of $(\tilde{\zeta},\Omega)$ and $(\tilde{\eta},\tilde{\mho})$ whenever Ω and ρ are disjoint.*

Proof. The proof is followed from Theorem 6 and Definition 9. □

Theorem 11. *Let $(\tilde{\xi},\Upsilon)\in\mathscr{FS}_{\mathscr{BCK}/\mathscr{BCI}}\mathscr{A}(\tilde{\mathscr{X}})$. If $(\tilde{\zeta},\Omega)$ and $(\tilde{\eta},\tilde{\mho})$ are $(\in,\in\vee q)$-fuzzy soft sub-BCK/BCI-algebras of $(\tilde{\xi},\Upsilon)$, then $(\tilde{\zeta},\Omega)\tilde{\wedge}(\tilde{\eta},\tilde{\mho})$ is an $(\in,\in\vee q)$-fuzzy soft sub-BCK/BCI-algebra of $(\tilde{\xi},\Upsilon)\tilde{\wedge}(\tilde{\xi},\Upsilon)$.*

Proof. The proof is followed from Theorem 7 and Definition 9. □

4. Conclusion

In this paper, we introduced the notions of $(\in,\in\vee q)$-fuzzy soft BCK/BCI-algebras and $(\in,\in\vee q)$-fuzzy soft sub-BCK/BCI-algebras and investigated their related properties. Also, we discussed relations between fuzzy soft BCK/BCI-algebras and $(\in,\in\vee q)$-fuzzy soft BCK/BCI-algebras. Moreover, conditions for an $(\in,\in\vee q)$-fuzzy soft BCK/BCI-algebra to be a fuzzy soft BCK/BCI-algebra are provided. Moreover, the union, the extended intersection, and the "AND"-operation of two $(\in,\in\vee q)$-fuzzy soft (sub-) BCK/BCI-algebras are discussed, and a characterization of an $(\in,\in\vee q)$-fuzzy soft BCK/BCI-algebra is established.

We hope that this work will provide a deep impact on the upcoming research in this field and other soft algebraic studies to open up new horizons of interest and innovations. To extend these results, one can further study these notions on different algebras such as rings, hemirings, LA-semigroups, semihypergroups, semihyperrings, BL-algebras, MTL-algebras, R_0-algebras, MV-algebras, EQ-algebras, d-algebras, Q-algebras, and lattice implication algebras. Some important issues for future work are (1) to develop strategies for obtaining more valuable results and (2) to apply these notions and results for studying related notions in other algebraic (soft) structures.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: New Operators of Cubic Picture Fuzzy Information with Applications

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] Tehreem, A. Gumaei, and A. Hussain, "New Operators of Cubic Picture Fuzzy Information with Applications," *Journal of Mathematics*, vol. 2021, Article ID 9938181, 16 pages, 2021.

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Research Article

New Operators of Cubic Picture Fuzzy Information with Applications

Tehreem,[1] Abdu Gumaei ,[2] and Amjad Hussain[1]

[1]*Department of Mathematics, Quaid-i-Azam University, Islamabad 45320, Pakistan*
[2]*Research Chair of Pervasive and Mobile Computing, Department of Information Systems, College of Computer and Information Sciences, King Saud University, Riyadh 11543, Saudi Arabia*

Correspondence should be addressed to Abdu Gumaei; agumaei.c@ksu.edu.sa

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The researcher has been facing problems while handling imprecise and vague information, i.e., the problems of networking, decision-making, etc. For encountering such complicated data, the notion of fuzzy sets (FS) has been considered an influential tool. The notion was extended to its generalizations by a number of researchers in different ways which helps to understand and assess even more complex issues. This article characterizes imprecision with four kinds of values of membership. In this work, we aim to define and examine cubic picture fuzzy sets and give an application on averaging aggregation operators. We first introduce the notion of a cubic picture fuzzy set, which is a pair of interval-valued picture fuzzy set and a picture fuzzy set by giving examples. Then, we define two kinds of ordering on these sets and also discuss some set-theoretical properties. Moreover, we introduce three kinds of averaging aggregation operators based on cubic picture fuzzy sets and, at the end, we illustrate the results with a decision-making problem by using one of the provided aggregation operators.

1. Introduction

In 1965, Zadeh generalized the classical set and perceived the idea of fuzzy sets [1] to deal with uncertainty. This idea allows creating some new dimensions in the field of research and has been applied in many fields such as decision-making, medical diagnosis, and pattern recognition [1–6]. But in fuzzy set only the membership degree is considered. The limitation of fuzzy sets is that the nonmembership degree cannot be defined independently. To overcome this limitation, several extensions have been made by many researchers such as interval-valued fuzzy sets [7], intuitionistic fuzzy sets (IFSs) [8], cubic sets [9], and neutrosophic sets [10]. Among these various extensions of fuzzy sets, cubic set is one of the most prominent extensions. Jun [9] presented the idea of cubic sets in terms of interval-valued fuzzy set and fuzzy set in 2012. The very basic properties of cubic sets were studied, and some useful operations were defined successfully in his paper. Khir et al. [11] presented the idea of fuzzy sets and fuzzy logic and their application. Later on, the idea of cubic sets was applied to various fields by many authors (see [12–17]).

In recent years, the notion of fuzzy sets was further generalized by Coung et al. and they proposed the concept of picture fuzzy sets [18, 19], and this idea gained more and more attention from the researchers. Several similarity measures, correlation coefficients, and entropy measures for picture fuzzy sets were defined by many authors and they applied these sets in various fields (see [20–28]). Recently, Coung et al. [29] have extended the picture fuzzy sets to the interval-valued picture fuzzy sets. For some works on picture fuzzy sets and several types of aggregation operators, we refer the reader to [24, 25, 30–35].

Inspiring from the above study, we propose the concept of cubic picture fuzzy sets, which is an extension of cubic sets, picture fuzzy sets, and interval-valued picture fuzzy sets.

The rest of the paper is organized as follows. In Section 2, some basic definitions and results which are necessary for the main sections are discussed. In Section 3, the concept of cubic picture fuzzy sets which is a mixture of an interval-valued picture fuzzy set and a picture fuzzy set is introduced, and some basic operations on these sets were defined by giving several examples. Then the related theorems are studied. In Section 4, three types of aggregation operators in the environment of cubic picture fuzzy sets are discussed and, finally, one of them is applied in decision-making problem in the last section.

2. Preliminaries

Definition 1 (see [1]). Let $\ddot{\breve{S}}$ be a nonempty set. Then $\mathbb{U} = \left\{\langle \ddot{\breve{s}}, \mathscr{U}_{\mathbb{U}}(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$ is called a fuzzy set, where $\mathscr{U}_{\mathbb{U}}$ is a membership function that maps each element of $\dot{S}$ in $[0,1]$. Here we say that $\mathbb{U}$ is a fuzzy subset of $\breve{S}$.

Definition 2 (see [8]). Consider closed subinterval $\mathscr{U}_{\mathbb{U}} = [\mathscr{U}_{\mathbb{U}}^{-}, \mathscr{U}_{\mathbb{U}}^{+}]$ of I where $I = [0,1]$ is called an interval number, where $0 \leq \mathscr{U}_{\mathbb{U}}^{-} \leq \mathscr{U}_{\mathbb{U}}^{+} \leq 1$. The set of all interval numbers is denoted by $[I]$. A function β: $\breve{S} \longrightarrow [I]$ is said to be an interval-valued fuzzy (IVF) set of $\breve{S}$. The set of all IVF sets of $\breve{S}$ is denoted by $[I]^{\breve{S}}$. For each $\mathscr{U}_{\mathbb{U}} \in [I]^{\breve{S}}$ and $\breve{s} \in \breve{S}$, $\mathscr{U}_{\mathbb{U}}(\ddot{\breve{s}}) = [\mathscr{U}_{\mathbb{U}}^{-}(\breve{s}), \mathscr{U}_{\mathbb{U}}^{+}(\breve{s})]$ is called the degree of membership of an element $\breve{s}$ to $\mathscr{U}_{\mathbb{U}}$; in this case $\mathscr{U}_{\mathbb{U}}^{-}$: $\breve{S} \longrightarrow I$ and $\mathscr{U}_{\mathbb{U}}^{+}$: $\breve{S} \longrightarrow I$ are fuzzy subsets of $\breve{S}$; these sets are known as lower fuzzy set and upper fuzzy subset of $\breve{S}$, respectively.

Definition 3 (see [9]). The cubic set of a nonempty set $\ddot{\breve{S}}$ is defined as follows: $\widehat{I} = \left\{\langle \ddot{\breve{s}}, \widehat{A}(\ddot{\breve{s}}), \mathscr{B}(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$, where $\widehat{A}$ is an interval-valued fuzzy (IVF) set of $\breve{S}$ and $\mathscr{B}$ is a fuzzy subset of $\breve{S}$. A cubic set is simply denoted by $\widehat{I} = \langle \widehat{A}, \mathscr{B}\rangle$.

Definition 4 (see [9]). A cubic set $\widehat{I} = \langle \widehat{A}, \mathscr{B}\rangle$ is known to be

(1) an internal cubic set (briefly, ICS) if $\widehat{A}^{-}(\ddot{\breve{s}}) \leq \mathscr{B}(\breve{s}) \leq \widehat{A}^{+}(\breve{s}), \forall \breve{s} \in \breve{S}$

(2) an external cubic set (briefly, ECS) if $\mathscr{B}(\ddot{\breve{s}}) \notin (\widehat{A}^{-}(\breve{s}), \widehat{A}^{+}(\ddot{\breve{s}})), \forall \breve{s} \in \breve{S}$

Example 1. If $\widehat{A}$ is an IVF set of $\ddot{\breve{S}}$, then $\mathbb{U} = \left\{\langle \ddot{\breve{s}}, \widehat{A}(\ddot{\breve{s}}), 1(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$, $\mathbb{V} = \left\{\ddot{\breve{s}}, \widehat{A}(\ddot{\breve{s}}), 0(\ddot{\breve{s}}) > | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$ and $C = \left\{\langle \ddot{\breve{s}}, \widehat{A}(\ddot{\breve{s}}), \mathscr{B}(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \varepsilon \ddot{\breve{S}}\right\}$, where $\lambda(\ddot{\breve{s}}) = (\widehat{A}^{-}(\ddot{\breve{s}}) + \widehat{A}^{+}(\ddot{\breve{s}})/2)$ are cubic sets of $\ddot{\breve{S}}$.

Example 2. Let $\mathbb{U} = \left\{\langle \ddot{\breve{s}}, \widehat{A}(\ddot{\breve{s}}), 1(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$ be a cubic set of $\breve{S}$ and $\widehat{A}(\ddot{\breve{s}}) = [0.3, 0.7]$ and $\mathscr{B}(\ddot{\breve{s}}) = 0.4$, for each $\ddot{\breve{s}} \in \breve{S}$. Then $\mathbb{U}$ is an ICS. If $\widehat{A}(\ddot{\breve{s}}) = [0.3, 0.7]$ and $\mathscr{B}(\ddot{\breve{s}}) = 0.8$, for each $\ddot{\breve{s}} \in \breve{S}$, then $\mathbb{U}$ is an ECS. If $\widehat{A}(\ddot{\breve{s}}) = [0.3, 0.7]$ and $\mathscr{B}(\ddot{\breve{s}}) = \ddot{\breve{s}}$, for each $\ddot{\breve{s}} \in \breve{S}$, then $\mathbb{U}$ is neither an ICS nor an ECS.

Definition 5 (see [9]). Let $\ddot{\breve{S}}$ be a nonempty set and let $\mathbb{U} = \langle \widehat{A}, \mathscr{B}\rangle$ and $\mathbb{V} = \langle J, K\rangle$ be two cubic sets of $\breve{S}$. Then the orderings are defined in the following way:

(1) (Equality) $\mathbb{U} = \mathbb{V} \Leftrightarrow \widehat{A} = J$ and $\mathscr{B} = K$

(2) (P-Order) $\mathbb{U} \subseteq_{P} \mathbb{V} \Leftrightarrow \widehat{A} \subseteq J$ and $\mathscr{B} \leq K$

(3) (R-Order) $\mathbb{U} \subseteq_{R} \mathbb{V} \Leftrightarrow \widehat{A} \subseteq J$ and $\mathscr{B} \geq K$

Definition 6 (see [9]). For arbitrary indexed family of cubic sets $\mathbb{U}_i = \left\{\langle \ddot{\breve{s}}, \widehat{A}_i(\ddot{\breve{s}}), \mathscr{B}_i(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \varepsilon \ddot{\breve{S}}\right\}$, where $i \varepsilon \wedge$, we define the P-union, P-intersection, R-union, and R-intersection as follows:

(1) $\cup_{P, i\varepsilon\wedge} \mathbb{U}_i = \left\{\langle \ddot{\breve{s}}, (\cup_{i\varepsilon\wedge} I_i)(\ddot{\breve{s}}), (\vee_{i\in\Gamma}\alpha_i)(\ddot{\breve{s}})\rangle \ | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$ $(P - \text{union})$

(2) $\cap_{P, i\varepsilon\wedge} \mathbb{U}_i = \left\{\langle \ddot{\breve{s}}, (\cap_{i\varepsilon\wedge} I_i)(\ddot{\breve{s}}), (\wedge_{i\varepsilon\wedge}\alpha_i)(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$ $(P - \text{intersection})$

(3) $\cup_{R, i\varepsilon\wedge} \mathbb{U}_i = \left\{\langle \ddot{\breve{s}}, (\cap_{i\varepsilon\wedge} I_i)(\ddot{\breve{s}}), (\wedge_{i\varepsilon\wedge}\alpha_i)(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$ $(R - \text{union})$

(4) $\cap_{R, i\varepsilon\wedge} \mathbb{U}_i = \left\{\langle \ddot{\breve{s}}, (\cap_{i\varepsilon\wedge} I_i)(\ddot{\breve{s}}), (\vee_{i\in\wedge}\alpha_i)(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$ $(R - \text{intersection})$

The complement of $\mathbb{U} = \langle I, \alpha\rangle$ is also a cubic set that is defined by $\mathbb{U}^c = \left\{\langle \ddot{\breve{s}}, I^c(\ddot{\breve{s}}), 1 - \alpha(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \varepsilon \ddot{\breve{S}}\right\}$. Obviously, $(\mathbb{U}^c)^c = \mathbb{U}$ for any indexed family of cubic sets $\mathbb{U}_i = \{\langle \ddot{\breve{s}}, I_i(\ddot{\breve{s}}), \alpha_i(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \varepsilon \ddot{\breve{S}}\} (i \varepsilon \wedge)$:

(1) $(\cup_{P, i\varepsilon\wedge} \mathbb{U}_i)^c = \cap_{P, i\varepsilon\wedge} (\mathbb{U}_i)^c$ and $(\cap_{P, i\varepsilon\wedge} \mathbb{U}_i)^c = \cup_{P, i\varepsilon\wedge} (\mathbb{U}_i)^c$

(2) $(\cup_{R, i\varepsilon\wedge} \mathbb{U}_i)^c = \cap_{R, i\varepsilon\wedge} (\mathbb{U}_i)^c$ and $\cap_{R, i\varepsilon\wedge} \mathbb{U}_i = \cup_{R, i\varepsilon\wedge} (\mathbb{U}_i)^c$

Definition 7 (see [18, 19]). A picture fuzzy set (briefly, PFS) $\mathbb{U}$ of a universe $\ddot{\breve{S}}$ is an object in the form of $\left\{\ddot{\breve{s}}, \mathscr{U}_{\mathbb{U}}(\ddot{\breve{s}}), \rho_{\mathbb{U}}(\ddot{\breve{s}}), \vartheta_{\mathbb{U}}(\ddot{\breve{s}}) | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$, where $\mathscr{U}_{\mathbb{U}}, \rho_{\mathbb{U}}, \vartheta_{\mathbb{U}}$: $\ddot{\breve{S}} \longrightarrow [0,1]$ are fuzzy sets that satisfy $0 \leq \mathscr{U}_{\mathbb{U}}(\ddot{\breve{s}}) + \rho_{\mathbb{U}}(\ddot{\breve{s}}) + \vartheta_{\mathbb{U}}(\ddot{\breve{s}}) \leq 1$ for each $\ddot{\breve{s}} \in \ddot{\breve{S}}$. Then the values $\mathscr{U}_{\mathbb{U}}(\ddot{\breve{s}})$, $\rho_{\mathbb{U}}(\ddot{\breve{s}}), \vartheta_{\mathbb{U}}(\ddot{\breve{s}})$ are called the degree of positive membership of $\ddot{\breve{s}}$ in $\mathbb{U}$, the degree of neutral membership of $\ddot{\breve{s}}$ in $\mathbb{U}$, and the degree of negative membership of $\ddot{\breve{s}}$ in $\mathbb{U}$, respectively. Now $(1 - \mathscr{U}_{\mathbb{U}}(\ddot{\breve{s}}) + \rho_{\mathbb{U}}(\ddot{\breve{s}}) + \vartheta_{\mathbb{U}}(\ddot{\breve{s}}))$ could be called the degree of refusal membership of $\ddot{\breve{s}}$ in $\mathbb{U}$. Let PFS$(\ddot{\breve{S}})$ represent the set of all picture fuzzy sets of a universe $\ddot{\breve{S}}$.

Definition 8 (see [18, 19]). Let $\mathbb{U}$ and $\mathbb{V}$ be the PFSs. Then the set of operations are defined as follows:

(1) $\mathbb{U} \subseteq \mathbb{V}$ if $f(\forall\ \ddot{\breve{s}} \in \ddot{\breve{S}}, \mathscr{U}_{\mathbb{U}}(\ddot{\breve{s}}) \leq \mathscr{U}_{\mathbb{V}}(\ddot{\breve{s}})$ and $\rho_{\mathbb{U}}(\ddot{\breve{s}}) \leq \rho_{\mathbb{V}}(\ddot{\breve{s}})$ and $\vartheta_{\mathbb{U}}(\ddot{\breve{s}}) \leq \vartheta_{\mathbb{V}}(\ddot{\breve{s}}))$

(2) $\mathbb{U} = \mathbb{V}$ if $f(\mathbb{U} \subseteq \mathbb{V}$ and $\mathbb{V} \subseteq \mathbb{U})$

(3) $\mathbb{U} \cup \mathbb{V} = \{\ddot{\breve{s}}, \operatorname{Max}(\mathscr{U}_{\mathbb{U}}(\ddot{\breve{s}}), \mathscr{U}_{\mathbb{V}}(\ddot{\breve{s}})), \operatorname{Min}(\rho_{\mathbb{U}}(\ddot{\breve{s}}), \rho_{\mathbb{V}}(\ddot{\breve{s}})), \operatorname{Min}(\vartheta_{\mathbb{U}}(\ddot{\breve{s}}), \vartheta_{\mathbb{V}}(\ddot{\breve{s}})) | \ddot{\breve{s}} \in \ddot{\breve{S}}\}$

(4) $\mathbb{U} \cap \mathbb{V} = \{(\ddot{\breve{s}}, \operatorname{Min}(\mathscr{U}_{\mathbb{U}}(\ddot{\breve{s}}), \mathscr{U}_{\mathbb{V}}(\ddot{\breve{s}})), \operatorname{Min}(\rho_{\mathbb{U}}(\ddot{\breve{s}}), \rho_{\mathbb{V}}(\ddot{\breve{s}})), \operatorname{Max}(\vartheta_{\mathbb{U}}(\ddot{\breve{s}}), \vartheta_{\mathbb{V}}(\ddot{\breve{s}})) | \ddot{\breve{s}} \in \ddot{\breve{S}}\}$

Now, a generalization of interval-valued fuzzy set $\mathbb{U}$ is proposed. Here $\mathrm{int}[0,1]$ stands for the set of all closed subintervals of $[0,1]$.

Definition 9 (see [29]). An interval-valued picture fuzzy set (briefly, IVPFS) $\mathbb{U}$ of a universe $\ddot{\breve{S}}$ is an object in the following form: $\mathbb{U} = \left\{(M_{\mathbb{U}}(\ddot{\breve{s}}), L_{\mathbb{U}}(\ddot{\breve{s}}), N_{\mathbb{U}}(\ddot{\breve{s}})) \mid \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$, where

$$\begin{aligned} M_{\mathbb{U}}\colon \ddot{\breve{S}} \longrightarrow \mathrm{int}([0,1]), M_{\mathbb{U}}(\ddot{\breve{s}}) &= \left[\left(M_{\mathbb{U}L}(\ddot{\breve{s}})\right), \left(M_{\mathbb{U}U}(\ddot{\breve{s}})\right)\right] \in \mathrm{int}([0,1]), L_{\mathbb{U}}\colon \ddot{\breve{S}} \longrightarrow \mathrm{int}([0,1]), L_{\mathbb{U}}(\ddot{\breve{s}}) \\ &= \left[\left(L_{\mathbb{U}L}(\ddot{\breve{s}})\right), \left(L_{\mathbb{U}U}(\ddot{\breve{s}})\right)\right] \in \mathrm{int}([0,1]), N_{\mathbb{U}}\colon \ddot{\breve{S}} \longrightarrow \mathrm{int}([0,1]), N_{\mathbb{U}}(\ddot{\breve{s}}) \\ &= \left[\left(N_{\mathbb{U}L}(\ddot{\breve{s}})\right), \left(N_{\mathbb{U}U}(\ddot{\breve{s}})\right)\right] \in \mathrm{int}([0,1]). \end{aligned} \tag{1}$$

The following condition is satisfied: $\sup M_{\mathbb{U}}(\ddot{\breve{s}}) + \sup L_{\mathbb{U}}(\ddot{\breve{s}}) + \sup N_{\mathbb{U}}(\ddot{\breve{s}}) \leq 1, (\forall \ddot{\breve{s}} \in \ddot{\breve{S}})$. The $\mathrm{IVPFS}(\ddot{\breve{S}})$ denotes the set of all interval-valued picture fuzzy sets of $\ddot{\breve{S}}$.

3. Cubic Picture Fuzzy Sets

In this section, we propose the notion of a cubic picture fuzzy set and investigate its set-theoretical operations and some basic properties by giving illustrative examples.

Definition 10. A cubic picture fuzzy set (briefly, CPFS) of $\ddot{\breve{S}}$ is denoted and defined by $C_P = \left\{\langle \ddot{\breve{s}}, \widehat{A}(\ddot{\breve{s}}), \mathscr{B}(\ddot{\breve{s}})\rangle \mid \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$, where $\widehat{A}$ is an interval-valued picture fuzzy set and $\mathscr{B}$ is a picture fuzzy set of $\ddot{\breve{S}}$. A CPFS $C_P = \left\{\langle \ddot{\breve{s}}, \widehat{A}(\ddot{\breve{s}}), \mathscr{B}(\ddot{\breve{s}})\rangle \mid \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$ is simply denoted by $C_P = \langle \widehat{A}, \mathscr{B}\rangle$.

Definition 11. A cubic picture fuzzy set $C_P = \langle \widehat{A}, \mathscr{B}\rangle$ of $\ddot{\breve{S}}$ is said to be

(1) positive internal CPFS if $\mathscr{B}_1^-(\ddot{\breve{s}}) \leq \mathscr{B}_1(\ddot{\breve{s}}) \leq \mathscr{B}_1^+(\ddot{\breve{s}})$, where $\mathscr{B}_1^-(\ddot{\breve{s}}), \mathscr{B}_1^+(\ddot{\breve{s}})$ are the lower and the upper positive degrees in $\ddot{\breve{S}}$, respectively

(2) negative internal CPFS if $\mathscr{B}_2^-(\ddot{\breve{s}}) \leq \mathscr{B}_2(\ddot{\breve{s}}) \leq \mathscr{B}_2^+(\ddot{\breve{s}})$, where $\mathscr{B}_2^-(\ddot{\breve{s}}), \mathscr{B}_2^+(\ddot{\breve{s}})$ are the lower and the upper negative degrees in $\ddot{\breve{S}}$, respectively

(3) indeterminacy internal CPFS if $\mathscr{B}_3^-(\ddot{\breve{s}}) \leq \mathscr{B}_3(\ddot{\breve{s}}) \leq \mathscr{B}_3^+(\ddot{\breve{s}})$, where $\mathscr{B}_3^-(\ddot{\breve{s}}), \mathscr{B}_3^+(\ddot{\breve{s}})$ are the lower and the upper indeterminacy degrees in $\ddot{\breve{S}}$, respectively

When conditions (1), (2), and (3) hold, then it is called an internal cubic picture fuzzy set (ICPFS) in $\ddot{\breve{S}}$.

Definition 12. A cubic picture fuzzy set $C_P = \langle \widehat{A}, \mathscr{B}\rangle$ is said to be an external cubic picture fuzzy set (ECPFS) if $\mathscr{B}_1(\ddot{\breve{s}}) \notin (\mathscr{B}_1^-(\ddot{\breve{s}}), \mathscr{B}_1^+(\ddot{\breve{s}})), \mathscr{B}_2(\ddot{\breve{s}}) \notin (\mathscr{B}_2^-(\ddot{\breve{s}}), \mathscr{B}_2^+(\ddot{\breve{s}}))$ and $\mathscr{B}_3(\ddot{\breve{s}}) \notin \mathscr{B}_3^-(\ddot{\breve{s}}), \mathscr{B}_3^+(\ddot{\breve{s}})$ for all $\ddot{\breve{s}} \in \ddot{\breve{S}}$.

Example 3. Let $\ddot{\breve{S}} = \left\{\ddot{\breve{s}}_1, \ddot{\breve{s}}_2, \ddot{\breve{s}}_3\right\}$ be given. Then, the CPFS,

$$\mathbb{U} = \left\{\begin{array}{l} \left(\ddot{\breve{s}}_1, [0.1, 0.3], [0.3, 0.4], [0.1, 0.3], 0.2, 0.3, 0.2\right), \\ \left(\ddot{\breve{s}}_2, [0.1, 0.3], [0.3, 0.5], [0.0, 0.2], 0.2, 0.4, 0.1\right), \\ \left(\ddot{\breve{s}}_3, [0.1, 0.3], [0.1, 0.4], [0.0, 0.2], 0.2, 0.3, 0.1\right) \end{array}\right\}, \tag{2}$$

is an internal cubic picture fuzzy set of $\ddot{\breve{S}}$.

Example 4. Let $\ddot{\breve{S}} = \left\{\ddot{\breve{s}}_1, \ddot{\breve{s}}_2, \ddot{\breve{s}}_3\right\}$ be given; then the CPFS,

$$\mathbb{U} = \left\{\begin{array}{l} \left(\ddot{\breve{s}}_1, [0.2, 0.4], [0.0, 0.2], [0.1, 0.3], 0.4, 0.3, 0.0\right), \\ \left(\ddot{\breve{s}}_2, [0.1, 0.3], [0.1, 0.4], [0.0, 0.3], 0.0, 0.5, 0.4\right), \\ \left(\ddot{\breve{s}}_3, [0.1, 0.3], [0.2, 0.5], [0.0, 0.2], 0.4, 0.1, 0.3\right) \end{array}\right\}, \tag{3}$$

is an external cubic picture fuzzy set of $\ddot{\breve{S}}$.

Theorem 1. *If $C_P = \langle \widehat{A}, \mathscr{B}\rangle$ is a cubic picture fuzzy set, which is not an ECPFS, then there exists $\ddot{\breve{s}} \in \ddot{\breve{S}}$ such that $\mathscr{B}_1(\ddot{\breve{s}}) \in (\mathscr{B}_1^-(\ddot{\breve{s}}), \mathscr{B}_1^+(\ddot{\breve{s}})), \mathscr{B}_2(\ddot{\breve{s}}) \in (\mathscr{B}_2^-(\ddot{\breve{s}}), \mathscr{B}_2^+(\ddot{\breve{s}}))$, and $\mathscr{B}_3(\ddot{\breve{s}}) \in \mathscr{B}_3^-(\ddot{\breve{s}}) \cap \mathscr{B}_3^+(\ddot{\breve{s}})$.*

Proof. The proof is straightforward and therefore is omitted. □

Theorem 2. *If $C_P = \langle \widehat{A}, \mathscr{B}\rangle \mathbb{R}$ is both ICPFS and ECPFS, then the following is satisfied for each $\ddot{\breve{s}} \in \ddot{\breve{S}}$: $\mathscr{B}_1(\ddot{\breve{s}}) \in (\mathscr{B}_1^-(\ddot{\breve{s}}) \cup \mathscr{B}_1^+(\ddot{\breve{s}}))$, $\mathscr{B}_2(\ddot{\breve{s}}) \in (\mathscr{B}_2^-(\ddot{\breve{s}}) \cup \mathscr{B}_2^+(\ddot{\breve{s}}))$, and $\mathscr{B}_3(\ddot{\breve{s}}) \in (\mathscr{B}_3^-(\ddot{\breve{s}}) \cup \mathscr{B}_3^+(\ddot{\breve{s}}))$.*

Proof. Assume that C_P is both ICPFS and ECPFS. Then, by using the definitions of ICPFS and ECPFS, we have $\mathscr{B}_1^-(\ddot{\breve{s}}) \leq \mathscr{B}_1(\ddot{\breve{s}}) \leq \mathscr{B}_1^+(\ddot{\breve{s}})$ and $\mathscr{B}_1(\ddot{\breve{s}}) \notin (\mathscr{B}_1^-(\ddot{\breve{s}}), \mathscr{B}_1^+(\ddot{\breve{s}}))$, $\mathscr{B}_2^-(\ddot{\breve{s}}) \leq \mathscr{B}_2(\ddot{\breve{s}}) \leq \mathscr{B}_2^+(\ddot{\breve{s}})$ and $\mathscr{B}_2(\ddot{\breve{s}}) \notin (\mathscr{B}_2^-(\ddot{\breve{s}}), \mathscr{B}_2^+(\ddot{\breve{s}}))$, $\mathscr{B}_3^-(\ddot{\breve{s}}) \leq \mathscr{B}_3(\ddot{\breve{s}}) \leq \mathscr{B}_3^+(\ddot{\breve{s}})$ and $\mathscr{B}_3(\ddot{\breve{s}}), \notin (\mathscr{B}_3^-(\ddot{\breve{s}}), \mathscr{B}_3^+(\ddot{\breve{s}}))$ for all $\ddot{\breve{s}} \in \ddot{\breve{S}}$. Thus $\mathscr{B}_1(\ddot{\breve{s}}) = \mathscr{B}_1^-(\ddot{\breve{s}})$ or $\mathscr{B}_1(\ddot{\breve{s}}) = \mathscr{B}_1^+(\ddot{\breve{s}})$, implying that $(\mathscr{B}_1(\ddot{\breve{s}}) \in (\mathscr{B}_1^-(\ddot{\breve{s}}) \cup \mathscr{B}_1^+(\ddot{\breve{s}})))$, $\mathscr{B}_2(\ddot{\breve{s}}) = \mathscr{B}_2^-(\ddot{\breve{s}})$ or $\mathscr{B}_2(\ddot{\breve{s}}) = \mathscr{B}_1^+(\ddot{\breve{s}})$ implying that $\mathscr{B}_2(\ddot{\breve{s}}) \in$

$(\mathscr{B}_2^-(\ddot{\breve{s}}) \cup \mathscr{B}_2^+(\ddot{\breve{s}})), \mathscr{B}_3(\ddot{\breve{s}}) = \mathscr{B}_3^-(\ddot{\breve{s}})$ or $\mathscr{B}_3(\ddot{\breve{s}}) = \mathscr{B}_3^+(\ddot{\breve{s}})$ implying that $\mathscr{B}_3(\ddot{\breve{s}}) \in \mathscr{B}_3^-(\ddot{\breve{s}}) \cup \mathscr{B}_3^+(\ddot{\breve{s}})$. □

Definition 13. If $\mathbb{U} = C_P = \langle \widehat{A}, \mathscr{B}\rangle$ and $\mathbb{V} = \mathscr{S}_p = (J, K)$ are the cubic picture fuzzy sets, then equality, P-order, and R-order are defined as follows:

(1) (Equality) $\mathbb{U} = \mathbb{V} \Leftrightarrow \widehat{A} = J$ and $\mathscr{B} = K$
(2) (P-order) $\mathbb{U} \subseteq_P \mathbb{V} \Leftrightarrow \widehat{A} \subseteq J$ and $\mathscr{B} \leq K$
(3) (R-order) $\mathbb{U} \subseteq_R \mathbb{V} \Leftrightarrow \widehat{A} \subseteq J$ and $\mathscr{B} \geq K$

Definition 14. For any indexed family of CPFSs $\mathbb{U}_i = \left\{\langle \ddot{\breve{s}}, \widehat{A}_i(\ddot{\breve{s}}), \mathscr{B}_i(\ddot{\breve{s}})\rangle | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\} (i \in \Lambda)$we define the following:

(1) $\cup_P \mathbb{U}_i = \left\{\langle \ddot{\breve{s}}, (\cup_{i\in I} \widehat{A}_i)(\ddot{\breve{s}}), (\vee_{i\in I}\mathscr{B}_i)(\ddot{\breve{s}})\rangle / \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$ (P-union)
(2) $\cap_P \mathbb{U}_i = \left\{\langle \ddot{\breve{s}}, (\cap_{i\in I} \widehat{A}_i)(\ddot{\breve{s}}), (\wedge_{i\in I}\mathscr{B}_i)(\ddot{\breve{s}}) / \ddot{\breve{s}} \in \ddot{\breve{S}}\rangle\right\}$ (P-intersection)
(3) $\cup_R \mathbb{U}_i = \left\{\langle \ddot{\breve{s}}, (\cup_{i\in I} \widehat{A}_i)(\ddot{\breve{s}}), (\wedge_{i\in I}\mathscr{B}_i)(\ddot{\breve{s}}) / \ddot{\breve{s}} \in \ddot{\breve{S}}\rangle\right\}$ (R-union)
(4) $\cap_R \mathbb{U}_i = \left\{\langle \ddot{\breve{s}}, (\cap_{i\in I} \widehat{A}_i)(\ddot{\breve{s}}), (\vee_{i\in I}\mathscr{B}_i)(\ddot{\breve{s}}) / \ddot{\breve{s}} \in \ddot{\breve{S}}\rangle\right\}$ (R-intersection)

The complement of $\mathbb{U} = \langle \widehat{A}, \mathscr{B}\rangle$ is also a cubic picture fuzzy set which is defined by $\mathbb{U}^c = \left\{\ddot{\breve{s}}, \widehat{A}^c(\ddot{\breve{s}}), \mathscr{B}^c(\ddot{\breve{s}}) | \ddot{\breve{s}} \in \ddot{\breve{S}}\right\}$.

Proposition 1. *For any CPFS $\mathbb{U} = \langle \widehat{A}, \mathscr{B}\rangle$, $\mathbb{V} = (J, K)$, $C = (L, M)$, and $D = (O, T)$, we have the following:*

(1) If $\mathbb{U} \subseteq_P \mathbb{V}$ and $\mathbb{V} \subseteq_P C$ then $\mathbb{U} \subseteq_P C$
(2) If $\mathbb{U} \subseteq_P \mathbb{V}$ and $\mathbb{U} \subseteq_P C$ then $\mathbb{U} \subseteq_P \mathbb{V} \cap_P C$
(3) If $\mathbb{U} \subseteq_P \mathbb{V}$ and $C \subseteq_P \mathbb{V}$ then $\mathbb{U} \cup_P C \subseteq_P \mathbb{V}$
(4) If $\mathbb{U} \subseteq_P \mathbb{V}$ and $C \subseteq_P D$ then $\mathbb{U} \cup_P C \subseteq_P \mathbb{V} \cup_P D$ and $\mathbb{U} \cap_P C \subseteq_P \mathbb{V} \cap_P D$
(5) If $\mathbb{U} \subseteq_R \mathbb{V}$ and $\mathbb{V} \subseteq_R C$ then $\mathbb{U} \subseteq_R C$
(6) If $\mathbb{U} \subseteq_R \mathbb{V}$ and $\mathbb{U} \subseteq_R C$ then $\mathbb{U} \subseteq_R \mathbb{V} \cap_R C$
(7) If $\mathbb{U} \subseteq_R \mathbb{V}$ and $C \subseteq_R \mathbb{V}$ then $\mathbb{U} \cup_R C \subseteq_R \mathbb{V}$
(8) If $\mathbb{U} \subseteq_R \mathbb{V}$ and $C \subseteq_R D$ then $\mathbb{U} \cup_R C \subseteq_R \mathbb{V} \cup_R D$ and $\mathbb{U} \cap_R C \subseteq_R \mathbb{V} \cap_R D$

Proof. The proof is straightforward and therefore is omitted. □

Remark 1. Te following are noted:

(1) If $\mathbb{U} \subseteq_P \mathbb{V}$, then $\mathbb{V}^c \nsubseteq_P \mathbb{U}^c$
(2) If $\mathbb{U} \subseteq_R \mathbb{V}$, then $\mathbb{V}^c \nsubseteq_R \mathbb{U}^c$

Example 5. Let

$$\begin{aligned} \mathbb{U} &= \{[0.1, 0.2], [0.2, 0.25], [0.2, 0.4], (0.1, 0.2, 0.2)\},\\ \mathbb{V} &= \{[0.1, 0.2], [0.2, 0.3], [0.2, 0.5], (0.1, 0.2, 0.3)\}, \end{aligned} \tag{4}$$

and then $\mathbb{U} \subseteq_P \mathbb{V}$.

Since

$$\begin{aligned} \mathbb{U}^c &= \{[0.2, 0.4], [0.2, 0.25], [0.1, 0.2], (0.2, 0.2, 0.1)\},\\ \mathbb{V}^c &= \{[0.2, 0.5], [0.2, 0.3], [0.1, 0.2], (0.3, 0.2, 0.1)\}, \end{aligned} \tag{5}$$

we obtain $\mathbb{V}^c \nsubseteq_p \mathbb{U}^c$.

Example 6. Let

$$\begin{aligned} \mathbb{U} &= \{[0.1, 0.2], [0.2, 0.25], [0.2, 0.4], (0.1, 0.2, 0.3)\},\\ \mathbb{V} &= \{[0.1, 0.2], [0.2, 0.3], [0.2, 0.5], (0.1, 0.2, 0.2)\}, \end{aligned} \tag{6}$$

and then $\mathbb{U} \subseteq_R \mathbb{V}$.

Since

$$\begin{aligned} \mathbb{U}^c &= \{[0.2, 0.4], [0.2, 0.25], [0.1, 0.2], (0.3, 0.2, 0.1)\},\\ \mathbb{V}^c &= \{[0.2, 0.5], [0.2, 0.3], [0.1, 0.2], (0.2, 0.2, 0.1)\}, \end{aligned} \tag{7}$$

we have $\mathbb{V}^c \nsubseteq_R \mathbb{U}^c$.

Theorem 3. *Let $\mathbb{U} = \langle \widehat{A}, \mathscr{B}\rangle$ be a cubic picture fuzzy set. If $\mathbb{U}$ is an ICPFS (resp., ECPFS), then $\mathbb{U}^c$ is an ICPFS (resp., ECPFS).*

Proof. The proof is straightforward and therefore is omitted. □

Theorem 4. *P-union and P-intersection of arbitrary indexed family of ICPFSs $\left\{\mathbb{U}_i = \langle \widehat{A}_i, \mathscr{B}_i\rangle | i \in \Lambda\right\}$ are ICPFSs.*

Proof. As $\mathbb{U}_i$ is an ICPFSs,

$$\begin{aligned} &\mathscr{B}_{1i}^-(\ddot{\breve{s}}) \leq \mathscr{B}_{1i}(\ddot{\breve{s}}) \leq \mathscr{B}_{1i}^+(\ddot{\breve{s}}), \mathscr{B}_{2i}^-(\ddot{\breve{s}}) \leq \mathscr{B}_{2i}(\ddot{\breve{s}}) \leq \mathscr{B}_{2i}^+(\ddot{\breve{s}}),\\ &\mathscr{B}_{3i}^-(\ddot{\breve{s}}) \leq \mathscr{B}_{3i}(\ddot{\breve{s}}) \leq \mathscr{B}_{3i}^+(\ddot{\breve{s}}), \end{aligned} \tag{8}$$

for each $i \in \Lambda$. This implies that

$$\begin{aligned} \bigcup_{i\in\Lambda} \mathscr{B}_{1i}^-(\ddot{\breve{s}}) &\leq \bigvee_{i\in\Lambda} \mathscr{B}_{1i}(\ddot{\breve{s}}) \leq \bigcup_{i\in\Lambda} \mathscr{B}_{1i}^+(\ddot{\breve{s}}),\\ \bigcup_{i\in\Lambda} \mathscr{B}_{2i}^-(\ddot{\breve{s}}) &\leq \bigvee_{i\in\Lambda} \mathscr{B}_{2i}(\ddot{\breve{s}}) \leq \bigcup_{i\in\Lambda} \mathscr{B}_{2i}^+(\ddot{\breve{s}}),\\ \bigcup_{i\in\Lambda} \mathscr{B}_{3i}^-(\ddot{\breve{s}}) &\leq \bigvee_{i\in\Lambda} \mathscr{B}_{3i}(\ddot{\breve{s}}) \leq \bigcup_{i\in\Lambda} \mathscr{B}_{3i}^+(\ddot{\breve{s}}), \end{aligned} \tag{9}$$

and, likewise,

$$\begin{aligned} \bigcap_{i\in\Lambda} \mathscr{B}_{1i}^-(\ddot{\breve{s}}) &\leq \bigwedge_{i\in\Lambda} \mathscr{B}_{1i}(\ddot{\breve{s}}) \leq \bigcap_{i\in\Lambda} \mathscr{B}_{1i}^+(\ddot{\breve{s}}),\\ \bigcap_{i\in\Lambda} \mathscr{B}_{2i}^-(\ddot{\breve{s}}) &\leq \bigwedge_{i\in\Lambda} \mathscr{B}_{2i}(\ddot{\breve{s}}) \leq \bigcap_{i\in\Lambda} \mathscr{B}_{2i}^+(\ddot{\breve{s}}),\\ \bigcap_{i\in\Lambda} \mathscr{B}_{3i}^-(\ddot{\breve{s}}) &\leq \bigwedge_{i\in\Lambda} \mathscr{B}_{3i}(\ddot{\breve{s}}) \leq \bigcap_{i\in\Lambda} \mathscr{B}_{3i}^+(\ddot{\breve{s}}). \end{aligned} \tag{10}$$

Hence, P-union and P-intersection of $\mathbb{U}_i$ are CPFSs. □

Remark 2. P-union and P-intersection of ECPFSs need not be an ECPFS.

Example 7. Let $\mathbb{U} = \langle \widehat{A}, \mathscr{B}\rangle$ and $\mathbb{V} = (J, K)$ be the ECPFSs of $I = [0, 1]$, where

$$\begin{aligned}&\widehat{A}(\ddot{\breve{s}})=\left\{\langle[0.2,0.3],[0.0,0.2],[0.0,0.2]\rangle,\mathscr{B}(\ddot{\breve{s}})=\langle0.1,0.3,0.3\rangle\right\},\\&J(\ddot{\breve{s}})=\left\{\langle[0.0,0.2],[0.1,0.3],[0.2,0.3]\rangle,K(\ddot{\breve{s}})=\langle0.3,0.0,0.1\rangle\right\},\end{aligned}\tag{11}$$

for all $\ddot{\breve{s}}\in I$.

(1) We know that $\mathbb{U}\cup_P\mathbb{V}=\left\{\langle\ddot{\breve{s}},J(\ddot{\breve{s}}),\mathscr{B}(\ddot{\breve{s}})|\ddot{\breve{s}}\in\ddot{\breve{S}}\rangle\right\}$ and $\mathscr{B}_1(\ddot{\breve{s}})\in(\mathscr{B}_1^-(\ddot{\breve{s}}),\ \mathscr{B}_1^+(\ddot{\breve{s}}))$, $\mathscr{B}_2(\ddot{\breve{s}})\in(\mathscr{B}_2^-(\ddot{\breve{s}}),\mathscr{B}_2^+(\ddot{\breve{s}}))$, $\mathscr{B}_3(\ddot{\breve{s}})\in(\mathscr{B}_3^-(\ddot{\breve{s}}),\mathscr{B}_3^+(\ddot{\breve{s}}))$ for all $\breve{s}\in\breve{S}$. Hence, $\mathbb{U}\cap_P\mathbb{V}$ is not an ECPFS.

(2) We know that $\mathbb{U}\cap_P\mathbb{V}=\left\{\ddot{\breve{s}},\widehat{A}(\ddot{\breve{s}}),K(\ddot{\breve{s}})|\ddot{\breve{s}}\in\ddot{\breve{S}}\right\}$ and $K_1(\ddot{\breve{s}})\in(K_1^-(\ddot{\breve{s}}),K_1^+(\ddot{\breve{s}}))$, $K_2(\ddot{\breve{s}})\in(K_2^-(\ddot{\breve{s}})\cup K_2^+(\ddot{\breve{s}}))$, $K_3(\ddot{\breve{s}})\in(K_3^-(\ddot{\breve{s}}),K_3^+(\ddot{\breve{s}}))$ for all $\ddot{\breve{s}}\in\breve{S}$. Hence $\mathbb{U}\cap_P\mathbb{V}$ is not an ECPFS.

The following example shows that the R-union and R-intersection of CPFSs need not be an CPFS.

Example 8. Let $\mathbb{U}=\langle\widehat{A},\mathscr{B}\rangle$ and $\mathbb{V}=(J,K)$ be CPFSs in $I=[0,1]$, where $\widehat{A}(\ddot{\breve{s}})=\{[0.0,0.2],[0.1,0.3],[0.2,0.3]\}$, $\mathscr{B}(\ddot{\breve{s}})=\ \{0.1,\ 0.3,0.3\}$ and $J(\ddot{\breve{s}})=\{[0.2,0.3],\ [0.0,0.2],[0.0,0.2]\}$, $K(\ddot{\breve{s}})=0.3,0.0,0.1$ for all $\ddot{\breve{s}}\in I$.

(1) We know that $\mathbb{U}\cup_R\mathbb{V}=\left\{\langle\ddot{\breve{s}},J(\ddot{\breve{s}}),Z(\ddot{\breve{s}})|\ddot{\breve{s}}\in I\rangle\right\}$ and $\mathscr{B}_1(\ddot{\breve{s}})\notin(\mathscr{B}_1^-(\ddot{\breve{s}}),\mathscr{B}_1^+(\ddot{\breve{s}}))$, $\mathscr{B}_2(\ddot{\breve{s}})\notin(\mathscr{B}_2^-(\ddot{\breve{s}}),\mathscr{B}_2^+(\ddot{\breve{s}}))$, $\mathscr{B}_3(\ddot{\breve{s}})\notin(\mathscr{B}_3^-(\ddot{\breve{s}}),\mathscr{B}_3^+(\ddot{\breve{s}}))$ for all $\ddot{\breve{s}}\in I$. Hence, $\mathbb{U}\cup_R\mathbb{V}$ is not a CPFS.

(2) We know that $\mathbb{U}\cap_R\mathbb{V}=\left\{\langle\ddot{\breve{s}},\widehat{A}(\ddot{\breve{s}}),K(\ddot{\breve{s}})|\ddot{\breve{s}}\in I\rangle\right\}$ and $K_1(\ddot{\breve{s}})\notin(K_1^-(\ddot{\breve{s}}),K_1^+(\ddot{\breve{s}})),K_2(\ddot{\breve{s}})\notin(K_2^-(\ddot{\breve{s}}),K_2^+(\ddot{\breve{s}})),(K_3(\ddot{\breve{s}})\notin(K_3^-(\ddot{\breve{s}}),K_3^+(\ddot{\breve{s}})))$ for all $\ddot{\breve{s}}\in I$. Hence, $\mathbb{U}\cap_R\mathbb{V}$ is not a CPFS.

The following example shows that "R-union" and "R-intersection" of ECPFS need not be an ECPFS.

Example 9. Let $\mathbb{U}=\langle\widehat{A},\mathscr{B}\rangle$ and $\mathbb{V}=(J,K)$ be ECPFSs of $I=[0,1]$ in which $\widehat{A}(\ddot{\breve{s}})=\{[0.0,0.2],[0.1,0.4],[0.2,0.3]$, $\mathscr{B}(\ddot{\breve{s}})=\{0.3,0.0,0.1\}$, and $J(\ddot{\breve{s}})=\{[0.0,0.3],[0.0,0.2],[0.1,0.2]\}$, and $K(\ddot{\breve{s}})=\langle0.4,0.3,0.0\rangle$ for all $\ddot{\breve{s}}\in I$.

(1) We know that $\mathbb{U}\cup_R\mathbb{V}=\left\{\langle\ddot{\breve{s}},J(\ddot{\breve{s}}),\mathscr{B}(\ddot{\breve{s}})|\ddot{\breve{s}}\in I\rangle\right\}$; clearly $\mathscr{B}_1(\ddot{\breve{s}})\in(\mathscr{B}_1^-(\ddot{\breve{s}}),\mathscr{B}_1^+(\ddot{\breve{s}}))$. Hence, $\mathbb{U}\cap_R\mathbb{V}$ is not an ECPFS in I.

(2) We know that $\mathbb{U}\cap_R\mathbb{V}=\left\{\langle\ddot{\breve{s}},\widehat{A}(\ddot{\breve{s}}),K(\ddot{\breve{s}})|\ddot{\breve{s}}\in I\rangle\right\}$; clearly $K_3(\widehat{A})\in(K_3^-(\ddot{\breve{s}}),K_3^+(\ddot{\breve{s}}))$ for all $\ddot{\breve{s}}\in I$. Hence, $\mathbb{U}\cap_R\mathbb{V}$ is not an ECPFS.

Theorem 5. *Let $\mathbb{U}=\langle\widehat{A},\mathscr{B}\rangle$ and $\mathbb{V}=(J,K)$ be the CPFSs, such that*

$$\begin{aligned}&\mathrm{Max}\left(\left(\mathscr{B}_1^-(\ddot{\breve{s}}),K_1^{'-}(\ddot{\breve{s}})\right)\le(\mathscr{B}_1\Lambda K_1)(\ddot{\breve{s}}),\\&\mathrm{Max}\left(\mathscr{B}_2^-(\ddot{\breve{s}}),K_2^{'-}(\ddot{\breve{s}})\right)\le(\mathscr{B}_2\Lambda K_2)(\ddot{\breve{s}}),\\&\mathrm{Max}\left(\left(\mathscr{B}_3^-(\ddot{\breve{s}}),K_3^{'-}(\ddot{\breve{s}})\right)\le(\mathscr{B}_3\Lambda K_3)(\ddot{\breve{s}}),\end{aligned}\tag{12}$$

for each $\ddot{\breve{s}}\in\ddot{\breve{S}}$. Then the "R-union" of $\mathbb{U}$ and $\mathbb{V}$ is a CPFS.

Proof. Let $\mathbb{U}=\langle\widehat{A},\mathscr{B}\rangle$ and $\mathbb{V}=(J,K)$ be two CPFSs, which satisfy the conditions given in Theorem 5; then we have $\mathscr{B}_1^-(\ddot{\breve{s}})\le\mathscr{B}_1(\ddot{\breve{s}})\le\mathscr{B}_1^+(\ddot{\breve{s}})$, $\mathscr{B}_2^-(\ddot{\breve{s}})\le\mathscr{B}_2(\ddot{\breve{s}})\le\mathscr{B}_2^+(\ddot{\breve{s}})$, $\mathscr{B}_3^-(\ddot{\breve{s}})\le\mathscr{B}_3(\ddot{\breve{s}})\le\mathscr{B}_3^+(\ddot{\breve{s}})$, and $K_1^{'-}(\ddot{\breve{s}})\le K_1(\ddot{\breve{s}})\le K_1^{'+}(\ddot{\breve{s}})$, $K_2^{'-}(\ddot{\breve{s}})\le K_2(\ddot{\breve{s}})\le K_2^{'+}(\ddot{\breve{s}}),K_3^{'-}(\ddot{\breve{s}})\le K_3(\ddot{\breve{s}})\le K_3^{'+}(\ddot{\breve{s}})$.

These imply that $(\mathscr{B}_1(\ddot{\breve{s}})\Lambda K_1(\ddot{\breve{s}}))\le(\mathscr{B}_1\cup K_1')^-(\ddot{\breve{s}})$, $(\mathscr{B}_2(\ddot{\breve{s}})\Lambda K_2(\ddot{\breve{s}}))\le(\mathscr{B}_2\cup K_2')^-(\ddot{\breve{s}})$, $(\mathscr{B}_3(\ddot{\breve{s}})\Lambda K_3(\ddot{\breve{s}}))\le(\mathscr{B}_3\cup K_3')^-(\ddot{\breve{s}})$. It follows from the assumption that

$$\begin{aligned}&(\mathscr{B}_1\cup K_1')^-(\ddot{\breve{s}})=\left\{\mathrm{Max}\left(\mathscr{B}_1^-(\ddot{\breve{s}}),K_1^{'-}(\ddot{\breve{s}})\right)\right\}\le(\mathscr{B}_1\Lambda K_1)(w)\le(\mathscr{B}_1\cup K_1')^+(\ddot{\breve{s}}),\\&(\mathscr{B}_2\cup K_2')^-(\ddot{\breve{s}})=\left\{\mathrm{Max}\left(\mathscr{B}_2^-(\ddot{\breve{s}}),K_2^{'-}(\ddot{\breve{s}})\right)\right\}\le(\mathscr{B}_2\Lambda K_2)(w)\le(\mathscr{B}_2\cup K_2')^+(\ddot{\breve{s}}),\\&(\mathscr{B}_3\cup K_3')^-(\ddot{\breve{s}})=\left\{\mathrm{Max}\left(\mathscr{B}_3^-(\ddot{\breve{s}}),K_3^{'-}(\ddot{\breve{s}})\right)\right\}\le(\mathscr{B}_3\Lambda K_3)(\ddot{\breve{s}})\le(\mathscr{B}_3\cup K_3')^+(\ddot{\breve{s}}),\end{aligned}\tag{13}$$

where $\mathbb{U}\cup_R\mathbb{V}=\left\{\langle\ddot{\breve{s}},(\widehat{A}\cup J)(\ddot{\breve{s}}),(\mathscr{B}VK)(\ddot{\breve{s}})|\ddot{\breve{s}}\in\ddot{\breve{S}}\rangle\right\}$ is a CPFS. For two ECPFSs $\mathbb{U}$ and $\mathbb{V}$ of $\breve{S}$, two CPFSs $\mathbb{U}^*$ and $\mathbb{V}^*$ derived from the given sets need not be CPFSs. □

Example 10. Let $\mathbb{U}=\langle\widehat{A},\mathscr{B}\rangle$ and $\mathbb{V}=(J,K)$ be ECPFSs of $I=[0,1]$, in which

$$\begin{aligned}
\widehat{A}\cdot(\ddot{\breve{s}}) &= \{[0.1, 0.2], [0.1, 0.3], [0.2, 0.4]\}, \mathscr{B}(\ddot{\breve{s}})\\
&= \{\langle 0.3, 0.0, 0.5\rangle\}, \quad J(\ddot{\breve{s}})\\
&= \{[0.0, 0.2], [0.0, 0.3], [0.1, 0.3]\}, \ K(\ddot{\breve{s}})\\
&= \{\langle 0.3, 0.4, 0.0\rangle\},
\end{aligned} \tag{14}$$

for all $\ddot{\breve{s}} \in I$.

It is seen that $\mathbb{U}^* = (\widehat{A}, K)$ and $\mathbb{V}^* = (J, \mathscr{B})$ are not CPFSs, because in $\mathbb{U}^*$, $\mathscr{B}_1(\ddot{\breve{s}}) \notin [0.1, 0.2]$, $\mathscr{B}_2(\ddot{\breve{s}}) \notin [0.1, 0.3]$, $\mathscr{B}_3(\ddot{\breve{s}}) \notin [0.2, 0.4]$ and in $\mathbb{V}^*, K_1(\ddot{\breve{s}}) \notin [0.0, 0.2]$, $K_3(\ddot{\breve{s}}) \notin [0.1, 0.3]$.

The following example shows that the "P-union" of two ECPFSs need not be a CPFS.

Example 11. Let $\mathbb{U} = \langle \widehat{A}, \mathscr{B}\rangle$ and $\mathbb{V} = (J, K)$ be ECPFSs of $I = [0, 1]$ in which $\widehat{A}\cdot(\ddot{\breve{s}}) = \{[0.2, 0.4], [0.1, 0.2], [0.0, 0.3]\}$, $\mathscr{B}(\ddot{\breve{s}}) = \{0.1, 0.3, 0.4\}$. $J(\ddot{\breve{s}}) = \{[0.0, 0.2], [0.1, 0.4], [0.2, 0.3]\}$ and $K(\ddot{\breve{s}}) = \{0.3, 0.0, 0.1\}$ for all $\ddot{\breve{s}} \in I$, $\mathbb{U} \cup_P \mathbb{V} = \left\{\ddot{\breve{s}}, J(\ddot{\breve{s}}), \mathscr{B}(\ddot{\breve{s}}) | \ddot{\breve{s}} \in I\right\}$; clearly, $0.4 \notin [0.2, 0.3]$. Hence $\mathbb{U} \cup_P \mathbb{V}$ is not a CPFS of I.

Theorem 6. *Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be the CPFSs in $\mathscr{W}$ satisfying the following inequalities:*

$$\begin{aligned}
\mathrm{Min}\left(\mathscr{B}_1^-(\ddot{\breve{s}}), K_1^{\prime -}(\ddot{\breve{s}})\right) &\geq (\mathscr{B}_1 \vee K_1)(\ddot{\breve{s}}),\\
\mathrm{Min}\left(\mathscr{B}_2^-(\ddot{\breve{s}}), K_2^{\prime -}(\ddot{\breve{s}})\right) &\geq (\mathscr{B}_2 \vee K_2)(\ddot{\breve{s}}),\\
\mathrm{Min}\left(\mathscr{B}_3^-(\ddot{\breve{s}}), K_{3i}^{\prime -}(\ddot{\breve{s}})\right) &\geq (\mathscr{B}_3 \vee K_3)(\ddot{\breve{s}}),
\end{aligned} \tag{15}$$

for all $\ddot{\breve{s}} \in \ddot{\breve{S}}$. Then the "R-intersection" of $\mathbb{U}$ and $\mathbb{V}$ is a CPFS.

Proof. The proof is straightforward and therefore is omitted. □

Theorem 7. *Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be the ECPFSs. if $\mathbb{U}^* = (\widehat{A}, K)$ and $\mathbb{V}^* = (J, \mathscr{B})$ are CPFSs, then P-union $(\mathbb{U} \cup_P \mathbb{V})$ of $\mathbb{U}$ and $\mathbb{V}$ is a CPFS.*

Proof. The proof is straightforward and therefore is omitted. □

Theorem 8. *Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be the ECPFSs. If $\mathbb{U}^* = (\widehat{A}, K)$ and $\mathbb{V}^* = (J, \mathscr{B})$ are CPFSs, then P-intersection $(\mathbb{U} \cup_P \mathbb{V})$ of $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ is a CPFS.*

Proof. The proof is straightforward and therefore is omitted. □

Remark 3. For two ECPFSs $\mathbb{U}$ and $\mathbb{V}$ of $\mathscr{W}$, the derived CPFSs $\mathbb{U}^*$ and $\mathbb{V}^*$ need not be ECPFSs.

Example 12. Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be ECPFSs of $I = [0, 1]$ in which

$$\begin{aligned}
\widehat{A}\cdot(\ddot{\breve{s}}) &= \{[0.0, 0.3], [0.1, 0.2], [0.2, 0.4]\}, \mathscr{B}(\ddot{\breve{s}})\\
&= \{0.4, 0.0, 0.1\}, J(\ddot{\breve{s}})\\
&= \{[0.1, 0.2], [0.0, 0.3], [0.2, 0.3]\}, K(\ddot{\breve{s}})\\
&= \{0.3, 0.4, 0.1\},
\end{aligned} \tag{16}$$

for all $\ddot{\breve{s}} \in I$. Now, from the above, we observe that $\mathbb{U}^* = (\widehat{A}, K)$ and $\mathbb{V}^* = (J, \mathscr{B})$ are not ECPFSs, because, in $\mathbb{U}^*, K_1(\ddot{\breve{s}}) \in [0.0, 0.3]$, and, in $\mathbb{V}^*, \mathscr{B}_2(\ddot{\breve{s}}) \notin [0.0, 0.3]$.

Theorem 9. *Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be two ECPFSs. If $\mathbb{U}^* = (\widehat{A}, K)$ and $\mathbb{V}^* = (J, \mathscr{B})$ are ECPFSs, then P-union $(\mathbb{U} \cup_P \mathbb{V})$ of $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ is an ECPFS.*

Proof. Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be ECPFSs, such that $\mathbb{U}^* = (\widehat{A}, K)$ and $\mathbb{V}^* = (J, \mathscr{B})$ are ECPFSs. Then we obtain that $\mathscr{B}_1(\ddot{\breve{s}}) \notin (\mathscr{B}_1^-(\ddot{\breve{s}}), \mathscr{B}_1^+(\ddot{\breve{s}}))$, $\mathscr{B}_2(\ddot{\breve{s}}) \notin (\mathscr{B}_2^-(\ddot{\breve{s}}), \mathscr{B}_2^+(\ddot{\breve{s}}))$, and $\mathscr{B}_3(\ddot{\breve{s}}) \notin (\mathscr{B}_3^-(\ddot{\breve{s}}), \mathscr{B}_3^+(\ddot{\breve{s}}))$, and $K_1(\ddot{\breve{s}}) \notin \left\{K_1^{\prime -}(\ddot{\breve{s}}), K_1^{\prime +}(\ddot{\breve{s}})\right\}$, $K_2(\ddot{\breve{s}}) \notin \left\{K_2^{\prime -}(\ddot{\breve{s}}), K_2^{\prime +}(\ddot{\breve{s}})\right\}$, and $K_3(\ddot{\breve{s}}) \notin \left\{K_3^{\prime -}(\ddot{\breve{s}}), K_3^{\prime +}(\ddot{\breve{s}})\right\}$.

Hence,

$$\begin{aligned}
(\mathscr{B}_1 \vee K_1)(\ddot{\breve{s}}) &\notin \left(\left\{\mathrm{Max}\left(\mathscr{B}_1^-(\ddot{\breve{s}}), K_1^{\prime -}(\ddot{\breve{s}})\right)\right\}, \mathrm{Max}\left(\mathscr{B}_1^+(\ddot{\breve{s}}), K_1^{\prime +}(\ddot{\breve{s}})\right)\right),\\
(\mathscr{B}_2 \vee K_2)(\ddot{\breve{s}}) &\notin \left(\left\{\mathrm{Max}\left(\mathscr{B}_2^-(\ddot{\breve{s}}), K_2^{\prime -}(\ddot{\breve{s}})\right)\right\}, \mathrm{Max}\left(\mathscr{B}_2^+(\ddot{\breve{s}}), K_2^{\prime +}(\ddot{\breve{s}})\right)\right),\\
(\mathscr{B}_3 \vee K_3)(\ddot{\breve{s}}) &\notin \left(\left\{\mathrm{Max}\left(\mathscr{B}_3^-(\ddot{\breve{s}}), K_3^{\prime -}(\ddot{\breve{s}})\right)\right\}, \mathrm{Max}\left(\mathscr{B}_3^+(\ddot{\breve{s}}), K_3^{\prime +}(\ddot{\breve{s}})\right)\right).
\end{aligned} \tag{17}$$

This implies that

$$\begin{aligned}&(\mathscr{B}_1 \vee K_1)(w) \notin \left((\mathscr{B}_1 \cup K_1')^- (\ddot{\breve{s}}), (\mathscr{B}_1 \cup K_1')^+ (\ddot{\breve{s}}) \right),\\&(\mathscr{B}_1 \vee K_1)(w) \notin \left((\mathscr{B}_2 \cup K_2')^- (\ddot{\breve{s}}), (\mathscr{B}_2 \cup K_2')^+ (\ddot{\breve{s}}) \right),\\&(\mathscr{B}_1 \vee K_1)(w) \notin \left((\mathscr{B}_3 \cup K_3')^- (\ddot{\breve{s}}), (\mathscr{B}_3 \cup K_3')^+ (\ddot{\breve{s}}) \right).\end{aligned} \tag{18}$$

Hence, $\mathbb{U} \cup_P \mathbb{V}$ is an ECPFS. □

Theorem 10. *Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be the ECPFSs of $\breve{S}$ such that*

$$\begin{aligned}&\mathrm{Min}\left(\left\{\mathrm{Max}\left(\mathscr{B}_1^+(\ddot{\breve{s}}), K_1'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Max}\left(\mathscr{B}_1^-(\ddot{\breve{s}}), K_1'^+(\ddot{\breve{s}})\right)\right\}\right)\\&\quad\geq (\mathscr{B}_1 \vee K_1)(\ddot{\breve{s}}) > \mathrm{Max}\left(\left\{\mathrm{Min}\left(\mathscr{B}_1^+(\ddot{\breve{s}}), K_1'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Min}\left(\mathscr{B}_1^-(\ddot{\breve{s}}), K_1'^+(\ddot{\breve{s}})\right)\right\}\right),\\&\mathrm{Min}\left(\left\{\mathrm{Max}\left(\mathscr{B}_2^+(\ddot{\breve{s}}), K_2'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Max}\left(\mathscr{B}_2^-(\ddot{\breve{s}}), K_2'^+(\ddot{\breve{s}})\right)\right\}\right)\\&\quad\geq (\mathscr{B}_1 \vee K_2)(\ddot{\breve{s}}) > \mathrm{Max}\left(\left\{\mathrm{Min}\left(\mathscr{B}_2^+(\ddot{\breve{s}}), K_1'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Min}\left(\mathscr{B}_1^-(\ddot{\breve{s}}), K_2'^+(\ddot{\breve{s}})\right)\right\}\right),\\&\mathrm{Min}\left(\left\{\mathrm{Max}\left(\mathscr{B}_3^+(\ddot{\breve{s}}), K_3'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Max}\left(\mathscr{B}_1^-(\ddot{\breve{s}}), K_1'^+(\ddot{\breve{s}})\right)\right\}\right)\\&\quad\geq (\mathscr{B}_3 \vee K_3)(w) > \mathrm{Max}\left(\left\{\mathrm{Min}\left(\mathscr{B}_3^+(\ddot{\breve{s}}), K_3'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Min}\left(\mathscr{B}_3^-(\ddot{\breve{s}}), K_3'^+(\ddot{\breve{s}})\right)\right\}\right),\end{aligned} \tag{19}$$

for all $\ddot{\breve{s}} \in \ddot{\breve{S}}$.

Then, the P-intersection of $\mathbb{U}$ and $\mathbb{V}$ is an ECPFS.

Proof. The proof is straightforward by the definitions in [11, 13]. □

Theorem 11. *Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be the CPFSs, such that the following implications are valid:*

$$\begin{aligned}&\mathrm{Min}\left(\left\{\mathrm{Max}\left(\mathscr{B}_1^+(\ddot{\breve{s}}), K_1'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Max}\left(\mathscr{B}_1^-(\ddot{\breve{s}}), K_1'^+(\ddot{\breve{s}})\right)\right\}\right) = (\mathscr{B}_1 \vee K_1)(\ddot{\breve{s}})\\&\quad= \mathrm{Max}\left(\left\{\mathrm{Min}\left(\mathscr{B}_1^+(\ddot{\breve{s}}), K_1'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Min}\left(\mathscr{B}_1^-(\ddot{\breve{s}}), K_1'^+(\ddot{\breve{s}})\right)\right\}\right),\\&\mathrm{Min}\left(\left\{\mathrm{Max}\left(\mathscr{B}_2^+(\ddot{\breve{s}}), K_2'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Max}\left(\mathscr{B}_2^-(\ddot{\breve{s}}), K_2'^+(\ddot{\breve{s}})\right)\right\}\right) = (\mathscr{B}_2 \vee K_2)(\ddot{\breve{s}})\\&\quad= \mathrm{Max}\left(\left\{\mathrm{Min}\left(\mathscr{B}_2^+(\ddot{\breve{s}}), K_2'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Min}\left(\mathscr{B}_2^-(\ddot{\breve{s}}), K_2'^+(\ddot{\breve{s}})\right)\right\}\right),\\&\mathrm{Min}\left(\left\{\mathrm{Max}\left(\mathscr{B}_3^+(\ddot{\breve{s}}), K_3'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Max}\left(\mathscr{B}_3^-(\ddot{\breve{s}}), K_3'^+(\ddot{\breve{s}})\right)\right\}\right) \geq (\mathscr{B}_3 \vee K_3)(\ddot{\breve{s}})\\&\quad= \mathrm{Max}\left(\left\{\mathrm{Min}\left(\mathscr{B}_3^+(\ddot{\breve{s}}), K_3'^-(\ddot{\breve{s}})\right)\right\}, \left\{\mathrm{Min}\left(\mathscr{B}_3^-(\ddot{\breve{s}}), K_3'^+(\ddot{\breve{s}})\right)\right\}\right),\end{aligned} \tag{20}$$

for all $\ddot{\breve{s}} \in \ddot{\breve{S}}$. Then the P-intersection of $\mathbb{U}$ and $\mathbb{V}$ is both an ECPFS and a CPFS.

Proof. It is straightforward by the definitions in [11, 13].

The following example shows that the P-union of two ECPFSs needs not be an ECPFS. □

Example 13. Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be two ECPFSs of $I = [0, 1]$ defined as follows:

$$\begin{aligned}&\hat{A} \cdot (\ddot{\breve{s}}) = \{[0.1, 0.3], [0.2, 0.3], [0.0, 0.2]\},\\&\mathscr{B}(\ddot{\breve{s}}) = \{0.0, 0.1, 0.3\},\\&J(\ddot{\breve{s}}) = \{[0.1, 0.2], [0.2, 0.4], [0.1, 0.3]\},\\&K(\ddot{\breve{s}}) = \{0.3, 0.1, 0.4\},\end{aligned} \tag{21}$$

for all $\ddot{\breve{s}} \in I$.

Since $\mathbb{U}\cup_P\mathbb{V}=\left\{\ddot{\breve{s}}, J(\ddot{\breve{s}}), \mathscr{B}(\ddot{\breve{s}})\ddot{\breve{s}}\in I\right\}$, clearly $0.3\in[0.1,0.3]$. Hence $\mathbb{U}\cup_P\mathbb{V}$ is not an ECPFS of I.

Theorem 12. *Let $\mathbb{U}=(\hat{A},\mathscr{B})$ and $\mathbb{V}=(J,K)$ be two ECPFSs, such that the following are satisfied:*

$$\begin{aligned}
&\operatorname{Min}\left(\left\{\operatorname{Max}\left(\mathscr{B}_1^{+}(\ddot{\breve{s}}), K_1^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Max}\left(\mathscr{B}_1^{-}(\ddot{\breve{s}}), K_1^{'+}(\ddot{\breve{s}})\right)\right\}\right) > (\mathscr{B}_1\vee K_1)(\ddot{\breve{s}})\\
&\quad\geq \operatorname{Max}\left(\left\{\operatorname{Min}\left(\mathscr{B}_1^{+}(\ddot{\breve{s}}), K_1^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Min}\left(\mathscr{B}_1^{-}(\ddot{\breve{s}}), K_1^{'+}(\ddot{\breve{s}})\right)\right\}\right),\\
&\operatorname{Min}\left(\left\{\operatorname{Max}\left(\mathscr{B}_2^{+}(\ddot{\breve{s}}), K_2^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Max}\left(\mathscr{B}_2^{-}(\ddot{\breve{s}}), K_2^{'+}(\ddot{\breve{s}})\right)\right\}\right) > (\mathscr{B}_2\vee K_2)(\ddot{\breve{s}})\\
&\quad\geq \operatorname{Max}\left(\left\{\operatorname{Min}\left(\mathscr{B}_2^{+}(\ddot{\breve{s}}), K_2^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Min}\left(\mathscr{B}_2^{-}(\ddot{\breve{s}}), K_2^{'+}(\ddot{\breve{s}})\right)\right\}\right),\\
&\operatorname{Min}\left(\left\{\operatorname{Max}\left(\mathscr{B}_3^{+}(\ddot{\breve{s}}), K_3^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Max}\left(\mathscr{B}_3^{-}(\ddot{\breve{s}}), K_3^{'+}(\ddot{\breve{s}})\right)\right\}\right) > (\mathscr{B}_3\vee K_3)(\ddot{\breve{s}})\\
&\quad\geq \operatorname{Max}\left(\left\{\operatorname{Min}\left(\mathscr{B}_3^{+}(\ddot{\breve{s}}), K_3^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Min}\left(\mathscr{B}_3^{-}(\ddot{\breve{s}}), K_3^{'+}(\ddot{\breve{s}})\right)\right\}\right),
\end{aligned}\tag{22}$$

for all $\ddot{\breve{s}}\in\ddot{\breve{S}}$. Then, P-union of $\mathbb{U}$ and $\mathbb{V}$ is an ECPFS.

Proof. The proof is straightforward and therefore is omitted. □

Theorem 13. *Let $\mathbb{U}=(\hat{A},\mathscr{B})$ and $\mathbb{V}=(J,K)$ be two ECPFSs, which satisfy the followings conditions:*

$$\begin{aligned}
&\operatorname{Min}\left(\left\{\operatorname{Max}\left(\mathscr{B}_1^{+}(\ddot{\breve{s}}), K_1^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Max}\left(\mathscr{B}_1^{-}(\ddot{\breve{s}}), K_1^{'+}(\ddot{\breve{s}})\right)\right\}\right) > (\mathscr{B}_1\vee K_1)(w)\\
&\quad\geq \operatorname{Max}\left(\left\{\operatorname{Min}\left(\mathscr{B}_1^{+}(\ddot{\breve{s}}), K_1^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Min}\left(\mathscr{B}_1^{-}(\ddot{\breve{s}}), K_1^{'+}(\ddot{\breve{s}})\right)\right\}\right),\\
&\operatorname{Min}\left(\left\{\operatorname{Max}\left(\mathscr{B}_2^{+}(\ddot{\breve{s}}), K_2^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Max}\left(\mathscr{B}_2^{-}(\ddot{\breve{s}}), K_2^{'+}(\ddot{\breve{s}})\right)\right\}\right) > (\mathscr{B}_2\vee K_2)(\ddot{\breve{s}})\\
&\quad\geq \operatorname{Max}\left(\left\{\operatorname{Min}\left(\mathscr{B}_2^{+}(\ddot{\breve{s}}), K_2^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Min}\left(\mathscr{B}_2^{-}(\ddot{\breve{s}}), K_2^{'+}(\ddot{\breve{s}})\right)\right\}\right),\\
&\operatorname{Min}\left(\left\{\operatorname{Max}\left(\mathscr{B}_3^{+}(\ddot{\breve{s}}), K_3^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Max}\left(\mathscr{B}_3^{-}(\ddot{\breve{s}}), K_3^{'+}(\ddot{\breve{s}})\right)\right\}\right) > (\mathscr{B}_3\vee K_3)(\ddot{\breve{s}})\\
&\quad\geq \operatorname{Max}\left(\left\{\operatorname{Min}\left(\mathscr{B}_3^{+}(\ddot{\breve{s}}), K_3^{'-}(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Min}\left(\mathscr{B}_3^{-}(\ddot{\breve{s}}), K_3^{'+}(\ddot{\breve{s}})\right)\right\}\right),
\end{aligned}\tag{23}$$

for all $\ddot{\breve{s}}\in\ddot{\breve{S}}$. Then R-union of $\mathbb{U}$ and $\mathbb{V}$ is an ECPFS.

Proof. The proof is straightforward. □

Theorem 14. *Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be the ECPFSs, such that the following are satisfied:*

$$\begin{aligned}
&\text{Min}\left(\left\{\text{Max}\left(\mathscr{B}_1^{+}(\ddot{\breve{s}}), K_1^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Max}\left(\mathscr{B}_1^{-}(\ddot{\breve{s}}), K_1^{\prime +}(\ddot{\breve{s}})\right)\right\}\right) \geq (\mathscr{B}_1 \vee K_1)(\ddot{\breve{s}}) \\
&\quad > Max\left(\left\{\text{Min}\left(\mathscr{B}_1^{+}(\ddot{\breve{s}}), K_1^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Min}\left(\mathscr{B}_1^{-}(\ddot{\breve{s}}), K_1^{\prime +}(\ddot{\breve{s}})\right)\right\}\right), \\
&\text{Min}\left(\left\{\text{Max}\left(\mathscr{B}_2^{+}(\ddot{\breve{s}}), K_2^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Max}\left(\mathscr{B}_2^{-}(\ddot{\breve{s}}), K_2^{\prime +}(\ddot{\breve{s}})\right)\right\}\right) \geq (\mathscr{B}_2 \vee K_2)(\ddot{\breve{s}}) \\
&\quad > Max\left(\left\{\text{Min}\left(\mathscr{B}_2^{+}(\ddot{\breve{s}}), K_2^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Min}\left(\mathscr{B}_2^{-}(\ddot{\breve{s}}), K_2^{\prime +}(\ddot{\breve{s}})\right)\right\}\right), \\
&\text{Min}\left(\left\{\text{Max}\left(\mathscr{B}_3^{+}(\ddot{\breve{s}}), K_3^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Max}\left(\mathscr{B}_3^{-}(\ddot{\breve{s}}), K_3^{\prime +}(\ddot{\breve{s}})\right)\right\}\right) \geq (\mathscr{B}_3 \vee K_3)(w) \\
&\quad > Max\left(\left\{\text{Min}\left(\mathscr{B}_3^{+}(\ddot{\breve{s}}), K_3^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Min}\left(\mathscr{B}_3^{-}(\ddot{\breve{s}}), K_3^{\prime +}(\ddot{\breve{s}})\right)\right\}\right),
\end{aligned} \tag{24}$$

for all $\ddot{\breve{s}} \in \ddot{\breve{S}}$. Then R-intersection of $\mathbb{U}$ and $\mathbb{V}$ is an ECPFS.

Proof. The proof is straightforward. □

Remark 4. Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be two ECPFSs, such that the following are satisfied:

$$\begin{aligned}
&\text{Min}\left(\left\{\text{Max}\left(\mathscr{B}_1^{+}(\ddot{\breve{s}}), K_1^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Max}\left(\mathscr{B}_1^{-}(\ddot{\breve{s}}), K_1^{\prime +}(\ddot{\breve{s}})\right)\right\}\right) > (\mathscr{B}_1 \vee K_1)(\ddot{\breve{s}}) \\
&\quad = \text{Max}\left(\left\{\text{Min}\left(\mathscr{B}_1^{+}(\ddot{\breve{s}}), K_1^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Min}\left(\mathscr{B}_1^{-}(\ddot{\breve{s}}), K_1^{\prime +}(\ddot{\breve{s}})\right)\right\}\right), \\
&\text{Min}\left(\left\{\text{Max}\left(\mathscr{B}_2^{+}(\ddot{\breve{s}}), K_2^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Max}\left(\mathscr{B}_2^{-}(\ddot{\breve{s}}), K_2^{\prime +}(\ddot{\breve{s}})\right)\right\}\right) > (\mathscr{B}_2 \vee K_2)(\ddot{\breve{s}}) \\
&\quad = \text{Max}\left(\left\{\text{Min}\left(\mathscr{B}_2^{+}(\ddot{\breve{s}}), K_2^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Min}\left(\mathscr{B}_2^{-}(\ddot{\breve{s}}), K_2^{\prime +}(\ddot{\breve{s}})\right)\right\}\right), \\
&\text{Min}\left(\left\{\text{Max}\left(\mathscr{B}_3^{+}(\ddot{\breve{s}}), K_3^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Max}\left(\mathscr{B}_3^{-}(\ddot{\breve{s}}), K_3^{\prime +}(\ddot{\breve{s}})\right)\right\}\right) > (\mathscr{B}_3 \vee K_3)(w) \\
&\quad = \text{Max}\left(\left\{\text{Min}\left(\mathscr{B}_3^{+}(\ddot{\breve{s}}), K_3^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Min}\left(\mathscr{B}_3^{-}(\ddot{\breve{s}}), K_3^{\prime +}(\ddot{\breve{s}})\right)\right\}\right),
\end{aligned} \tag{25}$$

for all $\ddot{\breve{s}} \in \ddot{\breve{S}}$. Then R-intersection of $\mathbb{U}$ and $\mathbb{V}$ may not be an ECPFS.

Theorem 15. *Let $\mathbb{U} = (\widehat{A}, \mathscr{B})$ and $\mathbb{V} = (J, K)$ be two ECPFSs, such that the following are satisfied:*

$$\begin{aligned}
&\text{Min}\left(\left\{\text{Max}\left(\mathscr{B}_1^{+}(\ddot{\breve{s}}), K_1^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Max}\left(\mathscr{B}_1^{-}(\ddot{\breve{s}}), K_1^{\prime +}(\ddot{\breve{s}})\right)\right\}\right) = (\mathscr{B}_1 \vee K_1)(\ddot{\breve{s}}) \\
&\quad = \text{Max}\left(\left\{\text{Min}\left(\mathscr{B}_1^{+}(\ddot{\breve{s}}), K_1^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Min}\left(\mathscr{B}_1^{-}(\ddot{\breve{s}}), K_1^{\prime +}(\ddot{\breve{s}})\right)\right\}\right), \\
&\text{Min}\left(\left\{\text{Max}\left(\mathscr{B}_2^{+}(\ddot{\breve{s}}), K_2^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Max}\left(\mathscr{B}_2^{-}(\ddot{\breve{s}}), K_2^{\prime +}(\ddot{\breve{s}})\right)\right\}\right) = (\mathscr{B}_2 \vee K_2)(\ddot{\breve{s}}) \\
&\quad = \text{Max}\left(\left\{\text{Min}\left(\mathscr{B}_2^{+}(\ddot{\breve{s}}), K_2^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Min}\left(\mathscr{B}_2^{-}(\ddot{\breve{s}}), K_2^{\prime +}(\ddot{\breve{s}})\right)\right\}\right), \\
&\text{Min}\left(\left\{\text{Max}\left(\mathscr{B}_3^{+}(\ddot{\breve{s}}), K_3^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Max}\left(\mathscr{B}_3^{-}(\ddot{\breve{s}}), K_3^{\prime +}(\ddot{\breve{s}})\right)\right\}\right) > (\mathscr{B}_3 \vee K_3)(w) \\
&\quad = \text{Max}\left(\left\{\text{Min}\left(\mathscr{B}_3^{+}(\ddot{\breve{s}}), K_3^{\prime -}(\ddot{\breve{s}})\right)\right\}, \left\{\text{Min}\left(\mathscr{B}_3^{-}(\ddot{\breve{s}}), K_3^{\prime +}(\ddot{\breve{s}})\right)\right\}\right),
\end{aligned} \tag{26}$$

for all $\ddot{\breve{s}} \in \ddot{\breve{S}}$. *Then R-intersection of* $\mathbb{U}$ *and* $\mathbb{V}$ *is both an ECPFS and a CPFS.*

Proof. The proof is straightforward. □

Theorem 16. *Let* $\mathbb{U} = (\widehat{A}, \mathscr{B})$ *and* $\mathbb{V} = (J, K)$ *be two CPFSs. If the implications are satisfied, for all* $\ddot{\breve{s}} \in \ddot{\breve{S}}$,

$$
\begin{aligned}
&(\mathscr{B}_1 \wedge K_1)(\ddot{\breve{s}}) \leq \operatorname{Max}\left(\mathscr{B}_1^-(\ddot{\breve{s}}), K_1^-(\ddot{\breve{s}})\right),\\
&(\mathscr{B}_2 \wedge K_2)(\ddot{\breve{s}}) \leq \operatorname{Max}\left(\mathscr{B}_2^-(\ddot{\breve{s}}), K_2^-(\ddot{\breve{s}})\right),\\
&(\mathscr{B}_3 \wedge K_3)(\ddot{\breve{s}}) \leq \operatorname{Max}\left(\mathscr{B}_3^-(\ddot{\breve{s}}), K_3^-(\ddot{\breve{s}})\right),
\end{aligned}
\tag{27}
$$

then the R-union of $\mathbb{U}$ *and* $\mathbb{V}$ *is an EPCFS.*

Proof. The proof is straightforward. □

Theorem 17. *Let* $\mathbb{U} = (\widehat{A}, \mathscr{B})$ *and* $\mathbb{V} = (J, K)$ *be two CPFSs. If the following implications are satisfied for all* $\ddot{\breve{s}} \in \ddot{\breve{S}}$, $(\mathscr{B}_1 \vee K_1)(\ddot{\breve{s}}) \geq Min\left\{\mathscr{B}_1^+(\ddot{\breve{s}}), K_1'^+(\ddot{\breve{s}})\right\}$, $(\mathscr{B}_2 \vee K_2)(w) \geq Min\ (\mathscr{B}_2^+(\ddot{\breve{s}}), K_2'^+(\ddot{\breve{s}}))$, *and* $(\mathscr{B}_3 \vee K_3)(w) \geq Min(\mathscr{B}_3^+(\ddot{\breve{s}}), K_3'^+(\ddot{\breve{s}}))$, *then R-intersection of* $\mathbb{U}$ *and* $\mathbb{V}$ *is an ECPFS.*

Proof. The proof is straightforward. □

Theorem 18. *Let* $\mathbb{U} = (\widehat{A}, \mathscr{B})$ *and* $\mathbb{V} = (J, K)$ *be two ECPFSs, such that the following implications hold:*

$$
\begin{aligned}
&\operatorname{Min}\left(\left\{\operatorname{Max}\left(\mathscr{B}_1^+(\ddot{\breve{s}}), K_1'^-(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Max}\left(\mathscr{B}_1^-(\ddot{\breve{s}}), K_1^+(\ddot{\breve{s}})\right)\right\}\right) \leq (\mathscr{B}_1 \wedge K_1)(\ddot{\breve{s}}) \leq \operatorname{Max}\left(\left\{\left(\mathscr{B}_1^+(\ddot{\breve{s}}), K_1'^+(\ddot{\breve{s}})\right)\right\}\right),\\
&\operatorname{Min}\left(\left\{\operatorname{Max}\left(\mathscr{B}_2^+(\ddot{\breve{s}}), K_2'^-(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Max}\left(\mathscr{B}_2^-(\ddot{\breve{s}}), K_2^+(\ddot{\breve{s}})\right)\right\}\right) \leq (\mathscr{B}_2 \wedge K_2)(\ddot{\breve{s}}) \leq \operatorname{Max}\left(\left\{\operatorname{Min}\left(\mathscr{B}_2^+(\ddot{\breve{s}}), K_2'^+(\ddot{\breve{s}})\right)\right\}\right),\\
&\operatorname{Min}\left(\left\{\operatorname{Max}\left(\mathscr{B}_3^+(\ddot{\breve{s}}), K_3'^-(\ddot{\breve{s}})\right)\right\}, \left\{\operatorname{Max}\left(\mathscr{B}_3^-(\ddot{\breve{s}}), K_3^+(\ddot{\breve{s}})\right)\right\}\right) \leq (\mathscr{B}_3 \wedge K_3)(\ddot{\breve{s}}) \leq \operatorname{Max}\left(\left\{\operatorname{Min}\left(\mathscr{B}_3^+(\ddot{\breve{s}}), K_3'^+(\ddot{\breve{s}})\right)\right\}\right),
\end{aligned}
\tag{28}
$$

for all $\ddot{\breve{s}} \in \ddot{\breve{S}}$. *Then, R-union of* $\mathbb{U}$ *and* $\mathbb{V}$ *is a CPFS.*

Proof. The proof is straightforward. □

4. Averaging Aggregation Operators

In this section, we present three types of new aggregation operators called cubic picture fuzzy weighted averaging, cubic picture fuzzy ordered weighted averaging, and cubic picture fuzzy hybrid weighted averaging operators based on cubic picture fuzzy sets. Let $\mathscr{CPF}$ denote the collection of all CPFSs.

Definition 15 (see [11]). A function T: $[0, 1] \times [0, 1] \longrightarrow [0, 1]$ is said to be a t-norm which satisfies the following:

(1) Boundary: $T(0, 0) = 0$; $T(x, 1) = T(1, x) = x$ for all $x \in [0, 1]$
(2) Monotonicity: If $x_1 \leq y_1$ and $x_2 \leq y_2$, then $T(x_1, x_2) \leq T(y_1, y_2)$
(3) Commutativity: $T(x_1, x_2) = T(x_2, x_1)$
(4) Associativity: $T(x_1, T(x_2, x_3)) = T(T(x_1, x_2), x_3)$

A function S defined by $S(x, y) = 1 - T(1 - x, 1 - y)$ is called t-co-norm. A decreasing function g generates a t-norm as $T(x, y) = g^{-1}(g(x) + g(y))$ such that $g(1) = 0$ and function h generates the t-co-norm as $S(x, y) = h^{-1}(h(x) + h(y))$, where $h(t) = g(1 - t)$. Based on these norms' generators, g and h will be used in the next theorems.

Definition 16. Let $C = ((M, L, N), (\mathscr{U}, \rho, \vartheta)), C_1 = ((M_1, L_1, N_1), (\mathscr{U}_1, \rho_1, \vartheta_1))$, and $C_2 = ((M_2, L_2, N_2), (\mathscr{U}_2, \rho_2, \vartheta_2))$ be three CPFSs. Then the operations $\oplus$, $\otimes$, λC, and C^λ are defined as follows:

(1) $C_1 \oplus C_2 = \left((h^{-1}(h(M_1) + h(M_2)), g^{-1}(g(L_1) + g(L_2)), g^{-1}(g(N_1) + g(N_2))), (h^{-1}(h(\mathscr{U}_1) + h(\mathscr{U}_2)), g^{-1}(g(\rho_1) + g(\rho_2)), g^{-1}(g(\vartheta_1) + g(\vartheta_2))) \right)$
(2) $C_1 \otimes C_2 = \left((g^{-1}(g(M_1) + g(M_2)), h^{-1}(h(L_1) + h(L_2)), h^{-1}(h(N_1) + h(N_2))), (g^{-1}(g(\mathscr{U}_1) + g(\mathscr{U}_2)), h^{-1}(h(\rho_1) + h(\rho_2)), h^{-1}(h(\vartheta_1) + h(\vartheta_2))) \right)$
(3) $\lambda C = \begin{pmatrix} (h^{-1}(\lambda h(M)), g^{-1}(\lambda g(L)), g^{-1}(\lambda g(N))), \\ (h^{-1}(\lambda h(\mathscr{U})), g^{-1}(\lambda g(\rho)), g^{-1}(\lambda g(\vartheta))) \end{pmatrix}$
(4) $C^\lambda = \begin{pmatrix} (g^{-1}(\lambda g(M)), h^{-1}(\lambda h(L)), h^{-1}(\lambda h(N))), \\ (g^{-1}(\lambda g(\mathscr{U})), h^{-1}(\lambda h(\rho)), h^{-1}(\lambda h(\vartheta))) \end{pmatrix}$

Theorem 19. *Let* C_1, C_2, *and* C_3 *be three CPFNs and* λ, λ_1, *and* $\lambda_2 > 0$. *Then, we have the following:*

(1) $C_1 \oplus C_2 = C_2 \oplus C_1$
(2) $C_1 \otimes C_2 = C_2 \otimes C_1$
(3) $\lambda(C_1 \oplus C_2) = \lambda C_1 \oplus \lambda C_2$
(4) $(C_1 \otimes C_2)^\lambda = C_1^\lambda \otimes C_2^\lambda$
(5) $\lambda_1 C \oplus \lambda_2 C = (\lambda_1 + \lambda_2) C$
(6) $C^{\lambda_1} \otimes C^{\lambda_1} = C^{\lambda_1 + \lambda_2}$

Proof. It is easily obtained by the above definition. □

4.1. Cubic Picture Fuzzy Weighted Averaging (CPFWA) Operators

Definition 17. Let $\{\mathbb{U}_i\}_{i\in\Lambda}$ be a collection of CPFSs. Then theCPFWA $\mathscr{CPF}^n \longrightarrow \mathscr{CPF}$ is defined as follows:

$$\begin{aligned}\mathrm{CPFWA}\left(\mathbb{U}_1,\mathbb{U}_2,\ldots,\mathbb{U}_n\right) &= \ddot{w}_1\,\mathbb{U}_1 \oplus \ddot{w}_2\,\mathbb{U}_2 \oplus \cdots \oplus \ddot{w}_n\mathbb{U}_n\\ &= \oplus_{i=1}^{n}\ddot{w}_i\mathbb{U}_i,\end{aligned} \tag{29}$$

where $\ddot{w} = \{\ddot{w}_1,\ddot{w}_2,\ldots,\ddot{w}_n\}^T$ is the weighted vector of $\mathbb{U}_i$, s.t. $\ddot{w}_i > 0$ and $\sum_{i=1}^{n}\ddot{w}_i = 1$.

Theorem 20. *f* $\mathbb{U}_i = (\widehat{A}_i(\ddot{\breve{s}}),\mathscr{B}_i(\ddot{\breve{s}})),\ \ddot{\breve{s}} \in \ddot{\breve{S}}$ *is the collection of CPFSs, then the averaging value by using CPFWA operator is still CPFS and is given by*

$$\begin{aligned}\mathrm{CPFWA}\left(\mathbb{U}_1,\mathbb{U}_2,\ldots,\mathbb{U}_n\right) = &\left(\left(h^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i h(M_i)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g(L_i)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g(N_i)\right)\right),\right.\\ &\left.\left(h^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i h(\mathscr{U}_i)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g(\rho_i)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g(\vartheta_i)\right)\right)\right).\end{aligned} \tag{30}$$

Proof. We shall prove the result by using the principle of mathematical induction on "n."

Step 1. For $n = 2$, we have $\mathbb{U}_1 = (\widehat{A}_1,\mathscr{B}_1), \mathbb{U}_2 = (\widehat{A}_2,\mathscr{B}_2)$; thus, by the operations of CPFSs, we get

$$\begin{aligned}w_1\mathbb{U}_1 &= \left(\left(h^{-1}\left(\ddot{w}_1 h(M_1)\right), g^{-1}\left(\ddot{w}_1 g(L_1)\right), g^{-1}\left(\ddot{w}_1 g(N_1)\right)\right),\right.\\ &\quad\left.\left(h^{-1}\left(\ddot{w}_1 h(\mathscr{U}_1)\right), g^{-1}\left(\ddot{w}_1 g(\rho_1)\right), g^{-1}\left(\ddot{w}_1 g(\vartheta_1)\right)\right)\right),\\ w_2\mathbb{U}_2 &= \left(\left(h^{-1}\left(\ddot{w}_2 h(M_2)\right), g^{-1}\left(\ddot{w}_2 g(L_2)\right), g^{-1}\left(\ddot{w}_2 g(N_2)\right)\right),\right.\\ &\quad\left.\left(h^{-1}\left(\ddot{w}_2 h(\mathscr{U}_2)\right), g^{-1}\left(\ddot{w}_2 g(\rho_2)\right), g^{-1}\left(\ddot{w}_2 g(\vartheta_2)\right)\right)\right).\end{aligned} \tag{31}$$

Hence, by additive properties of CPFSs, we get

$$\begin{aligned}\mathrm{CPFWA}\left(\mathbb{U}_1,\mathbb{U}_2\right) &= w_1\mathbb{U}_1 \oplus w_2\mathbb{U}_2\\ &= \left(\left(h^{-1}\left(\ddot{w}_1 h(M_1)\right), g^{-1}\left(\ddot{w}_1 g(L_1)\right), g^{-1}\left(\ddot{w}_1 g(N_1)\right)\right), \left(h^{-1}\left(\ddot{w}_1 h(\mathscr{U}_1)\right), g^{-1}\left(\ddot{w}_1 g(\rho_1)\right), g^{-1}\left(\ddot{w}_1 g(\vartheta_1)\right)\right)\right)\\ &\quad\oplus\left(\left(h^{-1}\left(\ddot{w}_2 h(M_2)\right), g^{-1}\left(\ddot{w}_2 g(L_2)\right), g^{-1}\left(\ddot{w}_2 g(N_2)\right)\right), \left(h^{-1}\left(\ddot{w}_2 h(\mathscr{U}_2)\right), g^{-1}\left(\ddot{w}_2 g(\rho_2)\right), g^{-1}\left(\ddot{w}_2 g(\vartheta_2)\right)\right)\right)\\ &= \left(h^{-1}\left\{h\left(h^{-1}\left(\ddot{w}_1 h(M_1)\right)\right) + h\left(h^{-1}\left(\ddot{w}_2 h(M_2)\right)\right)\right\}, g^{-1}\left\{g\left(g^{-1}\left(\ddot{w}_1 g(L_1)\right)\right) + g\left(g^{-1}\left(\ddot{w}_2 g(L_2)\right)\right)\right\},\right.\\ &\quad g^{-1}\left\{g\left(g^{-1}\left(\ddot{w}_1 g(N_1)\right)\right) + g\left(g^{-1}\left(\ddot{w}_2 g(N_2)\right)\right)\right\}\Big),\\ &\quad\left(h^{-1}\left\{h\left(h^{-1}\left(\ddot{w}_1 h(\mathscr{U}_1)\right)\right) + h\left(h^{-1}\left(\ddot{w}_2 h(\mathscr{U}_2)\right)\right)\right\}, g^{-1}\left\{g\left(g^{-1}\left(\ddot{w}_1 g(\rho_1)\right)\right) + g\left(g^{-1}\left(\ddot{w}_2 g(\rho_2)\right)\right)\right\},\right.\\ &\quad\left. g^{-1}\left\{g\left(g^{-1}\left(\ddot{w}_1 g(\vartheta_1)\right)\right) + g\left(g^{-1}\left(\ddot{w}_2 g(\vartheta_2)\right)\right)\right\}\right)\\ &= \left(\left(h^{-1}\left\{\left(\ddot{w}_1 h(M_1)\right) + \left(\ddot{w}_2 h(M_2)\right)\right\}, g^{-1}\left\{\left(\ddot{w}_1 g(L_1)\right) + \left(\ddot{w}_2 g(L_2)\right)\right\}, g^{-1}\left\{\left(\ddot{w}_1 g(N_1)\right) + \left(\ddot{w}_2 g(N_2)\right)\right\}\right),\right.\\ &\quad\left.\left(h^{-1}\left\{\left(\ddot{w}_1 h(\mathscr{U}_1)\right) + \left(\ddot{w}_2 h(\mathscr{U}_2)\right)\right\}, g^{-1}\left\{\left(\ddot{w}_1 g(\rho_1)\right) + \left(\ddot{w}_2 g(\rho_2)\right)\right\}, g^{-1}\left\{\left(\ddot{w}_1 g(\vartheta_1)\right) + \left(\ddot{w}_2 g(\vartheta_2)\right)\right\}\right)\right)\\ &= \left(\left(h^{-1}\sum_{i=1}^{2}\left(\ddot{w}_i h(M_i)\right), g^{-1}\sum_{i=1}^{2}\left(\ddot{w}_i g(L_i)\right), g^{-1}\sum_{i=1}^{2}\left(\ddot{w}_i g(N_i)\right)\right),\right.\\ &\quad\left.\left(h^{-1}\sum_{i=1}^{2}\left(\ddot{w}_i h(\mathscr{U}_i)\right), g^{-1}\sum_{i=1}^{2}\left(\ddot{w}_i g(\rho_i)\right), g^{-1}\sum_{i=1}^{2}\left(\ddot{w}_i g(\vartheta_i)\right)\right)\right).\end{aligned} \tag{32}$$

Then, the results hold for $n = 2$.

Step 2. If equation (30) holds for $n=k$, then, for $n=k+1$, we have

$$
\begin{aligned}
\mathrm{CPFWA}\left(\mathbb{U}_1,\mathbb{U}_2,\ldots,\mathbb{U}_{K+1}\right) &= \oplus_{i=1}^{K+1} w_i\mathbb{U}_i = w_1\mathbb{U}_1 \oplus \ddot{w}_2\mathbb{U}_2 \oplus \cdots \oplus \ddot{w}_{K+1}\mathbb{U}_{K+1}\\
&= \oplus_{i=1}^{K}\ddot{w}_i\mathbb{U}_i \oplus \ddot{w}_{K+1}\mathbb{U}_{K+1}\\
&= \left(\left(h^{-1}\sum_{i=1}^{K}(\ddot{w}_i h(M_i)), g^{-1}\sum_{i=1}^{K}(\ddot{w}_i g(L_i)), g^{-1}\sum_{i=1}^{K}(\ddot{w}_i g(N_i))\right),\right.\\
&\quad \left.\cdot\left(\left(h^{-1}\sum_{i=1}^{K}(\ddot{w}_i h(\mathcal{U}_i)), g^{-1}\sum_{i=1}^{K}(\ddot{w}_i g(\rho_i)), g^{-1}\sum_{i=1}^{K}(\ddot{w}_i g(\vartheta_i))\right)\right)\right)\\
&\quad \oplus\left(\left(h^{-1}(\ddot{w}_{K+1}h(M_{K+1})), g^{-1}(\ddot{w}_{K+1}g(L_{K+1})), g^{-1}(\ddot{w}_{K+1}g(N_{K+1}))\right), \left(h^{-1}(\ddot{w}_{K+1}h(\mathcal{U}_{K+1})),\right.\right.\\
&\quad \left.\left. g^{-1}(\ddot{w}_{K+1}g(\rho_{K+1})), g^{-1}(\ddot{w}_{K+1}g(\vartheta_{K+1}))\right)\right)\\
&= \left(\left(h^{-1}\left\{h\left(h^{-1}\sum_{i=1}^{K}(\ddot{w}_i h(M_i)) + h^{-1}(\ddot{w}_{K+1}h(M_{K+1}))\right)\right\},\right.\right.\\
&\quad g^{-1}\left\{g\left(g^{-1}\sum_{i=1}^{K}(\ddot{w}_i g(L_i)) + g^{-1}(\ddot{w}_{K+1}g(L_{K+1}))\right)\right\},\\
&\quad \left. g^{-1}\left\{g\left(g^{-1}\sum_{i=1}^{K}(\ddot{w}_i g(N_i)) + g^{-1}(\ddot{w}_{K+1}g(N_{k+1}))\right)\right\}\right),\\
&\quad \cdot\left(h^{-1}\left\{h\left(h^{-1}\sum_{i=1}^{K}(\ddot{w}_i h(\mathcal{U}_i)) + h^{-1}(\ddot{w}_{K+1}h(\mathcal{U}_{K+1}))\right)\right\},\right.\\
&\quad g^{-1}\left\{g\left(g^{-1}\sum_{i=1}^{K}(\ddot{w}_i g(\rho_i)) + g^{-1}(\ddot{w}_{K+1}g(\rho_{K+1}))\right)\right\},\\
&\quad \left.\left. g^{-1}\left\{g\left(g^{-1}\sum_{i=1}^{K}(\ddot{w}_i g(\vartheta_i)) + g^{-1}(\ddot{w}_{K+1}g(\vartheta_{K+1}))\right)\right\}\right)\right)\\
&= \left(\left(h^{-1}\left\{\sum_{i=1}^{K}(\ddot{w}_i h(M_i)) + \ddot{w}_{K+1}h(M_{K+1})\right\}, g^{-1}\left\{\sum_{i=1}^{K}(\ddot{w}_i g(L_i)) + \ddot{w}_{K+1}g(L_{K+1})\right\},\right.\right.\\
&\quad \left. g^{-1}\left\{\sum_{i=1}^{K}(\ddot{w}_i g(N_i)) + v_{K+1}g(N_{k+1})\right\}\right)\\
&\quad \cdot\left(h^{-1}\left\{\sum_{i=1}^{K}(\ddot{w}_i h(\mathcal{U}_i)) + \ddot{w}_{K+1}h(\mathcal{U}_{K+1})\right\}, g^{-1}\left\{\sum_{i=1}^{K}(\ddot{w}_i g(\rho_i)) + \ddot{w}_{K+1}g(\rho_{K+1})\right\},\right.\\
&\quad \left.\left. g^{-1}\left\{\sum_{i=1}^{K}(\ddot{w}_i g(\vartheta_i)) + \ddot{w}_{K+1}g(\vartheta_{K+1})\right\}\right)\right)\\
&= \left(\left(h^{-1}\left\{\sum_{i=1}^{K+1}(\ddot{w}_i h(M_i))\right\}, g^{-1}\left\{\sum_{i=1}^{K+1}(\ddot{w}_i g(L_i))\right\}, g^{-1}\left\{\sum_{i=1}^{K+1}(\ddot{w}_i g(N_i))\right\}\right),\right.\\
&\quad \left.\cdot\left(h^{-1}\left\{\sum_{i=1}^{K+1}(\ddot{w}_i h(\mathcal{U}_i))\right\}, g^{-1}\left\{\sum_{i=1}^{K+1}(\ddot{w}_i g(\rho_i))\right\}, g^{-1}\left\{\sum_{i=1}^{K+1}(\ddot{w}_i g(\vartheta_i))\right\}\right)\right).
\end{aligned}
\tag{33}
$$

Since the results hold for $n=k+1$, hence, by the principle of mathematical induction, the result given in equation (30) holds for all positive integers n. □

Remark 5. If $g(t)$ is taken to be $g(t) = -\log(t)$, then by equation (30) we have that

$$\mathrm{CPFWA}(\mathbb{U}_1, \mathbb{U}_2, \ldots, \mathbb{U}_n) = \left(\left(1 - \prod_{i=1}^{n}(1-(M_i))^{\ddot{w}_i}, \prod_{i=1}^{n}(L_i)^{\ddot{w}_i}, \prod_{i=1}^{n}(N_i)^{\ddot{w}_i}\right), \left(1 - \prod_{i=1}^{n}(1-(\mathscr{U}_i))^{\ddot{w}_i}, \prod_{i=1}^{n}(\rho_i)^{\ddot{w}_i}, \prod_{i=1}^{n}(\vartheta_i)^{\ddot{w}_i}\right)\right), \tag{34}$$

which is called cubic picture fuzzy Archimedean weighted averaging operator.

4.2. Cubic Picture Fuzzy Ordered Weighted Averaging (CPFOWA) Operator. In this section, we intend to take the idea of OWA into CPFWA operator and propose a new operator which is defined as follows.

Definition 18. Let $\{\mathbb{U}_i\}_{i\in\Lambda}$ be a collection of CPFSs. Then the CPFOWA $\mathscr{CPF}^n \longrightarrow \mathscr{CPF}$ is defined in the following way:

$$\begin{aligned}\mathrm{CPFOWA}(\mathbb{U}_1, \mathbb{U}_2, \ldots, \mathbb{U}_n) &= \ddot{w}_1 \mathbb{U}_{o(1)} \oplus \ddot{w}_2 \mathbb{U}_{o(2)} \oplus \cdots \oplus \ddot{w}_n \mathbb{U}_{o(n)}\\ &= \oplus_{i=1}^{n} \ddot{w}_i \mathbb{U}_{o(i)},\end{aligned} \tag{35}$$

where $\ddot{w} = \{\ddot{w}_1, \ddot{w}_2, \ldots, \ddot{w}_n\}^T$ is the weighted vector of U_i, such that $\ddot{w}_i > 0$ and $\sum_{i=1}^{n} \ddot{w}_i = 1$. Here $\{o(1), o(2), \ldots, o(n)\}$ is the permutation of $(1, 2, \ldots, n)$, such that $\mathbb{U}_{o(i-1)} \geq \mathbb{U}_{o(i)}$, and $\mathbb{U}_{o(i)}$ is the ith largest of CPFSs $\mathbb{U}_i\,(i\varepsilon\wedge)$.

Theorem 21. *Let $\{\mathbb{U}_i\}_{i\in\Lambda}$ be a collection of CPFSs. Then, based on the CPFOWA operator, the aggregated CPFSs can be expressed as follows:*

$$\begin{aligned}\mathrm{CPFOWA}(\mathbb{U}_1, \mathbb{U}_2, \ldots, \mathbb{U}_n) = \Bigg(&\left(h^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i h\left(M_{o(i)}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g\left(L_{o(i)}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g\left(N_{o(i)}\right)\right)\right),\\ &\left(h^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i h\left(\mathscr{U}_{o(i)}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g\left(\rho_{o(i)}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g\left(\vartheta_{o(i)}\right)\right)\right)\Bigg).\end{aligned} \tag{36}$$

In particular, if $L_{o(i)} = \rho_{o(i)} = 0$ for all i, then equation (36) reduces to

$$\begin{aligned}\mathrm{CPFOWA}(\mathbb{U}_1, \mathbb{U}_2, \ldots, \mathbb{U}_n) = \Bigg(&\left(h^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i h\left(M_{o(i)}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g\left(N_{o(i)}\right)\right)\right),\\ &\left(h^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i h\left(\mathscr{U}_{o(i)}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g\left(\vartheta_{o(i)}\right)\right)\right)\Bigg),\end{aligned} \tag{37}$$

which becomes cubic intuitionistic OWA operator.

Proof. The proof follows from Theorem 19. □

4.3. Cubic Picture Fuzzy Hybrid Averaging (CPFHA) Operator. CPFWA operator weighs the CPFSs only, while CPFOWA weighs the ordered positions of it. However, in order to combine these two aspects in one, we introduce CPFHA operator.

Definition 19. Let $\{\mathbb{U}_i\}_{i\in\Lambda}$ be a collection of CPFSs. Then theCPFHA $\mathscr{CPF}^n \longrightarrow \mathscr{CPF}$ is defined as follows:

$$\begin{aligned}\mathrm{CPFHA}(\mathbb{U}_1, \mathbb{U}_2, \ldots, \mathbb{U}_n) &= \ddot{w}_1 \dot{\mathbb{U}}_{o(1)} \oplus \ddot{w}_2 \dot{\mathbb{U}}_{o(2)} \oplus \cdots \oplus \ddot{w}_n \dot{\mathbb{U}}_{o(n)}\\ &= \oplus_{i=1}^{n} \ddot{w}_i \dot{\mathbb{U}}_{o(i)},\end{aligned} \tag{38}$$

where $\ddot{w} = \{\ddot{w}_1, \ddot{w}_2, \ldots, \ddot{w}_n\}^T$ is the standard weight vector of $\mathbb{U}_i$, such that $\ddot{w}_i > 0$ and $\sum_{i=1}^{n} \ddot{w}_i = 1$, $\dot{\mathbb{U}}_{o(i)}$ is the ith largest of the weighted CPFSs $\dot{\mathbb{U}}_i$ ($\dot{\mathbb{U}}_i = n\dot{\ddot{w}}_i\mathbb{U}_i, i = 1, 2, \ldots, n$), where n is the number of CPFSs. Then CPFHA is called cubic picture fuzzy hybrid averaging operator.

Theorem 22. *Let $\{\mathbb{U}_i\}_{i\in\Lambda}$ be a collection of CPFSs; then, based on CPFHA operator, the aggregated CPFSs can be expressed as*

$$\begin{aligned}\mathrm{CPFHA}(\mathbb{U}_1, \mathbb{U}_2, \ldots, \mathbb{U}_n) = {}&\left(\left(h^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i h\left(\dot{M}_{o(i)}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g\left(\dot{L}_{o(i)}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g\left(\dot{N}_{o(i)}\right)\right)\right),\right.\\ &\left.\left(h^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i h\left(\dot{\mathcal{U}}_{o(i)}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g\left(\dot{\rho}_{o(i)}\right)\right), g^{-1}\left(\sum_{i=1}^{n}\ddot{w}_i g\left(\dot{\vartheta}_{o(i)}\right)\right)\right)\right).\end{aligned} \tag{39}$$

Proof. The proof is similar to Theorem 20, so it is omitted here. □

5. MCDM Based on the Proposed Operation

In this section, we need the previous aggregation operators in a decision-making for CPFSs with illustrative example for evaluating the approach.

Let a set of m' alternatives denoted by $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_{m'})$ be found by the decision-maker under the set of the unlikely criteria $\breve{G} = (\breve{G}_1, \breve{G}_2, \ldots, \breve{G}_n)$ whose weight vector is $\ddot{w} = (\ddot{w}_1, \ddot{w}_2, \ldots, \ddot{w}_n)^T$ such that $\ddot{w}_i > 0$ and $\sum_{i=1}^{n}\ddot{w}_i = 1$.

Suppose that the ranking of an alternative x_j, $(j = 1, 2, \ldots, m')$ on the criteria $\breve{G}_i$, $(i = 1, 2, \ldots, m')$ is assessed by the decision-maker in the form of CPFSs $\mathbb{U}_{ij} = (\widehat{A}_{ij}, \mathcal{B}_{ij}), i, j \in (1, 2, \ldots, n)$, where $\widehat{A}_{ij}$ is the degree of VPFS and β_{ij} is the degree of PFS that the alternative $\mathcal{A}_i$ does not satisfy the attribute $\breve{G}_i$. So we develop an approach for evaluating the best alternative based on the proposed operators for MCGDM problem whose steps are as follows:

Step 1. Construct the decision matrix of CPFSs.

$\mathbb{U}_{ij} = (\widehat{A}_{ij}, \mathcal{B}_{ij})$, where $\widehat{A}_{ij} = (M_{ij}, L_{ij}, N_{ij})$ are VPFNs and $\beta_{ij} = (\mathcal{U}_{ij}, \rho_{ij}, \vartheta_{ij})$ are the PFSs towards the alternative $\mathcal{A}_i$ and hence construct a cubic picture fuzzy decision matrix $\mathcal{D} = (\ddot{P}_{ij})_{m'\times n}$.

Step 2. Normalized decision matrix, namely, cost (C) and benefits $i(B)$, so we normalize

$$r_{ij} = \left\{\begin{array}{ll}\ddot{P}_{ij}^{C}, & k \in B,\\ \ddot{P}_{ij}, & k \in C\end{array}\right\}, \tag{40}$$

where $\ddot{P}_{ij}^{C}$ is the complement of $\ddot{P}_{ij}$.

Step 3. Aggregated assessment of alternative, based on the decision matrix, as taken from step 2, all the aggregated values of the alternatives $\mathcal{A}_i$, $(i = 1, 2, \ldots, m')$ under the different criteria $\breve{G}_i$ are obtained by using either CPFWA or CPFOWA or CPFHA operator and we collect the value of r_i for each alternative $\mathcal{A}_i$, $(i = 1, 2, \ldots, m')$.

Step 4. We compute the score values of r_i $(i = 1, 2, \ldots, n)$.

Step 5. At last, we find that the rank of the alternatives $\mathcal{A}_i, i\,(i = 1, 2, \ldots, m')$ according to the descending value of the score value are most valuable.

6. Illustrative Example

In this section, we illustrate with the mathematical example for the decision-making studied as follows.

Suppose few companies design their financial strategy for the next fiscal year, and according to their plan of strategy, they are picking three alternatives defined as follows: $\mathcal{A}_1$: to invest in the "Chinese markets"; $\mathcal{A}_2$: to invest in the "Indian markets"; and $\mathcal{A}_3$: to invest in "USA markets." These proceed for finding the aspect as follows: $\breve{G}_1$: "the increases analysis," $\breve{G}_2$: "the decreases analysis," and $\breve{G}_3$: "the neutral analysis," whose weight vector $w = (0.5, 0.2, 0.3)^T$.

6.1. Example by the CPFWA Operator.

The example is applied in CPFWA operator to calculate the best one.

Step 1. These three alternatives $\mathcal{A}_i$, $(i = 1, 2, 3)$ are to be solved by an expert under the three aspects $\breve{G}_j$ $(j = 1, 2, 3)$ by using cubic picture fuzzy decision matrix $\mathcal{D} = (\ddot{P}_{ij})_{3\times 3} = ([\widehat{A}]_{ij}, \mathcal{B}_{ij})$ for $(i, j = 1, 2, 3)$.

Step 2. Since the criteria $\breve{G}_2$ and $\breve{G}_3$ are the porches criteria while $\breve{G}_1$ are losses criteria $\breve{R} = (r_{ij})_{3\times 3}$, equation (40) is used as follows:

$$\widetilde{R} = \begin{bmatrix} \left(\begin{pmatrix} [0.1, 0.3], \\ [0.2, 0.1], \\ [0.1, 0.1] \end{pmatrix}, (\langle 0.3, 0.2, 0.1 \rangle)\right) & \left(\begin{pmatrix} [0.1, 0.1], \\ [0.2, 0.1], \\ [0.3, 0.2] \end{pmatrix}, (\langle 0.4, 0.3, 0.2 \rangle)\right) & \left(\begin{pmatrix} [0.3, 0.4], \\ [0.2, 0.2], \\ [0.1, 0.3] \end{pmatrix}, (\langle 0.1, 0.2, 0.1 \rangle)\right) \\ \left(\begin{pmatrix} [0.3, 0.1], \\ [0.1, 0.2], \\ [0.1, 0.3] \end{pmatrix}, (\langle 0.1, 0.2, 0.3 \rangle)\right) & \left(\begin{pmatrix} [0.1, 0.2], \\ [0.1, 0.3], \\ [0.2, 0.3] \end{pmatrix}, (\langle 0.2, 0.3, 0.4 \rangle)\right) & \left(\begin{pmatrix} [0.4, 0.3], \\ [0.2, 0.1], \\ [0.3, 0.1] \end{pmatrix}, (\langle 0.1, 0.1, 0.2 \rangle)\right) \\ \left(\begin{pmatrix} [0.3, 0.1], \\ [0.2, 0.1], \\ [0.1, 0.0] \end{pmatrix}, (\langle 0.3, 0.4, 0.1 \rangle)\right) & \left(\begin{pmatrix} [0.1, 0.1], \\ [0.2, 0.2], \\ [0.3, 0.3] \end{pmatrix}, (\langle 0.1, 0.1, 0.1 \rangle)\right) & \left(\begin{pmatrix} [0.3, 0.3], \\ [0.2, 0.3], \\ [0.2, 0.1] \end{pmatrix}, (\langle 0.4, 0.3, 0.0 \rangle)\right) \end{bmatrix}. \tag{41}$$

Step 3. By following the CPFWA given in equation (30) with generator$g(t) = -\mathrm{Log}(t)$, we obtain the overall rating value of each alternative $\mathscr{A}_i$ as

$$\begin{aligned} r_1 &= \mathrm{CPFWA}(r_{11}, r_{12}, r_{13}) \\ &= (([0.15, 0.28], [0.20, 0.12], [0.14, 0.165]), (\langle 0.32, 0.19, 0.11 \rangle)), \\ r_2 &= \mathrm{CPFWA}(r_{21}, r_{22}, r_{23}) \\ &= (([0.28, 0.18], [0.12, 0.17], [0.16, 0.21]), (\langle 0.11, 0.18, 0.28 \rangle)), \\ r_3 &= \mathrm{CPFWA}(r_{21}, r_{22}, r_{23}) \\ &= (([0.25, 0.16], [0.19, 0.16], [0.15, 0]), (\langle 0.29, 0.27, 0 \rangle)). \end{aligned} \tag{42}$$

Step 4. The definitions of the score functions of $r_i (i = 1, 2, 3)$ are $S(r_1) = -0.0054$, $S(r_2) = -0.19$, and $S(r_3) = -0.11$.

Step 5. Since $S(r_1) > S(r_3) > S(r_2)$, we have $\mathscr{A}_1 > \mathscr{A}_3 > \mathscr{A}_2$. Hence, the gorgeous financial strategy is $\mathscr{A}_1$, that is, to invest in the Chinese markets.

7. Conclusion

The article is based on a novel approach to CPFSs as a generalization of two new strong concepts of CSs and PFSs. The basic operations for CPFSs are developed and exemplified. Some related results based on proposed operations are discussed. Several aggregation operators are defined for CPFSs and their properties are investigated. The proposed aggregation operators are subjected to a decision-making problem and the results are discussed. Furthermore, we developed multicriteria decision-making (MCDM) to prove the effectiveness and validity of the proposed methodology. A numerical example showed that the proposed operators can resolve decision-making more accurately. We compared these with predefined operators to show the validity and effectiveness of the proposed methodology.

In the future, some similarity measures for CPFS can be developed and can be applied in pattern recognition problems. We will define other methods with CPFS such as Dombi aggregation operators and introduce the idea of cubic picture fuzzy Dombi weighted average (CPFDWA), cubic picture fuzzy Dombi ordered weighted average (CPFDOWA), cubic picture fuzzy Dombi weighted geometric (CPFDWG), cubic picture fuzzy Dombi ordered weighted geometric (CPFDOWG), and generalized operators in multicriteria decision-making.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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Retraction

Retracted: Some Complex Intuitionistic Uncertain Linguistic Heronian Mean Operators and Their Application in Multiattribute Group Decision Making

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] H. Garg, Z. Ali, J. Gwak, T. Mahmood, and S. Aljahdali, "Some Complex Intuitionistic Uncertain Linguistic Heronian Mean Operators and Their Application in Multiattribute Group Decision Making," *Journal of Mathematics*, vol. 2021, Article ID 9986704, 31 pages, 2021.

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Research Article

Some Complex Intuitionistic Uncertain Linguistic Heronian Mean Operators and Their Application in Multiattribute Group Decision Making

Harish Garg,[1] **Zeeshan Ali**,[2] **Jeonghwan Gwak**,[3,4,5,6] **Tahir Mahmood**,[2] **and Sultan Aljahdali**[7]

[1]*School of Mathematics, Thapar Institute of Engineering and Technology (Deemed University), Patiala, Punjab, India*
[2]*Department of Mathematics and Statistics, International Islamic University Islamabad, Islamabad, Pakistan*
[3]*Department of Software, Korea National University of Transportation, Chungju 27469, Republic of Korea*
[4]*Department of Biomedical Engineering, Korea National University of Transportation, Chungju 27469, Republic of Korea*
[5]*Department of AI Robotics Engineering, Korea National University of Transportation, Chungju 27469, Republic of Korea*
[6]*Department of IT & Energy Convergence (BK21 FOUR), Korea National University of Transportation, Chungju 27469, Republic of Korea*
[7]*Department of Computer Science, College of Computers and Information Technology, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia*

Correspondence should be addressed to Jeonghwan Gwak; james.han.gwak@gmail.com

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In this paper, a new decision-making algorithm has been presented in the context of a complex intuitionistic uncertain linguistic set (CIULS) environment. CIULS integrates the concept the complex of a intuitionistic fuzzy set (CIFS) and uncertain linguistic set (ULS) to deal with uncertain and imprecise information in a more proactive manner. To investigate the interrelation between the pairs of CIULSs, we combine the concept of the Heronian mean (HM) and the complex intuitionistic uncertain linguistic (CIUL) to describe some new operators, namely, CIUL arithmetic HM (CIULAHM), CIUL weighted arithmetic HM (CIULWAHM), CIUL geometric HM (CIULGHM), and CIUL weighted geometric HM (CIULWGHM). The main advantage of these suggested operators is that they considered the interaction between pairs of objects during the formulation process. Also, a number of distinct brief cases and properties of the operators are analyzed. In addition, based on these operators, we have stated a MAGDM ("multiattribute group decision-making") problem-solving algorithm. The consistency of the algorithm is illustrated by a computational example that compares the effects of the algorithm with a number of well-known existing methods.

1. Introduction

MAGDM issues are the critical exploration aspects of the current judgement philosophy to deal with questionable and incorrect facts in time complications. If the reasons remain fuzzy, the signature values involved in decision-making problems are not continuously seen to be crisp artefacts, and some of them are extensively sufficient to be identified by a number of hypotheses. The fuzzy set (FS) theory is one of those that Zadeh [1] has built to handle with awkward and difficult facts. FS applies only to the term of the degree of truth limited to the unit interval. FS has gained a great deal of interest from various academics and has been exploited by a number of scientists in the nature of separate fields. For example, L-FS was investigated by Goguen [2]. L-FS is essentially a mixture of two theories, such as FS and lattice's ordered series, which is a useful strategy for dealing with difficult facts. In addition, Torra [3] reworked the FS theorem to explain the hesitant FS (HFS) principle, which covers the degree of truth in the form of the finite subset of the unit interval. Pawlak [4] looked at the rough sets and the FSs. Zhang [5] introduced the concept of bipolar FS (BFS) containing two degrees with a law that is the degree of truth belonging to [0, 1] and the degree of falsehood belonging to [−1, 0]. BFS has gained considerable attention from separate intellectuals and has been extensively used by many scientists in the world of various fields. For instance, the theory of bipolar soft set was developed by Mahmood [6].

FS is a major apparatus for dealing with troublesome and complex information in day-to-day natural life problems, and a number of researchers have made extensive use of it in different fields. However, in some cases, the theory of FS is not capable of dealing with such a kind of concern, for example, if an individual gives certain sources of knowledge, including the degree of truth and falsehood, then the theory of FS has failed. To deal with such problems, Atanassov [7] used the principle of intuitionistic FS (IFS) with the law that the totality of the degrees of each other lies inside the unit interval. IFS is a simplified version of FS to deal with uncomfortable experience of natural life problems. IFS has gained considerable recognition from various academics and has been employed by a number of scientists in distinct neighbourhoods. For example, Beg and Rashid [8] discussed the principle of intuitionistic HFS (IHFS) holding the degree of truth and the degree of falsehood in the form of a finite unit interval subset. The law of IHFS is that the absolute maximum (also for the least) of the truth and the minimum (also for the maximum) falsity is limited to the unit interval. In addition, Atanassov [9] introduced the principle of interval-valued IFS (IVIFS), which is the extension of the interval-valued FS (IVFS). IVIFS refers to the degree of truth and falsehood in the shape of a subinterval of the unit interval. The IFS and IVIFS have received large concentrations from separate intellectuals and have been extensively used by many scientists in the world in various fields [10–14].

Complex FS (CFS) theory is one of the most proficient techniques developed by Ramot et al. [15] to manage uncomfortable and difficult details. CFS covers only the term of the degree of truth in the structure of complex numbers relevant to the complex plane in the unit disc with a restriction that the true and imaginary portions of the degree of truth are limited to the unit interval. CFS has attracted considerable interest from a variety of researchers and has been exploited by a number of scientists in distinct fields. For example, the neuro fuzzy architecture used was investigated by Chen et al. [16]. Ramot et al. [17] has studied a dynamic fuzzy logic. Zhang et al. [18] investigated the activity properties of CFSs. The CFS theory has also been established by Nguyen et al. [19], Dick [20], and Tamir et al. [21]. Tamir et al. [22] presented a concept of generalized complex fuzzy propositional logic. The aggregation operators on the complex fuzzy information have been defined by the researchers in [23–25].

CFS is an important apparatus for dealing with troublesome and complex information in day-to-day natural life problems, and a number of researchers have made extensive use of it in different fields. However, in some cases, the theory of CFS is not capable of dealing with this kind of concern, for example, if an individual gives certain sources of knowledge, including the degree of truth and falsehood, then the theory of CFS has failed. To handle with such sort of troubles, Alkouri and Salleh [26] used the theory of complex IFS (CIFS) with a requirement that the totality of the real parts (also for imaginary parts) of both degrees is inside the unit interval. CIFS is a modified form of CFS to deal with awkward and convoluted awareness of natural world problems. The CIFS has attracted considerable interest from various academics and has been exploited by a number of scientists in separate fields. For example, Al-Qudah et al. [27] presented a decision-making approach under the complex multifuzzy soft set environment. Kumar and Bajaj [28] used the CIF concept in the soft set environment to investigate the dynamic intuitive fuzzy soft set. Garg and Rani [29] have established a number of knowledge measures for the CYPSs. Ngan et al. [30] looked at the quaternion number depending on the CIFS. Rani and Garg [31] presented preference relation for the complex intuitionistic fuzzy set in individual and group decision-making process. Ali et al. [32] studied the complex intuitionistic fuzzy groups. Garg and Rani have established the theory of aggregation operators for IFCS [33]. In addition, Rahman et al. [34] developed the hybrid model of the hypersoft set with complex fuzzy set and complex intuitionistic fuzzy set and neurtrosophic set. CIFS has received considerable attention from separate intellectuals and has been widely used by many scientists in the world in various fields [35–37].

However, in different real difficulties, it is not easy for decision makers to express their views in quantitative representations. For example, as a professional considering the applicant's degree of advanced expertise, the use of linguistic expressions, such as linguistic phrases, "very good," "good," or "medium" may be considered for being additionally suitable or familiar to convey his or her opinion. To handle such sorts of concerns, Zadeh [38] investigated the linguistic variable theory (LV) in order to describe the interests of decision makers. In addition, the principle of the two-fold linguistic set was established by Herrera and Martinez [39]. Liu and Jin [40] have studied the uncertain LV (ULV). Heronian mean operators based on the intuitionistic uncertain linguistic set (IULS) were developed in [41]. Liu and Liu [42] studied the partitioned Bonferroni mean IULS operators. In addition, Liu et al. [43] investigated the weighted Bonferroni order weighted average operators for IULS. Liu et al. [44] used the concept of Hamy as a mean operator for IULSs. The theory of Bonferroni mean IULS operators has been established by Liu and Zhang [45]. But, to date, no one has used these concepts in the CIULS setting, and to discover the interrelationship between some numbers of CIULS, HM operators are very useful for dealing with uncomfortable and troublesome knowledge in everyday difficulties.

(1) To investigate the CIULS and discuss their operational laws.

(2) To explore the CIULAHM, CIULWAHM, CIULGHM, and CIULWGHM operators and discuss their special cases with some properties.

(3) A MAGDM procedure is developed by using the explored operators based on CIULSs.

(4) Some numerical examples are illustrated with the help of investigated approaches.

(5) In order to determine the efficiency and competence of the developed operators, comparative analysis and graphic expressions are often used to demonstrate the superiority of the methods developed.

The remainder of the paper is presented as follows. In Section 2, we refer to some basic concepts, such as the CIFS and their operating rules. The current idea of LSs, ULVs, and their operations is also updated in this report. In addition, the definition of HM with parameters and without parameters is discussed. In Section 3, we investigated the CIULS and examined their operating rules. In Section 4, we examined the CIULAHM, the CIULWAHM, the CIULGHM, and the CIULWGHM operators and addressed their specific cases with those properties. Section 5 develops a MAGDM procedure by using CIULS-based explored operators. Some numerical examples are illustrated with the help of investigated approaches. To discover the consistency and expertise of the developed operators, comparative analysis and graphic expressions are often used to show the superiority of the methods developed. The end of the script is explored in Section 6.

2. Preliminaries

For better describing the investigated ideas, we recall some fundamental notions such as CIFSs and their operational laws. The existing idea of LSs, ULVs, and their operations is also revised in this study. Moreover, the idea of HMO with parameters and without parameters is also discussed. Throughout the article, the symbol $\widehat{\mathcal{X}_{\text{UNI}}}$ is used for fixed sets and the terms $\mathcal{J}_{\mathfrak{W}_{\text{CI}}}$ and $\mathcal{K}_{\mathfrak{W}_{\text{CI}}}$ are shown the grade of positive and the grade of negative.

Definition 1 (see [26]). A CIFS $\mathfrak{W}_{\text{CI}}$ is demonstrated by

$$\mathfrak{W}_{\text{CI}} = \left\{\left(\mathcal{J}_{\mathfrak{W}_{\text{CI}}}(\breve{\mathfrak{k}}), \mathcal{K}_{\mathfrak{W}_{\text{CI}}}(\breve{\mathfrak{k}})\right): \breve{\mathfrak{k}} \in \widehat{\mathcal{X}_{\text{UNI}}}\right\}, \tag{1}$$

where $\mathcal{J}_{\mathfrak{W}_{\text{CI}}}(\breve{\mathfrak{k}}) = \mathcal{J}_{\mathfrak{W}_{\text{RP}}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{IP}}(\breve{\mathfrak{k}}))}$ and $\mathcal{K}_{\mathfrak{W}_{\text{CI}}}(\breve{\mathfrak{k}}) = \mathcal{K}_{\mathfrak{W}_{\text{RP}}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{\text{IP}}}(\breve{\mathfrak{k}}))}$ with the rules such that $0 \le \mathcal{J}_{\mathfrak{W}_{\text{RP}}}(\breve{\mathfrak{k}}) + \mathcal{K}_{\mathfrak{W}_{\text{RP}}}(\breve{\mathfrak{k}}) \le 1$ and $0 \le \mathcal{J}_{\mathfrak{W}_{\text{IP}}}(\breve{\mathfrak{k}}) + \mathcal{K}_{\mathfrak{W}_{\text{IP}}}(\breve{\mathfrak{k}}) \le 1$. Furthermore, the refusal grade is demonstrated in the form of $\mathcal{L}_{\mathfrak{W}_{\text{CI}}}(\breve{\mathfrak{k}}) = \mathcal{L}_{\mathfrak{W}_{\text{RP}}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{L}_{\mathfrak{W}_{\text{IP}}}(\breve{\mathfrak{k}}))} = (1 - \mathcal{J}_{\mathfrak{W}_{\text{RP}}}(\breve{\mathfrak{k}}) - \mathcal{K}_{\mathfrak{W}_{\text{RP}}}(\breve{\mathfrak{k}}))e^{i2\pi(1 - \mathcal{J}_{\mathfrak{W}_{\text{IP}}}(\breve{\mathfrak{k}}) - \mathcal{K}_{\mathfrak{W}_{\text{IP}}}(\breve{\mathfrak{k}}))}$. In this paper, the complex intuitionistic fuzzy numbers (CIFNs) are represented by $\mathfrak{W}_{\text{CI}-i} = (\mathcal{J}_{\mathfrak{W}_{\text{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi}(\mathcal{J}_{\mathfrak{W}_{\text{IP}-i}}(\breve{\mathfrak{k}})), \mathcal{K}_{\mathfrak{W}_{\text{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{\text{IP}-i}}(\breve{\mathfrak{k}}))})$, $i = 1, 2, \ldots, \widehat{\Xi}$.

Definition 2 (see [33]). Based on any two CIFNs $\mathfrak{W}_{\text{CI}-i} = (\mathcal{J}_{\mathfrak{W}_{\text{RP}-i}}(\breve{\mathfrak{k}}) e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{\text{IP}-i}}(\breve{\mathfrak{k}}))}, \mathcal{K}_{\mathfrak{W}_{\text{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{\text{IP}-i}}(\breve{\mathfrak{k}}))}), i = 1, 2,$ then

(1) $\mathfrak{W}_{\text{CI}-1} \oplus \mathfrak{W}_{\text{CI}-2} = \Big((\mathcal{J}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}}) + \mathcal{J}_{\mathfrak{W}_{\text{RP}-2}}(\breve{\mathfrak{k}}) - \mathcal{J}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{\text{RP}-2}}(\breve{\mathfrak{k}}))e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}}) + \mathcal{J}_{\mathfrak{W}_{\text{IP}-2}}(\breve{\mathfrak{k}}) - \mathcal{J}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}})} \mathcal{J}_{\mathfrak{W}_{\text{IP}-2}}(\breve{\mathfrak{k}})), \mathcal{K}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}})\mathcal{K}_{\mathfrak{W}_{\text{RP}-2}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}})} \mathcal{K}_{\mathfrak{W}_{\text{IP}-2}}(\breve{\mathfrak{k}}))\Big)$.

(2) $\mathfrak{W}_{\text{CI}-1} \otimes \mathfrak{W}_{\text{CI}-2} = \left(\begin{array}{c} \mathcal{J}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{\text{RP}-2}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{\text{IP}-2}}(\breve{\mathfrak{k}}))}, \\ (\mathcal{K}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}}) + \mathcal{K}_{\mathfrak{W}_{\text{RP}-2}}(\breve{\mathfrak{k}}) - \mathcal{K}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}})\mathcal{K}_{\mathfrak{W}_{\text{RP}-2}}(\breve{\mathfrak{k}}))e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}}) + \mathcal{K}_{\mathfrak{W}_{\text{IP}-2}}(\breve{\mathfrak{k}}) - \mathcal{K}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}})\mathcal{K}_{\mathfrak{W}_{\text{IP}-2}}(\breve{\mathfrak{k}}))} \end{array}\right)$.

(3) $\Phi_{\text{SC}}\mathfrak{W}_{\text{CI}-1} = ((1 - (1 - \mathcal{J}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}}))^{\Phi_{\text{SC}}}) e^{i2\pi(1 - (1 - \mathcal{J}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}}))^{\Phi_{\text{SC}}})}, \mathcal{K}^{\Phi_{\text{SC}}}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}^{\Phi_{\text{SC}}}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}}))})$.

(4) $\mathfrak{W}^{\Phi_{\text{SC}}}_{\text{CI}-1} = (\mathcal{J}^{\Phi_{\text{SC}}}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}^{\Phi_{\text{SC}}}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}}))}, (1 - (1 - \mathcal{K}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}}))^{\Phi_{\text{SC}}})e^{i2\pi(1 - (1 - \mathcal{K}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}}))^{\Phi_{\text{SC}}})})$.

Definition 3 (see [33]). For two CIFNs $\mathfrak{W}_{\text{CI}-i} = (\mathcal{J}_{\mathfrak{W}_{\text{RP}-i}}(\breve{\mathfrak{k}}) e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{\text{IP}-i}}(\breve{\mathfrak{k}}))}, \mathcal{K}_{\mathfrak{W}_{\text{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{\text{IP}-i}}(\breve{\mathfrak{k}}))}), i = 1, 2,$ the score value and accuracy value are demonstrated by

$$\begin{aligned} \overline{\overline{\zeta}}(\mathfrak{W}_{\text{CI}-1}) &= \mathcal{J}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}}) - \mathcal{K}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}}) + \mathcal{J}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}}) - \mathcal{K}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}}), \\ \overline{\overline{\mathfrak{F}}}(\mathfrak{W}_{\text{CI}-1}) &= \mathcal{J}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}}) + \mathcal{K}_{\mathfrak{W}_{\text{RP}-1}}(\breve{\mathfrak{k}}) + \mathcal{J}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}}) + \mathcal{K}_{\mathfrak{W}_{\text{IP}-1}}(\breve{\mathfrak{k}}), \end{aligned} \tag{2}$$

where $\overline{\overline{\zeta}}(\mathfrak{W}_{\text{CI}-1}) \in [-1, 1]$ and $\overline{\overline{\mathfrak{F}}}(\mathfrak{W}_{\text{CI}-1}) \in [0, 1]$. To find the relationships between any two CIFNs, we use the following rules:

(1) If $\overline{\overline{\zeta}}(\mathfrak{W}_{\text{CI}-1}) > \overline{\overline{\zeta}}(\mathfrak{W}_{\text{CI}-2})$, then $\mathfrak{W}_{\text{CI}-1} > \mathfrak{W}_{\text{CI}-2}$.

(2) If $\overline{\overline{\zeta}}(\mathfrak{W}_{\text{CI}-1}) < \overline{\overline{\zeta}}(\mathfrak{W}_{\text{CI}-2})$, then $\mathfrak{W}_{\text{CI}-1} < \mathfrak{W}_{\text{CI}-2}$.

(3) If $\overline{\overline{\zeta}}(\mathfrak{W}_{\text{CI}-1}) = \overline{\overline{\zeta}}(\mathfrak{W}_{\text{CI}-2})$, then

  (1) If $\underline{\underline{\zeta}}(\mathfrak{W}_{\text{CI}-1}) > \underline{\underline{\zeta}}(\mathfrak{W}_{\text{CI}-2})$, then $\mathfrak{W}_{\text{CI}-1} > \mathfrak{W}_{\text{CI}-2}$.
  (2) If $\underline{\underline{\zeta}}(\mathfrak{W}_{\text{CI}-1}) < \underline{\underline{\zeta}}(\mathfrak{W}_{\text{CI}-2})$, then $\mathfrak{W}_{\text{CI}-1} < \mathfrak{W}_{\text{CI}-2}$.
  (3) If $\underline{\underline{\zeta}}(\mathfrak{W}_{\text{CI}-1}) = \underline{\underline{\zeta}}(\mathfrak{W}_{\text{CI}-2})$, then $\mathfrak{W}_{\text{CI}-1} = \mathfrak{W}_{\text{CI}-2}$.

Definition 4 (see [38]). A LS is demonstrated by

$$\psi = \left\{\psi_0, \psi_1, \psi_2, \ldots, \psi_{\overline{\overline{k_{\text{SC}}}}} - 1\right\}, \tag{3}$$

where $\overline{\overline{k_{\text{SC}}}}$ should be odd, which grips the ensuing circumstances:

(1) If $\overline{\overline{k_{\text{SC}}}} > \overline{\overline{k_{\text{SC}}}}'$, then $\psi_{\overline{\overline{k_{\text{SC}}}}} > \psi_{\overline{\overline{k_{\text{SC}}}}'}$.

(2) The negative operator $\text{neg}(\psi_{\overline{\overline{k_{\text{SC}}}}}) = \psi_{\overline{\overline{k_{\text{SC}}}}'}$ with a rule $\overline{\overline{k_{\text{SC}}}} + \overline{\overline{k_{\text{SC}}}}' = \overline{\overline{k_{\text{SC}}}} + 1$.

(3) If $\overline{\overline{k_{\text{SC}}}} \ge \overline{\overline{k_{\text{SC}}}}'$, $\max(\psi_{\overline{\overline{k_{\text{SC}}}}}, \psi_{\overline{\overline{k_{\text{SC}}}}'}) = \psi_{\overline{\overline{k_{\text{SC}}}}}$, and if $\overline{\overline{k_{\text{SC}}}} \le \overline{\overline{k_{\text{SC}}}}'$, $\max(\psi_{\overline{\overline{k_{\text{SC}}}}}, \psi_{\overline{\overline{k_{\text{SC}}}}'}) = \psi_{\overline{\overline{k_{\text{SC}}}}}$.

Likewise, $\widehat{\psi} = \{\psi_i: i \in R\}$ conveyed the LSs. A set $\psi = [\psi_{\varphi_i}, \psi_{\phi_{\mathfrak{s}}}], \psi_{\varphi_i}, \psi_{\phi_{\mathfrak{s}}} \in \widehat{\psi}(i \le \mathfrak{s})$, where $\psi_{\varphi_i}, \psi_{\phi_{\mathfrak{s}}}$ characterize the upper and lower limits of ψ is called ULVs [40]. By utilizing any two ULVs $\psi_1 = [\psi_{\varphi_1}, \psi_{\phi_1}]$ and $\psi_2 = [\psi_{\varphi_2}, \psi_{\phi_2}]$ belonging to $\widehat{\psi}_{[0,h]}$,

(1) $\psi_1 \oplus \psi_2 = [\psi_{\varphi_1}, \psi_{\phi_1}] \oplus [\psi_{\varphi_2}, \psi_{\phi_2}] = [\psi_{\varphi_1 + \varphi_2 - (\varphi_1\varphi_2/h)}, \psi_{\phi_1 + \phi_2 - (\phi_1\phi_2/h)}]$.

(2) $\psi_1 \otimes \psi_2 = [\psi_{\varphi_1}, \psi_{\phi_1}] \otimes [\psi_{\varphi_2}, \psi_{\phi_2}] = [\psi_{\varphi_1 \times \varphi_2/h}, \psi_{\phi_1 \times \phi_2/h}]$.

(3) $\Phi_{\text{SC}}\psi_1 = \Phi_{\text{SC}}[\psi_{\varphi_1}, \psi_{\phi_1}] = [\psi_{h(1 - (1 - (\varphi_1/h))^{\Phi_{\text{SC}}})}, \psi_{h(1 - (1 - (\phi_1/h))^{\Phi_{\text{SC}}})}]$.

(4) $\psi_1^{\Phi_{\text{SC}}} = [\psi_{h(\varphi_1/h)^{\Phi_{\text{SC}}}}, \psi_{h(\phi_1/h)^{\Phi_{\text{SC}}}}]$.

Definition 5 (see [41]). The HM operator $\mathrm{HM}^{p_{\mathrm{SC}},q_{\mathrm{SC}}}\colon \Theta^{\overbrace{\Xi}} \longrightarrow \Theta$ is demonstrated by

$$\mathrm{HM}^{p_{\mathrm{SC}},q_{\mathrm{SC}}}\left(\mathfrak{W}_{\mathrm{CI}-1},\mathfrak{W}_{\mathrm{CI}-2},\ldots,\mathfrak{W}_{\mathrm{CI}-\overbrace{\Xi}}\right)=\left(\frac{2}{\overbrace{\Xi}\left(\overbrace{\Xi}+1\right)}\sum_{i=1}^{\overbrace{\Xi}}\sum_{\mathfrak{s}=1}^{\overbrace{\Xi}}\mathfrak{W}_{\mathrm{CI}-i}^{p_{\mathrm{SC}}}\mathfrak{W}_{\mathrm{CI}-\mathfrak{s}}^{q_{\mathrm{SC}}}\right)^{\left(1/p_{\mathrm{SC}}+q_{\mathrm{SC}}\right)}. \tag{4}$$

If we define the HM operator without parameter, it is demonstrated by:

$$\begin{gathered}\mathrm{HM}^{p_{\mathrm{SC}},q_{\mathrm{SC}}}\colon \Theta^{\overbrace{\Xi}} \longrightarrow \Theta, \text{ by}\\ \mathrm{HM}\left(\mathfrak{W}_{\mathrm{CI}-1},\mathfrak{W}_{\mathrm{CI}-2},\ldots,\mathfrak{W}_{\mathrm{CI}-\overbrace{\Xi}}\right)=\left(\frac{2}{\overbrace{\Xi}\left(\overbrace{\Xi}+1\right)}\sum_{i=1}^{\overbrace{\Xi}}\sum_{\mathfrak{s}=1}^{\overbrace{\Xi}}\mathfrak{W}_{\mathrm{CI}-i}\mathfrak{W}_{\mathrm{CI}-\mathfrak{s}}\right).\end{gathered} \tag{5}$$

3. Complex Intuitionistic Uncertain Linguistic Variables

In this study, we elaborate the fundamental notions of CIULVs and their related principles by utilizing the remaining theories of ULVs and CIFSs.

Definition 6. A CIULV $\mathfrak{W}_{\mathrm{CIU}}$ is demonstrated by

$$\mathfrak{W}_{\mathrm{CIU}}=\left\{\left(\left[\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}\right],\left(\mathscr{J}_{\mathfrak{W}_{\mathrm{CIU}}}(\breve{\mathfrak{k}}),\mathscr{K}_{\mathfrak{W}_{\mathrm{CIU}}}(\breve{\mathfrak{k}})\right)\right)\colon \breve{\mathfrak{k}}\in\overbrace{\mathscr{X}_{\mathrm{UNI}}}\right\}, \tag{6}$$

where $\mathscr{J}_{\mathfrak{W}_{\mathrm{CIU}}}(\breve{\mathfrak{k}})=\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}}}(\breve{\mathfrak{k}}))}$ and $\mathscr{K}_{\mathfrak{W}_{\mathrm{CIU}}}(\breve{\mathfrak{k}})=\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}}}(\breve{\mathfrak{k}}))}$ with the rules such that $0\le\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}})+\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}})\le 1$ and $0\le\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}}}(\breve{\mathfrak{k}})+\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}}}(\breve{\mathfrak{k}})\le 1$ with $\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}\in\widehat{\psi}\,(i\le\mathfrak{s})$. Furthermore, the refusal grade is demonstrated in the form of $\mathscr{L}_{\mathfrak{W}_{\mathrm{CIU}}}(\breve{\mathfrak{k}})=\mathscr{L}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{L}_{\mathfrak{W}_{IP}}(\breve{\mathfrak{k}}))}=(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}})-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}}))e^{i2\pi(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}}}(\breve{\mathfrak{k}})-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}}}(\breve{\mathfrak{k}}))}$. In this paper, the complex intuitionistic uncertain linguistic numbers (CIULNs) are represented by

$$\mathfrak{W}_{\mathrm{CIU}-i}=\left(\left[\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}\right],\left(\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi\left(\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)},\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi\left(\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)}\right)\right),\quad i,\mathfrak{s}=1,2,\ldots,\overbrace{\Xi}. \tag{7}$$

Definition 7. For two CIULNs $\mathfrak{W}_{\mathrm{CIU}-i}=([\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}],(\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}}))},\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}}))})), i=1,2$, some operational laws are stated as

(1) $\mathfrak{W}_{\mathrm{CIU}-1}\oplus\mathfrak{W}_{\mathrm{CIU}-2}=\left([\psi_{\varphi_1+\varphi_2-(\varphi_1\varphi_2/h)},\psi_{\phi_1+\phi_2-(\phi_1\phi_2/h)}],\left((\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-1}}(\breve{\mathfrak{k}})+\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-2}}(\breve{\mathfrak{k}})-\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-1}}(\breve{\mathfrak{k}})\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-2}}(\breve{\mathfrak{k}}))e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-1}}(\breve{\mathfrak{k}})+\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-2}}(\breve{\mathfrak{k}})-\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-1}}(\breve{\mathfrak{k}})\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-2}}(\breve{\mathfrak{k}}))},\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-1}}(\breve{\mathfrak{k}})\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-2}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-1}}(\breve{\mathfrak{k}})\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-2}}(\breve{\mathfrak{k}}))}\right)\right)$.

(2) $\mathfrak{W}_{\mathrm{CIU}-1}\otimes\mathfrak{W}_{\mathrm{CIU}-2}=\left([\psi_{\varphi_1\times\varphi_2/h},\psi_{\phi_1\times\phi_2/h}],\left(\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-1}}(\breve{\mathfrak{k}})\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-2}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-1}}(\breve{\mathfrak{k}})\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-2}}(\breve{\mathfrak{k}}))},(\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-1}}(\breve{\mathfrak{k}})+\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-2}}(\breve{\mathfrak{k}})-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-1}}(\breve{\mathfrak{k}})\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-2}}(\breve{\mathfrak{k}}))e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-1}}(\breve{\mathfrak{k}})+\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-2}}(\breve{\mathfrak{k}})-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-1}}(\breve{\mathfrak{k}})\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-2}}(\breve{\mathfrak{k}}))}\right)\right)$.

(3) $\Phi_{\mathrm{SC}}\mathfrak{W}_{\mathrm{CIU}-1}=\left(\begin{array}{c}[\psi_{h(1-(1-(\varphi_1/h))^{\Phi_{\mathrm{SC}}})},\psi_{h(1-(1-(\phi_1/h))^{\Phi_{\mathrm{SC}}})}],\\ ((1-(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-1}}(\breve{\mathfrak{k}}))^{\Phi_{\mathrm{SC}}})e^{i2\pi(1-(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-1}}(\breve{\mathfrak{k}}))^{\Phi_{\mathrm{SC}}})},\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-1}}^{\Phi_{\mathrm{SC}}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-1}}^{\Phi_{\mathrm{SC}}}(\breve{\mathfrak{k}}))})\end{array}\right)$.

(4) $\mathfrak{W}_{\mathrm{CIU}-1}^{\Phi_{\mathrm{SC}}}=\left([\psi_{h(\varphi_1/h)^{\Phi_{\mathrm{SC}}}},\psi_{h(\phi_1/h)^{\Phi_{\mathrm{SC}}}}],(\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-1}}^{\Phi_{\mathrm{SC}}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-1}}^{\Phi_{\mathrm{SC}}}(\breve{\mathfrak{k}}))},(1-(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-1}}(\breve{\mathfrak{k}}))^{\Phi_{\mathrm{SC}}})e^{i2\pi(1-(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-1}}(\breve{\mathfrak{k}}))^{\Phi_{\mathrm{SC}}})})\right)$.

Proposition 1. *For two CIULNs* $\mathfrak{W}_{CIU-i}=([\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}],(\mathscr{J}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))},\mathscr{K}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))}))$, *the operations defined in Definition 7 are also CIULNs.*

Proof. For any two CIULNs $\mathfrak{W}_{\mathrm{CIU}-1}=([\psi_{\varphi_1},\psi_{\phi_1}],(\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-1}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-1}}(\breve{\mathfrak{k}}))},\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-1}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-1}}(\breve{\mathfrak{k}}))}))$ and $\mathfrak{W}_{\mathrm{CIU}-2}=([\psi_{\varphi_2},\psi_{\phi_2}],(\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-2}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-2}}(\breve{\mathfrak{k}}))},\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-2}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-2}}(\breve{\mathfrak{k}}))}))$, then by using the idea of T-norm and T-conorm such that

$$T\colon [0,1]\times[0,1]\longrightarrow[0,1]. \tag{8}$$

is called T-norm, if T holds the following conditions:

(1) Commutativity
(2) Monotonicity
(3) Associativity
(4) $T(x,1)=x$

And similarly, for T-conorm, we defined a function such that

$$S\colon [0,1]\times[0,1]\longrightarrow[0,1] \tag{9}$$

is called T-conorm, if S holds the following conditions:

(1) Commutativity
(2) Monotonicity
(3) Associativity
(4) $S(x,0)=x$

Then, we prove that the above four conditions.

(1) The addition of two linguistic number is again linguistic number such that $\varphi_1+\varphi_2-(\varphi_1\varphi_2/h)$ and $\phi_1+\phi_2-(\phi_1\phi_2/h)$ are also T-conorm, the real and imiginary parts of the truth are T-conorm which indicates that these two satisfy the conditions of T-conorm such that $\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})+\mathcal{J}_{\mathfrak{W}_{RP-2}}(\breve{\mathfrak{k}})-\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{RP-2}}(\breve{\mathfrak{k}})$ and $\mathcal{J}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}})+\mathcal{J}_{\mathfrak{W}_{IP-2}}(\breve{\mathfrak{k}})-\mathcal{J}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{IP-2}}(\breve{\mathfrak{k}})$, the real and imaginary parts of the truth are T-conorm which means that these two function satisfy the conditions of T-conorm such that $S(\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}}),0)=\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})+\mathcal{J}_{\mathfrak{W}_{RP-2}}(\breve{\mathfrak{k}})-\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{RP-2}}(\breve{\mathfrak{k}})=\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})+0-\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})0=\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})$, and thus, by using the definition of T-conorm, the values of the two should be in unit interval. Similarly, the function $\mathcal{K}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})\mathcal{K}_{\mathfrak{W}_{RP-2}}(\breve{\mathfrak{k}})$ and $\mathcal{K}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}})\mathcal{K}_{\mathfrak{W}_{IP-2}}(\breve{\mathfrak{k}})$ are in the form of T-conorm, which means that these two satisfy the conditions of T-norm such that $T(\mathcal{K}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}}),1)=\mathcal{K}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})\mathcal{K}_{\mathfrak{W}_{RP-2}}(\breve{\mathfrak{k}})=\mathcal{K}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})1=\mathcal{K}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})$. Therefore, from the above analysis, we get the result such that

$$\mathfrak{W}_{CIU-1}\oplus\mathfrak{W}_{CIU-2}=\left(\begin{array}{c}\left[\psi_{\varphi_1+\varphi_2-(\varphi_1\varphi_2/h)},\psi_{\phi_1+\phi_2-(\phi_1\phi_2/h)}\right],\\ \left(\begin{array}{c}\left(\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})+\mathcal{J}_{\mathfrak{W}_{RP-2}}(\breve{\mathfrak{k}})-\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{RP-2}}(\breve{\mathfrak{k}})\right)e^{i2\pi\left(\mathcal{J}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}})+\mathcal{J}_{\mathfrak{W}_{IP-2}}(\breve{\mathfrak{k}})-\mathcal{J}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{IP-2}}(\breve{\mathfrak{k}})\right)},\\ \mathcal{K}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})\mathcal{K}_{\mathfrak{W}_{RP-2}}(\breve{\mathfrak{k}})e^{i2\pi\left(\mathcal{K}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}})\mathcal{K}_{\mathfrak{W}_{IP-2}}(\breve{\mathfrak{k}})\right)}\end{array}\right)\end{array}\right). \tag{10}$$

The points 2–4 are similar. □

Definition 8. For CIULN $\mathfrak{W}_{CIU-1}=([\psi_{\varphi_1},\psi_{\phi_1}],(\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}}))},\mathcal{K}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}}))}))$, the score and accuracy functions are defined as

$$\begin{aligned}\bar{\bar{\zeta}}(\mathfrak{W}_{CIU-1})&=\frac{1}{4}(\varphi_1+\phi_1)\times\left(\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})-\mathcal{K}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})+\mathcal{J}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}})-\mathcal{K}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}})\right),\\ \bar{\bar{\mathfrak{F}}}(\mathfrak{W}_{CIU-1})&=\frac{1}{4}(\varphi_1+\phi_1)\times\left(\mathcal{J}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})+\mathcal{K}_{\mathfrak{W}_{RP-1}}(\breve{\mathfrak{k}})+\mathcal{J}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}})+\mathcal{K}_{\mathfrak{W}_{IP-1}}(\breve{\mathfrak{k}})\right),\end{aligned} \tag{11}$$

where $\bar{\bar{\zeta}}(\mathfrak{W}_{CIU-1})\in[-1,1]$ and $\bar{\bar{\mathfrak{F}}}(\mathfrak{W}_{CIU-1})\in[0,1]$.

An order relation between pairs of two CIULNs is stated as

(1) If $\bar{\bar{\zeta}}(\mathfrak{W}_{CIU-1})>\bar{\bar{\zeta}}(\mathfrak{W}_{CIU-2})$, then $\mathfrak{W}_{CIU-1}>\mathfrak{W}_{CIU-2}$.
(2) If $\bar{\bar{\zeta}}(\mathfrak{W}_{CIU-1})<\bar{\bar{\zeta}}(\mathfrak{W}_{CIU-2})$, then $\mathfrak{W}_{CIU-1}<\mathfrak{W}_{CIU-2}$.
(3) If $\bar{\bar{\zeta}}(\mathfrak{W}_{CIU-1})=\bar{\bar{\zeta}}(\mathfrak{W}_{CIU-2})$, then
   (1) If $\bar{\bar{\mathfrak{F}}}(\mathfrak{W}_{CIU-1})>\bar{\bar{\mathfrak{F}}}(\mathfrak{W}_{CIU-2})$, then $\mathfrak{W}_{CIU-1}>\mathfrak{W}_{CIU-2}$.
   (2) If $\bar{\bar{\mathfrak{F}}}(\mathfrak{W}_{CIU-1})<\bar{\bar{\mathfrak{F}}}(\mathfrak{W}_{CIU-2})$, then $\mathfrak{W}_{CIU-1}<\mathfrak{W}_{CIU-2}$.
   (3) If $\bar{\bar{\zeta}}(\mathfrak{W}_{CIU-1})=\bar{\bar{\zeta}}(\mathfrak{W}_{CIU-2})$, then $\mathfrak{W}_{CIU-1}=\mathfrak{W}_{CIU-2}$.

4. Complex Intuitionistic Uncertain Linguistic Heronian Mean Operators

In this study, we investigate the ideas of the CIULAHM operator, CIULWAHM operator, CIULGHM operator, and CIULWGHM operator and discuss their particular cases with the help of parameters. Some properties for investigated operators are developed such that idempotency, monotonicity, and boundedness are also explored.

Definition 9. For the families of CIULNs $\mathfrak{W}_{CIU-i}=([\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}],(\mathcal{J}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))},\mathcal{K}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))})), i,\mathfrak{s}=1,2,\ldots,\widehat{\Xi}$, the CIULAHM operator is mapping $\text{CIULAHM}^{p_{SC},q_{SC}}\colon\Theta^{\widehat{\Xi}}\longrightarrow\Theta$, defined by

$$\mathrm{CIULAHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)=\left(\frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\mathfrak{W}_{\mathrm{CIU}-i}^{p_{SC}}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}^{q_{SC}}\right)^{\left(1/p_{SC}+q_{SC}\right)}. \tag{12}$$

By using Definition 9, we investigate the following result.

Theorem 1. *For the families of CIULNs* $\mathfrak{W}_{CIU-i}=([\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}],(\mathcal{J}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))},\mathcal{K}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))}))$, $i,\mathfrak{s}=1,2,\ldots,\widehat{\Xi}$, *and by using Definitions 7 and 9, we obtain*

$$\begin{aligned}&\mathrm{CIULAHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\Xi}\right)\\&=\left\{\begin{array}{c}\left[\psi_{\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(\varphi_i^{p_{SC}}\varphi_S^{q_{SC}}/h\right)\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}},\psi_{\left(\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(\phi_i^{p_{SC}}\phi_S^{q_{SC}}/h\right)\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}\right)}\right]\\ \left\{\begin{array}{c}\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-i}}^{p_{SC}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-S}}^{q_{SC}}(\breve{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}\\ e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-i}}^{p_{SC}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-S}}^{q_{SC}}(\breve{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}},\\ \left(1-\left(1-\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\right)^{p_{SC}}\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-S}}(\breve{\mathfrak{k}})\right)^{q_{SC}}\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}\right)\\ e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)^{p_{SC}}\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-S}}(\breve{\mathfrak{k}})\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}}\end{array}\right\}\end{array}\right\},\end{aligned} \tag{13}$$

Proof. By using Definition 7, we obtain

$$\begin{aligned}\mathfrak{W}_{\mathrm{CIU}-i}^{p_{SC}}&=\left(\begin{array}{c}\left[\psi_{h\left(\varphi_i/h\right)^{p_{SC}}},\psi_{h\left(\phi_i/h\right)^{p_{SC}}}\right],\\ \left(\begin{array}{c}\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-i}}^{p_{SC}}(\breve{\mathfrak{k}})e^{i2\pi\left(\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-i}}^{p_{SC}}(\breve{\mathfrak{k}})\right)},\\ \left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\right)^{p_{SC}}\right)e^{i2\pi\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)^{p_{SC}}\right)}\end{array}\right)\end{array}\right),\\ \mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}^{p_{SC}}&=\left(\begin{array}{c}\left[\psi_{h\left(\varphi_s/h\right)^{p_{SC}}},\psi_{h\left(\phi_s/h\right)^{p_{SC}}}\right],\\ \left(\begin{array}{c}\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}^{p_{SC}}(\breve{\mathfrak{k}})e^{i2\pi\left(\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}^{p_{SC}}(\breve{\mathfrak{k}})\right)},\\ \left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)^{p_{SC}}\right)e^{i2\pi\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)^{p_{SC}}\right)}\end{array}\right)\end{array}\right).\end{aligned} \tag{14}$$

By using the above information, we obtain

$$\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\mathfrak{W}_{\mathrm{CIU}-i}^{p_{\mathrm{SC}}}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}^{q_{\mathrm{SC}}}=\left(\begin{array}{c}\left[\psi^{\left(1-\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(\varphi_i^{p_{\mathrm{SC}}}\varphi_{\mathfrak{s}}^{q_{\mathrm{SC}}}/h\right)\right)\right)},\psi^{\left(1-\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(\phi_i^{p_{\mathrm{SC}}}\phi_{\mathfrak{s}}^{q_{\mathrm{SC}}}/h\right)\right)\right)}\right],\\ \left(\begin{array}{c}\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-i}}^{p_{\mathrm{SC}}}(\breve{\mathfrak{k}})\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}^{q_{\mathrm{SC}}}(\breve{\mathfrak{k}})\right)\right)\right)e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-i}}^{p_{\mathrm{SC}}}(\breve{\mathfrak{k}})\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}^{q_{\mathrm{SC}}}(\breve{\mathfrak{k}})\right)\right)\right)},\\ \prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\right)^{p_{\mathrm{SC}}}\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)^{q_{\mathrm{SC}}}\right)e^{i2\pi\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)^{p_{\mathrm{SC}}}\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)^{q_{\mathrm{SC}}}\right)\right)}\end{array}\right)\end{array}\right), \tag{15}$$

and then

$$\frac{2}{\widehat{\Xi}(\widehat{\Xi}+1)}\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\mathfrak{W}_{\mathrm{CIU}-i}^{p_{\mathrm{SC}}}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}^{q_{\mathrm{SC}}}=\left(\begin{array}{c}\left[\psi^{1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(\varphi_i^{p_{\mathrm{SC}}}\varphi_{\mathfrak{s}}^{q_{\mathrm{SC}}}/h\right)\right)\right)^{2/\widehat{\Xi}(\widehat{\Xi}+1)}},\psi^{1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(\phi_i^{p_{\mathrm{SC}}}\phi_{\mathfrak{s}}^{q_{\mathrm{SC}}}/h\right)\right)\right)^{2/\widehat{\Xi}(\widehat{\Xi}+1)}}\right],\\ \left(\begin{array}{c}\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-i}}^{p_{\mathrm{SC}}}(\breve{\mathfrak{k}})\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-S}}^{q_{\mathrm{SC}}}(\breve{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}(\widehat{\Xi}+1)}\right)\\ e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-i}}^{p_{\mathrm{SC}}}(\breve{\mathfrak{k}})\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}^{q_{\mathrm{SC}}}(\breve{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}(\widehat{\Xi}+1)}\right)},\\ \left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\right)^{p_{\mathrm{SC}}}\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-S}}(\breve{\mathfrak{k}})\right)^{q_{\mathrm{SC}}}\right)\right)^{2/\widehat{\Xi}(\widehat{\Xi}+1)}\\ e^{i2\pi\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)^{p_{\mathrm{SC}}}\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-S}}(\breve{\mathfrak{k}})\right)^{q_{\mathrm{SC}}}\right)\right)^{2/\widehat{\Xi}(\widehat{\Xi}+1)}}\end{array}\right)\end{array}\right),$$

$$\mathrm{CIULAHM}^{p_{\mathrm{SC}},q_{\mathrm{SC}}}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)=\left(\frac{2}{\widehat{\Xi}(\widehat{\Xi}+1)}\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\mathfrak{W}_{\mathrm{CIU}-i}^{p_{\mathrm{SC}}}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}^{q_{\mathrm{SC}}}\right)^{\left(1/p_{\mathrm{SC}}+q_{\mathrm{SC}}\right)}$$

$$
=\left(\begin{array}{c}\left[\psi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(\varphi_i^{p_{SC}}\varphi_S^{q_{SC}}/h\right)\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)},\psi\left(\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(\phi_i^{p_{SC}}\phi_S^{q_{SC}}/h\right)\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}\right)\right],\\
\left(\begin{array}{c}\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\mathcal{J}_{\mathfrak{W}_{RP-i}}^{p_{SC}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{RP-S}}^{q_{SC}}(\breve{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}\\
e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\mathcal{J}_{\mathfrak{W}_{IP-i}}^{p_{SC}}(\breve{\mathfrak{k}})\mathcal{J}_{\mathfrak{W}_{IP-S}}^{q_{SC}}(\breve{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}},\\
\left(1-\left(1-\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})\right)^{p_{SC}}\left(1-\mathcal{K}_{\mathfrak{W}_{RP-S}}(\breve{\mathfrak{k}})\right)^{q_{SC}}\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}\right)\\
e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}})\right)^{p_{SC}}\left(1-\mathcal{K}_{\mathfrak{W}_{IP-S}}(\breve{\mathfrak{k}})\right)^{q_{SC}}\right)\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{\left(1/p_{SC}+q_{SC}\right)}}\end{array}\right)\end{array}\right). \tag{16}
$$

Moreover, by using Definition 7, we prove that certain properties for investigated ideas are similar to idempotency, monotonicity, and boundedness, which are stated below. □

Property 1. For the families of CIULNs $\mathfrak{W}_{CIU-i} = ([\psi_{\varphi_i}, \psi_{\phi_{\mathfrak{s}}}], (\mathcal{J}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))}, \mathcal{K}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))})), i, \mathfrak{s} = 1, 2, \ldots, \widehat{\Xi}$, we have

(1) If $\mathfrak{W}_{CIU-i} = \mathfrak{W}_{CIU}$, then

$$
\mathrm{CIULAHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}\right) = \mathfrak{W}_{CIU}. \tag{17}
$$

(2) For CIULN $\mathfrak{W}_{CIU-i}' = ([\psi_{\varphi_i}', \psi_{\phi_{\mathfrak{s}}}'], (\mathcal{J}_{\mathfrak{W}_{RP-i}}'(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{IP-i}}'(\breve{\mathfrak{k}}))}, \mathcal{K}_{\mathfrak{W}_{RP-i}}'(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{IP-i}}'(\breve{\mathfrak{k}}))})), i, \mathfrak{s} = 1, 2, \ldots, \widehat{\Xi}$, such that $\psi_{\varphi_i}' \leq \psi_{\varphi_i}$, $\psi_{\phi_{\mathfrak{s}}}' \leq \psi_{\phi_{\mathfrak{s}}}$, $\mathcal{J}_{\mathfrak{W}_{RP-i}}' \leq \mathcal{J}_{\mathfrak{W}_{RP-i}}$, $\mathcal{J}_{\mathfrak{W}_{IP-i}}' \leq \mathcal{J}_{\mathfrak{W}_{IP-i}}$, $\mathcal{K}_{\mathfrak{W}_{RP-i}}' \geq \mathcal{K}_{\mathfrak{W}_{RP-i}}$, and $\mathcal{K}_{\mathfrak{W}_{IP-i}}' \geq \mathcal{K}_{\mathfrak{W}_{IP-i}}$, we have

$$
\mathrm{CIULAHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{CIU-1}', \mathfrak{W}_{CIU-2}', \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}'\right) \leq \mathrm{CIULAHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}\right). \tag{18}
$$

(3) If $\mathfrak{W}_{CIU-A} = \min(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}})$ and $\mathfrak{W}_{CIU-B} = \max(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}})$, then

$$
\mathfrak{W}_{CIU-A} \leq \mathrm{CIULAHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}\right) \leq \mathfrak{W}_{CIU-B} \tag{19}
$$

Proof. We prove the above three equations, such that

(1) If $\mathfrak{W}_{CIU-i} = \mathfrak{W}_{CIU}, i = 1, 2, \ldots, \widehat{\Xi}$, then

$$\begin{aligned} \mathrm{CIULAHM}^{p_{SC}, q_{SC}}\left(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}\right) &= \left(\frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)} \sum_{i=1}^{\widehat{\Xi}} \sum_{\mathfrak{s}=1}^{\widehat{\Xi}} \mathfrak{W}_{CIU-i}^{p_{SC}} \mathfrak{W}_{CIU-\mathfrak{s}}^{q_{SC}}\right)^{\left(1/p_{SC}+q_{SC}\right)} \\ &= \left(\frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)} \sum_{i=1}^{\widehat{\Xi}} \sum_{\mathfrak{s}=1}^{\widehat{\Xi}} \mathfrak{W}_{CIU}^{p_{SC}} \mathfrak{W}_{CIU}^{q_{SC}}\right)^{\left(1/p_{SC}+q_{SC}\right)} \\ &= \left(\frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)} \sum_{i=1}^{\widehat{\Xi}} \sum_{\mathfrak{s}=1}^{\widehat{\Xi}} \mathfrak{W}_{CIU}^{p_{SC}+q_{SC}}\right)^{\left(1/p_{SC}+q_{SC}\right)} \\ &= \left(\mathfrak{W}_{CIU}^{p_{SC}+q_{SC}}\right)^{\left(1/p_{SC}+q_{SC}\right)} = \mathfrak{W}_{CIU}. \end{aligned} \tag{20}$$

(2) When $\psi_{\varphi_i}' \leq \psi_{\varphi_i}$, $\psi_{\phi_{\mathfrak{s}}}' \leq \psi_{\phi_{\mathfrak{s}}}$, $\mathcal{J}_{\mathfrak{W}_{RP-i}}' \leq \mathcal{J}_{\mathfrak{W}_{RP_T i}}$, $\mathcal{J}_{\mathfrak{W}_{IP-i}}' \leq \mathcal{J}_{\mathfrak{W}_{IP-i}}$, $\mathcal{K}_{\mathfrak{W}_{RP-i}}' \geq \mathcal{K}_{\mathfrak{W}_{RP-i}}$, and $\mathcal{K}_{\mathfrak{W}_{IP-i}}' \geq \mathcal{K}_{\mathfrak{W}_{IP-i}}$, then we have $\mathfrak{W}_{CIU-i}' \leq \mathfrak{W}_{CIU-i}, i = 1, 2, \ldots, \widehat{\Xi}$.

Thus, $\mathfrak{W}_{CIU-i}' \leq \mathfrak{W}_{CIU-i}$ and $\mathfrak{W}_{CIU-\mathfrak{s}}' \leq \mathfrak{W}_{CIU-\mathfrak{s}}$, such that

$$\begin{aligned} &\mathfrak{W}_{CIU-i}^{\prime p_{SC}} \mathfrak{W}_{CIU-\mathfrak{s}}^{\prime q_{SC}} \leq \mathfrak{W}_{CIU-i}^{p_{SC}} \mathfrak{W}_{CIU-\mathfrak{s}}^{q_{SC}} \\ &\Longrightarrow \frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)} \sum_{i=1}^{\widehat{\Xi}} \sum_{\mathfrak{s}=1}^{\widehat{\Xi}} \mathfrak{W}_{CIU-i}^{\prime p_{SC}} \mathfrak{W}_{CIU-\mathfrak{s}}^{\prime q_{SC}} \leq \frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)} \sum_{i=1}^{\widehat{\Xi}} \sum_{\mathfrak{s}=1}^{\widehat{\Xi}} \mathfrak{W}_{CIU-i}^{p_{SC}} \mathfrak{W}_{CIU-\mathfrak{s}}^{q_{SC}} \\ &\Longrightarrow \left(\frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)} \sum_{i=1}^{\widehat{\Xi}} \sum_{\mathfrak{s}=1}^{\widehat{\Xi}} \mathfrak{W}_{CIU-i}^{\prime p_{SC}} \mathfrak{W}_{CIU-\mathfrak{s}}^{\prime q_{SC}}\right)^{\left(1/p_{SC}+q_{SC}\right)} \leq \left(\frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)} \sum_{i=1}^{\widehat{\Xi}} \sum_{\mathfrak{s}=1}^{\widehat{\Xi}} \mathfrak{W}_{CIU-i}^{p_{SC}} \mathfrak{W}_{CIU-\mathfrak{s}}^{q_{SC}}\right)^{\left(1/p_{SC}+q_{SC}\right)}. \end{aligned} \tag{21}$$

Then, we obtain

$$\mathrm{CIULAHM}^{p_{SC}, q_{SC}}\left(\mathfrak{W}_{CIU-1}', \mathfrak{W}_{CIU-2}', \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}'\right) \leq \mathrm{CIULAHM}^{p_{SC}, q_{SC}}\left(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}\right). \tag{22}$$

(3) If $\mathfrak{W}_{CIU-A} = \min\left(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}\right)$ and $\mathfrak{W}_{CIU-B} = \max\left(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}\right)$, then by using Property 1, we get

$$\begin{aligned} &\mathfrak{W}_{CIU-A} \leq \mathrm{CIULAHM}^{p_{SC}, q_{SC}}\left(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}\right), \\ &\mathrm{CIULAHM}^{p_{SC}, q_{SC}}\left(\mathfrak{W}_{CIU-1}, \mathfrak{W}_{CIU-2}, \ldots, \mathfrak{W}_{CIU-\widehat{\Xi}}\right) \leq \mathfrak{W}_{CIU-B}, \end{aligned} \tag{23}$$

and then

$$\mathfrak{W}_{\mathrm{CIU}-A} \leq \mathrm{CIULAHM}^{p_{\mathrm{SC}},q_{\mathrm{SC}}}\left(\mathfrak{W}_{\mathrm{CIU}-1}, \mathfrak{W}_{\mathrm{CIU}-2}, \ldots, \mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right) \leq \mathfrak{W}_{\mathrm{CIU}-B}. \tag{24}$$

Moreover, by using the investigated operators, we discuss some cases of the explored operators, which are discussed below.

(1) For $q_{\mathrm{SC}} \longrightarrow 0$, the idea of CIULAHM operator is converted to CIUL generalized linear descending weighted mean (CIULGLDWM) operator, such that

$$\begin{aligned}
&\mathrm{CIULAHM}^{p_{\mathrm{SC}},0}\left(\mathfrak{W}_{\mathrm{CIU}-1}, \mathfrak{W}_{\mathrm{CIU}-2}, \ldots, \mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)\\
&= \lim_{q_{\mathrm{SC}} \longrightarrow 0}\left(\frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\mathfrak{W}_{\mathrm{CIU}-i}^{p_{\mathrm{SC}}}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}^{q_{\mathrm{SC}}}\right)^{\left(1/p_{\mathrm{SC}}+q_{\mathrm{SC}}\right)}\\
&= \left(\frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\sum_{i=1}^{\widehat{\Xi}}\mathfrak{W}_{\mathrm{CIU}-i}^{p_{\mathrm{SC}}}\right)^{\left(1/p_{\mathrm{SC}}\right)}\\
&= \left(\begin{array}{c}
\left[\psi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(\varphi_i^{p_{\mathrm{SC}}}/h\right)\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/p_{\mathrm{SC}}\right)}, \psi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(\phi_i^{p_{\mathrm{SC}}}/h\right)\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/p_{\mathrm{SC}}\right)}\right],\\
\left(\begin{array}{c}
\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-i}}^{p_{\mathrm{SC}}}\left(\check{\mathfrak{t}}\right)\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/p_{\mathrm{SC}}\right)}\\
e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-i}}^{p_{\mathrm{SC}}}\left(\check{\mathfrak{t}}\right)\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/p_{\mathrm{SC}}\right)}},\\
\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-i}}\left(\check{\mathfrak{t}}\right)\right)^{p_{\mathrm{SC}}}\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/p_{\mathrm{SC}}\right)}\right)\\
e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-i}}\left(\check{\mathfrak{t}}\right)\right)^{p_{\mathrm{SC}}}\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/p_{\mathrm{SC}}\right)}\right)}
\end{array}\right)
\end{array}\right).
\end{aligned} \tag{25}$$

(2) For $p_{\mathrm{SC}} \longrightarrow 0$, the idea of CIULAHM operator is converted to CIUL generalized linear ascending weighted mean (CIULGLAWM) operator, such that

$$
\begin{aligned}
& CIULAHM^{0,q_{SC}}\left(\mathfrak{W}_{\text{CIU}-1},\mathfrak{W}_{\text{CIU}-2},\ldots,\mathfrak{W}_{\text{CIU}-\widehat{\Xi}}\right)\\
&= \lim_{p_{SC}\longrightarrow 0}\left(\frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\mathfrak{W}_{\text{CIU}-i}^{p_{SC}}\mathfrak{W}_{\text{CIU}-\mathfrak{s}}^{q_{SC}}\right)^{\left(1/p_{SC}+q_{SC}\right)}\\
&= \left(\frac{2}{\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\sum_{i=1}^{\widehat{\Xi}}\mathfrak{W}_{\text{CIU}-i}^{q_{SC}}\right)^{\left(1/q_{SC}\right)}\\
&= \left(\begin{array}{c}
\left[\psi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(\varphi_i^{q_{SC}}/h\right)\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/q_{SC}\right)},\ \psi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(\phi_i^{q_{SC}}/h\right)\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/q_{SC}\right)}\right],\\
\left(\begin{array}{c}
\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\mathscr{J}_{\mathfrak{W}_{\text{RP}-i}}^{q_{SC}}\left(\check{\mathfrak{k}}\right)\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/q_{SC}\right)}\\
e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\mathscr{J}_{\mathfrak{W}_{\text{IP}-i}}^{q_{SC}}\left(\check{\mathfrak{k}}\right)\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/q_{SC}\right)}},\\
\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{K}_{\mathfrak{W}_{\text{RP}-i}}\left(\check{\mathfrak{k}}\right)\right)^{q_{SC}}\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/q_{SC}\right)}\right)\\
e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{K}_{\mathfrak{W}_{\text{IP}-i}}\left(\check{\mathfrak{k}}\right)\right)^{q_{SC}}\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/q_{SC}\right)}\right)}
\end{array}\right)
\end{array}\right).
\end{aligned}
\tag{26}
$$

(3) For $p_{SC}=q_{SC}=(1/2)$, the idea of CIULAHM operator is converted to CIUL basic HM (CIULBHM) operator, such that

$$\mathrm{CIULAHM}^{(1/2),(1/2)}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)$$

$$=\left(\begin{array}{c}\left[\psi^{\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\frac{\varphi_i^{(1/2)}\varphi_{\mathfrak{s}}^{(1/2)}}{h}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)},\psi^{\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\frac{\phi_i^{(1/2)}\phi_{\mathfrak{s}}^{(1/2)}}{h}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)}\right],\\ \left(\begin{array}{c}\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-i}}^{(1/2)}(\breve{\mathfrak{k}})\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}^{(1/2)}(\breve{\mathfrak{k}})\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)\\ e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-i}}^{(1/2)}(\breve{\mathfrak{k}})\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}^{(1/2)}(\breve{\mathfrak{k}})\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)},\\ \left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\right)^{(1/2)}\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)^{(1/2)}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\\ e^{i2\pi\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)^{(1/2)}\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)^{(1/2)}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}}\end{array}\right)\end{array}\right). \tag{27}$$

(4) For $p_{\mathrm{SC}}=q_{\mathrm{SC}}=1$, the idea of CIULAHM operator is converted to CIUL basic HM (CIULBHM) operator, such that

$$\mathrm{CIULAHM}^{1,1}\left(\mathfrak{W}_{CIU-1},\mathfrak{W}_{CIU-2},\ldots,\mathfrak{W}_{CIU-\widehat{\Xi}}\right)$$

$$=\left(\begin{array}{c}\left[\psi_{\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\frac{\varphi_i^1\varphi_{\mathfrak{s}}^1}{h}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/2)}},\psi_{\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\frac{\phi_i^1\phi_{\mathfrak{s}}^1}{h}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/2)}}\right],\\ \left(\begin{array}{c}\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathscr{J}^1_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\mathscr{J}^1_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/2)}\\ e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathscr{J}^1_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\mathscr{J}^1_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/2)}},\\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\right)^{1}\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)^{(1)}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/2)}\right)\\ e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)^{1}\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)^{(1)}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/2)}\right)}\end{array}\right)\end{array}\right). \tag{28}$$

$\square$

Definition 10. Based on any families of CIULNs $\mathfrak{W}_{\mathrm{CIU}-i}=([\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}],(\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}}))},\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}}))}))$, $i,\mathfrak{s}=1,2,\ldots,\widehat{\Xi}$, the CIULWAHM operator is a mapping $\mathrm{CIULWAHM}^{p_{\mathrm{SC}},q_{\mathrm{SC}}}:\Theta^{\widehat{\Xi}}\longrightarrow\Theta$ defined by

$$\mathrm{CIULWAHM}^{p_{\mathrm{SC}},q_{\mathrm{SC}}}\left(\mathfrak{W}_{CIU-1},\mathfrak{W}_{CIU-2},\ldots,\mathfrak{W}_{CIU-\widehat{\Xi}}\right)=\left(\frac{2}{\widehat{\Xi}(\widehat{\Xi}+1)}\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(\widehat{\Xi}\,\widehat{\Omega}_{W-i}\mathfrak{W}_{\mathrm{CIU}-i}\right)^{p_{\mathrm{SC}}}\left(\widehat{\Xi}\,\widehat{\Omega}_{W-\mathfrak{s}}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}\right)^{q_{\mathrm{SC}}}\right)^{(1/p_{\mathrm{SC}}+q_{\mathrm{SC}})}, \tag{29}$$

where $\widehat{\Omega}_W=\{\widehat{\Omega}_{W-1},\widehat{\Omega}_{W-2},\ldots,\widehat{\Omega}_{W-\widehat{\Xi}}\}$ expresses the weight vector with a condition that is $\sum_{i=1}^{\widehat{\Xi}}\widehat{\Omega}_{W-i}=1,\widehat{\Omega}_{W-i}\in[0,1]$. By using Definition 10, we investigate the following result.

Theorem 2. *For families of CIULNs* $\mathfrak{W}_{CIU-i}=([\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}],(\mathscr{J}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))},\mathscr{K}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))}))$, $i,\mathfrak{s}=1,2,\ldots,\widehat{\Xi}$, *by using Definitions 7 and 10, we obtain*

$$\mathrm{CIULGHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)$$

$$=\left(\begin{array}{c}\left[\begin{array}{l}\psi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(1-\left(\varphi_i/h\right)^{\widehat{\Xi}\widehat{\Omega}_{W-i}}\right)^{p_{SC}}\left(1-\left(1-\left(\varphi_s/h\right)\right)^{\widehat{\Xi}\widehat{\Omega}_{W-S}}\right)^{q_{SC}}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/p_{SC}+q_{SC})}\right),\\ \psi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(1-\left(\varphi_i/h\right)^{\widehat{\Xi}\widehat{\Omega}_{W-i}}\right)^{p_{SC}}\left(1-\left(1-\left(\varphi_s/h\right)\right)^{\widehat{\Xi}\widehat{\Omega}_{W-S}}\right)^{q_{SC}}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/p_{SC}+q_{SC})}\right)\end{array}\right],\\ \left(\begin{array}{c}\left(\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-i}\right)^{p_{SC}}\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-S}}(\breve{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{q_{SC}}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/p_{SC}+q_{SC})}\right)\right)\\ e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-i}\right)^{p_{SC}}\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-S}}(\breve{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{q_{SC}}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))\,(1/p_{SC}+q_{SC})}\right)}\\ \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-i}}^{\widehat{\Xi}}\widehat{\Omega}_{W-i}(\breve{\mathfrak{k}})\right)^{p_{SC}}\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-S}}^{\widehat{\Xi}}\widehat{\Omega}_{W-S}(\breve{\mathfrak{k}})\right)^{q_{SC}}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/p_{SC}+q_{SC})}\right)\\ e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-i}}^{\widehat{\Xi}}\widehat{\Omega}_{W-i}(\breve{\mathfrak{k}})\right)^{p_{SC}}\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-S}}^{q_{SC}}(\breve{\mathfrak{k}})\right)^{q_{SC}}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/p_{SC}+q_{SC})}\right)}\end{array}\right)\end{array}\right). \tag{30}$$

Proof. Trivial. □

Theorem 3. *The CIULAHM operator is a certain brief case of CIULWAHM operator.*

Proof. We know that

$$\mathrm{CIULWAHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{CIU-1},\mathfrak{W}_{CIU-2},\ldots,\mathfrak{W}_{CIU-\widehat{\Xi}}\right)=\left(\frac{2}{\widehat{\Xi}(\widehat{\Xi}+1)}\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(\widehat{\Xi}\,\widehat{\Omega}_{W-i}\mathfrak{W}_{\mathrm{CIU}-i}\right)^{p_{SC}}\left(\widehat{\Xi}\,\widehat{\Omega}_{W-\mathfrak{s}}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}\right)^{q_{SC}}\right)^{(1/p_{SC}+q_{SC})}. \tag{31}$$

If $\widehat{\Omega}_W=\left\{(1/\widehat{\Xi}),(1/\widehat{\Xi}),\ldots,(1/\widehat{\Xi})\right\}$, then

$$\begin{aligned}&\mathrm{CIULWAHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)\\&=\left(\frac{2}{\widehat{\Xi}(\widehat{\Xi}+1)}\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(\widehat{\Xi}\,\widehat{\Omega}_{W-i}\mathfrak{W}_{\mathrm{CIU}-i}\right)^{p_{SC}}\left(\widehat{\Xi}\,\widehat{\Omega}_{W-\mathfrak{s}}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}\right)^{q_{SC}}\right)^{(1/p_{SC}+q_{SC})}\\&=\left(\frac{2}{\widehat{\Xi}(\widehat{\Xi}+1)}\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(\widehat{\Xi}\frac{1}{\widehat{\Xi}}\mathfrak{W}_{\mathrm{CIU}-i}\right)^{p_{SC}}\left(\widehat{\Xi}\frac{1}{\widehat{\Xi}}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}\right)^{q_{SC}}\right)^{(1/p_{SC}+q_{SC})}=\left(\frac{2}{\widehat{\Xi}(\widehat{\Xi}+1)}\sum_{i=1}^{\widehat{\Xi}}\sum_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(\mathfrak{W}_{\mathrm{CIU}-i}\right)^{p_{SC}}\left(\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}\right)^{q_{SC}}\right)^{(1/p_{SC}+q_{SC})}\\&=CIULAHM^{p_{SC},q_{SC}}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right).\end{aligned} \tag{32}$$

□

Definition 11. For the families of CIULNs, the CIULGHM operator is mapping $\mathrm{CIULGHM}^{p_{SC},q_{SC}}: \Theta^{\frown \Xi} \longrightarrow \Theta$defined by

$$\mathrm{CIULGHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{\mathrm{CIU}-1}, \mathfrak{W}_{\mathrm{CIU}-2}, \ldots, \mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right) = \frac{1}{p_{SC} + q_{SC}}\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(p_{SC}\mathfrak{W}_{\mathrm{CIU}-i} + q_{SC}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}\right)\right)^{2/\widehat{\Xi}\,(\widehat{\Xi}+1)}. \tag{33}$$

By using Definition 11, we investigate the following result.

Theorem 4. *For families of CIULNs $\mathfrak{W}_{CIU-i} = ([\psi_{\varphi_i}, \psi_{\phi_{\mathfrak{s}}}], (\mathcal{J}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))}, \mathcal{K}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})e^{i2\pi\ (\mathcal{K}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}}))})), i, \mathfrak{s} = 1, 2, \ldots, \widehat{\Xi}\ (\widehat{\Xi}+1)$, and by using Definitions 7 and 11, we obtain*

$$\mathrm{CIULWAHM}^{p_{SC},q_{SC}}\left(\mathfrak{W}_{\mathrm{CIU}-1}, \mathfrak{W}_{\mathrm{CIU}-2}, \ldots, \mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)$$
$$= \left(\begin{array}{c} \left[\begin{array}{l} \psi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(\varphi_i/h\right)\right)^{p_{SC}}\left(1-\left(\varphi_S/h\right)\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}\,(\widehat{\Xi}+1)}\right)^{(1/p_{SC}+q_{SC})}\right), \\ \psi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(\phi_i/h\right)\right)^{p_{SC}}\left(1-\left(\phi_S/h\right)\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}\,(\widehat{\Xi}+1)}\right)^{(1/p_{SC}+q_{SC})}\right), \end{array}\right], \\ \left(\begin{array}{c} \left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})\right)^{p_{SC}}\left(1-\mathcal{J}_{\mathfrak{W}_{RP-S}}(\breve{\mathfrak{k}})\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}\,(\widehat{\Xi}+1)}\right)^{(1/p_{SC}+q_{SC})}\right) \\ e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{IP-i}}(\breve{\mathfrak{k}})\right)^{p_{SC}}\left(1-\mathcal{J}_{\mathfrak{W}_{IP-S}}(\breve{\mathfrak{k}})^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}\,(\widehat{\Xi}+1)}\right)^{(1/p_{SC}+q_{SC})}\right)} \\ \left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\mathcal{K}^{p_{SC}}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})\mathcal{K}^{q_{SC}}_{\mathfrak{W}_{RP-S}}(\breve{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}\,(\widehat{\Xi}+1)}\right)^{(1/p_{SC}+q_{SC})} \\ e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\mathcal{K}^{p_{SC}}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})\mathcal{K}^{q_{SC}}_{\mathfrak{W}_{RP-i}}(\breve{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}\,(\widehat{\Xi}+1)}\right)^{(1/p_{SC}+q_{SC})}} \end{array}\right) \end{array}\right). \tag{34}$$

Proof. Trivial.

Moreover, by using the investigated operators, we discuss some cases of the explored operators, which are discussed below.

(1) For $q_{SC} \longrightarrow 0$, the idea of CIULGHM operator is converted to CIUL generalized geometric linear descending weighted mean (CIULGGLDWM) operator, such that

$$\begin{aligned}
&\mathrm{CIULGHM}^{p_{SC},0}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)\\
&=\lim_{q_{SC}\longrightarrow 0}\left(\frac{1}{p_{SC}+q_{SC}}\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(p_{SC}\mathfrak{W}_{\mathrm{CIU}-i}+q_{SC}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}\right)\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)\\
&=\left(\frac{1}{p_{SC}}\left(\prod_{i=1}^{\widehat{\Xi}}p_{SC}\mathfrak{W}_{\mathrm{CIU}-i}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)\\
&=\left(\begin{array}{c}\left[\begin{array}{l}\psi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\left(\varphi_i/h\right)\right)^{p_{SC}}\right)\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)\right)^{\left(1/p_{SC}\right)},\\
\psi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\left(\phi_i/h\right)\right)^{p_{SC}}\right)\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)\right)^{\left(1/p_{SC}\right)}\end{array}\right],\end{array}\right)\\
&=\left(\left(\begin{array}{c}\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{J}_{\mathfrak{W}_{RP-i}}\left(\breve{\mathfrak{k}}\right)\right)^{p_{SC}}\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)\right)^{\left(1/p_{SC}\right)}\\
e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{J}_{\mathfrak{W}_{IP-i}}\left(\breve{\mathfrak{k}}\right)\right)^{p_{SC}}\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)\right)^{\left(1/p_{SC}\right)}},\\
\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\mathscr{K}_{\mathfrak{W}_{RP-i}}^{p_{SC}}\left(\breve{\mathfrak{k}}\right)\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/p_{SC}\right)}\\
e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}1-\mathscr{K}_{\mathfrak{W}_{IP-i}}^{p_{SC}}\left(\breve{\mathfrak{k}}\right)^{\left(\widehat{\Xi}+1-i\right)}\right)^{\left(2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)\right)}\right)^{\left(1/p_{SC}\right)}}\end{array}\right)\right).
\end{aligned}\tag{35}$$

(2) For $p_{SC}\longrightarrow 0$, the idea of CIULGHM operator is converted to CIUL generalized geometric linear ascending weighted mean (CIULGGLAWM) operator, such that

$$
\begin{aligned}
\mathrm{CIULGHM}^{0,q_{\mathrm{SC}}}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right) &= \lim_{p_{\mathrm{SC}}\longrightarrow 0}\left(\frac{1}{p_{\mathrm{SC}}+q_{\mathrm{SC}}}\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(p_{\mathrm{SC}}\mathfrak{W}_{\mathrm{CIU}-i}+q_{\mathrm{SC}}\mathfrak{W}_{\mathrm{CIU}-\mathfrak{s}}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)\\
&=\left(\frac{1}{q_{\mathrm{SC}}}\left(\prod_{i=1}^{\widehat{\Xi}}q_{\mathrm{SC}}\mathfrak{W}_{\mathrm{CIU}-i}\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)\\
&=\left(\begin{array}{c}
\left[\begin{array}{c}
\psi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\left(\varphi_i/h\right)\right)^{q_{\mathrm{SC}}}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/q_{\mathrm{SC}})}\right),\\
\psi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\left(\phi_i/h\right)\right)^{q_{\mathrm{SC}}}\right)\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/q_{\mathrm{SC}})}\right)
\end{array}\right],\\
\left(\begin{array}{c}
\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\right)^{q_{\mathrm{SC}}}\right)^{(\widehat{\Xi}+1-i)}\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/q_{\mathrm{SC}})}\right)\\
e^{i2\pi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)^{q_{\mathrm{SC}}}\right)^{(\widehat{\Xi}+1-i)}\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/q_{\mathrm{SC}})}\right)},\\
\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\left(1-\mathscr{K}^{q_{\mathrm{SC}}}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\right)^{(\widehat{\Xi}+1-i)}\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/q_{\mathrm{SC}})}\\
e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}1-\mathscr{K}^{q_{\mathrm{SC}}}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})^{(\widehat{\Xi}+1-i)}\right)^{(2/\widehat{\Xi}(\widehat{\Xi}+1))}\right)^{(1/q_{\mathrm{SC}})}}
\end{array}\right)
\end{array}\right).
\end{aligned}
\tag{36}
$$

(3) For $p_{\mathrm{SC}}=q_{\mathrm{SC}}=(1/2)$, the idea of CIULGHM operator is converted to CIUL basic geometric HM (CIULBGHM) operator, such that

$$\begin{aligned}
&\mathrm{CIULGHM}^{(1/2),(1/2)}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)\\
&=\left(\begin{array}{c}
\left[\begin{array}{l}
\psi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\left(\varphi_i/h\right)\right)^{(1/2)}\left(1-\left(\varphi_{\mathfrak{s}}/h\right)\right)^{(1/2)}\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)\right),\\
\psi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\left(\phi_i/h\right)\right)^{(1/2)}\left(1-\left(\phi_{\mathfrak{s}}/h\right)\right)^{(1/2)}\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)\right),
\end{array}\right],\\
\left(\begin{array}{c}
\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-i}}(\check{\mathfrak{k}})\right)^{(1/2)}\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}(\check{\mathfrak{k}})\right)^{(1/2)}\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\\
e^{i2\pi\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-i}}(\check{\mathfrak{k}})\right)^{(1/2)}\left(1-\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}(\check{\mathfrak{k}})\right)^{(1/2)}\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}},\\
\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-i}}^{(1/2)}(\check{\mathfrak{k}})\mathscr{K}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}^{(1/2)}(\check{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)\\
e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-i}}^{(1/2)}(\check{\mathfrak{k}})\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}^{(1/2)}(\check{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)}
\end{array}\right)
\end{array}\right).
\end{aligned}\tag{37}$$

(4) For $p_{SC}=q_{SC}=1$, the idea of CIULGHM operator is converted to CIUL geometric line HM (CIULGLHM) operator, such that

$$\operatorname{CIULGHM}^{1,1}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)$$

$$=\left(\begin{array}{c}\left[\begin{array}{c}\psi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\left(\varphi_i/h\right)\right)^{1}\left(1-\left(\varphi_{\mathfrak{s}}/h\right)\right)^{(1)}\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{(1/2)}\right),\\ \psi\left(1-\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\left(\phi_i/h\right)\right)^{1}\left(1-\left(\phi_{\mathfrak{s}}/h\right)\right)^{1}\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{(1/2)}\right)\end{array}\right],\\ \left(\begin{array}{c}\left(1-\left(1-\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\right)\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{(1/2)}\\ e^{i2\pi\left(1-\left(1-\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\right)\left(1-\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{(1/2)}},\\ \left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{(1/2)}\\ e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(1-\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}})\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-\mathfrak{s}}}(\breve{\mathfrak{k}})\right)\right)^{2/\widehat{\Xi}\left(\widehat{\Xi}+1\right)}\right)^{(1/2)}}\end{array}\right)\end{array}\right). \tag{38}$$

□

Property 2. For families of CIULNs $\mathfrak{W}_{\mathrm{CIU}-i}=([\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}],(\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}}))},\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-i}}(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-i}}(\breve{\mathfrak{k}}))})), i,\mathfrak{s}=1,2,\ldots,\widehat{\Xi}$, then

(1) If $\mathfrak{W}_{\mathrm{CIU}-i}=\mathfrak{W}_{\mathrm{CIU}}$, then

$$\operatorname{CIULGHM}^{p_{\mathrm{SC}},q_{\mathrm{SC}}}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right)=\mathfrak{W}_{\mathrm{CIU}}. \tag{39}$$

(2) For CIULN $\mathfrak{W}_{\mathrm{CIU}-i}'=([\psi_{\varphi_i}',\psi_{\phi_{\mathfrak{s}}}'],(\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-i}}'(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-i}}'(\breve{\mathfrak{k}}))},\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-i}}'(\breve{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-i}}'(\breve{\mathfrak{k}}))})), i,\mathfrak{s}=1,2,\ldots,\widehat{\Xi}$, such that $\psi_{\varphi_i}'\le\psi_{\varphi_i}$, $\psi_{\phi_{\mathfrak{s}}}'\le\psi_{\phi_{\mathfrak{s}}}$, $\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-i}}'\le\mathcal{J}_{\mathfrak{W}_{\mathrm{RP}-i}}$, $\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-i}}'\le\mathcal{J}_{\mathfrak{W}_{\mathrm{IP}-i}}$, $\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-i}}'\ge\mathcal{K}_{\mathfrak{W}_{\mathrm{RP}-i}}$, and $\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-i}}'\ge\mathcal{K}_{\mathfrak{W}_{\mathrm{IP}-i}}$, we have

$$\operatorname{CIULGHM}^{p_{\mathrm{SC}},q_{\mathrm{SC}}}\left(\mathfrak{W}_{\mathrm{CIU}-1}',\mathfrak{W}_{\mathrm{CIU}-2}',\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}'\right)\le\operatorname{CIULGHM}^{p_{\mathrm{SC}},q_{\mathrm{SC}}}\left(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}}\right). \tag{40}$$

(3) If $\mathfrak{W}_{\mathrm{CIU}-A}=\min(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}})$ and $\mathfrak{W}_{\mathrm{CIU}-B}=\max(\mathfrak{W}_{\mathrm{CIU}-1},\mathfrak{W}_{\mathrm{CIU}-2},\ldots,\mathfrak{W}_{\mathrm{CIU}-\widehat{\Xi}})$, then

$$\mathfrak{W}_{CIU-A} \leq CIULGHM^{p_{SC},q_{SC}}\left(\mathfrak{W}_{CIU-1},\mathfrak{W}_{CIU-2},\ldots,\mathfrak{W}_{CIU-\widehat{\Xi}}\right) \leq \mathfrak{W}_{CIU-B}. \tag{41}$$

Proof. Follows from the proof similar to Property 1. □

Definition 12. For the families of CIULNs, the CIULWGHM operator is mapping $CIULWGHM^{p_{SC},q_{SC}}: \Theta^{\widehat{\Xi}} \longrightarrow \Theta$stated by

$$CIULWGHM^{p_{SC},q_{SC}}\left(\mathfrak{W}_{CIU-1},\mathfrak{W}_{CIU-2},\ldots,\mathfrak{W}_{CIU-\widehat{\Xi}}\right) = \frac{1}{p_{SC}+q_{SC}}\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{\mathfrak{s}=1}^{\widehat{\Xi}}\left(\left(p_{SC}\mathfrak{W}_{CIU-i}\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-i}+\left(q_{SC}\mathfrak{W}_{CIU-\mathfrak{s}}\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-\mathfrak{s}}\right)\right)^{2/\widehat{\Xi}}\left(\widehat{\Xi}+1\right), \tag{42}$$

where $\widehat{\Omega}_W = \left\{\widehat{\Omega}_{W-1},\widehat{\Omega}_{W-2},\ldots,\widehat{\Omega}_{W-\widehat{\Xi}}\right\}$ expresses the weight vector with a condition that is $\sum_{i=1}^{\widehat{\Xi}}\widehat{\Omega}_{W-i} = 1, \widehat{\Omega}_{W-i} \in [0,1]$. By using Definition 12, we investigate the following result.

Theorem 5. *Based on any families of CIULNs* $\mathfrak{W}_{CIU-i} = \left([\psi_{\varphi_i},\psi_{\phi_{\mathfrak{s}}}],\ \left(\mathcal{J}_{\mathfrak{W}_{RP-i}}(\check{\mathfrak{k}})e^{i2\pi(\mathcal{J}_{\mathfrak{W}_{IP-i}}(\check{\mathfrak{k}}))},\mathcal{K}_{\mathfrak{W}_{RP-i}}(\check{\mathfrak{k}})e^{i2\pi(\mathcal{K}_{\mathfrak{W}_{IP-i}}(\check{\mathfrak{k}}))}\right)\right), i,\mathfrak{s} = 1,2,\ldots,\widehat{\Xi}$, *then by using Definitions 7 and 12, we obtain*

$$\begin{aligned}
&CIULGHM^{p_{SC},q_{SC}}\left(\mathfrak{W}_{CIU-1},\mathfrak{W}_{CIU-2},\ldots,\mathfrak{W}_{CIU-\widehat{\Xi}}\right)\\
&=\left(\begin{array}{c}
\left[\begin{array}{l}
\psi_{\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(1-(\varphi_i/h)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{p_{SC}}\left(1-\left(1-(\varphi_s/h)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}}(\widehat{\Xi}+1)\right)^{1/p_{SC}+q_{SC}}},\\
\psi_{\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(1-(\phi_i/h)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{p_{SC}}\left(1-\left(1-(\phi_s/h)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}}(\widehat{\Xi}+1)\right)^{1/p_{SC}+q_{SC}}}
\end{array}\right],\\
\left(\begin{array}{c}
\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{RP-i}}^{\widehat{\Xi}}\widehat{\Omega}_{W-i}(\check{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-i}\right)^{p_{SC}}\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{RP-S}}^{\widehat{\Xi}}\widehat{\Omega}_{W-S}(\check{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}}(\widehat{\Xi}+1)\right)^{1/p_{SC}+q_{SC}}\\
e^{i2\pi\left(1-\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{IP-i}}^{\widehat{\Xi}}\widehat{\Omega}_{W-i}(\check{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-i}\right)^{p_{SC}}\left(1-\left(1-\mathcal{J}_{\mathfrak{W}_{IP-S}}^{\widehat{\Xi}}\widehat{\Omega}_{W-S}(\check{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}}(\widehat{\Xi}+1)\right)^{1/p_{SC}+q_{SC}}}\\
\left(1-\left(\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{RP-i}}(\check{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{p_{SC}}\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{RP-S}}(\check{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}}(\widehat{\Xi}+1)\right)^{1/p_{SC}+q_{SC}}\right)\\
e^{i2\pi\left(1-\left(\left(\prod_{i=1}^{\widehat{\Xi}}\prod_{S=1}^{\widehat{\Xi}}\left(1-\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{IP-i}}(\check{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{p_{SC}}\left(1-\left(1-\mathcal{K}_{\mathfrak{W}_{IP-S}}(\check{\mathfrak{k}})\right)^{\widehat{\Xi}}\widehat{\Omega}_{W-S}\right)^{q_{SC}}\right)\right)^{2/\widehat{\Xi}}(\widehat{\Xi}+1)\right)^{1/p_{SC}+q_{SC}}}
\end{array}\right)
\end{array}\right).
\end{aligned} \tag{43}$$

Proof. Trivial. □

Theorem 6. *The CIULGHM operator is a certain brief case of CIULWGHM operator.*

Proof. Trivial. □

5. MADM Procedure Based on CIULSHM Operators

In genuine decision troubles, there occur the exchanges among the attributes. At the similar moment, due to the ambiguity of the attributes, they can be certainly shown by the CIULSs. So, by using the CIUL information (CIULI), it is essential to utilize various decision-making processes to sort out the exchanges among the characteristics.

In this analysis, we shall investigate a methodology to MAGDM procedure by using the CIULI by CIULWAHM operator or CIULWGHM operator. Reflect a MAGDM procedure by using the CIULI: let $\overline{\Phi}_{\mathrm{Al}} = \{\overline{\Phi}_{\mathrm{Al}-1}, \overline{\Phi}_{\mathrm{Al}-2}, \ldots, \overline{\Phi}_{\mathrm{Al}-\frown\Xi}\}$ be the family of alternatives and their attributes $\overline{\mathscr{L}}_{\mathrm{At}} = \left\{\overline{\mathscr{L}}_{\mathrm{At}-1}, \overline{\mathscr{L}}_{\mathrm{At}-2}, \ldots, \overline{\mathscr{L}}_{\mathrm{At}-\frown m}\right\}$. For this, we choose the weight vectors $\widehat{\Omega}_W = \left\{\widehat{\Omega}_{W-1}, \widehat{\Omega}_{W-2}, \ldots, \widehat{\Omega}_{W-\frown n}\right\}$ with a rule that is $\sum_{i=1}^{\frown n} \widehat{\Omega}_{W-i} = 1$. Moreover, we choose the family of decision makers such that $\overline{\mathscr{D}}_{Dm} = \left\{\overline{\mathscr{D}}_{Dm-1}, \overline{\mathscr{D}}_{Dm-2}, \ldots, \overline{\mathscr{D}}_{Dm-\frown\Xi}\right\}$, and $\widehat{\Omega}_W' = \left\{\widehat{\Omega}_{W-1}', \widehat{\Omega}_{W-2}', \ldots, \widehat{\Omega}_{W-\frown n}'\right\}$ with a rule that is $\sum_{i=1}^{\frown n} \widehat{\Omega}_{W-i}' = 1$ are expressing the weight vectors of decision makers. To resolve the above discussed issues, we choose the decision matrix $\breve{\mathscr{R}}^i, i = 1, 2, \ldots, \frown n$, whose every term in the form of CIULNs such that $\mathfrak{W}_{\mathrm{CIU}} = ([\psi_{\varphi_i}, \psi_{\phi_{\mathfrak{s}}}], (\mathscr{J}_{\mathfrak{W}_{\mathrm{CIU}}}(\breve{\mathfrak{k}}), \mathscr{K}_{\mathfrak{W}_{\mathrm{CIU}}}(\breve{\mathfrak{k}})))$, where $\mathscr{J}_{\mathfrak{W}_{\mathrm{CIU}}}(\breve{\mathfrak{k}}) = \mathscr{J}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}}) e^{i2\pi(\mathscr{J}_{\mathfrak{W}_{\mathrm{IP}}}(\breve{\mathfrak{k}}))}$ and $\mathscr{K}_{\mathfrak{W}_{\mathrm{CIU}}}(\breve{\mathfrak{k}}) = \mathscr{K}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}}) e^{i2\pi(\mathscr{K}_{\mathfrak{W}_{\mathrm{IP}}}(\breve{\mathfrak{k}}))}$ with the rules such that $0 \leq \mathscr{J}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}}) + \mathscr{K}_{\mathfrak{W}_{\mathrm{RP}}}(\breve{\mathfrak{k}}) \leq 1$ and $0 \leq \mathscr{J}_{\mathfrak{W}_{\mathrm{IP}}}(\breve{\mathfrak{k}}) + \mathscr{K}_{\mathfrak{W}_{\mathrm{IP}}}(\breve{\mathfrak{k}}) \leq 1$ with $\psi_{\varphi_i}, \psi_{\phi_{\mathfrak{s}}} \in \widehat{\psi}(i \leq \mathfrak{s})$. For resolving the aforementioned issues, we use the following MAGDM procedures:

Step 1: utilize the CIULWAHM operator to total the choice matrices which are given by decision makers with weighted vectors.

Step 2: utilize the CIULAHM operator, CIULWAHM operator, CIULGHM operator, and CIULWGHM operator to collect the choice matrices which are in Step 1.

Step 3: by using the score function, we analyze the score principles of the accumulated values in Step 2.

Step 4: rank all the options and discover the most excellent one.

Example 1. The MAGDM issue is cited from Ref. [41]. There is a speculation organization, which plans to pick the most ideal interest in some options. There are four potential alternatives for the speculation organization to browse: (1) a vehicle organization $\overline{\Phi}_{\mathrm{Al}-1}$; (2) a food organization $\overline{\Phi}_{\mathrm{Al}-2}$; (3) a PC organization $\overline{\Phi}_{\mathrm{Al}-3}$; and (4) a mobile organization $\overline{\Phi}_{\mathrm{Al}-4}$. The venture organization considers four criteria to settle on decisions: (1) the hazard investigation $\overline{\mathscr{L}}_{\mathrm{At}-1}$; (2) the development examination $\overline{\mathscr{L}}_{\mathrm{At}-2}$; (3) the natural impact investigation $\overline{\mathscr{L}}_{\mathrm{At}-3}$; and (4) social impact $\overline{\mathscr{L}}_{\mathrm{At}-4}$, where all criteria values are benefit type. The weight vector of the criteria is $\widehat{\Omega}_W' = (0.5, 0.4, 0.1)^T, \widehat{\Omega}_W = (0.4, 0.3, 0.2, 0.1)^T$. The four potential options are assessed regarding the four rules by the type of CIULNs, and complex intuitionistic uncertain linguistic decision matrices $\breve{\mathscr{R}}^i, i = 1, 2, 3$ are developed and listed in the form of Tables 1–3 , respectively.

For resolving the aforementioned issues, we use the following MAGDM procedures:

Step 1: by utilizing the CIULWAHM operator, we aggregated the decision matrices which are given by decision makers with weighted vectors. The aggregated decision matrix is discussed in the form of Table 4 for $p_{\mathrm{SC}}, q_{\mathrm{SC}} = 1$.

Step 2: utilize the CIULAHM operator, CIULWAHM operator, CIULGHM operator, and CIULWGHM operator to aggregate the decision matrices which are in Step 1, which are discussed in the form of Table 5 for $p_{\mathrm{SC}}, q_{\mathrm{SC}} = 0.4$.

Step 3: by using the score function, we compute their values which are listed in Table 6.

Step 4: rank all the options and invent the superlative one, which are discussed in the form of Table 7.

From the above analysis, we obtain different results by using the investigated operators such as CIULAHM operator, CIULWAHM operator, CIULGHM operator, and CIULWGHM operator. The best options are $\overline{\Phi}_{\mathrm{AI}-1}, \overline{\Phi}_{\mathrm{AI}-2}$, and $\overline{\Phi}_{\mathrm{AI}-4}$ by using different operators. The graphical interpretations of the information of Table 6 are discussed in the form of Figure 1.

5.1. Influence of Parameters.

To demonstrate the stability and validity of the investigated operators with the help of the parameters p_{SC} and q_{SC} are discussed by using the information of Example 1. The stability of the parameters by using the information of Example 1 is discussed in the form of Tables 8–10 by using the investigated CIULAHM, CIULWAHM, CIULGHM, and CIULWGHM operators with the help of parameter $q_{\mathrm{SC}} = 0.4$.

5.2. Comparative Analysis.

In addition, we want to enhance the excellence and quantity of the investigated operators centered on CIULSs with the help of comparative analysis between explored operators with certain prevailing operators to find the validity and capability of the investigated operators. The information related to existing ideas are discussed as follows: Heronian mean operators based on intuitionistic uncertain linguistic set (IULS) were developed by Liu et al. [41]. Liu and Liu [42] investigated the partitioned Bonferroni mean (PBM) operators for IULS. Moreover, Liu et al. [43] explored weighted Bonferroni ordered weighted averaging (WBOWA) operators for IULS. Liu et al. [44] utilized the idea of Hamy mean (HaM) operators for IULSs. The theory of Bonferroni mean (BM) operators for IULS was developed by Liu and Zhang [45].

TABLE 1: Decision matrix in terms of CIULNs provided by expert $\breve{\mathscr{R}}^{1}$.

Alternatives/attributes	$\overline{\mathscr{L}}_{\mathbf{At-1}}$	$\overline{\mathscr{L}}_{\mathbf{At-2}}$	$\overline{\mathscr{L}}_{\mathbf{At-3}}$	$\overline{\mathscr{L}}_{\mathbf{At-4}}$
$\overline{\Phi}_{\mathbf{Al-1}}$	$\left([\psi_5,\psi_5],\left(0.2e^{i2\pi(0.2)},0.7e^{i2\pi(0.7)}\right)\right)$	$\left([\psi_2,\psi_3],\left(0.4e^{i2\pi(0.4)},0.6e^{i2\pi(0.6)}\right)\right)$	$\left([\psi_5,\psi_6],\left(0.5e^{i2\pi(0.5)},0.5e^{i2\pi(0.5)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.2e^{i2\pi(0.2)},0.6e^{i2\pi(0.6)}\right)\right)$
$\overline{\Phi}_{\mathbf{Al-2}}$	$\left([\psi_4,\psi_5],\left(0.2e^{i2\pi(0.2)},0.6e^{i2\pi(0.6)}\right)\right)$	$\left([\psi_5,\psi_5],\left(0.4e^{i2\pi(0.4)},0.5e^{i2\pi(0.5)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.1e^{i2\pi(0.1)},0.8e^{i2\pi(0.8)}\right)\right)$	$\left([\psi_4,\psi_4],\left(0.5e^{i2\pi(0.5)},0.5e^{i2\pi(0.5)}\right)\right)$
$\overline{\Phi}_{\mathbf{Al-3}}$	$\left([\psi_3,\psi_4],\left(0.2e^{i2\pi(0.2)},0.7e^{i2\pi(0.7)}\right)\right)$	$\left([\psi_4,\psi_4],\left(0.2e^{i2\pi(0.2)},0.7e^{i2\pi(0.7)}\right)\right)$	$\left([\psi_4,\psi_5],\left(0.3e^{i2\pi(0.3)},0.7e^{i2\pi(0.7)}\right)\right)$	$\left([\psi_4,\psi_5],\left(0.2e^{i2\pi(0.2)},0.7e^{i2\pi(0.7)}\right)\right)$
$\overline{\Phi}_{\mathbf{Al-4}}$	$\left([\psi_6,\psi_6],\left(0.5e^{i2\pi(0.5)},0.4e^{i2\pi(0.4)}\right)\right)$	$\left([\psi_2,\psi_3],\left(0.2e^{i2\pi(0.2)},0.8e^{i2\pi(0.8)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.2e^{i2\pi(0.2)},0.6e^{i2\pi(0.6)}\right)\right)$	$\left([\psi_3,\psi_3],\left(0.3e^{i2\pi(0.3)},0.6e^{i2\pi(0.6)}\right)\right)$

TABLE 2: Decision matrix in terms of CIULNs provided by expert $\breve{\mathscr{R}}^{2}$.

Alternatives/attributes	$\overline{\mathscr{L}}_{\mathbf{At-1}}$	$\overline{\mathscr{L}}_{\mathbf{At-2}}$	$\overline{\mathscr{L}}_{\mathbf{At-3}}$	$\overline{\mathscr{L}}_{\mathbf{At-4}}$
$\overline{\Phi}_{\mathbf{Al-1}}$	$\left([\psi_5,\psi_5],\left(0.2e^{i2\pi(0.2)},0.5e^{i2\pi(0.5)}\right)\right)$	$\left([\psi_2,\psi_3],\left(0.4e^{i2\pi(0.4)},0.5e^{i2\pi(0.1)}\right)\right)$	$\left([\psi_5,\psi_6],\left(0.5e^{i2\pi(0.5)},0.3e^{i2\pi(0.2)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.2e^{i2\pi(0.2)},0.1e^{i2\pi(0.1)}\right)\right)$
$\overline{\Phi}_{\mathbf{Al-2}}$	$\left([\psi_4,\psi_5],\left(0.2e^{i2\pi(0.2)},0.5e^{i2\pi(0.3)}\right)\right)$	$\left([\psi_5,\psi_6],\left(0.5e^{i2\pi(0.5)},0.3e^{i2\pi(0.2)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.1e^{i2\pi(0.1)},0.6e^{i2\pi(0.5)}\right)\right)$	$\left([\psi_4,\psi_4],\left(0.5e^{i2\pi(0.5)},0.3e^{i2\pi(0.3)}\right)\right)$
$\overline{\Phi}_{\mathbf{Al-3}}$	$\left([\psi_3,\psi_4],\left(0.2e^{i2\pi(0.2)},0.4e^{i2\pi(0.7)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.1e^{i2\pi(0.1)},0.6e^{i2\pi(0.5)}\right)\right)$	$\left([\psi_4,\psi_5],\left(0.3e^{i2\pi(0.3)},0.4e^{i2\pi(0.7)}\right)\right)$	$\left([\psi_4,\psi_5],\left(0.2e^{i2\pi(0.2)},0.4e^{i2\pi(0.7)}\right)\right)$
$\overline{\Phi}_{\mathbf{Al-4}}$	$\left([\psi_6,\psi_6],\left(0.5e^{i2\pi(0.5)},0.4e^{i2\pi(0.4)}\right)\right)$	$\left([\psi_4,\psi_5],\left(0.3e^{i2\pi(0.3)},0.4e^{i2\pi(0.7)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.2e^{i2\pi(0.2)},0.3e^{i2\pi(0.6)}\right)\right)$	$\left([\psi_3,\psi_3],\left(0.3e^{i2\pi(0.3)},0.3e^{i2\pi(0.6)}\right)\right)$

TABLE 3: Decision matrix in terms of CIULNs provided by expert $\breve{\mathscr{R}}^{3}$.

Alternatives/attributes	$\overline{\mathscr{L}}_{\mathbf{At-1}}$	$\overline{\mathscr{L}}_{\mathbf{At-2}}$	$\overline{\mathscr{L}}_{\mathbf{At-3}}$	$\overline{\mathscr{L}}_{\mathbf{At-4}}$
$\overline{\Phi}_{\mathbf{Al-1}}$	$\left([\psi_5,\psi_5],\left(0.2e^{i2\pi(0.2)},0.7e^{i2\pi(0.7)}\right)\right)$	$\left([\psi_2,\psi_3],\left(0.4e^{i2\pi(0.4)},0.6e^{i2\pi(0.6)}\right)\right)$	$\left([\psi_5,\psi_6],\left(0.5e^{i2\pi(0.5)},0.5e^{i2\pi(0.5)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.2e^{i2\pi(0.2)},0.6e^{i2\pi(0.6)}\right)\right)$
$\overline{\Phi}_{\mathbf{Al-2}}$	$\left([\psi_5,\psi_5],\left(0.2e^{i2\pi(0.2)},0.5e^{i2\pi(0.5)}\right)\right)$	$\left([\psi_2,\psi_3],\left(0.4e^{i2\pi(0.4)},0.5e^{i2\pi(0.1)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.1e^{i2\pi(0.1)},0.8e^{i2\pi(0.8)}\right)\right)$	$\left([\psi_4,\psi_4],\left(0.5e^{i2\pi(0.5)},0.5e^{i2\pi(0.5)}\right)\right)$
$\overline{\Phi}_{\mathbf{Al-3}}$	$\left([\psi_4,\psi_5],\left(0.2e^{i2\pi(0.2)},0.5e^{i2\pi(0.3)}\right)\right)$	$\left([\psi_5,\psi_6],\left(0.5e^{i2\pi(0.5)},0.3e^{i2\pi(0.2)}\right)\right)$	$\left([\psi_2,\psi_3],\left(0.4e^{i2\pi(0.4)},0.6e^{i2\pi(0.6)}\right)\right)$	$\left([\psi_5,\psi_6],\left(0.5e^{i2\pi(0.5)},0.5e^{i2\pi(0.5)}\right)\right)$
$\overline{\Phi}_{\mathbf{Al-4}}$	$\left([\psi_3,\psi_4],\left(0.2e^{i2\pi(0.2)},0.4e^{i2\pi(0.7)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.1e^{i2\pi(0.1)},0.6e^{i2\pi(0.5)}\right)\right)$	$\left([\psi_2,\psi_3],\left(0.4e^{i2\pi(0.4)},0.5e^{i2\pi(0.1)}\right)\right)$	$\left([\psi_3,\psi_4],\left(0.1e^{i2\pi(0.1)},0.8e^{i2\pi(0.8)}\right)\right)$

Table 4: Aggregated decision matrix of the experts by CIULWAHM operator.

	$\overline{\mathscr{L}}_{\mathbf{At-1}}$	$\overline{\mathscr{L}}_{\mathbf{At-2}}$	$\overline{\mathscr{L}}_{\mathbf{At-3}}$	$\overline{\mathscr{L}}_{\mathbf{At-4}}$
$\overline{\Phi}_{\mathbf{Al-1}}$	$([\psi_{1.6489},\psi_{1.6489}],(0.6665e^{i2\pi(0.6665)},0.2189e^{i2\pi(0.2189)}))$	$([\psi_{0.6595},\psi_{0.9893}],(0.7947e^{i2\pi(0.7947)},0.1834e^{i2\pi(0.1393)}))$	$([\psi_{1.6489},\psi_{1.9786}],(0.8474e^{i2\pi(0.8474)},0.1188e^{i2\pi(0.1089)}))$	$([\psi_{0.9893},\psi_{1.3191}],(0.6665e^{i2\pi(0.6665)},0.1393e^{i2\pi(0.1393)}))$
$\overline{\Phi}_{\mathbf{Al-2}}$	$([\psi_{1.3693},\psi_{1.6489}],(0.6665e^{i2\pi(0.6665)},0.1769e^{i2\pi(0.141)}))$	$([\psi_{1.4874},\psi_{1.6737}],(0.8166e^{i2\pi(0.8166)},0.1188e^{i2\pi(0.0804)}))$	$([\psi_{0.9893},\psi_{1.3191}],(0.5891e^{i2\pi(0.5891)},0.2803e^{i2\pi(0.2596)}))$	$([\psi_{1.3191},\psi_{1.3191}],(0.8474e^{i2\pi(0.8474)},0.1188e^{i2\pi(0.1188)}))$
$\overline{\Phi}_{\mathbf{Al-3}}$	$([\psi_{1.0392},\psi_{1.3693}],(0.6665e^{i2\pi(0.6665)},0.1823e^{i2\pi(0.2493)}))$	$([\psi_{1.2247},\psi_{1.4177}],(0.6488e^{i2\pi(0.6488)},0.2217e^{i2\pi(0.1942)}))$	$([\psi_{1.2124},\psi_{1.5431}],(0.7409e^{i2\pi(0.7409)},0.191e^{i2\pi(0.2591)}))$	$([\psi_{1.3693},\psi_{1.6993}],(0.6824e^{i2\pi(0.6824)},0.1823e^{i2\pi(0.2544)}))$
$\overline{\Phi}_{\mathbf{Al-4}}$	$([\psi_{1.8187},\psi_{1.8735}],(0.8212e^{i2\pi(0.8212)},0.107e^{i2\pi(0.1313)}))$	$([\psi_{0.9447},\psi_{1.2845}],(0.6854e^{i2\pi(0.6854)},0.2186e^{i2\pi(0.2867)}))$	$([\psi_{0.9367},\psi_{1.2669}],(0.6786e^{i2\pi(0.6786)},0.141e^{i2\pi(0.1916)}))$	$([\psi_{0.9893},\psi_{1.0392}],(0.7167e^{i2\pi(0.7167)},0.1765e^{i2\pi(0.2221)}))$

Table 5: Aggregated values of the alternatives by CIULAHM, CIULWAHM, CIULGHM, and CIULWGHM operators.

	CIULAHM	**CIULWAHM**	**CIULGHM**	**CIULWGHM**
$\overline{\Phi}_{\mathbf{Al-1}}$	$([\psi_{1.2043},\psi_{1.5056}],(0.745e^{i2\pi(0.745)},0.0808e^{i2\pi(0.0723)}))$	$([\psi_{0.9328},\psi_{1.1553}],(0.9873e^{i2\pi(0.9873)},0.0014e^{i2\pi(0.001)}))$	$([\psi_{1.147},\psi_{1.4558}],(0.6082e^{i2\pi(0.6082)},0.1622e^{i2\pi(0.1482)}))$	$([\psi_{1.0985},\psi_{1.246}],(0.1072e^{i2\pi(0.1072)},0.7938e^{i2\pi(0.7828)}))$
$\overline{\Phi}_{\mathbf{Al-2}}$	$([\psi_{1.2632},\psi_{1.4188}],(0.732e^{i2\pi(0.732)},0.0852e^{i2\pi(0.0699)}))$	$([\psi_{0.9789},\psi_{1.1316}],(0.9853e^{i2\pi(0.9853)},0.0015e^{i2\pi(0.0009)}))$	$([\psi_{1.2462},\psi_{1.4112}],(0.5974e^{i2\pi(0.5974)},0.1663e^{i2\pi(0.1409)}))$	$([\psi_{1.1493},\psi_{1.2644}],(0.0959e^{i2\pi(0.0959)},0.7799e^{i2\pi(0.7576)}))$
$\overline{\Phi}_{\mathbf{Al-3}}$	$([\psi_{1.2662},\psi_{1.5712}],(0.6847e^{i2\pi(0.6847)},0.1001e^{i2\pi(0.1307)}))$	$([\psi_{0.9406},\psi_{1.1606}],(0.9786e^{i2\pi(0.9786)},0.0019e^{i2\pi(0.0033)}))$	$([\psi_{1.2638},\psi_{1.5664}],(0.5324e^{i2\pi(0.5324)},0.1939e^{i2\pi(0.2383)}))$	$([\psi_{1.1212},\psi_{1.2707}],(0.0796e^{i2\pi(0.0796)},0.8017e^{i2\pi(0.8218)}))$
$\overline{\Phi}_{\mathbf{Al-4}}$	$([\psi_{1.0178},\psi_{1.223}],(0.7258e^{i2\pi(0.7258)},0.078e^{i2\pi(0.1085)}))$	$([\psi_{0.7861},\psi_{0.9838}],(0.9877e^{i2\pi(0.9877)},0.001e^{i2\pi(0.0022)}))$	$([\psi_{1.0313},\psi_{1.2197}],(0.582e^{i2\pi(0.582)},0.1572e^{i2\pi(0.203)}))$	$([\psi_{1.0612},\psi_{1.1872}],(0.1009e^{i2\pi(0.1009)},0.7749e^{i2\pi(0.8036)}))$

Table 6: Score values of each alternative from the aggregated values by different operators.

	CIULAHM	CIULWAHM	CIULGHM	CIULWGHM
$\overline{\Phi}_{\mathrm{Al-1}}$	0.1971	1.0359	1.0944	−0.896
$\overline{\Phi}_{\mathrm{Al-2}}$	0.1797	1.045	1.0732	−0.934
$\overline{\Phi}_{\mathrm{Al-3}}$	−0.0260	1.0353	1.0701	−0.9700
$\overline{\Phi}_{\mathrm{Al-4}}$	0.0295	0.8747	0.8849	−0.888

Table 7: Ranking values of the alternatives based on score values by different operators.

Methods	Ranking values
CIULAHM operator	$\overline{\Phi}_{\mathrm{Al-1}} \geq \overline{\Phi}_{\mathrm{Al-2}} \geq \overline{\Phi}_{\mathrm{Al-4}} \geq \overline{\Phi}_{\mathrm{Al-3}}$
CIULWAHM operator	$\overline{\Phi}_{\mathrm{Al-2}} \geq \overline{\Phi}_{\mathrm{Al-1}} \geq \overline{\Phi}_{\mathrm{Al-3}} \geq \overline{\Phi}_{\mathrm{Al-4}}$
CIULGHM operator	$\overline{\Phi}_{\mathrm{Al-1}} \geq \overline{\Phi}_{\mathrm{Al-2}} \geq \overline{\Phi}_{\mathrm{Al-3}} \geq \overline{\Phi}_{\mathrm{Al-4}}$
CIULWGHM operator	$\overline{\Phi}_{\mathrm{Al-4}} \geq \overline{\Phi}_{\mathrm{Al-1}} \geq \overline{\Phi}_{\mathrm{Al-2}} \geq \overline{\Phi}_{\mathrm{Al-3}}$

The comparative analyses of the investigated operators with certain remaining operators are discussed in the form of Table 10, by using the information of Example 1.

The graphical interpretations of the information of Table 10 are discussed in the form of Figure 2.

From the obtained results, we acquire the effect; if we choose the complex intuitionistic uncertain linguistic type of information, then the existing operators centered on IULVs are not able to cope with it. But, if we prefer the intuitionistic uncertain linguistic type of knowledge, then the proposed operators centered on IUL variables can cope with it. For this, we choose the intuitionistic uncertain linguistic type of knowledge and resolve it by utilizing scrutinized and accessible operators to discover the consistency and efficiency of the offered approaches.

Example 2. The information of this example is taken from Ref. [41]. There is a speculation organization, which plans to pick the most ideal interest in some options. There are four potential alternatives for the speculation organization to browse: (1) a vehicle organization $\overline{\Phi}_{\mathrm{Al-1}}$; (2) a food organization $\overline{\Phi}_{\mathrm{Al-2}}$; (3) a PC organization $\overline{\Phi}_{\mathrm{Al-3}}$; and (4) a mobile organization $\overline{\Phi}_{\mathrm{Al-4}}$. The venture organization considers four

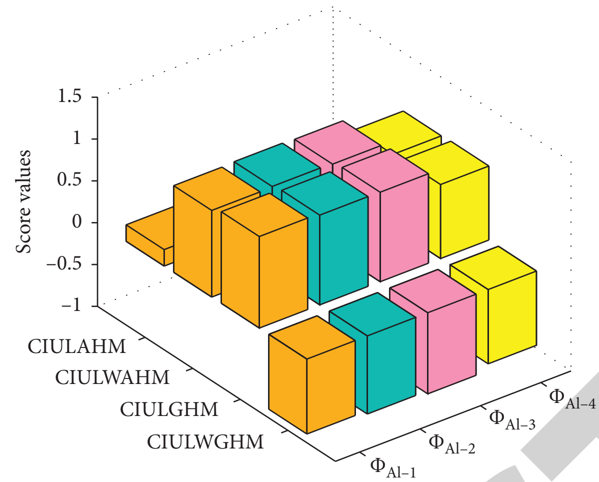

FIGURE 1: Geometrical expressions of the information given in Table 6.

TABLE 8: Influence of the parameters p_{SC} on the ranking of the alternatives with $q_{SC} = 0.4$.

p_{SC}	Operators	Score values	Ranking values
0.1	AHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.8313, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.8962, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.8114, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.4714$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WAHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.3543, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.4224, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.427, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.3226$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	GHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.1441, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.1348, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.034, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.0354$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WGHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = -0.233, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = -0.295, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.331, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = -0.254$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
0.2	AHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.4547, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.5372, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.4055, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.3124$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WAHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.4426, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.5227, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.4544, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.4137$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	GHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.3201, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.3139, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.1428, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.2552$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WGHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = -0.21, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = -0.251, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.248, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = -0.235$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
0.5	AHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.4353, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.5036, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.1744, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 1.5009$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WAHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.1718, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.2607, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.1246, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.1527$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	GHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.211, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.233,$ $\underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.0433, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.121$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WGHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = -0.314, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = -0.352, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.335, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = -0.303$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
1	AHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.0441, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.1156, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.4113, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.0221$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WAHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.4053, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.4044, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.3418, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.4539$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	GHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.5304, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 1.0322, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.3434, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.4446$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WGHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = -0.312, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = -0.344, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.323, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = -0.413$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
2	AHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.1345, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.1343, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.041, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.1431$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WAHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 1.0332, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 1.1143,$ $\underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.4513, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 1.2151$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	GHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 1.0352, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 1.2243, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.5315, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 1.0533$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WGHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = -0.341, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = -0.442 \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.342, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = -0.404$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$

TABLE 9: Influence of the parameter q_{SC} on the ranking of the alternatives with $p_{SC} = 0.4$.

q_{SC}	Operators	Score values	Ranking values
0.1	AHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.535, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 1.0244,$ $\underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.432, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 1.0034$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WAHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.4201, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.4541, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.3452, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.4557$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	GHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.1039, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.1034, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.042, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.0533$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WGHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = -0.333, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = -0.321, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.345, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = -0.332$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
0.2	AHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.3007, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.5451, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.4005, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.4342$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WAHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.5032, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.5695, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.4581, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.5416$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	GHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = 0.4981, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = 0.5053, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = 0.1559, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = 0.4139$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	WGHM	$\underline{\overline{\zeta}}(\overline{\Phi}_{Al-1}) = -0.441, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-2}) = -0.506, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-3}) = -0.463, \underline{\overline{\zeta}}(\overline{\Phi}_{Al-4}) = -0.529$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$

Table 9: Continued.

q_{SC}	Operators	Score values	Ranking values
0.5	AHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 0.5501, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 0.6017,$ $\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = 0.2114, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.213$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
	WAHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 0.1353, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 0.2332, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = 0.1344, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.1244$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
	GHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 0.3308, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 0.3387, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = 0.0881, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.1706$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
	WGHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = -0.372, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = -0.436, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = -0.418, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = -0.434$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
1	AHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 0.0311 \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 0.0204,$ $\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = 0.6622, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.6731$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
	WAHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 0.5365, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 0.6144, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = 0.5654, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.4057$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
	GHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 0.8633, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 0.8637, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = 0.6737, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.5025$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
	WGHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = -0.316, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = -0.380, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = -0.388, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = -0.351$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
2	AHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 0.2822, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 0.2485,$ $\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = -0.042, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.049$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
	WAHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 1.0817, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 1.08,$ $\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = 1.0606, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.6393$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
	GHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 1.0444, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 1.0233,$ $\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = 1.0203, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.4354$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$
	WGHM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = -0.379, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = -0.442, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = -0.475, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = -0.386$	$\overline{\Phi}_{\mathbf{Al-2}} \geq \overline{\Phi}_{\mathbf{Al-1}} \geq \overline{\Phi}_{\mathbf{Al-4}} \geq \overline{\Phi}_{\mathbf{Al-3}}$

Table 10: Comparative analysis of the proposed and existing operators for Example 1.

Methods	Operators	Score values	Ranking values
Liu et al. [41]	HM	Failed	Failed
Liu and Liu [42]	PBM	Failed	Failed
Liu et al. [43]	WBOWA	Failed	Failed
Liu et al. [44]	HaM	Failed	Failed
Liu and Zhang [45]	BM	Failed	Failed
Proposed operators	CIULAHM operator	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 0.1971, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 0.1797, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = -0.026, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.0295$	$\overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
	CIULWAHM operator	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 1.0359, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 1.045, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = 1.0353, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.8747$	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-3} \geq \overline{\Phi}_{Al-4}$
	CIULGHM operator	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = 1.0944, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = 1.0732, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = 1.0701, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = 0.8849$	$\overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-3} \geq \overline{\Phi}_{Al-4}$
	CIULWGHM operator	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-1}}) = -0.896, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-2}}) = -0.934, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-3}}) = -0.97, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{Al-4}}) = -0.888$	$\overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-3}$

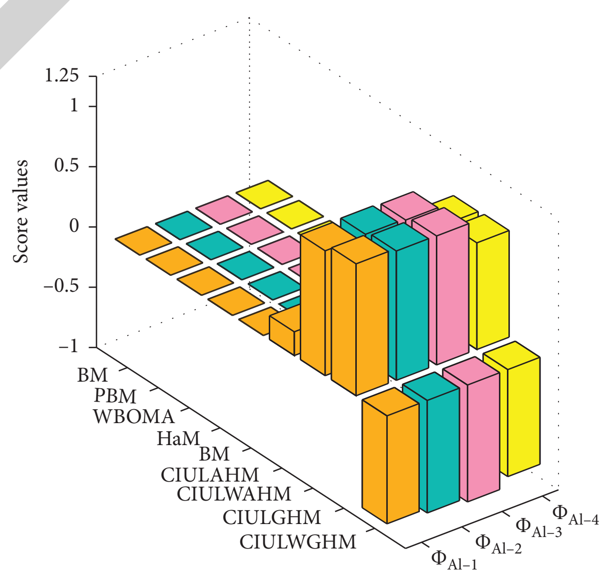

Figure 2: Geometrical interpretation of the information given in Table 10.

Table 11: Decision matrix provided by expert $\breve{\mathscr{R}}^1$ in terms of CIULNs.

Alternatives/attributes	$\overline{\mathscr{L}}_{\mathbf{At}-1}$	$\overline{\mathscr{L}}_{\mathbf{At}-2}$	$\overline{\mathscr{L}}_{\mathbf{At}-3}$	$\overline{\mathscr{L}}_{\mathbf{At}-4}$
$\overline{\Phi}_{\mathbf{Al}-1}$	$\begin{pmatrix}[\psi_5,\psi_5],\\(0.2,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_2,\psi_3],\\(0.4,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_5,\psi_6],\\(0.5,0.5)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.2,0.6)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-2}$	$\begin{pmatrix}[\psi_4,\psi_5],\\(0.4,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_5,\psi_5],\\(0.4,0.5)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.1,0.8)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_4],\\(0.5,0.5)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-3}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.2,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_4],\\(0.2,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_5],\\(0.3,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_5],\\(0.2,0.7)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-4}$	$\begin{pmatrix}[\psi_6,\psi_6],\\(0.5,0.4)\end{pmatrix}$	$\begin{pmatrix}[\psi_2,\psi_3],\\(0.2,0.8)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.2,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_3],\\(0.3,0.6)\end{pmatrix}$

Table 12: Decision matrix provided by expert $\breve{\mathscr{R}}^2$ in terms of CIULNs.

Alternatives/attributes	$\overline{\mathscr{L}}_{\mathbf{At}-1}$	$\overline{\mathscr{L}}_{\mathbf{At}-2}$	$\overline{\mathscr{L}}_{\mathbf{At}-3}$	$\overline{\mathscr{L}}_{\mathbf{At}-4}$
$\overline{\Phi}_{\mathbf{Al}-1}$	$\begin{pmatrix}[\psi_4,\psi_4],\\(0.1,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.2,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.2,0.8)\end{pmatrix}$	$\begin{pmatrix}[\psi_6,\psi_6],\\(0.4,0.5)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-2}$	$\begin{pmatrix}[\psi_5,\psi_6],\\(0.4,0.5)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.3,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_5],\\(0.2,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.2,0.7)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-3}$	$\begin{pmatrix}[\psi_4,\psi_5],\\(0.2,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_4],\\(0.2,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_2,\psi_3],\\(0.4,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.3,0.7)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-4}$	$\begin{pmatrix}[\psi_5,\psi_5],\\(0.3,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_5],\\(0.4,0.5)\end{pmatrix}$	$\begin{pmatrix}[\psi_2,\psi_3],\\(0.3,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_4],\\(0.2,0.6)\end{pmatrix}$

Table 13: Decision matrix provided by expert $\breve{\mathscr{R}}^3$ in terms of CIULNs.

Alternatives/attributes	$\overline{\mathscr{L}}_{\mathbf{At}-1}$	$\overline{\mathscr{L}}_{\mathbf{At}-2}$	$\overline{\mathscr{L}}_{\mathbf{At}-3}$	$\overline{\mathscr{L}}_{\mathbf{At}-4}$
$\overline{\Phi}_{\mathbf{Al}-1}$	$\begin{pmatrix}[\psi_5,\psi_5],\\(0.2,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.3,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_5],\\(0.4,0.5)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_4],\\(0.2,0.7)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-2}$	$\begin{pmatrix}[\psi_4,\psi_5],\\(0.3,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_5,\psi_5],\\(0.3,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_2,\psi_3],\\(0.1,0.8)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.4,0.6)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-3}$	$\begin{pmatrix}[\psi_4,\psi_4],\\(0.2,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_5,\psi_5],\\(0.3,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_1,\psi_3],\\(0.1,0.8)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_4],\\(0.2,0.7)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-4}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.2,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_3,\psi_4],\\(0.1,0.7)\end{pmatrix}$	$\begin{pmatrix}[\psi_4,\psi_5],\\(0.3,0.6)\end{pmatrix}$	$\begin{pmatrix}[\psi_5,\psi_5],\\(0.4,0.5)\end{pmatrix}$

Table 14: Aggregated decision matrix of the expert by CIULWAHM operator.

	$\overline{\mathscr{L}}_{\mathbf{At}-1}$	$\overline{\mathscr{L}}_{\mathbf{At}-2}$	$\overline{\mathscr{L}}_{\mathbf{At}-3}$	$\overline{\mathscr{L}}_{\mathbf{At}-4}$
$\overline{\Phi}_{\mathbf{Al}-1}$	$\begin{pmatrix}[\psi_{1.6377},\psi_{1.6377}],\\(0.6394,0.2512)\end{pmatrix}$	$\begin{pmatrix}[\psi_{0.9178},\psi_{1.2718}],\\(0.7743,0.2101)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.4011},\psi_{1.7561}],\\(0.6645,0.249)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.4746},\psi_{1.6219}],\\(0.7514,0.1921)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-2}$	$\begin{pmatrix}[\psi_{1.5179},\psi_{1.8703}],\\(0.7318,0.2452)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.508},\psi_{1.6377}],\\(0.7548,0.1879)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.0473},\psi_{1.4029}],\\(0.6832,0.2512)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.1758},\psi_{1.4063}],\\(0.6785,0.2613)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-3}$	$\begin{pmatrix}[\psi_{1.2718},\psi_{1.5179}],\\(0.7682,0.2164)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.51},\psi_{1.51}],\\(0.6125,0.2971)\end{pmatrix}$	$\begin{pmatrix}[\psi_{0.7869},\psi_{1.2855}],\\(0.7058,0.2701)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.285},\psi_{1.5289}],\\(0.7056,0.2076)\end{pmatrix}$
$\overline{\Phi}_{\mathbf{Al}-4}$	$\begin{pmatrix}[\psi_{1.6432},\psi_{1.7664}],\\(0.7029,0.2101)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.0194},\psi_{1.377}],\\(0.7682,0.2084)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.03},\psi_{1.3856}],\\(0.6857,0.2678)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.3704},\psi_{1.3704}],\\(0.7254,0.193)\end{pmatrix}$

Table 15: Aggregated values of the alternatives by CIULAHM, CIULWAHM, CIULGHM, and CIULWGHM operators.

	CIULAHM	CIULWAHM	CIULGHM	CIULWGHM
$\overline{\Phi}_{\mathrm{Al}-1}$	$\begin{pmatrix}[\psi_{1.349},\psi_{1.5684}],\\(0.7112,0.6388)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.0074},\psi_{1.1614}],\\(0.9854,0.0032)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.3489},\psi_{1.5684}],\\(0.9575,0.0305)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.2581},\psi_{1.3935}],\\(0.0893,0.8493)\end{pmatrix}$
$\overline{\Phi}_{\mathrm{Al}-2}$	$\begin{pmatrix}[\psi_{1.3068},\psi_{1.5753}],\\(0.729,0.6321)\end{pmatrix}$	$\begin{pmatrix}[\psi_{0.9926},\psi_{1.1902}],\\(0.9898,0.0029)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.3068},\psi_{1.5753}],\\(0.9628,0.0247)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.2501},\psi_{1.4363}],\\(0.1025,0.838)\end{pmatrix}$
$\overline{\Phi}_{\mathrm{Al}-3}$	$\begin{pmatrix}[\psi_{1.2038},\psi_{1.4594}],\\(0.6848,0.672)\end{pmatrix}$	$\begin{pmatrix}[\psi_{0.9192},\psi_{1.0962}],\\(0.9833,0.0044)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.2038},\psi_{1.4594}],\\(0.9491,0.2595)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.1739},\psi_{1.3417}],\\(0.0807,0.859)\end{pmatrix}$
$\overline{\Phi}_{\mathrm{Al}-4}$	$\begin{pmatrix}[\psi_{1.2568},\psi_{1.4716}],\\(0.7154,0.6171)\end{pmatrix}$	$\begin{pmatrix}[\psi_{0.9543},\psi_{1.1079}],\\(0.9875,0.0025)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.2567},\psi_{1.4716}],\\(0.9587,0.213)\end{pmatrix}$	$\begin{pmatrix}[\psi_{1.216},\psi_{1.3588}],\\(0.0934,0.8385)\end{pmatrix}$

Table 16: Score values of the given alternatives.

	CIULAHM	CIULWAHM	CIULGHM	CIULWGHM
$\overline{\Phi}_{\mathrm{Al}-1}$	0.0528	0.5325	0.5302	−0.5038
$\overline{\Phi}_{\mathrm{Al}-2}$	0.0698	0.5386	0.5318	−0.494
$\overline{\Phi}_{\mathrm{Al}-3}$	0.0085	0.4932	0.4591	−0.4895
$\overline{\Phi}_{\mathrm{Al}-4}$	0.067	0.5078	0.5087	−0.4796

criteria to settle on decisions: (1) the hazard investigation $\overline{\mathscr{L}}_{\mathrm{At}-1}$; (2) the development examination $\overline{\mathscr{L}}_{\mathrm{At}-2}$; (3) the natural impact investigation $\overline{\mathscr{L}}_{\mathrm{At}-3}$; and (4) social impact $\overline{\mathscr{L}}_{\mathrm{At}-4}$, where all criteria values are benefit type. The weight vector of the criteria is $\widehat{\Omega}_W' = (0.4, 0.32, 0.28)^T$, $\widehat{\Omega}_W = (0.32, 0.26, 0.18, 0.24)^T$. The four potential options are assessed regarding the four rules by the type of CIULNs, and complex intuitionistic uncertain linguistic decision matrices $\mathscr{R}^i, i = 1, 2, 3$ are developed and listed in the form of Tables 11–13 .

For resolving the aforementioned issues, we use the following MAGDM procedures:

Step 1: by utilizing the CIULWAHM operator, we aggregated the decision matrices which are given by decision makers with weighted vectors. The aggregated decision matrix is discussed in the form of Table 14 for $p_{\mathrm{SC}}, q_{\mathrm{SC}} = 1$.

Step 2: utilize the CIULAHM operator, CIULWAHM operator, CIULGHM operator, and CIULWGHM operator to aggregate the decision matrices which are in Step 1, which are discussed in the form of Table 15 for $p_{\mathrm{SC}}, q_{\mathrm{SC}} = 1$.

Step 3: the score values of the given alternatives are computed and results are listed in Table 16.

Step 4: rank all the alternatives and find the best one, which are discussed in the form of Table 17.

From the above analysis, we obtain different results by using the investigated operators such as CIULAHM operator, CIULWAHM operator, CIULGHM operator, and CIULWGHM operator. The best options are $\overline{\Phi}_{\mathrm{Al}-2}$ and $\overline{\Phi}_{\mathrm{Al}-4}$. The graphical interpretations of the information of Table 16 are discussed in the form of Figure 3.

The comparative analysis of the investigated operators with some existing operators is discussed in the form of Table 18 by using the information of Example 2.

Table 17: Ordering of the given alternatives.

Methods	Ranking values
CIULAHM operator	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-3}$
CIULWAHM operator	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
CIULGHM operator	$\overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1} \geq \overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3}$
CIULWGHM operator	$\overline{\Phi}_{Al-4} \geq \overline{\Phi}_{Al-3} \geq \overline{\Phi}_{Al-2} \geq \overline{\Phi}_{Al-1}$

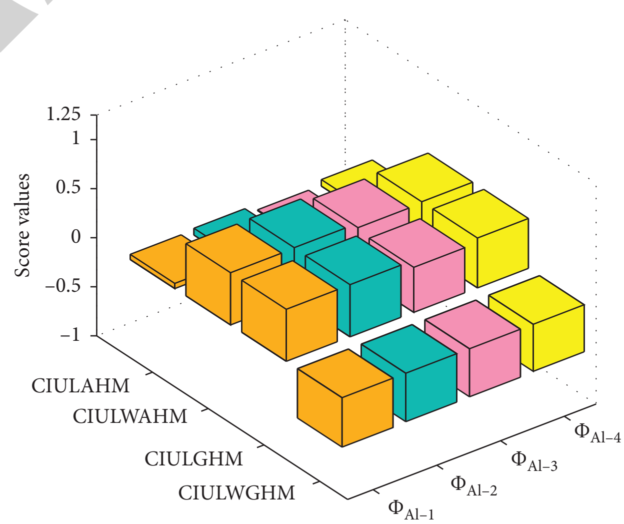

Figure 3: Geometrical interpretation of the information given in Table 16.

From this, we acquire the result; if we choose the complex intuitionistic uncertain linguistic type of knowledge, then the existing operators grounded on IULVs are not able to cope with it. But, if we choose the intuitionistic uncertain linguistic type of information, then the proposed operators based on CIUL variables can cope with it. Therefore, the proposed operators are extensively powerful and more reliable than the existing ideas [41–45]. The

Table 18: Comparative analysis of the proposed and existing operators for Example 2.

Methods	Operators	Score values	Ranking values
Liu et al. [41]	HM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-1}}) = 0.0033, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-2}}) = 0.0304, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-3}}) = 0.0141, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-4}}) = 0.0231$	$\overline{\Phi}_{AI-2} \geq \overline{\Phi}_{AI-4} \geq \overline{\Phi}_{AI-1} \geq \overline{\Phi}_{AI-3}$
Liu and Liu [42]	PBM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-1}}) = 0.0133, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-2}}) = 0.0354, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-3}}) = 0.0241, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-4}}) = 0.0321$	$\overline{\Phi}_{AI-2} \geq \overline{\Phi}_{AI-4} \geq \overline{\Phi}_{AI-1} \geq \overline{\Phi}_{AI-3}$
Liu et al. [43]	WBOWA	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-1}}) = 0.0023, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-2}}) = 0.0244, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-3}}) = 0.0031, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-4}}) = 0.0221$	$\overline{\Phi}_{AI-2} \geq \overline{\Phi}_{AI-4} \geq \overline{\Phi}_{AI-1} \geq \overline{\Phi}_{AI-3}$
Liu et al. [44]	HaM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-1}}) = 0.0036, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-2}}) = 0.0277, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-3}}) = 0.0041, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-4}}) = 0.0211$	$\overline{\Phi}_{AI-2} \geq \overline{\Phi}_{AI-4} \geq \overline{\Phi}_{AI-1} \geq \overline{\Phi}_{AI-3}$
Liu and Zhang [45]	BM	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-1}}) = 0.1016, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-2}}) = 0.1357 \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-3}}) = 0.0131, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-4}}) = 0.1301$	$\overline{\Phi}_{AI-2} \geq \overline{\Phi}_{AI-4} \geq \overline{\Phi}_{AI-1} \geq \overline{\Phi}_{AI-3}$
Proposed operators	CIULAHM operator	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-1}}) = 0.0528, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-2}}) = 0.0698, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-3}}) = 0.0085, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-4}}) = 0.067$	$\overline{\Phi}_{AI-2} \geq \overline{\Phi}_{AI-4} \geq \overline{\Phi}_{AI-1} \geq \overline{\Phi}_{AI-3}$
	CIULWAHM operator	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-1}}) = 0.5325, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-2}}) = 0.5386, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-3}}) = 0.4932, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-4}}) = 0.5078$	$\overline{\Phi}_{AI-2} \geq \overline{\Phi}_{AI-1} \geq \overline{\Phi}_{AI-4} \geq \overline{\Phi}_{AI-3}$
	CIULGHM operator	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-1}}) = 0.5302, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-2}}) = 0.5318, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-3}}) = 0.4591, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-4}}) = 0.5087$	$\overline{\Phi}_{AI-2} \geq \overline{\Phi}_{AI-1} \geq \overline{\Phi}_{AI-4} \geq \overline{\Phi}_{AI-3}$
	CIULWGHM operator	$\overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-1}}) = -0.5038, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-2}}) = -0.494, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-3}}) = -0.4895, \overline{\overline{\zeta}}(\overline{\Phi}_{\mathbf{AI-4}}) = -0.4796$	$\overline{\Phi}_{AI-4} \geq \overline{\Phi}_{AI-3} \geq \overline{\Phi}_{AI-2} \geq \overline{\Phi}_{AI-1}$

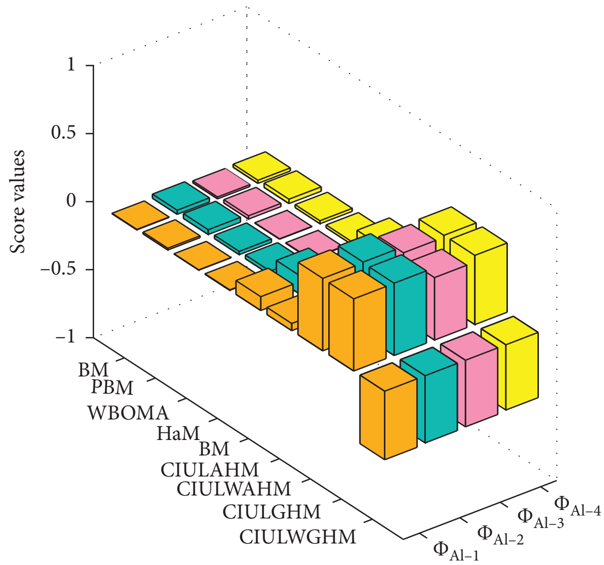

Figure 4: Geometrical interpretation of the information shown in Table 18.

graphical interpretations of the information of Table 18 are discussed in the form of Figure 4.

6. Conclusion

The idea of CIULS is developed, and their fundamental laws are discussed. CIULS covers the uncertain linguistic terms; the degree of truth and the degree of falsity are in the form of complex number, whose sum of the real parts (Imaginary parts) is restarted to unit interval. In addition, to analyze the interrelation between any numbers of CIULS, we use the concept of CIULS and HM operators being formed by CIULAHM operator, CIULWAHM operator, CIULGHM operator, and CIULWGHM operator. The major advantages of utilizing the HM operator in the given pairs of CIULNs are that it can interact the different pairs of the argument at the same time. Also, the stated operators have well handled the pairs of the linguistic values along with their membership degrees. Certain higher accidents and the characteristics of the operators under investigation are often illustrated by the use of parameters. In comparison, the MAGDM procedure is built through the use of CIULS-based explored operators. A number of numerical representations are demonstrated with the aid of the methods examined. In order to discover the continuity and experience of the operator's generated, comparative analysis and graphic expressions are often used to show the predominance of residential approaches. Based on the different pairs of the stated operators and their associated parameters, a decision maker can select their required task as per their preferences. Also, they can analyze their decision impact on the optimal alternatives by varying the parameters used in the decision-making process. Therefore, the suggested decision-making approach is beneficial for an expert to handle the decision-making problem in an uncertain and vague environment. Future work can focus on extending the proposed approach in different fuzzy environments to solve the problems related to decision making, medical diagnosis, pattern recognition, and so on [46–50].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Evidence Theory in Picture Fuzzy Set Environment

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Evidence Theory in Picture Fuzzy Set Environment

Harish Garg,[1] **R. Sujatha**,[2] **D. Nagarajan**,[3] **J. Kavikumar**,[4] **and Jeonghwan Gwak**[5,6,7,8]

[1]*School of Mathematics, Thapar Institute of Engineering and Technology, Deemed University, Patiala, Punjab, India*
[2]*Department of Mathematics, SSN College of Engineering, Chennai, India*
[3]*Department of Mathematics, Hindustan Institute of Technology & Science, Chennai, India*
[4]*Fuzzy Mathematics & Applications, Faculty of Applied Sciences & Technology, Universiti Tun Hussein Onn Malaysia, Johor, Malaysia*
[5]*Department of Software, Korea National University of Transportation, Chungju 27469, Republic of Korea*
[6]*Department of Biomedical Engineering, Korea National University of Transportation, Chungju 27469, Republic of Korea*
[7]*Department of AI Robotics Engineering, Korea National University of Transportation, Chungju 27469, Republic of Korea*
[8]*Department of IT & Energy Convergence (BK21 FOUR), Korea National University of Transportation, Chungju 27469, Republic of Korea*

Correspondence should be addressed to Jeonghwan Gwak; james.han.gwak@gmail.com

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Picture fuzzy set is the most widely used tool to handle the uncertainty with the account of three membership degrees, namely, positive, negative, and neutral such that their sum is bound up to 1. It is the generalization of the existing intuitionistic fuzzy and fuzzy sets. This paper studies the interval probability problems of the picture fuzzy sets and their belief structure. The belief function is a vital tool to represent the uncertain information in a more effective manner. On the other hand, the Dempster–Shafer theory (DST) is used to combine the independent sources of evidence with the low conflict. Keeping the advantages of these, in the present paper, we present the concept of the evidence theory for the picture fuzzy set environment using DST. Under this, we define the concept of interval probability distribution and discuss its properties. Finally, an illustrative example related to the decision-making process is employed to illustrate the application of the presented work.

1. Introduction

Decision making is based on experts opinion, and often experts have to take decisions based on limited data or knowledge. Thus, in any decision-making process, two types of uncertainties arise. Epistemic uncertainties occur due to lack of knowledge, insufficient data, and ambiguity, whereas aleatory uncertainty is due to the randomness of the physical system under study [1–6]. Probability theory is proposed to deal with randomness and is not effective in dealing with epistemic uncertainties. Theory based on evidence to handle uncertainty is Dempster–Shefer theory [7, 8]. It has vast applications [9]. In a finite discrete space, Dempster–Shafer theory can be interpreted as a generalization of probability theory where probabilities are assigned to sets as opposed to mutually exclusive singletons. In traditional probability theory, evidence is associated with only one possible event. In Dempster–Shafer theory, evidence can be associated with multiple possible events, in contrast to one single event. DS theory was extended to fuzzy sets by Zadeh [10, 11]. DS theory in fuzzy sets is presented in [12–16]. DS theory in intuitionistic domain was introduced by Grzegorzewski and Mrowka in [17]. Representation theorem was proved by Riecan [18, 19]. Gerstenkorn and Manko defined intuitionistic probability in two ways [20]. Also, Feng et al. [21] used intuitionistic fuzzy lower and upper approximation operators. Fuzzy clustering based on DS theory was applied for breast cancer cell detection in [22]. Pavement condition distress index was calculated using DS theory in [23], and the combination of quantum theory and DS theory has been investigated in [24]. Belief degrees and belief structures are required for Dempster–Shafer theory.

Cong and Kreinovich [25] introduced picture fuzzy sets and developed some operations and relations on them. Temporal picture fuzzy soft set and related concepts were developed in [26]. Picture fuzzy geometric operators were proposed, and using it, the multiple attribute decision-making problem was addressed in [27]. P-order and R-order union and the intersection of internal (external) cubic picture fuzzy sets were discussed in [28]. In the interval-valued picture fuzzy domain, the similarity measures are developed in [29]. Einstein information-based aggregation operators applied in group decision-making problem were dealt in [30]. Distance measure and dissimilarity measure are defined in [31]. Generalized weighted distance measure, the generalized weighted Hausdorff distance measure, and the generalized hybrid weighted distance measure between LPFSs and their properties are discussed and applied to TOPSIS [32]. Decomposition theorems for PFS are proved in [33].

The estimation of the probability of belief function in the environment of fuzzy events and intuitionistic fuzzy events exist in literature. Dempster–Shafer theory is based on belief degrees and structures with precision, but in decision-making situations, the data are incomplete, and there is a lack of information. In face recognition, when two persons have highly similar features, a classifier may be unable to give a precise decision. In such situations, its belief degree may be imprecise. Thus, in the decision-making process involving elucidation of multiple experts' opinion, interval-valued belief degree is appropriate. Interval probability distribution based on Dempster–Shafer evidence theory in fuzzy and intuitionistic fuzzy environment is discussed by the authors. Fuzzy theory takes into account membership function, in contrast to intuitionistic fuzzy which accommodates non-membership. In some decision-making situations, some experts prefer neutral membership. Picture fuzzy assents positive, negative, and neutral memberships, thereby providing refusal degree. Thus, picture fuzzy sets are more apposite in decision making, indicating the need for extension of evidence theory for picture fuzzy sets. The main motivation of this paper is to frame probability distribution based on evidence theory for picture fuzzy sets and illustrate it through a suitable example.

2. Background

Dempster–Shafer theory of evidence is based on a finite set of mutually exclusive elements, called the frame of discernment denoted by Ω. 2^{Ω} is the power set of Ω, and it contains all possible unions of the sets in Ω. Atomic sets are the singleton sets in a frame of discernment.

DS theory can express and deal with uncertainty in crisp sets. However, it fails to handle vague information and linguistic terms. Thus, fuzzy evidence theory was developed, and it was extended to intuitionistic fuzzy sets. In this paper, we define evidence theory using picture fuzzy sets. The probability distribution is expressed as an interval. The following definitions are provided as background for this paper.

(1) *Definition* [7, 8]: let $\Omega = A_1, A_2, \ldots, A_n$ be the frame of discernment. A basic belief assignment or basic probability assignment (BPA) is a function m: $2^{\Omega} \longrightarrow [0,1]$ satisfying the conditions: $m(\varnothing) = 0; \sum_{A\subseteq\Omega} m(A) = 1$. For each subset $A\subseteq\Omega$, the value taken by the BPA at A is called the basic probability assigned to A and denoted by $m(A)$. A subset A of Ω is called the focal element of a belief function m if $m(A) > 0$.

(2) *Dempster's rule of combination* [7, 8]: let m_1 and m_2 be two basic probability assignments (BPAs) on the frame of discernment Ω, where the BPAs m_1 and m_2 are independent. The orthogonal sum based on Dempster's rule of combination defined by $m = m_1 \oplus m_2$ is $m(A) = (1/1 - K)\sum_{B\cap C=A} m_1(B) m_2(C)$. The conflict between the BPAs m_1 and m_2 is $K = \sum_{B\cap C=\varnothing} m_1(B) m_2(C)$.

(3) *Definition* [25]: a picture fuzzy set A on a universe X is of the form $A = x, P_A(x), Nu_A(x), Ng_A(x) | x \in X$ where $P_A(x) \in [0,1]$ is the degree of positive membership of x in A, $Nu_A(x) \in [0,1]$ is the degree of neutral membership of x in A, and $Ng_A(x) \in [0,1]$ is the degree of negative membership of x in A. These memberships satisfy the condition $P_A(x) + Nu_A(x) + Ng_A(x) = 1$. Further, the refusal degree of x in A is $R_A(x) = 1 - (P_A(x) + Nu_A(x) + Ng_A(x))$.

(4) *Arithmetic operations on intervals:* consider $[a_1, a_2]$ and $[b_1, b_2]$, with $a_1, b_1 > 0$. Then, the arithmetic operations on these intervals are given by

  (a) Addition: $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$
  (b) Subtraction: $[a_1, a_2] - [b_1, b_2] = [a_1 - b_2, a_2 - b_1]$
  (c) Multiplication: $[a_1, a_2] \cdot [b_1, b_2] = [a_1 \cdot b_1, a_2 \cdot b_2]$
  (d) Division: $([a_1, a_2]/[b_1, b_2]) = [(a_1/b_2), (a_2/b_1)]$

3. Picture Fuzzy Interval Probability (PFIP)

Probability distribution in the framework of picture fuzzy sets is introduced in this section. The probability distribution is in the form of an interval. The validation of this interval probability distribution is examined. Further, this definition coincides with fuzzy and intuitionistic interval probabilities when the negative, neutral, and refusal degrees are zero for the former and neutral membership is zero for the latter. Let $X = x_1, x_2, \ldots, x_n$ be a universe of discourse and F be the set of all focal elements. A picture fuzzy belief function m is given as $\left\{\langle A_i^{\mathrm{PF}}, m(A_i^{\mathrm{PF}}), P_{A_i}^{\mathrm{PF}}(x_j), Ng_{A_i}^{\mathrm{PF}}(x_j), Nu_{A_i}^{\mathrm{PF}}(x_j)\rangle\right\}$ where $A_i^{\mathrm{PF}} \in F, x_j \in X$. Then, the probability of x_j, $j = 1, 2, \ldots, n$ is defined as $\overline{P}(x_j) = [\overline{a_j}, \overline{b_j}]$, where $\overline{a_j}$ and $\overline{b_j}$ are given by

$$\overline{a_j} = \sum_{A_i^{\mathrm{PF}} \in F} \frac{m\left(A_i^{\mathrm{PF}}\right) P_{A_i}^{\mathrm{PF}}\left(x_j\right)}{\left[\sum_{j=1}^{n}\left(-Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)\right] - R_{A_i}^{\mathrm{PF}}\left(x_j\right)}, \tag{1}$$

$$\overline{b_j} = \sum_{A_i^{\mathrm{PF}} \in F} \frac{m\left(A_i^{\mathrm{PF}}\right)\left(1 - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)}{\left[\sum_{j=1}^{n}\left(P_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)\right] + R_{A_i}^{\mathrm{PF}}\left(x_j\right)}. \tag{2}$$

Theorem 1. *The picture fuzzy set interval probability estimation* $\overline{P}(x_j) = [\overline{a_j}, \overline{b_j}]$, $j = 1, 2, \ldots, n$ *forms a valid interval-valued probability distribution in* $X = x_1, x_2, \ldots, x_n$.

Proof. To prove that $\overline{P}(x_j) = [\overline{a_j}, \overline{b_j}]$, $j = 1, 2, \ldots, n$ is a valid interval-valued probability distribution, we need to prove that

$$\begin{aligned} &\left(\sum_{j=1}^{n} \overline{a_j}\right) + \left(\overline{b_q} - \overline{a_q}\right) \leq 1, \\ &\left(\sum_{j=1}^{n} \overline{b_j}\right) - \left(\overline{b_q} - \overline{a_q}\right) \geq 1, \quad \forall q \in 1, 2, \ldots, n. \end{aligned} \tag{3}$$

The picture fuzzy set interval probability estimation is defined on picture fuzzy set. In picture fuzzy set, the rejection membership is given by $R_{A_i}^{\mathrm{PF}}(x_j) = 1 - P_{A_i}^{\mathrm{PF}}(x_j) - Ng_{A_i}^{\mathrm{PF}}(x_j) - Nu_{A_i}^{\mathrm{PF}}(x_j)$. Consider equation (1):

$$\overline{a_j} = \sum_{A_i^{\mathrm{PF}} \in F} \frac{m\left(A_i^{\mathrm{PF}}\right) P_{A_i}^{\mathrm{PF}}\left(x_j\right)}{\left[\sum_{j=1}^{n}\left(1 - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)\right] - R_{A_i}^{\mathrm{PF}}\left(x_j\right)}, \tag{4}$$

where $\overline{a_j}$ can be rewritten as

$$\begin{aligned} \overline{a_j} &= \sum_{A_j^{\mathrm{PF}} \in F} \frac{m\left(A_j^{\mathrm{PF}}\right) P_{A_j}^{\mathrm{PF}}\left(x_j\right)}{\left[\sum_{\substack{k=1 \\ k \neq j}}^{n}\left(1 - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)\right] - \left(1 - Nu_{A_i}^{\mathrm{PF}}\left(x_k\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right) - R_{A_i}^{\mathrm{PF}}\left(x_j\right)}, \\ \overline{a_j} &= \sum_{A_j^{\mathrm{PF}} \in F} \frac{m\left(A_j^{\mathrm{PF}}\right) P_{A_j}^{\mathrm{PF}}\left(x_j\right)}{\left[\sum_{\substack{k=1 \\ k \neq j}}^{n}\left(1 - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right) + P_{A_j}^{\mathrm{PF}}\left(x_j\right)\right]}. \end{aligned} \tag{5}$$

Now,

$$\begin{aligned} \overline{b_j} &= \sum_{A_i^{\mathrm{PF}} \in F} \frac{m\left(A_i^{\mathrm{PF}}\right)\left(1 - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)}{\left[\sum_{j=1}^{n}\left(P_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)\right] + R_{A_i}^{\mathrm{PF}}\left(x_j\right)} \\ &= \sum_{A_i^{\mathrm{PF}} \in F} \frac{m\left(A_i^{\mathrm{PF}}\right)\left(1 - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)}{\left[\sum_{j=1}^{n}\left(P_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)\right] + \left(1 - P_{A_i}^{\mathrm{PF}}\left(x_j\right) - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)} \\ &= \sum_{A_i^{\mathrm{PF}} \in F} \frac{m\left(A_i^{\mathrm{PF}}\right)\left(1 - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)}{\left[\sum_{\substack{k=1 \\ k \neq j}}^{n}\left(P_{A_i}^{\mathrm{PF}}\left(x_k\right)\right) + \left(1 - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right)\right]}. \end{aligned} \tag{6}$$

Table 1: Comparison of the proposed approach with DST, fuzzy, and intuitionistic approaches.

S. no.	Dempster–Shafer theory	Evidence theory for fuzzy sets	Evidence theory for intuitionistic fuzzy sets	Evidence theory for picture fuzzy sets
1	Based on probability distribution	Based on fuzzy probability distribution	Based on intuitionistic probability distribution	Based on picture fuzzy probability distribution
2	Deals with aleatory uncertainties	Deals with epistemic uncertainties	Deals with epistemic uncertainties along with hesitancy in experts' opinion	Deals with epistemic uncertainties along with neutral in experts' opinion
3	Can be provided as interval probability distribution [34]	Can be provided as interval probability distribution [35]	Can be provided as interval probability distribution [9]	Can be provided as interval probability distribution *(present study)*
4	—	Supports degree of membership for belief functions	Supports degrees of membership and nonmembership for belief functions	Supports degrees of positive, negative, and neutral for belief functions
5	—	—	Accommodates hesitancy degree	Accommodates refusal degree
6	—	Reduces to crisp case interval probability distribution when degree	Reduces to fuzzy interval probability distribution when degree of nonmembership	Reduces to fuzzy interval probability distribution when degree negative, neutral, and refusal memberships are zero; reduces to intuitionistic interval probability distribution when neutral membership is zero

Again, for $\forall q \in \{1, 2, \ldots, n\}$,

$$
\begin{aligned}
&\sum_{\substack{j=1\\ j\neq q}}^{n} \frac{P_{A_i}^{\mathrm{PF}}(x_j)}{\left[\sum_{\substack{k=1\\ k\neq j}}^{n}\left(1-Nu_{A_i}^{\mathrm{PF}}(x_k)-Ng_{A_i}^{\mathrm{PF}}(x_k)\right)\right]+P_{A_i}^{\mathrm{PF}}(x_j)}\\
&\leq \sum_{\substack{j=1\\ j\neq q}}^{n} \frac{P_{A_i}^{\mathrm{PF}}(x_j)}{1-P_{A_i}^{\mathrm{PF}}(x_q)+\left[\sum_{\substack{k=1\\ k\neq q}}^{n}\left(P_{A_i}^{\mathrm{PF}}(x_k)\right)\right]} = \frac{\sum_{\substack{j=1\\ j\neq q}}^{n} P_{A_i}^{\mathrm{PF}}(x_j)}{1-P_{A_i}^{\mathrm{PF}}(x_q)+\sum_{\substack{k=1\\ k\neq q}}^{n} n\left(P_{A_i}^{\mathrm{PF}}(x_k)\right)}\\
&= 1-\frac{1-P_{A_i}^{\mathrm{PF}}(x_q)}{1-P_{A_i}^{\mathrm{PF}}(x_q)+\left[\sum_{\substack{k=1\\ k\neq q}}^{n}\left(P_{A_i}^{\mathrm{PF}}(x_k)\right)\right]}.
\end{aligned}
\tag{7}
$$

Consider

$$\sum_{\substack{j=1\\ j\neq q}} \frac{1 - Nu^{\mathrm{PF}}_{A_i}(x_j) - Ng^{\mathrm{PF}}_{A_i}(x_j)}{\left[\sum_{\substack{k=1\\ k\neq q}}^{n} \left(P^{\mathrm{PF}}_{A_i}(x_k)\right)\right] + \left(1 - Nu^{\mathrm{PF}}_{A_i}(x_j) - Ng^{\mathrm{PF}}_{A_i}(x_j)\right)} \geq \sum_{\substack{j=1\\ j\neq q}} \frac{1 - Nu^{\mathrm{PF}}_{A_i}(x_j) - Ng^{\mathrm{PF}}_{A_i}(x_j)}{\left[\sum_{\substack{k=1\\ k\neq q}}^{n} \left(1 - Nu^{\mathrm{PF}}_{A_i}(x_j) - Ng^{\mathrm{PF}}_{A_i}(x_j)\right)\right] + P^{\mathrm{PF}}_{A_i}(x_q)}$$
$$= 1 - \frac{P^{\mathrm{PF}}_{A_i}(x_q)}{\left[\sum_{\substack{k=1\\ k\neq q}}^{n} \left(1 - Nu^{\mathrm{PF}}_{A_i}(x_j) - Ng^{\mathrm{PF}}_{A_i}(x_j)\right)\right] + P^{\mathrm{PF}}_{A_i}(x_q)}, \tag{8}$$

$$\begin{aligned}
\sum_{\substack{j=1\\ j\neq q}}^{n} \overline{a_q} &= \sum_{A_i} \sum_{\substack{j=1\\ j\neq q}}^{n} \frac{m\left(A_i^{\mathrm{PF}}\right) P^{\mathrm{PF}}_{A_i}(x_j)}{\sum_{\substack{k=1\\ k\neq j}}^{n} \left[1 - Nu^{\mathrm{PF}}_{A_i}(x_j) - Ng^{\mathrm{PF}}_{A_i}(x_j)\right] + P^{\mathrm{PF}}_{A_i}(x_j)}\\
&\leq \sum m\left(A_i^{\mathrm{PF}}\right)\left(1 - \frac{1 - Nu^{\mathrm{PF}}_{A_i}(x_k) - Ng^{\mathrm{PF}}_{A_i}(x_k)}{1 - Nu^{\mathrm{PF}}_{A_i}(x_q) - Ng^{\mathrm{PF}}_{A_i}(x_q) + \sum_{\substack{j=1\\ j\neq q}}^{n} \left(P^{\mathrm{PF}}_{A_i}(x_j)\right)}\right)\\
&= \sum m\left(A_i^{\mathrm{PF}}\right) - \sum \frac{m\left(A_i^{\mathrm{PF}}\right)\left(1 - Nu^{\mathrm{PF}}_{A_i}(x_k) - Ng^{\mathrm{PF}}_{A_i}(x_k)\right)}{1 - Nu^{\mathrm{PF}}_{A_i}(x_q) - Ng^{\mathrm{PF}}_{A_i}(x_q) + \sum_{\substack{j=1\\ k\neq q}}^{n} \left(P^{\mathrm{PF}}_{A_i}(x_j)\right)} = 1 - \overline{b_q},
\end{aligned} \tag{9}$$

$$\begin{aligned}
\sum_{\substack{j=1\\ j\neq q}}^{n} \overline{b_q} &= \sum_{\substack{j=1\\ j\neq q}}^{n} \sum_{A_i} \frac{m\left(A_i^{\mathrm{PF}}\right)\left(1 - Nu^{\mathrm{PF}}_{A_i}(x_j) - Ng^{\mathrm{PF}}_{A_i}(x_j)\right)}{\sum_{\substack{k=1\\ k\neq j}}^{n} \left[P^{\mathrm{PF}}_{A_i}(x_j)\right] + \left(1 - Nu^{\mathrm{PF}}_{A_i}(x_j) - Ng^{\mathrm{PF}}_{A_i}(x_j)\right)}\\
&= \sum_{A_i} m\left(A_i^{\mathrm{PF}}\right) \sum_{\substack{j=1\\ j\neq q}}^{n} \frac{1 - Nu^{\mathrm{PF}}_{A_i}(x_j) - Ng^{\mathrm{PF}}_{A_i}(x_j)}{\sum_{\substack{k=1\\ k\neq q}}^{n} \left(P^{\mathrm{PF}}_{A_i}(x_j)\right) + \left(1 - Nu^{\mathrm{PF}}_{A_i}(x_j) - Ng^{\mathrm{PF}}_{A_i}(x_j)\right)}\\
&\geq \sum_{A_i} m\left(A_i^{\mathrm{PF}}\right)\left(1 - \frac{P^{\mathrm{PF}}_{A_i}(x_q)}{P^{\mathrm{PF}}_{A_i}(x_q) + \sum_{k=1 k\neq q}^{n} \left[1 - Nu^{\mathrm{PF}}_{A_i}(x_k) - Ng^{\mathrm{PF}}_{A_i}(x_k)\right]}\right)\\
&= \sum_{A_i} m\left(A_i^{\mathrm{PF}}\right) - \sum \frac{m\left(A_i^{\mathrm{PF}}\right) P^{\mathrm{PF}}_{A_i}(x_q)}{P^{\mathrm{PF}}_{A_i}(x_q) + \sum_{\substack{k=1\\ k\neq q}}^{n} \left[1 - Nu^{\mathrm{PF}}_{A_i}(x_k) - Ng^{\mathrm{PF}}_{A_i}(x_k)\right]} = 1 - \overline{a_q}.
\end{aligned} \tag{10}$$

Hence, finally by equations (9) and (10),

$$\sum_{\substack{j=1 \\ j\neq q}}^{n} \overline{a_q} \le 1 - \overline{b_q} \sum_{\substack{j=1 \\ j\neq q}}^{n} \overline{b_q} \ge 1 - \overline{a_q} \tag{11}$$

$$\text{Thus, } \sum_{j=1}^{b} \overline{a_j} + \left(\overline{b_q} - \overline{a_q}\right) \le 1 \sum_{j=1}^{n} \overline{a_j} + \left(\overline{b_q} - \overline{a_q}\right) \ge 1, \quad \forall q \in \{1, 2, \ldots, n\}.$$

Therefore, $\overline{P_{A_i}^{\mathrm{PF}}(x_j)} = [\overline{a_j}, \overline{b_j}]$ is an interval-valued probability distribution in X. □

Theorem 2. *The interval picture fuzzy probability estimation $A_i^{PF} = \overline{P_{A_i}^{PF}(x_j)} = [\overline{a_j}, \overline{b_j}]$ if it reduces to interval intuitionistic probability estimation A_i^{IF}. Further, it also reduces to interval fuzzy probability estimation.*

Proof. In intuitionistic fuzzy set, the neutral membership $Nu_{A_i}^{\mathrm{PF}}(x_j) = 0$:

$$\begin{aligned}
\overline{a_j} &= \sum \frac{m\left(A_i^{\mathrm{PF}}\right) P_{A_i}^{\mathrm{PF}}\left(x_j\right)}{\sum_{j=1}^{n}\left[1 - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right] - R_{A_i}^{\mathrm{PF}}\left(x_j\right)} \\
&= \sum \frac{m\left(A_i^{\mathrm{PF}}\right) P_{A_i}^{\mathrm{PF}}\left(x_j\right)}{\sum_{j=1}^{n}\left[1 - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)\right] - R_{A_i}^{\mathrm{PF}}\left(x_j\right)}, \\
\overline{b_j} &= \sum \frac{m\left(A_i^{\mathrm{PF}}\right) 1 - Nu_{A_i}^{\mathrm{PF}}\left(x_j\right) - Ng_{A_i}^{\mathrm{PF}}\left(x_j\right)}{\sum_{j=1}^{n}\left[P_{A_i}^{\mathrm{PF}}\left(x_j\right)\right] + R_{A_i}^{\mathrm{PF}}\left(x_j\right)}.
\end{aligned} \tag{12}$$

In fuzzy sets, $Nu_{A_i}^{\mathrm{PF}}(x_j) = 0 = R_{A_i}^{\mathrm{PF}}(x_j)$:

$$\begin{aligned}
\overline{a_j} &= \sum \frac{m\left(A_i^{\mathrm{PF}}\right) P_{A_i}^{\mathrm{PF}}\left(x_j\right)}{\sum_{j=1}^{n} P_{A_i}^{\mathrm{PF}}\left(x_j\right)}, \\
\overline{b_j} &= \sum \frac{m\left(A_i^{\mathrm{PF}}\right) P_{A_i}^{\mathrm{PF}}\left(x_j\right)}{\sum_{j=1}^{n} P_{A_i}^{\mathrm{PF}}\left(x_j\right)}.
\end{aligned} \tag{13}$$

The interval probability estimation based on evidence theory for picture fuzzy sets is established. The similarities and contrasts against the fuzzy and intuitionistic fuzzy sets are provided in Table 1. □

3.1. Example 1. Submarines form a significant and crucial part of the navy of any country. The overall cost of the submarine includes the cost of paint used to coat it. Properly formulated protective coatings are important to the durability and performance of the submarine. Coating systems for the underwater parts of a ship should be corrosion-inhibiting, antifouling, abrasion-resistant, smooth, and compatible with cathodic protection. To minimise bunker (fuel) costs, the underwater hull should remain smooth during service. Consequently, a coating system should be applied as evenly as possible, and it should provide long-term protection against corrosion and fouling. Increased hull friction due to fouling can result in up to 40% more fuel consumption compared to a clean hull and greater air pollution because of the extra fuel burned to maintain a ship's speed. Systems for the underwater hull/boottop areas consist of anticorrosive paint and antifouling paint on top of it. Thus, choosing a cost-effective paint with anticorrosive and antifouling properties is important. Often the estimated cost is based on experts' opinion as it decides the cost of the submarine. Let the universe of discourse be $X = \{80, 90, 100\}$. The assessment result from some experts is that the probability of "assigning about 90 thousand rupees for paint" is 0.5, the probability of "assigning a small amount of money for paint" is 0.3, and the probability of "assigning a large amount of money for paint" is 0.2. The linguistic terms "about 90 thousand," "small amount of money," and "large amount of money" can be captured using picture fuzzy sets, as the opinion of experts often differs and uncertainty is involved. Let these linguistic terms be expressed by three picture fuzzy events A, B, and C, respectively. These focal elements can be expressed as

$$\begin{aligned}
A &= \{(80, 0.7, 0.1, 0.1), (90, 1, 0, 0), (100, 0.7, 0.1, 0.1)\}, \\
B &= \{(80, 0.7, 0.2, 0.1), (90, 0.5, 0.1, 0.3), (100, 0.3, 0.3, 0.3)\}, \\
C &= \{(80, 0.5, 0.2, 0.2), (90, 0.6, 0.1, 0.1), (100, 1, 0, 0)\}.
\end{aligned} \tag{14}$$

The picture fuzzy interval probability distribution is calculated using the proposed approach as

$$\begin{aligned}
\overline{P}(80) &= [0.307, 0.3545], \\
\overline{P}(90) &= [0.34055, 0.40915], \\
\overline{P}(100) &= [0.27955, 0.3302].
\end{aligned} \tag{15}$$

To make a decision on the assigning money to paint is obtained by comparing the picture fuzzy interval probability distribution. Based on comparison of intervals using their centers, $\overline{P}(100) \le \overline{P}(80) \le \overline{P}(90)$. Thus, the decision of assigning 90 thousand for paint can be inferred.

3.2. Example 2. Let us continue with the previous case study of choosing suitable paint for the submarine. Decision is often based on more than one variety of paints. For each paint variety, more than one expert opinion is obtained to take a decision as the cost of paint for the submarine is higher. Suppose the independent opinion of two experts for a variety of paint is obtained based on four main factors of corrosion-inhibiting, antifouling, abrasion-resistant, and smoothness. These factors can be taken as the frame of discernment. Let this frame of discernment be

Table 2: Basic probability assignments by two experts.

S. no.	BPA m_1	BPA m_2
1	$\{x_1, x_3\} = [0.045, 0.514]$	$\{x_1, x_3\} = [0.181, 0.25]$
2	$\{x_3\} = [0.081, 0.172]$	$\{x_1, x_2, x_4\} = [0.016, 0.28]$
3	$\{x_1, x_2, x_3\} = [0.46, 0.57]$	$\{x_1, x_4\} = [0.2, 0.25]$
4	$\{x_2, x_4\} = [0.32, 0.48]$	$\{x_1, x_2, x_3, x_4\} = [0.21, 0.38]$

Table 3: Midpoint of combined IPFP.

S. no.	Combined IPFP m_{12}	Midpoints
1	$\{x_1, x_2\} = [0.0736, 0.159605]$	0.1166025
2	$\{x_1, x_3\} = [0.08326, 0.14250456]$	0.11288228
3	$\{x_2, x_4\} = [0.0512, 0.1344043]$	0.09280215
4	$\{x_1, x_2, x_3\} = [0.0966, 0.216606]$	0.156603

$\Omega = \{x_1, x_2, x_3, x_4\}$. Let m_1 and m_2 be two BPAs defined on Ω, given by Table 2.

By Dempster rule of combination, the combined opinion of the two experts can be estimated. The combined opinion of interval picture fuzzy probability is given by

$$\begin{aligned}
m_{12}(x_1) &= [0.092, 0.1425],\\
m_{12}(x_2) &= [7.2000072E-3, 0.1439246],\\
m_{12}(x_3) &= [0.22806, 0.171505],\\
m_{12}(x_4) &= [0.064, 0.12],\\
m_{12}(x_1, x_2) &= [0.0736, 0.159605],\\
m_{12}(x_1, x_3) &= [0.08326, 0.14250456],\\
m_{12}(x_2, x_4) &= [0.0512, 0.1344043],\\
m_{12}(x_1, x_2, x_3) &= [0.0966, 0.216606].
\end{aligned} \tag{16}$$

The decision of selecting this variety of paint is suitable for submarine is decided based on the interval picture fuzzy probability distribution by comparison of these interval values with more than one characteristic. The midpoints of the intervals are given in Table 3.

Based on Table 3, the paint has three attributes, namely, corrosion-inhibiting, antifouling, and abrasion-resistant, but it lacks smoothness. Thus, the paint variety can be used with less smoothness in the finish.

4. Conclusion

In this paper, we have utilized the picture fuzzy set to address the uncertainty and vagueness in the data. The picture fuzzy set captures the uncertainty of the element with respect to the three membership degrees such that their sum is bounded by 1. In this paper, we reviewed the definition and properties of the interval probability distribution for the picture fuzzy information using the belief function and DST. The proof of their validation is also given in the work. By employing the belief functions on picture fuzzy information systems, the interval probability can be estimated and hence ranking of the number can be accessed. The functionality of the structure is also explained with two numerical examples. Also, the combined IPFP is used to compare two experts' opinions of the choice of paint.

In the future, we will utilize the belief function to address the decision-making problems arising under the different environmental issues such as greenhouse gas emissions, healthcare, green supplier selection, and so on. Also, we have established some generalized measures to combine the different preference values, and therefore, in the future work, we will try to develop different information measures for determining the nature of the decision-making process [36–38].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Retraction

Retracted: Graphical Structures of Cubic Intuitionistic Fuzzy Information

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Graphical Structures of Cubic Intuitionistic Fuzzy Information

Sami Ullah Khan,[1] Naeem Jan ,[1] Kifayat Ullah ,[2] and Lazim Abdullah [3]

[1]*Department of Mathematics, Institute of Numerical Sciences, Gomal University D. I. Khan, Dera Ismail Khan, Pakistan*
[2]*Department of Mathematics, Riphah Institute of Computing and Applied Sciences, Riphah International University Lahore, Lahore 54000, Pakistan*
[3]*Department of Mathematics, Faculty of Ocean Engineering Technology and Informatics, University of Malaysia Terengganu, Kuala Nerus 2103, Malaysia*

Correspondence should be addressed to Naeem Jan; naeem.phdma73@iiu.edu.pk

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The theory developed in this article is based on graphs of cubic intuitionistic fuzzy sets (CIFS) called cubic intuitionistic fuzzy graphs (CIFGs). This graph generalizes the structures of fuzzy graph (FG), intuitionistic fuzzy graph (IFG), and interval-valued fuzzy graph (IVFG). Moreover, several associated concepts are established for CIFG, such as the idea subgraphs, degree of CIFG, order of CIFG, complement of CIFG, path in CIFG, strong CIFG, and the concept of bridges for CIFGs. Furthermore, the generalization of CIFG is proved with the help of some remarks. In addition, the comparison among the existing and the proposed ideas is carried out. Finally, an application of CIFG in decision-making problem is studied, and some future study is proposed.

1. Introduction

Jun et al. [1] proposed cubic set (CS) and started a new research area. A CS is a mixture of two concepts known as fuzzy set (FS) and interval-valued fuzzy set (IVFS). The concept of CS draws the attentions of researchers and some potential works in this direction have been done; for example, the idea of CS was proposed in semigroup theory by Khan et al. [2], as well as some KU-ideal by Yaqoob et al. [3], and KU-algebras are developed for CS by Lu and Ye [4]; the similarity measures of CSs have been proposed and applied in decision-making problem. The framework of cubic neutrosophic sets is proposed by Jun et al. [5], while some pattern recognition problems are solved using neutrosophic sets by Ali et al. [6]. The concept of cubic soft sets was proposed by Muhiuddin and Al-roqi [7], which was further utilized by Muhiuddin et al. [8]. The theory of G-algebras is studied by Jun and Khan in [9] and by Jana and Senapati [10] along with the concepts of ideal in semigroups. Some other works in this direction are given in [11–14].

The theory of intuitionistic fuzzy set (IFS) was developed by Atanassov [15] as a generalization of FS by Rosenfeld [16]. An IFS described the membership and nonmembership degree of an element by two characteristic functions and can model phenomena of yes or no type easily. Garg and Kaur [17] initiated the concept of cubic intuitionistic fuzzy sets (CIFSs) and discussed their properties. Atanassov model of IFS provided a motivation for the concept of intuitionistic fuzzy graphs (IFGs) defined by Parvathi and Karunambigai [18]. The concept of IFG was a generalization of fuzzy graphs (FGs) proposed by Kauffman and Rosenfeld [19, 20] after Zadeh's exemplary work in [16]. FG theory has a potential role in application point of view as described by Chan and Cheung [21] who studied an approach to clustering algorithm using the concepts of FGs. Some FG problems are solved by a novel technique in [22, 23] by discussing the domination of FGs in pattern recognitions. Mathew and Sunitha [24] worked on fuzzy attribute graphs applied to Chinese character recognitions, and Bhattacharya [25] used FGs in image classifications and so forth. For some other works on FG, one may refer to [26–31].

The theory of IFG received great attention as Parvathi and Thamizhendhi [32] introduced the concept of strong IFGs; Akram and Dudek [33] discussed the order, degree, and size of IFGs; Akram and Alshehri [34] developed operations for IFGs; Karunambigai [35] worked on the domination of IFGs; Pasi et al. [36] developed the theory of intuitionistic fuzzy hypergraphs; Karunambigai et al. [37] studied the concepts of trees and cycles for IFGs; Parvathi [38] developed the idea of balanced IFGs, a multicriteria and multiperson decision-making based on IFGs was discussed by Chountas [39]; Akram and Dudek [40] studied constant IFGs; Mathew [41] discussed IF hypergraphs; and the authors of [42] discussed the matrix representation of IFGs. Interval-valued FGs have also been studied extensively after Akram [43] proposed interval-valued FGs, Rashmanlou and Pal [44] discussed the results proposed by [43], complete interval-valued FGs developed interval-valued fuzzy line graphs are discussed by Rashmanlou and Pal [45, 46], and Pramanik et al. [47] proposed balanced interval-valued FGs. Xiao et al. [48] worked on green supplier selection in steel industry with intuitionistic fuzzy Taxonomy method, Zhao et al. [49] proposed an extended CPT-TODIM method for IVIF MAGDM and applied it to urban ecological risk assessment, and Wu et al. [50] presented VIKOR method for financing risk assessment of rural tourism under IVIF environment. Further, for some works on interval-valued FGs, one may refer to [51–55]. Motivated by the existing theory, we proposed the framework of cubic intuitionistic fuzzy sets (CIFSs) and cubic intuitionistic fuzzy graphs (CIFGs). Several graphical and theoretical terms are illustrated with the help of examples and some results.

The manuscript is organized as follows: In Section 1, a brief introduction about existing concepts is presented. In Section 2, some basic definitions from the theories of FG, IFG, and IVFG are defined. The concept of CIFG is proposed in Section 3 along with some other related terms and results including the concepts of subgraphs, degrees, orders, and bridges in CIFGs. Section 4 is based on operations on CIFGs and their results. The applications of CIFG in decision-making problems are discussed in Section 5. Section 6 provides a comparison of CIFG with existing concepts, and Section 7 provides a brief discussion and concluding remarks.

2. Preliminaries

In this section, we introduce some basic concepts about fuzzy set, fuzzy graph, intuitionistic fuzzy set, and intuitionistic fuzzy graph, which provide a base for our graphical work on CIFG. Throughout this manuscript, X denotes the universe of discourse and $\mathrm{M}, \mathrm{Ŋ}$ are considered to be two mappings on $[0,1]$ intervals denoting the membership and nonmembership grades, respectively, of an element.

Definition 1 (see [13]). A *FS* on $\dot{\mathrm{X}}$ is defined as $A = \{u, (M_A(u)/u \in \dot{X})\}$, where $M_A(1/2)$ is a map on $[0,1]$.

Definition 2 (see [20]). A pair $\breve{G}^* = (\mathscr{V}, E)$ is known as FG if

(i) $\mathscr{V} = \{M_i \colon i \in I\}$ and $M_1 \colon \mathscr{V} \longrightarrow [0,1]$ is the association degree of $M_i \in \mathscr{V}$

(ii) $E = \{(u_i, u_j) \colon (u_i, u_j) \in \mathscr{V} \times \mathscr{V}\}$ and $M_2 \colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1]$ where $M_2(u_i, u_j) \le \min[M_1(u_i), M_1(u_j)]$ for all $(u_i, u_j) \in E$.

Definition 3 (see [15]). An IFS A on X is defined as $A = \{\langle\langle u, M_A(u), \mathrm{Ŋ}_A(u)\rangle/u \in \dot{X}\rangle\}$, where M_A and $\mathrm{Ŋ}_A$ are mappings on 0,1 interval such that $0 \le M_A + \mathrm{Ŋ}_A \le 1$.

Definition 4 (see [18]). A Pair $\breve{G}*=(V, \ddot{\hat{E}})$ is known as IFG if

(i) V is the collection of nodes such that M_1 and $\mathrm{Ŋ}_1$ are mappings on unit intervals from V with a condition $0 \le M_1(ui) + \mathrm{Ŋ}_1(ui) \le 1$ for all $u_i \in V$, $i \in I$

(ii) $E \subseteq \mathscr{V} \times \mathscr{V}$, where M_2 and $\mathrm{Ŋ}_2$ are mappings that associate some grade to each $(u_i, u_j) \in E$ from $[0,1]$ interval such that $M_2(u_i, u_j) \le \min\{M_1(u_i), M_1(u_j)\}$ and $\mathrm{Ŋ}_2(u_i, u_j) \le \max\{\mathrm{Ŋ}_1(u_i), \mathrm{Ŋ}_1((u_j))\}$ with a condition $0 \le \mathrm{M}_2 + \mathrm{Ŋ}_2 \le 1$

Example 1. The graph in Figure 1 is an IFG having four vertices and four edges.

Definition 5 (see [33]). The complement of an IFG $\breve{G}^* = (\mathscr{V}, E)$ is $\breve{G}^{*c} = (\mathscr{V}^c, E^c)$, where

(i) $V_c = V$

(ii) $M_A(u_i)c = M_A(u_i)$, $\mathrm{Ŋ}_A(u_i)c = \mathrm{Ŋ}_A(u_i), \forall\ u_i \in V$

(iii) $M_B(u_i, u_j)^c = \min[M_B(u_i), M_B(u_j)] - M_B(u_i, u_j)$, $\mathrm{Ŋ}_B(u_i, u_j)^c = \max[\mathrm{Ŋ}_B(u_i), \mathrm{Ŋ}_B(u_j)] - \mathrm{Ŋ}_B(u_i, u_j)$, for all $(u_i, u_j) \in E$

Here $(u_i, M_A, \mathrm{Ŋ}_A)$ represent the vertices and $(e_{ij}, M_B, \mathrm{Ŋ}_B)$ represent the edges.

Definition 6 (see [32]). A Pair $\breve{G}^* = (\mathscr{V}, E)$ is known as strong IFG if

(i) $\mathscr{V}$ is the collection of nodes such that M_1 and $\mathrm{Ŋ}_1$ are mappings on unit intervals from $\mathscr{V}$ with a condition $0 \le \mathrm{M}_1(u_i) + \mathrm{Ŋ}_1(u_i) \le 1$ for all $u_i \in \mathscr{V}$ $(i \in I)$

(ii) $E \subseteq \mathscr{V} \times \mathscr{V}$, where M_2 and $\mathrm{Ŋ}_2$ are mappings that associate some grade to each $(u_i, u_j) \in E$ from $[0,1]$ interval such that $M_2(u_i, u_j) = \min\{M_1(u_i), M_1(u_j)\}$ and $\mathrm{Ŋ}_2(u_i, u_j) = \max\{\mathrm{Ŋ}_1(u_i), \mathrm{Ŋ}_1(u_j)\}$ with a condition $0 \le M_2 + \mathrm{Ŋ}_2 \le 1$

Remark 1 (see [32]). If $\breve{G}^* = (\mathscr{V}, E)$ is an *IFG*, then by the above definition $(\breve{G}^{*c})^c = \breve{G}^*$ and it is called self-complementary.

Proposition 1 (see [32]). *If $\breve{G}^*$ is strong IFG, then it preserves self-complementary law.*

Example 2. Figures 2(a) and 2(b) provide a verification of Proposition 1.

Clearly $(\breve{G}^{*c})^c = \breve{G}^*$ is self-complementry.

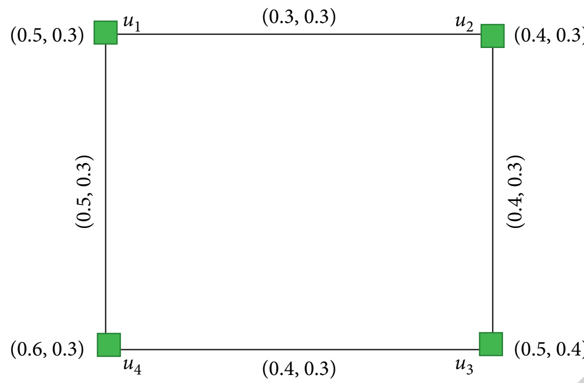
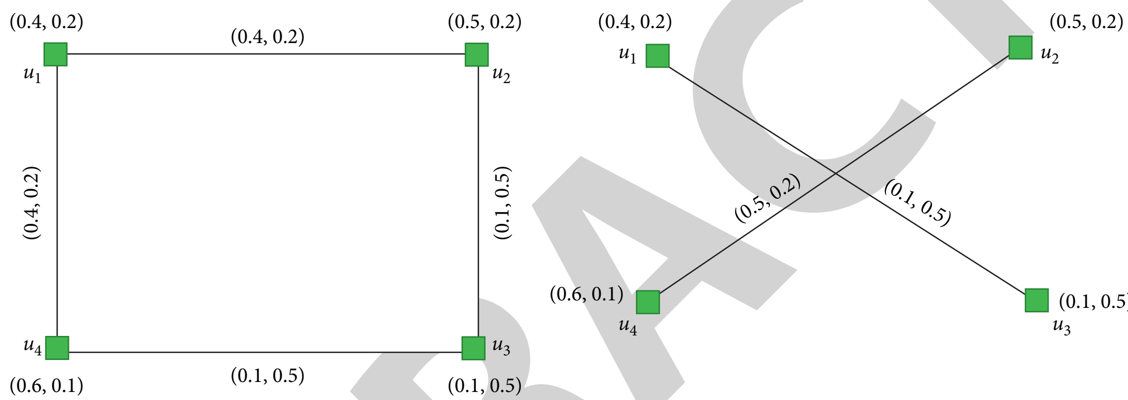


Figure 1: Intuitionistic fuzzy graph.



(a) (b)

Figure 2: (a) Intuitionistic fuzzy graph. (b) Complement of intuitionistic fuzzy graph.

Definition 7 (see [55]). A pair $\breve{G} = (A, \mathscr{B})$ of a graph $\breve{G}^* = (\mathscr{V}, E)$ is known as IVIFG, where $A = \{([M_{AL}, M_{AU}], [\eta_{AL}, \eta_{AU}])\}$ is IVFS on $\mathscr{V}$, and $\mathscr{B} = \{([M_{\mathscr{B}L}, M_{\mathscr{B}U}], [\eta_{\mathscr{B}L}, \eta_{\mathscr{B}U}])\}$ is the IVF relation on E satisfying the following conditions:

(i) $\mathscr{V} = \{u_1, u_2, u_3, \ldots, u_n\}$such that $M_{AL}\colon \mathscr{V} \longrightarrow [0,1], M_{AU}\colon \mathscr{V} \longrightarrow [0,1]$ and $\eta_{AL}\colon \mathscr{V} \longrightarrow [0,1]$, $\eta_{AU}\colon \mathscr{V} \longrightarrow [0,1]$ represent the degrees of membership and nonmembership of the element $u \in \mathscr{V}$, respectively, and $0 \leq M_A + \eta_A \leq 1$ for all $u_i \in \mathscr{V}$ $(i = 1, 2, \ldots, n)$

(ii) The functions $M_{\mathscr{B}L}\colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1]$, $M_{\mathscr{B}U}\colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1]$, $\eta_{\mathscr{B}L}\colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1]$, and $\eta_{\mathscr{B}U}\colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1]$ are such that $M_{\mathscr{B}L}(u, y) \leq \min(M_{AL}(u), M_{AL}(y))$, $\eta_{\mathscr{B}L}(u, y) \leq \max(\eta_{AL}(u), \eta_{AL}(y))$ $M_{\mathscr{B}U}(u, y) \leq \min(M_{AU}(u), M_{AU}(y))$, and $\eta_{\mathscr{B}U}(u, y) \leq \max(\eta_{AU}(u), \eta_{AU}(y))$; $0 \leq M_{\mathscr{B}}(u, y) + \eta_{\mathscr{B}}(u, y) \leq 1$ for all $(u_i, y_j) \in E$ $(i, j = 1, 2, \ldots, n)$

Example 3. Let $\breve{G}^* = (\mathscr{V}, E)$ be a graph, where $\mathscr{V} = \{u_1, u_2, u_3\}$ is the set of vertices and $E = \{u_1u_2, u_2u_3, u_3u_1\}$ is the set of edges.

3. Cubic Intuitionistic Fuzzy Graphs

In this section, we discussed the basic concept of CIFG-like complement of CIFG, degree of CIFG, and bridge and cut vertex of CIFG with the help of examples and several results (Figures 3 and 4).

Definition 8. A pair $\breve{G} = (A, \mathscr{B})$ of a graph $\breve{G}^* = (\mathscr{V}, E)$ is known as cubic IFG, where $A = \{([M_{AL}, M_{AU}], [\eta_{AL}, \eta_{AU}]), (M_A, \eta_A)\}$ is a cubic IFS on $\mathscr{V}$, and $\mathscr{B} = \{([M_{\mathscr{B}L}, M_{\mathscr{B}U}], [\eta_{\mathscr{B}L}, \eta_{\mathscr{B}U}]), (M_{\mathscr{B}}, \eta_{\mathscr{B}})\}$ is the cubic IF relation on E satisfying the following conditions:

(iii) $\mathscr{V} = \{u_1, u_2, u_3, \ldots, u_n\}$ such that $M_{AL}\colon \mathscr{V} \longrightarrow [0,1], M_{AU}\colon \mathscr{V} \longrightarrow [0,1]$ and $\eta_{AL}\colon \mathscr{V} \longrightarrow [0,1]$, $\eta_{AU}\colon \mathscr{V} \longrightarrow [0,1]$ and $M_A\colon \mathscr{V} \longrightarrow [0,1]$, $\eta_A\colon \mathscr{V} \longrightarrow [0,1]$⊸ represent the degrees of membership and nonmembership of the element $u \in \mathscr{V}$, respectively, and $0 \leq M_A + \eta_A \leq 1$ for all $u_i \in \mathscr{V}$ $(i = 1, 2, \ldots, n)$

(iv) The functions $M_{\mathscr{B}L}\colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1]$, $M_{\mathscr{B}U}\colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1]$, $\eta_{\mathscr{B}L}\colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1]$, $\eta_{\mathscr{B}U}\colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1]$ and $M_{\mathscr{B}}\colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1], \eta_{\mathscr{B}}\colon \mathscr{V} \times \mathscr{V} \longrightarrow [0,1]$ are such that $M_{\mathscr{B}L}(u, y) \leq \min(M_{AL}(u), M_{AL}(y))$, $\eta_{\mathscr{B}L}(u, y) \leq$

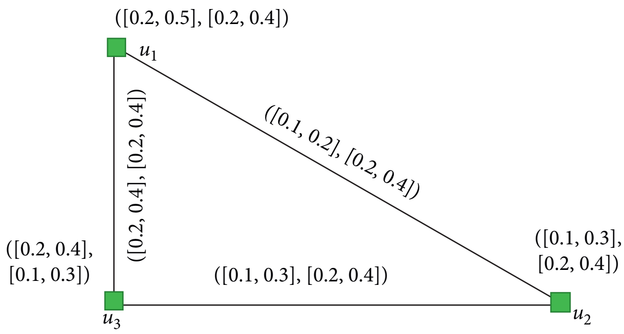


Figure 3: Interval-valued intuitionistic fuzzy graph.

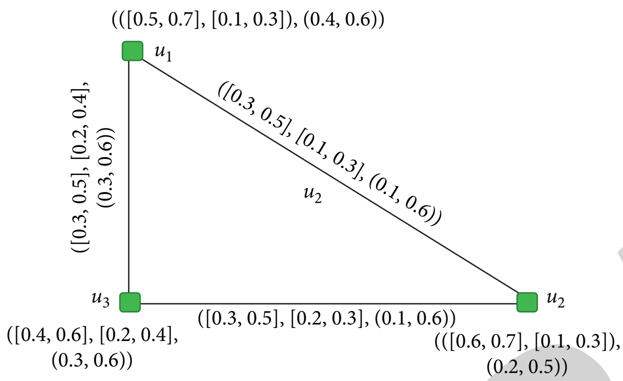


Figure 4: Cubic intuitionistic fuzzy graph.

$\max(\eta_{AL}(u), \eta_{AL}(y))$ $M_{\mathscr{B}U}(u, y) \leq \min(M_{AU}(u), M_{AU}(y))$, and $\eta_{\mathscr{B}U}(u, y) \leq \max(\eta_{AU}(u), \eta_{AU}(y))$; and $M_{\mathscr{B}}(u, y) \leq \min(M_A(u), M_A(y))$ and $\eta_{\mathscr{B}}(u, y) \leq \max(M_A(u), M_A(y))$ such that $0 \leq M_{\mathscr{B}}(u, y) + \eta_{\mathscr{B}}(u, y) \leq 1$ for all $(u_i, y_j) \in E$ $(i, j = 1, 2, \ldots, n)$

Example 4. Consider a graph $\check{G}^* = (\mathscr{V}, E)$, where $\mathscr{V} = \{u_1, u_2, u_3\}$ is the set of vertices and $E = \{u_1u_2, u_2u_3, u_3u_1\}$ is the set of edges.

Definition 9. A pair $\check{G} = (A, \mathscr{B})$ of a graph $\check{G}^* = (\mathscr{V}, E)$ is known as strong cubic IFG, where $A = \{([M_{AL}, M_{AU}], [\eta_{AL}, \eta_{AU}]), (M_A, \eta_A)\}$ is a cubic IFS on $\mathscr{V}$, and $\mathscr{B} = \{([M_{\mathscr{B}L}, M_{\mathscr{B}U}], [\eta_{\mathscr{B}L}, \eta_{\mathscr{B}U}]), (M_{\mathscr{B}}, \eta_{\mathscr{B}})\}$ is a cubic IF relation on E satisfying the following conditions:

(i) $\mathscr{V} = \{u_1, u_2, u_3, \ldots, u_n\}$ such that $M_{AL}: \mathscr{V} \longrightarrow [0, 1]$, $M_{AU}: \mathscr{V} \longrightarrow [0, 1]$ and $\eta_{AL}: \mathscr{V} \longrightarrow [0, 1]$, $\eta_{AU}: \mathscr{V} \longrightarrow [0, 1]$ and $M_A: \mathscr{V} \longrightarrow [0, 1]$, $\eta_A: \mathscr{V} \longrightarrow [0, 1]$ represent the degrees of membership and nonmembership of the element $u \in \mathscr{V}$, respectively, and $0 \leq M_A + \eta_A \leq 1$ for all $u_i \in \mathscr{V}$ $(i = 1, 2, \ldots, n)$

(ii) The functions $M_{\mathscr{B}L}: \mathscr{V} \times \mathscr{V} \longrightarrow [0, 1]$, $M_{\mathscr{B}U}: \mathscr{V} \times \mathscr{V} \longrightarrow [0, 1]$, $\eta_{\mathscr{B}L}: \mathscr{V} \times \mathscr{V} \longrightarrow [0, 1]$, $\eta_{\mathscr{B}U}: \mathscr{V} \times \mathscr{V} \longrightarrow [0, 1]$, and $M_{\mathscr{B}}: \mathscr{V} \times \mathscr{V} \longrightarrow [0, 1]$, $\eta_{\mathscr{B}}: \mathscr{V} \times \mathscr{V} \longrightarrow [0, 1]$ are such that $M_{\mathscr{B}L}(u, y) = \min(M_{AL}(u), M_{AL}(y))$, $\eta_{\mathscr{B}L}(u, y) = \max(\eta_{AL}(u), \eta_{AL}(y))$ $M_{\mathscr{B}U}(u, y) = \min(M_{AU}(u), M_{AU}(y))$, and $\eta_{\mathscr{B}U}(u, y) = \max(\eta_{AU}(u), \eta_{AU}(y))$; and $M_{\mathscr{B}}(u, y) = \min(M_A(u), M_A(y))$ and $\eta_{\mathscr{B}}(u, y) = \max(M_A(u), M_A(y))$ such that $0 \leq M_{\mathscr{B}}(u, y) + \eta_{\mathscr{B}}(u, y) \leq 1$ for all $(u_i, y_j) \in E$ $(i, j = 1, 2, \ldots, n)$

Definition 10. A cubic IFG $H = (\mathscr{V}^{\curlyvee}, E^{\curlyvee})$ is said to be cubic IFG subgraph of $\check{G}^* = (\mathscr{V}, E)$ if $\mathscr{V}^{\curlyvee} \subseteq \mathscr{V}$ and $E^{\curlyvee} \subseteq E$. In other words, $[M_{ALi}, M_{AUi}]^{\curlyvee} \leq [M_{ALi}, M_{AUi}]$, $[\eta_{ALi}, \eta_{AUi}]^{\curlyvee} \leq [\eta_{ALi}, \eta_{AUi}]$, and $(M_{Ai}, \eta_{Ai})^{\curlyvee} \leq (M_{Ai}, \eta_{Ai})$ and $[M_{\mathscr{B}Lij}, M_{\mathscr{B}Uij}]^{\curlyvee} \leq [M_{\mathscr{B}Lij}, M_{\mathscr{B}Uij}]$, $[\eta_{\mathscr{B}Lij}, \eta_{\mathscr{B}Uij}]^{\curlyvee} \leq [\eta_{\mathscr{B}Lij}, \eta_{\mathscr{B}Uij}]$, and $(M_{\mathscr{B}ij}, \eta_{\mathscr{B}ij})^{\curlyvee} \leq (M_{\mathscr{B}ij}, \eta_{\mathscr{B}ij})$ for $i, j = 1, 2, \ldots, n$.

Definition 11. The order of cubic IFG $\check{G}^* = (\mathscr{V}, \hat{E})$ is denoted and defined by

$$O\left(\check{G}^*\right) = \left(\left(\sum_{u\in\mathscr{V}} M_{A\underset{\frown}{L}}(u), \sum_{u\in\mathscr{V}} M_{A\hat{U}}(u), \sum_{u\in\mathscr{V}} \eta_{A\underset{\frown}{L}}(u), \sum_{u\in\mathscr{V}} \eta_{A\hat{U}}(u)\right), \left(\sum_{u\in\mathscr{V}} M_A(u), \sum_{u\in\mathscr{V}} \eta_A(u)\right)\right), \tag{1}$$

and the size of cubic IFG is

$$S(G) = \left(\left(\sum_{\substack{u\neq y\\ u,y\in V}} M_{\mathscr{B}L}(uy) \sum_{\substack{u\neq y\\ u,y\in V}} M_{\mathscr{B}A\hat{U}}(uy), \sum_{\substack{u\neq y\\ u,y\in V}} \eta_{\mathscr{B}L}(uy), \sum_{\substack{u\neq y\\ u,y\in V}} \eta_{\mathscr{B}A\hat{U}}(uy)\right), \left(\sum_{\substack{u\neq y\\ u,y\in V}} M_{\mathscr{B}}(uy), \sum_{\substack{u\neq y\\ u,y\in V}} \eta_{\mathscr{B}}(uy)\right)\right). \tag{2}$$

Definition 12. The degree of a vertex in a cubic IFG $\breve{G}^* = (\mathscr{V}, E)$ is denoted and defined by

$$d(u) = ((dM_{AL}(u), dM_{AU}(u), dŊ_{AU}(u), dŊ_{AU}(u)),\\ (d(M_A)(u), d(Ŋ_A)(u))), \tag{3}$$

where

$$\begin{aligned} dM_{AL}(u) &= \sum_{\substack{u\neq y\\ u\in V}} M_{\mathscr{B}L}(uy),\\ dM_{AU}(u) &= \sum_{\substack{u\neq y\\ u\in V}} M_{\mathscr{B}U}(uy),\\ dŊ_{AL}(u) &= \sum_{\substack{u\neq y\\ u,y\in V}} Ŋ_{\mathscr{B}L}(uy),\\ dŊ_{AU}(u) &= \sum_{\substack{u\neq y\\ u,y\in V}} Ŋ_{\mathscr{B}U}(uy),\\ d(M_A)(u) &= \sum_{\substack{u\neq y\\ u,y\in V}} M_{\mathscr{B}}(uy),\\ d(Ŋ_A)(u) &= \sum_{\substack{u\neq y\\ u,y\in V}} Ŋ_{\mathscr{B}}(uy). \end{aligned} \tag{4}$$

Example 5. Let Figure 5 be a graph $\breve{G}^* = (\mathscr{V}, E)$, where $\mathscr{V} = \{u_1, u_2, u_3, u_4\}$ is the set of vertices and $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$ is the set of edges.

The degrees of vertices are

$$\begin{aligned} d(u_1) &= ([0.3, 0.6], [0.5, 0.8], (0.3, 0.8)),\\ d(u_2) &= ([0.4, 0.7], [0.5, 0.8], (0.3, 0.8)),\\ d(u_3) &= ([0.3, 0.7], [0.4, 0.8], (0.2, 0.8)),\\ d(u_4) &= ([0.2, 0.6], [0.4, 0.8], (0.2, 0.9)). \end{aligned} \tag{5}$$

Definition 13. The complement of a cubic IFG $\breve{G} = (A, \mathscr{B})$ on $\breve{G}^* = (\mathscr{V}, E)$ is defined as follows:

(i) $\overline{A} = A$

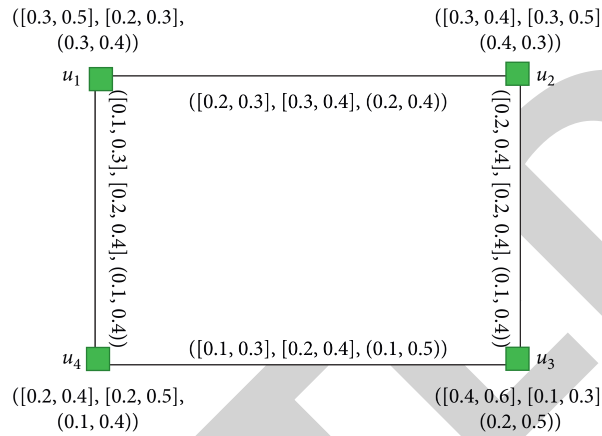


Figure 5: Cubic intuitionistic fuzzy graph.

(ii) $\overline{M_{AL}}(u_i) = M_{AL}(u_i)$, $\overline{M_{AU}}(u_i) = M_{AU}(u_i)$, $\overline{Ŋ_{AL}}(u_i) = Ŋ_{AL}(u_i)$, $\overline{Ŋ_{AU}}(u_i) = Ŋ_{AU}(u_i)$ and $\overline{M_A}(u_i) = M_A(u_i)$, $\overline{Ŋ_A}(u_i) = Ŋ_A(u_i)$ for all $u_i \in \mathscr{V}$

(iii) $\overline{M_{\mathscr{B}L}}(u_i, u_j) = \min[M_{AL}(u_i), M_{AL}(u_j)] - M_{\mathscr{B}U}(u_i, u_j)$, $\overline{M_{\mathscr{B}U}}(u_i, u_j) = \min[M_{AU}(u_i), M_{AU}(u_j)] - M_{\mathscr{B}U}(u_i, u_j)$, $\overline{Ŋ_{\mathscr{B}L}}(u_i, u_j) = \max[Ŋ_{AL}(u_i), Ŋ_{AL}(u_j)] - Ŋ_{\mathscr{B}U}(u_i, u_j)$, $\overline{Ŋ_{\mathscr{B}U}}(u_i, u_j) = (1/2)\max[Ŋ_{AL}(u_i), Ŋ_{AL}(u_j)] - Ŋ_{\mathscr{B}U}(u_i, u_j)$ for all $(u_i, u_j) \in E$

Proposition 2. *$\breve{G} = \overline{\overline{G}}$ if and if $\breve{G}$ is strong cubic IF graph.*

Proof. The proof is straightforward. □

Definition 14. A strong IFG is said to be self-complementary if $\breve{G} \cong \overline{\overline{G}}$, where $\overline{\overline{G}}$ is the complement of IFG $\breve{G}$.

Example 6. Let Figures 6 and 7 be two graphs of $\breve{G}^* = (\mathscr{V}, E)$, where $\mathscr{V} = \{u_1, u_2, u_3, u_4\}$ is the set of vertices and $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$ is the set of edges.

Clearly $\breve{G} = \overline{\overline{G}}$; hence, $\breve{G}$ is self-complementary.

Definition 15. The power of edge relation in a cubic IFG is defined as

$$\begin{aligned} e^1_{ij} &= (e_{ij}, (([M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}], [Ŋ_{\mathscr{B}ijL}, Ŋ_{\mathscr{B}ijU}]), (M_{\mathscr{B}ij}, Ŋ_{\mathscr{B}ij})))\\ e^2_{ij} &= e^*_{ij}e_{ij} = \left(e_{ij}, [M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]^2, [Ŋ_{\mathscr{B}ijL}, Ŋ_{\mathscr{B}ijU}]^2, (M^2_{\mathscr{B}ij}, Ŋ^2_{\mathscr{B}ij})\right)\\ e^3_{ij} &= e^*_{ij}e^*_{ij}e_{ij} = \left(e_{ij}, [M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]^3, [Ŋ_{\mathscr{B}ijL}, Ŋ_{\mathscr{B}ijU}]^3, (M^3_{\mathscr{B}ij}, Ŋ^3_{\mathscr{B}ij})\right). \end{aligned} \tag{6}$$

Also,

$$e^\infty_{ij} = \left(e_{ij}, [M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]^\infty, [Ŋ_{\mathscr{B}ijL}, Ŋ_{\mathscr{B}ijU}]^\infty, (M^\infty_{\mathscr{B}ij}, Ŋ^\infty_{\mathscr{B}ij})\right). \tag{7}$$

Here, $[M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]^\infty = \max(\{[M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]^k\}$, $M^\infty_{\mathscr{B}ij} = \max\{M^k_{\mathscr{B}ij}\})$ and $[Ŋ_{\mathscr{B}ijL}, Ŋ_{\mathscr{B}ijU}]^\infty = \min\{[Ŋ_{\mathscr{B}ijL}, Ŋ_{\mathscr{B}ijU}]^k\}$, $Ŋ^\infty_{\mathscr{B}ij} = \min\{Ŋ^k_{\mathscr{B}ij}]$ are the M – strength

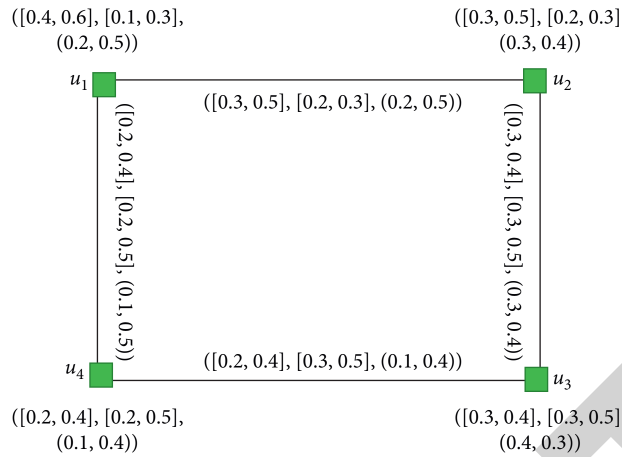


Figure 6: Cubic strong intuitionistic fuzzy graph.

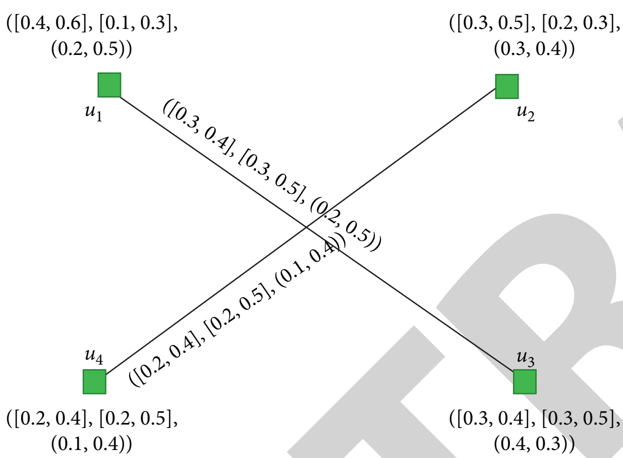


Figure 7: Complement of cubic strong intuitionistic fuzzy graph.

and η – strength of the connectedness between the two vertices (y_i, y_j).

Definition 16. An edge in a cubic IFG $\breve{G}^* = (\mathscr{V}, E)$ is said to be bridge, if deleting that edge reduces the strength of connectedness between some pair of vertices.

Example 7. Let Figure 8 be a graph $\breve{G}^* = (\mathscr{V}, E)$, where $\mathscr{V} = \{u_1, u_2, u_3, u_4\}$ is the set of vertices and $E = \{u_1u_2, u_2u_3, u_2u_4, u_4u_1\}$ is the set of edges.

The strength of (u_1, u_4) is $([0.1, 0.4], [0.3, 0.5], (0.1, 0.4))$, so (u_1, u_4) is a bridge because when deleteing (u_1, u_4) the strength of the connectedness between u_1 and u_4 is decreased.

Theorem 1. *If $\breve{G}^* = (\mathscr{V}, E)$ is a cubic IFG, then, for any two vertices y_i and y_j, the following are equivalent:*

(i) (y_i, y_j) is a bridge

(ii) $[M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]'^{\infty} < [M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}], M'^{\infty}_{\mathscr{B}ij} < M_{\mathscr{B}ij}$ and $[\eta_{\mathscr{B}ijL}, \eta_{\mathscr{B}ijU}]'^{\infty} > [\eta_{\mathscr{B}ijL}, \eta_{\mathscr{B}ijU}], \eta'^{\infty}_{\mathscr{B}ij} > \eta_{\mathscr{B}ij}$

(iii) (y_i, y_j) is not an edge of any cycle

Proof. (ii) $\Longrightarrow$ (i).

Consider $[M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]'^{\infty} < [M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}], M'^{\infty}_{\mathscr{B}ij} < M_{\mathscr{B}ij}$ and $[\eta_{\mathscr{B}ijL}, \eta_{\mathscr{B}ijU}]'^{\infty} > [\eta_{\mathscr{B}ijL}, \eta_{\mathscr{B}ijU}], \eta'^{\infty}_{\mathscr{B}ij} > \eta_{\mathscr{B}ij}$ to show that (y_i, y_j) is a bridge; then $[M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]'^{\infty} = [M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]^{\infty} \geq [M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]$, $M'^{\infty}_{\mathscr{B}ij} = M^{\infty}_{\mathscr{B}ij} \geq M_{\mathscr{B}ij}$ and $[\eta_{\mathscr{B}ijL}, \eta_{\mathscr{B}ijU}]'^{\infty} = [\eta_{\mathscr{B}ijL}, \eta_{\mathscr{B}ijU}]^{\infty} \leq [\eta_{\mathscr{B}ijL}, \eta_{\mathscr{B}ijU}], \eta'^{\infty}_{\mathscr{B}ij} = \eta^{\infty}_{\mathscr{B}ij} \leq \eta_{\mathscr{B}ij}$.

$[M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}]'^{\infty} \geq [M_{\mathscr{B}ijL}, M_{\mathscr{B}ijU}], M'^{\infty}_{\mathscr{B}ij} \geq M_{\mathscr{B}ij}$ and $[\eta_{\mathscr{B}ijL}, \eta_{\mathscr{B}ijU}]'^{\infty} \leq [\eta_{\mathscr{B}ijL}, \eta_{\mathscr{B}ijU}], \eta^{\infty}_{\mathscr{B}ij}\eta'^{\infty}_{\mathscr{B}ij}$, which is a contradiction. Hence, (y_i, y_j) is a bridge.

(i) $\Longrightarrow$ (iii).

Suppose that (y_i, y_j) is a bridge to show that (y_i, y_j) is not an edge of any cycle. If (y_i, y_j) is an edge of cycle, then any path involving the edge (y_i, y_j) can be converted into a path not involving (y_i, y_j) by using the rest of the cycle as a path from y_i to y_j. This implies that (y_i, y_j) cannot be a bridge, which is a contradiction to our supposition. Hence, (y_i, y_j) is not an edge of any cycle.

(iii) $\Longrightarrow$ (i).

The proof is straightforward. □

Definition 17. A vertex u_i in a cubic IFG $\breve{G}^*$ is said to be cut-vertex if deleting a vertex u_i reduces the strength of connectedness between some pair of vertices.

Example 8. Consider a graph $\breve{G}^* = (\mathscr{V}, E)$, where $\mathscr{V} = \{u_1, u_2, u_3, u_4, u_5\}$ is the set of vertices and $E = \{u_1u_2, u_2u_4, u_4u_3, u_4u_5, u_4u_1\}$ is the set of edges.

In Figure 9, u_1 is a cut-vertex.

4. Operations on Cubic IFG

In this section, the operations of CIFG-like Cartesian product of CIFG, union of CIFG, joint operation of CIFG, and so forth with the help of examples are discussed and some interesting results related to these operations are proved.

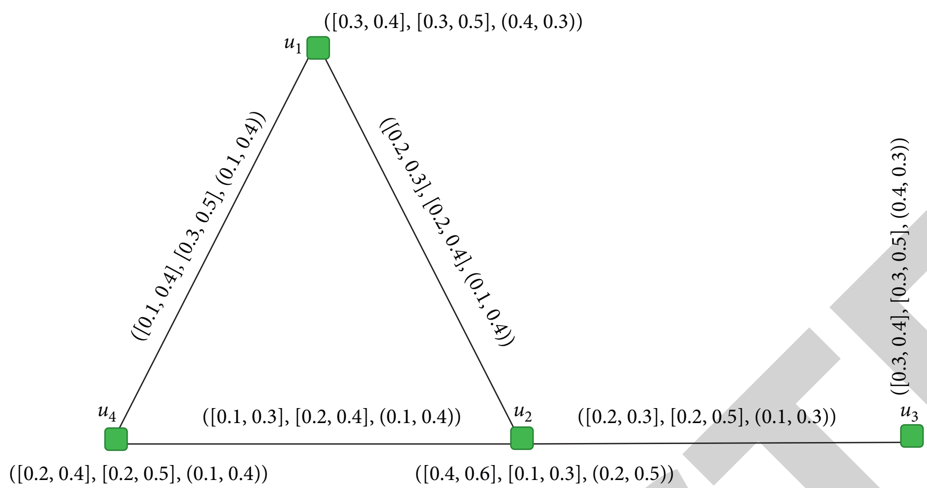


Figure 8: Cubic intuitionistic fuzzy graph.

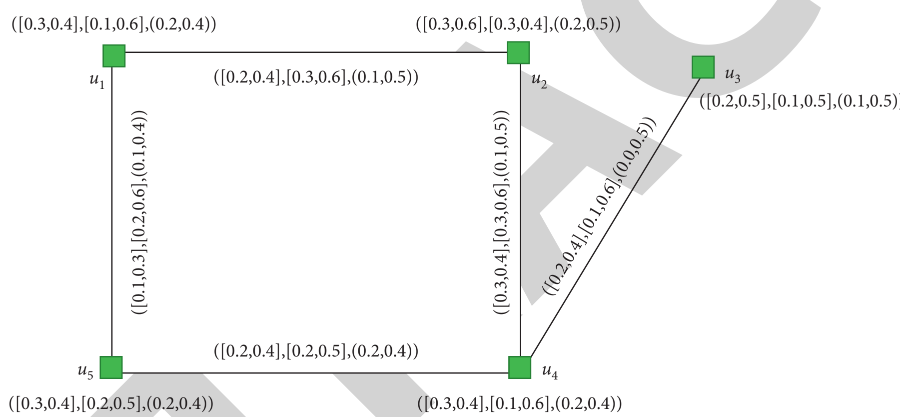


Figure 9: Cubic intuitionistic fuzzy graph.

Definition 18. The Cartesian product $\check{G} = \check{G}_1 \times \check{G}_2 = (A_1 \times A_2, \mathscr{B}_1 \times \mathscr{B}_2)$of two cubic IFGs $\check{G}_1 = (A_1, \mathscr{B}_1)$ and $\check{G}_2 = (A_2, \mathscr{B}_2)$ of the graphs $\check{G}_1^* = (\mathscr{V}_1, E_1)$ and $\check{G}_2^* = (\mathscr{V}_2, E_2)$ is defined as follows:

(i)

$$
\begin{aligned}
(M_{A1L} \times M_{A2L})(u_1, u_2) &= \min(M_{A1L}(u_1), M_{A2L}(u_2)),\\
(M_{A1U} \times M_{A2U})(u_1, u_2) &= \min(M_{A1U}(u_1), M_{A2UM}(u_2)),\\
(Ŋ_{A1L} \times Ŋ_{A2L})(u_1, u_2) &= \max(Ŋ_{A1L}(u_1), Ŋ_{A2L}(u_2)),\\
(Ŋ_{A1U} \times Ŋ_{A2U})(u_1, u_2) &= \max(Ŋ_{A1U}(u_1), Ŋ_{A2U}(u_2)),\\
(M_{A1} \times M_{A2})(u_1, u_2) &= \min(M_{A1}(u_1), M_{A2}(u_2)),\\
(Ŋ_{A1} \times Ŋ_{A2})(u_1, u_2) &= \max(Ŋ_{A1}(u_1), Ŋ_{A2}(u_2)), \quad \text{for all } u_1, u_2 \in \mathscr{V}.
\end{aligned}
\tag{8}
$$

(ii)

$$\begin{aligned}
(M_{\mathscr{B}1L} \times M_{\mathscr{B}2L})(u, u_2)(u, y_2) &= \min(M_{A1L}(u), M_{\mathscr{B}2L}(u_2 y_2)),\\
(M_{\mathscr{B}1U} \times M_{\mathscr{B}2U})(u, u_2)(u, y_2) &= \min(M_{A1U}(u), M_{\mathscr{B}2U}(u_2 y_2)),\\
(\eta_{\mathscr{B}1L} \times \eta_{\mathscr{B}2L})(u, u_2)(u, y_2) &= \max(\eta_{A1L}(u), \eta_{\mathscr{B}2L}(u_2 y_2)),\\
(\eta_{\mathscr{B}1U} \times \eta_{\mathscr{B}2U})(u, u_2)(u, y_2) &= \max(\eta_{A1U}(u), \eta_{\mathscr{B}2U}(u_2 y_2)),\\
(M_{\mathscr{B}1} \times M_{\mathscr{B}2})(u, u_2)(u, y_2) &= \min(M_{A1}(u), M_{\mathscr{B}2}(u_2 y_2)),\\
(\eta_{\mathscr{B}1} \times \eta_{\mathscr{B}2})(u, u_2)(u, y_2) &= \max(\eta_{A1}(u), \eta_{\mathscr{B}2}(u_2 y_2)), \quad \text{for all } u \in \mathscr{V}_1 \text{ and } u_2 y_2 \in E_2.
\end{aligned} \tag{9}$$

(iii)

$$\begin{aligned}
(M_{\mathscr{B}1L} \times M_{\mathscr{B}2L})(u_1, z)(y_1, z) &= \min(M_{\mathscr{B}1L}(u_1 y_1), M_{A2L}(z)),\\
(M_{\mathscr{B}1U} \times M_{\mathscr{B}2U})(u_1, z)(y_1, z) &= \min(\mathscr{B}_{A1U}(u_1 y_1), M_{A2U}(z)),\\
(\eta_{\mathscr{B}1L} \times \eta_{\mathscr{B}2L})(u_1, z)(y_1, z) &= \max(\eta_{\mathscr{B}1L}(u_1 y_1), \eta_{A2L}(z)),\\
(\eta_{\mathscr{B}1U} \times \eta_{\mathscr{B}2U})(u_1, z)(y_1, z) &= \max(\eta_{\mathscr{B}1U}(u_1 y_1), \eta_{A2U}(z)),\\
(M_{\mathscr{B}1} \times M_{\mathscr{B}2})(u_1, z)(y_1, z) &= \min(M_{\mathscr{B}1}(u_1 y_1), M_{A2}(z)),\\
(\eta_{\mathscr{B}1} \times \eta_{\mathscr{B}2})(u_1, z)(y_1, z) &= \max(\eta_{\mathscr{B}1}(u_1 y_1), \eta_{A2}(z)), \quad \text{for all } z \in \mathscr{V}_2 \text{ and } u_1 y_1 \in E_1.
\end{aligned} \tag{10}$$

Example 9. Let $\breve{G}^* = (\mathscr{V}, E)$ be a graph, where $\mathscr{V}$ is the set of vertices and E is the set of edges; then the product of two cubic IFGs in Figures 10–12 is given below.

Proposition 3. *If $\breve{G}_1$ and $\breve{G}_2$ are strong cubic IFGs, then the Cartesian product $\breve{G}_1 \times \breve{G}_2$ is also strong cubic IFG.*

Proof. Suppose that $\breve{G}_1$ and $\breve{G}_2$ are strong cubic IFGs; then there exist $u_i, y_i \in E_i$ such that

$$\begin{aligned}
M_{\mathscr{B}L}(u_i, y_i) &= \min(M_{AL}(u_i), M_{AL}(y_i)),\\
\eta_{\mathscr{B}L}(u_i, y_i) &= \max(\eta_{AL}(u_i), \eta_{AL}(y_i)),\\
M_{\mathscr{B}U}(u_i, y_i) &= \min(M_{AU}(u_i), M_{AU}(y_i)),\\
\eta_{\mathscr{B}U}(u_i, y_i) &= \max(\eta_{AU}(u_i), \eta_{AU}(y_i)),\\
M_{\mathscr{B}}(u_i, y_i) &= \min(M_A(u_i), M_A(y)),\\
\eta_{\mathscr{B}}(u_i, y_i) &= \max(M_A(u_i), M_A(y_i)).
\end{aligned} \tag{11}$$

Consider $E = \{(u, u_2)(u, y_2)/u_2 \in \mathscr{V}_1, \quad u_2 y_2 \in E_2\} \cup \{(u_1, z)\ (y_1, z)/z \in \mathscr{V}_2, u_1 y_1 \in E_1\}$.

Let $(u, u_2)(u, y_2) \in E$; then

$$\begin{aligned}
(M_{\mathscr{B}1L} \times M_{\mathscr{B}2L})(u, u_2)(u, y_2) &= \min(M_{A1L}(u), M_{\mathscr{B}2L}(u_2 y_2))\\
&= \min(M_{A1L}(u), M_{A2L}(u_2), M_{A2L}(y_2)).
\end{aligned} \tag{12}$$

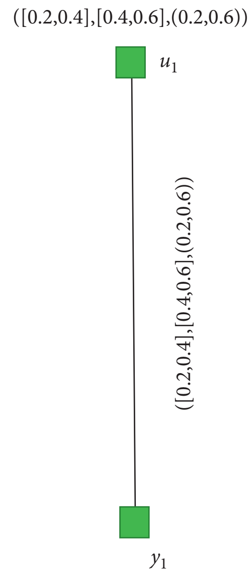


Figure 10: Cubic intuitionistic fuzzy graph.

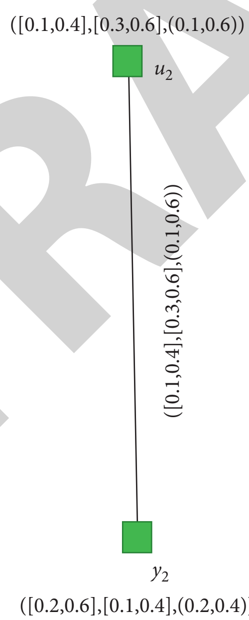


Figure 11: Cubic intuitionistic fuzzy graph.

Similarly,

$$
\begin{aligned}
(M_{\mathscr{B}1L} \times M_{\mathscr{B}2L})(u,u_2)(u,y_2) &= \min(M_{A1L}(u), M_{\mathscr{B}2L}(u_2 y_2)) = \min(M_{A1U}(u), M_{A2U}(u_2), M_{A2U}(y_2)),\\
(M_{A1L} \times M_{A2L})(u_1,u_2) &= \min(M_{A1L}(u_1), M_{A2L}(u_2)),\\
(M_{A1L} \times M_{A2L})(u_1,u_2) &= \min(M_{A1L}(u_1), M_{A2LM}(u_2)),\\
(M_{A1U} \times M_{A2U})(u_1,y_2) &= \min(M_{A1U}(u_1), M_{A2U}(y_2)),\\
(M_{A1U} \times M_{A2U})(u_1,y_2) &= \min(M_{A1U}(u_1), M_{A2U}(y_2)),\\
&= \min((M_{A1U} \times M_{A2U})(u,u_2), (M_{A1U} \times M_{A2U})(u,y_2))\\
&= \min(\min(M_{A1U}(u), M_{A2U}(u_2)), \min(M_{A1U}(u), M_{A2U}(y_2)))\\
&= \min((M_{A1U}(u), M_{A2U}(u_2), M_{A2U}(y_2))).
\end{aligned}
\tag{13}
$$

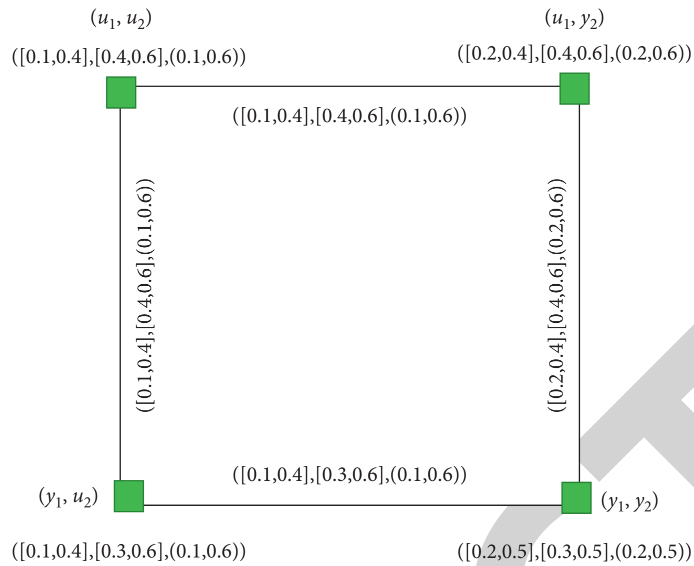


Figure 12: Cartesian product of cubic intuitionistic fuzzy graph.

Hence,

$$\begin{aligned}(M_{\mathscr{B}1L}\times M_{\mathscr{B}2L})(u,u_2)(u,y_2)&=\min((M_{A1L}\times M_{A2L})(u,u_2),(M_{A1L}\times M_{A2L})(u,y_2)),\\(M_{\mathscr{B}1U}\times M_{\mathscr{B}2U})(u,u_2)(u,y_2)&=\min((M_{A1U}\times M_{A2U})(u,u_2),(M_{A1U}\times M_{A2U})(u,y_2)).\end{aligned}\tag{14}$$

Similarly, we can show that

$$\begin{aligned}(\eta_{\mathscr{B}1L}\times \eta_{\mathscr{B}2L})(u,u_2)(u,y_2)&=\max((\eta_{A1L}\times \eta_{A2L})(u,u_2),(\eta_{A1L}\times \eta_{A2L})(u,y_2)),\\(\eta_{\mathscr{B}1U}\times \eta_{\mathscr{B}2U})(u,u_2)(u,y_2)&=\max((\eta_{A1U}\times \eta_{A2U})(u,u_2),(\eta_{A1U}\times \eta_{A2U})(u,y_2)),\\(M_{\mathscr{B}1}\times M_{\mathscr{B}2})(u,u_2)(u,y_2)&=\min((M_{A1}\times M_{A2})(u,u_2),(M_{A1}\times M_{A2})(u,y_2)),\\(\eta_{\mathscr{B}1}\times \eta_{\mathscr{B}2})(u,u_2)(u,y_2)&=\max((\eta_{A1}\times \eta_{A2})(u,u_2),(\eta_{A1}\times \eta_{A2})(u,y_2)).\end{aligned}\tag{15}$$

□

Proposition 4. *If $\breve{G}_1\times\breve{G}_2$ is a strong cubic IFG, then at least $\breve{G}_1$ or $\breve{G}_2$ must be strong.*

Proof. Suppose that $\breve{G}_1$ and $\breve{G}_2$ are not strong cubic IFGs, then there exist $u_i, y_i \in E_i$ such that

$$\begin{aligned}M_{\mathscr{B}L}(u_i,y_i)&<\min(M_{AL}(u_i),M_{AL}(y_i)),\\\eta_{\mathscr{B}L}(u_i,y_i)&>\max(\eta_{AL}(u_i),\eta_{AL}(y_i)),\\M_{\mathscr{B}U}(u_i,y_i)&<\min(M_{AU}(u_i),M_{AU}(y_i)),\\\eta_{\mathscr{B}U}(u_i,y_i)&>\max(\eta_{AU}(u_i),\eta_{AU}(y_i)),\\M_{\mathscr{B}}(u_i,y_i)&<\min(M_A(u_i),M_A(y)),\\\eta_{\mathscr{B}}(u_i,y_i)&>\max(M_A(u_i),M_A(y_i)).\end{aligned}\tag{16}$$

Consider $E=\{(u,u_2)(u,y_2)/u_2\in\mathscr{V}_1,\ \ u_2y_2\in E_2\}\cup\{(u_1,z)(y_1,z)/z\in\mathscr{V}_2,u_1y_1\in E_1\}$.

Let $(u,u_2)(u,y_2)\in E$, then

$$\begin{aligned}&(M_{\mathscr{B}1L}\times M_{\mathscr{B}2L})(u,u_2)(u,y_2)\\&\quad=\min(M_{A1L}(u),M_{\mathscr{B}2L}(u_2y_2))\\&\quad<\min(M_{A1L}(u),M_{A2L}(u_2),M_{A2L}(y_2)).\end{aligned}\tag{17}$$

Similarly,

$$
\begin{aligned}
\left(M_{\mathscr{B}1L}\times M_{\mathscr{B}2l}\right)\left(u,u_2\right)\left(u,y_2\right) &= \min\left(M_{A1L}(u), M_{\mathscr{B}2L}\left(u_2 y_2\right)\right) < \min\left(M_{A1U}(u), M_{A2U}\left(u_2\right), M_{A2U}\left(y_2\right)\right),\\
\left(M_{A1L}\times M_{A2L}\right)\left(u_1,u_2\right) &= \min\left(M_{A1L}\left(u_1\right), M_{A2L}\left(u_2\right)\right),\\
\left(M_{A1U}\times M_{A2UM}\right)\left(u_1,u_2\right) &= \min\left(M_{A1U}\left(u_1\right), M_{A2U}\left(u_2\right)\right),\\
\left(M_{A1L}\times M_{A2L}\right)\left(u_1,y_2\right) &= \min\left(M_{A1L}\left(u_1\right), M_{A2L}\left(y_2\right)\right),\\
\left(M_{A1U}\times M_{A2U}\right)\left(u_1,y_2\right) &= \min\left(M_{A1U}\left(u_1\right), M_{A2U}\left(y_2\right)\right)\\
&= \min\left(\left(M_{A1U}\times M_{A2U}\right)\left(u,u_2\right), \left(\left(M_{A1U}\times M_{A2U}\right)\left(u,y_2\right)\right)\right.\\
&= \min\left(\min\left(M_{A1U}(u), M_{A2U}\left(u_2\right), \min\left(\left(M_{A1U}(u), M_{A2U}\left(y_2\right)\right)\right)\right)\right.\\
&= \min\left(M_{A1U}(u), M_{A2U}\left(u_2\right), M_{A2U}\left(y_2\right)\right).
\end{aligned}
\tag{18}
$$

Hence,

$$
\begin{aligned}
\left(M_{\mathscr{B}1L}\times M_{\mathscr{B}2L}\right)\left(u,u_2\right)\left(u,y_2\right) &< \min\left(\left(M_{A1L}\times M_{A2L}\right)\left(u,u_2\right), \left(M_{A1L}\times M_{A2L}\right)\left(u,y_2\right)\right),\\
\left(M_{\mathscr{B}1U}\times M_{\mathscr{B}2U}\right)\left(u,u_2\right)\left(u,y_2\right) &< \min\left(\left(M_{A1U}\times M_{A2U}\right)\left(u,u_2\right), \left(M_{A1U}\times M_{A2U}\right)\left(u,y_2\right)\right).
\end{aligned}
\tag{19}
$$

Similarly, we can show that

$$
\begin{aligned}
\left(\text{Ŋ}_{\mathscr{B}1L}\times \text{Ŋ}_{\mathscr{B}2L}\right)\left(u,u_2\right)\left(u,y_2\right) &> \max\left(\left(\text{Ŋ}_{A1L}\times \text{Ŋ}_{A2L}\right)\left(u,u_2\right), \left(\text{Ŋ}_{A1L}\times \text{Ŋ}_{A2L}\right)\left(u,y_2\right)\right),\\
\left(\text{Ŋ}_{\mathscr{B}1U}\times \text{Ŋ}_{\mathscr{B}2U}\right)\left(u,u_2\right)\left(u,y_2\right) &> \max\left(\left(\text{Ŋ}_{A1U}\times \text{Ŋ}_{A2U}\right)\left(u,u_2\right), \left(\text{Ŋ}_{A1U}\times \text{Ŋ}_{A2U}\right)\left(u,y_2\right)\right),\\
\left(M_{\mathscr{B}1}\times M_{\mathscr{B}2}\right)\left(u,u_2\right)\left(u,y_2\right) &< \min\left(\left(M_{A1}\times M_{A2}\right)\left(u,u_2\right), \left(M_{A1}\times M_{A2}\right)\left(u,y_2\right)\right),\\
\left(\text{Ŋ}_{\mathscr{B}1}\times \text{Ŋ}_{\mathscr{B}2}\right)\left(u,u_2\right)\left(u,y_2\right) &> \max\left(\left(\text{Ŋ}_{A1}\times \text{Ŋ}_{A2}\right)\left(u,u_2\right), \left(\text{Ŋ}_{A1}\times \text{Ŋ}_{A2}\right)\left(u,y_2\right)\right).
\end{aligned}
\tag{20}
$$

Therefore, $\check{G}_1 \times \check{G}_2$ is not a strong cubic IFG, which is a contradiction. This completes the proof. □

Definition 19. The composition $\check{G}_1[\check{G}_2] = \check{G}_1 \circ \check{G}_2 = (A_1 \circ A_2, \mathscr{B}_1 \circ \mathscr{B}_2)$ of two cubic IFGs $\check{G}_1 = (A_1, \mathscr{B}_1)$ and $\check{G}_2 = (A_2, \mathscr{B}_2)$ of the graphs $\check{G}_1^* = (\mathscr{V}_1, E_1)$ and $\check{G}_2^* = (\mathscr{V}_2, E_2)$ is defined as follows:

(i)

$$
\begin{aligned}
\left(M_{A1L}\circ \mathrm{M}_{A2L}\right)\left(u_1,u_2\right) &= \min\left(M_{A1L}\left(u_1\right), M_{A2L}\left(u_2\right)\right),\\
\left(M_{A1U}\circ \mathrm{M}_{A2U}\right)\left(u_1,u_2\right) &= \min\left(M_{A1U}\left(u_1\right), M_{A2U}\left(u_2\right)\right),\\
\left(\text{Ŋ}_{A1L}\circ \text{Ŋ}_{A2L}\right)\left(u_1,u_2\right) &= \max\left(\text{Ŋ}_{A1L}\left(u_1\right), \text{Ŋ}_{A2L}\left(u_2\right)\right),\\
\left(\text{Ŋ}_{A1U}\circ \text{Ŋ}_{A2U}\right)\left(u_1,u_2\right) &= \max\left(\text{Ŋ}_{A1U}\left(u_1\right), \text{Ŋ}_{A2U}\left(u_2\right)\right),\\
\left(M_{A1}\circ \mathrm{M}_{A2}\right)\left(u_1,u_2\right) &= \min\left(M_{A1}\left(u_1\right), M_{A2}\left(u_2\right)\right),\\
\left(\text{Ŋ}_{A1}\circ \text{Ŋ}_{A2}\right)\left(u_1,u_2\right) &= \max\left(\text{Ŋ}_{A1}\left(u_1\right), \text{Ŋ}_{A2}\left(u_2\right)\right), \quad \text{for all } u_1, u_2 \in \mathscr{V}
\end{aligned}
\tag{21}
$$

(ii)

$$
\begin{aligned}
\left(M_{\mathscr{B}1L}\circ \mathrm{M}_{\mathscr{B}2L}\right)\left(u,u_2\right)\left(u,y_2\right) &= \min\left(M_{A1L}(u), M_{\mathscr{B}2L}\left(u_2 y_2\right)\right),\\
\left(M_{\mathscr{B}1U}\circ \mathrm{M}_{\mathscr{B}2U}\right)\left(u,u_2\right)\left(u,y_2\right) &= \min\left(M_{A1U}(u), M_{\mathscr{B}2U}\left(u_2 y_2\right)\right),\\
\left(\text{Ŋ}_{\mathscr{B}1L}\circ \text{Ŋ}_{\mathscr{B}2L}\right)\left(u,u_2\right)\left(u,y_2\right) &= \max\left(\text{Ŋ}_{A1L}(u), \text{Ŋ}_{\mathscr{B}2L}\left(u_2 y_2\right)\right),\\
\left(\text{Ŋ}_{\mathscr{B}1U}\circ \text{Ŋ}_{\mathscr{B}2U}\right)\left(u,u_2\right)\left(u,y_2\right) &= \max\left(\text{Ŋ}_{A1U}(u), \text{Ŋ}_{\mathscr{B}2U}\left(u_2 y_2\right)\right),\\
\left(M_{\mathscr{B}1}\circ \mathrm{M}_{\mathscr{B}2}\right)\left(u,u_2\right)\left(u,y_2\right) &= \min\left(M_{A1}(u), M_{\mathscr{B}2}\left(u_2 y_2\right)\right),\\
\left(\text{Ŋ}_{\mathscr{B}1}\circ \text{Ŋ}_{\mathscr{B}2}\right)\left(u,u_2\right)\left(u,y_2\right) &= \max\left(\text{Ŋ}_{A1}(u), \text{Ŋ}_{\mathscr{B}2}\left(u_2 y_2\right)\right), \quad \text{for all } u \in \mathscr{V}_1 \text{ and } u_2 y_2 \in E_2.
\end{aligned}
\tag{22}
$$

(iii)

$$
\begin{aligned}
(M_{\mathscr{B}1L}{\circ}\mathrm{M}_{\mathscr{B}2\mathrm{L}})(u_1,z)(y_1,z) &= \min(M_{\mathscr{B}1L}(u_1y_1), M_{A2L}(z)),\\
(M_{\mathscr{B}1U}{\circ}\mathrm{M}_{\mathscr{B}2\mathrm{U}})(u_1,z)(y_1,z) &= \min(\mathscr{B}_{A1U}(u_1y_1), M_{A2U}(z)),\\
(\eta_{\mathscr{B}1L}{\circ}\eta_{\mathscr{B}2L})(u_1,z)(y_1,z) &= \max(\eta_{\mathscr{B}1L}(u_1y_1), \eta_{A2L}(z)),\\
(\eta_{\mathscr{B}1U}{\circ}\eta_{\mathscr{B}2U})(u_1,z)(y_1,z) &= \max(\eta_{\mathscr{B}1U}(u_1y_1), \eta_{A2U}(z)),\\
((M_{\mathscr{B}1}{\circ}\mathrm{M}_{\mathscr{B}2})(u_1,z)(y_1,z) &= \min(M_{\mathscr{B}1}(u_1y_1), M_{A2}(z)),\\
(\eta_{\mathscr{B}1}{\circ}\eta_{\mathscr{B}2})(u_1,z)(y_1,z) &= \max(\eta_{\mathscr{B}1}(u_1y_1), \eta_{A2}(z)), \quad \text{for all } z \in \mathscr{V}_2 \text{ and } u_1y_1 \in E_1.
\end{aligned}
\tag{23}
$$

(iv)

$$
\begin{aligned}
(M_{\mathscr{B}1L}{\circ}\mathrm{M}_{\mathscr{B}2\mathrm{L}})(u_1,u_2)(y_1,y_2) &= \min(M_{A2L}(u_2), M_{A2L}(y_2), M_{\mathscr{B}1L}(u_1y_1)),\\
(M_{\mathscr{B}1U}{\circ}\mathrm{M}_{\mathscr{B}2\mathrm{U}})(u_1,u_2)(y_1,y_2) &= \min(M_{A2U}(u_2), M_{A2U}(y_2), M_{\mathscr{B}1U}(u_1y_1)),\\
(\eta_{\mathscr{B}1L}{\circ}\eta_{\mathscr{B}2L})(u_1,u_2)(y_1,y_2) &= \max(\eta_{A2L}(u_2), \eta_{A2L}(y_2), \eta_{\mathscr{B}1L}(u_1y_1)),\\
(\eta_{\mathscr{B}1U}{\circ}\eta_{\mathscr{B}2U})(u_1,u_2)(y_1,y_2) &= \max(\eta_{A2U}(u_2), \eta_{A2U}(y_2), \eta_{\mathscr{B}1U}(u_1y_1)),\\
(M_{\mathscr{B}1}{\circ}\mathrm{M}_{\mathscr{B}2})(u_1,u_2)(y_1,y_2) &= \min(M_{A2}(u_2), M_{A2}(y_2), M_{\mathscr{B}1}(u_1y_1)),\\
(\eta_{\mathscr{B}1}{\circ}\eta_{\mathscr{B}2})(u_1,u_2)(y_1,y_2) &= \max(\eta_{A2}(u_2), \eta_{A2}(y_2), \eta_{\mathscr{B}1}(u_1y_1)), \quad \text{for all } (u_1,u_2)(y_1,y_2) \in E^{\circ} - E.
\end{aligned}
\tag{24}
$$

Proof. The proof is straightforward. □

Example 10. Let $\breve{G}^* = (\mathscr{V}, E)$ be a graph; then the compositions of two cubic IFGs in Figures 13–15 are given as follows.

Proposition 5. *The composition $\breve{G}_1[\breve{G}_2]$ of cubic IFG for the graphs $\breve{G}_1$ and $\breve{G}_2$ of the graphs $\breve{G}_1^*$ and $\breve{G}_2^*$ is a cubic IFG of $\breve{G}_1^*[\breve{G}_2^*]$.*

Definition 20. The union $\breve{G}_1 \cup \breve{G}_2 = (A_1 \cup A_2, \mathscr{B}_1 \cup \mathscr{B}_2)$of two cubic IFGs $\breve{G}_1 = (A_1, \mathscr{B}_1)$ and $\breve{G}_2 = (A_2, \mathscr{B}_2)$ of the graphs $\breve{G}_1^* = (\mathscr{V}_1, E_1)$ and $\breve{G}_2^* = (\mathscr{V}_2, E_2)$ is defined as follows:

(i)

$$
\begin{cases}
(M_{A1L} \cup M_{A2L})(u) = M_{A1L}(u), & \text{if } u \in \mathscr{V}_1 - \mathscr{V}_2,\\
(M_{A1L} \cup M_{A2L})(u) = M_{A2L}(u), & \text{if } u \in \mathscr{V}_2 - \mathscr{V}_1,\\
(M_{A1L} \cup M_{A2L})(u) = \max(M_{A1L}(u), M_{A2L}(u)), & \text{if } u \in \mathscr{V}_1 \cap \mathscr{V}_2.
\end{cases}
\tag{25}
$$

(ii)

$$
\begin{cases}
(M_{A1U} \cup M_{A2U})(u) = M_{A1U}(u), & \text{if } u \in \mathscr{V}_1 - \mathscr{V}_2,\\
(M_{A1U} \cup M_{A2U})(u) = M_{A2U}(u), & \text{if } u \in \mathscr{V}_2 - \mathscr{V}_1,\\
(M_{A1U} \cup M_{A2U})(u) = \max(M_{A1U}(u), M_{A2U}(u)), & \text{if } u \in \mathscr{V}_1 \cap \mathscr{V}_2.
\end{cases}
\tag{26}
$$

(iii)

$$
\begin{cases}
(\eta_{A1L} \cap \eta_{A2L})(u) = \eta_{A1L}(u), & \text{if } u \in \mathscr{V}_1 - \mathscr{V}_2,\\
(\eta_{A1L} \cap \eta_{A2L})(u) = \eta_{A2L}(u), & \text{if } u \in \mathscr{V}_2 - \mathscr{V}_1,\\
(\eta_{A1L} \cap \eta_{A2L})(u) = \min(\eta_{A1L}(u), \eta_{A2L}(u)), & \text{if } u \in \mathscr{V}_1 \cap \mathscr{V}_2.
\end{cases}
\tag{27}
$$

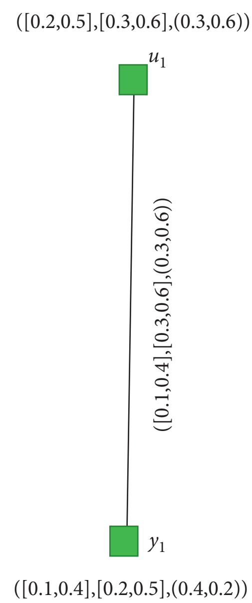


FIGURE 13: Cubic intuitionistic fuzzy graph.

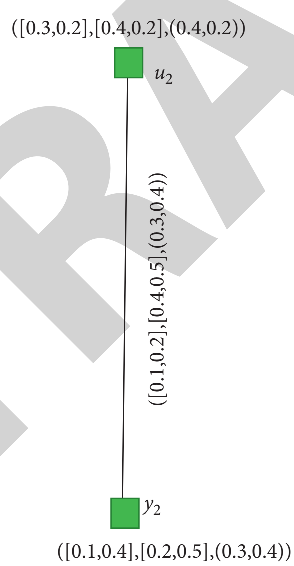


FIGURE 14: Cubic intuitionistic fuzzy graph.

(iv)

$$\begin{cases} (\text{Ŋ}_{A1U} \cap \text{Ŋ}_{A2U})(u) = \text{Ŋ}_{A1U}(u), & \text{if } u \in \mathscr{V}_1 - \mathscr{V}_2, \\ (\text{Ŋ}_{A1U} \cap \text{Ŋ}_{A2U})(u) = \text{Ŋ}_{A2U}(u) & \text{if } u \in \mathscr{V}_2 - \mathscr{V}_1, \\ (\text{Ŋ}_{A1U} \cap \text{Ŋ}_{A2U})(u) = min(\text{Ŋ}_{A1U}(u), \text{Ŋ}_{A2U}(u)), & \text{if } u \in \mathscr{V}_1 \cap \mathscr{V}_2. \end{cases} \tag{28}$$

(v)

$$\begin{cases} (M_{A1} \cup M_{A2})(u) = M_{A1}(u), & \text{if } u \in \mathscr{V}_1 - \mathscr{V}_2, \\ (M_{A1} \cup M_{A2})(u) = M_{A2}(u), & \text{if } u \in \mathscr{V}_2 - \mathscr{V}_1, \\ (M_{A1} \cup M_{A2})(u) = \max(M_{A1}(u), M_{A2}(u)), & \text{if } u \in \mathscr{V}_1 \cap \mathscr{V}_2. \end{cases} \tag{29}$$

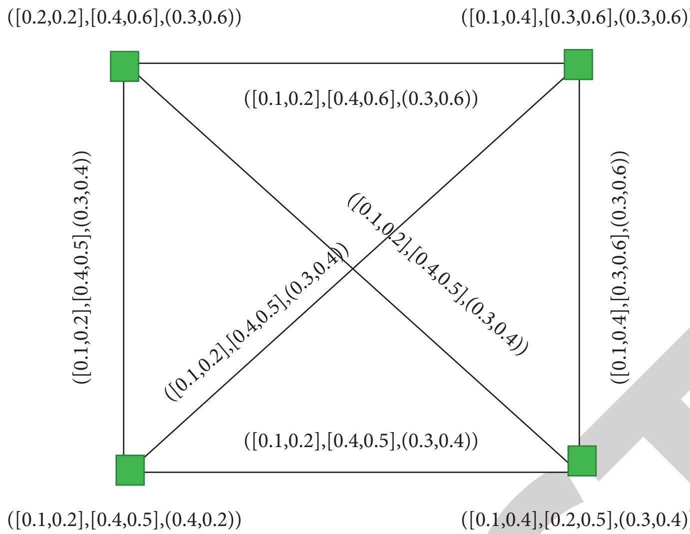


Figure 15: Composition of cubic intuitionistic fuzzy graph.

(vi)

$$\begin{cases} (\eta_{A1} \cap \eta_{A2})(u) = \eta_{A1}(u), & \text{if } u \in \mathscr{V}_1 - \mathscr{V}_2, \\ (\eta_{A1} \cap \eta_{A2})(u) = \eta_{A2}(u), & \text{if } u \in \mathscr{V}_2 - \mathscr{V}_1, \\ (\eta_{A1} \cap \eta_{A2})(u) = \min(\eta_{A1}(u), \eta_{A2}(u)), & \text{if } u \in \mathscr{V}_1 \cap \mathscr{V}_2. \end{cases} \tag{30}$$

(vii)

$$\begin{cases} (M_{\mathscr{B}1L} \cup M_{\mathscr{B}2L})(uy) = M_{\mathscr{B}1L}(uy), & \text{if } uy \in E_1 - E_2, \\ (M_{\mathscr{B}1L} \cup M_{\mathscr{B}2L})(uy) = M_{\mathscr{B}2L}(uy), & \text{if } uy \in E_2 - E_1, \\ (M_{\mathscr{B}1L} \cup M_{\mathscr{B}2L})(uy) = \max(M_{\mathscr{B}1L}(uy), M_{\mathscr{B}2L}(uy)), & \text{if } y \in E_1 \cap E_2. \end{cases} \tag{31}$$

(viii)

$$\begin{cases} (M_{\mathscr{B}1U} \cup M_{\mathscr{B}2U})(uy) = M_{\mathscr{B}1U}(uy), & \text{if } uy \in E_1 - E_2, \\ (M_{\mathscr{B}1U} \cup M_{\mathscr{B}2U})(uy) = M_{\mathscr{B}2U}(uy), & \text{if } uy \in E_2 - E_1, \\ (M_{\mathscr{B}1U} \cup M_{\mathscr{B}2U})(uy) = \max(M_{\mathscr{B}1U}(uy), M_{\mathscr{B}2U}(uy)), & \text{if } uy \in E_1 \cap E_2. \end{cases} \tag{32}$$

(ix)

$$\begin{cases} (\eta_{\mathscr{B}1L} \cap \eta_{\mathscr{B}2L})(uy) = \eta_{\mathscr{B}1L}(uy), & \text{if } uy \in E_1 - E_2, \\ (\eta_{\mathscr{B}1L} \cap \eta_{\mathscr{B}2L})(uy) = \eta_{\mathscr{B}2L}(uy), & \text{if } uy \in E_2 - E_1, \\ (\eta_{\mathscr{B}1L} \cap \eta_{\mathscr{B}2L})(uy) = \min(\eta_{\mathscr{B}1L}(uy), \eta_{\mathscr{B}2L}(uy)), & \text{if } uy \in E_1 \cap E_2. \end{cases} \tag{33}$$

(x)

$$\begin{cases} (\eta_{\mathscr{B}1U} \cap \eta_{\mathscr{B}2U})(uy) = \eta_{\mathscr{B}1U}(uy), & \text{if } uy \in E_1 - E_2, \\ (\eta_{\mathscr{B}1U} \cap \eta_{\mathscr{B}2U})(uy) = \eta_{\mathscr{B}2U}(uy), & \text{if } uy \in E_2 - E_1, \\ (\eta_{\mathscr{B}1U} \cap \eta_{\mathscr{B}2U})(uy) = \min(\eta_{\mathscr{B}1U}(uy), \eta_{\mathscr{B}2U}(uy)), & \text{if } uy \in E_1 \cap E_2. \end{cases} \tag{34}$$

(xi)

$$\begin{cases} (M_{\mathscr{B}1} \cup M_{\mathscr{B}2})(uy) = M_{\mathscr{B}1}(uy), & \text{if } uy \in E_1 - E_2, \\ (M_{\mathscr{B}1} \cup M_{\mathscr{B}2})(uy) = M_{\mathscr{B}2}(uy), & \text{if } uy \in E_2 - E_1, \\ (M_{\mathscr{B}1} \cup M_{\mathscr{B}2})(uy) = \max(M_{\mathscr{B}1}(uy), M_{\mathscr{B}2}(uy)), & \text{if } uy \in E_1 \cap E_2. \end{cases} \tag{35}$$

(xii)

$$\begin{cases} (\eta_{\mathscr{B}1} \cap \eta_{\mathscr{B}2})(uy) = \eta_{\mathscr{B}1}(uy), & \text{if } uy \in E_1 - E_2, \\ (\eta_{\mathscr{B}1} \cap \eta_{\mathscr{B}2})(uy) = \eta_{\mathscr{B}2}(uy), & \text{if } uy \in E_2 - E_1, \\ (\eta_{\mathscr{B}1} \cap \eta_{\mathscr{B}2})(uy) = \min(\eta_{\mathscr{B}1}(uy), \eta_{\mathscr{B}2}(uy)), & \text{if } uy \in E_1 \cap E_2. \end{cases} \tag{36}$$

Example 11. Let $\breve{G}^* = (\mathscr{V}, E)$ be a graph; then the union of two cubic IFGs is given below.

In Figures 16–18 the union of two CIFGs is defined.

Proposition 6. *The union of two cubic IFGs is a cubic IFG.*

Proof. Let $\breve{G}_1 = (A_1, \mathscr{B}_1)$ and $\breve{G}_2 = (A_2, \mathscr{B}_2)$ be the cubic IFGs $\breve{G}_1^*$ and $\breve{G}_2^*$, respectively. Then, we have to prove $\breve{G}_1 \cup \breve{G}_2 = (A_1 \cup A_2, \mathscr{B}_1 \cup \mathscr{B}_2)$is a cubic IFG and of the graphs $\breve{G}_1^* \cup \breve{G}_2^*$. As all the conditions of $A_1 \cup A_2$ are satisfied, we only have to verify the conditions of $\mathscr{B}_1 \cup \mathscr{B}_2$.

First assume that $uy \in E_1 \cap E_2$. Then,

$$\begin{aligned}
(M_{\mathscr{B}1L} \cup M_{\mathscr{B}2L})(uy) &= \max(M_{\mathscr{B}1L}(uy), M_{\mathscr{B}2L}(uy)) \\
&\leq \max(\min(M_{A1L}(u), M_{A1L}(y)), \min(M_{A2L}(u), M_{A2L}(y))), \\
&= \min(\max(M_{A1L}(u), M_{A2L}(u)), \max(M_{A1L}(y), M_{A2L}(y))), \\
&= \min(M_{A1L} \cup M_{A2L})(u), (M_{A1L} \cup M_{A2L})(y), \\
(M_{\mathscr{B}1U \cup} M_{\mathscr{B}2U})(uy) &= \max(M_{\mathscr{B}1U}(uy), M_{\mathscr{B}2U}(uy)) \\
&\leq \max(\min(M_{A1U}(u), M_{A1U}(y)), \min(M_{A2U}(u), M_{A2L}(y))) \\
&= \min(\max(M_{A1U}(u), M_{A2U}(u)), \max(M_{A1UM}(y), M_{A2U}(y))) \\
&= \min(M_{A1U} \cup M_{A2U})(u), (M_{A1U} \cup M_{A2U})(y), \\
(\eta_{\mathscr{B}1L} \cup \eta_{\mathscr{B}2L})(uy) &= \min(\eta_{\mathscr{B}1L}(uy), \eta_{\mathscr{B}2L}(uy)) \\
&\leq \min(\max(\eta_{A1L}(u), \eta_{A1L}(y)), \max(\eta_{A2L}(u), \eta_{A2L}(y))), \\
&= \max(\min(\eta_{A1L}(u), \eta_{A2L}(u)), \min(\eta_{A1L}(y), \eta_{A2L}(y))), \\
&= \max(\eta_{A1L} \cup \eta_{A2L})(u), (\eta_{A1L} \cup \eta_{A2L})(y), \\
(\eta_{\mathscr{B}1U} \cup \eta_{\mathscr{B}2U})(uy) &= \min(\eta_{\mathscr{B}1U}(uy), \eta_{\mathscr{B}2U}(uy)) \\
&\leq \min(\max(\eta_{A1U}(u), \eta_{A1U}(y)), \max(\eta_{A2U}(u), \eta_{A2U}(y))), \\
&= \max(\min(\eta_{A1U}(u), \eta_{A2U}(u)), \min(\eta_{A1U}(y), \eta_{A2U}(y))), \\
&= \max(\eta_{A1U} \cup \eta_{A2U})(u), (\eta_{A1U} \cup \eta_{A2U})(y), \\
(M_{\mathscr{B}1} \cup M_{\mathscr{B}2})(uy) &= \max(M_{\mathscr{B}1}(uy), M_{\mathscr{B}2}(uy)) \\
&\leq \max(\min(M_{A1}(u), M_{A1}(y)), \min(M_{A2}(u), M_{A2}(y))), \\
&= \min(\max(M_{A1}(u), M_{A2}(u)), \max(M_{A1}(y), M_{A2}(y))), \\
&= \min(M_{A1} \cup M_{A2})(u), (M_{A1} \cup M_{A2})(y), \\
(\eta_{\mathscr{B}1} \cup \eta_{\mathscr{B}2})(uy) &= \min(\eta_{\mathscr{B}1}(uy), \eta_{\mathscr{B}2}(uy)) \\
&\leq \min(\max(\eta_{A1}(u), \eta_{A1}(y)), \max(\eta_{A2}(u), \eta_{A2}(y))) \\
&= \max(\min(\eta_{A1}(u), \eta_{A2}(u)), \min(\eta_{A1}(y), \eta_{A2}(y))) \\
&= \max(\eta_{A1} \cup \eta_{A2})(u), (\eta_{A1} \cup \eta_{A2})(y).
\end{aligned} \tag{37}$$

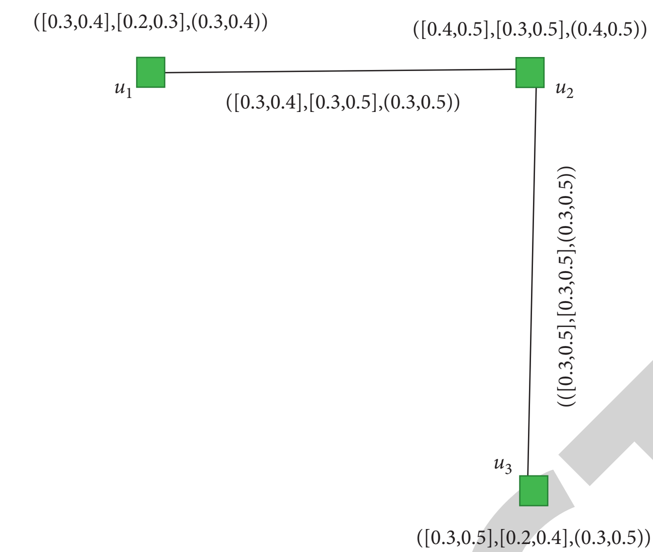


Figure 16: Cubic intuitionistic fuzzy graph.

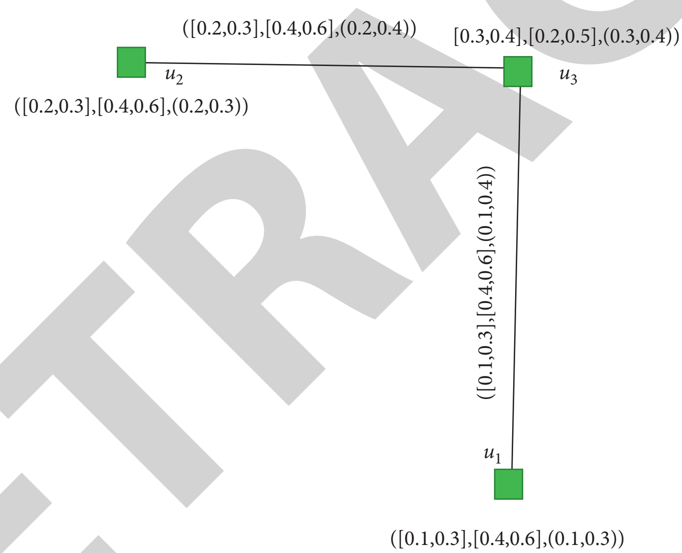


Figure 17: Cubic intuitionistic fuzzy graph.

If $uy \in E_1$ and $uy \notin E_2$, then

$$
\begin{aligned}
(M_{\mathscr{B}1L} \cup M_{\mathscr{B}2L})(uy) &\leq \min((M_{A1L} \cup M_{A2L})(u), (M_{A1L} \cup M_{A2L})(y)),\\
(M_{\mathscr{B}1U} \cup M_{\mathscr{B}2U})(uy) &\leq \min((M_{A1U} \cup M_{A2U})(u), (M_{A1U} \cup M_{A2U})(y)),\\
(\eta_{\mathscr{B}1L} \cup \eta_{\mathscr{B}2L})(uy) &\leq \max((\eta_{A1L} \cup \eta_{A2L})(u), (\eta_{A1L} \cup \eta_{A2L})(y)),\\
(\eta_{\mathscr{B}1U} \cup \eta_{\mathscr{B}2U})(uy) &\leq \max((\eta_{A1U} \cup \eta_{A2U})(u), (\eta_{A1U} \cup \eta_{A2U})(y)),\\
(M_{\mathscr{B}1} \cup M_{\mathscr{B}2})(uy) &\leq \min((M_{A1} \cup M_{A2})(u), (M_{A1} \cup M_{A2})(y)),\\
(\eta_{\mathscr{B}1} \cup \eta_{\mathscr{B}2})(uy) &\leq \max((\eta_{A1} \cup \eta_{A2})(u), (\eta_{A1} \cup \eta_{A2})(y)).
\end{aligned} \tag{38}
$$

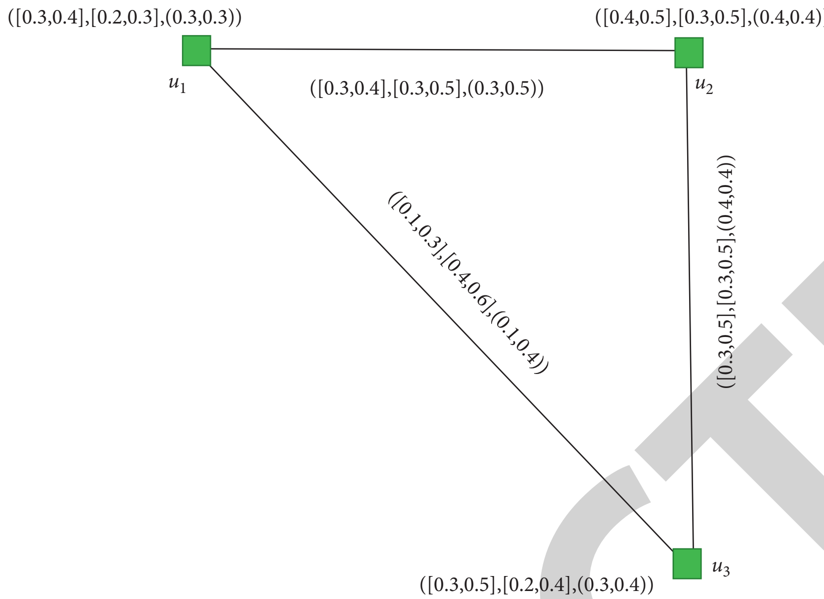


FIGURE 18: Union of cubic intuitionistic fuzzy graphs.

If $uy \notin E_1$ and $uy \in E_2$, then

$$\begin{aligned}
&(M_{\mathscr{B}1L} \cup M_{\mathscr{B}2L})(uy) \leq \min((M_{A1L} \cup M_{A2L})(u), (M_{A1L} \cup M_{A2L})(y)),\\
&(M_{\mathscr{B}1U} \cup M_{\mathscr{B}2U})(uy) \leq \min((M_{A1U} \cup M_{A2U})(u), (M_{A1U} \cup M_{A2U})(y)),\\
&(Ŋ_{\mathscr{B}1L} \cup Ŋ_{\mathscr{B}2L})(uy) \leq \max((Ŋ_{A1L} \cup Ŋ_{A2L})(u), (Ŋ_{A1L} \cup Ŋ_{A2L})(y)),\\
&(Ŋ_{\mathscr{B}1U} \cup Ŋ_{\mathscr{B}2U})(uy) \leq \max((Ŋ_{A1U} \cup Ŋ_{A2U})(u), (Ŋ_{A1U} \cup Ŋ_{A2U})(y)),\\
&(M_{\mathscr{B}1} \cup M_{\mathscr{B}2})(uy) \leq \min((M_{A1} \cup M_{A2})(u), (M_{A1} \cup M_{A2})(y)),\\
&(Ŋ_{\mathscr{B}1} \cup Ŋ_{\mathscr{B}2})(uy) \leq \max((Ŋ_{A1} \cup Ŋ_{A2})(u), (Ŋ_{A1} \cup Ŋ_{A2})(y)).
\end{aligned} \tag{39}$$

This completes the proof. □

Definition 21. The joint $\check{G}_1 + \check{G}_2 = (A_1 + A_2, \mathscr{B}_1 + \mathscr{B}_2)$ of two cubic IFGs $\check{G}_1 = (A_1, \mathscr{B}_1)$ and $\check{G}_2 = (A_2, \mathscr{B}_2)$ of the graphs $\check{G}_1^* = (\mathscr{V}_1, E_1)$ and $\check{G}_2^* = (\mathscr{V}_2, E_2)$ is defined as follows:

(i)

$$\begin{aligned}
&(M_{A1L} + M_{A2L})(u) = (M_{A1L} \cup M_{A2L})(u),\\
&(M_{A1U} + M_{A2U})(u) = (M_{A1U} \cup M_{A2U})(u)\\
&(Ŋ_{A1L} + Ŋ_{A2L})(u) = (Ŋ_{A1L} \cup Ŋ_{A2L})(u),\\
&(Ŋ_{A1U} + Ŋ_{A2U})(u) = (M_{A1U} \cup M_{A2U})(u),\\
&(M_{A1} + M_{A2})(u) = (M_{A1} \cup M_{A2})(u)\\
&(Ŋ_{A1} + Ŋ_{A2})(u) = (Ŋ_{A1} \cup Ŋ_{A2})(u).
\end{aligned} \tag{40}$$

If $u \in \mathscr{V}_1 \cup \mathscr{V}_2$,

(ii)

$$\begin{aligned}
&(M_{\mathscr{B}1L} + M_{\mathscr{B}2L})(uy) = (M_{\mathscr{B}1L} \cup M_{\mathscr{B}2L})(uy),\\
&(M_{\mathscr{B}1U} + M_{\mathscr{B}2U})(uy) = (M_{\mathscr{B}1U} \cup M_{\mathscr{B}2U})(uy),\\
&(Ŋ_{\mathscr{B}1L} + Ŋ_{\mathscr{B}2L})(uy) = (Ŋ_{\mathscr{B}1L} \cup Ŋ_{\mathscr{B}2L})(uy),\\
&(Ŋ_{\mathscr{B}1U} + Ŋ_{\mathscr{B}2U})(uy) = (M_{\mathscr{B}1U} \cup M_{\mathscr{B}2U})(uy),\\
&(M_{\mathscr{B}1} + M_{\mathscr{B}2})(uy) = (M_{\mathscr{B}1} \cup M_{\mathscr{B}2})(uy),\\
&(Ŋ_{\mathscr{B}1} + Ŋ_{\mathscr{B}2})(uy) = (Ŋ_{\mathscr{B}1} \cup Ŋ_{\mathscr{B}2})(uy)
\end{aligned} \tag{41}$$

$uy \in E_1 \cap E_2$, and then

$$\begin{aligned}
&(M_{\mathscr{B}1L} + M_{\mathscr{B}2L})(uy) = \min(M_{A1L}(u), M_{A2L}(y)),\\
&(M_{\mathscr{B}1U} + M_{\mathscr{B}2U})(uy) = \min(M_{A1U}(u), M_{A2U}(y)),\\
&(Ŋ_{\mathscr{B}1L} + Ŋ_{\mathscr{B}2L})(uy) = \max(Ŋ_{A1L}(u), Ŋ_{A2L}(y)),\\
&(Ŋ_{\mathscr{B}1U} + Ŋ_{\mathscr{B}2U})(uy) = \max(Ŋ_{A1U}(u), Ŋ_{A2U}(y)),\\
&(M_{\mathscr{B}1} + M_{\mathscr{B}2})(uy) = \min(M_{A1}(u), M_{A2}(y)),\\
&(Ŋ_{\mathscr{B}1} + Ŋ_{\mathscr{B}2})(uy) = \max(Ŋ_{A1}(u), Ŋ_{A2}(y)),
\end{aligned} \tag{42}$$

$uy \in E'$, where E' is the set of all edges joining the nodes of $\mathscr{V}_1$ and $\mathscr{V}_2$.

Proposition 7. *The joint of two cubic IFGs is a cubic IFG.*

Proof. Assume that $\breve{G}_1 = (A_1, \mathscr{B}_1)$ and $\breve{G}_2 = (A_2, \mathscr{B}_2)$ are two cubic IFGs of the graphs $\breve{G}_1^* = (\mathscr{V}_1, E_1)$ and $\breve{G}_2^* = (\mathscr{V}_2, E_2)$. Then, we have to prove $\breve{G}_1 + \breve{G}_2 = (A_1 + A_2, \mathscr{B}_1 + \mathscr{B}_2)$ is a cubic IFG. In view of proposition 6 is sufficient to verify the case when $uy \in E\prime$. In this case, we have

$$
\begin{aligned}
(M_{\mathscr{B}1L} \cup M_{\mathscr{B}2L})(uy) &= \min((M_{A1L}(u)), (M_{A2U}(y)))\\
(M_{\mathscr{B}1L} \cup M_{\mathscr{B}2L})(uy) &= \min((M_{A1L}(u)), (M_{A2U}(y)))\\
&= \min((M_{A1L} + M_{A2L})(u), (M_{A1L} + M_{A2L})(y)),\\
(M_{\mathscr{B}1U} \cup M_{\mathscr{B}2U})(uy) &= \min((M_{A1U}(u)), (M_{A2U}(y)))\\
&\le \min((M_{A1U} \cup M_{A2U})(u), (M_{A1U} \cup M_{A2U})(y))\\
&= \min((M_{A1U} + M_{A2U})(u), (M_{A1U} + M_{A2U})(y)),\\
(\eta_{\mathscr{B}1L} \cup \eta_{\mathscr{B}2L})(uy) &= \max((\eta_{A1L}(u)), (\eta_{A2L}(y)))\\
&\le \max((\eta_{A1L} \cup \eta_{A2L})(u), (\eta_{A1L} \cup \eta_{A2L})(y))\\
&= \max((\eta_{A1L} + \eta_{A2L})(u), (\eta_{A1L} + \eta_{A2L})(y)),\\
(\eta_{\mathscr{B}1U} \cup \eta_{\mathscr{B}2U})(uy) &= \max((\eta_{A1U}(u)), (\eta_{A2U}(y)))\\
&\le \max((\eta_{A1U} \cup \eta_{A2U})(u), (\eta_{A1U} \cup \eta_{A2U})(y))\\
&= \max((\eta_{A1U} + \eta_{A2U})(u), (\eta_{A1U} + \eta_{A2U})(y)),\\
(M_{\mathscr{B}1} \cup M_{\mathscr{B}2})(uy) &= \min((M_{A1}(u)), (M_{A2}(y)))\\
&\le \min((M_{A1} \cup M_{A2})(u), (M_{A1} \cup M_{A2})(y))\\
&= \min((M_{A1} + M_{A2})(u), (M_{A1} + M_{A2})(y)),\\
(\eta_{\mathscr{B}1} \cup \eta_{\mathscr{B}2})(uy) &= \max((\eta_{A1}(u)), (\eta_{A2}(y)))\\
&\le \max((\eta_{A1} \cup \eta_{A2})(u), (\eta_{A1} \cup \eta_{A2})(y))\\
&= \max((\eta_{A1} + \eta_{A2})(u), (\eta_{A1} + \eta_{A2})(y)).
\end{aligned}
\tag{43}
$$

This completes the proof. □

5. Application

In this section, we apply the concept of CIFGs in multi-attribute decision-making problem, where the selection of suitable subjects has been carried out.

There are many career options for the students of present times. Moreover, some of the courses are usually chosen where all the available choices remain superior and best choices until a single student has to choose a field of his interest by keeping in view his preferences. At the finishing of college level education requires selecting their first choice of career planning. During this time, pupils must be given enough information about choosing career according to their interest. According to the survey of random sample of 100 pupils of class X carried out in this part, pupils with favour of interests and no favouring of choices of a specific subject up to class X are measured and given below. Based on the data, cubic nonrational fuzzy graph is used as a tool as it makes the level of membership (interval-valued membership) (percentage of students who favour a subject or a pair of subjects) and level of nonmembership (interval-valued nonmembership) (percentage of students who disfavour a subject or a pair of subjects). Employing CIFS, the best subject's combination may be evaluated that are the class having subjects that could be productive to most students and have best academic performance of most of the students.

Let $S = \{\text{English (E), Language (L), Maths (M), Science (S), Social Sciences (SS)}\}$ be the set of vertices. Tables 1 and 2 illustrate the percentages of students with interest/disinterest towards a subject or a pair of subjects.

Based on the above information, we generate an CIFG as follows (Figure 19).

In every vertex of the graph, the degree of membership shows the percentage of students with zeal for a specific subject and the degree of nonmembership is the percentage of students with no zeal in subject from a random sample of 100 students of class X chosen for survey. Also, the corners of graph of both membership and nonmembership show the favour and disfavour of students to study the combined two subjects at higher secondary corner. From the given graph, the corner $(L-SS)$ possesses high degree of nonmembership, which shows that majority of pupils do not like to study the combined subjects Language and Social Science, and the corner $(M-S)$ possesses high degree of membership, which shows that majority of pupils have zeal for studying the

Table 1: Subject combination.

Subject combination	Interest percentage	Disinterest percentage
E	$[0.3, 0.4], 0.3$	$[0.4, 0.5], 0.7$
L	$[0.2, 0.4], 0.4$	$[0.55, 0.6], 0.6$
M	$[0.2, 0.3], 0.3$	$[0.6, 0.7], 0.5$
S	$[0.1, 0.4], 0.5$	$[0.5, 0.6], 0.4$
SS	$[0.2, 0.3], 0.7$	$[0.3, 0.6], 0.3$

Table 2: Subjects combinations.

Subjects combination	Interest percentage	Disinterest percentage
$E - M$	$[0.2, 0.3], 0.3$	$[0.6, 0.7], 0.7$
$E - L$	$[0.2, 0.4], 0.3$	$[0.55, 0.6], 0.7$
$E - S$	$[0.1, 0.4], 0.3$	$[0.5, 0.6], 0.7$
$E - SS$	$[0.2, 0.3], 0.3$	$[0.4, 0.6], 0.7$
$L - M$	$[0.2, 0.3], 0.3$	$[0.6, 0.7], 0.6$
$L - S$	$[0.1, 0.4], 0.4$	$[0.55, 0.6], 0.6$
$L - SS$	$[0.2, 0.3], 0.4$	$[0.55, 0.6], 0.6$
$M - S$	$[0.1, 0.3], 0.3$	$[0.6, 0.7], 0.5$
$M - SS$	$[0.2, 0.3], 0.3$	$[0.6, 0.7], 0.5$
$S - SS$	$[0.1, 0.3], 0.5$	$[0.5, 0.6], 0.4$

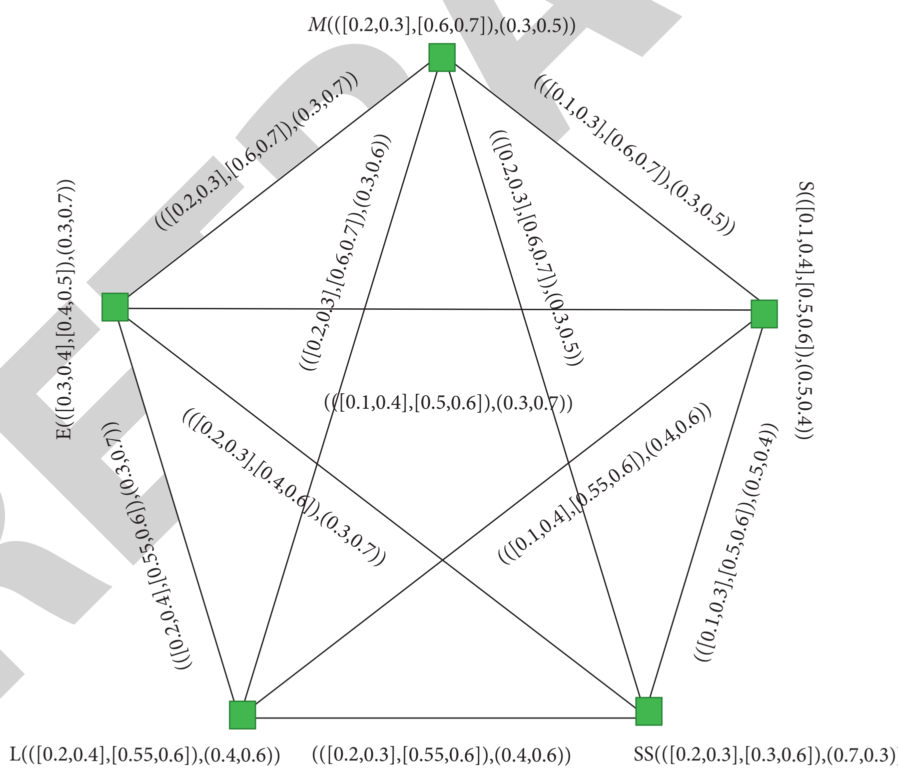


Figure 19: Cubic intuitionistic fuzzy graph.

combined subjects of Math and Science. There is disfavour to study the combined subjects of Tamil and Math, which indicates that these subjects do not require to be combined. Therefore, a high (low) level of membership of any corner shows the high (low) weightage of combined subjects at higher studies.

6. Comparison

Proposition 8. *A cubic IFG is a generalization of cubic FG.*

Proof. Let $\check{G}^* = (\mathscr{V}, E)$ be a cubic IFG. Then if we put the value of nonmembership of the vertex set and edge set as zero in the IVFS and FS, then the cubic IFG reduces to cubic FG. □

Proposition 9. *An IVIFG is a generalization of IVFG.*

Proof. Let $\breve{G}^* = (\mathscr{V}, E)$ be an IVIFG. If we put the value of nonmembership of the vertex set and edge set as zero, then the IVIFG reduces to IVFG. □

Proposition 10. *An IFG is a generalization of FG.*

Proof. Let $\breve{G}^* = (\mathscr{V}, E)$ be an IFG. If we put the value of nonmembership of the vertex set and edge set as zero, then the IFG reduces to FG. □

7. Conclusion

In this article, we developed a novel concept of CIFG as a generalization of IFGs. The graph theoretic terms like subgraphs, complements, degree of vertices, strength of graphs, paths, and cycle are briefly presented with the help of examples. Some related results and properties of the defined concepts are discussed. The generalization of CIFG is proved by some examples and remarks. A comparison of CIFG with IFG and other related concepts is given. The theory of CIFG is a generalization of IFG and can be applied to many real-life problems such as shortest path problem, communication problem, cluster analysis, and traffic signal problems. In the future, the graphs of the cubic Pythagorean fuzzy sets, cubic q-rung orthopair fuzzy sets, and cubic spherical fuzzy sets can be developed and different aggregation operators are defined for better decision-making.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Retraction

Retracted: Belong and Nonbelong Relations on Double-Framed Soft Sets and Their Applications

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Belong and Nonbelong Relations on Double-Framed Soft Sets and Their Applications

Tareq M. Al-shami [1] **and Abdelwaheb Mhemdi** [2]

[1] *Department of Mathematics, Sana'a University, Sana'a, Yemen*
[2] *Department of Mathematics, College of Sciences and Humanities in Aflaj Prince Sattam Bin Abdulaziz University, Riyadh, Saudi Arabia*

Correspondence should be addressed to Tareq M. Al-shami; tareqalshami83@gmail.com

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We aim through this paper to achieve two goals: first, we define some types of belong and nonbelong relations between ordinary points and double-framed soft sets. These relations are one of the distinguishing characteristics of double-framed soft sets and are somewhat expression of the degrees of membership and nonmembership. We explore their main properties and determine the conditions under which some of them are equivalent. Also, we introduce the concept of soft mappings between two classes of double-framed soft sets and investigate the relationship between an ordinary point and its image and preimage with respect to the different types of belong and nonbelong relations. By the notions presented herein, many concepts can be studied on double-framed soft topology such as soft separation axioms and cover properties. Second, we give an educational application of optimal choices using the idea of double-framed soft sets. We provide an algorithm of this application with an example to show how this algorithm is carried out.

1. Introduction

The (crisp) set theory is a main mathematical approach to deal with a class of problems that are characterized by precision, exactness, specificity, perfection, and certainty. However, many problems in the real-life inherently involve inconsistency, imprecision, ambiguity, and uncertainties. In particular, such classes of problems arise in engineering, economics, medical sciences, environmental sciences, social sciences, and many different scopes. The crisp (classical) mathematical tools fail to model or solve these types of problems.

In the course of time, mathematicians, engineers, and scientists, particularly those who focus on artificial intelligence, are seeking for alternative mathematical approaches to solve the problems that contain uncertainty or vagueness. They initiated several set theories such as probability theory, fuzzy set [1], intuitionistic fuzzy set [2], and rough set [3].

In 1999, Molodtsov [4] proposed the concept of soft sets as a new mathematical tool to cope with uncertainties. He investigated the efficiency of soft sets to deal with complicated problems compared with the probability theory and fuzzy set theory. After Molodtsov's work, many researchers have studied several operations and relations between soft sets (see, for example, [5–10]). Soft sets were applied in various domains such as algebraic structures (see, for example, [11–13]), soft topological spaces (see, for example, [14–16]), and decision-making problems (see, for example, [17–25]). Also, the relationship among soft sets, rough sets, and fuzzy sets was the goal of some papers such as [17, 26, 27].

In the last few years, a number of scholars have extensively studied some extensions of soft set. These studies go into two ways: the first one is initiated by giving some generalizations of the structure of soft sets. This leads to define binary soft set [28], N-soft set [29], double-framed soft set [30], and bipolar soft set [31] (several relations between bipolar soft sets and ordinary points were presented in [32]). The second one is coming from the combination of soft set (or its updating forms) with rough set or fuzzy set or both. This leads to define fuzzy soft set [33], fuzzy bipolar soft set [34], bipolar fuzzy soft set [35], soft rough set [26], bipolar soft rough set [36], and modified rough bipolar soft set [37].

Soft set was formulated over an initial universal set X by using a map from a set of parameters A into the power set of X. However, we need sometimes to define two maps from A into the power set of X; for example, if we schedule students' results in **n** subjects, we define **n** different maps over the same sets X and A. For this purpose, Jun and Ahn [30] initiated the notion of double-framed soft sets and applied in BCK/BCI algebras. In 2014, Muhiuddin and Al-Roqi [38] studied the concept of double-framed soft hypervector spaces, and in 2015, Naz [39] revealed some algebraic properties of double-framed soft set. In 2017, Khana et al. [40] introduced the concept of double-framed soft LA-semigroups. In the same year, Shabir and Samreena [41] made use of a double-framed soft set to define a new soft structure called a double-framed soft topological space. They initiated its basic notions such as DFS open and closed sets and DFS neighborhoods. In 2018, Iftikhar and Mahmood [42] presented some results on lattice-ordered double-framed soft semirings; and Park [43] discussed double-framed soft deductive system of subtraction algebras. Bordbar et al. [44] applied double-framed soft set theory to hyper-*BCK* algebras. Saeed et al. [45] formulated the concepts of N-framed soft set and then defined the soft union and intersection of two double-framed soft sets. They also provided an example to elucidate an application of N-framed soft set.

The motivation for this work is to define new types of belong and nonbelong relations between ordinary points and double-framed soft sets which create new degrees of membership and nonmembership for the ordinary points. In fact, this leads to initiate novel concepts on double-framed soft topology, in particular in the areas of soft separation axioms and cover properties.

We organize the rest of this paper as follows. Section 2 recalls some operations between double-framed soft sets. In Section 3, we formulate four types of belong relations between ordinary points and double-framed soft sets called weakly partial belong, strongly partial belong, weakly total belong, and strongly total belong relations and formulate four types of nonbelong relations between ordinary points and double-framed soft sets called weakly partial nonbelong, strongly partial nonbelong, weakly total nonbelong, and strongly total nonbelong relations. Then, we examine their behaviours under the operations of soft intersection and union. Also, we study soft mappings with respect to the classes of double-framed soft sets and prob the relationships between ordinary points and their images and preimages. In Section 4, we propose a method of optimum choice based on double-framed soft sets. We provide an example to illustrate how this method can be applied to model some real-life problems. Finally, we summarize the main obtained results and present some future works in Section 5.

2. Preliminaries

In this part, we mention some definitions and results of double-framed soft sets.

In this article, the sets of parameters are denoted by A, B, C, D, E, M, N; the initial universal sets are denoted by X, Y; and the power set of X is denoted by 2^X.

Definition 1 (see [4]). A soft set over X, denoted by (h, A), is a map h from A to 2^X. We call X an initial universal set and A a set of parameters.

Usually, we write (h, A) as a set of ordered pairs:

$$(h, A) = \left\{(a, h(a)) \colon a \in A \,\text{and}\, h(a) \in 2^X\right\}. \tag{1}$$

Definition 2 (see [30]). Let h, k be two mappings from A to 2^X. A double-framed soft set over X, determined by h and k, is the set $\{(a, h(a), k(a)) \colon a \in A\}$.

We will denote this double-framed soft set by (h, k, A). The set X is called the initial universal set, and the set A is called the set of parameters.

A class of all double-framed soft sets defined over X with all parameters subsets of A is denoted by $C(X_A)$.

In a similar way, one define the concepts of triple-framed soft set, quadruple-framed soft set, quintuple-framed soft set, sextuple-framed soft set, septuple-framed soft set,..., and N-framed soft set.

Definition 3 (see [45]). $(h_1, h_2, \ldots, h_n, A)$ is said to be an N-framed soft set over a nonempty set X, where h_i is a map from A into 2^X for $i = 1, 2, \ldots, n$, X is an initial universal set, and A is a set of parameters.

An N-framed soft set is expressed as follows:

$$(h_1, h_2, \ldots, h_n, A) = \left\{(a, h_1(a), h_2(a), \ldots, h_n(a)) \colon a \in A \,\text{and}\, h_i(a) \in 2^X \text{for each}\, i = 1, 2, \ldots, n\right\}. \tag{2}$$

Henceforth, we assume that the initial universal set of every double-framed soft set in this paper is nonempty.

Example 1. Let $X = \{x_1, x_2, \ldots, x_{50}\}$ be the universal set of third graders and $A = \{a_1, a_2, a_3, a_4\}$ be a set of parameters, where a_1 represents the students holding first rank, a_2 represents the students holding second rank, a_3 represents the students holding third rank, and a_4 represents the students holding fourth rank.

Let $h \colon A \longrightarrow 2^X$ be a map of ranking students in mathematics subject and $k \colon A \longrightarrow 2^X$ be a map of ranking students in physics subject.

Suppose that h and k are given as follows:

$$\begin{aligned} h(a_1) &= \{x_{14}\},\\ k(a_1) &= \{x_3, x_{14}\},\\ h(a_2) &= \{x_{19}\},\\ k(a_2) &= \{x_7\},\\ h(a_3) &= \{x_7, x_{21}, x_{26}\},\\ k(a_3) &= \{x_{35}\},\\ h(a_4) &= \{x_2\},\\ k(a_4) &= \{x_2, x_{43}\}. \end{aligned} \tag{3}$$

Now, we can describe this system using a double-framed soft set as follows:

$$(h,k,A) = \{(a_1, \{x_{14}\}, \{x_3, x_{14}\}), (a_2, \{x_{19}\}, \{x_7\}), (a_3, \{x_7, x_{21}, x_{26}\}, \{x_{35}\}), (a_4, \{x_2\}, \{x_2, x_{43}\})\}. \tag{4}$$

If there are three maps of subjects, a system is described using a triple-framed soft set; and if there are four maps of subjects, a system is described using a quadruple-framed soft set and so on.

Definition 4 (see [41]). Let (h,k,A) be a double-framed soft set and $x \in X$. We say that $x \in (h,k,A)$ if $x \in h(a)$ and $x \in k(a)$ for all $a \in A$ and $x \notin (h,k,A)$ if $x \notin h(a)$ or $x \notin k(a')$ for some $a, a' \in A$.

Definition 5 (see [39]). A double-framed soft set (h,k,A) is said to be a null double-framed soft set (resp., an absolute double-framed soft set) if $h(a), k(a)$ equals to the empty (resp., universal) set for each $a \in A$.

Henceforth, the null and absolute double-framed soft sets are symbolized by $(\widetilde{\Phi_A}, \widetilde{\Phi_A})$ and $(\widetilde{X_A}, \widetilde{X_A})$, respectively.

Definition 6 (see [45]). The intersection of two double-framed soft sets (h_1, h_2, A) and (k_1, k_2, B) is a double-framed soft set (f_1, f_2, C) such that $C = A \cap B \neq \varnothing$ and $f_1\colon C \longrightarrow 2^X$ and $f_2\colon C \longrightarrow 2^X$ are defined by $f_1(c) = h_1(c) \cap k_1(c)$ and $f_2(c) = h_2(c) \cap k_2(c)$.

It is symbolized by $(h_1, h_2, A) \widetilde{\cap} (k_1, k_2, B)$.

Definition 7 (see [45]). The soft union of two double-framed soft sets (h_1, h_2, A) and (k_1, k_2, B) is a double-framed soft set (f_1, f_2, C), where $C = A \cup B$ and $f_1\colon C \longrightarrow 2^X$ and $f_2\colon C \longrightarrow 2^X$ are defined by

$$f_i(c) = \begin{cases} h_i(c), & : c \in A - B,\\ k_i(c), & : c \in B - A,\\ h_i(c) \cup k_i(c), & : c \in A \cap B. \end{cases} \tag{5}$$

It is symbolized by $(h_1, h_2, A) \widetilde{\cup} (k_1, k_2, B)$.

Definition 8 (see [30]). A double-framed soft set (h_1, h_2, A) is called a subset of a double-framed soft set (k_1, k_2, B), denoted by $(h_1, h_2, A) \widetilde{\subseteq} (k_1, k_2, B)$, if $A \subseteq B$, and $h_1(a) \subseteq k_1(a)$ and $h_2(a) \subseteq k_2(a)$ holds true for all $a \in A$.

The double-framed soft sets (h_1, h_2, A) and (k_1, k_2, B) are called equal if $(h_1, h_2, A) \widetilde{\subseteq} (k_1, k_2, B)$ and $(k_1, k_2, B) \widetilde{\subseteq} (h_1, h_2, A)$.

Definition 9 (see [39]). The relative complement of a double-framed soft set (h,k,A) is a double-framed soft set $(h,k,A)^c = (h^c, k^c, A)$, where h^c and k^c are two maps from A to 2^X defined as follows:

$$\begin{aligned} h^c(a) &= X - h(a),\\ k^c(a) &= X - k(a). \end{aligned} \tag{6}$$

Proposition 1 (see [39]). *ie operations of soft union and soft intersection of double-framed soft sets are commutative and associative.*

Proposition 2 (see [39]). *We have the following results for two double-framed soft sets:*

(i) $[(h,k,A) \widetilde{\cup} (p,t,A)]^c = (h,k,A)^c \widetilde{\cap} (p,t,A)^c$.

(ii) $[(h,k,A) \widetilde{\cap} (p,t,A)]^c = (h,k,A)^c \widetilde{\cup} (p,t,A)^c$.

3. Belong and Nonbelong Relations on Double-Framed Soft Sets

We dedicate this section to establish four types of memberships and four types of nonmemberships between an ordinary point and double-framed soft set and lay the foundations of them. We obtain some results that concern the soft intersection and union operators, the product of double-framed soft sets and soft mappings.

Definition 10. Let (h,k,A) be a double-framed soft set and $\delta \in X$. We say that

(i) $\delta \in_w (h,k,A)$, reading as δ weakly partial belongs to (h,k,A), if $\delta \in h(a)$ or $\delta \in k(a')$ for some $a, a' \in A$.

(ii) $\delta \Subset_s (h,k,A)$, reading as δ strongly partial belongs to (h,k,A), if $\delta \in h(a)$ and $\delta \in k(a')$ for some $a, a' \in A$.

(iii) $\delta \in_w (h,k,A)$, reading as δ weakly total belongs to (h,k,A), if $\delta \in h(a)$ or $\delta \in k(a)$ for all $a \in A$.

(iv) $\delta \in_s (h,k,A)$, reading as δ strongly total belongs to (h,k,A), if $\delta \in h(a)$ and $\delta \in k(a)$ for all $a \in A$.

Definition 11. Let (h,k,A) be a double-framed soft set and $\delta \in X$. We say that

(i) $\delta \not\Subset_w (h,k,A)$, reading as δ weakly partial belong to (h,k,A), if $\delta \in h(a)$ or $\delta \in k(a')$ for some $a, a' \in A$.

(ii) $\delta \not\Subset_s (h,k,A)$, reading as δ strongly partial belong to (h,k,A), if $\delta \in h(a)$ and $\delta \in k(a')$ for some $a, a' \in A$.

(iii) $\delta \notin_w (h,k,A)$, reading as δ does not weakly total belong to (h,k,A), if $\delta \notin h(a)$ or $\delta \notin k(a)$ for all $a \in A$.

(iv) $\delta \notin_s (h,k,A)$, reading as δ does not strongly total belong to (h,k,A), if $\delta \notin h(a)$ and $\delta \notin k(a)$ for all $a \in A$.

Remark 1. The relations of strongly total belong and weakly partial nonbelong were introduced in [41] (see Definition 4).

Proposition 3. *For a double-framed soft set (h,k,A) and $\delta \in X$, we have the following results:*

(i) $\delta \Subset_w (h,k,A)$ *iff* $\delta \not\Subset_w (h^c, k^c, A)$.

(ii) $\delta \Subset_s (h,k,A)$ *iff* $\delta \not\Subset_s (h^c, k^c, A)$.

(iii) $\delta \in_w (h,k,A)$ *iff* $\delta \notin_w (h^c, k^c, A)$.

(iv) $\delta \in_s (h,k,A)$ *iff* $\delta \notin_s (h^c, k^c, A)$.

Proof. We will just prove (i) and (iv).

(i) $\delta \Subset_w (h,k,A) \Leftrightarrow \delta \in h(a)$ or $\delta \in k(a')$ for some $a, a' \in A \Leftrightarrow \delta \notin X - h(a) = h^c(a)$ or $\delta \notin X - k(a') = k^c(a')$ for some $a, a' \in A \Leftrightarrow \delta \not\Subset_w (h^c, k^c, A)$.

(ii) $\delta \in_s (h,k,A) \Leftrightarrow \delta \in h(a)$ and $\delta \in k(a)$ for all $a \in A \Leftrightarrow \delta \notin X - h(a) = h^c(a)$ and $\delta \notin X - k(a') = k^c(a')$ for all $a \in A \Leftrightarrow \delta \notin_s (h^c, k^c, A)$.

The following proposition is a direct result of Definition 10. □

Proposition 4. *Let (h,k,A) be a double-framed soft set and $\delta \in X$. Then,*

(i) $\delta \in_s (h,k,A) \Rightarrow \delta \in_w (h,k,A) \Rightarrow \delta \Subset_w (h,k,A)$.

(ii) $\delta \in_s (h,k,A) \Rightarrow \delta \Subset_s (h,k,A) \Rightarrow \delta \Subset_w (h,k,A)$.

(iii) $\delta \notin_s (h,k,A) \Rightarrow \delta \notin_w (h,k,A) \Rightarrow \delta \not\Subset_w (h,k,A)$.

(iv) $\delta \notin_s (h,k,A) \Rightarrow \delta \not\Subset_s (h,k,A) \Rightarrow \delta \not\Subset_w (h,k,A)$.

Example below is given to clarify that the converse of Proposition 4 fails. Also, it shows that the relations of strongly partial belong and weakly total belong (the relations of weakly total nonbelong and strongly partial nonbelong) are independent of each other.

Example 2. Let $A = \{a_1, a_2, a_3\}$ be a set of parameters and (h,k,A) double-framed soft set over $X = \{x_1, x_2, \ldots, x_{10}\}$ be defined as follows:

$$(h,k,A) = \{(a_1, \{x_1\}, \{x_2, x_4, x_{10}\}), (a_2, \varnothing, \{x_4, x_{10}\}), (a_3, \{x_2, x_3\}, \{x_4, x_5, x_{10}\})\}. \tag{7}$$

We find the next relations:

(i) $x_1 \Subset_w (h,k,A)$, but $x_1 \in_w (h,k,A)$ and $x_1 \Subset_s (h,k,A)$ do not hold.

(ii) $x_4 \in_w (h,k,A)$, but $x_4 \in_s (h,k,A)$ does not hold.

(iii) $x_2 \Subset_s (h,k,A)$, but $x_2 \in_s (h,k,A)$ does not hold.

(iv) $x_2 \not\Subset_s (h,k,A)$, but $x_2 \notin_w (h,k,A)$ and $x_2 \notin_s (h,k,A)$ do not hold.

(v) $x_4 \in_w (h,k,A)$, but $x_4 \Subset_s (h,k,A)$ does not hold. Also, $x_2 \Subset_s (h,k,A)$, but $x_2 \in_w (h,k,A)$ does not hold.

(vi) $x_3 \notin_w (h,k,A)$, but $x_2 \notin_s (h,k,A)$ does not hold.

Remark 2. It is well-known in the Quantum physics the possibility of existence and nonexistence of an electron in the same place. This matter also occurs here with respect to weakly partial belong and weakly partial nonbelong relations; strongly partial belong and strongly partial nonbelong relations; and weakly total belong and weakly total nonbelong relations. To illustrate that it can be seen from Example 2 that

$$\begin{aligned} & x_5 \Subset_w (h,k,A), \\ & x_5 \not\Subset_w (h,k,A), \\ & x_2 \Subset_s (h,k,A), \\ & x_2 \not\Subset_s (h,k,A), \\ & x_{10} \in_w (h,k,A), \\ & x_{10} \notin_w (h,k,A). \end{aligned} \tag{8}$$

Proposition 5. *Let (h,k,A) and (p,t,A) be double-framed soft sets such that $(h,k,A) \widetilde{\subseteq} (p,t,A)$. Then,*

(i) If $\delta \Subset_w (h,k,A)$ *(resp.,* $\delta \Subset_s (h,k,A)$, $\delta \in_w (h,k,A)$, $\delta \in_s (h,k,A)$*), then* $\delta \Subset_w (p,t,A)$ *(resp.,* $\delta \Subset_s (p,t,A)$, $\delta \in_w (p,t,A)$, $\delta \in_s (p,t,A)$*).*

(ii) If $\delta\not\Subset_w(p,t,A)$ *(resp.,* $\delta\not\Subset_s(p,t,A)$, $\delta\notin_w(p,t,A)$, $\delta\notin_s(p,t,A)$*), then* $\delta\not\Subset_w(h,k,A)$ *(resp.,* $\delta\not\Subset_s(h,k,A)$, $\delta\notin_w(h,k,A)$, $\delta\notin_s(h,k,A)$*).*

Proof. Straightforward. □

Remark 3. Note that satisfying the two conditions (i) and (ii) of the above proposition does not imply $(h,k,A)\widetilde{\subseteq}(p,t,A)$. To illustrate this fact, consider Example 2 and let $(p,t,A)=\{(a_1,\{x_2,x_4,x_{10}\},$ $\{x_1\}),(a_2\varnothing,\{x_4,x_{10}\}),(a_3,\{x_4,x_5,x_{10}\}\{x_2,x_3\})\}$. It is clear that $\delta\Subset_w(h,k,A)$ (resp., $\delta\Subset_s(h,k,A)$, $\delta\in_w(h,k,A)$, $\delta\in_s(h,k,A)$) if and only if $\delta\Subset_w(p,t,A)$ (resp., $\delta\Subset_s(p,t,A)$, $\delta\in_w(p,t,A)$, $\delta\in_s(p,t,A)$). However, $(h,k,A)\widetilde{\nsubseteq}(p,t,A)$ and $(p,t,A)\widetilde{\nsubseteq}(h,k,A)$.

Proposition 6. *For two double-framed soft sets* (h,k,A) *and* (p,t,A) *and* $\delta\in X$*, we have the following results:*

(i) $\delta\Subset_w(h,k,A)$ *or* $\delta\Subset_w(p,t,A)\Leftrightarrow\delta\Subset_w(h,k,A)\widetilde{\cup}(p,t,A)$.

(ii) $\delta\Subset_s(h,k,A)$ *or* $\delta\Subset_s(p,t,A)\Rightarrow\delta\Subset_s(h,k,A)\widetilde{\cup}(p,t,A)$.

(iii) $\delta\in_w(h,k,A)$ *or* $\delta\in_w(p,t,A)\Rightarrow\delta\in_w(h,k,A)\widetilde{\cup}(p,t,A)$.

(iv) $\delta\in_s(h,k,A)$ *or* $\delta\in_s(p,t,A)\Rightarrow\delta\in_s(h,k,A)\widetilde{\cup}(p,t,A)$.

(v) $\delta\Subset_w(h,k,A)\widetilde{\cap}(p,t,A)\Rightarrow\delta\Subset_w(h,k,A)$ *and* $\delta\Subset_w(p,t,A)$.

(vi) $\delta\Subset_s(h,k,A)\widetilde{\cap}(p,t,A)\Rightarrow\delta\Subset_s(h,k,A)$ *and* $\delta\Subset_s(p,t,A)$.

(vii) $\delta\in_w(h,k,A)\widetilde{\cap}(p,t,A)\Rightarrow\delta\in_w(h,k,A)$ *and* $\delta\in_w(p,t,A)$.

(viii) $\delta\in_s(h,k,A)\widetilde{\cap}(p,t,A)\Leftrightarrow\delta\in_s(h,k,A)$ *and* $\delta\in_s(p,t,A)$.

Proof. Since (h,k,A) and (p,t,A) are subsets of $(h,k,A)\widetilde{\cup}(p,t,A)$, then the necessary parts of (i) to (iv) hold; and since $(h,k,A)\widetilde{\cap}(p,t,A)$ are subsets of (h,k,A) and (p,t,A), then the necessary parts of (v) to (viii) hold.

To prove the sufficient part of (i), let $\delta\Subset_w(h,k,A)\widetilde{\cup}(p,t,A)$. Then, $\delta\in h(a)\cup p(a)$ or $\delta\in k(a')\cup t(a')$ for some $a,a'\in A$. Say $\delta\in h(a)\cup p(a)$ for some $a\in A$. Therefore, $\delta\in h(a)$ or $p(a)$ for some $a\in A$, and hence, $\delta\Subset_w(h,k,A)$ or $\delta\Subset_w(p,t,A)$.

To prove the sufficient part of (viii), let $\delta\in_s(h,k,A)$ and $\delta\in_s(p,t,A)$. Then, for all $a\in A$, we have $\delta\in h(a)$ and $\delta\in k(a)$ and $\delta\in p(a)$ and $\delta\in t(a)$. Therefore, $\delta\in h(a)\cap p(a)$ and $\delta\in k(a)\cap t(a)$ for all $a\in A$, and hence, $\delta\in_s(h,k,A)\widetilde{\cap}(p,t,A)$.

Example below is given to clarify that the converse of the results (ii) to (iv) and (v) to (vii) of Proposition 6 fails. □

Example 3. Let $A=\{a_1,a_2\}$ be a set of parameters and $(h,k,A),(p,t,A)$ double-framed soft sets over $X=\{x_1,x_2,x_3,x_4,x_5\}$ defined as follows:

$$\begin{aligned}(h,k,A)&=\{(a_1,\{x_1,x_3\},\varnothing),(a_2,\{x_3,x_4\},\{x_4,x_5\})\},\\(p,t,A)&=\{(a_1,\{x_4\},\{x_3,x_4,x_5\}),(a_2,\{x_2\},\{x_1,x_3\})\}.\end{aligned}\tag{9}$$

Then, $(h,k,A)\widetilde{\cup}(p,t,A)=\{(a_1,\ \{x_1,x_3,x_4\},\ \{x_3,x_4,x_5\}),(a_2,\{x_2,x_3,x_4\},\{x_1,x_3,x_4,x_5\})\}$ and $(h,k,A)\widetilde{\cap}(p,t,A)=\widetilde{\Phi}$.

We note the following:

(i) $x_1\Subset_s(h,k,A)\widetilde{\cup}(p,t,A)$, but $x_1\Subset_s(h,k,A)$ or $x_1\Subset_s(p,t,A)$ does not hold.

(ii) $x_5\in_w(h,k,A)\widetilde{\cup}(p,t,A)$, but $x_5\in_w(h,k,A)$ or $x_5\in_w(p,t,A)$ does not hold.

(iii) $x_4\in_s(h,k,A)\widetilde{\cup}(p,t,A)$, but $x_4\in_s(h,k,A)$ or $x_4\in_s(p,t,A)$ does not hold.

(iv) $x_4\Subset_w(h,k,A)$ and $x_4\Subset_w(p,t,A)$, but $x_4\Subset_w(h,k,A)\widetilde{\cap}(p,t,A)$ does not hold.

(v) $x_4\Subset_s(h,k,A)$ and $x_4\Subset_s(p,t,A)$, but $x_4\Subset_s(h,k,A)\widetilde{\cap}(p,t,A)$ does not hold.

(vi) $x_3\in_w(h,k,A)$ and $x_3\in_w(p,t,A)$, but $x_3\in_w(h,k,A)\widetilde{\cap}(p,t,A)$ does not hold.

Similarly, it can be proved the following result.

Proposition 7. *For two double-framed soft sets* (h,k,A) *and* (p,t,A) *over* X *and* $\delta\in X$*, we have the following results:*

(i) $\delta\not\Subset_w(h,k,A)\widetilde{\cup}(p,t,A)\Rightarrow\delta\not\Subset_w(h,k,A)$ *and* $\delta\not\Subset_w(p,t,A)$.

(ii) $\delta\not\Subset_s(h,k,A)\widetilde{\cup}(p,t,A)\Rightarrow\delta\not\Subset_s(h,k,A)$ *and* $\delta\not\Subset_s(p,t,A)$.

(iii) $\delta\notin_w(h,k,A)\widetilde{\cup}(p,t,A)\Rightarrow\delta\notin_w(h,k,A)$ *and* $\delta\notin_w(p,t,A)$.

(iv) $\delta\notin_s(h,k,A)\widetilde{\cup}(p,t,A)\Leftrightarrow\delta\notin_s(h,k,A)$ *and* $\delta\notin_s(p,t,A)$.

(v) $\delta\not\Subset_w(h,k,A)$ *or* $\delta\not\Subset_w(p,t,A)\Rightarrow\delta\not\Subset_w(h,k,A)\widetilde{\cap}(p,t,A)$.

(vi) $\delta\not\Subset_s(h,k,A)$ *or* $\delta\not\Subset_s(p,t,A)\Rightarrow\delta\not\Subset_s(h,k,A)\widetilde{\cap}(p,t,A)$.

(vii) $\delta\notin_w(h,k,A)$ *or* $\delta\notin_w(p,t,A)\Rightarrow\delta\notin_w(h,k,A)\widetilde{\cap}(p,t,A)$.

(viii) $\delta\notin_s(h,k,A)$ *or* $\delta\notin_s(p,t,A)\Rightarrow\delta\notin_s(h,k,A)\widetilde{\cap}(p,t,A)$.

Definition 12. A double-framed soft set (h,k,A) is said to be 2-stable if $h(a)=U\subseteq X$ and $k(a)=V\subseteq X$ for each $a\in A$. If $U=V$, then (h,k,A) is said to be 1-stable.

Obviously, a 1-stable double-framed soft set is 2-stable, but the converse is not always true.

Proposition 8. *Let* (h,k,A) *be a 1-stable double-framed soft set. Then,* $\delta\Subset_w(h,k,A)\Leftrightarrow\delta\Subset_s(h,k,A)\Leftrightarrow\delta\in_w(h,k,A)\Leftrightarrow\delta\in_s(h,k,A)$.

Proof. Since (h,k,A) is a 1-stable double-framed soft set, there is a subset U of X such that $h(a)=k(a)=U$ for each

$a \in A$. This means that $\delta \in h(a)$ or $\delta \in k(a)$ for some $a \in A$ iff $\delta \in h(a)$ and $\delta \in k(a)$ for each $a \in A$. Hence, the desired result is proved. □

Corollary 1. *Let (h, k, A) be a 1-stable double-framed soft set. Then, $\delta \not\Subset_w (h, k, A) \Leftrightarrow \delta \not\Subset_s (h, k, A) \Leftrightarrow \delta \notin_w (h, k, A) \Leftrightarrow \delta \notin_s (h, k, A)$.*

Proposition 9. *Let (h, k, A) be a 2-stable double-framed soft set. Then,*

(i) $\delta \Subset_w (h, k, A) \Leftrightarrow \delta \in_w (h, k, A)$.

(ii) $\delta \Subset_s (h, k, A) \Leftrightarrow \delta \in_s (h, k, A)$.

Proof. Since (h, k, A) is a 2-stable double-framed soft set, there exist two subsets U, V of X such that $h(a) = U$ and $k(a) = V$ for each $a \in A$. Now, we have the following two cases:

Case 1: $\delta \in h(a)$ or $\delta \in k(a')$ for some $a, a' \in A$ if and only if $\delta \in h(a)$ or $\delta \in k(a')$ for all $a, a' \in A$.

Case 2: $\delta \in h(a)$ and $\delta \in k(a')$ for some $a, a' \in A$ if and only if $\delta \in h(a)$ and $\delta \in k(a')$ for all $a, a' \in A$.

Hence, the desired results are proved. □

Corollary 2. *Let (h, k, A) be a 2-stable double-framed soft set. Then,*

(i) $\delta \not\Subset_w (h, k, A) \Leftrightarrow \delta \notin_w (h, k, A)$.

(ii) $\delta \not\Subset_s (h, k, A) \Leftrightarrow \delta \notin_s (h, k, A)$.

Definition 14. The Cartesian product of two double-framed soft sets (h, k, A) and (p, t, B), denoted by $(h \times p, k \times t, A \times B)$, is defined as $(h \times p)(e, e') = h(e) \times p(e')$ and $(k \times t)(e, e') = k(e) \times t(e')$ for each $(e, e') \in A \times B$.

Proposition 10.

(i) $(\delta, \zeta) \Subset_s (h, k, A) \times (p, t, B)$ if and only if $\delta \Subset_s (h, k, A)$ and $\xi \Subset_s (p, t, B)$.

(ii) If $(\delta, \zeta) \Subset_w (h, k, A) \times (p, t, B)$, then $\delta \Subset_w (h, k, A)$ and $\xi \Subset_w (p, t, B)$.

(iii) $(\delta, \zeta) \in_s (h, k, A) \times (p, t, B)$ if and only if $\delta \in_s (h, k, A)$ and $\xi \in_s (p, t, B)$.

(iv) If $(\delta, \zeta) \in_w (h, k, A) \times (p, t, B)$, then $\delta \in_w (h, k, A)$ and $\xi \in_w (p, t, B)$.

Proof. (i) $(\delta, \zeta) \Subset_s (h, k, A) \times (p, t, B) = (h \times p, k \times t, A \times B)$.

$\Leftrightarrow (\delta, \zeta) \in (h \times p)(a, b) = h(a) \times p(b)$ and $(\delta, \zeta) \in (k \times t)(a', b') = k(a') \times t(b')$ for some $(a, b), (a', b') \in A \times B$.

$\Leftrightarrow \delta \in h(a)$ and $\xi \in p(b)$ for some $a \in A$ and $b \in B$ and $\delta \in k(a')$ and $\xi \in t(b')$ for some $a' \in A$ and $b' \in B$.

$\Leftrightarrow \delta \in h(a)$ and $\delta \in k(a')$ for some $a, a' \in A$ and $\xi \in p(b)$ and $\xi \in t(b')$ for some $b, b' \in B$.

$\Leftrightarrow \delta \Subset_s (h, k, A)$ and $\xi \Subset_s (p, t, B)$.

The other cases can be achieved similarly.

The following example explains that the converses of (ii) and (iv) of the above proposition fail. □

Example 4. Let $A = \{a_1, a_2\}$ be a set of parameters and (h, k, A), (p, t, A) double-framed soft sets over $X = \{x_1, x_2, x_3, x_4\}$ defined as follows:

$$\begin{aligned} (h, k, A) &= \{(a_1, \{x_1, x_2\}, \varnothing), (a_2, \{x_2\}, \{x_4\})\}, \\ (p, t, A) &= \{(a_1, \{x_1\}, \{x_3\}), (a_2, \{x_4\}, \{x_3\})\}. \end{aligned} \tag{10}$$

Then, $(h, k, A) \times (p, t, A) = \{((a_1, a_1), \{(x_1, x_1), (x_2, x_1)\}, \varnothing), ((a_1, a_2), \{(x_1, x_4), (x_2, x_4)\}, \varnothing), ((a_2, a_1), \{(x_2, x_1)\}, \{(x_4, x_3)\}), ((a_2, a_2), \{(x_2, x_4)\}, \{(x_4, x_3)\})\}$.

We find the following relations:

(i) $x_1 \Subset_w (h, k, A)$ and $x_3 \Subset_w (p, t, B)$; however, $(x_1, x_3) \Subset_w (h, k, A) \times (p, t, B)$ does not hold true.

(ii) $x_2 \in_w (h, k, A)$ and $x_3 \in_w (p, t, B)$; however, $(x_2, x_3) \in_w (h, k, A) \times (p, t, B)$ does not hold true.

Definition 15. A soft mapping π_φ from $C(X_A)$ into $C(Y_B)$ is a pair (π, φ) of crisp mappings such that $\pi\colon X \longrightarrow Y$ and $\varphi\colon A \longrightarrow B$ and is defined as follows: the image of a double-framed soft set (f_1, f_2, M) in $C(X_A)$ is a double-framed soft set $\pi_\varphi(f_1, f_2, U) = (\pi_{f_1}, \pi_{f_2}, E)$ in $C(Y_B)$ such that $E = \varphi(\beta) \subseteq B$ and π_{f_1} and π_{f_2} are two maps defined as

$$\pi_{f_i}(e) = \pi\left(\bigcup_{\varepsilon \in \varphi^{-1}(e) \cap \beta} f_i(\varepsilon)\right), \tag{11}$$

for each $e \in E$ and $i = 1, 2$.

Definition 16. A soft map $\pi_\varphi\colon C(X_A) \longrightarrow C(Y_B)$ is said to be injective (resp., surjective and bijective) if π and φ are injective (resp., surjective and bijective).

Definition 17. Let $\pi_\varphi\colon C(X_A) \longrightarrow C(Y_B)$ be a soft mapping. Then, the preimage of a double-framed soft set (g_1, g_2, N) in $C(Y_B)$ is a double-framed soft set $\pi_\varphi^{-1}(g_1, g_2, N) = (\pi_{g_1}^{-1}, \pi_{g_2}^{-1}, D)$ in $C(X_A)$ such that $D = \varphi^{-1}(N) \subseteq A$ and $\pi_{g_1}^{-1}$ and $\pi_{g_2}^{-1}$ are two maps defined as

$$\pi_{g_i}^{-1}(d) = \pi^{-1}(g_i \varphi(d)), \tag{12}$$

for each $d \in D$ and $i = 1, 2$.

Proposition 11. *Let $\pi_\varphi\colon C(X_A) \longrightarrow C(Y_B)$ be a soft mapping, and let (f_1, f_2, β) and (h_1, h_2, β') be two double-framed soft sets in $C(X_A)$. Then,*

(i) $\pi_\varphi(\widetilde{\Phi_A}, \widetilde{\Phi_A}) \widetilde{\subseteq} (\widetilde{\Phi_B}, \widetilde{\Phi_B})$. The equality holds if φ is surjective.

(ii) $\pi_\varphi(\widetilde{X_A}, \widetilde{X_A}) \widetilde{\subseteq} (\widetilde{Y_B}, \widetilde{Y_B})$. The equality holds if π and φ are surjective.

(iii) If $(f_1, f_2, \beta) \widetilde{\subseteq} (h_1, h_2, \beta')$, then $\pi_\varphi(f_1, f_2, \beta) \widetilde{\subseteq} \pi_\varphi(h_1, h_2, \beta')$.

(iv) $\pi_\varphi[(f_1, f_2, \beta)\widetilde{\cup}(h_1, h_2, \beta')] = \pi_\varphi(f_1, f_2, \beta)\widetilde{\cup}\pi_\varphi(h_1, h_2, \beta')$.

(v) $\pi_\varphi[(f_1, f_2, \beta)\widetilde{\cap}(h_1, h_2, \beta')]\widetilde{\subseteq}\pi_\varphi(f_1, f_2, \beta)\widetilde{\cap}\pi_\varphi(h_1, h_2, \beta')$.

The equality holds if π and φ are injective.

Proof. To prove (i), let $\pi_\varphi(\widetilde{\Phi_A}, \widetilde{\Phi_A}) = \pi_\varphi(u, u, A) = (v, v, E)$, where $u(a) = \varnothing$ for each $a \in A$ and $E = \varphi(A)$. Then, $v(e) = \pi(\cup_{\varepsilon\in\varphi^{-1}(e)} u(\varepsilon)) = \pi(\varnothing) = \varnothing$ for each $e \in E$. Therefore, $(v, v, E) = (\widetilde{\Phi_E}, \widetilde{\Phi_E})$. Since $E \subseteq B$, then $\pi_\varphi(\widetilde{\Phi_A}, \widetilde{\Phi_A}) = (\widetilde{\Phi_E}, \widetilde{\Phi_E})\widetilde{\subseteq}(\widetilde{\Phi_B}, \widetilde{\Phi_B})$.

If φ is surjective, then $E = \varphi(A) = B$. Hence, $\pi_\varphi(\widetilde{\Phi_A}, \widetilde{\Phi_A}) = (\widetilde{\Phi_E}, \widetilde{\Phi_E}) = (\widetilde{\Phi_B}, \widetilde{\Phi_B})$.

To prove (ii), let $\pi_\varphi(\widetilde{X_A}, \widetilde{X_A}) = \pi_\varphi(u, u, A) = (u, u, E)$, where $u(a) = X$ for each $a \in A$ and $E = \varphi(A)$. Then, $v(e) = \pi(\cup_{\varepsilon\in\varphi^{-1}(e)} u(\varepsilon)) = \pi(X) \subseteq Y$ for each $e \in E$. Therefore, $(v, v, E)\widetilde{\subseteq}(\widetilde{Y_B}, \widetilde{Y_B})$.

If φ and π are surjective, then $E = \varphi(A) = B$ and $\pi(X) = Y$. Hence, $\pi_\varphi(\widetilde{X_A}, \widetilde{X_A}) = (\widetilde{Y_B}, \widetilde{Y_B})$.

One can prove (iii) easily.

To prove (iv), first, let $\pi_\varphi[(f_1, f_2, \beta)\widetilde{\cup}(h_1, h_2, \beta')] = \pi_\varphi(u_1, u_2, \beta\cup\beta') = (v_1, v_2, E)$, where $E = \varphi(\beta\cup\beta')$. Now, for each $e \in E$, we have $v_i(e) = \pi(\cup_{\varepsilon\in\varphi^{-1}(e)\cap E} u_i(\varepsilon))$. Since

$$u_i(\varepsilon) = \begin{cases} f_i(\varepsilon), & : \varepsilon \in \beta - \beta', \\ h_i(\varepsilon), & : \varepsilon \in \beta' - \beta, \\ f_i(\varepsilon) \cup h_i(\varepsilon), & : \varepsilon \in \beta \cap \beta', \end{cases} \tag{13}$$

then

$$\pi\left(\bigcup_{\varepsilon\in\varphi^{-1}(e)\cap E} u_i(\varepsilon)\right) = \pi\left(\cup\begin{cases} f_i(\varepsilon) & : \varepsilon \in (\beta-\beta')\cap\varphi^{-1}(e) \\ h_i(\varepsilon) & : \varepsilon \in (\beta'-\beta)\cap\varphi^{-1}(e) \\ f_i(\varepsilon)\cup h_i(\varepsilon) & : \varepsilon \in (\beta\cap\beta')\cap\varphi^{-1}(e) \end{cases}\right). \tag{14}$$

Second, let $\pi_\varphi(f_1, f_2, \beta)\widetilde{\cup}\pi_\varphi(h_1, h_2, \beta') = (w_1, w_2, N)$, where $N = \varphi(\beta)\cup\varphi(\beta')$. Now, for each $n \in N$, we have

$$\begin{aligned} w_i(n) &= \pi\left(\bigcup_{\varepsilon\in\varphi^{-1}(n)\cap N} f_i(\varepsilon)\right) \cup \pi\left(\bigcup_{\varepsilon\in\varphi^{-1}(n)\cap N} h_i(\varepsilon)\right) \\ &= \pi\left(\bigcup_{\varepsilon\in\varphi^{-1}(n)\cap N} f_i(\varepsilon) \cup \bigcup_{\varepsilon\in\varphi^{-1}(n)\cap N} h_i(\varepsilon)\right) \\ &= \pi\left(\cup\begin{cases} f_i(\varepsilon) & : \varepsilon \in (\beta-\beta')\cap\varphi^{-1}(n) \\ h_i(\varepsilon) & : \varepsilon \in (\beta'-\beta)\cap\varphi^{-1}(n) \\ f_i(\varepsilon)\cup h_i(\varepsilon) & : \varepsilon \in (\beta\cap\beta')\cap\varphi^{-1}(n) \end{cases}\right). \end{aligned} \tag{15}$$

Since $\varphi(\beta\cup\beta') = \varphi(\beta)\cup\varphi(\beta')$, then $E = N$. Thus, $v_i(e) = w_i(e)$ for each $e \in E = N$. Hence, we obtain the desired result.

One can prove (v) similarly.

By using a similar technique, one can prove the following result. □

Proposition 12. *Let π_φ: $C(X_A) \longrightarrow C(Y_B)$ be a soft mapping and let (g_1, g_2, N) and (l_1, l_2, N') be two double-framed soft sets in $C(Y_B)$. Then, we have the following results:*

(i) $\pi_\varphi^{-1}(\widetilde{\Phi_B}, \widetilde{\Phi_B}) = (\widetilde{\Phi_A}, \widetilde{\Phi_A})$.

(ii) $\pi_\varphi^{-1}(\widetilde{Y_B}, \widetilde{Y_B}) = (\widetilde{X_A}, \widetilde{X_A})$.

(iii) If $(g_1, g_2, N)\widetilde{\subseteq}(l_1, l_2, N')$, then $\pi_\varphi^{-1}(g_1, g_2, N)\widetilde{\subseteq}\pi_\varphi^{-1}(l_1, l_2, N')$.

(iv) $\pi_\varphi^{-1}[(g_1, g_2, N)\widetilde{\cup}(l_1, l_2, N')] = \pi_\varphi^{-1}(g_1, g_2, N)\widetilde{\cup}\pi_\varphi^{-1}(l_1, l_2, N')$.

(v) $\pi_\varphi^{-1}[(g_1, g_2, N)\widetilde{\cap}(l_1, l_2, N')] = \pi_\varphi^{-1}(g_1, g_2, N)\widetilde{\cap}\pi_\varphi^{-1}(l_1, l_2, N')$.

Proposition 13. *Let π_φ: $C(X_A) \longrightarrow C(Y_B)$ be a soft mapping, and let (h, k, M) be a double-framed soft set in $C(X_A)$. Then, we have the following results:*

(i) If $\delta\Subset_w(h, k, M)$, then $\pi(\delta)\Subset_w\pi_\varphi(h, k, M)$.

(ii) If $\delta\Subset_s(h, k, M)$, then $\pi(\delta)\Subset_s\pi_\varphi(h, k, M)$.

(iii) If $\delta\in_w(h, k, M)$, then $\pi(\delta)\in_w\pi_\varphi(h, k, M)$.

(iv) If $\delta\in_s(h, k, M)$, then $\pi(\delta)\in_s\pi_\varphi(h, k, M)$.

(v) If $\delta\not\Subset_w(h, k, M)$ and φ is injective, then $\pi(\delta)\not\Subset_w\pi_\varphi(h, k, M)$.

(vi) If $\delta\not\Subset_s(h, k, M)$ and φ is injective, then $\pi(\delta)\not\Subset_s\pi_\varphi(h, k, M)$.

(vii) If $\delta \notin_w (h, k, M)$, then $\pi(\delta) \notin_w \pi_\varphi(h, k, M)$.

(viii) If $\delta \notin_s (h, k, M)$, then $\pi(\delta) \notin_s \pi_\varphi(h, k, M)$.

Proof. We only prove (i), (ii), (v), and (viii). The other cases can be made similarly.

To prove (i), let $\delta\Subset_w(h, k, M)$, then there exist parameters $a, a' \in M \subseteq A$ such that $\delta \in h(a)$ or $\delta \in k(a')$. Without loss of generality, consider $\delta \in h(a)$. Now, there is a parameter $b \in \varphi(M) \subseteq B$ such that $a \in \varphi^{-1}(b)$. Obviously, $a \in \varphi^{-1}(b) \cap M$, so that it follows from Definition 15 that $\pi(\delta) \in \pi_h(b) = \pi(\cup_{\varepsilon\in\varphi^{-1}(b)\cap M} h(\varepsilon))$. Therefore, $\pi(\delta)\Subset_w(\pi_h, \pi_k, \varphi(M)) = \pi_\varphi(h, k, M)$, as required.

To prove (ii), let $\delta\Subset_s(h, k, M)$. Then, there exist parameters $a, a' \in M \subseteq A$ such that $\delta \in h(a)$ and $\delta \in k(a')$. Without loss of generality, suppose that there exist two distinct parameters $b, b' \in \varphi(M) \subseteq B$ such that $a \in \varphi^{-1}(b)$ and $a' \in \varphi^{-1}(b')$. Obviously, $a \in \varphi^{-1}(b) \cap M$ and $a' \in \varphi^{-1}(b') \cap M$ so that it follows from Definition 15 that $\pi(\delta) \in \pi_h(b) = \pi(\cup_{\varepsilon\in\varphi^{-1}(b)\cap M} h(\varepsilon))$ and $\pi(\delta) \in \pi_k(b') = \pi(\cup_{\varepsilon\in\varphi^{-1}(b')\cap M} k(\varepsilon))$. Therefore, $\pi(\delta)\Subset_s(\pi_h, \pi_k, \varphi(M)) = \pi_\varphi(h, k, M)$, as required.

To prove (v), let $\delta\not\Subset_w(h, k, M)$. Then, there exist parameters $a, a' \in M \subseteq A$ such that $\delta \notin h(a)$ or $\delta \notin k(a')$. Say $\delta \notin h(a)$. Then, there is a parameter $b \in \varphi(M) \subseteq B$ such that $a \in \varphi^{-1}(b)$. Since φ is injective, then $a = \varphi^{-1}(b)$. This means that $\{a\} = \varphi^{-1}(b) \cap M$. Therefore, $\pi(\delta) \notin \pi_h(b) = \pi(\cup_{\varepsilon\in\varphi^{-1}(b)\cap M} h(\varepsilon)) = \pi(h(a))$. Therefore, $\pi(\delta)\not\Subset_w(\pi_h, \pi_k, \varphi(M)) = \pi_\varphi(h, k, M)$, as required.

To prove (viii), let $\delta \notin_s (h, k, M)$. Then, $\delta \notin h(a)$ and $\delta \notin k(a)$ for all $a \in M \subseteq A$. Therefore, for each parameter $b \in \varphi(M) \subseteq B$, there is $a \in M$ such that $a \in \varphi^{-1}(b)$. Thus, for each $b \in \varphi(M)$, we obtain $\pi(\delta) \notin \pi_h(b) = \pi(\cup_{\varepsilon \in \varphi^{-1}(b) \cap M} h(\varepsilon)) = \pi(h(a))$ and $\pi(\delta) \notin \pi_k(b) = \pi(\cup_{\varepsilon \in \varphi^{-1}(b) \cap M} k(\varepsilon)) = \pi(h(a))$. Hence, $\pi(\delta) \notin_s (\pi_h, \pi_k, \varphi(M)) = \pi_\varphi(h, k, M)$, as required. □

Proposition 14. *Let $\pi_\varphi: C(X_A) \longrightarrow C(Y_B)$ be a soft mapping and let (p, t, N) be a double-framed soft set in $C(Y_B)$. If φ is surjective, then we have the following results:*

(i) If $\xi \Subset_w (p, t, N)$, then $\delta \Subset_w \pi_\varphi^{-1}(p, t, N)$ for each $\delta \in \pi^{-1}(\xi)$.

(ii) If $\xi \Subset_s (p, t, N)$, then $\delta \Subset_s \pi_\varphi^{-1}(p, t, N)$ for each $\delta \in \pi^{-1}(\xi)$.

(iii) If $\xi \in_w (p, t, N)$, then $\delta \in_w \pi_\varphi^{-1}(p, t, N)$ for each $\delta \in \pi^{-1}(\xi)$.

(iv) If $\xi \in_s (p, t, N)$, then $\delta \in_s \pi_\varphi^{-1}(p, t, N)$ for each $\delta \in \pi^{-1}(\xi)$.

(v) If $\xi \not\Subset_w (p, t, N)$ such that π is injective, then $\pi^{-1}(\xi) \not\Subset_w \pi_\varphi^{-1}(p, t, N)$.

(vi) If $\xi \not\Subset_s (p, t, N)$ such that π is injective, then $\pi^{-1}(\xi) \not\Subset_s \pi_\varphi^{-1}(p, t, N)$.

(vii) If $\xi \notin_w (p, t, N)$ such that π is injective, then $\pi^{-1}(\xi) \notin_w \pi_\varphi^{-1}(p, t, N)$.

(viii) If $\xi \notin_s (p, t, N)$ such that π is injective, then $\pi^{-1}(\xi) \notin_s \pi_\varphi^{-1}(p, t, N)$.

Proof. We only prove (i), (ii), (v), and (viii). The other cases can be made similarly.

To prove (i), let $\xi \Subset_w (p, t, N)$. Then, there exist parameters $b, b' \in N \subseteq B$ such that $\xi \in p(b)$ or $\xi \in t(b')$. Without loss of generality, consider $\xi \in p(b)$. Since φ is surjective, then there is a parameter $a \in \varphi^{-1}(N) \subseteq A$ such that $\varphi(a) = b$. It follows from Definition 17 that $\pi_h^{-1}(a) = \pi^{-1}(p\varphi(a)) = \pi^{-1}(p(b))$. Now, for each $\delta \in \pi^{-1}(\xi)$, we obtain $\delta \Subset_w (\pi_p^{-1}, \pi_t^{-1}, \varphi^{-1}(N)) = \pi_\varphi^{-1}(p, t, N)$, as required.

To prove (ii), let $\xi \Subset_s (p, t, N)$. Then, there exist parameters $b, b' \in N \subseteq B$ such that $\xi \in p(b)$ and $\xi \in t(b')$. Since φ is surjective, then there are two parameters $a, a' \in \varphi^{-1}(N) \subseteq A$ such that $\varphi(a) = b$ and $\varphi(a') = b'$. It follows from Definition 17 that $\pi_h^{-1}(a) = \pi^{-1}(p\varphi(a)) = \pi^{-1}(p(b))$ and $\pi_l^{-1}(a') = \pi^{-1}(t\varphi(a')) = \pi^{-1}(t(b'))$. Now, for each $\delta \in \pi^{-1}(\xi)$, we obtain $\delta \Subset_s (\pi_p^{-1}, \pi_t^{-1}, \varphi^{-1}(N)) = \pi_\varphi^{-1}(p, t, N)$, as required.

To prove (v), let $\xi \not\Subset_w (p, t, N)$. Then, there exist parameters $b, b' \in N \subseteq B$ such that $\xi \notin p(b)$ or $\xi \notin t(b')$. Say $\xi \notin p(b)$. Since φ is surjective, then there exists a parameter $a \in \varphi^{-1}(N) \subseteq A$ such that $\varphi(a) = b$. It follows from Definition 17 that $\pi_h^{-1}(a) = \pi^{-1}(p\varphi(a)) = \pi^{-1}(p(b))$. Since π is injective, then $\pi^{-1}(\xi) \not\Subset_s (\pi_p^{-1}, \pi_t^{-1}, \varphi^{-1}(N)) = \pi_\varphi^{-1}(p, t, N)$, as required.

To prove (viii), let $\xi \notin_s (p, t, N)$. Then, $\xi \notin p(b)$ and $\xi \notin t(b)$ for all $b \in N \subseteq B$. Since φ is surjective, then there exists a parameter $a \in \varphi^{-1}(N) \subseteq A$ such that $\varphi(a) = b$. It follows from Definition 17 that $\pi_h^{-1}(a) = \pi^{-1}(p\varphi(a)) = \pi^{-1}(p(b))$ and $\pi_l^{-1}(a) = \pi^{-1}(t\varphi(a)) = \pi^{-1}(t(b))$. Since π is injective, then $\pi^{-1}(\xi) \not\Subset_s (\pi_p^{-1}, \pi_t^{-1}, \varphi^{-1}(N)) = \pi_\varphi^{-1}(p, t, N)$, as required. □

Proposition 15. *Let $\pi_\varphi: C(X_A) \longrightarrow C(Y_B)$ be a soft mapping and let (h, k, M) and (p, t, N) be two double-framed soft sets in $C(X_A)$ and $C(Y_B)$, respectively. Then, the following holds:*

(i) If π is bijective, then $\pi_\varphi((h, k, M)^c) = [\pi_\varphi(h, k, M)]^c$.

(ii) $\pi_\varphi^{-1}((p, t, N)^c) = [\pi_\varphi^{-1}(p, t, N)]^c$.

Proof. We only prove (i).

It is clear that $\pi_\varphi((h, k, M)^c) = \pi_\varphi(h^c, k^c, M)$, where $\pi_{h^c}(e) = \pi(\cup_{\varepsilon \in \varphi^{-1}(e) \cap M} h^c(\varepsilon))$ and $\pi_{k^c}(e) = \pi(\cup_{\varepsilon \in \varphi^{-1}(e) \cap M} k^c(\varepsilon))$ for each $e \in \varphi(M)$. Since π is bijective, then $\pi_{h^c}(e) = \pi(\cup_{\varepsilon \in \varphi^{-1}(e) \cap M} h^c(\varepsilon)) = (\pi(\cup_{\varepsilon \in \varphi^{-1}(e) \cap M} h(\varepsilon)))^c$ and $\pi_{k^c}(e) = \pi(\cup_{\varepsilon \in \varphi^{-1}(e) \cap M} k^c(\varepsilon)) = (\pi(\cup_{\varepsilon \in \varphi^{-1}(e) \cap M} k(\varepsilon)))^c$ □

4. Application of Double-Framed Soft Sets

In this section, we present an application of optimal choices using the idea of double-framed soft sets. The idea of this application is based on the evaluation of rank of the applicants in the different disciplines under study, not on the total summation of marks obtained by the applicant. The philosophy of this method is based on comprehensive evaluation, in other words, confirming the ability of applicants of satisfying high levels for all testing criteria.

Now, we provide an example to demonstrate: how we make optimal choices? Then, we construct an algorithm of this method.

Example 5. Ministry of education advertises of five scholarships supported from the government for the students who finished secondary stage. The trade-off between applicants is based on the examinations of two subjects: maths and physics.

Twenty students $S = \{s_i: i = 1, 2, \ldots, 20\}$ applied to compete with each other to gain one of these scholarships. They carried out the examination of the two subjects. Then, we input subjects' marks of all students in Table 1.

Now, we determine the ranks of the students for each subject. In fact, this step will depend on the content of the application or the desire of those in charge of work. Regarding our example, we put a set $A = \{a_i: i = 1, 2, \ldots, 10\}$ expressing ten levels of ranks:

a_1 stands for the students with the first rank.

a_2 stands for the students with the second rank.

$\vdots$

a_n stands for the students with the n-th rank.

From Table 1, we complete Table 2 by constructing a double-framed soft set $(f_{\text{Maths}}, f_{\text{Physics}}, A)$ over S, where the maps f_{Maths} and f_{Physics} from A into the power set of S are given by $f_{\text{Maths}}(a_i)$ = the set of students who rank are a_i in maths subject and $f_{\text{Physics}}(a_i)$ = the set of students who rank are a_i in physics subject.

Table 1: Subjects' marks of twenty students.

Student	Subjects Maths	Physics
s_1	35	31
s_2	28	25
s_3	42	48
s_4	22	19
s_5	49	47
s_6	33	36
s_7	18	23
s_8	34	34
s_9	50	37
s_{10}	21	25
s_{11}	20	18
s_{12}	27	32
s_{13}	11	17
s_{14}	30	25
s_{15}	49	40
s_{16}	50	41
s_{17}	36	44
s_{18}	14	16
s_{19}	16	25
s_{20}	46	38

Table 2: Maps of subjects.

A	Maps f_{Maths}	f_{Physics}
a_1	$\{s_9, s_{16}\}$	$\{s_3\}$
a_2	$\{s_5, s_{15}\}$	$\{s_5\}$
a_3	$\{s_1\}$	$\{s_{20}, s_{17}\}$
a_4	$\{s_3\}$	$\{s_{16}\}$
a_5	$\{s_{17}\}$	$\{s_{15}\}$
a_6	$\{s_{20}\}$	$\{s_{19}\}$
a_7	$\{s_8\}$	$\{s_9\}$
a_8	$\{s_6\}$	$\{s_6\}$
a_9	$\{s_{14}\}$	$\{s_8\}$
a_{10}	$\{s_2\}$	$\{s_{12}\}$

Finally, we give each rank a standard score. Regarding our example, we consider the following standard score of each rank a_i:

Rank a_1 takes 10 standard scores of each subject.

Rank a_2 takes 9 standard scores of each subject.

⋮

Rank a_{10} takes 1 standard score of each subject.

Any rank a_m such that $m > 10$ takes standard zero score of each subject.

For each map f_j of a double-framed soft set $(f_{\text{Maths}}, f_{\text{Physics}}, A)$ and each student $s_i \in S$, we calculate the value of each pair (s_i, f_j) of Table 3 by the following rule:

$$\left(s_i, f_j\right) = \begin{cases} \text{the standard score } a_m, & s_i \in f_j\left(a_m\right), \\ 0, & s_i \notin \left(f_j, A\right). \end{cases} \tag{16}$$

We sum the standard scores of all subjects for each student and then decide the student's rank depending on the summation of his/her standard scores.

Table 3 illustrates this step.

Table 3: Students' rank.

Student	f_j f_{Maths}	f_{Physics}	Total	Rank
s_1	5	0	5	7^{th}
s_2	1	0	1	9^{th}
s_3	7	10	17	2^{nd}
s_4	0	0	0	10^{th}
s_5	9	9	18	1^{st}
s_6	3	3	6	6^{th}
s_7	0	5	5	7^{th}
s_8	4	2	6	6^{th}
s_9	10	4	14	4^{th}
s_{10}	0	0	0	10^{th}
s_{11}	0	0	0	10^{th}
s_{12}	0	1	1	8^{th}
s_{13}	0	0	0	13^{th}
s_{14}	2	0	2	8^{th}
s_{15}	9	6	15	3^{rd}
s_{16}	10	7	17	2^{nd}
s_{17}	6	8	14	4^{th}
s_{18}	0	0	0	10^{th}
s_{19}	0	5	5	7^{th}
s_{20}	5	8	13	5^{th}

One can note from the above table that we can decide four wining students: s_5 is the first, s_3 and s_{16} are the second, and s_{15} is the third. However, the last wining student is chosen from the set $\{s_9, s_{17}\}$. The method of choosing them can be done by ways such as interview, total marks, or random lottery.

In the following, we present an algorithm of determining the wining students.

On the contrary, if the subjects f_j are not of equal significance, that is, Ministry of education imposes weights on the subjects, i.e., corresponding to each subject f_j, there is a weight $w_i \in 0, 1$.

Step 1. Examine the applicants in the specified subjects.
Step 2. Input the marks of each applicant in the specified subjects (see Table 1).
Step 3. Determine the range of rank a_i: $i = 1, 2, \ldots, n$
Step 4. Classify the students according to the proposed range rank of each subject (see Table 2).
Step 5. Give each rank a standard score.
Step 6. Sum the standard scores of all subjects for each student (see Table 3).
Step 7. Order the column of the total standard scores in descending order.
Step 8. Choose the first students according to the permissible range, if there are more than one student in the last chosen rank, then you can compare between them by interview, or total marks, or random lottery.

Algorithm 1: Algorithm of determining the winning students in the case of equal significance.

Step 1. Repeat Step 1–Step 5 of Algorithm 1.
Step 2. Find a weighted table of the subjects f_j according to the weights decided by the organizer of the competition, and the weights are denoted by w_i: $i = 1, 2, \ldots, m$.
Step 3. Multiple each standard score with its corresponding weight (see Table 4).
Step 4. Sum the weight standard scores of all subjects for each student.
Step 5. Order the column of the total standard scores in descending order.
Step 6. Choose the first students according to the permissible range, if there are more than one student in the last chosen rank, then you can compare between them by interview, or total marks, or random lottery.

Algorithm 2: Algorithm of determining the winning students in the case of different significance.

Table 4: Students' weight rank.

	f_j, w_i			
S	$f_{\text{Maths}}, W_1 = 30\%$	$H_{\text{Physics}}, W_2 = 70\%$	Total	Rank
s_1	1.5	0	1.5	11th
s_2	0.3	0	0.3	15th
s_3	2.1	7	9.1	1st
s_4	0	0	0	16th
s_5	2.7	6.3	9	2nd
s_6	0.9	2.1	3	9th
s_7	0	3.5	3.5	8th
s_8	1.2	1.4	2.6	10th
s_9	3	2.8	5.8	7th
s_{10}	0	0	0	16th
s_{11}	0	0	2.6	10th
s_{12}	0	0.7	0.7	12th
s_{13}	0	0	0	16th
s_{14}	0.6	0	0.6	13th
s_{15}	2.7	4.2	6.9	6th
s_{16}	3	4.9	7.9	3rd
s_{17}	1.8	5.6	7.4	4th
s_{18}	0	0	0.4	14th
s_{19}	0	0	3.5	8th
s_{20}	1.5	5.6	7.1	5th

In this case, we modify the previous algorithm to be convenient for weighted selection.

With respect to our example (Algorithm 2), suppose that the weights 30% and 70% are, respectively, corresponding to maths and physics subjects. Then, we update Table 3 to be as follows.

Now, one can note from the above table that the five wining students are as follows: s_3 is the first, s_5 is the second, s_{16} is the third, s_{17} is the fourth, and s_{20} is the fifth.

5. Conclusions

In this article, we have initiated four types of belong relations and four types of nonbelong relations between an ordinary point and double-framed soft sets. These relations are primary indicator of the degree of membership and nonmembership of an element. Then, we have defined soft mappings between two classes of double-framed soft sets and determine the conditions under which an ordinary point and its image and preimage are preserved with respect to the different types of belong and nonbelong relations. In the end, we have exploited the idea of double-framed soft sets to investigate an educational application of choosing the best students in terms of their performance rank in all testing criteria. An algorithm of the application was explained with the aid of an illustrative example.

We draw attention to that the different types of belong and nonbelong relations classify the relationships between elements and double-framed soft sets into eight levels as well as classify the stability into two levels. One of the unique properties of these relations is the possibility of belonging and nonbelonging of the element to the same double-framed soft sets with respect to weakly partial belong and weakly partial nonbelong relations, strongly partial belong and strongly partial nonbelong relations, and weakly total belong and weakly total nonbelong relations. This matter leads to new relations between belonging and nonbelonging of the ordinary points and the soft intersection and union of double-framed soft sets.

As future works, we shall apply the relations presented in this work to formulate several types of soft separation axioms and compact spaces on double-framed soft topological spaces. To simplify and clarify this idea, we define four types of covers of a double-framed soft topological space using weakly partial belong, strongly partial belong, weakly total belong, and strongly total belong relations. In addition, we try to model some natural phenomena using the idea of N-framed soft set. It is worthy to note that one can extend this work by studying the belong and nonbelong relations introduced herein with respect to N-framed soft sets, where $N = 3, 4, \ldots$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Certain Notions of Picture Fuzzy Information with Applications

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Certain Notions of Picture Fuzzy Information with Applications

Rukhshanda Anjum,[1] Abdu Gumaei ,[2] and Abdul Ghaffar[3]

[1]*Deaprtament of Mathematics and Statistics, University of Lahore, Lahore, Pakistan*
[2]*Research Chair of Pervasive and Mobile Computing, Department of Information Systems, College of Computer and Information Sciences, King Saud University, Riyadh 11543, Saudi Arabia*
[3]*Department of Mathematics, Ghazi University, DG Khan 32200, Pakistan*

Correspondence should be addressed to Abdu Gumaei; agumaei.c@ksu.edu.sa

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In this manuscript, the theory of constant picture fuzzy graphs (CPFG) is developed. A CPFG is a generalization of constant intuitionistic fuzzy graph (CIFG) and a special case of picture fuzzy graph (PFG). Additionally, the article includes some basic definitions of CPFG such as totally constant picture fuzzy graphs (TCPFGs), constant function, bridge of CPFG, and their related results. Also, an application of CPFG in Wi-Fi network system is discussed. Finally, a comparison of CPFG is established with that of the CIFG which exhibits the superiority of the proposed idea over the existing ones is discussed.

1. Introduction

Wi-Fi systems and the analysis of their signals have been under discussion during the last decades [1, 2]. To provide signals effectively, potential research has been carried out in [3, 4]. A Wi-Fi device within the range can either be connected, disconnected, or fluctuate between the state of connected and disconnected or it could be out of range. Such uncertain situations can be dealt by the idea of PFG which proves to be helpful in such cases.

Zadeh [5] proposed the theory of fuzzy sets (FSs) that is very popular tool and is considered the superior tool till now. Kaufman defined fuzzy graph (FG) in [6]. A detailed study is contributed by Rosenfeld in his article [7]. Since then theory of FGs has been extensively applied to many fields such as clustering [8–10], networking [11, 12] and communication problems [13–15].

Atanassov [16] proposed intuitionistic fuzzy set (IFS) as a generalization of fuzzy set (FS). The concept of intuitionistic fuzzy relations has also been discussed in [16] providing fundamentals of the theory of IFGs. Parvathi and Karunambigai [17] defined IFGs as generalization of FGs and discussed various graph theoretic concepts. For detailed work in the course of IFGs, one may refer to [18–26]. The structure of IFGs is diverse than that of FGs and it is applied to many problems such as radio coverage networking [22], decision making and shortest path problems [20, 27–31], and social networks [32].

In Wi-Fi networks, we usually face more situations that we could not handle by FGs and IFGs. Therefore, in this article, the idea of PFG and consequently CPFG is introduced as a generalization of constant IFGs. The properties and results of CPFG are discussed and illustrated with examples. In addition, a Wi-Fi network problem is modeled using CPFGs.

The article starts with introduction followed by the section that discusses some basic ideas. The third section is based on concepts of PFGs while section four is based on CPFGs and its related theory. In section five, an application is discussed thoroughly with some numerical explanations. Finally, the concluding statements are added to the manuscript.

2. Preliminaries

This section discusses some basic ideas of graph theory including the ideas of FGs and IFGs. These concepts of FGs and IFGs are illustrated with the help of examples.

Definition 1 (see [7]). An FG is a pair $\breve{G} = (V, \check{E})$ such that

(I) V is the set of vertices and T_1 maps on $[0, 1]$ are the association degree of $v_i \in V$.

(II) $\check{E} = \{(v_i, v_j)\colon (v_i, v_j) \in V \times V\}$ and $T_2\colon V \times V \longrightarrow [0, 1]$,
where $T_2(v_i, v_j) \leq \min\{T_1(v_i), T_1(v_j)\}$ for all $(v_i, v_j) \in \check{E}$.

Example 1. An FG $\breve{G} = (V, \check{E})$ with the collection of vertices V and the collection of edges $\check{E}$ is depicted in Figure 1.

Definition 2 (see [17]). An IFG is a pair $\breve{G} = (V, \check{E})$ such that

(i) V is the set of vertices such that T_1 and F_1 maps on the closed interval $[0, 1]$ represent the grads of membership and nonmembership of the vertex elements $v_i \in V$, respectively, with a condition $0 \leq T_1 + F_1 \leq 1$ for all $v_i \in V$, $(i \in I)$.

(ii) $\check{E} \subseteq V \times V$ where $T_2, F_2\colon V \times V \longrightarrow [0, 1]$ represent the grads of membership and nonmembership of the edge elements $(v_i, v_j) \in \check{E}$ such that $T_2(v_i, v_j) \leq \min\{T_1(v_i), T_1(v_j)\}$ and $F_2(v_i, v_j) \leq \max\{F_1(v_i), F_1(v_j)\}$ with a condition $0 \leq T_2(v_i, v_j) + F_2(v_i, v_j) \leq 1$ for all $(v_i, v_j) \in \check{E}$, $(i \in I)$.

Example 2. Consider an IFG $\breve{G} = (V, \check{E})$ depicted in Figure 2.

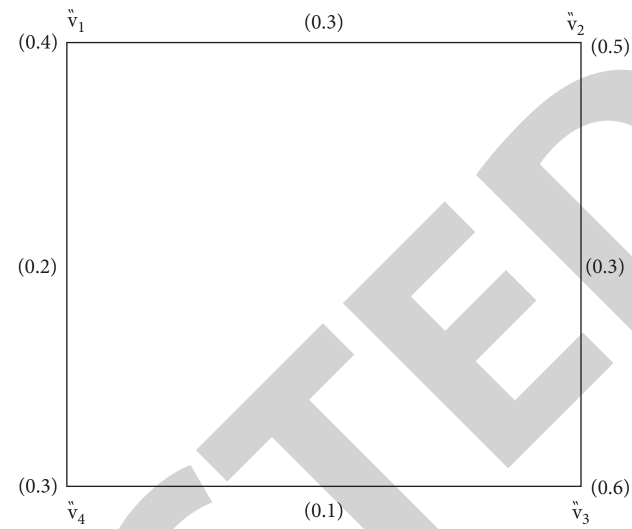

Figure 1: (FG).

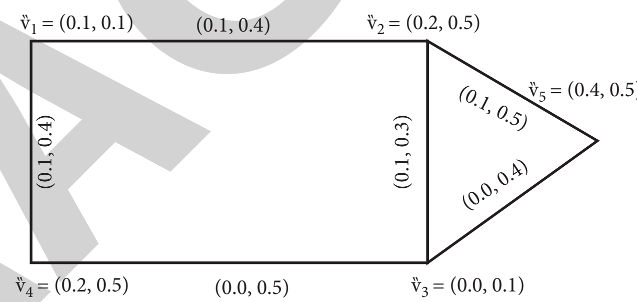

Figure 2: (IFG).

3. Picture Fuzzy Graphs

This section is based on some very basic concepts related to PFGs including its definition, and some of its associated terms such as degree of PFGs and completeness of PFGs are discussed.

Definition 3. A PFG is a pair $\breve{G} = (V, \check{E})$ such that

(i) V is the collection of vertices such that $T_1, \Gamma_1, F_1\colon V \longrightarrow [0, 1]$ represent the grads of membership, abstinence, and nonmembership of the vertex elements $v_i \in V$, respectively, so long as $0 \leq T_1 + \Gamma_1 + F_1 \leq 1$ for all $v_i \in V$, $(i \in I)$.

(ii) $\check{E} \subseteq V \times V$ where $T_2, \Gamma_2, F_2\colon V \times V \longrightarrow [0, 1]$ represent the grads of membership, abstinence, and nonmembership of the edge elements $(v_i, v_j) \in \check{E}$ such that $T_2(v_i, v_j) \leq \min\{T_1(v_i), T_1(v_j)\}$, $\Gamma_2(v_i, v_j) \leq \min\{\Gamma_1(v_i), \Gamma_1(v_j)\}$, and $F_2(v_i, v_j) \leq \max\{F_1(v_i), F_1(v_j)\}$ as long as $0 \leq T_2(v_i, v_j) + \Gamma_2(v_i, v_j) \in + F_2(v_i, v_j) \leq 1$ for all $(v_i, v_j) \in \check{E}$, $(i \in I)$.

Moreover, $1 - (T_{1i} + \Gamma_{1i} + F_{1i})$ represent refusal degree.

Example 3. A PFG $\breve{G} = (V, \check{E})$ is depicted in Figure 3.

Definition 4. Let $\breve{G} = (V, \check{E})$ be PFG. Then, the degree of any vertex v is defined by $d(v) = (d_T(v), d_\Gamma(v), d_{\mathscr{D}}(v))$, where $d_T(v) = \sum_{u \neq v} T_2(v, u)$, $d_\Gamma(v) = \sum_{u \neq v} \Gamma_2(v, u)$, and $d_F(v) = \sum_{u \neq v} F_2(v, u)$.

Example 4. A PFG $\breve{G} = (V, \check{E})$ depicted in Figure 4 is calculated as follows.

Degree of vertices is

$$\begin{aligned} d(v_1) &= (0.3, 0.3, 0.8), \\ d(v_2) &= (0.2, 0.3, 0.8), \\ d(v_3) &= (0.0, 0.3, 0.8), \\ d(v_4) &= (0.1, 0.3, 0.8). \end{aligned} \tag{1}$$

Definition 5. The complement $\breve{G}'$ of PFG $\breve{G} = (V, \check{E})$ is as follows:

(1) $T_1(v_i)' = T_1(v_i), \Gamma_1(v_i)' = \Gamma_1(v_i), F_1(v_i)' = F_1(v_i)$, $\forall v_i \in V$.

(2) $T_2(v_i, v_j)' = \min[T_1(v_i), T_1(v_j)] - T_2(v_i, v_j), \Gamma_2(v_i, v_j)' = \min[\Gamma_1(v_i), \Gamma_1(v_j)] - \Gamma_2(v_i, v_j)$ and $F_2(v_i, v_j)' = \max[F_2(v_i), F_2(v_j)] - F_2(v_i, v_j) \forall v_i, v_j \in \check{E}$.

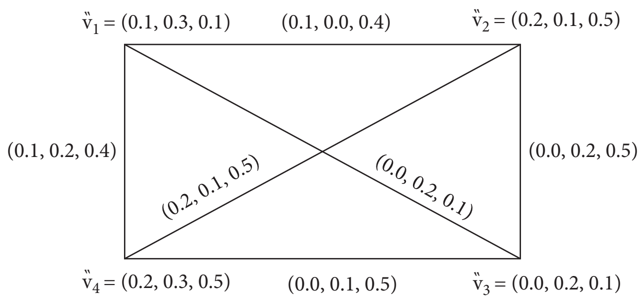

Figure 3: (PFG).

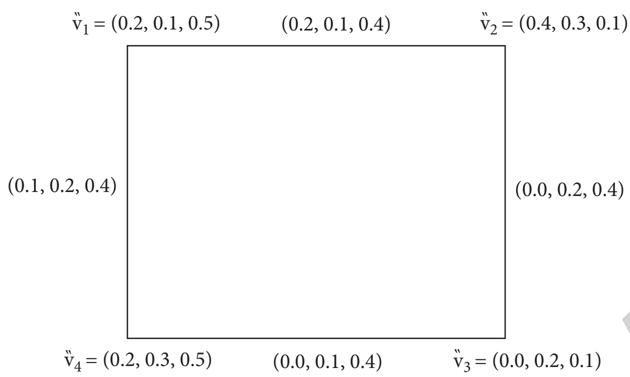

Figure 4: (PFG).

Remark 1. According to definition of a compliment, for a PFG, $\breve{G} = (V, \check{E})$, the graph $\breve{G}'' = (V'', \check{E}'') = \breve{G}$.

Proposition 1. *$\breve{G} = \breve{G}'' \Leftrightarrow \breve{G}$ is a strong PFG.*

Proof. According to the definition of $\breve{G}'$, the result and the proof are straight forward. □

Example 5. Figures 5 and 6 provide a verification of Proposition 1.

Definition 6. A PFG $\breve{G}$ is called a self-complementary graph if $\breve{G} = \breve{G}''$.

Definition 7. A PFG is said to be a complete PFG if $T_2(v_i, v_j) = \min\{T_1(v_i), T_1(v_j)\}$, $\Gamma_2(v_i, v_j) = \min\{\Gamma_1(v_i), \Gamma_1(v_j)\}$, and $F_2(v_i, v_j) = \max\{F_1(v_i), F_1(v_j)\}$.

Example 6. A complete PFG is depicted in Figure 7.

Definition 8. For any pair of different vertices (v_i, v_j) in a PFG, $\breve{G} = (V, \check{E})$, if deleting the edge (v_i, v_j) lessens the strength between that pair of vertices, then this edge is called the bridge in graph $\breve{G}$.

Example 7. A PFG $\breve{G} = (V, \check{E})$ is depicted in Figure 8 and explained as follows.

In Figure 8, the strength of $v_1 v_4$ is (0.1, 0.3, 0.4). Since the removal of (v_1, v_4) from G lessens the strength between the vertices v_1 and v_4 in G, therefore, (v_1, v_4) is a bridge.

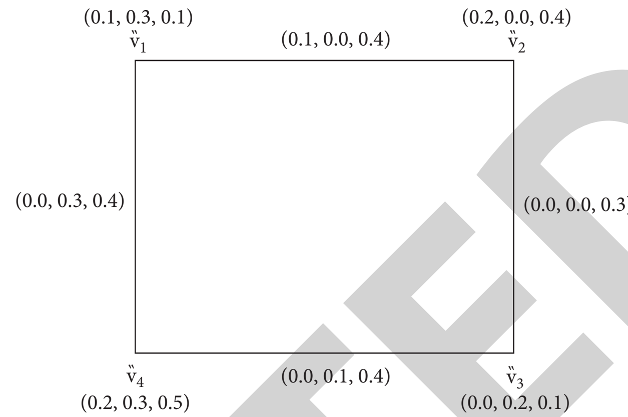

Figure 5: (PFG).

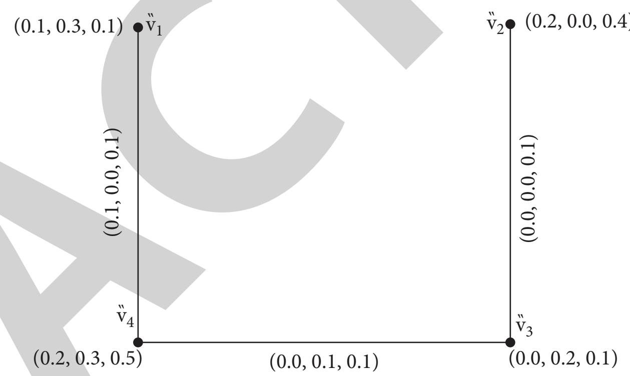

Figure 6: (Complement of Figure 5).

Definition 9. For a PFG G, If we remove a vertex v_i in $\breve{G}$ which decreases the strength of connectedness among some pairs of vertices, then it is called cut vertex of $\breve{G}$.

4. Constant Picture Fuzzy Graph

Definition 10. A PFG $\breve{G} = \{(v_i, T_{1i}, \Gamma_{1i}, F_{1i}), (\grave{e}_{ij}, T_{2ij}, \Gamma_{2ij}, F_{2ij})\}$ is known as CPFG of degree (k_i, k_j, k_k) or $(k_i, k_j, k_k) -$ PFG. If

$$\begin{aligned} d_T(v_i) &= k_i, \\ d_\Gamma(v_j) &= k_j, \qquad v_j, v_k \in V. \\ d_f(v_k) &= k_k \; \forall\, v_i, \end{aligned} \tag{2}$$

Example 8. A $\breve{G} = (V, \check{E})$. Then, the CPFG is depicted in Figure 9.

Example 9. A complete PFG needs not be a CPFG depicted in Figure 10 and explained as follows.

Figure 8 clearly shows that it is a complete PFG but not constant.

Definition 11. The total degree (τ_1, τ_2, τ_3) of a vertex $v \in V$ in PFG G is defined as

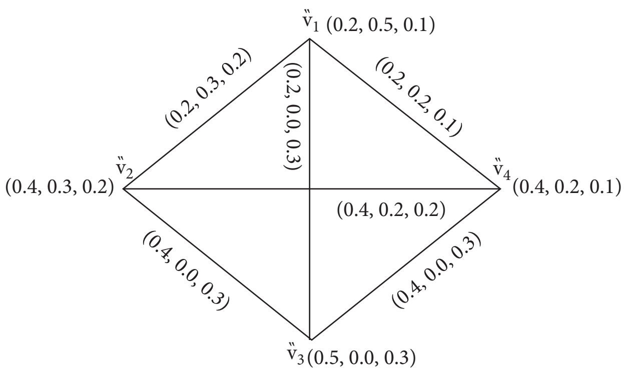

Figure 7: (Complete PFG).

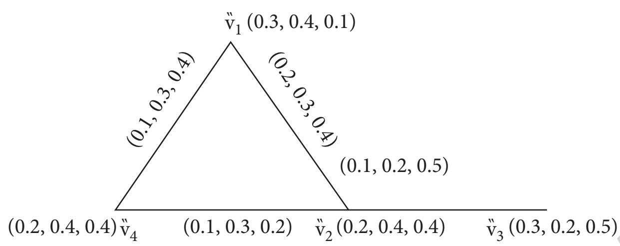

Figure 8: (PFG).

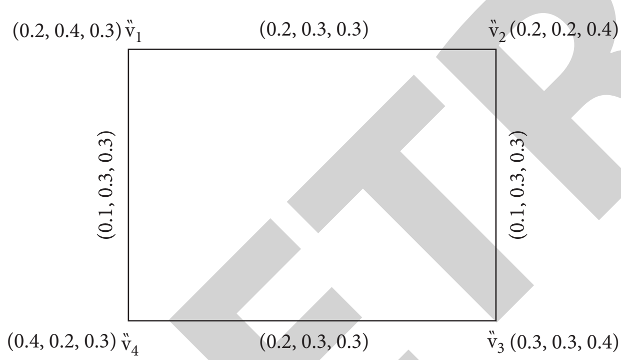

Figure 9: (CPFG). The degree of the vertices v_1, v_2, v_3, and v_4 is (0.3, 0.6, 0.6).

$$\mathrm{td}(v) = \left[\sum_{v\in \breve{E}} d_{T_2}(v) + T_1(v), \sum_{v\in \breve{E}} d_{\Gamma_2}(v) + \Gamma_1(v), \sum_{v\in \breve{E}} d_{F_2}(v) + F_1(v)\right]. \tag{3}$$

If total degree of each vertex of $\breve{G}$ is same, then $\breve{G}$ is called PFG of total degree (τ_1, τ_2, τ_3) or (τ_1, τ_2, τ_3)-TCP.

Example 10. Consider a TCPFG depicted in Figure 11.

Theorem 1. (T_1, Γ_1, F_1) *is a constant function (CF) in a PFG* $\breve{G}$ *iff the following are equivalent:*

(i) $\breve{G}$ *is a constant PFG.*

(ii) $\breve{G}$ *is totally PFG.*

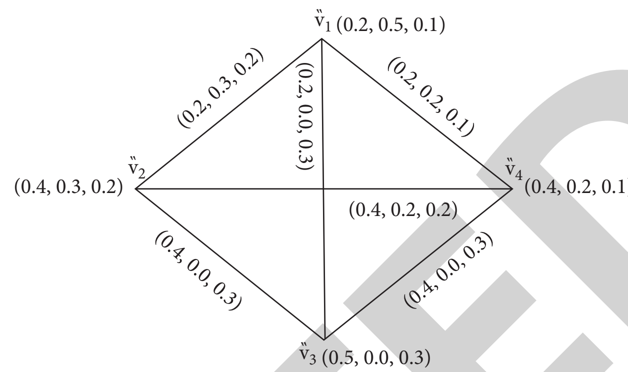

Figure 10: (Complete PFG).

Figure 11: (Totally constant PFG).

Proof. $(i) \Longrightarrow (ii)$ Consider (T_1, Γ_1, F_1) is a constant function. Suppose $T_1(v_i) = c_1, \Gamma_1(v_i) = c_2$ and $F_1(v_i) = c_3 \,\forall v_i \in V$ where c_1, c_2, and c_3 are constants. Let G be a constantPFG. Then, $d_F(v_i) = v_1, d_\Gamma(v_i) = v_2$ and $d_F(v_i) = k_3 \,\forall v_i \in V$. So, $\mathrm{td}_T(v_i) = d_F(v_i) + T_1(v_i), \mathrm{td}_\Gamma(v_i) = d_\Gamma(v_i) + \Gamma_1(v_i)$ and $d_F(v_i) = d_F(v_i) + F_1(v_i) \;\forall v_i \in V, \mathrm{td}_T(v_i) = k_1 + c_1$, $\mathrm{td}_\Gamma(v_i) = k_2 + c_2$, $\mathrm{td}_F(v_i) = k_3 + c_3 \,\forall v_i \in V$. Hence, (ii) is proved. $(ii) \Rightarrow (i)$ Assume that G is a (τ_1, τ_2, τ_3)-TCPFG. Then, $\mathrm{td}_T(v_i) = \tau_1$, $\mathrm{td}_\Gamma(v_i) = \tau_2$ and $\mathrm{td}_F(v_i) = \tau_3 \;\forall v_i \in V$ $d_T(v_i) + c_1(v_i) = \tau_1$, $d_T(v_i) + c_1 = \tau_1$, $d_T(v_i) = \tau_1 - c_1$, $d_\Gamma(v_i) + \Gamma_1(v_i) = \tau_1$, $d_\Gamma(v_i) + c_2 = \tau_2, d_\Gamma(v_i) = \tau_2 - c_2$, and $d_F(v_i) + c_3 = \tau_3, d_F(v_i) = \tau_3 - c_3$. So, G is CPFG. Conversely, if (i) and (ii) are equivalent, then (T_1, Γ_1, F_1) is a constant function. Now, (T_1, Γ_1, F_1) is a constant function iff (T_1, Γ_1, F_1) is a TCPFG. Assume that (T_1, Γ_1, F_1) is not a constant function. Then, $T_1(v_1) \neq T_2(v_2)$, $\Gamma_1(v_1) \neq \Gamma_2(v_2), F_1(v_1) \neq F_2(v_2)$ for $v_1, v_2 \in V$ and if (T_1, Γ_1, F_1) is a constant function, then $T_1(v_1) = T_2(v_2) = k_1$, $\Gamma_1(v_1) = \Gamma_2(v_2) = k_2$, $F_1(v_1) = F_2(v_2) = k_3$. So, $\mathrm{td}_T(v_1) = d_T(v_1) + T_1(v_1) = k_1 + T_1(v_1)$ and $\mathrm{td}_T(v_2) = k_1 + T_1(v_2)$, $\mathrm{td}_\Gamma(v_1) = d_\Gamma(v_1) + \Gamma_1(v_1) = k_2 + \Gamma_1(v_1)$ and $\mathrm{td}_\Gamma(v_2) = k_2 + \Gamma_1(v_2)$, $\mathrm{td}_F(v_1) = d_F(v_1) + F_1(v_1) = k_3 + F_1(v_1)$ and $\mathrm{td}_F(v_2) = k_3 + F_1(v_2)$. Hence, $T_1(v_1) \neq T_1(v_2), \Gamma_1(v_1) \neq \Gamma_1(v_2), F_1(v_1) \neq F_1(v_2)$ implies $\mathrm{td}_T(v_1) \neq \mathrm{td}_T(v_2), \mathrm{td}_\Gamma(v_1) \neq \mathrm{td}_\Gamma(v_2), \mathrm{td}_F(v_1) \neq \mathrm{td}_F(v_2)$ implies $\breve{G}$ is not TCPFG which is leading to contradiction. Now, if G is TCPFG, then, by contrary, we can easily see that $d_T(v_1) \neq d_T(v_2)$, $d_\Gamma(v_1) \neq d_\Gamma(v_2), d_F(v_1) \neq d_F(v_2)$. Therefore, (T_1, Γ_1, F_1) is a CF. □

Example 11. A PFG $G = (V, \check{E})$ is CPFG and TCPFG. Figure 12 explains the defined concept.

Theorem 2. *A constant and totally constant graph $\breve{G}$ implies that (T_1, Γ_1, F_1) is CF.*

Proof. Suppose $\breve{G}$ is CPFG and TCPFG. Then, $d_T(v_1) = k_1, d_\Gamma(v_1) = k_2$ and $d_F(v_1) = k_3$ and $\mathrm{td}_T(v_1) = \tau_1$, $\mathrm{td}_\Gamma(v_1) = \tau_2$, $\mathrm{td}_F(v_1) = \tau_2$. As $\mathrm{td}_T(v_1) = \tau_1$ where $v \in V$, then $d_T(v_1) + T_1(v_1) = \tau_1, \forall v \in V$. $k_1 + T_1(v_1) = \tau_1, \forall v \in V$ implies $T_1(v_1) = \tau_1 - k_1, \forall v \in V$. Therefore, $\Gamma_1(v_1)$ is a constant function. Likewise, $\Gamma_1(v_1) = \tau_2 - k_2$ and $F_1(v_1) = \tau_3 - k_3, \forall v \in V$. □

Remark 2. Converse of the above theorem is not true in general.

Example 12. A PFG is not CPFG and not TCPFG. Figure 13 explains the defined concept.

Theorem 3. *If a crisp graph G is an odd cycle and $\breve{G}$ is aPFG, then $\breve{G}$ is CPFG $\Leftrightarrow$ (T_2, Γ_2, F_2) which is a CF.*

Proof. Assume that (T_2, Γ_2, F_2) is a constant function that implies $T_2 = c_1, \Gamma_2 = c_2, F_2 = c_3 \quad \forall v_i, v_j \in \check{E}$, and implies $d_T(v_1) = 2c_1$, $d_\Gamma(v_1) = 2c_2$, and $d_F(v_1) = 2c_3$, for any $v_i \in \check{E}$, therefore, G is a CPFG.

Conversely, assume that $\breve{G}$ is a (k_1, k_2, k_3)-regular PFG. Consider $\grave{e}_1, \grave{e}_2, \grave{e}_3, \ldots, \grave{e}_{n+1}$ represented the edges of $\breve{G}$ in order. Suppose $T_2(\grave{e}_1) = c_1, T_2(\grave{e}_2) = k_1 - c_1, T_2(e_3) = k_1 - (k_1 - c_1) = c_1, T_2(e_4) = k_1 - c_1$, and so on. Likewise, $\Gamma_2(\grave{e}_1) = c_1, \Gamma_2(\grave{e}_2) = k_1 - c_1, \Gamma_2(\grave{e}_3) = k_1 - (k_1 - c_1) = c_1$, $\Gamma_2(\grave{e}_4) = k_1 - c_1$, and so on; $F_2(\grave{e}_1) = c_1, F_2(\grave{e}_2) = k_1 - c_1, F_2(\grave{e}_3) = k_1 - (k_1 - c_1) = c_1, F_2(e_4) = k_1 - c_1$, and so on.

Hence, $T_2(\grave{e}_i) = \left\{ \begin{array}{ll} c_1, & \text{if } i \text{ is odd,} \\ k_1 - c_1, & \text{if } i \text{ is even.} \end{array} \right\}$.

Therefore, $T_2(\grave{e}_1) = T(\grave{e}_{2n+1}) = c_1$. Consequently, if $\grave{e}_1$ and $\grave{e}_{2n+1}$ connected at a vertex v_1, then $d_T(v_1) = k_1, d(\grave{e}_1) + d(\grave{e}_{2n+1}) = k_1, c_1 + c_1 = k_1, 2c_1 = k_1/2$. □

Remark 3. For TCPFG, the above theorem does not hold.

Example 13. The following PFG supports the above remark. In Figure 14, the defined concept is explained.

Theorem 4. *Let G be a crisp graph and $\breve{G}$ be an even cycle. Then, $\breve{G}$ is CPFG $\Leftrightarrow$ (T_2, Γ_2, F_2) which is a CF or different edges have same truth membership, abstinence membership, and false membership values.*

Proof. Assume (T_2, Γ_2, F_2) is a CF, then obviously $\breve{G}$ is a constant PFG. Conversely, suppose that $\breve{G}$ is (k_1, k_2, k_3)CPFG. Consider $\grave{e}_1, \grave{e}_2, \grave{e}_3, \ldots, \grave{e}_{2n}$ to be the edges of even cycle G in that order. By theorem (3.3),

$$T_2(\grave{e}_i) = \left\{ \begin{array}{ll} c_1, & \text{if } i \text{ is odd,} \\ k_1 - c_1, & \text{if } i \text{ is even.} \end{array} \right\}. \tag{4}$$

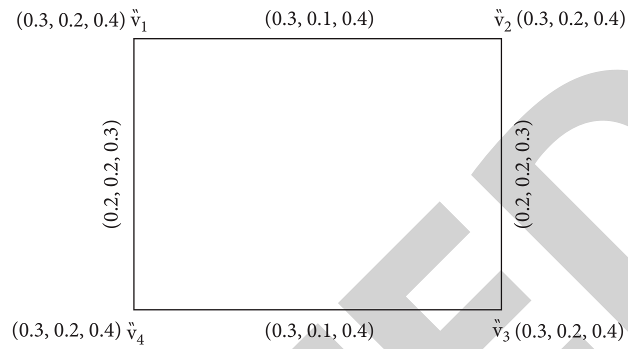

Figure 12: (T_1, Γ_1, F_1) is CF, then $\breve{G}$ is constant and totally constant).

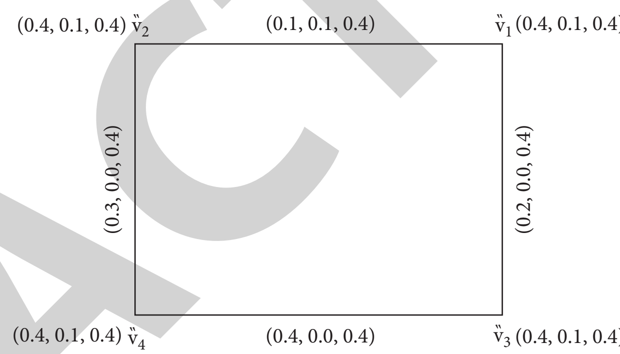

Figure 13: ((T_1, Γ_1, F_1) is CF, then $\breve{G}$ is not a CPFG nor a TCPFG).

Likewise,

$$\begin{aligned} \Gamma_2(\grave{e}_i) &= \left\{ \begin{array}{ll} c_1, & \text{if } i \text{ is odd,} \\ k_1 - c_1, & \text{if } i \text{ is even,} \end{array} \right\}, \\ F_2(\grave{e}_i) &= \left\{ \begin{array}{ll} c_1, & \text{if } i \text{ is odd,} \\ k_1 - c_1, & \text{if } i \text{ is even.} \end{array} \right\}. \end{aligned} \tag{5}$$

If $c_1 = k_1 - c_1$, then (T_2, Γ_2, F_2) is a constant function. If $c_1 \neq k_1 - c_1$, then different edges have same truth membership, abstinence membership, and false membership values. □

Remark 4. The above theorem does not hold for TCPFG.

Example 14. The following PFG graph supports that a PFG is constant but not totally constant. Figure 15 explains the defined concept.

4.1. Properties of Constant PFG

Theorem 5. *If a c CPFG is an odd cycle, then there is no PF bridge and no PF cut vertex.*

Proof. Suppose G is a crisp graph having odd cycle and $\breve{G}$ is a constant PFG. Then, (T_2, Γ_2, F_2) is a CF. Consequently, deleting any vertex does not decrease the strength of

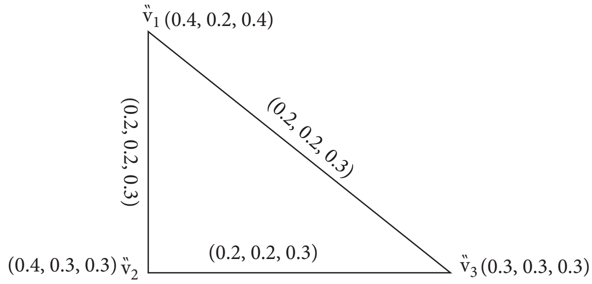

Figure 14: ((T_2, Γ_2, F_2) is constant function but no totally constant PFG).

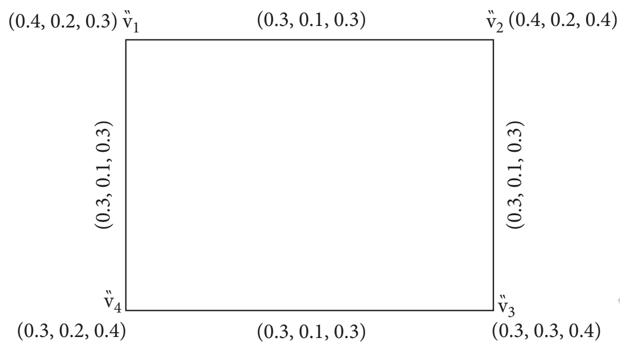

Figure 15: ((T_2, Γ_2, F_2) is CF, then $\breve{G}$ is CPFG, but no TCPFG).

connectedness between any pair of vertices. Therefore, $\breve{G}$ is no bridge and no PF cut vertex. □

Theorem 6. *If a CPFG is an even cycle, then there is no PF bridge and no PF cut vertex.*

Proof. Suppose G is a crisp graph having even cycle and $\breve{G}$ is a CPFG. Then, by Theorem 5, (T_2, Γ_2, F_2) is a CF or different edges have same truth membership, abstinence membership, and false membership values.

Case (i). If (T_2, Γ_2, F_2) is CF, then deleting any vertex does not decrease the strength of connectedness between any pair of vertices. Therefore, $\breve{G}$ is no bridge and no PF cut vertex

Case (ii). Straight forward. □

Remark 5. For TPFG, the above theorem does not hold.

Example 15. Figure 16 supports the above remark 5 in which the PFG constant is neither bridge nor cut vertex. Figure 16 explains the defined concept.

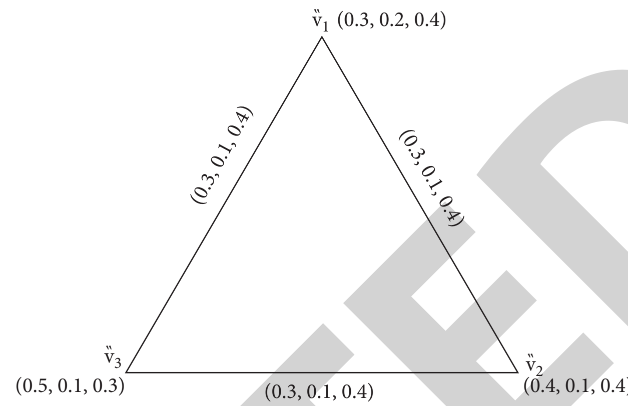

Figure 16: ((T_2, Γ_2, F_2) is constant but there is no PF bridge and no cut vertex).

5. Application

In this section, the application of CPFG in Wi-Fi network system is discussed.

The Wi-Fi technology offers Internet access through a wireless network linked to the Internet to the electronic devices and machines that are in its range. The broadcasting of one or more interconnected access points (hotspots) can extend the range of the connection from a small area of a few rooms to a vast area of many square kilometers. The range of Wi-Fi signals depends on the frequency band, radio power output, and the modulation technique. Although the Wi-Fi connection provides easy access to the Internet, it is also a security risk as compared to the wired connection called Ethernet. For gaining access to Internet connection in a wired network connection, it is necessary to gain physical access to a building that has got the Internet connection or break through an external firewall. On the other hand, in a wireless Wi-Fi connection, the requirement for accessing the Internet is just to get within the range of the Wi-Fi. There are two types of Wi-Fi networks, namely, indoor and outdoor Wi-Fi networks. A compact Wi-Fi hotspot device is called an indoor coin Wi-Fi that intends to facilitate all the indoor owners to access the Internet. These provide Wi-Fi signals ranging at 100 meters (outdoor)/30 meters (indoor). This type of Wi-Fi network is discussed and modeled with the help of CPFG.

Since there are four values to deal with, therefore, the CPFG has been applied to a Wi-Fi network. The first value represents the state of connectedness, the second value describes the fluctuating state of the connection of the device amid the connectedness and disconnectedness states, the third value shows the disconnection, and the last value shows that the device is not in the range. Since the structure of an IFG is limited to just two values, i.e., state of connection and disconnection, therefore, a Wi-Fi system is almost impossible to model through the concept of IFG, whereas the CPFG discusses more than these two situations. Consider an outdoor Wi-Fi system that contains four vertices representing the Wi-Fi devices in such a way that there is a block between every two routers and both routers have been giving signals to the block together, as shown in Figure 17. With the help of CPFG, the devices can give a constant signal to each block.

The four vertices in Figure 17 represent four different routers. The edge between each pair of routers shows the strength of the signals of the routers. Each edge and vertex are in the form of a picture fuzzy number where the first value represents the connectivity. The second one describes the fluctuating state of the device, i.e., the device is in range but fluctuates between the connected and disconnected

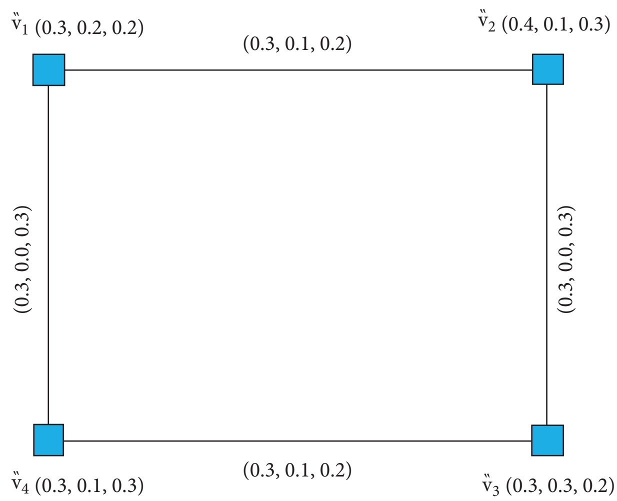

Figure 17: (PFG Wi-Fi network).

Table 1: (Vertices and their degrees).

Vertex	Degree
v_1	$(0.6, 0.1, 0.5)$
v_2	$(0.6, 0.1, 0.5)$
v_3	$(0.6, 0.1, 0.5)$
v_4	$(0.6, 0.1, 0.5)$

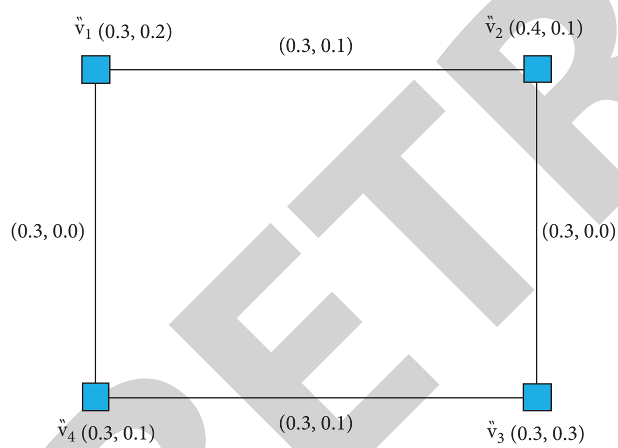

Figure 18: (IFG-Wi-Fi network).

states, the third value shows disconnection, and the last value indicates that the device is out of the range. The degree of each vertex is calculated using Definition 4. In this case, the degree of every router is same, which interprets that every router has been giving the same signals. It means that each router is providing the same signal to the block. Thus, the idea of CPFG has been successfully applied to practical problems showing its significance.

Table 1 shows the degree of the vertices in Figure 17.

5.1. Advantages of PFG. The advantage of PFGs over existing concept of IFGs is that IFGs cannot be used to model the Wi-Fi network systems as it allows to only deal with just two states, i.e., the state of connectedness and the state of disconnectedness only. The diverse structure of PFGs enables us to deal with uncertain situations with additional types of states, as presented in the application section. The block together is shown in Figure 18. With the help of IFG, the devices can give a constant signal to each block. But that IFGs cannot be used to model the Wi-Fi network system because it only allows to deal with two states, i.e., the state of connectedness and the state of disconnectedness only.

6. Conclusion

This manuscript proposes the ideas of PFG and CPFG. Some fundamental graph theoretic concepts are discussed and illustrated with the help of examples. Moreover, the comparison between PFG and IFG is carried out that shows the significance of the proposed concept. Furthermore, the proposed concept is applied to a practical problem of Wi-Fi network system, and results are discussed. More applications in the different fields can be discussed in the proposed framework, such as in engineering and computer sciences.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare no conflicts of interest about the publication of the research article.

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Retraction

Retracted: Analysis of Social Networks by Using Pythagorean Cubic Fuzzy Einstein Weighted Geometric Aggregation Operators

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] Tehreem, A. Hussain, J. R. Lee, M. S. Ali Khan, and D. Y. Shin, "Analysis of Social Networks by Using Pythagorean Cubic Fuzzy Einstein Weighted Geometric Aggregation Operators," *Journal of Mathematics*, vol. 2021, Article ID 5516869, 18 pages, 2021.

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Research Article

Analysis of Social Networks by Using Pythagorean Cubic Fuzzy Einstein Weighted Geometric Aggregation Operators

Tehreem,[1] Amjad Hussain,[1] Jung Rye Lee,[2] Muhammad Sajjad Ali Khan,[3] and Dong Yun Shin[4]

[1]*Department of Mathematics, Quaid-i-Azam University, Islamabad 45320, Pakistan*
[2]*Department of Data Science, Daejin University, Kyunggi 11159, Republic of Korea*
[3]*Institute of Numerical Sciences Kohat University of Science and Technology, Kohat, Khyber Pakhtunkhwa, Pakistan*
[4]*Department of Mathematics, University of Seoul, Seoul 02504, Republic of Korea*

Correspondence should be addressed to Jung Rye Lee; jrlee@daejin.ac.kr

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Pythagorean cubic set (PCFS) is the combination of the Pythagorean fuzzy set (PFS) and interval-valued Pythagorean fuzzy set (IVPFS). PCFS handle more uncertainties than PFS and IVPFS and thus are more extensive in their applications. The objective of this paper is under the PCFS to establish some novel operational laws and their corresponding Einstein weighted geometric aggregation operators. We describe some novel Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operators to handle multiple attribute group decision-making problems. The desirable relationship and the characteristics of the proposed operator are discussed in detail. Finally, a descriptive case is given to describe the practicality and the feasibility of the methodology established.

1. Introduction

Multicriteria decision-making (MCDM) is a process that can give the ranking result of finite alternatives according to the attribute value of different alternatives, and it is an important aspect of decision sciences. A significant part of the decision-making model that has been commonly used in human impacts is MCDM (or MCGDM) [1]. The assessment information is generally fuzzy because the real decision-making issues have always been created from a complicated context. In general, fuzzy data take two models: one quantitatively and one qualitatively. Fuzzy set (FS) [2], intuitionistic fuzzy set (IFS) [3], Pythagorean fuzzy set (PFS) [4], and so on, can express quantitative fuzzy knowledge. The theory of FS suggested by Zadeh [2] was used to explain fuzzy quantitative knowledge containing only a degree of membership. On this basis, Atanassov [5] proposed the idea of IFS as a generalization of FS; the important aspect is that it has two fuzzy values: the first is called membership grade and the second is called nonmembership grade. Sometimes, meanwhile, the two degrees do not satisfy the limit, so the square sum is less than or equal to one. The PFS was introduced by Yager [4] in which the sum of squares of membership and nonmembership is equal to or less than one. In certain conditions, PFS is capable of expressing the fuzzy data compared to the IFS. For instance, PFS improved the concept of IFS by enlarging its domain. To define this decision information, IFS is invalid, but it can be efficiently defined by PFS. In the Pythagorean fuzzy set, Peng et al. [6] introduced some characteristics, which are division, subtraction, and other significant properties.

To understand multicriteria problems in group decision-making in the Pythagorean fuzzy setting, authors are concerned with the methods of dominance and a ranking of dependencies. For multicriteria decision-making based on Pythagorean fuzzy sets, Khan et al. established prioritized aggregation operators in [7]. Peng et al. [8] advanced linguistic Pythagorean fuzzy sets (LPFSs) and the Pythagorean fuzzy linguistic numbers' operating laws and score function. An optimizing variance technique was developed by Wei et al. [9] to clarify problems involving decision-making depending on Pythagorean fuzzy environments valued at intervals. The Pythagorean fuzzy numbers (PFNs) subtraction and division acts were intended by Gou et al. [10]. The notion of the obvious concept of the Pythagorean fuzzy distance degree was provided by Pend et al. [11], which is categorized by a Pythagorean fuzzy number that will minimize a drawback of data additionally proceeding to provide imaginative proof. The well-known definition of the novel score function is also well defined. Liang et al. [12] introduced the Bonferroni weighted Pythagorean fuzzy geometric (BWPFG) operator.

In [13], Garg introduced an interval-valued Pythagorean fuzzy geometric (IVPFG) operator and discussed a new precision function. Khan et al. improved the definition of the multiattribute decision-making TOPSIS system as well as established the integral Choquet method of TOPSIS on the basis of IVPFNs [14]. In [15], Khan suggested the GRA method for making multicriteria decisions under the Pythagorean fuzzy condition valued at intervals. The authors first developed the Choquet integral average interval-valued Pythagorean operator and then developed a system for making multiattribute decisions dependent on the GRA technique. An Einstein geometric intuitionistic fuzzy (EGIF) operator was introduced by Wang [16] and an ordered weighted Einstein geometric intuitionistic fuzzy (OWEGIF) operator.

The definition of the intuitionistic fuzzy Einstein weighted averaging operator was introduced by Wang and Liu [17] and an ordered weighted Einstein average intuitionistic fuzzy (OWEAIF) operator. Einstein operations can be divided into two categories: Einstein sum and product. In [18], Garg implemented the Einstein sum definition of the Pythagorean fuzzy mean aggregation operators such as the average operator of Pythagorean fuzzy Einstein, the weighted average operator of Pythagorean fuzzy Einstein, the geometric operator of Pythagorean fuzzy Einstein, and the ordered geometric weighted operator of Pythagorean fuzzy Einstein. For more related work, one may refer to [19–39].

We will use the Einstein product in this article and present the Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operator definition. Under Pythagorean fuzzy data, these two are new decision-making methods, but the Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operator is more reliable than mean aggregation operators.

This paper is composed of nine sections. We begin with a brief overview relevant to the literature review in Section 1. We provide essential concepts and consequences in Section 2 that we can include in the following aspects. In Section 3, we define the Pythagorean cubic fuzzy number and their properties. We propose Pythagorean cubic fuzzy Einstein operations in Section 4 and examine some excellent features of the suggested operations. We present a Pythagorean cubic fuzzy Einstein weighted geometric aggregation operator (PCFEWG) in Section 5. With Pythagorean cubic fuzzy data, we apply the (PCFEWG) operator to MADM in Section 6 and we also give a case of numerical development (PFEWG) operator in Section 7. In Section 8, the comparative analysis is given and the conclusion is in Section 9.

2. Preliminaries

We introduce a basic definition and essential characteristics in this section.

Definition 1 (see [8]). Let $\widehat{X}$ be a universal set, then the fuzzy set (FS) $\widehat{F}$ is defined as follows:

$$\widehat{F} = \left\{ \left\langle \widehat{x}, \widehat{\mu}_{\widehat{F}}(\widehat{x}) | \widehat{x} \in \widehat{X} \right\rangle \right\}, \quad (1)$$

where $\widehat{\mu}_{\widehat{F}}(\widehat{x})$ is a mapping from $\widehat{X}$ to [0,1] and $\widehat{\mu}_{\widehat{F}}(\widehat{x})$ is known as the membership function of $\widehat{x} \in \widehat{X}$.

Definition 2 (see [3]). Let $\widehat{X}$ be a universal set, then the intuitionistic fuzzy set (IFS) $\widehat{I}$ is defined as follows:

$$\widehat{I} = \left\{ \left\langle \widehat{x}, \widehat{\mu}_{\widehat{I}}(\widehat{x}), \widehat{\nu}_{\widehat{I}}(\widehat{x}) | \widehat{x} \in \widehat{X} \right\rangle \right\}, \quad (2)$$

where $\widehat{\mu}_{\widehat{I}}(\widehat{x})$ and $\widehat{\nu}_{\widehat{I}}(\widehat{x})$ are a mapping from $\widehat{X}$ to [0,1] also satisfy the condition $0 \leq \widehat{\mu}_{\widehat{I}} \leq 1, 0 \leq \widehat{\nu}_{\widehat{I}} \leq 1$ for all $\widehat{x} \in \widehat{X}$ and represent the membership and nonmembership function of $\widehat{x}$ in $\widehat{X}$.

Definition 3 (see [19]). Let $\widehat{X}$ be a universal set, then the Pythagorean fuzzy set (PFS) $\widehat{P}$ is defined as follows:

$$\widehat{P} = \left\{ \left\langle \widehat{x}, \widehat{\mu}_{\widehat{P}}(\widehat{x}), \widehat{\nu}_{\widehat{P}}(\widehat{x}) | \widehat{x} \in \widehat{X} \right\rangle \right\}, \quad (3)$$

where $\widehat{\mu}_{\widehat{P}}$ and $\widehat{\nu}_{\widehat{P}}$ are a mapping from $\widehat{X}$ to [0,1] also satisfying the conditions $0 \leq \widehat{\mu}_{\widehat{P}} \leq 1, 0 \leq \widehat{\nu}_{\widehat{P}} \leq 1$, and $0 \leq (\widehat{\mu}_{\widehat{P}})^2 \leq 1, 0 \leq (\widehat{\nu}_{\widehat{P}})^2 \leq 1$, for all $\widehat{x} \in \widehat{X}$ and characterize the membership and nonmembership degree to set $\widehat{P}$. Let $\widehat{\pi}_{\widehat{P}}(\widehat{x}) = \sqrt{1 - (\widehat{\mu}_{\widehat{P}})^2 - (\widehat{\nu}_{\widehat{P}})^2}$, then it is known as the Pythagorean fuzzy index of $\widehat{x} \in \widehat{X}$ to set $\widehat{P}$, representing the degree of indeterminacy of $\widehat{P}$. Also, for every $\widehat{x} \in \widehat{X}$, we represent the Pythagorean fuzzy number (PFN) by $\widetilde{B} = \left\langle \Lambda_{\widetilde{B}}, \Gamma_{\widetilde{B}} \right\rangle$.

Definition 4 (see [19]). Let $\widetilde{B}_1 = \left\langle \Lambda_{\widetilde{B}_1}, \Gamma_{\widetilde{B}_1} \right\rangle$, $\widetilde{B}_2 = \left\langle \Lambda_{\widetilde{B}_2}, \Gamma_{\widetilde{B}_2} \right\rangle$, and $\widetilde{B} = \left\langle \Lambda_{\widetilde{B}}, \Gamma_{\widetilde{B}} \right\rangle$be three (PFNs) and $\widehat{\lambda} > 0$, then we have

(1) $\widetilde{B}_1 \oplus \widetilde{B}_2 = (\sqrt{\widehat{\mu}^2_{\widetilde{B}_1} + \widehat{\mu}^2_{\widetilde{B}_2} - \widehat{\mu}^2_{\widetilde{B}_1} \widehat{\mu}^2_{\widetilde{B}_2}}, \widehat{\nu}_{\widetilde{B}_1} \widehat{\nu}_{\widetilde{B}_2})$;

(2) $\widetilde{B}_1 \otimes \widetilde{B}_2 = (\widehat{\mu}_{\widetilde{B}_1} \widehat{\mu}_{\widetilde{B}_2}, \sqrt{\widehat{\nu}^2_{\widetilde{B}_1} + \widehat{\nu}^2_{\widetilde{B}_2} - \widehat{\nu}^2_{\widetilde{B}_1} \widehat{\nu}^2_{\widetilde{B}_2}})$;

(3) $\widehat{\lambda} \widetilde{B} = (\sqrt{1 - (1 - \widehat{\mu}^2_{\widetilde{B}})^{\widehat{\lambda}}}, (\widehat{\nu}_{\widetilde{B}})^{\widehat{\lambda}})$;

(4) $\widetilde{B}^{\widehat{\lambda}} = ((\widehat{\mu}_{\widetilde{B}})^{\widehat{\lambda}}, \sqrt{1 - (1 - \widehat{\nu}^2_{\widetilde{B}})^{\widehat{\lambda}}})$;

(5) $\widetilde{B}^c = (\widehat{\nu}_{\widetilde{B}}, \widehat{\mu}_{\widetilde{B}})$.

Definition 5 (see [20]). Let $\widehat{X}$ be a universal set, then the object with the following formulation is an IVPFS set $\widehat{R}$:

$$\widehat{R} = \left\{ \left\langle \widehat{x}, \widehat{\mu}_{\widehat{R}}(\widehat{x}), \widehat{\nu}_{\widehat{R}}(\widehat{x}) \,|\, \widehat{x} \in \widehat{X} \right\rangle \right\}, \quad (4)$$

Where $\widehat{\mu}_{\widehat{R}}(\widehat{x}) = [\widehat{\mu}^L_{\widehat{R}}(\widehat{x}), \widehat{\mu}^L_{\widehat{R}}(\widehat{x})] \subseteq [0,1]$ and $\widehat{\nu}_{\widehat{R}}(\widehat{x}) = [\widehat{\nu}^L_{\widehat{R}}(\widehat{x}), \widehat{\nu}^L_{\widehat{R}}(\widehat{x})] \subseteq [0,1]$ are the intervals, and $\widehat{\mu}^L_{\widehat{R}}(\widehat{x}) = \inf \widehat{\mu}_{\widehat{R}}(\widehat{x})$ and $\widehat{\mu}^U_{\widehat{R}}(\widehat{x}) = \mathrm{Sup}\,\widehat{\mu}_{\widehat{R}}(\widehat{x})$; similarly, $\widehat{\nu}^L_{\widehat{R}}(\widehat{x}) = \inf \widehat{\nu}_{\widehat{R}}(\widehat{x})$ and $\widehat{\nu}^U_{\widehat{R}}(\widehat{x}) = \mathrm{Sup}\,\widehat{\nu}_{\widehat{R}}(\widehat{x})$, for all $\widehat{x} \in \widehat{X}$.Also, $0 \le (\widehat{\mu}^U_{\widehat{R}}(\widehat{x}))^2 + (\widehat{\nu}^U_{\widehat{R}}(\widehat{x}))^2 \le 1$. Let $\widehat{\pi}_{\widehat{R}}(\widehat{x}) = [\widehat{\pi}^L_{\widehat{R}}(\widehat{x}), \widehat{\pi}^L_{\widehat{R}}(\widehat{x})]$, for all $\widehat{x} \in \widehat{X}$, then it is known as the interval-valued Pythagorean fuzzy index of $\widehat{x}$ to $\widehat{R}$, where $\widehat{\pi}^L_{\widehat{R}}(\widehat{x}) = \sqrt{1 - (\widehat{\mu}^L_{\widehat{R}}(\widehat{x}))^2 + (\widehat{\nu}^L_{\widehat{R}}(\widehat{x}))^2}$ and $\widehat{\pi}^U_{\widehat{R}}(\widehat{x}) = \sqrt{1 - (\widehat{\mu}^U_{\widehat{R}}(\widehat{x}))^2 + (\widehat{\nu}^U_{\widehat{R}}(\widehat{x}))^2}$ which meet the requirements of the following relationship:

(1) If $\widehat{\mu}^L_{\widehat{R}}(\widehat{x}) = \widehat{\mu}^U_{\widehat{R}}(\widehat{x})$ and $\widehat{\nu}^L_{\widehat{R}}(\widehat{x}) = \widehat{\nu}^U_{\widehat{R}}(\widehat{x})$, then an IVPFS set becomes a PFS set.

(2) If $\widehat{\mu}^U_{\widehat{R}}(\widehat{x}) + \widehat{\nu}^U_{\widehat{R}}(\widehat{x}) \le 1$, then an IVPFS becomes an IVIFS.

Definition 6 (see [21]). Let $\widehat{A} = ([\widehat{\mu}^L_{\widehat{A}}, \widehat{\mu}^U_{\widehat{A}}], [\widehat{\nu}^L_{\widehat{A}}, \widehat{\nu}^U_{\widehat{A}}])$, $\widehat{A}_1 = ([\widehat{\mu}^L_{\widehat{A}_1}, \widehat{\mu}^U_{\widehat{A}_1}], [\widehat{\nu}^L_{\widehat{A}_1}, \widehat{\nu}^U_{\widehat{A}_1}])$, and $\widehat{A}_2 = ([\widehat{\mu}^L_{\widehat{A}_2}, \widehat{\mu}^U_{\widehat{A}_2}], [\widehat{\nu}^L_{\widehat{A}_2}, \widehat{\nu}^U_{\widehat{A}_2}])$ arethree IVPFNs and $\widehat{\lambda} > 0$, then we have the following:

(1) $\widehat{\lambda}\widehat{A} = ([\sqrt{1 - (1 - (\widehat{\mu}^L_{\widehat{A}}))^{\widehat{\lambda}}}, \sqrt{1 - (1 - (\widehat{\mu}^U_{\widehat{A}}))^{\widehat{\lambda}}}], [\widehat{\nu}^L_{\widehat{A}}, \widehat{\nu}^U_{\widehat{A}}])$,

(2) $\widehat{A}^{\widehat{\lambda}} = ([\widehat{\mu}^L_{\widehat{A}}, \widehat{\mu}^U_{\widehat{A}}], [\sqrt{1 - (1 - (\widehat{\nu}^L_{\widehat{A}}))^{\widehat{\lambda}}}, \sqrt{1 - (1 - (\widehat{\nu}^U_{\widehat{A}}))^{\widehat{\lambda}}}])$,

(3) $\widehat{A}_1 \oplus \widehat{A}_2 = \left(\begin{array}{c} [\sqrt{(\widehat{\mu}^L_{\widehat{A}_1})^2 + (\widehat{\mu}^L_{\widehat{A}_2})^2 - (\widehat{\mu}^L_{\widehat{A}_1})^2 (\widehat{\mu}^L_{\widehat{A}_2})^2}, \sqrt{(\widehat{\mu}^U_{\widehat{A}_1})^2 + (\widehat{\mu}^U_{\widehat{A}_2})^2 - (\widehat{\mu}^U_{\widehat{A}_1})^2 (\widehat{\mu}^U_{\widehat{A}_2})^2}], \\ [\widehat{\nu}^L_{\widehat{A}_1} \widehat{\nu}^L_{\widehat{A}_2}, \widehat{\nu}^U_{\widehat{A}_1} \widehat{\nu}^U_{\widehat{A}_2}] \end{array} \right)$,

(4) $\widehat{A}_1 \otimes \widehat{A}_2 = \left(\begin{array}{c} [\widehat{\mu}^L_{\widehat{A}_1} \widehat{\mu}^L_{\widehat{A}_2}, \widehat{\mu}^U_{\widehat{A}_1} \widehat{\mu}^U_{\widehat{A}_2}], \\ [\sqrt{(\widehat{\nu}^L_{\widehat{A}_1})^2 + (\widehat{\nu}^L_{\widehat{A}_2})^2 - (\widehat{\nu}^L_{\widehat{A}_1})^2 (\widehat{\nu}^L_{\widehat{A}_2})^2}, \sqrt{(\widehat{\nu}^U_{\widehat{A}_1})^2 + (\widehat{\nu}^U_{\widehat{A}_2})^2 - (\widehat{\nu}^U_{\widehat{A}_1})^2 (\widehat{\nu}^U_{\widehat{A}_2})^2}] \end{array} \right)$.

Definition 7 (see [21]). Let $\widehat{A} = ([\widehat{\mu}^L_{\widehat{A}}, \widehat{\mu}^U_{\widehat{A}}], [\widehat{\nu}^L_{\widehat{A}}, \widehat{\nu}^U_{\widehat{A}}])$; the score function of $\widehat{A}$ can be defined as follows using the IVPFN $\widehat{A}$:

$$S(\widehat{A}) = \frac{1}{2}\left[\left(\widehat{\mu}^L_{\widehat{A}}\right)^2 + \left(\widehat{\mu}^U_{\widehat{A}}\right)^2 - \left(\widehat{\nu}^L_{\widehat{A}}\right)^2 - \left(\widehat{\nu}^U_{\widehat{A}}\right)^2\right], \quad \text{where } S(\widehat{A}) \in [0,1]. \tag{5}$$

Definition 8 (see [23]). Let $\widehat{A} = ([\widehat{\mu}^L_{\widehat{A}}, \widehat{\mu}^U_{\widehat{A}}], [\widehat{\nu}^L_{\widehat{A}}, \widehat{\nu}^U_{\widehat{A}}])$; the accuracy function of $\widehat{A}$ can be defined as follows using the IVPFN $\widehat{A}$:

$$H(\widehat{A}) = \frac{1}{2}\left[\left(\widehat{\mu}^L_{\widehat{A}}\right)^2 + \left(\widehat{\mu}^U_{\widehat{A}}\right)^2 + \left(\widehat{\nu}^L_{\widehat{A}}\right)^2 + \left(\widehat{\nu}^U_{\widehat{A}}\right)^2\right], \quad \text{where } H(\widehat{A}) \in [0,1]. \tag{6}$$

Definition 9 (see [21]). Let $\widehat{A} = ([\widehat{\mu}^L_{\widehat{A}}, \widehat{\mu}^U_{\widehat{A}}], [\widehat{\nu}^L_{\widehat{A}}, \widehat{\nu}^U_{\widehat{A}}])$ and $\widehat{A}_1 = ([\widehat{\mu}^L_{\widehat{A}_1}, \widehat{\mu}^U_{\widehat{A}_1}], [\widehat{\nu}^L_{\widehat{A}_1}, \widehat{\nu}^U_{\widehat{A}_1}])$ be two IVPFNs, then

$$\begin{aligned} S(\widehat{A}) &= \frac{1}{2}\left[\left(\widehat{\mu}^L_{\widehat{A}}\right)^2 + \left(\widehat{\mu}^U_{\widehat{A}}\right)^2 - \left(\widehat{\nu}^L_{\widehat{A}}\right)^2 - \left(\widehat{\nu}^U_{\widehat{A}}\right)^2\right], \\ S(\widehat{A}_1) &= \frac{1}{2}\left[\left(\widehat{\mu}^L_{\widehat{A}_1}\right)^2 + \left(\widehat{\mu}^U_{\widehat{A}_1}\right)^2 - \left(\widehat{\nu}^L_{\widehat{A}_1}\right)^2 - \left(\widehat{\nu}^U_{\widehat{A}_1}\right)^2\right] \end{aligned} \tag{7}$$

are the score of $\widehat{A}$ and $\widehat{A}_1$, separately, while

$$\begin{aligned} H(\widehat{A}) &= \frac{1}{2}\left[\left(\widehat{\mu}^L_{\widehat{A}}\right)^2 + \left(\widehat{\mu}^U_{\widehat{A}}\right)^2 + \left(\widehat{\nu}^L_{\widehat{A}}\right)^2 + \left(\widehat{\nu}^U_{\widehat{A}}\right)^2\right], \\ H(\widehat{A}_1) &= \frac{1}{2}\left[\left(\widehat{\mu}^L_{\widehat{A}_1}\right)^2 + \left(\widehat{\mu}^U_{\widehat{A}_1}\right)^2 + \left(\widehat{\nu}^L_{\widehat{A}_1}\right)^2 + \left(\widehat{\nu}^U_{\widehat{A}_1}\right)^2\right] \end{aligned} \tag{8}$$

are the accuracy of A and A_1, separately, which meet the following criteria:

(1) If $S(\widehat{A}) < S(\widehat{A}_1)$, then $\widehat{A} < \widehat{A}_1$;

(2) If $S(\widehat{A}) > S(\widehat{A}_1)$, then $\widehat{A} > \widehat{A}_1$;

(3) If $S(\widehat{A}) = S(\widehat{A}_1)$, we have the following:

(a) If $H(\widehat{A}) = H(\widehat{A}_1)$, then $\widehat{A} = \widehat{A}_1$,
(b) If $H(\widehat{A}) < H(\widehat{A}_1)$, then $\widehat{A} < \widehat{A}_1$,
(c) If $H(\widehat{A}) > H(\widehat{A}_1)$, then $\widehat{A} > \widehat{A}_1$.

Definition 10 (see [22]). Let $\widehat{X}$ be a universal set. Then, a cubic set can be stated:

$$C = \left\{\left\langle \widehat{x}, \overline{\mu}_C(\widehat{x}), \widehat{\nu}_C(\widehat{x}) | \widehat{x} \in \widehat{X} \right\rangle\right\}, \tag{9}$$

where $\overline{\mu}_C$ is an interval-valued fuzzy set in $\widehat{X}$ and $\widehat{\nu}_C$ is a fuzzy set in $\widehat{X}$.

Definition 11 (see [19]). Let p_1 and p_2 be two PFNs, then the distance between p_1 and p_2 can be described as

$$\begin{aligned} d(p_1, p_2) = \frac{1}{2}\Big(&\left|\left(\widehat{\mu}_{p_1}\right)^2 - \left(\widehat{\mu}_{p_2}\right)^2\right| \\ &+ \left|\left(\widehat{\nu}_{p_1}\right)^2 - \left(\widehat{\nu}_{p_2}\right)^2\right| + \left|\left(\widehat{\pi}_{p_1}\right)^2 - \left(\widehat{\pi}_{p_2}\right)^2\right|\Big). \end{aligned} \tag{10}$$

Definition 12 (see [23]). Let $p_i = (\langle [a_i, b_i], [c_i, d_i] \rangle)$, $(i = 1, 2)$, be two IVPFNs, then the distance between $p_1$and $p_2$is defined as follows:

$$\begin{aligned} d(p_1, p_2) = \frac{1}{4}\Big(&\left|(a_1)^2 - (a_2)^2\right| + \left|(b_1)^2 - (b_2)^2\right| \\ &+ \left|(c_1)^2 - (c_2)^2\right| + \left|(d_1)^2 - (d_2)^2\right| \\ &+ \left|(\pi_1)^2 - (\pi_2)^2\right| + \left|(\psi_1)^2 - (\psi_2)^2\right|\Big), \end{aligned} \tag{11}$$

where $[\pi_1, \psi_1] = [\sqrt{1 - (a_1)^2 - (c_1)^2}, \sqrt{1 - (b_1)^2 - (d_1)^2}]$ and $[\pi_2, \psi_2] = [\sqrt{1 - (a_2)^2 - (c_2)^2}, \sqrt{1 - (b_2)^2 - (d_2)^2}]$.

Definition 13 (see [24]). Let $p_i = (\langle [a_i, b_i], [c_i, d_i] \rangle)$, $(i = 1, 2, 3, \ldots, n)$, be the collection of IVPFNs, then IVPFWG operator is defined as

$$\text{IVPFWG}_w(p_1, p_2, \ldots, p_n) = \left(\left[\prod_{i=1}^{n}(a_i)^{w_i}, \prod_{i=1}^{n}(b_i)^{w_i}\right], \left[\sqrt{1-\prod_{i=1}^{n}\left(1-(c_i)^2\right)^{w_i}}, \sqrt{1-\prod_{i=1}^{n}\left(1-(d_i)^2\right)^{w_i}}\right]\right), \tag{12}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $p_i\,(i = 1, 2, 3, \ldots, n)$ and $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$.

Definition 14 (see [24]). Let $p_i = (\langle [a_i, b_i], [c_i, d_i]\rangle)$, $(i = 1, 2, 3, \ldots, n)$, be the collection of IVPFNs, then IVPFOWG operator is defined as

$$\text{IVPFOWG}_w(p_1, p_2, \ldots, p_n) = \left(\begin{array}{c}\left[\prod_{i=1}^{n}\left(a_{\sigma(i)}\right)^{w_i}, \prod_{i=1}^{n}\left(b_{\sigma(i)}\right)^{w_i}\right], \\ \left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(c_{\sigma(i)}\right)^2\right)^{w_i}}, \sqrt{1-\prod_{i=1}^{n}\left(1-\left(d_{\sigma(i)}\right)^2\right)^{w_i}}\right]\end{array}\right), \tag{13}$$

where $p_{\sigma(i)}$ is the i-th largest value and $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $p_i\,(i = 1, 2, 3, \ldots, n)$.

Definition 15 (see [24]). Let $p_i = (\langle [a_i, b_i], [c_i, d_i]\rangle)$, $(i = 1, 2, 3, \ldots, n)$, be the collection of IVPFNs, then IVPFHWG operator is defined as

$$\text{IVPFHG}_w(p_1, p_2, \ldots, p_n) = \left(\begin{array}{c}\left[\prod_{i=1}^{n}\left(a_{\tau(i)}\right)^{w_i}, \prod_{i=1}^{n}\left(b_{\tau(i)}\right)^{w_i}\right], \\ \left[\sqrt{1-\prod_{i=1}^{n}\left(1-\left(c_{\tau(i)}\right)^2\right)^{w_i}}, \sqrt{1-\prod_{i=1}^{n}\left(1-\left(d_{\tau(i)}\right)^2\right)^{w_i}}\right]\end{array}\right), \tag{14}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of $p_i\,(i = 1, 2, 3, \ldots, n)$.

Definition 16 (see [25, 26]). Let $p_i = (\langle [a_i, b_i], [c_i, d_i]\rangle)$, $(i = 1, 2, 3, \ldots, n)$, be the collection of IVPFNs, and $\delta > 0$, then the following operational laws are satisfied:

(1) $\delta p_1 = ([\sqrt{1-(1-(a_1)^2)^{\delta}}, \sqrt{1-(1-(b_1)^2)^{\delta}}], [(c_1)^{\delta}, (d_1)^{\delta}])$,

(2) $(p_1)^{\delta} = ([(a_1)^{\delta}, (b_1)^{\delta}], [\sqrt{1-(1-(c_1)^2)^{\delta}}, \sqrt{1-(1-(d_1)^2)^{\delta}}])$,

(3) $p_1 \otimes p_2 = ([a_1 a_2, b_1 b_2], [\sqrt{(c_1)^2 + (c_2)^2 - (c_1)^2 (c_2)^2}, \sqrt{(d_1)^2 + (d_2)^2 - (d_1)^2 (d_2)^2}])$,

(4) $p_1 \oplus p_2 = ([\sqrt{(a_1)^2 + (a_2)^2 - (a_1)^2 (a_2)^2}, \sqrt{(b_1)^2 + (b_2)^2 - (b_1)^2 (b_2)^2}], [c_1 c_2, d_1 d_2])$.

3. Pythagorean Cubic Fuzzy Numbers and Their Characteristics

In this unit, we define some new concepts of the Pythagorean cubic fuzzy set and discuss the characteristics of the Pythagorean cubic fuzzy set that is not an intuitionistic cubic fuzzy set with the help of illustrations. In this article, p_c stands for a Pythagorean cubic fuzzy set.

Definition 17 (see [27]). Let $\widehat{X}$ be a fixed set, then a Pythagorean cubic fuzzy set can be defined as

$$p_c = \left\{\left\langle x, \mu_{c_1}(x), \nu_{c_1}(x) | x \in \widehat{X}\right\rangle\right\}, \tag{15}$$

where

$$\begin{aligned}\mu_{c_1}(x) &= \langle A(x), \lambda(x)\rangle, \\ \nu_{c_1}(x) &= \left\langle \widetilde{A}(x), \mu(x)\right\rangle, \\ 0 &\leq \left(\mu_{c_1}(x)\right)^2 + \left(\nu_{c_1}(x)\right)^2 \leq [\widehat{1}, 1].\end{aligned} \tag{16}$$

The preceding condition may also be written as follows:

$$\begin{aligned}0 &\leq (\sup(A(x)))^2 + (\sup(\widetilde{A}(x)))^2 \leq 1, \\ 0 &\leq \lambda^2(x) + \mu^2(x) \leq 1.\end{aligned} \tag{17}$$

For a Pythagorean cubic set, the degree of indeterminacy is classified as

$$\begin{aligned}\pi_{p_c} = \Big\langle &\sqrt{1-(\sup(A(x)))^2 - (\sup(\widetilde{A}(x)))^2}, \\ &\cdot \sqrt{1-\lambda^2(x) - \mu^2(x)}\Big\rangle.\end{aligned} \tag{18}$$

For simplicity, we call (μ_{c_1}, ν_{c_2}) a Pythagorean cubic fuzzy number (PCFN) denoted by $P_c = (\mu_{c_1}, \nu_{c_2})$.

Example 1. Let $X = \{x_1, x_2, x_3\}$ be a fixed set and consider a set in X by

$$p_c = \begin{pmatrix} \langle(x_1, [0.5, 0.6], 0.7), ([0.6, 0.7], 0.5)\rangle, \\ (\langle(x_2, [0.4, 0.7], 0.6), ([0.5, 0.4], 0.6)\rangle), \\ (\langle(x_3, [0.4, 0.6], 0.7), ([0.5, 0.2], 0.6)\rangle) \end{pmatrix}. \tag{19}$$

Then, also $(0.7)^2 + (0.5)^2 = 0.49 + 0.25 = 0.74 < 1$; similarly, we can calculate the other cases. Thus, p_{c_1}, p_{c_2}, and p_{c_3} are (PCFNs). Therefore, p_c are PCFS.

Definition 18. Let $p_{c_1} = (\langle A_1, \lambda_1\rangle, \langle \widetilde{A}_1, \mu_1\rangle)$, $p_{c_2} = (\langle A_2, \lambda_2\rangle, \langle \widetilde{A}_2, \mu_2\rangle)$ and $p_c = (\langle A, \lambda\rangle, \langle \widetilde{A}, \mu\rangle)$ be three PCFNs and $\delta > 0$, where $A_1 = [a_1, b_1], \widetilde{A}_1 = [\widetilde{a}_1, \widetilde{b}_1], A_2 = [a_2, b_2], \widetilde{A}_2 = [\widetilde{a}_2, \widetilde{b}_2], A = [a, b]$, and $\widetilde{A} = [\widetilde{a}, \widetilde{b}]$; the operational laws are as follows:

(1) $p_{c_1} \oplus p_{c_2} = (\langle [\sqrt{a_1^2 + a_2^2 - a_1^2 a_2^2}, \sqrt{b_1^2 + b_2^2 - b_1^2 b_2^2}], \sqrt{\lambda_1^2 + \lambda_2^2 - \lambda_1^2 \lambda_2^2}\rangle, \langle [\widetilde{a}_1, \widetilde{a}_2], \mu_1 \mu_2\rangle)$,

(2) $p_{c_1} \otimes p_{c_2} = (\langle [a_1, a_2], \lambda_1 \lambda_2\rangle, \langle [\sqrt{\widetilde{a}_1^2 + \widetilde{a}_2^2 - \widetilde{a}_1^2 \widetilde{a}_2^2}, \sqrt{\widetilde{b}_1^2 + \widetilde{b}_2^2 - \widetilde{b}_1^2 \widetilde{b}_2^2}], \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2 \mu_2^2}\rangle)$,

(3) $\delta p_{c_1} = ((\langle [\sqrt{1 - (1 - a_1^2)^\delta}, \sqrt{1 - (1 - b_1^2)^\delta}], \sqrt{1 - (1 - \lambda_1^2)^\delta}), ([(\widetilde{a}_1 \widetilde{a}_2)^\delta, (\widetilde{b}_1 \widetilde{b}_2)^\delta], (\mu_1 \mu_2)^\delta\rangle))$,

(4) $p_{c_1}^\delta = ([(a_1 a_2)^\delta, (b_1 b_2)^\delta], (\lambda_1 \lambda_2)^\delta), ((\langle [\sqrt{1 - (1 - \widetilde{a}_1^2)^\delta}, \sqrt{1 - (1 - \widetilde{b}_1^2)^\delta}], \sqrt{1 - (1 - \mu_1^2)^\delta}\rangle))$,

(5) $p_{c_1}^c = \langle \widetilde{A}_1, \mu_1\rangle, \langle A_1, \lambda_1\rangle$.

Theorem 1. *Let* $p_{c_1} = (\langle A_1, \lambda_1\rangle, \langle \widetilde{A}_1, \mu_1\rangle)$, $p_{c_2} = (\langle A_2, \lambda_2\rangle, \langle \widetilde{A}_2, \mu_2\rangle)$, *and* $p_c = (\langle A, \lambda\rangle, \langle \widetilde{A}, \mu\rangle)$ *be three PCFNs and* $\delta > 0, \delta_1 > 0$, *and* $\delta_2 > 0$, *where* $A_1 = [a_1, b_1], \widetilde{A}_1 = [\widetilde{a}_1, \widetilde{b}_1]$, $A_2 = [a_2, b_2]$, $\widetilde{A}_2 = [\widetilde{a}_2, \widetilde{b}_2], A = [a, b]$, *and* $\widetilde{A} = [\widetilde{a}, \widetilde{b}]$, *then the following will hold:*

(1) $p_{c_1} \oplus p_{c_2} = p_{c_2} \oplus p_{c_1}$,

(2) $p_{c_1} \otimes p_{c_2} = p_{c_2} \otimes p_{c_1}$,

(3) $\delta(p_{c_1} \oplus p_{c_2}) = \delta(p_{c_1}) \oplus \delta(p_{c_2})$,

(4) $(\delta_1 + \delta_2) p_c = \delta_1 p_c \oplus \delta_2 p_c$,

(5) $(p_{q_1} \otimes p_{c_2})^\delta = (p_{q_1})^\delta \otimes (p_{c_2})^\delta$,

(6) $p_c^{(\delta_1 + \delta_2)} = p_c^{\delta_1} \otimes p_c^{\delta_2}$.

Proof. The proof is obvious.

We describe a score function and its basic properties to equate two PCFNs. □

Definition 19. Let $p_c = (\langle A_1, \lambda_1\rangle, \langle \widetilde{A}_1, \mu_1\rangle)$ be a PCFN, where $A_1 = [a_1, b_1], \widetilde{A}_1 = [\widetilde{a}_1, \widetilde{b}_1]$. We can introduce the score function of p_c as

$$S(p_c) = \left(\frac{a_1 + b_1 - \lambda_1}{3}\right)^2 - \left(\frac{\widetilde{a}_1 + \widetilde{b}_1 - \mu_1}{3}\right)^2, \tag{20}$$

where $S(p_c) \in [-1, 1]$.

Definition 20. Let $p_{c_1} = (\langle A_1, \lambda_1\rangle, \langle A_1, \mu_1\rangle)$ and $p_{c_2} = (\langle A_2, \lambda_2\rangle, \langle \widetilde{A}_2, \mu_2\rangle)$ be two PCFNs, $S(p_{c_1})$be the score function of p_{c_1}, and $S(p_{c_2})$ be the score function of p_{c_2}. Then,

(1) If $S(p_{c_1}) < S(p_{c_2})$, then $p_{c_1} < p_{c_2}$.

(2) If $S(p_{c_1}) > S(p_{c_2})$, then $p_{c_1} > p_{c_2}$.

(3) If $S(p_{c_1}) = S(p_{c_2})$, then $p_{c_1} \sim p_{c_2}$.

Example 2. Let $p_{c_1} = (([0.5, 0.7], 0.9), ([0.1, 0.5], 0.6))$, $p_{c_2} = (([0.4, 0.7], 0.6), ([0.2, 0.4], 0.6))$, and $p_{c_3} = (([0.03, 0.8], 0.9), ([0.0, 0.3], 0.7))$ be three PCFNs. Then, by Definition 18, we have $S(p_q) = 0.01, S(p_{c_2}) = 0.027$ and $S(p_{c_3}) = -0.0173$. Thus, $S(p_{c_2}) > S(p_{c_1}) > S(p_{c_3})$. Let $p_{c_1} = (([0.5, 0.7], 0.9), ([0.1, 0.5], 0.6))$ and $p_{c_2} = (([0.4, 0.7], 0.6), ([0.2, 0.4], 0.7))$ be two PCFNs. Then by Definition 19, we have $S(p_q) = 0.01$ and $S(p_{c_2}) = 0.01$ Thus, $S(p_q) = S(p_{c_2})$.

Therefore, by Definition 20, we cannot get information from P_{c_1} and P_{c_2}. Usually, such a case grows in preparation. It is clear from Definition 20 that we are unable to consider the requirement that two PCFNs have the same ranking. On the other side, deviancy may be changed. The consistency property of all the components to the average number in a PCFNs returns that they may accept. For the comparison of two PCFNs, we present a definition of accuracy degree.

Definition 21. Let $p_c = (\langle A_1, \lambda_1\rangle, \langle A_1, \mu_1\rangle)$ be a PCFN. Then, we define the accuracy degree of p_c which is denoted by $\alpha(p_c)$, where $A_1 = [a_1, b_1], \widetilde{A}_1 = [\widetilde{a}_1, \widetilde{b}_1]$ can be defined as

$$\alpha(p_c) = \left(\frac{a_1 + b_1 - \lambda_1}{3}\right)^2 + \left(\frac{\widetilde{a}_1 + \widetilde{b}_1 - \mu_1}{3}\right)^2, \tag{21}$$

where $\alpha(p_c) \in [0, 1]$.

Definition 22. Let $p_{c_1} = (\langle A_1, \lambda_1\rangle, \langle A_1, \mu_1\rangle)$ and $p_{c_2} = (\langle A_2, \lambda_2\rangle, \langle A_2, \mu_2\rangle)$ be two PCFNs, $\alpha(p_{c_1})$be the accuracy degree of p_{c_1}, and $\alpha(p_{c_2})$ be the accuracy degree of p_{c_2}. Then,

(1) If $\alpha(p_{c_1}) < \alpha(p_{c_2})$, then $p_{c_1} < p_{c_2}$.

(2) If $\alpha(p_{c_1}) > \alpha(p_{c_2})$, then $p_{c_1} > p_{c_2}$.

(3) If $\alpha(p_{c_1}) = \alpha(p_{c_2})$, then $p_{c_1} - p_{c_2}$.

Example 3. From example 2, since $S(p_{c_1}) = 0.01$ and $S(p_{c_2}) = 0.01$, thus, $S(p_{c_1}) = S(p_{c_2})$..So, we have $\alpha(p_{c_1}) = 0.01$ and $\alpha(p_{c_2}) = 0.044$. Thus, $\alpha(p_{c_1}) > \alpha(p_{c_2})$. Hence, $p_{c_1} > p_{c_2}$. As a result, the condition when two PCFNs have the same score has been resolved.

Definition 23. Let P_{c_1} and p_{c_2} be any two PCFNs on a set $X = \{x_1, x_2, \ldots, x_n\}$. The following is a definition of the distance measure between P_{c_1} and P_{c_2}:

$$D(p_{c_1}, p_{c_2}) = \frac{1}{6}\Big[|a_1^2 - a_1^2| + |b_1^2 - b_1^2| + |\widetilde{a}_1^2 - \widetilde{a}_1^2| + |\widetilde{b}_1^2 - \widetilde{b}_1^2| + |\lambda_1^2 - \lambda_1^2| + |\mu_1^2 - \mu_1^2|\Big]. \tag{22}$$

Example 4. Let $p_{c_1} = (\langle [0.6, 0.7], 0.3\rangle, \langle [0.5, 0.7], 0.8\rangle)$ and $p_{c_2}(\langle [0.5, 0.6], 0.4\rangle, \langle [0.4, 0.7], 0.5\rangle)$be two PCFNs. Then,

$$D\left(p_{c_1}, p_{c_2}\right) = \frac{1}{6}\left[|0.36 - 0.25| + |0.49 - 0.36| + |0.25 - 0.16| + |0.49 - 0.49| + |0.9 - 0.16| + |0.64 - 0.25|\right],$$
$$D\left(p_{c_1}, p_{c_2}\right) = \frac{1}{6}\left[|0.11| + |0.13| + |0.09| + |0| + |0.74| + |0.39|\right] = 0.2433. \tag{23}$$

4. Einstein Operations of Pythagorean Cubic Fuzzy Sets

In this section, we defined the Einstein product $(p_{c_1} \otimes \varepsilon p_{c_2})$ and the Einstein sum $(p_{c_1} \oplus \varepsilon p_{c_2})$on two PCFSs p_{c_1} and p_{c_2} which can be defined in the following forms.

Definition 24. Let $p_{c_1} = (\langle A_1, \lambda_1\rangle, \langle \widetilde{A}_1, \mu_1\rangle)$ and $p_{c_2} = (\langle A_2, \lambda_2\rangle, \langle \widetilde{A}_2, \mu_2\rangle)$ be two PCFNs, where $A_1 = [a_1, b_1]$, $\widetilde{A}_1 = [\widetilde{a}_1, \widetilde{b}_1]$, $A_2 = [a_2, b_2]$, and $\widetilde{A}_2 = [\widetilde{a}_2, \widetilde{b}_2]$, then

$$p_{c_1} \otimes \varepsilon p_{c_2} = \left\{ \frac{\left(\left[a_1^2, b_1^2\right], \lambda_I^2\right) \cdot \left(\left[a_2^2, b_2^2\right], \lambda_2^2\right)}{\sqrt{1 + \left(1 - \left[a_1^2, b_1^2\right], \lambda_I^2\right)\left(1 - \left[a_1^2, b_1^2\right], \lambda_1^2\right)}}, \frac{\sqrt{\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \mu_1^2 + \left[\widetilde{a}_2^2, \widetilde{b}_2^2\right], \mu_2^2)}}{\sqrt{1 + \left(\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \mu_1^2\right)\left(\left[\widetilde{a}_2^2, \widetilde{b}_2^2\right], \mu^2\right)}} \right\},$$
$$p_{c_1} \oplus \varepsilon p_{c_2} = \left\{ \frac{\sqrt{\left(\left[a_1^2, b_1^2\right], \lambda_1^2\right) + \left(\left[a_2^2, b_2^2\right], \lambda_2^2\right)}}{\sqrt{1 + \left(\left[a_1^2, b_1^2\right], \lambda_1^2\right) \cdot \left(\left[a_1^2, b_1^2\right], \lambda_1^2\right)}}, \frac{\left(\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \mu^2\right) \cdot \left(\left[\widetilde{a}_2^2, \widetilde{b}_2^2\right], \mu^2\right)_2\Big)}{\sqrt{1 + \left(1 - \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \mu_1^2\right) \cdot \left(1 - \left[\widetilde{a}_2^2, \widetilde{b}_2^2\right], \mu^2\right)}} \right\}. \tag{24}$$

Theorem 2. *Let n be any positive integer and p_{c_1} is a PCFS, then the exponentiation operation $p_{c_1} \wedge \varepsilon^n$ is a mapping from $Z^+ \times \diamond \longrightarrow \diamond$:*

$$p_{c_1} \wedge \varepsilon^n = \left\{ \frac{\sqrt{2\left(\left[a_1^2, b_1^2\right], \lambda_4^2\right)^n}}{\sqrt{\left(2 - \left[a_1^2, b_1^2\right], \lambda_1^2\right)^n + \left(\left[a_1^2, b_1^2\right], \lambda_1^2\right)^n}}, \frac{\sqrt{\left(1 + \left(\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)\right)^n - \left(1 - \left(\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)\right)^n}}{\sqrt{\left(1 + \left(\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)\right)^n + \left(1 - \left(\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)\right)^n}} \right\}, \tag{25}$$

where $p_{c_1} \wedge \varepsilon^n = \overbrace{p_{c_1} \otimes \varepsilon p_{c_1} \otimes \varepsilon p_{c_1}}^{n} \otimes \varepsilon, \ldots, p_{c_1} \otimes \varepsilon p_{c_1}$. Moreover, $p_{c_1} \wedge \varepsilon^n$ is a Pythagorean cubic fuzzy set (PCFS), even if $n \in \mathbb{R}^+$.

Proof. We may prove that equation (25) holds for all positive integers n using mathematical induction. First, it holds for $n = 1$.

$$p_{c_1} \wedge \varepsilon^1 = \left\{ \frac{\sqrt{2\left(\left[a_1^2, b_1^2\right], \lambda_1^2\right)^1}}{\sqrt{\left(2 - \left[a_1^2, b_1^2\right], \lambda_1^2\right)^1 + \left(\left[q_1^2, b_1^2\right], \lambda_1^2\right)^1}}, \frac{\sqrt{\left(1 + \left(\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)\right)^1 - \left(1 - \left(\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)\right)^1}}{\sqrt{\left(1 + \left(\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)\right)^1 + \left(1 - \left(\left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)\right)^1}} \right\}. \tag{26}$$

Taking the left-hand side of the equation above,

$$p_{c_1}\wedge\varepsilon^1 = p_{c_1} = \left\{\left(x,\mu_{p_q},\nu_{p_c}\right)|x\in X\right\}. \tag{27}$$

Taking the right-hand side of the equation above,

$$= \left\{ \frac{\sqrt{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^1}}{\sqrt{\left(2-\left[a_1^2,b_1^2\right],\lambda_1^2\right)^1+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^1}} \frac{\sqrt{\left(1+\left(\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^1-\left(1-\left(\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^1}}{\sqrt{\left(1+\left(\left[\tilde{a}_1^2\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^1+\left(1-\left(\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^1}} \right\}$$
$$= p_{c_1} = \left\{\left(x,\mu_{p_q},\nu_{p_q}\right)|x\in X\right\}. \tag{28}$$

From equations (25) and (27), we have equation (25) which holds for $n = 1$. Next, we show that equation (25) holds for $n = k$. If equation (25) holds for $n = k$, then equation (25) also holds for $n = k + 1$.

$$\left\{ \frac{\sqrt{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^k}}{\sqrt{\left(2-\left[a_1^2,b_1^2\right],\lambda_1^2\right)^k+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^k}}, \frac{\sqrt{\left(1+\left(\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^k-\left(1-\left(\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^k}}{\sqrt{\left(1+\left(\left[\tilde{a}_1^2\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^k+\left(1-\left(\left[\tilde{a}_1^2\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^k}} \right\}$$
$$\otimes\varepsilon p_{c_1} = \left\{\left(x,\mu_{p_1},\nu_{p_q}\right)|x\in X\right\} \tag{29}$$
$$= \left\{ \frac{\sqrt{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^{k+1}}}{\sqrt{\left(2-\left[a_1^2,b_1^2\right],\lambda_1^2\right)^{k+1}+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^{k+1}}} \frac{\sqrt{\left(1+\left(\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^{k+1}-\left(1-\left(\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^{k+1}}}{\sqrt{\left(1+\left(\left[\tilde{a}_1^2\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^{k+1}+\left(1-\left(\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)\right)^{k+1}}} \right\}.$$

Now, we'll show that equation (25) is valid for every positive integer n,

$$0\le \left(\left(\frac{\sqrt{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}}{\sqrt{\left(2-\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}}\right)^2, \left(\frac{\sqrt{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n-\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}}{\sqrt{\left(1+\left[\tilde{a}_1^2\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n+\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}}\right)^2\right)\le 1, \tag{30}$$

even if $n\in\mathbb{R}^+$. Since $0\le\mu_{p_{q_2}}(x)\le 1, 0\le\nu_{p_{q_2}}(x)\le 1, 0\le \mu^2_{p_{q_1}}(x)+\nu^2_{p_q}(x)\le 1$, then $1-\mu^2_{p_{c_1}}(x)\ge\nu^2_{p_{c_1}}(x)\ge 0$,.so

$$0\le \left(\frac{\sqrt{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}}{\sqrt{\left(2-\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}}\right)^2, \left(\frac{\sqrt{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)^{w_1}\right)^n-\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)^{w_1}\right)^n}}{\sqrt{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)^{w_1}\right)^n+\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)^{m_1}\right)^n}}\right)^2\right)\le 1$$
$$0\le \left(\frac{\sqrt{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}}{\sqrt{\left(2-\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}}, \frac{\sqrt{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)^{\nu_1}\right)^n-\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)^{\nu_1}\right)^n}}{\sqrt{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)^{\nu_1}\right)^n+\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)^{\nu_1}\right)^n}}\right)\le 1. \tag{31}$$

Since

$$\left(\begin{array}{c}\dfrac{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}{\left(2-\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}\\[2ex] =\dfrac{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}{1+\left(1-\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}\\[2ex] \leq\dfrac{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}{1+\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],u_1^2\right)^n+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}\leq\dfrac{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}\left(2-\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n\\[2ex] \leq\dfrac{1}{1+\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],u_1^2\right)^n+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}\end{array}\right), \tag{32}$$

again

$$\begin{aligned}1-\nu_{p_{p_{c_1}}}^2(x)\geq\mu_{p_{c_1}}^2(x)\geq 0&\frac{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n-\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n+\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}\\ &=\frac{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n-\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n+\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}\\ &\leq\frac{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n-\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n+\left(\left[a_1^2,b_1^2\right],\left(\lambda_1^2\right)\right)^n}.\end{aligned} \tag{33}$$

Thus,

$$\frac{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n-\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n+\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}\leq\frac{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n-\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n+\left(\left[a_1^2,b_1^2\right],\left(\lambda_1^2\right)\right)^n}. \tag{34}$$

From equations (14) and (34), we have

$$=\frac{2\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}{1+\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],u_1^2\right)^n+\left(\left[a_1^2,b_1^2\right],\lambda_1^2\right)^n}+\frac{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n-\left(1-\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n}{\left(1+\left[\tilde{a}_1^2,\tilde{b}_1^2\right],\left(\mu_1^2\right)\right)^n+\left(\left[a_1^2,b_1^2\right],\left(\lambda_1^2\right)\right)^n}$$

$$= \frac{2\left(\left[a_1^2, b_1^2\right], \lambda_1^2\right)^n}{1 + \left(1 - \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], u_1^2\right)^n + \left(\left[a_1^2, b_1^2\right], \lambda_1^2\right)^n} + \frac{\left(1 + \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)^n - \left(1 - \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)^n}{\left(1 + \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)^n + \left(\left[a_1^2, b_1^2\right], \left(\lambda_1^2\right)\right)^n}$$

$$\leq \frac{2\left(\left[a_1^2, b_1^2\right], \lambda_1^2\right)^n + \left(1 + \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)^n - \left(\left[a_1^2, b_1^2\right], \lambda_1^2\right)^n}{1 + \left(1 - \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], u_1^2\right)^n + \left(\left[a_1^2, b_1^2\right], \lambda_1^2\right)^n} \leq 1. \tag{35}$$

Thus,

$$0 \leq \frac{2\left(\left[a_1^2, b_1^2\right], \lambda_1^2\right)^n}{1 + \left(1 - \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], u_1^2\right)^n + \left(\left[a_1^2, b_1^2\right], \lambda_1^2\right)^n} + \frac{\left(1 + \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)^n - \left(1 - \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)^n}{\left(1 + \left[\widetilde{a}_1^2, \widetilde{b}_1^2\right], \left(\mu_1^2\right)\right)^n + \left(\left[a_1^2, b_1^2\right], \left(\lambda_1^2\right)\right)^n} \leq 1. \tag{36}$$

Thus, a PCFS $p_{c_i}^{\varepsilon}$ defined above is a PCFS for any $n \in \mathbb{R}^+$. □

5. Pythagorean Cubic Fuzzy Einstein Weighted Geometric Aggregation Operator

Definition 25. Let $p_{c_j} = (\mu_{p_{cj}}, v_{p_j})$, $(j = 1, \ldots, m)$, be the collection of PCFV_S with $\leq L$, then a $\mathrm{PCFWG}_{\varepsilon}$ operator of dimension n is a mapping $\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}$: $\phi^m \longrightarrow \phi$, and

$$\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right) = p_{c_1}^{\varepsilon\widetilde{w}_1} \otimes_{\varepsilon} p_{c_2}^{\varepsilon_2} \otimes_{\varepsilon}, \ldots, \otimes_{\varepsilon} p_{c_m}^{\varepsilon_m}, \tag{37}$$

where $\widetilde{w} = (\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_m)^T$ is the weighted vector of P_{cj} $(j = 1, \ldots, n)$ such that $\widetilde{w}_j \in [0, 1]$and $\sum_{j=1}^{m} \widetilde{w}_j = 1$.

Theorem 3. *Let $p_{c_j} = (\mu_{p_{c_j}}, v_{p_{t_j}})$, $(j = 1, \ldots, n)$, be the collection of PCFVs with $\leq L$, then their aggregated value by using the $\mathrm{PCFWG}_{\varepsilon}$ operator is also a PCFV, and let $p_{c_1} = (\langle A_1, \lambda_1 \rangle, \langle \widetilde{A}_1, \mu_1 \rangle)$ and $p_{c_2} = (\langle A_2, \lambda_2 \rangle, \langle \widetilde{A}_2, \mu_2 \rangle)$, where $A_1 = [a_1, b_1]$, $\widetilde{A}_1 = [\widetilde{a}_1, \widetilde{b}_1]$, $A_2 = [a_2, b_2]$, and $\widetilde{A}_2 = [\widetilde{a}_2, \widetilde{b}_2]$, then*

$$\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right) = \left(\begin{array}{c} \left\langle \left[\dfrac{\sqrt{2\prod_{j=1}^{m}\left(a_j^{2\widetilde{w}_j}\right)}}{\sqrt{\prod_{j=1}^{m}\left(2 - a_j^2\right)^{\widetilde{w}_j} + \prod_{j=1}^{m}\left(a_j^2\right)^{\widetilde{w}_j}}}, \dfrac{\sqrt{2\prod_{j=1}^{m}\left(b_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(2 - b_j^2\right)^{\widetilde{w}_j} + \prod_{j=1}^{m}\left(b_j^2\right)^{\widetilde{w}_j}}} \right], \dfrac{\sqrt{2\prod_{j=1}^{m}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(2 - \lambda_j^2\right)^{\widetilde{w}_j} + \prod_{j=1}^{m}\left(\lambda_j^2\right)^{\widetilde{w}_j}}} \right\rangle; \\ \left\langle \left[\dfrac{\sqrt{\prod_{j=1}^{m}\left(1 + \widetilde{a}_j^2\right)^{\widetilde{w}_j} + \prod_{j=1}^{m}\left(1 - \widetilde{a}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(1 + \widetilde{a}_j^2\right)^{\widetilde{w}_j} + \prod_{j=1}^{m}\left(1 - \widetilde{a}_j^2\right)^{\widetilde{w}_k}}}, \dfrac{\sqrt{\prod_{j=1}^{m}\left(1 + \widetilde{b}_j^2\right)^{\widetilde{w}_j} - \prod_{j=1}^{m}\left(1 - \widetilde{b}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(1 + \widetilde{b}_j^2\right)^{\widetilde{w}_j} + \prod_{j=1}^{m}\left(1 - \widetilde{b}_j^2\right)^{\widetilde{w}_j}}} \right], \dfrac{\sqrt{\prod_{j=1}^{m}\left(1 + \mu_j^2\right)^{\widetilde{w}_j} - \prod_{j=1}^{m}\left(1 - \mu_j^2\right)^{\widetilde{w}_k}}}{\sqrt{\prod_{j=1}^{m}\left(1 + \mu_j^2\right)^{\widetilde{w}_j} + \prod_{j=1}^{m}\left(1 - \mu_j^2\right)^{\widetilde{w}_k}}} \right\rangle \end{array} \right), \tag{38}$$

where $\widetilde{w} = (\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_m)^T$ is the weighted vector of Pc_j (j = 1,...,n) such that $\widetilde{w}_j \in [0, 1]$ and $\sum_{j=1}^{m} \widetilde{w}_j = 1$.

Proof. Mathematical induction may be used to prove this theorem. To begin, we prove that equation (38) holds for $m = 1$. Taking the left side,

$$\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right) = \mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_c\right) = p_c^{\varepsilon\widetilde{w}}. \tag{39}$$

Now, taking right-hand side,

$$= \begin{pmatrix} \left\langle \left[\frac{\sqrt{2\prod_{j=1}^{m}\left(a_j^{2\widetilde{w}_j}\right)}}{\sqrt{\prod_{j=1}^{m}\left(2-a_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(a_j^2\right)^{\widetilde{w}_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(b_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(2-b_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(b_j^2\right)^{\widetilde{w}_j}}} \right], \frac{\sqrt{2\prod_{j=1}^{m}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(2-\lambda_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(\lambda_j^2\right)^{\widetilde{w}_j}}} \right\rangle; \\ \left\langle \left[\sqrt{\frac{\prod_{j=1}^{m}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{m}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}{\prod_{j=1}^{m}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_k}}}, \sqrt{\frac{\prod_{j=1}^{m}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{m}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}{\prod_{j=1}^{m}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}} \right], \sqrt{\frac{\prod_{j=1}^{m}\left(1+\mu_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{m}\left(1-\mu_j^2\right)^{\widetilde{w}_k}}{\prod_{j=1}^{m}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(1-\mu_j^2\right)^{\widetilde{w}_k}}} \right\rangle \end{pmatrix}. \tag{40}$$

From equations (39) and (40), we have equation (38) which holds for $m=1$. Now, we show that equation (38) holds for $m=k$.

$$\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1},p_{c_2},\ldots,p_{c_k}\right) = \begin{pmatrix} \left\langle \left[\frac{\sqrt{2\prod_{j=1}^{k}\left(a_j^{2\widetilde{w}_j}\right)}}{\sqrt{\prod_{j=1}^{k}\left(2-a_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(a_j^2\right)^{\widetilde{w}_j}}}, \frac{\sqrt{2\prod_{j=1}^{k}\left(b_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-b_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(b_j^2\right)^{\widetilde{w}_j}}} \right], \frac{\sqrt{2\prod_{j=1}^{k}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-\lambda_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(\lambda_j^2\right)^{\widetilde{w}_j}}} \right\rangle; \\ \left\langle \left[\sqrt{\frac{\prod_{j=1}^{k}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}{\prod_{j=1}^{k}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_k}}}, \sqrt{\frac{\prod_{j=1}^{k}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}{\prod_{j=1}^{k}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}} \right], \sqrt{\frac{\prod_{j=1}^{k}\left(1+\mu_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\mu_j^2\right)^{\widetilde{w}_k}}{\prod_{j=1}^{k}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\mu_j^2\right)^{\widetilde{w}_k}}} \right\rangle \end{pmatrix}. \tag{41}$$

Next, we are going to show that equation (38) holds for m $=k+1$.

$$\begin{aligned} &\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1},p_{c_2},\ldots,p_{c_{k+1}}\right) \\ &= \begin{pmatrix} \left\langle \left[\frac{\sqrt{2\prod_{j=1}^{k}\left(a_j^{2\widetilde{w}_j}\right)}}{\sqrt{\prod_{j=1}^{k}\left(2-a_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(a_j^2\right)^{\widetilde{w}_j}}}, \frac{\sqrt{2\prod_{j=1}^{k}\left(b_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-b_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(b_j^2\right)^{\widetilde{w}_j}}} \right], \frac{\sqrt{2\prod_{j=1}^{k}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-\lambda_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(\lambda_j^2\right)^{\widetilde{w}_j}}} \right\rangle; \\ \left\langle \left[\sqrt{\frac{\prod_{j=1}^{k}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}{\prod_{j=1}^{k}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}}, \sqrt{\frac{\prod_{j=1}^{k}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}{\prod_{j=1}^{k}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}} \right], \sqrt{\frac{\prod_{j=1}^{k}\left(1+\mu_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\mu_j^2\right)^{\widetilde{w}_k}}{\prod_{j=1}^{k}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\mu_j^2\right)^{\widetilde{w}_k}}} \right\rangle \end{pmatrix} \\ &\otimes_{\varepsilon} \begin{pmatrix} \left\langle \left[\frac{\sqrt{2\prod_{j=1}^{k+1}\left(a_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-a_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(a_j^2\right)^{\widetilde{w}_j}}}, \frac{\sqrt{2\prod_{j=1}^{k+1}\left(b_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-b_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(b_j^2\right)^{\widetilde{w}_j}}} \right], \frac{\sqrt{2\prod_{j=1}^{k+1}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-\lambda_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(\lambda_j^2\right)^{\widetilde{w}_j}}} \right\rangle; \\ \left\langle \left[\sqrt{\frac{\prod_{j=1}^{k+1}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k+1}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}{\prod_{j=1}^{k+1}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}}, \sqrt{\frac{\prod_{j=1}^{k+1}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k+1}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}{\prod_{j=1}^{k+1}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}} \right], \sqrt{\frac{\prod_{j=1}^{k+1}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\mu_j^2\right)^{\widetilde{w}_j}}{\prod_{j=1}^{k+1}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\mu_j^2\right)^{\widetilde{w}_j}}} \right\rangle \end{pmatrix}. \end{aligned} \tag{42}$$

Let

$$\begin{aligned}
s_1 &= \left[\frac{\sqrt{2\prod_{j=1}^{k}\left(a_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-a_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(a_j^2\right)^{\widetilde{w}_j}}},\frac{\sqrt{2\prod_{j=1}^{k}\left(b_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-b_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(b_j^2\right)^{\widetilde{w}_j}}}\right],\\
s_2 &= \frac{\sqrt{2\prod_{j=1}^{k}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-\lambda_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(\lambda_j^2\right)^{\widetilde{w}_j}}},\\
s_3 &= \left[\frac{\sqrt{\prod_{j=1}^{k}\left(1+\tilde{a}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\tilde{a}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+\tilde{a}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\tilde{a}_j^2\right)^{\widetilde{w}_j}}},\frac{\sqrt{\prod_{j=1}^{k}\left(1+\tilde{b}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\tilde{b}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+\tilde{b}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\tilde{b}_j^2\right)^{\widetilde{w}_j}}}\right],\\
s_4 &= \frac{\sqrt{\prod_{j=1}^{k}\left(1+\mu_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\mu_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\mu_j^2\right)^{\widetilde{w}_j}}},\\
u_1 &= \left[\frac{\sqrt{2\prod_{j=1}^{k+1}\left(a_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-a_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(a_j^2\right)^{\widetilde{w}_j}}},\frac{\sqrt{2\prod_{j=1}^{k+1}\left(b_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-b_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(b_j^2\right)^{\widetilde{w}_j}}}\right],\\
u_2 &= \frac{\sqrt{2\prod_{j=1}^{k+1}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-\lambda_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(\lambda_j^2\right)^{\widetilde{w}_j}}},\\
u_3 &= \left[\frac{\sqrt{\prod_{j=1}^{k+1}\left(1+\tilde{a}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k+1}\left(1-\tilde{a}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+\tilde{a}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\tilde{a}_j^2\right)^{\widetilde{w}_j}}},\frac{\sqrt{\prod_{j=1}^{k+1}\left(1+\tilde{b}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k+1}\left(1-\tilde{b}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+\tilde{b}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\tilde{b}_j^2\right)^{\widetilde{w}_j}}}\right],\\
u_4 &= \frac{\sqrt{\prod_{j=1}^{k+1}\left(1+\mu_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k+1}\left(1-\mu_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\mu_j^2\right)^{\widetilde{w}_j}}}.
\end{aligned}\tag{43}$$

Now, putting these values in equation (40), we have

$$\begin{aligned}
\mathrm{PCFWG}_w^{\varepsilon}\left(p_{c_1},p_{c_2},\ldots,p_{c_{k+1}}\right) &= \left(\frac{s_1}{s_2},\frac{s_3}{s_4}\right)\otimes_z\left(\frac{u_1}{u_2},\frac{u_3}{u_4}\right)\\
&\quad\cdot\left(\frac{s_1u_1}{\sqrt{2s_2^2u_2^2+s_1^2u_1^2-s_2^2u_1^2-s_1^2u_2^2}}\right).
\end{aligned}\tag{44}$$

Now, putting the values in equation (42), we have

$$
\begin{aligned}
&\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_{k+1}}\right)\\
&= \left(\begin{array}{c}
\left\langle \left[\dfrac{\sqrt{2\prod_{j=1}^{k}\left(a_j^{2\widetilde{w}_j}\right)}}{\sqrt{\prod_{j=1}^{k}\left(2-a_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(a_j^2\right)^{\widetilde{w}_j}}}, \dfrac{\sqrt{2\prod_{j=1}^{k}\left(b_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-b_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(b_j^2\right)^{\widetilde{w}_j}}} \right], \dfrac{\sqrt{2\prod_{j=1}^{k}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(2-\lambda_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(\lambda_j^2\right)^{\widetilde{w}_j}}} \right\rangle;\\
\left\langle \left[\dfrac{\sqrt{\prod_{j=1}^{k}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}}, \dfrac{\sqrt{\prod_{j=1}^{k}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}} \right], \dfrac{\sqrt{\prod_{j=1}^{k}\left(1+\mu_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k}\left(1-\mu_j^2\right)^{\widetilde{w}_k}}}{\sqrt{\prod_{j=1}^{k}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k}\left(1-\mu_j^2\right)^{\widetilde{w}_k}}} \right\rangle
\end{array} \right)\\
&\otimes_{\varepsilon} \left(\begin{array}{c}
\left\langle \left[\dfrac{\sqrt{2\prod_{j=1}^{k+1}\left(a_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-a_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(a_j^2\right)^{\widetilde{w}_j}}}, \dfrac{\sqrt{2\prod_{j=1}^{k+1}\left(b_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-b_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(b_j^2\right)^{\widetilde{w}_j}}} \right], \dfrac{\sqrt{2\prod_{j=1}^{k+1}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(2-\lambda_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(\lambda_j^2\right)^{\widetilde{w}_j}}} \right\rangle\\
\left\langle \left[\dfrac{\sqrt{\prod_{j=1}^{k+1}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k+1}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}}, \dfrac{\sqrt{\prod_{j=1}^{k+1}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{k+1}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}} \right], \dfrac{\sqrt{\prod_{j=1}^{k+1}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\mu_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{k+1}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{k+1}\left(1-\mu_j^2\right)^{\widetilde{w}_j}}} \right\rangle
\end{array} \right).
\end{aligned}
\tag{45}
$$

Equation (38) holds for $m = k+1$. Thus, equation (38) holds for all m. □

Lemma 1. *Let $p_{c_j} > 0, \widetilde{w}_j > 0$, $(j = 1, \ldots, n)$ and $\sum_{j=1}^{m} \widetilde{w}_j = 1$. Then,*

$$\prod_{j=1}^{m}\left(p_{c_j}\right)^{\widetilde{w}_j} \leq \sum_{j=1}^{m} \widetilde{w}_j p_{c_j}, \tag{46}$$

where the equality holds if and only if $p_{c_1} = p_{c_2} = \cdots = p_{c_m}$.

Theorem 4. *Let $p_{c_j} = (\mu_{p_{c_j}}, v_{p_{t_j}})$, $(j = 1, \ldots, n)$, be the collection of PCFVs with ≤L, then*

$$\mathrm{PCFWG}_{\widetilde{w}}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right) \leq \mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right), \tag{47}$$

where $\widetilde{w} = (\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_m)^T$ is the weighted vector of p_{c_j} (j = 1,…, n) such that $\widetilde{w}_j \in [0,1]$ and $\sum_{j=1}^{m} \widetilde{w}_j = 1$.

ProofStraight. forward. □

Theorem 5. *Let p_{c_j}(j = 1,…,n) be the collection of PCFVs with ≤L, where $\widetilde{w} = (\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_m)^T$ is the weighted vector of (j = 1,…,n) such that $\widetilde{w}_j \in [0,1]$ and $i\sum_{j=1}^{m} \widetilde{w}_j = 1$. Then,*

(1) *Idempotency: if all P_{c_j} (j = 1,…,n) are equal, i.e., P_{c_j} (j = 1,…,n) = P_{c_j}, then*

$$\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right) = p_{c_j}. \tag{48}$$

(2) *Boundary:*

$$p_{\min} \leq \mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right) \leq p_{\max} \text{ for every } \widetilde{w}. \tag{49}$$

(3) *Monotonicity: let $p_{c_j}^* = (\mu_{p_{c_j}}^*, v_{p_{c_j}}^*)$, $(j = 1, \ldots, n)$ be the collection of PCFVs with ≤L, and $\mu_{p_{c_j}} \leq \mu_{p_{c_j}}^*$, $v_{p_{c_j}} \leq v_{p_{c_j}}^*$, for all, then*

$$
\begin{aligned}
&\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right)\\
&\quad \leq \mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}^*, p_2^*, \ldots, p_{c_m}^*\right) \text{ for every } \widetilde{w}.
\end{aligned}
\tag{50}
$$

Proof. (1) Idempotency: since

$$\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right) = \left(\begin{array}{c} \left\langle\left[\frac{\sqrt{2\prod_{j=1}^{m}\left(a_j^{2\widetilde{w}_j}\right)}}{\sqrt{\prod_{j=1}^{m}\left(2-a_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(a_j^2\right)^{\widetilde{w}_j}}}, \frac{\sqrt{2\prod_{j=1}^{m}\left(b_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(2-b_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(b_j^2\right)^{\widetilde{w}_j}}}\right], \frac{\sqrt{2\prod_{j=1}^{m}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(2-\lambda_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(\lambda_j^2\right)^{\widetilde{w}_j}}}\right\rangle; \\ \left\langle\left[\frac{\sqrt{\prod_{j=1}^{m}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{m}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(1+\widetilde{a}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(1-\widetilde{a}_j^2\right)^{\widetilde{w}_k}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{m}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(1+\widetilde{b}_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(1-\widetilde{b}_j^2\right)^{\widetilde{w}_j}}}\right], \frac{\sqrt{\prod_{j=1}^{m}\left(1+\mu_j^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{m}\left(1-\mu_j^2\right)^{\widetilde{w}_k}}}{\sqrt{\prod_{j=1}^{m}\left(1+\mu_j^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(1-\mu_j^2\right)^{\widetilde{w}_k}}}\right\rangle \end{array}\right), \tag{51}$$

For simplicity, we use the notation of PCFSs. Let $p_{c_j} = (\mu_{p_{cj}}, \nu_{p_{t_j}})$ where $\mu_{p_q} = (\langle A_1, \lambda_1\rangle, \langle \widetilde{A}_1, \mu_1\rangle)$ and $\mu_{p_{c_2}} = (\langle A_2, \lambda_2\rangle, \langle \widetilde{A}_2, \mu_2\rangle)$, where $A_1 = [a_1, b_1]$, $\widetilde{A}_1 = [\widetilde{a}_1, \widetilde{b}_1]$, $A_2 = [a_2, b_2]$, and $\widetilde{A}_2 = [\widetilde{a}_2, \widetilde{b}_2]$, then the above equation can be written in the following form:

$$\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right) = \left(\frac{\sqrt{2\prod_{j=1}^{m}\left(\mu_{p_{c_j}}^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(2-\mu_{p_{c_j}}^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(\mu_{p_{c_j}}^2\right)^{\widetilde{w}_j}}}, \frac{\sqrt{\prod_{j=1}^{m}\left(1+\widetilde{\nu}_{p_{c_j}}^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{m}\left(1-\widetilde{\nu}_{p_{c_j}}^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(1+\widetilde{\nu}_{p_{c_j}}^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(1-\widetilde{\nu}_{c_j}\right)^{\widetilde{w}_j}}}\right). \tag{52}$$

Now, $p_{cj}(j = 1,\ldots,n) = p_c$. Then, equation (26) can be written as

$$\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right) = \left(\frac{\sqrt{2\left(\mu_{p_{c_j}}^2\right)}}{\sqrt{\left(2-\mu_{p_{c_j}}^2\right)+\left(\mu_{p_{c_j}}^2\right)}}, \frac{\sqrt{\left(1+\widetilde{\nu}_{p_{c_j}}^2\right)-\left(1-\widetilde{\nu}_{p_{c_j}}^2\right)}}{\sqrt{\left(1+\widetilde{\nu}_{p_{c_j}}^2\right)+\left(1-\widetilde{\nu}_{c_j}\right)}}\right). \tag{53}$$

(2) Boundary:

$$p_{\min} \le \mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1}, p_{c_2}, \ldots, p_{c_m}\right) \le p_{\max} \text{ for every } \widetilde{w}, \tag{54}$$

where $p_{\min} = \min(p_{c_j})$ and $p_{\max} = \max(p_{c_j})$. Let $f(x) = \sqrt{(2 - x^2/x^2)}, x \in [0,1]$, then $f(x) = (-2/x^3)\sqrt{(x^2/2 - x^2)} < 0$. So, $f(x)$ is the decreasing function on $(0,1]$. Since $\mu_{p_c \min} \le \mu_{p_{cj}} \le \mu_{p_c \max}$ for all j, then $f(\mu_{p_c \max}) \le f(\mu_{p_c}) \le f(\mu_{p_c \min})$ $(j = 1, \ldots n)$ $\sqrt{(2-\mu_{p_c \max}^2/\mu_{p_c \max}^2)} \le \sqrt{(2-\mu_{p_c}^2/\mu_{p_c}^2)} \le \sqrt{(2-\mu_{p_c \min}^2/\mu_{p_c \min}^2)}$ where $\widetilde{w} = (\widetilde{w}_1, \widetilde{w}_2, \ldots, \widetilde{w}_m)^T$ is the weighted vector of p_{c_j} $(j = 1, \ldots, n)$ such that $\widetilde{w}_j \in [0,1]$ and $\sum_{j=1}^{m} \widetilde{w}_j = 1$. Then, we have

$$\Longleftrightarrow \sqrt{\prod_{j=1}^{m}\left(\frac{2-\mu_{p_c \max}^2}{\mu_{p_c \max}^2}\right)^{\widetilde{w}_j}} \le \sqrt{\prod_{j=1}^{m}\left(\frac{2-\mu_{p_c}^2}{\mu_{p_c}^2}\right)^{\widetilde{w}_j}} \le \sqrt{\prod_{j=1}^{m}\left(\frac{2-\mu_{p_c \min}^2}{\mu_{p_c \min}^2}\right)^{\widetilde{w}_j}}$$

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\frac{2-\mu_{p_c \max}^2}{\mu_{p_c \max}^2}\right)^{\sum_{j=1}^{m}\widetilde{w}_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\frac{2-\mu_{p_c}^2}{\mu_{p_c}^2}\right)^{\sum_{j=1}^{m}\widetilde{w}_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\frac{2-\mu_{p_c \min}^2}{\mu_{p_c \min}^2}\right)^{\sum_{j=1}^{m}\widetilde{w}_j}}$$

$$\Leftrightarrow \nu_{p_c \min} \leq \frac{\sqrt{\prod_{j=1}^{m}\left(\mu_{p_c}^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(2-\mu_{p_c}^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(\mu_{p_c}^2\right)^{\widetilde{w}_j}}} \leq \mu_{p_c \max}. \tag{55}$$

Again, let $h(y)=\sqrt{(1-y^2/1+y^2)}, x\in[0,1]$, then $h(y)=(-2y/(1-y^3)^2)\sqrt{(1+y^2/1-y^2)}<0$. So, $h(y)$ is the decreasing function on $(0,1]$. Since $\nu_{p_c \min}\leq \nu_{p_{cj}} \leq \nu_{p_c \max}$for all j.Then, $h(\nu_{p_c \max})\leq h\ (\nu_{p_c})\leq\ h(\nu_{p_c \min})$ for all $j\sqrt{(1-\nu_{p_c \max}^2/1+\nu_{p_c \max}^2)}\leq\sqrt{(1-\nu_{p_c}^2/\nu_{p_c}^2)}\leq \sqrt{(1-\nu_{p_c \min}^2/\nu_{p_c \min}^2)}$, where $\widetilde{w}=(\widetilde{w}_1,\widetilde{w}_2,\ldots,\widetilde{w}_m)^T$ is the weighted vector of $p_{c_j}(j=1,\ldots,n)$ such that $\widetilde{w}_j\in[0,1]$and $\sum_{j=1}^{m}\widetilde{w}_j=1$. Then, we have

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\frac{1-\nu_{p_c \max}^2}{1+\nu_{p_c \max}^2}\right)^{\widetilde{w}_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\frac{1-\nu_{p_c}^2}{1+\nu_{p_c}^2}\right)^{\widetilde{w}_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\frac{1-\nu_{p_c \min}^2}{1+\nu_{p_c \min}^2}\right)^{\widetilde{w}_j}}$$

$$\Leftrightarrow \sqrt{\prod_{j=1}^{m}\left(\frac{1-\nu_{p_c \max}^2}{1+\nu_{p_c \max}^2}\right)^{\sum_{j=1}^{m}\widetilde{w}_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\frac{1-\nu_{p_c}^2}{1+\nu_{p_c}^2}\right)^{\sum_{j=1}^{m}\widetilde{w}_j}} \leq \sqrt{\prod_{j=1}^{m}\left(\frac{1-\nu_{p_c \min}^2}{1+\nu_{p_c \min}^2}\right)^{\sum_{j=1}^{m}\widetilde{w}_j}} \tag{56}$$

$$\Leftrightarrow \nu_{p_c \max} \leq \frac{\sqrt{\prod_{j=1}^{m}\left(1+\nu_{p_c}^2\right)^{\widetilde{w}_j}-\prod_{j=1}^{m}\left(1-\nu_{p_c}^2\right)^{\widetilde{w}_j}}}{\sqrt{\prod_{j=1}^{m}\left(1-\nu_{p_c}^2\right)^{\widetilde{w}_j}+\prod_{j=1}^{m}\left(1+\nu_{p_c}^2\right)^{\widetilde{w}_j}}} \leq \mu_{p_c \min}.$$

Let $\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}(p_{c_1},p_{c_2},\ldots,p_{c_m})=p_c$. Then, equations (52) and (55) can be written as $\mu_{p_c \min}\leq\mu_{p_{cj}}\leq\mu_{p_c \max}$ and $\nu_{p_c \min}\leq\nu_{p_{cj}}\leq\nu_{p_c \max}$, respectively. Thus, $S(p_c)=\mu_{p_c}^2-\nu_{p_c}^2\leq\mu_{p_c \max}^2-\nu_{p_c \max}^2=S(p_{c\max})$ and $S(p_c)=\mu_{p_c}^2-\nu_{p_c}^2\geq\mu_{p_c \min}^2-\nu_{p_c \min}^2=S(p_{c\min})$.

If $S(p_c)<S(p_{c\max})$ and $S(p_c)>S(p_{c\min})$, then

$$p_{c\min}<\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1},p_{c_2}\ldots,p_{c_m}\right)<p_{c\max}\ \text{for all}\ \widetilde{w}. \tag{57}$$

If $S(p_c)=S(p_{c\max})$, then $\mu_{p_c}^2=\mu^2 p_{c_c \max}$ and $\nu_{p_c}^2=\nu_{p_c \max}^2$.

Thus,
$H(p_c)=\mu_{p_c}^2+\nu_{p_c}^2=\mu_{p_c \max}^2+\nu_{p_{c\max}}^2=H(p_{c\max})$.
Then, we have

$$\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1},p_{c_2},\ldots,p_{c_m}\right)=p_{c\max},\quad \text{for every}\ \widetilde{w}. \tag{58}$$

If $S(p_c)=S(p_{c\min})$, then $\mu_{p_c}^2-\nu_{p_c}^2=\mu_{p_c \min}^2-\nu_{p_c \min}^2$, then $\mu_{p_c}^2=\mu_{p_c \min}^2$ and $\nu_{p_c}^2=\nu_{p_c \min}^2$. Thus, $H(p_c)=\mu_{p_c}^2+\nu_{p_c}^2=\mu_{p_c \min}^2+\nu_{p_{c\min}}^2=H(p_{c\min})$. Then, we have

$$\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}\left(p_{c_1},p_{c_2},\ldots,p_{c_m}\right)<p_{c\min}\ \text{for every}\ \widetilde{w}. \tag{59}$$

Thus, from equations (55) to (57), we have $p_{c\min}<\mathrm{PCFWG}_{\widetilde{w}}^{\varepsilon}(p_{c_1},p_{c_2},\ldots,p_{c_m})<p_{c\max}$ for every $\widetilde{w}$.

(3) Monotonicity:

The proof follows from (2). □

6. An Application of the Pythagorean Cubic Fuzzy Einstein Weighted Geometric (PCFEWG) Aggregation Operator to Group Decision-Making Problems

In this unit, we develop an application of Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operator to multicriteria decision-making problem.

Algorithm. Let $F=\{F_1,F_2,\ldots,F_n\}$ be the set of n alternatives, $H=\{H_1,H_2,\ldots,H_m\}$ be the set of m attributes, and $\overrightarrow{D}=\left\{\overrightarrow{D}_1,\overrightarrow{D}_2,\ldots,\overrightarrow{D}_k\right\}$ be the set of k decision makers. Let $w=(w_1,w_2,\ldots,w_m)^T$ be the weighted vector of the attributes $H_i\,(i=1,2,\ldots,m)$, such that $w_i\in[0,1]$ and $\sum_{i=1}^{m}w_i=$

1. Let $\overrightarrow{\eta} = (\overrightarrow{\eta}_1, \overrightarrow{\eta}_2, \overrightarrow{\eta}_3, \ldots, \overrightarrow{\eta}_k)^T$ be the weighted vector of the decision makers $\overrightarrow{D}_s (s = 1, 2, \ldots, k)$, such that $\overrightarrow{\eta}_s \in [0, 1]$ and $\sum_{s=1}^{k} \overrightarrow{\eta}_s = 1$. This method has the following steps:

Step 1. In this step, we construct the Pythagorean cubic fuzzy decision-making matrices, $\overrightarrow{D}^s = [\overrightarrow{\alpha}_{ji}^{(s)}]_{n\times m} (s = 1, 2, \ldots, k)$. If the criteria have two types, such as benefit criteria and cost criteria, then the Pythagorean cubic fuzzy decision matrices, $\overrightarrow{D}^s = [\overrightarrow{\alpha}_{ji}^{(s)}]_{n\times m}$ can be converted into the normalized Pythagorean cubic fuzzy decision matrices, $\overrightarrow{R}^s = (\varepsilon^{(s)})_{m\times n}$, where $\varepsilon_{ij}^{(s)}$. If all the criteria have the same type, then there is no need of normalization.

Step 2. We use the Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operator to aggregate all the individual normalized Pythagorean cubic fuzzy decision matrices, $\overrightarrow{R}^s = (\varepsilon^{(s)})_{m\times n} (s = 1, 2, \ldots, k)$, into the single Pythagorean cubic fuzzy decision matrix, $\overrightarrow{R} = (\varepsilon_{ij})_{m\times n}$.

Step 3. We aggregate all the preference values $\varepsilon_{ij} (j = 1, 2, 3, \ldots, n, i = 1, 2, \ldots, m)$ by using the PCFEWG operator and get the overall preference values $\varepsilon_j (j = 1, 2, \ldots, n)$ corresponding to the alternatives $F_j (j = 1, \ldots, n)$.

Step 4. We calculate the scores of $\varepsilon_{ij} (j = 1, 2, 3, \ldots, n)$. If there is no difference between two or more than two scores, then we must have to find out the accuracy degrees of the collective overall preference values.

Step 5. We arrange the scores of all the alternatives in the form of descending order and select that alternative which has the highest score function.

7. Numerical Example

In Pakistan's stock exchange, listed Internet companies play an important role. The performance of listed companies affects capital market resource allocation and has become a common concern of shareholders, creditors, government bodies, and other stakeholders. An investment firm would like to invest a sum of money in stocks on the Internet. So, the investment bank employs three kinds of experts to determine the possible investment value: market maker, dealer, and finder. Three Internet stocks are chosen in which the earnings ratio is higher than other stocks: (1) is PTCL; (2) is NayaTel; (3) is Wi-Tribe out of three characteristics: (1) is the trend in the stock market; (2) is in the course of policy; (3) is the annual results. About the attributes Aj ($j = 1, 2, 3$), the three experts test Internet stocks xi (I = 1, 2, 3) and create the following three Pythagorean cubic fuzzy decision matrices in Table 1. Tablesss 2 and 3 display the expert weights and attribute weights, which all take the form of PCFEs, respectively. Then, to get the most desirable alternative(s), which includes the following steps, we use the approach developed in Section 6:

Step 1. The decision maker gives his decision in Tables 1–3.

Step 2. We apply the Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operator to aggregate all the individual normalized Pythagorean cubic fuzzy decision matrices $\overrightarrow{R}^s = (\varepsilon^{(s)})_{m\times n}$, $(s = 1, 2, \ldots, k)$, into the single Pythagorean cubic fuzzy decision matrix, $\overrightarrow{R} = (\varepsilon_{ij})_{m\times n}$.

Aggregated Pythagorean cubic fuzzy decision matrix D_1,

$$\begin{aligned} X_1 &= ([.4660, .6120], .7323; [.5760, .7056], .5726),\\ X_2 &= ([.5780, .7306], .5360; [.4810, .6235], .7283),\\ X_3 &= ([.6662, .7908], .5691; [.4661, .5657], .7381). \end{aligned} \tag{60}$$

Aggregated Pythagorean cubic fuzzy decision matrix D_2,

$$\begin{aligned} X_1 &= ([.5700, .6742], .6364; [.5532, .7443], .5563),\\ X_2 &= ([.5095, .6564], .5333; [.5972, .6994], .5637),\\ X_3 &= ([.6733, .7002], .6155; [.4523, .5812], .5810). \end{aligned} \tag{61}$$

Aggregated Pythagorean cubic fuzzy decision matrix D_3,

$$\begin{aligned} X_1 &= ([.6204, .7207], .5655; [.4523, .5810], .5703),\\ X_2 &= ([.6784, .7788], .7024; [.4233, .5232], .5434),\\ X_3 &= ([.6291, .7306], .4948; [.4765, .6040], .6994). \end{aligned} \tag{62}$$

Step 3. We aggregate all the preference values, which are

$$\begin{aligned} X_1 &= ([.6808, .7692], .7472; [.4468, .5859], .4777),\\ X_2 &= ([.7041, .8039], .7070; [.4312, .5310], .5221),\\ X_3 &= ([.7591, .8173], .6866; [.3902, .4940], .5743). \end{aligned} \tag{63}$$

Step 4. We calculate the scores of $X_j (j = 1, 2, 3)$.

$$\begin{aligned} S(X_1) &= .0207,\\ S(X_2) &= .0532,\\ S(X_3) &= .0773. \end{aligned} \tag{64}$$

Step 5. We organize the scores of the alternatives in descending order and choose the highest score function. Hence, $X_3 > X_2 > X_1$. Thus, the most wanted alternative is X_3.

8. Comparison Analysis

The same numerical example is solved by using other aggregation operators, including IFEWG (intuitionistic fuzzy Einstein weighted geometric) operator, IFEOWG (intuitionistic fuzzy Einstein ordered weighted geometric) operator, PFEWG (picture fuzzy Einstein weighted geometric) operator, PFEOWG (picture fuzzy Einstein ordered weighted geometric) operator, PyFEWG (Pythagorean fuzzy

Table 1: Decision matrix $C$by decision maker D_1.

	A_1	A_2	A_3	A_4
X_1	$\begin{pmatrix}\langle[.5,.7];.5\rangle\\\langle[.4,.5];.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.3,.4];.8\rangle\\\langle[.8,.9];.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.5,.7];.8\rangle\\\langle[.4,.6];.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.7];.8\rangle\\\langle[.5,.6];.5\rangle\end{pmatrix}$
X_2	$\begin{pmatrix}\langle[.5,.6];.8\rangle\\\langle[.4,.7];.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.7];.6\rangle\\\langle[.5,.6];.7\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.8];.2\rangle\\\langle[.5,.6];.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.8];.7\rangle\\\langle[.5,.6];.6\rangle\end{pmatrix}$
X_3	$\begin{pmatrix}\langle[.8,.9];.3\rangle\\\langle[.3,.4];.9\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.7,.8];.9\rangle\\\langle[.5,.6];.3\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.8];.5\rangle\\\langle[.5,.6];.7\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.7];.6\rangle\\\langle[.5,.6];.8\rangle\end{pmatrix}$

Table 2: Decision matrix $C$by decision maker D_2.

	A_1	A_2	A_3	A_4
X_1	$\begin{pmatrix}\langle[.5,.6];.8\rangle\\\langle[.7,.8];.5\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.4,.5];.8\rangle\\\langle[.7,.9];.5\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.7,.8];.4\rangle\\\langle[.4,.5];.7\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.7,.8];.6\rangle\\\langle[.3,.6];.5\rangle\end{pmatrix}$
X_2	$\begin{pmatrix}\langle[.7,.8];.4\rangle\\\langle[.4,.5];.7\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.4,.5];.7\rangle\\\langle[.7,.8];.5\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.5,.8];.7\rangle\\\langle[.6,.7];.4\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.5,.6];.4\rangle\\\langle[.5,.7];.6\rangle\end{pmatrix}$
X_3	$\begin{pmatrix}\langle[.7,.8];.6\rangle\\\langle[.4,.5];.5\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.7];.7\rangle\\\langle[.5,.6];.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.7,.5];.7\rangle\\\langle[.5,.7];.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.7,.8];.5\rangle\\\langle[.4,.5];.6\rangle\end{pmatrix}$

Table 3: Decision matrix $C$by decision maker D_3.

	A_1	A_2	A_3	A_4
X_1	$\begin{pmatrix}\langle[.7,.8];.4\rangle\\\langle[.4,.5];.7\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.7];.4\rangle\\\langle[.5,.6];.7\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.7];.8\rangle\\\langle[.5,.6];.4\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.7];.7\rangle\\\langle[.4,.6];.4\rangle\end{pmatrix}$
X_2	$\begin{pmatrix}\langle[.6,.7];.7\rangle\\\langle[.5,.6];.6\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.7,.8];.5\rangle\\\langle[.4,.5];.7\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.7,.8];.7\rangle\\\langle[.4,.5];.5\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.7,.8];.9\rangle\\\langle[.4,.5];.3\rangle\end{pmatrix}$
X_3	$\begin{pmatrix}\langle[.5,.6];.6\rangle\\\langle[.6,.7];.5\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.6,.7];.5\rangle\\\langle[.5,.6];.8\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.7,.8];.4\rangle\\\langle[.4,.5];.7\rangle\end{pmatrix}$	$\begin{pmatrix}\langle[.7,.8];.5\rangle\\\langle[.4,.6];.7\rangle\end{pmatrix}$

Table 4: Final ranking comparative study with existing aggregation operators.

Models	Aggegation operators	Ranking
IFEWG	Intuitionistic fuzzy Einstein weighted geometric [18]	$X_3 > X_1 > X_2$
IFEOWG	Intuitionistic fuzzy Einstein ordered weighted geometric [18]	$X_3 > X_1 > X_2$
PFEWG	Picture fuzzy Einstein weighted geometric [28]	$X_3 > X_2 > X_1$
PFEOWG	Picture fuzzy Einstein ordered weighted geometric [28]	$X_3 > X_2 > X_1$
PyFEWG	Pythagorean fuzzy Einstein weighted geometric [29]	$X_3 > X_2 > X_1$
PyFEOWG	Pythagorean fuzzy Einstein ordered weighted geometric [29]	$X_3 > X_2 > X_1$
ICFEWG	Intuitionistic cubic fuzzy Einstein weighted geometric [22]	$X_3 > X_2 > X_1$
ICFEOWG	Intuitionistic cubic fuzzy Einstein ordered weighted geometric [22]	$X_3 > X_2 > X_1$
PCFEWG	Pythagorean cubic fuzzy Einstein weighted geometric (proposed)	$X_3 > X_2 > X_1$
PCFEOWG	Pythagorean cubic fuzzy Einstein ordered weighted geometric (proposed)	$X_3 > X_2 > X_1$

Einstein weighted geometric) operator, PyFEOWG (Pythagorean fuzzy Einstein ordered weighted geometric) operator, ICFEWG (intuitionistic cubic fuzzy Einstein weighted geometric) operator, ICFEOWG (intuitionistic cubic fuzzy Einstein ordered weighted geometric) operator, CPFEWG (cubic picture fuzzy Einstein weighted geometric) operator, and CPFEOWG (cubic picture fuzzy Einstein ordered weighted geometric) operator to demonstrate the efficiency and eminent benefits of the proven aggregation operators, by ignoring the additional preference matrix in some existing operators. Different aggregation operators have distinct strategic classifications so that, in compliance with their consultation, they may retain a small disparity. By contrast, the appropriate choice developed by any aggregation operator is important and recognizes the proposed solution's feasibility and effectiveness of aggregation operators. Table 4 gives a comparative study of the final rankings of all aggregation operators.

9. Conclusion

We introduced the Pythagorean cubic fuzzy set, which is a generalization of the interval-valued Pythagorean fuzzy set, in this paper. Einstein's Pythagorean cubic fuzzy weighted geometric operator has been described (PCFEWG). We also discussed some of the fundamental properties of this operator, such as idempotency, boundary, and monotonicity. The Pythagorean cubic fuzzy Einstein weighted geometric (PCFEWG) operator was then used to deal with different parameters for decision-making problems under Pythagorean cubic fuzzy details. We developed a multicriteria decision-making algorithm for Pythagorean cubic fuzzy Einstein weighted geometric problems (PCFEWG). Finally, we put together a numerical example of a decision-making problem.

In future, we can extend this concept for spherical cubic fuzzy sets and their application in multicriteria group decision-making, pattern recognition, and cluster analysis. We can also extend Pythagorean cubic fuzzy sets for various aggregation operators such as Hamacher, Dombi, Haronian mean, Bonferroni mean, TOPSIS, and their applications in group decision-making.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: More on $\mathscr{D}\alpha$-Closed Sets in Topological Spaces

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

More on $\mathscr{D}\alpha$-Closed Sets in Topological Spaces

Xiao-Yan Gao[1] and Ahmed Mostafa Khalil[2]

[1]*School of Mathematics and Statistics, Yulin University, Yulin 719000, China*
[2]*Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt*

Correspondence should be addressed to Ahmed Mostafa Khalil; a.khalil@azhar.edu.eg

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The aim of this paper is to present and study topological properties of $\mathscr{D}\alpha$-derived, $\mathscr{D}\alpha$-border, $\mathscr{D}\alpha$-frontier, and $\mathscr{D}\alpha$-exterior of a set based on the concept of $\mathscr{D}\alpha$-open sets. Then, we introduce new separation axioms (i.e., $\mathscr{D}\alpha - R_0$ and $\mathscr{D}\alpha - R_1$) by using the notions of $\mathscr{D}\alpha$-open set and $\mathscr{D}\alpha$-closure. The space of $\mathscr{D}\alpha - R_0$ (resp., $\mathscr{D}\alpha - R_1$) is strictly between the spaces of $\alpha - R_0$ (resp., $\alpha - R_1$) and $g - R_0$ (resp., $g - R_1$). Further, we present the notions of $\mathscr{D}\alpha$-kernel and $\mathscr{D}\alpha$-convergent to a point and discuss the characterizations of interesting properties between $\mathscr{D}\alpha$-closure and $\mathscr{D}\alpha$-kernel. Finally, several properties of weakly $\mathscr{D}\alpha - R_0$ space are investigated.

1. Introduction and Preliminaries

Many researchers (see [1–9]) were interested in general topology-like family (e.g., the family of all α-open sets) and also the notion of generalized closed (briefly, g-closed) subset of a topological space [10–14]. In 1982, Dunham [14] used the generalized closed sets to define a novel closure operator and consequently a novel topology τ^*, on the space, and discussed several of the properties of this novel topology. Sayed and Khalil [15] introduced and studied a novel type of sets called $\mathscr{D}\alpha$-open sets in topological spaces and studied the notions of $\mathscr{D}\alpha$-continuous, $\mathscr{D}\alpha$-open, and $\mathscr{D}\alpha$-closed functions between topological spaces. Further, they investigated several properties of $\mathscr{D}\alpha$-closed and strongly $\mathscr{D}\alpha$-closed graphs. In fact, research on spaces analogous to topological spaces and generalized closed sets among topological spaces may have certain driving effect on research on theory of rough set, soft set, spatial reasoning, implicational spaces and knowledge spaces, and logic (see [16–18]). For this reason, we will define the notions of $\mathscr{D}\alpha$-derived, $\mathscr{D}\alpha$-border, $\mathscr{D}\alpha$-frontier, and $\mathscr{D}\alpha$-exterior of a set based on the notion of $\mathscr{D}\alpha$-open sets. We will also discuss new separation axioms ($\mathscr{D}\alpha - R_0$ and $\mathscr{D}\alpha - R_1$) by using the notions of $\mathscr{D}\alpha$-open set and $\mathscr{D}\alpha$-closure operator.

The rest of this article is arranged as follows. In this section, we briefly recall several notions: α-open set, an α-closed set, generalized open set, generalized closed set, $\alpha - R_0$ space, $g - R_0$ space, $\alpha - R_1$ space, $g - R_0$ space, α-derived, α-border, α-frontier, and α-exterior of a set, which are used in the sequel. In Section 2, we define the notions of $\mathscr{D}\alpha$-derived, $\mathscr{D}\alpha$-border, $\mathscr{D}\alpha$-frontier, and $\mathscr{D}\alpha$-exterior of a set based on $\mathscr{D}\alpha$-open sets. In Section 3, we present the notions $\mathscr{D}\alpha - R_0$, $\mathscr{D}\alpha - R_1$, $\mathscr{D}\alpha$-kernel, and $\mathscr{D}\alpha$-convergent to a point and introduce the characterizations of interesting properties between $\mathscr{D}\alpha$-closure and $\mathscr{D}\alpha$-kernel. In Section 4, we define the weakly $\mathscr{D}\alpha - R_0$ space and investigate some properties of weakly $\mathscr{D}\alpha - R_0$ space.

Throughout the present paper, two subsets A of a space (X, τ), $\mathscr{C}(A)$ and $\mathscr{I}(A)$, denote the closure and the interior of A, respectively. Since we require the following known definitions, notations, and some properties, we recall in this section.

Definition 1. Let (X, τ) be a topological space and $A \subseteq X$. Then,

(1) A is α-open [1] if $A \subseteq \mathscr{I}\mathscr{C}\mathscr{I}(A)$ and α-closed [1] if $\mathscr{C}(\mathscr{I}(\mathscr{C}(A))) \subseteq A$

(2) A is generalized closed (briefly, g-closed) [10] if $\mathscr{C}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X

(3) A is generalized open (briefly, g-open) [10] if $X \backslash A$ is g-closed

(4) A is $\mathscr{D}\alpha$-open [15] if $A \subseteq \mathscr{I}^* \mathscr{C} \mathscr{I}^*(A)$ and $D\alpha$-closed [15] if $\mathscr{C}^*(\mathscr{I}(\mathscr{C}^*(A))) \subseteq A$

The α-closure of a subset A of X [2] is the intersection of all α-closed sets containing A and is denoted by $\mathscr{C}_\alpha(A)$. The α-interior of a subset A of X [2] is the union of all α-open sets contained in A and is denoted by $\mathscr{I}_\alpha(A)$. The intersection of all g-closed sets containing A [14] is called the g-closure of A and is denoted by $\mathscr{C}^*(A)$ and the g-interior of A [19] is the union of all g-open sets contained in A and is denoted by $\mathscr{I}^*(A)$. The intersection of all $\mathscr{D}\alpha$-closed sets containing A [15] is called the $\mathscr{D}\alpha$-closure of A and is denoted by $\mathscr{C}_\alpha^{\mathscr{D}}(A)$ and the $\mathscr{D}\alpha$-interior of A [15] is the union of all $\mathscr{D}\alpha$-open sets contained in A and is denoted by $\mathscr{I}_\alpha^{\mathscr{D}}(A)$.

We need the following notations:

(i)$\alpha O(X)$ (resp., $\alpha\mathscr{C}(X)$) denotes the family of all α-open sets (resp., α-closed sets) in (X, τ)

(ii) $GO(X)$ (resp., $G\mathscr{C}(X)$) denotes the family of all generalized open sets (resp., generalized closed sets) in (X, τ)

(iii) $\mathscr{D}\alpha O(X)$ (resp., $\mathscr{D}\alpha\mathscr{C}(X)$) denotes the family of all $\mathscr{D}\alpha$-open sets (resp., $\mathscr{D}\alpha$-closed sets) in (X, τ)

(iv) $\alpha O(X, x) = \{U | x \in U \in \alpha O(X, \tau)\}$, $O(X, x) = \{U | x \in U \in \tau\}$, and $\alpha\mathscr{C}(X, x) = \{U | x \in U \in \alpha\mathscr{C}(X, \tau)\}$

(v) $\mathscr{D}\alpha O(X, x) = \{U | x \in U \in \mathscr{D}\alpha O(X, \tau)\}$ and $\mathscr{D}\alpha\mathscr{C}(X, x) = \{U | x \in U \in \mathscr{D}\alpha\mathscr{C}(X, \tau)\}$

Definition 2. A topological space (X, τ) is said to be

(1) $\alpha - R_0$ space [20] (resp., $g - R_0$ space [21]) if every α-open (resp., g-open) set contains the α-closure (resp., g-closure) of each of its singletons

(2) $\alpha - R_1$ space [20] (resp., $g - R_1$ space [21]) if, for x, y in X with $\mathscr{C}_\alpha(\{x\}) \neq \mathscr{C}_\alpha(\{y\})$ (resp., $\mathscr{C}^*(\{x\}) \neq \mathscr{C}^*(\{y\})$), there exist disjoint α-open (resp., g-open) sets U and V such that $\mathscr{C}_\alpha(\{x\})$ (resp., $\mathscr{C}^*(\{x\})$) is a subset of U and $\mathscr{C}_\alpha(\{y\})$ (resp., $\mathscr{C}^*(\{y\})$) is a subset of V

Definition 3 (see [22]). A point $x \in X$ is said to be α-limit point of A in topological space (X, τ) if, for each α-open set U containing $x, U \cap (A \smallsetminus \{x\}) \neq \phi$. The set of all α-limit points of A is called an α-derived set of A.

Definition 4 (see [22]). Let A be a subset of a space X:

(1) An α-border of A is defined by $b_\alpha(A) = A \smallsetminus \mathscr{I}_\alpha(A)$

(2) An α-frontier of A is defined by $F_\alpha(A) = \mathscr{C}_\alpha(A) \smallsetminus \mathscr{I}_\alpha(A)$

(3) An α-exterior of A is defined by $\mathrm{Ext}_\alpha(A) = \mathscr{I}_\alpha(X \smallsetminus A)$

2. A $\mathscr{D}\alpha$-Derived, $\mathscr{D}\alpha$-Border, $\mathscr{D}\alpha$-Frontier, and $\mathscr{D}\alpha$-Exterior of a Set

Definition 5. Let A be a subset of a space X. A point $x \in X$ is said to be $\mathscr{D}\alpha$-limit point of A if it satisfies the following assertion:

$$\forall U \in \mathscr{D}\alpha O(X) (x \in U \Rightarrow U \cap (A \smallsetminus \{x\}) \neq \phi) \quad (1)$$

The set of all $\mathscr{D}\alpha$-limit points of A is called a $\mathscr{D}\alpha$-derived set of A and is denoted by $d_\alpha^{\mathscr{D}}(A)$.

Note that, for a subset A of X, a point $x \in X$ is not a $\mathscr{D}\alpha$-limit point of A if and only if there exists a $\mathscr{D}\alpha$-open set U in X such that

$$x \in U, U \cap (A \smallsetminus \{x\}) = \phi, \quad (2)$$

or, equivalently,

$$x \in U, U \cap A = \phi, U \cap A = \{x\}, \quad (3)$$

or equivalently,

$$x \in U, U \cap A \subseteq \{x\}. \quad (4)$$

Theorem 1. *Let A and B be subsets of a topological space X. Then, the following results hold:*

(1) $d_\alpha^{\mathscr{D}}(A) \subseteq d_\alpha(A)$, where $d_\alpha(A)$ is the α-derived set ([22], Definition 2.1) of A

(2) If $A \subseteq B$, then $d_\alpha^{\mathscr{D}}(A) \subseteq d_\alpha^{\mathscr{D}}(B)$

(3) $d_\alpha^{\mathscr{D}}(A) \cup d_\alpha^{\mathscr{D}}(B) \subseteq d_\alpha^{\mathscr{D}}(A \cup B)$ and $d_\alpha^{\mathscr{D}}(A \cap B) \subseteq d_\alpha^{\mathscr{D}}(A) \cap d_\alpha^{\mathscr{D}}(B)$

(4) $d_\alpha^{\mathscr{D}}(d_\alpha^{\mathscr{D}}(A)) \smallsetminus A \subseteq d_\alpha^{\mathscr{D}}(A)$

(5) $d_\alpha^{\mathscr{D}}(A \cup d_\alpha^{\mathscr{D}}(A)) \subseteq A \cup d_\alpha^{\mathscr{D}}(A)$

Proof. (1) It follows from ([15], Theorem 3.6 (i)).

(2) Let $x \in d_\alpha^{\mathscr{D}}(A)$ and $U \in \mathscr{D}\alpha O(X)$ with $x \in U$. Then $(U \cap A) \smallsetminus \{x\} \neq \phi$. Since $A \subseteq B$, it follows that $(U \cap B) \smallsetminus \{x\} \neq \phi$. Therefore $x \in d_\alpha^{\mathscr{D}}(B)$.

(3) It follows from (2) above.

(4) Let $x \in d_\alpha^{\mathscr{D}}(d_\alpha^{\mathscr{D}}(A)) \smallsetminus A$ and $U \in \mathscr{D}\alpha O(X)$ with $x \in U$. Then $U \cap (d_\alpha^{\mathscr{D}}(A) \smallsetminus \{x\}) \neq \phi$. Let $y \in U \cap (d_\alpha^{\mathscr{D}}(A) \smallsetminus \{x\})$. Then $y \in U$ and $y \in d_\alpha^{\mathscr{D}}(A)$, and so $U \in (A \smallsetminus \{y\}) \neq \phi$. If we take $z \in U \cap (A \smallsetminus \{y\})$, then $z \neq x$ for $z \in A$ and $x \notin A$. Hence, $U \in (A \smallsetminus \{y\}) \neq \phi$. Therefore $x \in d_\alpha^{\mathscr{D}}(A)$.

(5) Let $x \in d_\alpha^{\mathscr{D}}(A \cup d_\alpha^{\mathscr{D}}(A))$. If $x \in A$, the result is obvious. Suppose that $x \notin A$. Then $U \cap ((A \cup d_\alpha^{\mathscr{D}}(A)) \smallsetminus \{x\}) \neq \phi$ for all $U \in \mathscr{D}\alpha O(X)$ with $x \in U$. Hence, $(U \cap A) \smallsetminus \{x\} \neq \phi$ or $U \cap (d_\alpha^{\mathscr{D}}(A) \smallsetminus \{x\}) \neq \phi$. The first case implies that $x \in d_\alpha^{\mathscr{D}}(A)$. If $U \cap (d_\alpha^{\mathscr{D}}(A) \smallsetminus \{x\}) \neq \phi$, then $x \in d_\alpha^{\mathscr{D}}(d_\alpha^{\mathscr{D}}(A))$. Since $x \notin A$, it follows similarly

from (4) that $x \in d_\alpha^{\mathscr{D}}(d_\alpha^{\mathscr{D}}(A)) \setminus A \subseteq d_\alpha^{\mathscr{D}}(A)$. Therefore, $d_\alpha^{\mathscr{D}}(A \cup d_\alpha^{\mathscr{D}}(A)) \subseteq A \cup d_\alpha^{\mathscr{D}}(A)$ holds. □

Theorem 2. *Let A be a subset of a topological space X. Then* $\mathscr{C}_\alpha^{\mathscr{D}}(A) = A \cup d_\alpha^{\mathscr{D}}(A)$.

Proof. Let $x \in \mathscr{C}_\alpha^{\mathscr{D}}(A)$. If $x \in A$, then the proof is complete. If $x \notin A$ and $U \in \mathscr{D}\alpha O(X)$ with $x \in U$, then $(U \cap A) \setminus \{x\} \neq \phi$, and so $x \in d_\alpha^{\mathscr{D}}(A)$. Hence, $\mathscr{C}_\alpha^{\mathscr{D}}(A) \subseteq A \cup d_\alpha^{\mathscr{D}}(A)$. The converse follows from ([15], Theorem 2.14 (i)) and $d_\alpha^{\mathscr{D}}(A) \subseteq \mathscr{C}_\alpha^{\mathscr{D}}(A)$. Thus, $A \cup d_\alpha^{\mathscr{D}}(A) \subseteq \mathscr{C}_\alpha^{\mathscr{D}}(A)$. Therefore $\mathscr{C}_\alpha^{\mathscr{D}}(A) = A \cup d_\alpha^{\mathscr{D}}(A)$. □

Corollary 1. *A subset A is a $\mathscr{D}\alpha$-closed set if and only if it contains the set of the $\mathscr{D}\alpha$-limit points*

Theorem 3. *Let A and B be subsets of X. If A is $\mathscr{D}\alpha$-closed, then* $\mathscr{C}_\alpha^{\mathscr{D}}(A \cap B) \subseteq A \cap \mathscr{C}_\alpha^{\mathscr{D}}(B)$.

Proof. It follows from Theorems 2.13 and 2.14 (vi) in [15]. □

Lemma 1. *Let A be a subset of a topological space X. If A is $\mathscr{D}\alpha$-closed set, then* $d_\alpha^{\mathscr{D}}(A) \subseteq A$.

Proof. Suppose that A is a $\mathscr{D}\alpha$-closed set. Let $x \notin A$; that is, $x \in X \setminus A$. Since it is a $\mathscr{D}\alpha$-open, x is not a $\mathscr{D}\alpha$-limit point of A, that is, $x \notin d_\alpha^{\mathscr{D}}(A)$, because $(X \setminus A) \cap (A \setminus \{x\}) = \phi$. Hence, $d_\alpha^{\mathscr{D}}(A) \subseteq A$. □

Theorem 4. *Let A be a subset of a topological space X. If F is a $\mathscr{D}\alpha$-closed set of A, then* $d_\alpha^{\mathscr{D}}(A) \subseteq F$.

Proof. By Theorem 1 (2) and Lemma 1, $A \subseteq F$ implies that $d_\alpha^{\mathscr{D}}(A) \subseteq d_\alpha^{\mathscr{D}}(F) \subseteq F$. □

Theorem 5. *Let A be a subset of a topological space X. If a point $x \in X$ is a $\mathscr{D}\alpha$-limit point of A, then x is also a $\mathscr{D}\alpha$-limit point of $A \setminus \{x\}$.*

Proof. The proof is obvious. □

Definition 6. Let A be a subset of a topological space X. The $\mathscr{D}\alpha$-border of A, denoted by $b_\alpha^{\mathscr{D}}(A)$, is defined as $b_\alpha^{\mathscr{D}}(A) = A \setminus \mathscr{I}_\alpha^{\mathscr{D}}(A)$.

Theorem 6. *Let A be a subset of a topological space X. Then, the following results hold:*

(1) $b_\alpha^{\mathscr{D}}(A) \subseteq b_\alpha(A)$, *where $b_\alpha(A)$ is the α-border ([22], Definition 2.8) of A*
(2) $A = \mathscr{I}_\alpha^{\mathscr{D}}(A) \cup b_\alpha^{\mathscr{D}}(A)$
(3) $\mathscr{I}_\alpha^{\mathscr{D}}(A) \cap b_\alpha^{\mathscr{D}}(A) = \phi$
(4) *A is a $\mathscr{D}\alpha$-open set if and only if* $b_\alpha^{\mathscr{D}}(A) = \phi$
(5) $b_\alpha^{\mathscr{D}}(\mathscr{I}_\alpha^{\mathscr{D}}(A)) = \phi$
(6) $b_\alpha^{\mathscr{D}}(A) = A \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \setminus A)$
(7) $b_\alpha^{\mathscr{D}}(A) = A \cap d_\alpha^{\mathscr{D}}(X \setminus A)$

Proof. (1) Since $\mathscr{I}_\alpha(A) \subseteq \mathscr{I}_\alpha^{\mathscr{D}}(A)$ ([1], Theorem 3.15 (i)), we have

$$b_\alpha^{\mathscr{D}}(A) = A \setminus \mathscr{I}_\alpha^{\mathscr{D}}(A) \subseteq A \setminus \mathscr{I}_\alpha(A) = b_\alpha(A), \tag{5}$$

(2) and (3) are obvious.

(4) Itfollows from Theorems 3.14 and 3.15 (i) in [15].

(5) Since $\mathscr{I}_\alpha^{\mathscr{D}}(A)$ is a $\mathscr{D}\alpha$-open, it follows from (4) that $b_\alpha^{\mathscr{D}}(\mathscr{I}_\alpha^{\mathscr{D}}(A)) = \phi$.

(6) Using ([15], Lemma 3.13 (i)), we have

$$\begin{aligned} b_\alpha^{\mathscr{D}}(A) &= A \setminus \mathscr{I}_\alpha^{\mathscr{D}}(A) = A \setminus \left(X \setminus \mathscr{C}_\alpha^{\mathscr{D}}(X \setminus A)\right) \\ &= A \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \setminus A). \end{aligned} \tag{6}$$

(7) Applying (6) and Theorem 3, we have

$$\begin{aligned} b_\alpha^{\mathscr{D}}(A) &= A \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \setminus A) = A \cap \left((X \setminus A) \cup d_\alpha^{\mathscr{D}}(X \setminus A)\right) \\ &= A \cap d_\alpha^{\mathscr{D}}(X \setminus A). \end{aligned} \tag{7}$$

□

The converse of (1) of Theorem 6 is not true in general as shown in the following example.

Example 1. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$. Then, $F_X = \{\phi, \{b, c\}, X\}$, $\alpha O(X) = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\alpha\mathscr{C}(X) = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}$, $\mathscr{D}\alpha O(X) = \mathscr{D}\alpha\mathscr{C}(X) = P(X)$. Let $A = \{c\}$. Then $b_\alpha(A) = \{c\} \nsubseteq b_\alpha^{\mathscr{D}}(A) = \phi$.

Definition 7. Let A be a subset of a topological space X. The $\mathscr{D}\alpha$-frontier of A, denoted by $Fr_\alpha^{\mathscr{D}}(A)$, is defined as $Fr_\alpha^{\mathscr{D}}(A) = \mathscr{C}_\alpha^{\mathscr{D}}(A) \setminus \mathscr{I}_\alpha^{\mathscr{D}}(A)$.

Lemma 2. *Let A be a subset of X. If A is a $\mathscr{D}\alpha$-closed subset of X, then* $b_\alpha^{\mathscr{D}}(A) = Fr_\alpha^{\mathscr{D}}(A)$.

Proof. It follows from ([15], Theorem 2.13). □

Theorem 7. *Let A be a subset of a topological space X. Then the following results hold:*

(1) $Fr_\alpha^{\mathscr{D}}(A) \subseteq Fr_\alpha(A)$, *where $Fr_\alpha(A)$ is the α-frontier ([22], Definition 2.11) of A.*
(2) $\mathscr{C}_\alpha^{\mathscr{D}}(A) = \mathscr{I}_\alpha^{\mathscr{D}}(A) \cup Fr_\alpha^{\mathscr{D}}(A)$.
(3) $\mathscr{I}_\alpha^{\mathscr{D}}(A) \cap Fr_\alpha^{\mathscr{D}}(A) = \phi$.
(4) $b_\alpha^{\mathscr{D}}(A) \subseteq Fr_\alpha^{\mathscr{D}}(A)$.
(5) $Fr_\alpha^{\mathscr{D}}(A) = b_\alpha^{\mathscr{D}}(A) \cup (d_\alpha^{\mathscr{D}}(A) \setminus \mathscr{I}_\alpha^{\mathscr{D}}(A))$.
(6) *If A is a $\mathscr{D}\alpha$-open set, then* $Fr_\alpha^{\mathscr{D}}(A) = b_\alpha^{\mathscr{D}}(X \setminus A)$.
(7) $Fr_\alpha^{\mathscr{D}}(A) = \mathscr{C}_\alpha^{\mathscr{D}}(A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \setminus A)$.
(8) $Fr_\alpha^{\mathscr{D}}(A) = Fr_\alpha^{\mathscr{D}}(X \setminus A)$.
(9) $Fr_\alpha^{\mathscr{D}}(A)$ *is a $\mathscr{D}\alpha$-closed set.*
(10) $Fr_\alpha^{\mathscr{D}}(Fr_\alpha^{\mathscr{D}}(A)) \subseteq Fr_\alpha^{\mathscr{D}}(A)$.

(11) $Fr_\alpha^{\mathscr{D}}(\mathscr{I}_\alpha^{\mathscr{D}}(A)) \subseteq Fr_\alpha^{\mathscr{D}}(A)$.

(12) $Fr_\alpha^{\mathscr{D}}(\mathscr{C}_\alpha^{\mathscr{D}}(A)) \subseteq Fr_\alpha^{\mathscr{D}}(A)$.

(13) $\mathscr{I}_\alpha^{\mathscr{D}}(A) = A \smallsetminus Fr_\alpha^{\mathscr{D}}(A)$.

(14) $Fr_\alpha^{\mathscr{D}}(A) = \mathscr{C}_\alpha^{\mathscr{D}}(A) \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)$.

(15) $\mathscr{C}_\alpha^{\mathscr{D}}(A) \subseteq A \cup Fr_\alpha^{\mathscr{D}}(A)$.

(16) $X \smallsetminus Fr_\alpha^{\mathscr{D}}(A) = \mathscr{I}_\alpha^{\mathscr{D}}(A) \cup \mathscr{I}_\alpha^{\mathscr{D}}(X \smallsetminus A)$.

Proof

(1) Since $\mathscr{C}_\alpha^{\mathscr{D}}(A) \subseteq \mathscr{C}_\alpha(A)$ ([15], Theorem 2.14 (i)) and $\mathscr{I}_\alpha(A) \subseteq \mathscr{I}_\alpha^{\mathscr{D}}(A)$ ([15], Theorem 3.15 (i)), we have $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \mathscr{C}_\alpha^{\mathscr{D}}(A) \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A) \subseteq \mathscr{C}_\alpha(A) \smallsetminus \mathscr{I}_\alpha(A) = \mathrm{Fr}_\alpha(A)$.

(2) It is obvious.

(3) $\mathscr{I}_\alpha^{\mathscr{D}}(A) \cap \mathrm{Fr}_\alpha^{\mathscr{D}}(A) = (\mathscr{C}_\alpha^{\mathscr{D}}(A) \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)) = \phi$.

(4) Since $A \subseteq \mathscr{C}_\alpha^{\mathscr{D}}(A)$ ([15], Theorem 2.14 (i)), we have $b_\alpha^{\mathscr{D}}(A) = A \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A) \subseteq \mathscr{C}_\alpha^{\mathscr{D}}(A) \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A) = \mathrm{Fr}_\alpha(A)$.

(5) Using Theorem 2, we have

$$\begin{aligned}\mathrm{Fr}_\alpha^{\mathscr{D}}(A) &= \mathscr{C}_\alpha^{\mathscr{D}}(A) \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)\\ &= \left(A \cup d_\alpha^{\mathscr{D}}(A)\right) \cap \left(X \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)\right)\\ &= \left(A \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)\right) \cup \left(d_\alpha^{\mathscr{D}}(A) \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)\right)\\ &= b_\alpha^{\mathscr{D}}(A) \cup \left(d_\alpha^{\mathscr{D}}(A) \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)\right).\end{aligned} \tag{8}$$

(6) It follows from (5) above, Theorem 6 (4), (7), and ([15], Theorem 3.14).

(7) It follows from ([15], Lemma 3.13 (ii)).

(8) It follows from (7) above.

(9) $\mathscr{C}_\alpha^{\mathscr{D}}(\mathrm{Fr}_\alpha^{\mathscr{D}}(A)) = \mathscr{C}_\alpha^{\mathscr{D}}(\mathscr{C}_\alpha^{\mathscr{D}}(A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A)) \subseteq \mathscr{C}_\alpha^{\mathscr{D}}(\mathscr{C}_\alpha^{\mathscr{D}}(A)) \cap \mathscr{C}_\alpha^{\mathscr{D}}(\mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A)) = \mathscr{C}_\alpha^{\mathscr{D}}(A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A) = \mathrm{Fr}_\alpha^{\mathscr{D}}(A)$

Obviously, $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) \subseteq \mathscr{C}_\alpha^{\mathscr{D}}(\mathrm{Fr}_\alpha^{\mathscr{D}}(A))$ ([15], Theorem 2.14 (i)), and so $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \mathscr{C}_\alpha^{\mathscr{D}}(\mathrm{Fr}_\alpha^{\mathscr{D}}(A))$. Hence, $\mathrm{Fr}_\alpha^{\mathscr{D}}(A)$ is a $\mathscr{D}\alpha$-closed set.

(10) It follows from (9) above and Lemma 2.

(11) It follows from Definition 7 and ([15], Theorem 3.15 (vi)).

(12) It follows from Definition 7 and ([15], Theorem 2.14 (vi)).

(13) $A \smallsetminus \mathrm{Fr}_\alpha^{\mathscr{D}} = A \smallsetminus (\mathscr{C}_\alpha^{\mathscr{D}}(A) \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)) = A \cap ((X \smallsetminus \mathscr{C}_\alpha^{\mathscr{D}}(A)) \cup \mathscr{I}_\alpha^{\mathscr{D}}(A)) = \phi \cup (A \cup \mathscr{I}_\alpha^{\mathscr{D}}(A)) = \mathscr{I}_\alpha^{\mathscr{D}}(A)$

(14) It follows from (7) above and ([15], Lemma 3.13 (ii)).

(15) $A \cup \mathrm{Fr}_\alpha^{\mathscr{D}}(A) = A \cup (\mathscr{C}_\alpha^{\mathscr{D}}(A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A)) = (A \cup \mathscr{C}_\alpha^{\mathscr{D}}(A)) \cap (A \cup \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A)) = \mathscr{C}_\alpha^{\mathscr{D}}(A) \cap (A \cup \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A)) \supseteq \mathscr{C}_\alpha^{\mathscr{D}}(A) \cap (A \cup (X \smallsetminus A)) = \mathscr{C}_\alpha^{\mathscr{D}}(A) \cap X = \mathscr{C}_\alpha^{\mathscr{D}}(A)$

(16) $\mathscr{I}_\alpha^{\mathscr{D}}(A) \cup \mathscr{I}_\alpha^{\mathscr{D}}(X \smallsetminus A) = (X \smallsetminus (X \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A))) \cup (X \smallsetminus (X \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(X \smallsetminus A))) = X \smallsetminus ((X \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)) \cap \mathscr{I}_\alpha^{\mathscr{D}}(X \smallsetminus A)) = X \smallsetminus (\mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(A)) = X \smallsetminus \mathrm{Fr}_\alpha^{\mathscr{D}}(A)$ ☐

The converse of (1) and (4) of Theorem 7 is not true as shown in the following examples.

Example 2. Consider the topological space (X, τ) which is given in Example 1. Let $A = \{c\}$. Then $\mathrm{Fr}_\alpha(A) = \{c\} \nsubseteq \mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \phi$.

Example 3. Let (X, τ) be a topological space, where $X = \{a, b, c\}$ and $\tau = \{\phi, \{a, b\}, X\}$. Then $F_X = \{\phi, \{c\}, X, \mathscr{D}\alpha O(X) = \phi, \{a\}, \{b\}, a, b, \{a, c, \{b, c, X, \mathscr{D}\alpha\mathscr{C}(X) = \phi, \{a\}, \{b\}, \{c\}, a, c, b, c, X$. Let $A = \{a, b\}$. Then $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \{c\} \nsubseteq br_\alpha^{\mathscr{D}}(A) = \phi$.

Theorem 8. *Let A be a subset of a topological space X. Then $Fr_\alpha^{\mathscr{D}}(A) = \phi$ if and only if A is a $\mathscr{D}\alpha$-closed set and a $\mathscr{D}\alpha$-open set.*

Proof. Suppose that $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \phi$. First, we prove that A is a $\mathscr{D}\alpha$-closed set. We have $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \phi$ or $\mathscr{C}_\alpha^{\mathscr{D}}(A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A) = \phi$. Hence, $\mathscr{C}_\alpha^{\mathscr{D}}(A) \subseteq X \smallsetminus \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A) = \mathscr{I}_\alpha^{\mathscr{D}}(A)$. Therefore, $\mathscr{C}_\alpha^{\mathscr{D}}(A) \subseteq A$ and so A is a $\mathscr{D}\alpha$-closed set. Now, we prove that A is a $\mathscr{D}\alpha$-open set. Indeed, we have $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \phi$ or $\mathscr{C}_\alpha^{\mathscr{D}}(A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A) = \phi$. Hence, $A \cap (X \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)) = \phi$ and so $A \subseteq \mathscr{I}_\alpha^{\mathscr{D}}(A)$. Therefore, A is a $\mathscr{D}\alpha$-open set. Conversely, suppose that A is a $\mathscr{D}\alpha$-closed set and a $\mathscr{D}\alpha$-open set. Then $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \mathscr{C}_\alpha^{\mathscr{D}}(A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A) = \mathscr{C}_\alpha^{\mathscr{D}}(A) \cap (X \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A)) = A \cap (X \smallsetminus A) = \phi$. ☐

Theorem 9. *Let A be a subset of a topological space X. Then,*

(1) A is a $\mathscr{D}\alpha$-open set if and only if $A \cap Fr_\alpha^{\mathscr{D}}(A) = \phi$;

(2) A is a $\mathscr{D}\alpha$-closed set if and only if $Fr_\alpha^{\mathscr{D}}(A) \subseteq A$.

Proof

(1) Let A be a $\mathscr{D}\alpha$-open set. Then $\mathscr{I}_\alpha^{\mathscr{D}}(A) = A$ implies that $A \cap \mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \mathscr{I}_\alpha^{\mathscr{D}}(A) \cap \mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \phi$ (by Theorem 7 (3)). Conversely, suppose that $A \cap \mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \phi$. Then $A \cap (\mathscr{C}_\alpha^{\mathscr{D}}(A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A)) = \phi$ or $A \cap \mathscr{C}_\alpha^{\mathscr{D}}(A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A) = \phi \Rightarrow A \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A) = \phi$, which implies that $A \subseteq X \smallsetminus \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A) = \mathscr{I}_\alpha^{\mathscr{D}}(A)$. Moreover, $\mathscr{I}_\alpha^{\mathscr{D}}(A) \subseteq A$. Therefore, $\mathscr{I}_\alpha^{\mathscr{D}}(A) = A$ and thus A is a $\mathscr{D}\alpha$-open set.

(2) Let A be a $\mathscr{D}\alpha$-closed set. Then $\mathscr{C}_\alpha^{\mathscr{D}}(A) = A$. Now, $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \mathscr{C}_\alpha^{\mathscr{D}}(A) \cap \mathscr{C}_\alpha^{\mathscr{D}}(X \smallsetminus A) \subseteq \mathscr{C}_\alpha^{\mathscr{D}}(A) = A$. That is, $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) \subseteq A$. Conversely, suppose that $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) \subseteq A$. Then $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) \cap (X \smallsetminus A) = \phi$. Since $\mathrm{Fr}_\alpha^{\mathscr{D}}(A) = \mathrm{Fr}_\alpha^{\mathscr{D}}(X \smallsetminus A)$ (by Theorem 7 (8)), we have $\mathrm{Fr}_\alpha^{\mathscr{D}} X \smallsetminus A \cap X \smallsetminus A = \phi$. By (1), $X \smallsetminus A$ is a $\mathscr{D}\alpha$-open set. Hence, A is a $\mathscr{D}\alpha$-closed set. ☐

Lemma 3. *Let A be a subset of a topological space X. If A is a $\mathscr{D}\alpha$-closed set, then $A \smallsetminus \mathscr{I}_\alpha^{\mathscr{D}}(A) = Fr_\alpha^{\mathscr{D}}(A)$.*

Proof. It follows from ([15], Theorem 2.13) and Theorem 7 (14). ☐

Theorem 10. *Let A and B be subsets of X. Then, the following results hold:*

(1) $Fr_{\alpha}^{\mathscr{D}}(A\cup B)\subseteq Fr_{\alpha}^{\mathscr{D}}(A)\cup Fr_{\alpha}^{\mathscr{D}}(B)$.

(2) $Fr_{\alpha}^{\mathscr{D}}(A\cap B)\subseteq [Fr_{\alpha}^{\mathscr{D}}(A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(B)]\cup [Fr_{\alpha}^{\mathscr{D}}(B)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(A)]$.

(3) $Fr_{\alpha}^{\mathscr{D}}(Fr_{\alpha}^{\mathscr{D}}(Fr_{\alpha}^{\mathscr{D}}(A))) = Fr_{\alpha}^{\mathscr{D}}(Fr_{\alpha}^{\mathscr{D}}(A))$.

Proof.

$$
\begin{aligned}
&(1)\ \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A\cup B) = \mathscr{C}_{\alpha}^{\mathscr{D}}(A\cup B)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus A\cup B)\\
&=\left(\mathscr{C}_{\alpha}^{\mathscr{D}}(A)\cup \mathscr{C}_{\alpha}^{\mathscr{D}}(B)\right)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}((X\smallsetminus A)\cap (X\smallsetminus B))\\
&\subseteq \left(\mathscr{C}_{\alpha}^{\mathscr{D}}(A)\cup \mathscr{C}_{\alpha}^{\mathscr{D}}(B)\right)\cap \left[\mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus B)\right]\\
&=\left(\mathscr{C}_{\alpha}^{\mathscr{D}}(A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus A)\right)\cap \left(\mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus B)\cup \mathscr{C}_{\alpha}^{\mathscr{D}}(B)\right)\cap \left[\mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus B)\right]\\
&=\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus B)\right)\cup \left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(B)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus A)\right)\subseteq \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\cup \mathrm{Fr}_{\alpha}^{\mathscr{D}}(B)\\
&(2)\ \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A\cap B) = \mathscr{C}_{\alpha}^{\mathscr{D}}(A\cap B)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus A\cap B)\\
&\subseteq \left[\mathscr{C}_{\alpha}^{\mathscr{D}}(A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(B)\right]\cap \left[\mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus A)\cup (X\smallsetminus B)\right]\\
&=\left[\mathscr{C}_{\alpha}^{\mathscr{D}}(A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(B)\right]\cap \left[\mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus A)\cup \mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus B)\right]\\
&=\left[\mathscr{C}_{\alpha}^{\mathscr{D}}(A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(B)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus A)\right]\cup \left[\mathscr{C}_{\alpha}^{\mathscr{D}}(A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(B)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(X\smallsetminus B)\right]\\
&=\left[\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}(B)\right]\cup \left[\mathscr{C}_{\alpha}^{\mathscr{D}}(A)\cap \mathrm{Fr}_{\alpha}^{\mathscr{D}}(B)\right]\\
&(3)\ \mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right) = \mathscr{C}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{D}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right)\\
&= \mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right)\ (\mathrm{i}).
\end{aligned}
\tag{9}
$$

Now consider

$$
\begin{aligned}
X\smallsetminus\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right) &= X\smallsetminus\left[\mathscr{C}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right]\\
&= X\smallsetminus\left[\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\cap \mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right]\\
&=\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\cup \left(X\smallsetminus \mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right).
\end{aligned}
\tag{10}
$$

Therefore,

$$
\begin{aligned}
\mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right) &= \mathscr{C}_{\alpha}^{\mathscr{D}}\left[\mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\cup X\smallsetminus \mathscr{C}_{\alpha}^{\mathscr{D}} X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right]\\
&= \mathscr{C}_{\alpha}^{\mathscr{D}}\left(\mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right)\cup \mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right)\\
&= B\cup \left(X\smallsetminus \mathscr{C}_{\alpha}^{\mathscr{D}}\left(X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}(B)\right)\right) = X\ (\mathrm{ii}),
\end{aligned}
\tag{11}
$$

where $B = \mathscr{C}_{\alpha}^{\mathscr{D}}\mathscr{C}_{\alpha}^{\mathscr{D}} X\smallsetminus \mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)$. From (i) and (ii), we have

$$
\mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\right) = \mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right)\cap X = \mathrm{Fr}_{\alpha}^{\mathscr{D}}\left(\mathrm{Fr}_{\alpha}^{\mathscr{D}}(A)\right).
\tag{12}
$$

□

Definition 8. Let A be a subset of a topological space X. The $\mathscr{D}\alpha$-exterior of A, denoted by $\mathrm{Ext}_{\alpha}^{\mathscr{D}}(A)$, is defined as $\mathrm{Ext}_{\alpha}^{\mathscr{D}}(A) = \mathscr{I}_{\alpha}^{\mathscr{D}}(X\smallsetminus A)$.

Theorem 11. *Let A and B be subsets of X. Then the following results hold:*

(1) $Ext_{\alpha}(A)\subseteq Ext_{\alpha}^{\mathscr{D}}(A)$, *where* $Ext_{\alpha}(A)$ *is the* α*-exterior ([22], Definition 2.16) of A.*

(2) $Ext_{\alpha}^{\mathscr{D}}(A) = X\smallsetminus \mathscr{C}_{\alpha}^{\mathscr{D}}(A)$.

(3) $Ext_{\alpha}^{\mathscr{D}}(Ext_{\alpha}^{\mathscr{D}}(A)) = \mathscr{I}_{\alpha}^{\mathscr{D}}(\mathscr{C}_{\alpha}^{\mathscr{D}}(A))$.

(4) *If* $A\subseteq B$, *then* $Ext_{\alpha}^{\mathscr{D}}(B)\subseteq Ext_{\alpha}^{\mathscr{D}}(A)$.

(5) $Ext_{\alpha}^{\mathscr{D}}(A\cup B)\subseteq Ext_{\alpha}^{\mathscr{D}}(A)\cap Ext_{\alpha}^{\mathscr{D}}(B)$.

(6) $Ext_{\alpha}^{\mathscr{D}}(A\cap B)\supseteq Ext_{\alpha}^{\mathscr{D}}(A)\cup Ext_{\alpha}^{\mathscr{D}}(B)$.

(7) $Ext_{\alpha}^{\mathscr{D}}(X) = \phi$ *and* $Ext_{\alpha}^{\mathscr{D}}(\phi) = X$.

(8) $Ext_{\alpha}^{\mathscr{D}}(A) = Ext_{\alpha}^{\mathscr{D}}(X\smallsetminus Ext_{\alpha}^{\mathscr{D}}(A))$.

(9) $X = \mathscr{I}_{\alpha}^{\mathscr{D}}(A)\cup Ext_{\alpha}^{\mathscr{D}}(A)\cup Fr_{\alpha}^{\mathscr{D}}(A)$.

Proof.

(1) It follows from ([15], Theorem 3.15 (i)).

(2) It follows from ([15], Lemma 3.13 (i)).

(3) It follows from ([15], Lemma 3.13 (ii)).

(4) It follows from ([15], Theorem 3.15 (iii)).

(5) It follows from ([15], Theorem 3.15 (vi)).

(6) It follows from ([15], Theorem 3.15 (v)).

(7) It is obvious.

(8) It follows from ([15], Theorem 3.15 (iv)).

(9) It is obvious. □

The opposite of (1) and (4) of Theorem 11 is not true as shown in the following examples.

Example 4. Consider the topological space (X, τ) which is given in Example 1. Let $A = \{a\}$. Then $\mathrm{Ext}^{\mathscr{D}}_{\alpha}(A) = \{b, c\} \nsubseteq \mathrm{Ext}_{\alpha}(A) = \phi$.

Example 5. Consider the topological space (X, τ) which is given in Example 3. Let $A = \{a\}$ and $B = \{a, b\}$. Then $\mathrm{Ext}^{\mathscr{D}}_{\alpha}(A) = b, c \nsubseteq \mathrm{Ext}^{\mathscr{D}}_{\alpha}(B) = \phi$.

Remark 1. The equality in statements (5) of Theorem 11 need not be true as seen from Example 3. Let $A = \{a\}, B = \{b\}$, and $A \cup B = \{a, b\}$. Then $\mathrm{Ext}^{\mathscr{D}}_{\alpha}(A \cup B) = \phi \neq \{c\} = \mathrm{Ext}^{\mathscr{D}}_{\alpha}(A) \cap \mathrm{Ext}^{\mathscr{D}}_{\alpha}(B)$. Furthermore, the equality in statement (6) of the above theorem need not be true as seen from Example 3. Let $A = \{a, b\}, B = \{c\}$, and $A \cap B = \phi$. Then $\mathrm{Ext}^{\mathscr{D}}_{\alpha}(A \cap B) = X \neq \{a, b\} = \mathrm{Ext}^{\mathscr{D}}_{\alpha}(A) \cup \mathrm{Ext}^{\mathscr{D}}_{\alpha}(B)$.

3. $D\alpha - R_0$ and $\mathscr{D}\alpha - R_1$ Spaces

Definition 9. Let A be a subset of a topological space X. The $\mathscr{D}\alpha$-kernel of A, denoted by $\mathrm{Ker}^{\mathscr{D}}_{\alpha}(A)$, is defined as $\mathrm{Ker}^{\mathscr{D}}_{\alpha}(A) = \cap\{U \in \mathscr{D}\alpha O(X) | A \subset U\}$.

Definition 10. Let x be a point of a topological space X. The $\mathscr{D}\alpha$-kernel of x, denoted by $\mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\})$, is defined as $\mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\}) = \cap\{U \in \mathscr{D}\alpha O(X) | x \in U\}$.

Lemma 4. *Let (X, τ) be a topological space and $x \in X$. Then,*

(1) $y \in \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\})$ if and only if $x \in \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$;

(2) $\mathrm{Ker}^{\mathscr{D}}_{\alpha}(A) = \cap\left\{x \in X | \mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \cap A \neq \phi\right\}$.

Proof

(1) Suppose that $y \notin \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\})$. Then there exists a $\mathscr{D}\alpha$-open set V containing x such that $y \notin V$. Therefore, we have $x \notin \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$. The proof of the opposite case can be done similarly.

(2) Let $x \in \mathrm{Ker}^{\mathscr{D}}_{\alpha}(A)$ and $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \cap A = \phi$. Hence, $x \notin X - \mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\})$ which is a $\mathscr{D}\alpha$-open set containing A. This is impossible, since $x \in \mathrm{Ker}^{\mathscr{D}}_{\alpha}(A)$. Consequently, $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \cap A \neq \phi$. Let $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \cap A \neq \phi$ and $x \notin \mathrm{Ker}^{\mathscr{D}}_{\alpha}(A)$. Then, there exists a $\mathscr{D}\alpha$-open set W containing A and $x \notin W$. Let $y \notin \mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \cap A$. Hence, W is a $\mathscr{D}\alpha$-neighborhood of y where $x \notin W$. By this contradiction, $x \in \mathrm{Ker}^{\mathscr{D}}_{\alpha}(A)$ and the proof is completed. □

Lemma 5. *The following statements are equivalent for any points x and y in a topological space (X, τ):*

(1) $\mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\}) \neq \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{y\})$.

(2) $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \neq \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$.

Proof

(i) (1)⇒(2) Suppose that $\mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\}) \neq \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{y\})$. Then there exists a point z in X such that $z \in \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\})$ and $z \notin \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{y\})$. It follows from $z \in \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\})$ that $\{x\} \cap \mathscr{C}^{\mathscr{D}}_{\alpha}(\{z\}) \neq \phi$. This implies that $x \in \mathscr{C}^{\mathscr{D}}_{\alpha}(\{z\})$. By $z \notin \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{y\})$, we have $\{y\} \cap \mathscr{C}^{\mathscr{D}}_{\alpha}(\{z\}) = \phi$. Since $x \in \mathscr{C}^{\mathscr{D}}_{\alpha}(\{z\})$, $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \subset \mathscr{C}^{\mathscr{D}}_{\alpha}(\{z\})$ and $\{y\} \cap \mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) = \phi$, $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \neq \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$. Now, $\mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\}) \neq \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{y\})$ implies that $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \neq \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$.

(ii) (2)⇒(1) Suppose that $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \neq \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$. Then there exists a point z in X such that $z \in \mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\})$ and $z \notin \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$. Then, there exists a $\mathscr{D}\alpha$-open set containing z and therefore x but not y, that is, $y \notin \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\})$. Hence, $\mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{x\}) \neq \mathrm{Ker}^{\mathscr{D}}_{\alpha}(\{y\})$. □

Definition 11. A topological space (X, τ) is said to be a $\mathscr{D}\alpha - R_0$ space if every $\mathscr{D}\alpha$-open set contains the $\mathscr{D}\alpha$-closure of each of its singletons.

Theorem 12. *Let (X, τ) be a topological space. Then,*

(1) every $\alpha - R_0$ space is $\mathscr{D}\alpha - R_0$

(2) every $g - R_0$ space is $\mathscr{D}\alpha - R_0$

Proof. It is obvious from ([15], Theorem 3.6). □

From the above discussions, we have the following diagram in which the opposites of implications need not be true.

$$\alpha - R_0 \longrightarrow \mathscr{D}\alpha - R_0 \longleftarrow g - R_0. \qquad (13)$$

Theorem 13. *A topological space (X, τ) is a $\mathscr{D}\alpha - R_0$ space if and only if, for any x and y in X, $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \neq \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$ implies that $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \cap \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\}) = \phi$.*

Proof.

Necessity. Suppose that (X, τ) is a $\mathscr{D}\alpha - R_0$ and $x, y \in X$ such that $\mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\}) \neq \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$. Then, there exists $z \in \mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\})$ such that $z \notin \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$ (or $z \in \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$ such that $z \notin \mathscr{C}^{\mathscr{D}}_{\alpha}(\{x\})$). There exists $U \in \mathscr{D}\alpha O(X)$ such that $y \notin U$ and $z \in U$; hence, $x \in U$. Therefore, we have $x \notin \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\})$. Thus, $x \in X - \mathscr{C}^{\mathscr{D}}_{\alpha}(\{y\}) \in \mathscr{D}\alpha O(X)$, which implies that

$\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset X - \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$ and $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) = \phi$. The proof for otherwise is similar.

(i) Sufficiency. Let $U \in \mathscr{D}\alpha O(X)$ and $x \in U$. We will show that $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset U$. Let $y \notin U$; that is, $y \in X - U$. Then $x \neq y$ and $x \notin \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. This shows that $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \neq \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. By assumption, $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) = \phi$. Hence, $y \notin \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ and therefore $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset U$. □

Theorem 14. *A topological space (X, τ) is a $\mathscr{D}\alpha - R_0$ space if and only if, for any x and y in X, $Ker_\alpha^{\mathscr{D}}(\{x\}) \neq Ker_\alpha^{\mathscr{D}}(\{y\})$ implies that $Ker_\alpha^{\mathscr{D}}(\{x\}) \cap Ker_\alpha^{\mathscr{D}}(\{x\}) = \phi$.*

Proof. Suppose that (X, τ) is a $\mathscr{D}\alpha - R_0$ space. Then, by Lemma 5, for any points x and y in X if $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \neq \mathrm{Ker}_\alpha^{\mathscr{D}}(\{y\})$, then $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \neq \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. Now, we prove that $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathrm{Ker}_\alpha^{\mathscr{D}}(\{y\}) = \phi$. Assume that $z \in \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathrm{Ker}_\alpha^{\mathscr{D}}(\{y\})$. By $z \in Ker_\alpha^{\mathscr{D}}(\{x\})$ and Lemma 4 (1), it follows that $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{z\})$. Since $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$, by Theorem 13 $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) = \mathscr{C}_\alpha^{\mathscr{D}}(\{z\})$. Similarly, we have $\mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) = \mathscr{C}_\alpha^{\mathscr{D}}(\{z\}) = \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. This is a contradiction. Therefore, we have $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathrm{Ker}_\alpha^{\mathscr{D}}(\{y\}) = \phi$. Conversely, let (X, τ) be a topological space such that, for any points x and y in X, $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \neq \mathrm{Ker}_\alpha^{\mathscr{D}}(\{y\})$ implies that $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathrm{Ker}_\alpha^{\mathscr{D}}(\{y\}) = \phi$. If $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \neq \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$, then, by Lemma 5, $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \neq \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\})$. Hence, $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathrm{Ker}_\alpha^{\mathscr{D}}(\{y\}) = \phi$, which implies that $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) = \phi$. Because $z \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ implies that $x \in \mathrm{Ker}_\alpha^{\mathscr{D}}(\{z\})$, $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathrm{Ker}_\alpha^{\mathscr{D}}(\{z\}) \neq \phi$. By hypothesis, we have $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) = \mathrm{Ker}_\alpha^{\mathscr{D}}(\{z\})$. Then $z \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ implies that $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) = \mathrm{Ker}_\alpha^{\mathscr{D}}(\{z\}) = \mathrm{Ker}_\alpha^{\mathscr{D}}(\{y\})$. This is a contradiction. Hence, $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) = \phi$. By Theorem 13, we have that (X, τ) is a $\mathscr{D}\alpha - R_0$ space. □

Theorem 15. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is a $\mathscr{D}\alpha - R_0$ space.

(2) For any $A \neq \phi$ and $G \in \mathscr{D}\alpha O(X)$ such that $A \cap G \neq \phi$, there exists $F \in \mathscr{D}\alpha\mathscr{C}(X)$ such that $A \cap F \neq \phi$ and $F \subset G$.

(3) Any $A \in \mathscr{D}\alpha O(X)$, $G = \cup\{F \in \mathscr{D}\alpha\mathscr{C}(X) | F \subset G\}$.

(4) Any $F \in \mathscr{D}\alpha O(X)$, $F = \cap\{G \in \mathscr{D}\alpha O(X) | F \subset G\}$.

(5) For any $x \in X$, $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset Ker_\alpha^{\mathscr{D}}(\{x\})$.

Proof. (1)⇒(2) Let A be a nonempty set of X and $G \in \mathscr{D}\alpha O(X)$ such that $A \cap G \neq \phi$. There exists $x \in A \cap G$. Since $x \in G \in \mathscr{D}\alpha O(X)$, $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset G$. Set $F = \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$; then $F \in \mathscr{D}\alpha\mathscr{C}(X)$, $F \subset G$, and $A \cap F \neq \phi$.

(2)⇒(3) Let $G \in \mathscr{D}\alpha O(X)$. Then $G^{\cup\{F \in \mathscr{D}\alpha\mathscr{C}(X) | F \subset G\}}$. Let x be any point of G. There exists $F \in \mathscr{D}\alpha\mathscr{C}(X)$ such that $x \in F$ and $F \subset G$. Therefore, we have $x \in F \subset \cup\{F \in \mathscr{D}\alpha\mathscr{C}(X) | F \subset G\}$ and hence $G = \cup\{F \in \mathscr{D}\alpha\mathscr{C}(X) | F \subset G\}$.

(3)⇒(4) This is obvious.

(4)⇒(5) Let x be any point of X and $y \notin \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\})$. There exists $U \in \mathscr{D}\alpha O(X)$ such that $x \in U$ and $y \notin U$. Hence, $\mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) \cap U = \phi$. By (4) $\cap G \in \mathscr{D}\alpha O(X) | \mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) \subset G \cap U = \phi$. There exists $G \in \mathscr{D}\alpha O(X)$ such that $x \notin G$ and $\mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) \subset G$. Therefore, $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \cap G = \phi$ and $y \notin \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. Consequently, we obtain $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\})$.

(5)⇒(1) Let $G \in \mathscr{D}\alpha O(X)$ and $x \in G$. Suppose that $y \in \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\})$. Then $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$ and $y \in G$. This implies that $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \subset G$. Therefore, (X, τ) is a $\mathscr{D}\alpha - R_0$ space. □

Corollary 2. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is a $\mathscr{D}\alpha - R_0$ space.

(2) $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) = Ker_\alpha^{\mathscr{D}}(\{x\})$ for all $x \in X$.

Proof

(1)⇒(2) Suppose that (X, τ) is a $\mathscr{D}\alpha - R_0$ space. By Theorem 15, $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\})$ for each $x \in X$. Let $y \in \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\})$. Then $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ and so $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) = \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. Therefore, $y \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ and hence $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \subset \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. This shows that $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) = \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\})$.

(2)⇒(1) This is obvious by Theorem 15. □

Theorem 16. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is a $\mathscr{D}\alpha - R_0$ space.

(2) $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$ if and only if $y \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$, for any points x and y in X.

Proof

(1)⇒(2) Assume that (X, τ) is a $\mathscr{D}\alpha - R_0$ space. Let $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$ and let W be any $\mathscr{D}\alpha$-open set such that $y \in W$. Now, by hypothesis, $x \in W$. Therefore, every $\mathscr{D}\alpha$-open set containing y contains x. Hence, $y \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$.

(2)⇒(1) Let U be a $\mathscr{D}\alpha$-open set and $x \in U$. If $y \notin U$, then $x \notin \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$ and hence $y \notin \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. This implies that $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset U$. Hence, (X, τ) is a $\mathscr{D}\alpha - R_0$ space. □

Theorem 17. *For a topological space (X, τ), the following properties are equivalent:*

(1) (X, τ) is a $\mathscr{D}\alpha - R_0$ space.

(2) If F is $\mathscr{D}\alpha$-closed, then $F = Ker_\alpha^{\mathscr{D}}(F)$.

(3) If F is $\mathscr{D}\alpha$-closed and $x \in F$, then $Ker_\alpha^{\mathscr{D}}(\{x\}) \subset F$.

(4) If $x \in X$, then $Ker_\alpha^{\mathscr{D}}(\{x\}) \subset \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$.

Proof

(1)$\Rightarrow$(2) Let F be $\mathscr{D}\alpha$-closed and $x \notin F$. Thus, $X - F$ is $\mathscr{D}\alpha$-open and contains x. Since (X, τ) is $\mathscr{D}\alpha - R_0$, $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset X - F$. Thus $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \cap F = \phi$ and by Lemma 4 (2) $x \notin \mathrm{Ker}_\alpha^{\mathscr{D}}(F)$. Therefore, $\mathrm{Ker}_\alpha^{\mathscr{D}}(F) = F$.

(2)$\Rightarrow$(3) In general, $A \subset B$ implies that $\mathrm{Ker}_\alpha^{\mathscr{D}}(A) \subset \mathrm{Ker}_\alpha^{\mathscr{D}}(B)$. Therefore, it follows from (2) that $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \subset \mathrm{Ker}_\alpha^{\mathscr{D}}(F) = F$.

(3)$\Rightarrow$(4) Since $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ and $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ is $\mathscr{D}\alpha$-closed, by (3), $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \subset \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$.

(4)$\Rightarrow$(1) We show the implication by using Theorem 16. Let $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. Then, by Lemma 4 (1), $y \in \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\})$. Since $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ and $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ is $\mathscr{D}\alpha$-closed, by (4), we obtain $y \in \mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \subset \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. Therefore, $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$ implies that $y \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. The opposite is obvious and (X, τ) is a $\mathscr{D}\alpha - R_0$ space. □

Definition 12. A filter base F is called $\mathscr{D}\alpha$-convergent to a point x in X, if, for any $\mathscr{D}\alpha$-open set U of X containing x, there exists B in F such that B is a subset of U.

Lemma 6. *Let (X, τ) be a topological space and x and y are any two points in X such that every net in $X\mathscr{D}\alpha$-converging to $y\mathscr{D}\alpha$-converges to x. Then $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$.*

Proof. Suppose that $x_n = y$ for each $n \in \mathbf{N}$. Then $\{x_n\}_{n\in\mathbf{N}}$ is a net in $\mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. Since $\{x_n\}_{n\in\mathbf{N}}$ $\mathscr{D}\alpha$-converges to y, $\{x_n\}_{n\in\mathbf{N}}\mathscr{D}\alpha$-converges to x and this implies that $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. □

Theorem 18. *For a topological space (X, τ), the following statements are equivalent:*

(1) (X, τ) is a $\mathscr{D}\alpha - R_0$ space.

(2) If $x, y \in X$, then $y \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ if and only if every net in $X\mathscr{D}\alpha$-converging to $y\mathscr{D}\alpha$-converges to x.

Proof

(1)$\Rightarrow$(2) Let $x, y \in X$ such that $y \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. Suppose that $\{x_\alpha\}_{\alpha\in\mathbf{N}}$ is a net in X such that $\{x_\alpha\}_{\alpha\in\mathbf{N}}\mathscr{D}\alpha$-converges to y. Since $y \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ and by Theorem 15, we have $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) = \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. Therefore, $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. This means that $\{x_\alpha\}_{\alpha\in\Lambda}\mathscr{D}\alpha$-converges to x. Conversely, let $x, y \in X$ such that every net in $X\mathscr{D}\alpha$-converging to $y\mathscr{D}\alpha$-converges to x. Then $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$ by Lemma 4 (2). By Theorem 15, we have $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) = \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. Therefore, $y \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$.

(2)$\Rightarrow$(1) Assume that x and y are any two points of X such that $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \cap \mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) \neq \phi$. Let $z \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \cap C_\alpha^{\mathscr{D}}(\{y\})$. So there exists a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ such that $\{x_\alpha\}_{\alpha\in\Lambda}\mathscr{D}\alpha$-converges to z. Since $z \in \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$, we have $\{x_\alpha\}_{\alpha\in\Lambda}\mathscr{D}\alpha$-converges to y. It follows that $y \in \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. Similarly, we obtain $x \in \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. Therefore, $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) = \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$ and, by Theorem 15, (X, τ) is a $\mathscr{D}\alpha - R_0$ space. □

Definition 13. A topological space (X, τ) is said to be $\mathscr{D}\alpha - R_1$ space if, for x, y in X with $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \neq \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$, there exist disjoint $\mathscr{D}\alpha$-open sets U and V such that $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$ is a subset of U and $\mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$ is a subset of V.

Theorem 19. *Let (X, τ) be a topological space. Then,*

(1) every $\alpha - R_1$ space is $\mathscr{D}\alpha - R_1$

(2) every $g - R_1$ space is $\mathscr{D}\alpha - R_1$

Proof. It is obvious. □

From the above discussions, we have the following diagram in which the opposite of implications need not be true.

$$\alpha - R_1 \longrightarrow \mathscr{D}\alpha - R_1 \longleftarrow g - R_1. \tag{14}$$

Theorem 20. *If (X, τ) is a $\mathscr{D}\alpha - R_1$ space, then (X, τ) is a $\mathscr{D}\alpha - R_0$ space.*

Proof. Let U be $\mathscr{D}\alpha$-open such that $x \in U$. If $y \notin U$, then since $x \notin \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$, $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \neq \mathscr{C}_\alpha^{\mathscr{D}}(\{y\})$. Hence, there exists $\mathscr{D}\alpha$-open V_y such that $\mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) \subset V_y$ and $x \notin V_y$, which implies that $y \notin \mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. Thus, $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset U$. Therefore, (X, τ) is a $\mathscr{D}\alpha - R_0$ space. □

Theorem 21. *A topological space (X, τ) is said to be a $\mathscr{D}\alpha - R_1$ space if and only if $x, y \in X$, $Ker_\alpha^{\mathscr{D}}(\{x\}) \neq Ker_\alpha^{\mathscr{D}}(\{y\})$, and there exist disjoint $\mathscr{D}\alpha$-open sets U and V such that $\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) \subset U$ and $\mathscr{C}_\alpha^{\mathscr{D}}(\{y\}) \subset V$.*

Proof. It follows from Lemma 4 (1). □

4. Weakly $\mathscr{D}\alpha - R_0$ Space

Definition 14. A topological space (X, τ) is said to be weakly $\mathscr{D}\alpha - R_0$ space if $\cap_{x\in X}\mathscr{C}_\alpha^{\mathscr{D}}(\{x\}) = \phi$.

Theorem 22. *A topological space (X, τ) is weakly $\mathscr{D}\alpha - R_0$ space if and only if $Ker_\alpha^{\mathscr{D}}(\{x\}) \neq X$ for every $x \in X$.*

Proof. Assume that the space (X, τ) is weakly $\mathscr{D}\alpha - R_0$ space. Suppose that there is a point y in X such that $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{y\}) = X$. Then $y \notin O$, where O is some proper $\mathscr{D}\alpha$-open subset of X. This implies that $y \in \cap_{x\in X}\mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$. But this is a contradiction. Now suppose that $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) \neq X$ for every $x \in X$. If there exists a point $y \in X$ such that $y \in \cap_{x\in X}\mathscr{C}_\alpha^{\mathscr{D}}(\{x\})$, then every $\mathscr{D}\alpha$-open set containing y must contain every point of X. This implies that the space X is the unique $\mathscr{D}\alpha$-open set containing y. Thus, $\mathrm{Ker}_\alpha^{\mathscr{D}}(\{x\}) = X$, which is a contradiction. Hence, (X, τ) is a weakly $\mathscr{D}\alpha - R_0$ space. □

Theorem 23. *A topological space (X, τ) is a weakly $\mathscr{D}\alpha - R_0$ space if and only if $Ker_\alpha^{\mathscr{D}}(\{x\}) \neq X$ for every $x \in X$.*

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Retraction

Retracted: Some Fixed Point Results in Function Weighted Metric Spaces

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] A. Asif, N. Hussain, H. Al-Sulami, and M. Arshad, "Some Fixed Point Results in Function Weighted Metric Spaces," *Journal of Mathematics*, vol. 2021, Article ID 6636504, 9 pages, 2021.

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Research Article

Some Fixed Point Results in Function Weighted Metric Spaces

Awais Asif,[1] **Nawab Hussain**,[2] **Hamed Al-Sulami**,[2] **and Muahammad Arshad**[1]

[1]*Department of Math & Stats, International Islamic University Islamabad, Islamabad, Pakistan*
[2]*Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

Correspondence should be addressed to Nawab Hussain; nhusain@kau.edu.sa

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After the establishment of the Banach contraction principle, the notion of metric space has been expanded to more concise and applicable versions. One of them is the conception of $\mathscr{F}$-metric, presented by Jleli and Samet. Following the work of Jleli and Samet, in this article, we establish common fixed points results of Reich-type contraction in the setting of $\mathscr{F}$-metric spaces. Also, it is proved that a unique common fixed point can be obtained if the contractive condition is restricted only to a subset closed ball of the whole $\mathscr{F}$-metric space. Furthermore, some important corollaries are extracted from the main results that describe fixed point results for a single mapping. The corollaries also discuss the iteration of fixed point for Kannan-type contraction in the closed ball as well as in the whole $\mathscr{F}$-metric space. To show the usability of our results, we present two examples in the paper. At last, we render application of our results.

1. Introduction and Preliminaries

In recent years, along with $\mathscr{F}$-metric presented by Jleli et al. [1], many authors presented interesting generalizations of metric spaces [2–9]. Jleli and Samet introduced generalized metric spaces, known as $\mathscr{F}$-metric spaces, and proved their generality to metric spaces with the help of concrete examples. The idea of $\mathscr{F}$-metric spaces was compared with b-metric and s-relaxed metric spaces, and hence, the Banach contraction principle was established in the frame of $\mathscr{F}$-metric spaces.

Banach contraction principle states that any contraction on a complete metric space has a unique fixed point. This principle guarantees the existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, the Banach contraction principle has been extended and generalized in many directions [10, 11]. The fixed point theory of multivalued contraction mappings using the Hausdorff metric was initiated by Nadler [12], who extended the Banach contraction principle to multivalued mappings. Since then, many authors have studied various fixed point results for multivalued mappings. Nazam et al. [13] proved fixed point theorems for Kannan-type contractions on closed balls in complete partial metric spaces. The abovementioned results and its generalizations are recently investigated for fixed point in the setting of F-metric space (see [14–16]).

In this article, we prove fixed point and common fixed points results of Reich-type contractions for single-valued mappings in $\mathscr{F}$-metric spaces.

This article is organized into three sections. Section 2 contains a short history of the previous literature that becomes a motivation for this article. There are some basic definitions which help readers to understand our results easily. In Section 3, we established theorems of fixed points and common fixed points of single-valued Reich contractions in $\mathscr{F}$-metric spaces. An example is provided to explain our results. Section 4 deals with fixed point theorems of contractions with respect to closed balls in $\mathscr{F}$-metric spaces along with an example.

2. Basic Relevant Notions

Definition 1 (see [1]). A self-mapping g on a nonempty set A is said to be Kannan contraction if there exists a number k, $0 < k < (1/2)$, such that, for each $a, b \in A$, we have

$$d(g(a), g(b)) \leq k[d(a, g(a)) + d(b, g(b))]. \quad (1)$$

Let $f\colon (0, \infty) \longrightarrow \mathbb{R}$ with following characteristics:

(F1) f is strictly increasing

(F2) For any sequence$\{t_n\} \subset (0, \infty)$, we have

$$\lim_{n \longrightarrow \infty} t_n = 0 \Leftrightarrow \lim_{n \longrightarrow \infty} f(t_n) = -\infty. \quad (2)$$

The collection of all such functions satisfying (F1) and (F2) is denoted by $\mathscr{F}$. The concept of $\mathscr{F}$-metric is generalized as follows:

Definition 2 (see [1]). Suppose$A$is a nonempty set and $(f, \alpha) \in \mathscr{F} \times [0, \infty)$. Let the function $d\colon A \times A \longrightarrow [0, \infty)$ be such that

(d1) For all $(a, b) \in A \times A, d(a, b) = 0 \Leftrightarrow a = b$

(d2) For all $(a, b) \in A \times A$, $d(a, b) = d(b, a)$

(d3) {tn}i=1n⊂X For every $(a, b) \in A \times A$,for each $N' \in \mathbb{N}, N' \geq 2$ and for every with $(t_1, t_{N'}) = (a, b)$, we have

$$d(a, b) > 0 \Rightarrow f(d(a, b)) \leq f\left(\sum_{i=1}^{N'-1} d(t_i, t_{i+1})\right) + \alpha. \quad (3)$$

Then, d is known as an$\mathscr{F}$-metric on A, and the pair (A, d) is called an $\mathscr{F}$-metric space.

Example 1 (see [1]). Let $A = \mathbb{N}$ (set of natural numbers) and $d\colon A \times A \longrightarrow (0, \infty)$ be defined by

$$d(a, b) = \begin{cases} (a - b)^2, & \text{if } (a, b) \in [0, 3] \times [0, 3], \\ |a - b|, & \text{if } (a, b) \notin [0, 3] \times [0, 3], \end{cases} \quad (4)$$

for all $(a, b) \in A \times A$. It can easily be seen that d is an $\mathscr{F}$-metric with $f(x) = \ln(x)$.

Example 2 (see [1]). Let $A = \mathbb{N}$ and $d\colon A \times A \longrightarrow (0, \infty)$ is defined as

$$d(a, b) = \begin{cases} 0, & \text{if } a = b, \\ e^{|a-b|}, & \text{if } a \neq b, \end{cases} \quad (5)$$

for all $(a, b) \in A \times A$. Then, d is $\mathscr{F}$-metric on A.

Definition 3 (see [1]). Suppose$\{a_n\}$is a sequence in A. Then,

(i) $\{a_n\}$ is $\mathscr{F}$-convergent to a point $a \in A$ if $\lim_{n \longrightarrow \infty} d(a_n, a) = 0$

(ii) $\{a_n\}$ is an $\mathscr{F}$-Cauchy sequence if $\lim_{n,m \longrightarrow \infty} d(a_n, a_m) = 0$

(iii) The space (A, d) is $\mathscr{F}$-complete if every $\mathscr{F}$-Cauchy sequence$\{a_n\} \subset A$ is $\mathscr{F}$-convergent to a point $a \in A$

Definition 4 (see [1]). Let (A, d)be an $\mathscr{F}$-metric space. A subset $O$of A is said to be $\mathscr{F}$-open if, for every $a \in O$, there is some $r > 0$ such that $B(a, r) \subset O$, where

$$B(a, r) = \{b \in A\colon d(a, b) < r\}. \quad (6)$$

We say that a subset C of A is $\mathscr{F}$-closed if $A \backslash C$ is $\mathscr{F}$-open.

Definition 5 (see [1]). Let (A, d)be an $\mathscr{F}$-metric space and$B$ be a nonempty subset of A. Then, the following statements are equivalent:

(i) B is $\mathscr{F}$-closed.

(ii) For any sequence $\{a_n\} \subset B$, we have

$$\lim_{n \longrightarrow \infty} d(a_n, a) = 0, \quad a \in A \Longrightarrow a \in B. \quad (7)$$

Theorem 1 (see [1]). *Suppose $(f, \alpha) \in \mathscr{F} \times [0, \infty)$ and (A, d) is an $\mathscr{F}$-complete $\mathscr{F}$-metric space. Let $g\colon A \longrightarrow A$ be a given mapping. Suppose that there exists $k \in (0, 1)$ such that*

$$d(g(a), g(b)) \leq k\ d(a, b), \quad (a, b) \in A \times A. \quad (8)$$

Then, g has a unique fixed point $a^* \in A$. Moreover, for any $a_0 \in A$, the sequence$\{a_n\} \subset A$ defined by $a_{n+1} = g(a_n), n \in \mathbb{N}$ is F-convergent to a^*.

Theorem 2 (see [17]). *Suppose A is a complete metric space with metric d, and let $g\colon A \longrightarrow A$ be a function such that*

$$d(g(a), g(b)) \leq \alpha\ d(a, b) + \beta\ d(a, g(a)) + \gamma\ d(b, g(b)), \quad (9)$$

for all $a, b \in A$, where α, β, and γ are nonnegative integers and satisfy $\alpha + \beta + \gamma < 1$. Then, g has a unique fixed point.

Lemma 1 (see [18]). *ie Banach space $(B(W), \|\cdot\|)$along with the metric d defined by*

$$d(g, h) = \|g - h\| = \max_{a \in W} |g(a) - h(a)|, \quad g, h \in B(W). \quad (10)$$

is an $\mathscr{F}$-metric space.

3. Fixed Points of Reich-Type Contractions in $\mathscr{F}$ – Metric Spaces

In this section, we construct fixed point and common fixed points results for single-valued Reich-type and Kannan-type contractions in the setting of $\mathscr{F}$-metric space.

Theorem 3. *Suppose $(f, \alpha) \in \mathscr{F} \times [0, \infty)$ and (X, d) is an $\mathscr{F}$-complete $\mathscr{F}$-metric space. Let $S, T\colon X \longrightarrow X$be self-mappings such that*

$$d(Sx, Ty) \leq a\, d(x, y) + b\, d(x, Sx) + c\, d(y, Ty), \quad (11)$$

for $a, b, c \in [0, \infty)$ *such* $a + b + c < 1$, *for all* $(x, y) \in X \times X$. *Then, S and T have at most one common fixed point in X.*

Proof. Suppose x_0 is an arbitrary point and define a sequence (x_n) by

$$\begin{aligned} Sx_{2j} &= x_{2j+1}, \\ Tx_{2j+1} &= x_{2j+2}, \quad j = 0, 1, 2, \ldots, \end{aligned} \quad (12)$$

Using (11) and (12), we can write

$$\begin{aligned} d\left(x_{2j+1}, x_{2j+2}\right) = d\left(Sx_{2j}, Tx_{2j+1}\right) &\leq a\, d\left(x_{2j}, x_{2j+1}\right) + b\, d\left(x_{2j}, Sx_{2j}\right) + c\, d\left(x_{2j+1}, Tx_{2j+1}\right) \\ &= a\, d\left(x_{2j}, x_{2j+1}\right) + b\, d\left(x_{2j}, x_{2j+1}\right) + c\, d\left(x_{2j+1}, x_{2j+2}\right). \end{aligned} \quad (13)$$

This implies

$$\begin{aligned} (1 - c) d\left(x_{2j+1}, x_{2j+2}\right) &\leq (a + b) d\left(x_{2j}, x_{2j+1}\right) \\ d\left(x_{2j+1}, x_{2j+2}\right) &< \frac{a + b}{1 - c} d\left(x_{2j}, x_{2j+1}\right) = \lambda\, d\left(x_{2j}, x_{2j+1}\right), \end{aligned} \quad (14)$$

where $((a + b)/(1 - c)) = \lambda$

Similarly,

$$d\left(x_{2j+2}, x_{2j+3}\right) < \frac{a + b}{1 - c} d\left(x_{2j+1}, x_{2j+2}\right) = \lambda\, d\left(x_{2j+1}, x_{2j+2}\right). \quad (15)$$

Continuing this way, we get

$$d(x_n, x_{n+1}) < \lambda\, d(x_{n-1}, x_n), \quad \text{for all } n \varepsilon \mathbb{N}, \quad (16)$$

which yields

$$d(x_n, x_{n+1}) < \lambda\, d(x_{n-1}, x_n) < \lambda^2 d(x_{n-2}, x_{n-1}) < \cdots < \lambda^n d(x_0, x_1). \quad (17)$$

Hence,

$$d(x_n, x_{n+1}) < \lambda^n d(x_0, x_1), \quad n \in \mathbb{N}. \quad (18)$$

Using (18), we can write

$$\begin{aligned} \sum_{k=n}^{m-1} d(x_k, x_{k+1}) &= d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &< \lambda^n d(x_0, x_1) + \lambda^{n+1} + \cdots + \lambda^{m-1} d(x_0, x_1) \\ &< \lambda^n \left[1 + \lambda + \lambda^2 + \cdots + \lambda^{m-n-1}\right] d(x_0, x_1) \\ &\leq \frac{\lambda^n}{1 - \lambda} d(x_0, x_1), \quad m > n. \end{aligned} \quad (19)$$

Since $\lim_{n \longrightarrow \infty} (\lambda^n/(1 - \lambda)) d(x_0, x_1) = 0$, for any $\delta > 0$, there exists some $n' \in \mathbb{N}$ such that

$$0 < \frac{\lambda^n}{1 - \lambda} d(x_0, x_1) < \delta, \quad n \geq n'. \quad (20)$$

Furthermore, suppose $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ satisfies (d3) and $\varepsilon > 0$ is fixed. By $(F2)$, there is some $\delta > 0$ such that

$$0 < t < \delta \Longrightarrow f(t) < f(\varepsilon) - \alpha. \quad (21)$$

By (21), we write

$$f\left(\frac{\lambda^n}{1 - \lambda} d(x_0, x_1)\right) < f(\varepsilon) - \alpha, \quad m > n \geq n'. \quad (22)$$

Using (20), we write

$$f\left(\sum_{k=n}^{m-1} d(x_k, x_{k+1})\right) \leq f\left(\frac{\lambda^n}{1 - \lambda} d(x_0, x_1)\right) < f(\varepsilon) - \alpha, \quad m > n \geq n'. \quad (23)$$

By (d3) and above equation, we obtain

$$d(x_n, x_m) > 0, \quad m > n > n' \Longrightarrow f(d(x_n, x_m)) < f(\varepsilon). \quad (24)$$

This shows that

$$d(x_n, x_m) < \varepsilon, \quad m > n \geq n'. \quad (25)$$

Hence, we showed that (x_n) is an $\mathcal{F}$-Cauchy sequence in X. Since (X, d) is $\mathcal{F}$-complete, there exists $z^* \in X$ such that (x_n) is $\mathcal{F}$-convergent to z^*, i.e.,

$$\lim_{n \longrightarrow \infty} d(x_n, z^*) = 0. \quad (26)$$

To prove that z^* is the fixed point of S, assume $d(Sz^*, z^*) > 0$. Then,

$$\begin{aligned} d\left(Sz^*, x_{2j+2}\right) &= d\left(Sz^*, Tx_{2j+1}\right) \\ &\leq a\, d\left(z^*, x_{2j+1}\right) + b\, d(z^*, Sz^*) + c\, d\left(x_{2j+1}, Tx_{2j+1}\right) \\ &= a\, d\left(z^*, x_{2j+1}\right) + b\, d(z^*, Sz^*) + c\, d\left(x_{2j+1}, x_{2j+2}\right), \end{aligned} \quad (27)$$

which implies $(1 - b) d(Sz^*, z^*) < 0$, which is a contradiction. Hence, $d(Sz^*, z^*) = 0$, i.e., $Sz^* = z^*$. Similarly, suppose $d(z^*, Tz^*) > 0$:

$$\begin{aligned} d\left(Tz^*, x_{2j+1}\right) &= d\left(Tz^*, Sx_{2j}\right) \\ &\leq a\, d\left(z^*, x_{2j}\right) + b\, d\left(x_{2j}, Sx_{2j}\right) + c\, d(z^*, Tz^*) \\ &= a\, d\left(z^*, x_{2j}\right) + b\, d(z^*, Sz^*) + c\, d\left(x_{2j}, x_{2j+1}\right), \end{aligned} \quad (28)$$

i.e.,

$$(1-c)d\left(z^*, Tz^*\right) < 0, \tag{29}$$

which is contradiction to the assumption. Therefore, we get $Tz^* = z^*$. Hence, $Tz^* = Sz^* = z^*$.

Uniqueness. Assume that$z^{**}$is also a common fixed point of S and T and $z^* \neq z^{**}$. Then,

$$\begin{aligned} d\left(z^*, z^{**}\right) &= d\left(Sz^*, Tz^{**}\right) \leq a\, d\left(z^*, z^{**}\right) \\ &\quad + b\, d\left(z^*, Sz^*\right) + c\, d\left(z^{**}, Tz^{**}\right) \\ &= a\, d\left(z^*, z^{**}\right) + b\, d\left(z^*, z^*\right) + c\, d\left(z^{**}, z^{**}\right). \end{aligned} \tag{30}$$

We get (1-a)$d\left(z^*, z^{**}\right) < 0$, which is a contradiction. Hence, $z^* = z^{**}$ □

Example 3. Suppose

$$\begin{aligned} Y &= \left\{Y_j := \frac{6j+1}{2}, j \in \mathbb{N}\right\}, \\ d(x, y) &= \begin{cases} 0, & \text{if } x = y \\ e^{|x-y|}, & \text{if } x \neq y \end{cases} \end{aligned} \tag{31}$$

Let $f(x) = \ln x$ and $S, T\colon X \longrightarrow X$ are defined by

$$\begin{aligned} T\left(Y_j\right) &= \begin{cases} Y_1 & \text{if } j = 1, 2, \\ Y_{j-1} & \text{if } j > 2, \end{cases} \\ S\left(Y_j\right) &= \begin{cases} Y_1 & \text{if } j = 1, \\ Y_2 & \text{if } j = 2, \\ Y_{j-2} & \text{if } j > 2. \end{cases} \end{aligned} \tag{32}$$

It can be easily verified that d is an $\mathcal{F}$-metric and f satisfies $(F1)-(F2)$. Fix $b = c = 0$ and $(x, y) \in X \times X$. Suppose $i \neq j$, then

$$\begin{aligned} d\left(SY_j, TY_i\right) &= d\left(Y_{j-2}, Y_{i-1}\right) = e^{\left|Y_{j-2} - Y_{i-1}\right|} \\ &= e^{|((6j-12+1)/2)-((6i-6+1)/2)|} \\ &= e^{|3(j-i)-3|} < e^{-2} \cdot e^{|3(j-i)|} = a\, d\left(Y_j, Y_i\right) \\ &= a\, d\left(Y_j, Y_i\right) + b\, d\left(Y_j, SY_j\right) + c\, d\left(Y_i, TY_i\right), \end{aligned} \tag{33}$$

where $a = e^{-2}$. The inequality (11) holds true. Moreover, it is clear that Y_1 is the only common fixed point of S and T.

Taking $a = 0$ in Theorem 1, we get the following result of Kannan contractions.

Replacing S in Theorem 3, we get the following corollary.

Corollary 1. *Suppose $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and (X, d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $T\colon X \longrightarrow X$is a self-mapping such that*

$$d(Tx, Ty) \leq a\ d(x, y) + b\ d(x, Tx) + c\ d(y, Ty), \tag{34}$$

for $a, b, c \in [0, \infty)$ such $a + b + c < 1$,for all $(x, y) \in X \times X$. Then, T has at most one fixed point in X.

Taking $b = c = 0$ in Corollary 1, we get the following result.

Corollary 2. *Suppose $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and (X, d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $T\colon X \longrightarrow X$is a self-mapping such that*

$$d(Tx, Ty) \leq a\, d(x, y), \tag{35}$$

for $a \in (0, \infty)$and $(x, y) \in X \times X$. Then, T has at most one fixed point in X.

Besides the above important results, Theorem 3 also led us to the following fixed point result of Kannan-type contraction.

Corollary 3. *Suppose$(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and (X, d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $S, T\colon X \longrightarrow X$be self-mappings. Suppose that, for $k \in [0, 1)$such that*

$$d(Sx, Ty) \leq \frac{k}{2}\left(d(x, Sx) + d(y, Ty)\right), \tag{36}$$

for all $(x, y) \in X \times X$, then S and T have at most one common fixed point in X.

Proof. Suppose x_0 is an arbitrary point and define a sequence (x_n) by $Sx_{2j} = x_{2j+1}$ and $Tx_{2j+1} = x_{2j+2}$; $j = 0, 1, 2, \ldots,$

Using the contraction and the iteration given above, we can write

$$\begin{aligned} d\left(x_{2j+1}, x_{2j+2}\right) &= d\left(Sx_{2j}, Tx_{2j+1}\right) \\ &= \frac{k}{2}\left[d\left(x_{2j}, Sx_{2j}\right) + d\left(x_{2j+1}, Tx_{2j+1}\right)\right] \\ &= \frac{k}{2}\left[d\left(x_{2j}, x_{2j+1}\right) + d\left(x_{2j+1}, x_{2j+2}\right)\right]. \end{aligned} \tag{37}$$

This implies

$$\left(1 - \frac{k}{2}\right) d\left(x_{2j+1}, x_{2j+2}\right) \leq \frac{k}{2} d\left(x_{2j}, Sx_{2j}\right). \tag{38}$$

or

$$\begin{aligned} d\left(x_{2j+1}, x_{2j+2}\right) &\leq \frac{k}{2-k} d\left(x_{2j}, Sx_{2j}\right) \\ d\left(x_{2j+1}, x_{2j+2}\right) &\leq \lambda\, d\left(x_{2j}, x_{2j+1}\right), \end{aligned} \tag{39}$$

where $(k/(2-k)) = \lambda$. Similarly,

$$d\left(x_{2j+2}, x_{2j+3}\right) < \frac{k}{2-k} d\left(x_{2j+1}, x_{2j+2}\right) = \lambda\, d\left(x_{2j+1}, x_{2j+2}\right). \tag{40}$$

Continuing the same way as in Theorem 3, we get the common fixed point of S and T.

Replacing S with T, we get the following result of single mapping. □

Corollary 4. *Suppose* $(f,\alpha) \in \mathscr{F} \times [0,\infty)$ *and* (X,d) *is an* $\mathscr{F}$*-complete* $\mathscr{F}$*-metric space. Let* $T\colon X \longrightarrow X$*be a self-mapping. Suppose that, for*$k \in [0,1)$*such that*

$$d(Tx,Ty) \leq \frac{k}{2}(d(x,Tx)+d(y,Ty)), \tag{41}$$

for all $(x,y) \in X \times X$*, then* T*has at most one fixed point in* X.

4. Fixed Points of Reich-Type Contractions on $\mathscr{F}$-Closed Balls

This portion of the paper deals with the fixed points theorems of Reich-type contractions that hold true only on the closed balls rather than on the whole space X.

Definition 6. Let (X,d) be an $\mathscr{F}$-complete $\mathscr{F}$-metric space and $S,T\colon X \longrightarrow X$be self-mappings. Suppose that $a+b+c<1$ for$a,b,c \in [0,\infty)$. Then, the mappings S and T are called Reich-type contractions on $B(x_0,r) \subseteq X$ such that

$$d(Sx,Ty) \leq a\,d(x,y) + b\,d(x,Sx) + c\,d(y,Ty), \quad \forall x,y \in B(x_0,r). \tag{42}$$

Theorem 4. *Suppose* $(f,\alpha) \in \mathscr{F} \times [0,\infty)$ *and* (X,d) *is an* $\mathscr{F}$*-complete* $\mathscr{F}$*-metric space. Let* S *and* T*be Reich-type F-contractions on* $B(x_0,r)$*. Suppose that for* $x_0 \in X$ *and* $r>0$*, the following conditions are satisfied:*

(a) $B(x_0,r)$ *is* $\mathscr{F}$*-closed*
(b) $d(x_0,x_1) \leq (1-\lambda)r$, *for* $x_1 \in X$ *and* $\lambda = ((a+b)/(1-c))$
(c) *There exist* $0<\epsilon<r$ *such as* $f((1-\lambda^{k+1})r) \leq f(\varepsilon) - \alpha$*, where* $k \in \mathbb{N}$

Then, S *and* T*have at most one common fixed point in* $B(x_0,r)$.

Proof. Suppose x_0 is an arbitrary point and define a sequence (x_n) by $T(x_{2j}) = x_{2j+1}$ and $S(x_{2j+1}) = x_{2j+2}$; $j = 0,1,2,\ldots,$.

We need to show that x_n is in $B(x_0,r)$ for all $n \in \mathbb{N}$. We show it by mathematical induction. By (b), we write

$$d(x_0,x_1) < r. \tag{43}$$

Therefore, $x_1 \in B(x_0,r)$. We know by previous theorems that

$$d(x_1,x_2) \leq \lambda\, d(x_0,x_1). \tag{44}$$

Now,

$$\begin{aligned} f(d(x_0,x_2)) &\leq f(d(x_0,x_1)+d(x_1,x_2)) + \alpha \\ &\leq f(d(x_0,x_1)+\lambda\, d(x_0,x_1)) + \alpha \\ &= f((1+\lambda)d(x_0,x_1)) + \alpha = f((1+\lambda)(1-\lambda)r) + \alpha \\ &= f\left(\left(1-\lambda^2\right)r\right) + \alpha \leq f(\varepsilon) < f(r). \end{aligned} \tag{45}$$

This implies that

$$d(x_0,x_2) < r, \tag{46}$$

i.e., $x_2 \in B(x_0,r)$. Suppose $x_3,\ldots,x_k \in B(x_0,r)$ for some $k \in \mathbb{N}$. Now, if $x_{2j+1} \leq x_k$, then by (42), we can write

$$\begin{aligned} d\left(x_{2j},x_{2j+1}\right) &= d\left(Sx_{2j-1},Tx_{2j}\right) \leq a\,d\left(x_{2j-1},x_{2j}\right) \\ &\quad + b\,d\left(x_{2j-1},Sx_{2j-1}\right) + c\,d\left(x_{2j},Tx_{2j}\right) \\ &= a\,d\left(x_{2j-1},x_{2j}\right) + b\,d\left(x_{2j-1},x_{2j}\right) + c\,d\left(x_{2j},x_{2j+1}\right). \end{aligned} \tag{47}$$

This implies

$$(1-c)d\left(x_{2j},x_{2j+1}\right) \leq (a+b)d\left(x_{2j-1},x_{2j}\right), \tag{48}$$

or

$$d\left(x_{2j},x_{2j+1}\right) < \frac{a+b}{1-c}d\left(x_{2j-1},x_{2j}\right). \tag{49}$$

Let $((a+b)/(1-c)) = \lambda$, we get

$$d\left(x_{2j},x_{2j+1}\right) < \lambda\, d\left(x_{2j-1},x_{2j}\right). \tag{50}$$

Similarly, if $x_{2j} \leq x_k$, then

$$d\left(x_{2j-1},x_{2j}\right) < \frac{a+b}{1-c}d\left(x_{2j-2},x_{2j-1}\right) = \lambda\, d\left(x_{2j-2},x_{2j-1}\right). \tag{51}$$

Therefore, from inequality (50) and (51), we write

$$d\left(x_{2j},x_{2j+1}\right) < \lambda\, d\left(x_{2j-1},x_{2j}\right) < \cdots < \lambda^{2j} d(x_0,x_1) \tag{52}$$

and

$$d\left(x_{2j-1},x_{2j}\right) < \lambda\, d\left(x_{2j-2},x_{2j-1}\right) < \cdots < \lambda^{2j-1} d(x_0,x_1). \tag{53}$$

From (52) and (53), we write

$$d(x_k,x_{k+1}) \leq \lambda^k d(x_0,x_1), \quad \text{for some } k \in \mathbb{N}. \tag{54}$$

Now, using (54), we have

$$\begin{aligned} f(d(x_0,x_{k+1})) &\leq f\left(\sum_{i=1}^{k+1} d(x_{i-1},x_i)\right) + \alpha \\ &= f(d(x_0,x_1) + \cdots + d(x_k,x_{k+1})) + \alpha \\ &\leq f\left[\left(1+\lambda+\lambda^2+\cdots+\lambda^k\right)d(x_0,x_1)\right] + \alpha \\ &= f\left[\frac{1-\lambda^{k+1}}{1-\lambda}d(x_0,x_1)\right] + \alpha. \end{aligned} \tag{55}$$

Using (b), we write

$$f(d(x_0,x_{k+1})) \leq f\left[\frac{1-\lambda^{k+1}}{1-\lambda}(1-\lambda)r\right] + \alpha = f\left(\left(1-\lambda^{k+1}\right)r\right) + \alpha. \tag{56}$$

Using (c), we deduce that

$$f\left(d\left(x_0, x_{k+1}\right)\right) \leq f(\varepsilon) < f(r). \tag{57}$$

Hence, by (F1), we notice that

$$d\left(x_0, x_{k+1}\right) \leq r. \tag{58}$$

This implies that $x_{k+1} \in B(x_0, r)$. Therefore, $x_n \in B(x_0, r)$ for all $n \in \mathbb{N}$. Now, we have by (42)

$$\begin{aligned} d\left(x_{2i+1}, x_{2i+2}\right) &= d\left(Sx_{2i}, Tx_{2i+1}\right) \\ &\leq a\, d\left(x_{2i}, x_{2i+1}\right) + b\, d\left(x_{2i}, Sx_{2i}\right) + c\left(x_{2i+1}, Tx_{2i+1}\right) \\ &= a\, d\left(x_{2i}, x_{2i+1}\right) + b\, d\left(x_{2i}, x_{2i+1}\right) + c\, d\left(x_{2i+1}, x_{2i+2}\right). \end{aligned} \tag{59}$$

Following the same steps of proof of Theorem 3 and using (a), we obtain that the sequence (x_n) is $\mathcal{F}$-convergent to some z^* in $B(x_0, r) \cdot z^*$ can be proved as common fixed point of S and T in the same way as in Theorem 3.

Taking $S = T$ in Theorem 4, we get the following result of single mappings. □

Corollary 5. *Suppose $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, (X, d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space and $T: X \longrightarrow X$is a self-mapping. Suppose that $a + b + c < 1$,for$a, b, c \in [0, \infty)$. Suppose that for $x_0 \in X$ and $r > 0$, the following conditions are satisfied:*

(a) $B(x_0, r) \subseteq X$ is $\mathcal{F}$-closed

(b) $d(Tx, Ty) \leq a\ d(x, y) + b\ d(x, Tx) + c\ d(y, Ty)$, for all $x, y \in B(x_0, r)$

(c) $d(x_0, x_1) \leq (1 - \lambda)r$, for $x_1 \in X$ and $\lambda = ((a + b)/(1 - c))$

(d) There exists $0 < \varepsilon < r$ such as $f((1 - \lambda^{k+1})r) \leq f(\varepsilon) - \alpha$, where $k \in \mathbb{N}$

Then, Thas at most one fixed point in $B(x_0, r)$.

Example 4. Let $X = [0, \infty)$ and $f(x) = \ln x$. Define $T: X \longrightarrow X$ by

$$Tx = \begin{cases} \dfrac{x}{3}, & \text{if } x \in [0, 1], \\ x^3 & \text{if } x \in (1, \infty), \end{cases} \tag{60}$$

and define d by

$$d(x, y) = \begin{cases} (x - y)^2, & \text{if } (x, y) \in [0, 1] \times [0, 1], \\ |x - y|, & \text{if } (x, y) \notin [0, 1] \times [0, 1]. \end{cases} \tag{61}$$

It can be easily verified that d is an $\mathcal{F}$-metric and function f satisfies $(F1) - (F2)$. Fix $x_0 = r = (1/4)$, then $B(x_0, r) = [0, (1/2)]$. Clearly, $B(x_0, r)$ is $\mathcal{F}$-closed so condition (a) of Corollary 5 is satisfied. Now, if $a = (3/4), b = c = 0$, then $\lambda = a$ and

$$\begin{aligned} d\left(x_0, x_1\right) = d\left(x_0, Tx_0\right) &= \left(\frac{1}{4} - \frac{1}{12}\right)^2 = \frac{1}{36} \\ &< \left(1 - \frac{3}{4}\right)\frac{1}{4} = (1 - \lambda)r. \end{aligned} \tag{62}$$

This shows that condition (b) is fulfilled. Furthermore, suppose $k = 1$, then

$$f\left(\left(1 - \lambda^{k+1}\right)r\right) = \ln\left(\left(1 - \left(\frac{3}{4}\right)^2\right)\frac{1}{4}\right) = \ln\left(\frac{7}{64}\right)$$
$$\ln\left(\frac{8}{64}\right) - \ln\left(\frac{8}{7}\right) = f(\varepsilon) - \alpha, \tag{63}$$

i.e.,

$$f\left(\left(1 - \lambda^{k+1}\right)r\right) = f(\varepsilon) - \alpha. \tag{64}$$

Hence, condition (d) is satisfied for $\varepsilon = (8/64) \leq (1/4) = r$ and $\alpha = \ln(8/7)$ Similarly, for all values of $k \in N$, we can find some $0 < \varepsilon < r$ and α such that condition (d) is fulfilled. Now, checking for condition (b), we have two cases:

(i) If $(x, y) \in B(x_0, r) \times B(x_0, r)$, then

$$\begin{aligned} d(Tx, Ty) &= \left(\frac{x}{3} - \frac{y}{3}\right)^2 = \frac{1}{9}(x - y)^2 < \left(\frac{3}{4}(x - y)^2\right) \\ &= a\, d(x, y) + 0.\ d(x, Tx) + 0.\ d(y, Ty) \\ &= a\, d(x, y) + b\, d(x, Tx) + c\, d(y, Ty), \end{aligned} \tag{65}$$

as $b = c = 0$.

Therefore, for all $(x, y) \in B(x_0, r) \times B(x_0, r)$, condition (d) is also satisfied.

(ii) If $(x, y) \notin B(x_0, r) \times B(x_0, r)$, e.g., $x = 2$ and $y = 3$, then

$$\begin{aligned} d(Tx, Ty) &= \left|2^3 - 3^3\right| > \left(\frac{3}{4}|(2 - 3)|\right) \\ &= a\, d(x, y) \\ &= a\, d(x, y) + 0.\ d(x, Tx) + 0.\ d(y, Ty) \\ &= a\, d(x, y) + b\, d(x, Tx) + c\, d(y, Ty). \end{aligned} \tag{66}$$

Hence, condition (b) holds only for $B(x_0, r)$ and not on $X \times X$. Moreover, $0 \in B(x_0, r)$ is the fixed point of T.

Corollary 6. *Suppose $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ and (X, d) is an $\mathcal{F}$-complete $\mathcal{F}$-metric space. Let $S, T: X \longrightarrow X$are self-mappings and $k \in [0, 1)$, assume that, for $x_0 \in X$ and $r > 0$, the following conditions are satisfied:*

(a) $B(x_0, r) \subseteq X$ is $\mathcal{F}$-closed

(b) $d(Sx, Ty) \leq (k/2)(d(x, Sx) + d(y, Ty))$, for all $x, y \in B(x_0, r)$

(c) $d(x_0, x_1) \leq (1 - \lambda)r$, for $x_1 \in X$ and $\lambda = (k/(2 - k))$

(d) there exist $0 < \varepsilon < r$ such as $f((1 - \lambda^{k+1})r) \leq f(\varepsilon) - \alpha$, where $k \in N$

Then, S and $T\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *have at most one common fixed point in* $B(x_0, r)$.

Corollary 7. *Suppose* $(f, \alpha) \in \mathscr{F} \times [0, \infty)$ *and* (X, d) *is an* $\mathscr{F}$*-complete* $\mathscr{F}$*-metric space. Let* $S, T\colon X \longrightarrow X$*are self-mappings and* $k \in [0, 1)$*, assume that, for* $x_0 \in X$ *and* $r > 0$*, the following conditions are satisfied:*

(a) $B(x_0, r) \subseteq X$ *is* $\mathscr{F}$*-closed*

(b) $d(Sx, Ty) \leq k\, d(x, y)$*, for all* $x, y \in B(x_0, r)$

(c) $d(x_0, x_1) \leq (1 - \lambda)r$*, for* $x_1 \in X$ *and* $\lambda = (k/(2 - k))$

(d) there exist $0 < \epsilon < r$ *such as* $f((1 - \lambda^{k+1})r) \leq f(\varepsilon) - \alpha$*, where* $k \in N$

Then, S and Thave at most one common fixed point in $B(x_0, r)$.

An example can be proved in a similar way as that to the previous examples.

5. Application

This section is concerned with the application of the main result proved in Section 2, in finding a unique common solution of the functional equations that are used in dynamic programming.

The two main components of dynamic programming are decision space (DS) and a state space (SS). The SS includes different states such as transitional states, initial, and action states, while the DS is composed of the steps that are taken for locating the possible solution point of the problem. Optimization and computer programming are based on this system. In particular, a problem of dynamic programming is converted to functional equations as

$$p(u) = \max_{v \in V}\{F(u, v) + f_1(u, v, p(\eta(u, v)))\}, \quad \text{for } u \in U, \tag{67}$$

$$x(u) = \max_{v \in V}\{F(u, v) + f_2(u, v, p(\eta(u, v)))\}, \quad \text{for } u \in U, \tag{68}$$

where Y and Z are Banach spaces such as $U \subseteq Y$ and $V \subseteq Z$ and

$$\begin{aligned} &\eta\colon U \times V \longrightarrow U \\ &F\colon U \times V \longrightarrow R, \\ &f_1, f_2\colon U \times V \times R \longrightarrow R. \end{aligned} \tag{69}$$

Suppose U and V are the DS and SS, respectively. We aim to locate a single common solution point for equations (67) and (68). We denote the set of all bounded real-valued mappings on U by $W(U)$. Let j be arbitrary member of $W(U)$ and say $\|j\| = \max_{u \in U}|j(u)|$. Then, the duplet $(W(U), \|\cdot\|)$ is a Banach space with $d$defined by

$$d(j, k) = \max_{u \in U}|j(u) - k(u)|. \tag{70}$$

Let the following conditions holds true:

(C1) F, f_1, f_2 are bounded.

(C2)For $u \in U$ and $j \in W(U)$, define $S, T\colon W(U) \longrightarrow W(U)$ by

$$\begin{aligned} Sj(u) &= \max_{v \in V}\{F(u, v) + f_1(u, v, j(\eta(u, v)))\}, \quad \text{for } u \in U, \\ Tj(u) &= \max_{v \in V}\{F(u, v) + f_2(u, v, j(\eta(u, v)))\}, \quad \text{for } u \in U. \end{aligned} \tag{71}$$

Observe that, S and T are well-defined whenever the functions F, f_1 and f_2 are bounded.

(C3) For $(u, v) \in U \times V$, $j, k \in W(U)$ and $l \in U$, we write

$$\left|f_1(u, v, j(l)) - f_1(u, v, k(l))\right| \leq M(j, k), \tag{72}$$

where

$$M(j, k) = \alpha\, d(j, k) + \beta\, d(j, Sj) + \gamma\, d(k, Tk), \tag{73}$$

for $\alpha, \beta, \gamma \in [0, \infty)$ and $\alpha + 2\beta + 2\gamma < 1$

Now, we develop the following theorem.

Theorem 5. *Suppose conditions* (C_1)–(C_3) *hold true, then there exists a single bounded common solution of equations (67) and (68).*

Proof. From Lemma 1.10, we have $(W(U), d)$ is an F-complete F–MS. d is defined by (70), and from (C_1), we deduce that S and T are self-mappings on $W(U)$. Let ω be an arbitrary positive number and $j_1, j_2 \in W(U)$. Take $u \in U$ and $v_1, v_2 \in V$ such as

$$Sj_x < F(u, v_x) + f_1\big(u, v_{x,} j_x(\eta(u, v_x))\big) + \omega, \tag{74}$$

$$Tj_x < F(u, v_x) + f_2\big(u, v_{x,} j_x(\eta(u, v_x))\big) + \omega, \tag{75}$$

and

$$Sj_1 \geq F(u, v_2) + f_1\big(u, v_{2,} j_1(\eta(u, v_2))\big), \tag{76}$$

$$Tj_2 \geq F(u, v_1) + f_1(u, v, j_2(\eta(u, v_1))). \tag{77}$$

Then, using (74) and (77), we obtain

$$\begin{aligned} Sj_1(u) - Tj_2(u) &< f_1(u, v_1, j_1(\eta(u, v_1))) - f_1(u, v_1, j_2(\eta(u, v_1))) + \omega \leq \left|f_1(u, v_1, j_1(\eta(u, v_1))) - f_1(u, v_1, j_2(\eta(u, v_1)))\right| + \omega \\ &\leq M(j_1(u), j_2(u)) + \omega. \end{aligned} \tag{78}$$

Also, from (75) and (76), we get

$$Tj_2(u) - Sj_1(u) < M(j_1(u), j_2(u)) + \omega. \tag{79}$$

Merging the above two inequalities, we write

$$|Sj_1(u) - Tj_2(u)| < M(j_1(u), j_2(u)) + \omega, \tag{80}$$

for all $\omega > 0$. Thus,

$$d(Sj_1(u), Tj_2(u)) \le M(j_1(u), j_2(u)), \tag{81}$$

i.e.,

$$d(Sj_1, Tj_2) \le M(j_1, j_2), \tag{82}$$

for every$\in\in U$. All the requirements of Theorem 3 are fulfilled. Therefore, by using Theorem 3, S and T have a unique bounded and common solution for equations (67) and (68). □

6. Conclusion

This article has furthered the idea of F-metric space and fixed point and common fixed point results are elaborated in the setting of F-metric space. It is obtained that the fixed point and common fixed point of a contraction mapping can be availed even if the contractive condition is restricted to only a subset closed ball of the whole F-metric space. Examples have been provided for both locally and globally contractions and a comparison between them is made for better understanding. Some important corollaries have been developed from the proved results. At last, application of the main result in finding a unique solution of the functional equation is given. In future, we opt to explore similar results in the frame of fuzzy cone metric space. Fixed point of Reich-type contractions will be investigated in picture fuzzy metric space, fuzzy soft sets, and other applicable abstract spaces. The proposed research will be primarily based upon some existing literature on the topics ([19–22]).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally to the research.

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Retraction

Retracted: TOPSIS Method for Teaching Effect Evaluation of College English with Interval-Valued Intuitionistic Fuzzy Information

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] F. Wang, "TOPSIS Method for Teaching Effect Evaluation of College English with Interval-Valued Intuitionistic Fuzzy Information," *Journal of Mathematics*, vol. 2021, Article ID 5517198, 9 pages, 2021.

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Research Article

TOPSIS Method for Teaching Effect Evaluation of College English with Interval-Valued Intuitionistic Fuzzy Information

Fengling Wang

School of Foreign Languages, Xianyang Normal University, Xianyang 712000, Shaanxi, China

Correspondence should be addressed to Fengling Wang; wangfengling6@126.com

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Teaching effect evaluation of College English is frequently considered as a multiattribute group decision-making (MAGDM) issue. Thus, a novel MAGDM method is needed to tackle it. Depending on the classical TOPSIS method and interval-valued intuitionistic fuzzy sets (IVIFSs), this paper designs a novel intuitive distance-based IVIF-TOPSIS method for teaching effect evaluation of College English. First of all, a related literature review is conducted. Furthermore, some necessary theories related to IVIFSs are briefly reviewed. In addition, the weights of attribute are decided objectively by using the CRITIC method. Afterwards, relying on novel distance measures between IVIFSs, the conventional TOPSIS method is extended to the IVIFSs to calculate closeness degree of each alternative from the interval-valued intuitionistic fuzzy positive ideal solution (IVIF-PIS). Finally, an empirical example about teaching effect evaluation of College English and some comparative analyses have been given. The results show that the designed method is useful for teaching effect evaluation of College English.

1. Introduction

Since the process of making decision is filled with uncertainty and ambiguity [1–7], in order to cope with the accuracy of decision-making [8–14], Zadeh [15] defined the fuzzy sets (FSs). Atanassov [16] defined the concept of intuitionistic fuzzy sets (IFSs). Liu et al. [17] built some intuitionistic fuzzy BM fused operators with Dombi operations. Gupta et al. [18] extended the fuzzy entropy to IFSs. He et al. [19] integrated the power averaging with IFSs. Garg [20] presented a method related to MAGDM on the basis of intuitionistic fuzzy multiplicative preference and defined several geometric operators. Chen et al. [21] developed TOPSIS method and similarity measures under IFSs. Rouyendegh [22] used the ELECTRE method in IFSs to tackle some MCDM issues. Gan and Luo [23] used the hybrid method with DEMATEL and IFSs. Jin et al. [24] defined two GDM methods which can obtain the normalized intuitionistic fuzzy priority weights from IFPRs on the basis of the order consistency and the multiplicative consistency. Xiao et al. [25] defined the intuitionistic fuzzy Taxonomy method. Zhao et al. [26] defined TODIM method for IF-MAGDM based on CPT. Cali and Balaman [27] extended ELECTRE I with VIKOR method in IFSs to reflect the decision-makers' preferences. Hao et al. [28] presented a theory of decision field for IFSs. Gupta et al. [29] modified the SIR method and combined it with IFSs. Li et al. [30] gave a grey target decision-making with IFNs. Gou et al. [31] defined some exponential operational law for IFNs. Khan and Lohani [32] defined similarity measure about IFNs. Bao et al. [33] defined prospect theory and evidential reasoning method under IFSs. Oztaysi et al. [34] solved the research proposals evaluation for grant funding using IVIFSs. Sahu et al. [35] defined the hierarchical clustering of IVIFSs. Xian et al. [36] defined combined weighted averaging operator for GDM under IVIFSs. Zhang et al. [37] defined the programming technique for MAGDM based on Shapley values and incomplete information. Zhang [38] proposed some Frank aggregation operators under IVIFSs. An et al. [39] gave the project delivery system selection with IVIF-MAGDM method. Zeng et al. [40] solved IVIF-MADM based on nonlinear programming methodology and TOPSIS method. Zhao et al. [41] defined the CPT-TODIM method for interval-valued intuitionistic fuzzy MAGDM. Wang and

Mendel [42] solved the aggregation methodology for IVIF-MADM with a prioritization of criteria.

TOPSIS was initially developed by Hwang and Yoon [43] to solve MAGDM issues. Compared with other MAGDM, TOPSIS method can consider the distances degree of every alternative from PIS and NIS. This method has been used in various fuzzy settings [44–49]. This paper's goal is to use TOPSIS method in IVIFSs and build a new decision-making model for actual MADM problems. Thus, the motivation of this study is the following: (1) the weights of attributes are decided objectively by CRITIC method; (2) an empirical example about teaching effect evaluation of College English and some comparative analyses have been given. In order to do so, the reminder of this paper is organized as follows: Some concepts of IVIFSs are reviewed in Section 2. The improved TOPSIS method is defined with IVIFSs and the calculating steps is simply listed in Section 3. An empirical application about teaching effect evaluation of College English is given to show the superiority of this designed approach and some comparative analyses are given to prove the merits of such method in Section 4. At last, we make an overall conclusion of such work in Section 5.

2. Preliminaries

2.1. IVIFSs

Definition 1 (see [50]). The interval-valued IFS (IVIFS) on X is

$$I = \{\langle x, \widetilde{\mu}_I(x), \widetilde{\nu}_I(x)\rangle, |x \in X\}, \tag{1}$$

where $\widetilde{\mu}_I(x) \subset [0,1]$ is named as "membership degree of I" and $\widetilde{\nu}_I(x) \subset [0,1]$ is called "non-membership degree of I," and $\widetilde{\mu}_I(x)$ and $\widetilde{\nu}_I(x)$ meet the following condition: $0 \leq \sup\widetilde{\mu}_I(x) + \sup\widetilde{\nu}_I(x) \leq 1$, $\forall x \in X$. For convenience, we call $I = ([\mu^L, \mu^R], [\nu^L, \nu^R])$ an IVIFN.

Definition 2 (see [51]). Let $I_1 = ([\mu_1^L, \mu_1^R], [\nu_1^L, \nu_1^R])$ and $I_2 = ([\mu_2^L, \mu_2^R], [\nu_2^L, \nu_2^R])$ be two IVIFNs; the operation formula of them can be defined:

$$\begin{aligned}
I_1 \oplus I_2 &= \left(\left[\mu_1^L + \mu_2^L - \mu_1^L\mu_2^L, \mu_1^R + \mu_2^R - \mu_1^R\mu_2^R\right], \left[\nu_1^L\nu_2^L, \nu_1^R\nu_2^R\right]\right),\\
I_1 \otimes I_2 &= \left(\left[\mu_1^L\mu_2^L, \mu_1^R\mu_2^R\right], \left[\nu_1^L + \nu_2^L - \nu_1^L\nu_2^L, \nu_1^R + \nu_2^R - \nu_1^R\nu_2^R\right]\right),\\
\lambda I_1 &= \left(\left[1 - \left(1 - \mu_1^L\right)^\lambda, 1 - \left(1 - \mu_1^R\right)^\lambda\right], \left[\left(\nu_1^L\right)^\lambda, \left(\nu_1^R\right)^\lambda\right]\right), \quad \lambda > 0,\\
I_1^\lambda &= \left(\left[\left(\mu_1^L\right)^\lambda, \left(\mu_1^R\right)^\lambda\right], \left[1 - \left(1 - \lambda_1^L\right)^\lambda, 1 - \left(1 - \lambda_1^R\right)^\lambda\right]\right), \quad \lambda > 0.
\end{aligned} \tag{2}$$

Definition 3 (see [52]). Let $I_1 = ([\mu_1^L, \mu_1^R], [\nu_1^L, \nu_1^R])$ and $I_2 = ([\mu_2^L, \mu_2^R], [\nu_2^L, \nu_2^R])$ be IVIFNs; the score and accuracy values of I_1 and I_2 can be defined:

$$\begin{aligned}
S(I_1) &= \frac{\mu_1^L + \mu_1^L\left(1 - \mu_1^L - \nu_1^L\right) + \mu_1^R + \mu_1^R\left(1 - \mu_1^R - \nu_1^R\right)}{2},\\
S(I_2) &= \frac{\mu_2^L + \mu_2^L\left(1 - \mu_2^L - \nu_2^L\right) + \mu_2^R + \mu_2^R\left(1 - \mu_2^R - \nu_2^R\right)}{2},\\
H(I_1) &= \frac{\mu_1^L + \nu_1^L + \mu_1^R + \nu_1^R}{2},\\
H(I_2) &= \frac{\mu_2^L + \nu_2^L + \mu_2^R + \nu_2^R}{2}.
\end{aligned} \tag{3}$$

For two IVIFNs I_1 and I_2, according to Definition 3, we have the following:

(1) if $s(I_1) < s(I_2)$, then $I_1 < I_2$
(2) if $s(I_1) = s(I_2), h(I_1) < h(I_2)$, then $I_1 > I_2$
(3) if $s(I_1) = s(I_2), h(I_1) = h(I_2)$, then $I_1 = I_2$

Definition 4 (see [53]). Let $I_1 = ([\mu_1^L, \mu_1^R], [\nu_1^L, \nu_1^R])$ and $I_2 = ([\mu_2^L, \mu_2^R], [\nu_2^L, \nu_2^R])$ be IVIFNs; the Euclidean distance between two IVIFNs can be given as follows:

$$\mathrm{IVIFED}(I_1, I_2) = \sqrt{\frac{1}{4}\left[\left(\mu_1^L - \mu_2^L\right)^2 + \left(\mu_1^R - \mu_2^R\right)^2 + \left(\nu_1^L - \nu_2^L\right)^2 + \left(\nu_1^R - \nu_2^R\right)^2\right]}. \tag{4}$$

2.2. Two Aggregation Operators under IVIFSs. Under the IVIFSs, some fused operators will be introduced in this section, including IVIFWA fused operator and IVIFWG fused operator.

Definition 5 (see [54]). Let $I_j = ([\mu_{I_j}^L, \mu_{I_j}^R], [\nu_{I_j}^L, \nu_{I_j}^R])$ $(j = 1, 2, \ldots, n)$ be a set of IVIFNs; the IFWA operator is

$$\begin{aligned}
\mathrm{IVIFWA}_\omega(I_1, I_2, \ldots, I_n) &= \mathop{\oplus}_{j=1}^{n}\left(\omega_j I_j\right)\\
&= \left(\left[1 - \prod_{j=1}^{n}\left(1 - \mu_{I_j}^L\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - \mu_{I_j}^R\right)^{\omega_j}\right], \left[\prod_{j=1}^{n}\left(\nu_{I_j}^L\right)^{\omega_j}, \prod_{j=1}^{n}\left(\nu_{I_j}^R\right)^{\omega_j}\right]\right),
\end{aligned} \tag{5}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of I_j $(j = 1, 2, \ldots, n)$ and $\omega_j > 0, \sum_{j=1}^{n} \omega_j = 1$.

Definition 6 (see [51]). Let $I_j = ([\mu_{I_j}^L, \mu_{I_j}^R], [\nu_{I_j}^L, \nu_{I_j}^R])$ $(j = 1, 2, \ldots, n)$ be a set of IVIFNs; the IVIFWG operator is

$$\begin{aligned}\mathrm{IVIFWG}_{\omega}(I_1, I_2, \ldots, I_n) &= \mathop{\otimes}_{j=1}^{n} (I_j)^{\omega_j} \\ &= \left(\left[\prod_{j=1}^{n}\left(\mu_{I_j}^L\right)^{\omega_j}, \prod_{j=1}^{n}\left(\mu_{I_j}^R\right)^{\omega_j}\right], \left[1 - \prod_{j=1}^{n}\left(1 - \nu_{I_j}^L\right)^{\omega_j}, 1 - \prod_{j=1}^{n}\left(1 - \nu_{I_j}^R\right)^{\omega_j}\right]\right),\end{aligned} \tag{6}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of I_j $(j = 1, 2, \ldots, n)$ and $\omega_j > 0, \sum_{j=1}^{n} \omega_j = 1$.

3. TOPSIS Method for IVIF-MAGDM with the CRITIC Method

In this section, we build the IVIF-TOPSIS method for MAGDM. The calculating steps of the designed method can be described subsequently. Let $R = \{R_1, R_2, \ldots, R_n\}$ be the group of attributes, and let $r = \{r_1, r_2, \ldots, r_n\}$ be the weight of attributes R_j, where $r_j \in [0, 1]$, $j = 1, 2, \ldots, n$, $\sum_{j=1}^{n} r_j = 1$. Assume that $H = \{H_1, H_2, \ldots, H_l\}$ is a set of DMs that have degree of $h = \{h_1, h_2, \ldots, h_l\}$, where $h_k \in [0, 1]$, $k = 1, 2, \ldots, l$. $\sum_{k=1}^{l} h_k = 1$. Let $F = \{F_1, F_2, \ldots, F_m\}$ be a set of alternatives. $Q = (q_{ij})_{m\times n}$ is the matrix with IVIFNs, where q_{ij} means F_i for R_j. Subsequently, the specific calculating steps will be depicted.

Step 1. Build each DM's matrix $Q^{(k)} = (q_{ij}^k)_{m\times n}$ with IVIFNs and calculate the overall IVIF decision matrix $Q = (q_{ij})_{m\times n}$.

$$\begin{aligned}Q^{(k)} &= \left[q_{ij}^k\right]_{m\times n} = \begin{bmatrix} q_{11}^k & q_{12}^k & \cdots & q_{1n}^k \\ q_{21}^k & q_{22}^k & \cdots & q_{2n}^k \\ \vdots & \vdots & \vdots & \vdots \\ q_{m1}^k & q_{m2}^k & \cdots & q_{mn}^k \end{bmatrix}, \\ Q &= \left[q_{ij}\right]_{m\times n} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ q_{m1} & q_{m2} & \cdots & q_{mn} \end{bmatrix}, \\ q_{ij} &= \left(\left[1 - \prod_{k=1}^{l}\left(1 - \mu_{q_{ij}^k}^L\right)^{d_k}, 1 - \prod_{k=1}^{l}\left(1 - \mu_{q_{ij}^k}^R\right)^{d_k}\right], \left[\prod_{k=1}^{l}\left(\nu_{q_{ij}^k}^L\right)^{d_k}, \prod_{k=1}^{l}\left(\nu_{q_{ij}^k}^R\right)^{d_k}\right]\right),\end{aligned} \tag{7}$$

where q_{ij}^k is the assessment value of F_i $(i = 1, 2, \ldots, m)$ on the basis of the attribute R_j $(j = 1, 2, \ldots, n)$ and the DM H_k $(k = 1, 2, \ldots, l)$.

Step 2. Normalize the overall matrix $Q = (q_{ij})_{m\times n}$ with IFNs to $Q^N = [q_{ij}^N]_{m\times n}$.

$$q_{ij}^N = \begin{cases} \left(\left[\mu_{ij}^L, \mu_{ij}^R\right], \left[\nu_{ij}^L, \nu_{ij}^R\right]\right), & Z_j \text{ is a benefit criterion,} \\ \left(\left[\nu_{ij}^L, \nu_{ij}^R\right], \left[\mu_{ij}^L, \mu_{ij}^R\right]\right), & Z_j \text{ is a cost criterion.} \end{cases} \tag{8}$$

Step 3. Employ CRITIC method to determine the weighting of attributes.

CRiteria Importance Through Intercriteria Correlation (CRITIC) method [55] will be proposed in this part, which is utilized to decide attributes' weights.

(1) Depending on the normalized overall matrix $Q^N = (q_{ij}^N)_{m\times n}$ with IVIFNs, the correlation coefficient between attributes can be defined.

$$\mathrm{IVIFCC}_{jr} = \frac{\sum_{i=1}^{m}\left(S\left(q_{ij}^N\right) - S\left(q_j^N\right)\right)\left(S\left(q_{ir}^N\right) - S\left(q_r^N\right)\right)}{\sqrt{\sum_{i=1}^{m}\left(S\left(q_{ij}^N\right) - S\left(q_j^N\right)\right)^2}\sqrt{\sum_{i=1}^{m}\left(S\left(q_{ir}^N\right) - S\left(q_r^N\right)\right)^2}}, \quad j, r = 1, 2, \ldots, n, \tag{9}$$

where $S(q_j^N) = (1/m)\sum_{i=1}^m S(q_{ij}^N)$ and $S(q_t^N) = (1/m)\sum_{i=1}^m S(q_{it}^N)$.

(2) Obtain attributes' standard deviation.

$$\mathrm{IVIFSD}_j = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(S\left(q_{ij}^N\right) - S\left(q_j^N\right)\right)^2}, \quad j = 1, 2, \ldots, n, \tag{10}$$

where $S(q_j^N) = (1/m)\sum_{i=1}^m S(q_{ij}^N)$.

(3) Obtain the attributes' weights.

$$r_j = \frac{\mathrm{IVIFSD}_j \sum_{t=1}^n \left(1 - \mathrm{IVIFCC}_{jt}\right)}{\sum_{j=1}^n \left(\mathrm{IVIFSD}_j \sum_{t=1}^n \left(1 - \mathrm{IVIFCC}_{jt}\right)\right)}, \quad j = 1, 2 \ldots, n, \tag{11}$$

where $r_j \in [0, 1]$ and $\sum_{j=1}^n r_j = 1$.

Step 4. Define the interval-valued intuitionistic fuzzy PIS (IVIF-PIS) A_j^+ and the interval-valued intuitionistic fuzzy NIS (IVIF-NIS) A_j^- as

$$\begin{aligned}\mathrm{IVIFPIS}_j &= \left(\left[\mu_j^{L+}, \mu_j^{R+}\right], \left[\nu_j^{L+}, \nu_j^{R+}\right]\right),\\ \mathrm{IVIFNIS}_j &= \left(\left[\mu_j^{L+}, \mu_j^{R+}\right], \left[\nu_j^{L+}, \nu_j^{R+}\right]\right),\end{aligned} \tag{12}$$

where$\mathrm{IVIFPIS}_j = ([\max_j(\mu_{ij}^L), \max_j(\mu_{ij}^R)], [\min_j(\nu_{ij}^L), \min_j(\nu_{ij}^R)])$ and $\mathrm{IVIFNIS}_j = ([\min_j(\mu_{ij}^L), \min_j(\mu_{ij}^R)], [\max_j(\nu_{ij}^L), \max_j(\nu_{ij}^R)])$.

Step 5. Compute the positive distances d_i^+ between each alternative and IVIF-PIS and the negative distances d_i^- between each alternative and IVIF-NIS as

$$\begin{aligned}d_i^+ &= \sum_{j=1}^n r_j \mathrm{IVIFED}\left(q_{ij}^N, A_j^+\right), \quad i = 1, 2, \ldots, m,\\ d_i^- &= \sum_{j=1}^n r_j \mathrm{IVIFED}\left(q_{ij}^N, A_j^-\right), \quad i = 1, 2, \ldots, m,\end{aligned} \tag{13}$$

where $\mathrm{IVIFED}(q_{ij}^N, A_j^+)$ and $\mathrm{IVIFED}(q_{ij}^N, A_j^-)$ denote the IVIF Euclidean distances given in Definition 4, and r_j is the weight of attributes.

Step 6. Compute each alternative's closeness degree from IVIF-PIS as

$$C_i = \frac{d_i^-}{d_i^- + d_i^+}, \quad i = 1, 2, \ldots, m. \tag{14}$$

Step 7. According to the value, C_i $(i = 1, 2, \ldots, m)$. The highest value of C_i $(i = 1, 2, \ldots, m)$ is the optimal alternative which is designed.

4. The Empirical Example and Comparative Analysis

4.1. Empirical Example. With the increasing development of economy and more frequent communication between countries, English, as an international language, has more important position and plays a greater role. Accordingly, the requirements for English teaching and learning become higher. Although experts and scholars have been trying to reform the English teaching approaches, the result is not satisfactory. In particular, the recent increasing enrollment has challenged the teaching of College English greatly. The increasing number of students and the lack of faculties lead to the larger number of students in English class. So how to improve the quality of large-class English teaching is the great concern of teachers and students, which is also the ultimate purpose of this research. It is evident that the traditional teacher-centered teaching approach cannot meet the demands of the development. At this moment, the popular cooperative learning approach has gained wide attention. Cooperative learning theories and methods have been researched deeply and are adopted widely in many countries all over the world. The core of the cooperative learning is the group work. It emphasizes the student as center and the teacher as designer, instructor, monitor, etc. By means of such instruments as questionnaires, tests, interviews, and classroom observations, the research on the effect of cooperative learning on the large-class College English teaching is conducted. The result of the research shows that the cooperative learning theories and methods are suitable for the large-class English teaching and are helpful to improve the quality of the teaching. The cooperative learning's heterogeneous group, positive interdependence, individual accountability and group work, and so forth make the classroom atmosphere relaxed and greatly improve students' positivity of participation and interest in learning. Students make great progress not only in academic performance but also in communication skills, self-confidence, self-esteem, and so forth. Through this research, some disadvantages of cooperative learning in large-class College English teaching are found, such as students' inadequate preparations for the group work and unequal opportunities and time for participation of group members. On the basis of these findings in the research, some pedagogical implications are put forward to improve the effect of cooperative learning and the quality of large-class College English teaching. In this chapter, an empirical application about teaching effect evaluation of College English will be provided by making use of IVIF-TOPSIS method. There are five potential College English teaching methods F_i $(i = 1, 2, 3, 4, 5)$ preparing to evaluate their investment environment. In order to assess the effect of College English teaching methods fairly, three experts $H = \{H_1, H_2, H_3\}$ (expert's weight $h = (0.35, 0.32, 0.33)$) are invited. All experts depict their assessment information through four subsequent attributes: ① R_1 denotes teaching attitude; ② R_2 denotes the teaching methods; ③ R_3 denotes student feedback; ④ R_4 denotes peer recognition. The decision-making matrices are given in Tables 1–3.

Then, we shall use the defined TOPSIS method for teaching effect evaluation of College English.:

Step 1. Based on the decision-making information $Q^{(k)} = (q_{ij}^k)_{m\times n}$ $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ given in

Table 1: Decision-making information given by H_1.

	R_1	R_2	R_3	R_4
F_1	([0.16, 0.22], [0.65, 0.78])	([0.33, 0.42], [0.50, 0.58])	([0.24, 0.30], [0.65, 0.70])	([0.47, 0.55], [0.40, 0.45])
F_2	([0.32, 0.40], [0.55, 0.60])	([0.17, 0.25], [0.70, 0.75])	([0.71, 0.80], [0.14, 0.20])	([0.60, 0.70], [0.25, 0.30])
F_3	([0.43, 0.47], [0.50, 0.53])	([0.32, 0.40], [0.55,0.60])	([0.57, 0.62], [0.30, 0.38])	([0.29, 0.36], [0.58, 0.64])
F_4	([0.32, 0.39], [0.41, 0.61])	([0.27, 0.36], [0.57, 0.64])	([0.34, 0.40], [0.50, 0.60])	([0.32, 0.40], [0.55, 0.60])
F_5	([0.25, 0.30], [0.55, 0.70])	([0.44, 0.48], [0.50, 0.52])	([0.62, 0.70], [0.25, 0.30])	([0.60, 0.65], [0.30, 0.35])

Table 2: Decision-making information given by H_2.

	R_1	R_2	R_3	R_4
F_1	([0.36, 0.41], [0.56, 0.59])	([0.41, 0.45], [0.50, 0.55])	([0.74, 0.80], [0.15, 0.20])	([0.52, 0.62], [0.32, 0.38])
F_2	([0.70, 0.80], [0.15, 0.20])	([0.36, 0.40], [0.57, 0.60])	([0.59, 0.65], [0.30, 0.35])	([0.66, 0.75], [0.20, 0.25])
F_3	([0.55, 0.62], [0.27, 0.38])	([0.29, 0.35], [0.60, 0.65])	([0.57, 0.62], [0.32, 0.38])	([0.60, 0.65], [0.30, 0.35])
F_4	([0.28, 0.46], [0.50, 0.54])	([0.53, 0.60], [0.35, 0.40])	([0.68, 0.75], [0.20, 0.25])	([0.35, 0.40], [0.55, 0.60])
F_5	([0.52, 0.60], [0.35, 0.40])	([0.46, 0.52], [0.40, 0.48])	([0.41, 0.52], [0.40, 0.48])	([0.58, 0.63], [0.30, 0.37])

Table 3: Decision-making information given by H_3.

	R_1	R_2	R_3	R_4
F_1	([0.59, 0.62], [0.26, 0.38])	([0.63, 0.70], [0.25, 0.30])	([0.37, 0.45], [0.50, 0.55])	([0.55, 0.60], [0.32, 0.40])
F_2	([0.65, 0.75], [0.20, 0.25])	([0.35, 0.40], [0.55, 0.60])	([0.70, 0.80], [0.10, 0.20])	([0.52, 0.62], [0.30, 0.38])
F_3	([0.37, 0.40], [0.53, 0.60])	([0.42, 0.48], [0.50, 0.52])	([0.19, 0.25], [0.70, 0.75])	([0.59, 0.65], [0.30, 0.35])
F_4	([0.61, 0.65], [0.30, 0.35])	([0.38, 0.42], [0.52, 0.58])	([0.62, 0.70], [0.25, 0.30])	([0.37, 0.45], [0.55, 0.60])
F_5	([0.35, 0.45], [0.50, 0.55])	([0.61, 0.65], [0.30, 0.35])	([0.36, 0.40], [0.55, 0.60])	([0.55, 0.62], [0.28, 0.38])

Tables 1–3 and the expert's weights$h = (0.35, 0.32, 0.33)$, we can derive the overall matrix $Q = (q_{ij})_{m\times n}$ $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$ according to equation (10), and the computing results are listed in Table 4.

Step 2. All the attributes are beneficial attributes; thus, this step is omitted.

Step 3. Decide the attribute weights r_j $(j = 1, 2, \ldots, n)$ by CRITIC method as listed in Table 5.

Step 4. Calculate the IVIF-PIS A_j^+ and the IVIF-NIS A_j^- according to equations (20) and (21).

$$\begin{aligned} A_j^+ &= \left\{ \begin{array}{l} (0.6862, 0.1569), (0.4924, 0.2844), \\ (0.4413, 0.1625), (0.5054, 0.2042) \end{array} \right\}, \\ A_j^- &= \left\{ \begin{array}{l} (0.3169, 0.2763), (0.1986, 0.5885), \\ (0.2051, 0.3945), (0.3078, 0.4041) \end{array} \right\}. \end{aligned} \tag{15}$$

Step 5. Compute the distances d_i^+ and d_i^-; the results are as follows:

$$\begin{aligned} &d_1^+ = 0.1823, d_2^+ = 0.1978, d_2^+ = 0.1043, d_2^+ = 0.2123, d_2^+ = 0.2213; \\ &d_1^- = 0.1246, d_2^- = 0.1623, d_2^- = 0.2509, d_2^- = 0.1366, d_2^- = 0.1735. \end{aligned} \tag{16}$$

Step 6. Compute each alternative's closeness degree C_i from IVIF-PIS by equation (14); the results are as follows:

$$\begin{aligned} C_1 &= 0.3709, \\ C_2 &= 0.4982, \\ C_3 &= 0.6976, \\ C_4 &= 0.3869, \\ C_5 &= 0.3916. \end{aligned} \tag{17}$$

Step 7. Relying on C_i, all the alternatives can be ordered, and the higher the value of C_i is, the best alternative selected will be. Evidently, the order is $F_3 > F_2 > F_5 > F_4 > F_1$ and F_3 is the optimal College English teaching method.

4.2. Comparison Analysis. In this section, our defined method is compared with some other methods to show its superiority.

First of all, our defined method is compared with IVIFWA and IVIFWG operators [54]. For the IVIFWA operator, the calculating result is $S(F_1) = 0.0795, S(F_2) = 0.1508, S(F_3) = 0.3435, S(F_4) = 0.0498, S(F_5) = 0.0421$. Thus, the ranking order is $F_3 > F_2 > F_1 > F_4 > F_5$. For the

Table 4: The overall matrix with IVIFNs.

	R_1	R_2	R_3	R_4
F_1	([0.5265, 0.5879], [0.2908, 0.4079])	([0.4478, 0.5187], [0.3848, 0.4723])	([0.5356, 0.6149], [0.4550, 0.3636])	([0.4872, 0.5702], [0.4127, 0.4153])
F_2	([0.5623, 0.6406], [0.3011, 0.3594])	([0.3034, 0.3589], [0.5872, 0.6411])	([0.6589, 0.7498], [0.1660, 0.2502])	([0.5875, 0.6829], [0.2511, 0.3171])
F_3	([0.4125, 0.4685], [0.4625, 0.5315])	([0.3638, 0.4299], [0.5243, 0.5701])	([0.5273, 0.5805], [0.3471, 0.4195])	([0.5144, 0.5866], [0.3420, 0.4134])
F_4	([0.4759, 0.5699], [0.3497, 0.4301])	([0.4333, 0.5053], [0.4275, 0.4947])	([0.5314, 0.6012], [0.3298, 0.3988])	([0.3010, 0.3678], [0.5677, 0.6322])
F_5	([0.3160, 0.4106], [0.5152, 0.5894])	([0.4702, 0.5165], [0.4317, 0.4835])	([0.5080, 0.5864], [0.3567, 0.4136])	([0.4932, 0.5582], [0.3658, 0.4418])

Table 5: The attributes weights r_j.

	R_1	R_2	R_3	R_4
r_j	0.2778	0.2192	0.2612	0.2418

Table 6: Evaluation results of these methods.

Methods	Ranking order	The best alternative	The worst alternative
IVIFWA operator [54]	$F_3 > F_2 > F_1 > F_4 > F_5$	F_3	F_5
IVIFWG operator [54]	$F_3 > F_2 > F_4 > F_5 > F_1$	F_3	F_1
IVIF-VIKOR method [56]	$F_3 > F_2 > F_4 > F_1 > F_5$	F_3	F_5
IVIF-CODAS method [57]	$F_3 > F_2 > F_4 > F_5 > F_1$	F_3	F_1
The developed method	$F_3 > F_2 > F_5 > F_4 > F_1$	F_3	F_1

IVIFWG operator, the calculating result is $S(F_1) = -0.0116$, $S(F_2) = 0.1239$, $S(F_3) = 0.3213$, $S(F_4) = 0.0368$, $S(F_5) = 0.0087$. So the ranking order is $F_3 > F_2 > F_4 > F_5 > F_1$.

Furthermore, our defined method is compared with the IVIF-VIKOR method [56]. Then we can obtain the calculating result. The closest ideal score values are the following: $\mathrm{CI}^*(F_1) = 0.9034$, $\mathrm{CI}^*(F_2) = 0.6714$, $\mathrm{CI}^*(F_3) = 0.0000$, $\mathrm{CI}^*(F_4) = 0.9854$, and $\mathrm{CI}^*(F_5) = 0.9509$; and the farthest worst score values are the following: $\mathrm{CI}^-(F_1) = 0.0134$, $\mathrm{CI}^-(F_2) = 0.3467$, $\mathrm{CI}^-(F_3) = 1.0000$, $\mathrm{CI}^-(F_4) = 0.0176$, and $\mathrm{CI}^-(F_5) = 0.0000$. Then the alternatives' relative closeness is calculated as follows: $\mathrm{DRC}_1 = 0.9859$, $\mathrm{DRC}_2 = 0.6656$, $\mathrm{DRC}_3 = 0.0000$, $\mathrm{DRC}_4 = 0.9796$, and $\mathrm{DRC}_5 = 1.0000$. Hence, the order is $F_3 > F_2 > F_4 > F_1 > F_5$.

In the end, our defined method is also compared with IVIF-CODAS method [57]. Then we can have the calculating result. The total assessment score (AS) of each alternative is calculated as follows: $\mathrm{AS}_1 = -0.8023$, $\mathrm{AS}_2 = 0.1650$, $\mathrm{AS}_3 = 1.4827$, $\mathrm{AS}_4 = -0.3976$, and $\mathrm{AS}_5 = -0.4436$. Therefore, the order is $F_3 > F_2 > F_4 > F_5 > F_1$.

Eventually, the results of these methods are in Table 6.

From Table 6, it is evident that the best alternative is F_3, while the worst alternative is F_1 in most situations. In other words, these methods' order is slightly different. Different methods can tackle MAGDM from different angles.

5. Conclusion

With the development of multimedia technology and the wide use of the Internet and computer, College English teaching is becoming more and more multimodal. The rapid development of information technology promotes the change in the ways of communication and the education idea. However, the traditional teaching mode is not adapted to the requirements of the times. This paper offers an effective solution for this issue, since it designs a novel intuitive distance based IVIF-TOPSIS method to build the teaching effect evaluation of College English. Then a numerical example has been given to confirm that this novel method is reasonable. What is more, to verify the validity and feasibility of the developed method, some comparative analysis is also given. However, the main drawback of this paper is that the numbers of DMs and attributes are small, and interdependency of attributes is not taken into consideration, which may limit the application scope of the developed method to some extent. Thus, the highlights of this study are the following: (1) the weights of attributes are derived objectively by CRITIC method; (2) an empirical example about teaching effect evaluation of College English and some comparative analyses have been given to show the effectiveness of the designed IVIF-TOPSIS method in MAGDM issues. In our future works, the designed model and algorithm will be needful and meaningful to apply to solve other real MADM or MAGDM problems [58–62], and the designed methods can also be extended to other uncertain settings [63–68].

Data Availability

The data used to support the findings of this study are included in the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Retraction

Retracted: Fire Safety Evaluation for Scenic Spots: An Evidential Best-Worst Method

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] D. Chen and H. Xie, "Fire Safety Evaluation for Scenic Spots: An Evidential Best-Worst Method," *Journal of Mathematics*, vol. 2021, Article ID 5592150, 10 pages, 2021.

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Review Article

Fire Safety Evaluation for Scenic Spots: An Evidential Best-Worst Method

Dongjun Chen [1,2] and Hongbin Xie [1,2]

[1] *College of Geography Science, Fujian Normal University, Fuzhou 350007, China*
[2] *National Laboratory of Humid Subtropical Eco-geographical Process, Ministry of Education, Fuzhou 350007, China*

Correspondence should be addressed to Hongbin Xie; xiehongbin933@sina.com

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Fire safety plays a vital role in tourism management, which can cause significant loss of life and property. It is necessary to present an efficient fire safety evaluation for scenic spots. However, some key issues are not well addressed in existing methods. For example, how to model experts' opinions and how to combine them are still open problems. To address these issues, a new evaluation method based on the Dempster–Shafer evidence theory and best-worst method is presented. First, a fire safety evaluation index system is constructed using the domino model. Domain experts can evaluate different factors with linguistic assessments. The best-worst method is used to determine the weights of different factors. Then, these weighted linguistic assessments are efficiently fused by Dempster's combination rule to obtain the evaluation result. Finally, a case study is illustrated to demonstrate the efficiency of the proposed method in fire safety evaluation for scenic spots. The main contribution of the proposed method is to represent and handle the uncertainty in experts' linguistic assessments, so as to decrease the uncertainty and improve decision making. In addition, the weight determination method BWM is easier and more reliable than the existing method AHP.

1. Introduction

Safety is of great significance for the development of tourism. A tourism accident can cause a great damage to the economy of tourism and tourists' lives and properties. Tourism safety has been an unavoidable problem since accidents occur frequently. Studies concerning tourism safety mainly focus on food safety [1], terrorist incidents [2], natural disasters [3], social crime [4], and road accidents [5]. However, there is not enough focus on fire safety.

Fire accident in tourism reminds us of the importance of fire safety in scenic spots. For example, on September 30, 2018, Longji Terraced Fields Scenic Area, a famous scenic spot in Guilin, Guangxi Zhuang Autonomous Region, China, caught fire. 48 ancient houses burned down and the fire almost destroyed the whole scenic area. Scenic spots afford a remarkable tourism service and have capacity for a great number of tourists, so it is necessary and significant to establish a fire safety evaluation system for scenic spots.

Reviewing the existing literature, we find that there are very few articles concerning fire safety in tourism, but only some research on related situations has been carried out. For example, Spyrou et al. [6] proposed a general risk model for evaluating the fire safety of passenger ships. Chen et al. [7] carried out a quantitative risk assessment of cotton storage fire accidents. Brzezinska et al. [8] presented a new evaluation and indicating approach for sustainable fire safety in the process industry. However, it is still an open problem to handle uncertainty in evaluation models.

A common idea in evaluating a fire system is to ask authoritative experts for opinions and then to make a comprehensive consideration for decision. Hence, the fire system evaluation problem is a kind of multicriteria decision-making (MCDM) problem. One of the important problems in MCDM is to deal with uncertainty [9,10]. Many mathematical tools such as fuzzy sets [11–14], neutrosophic sets [15], and Z-numbers [16] are used. For example, Xue et al. [17] addressed the uncertain database retrieval problem based on intuitionistic fuzzy set. Li et al. [18] proposed a new uncertainty measure of discrete Z-numbers and applied it to solve MCDM problems. Harish and Gagandeep [19] developed a method to solve MCDM problems under the probabilistic dual hesitant fuzzy set environment.

In view of the uncertainty in experts' linguistic assessments and the need for opinion integration, a suitable tool, named Dempster–Shafer evidence theory (DSET) [20–22], can solve the two problems perfectly. DSET is an efficient tool for decision making under uncertain environment and provides a combination rule for information fusion [23, 24]. In addition, in this paper, we apply the best-worst method (BWM) [25, 26] to determine different weights of factors. Actually, there are some research studies on the combination of DSET and BWM. For example, Fei et al. [27] extended BWM by belief functions in DSET and implemented the algorithm in hospital service evaluation. Liu et al. [28] developed a MCDM method in combination with DSET and BWM to solve the sustainable development alternative selection problem.

Therefore, based on DSET and BWM, this paper presents an evidential fire safety evaluation method for scenic spots, which can help managers to assess fire risk and take measures to prevent fire accidents. The proposed model can well deal with the uncertainty flexibly. In summary, the main contribution of the proposed method is that it can well deal with the uncertainty flexibly by modeling and fusing uncertain information, which is useful to reduce uncertainty and improve decision making. In addition, the weight determination method BWM is easier and more reliable than the existing method AHP (analytic hierarchy process).

The remainder of the paper is organized as follows. Section 2 introduces the accident model and the related theory including DSET and BWM. In Section 3, we present the index system and the evidential BWM for fire safety. Section 4 illustrates a case of fire safety evaluation. Finally, we conclude this paper in Section 5.

2. Preliminaries

2.1. Accident Model. The accident model plays a critical part in processing safety management, since it can provide a better understanding of accident scenarios and describe the relation between causes and consequences [29]. Heinrich [30] originally proposed the domino theory to illustrate sequential aspects of accident occurrence. This considers the accident as the outcome of series of successive events, rather than an isolated incident, and distinguishes five stages or factors in an accident, including ancestry and social environment, fault of person, unsafe act and/or mechanical or physical hazard, accident, and injury [31].

Updating and modifying the domino theory, which stressed on inherent shortcomings of humans, the loss causation model was proposed and many different variations appeared subsequently [32]. The loss causation model applied in the current work is usually called the domino model [33], which places more emphasis on management and organizational factors. In the domino model, an accident is directly caused by human's unsafe behaviors and objects' insecurity state and indirectly caused by personal factors and work-related factors. Management deficiency is the root cause, namely, the problem or deficiency in management leads to the remote cause, which then results in the immediate cause and ultimately brings about an accident. In summary, person, work-related object, and management comprise the accident model.

The domino model is appropriate for tourism, and this article applies it to construct a safety evaluation index system for touristic scenic spots (Figure 1).

2.2. Dempster–Shafer Evidence Theory. Dempster–Shafer evidence theory, abbreviated as DSET, was first proposed by Dempster [34] and then developed by Shafer [35]. DSET has two unique characteristics: one is to assign belief values to multi-subset propositions and the other is to fuse bodies of evidence. However, DSET still has some unresolved issues, like conflict management [36–39], dependence evidence combination [40–42], and belief entropy [43–45]. Considering its superiority under uncertain environment and its practicability in engineering [46–48], DSET has a broad application in many areas, such as risk assessment [49–52], fault diagnosis [53–55], and classification and clustering [56–58].

Assume a random variable X taking values from $\Theta = \{\Theta_1, \Theta_2, \ldots, \Theta_n\}$, where Θ is called a frame of discernment (FOD). 2^Θ denotes the power set of Θ.

A mass function (also called a basic probability assignment, BPA) is a mapping from 2^Θ to [0, 1], formally defined by [34, 35]

$$m\colon 2^\Theta \longrightarrow [0,1], \tag{1}$$

satisfying the following condition:

$$\sum_{A\in 2^\Theta} m(A) = 1, \quad m(\varnothing) = 0, \tag{2}$$

where $m(A)$ represents the belief value that supports A.

Assuming two BPAs m_1 and m_2 are from two pieces of dependent evidence, Dempster's combination rule, represented as $m = m_1 \oplus m_2$, is defined as follows [34, 35]:

$$m(A) = \frac{\sum_{A_1\cap A_2=A} m_1(A_1)m_2(A_2)}{1-K}, \tag{3}$$

where K is called the conflict coefficient of two BPAs and is calculated by $\sum_{A_1\cap A_2=\varnothing} m_1(A_1)m_2(A_2)$.

2.3. Best-Worst Method. The best-worst method (BWM) was put forward by Rezaei to manage multicriteria decision problems [59, 60]. One of the common applications of BWM is to determine weights. By making comparisons between the most important (best)/least important (worst) criteria and the others, BWM establishes a minimum optimization problem to derive weights. BWM has been broadly applied in many fields such as medical system [61, 62], risk assessment [63, 64], and supplier selection [65]. The detailed procedures of BWM are described below.

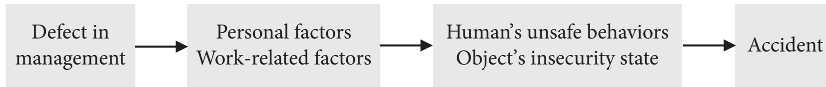

Figure 1: The domino model.

Given a list of l criteria, experts should identify the most and least important criteria, respectively, and then give a preference judgment of the most important criteria over all criteria with a scale from 1 to 9. A larger number means more preference. The results are stored in two vectors U_{BO} and V_{OW} as follows [59]:

$$\begin{aligned} U_{BO} &= (u_{B1}, u_{B2}, \ldots, u_{Bl}), \\ V_{OW} &= (v_{1W}, v_{2W}, \ldots, v_{lW}), \end{aligned} \tag{4}$$

where u_{Bj} means the preference of the best criteria B over the criteria j, and $u_{BB} = 1$. v_{jW} means the preference of the criteria j over the worst criteria W, and $v_{WW} = 1$.

With the two vectors obtained, an optional linear programming model is established to determine the optional weight $(w_1^*, w_2^*, \ldots, w_l^*)$ [59]:

$$\min \varepsilon$$

$$\text{s.t.} \quad \begin{cases} \left\| \dfrac{w_B}{w_j} - u_{Bj} \right\| \leq \varepsilon & \text{for all } j \\ \left\| \dfrac{w_j}{w_W} - v_{jW} \right\| \leq \varepsilon & \text{for all } j \\ \sum_j w_j = 1 & \\ w_j \geq 0 & \text{for all } j. \end{cases} \tag{5}$$

Furthermore, to guarantee the consistency of comparison, the definition of consistency ratio (CR) is given by BWM [59]:

$$\mathrm{CR} = \frac{\varepsilon^*}{\mathrm{CI}}, \tag{6}$$

where ε^* is the best solution of ε corresponding to equation (5) and CI is determined by u_{BW} (the preference of the most important criteria B over the least important one W), as shown in Table 1. The range of CR is [0, 1]; the larger CR is, the more inconsistent the comparison vector is. In general, $\mathrm{CR} \leq 0.1$ shows that the comparison vector is acceptable.

3. Evaluation Methodology

3.1. Establishing the Evaluation Index System. The domino model is applicable to evaluate fire safety in tourism scenic spots, wherein person includes tourists and staff, work-related object contains firefighting equipment and surroundings of scenic spots, and management means how to mobilize persons and work-related objects to be out of fire danger.

Table 1: Consistency index.

u_{BW}	1	2	3	4	5	6	7	8	9
CI	0.00	0.44	1.00	1.63	2.30	3.00	3.73	4.47	5.23

Combining this with the general actual situation of tourism scenic spots and reference to the prior related studies [66–68], we establish a fire safety evaluation index system as shown in Table 2. As we can see, the system is a hierarchical structure, including the target level, the base level, and the criteria level. The target level contains three factors: management (η_1), person (η_2), and work-related job (η_3). The base level is in the second level, such as inner management (η_{11}). There are 6 factors at the base level. The criteria level contains the most specific factors, such as fire safety education and training (η_{111}). There are 14 indexes at the criteria level.

3.2. Evaluation Method. Following the construction of the evaluation index system for fire safety, this section presents an evaluation model using DSET and BWM. The main idea is to use BWM to weight different factors at all levels, and then based on experts' linguistic evaluations on fire safety levels, the corresponding mass functions of factors at the target level can be constructed for each expert; finally, using the combination rule to fuse experts' opinions, we can determine the fire security level of scenic spots, along with the confidence degree of the assessment result. The flowchart of the proposed model is given in Figure 2. The procedure of the fire safety evaluation method is divided into six steps.

Step 1: expert evaluation on the fire security level.

Based on the fire safety evaluation index system in Table 2, experts should evaluate the fire safety status of the project on factors at the criteria level and then give linguistic assessments with a range of [0, 1]. A value closer to 0 means that the safety status of the corresponding project is more dangerous, or the working ability of the project is lower, and hence the possibility of a fire is greater. The linguistic evaluation corresponding to different scores is shown in Table 3. A_{ijz}^h means the value of the factor η_{ijz} assessed by the hth expert, $i \in [1, 3]$, $j \in [1, 2]$, $z \in [1, 3]$.

Step 2: weight all factors at different levels based on the BWM.

According to the fire safety evaluation index system in Table 2, BWM is applied to compute the weight of indexes on the same level. Here we use w_{ijz} to represent the weight of the criteria η_{ijz}. For example, w_2 means the weight of person (η_2) and w_{21} means the weight of tourists (η_{21}).

Table 2: The fire safety evaluation index system.

Target level	Base level	Criteria level
Management (η_1)	Inner management (η_{11})	Fire safety education and training (η_{111})
		Fire emergency response plan (η_{112})
	External management (η_{12})	Regular fire safety inspection (η_{121})
		Inspection for inflammable and explosive dangerous goods before entering the scenic spot (η_{122})
Person (η_2)	Tourists (η_{21})	Tourists' fire safety awareness and firefighting skills (η_{211})
		Tourist quantity (η_{212})
	Staff members (η_{22})	Staff's fire safety awareness and firefighting skills (η_{221})
		Firefighting quantity (η_{222})
Work-related object (η_3)	Firefighting device (η_{31})	Fire detection alarm system (η_{311})
		Fire extinguishing equipment (η_{312})
		Evacuation equipment (η_{313})
	Surroundings of scenic spots (η_{32})	Distance from the nearest fire station (η_{321})
		Fire resistance of building materials (η_{322})
		Traffic planning and spatial layout of scenic area (η_{323})

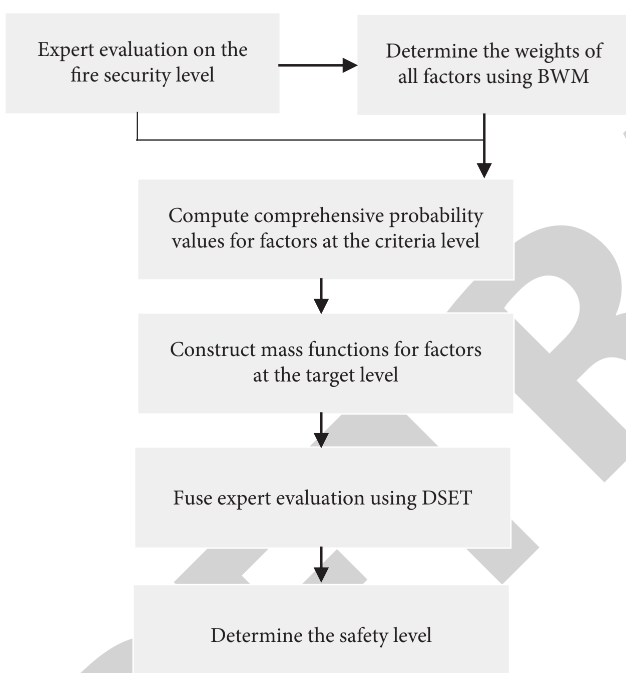

Figure 2: The flowchart of the proposed method.

Table 3: The linguistic assessment on fire safety levels.

Fire safety status	Dangerous	General	Subsafe	Safe
Scale	$0\sim<0.25$	$0.25\sim<0.5$	$0.5\sim<0.75$	$0.75\sim<1$

Example 1. Suppose experts are requested to evaluate the weight of three factors η_{321}, η_{322}, and η_{323}. After their discussion, they identify η_{323} and η_{321} as the most important and the least important factors, respectively. $U_{BO}=(8,2,1)$, $V_{OW}=(1,5,8)$. The optional linear programming model is established based on equation (6):

$$\min \varepsilon$$
$$\text{s.t.}\quad \begin{cases} \left\| \dfrac{w_3}{w_1} - 8 \right\| \leq \varepsilon \\ \left\| \dfrac{w_3}{w_2} - 2 \right\| \leq \varepsilon \\ \left\| \dfrac{w_2}{w_1} - 5 \right\| \leq \varepsilon \\ \sum\limits_{i=1}^{3} w_i = 1 \\ w_i \geq 0. \end{cases} \tag{7}$$

Using Matlab R2018a to solve this model, we can get $w_{321}^* = 0.0714$, $w_{322}^* = 0.3387$, $w_{323}^* = 0.5589$, $\varepsilon^* = 0.26$, and $\mathrm{CR} = (0.26/4.47) = 0.058 < 0.1$, which means a good consistency. That is, $w_{321} = 0.0714$, $w_{322} = 0.3387$, and $w_{323} = 0.5589$.

Step 3: compute weighted probability values for factors at the criteria level.

Based on the level of each factor, the weighted probability value for the hth expert is computed as follows:

$$p^h(\eta_{ijz}) = A_{ijz}^h * w_{ijz} * w_{ij} * w_i. \tag{8}$$

Example 2. As shown in Table 1, we take the subsystem of management (η_1) as an example. It has two factors on the base level, η_{11} and η_{12}. Each base factor includes two criteria factors, that is, η_{111}, η_{112} and η_{121}, η_{122}. Supposing that two experts participate in the evaluation, the weights of factors and the experts' scale are shown in Tables 4 and 5, respectively.

Table 4: Weights of factors.

Factor	η_1	η_{11}	η_{12}	η_{111}	η_{112}	η_{121}	η_{122}
Weight	0.1	0.6	0.4	0.5	0.5	0.2	0.8

Table 5: The experts' evaluation.

	Expert 1's evaluation	Expert 2's evaluation	Weighted probability values
η_{111}	0.3	0.5	$p^1(\eta_{111}) = 0.009,\ p^2(\eta_{111}) = 0.015$
η_{112}	0.7	0.7	$p^1(\eta_{112}) = 0.021,\ p^2(\eta_{112}) = 0.021$
η_{121}	0.6	0.5	$p^1(\eta_{121}) = 0.048,\ p^2(\eta_{121}) = 0.016$
η_{122}	0.4	0.8	$p^1(\eta_{122}) = 0.016,\ p^2(\eta_{122}) = 0.0256$

Hence, the weighted probability values for η_{111} are calculated by $p^1(\eta_{111}) = 0.3^*0.5^*0.6^*0.1 = 0.009$. $p^2(\eta_{111}) = 0.5^*0.5^*0.6^*0.1 = 0.015$. Other results are shown in the fourth column of Table 5.

Step 4: construct mass functions for factors at the target level.

Under the framework of DSET, mass functions are obtained by the sum of weighted probability values of factors at the criteria level. The specified formula is shown as follows:

$$m^h(\eta_i) = \sum_{z=1}^{f}\sum_{j=1}^{f} m^h(\eta_{ijz}), \tag{9}$$

$$m^h(\Theta) = 1 - \sum_i m^h(\eta_i), \tag{10}$$

where f is the number of all factors at the criteria level and $m^h(\Theta)$ can be regarded as the uncertainty of experts' evaluation.

Example 3. Calculate the mass functions of the target factor η_1 in Table 5:

$$\begin{aligned} m^1(\eta_1) &= p^1(\eta_{111}) + p^1(\eta_{112}) + p^1(\eta_{121}) + p^1(\eta_{122}) = 0.094,\\ m^2(\eta_1) &= p^2(\eta_{111}) + p^2(\eta_{112}) + p^2(\eta_{121}) + p^2(\eta_{122}) = 0.0776. \end{aligned} \tag{11}$$

Step 5: fuse experts' evaluations using DSET.

According to Dempster's combination rule (see equation (3)), the fused experts' evaluation can be obtained. The specified formula is shown below.

$$m(\eta_i) = m^1(\eta_i) \oplus m^2(\eta_i) \oplus \cdots \oplus m^n(\eta_i). \tag{12}$$

Step 6: determine the safety level.

The comprehensive belief value for safety assessment can be obtained by equation (13), i.e., the sum of the fused belief value for each factor at the target value. According to the criteria in Table 2, the safe level is finally determined.

$$p = \sum_i m(\eta_i). \tag{13}$$

The confidence of the assessment result is calculated by $1 - m(\Theta)$.

4. An Evaluation Case

In this paper, a case study of fire safety evaluation for one scenic spot is provided based on the proposed model. Assume that 3 experts are required to evaluate the security level on all factors at the criteria level based on Table 2; the detailed assessments are shown in Table 6. The weights of different factors based on BWM are given in Table 7.

According to Step 3 of this model, with the weights of factors at all levels and experts' linguistic assessments on security status, we can calculate the weighted probability values of indexes at the criteria level using equation (8). The results are shown in Table 8.

Based on Step 4 and Step 5 of the model, we can construct mass functions using equations (9) and (10) for each expert and then fuse them using equation (12). The results are shown below:

$$\begin{aligned} m^1(\eta_1) &= 0.2174, m^1(\eta_2) = 0.1743, m^1(\eta_3) = 0.1073, m^1(\Theta) = 0.5010,\\ m^2(\eta_1) &= 0.1952, m^2(\eta_2) = 0.1766, m^2(\eta_3) = 0.1444, m^2(\Theta) = 0.4838,\\ m^3(\eta_1) &= 0.2350, m^3(\eta_2) = 0.1649, m^3(\eta_3) = 0.1666, m^3(\Theta) = 0.4335,\\ m(\eta_1) &= 0.3611, m(\eta_2) = 0.2643, m(\eta_3) = 0.2029, m(\Theta) = 0.1716. \end{aligned} \tag{14}$$

Table 6: Experts' assessments on security level.

	Expert 1	Expert 2	Expert 3
η_{111}	0.75	0.60	0.70
η_{112}	0.80	0.75	0.80
η_{121}	0.45	0.55	0.65
η_{122}	0.65	0.50	0.70
η_{211}	0.65	0.70	0.65
η_{212}	0.70	0.75	0.85
η_{221}	0.60	0.65	0.50
η_{222}	0.35	0.30	0.25
η_{311}	0.15	0.35	0.45
η_{312}	0.35	0.50	0.60
η_{313}	0.45	0.35	0.50
η_{321}	0.80	0.70	0.65
η_{322}	0.70	0.80	0.75
η_{323}	0.30	0.50	0.45

Table 7: The weights of different factors based on BWM.

Target level	Base level	Criteria level
$\eta_1 = 0.333$	$\eta_{11} = 0.456$	$\eta_{111} = 0.655$
		$\eta_{112} = 0.345$
	$\eta_{12} = 0.544$	$\eta_{121} = 0.5$
		$\eta_{122} = 0.5$
$\eta_2 = 0.333$	$\eta_{21} = 0.450$	$\eta_{211} = 0.604$
		$\eta_{212} = 0.396$
	$\eta_{22} = 0.550$	$\eta_{221} = 0.215$
		$\eta_{222} = 0.785$
$\eta_3 = 0.333$	$\eta_{31} = 0.701$	$\eta_{311} = 0.595$
		$\eta_{312} = 0.083$
		$\eta_{313} = 0.321$
	$\eta_{32} = 0.299$	$\eta_{321} = 0.071$
		$\eta_{322} = 0.339$
		$\eta_{323} = 0.560$

Table 8: Weighted probability values of factors at the criteria level.

	$p^1_{\eta_{ijz}}$	$p^2_{\eta_{ijz}}$	$p^3_{\eta_{ijz}}$
η_{111}	0.0746	0.0597	0.0696
η_{112}	0.0431	0.0404	0.0431
η_{121}	0.0408	0.0498	0.0589
η_{122}	0.0589	0.0453	0.0634
η_{211}	0.0588	0.0634	0.0588
η_{212}	0.0415	0.0445	0.0504
η_{221}	0.0236	0.0256	0.0197
η_{222}	0.0503	0.0431	0.0359
η_{311}	0.0208	0.0486	0.0625
η_{312}	0.0068	0.0097	0.0116
η_{313}	0.0337	0.0262	0.0375
η_{321}	0.0057	0.0049	0.0046
η_{322}	0.0236	0.0270	0.0253
η_{323}	0.0167	0.0279	0.0251

Table 9: The weight results of two groups of factors by BWM and AHP.

	Method	Comparison data	Weights	CR
$\{\eta_{311}, \eta_{312}, \eta_{313}\}$	BWM	$U_{BO} = (1, 7, 2)$ $V_{OW} = (7, 1, 4)$	$w = \{0.5954, 0.0833, 0.3213\}$	$0.04 < 0.1$
	AHP	$\begin{bmatrix} 1 & 7 & 2 \\ 1/7 & 1 & 1/4 \\ 1/2 & 4 & 1 \end{bmatrix}$	$w = \{0.6026, 0.0823, 0.3150\}$	$0.0017 < 0.1$
$\{\eta_{321}, \eta_{322}, \eta_{323}\}$	BWM	$U_{BO} = (8, 2, 1)$ $V_{OW} = (1, 5, 8)$	$w = \{0.0714, 0.3387, 0.5589\}$	$0.058 < 0.1$
	AHP	$\begin{bmatrix} 1 & 1/5 & 1/8 \\ 5 & 1 & 1/2 \\ 8 & 2 & 1 \end{bmatrix}$	$w = \{0.0701, 0.3255, 0.6044\}$	$0.0048 < 0.1$

Table 10: The fire safety evaluation result by Chen and Deng's method [69].

The fused experts' result	$m(\eta_1) = 0.3607, m(\eta_2) = 0.2641, m(\eta_3) = 0.2039, m(\Theta) = 0.1712$
The final belief value	$p = 0.3607 + 0.2641 + 0.2039 = 0.8287$ (The safe level)
The confidence of the result	$1 - 0.1712 = 0.8288$

Finally, the overall belief value of safety assessment can be computed using equation (13):

$$p = m(\eta_1) + m(\eta_2) + m(\eta_3) = 0.8283. \tag{15}$$

Hence, according to Table 2, we can conclude that the comprehensive evaluation result is safe, and the confidence of the result is $1 - 0.1716 = 0.8284$.

In addition, we compare the proposed method with Chen and Deng's method [69]. An evidential AHP method was presented in [69] to evaluate sustainable transport solutions, where AHP was applied to determine weights and Dempster–Shafer evidence theory was used for handling uncertain information. The two methods are different in the weight determination methods by AHP and BWM, respectively, which will be analyzed next.

From Table 2, we can find that in most situations, there are just two factors that can be compared to determine weights, like η_{11} and η_{12}. It is easy for experts to determine the relative importance between them and get reasonable weights. Therefore, two groups with three compared factors $\{\eta_{311}, \eta_{312}, \eta_{313}\}$ and $\{\eta_{321}, \eta_{322}, \eta_{323}\}$ are taken as examples to show the advantages of BWM. It should be noted that for simplicity and fair comparison, the relative importance in the comparison matrix in AHP refers to the values in comparison vectors in BWM. Table 9 shows the weight results by AHP and BWM, respectively.

As shown in Table 9, the weights of factors for each group are very similar, and their consistency ratios (CRs) are all less than 0.1. On the one hand, the third column gives the comparison data required by BWM and AHP, which shows that BWM needs less comparison data than AHP. Exactly, BWM needs to have $(2n - 3)$ comparisons while for AHP, $n(n-1)/2$ comparisons are needed, where n means the number of factors. On the other hand, the fifth column gives CR values computed by BWM and AHP. In AHP, the values of CR are far smaller than 0.1; it means that the comparison relationship given by BWM performs well in AHP. This shows that BWM provides consistent comparison data. Since BWM considers the best and worst factors to compare, the weights derived by BWM are highly reliable as it provides more consistent comparison data compared to AHP. Actually, Rezaei has analyzed the two advantages of BWM compared to AHP in [59]. In conclusion, BWM is easier and more reliable than AHP.

Based on the weights determined by AHP in Table 9, the safety evaluation result by Chen and Deng's method [69] can be obtained. For simplicity we just display the results of the last two steps, as shown in Table 10. Compared with the proposed method, both suggest that the comprehensive evaluation result is safe, the belief values of safety assessment are very similar (0.8283 vs 0.8287), and so is the confidence of results (0.8284 vs 0.8288), which shows the effectiveness of the proposed method. However, considering the superiority of BWM compared to AHP, the proposed method is easier and more reliable than Chen and Deng's method [69].

5. Conclusion

In tourism management, fire safety is a significant problem worthy of attention, since a fire security accident will have a great negative impact on the loss of life and property. In this paper, a new evidential BWM is presented to address the fire safety evaluation for scenic spots. First, a fire safety evaluation index system is constructed using the domino model, which constructs a three-level hierarchical structure of factors. Based on the established index model, experts are required to assess the safety level using fuzzy linguistic variables. Combined with the weights determined by BWM, these linguistic assessments can be transformed into mass functions in DSET. Finally Dempster's combination rule is applied to fuse mass functions to obtain the overall belief value of safety level. The proposed method has an advantage to represent and handle the uncertainty in experts' linguistic assessments, so as to decrease the uncertainty and improve decision making. Furthermore, the weight determination method BWM is easier and more reliable than the existing method AHP. In conclusion, the proposed method can address the fire safety evaluation issue for scenic spots under uncertain environment, which is useful to lower the risk of fire and prevent fire accidents.

However, there still exist some problems to be considered. One is that the weights of experts can also be considered into the model, such as using the belief entropy [70]. The other is that using interval-valued linguistic variables as in [71, 72] may be a more flexible way to represent uncertain information.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Control Fuzzy Metric Spaces via Orthogonality with an Application

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] F. Uddin, K. Javed, H. Aydi, U. Ishtiaq, and M. Arshad, "Control Fuzzy Metric Spaces via Orthogonality with an Application," *Journal of Mathematics*, vol. 2021, Article ID 5551833, 12 pages, 2021.

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Research Article

Control Fuzzy Metric Spaces via Orthogonality with an Application

Fahim Uddin,[1] Khalil Javed ,[1] Hassen Aydi ,[2,3,4] Umar Ishtiaq,[1] and Muhammad Arshad[5]

[1]*Department of Math and Stats, International Islamic University Islamabad, Islamabad, Pakistan*
[2]*Institut Supérieur d'Informatique et des Techniques de Communication, Université de Sousse, H. Sousse 4000, Tunisia*
[3]*China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[4]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa*
[5]*Department of Mathematics and Statistics, International Islamic University, H-10, Islamabad 44000, Pakistan*

Correspondence should be addressed to Khalil Javed; khalil.msma551@iiu.edu.pk and Hassen Aydi; hassen.aydi@isima.rnu.tn

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In this article, we are generalizing the concept of control fuzzy metric spaces by introducing orthogonal control fuzzy metric spaces. We prove some fixed point results in this setting. We provide nontrivial examples to show the validity of our main results and the introduced concepts. An application to fuzzy integral equations is also included. Our results generalize and improve several developments from the existing literature.

1. Introduction and Preliminaries

Many authors studied fixed point theory explicitly, introduced and popularized lot of spaces, and made the area of fixed point theory more fascinating. In this connectedness, Bakhtin [1] and Czerwik [2] provided a generalization of metric spaces, named as a b-metric space. Zadeh [3] originated fuzzy sets. The formulation of metric spaces and fuzzy sets, named as fuzzy metric spaces, helped many authors in various ways. Nădăban [4] originated fuzzy b-metric spaces. Many authors [5–7] worked in fuzzy b-metric spaces.

In [8], the authors introduced the concept of an extended fuzzy b-metric space as a generalization of fuzzy b-metric spaces. The work [9] originates the concept of controlled metric type spaces (see also [10]). Recently, in [11], the notion of controlled-type metric spaces has been generalized by a formulation of controlled fuzzy metric spaces, which are also generalizations of extended fuzzy b-metric spaces.

Eshaghi et al. [12] introduced the notion of an orthogonal set and proved the Banach fixed point result. Many of the authors [13–15] continued working on orthogonal spaces. In this article, we are generalizing the concept of control fuzzy metric spaces [11]. Namely, we initiate the notion of orthogonal control fuzzy metric spaces.

Let us first recall some basic definitions related to this manuscript.

Definition 1 (see [4]). A 4-tuple $(Z, \Delta, *, u)$ is called a fuzzy b-metric space if Z is an arbitrary (nonempty) set, $*$ is a continuous t-norm, and Δ is a fuzzy set on $Z \times Z \times (0, \infty)$ satisfying the following conditions, for all $\nu, \omega, \varkappa \in Z, r, s > 0$ and for a given real number$u \geq 1$:

(B1) $\Delta(\nu, \omega, r) > 0$

(B2) $\Delta(\nu, \omega, r) = 1$ if and only if $\nu = \omega$

(B3) $\Delta(\nu, \omega, r) = \Delta(\omega, \nu, r)$

(B4) $\Delta(\nu, \varkappa, u(r + s)) \geq \Delta(\nu, \omega, r) * \Delta(\omega, \varkappa, s)s$

(B5) $\Delta(\nu, \omega, \cdot)\colon (0, \infty) \longrightarrow [0, 1]$ is continuous

Definition 2 (see [8]). A 4-tuple $(Z, \Delta_\alpha, *, \alpha)$ is called an extended fuzzy b-metric space if Z is a (nonempty) set, where $\alpha\colon Z \times Z \longrightarrow [1, \infty)$, $*$ is a continuous t-norm, and Δ_α is a fuzzy set on $Z \times Z \times (0, \infty)$, satisfying the following conditions, for all $\nu, \omega, \varkappa \in Z$ and $r, s > 0$:

($\Delta 1$) $\Delta_\alpha(\nu, \omega, 0) = 0$

($\Delta 2$) $\Delta_\alpha(\nu, \omega, r) = 1 \Longleftrightarrow \nu = \omega$

($\Delta 3$) $\Delta_\alpha(\nu, \omega, r) = \Delta_\alpha(\omega, \nu, r)$

($\Delta 4$) $\Delta_\alpha(\nu, \varkappa, \alpha(\nu, \varkappa)(r + s)) \geq \Delta_\alpha(\nu, \omega, r) * \Delta_\alpha(\omega, \varkappa, s)$

($\Delta 5$) $\Delta_\alpha(\nu, \omega, \cdot)\colon (0, \infty) \longrightarrow [0, 1]$ is continuous

Definition 3 (see [11]). A 4-tuple $(Z, \Delta_\gamma, *)$ is called a control fuzzy metric space if Z is a (nonempty) set, $\gamma\colon Z \times Z \longrightarrow [1, \infty)$, where $*$ is a continuous t-norm and Δ_γ is a fuzzy set on $Z \times Z \times (0, \infty)$, satisfying the following conditions, for all $\nu, \omega, \varkappa \in Z$ and $r, s > 0$:

($\Delta 1$) $\Delta_\gamma(\nu, \omega, 0) = 0$

($\Delta 2$) $\Delta_\gamma(\nu, \omega, r) = 1 \Longleftrightarrow \nu = \omega$

($\Delta 3$) $\Delta_\gamma(\nu, \omega, r) = \Delta_\gamma(\omega, \nu, r)$

($\Delta 4$) $\Delta_\gamma(\nu, \varkappa, r + s) \geq \Delta_\gamma(\nu, \omega, (r/(\gamma(\nu, \omega)))) * \Delta_\gamma(\omega, \varkappa, (s/(\gamma(\omega, \varkappa))))$

($\Delta 5$) $\Delta_\gamma(\nu, \omega, \cdot)\colon (0, \infty) \longrightarrow [0, 1]$ is continuous

Definition 4 (see [11]). Let Z be a set and let $\zeta\colon Z \longrightarrow Z$ and $O(\nu) = \{\nu_0, \zeta\nu_0, \zeta^2\nu_0, \ldots\}$, for some $\nu_0 \in Z$, be the orbit of ζ. A function $T\colon Z \longrightarrow Z$ is said to be ζ-orbitally lower semicontinuous at $u \in Z$ if $\{\nu_n\} \in O(\nu_0)$ such that $\nu_n \longrightarrow u$, then we get $T(u) \geq \lim_{n \longrightarrow \infty} \inf T(\nu_n)$.

2. Main Results

In this section, we introduce orthogonal control fuzzy metric spaces and prove some fixed point results.

Definition 5. A 4-tuple $(Z, \theta_\gamma, *, \perp)$ is called an orthogonal control fuzzy metric space if Z is an (nonempty) orthogonal set, $\gamma\colon Z \times Z \longrightarrow [1, \infty)$, where $*$ is a continuous t-norm and θ_γ is a fuzzy set on $Z \times Z \times (0, \infty)$, satisfying the following conditions:

($\theta\gamma 1$) $\theta_\gamma(\nu, \omega, r) > 0$, $\forall \nu, \omega \in Z$, $r > 0$ such that $\nu \perp \omega$ and $\omega \perp \nu$

($\theta\gamma 2$) $\theta_\gamma(\nu, \omega, r) = 1 \Longleftrightarrow \nu = \omega$, $\forall \nu, \omega \in Z$, $r > 0$ such that $\nu \perp \omega$ and $\omega \perp \nu$

($\theta\gamma 3$) $\theta_\gamma(\nu, \omega, r) = \theta_\gamma(\omega, \nu, r)$, $\forall \nu, \omega \in Z$, $r > 0$ such that $\nu \perp \omega$ and $\omega \perp \nu$

($\theta\gamma 4$) $\theta_\gamma(\nu, \varkappa, r + s) \geq \theta_\gamma(\nu, \omega, (r/\gamma(\nu, \omega))) * \theta_\gamma(\omega, \varkappa, (s/\gamma(\omega, \varkappa)))$, or $\theta_\gamma(\nu, \varkappa, \gamma(\nu, \omega)\gamma(\omega, \varkappa)(r + s)) \geq \theta_\gamma(\nu, \omega, r) * \theta_\gamma(\omega, \varkappa, s)$, $\forall \nu, \omega, \varkappa \in Z$, $r, s > 0$ such that $\nu \perp \omega$, $\omega \perp \varkappa$, and $\nu \perp \varkappa$

($\theta\gamma 5$) $\theta_\gamma(\nu, \omega, \cdot)\colon (0, \infty) \longrightarrow [0, 1]$ is continuous, $\forall \nu, \omega \in Z$ such that $\nu \perp \omega$ and $\omega \perp \nu$

Now, we show that the following are equivalent:

(i) $\theta_\gamma(\nu, \varkappa, r + s) \geq \theta_\gamma(\nu, \omega, (r/\gamma(\nu, \omega))) * \theta_\gamma(\omega, \varkappa, (s/\gamma(\omega, \varkappa)))$

(ii) $\theta_\gamma(\nu, s, \gamma(\nu, \omega)\gamma(\omega, \varkappa)(r + s)) \geq \theta_\gamma(\nu, \omega, r) * \theta_\gamma(\omega, \varkappa, s)$

Proof. (ii) $\Longrightarrow$ (i)

$$\begin{aligned}
&\theta_\gamma(\nu, \varkappa, \gamma(\nu, \omega)\gamma(\omega, \varkappa)(r + s)) \\
&\quad = \theta_\gamma(\nu, \varkappa, \gamma(\nu, \omega)\gamma(\omega, \varkappa)r + \gamma(\nu, \omega)\gamma(\omega, \varkappa)s) \\
&\quad \geq \theta_\gamma\left(\nu, \omega, \frac{\gamma(\nu, \omega)\gamma(\omega, \varkappa)r}{\gamma(\nu, \omega)\gamma(\omega, \varkappa)}\right) * \theta_\gamma\left(\omega, \varkappa, \frac{\gamma(\nu, \omega)\gamma(\omega, \varkappa)s}{\gamma(\nu, \omega)\gamma(\omega, \varkappa)}\right) \\
&\quad = \theta_\gamma(\nu, \omega, r) * \theta_\gamma(\omega, \varkappa, s).
\end{aligned} \tag{1}$$

Similarly, we can easily prove (i)$\Rightarrow$ (ii).

Example 1. Let $Z = \{-1, 1, 2, 3, 4, \ldots\} = A \bigcup B$, where $A = \{-1, 1\}$ and $B = \mathbb{N} \backslash \{1\}$. Define a binary relation $\perp$ by $\nu \perp \omega \Longleftrightarrow \nu, \omega \in \{|\nu|, |\omega|\}$. Given $\theta_\gamma\colon Z \times Z \times [0, \infty) \longrightarrow [0, 1]$ as

$$\theta_\gamma(\nu, \omega, r) = \begin{cases} 1, & \text{if } \nu = \omega, \\ \dfrac{r + (1/\nu)}{r + (1/\omega)}, & \text{if } \nu \in B \text{ and } \omega \in A, \\ \dfrac{r + (1/\omega)}{r + (1/\nu)}, & \text{if } \nu \in A \text{ and } \omega \in B, \\ \dfrac{r + (1/\max\{\nu, \omega\})}{r + (1/\min\{\nu, \omega\})}, & \text{if otherwise}, \end{cases} \tag{2}$$

with a continuous t-norm $*$ defined by $r_1 * r_2 = r_1 \cdot r_2$. Given $\gamma\colon Z \times Z \longrightarrow [1, \infty)$ as

$$\gamma(\nu, \omega) = \begin{cases} 1, & \text{if } \nu, \omega \in A, \\ \max\{\nu, \omega\}, & \text{otherwise}. \end{cases} \tag{3}$$

Then, $(Z, \theta_\gamma, *, \perp)$ is an orthogonal control fuzzy metric space, but it is not a control fuzzy metric space.

Proof. ($\theta\gamma 1$), ($\theta\gamma 2$), ($\theta\gamma 3$), and ($\theta\gamma 5$) are obvious. Here, we prove ($\theta\gamma 4$):

($\theta\gamma 4$) $\theta_\gamma(\nu, \varkappa, r + s) \geq \theta_\gamma(\nu, \omega, (r/\gamma(\nu, \omega))) * \theta_\gamma(\omega, \varkappa, (s/\gamma(\omega, \varkappa)))$, $\forall \nu, \omega, \varkappa \in Z$, $r, s > 0$, such that $\nu \perp \omega$, $\omega \perp \varkappa$, and $\nu \perp \varkappa$

We have the following cases to prove ($\theta\gamma 4$).

Case 1. If $\varkappa = \nu$, then $\theta_\gamma(\nu, \varkappa, r + s) = 1$. Also, $\theta_\gamma(\nu, \omega, (r/\gamma(\nu, \omega))) \leq 1$ and $\theta_\gamma(\omega, \varkappa, (s/\gamma(\omega, \varkappa))) \leq 1$.

This implies

$$\theta_\gamma\left(\nu, \omega, \frac{r}{\gamma(\nu, \omega)}\right) * \theta_\gamma\left(\omega, \varkappa, \frac{s}{\gamma(\omega, \varkappa)}\right) \leq 1. \tag{4}$$

Case 2. If $\varkappa = \omega$, then $\theta_\gamma(\omega, \varkappa, (s/\gamma(\omega, \varkappa))) = 1$, and clearly, $\theta_\gamma(\nu, \varkappa, r+s) \geq \theta_\gamma(\nu, \omega, (r/\gamma(\nu, \omega)))$. This implies

$$\theta_\gamma(\nu, \varkappa, r+s) \geq \theta_\gamma\left(\nu, \omega, \frac{r}{\gamma(\nu, \omega)}\right) * \theta_\gamma\left(\omega, \varkappa, \frac{s}{\gamma(\omega, \varkappa)}\right). \tag{5}$$

Case 3. If $\varkappa \neq \nu$, $\varkappa \neq \omega$, and $\nu = \omega$, then $\theta_\gamma(\nu, \omega, (r/\gamma(\nu, \omega))) = 1$ and clearly,

$$\theta_\gamma(\nu, \varkappa, r+s) \geq \theta_\gamma\left(\omega, \varkappa, \frac{s}{\gamma(\omega, \varkappa)}\right). \tag{6}$$

This implies

$$\theta_\gamma(\nu, \varkappa, r+s) \geq \theta_\gamma\left(\nu, \omega, \frac{r}{\gamma(\nu, \omega)}\right) * \theta_\gamma\left(\omega, \varkappa, \frac{s}{\gamma(\omega, \varkappa)}\right). \tag{7}$$

Case 4. If $\varkappa \neq \nu$, $\varkappa \neq \omega$, and$\nu \neq \omega$, then we have the following cases:

(1) $\nu, \varkappa \in A$ and $\omega \in B$
(2) $\omega \in A$ and $\nu, \varkappa \in B$
(3) $\omega, \varkappa \in A$ and $\nu \in B$
(4) $\nu, \omega \in A$ and $\varkappa \in B$
(5) $\varkappa \in A$ and $\nu, \omega \in B$
(6) $\nu \in A$ and $\omega, \varkappa \in B$
(7) $\nu, b, \omega \in A$
(8) $\nu, b, \omega \in B$

Proof of (1). If $\nu, \varkappa \in A$ and $\omega \in B$, then

$$\theta_\gamma(\nu, \varkappa, r+s) = \frac{r+s+(1/\max\{\nu, \varkappa\})}{r+s+(1/\min\{\nu, \varkappa\})}. \tag{8}$$

Observe that $\max\{\nu, \varkappa\} = \min\{\nu, \varkappa\} = 1$. This implies $\theta_\gamma(\nu, \varkappa, r+s) = 1$.

On the contrary,

$$\theta_\gamma\left(\nu, \omega, \frac{r}{\gamma(\nu, \omega)}\right) = \frac{(r/\gamma(\nu, \omega)) + (1/\omega)}{(r/\gamma(\nu, \omega)) + (1/\nu)}. \tag{9}$$

Observe that $\gamma(\nu, \omega) = \omega$; then,

$$\begin{aligned} \theta_\gamma\left(\nu, \omega, \frac{r}{\gamma(\nu, \omega)}\right) &= \frac{\nu r + \nu}{\nu r + \omega} < 1, \\ \theta_\gamma\left(\omega, \varkappa, \frac{s}{\gamma(\omega, \varkappa)}\right) &= \frac{(s/\gamma(\omega, \varkappa)) + (1/\omega)}{(s/(\gamma(\omega, \varkappa))) + (1/\varkappa)}. \end{aligned} \tag{10}$$

Note that $\gamma(\omega, \varkappa) = \omega$; then,

$$\theta_\gamma\left(\omega, \varkappa, \frac{s}{\gamma(\omega, \varkappa)}\right) = \frac{\varkappa s + \varkappa}{\varkappa s + \omega} < 1. \tag{11}$$

This implies

$$\theta_\gamma(\nu, \varkappa, r+s) \geq \theta_\gamma\left(\nu, \omega, \frac{r}{\gamma(\nu, \omega)}\right) * \theta_\gamma\left(\omega, \varkappa, \frac{s}{\gamma(\omega, \varkappa)}\right). \tag{12}$$

Similarly, (2)–(8) are easily satisfied.

Now, we show that θ_γ is not a control fuzzy metric space. Let $\nu, \omega, \varkappa \in A$. Also, let $\nu = \varkappa = 1$, $\omega = -1$, and $r, s > 1$; then,

$$\theta_\gamma(\nu, \varkappa, r+s) = 1. \tag{13}$$

On the contrary,

$$\begin{aligned} \theta_\gamma\left(\nu, \omega, \frac{r}{\gamma(\nu, \omega)}\right) &= \frac{r+1}{r-1}, \quad (r \neq 1), \\ \theta_\gamma\left(\omega, \varkappa, \frac{s}{\gamma(\omega, \varkappa)}\right) &= \frac{s+1}{s-1}, \quad (s \neq 1). \end{aligned} \tag{14}$$

This implies

$$1 \geq \frac{r+1}{r-1} + \frac{s+1}{s-1}. \tag{15}$$

This fails $(\theta\gamma 4)$.

Example 2. Let $\mathbb{Z} = A \bigcup B$, where $A = \{-1, -2, -3, \ldots\}$ and $B = \{0, 1, 2, 3, \ldots\}$. Define a binary relation $\perp$ by $\nu \perp \omega \Longleftrightarrow \nu + \omega \geq 0$. Define θ_γ: $\mathrm{Z} \times \mathrm{Z} \times [0, \infty) \longrightarrow [0, 1]$ by

$$\theta_\gamma(\nu, \omega, r) = \frac{r}{r + \max\{\nu, \omega\}}, \tag{16}$$

for all $r > 0$ and $\nu, \omega \in \mathrm{Z}$ with a continuous t-norm $*$ defined by $r_1 * r_2 = r_1 \cdot r_2$. Given γ: $\mathrm{Z} \times \mathrm{Z} \longrightarrow [1, \infty)$ as

$$\gamma(\nu, \omega) = \begin{cases} 1, & \text{if } \nu, \omega \in A \text{ or } \nu = 0 \text{ or } \omega = 0, \\ \max\{\nu, \omega\}, & \text{otherwise.} \end{cases} \tag{17}$$

Then, $(\mathrm{Z}, \theta_\gamma, *, \perp)$ is an orthogonal control fuzzy metric space, but it is not a control fuzzy metric space.

Proof. First, we show that θ_γ is an orthogonal control fuzzy metric space. $(\theta\gamma 1)$, $(\theta\gamma 3)$, and $(\theta\gamma 5)$ are obvious. Here, we prove $(\theta\gamma 2)$ and $(\theta\gamma 4)$:

$(\theta\gamma 2)$ $\theta_\gamma(\nu, \omega, r) = 1 \Longleftrightarrow \nu = \omega$, $\forall \nu, \omega \in \mathrm{Z}, r > 0$ such that $\nu \perp \omega$ and $\omega \perp \nu$:

$$\begin{aligned} &\theta_\gamma(\nu, \omega, r) = 1, \\ &\Longleftrightarrow \frac{r}{r + \max\{\nu, \omega\}} = 1, \\ &\Longleftrightarrow r = r + \max\{\nu, \omega\}, \\ &\Longleftrightarrow \max\{\nu, \omega\} = 0, \\ &\Longleftrightarrow \nu = \omega. \end{aligned} \tag{18}$$

$(\theta\gamma 3)$ $\theta_\gamma(\nu, \omega, r) = \theta_\gamma(\omega, \nu, r)$, $\forall \nu, \omega \in \mathrm{Z}, r > 0$ such that $\nu \perp \omega$ and $\omega \perp \nu$:

$$\theta_\gamma(\nu,\omega,r)=\frac{r}{r+\max\{\nu,\omega\}}=\frac{r}{r+\max\{\omega,\nu\}}=\theta_\gamma(\omega,\nu,r). \tag{19}$$

($\theta\gamma 4$) $\theta_\gamma(\nu,\varkappa,\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s))\geq\theta_\gamma(\nu,\omega,r)*\theta_\gamma(\omega,\varkappa,s)$, $\forall\nu,\omega,\varkappa\in Z, r,s>0$, such that $\nu\perp\omega$, $\omega\perp\varkappa$, and $\nu\perp\varkappa$:

$$\begin{aligned}
&\Rightarrow\max\{\nu,\varkappa\}\leq\gamma(\nu,\omega)[\max\{\nu,\omega\}]+\gamma(\omega,\varkappa)[\max\{\omega,\varkappa\}]\\
&\Rightarrow rs\max\{\nu,\varkappa\}\leq\gamma(\nu,\omega)\left(rs+s^2\right)[\max\{\nu,\omega\}]+\gamma(\omega,\varkappa)\left(rs+r^2\right)[\max\{\omega,\varkappa\}]\\
&\Rightarrow rs\max\{\nu,\varkappa\}\leq\gamma(\nu,\omega)(r+s)s[\max\{\nu,\omega\}]+\gamma(\omega,\varkappa)(s+r)r[\max\{\omega,\varkappa\}]\\
&\Rightarrow rs\max\{\nu,\varkappa\}\leq\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)\left[\frac{s\max\{\nu,\omega\}}{\gamma(\omega,\varkappa)}+\frac{r\max\{\omega,\varkappa\}}{\gamma(\nu,\omega)}\right]\\
&\Rightarrow rs\max\{\nu,\varkappa\}\leq\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)[s\max\{\nu,\omega\}+r\max\{\omega,\varkappa\}]\\
&\Rightarrow rs\max\{\nu,\varkappa\}\leq\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)[s\max\{\nu,\omega\}+r\max\{\omega,\varkappa\}+\max\{\nu,\omega\}\max\{\omega,\varkappa\}]\\
&\Rightarrow\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)rs+rs\max\{\nu,\varkappa\}\leq\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)rs+\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)[s\max\{\nu,\omega\}\\
&\quad+r\max\{\omega,\varkappa\}+\max\{\nu,\omega\}\max\{\omega,\varkappa\}]\\
&\Rightarrow\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)rs+rs\max\{\nu,\varkappa\}\leq\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)[rs+s\max\{\nu,\omega\}+r\max\{\omega,\varkappa\}+\max\{\nu,\omega\}\max\{\omega,\varkappa\}]\\
&\Rightarrow rs[\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)+\max\{\nu,\varkappa\}]\leq\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)[r+\max\{\nu,\omega\}][s+\max\{\omega,\varkappa\}]\\
&\Rightarrow\frac{\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)}{\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)+\max\{\nu,\varkappa\}}\geq\frac{rs}{[r+\max\{\nu,\omega\}][s+\max\{\omega,\varkappa\}]}\\
&\Rightarrow\frac{\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)}{\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)+\max\{\nu,\varkappa\}}\geq\frac{r}{r+\max\{\nu,\omega\}}\cdot\frac{s}{s+\max\{\omega,\varkappa\}}\\
&\Rightarrow\theta_\gamma(\nu,\varkappa,\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s))\geq\theta_\gamma(\nu,\omega,r)*\theta_\gamma(\omega,\varkappa,s).
\end{aligned} \tag{20}$$

Now, we show that θ_γ is not a control fuzzy metric space. Indeed,

$$\begin{aligned}
\theta_\gamma(\nu,\varkappa,\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s))&=\frac{\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)}{\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)+\max\{\nu,\varkappa\}}\\
\theta_\gamma(\nu,\omega,r)&=\frac{r}{r+\max\{\nu,\omega\}}\\
\theta_\gamma(\omega,\varkappa,s)&=\frac{s}{s+\max\{\omega,\varkappa\}}.
\end{aligned} \tag{21}$$

This implies

$$\frac{\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)}{\gamma(\nu,\omega)\gamma(\omega,\varkappa)(r+s)+\max\{\nu,\varkappa\}}\geq\frac{r}{r+\max\{\nu,\omega\}}\cdot\frac{s}{s+\max\{\omega,\varkappa\}}. \tag{22}$$

Now, let$\nu=\omega=\varkappa=-1$; then, $\gamma(\nu,\omega)=\gamma(\omega,\varkappa)=1$ and $\max\{\nu,\varkappa\}=\max\{\nu,\omega\}=\max\{\omega,\varkappa\}=-1$. This implies that

$$\frac{r+s}{r+s-1}\geq\frac{r}{r-1}\cdot\frac{s}{s-1}=\frac{rs}{(r-1)(s-1)},\quad r,s\neq 1. \tag{23}$$

Taking $r=s=2$, we get a contradiction.

Remark 1. Every control fuzzy metric space is an orthogonal control fuzzy metric space, but the converse is not true.

Remark 2. Note that Example 2 also holds for the t-norm $: r_1 * r_2 = \min\{r_1, r_2\}$.

Definition 6. Let $(Z, \theta_\gamma, *, \perp)$ be an orthogonal control fuzzy metric space. Then, a sequence $\{\nu_n\}$ is said to be G-convergent to ν, where $\nu, \{\nu_n\} \in Z$ if and only if $\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n, \nu, r) = 1$ for any $n > 0$ and for all $r > 0$.

Definition 7. Let $(Z, \theta_\gamma, *, \perp)$ be an orthogonal control fuzzy metric space. Then, a sequence $\{\nu_n\}$ is said to be a G-Cauchy sequence with $\{\nu_n\} \in Z$ if and only if $\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n, \nu_{n+m}, r) = 1$ for all $m > 0$ and $r > 0$.

Definition 8. Let $(Z, \theta_\gamma, *, \perp)$ be an orthogonal control fuzzy metric space; then, it is G-complete if and only if every G-Cauchy sequence is convergent.

Definition 9. ζ: $Z \longrightarrow Z$ is $\perp$-continuous at $\nu \in Z$ in an orthogonal control fuzzy metric space $(Z, \theta_\gamma, *, \perp)$ if for each orthogonal sequence $\{\nu_n\}$ in Z so that if $\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n, \nu, r)$ exists and is finite for all $r > 0$, then $\lim_{n\longrightarrow\infty}\theta_\gamma(\zeta\nu_n, \zeta\nu, r)$ again exists and is finite for all $r > 0$. Furthermore, ζ is $\perp$-continuous if ζ is $\perp$-continuous at each $\nu \in Z$. Also, ζ is $\perp$-preserving if $\zeta\nu \perp \zeta\omega$; hence, $\nu \perp \omega$.

Remark 3. It is not necessary that the limit of a convergent sequence will be unique in an orthogonal control fuzzy metric space.

For this, take a sequence $\{\nu_n\}$ defined by $\nu_n = 1 - (1/n)$ for each integer n, and define an orthogonal control fuzzy metric space as in Example 2 with $\nu \geq 1$. Also, in particular, take $\gamma(\nu, \omega) = \gamma(\omega, \varkappa) = 1$; then,

$$\begin{aligned}\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n, \nu, r) &= \lim_{n\longrightarrow\infty}\frac{r}{r + \max\{\nu_n, \nu\}} = \lim_{n\longrightarrow\infty}\frac{r}{r + \nu} \\ &= \theta_\gamma(\nu, \nu, r),\end{aligned} \tag{24}$$

for all $r > 0$. Observe that the sequence $\{\nu_n\}$ converges to all $\nu \in Z$ with $\nu \geq 1$.

Remark 4. It is not necessary that the convergent sequence will be a Cauchy sequence in an orthogonal control fuzzy metric space.

For this, take a sequence $\{\nu_n\}$ defined by $\nu_n = 1 + (-1)^n$ for each integer n, and define an orthogonal control fuzzy metric space as in Example 2 with $\nu \geq 2$. Also, in particular, take $\gamma(\nu, \omega) = \gamma(\omega, \varkappa) = 1$; then,

$$\begin{aligned}\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n, \nu, r) &= \lim_{n\longrightarrow\infty}\frac{r}{r + \max\{\nu_n, \nu\}} \\ &= \lim_{n\longrightarrow\infty}\frac{r}{r + \nu} = \theta_\gamma(\nu, \nu, r),\end{aligned} \tag{25}$$

for all $r > 0$. Observe that the sequence $\{\nu_n\}$ converges to all $\nu \in Z$ with $\nu \geq 2$. However, $\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n, \nu_{n+m}, r)$ does not exist.

Mihet [16] introduced a control function ψ. We generalize it as follows.

Definition 10. Let ψ be the class of all mappings Ψ: $[0,1] \longrightarrow [0,1]$ such that Ψ is orthogonal continuous, nondecreasing, and $\Psi(E) > E$, for all $E \in (0,1)$. If $\Psi \in \psi$, then $\Psi(1) = 1$ and $\lim_{n\longrightarrow\infty}\Psi^n(E) = 1$, for all $E \in (0,1)$.

Theorem 10. *Let $(Z, \theta_\gamma, *, \perp)$ be an orthogonal G-complete control fuzzy metric space with γ: $Z \times Z \longrightarrow [1,\infty)$ such that*

$$\lim_{r\longrightarrow\infty}\theta_\gamma(\nu, \omega, r) = 1, \tag{26}$$

for all $\nu \in Z$. Suppose that ζ: $Z \longrightarrow Z$ is an $\perp$-continuous, $\perp$-contraction, and $\perp$-preserving mapping so that

$$\theta_\gamma(\zeta\nu, \zeta\omega, kr) \geq \theta_\gamma(\nu, \omega, r), \tag{27}$$

for all $\nu, \omega \in Z, r > 0$, where $k \in (0,1)$. Also, assume that, for every $\nu \in Z$,

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\gamma(\nu_n, \omega), \\ &\lim_{n\longrightarrow\infty}\gamma(\omega, \nu_n),\end{aligned} \tag{28}$$

exist and are finite. Then, ζ has a unique fixed point in Z. Furthermore,

$$\lim_{n\longrightarrow\infty}\theta_\gamma(\zeta^n u, u, r) = \theta_\gamma(u, u, r), \quad \text{for all } u \in Z \text{ and } r > 0. \tag{29}$$

Proof. Since $(Z, \theta_\gamma, *, \perp)$ is an orthogonal G-complete control fuzzy metric space, there exists $\nu_0 \in Z$ such that

$$\nu_0 \perp \omega, \quad \text{for all } \omega \in Z. \tag{30}$$

This yields that $\nu_0 \perp \zeta\nu_0$. Consider

$$\nu_1 = \zeta\nu_0, \nu_2 = \zeta^2\nu_0 = \zeta\nu_1, \ldots, \nu_n = \zeta^n\nu_0 = \zeta\nu_{n-1}. \tag{31}$$

If $\nu_n = \nu_{n-1}$, then ν_n is a fixed point of ζ. Suppose that $\nu_n \neq \nu_{n-1}$ for all $n \in \mathbb{N}$. Since ζ is $\perp$-preserving, $\{\nu_n\}$ is an orthogonal sequence. Since ζ is an $\perp$-contraction, we have

$$\begin{aligned}\theta_\gamma(\nu_n, \nu_{n+1}, r) &= \theta_\gamma(\zeta\nu_{n-1}, \zeta\nu_n, r) \\ &\geq \theta_\gamma\left(\nu_{n-2}, \nu_{n-1}, \frac{r}{k}\right) \\ &\geq \cdots \geq \theta_\gamma\left(\nu_0, \nu_1, \frac{r}{k^{n-1}}\right).\end{aligned} \tag{32}$$

Now, from $(\theta_\gamma 4)$, we have

$$
\begin{aligned}
\theta_\gamma(\nu_n,\nu_{n+m},r) &\geq \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \theta_\gamma\left(\nu_{n+1},\nu_{n+m},\frac{r}{2\gamma(\nu_{n+1},\nu_{n+m})}\right)\\
&\geq \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \theta_\gamma\left(\nu_{n+1},\nu_{n+2},\frac{r}{(2)^2\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+1},\nu_{n+2})}\right)\\
&\quad * \theta_\gamma\left(\nu_{n+2},\nu_{n+m},\frac{r}{(2)^2\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+m})}\right)\\
&\geq \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \theta_\gamma\left(\nu_{n+1},\nu_{n+2},\frac{r}{(2)^2\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+1},\nu_{n+2})}\right)\\
&\quad * \theta_\gamma\left(\nu_{n+2},\nu_{n+3},\frac{r}{(2)^3\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+3})}\right)\\
&\quad * \theta_\gamma\left(\nu_{n+3},\nu_{n+m},\frac{r}{(2)^3\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+m})\gamma(\nu_{n+3},\nu_{n+m})}\right)\\
&\geq \ldots \geq \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \left[*_{i=n+1}^{n+m-2}\theta_\gamma\left(\nu_i,\nu_{i+1},\frac{r}{(2)^{m-2}\left(\prod_{j=n+1}^{i}\left(\gamma\left(\nu_j,\nu_{n+m}\right)\gamma(\nu_i,\nu_{i+1})\right)\right)}\right)\right]\\
&\quad * \left[\theta_\gamma\left(\nu_{n+m-1},\nu_{n+m},\frac{r}{(2)^{m-1}\left(\prod_{i=n+1}^{n+m-1}\gamma(\nu_i,\nu_{n+m})\right)}\right)\right]\\
&\geq \theta_\gamma\left(\nu_0,\nu_1,\frac{r}{2k^{n-1}\gamma(\nu_n,\nu_{n+1})}\right) * \left[*_{i=n+1}^{n+m-2}\theta_\gamma\left(\nu_0,\nu_1,\frac{r}{(2)^{m-1}k^{i-1}\left(\prod_{j=n+1}^{i}\left(\gamma\left(\nu_j,\nu_{n+m}\right)\gamma(\nu_i,\nu_{i+1})\right)\right)}\right)\right]\\
&\quad * \left[\theta_\gamma\left(\nu_0,\nu_1,\frac{r}{(2)^{m-1}k^{n+m-1}\left(\prod_{i=n+1}^{n+m-1}\gamma(\nu_i,\nu_{n+m})\right)}\right)\right].
\end{aligned}
\tag{33}
$$

Now, taking limit as $n \longrightarrow \infty$ in (33), in (32) together with (26), we have

$$\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n,\nu_{n+m},r) \geq 1 * 1 * \cdots * 1 = 1, \tag{34}$$

for all $r>0$ and $m\in\mathbb{N}$. Thus, $\{\nu_n\}$ is an orthogonal G-Cauchy sequence in Z. The completeness of $(Z,\theta_\gamma, *, \perp)$ implies the existence of $u\in Z$ such that

$$\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n,u,r) = 1, \tag{35}$$

for all $r>0$. Now, since ζ is an$\perp$-continuous mapping, one writes $\lim\theta_\gamma(\nu_{n+1},\zeta u,r) = \lim\theta_\gamma(\zeta\nu_n,\zeta u,r) = 1$. For $r>0$ and from $(\theta_\gamma 4)$, we have

$$
\begin{aligned}
\theta_\gamma(u,\zeta u,r) &\geq \theta_\gamma\left(u,\nu_{n+1},\frac{r}{2\gamma(u,\nu_{n+1})}\right)\\
&\quad * \theta_\gamma\left(\nu_{n+1},\zeta u,\frac{r}{2\gamma(\nu_{n+1},\zeta u)}\right)\\
&= \theta_\gamma\left(u,\nu_{n+1},\frac{r}{2\gamma(u,\nu_{n+1})}\right)\\
&\quad * \theta_\gamma\left(\zeta\nu_n,\zeta u,\frac{r}{2\gamma(\nu_{n+1},\zeta u)}\right).
\end{aligned}
\tag{36}
$$

Taking $n\longrightarrow\infty$ in (36) and using (35), we get $\theta_\gamma(u,\zeta u,r) = 1$ for all $r>0$, that is, $\zeta u = u$.

Now, for uniqueness, let $w \in Z$ be another fixed point for ζ and let there exist $r > 0$ such that $\theta_\gamma(u, w, r) \neq 1$. We can obtain

$$\begin{aligned} &\nu_0 \perp u, \\ &\nu_0 \perp w. \end{aligned} \tag{37}$$

Since ζ is an $\perp$-preserving, this implies that

$$\begin{aligned} &\zeta^n \nu_0 \perp \zeta^n u, \\ &\zeta^n \nu_0 \perp \zeta^n w, \end{aligned} \quad \text{for all } n \in \mathbb{N}. \tag{38}$$

From (27), we can derive

$$\begin{aligned} &\theta_\gamma(\zeta^n \nu_0, \zeta^n u, r) \geq \theta_\gamma(\zeta^n \nu_0, \zeta^n u, kr) \geq \theta_\gamma\left(\nu_0, u, \frac{r}{k^n}\right) \\ &\theta_\gamma(\zeta^n \nu_0, \zeta^n w, r) \geq \theta_\gamma(\zeta^n \nu_0, \zeta^n w, kr) \geq \theta_\gamma\left(\nu_0, w, \frac{r}{k^n}\right). \end{aligned} \tag{39}$$

We can write

$$\begin{aligned} \theta_\gamma(u, w, r) = \theta_\gamma(\zeta^n u, \zeta^n w, r) &\geq \theta_\gamma\left(\zeta^n \nu_0, \zeta^n u, \frac{r}{2\gamma(\nu_0, u)}\right) \\ &\quad * \theta_\gamma\left(\zeta^n \nu_0, \zeta^n w, \frac{r}{2\gamma(\nu_0, w)}\right) \\ &\geq \theta_\gamma\left(\nu_0, u, \frac{r}{k^n 2\gamma(\nu_0, u)}\right) \\ &\quad * \theta_\gamma\left(\nu_0, w, \frac{r}{k^n 2\gamma(\nu_0, w)}\right), \end{aligned} \tag{40}$$

for all $n \in \mathbb{N}$. By taking limit as $n \longrightarrow \infty$, we get $\theta_\gamma(u, w, r) = 1$, for all $r > 0$; hence, $u = w$.

Corollary 1. *Let $(Z, \theta_\gamma, *, \perp)$ be an orthogonal G-complete control fuzzy metric space. Let $\zeta: Z \longrightarrow Z$ be $\perp$-contraction and $\perp$-preserving. Also, assume that if $\{\nu_n\}$ is an O-sequence with $\nu_n \longrightarrow \nu \in Z$, then $\nu \perp \nu_n$ for all $n \in \mathbb{N}$. Therefore, ζ has a unique fixed point $\nu_* \in Z$. Furthermore, $\lim_{n \longrightarrow \infty} \theta_\gamma(\zeta^n \nu, \nu_*, r) = \theta_\gamma(\nu_*, \nu_*, r)$, for all $\nu \in Z$ and $r > 0$.*

Proof. We can prove alike as in the proof of Theorem 1 that $\{\nu_n\}$ is a G-Cauchy sequence and converges to $\nu_* \in Z$. Hence, $\nu_* \perp \nu_n$ for all $n \in \mathbb{N}$. We get from (26) that

$$\begin{aligned} \theta_\gamma(\zeta \nu_*, \nu_{n+1}, r) = \theta_\gamma(\zeta \nu_*, \zeta \nu_n, r) &\geq \theta_\gamma(\zeta \nu_*, \zeta \nu_n, kr) \\ &\geq \theta_\gamma(\nu_*, \nu_n, r), \\ \lim_{n \longrightarrow \infty} \theta_\gamma(\zeta \nu_*, \nu_{n+1}, r) = 1. \end{aligned} \tag{41}$$

Then, we can write

$$\begin{aligned} \theta_\gamma(\nu_*, \zeta \nu_*, r) &\geq \theta_\gamma\left(\nu_*, \nu_{n+1}, \frac{r}{2\gamma(\nu_*, \nu_{n+1})}\right) \\ &\quad * \theta_\gamma\left(\nu_{n+1}, \zeta \nu_*, \frac{r}{2\gamma(\nu_{n+1}, \zeta \nu_*)}\right). \end{aligned} \tag{42}$$

Taking limit as $n \longrightarrow \infty$, we get $\theta_\gamma(\nu_*, \zeta \nu_*, r) = 1 * 1 = 1$, and hence, $\zeta \nu_* = \nu_*$. The rest of proof is similar as in Theorem 1.

Theorem 2. *Let $(Z, \theta_\gamma, *, \perp)$ be an orthogonal G-complete control fuzzy metric space with $\gamma: Z \times Z \longrightarrow [1, \infty)$ so that*

$$\lim_{r \longrightarrow \infty} \theta_\gamma(\nu, \omega, r) = 1, \tag{43}$$

for all $\nu \in Z$. If $\zeta: Z \longrightarrow Z$ is an $\perp$-contraction and $\perp$-preserving and satisfies

$$\theta_\gamma(\zeta \nu, \zeta^2 \nu, kr) \geq \theta_\gamma(\nu, \zeta \nu, r), \tag{44}$$

for all $\nu \in O(\nu), r > 0$, where $k \in (0, 1)$, then $\zeta^n \nu_0 \longrightarrow u$. Furthermore, u is a fixed point of ζ if and only if $\zeta \nu = \theta_\gamma(\nu, \zeta \nu, r)$ is ζ-orbitally lower semicontinuous at u.

Proof. Since $(Z, \theta_\gamma, *, \perp)$ is an orthogonal G-complete control fuzzy metric space, there exists $\nu_0 \in Z$ such that

$$\nu_0 \perp \omega, \quad \text{for all } \omega \in Z. \tag{45}$$

This says that $\nu_0 \perp \zeta \nu_0$. Consider

$$\nu_1 = \zeta \nu_0, \nu_2 = \zeta^2 \nu_0 = \zeta \nu_1, \ldots, \nu_n = \zeta^n \nu_0 = \zeta \nu_{n-1}. \tag{46}$$

If $\nu_n = \nu_{n-1}$, then ν_n is a fixed point of ζ. Suppose that $\nu_n \neq \nu_{n-1}$ for all $n \in \mathbb{N}$. Since ζ is $\perp$-preserving, $\{\nu_n\}$ is an orthogonal sequence. Since ζ is an $\perp$-contraction, we have

$$\begin{aligned} \theta_\gamma(\zeta^n \nu_0, \zeta^{n+1} \nu_0, kr) &= \theta_\gamma(\nu_n, \nu_{n+1}, kr) \\ &\geq \theta_\gamma\left(\nu_{n-1}, \nu_n, \frac{r}{k}\right) \\ &\geq \cdots \geq \theta_\gamma\left(\nu_0, \nu_1, \frac{r}{k^{n-1}}\right). \end{aligned} \tag{47}$$

Now, from $(\theta_\gamma 4)$, we have

$$\begin{aligned}
\theta_\gamma(\nu_n,\nu_{n+m},r) &\geq \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \theta_\gamma\left(\nu_{n+1},\nu_{n+m},\frac{r}{2\gamma(\nu_{n+1},\nu_{n+m})}\right)\\
&\geq \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \theta_\gamma\left(\nu_{n+1},\nu_{n+2},\frac{r}{(2)^2\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+1},\nu_{n+2})}\right)\\
&\quad * \theta_\gamma\left(\nu_{n+2},\nu_{n+m},\frac{r}{(2)^2\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+m})}\right)\\
&\geq \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \theta_\gamma\left(\nu_{n+1},\nu_{n+2},\frac{r}{(2)^2\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+1},\nu_{n+2})}\right)\\
&\quad * \theta_\gamma\left(\nu_{n+2},\nu_{n+3},\frac{r}{(2)^3\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+3})}\right)\\
&\quad * \theta_\gamma\left(\nu_{n+3},\nu_{n+m},\frac{r}{(2)^3\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+m})\gamma(\nu_{n+3},\nu_{n+m})}\right)\\
&\geq \cdots \geq \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \left[*_{i=n+1}^{n+m-2}\theta_\gamma\left(\nu_i,\nu_{i+1},\frac{r}{(2)^{m-2}\left(\prod_{j=n+1}^{i}\left(\gamma\left(\nu_j,\nu_{n+m}\right)\gamma(\nu_i,\nu_{i+1})\right)\right)}\right)\right]\\
&\quad * \left[\theta_\gamma\left(\nu_{n+m-1},\nu_{n+m},\frac{r}{(2)^{m-1}\left(\prod_{i=n+1}^{n+m-1}\gamma(\nu_i,\nu_{n+m})\right)}\right)\right]\\
&\geq \theta_\gamma\left(\nu_0,\nu_1,\frac{r}{2k^{n-1}\gamma(\nu_n,\nu_{n+1})}\right) * \left[*_{i=n+1}^{n+m-2}\theta_\gamma\left(\nu_0,\nu_1,\frac{r}{(2)^{m-1}k^{i-1}\left(\prod_{j=n+1}^{i}\left(\gamma\left(\nu_j,\nu_{n+m}\right)\gamma(\nu_i,\nu_{i+1})\right)\right)}\right)\right]\\
&\quad * \left[\theta_\gamma\left(\nu_0,\nu_1,\frac{r}{(2)^{m-1}k^{n+m-1}\left(\prod_{i=n+1}^{n+m-1}\gamma(\nu_i,\nu_{n+m})\right)}\right)\right].
\end{aligned} \tag{48}$$

Now, taking limit as $n \longrightarrow \infty$ in (48), we have

$$\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n,\nu_{n+m},r) \geq 1 * 1 * \cdots * 1 = 1, \tag{49}$$

for all $r>0$ and $m\in\mathbb{N}$. Thus, $\{\nu_n\}$ is an orthogonal G-Cauchy sequence in Z. From the completeness of $(Z,\theta_\gamma,*,\perp)$, there is $\nu_n \longrightarrow \zeta^n\nu_0 = u$. Assume that ζ is ζ-orbitally lower semicontinuous at $u\in Z$; then, we have

$$\begin{aligned}
\theta_\gamma(u,\zeta u,kr) &= \lim_{n\longrightarrow\infty}\sup\theta_\gamma\left(\zeta^n\nu_0,\zeta^{n+1}\nu_0,kr\right)\\
&\geq \lim_{n\longrightarrow\infty}\sup\theta_\gamma\left(\nu_0,\nu_1,\frac{r}{k^{n-1}}\right) = 1.
\end{aligned} \tag{50}$$

Conversely, let $u=\zeta u$ and $\nu_n\in Z$ with $\nu_n\longrightarrow u$; then, we obtain

$$\begin{aligned}
\zeta(u) = \theta_\gamma(u,\zeta u,kr) = 1 &\geq \lim_{n\longrightarrow\infty}\sup\zeta(\nu_n)\\
&= \theta_\gamma\left(\zeta^n\nu_0,\zeta^{n+1}\nu_0,kr\right).
\end{aligned} \tag{51}$$

Theorem 3. *Let* $(Z,\theta_\gamma,*,\perp)$ *be an orthogonal G-complete control fuzzy metric space and* $\zeta\colon Z\longrightarrow Z$ *be an* $\perp$-*-continuous,* $\perp$*-contraction, and* $\perp$*-preserving mapping so that*

$$\theta_\gamma(\nu,\omega,r)>0 \Rightarrow \theta_\gamma(\zeta\nu,\zeta\omega,r) \geq \Psi\left(\theta_\gamma(\nu,\omega,r)\right), \tag{52}$$

for all $\nu,\omega\in Z$ *and* $r>0$. *Then,* ζ *has a unique fixed point in* Z.

Proof. Since $(Z,\theta_\gamma,*,\perp)$ is an orthogonal G-complete control fuzzy metric space, there exists $\nu_0\in Z$ such that

$$\nu_0\perp\omega, \quad \text{for all } \omega\in Z. \tag{53}$$

Thus, $\nu_0\perp\zeta\nu_0$. Assume

$$\begin{aligned}
&\nu_1 = \zeta\nu_0,\\
&\nu_2 = \zeta^2\nu_0 = \zeta\nu_1,\ldots,\nu_n = \zeta^n\nu_0 = \zeta\nu_{n-1}.
\end{aligned} \tag{54}$$

If $\nu_n = \nu_{n-1}$, then ν_n is a fixed point of ζ. Suppose that $\nu_n\neq\nu_{n-1}$ for all $n\in\mathbb{N}$. Since ζ is $\perp$-preserving, $\{\nu_n\}$ is an orthogonal sequence. Since ζ is an $\perp$-contraction, we have

$$\begin{aligned}
\theta_\gamma(\nu_n,\nu_{n+1},r) &= \theta_\gamma(\zeta\nu_{n-1},\zeta\nu_n,r)\\
&\geq \Psi\left(\theta_\gamma(\nu_{n-2},\nu_{n-1},r)\right)\\
&\geq \cdots \geq \Psi^n\left(\theta_\gamma(\nu_0,\nu_1,r)\right).
\end{aligned} \tag{55}$$

Now, from $(\theta_\gamma 4)$, we have

$$
\begin{aligned}
\theta_\gamma(\nu_n,\nu_{n+m},r) &\ge \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \theta_\gamma\left(\nu_{n+1},\nu_{n+m},\frac{r}{2\gamma(\nu_{n+1},\nu_{n+m})}\right)\\
&\ge \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \theta_\gamma\left(\nu_{n+1},\nu_{n+2},\frac{r}{(2)^2\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+1},\nu_{n+2})}\right)\\
&\quad * \theta_\gamma\left(\nu_{n+2},\nu_{n+m},\frac{r}{(2)^2\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+m})}\right)\\
&\ge \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \theta_\gamma\left(\nu_{n+1},\nu_{n+2},\frac{r}{(2)^2\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+1},\nu_{n+2})}\right)\\
&\quad * \theta_\gamma\left(\nu_{n+2},\nu_{n+3},\frac{r}{(2)^3\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+3})}\right)\\
&\quad * \theta_\gamma\left(\nu_{n+3},\nu_{n+m},\frac{r}{(2)^3\gamma(\nu_{n+1},\nu_{n+m})\gamma(\nu_{n+2},\nu_{n+m})\gamma(\nu_{n+3},\nu_{n+m})}\right)\\
&\ge \cdots \ge \theta_\gamma\left(\nu_n,\nu_{n+1},\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right) * \left[*_{i=n+1}^{n+m-2}\theta_\gamma\left(\nu_i,\nu_{i+1},\frac{r}{(2)^{m-2}\left(\prod_{j=n+1}^{i}\left(\gamma(\nu_j,\nu_{n+m})\gamma(\nu_i,\nu_{i+1})\right)\right)}\right)\right]\\
&\quad * \left[\theta_\gamma\left(\nu_{n+m-1},\nu_{n+m},\frac{r}{(2)^{m-1}\left(\prod_{i=n+1}^{n+m-1}\gamma(\nu_i,\nu_{n+m})\right)}\right)\right]\\
&\ge \Psi^n\left[\theta_\gamma\left(\nu_0,\nu_1,\frac{r}{2\gamma(\nu_n,\nu_{n+1})}\right)\right] * \left[*_{i=n+1}^{n+m-2}\Psi^i\left(\theta_\gamma\left(\nu_0,\nu_1,\frac{r}{(2)^{m-1}\left(\prod_{j=n+1}^{i}\left(\gamma(\nu_j,\nu_{n+m})\gamma(\nu_i,\nu_{i+1})\right)\right)}\right)\right)\right]\\
&\quad * \left[\Psi^{n+m-1}\left(\theta_\gamma\left(\nu_0,\nu_1,\frac{r}{(2)^{m-1}\left(\prod_{i=n+1}^{n+m-1}\gamma(\nu_i,\nu_{n+m})\right)}\right)\right)\right].
\end{aligned}
\tag{56}
$$

Now, taking limit as $n \longrightarrow \infty$ in (55 and 56), we have

$$\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n,\nu_{n+m},r) \ge 1 * 1 * \cdots * 1 = 1, \tag{57}$$

for all $r>0$ and $m \in \mathbb{N}$. Thus, $\{\nu_n\}$ is an orthogonal G-Cauchy sequence in Z. From the completeness of $(Z,\theta_\gamma, *, \perp)$, there exists $u \in Z$ such that

$$\lim_{n\longrightarrow\infty}\theta_\gamma(\nu_n,u,r) = 1, \tag{58}$$

for all $r>0$. Now, since ζ is an $\perp$-continuous mapping, one gets $\lim\theta_\gamma(\nu_{n+1},\zeta u,r) = \lim\theta_\gamma(\zeta\nu_n,\zeta u,r) = 1$ as $n \longrightarrow \infty$. For $r>0$ and from $(\theta_\gamma 4)$, we have

$$
\begin{aligned}
\theta_\gamma(u,\zeta u,r) &\ge \theta_\gamma\left(u,\nu_{n+1},\frac{r}{2\gamma(u,\nu_{n+1})}\right) * \theta_\gamma\left(\nu_{n+1},\zeta u,\frac{r}{2\gamma(\nu_{n+1},\zeta u)}\right)\\
&= \theta_\gamma\left(u,\nu_{n+1},\frac{r}{2\gamma(u,\nu_{n+1})}\right) * \theta_\gamma\left(\zeta\nu_n,\zeta u,\frac{r}{2\gamma(\nu_{n+1},\zeta u)}\right)\\
&\ge \theta_\gamma\left(u,\nu_{n+1},\frac{r}{2\gamma(u,\nu_{n+1})}\right) * \Psi\left(\theta_\gamma\left(\nu_n,u,\frac{r}{2\gamma(\nu_{n+1},\zeta u)}\right)\right).
\end{aligned}
\tag{59}
$$

Taking $n \longrightarrow \infty$ in (59) and using (58), we get $\theta_\gamma(u,\zeta u,r) = 1$ for all $r>0$, that is, $\zeta u = u$.

Now, for uniqueness, let $w \in Z$ be another fixed point for ζ and let there exist $r>0$ such that $u \neq w$. We can obtain

$$
\begin{aligned}
&\nu_0 \perp u,\\
&\nu_0 \perp w.
\end{aligned}
\tag{60}
$$

Since ζ is an $\perp$-preserving, this implies

$$\left.\begin{array}{l}\zeta^n\nu_0 \perp \zeta^n u,\\ \zeta^n\nu_0 \perp \zeta^n w,\end{array}\right. \quad \text{for all } n \in \mathbb{N}. \tag{61}$$

We can derive

$$\begin{aligned}&\theta_\gamma(\zeta^n\nu_0, \zeta^n u, r) \geq \theta_\gamma(\zeta^n\nu_0, \zeta^n u, kr) \geq \Psi\big(\theta_\gamma(\nu_0, u, r)\big),\\ &\theta_\gamma(\zeta^n\nu_0, \zeta^n w, r) \geq \theta_\gamma(\zeta^n\nu_0, \zeta^n w, kr) \geq \Psi\big(\theta_\gamma(\nu_0, w, r)\big).\end{aligned} \tag{62}$$

We can write

$$\begin{aligned}\theta_\gamma(u, w, r) = \theta_\gamma(\zeta^n u, \zeta^n w, r) &\geq \theta_\gamma(\zeta^n\nu_0, \zeta^n u, kr)\\ &\quad * \theta_\gamma(\zeta^n\nu_0, \zeta^n w, kr)\\ &\geq \Psi\big(\theta_\gamma(\nu_0, u, r)\big) * \Psi\big(\theta_\gamma(\nu_0, w, r)\big)\\ &\geq \theta_\gamma(\nu_0, u, r) * \theta_\gamma(\nu_0, w, r),\end{aligned} \tag{63}$$

for all $n \in \mathbb{N}$. This is a contradiction; hence, $u = w$.

Example 3. Let $Z = \mathbb{Z} = A \bigcup B$, where $A = \{-1, -2, -3, \ldots\} \bigcup \{0, 1\}$ and $B = \{2, 3, 4, \ldots\}$. Define a binary relation $\perp$ by$\nu \perp \omega \Longleftrightarrow \nu, \omega \in \{|\nu|, |\omega|\}$. Define $\theta_\gamma\colon Z \times Z \times [0, \infty) \longrightarrow [0, 1]$ by

$$\theta_\gamma(\nu, \omega, r) = \frac{r}{r + \max\{\nu, \omega\}}, \tag{64}$$

for all $r > 0$ and $\nu, \omega \in Z$ with a continuous t-norm $*$ defined by: $r_1 * r_2 = r_1 \cdot r_2$. Define $\gamma\colon Z \times Z \longrightarrow [1, \infty)$ by

$$\gamma(\nu, \omega) = \begin{cases} 1, & \text{if } \nu, \omega \in A \text{ or } \nu = 0 \text{ or } \omega = 0,\\ \max\{\nu, \omega\}, & \text{otherwise.}\end{cases} \tag{65}$$

Then, $(Z, \theta_\gamma, *, \perp)$ is an orthogonal G-complete control fuzzy metric space. Observe that

$$\lim_{r \longrightarrow \infty} \theta_\gamma(\nu, \omega, r) = 1. \tag{66}$$

Now, we define $\zeta\colon Z \longrightarrow Z$ by

$$\zeta\nu = \begin{cases} \dfrac{\nu}{2}, & \text{if } \nu \in A,\\ 1, & \text{if } \nu \in B,\end{cases} \tag{67}$$

for all $\nu \in Z$.

Proof. Observe that if $\nu \perp \omega$, then clearly $\zeta\nu \perp \zeta\omega$. Now, there are some cases to prove that the contraction is orthogonal for $k \in [(1/2), 1)$.

(1) If $\nu, \omega \in A$, then $\zeta\nu = \nu/2$ and $\zeta\omega = \omega/2$. We have

$$\begin{aligned}\theta_\gamma(\zeta\nu, \zeta\omega, kr) = \theta_\gamma\left(\frac{\nu}{2}, \frac{\omega}{2}, kr\right) &= \frac{kr}{kr + \max\{(\nu/2), (\omega/2)\}}\\ &\geq \frac{r}{r + \max\{\nu, \omega\}}\\ &= \theta_\gamma(\nu, \omega, r).\end{aligned} \tag{68}$$

(2) If $\nu, \omega \in B$, then $\zeta\nu = 1$ and $\zeta\omega = 1$. In this case,

$$\begin{aligned}\theta_\gamma(\zeta\nu, \zeta\omega, kr) = \theta_\gamma(1, 1, kr) &= \frac{kr}{kr + \max\{1, 1\}}\\ &\geq \frac{r}{r + \max\{\nu, \omega\}} = \theta_\gamma(\nu, \omega, r).\end{aligned} \tag{69}$$

(3) If $\nu \in A$ and $\omega \in B$, then $\zeta\nu = \nu/2$ and $\zeta\omega = 1$. Here,

$$\begin{aligned}\theta_\gamma(\zeta\nu, \zeta\omega, kr) = \theta_\gamma\left(\frac{\nu}{2}, 1, kr\right) &= \frac{kr}{kr + \max\{(\nu/2), 1\}}\\ &\geq \frac{r}{r + \max\{\nu, \omega\}} = \theta_\gamma(\nu, \omega, r).\end{aligned} \tag{70}$$

(4) If $\nu \in B$ and $\omega \in A$, then $\zeta\nu = 1$ and $\zeta\omega = \omega/2$. This implies that

$$\begin{aligned}\theta_\gamma(\zeta\nu, \zeta\omega, kr) = \theta_\gamma\left(1, \frac{\omega}{2}, kr\right) &= \frac{kr}{kr + \max\{1, (\omega/2)\}}\\ &\geq \frac{r}{r + \max\{\nu, \omega\}} = \theta_\gamma(\nu, \omega, r).\end{aligned} \tag{71}$$

Hence, it is an $\perp$-contraction. Now, we show that it is not a contraction. Let $\nu, \omega \in A$, then $\zeta\nu = \nu/2$ and $\zeta\omega = \omega/2$. Here,

$$\theta_\gamma(\zeta\nu, \zeta\omega, kr) = \theta_\gamma\left(\frac{\nu}{2}, \frac{\omega}{2}, kr\right) = \frac{kr}{kr + \max\{(\nu/2), (\omega/2)\}}. \tag{72}$$

Let $\nu = \omega = -2, k = (9/10)$ and $r = 10$, so

$$\begin{aligned}\theta_\gamma(\zeta\nu, \zeta\omega, kr) &= \frac{9}{9 + \max\{-1, -1\}} \leq \frac{10}{10 + \max\{-2, -2\}}\\ &= \theta_\gamma(\nu, \omega, r),\end{aligned} \tag{73}$$

which implies $\theta_\gamma(\zeta\nu, \zeta\omega, kr) \leq \theta_\gamma(\nu, \omega, r)$. This is wrong.

If $\lim_{n \longrightarrow \infty} \theta_\gamma(\nu_n, \nu, r)$ is finite and exists, then also $\lim_{n \longrightarrow \infty} \theta_\gamma(\zeta\nu_n, \zeta\nu, r)$ is finite and exists. This implies that it is $\perp$-continuous. Also, observe that

$$\begin{aligned}&\lim_{n \longrightarrow \infty} \gamma(\nu_n, \omega),\\ &\lim_{n \longrightarrow \infty} \gamma(\omega, \nu_n),\end{aligned} \tag{74}$$

are finite and exist. All circumstances of Theorem 1 are fulfilled and 0 is the unique fixed point of ζ.

3. An Application to a Fuzzy Integral Equation

In this section, we utilize Theorem 1 to examine the existence and uniqueness of a solution of a fuzzy Fredholm-type integral equation of second kind.

Let $Z = C([e, g], \mathbb{R})$ be the set of all continuous real-valued functions defined on $[e, g]$.

Now, we consider the fuzzy Fredholm-type integral equation of the second kind:

$$\nu(l) = f(j) + \beta \int_e^g F(l, j)\nu(l)\mathrm{d}j, \quad \text{for } l, j \in [e, g]. \tag{75}$$

where $\beta > 0$, $f(j)$ is a fuzzy function of $j \in [e, g]$ and $F \in \mathrm{Z}$. Define θ_γ by

$$\theta_\gamma(\nu(l), \omega(l), r) = \sup_{l\in[e,g]} \frac{r}{r + \max\{\nu(l), \omega(l)\}}, \quad \text{for } \nu, \omega \in \mathrm{Z} \text{ and } r > 0, \tag{76}$$

with a continuous t-norm $*$ defined by $r_1 * r_2 = r_1 \cdot r_2$. Define γ: $\mathrm{Z} \times \mathrm{Z} \longrightarrow [1, \infty)$ by

$$\gamma(\nu, \omega) = \begin{cases} 1, & \text{if } \nu, \omega \in A \text{ or } \nu = 0 \text{ or } \omega = 0, \\ \max\{\nu, \omega\}, & \text{otherwise.} \end{cases} \tag{77}$$

Then, $(\mathrm{Z}, \theta_\gamma, * \perp)$ is an orthogonal G-complete control fuzzy metric space.

Theorem 4. *Assume that* $\max\{F(l, j)\nu(l), F(l, j)\omega(l)\} \le \max\{\nu(l), \omega(l)\}$ *for* $\nu, \omega \in \mathrm{Z}$, $k \in (0, 1)$, *and* $\forall l, j \in [e, g]$.

Also, consider $\int_e^g \mathrm{d}j = g - e \le k < 1$. *Let* ζ: $\mathrm{Z} \longrightarrow \mathrm{Z}$ *be*

(i) $\perp$*-preserving*

(ii) $\perp$*-contraction*

(iii) $\perp$*-continuous*

Then, the fuzzy Fredholm-type integral equation of second kind in equation (75) has a unique solution.

Proof. Define ζ: $\mathrm{Z} \longrightarrow \mathrm{Z}$ by

$$\zeta\nu(l) = f(j) + \beta \int_e^g F(l, j)e(l)\mathrm{d}j, \quad \text{for all } l, j \in [e, g]. \tag{78}$$

(i) Take orthogonality as $\nu(l) \perp \omega(l) \Longleftrightarrow \nu(l)\omega(l) \in \{|\nu(l)|, |\omega(l)|\}$. We see that $\nu(l)$ and $\zeta\nu(l)$ belong to Z. So, if $\nu(l) \perp \omega(l)$, then clearly $\zeta\nu(l) \perp \zeta\omega(l)$.

(ii) Observe that the existence of a fixed point of the operator ζ is equivalent to the existence of a solution of the Fredholm-type integral Equation (75).

(iii) Note that

$$\begin{aligned} &\max\{F(l, j)\nu(l), F(l, j)\omega(l)\} \le \max\{\nu(l), \omega(l)\} \\ &\Rightarrow f(j) + \beta \int_e^g \max\{F(l, j)\nu(l), F(l, j)\omega(l)\} \\ &\le f(j) + \beta \int_e^g \max\{\nu(l), \omega(l)\}. \end{aligned} \tag{79}$$

(iv)Now, for all $\nu, \omega \in \mathrm{Z}$, we have

$$\begin{aligned} \theta_\gamma(\zeta\nu(l), \zeta\omega(l), kr) &= \sup_{l\in[e,g]} \frac{kr}{kr + \max\{\zeta\nu(l), \zeta\omega(l)\}} \\ &= \sup_{l\in[e,g]} \frac{kr}{kr + \max\left\{\int_e^g F(l, j)\nu(l)\mathrm{d}j, \int_e^g F(l, j)\omega(l)\mathrm{d}j\right\}} \\ &= \sup_{l\in[e,g]} \frac{kr}{kr + \int_e^g \max\{F(l, j)\nu(l), F(l, j)\omega(l)\}\mathrm{d}j} \\ &\ge \sup_{l\in[e,g]} \frac{kr}{kr + \int_e^g \max\{\nu(l), \omega(l)\}\mathrm{d}j} \\ &= \sup_{l\in[e,g]} \frac{kr}{kr + \max\{\nu(l), \omega(l)\} \int_e^g \mathrm{d}j} \\ &\ge \sup_{l\in[e,g]} \frac{kr}{kr + k\max\{\nu(l), \omega(l)\}} \\ &\ge \frac{r}{r + \max\{\nu(l), \omega(l)\}} \\ &= \theta_\gamma(\nu(l), \omega(l), r). \end{aligned} \tag{80}$$

(v) Hence, ζ is an $\perp$-contraction.

(vi) Suppose $\{\nu_n\}$ is an orthogonal sequence in Z such that $\{\nu_n\}$ converges to $\nu \in \mathrm{Z}$. Because ζ is $\perp$-preserving, $\{\zeta\nu_n\}$ is an orthogonal sequence for each $n \in \mathbb{N}$. From (ii), we have

$$\theta_\gamma(\nu(l), \omega(l), kr) \ge \theta_\gamma(\nu(l), \omega(l), r). \tag{81}$$

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Retraction

Retracted: Topological Structures of Lower and Upper Rough Subsets in a Hyperring

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

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Research Article

Topological Structures of Lower and Upper Rough Subsets in a Hyperring

Nabilah Abughazalah,[1] **Naveed Yaqoob**,[2] **and Kiran Shahzadi**[2]

[1] *Mathematical Sciences Department, College of Science, Princess Nourah Bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia*
[2] *Department of Mathematics and Statistics, Riphah International University, I-14, Islamabad, Pakistan*

Correspondence should be addressed to Nabilah Abughazalah; nhabughazala@pnu.edu.sa

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In this paper, we study the connection between topological spaces, hyperrings (semi-hypergroups), and rough sets. We concentrate here on the topological parts of the lower and upper approximations of hyperideals in hyperrings and semi-hypergroups. We provide the conditions for the boundary of hyp-ideals of a hyp-ring to become the hyp-ideals of hyp-ring.

1. Introduction

Algebraic hyp-structure (hyperstructure) represents a real extension of classical algebraic structure. Algebraic hyp-structures depend on hyperoperations and their properties. Sm-hyp-group (semi-hypergroup) was first introduced by French Mathematician Marty [1] in 1934. The sm-hyp-group concept is the generalization of sm-group (semigroup) concept, likewise the hyp-ring (hyperring) concept is the generalization of ring concept. In [2, 3], authors provided many applications of hyp-structures. There are several creators who added numerous outcomes to the hypothesis of algebraic hyp-structures, for instance, Hila and Dine [4] studied the hyperideals of left almost semi-hypergroups. Tang et al. [5] introduced the idea of hyperfilters in ordered semi-hypergroups, also see [6, 7].

In 1982, Pawlak [8] introduced R-sets (rough sets) for the very first time. R-set theory has been a knowledge discovery in rational databases. Set approximation is divided into two parts, i.e., lower approximation and upper approximation. The applications of R-sets are considered in finance, pattern recognition, industries, information processing, and business. It provides a mathematical tool to find out pattern hidden in data. The major advantages of R-set approach is that it does not need any primary/secondary information about the data like the theory of probability in statistics and the grade of membership in the theory of fuzzy set. It gives systematic procedures, tools, and algorithms to find out hidden patterns in data, and it permits generating in mechanized way the sets of decision rules from data. Thivagar and Devi [9] introduced the concept of nanotopology via ring structure. R-set theory has been studied by several authors in algebraic structures and also in algebraic hyperstructures. Ahn and Kim applied R-set theory to BE-algebras [10]. Ali et al. [11] studied generalized roughness in $(\varepsilon, \varepsilon \vee q_k)$-fuzzy filters of ordered semigroups. Biswas and S. Nanda [12] applied R-set theory to groups. Shabir and Irshad [13] applied roughness in ordered semigroups. In [14–22], authors studied roughness in different hyperstructures. Fuzzy sets were also considered by many authors, for instance, Fotea and Davvaz [23] studied fuzzy hyperrings. Ameri and Motameni [24] applied fuzzy set theory to the hyperideals of fuzzy hyperrings. Bayrak and Yamak [25] introduced some results on the lattice of fuzzy hyperideals of a hyperring. Davvaz [26] studied fuzzy Krasner (m, n)-hyperrings. Connections between fuzzy sets and topology are considered in [27–29].

2. Preliminaries and Notations

Definition 1. A topological space refers to a pair (F, τ), where F is a nonempty set and τ is a topology on F.

Definition 2. A hyp-groupoid (hypergroupoid) $(\mathcal{F}, \widehat{\circ})$ is called a sm-hyp-group if, for all a, b, c of $\mathcal{F}$, we have $(a\widehat{\circ}b)\widehat{\circ}c = a\widehat{\circ}(b\widehat{\circ}c)$, which means that

$$\bigcup_{d\varepsilon\, a\widehat{\circ}b} d\widehat{\circ}c = \bigcup_{e\varepsilon b\widehat{\circ}c} a\widehat{\circ}e. \tag{1}$$

Definition 3. A subset I of a sm-hyp-group $\mathcal{F}$ is called right hyp-ideal (resp., left hyp-ideal) if

(i) $I\widehat{\circ}I \subseteq I$

(ii) $I\widehat{\circ}\mathcal{F} \subseteq I$ (resp., $\mathcal{F}\widehat{\circ}I \subseteq I$)

A left and right hyp-ideal I of $\mathcal{F}$ is known as hyp-ideal of $\mathcal{F}$.

Definition 4. (lower approximation of a subset, see [8]). The l-approximation (lower approximation) of $\Upsilon \subseteq U$ w.r.t E (E is an equivalence relation) is a set of all those objects, which are contained in Υ. From the diverse representations of an E-relation, we attain three productive definitions of l-approximation:

(i) $\underline{E_{\text{Lower}}}(\Upsilon) = \{a\varepsilon U\colon\ [a]_E \subseteq \Upsilon\}$

(ii) $\underline{E_{\text{Lower}}}(\Upsilon) = \cup_{[a]_E \subseteq \Upsilon}[a]_E$

(iii) $\underline{E_{\text{Lower}}}(\Upsilon) = \bigcup\{A\varepsilon U|E\colon A \subseteq \Upsilon\}$, where $[a]_E = \{q\colon qEa\}$

(i) is element-based definition, (ii) is granule-based definition, and (iii) is subsystem-based definition.

Definition 5. (upper approximation of a subset, see [8]). The u-approximation (upper approximation) of a set Υ w.r.t E is a set of all those objects which have nonempty intersection with Υ. From the unlike representations of an E-relation, we obtain three constructive definitions of u-approximation:

(i) $\overline{E_{\text{Upper}}}(\Upsilon) = \{a\varepsilon U\colon\ [a]_E \cap \Upsilon \neq \varnothing\}$

(ii) $\overline{E_{\text{Upper}}}(\Upsilon) = \cup_{[a]_E \cap \Upsilon \neq \varnothing}[a]_E$

(iii) $\overline{E_{\text{Upper}}}(\Upsilon) = \cap\{A\varepsilon U/E\colon A \cap \Upsilon \neq \varnothing\}$, where $[a]_E = \{q\colon qEa\}$

The following properties hold in approximation space [8]:

(1) $\underline{E_{\text{Lower}}}(\Upsilon) \subseteq \Upsilon \subseteq \overline{E_{\text{Upper}}}(\Upsilon)$

(2) $\underline{E_{\text{Lower}}}(\varnothing) = \varnothing = \overline{E_{\text{Upper}}}(\varnothing)$; $\underline{E_{\text{Lower}}}(U) = U = \overline{E_{\text{Upper}}}(U)$

(3) $\overline{E_{\text{Upper}}}(\Upsilon_1 \cup \Upsilon_2) = \overline{E_{\text{Upper}}}(\Upsilon_1) \cup \overline{E_{\text{Upper}}}(\Upsilon_2)$

(4) $\underline{E_{\text{Lower}}}(\Upsilon_1 \cup \Upsilon_2) \supseteq \underline{E_{\text{Lower}}}(\Upsilon_1) \cup \underline{E_{\text{Lower}}}(\Upsilon_2)$

(5) $\overline{E_{\text{Upper}}}(\Upsilon_1 \cap \Upsilon_2) \subseteq \overline{E_{\text{Upper}}}(\Upsilon_1) \cap \overline{E_{\text{Upper}}}(\Upsilon_2)$

(6) $\underline{E_{\text{Lower}}}(\Upsilon_1 \cap \Upsilon_2) = \underline{E_{\text{Lower}}}(\Upsilon_1) \cap \underline{E_{\text{Lower}}}(\Upsilon_2)$

(7) $\Upsilon_1 \subseteq \Upsilon_2$ implies $\underline{E_{\text{Lower}}}(\Upsilon_1) \subseteq \underline{E_{\text{Lower}}}(\Upsilon_2)$, $\overline{E_{\text{Upper}}}(\Upsilon_1) \subseteq \overline{E_{\text{Upper}}}(\Upsilon_2)$

(8) $\underline{E_{\text{Lower}}}(\Upsilon) = \overline{E_{\text{Upper}}}(\Upsilon)$

(9) $\overline{E_{\text{Upper}}}(\Upsilon) = \underline{E_{\text{Lower}}}(\Upsilon)$

(10) $\underline{E_{\text{Lower}}}\,\underline{E_{\text{Lower}}}(\Upsilon) = \overline{E_{\text{Upper}}}\,\underline{E_{\text{Lower}}}(\Upsilon) = \underline{E_{\text{Lower}}}(\Upsilon)$

(11) $\overline{E_{\text{Upper}}E_{\text{Upper}}}(\Upsilon) = \underline{E_{\text{Lower}}}\,\overline{E_{\text{Upper}}}(\Upsilon) = \overline{E_{\text{Upper}}}(\Upsilon)$

3. T-Structures of R-Sets Based on Sm-Hyp-Groups

In this section, we develop some concepts related to topology of R-sets based on sm-hyp-groups.

Definition 6. Let $\mathcal{F}$ be a sm-hyp-group, $\Upsilon \subseteq \mathcal{F}$, and ξ be a REG-relation (regular relation) on $\mathcal{F}$. Then, the (l–) u-approximations and boundary of Υ with respect to the REG-relation ξ are given as follows:

(i) $\underline{\xi_{\text{Lower}}}(\Upsilon) = \{x\,\varepsilon\,\mathcal{F}\colon\ \xi(x) \subseteq \Upsilon\}$

(ii) $\overline{\xi_{\text{Upper}}}(\Upsilon) = \{x\,\varepsilon\,\mathcal{F}\colon\ \xi(x) \cap \Upsilon \neq \varnothing\}$

(iii) $\xi^B(\Upsilon) = \overline{\xi_{\text{Upper}}}(\Upsilon) - \underline{\xi_{\text{Lower}}}(\Upsilon)$

The family of sets

$$\xi^\tau(\Upsilon) = \left\{\mathcal{F}, \varnothing, \underline{\xi_{\text{Lower}}}(\Upsilon), \overline{\xi_{\text{Upper}}}(\Upsilon), \xi^B(\Upsilon)\right\} \tag{2}$$

forms a topology on $\mathcal{F}$.

Example 1. Let $\mathcal{F} = \{a_{\mathcal{F}}, b_{\mathcal{F}}, c_{\mathcal{F}}, d_{\mathcal{F}}\}$ be a sm-hyp-group under the binary hyperoperation "$\widehat{\circ}$" defined in Cayley (Table 1).

Let

$$\begin{aligned}\xi = \{&(a_{\mathcal{F}}, a_{\mathcal{F}}), (a_{\mathcal{F}}, b_{\mathcal{F}}), (a_{\mathcal{F}}, c_{\mathcal{F}}), (b_{\mathcal{F}}, a_{\mathcal{F}}), (b_{\mathcal{F}}, b_{\mathcal{F}}),\\ &(b_{\mathcal{F}}, c_{\mathcal{F}}), (c_{\mathcal{F}}, a_{\mathcal{F}}), (c_{\mathcal{F}}, b_{\mathcal{F}}), (c_{\mathcal{F}}, c_{\mathcal{F}}), (d_{\mathcal{F}}, d_{\mathcal{F}})\}\end{aligned} \tag{3}$$

be a REG-relation on the sm-hyp-group $\mathcal{F}$ with the following regular classes:

$$\xi(a_{\mathcal{F}}) = \xi(b_{\mathcal{F}}) = \xi(c_{\mathcal{F}}) = \{a_{\mathcal{F}}, b_{\mathcal{F}}, c_{\mathcal{F}}\} \text{ and } \xi(d_{\mathcal{F}}) = \{d_{\mathcal{F}}\}. \tag{4}$$

Now, let $\Upsilon = \{a_{\mathcal{F}}, b_{\mathcal{F}}, d_{\mathcal{F}}\} \subseteq \mathcal{F}$. Then, $\underline{\xi_{\text{Lower}}}(\Upsilon) = \{d_{\mathcal{F}}\}$, $\overline{\xi_{\text{Upper}}}(\Upsilon) = \mathcal{F}$, and $\xi^B(\Upsilon) = \{a_{\mathcal{F}}, b_{\mathcal{F}}, c_{\mathcal{F}}\}$. Hence, $\xi^\tau(\Upsilon) = \{\mathcal{F}, \varnothing, \{d_{\mathcal{F}}\}, \{a_{\mathcal{F}}, b_{\mathcal{F}}, c_{\mathcal{F}}\}\}$, which is clearly a topology on $\mathcal{F}$.

Remark 1. Let $\mathcal{F}$ be a sm-hyp-group, ξ be a REG-relation on $\mathcal{F}$, and $\Upsilon \subseteq \mathcal{F}$.

(i) If $\underline{\xi_{\text{Lower}}}(\Upsilon) = \varnothing$ and $\overline{\xi_{\text{Upper}}}(\Upsilon) = \mathcal{F}$, then $\xi^\tau(\Upsilon) = \{\mathcal{F}, \varnothing\}$ is called the indiscrete topology on $\mathcal{F}$.

(ii) If $\underline{\xi_{\text{Lower}}}(\Upsilon) = \overline{\xi_{\text{Upper}}}(\Upsilon) = \Upsilon$, then the topology

$$\begin{aligned}\xi^\tau(\Upsilon) &= \left\{\mathcal{F}, \varnothing, \underline{\xi_{\text{Lower}}}(\Upsilon)\right\} = \left\{\mathcal{F}, \varnothing, \overline{\xi_{\text{Upper}}}(\Upsilon)\right\}\\ &= \{\mathcal{F}, \varnothing, \Upsilon\}.\end{aligned} \tag{5}$$

(iii) If $\underline{\xi_{\text{Lower}}}(\Upsilon) = \varnothing$ and $\overline{\xi_{\text{Upper}}}(\Upsilon) \neq \mathcal{F}$, then $\xi^\tau(\Upsilon) = \left\{\mathcal{F}, \varnothing, \overline{\xi_{\text{Upper}}}(\Upsilon)\right\}$.

Table 1: Tabular form of the hyperoperation " $\widehat{\circ}$" defined in Example 1.

$\widehat{\circ}$	$a_{\mathcal{F}}$	$b_{\mathcal{F}}$	$c_{\mathcal{F}}$	$d_{\mathcal{F}}$
$a_{\mathcal{F}}$	$a_{\mathcal{F}}$	$b_{\mathcal{F}}$	$\{a_{\mathcal{F}}, c_{\mathcal{F}}\}$	$d_{\mathcal{F}}$
$b_{\mathcal{F}}$	$b_{\mathcal{F}}$	$b_{\mathcal{F}}$	$b_{\mathcal{F}}$	$d_{\mathcal{F}}$
$c_{\mathcal{F}}$	$\{a_{\mathcal{F}}, c_{\mathcal{F}}\}$	$b_{\mathcal{F}}$	$c_{\mathcal{F}}$	$d_{\mathcal{F}}$
$d_{\mathcal{F}}$	$d_{\mathcal{F}}$	$d_{\mathcal{F}}$	$d_{\mathcal{F}}$	$d_{\mathcal{F}}$

(iv) If $\underline{\xi_{\text{Lower}}}(\Upsilon) \neq \varnothing$ and $\overline{\xi_{\text{Upper}}}(\Upsilon) = \mathcal{F}$, then $\xi^{\tau}(\Upsilon) = \{\mathcal{F}, \varnothing, \xi^{B}(\Upsilon)\}$.

(v) If $\underline{\xi_{\text{Lower}}}(\Upsilon) \neq \overline{\xi_{\text{Upper}}}(\Upsilon)$, where $\underline{\xi_{\text{Lower}}}(\Upsilon) \neq \varnothing$, then $\xi^{\tau}(\Upsilon) = \{\mathcal{F}, \varnothing, \underline{\xi_{\text{Lower}}}(\Upsilon), \overline{\xi_{\text{Upper}}}(\Upsilon), \xi^{B}(\Upsilon)\}$ is the discrete topology on $\mathcal{F}$.

Theorem 1. *Let $\mathcal{F}$ be a sm-hyp-group, ξ be a REG-relation on $\mathcal{F}$, and $\Upsilon \subseteq \mathcal{F}$. Then,*

(i) $\underline{\xi_{Lower}}(\Upsilon) \subseteq \Upsilon \subseteq \overline{\xi_{Upper}}(\Upsilon)$

(ii) $\underline{\xi_{Lower}}(\varnothing) = \varnothing = \overline{\xi_{Upper}}(\varnothing)$

(iii) $\underline{\xi_{Lower}}(\mathcal{F}) = \mathcal{F} = \overline{\xi_{Upper}}(\mathcal{F})$

Proof

(i) We have to prove that $\underline{\xi_{\text{Lower}}}(\Upsilon) \subseteq \Upsilon \subseteq \overline{\xi_{\text{Upper}}}(\Upsilon)$. First, we prove that $\underline{\xi_{\text{Lower}}}(\Upsilon) \subseteq \Upsilon$.
Let

$$x\,\varepsilon\,\underline{\xi_{\text{Lower}}}(\Upsilon) \Rightarrow \xi(x) \subseteq \Upsilon. \tag{6}$$

As $\xi(x)$ is a regular class of x, so $x\varepsilon\xi(x)$. However, as $\xi(x) \subseteq \Upsilon$, thus $x\varepsilon\Upsilon$. Now, we prove that $\Upsilon \subseteq \overline{\xi_{\text{Upper}}}(\Upsilon)$. Let $y\,\varepsilon\Upsilon$. As $\xi(y)$ is a regular class of y, so $y\,\varepsilon\xi(y)$. Thus,

$$y\varepsilon\xi(y) \cap \Upsilon \Rightarrow \xi(y) \cap \Upsilon \neq \varnothing. \tag{7}$$

Thus, $y\varepsilon\overline{\xi_{\text{Upper}}}(\Upsilon)$.

(ii) The proof of this part is straightforward.

(iii) The proof of this part is straightforward.

It is easy to see from Example 1 that $\overline{\xi_{\text{Upper}}}(\Upsilon) \nsubseteq \Upsilon \nsubseteq \underline{\xi_{\text{Lower}}}(\Upsilon)$. ☐

Proposition 1. *Let $\mathcal{F}$ be a sm-hyp-group, ξ be a REG-relation on $\mathcal{F}$, and Υ_1 and Υ_2 two subsets of $\mathcal{F}$ such that $\Upsilon_1 \subseteq \Upsilon_2$. Then,*

(i) $\underline{\xi_{Lower}}(\Upsilon_1) \subseteq \underline{\xi_{Lower}}(\Upsilon_2)$

(ii) $\overline{\xi_{Upper}}(\Upsilon_1) \subseteq \overline{\xi_{Upper}}(\Upsilon_2)$

(iii) $\xi^{B}(\Upsilon_1) \subseteq \xi^{B}(\Upsilon_2)$

Proof

(i) Given $\Upsilon_1 \subseteq \Upsilon_2$ and $x\varepsilon\,\underline{\xi_{\text{Lower}}}(\Upsilon_1)$, by definition

$$\begin{aligned}&\Rightarrow \xi(x) \subseteq \Upsilon_1 \quad \text{for all } x\varepsilon\mathcal{F}\\&\Rightarrow \xi(x) \subseteq \Upsilon_1 \subseteq \Upsilon_2 \Rightarrow \xi(x) \subseteq \Upsilon_2 \quad \text{for all } x\varepsilon\mathcal{F}.\end{aligned} \tag{8}$$

Thus, $\underline{\xi_{\text{Lower}}}(\Upsilon_1) \subseteq \underline{\xi_{\text{Lower}}}(\Upsilon_2)$.

(ii) Let $x\varepsilon\overline{\xi_{\text{Upper}}}(\Upsilon_1) \Rightarrow \xi(x) \cap \Upsilon_1 \neq \varnothing$. Let

$$\begin{aligned}&y\varepsilon\xi(x) \cap \Upsilon_1\\&\Rightarrow y\varepsilon\xi(x) \text{ and } y\varepsilon\Upsilon_1\\&\Rightarrow y\varepsilon\xi(x) \text{ and } y\varepsilon\Upsilon_1 \subseteq \Upsilon_2\\&\Rightarrow y\varepsilon\xi(x) \cap \Upsilon_2 y\varepsilon\xi(x) \text{ and } y\varepsilon\Upsilon_2\\&\Rightarrow \xi(x) \cap \Upsilon_2 \neq \varnothing\\&\Rightarrow x\varepsilon\overline{\xi_{\text{Upper}}}(\Upsilon_2).\end{aligned} \tag{9}$$

Hence, we get $\overline{\xi_{\text{Upper}}}(\Upsilon_1) \subseteq \overline{\xi_{\text{Upper}}}(\Upsilon_2)$.

(iii) From (i) and (ii),

$$\overline{\xi_{\text{Upper}}}(\Upsilon_1) - \underline{\xi_{\text{Lower}}}(\Upsilon_1) \subseteq \overline{\xi_{\text{Upper}}}(\Upsilon_2) - \underline{\xi_{\text{Lower}}}(\Upsilon_2). \tag{10}$$

Thus, we have $\xi^{B}(\Upsilon_1) \subseteq \xi^{B}(\Upsilon_2)$. ☐

Theorem 2. *Let $\mathcal{F}$ be a sm-hyp-group and ξ be a REG-relation on $\mathcal{F}$, $\Upsilon_1, \Upsilon_2 \subseteq \mathcal{F}$ such that $\Upsilon_1 \subseteq \Upsilon_2$. Then, $\xi^{\tau}(\Upsilon_1) \subseteq \xi^{\tau}(\Upsilon_2)$.*

Proof. Since $\Upsilon_1 \subseteq \Upsilon_2 \subseteq \mathcal{F}$, the approximations with respect to the sm-hyp-group satisfy

$$\begin{aligned}&\underline{\xi_{\text{Lower}}}(\Upsilon_1) \subseteq \underline{\xi_{\text{Lower}}}(\Upsilon_2),\\&\overline{\xi_{\text{Upper}}}(\Upsilon_1) \subseteq \overline{\xi_{\text{Upper}}}(\Upsilon_2) \text{ and}\\&\xi^{B}(\Upsilon_1) \subseteq \xi^{B}(\Upsilon_2),\end{aligned} \tag{11}$$

which implies that $\xi^{\tau}(\Upsilon_1) \subseteq \xi^{\tau}(\Upsilon_2)$. ☐

Proposition 2. *Suppose ξ and γ are two REG-relations on $\mathcal{F}$ such that $\xi \subseteq \gamma$, and let Υ_1 be the nonempty subset of $\mathcal{F}$. Then,*

(i) $\underline{\gamma_{Lower}}(\Upsilon_1) \subseteq \underline{\xi_{Lower}}(\Upsilon_1)$

(ii) $\overline{\xi_{Upper}}(\Upsilon_1) \subseteq \overline{\gamma_{Upper}}(\Upsilon_1)$

(iii) $\xi^{B}(\Upsilon_1) \subseteq \gamma^{B}(\Upsilon_1)$

Proof. Suppose ξ and γ are two REG-relations on $\mathcal{F}$ such that $\xi \subseteq \gamma$, and let Υ_1 be the nonempty subset of $\mathcal{F}$.

(i) Let $x\varepsilon\,\underline{\gamma_{\text{Lower}}}(\Upsilon_1)$. Then, $\gamma(x) \subseteq \Upsilon_1$. Now, as $\xi \subseteq \gamma$, so $\xi(x) \subseteq \gamma(x)$ for any $x\,\epsilon\,\mathcal{F}$. Then, we get $\xi(x) \subseteq \Upsilon_1$. Hence, $x\varepsilon\,\underline{\xi_{\text{Lower}}}(\Upsilon_1)$.

(ii) Let $x\varepsilon\overline{\xi_{\text{Upper}}}(\Upsilon_1)$. Then, $\xi(x) \cap \Upsilon_1 \neq \varnothing$. Now, as $\xi \subseteq \gamma$, so

$$\begin{aligned}&\xi(x) \subseteq \gamma(x) \quad \text{for any } x\varepsilon\mathcal{F}\\&\Rightarrow \xi(x) \cap \Upsilon_1 \subseteq \gamma(x) \cap \Upsilon_1 \quad \text{for any } x\,\varepsilon\,\mathcal{F}.\end{aligned} \tag{12}$$

As $\varnothing \neq \xi(x) \cap \Upsilon_1 \subseteq \gamma(x) \cap \Upsilon_1$. Thus, $\gamma(x) \cap \Upsilon_1 \neq \varnothing$. Hence, $x\varepsilon\overline{\gamma_{\text{Upper}}}(\Upsilon_1)$.

(iii) The proof of this part implies from (i) and (ii). □

Theorem 3. *Let $\mathscr{F}$ be a sm-hyp-group and ξ and γ be the REG-relations on $\mathscr{F}$ such that $\xi \subseteq \gamma$, and let Υ_1 be the nonempty subset of $\mathscr{F}$. Then, $\xi^{\tau}(\Upsilon_1) \neq \gamma^{\tau}(\Upsilon_1)$.*

Proof. Since ξ and γ are the REG-relations on $\mathscr{F}$ such that $\xi \subseteq \gamma$, then

$$\begin{gathered}\underline{\gamma_{\text{Lower}}}(\Upsilon_1) \subseteq \underline{\xi_{\text{Lower}}}(\Upsilon_1),\\ \overline{\xi_{\text{Upper}}}(\Upsilon_1) \subseteq \overline{\gamma_{\text{Upper}}}(\Upsilon_1)\ \text{and}\\ \xi^{B}(\Upsilon_1) \subseteq \gamma^{B}(\Upsilon_1),\end{gathered}\tag{13}$$

which implies that $\xi^{\tau}(\Upsilon_1) \neq \gamma^{\tau}(\Upsilon_1)$. □

4. T-Structures of R-Sets Based on Hyp-Rings

In this section, we develop some concepts related to topology of R-sets based on hyp-rings.

Definition 7. Let $\mathfrak{R}$ be a hyp-ring, $\Upsilon \subseteq \mathfrak{R}$, and $\mathscr{F}$ be a hyperideal of $\mathfrak{R}$. Then, the (l-) u-approximations and boundary of Υ with respect to the hyp-ideal $\mathscr{F}$ are given as follows:

(i) $\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon) = \{x\varepsilon\mathfrak{R}\colon x \oplus \mathscr{F} \subseteq \Upsilon\}$
(ii) $\overline{\mathscr{F}_{\text{Upper}}}(\Upsilon) = \{x\varepsilon\mathfrak{R}\colon (x \oplus \mathscr{F}) \cap \Upsilon \neq \varnothing\}$
(iii) $\mathscr{F}^{B}(\Upsilon) = \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon) - \underline{\mathscr{F}_{\text{Lower}}}(\Upsilon)$

The family of sets

$$\mathscr{F}^{\tau}(\Upsilon) = \left\{\mathfrak{R}, \varnothing, \underline{\mathscr{F}_{\text{Lower}}}(\Upsilon), \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon), \mathscr{F}^{B}(\Upsilon)\right\},\tag{14}$$

forms a topology on $\mathfrak{R}$ with respect to $\mathscr{F}$.

Example 2. Let $\mathfrak{R} = \{a_{\mathfrak{R}}, b_{\mathfrak{R}}, c_{\mathfrak{R}}, d_{\mathfrak{R}}, e_{\mathfrak{R}}, f_{\mathfrak{R}}\}$ be a hyp-ring under the binary hyperoperations $\oplus$ and $\hat{\circ}$ defined in the Cayley (Tables 2 and 3).

Let $\mathscr{F} = \{a_{\mathfrak{R}}, e_{\mathfrak{R}}\}$ be a hyp-ideal of $\mathfrak{R}$. Consider $\Upsilon = \{a_{\mathfrak{R}}, c_{\mathfrak{R}}, d_{\mathfrak{R}}, f_{\mathfrak{R}}\} \subseteq \mathfrak{R}$. Then,

$$\begin{gathered}\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon) = \{c_{\mathfrak{R}}, d_{\mathfrak{R}}\}, \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon) = \mathfrak{R},\\ \mathscr{F}^{B}(\Upsilon) = \{a_{\mathfrak{R}}, b_{\mathfrak{R}}, e_{\mathfrak{R}}, f_{\mathfrak{R}}\}.\end{gathered}\tag{15}$$

Hence, $\mathscr{F}^{\tau}(\Upsilon) = \{\mathfrak{R}, \varnothing, \{c_{\mathfrak{R}}, d_{\mathfrak{R}}\}, \{a_{\mathfrak{R}}, b_{\mathfrak{R}}, e_{\mathfrak{R}}, f_{\mathfrak{R}}\}\}$, which is clearly a topology on $\mathfrak{R}$.

Remark 2. Let $\mathfrak{R}$ be a hyp-ring, $\mathscr{F}$ be a hyp-ideal of $\mathfrak{R}$, and $\Upsilon \subseteq \mathfrak{R}$.

(i) If $\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon) = \varnothing$ and $\overline{\mathscr{F}_{\text{Upper}}}(\Upsilon) = \mathfrak{R}$, then $\mathscr{F}^{\tau}(\Upsilon) = \{\mathfrak{R}, \varnothing\}$ is called the indiscrete topology on $\mathfrak{R}$.
(ii) If $\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon) = \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon) = \Upsilon$, then the topology

$$\begin{aligned}\mathscr{F}^{\tau}(\Upsilon) &= \left\{\mathfrak{R}, \varnothing, \underline{\mathscr{F}_{\text{Lower}}}(\Upsilon)\right\}\\ &= \left\{\mathfrak{R}, \varnothing, \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon)\right\}\\ &= \{\mathfrak{R}, \varnothing, \Upsilon\}.\end{aligned}\tag{16}$$

(iii) If $\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon) = \varnothing$ and $\overline{\mathscr{F}_{\text{Upper}}}(\Upsilon) \neq \mathfrak{R}$, then $\mathscr{F}^{\tau}(\Upsilon) = \left\{\mathfrak{R}, \varnothing, \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon)\right\}$.
(iv) If $\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon) \neq \varnothing$ and $\overline{\mathscr{F}_{\text{Upper}}}(\Upsilon) = \mathfrak{R}$, then $\mathscr{F}^{\tau}(\Upsilon) = \{\mathfrak{R}, \varnothing, \mathscr{F}^{B}(\Upsilon)\}$.
(v) If $\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon) \neq \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon)$ where $\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon) \neq \varnothing$, then $\mathscr{F}^{\tau}(\Upsilon) = \left\{\mathfrak{R}, \varnothing, \underline{\mathscr{F}_{\text{Lower}}}(\Upsilon), \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon),\right.$ $\mathscr{F}^{B}(\Upsilon)\}$ is the discrete topology on $\mathfrak{R}$.

Theorem 4. *Let $\mathfrak{R}$ be a hyp-ring, $\mathscr{F}$ be a hyp-ideal of $\mathfrak{R}$, and $\Upsilon \subseteq \mathfrak{R}$. Then,*

(i) $\underline{\mathscr{F}_{Lower}}(\Upsilon) \subseteq \Upsilon \subseteq \overline{\mathscr{F}_{Upper}}(\Upsilon)$
(ii) $\underline{\mathscr{F}_{Lower}}(\varnothing) = \varnothing = \overline{\mathscr{F}_{Upper}}(\varnothing)$
(iii) $\underline{\mathscr{F}_{Lower}}(\mathfrak{R}) = \mathfrak{R} = \overline{\mathscr{F}_{Upper}}(\mathfrak{R})$

Proposition 3. *Let $\mathfrak{R}$ be a hyp-ring, $\mathscr{F}$ be a hyp-ideal of $\mathfrak{R}$, and Υ_1 and Υ_2 two subsets of $\mathfrak{R}$ such that $\Upsilon_1 \subseteq \Upsilon_2$. Then,*

(i) $\underline{\mathscr{F}_{Lower}}(\Upsilon_1) \subseteq \underline{\mathscr{F}_{Lower}}(\Upsilon_2)$
(ii) $\overline{\mathscr{F}_{Upper}}(\Upsilon_1) \subseteq \overline{\mathscr{F}_{Upper}}(\Upsilon_2)$
(iii) $\mathscr{F}^{B}(\Upsilon_1) \subseteq \mathscr{F}^{B}(\Upsilon_2)$

Theorem 5. *Let $\mathfrak{R}$ be a hyp-ring and $\mathscr{F}$ be a hyp-ideal of $\mathfrak{R}$, and $\Upsilon_1, \Upsilon_2 \subseteq \mathfrak{R}$ such that $\Upsilon_1 \subseteq \Upsilon_2$. Then, $\mathscr{F}^{\tau}(\Upsilon_1) \subseteq \mathscr{F}^{\tau}(\Upsilon_2)$.*

Proposition 4. *Suppose $\mathscr{F}, W$ are two hyp-ideals of $\mathfrak{R}$ such that $\mathscr{F} \subseteq W$, and let Υ_1 be the nonempty subset of $\mathfrak{R}$. Then,*

(i) $\underline{W_{Lower}}(\Upsilon_1) \subseteq \underline{\mathscr{F}_{Lower}}(\Upsilon_1)$
(ii) $\overline{\mathscr{F}_{Upper}}(\Upsilon_1) \subseteq \overline{W_{Upper}}(\Upsilon_1)$
(iii) $\mathscr{F}^{B}(\Upsilon_1) \subseteq W^{B}(\Upsilon_1)$

Theorem 6. *Let $\mathfrak{R}$ be a hyp-ring and $\mathscr{F}, W$ be the hyp-ideals of $\mathfrak{R}$ such that $\mathscr{F} \subseteq W$ and let Υ_1 be the non-empty subset of $\mathfrak{R}$. Then $\mathscr{F}^{\tau}(\Upsilon_1) \neq W^{\tau}(\Upsilon_1)$.*

The following theorem can also be seen in [17].

Theorem 7. *Let $\mathscr{F}$ and Υ_2 be two hyp-ideals of $\mathfrak{R}$. Then,*

(i) $\underline{\mathscr{F}_{Lower}}(\Upsilon_2)$ *is, if it is nonempty, a hyp-ideal of* $\mathfrak{R}$
(ii) $\overline{\mathscr{F}_{Upper}}(\Upsilon_2)$ *is a hyp-ideal of* $\mathfrak{R}$

Proof

(i) Suppose $x, y\varepsilon\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon_2)$ and $r\varepsilon\mathfrak{R}$; then,

$$x \oplus \mathscr{F} \subseteq \Upsilon_2\ \text{and}\ y \oplus \mathscr{F} \subseteq \Upsilon_2.\tag{17}$$

This implies that $(x \oplus y \oplus \mathscr{F}) \subseteq \Upsilon_2$ and $-y \oplus \mathscr{F} \subseteq \Upsilon_2$. Also, $(r\hat{\circ}x \oplus \mathscr{F}) \subseteq \Upsilon_2$ and $(x\hat{\circ}r \oplus \mathscr{F}) \subseteq \Upsilon_2$. This implies that

$$x \oplus y \subseteq \underline{\mathscr{F}_{\text{Lower}}}(\Upsilon_2)\ \text{and}\ -y\varepsilon\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon_2).\tag{18}$$

Table 2: Tabular form of the hyperoperation "$\oplus$" defined in Example 2.

$\oplus$	$a_{\mathfrak{R}}$	$b_{\mathfrak{R}}$	$c_{\mathfrak{R}}$	$d_{\mathfrak{R}}$	$e_{\mathfrak{R}}$	$f_{\mathfrak{R}}$
$a_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$b_{\mathfrak{R}}$	$c_{\mathfrak{R}}$	$d_{\mathfrak{R}}$	$e_{\mathfrak{R}}$	$f_{\mathfrak{R}}$
$b_{\mathfrak{R}}$	$b_{\mathfrak{R}}$	$\{a_{\mathfrak{R}}, b_{\mathfrak{R}}\}$	$d_{\mathfrak{R}}$	$\{c_{\mathfrak{R}}, d_{\mathfrak{R}}\}$	$f_{\mathfrak{R}}$	$\{e_{\mathfrak{R}}, f_{\mathfrak{R}}\}$
$c_{\mathfrak{R}}$	$c_{\mathfrak{R}}$	$d_{\mathfrak{R}}$	$\{a_{\mathfrak{R}}, c_{\mathfrak{R}}, e_{\mathfrak{R}}\}$	$\{b_{\mathfrak{R}}, d_{\mathfrak{R}}, f_{\mathfrak{R}}\}$	$c_{\mathfrak{R}}$	$d_{\mathfrak{R}}$
$d_{\mathfrak{R}}$	$d_{\mathfrak{R}}$	$\{c_{\mathfrak{R}}, d_{\mathfrak{R}}\}$	$\{b_{\mathfrak{R}}, d_{\mathfrak{R}}, f_{\mathfrak{R}}\}$	$\mathfrak{R}$	$d_{\mathfrak{R}}$	$\{c_{\mathfrak{R}}, d_{\mathfrak{R}}\}$
$e_{\mathfrak{R}}$	$e_{\mathfrak{R}}$	$f_{\mathfrak{R}}$	$c_{\mathfrak{R}}$	$d_{\mathfrak{R}}$	$\{a_{\mathfrak{R}}, e_{\mathfrak{R}}\}$	$\{b_{\mathfrak{R}}, f_{\mathfrak{R}}\}$
$f_{\mathfrak{R}}$	$f_{\mathfrak{R}}$	$\{e_{\mathfrak{R}}, f_{\mathfrak{R}}\}$	$d_{\mathfrak{R}}$	$\{c_{\mathfrak{R}}, d_{\mathfrak{R}}\}$	$\{b_{\mathfrak{R}}, f_{\mathfrak{R}}\}$	$\{a_{\mathfrak{R}}, b_{\mathfrak{R}}, e_{\mathfrak{R}}, f_{\mathfrak{R}}\}$

Table 3: Tabular form of the hyperoperation "$\hat{\circ}$" defined in Example 2.

$\hat{\circ}$	$a_{\mathfrak{R}}$	$b_{\mathfrak{R}}$	$c_{\mathfrak{R}}$	$d_{\mathfrak{R}}$	$e_{\mathfrak{R}}$	$f_{\mathfrak{R}}$
$a_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$a_{\mathfrak{R}}$
$b_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$b_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$b_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$b_{\mathfrak{R}}$
$c_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$c_{\mathfrak{R}}$	$c_{\mathfrak{R}}$	$e_{\mathfrak{R}}$	$e_{\mathfrak{R}}$
$d_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$b_{\mathfrak{R}}$	$c_{\mathfrak{R}}$	$d_{\mathfrak{R}}$	$e_{\mathfrak{R}}$	$f_{\mathfrak{R}}$
$e_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$e_{\mathfrak{R}}$	$e_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$a_{\mathfrak{R}}$
$f_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$b_{\mathfrak{R}}$	$e_{\mathfrak{R}}$	$f_{\mathfrak{R}}$	$a_{\mathfrak{R}}$	$b_{\mathfrak{R}}$

Also,

$$r\hat{\circ}x \subseteq \underline{\mathscr{F}_{\text{Lower}}}(\Upsilon_2) \text{ and } x\hat{\circ}r \subseteq \underline{\mathscr{F}_{\text{Lower}}}(\Upsilon_2). \tag{19}$$

Therefore, $\underline{\mathscr{F}_{\text{Lower}}}(\Upsilon_2)$ is a hyp-ideal of $\mathfrak{R}$.

(ii) Suppose $x, y \varepsilon \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon_2)$ and $r\varepsilon\mathfrak{R}$; then,

$$(x \oplus \mathscr{F}) \cap \Upsilon_2 \neq \varnothing \text{ and } (y \oplus \mathscr{F}) \cap \Upsilon_2 \neq \varnothing. \tag{20}$$

So, there exists

$$p\varepsilon (x \oplus \mathscr{F}) \cap \Upsilon_2 \text{ and } q\varepsilon (y \oplus \mathscr{F}) \cap \Upsilon_2. \tag{21}$$

Since Υ_2 is a hyp-ideal of $\mathfrak{R}$, we have $p \oplus q \subseteq \Upsilon_2$ and $-q\varepsilon\Upsilon_2$; also,

$$p \oplus q \subseteq (x \oplus \mathscr{F}) \oplus (y \oplus \mathscr{F}) = x \oplus y \oplus \mathscr{F} \text{ and } -q\varepsilon - y \oplus \mathscr{F}. \tag{22}$$

Hence, $(x \oplus y \oplus \mathscr{F}) \cap \Upsilon_2 \neq \varnothing$ and $(-y \oplus \mathscr{F}) \cap \Upsilon_2 \neq \varnothing$, which implies that

$$x \oplus y \subseteq \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon_2) \text{ and } -y\varepsilon\overline{\mathscr{F}_{\text{Upper}}}(\Upsilon_2). \tag{23}$$

Also, we have $r \cdot p\varepsilon\Upsilon_2$ and

$$r\hat{\circ}p \subseteq r\hat{\circ}(x \oplus \mathscr{F}) = (r\hat{\circ}x) \oplus \mathscr{F}. \tag{24}$$

So, $(r\hat{\circ}x \oplus \mathscr{F}) \cap \Upsilon_2 \neq \varnothing$, which implies $r\hat{\circ}x \subseteq \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon_2)$. Similarly, we can prove that $x\hat{\circ}r \subseteq \overline{\mathscr{F}_{\text{Upper}}}(\Upsilon_2)$. Therefore, $\overline{\mathscr{F}_{\text{Upper}}}(\Upsilon_2)$ is a hyp-ideal of $\mathfrak{R}$. □

Theorem 8. *Let $\mathscr{F}$ and Υ_2 be two hyp-ideals of $\mathfrak{R}$. Then,*

(i) $\mathscr{F}^B(\Upsilon_2)$ is not a hyp-ideal of $\mathfrak{R}$ if $\underline{\mathscr{F}_{Lower}}(\Upsilon_2) \neq \varnothing$

(ii) $\mathscr{F}^B(\Upsilon_2)$ is a hyp-ideal of $\mathfrak{R}$ if $\underline{\mathscr{F}_{Lower}}(\Upsilon_2) = \varnothing$

Corollary 1. *Let $\mathscr{F}, \Upsilon$ be two hyp-ideals of $\mathfrak{R}$. Then,*

(i) $\underline{\Upsilon_{Lower}}(\mathscr{F})$ is also a hyp-ideal of $\mathfrak{R}$, where $\underline{\Upsilon_{Lower}}(\mathscr{F}) \neq \varnothing$

(ii) $\overline{\Upsilon_{Upper}}(\mathscr{F})$ is also a hyp-ideal of $\mathfrak{R}$

(iii) $\Upsilon^B(\mathscr{F})$ is a hyp-ideal of $\mathfrak{R}$, when $\underline{\Upsilon_{Lower}}(\mathscr{F}) = \varnothing$

Theorem 9. *Let $\mathfrak{R}$ and S be two hyp-rings and f be a homomorphism from $\mathfrak{R}$ to S. If Υ_1 is a nonempty subset of $\mathfrak{R}$, then*

(i) $f(\overline{\ker f_{Upper}}(\Upsilon_1)) = f(\Upsilon_1)$

(ii) $f(\underline{\ker f_{Lower}}(\Upsilon_1)) \subset f(\Upsilon_1)$

Proof

(i) Since $\Upsilon_1 \subseteq \overline{\ker f_{\text{Upper}}}(\Upsilon_1)$, it follows that $f(\Upsilon_1) \subseteq f(\overline{\ker f_{\text{Upper}}}(\Upsilon_1))$. Conversely, let $y\varepsilon f(\overline{\ker f_{\text{Upper}}}(\Upsilon_1))$. Then, there exist an element $x \epsilon \overline{\ker f_{\text{Upper}}}(\Upsilon_1)$ such that $f(x) = y$, so we have $(x \oplus \ker f) \cap \Upsilon_1 \neq \varnothing$. Then, there exists an element $a\varepsilon(x \oplus \ker f) \cap \Upsilon_1$. Then, $a = x \oplus b$ for some $b\varepsilon\ker f$, that is, $x = a - b$. Then, we have

$$\begin{aligned} y &= f(x) = f(a - b) \\ &= f(a) - f(b) \\ &= f(a)\varepsilon f(\Upsilon_1), \end{aligned} \tag{25}$$

and so $f(\overline{\ker f_{\text{Upper}}}(\Upsilon_1)) = f(\Upsilon_1)$.

(ii) The proof is easy. □

5. Conclusion and Future Work

Relations between R-sets, hyp-rings, and topological structures are considered in this paper. In place of universal set, we added sm-hyp-groups and hyp-rings. In future, this work can be extended to soft set theory [30], bipolar fuzzy sets [31], intuitionistic fuzzy sets [32], or neutrosophic sets [33].

Data Availability

No data were used to support this study.

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Retraction

Retracted: Rational Type Fuzzy-Contraction Results in Fuzzy Metric Spaces with an Application

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] S. U. Rehman, R. Chinram, and C. Boonpok, "Rational Type Fuzzy-Contraction Results in Fuzzy Metric Spaces with an Application," *Journal of Mathematics*, vol. 2021, Article ID 6644491, 13 pages, 2021.

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Research Article

Rational Type Fuzzy-Contraction Results in Fuzzy Metric Spaces with an Application

Saif Ur Rehman,[1] **Ronnason Chinram**,[2] **and Chawalit Boonpok**[3]

[1]*Department of Mathematics, Gomal University, Dera Ismail Khan 29050, Pakistan*
[2]*Algebra and Applications Research Unit, Division of Computational Science, Faculty of Science, Prince of Songkla University, Songkhla 90110, Thailand*
[3]*Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham 44150, Thailand*

Correspondence should be addressed to Chawalit Boonpok; chawalit.b@msu.ac.th

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This paper aims to introduce the new concept of rational type fuzzy-contraction mappings in fuzzy metric spaces. We prove some fixed point results under the rational type fuzzy-contraction conditions in fuzzy metric spaces with illustrative examples to support our results. This new concept will play a very important role in the theory of fuzzy fixed point results and can be generalized for different contractive type mappings in the context of fuzzy metric spaces. Moreover, we present an application of a nonlinear integral type equation to get the existing result for a unique solution to support our work.

1. Introduction

The theory of fixed point is one of the most interesting areas of research in mathematics. In the last decades, a lot of work was dedicated to the theory of fixed point. A point μ belonging to a nonempty set U is called a fixed point of a mapping $\ell: U \longrightarrow U$ if and only if $\ell\mu = \mu$. In 1922, Stefan Banach, a well-known mathematician, proved a Banach contraction principle in [1], which is stated as "A self-mapping in a complete metric space satisfying the contraction condition has a unique fixed point." After the publication of this principle, many researchers contributed their ideas to the theory of fixed point and proved different contractive type mapping results for single and multivalued mappings in the context of metric spaces for fixed point, coincidence point, and common fixed point. Some of these results can be found in [2–13].

In 1965, the theory of fuzzy set was introduced by Zadeh [14]. Recently, this theory is used, investigated, and applied in many directions. One direction is the evaluation of test results which is the application of fuzzy logic in the processing of students evaluation; moreover, the application is expected to represent the mechanisms of human thought processes capable of resolving the problem of evaluation of students, which can be directly monitored by the teacher (for example, see [15–19]). Many researchers have extensively developed the theory of fuzzy sets and their applications in different fields. Some of their results can be found in [20–29] the references therein.

The other direction is the generalization of metric spaces to fuzzy metric spaces. In [30], Kramosil and Michalek introduced the concept of fuzzy metric spaces (FM-space) and some more notions. Later on, the stronger form of the metric fuzziness was given by George and Veeramani [31]. In 2002, Gregory and Sapena [32] proved some contractive type fixed point theorems in FM-spaces. Some more fixed point results in the said space can be found in [33–41].

This research work aims to present the new concept of rational type fuzzy-contraction mappings in G-complete FM-spaces. We use the concept of Gregory and Sapena [32] and the "triangular property of fuzzy metric" presented by Bari and Vetro [33] and prove some unique fixed point theorems under the rational type fuzzy-contraction conditions in G-complete FM-spaces with some illustrative examples. This new theory will play a very important role in the theory of fuzzy fixed point results and can be generalized for different contractive type mappings in the context of fuzzy metric spaces. Moreover, we present an integral type application in the sense of Jabeen et al. [42] to prove a result for a unique solution to support our work. The application section of the paper is more important; one can use this concept and present different types of nonlinear integral type equations for the existence of unique solutions for their results. Some integral type application results in the theory of fixed point can be found in [43–46].

2. Preliminaries

Definition 1 (see [47]). An operation $*: [0,1]^2 \longrightarrow [0,1]$ is called a continuous t-norm, if

(i) $*$ is commutative, associative, and continuous.

(ii) $1 * \xi_1 = \xi_1$ and $\xi_1 * \xi_2 \leq \xi_3 * \xi_4$, whenever $\xi_1 \leq \xi_3$ and $\xi_2 \leq \xi_4$, $\forall \xi_1, \xi_2, \xi_3, \xi_4 \in [0,1]$.

The basic t-norms, the minimum, the product, and the Lukasiewicz continuous t-norms are defined as follows (see [47]):

$$\begin{aligned} &\xi_1 * \xi_2 = \min\{\xi_1, \xi_2\}, \quad \xi_1 * \xi_2 = \xi_1\xi_2, \\ &\xi_1 * \xi_2 = \max\{\xi_1 + \xi_2 - 1, 0\}. \end{aligned} \tag{1}$$

Definition 2 (see [31]). A 3-tuple $(U, M_r, *)$ is said to be a FM-space if U is an arbitrary set, $*$ is a continuous t-norm, and M_r is a fuzzy set on $U^2 \times (0,\infty)$ satisfying the following conditions:

(i) $M_r(\mu_1, \mu^*, t) > 0$ and $M_r(\mu_1, \mu^*, t) = 1 \Longleftrightarrow \mu_1 = \mu^*$

(ii) $M_r(\mu_1, \mu^*, t) = M_r(\mu^*, \mu_1, t)$

(iii) $M_r(\mu_1, \mu, t) * M_r(\mu, \mu^*, s) \leq M_r(\mu_1, \mu^*, t+s)$

(iv) $M_r(\mu_1, \mu^*, .)\colon (0,\infty) \longrightarrow [0,1]$ is continuous,

$\forall \mu, \mu_1, \mu^* \in U$ and $t, s \in (0,\infty)$.

Lemma 1 (see [31]). *$M_r(\mu_1, \mu^*, *)$ is nondecreasing $\forall \mu_1, \mu^* \in U$.*

Definition 3 (see [31]). Let $(U, M_r, *)$ be a FM-space, $\nu_1 \in U$, and a sequence (μ_j) in U is

(i) Converges to ν_1 if $\xi \in (0,1)$ and $t > 0 \exists j_1 \in \mathbb{N}$, such that $M_r(\mu_j, \nu_1, t) > 1 - \xi$, $\forall j \geq j_1$. We may write this $\lim_{j\longrightarrow\infty}\mu_j = \nu_1$ or $\mu_j \longrightarrow \mu_1$ as $j \longrightarrow \infty$.

(ii) Cauchy sequence if $\xi \in (0,1)$ and $t > 0 \exists j_1 \in \mathbb{N}$ such that $M_r(\mu_j, \mu_k, t) > 1 - \xi$, $\forall j, k \geq j_1$.

(iii) $(U, M_r, *)$ is complete if every Cauchy sequence is convergent in U.

(iv) [32] fuzzy-contractive if $\exists a \in (0,1)$ such that

$$\frac{1}{M_r(\mu_j, \mu_{j+1}, t)} - 1 \leq a\left(\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1\right), \quad \text{for } t > 0,\ j \geq 1. \tag{2}$$

In the sense of Gregori and Sapena [32], a sequence (μ_j) in a FM-space is said to be G-Cauchy if $\lim_j M_r(\mu_j, \mu_{j+p}, t) = 1$, for $t > 0$ and $p > 0$. A FM-space $(U, M_r, *)$ is called G-complete if every G-Cauchy sequence is convergent.

Throughout this paper, $\mathbb{N}$ represents the set of natural numbers.

Lemma 2 (see [31]). *Let $(U, M_r, *)$ be a FM-space and let a sequence (μ_j) in U converge to a point $\nu_1 \in U$ iff $M_r(\mu_j, \nu_1, t) \longrightarrow 1$, as $j \longrightarrow \infty$, for $t > 0$.*

Definition 4 (see [33]). Let $(U, M_r, *)$ be a FM-space. The fuzzy metric M_r is triangular, if

$$\begin{aligned} \frac{1}{M_r(\mu_1, \mu^*, t)} - 1 \leq &\left(\frac{1}{M_r(\mu_1, \mu, t)} - 1\right) \\ &+ \left(\frac{1}{M_r(\mu, \mu^*, t)} - 1\right), \quad \forall \mu, \mu_1, \mu^* \in U, t > 0. \end{aligned} \tag{3}$$

Definition 5 (see [32]). Let $(U, M_r, *)$ be a FM-space and $\ell\colon U \longrightarrow U$. Then, ℓ is said to be fuzzy-contractive if $\exists a \in (0,1)$ such that

$$\frac{1}{M_r(\ell\mu_1, \ell\mu^*, t)} - 1 \leq a\left(\frac{1}{M_r(\mu_1, \mu^*, t)} - 1\right), \quad \forall \mu_1, \mu^*, \in U, t > 0. \tag{4}$$

In the following, we present some rational type fixed point results under the rational type fuzzy-contraction conditions in G-complete FM-spaces by using the "triangular property of fuzzy metric." We present illustrative examples to support our results. In the last section of this paper, we present an integral type application for a unique solution to support our work.

3. Main Result

In this section, we define rational type fuzzy-contraction maps and prove some unique fixed point theorems under the rational type fuzzy-contraction mappings in G-complete FM-spaces.

Definition 6. Let $(U, M_r, *)$ be a FM-space; a mapping $\ell: U \longrightarrow U$ is called a rational type fuzzy-contraction if $\exists a, b \in [0,1)$ such that

$$\frac{1}{M_r(\ell\mu_1, \ell\mu^*, t)} - 1 \leq a\left(\frac{1}{M_r(\mu_1, \mu^*, t)} - 1\right) + b\left(\frac{M_r(\mu_1, \mu^*, t)}{M_r(\mu_1, \ell\mu_1, t) * M_r(\mu^*, \ell\mu_1, 2t)} - 1\right), \tag{5}$$

$\forall \mu_1, \mu^*, \in U,\ t > 0$.

Theorem 1. *Let $(U, M_r, *)$ be a G-complete FM-space in which M_r is triangular and a mapping $\ell: U \longrightarrow U$ is a rational type fuzzy-contraction satisfying (5) with $a + b < 1$. Then, ℓ has a unique fixed point in U.*

Proof. Fix $\mu_0 \in U$ and $\mu_{j+1} = \ell\mu_j$, $j \geq 0$. Then, by (5), for $t > 0$, $j \geq 1$,

$$\begin{aligned}
\frac{1}{M_r(\mu_j, \mu_{j+1}, t)} - 1 &= \frac{1}{M_r(\ell\mu_{j-1}, \ell\mu_j, t)} - 1 \\
&\leq a\left(\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1\right) \\
&\quad + b\left(\frac{M_r(\mu_{j-1}, \mu_j, t)}{M_r(\mu_{j-1}, \ell\mu_{j-1}, t) * M_r(\mu_j, \ell\mu_{j-1}, 2t)} - 1\right) \\
&= a\left(\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1\right) \\
&\quad + b\left(\frac{M_r(\mu_{j-1}, \mu_j, t)}{M_r(\mu_{j-1}, \mu_j, t) * M_r(\mu_j, \mu_j, 2t)} - 1\right),
\end{aligned} \tag{6}$$

and after simplification,

$$\frac{1}{M_r(\mu_j, \mu_{j+1}, t)} - 1 \leq a\left(\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1\right), \quad \text{for } t > 0. \tag{7}$$

Similarly,

$$\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1 \leq a\left(\frac{1}{M_r(\mu_{j-2}, \mu_{j-1}, t)} - 1\right), \quad \text{for } t > 0. \tag{8}$$

Now, from (7) and (8) and by induction, for $t > 0$, we have that

$$\begin{aligned}
\frac{1}{M_r(\mu_j, \mu_{j+1}, t)} - 1 &\leq a\left(\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1\right) \\
&\leq a^2\left(\frac{1}{M_r(\mu_{j-2}, \mu_{j-1}, t)} - 1\right) \\
&\leq \cdots \leq a^j\left(\frac{1}{M_r(\mu_0, \mu_1, t)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{9}$$

Hence, (μ_j) is a fuzzy-contractive sequence in $(U, M_r, *)$; therefore,

$$\lim_{j \longrightarrow \infty} M_r(\mu_j, \mu_{j-1}, t) = 1, \quad \text{for } t > 0. \tag{10}$$

Now, we show that (μ_j) is a G-Cauchy sequence; let $j \in \mathbb{N}$, and there is a fixed $q \in \mathbb{N}$ such that

$$\begin{aligned}
M_r(\mu_j, \mu_{j+q}, t) &= M_r\left(\mu_j, \mu_{j+q}, \Bigl(\underbrace{\frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}}_{q\text{-times}}\Bigr)t\right) \\
&\geq M_r\left(\mu_j, \mu_{j+1}, \frac{t}{q}\right) * M_r\left(\mu_{j+1}, \mu_{j+2}, \frac{t}{q}\right) * \cdots * M_r\left(\mu_{j+q-1}, \mu_{j+q}, \frac{t}{q}\right) \\
&\longrightarrow \underbrace{1 * 1 * \cdots * 1}_{q\text{-times}} = 1, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{11}$$

Hence, it is proved that (μ_j) is a G-Cauchy sequence. Since $(U, M_r, *)$ is G-complete, $\exists \nu_1 \in U$ such that $\mu_j \longrightarrow \nu_1$, as $j \longrightarrow \infty$, i.e.,

$$\lim_{j \longrightarrow \infty} M_r(\mu_j, \nu_1, t) = 1, \quad \text{for } t > 0. \tag{12}$$

Since M_r is triangular, from (5), (10), and (12), for $t > 0$, we have

$$
\begin{aligned}
\frac{1}{M_r(v_1, \ell v_1, t)} - 1 &\leq \left(\frac{1}{M_r(v_1, \mu_{j+1}, t)} - 1\right) + \left(\frac{1}{M_r(\ell\mu_j, \ell v_1, t)} - 1\right) \\
&\leq \left(\frac{1}{M_r(v_1, \mu_{j+1}, t)} - 1\right) + a\left(\frac{1}{M_r(\mu_j, v_1, t)} - 1\right) + b\left(\frac{M_r(\mu_j, v_1, t)}{M_r(\mu_j, \ell\mu_j, t) * M_r(v_1, \ell\mu_j, 2t)} - 1\right) \\
&= \left(\frac{1}{M(v_1, \mu_{j+1}, t)} - 1\right) + a\left(\frac{1}{M_r(\mu_j, v_1, t)} - 1\right) + b\left(\frac{M_r(\mu_j, v_1, t)}{M_r(\mu_j, \mu_{j+1}, t) * M_r(v_1, \mu_{j+1}, 2t)} - 1\right) \\
&\longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{13}
$$

Hence, $M_r(v_1, \ell v_1, t) = 1 \Rightarrow \ell v_1 = u_1$, for $t > 0$.

Uniqueness. Let $\exists z_1 \in U$ such that $\ell z_1 = z_1$ and $\ell v_1 = v_1$; then, from (5) and by using Definition 2 (iii), for $t > 0$, we have

$$
\begin{aligned}
\frac{1}{M_r(v_1, z_1, t)} - 1 &= \frac{1}{M_r(\ell v_1, \ell z_1, t)} - 1 \\
&\leq a\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) + b\left(\frac{M_r(v_1, z_1, t)}{M_r(v_1, \ell v_1, t) * M_r(z_1, \ell v_1, 2t)} - 1\right) \\
&\leq a\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) + b\left(\frac{M_r(v_1, z_1, t)}{M_r(z_1, v_1, t) * M_r(v_1, v_1, t)} - 1\right) \\
&= a\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) = a\left(\frac{1}{M_r(\ell v_1, \ell z_1, t)} - 1\right) \\
&\leq a^2\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) \leq \cdots \leq a^j\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) \longrightarrow 0, \quad \text{as } j \longrightarrow \infty.
\end{aligned} \tag{14}
$$

Hence, it is proved that $M_r(v_1, z_1, t) = 1$, and this implies that $v_1 = z_1$. □

Corollary 1 (fuzzy Banach contraction principle). *Let* $(U, M_r, *)$ *be a G-complete FM-space in which* M_r *is triangular and a mapping* $\ell: U \longrightarrow U$ *is a fuzzy-contraction satisfying (4) with* $a \in (0, 1)$*. Then,* ℓ *has a unique fixed point in* U*.*

Example 1. Let $U = [0, \infty)$, $*$ be a continuous t-norm, and $M_r: U^2 \times (0, \infty) \longrightarrow [0, 1]$ be defined as

$$M_r(\mu_1,\mu^*,t) = \frac{t}{t+\left|(4\mu_1 - 4\mu^*)/5\right|}, \quad \forall \mu_1,\mu^* \in U, t>0. \tag{15}$$

Then, one can easily verify that M_r is triangular and $(U, M_r, *)$ is a G-complete FM-space. Now we define a mapping $\ell: U \longrightarrow U$ as

$$\ell(\mu_1) = \begin{cases} \dfrac{3\mu_1}{4}, & \text{if } \mu_1 \in [0,1], \\ \dfrac{2\mu_1}{3} + 8, & \text{if } \mu_1 \in (1,\infty). \end{cases} \tag{16}$$

Then, we have

$$\frac{1}{M_r(\ell\mu_1,\ell\mu^*,t)} - 1 = \frac{3}{4}\left(\frac{1}{M_r(\mu_1,\mu^*,t)} - 1\right), \quad \forall \mu_1,\mu^* \in U, t>0. \tag{17}$$

Hence, a mapping ℓ is a fuzzy contraction. Now, from Definition 2 (iii), for $t>0$,

$$\begin{aligned} \frac{M_r(\mu_1,\mu^*,t)}{M_r(\mu_1,\ell\mu_1,t) * M_r(\mu^*,\ell\mu_1,2t)} - 1 &\le \frac{M_r(\mu_1,\mu^*,t)}{M_r(\mu_1,\ell\mu_1,t) * M_r(\mu^*,\mu_1,t) * M_r(\mu_1,\ell\mu_1,t)} - 1 \\ &= \frac{1}{M_r(\mu_1,\ell\mu_1,t) * M_r(\mu_1,\ell\mu_1,t)} - 1 \\ &= \left(\frac{1}{(M_r(\mu_1,\ell\mu_1,t))^2} - 1\right) = \frac{2\mu_1}{5t^2}\left(\frac{\mu_1}{5} + t\right). \end{aligned} \tag{18}$$

Hence, all the conditions of Theorem 1 are satisfied with $a = (3/4)$ and $b = (2/9)$. A mapping ℓ has a fixed point, i.e., $\ell(24) = 24 \in [0,\infty)$.

Next, we present a generalized rational type fuzzy-contraction theorem.

Theorem 2. *Let $(U, M_r, *)$ be a G-complete FM-space in which M_r is triangular and a mapping $\ell: U \longrightarrow U$ satisfies*

$$\begin{aligned} \frac{1}{M_r(\ell\mu_1,\ell\mu^*,t)} - 1 \le\ & a\left(\frac{1}{M_r(\mu_1,\mu^*,t)} - 1\right) + b\left(\frac{M_r(\mu_1,\mu^*,t) * M_r(\mu^*,\ell\mu^*,t)}{M_r(\mu_1,\ell\mu_1,t) * M_r(\mu_1,\ell\mu^*,2t)} - 1\right) \\ &+ c\left(\frac{M_r(\mu_1,\ell\mu_1,t)}{M_r(\mu_1,\ell\mu^*,2t)} - 1 + \frac{M_r(\mu^*,\ell\mu^*,t)}{M_r(\mu_1,\ell\mu^*,2t)} - 1\right) \\ &+ d\left(\frac{1}{M_r(\mu_1,\ell\mu_1,t)} - 1 + \frac{1}{M_r(\mu^*,\ell\mu^*,t)} - 1\right), \end{aligned} \tag{19}$$

$\forall \mu_1, \mu^* \in U, t > 0, a, b, c, d \geq 0$ with $(a + b + 2c + 2d) < 1$. Then, ℓ has a unique fixed point.

Proof. Fix $\mu_0 \in U$ and $\mu_{j+1} = \ell \mu_j$, $j \geq 0$. Then, by (19), for $t > 0$, $j \geq 1$,

$$\begin{aligned}
\frac{1}{M_r(\mu_j, \mu_{j+1}, t)} - 1 &= \frac{1}{M_r(\ell\mu_{j-1}, \ell\mu_j, t)} - 1 \\
&\leq a\left(\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1\right) + b\left(\frac{M_r(\mu_{j-1}, \mu_j, t) * M_r(\mu_j, \ell\mu_j, t)}{M_r(\mu_{j-1}, \ell\mu_{j-1}, t) * M_r(\mu_{j-1}, \ell\mu_j, 2t)} - 1\right) \\
&+ c\left(\frac{M_r(\mu_{j-1}, \ell\mu_{j-1}, t)}{M_r(\mu_{j-1}, \ell\mu_j, 2t)} - 1 + \frac{M_r(\mu_j, \ell\mu_j, t)}{M_r(\mu_{j-1}, \ell\mu_j, 2t)} - 1\right) \\
&+ d\left(\frac{1}{M_r(\mu_{j-1}, \ell\mu_{j-1}, t)} - 1 + \frac{1}{M_r(\mu_j, \ell\mu_j, t)} - 1\right) \\
&= a\left(\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1\right) + b\left(\frac{M_r(\mu_{j-1}, \mu_j, t) * M_r(\mu_j, \mu_{j+1}, t)}{M_r(\mu_{j-1}, \mu_j, t) * M_r(\mu_{j-1}, \mu_{j+1}, 2t)} - 1\right) \\
&+ c\left(\frac{M_r(\mu_{j-1}, \mu_j, t)}{M_r(\mu_{j-1}, \mu_{j+1}, 2t)} - 1 + \frac{M_r(\mu_j, \mu_{j+1}, t)}{M_r(\mu_{j-1}, \mu_{j+1}, 2t)} - 1\right) \\
&+ d\left(\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1 + \frac{1}{M_r(\mu_j, \mu_{j+1}, t)} - 1\right).
\end{aligned} \tag{20}$$

From Definition 2 (iii), $M_r(\mu_{j-1}, \mu_{j+1}, 2t) \geq M_r(\mu_{j-1}, \mu_j, t) * M_r(\mu_j, \mu_{j+1}, t)$, for $t > 0$, and after simplification, we have

$$\frac{1}{M_r(\mu_j, \mu_{j+1}, t)} - 1 \leq \beta\left(\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1\right), \quad \text{where } \beta = \frac{a + b + c + d}{1 - c - d} < 1. \tag{21}$$

Similarly, for $t > 0$, we have

$$\frac{1}{M_r(\mu_{j-1}, \mu_j, t)} - 1 \leq \beta\left(\frac{1}{M_r(\mu_{j-2}, \mu_{j-1}, t)} - 1\right), \quad \text{where } \beta = \frac{a + b + c + d}{1 - c - d} < 1. \tag{22}$$

Now, from (21) and (22) and by induction, for $t > 0$, we have

$$\begin{aligned}\frac{1}{M_r(\mu_j,\mu_{j+1},t)}-1 &\le \beta\left(\frac{1}{M_r(\mu_{j-1},\mu_j,t)}-1\right)\le \beta^2\left(\frac{1}{M_r(\mu_{j-2},\mu_{j-1},t)}-1\right)\\ &\le \cdots \le \beta^j\left(\frac{1}{M_r(\mu_0,\mu_1,t)}-1\right)\longrightarrow 0,\quad \text{as } j\longrightarrow\infty.\end{aligned}\tag{23}$$

Hence, (μ_j) is a rational type fuzzy-contractive sequence in U such that

$$\lim_{j\longrightarrow\infty} M_r(\mu_j,\mu_{j+1},t)=1,\quad \text{for } t>0.\tag{24}$$

Now we have to show that (μ_j) is a G-Cauchy sequence; let $j\in\mathbb{N}$, and there is a fixed $q\in\mathbb{N}$ such that

$$\begin{aligned}M_r(\mu_j,\mu_{j+q},t) &= M_r\left(\mu_j,\mu_{j+q},\left(\underbrace{\frac{1}{q}+\frac{1}{q}+\cdots+\frac{1}{q}}_{q\text{-times}}\right)t\right)\\ &\ge M_r\left(\mu_j,\mu_{j+1},\frac{t}{q}\right)*M_r\left(\mu_{j+1},\mu_{j+2},\frac{t}{q}\right)\\ &\quad *\cdots *M_r\left(\mu_{j+q-1},\mu_{j+q},\frac{t}{q}\right)\\ &\quad \cdot\longrightarrow \underbrace{1*1*\cdots *1}_{q\text{-times}}=1,\quad \text{as } j\longrightarrow\infty.\end{aligned}\tag{25}$$

Hence, it is proved that (μ_j) is a G-Cauchy sequence. Since $(U,M_r,*)$ is G-complete, then $\exists v_1\in U$ such that $\mu_j\longrightarrow v_1$, as $j\longrightarrow\infty$, i.e.,

$$\lim_{j\longrightarrow\infty} M_r(\mu_j,v_1,t)=1,\quad \text{for } t>0.\tag{26}$$

Since M_r is triangular,

$$\begin{aligned}\frac{1}{M_r(v_1,\ell v_1,t)}-1 &\le \left(\frac{1}{M_r(v_1,\mu_{j+1},t)}-1\right)\\ &\quad +\left(\frac{1}{M_r(\mu_{j+1},\ell v_1,t)}-1\right),\quad \text{for } t>0.\end{aligned}\tag{27}$$

Now from (19), (24), and (26), for $t > 0$, we have

$$\begin{aligned}\frac{1}{M_r(\mu_{j+1},\ell v_1,t)}-1 &= \frac{1}{M_r(\ell\mu_j,\ell v_1,t)}-1\\ &\le a\left(\frac{1}{M_r(\mu_j,v_1,t)}-1\right)+b\left(\frac{M_r(\mu_j,v_1,t)*M_r(v_1,\ell v_1,t)}{M_r(\mu_j,\ell\mu_j,t)*M_r(\mu_j,\ell v_1,2t)}-1\right)\\ &\quad +c\left(\frac{M_r(\mu_j,\ell\mu_j,t)}{M_r(\mu_j,\ell v_1,2t)}-1+\frac{M_r(v_1,\ell v_1,t)}{M_r(\mu_j,\ell v_1,2t)}-1\right)\\ &\quad +d\left(\frac{1}{M_r(\mu_j,\ell\mu_j,t)}-1+\frac{1}{M_r(v_1,\ell v_1,t)}-1\right)\\ &= a\left(\frac{1}{M_r(\mu_j,v_1,t)}-1\right)+b\left(\frac{M_r(\mu_j,v_1,t)*M_r(v_1,\ell v_1,t)}{M_r(\mu_j,\mu_{j+1},t)*M_r(\mu_j,\ell v_1,2t)}-1\right)\\ &\quad +c\left(\frac{M_r(\mu_j,\mu_{j+1},t)}{M_r(\mu_j,\ell v_1,2t)}-1+\frac{M_r(v_1,\ell v_1,t)}{M_r(\mu_j,\ell v_1,2t)}-1\right)\\ &\quad +d\left(\frac{1}{M_r(\mu_j,\mu_{j+1},t)}-1+\frac{1}{M_r(v_1,\ell v_1,t)}-1\right).\end{aligned}\tag{28}$$

From Definition 2 (iii), $M_r(\mu_j, \ell v_1, 2t) \geq M_r(\mu_j, v_1, t) * M_r(v_1, \ell v_1, t)$, for $t > 0$, and we have

$$
\begin{aligned}
\frac{1}{M_r(\mu_{j+1}, \ell v_1, t)} - 1 &\leq a\left(\frac{1}{M_r(\mu_j, v_1, t)} - 1\right) + b\left(\frac{M_r(\mu_j, v_1, t) * M_r(v_1, \ell v_1, t)}{M_r(\mu_j, \mu_{j+1}, t) * M_r(\mu_j, v_1, t) * M_r(v_1, \ell v_1, t)} - 1\right) \\
&\quad + c\left(\frac{M_r(\mu_j, \mu_{j+1}, t)}{M_r(\mu_j, v_1, t) * M_r(v_1, \ell v_1, t)} - 1 + \frac{M_r(v_1, \ell v_1, t)}{M_r(\mu_j, v_1, t) * M_r(v_1, \ell v_1, t)} - 1\right) \\
&\quad + d\left(\frac{1}{M_r(\mu_j, \mu_{j+1}, t)} - 1 + \frac{1}{M_r(v_1, \ell v_1, t)} - 1\right) \\
&\quad \longrightarrow (c + d)\left(\frac{1}{M_r(v_1, \ell v_1, t)} - 1\right), \quad \text{as } j \longrightarrow \infty.
\end{aligned}
\tag{29}
$$

Then,

$$
\limsup_{j \longrightarrow \infty}\left(\frac{1}{M_r(\mu_{j+1}, \ell v_1, t)} - 1\right) \leq (c + d)\left(\frac{1}{M_r(v_1, \ell v_1, t)} - 1\right), \quad \text{for } t > 0.
\tag{30}
$$

Now, from (26), (27), and (30), as $j \longrightarrow \infty$, we get that

$$
\frac{1}{M_r(v_1, \ell v_1, t)} - 1 \leq (c + d)\left(\frac{1}{M_r(v_1, \ell v_1, t)} - 1\right), \quad \text{for } t > 0,
\tag{31}
$$

and $c + d < 1$ where $(a + b + 2c + 2d) < 1$, and hence $M_r(v_1, \ell v_1, t) = 1$, i.e., $\ell v_1 = v_1$, for $t > 0$.

Uniqueness. Let $\exists z_1 \in U$ such that $\ell z_1 = z_1$ and $\ell v_1 = v_1$. Then, from (19) and from Definition 2 (iii), for $t > 0$, we have

$$
\begin{aligned}
\frac{1}{M_r(v_1, z_1, t)} - 1 &= \frac{1}{M_r(\ell v_1, \ell z_1, t)} - 1 \\
&\leq a\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) + b\left(\frac{M_r(v_1, z_1, t) * M_r(z_1, \ell z_1, t)}{M_r(v_1, \ell v_1, t) * M_r(v_1, \ell z_1, 2t)} - 1\right) \\
&\quad + c\left(\frac{M_r(v_1, \ell v_1, t)}{M_r(v_1, \ell z_1, 2t)} - 1 + \frac{M_r(z_1, \ell z_1, t)}{M_r(v_1, \ell z_1, 2t)} - 1\right) \\
&\quad + d\left(\frac{1}{M_r(v_1, \ell v_1, t)} - 1 + \frac{1}{M_r(z_1, \ell z_1, t)} - 1\right) \\
&= a\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) + b\left(\frac{M_r(v_1, z_1, t)}{M_r(v_1, z_1, 2t)} - 1\right) \\
&\quad + c\left(\frac{1}{M_r(v_1, z_1, 2t)} - 1 + \frac{1}{M_r(v_1, z_1, 2t)} - 1\right) \\
&= a\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) + b\left(\frac{M_r(v_1, z_1, t)}{M_r(v_1, z_1, t) * M_r(z_1, z_1, t)} - 1\right) \\
&\quad + c\left(\frac{1}{M_r(v_1, z_1, t) * M - r(z_1, z_1, t)} - 1 + \frac{1}{M_r(v_1, z_1, t) * M_r(z_1, z_1, t)} - 1\right) \\
&= (a + 2c)\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) = (a + 2c)\left(\frac{1}{M_r(\ell v_1, \ell z_1, t)} - 1\right) \\
&\leq (a + 2c)^2\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) \leq \cdots \leq (a + 2c)^j\left(\frac{1}{M_r(v_1, z_1, t)} - 1\right) \\
&\longrightarrow 0, \quad \text{as } j \longrightarrow \infty, \text{where } (a + 2c) < 1.
\end{aligned}
\tag{32}
$$

Hence, $M_r(v_1, z_1, t) = 1$, and this implies that $v_1 = z_1$, for $t > 0$. □

Corollary 2. *Let $(U, M_r, *)$ be a G-complete FM-space in which M_r is triangular and a mapping $\ell: U \longrightarrow U$ satisfies*

$$\frac{1}{M_r(\ell\mu_1, \ell\mu^*, t)} - 1 \le a\left(\frac{1}{M_r(\mu_1, \mu^*, t)} - 1\right) + b\left(\frac{M_r(\mu_1, \mu^*, t) * M_r(\mu^*, \ell\mu^*, t)}{M_r(\mu_1, \ell\mu_1, t) * M_r(\mu_1, \ell\mu^*, 2t)} - 1\right) + d\left(\frac{1}{M_r(\mu_1, \ell\mu_1, t)} - 1 + \frac{1}{M_r(\mu^*, \ell\mu^*, t)} - 1\right), \quad (33)$$

$\forall \mu_1, \mu^* \in U$, $t > 0$, $a, b, d \ge 0$ with $a + b + 2d < 1$. Then, ℓ has a unique fixed point.

Corollary 3. *Let $(U, M_r, *)$ be a G-complete FM-space in which M_r is triangular and a mapping $\ell: U \longrightarrow U$ satisfies*

$$\frac{1}{M_r(\ell\mu_1, \ell\mu^*, t)} - 1 \le a\left(\frac{1}{M_r(\mu_1, \mu^*, t)} - 1\right) + c\left(\frac{M_r(\mu_1, \ell\mu_1, t)}{M_r(\mu_1, \ell\mu^*, 2t)} - 1 + \frac{M_r(\mu^*, \ell\mu^*, t)}{M_r(\mu_1, \ell\mu^*, 2t)} - 1\right) + d\left(\frac{1}{M_r(\mu_1, \ell\mu_1, t)} - 1 + \frac{1}{M_r(\mu^*, \ell\mu^*, t)} - 1\right), \quad (34)$$

$\forall \mu_1, \mu^* \in U$, $t > 0$, $a, c, d \ge 0$ with $a + 2c + 2d < 1$. Then, ℓ has a unique fixed point.

Corollary 4. *Let $(U, M_r, *)$ be a G-complete FM-space in which M_r is triangular and a mapping $\ell: U \longrightarrow U$ satisfies*

$$\frac{1}{M_r(\ell\mu_1, \ell\mu^*, t)} - 1 \le a\left(\frac{1}{M_r(\mu_1, \mu^*, t)} - 1\right) + d\left(\frac{1}{M_r(\mu_1, \ell\mu_1, t)} - 1 + \frac{1}{M_r(\mu^*, \ell\mu^*, t)} - 1\right), \quad (35)$$

$\forall \mu_1, \mu^* \in U$, $t > 0$, $a, d \ge 0$ with $a + 2d < 1$. Then, ℓ has a unique fixed point.

Example 2. From Example 1, we define M_r as

$$M_r(\mu_1, \mu^*, t) = \frac{t}{t + |(\mu_1 - \mu^*)/2|}, \quad \forall \mu_1, \mu^* \in U, t > 0. \quad (36)$$

Then, one can easily show that M_r is triangular and $(U, M_r, *)$ is G-complete FM-space. Now we define a mapping $\ell: U \longrightarrow U$ as

$$\ell(\mu_1) = \begin{cases} \frac{3\mu_1}{7}, & \text{if } \mu_1 \in [0, 1], \\ \frac{3\mu_1}{4} + 1, & \text{if } \mu_1 \in (1, \infty). \end{cases} \quad (37)$$

Then, we have

$$\frac{1}{M_r(\ell\mu_1, \ell\mu^*, t)} - 1 = \frac{3}{7}\left(\frac{1}{M_r(\mu_1, \mu^*, t)} - 1\right), \quad \forall \mu_1, \mu^* \in U, t > 0. \quad (38)$$

A mapping ℓ satisfies (4), and hence ℓ is a fuzzy contraction. Now, from Definition 2 (iii), $M_r(\mu_1, \ell\mu^*, 2t) \ge M_r(\mu_1, \mu^*, t) * M_r(\mu^*, \ell\mu^*, t)$ for $t > 0$, and after simplification, we get the following:

$$\begin{aligned} &\frac{M_r(\mu_1, \mu^*, t) * M_r(\mu^*, \ell\mu^*, t)}{M_r(\mu_1, \ell\mu_1, t) * M_r(\mu_1, \ell\mu^*, 2t)} - 1 \le \left(\frac{1}{M_r(\mu_1, \ell\mu_1, t)} - 1\right) = \frac{2\mu_1}{7t}, \\ &\left(\frac{M_r(\mu_1, \ell\mu_1, t)}{M_r(\mu_1, \ell\mu^*, 2t)} - 1 + \frac{M_r(\mu^*, \ell\mu^*, t)}{M_r(\mu_1, \ell\mu^*, 2t)} - 1\right) \le \frac{10}{7}\left(\frac{1}{M_r(\mu_1, \mu^*, t)} - 1\right) = \frac{5|\mu_1 - \mu^*|}{7t}, \\ &\left(\frac{1}{M_r(\mu_1, \ell\mu_1, t)} - 1 + \frac{1}{M_r(\mu^*, \ell\mu^*, t)} - 1\right) = \frac{2|\mu_1 + \mu^*|}{7t}. \end{aligned} \quad (39)$$

Hence, all the conditions of Theorem 2 are satisfied with $a = (3/7)$, $b = c = (1/9)$, and $d = (1/12)$, and ℓ has a unique fixed point, i.e., $\ell(4) = 4 \in [0, \infty)$.

4. Application

In this section, we present an integral type application to support our work. Let $U = C([0, \eta], \mathbb{R})$ be the space of all $\mathbb{R}$-valued continuous functions on the interval $[0, \eta]$, where $0 < \eta \in \mathbb{R}$. The nonlinear integral equation is

$$\mu_1(\tau) = \int_0^\tau \Gamma(\tau, \nu, \mu_1(\nu))\mathrm{d}\nu, \quad \forall \mu_1 \in U, \tag{40}$$

where $\tau, \nu \in [0, \eta]$ and $\Gamma\colon [0, \eta] \times [0, \eta] \times \mathbb{R} \longrightarrow \mathbb{R}$. The induced metric $m\colon U^2 \longrightarrow \mathbb{R}$ can be defined as

$$\begin{gathered} m(\mu_1, \mu^*) = \sup_{\tau \in [0, \eta]} \left|\mu_1(\tau) - \mu^*(\tau)\right| = \left\|\mu_1 - \mu^*\right\|, \\ \text{where } \mu_1, \mu^* \in C([0, \eta], \mathbb{R}) = U. \end{gathered} \tag{41}$$

The binary operation $*$ is defined by $\alpha * \lambda = \alpha\lambda$, $\forall \alpha, \lambda \in [0, \eta]$. A standard fuzzy metric $M_r\colon U^2 \times (0, \infty) \longrightarrow [0, 1]$ can be defined as

$$M_r(\mu_1, \mu^*, t) = \frac{t}{t + m(\mu_1, \mu^*)}, \quad \text{for } t > 0, \forall \mu_1, \mu^* \in U. \tag{42}$$

Then, one can easily verify that M_r is triangular and $(U, M_r, *)$ is a G-complete FM-space.

Theorem 3. *Let the integral equation be defined in (40), and there exists $\beta \in (0, 1)$, satisfying*

$$m(\ell\mu_1, \ell\mu^*) \leq \beta N(\ell, \mu_1, \mu^*)) \quad \forall \mu_1, \mu^* \in U, \tag{43}$$

where

$$N(\ell, \mu_1, \mu^*) = \max\left\{\left\|\mu_1 - \mu^*\right\|, 2\left\|\mu_1 - \ell\mu_1\right\|\right\}, \quad \forall \mu_1, \mu^* \in U. \tag{44}$$

Then, the integral equation in (40) has a unique solution in U.

Proof. Define the integral operator $\ell\colon U \longrightarrow U$ by

$$\ell\mu_1(\tau) = \int_0^\tau \Gamma(\tau, \nu, \mu_1(\nu))\mathrm{d}\nu, \quad \forall \mu_1 \in U. \tag{45}$$

Notice that ℓ is well defined and (40) has a unique solution if and only if ℓ has a unique fixed point in U. Now we have to show that Theorem 1 applies to the integral operator ℓ. Then, $\forall \mu_1, \mu^* \in U$, we have the following two cases:

(a) If $N(\ell, \mu_1, \mu^*) = \|\mu_1 - \mu^*\|$ in (44), then, from (42) and (43), we have

$$\begin{aligned} \frac{1}{M_r(\ell\mu_1, \ell\mu^*, t)} - 1 &= \frac{m(\ell\mu_1, \ell\mu^*)}{t} \\ &\leq \beta\frac{N(\ell, \mu_1, \mu^*)}{t} \\ &= \beta\frac{\left\|\mu_1 - \mu^*\right\|}{t} \\ &= \beta\left(\frac{1}{M_r(\mu_1, \mu^*, t)} - 1\right), \end{aligned} \tag{46}$$

and this implies that

$$\frac{1}{M_r(\ell\mu_1, \ell\mu^*, t)} - 1 \leq \beta\left(\frac{1}{M_r(\mu_1, \mu^*, t)} - 1\right), \quad \text{for } t > 0, \tag{47}$$

$\forall \mu_1, \mu^* \in U$ such that $\ell\mu_1 \neq \ell\mu^*$. Inequality (47) holds if $\ell\mu_1 = \ell\mu^*$. Thus, the integral operator ℓ satisfies all the conditions of Theorem 1 with $\beta = a$ and $b = 0$ in (5). The integral operator ℓ has a unique fixed point, i.e., (40) has a solution in U.

(b) If $N(\ell, \mu_1, \mu^*) = \|\mu_1 - \ell\mu_1\|$ in (44), then, from (42) and (43), we have

$$\begin{aligned} \frac{1}{M_r(\ell\mu_1, \ell\mu^*, t)} - 1 &= \frac{m(\ell\mu_1, \ell\mu^*)}{t} \\ &\leq \beta\frac{N(\ell, \mu_1, \mu^*)}{t} \\ &= \beta\frac{\left\|\mu_1 - \ell\mu_1\right\|}{t} \\ &\leq 2\beta\frac{\left\|\mu_1 - \ell\mu_1\right\|}{t}, \end{aligned} \tag{48}$$

and this implies that

$$\frac{1}{M_r(\ell\mu_1, \ell\mu^*, t)} - 1 \leq 2\beta\frac{\left\|\mu_1 - \ell\mu_1\right\|}{t}, \quad \text{for } t > 0. \tag{49}$$

Here, we simplify the term $(M_r(\mu_1, \mu^*, t)/M_r(\mu_1, \ell\mu_1, t) * M_r(\mu^*, \ell\mu_1, 2t)) - 1$, and by using Definition 2 (iii) and (42), for $t > 0$, we have

$$\begin{aligned}\frac{M_r(\mu_1,\mu^*,t)}{M_r(\mu_1,\ell\mu_1,t)*M_r(\mu^*,\ell\mu_1,2t)}-1 &\leq \frac{M_r(\mu_1,\mu^*,t)}{M_r(\mu_1,\ell\mu_1,t)*M_r(\mu^*,\mu_1,t)*M_r(\mu_1,\ell\mu_1,t)}-1\\ &= \frac{1}{(M_r(\mu_1,\ell\mu_1,t))^2}-1=\frac{(t+m(\mu_1,\ell\mu_1))^2-t^2}{t^2}\\ &= \frac{2m(\mu_1,\ell\mu_1)}{t}+\left(\frac{m(\mu_1,\ell\mu_1)}{t}\right)^2\\ &= \frac{2\|\mu_1-\ell\mu_1\|}{t}+\left(\frac{\|\mu_1-\ell\mu_1\|}{t}\right)^2,\end{aligned} \tag{50}$$

and this implies that

$$\frac{M_r(\mu_1,\mu^*,t)}{M_r(\mu_1,\ell\mu_1,t)*M_r(\mu^*,\ell\mu_1,2t)}-1\leq \frac{2\|\mu_1-\ell\mu_1\|}{t}+\left(\frac{\|\mu_1-\ell\mu_1\|}{t}\right)^2, \quad \text{for } t>0. \tag{51}$$

Now from (49) and (51), we have

$$\frac{1}{M_r(\ell\mu_1,\ell\mu^*,t)}-1\leq \beta\left(\frac{M_r(\mu_1,\mu^*,t)}{M_r(\mu_1,\ell\mu_1,t)*M_r(\mu^*,\ell\mu_1,2t)}-1\right), \quad \text{for } t>0, \tag{52}$$

$\forall \mu_1,\mu^*\in U$ such that $\ell\mu_1\neq \ell\mu^*$. Inequality (52) holds if $\ell\mu_1=\ell\mu^*$. Thus, the integral operator ℓ satisfies all the conditions of Theorem 1 with $\beta=b$ and $a=0$ in (5). The integral operator ℓ has a unique fixed point, i.e., (40) has a solution in U. □

5. Conclusion

In this paper, we have presented the concept of rational type fuzzy-contraction maps in FM-spaces and proved some rational type fixed point theorems in G-complete FM-spaces under the rational type fuzzy-contraction conditions by using the "triangular property of fuzzy metric." In the last section, we presented an integral type application for rational type fuzzy-contraction maps and proved a result of a unique solution for an integral operator in FM-space. In this direction, one can prove more rational type fuzzy-contraction results in G-complete FM-spaces with different types of applications.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study.

Acknowledgments

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Retraction

Retracted: Covering Fuzzy Rough Sets via Variable Precision

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Covering Fuzzy Rough Sets via Variable Precision

Mohammed Atef [1] and A. A. Azzam [2,3]

[1]Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Menoufia, Egypt
[2]Department of Mathematics, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia
[3]Department of Mathematics, Faculty of Science, New Valley University, Elkharga 72511, Egypt

Correspondence should be addressed to A. A. Azzam; azzam0911@yahoo.com

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Lately, covering fuzzy rough sets via variable precision according to a fuzzy γ-neighborhood were established by Zhan et al. model. Also, Ma et al. gave the definition of complementary fuzzy γ-neighborhood with reflexivity. In a related context, we used the concepts by Ma et al. to construct three new kinds of covering-based variable precision fuzzy rough sets. Furthermore, we establish the relevant characteristics. Also, we study the relationships between Zhan's model and our three models. Finally, we introduce a MADM approach to make a decision on a real problem.

1. Introduction

Pawlak [1, 2] presented the classical definition of rough sets as a valuable mathematical method to deal with the vagueness and granularity of information systems and data processing. His theory and its generalizations since then have produced applications in different areas [3–13].

One of the most elaborated generalizations of rough sets is potentially covering-based rough sets (CRS). There are several scholars working on CRS with various views in previous years, see, for more information, [14–22]. After that, the definition of a fuzzy β-neighborhood was seen by Ma [23] and the fuzzy complementary β-neighborhood by Yang and Hu [24]. Also, Yang and Hu [25, 26] introduced the concepts of fuzzy β-minimal description and fuzzy β-maximal description. They used these definitions to construct a fuzzy β covering approximation space (FβCAS). D'eer et al. [27] studied fuzzy neighborhoods based on fuzzy coverings.

The definition of rough fuzzy sets and fuzzy rough sets was found by Dubois and Prade [28]. Different research studies on covering-based rough set and fuzzy rough set have recently been investigated [29–33].

Variable precision rough sets' (VPRSs) notion was obtained by Ziarko [34] and variable precision fuzzy rough sets (VPFRSs) were built by Zhao et al. [35]. In addition, the PROMETHEE II approach based on variable precision fuzzy rough sets was also proposed by Jiang et al. [36]. Different kinds of variable precision were further applied in various areas [37–40].

One of the standard decision-making processes is TOPSIS (technique for order preference by similarity to an ideal solution). Yoon and Hwang [41] indicated that TOPSIS will solve the problem of multiattribute decision-making (MADM), where the aim is to obtain an object with the highest effect value (PIS) and the lowest effect value (NIS). There are several papers concerning TOPSIS published in different fields [42–51].

Zhan et al. [52] put the definition of fuzzy γ-neighborhoods and also studied the covering-based variable precision fuzzy rough sets (CVPFRSs). Furthermore, Ma et al. [53] defined the complementary fuzzy γ-neighborhoods and presented another two types of neighborhoods by merging the fuzzy γ-neighborhoods and the complementary fuzzy γ-neighborhoods. Based on these kinds of fuzzy γ-neighborhoods, this paper proposes to introduce three new kinds of CVPFRSs models as a generalization of the Zhan et al. [52] method. Thus, we discuss some of their properties. The relationships between these methods are also established. Then, we present and explain the methodology to solve MADM problems. The paper structure is as follows. Section 2 gives the basic notions. Section 3 establishes three novel types of CVPFRSs. A decision-making process to explain the theoretical study is advanced in Section 4. We deduced in Section 5.

2. Preliminaries

We extend a short scanning of some concepts utilized over the paper in this section. In this article, we work on $\mathscr{R}$-implication operator, in particular, $\mathscr{I} = \mathscr{I}_{\mathscr{L}}$, i.e., $\mathscr{I}_{\mathscr{L}}(a,b) = 1 \wedge (1 - a + b)$, and $\mathscr{T} = \mathscr{T}_{\mathscr{L}}$, i.e., $\mathscr{T}_{\mathscr{L}}(a,b) = 0 \vee (a + b - 1)$. To get more information, see [54].

Definition 1 (see [32, 55]). Suppose that Ω is the universal arbitrary set and $\mathscr{F}(\Omega)$ is the fuzzy power set of Ω. We mean $\widehat{\mathscr{C}} = \left\{\widehat{\mathscr{C}_1}, t\widehat{\mathscr{C}_2}n, q \ldots h_{\widehat{\mathscr{C}_m}\}}\right.$, for $\widehat{\mathscr{C}_i} \in \mathscr{F}(\Omega)$ $(i = 1, 2, \ldots, m)$, a fuzzy covering of Ω if $(\cup_{i=1}^{m} \widehat{\mathscr{C}_i})(a) = 1$, $\forall a \in \Omega$.

The notion of a fuzzy β-covering was considered by Ma [23] via substituting 1 for the threshold $\beta (0 < \beta \leq 1)$, i.e., we mean $\widehat{\mathscr{C}} = \left\{\widehat{\mathscr{C}_1}, t\widehat{\mathscr{C}_2}n, q \ldots h_{\widehat{\mathscr{C}_m}\}}\right.$, for $\widehat{\mathscr{C}_i} \in \mathscr{F}(\Omega)$ $(i = 1, 2, \ldots, m)$, a fuzzy β-covering of Ω if $(\cup_{i=1}^{m} \widehat{\mathscr{C}_i})(a) = 1$, $\forall a \in \Omega$,. In addition, $(\Omega, \widehat{\mathscr{C}})$ is referred to as the a fuzzy β-covering approximation space (briefly, FβCAS).

Definition 2 (see [23–26]). Assume that $(\Omega, \widehat{\mathscr{C}})$ is a FβCAS for some $\beta \in (0, 1]$. For each $a \in \Omega$, the fuzzy β-neighborhood (resp., the fuzzy complementary β-neighborhood and the fuzzy β-minimal description) of a is defined by

$$
\begin{aligned}
\mathscr{N}_a^{\beta} &= \cap \left\{\widehat{\mathscr{C}_i} \in t\widehat{\mathscr{C}}n \colon q\widehat{\mathscr{C}_i}h \geq_{\beta}\right\},\\
\mathscr{M}_a^{\beta}(b) &= \mathscr{N}_b^{\beta}(a),\\
\mathscr{M}d_{\widehat{\mathscr{C}}}^{\beta}(a) &= \left\{\widehat{\mathscr{C}} \in t\widehat{\mathscr{C}}n \colon q(\widehat{\mathscr{C}}(a) \geq \beta)h \wedge_{(} \forall \widehat{\mathscr{D}} \in \widehat{\mathscr{C}} \wedge \widehat{\mathscr{D}}(a) \geq \beta \wedge \widehat{\mathscr{D}} \subseteq \widehat{\mathscr{C}} \longrightarrow \widehat{\mathscr{D}} = \widehat{\mathscr{C}})\right\}.
\end{aligned}
\tag{1}
$$

Zhan et al. [52] presented a new definition called fuzzy γ-neighborhood with reflexivity. Using these definitions, they describe the notion of a CVPFRSs based on this definition and solve problems in MADM. The $(\Omega, \widehat{\mathscr{C}})$ pair produced by this neighborhood is called a fuzzy γ-covering approximation space (FγCAS for short) and $\widehat{\mathscr{C}}$ is called a fuzzy γ-covering [51].

Definition 3 (see [52]). Suppose that $(\Omega, \widehat{\mathscr{C}})$ is a FγCAS and $\widehat{\mathscr{C}} = \left\{\widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \ldots, \widehat{\mathscr{C}_m}\right\}$. For every $a, b \in \Omega$, the fuzzy γ-neighborhood of a is as follows:

$$
\mathbb{N}_1^{\gamma}(a)(b) = \bigwedge_{\widehat{\mathscr{C}} \in \mathscr{M}d^{\gamma}(a)} \mathscr{I}(\widehat{\mathscr{C}}(a), \widehat{\mathscr{C}}(b)). \tag{2}
$$

According to the above definition, we have the following result.

Assume that $(\Omega, \widehat{\mathscr{C}})$ is a FγCAS and the variable precision parameter is $\xi \in [0, 1]$. For every $a \in \Omega$ and $\widehat{\mathscr{A}} \in \mathscr{F}(\Omega)$, the first model of a covering-based variable precision fuzzy rough lower and upper approximation which are denoted by 1-CVPFRLA and 1-CVPFRUA, respectively, are given as follows.

Model 1:

$$
\begin{aligned}
\mathscr{O}^{-1}(\widehat{\mathscr{A}})(a) &= \bigwedge_{b \in \Omega} \mathscr{I}\left(\mathbb{N}_1^{\gamma}(a)(b), \xi \vee \widehat{\mathscr{A}}(b)\right),\\
\mathscr{O}^{+1}(\widehat{\mathscr{A}})(a) &= \bigwedge_{b \in \Omega} \mathscr{T}\left(\mathbb{N}_1^{\gamma}(a)(b), \xi \wedge \widehat{\mathscr{A}}(b)\right).
\end{aligned}
\tag{3}
$$

If $\mathscr{O}^{-1}(\widehat{\mathscr{A}}) \neq \mathscr{O}^{+1}(\widehat{\mathscr{A}})$, then $\widehat{\mathscr{A}}$ is said to be a covering-based variable precision fuzzy rough set (briefly, 1-CVPFRS); otherwise it is definable [52].

Ma et al. [53] generalizes Zhan's model by introducing three kinds of neighborhoods as follows.

Definition 4. Assume that $(\Omega, \widehat{\mathscr{C}})$ is a FγCAS. For any $a, b \in \Omega$, three types of the fuzzy γ-neighborhoods of x are as follows:

(1) $\mathbb{N}_2^{\gamma}(a)(b) = \mathbb{N}_1^{\gamma}(b)(a)$
(2) $\mathbb{N}_3^{\gamma}(a)(b) = \mathbb{N}_1^{\gamma}(a)(b) \wedge \mathbb{N}_2^{\gamma}(a)(b)$
(3) $\mathbb{N}_4^{\gamma}(a)(b) = \mathbb{N}_1^{\gamma}(a)(b) \vee \mathbb{N}_2^{\gamma}(a)(b)$

To explain the comparisons between these four kinds of neighborhoods, we give the next example.

Example 1. If $(\Omega, \widehat{\mathscr{C}})$ is a FγCAS, $\Omega = \{a_1, a_2, a_3, a_4\}$ and $\widehat{\mathscr{C}} = \left\{\widehat{\mathscr{C}_1}, t\widehat{\mathscr{C}_2}n, q\widehat{\mathscr{C}_3}\right\}$ is a three fuzzy γ covering on Ω set as follows:

$$
\begin{aligned}
\widehat{\mathscr{C}_1} &= \frac{0.89}{a_1} + \frac{0.88}{a_2} + \frac{0.79}{a_3} + \frac{0.56}{a_4},\\
\widehat{\mathscr{C}_2} &= \frac{0.77}{a_1} + \frac{0.85}{a_2} + \frac{0.67}{a_3} + \frac{0.84}{a_4},\\
\widehat{\mathscr{C}_3} &= \frac{0.69}{a_1} + \frac{0.78}{a_2} + \frac{0.93}{a_3} + \frac{0.63}{a_4}.
\end{aligned}
\tag{4}
$$

Let $\gamma = 0.74$ and $\mathscr{I} = \mathscr{I}_{\mathscr{L}}$. Then, the following values hold for each point on Ω for the three types of neighborhoods which are set in Definition 4.

Firstly, we compute the results for $\mathbb{N}_2^{\gamma}(a_r)\ \forall i \in \{1, 2, 3, 4\}$:

$$\begin{aligned}
\mathbb{N}_2^{\gamma}(a_1) &= \frac{1}{a_1} + \frac{0.91}{a_2} + \frac{0.76}{a_3} + \frac{0.93}{a_4},\\
\mathbb{N}_2^{\gamma}(a_2) &= \frac{0.99}{a_1} + \frac{1}{a_2} + \frac{0.85}{a_3} + \frac{1}{a_4},\\
\mathbb{N}_2^{\gamma}(a_3) &= \frac{0.9}{a_1} + \frac{0.82}{a_2} + \frac{1}{a_3} + \frac{0.83}{a_4},\\
\mathbb{N}_2^{\gamma}(a_4) &= \frac{0.67}{a_1} + \frac{0.68}{a_2} + \frac{0.7}{a_3} + \frac{1}{a_4}.
\end{aligned} \tag{5}$$

Secondly, we compute the results for $\mathbb{N}_3^{\gamma}(a_r)\ \forall i \in \{1, 2, 3, 4\}$:

$$\begin{aligned}
\mathbb{N}_3^{\gamma}(a_1) &= \frac{1}{a_1} + \frac{0.91}{a_2} + \frac{0.76}{a_3} + \frac{0.67}{a_4},\\
\mathbb{N}_3^{\gamma}(a_2) &= \frac{0.91}{a_1} + \frac{1}{a_2} + \frac{0.82}{x_3} + \frac{0.68}{a_4},\\
\mathbb{N}_3^{\gamma}(a_3) &= \frac{0.76}{a_1} + \frac{0.82}{a_2} + \frac{1}{a_3} + \frac{0.7}{a_4},\\
\mathbb{N}_3^{\gamma}(a_4) &= \frac{0.67}{a_1} + \frac{0.68}{a_2} + \frac{0.7}{a_3} + \frac{1}{a_4}.
\end{aligned} \tag{6}$$

Finally, we compute the results for $\mathbb{N}_4^{\gamma}(a_r)\ \forall i \in \{1, 2, 3, 4\}$:

$$\begin{aligned}
\mathbb{N}_4^{\gamma}(a_1) &= \frac{1}{a_1} + \frac{0.99}{a_2} + \frac{0.9}{a_3} + \frac{0.93}{a_4},\\
\mathbb{N}_4^{\gamma}(a_2) &= \frac{0.99}{a_1} + \frac{1}{a_2} + \frac{0.85}{a_3} + \frac{1}{a_4},\\
\mathbb{N}_4^{\gamma}(a_3) &= \frac{0.9}{a_1} + \frac{0.85}{a_2} + \frac{1}{a_3} + \frac{0.83}{a_4},\\
\mathbb{N}_4^{\gamma}(a_4) &= \frac{0.93}{a_1} + \frac{1}{a_2} + \frac{0.83}{a_3} + \frac{1}{a_4}.
\end{aligned} \tag{7}$$

From the above example, you can see the differences between these kinds of neighborhoods. Also, you can conclude that $\mathbb{N}_3^{\gamma}(a_r)$ is considered as the union between $\mathbb{N}_1^{\gamma}(a_r)$ and $\mathbb{N}_2^{\gamma}(a_r)$. Furthermore, $\mathbb{N}_4^{\gamma}(a_r)$ is considered as the intersection between $\mathbb{N}_1^{\gamma}(a_r)$ and $\mathbb{N}_2^{\gamma}(a_r)$. Therefore, it is easy to say that the third neighborhood $\mathbb{N}_3^{\gamma}(a_r)$ is better than others.

3. Three New Models of Covering Fuzzy Rough Sets via Variable Precision

Now, we are implementing three CVPFRSs' models based on different kinds of a reflexive fuzzy γ-neighborhood.

Assume that $(\Omega, \widehat{\mathscr{C}})$ is a FγCAS and the parameter $\xi \in [0, 1]$. For every $a \in \Omega$ and $\widehat{\mathscr{A}} \in \mathscr{F}(\Omega)$, three models of CVPFRSs are defined as follows.

Model 2:

$$\begin{aligned}
\mathcal{O}^{-2}(\widehat{\mathscr{A}})(a) &= \bigwedge_{b\in\Omega} \mathscr{I}\left(\mathbb{N}_2^{\gamma}(a)(b), \xi \vee \widehat{\mathscr{A}}(b)\right),\\
\mathcal{O}^{+2}(\widehat{\mathscr{A}})(a) &= \bigvee_{b\in\Omega} \mathscr{T}\left(\mathbb{N}_2^{\gamma}(a)(b), \xi \wedge \widehat{\mathscr{A}}(b)\right).
\end{aligned} \tag{8}$$

Model 3:

$$\begin{aligned}
\mathcal{O}^{-3}(\widehat{\mathscr{A}})(a) &= \bigwedge_{b\in\Omega} \mathscr{I}\left(\mathbb{N}_3^{\gamma}(a)(b), \xi \vee \widehat{\mathscr{A}}(b)\right),\\
\mathcal{O}^{+3}(\widehat{\mathscr{A}})(a) &= \bigvee_{b\in\Omega} \mathscr{T}\left(\mathbb{N}_3^{\gamma}(a)(b), \xi \wedge \widehat{\mathscr{A}}(b)\right).
\end{aligned} \tag{9}$$

Model 4:

$$\begin{aligned}
\mathcal{O}^{-4}(\widehat{\mathscr{A}})(a) &= \bigwedge_{b\in\Omega} \mathscr{I}\left(\mathbb{N}_4^{\gamma}(a)(b), \xi \vee \widehat{\mathscr{A}}(b)\right),\\
\mathcal{O}^{+4}(\widehat{\mathscr{A}})(a) &= \bigvee_{b\in\Omega} \mathscr{T}\left(\mathbb{N}_4^{\gamma}(a)(b), \xi \wedge \widehat{\mathscr{A}}(b)\right),
\end{aligned} \tag{10}$$

where the three models are called the 2-CVPFRLA (resp., 3-CVPFRLA and 4-CVPFRLA) and the 2-CVPFRUA (resp., 3-CVPFRUA and 4-CVPFRUA), respectively.

If $\mathcal{O}^{-2}(\widehat{\mathscr{A}})$ (resp., $\mathcal{O}^{-3}(\widehat{\mathscr{A}})$, $\mathcal{O}^{-4}(\widehat{\mathscr{A}})$) $\neq \mathcal{O}^{+2}(\widehat{\mathscr{A}})$ (resp., $\mathcal{O}^{+3}(\widehat{\mathscr{A}})$, $\mathcal{O}^{+4}(\widehat{\mathscr{A}})$), then $\widehat{\mathscr{A}}$ is called a 2-CVPFRS (resp., 3-CVPFRS, 4-CVPFRS)), otherwise it is definable.

The next example clarifies the above.

Example 2. (continued from Example 1). Suppose that $\widehat{\mathscr{A}} = (0.58/a_1) + (0.65/a_2) + (0.77/a_3) + (0.76/a_4)$. Then, we have the following results for the above four models (i.e., 1-CVPFRS, 2-CVPFRS, 3-CVPFRS, and 4-CVPFRS).

Model 1:

$$\begin{aligned}
\mathcal{O}^{-1}(\widehat{\mathscr{A}}) &= \frac{0.58}{a_1} + \frac{0.65}{a_2} + \frac{0.77}{a_3} + \frac{0.65}{a_4},\\
\mathcal{O}^{+1}(\widehat{\mathscr{A}}) &= \frac{0.67}{a_1} + \frac{0.65}{a_2} + \frac{0.77}{a_3} + \frac{0.76}{a_4}.
\end{aligned} \tag{11}$$

Model 2:

$$\begin{aligned}
\mathcal{O}^{-2}(\widehat{\mathscr{A}}) &= \frac{0.58}{a_1} + \frac{0.59}{a_2} + \frac{0.68}{a_3} + \frac{0.76}{a_4},\\
\mathcal{O}^{+2}(\widehat{\mathscr{A}}) &= \frac{0.69}{a_1} + \frac{0.76}{a_2} + \frac{0.77}{a_3} + \frac{0.76}{a_4}.
\end{aligned} \tag{12}$$

Model 3:

$$\begin{aligned}
\mathcal{O}^{-3}(\widehat{\mathscr{A}}) &= \frac{0.58}{a_1} + \frac{0.65}{a_2} + \frac{0.77}{a_3} + \frac{0.76}{a_4},\\
\mathcal{O}^{+3}(\widehat{\mathscr{A}}) &= \frac{0.67}{a_1} + \frac{0.65}{a_2} + \frac{0.77}{a_3} + \frac{0.76}{a_4}.
\end{aligned} \tag{13}$$

Model 4:

$$\mathscr{O}^{-4}(\widehat{\mathscr{A}}) = \frac{0.58}{a_1} + \frac{0.59}{a_2} + \frac{0.68}{a_3} + \frac{0.65}{a_4},$$
$$\mathscr{O}^{+4}(\widehat{\mathscr{A}}) = \frac{0.69}{a_1} + \frac{0.76}{a_2} + \frac{0.77}{a_3} + \frac{0.76}{a_4}. \tag{14}$$

Remark 1. From Example 2, it is easy to see that

(1) $\mathscr{O}^{-2}(\widehat{\mathscr{A}}) \nsubseteq \mathscr{O}^{-1}(\widehat{\mathscr{A}})$ and $\mathscr{O}^{-1}(\widehat{\mathscr{A}}) \nsubseteq \mathscr{O}^{-2}(\widehat{\mathscr{A}})$
(2) $\mathscr{O}^{+2}(\widehat{\mathscr{A}}) \subseteq \mathscr{O}^{+1}(\widehat{\mathscr{A}})$ and $\mathscr{O}^{+1}(\widehat{\mathscr{A}}) \nsubseteq \mathscr{O}^{+2}(\widehat{\mathscr{A}})$

The 1-CVPFRS model and the 2-CVPFRS model are clearly not capable of containing each other.

Next, if $r = 1$, we propose Theorem 1, and also, it meets in case $r = 2, 3, 4$.

Theorem 1. *Assume that $(\Omega, \widehat{\mathscr{C}})$ is a FγCAS and the parameter is $\xi \in [0,1]$. For any $\widehat{\mathscr{A}}, \widehat{\mathscr{B}} \in \mathscr{F}(\Omega)$ and $\xi, \varepsilon \in [0,1)$ ($\forall r \in \{1,2,3,4\}$), the following properties hold:*

(1) $\mathscr{O}^{-r}(\widehat{\mathscr{A}}^c) = (\mathscr{O}^{+r}(\widehat{\mathscr{A}}))^c$
(2) $\mathscr{O}^{+r}(\widehat{\mathscr{A}}^c) = (\mathscr{O}^{-r}(\widehat{\mathscr{A}}))^c$
(3) $\mathscr{O}^{-r}(\Omega) = \Omega$
(4) $\mathscr{O}^{+r}(\varnothing) = \varnothing$
(5) If $\widehat{\mathscr{A}} \le \widehat{\mathscr{B}}$, then $\mathscr{O}^{-r}(\widehat{\mathscr{A}}) \le \mathscr{O}^{-r}(\widehat{\mathscr{B}})$
(6) If $\widehat{\mathscr{A}} \le \widehat{\mathscr{B}}$, then $\mathscr{O}^{+r}(\widehat{\mathscr{A}}) \le \mathscr{O}^{+r}(\widehat{\mathscr{B}})$
(7) $\mathscr{O}^{-r}(\widehat{\mathscr{A}} \wedge \widehat{\mathscr{B}}) = \mathscr{O}^{-r}(\widehat{\mathscr{A}}) \wedge \mathscr{O}^{-r}(\widehat{\mathscr{B}})$
(8) $\mathscr{O}^{+r}(\widehat{\mathscr{A}} \wedge \widehat{\mathscr{B}}) \le \mathscr{O}^{+r}(\widehat{\mathscr{A}}) \wedge \mathscr{O}^{+r}(\widehat{\mathscr{B}})$
(9) $\mathscr{O}^{-r}(\widehat{\mathscr{A}} \vee \widehat{\mathscr{B}}) \ge \mathscr{O}^{-r}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{-r}(\widehat{\mathscr{B}})$
(10) $\mathscr{O}^{+r}(\widehat{\mathscr{A}} \vee \widehat{\mathscr{B}}) = \mathscr{O}^{+r}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{+r}(\widehat{\mathscr{B}})$
(11) If $\xi \le \varepsilon$, then $\mathscr{O}_{\xi}^{-r}(\widehat{\mathscr{A}}) \le \mathscr{O}_{\varepsilon}^{-r}(\widehat{\mathscr{B}})$
(12) If $\xi \le \varepsilon$, then $\mathscr{O}_{\varepsilon}^{+r}(\widehat{\mathscr{A}}) \le \mathscr{O}_{\xi}^{+r}(\widehat{\mathscr{B}})$

Proof. We shall only prove (1), (3), (5), (7), (9), and (11).

(1) $(\mathscr{O}^{-1}(\widehat{\mathscr{A}}^c)) = \wedge_{b\in\Omega}\mathscr{I}(\mathbb{N}_1^{\gamma}(a)(b), \xi \vee \mathscr{N}(\widehat{\mathscr{A}}(b))) = \wedge_{b\in\Omega}\mathscr{N}(\mathscr{T}(\mathbb{N}_1^{\gamma}(a)(b), \mathscr{N}(\xi) \wedge \widehat{\mathscr{A}}(b))) = \mathscr{N}(\wedge_{b\in\Omega}(\mathscr{T}(\mathbb{N}_1^{\gamma}(a)(b), \mathscr{N}(\xi) \wedge \widehat{\mathscr{A}}(b)))) = (\mathscr{O}^{+1}(\widehat{\mathscr{A}}))^c$.

(3) As $\mathscr{I}$ is left monotonic and $\Omega(a) = 1$, for every $a \in \Omega$. Then, we have $\mathscr{O}^{-1}(\Omega) = \wedge_{b\in\Omega}\mathscr{I}(\mathbb{N}_1^{\gamma}(a)(b), \xi \vee \Omega(b)) = \wedge_{b\in\Omega}\mathscr{I}(\mathbb{N}_1^{\gamma}(a)(b), \Omega(b)) = \wedge_{b\in\Omega}\mathscr{I}(\mathbb{N}_1^{\gamma}(a)(b), 1) = 1 = \Omega(a)$.

(5) $\mathscr{I}$ is right monotonic and for every $a \in \Omega$. If $\widehat{\mathscr{A}} \le \widehat{\mathscr{B}}$, then we get the following result. $\mathscr{O}^{-1}(\widehat{\mathscr{A}})(a) = \wedge_{b\in\Omega}\mathscr{I}(\mathbb{N}_1^{\gamma}(a)(b), \xi \vee \widehat{\mathscr{A}}(b)) \le \wedge_{b\in\Omega}\mathscr{I}(\mathbb{N}_1^{\gamma}(a)(b), \xi \vee \widehat{\mathscr{B}}(b)) = \mathscr{O}^{-1}(\widehat{\mathscr{B}})(a)$.

(7) $\mathscr{I}$ is right monotonic and for all $a \in \Omega$. Then, we have $\mathscr{O}^{-1}(\widehat{\mathscr{A}} \wedge \widehat{\mathscr{B}})(a) = \wedge_{b\in\Omega}\mathscr{I}(\mathbb{N}_1^{\gamma}(a)(b), \xi \vee (\widehat{\mathscr{A}} \wedge \widehat{\mathscr{B}})(b)) = \wedge_{b\in\Omega}\mathscr{I}(\mathbb{N}_1^{\gamma}(a)(b), \xi \vee \widehat{\mathscr{A}}(b)) \wedge \wedge_{b\in\Omega}\mathscr{I}(\mathbb{N}_1^{\gamma}(a)(b), \xi \vee \widehat{\mathscr{B}}(b)) = \mathscr{O}^{-1}(\widehat{\mathscr{A}})(a) \wedge \mathscr{O}^{-1}(\widehat{\mathscr{B}})(a)$.

(9) As $\mathscr{I}$ is right monotonic, $\widehat{\mathscr{A}} \le \widehat{\mathscr{A}} \vee \widehat{\mathscr{B}}$ and $\widehat{\mathscr{B}} \le \widehat{\mathscr{A}} \vee \widehat{\mathscr{B}}$. Then, by (3), we obtain the following $\mathscr{O}^{-1}(\widehat{\mathscr{A}}) \le \mathscr{O}^{-1}(\widehat{\mathscr{A}} \vee \widehat{\mathscr{B}})$ and $\mathscr{O}^{-1}(\widehat{\mathscr{B}}) \le \mathscr{O}^{-1}(\widehat{\mathscr{A}} \vee \widehat{\mathscr{B}})$. Thus, $\mathscr{O}^{-1}(\widehat{\mathscr{A}} \vee \widehat{\mathscr{B}}) \ge \mathscr{O}^{-1}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{-1}(\widehat{\mathscr{B}})$.

(11) It is obtained directly from Definition of Model 1. □

The relationships between our models and the Zhan model in [52] are defined as follows. The following characteristics are clear and will be seen without proof.

Proposition 1. *Assume that $(\Omega, \widehat{\mathscr{C}})$ is a FγCAS of Ω. For every $\widehat{\mathscr{A}} \in \mathscr{F}(\Omega)$ and $\forall r \in \{1,2,3,4\}$, we have the following properties:*

(1) $\mathscr{O}^{-r}(\widehat{\mathscr{A}}) \subseteq \widehat{\mathscr{A}} \subseteq \mathscr{O}^{+r}(\widehat{\mathscr{A}})$
(2) $\mathscr{O}^{-3}(\widehat{\mathscr{A}}) = \mathscr{O}^{-1}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{-2}(\widehat{\mathscr{A}})$ and $\mathscr{O}^{-3}(\widehat{\mathscr{A}}) = \mathscr{O}^{+1}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{+2}(\widehat{\mathscr{A}})$
(3) $\mathscr{O}^{-4}(\widehat{\mathscr{A}}) = \mathscr{O}^{-1}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{-2}(\widehat{\mathscr{A}})$ and $\mathscr{O}^{+4}(\widehat{\mathscr{A}}) = \mathscr{O}^{+1}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{+2}(\widehat{\mathscr{A}})$
(4) $\mathscr{O}^{-4}(\widehat{\mathscr{A}}) = \mathscr{O}^{-1}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{-3}(\widehat{\mathscr{A}})$ and $\mathscr{O}^{+3}(\widehat{\mathscr{A}}) = \mathscr{O}^{+1}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{+4}(\widehat{\mathscr{A}})$
(5) $\mathscr{O}^{-4}(\widehat{\mathscr{A}}) = \mathscr{O}^{-2}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{-3}(\widehat{\mathscr{A}})$ and $\mathscr{O}^{+3}(\widehat{\mathscr{A}}) = \mathscr{O}^{+2}(\widehat{\mathscr{A}}) \vee \mathscr{O}^{+4}(\widehat{\mathscr{A}})$

Proposition 2. *Suppose that $(\Omega, \widehat{\mathscr{C}})$ is a FγCAS of Ω. For any $\widehat{\mathscr{A}} \in \mathscr{F}(\Omega)$ and $\forall a \in \Omega$.*

Then, $\mathbb{N}_1^{\gamma}(a) = \mathbb{N}_2^{\gamma}(a) \Leftrightarrow$ either $\mathscr{O}^{-1}(\widehat{\mathscr{A}}) = \mathscr{O}^{-2}(\widehat{\mathscr{A}}) = \mathscr{O}^{-3}(\widehat{\mathscr{A}}) = \mathscr{O}^{-4}(\widehat{\mathscr{A}})$ or $\mathscr{O}^{+1}(\widehat{\mathscr{A}}) = \mathscr{O}^{+2}(\widehat{\mathscr{A}}) = \mathscr{O}^{+3}(\widehat{\mathscr{A}}) = \mathscr{O}^{+4}(\widehat{\mathscr{A}})$.

4. Decision-Making Approach to MADM Based on CVPFRS

This section introduces a new decision-making method to solve MADM problems by using CVPFRSs' models.

4.1. Description and Process. In medicine, some types of drugs exist for the treatment of a disease, such as viral fever, dysentery, and chest problems. Assume that $\Omega = \{a_1, a_2, \ldots, a_n\}$ is n kinds of drugs (alternatives) and $\widehat{\mathscr{C}} = \{\widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \ldots, \widehat{\mathscr{C}_m}\}$ is m symptoms (attributes). According to the decision assessment, maker's efficacy effect of the drug x_i on the symptoms $\widehat{\mathscr{C}_r}$ ($\forall r = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, n$) has been determined. Hence, $(\Omega, \widehat{\mathscr{C}})$ establishes an FγCAS. According to the presented work, in the next steps, we introduce a decision-making algorithm that finds the most effective drug.

Step 1 : fuzzy decision matrix $\mathscr{F}$ of medicine evaluations set as below:

$$\mathscr{F} = \begin{pmatrix} \frac{\Omega}{\widehat{\mathscr{C}}} & \widehat{\mathscr{C}_1} & \widehat{\mathscr{C}_2} & \ldots & \widehat{\mathscr{C}_m} \\ a_1 & d_{11} & d_{12} & \ldots & d_{1m} \\ a_2 & d_{21} & d_{22} & \ldots & d_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & d_{n1} & d_{n2} & \ldots & d_{nm} \end{pmatrix}. \tag{15}$$

Step 2 : calculate the lower and upper approximations of $\widehat{\mathscr{C}_r}$ and evaluate the lower and upper fuzzy decision-making matrix of medicine evaluations:

$$\begin{aligned}\mathcal{O}^{-k}\left(\widehat{\mathscr{C}_r}\right)(a) &= \bigwedge_{b\in\Omega} \mathscr{I}\left(\mathbb{N}_q^{\gamma}(a)(b), \xi\vee\widehat{\mathscr{C}_r}(b)\right),\\ \mathcal{O}^{+k}\left(\widehat{\mathscr{C}_r}\right)(a) &= \bigvee_{b\in\Omega} \mathscr{T}\left(\mathbb{N}_q^{\gamma}(a)(b), \xi\wedge\widehat{\mathscr{C}_r}(b)\right), \quad (\forall k, q \in \{1,2,3,4\}).\end{aligned} \tag{16}$$

Step 3 : three deflections among the estimations of any two alternatives are called the deflections among drugs $\widehat{\mathscr{D}}_r$, the lower deflections among drugs $\widehat{\mathscr{D}}_r^{\ominus}$, and the upper deflections among drugs $\widehat{\mathscr{D}}_r^{\oplus}$, respectively. These three deviations are computed as follows:

$$\begin{aligned}\widehat{\mathscr{D}}_r\left(a_i, a_j\right) &= \widehat{\mathscr{C}_r}\left(a_i\right) - \widehat{\mathscr{C}_r}\left(a_j\right)\\ \widehat{\mathscr{D}}_r^{\ominus}\left(a_i, a_j\right) &= \mathcal{O}^{-k}\left(\widehat{\mathscr{C}_r}\right)\left(a_i\right) - \mathcal{O}^{-k}\left(\widehat{\mathscr{C}_r}\right)\left(a_j\right),\\ \widehat{\mathscr{D}}_r^{\oplus}\left(a_i, a_j\right) &= \mathcal{O}^{+k}\left(\widehat{\mathscr{C}_r}\right)\left(a_i\right) - \mathcal{O}^{+k}\left(\widehat{\mathscr{C}_r}\right)\left(a_j\right),\end{aligned} \tag{17}$$

where $k \in \{1,2,3,4\}$.

Step 4 : according to the three deviations, three drug preference values are referred to as drug preference values $\widehat{\mathscr{P}}_r$, lower drug preference values $\widehat{\mathscr{P}}_r^{\ominus}$, and upper drug preference values $\widehat{\mathscr{P}}_r^{\oplus}$. These three values of choice among alternatives are therefore computed as follows:

$$\begin{aligned}\widehat{\mathscr{P}}_r\left(a_i, a_j\right) &= \begin{cases} 0 & \text{if } \widehat{\mathscr{D}}_r\left(a_i, a_j\right) < 0,\\ \dfrac{\widehat{\mathscr{D}}_r\left(a_i, a_j\right)}{\alpha} & \text{if } 0 \le \widehat{\mathscr{D}}_r\left(a_i, a_j\right) \le \alpha,\\ 1 & \text{if } \widehat{\mathscr{D}}_r\left(a_i, a_j\right) > \alpha,\end{cases}\\ \widehat{\mathscr{P}}_r^{\ominus}\left(a_i, a_j\right) &= \begin{cases} 0 & \text{if } \widehat{\mathscr{D}}_r^{\ominus}\left(a_i, a_j\right) < 0,\\ \dfrac{\widehat{\mathscr{D}}_r^{\ominus}\left(a_i, a_j\right)}{\alpha} & \text{if } 0 \le \widehat{\mathscr{D}}_r^{\ominus}\left(a_i, a_j\right) \le \alpha,\\ 1 & \text{if } \widehat{\mathscr{D}}_r^{\ominus}\left(a_i, a_j\right) > \alpha,\end{cases}\\ \widehat{\mathscr{P}}_r^{\oplus}\left(a_i, a_j\right) &= \begin{cases} 0 & \text{if } \widehat{\mathscr{D}}_r^{\oplus}\left(a_i, a_j\right) < 0,\\ \dfrac{\widehat{\mathscr{D}}_r^{\oplus}\left(a_i, a_j\right)}{\alpha} & \text{if } 0 \le \widehat{\mathscr{D}}_r^{\oplus}\left(a_i, a_j\right) \le \alpha,\\ 1 & \text{if } \widehat{\mathscr{D}}_r^{\oplus}\left(a_i, a_j\right) > \alpha.\end{cases}\end{aligned} \tag{18}$$

where α denotes the value of preference threshold.

Step 5 : calculate three general drug preference indices, referred to as the overall drug preference indices for alternatives $\widehat{\mathcal{O}}$, the overall lower drug preference indices for alternatives $\widehat{\mathcal{O}}^{\ominus}$, and the overall upper drug preference indices for alternatives $\widehat{\mathcal{O}}^{\oplus}$, as follows:

$$\begin{aligned}\widehat{\mathcal{O}}\left(a_i, a_j\right) &= \sum_{r=1}^{m} \widehat{\mathscr{W}}_r \widehat{\mathscr{P}}_r\left(a_i, a_j\right),\\ \widehat{\mathcal{O}}^{\ominus}\left(a_i, a_j\right) &= \sum_{r=1}^{m} \widehat{\mathscr{W}}_r \widehat{\mathscr{P}}_r^{\ominus}\left(a_i, a_j\right),\\ \widehat{\mathcal{O}}^{\oplus}\left(a_i, a_j\right) &= \sum_{r=1}^{m} \widehat{\mathscr{W}}_r \widehat{\mathscr{P}}_r^{\oplus}\left(a_i, a_j\right),\end{aligned} \tag{19}$$

where $\widehat{\mathbb{W}} = (\widehat{\mathscr{W}}_1, \widehat{\mathscr{W}}_2, \ldots, \widehat{\mathscr{W}}_m)$ is the vector of the weight of attributes such that $\sum_{r=1}^{m} \widehat{\mathscr{W}}_r = 1$ and $\widehat{\mathscr{W}}_r \in [0,1]$.

Step 6 : three outflows of medicines are referred to as the outflows of alternatives $\mathscr{L}_{\bullet}$, the lower outflows of alternatives $\mathscr{L}_{\bullet}^{\ominus}$, and the upper outflows of medicines $\mathscr{L}_{\bullet}^{\oplus}$. These flows are thus constructed as follows:

$$\begin{aligned}\mathscr{L}_{\bullet}\left(a_i, a_j\right) &= \sum_{r=1}^{m} \widehat{\mathcal{O}}\left(a_i, a_j\right),\\ \mathscr{L}_{\bullet}^{\ominus}\left(a_i, a_j\right) &= \sum_{r=1}^{m} \widehat{\mathcal{O}}^{\ominus}\left(a_i, a_j\right),\\ \mathscr{L}_{\bullet}^{\oplus}\left(a_i, a_j\right) &= \sum_{r=1}^{m} \widehat{\mathcal{O}}^{\oplus}\left(a_i, a_j\right).\end{aligned} \tag{20}$$

We also create three input flows of drugs called the input flows of drugs $\mathscr{L}_{\circ}$, the lower input flows of drugs $\mathscr{L}_{\circ}^{\ominus}$, and the upper input flows of drugs $\mathscr{L}_{\circ}^{\oplus}$, respectively, as follows:

$$\begin{aligned}\mathscr{L}_{\circ}\left(a_i, a_j\right) &= \sum_{r=1}^{m} \widehat{\mathcal{O}}\left(a_i, a_j\right),\\ \mathscr{L}_{\circ}^{\ominus}\left(a_i, a_j\right) &= \sum_{r=1}^{m} \widehat{\mathcal{O}}^{\ominus}\left(a_i, a_j\right),\\ \mathscr{L}_{\circ}^{\oplus}\left(a_i, a_j\right) &= \sum_{r=1}^{m} \widehat{\mathcal{O}}^{\oplus}\left(a_i, a_j\right).\end{aligned} \tag{21}$$

Step 7 : the next formula computes the net flow of alternatives:

$$\mathscr{L}\left(a_i, a_j\right) = \left(\mathscr{L}_{\bullet} + \mathscr{L}_{\bullet}^{\ominus} + \mathscr{L}_{\bullet}^{\oplus}\right) - \left(\mathscr{L}_{\circ} + \mathscr{L}_{\circ}^{\ominus} + \mathscr{L}_{\circ}^{\oplus}\right), \tag{22}$$

Input: the $\mathscr{F}$ fuzzy decision matrix, the α choice threshold, and the ξ parameter.
Output: decision-aking.
(1)Compute the lower and upper approximations by using Model 3
(2)Compute three deflections among drugs (i.e., $\widehat{\mathscr{D}}_r, \widehat{\mathscr{D}}_r^{\ominus}$, and $\widehat{\mathscr{D}}_r^{\oplus}$)
(3)Compute three preference values among drugs (i.e., $\widehat{\mathscr{P}}_r, \widehat{\mathscr{P}}_r^{\ominus}$, and $\widehat{\mathscr{P}}_r^{\oplus}$)
(4)Compute three overall preference indices among drugs (i.e., $\widehat{\mathscr{O}}, \widehat{\mathscr{O}}^{\ominus}$, and $\widehat{\mathscr{O}}^{\oplus}$)
(5)Compute three leaving flows of drugs (i.e., $\mathscr{L}_{\bullet}, \mathscr{L}_{\bullet}^{\ominus}$, and $\mathscr{L}_{\bullet}^{\oplus}$)
(6)Compute three entering flows of drugs (i.e., $\mathscr{L}_{\circ}, \mathscr{L}_{\circ}^{\ominus}$, and $\mathscr{L}_{\circ}^{\oplus}$)
(7)Compute the net flow of alternatives $\mathscr{L}$
(8) Ranking the alternatives and obtain the decision

Algorithm 1: Algorithm for the presented drug selections.

hence ranking the alternatives.

In accordance with these steps, we include an algorithm based on Model 3 (3-CVPFRS) to solve decision-making issues. Algorithm 1 summarizes the measures leading to it.

4.2. A Numerical Example. The steps aforementioned have been illustrated as follows with a check instance.

Example 3. Alternatives (medicines) construct a set $\Omega = \{a_1, a_2, \ldots, a_6\}$ which are treated a diseases $\widehat{\mathscr{A}}$, and their symptoms are gathered by the attribute set $\widehat{\mathscr{C}}$ = fever $(\widehat{\mathscr{C}_1})$, cough $(\widehat{\mathscr{C}_2})$, headache $(\widehat{\mathscr{C}_3})$, stomachaches $(\widehat{\mathscr{C}_4})$, dizzy giddy $(\widehat{\mathscr{C}_5})$. Here, the following steps of the algorithm mentioned are implemented.

Step 1: over the set of symptoms, experts analyze each medication and present its conclusions with acceptable values set out in Table 1.

Step 2: let us fix $\mathscr{I}_{\mathscr{L}}$ and $\mathscr{T}_{\mathscr{L}}$. Then, by 2-CVPFRS, we have the following:

$$
\begin{aligned}
\mathbb{N}_3^{\gamma}(a_1) &= \frac{1}{a_1} + \frac{0.89}{a_2} + \frac{0.82}{a_3} + \frac{0.77}{a_4} + \frac{0.54}{a_5} + \frac{0.49}{a_6},\\
\mathbb{N}_3^{\gamma}(a_2) &= \frac{0.89}{a_1} + \frac{1}{a_2} + \frac{0.83}{a_3} + \frac{0.74}{a_4} + \frac{0.65}{a_5} + \frac{0.55}{a_6},\\
\mathbb{N}_3^{\gamma}(a_3) &= \frac{0.82}{a_1} + \frac{0.83}{a_2} + \frac{1}{a_3} + \frac{0.87}{a_4} + \frac{0.48}{a_5} + \frac{0.64}{a_6},\\
\mathbb{N}_3^{\gamma}(a_4) &= \frac{0.77}{a_1} + \frac{0.74}{a_2} + \frac{0.87}{a_3} + \frac{1}{a_4} + \frac{0.73}{a_5} + \frac{0.51}{a_6},\\
\mathbb{N}_3^{\gamma}(a_5) &= \frac{0.54}{a_1} + \frac{0.65}{a_2} + \frac{0.48}{a_3} + \frac{0.73}{a_4} + \frac{1}{a_5} + \frac{0.51}{a_6},\\
\mathbb{N}_3^{\gamma}(a_6) &= \frac{0.49}{a_1} + \frac{0.55}{a_2} + \frac{0.64}{a_3} + \frac{0.51}{a_4} + \frac{0.51}{a_5} + \frac{1}{a_6}.
\end{aligned}
\tag{23}
$$

Thus, the 3-CVPFRLA and 3-CVPFRUA are obtained as follows:

$$
\begin{aligned}
\mathscr{O}^{-3}(\widehat{\mathscr{C}_1}) &= \frac{0.92}{a_1} + \frac{0.91}{a_2} + \frac{0.86}{a_3} + \frac{0.73}{a_4} + \frac{0.56}{a_5} + \frac{1}{a_6},\\
\mathscr{O}^{+3}(\widehat{\mathscr{C}_1}) &= \frac{0.9}{a_1} + \frac{0.9}{a_2} + \frac{0.9}{a_3} + \frac{0.77}{a_4} + \frac{0.56}{a_5} + \frac{1}{a_6},\\
\mathscr{O}^{-3}(\widehat{\mathscr{C}_2}) &= \frac{0.54}{a_1} + \frac{0.65}{a_2} + \frac{0.48}{a_3} + \frac{0.61}{a_4} + \frac{1}{a_5} + \frac{0.51}{a_6},\\
\mathscr{O}^{+3}(\widehat{\mathscr{C}_2}) &= \frac{0.61}{a_1} + \frac{0.65}{a_2} + \frac{0.71}{a_3} + \frac{0.84}{a_4} + \frac{0.9}{a_5} + \frac{0.51}{a_6},\\
\mathscr{O}^{-3}(\widehat{\mathscr{C}_3}) &= \frac{0.95}{a_1} + \frac{0.94}{a_2} + \frac{0.82}{a_3} + \frac{0.95}{a_4} + \frac{0.82}{a_5} + \frac{0.49}{a_6},\\
\mathscr{O}^{+3}(\widehat{\mathscr{C}_3}) &= \frac{0.9}{a_1} + \frac{0.9}{a_2} + \frac{0.82}{a_3} + \frac{0.9}{a_4} + \frac{0.82}{a_5} + \frac{0.49}{a_6},\\
\mathscr{O}^{-3}(\widehat{\mathscr{C}_4}) &= \frac{0.48}{a_1} + \frac{0.52}{a_2} + \frac{0.57}{a_3} + \frac{0.44}{a_4} + \frac{0.52}{a_5} + \frac{0.93}{a_6},\\
\mathscr{O}^{+3}(\widehat{\mathscr{C}_4}) &= \frac{0.48}{a_1} + \frac{0.52}{a_2} + \frac{0.57}{a_3} + \frac{0.44}{a_4} + \frac{0.52}{a_5} + \frac{0.9}{a_6},\\
\mathscr{O}^{-3}(\widehat{\mathscr{C}_5}) &= \frac{0.28}{a_1} + \frac{0.25}{a_2} + \frac{0.42}{a_3} + \frac{0.51}{a_4} + \frac{0.24}{a_5} + \frac{0.28}{a_6},\\
\mathscr{O}^{+3}(\widehat{\mathscr{C}_5}) &= \frac{0.28}{a_1} + \frac{0.29}{a_2} + \frac{0.46}{a_3} + \frac{0.51}{a_4} + \frac{0.24}{a_5} + \frac{0.28}{a_6}.
\end{aligned}
\tag{24}
$$

Steps 3 and 4: by using the previous data, it is easy to compute the three deflections among the estimations of any two alternatives and the three preference values among drugs.

Step 5: from this information, we construct the values for three overall preference indices among drugs as set in Tables 2–4.

Step 6: the three leaving flows of drugs are calculated as follows:

$$
\begin{aligned}
&\mathscr{L}_{\bullet}\left(a_i,a_j\right)=\frac{0.66895}{a_1}+\frac{0.9367}{a_2}+\frac{0.6456}{a_3}+\frac{1.411}{a_4}+\frac{0.8467}{a_5}+\frac{0.58875}{a_6},\\
&\mathscr{L}_{\bullet}^{\ominus}\left(a_i,a_j\right)=\frac{0.61255}{a_1}+\frac{0.70305}{a_2}+\frac{0.6116}{a_3}+\frac{0.9596}{a_4}+\frac{0.9227}{a_5}+\frac{0.64875}{a_6},\\
&\mathscr{L}_{\bullet}^{\oplus}\left(a_i,a_j\right)=\frac{0.45795}{a_1}+\frac{0.5091}{a_2}+\frac{0.7696}{a_3}+\frac{1.04425}{a_4}+\frac{0.5787}{a_5}+\frac{0.58775}{a_6},\\
&\mathscr{L}_{\circ}\left(a_i,a_j\right)=\frac{0.6219}{a_1}+\frac{0.3413}{a_2}+\frac{0.975}{a_3}+\frac{0.5367}{a_4}+\frac{1.0578}{a_5}+\frac{1.565}{a_6},\\
&\mathscr{L}_{\circ}^{\ominus}\left(a_i,a_j\right)=\frac{0.4894}{a_1}+\frac{0.456}{a_2}+\frac{0.65545}{a_3}+\frac{0.4811}{a_4}+\frac{0.9353}{a_5}+\frac{1.441}{a_6},\\
&\mathscr{L}_{\circ}^{\oplus}\left(a_i,a_j\right)=\frac{0.51755}{a_1}+\frac{0.4273}{a_2}+\frac{0.32625}{a_3}+\frac{0.3077}{a_4}+\frac{0.86605}{a_5}+\frac{1.5025}{a_6}.
\end{aligned}
\tag{25}
$$

Step 7: the values of the net flow of alternatives are computed as follows:

$$
\begin{aligned}
&\mathscr{L}\left(a_1\right)=0.1106,\\
&\mathscr{L}\left(a_2\right)=0.92425,\\
&\mathscr{L}\left(a_3\right)=0.0701,\\
&\mathscr{L}\left(a_4\right)=2.08935,\\
&\mathscr{L}\left(a_5\right)=-0.51105,\\
&\mathscr{L}\left(a_6\right)=-2.68325.
\end{aligned}
\tag{26}
$$

Thus, the drugs' ranking is as follows:

$$
a_4\geq a_2\geq a_1\geq a_3\geq a_5\geq a_6. \tag{27}
$$

4.3. Comparative Analysis. Here, we give the differences between the proposed method (i.e., 2-CVPFRS, 3-CVPFRS, and 4-CVPFRS) and the previous methods (i.e., Jiang's method [36], PROMETHEE II [56], TOPSIS [57], WAA [58], OWA [59], and VIKOR [60]). Based on the sorting values of various decision-making approaches summarized in Table 5, our approach is therefore rational and effective.

According to Table 5, (1) the best position of our presented method, Jiang's method [36], PROMETHEE II [56], TOPSIS [57], WAA [58], OWA [59], and VIKOR [60], is still consistent, that is, a_4 is the best drug. Thus, our suggested approach is rational and efficient from the point of view of the decision outcome (the best option in the decision-making process). (2) Five drug classifications based on various methods are not precisely the same in [36], meaning that the best drug is equal (i.e., the drug $a4$). However, operating on the fuzzy γ-neighborhoods without reflexivity in Jiang's [36] process, our methodology relies on the fuzzy γ-neighborhoods with reflexivity, which makes our approach proposed more rational and effective.

The best way to clarify these results, you can see Figures 1 and 2 which simplify the comparisons between the presented method and others.

Figure 1 explains the comparisons between the lower approximation for the four models (i.e., 1-CVPFRLA, 2-CVPFRLA, 3-CVPFRLA, and 4-CVPFRLA). This figure clarifies that the 3-CVPFRLA is larger than the others.

Figure 2 clarifies the comparisons between the upper approximation for the four models (i.e., 1-CVPFRUA, 2-CVPFRUA, 3-CVPFRUA, and 4-CVPFRUA). This figure shows that 3-CVPFRUA is smaller than the others.

(1) Two documented issues with fuzzy γ-neighborhoods are conquered by our presented methods. However, not all techniques can escape the obstacles that are not reflexive operators in fuzzy γ-neighborhoods and that the lower approximations they describe are not usually included in the corresponding upper approximation. For this reason, our approach for solving MADM issues is based on the CVPFRS models (i.e., 1-CVPFRS, 2-CVPFRS, and 3-CVPFRS). Moreover, by a comparative study in Section 4.3, by using fuzzy γ-neighborhoods, the proposed models are more freely used than the classical models.

(2) We can see in Section 4 that our presented models (i.e., Algorithm 1) are elastic and scalable, whereby decision makers can use fuzzy γ-neighborhoods to pick various logical operators and parameters according to current status.

(3) We can easily observe from a comparative study that our models presented are superior to Jiang's method [36], PROMETHEE II [56], TOPSIS [57], WAA [58], OWA [59], and VIKOR [60]. This implies that the innovative decision-making approaches suggested are rational and feasible.

Table 1: Decision-making matrix $\mathscr{F}$ with fuzzy information.

	a_1	a_2	a_3	a_4	a_5	a_6
$\widetilde{\mathscr{C}_1}$	0.92	1	1	0.73	0.56	1
$\widetilde{\mathscr{C}_2}$	0.54	0.65	0.48	0.84	1	0.51
$\widetilde{\mathscr{C}_3}$	1	0.94	0.82	1	0.82	0.49
$\widetilde{\mathscr{C}_4}$	0.48	0.52	0.57	0.44	0.52	0.93
$\widetilde{\mathscr{C}_5}$	0.28	0.25	0.46	0.51	0.24	0.28

Table 2: The overall preference indices among drugs.

$\widehat{\mathscr{O}}(a_i, a_j)$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	0	0.0622	0.1095	0.195	0.1607	0.0945
a_2	0.0425	0	0.0083	0.094	0.116	0.0805
a_3	0.11	0.2035	0	0.4175	0.174	0.07
a_4	0.0547	0.188	0.0775	0	0.067	0.1495
a_5	0.19675	0.212	0.2103	0.2445	0	0.19425
a_6	0.265	0.271	0.24	0.46	0.329	0

Table 3: The overall lower preference indices among drugs.

$\widehat{\mathscr{O}}^{\ominus}(a_i, a_j)$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	0	0.0427	0.0725	0.119	0.1607	0.0945
a_2	0.0431	0	0.0783	0.1126	0.116	0.106
a_3	0.101	0.12845	0	0.1475	0.174	0.1045
a_4	0.0547	0.0714	0.0625	0	0.143	0.1495
a_5	0.17275	0.1895	0.1583	0.2205	0	0.19425
a_6	0.241	0.271	0.24	0.36	0.329	0

Table 4: The overall upper preference indices among drugs.

$\widehat{\mathscr{O}}^{\oplus}(a_i, a_j)$	a_1	a_2	a_3	a_4	a_5	a_6
a_1	0	0.02435	0.124	0.171	0.1027	0.0955
a_2	0	0	0.0983	0.1565	0.084	0.0885
a_3	0.039	0.039	0	0.10375	0.064	0.0805
a_4	0.0397	0.046	0.055	0	0.033	0.134
a_5	0.14125	0.14125	0.1863	0.204	0	0.18925
a_6	0.238	0.2545	0.306	0.409	0.295	0

Table 5: Table for the ranking results for different methods.

Different models	Obtain a decision
Our model	$a_4 \geq a_2 \geq a_1 \geq a_3 \geq a_5 \geq a_6$
Jiang model [36]	$a_4 \geq a_2 \geq a_3 \geq a_1 \geq a_5 \geq a_6$
PROMETHEE II [56]	$a_4 \geq a_2 \geq a_1 = a_3 \geq a_5 \geq a_6$
TOPSIS [57]	$a_4 \geq a_2 \geq a_1 \geq a_5 \geq a_3 \geq a_6$
WAA [58]	$a_4 \geq a_2 \geq a_1 = a_3 \geq a_5 \geq a_6$
OWA [59]	$a_4 \geq a_3 \geq a_2 \geq a_1 \geq a_6 \geq a_5$
VIKOR [60]	$a_4 \geq a_3 \geq a_1 \geq a_2 \geq a_5 \geq a_6$

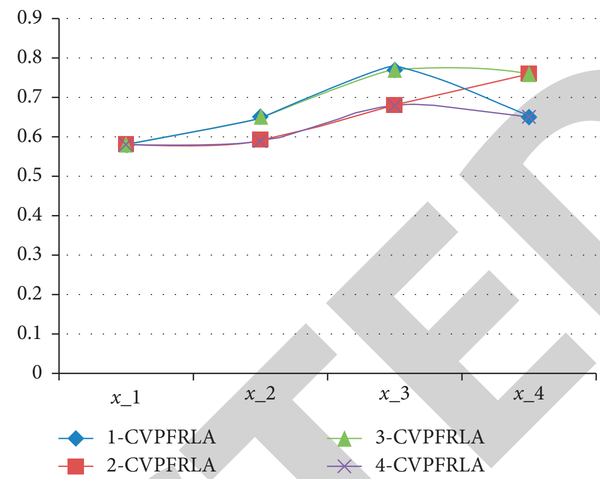

Figure 1: The presentation of lower approximations by using our models and the previous model.

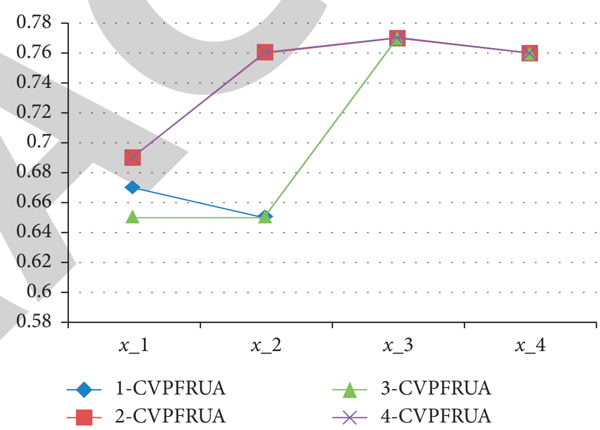

Figure 2: The presentation of upper approximations by using our models and the previous model.

5. Conclusion

As an improvement of the Zhan et al. method [52] and by using the concepts of neighborhoods by Ma et al. in [53], we then established new three kinds of covering-based variable precision fuzzy rough sets (i.e., 2-CVPFRS, 3-CVPFRS, and 4-CVPFRS). Relationship between these three paradigms and the paradigm of Zhan is also dealt with. This correlation indicates that the 3-CVPFRS is better than other models (i.e., the lower approximation is greater than others and the upper approximation is lower than others, as can be seen in Figures 1 and 2 based on Example 3). Finally, we set up an application for MADM to solve a problem. In the existing decision-making principles of interval-valued q-rung orthopair fuzzy sets [61] and linguistic interval-valued Pythagorean fuzzy sets [62], we hope this fuzzy rough concept can be incorporated.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Acknowledgments

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Retraction

Retracted: Some Generalized T-Spherical and Group-Generalized Fuzzy Geometric Aggregation Operators with Application in MADM Problems

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] Y. Chen, M. Munir, T. Mahmood, A. Hussain, and S. Zeng, "Some Generalized T-Spherical and Group-Generalized Fuzzy Geometric Aggregation Operators with Application in MADM Problems," *Journal of Mathematics*, vol. 2021, Article ID 5578797, 17 pages, 2021.

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Research Article

Some Generalized T-Spherical and Group-Generalized Fuzzy Geometric Aggregation Operators with Application in MADM Problems

Yujuan Chen,[1] Muhammad Munir ,[2] Tahir Mahmood ,[2] Azmat Hussain,[2] and Shouzhen Zeng[3,4]

[1]*School of Data Sciences, Zhejiang University of Finance & Economics, Hangzhou 310018, China*
[2]*Department of Mathematics and Statistics, International Islamic University, Islamabad, Pakistan*
[3]*School of Business, Ningbo University, Ningbo 315211, China*
[4]*College of Statistics and Mathematics, Zhejiang Gongshang University, Hangzhou 310018, China*

Correspondence should be addressed to Muhammad Munir; munir.phdma78@iiu.edu.pk

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In this article, the generalized parameter is involved in T-spherical fuzzy set (TSFS), and with the help of this generalized parameter, some generalized geometric aggregation operators for TSFSs are proposed. Then these operators are extended for group-generalized parameter. By using proposed operators, an algorithm is developed for the MADM problem. To check the validity of proposed operators, a numerical example is also investigated. In a comparative analysis, it is discussed that, under some conditions, the proposed work can be reduced to other fuzzy structures. An example is also solved in which it is shown that our proposed technique is superior to the existing technique.

1. Introduction

The concept of fuzzy set (FS) was introduced by Zadeh [1] which tells the membership grade (MG) of an object. FS plays an important role in solving problems in imprecise and uncertain environment. A generalization of the FS called an intuitionistic fuzzy set (IFS) was introduced by Atanassov [2], which tells the MG and nonmembership grade (NMG) of an object with a restriction that the sum of MG and NMG mu$[0, 1]$st belong to . IFS fails when the sum of MG and NMG exceeds 1. To overcome this issue, an extension of IFS was introduced by Yager [3] called Pythagorean fuzzy set (PyFS). In PyFS, the condition was relaxed to that the sum of squares of MG and NMG must belong to $[0, 1]$.

IFS and PyFS fail when the third degree of abstinence is involved. To deal with this type of data, the idea of picture fuzzy set (PFS) was given by Cuong [4]. In PFS, there are four grades known as MG, abstinence, NMG, and refusal. PFS has a restriction that the sum of MG, abstinence, and NMG belongs to [0, 1]. PFS fails when their sum exceeds 1. To overcome this issue, Mahmood et al. [5] proposed the notions of spherical fuzzy set (SFS) and TSFS. SFS has a restriction that the square sum of MG, abstinence, and NMG must belong to $[0, 1]$, and in TSFS, the experts have the flexibility that the sum of any integral power of MG, abstinence, and NMG must belong to $[0, 1]$.

Many authors defined different aggregation operators for these tools of uncertainty. Xu [6] defined intuitionistic fuzzy (IF) averaging operators. Xu and Yager [7] defined IF geometric operators and applied them to solve the MADM problem. Liu and Chen [8] proposed Heronian operators for IFSs. Liu [9] proposed several intuitionistic fuzzy power Heronian operators. Hayat et al. [10] proposed some aggregation operators on group-based generalized intuitionistic fuzzy soft sets. Based on the conception of entropy, some IF power operators are proposed by Jiang et al. [11]. Some MADM problems were investigated using IFSs in [12–16]. Jana et al. [17] proposed Pythagorean fuzzy Dombi aggregation operators and investigated their usefulness in the MADM. Teng et al. [18] introduced some power Maclaurin symmetric mean aggregation operators for PyFS. Liu et al. [19] extended Bonferroni mean operators to study the MADM problem for PyFSs. Jana et al. [20] proposed some Dombi aggregation operators for q-rung orthopair fuzzy set and investigated the MADM problem. Joshi [21] proposed group-generalized averaging aggregation operators for PyFSs and solved the MADM problem. Some MADM problems were solved using PyFSs in [22–25].

Wei [26] proposed averaging and geometric aggregation operators for PFSs and studied their usefulness in MADM. Garg [27] investigated decision-making problem using averaging operators for PFSs. Jana et al. [28] investigated the MADM problem by utilizing picture fuzzy Dombi aggregation operators. Some MADM problems were investigated using PFSs in [29–31]. Zeng et al. [32] investigated the decision-making problem by utilizing the idea of spherical fuzzy covering-based rough set model. Jin et al. [33] introduced logarithmic aggregation operators for SFSs. Donyatalab et al. [34] proposed harmonic mean aggregation operators and investigated their applications in MADM. Munir et al. [35] studied the MADM problem using TSF Einstein operators. Guleria and Bajaj [36] studied the MADM problem using aggregation operators for T-spherical fuzzy soft sets. Gündoğdu and Kahraman [37] investigated the MADM problem for SF VIKOR method. More studies on MADM problems with complex fuzzy tools can be found in [38–40].

If a pharmacist suggests a medicine only on symptoms provided by the patient, then he may not be cured because a patient may have more than one disease due to which he may not be able to express the symptoms more clearly. For example, pain is the main symptom of a heart attack if a patient suffering from a congenital disease has a heart attack, then he is unable to express pain. If junior doctors give the treatment only on symptoms provided by a patient without consulting specialist/senior doctor, then the patient may lead to death. So, it is necessary to consult with some specialist/senior doctor for good treatment. Another example in which expert opinion is involved is the construction of a house/building. If labor constructs a house/building only following the instructions of the owner, then it may be beautiful but not durable. So, for making a house more durable, an opinion of the engineer is necessary. By keeping this type of problem in mind, some generalized and group-generalized geometric aggregation operators are proposed in which the opinion of an expert is also involved in decision making.

In this article, by utilizing the most generalized fuzzy structure called TSFS, some aggregation operators based on generalized and group-generalized parameters are proposed. In these aggregation operators, the decision makers have a huge space for assigning the values to membership, abstinence, and nonmembership grades. In these aggregation operators, the opinion of an expert is also involved due to which these aggregation operators are more reliable.

The purposes of writing this manuscript are as follows:

(i) To define generalized parameter (GP) for TSFSs

(ii) To propose generalized geometric aggregation operators for TSFSs

(iii) To propose group-generalized geometric aggregation operators for TSFSs

(iv) To develop an algorithm for solving MADM problem using proposed operators

(v) To discuss the advantages of proposed operators

The manuscript can be concluded as follows. Section 2 reviews some basic definitions. In Section 3, a GP is defined for TSFSs. In Section 4, some generalized geometric operators are proposed for TSFSs. In Section 5, some group-generalized geometric operators are proposed for TSFSs. In Section 6, an approach to solve the MADM problem is proposed. In Section 7, a comparative analysis is developed in which it is described that the newly defined operators can be reduced to other fuzzy structures by using some conditions. The whole article is concluded in Section 8.

2. Preliminaries

In this section, some basic notions will be discussed which help in further study.

Definition 1. (see [4]). For a nonempty set X, TSFS is

$$T = \{(x, h(x), o(x), s(x)\colon x \in X)\}, \tag{1}$$

where $h, o, s\colon X \longrightarrow [0,1]$ having a condition that $0 \le h^t(x) + o^t(x) + s^t(x) \le 1$ for any positive integer t and the refusal degree will be $r(x) = \sqrt[t]{1 - (h^t(x) + o^t(x) + s^t(x))}$.

Remark 1. Definition 1 can be reduced to SFSs, PFSs, PyFSs, IFSs, and FSs by using the following conditions:

(i) $t = 2$ reduced it to SFSs

(ii) $t = 1$ reduced it to PFSs

(iii) $t = 2$, $o = 0$ reduced it to PyFSs

(iv) $t = 1$, $o = 0$ reduced it to IFSs

(v) $t = 1$, $o = 0$, $s = 0$ reduced it to FSs

Definition 2. (see [4]). Consider any two TSFNs $T_1 = (h_1, o_1, s_1)$ and $T_2 = (h_2, o_2, s_2)$, then some operations on these will be defined as follows:

(i) $T_1 \oplus T_2 = (\sqrt[t]{1 - (1 - h_1^t)(1 - h_2^t)}, o_1 o_2, s_1 s_2)$

(ii) $T_1 \otimes T_2 = (h_1 h_2, \sqrt[t]{1 - (1 - o_1^t)(1 - o_2^t)}, \sqrt[t]{1 - (1 - s_1^t)(1 - s_2^t)})$

(iii) $\tau T_1 = (\sqrt[t]{1 - (1 - h_1^t)^\tau}, (o_1)^\tau, (s_1)^\tau)$, $\tau > 0$

(iv) $T_1^\tau = ((h_1)^\tau, \sqrt[t]{1 - (1 - o_1^t)^\tau}, \sqrt[t]{1 - (1 - s_1^t)^\tau})$, $\tau > 0$

Definition 3. (see [4]). For any collection of TSFNs $T_j = (h_j, o_j, s_j)$ $(j = 1, 2, \ldots, m)$, TSFWG operator is defined as follows:

$$TSFWG_{\varpi}(T_1, T_2, \ldots, T_m) = \left(\prod_{j=1}^{m} (h_j)^{\varpi_j}, \sqrt[t]{1 - \prod_{j=1}^{m} \left(1 - o_j^t\right)^{\varpi_j}}, \sqrt[t]{1 - \prod_{j=1}^{m} \left(1 - s_j^t\right)^{\varpi_j}} \right), \tag{2}$$

where the weight vector $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)$ satisfies $\varpi \in [0,1]$ and $\sum_{j=1}^{m} \varpi_j = 1$.

Definition 4. The score and accuracy functions for any TSFN $T = (h, o, s)$ are defined as follows:

$$\begin{aligned} Sc(T) &= h^t - o^t - s^t, \\ Ac(T) &= h^t + o^t + s^t. \end{aligned} \tag{3}$$

(1) If $Sc(T_1) < Sc(T_2)$, then T_2 is greater than T_1

(2) If $Sc(T_1) = Sc(T_2)$, then we have to check accuracy, if, then T_2 is greater than T_1, and if again $Ac(T_1) = Ac(T_2)$, then both numbers will be equal

3. Generalized Parameter for T-Spherical Fuzzy Sets

In a medical diagnosis problem, a patient goes to a doctor and provides the symptoms based on his perception. If a disease is only diagnosed on symptoms provided by the patient, then he may not be cured, e.g., if a person who is also a patient of congenital disease (not feeling pain) has a stress. The preferences of the patient will be

$$\begin{aligned} T = \big(&(0.7, 0.2, 0.4)_{\text{Low energy}}, (0.8, 0.4, 0.3)_{\text{Upset stomach}}, \\ &(0.0, 0.1, 0.5)_{\text{Pains}}, (0.7, 0.3, 0.1)_{\text{Insomnia}} \big). \end{aligned} \tag{4}$$

Here, the patient gives 0 membership value to pains because he does not feel pain. If a doctor provides a treatment, then he may not be cured. So for a better treatment, it is required to get an opinion from an expert. To achieve this concept, generalized parameter is proposed.

Definition 5. For a nonempty set$X$, generalized T-spherical fuzzy set (GTSFS) is defined as follows:

$$T = \left\{ \left((x, h(x), o(x), s(x)) \left(h_g, o_g, s_g\right) \right) : x \in X \right\}, \tag{5}$$

where $h, o, s\colon X \longrightarrow [0,1]$ having a condition that $0 \leq h^t(x) + o^t(x) + s^t(x) \leq 1$ for any positive integer t and $h_g, o_g, s_g \in [0,1]$ denote the expert opinion with a condition $0 \leq h_g^t + o_g^t + s_g^t \leq 1$. (h_g, o_g, s_g) is called GP for TSFS.

4. Generalized T-Spherical Fuzzy Geometric Aggregation Operators

In this section, the generalized TSF weighted geometric (GTSFWG) operator, generalized TSF ordered weighted geometric (GTSFOWG) operator, and generalized TSF hybrid geometric (GTSFHG) operator are defined. Some basic results of these operators are also proved.

4.1. Generalized T-Spherical Fuzzy Weighted Geometric Operator

Definition 6. Considering the GP $T_g = (h_g, o_g, s_g)$ for the TSFNs $T_j = (h_j, o_j, s_j)\,(j = 1, 2, \ldots, m)$, then the GTSFWG operator is defined as

$$\begin{aligned} &GTSFWG\left(\langle T_1, T_2, \ldots, T_m\rangle, T_g\right) \\ &\quad = T_g \oplus TSFWG\langle T_1, T_2, \ldots, T_m\rangle. \end{aligned} \tag{6}$$

Theorem 1. *Considering a collection of TSFNs $T_j = (h_j, o_j, s_j)\,(j = 1, 2, \ldots, m)$ with GP $T_g = (h_g, o_g, s_g)$ having a weight vector $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)^T$ such that $\varpi \in [0,1]$ and $\sum_{j=1}^{m} \varpi_j = 1$, then the GTSFWG operator is given by*

$$\begin{aligned} GTSFWG\left(\langle T_1, T_2, \ldots, T_m\rangle, T_g\right) &= T_g \oplus \left(\otimes_{j=1}^{m} T_j^{\varpi_j} \right) \\ &= \left(\sqrt[t]{h_g^t + \left(1 - h_g^t\right) \prod_{j=1}^{m} \left(h_j^t\right)^{\varpi_j}}, o_g \cdot \sqrt[t]{1 - \prod_{j=1}^{m} \left(1 - o_j^t\right)^{\varpi_j}}, s_g \cdot \sqrt[t]{1 - \prod_{j=1}^{m} \left(1 - s_j^t\right)^{\varpi_j}} \right). \end{aligned} \tag{7}$$

Proof. By using mathematical induction, this proof can be done.

For $m = 2$,

$$\begin{aligned} GTSFWG\left(\langle T_1, T_2\rangle, T_g\right) &= T_g \oplus \left(T_1^{\varpi_1} \otimes T_2^{\varpi_2}\right) \\ &= \left(h_g, o_g, s_g\right) \oplus \left(\left(h_1^{\varpi_1}, \sqrt[t]{1 - \left(1 - o_1^t\right)^{\varpi_1}}, \sqrt[t]{1 - \left(1 - s_1^t\right)^{\varpi_1}} \right) \otimes \left(h_2^{\varpi_2}, \sqrt[t]{1 - \left(1 - o_2^t\right)^{\varpi_2}}, \sqrt[t]{1 - \left(1 - s_2^t\right)^{\varpi_2}} \right) \right) \end{aligned}$$

$$=\left(h_g,o_g,s_g\right)\oplus\left(h_1^{\varpi_1}h_2^{\varpi_2},\sqrt[t]{1-\left(1-o_1^t\right)^{\varpi_1}\left(1-o_2^t\right)^{\varpi_2}},\sqrt[t]{1-\left(1-s_1^t\right)^{\varpi_1}\left(1-s_2^t\right)^{\varpi_2}}\right)$$

$$=\left(\sqrt[t]{h_g^t+\left(h_1^t\right)^{\varpi_1}\left(h_2^t\right)^{\varpi_2}-h_g^t\left(h_1^t\right)^{\varpi_1}\left(h_2^t\right)^{\varpi_2}},o_g.\sqrt[t]{1-\left(1-o_1^t\right)^{\varpi_1}\left(1-o_2^t\right)^{\varpi_2}},s_g.\sqrt[t]{1-\left(1-s_1^t\right)^{\varpi_1}\left(1-s_2^t\right)^{\varpi_2}}\right)$$

$$=\left(\sqrt[t]{h_g^t+\left(1-h_g^t\right)\left(h_1^t\right)^{\varpi_1}\left(h_2^t\right)^{\varpi_2}},o_g.\sqrt[t]{1-\left(1-o_1^t\right)^{\varpi_1}\left(1-o_2^t\right)^{\varpi_2}},s_g.\sqrt[t]{1-\left(1-s_1^t\right)^{\varpi_1}\left(1-s_2^t\right)^{\varpi_2}}\right). \tag{8}$$

This shows that results hold for $m=2$. Let us consider that the result is true for $m=l$,

$$GTSFWG\left(\langle T_1,T_2,\ldots,T\rangle_l,T_g\right)=\left(\sqrt[t]{h_g^t+\left(1-h_g^t\right)\prod_{j=1}^{l}\left(h_j^t\right)^{\varpi_j}},o_g.\sqrt[t]{1-\prod_{j=1}^{l}\left(1-o_j^t\right)^{\varpi_j}},s_g.\sqrt[t]{1-\prod_{j=1}^{l}\left(1-s_j^t\right)^{\varpi_j}}\right). \tag{9}$$

Now

$$\begin{aligned}&GTSFWG\left(\langle T_1,T_2,\ldots,T_l\rangle,T_{l+1},T_g\right)\\&=\left(\sqrt[t]{h_g^t+\left(1-h_g^t\right)\left(h_{l+1}^t\right)^{\varpi_{k+1}}\prod_{j=1}^{l}\left(h_j^t\right)^{\varpi_j}},o_g.\sqrt[t]{1-\left(1-o_{l+1}^t\right)^{\varpi_{k+1}}\prod_{j=1}^{l}\left(1-o_j^t\right)^{\varpi_j}},s_g.\sqrt[t]{1-\left(1-s_{l+1}^t\right)^{\varpi_{k+1}}\prod_{j=1}^{l}\left(1-s_j^t\right)^{\varpi_j}}\right),\\&GTSFWG\left(\langle T_1,T_2,\ldots,T_{l+1}\rangle,T_g\right),\\&=\left(\sqrt[t]{h_g^t+\left(1-h_g^t\right)\prod_{j=1}^{l+1}\left(h_j^t\right)^{\varpi_j}},o_g.\sqrt[t]{1-\prod_{j=1}^{l+1}\left(1-o_j^t\right)^{\varpi_j}},s_g.\sqrt[t]{1-\prod_{j=1}^{l+1}\left(1-s_j^t\right)^{\varpi_j}}\right).\end{aligned} \tag{10}$$

□

Theorem 2. *Considering a collection of TSFNs $T_j=(h_j,o_j,s_j)(j=1,2,\ldots,m)$ with GP $T_g=(h_g,o_g,s_g)$ having a weight vector $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_m)^T$ such that $\varpi\in[0,1]$ and $\sum_{j=1}^{m}\varpi_j=1$, then the following properties hold:*

(i) If $T_j=T_0$ for all $(j=1,2,\ldots,m)$, then $GTSFWG(\langle T_1,T_2,\ldots,T_m\rangle,T_g)=T_g\oplus T_0$

(ii) If $T_j^L=(\min h_{T_g\oplus T_j},\min o_{T_g\oplus T_j},\max s_{T_g\oplus T_j})$ and $T_j^U=(\max h_{T_g\oplus T_j},\max o_{T_g\oplus T_j},\min s_{T_g\oplus T_j})$, then $T_j^L\le GTSFWG(\langle T_1,T_2,\ldots,T_m\rangle,T_g)\le T_j^U$

(iii) Considering a collection of TSFNs $T'_j=(h'_j,o'_j,s'_j)(j=1,2,\ldots,m)$ such that $h_j\le h'_j$, $o_j\le o'_j$ and $s_j\ge s'_j$ for all j, then $GTSFWG(\langle T_1,T_2,\ldots,T_m\rangle,T_g)\le GTSFWG(\langle T'_1,T'_2,\ldots,T'_m\rangle,T_g)$

Proof.

(i) If $T_j=T_0=(h_0,o_0,s_0)$ for all $(j=1,2,\ldots,m)$, then from the definition of GTSFWG operator

$$\begin{aligned}GTSFWG\left(T_1,T_2,\ldots,T_m,T_g\right)&=\left(\sqrt[t]{h_g^t+\left(1-h_g^t\right)\prod_{j=1}^{m}\left(h_j^t\right)^{\varpi_j}},o_g.\sqrt[t]{1-\prod_{j=1}^{m}\left(1-o_g^t\right)^{\varpi_j}},s_g.\sqrt[t]{1-\prod_{j=1}^{m}\left(1-s_g^t\right)^{\varpi_j}}\right)\\&=\left(\sqrt[t]{h_g^t+\left(1-h_g^t\right)\left(h_j^t\right)^{\sum_{j=1}^{m}\varpi_j}},o_g.\sqrt[t]{1-\left(1-o_j^t\right)^{\sum_{j=1}^{m}\varpi_j}},s_g.\sqrt[t]{1-\left(1-s_j^t\right)^{\sum_{j=1}^{m}\varpi_j}}\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(\sqrt[t]{h_g^t + \left(1 - h_g^t\right) h_0^t}, o_g . \sqrt[t]{1 - \left(1 - o_0^t\right)}, s_g . \sqrt[t]{1 - \left(1 - s_0^t\right)} \right) \\
&= \left(\sqrt[t]{h_g^t + h_0^t - h_0^t h_g^t}, o_g . o_0, s_g . s_0 \right).
\end{aligned} \tag{11}
$$

(ii) Consider $T_j^L = (\min h_{T_g \oplus T_j}, \min o_{T_g \oplus T_j}, \max s_{T_g \oplus T_j})$ and $T_j^U = (\max h_{T_g \oplus T_j}, \max o_{T_g \oplus T_j}, \min s_{T_g \oplus T_j})$ where $\min h_{T_g \oplus T_j} = \sqrt[t]{h_g^t + (1 - h_g^t)(\min h_j)^t}$, $\max h_{T_g \oplus T_j} = \sqrt[t]{h_g^t + (1 - h_g^t)(\max h_j)^t}$, $\min o_{T_g \oplus T_j} = o_g . \min o_j$, $\max o_{T_g \oplus T_j} = o_g . \max o_j$, $\min s_{T_g \oplus T_j} = s_g . \min s_j$, and $\max s_{T_g \oplus T_j} = s_g . \max s_j$. Then, for every $j = 1, 2, \ldots, m$

$$
\begin{aligned}
&\min h_j \le h_j \le \max h_j, \\
&\left(\min h_j\right)^t \le \prod_{j=1}^{m} \left(h_j^t\right)^{\varpi_j} \le \left(\max h_j\right)^t.
\end{aligned} \tag{12}
$$

For every $0 \le h_g \le 1$

$$
\begin{aligned}
&\left(1 - h_g^t\right)\left(\min h_j\right)^t \le \left(1 - h_g^t\right) \prod_{j=1}^{m} \left(h_j^t\right)^{\varpi_j} \le \left(1 - h_g^t\right)\left(\max h_j\right)^t, \\
&h_g^t + \left(1 - h_g^t\right)\left(\min h_j\right)^t \le h_g^t + \left(1 - h_g^t\right) \prod_{j=1}^{m} \left(h_j^t\right)^{\varpi_j} \le h_g^t + \left(1 - h_g^t\right)\left(\max h_j\right)^t, \\
&\sqrt[t]{h_g^t + \left(1 - h_g^t\right)\left(\min h_j\right)^t} \le \sqrt[t]{h_g^t + \left(1 - h_g^t\right) \prod_{j=1}^{m} \left(h_j^t\right)^{\varpi_j}} \le \sqrt[t]{h_g^t + \left(1 - h_g^t\right)\left(\max h_j\right)^t}, \\
&\min h_{T_g \oplus T_j} \le \sqrt[t]{h_g^t + \left(1 - h_g^t\right) \prod_{j=1}^{m} \left(h_j^t\right)^{\varpi_j}} \le \max h_{T_g \oplus T_j}.
\end{aligned} \tag{13}
$$

Furthermore, $\min o_j \le o_j \le \max o_j$

$$
\begin{aligned}
&\min o_j^t \le o_j^t \le \max o_j^t, \\
&1 - \max o_j^t \le 1 - o_j^t \le 1 - \min o_j^t, \\
&\prod_{j=1}^{m} \left(1 - \max o_j^t\right)^{\varpi_j} \le \prod_{j=1}^{m} \left(1 - o_j^t\right)^{\varpi_j} \le \prod_{j=1}^{m} \left(1 - \min o_j^t\right)^{\varpi_j}, \\
&\left(1 - \max o_j^t\right)^{\sum_{j=1}^{m} \varpi_j} \le \prod_{j=1}^{m} \left(1 - o_j^t\right)^{\varpi_j} \le \left(1 - \min o_j^t\right)^{\sum_{j=1}^{m} \varpi_j}, \\
&1 - \left(1 - \min o_j^t\right) \le 1 - \prod_{j=1}^{m} \left(1 - o_j^t\right)^{\varpi_j} \le 1 - \left(1 - \max o_j^t\right), \\
&\sqrt[t]{1 - \left(1 - \min o_j^t\right)} \le \sqrt[t]{1 - \prod_{j=1}^{m} \left(1 - o_j^t\right)^{\varpi_j}} \le \sqrt[t]{1 - \left(1 - \max o_j^t\right)}, \\
&\min o_j \le \sqrt[t]{1 - \prod_{j=1}^{m} \left(1 - o_j^t\right)^{\varpi_j}} \le \max o_j.
\end{aligned} \tag{14}
$$

For every $0 \leq o_g \leq 1$,

$$o_g\left(\min o_j\right) \leq o_g \cdot \sqrt[t]{1-\prod_{j=1}^{m}\left(1-o_j^t\right)^{\varpi_j}} \leq o_g\left(\max o_j\right),$$
$$\min o_{T_g \oplus T_j} \leq o_g \cdot \sqrt[t]{1-\prod_{j=1}^{m}\left(1-o_j^t\right)^{\varpi_j}} \leq \max o_{T_g \oplus T_j}. \tag{15}$$

Now, $\min s_j \leq s_j \leq \max s_j$

$$\begin{gathered}
\min s_j^t \leq s_j^t \leq \max s_j^t,\\
1-\max s_j^t \leq 1-s_j^t \leq 1-\min s_j^t,\\
\prod_{j=1}^{m}\left(1-\max s_j^t\right)^{\varpi_j} \leq \prod_{j=1}^{m}\left(1-s_j^t\right)^{\varpi_j} \leq \prod_{j=1}^{m}\left(1-\min s_j^t\right)^{\varpi_j},\\
\left(1-\max s_j^t\right)^{\sum_{j=1}^{m}\varpi_j} \leq \prod_{j=1}^{m}\left(1-s_j^t\right)^{\varpi_j} \leq \left(1-\min s_j^t\right)^{\sum_{j=1}^{m}\varpi_j},\\
1-\left(1-\min s_j^t\right) \leq 1-\prod_{j=1}^{m}\left(1-s_j^t\right)^{\varpi_j} \leq 1-\left(1-\max s_j^t\right),\\
\sqrt[t]{1-\left(1-\min s_j^t\right)} \leq \sqrt[t]{1-\prod_{j=1}^{m}\left(1-s_j^t\right)^{\varpi_j}} \leq \sqrt[t]{1-\left(1-\max s_j^t\right)},\\
\min s_j \leq \sqrt[t]{1-\prod_{j=1}^{m}\left(1-s_j^t\right)^{\varpi_j}} \leq \max s_j.
\end{gathered} \tag{16}$$

For every $0 \leq s_g \leq 1$,

$$s_g\left(\min s_j\right) \leq s_g \cdot \sqrt[t]{1-\prod_{j=1}^{m}\left(1-s_j^t\right)^{\varpi_j}} \leq s_g\left(\max s_j\right),$$
$$\min s_{T_g \oplus T_j} \leq s_g \cdot \sqrt[t]{1-\prod_{j=1}^{m}\left(1-s_j^t\right)^{\varpi_j}} \leq \max s_{T_g \oplus T_j}. \tag{17}$$

(iii) This can be proved by following part (ii). □

4.2. Generalized T-Spherical Fuzzy Ordered Weighted Geometric Operator

Definition 7. Considering the GP $T_g = (h_g, o_g, s_g)$ for the TSFNs $T_j = (h_j, o_j, s_j)\,(j = 1, 2, \ldots, m)$, then the GTSFOWG operator is defined as

$$GTSFOWG\left(\langle T_1, T_2, \ldots, T_m\rangle, T_g\right) = T_g \oplus TSFOWG\langle T_1, T_2, \ldots, T_m\rangle. \tag{18}$$

Theorem 3. *Considering a collection of TSFNs $T_j = (h_j, o_j, s_j)\,(j = 1, 2, \ldots, m)$ with GP $T_g = (h_g, o_g, s_g)$ having a associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ such that $\varpi \in [0,1]$ and $\sum_{j=1}^{m} \varpi_j = 1$, then the GTSFOWG operator is given by*

$$GTSFOWG\left(\langle T_1, T_2, \ldots, T_m\rangle, T_g\right) = T_g \oplus \left(\otimes_{j=1}^{m} T_{\varsigma(j)}^{\omega_j} \right)$$
$$= \left(\sqrt[t]{h_g^t + \left(1 - h_g^t\right) \prod_{j=1}^{m} \left(h_{\varsigma(j)}^t\right)^{\omega_j}}, o_g.\sqrt[t]{1 - \prod_{j=1}^{m} \left(1 - o_{\varsigma(j)}^t\right)^{\omega_j}}, s_g.\sqrt[t]{1 - \prod_{j=1}^{m} \left(1 - s_{\varsigma(j)}^t\right)^{\omega_j}} \right). \tag{19}$$

Theorem 4. *Considering a collection of TSFNs $T_j = (h_j, o_j, s_j)\,(j = 1, 2, \ldots, m)$ with GP $T_g = (h_g, o_g, s_g)$ having a associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ such that $\varpi \in [0,1]$ and $\sum_{j=1}^{m} \varpi_j = 1$, then the following properties hold:*

(i) If $T_j = T_0$ for all $(j = 1, 2, \ldots, m)$, then $GTSFOWG(\langle T_1, T_2, \ldots, T_m\rangle, T_g) = T_g \oplus T_0$

(ii) If $T_j^L = (\min h_{T_g \oplus T_j}, \min o_{T_g \oplus T_j}, \max s_{T_g \oplus T_j})$ and $T_j^U = (\max h_{T_g \oplus T_j}, \max o_{T_g \oplus T_j}, \min s_{T_g \oplus T_j})$, then $T_j^L \leq GTSFOWG(\langle T_1, T_2, \ldots, T_m\rangle, T_g) \leq T_j^U$

(iii) Considering a collection of TSFNs $T_j' = (h_j', o_j', s_j')\,(j = 1, 2, \ldots, m)$ such that $h_j \leq h_j'$, $o_j \leq o_j'$ and $s_j \geq s_j'$ for all j, then $GTSFOWG(\langle T_1, T_2, \ldots, T_m\rangle, T_g) \leq GTSFOWG(\langle T_1', T_2', \ldots, T_m'\rangle, T_g)$

Proof. The proof is as in Theorem 2. □

4.3. Generalized T-Spherical Fuzzy Hybrid Weighted Geometric Operator. In this section, the GTSFHG operator which weights both TSFNs and their ordered positions is proposed. Some of its basic properties are also proved.

Definition 8. Considering the GP $T_g = (h_g, o_g, s_g)$ for the TSFNs $T_j = (h_j, o_j, s_j)\,(j = 1, 2, \ldots, m)$, then the GTSFHG operator is defined as

$$GTSFHG\left(\langle T_1, T_2, \ldots, T_m\rangle, T_g\right) = T_g \oplus TSFHG\langle T_1, T_2, \ldots, T_m\rangle. \tag{20}$$

Theorem 5. *Considering a collection of TSFNs $T_j = (h_j, o_j, s_j)\,(j = 1, 2, \ldots, m)$ with GP $T_g = (h_g, o_g, s_g)$ having a weight vector $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)^T$ and associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ such that $\varpi, \omega \in [0,1]$, $\sum_{j=1}^{m} \varpi_j = 1$ and $\sum_{j=1}^{m} \omega_j = 1$, then the GTSFHG operator is given by*

$$GTSFHG\left(T_1, T_2, \ldots, T_m, T_g\right) = T_g \oplus \left(\otimes_{j=1}^{m} \widetilde{T}_{\varsigma(j)}^{\omega_j} \right)$$
$$= \left(\sqrt[t]{h_g^t + \left(1 - h_g^t\right) \prod_{j=1}^{m} \left(\widetilde{h}_{\varsigma(j)}^t\right)^{\omega_j}}, o_g.\sqrt[t]{1 - \prod_{j=1}^{m} \left(1 - \widetilde{o}_{\varsigma(j)}^t\right)^{\omega_j}}, s_g.\sqrt[t]{1 - \prod_{j=1}^{m} \left(1 - \widetilde{s}_{\varsigma(j)}^t\right)^{\omega_j}} \right), \tag{21}$$

where $\widetilde{T}_{\varsigma(j)} = T_j^{m\varpi_j}$ $(j = 1, \ldots, m)$ is the permutation and m is the balancing coefficient.

Theorem 6. *Considering a collection of TSFNs $T_j = (h_j, o_j, s_j)\,(j = 1, 2, \ldots, m)$ with GP $T_g = (h_g, o_g, s_g)$ having a weight vector $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)^T$ and associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ such that $\varpi, \omega \in [0,1]$, and $\sum_{j=1}^{m} \omega_j = 1$, then the following properties hold:*

(i) If $T_j = T_0$ for all $(j = 1, 2, \ldots, m)$, then $GTSFHG(\langle T_1, T_2, \ldots, T_m\rangle, T_g) = T_g \oplus T_0$

(ii) If $T_j^L = (\min h_{T_g \oplus T_j}, \min o_{T_g \oplus T_j}, \max s_{T_g \oplus T_j})$ and $T_j^U = (\max h_{T_g \oplus T_j}, \max o_{T_g \oplus T_j}, \min s_{T_g \oplus T_j})$, then $T_j^L \leq GTSFHG(\langle T_1, T_2, \ldots, T_m\rangle, T_g) \leq T_j^U$

(iii) Considering a collection of TSFNs $T_j' = (h_j', o_j', s_j')\,(j = 1, 2, \ldots, m)$ such that $h_j \leq h_j'$, $o_j \leq o_j'$ and $s_j \geq s_j'$ for all j, then $GTSFHG(\langle T_1, T_2, \ldots, T_m\rangle, T_g) \leq GTSFHG(\langle T_1', T_2', \ldots, T_m'\rangle, T_g)$

Proof. The proof is as in Theorem 2. □

5. Group-Generalized T-Spherical Fuzzy Geometric Aggregation Operators

In this section, the group-generalized TSF weighted geometric (GGTSFWG) operator, group-generalized TSF ordered weighted geometric (GGTSFOWG) operator, and group-generalized TSF hybrid geometric (GGTSFHG) operator are defined. Some basic results of these operators are also discussed.

5.1. Group-Generalized T-Spherical Fuzzy Weighted Geometric Operator

Definition 9. Considering the $T_{g_k} = (h_{g_k}, o_{g_k}, s_{g_k})(k = 1, \ldots, n)$ be the expert preferences for the TSFNs $T_j = (h_j, o_j, s_j)(j = 1, 2, \ldots, m)$, then the GGTSFWG operator is defined as

$$GGTSFWG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) = TSFWG\langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle \oplus TSFWG\langle T_1, T_2, \ldots, T_m\rangle. \tag{22}$$

Theorem 7. *Considering a $T_{g_k} = (h_{g_k}, o_{g_k}, s_{g_k})$ $(k = 1, 2, \ldots, n)$ with a weight vector $\varpi' = (\varpi'_1, \varpi'_2, \ldots, \varpi'_n)^T$ be the expert preferences for TSFNs $T_j = (h_j, o_j, s_j)(j = 1, 2, \ldots, m)$ having a weight vector $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)^T$, then the GGTSFWG operator is given by*

$$\begin{aligned}
&GGTSFWG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) = \left(\otimes_{k=1}^{n} T_{g_k}^{\varpi'_k}\right) \oplus \left(\otimes_{j=1}^{m} T_j^{\varpi_j}\right) \\
&= \left(\sqrt[t]{\prod_{k=1}^{n}\left(h_{g_k}^t\right)^{\varpi'_k} + \prod_{j=1}^{m}\left(h_j^t\right)^{\varpi_j} - \prod_{k=1}^{n}\left(h_{g_k}^t\right)^{\varpi'_k} \cdot \prod_{j=1}^{m}\left(h_j^t\right)^{\varpi_j}}, \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - o_{g_k}^t\right)^{\varpi'_k} \cdot \sqrt[t]{1 - \prod_{j=1}^{m}\left(1 - o_j^t\right)^{\varpi_j}}}, \sqrt[t]{1 - \prod_{j=1}^{m}\left(1 - s_j^t\right)^{\varpi'_k} \cdot \sqrt[t]{1 - \prod_{j=1}^{m}\left(1 - s_j^t\right)^{\varpi_j}}}\right).
\end{aligned} \tag{23}$$

Proof. By using mathematical induction, this proof can be done.

For $m = 2$,

$$\begin{aligned}
GGTSFWG\left(\langle T_1, T_2\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) &= \left(\otimes_{k=1}^{n} T_{g_k}^{\varpi'_k}\right) \oplus \left(T_1^{\varpi_1} \otimes T_2^{\varpi_2}\right) \\
&= \left(\prod_{k=1}^{n}\left(h_{g_k}\right)^{\varpi'_k}, \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - o_{g_k}^t\right)^{\varpi'_k}}, \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - s_{g_k}^t\right)^{\varpi'_k}}\right) \\
&\quad \oplus \left(\left(h_1^{\varpi_1}, \sqrt[t]{1 - \left(1 - o_1^t\right)^{\varpi_1}}, \sqrt[t]{1 - \left(1 - s_1^t\right)^{\varpi_1}}\right) \otimes \left(h_2^{\varpi_2}, \sqrt[t]{1 - \left(1 - o_2^t\right)^{\varpi_2}}, \sqrt[t]{1 - \left(1 - s_2^t\right)^{\varpi_2}}\right)\right) \\
&= \left(\prod_{k=1}^{n}\left(h_{g_k}\right)^{\varpi'_k}, \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - o_{g_k}^t\right)^{\varpi'_k}}, \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - s_{g_k}^t\right)^{\varpi'_k}}\right) \\
&\quad \oplus \left(h_1^{\varpi_1} h_2^{\varpi_2}, \sqrt[t]{1 - \left(1 - o_1^t\right)^{\varpi_1}\left(1 - o_2^t\right)^{\varpi_2}}, \sqrt[t]{1 - \left(1 - s_1^t\right)^{\varpi_1}\left(1 - s_2^t\right)^{\varpi_2}}\right) \\
&= \left(\sqrt[t]{\prod_{k=1}^{n}\left(h_{g_k}^t\right)^{\varpi'_k} + \left(h_1^t\right)^{\varpi_1}\left(h_2^t\right)^{\varpi_2} - \prod_{k=1}^{n}\left(h_{g_k}^t\right)^{\varpi'_k}\left(h_1^t\right)^{\varpi_1}\left(h_2^t\right)^{\varpi_2}}, \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - o_{g_k}^t\right)^{\varpi'_k}}\right. \\
&\quad \left. \cdot \sqrt[t]{1 - \left(1 - o_1^t\right)^{\varpi_1}\left(1 - o_2^t\right)^{\varpi_2}}, \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - s_{g_k}^t\right)^{\varpi'_k}} \cdot \sqrt[t]{1 - \left(1 - s_1^t\right)^{\varpi_1}\left(1 - s_2^t\right)^{\varpi_2}}\right).
\end{aligned} \tag{24}$$

This shows that results hold for $m = 2$. Let us consider that result is true for $m = l$,

$$
\begin{aligned}
&GGTSFWG\left(\langle T_1, T_2, \ldots, T_l\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) \\
&= \left(\sqrt[t]{\prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi_k'} + \prod_{j=1}^{l}\left(h_j^{t}\right)^{\varpi_j} - \prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi_k'} \cdot \prod_{j=1}^{l}\left(h_j^{t}\right)^{\varpi_j}}, \sqrt[t]{1-\prod_{k=1}^{n}\left(1-o_{g_k}^{t}\right)^{\varpi_k'}} \cdot \sqrt[t]{1-\prod_{j=1}^{l}\left(1-o_j^{t}\right)^{\varpi_j}}, \right. \\
&\left. \sqrt[t]{1-\prod_{k=1}^{n}\left(1-s_{g_k}^{t}\right)^{\varpi_k'}} \cdot \sqrt[t]{1-\prod_{j=1}^{l}\left(1-s_j^{t}\right)^{\varpi_j}} \right).
\end{aligned}
\tag{25}
$$

Now

$$
\begin{aligned}
&GGTSFWG\left(\langle T_1, T_2, \ldots, T_l, T_{l+1}\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) \\
&= \left(\sqrt[t]{1-\prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi_k'} + \left(h_{l+1}^{t}\right)^{\varpi_{l+1}} \prod_{j=1}^{l}}\left(h_{l+1}^{t}\right)^{\varpi_j} - \prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi_k'} \cdot \left(h_{l+1}^{t}\right)^{\varpi_{l+i}} \prod_{j=1}^{l}\left(h_{l+1}^{t}\right)^{\varpi_j} \sqrt[t]{\prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi_k'} + \left(h_{l+1}^{t}\right)^{\varpi_{l+i}} \prod_{j=1}^{l}\left(h_{l+1}^{t}\right)^{\varpi_j}} \right. \\
&- \prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi_k'} \cdot \left(h_{l+1}^{t}\right)^{\varpi_{l+i}} \prod_{j=1}^{l}\left(h_{l+1}^{t}\right)^{\varpi_j}, \sqrt[t]{1-\prod_{k=1}^{n}\left(1-o_{g_k}^{t}\right)^{\varpi_k'}} \cdot \sqrt[t]{1-\left(1-o_{l+1}^{t}\right)^{\varpi_{l+1}} \prod_{j=1}^{l}}\left(1-o_j^{t}\right), \sqrt[t]{1-\prod_{k=1}^{n}\left(1-s_{g_k}^{t}\right)^{\varpi_k'}} \\
&\left. \cdot \sqrt[t]{1-\left(1-s_{l+1}^{t}\right)^{\varpi_{l+1}} \prod_{j=1}^{l}\left(1-s_j^{t}\right)^{\varpi_j}} \right), \\
&GGTSFWG\left(\langle T_1, T_2, \ldots, T_l, T_{l+1}\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) \\
&= \left(\sqrt[t]{\prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi_k'} + \prod_{j=1}^{l+1}\left(h_j^{t}\right)^{\varpi_j} - \prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi_k'} \cdot \prod_{j=1}^{l+1}\left(h_j^{t}\right)^{\varpi_j}}, \sqrt[t]{1-\prod_{k=1}^{n}\left(1-o_{g_k}^{t}\right)^{\varpi_k'}} \right. \\
&\left. \cdot \sqrt[t]{1-\prod_{j=1}^{l+1}\left(1-o_j^{t}\right)^{\varpi_j}}, \sqrt[t]{1-\prod_{k=1}^{n}\left(1-s_{g_k}^{t}\right)^{\varpi_k'}} \cdot \sqrt[t]{1-\prod_{j=1}^{l+1}\left(1-s_j^{t}\right)^{\varpi_j}} \right).
\end{aligned}
\tag{26}
$$

Thus, results hold for all m. □

Theorem 8. *Consider a collection of TSFNs $T_j = (h_j, o_j, s_j)\,(j = 1, 2, \ldots, m)$ and the expert preferences $T_{g_k} = (h_{g_k}, o_{g_k}, s_{g_k})\,(k = 1, 2, \ldots, n)$ having a weight vectors $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)^T$ and $\varpi' = (\varpi_1', \varpi_2', \ldots, \varpi_n')^T$, respectively. Then, the following properties hold:*

(i) If $T_j = T_0$ for all $(j = 1, 2, \ldots, m)$ and $T_{g_k} = T_{g_0}$ for all $(k = 1, 2, \ldots, n)$, then $GGTSFWG(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle) = T_{g_0} \oplus T_0$

(ii) If $T_j^L = (\min h_{T_{g_k} \oplus T_j}, \min o_{T_{g_k} \oplus T_j}, \max s_{T_{g_k} \oplus T_j})$ and $T_j^U = (\max h_{T_{g_k} \oplus T_j}, \max o_{T_{g_k} \oplus T_j}, \min s_{T_{g_k} \oplus T_j})$, then

$$
T_j^L \leq GGTSFWG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) \leq T_j^U. \tag{27}
$$

(iii) Considering a collection of TSFNs $T_j' = (h_j', o_j', s_j')\,(j = 1, 2, \ldots, m)$ such that $h_j \leq h_j'$, $o_j \leq o_j'$ and $s_j \geq s_j'$ for all j, then

$$
GGTSFWG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) \leq GGTSFWG\left(\langle T_1', T_2', \ldots, T_m'\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right). \tag{28}
$$

Proof. The proof is as in Theorem 2. □

5.2. Group-Generalized T-Spherical Fuzzy Ordered Weighted Geometric Operator

Definition 10. Considering the $T_{g_k} = (h_{g_k}, o_{g_k}, s_{g_k})(k = 1, \ldots, n)$ be the expert preferences for the TSFNs $T_j = (h_j, o_j, s_j)(j = 1, 2, \ldots, m)$, then the GGTSFOWG operator is defined as

$$GGTSFOWG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) = TSFWG\langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle \oplus TSFOWG\langle T_1, T_2, \ldots, T_m\rangle. \tag{29}$$

Theorem 9. *Considering a $T_{g_k} = (h_{g_k}, o_{g_k}, s_{g_k})(k = 1, 2, \ldots, n)$ with weight vector $\varpi' = (\varpi_1', \varpi_2', \ldots, \varpi_n')^T$ be the expert preferences for TSFNs $T_j = (h_j, o_j, s_j)(j = 1, 2, \ldots, m)$ having associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$, then the GGTSFOWG operator is given by*

$$\begin{aligned} &GGTSFOWG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) = \left(\otimes_{k=1}^{n} T_{g_k}^{\varpi_k'}\right) \oplus \left(\otimes_{j=1}^{m} T_{\varsigma(j)}^{\omega_j}\right) \\ &= \left(\sqrt[t]{\prod_{k=1}^{n}\left(h_{g_k}^t\right)^{\varpi_k'} + \prod_{j=1}^{m}\left(h_{\varsigma(j)}^t\right)^{\omega_j} - \prod_{k=1}^{n}\left(h_{g_k}^t\right)^{\varpi_k'} . \prod_{j=1}^{m}\left(h_{\varsigma(j)}^t\right)^{\omega_j}} \cdot \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - o_{g_k}^t\right)^{\varpi_k'}} . \sqrt[t]{1 - \prod_{j=1}^{m}\left(1 - o_{\varsigma(j)}^t\right)^{\omega_j}}, \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - s_{g_k}^t\right)^{\varpi_k'}} . \sqrt[t]{1 - \prod_{j=1}^{m}\left(1 - s_{\varsigma(j)}^t\right)^{\omega_j}} \right). \end{aligned} \tag{30}$$

Theorem 10. *Considering a collection of TSFNs $T_j = (h_j, o_j, s_j)(j = 1, 2, \ldots, m)$ and the expert preferences $T_{g_k} = (h_{g_k}, o_{g_k}, s_{g_k})(k = 1, 2, \ldots, n)$ having associated weight vectors $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$ and weight vector $\varpi' = (\varpi_1', \varpi_2', \ldots, \varpi_n')^T$, respectively, with a condition that each weight vector must belong to $[0, 1]$ and the sum of all weights must be equal to 1, then the following properties hold:*

(i) If $T_j = T_0$ for all $(j = 1, 2, \ldots, m)$ and $T_{g_k} = T_{g_0}$ for all $(k = 1, 2, \ldots, n)$, then $GGTSFOWG(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle) = T_{g_0} \oplus T_0$

(ii) If $T_j^L = (\min h_{T_{g_k} \oplus T_j}, \min o_{T_{g_k} \oplus T_j}, \max s_{T_{g_k} \oplus T_j})$ and $T_j^U = (\max h_{T_{g_k} \oplus T_j}, \max o_{T_{g_k} \oplus T_j}, \min s_{T_{g_k} \oplus T_j})$, then

$$T_j^L \leq GGTSFOWG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) \leq T_j^U. \tag{31}$$

(iii) Considering a collection of TSFNs $T_j' = (h_j', o_j', s_j')(j = 1, 2, \ldots, m)$ such that $h_j \leq h_j'$, $o_j \leq o_j'$ and $s_j \geq s_j'$ for all j, then

$$GGTSFOWG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) \leq GGTSFOWG\left(\langle T_1', T_2', \ldots, T_m'\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right). \tag{32}$$

Proof. The proof is as in Theorem 2. □

5.3. Group-Generalized T-Spherical Fuzzy Hybrid Weighted Geometric Operator

In this section, the GGTSFHG operator which weights both TSFNs and their ordered positions is proposed. Some of its basic properties are also discussed.

Definition 11. Considering the $T_{g_k} = (h_{g_k}, o_{g_k}, s_{g_k})(k = 1, \ldots, n)$ be the expert preferences for the TSFNs $T_j = (h_j, o_j, s_j)(j = 1, 2, \ldots, m)$, then the GGTSFHG operator is defined as

$$GGTSFHG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) = TSFWG\langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle \oplus TSFHG\langle T_1, T_2, \ldots, T_m\rangle. \tag{33}$$

Theorem 11. *Considering a $T_{g_k} = (h_{g_k}, o_{g_k}, s_{g_k})$ $(k = 1, 2, \ldots, n)$ with weight vector $\varpi' = (\varpi_1', \varpi_2', \ldots, \varpi_n')^T$ be the expert preferences for TSFNs $T_j = (h_j, o_j, s_j)(j = 1, 2, \ldots, m)$ having a weight vector $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)^T$ and associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$, GGTSFHG operator is given by*

$$
\begin{aligned}
GGTSFHG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) &= \left(\otimes_{k=1}^{n} T_{g_k}^{\varpi'_k}\right) \oplus \left(\otimes_{j=1}^{m} \widetilde{T}_{\varsigma(j)}^{\omega_j}\right) \\
&= \left(\sqrt[t]{\prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi'_k} + \prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}^{t}\right)^{\omega_j} - \prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi'_k} \cdot \prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}^{t}\right)^{\omega_j}}, \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - o_{g_k}^{t}\right)^{\varpi'_k}} \cdot \sqrt[t]{1 - \prod_{j=1}^{m}\left(1 - \widetilde{o}_{\varsigma(j)}^{t}\right)^{\omega_j}} \right. \\
&\quad \left. \cdot \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - s_{g_k}^{t}\right)^{\varpi'_k}} \cdot \sqrt[t]{1 - \prod_{j=1}^{m}\left(1 - \widetilde{s}_{\varsigma(j)}^{t}\right)^{\omega_j}} \right),
\end{aligned}
\tag{34}
$$

where $\widetilde{T}_{\varsigma(j)} = T_j^{m\varpi_j}$ $(j = 1, \ldots, m)$ is the permutation and m is the balancing coefficient.

Theorem 12. *Considering a $T_{g_k} = (h_{g_k}, o_{g_k}, s_{g_k})(k = 1, 2, \ldots, n)$ with weight vector $\varpi' = (\varpi'_1, \varpi'_2, \ldots, \varpi'_n)^T$ be the expert preferences for TSFNs $T_j = (h_j, o_j, s_j)(j = 1, 2, \ldots, m)$ having a weight vector $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)^T$ and associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)^T$, then the following properties hold:*

(i) If $T_j = T_0$ for all $(j = 1, 2, \ldots, m)$ and $T_{g_k} = T_{g_0}$ for all $(k = 1, 2, \ldots, n)$, then GGTSFHG $(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle) = T_{g_0} \oplus \widetilde{T}_0$

(ii) If $T_j^L = (\min h_{T_{g_k} \oplus T_j}, \min o_{T_{g_k} \oplus T_j}, \max s_{T_{g_k} \oplus T_j})$ and $T_j^U = (\max h_{T_{g_k} \oplus T_j}, \max o_{T_{g_k} \oplus T_j}, \min s_{T_{g_k} \oplus T_j})$, then

$$
T_j^L \leq GGTSFHG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) \leq T_j^U. \tag{35}
$$

(iii) Considering a collection of TSFNs $T'_j = (h'_j, o'_j, s'_j)(j = 1, 2, \ldots, m)$ such that $h_j \leq h'_j$, $o_j \leq o'_j$ and $s_j \geq s'_j$ for all j, then

$$
GGTSFHG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) \leq GGTSFHG\left(\langle T'_1, T'_2, \ldots, T'_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right). \tag{36}
$$

Proof. The proof is as in Theorem 2. □

6. Approach to MADM Problem Using Proposed Operators

Let $\{T_1, T_2, \ldots, T_l\}$ be the set of alternatives and $\{G_1, G_2, \ldots, G_m\}$ be the set of attributes with a weight vector $\varpi = (\varpi_1, \varpi_2, \ldots, \varpi_m)^T$ satisfying $\varpi_j \in [0, 1]$ and $\sum_{j=1}^{m} \varpi_j = 1$. A group of experts $\{T_{g_1}, T_{g_2}, \ldots, T_{g_n}\}$ with a weight vector $\varpi' = (\varpi'_1, \varpi'_2, \ldots, \varpi'_m)^T$ satisfying $\varpi'_k \in [0, 1]$ and $\sum_{k=1}^{n} \varpi'_k = 1$ evaluates each alternative against each attribute. Each expert rates alternatives in the form of TSFN. Then, an algorithm for solving the MADM problem is proposed as follows:

Step 1: the expert evaluates the alternatives by considering the attributes in terms of TSFNs and summarizes them in the decision matrix as

$$
T = \begin{pmatrix} (h_{11}, o_{11}, s_{11}) & \cdots & (h_{1m}, o_{1m}, s_{1m}) \\ \vdots & \ddots & \vdots \\ (h_{l1}, o_{l1}, s_{l1}) & \cdots & (h_{lm}, o_{lm}, s_{lm}) \end{pmatrix}. \tag{37}
$$

Step 2: convert cost type data into benefit type data and normalize the decision matrix by using

$$
\tau_m = \begin{cases} (h_{lm}, o_{lm}, s_{lm}), & \text{if attribute is of benefit type,} \\ (s_{lm}, o_{lm}, h_{lm}), & \text{if attribute is of cost type.} \end{cases} \tag{38}
$$

Step 3: calculate t for which the given information lies in TSF environment.

Step 4: aggregate the given information using GGTSFWG (GTSFWG) operator with weight vector ϖ and ϖ'.

Step 5: order the aggregated values in descending order with respect to score function.

Step 6: aggregate the ordered information using GGTSFHG (GTSFHG) operator with associated weight vector $\omega = (\omega_1, \omega_2, \ldots, \omega_m)$.

Step 7: find out the best option using score function.

Example 1. A construction company wants to construct new apartments. The company wants to select a place from the set $\{T_1, T_2, T_3, T_4\}$ on the basis of following attributes $\{G_1, G_2, G_3, G_4\}$, where G_1: cost of land, G_2: surroundings, G_3: technological, G_4: rental value, with a weight vector $(0.2, 0.1, 0.3, 0.4)^T$. An expert evaluates all given alternatives on the basis of given attributes as given in Table 1.

The normalized decision matrix is shown in Table 2.

A group of senior experts $\{T_{g_1}, T_{g_2}, T_{g_3}\}$ with weight vector $(0.3, 0.3, 0.4)^T$ assesses the alternatives listed in Table 3.

Table 4 is obtained by combining Tables 2 and 3.

As $0.83 + 0.44 + 0.35 = 1.62 \notin [0,1]$, $0.83^2 + 0.44^2 + 0.35^2 = 1.005 \notin [0,1]$, $0.83^3 + 0.44^3 + 0.35^3 = 0.6999 \in [0,1]$. Similarly, all values in Table 3 belong to $[0,1]$ for $t = 3$.

After aggregating the values of Table 4 by utilizing GGTSFWG operators, the results will be as shown in Table 5.

The corresponding scores of aggregated values of Table 5 are as shown in Table 6.

In Table 7, the aggregated values are ordered on the basis of descending order of score function.

Aggregated values of Table 7 by utilizing the GGTSFHG operators will be as follows:

$$\begin{aligned}
\widetilde{T}_{\varsigma(1)} &= (0.7821, 0.1739, 0.2230),\\
\widetilde{T}_{\varsigma(2)} &= (0.6903, 0.0892, 0.1369),\\
\widetilde{T}_{\varsigma(3)} &= (0.5944, 0.0584, 0.2448),\\
\widetilde{T}_{\varsigma(4)} &= (0.8594, 0.1008, 0.1131).
\end{aligned} \tag{39}$$

The score values of these aggregated values are

$$\begin{aligned}
SC\left(\widetilde{T}_{\varsigma(1)}\right) &= 0.4620,\\
SC\left(\widetilde{T}_{\varsigma(2)}\right) &= 0.3257,\\
SC\left(\widetilde{T}_{\varsigma(3)}\right) &= 0.1951,\\
SC\left(\widetilde{T}_{\varsigma(4)}\right) &= 0.6322.
\end{aligned} \tag{40}$$

The score value of T_4 is highest. So T_4 is the best option for a company to construct new apartments.

7. Comparative Analysis

In this section, some conditions are studied under which the defined work can be reduced to other fuzzy structures, and the significance of proposed operators is proved by solving an example of the existing literature by using proposed operators.

Consider

$$\begin{aligned}
&GGTSFHG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right)\\
&= \left(\sqrt[t]{\prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi'_k} + \prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}^{t}\right)^{\omega_j} - \prod_{k=1}^{n}\left(h_{g_k}^{t}\right)^{\varpi'_k} \cdot \prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}^{t}\right)^{\omega_j}},\right.\\
&\left. \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - o_{g_k}^{t}\right)^{\varpi'_k}} \cdot \sqrt[t]{1 - \prod_{j=1}^{m}\left(1 - \widetilde{o}_{\varsigma(j)}^{t}\right)^{\omega_j}}, \sqrt[t]{1 - \prod_{k=1}^{n}\left(1 - s_{g_k}^{t}\right)^{\varpi'_k}} \cdot \sqrt[t]{1 - \prod_{j=1}^{m}\left(1 - \widetilde{s}_{\varsigma(j)}^{t}\right)^{\omega_j}} \right).
\end{aligned} \tag{41}$$

(i) For $t = 2$, (1) can be reduced to group-generalized spherical fuzzy hybrid geometric (GGSFHG) operators

$$\begin{aligned}
&GGSFHG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right)\\
&= \left(\prod_{k=1}^{n}\left(h_{g_k}^{2}\right)^{\varpi'_k} + \prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}^{2}\right)^{\omega_j} - \prod_{k=1}^{n}\left(h_{g_k}^{2}\right)^{\varpi'_k} \cdot \prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}^{2}\right)^{\omega_j},\right.\\
&\left. \left(1 - \prod_{k=1}^{n}\left(1 - o_{g_k}^{2}\right)^{\varpi'_k}\right) \cdot \left(1 - \prod_{j=1}^{m}\left(1 - \widetilde{o}_{\varsigma(j)}^{2}\right)^{\omega_j}\right), \left(1 - \prod_{k=1}^{n}\left(1 - s_{g_k}^{2}\right)^{\varpi'_k}\right) \cdot \left(1 - \prod_{j=1}^{m}\left(1 - \widetilde{s}_{\varsigma(j)}^{2}\right)^{\omega_j}\right) \right).
\end{aligned} \tag{42}$$

(ii) For $t = 1$, (1) can be reduced to group-generalized picture fuzzy hybrid geometric (GGPFHG) operators

Table 1: Information given by an expert in TSFSs.

	$\mathbf{G}_1$	$\mathbf{G}_2$	$\mathbf{G}_3$	$\mathbf{G}_4$
$\mathbf{T}_1$	(0.43, 0.20, 0.61)	(0.54, 0.35, 0.63)	(0.81, 0.62, 0.11)	(0.18, 0.33, 0.66)
$\mathbf{T}_2$	(0.14, 0.32, 0.74)	(0.26, 0.17, 0.26)	(0.77, 0.23, 0.55)	(0.61, 0.34, 0.57)
$\mathbf{T}_3$	(0.75, 0.12, 0.41)	(0.59, 0.29, 0.13)	(0.56, 0.22, 0.36)	(0.11, 0.14, 0.45)
$\mathbf{T}_4$	(0.35, 0.44, 0.83)	(0.91, 0.12, 0.49)	(0.63, 0.11, 0.27)	(0.31, 0.36, 0.84)

Table 2: Normalized decision matrix.

	$\mathbf{G}_1$	$\mathbf{G}_2$	$\mathbf{G}_3$	$\mathbf{G}_4$
$\mathbf{T}_1$	(0.61, 0.20, 0.43)	(0.54, 0.35, 0.63)	(0.81, 0.62, 0.11)	(0.66, 0.33, 0.18)
$\mathbf{T}_2$	(0.74, 0.32, 0.14)	(0.26, 0.17, 0.26)	(0.77, 0.23, 0.55)	(0.57, 0.34, 0.61)
$\mathbf{T}_3$	(0.41, 0.12, 0.75)	(0.59, 0.29, 0.13)	(0.56, 0.22, 0.36)	(0.45, 0.14, 0.11)
$\mathbf{T}_4$	(0.83, 0.44, 0.35)	(0.91, 0.12, 0.49)	(0.63, 0.11, 0.27)	(0.84, 0.36, 0.31)

$$GGPFHG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) = \left(\prod_{k=1}^{n}\left(h_{g_k}\right)^{\varpi'_k} + \prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}\right)^{\omega_j} - \prod_{k=1}^{n}\left(h_{g_k}\right)^{\varpi'_k}.\prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}\right)^{\omega_j}, \left(1 - \prod_{k=1}^{n}\left(1 - o_{g_k}\right)^{\varpi'_k}\right).\left(1 - \prod_{j=1}^{m}\left(1 - \widetilde{o}_{\varsigma(j)}\right)^{\omega_j}\right), \left(1 - \prod_{k=1}^{n}\left(1 - s_{g_k}\right)^{\varpi'_k}\right).\left(1 - \prod_{j=1}^{m}\left(1 - \widetilde{s}_{\varsigma(j)}\right)^{\omega_j}\right)\right). \tag{43}$$

(iii) For $t = 2$ and $o = 0$, (1) can be reduced to group-generalized Pythagorean fuzzy hybrid geometric (GGPyFHG) operators

$$GGPyFHG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) = \left(\sqrt{\prod_{k=1}^{n}\left(h_{g_k}^2\right)^{\varpi'_k} + \prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}^2\right)^{\omega_j} - \prod_{k=1}^{n}\left(h_{g_k}^2\right)^{\varpi'_k}.\prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}^2\right)^{\omega_j}}, \sqrt{1 - \prod_{k=1}^{n}\left(1 - s_{g_k}^2\right)^{\varpi'_k}}.\sqrt{1 - \prod_{j=1}^{m}\left(1 - \widetilde{s}_{\varsigma(j)}^2\right)^{\omega_j}}\right). \tag{44}$$

(iv) For $t = 1$ and $o = 0$, (1) can be reduced to group-generalized intuitionistic fuzzy hybrid geometric (GGIFHG) operators

$$GGIFHG\left(\langle T_1, T_2, \ldots, T_m\rangle, \langle T_{g_1}, T_{g_2}, \ldots, T_{g_n}\rangle\right) = \left(\prod_{k=1}^{n}\left(h_{g_k}\right)^{\varpi'_k} + \prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}\right)^{\omega_j} - \prod_{k=1}^{n}\left(h_{g_k}\right)^{\varpi'_k}.\prod_{j=1}^{m}\left(\widetilde{h}_{\varsigma(j)}\right)^{\omega_j}, \left(1 - \prod_{k=1}^{n}\left(1 - s_{g_k}\right)^{\varpi'_k}\right).\left(1 - \prod_{j=1}^{m}\left(1 - \widetilde{s}_{\varsigma(j)}\right)^{\omega_j}\right)\right). \tag{45}$$

Similarly, all other defined operators can be reduced to other fuzzy structures by using these conditions.

Example 2. Consider$T_g = (0.4, 0.7)$ be the GP of $T_1 = (0.4, 0.5)$, $T_2 = (0.6, 0.6)$, $T_3 = (0.8, 0.3)$ and $T_4 = (0.7, 0.6)$ have a weight vector $\varpi = (0.4, 0.3, 0.1, 0.2)^T$. Then find the aggregated value by using GTSFWG operator.

Solution. The given information can be written in TSF environment as $T_g = (0.4, 0.0, 0.7)$, $T_1 = (0.4, 0.0, 0.5)$, $T_2 = (0.6, 0.0, 0.6)$, $T_3 = (0.8, 0.0, 0.3)$, and $T_4 = (0.7, 0.0, 0.6)$.

As$0.4 + 0.0 + 0.7 = 1.1 \notin [0, 1]$, $0.6^2 + 0.0^2 + 0.5^2 = 0.65 \in [0, 1]$. Similarly, all values lie in TSF environment for$t = 2$.

Table 3: Information given by senior experts in TSFSs.

	$\mathbf{T}_{g_1}$	$\mathbf{T}_{g_2}$	$\mathbf{T}_{g_3}$
$\mathbf{T}_1$	$(0.71, 0.30, 0.40)$	$(0.58, 0.21, 0.79)$	$(0.49, 0.52, 0.43)$
$\mathbf{T}_2$	$(0.74, 0.41, 0.25)$	$(0.34, 0.24, 0.23)$	$(0.44, 0.19, 0.28)$
$\mathbf{T}_3$	$(0.32, 0.29, 0.69)$	$(0.67, 0.35, 0.21)$	$(0.56, 0.22, 0.36)$
$\mathbf{T}_4$	$(0.78, 0.46, 0.39)$	$(0.87, 0.13, 0.17)$	$(0.53, 0.21, 0.37)$

Table 4: Combination of Tables 1 and 2.

	$\mathbf{G}_1$	$\mathbf{G}_2$	$\mathbf{G}_3$	$\mathbf{G}_4$	$\mathbf{T}_{g_1}$	$\mathbf{T}_{g_2}$	$\mathbf{T}_{g_3}$
$\mathbf{T}_1$	$(0.61, 0.2, 0.43)$	$(0.54, 0.35, 0.63)$	$(0.81, 0.62, 0.11)$	$(0.66, 0.33, 0.18)$	$(0.71, 0.30, 0.40)$	$(0.58, 0.21, 0.79)$	$(0.49, 0.52, 0.43)$
$\mathbf{T}_2$	$(0.74, 0.32, 0.14)$	$(0.26, 0.17, 0.26)$	$(0.77, 0.23, 0.55)$	$(0.57, 0.34, 0.61)$	$(0.74, 0.41, 0.25)$	$(0.34, 0.24, 0.23)$	$(0.44, 0.19, 0.28)$
$\mathbf{T}_3$	$(0.41, 0.12, 0.75)$	$(0.59, 0.29, 0.13)$	$(0.56, 0.22, 0.36)$	$(0.45, 0.14, 0.11)$	$(0.32, 0.29, 0.69)$	$(0.67, 0.35, 0.21)$	$(0.56, 0.22, 0.36)$
$\mathbf{T}_4$	$(0.83, 0.44, 0.35)$	$(0.91, 0.12, 0.49)$	$(0.63, 0.11, 0.27)$	$(0.84, 0.36, 0.31)$	$(0.78, 0.46, 0.39)$	$(0.87, 0.13, 0.17)$	$(0.53, 0.21, 0.37)$

Table 5: Aggregated values by utilizing the GGTSFWG operator.

	$\mathbf{G}_1$	$\mathbf{G}_2$	$\mathbf{G}_3$	$\mathbf{G}_4$	$\mathbf{T}_{g_1}$	$\mathbf{T}_{g_2}$	$\mathbf{T}_{g_3}$
$\mathbf{T}_1$	$(0.6734, 0.1857, 0.4003)$	$(0.7816, 0.2509, 0.4773)$	$(0.7766, 0.6532, 0.1169)$	$(0.5144, 0.3846, 0.2104)$	$(0.7347, 0.2898, 0.3866)$	$(0.6125, 0.2028, 0.7705)$	$(0.4248, 0.5499, 0.4557)$
$\mathbf{T}_2$	$(0.7859, 0.2974, 0.1300)$	$(0.5834, 0.1253, 0.1919)$	$(0.7308, 0.2443, 0.5811)$	$(0.4068, 0.3961, 0.6963)$	$(0.7626, 0.3963, 0.2414)$	$(0.3787, 0.2318, 0.2221)$	$(0.3734, 0.2019, 0.2973)$
$\mathbf{T}_3$	$(0.4900, 0.1114, 0.7080)$	$(0.8097, 0.2142, 0.0958)$	$(0.4987, 0.2337, 0.3820)$	$(0.2787, 0.1637, 0.1286)$	$(0.3586, 0.2801, 0.6703)$	$(0.6974, 0.3382, 0.2028)$	$(0.4987, 0.2337, 0.3820)$
$\mathbf{T}_4$	$(0.8615, 0.4097, 0.3254)$	$(0.9630, 0.0884, 0.3655)$	$(0.5744, 0.1169, 0.2867)$	$(0.7566, 0.4191, 0.3615)$	$(0.7996, 0.4449, 0.3769)$	$(0.8822, 0.1255, 0.1642)$	$(0.4668, 0.2231, 0.3925)$

Table 6: Score values.

	$\mathbf{G}_1$	$\mathbf{G}_2$	$\mathbf{G}_3$	$\mathbf{G}_4$	$\mathbf{T}_{g_1}$	$\mathbf{T}_{g_2}$	$\mathbf{T}_{g_3}$
$\mathbf{T}_1$	0.2348	0.3513	0.1880	0.0699	0.3145	−0.2360	−0.1842
$\mathbf{T}_2$	0.4570	0.1896	0.1795	−0.3324	0.3672	0.0309	0.0175
$\mathbf{T}_3$	−0.2386	0.5202	0.0555	0.0151	−0.2771	0.2922	0.0555
$\mathbf{T}_4$	0.5362	0.8435	0.1643	0.3122	0.3697	0.6802	0.0301

Table 7: Ordered aggregated values.

	$\mathbf{G}_1$	$\mathbf{G}_2$	$\mathbf{G}_3$	$\mathbf{G}_4$	$\mathbf{T}_{g_1}$	$\mathbf{T}_{g_2}$	$\mathbf{T}_{g_3}$
$\mathbf{T}_{\varsigma(1)}$	(0.7816, 0.2509, 0.4773)	(0.6734, 0.1857, 0.4003)	(0.7766, 0.6532, 0.1169)	(0.5144, 0.3846, 0.2104)	(0.7347, 0.2898, 0.3866)	(0.4248, 0.5499, 0.4557)	(0.6125, 0.2028, 0.7705)
$\mathbf{T}_{\varsigma(2)}$	(0.7859, 0.2974, 0.1300)	(0.5834, 0.1253, 0.1919)	(0.7308, 0.2443, 0.5811)	(0.4068, 0.3961, 0.6963)	(0.7626, 0.3963, 0.2414)	(0.3787, 0.2318, 0.2221)	(0.3734, 0.2019, 0.2973)
$\mathbf{T}_{\varsigma(3)}$	(0.8097, 0.2142, 0.0958)	(0.4987, 0.2337, 0.3820)	(0.2787, 0.1637, 0.1286)	(0.4900, 0.1114, 0.7080)	(0.6974, 0.3382, 0.2028)	(0.4987, 0.2337, 0.3820)	(0.3586, 0.2801, 0.6703)
$\mathbf{T}_{\varsigma(4)}$	(0.9630, 0.0884, 0.3655)	(0.8615, 0.4097, 0.3254)	(0.7566, 0.4191, 0.3615)	(0.5744, 0.1169, 0.2867)	(0.8822, 0.1255, 0.1642)	(0.7996, 0.4449, 0.3769)	(0.4668, 0.2231, 0.3925)

$$
\begin{aligned}
&\sqrt{h_g^2+\left(1-h_g^2\right)\prod_{j=1}^{4}\left(h_j^2\right)^{\varpi_j}} = \sqrt{0.4^2+\left(1-0.4^2\right)\left(0.4^2\right)^{0.4}\left(0.6^2\right)^{0.3}\left(0.8^2\right)^{0.1}\left(0.7^2\right)^{0.2}} = 0.6374,\\
&o_g.\sqrt[t]{1-\prod_{j=1}^{4}\left(1-o_g^2\right)^{\varpi_j}} = 0.0,\\
&s_g.\sqrt[t]{1-\prod_{j=1}^{4}\left(1-s_g^2\right)^{\varpi_j}} = 0.7\sqrt{1-\left(1-0.5^2\right)^{0.4}\left(1-0.6^2\right)^{0.3}\left(1-0.3^2\right)^{0.1}\left(1-0.6^2\right)^{0.2}} = 0.7393.
\end{aligned}
\tag{46}
$$

Now we have

$$GTSFWG\left(\langle T_1, T_2, \ldots, T_m\rangle, T_g\right) = (0.6374, 0.0, 0.7393). \tag{47}$$

8. Conclusion

In this manuscript, it is pointed out that existing geometric aggregation operators fail when the opinion of a senior expert is also involved with moderator's opinion because all decision makers are not much familiar with alternatives that is why an opinion of expert is necessary. In it, a generalized parameter is defined for TSFSs. Then, by using this, generalized TSF geometric operators are proposed. Then, these operators are extended to group-generalized TSF geometric operators which deal with a group of experts' opinion. Then, an algorithm is developed to solve MADM problem. The validity of defined operators is checked by a numerical example. A comparative analysis is also constructed in which the defined operators are reduced to other fuzzy structures such as SFSs, PFSs, PyFSs, and IFSs by using some conditions. An example is also solved by using proposed operators in which information is given in the form of PyFS. In future, it would be interesting to extend the concept generalized and group-generalized parameter to other aggregation operators and other structures like soft sets.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Types of Complex Fuzzy Relations with Applications in Future Commission Market

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Types of Complex Fuzzy Relations with Applications in Future Commission Market

Madad Khan,[1] Muhammad Zeeshan ,[1] Seok-Zun Song ,[2] and Sohail Iqbal[3]

[1]*Department of Mathematics, COMSATS University Islamabad, Abbottabad Campus, Islamabad, Pakistan*
[2]*Department of Mathematics, Jeju National University, Jeju 63243, Republic of Korea*
[3]*Department of Mathematics, COMSATS University Islamabad, Islamabad Campus, Islamabad, Pakistan*

Correspondence should be addressed to Seok-Zun Song; szsong@jejunu.ac.kr

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In this paper, we introduce types of relations on complex fuzzy sets such as the complex fuzzy (CF) inverse relation, complex fuzzy reflexive relation, complex fuzzy symmetric relation, complex fuzzy antisymmetric relation, complex fuzzy transitive relation, complex fuzzy irreflexive relation, complex fuzzy asymmetric relation, complex fuzzy equivalence relation, and complex fuzzy-order relation. We study some basic results and particular examples of these relations. Moreover, we discuss the applications of complex fuzzy relations in Future Commission Market (FCM). We show that the introduction of CF relations to applications of FCMs can give a significant method for describing the temporal dependence between parameters of a Future Commission Market.

1. Introduction

Models reflecting the phenomena of real life with just choices of truth and falsehood are not enough to reflect the true reality of the problems. The explanation for this is that the models have many complications, which is why a framework needs to be built to deal with the models' ill-defined situations. There are now two ways to deal with these kinds of situations, one is to find the problems' numerical solutions and the other is to create a numerical model. We get numerical solutions to the problems in both cases. The second is about the fuzzy set theory, which includes the theory of probability, the theory of fuzzy soft sets, the theory of intuitionist fuzzy sets, and most specifically, the theory of neutrosophic sets. The later theory for dealing with problems involving complexities is more generalized. One of the acceptable examples of these theories is the fuzzy differential equations theory, which is more generalized than the differential equations to solve problems of everyday life with greater precision.

Zadeh [1] gave the description of a fuzzy set (FS) in 1965, which is similar to a probability function. For models of real-world problems in different branches of science, a fuzzy set plays a vital role. Fuzzy set theory has many applications in operational research, decision making, medicine, engineering design, psychology, quantum physics, image processing, mathematical chemistry, biological classification, thermodynamics, economics, and nonequilibrium. Dubois et al. in [2] discussed the applications of fuzzy sets in approximate reasoning and information systems, pattern recognition and image processing, decision analysis, operation research and statistics, and modeling and control of systems. Ngan et al. in [3] provided two numerical examples of applying the complex t-norm and t-conorm to multi-criteria decision making in the context of medicine-related problems using medical datasets. Nisren et al. introduced the concept of complex multifuzzy soft expert set (CMFSES) and discussed the application of a complex multifuzzy expert soft set in decision-making problems [4]. Poodeh studied and evaluated a randomized-learning approach to train this neurofuzzy system and proposed a machine-learning algorithm, which is designed for fast training of a compact, accurate forecasting model [5]. Singh in [6] introduced a method to provide an effective way to analyze the uncertainty and vagueness in a complex (or dynamic) dataset using a complex vague concept lattice. Xindong et al. in [7] discussed the relationship between the distance measure, the similarity measure, the entropy, and the inclusion measure for Pythagorean fuzzy sets. They showed the efficiency of the proposed similarity measure in pattern recognition, clustering analysis, and medical diagnosis. Moreover, Xindong et al. in [8] studied deeper insights into the decision-making problem based on the interval-valued fuzzy soft set. Xindong et al. provided two novel algorithms in decision-making problems under a Pythagorean fuzzy environment [9]. Naz et al. developed a new decision-making approach based on graph theory to deal with the multiattribute decision-making problems. They utilized the numerical examples concerning the energy project selection and software evaluation to show the detailed implementation procedure and reliability of our method in solving multiattribute decision-making problems under hesitant fuzzy, interval-valued hesitant fuzzy, and a hesitant triangular fuzzy environment [10].

Ramot et al. in [11] first gave the concept of a complex fuzzy set (CFS). The generalization of a real number set introduced by Gauss in 1795 is the complex number set. Accordingly, a CFS is the extension of a fuzzy set, the range of which extends from a closed interval [0, 1] to a disc of radius one in a complex plane. The membership function of CFS C is denoted as $\lambda_C(u)$ and defined on the universal U as for any $u \in U$ a complex value in the disc of radius one in a complex plane. Thus, all values of $\lambda_C(u)$ exist inside a circle of radius one in a complex plane and $\lambda_C(u) = a_C(u)e^{ip_C(u)}$, where $i = \sqrt{-1}$. The term $p_C(u)$ is said to be phase term, $a_C(u)$ is said to be an amplitude term, and both of these are real valued with $a_C(u) \in [0, 1]$. The CFS C is represented as $\{(u, \lambda_C(u)) \mid u \in U\}$.

Imprecise, inconsistent, and incomplete knowledge of the periodic nature cannot be treated by fuzzy sets and intuitionistic fuzzy sets. These theories refer to various fields of research, but in both sets, there is one significant weakness, that is, a lack of capacity to discuss two-dimensional phenomena. Ramot presented a complex fuzzy set to address this challenge. The phase term of the CFS plays a crucial role in defining the functionality of the complex fuzzy set model. This term differentiates a model of the CFS from all other models available in the literature. The ability of a complex fuzzy set to depict two-dimensional phenomena makes it superior to the handling of vague and intuitive details prevalent in time-periodic phenomena. Complex fuzzy sets, their classes, and logic play an important role in applications including periodic event prediction and advanced control systems. A complex fuzzy set is somewhat similar to a Fourier transform; in reality, it is the particular form of the Fourier transformation by limiting the range of the Fourier transformation to a complex disc unit. Fourier transform has a lot of applications in various fields such as in signals and systems, communication, astronomy, geology, and optics. A complex fuzzy set can also be used in models such as the Fourier transform. Several other real-life phenomena are vague and cannot be modeled using one-dimensional variables. For example, objects can be represented as a collection of measurements in pattern recognition and are seen as vectors in a multidimensional space. These multidimensional variables cannot be expressed through a simple combination of variables, particularly the consideration of fuzzy sets. These types of sets can be expressed via complex classes. For periodic phenomena, a complex fuzzy set is very useful. Ramot et al. proposed that the intermittent problems or repeated-problem phenomenon be more precisely modeled using the phase term of the complex fuzzy set membership, such as describing the effect of two countries' financial measures on each other over time. He suggested that signal processing is yet another attractive area of operation for a complex fuzzy set. Xueling et al. in [12] proposed the model for identifying the reference signal out of largely interested signals by using complex fuzzy sets. In addition, it is used to convey solar activity (solar maximum and solar minimum) by means of the average sunspot number [11]. Dick suggested that one of the beneficial applications of complex fuzzy sets is to use it to represent relatively periodic behavioral phenomena [13]. Traffic congestions in a big city are aperiodic phenomena that never repeat themselves. Complex fuzzy logic can also be used to solve those forms of issues more easily and reliably than fuzzy logic. Akram et al. [14] introduced the concept of competition graphs under a complex fuzzy environment. They described an application in the ecosystem. Moreover, Akram et al. [15] discussed the complex Pythagorean Dombi fuzzy graph (CPDFG). They utilized CPDFAA and CPDFGA operators in solving a decision-making numerical example. Akram et al. [16] gave the notion of the complex Pythagorean fuzzy planar graph (CPFPG), and an extension of a Pythagorean fuzzy planar graph is presented to study the planarity. Moreover, Akram et al. proposed a new graph, a complex Pythagorean fuzzy competition graph by combining the complex Pythagorean fuzzy information with a competition graph. They also investigate the two extensions of complex Pythagorean fuzzy competition graphs, namely, complex Pythagorean fuzzy k-competition and complex Pythagorean fuzzy p-competition graphs [17].

The idea of relations is one of the most important notions in pure and applied science. Science has been defined as the discovery of the relation between events, objects, and states. Relations are associations that remain at the very core of the majority of science and engineering methodological approaches. Fuzzy relations in fuzzy theory are important concepts and have been commonly used in many fields, such as fuzzy control, fuzzy clustering, and uncertainty reasoning. In fuzzy diagnosis and fuzzy modeling, they also play a significant role. How to estimate and compare them is a significant issue when fuzzy relations are used in practice. Some researchers have carried out ambiguous measurements of fuzzy relations.

Klir studied the crisp relations in [18]. A crisp relation shows the existence or absence of association, interconnectedness, or interaction between the parameters. The relation between two sets is denoted by $R(M, N)$, and its membership function is represented by $\lambda_R(m, n)$, where $m \in M$ and $n \in N$. The membership function $\lambda_R(m, n)$ has two values either 1 or 0. The generalization of a crisp relation is the fuzzy relation. The fuzzy relation was discussed by Mendel in [19]. Fuzzy relations show a degree of the presence or absence of association, interaction, or interconnectedness between the elements of two or more fuzzy sets. A fuzzy relation between two fuzzy sets M and N is denoted by $R(M, N)$, and its membership function is represented by $\lambda_R(m, n)$, where $m \in M$ and $n \in N$. All the values of $\lambda_R(m, n) \in [0, 1]$. Fuzzy relations play a vital role in a fuzzy logic system. Triapathi et al. in [20] used the complex fuzzy relations to obtain diagnostic conclusions about diabetes by restricting grade values to symptoms of a disease from 0 to 1. Majid in [21] discussed some important compositions of fuzzy relations for predicting scores in cricket. Moreover, it studied the restoration and the identification of the causes (diagnosis) through the observed effects (symptoms) on the basis of fuzzy relations.

In crisp relations and fuzzy relations, there is one significant weakness, which is a lack of capacity to examine two-dimensional phenomena. Ramot discussed the complex fuzzy relations in [11], which is the generalization of crisp relations and fuzzy relations. Complex fuzzy relations represent both the degree of the presence or absence of association, interaction, or interconnectedness and the phase of association, interaction, or interconnectedness between the elements of two or more crisp sets. For any two crisp sets M and N, the complex fuzzy relation is denoted by $R(M, N)$. The relation $R(M, N)$ may be represented as the set of ordered pairs $R(M, N) = \{(m, n), \lambda_R(m, n)/(m, n)) \in tMn \times qN\}$. The membership function of a complex fuzzy relation is denoted by $\lambda_R(m, n)$, and all its values lie within the unit circle in the complex plane. Ramot et al. in [11] discussed the applications of complex fuzzy relations in Future Commission Merchant.

The purpose of this article is two-fold. The first half aims to present the theoretical foundations of the types of complex fuzzy relations. In any field of mathematics, we have needed such relations to solve problems. We can solve lots of problems with the help of these relations easily. The second half aims to present these theoretical foundations and key techniques in Future Commission Market, decision making, and the principle of the types of complex fuzzy relations in a coherent manner. The purpose of these innovative concepts is to provide a new approach with useful mathematical tools to address the fundamental problem of decision making (FCM problem). The generality of these new concepts is given special importance, illustrating how many interesting uncertainty problems can be formulated easily. These applied contexts provide solid evidence of the wide applications of the complex fuzzy relation approach to the model. This article will stimulate the interest in types of complex fuzzy relations and their application in various fields of science.

Now in this paper, we define some types of complex fuzzy relations such as the complex fuzzy inverse relation, complex fuzzy reflexive relation, complex fuzzy symmetric relation, complex fuzzy antisymmetric relation, complex fuzzy transitive relation, complex fuzzy irreflexive relation, complex fuzzy asymmetric relation, complex fuzzy equivalence relation, complex fuzzy-order relation, and complex fuzzy equivalence class and discuss the particular examples of these relations. We also study some basic results. Moreover, we discuss the applications of complex fuzzy relations in Future Commission Merchant.

2. Preliminaries

We will discuss here the types of complex fuzzy relations and also discuss particular examples of these relations.

Definition 1 (See [11]). *A CFS S, defined on a universal set U, is represented by a grade value $\lambda_C(u)$ whose codomain is the disc of the radius on in the complex plane. Mathematically, the grade value of CFS C can be represented by $\lambda_C(u) = a_C(u)e^{ip_C(u)}$ where $a_C(u)$ and $p_C(u)$ are known as an amplitude term and phase term, respectively. These two terms are real valued and $a_C(u) \in [0, 1]$.*

Mathematically, CFS can be expressed as a set of ordered pairs given by

$$C = \{(u; \lambda_C(u)) \colon u \in U\}. \tag{1}$$

Definition 2 (See [22]). *Let X_m, $m = 1, 2, 3, \ldots, M$ be M CFS defined on U and $\lambda_{C_m}(u) = a_{C_m}(u)e^{ip_{C_m}(u)}$ their membership functions. The complex fuzzy Cartesian product of C_m denoted by $C_1 \times C_2 \times C_3 \times \cdots \times C_m$ is specified by a function*

$$\begin{aligned}\lambda_{C_1\times C_2\times C_3\times\cdots\times C_m}(u) &= a_{C_1\times C_2\times C_3\times\cdots\times C_m}(u)e^{ip_{C_1\times C_2\times C_3\times\cdots\times C_m}(u)},\\ &= \min\left(a_{C_1}(u_1), a_{C_2}(u_2), \ldots, a_{C_m}\right.\\ &\quad \left.(u_m)\right)e^{i\min\left(p_{C_1}(u_1), p_{C_2}(u_2), \ldots, p_{C_m}(u_m)\right)}.\end{aligned} \tag{2}$$

Definition 3 (See [11]). *For any two crisps sets X and Y, the fuzzy relation $R(X, Y)$ is a fuzzy subset of the product space $X \times Y$. The grade value of the fuzzy relation is represented by $\lambda_{X\times Y}(x, y)$, where $x \in X$ and $y \in Y$. All the values of grade value lie in the closed interval $[0, 1]$.*

The fuzzy relation may be represented as the set of ordered pairs

$$R(X, Y) = \left\{\frac{(x, y), \lambda_{X\times Y}(x, y))}{(x, y)} \in X \times Y\right\}. \tag{3}$$

Definition 4 (See [11]). *For any two crisp sets X and Y, the complex fuzzy relation $R(X, Y)$ is a complex fuzzy subset of the product space $X \times Y$. The grade value of the complex fuzzy relation is represented by $\lambda_{X\times Y}(x, y)$, where $x \in X$ and $y \in Y$. All the values of the grade value of the complex fuzzy relation lie in unit disc in a complex plane.*

The complex fuzzy relation may be represented as the set of ordered pairs:

$$R(X, Y) = G = \left\{\frac{(x, y), \lambda_{X\times Y}(x, y))}{(x, y)} \in X \times Y\right\}, \tag{4}$$

where

$$\lambda_{X\times Y}(x,y) = \min\left(a_X(x), a_X(y)\right)\frac{e^{i\min\left(p_X(x),p_Y(y)\right)}}{(x,y)}. \quad (5)$$

Example 1. For any complex fuzzy set $C = (0.8e^{i\pi}/1 + 1e^{i2\pi})/2$, the product space $X \times X$ is

$$C\times C = \frac{0.8e^{i\pi}}{(1,1)} + \frac{0.8e^{i\pi}}{(1,2)} + \frac{0.8e^{i\pi}}{(2,1)} + \frac{1e^{i2\pi}}{(2,2)}. \quad (6)$$

So, the complex fuzzy relation on C is subset of $C\times C$ is given by

$$R(C,C) = G = \frac{0.8e^{i\pi}}{(1,1)} + \frac{0.8e^{i\pi}}{(1,2)} + \frac{1e^{i2\pi}}{(2,2)}. \quad (7)$$

3. Types of Complex Fuzzy Relations

Definition 5. If G is a complex fuzzy relation on a complex fuzzy set C, then the inverse relation G^{-1} in C is defined by

$$G^{-1} = \left\{\frac{\lambda_{C\times C}(v,u)}{\lambda_{C\times C}(u,v)} \in G\right\}. \quad (8)$$

Definition 6. If G is a complex fuzzy relation on a complex fuzzy set C, G is said to be complex fuzzy reflexive relation if for all $u \in U$ and $\lambda_C(u) \in C \Longrightarrow \lambda_{C\times C}(u,u) \in G$.

Example 2. For any complex fuzzy set $C = (0.4e^{i2\pi}/1) + (1e^{i(\pi/2)}/2) + (0.5e^{i\pi}/3)$ defined on any universal set $U = \{1,2,3\}$, the product space $C\times C$ is

$$C\times C = \frac{0.4e^{i2\pi}}{(1,1)} + \frac{0.4e^{i(\pi/2)}}{(1,2)} + \frac{0.4e^{i\pi}}{(1,3)} + \frac{0.4e^{i(\pi/2)}}{(2,1)} + \frac{1e^{i(\pi/2)}}{(2,2)} + \frac{0.5e^{i(\pi/2)}}{(2,3)} + \frac{0.4e^{i\pi}}{(3,1)} + \frac{0.5e^{i(\pi/2)}}{(3,2)} + \frac{0.5e^{i\pi}}{(3,3)}. \quad (9)$$

So, the complex fuzzy reflexive relation on C is subset of $C\times C$ is given by

$$G = \frac{0.4e^{i2\pi}}{(1,1)} + \frac{1e^{i(\pi/2)}}{(2,2)} + \frac{0.5e^{i\pi}}{(3,3)}. \quad (10)$$

Definition 7. If G is a complex fuzzy relation on a complex fuzzy set C, G is said to be a complex fuzzy symmetric relation if $\lambda_{C\times C}(u,v) \in G \Longrightarrow \lambda_{C\times C}(v,u) \in G$.

Example 3. For any complex fuzzy set $C = (0.4e^{i2\pi}/a) + (1e^{i(\pi/2)}/b) + (0.5e^{i\pi}/c)$ defined on any universal set $U = \{a,b,c\}$, the product space $C\times C$ is

$$C\times C = \frac{0.4e^{i2\pi}}{(a,a)} + \frac{0.4e^{i(\pi/2)}}{(a,b)} + \frac{0.4e^{i\pi}}{(a,c)} + \frac{0.4e^{i(\pi/2)}}{(b,a)} + \frac{1e^{i(\pi/2)}}{(b,b)} + \frac{0.5e^{i(\pi/2)}}{(b,c)} + \frac{0.4e^{i\pi}}{(c,a)} + \frac{0.5e^{i(\pi/2)}}{(c,b)} + \frac{0.5e^{i\pi}}{(c,c)}. \quad (11)$$

So, the complex fuzzy symmetric relation on C is given by

$$G = \frac{0.4e^{i2\pi}}{(a,a)} + \frac{1e^{i(\pi/2)}}{(b,b)} + \frac{0.5e^{i\pi}}{(c,c)} + \frac{0.4e^{i(\pi/2)}}{(a,b)} + \frac{0.4e^{i(\pi/2)}}{(b,a)}. \quad (12)$$

Definition 8. If G is a complex fuzzy relation on a complex fuzzy set C, G is said to be a complex fuzzy antisymmetric relation if $\lambda_{C\times C}(u,v) \in G$ and $\lambda_{C\times C}(v,u) \in G \Longrightarrow \lambda_{C\times C}(u,v) = \lambda_{C\times C}(v,u)$.

Example 4. For any complex fuzzy set $C = (0.5e^{i2\pi}/x) + (1e^{i(\pi/2)}/y) + (0e^{i\pi}/z)$ defined on any universal set $U = \{a,b,c\}$, the product space $C\times C$ is

$$C\times C = \frac{0.5e^{i2\pi}}{(x,x)} + \frac{0.5e^{i(\pi/2)}}{(x,y)} + \frac{0e^{i\pi}}{(x,z)} + \frac{0.5e^{i(\pi/2)}}{(y,x)} + \frac{1e^{i(\pi/2)}}{(y,y)} + \frac{0e^{i(\pi/2)}}{(y,z)} + \frac{0e^{i\pi}}{(z,x)} + \frac{0e^{i(\pi/2)}}{(z,y)} + \frac{0e^{i\pi}}{(z,z)}. \quad (13)$$

So, the complex fuzzy antisymmetric relation on C is given by

$$G = \frac{0.5e^{i2\pi}}{(x,x)} + \frac{1e^{i(\pi/2)}}{(y,y)} + \frac{0e^{i\pi}}{(z,z)}. \quad (14)$$

Definition 9. If G is a complex fuzzy relation on a complex fuzzy set C, G is said to be complex fuzzy transitive relation if $\lambda_{C\times C}(u,v) \in G$ and $\lambda_{C\times C}(v,w) \in G \Longrightarrow \lambda_{C\times C}(u,w) \in G$.

Example 5. For any complex fuzzy set $C = (0.5e^{i2\pi}/x) + (1e^{i(\pi/2)}/y) + (0e^{i\pi}/z)$ defined on any universal set $U = \{a,b,c\}$, the product space $C\times C$ is

$$C\times C = \frac{0.5e^{i2\pi}}{(x,x)} + \frac{0.5e^{i(\pi/2)}}{(x,y)} + \frac{0e^{i\pi}}{(x,z)} + \frac{0.5e^{i(\pi/2)}}{(y,x)} + \frac{1e^{i(\pi/2)}}{(y,y)} + \frac{0e^{i(\pi/2)}}{(y,z)} + \frac{0e^{i\pi}}{(z,x)} + \frac{0e^{i(\pi/2)}}{(z,y)} + \frac{0e^{i\pi}}{(z,z)}. \quad (15)$$

So, the complex fuzzy transitive relation on C is given by

$$G = \frac{0.5e^{i2\pi}}{(x,x)} + \frac{1e^{i(\pi/2)}}{(y,y)} + \frac{0e^{i\pi}}{(z,z)} + \frac{0.5e^{i(\pi/2)}}{(x,y)} + \frac{0e^{i\pi}}{(x,z)}. \quad (16)$$

Definition 10. If G is a complex fuzzy relation on a complex fuzzy set C, G is said to be a complex fuzzy irreflexive relation if for all $u \in U$ and $\lambda_C(u) \in C \Longrightarrow \lambda_{C\times C}(u,u) \notin G$.

Example 6. For any complex fuzzy set $C = (0.4e^{i2\pi}/1) + (1e^{i(\pi/2)}/2) + (0.5e^{i\pi}/3)$ defined on any universal set $U = \{1,2,3\}$, the product space $C \times C$ is

$$C\times C = \frac{0.4e^{i2\pi}}{(1,1)} + \frac{0.4e^{i(\pi/2)}}{(1,2)} + \frac{0.4e^{i\pi}}{(1,3)} + \frac{0.4e^{i(\pi/2)}}{(2,1)} + \frac{1e^{i(\pi/2)}}{(2,2)} + \frac{0.5e^{i(\pi/2)}}{(2,3)} + \frac{0.4e^{i\pi}}{(3,1)} + \frac{0.5e^{i(\pi/2)}}{(3,2)} + \frac{0.5e^{i\pi}}{(3,3)}. \tag{17}$$

So, the complex fuzzy irreflexive relation on C is subset of $C \times C$ is given by

$$G = \frac{0.4e^{i2\pi}}{(1,1)} + \frac{1e^{i(\pi/2)}}{(2,2)} + \frac{0.4e^{i\pi}}{(1,3)} + \frac{0.4e^{i(\pi/2)}}{(2,1)}. \tag{18}$$

Definition 11. If G is a complex fuzzy relation on a complex fuzzy set C, G is said to be a complex fuzzy asymmetric relation if $\lambda_{C\times C}(u,v) \in G \Longrightarrow \lambda_{C\times C}(v,u) \notin G$.

Example 7. For any complex fuzzy set $C = (0.4e^{i2\pi}/a) + (1e^{i(\pi/2)}/b) + (0.5e^{i\pi}/c)$ defined on any universal set $U = \{a,b,c\}$, the product space $C \times C$ is

$$C\times C = \frac{0.4e^{i2\pi}}{(a,a)} + \frac{0.4e^{i(\pi/2)}}{(a,b)} + \frac{0.4e^{i\pi}}{(a,c)} + \frac{0.4e^{i(\pi/2)}}{(b,a)} + \frac{1e^{i(\pi/2)}}{(b,b)} + \frac{0.5e^{i(\pi/2)}}{(b,c)} + \frac{0.4e^{i\pi}}{(c,a)} + \frac{0.5e^{i(\pi/2)}}{(c,b)} + \frac{0.5e^{i\pi}}{(c,c)}. \tag{19}$$

So, the complex fuzzy asymmetric relation on C is given by

$$G = \frac{0.4e^{i2\pi}}{(a,a)} + \frac{1e^{i(\pi/2)}}{(b,b)} + \frac{0.5e^{i\pi}}{(c,c)} + \frac{0.4e^{i(\pi/2)}}{(a,b)}. \tag{20}$$

Definition 12. A relation G is said to be a complex fuzzy equivalence relation if it satisfies the following conditions:

(i) G is reflexive
(ii) G is symmetric
(iii) G is transitive

Example 8. For any complex fuzzy set $C = (0.8e^{i2\pi}/u) + (0.5e^{i(\pi/2)}/v) + (0.2e^{i\pi}/v)$ defined on any universal set $U = \{1,2,3\}$, the product space $C \times C$ is given by

$$C\times C = \frac{0.8e^{i2\pi}}{(u,u)} + \frac{0.5e^{i(\pi/2)}}{(u,v)} + \frac{0.2e^{i\pi}}{(u,w)} + \frac{0.5e^{i(\pi/2)}}{(v,u)} + \frac{0.5e^{i(\pi/2)}}{(v,v)} + \frac{0.2e^{i(\pi/2)}}{(v,w)} + \frac{0.2e^{i\pi}}{(w,u)} + \frac{0.2e^{i(\pi/2)}}{(w,v)} + \frac{0.2e^{i\pi}}{(w,w)}. \tag{21}$$

So, the complex fuzzy equivalence relation on C is given by

$$G = \frac{0.8e^{i2\pi}}{(u,u)} + \frac{0.5e^{i(\pi/2)}}{(v,v)} + \frac{0.2e^{i\pi}}{(w,w)} + \frac{0.2e^{i(\pi/2)}}{(v,w)} + \frac{0.2e^{i(\pi/2)}}{(w,v)}. \tag{22}$$

Definition 13. A relation G is said to be a complex fuzzy-order relation if it satisfies the following conditions:

(i) G is reflexive
(ii) G is antisymmetric
(iii) G is transitive

Example 9. For any complex fuzzy set $C = (0.8e^{i2\pi}/u) + (0.5e^{i(\pi/2)}/v) + (0.2e^{i\pi}/v)$ defined on any universal set $U = \{1,2,3\}$, the product space $C \times C$ is given by

$$C\times C = \frac{0.8e^{i2\pi}}{(u,u)} + \frac{0.5e^{i(\pi/2)}}{(u,v)} + \frac{0.2e^{i\pi}}{(u,w)} + \frac{0.5e^{i(\pi/2)}}{(v,u)} + \frac{0.5e^{i(\pi/2)}}{(v,v)} + \frac{0.2e^{i(\pi/2)}}{(v,w)} + \frac{0.2e^{i\pi}}{(w,u)} + \frac{0.2e^{i(\pi/2)}}{(w,v)} + \frac{0.2e^{i\pi}}{(w,w)}. \tag{23}$$

So, the complex fuzzy-order relation on C is given by

$$G = \frac{0.8e^{i2\pi}}{(u,u)} + \frac{0.5e^{i(\pi/2)}}{(v,v)} + \frac{0.2e^{i\pi}}{(w,w)} + \frac{0.5e^{i(\pi/2)}}{(u,v)} + \frac{0.2e^{i(\pi/2)}}{(v,w)}. \tag{24}$$

Definition 14. Let C be a complex fuzzy set and G be a complex fuzzy equivalence relation in C. If $\lambda_C(u) \in C$, then the complex fuzzy equivalence class of $\lambda_C(u)$ modulo G is the set $G_{\lambda_C(u)}$ defined by

$$G_{\lambda_C(u)} = \left\{\frac{\lambda_C(v)}{\lambda_C(v,u)} \in G\right\}. \tag{25}$$

Example 10. For any complex fuzzy set $C = (1e^{i1.2\pi}/1) + (0.5e^{i\pi}/2) + (0e^{i(\pi/2)}/3)$ defined on any universal set $U = \{1,2,3\}$, the product space $C \times C$ is given by

$$C\times C = \frac{1e^{i1.2\pi}}{(1,1)} + \frac{0.5e^{i\pi}}{(1,2)} + \frac{0e^{i(\pi/2)}}{(1,3)} + \frac{0.5e^{i\pi}}{(2,1)} + \frac{0.5e^{i\pi}}{(2,2)} + \frac{0e^{i(\pi/2)}}{(2,3)} + \frac{0e^{i(\pi/2)}}{(3,1)} + \frac{0e^{i(\pi/2)}}{(3,2)} + \frac{0e^{i(\pi/2)}}{(3,3)}. \tag{26}$$

Also, let

$$G = \frac{1e^{i1.2\pi}}{(1,1)} + \frac{0.5e^{i\pi}}{(2,2)} + \frac{0.5e^{i\pi}}{(2,1)} + \frac{0e^{i(\pi/2)}}{(3,2)} + \frac{0e^{i(\pi/2)}}{(3,3)}, \tag{27}$$

be the complex fuzzy relation in C. Then, the complex fuzzy equivalence class of $\lambda_C(1)$ is given by

$$G_{\lambda_C(1)} = \{\lambda_C(1), \lambda_C(2)\}. \tag{28}$$

The complex fuzzy equivalence class of $\lambda_C(2)$ is given by

$$G_{\lambda_C(2)} = \{\lambda_C(2), \lambda_C(3)\}. \tag{29}$$

Also, the complex fuzzy equivalence class of $\lambda_C(3)$ is given by

$$G_{\lambda_C(3)} = \{\lambda_C(3)\}. \tag{30}$$

Definition 15. Let C be a complex fuzzy set and G be a complex fuzzy relation in C. For any $\lambda_{C\times C}(u,v) \in G$ and $\lambda_{C\times C}(v,w) \in G$, $\lambda_{C\times C}(u,w) \in G^\circ G$ for all $u, v, w \in U$ (universal set). $G^\circ G$ is called the complex fuzzy composition relations.

4. Main Results

Proposition 1. *If G and H are symmetric relations in a complex fuzzy set C, then $G \cap H$ is also a complex fuzzy symmetric relation in C.*

Proof. Suppose that G and H are complex fuzzy symmetric relations in a complex fuzzy set C. Since $G \subseteq C \times C$ and $H \subseteq C \times C$, $G \cap H \subseteq C \times C$. Therefore, $G \cap H$ is a complex fuzzy relation in C.

Let $\lambda_{C\times C}(u,v) \in G \cap H$; then, $\lambda_{C\times C}(u,v) \in G$ and $\lambda_{C\times C}(u,v) \in H$. But, G and H are symmetric. Therefore $\lambda_{C\times C}(v,u) \in G$ and $\lambda_{C\times C}(v,u) \in H$, so that $\lambda_{C\times C}(v,u) \in G \cap H$. □

Proposition 2. *Let G be a complex fuzzy relation in a complex fuzzy set C; then, G is symmetric if $G = G^{-1}$.*

Proof. Suppose G is a complex fuzzy symmetric relation; then,

$$\lambda_{C\times C}(u,v) \in G \Longleftrightarrow \lambda_{C\times C}(v,u) \in G \Longleftrightarrow \lambda_{C\times C}(u,v) \in G^{-1}. \tag{31}$$

Thus,

$$G = G^{-1}. \tag{32}$$

Conversely, let $G = G^{-1}$; then,

$$\lambda_{C\times C}(u,v) \in G \Longleftrightarrow \lambda_{C\times C}(u,v) \in G^{-1} \Longleftrightarrow \lambda_{C\times C}(v,u) \in G. \tag{33}$$

Thus, G is a complex fuzzy symmetric relation. □

Proposition 3. *Let G be a complex fuzzy relation in a complex fuzzy set C; then, G is transitive if $G^\circ G \subseteq G$.*

Proof. Suppose G is transitive. Assume that $\lambda_{C\times C}(u,w) \in G^\circ G$. Then, there exists $v \in U$ such that $\lambda_{C\times C}(u,v) \in G$ and $\lambda_{C\times C}(v,w) \in G$. Since G is complex fuzzy transitive relation and, hence, by the transitive property, $\lambda_{C\times C}(u,w) \in G$.

Thus,

$$G^\circ G \subseteq G. \tag{34}$$

Conversely, suppose that $G^\circ G \subseteq G$. Then, $\lambda_{C\times C}(u,v) \in G$ and $\lambda_{C\times C}(v,w) \in G \Longleftrightarrow \lambda_{C\times C}(u,w) \in G^\circ G \subseteq G$. Thus, $\lambda_{C\times C}(u,w) \in G$, and hence, G is transitive. □

Proposition 4. *If G is a complex fuzzy equivalence relation in a complex fuzzy set C, then $G^\circ G = G$.*

Proof. Let $\lambda_{C\times C}(u,v) \in G^\circ G$; then, there exists $w \in U$ such that $\lambda_{C\times C}(u,w) \in G$ and $\lambda_{C\times C}(w,v) \in G$, but G is a complex fuzzy equivalence relation in C, so G is a complex fuzzy transitive relation. Therefore, by the transitive property, $\lambda_{C\times C}(u,v) \in G$. Thus,

$$G^\circ G \subseteq G. \tag{35}$$

Conversely, assume that $\lambda_{C\times C}(u,v) \in G$. Since G is a complex fuzzy reflexive relation, for $v \in U \Longrightarrow \lambda_{C\times C}(v,v) \in G$. Now, $\lambda_{C\times C}(u,v) \in G$ and $\lambda_{C\times C}(v,v) \in G$. Since G is a complex fuzzy transitive relation, so by the transitive property, $\lambda_{C\times C}(u,v) \in G^\circ G$. Thus,

$$G \subseteq G^\circ G. \tag{36}$$

From (35) and (36), we have

$$G = G^\circ G. \tag{37}$$

□

Proposition 5. *The inverse of a complex fuzzy-order relation in a complex fuzzy set C is also a complex fuzzy-order relation in C.*

Proof. Let G be a complex fuzzy-order relation in a complex fuzzy set C. To show that G^{-1} is also a complex fuzzy-order relation in C, we have to satisfy the three conditions of a complex fuzzy-order relation.

Since G is a complex fuzzy-order relation, for any $u \in U \Longrightarrow \lambda_{C\times C}(u,u) \in G \Longrightarrow \lambda_{C\times C}(u,u) \in G^{-1}$. Thus, G^{-1} is a complex fuzzy reflexive relation.

Let $\lambda_{C\times C}(u,v) \in G^{-1}$ and $\lambda_{C\times C}(v,u) \in G^{-1}$. Then, $\lambda_{C\times C}(v,u) \in G$ and $\lambda_{C\times C}(u,v) \in G$, but G is a complex fuzzy-order relation in C. Therefore, $\lambda_{C\times C}(u,v) = \lambda_{C\times C}(v,u)$, and hence, G^{-1} is antisymmetric.

Assume that $\lambda_{C\times C}(u,v) \in G^{-1}$ and $\lambda_{C\times C}(v,w) \in G^{-1}$. Then, $\lambda_{C\times C}(v,u) \in G$ and $\lambda_{C\times C}(w,v) \in G$. Since G is a complex fuzzy transitive relation. By the transitive property, $\lambda_{C\times C}(w,u) \in G$. Thus, $\lambda_{C\times C}(u,w) \in G^{-1}$, and hence, G^{-1} is a complex fuzzy transitive relation in C. Since G^{-1} satisfies all the properties of a complex fuzzy-order relation, G^{-1} is a complex fuzzy-order relation. □

Theorem 1. *Let G be any complex fuzzy equivalence relation in a complex fuzzy set C. Then,* $\lambda_{C\times C}(v,u) \in G$ *if and only if* $G_{\lambda_C(u)} = G_{\lambda_C(v)}$.

Proof. Suppose that $\lambda_{C\times C}(u,v) \in G$. Let $\lambda_C(w) \in G_{\lambda_C(u)}$. Then, $\lambda_{C\times C}(w,u) \in G$. Now, $\lambda_{C\times C}(w,u) \in G$ and $\lambda_{C\times C}(u,v) \in G$. But, since G is a complex fuzzy equivalence relation in a complex fuzzy set C, by the transitive property, $\lambda_{C\times C}(w,v) \in G$, so that $\lambda_C(w) \in G_{\lambda_C(v)}$. Thus,

$$G_{\lambda_C(u)} \subseteq G_{\lambda_C(v)}. \tag{38}$$

Let $\lambda_C(w) \in G_{\lambda_C(v)}$; then, $\lambda_{C\times C}(w,v) \in G$ and also $\lambda_{C\times C}(u,v) \in G$. Since G is a complex fuzzy symmetric relation, so by the symmetric property, $\lambda_{C\times C}(v,u) \in G$. Also, G is a complex fuzzy equivalence relation. Therefore, by the transitive property, $\lambda_{C\times C}(w,u) \in G$. Thus, $\lambda_C(w) \in G_{\lambda_C(u)}$, and hence,

$$G_{\lambda_C(v)} \subseteq G_{\lambda_C(u)}. \tag{39}$$

From (38) and (39), we have

$$G_{\lambda_C(v)} = G_{\lambda_C(u)}. \tag{40}$$

Conversely, suppose that $G_{\lambda_C(v)} = G_{\lambda_C(u)}$. Since G is a complex fuzzy equivalence relation, so by the reflexive property, $\lambda_{C\times C}(u,u) \in G$. By definition of a complex fuzzy equivalence class of modulo G,

$$\lambda_C(u) \in G_{\lambda_C(u)} = G_{\lambda_C(v)} \lambda_C(u) \in G_{\lambda_C(v)}. \tag{41}$$

Hence, $\lambda_{C\times C}(u,v) \in G$. □

5. Applications

In this section, we will discuss the application of complex fuzzy relations in Future Commission Market.

We are going to discuss a real-life application of newly defined types/properties of complex fuzzy relations. Ramot et al. in [11] discussed the application of the complex fuzzy relation in Future Commission Merchant. The physical meaning of the Ramot and proposed model is the same, but here, we will show that how the types of complex fuzzy relations play a significant role in real-life applications.

5.1. Decision-Making Method. Let U be a collection of financial indicators or indexes of any country. Possible components of this collection are import, export, agriculture, unemployment, and development rate, that is, $U = \{\text{import, export, agriculture, unemployment, devlopment rate}\}$. Let

$$\begin{aligned} C = \{&\lambda_C(\text{import}), \lambda_C(\text{export}), \lambda_C(\text{agriculture}),\\ &\lambda_C(\text{unemployment}), \lambda_C(\text{development rate})\} \end{aligned} \tag{42}$$

represent the complex fuzzy sets and $\lambda_{C\times C}(x,y)$ represent the membership function of the complex fuzzy relations.

The membership function $\lambda_{C\times C}(x,y)$ is complex-valued, with a phase term and amplitude term. The amplitude term shows the grade of influence of one parameter on another parameter. The amplitude term with a value close to zero shows more influence, while a value close to zero shows a small influence. The phase term refers to the "phase" of control or time lag that characterizes one parameter's effect on another parameter.

Consider, for example, we find the relation set G for the complex fuzzy set C, that is, the cross product of C.

$$\begin{aligned} G = \{&\lambda_{C\times C}(\text{import, import}), \lambda_{C\times C}(\text{import, export}), \lambda_{C\times C}(\text{import, agriculture}), \lambda_{C\times C}(\text{import, unemployment}),\\ &\lambda_{C\times C}(\text{import, development rate}),\\ &\lambda_{C\times C}(\text{export, import}), \lambda_{C\times C}(\text{export, export}), \lambda_{C\times C}(\text{export, agriculture}),\\ &\lambda_{C\times C}(\text{export, unemployment}), \lambda_{C\times C}(\text{export, development rate}),\\ &\lambda_{C\times C}(\text{agriculture, import}), \lambda_{C\times C}(\text{agriculture, export}),\\ &\lambda_{C\times C}(\text{agriculture, agriculture}), \lambda_{C\times C}(\text{agriculture, unemployment}),\\ &\lambda_{C\times C}(\text{agriculture, development rate}), \lambda_{C\times C}(\text{unemployment, import}),\\ &\lambda_{C\times C}(\text{unemployment, export}), \lambda_{C\times C}(\text{unemployment, agriculture}),\\ &\lambda_{C\times C}(\text{unemployment, unemployment}), \lambda_{C\times C}(\text{unemployment, development rate}),\\ &\lambda_{C\times C}(\text{development rate, import}), \lambda_{C\times C}(\text{development rate, export}),\\ &\lambda_{C\times C}(\text{development rate, agriculture}), \lambda_{C\times C}(\text{development rate, unemployment}),\\ &\lambda_{C\times C}(\text{development rate, development rate})\}. \end{aligned} \tag{43}$$

The relation G is an equivalence relation, that is, G is a complex fuzzy reflexive, complex fuzzy symmetric, and complex fuzzy transitive relation. From this relation, we can easily determine the influence of one parameter on another parameter. For example, we take $\lambda_{C\times C}(\text{import, agriculture}) = (\min \{a_C(\text{import}), a_C(\text{agriculture})\} e^{i \min\{p_C(\text{import}), p_C(\text{agriculture})\}} / (\text{import, agriculture}))$.

The $\min\{a_C(\text{import}), a_C(\text{agriculture})\}$ shows the degree of influence of import on agriculture or agriculture on import of a country with respect to time.

$$\min\{p_C(\text{import}), p_C(\text{agriculture})\}. \tag{44}$$

This means that the degree of influence depends on the min value of the parameter. Moreover, if we know the influence of import on agriculture and agriculture on development rate, then by the complex fuzzy transitive relation, we can determine the influence of import on the development rate. Similarly, we have to find the influence of each parameter on the other. The more significant relation is the complex fuzzy transitive relation because if we know the degree of influence of the first parameter on the second parameter and second parameter on the third parameter, then by the complex fuzzy transitive relation, we can find the degree of influence of the first parameter on the third parameter. Moreover, the complex fuzzy symmetric relation shows that the degree of influence of the first parameter on the second parameter and the second parameter on the first parameter is the same.

Example 11. Consider a complex fuzzy set

$$C = \left\{\frac{0.4e^{i\pi}}{\text{development rate}} + \frac{0.8e^{i(\pi/2)}}{\text{agriculture}} + \frac{0.2e^{i2\pi}}{\text{export}}\right\}. \tag{45}$$

Then, the relation G on $C \times C$ is

$$\begin{aligned}
G = C \times C = \Bigg\{ & \frac{\min\{0.4, 0.4\}e^{i\min\{\pi,\pi\}}}{(\text{development rate, development rate})} + \\
& \frac{\min\{0.4, 0.8\}e^{i\min\{\pi,(\pi/2)\}}}{(\text{development rate, dgriculture})} + \\
& \frac{\min\{0.4, 0.2\}e^{i\min\{\pi,2\pi\}}}{(\text{development rate, export})} + \\
& \frac{\min\{0.8, 0.4\}e^{i\min\{(\pi/2),\pi\}}}{(\text{agriculture, development rate})} + \\
& \frac{\min\{0.8, 0.8\}e^{i\min\{(\pi/2),(\pi/2)\}}}{(\text{agriculture, agriculture})} + \\
& \frac{\min\{0.8, 0.2\}e^{i\min\{\pi/2,t2n\pi\}}}{(\text{agriculture, export})} + \\
& \frac{\min\{0.2, 0.4\}e^{i\min\{2\pi,\pi\}}}{(\text{export, development rate})} + \\
& \frac{\min\{0.2, 0.8\}e^{i\min\{2\pi,(\pi/2)\}}}{(\text{export, agriculture})} + \\
& \frac{\min\{0.2, 0.2\}e^{i\min\{(\pi/2),\pi\}}}{(\text{export, export})}\Bigg\}, \\
G = C \times C = \Bigg\{ & \frac{0.4e^{i\pi}}{(\text{development rate, development rate})} + \\
& \frac{0.4, e^{i(\pi/2)}}{(\text{development rate, dgriculture})} + \frac{0.2e^{i\pi}}{(\text{development rate, export})} \\
& + \frac{0.4e^{i(\pi/2)}}{(\text{agriculture, development rate})} + \frac{0.8e^{i(\pi/2)}}{(\text{agriculture, agriculture})} \\
& + \frac{0.2e^{i(\pi/2)}}{(\text{agriculture, export})} + \frac{0.2e^{i\pi}}{(\text{export, development rate})} \\
& + \frac{0.2e^{i(\pi/2)}}{(\text{export, agriculture, })} + \frac{0.2e^{i(\pi/2)}}{(\text{export, export})}\Bigg\}.
\end{aligned} \tag{46}$$

From the abovementioned relation G (complex fuzzy equivalence relation), we can determine the influence of the development rate, agriculture, and export on each other. For example,

$$\lambda_{C\times C}(\text{agriculture, development rate}) = 0.4e^{i(\pi/2)} \in G. \tag{47}$$

Here, 0.4 shows the degree of influence of agriculture on the development rate with respect to half a month. Moreover, the membership function $\lambda_{C\times C}$(agriculture, export), that is,

$$\lambda_{C\times C}(\text{agriculture, export}) = 0.2e^{i(\pi/2)} \in G, \tag{48}$$

shows that the degree of influence of agriculture on export is 0.2 with respect to the half a month.

If we compare the influence of agriculture on the development rate and export, we find that the degree of influence of agriculture on export is less than the degree of influence on the development rate.

From the abovementioned relation G, we have the same degree of influence of agriculture on the development rate and the development rate on agriculture.

Moreover, if we have the degree of influence of agriculture on the development rate and the degree of influence of the development rate on export, then by complex fuzzy transitive relations, we can obtain the degree of influence of agriculture on export.

In the second example, we will discuss the degree of influence of American financial indexes on China's financial indexes and China's financial indexes on Saudi Arabia's financial indexes.

Example 12. Let C_1, C_2, and C_3 represent the set of America, China, and Saudi Arabia's financial indexes. A possible collection of these sets are import, export, growth rate, interest rate, and unemployment rate. They are all complex-valued functions. Let the complex fuzzy relation $\lambda_{C_1\times C_2}(x,y)$ represent the relation of influence of American financial indexes on China financial indexes and the complex fuzzy relation $\lambda_{C_2\times C_3}(x,y)$ represent the relation of influence of China financial indexes on Saudi Arabia financial indexes, where x, y represent any two parameters.

Consider

$$\begin{aligned}
C_1 &= \left\{\lambda_{C_1}(\text{import}), \lambda_{C_1}(\text{export}), \lambda_{C_1}(\text{growth rate}), \lambda_{C_1}(\text{interest rate})\lambda_{C_1}(\text{unemployment rate})\right\},\\
C_2 &= \left\{\lambda_{C_2}(\text{import}), \lambda_{C_2}(\text{export}), \lambda_{C_2}(\text{growth rate}), \lambda_{C_2}(\text{interest rate})\lambda_{C_2}(\text{unemployment rate})\right\},\\
C_3 &= \left\{\lambda_{C_3}(\text{import}), \lambda_{C_3}(\text{export}), \lambda_{C_3}(\text{growth rate}), \lambda_{C_3}(\text{interest rate})\lambda_{C_3}(\text{unemployment rate})\right\}.
\end{aligned} \tag{49}$$

Now, the complex fuzzy relation G_1 between C_1 and C_2 is given by

$$\begin{aligned}
G_1 = C_1\times C_2 = \{&\lambda_{C_1\times C_2}(\text{import, import}), \lambda_{C_1\times C_2}(\text{import, export}),\\
&\lambda_{C_1\times C_2}(\text{import, growth rate}), \lambda_{C_1\times C_2}(\text{import, interest rate}),\\
&\lambda_{C_1\times C_2}(\text{import, unemployment rate}), \lambda_{C_1\times C_2}(\text{export, import}),\\
&\lambda_{C_1\times C_2}(\text{export, export}), \lambda_{C_1\times C_2}(\text{export, growth rate}),\\
&\lambda_{C_1\times C_2}(\text{export, interest rate}), \lambda_{C_1\times C_2}(\text{export, unemployment rate}),\\
&\lambda_{C_1\times C_2}(\text{growth rate, import}), \lambda_{C_1\times C_2}(\text{growth rate, export}),\\
&\lambda_{C_1\times C_2}(\text{growth rate, growth rate}), \lambda_{C_1\times C_2}(\text{growth rate, interest rate}),\\
&\lambda_{C_1\times C_2}(\text{growth rate, unemployment rate}), \lambda_{C_1\times C_2}(\text{interest rate, import}),\\
&\lambda_{C_1\times C_2}(\text{interest rate, export}), \lambda_{C_1\times C_2}(\text{interest rate, growth rate}),\\
&\lambda_{C_1\times C_2}(\text{interest rate, interest rate}), \lambda_{C_1\times C_2}(\text{interest rate, unemployment rate}),\\
&\lambda_{C_1\times C_2}(\text{unemployment rate, import}), \lambda_{C_1\times C_2}(\text{unemployment rate, export}),\\
&\lambda_{C_1\times C_2}(\text{unemployment rate, growth rate}), \lambda_{C_1\times C_2}(\text{unemployment rate, interest rate}),\\
&\lambda_{C_1\times C_2}(\text{unemployment rate, unemployment rate})\}.
\end{aligned} \tag{50}$$

The relation G_1 shows the relation of influence of American financial indexes on China financial indexes.

The complex fuzzy relation between C_2 and C_3 is given by

$$\begin{aligned}
G_2 = C_2 \times C_3 = \{&\lambda_{C_2\times C_3}(\text{import, import}), \lambda_{C_2\times C_3}(\text{import, export}),\\
&\lambda_{C_2\times C_3}(\text{import, growth rate}), \lambda_{C_2\times C_3}(\text{import, interest rate}),\\
&\lambda_{C_2\times C_3}(\text{import, unemployment rate}), \lambda_{C_2\times C_3}(\text{export, import}),\\
&\lambda_{C_2\times C_3}(\text{export, export}), \lambda_{C_2\times C_3}(\text{export, growth rate}),\\
&\lambda_{C_2\times C_3}(\text{export, interest rate}), \lambda_{C_2\times C_3}(\text{export, unemployment rate}),\\
&\lambda_{C_2\times C_3}(\text{growth rate, import}), \lambda_{C_2\times C_3}(\text{growth rate, export}),\\
&\lambda_{C_2\times C_3}(\text{growth rate, growth rate}), \lambda_{C_2\times C_3}(\text{growth rate, interest rate}),\\
&\lambda_{C_2\times C_3}(growth\ rate,\ unemployment\ rate), \lambda_{C_2\times C_3}(\text{interest rate, import}),\\
&\lambda_{C_2\times C_3}(\text{interest rate, export}), \lambda_{C_2\times C_3}(\text{interest rate, growth rate}),\\
&\lambda_{C_2\times C_3}(\text{interest rate, interest rate}), \lambda_{C_2\times C_3}(\text{interest rate, unemployment rate}),\\
&\lambda_{C_2\times C_3}(\text{unemployment rate, import}), \lambda_{C_2\times C_3}(\text{unemployment rate, export}),\\
&\lambda_{C_2\times C_3}(\text{unemployment rate, growth rate}), \lambda_{C_2\times C_3}(\text{unemployment rate, interest rate}),\\
&\lambda_{C_2\times C_3}(\text{unemployment rate, unemployment rate})\}.
\end{aligned} \tag{51}$$

The relation G_2 represents the relation of influence of China's financial indexes on Saudi Arabia's financial indexes.

From relation G_1 and G_2, we have the relation of influence of American financial indexes on China's financial indexes and China financial indexes on Saudi Arabia financial indexes. By the complex fuzzy composition relation, we can find the relation of the influence of American financial indexes on Saudi Arabia's financial indexes.

The relation G represents the relation of influence of American financial indexes on Saudi Arabia financial indices, that is,

$$\begin{aligned}
G_1 = C_1 \times C_3 = \{&\lambda_{C_1\times C_3}(\text{import, import}), \lambda_{C_1\times C_3}(\text{import, export}),\\
&\lambda_{C_1\times C_3}(\text{import, growth rate}), \lambda_{C_1\times C_3}(\text{import, interest rate}),\\
&\lambda_{C_1\times C_3}(\text{import, unemployment rate}), \lambda_{C_1\times C_3}(\text{export, import}),\\
&\lambda_{C_1\times C_3}(\text{export, export}), \lambda_{C_1\times C_3}(\text{export, growth rate}),\\
&\lambda_{C_1\times C_3}(\text{export, interest rate}), \lambda_{C_1\times C_3}(\text{export, unemployment rate}),\\
&\lambda_{C_1\times C_3}(\text{growth rate, import}), \lambda_{C_1\times C_3}(\text{growth rate, export}),\\
&\lambda_{C_1\times C_3}(\text{growth rate, growth rate}), \lambda_{C_1\times C_3}(\text{growth rate, interest rate}),\\
&\lambda_{C_1\times C_3}(\text{growth rate, unemployment rate}), \lambda_{C_1\times C_3}(\text{interest rate, import}),\\
&\lambda_{C_1\times C_3}(\text{interest rate, export}), \lambda_{C_1\times C_3}(\text{interest rate, growth rate}),\\
&\lambda_{C_1\times C_3}(\text{interest rate, interest rate}), \lambda_{C_1\times C_3}(\text{interest rate, unemployment rate}),\\
&\lambda_{C_1\times C_3}(\text{unemployment rate, import}), \lambda_{C_1\times C_3}(\text{unemployment rate, export}),\\
&\lambda_{C_1\times C_3}(\text{unemployment rate, growth rate}), \lambda_{C_1\times C_3}(\text{unemployment rate, interest rate}),\\
&\lambda_{C_1\times C_3}(\text{unemployment rate, unemployment rate})\}.
\end{aligned} \tag{52}$$

For example, the membership function $\lambda_{C_1\times C_2}$ (import, export) $\in G_1$ shows the degree of influence of American import on a China export and the membership function $\lambda_{C_2\times C_3}$(export, growth rate) $\in G_2$ shows the degree of influence of China export on a Saudi Arabia growth rate. By the complex fuzzy composition relation of these two relations, we have the membership function $\lambda_{C_1 \times C_3}$(import, growth rate)$\in\in G$, which is more significant and gives the degree of influence of American import on the Saudi Arabia growth rate.

Moreover, the types of complex fuzzy relations play a vital role in applications. If we have known the degree of influence of American financial indexes on China financial indexes, then by the inverse complex fuzzy relations, we can find the degree of China financial indexes on American financial indexes.

Similarly, the complex fuzzy transitive relation is the most important type of complex fuzzy relations and plays a major role in applications. For example, if we have the degree of the influence of import on the interest rate and the interest rate on the unemployment rate, that is, $\lambda_{C_1 \times C_3}$(import, interest rate) and $\lambda_{C_1 \times C_3}$(interest rate, unemployment rate), then by complex fuzzy transitive relations, we can easily find the degree of influence of import on interest rate $\lambda_{C_1 \times C_3}$(import, interest rate).

The abovementioned process may be applied to discuss the degree of financial indexes of more than three countries.

Example 13. Let $C_1 = \{0.8e^{i(\pi} \quad /2)//\text{import},(0.5e^{i\pi}$ /export), $(1e^{i2\pi}$/interestrate), $(0.7e^{i(3\pi/2)}$/unemploymnt rate *rate*)}, $C_2 = \{0.5e^{i(3\pi/2)}$/import, $0.9e^{i2\pi}$ /export, $(0.3e^{i\pi}$/ interest rate), $0.4e^{i(\pi/2)}$/unemployment rate, , *and* $C_3 = \{0.6e^{i(5\pi/2)}$/ import, $1e^{i(\pi/2)}$/export $0.7e^{i(\pi/3)}$/interest rate, $0.5e^{i\pi}$/unemployment rate, *represent the sets of American, China, and Saudi Arabia's financial indexes. Then, the relation* G_1 *on* $C_1 \times C_2$ *is*

$$\begin{aligned} G_1 = C_1 \times C_2 = \Bigg\{ & \frac{0.5e^{i(\pi/2)}}{(\text{import, import})}, \frac{0.8e^{i(\pi/2)}}{(\text{import, export})}, \\ & \frac{0.3e^{i(\pi/2)}}{(\text{import, interest rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{import, unemployment rate})}, \\ & \frac{0.5e^{i(3\pi/2)}}{(\text{export, import})}, \frac{0.5e^{i\pi}}{(\text{export, export})}, \\ & \frac{0.3e^{i\pi}}{(\text{export, interest rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{export, unemployment rate})}, \\ & \frac{0.5e^{i(3\pi/2)}}{(\text{interest rate, import})}, \frac{0.9e^{i2\pi}}{(\text{interest rate, export})}, \\ & \frac{0.3e^{i\pi}}{(\text{interest rate, interest rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{interest rate, unemployment rate}),} \\ & \frac{0.5e^{i(3\pi/2)}}{(\text{unemployment rate, import})}, \frac{0.7e^{i(3\pi/2)}}{(\text{unemployment rate, export})}, \\ & \frac{0.3e^{i\pi}}{(\text{unemployment rate, interest rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{unemployment rate, unemployment rate})} \Bigg\}. \end{aligned} \tag{53}$$

The relation G_1 shows the relation of influence of American financial indexes on China financial indexes.

The complex fuzzy relation between C_2 and C_3 is given by

$$G_2 = C_2 \times C_3 = \left\{ \frac{0.5e^{i(3\pi/2)}}{(\text{import, import})}, \frac{0.5e^{i(\pi/2)}}{(\text{import, export})}, \frac{0.5e^{i(\pi/3)}}{(\text{import, interest rate})}, \frac{0.5e^{i(3\pi/2)}}{(\text{import, unemployment rate})}, \frac{0.6e^{i(5\pi/2)}}{(\text{export, import})}, \frac{0.9e^{i(\pi/2)}}{(\text{export, export})}, \frac{0.7e^{i(\pi/3)}}{(\text{export, interest rate})}, \frac{0.5e^{i\pi}}{(\text{export, unemployment rate})}, \frac{0.3e^{i(5\pi/2)}}{(\text{interest rate, import})}, \frac{0.3e^{i(\pi/2)}}{(\text{interest rate, export})}, \frac{0.3e^{i(\pi/3)}}{(\text{interest rate, interest rate})}, \frac{0.3e^{i\pi}}{(\text{interest rate, unemployment rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{unemployment rate, import})}, \frac{0.4e^{i(\pi/2)}}{(\text{unemployment rate, export})}, \frac{0.4e^{i(\pi/3)}}{(\text{unemployment rate, interest rate})}, \frac{0.4e^{i(\pi/2)}}{(\text{unemployment rate, unemployment rate})} \right\}. \tag{54}$$

The relation G_2 represents the relation of influence of China's financial indexes on Saudi Arabia's financial indexes. By the complex fuzzy composition relation, we can find the relation of the influence of American financial indexes on Saudi Arabia's financial indexes. The relation G_3 represents the relation of influence of American financial indexes on Saudi Arabia financial indices, that is,

$$G_3 = C_1 \times C_3 = \left\{ \frac{0.6e^{i(\pi/2)}}{(\text{import, import})}, \frac{0.8e^{i(\pi/2)}}{(\text{import, export})}, \frac{0.7e^{i(\pi/2)}}{(\text{import, interest rate})}, \frac{0.5e^{i(\pi/2)}}{(\text{import, unemployment rate})}, \frac{0.5e^{i\pi}}{(\text{export, import})}, \frac{0.5e^{i(\pi/2)}}{(\text{export, export})}, \frac{0.5e^{i(\pi/3)}}{(\text{export, interest rate})}, \frac{0.5e^{i\pi}}{(\text{export, unemployment rate})}, \frac{0.6e^{i2\pi}}{(\text{interest rate, import})}, \frac{1e^{i(\pi/2)}}{(\text{interest rate, export})}, \frac{0.7e^{i(\pi/3)}}{(\text{interest rate, interest rate})}, \frac{0.5e^{i\pi}}{(\text{interest rate, unemployment rate})}, \frac{0.6e^{i(3\pi/2)}}{(\text{unemployment rate, import})}, \frac{0.7e^{i(\pi/2)}}{(\text{unemployment rate, export})}, \frac{0.7e^{i(\pi/3)}}{(\text{unemployment rate, interest rate})}, \frac{0.5e^{i\pi}}{(\text{unemployment rate, unemployment rate})} \right\}. \tag{55}$$

The abovementioned three relations G_1, G_2, and G_3 show the degree of influence of American financial indexes on China financial indexes, China financial indexes on Saudi Arabia financial indexes, and American financial indexes on Saudi Arabia financial indexes, respectively. Moreover, by the inverse complex fuzzy relations, we can obtain the degree of influence of China financial indexes on American financial indexes, Saudi Arabia financial indexes on China financial indexes, and Saudi Arabia financial indexes on American financial indexes.

Moreover, if we have the degree of the influence of import on the interest rate and the interest rate on the unemployment rate, then by complex fuzzy transitive relations, we can easily find the degree of influence of import on the interest rate. For example,

$0.5e^{i(\pi/3)}$/import, interest rate, $\in\in G$ and $0.3e^{i\pi}$/(interest rate, unemployment rate)$\in\in G$; then, by complex fuzzy transitive relations, we have $0.5e^{i(3\pi/2)}$/(import, unemployment rate).

6. Comparison

There are many applications of crisp relations and fuzzy relations, particularly in fuzzy logic systems, diagnostic of symptoms, and decision making. But, there is one significant weakness, which is a lack of capacity to examine two-dimensional phenomena. They cannot deal with two-dimensional parameters. Ramot et al. in [11] introduced complex fuzzy relations which is the generalization of a fuzzy relation. Complex fuzzy relations show a degree of the presence or absence of association, interaction, or interconnectedness between two-dimensional parameters. Ramot et al. in [11] discussed the application of the complex fuzzy relation in Future Commission Merchant. He studied the degree of influence of parameters with respect to complex fuzzy relations. We explore this concept in detail and used the types of complex fuzzy relations such as the complex fuzzy transitive relation and complex fuzzy equivalence relation. The method we proposed here gives the degree of influence of the financial indexes of the three countries. This method can be used to find the degree of influence of financial indexes of more than three countries by using complex fuzzy transitive relations or composition of complex fuzzy relations. Moreover, this approach provides the degree of influence of the two countries financial indexes that do not have a direct relation by using the types of complex fuzzy relations. Similarly, by the inverse complex fuzzy relations, we can obtain the degree of influence of financial indexes on each other. However, our designed model is not complete, but it is stuck with a shortage of theoretical support. For applications, the types of complex fuzzy relations can be useful. Therefore, it will be significant for future work.

7. Conclusions

In this paper, we have discussed some new types of complex fuzzy relations such as the complex fuzzy inverse relation, complex fuzzy reflexive relation, complex fuzzy symmetric relation, complex fuzzy antisymmetric relation, complex fuzzy transitive relation, complex fuzzy irreflexive relation, complex fuzzy asymmetric relation, complex fuzzy equivalence relation, complex fuzzy-order relation, and complex fuzzy equivalence class. We have presented some basic results and examples of these relations. Moreover, we have discussed the application of complex fuzzy relations in Future Commission Market. The complex fuzzy relation may be used in geology, signals and systems, and engineering fields for the identification of reference signals [23].

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Retraction

Retracted: IF-MABAC Method for Evaluating the Intelligent Transportation System with Intuitionistic Fuzzy Information

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] Y. Li, "IF-MABAC Method for Evaluating the Intelligent Transportation System with Intuitionistic Fuzzy Information," *Journal of Mathematics*, vol. 2021, Article ID 5536751, 10 pages, 2021.

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Research Article

IF-MABAC Method for Evaluating the Intelligent Transportation System with Intuitionistic Fuzzy Information

Yanping Li

Information Engineering School, ZhengZhou ShengDa University, ZhengZhou, HeNan 451191, China

Correspondence should be addressed to Yanping Li; 101735@shengda.edu.cn

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Intelligent transportation system (ITS) is the development direction of the future traffic system. ITS can effectively employ the existing traffic facilities and ensure the safety of traffic, urban traffic, and public security management for effective control in order to satisfy people's travel demand. Therefore, the results of the system in-depth understanding and objective evaluation are very necessary. And it is frequently regarded as a multiattribute group decision-making (MAGDM) issue. Thus, a novel MAGDM method is required to tackle it. Depending on the conventional multiattributive border approximation area comparison (MABAC) method and intuitionistic fuzzy sets (IFSs), this article designs a novel intuitive distance-based IF-MABAC method to assess the performance of financial management. First of all, a related literature review is conducted. Furthermore, some necessary theories related to IFSs are briefly reviewed. In addition, since subjective randomness frequently exists in determining criteria weights, the weights of criteria are decided objectively by utilizing the maximizing deviation method. Afterwards, relying on novel distance measures between intuitionistic fuzzy numbers (IFNs), the conventional MABAC method is extended to the IFSs to calculate the final value of each enterprise. Therefore, all enterprises can be ranked, and the one with the best environmental behaviors and awareness can be identified. Eventually, an application for evaluating the intelligent transportation system and some comparative analyses have been given. The results illustrate that the designed algorithm is useful for assessing the performance of financial management.

1. Introduction

With our rapid economic development, accelerating urbanization, and the rapid rise of motor vehicle ownership, existing roads' hardware facilities have failed to meet the demand of swelling traffic. Traffic congestion, frequent accidents, and serious environmental pollution have become increasingly serious problems. It is not a good and effective way to solve them by limiting demand, increasing supply, and expanding the scale of the road. The best strategy to ensure the sustainable development of the urban traffic is adopting modern technology to transform the existing transportation system and grasp the real-time traffic conditions. It can be called ITSs (intelligent transportation systems). The key of intelligent transportation systems is to obtain comprehensive, real-time, accurate, and dynamic traffic information.

Like most other phenomena in organizational research, the intelligent transportation system cannot be observed directly. Thus, for enterprises, evaluating the intelligent transportation system can be regarded as a significant strategic issue and great challenge. To overcome it, a novel intuitionistic fuzzy MAGDM method on the basis of the improved MABAC method is designed to tackle this issue. Our work's contributions can be listed as follows:

(1) Although Liang, He, Wang, Chen, and Li [1] extended the MABAC to the intuitionistic fuzzy environment on the basis of novel generalized measures, these measures may generate situations which do not consider wavering in IFSs. Opposite, depending on the distance measures introduced in this paper, our method can reflect intuitionistic fuzzy information more comprehensively. Besides, the calculation process of our method is simpler.

(2) There are various criteria in the intelligent transportation system evaluation which frequently have

different weights. Since the DMs are restrained through their limited knowledge, it not easy to assign the criteria weights correctly. In this paper, an objectively weight-determining method is built to calculate the values of weight.

The remainder of this paper proceeds as follows. A literature review is given in Section 2. The knowledge of IFSs is concisely listed in Section 3. The improved MABAC method with IFSs is defined for MAGDM in Section 4. An empirical application for evaluating the intelligent transportation system is given and some comparative analyses are also offered in Section 5. At last, the conclusion of this work is given in Section 6.

2. Literature Review

Since the process of evaluating the intelligent transportation system is filled with uncertainty and ambiguity [2, 3], thus, in order to improve the accuracy of MAGDM, Zadeh [4] built the fuzzy sets (FSs). Atanassov [5] built the intuitionistic fuzzy sets (IFSs). Garg [6] presented the intuitionistic fuzzy multiplicative preference relations and defined several geometric operators. Gou, Xu, and Lei [7] built the exponential operational law of IFNs. Garg [8] defined the intuitionistic fuzzy averaging fused operators with hesitation degrees. He, He, and Huang [9] integrated the power operators with IFSs. Liu, Liu, and Chen [10] built the BM operator and Dombi operations under IFSs. Gupta, Arora, and Tiwari [11] built the fuzzy entropy through IFSs and parameter alpha. Li and Wu [12] presented the intuitionistic fuzzy cross-entropy distance and grey correlation analysis method. Khan and Lohani [13] defined the similarity measure of IFNs through the distance measure of bounded variation. Li, Liu, Liu, Su, and Wu [14] built the grey target decision-making for IFNs. Bao, Xie, Long, and Wei [15] defined the prospect theory and the evidential reasoning method under IFSs. Chen, Cheng, and Lan [16] built the TOPSIS method for MCDM through similarity measures under IFSs. Gan and Luo [17] used a hybrid method with the decision-making trial and evaluation laboratory (DEMATEL) and IFSs. Gupta, Mehlawat, Grover, and Chen [18] defined the superiority and inferiority ranking (SIR) method under IFSs. Hao, Xu, Zhao, and Zhang [19] defined the intuitionistic fuzzy method through the decision field. Krishankumar, Arvinda, Amrutha, Premaladha, and Ravichandran [20] integrated AHP with IFSs to design a GDM method for effective cloud vendor selection. Krishankumar, Ravichandran, and Saeid [21] built the IF-PROMETHEE method. Luo and Wang [22] built the VIKOR method with distance measure for IFSs. Rouyendegh [23] integrated the ELECTRE method under IFSs to tackle some MCDM issues. Cali and Balaman [24] extended ELECTRE I with the VIKOR method in the context of intuitionistic fuzzy to reflect the decision makers' preferences. Phochanikorn and Tan [25] incorporated DEMATEL with ANP to determine uncertainties and interdependencies among criteria and modified VIKOR to evaluate the sustainable supplier performance's desired level under the intuitionistic fuzzy context. Liu [26] researched on the teaching quality evaluation of physical education with the intuitionistic fuzzy TOPSIS method.

MABAC method was initially developed through Pamucar and Cirovic [27] to solve MAGDM. Compared with other MAGDM models, MABAC method is used to obtain the alternatives' order by calculating the potential values of gains and losses. This method has been extended to various fuzzy environments. For example, Sahin and Altun [28] integrated MABAC with the probabilistic neutrosophic hesitant fuzzy environment. Wei, He, Lei, Wu, and Wei [29] defined the probabilistic uncertain linguistic MABAC. Wei et al. [30] defined the uncertain probabilistic linguistic MABAC method. Xu, Shi, Zhang, and Liu [31] designed the MABAC with heterogeneous criteria information. Liang, He, Wang, Chen, and Li [1] put forward some novel distance measures of IFSs and combined them with the MABAC method to tackle MCGDM issues. Jia, Liu, and Wang [32] designed two models which were an IF-MABAC and an IFRN-MABAC model, respectively. Liang, Zhao, Wu, and Dai [33] defined the MABAC method related to TFNs.

3. Preliminaries

3.1. IFSs

Definition 1 (see [5]). An IFS on the universe X is defined:

$$I = \{\langle x, \mu_I(x), \nu_I(x)\rangle | x \in X\}, \tag{1}$$

where $\mu_I(x) \in [0,1]$ is called the "membership degree of I" and $\nu_I(x) \in [0,1]$ is called the "nonmembership degree of I," and $\mu_I(x)$, $\nu_I(x)$ meet the mathematical condition: $0 \le \mu_I(x) + \nu_I(x) \le 1$, $\forall x \in X$.

Definition 2 (see [34]). Let $I_1 = (\mu_1, \nu_1)$ and $I_2 = (\mu_2, \nu_2)$ be two IFNs; the operation of them is defined:

$$I_1 \oplus I_2 = (\mu_1 + \mu_2 - \mu_1\mu_2, \nu_1\nu_2), \tag{2}$$

$$I_1 \otimes I_2 = (\mu_1\mu_2, \nu_1 + \nu_2 - \nu_1\nu_2), \tag{3}$$

$$\lambda I_1 = \left(1 - (1-\mu_1)^\lambda, \nu_1^\lambda\right), \quad \lambda > 0, \tag{4}$$

$$I_1^\lambda = \left(\mu_1^\lambda, 1 - (1-\nu_1)^\lambda\right), \quad \lambda > 0. \tag{5}$$

Definition 3 (see [35]). Let $I_1 = (\mu_1, \nu_1)$ and $I_2 = (\mu_2, \nu_2)$ be IFNs; the score and accuracy functions of I_1 and I_2 can be expressed:

$$\begin{aligned} S(I_1) &= \mu_1 + \mu_1(1-\mu_1-\nu_1), \\ S(I_2) &= \mu_2 + \mu_2(1-\mu_2-\nu_2), \end{aligned} \tag{6}$$

$$H(I_1) = \mu_1 + \nu_1, H(I_2) = \mu_2 + \nu_2. \tag{7}$$

For two IFNs I_1 andI_2, according to Definition 3,

(i) If $s(I_1) < s(I_2)$, then $I_1 < I_2$

(ii) If $s(I_1) > s(I_2)$, then $I_1 > I_2$
(iii) If $s(I_1) = s(I_2)$ and $h(I_1) < h(I_2)$, then $I_1 < I_2$
(iv) If $s(I_1) = s(I_2)$ and $h(I_1) > h(I_2)$, then $I_1 > I_2$
(v) If $s(I_1) = s(I_2)$ and $h(I_1) = h(I_2)$, then $I_1 = I_2$

Definition 4 (see [22]). Let $I_1 = (\mu_1, \nu_1)$ and $I_2 = (\mu_2, \nu_2)$ be IFNs; the Hamming distance between two IFNs is defined:

$$\mathrm{IFHD}(I_1, I_2) = \frac{1}{6}(\ell_1 + \ell_2 + \ell_3), \tag{8}$$

where

$$\begin{aligned}
\ell_1 &= \frac{|\mu_1 - \mu_2| + |\nu_1 - \nu_2| + |(\mu_1 + 1 - \nu_1) - (\mu_2 + 1 - \nu_2)|}{2},\\
\ell_2 &= \frac{\pi_1 + \pi_2}{2},\\
\ell_3 &= \max\left(|\mu_1 - \mu_2|, |\nu_1 - \nu_2|, \frac{|\pi_1 - \pi_2|}{2}\right).
\end{aligned} \tag{9}$$

3.2. Intuitionistic Fuzzy Aggregation Operators. Under the context of the IFSs, some operators are introduced, including intuitionistic fuzzy weighted averaging (IFWA) and intuitionistic fuzzy weighted geometric (IFWG) operator.

Definition 5 (see [34]). Let $I_j = (\mu_{I_j}, \nu_{I_j})\,(j = 1, 2, \ldots, n)$ be a set of IFNs; the intuitionistic fuzzy weighted averaging (IFWA) operator is defined:

$$\mathrm{IFWA}_\omega(I_1, I_2, \ldots, I_n) = \overset{n}{\underset{j=1}{\oplus}}\left(\omega_j I_j\right), \tag{10}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight of $I_j\,(j = 1, 2, \ldots, n)$ and $\omega_j > 0, \sum_{j=1}^n \omega_j = 1$.

From Definition 5, the following theorem can be obtained.

Theorem 1. *The fused value by the IFWA operator is also a IFN, where*

$$\begin{aligned}
\mathrm{IFWA}_\omega(I_1, I_2, \ldots, I_n) &= \overset{n}{\underset{j=1}{\oplus}}\left(\omega_j I_j\right)\\
&= \left(1 - \prod_{j=1}^n \left(1 - \mu_{I_j}\right)^{\omega_j}, \prod_{j=1}^n \left(\nu_{I_j}\right)^{\omega_j}\right),
\end{aligned} \tag{11}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight of $I_j\,(j = 1, 2, \ldots, n)$ and $\omega_j > 0, \sum_{j=1}^n \omega_j = 1$.

Definition 6 (see [34]). Let $I_j\,(j = 1, 2, \ldots, n)$ be a set of IFNs; the IFWG operator is defined:

$$\mathrm{IFWG}_\omega(I_1, I_2, \ldots, I_n) = \overset{n}{\underset{j=1}{\otimes}}\left(I_j\right)^{\omega_j}, \tag{12}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight of $I_j\,(j = 1, 2, \ldots, n)$ and $\omega_j > 0, \sum_{j=1}^n \omega_j = 1$.

From Definition 6, the following theorem can be obtained.

Theorem 2. *The fused value by the IFWG operator is also an IFN, where*

$$\begin{aligned}
\mathrm{IFWG}_\omega(I_1, I_2, \ldots, I_n) &= \overset{n}{\underset{j=1}{\otimes}}\left(I_j\right)^{\omega_j}\\
&= \left(\prod_{j=1}^n \left(\mu_{I_j}\right)^{\omega_j}, 1 - \prod_{j=1}^n \left(1 - \nu_{I_j}\right)^{\omega_j}\right),
\end{aligned} \tag{13}$$

where $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ is the weight vector of $I_j\,(j = 1, 2, \ldots, n)$ and $\omega_j > 0, \sum_{j=1}^n \omega_j = 1$.

4. MABAC Method for MAGDM with Intuitionistic Fuzzy Information

Integrating the MABAC method with IFSs, the IF-MABAC method is given by IFNs. The calculating procedures of the designed method can be listed subsequently. Let $Z = \{Z_1, Z_2, \ldots, Z_n\}$ be a set of attributes and $z = \{z_1, z_2, \ldots z_n\}$ be the weight vector of attributes Z_j, where $r_j \in [0, 1]$, $j = 1, 2, \ldots, n, \sum_{j=1}^n r_j = 1$. Assume $H = \{H_1, H_2, \ldots, H_l\}$ is a set of DMs that have a significant degree of $h = \{h_1, h_2, \ldots, h_l\}$, where $h_k \in [0, 1]$, $k = 1, 2, \ldots, l$, $\sum_{k=1}^l h_k = 1$. Let $P = \{P_1, P_2, \ldots, P_m\}$ be a set of alternatives. And $Q = (q_{ij})_{m\times n}$ is the overall decision matrix, and q_{ij} means the value of alternative F_i regarding the attribute R_j with IFNs. Subsequently, the corresponding calculating steps will be depicted:

Step 1: build the decision maker's decision matrix $Q^{(k)} = (q_{ij}^k)_{m\times n}$ and calculate the overall decision matrix$Q = (q_{ij})_{m\times n}$:

$$Q^{(k)} = \left[q_{ij}^k\right]_{m\times n} = \begin{bmatrix} q_{11}^k & q_{12}^k & \cdots & q_{1n}^k\\ q_{21}^k & q_{22}^k & \cdots & q_{2n}^k\\ \vdots & \vdots & \vdots & \vdots\\ q_{m1}^k & q_{m2}^k & \cdots & q_{mn}^k \end{bmatrix}, \tag{14}$$

$$Q = \left[q_{ij}\right]_{m\times n} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n}\\ q_{21} & q_{22} & \cdots & q_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ q_{m1} & q_{m2} & \cdots & q_{mn} \end{bmatrix}, \tag{15}$$

$$q_{ij} = \left(1 - \prod_{k=1}^l \left(1 - \mu_{q_{ij}^k}\right)^{h_k}, \prod_{k=1}^l \left(\nu_{q_{ij}^k}\right)^{h_k}\right), \tag{16}$$

where q_{ij}^k is the assessment value of the alternative $P_i\,(i = 1, 2, \ldots, m)$ for attribute $Z_j\,(j = 1, 2, \ldots, n)$ and DM $H_k\,(k = 1, 2, \ldots, l)$.

Step 2: normalize the overall intuitionistic fuzzy matrix $Q = (q_{ij})_{m\times n}$ to $Q^N = [q_{ij}^N]_{m\times n}$:

$$q_{ij}^{N}=\begin{cases}\left(\mu_{ij},\nu_{ij}\right), & Z_j \text{ is a benefit criterion,}\\ \left(\nu_{ij},\mu_{ij}\right), & Z_j \text{is a cost criterion.}\end{cases} \tag{17}$$

Step 3: utilize the maximizing deviation method to determine the weighting matrix of attributes.

The maximizing deviation method will be integrated with IFSs in this part to determine each attribute's weight with completely unknown information. This method was initially put forward by Wang [36] which took the differences among all alternatives' performance values into consideration. Subsequently, the calculating procedures of this method are presented:

(1) Depending on the normalized overall decision matrix $Q^N = (q_{ij}^N)_{m\times n}$, the deviation of P_i to all the other alternatives could be calculated.

$$\mathrm{IFD}_{ij}=\sum_{t=1}^{m} z_j\cdot d\left(q_{ij}^{N},q_{tj}^{N}\right), \tag{18}$$

where

$$d\left(q_{ij}^{N},q_{tj}^{N}\right)=\frac{1}{6}\left(\frac{\left|\mu_{ij}-\mu_{tj}\right|+\left|\nu_{ij}-\nu_{tj}\right|+\left|\left(\mu_{ij}+1-\nu_{ij}\right)-\left(\mu_{tj}+1-\nu_{tj}\right)\right|}{2}+\frac{\pi_{ij}+\pi_{tj}}{2}+\max\left(\left|\mu_{ij}-\mu_{tj}\right|,\left|\nu_{ij}-\nu_{tj}\right|,\frac{\left|\pi_{ij}-\pi_{tj}\right|}{2}\right)\right). \tag{19}$$

(2) Calculate the total weighted deviation values of all alternatives:

$$\begin{aligned}\mathrm{IFD}_{j}(z)=\sum_{i=1}^{m}\mathrm{IFD}_{ij}(z)=\sum_{i=1}^{m}\sum_{t=1}^{m}z_j\Bigg(\frac{1}{6}\Bigg(&\frac{\left|\mu_{ij}-\mu_{tj}\right|+\left|\nu_{ij}-\nu_{tj}\right|+\left|\left(\mu_{ij}+1-\nu_{ij}\right)-\left(\mu_{tj}+1-\nu_{tj}\right)\right|}{2}+\frac{\pi_{ij}+\pi_{tj}}{2}\\&+\max\left(\left|\mu_{ij}-\mu_{tj}\right|,\left|\nu_{ij}-\nu_{tj}\right|,\frac{\left|\pi_{ij}-\pi_{tj}\right|}{2}\right)\Bigg)\Bigg).\end{aligned} \tag{20}$$

(3) Construct a nonlinear programming model with IFNs.

$$(M-1)\begin{cases}\max D(z)=\displaystyle\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{t=1}^{m}z_j\Bigg(\frac{1}{6}\Bigg(\frac{\left|\mu_{ij}-\mu_{tj}\right|+\left|\nu_{ij}-\nu_{tj}\right|+\left|\left(\mu_{ij}+1-\nu_{ij}\right)-\left(\mu_{tj}+1-\nu_{tj}\right)\right|}{2}+\frac{\pi_{ij}+\pi_{tj}}{2}\\ +\max\left(\left|\mu_{ij}-\mu_{tj}\right|,\left|\nu_{ij}-\nu_{tj}\right|,\dfrac{\left|\pi_{ij}-\pi_{tj}\right|}{2}\right)\Bigg)\Bigg),\\ \text{s.t. } z_j\geq 0,\ j=1,2,\ldots,n,\ \displaystyle\sum_{j=1}^{n}z_j^2=1.\end{cases} \tag{21}$$

To solve this model, the Lagrange function can be utilized:

$$\begin{aligned}L(z,\xi)=\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{t=1}^{m}z_j\Bigg(\frac{1}{6}\Bigg(&\frac{\left|\mu_{ij}-\mu_{tj}\right|+\left|\nu_{ij}-\nu_{tj}\right|+\left|\left(\mu_{ij}+1-\nu_{ij}\right)-\left(\mu_{tj}+1-\nu_{tj}\right)\right|}{2}+\frac{\pi_{ij}+\pi_{tj}}{2}\\&+\max\left(\left|\mu_{ij}-\mu_{tj}\right|,\left|\nu_{ij}-\nu_{tj}\right|,\frac{\left|\pi_{ij}-\pi_{tj}\right|}{2}\right)\Bigg)\Bigg)+\frac{\xi}{2}\left(\sum_{j=1}^{n}z_j^2-1\right),\end{aligned} \tag{22}$$

where ξ is the Lagrange multiplier. Then, the partial derivatives of L can be calculated:

$$\begin{cases} \dfrac{\partial L}{\partial z_j} = \displaystyle\sum_{i=1}^{m}\sum_{t=1}^{m}\left(\frac{1}{6}\left(\frac{\left|\mu_{ij}-\mu_{tj}\right|+\left|\nu_{ij}-\nu_{tj}\right|+\left|\left(\mu_{ij}+1-\nu_{ij}\right)-\left(\mu_{tj}+1-\nu_{tj}\right)\right|}{2}+\frac{\pi_{ij}+\pi_{tj}}{2}\right.\right. \\ \left.\left.+\max\left(\left|\mu_{ij}-\mu_{tj}\right|,\left|\nu_{ij}-\nu_{tj}\right|,\frac{\left|\pi_{ij}-\pi_{tj}\right|}{2}\right)\right)\right)+\xi z_j=0, \\ \dfrac{\partial L}{\partial \xi}=\dfrac{1}{2}\left(\displaystyle\sum_{j=1}^{n} z_j^2-1\right)=0. \end{cases} \tag{23}$$

And then, a simple formula for determining the weight can be obtained by solving the above equations:

$$z_j^* = \frac{\sum_{i=1}^{m}\sum_{t=1}^{m}\left(1/6\left|\mu_{ij}-\mu_{tj}\right|+\left|\nu_{ij}-\nu_{tj}\right|+|\left(\mu_{ij}+1-\nu_{ij}\right)-\left(\mu_{tj}+1-\nu_{tj}\right)|/2\right)\pi_{ij}+\pi_{tj}/2+\max\left(\left|\mu_{ij}-\mu_{tj}\right|,\left|\nu_{ij}-\nu_{tj}\right|,\left|\pi_{ij}-\pi_{tj}\right|/2\right)}{\sqrt{\sum_{j=1}^{n}\left(\sum_{i=1}^{m}\sum_{t=1}^{m}\left(1/6\left(\left|\mu_{ij}-\mu_{tj}\right|+\left|\nu_{ij}-\nu_{tj}\right|+\left|\left(\mu_{ij}+1-\nu_{ij}\right)-\left(\mu_{tj}+1-\nu_{tj}\right)\right|/2\pi_{ij}+\pi_{tj}/2+\max\left(\left|\mu_{ij}-\mu_{tj}\right|,\left|\nu_{ij}-\nu_{tj}\right|,\left|\pi_{ij}-\pi_{tj}\right|/2\right)\right)\right)\right)^2}}. \tag{24}$$

Finally, the normalized weights can be determined:

$$z_j = \frac{\sum_{i=1}^{m}\left(\sum_{t=1}^{m}1/6\left(\left|\mu_{ij}-\mu_{tj}\right|+\left|\nu_{ij}-\nu_{tj}\right|+\left|\left(\mu_{ij}+1-\nu_{ij}\right)-\left(\mu_{tj}+1-\nu_{tj}\right)\right|/2\pi_{ij}+\pi_{tj}/2+\max\left(\left|\mu_{ij}-\mu_{tj}\right|,\left|\nu_{ij}-\nu_{tj}\right|,\left|\pi_{ij}-\pi_{tj}\right|/2\right)\right)\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m}\left(\sum_{t=1}^{m}1/6\left(\left|\mu_{ij}-\mu_{tj}\right|+\left|\nu_{ij}-\nu_{tj}\right|+\left|\left(\mu_{ij}+1-\nu_{ij}\right)-\left(\mu_{tj}+1-\nu_{tj}\right)\right|/2\pi_{ij}+\pi_{tj}/2+\max\left(\left|\mu_{ij}-\mu_{tj}\right|,\left|\nu_{ij}-\nu_{tj}\right|,\left|\pi_{ij}-\pi_{tj}\right|/2\right)\right)\right)}. \tag{25}$$

Step 4: calculate the weighted matrix $O=(o_{ij})_{m\times n}$ by equation (12):

$$o_{ij}=z_j\cdot q_{ij}^N=\left(1-\left(1-\mu_{q_{ij}^N}\right)^{z_j},\nu_{q_{ij}^N}^{z_j}\right). \tag{26}$$

Step 5: compute the border approximation area matrix $G=(g_i)_{1\times n}$. The border approximation area (BAA) for every attribute is obtained from the following equation:

$$g_j=\prod_{i=1}^{m}\left(o_{ij}\right)^{1/m}=\left(\prod_{i=1}^{m}\left(\mu_{o_{ij}}\right)^{1/m},1-\prod_{i=1}^{m}\left(1-\nu_{o_{ij}}\right)^{1/m}\right). \tag{27}$$

Step 6: calculate the distance matrix $D=(d_{ij})_{m\times n}$. The alternatives' distances from the BAA are derived with the following equation:

$$d_{ij}=\begin{cases}\left(d\left(o_{ij},g_j\right)\right)^{\vartheta}, & \text{if } S\left(o_{ij}\right)\geq S\left(g_j\right),\\ -\rho\left(d\left(o_{ij},g_j\right)\right)^{\varsigma}, & \text{if } S\left(o_{ij}\right)<S\left(\mathrm{g}_j\right),\end{cases} \tag{28}$$

where the distance measure is defined as equation (8). ϑ and ς are the parameters of DMs' risk attitudes, and ρ is the loss aversion's parameter. In this article, $\vartheta=0.88$, $\varsigma=0.88$, and $\rho=2.25$. The values come from Tversky and Kahneman [37] who conducted an experiment to determine the most acceptable values from numerous researchers.

Now, if $d_{ij}=0$, the alternative P_i will belong to the border approximation area (G). If $d_{ij}>0$, P_i belongs to the upper approximation area (G^+). And if $d_{ij}<0$, P_i belongs to the lower approximation area (G^-). G^+ is the area involving the positive alternative (P^+), whereas G^- is the area involving the negative alternative (P^-).

Step 7: calculate the final value of criterion functions F_i:

$$F_i = \sum_{j=1}^{n} d_{ij}, \quad i = 1, 2, \ldots, m, \ j = 1, 2, \ldots, n. \tag{29}$$

Step 8: depending on the calculating results of F_i, all the alternatives could be ranked. The larger the value of F_i is, the optimal the alternative will be.

5. Numerical Example and Comparative Analysis

5.1. Numerical Example. Intelligent transportation system is the development direction of the future traffic system. It is the advanced information technology, data communication transmission technology, electronic sensor technology, control technology, and computer technology to effectively integrate with the whole ground traffic management system and establish a large-range, all-round function, real-time, accurate, and efficient integrated transportation management system. Not only that, the high-tech project is a process full of unknown by its size, complexity of technology, economic investment, the degree of market demand, and other aspects of influence and restriction. Therefore, the project evaluation plays an important role during the process of investment to project the overall technology evaluation, market evaluation, and economic evaluation; risk forecast has a great impact on the project decision makers for the project development scheme and is also the key to the success of a project. Intelligent transportation system evaluation could be regarded as the MADM or MAGDM issues [38–45]. In this section, an empirical application of evaluating the intelligent transportation system is provided with the IF-MABAC method. There are five potential cities $P_i\,(i = 1, 2, 3, 4, 5)$ preparing to evaluate their intelligent transportation system. In order to assess these cities fairly, five experts $H = \{H_1, H_2, H_3, H_4, H_5\}$ (expert's weight $h = (0.20, 0.20, 0.20, 0.20, 0.20)$ are invited. All experts could give their assessment information through four subsequent attributes: ① Z_1 is the intelligent transportation environment; ② Z_2 is the intelligent transportation cost; ③ Z_3 is the intelligent transportation safety; and ④ Z_4 is the intelligent transportation equipment investment. Evidently, Z_2 is the cost attribute, while Z_1, Z_3, and Z_4 are the benefit attributes.

Step 1: build each DM's matrix $Q^{(k)} = (q_{ij}^{k})_{m\times n}$ as in Tables 1–5. Derived from the tables and equations (14)–(16), the overall decision matrix could be calculated. The results are recorded in Table 6.

Step 2: normalize the matrix $Q = [q_{ij}]_{m\times n}$ to $Q^N = [q_{ij}^N]_{m\times n}$ (see Table 7).

Step 3: decide the attribute weights $z_j\,(j = 1, 2, \ldots, n)$ through the maximizing deviation method (see Table 8).

Step 4: calculate the weighted matrix $O = (o_{ij})_{m\times n}$ by utilizing equation (26) (Table 9).

Step 5: determine the BAA matrix $G = (g_j)_{1\times n}$ (Table 10).

Table 1: Intuitionistic fuzzy matrix by H_1.

	Z_1	Z_2	Z_3	Z_4
P_1	(0.63, 0.15)	(0.45, 0.50)	(0.57, 0.31)	(0.26, 0.63)
P_2	(0.70, 0.30)	(0.21, 0.69)	(0.72, 0.28)	(0.64, 0.22)
P_3	(0.39, 0.51)	(0.38, 0.48)	(0.50, 0.40)	(0.61, 0.30)
P_4	(0.53, 0.37)	(0.42, 0.51)	(0.35, 0.56)	(0.55, 0.34)
P_5	(0.26, 0.69)	(0.58, 0.35)	(0.55, 0.35)	(0.69, 0.13)

Table 2: Intuitionistic fuzzy matrix by H_2.

	Z_1	Z_2	Z_3	Z_4
P_1	(0.56, 0.33)	(0.21, 0.53)	(0.49, 0.35)	(0.57, 0.43)
P_2	(0.56, 0.33)	(0.28, 0.63)	(0.75, 0.25)	(0.67, 0.25)
P_3	(0.52, 0.37)	(0.16, 0.68)	(0.49, 0.51)	(0.58, 0.35)
P_4	(0.71, 0.18)	(0.35, 0.57)	(0.45, 0.47)	(0.56, 0.34)
P_5	(0.59, 0.39)	(0.26, 0.65)	(0.46, 0.52)	(0.71, 0.11)

Table 3: Intuitionistic fuzzy matrix by H_3.

	Z_1	Z_2	Z_3	Z_4
P_1	(0.19, 0.65)	(0.30, 0.60)	(0.54, 0.37)	(0.54, 0.35)
P_2	(0.80, 0.20)	(0.24, 0.58)	(0.75, 0.15)	(0.77, 0.23)
P_3	(0.58, 0.39)	(0.19, 0.66)	(0.44, 0.51)	(0.49, 0.39)
P_4	(0.48, 0.47)	(0.23, 0.53)	(0.63, 0.30)	(0.67, 0.20)
P_5	(0.54, 0.35)	(0.26, 0.55)	(0.41, 0.57)	(0.69, 0.15)

Table 4: Intuitionistic fuzzy matrix by H_4.

	Z_1	Z_2	Z_3	Z_4
P_1	(0.56, 0.25)	(0.32, 0.58)	(0.59, 0.35)	(0.58, 0.25)
P_2	(0.66, 0.20)	(0.36, 0.64)	(0.55, 0.25)	(0.52, 0.33)
P_3	(0.53, 0.31)	(0.43, 0.51)	(0.34, 0.41)	(0.41, 0.35)
P_4	(0.43, 0.37)	(0.29, 0.63)	(0.55, 0.30)	(0.49, 0.51)
P_5	(0.59, 0.29)	(0.39, 0.55)	(0.27, 0.67)	(0.63, 0.19)

Table 5: Intuitionistic fuzzy matrix by H_5.

	Z_1	Z_2	Z_3	Z_4
P_1	(0.39, 0.55)	(0.26, 0.68)	(0.47, 0.38)	(0.58, 0.27)
P_2	(0.72,0.15)	(0.32, 0.64)	(0.64, 0.25)	(0.70, 0.30)
P_3	(0.48, 0.51)	(0.23, 0.58)	(0.54, 0.41)	(0.44, 0.55)
P_4	(0.58, 0.33)	(0.36, 0.53)	(0.60, 0.30)	(0.25, 0.61)
P_5	(0.44, 0.55)	(0.29, 0.65)	(0.51, 0.39)	(0.39, 0.59)

Table 6: Overall intuitionistic fuzzy matrix.

	Z_1	Z_2	Z_3	Z_4
P_1	(0.4874, 0.3382)	(0.3130, 0.5747)	(0.5342, 0.3512)	(0.5187, 0.3641)
P_2	(0.7184, 0.2087)	(0.2840, 0.6350)	(0.6906, 0.2309)	(0.6696, 0.2628)
P_3	(0.5039, 0.4103)	(0.2863, 0.5766)	(0.4662, 0.4452)	(0.5123, 0.3796)
P_4	(0.5575, 0.3284)	(0.3331, 0.5524)	(0.5265, 0.3718)	(0.5219, 0.3727)
P_5	(0.4975, 0.4319)	(0.3695, 0.5372)	(0.4479, 0.4860)	(0.6371, 0.1889)

Table 7: The normalized intuitionistic fuzzy matrix.

	Z_1	Z_2	Z_3	Z_4
P_1	(0.4874, 0.3382)	(0.5747, 0.3130)	(0.5342, 0.3512)	(0.5187, 0.3641)
P_2	(0.7184, 0.2087)	(0.6350, 0.2840)	(0.6906, 0.2309)	(0.6696, 0.2628)
P_3	(0.5039, 0.4103)	(0.5766, 0.2863)	(0.4662, 0.4452)	(0.5123, 0.3796)
P_4	(0.5575, 0.3284)	(0.5524, 0.3331)	(0.5265, 0.3718)	(0.5219, 0.3727)
P_5	(0.4975, 0.4319)	(0.5372, 0.3695)	(0.4479, 0.4860)	(0.6371, 0.1889)

Table 8: The attribute weights r_j.

	Z_1	Z_2	Z_3	Z_4
z_j	0.2793	0.1699	0.2845	0.2663

Table 9: Intuitionistic fuzzy weighted normalized performance values of alternatives.

	Z_1	Z_2	Z_3	Z_4
P_1	(0.1703, 0.7387)	(0.1352, 0.8209)	(0.1954, 0.7425)	(0.1769, 0.7641)
P_2	(0.2981,0.6456)	(0.1574, 0.8074)	(0.2838, 0.6590)	(0.2554, 0.7005)
P_3	(0.1778, 0.7797)	(0.1359, 0.8085)	(0.1635, 0.7944)	(0.1740, 0.7727)
P_4	(0.2036, 0.7327)	(0.1277, 0.8296)	(0.1916, 0.7547)	(0.1784, 0.7689)
P_5	(0.1749, 0.7910)	(0.1227, 0.8444)	(0.1555, 0.8144)	(0.2366, 0.6416)

Table 10: BAA.

	BAA
Z_1	(0.2002, 0.7422)
Z_2	(0.1353, 0.8227)
Z_3	(0.1933, 0.7585)
Z_4	(0.2015, 0.7341)

Step 6: calculate the distance matrix $D = (d_{ij})_{m\times n}$ (see Table 11).

Step 7: sum up each row's elements, and each alternative's final value F_i can be determined as in Table 12.

Step 8: relying on F_i, all the alternatives could be ranked; the larger the value of F_i is, the optimal the alternative will be. Evidently, the rank of all alternatives is $P_2 > P_1 > P_4 > P_3 > P_5$, and P_2 is the optimal city.

5.2. Comparative Analysis. First of all, the designed method is compared with IFWA and IFWG operators [34]. For the IFWA operator, the calculating result is $S(P_1) = 0.5936$, $S(P_2) = 0.7358$, $S(P_3) = 0.5620$, $S(P_4) = 0.5971$, and $S(P_5) = 0.5961$. Thus, the ranking order is$P_2 > P_4 > P_5 > P_1 > P_3$. For the IFWG operator, the calculating result is $S(P_1) = 0.5922$, $S(P_2) = 0.7336$, $S(P_3) = 0.5573$, $S(P_4) = 0.5963$, and $S(P_5) = 0.5724$. So, the ranking order is $P_2 > P_4 > P_1 > P_5 > P_3$.

Furthermore, the designed method is compared with the modified IF-VIKOR method [46]. Then, we can obtain the calculating result. Then, each alternatives' relative closeness is calculated as $\mathrm{DRC}_1 = 0.8683$, $\mathrm{DRC}_2 = 0.0000$, $\mathrm{DRC}_3 = 1.0000$, $\mathrm{DRC}_4 = 0.8878$, and $\mathrm{DRC}_5 = 0.9366$. Hence, the order is $P_2 > P_1 > P_4 > P_5 > P_3$.

Besides, the designed method is compared with the IF-GRA method [47]. Then, we can obtain the calculating result. The grey relational grades of every alternative are $\gamma_1 = 0.8065$, $\gamma_2 = 0.9800$, $\gamma_3 = 0.7847$, $\gamma_4 = 0.8274$, and $\gamma_5 = 0.8342$. Therefore, the order is $P_2 > P_5 > P_4 > P_1 > P_3$.

In the end, the designed method is also compared with the IF-MABAC method [1]. Then, we can obtain the calculating result. The overall value of every alternative is $I_1 = 2.9135$, $I_2 = 3.3834$, $I_3 = 1.3719$, $I_4 = 2.8685$, and $I_5 = 1.0845$. Therefore, the order is $P_2 > P_1 > P_4 > P_3 > P_5$.

Eventually, the results of these methods are depicted in Table 13.

From Table 13, it is evident that the optimal enterprise is P_2, while the worst is P_3 in most cases. In other words, these methods' order is slightly different. These methods can effectively solve MAGDM from different angles.

Table 11: Distance matrix.

	Z_1	Z_2	Z_3	Z_4
P_1	−0.0948	0.0144	0.0289	−0.1133
P_2	0.1209	−0.0801	0.1186	−0.1532
P_3	−0.1165	−0.0572	−0.1189	−0.1273
P_4	−0.0591	−0.0498	−0.0404	−0.1174
P_5	−0.1338	−0.0759	−0.1529	−0.2261

Table 12: The final value.

Alternative	Final value
P_1	−0.1647
P_2	0.0063
P_3	−0.4200
P_4	−0.2667
P_5	−0.5888

Table 13: Evaluation results of these methods.

Methods	Ranking order	The optimal alternative	The worst alternative
IFWA operator [34]	$P_2 > P_4 > P_5 > P_1 > P_3$	P_2	P_3
IFWG operator [34]	$P_2 > P_4 > P_1 > P_5 > P_3$	P_2	P_3
IF-VIKOR method [46]	$P_2 > P_1 > P_4 > P_5 > P_3$	P_2	P_3
IF-GRA method [47]	$P_2 > P_5 > P_4 > P_1 > P_3$	P_2	P_3
IF-MABAC method [1]	$P_2 > P_1 > P_4 > P_3 > P_5$	P_2	P_5
The designed method	$P_2 > P_1 > P_4 > P_3 > P_5$	P_2	P_5

6. Conclusion

ITS is the trend of future traffic development. The problem of traffic jam exists in all the big cities around the world. Intelligent transportation project has made the world attach great importance in the development of the intelligent transportation system, which domestic and foreign scholars in succession of the intelligent transportation management project and related research work on performance appraisal. The performance appraisal of our national public program currently has not formed a set of appraising systems of standard and systemization and has problems of insufficient technology system, appraising subjective color, and public participation intensity. With respect to the intelligent transportation project, carrying on the project expenditure performance appraisal of the intellectual traffic has the vital significance. This paper designs an effective method for this issue since it designs a novel intuitive distance-based IF-MABAC method for evaluating the intelligent transportation system. And then, a numerical example for evaluating the intelligent transportation system has been given to confirm that this novel method is reasonable. Furthermore, to show the validity and feasibility of the developed method, some comparative analyses are also conducted. However, the main drawback of this paper is that the number of DMs and attributes is small, and interdependency of criteria is not taken into consideration, which may limit the application scope of the developed method to some extent. Furthermore, the developed method can be utilized to tackle many other MAGDM issues such as risk evaluation, project selection, and site selection [48–59].

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Retraction

Retracted: Hamacher Interactive Hybrid Weighted Averaging Operators under Fermatean Fuzzy Numbers

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Hamacher Interactive Hybrid Weighted Averaging Operators under Fermatean Fuzzy Numbers

Gulfam Shahzadi,[1] G. Muhiuddin ,[2] Muhammad Arif Butt,[3] and Ather Ashraf[3]

[1]*Department of Mathematics, University of the Punjab New Campus, Lahore, Pakistan*
[2]*Department of Mathematics, University of Tabuk, P. O. Box 741, Tabuk 71491, Saudi Arabia*
[3]*Punjab University College of Information Technology, University of the Punjab Old Campus, Lahore-54000, Pakistan*

Correspondence should be addressed to G. Muhiuddin; chishtygm@gmail.com

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A Fermatean fuzzy set is a more powerful tool to deal with uncertainties in the given information as compared to intuitionistic fuzzy set and Pythagorean fuzzy set and has energetic applications in decision-making. Aggregation operators are very helpful for assessing the given alternatives in the decision-making process, and their purpose is to integrate all the given individual evaluation values into a unified form. In this research article, some new aggregation operators are proposed under the Fermatean fuzzy set environment. Some deficiencies of the existing operators are discussed, and then, new operational law, by considering the interaction between the membership degree and nonmembership degree, is discussed to reduce the drawbacks of existing theories. Based on Hamacher's norm operations, new averaging operators, namely, Fermatean fuzzy Hamacher interactive weighted averaging, Fermatean fuzzy Hamacher interactive ordered weighted averaging, and Fermatean fuzzy Hamacher interactive hybrid weighted averaging operators, are introduced. Some interesting properties related to these operators are also presented. To get the optimal alternative, a multiattribute group decision-making method has been given under proposed operators. Furthermore, we have explicated the comparison analysis between the proposed and existing theories for the exactness and validity of the proposed work.

1. Introduction

The process of multiattribute group decision-making (MAGDM) yields the best alternative when the list of all possible alternatives has been compiled according to some certain attributes. Previously, the data about alternatives corresponding to attributes and their weights were given in crisp values. However, nowadays, uncertainties play an important part in the decision-making (DM) approach. Each alternative is allotted a preference to some certain degree to deal with the complicated system. However, information regarding real-world system is indefinite and fuzzy with a lot of ambiguities. Such type of conditions is appropriately explained by fuzzy set (FS) [1] and intuitionistic fuzzy set (IFS) [2] rather than crisp values. IFS is a more efficient tool to deal with vague information because it has both the membership degree (MD) and nonmembership degree (NMD), but there are some drawbacks. The sum of MD and NMD is constrained to unit interval in IFS's model. Pythagorean fuzzy set (PFS) was introduced by Yager [3] to tackle vague decisions more effectively. However, this model also has some restrictions; if MD of an element is 0.8 and NMD is 0.76, then $0.8^2 + 0.76^2 > 1$. Therefore, Yager [4] narrated the theory of q-rung orthopair fuzzy set (q-ROFS) with condition $0 \leq \varrho^q + \sigma^q \leq 1$. The basic notions about Fermatean fuzzy set (FFS) were studied by Senapati and Yager [5].

The idea of aggregation operators (AOs) performs a crucial role in getting an optimal solution when there are a lot of choices for one given problem. The idea of aggregation of infinite sequences was presented by Mesiar and Pap [6]. Xu [7] gave the theory of intuitionistic fuzzy (IF) AOs.

Zhao et al. [8] developed the theory of generalized AOs for IFS. The Einstein hybrid AOs under IF environment were studied by Zhao and Wei [9]. The concept of IF AOs using Einstein operations were discussed by Wang and Liu [10]. Garg [11] combined the theories of IFS and interactive averaging AOs. Garg et al. [12] gave the idea of Choquet integral aggregation operators for interval-valued IFS. Garg [13] introduced IF Hamacher AOs with entropy weight. Alcantud et al. [14] elaborated the idea of aggregation of infinite chains of IFS. Wu and Wei [15] gave the theory of Pythagorean fuzzy (PF) Hamacher AOs. Wei [16] proposed the PF interaction AOs. Shahzadi and Akram [17] combined the concept of PF numbers and Yager operators. The theory of novel interactive hybrid weighted AOs with PF environment was studied by Li et al. [18]. The idea of q-ROF power Maclaurin symmetric mean operators was narrated by Liu et al. [19]. q-rung orthopair fuzzy (q-ROF) weighted AOs were expressed by Liu and Wang [20]. The exponential aggregation operators for q-ROFS were defined by Peng et al. [21]. Some confidence levels about q-ROF AOs were studied by Joshi and Gegov [22]. The hybrid DM model under q-ROF Yager AOs was developed by Akram and Shahzadi [23]. Akram et al. [24] presented the Einstein geometric operators for q-ROF information. Akram et al. [25] gave the protraction of Einstein operators under q-ROF environment. Darko and Liang [26] examined q-ROF Hamacher AOs and their application in MAGDM with modified EDAS method. Senapati and Yager [27] elaborated the theory of Fermatean fuzzy (FF) averaging/geometric operators. Senapati and Yager [28] studied subtraction, division, and Fermatean arithmetic mean operations over FFS. Many new operations for FFS were defined by Senapati and Yager [28]. Garg et al. [29] developed the theory for the choice of a most suitable laboratory for COVID-19 test under FF environment. The effectiveness of a sanitizer in COVID-19 was discussed by Akram et al. [30]. For more knowledge and applications, the readers are suggested to study [31–44].

1.1. Motivations of Proposed Work

(i) The proposed operators have the ability to deal with the interaction between the MD and NMD.

(ii) The proposed theory shows that the change in MD will affect the NMD.

(iii) The developed operators show that there will be nonzero NMD of the whole aggregated FF numbers (FFNs) even if at least one of them is zero. Therefore, the others grades of nonmembership function of FFNs perform a significant role in the aggregation process (AP).

1.2. Contributions of Proposed Work

(i) Some novel operators such as Fermatean fuzzy Hamacher interactive weighted averaging (FFHIWA), Fermatean fuzzy Hamacher interactive ordered weighted averaging (FFHIOWA), and Fermatean fuzzy Hamacher interactive hybrid weighted averaging (FFHIHWA) operators are explored here.

(ii) Some special cases of these operators along with their attractive properties are discussed, which reduce the shortcomings of the existing operators.

(iii) Some basic steps for MAGDM under proposed operators are explained with the help of a numerical example.

(iv) The comparison analysis with other developed approaches shows the validity of proposed theory.

1.3. Framework and Organization of the Paper. The remaining paper is arranged as follows: Section 2 recalls some elementary definitions. Section 3 defines the hybrid structure of Hamacher, interactive operators, and FFNs such as FFHIWA operator along with some fundamental properties. In Section 4, we elaborate the idea of FFHIOWA operator with some attractive properties. Section 5 presents the notion of FFHIHWA operator. Section 6 discusses an algorithm to deal with MAGDM along with a numerical example. Section 7 gives a comparison analysis with FF Einstein weighted averaging (FFEWA) operator for the validity and importance of proposed theory. In Section 8, we have summarized the results.

2. Preliminaries

In this section, we recall some basic definitions.

Definition 1 (see [5]). A FFS $\mathfrak{P}$ on nonempty set $\mathfrak{D}$ is given by

$$\mathfrak{P} = \left\{\left\langle \mathfrak{r}, \varrho_{\mathfrak{P}}(\mathfrak{r}), \sigma_{\mathfrak{P}}(\mathfrak{r})\right\rangle\right\}, \tag{1}$$

where $\varrho_{\mathfrak{P}}\colon \mathfrak{D} \longrightarrow [0,1]$, $\sigma_{\mathfrak{P}}\colon \mathfrak{D} \longrightarrow [0,1]$, and $\varpi_{\mathfrak{P}}(\mathfrak{r}) = \sqrt[3]{1 - (\varrho_{\mathfrak{P}}(\mathfrak{r}))^3 - (\sigma_{\mathfrak{P}}(\mathfrak{r}))^3}$ indicate MD, NMD, and indeterminacy degree (InD), respectively.

Definition 2 (see [5]). For FFN $\mathfrak{P} = (\varrho_{\mathfrak{P}}, \sigma_{\mathfrak{P}})$, the score function and accuracy function are given as

$$\begin{aligned} S(\mathfrak{P}) &= \varrho_{\mathfrak{P}}^3 - \sigma_{\mathfrak{P}}^3,\ S(\mathfrak{P}) \in [-1,1],\\ \mathscr{A}(\mathfrak{P}) &= \varrho_{\mathfrak{P}}^3 + \sigma_{\mathfrak{P}}^3,\ \mathscr{A}(\mathfrak{P}) \in [0,1]. \end{aligned} \tag{2}$$

Definition 3 (see [5]). Consider two FFNs $\mathfrak{P}_1 = \left\langle \varrho_{\mathfrak{P}_1}, \sigma_{\mathfrak{P}_1}\right\rangle$ and $\mathfrak{P}_2 = \left\langle \varrho_{\mathfrak{P}_2}, \sigma_{\mathfrak{P}_2}\right\rangle$. Then, the following holds:

(1) If $S(\mathfrak{P}_1) < S(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \prec \mathfrak{P}_2$.

(2) If $S(\mathfrak{P}_1) > S(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \succ \mathfrak{P}_2$.

(3) If $S(\mathfrak{P}_1) = S(\mathfrak{P}_2)$, then

(a) If $\mathscr{A}(\mathfrak{P}_1) < \mathscr{A}(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \prec \mathfrak{P}_2$.
(b) If $\mathscr{A}(\mathfrak{P}_1) > \mathscr{A}(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \succ \mathfrak{P}_2$.
(c) If $\mathscr{A}(\mathfrak{P}_1) = \mathscr{A}(\mathfrak{P}_2)$, then $\mathfrak{P}_1 \sim \mathfrak{P}_2$.

$$T(\mathfrak{r},\mathfrak{s}) = \frac{\mathfrak{r}\mathfrak{s}}{\delta + (1-\delta)(\mathfrak{r}+\mathfrak{s}-\mathfrak{r}\mathfrak{s})},$$
$$T^*(\mathfrak{r},\mathfrak{s}) = \frac{\mathfrak{r}+\mathfrak{s}-\mathfrak{r}\mathfrak{s}-(1-\delta)\mathfrak{r}\mathfrak{s}}{1-(1-\delta)\mathfrak{r}\mathfrak{s}}. \tag{3}$$

(i) For $\delta = 1$, these operations become algebraic t-norm and t-conorm $T(\mathfrak{r},\mathfrak{s})) = \mathfrak{r}\mathfrak{s}$ and $T^*(\mathfrak{r},\mathfrak{s}) = \mathfrak{r} + \mathfrak{s} - \mathfrak{r}\mathfrak{s}$.

(ii) For $\delta = 2$, these operations become Einstein t-norm and t-conorm $T(\mathfrak{r},\mathfrak{s}) = \mathfrak{r}\mathfrak{s}/1 + (1-\mathfrak{r})(1-\mathfrak{s})$ and $T^*(\mathfrak{r},\mathfrak{s}) = \mathfrak{r} + \mathfrak{s}/1 + \mathfrak{r}\mathfrak{s}$.

3. Fermatean Fuzzy Hamacher Interactive Average Operators

Definition 4. Let $\mathfrak{P}_1 = \langle \varrho_1, \sigma_1\rangle$, $\mathfrak{P}_2 = \langle \varrho_2, \sigma_2\rangle$, and $\mathfrak{P} = \langle \varrho, \sigma\rangle$ be three FFNs and $\beta > 0$. Then, some arithmetic operations between them by using Hamacher norms are as follows:

(i) $\mathfrak{P}_1 \oplus \mathfrak{P}_2 = \left\langle \sqrt[3]{\prod_{\mathfrak{i}=1}^{2}(1+(\delta-1)\varrho_{\mathfrak{i}}^3) - \prod_{\mathfrak{i}=1}^{2}(1-\varrho_{\mathfrak{i}}^3)/\prod_{\mathfrak{i}=1}^{2}(1+(\delta-1)\varrho_{\mathfrak{i}}^3) + (\delta-1)\prod_{\mathfrak{i}=1}^{2}(1-\varrho_{\mathfrak{i}}^3)}, \sqrt[3]{\delta\left\{\prod_{\mathfrak{i}=1}^{2}(1-\varrho_{\mathfrak{i}}^3) - \prod_{\mathfrak{i}=1}^{2}(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3)\right\}/\prod_{\mathfrak{i}=1}^{2}(1+(\delta-1)\varrho_{\mathfrak{i}}^3) + (\delta-1)\prod_{\mathfrak{i}=1}^{2}(1-\varrho_{\mathfrak{i}}^3)}\right\rangle$,

(ii) $\beta.\mathfrak{P} = \left\langle \sqrt[3]{(1+(\delta-1)\varrho^3)^\beta - (1-\varrho^3)^\beta/(1+(\delta-1)\varrho^3)^\beta + (\delta-1)(1-\varrho^3)^\beta}, \sqrt[3]{\delta\left\{(1-\varrho^3) - (1-\varrho^3-\sigma^3)^\beta\right\}/(1+(\delta-1)\varrho^3)^\beta + (\delta-1)(1-\varrho^3)^\beta}\right\rangle$.

3.1. Weighted Average Aggregation Operators

3.1. Weighted Average Aggregation Operators. Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})(\mathfrak{i} = 1, 2, \ldots, \mathfrak{y})$ be a collection of FFNs and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\mathfrak{y}})^T$ be its weight vector (WV) such that $\kappa_{\mathfrak{i}} > 0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}} \kappa_{\mathfrak{i}} = 1$, then FFHIWA: $\Omega^{\mathfrak{y}} \longrightarrow \Omega$ is defined as

$$\mathrm{FFHIWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}) = \kappa_1\mathfrak{P}_1 \oplus \kappa_2\mathfrak{P}_2 \oplus \cdots \kappa_{\mathfrak{y}}\mathfrak{P}_{\mathfrak{y}}. \tag{4}$$

Theorem 1. *Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs, then*

$$\mathrm{FFHIWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}) = \left\langle \sqrt{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}/\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}3}, \sqrt{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}/\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}3}\right\rangle. \tag{5}$$

Proof. For $\mathfrak{y} = 1, \kappa = \kappa_1 = 1$,

$$\begin{aligned}\mathrm{FFHIWA}(\mathfrak{P}_1) &= \kappa_1\mathfrak{P}_1\\ &= \mathfrak{P}_1\\ &= (\varrho_1, \sigma_1)\\ &= \left\langle \sqrt[3]{\frac{\left(1+(\delta-1)\varrho_1^3\right) - \left(1-\varrho_1^3\right)}{\left(1+(\delta-1)\varrho_1^3\right) + (\delta-1)\left(1-\varrho_1^3\right)}}, \sqrt[3]{\frac{\delta\left\{\left(1-\varrho_1^3\right) - \left(1-\varrho_1^3-\sigma_1^3\right)\right\}}{\left(1+(\delta-1)\varrho_1^3\right)(\delta-1)\left(1-\varrho_1^3\right)}}\right\rangle.\end{aligned} \tag{6}$$

Thus, the result holds for $\mathfrak{y} = 1$. Suppose that result holds for $\mathfrak{y} = \mathfrak{p}$, i.e.,

$$\mathrm{FFHIWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{p}}) = \left\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}1+(\delta-1)\varrho_{\mathfrak{i}}^3 + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}}, \cdot\sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{p}}1+(\delta-1)\varrho_{\mathfrak{i}}^3 + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{p}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}}\right\rangle. \tag{7}$$

Now, for $\mathfrak{y} = \mathfrak{p} + 1$,

$$\begin{aligned}
\mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{p}+1}\right) &= \overset{\mathfrak{p}+1}{\underset{i=1}{\oplus}} \kappa_i \mathfrak{P}_i \\
&= \left\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{p}}\left(1-\varrho_i^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{p}}\left(1-\varrho_i^3\right)^{\kappa_i}}}, \sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{p}}\left(1-\varrho_i^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{p}}\left(1-\varrho_i^3-\sigma_i^3\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{p}}\left(1-\varrho_i^3\right)^{\kappa_i}}} \right\rangle \\
&\oplus \left\langle \sqrt[3]{\frac{\left(1+(\delta-1)\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}} - \left(1-\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}}}{\left(1+(\delta-1)\varrho_{\mathfrak{p}+1}^3\right) + (\delta-1)\left(1-\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}}}}, \sqrt[3]{\frac{\delta\left\{\left(1-\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}} - \left(1-\varrho_{\mathfrak{p}+1}^3-\sigma_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}}\right\}}{\left(1+(\delta-1)\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}} + (\delta-1)\left(1-\varrho_{\mathfrak{p}+1}^3\right)^{\kappa_{\mathfrak{p}+1}}}} \right\rangle \\
&= \left\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{p}}\left(1-\varrho_i^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{p}}\left(1-\varrho_i^3\right)^{\kappa_i}}}, \sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{p}}\left(1-\varrho_i^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{p}}\left(1-\varrho_i^3-\sigma_i^3\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{p}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{p}}\left(1-\varrho_i^3\right)^{\kappa_i}}} \right\rangle.
\end{aligned} \tag{8}$$

$\Rightarrow$ Result holds $\forall\, \mathfrak{y}$. □

Remark 1. We elaborate two cases of the FFHIWA operator:

(i) For $\delta = 1$, FFHIWA operator becomes FF interactive weighted averaging (FFIWA) operator:

$$\mathrm{FFIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \left\langle \sqrt[3]{1 - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}}, \sqrt[3]{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3-\sigma_i^3\right)^{\kappa_i}} \right\rangle. \tag{9}$$

(ii) For $\delta = 2$, FFHIWA operator becomes FF Einstein interactive weighted averaging (FFEIWA) operator:

$$\begin{aligned}
\mathrm{FFEIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) &= \left\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i} + \prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i}}}, \right. \\
&\left. \cdot \sqrt[3]{\frac{2\left\{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3-\sigma_i^3\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i} + \prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i}}} \right\rangle.
\end{aligned} \tag{10}$$

Theorem 2. *Let $\mathfrak{P}_i = (\varrho_i, \sigma_i)$ be FFNs, then the accumulated value by using FFHIWA operator is a FFN, i.e.,*

$$\mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) \in \mathrm{FFN}. \tag{11}$$

Proof. As $\mathfrak{P}_i's$ are FFNs, $0 \le \varrho_i, \sigma_i \le 1$ and $0 \le \varrho_i^3 + \sigma_i^3 \le 1$. Therefore,

$$\begin{aligned}
\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i} - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}} &= 1 - \frac{\delta\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i} + (\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}} \\
&\le 1 - \prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i} \le 1.
\end{aligned} \tag{12}$$

Also, $(1+(\delta-1)\varrho_{\mathfrak{i}}^3)\geq(1-\varrho_{\mathfrak{i}}^3)\Rightarrow\prod_{\mathfrak{i}=1}^{\mathfrak{y}}(1+(\delta-1)\varrho_{\mathfrak{i}}^3)-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}(1-\varrho_{\mathfrak{i}}^3)\geq 0$. Therefore,

$$\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}\geq 0,$$

$$\Rightarrow\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}}\geq 0. \tag{13}$$

Thus,
Moreover,

$$\begin{aligned}&\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}\\
&\leq\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}\\
&\leq\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\leq 1.\end{aligned} \tag{14}$$

Also,

$$\begin{aligned}&\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\geq 0,\\
&\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}\geq 0,\\
&\sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}}}\geq 0.\end{aligned} \tag{15}$$

Thus, □

Property 1. (idempotency). If $\mathfrak{P}_{\mathfrak{i}}=\mathfrak{P}_o=(\varrho_o,\sigma_o)$, $\forall\,\mathfrak{i}$, then

$$\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)=\mathfrak{P}_o. \tag{16}$$

Proof. Since $\mathfrak{P}_{\mathfrak{i}}=\mathfrak{P}_o=(\varrho_o,\sigma_o)\,(\forall\,\mathfrak{i}=1,2.\ldots,\mathfrak{y})$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}=1$, by Theorem 1,

$$\begin{aligned}\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)&=\left\langle\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}}},\sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_o^3-\sigma_o^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_o^3\right)^{\kappa_{\mathfrak{i}}}}}\right\rangle\\
&=\left\langle\sqrt[3]{\frac{\left(1+(\delta-1)\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}-\left(1-\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}}{\left(1+(\delta-1)\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}+(\delta-1)\left(1-\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}}},\sqrt[3]{\frac{\delta\left\{\left(1-\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}-\left(1-\varrho_o^3-\sigma_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}\right\}}{\left(1+(\delta-1)\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}+(\delta-1)\left(1-\varrho_o^3\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}}}}\right\rangle\\
&=\left\langle\sqrt[3]{\frac{\left(1+(\delta-1)\varrho_o^3\right)-\left(1-\varrho_o^3\right)}{\left(1+(\delta-1)\varrho_o^3\right)+(\delta-1)\left(1-\varrho_o^3\right)}},\sqrt[3]{\frac{\delta\left\{\left(1-\varrho_o^3\right)-\left(1-\varrho_o^3-\sigma_o^3\right)\right\}}{\left(1+(\delta-1)\varrho_o^3\right)+(\delta-1)\left(1-\varrho_o^3\right)}}\right\rangle\\
&=(\varrho_o,\sigma_o).\end{aligned} \tag{17}$$

□

Property 2. (boundedness). Let $\mathfrak{P}^-=(\min_{\mathfrak{i}}(\varrho_{\mathfrak{i}}),\max_{\mathfrak{i}}(\sigma_{\mathfrak{i}}))$ and $\mathfrak{P}^+=(\min_{\mathfrak{i}}(\varrho_{\mathfrak{i}}),\max_{\mathfrak{i}}(\sigma_{\mathfrak{i}}))$, then

$$\mathfrak{P}^-\leq\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)\leq\mathfrak{P}^+. \tag{18}$$

Proof. Let $f(\mathfrak{r})=1-\mathfrak{r}/1+(\delta-1)\mathfrak{r}$, $\mathfrak{r}\in[0,1]$, then $f'(\mathfrak{r})=-\delta/(1+(\delta-1)\mathfrak{r})^2<0$, so $f(\mathfrak{r})$ is a decreasing function (DF). As $\varrho_{\mathfrak{i},\min}^3\leq\varrho_{\mathfrak{i}}^3\leq\varrho_{\mathfrak{i},\max}^3$, $\forall\,\mathfrak{i}=1,2,\ldots,\mathfrak{y}$, then $f(\varrho_{\mathfrak{i},\max}^3)\leq f(\varrho_{\mathfrak{i}}^3)\leq f(\varrho_{\mathfrak{i},\min}^3)$, $\forall\,\mathfrak{i}$; that is, $1-\varrho_{\mathfrak{i},\max}^3/1+(\delta-1)\varrho_{\mathfrak{i},\max}^3\leq 1-\varrho_{\mathfrak{i}}^3/1+(\delta-1)\varrho_{\mathfrak{i}}^3\leq 1-\varrho_{\mathfrak{i},\min}^3/1+(\delta-1)\varrho_{\mathfrak{i},\min}^3$, $\forall\,\mathfrak{i}$. Let $\kappa_{\mathfrak{i}}\in[0,1]$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}=1$, we have

$$\begin{aligned}&\left(\frac{1-\varrho_{\mathfrak{i},\max}^3}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^3}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{1-\varrho_{\mathfrak{i}}^3}{1+(\delta-1)\varrho_{\mathfrak{i}}^3}\right)^{\kappa_{\mathfrak{i}}}\leq\left(\frac{1-\varrho_{\mathfrak{i},\min}^3}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^3}\right)^{\kappa_{\mathfrak{i}}}\\
&\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i},\max}^3}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^3}\right)^{\kappa_{\mathfrak{i}}}\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i}}^3}{1+(\delta-1)\varrho_{\mathfrak{i}}^3}\right)^{\kappa_{\mathfrak{i}}}\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i},\min}^3}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^3}\right)^{\kappa_{\mathfrak{i}}}\end{aligned}$$

$$
\begin{aligned}
&\Leftrightarrow\left(\frac{1-\varrho_{\mathfrak{i},\max}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}}\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}} \leq \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i}}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i}}^{3}}\right)^{\kappa_{\mathfrak{i}}} \leq\left(\frac{1-\varrho_{\mathfrak{i},\min}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}}\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\varpi_{\mathfrak{i}}} \\
&\Leftrightarrow\left(\frac{1-\varrho_{\mathfrak{i},\max}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}}\right) \leq \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i}}^{3}}{1+(\delta-1)\varrho_{\text{math fraki}}^{3}}\right)^{\kappa_{\mathfrak{i}}} \leq\left(\frac{1-\varrho_{\mathfrak{i},\min}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}}\right) \\
&\Leftrightarrow(\delta-1)\left(\frac{1-\varrho_{\mathfrak{i},\max}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}}\right) \leq(\delta-1) \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i}}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i}}^{3}}\right)^{\varpi_{\mathfrak{i}}} \leq(\delta-1)\left(\frac{1-\varrho_{\mathfrak{i},\min}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}}\right) \\
&\Leftrightarrow\left(\frac{\delta}{1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}}\right) \leq 1+(\delta-1) \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{1-\varrho_{\mathfrak{i}}^{3}}{1+(\delta-1)\varrho_{\mathfrak{i}}^{3}}\right)^{\kappa_{\mathfrak{i}}} \leq\left(\frac{\delta}{1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}}\right) \\
&\Leftrightarrow\left(\frac{1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}}{\delta}\right) \leq \frac{1}{1+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}/1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}} \leq\left(\frac{1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}}{\delta}\right) \\
&\Leftrightarrow\left(1+(\delta-1)\varrho_{\mathfrak{i},\min}^{3}\right) \leq \frac{\delta}{1+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}/1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}} \leq\left(1+(\delta-1)\varrho_{\mathfrak{i},\max}^{3}\right) \\
&\Leftrightarrow(\delta-1)\varrho_{\mathfrak{i},\min}^{3} \leq \frac{\delta}{1+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}/1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}-1 \leq(\delta-1)\varrho_{\mathfrak{i},\max}^{3} \\
&\Leftrightarrow\varrho_{\mathfrak{i},\min}^{3} \leq \frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\varpi_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\varpi_{\mathfrak{i}}}} \leq \varrho_{\mathfrak{i},\max}^{3}.
\end{aligned}
\tag{19}
$$

Thus,

$$
\varrho_{\mathfrak{i},\min} \leq \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\prod\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^{3}\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^{3}\right)^{\varpi_{\mathfrak{i}}}}} \leq \varrho_{\mathfrak{i},\max}. \tag{20}
$$

Consider $g(\mathfrak{s})=\delta-(\delta-1)\mathfrak{s}/(\delta-1)\mathfrak{s}$, $\mathfrak{s}\in(0,1]$, then $g'(\mathfrak{s})=-\delta/(\delta-1)\mathfrak{s}^{2}$; i.e., $g(\mathfrak{s})$ is a DF on $(0,1]$. Since $1-\varrho_{\mathfrak{i},\max}^{3}\leq 1-\varrho_{\mathfrak{i}}^{3}\leq 1-\varrho_{\mathfrak{i},\min}^{3}$, $\forall\,\mathfrak{i}$, then $g(1-\varrho_{i,\max}^{3})\leq g(1-\varrho_{\mathfrak{i}}^{3})\leq g(1-\varrho_{\mathfrak{i},\max}^{3})$, $\forall\,\mathfrak{i}$, that is, $\delta-(\delta-1)(1-\varrho_{\mathfrak{i},\min}^{3})/(\delta-1)(1-\varrho_{\mathfrak{i},\min}^{3})\leq\delta-(\delta-1)(1-\varrho_{\mathfrak{i}}^{3})/(\delta-1)(1-\varrho_{\mathfrak{i}}^{3})\leq\delta-(\delta-1)(1-\varrho_{\mathfrak{i},\max}^{3})/(\delta-1)(1-\varrho_{\mathfrak{i},\max}^{3})$. Then,

$$
\begin{aligned}
&\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}} \leq\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}} \leq\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}} \\
&\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}} \leq \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}} \leq \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}} \\
&\Rightarrow\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}} \leq \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}} \leq\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}\right)^{\sum_{\mathfrak{i}=1}^{\mathfrak{y}}\kappa_{\mathfrak{i}}} \\
&\Rightarrow\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^{3}\right)}\right) \leq \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i}}^{3}\right)}\right)^{\kappa_{\mathfrak{i}}} \leq\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^{3}\right)}\right)
\end{aligned}
$$

$$\begin{aligned}
&\Rightarrow\left(\frac{\delta}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^3\right)}\right)\leq\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\frac{\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i}}^3\right)}{(\delta-1)\left(1-\varrho_{\mathfrak{i}}^3\right)}\right)^{\kappa_{\mathfrak{i}}}+1\leq\left(\frac{\delta}{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^3\right)}\right)\\
&\Rightarrow\left(\frac{(\delta-1)\left(1-\varrho_{\mathfrak{i},\max}^3\right)}{\delta}\right)\leq\frac{1}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i}}^3\right)/(\delta-1)\left(1-\varrho_{\mathfrak{i}}^3\right)\right)^{\kappa_{\mathfrak{i}}}+1}\leq\left(\frac{(\delta-1)\left(1-\varrho_{\mathfrak{i},\min}^3\right)}{\delta}\right)\\
&\Rightarrow\left(1-\varrho_{\mathfrak{i},\max}^3\right)\leq\frac{\delta}{(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\delta-(\delta-1)\left(1-\varrho_{\mathfrak{i}}^3\right)/(\delta-1)\left(1-\varrho_{\mathfrak{i}}^3\right)\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)}\leq\left(1-\varrho_{\mathfrak{i},\min}^3\right)\\
&\Rightarrow\left(1-\varrho_{\mathfrak{i},\max}^3\right)\leq\frac{\delta}{(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\delta-(\delta-1)\sigma_{\mathfrak{i}}^3/(\delta-1)\sigma_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)}\leq\left(1-\varrho_{\mathfrak{i},\min}^3\right).
\end{aligned}\tag{21}$$

Let FFHIWA $(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}})=\mathfrak{P}=\langle\varrho_{\mathfrak{P}},\sigma_{\mathfrak{P}}\rangle$, then from inequalities (20) and (21), $\varrho_{\min}\leq\varrho_{\mathfrak{P}}\leq\varrho_{\max}$, $\sigma_{\max}\leq\sigma_{\mathfrak{P}}\leq\sigma_{\min}$, where $\varrho_{\min}=\min_i\{\varrho_{\mathfrak{i}}\}$, $\varrho_{\max}=\max_{\mathfrak{i}}\{\varrho_{\mathfrak{i}}\}$, $\sigma_{\min}=\min_{\mathfrak{i}}\{\sigma_{\mathfrak{i}}\}$, and $\sigma_{\max}=\max_{\mathfrak{i}}\{\sigma_{\mathfrak{i}}\}$. So $S(\mathfrak{P})=\varrho_{\mathfrak{P}}^3-\sigma_{\mathfrak{P}}^3\leq\varrho_{\max}^3-\sigma_{\max}^3=S(\mathfrak{P}^+)$ and $S(\mathfrak{P})=\varrho_{\mathfrak{P}}^3-\sigma_{\mathfrak{P}}^3\geq\varrho_{\min}^3-\sigma_{\min}^3=S(\mathfrak{P}^-)$. As $S(\mathfrak{P})<S(\mathfrak{P}^+)$ and $S(\mathfrak{P})>S(\mathfrak{P}^-)$,

$$\mathfrak{P}^-\leq\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)\leq\mathfrak{P}^+.\tag{22}$$

□

Property 3. (monotonicity). When $\mathfrak{P}_{\mathfrak{i}}\leq\mathscr{T}_{\mathfrak{i}}$, $\forall\,\mathfrak{i}$, then

$$\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)\leq\text{FFHIWA}\left(\mathscr{T}_1,\mathscr{T}_2,\ldots,\mathscr{T}_{\mathfrak{y}}\right).\tag{23}$$

Proof. It is similar to above. □

Property 4. (shift invariance). If $\mathscr{T}=(\varrho_{\mathscr{T}},\sigma_{\mathscr{T}})$ is another FFN, then

$$\begin{aligned}
&\text{FFHIWA}\left(\mathfrak{P}_1\oplus\mathscr{T},\mathfrak{P}_2\oplus\mathscr{T},\ldots,\mathfrak{P}_{\mathfrak{y}}\oplus\mathscr{T}\right)\\
&\quad=\text{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)\oplus\mathscr{T}.
\end{aligned}\tag{24}$$

Proof. As $\mathfrak{P}_{\mathfrak{i}},\mathscr{T}\in$ FFNs,

$$\begin{aligned}
\mathfrak{P}_{\mathfrak{i}}\oplus\mathscr{T}=&\left\langle\sqrt[3]{\frac{\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)-\left(1-\varrho_{\mathfrak{i}}^3\right)\left(1-\varrho_{\mathscr{T}}^3\right)}{\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)+(\delta-1)\left(1-\varrho_{\mathfrak{i}}^3\right)\left(1-\varrho_{\mathscr{T}}^3\right)}},\right.\\
&\left.\cdot\sqrt[3]{\frac{\delta\left\{\left(1-\varrho_{\mathfrak{i}}^3\right)\left(1-\varrho_{\mathscr{T}}^3\right)-\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)\left(1-\varrho_{\mathscr{T}}^3-\sigma_{\mathscr{T}}^3\right)\right\}}{\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)+(\delta-1)\left(1-\varrho_{\mathfrak{i}}^3\right)\left(1-\varrho_{\mathscr{T}}^3\right)}}\right\rangle
\end{aligned}\tag{25}$$

Therefore,

$$\begin{aligned}
\text{FFHIWA}\left(\mathfrak{P}_1\oplus\mathscr{T},\mathfrak{P}_2\oplus\mathscr{T},\ldots,\mathfrak{P}_{\mathfrak{y}}\oplus\mathscr{T}\right)=&\left\langle\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{i}}^3\right)\left(1-\varrho_{\mathscr{T}}^3\right)\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{i}}^3\right)\left(1-\varrho_{\mathscr{T}}^3\right)\right)^{\kappa_{\mathfrak{i}}}}},\right.\\
&\left.\cdot\sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{i}}^3\right)\left(1-\varrho_{\mathscr{T}}^3\right)\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{i}}^3-\sigma_{\mathfrak{i}}^3\right)\left(1-\varrho_{\mathscr{T}}^3-\sigma_{\mathscr{T}}^3\right)\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{i}}^3\right)\left(1-\varrho_{\mathscr{T}}^3\right)\right)^{\kappa_{\mathfrak{i}}}}}\right\rangle\\
=&\left\langle\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\left(1-\varrho_{\mathscr{T}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\left(1+(\delta-1)\varrho_{\mathscr{T}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{i}}^3\right)^{\kappa_{\mathfrak{i}}}\left(1-\varrho_{\mathscr{T}}^3\right)^{\kappa_{\mathfrak{i}}}}},\right.
\end{aligned}$$

$$
\begin{aligned}
&\cdot\sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\left(1-\varrho_{\mathcal{T}}^3\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3-\sigma_i^3\right)^{\kappa_i}\left(1-\varrho_{\mathcal{T}}^3-\sigma_{\mathcal{T}}^3\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}\left(1+(\delta-1)\varrho_{\mathcal{T}}^3\right)^{\kappa_i}+(\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\left(1-\varrho_{\mathcal{T}}^3\right)^{\kappa_i}}}\Bigg\rangle\\
&=\Bigg\langle\sqrt[3]{\frac{\left\{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}\right\}\left(1+(\delta-1)\varrho_{\mathcal{T}}^3\right)-\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\right\}\left(1-\varrho_{\mathcal{T}}^3\right)}{\left\{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}\right\}\left(1+(\delta-1)\varrho_{\mathcal{T}}^3\right)+(\delta-1)\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\right\}\left(1-\varrho_{\mathcal{T}}^3\right)}},\\
&\cdot\sqrt[3]{\frac{\delta\left(\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\right\}\left(1-\varrho_{\mathcal{T}}^3\right)-\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3-\sigma_i^3\right)^{\kappa_i}\right\}\left(1-\varrho_{\mathcal{T}}^3-\sigma_{\mathcal{T}}^3\right)\right)}{\left\{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}\right\}\left(1+(\delta-1)\varrho_{\mathcal{T}}^3\right)+(\delta-1)\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}\right\}\left(1-\varrho_{\mathcal{T}}^3\right)}}\Bigg\rangle\\
&=\mathrm{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right)\oplus\mathcal{T}.
\end{aligned}
\tag{26}
$$

□

Property 5. (homogeneity). Let $\beta>0$, then

$$
\begin{aligned}
&\mathrm{FFHIWA}\left(\beta\mathfrak{P}_1,\beta\mathfrak{P}_2,\ldots,\beta\mathfrak{P}_{\mathfrak{y}}\right)\\
&\quad=\beta\mathrm{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right).
\end{aligned}
\tag{27}
$$

Proof. Since $\mathfrak{P}_i=(\varrho_i,\sigma_i)$ are FFNs, for $\beta>0$,

$$
\beta\mathfrak{P}_i=\left\langle\sqrt[3]{\frac{\left(1+(\delta-1)\varrho^3\right)^{\beta}-\left(1-\varrho^3\right)^{\beta}}{\left(1+(\delta-1)\varrho^3\right)^{\beta}+(\delta-1)\left(1-\varrho^3\right)^{\beta}}},\sqrt[3]{\frac{\delta\left\{\left(1-\varrho^3\right)^{\beta}-\left(1-\varrho^3-\sigma^3\right)^{\beta}\right\}}{\left(1+(\delta-1)\varrho^3\right)^{\beta}+(\delta-1)\left(1-\varrho^3\right)^{\beta}}}\right\rangle.
\tag{28}
$$

Therefore,

$$
\begin{aligned}
\mathrm{FFHIWA}\left(\beta\mathfrak{P}_1,\beta\mathfrak{P}_2,\ldots,\beta\mathfrak{P}_{\mathfrak{y}}\right)&=\Bigg\langle\sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho^3\right)^{\beta}\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho^3\right)^{\beta}\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho^3\right)^{\beta}\right)^{\kappa_i}+(\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho^3\right)^{\beta}\right)^{\kappa_i}}},\\
&\cdot\sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho^3\right)^{\beta}\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho^3-\sigma^3\right)^{\beta}\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho^3\right)^{\beta}\right)^{\kappa_i}+(\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(\left(1-\varrho^3\right)^{\beta}\right)^{\kappa_i}}}\Bigg\rangle\\
&=\Bigg\langle\sqrt[3]{\frac{\left(\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho^3\right)^{\kappa_i}\right)^{\beta}-\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho^3\right)^{\kappa_i}\right)^{\beta}}{\left(\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho^3\right)^{\kappa_i}\right)^{\beta}+(\delta-1)\left(\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho^3\right)^{\kappa_i}\right)\right)^{\beta}}},\\
&\cdot\sqrt[3]{\frac{\delta\left\{\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho^3\right)^{\kappa_i}\right)^{\beta}-\left(\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho^3\right)^{\kappa_i}\right)^{\beta}\right\}}{\left(\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho^3\right)^{\kappa_i}\right)^{\beta}+(\delta-1)\left(\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho^3\right)^{\kappa_i}\right)^{\beta}}}\Bigg\rangle\\
&=\Bigg\langle\beta\sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}+(\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}}},\\
&\cdot\sqrt[3]{\frac{\delta\left\{\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3-\sigma_i^3\right)^{\kappa_i}\right\}}{\prod_{i=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_i^3\right)^{\kappa_i}+(\delta-1)\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_i^3\right)^{\kappa_i}}}\Bigg\rangle\\
&=\beta\mathrm{FFHIWA}\left(\mathfrak{P}_1,\mathfrak{P}_2,\ldots,\mathfrak{P}_{\mathfrak{y}}\right).
\end{aligned}
\tag{29}
$$

□

Property 6. Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{P}_{\mathfrak{i}}}, \sigma_{\mathfrak{P}_{\mathfrak{i}}})$ and $\mathscr{T}_{\mathfrak{i}} = (\varrho_{\mathscr{T}_{\mathfrak{i}}}, \sigma_{\mathscr{T}_{\mathfrak{i}}})$ be two collections of FFNs, then

$$\begin{aligned}&\mathrm{FFHIWA}\left(\mathfrak{P}_1 \oplus \mathscr{T}_1, \mathfrak{P}_2 \oplus \mathscr{T}_2, \ldots, \mathfrak{P}_{\mathfrak{y}} \oplus \mathscr{T}_{\mathfrak{y}}\right)\\ &\quad = \mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) \oplus \mathrm{FFHIWA}\left(\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_{\mathfrak{y}}\right).\end{aligned} \tag{30}$$

Proof. As $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{P}_{\mathfrak{i}}}, \sigma_{\mathfrak{P}_{\mathfrak{i}}})$ and $\mathscr{T}_{\mathfrak{i}} = (\varrho_{\mathscr{T}_{\mathfrak{i}}}, \sigma_{\mathscr{T}_{\mathfrak{i}}})$ are two collections of FFNs, then

$$\begin{aligned}\mathfrak{P}_{\mathfrak{i}} \oplus \mathscr{T}_{\mathfrak{i}} = \Bigg\langle &\sqrt[3]{\frac{\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)-\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)}{\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)+(\delta-1)\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)}},\\ &\cdot\sqrt[3]{\frac{\delta\left\{\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)-\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3-\sigma_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3-\sigma_{\mathscr{T}_{\mathfrak{i}}}^3\right)\right\}}{\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)+(\delta-1)\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)}}\Bigg\rangle\end{aligned} \tag{31}$$

Therefore,

$$\begin{aligned}&\mathrm{FFHIWA}\left(\mathfrak{P}_1 \oplus \mathscr{T}_1, \mathfrak{P}_2 \oplus \mathscr{T}_2, \ldots, \mathfrak{P}_{\mathfrak{y}} \oplus \mathscr{T}_{\mathfrak{y}}\right)\\
&= \Bigg\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)\right)^{\kappa_{\mathfrak{i}}}}},\\
&\quad\cdot\sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3-\sigma_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3-\sigma_{\mathscr{T}_{\mathfrak{i}}}^3\right)\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)\right)^{\kappa_{\mathfrak{i}}}}}\Bigg\rangle\\
&= \Bigg\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}}},\\
&\quad\cdot\sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3-\sigma_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3-\sigma_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}}}\Bigg\rangle\\
&= \Bigg\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}}}, \sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3-\sigma_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathfrak{P}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}}}\Bigg\rangle\\
&\quad\oplus\Bigg\langle \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}}}, \sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}-\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3-\sigma_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}+(\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\mathscr{T}_{\mathfrak{i}}}^3\right)^{\kappa_{\mathfrak{i}}}}}\Bigg\rangle\\
&= \mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) \oplus \mathrm{FFHIWA}\left(\mathscr{T}_1, \mathscr{T}_2, \ldots,\right.\end{aligned} \tag{32}$$

□

Property 7. Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ and $\mathscr{T} = (\varrho, \sigma)$ be FFNs and $\eta > 0$, then

$$\mathrm{FFHIWA}\left(\eta\mathfrak{P}_1 \oplus \mathscr{T}, \eta\mathfrak{P}_2 \oplus \mathscr{T}, \ldots, \eta\mathfrak{P}_{\mathfrak{y}} \oplus \mathscr{T}\right) = \eta\mathrm{FFHIWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) \oplus \mathscr{T}. \tag{33}$$

Proof. By applying the Properties 1, 5, and 6, we can proof it. □

4. Ordered Weighted Averaging Operator

Definition 5. Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\mathfrak{y}})^T$ be its WV such that $\kappa_{\mathfrak{i}} > 0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}} \kappa_{\mathfrak{i}} = 1$, then FFHIOWA: $\Omega^{\mathfrak{y}} \longrightarrow \Omega$ is defined as

$$\mathrm{FFHIWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}) = \kappa_1 \mathfrak{P}_{\sigma(1)} \oplus \kappa_2 \mathfrak{P}_{\sigma(2)} \oplus \cdots \kappa_{\mathfrak{y}} \mathfrak{P}_{\sigma(\mathfrak{y})}, \tag{34}$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(\mathfrak{y}))$ is a permutation of $(1, 2, \ldots, \mathfrak{y})$ such that $\sigma(\mathfrak{i}-1) \geq \sigma(\mathfrak{i})$ for any $\mathfrak{i}$.

Theorem 3. *Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs, then*

$$\begin{aligned}\mathrm{FFHIOWA}((\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}) = \Bigg\langle & \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}}, \\ & \cdot \sqrt[3]{\frac{\delta\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3-\sigma_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}}\Bigg\rangle\end{aligned} \tag{35}$$

Proof. It is similar to Theorem 1. □

Remark 2. We elaborate two cases of the FFHIOWA operator.

(i) For $\delta = 1$, FFHIOWA operator becomes FF interactive ordered weighted averaging (FFIOWA) operator:

$$\mathrm{FFIOWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}) = \left\langle \sqrt[3]{1 - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}, \sqrt[3]{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3-\sigma_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}} \right\rangle. \tag{36}$$

(ii) For $\delta = 2$, FFHIOWA operator becomes FF Einstein interactive ordered weighted averaging (FFEIOWA) operator:

$$\begin{aligned}\mathrm{FFEIOWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}) = \Bigg\langle & \sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} + \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}}, \\ & \cdot \sqrt[3]{\frac{2\left\{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3-\sigma_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}\right\}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} + \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}}.\Bigg\rangle\end{aligned} \tag{37}$$

Property 8. Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\mathfrak{y}})^T$ be its WV such that $\kappa_{\mathfrak{i}} > 0$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}} \kappa_{\mathfrak{i}} = 1$.

(i) Idempotency: if $\mathfrak{P}_{\mathfrak{i}} = \mathfrak{P}_o = (\varrho_o, \sigma_o)$, $\forall\, \mathfrak{i}$, then

$$\mathrm{FFHIOWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}) = \mathfrak{P}_o. \tag{38}$$

(ii) Boundedness: let $\mathfrak{P}^- = (\min_{\mathfrak{i}}(\varrho_{\mathfrak{i}}), \max_{\mathfrak{i}}(\sigma_{\mathfrak{i}}))$ and $\mathfrak{P}^+ = (\max_{\mathfrak{i}}(\varrho_{\mathfrak{i}}), \min_{\mathfrak{i}}(\sigma_{\mathfrak{i}}))$, then

$$\mathfrak{P}^- \leq \mathrm{FFHIOWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}) \leq \mathfrak{P}^+. \tag{39}$$

(iii) Monotonicity: when $\mathfrak{P}_{\mathfrak{i}} \leq \mathscr{T}_{\mathfrak{i}}, \forall\, \mathfrak{i}$, then

$$\begin{aligned}&\mathrm{FFHIOWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}) \\ &\leq \mathrm{FFHIOWA}(\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_{\mathfrak{y}}).\end{aligned} \tag{40}$$

(iv) Shift invariance: if $\mathscr{T} = (\varrho_{\mathscr{T}}, \sigma_{\mathscr{T}})$ is another FFN, then

$$\begin{aligned}&\mathrm{FFHIOWA}(\mathfrak{P}_1 \oplus \mathscr{T}, \mathfrak{P}_2 \oplus \mathscr{T}, \ldots, \mathfrak{P}_{\mathfrak{y}} \oplus \mathscr{T}) \\ &= \mathrm{FFHIOWA}(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}) \oplus \mathscr{T}.\end{aligned} \tag{41}$$

(v) Homogeneity: let $\beta > 0$, then

$$\begin{aligned}&\mathrm{FFHIOWA}\left(\beta\mathfrak{P}_1, \beta\mathfrak{P}_2, \ldots, \beta\mathfrak{P}_{\mathfrak{y}}\right)\\ &= \beta\mathrm{FFHIOWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right).\end{aligned} \tag{42}$$

Proof. It is similar to the FFHIWA properties. □

5. Hybrid Weighted Averaging Operator

Definition 6. Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs, then FFHIHWA: $\Omega^{\mathfrak{y}} \longrightarrow \Omega$ is defined as

$$\mathrm{FFHIHWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \kappa_1\dot{\mathfrak{P}}_1 \oplus \kappa_2\dot{\mathfrak{P}}_2 \oplus \cdots \kappa_{\mathfrak{y}}\dot{\mathfrak{P}}_{\mathfrak{y}}, \tag{43}$$

where $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_{\mathfrak{y}})^T$ is the WV associated with FFHIHWA operator and $\phi = (\phi_1, \phi_2, \ldots, \phi_{\mathfrak{y}})^T$ is the WV of $\mathfrak{P}_{\mathfrak{i}}$ such that $\phi_{\mathfrak{i}} \in [0,1]$ and $\sum_{\mathfrak{i}=1}^{\mathfrak{y}} \phi_{\mathfrak{i}} = 1$. Let $\dot{\mathfrak{P}}$ is the $\mathfrak{i}th$ largest of the weighted FFNs $((\dot{\mathfrak{P}} = \mathfrak{y}\phi_{\mathfrak{i}}\mathfrak{P}_{\mathfrak{i}})$ and $(\sigma(1), \sigma(2), \ldots, \sigma(\mathfrak{y}))$ is a permutation of $(1, 2, \ldots, \mathfrak{y})$ such that $\sigma(\mathfrak{i}-1) \geq \sigma(\mathfrak{i})$ for any $\mathfrak{i}$.

Theorem 4. *Let $\mathfrak{P}_{\mathfrak{i}} = (\varrho_{\mathfrak{i}}, \sigma_{\mathfrak{i}})$ be a collection of FFNs, then*

$$\begin{aligned}\mathrm{FFHIHWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \Bigg\langle &\sqrt[3]{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}},\\ &\cdot\sqrt[3]{\delta\left\{\frac{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} - \prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\dot{\varrho}_{\sigma(\mathfrak{i})}^3 - \dot{\sigma}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}{\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1+(\delta-1)\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}} + (\delta-1)\prod_{\mathfrak{i}=1}^{\mathfrak{y}}\left(1-\dot{\varrho}_{\sigma(\mathfrak{i})}^3\right)^{\kappa_{\mathfrak{i}}}}\right\}}\Bigg\rangle.\end{aligned} \tag{44}$$

Proof. It is similar to Theorem 1. □

Remark 3. FFHIHWA operator also satisfies the same properties as given in Property 8.

6. MAGDM under Fermatean Fuzzy Environment

In MAGDM problem, it is a biggest challenge for decision makers (DMrs) to choose the best alternative among the list of possible alternatives. Let $\{\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_{\mathfrak{y}}\}$ be $\mathfrak{y}$ distinct alternatives which can be classified under the set of m different attributes $\{c_1, c_2, \ldots, c_m\}$ by the DMrs. Suppose that DMrs give their preferences in terms of FFNs $\alpha_{\mathfrak{i}j} = (\varrho_{\mathfrak{i}j}, \sigma_{\mathfrak{i}j})(\mathfrak{i} = 1, 2, \ldots, \mathfrak{y}; j = 1, 2, \ldots, m)$, where $\varrho_{\mathfrak{i}j}$ and $\sigma_{\mathfrak{i}j}$ are the satisfaction and dissatisfaction degrees, respectively, of the alternative corresponding to given parameter given by the DMrs such that $0 \leq {\varrho_{\mathfrak{i}j}}^3 + {\sigma_{\mathfrak{i}j}}^3 \leq 1$. The different steps for MAGDM problem are given as follows:

Step 1. Attain the normalize FF decision matrix by exchanging the assessment value of cost parameter (CP) into benefit parameter (BP) [40], i.e.,

$$\mathscr{P}_{\mathfrak{i}j} = \begin{cases}\alpha_{\mathfrak{i}j}^c; & \text{for CP},\\ \alpha_{\mathfrak{i}j}; & \text{for BP}.\end{cases} \tag{45}$$

Step 2. By using the decision matrix of step 1, the overall aggregated value of alternative $\mathfrak{S}_{\mathfrak{i}}$ under the distinct choices of attributes c_j is obtained by using FFHIWA or FFHIOWA or FFHIHWA operator and get the overall value of them.

Step 3. By using the score function, calculate the score values of all alternatives.

Step 4. Rank the alternatives $\{\mathfrak{S}_1, \mathfrak{S}_2, \ldots, \mathfrak{S}_{\mathfrak{y}}\}$ in the descending order of score values and then select the most suitable alternative.

6.1. Numerical Example. To classify the air quality (AQ) of Guangzhou for the 16th Asian Olympic Games [41] held during November 12-27, 2010, the AQ data in Guangzhou for November 2006, November 2007, November 2008, and November 2009 are collected to find out the trends in the AQ. Suppose that there are three AQ monitoring stations E_1, E_2, and E_3, which are considered as DMrs and suppose that the 0.314, 0.355, and 0.331 are weights of E_1, E_2, and E_3, respectively. There are three measured indexes, namely, $SO_2\,(c_1)$, $NO_2\,(c_2)$, and $PM_10\,(c_3)$, and their weight is $\kappa = (0.40, 0.20, 0.40)^T$. Let $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$, and $\mathfrak{S}_4$ be alternatives, where $\mathfrak{S}_1$ = November 2006, $\mathfrak{S}_2$ = November 2007, $\mathfrak{S}_3$ = November 2008, and $\mathfrak{S}_4$ = November 2009. Suppose that the measured values obtained from the AQ monitoring stations E_1, E_2, and E_3 under the measured indexes SO_2 (i.e., the attribute c_1), NO_2 (i.e., the attribute c_2), and PM_10 (i.e., the attribute c_3) in the form of FFNs are shown in Tables 1–3, respectively. We rank the AQ from 2006–2009 by using the proposed method.

By FFHIWA operator, the steps are as follows:

Step 1: as all criteria are of same type, decision matrix cannot be normalized. The aggregated decision matrix by using FF weighted averaging operator with WV $\lambda = (0.314, 0.355, 0.331)^T$ is shown in Table 4.

Step 2: to find the overall assessment of each alternative, we apply the FFHIWA operator for $\delta = 1$ as follows.

For $\mathscr{P}_1$,

Table 1: Air quality data by expert E_1.

E_1	c_1	c_2	c_3
$\mathfrak{S}_1$	(0.5, 0.7)	(0.9, 0.6)	(0.8, 0.7)
$\mathfrak{S}_2$	(0.3, 0.7)	(0.1, 0.9)	(0.66, 0.8)
$\mathfrak{S}_3$	(0.8, 0.3)	(0.7, 0.2)	(0.8, 0.4)
$\mathfrak{S}_4$	(0.9, 0.1)	(0.8, 0.1)	(0.7, 0.3)

Table 2: Air quality data by expert E_2.

E_1	c_1	c_2	c_3
$\mathfrak{S}_1$	(0.6, 0.88)	(0.3, 0.57)	(0.46, 0.76)
$\mathfrak{S}_2$	(0.3, 0.7)	(0.1, 0.99)	(0.4, 0.77)
$\mathfrak{S}_3$	(0.96, 0.3)	(0.7, 0.22)	(0.6, 0.01)
$\mathfrak{S}_4$	(0.88, 0.2)	(0.86, 0.01)	(0.96, 0.33)

Table 3: Air quality data by expert E_3.

E_3	c_1	c_2	c_3
$\mathfrak{S}_1$	(0.1, 0.93)	(0.7, 0.3)	(0.5, 0.3)
$\mathfrak{S}_2$	(0.9, 0.4)	(0.3, 0.56)	(0.47, 0.66)
$\mathfrak{S}_3$	(0.93, 0.3)	(0.76, 0.2)	(0.76, 0.1)
$\mathfrak{S}_4$	(0.97, 0.4)	(0.88, 0.3)	(0.89, 0.5)

Table 4: Aggregated FF decision matrix.

	c_1	c_2	c_3
$\mathfrak{S}_1$	(0.4031, 0.8400)	(0.6208, 0.4901)	(0.5800, 0.5889)
$\mathfrak{S}_2$	(0.4986, 0.6007)	(0.1662, 0.7252)	(0.5048, 0.7430)
$\mathfrak{S}_3$	(0.8998, 0.3000)	(0.7199, 0.2071)	(0.7158, 0.1623)
$\mathfrak{S}_4$	(0.9161, 0.2348)	(0.8478, 0.1343)	(0.8552, 0.3769)

$$
\begin{aligned}
&= \mathrm{FFHIWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{1-\left(1-0.4031^3\right)^{0.40}\left(1-0.6208^3\right)^{0.20}\left(1-0.5800^3\right)^{0.40}},\right.\\
&\cdot \sqrt[3]{\left(1-0.4031^3\right)^{0.40}\left(1-0.6208^3\right)^{0.20}\left(1-0.5800^3\right)^{0.40}-\left(1-0.4031^3-0.8400^3\right)^{0.40}\left(1-0.6208^3-0.4901^3\right)^{0.20}\left(1-0.5800^3-0.5889^3\right)^{0.40}}\\
&= (0.5374, 0.7106).
\end{aligned}
\tag{46}
$$

For $\mathscr{P}_2$,

$$
\begin{aligned}
&= \mathrm{FFHIWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{1-\left(1-0.4986^3\right)^{0.40}\left(1-0.1662^3\right)^{0.20}\left(1-0.5048^3\right)^{0.40}},\right.\\
&\cdot \left.\sqrt[3]{\left(1-0.0.4986^3\right)^{0.40}\left(1-0.1662^3\right)^{0.20}\left(1-0.5048^3\right)^{0.40}-\left(1-0.4986^3-0.6007^3\right)^{0.40}\left(1-0.1662^3-0.7252^3\right)^{0.20}\left(1-0.5048^3-0.7430^3\right)^{0.40}}\right\rangle\\
&= (0.4691, 0.6934).
\end{aligned}
\tag{47}
$$

For $\mathscr{P}_3$,

$$\begin{aligned}
&= \mathrm{FFHIWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{1-\left(1-0.8998^3\right)^{0.40}\left(1-0.7199^3\right)^{0.20}\left(1-0.7158^3\right)^{0.40}},\right.\\
&\cdot \left.\sqrt[3]{\left(1-0.8998^3\right)^{0.40}\left(1-0.7199^3\right)^{0.20}\left(1-0.7158^3\right)^{0.40}-\left(1-0.8998^3-0.3000^3\right)^{0.40}\left(1-0.7199^3-0.2071^3\right)^{0.20}\left(1-0.7158^3-0.1623^3\right)^{0.40}}\right\rangle\\
&= (0.8191, 0.2754).
\end{aligned} \tag{48}$$

For $\mathscr{P}_4$,

$$\begin{aligned}
&= \mathrm{FFHIWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{1-\left(1-0.9161^3\right)^{0.40}\left(1-0.8478^3\right)^{0.20}\left(1-0.8552^3\right)^{0.40}},\right.\\
&\cdot \left.\sqrt[3]{\left(1-0.9161^3\right)^{0.40}\left(1-0.8478^3\right)^{0.20}\left(1-0.8552^3\right)^{0.40}-\left(1-0.9161^3-0.2348^3\right)^{0.40}\left(1-0.8478^3-0.1343^3\right)^{0.20}\left(1-0.8552^3-0.3769^3\right)^{0.40}}\right\rangle\\
&= (0.8831, 0.2951).
\end{aligned} \tag{49}$$

Step 3: the score values for alternatives are

$$\begin{aligned}
S\left(\mathfrak{S}_1\right) &= -0.2036, S\left(\mathfrak{S}_2\right) = -0.2302, S\left(\mathfrak{S}_3\right)\\
&= 0.5287, S\left(\mathfrak{S}_4\right) = 0.6630.
\end{aligned} \tag{50}$$

Step 4: as $\mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2$, the best AQ in Guangzhou is November of 2009.

The whole method which we have adopted in this application is given in Figure 1.

7. Comparison Analysis

For the validity and importance of proposed operators, we aggregate the same information using different operator, namely, FFEWA or FFEOWA operator [30].

Definition 7. (see [30]). The FFEWA operator is as follows:

$$\mathrm{FFEWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \left\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i}+\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_i^3\right)^{\kappa_i}}}, \frac{\sqrt[3]{2}\prod_{i=1}^{\mathfrak{y}}\sigma_i^{\kappa_i}}{\sqrt[3]{\prod_{i=1}^{\mathfrak{y}}\left(2-\sigma_i^3\right)^{\kappa_i}+\prod_{i=1}^{\mathfrak{y}}\left(\sigma_i^3\right)^{\kappa_i}}} \right\rangle. \tag{51}$$

The FF Einstein ordered weighted averaging (FFEOWA) operator is

$$\mathrm{FFEOWA}\left(\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_{\mathfrak{y}}\right) = \left\langle \sqrt[3]{\frac{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_{\sigma(i)}^3\right)^{\kappa_i}-\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(i)}^3\right)^{\kappa_i}}{\prod_{i=1}^{\mathfrak{y}}\left(1+\varrho_{\sigma(i)}^3\right)^{\kappa_i}+\prod_{i=1}^{\mathfrak{y}}\left(1-\varrho_{\sigma(i)}^3\right)^{\kappa_i}}}, \frac{\sqrt[3]{2}\prod_{i=1}^{\mathfrak{y}}\sigma_{\sigma(i)}^{\kappa_i}}{\sqrt[3]{\prod_{i=1}^{\mathfrak{y}}\left(2-\sigma_{\sigma(i)}^3\right)^{\kappa_i}+\prod_{i=1}^{\mathfrak{y}}\left(\sigma_{\sigma(i)}^3\right)^{\kappa_i}}} \right\rangle. \tag{52}$$

By FFEWA operator, the steps are as follows:

Step 1: same as above.

Step 2: to find the overall assessment of each alternative, we apply the FFEWA operator as

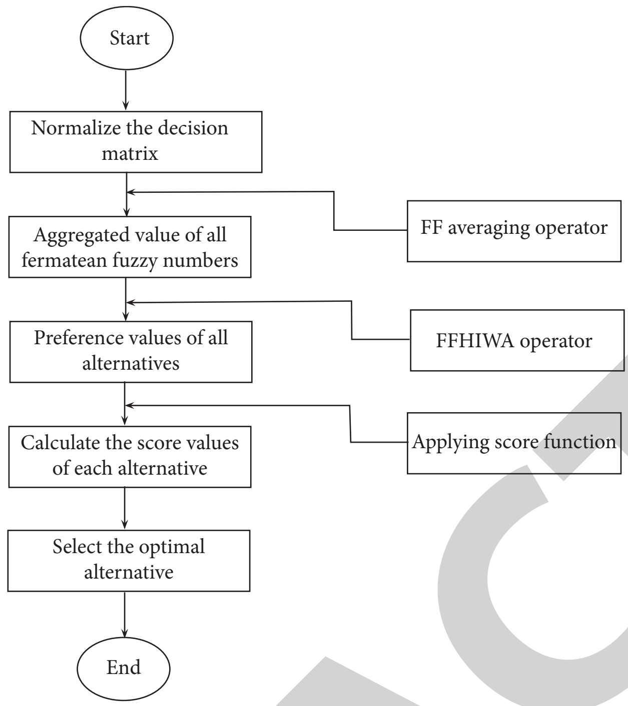

Figure 1: Flow chart for the classification of the air quality of Guangzhou.

$$
\begin{aligned}
\mathscr{P}_1 &= \mathrm{FFEWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{\frac{\left(1+0.4031^3\right)^{0.40}\left(1+0.6208^3\right)^{0.20}\left(1+0.5800^3\right)^{0.40}-\left(1-0.4031^3\right)^{0.40}\left(1-0.6208^3\right)^{0.20}\left(1-0.5800^3\right)^{0.40}}{\left(1+0.4031^3\right)^{0.40}\left(1+0.6208^3\right)^{0.20}\left(1+0.5800^3\right)^{0.40}+\left(1-0.4031^3\right)^{0.40}\left(1-0.6208^3\right)^{0.20}\left(1-0.5800^3\right)^{0.40}}},\right.\\
&\left.\frac{\sqrt[3]{2}\left((0.8400)^{0.40}(0.4901)^{0.20}(0.5889)^{0.40}\right)}{\left(2-0.8400^3\right)^{0.40}\left(2-0.4901^3\right)^{0.20}\left(2-0.5889^3\right)^{0.40}+\left(0.8400^3\right)^{0.40}\left(0.4901^3\right)^{0.20}\left(0.5889^3\right)^{0.40}}\right\rangle\\
&= (0.5347, 0.6628).
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{P}_2 &= \mathrm{FFEWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{\left(\frac{\left(1+0.4986^3\right)^{0.40}\left(1+0.1662^3\right)^{0.20}\left(1+0.5048^3\right)^{0.40}-\left(1-0.4986^3\right)^{0.40}\left(1-0.1662^3\right)^{0.20}\left(1-0.5048^3\right)^{0.40}}{\left(1+0.4986^3\right)^{0.40}\left(1+0.1662^3\right)^{0.20}\left(1+0.5048^3\right)^{0.40}+\left(1-0.4986^3\right)^{0.40}\left(1-0.1662^3\right)^{0.20}\left(1-0.5048^3\right)^{0.40}}\right)},\right.\\
&\left.\cdot\frac{\sqrt[3]{2}\left((0.6007)^{0.40}(0.7252)^{0.20}(0.7430)^{0.40}\right)}{\left(2-0.6007^3\right)^{0.40}\left(2-0.7252^3\right)^{0.20}\left(2-0.7430^3\right)^{0.40}+\left(0.6007^3\right)^{0.40}\left(0.7252^3\right)^{0.20}\left(0.7430^3\right)^{0.40}}\right\rangle\\
&= (0.4674, 0.6810).
\end{aligned}
$$

$$
\mathscr{P}_3 = \mathrm{FFEWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)
$$

$$
\begin{aligned}
&= \left\langle \sqrt[3]{\frac{\left(1+0.8998^3\right)^{0.40}\left(1+0.7199^3\right)^{0.20}\left(1+0.7158^3\right)^{0.40} - \left(1-0.8998^3\right)^{0.40}\left(1-0.7199^3\right)^{0.20}\left(1-0.7158^3\right)^{0.40}}{\left(1+0.8998^3\right)^{0.40}\left(1+0.7199^3\right)^{0.20}\left(1+0.7158^3\right)^{0.40} + \left(1-0.8998^3\right)^{0.40}\left(1-0.7199^3\right)^{0.20}\left(1-0.7158^3\right)^{0.40}}}, \right.\\
&\quad \left. \frac{\sqrt[3]{2}\left((0.3000)^{0.40}(0.2071)^{0.20}(0.1623)^{0.40}\right)}{\left(2-0.3000^3\right)^{0.40}\left(2-0.2071^3\right)^{0.20}\left(2-0.1623^3\right)^{0.40} + \left(0.3000^3\right)^{0.40}\left(0.2071^3\right)^{0.20}\left(0.1623^3\right)^{0.40}} \right\rangle\\
&= (0.8137, 0.2180).\\
&\mathscr{P}_4 = \mathrm{FFEWA}\left(\mathfrak{S}_{11}, \mathfrak{S}_{12}, \mathfrak{S}_{13}\right)\\
&= \left\langle \sqrt[3]{\frac{\left(1+0.9161^3\right)^{0.40}\left(1+0.8478^3\right)^{0.20}\left(1+0.8552^3\right)^{0.40} - \left(1-0.9161^3\right)^{0.40}\left(1-0.8478^3\right)^{0.20}\left(1-0.8552^3\right)^{0.40}}{\left(1+0.9161^3\right)^{0.40}\left(1+0.8478^3\right)^{0.20}\left(1+0.8552^3\right)^{0.40} + \left(1-0.9161^3\right)^{0.40}\left(1-0.8478^3\right)^{0.20}\left(1-0.8552^3\right)^{0.40}}}, \right.\\
&\quad \left. \frac{\sqrt[3]{2}\left((0.2348)^{0.40}(0.1343)^{0.20}(0.3769)^{0.40}\right)}{\left(2-0.2348^3\right)^{0.40}\left(2-0.1343^3\right)^{0.20}\left(2-0.3769^3\right)^{0.40} + \left(0.2348^3\right)^{0.40}\left(0.1343^3\right)^{0.20}\left(0.3769^3\right)^{0.40}} \right\rangle\\
&= (0.8831, 0.2951).
\end{aligned}
\tag{53}
$$

Step 3: the score values for alternatives are

$$
\begin{aligned}
S\left(\mathfrak{S}_1\right) &= -0.1383, S\left(\mathfrak{S}_2\right) = -0.2137, S\left(\mathfrak{S}_3\right)\\
&= 0.5284, S\left(\mathfrak{S}_4\right) = 0.6704.
\end{aligned}
\tag{54}
$$

Step 4: as $\mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2$, the best AQ in Guangzhou is November of 2009.

The results obtained from these operators are shown in Table 5 and Figure 2. It is clear that the most suitable alternative obtained by using FFHIWA and FFEWA operators is the same. This implies that our proposed methods are accurate and can be utilized in DM problems.

Advantages of proposed operators: the main reason behind proposed approach is that

(i) We can see the effect of other grades of nonmembership in the aggregated value even if nonmembership of any one alternative is zero.

(ii) We can see that there is a proper interaction between the MD and NMD.

The operators defined in [30] are very concise and have been extensively used, but these operators have certain drawbacks. Few of them have been highlighted as follows:

(1)Let $\mathfrak{P}_1 = (0.9, 0)$, $\mathfrak{P}_2 = (0.77, 0.45)$, $\mathfrak{P}_3 = (0.80, 0.63)$, and $\mathfrak{P}_4 = (0.58, 0.67)$ be four FFNs and $\kappa = (0.3, 03, 0.2, 0.2)^T$ is the WV corresponding to FFNs. By applying the FFEWA operator, we get FFEWA $(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4) = (0.88, 0)$. This shows that NMD of a FFN is independent of the NMD of others FFNs (which are nonzero in $\mathfrak{P}_i' s$) and hence does not play a significant role during the AP. The aggregated FFNs as $\mathrm{FFHIWA}(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4) = (0.81, 0.49)$ for $\delta = 1$ and for $\delta = 2$ $\mathrm{FFHIWA}(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4) = (0.88, 0.43)$. It can be seen that NMD is nonzero of the whole aggregated FFNs even if at least one of the NMD of FFNs is zero. Thus, the others nonmembership values of FFNs play a predominant role during the AP in the proposed operator.

(2) Let $\mathfrak{P}_1 = (0.53, 0.42)$, $\mathfrak{P}_2 = (0.98, 0.34)$, $\mathfrak{P}_3 = (0.61, 0.54)$, and $\mathfrak{P}_4 = (0.71, 0.46)$ be four FFNs and $\kappa = (0.4, 0.2, 0.3, 0.1)^T$ is the WV corresponding to FFNs. By applying the FFEWA operator, we get FFEWA $(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4) = (0.78, 0.44)$. If we replace FFNs $\mathfrak{P}_2$ and $\mathfrak{P}_3$ with $\mathscr{T}_2 = (0.78, 0.34)$ and $\mathscr{T}_3 = (0.67, 0.54)$, then their corresponding aggregated FFN become $(0.66, 0.44)$. Hence, the NMD part of aggregated FFN becomes independent of the change in MD's values. That is why it is incompatible and does not produce an accurate information to the decision maker. The aggregated FFNs as $\mathrm{FFHIWA}(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4) = (0.80, 0.53)$ for $\delta = 1$ and $\mathrm{FFHIWA}(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3, \mathfrak{P}_4) = (0.78, 0.55)$ for $\delta = 2$, and if we consider modified FFNs, then $FFHIWA(\mathfrak{P}_1, \mathscr{T}_2, \mathscr{T}_3, \mathfrak{P}_4) = (0.67, 0.45)$ for $\delta = 1$ and $\mathrm{FFHIWA}(\mathfrak{P}_1, \mathscr{T}_2, \mathscr{T}_3, \mathfrak{P}_4) = (0.66, 0.45)$ for $\delta = 2$. It can be seen that the modification in membership function will affect aggregated value of nonmembership function and is nonzero. That is why, there is a proper interaction between the MD and NMD, and hence, the results are unchangeable and more realistic than the existing operators results.

Table 5: Comparison analysis with FFEWA operator.

Methods	$S(\mathfrak{S}_1)$	$S(\mathfrak{S}_2)$	$S(\mathfrak{S}_3)$	$S(\mathfrak{S}_4)$	Ranking order
FFEWA operator	−0.1383	−0.2137	0.5284	0.6704	$\mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2$
FFHIWA operator (proposed)	−0.2036,	−0.2302	0.5287	0.6630	$\mathfrak{S}_4 \succ \mathfrak{S}_3 \succ \mathfrak{S}_1 \succ \mathfrak{S}_2$

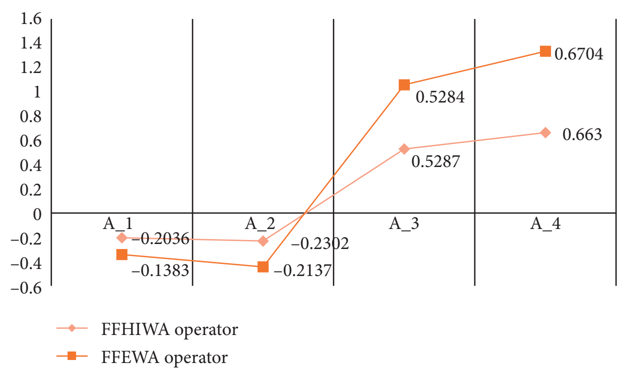

Figure 2: Comparison with FFEWA operator.

8. Conclusions

FFS is a generalized structure of IFS and PFS. It is more powerful tool to solve DM problems involving uncertainty and satisfies the condition $0 \leq \varrho^3 + \sigma^3 \leq 1$. The structure of Hamacher's t-norm and t-conorm is more generalized that effectively integrates the complex information. The shortcomings of the existing methods and beneficial characteristics of Hamacher AOs motivate us to endeavor for the development of a fruitful fusion with FFNs. In this research article, we have developed a group of novel FF Hamacher interactive averaging AOs, such as FFHIWA, FFHIOWA, and FFHIHWA operators. These proposed operators have the characteristic of idempotency, boundedness, monotonicity, homogeneity, and shift invariance. These operators reduce the shortcomings of FFEWA operators. We have also discussed some particular cases of proposed operators. Moreover, the developed operators study the interaction between membership and non-membership grades. We have presented an algorithm to deal with MAGDM problems. For the validity and flexibility of proposed work, we have given the comparison analysis. In short, this work focuses on role of Hamacher interactive AOs as well as the propitious characteristics of FFNs. It is concluded that the new model of uncertain data is flexible which aptly depicts imprecise and inexact information in complicated scenarios. Thus, the operators serve as a powerful tool with further applications due to their highly adaptable nature. In future, we will work on the following topics:

(1) Neutrality aggregation operators for Fermatean fuzzy sets.

(2) Fermatean fuzzy power aggregation operators.

(3) Fermatean fuzzy Hamy mean aggregation operators and their application in multiattribute decision making.

(4) Fermatean fuzzy soft Dombi aggregation operators.

Data Availability

No data were used to support this study.

Disclosure

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Hybrid Decision-Making Frameworks under Complex Spherical Fuzzy *N*-Soft Sets

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] M. Akram, M. Shabir, A. N. Al-Kenani, and J. C. R. Alcantud, "Hybrid Decision-Making Frameworks under Complex Spherical Fuzzy *N*-Soft Sets," *Journal of Mathematics*, vol. 2021, Article ID 5563215, 46 pages, 2021.

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Research Article

Hybrid Decision-Making Frameworks under Complex Spherical Fuzzy N-Soft Sets

Muhammad Akram [1], **Maria Shabir** [1], **Ahmad N. Al-Kenani** [2] **and José Carlos R. Alcantud** [3]

[1] *Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan*
[2] *Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80219, Jeddah 21589, Saudi Arabia*
[3] *BORDA Research Unit and IME, University of Salamanca, 37007 Salamanca, Spain*

Correspondence should be addressed to Muhammad Akram; m.akram@pucit.edu.pk

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This paper presents the novel concept of complex spherical fuzzy N-soft set ($CSFNS_fS$) which is capable of handling two-dimensional vague information with parameterized ranking systems. First, we propose the basic notions for a theoretical development of $CSFNS_fSs$, including ranking functions, comparison rule, and fundamental operations (complement, union, intersection, sum, and product). Furthermore, we look into some properties of $CSFNS_fSs$. We then produce three algorithms for multiattribute decision-making that take advantage of these elements. We demonstrate their applicability with the assistance of a numerical problem (selection of best third-party app of the year). A comparison with the performance of Pythagorean N-soft sets speaks for the superiority of our approach. Moreover, with an aim to expand the range of techniques for multiattribute group decision-making problems, we design a $CSFNS_f$-TOPSIS method. We use a complex spherical fuzzy N-soft weighted average operator in order to aggregate the decisions of all experts according to the power of the attributes and features of alternatives. We present normalized-Euclidean distances (from the alternatives to both the $CSFNS_f$ positive and negative ideal solutions, respectively) and revised closeness index in order to produce a best feasible alternative. As an illustration, we design a mathematical model for the selection of the best physiotherapist doctor of Mayo hospital, Lahore. We conduct a comparison with the existing complex spherical fuzzy TOPSIS method that confirms the stability of the proposed model and the reliability of its results.

1. Introduction

Multiattribute decision-making (MADM) and multiattribute group decision-making (MAGDM) methods are broad sections in the field of decision-making. Researchers and practitioners have resorted to them in order to evaluate optimal solutions among a finite number of choices under several attributes. For the purpose of improving the flexibility of the evaluations that support the decision-making process, Zadeh [1] proposed fuzzy set (FS) theory that reshaped the field of decision-making and related disciplines such as mathematical social sciences [2, 3]. In FS theory, a membership degree belongs to the interval $[0, 1]$; thus, when assigned to an object, it represents its degree of belongingness to a mathematical object (a fuzzy set); in formal logic, it means a degree of truth. This means an extension of binary valuations, which is, henceforth, referred to as crisp evaluations. Concerning its use for solving MADM and MAGDM problems in fuzzy environments, Song et al. [4] gave an algorithm based on arithmetic operators, and Chen [5] built up a theory for a fuzzy-TOPSIS method. No doubt, FS theory produced a turn of direction in the field of decision-making. However, it was not designed to look at the dissatisfaction nature of humans in decision-making. This drawback prompted Atanassov [6] to present intuitionistic fuzzy sets (IFS) in 1986. They allocate both degrees of satisfaction and dissatisfaction to an object.

Other extensions soon followed. Yager [7] further extended the IFS to Pythagorean fuzzy set (P_yFS) in which the sums of the squares of degree of satisfaction and dissatisfaction should be within the closed unit interval. Later on, Cuong [8] introduced picture fuzzy set (PFS) keeping in view the existence of neutral positions under natural circumstances. For example, in case of voting systems a candidate could be either satisfied, remain neutral, and disagree with any given participant [9]. Many researchers chose this environment for solving decision-making problems, but others pointed out that PFS has the limitation that it is not applicable in situations where the sum of the degrees of satisfaction, neutrality, and dissatisfaction exceeds 1. This is the origin of spherical fuzzy sets and the spherical fuzzy TOPSIS method presented by Gundogdu and Kahraman [10] in 2019. Similarly and also motivated by spherical fuzzy sets, Kahraman et al. [11] used a spherical TOPSIS method to find the best location for hospital. Later on, Mahmood et al. [12] proposed T-spherical fuzzy sets, a generalization of spherical fuzzy sets, which are less restrictive. They overcame all the limitations of the existing models except in the presence of 2-dimensional problems. Such 2-dimensional problems in MADM and MAGDM can now be analyzed with the tool developed by Ramot et al. [13], who introduced complex fuzzy set in which the degree of satisfaction belongs to the complex unit circle and consists of a periodic term as well as the amplitude term which belong to the unit closed interval. Akram and Bashir [14] extended the averaging operators in the framework of complex fuzzy sets. Alkouri and Salleh [15] presented the idea of complex intuitionistic fuzzy set (CIFS), which describes both degree of satisfaction and dissatisfaction within the complex unit circle, where the sum of amplitude and periodic terms of the satisfaction and dissatisfaction degrees should be within the unit interval $[0, 1]$.

Recently, Akram et al. [16] introduced the concept of complex spherical fuzzy set and extended the TOPSIS method to that setting. As an application, a model for the selection of best water supply strategy for Nohoor village in Iran was considered. This novel concept contains degrees of satisfaction, neutrality, and dissatisfaction which lie in the complex unit circle. They are further restricted by the condition that the sum of the squares of their amplitude and phase terms should be less than or equal to 1.

There is a widespread handicap in the aforementioned models and methods: they discard the frameworks that are characterized by the satisfaction of certain attributes or the fulfilment of properties. Soft set theory, launched in 1999, accommodates all type of parameters [17]. Alkouri and Salleh [15] introduced some new operators on soft set theory which soon found applications in the fields of operations research, game theory, stability, regularization, medicine, and obviously in decision-making. Following this trend, researchers brought up many models and methods for soft sets and its extensions, inclusive of a new decision-making method for valuation fuzzy soft sets introduced by Alcantud et al. [18]. Despite these improvements there were still problems in real life that could not be solved using the existing MADM and MAGDM methods, for example, because the objects are evaluated using a ranking system or a nonbinary scale. When we check out from hotels, hotel staff ask for our feedback, which we give, for example, in the form of 4 stars, 3 stars, 2 stars, 1 star, and big dot: 4 stars mean "outstanding," 3 stars mean "superb," 2 stars mean "good," 1 star means "satisfactory," and big dot means "unacceptable." Similarly, nonbinary rates are given to third-party apps, whether we use a transportation service (Uber, Cabify, etc.) or online shopping facilities. As technology improved and extended, people have become accostumed to such types of ranking systems due to their ease of use and widespread utilization. For this reason, many researchers have become interested in formal models for nonbinary evaluations. The idea presented by Fatima et al. [19], namely, N-soft set and their decision-making methods, stirred up new decision-making methodologies. Very soon and keeping in view the possible fuzziness of the parameters, Akram et al. [20] combined the concept of N-soft with a fuzzy definition of the attributes thus producing fuzzy N-soft sets (FNS_fS). This novel prescription involves a finite number of ordered grades as well as fuzziness in the conception of the attributes that are used for decision-making. Still another hybrid model called hesitant N-soft set was introduced by Akram et al. [21] in order to allow for hesitancy in the allocation of grades. Hesitant fuzzy N-soft sets [22] combine the features of these two models. Akram et al. [23] extended the idea of fuzzy N-soft set in another direction. They conceived intuitionistic fuzzy N-soft sets ($IFNS_fS$) that describe the dissatisfactory part separately, with the usual constraint that the sums of the degrees of membership and nonmembership always belong to $[0, 1]$. Finally, so far, Zhang et al. [24] extended $IFNS_fS$ to Pythagorean fuzzy N-soft set ($PFNS_fS$) which is more flexible than the existing models.

The motivation of this article depends on the following facts:

(1) The existing models $IFNS_fS$ and $PFNS_fS$ make decisions based on degrees of membership and nonmembership; however, they are unable to incorporate a neutral part of judgement.

(2) The decision-making techniques based on existing models FNS_fS, $IFNS_fS$, and $PFNS_fS$ can solve only problems of the 1-dimensional type. Neither of these models can operate in the presence of a periodic term or 2-dimensional type problems.

(3) Although CSFSs deal with 2-dimensional problems of real life, they are unable to describe parameterized information as well as finitely many ranked grades of association of the alternatives with the pertinent parameters.

(4) These limitations motivated us to put forward a new model called $CSFNS_fS$ which efficiently deals with abstention (together with degrees of satisfaction and dissatisfaction) as well as the periodic term of 2-dimensional decision-making problems. At the same time, $CSFNS_fS$ competently handles the ordered grades of the alternatives according to the different attributes.

The main contributions of this article are as follows:

(1) The proposed model, $CSFNS_fS$, allows for neutral opinions in the framework of 2-dimensional problems. In this way, it can manipulate conditions on amplitude and periodic terms with more flexibility.

(2) This model establishes a modern theory that captures a new perspective of decision-making. It is based on ratings or ranking systems including ordered grades of elements according to related attributes.

(3) The algorithms and $CSFNS_f$-TOPSIS method defined in this article solve MADM and MAGDM problems, respectively. They apply to more general situations than the existing algorithms and TOPSIS Method. These methods for decision-making under the framework of $CSFNS_f$ are illustrated with numerical examples.

(4) The comparative study with $PFNS_f$ algorithms and the CSF-TOPSIS method shows their ability and significance.

The rest of the paper is organized as follows. Section 2 contains some definitions from existing models. In Section 3, we propose the novel concept of $CSFNS_fS$ which is then followed by the operations on $CSFNS_fSs$ and $CSFNS_fNs$. Section 3 describes three algorithms for making decisions and performs a comparison with a $PFNS_f$ method. In Section 4, we develop a theatrical foundation for the $CSFNS_fS$-TOPSIS method. In Section 5, we present the mathematical algorithms of these decision-making mechanisms that are applied to some numerical examples. Section 6 describes the comparison analysis with CSF-TOPSIS method. In Section 7, we conclude the paper and provide future directions of research.

2. Preliminaries

Definition 1 (see [10]). A *spherical fuzzy set* (SFS) Υ on a universe of discourse U has the form

$$\Upsilon = \langle u, \mu_\Upsilon(u), \eta_\Upsilon(u), \nu_\Upsilon(u) | u \in U\rangle, \tag{1}$$

where $\mu_\Upsilon(u)$, $\eta_\Upsilon(u)$, and $\nu_\Upsilon(u)$, which lie within the unit interval, are called the grade of the positive, neutral, and negative membership, respectively; and they are restricted by the condition $\mu_\Upsilon(u)^2 + \eta_\Upsilon(u)^2 + \nu_\Upsilon(u)^2 \leq 1$, for every $u \in U$. The degree of refusal of u in U is defined as

$$\Theta_\Upsilon(u) = \sqrt{1 - \left(p_\Upsilon(u)^2 + \nu_\Upsilon(u)^2 + r_\Upsilon(u)^2\right)}. \tag{2}$$

The triplet $(\mu_\Upsilon(u), \eta_\Upsilon(u), \nu_\Upsilon(u))$ is called spherical fuzzy number (SFN).

Definition 2 (see [25]). A *complex T-spherical fuzzy set* (*CTSFS*) Υ on the universe U is defined as

$$\Upsilon = \langle (u, \mu_\Upsilon(u), \eta_\Upsilon(u), \nu_\Upsilon(u)) | u \in U\rangle, \tag{3}$$

where $\mu_\Upsilon(u) = p_\Upsilon(u) e^{i2\pi\phi_\Upsilon(u)}$, $\eta_\Upsilon(u) = \nu_\Upsilon(u) e^{i2\pi\delta_\Upsilon(u)}$, and $\nu_\Upsilon(u) = r_\Upsilon(u) e^{i2\pi\lambda_\Upsilon(u)}$, which denote the positive, neutral, and negative degree of membership, respectively. They are restricted by the conditions $p_\Upsilon(u)^q + \nu_\Upsilon(u)^q + r_\Upsilon(u)^q \leq 1$ and $\phi_\Upsilon(u)^q + \delta_\Upsilon(u)^q + \lambda_\Upsilon(u)^q \leq 1$, for each $u \in U$, where $i = \sqrt{-1}$, and p_Υ, ν_Υ, r_Υ, ϕ_Υ, δ_Υ, $\lambda_\Upsilon \in [0, 1]$. The degree of refusal of u in U is defined as

$$\begin{aligned}\Pi_\Upsilon(u) = &\sqrt{1 - \left(p_\Upsilon(u)^q + \nu_\Upsilon(u)^q + r_\Upsilon(u)^q\right)} \\ &\cdot e^{i2\pi\sqrt{1-\left(\phi_\Upsilon(u)^q + \delta_T(u)^q + \lambda_\Upsilon(u)^q\right)}}.\end{aligned} \tag{4}$$

The triplet $(\mu_\Upsilon, \eta_\Upsilon, \nu_\Upsilon) = (p_\Upsilon e^{i2\pi\phi_\Upsilon}, \nu_\Upsilon e^{i2\pi\delta_\Upsilon}, r_\Upsilon e^{i2\pi\lambda_\Upsilon})$ is called *CTSFN*.

Particular case: When $T = 2$, a *CTSFS* becomes a complex spherical fuzzy set (CSFS).

Definition 3 (see [26]). Let W be a nonempty set and R be a set of attributes and $Z \subseteq R$. A *soft set* S_fS over W is a pair (Γ, Z), where Γ is a set-valued function from Z to the set of all subsets of W, which is denoted as

$$(\Gamma, Z) = \left\{\langle z, \Gamma(z)\rangle | z \in Z, \Gamma(z) \in 2^W\right\}. \tag{5}$$

Definition 4. Let W be a nonempty set and R be a set of attributes, $Z \subseteq R$. A *complex spherical fuzzy soft set* ($CSFS_fS$) over W is a pair (Λ, Z), where Λ is a function from Z to the set of all subsets of CSFSs of W, which is denoted as

$$\begin{aligned}(\Lambda, Z) &= \left\{\langle z, \Lambda(z)\rangle | z \in Z, \Lambda(z) \in CSFS^W\right\} \\ &= \left\{\langle z, (w, (\mu_z(w), \eta_z(w), \nu_z(w)))\rangle\right\} \\ &= \left\{\langle z, \left(w, p_z(w) e^{i2\pi\phi_z(w)}, \nu_z(w) e^{i2\pi\delta_z(w)}, r_z(w) e^{i2\pi\lambda_z(w)}\right)\rangle\right\},\end{aligned} \tag{6}$$

where p_z, ν_z, r_z, $\phi_z, \delta_z, \lambda_z \in [0, 1]$ are restricted by the conditions

$$\begin{aligned}&0 \leq p_z(w)^2 + \nu_z(w)^2 + r_z(w)^2 \leq 1, \\ &0 \leq \phi_z(w)^2 + \delta_z(w)^2 + \lambda_z(w)^2 \leq 1, \\ &\forall w \in W.\end{aligned} \tag{7}$$

Definition 5 (see [19]). Let W be a nonempty set and R be a set of attributes. Let $Z \subseteq R$ and $G = \{0, 1, 2, \ldots, N - 1\}$ be a set of ordered grades with $N \in \{2, 3, \ldots\}$. A triple (F, Z, N) is called *N-soft set* (NS_fS) over W if F is a mapping from Z to $2^{U\times G}$, with the property that, for each $z \in Z$ and $w \in W$, there exist a unique $(w, g_z^w) \in W \times G$ such that $(w, g_z^w) \in F(z)$, $w \in W$, $g_z^w \in G$ [27–40].

3. Complex Spherical Fuzzy N-Soft Sets

Definition 6. Let W be a nonempty set and R be a set of attributes. Let $Z \subseteq R$ and $G = \{0, 1, 2, \ldots, N - 1\}$ be a set of ordered grades with $N \in \{2, 3, \ldots\}$. A triple (F_J, Z, N) is called a *complex spherical fuzzy N-soft set* ($CSFNS_fS$) on Z, when (F, Z, N) $(F\colon Z \longrightarrow 2^{W\times G})$ is an NS_fS on W, if $F_J\colon Z \longrightarrow 2^{W\times G} \times \text{CSFN}$ is a mapping, which is defined as

$$\begin{aligned}(F_J, Z, N) &= \{\langle z, F(z), J(z)\rangle | z \in Z, (F(z), J(z)) \in 2^{W\times G} \times \text{CSFN}\}\\ &= \{\langle z, ((w, g_z^w), (\mu_z(w), \eta_z(w), \nu_z(w)))\rangle\}\\ &= \{\langle z, ((w, g_z^w), p_z(w)e^{i2\pi\phi_z(w)}, \nu_z(w)e^{i2\pi\delta_z(w)}, r_z(w)e^{i2\pi\lambda_z(w)})\rangle\},\end{aligned} \tag{8}$$

where J: $Z \longrightarrow$ CSFN, CSFN denotes the collection of all complex spherical fuzzy numbers of W, g_z^w denotes the level of attribute for the element w and $p_z, \nu_z, r_z, \phi_z, \delta_z, \lambda_z \in [0, 1]$, restricted with conditions

$$\begin{aligned}&0 \leq p_z(w)^2 + \nu_z(w)^2 + r_z(w)^2 \leq 1,\\ &0 \leq \phi_z(w)^2 + \delta_z(w)^2 + \lambda_z(w)^2 \leq 1, \quad \text{for all } w \text{ belongs to } W.\end{aligned} \tag{9}$$

Definition 7. Let $F_J(z_k) = ((w_j, g_k^j), p_{kj}e^{i2\pi\phi_{kj}}, \nu_{kj}e^{i2\pi\delta_{kj}} r_{kj}e^{i2\pi\lambda_{kj}})$ be a $CSFNS_fS$. Then, the *complex spherical fuzzy N-soft number* ($CSFNS_fN$) is defined as

$$\begin{aligned}\Upsilon_{kj} &= (g_k^j, p_{kj}e^{i2\pi\phi_{kj}}, \nu_{kj}e^{i2\pi\delta_{kj}}, r_{kj}e^{i2\pi\lambda_{kj}}),\\ \Omega_{\Upsilon_{kj}} &= \sqrt{1 - (p_{kj}^2 + w_{kj}^2 + r_{kj}^2)}e^{i2\pi\sqrt{1-(\phi_{kj}^2+\delta_{kj}^2+\lambda_{kj}^2)}},\end{aligned} \tag{10}$$

is the hesitancy degree, where $p_{kj}, \nu_{kj}, r_{kj}, \phi_{kj}, \delta_{kj}$, and λ_{kj} represent $p_{z_k}(w_j), \nu_{z_k}(w_j), r_{z_k}(w_j), \phi_{z_k}(w_j), \delta_{z_k}(w_j)$, and $\lambda_{z_k}(w_j)$, respectively.

Definition 8. Consider a $CSFNS_fN\, \Upsilon_{kj} = (g_k^j, p_{kj}e^{i2\pi\phi_{kj}}, \nu_{kj}e^{i2\pi\delta_{kj}} r_{kj}e^{i2\pi\lambda_{kj}})$. The score function $S(\Upsilon_{kj})$ is

$$S_{\Upsilon_{kj}} = \left(\frac{g_k^j}{N-1}\right)^2 + (p_{kj}^2 - w_{kj}^2 - r_{kj}^2) + [\phi_{kj}^2 - \delta_{kj}^2 - \lambda_{kj}^2], \tag{11}$$

where $S_{\Upsilon_{kj}} \in [-2, 3]$. The accuracy function $A(\Upsilon_{kj})$ is

$$A_{\Upsilon_{kj}} = \left(\frac{g_k^j}{N-1}\right)^2 + (p_{kj}^2 + w_{kj}^2 + r_{kj}^2) + [\phi_{kj}^2 + \delta_{kj}^2 + \lambda_{kj}^2], \tag{12}$$

where $A_{\Upsilon_{kj}} \in [0, 3]$, respectively.

Definition 9. Let $\Upsilon_{lj} = (g_l^j, p_{lj}e^{i2\pi\phi_{lj}}, \nu_{lj}e^{i2\pi\delta_{lj}}, r_{lj}e^{i2\pi\lambda_{lj}})$ and $\Upsilon_{kj} = (g_k^j, p_{kj}e^{i2\pi\phi_{kj}}, \nu_{kj}e^{i2\pi\delta_{kj}}, r_{kj}e^{i2\pi\lambda_{kj}})$ be two $CSFNS_fNs$:

(1) If $S_{\Upsilon_{lj}} < S_{\Upsilon_{kj}}$, then $\Upsilon_{lj} \prec \Upsilon_{kj}$ (Υ_{lj} is inferior to Υ_{kj}).

(2) If $S_{\Upsilon_{lj}} > S_{\Upsilon_{kj}}$, then $\Upsilon_{lj} \succ \Upsilon_{kj}$ (Υ_{lj} is superior to Υ_{kj}).

(3) If $S_{\Upsilon_{lj}} = S_{\Upsilon_{kj}}$, then

  (i) $A_{\Upsilon_{lj}} < A_{\Upsilon_{kj}}$, then $\Upsilon_{lj} \prec \Upsilon_{kj}$ (Υ_{lj} is inferior to Υ_{kj})

  (ii) $A_{\Upsilon_{lj}} > A_{\Upsilon_{kj}}$, then $\Upsilon_{lj} \succ \Upsilon_{kj}$ (Υ_{lj} is superior to Υ_{kj})

  (iii) $A_{\Upsilon_{lj}} = A_{\Upsilon_{kj}}$, then $\Upsilon_{lj} \sim \Upsilon_{kj}$ (Υ_{lj} is equivalent to Υ_{kj})

Remark 1. We see that

(1) For $N = 2$, $CSFNS_fS$ becomes complex spherical fuzzy soft set

(2) When $|Z| = 1$, $CSFNS_fS$ becomes complex spherical fuzzy set

(3) When $\phi_z = \delta_z = \lambda_z = 0$, $CSFNS_fS$ becomes spherical fuzzy N-soft set

Example 1. In a city, a parent wants to choose the best school for their child. It is necessary to go after the advice of experts, for the selection of a school based on rankings and ratings. Let $W = \{w_1, w_2, w_3\}$ be the family of three schools under consideration and $Z = \{z_1 = \text{size of school}, z_2 = \text{location}, z_3 = \text{academic performance}, z_4 = \text{services}\}$ be the attributes which are used to assign rankings to schools by the experts. In a relation to these parameters, a 5-soft set is given in Table 1, where

Four diamonds means "Outstanding"

Three diamonds means "Super"

Two diamonds means "Good"

One diamond means "Satisfactory"

Big dot means "Acceptable"

This level assessment by diamonds can be represented by numbers as $G = \{0, 1, 2, 3, 4\}$, where

0 means "•"

1 means "◇"

2 means "◇◇"

3 means "◇◇◇"

4 means "◇◇◇◇"

Table 2 can be adopted as natural convention of 5-soft set model.

By Definition 6, when the data is vague and uncertain, we need $CSFNS_fSs$ which provides us information on how these grades are given to schools. The evaluation of schools by experts follows the following grading:

$$\begin{aligned}&\text{when } g_z^w = 0, \quad -2.00 \leq S_J < -1.85,\\ &\text{when } g_z^w = 1, \quad -1.85 \leq S_J < -1.30,\\ &\text{when } g_z^w = 2, \quad -1.30 \leq S_J < 0.15,\\ &\text{when } g_z^w = 3, \quad 0.15 \leq S_J < 1.30,\\ &\text{when } g_z^w = 4, \quad 1.30 \leq S_J < 2.00.\end{aligned} \tag{13}$$

According to the above criteria, we can obtain Table 3. At last, $CSF5S_fS$ is defined as

Table 1: Evaluation data provided by the experts.

W/Z	z_1	z_2	z_3	z_4
w_1	◇◇	◇◇◇	•	◇◇
w_2	•	◇◇◇	◇	◇
w_3	◇◇◇◇	◇	◇◇◇	◇◇

Table 2: Tabular representation of 5-soft set.

W/Z	z_1	z_2	z_3	z_4
w_1	2	3	0	2
w_2	0	3	1	1
w_3	4	1	3	2

Table 3: Grading criteria.

g_z^w/J	Positive membership		Neutral membership		Negative membership	
Grades	p_z	$2\pi\phi_z$	ν_z	$2\pi\delta_z$	r_z	$2\pi\lambda_z$
$g_z^w = 0$	$[0, 0.15]$	$[0, 0.3\pi)$	$[0, 0.017)$	$[0, 0.034\pi)$	$[0.98, 1)$	$[1.96\pi, 2\pi]$
$g_z^w = 1$	$(0.15, 0.35)$	$[0.3\pi, 0.7\pi)$	$[0.017, 0.1031)$	$[0.034\pi, 0.2062\pi)$	$(0.88, 0.98)$	$[1.76\pi, 1.96\pi)$
$g_z^w = 2$	$[0.35, 0.65)$	$[0.7\pi, 1.3\pi)$	$[0, 0.017)$	$[0, 0.034\pi)$	$(0.58, 0.88]$	$[1.16\pi, 1.76\pi)$
$g_z^w = 3$	$[0.65, 0.85]$	$[1.3\pi, 1.7\pi]$	$[0.017, 0.1031)$	$[0.034\pi, 0.2062\pi)$	$[0.28, 0.58]$	$[0.56\pi, 1.16\pi)$
$g_z^w = 4$	$(0.85, 1]$	$(1.7\pi, 2\pi]$	$[0, 0.017)$	$[0, 0.034\pi)$	$[0, 0.28)$	$[0, 0.56\pi)$

$$
\begin{aligned}
\left(\mu_{z_1}, \eta_{z_1}, \nu_{z_1}\right) &= \Big\{\left((w_1, 2), \left(0.4e^{i0.82\pi}, 0.017e^{i0.0356\pi}, 0.6e^{i1.22\pi}\right)\right), \left((w_2, 0), \left(0.02e^{i0.06\pi}, 0.012e^{i0.026\pi}, 0.98e^{i1.962\pi}\right)\right),\\
&\quad ((w_3, 4), (1, 0, 0))\Big\},\\
\left(\mu_{z_2}, \eta_{z_2}, \nu_{z_2}\right) &= \Big\{\left((w_1, 3), \left(0.65e^{i1.32\pi}, 0.018e^{i0.038\pi}, 0.28e^{i0.58\pi}\right)\right), \left((w_2, 3), \left(0.7e^{i1.42\pi}, 0.019e^{i0.04\pi}, 0.3e^{i0.56\pi}\right)\right),\\
&\quad \left((w_3, 1), \left(0.16e^{i0.34\pi}, 0.1e^{i0.204\pi}, 0.89e^{1.784\pi}\right)\right)\Big\},\\
\left(\mu_{z_3}, \eta_{z_3}, \nu_{z_3}\right) &= \Big\{\left((w_1, 0), \left(0.1e^{i0.24\pi}, 0.012e^{i0.022\pi}, 0.985e^{i1.964\pi}\right)\right), \left((w_2, 1), \left(0.2e^{i0.36\pi}, 0.027e^{i0.05\pi}, 0.91e^{i1.824\pi}\right)\right),\\
&\quad \left((w_3, 3), \left(0.69e^{i1.384\pi}, 0.101e^{i0.204\pi}, 0.32e^{0.62\pi}\right)\right)\Big\},\\
\left(\mu_{z_4}, \eta_{z_4}, \nu_{z_4}\right) &= \Big\{\left((w_1, 2), \left(0.5e^{i1.1\pi}, 0.1e^{i0.18\pi}, 0.59e^{1.28i\pi}\right)\right), \left((w_2, 1), \left(0.3e^{i0.56\pi}, 0.019e^{i0.042\pi}, 0.885e^{1.72i\pi}\right)\right),\\
&\quad \left((w_3, 2), \left(0.45e^{i0.86\pi}, 0.015e^{i0.022\pi}, 0.78e^{i1.566\pi}\right)\right)\Big\}.
\end{aligned}
\tag{14}
$$

The tabular representation of $CSF5S_fS$ is shown by Table 4.

Definition 10. A $CSFNS_fS$ (F_J, Z, N) over a nonempty set W is said to be *efficient*, where (F, Z, N) is an NS_fS if $F_J(z) = \langle (w, N-1), 1, 0, 0\rangle$ for some $z \in Z$, $w \in W$.

Example 2. Let $(F_J, Z, 5)$ be $CSF5S_fS$, as in Example 1. It is easy to check from Table 4 that $F_J(z_1) = ((w_3, 4), 1, 0, 0)$, i.e., Example 1 is efficient.

Definition 11. Let (F_J, Z, N_1) and (H_A, B, N_2) be two $CSFNS_fSs$ on a universe of discourse W. Then, they are said to be *equal* if and only if $F = H$, $J = A$, $Z = B$, and $N_1 = N_2$.

Definition 12. Let (F_J, Z, N) be $CSFNS_fS$ on W. The *weak complement* of $CSFNS_fS$ is defined as the weak complement of the N-soft set (F, Z, N), that is, any N-soft set such that $F^c(z) \cap F(z) = \varnothing$ for all $z \in Z$. The weak complement of $CSFNS_fS$ of (F_J, Z, N) is represented as (F_J^c, Z, N).

Example 3. Let $(F_J, Z, 5)$ be $CSF5S_fS$, as in Example 1. The weak complement (F_J^c, Z, N) is given in Table 5.

Definition 13. Let (F_J, Z, N) be $CSFNS_fS$ on W. The *complex spherical fuzzy complement* of $CSFNS_fS$ is denoted as (F_{J^c}, Z, N) and is defined as

Table 4: Tabular representation of the $CSF5S_fS$ $(F_J, Z, 5)$.

$(F_J, Z, 5)$	z_1	z_2	z_3	z_4
w_1	$(2, (0.4e^{i0.82\pi}, 0.017e^{i0.0356\pi}, 0.6e^{i1.22\pi}))$	$(3, (0.65e^{i1.32\pi}, 0.018e^{i0.038\pi}, 0.28e^{i0.58\pi}))$	$(0, (0.1e^{i0.24\pi}, 0.012e^{i0.022\pi}, 0.985e^{i1.964\pi}))$	$(2, (0.5e^{i1.1\pi}, 0.1e^{i0.18\pi}, 0.59e^{1.28i\pi}))$
w_2	$(0, (0.02e^{i0.06\pi}, 0.012e^{i0.026\pi}, 0.98e^{i1.962\pi}))$	$(3, (0.7e^{i1.42\pi}, 0.019e^{i0.04\pi}, 0.3e^{i0.56\pi}))$	$(1, (0.2e^{i0.36\pi}, 0.027e^{i0.05\pi}, 0.91e^{i1.824\pi}))$	$(1, (0.3e^{i0.56\pi}, 0.019e^{i0.042\pi}, 0.885e^{1.72i\pi}))$
w_3	$(4, (1, 0, 0))$	$(1, (0.16e^{i0.34\pi}, 0.1e^{i0.204\pi}, 0.89e^{1.784\pi}))$	$(3, (0.69e^{i1.384\pi}, 0.101e^{i0.204\pi}, 0.32e^{0.62\pi}))$	$(2, (0.45e^{i0.86\pi}, 0.015e^{i0.022\pi}, 0.78e^{i1.566\pi}))$

Table 5: A weak complement of the $CSF5S_fS$ $(F_J, Z, 5)$ in Example 1.

$(F_J^c, Z, 5)$	z_1	z_2	z_3	z_4
w_1	$(3, (0.4e^{i0.82\pi}, 0.017e^{i0.0356\pi}, 0.6e^{i1.22\pi}))$	$(0, (0.65e^{i1.32\pi}, 0.018e^{i0.038\pi}, 0.28e^{i0.58\pi}))$	$(1, (0.1e^{i0.24\pi}, 0.012e^{i0.022\pi}, 0.985e^{i1.964\pi}))$	$(3, (0.5e^{i1.1\pi}, 0.1e^{i0.18\pi}, 0.59e^{1.28i\pi}))$
w_2	$(3, (0.02e^{i0.06\pi}, 0.012e^{i0.026\pi}, 0.98e^{i1.962\pi}))$	$(1, (0.7e^{i1.42\pi}, 0.019e^{i0.04\pi}, 0.3e^{i0.56\pi}))$	$(4, (0.69e^{i1.384\pi}, 0.101e^{i0.204\pi}, 0.32e^{0.62\pi}))$	$(2, (0.3e^{i0.56\pi}, 0.019e^{i0.042\pi}, 0.885e^{1.72i\pi}))$
w_3	$(2, (1, 0, 0))$	$(3, (0.16e^{i0.34\pi}, 0.1e^{i0.204\pi}, 0.89e^{1.784\pi}))$	$(1, (0.69e^{i1.384\pi}, 0.101e^{i0.204\pi}, 0.32e^{0.62\pi}))$	$(4, (0.45e^{i0.86\pi}, 0.015e^{i0.022\pi}, 0.78e^{i1.566\pi}))$

$$
\begin{aligned}
F_{J^c}(z) &= \langle \nu_z(w), \eta_z(w), \mu_z(w) \rangle \\
&= \langle (w, g_z^w)\big(r_z(w)e^{i2\pi\lambda_z(w)}, \nu_z(w)e^{i2\pi\delta_z(w)}, \\
&\quad p_z(w)e^{i2\pi\phi_z(w)}\big)\rangle.
\end{aligned}
\tag{15}
$$

Example 4. Let $(F_J, Z, 5)$ be $CSF5S_fS$, as in Example 1. The complex spherical fuzzy complement (F_{J^c}, Z, N) is given in Table 6.

Definition 14. Let (F_J, Z, N) be a $CSFNS_fS$ on W. $(F^c_{J^c}, Z, N)$ is referred to as a *weak complex spherical fuzzy complement* of (F_J, Z, N) if and only if (F^c_J, Z, N) is a weak complement and (F_{J^c}, Z, N) is a complex spherical fuzzy complement of (F_J, Z, N).

Example 5. Let $(F_J, Z, 5)$ be $CSF5S_fS$, as in Example 1. The weak complex spherical fuzzy complement $(F^c_{J^c}, Z, N)$ is given in Table 7.

Definition 15. Let (F_J, Z, N) be a $CSFNS_fS$ on W; then, the *top weak complex spherical fuzzy complement* $(F_J^{>}, Z, N)$ is defined as

$$
\left(F_J^{>}, Z, N\right) = \begin{cases}
F_J(z_k) = \left\langle \left(w_j, N-1\right), r_{z_k}\left(w_j\right)e^{i2\pi\lambda_{z_k}\left(w_j\right)}, \nu_{z_k}\left(w_j\right)e^{i2\pi\delta_{z_k}\left(w_j\right)} p_{z_k}\left(w_j\right)e^{i2\pi\phi_{z_k}\left(w_j\right)} \right\rangle, & \text{if } g_k^j < N-1, \\
F_J(z_k) = \left\langle \left(w_j, 0\right), r_{z_k}\left(w_j\right)e^{i2\pi\lambda_{z_k}\left(w_j\right)}, \nu_{z_k}\left(w_j\right)e^{i2\pi\delta_{z_k}\left(w_j\right)} p_{z_k}\left(w_j\right)e^{i2\pi\phi_{z_k}\left(w_j\right)} \right\rangle, & \text{if } g_k^j = N-1.
\end{cases}
\tag{16}
$$

Example 6. Let $(F_J, Z, 5)$ be $CSF5S_fS$, as in Example 1. The top weak complex spherical fuzzy complement $(F_J^{>}, Z, N)$, is given in Table 8.

Definition 16. Let (F_J, Z, N) be $CSFNS_fS$ on W; then, the *bottom weak complex spherical fuzzy complement* $(F_J^{<}, Z, N)$ is defined as

$$
\left(F_J^{<}, Z, N\right) = \begin{cases}
F_J(z_k) = \left\langle \left(w_j, 0\right), r_{z_k}\left(w_j\right)e^{i2\pi\lambda_{z_k}\left(w_j\right)}, \nu_{z_k}\left(w_j\right)e^{i2\pi\delta_{z_k}\left(w_j\right)} p_{z_k}\left(w_j\right)e^{i2\pi\phi_{z_k}\left(w_j\right)} \right\rangle, & \text{if } g_k^j > 0, \\
F_J(z_k) = \left\langle \left(w_j, N-1\right), r_{z_k}\left(w_j\right)e^{i2\pi\lambda_{z_k}\left(w_j\right)}, \nu_{z_k}\left(w_j\right)e^{i2\pi\delta_{z_k}\left(w_j\right)} p_{z_k}\left(w_j\right)e^{i2\pi\phi_{z_k}\left(w_j\right)} \right\rangle, & \text{if } g_k^j = 0.
\end{cases}
\tag{17}
$$

Example 7. Let $(F_J, Z, 5)$ be $CSF5S_fS$, as in Example 1. The bottom weak complex spherical fuzzy complement $(F_J^{<}, Z, N)$ is given in Table 9.

Definition 17. Let W be a nonempty set and (F_J, Z, N_1) and (H_A, B, N_2) be $CSFN_1S_fS$ and $CSFN_2S_fSs$ on W, respectively, and their *restricted intersection* is defined as $(K_L, M, S) = (F_J, Z, N_1) \cap_R (H_A, B, N_2)$, where $K_L = F_J \cap_R H_A$, $M = Z \cap B$, $S = \min(N_1, N_2)$, i.e., $\forall u_k \in M$ and $w_J \in W$, $(g_k^j, (\mu_{ij}, \eta_{ij}, \nu_{ij})) \in K_L(u_j)$, $g_k^j = \min(g_k^1, g_k^2)$, $\mu_{ij}(u_j) = \min(\mu^1_{ij}(u^1_k), \mu^2_{ij}(u^2_k)) = \min(p^1_{ij}(u^1_k), p^2_{ij}(u^2_k)) e^{i2\pi(\min(\phi^1_{ij}(u^1_k), \phi^2_{ij}(u^2_k)))}$, $\eta_{ij}(u_j) = \max(\eta^1_{ij}(u^1_k), \eta^2_{ij}(u^2_k)) = \max(\nu^1_{ij}(u^1_k), \nu^2_{ij}(u^2_k))e^{i2\pi(\max(\delta^1_{ij}(u^1_k), \delta^2_{ij}(u^2_k)))}$, $\nu_{ij}(u_j) = \max(\nu^1_{ij}(u^1_k), \nu^2_{ij}(u^2_k)) = \max(r^1_{ij}(u^1_k), r^2_{ij}(u^2_k))e^{i2\pi(\max(\lambda^1_{ij}(u^1_k), \lambda^2_{ij}(u^2_k)))}$, where, $(g^1_k, (\mu^1_{ij}(u^1_k), \eta^1_{ij}(u^1_k), \nu^1_{ij}(u^1_k))) \in (\mu_{F_J}(u^1_k), \eta_{F_J}(u^1_k), \nu_{F_J}(u^1_k))$, and $(g^2_k, (\mu^2_{ij}(u^2_k), \eta^2_{ij}(u^2_k), \nu^2_{ij}(u^2_k))) \in (\mu_{H_A}(u^2_k), \eta_{H_A}(u^2_k), \nu_{H_A}(u^2_k))$, with $u^1_k \in Z$ and $u^2_k \in B$.

Example 8. Let $(E_P, Z, 5)$ and $(H_A, B, 6)$ be two $CSF5S_fS$ and $CSF6S_fS$, given in Tables 10 and 11, respectively. Their restricted intersection $(K_L, M, 5) = (E_P, Z, 5) \cap_R (H_A, B, 6)$ is shown in Table 12.

Definition 18. Let W be a nonempty set and (F_J, Z, N_1) and (H_A, B, N_2) be two $CSFNS_fSs$ on W; their *extended intersection* is defined as $(\mathcal{Q}_D, C, Y) = (F_J, Z, N_1) \cap_E (H_A, B, N_2)$, where $\mathcal{Q}_D = F_J \cap_E H_A$, $C = Z \cup B$, and $Y = \max(N_1, N_2)$, that is, $\forall u_k \in C$ and $w_j \in W$, $(g_k^j, (\mu_{kj}, \eta_{kj}, \nu_{kj})) \in \mathcal{Q}_D(u_k)$, with

Table 6: The complex spherical fuzzy complement (F_{J^c}, Z, N) of the $CSF5S_fS$ in Example 1.

$(F_{J^c}, Z, 5)$	z_1	z_2	z_3	z_4
w_1	$(2, (0.6e^{i1.22\pi}, 0.017e^{i0.0356\pi}, 0.4e^{i0.82\pi}))$	$(3, (0.28e^{i0.58\pi}, 0.018e^{i0.038\pi}, 0.65e^{i1.32\pi}))$	$(0, (0.985e^{i1.964\pi}, 0.012e^{i0.022\pi}, 0.1e^{i0.24\pi}))$	$(2, (0.59e^{1.28i\pi}, 0.1e^{i0.18\pi}, 0.5e^{i1.1\pi}))$
w_2	$(0, (0.98e^{i1.962\pi}, 0.012e^{i0.026\pi}, 0.02e^{i0.06\pi}))$	$(3, (0.3e^{i0.56\pi}, 0.019e^{i0.04\pi}, 0.7e^{i1.42\pi}))$	$(1, (0.32e^{0.62\pi}, 0.101e^{i0.204\pi}, 0.69e^{i1.384\pi}))$	$(1, (0.885e^{1.72i\pi}, 0.019e^{i0.042\pi}, 0.3e^{i0.56\pi}))$
w_3	$(4, (0, 0, 1))$	$(1, (0.89e^{1.784\pi}, 0.1e^{i0.204\pi}, 0.16e^{i0.34\pi}))$	$(3, (0.32e^{0.62\pi}, 0.101e^{i0.204\pi}, 0.69e^{i1.384\pi}))$	$(2, (0.78e^{i1.566\pi}, 0.015e^{i0.022\pi}, 0.45e^{i0.86\pi}))$

Table 7: The weak complex spherical fuzzy complement $(F^c_{J^c}, Z, N)$ of the $CSF5S_fS$ in Example 1.

$(F^c_{J^c}, Z, 5)$	z_1	z_2	z_3	z_4
w_1	$(3, (0.6e^{i1.22\pi}, 0.017e^{i0.0356\pi}, 0.4e^{i0.82\pi}))$	$(0, (0.28e^{i0.58\pi}, 0.018e^{i0.038\pi}, 0.65e^{i1.32\pi}))$	$(1, (0.985e^{i1.964\pi}, 0.012e^{i0.022\pi}, 0.1e^{i0.24\pi}))$	$(3, (0.59e^{1.28i\pi}, 0.1e^{i0.18\pi}, 0.5e^{i1.1\pi}))$
w_2	$(3, (0.98e^{i1.962\pi}, 0.012e^{i0.026\pi}, 0.02e^{i0.06\pi}))$	$(1, (0.3e^{i0.56\pi}, 0.019e^{i0.04\pi}, 0.7e^{i1.42\pi}))$	$(4, (0.32e^{0.62\pi}, 0.101e^{i0.204\pi}, 0.69e^{i1.384\pi}))$	$(2, (0.885e^{1.72i\pi}, 0.019e^{i0.042\pi}, 0.3e^{i0.56\pi}))$
w_3	$(2, (0, 0, 1))$	$(3, (0.89e^{1.784\pi}, 0.1e^{i0.204\pi}, 0.16e^{i0.34\pi}))$	$(1, (0.32e^{0.62\pi}, 0.101e^{i0.204\pi}, 0.69e^{i1.384\pi}))$	$(4, (0.78e^{i1.566\pi}, 0.015e^{i0.022\pi}, 0.45e^{i0.86\pi}))$

Table 8: The top weak complex spherical fuzzy complement $(F_J^{>}, Z, N)$ of the $CSF5S_fS$ set in Example 1.

$(F_J^{>}, Z, 5)$	z_1	z_2	z_3	z_4
w_1	$(4, (0.6e^{i1.22\pi}, 0.017e^{i0.0356\pi}, 0.4e^{i0.82\pi}))$	$(4, (0.28e^{i0.58\pi}, 0.018e^{i0.038\pi}, 0.65e^{i1.32\pi}))$	$(4, (0.985e^{i1.964\pi}, 0.012e^{i0.022\pi}, 0.1e^{i0.24\pi}))$	$(4, (0.59e^{1.28i\pi}, 0.1e^{i0.18\pi}, 0.5e^{i1.1\pi}))$
w_2	$(4, (0.98e^{i1.962\pi}, 0.012e^{i0.026\pi}, 0.02e^{i0.06\pi}))$	$(4, (0.3e^{i0.56\pi}, 0.019e^{i0.04\pi}, 0.7e^{i1.42\pi}))$	$(4, (0.32e^{0.62\pi}, 0.101e^{i0.204\pi}, 0.69e^{i1.384\pi}))$	$(4, (0.885e^{1.72i\pi}, 0.019e^{i0.042\pi}, 0.3e^{i0.56\pi}))$
w_3	$(0, (0, 0, 1))$	$(4, (0.89e^{1.784\pi}, 0.1e^{i0.204\pi}, 0.16e^{i0.34\pi}))$	$(4, (0.32e^{0.62\pi}, 0.101e^{i0.204\pi}, 0.69e^{i1.384\pi}))$	$(4, (0.78e^{i1.566\pi}, 0.015e^{i0.022\pi}, 0.45e^{i0.86\pi}))$

TABLE 9: The bottom weak complex spherical fuzzy complement $(F_J^{<}, Z, N)$ of the $CSF5S_fS$ in Example 1.

$(F_J^{<}, Z, 5)$	z_1	z_2	z_3	z_4
w_1	$(0, (0.6e^{i1.22\pi}, 0.017e^{i0.0356\pi}, 0.4e^{i0.82\pi}))$	$(0, (0.28e^{i0.58\pi}, 0.018e^{i0.038\pi}, 0.65e^{i1.32\pi}))$	$(4, (0.985e^{i1.964\pi}, 0.012e^{i0.022\pi}, 0.1e^{i0.24\pi}))$	$(0, (0.59e^{1.28i\pi}, 0.1e^{i0.18\pi}, 0.5e^{i1.1\pi}))$
w_2	$(4, (0.98e^{i1.962\pi}, 0.012e^{i0.026\pi}, 0.02e^{i0.06\pi}))$	$(0, (0.3e^{i0.56\pi}, 0.019e^{i0.04\pi}, 0.7e^{i1.42\pi}))$	$(0, (0.32e^{0.62\pi}, 0.101e^{i0.204\pi}, 0.69e^{i1.384\pi}))$	$(0, (0.885e^{1.72i\pi}, 0.019e^{i0.042\pi}, 0.3e^{i0.56\pi}))$
w_3	$(0, (0, 0, 1))$	$(0, (0.89e^{1.784\pi}, 0.1e^{i0.204\pi}, 0.16e^{i0.34\pi}))$	$(0, (0.32e^{0.62\pi}, 0.101e^{i0.204\pi}, 0.69e^{i1.384\pi}))$	$(0, (0.78e^{i1.566\pi}, 0.015e^{i0.022\pi}, 0.45e^{i0.86\pi}))$

Table 10: Tabular representation of the $CSF5S_fS$ $(E_P, Z, 5)$.

$(E_P, Z, 5)$	z_1	z_2	z_3
w_1	$(2, (0.4e^{i0.82\pi}, 0.017e^{i0.0356\pi}, 0.6e^{i1.22\pi}))$	$(3, (0.65e^{i1.32\pi}, 0.018e^{i0.038\pi}, 0.28e^{i0.58\pi}))$	$(0, (0.1e^{i0.24\pi}, 0.012e^{i0.022\pi}, 0.985e^{i1.964\pi}))$
w_2	$(0, (0.02e^{i0.06\pi}, 0.012e^{i0.026\pi}, 0.98e^{i1.962\pi}))$	$(3, (0.7e^{i1.42\pi}, 0.019e^{i0.04\pi}, 0.3e^{i0.56\pi}))$	$(1, (0.2e^{i0.36\pi}, 0.027e^{i0.05\pi}, 0.91e^{i1.824\pi}))$
w_3	$(4, (1, 0, 0))$	$(1, (0.16e^{i0.34\pi}, 0.1e^{i0.204\pi}, 0.89e^{1.784\pi}))$	$(3, (0.69e^{i1.384\pi}, 0.101e^{i0.204\pi}, 0.32e^{0.62\pi}))$

Table 11: Tabular representation of $CSF6S_fS$ $(H_A, B, 6)$.

$(H_A, B, 6)$	z_1	z_2	z_6
w_1	$(1, (0.23e^{i0.44\pi}, 0.019e^{i0.036\pi}, 0.92e^{i1.85\pi}))$	$(5, (0.95e^{i1.88\pi}, 0.03e^{i0.062\pi}, 0.14e^{i0.29\pi}))$	$(3, (0.69e^{i1.39\pi}, 0.04e^{i0.084\pi}, 0.65e^{i1.36\pi}))$
w_2	$(0, (0.03e^{i0.1\pi}, 0.015e^{i0.032\pi}, 0.983e^{i1.968\pi}))$	$(2, (0.35e^{i0.66\pi}, 0.014e^{i0.029\pi}, 0.8e^{i1.6\pi}))$	$(4, (0.87e^{1.72\pi}, 0.035e^{i0.068\pi}, 0.93e^{i1.862\pi}))$
w_3	$(2, (0.4e^{i0.82\pi}, 0.012e^{i0.026\pi}, 0.77e^{i1.52\pi}))$	$(3, (0.6e^{1.22\pi}, 0.02e^{i0.06\pi}, 0.49e^{i\pi}))$	$(1, (0.17e^{0.344\pi}, 0.035e^{i0.068\pi}, 0.93e^{i1.862\pi}))$

Table 12: Tabular representation of restricted intersection $(K_L, M, 5)$.

$(K_L, M, 5)$	z_1	z_2
w_1	$(1, (0.23e^{i0.44\pi}, 0.019e^{i0.036\pi}, 0.92e^{i1.85\pi}))$	$(3, (0.65e^{i1.32\pi}, 0.03e^{i0.062\pi}, 0.28e^{i0.58\pi}))$
w_2	$(0, (0.02e^{i0.06\pi}, 0.015e^{i0.032\pi}, 0.983e^{i1.968\pi}))$	$(2, (0.35e^{i0.66\pi}, 0.019e^{i0.04\pi}, 0.8e^{i1.6\pi}))$
w_3	$(2, (0.4e^{i0.82\pi}, 0.012e^{i0.026\pi}, 0.77e^{i1.52\pi}))$	$(1, (0.16e^{i0.34\pi}, 0.1e^{i0.204\pi}, 0.89e^{1.784\pi}))$

$$\mathcal{Q}_D(u_k) = \begin{cases} \left(g_k^1, \left(\mu_{kj}^1, \eta_{kj}^1, \nu_{kj}^1\right)\right), & \text{if } u_k \in Z - B, \\ \left(g_k^2, \left(\mu_{kj}^2, \eta_{kj}^2, \nu_{kj}^2\right)\right), & \text{if } u_k \in B - Z, \\ \left(g_k^j, \left(\mu_{kj}, \eta_{kj}, \nu_{kj}\right)\right), & \text{such that } g_k^j = \min\left(g_k^1, g_k^2\right), \\ \mu_{kj}(u_k) = \min\left(\mu_{kj}^1\left(u_k^1\right), \mu_{kj}^2\left(u_k^2\right)\right) = \min\left(p_{kj}^1\left(u_k^1\right), p_{kj}^2\left(u_k^2\right)\right) e^{i2\pi\left(\min\left(\phi_{kj}^1\left(u_k^1\right), \phi_{kj}^2\left(u_k^2\right)\right)\right)}, \\ \eta_{kj}(u_k) = \max\left(\eta_{kj}^1\left(u_k^1\right), \eta_{kj}^2\left(u_k^2\right)\right) = \max\left(\nu_{kj}^1\left(u_k^1\right), \nu_{kj}^2\left(u_k^2\right)\right) e^{i2\pi\left(\max\left(\delta_{kj}^1\left(u_k^1\right), \delta_{kj}^2\left(u_k^2\right)\right)\right)}, \\ = \nu_{ij}(u_k) = \max\left(\nu_{kj}^1\left(u_k^1\right), \nu_{kj}^2\left(u_k^2\right)\right) = \max\left(r_{kj}^1\left(u_k^1\right), r_{kj}^2\left(u_k^2\right)\right) e^{i2\pi\left(\min\left(\lambda_{kj}^1\left(u_k^1\right), \lambda_{kj}^2\left(u_k^2\right)\right)\right)}, \\ \text{where } \left(g_k^1, \left(\mu_{kj}^1\left(u_k^1\right), \eta_{kj}^1\left(u_k^1\right), \nu_{kj}^1\left(u_k^1\right)\right)\right) \in \left(\mu_{F_J}\left(u_k^1\right), \eta_{F_J}\left(u_k^1\right), \nu_{F_J}\left(u_k^1\right)\right), \\ \text{and } \left(g_k^2, \left(\mu_{kj}^2\left(u_k^2\right), \eta_{kj}^2\left(u_k^2\right), \nu_{kj}^2\left(u_k^2\right)\right)\right) \in \left(\mu_{H_A}\left(u_k^2\right), \eta_{H_A}\left(u_k^2\right), \nu_{H_A}\left(u_k^2\right)\right), \\ \text{with } u_k^1 \in Z \text{ and } u_k^2 \in B. \end{cases} \tag{18}$$

Example 9. Let $(E_P, Z, 5)$ and $(H_A, B, 6)$ be two $CSF5S_fS$ and $CSF6S_fS$, given in Tables 10 and 11, respectively. Their extended intersection $(\mathcal{Q}_D, C, Y) = (E_P, Z, N_1) \cap_E (H_A, B, N_2)$ is shown in Table 13.

Definition 19. Let W be a nonempty set and (F_J, Z, N_1) and (H_A, B, N_2) be two $CSFNS_fSs$ on W; their *restricted union* is defined as $(\mathcal{R}_T, M, S) = (F_J, Z, N_1) \cup_R (H_A, B, N_2)$, where $\mathcal{R}_T = F_J \cup_R H_A$, $M = Z \cap B$, $S = \max(N_1, N_2)$, i.e., $\forall u_k \in M$ and $w_j \in W$, $(g_k^j, (\mu_{kj}, \eta_{kj}, \nu_{kj})) \in \mathcal{R}_T(u_k)$, $\mathbf{g}_k^j = \max(g_k^1, g_k^2)$, $\mu_{kj}(u_k) = \max(\mu_{kj}^1(u_k^1), \mu_{kj}^2(u_k^2)) = \max(p_{kj}^1(u_k^1), p_{kj}^2(u_k^2)) e^{i2\pi(\max(\phi_{kj}^1(u_k^1), \phi_{kj}^2(u_k^2)))}$, $\eta_{kj}(u_k) = \min(\eta_{kj}^1(u_k^1), \eta_{kj}^2(u_k^2)) = \min(\nu_{kj}^1(u_k^1), \nu_{kj}^2(u_k^2)) e^{i2\pi(\min(\delta_{kj}^1(u_k^1), \delta_{kj}^2(u_k^2)))}$, $\nu_{kj}(u_k) = \min(\nu_{kj}^1(u_k^1), \nu_{kj}^2(u_k^2)) = \min(r_{kj}^1(u_k^1), r_{kj}^2(u_k^2)) e^{i2\pi(\min(\lambda_{kj}^1(u_k^1), \lambda_{kj}^2(u_k^2)))}$, where, $(g_k^1, (\mu_{kj}^1(u_k^1), \eta_{kj}^1(u_k^1), \nu_{kj}^1(u_k^1))) \in (\mu_{F_J}(u_k^1), \eta_{F_J}(u_k^1), \nu_{F_J}(u_k^1))$, and $(g_k^2, (\mu_{kj}^2(u_k^2), \eta_{kj}^2(u_k^2), \nu_{kj}^2(u_k^2))) \in (\mu_{H_A}(u_k^2), \eta_{H_A}(u_k^2), \nu_{H_A}(u_k^2))$, with $u_k^1 \in Z$ and $u_k^2 \in B$.

Example 10. Let $(E_P, Z, 5)$ and $(H_A, B, 6)$ be two $CSF5S_fS$ and $CSF6S_fS$, given in Tables 10 and 11, respectively. Their restricted union $(\mathcal{Q}_D, C, Y) = (E_P, Z, N_1) \cap_E (H_A, B, N_2)$ is shown in Table 14.

Definition 20. Let W be a nonempty set (F_J, Z, N_1) and (H_A, B, N_2) be $CSFN_1S_fSs$ and $CSFN_2S_fSs$ on W; their *extended union* is defined as $(\mathfrak{O}_X, C, Y) = (F_J, Z, N_1) \cup_E (H_A, B, N_2)$, where $\mathfrak{O}_X = F_J \cup_E H_A$, $C = Z \cup B$, $Y = \max(N_1, N_2)$, that is, $\forall u_k \in C$ and $w_j \in W$, $(g_k^j, (\mu_{kj}, \eta_{kj}, \nu_{kj})) \in \mathfrak{O}_X(u_k)$, with

Table 13: Tabular representation of extended intersection $(\mathcal{Q}_D, C, 6)$.

$(\mathcal{Q}_D, C, 6)$	z_1	z_2	z_3	z_6
w_1	$(1, (0.23e^{i0.44\pi}, 0.019e^{i0.036\pi}, 0.92e^{i1.85\pi}))$	$(3, (0.65e^{i1.32\pi}, 0.03e^{i0.062\pi}, 0.28e^{i0.58\pi}))$	$(0, (0.1e^{i0.24\pi}, 0.04e^{i0.084\pi}, 0.985e^{i1.964\pi}))$	$(3, (0.69e^{i1.39\pi}, 0.04e^{i0.084\pi}, 0.65e^{i1.36\pi}))$
w_2	$(0, (0.02e^{i0.06\pi}, 0.015e^{i0.032\pi}, 0.983e^{i1.968\pi}))$	$(2, (0.35e^{i0.66\pi}, 0.019e^{i0.04\pi}, 0.8e^{i1.6\pi}))$	$(1, (0.2e^{i0.36\pi}, 0.027e^{i0.05\pi}, 0.91e^{i1.824\pi}))$	$(4, (0.87e^{1.72\pi}, 0.035e^{i0.068\pi}, 0.93e^{i1.862\pi}))$
w_3	$(2, (0.4e^{i0.82\pi}, 0.012e^{i0.026\pi}, 0.77e^{i1.52\pi}))$	$(1, (0.16e^{i0.34\pi}, 0.1e^{i0.204\pi}, 0.89e^{1.784\pi}))$	$(3, (0.69e^{i1.384\pi}, 0.101e^{i0.204\pi}, 0.32e^{0.62\pi}))$	$(1, (0.17e^{0.344\pi}, 0.035e^{i0.068\pi}, 0.93e^{i1.862\pi}))$

Table 14: Tabular representation of restricted union $(\mathscr{R}_T, M, 6)$.

$(\mathscr{R}_T, M, 6)$	z_1	z_2
w_1	$(2, (0.4e^{i0.82\pi}, 0.017e^{i0.0356\pi}, 0.6e^{i1.22\pi}))$	$(5, (0.95e^{i1.88\pi}, 0.018e^{i0.038\pi}, 0.14e^{i0.29\pi}))$
w_2	$(0, (0.03e^{i0.1\pi}, 0.012e^{i0.026\pi}, 0.98e^{i1.962\pi}))$	$(3, (0.7e^{i1.42\pi}, 0.014e^{i0.029\pi}, 0.3e^{i0.56\pi}))$
w_3	$(4, (1, 0, 0))$	$(3, (0.6e^{1.22\pi}, 0.02e^{i0.06\pi}, 0.49e^{i\pi}))$

$$\mathfrak{O}_X(u_k) = \begin{cases} \left(g_k^1, \left(\mu_{kj}^1, \eta_{kj}^1, \nu_{kj}^1\right)\right), & \text{if } u_k \in Z - B, \\ \left(g_k^2, \left(\mu_{kj}^2, \eta_{kj}^2, \nu_{kj}^2\right)\right), & \text{if } u_k \in B - Z, \\ \left(g_k^j, \left(\mu_{kj}, \eta_{kj}, \nu_{kj}\right)\right), & \text{such that } g_k^j = \max\left(g_k^1, g_k^2\right), \\ \mu_{kj}(u_k) = \max\left(\mu_{kj}^1\left(u_k^1\right), \mu_{kj}^2\left(u_k^2\right)\right) = \max\left(p_{kj}^1\left(u_k^1\right), p_{kj}^2\left(u_k^2\right)\right)e^{i\left(\max\left(\phi_{kj}^1\left(u_k^1\right), \phi_{kj}^2\left(u_k^2\right)\right)\right)}, \\ \eta_{kj}(u_k) = \min\left(\eta_{kj}^1\left(u_k^1\right), \eta_{kj}^2\left(u_k^2\right)\right) = \min\left(v_{kj}^1\left(u_k^1\right), v_{kj}^2\left(u_k^2\right)\right)e^{i\left(\min\left(\delta_{kj}^1\left(u_k^1\right), \delta_{kj}^2\left(u_k^2\right)\right)\right)}, \\ \nu_{kj}(u_k) = \min\left(\nu_{kj}^1\left(u_k^1\right), \nu_{kj}^2\left(u_k^2\right)\right) = \min\left(r_{kj}^1\left(u_k^1\right), r_{kj}^2\left(u_k^2\right)\right)e^{i\left(\min\left(\lambda_{kj}^1\left(u_k^1\right), \lambda_{kj}^2\left(u_k^2\right)\right)\right)}, \\ \text{where } \left(g_k^1, \left(\mu_{kj}^1\left(u_k^1\right), \eta_{kj}^1\left(u_k^1\right), \nu_{kj}^1\left(u_k^1\right)\right)\right) \in \left(\mu_{F_J}\left(u_k^1\right), \eta_{F_J}\left(u_k^1\right), \nu_{F_J}\left(u_k^1\right)\right), \\ \text{and } \left(g_k^2, \left(\mu_{kj}^2\left(u_k^2\right), \eta_{kj}^2\left(u_k^2\right), \nu_{kj}^2\left(u_k^2\right)\right)\right) \in \left(\mu_{H_A}\left(u_k^2\right), \eta_{H_A}\left(u_k^2\right), \nu_{H_A}\left(u_k^2\right)\right), \\ \text{with } u_k^1 \in Z \text{ and } u_k^2 \in B. \end{cases} \tag{19}$$

Example 11. Let $(E_P, Z, 5)$ and $(H_A, B, 6)$ be two $CSF5S_fS$ and $CSF6S_fS$, given in Tables 10 and 11, respectively. Their restricted union $(\mathcal{Q}_D, C, Y) = (E_P, Z, N_1) \cap_E (H_A, B, N_2)$ is shown in Table 15.

We state the following properties without their proofs.

Theorem 1. *Let (F_J, Z, N) be $CSFNS_fS$ over a nonempty set W. Then,*

(1) $(F_J, Z, N) \cap_R (F_J, Z, N) = (F_J, Z, N)$
(2) $(F_J, Z, N) \cap_E (F_J, Z, N) = (F_J, Z, N)$
(3) $(F_J, Z, N) \cup_R (F_J, Z, N) = (F_J, Z, N)$
(4) $(F_J, Z, N) \cup_E (F_J, Z, N) = (F_J, Z, N)$

We state the following properties without their proofs.

Theorem 2. *Let (F_J, Z, N_1) and (H_A, B, N_2) be two $CSFNS_fS$s over the same universe W; then, the absorption properties hold:*

(1) $((F_J, Z, N_1) \cup_E (H_A, B, N_2)) \cap_R (F_J, Z, N_1) = (F_J, Z, N_1)$
(2) $(F_J, Z, N_1) \cup_E ((H_A, B, N_2) \cap_R (F_J, Z, N_1) = (F_J, Z, N_1))$
(3) $((F_J, Z, N_1) \cap_R (H_A, B, N_2)) \cup_E (F_J, Z, N_1) = (F_J, Z, N_1)$
(4) $(F_J, Z, N_1) \cap_R ((H_A, B, N_2) \cup_E (F_J, Z, N_1)) = (F_J, Z, N_1)$

We state the following properties without their proofs.

Theorem 3. *Let (F_J, Z, N_1), (H_A, B, N_2) and $(D_\wp, \mathfrak{R}, N_2)$ be any three $CSFNS_fS$s over the same universe W; then, the following properties hold:*

(1) $(F_J, Z, N_1) \cup_E (H_A, B, N_2) = (H_A, B, N_2) \cup_E (F_J, Z, N_1)$
(2) $(F_J, Z, N_1) \cup_R (H_A, B, N_2) = (H_A, B, N_2) \cup_R (F_J, Z, N_1)$
(3) $(F_J, Z, N_1) \cap_E (H_A, B, N_2) = (H_A, B, N_2) \cap_E (F_J, Z, N_1)$
(4) $(F_J, Z, N_1) \cap_R (H_A, B, N_2) = (H_A, B, N_2) \cap_R (F_J, Z, N_1)$
(5) $((F_J, Z, N_1) \cup_E (H_A, B, N_2)) \cup_E (D_\wp, \mathfrak{R}, N_3) = (F_J, Z, N_1) \cup_E ((H_A, B, N_2) \cup_E (D_\wp, \mathfrak{R}, N_3))$
(6) $((F_J, Z, N_1) \cup_R (H_A, B, N_2)) \cup_R (D_\wp, \mathfrak{R}, N_3) = (F_J, Z, N_1) \cup_R ((H_A, B, N_2) \cup_R (D_\wp, \mathfrak{R}, N_3))$
(7) $((F_J, Z, N_1) \cap_E (H_A, B, N_2)) \cap_E (D_\wp, \mathfrak{R}, N_3) = (F_J, Z, N_1) \cap_E ((H_A, B, N_2) \cap_E (D_\wp, \mathfrak{R}, N_3))$
(8) $((F_J, Z, N_1) \cap_R (H_A, B, N_2)) \cap_R (D_\wp, \mathfrak{R}, N_3) = (F_J, Z, N_1) \cap_R ((H_A, B, N_2) \cap_R (D_\wp, \mathfrak{R}, N_3))$
(9) $(F_J, Z, N_1) \cup_E ((H_A, B, N_2) \cap_R (D_\wp, \mathfrak{R}, N_3)) = ((F_J, Z, N_1) \cup_E (H_A, B, N_2)) \cap_R ((F_J, Z, N_1) \cup_E (D_\wp, \mathfrak{R}, N_3))$

Table 15: Tabular representation of extended union $(\mathfrak{D}_X, C, 6)$.

$(\mathfrak{D}_X, C, 6)$	z_1	z_2	z_3	z_6
w_1	$(2, (0.4e^{i0.82\pi}, 0.017e^{i0.0356\pi}, 0.6e^{i1.22\pi}))$	$(5, (0.95e^{i1.88\pi}, 0.018e^{i0.038\pi}, 0.14e^{i0.29\pi}))$	$(0, (0.1e^{i0.24\pi}, 0.04e^{i0.084\pi}, 0.985e^{i1.964\pi}))$	$(3, (0.69e^{i1.39\pi}, 0.04e^{i0.084\pi}, 0.65e^{i1.36\pi}))$
w_2	$(0, (0.03e^{i0.1\pi}, 0.012e^{i0.026\pi}, 0.98e^{i1.962\pi}))$	$(3, (0.7e^{i1.42\pi}, 0.014e^{i0.029\pi}, 0.3e^{i0.56\pi}))$	$(1, (0.2e^{i0.36\pi}, 0.027e^{i0.05\pi}, 0.91e^{i1.824\pi}))$	$(4, (0.87e^{1.72\pi}, 0.035e^{i0.068\pi}, 0.93e^{i1.862\pi}))$
w_3	$(4, (1, 0, 0))$	$(3, (0.6e^{1.22\pi}, 0.02e^{i0.06\pi}, 0.49e^{i\pi}))$	$(3, (0.69e^{i1.384\pi}, 0.101e^{i0.204\pi}, 0.32e^{0.62\pi}))$	$(1, (0.17e^{0.344\pi}, 0.035e^{i0.068\pi}, 0.93e^{i1.862\pi}))$

(10) $(F_J, Z, N_1) \cap_E ((H_A, B, N_2) \cup_R (D_\wp, \mathfrak{R}, N_3)) = ((F_J, Z, N_1) \cap_E (H_A, B, N_2)) \cup_R ((F_J, Z, N_1) \cup_E (D_\wp, \mathfrak{R}, N_3))$

(11) $(F_J, Z, N_1) \cup_R ((H_A, B, N_2) \cap_E (D_\wp, \mathfrak{R}, N_3)) = ((F_J, Z, N_1) \cup_R (H_A, B, N_2)) \cap_E ((F_J, Z, N_1) \cup_R (D_\wp, \mathfrak{R}, N_3))$

(12) $(F_J, Z, N_1) \cap_R ((H_A, B, N_2) \cup_E (D_\wp, \mathfrak{R}, N_3)) = ((F_J, Z, N_1) \cap_R (H_A, B, N_2)) \cup_E ((F_J, Z, N_1) \cap_R (D_\wp, \mathfrak{R}, N_3))$

Definition 21. Let (F_J, Z, N) be a $CSFNS_fS$, where (F, Z, N) is NS_fS over the universe W, and $0 < L < N$ be a threshold. $CSFS_fS$ over W associated with (F, Z, N) and L, denoted by (F_J^L, Z), is defined as follows:

$$F_J^L(z) = \begin{cases} \left(\mu_{kj}(z), \eta_{kj}(z), \nu_{kj}(z)\right), & \text{if } (w, g_z^w) \in F(z) \text{ and } g_z^w \geq L, \\ (0, 0, 1), & \text{otherwise.} \end{cases} \tag{20}$$

Example 12. Let (E_J, Z, N) be a $CSF5S_fS$ given in Table 10. Then, $CSFS_fS$ associated with the thresholds 1, 2, 3, and 4 are shown in Tables 16–19, respectively.

Definition 22. Let (F_J, Z, N) be $CSFNS_fS$, where (F, Z, N) is NS_fS over the universe W. Let $0 < L < N$ and $\alpha \in [-2, 2]$ be two thresholds. S_fS over W associated with (F_J, Z, N) and (L, α), denoted by $(F_J^{(L,\alpha)}, Z)$, is defined as follows:

$$F_J^{(L,\alpha)}(z) = \left\{ S_z^{F_J^L}(w) > \alpha \colon w \in W, \forall z \in Z \right\}, \tag{21}$$

where $S_z^{F_J^L}$ represents the score function of $F_J^L(z)$.

Definition 23. Let $T_{lj} = (g_l^j, p_{lj} e^{i2\pi\phi_{lj}}, \nu_{lj} e^{i2\pi\delta_{lj}}, r_{lj} e^{i2\pi\lambda_{lj}})$ and $T_{kj} = (g_k^j, p_{kj} e^{i2\pi\phi_{kj}}, \nu_{kj} e^{i2\pi\delta_{kj}}, r_{kj} e^{i2\pi\lambda_{kj}})$ be two $CSFNS_fNs$ and $\sigma > 0$. Some operation for $CSFNS_fNs$ are

$$\begin{aligned}
\sigma T_{lj} &= \left(g_l^j, \left[1 - \left(1 - p_{lj}^2\right)^\sigma\right] e^{i2\pi\left[1 - \left(1 - \phi_{lj}^2\right)^\sigma\right]}, \nu_{lj}^\sigma e^{i2\pi\delta_{lj}^\sigma}, r_{lj}^\sigma e^{i2\pi\lambda_{lj}^\sigma} \right), \\
T_{lj}^\sigma &= \left(g_l^j, p_{lj}^\sigma e^{i2\pi\phi_{lj}^\sigma}, \left[1 - \left(1 - \nu_{lj}^2\right)^\sigma\right] e^{i2\pi\left[1 - \left(1 - \delta_{lj}^2\right)^\sigma\right]}, \left[1 - \left(1 - r_{lj}^2\right)^\sigma\right] e^{i2\pi\left[1 - \left(1 - \lambda_{lj}^2\right)^\sigma\right]} \right), \\
T_{lj} \oplus T_{kj} &= \left(\max\left(g_l^j, g_k^j\right), \sqrt{p_{lj}^2 + p_{kj}^2 - p_{lj}^2 p_{kj}^2}\, e^{i2\pi\sqrt{\phi_{lj}^2 + \phi_{kj}^2 - \phi_{lj}^2\phi_{kj}^2}}, \nu_{lj}\nu_{kj} e^{i2\pi\delta_{lj}\delta_{kj}}, r_{lj} r_{kj} e^{i2\pi\lambda_{lj}\lambda_{kj}} \right), \\
T_{lj} \otimes T_{kj} &= \left(\min\left(g_l^j, g_k^j\right), p_{lj} p_{kj} e^{i2\pi\phi_{lj}\phi_{kj}}, \sqrt{\nu_{lj}^2 + \nu_{kj}^2 - \nu_{lj}^2 \nu_{kj}^2}\, e^{i2\pi\sqrt{\delta_{lj}^2 + \delta_{kj}^2 - \delta_{lj}^2\delta_{kj}^2}}, \sqrt{r_{lj}^2 + r_{kj}^2 - r_{lj}^2 r_{kj}^2}\, e^{i2\pi\sqrt{\lambda_{lj}^2 + \lambda_{kj}^2 - \lambda_{lj}^2\lambda_{kj}^2}} \right).
\end{aligned} \tag{22}$$

4. $CSFNS_f$-TOPSIS Method for MAGDM

In this section, we combine $CSFNS_fS$ with the TOPSIS method. The main idea of this methodology is the selection of a best alternative using both the positive ideal solution (PIS) and the negative ideal solution (NIS). Therefore, we present the corresponding $CSFNS_f$-TOPSIS method in order to solve MAGDM problems in a $CSFNS_f$ environment under such methodology. The elements and steps of this algorithm for MAGDM are as follows.

Let $W = \{w_1, w_2, w_3, \ldots, w_q\}$ denote the set of alternatives that are evaluated by s experts $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3, \ldots, \widetilde{E}_s$. According to the needs of MAGDM problems, set of m attributes $Z = \{z_1, z_2, z_3, \ldots, z_m\}$ are assigned to these alternatives by the experts. Let $\sigma = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_s)^T$ be the weight vector, which represents the weightage of experts such that $\sum_{d=1}^s \sigma_d = 1$, where $\sigma_d \in [0, 1]$. The step by step Algorithm 1 of $CSFNS_f$-TOPSIS method is presented in Section 5.1, and its theoretical description is as follows:

Step 1: according to the MAGDM problem and attributes related to the alternatives, each expert assigns ratings to them. There is a linguistic term corresponding with each rating, which could be a number of stars (such as "three stars," "two stars," and "one star" in MAGDM), numerical labels (such as 3 as a label for "high," 2 for "medium," and 0 for "low"). In such a way, NS_fS (F^d, Z, N) is found corresponding to each expert $\widetilde{E}_d$ with $G = \{0, 1, 2, 3, \ldots, N-1\}$ as the set of grades, where $N \in \{1, 2, 3, \ldots\}$ and $d \in \{1, 2, 3, \ldots, s\}$. Now, $CSFNS_fN$ is assigned by the dth expert $\widetilde{E}_d$, corresponding to each rank in the NS_fS (F^d, Z, N), according to the grading criteria defined for the MAGDM problem. Similarly, we get $s\,CSFNS_fSs$ by s experts, respectively. The complex spherical fuzzy N-soft decision matrix $CSFNS_fDM$ of the dth expert $\widetilde{E}_d$ is as follows:

TABLE 16: $CSFS_fS$ related with $(E, Z, 5)$ and threshold 1.

$(E^1, Z, 5)$	z_1	z_2	z_3
w_1	$(0.4e^{i0.82\pi}, 0.017e^{i0.0356\pi}, 0.6e^{i1.22\pi})$	$(0.65e^{i1.32\pi}, 0.018e^{i0.038\pi}, 0.28e^{i0.58\pi})$	$(0, 0, 1)$
w_2	$(0, 0, 1)$	$(0.7e^{i1.42\pi}, 0.019e^{i0.04\pi}, 0.3e^{i0.56\pi})$	$(0.2e^{i0.36\pi}, 0.027e^{i0.05\pi}, 0.91e^{i1.824\pi})$
w_3	$(1, 0, 0)$	$(0.16e^{i0.34\pi}, 0.1e^{i0.204\pi}, 0.89e^{1.784\pi})$	$(0.69e^{i1.384\pi}, 0.101e^{i0.204\pi}, 0.32e^{0.62\pi})$

TABLE 17: $CSFS_fS$ related with $(E, Z, 5)$ and threshold 2.

$(E^2, Z, 5)$	z_1	z_2	z_3
w_1	$(0.4e^{i0.82\pi}, 0.017e^{i0.0356\pi}, 0.6e^{i1.22\pi})$	$(0.65e^{i1.32\pi}, 0.018e^{i0.038\pi}, 0.28e^{i0.58\pi})$	$(0, 0, 1)$
w_2	$(0, 0, 1)$	$(0.7e^{i1.42\pi}, 0.019e^{i0.04\pi}, 0.3e^{i0.56\pi})$	$(0, 0, 1)$
w_3	$(1, 0, 0)$	$(0, 0, 1)$	$(0, 0, 1)$

TABLE 18: $CSFS_fS$ related with $(E_J, Z, 5)$ and threshold 3.

$(E^3, Z, 5)$	z_1	z_2	z_3
w_1	$(0, 0, 1)$	$(0.65e^{i1.32\pi}, 0.018e^{i0.038\pi}, 0.28e^{i0.58\pi})$	$(0, 0, 1)$
w_2	$(0, 0, 1)$	$(0.7e^{i1.42\pi}, 0.019e^{i0.04\pi}, 0.3e^{i0.56\pi})$	$(0, 0, 1)$
w_3	$(1, 0, 0)$	$(0, 0, 1)$	$(0, 0, 1)$

TABLE 19: $CSFS_fS$ related with $(E, Z, 5)$ and threshold 4.

$(E^4, Z, 5)$	z_1	z_2	z_3
w_1	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$
w_2	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$
w_3	$(1, 0, 0)$	$(0, 0, 1)$	$(0, 0, 1)$

$$\mathscr{P}^{(d)} = \begin{pmatrix} \left(g_1^{1(d)}, \mu_{11}^{(d)}, \eta_{11}^{(d)}, \nu_{11}^{(d)}\right) & \left(g_2^{1(d)}, \mu_{12}^{(d)}, \eta_{12}^{(d)}, \nu_{12}^{(d)}\right) & \cdots & \left(g_m^{1(d)}, \mu_{1m}^{(d)}, \eta_{1m}^{(d)}, \nu_{1m}^{(d)}\right) \\ \left(g_1^{2(d)}, \mu_{21}^{(d)}, \eta_{21}^{(d)}, \nu_{21}^{(d)}\right) & \left(g_2^{2(d)}, \mu_{22}^{(d)}, \eta_{22}^{(d)}, \nu_{22}^{(d)}\right) & \cdots & \left(g_m^{2(d)}, \mu_{2m}^{(d)}, \eta_{2m}^{(d)}, \nu_{2m}^{(d)}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(g_1^{q(d)}, \mu_{q1}^{(d)}, \eta_{q1}^{(d)}, \nu_{q1}^{(d)}\right) & \left(g_2^{q(d)}, \mu_{q2}^{(d)}, \eta_{q2}^{(d)}, \nu_{q2}^{(d)}\right) & \cdots & \left(g_m^{q(d)}, \mu_{qm}^{(d)}, \eta_{qm}^{(d)}, \nu_{qm}^{(d)}\right) \end{pmatrix}, \tag{23}$$

where $\mathscr{P}_{jk}^{(d)} = ((g_k^j)^{(d)}, \mu_{jk}^{(d)}, \eta_{jk}\ (d), \nu_{jk}^{(d)}) = ((g_k^j)^{(d)}, p_{jk}^{(d)} e^{i2\pi\phi_{jk}^{(d)}}, \nu_{jk}^{(d)} e^{i2\pi\delta_{jk}^{(d)}}, r_{jk}^{(d)} e^{i2\pi\lambda_{jk}^{(d)}})$, $j = \{1, 2, 3, \ldots, q\}$, $k = \{1, 2, 3, \ldots, m\}$, and $d = \{1, 2, 3, \ldots, s\}$.

Step 2: to formulate the aggregate complex spherical fuzzy N-soft decision matrix ($ACSFNS_fDM$), $\mathscr{P} = (\mathscr{P}_{jk})_{q\times m}$, $CSFNS_fDM$ of all experts are assembled with the help of $CSFNS_fWA$ operator. For $ACSFNS_fDM$, $CSFNS_fWA$ operator is defined as

$$\begin{aligned} \mathscr{P}_{jk} &= CSFNS_fWA\left(\mathscr{P}_{jk}^{(1)}, \mathscr{P}_{jk}^{(2)}, \ldots, \mathscr{P}_{jk}^{(s)}\right) \\ &= \sigma_1 \mathscr{P}_{jk}^{(1)} \oplus \sigma_{(2)} \mathscr{P}_{jk}^{(2)} \oplus \cdots \oplus \sigma_s \mathscr{P}_{jk}^{(s)} \\ &= \left(\max_1^{d=s}\left(\left(g_k^j\right)^{(d)}\right), \sqrt{1 - \prod_1^{d=s}\left(1 - \left(p_{jk}^{(d)}\right)^2\right)^{\sigma_d}}\, e^{i2\pi\sqrt{1-\prod_1^{d=s}\left(1-\left(\phi_{jk}^{(d)}\right)^2\right)^{\sigma_d}}}, \prod_1^{d=s} \nu_{jk}^{(d)} e^{i2\pi\prod_1^{d=s}\delta_{jk}^{(d)}}, \prod_1^{d=s} r_{jk}^{(d)} e^{i2\pi\prod_1^{d=s}\lambda_{jk}^{(d)}} \right) \\ &= \left(g_k^j, p_{jk} e^{i2\pi\phi_{jk}}, \nu_{jk} e^{i2\pi\delta_{jk}}, r_{jk} e^{i2\pi\lambda_{jk}}\right). \end{aligned} \tag{24}$$

Using these entities, we can form $ACSFNS_fDM$ as

$$\mathscr{P} = \begin{pmatrix} \left(g_1^1, \mu_{11}, \eta_{11}, \nu_{11}\right) & \left(g_2^1, \mu_{12}, \eta_{12}, \nu_{12}\right) & \cdots & \left(g_m^1, \mu_{1m}, \eta_{1m}, \nu_{1m}\right) \\ \left(g_1^2, \mu_{21}, \eta_{21}, \nu_{21}\right) & \left(g_2^2, \mu_{22}, \eta_{22}, \nu_{22}\right) & \cdots & \left(g_m^2, \mu_{2m}, \eta_{2m}, \nu_{2m}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(g_1^q, \mu_{q1}, \eta_{q1}, \nu_{q1}\right) & \left(g_2^q, \mu_{q2}, \eta_{q2}, \nu_{q2}\right) & \cdots & \left(g_m^q, \mu_{qm}, \eta_{qm}, \nu_{qm}\right) \end{pmatrix}. \tag{25}$$

Step 3: in MAGDM problem, each attribute has it is own worth. Therefore, each expert $\widetilde{E}_d$ assigns rank as weightage of each attribute z_k relative to their importance in MAGDM problem. Furthermore, $CSFNS_fNs$ are assigned to the weights, according to the grading criteria, by the experts. Let $\chi_k^{(d)} = \left(g_k^{(d)}, p_k^{(d)} e^{i2\pi\phi_k^{(d)}}, v_k^{(d)} e^{i2\pi\delta_k^{(d)}}, r_k^{(d)} e^{i2\pi\lambda_k^{(d)}}\right)$ be the weightage of kth attribute given by the dth expert. To find out the weight vector $\chi = (\chi_1, \chi_2, \ldots, \chi_m)^T$, we aggregated them, as follows:

$$\begin{aligned} \chi_k &= CSFNS_fWA\left(\chi_1^{(1)}, \chi_2^{(2)}, \ldots, \chi_m^{(s)}\right) \\ &= \sigma_1\chi_k^1 \oplus \sigma_2\chi_k^2 \oplus \cdots \oplus \sigma_s\chi_k^s \\ &= \left(\max_{d=1}^s\left(\left(g_k^j\right)^{(d)}\right), \sqrt{1 - \prod_1^{d=s}\left(1 - \left(p_k^{(d)}\right)^2\right)^{\sigma_d}}\, e^{i2\pi\sqrt{1 - \prod_1^{d=s}\left(1 - \left(\phi_k^{(d)}\right)^2\right)^{\sigma_d}}}, \prod_1^{d=s} v_k^{(d)} e^{i2\pi\prod_1^{d=s}\delta_k^{(d)}}, \prod_1^{d=s} r_k^{(d)} e^{i2\pi\prod_1^{d=s}\lambda_k^{(d)}}\right) \\ &= \left(g_k, p_k e^{i2\pi\phi_k}, v_k e^{i2\pi\delta_k}, r_k e^{i2\pi\lambda_k}\right). \end{aligned} \tag{26}$$

Step 4: calculate the aggregated weighted complex spherical fuzzy N-soft decision matrix ($AWCSFNS_fDM$) using $ACSFNS_fDM\, \mathscr{Y}_{jk}$ and the weight vector of attribute χ_k as follows:

$$\begin{aligned} \overline{\mathscr{P}_{jk}} &= \mathscr{P}_{jk} \otimes \chi_k \\ &= \left(\min\left(\left(g_k^j\right), g_k\right), p_{jk}p_k e^{i2\pi\phi_{jk}\phi_k}, \sqrt{v_{jk}^2 + v_k^2 - v_{jk}^2 v_k^2}\, e^{i2\pi\sqrt{\delta_{jk}^2 + \delta_k^2 - \delta_{jk}^2\delta_k^2}}, \sqrt{r_{jk}^2 + r_k^2 - r_{jk}^2 r_k^2}\, e^{i2\pi\sqrt{\lambda_{jk}^2 + \lambda_k^2 - \lambda_{jk}^2\lambda_k^2}}\right) \\ &= \left(\overline{g}_k^j, \overline{\mu}_{jk}, \overline{\eta}_{jk}, \overline{\nu}_{jk}\right) \\ &= \left(\overline{g}_k^j, \overline{p}_{jk} e^{i2\pi\overline{\phi}_{jk}}, \overline{v}_{jk} e^{i2\pi\overline{\delta}_{jk}}, \overline{r}_{jk} e^{i2\pi\overline{\lambda}_{jk}}\right). \end{aligned} \tag{27}$$

Using these entities, we can form $AWCSFNS_fDM$ as

$$\overline{\mathscr{P}} = \begin{pmatrix} \left(\overline{g}_1^1, \overline{\mu}_{11}, \overline{\eta}_{11}, \overline{\nu}_{11}\right) & \left(\overline{g}_{12}^1, \overline{\mu}_{12}, \overline{\eta}_{12}, \overline{\nu}_{12}\right) & \cdots & \left(\overline{g}_m^1, \overline{\mu}_{1m}, \overline{\eta}_{1m}, \overline{\nu}_{1m}\right) \\ \left(\overline{g}_1^2, \overline{\mu}_{21}, \overline{\eta}_{21}, \overline{\nu}_{21}\right) & \left(\overline{g}_2^2, \overline{\mu}_{22}, \overline{\eta}_{22}, \overline{\nu}_{22}\right) & \cdots & \left(\overline{g}_m^2, \overline{\mu}_{2m}, \overline{\eta}_{2m}, \overline{\nu}_{2m}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\overline{g}_1^q, \overline{\mu}_{q1}, \overline{\eta}_{q1}, \overline{\nu}_{q1}\right) & \left(\overline{g}_2^q, \overline{\mu}_{q2}, \overline{\eta}_{q2}, \overline{\nu}_{q2}\right) & \cdots & \left(\overline{g}_m^q, \overline{\mu}_{qm}, \overline{\eta}_{qm}, \overline{\nu}_{qm}\right) \end{pmatrix}. \tag{28}$$

Step 5: let $\mathscr{P}_B$ and $\mathscr{P}_C$ be the collection of benefit-type attribute and cost-type attribute, respectively. $CSFNS_f$-PIS related to the attribute z_k can be taken as follows:

$$\widehat{\mathscr{P}}_k = \begin{cases} \max_{j=1}^q \overline{\mathscr{P}}_{jk}, & \text{if } z_k \in \mathscr{P}_B, \\ \min_{j=1}^q \overline{\mathscr{P}}_{jk}, & \text{if } z_k \in \mathscr{P}_C. \end{cases} \tag{29}$$

Now, $CSFNS_f$-NIS related to the attribute z_k can be taken as follows:

$$\check{\mathscr{P}}_k = \begin{cases} \max_{j=1}^{q} \overline{\mathscr{P}}_{jk}, & \text{if } z_k \in \mathscr{P}_C, \\ \min_{j=1}^{q} \overline{\mathscr{P}}_{jk}, & \text{if } z_k \in \mathscr{P}_B. \end{cases} \tag{30}$$

To evaluate $\max \overline{\mathscr{P}}_{jk}$ and $\min \overline{\mathscr{P}}_{jk}$, we use the score value and accuracy value of $CSFNS_f N$. $CSFNS_f$-PIS and $CSFNS_f$-NIS are denoted as follows: $\widehat{\mathscr{P}}_k = (\widehat{g}_k, \widehat{\mu}_k, \widehat{\eta}_k, \widehat{\nu}_k) = (\widehat{g}_k, \widehat{p}_k e^{i2\pi\widehat{\phi}_k}, \widehat{v}_k e^{i2\pi\widehat{\delta}_k}, \widehat{r}_k e^{i2\pi\widehat{\lambda}_k})$ and $\check{\mathscr{P}}_k = (\check{g}_k, \check{\mu}_k, \check{\eta}_k, \check{\nu}_k) = (\check{g}_k, \check{p}_k e^{i2\pi\check{\phi}_k}, \check{v}_k e^{i2\pi\check{\delta}_k}, \check{r}_k e^{i2\pi\check{\lambda}_k})$, respectively.

Step 6: calculate the normalized Euclidean distance of each alternative w_j from $CSFNS_f$-PIS and $CSFNS_f$-NIS. In this way, we get the best alternative that is nearer to $CSFNS_f$-PIS and far from $CSFNS_f$-NIS. The normalized Euclidean distance between $CSFNS_f$-PIS and any of the alternative w_j can be formulated as follows:

$$d\left(\widehat{\mathscr{P}}_k, w_j\right) = \left(\frac{1}{4k}\sum_{k=1}^{m}\left[\left(\left(\frac{\widehat{g}_k}{N-1}\right)^2 - \left(\frac{\overline{g}_k^j}{N-1}\right)^2\right)^2 + \left(\widehat{p}_k^2 - \overline{p}_{jk}^2\right)^2 + \left(\widehat{v}_k^2 - \overline{v}_{jk}^2\right)^2 + \left(\widehat{r}_k^2 - \overline{r}_{jk}^2\right)^2 + \left(\widehat{\phi}_k^2 - \overline{\phi}_{jk}^2\right)^2 + \left(\widehat{\delta}_k^2 - \overline{\delta}_{jk}^2\right)^2 + \left(\widehat{\lambda}_k^2 - \overline{\lambda}_{jk}^2\right)^2\right]\right)^{1/2}. \tag{31}$$

Similarly, the normalized Euclidean distance between $CSFNS_f$-NIS and any of the alternative w_j, can be formulated as follows:

$$d\left(\check{\mathscr{P}}_k, w_j\right) = \left(\frac{1}{4k}\sum_{k=1}^{m}\left[\left(\left(\frac{\check{g}_k}{N-1}\right)^2 - \left(\frac{\overline{g}_k^j}{N-1}\right)^2\right)^2 + \left(\check{p}_k^2 - \overline{p}_{jk}^2\right)^2 + \left(\check{v}_k^2 - \overline{v}_{jk}^2\right)^2 + \left(\check{r}_k^2 - \overline{r}_{jk}^2\right)^2 + \left(\check{\phi}_k^2 - \overline{\phi}_{jk}^2\right)^2 + \left(\check{\delta}_k^2 - \overline{\delta}_{jk}^2\right)^2 + \left(\check{\lambda}_k^2 - \overline{\lambda}_{jk}^2\right)^2\right]\right)^{1/2}. \tag{32}$$

Step 7: to chose one of the most appropriate alternative, we have to use some ranking index. For this purpose, the revised closeness index corresponding to the alternative w_k is evaluated using the formula [10]

$$\mathfrak{I}\left(w_j\right) = \frac{d\left(\widehat{\mathscr{P}}_k, w_j\right)}{\min_j d\left(\widehat{\mathscr{P}}_k, w_j\right)} - \frac{d\left(\check{\mathscr{P}}_k, w_j\right)}{\max_j d\left(\check{\mathscr{P}}_k, w_j\right)}, \tag{33}$$

where $k = 1, 2, \ldots, m$.

Step 8: the alternative with the minimum value of revised closeness index would be the best solution for the MAGDM problem. Therefore, the ascending order of the revised closeness index gives the ranking of the alternatives.

5. Development of Algorithms and Numerical Examples

In this section, we describe multiattribute decision-making (MADM) methods that work on models to identify the best alternative. Therefore, we characterize respective algorithms for the MADM problems in $CSFNS_f$ environment, as well as we present Algorithm 1 for $CSFNS_f$-TOPSIS method described in Section 4. Let $W = \{w_1, w_2, w_3, \ldots, w_q\}$ be a set, representing the available alternatives with a set of attributes $Z = \{z_1, z_2, z_3, \ldots, z_m\}$ having weight vector $\sigma = (\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_m)^T$ describing the worth of attributes according to the MADM problem, where $\sum_{k=1}^{m} \sigma_k = 1$ and $\sigma_k \in [0, 1]$.

The algorithm for $CSFNS_f$-TOPSIS method is described in Algorithm 1.

Let us now introduce some explicit MADM and MAGDM problems and solve them using Algorithms 1– 4, respectively. We apply Algorithms 2– 4 to solve the MADM problem defined in Section 5.1 and Algorithm 1 is used to solve the MAGDM problem defined in Section 5.2 which show their importance and feasibility in the field of decision-making.

5.1. Selection of Best Third-Party App of the Year. A third-party app is a software application made by someone other than the manufacturer of a mobile device or its operating system. This world is full of gadgets and gadgets are full of apps. We can access the world if we have these apps. Therefore, selecting one of the best third-party app of the

Input: $W = \{w_1, w_2, w_3, \ldots, w_q\}$ as universal element.
$Z = \{z_1, z_2, z_3, \ldots, z_m\}$ as set of attributes.
$NS_fS\,(F, Z, N)$ with $G = \{0, 1, 2, 3, \ldots, N-1\}$, $N \in \{1, 2, 3, \ldots\}$, $(\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_s)^T$ as weight vector of experts $\widetilde{E}_d$.
(1) Construct $CSFNS_fDM\ \mathscr{P}^{(d)}$, corresponding to each level of attribute for the element w_j.
(2) Compute $ACSFNS_fDM$ using equation $\mathscr{P}_{jk} = (\max_1^{d=s}((g_k^j)^{(d)}), \sqrt{1-\prod_1^{d=s}(1-(p_{jk}^{(d)})^2)^{\sigma_d}}\ e^{i2\pi\sqrt{1-\prod_1^{d=s}(1-(\phi_{jk}^{(d)})^2)^{\sigma_d}}}, \prod_1^{d=s} v_{jk}^{(d)} e^{i2\pi\prod_1^{d=s}\delta_{jk}^{(d)}}, \prod_1^{d=s} r_{jk}^{(d)} e^{i2\pi\prod_1^{d=s}\lambda_{jk}^{(d)}})$.
(3) Evaluate the weight vector $\chi = (\chi_1, \chi_2, \ldots, \chi_m)^T$ as follows: $\chi_k = (\max_{d=1}^{s}((g_k^j)^{(d)}), \sqrt{1-\prod_1^{d=s}(1-(p_k^{(d)})^2)^{\sigma_d}}\ e^{i2\pi\sqrt{1-\prod_1^{d=s}(1-(\phi_k^{(d)})^2)^{\sigma_d}}}, \prod_1^{d=s} v_k^{(d)} e^{i2\pi\prod_1^{d=s}\delta_k^{(d)}}, \prod_1^{d=s} r_k^{(d)} e^{i2\pi\prod_1^{d=s}\lambda_k^{(d)}})$.
(4) Calculate $AWCSFNS_fDM$ using $ACSFNS_fDM$ and the weight vector of attributes as follows: $\overline{\mathscr{P}}_{jk} = (\min((g_k^j), g_k), p_{jk}p_k e^{i2\pi\phi_{jk}\phi_k}, \sqrt{v_{jk}^2 + v_k^2 - v_{jk}^2 v_k^2}\ e^{i2\pi\sqrt{\delta_{jk}^2+\delta_k^2-\delta_{jk}^2\delta_k^2}}, \sqrt{r_{jk}^2 + r_k^2 - r_{jk}^2 r_k^2}\ e^{i2\pi\sqrt{\lambda_{jk}^2+\lambda_k^2-\lambda_{jk}^2\lambda_k^2}})$
(5) Calculate $CSFNS_f$ PIS and $CSFNS_f$ NIS, using equations (29) and (30).
(6) Calculate the normalized Euclidean distance of $CSFNS_f$ PIS and $CSFNS_f$ NIS from each alternative, by utilizing equations (31) and (32), respectively.
(7) Calculate the revised closeness index corresponding to each alternative using the formula from [10] $\mathfrak{J}(w_j) = (d(\widehat{\mathscr{P}}_k, w_j)/\min_j d(\widehat{\mathscr{P}}_k, w_j)) - (d(\check{\mathscr{P}}_k, w_j)/\max_j d(\check{\mathscr{P}}_k, w_j))$.
(8) Identify the alternative with minimum revised closeness index.

Algorithm 1: The algorithm of $CSFNS_f$-TOPSIS method.

(1) $h!$
Input: $W = \{w_1, w_2, w_3, \ldots, w_q\}$ as universal element.
$Z = \{z_1, z_2, z_3, \ldots, z_m\}$ as set of attributes.
$NS_fS\,(F, Z, N)$ with $G = \{0, 1, 2, 3, \ldots, N-1\}$, $N \in \{1, 2, 3, \ldots\}$.
(2) Construct the $CSFNS_fN\,\Upsilon_{kj}$, corresponding to each level of attribute for the element w_j.
(3) Compute $X_j = \oplus_{k=1}^{m}\Upsilon_{kj}$ using equation (22), where $\Upsilon_{kj} = (g_k^j, p_{kj}e^{i2\pi\phi_{kj}}, v_{kj}e^{i2\pi\delta_{kj}}, r_{kj}e^{i2\pi\lambda_{kj}})$, $(w_j, g_k^j) \in F(z)$.
(4) Calculate the score function S_{X_j}, using equation (11) for all $j = \{1, 2, 3, \ldots, q\}$.
(5) **if** $S_j = S_p$, *for some* $j, p \in \{1, 2, 3, \ldots, q\}$, **then**
(6) Use accuracy degree and identify alternative with maximum accuracy value
(7) **else**
(8) Identify the alternative with maximum score value.
(9)

Algorithm 2: The algorithm of choice values of $CSFNS_fSs$.

Input: $W = \{w_1, w_2, w_3, \ldots, w_q\}$ as universal element.
$Z = \{z_1, z_2, z_3, \ldots, z_m\}$ as set of attributes.
$NS_fS\,(F, Z, N)$ with $G = \{0, 1, 2, 3, \ldots, N-1\}$, $N \in \{1, 2, 3, \ldots\}$, $(\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_k)^T$ as weight vector.
(1) Construct $CSFNS_fN\,\Upsilon_{kj}$, corresponding to each level of attribute for the element w_j.
(2) Compute $X_j^\sigma = \oplus_{k=1}^{m}\sigma_k\Upsilon_{kj}$, where $\Upsilon_{kj} = (g_k^j, p_{kj}e^{i2\pi\phi_{kj}}, v_{kj}e^{i2\pi\delta_{kj}}, r_{kj}e^{i2\pi\lambda_{kj}})$, $(w_j, g_k^j) \in F(z)$.
(3) Calculate the score function S_{X_j}, for all $j = \{1, 2, 3, \ldots, q\}$. Calculate all the indices j for which $S_j = \max_j S_{X_j^\sigma}$.
(4) **if** $S_j = S_p$, *for some* $j, p \in \{1, 2, 3, \ldots, q\}$, **then**
(5) Use accuracy degree and identify alternative with maximum accuracy value
(6) **else**
(7) Identify the alternative with maximum score value.
(8)

Algorithm 3: The algorithm of weighted choice values of $CSFNS_fSs$.

year and keeping in view the priorties of people is a very difficult task. For this purpose, the data has been collected from the websites http://www.makeawebsitehub.com and http://www.trustraduis.com regarding to each third-party app. To find out the best app of the year, we will use $CSFNS_fS$.

Let $\mathbb{A} = \{\mathbb{a}_1 = \text{Facebook}, \mathbb{a}_2 = \text{Skype}, \mathbb{a}_3 = \text{Viber}, \mathbb{a}_4 = \text{Twitter}, \mathbb{a}_5 = \text{Whatsapp}\}$ be universe of third-party apps and $Z = \{z_1 = \text{telecom framework}, z_2 = \text{reliability}, z_3 = \text{worldwide contact}, z_4 = \text{data usage}\}$ be the attributes. According to these attributes, a 6-soft set is modeled in Table 20, where

Input: $W = \{w_1, w_2, w_3, \ldots, w_q\}$ as universal element.
$Z = \{z_1, z_2, z_3, \ldots, z_m\}$ as set of attributes.
$NS_f S\ (F, Z, N)$ with $G = \{0, 1, 2, 3, \ldots, N-1\}$,
$N \in \{1, 2, 3, \ldots\}$,
Input $0 < L < N-1$, threshold.
(1) Work out for $F_J^L(z) = \begin{cases} (\mu_{kj}(z), \eta_{kj}(z), \nu_{kj}(z)), & \text{if } (w, g_z^w) \in F(z) \text{ and } g_z^w \geq L, \\ (0, 0, 1), & \text{otherwise.} \end{cases}$
(2) First, compute $X_j^L = \oplus_{k=1}^m \Upsilon_{kj}^L$, where $\Upsilon_{kj}^L = (p_{kj}e^{i2\pi\phi_{kj}}, \nu_{kj}e^{i2\pi\delta_{kj}}, r_{kj}e^{i2\pi\lambda_{kj}})$, $(w_j, g_k^j) \in F(z)$.
(3) Calculate the score function $S_{X_j^L}$, for all $j = \{1, 2, 3, \ldots, q\}$.
(4) Calculate all the indices j for which $S_j^L = \max_j S_{X_j^L}$.
(5) **if** $S_j = S_p$, *for some* $j, p \in \{1, 2, 3, \ldots, q\}$, **then**
(6) Use accuracy degree and identify alternative with maximum accuracy value
(7) **else**
(8) Identify the alternative with maximum score value.
(9)

Algorithm 4: The algorithm of L-choice values of $CSFNS_f Ss$.

Five diamonds mean "Marvellous"

Four diamonds mean "Outstanding"

Three diamonds mean "Super"

Two diamonds mean "Good"

One diamond means "Satisfactory"

Big dot means "Acceptable."

This level assessment by diamonds can be represented by numbers as $G = \{0, 1, 2, 3, 4, 5\}$, where

0 means "•"

1 means "◇"

2 means "◇◇"

3 means "◇◇◇,"

4 means "◇◇◇◇"

5 means "◇◇◇◇◇"

Thus, tabular representation of 6-soft set is shown in Table 21 and the tabular representation of $CSF6S_f S\ (F_J, Z, 6)$ is shown in Table 22.

5.1.1. Choice Values of $CSF6S_f S$. The choice values of $CSF6S_f S$ is evaluated using the steps defined in Algorithm 2. Table 23 presents the calculated choice values of $CSF6S_f S$ for the selection of the third-party app. We can observe from Table 23 that, according to the choice values, the ranking of third-party apps is as follows: $\mathbb{a}_1 > \mathbb{a}_4 > \mathbb{a}_5 > \mathbb{a}_2 > \mathbb{a}_3$, which shows that $\mathbb{a}_1$ = Facebook has maximum choice value. Therefore, Facebook is selected as best third-party app of the year.

5.1.2. Weighted Choice Values of $CSF6S_f S$. Let $\sigma_1 = 0.4, \sigma_2 = 0.3, \sigma_3 = 0.2$, and $\sigma_4 = 0.1$ be the weights for each attribute z_k, $k = 1, 2, 3, 4$. Using these weights in Algorithm 3, we can compute weighted choice values of $CSF6S_f S$, which are given by Table 24.

It is clear from Table 24 that G_4 has maximum score; therefore, $\mathbb{a}_4$ = Facebook is selected as best third-party app of the year. According to the weighted choice values, ranking of third-party apps is as follows: $\mathbb{a}_1 > \mathbb{a}_4 > \mathbb{a}_5 > \mathbb{a}_2 > \mathbb{a}_3$.

5.1.3. L-Choice Values of $CSF6S_f S$. The L-choice values of $CSF6S_f S$ are evaluated using Algorithm 4 to find out the best alternative for the proposed MADM problem. Let $L = 4$ be threshold; then, 4-choice values of $CSFS_f S$ is shown in Table 25. We can observe that, from Table 25, the ranking of third-party apps according to 4-choice values is as follows: $\mathbb{a}_1 > \mathbb{a}_4 > \mathbb{a}_5 > \mathbb{a}_2 \geq \mathbb{a}_3$, which shows that $\mathbb{a}_1$ = Facebook has maximum choice value so that Facebook is selected as the best third-party app of the year.

5.2. Selection of the Best Physiotherapist Doctor of Mayo Hospital in Lahore. Physiotherapy helps to restore movement and function when people are affected by injury or disability. A physiotherapist treats such kind of people and helps them through exercise, manual therapy, education, and advice. A physiotherapist is very helpful in maintaining the health of people of all ages as well as encourages them for happy life. A physiotherapist must have patience, communication skills, and ability to establish a good relationship with patients and their families. The motive of this study is to select the best physiotherapist doctor in Lahore relative to their attributes under the environment of $CSFNS_f$. For this purpose, the data has been collected from the students of Mayo Hospital, Lahore, enact here as experts $\tilde{E}_1$, $\tilde{E}_2$, $\tilde{E}_3$, and $\tilde{E}_4$ whose weight vectors are $\sigma = (0.4, 0.2, 0.1, 0.3)^T$. The following physiotherapists of Mayo Hospital are treated as alternatives in this MAGDM problem:

w_1: Dr. Amna

w_2: Dr. Rizwan

w_3: Dr. Akmal

w_4: Dr. Sidra

w_5: Dr. Saleem

Five attributes considered as key factors for a physiotherapist are as follows:

z_1: knowledge and experience.

z_2: behavioral (positivity, patience, and humbleness).

z_3: availability and flexibility.

Table 20: Evaluation of data from the websites.

$\mathbb{A}/Z$	z_1	z_2	z_3	z_4
$\mathbb{a}_1$	◇◇◇◇◇	◇◇◇◇	◇◇◇◇	◇◇
$\mathbb{a}_2$	◇◇◇	◇◇	◇◇◇	◇◇◇
$\mathbb{a}_3$	◇◇	◇◇	◇◇	◇◇◇
$\mathbb{a}_4$	◇◇	◇◇◇	◇◇◇◇◇	◇◇◇◇
$\mathbb{a}_5$	◇◇◇	◇◇◇◇	◇◇◇	◇◇

Table 21: Tabular representation of 6-soft set.

$\mathbb{A}/Z$	z_1	z_2	z_3	z_4
$\mathbb{a}_1$	5	4	4	2
$\mathbb{a}_2$	3	2	3	3
$\mathbb{a}_3$	2	2	2	3
$\mathbb{a}_4$	2	3	5	4
$\mathbb{a}_5$	3	4	3	2

z_4: master of skills (communication, organizational, or problem-solving skills).

z_5: session fee.

We solve this MAGDM problem by following the $CSFNS_f$-TOPSIS method.

Step 1: according to these attributes, each expert model 6-soft set is in Table 26, where

Five stars mean "Marvellous"
Four stars mean "Outstanding"
Three stars mean "Super"
Two stars mean "Good"
One star mean "Satisfactory"
Big dot means "Acceptable"

Table 3 represents the grading criteria, used for assigning the $CSFNS_f N$ corresponding to each rank by the expert $\widetilde{E}_1$, $\widetilde{E}_2$, $\widetilde{E}_3$, and $\widetilde{E}_4$ tabulated in Tables 27–30, respectively.

Step 2: using equation (24), we can put together the opinions of all experts. $ACSFNS_f DM$ formed by aggregation is given in Table 31.

Step 3: to demonstrate the importance of attributes in the MAGDM problem, experts rank them and associate $CSFNS_f N$ to each attribute which are arranged in Table 32. We cumulated the weights given by experts using equation (26) to form $CSFNS_f$ weight vector χ of attributes, i.e.,

$$\chi = \begin{pmatrix} \left(4, \left(0.89e^{i1.72\pi}, 0.017e^{i0.034\pi}, 0.23e^{0.5i\pi}\right)\right) \\ \left(4, \left(0.91e^{i1.86\pi}, 0.016e^{i0.034\pi}, 0.09e^{0.2i\pi}\right)\right) \\ \left(3, \left(0.62e^{i1.24\pi}, 0.016e^{i0.028\pi}, 0.53e^{1.1i\pi}\right)\right) \\ \left(3, \left(0.53e^{i1.06\pi}, 0.02e^{i0.04\pi}, 0.73e^{1.42i\pi}\right)\right) \\ \left(2, \left(0.55e^{i1.14\pi}, 0.019e^{i0.042\pi}, 0.67e^{1.34i\pi}\right)\right) \end{pmatrix}. \tag{34}$$

Step 4: by utilizing $ACSFNS_f DM$ and weight vector χ of attribute in equation (34), $AWCSFNS_f DM$ is evaluated and summarized in Table 33.

Step 5: in the MAGDM problem, all the attributes' knowledge and experience, behavior, availability and flexibility, and master of skills are benefit-type attributes except the session fee, which is a cost-type attribute. According to the nature of attributes and applying equation (29) and (30), $CSFNS_f$-PIS and $CSFNS_f$-NIS are evaluated and arranged in Table 34

Step 6: Table 35 represents the normalized Euclidean distance from each alternative to $CSFNS_f$-PIS and $CSFNS_f$-NIS using equations (31) and (32), respectively.

Step 7: the revised closeness index of each alternative is calculated by utilizing equation (33) and given in Table 36.

Step 8: since w_1 has least revised closeness index, therefore, Dr. Amna is the best physiotherapist in Mayo Hospital, Lahore. The ranking of alternatives is shown in Table 37.

6. Comparative Analysis

We now compare our proposed model with Pythagorean fuzzy N-soft set ($PFNS_f S$) that was discussed by Zhang et al. [24].

(1) Table 38 represents the ratings of MADM problem as shown in Section5.1. in $PFNS_f Ns$.

(2) Table 39 presents the calculated choice values of $PF6S_f S$ using the algorithm defined in [24] for the selection of the third-party app. Clearly, from Table 39, $\mathbb{a}_1$ is the best choice, and the ranking of third-party apps is as follows: $\mathbb{a}_1 > \mathbb{a}_4 > \mathbb{a}_5 > \mathbb{a}_2 > \mathbb{a}_3$.

(3) Let $L = 4$ be threshold; then, 4-choice values of $PFS_f S$ are shown in Table 40. The ranking of third-party apps according to 4-choice values is as follows:

Table 22: Tabular representation of the $CSF6S_fS$ $(F_J, Z, 6)$.

$(F_J, Z, 6)$	z_1	z_2	z_3	z_4
a_1	$(5, (0.95e^{i1.88\pi}, 0.0172e^{i0.0346\pi}, 0.13e^{i0.2\pi}))$	$(4, (0.87e^{i1.76\pi}, 0.01e^{i0.024\pi}, 0.21e^{i0.38\pi}))$	$(4, (0.88e^{i1.78\pi}, 0.012e^{i0.026\pi}, 0.2e^{i0.394\pi}))$	$(2, (0.48e^{i0.98\pi}, 0.012e^{i0.022\pi}, 0.77e^{1.52i\pi}))$
a_2	$(3, (0.7e^{i1.34\pi}, 0.0173e^{i0.0344\pi}, 0.49e^{i\pi}))$	$(2, (0.32e^{i0.66\pi}, 0.015e^{i0.03\pi}, 0.9e^{i1.78\pi}))$	$(3, (0.6e^{i1.24\pi}, 0.081e^{i0.16\pi}, 0.69e^{1.39\pi}))$	$(3, (0.62e^{i1.23\pi}, 0.079e^{i0.15\pi}, 0.73e^{1.48i\pi}))$
a_3	$(2, (0.41e^{i0.84\pi}, 0.014e^{i0.026\pi}, 0.77e^{i1.52\pi}))$	$(2, (0.33e^{i0.64\pi}, 0.016e^{i0.028\pi}, 0.89e^{1.8\pi}))$	$(2, (0.35e^{i0.8\pi}, 0.016e^{i0.033\pi}, 0.84e^{1.7\pi}))$	$(3, (0.53e^{i1.1\pi}, 0.063e^{i0.13\pi}, 0.67e^{i1.38\pi}))$
a_4	$(2, (0.47e^{i0.92\pi}, 0.0169e^{i0.0338\pi}, 0.9e^{i1.8\pi}))$	$(3, (0.72e^{i1.4\pi}, 0.05e^{i0.88\pi}, 0.59e^{i1.22\pi}))$	$(5, (0.9e^{i1.82\pi}, 0.08e^{i0.158\pi}, 0.18e^{0.358\pi}))$	$(4, (0.77e^{i1.52\pi}, 0.017e^{i0.032\pi}, 0.35e^{0.66i\pi}))$
a_5	$(3, (0.71e^{i1.44\pi}, 0.0174e^{i0.0342\pi}, 0.51e^{i0.98\pi}))$	$(4, (0.89e^{i1.784\pi}, 0.013e^{i0.0278\pi}, 0.145e^{i0.4\pi}))$	$(3, (0.55e^{i1.2\pi}, 0.06e^{i0.118\pi}, 0.7e^{i1.42\pi}))$	$(2, (0.49e^{i0.96\pi}, 0.011e^{i0.022\pi}, 0.76e^{1.54i\pi}))$

Table 23: Tabular representation of the choice values of $(F_J, Z, 6)$.

$(F_J, Z, 6)$	z_1	z_2	z_3	z_4	X_j	S_{X_j}
$\mathbb{a}_1$	$(5, (0.95e^{i1.88\pi}, 0.0172e^{i0.0346\pi}, 0.13e^{i0.2\pi}))$	$(4, (0.87e^{i1.76\pi}, 0.01e^{i0.024\pi}, 0.21e^{i0.38\pi}))$	$(4, (0.88e^{i1.78\pi}, 0.012e^{i0.026\pi}, 0.2e^{i0.394\pi}))$	$(2, (0.48e^{i0.98\pi}, 0.012e^{i0.022\pi}, 0.77e^{1.52i\pi}))$	$(5, (0.9979e^{i1.9958\pi}, 2.47\times 10^{-8}e^{i6.34\times 10^{-8}\pi}, 4.21\times 10^{-3}e^{5.62\times 10^{-3}i\pi}))$	2.6860
$\mathbb{a}_2$	$(3, (0.7e^{i1.34\pi}, 0.0173e^{i0.0344\pi}, 0.49e^{i\pi}))$	$(2, (0.32e^{i0.66\pi}, 0.015e^{i0.03\pi}, 0.9e^{i1.78\pi}))$	$(3, (0.6e^{i1.24\pi}, 0.081e^{i0.16\pi}, 0.69e^{1.39\pi}))$	$(3, ((0.62e^{i1.23\pi}, 0.079e^{i0.15\pi}, 0.73e^{1.48i\pi}))$	$(3, (0.9053e^{i1.8024\pi}, 1.66\times 10^{-6}e^{i5.02\times 10^{-7}\pi}, 0.22e^{0.44i\pi}))$	1.78227
$\mathbb{a}_3$	$(2, (0.41e^{i0.84\pi}, 0.014e^{i0.026\pi}, 0.77e^{i1.52\pi}))$	$(2, (0.33e^{i0.64\pi}, 0.016e^{i0.028\pi}, 0.89e^{1.8\pi}))$	$(2, (0.35e^{i0.8\pi}, 0.016e^{i0.033\pi}, 0.84e^{1.7\pi}))$	$(3, (0.53e^{i1.1\pi}, 0.063e^{i0.13\pi}, 0.67e^{1.38i\pi}))$	$(3, (0.7295e^{i1.53\pi}, 2.25\times 10^{-7}e^{i18.93\times 10^{-7}\pi}, 0.3857e^{0.802i\pi}))$	1.0606
$\mathbb{a}_4$	$(2, (0.47e^{i0.92\pi}, 0.0169e^{i0.0338\pi}, 0.9e^{i1.8\pi}))$	$(3, (0.72e^{i1.4\pi}, 0.05e^{i0.88\pi}, 0.59e^{i1.22\pi}))$	$(5, (0.9e^{i1.82\pi}, 0.08e^{i0.158\pi}, 0.18e^{0.358\pi}))$	$(4, (0.77e^{i1.52\pi}, 0.017e^{i0.032\pi}, 0.35e^{0.66i\pi}))$	$(5, (0.9853e^{i1.9704\pi}, 1.149\times 10^{-6}e^{i18.78\times 10^{-6}\pi}, 0.033e^{0.064i\pi}))$	2.6337
$\mathbb{a}_5$	$(3, (0.71e^{i1.44\pi}, 0.0174e^{i0.0342\pi}, 0.51e^{i0.98\pi}))$	$(4, (0.89e^{i1.784\pi}, 0.013e^{i0.0278\pi}, 0.145e^{i0.4\pi}))$	$(3, (0.55e^{i1.2\pi}, 0.06e^{i0.118\pi}, 0.7e^{i1.42\pi}))$.	$(2, (0.49e^{i0.96\pi}, 0.011e^{i0.022\pi}, 0.76e^{1.54i\pi}))$	$(4, (0.9723e^{i1.9506\pi}, 1.493\times 10^{-7}e^{i3.08\times 10^{-7}\pi}, 0.053e^{0.106i\pi}))$	2.3353

Table 24: Tabular representation of the weighted choice values of $(F_J, Z, 6)$.

$(F_J, Z, 6)$	z_1	z_2	z_3	z_4	X_j^σ	$S_{X_j^\sigma}$
$\mathbb{a}_1$	$(5, (0.95e^{i1.88\pi}, 0.0172e^{i0.0346\pi}, 0.13e^{i0.2\pi}))$	$(4, (0.87e^{i1.76\pi}, 0.01e^{i0.024\pi}, 0.21e^{i0.38\pi}))$	$(4, (0.88e^{i1.78\pi}, 0.012e^{i0.026\pi}, 0.2e^{i0.394\pi}))$	$(2, (0.48e^{i0.98\pi}, 0.012e^{i0.022\pi}, 0.77e^{1.52i\pi}))$	$(5, (0.9019e^{i1.7974\pi}, 0.0131e^{i0.0278\pi}, 0.195e^{0.3402i\pi}))$	2.2482
$\mathbb{a}_2$	$(3, (0.7e^{i1.34\pi}, 0.0173e^{i0.0344\pi}, 0.49e^{i\pi}))$	$(2, (0.32e^{i0.66\pi}, 0.015e^{i0.03\pi}, 0.9e^{i1.78\pi}))$	$(3, (0.6e^{i1.24\pi}, 0.081e^{i0.16\pi}, 0.69e^{1.39\pi}))$	$(3, (0.62e^{i1.23\pi}, 0.079e^{i0.15\pi}, 0.73e^{1.48i\pi}))$	$(3, (0.5963e^{i1.168\pi}, 0.0263e^{i0.052\pi}, 0.655e^{1.319i\pi}))$	0.081
$\mathbb{a}_3$	$(2, (0.41e^{i0.84\pi}, 0.014e^{i0.026\pi}, 0.77e^{i1.52\pi}))$	$(2, (0.33e^{i0.64\pi}, 0.016e^{i0.028\pi}, 0.89e^{1.8\pi}))$	$(2, (0.35e^{i0.8\pi}, 0.016e^{i0.033\pi}, 0.84e^{1.7\pi}))$	$(3, (0.53e^{i1.1\pi}, 0.063e^{i0.13\pi}, 0.67e^{1.38i\pi}))$	$(3, (0.3925e^{i0.813\pi}, 0.015e^{i0.032\pi}, 0.807e^{1.61i\pi}))$	−0.7378
$\mathbb{a}_4$	$(2, (0.47e^{i0.92\pi}, 0.0169e^{i0.0338\pi}, 0.9e^{i1.8\pi}))$	$(3, (0.72e^{i1.4\pi}, 0.05e^{i0.88\pi}, 0.59e^{i1.22\pi}))$	$(5, (0.9e^{i1.82\pi}, 0.08e^{i0.158\pi}, 0.18e^{0.358\pi}))$	$(4, (0.77e^{i1.52\pi}, 0.017e^{i0.032\pi}, 0.35e^{0.66i\pi}))$	$(5, (0.7235e^{i1.443\pi}, 0.0319e^{i0.121\pi}, 0.5229e^{1.049i\pi}))$	1.185
$\mathbb{a}_5$	$(3, (0.71e^{i1.44\pi}, 0.0174e^{i0.0342\pi}, 0.51e^{i0.98\pi}))$	$(4, (0.89e^{i1.784\pi}, 0.013e^{i0.0278\pi}, 0.145e^{i0.4\pi}))$	$(3, (0.55e^{i1.2\pi}, 0.06e^{i0.118\pi}, 0.7e^{i1.42\pi}))$	$(2, (0.49e^{i0.96\pi}, 0.011e^{i0.022\pi}, 0.76e^{1.54i\pi}))$	$(4, (0.7570e^{i1.532\pi}, 0.0195e^{i0.0394\pi}, 0.5503e^{0.843i\pi}))$	1.123

Table 25: Tabular representation of the 4-choice values of $(F_J, Z, 6)$.

$(F_J^4, Z, 6)$	z_1	z_2	z_3	z_4	X_j	S_{X_j}
$\mathbb{a}_1$	$(0.95e^{i1.88\pi}, 0.0172e^{i0.0346\pi}, 0.13e^{i0.2\pi}))$	$(0.87e^{i1.76\pi}, 0.01e^{i0.024\pi}, 0.21e^{i0.38\pi})$	$(0.88e^{i1.78\pi}, 0.012e^{i0.026\pi}, 0.2e^{i0.394\pi})$	$(0, 0, 1)$	$(0.9973e^{i1.9944\pi}, 2.064 \times 10^{-6}e^{i1.45\times10^{-11}\pi}, 5.46 \times 10^{-3}e^{2.8\times10^{-5}i\pi})$	1.989
$\mathbb{a}_2$	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$	−1
$\mathbb{a}_3$	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$	$(0, 0, 1)$	−1
$\mathbb{a}_4$	$(0, 0, 1)$	$(0, 0, 1)$	$(0.9e^{i1.82\pi}, 0.08e^{i0.158\pi}, 0.18e^{0.358\pi})$	$(0.77e^{i1.52\pi}, 0.017e^{i0.032\pi}, 0.35e^{0.66i\pi})$	$(0.9605e^{i1.926\pi}, 1.36 \times 10^{-3}e^{i3.18\times10^{-7}\pi}, 0.063e^{i6.978\times10^{-3}\pi})$	1.8459
$\mathbb{a}_5$	$(0, 0, 1)$	$(0.89e^{i1.784\pi}, 0.013e^{i0.0278\pi}, 0.145e^{i0.4\pi})$	$(0, 0, 1)$	$(0, 0, 1)$	$(0.89e^{i1.78\pi}, 0.013e^{i0.0278\pi}, 0.145e^{0.4i\pi})$	1.5263

Table 26: Experts' rating according to attributes.

Attributes	Alternatives	$\widetilde{E}_1$	$\widetilde{E}_2$	$\widetilde{E}_3$	$\widetilde{E}_4$
	w_1	**** = 4	*** = 3	*** = 3	**** = 4
	w_2	*** = 3	*** = 3	*** = 3	*** = 3
z_1	w_3	* = 1	• = 0	** = 2	* = 1
	w_4	** = 2	*** = 3	**** = 4	** = 2
	w_5	** = 2	* = 1	** = 2	• = 0
	w_1	**** = 4	**** = 4	**** = 4	*** = 3
	w_2	** = 2	*** = 3	** = 2	* = 1
z_2	w_3	* = 1	* = 1	* = 1	** = 2
	w_4	*** = 3	** = 2	*** = 3	**** = 4
	w_5	* = 1	** = 2	** = 2	* = 1
	w_1	**** = 4	**** = 4	**** = 4	**** = 4
	w_2	* = 1	** = 2	** = 2	• = 0
z_3	w_3	** = 2	** = 2	** = 2	** = 2
	w_4	*** = 3	* = 1	• = 0	** = 2
	w_5	*** = 3	** = 2	**** = 4	* = 1
	w_1	*** = 3	*** = 3	** = 2	** = 2
	w_2	**** = 4	**** = 4	*** = 3	*** = 3
z_4	w_3	* = 1	*** = 3	**** = 4	** = 2
	w_4	** = 2	*** = 3	*** = 3	*** = 3
	w_5	* = 1	* = 1	** = 2	** = 2
	w_1	**** = 4	*** = 3	*** = 3	** = 2
	w_2	** = 2	** = 2	*** = 3	** = 2
z_5	w_3	* = 1	** = 2	* = 1	* = 1
	w_4	*** = 3	** = 2	* = 1	• = 0
	w_5	** = 2	• = 0	*** = 3	* = 1

$\mathbb{a}_1 > \mathbb{a}_4 > \mathbb{a}_5 > \mathbb{a}_2 \geq \mathbb{a}_3$, which further shows that $\mathbb{a}_1$ = Facebook has maximum choice value.

(4) We conclude the same results from both choice values and L-choice values of $PFNS_fS$ [24], which shows the reliability of our proposed method, and it can be applied to any MADM problem.

(5) The data arranged in Table 23 is able to handle more real-life problems compared to Pythagorean N-soft set and intuitionistic N-soft set as it includes the neutral membership degree as well as it could deal with 2-dimensional data.

(6) The proposed model would provide the same results under spherical fuzzy N-soft environment by taking the periodic terms equal to zero.

6.1. Comparison with Complex Spherical Fuzzy TOPSIS Method. In this section, we solve the MAGDM problem "selection of best physiotherapist doctor of Mayo Hospital in Lahore" by complex spherical fuzzy TOPSIS method, proposed by Akram et al. [16], to demonstrate the importance and superiority of the proposed model. The solution by the complex spherical fuzzy TOPSIS method is as follows:

Step 1: the linguistic term corresponding to each rank assessed by the experts are the same as given in Table 26. To apply the CSF TOPSIS method, the grading part is excluded from $CSFNS_fN$ and CSFNs are assigned by each expert $\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3$, and $\widetilde{E}_4$, which are arranged in Tables 41–44, respectively, according to the grading criteria defined in Table 3.

Step 2: using the weight vector of experts $\sigma = \{0.4, 0.2, 0.1, 0.3\}^T$ and complex spherical fuzzy weighted average (CSFWA) operator [16], we can calculate the aggregated complex spherical fuzzy decision matrix (ACSFDM), whose entries are evaluated by the formula defined as follows [16]:

$$\mathscr{P}_{jk} = \left(\sqrt{1 - \prod_1^{d=s} \left(1 - \left(p_{jk}^{(d)}\right)^2\right)^{\sigma_d}} e^{i2\pi \sqrt{1 - \prod_1^{d=s} \left(1 - \left(\phi_{jk}^{(d)}\right)^2\right)^{\sigma_d}}}, \prod_1^{d=s} v_{jk}^{(d)} e^{i2\pi \prod_1^{d=s} \delta_{jk}^{(d)}}, \prod_1^{d=s} r_{jk}^{(d)} e^{i2\pi \prod_1^{d=s} \lambda_{jk}^{(d)}} \right). \tag{35}$$

ACSFDM is summarized in Table 45.

Step 3: the experts' opinion about the importance of attributes are given in Table 46. The experts' opinion are combined using (CSFWA) operator [16], to formulate

Table 27: Tabular representation of $CSFNS_f DM$ of expert $\widetilde{E}_1$.

$(F_J^{(1)}, Z, 5)$	z_1	z_2	z_3	z_4	z_5
w_1	$(4, (0.98e^{i1.98\pi}, 0.01e^{i0.024\pi}, 0.02e^{i0.06\pi}))$	$(4, (0.99e^{i1.94\pi}, 0.012e^{i0.02\pi}, 0.1e^{i0.26\pi}))$	$(4, (0.97e^{i1.96\pi}, 0.013e^{i0.02\pi}, 0.05e^{i0.08\pi}))$	$(3, (0.84e^{i1.66\pi}, 0.019e^{i0.04\pi}, 0.29e^{0.6i\pi}))$	$(4, (0.98e^{i1.92\pi}, 0.012e^{i0.026\pi}, 0.03e^{0.01i\pi}))$
w_2	$(3, (0.84e^{i1.66\pi}, 0.02e^{i0.042\pi}, 0.29e^{i0.62\pi}))$	$(2, (0.37e^{i0.78\pi}, 0.015e^{i0.026\pi}, 0.87e^{i1.72\pi}))$	$(1, (0.16e^{i0.34\pi}, 0.018e^{i0.038\pi}, 0.89e^{1.78\pi}))$	$(4, (0.96e^{i1.94\pi}, 0.013e^{i0.024\pi}, 0.04e^{0.06i\pi}))$	$(2, (0.36e^{i0.74\pi}, 0.016e^{i0.03\pi}, 0.59e^{1.2i\pi}))$
w_3	$(1, (0.17e^{i0.36\pi}, 0.02e^{i0.038\pi}, 0.91e^{i1.84\pi}))$	$(1, (0.19e^{i0.36\pi}, 0.021e^{i0.04\pi}, 0.93e^{1.82\pi}))$	$(2, (0.59e^{i0.12\pi}, 0.0155e^{i0.032\pi}, 0.82e^{1.6\pi}))$	$(1, (0.2e^{i0.38\pi}, 0.022e^{i0.042\pi}, 0.92e^{1.86i\pi}))$	$(1, (0.21e^{i0.44\pi}, 0.024e^{i0.05\pi}, 0.93e^{1.88i\pi}))$
w_4	$(2, (0.58e^{i1.12\pi}, 0.015e^{i0.032\pi}, 0.82e^{i1.66\pi}))$	$(3, (0.66e^{i1.34\pi}, 0.1e^{i0.18\pi}, 0.56e^{i1.14\pi}))$	$(3, (0.67e^{i1.36\pi}, 0.021e^{i0.044\pi}, 0.31e^{0.64\pi}))$	$(2, (0.55e^{i1.14\pi}, 0.014e^{i0.028\pi}, 0.8e^{1.62i\pi}))$	$(3, (0.67e^{i1.38\pi}, 0.025e^{i0.048\pi}, 0.33e^{0.68i\pi}))$
w_5	$(2, (0.52e^{i1.02\pi}, 0.012e^{i0.026\pi}, 0.74e^{i1.5\pi}))$	$(1, (0.21e^{i0.4\pi}, 0.021e^{i0.046\pi}, 0.95e^{i1.88\pi}))$	$(1, (0.7e^{i1.36\pi}, 0.026e^{i0.05\pi}, 0.35e^{i0.68\pi}))$	$(1, (0.16e^{i0.42\pi}, 0.018e^{i0.038\pi}, 0.89e^{1.88i\pi}))$	$(2, (0.58e^{i1.14\pi}, 0.01e^{i0.024\pi}, 0.82e^{1.62i\pi}))$

Table 28: Tabular representation of $CSFNS_f DM$ of expert $\widetilde{E}_2$.

$(F_J^{(2)}, Z, 5)$	z_1	z_2	z_3	z_4	z_5
w_1	$(3, (0.83e^{i1.64\pi}, 0.019e^{i0.036\pi}, 0.3e^{i0.58\pi}))$	$(4, (0.98e^{i1.94\pi}, 0.013e^{i0.024\pi}, 0.1e^{i0.26\pi}))$	$(4, (0.98e^{i1.98\pi}, 0.01e^{i0.024\pi}, 0.02e^{i0.06\pi}))$	$(3, (0.84e^{i1.66\pi}, 0.02e^{i0.036\pi}, 0.31e^{0.63i\pi}))$	$(3, (0.8e^{i1.62\pi}, 0.022e^{i0.042\pi}, 0.32e^{0.6i\pi}))$
w_2	$(3, (0.71e^{i1.38\pi}, 0.028e^{i0.058\pi}, 0.37e^{i0.72\pi}))$	$(3, (0.72e^{i1.42\pi}, 0.03e^{i0.062\pi}, 0.38e^{i0.76\pi}))$	$(2, (0.57e^{i1.1\pi}, 0.012e^{i0.028\pi}, 0.81e^{1.6\pi}))$	$(4, (0.86e^{i1.74\pi}, 0.0169e^{i0.032\pi}, 0.027e^{0.052i\pi}))$	$(2, (0.56e^{i1.1\pi}, 0.015e^{i0.07\pi}, 0.8e^{1.58i\pi}))$
w_3	$(0, (0.05e^{i0.012\pi}, 0.01e^{i0.06\pi}, 0.985e^{i1.972\pi}))$	$(1, (0.21e^{i0.46\pi}, 0.024e^{i0.52\pi}, 0.94e^{1.86\pi}))$	$(2, (0.5e^{i1.02\pi}, 0.01e^{i0.02\pi}, 0.7e^{1.4\pi}))$	$(3, (0.73e^{i1.48\pi}, 0.032e^{i0.062\pi}, 0.39e^{0.76i\pi}))$	$(2, (0.55e^{i1.08\pi}, 0.01e^{i0.02\pi}, 0.79e^{1.56i\pi}))$
w_4	$(3, (0.74e^{i1.5\pi}, 0.035e^{i0.068\pi}, 0.4e^{i0.78\pi}))$	$(2, (0.54e^{i1.04\pi}, 0.015e^{i0.026\pi}, 0.78e^{i1.54\pi}))$	$(1, (0.24e^{i0.44\pi}, 0.03e^{i0.056\pi}, 0.94e^{1.84\pi}))$	$(3, (0.71e^{i1.38\pi}, 0.028e^{i0.058\pi}, 0.37e^{0.72i\pi}))$	$(2, (0.53e^{i1.04\pi}, 0.013e^{i0.026\pi}, 0.77e^{1.52i\pi}))$
w_5	$(1, (0.24e^{i0.42\pi}, 0.028e^{i0.062\pi}, 0.91e^{i1.84\pi}))$	$(2, (0.51e^{i1.04\pi}, 0.013e^{i0.06\pi}, 0.76e^{i1.5\pi}))$	$(2, (0.52e^{i1.02\pi}, 0.013e^{i0.06\pi}, 0.75e^{i1.52\pi}))$	$(1, (0.23e^{i0.5\pi}, 0.035e^{i0.066\pi}, 0.89e^{1.82i\pi}))$	$(0, (0.1e^{i0.24\pi}, 0.012e^{i0.026\pi}, 0.99e^{1.972i\pi}))$

TABLE 29: Tabular representation of $CSFNS_f DM$ of expert $\widetilde{E}_3$.

$(F_J^{(3)}, Z, 5)$	z_1	z_2	z_3	z_4	z_5
w_1	$(3, (0.84e^{i1.66\pi}, 0.019e^{i0.04\pi}, 0.29e^{i0.6\pi}))$	$(4, (0.97e^{i1.98\pi}, 0.013e^{i0.028\pi}, 0.14e^{i0.26\pi}))$	$(4, (0.96e^{i1.98\pi}, 0.01e^{i0.04\pi}, 0.04e^{i0.04\pi}))$	$(2, (0.5e^{i1.02\pi}, 0.014e^{i0.026\pi}, 0.74e^{1.46i\pi}))$	$(3, (0.8e^{i1.62\pi}, 0.018e^{i0.038\pi}, 0.29e^{0.6i\pi}))$
w_2	$(3, (0.72e^{i1.46\pi}, 0.036e^{i0.068\pi}, 0.41e^{i0.8\pi}))$	$(2, (0.49e^{i\pi}, 0.013e^{i0.028\pi}, 0.71e^{i1.44\pi}))$	$(2, (0.48e^{i0.98\pi}, 0.015e^{i0.032\pi}, 0.7e^{1.36\pi}))$	$(3, (0.74e^{i1.44\pi}, 0.034e^{i0.07\pi}, 0.42e^{0.82i\pi}))$	$(3, (0.75e^{i1.48\pi}, 0.035e^{i0.072\pi}, 0.43e^{0.82i\pi}))$
w_3	$(2, (0.47e^{i0.96\pi}, 0.015e^{i0.032\pi}, 0.7e^{i1.36\pi}))$	$(1, (0.27e^{i0.5\pi}, 0.034e^{i0.066\pi}, 0.94e^{1.92\pi}))$	$(2, (0.46e^{i0.94\pi}, 0.014e^{i0.03\pi}, 0.68e^{1.38\pi}))$	$(4, (0.88e^{i1.78\pi}, 0.015e^{i0.032\pi}, 0.25e^{0.46i\pi}))$	$(1, (0.29e^{i0.56\pi}, 0.04e^{i0.078\pi}, 0.96e^{1.9i\pi}))$
w_4	$(4, (0.98e^{i1.94\pi}, 0.013e^{i0.06\pi}, 0.1e^{i0.26\pi}))$	$(3, (0.76e^{i1.5\pi}, 0.036e^{i0.074\pi}, 0.44e^{i0.86\pi}))$	$(0, (0.13e^{i0.6\pi}, 0.014e^{i0.03\pi}, 0.986e^{1.97\pi}))$	$(3, (0.77e^{i1.52\pi}, 0.038e^{i0.078\pi}, 0.45e^{0.88i\pi}))$	$(1, (0.3e^{i0.58\pi}, 0.042e^{i0.082\pi}, 0.925e^{1.86i\pi}))$
w_5	$(2, (0.44e^{i0.92\pi}, 0.012e^{i0.028\pi}, 0.66e^{i1.34\pi}))$	$(2, (0.43e^{i0.88\pi}, 0.015e^{i0.028\pi}, 0.66e^{i1.34\pi}))$	$(4, (0.87e^{i1.78\pi}, 0.016e^{i0.028\pi}, 0.21e^{i0.43\pi}))$	$(2, (0.42e^{i0.86\pi}, 0.012e^{i0.02\pi}, 0.65e^{1.32i\pi}))$	$(3, (0.78e^{i1.54\pi}, 0.04e^{i0.076\pi}, 0.46e^{0.9i\pi}))$

Table 30: Tabular representation of $CSFNS_f DM$ of expert $\widetilde{E}_4$.

$(F_J^{(4)}, Z, 5)$	z_1	z_2	z_3	z_4	z_5
w_1	$(4, (0.95e^{i1.94\pi}, 0.012e^{i0.02\pi}, 0.13e^{i0.2\pi}))$	$(3, (0.8e^{i1.62\pi}, 0.018e^{i0.038\pi}, 0.31e^{i0.6\pi}))$	$(4, (0.97e^{i1.96\pi}, 0.01e^{i0.02\pi}, 0.15e^{i0.32\pi}))$	$(2, (0.63e^{i1.28\pi}, 0.012e^{i0.02\pi}, 0.6e^{1.18i\pi}))$	$(2, (0.64e^{i1.26\pi}, 0.01e^{i0.024pi}, 0.59e^{1.2i\pi}))$
w_2	$(3, (0.77e^{i1.56\pi}, 0.041e^{i0.08\pi}, 0.47e^{i0.92\pi}))$	$(1, (0.31e^{i0.64\pi}, 0.022e^{i0.04\pi}, 0.89e^{i1.8\pi}))$	$(0, (0.06e^{i0.14\pi}, 0.012e^{i0.02\pi}, 0.986e^{1.97\pi}))$	$(3, (0.8e^{i1.58\pi}, 0.05e^{i0.102\pi}, 0.48e^{0.94i\pi}))$	$(2, (0.37e^{i1.78\pi}, 0.014e^{i0.024\pi}, 0.86e^{1.74i\pi}))$
w_3	$(1, (0.3e^{i0.64\pi}, 0.05e^{i0.012\pi}, 0.91e^{i1.84\pi}))$	$(2, (0.41e^{i0.84\pi}, 0.01e^{i0.06\pi}, 0.64e^{1.3\pi}))$	$(2, (0.4e^{i0.84\pi}, 0.015e^{i0.032\pi}, 0.63e^{1.28\pi}))$	$(2, (0.39e^{i0.8\pi}, 0.014e^{i0.03\pi}, 0.62e^{1.26i\pi}))$	$(1, (0.33e^{i0.64\pi}, 0.07e^{i0.1\pi}, 0.93e^{1.88i\pi}))$
w_4	$(2, (0.38e^{i0.78\pi}, 0.013e^{i0.07\pi}, 0.61e^{i1.24\pi}))$	$(4, (0.9e^{i1.84\pi}, 0.014e^{i0.03\pi}, 0.19e^{i0.36\pi}))$	$(2, (0.37e^{i0.76\pi}, 0.012e^{i0.026\pi}, 0.6e^{1.22\pi}))$	$(3, (0.81e^{i1.6\pi}, 0.051e^{i0.104\pi}, 0.49e^{i\pi}))$	$(0, (0.13e^{i0.28\pi}, 0.014e^{i0.026\pi}, 0.982e^{1.966i\pi}))$
w_5	$(0, (0.07e^{i0.012\pi}, 0.016e^{i0.03\pi}, 0.991e^{i1.984\pi}))$	$(1, (0.34e^{i0.64\pi}, 0.065e^{i0.132\pi}, 0.95e^{i1.92\pi}))$	$(1, (0.3e^{i0.58\pi}, 0.014e^{i0.03\pi}, 0.93e^{i1.88\pi}))$	$(2, (0.36e^{i0.74\pi}, 0.01e^{i0.024\pi}, 0.59e^{1.2i\pi}))$	$(1, (0.21e^{i0.46\pi}, 0.024e^{i0.052\pi}, 0.94e^{1.86i\pi}))$

Table 31: Tabular representation of $ACSFNS_f DM$.

$(F_J^{(1)}, Z, 5)$	z_1	z_2	z_3	z_4	z_5
w_1	$(4, (0.95e^{i1.94\pi}, 0.013e^{i0.026\pi}, 0.07e^{i0.18\pi}))$	$(4, (0.97e^{i1.9\pi}, 0.013e^{i0.026\pi}, 0.15e^{i0.34\pi}))$	$(4, (0.97e^{i1.96\pi}, 0.01e^{i0.02\pi}, 0.05e^{i0.1\pi}))$	$(3, (0.77e^{i1.54\pi}, 0.016e^{i0.03\pi}, 0.4e^{0.82i\pi}))$	$(4, (0.91e^{i1.76\pi}, 0.013e^{i0.03\pi}, 0.15e^{0.36i\pi}))$
w_2	$(3, (0.79e^{i1.56\pi}, 0.028e^{i0.058\pi}, 0.36e^{i0.74\pi}))$	$(3, (0.48e^{i0.96\pi}, 0.019e^{i0.034\pi}, 0.73e^{i1.46\pi}))$	$(2, (0.33e^{i0.66\pi}, 0.01e^{i0.02\pi}, 0.88e^{1.76\pi}))$	$(4, (0.9e^{i1.82\pi}, 0.02e^{i0.04\pi}, 0.15e^{0.26i\pi}))$	$(3, (0.47e^{i0.96\pi}, 0.016e^{i0.03\pi}, 0.68e^{1.36i\pi}))$
w_3	$(2, (0.25e^{i0.52\pi}, 0.022e^{i0.048\pi}, 0.9e^{i1.74\pi}))$	$(2, (0.29e^{i0.58\pi}, 0.018e^{i0.038\pi}, 0.83e^{1.66\pi}))$	$(2, (0.5e^{i1.04\pi}, 0.013e^{i0.028\pi}, 0.72e^{1.42\pi}))$	$(4, (0.55e^{i1.12\pi}, 0.029e^{i0.038\pi}, 0.6e^{1.2i\pi}))$	$(2, (0.35e^{i0.7\pi}, 0.029e^{i0.054\pi}, 0.9e^{1.82i\pi}))$
w_4	$(4, (0.7e^{i1.18\pi}, 0.016e^{i0.034\pi}, 0.63e^{i1.28\pi}))$	$(4, (0.77e^{i1.56\pi}, 0.034e^{i0.064\pi}, 0.42e^{i0.84\pi}))$	$(3, (0.51e^{i1.02\pi}, 0.01e^{i0.02\pi}, 0.53e^{1.06\pi}))$	$(3, (0.71e^{i1.4\pi}, 0.026e^{i0.054\pi}, 0.55e^{1.12i\pi}))$	$(3, (0.52e^{i1.06\pi}, 0.019e^{i0.038\pi}, 0.6e^{1.22i\pi}))$
w_5	$(2, (0.39e^{i0.74\pi}, 0.015e^{i0.032\pi}, 0.83e^{i1.68\pi}))$	$(2, (0.35e^{i0.7\pi}, 0.025e^{i0.052\pi}, 0.87e^{i1.74\pi}))$	$(4, (0.63e^{i1.26\pi}, 0.02e^{i0.034\pi}, 0.51e^{i1.02\pi}))$	$(2, (0.28e^{i0.6\pi}, 0.016e^{i0.034\pi}, 0.76e^{1.58i\pi}))$	$(3, (0.49e^{i0.96\pi}, 0.015e^{i0.034\pi}, 0.83e^{1.6i\pi}))$

Table 32: Experts opinion related to each attribute.

$(F_J, Z, 6)$	$\widetilde{E}_1$	$\widetilde{E}_2$	$\widetilde{E}_3$	$\widetilde{E}_4$
z_1	$(3, (0.85e^{i1.66\pi}, 0.019e^{i0.042\pi}, 0.29e^{i0.6\pi}))$	$(4, (0.99e^{i1.94\pi}, 0.012e^{i0.02\pi}, 0.1e^{i0.26\pi}))$	$(4, (0.88e^{i1.78\pi}, 0.015e^{i0.032\pi}, 0.25e^{i0.46\pi}))$	$(3, (0.67e^{i1.36\pi}, 0.021e^{i0.044\pi}, 0.31e^{0.64i\pi}))$
z_2	$(3, (0.66e^{i1.34\pi}, 0.021e^{i0.044\pi}, 0.31e^{i0.64\pi}))$	$(4, (0.96e^{i1.94\pi}, 0.013e^{i0.024\pi}, 0.04e^{i0.06\pi}))$	$(3, (0.78e^{i1.54\pi}, 0.04e^{i0.076\pi}, 0.46e^{0.9\pi}))$	$(4, (0.98e^{i1.98\pi}, 0.01e^{i0.024\pi}, 0.02e^{0.06i\pi}))$
z_3	$(2, (0.63e^{i1.28\pi}, 0.012e^{i0.02\pi}, 0.59e^{i1.2\pi}))$	$(3, (0.72e^{i1.42\pi}, 0.03e^{i0.062\pi}, 0.38e^{0.78\pi}))$	$(3, (0.75e^{i1.48\pi}, 0.035e^{i0.072\pi}, 0.43e^{0.82\pi}))$	$(2, (0.42e^{i0.86\pi}, 0.012e^{i0.02\pi}, 0.65e^{i1.32\pi}))$
z_4	$(1, (0.29e^{i0.6\pi}, 0.014e^{i0.03\pi}, 0.94e^{i1.86\pi}))$	$(1, (0.21e^{i0.46\pi}, 0.024e^{i0.052\pi}, 0.94e^{i1.86\pi}))$	$(2, (0.37e^{i0.76\pi}, 0.012e^{i0.026\pi}, 0.6e^{1.22\pi}))$	$(3, (0.78e^{i1.54\pi}, 0.04e^{i0.076\pi}, 0.46e^{0.9i\pi}))$
z_5	$(2, (0.62e^{i1.28\pi}, 0.01e^{i0.024\pi}, 0.58e^{i1.2\pi}))$	$(2, (0.66e^{i1.34\pi}, 0.1e^{i0.18\pi}, 0.56e^{i1.14\pi}))$	$(2, (0.64e^{i1.26\pi}, 0.01e^{i0.024\pi}, 0.59e^{i1.2\pi}))$	$(1, (0.21e^{i0.4\pi}, 0.021e^{i0.046\pi}, 0.95e^{1.88i\pi}))$

Table 33: Tabular representation of $AWCSFNS_f DM$.

	z_1	z_2	z_3	z_4	z_5
w_1	$(4, (0.85e^{i1.66\pi}, 0.021e^{i0.042\pi}, 0.23e^{i0.52\pi}))$	$(4, (0.88e^{i1.76\pi}, 0.021e^{i0.042\pi}, 0.174e^{i0.38\pi}))$	$(3, (0.601e^{i1.22\pi}, 0.019e^{i0.034\pi}, 0.053e^{i1.1\pi}))$	$(3, (0.41e^{i0,82\pi}, 0.025e^{i0.05\pi}, 0.77e^{1.54i\pi}))$	$(2, (0.5e^{i\pi}, 0.023e^{i0.05\pi}, 0.67e^{1.36i\pi}))$
w_2	$(3, (0.7e^{i1.34\pi}, 0.03e^{i0.066\pi}, 0.41e^{i0.86\pi}))$	$(3, (0.43e^{i0.9\pi}, 0.025e^{i0.048\pi}, 0.73e^{i1.46\pi}))$	$(2, (0.2e^{i0.4\pi}, 0.019e^{i0.034\pi}, 0.91e^{1.82\pi}))$	$(3, (0.47e^{i0.96\pi}, 0.028e^{i0.056\pi}, 0.73e^{1.42i\pi}))$	$(2, (0.25e^{i0.54\pi}, 0.025e^{i0.05\pi}, 0.83e^{1.66i\pi}))$
w_3	$(2, (0.22e^{i0.44\pi}, 0.027e^{i0.058\pi}, 0.91e^{i1.74\pi}))$	$(2, (0.26e^{i0.54\pi}, 0.024e^{i0.025\pi}, 0.83e^{1.66\pi}))$	$(2, (0.32e^{i0.64\pi}, 0.021e^{i0.038\pi}, 0.8e^{1.62\pi}))$	$(3, (0.29e^{i0.58\pi}, 0.035e^{i0.068\pi}, 0.73e^{1.42i\pi}))$	$(2, (0.19e^{i0.34\pi}, 0.035e^{i0.068\pi}, 0.95e^{1.9i\pi}))$
w_4	$(4, (0.62e^{i1.02\pi}, 0.023e^{i0.048\pi}, 0.65e^{i1.34\pi}))$	$(4, (0.70e^{i1.46\pi}, 0.037e^{i0.072\pi}, 0.43e^{i0.86\pi}))$	$(3, (0.32e^{i0.64\pi}, 0.019e^{i0.034\pi}, 0.69e^{1.42\pi}))$	$(3, (0.37e^{i0.74\pi}, 0.033e^{i0.066\pi}, 0.82e^{1.62i\pi}))$	$(2, (0.29e^{i0.6\pi}, 0.027e^{i0.056\pi}, 0.8e^{1.62i\pi}))$
w_5	$(2, (0.35e^{i0.62\pi}, 0.022e^{i0.046\pi}, 0.83e^{i1.7\pi}))$	$(2, (0.31e^{i0.66\pi}, 0.029e^{i0.06\pi}, 0.87e^{i1.74\pi}))$	$(3, (0.39e^{i0.78\pi}, 0.025e^{i0.044\pi}, 0.68e^{i1.38\pi}))$	$(2, (0.15e^{i0.318\pi}, 0.025e^{i0.052\pi}, 0.89e^{1.8i\pi}))$	$(2, (0.27e^{i0.54\pi}, 0.024e^{i0.042\pi}, 0.91e^{1.82i\pi}))$

Table 34: Tabular representation of $CSFNS_f$-PIS and $CSFNS_f$-NIS.

Attribute	$CSFNS_f$-PIS	$CSFNS_f$-NIS
z_1	$(4, (0.85e^{i1.66\pi}, 0.021e^{i0.042\pi}, 0.23e^{i0.52\pi}))$	$(2, (0.22e^{i0.44\pi}, 0.027e^{i0.058\pi}, 0.91e^{i1.74\pi}))$
z_2	$(4, (0.88e^{i1.76\pi}, 0.021e^{i0.042\pi}, 0.174e^{i0.038\pi}))$	$(2, (0.31e^{i0.66\pi}, 0.029e^{i0.06\pi}, 0.87e^{i1.74\pi}))$
z_3	$(3, (0.601e^{i1.22\pi}, 0.019e^{i0.034\pi}, 0.53e^{i1.1\pi}))$	$(2, (0.2e^{i0.4\pi}, 0.019e^{i0.034\pi}, 0.91e^{1.82\pi}))$
z_4	$(3, (0.47e^{i0.96\pi}, 0.028e^{i0.056\pi}, 0.73e^{1.42\pi}))$	$(2, (0.15e^{i0.318\pi}, 0.025e^{i0.052\pi}, 0.89e^{1.8\pi}))$
z_5	$(2, (0.19e^{i0.38\pi}, 0.035e^{i0.068\pi}, 0.95e^{i1.9\pi}))$	$(2, (0.5e^{i\pi}, 0.023e^{i0.046\pi}, 0.67e^{i1.36\pi}))$

Table 35: Tabular representation of normalized Euclidean distance from ideal solution.

Alternative	$d(\widehat{\mathscr{P}}_k, w_j)$	$d(\check{\mathscr{P}}_k, w_j)$
w_1	0.20559	0.55439
w_2	0.36677	0.311911
w_3	0.528106	0.18778
w_4	0.254649	0.396793
w_5	0.50649	0.1936

Table 36: Tabular representation of revised closeness index of each alternative.

Alternative	$\mathfrak{I}(w_j)$
w_1	0
w_2	1.22136
w_3	2.236816
w_4	0.485238
w_5	2.1143

Table 37: Tabular representation of revised closeness index of each alternative.

Alternative	Ranking
w_1	1
w_2	3
w_3	5
w_4	2
w_5	4

the weight vector χ for the attributes, and are defined as follows:

$$\chi_k = \left(\sqrt{1 - \prod_1^{d=s}\left(1 - \left(p_k^{(d)}\right)^2\right)^{\sigma_d}}\, e^{i2\pi\sqrt{1 - \prod_1^{d=s}\left(1 - \left(\phi_k^{(d)}\right)^2\right)^{\sigma_d}}}, \prod_1^{d=s} \nu_k^{(d)} e^{i2\pi \prod_1^{d=s} \delta_k^{(d)}}, \prod_1^{d=s} r_k^{(d)} e^{i2\pi \prod_1^{d=s} \lambda_k^{(d)}} \right). \quad (36)$$

Thus, we have

$$\chi = \begin{pmatrix} \left(0.89e^{i1.72\pi}, 0.017e^{i0.034\pi}, 0.23e^{0.5i\pi}\right) \\ \left(0.91e^{i1.86\pi}, 0.016e^{i0.034\pi}, 0.09e^{0.2i\pi}\right) \\ \left(0.62e^{i1.24\pi}, 0.016e^{i0.028\pi}, 0.53e^{1.1i\pi}\right) \\ \left(0.53e^{i1.06\pi}, 0.02e^{i0.04\pi}, 0.73e^{1.42i\pi}\right) \\ \left(0.55e^{i1.14\pi}, 0.019e^{i0.042\pi}, 0.67e^{1.34i\pi}\right) \end{pmatrix}. \quad (37)$$

Step 4: the aggregated weighted complex spherical decision matrix (AWCSFDM) is arranged in Table 47, where the entries of AWCSFDM are calculated using the formula [16]

$$\overline{\mathscr{P}_{jk}} = \left(p_{jk} p_k e^{i2\pi\phi_{jk}\phi_k}, \sqrt{\nu_{jk}^2 + \nu_k^2 - \nu_{jk}^2\nu_k^2}\, e^{i2\pi\sqrt{\delta_{jk}^2 + \delta_k^2 - \delta_{jk}^2\delta_k^2}}, \sqrt{r_{jk}^2 + r_k^2 - r_{jk}^2 r_k^2}\, e^{i2\pi\sqrt{\lambda_{jk}^2 + \lambda_k^2 - \lambda_{jk}^2\lambda_k^2}} \right). \quad (38)$$

Step 5: to compute the complex spherical fuzzy positive ideal solution (CSF-PIS) and negative ideal solution (CSF-NIS), we evaluate the score degree of all CSFNs in AWCSFDM, using the formula

$$Sc\left(\overline{\mathscr{P}_{jk}}\right) = \left(p_{kj}^2 - w_{kj}^2 - r_{kj}^2\right) + \left[\phi_{kj}^2 - \delta_{kj}^2 - \lambda_{kj}^2\right]. \quad (39)$$

Table 48 represents the CSF-PIS and CSF-NIS with the help of equation (26).

Step 6: the complex spherical fuzzy normalized Euclidean distance of each alternative is given in Table 49 and computed by the formula [16], from CSF-PIS:

TABLE 38: Tabular representation of the $PF6S_fS$ defined from the problem proposed in Section 5.1.

	z_1	z_2	z_3	z_4
$\mathbb{a}_1$	$(5, (0.95, 0.13))$	$(4, (0.87, 0.21))$	$(4, (0.88, 0.2))$	$(2, (0.48, 0.77))$
$\mathbb{a}_2$	$(3, (0.7, 0.49))$	$(2, (0.32, 0.9))$	$(3, (0.6, 0.69))$	$(3, (0.62, 0.73))$
$\mathbb{a}_3$	$(2, (0.41, 0.77))$	$(2, (0.33, 0.89))$	$(2, (0.35, 0.84))$	$(3, (0.53, 0.67))$
$\mathbb{a}_4$	$(2, (0.4, 0.9))$	$(3, (0.72, 0.59))$	$(5, (0.9, 0.18))$	$(4, (0.77, 0.35))$
$\mathbb{a}_5$	$(3, (0.71, 0.51))$	$(4, (0.89, 0.145))$	$(3, (0.55, 0.7))$	$(2, (0.49, 0.76))$

TABLE 39: Tabular representation of Choice value of $PF6S_fS$ in Section 5.1.

	z_1	z_2	z_3	z_4	H_i
$\mathbb{a}_1$	$(5, (0.95, 0.13))$	$(4, (0.87, 0.21))$	$(4, (0.88, 0.2))$	$(2, (0.48, 0.77))$	$(15, 2.83162)$
$\mathbb{a}_2$	$(3, (0.7, 0.49))$	$(2, (0.32, 0.9))$	$(3, (0.6, 0.69))$	$(3, (0.62, 0.73))$	$(11, 1.6754702)$
$\mathbb{a}_3$	$(2, (0.41, 0.77))$	$(2, (0.33, 0.89))$	$(2, (0.35, 0.84))$	$(3, (0.53, 0.67))$	$(9, 1.3498273)$
$\mathbb{a}_4$	$(2, (0.4, 0.9))$	$(3, (0.72, 0.59))$	$(5, (0.9, 0.18))$	$(4, (0.77, 0.35))$	$(14, 2.30656384)$
$\mathbb{a}_5$	$(3, (0.71, 0.51))$	$(4, (0.89, 0.145))$	$(3, (0.55, 0.7))$	$(2, (0.49, 0.76))$	$(12, 2.231012)$

TABLE 40: Tabular representation of 4 choice value of $PF6S_fS$ Section 5.1.

	z_1	z_2	z_3	z_4	H_i^4
$\mathbb{a}_1$	$(0.95, 0.13)$	$(0.87, 0.21)$	$(0.88, 0.2)$	$(0, 1)$	0.33532
$\mathbb{a}_2$	$(0, 0.5)$	$(0, 1)$	$(0, 0.5)$	$(0, 0.5)$	−0.4375
$\mathbb{a}_3$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0, 0.5)$	−0.8125
$\mathbb{a}_4$	$(0, 1)$	$(0, 0.5)$	$(0.9, 0.18)$	$(0.77, 0.35)$	−0.0005
$\mathbb{a}_5$	$(0, 0.5)$	$(0.89, 0.145)$	$(0, 0.5)$	$(0, 1)$	−0.18223125

$$d\left(\widehat{\mathscr{P}}_k, w_j\right) = \left(\frac{1}{3k}\sum_{k=1}^{m}\left[\left(\widehat{p}_k^2 - \overline{p}_{jk}^2\right)^2 + \left(\widehat{v}_k^2 - \overline{v}_{jk}^2\right)^2 + \left(\widehat{r}_k^2 - \overline{r}_{jk}^2\right)^2 + \left(\widehat{\phi}_k^2 - \overline{\phi}_{jk}^2\right)^2 + \left(\widehat{\delta}_k^2 - \overline{\delta}_{jk}^2\right)^2 + \left(\widehat{\lambda}_k^2 - \overline{\lambda}_{jk}^2\right)^2\right]\right)^{1/2}. \tag{40}$$

Similarly, the normalized Euclidean distance between the CSF-NIS and any of the alternative w_j can be formulated as follows:

$$d\left(\check{\mathscr{P}}_k, w_j\right) = \left(\frac{1}{3k}\sum_{k=1}^{m}\left[\left(\check{p}_k^2 - \overline{p}_{jk}^2\right)^2 + \left(\check{v}_k^2 - \overline{v}_{jk}^2\right)^2 + \left(\check{r}_k^2 - \overline{r}_{jk}^2\right)^2 + \left(\check{\phi}_k^2 - \overline{\phi}_{jk}^2\right)^2 + \left(\check{\delta}_k^2 - \overline{\delta}_{jk}^2\right)^2 + \left(\check{\lambda}_k^2 - \overline{\lambda}_{jk}^2\right)^2\right]\right)^{1/2}. \tag{41}$$

Step 7: Equation (33) is used to calculate the revised closeness index of each alternative, as given in Table 50.

Step 8: revised closeness index in Table 50 reveals that w_1 is the best alternative within the ranking $w_1 > w_4 > w_2 > w_5 > w_3$.

6.1.1. Discussion

(1) We now compare the proposed model $CSFNS_f$-TOPSIS method with the existing technique CSF-TOPSIS method to evaluate the accuracy of the result. The same result concludes from both methods as well as the ranking of the alternatives also same.

(2) We also apply technique on SF-TOPSIS methods [10, 11], to select the most appropriate physiotherapist. The same results including the ranking and best solution are organized in Table 51, which enhance the credibility of the proposed method.

(3) The proposed model deals not only with 2-dimensional uncertainties but also with the level of attribute for the alternative. The existing models are unable to handle MAGDM problems, but the proposed model has the ability to tackle those real-life DM problems having ranking system and parameterized information.

(4) The proposed model, $CSFNS_f$-TOPSIS method, could be efficiently applied to the environment of

Table 41: Tabular representation of CSFDM of expert $\widetilde{E}_1$.

	z_1	z_2	z_3	z_4	z_5
w_1	$(0.98e^{i1.98\pi}, 0.01e^{i0.024\pi}, 0.02e^{i0.06\pi})$	$(0.99e^{i1.94\pi}, 0.012e^{i0.02\pi}, 0.1e^{i0.26\pi})$	$(0.97e^{i1.96\pi}, 0.013e^{i0.02\pi}, 0.05e^{i0.08\pi})$	$(0.84e^{i1.66\pi}, 0.019e^{i0.04\pi}, 0.29e^{0.6i\pi})$	$(0.98e^{i1.92\pi}, 0.012e^{i0.026\pi}, 0.03e^{0.01i\pi})$
w_2	$(0.84e^{i1.66\pi}, 0.02e^{i0.042\pi}, 0.29e^{i0.62\pi})$	$(0.37e^{i0.78\pi}, 0.015e^{i0.026\pi}, 0.87e^{i1.72\pi})$	$(0.16e^{i0.34\pi}, 0.018e^{i0.038\pi}, 0.89e^{1.78\pi})$	$(0.96e^{i1.94\pi}, 0.013e^{i0.024\pi}, 0.04e^{0.06i\pi})$	$(0.36e^{i0.74\pi}, 0.016e^{i0.03\pi}, 0.59e^{1.2i\pi})$
w_3	$(0.17e^{i0.36\pi}, 0.02e^{i0.038\pi}, 0.91e^{i1.84\pi})$	$(0.19e^{i0.36\pi}, 0.021e^{i0.04\pi}, 0.93e^{1.82\pi})$	$(0.59e^{i0.12\pi}, 0.0155e^{i0.032\pi}, 0.82e^{1.6\pi})$	$(0.2e^{i0.38\pi}, 0.022e^{i0.042\pi}, 0.92e^{1.86i\pi})$	$(0.21e^{i0.44\pi}, 0.024e^{i0.05\pi}, 0.93e^{1.88i\pi})$
w_4	$(0.58e^{i1.12\pi}, 0.015e^{i0.032\pi}, 0.82e^{i1.66\pi})$	$(0.66e^{i1.34\pi}, 0.1e^{i0.18\pi}, 0.56e^{i1.14\pi})$	$(0.67e^{i1.36\pi}, 0.021e^{i0.044\pi}, 0.31e^{0.64\pi})$	$(0.55e^{i1.14\pi}, 0.014e^{i0.028\pi}, 0.8e^{1.62i\pi})$	$(0.67e^{i1.38\pi}, 0.025e^{i0.048\pi}, 0.33e^{0.68i\pi})$
w_5	$(0.52e^{i1.02\pi}, 0.012e^{i0.026\pi}, 0.74e^{i1.5\pi})$	$(0.21e^{i0.4\pi}, 0.021e^{i0.046\pi}, 0.95e^{i1.88\pi})$	$(0.7e^{i1.36\pi}, 0.026e^{i0.05\pi}, 0.35e^{i0.68\pi})$	$(0.16e^{i0.42\pi}, 0.018e^{i0.038\pi}, 0.89e^{1.88i\pi})$	$(0.58e^{i1.14\pi}, 0.01e^{i0.024\pi}, 0.82e^{1.62i\pi})$

Table 42: Tabular representation of CSFDM of expert $\widetilde{E}_2$.

	z_1	z_2	z_3	z_4	z_5
w_1	$(0.83e^{i1.64\pi}, 0.019e^{i0.036\pi}, 0.3e^{i0.58\pi})$	$(0.98e^{i1.94\pi}, 0.013e^{i0.024\pi}, 0.1e^{i0.26\pi})$	$(0.98e^{i1.98\pi}, 0.01e^{i0.024\pi}, 0.02e^{i0.06\pi})$	$(0.84e^{i1.66\pi}, 0.02e^{i0.036\pi}, 0.31e^{0.63i\pi})$	$(0.8e^{i1.62\pi}, 0.022e^{i0.042\pi}, 0.32e^{0.6i\pi})$
w_2	$(0.71e^{i1.38\pi}, 0.028e^{i0.058\pi}, 0.37e^{i0.72\pi})$	$(0.72e^{i1.42\pi}, 0.03e^{i0.062\pi}, 0.38e^{i0.76\pi})$	$(0.57e^{i1.1\pi}, 0.012e^{i0.028\pi}, 0.81e^{1.6\pi})$	$(0.86e^{i1.74\pi}, 0.0169e^{i0.032\pi}, 0.027e^{0.052i\pi})$	$(2, (0.56e^{i1.1\pi}, 0.015e^{i0.07\pi}, 0.8e^{1.58i\pi}))$
w_3	$(0.05e^{i0.012\pi}, 0.01e^{i0.06\pi}, 0.985e^{i1.972\pi})$	$(0.21e^{i0.46\pi}, 0.024e^{i0.52\pi}, 0.94e^{1.86\pi})$	$(0.5e^{i1.02\pi}, 0.01e^{i0.02\pi}, 0.7e^{1.4\pi})$	$(0.73e^{i1.48\pi}, 0.032e^{i0.062\pi}, 0.39e^{0.76i\pi})$	$(2, (0.55e^{i1.08\pi}, 0.01e^{i0.02\pi}, 0.79e^{1.56i\pi}))$
w_4	$(0.74e^{i1.5\pi}, 0.035e^{i0.068\pi}, 0.4e^{i0.78\pi})$	$(0.54e^{i1.04\pi}, 0.015e^{i0.026\pi}, 0.78e^{i1.54\pi})$	$(1, (0.24e^{i0.44\pi}, 0.03e^{i0.056\pi}, 0.94e^{1.84\pi}))$	$(0.71e^{i1.38\pi}, 0.028e^{i0.058\pi}, 0.37e^{0.72i\pi})$	$(2, (0.53e^{i1.04\pi}, 0.013e^{i0.026\pi}, 0.77e^{1.52i\pi}))$
w_5	$(0.24e^{i0.42\pi}, 0.028e^{i0.062\pi}, 0.91e^{i1.84\pi})$	$(0.51e^{i1.04\pi}, 0.013e^{i0.06\pi}, 0.76e^{i1.5\pi})$	$(0.52e^{i1.02\pi}, 0.013e^{i0.06\pi}, 0.75e^{i1.52\pi})$	$(0.23e^{i0.5\pi}, 0.035e^{i0.066\pi}, 0.89e^{1.82i\pi})$	$(0.1e^{i0.24\pi}, 0.012e^{i0.026\pi}, 0.99e^{1.972i\pi})$

Table 43: Tabular representation of CSFDM of expert $\widetilde{E}_3$.

	z_1	z_2	z_3	z_4	z_5
w_1	$(0.84e^{i1.66\pi}, 0.019e^{i0.04\pi}, 0.29e^{i0.6\pi})$	$(0.97e^{i1.98\pi}, 0.013e^{i0.028\pi}, 0.14e^{i0.26\pi})$	$(0.96e^{i1.98\pi}, 0.01e^{i0.04\pi}, 0.04e^{i0.04\pi})$	$(0.5e^{i1.02\pi}, 0.014e^{i0.026\pi}, 0.74e^{1.46i\pi})$	$(0.8e^{i1.62\pi}, 0.018e^{i0.038\pi}, 0.29e^{0.6i\pi})$
w_2	$(0.72e^{i1.46\pi}, 0.036e^{i0.068\pi}, 0.41e^{i0.8\pi})$	$(0.49e^{i\pi}, 0.013e^{i0.028\pi}, 0.71e^{i1.44\pi})$	$(0.48e^{i0.98\pi}, 0.015e^{i0.032\pi}, 0.7e^{1.36\pi})$	$(0.74e^{i1.44\pi}, 0.034e^{i0.07\pi}, 0.42e^{0.82i\pi})$	$(0.75e^{i1.48\pi}, 0.035e^{i0.072\pi}, 0.43e^{0.82i\pi})$
w_3	$(0.47e^{i0.96\pi}, 0.015e^{i0.032\pi}, 0.7e^{i1.36\pi})$	$(0.27e^{i0.5\pi}, 0.034e^{i0.066\pi}, 0.94e^{1.92\pi})$	$(0.46e^{i0.94\pi}, 0.014e^{i0.03\pi}, 0.68e^{1.38\pi})$	$(0.88e^{i1.78\pi}, 0.015e^{i0.032\pi}, 0.25e^{0.46i\pi})$	$(0.29e^{i0.56\pi}, 0.04e^{i0.078\pi}, 0.96e^{1.9i\pi})$
w_4	$(0.98e^{i1.94\pi}, 0.013e^{i0.06\pi}, 0.1e^{i0.26\pi})$	$(0.76e^{i1.5\pi}, 0.036e^{i0.074\pi}, 0.44e^{i0.86\pi})$	$(0.13e^{i0.6\pi}, 0.014e^{i0.03\pi}, 0.986e^{1.97\pi})$	$(0.77e^{i1.52\pi}, 0.038e^{i0.078\pi}, 0.45e^{0.88i\pi})$	$(0.3e^{i0.58\pi}, 0.042e^{i0.082\pi}, 0.925e^{1.86i\pi})$
w_5	$(0.44e^{i0.92\pi}, 0.012e^{i0.028\pi}, 0.66e^{i1.34\pi})$	$(0.43e^{i0.88\pi}, 0.015e^{i0.028\pi}, 0.66e^{i1.34\pi})$	$(0.87e^{i1.78\pi}, 0.016e^{i0.028\pi}, 0.21e^{i0.43\pi})$	$(0.42e^{i0.86\pi}, 0.012e^{i0.02\pi}, 0.65e^{1.32i\pi})$	$(0.78e^{i1.54\pi}, 0.04e^{i0.076\pi}, 0.46e^{0.9i\pi})$

Table 44: Tabular representation of CSFDM of expert $\widetilde{E}_4$.

	z_1	z_2	z_3	z_4	z_5
w_1	$(0.95e^{i1.94\pi}, 0.012e^{i0.02\pi}, 0.13e^{i0.2\pi})$	$(0.8e^{i1.62\pi}, 0.018e^{i0.038\pi}, 0.31e^{i0.6\pi})$	$(0.97e^{i1.96\pi}, 0.01e^{i0.02\pi}, 0.15e^{i0.32\pi})$	$(0.63e^{i1.28\pi}, 0.012e^{i0.02\pi}, 0.6e^{1.18i\pi})$	$(0.64e^{i1.26\pi}, 0.01e^{i0.024pi}, 0.59e^{1.2i\pi})$
w_2	$(0.77e^{i1.56\pi}, 0.041e^{i0.08\pi}, 0.47e^{i0.92\pi})$	$(0.31e^{i0.64\pi}, 0.022e^{i0.04\pi}, 0.89e^{i1.8\pi})$	$(0.06e^{i0.14\pi}, 0.012e^{i0.02\pi}, 0.986e^{1.97\pi})$	$(0.8e^{i1.58\pi}, 0.05e^{i0.102\pi}, 0.48e^{0.94i\pi})$	$(0.37e^{i1.78\pi}, 0.014e^{i0.024\pi}, 0.86e^{1.74i\pi})$
w_3	$(0.3e^{i0.64\pi}, 0.05e^{i0.012\pi}, 0.91e^{i1.84\pi})$	$(0.41e^{i0.84\pi}, 0.01e^{i0.06\pi}, 0.64e^{1.3\pi})$	$(0.4e^{i0.84\pi}, 0.015e^{i0.032\pi}, 0.63e^{1.28\pi})$	$(0.39e^{i0.8\pi}, 0.014e^{i0.03\pi}, 0.62e^{1.26i\pi})$	$(1, (0.33e^{i0.64\pi}, 0.07e^{i0.1\pi}, 0.93e^{1.88i\pi}))$
w_4	$(0.38e^{i0.78\pi}, 0.013e^{i0.07\pi}, 0.61e^{i1.24\pi})$	$(0.9e^{i1.84\pi}, 0.014e^{i0.03\pi}, 0.19e^{i0.36\pi})$	$(0.37e^{i0.76\pi}, 0.012e^{i0.026\pi}, 0.6e^{1.22\pi})$	$(0.81e^{i1.6\pi}, 0.051e^{i0.104\pi}, 0.49e^{i\pi})$	$(0.13e^{i0.28\pi}, 0.014e^{i0.026\pi}, 0.982e^{1.966i\pi})$
w_5	$(0.07e^{i0.012\pi}, 0.016e^{i0.03\pi}, 0.991e^{i1.984\pi})$	$(0.34e^{i0.64\pi}, 0.065e^{i0.132\pi}, 0.95e^{i1.92\pi})$	$(0.3e^{i0.58\pi}, 0.014e^{i0.03\pi}, 0.93e^{i1.88\pi})$	$(0.36e^{i0.74\pi}, 0.01e^{i0.024\pi}, 0.59e^{1.2i\pi})$	$(0.21e^{i0.46\pi}, 0.024e^{i0.052\pi}, 0.94e^{1.86i\pi})$

Table 45: Tabular representation of ACSFDM.

	z_1	z_2	z_3	z_4	z_5
w_1	$(0.95e^{i1.94\pi}, 0.013e^{i0.026\pi}, 0.07e^{i0.18\pi})$	$(0.97e^{i1.9\pi}, 0.013e^{i0.026\pi}, 0.15e^{i0.34\pi})$	$(0.97e^{i1.96\pi}, 0.01e^{i0.02\pi}, 0.05e^{i0.1\pi})$	$(0.77e^{i1.54\pi}, 0.016e^{i0.03\pi}, 0.4e^{0.82i\pi})$	$(0.91e^{i1.76\pi}, 0.013e^{i0.03\pi}, 0.15e^{0.36i\pi})$
w_2	$(0.79e^{i1.56\pi}, 0.028e^{i0.058\pi}, 0.36e^{i0.74\pi})$	$(0.48e^{i0.96\pi}, 0.019e^{i0.034\pi}, 0.73e^{i1.46\pi})$	$(0.33e^{i0.66\pi}, 0.01e^{i0.02\pi}, 0.88e^{1.76\pi})$	$(0.9e^{i1.82\pi}, 0.02e^{i0.04\pi}, 0.15e^{0.26i\pi})$	$(0.47e^{i0.96\pi}, 0.016e^{i0.03\pi}, 0.68e^{1.36i\pi})$
w_3	$(0.25e^{i0.52\pi}, 0.022e^{i0.048\pi}, 0.9e^{i1.74\pi})$	$(0.29e^{i0.58\pi}, 0.018e^{i0.038\pi}, 0.83e^{1.66\pi})$	$(0.5e^{i1.04\pi}, 0.013e^{i0.028\pi}, 0.72e^{1.42\pi})$	$(0.55e^{i1.12\pi}, 0.029e^{i0.038\pi}, 0.6e^{1.2i\pi})$	$(0.35e^{i0.7\pi}, 0.029e^{i0.054\pi}, 0.9e^{1.82i\pi})$
w_4	$(0.7e^{i1.18\pi}, 0.016e^{i0.034\pi}, 0.63e^{i1.28\pi})$	$(0.77e^{i1.56\pi}, 0.034e^{i0.064\pi}, 0.42e^{i0.84\pi})$	$(0.51e^{i1.02\pi}, 0.01e^{i0.02\pi}, 0.53e^{1.06\pi})$	$(0.71e^{i1.4\pi}, 0.026e^{i0.054\pi}, 0.55e^{1.12i\pi})$	$(0.52e^{i1.06\pi}, 0.019e^{i0.038\pi}, 0.6e^{1.22i\pi})$
w_5	$(0.39e^{i0.74\pi}, 0.015e^{i0.032\pi}, 0.83e^{i1.68\pi})$	$(0.35e^{i0.7\pi}, 0.025e^{i0.052\pi}, 0.87e^{i1.74\pi})$	$(0.63e^{i1.26\pi}, 0.02e^{i0.034\pi}, 0.51e^{i1.02\pi})$	$(0.28e^{i0.6\pi}, 0.016e^{i0.034\pi}, 0.76e^{1.58i\pi})$	$(0.49e^{i0.96\pi}, 0.015e^{i0.034\pi}, 0.83e^{1.6i\pi})$

Table 46: Experts opinion related to each attribute.

	$\widetilde{E}_1$	$\widetilde{E}_2$	$\widetilde{E}_3$	$\widetilde{E}_4$
z_1	$(0.85e^{i1.66\pi}, 0.019e^{i0.042\pi}, 0.29e^{i0.6\pi})$	$(0.99e^{i1.94\pi}, 0.012e^{i0.02\pi}, 0.1e^{i0.26\pi})$	$(0.88e^{i1.78\pi}, 0.015e^{i0.032\pi}, 0.25e^{i0.46\pi})$	$(0.67e^{i1.36\pi}, 0.021e^{i0.044\pi}, 0.31e^{0.64i\pi})$
z_2	$(0.66e^{i1.34\pi}, 0.021e^{i0.044\pi}, 0.31e^{i0.64\pi})$	$(0.96e^{i1.94\pi}, 0.013e^{i0.024\pi}, 0.04e^{i0.06\pi})$	$(0.78e^{i1.54\pi}, 0.04e^{i0.076\pi}, 0.46e^{0.9\pi})$	$(0.98e^{i1.98\pi}, 0.01e^{i0.024\pi}, 0.02e^{0.06i\pi})$
z_3	$(0.63e^{i1.28\pi}, 0.012e^{i0.02\pi}, 0.59e^{i1.2\pi})$	$(0.72e^{i1.42\pi}, 0.03e^{i0.062\pi}, 0.38e^{0.78\pi})$	$(0.75e^{i1.48\pi}, 0.035e^{i0.072\pi}, 0.43e^{0.82\pi})$	$(0.42e^{i0.86\pi}, 0.012e^{i0.02\pi}, 0.65e^{i1.32\pi})$
z_4	$(0.29e^{i0.6\pi}, 0.014e^{i0.03\pi}, 0.94e^{i1.86\pi})$	$(0.21e^{i0.46\pi}, 0.024e^{i0.052\pi}, 0.94e^{i1.86\pi})$	$(0.37e^{i0.76\pi}, 0.012e^{i0.026\pi}, 0.6e^{1.22\pi})$	$(0.78e^{i1.54\pi}, 0.04e^{i0.076\pi}, 0.46e^{0.9i\pi})$
z_5	$(0.62e^{i1.28\pi}, 0.01e^{i0.024\pi}, 0.58e^{i1.2\pi})$	$(0.66e^{i1.34\pi}, 0.1e^{i0.18\pi}, 0.56e^{i1.14\pi})$	$(0.64e^{i1.26\pi}, 0.01e^{i0.024\pi}, 0.59e^{i1.2\pi})$	$(0.21e^{i0.4\pi}, 0.021e^{i0.046\pi}, 0.95e^{1.88i\pi})$

Table 47: Tabular representation of AWCSFDM.

	z_1	z_2	z_3	z_4	z_5
w_1	$(0.85e^{i1.66\pi}, 0.021e^{i0.042\pi}, 0.23e^{i0.52\pi})$	$(0.88e^{i1.76\pi}, 0.021e^{i0.042\pi}, 0.174e^{i0.38\pi})$	$(0.601e^{i1.22\pi}, 0.019e^{i0.034\pi}, 0.053e^{i1.1\pi})$	$(0.41e^{i0,82\pi}, 0.025e^{i0.05\pi}, 0.77e^{1.54i\pi})$	$(0.5e^{i\pi}, 0.023e^{i0.05\pi}, 0.67e^{1.36i\pi})$
w_2	$(0.7e^{i1.34\pi}, 0.03e^{i0.066\pi}, 0.41e^{i0.86\pi})$	$(0.43e^{i0.9\pi}, 0.025e^{i0.048\pi}, 0.73e^{i1.46\pi})$	$(0.2e^{i0.4\pi}, 0.019e^{i0.034\pi}, 0.91e^{1.82\pi})$	$(0.47e^{i0.96\pi}, 0.028e^{i0.056\pi}, 0.73e^{1.42i\pi})$	$(0.25e^{i0.54\pi}, 0.025e^{i0.05\pi}, 0.83e^{1.66i\pi})$
w_3	$(0.22e^{i0.44\pi}, 0.027e^{i0.058\pi}, 0.91e^{i1.74\pi})$	$(0.26e^{i0.54\pi}, 0.024e^{i0.025\pi}, 0.83e^{1.66\pi})$	$(0.32e^{i0.64\pi}, 0.021e^{i0.038\pi}, 0.8e^{1.62\pi})$	$(0.29e^{i0.58\pi}, 0.035e^{i0.068\pi}, 0.73e^{1.42i\pi})$	$(0.19e^{i0.34\pi}, 0.035e^{i0.068\pi}, 0.95e^{1.9i\pi})$
w_4	$(0.62e^{i1.02\pi}, 0.023e^{i0.048\pi}, 0.65e^{i1.34\pi})$	$(0.70e^{i1.46\pi}, 0.037e^{i0.072\pi}, 0.43e^{i0.86\pi})$	$(0.32e^{i0.64\pi}, 0.019e^{i0.034\pi}, 0.69e^{1.42\pi})$	$(0.37e^{i0.74\pi}, 0.033e^{i0.066\pi}, 0.82e^{1.62i\pi})$	$(0.29e^{i0.6\pi}, 0.027e^{i0.056\pi}, 0.8e^{1.62i\pi})$
w_5	$(2, (0.35e^{i0.62\pi}, 0.022e^{i0.046\pi}, 0.83e^{i1.7\pi}))$	$(0.31e^{i0.66\pi}, 0.029e^{i0.06\pi}, 0.87e^{i1.74\pi})$	$(0.39e^{i0.78\pi}, 0.025e^{i0.044\pi}, 0.68e^{i1.38\pi})$	$(0.15e^{i0.318\pi}, 0.025e^{i0.052\pi}, 0.89e^{1.8i\pi})$	$(0.27e^{i0.54\pi}, 0.024e^{i0.042\pi}, 0.91e^{1.82i\pi})$

Table 48: Tabular representation of CSF-PIS and CSF-NIS.

Attribute	CSF-PIS	CSFN-NIS
z_1	$(0.85e^{i1.66\pi}, 0.021e^{i0.042\pi}, 0.23e^{i0.52\pi})$	$(0.22e^{i0.44\pi}, 0.027e^{i0.058\pi}, 0.91e^{i1.74\pi})$
z_2	$(0.88e^{i1.76\pi}, 0.021e^{i0.042\pi}, 0.174e^{i0.038\pi})$	$(0.31e^{i0.66\pi}, 0.029e^{i0.06\pi}, 0.87e^{i1.74\pi})$
z_3	$(0.601e^{i1.22\pi}, 0.019e^{i0.034\pi}, 0.53e^{i1.1\pi})$	$(0.2e^{i0.4\pi}, 0.019e^{i0.034\pi}, 0.91e^{i1.82\pi})$
z_4	$(0.47e^{i0.96\pi}, 0.028e^{i0.056\pi}, 0.73e^{i1.42\pi})$	$(0.15e^{i0.318\pi}, 0.025e^{i0.052\pi}, 0.89e^{i1.8\pi})$
z_5	$(0.19e^{i0.38\pi}, 0.035e^{i0.068\pi}, 0.95e^{i1.9\pi})$	$(0.5e^{i\pi}, 0.023e^{i0.046\pi}, 0.67e^{i1.36\pi})$

Table 49: Normalized Euclidean distance from ideal solution.

Alternative	$d(\widehat{\mathscr{P}}_k, w_j)$	$d(\check{\mathscr{P}}_k, w_j)$
w_1	0.182947	0.565764
w_2	0.383839	0.330875
w_3	0.5388422	0.200388
w_4	0.266370	0.349150
w_5	0.510455	0.208933

Table 50: Revised closeness index of each alternative.

Alternative	$\mathfrak{I}(w_j)$
w_1	0
w_2	1.5132
w_3	2.5911
w_4	0.838853
w_5	2.42088

Table 51: Comparison.

Method	Ranking	Best physiotherapist
$CSFNS_f$-TOPSIS	$w_1 > w_4 > w_2 > w_5 > w_3$	w_1
CSF-TOPSIS [16]	$w_1 > w_4 > w_2 > w_5 > w_3$	w_1
SF-TOPSIS [10]	$w_1 > w_4 > w_2 > w_5 > w_3$	w_1
SF-TOPSIS [11]	$w_1 > w_4 > w_2 > w_5 > w_3$	w_1

$SFNS_f$, $CSFS_f$, and SFS_f, by substituting periodic terms equal to zero and $N = 2$.

7. Conclusions

Complex spherical fuzzy N-soft sets ($CSFNS_fS$) broaden the families of both fuzzy sets and N-soft sets. This novel concept has allowed us to propose techniques in a wide environment that have a large ability of solving real-life MADM and MAGDM problems. The model of $CSFNS_fS$ described in this paper copes with 2-dimensional fuzziness, parameterized information, and ordinal ranking systems. In addition to the notion of $CSFNS_fS$, we have defined score and accuracy functions for the purpose of comparing two $CSFNS_fNs$. We have defined useful operations on $CSFNS_fS$ and given relevant examples. We developed three direct algorithms and, furthermore, a $CSFNS_f$-TOPSIS Method to solve decision-making problems. We compared them with existing methods for Pythagorean fuzzy N-soft sets and with the complex spherical fuzzy TOPSIS Method, respectively. For the purpose of extending the theoretical background of TOPSIS methods to the new $CSFNS_f$-TOPSIS method, we have defined complex spherical fuzzy N-soft weighted averaging operator $CSFNS_fWA$ which produces an aggregate complex spherical fuzzy N-soft decision matrix $ACSFNS_fDM$ and aggregates the weight vectors of attributes given by experts. Similarly, we have defined a normalized Euclidean distance in $CSFNS_f$ environment that simultaneously evaluates the distances of alternatives from $CSFNS_f$-PIS and $CSFNS_f$-NIS. This is required to find a revised closeness index. The ascending order of such an index gives us a ranking of the alternatives, where the smallest revised closeness index indicates a best solution. In the future, we intend to pursue the formalization of other methods (ELECTRE I, II, and III and VIKOR methodologies), under the framework of $CSFNS_f$. We can also extend this theory to accommodate T-spherical fuzzy soft sets, T-spherical fuzzy N-soft sets, and complex T-spherical fuzzy N-soft sets.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: On Multivalued Fuzzy Contractions in Extended *b*-Metric Spaces

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] N. Alamgir, Q. Kiran, H. Aydi, and Y. U. Gaba, "On Multivalued Fuzzy Contractions in Extended *b*-Metric Spaces," *Journal of Mathematics*, vol. 2021, Article ID 5579991, 11 pages, 2021.

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Research Article

On Multivalued Fuzzy Contractions in Extended *b*-Metric Spaces

Nayab Alamgir,[1] Quanita Kiran,[2] Hassen Aydi [3,4,5] and Yaé Ulrich Gaba [4,6,7]

[1]*School of Natural Sciences, National University of Sciences and Technology (NUST), Sector H-12, Islamabad, Pakistan*
[2]*School of Electrical Engineering and Computer Science (SEECS), National University of Sciences and Technology (NUST), Sector H-12, Islamabad, Pakistan*
[3]*Université de Sousse, Institut Supérieur d'Informatique et des Techniques de Communication, H. Sousse 4000, Tunisia*
[4]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa*
[5]*China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[6]*Institut de Mathématiques et de Sciences Physiques (IMSP/UAC), Laboratoire de Topologie Fondamentale, Computationnelle et leurs Applications (Lab-ToFoCApp), BP 613, Porto-Novo, Benin*
[7]*Quantum Leap Africa (QLA), AIMS Rwanda Centre, Remera Sector KN 3, Kigali, Rwanda*

Correspondence should be addressed to Hassen Aydi; hassen.aydi@isima.rnu.tn

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In this paper, we establish a Hausdorff metric over the family of nonempty closed subsets of an extended b-metric space. Thereafter, we introduce the concept of multivalued fuzzy contraction mappings and prove related α-fuzzy fixed point theorems in the context of extended b-metric spaces that generalize Nadler's fixed point theorem as well as many preexisting results in the literature. Further, we establish α-fuzzy fixed point theorems for Ćirić type fuzzy contraction mappings as a generalization of previous results. Moreover, we give some examples to support the obtained results.

1. Introduction

In 1928, Von Neumann [1] introduced the concept of fixed points for multivalued mappings because of its applications in several branches of mathematics. The development of the geometric fixed point theory for multivalued mappings was initiated by the work of Nadler [2]. He used the Pompeiu–Hausdorff metric to prove the multivalued contraction principle over the collection of nonempty closed and bounded subsets of a metric space. After that, several researchers have studied and generalized Nadler's contraction principle in many directions.

In 1965, Zadeh [3] initiated the concept of fuzzy set theory. After that, several authors extended the Banach contraction principle for single and multivalued mappings in the context of fuzzy sets. In 1981, Heilpern [4] proved a fixed point theorem for fuzzy contraction mappings as a generalization of Nadler's contraction principle. Consequently, several authors studied and generalized fuzzy fixed point theorems in many directions (see [5–11]). In 2015, Phiangsungnoen and Kumam [12] established the concept of multivalued fuzzy contraction mappings in b-metric spaces and proved a related α-fuzzy fixed point theorem. In [13], Anita extended α-fuzzy fixed point theorems involving Ćirić type fuzzy contraction mappings.

Meanwhile, Kamran in [14] introduced the concept of an extended b-metric, as a generalization of a b-metric, and proved fixed point results on such space. Thereafter, many researchers have studied and generalized fixed point results for single and multivalued mappings (see [15–24]).

In this paper, we establish a Hausdorff metric over the family of nonempty closed subsets of an extended b-metric space. After that, we introduce the concept of multivalued fuzzy contraction mappings and prove α-fuzzy fixed point theorems for such mappings in the context of extended b-metric spaces that generalize many preexisting results in the literature. To justify our results, we give some examples. In the last section, we further establish the fact that, by utilizing these concepts, we can also derive results for multivalued mappings. Throughout this paper, we will denote by $CLB(\mathcal{U})$ the collection of nonempty closed and bounded subsets of $\mathcal{U}$ and by $CLD(\mathcal{U})$ the collection of nonempty closed subsets of $\mathcal{U}$.

Definition 1 (see [14]). Let $\mathcal{U}$ be a nonempty set with $\theta\colon \mathcal{U}\times\mathcal{U}\longrightarrow[1,\infty)$. Then, a mapping $d_\theta\colon \mathcal{U}\times\mathcal{U}\longrightarrow[0,\infty)$ is called an extended b-metric, if for all $\mu,\nu,\omega\in\mathcal{U}$, it satisfies the following:

(1) $d_\theta(\mu,\nu)=0$ *iff* $\mu=\nu$

(2) $d_\theta(\mu,\nu)=d_\theta(\nu,\mu)$

(3) $d_\theta(\mu,\omega)\le\theta(\mu,\omega)[d_\theta(\mu,\nu)+d_\theta(\nu,\omega)]$

Clearly, every b-metric space is an extended b-metric space with $\theta(\mu,\nu)=s\ge 1$.

Example 1 (see [14]). Let $\mathcal{U}=[0,\infty)$. Define $d_\theta\colon \mathcal{U}\times\mathcal{U}\longrightarrow[0,\infty)$ by

$$d_\theta(\mu,\nu)=(\mu-\nu)^2, \tag{1}$$

for all $\mu,\nu\in\mathcal{U}$. Thus, d_θ is an extended b-metric on $\mathcal{U}$, where $\theta\colon \mathcal{U}\times\mathcal{U}\longrightarrow[1,\infty)$ is defined as $\theta(\mu,\nu)=\mu+\nu+1$.

Samreen et al. in [25] established the concept of an extended b-comparison function as an extension of a b-comparison function and generalized the concept of α-ψ-contraction mappings in the framework of extended b-metric spaces.

Definition 2 (see [25]). Let $(\mathcal{U},d_\theta)$ be an extended b-metric space. Then, a function $\varphi\colon[0,\infty)\longrightarrow[0,\infty)$ is called an extended b-comparison function if it is increasing, and there exists a mapping $f\colon D\subset\mathcal{U}\longrightarrow\mathcal{U}$ such that, for some $\mu_0\in D$, $O(\mu_0)\subset D$, $\sum_{n=0}^{\infty}\varphi^n(t)\prod_{i=1}^{n}\theta(\mu_i,\mu_m)$ converges for all $t\in[0,\infty)$ and for every $m\in\mathbb{N}$. Here, $\mu_n=f^n\mu_0$ for $n=1,2,\ldots$ and $O(\mu_0)$ is an orbit at a point $\mu_0\in\mathcal{U}$. We say that φ is an extended b-comparison function for f at μ_0. It is known that, for each extended b-comparison function φ, we have $\varphi(t)<t$ for all $t>0$ and $\varphi(0)=0$ for $t=0$.

We denote by Ψ_θ the collection of all extended b-comparison functions. If we put $\theta(\mu_i,\mu_m)=s$ in Definition 2, we get $\sum_{n=0}^{\infty}\varphi^n(t)\prod_{i=1}^{n}s^n<\infty$, which is a b-comparison function. We denote by Ψ_b the collection of all b-comparison functions.

In [17], Subashi and Gjini introduced the concept of a Pompeiu–Hausdorff metric on the collection of all compact subsets of extended b-metric spaces. On the other hand, Subashi in [18] initiated Pompeiu–Hausdorff metric on the collection of all nonempty closed and bounded subsets of extended b-metric spaces.

Next, recall definitions of fuzzy sets, fuzzy mappings, α-fuzzy fixed point, Ćirić type contraction, and related fixed point theorem in b-metric spaces from [12, 13, 26].

Let $(\mathcal{U},d_\theta)$ be a b-metric space. Then a fuzzy set A in $\mathcal{U}$ is characterized by a membership function

$$f_A\colon \mathcal{U}\longrightarrow[0,1], \tag{2}$$

which assigns every member of $\mathcal{U}$ a membership grade in A. Denote by $F(\mathcal{U})$ the collection of all fuzzy sets in $\mathcal{U}$. Let us take $A\in F(\mathcal{U})$ and $\alpha\in[0,1]$. The α-level set of A is denoted by $[A]_\alpha$ and is defined as follows:

$$\begin{aligned}[A]_\alpha&=\{\mu\in\mathcal{U}\colon A(\mu)\ge\alpha\},\quad \alpha\in(0,1],\\ [A]_0&=\overline{\{\mu\in\mathcal{U}\colon A(\mu)>0\}},\end{aligned} \tag{3}$$

where $\overline{A}$ denotes the closure of A. Clearly, $[A]_\alpha$ and $[A]_0$ are subsets of $\mathcal{U}$.

For $A,B\in F(\mathcal{U})$, a fuzzy set A is said to be more accurate than a fuzzy set B and is denoted by $A\subset B$, if and only if $f_A(\mu)\le f_B(\mu)$ for each $\mu\in\mathcal{U}$. Now, for $\mu\in\mathcal{U}$, $A,B\in F(\mathcal{U})$, $\alpha\in[0,1]$, and $[A]_\alpha,[B]_\alpha\in CLD(\mathcal{U})$, define

$$\begin{aligned}\rho_\alpha(\mu,[A]_\alpha)&=\inf\{d(\mu,a)\colon a\in[A]_\alpha\},\\ \rho_\alpha([A]_\alpha,[B]_\alpha)&=\inf\{d(a,b)\colon a\in[A]_\alpha,b\in[B]_\alpha\},\\ \rho([A]_\alpha,[B]_\alpha)&=\sup_\alpha\rho_\alpha([A]_\alpha,[B]_\alpha).\end{aligned} \tag{4}$$

Then Hausdorff fuzzy b-metric is denoted by $H([A]_\alpha,[B]_\alpha)$ and is defined as follows:

$$H([A]_\alpha,[B]_\alpha)=\max\left\{\sup_{a\in[A]_\alpha}d(a,[B]_\alpha),\ \sup_{b\in[B]_\alpha}d(b,[A]_\alpha)\right\}. \tag{5}$$

Remark 1 (see [12]). The function $H\colon CLD(\mathcal{U})\times CLD(\mathcal{U})\longrightarrow F(\mathcal{U})$ is a generalized Hausdorff fuzzy b-metric induced by d and is defined as follows:

$$H([A]_\alpha,[B]_\alpha)=\begin{cases}\max\left\{\sup_{a\in[A]_\alpha}d(a,[B]_\alpha),\ \sup_{b\in[B]_\alpha}d(b,[A]_\alpha)\right\}, & \text{if the maximum exists;}\\ \infty, & \text{otherwise,}\end{cases} \tag{6}$$

where $[A]_\alpha,[B]_\alpha\in CLD(\mathcal{U})$.

Definition 3 (see [12]). Let $\mathcal{U}$ be a nonempty set and $\mathcal{V}$ be a b-metric space. Then

(1) A mapping $\mathfrak{R}\colon\mathcal{U}\longrightarrow F(\mathcal{V})$ is called a fuzzy mapping

(2) For a fuzzy mapping $\mathfrak{R}\colon\mathcal{U}\longrightarrow F(\mathcal{U})$, an element $\xi\in\mathcal{U}$ is called a α fuzzy fixed point of $\mathfrak{R}$, if $\xi\in[\mathfrak{R}\xi]_\alpha$

Theorem 1 (see [12]). *Let $(\mathcal{U}, d)$ be a complete b-metric space with $s \geq 1$. Let $\mathfrak{R}\colon \mathcal{U} \longrightarrow F(\mathcal{U})$ and $\alpha \in (0, 1]$ such that, for each $\mu \in \mathcal{U}$, $[\mathfrak{R}\mu]_\alpha$ is a nonempty closed subset of $\mathcal{U}$ and $\varphi \in \Psi_b$ such that*

$$H([\mathfrak{R}\mu]_\alpha, [\mathfrak{R}\nu]_\alpha) \leq \varphi(d(\mu, \nu)), \tag{7}$$

for all $\mu, \nu \in \mathcal{U}$. Then, $\mathfrak{R}$ has an α-fuzzy fixed point.

Definition 4 (see [26]). A self-mapping $\mathfrak{R}\colon \mathcal{U} \longrightarrow \mathcal{U}$ on a metric space $(\mathcal{U}, d)$ is called a Ćirić type contraction if and only if for all $\mu, \nu \in \mathcal{U}$, there exists $h < 1$ such that

$$d(\mathfrak{R}\mu, \mathfrak{R}\nu) \leq h \max\left\{d(\mu, \nu), d(\mu, \mathfrak{R}\mu), d(\nu, \mathfrak{R}\nu), \frac{d(\mu, \mathfrak{R}\nu) + d(\nu, \mathfrak{R}\mu)}{2}\right\}. \tag{8}$$

Theorem 2 (see [13]). *Let $(\mathcal{U}, d)$ be a complete b-metric space with $s \geq 1$. Let $\mathfrak{R}\colon \mathcal{U} \longrightarrow F(\mathcal{U})$ and $\alpha \in (0, 1]$ such that, for each $\mu \in \mathcal{U}$, $[\mathfrak{R}\mu]_\alpha$ is a nonempty closed subset of $\mathcal{U}$ and $\varphi \in \Psi_b$ such that*

$$H([\mathfrak{R}\mu]_\alpha, [\mathfrak{R}\nu]_\alpha) \leq \varphi(M(\mu, \nu)), \tag{9}$$

where

$$M(\mu, \nu) = \max\left\{d(\mu, \nu), d(\mu, [\mathfrak{R}\mu]_\alpha), d(\nu, [\mathfrak{R}\nu]_\alpha), \frac{d(\mu, [\mathfrak{R}\nu]_\alpha) + d(\nu, [\mathfrak{R}\mu]_\alpha)}{2s}\right\}, \tag{10}$$

for all $\mu, \nu \in \mathcal{U}$. Then, $\mathfrak{R}$ has an α-fuzzy fixed point.

Remark 2. If $M(\mu, \nu) = d(\mu, \nu)$ in Theorem 2, we get Theorem 1. Hence, Theorem 2 is an extension of Theorem 1.

2. Main Results

For $A \subset U$, $\theta(\mu, A) = \inf_{a\in A}\theta(\mu, a)$ and $d_\theta(\mu, A) = \inf_{a\in A} d_\theta(\mu, a)$. The following lemma is essential in the sequel.

Lemma 1. *Let $(\mathcal{U}, d_\theta)$ be an extended b-metric space. Then*

$$d_\theta(\mu, A) \leq \theta(\mu, A) d_\theta(\mu, \nu) + \theta(\nu, A) d_\theta(\nu, A), \tag{11}$$

for all $\mu, \nu \in \mathcal{U}$ and $a \in A \subset \mathcal{U}$, where $\theta(\mu, A) = \inf_{a\in A}\theta(\mu, a)$.

Proof. Since $(\mathcal{U}, d_\theta)$ is an extended b-metric space, therefore by using the triangle inequality one writes

$$d_\theta(\mu, a) \leq \theta(\mu, a)[d_\theta(\mu, \nu) + d_\theta(\nu, a)], \quad \text{for all } \mu, \nu, a \in \mathcal{U}. \tag{12}$$

This implies that

$$d_\theta(\mu, a) \leq \theta(\mu, a) d_\theta(\mu, \nu) + \theta(\mu, a) d_\theta(\nu, a), \tag{13}$$

for all $\mu, \nu, a \in \mathcal{U}$. By taking infimum over A in equation (13), we obtain

$$\begin{aligned}\inf_{a\in A} d_\theta(\mu, a) &\leq \inf_{a\in A}\theta(\mu, a) \inf_{a\in A}[d_\theta(\mu, \nu) + d_\theta(\nu, a)] \\ &= \inf_{a\in A}\theta(\mu, a) d_\theta(\mu, \nu) + \inf_{a\in A}\theta(\mu, a) \inf_{a\in A} d_\theta(\nu, a).\end{aligned} \tag{14}$$

Since $d_\theta(\mu, A) = \inf_{a\in A} d_\theta(\mu, a)$ and $\theta(\mu, A) = \inf_{a\in A}\theta(\mu, a)$, thus from equation (14), we have

$$d_\theta(\mu, A) \leq \theta(\mu, A) d_\theta(\mu, \nu) + \theta(\nu, A) d_\theta(\nu, A). \tag{15}$$

□

Lemma 2. *Let $\{\mu_k\}_{k=0}^n \subset \mathcal{U}$. Then*

$$\begin{aligned} d_\theta(\mu_0, \mu_n) \leq{}& \theta(\mu_0, \mu_n) d_\theta(\mu_0, \mu_1) + \theta(\mu_0, \mu_n)\theta(\mu_1, \mu_n) d_\theta(\mu_1, \mu_2) \\ &+ \cdots + \theta(\mu_0, \mu_n)\theta(\mu_1, \mu_n) + \cdots + \theta(\mu_{n-1}, \mu_n) d_\theta(\mu_{n-1}, \mu_n). \end{aligned} \tag{16}$$

Proof. From triangle inequality, we have

$$d_\theta(\mu_0, \mu_n) \leq \theta(\mu_0, \mu_n)[d_\theta(\mu_0, \mu_1) + d_\theta(\mu_1, \mu_n)]. \tag{17}$$

This implies that

$$d_\theta(\mu_0, \mu_n) \leq \theta(\mu_0, \mu_n) d_\theta(\mu_0, \mu_1) + \theta(\mu_0, \mu_n) d_\theta(\mu_1, \mu_n). \tag{18}$$

Again, by the triangle inequality,

$$d_\theta(\mu_0,\mu_n) \le \theta(\mu_0,\mu_n) d_\theta(\mu_0,\mu_1) + \theta(\mu_0,\mu_n)\{\theta(\mu_1,\mu_n)[d_\theta(\mu_1,\mu_2) + d_\theta(\mu_2,\mu_n)]\},$$
$$d_\theta(\mu_0,\mu_n) \le \theta(\mu_0,\mu_n) d_\theta(\mu_0,\mu_1) + \theta(\mu_0,\mu_n)\theta(\mu_1,\mu_n) d_\theta(\mu_1,\mu_2) + \theta(\mu_0,\mu_n)\theta(\mu_1,\mu_n) d_\theta(\mu_2,\mu_n). \tag{19}$$

By continuing in this fashion, we have

$$\begin{aligned} d_\theta(\mu_0,\mu_n) \le{}& \theta(\mu_0,\mu_n) d_\theta(\mu_0,\mu_1) + \theta(\mu_0,\mu_n)\theta(\mu_1,\mu_n) d_\theta(\mu_1,\mu_2) + \\ & \cdots + \theta(\mu_0,\mu_n)\theta(\mu_1,\mu_n) + \cdots + \theta(\mu_{n-1},\mu_n) d_\theta(\mu_{n-1},\mu_n). \end{aligned} \tag{20}$$

□

Theorem 3. *Let $\{\mu_n\}$ be a sequence in an extended b-metric space with the property that, for all n, $d_\theta(\mu_n,\mu_{n+1}) < (\varepsilon/k^n)$ and $\lim_{n\longrightarrow\infty}\theta(\mu_n,\mu_{n+1})k < 1$, where $k \ge 1$ is a real constant. Then, $\{\mu_n\}$ is a Cauchy sequence.*

Proof. Let $\varepsilon > 0$ and choose a positive integer N. Then, from the triangle inequality for all $m > n \ge N$, we have

$$\begin{aligned} d_\theta(\mu_n,\mu_m) \le{}& \theta(\mu_n,\mu_m) d_\theta(\mu_n,\mu_{n+1}) + \theta(\mu_n,\mu_m)\theta(\mu_{n+1},\mu_m) d_\theta(\mu_{n+1},\mu_{n+2}) \\ & + \cdots + \theta(\mu_n,\mu_m)\theta(\mu_{n+1},\mu_m) + \cdots + \theta(\mu_{m-1},\mu_m) d_\theta(\mu_{m-1},\mu_m), \\ d_\theta(\mu_n,\mu_m) \le{}& \theta(\mu_n,\mu_m)\frac{\varepsilon}{k^n} + \theta(\mu_n,\mu_m)\theta(\mu_{n+1},\mu_m)\frac{\varepsilon}{k^{n+1}} + \cdots \\ & + \theta(\mu_n,\mu_m)\theta(\mu_{n+1},\mu_m) + \cdots + \theta(\mu_{m-1},\mu_m)\frac{\varepsilon}{k^{m-1}}, \\ d_\theta(\mu_n,\mu_m) \le{}& \theta(\mu_1,\mu_m)\theta(\mu_2,\mu_m) + \cdots + \theta(\mu_n,\mu_m)\frac{\varepsilon}{k^n} + \theta(\mu_1,\mu_m)\theta(\mu_2,\mu_m) + \cdots + \theta(\mu_{n+1},\mu_m)\frac{\varepsilon}{k^{n+1}} \\ & + \cdots + \theta(\mu_1,\mu_m)\theta(\mu_2,\mu_m) + \cdots + \theta(\mu_n,\mu_m)\theta(\mu_{n+1},\mu_m) + \cdots + \theta(\mu_{m-1},\mu_m)\frac{\varepsilon}{k^{m-1}}. \end{aligned} \tag{21}$$

Since $\lim_{n,m\longrightarrow\infty}\theta(\mu_{n+1},\mu_m)k < 1$, the series $\sum_{j=1}^{\infty} \varepsilon/k^n \prod_{i=1}^{j} \theta(\mu_i,\mu_m)$ converges by the ratio test for each $m \in \mathbb{N}$. Let

$$\begin{aligned} S &= \sum_{p=1}^{\infty} \frac{\varepsilon}{k^p} \prod_{q=1}^{p} \theta(\mu_q,\mu_m), \\ S_n &= \sum_{p=1}^{n} \frac{\varepsilon}{k^p} \prod_{q=1}^{p} \theta(\mu_q,\mu_m). \end{aligned} \tag{22}$$

Thus, for $m > n$, the above inequality implies

$$d_\theta(\mu_n,\mu_m) \le [S_{m-1} - S_n]. \tag{23}$$

Let us take $n \longrightarrow \infty$. Hence, we conclude that $\{\mu_n\}$ is a Cauchy sequence. □

Now we will introduce the Pompeiu–Hausdorff metric.

Definition 5. Consider nonempty subsets A, B of extended b-metric space $(\mathcal{U}, d_\phi)$, and we define

$$H_\phi(A,B) = \begin{cases} \max\left\{\sup_{a\in A} d_\theta(a,B), \sup_{b\in B} d_\theta(b,A)\right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise.} \end{cases} \tag{24}$$

By following the same procedure as Lemma 2 of [27], we state the following lemma.

Lemma 3. *For all $A, B, C \subset \mathcal{U}$, we have*

$$\begin{aligned} H_\theta(A,C) \le{}& \max\left\{\sup_{a\in A}\theta(a,C), \sup_{c\in C}\theta(c,A)\right\} H_\theta(A,B) \\ & + \max\left\{\sup_{a\in A}\theta(a,C), \sup_{c\in C}\theta(c,A)\right\} H_\theta(B,C). \end{aligned} \tag{25}$$

Same as Theorem 2.1 of [27], we have the following theorem.

Theorem 4. *Let $(\mathscr{U}, d_\theta)$ is an extended b-metric space, then the function H_θ: $CLD(\mathscr{U}) \times CLD(\mathscr{U}) \longrightarrow [0, \infty]$ is a generalized extended b-metric space in $CLD(\mathscr{U})$.*

Definition 6. $a \in \overline{A}$, where $\overline{A}$ is the closure of a set $A \subset \mathscr{U}$, if and only if there exists a sequence $\{a_n\}$ in A such that $a = \lim_{n\longrightarrow\infty} a_n$.

Denote for $\varepsilon > 0$ and $A \subset \mathscr{U}$

$$\mathfrak{B}(\varepsilon, A) = \{\mu \in \mathscr{U}\colon d_\theta(\mu, A) \leq \varepsilon\}. \tag{26}$$

Lemma 4. *If $\mu \in \overline{B(\varepsilon, A)}$, then $d_\theta(\mu, A) \leq \theta(\mu, A)\varepsilon$, where*

$$\theta(\mu, A) = \inf_{a\in A} \theta(\mu, a). \tag{27}$$

Proof. Let $\mu \in \overline{B(\varepsilon, A)}$, then there exists a sequence $\{\mu_n\}$ in $B(\varepsilon, A)$, where $n = 1, 2, 3, \ldots$ such that $\mu = \lim_{n\longrightarrow\infty}\mu_n$. Hence, by Lemma 1,

$$d_\theta(\mu, A) \leq \theta(\mu, A) d_\theta(\mu, \mu_n) + \theta(\mu, A) d_\theta(u_n, A) \leq \theta(\mu, A)\varepsilon. \tag{28}$$

This implies that

$$d_\theta(\mu, A) \leq \theta(\mu, A)\varepsilon, \tag{29}$$

which proves the lemma. □

Definition 7. The upper topological limit of a sequence $\{A_k\}_{k=1}^{\infty}$ in the extended b-metric space $\mathscr{U}$ is denoted by $\overline{\mathrm{l}}\mathrm{t}A_k$ which is determined by

$$a \in \overline{\mathrm{l}}\mathrm{t}A_k, \text{iff} \lim_{k\longrightarrow\infty} \inf d_\theta(a, A_k) = 0. \tag{30}$$

Following Theorem 2.2 of [27], we have the following.

Theorem 5. *A point $a \in \overline{l}tA_k$ if and only if there exists a subsequence $\{a_{n_k}\} \subset A$ such that $\lim_{k\longrightarrow\infty} a_{n_k} = a$ and $a_{n_k} \in A_{n_k}$, for $k = 1, 2, 3, \ldots$*

As Theorem 2.3 of [27], we have the following theorem.

Theorem 6. *$L = \overline{l}tA_k$ is closed.*

Similar to Corollary 2.1 of [27], we state the following.

Corollary 1.

$$\overline{\mathrm{l}}\mathrm{t}A_k = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n=0}^{\infty} A_{k+n}}. \tag{31}$$

Again, as Corollary 2.2 of [27], we have the following.

Corollary 2.

$$\lim_{k\longrightarrow\infty} A_k = \overline{\overline{\mathrm{l}}\mathrm{t}A_k} = \overline{\mathrm{l}\mathrm{t}A_k}. \tag{32}$$

By applying the same procedure as Theorem 2.4 of [27], we state the following.

Theorem 7. *Let us consider a complete extended b-metric space $(\mathscr{U}, d_\theta)$ with $\lim_{n,m\longrightarrow\infty}\theta(\mu_n, \mu_m)k < 1$ for all $\mu_m, \mu_n \in \mathscr{U}$, where $k \geq 1$. Then, $(CLD(\mathscr{U}), H_\theta)$ is complete.*

Definition 8. Let $(\mathscr{U}, d_\theta)$ be an extended b-metric space with $\theta\colon \mathscr{U} \times \mathscr{U} \longrightarrow [1, \infty)$. Then, a fuzzy set $\mathscr{A}_\theta$ in $\mathscr{U}$ is characterized by a membership function

$$f_{\mathscr{A}_\theta}\colon \mathscr{U} \longrightarrow [0, 1], \tag{33}$$

which assigns every member of $\mathscr{U}$ a membership grade in $\mathscr{A}_\theta$.

We denote by $F_\theta(\mathscr{U})$ the collection of all fuzzy sets in $\mathscr{U}$. Let us take $\mathscr{A}_\theta \in F_\theta(\mathscr{U})$ and $\alpha \in [0, 1]$. The α-level set of $\mathscr{A}_\theta$ is denoted by $[\mathscr{A}_\theta]_\alpha$ and is defined as follows:

$$\begin{aligned} &[\mathscr{A}_\theta]_\alpha = \{\mu \in \mathscr{U}\colon \mathscr{A}_\theta(\mu) \geq \alpha\}, \quad \alpha \in (0, 1], \\ &[\mathscr{A}_\theta]_0 = \overline{\{\mu \in \mathscr{U}\colon \mathscr{A}_\theta(\mu) > 0\}}, \end{aligned} \tag{34}$$

where $\overline{B}$ denotes the closure of B. Clearly, $[\mathscr{A}_\theta]_\alpha$ and $[\mathscr{A}_\theta]_0$ are subsets of the extended b-metric space $\mathscr{U}$. For $\mathscr{A}_\theta, \mathscr{B}_\theta \in F_\theta(\mathscr{U})$, a fuzzy set $\mathscr{A}_\theta$ is said to be more accurate than a fuzzy set $\mathscr{B}_\theta$ and is denoted by $\mathscr{A}_\theta \subset \mathscr{B}_\theta$, if and only if $f_{\mathscr{A}_\theta}(\mu) \leq f_{\mathscr{B}_\theta}(\mu)$ for each $\mu \in \mathscr{U}$. Now, for $\mu \in \mathscr{U}$, $\mathscr{A}_\theta, \mathscr{B}_\theta \in F_\theta(\mathscr{U})$, $\alpha \in [0, 1]$, and $[\mathscr{A}_\theta]_\alpha, [\mathscr{B}_\theta]_\alpha \in \mathrm{CLD}(\mathscr{U})$, define

$$\begin{aligned} \rho_\alpha(\mu, [\mathscr{A}_\theta]_\alpha) &= \inf\{d_\theta(\mu, \mathscr{B}_\theta)\colon a \in [\mathscr{A}_\theta]_\alpha\}, \\ \rho_\alpha([\mathscr{A}_\theta]_\alpha, [\mathscr{B}_\theta]_\alpha) &= \inf\{d_\theta(a, b)\colon a \in [\mathscr{A}_\theta]_\alpha, b \in [\mathscr{B}_\theta]_\alpha\}, \\ \rho([\mathscr{A}_\theta]_\alpha, [\mathscr{B}_\theta]_\alpha) &= \sup_\alpha \rho_\alpha([\mathscr{A}_\theta]_\alpha, [\mathscr{B}_\theta]_\alpha). \end{aligned} \tag{35}$$

Remark 3. From Theorem 4, the function H_θ: $\mathrm{CLD}(\mathscr{U}) \times \mathrm{CLD}(\mathscr{U}) \longrightarrow [0, \infty]$ defined by

$$H_\theta([\mathscr{A}_\theta]_\alpha, [\mathscr{B}_\theta]_\alpha) = \begin{cases} \max\left\{\sup\limits_{a\in[\mathscr{A}_\theta]_\alpha} d_\theta(a, [\mathscr{B}_\theta]_\alpha), \sup\limits_{b\in[\mathscr{B}_\theta]_\alpha} d_\theta(b, [\mathscr{A}_\theta]_\alpha)\right\}, & \text{if the maximum exists;} \\ \infty, & \text{otherwise} \end{cases} \tag{36}$$

is a Hausdorff extended fuzzy b-metric on $\mathrm{CLD}(\mathscr{U})$.

Theorem 8. *Let $(\mathscr{U}, d_\theta)$ be a complete extended b-metric space with $\theta\colon \mathscr{U} \times \mathscr{U} \longrightarrow [1, \infty)$. Let $\mathfrak{R}\colon \mathscr{U} \longrightarrow F(\mathscr{U})$ be a fuzzy mapping and $\alpha\colon \mathscr{U} \longrightarrow (0, 1]$ such that, for each $\mu \in \mathscr{U}$, $[\mathfrak{R}\mu]_\alpha$ is a nonempty closed subset of $\mathscr{U}$ and $\varphi \in \Psi_\theta$ such that*

$$H_\theta([\mathfrak{R}\mu]_\alpha, [\mathfrak{R}\nu]_\alpha) \le \varphi(d_\theta(\mu, \nu)), \tag{37}$$

for all $\mu, \nu \in \mathscr{U}$. Then, $\mathfrak{R}$ has an α-fuzzy fixed point.

Proof. Let us take an arbitrary point $\mu_0 \in \mathscr{U}$. Suppose there exists $\mu_1 \in [\mathfrak{R}\mu_0]_\alpha$. As $[\mathfrak{R}\mu_0]_\alpha$ is a nonempty closed subset of $\mathscr{U}$, thus clearly if $\mu_0 = \mu_1$ and $\mu_1 \in [\mathfrak{R}\mu_1]_\alpha$, we get μ_1 as an α-fuzzy fixed point of T and the proof is complete. Hence, throughout our proof, we will assume $\mu_0 \ne \mu_1$ and $\mu_1 \notin [\mathfrak{R}\mu_1]_\alpha$. Thus, $d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha) > 0$. From the definition of Hausdorff metric, equation (37), and $\varphi \in \Psi_\theta$, we have

$$\begin{aligned} 0 < d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha) &\le H_\theta([\mathfrak{R}\mu_0]_\alpha, [\mathfrak{R}\mu_1]_\alpha) \\ &\le \varphi(d_\theta(\mu_0, \mu_1)) \\ &< \varphi(r d_\theta(\mu_0, \mu_1)), \end{aligned} \tag{38}$$

where $r > 1$ is a real number. Suppose there exists $\mu_2 \in [\mathfrak{R}\mu_1]_\alpha$ with $\mu_1 \ne \mu_2$ such that

$$0 < d_\theta(\mu_1, \mu_2) \le d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha) \le H_\theta([\mathfrak{R}\mu_0]_\alpha, [\mathfrak{R}\mu_1]_\alpha) \le \varphi(d_\theta(\mu_0, \mu_1)) < \varphi(r d_\theta(\mu_0, \mu_1)). \tag{39}$$

As $[\mathfrak{R}\mu_2]_\alpha$ is a nonempty closed subset of $\mathscr{U}$, therefore we assume $\mu_2 \notin [\mathfrak{R}\mu_2]_\alpha$, so $d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha) > 0$. From the definition of Hausdorff metric, equation (37), and $\varphi \in \Psi_\theta$, we have

$$\begin{aligned} 0 < d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha) &\le H_\theta([\mathfrak{R}\mu_1]_\alpha, [\mathfrak{R}\mu_2]_\alpha) \\ &\le \varphi(d_\theta(\mu_1, \mu_2)) \\ &< \varphi^2(r d_\theta(\mu_0, \mu_1)). \end{aligned} \tag{40}$$

Suppose there exists $\mu_3 \in [\mathfrak{R}\mu_2]_\alpha$ with $\mu_2 \ne \mu_3$ such that

$$0 < d_\theta(\mu_2, \mu_3) \le \varphi(d_\theta(\mu_1, \mu_2)) < \varphi^2(r d_\theta(\mu_0, \mu_1)). \tag{41}$$

As $[\mathfrak{R}\mu_3]_\alpha$ is a nonempty closed subset of $\mathscr{U}$, so we assume $\mu_3 \notin [\mathfrak{R}\mu_3]_\alpha$, so $d_\theta(\mu_3, [\mathfrak{R}\mu_3]_\alpha) > 0$. From the definition of Hausdorff metric, equation (37), and $\varphi \in \Psi_\theta$, we have

$$\begin{aligned} 0 < d_\theta(\mu_3, [\mathfrak{R}\mu_3]_\alpha) &\le H_\theta([\mathfrak{R}\mu_2]_\alpha, [\mathfrak{R}\mu_3]_\alpha) \\ &\le \varphi(d_\theta(\mu_2, \mu_3)) \\ &< \varphi^3(r d_\theta(\mu_0, \mu_1)). \end{aligned} \tag{42}$$

By induction, we can construct a sequence $\{\mu_n\}_{n=0}^{\infty}$ in $\mathscr{U}$ such that $\mu_n \notin [\mathfrak{R}\mu_n]_\alpha$, $\mu_{n+1} \in [\mathfrak{R}\mu_n]_\alpha$, and

$$\begin{aligned} 0 < d_\theta(\mu_n, [\mathfrak{R}\mu_n]_\alpha) &\le H_\theta([\mathfrak{R}\mu_{n-1}]_\alpha, [\mathfrak{R}\mu_n]_\alpha) \\ &\le \varphi(d_\theta(\mu_{n-1}, \mu_n)) \\ &< \varphi^n(r d_\theta(\mu_0, \mu_1)), \end{aligned} \tag{43}$$

for all $n \in \mathbb{N}$. From the triangle inequality, for all $m > n$, we have

$$\begin{aligned} d_\theta(\mu_n, \mu_m) &\le \theta(\mu_n, \mu_m) d_\theta(\mu_n, \mu_{n+1}) + \theta(\mu_n, \mu_m)\theta(\mu_{n+1}, \mu_m) d_\theta(\mu_{n+1}, \mu_{n+2}) + \\ &\quad \cdots + \theta(\mu_n, \mu_m)\theta(\mu_{n+1}, \mu_m) + \cdots + \theta(\mu_{m-1}, \mu_m) d_\theta(\mu_{m-1}, \mu_m) \\ &\le \theta(\mu_n, \mu_m)\varphi^n(r d_\theta(\mu_0, \mu_1)) + \theta(\mu_n, \mu_m)\theta(\mu_{n+1}, \mu_m)\varphi^{n+1}(r d_\theta(\mu_0, \mu_1)) + \cdots \\ &\quad + \theta(\mu_n, \mu_m)\theta(\mu_{n+1}, \mu_m) + \cdots + \theta(\mu_{m-1}, \mu_m)\varphi^{m-1}(r d_\theta(\mu_0, \mu_1)). \end{aligned} \tag{44}$$

Since the series $\sum_{i=0}^{\infty} \theta(\mu_j, \mu_m)\varphi^j(r d_\theta(\mu_0, \mu_1))$ converges, therefore $\{\mu_n\}_{n=0}^{\infty}$ is a Cauchy sequence. As $\mathscr{U}$ is complete, so there exists $\mu \in \mathscr{U}$ such that $\lim_{n \longrightarrow \infty} \mu_n = \mu$. Next, we will show that μ is an α-fuzzy fixed point. From the triangle inequality, we have

$$\begin{aligned} d_\theta(\mu, [\mathfrak{R}\mu]_\alpha) &\le \theta(\mu, [\mathfrak{R}\mu]_\alpha)[d_\theta(\mu, \mu_{n+1}) + d_\theta(\mu_{n+1}, [\mathfrak{R}\mu]_\alpha)] \\ &\le \theta(\mu, [\mathfrak{R}\mu]_\alpha)[d_\theta(\mu, \mu_{n+1}) + H_\theta([\mathfrak{R}\mu_n]_\alpha, [\mathfrak{R}\mu]_\alpha)] \\ &\le \theta(\mu, [\mathfrak{R}\mu]_\alpha)[d_\theta(\mu, \mu_{n+1}) + \varphi(d_\theta(\mu_n, \mu))]. \end{aligned} \tag{45}$$

By letting $n \longrightarrow \infty$ and $\varphi(0) = 0$, we get $d_\theta(\mu, [\mathfrak{R}\mu]_\alpha) = 0$. Since $[\mathfrak{R}\mu]_\alpha$ is closed, we get $\mu \in [\mathfrak{R}\mu]_\alpha$. Hence, μ is an α-fuzzy fixed point of $\mathfrak{R}$. □

By putting $\varphi(\mathfrak{R}) = \kappa\mathfrak{R}$, where $\kappa \in (0,1)$ in Theorem 8, we get the following corollary.

Corollary 3. *Let $(\mathscr{U}, d_\theta)$ be a complete extended b-metric space with θ: $\mathscr{U} \times \mathscr{U} \longrightarrow [1, \infty)$. Let $\mathfrak{R}$: $\mathscr{U} \longrightarrow F(\mathscr{U})$ be a fuzzy mapping and α: $\mathscr{U} \longrightarrow (0,1]$ such that, for each $\mu \in \mathscr{U}$, $[\mathfrak{R}\mu]_\alpha$ is a nonempty closed subset of $\mathscr{U}$ and $\varphi \in \Psi_\theta$ such that*

$$H_\theta([\mathfrak{R}\mu]_\alpha, [\mathfrak{R}\nu]_\alpha) \le k(d_\theta(\mu, \nu)), \tag{46}$$

for all $\mu, \nu \in \mathscr{U}$, where $k \in (0,1)$. Assume $k < (1/r)$, where $r \ge 1$ is a real constant. Then, $\mathfrak{R}$ has an α-fuzzy fixed point.

Remark 4. Theorem 8 generalizes Theorem 3.1 of [12]. Also, Corollary 3 generalizes Corollary 3.2 and 3.4 of [12].

Example 2. Let $\mathscr{U} = [0,1]$. Define d_θ: $\mathscr{U} \times \mathscr{U} \longrightarrow [0, \infty)$ by $d_\theta(\mu, \nu) = (\mu - \nu)^2$ and θ: $\mathscr{U} \times \mathscr{U} \longrightarrow [1, \infty)$ by $\theta(\mu, \nu) = \mu + \nu + 2$, for all $\mu, \nu \in \mathscr{U}$. Then, $(\mathscr{U}, d_\theta)$ is a complete extended b-metric space, which is not a b-metric space. Define $\mathfrak{R}$: $\mathscr{U} \longrightarrow F(\mathscr{U})$ by

$$(\mathfrak{R}\mu)(t) = \begin{cases} 0, & \text{if } 0 \le t < \dfrac{4}{5}; \\[2ex] \dfrac{4}{5}, & \text{if } \dfrac{4}{5} \le t \le \dfrac{4(\mu+1)}{5}; \\[2ex] \dfrac{4}{7}, & \text{if } \dfrac{4(\mu+1)}{5} < t \le 1. \end{cases} \tag{47}$$

Define α: $\mathscr{U} \longrightarrow (0,1]$ by $\alpha(\mu) = (4/5)$, for all $\mu \in \mathscr{U}$. Clearly, we can see that, for all $\mu \in \mathscr{U}$,

$$[\mathfrak{R}\mu]_\alpha = \left[\frac{4}{5}, \frac{4(\mu+1)}{5}\right]. \tag{48}$$

Hence, $[\mathfrak{R}\mu]_\alpha$ is a nonempty closed subset of $\mathscr{U}$. Also, $H_\theta([\mathfrak{R}\mu]_\alpha, [\mathfrak{R}\nu]_\alpha) = (16/25)(\mu - \nu)^2 = (16/25)(d_\theta(\mu, \nu))$, where $\varphi(t) = (16/25)t$. Therefore, all the conditions of Theorem 8 are satisfied, and hence, $(4/5) \in \mathscr{U}$ is an α-fuzzy fixed point of $\mathfrak{R}$.

Next, we will prove the existence of α-fuzzy fixed point for multivalued fuzzy contraction mapping under Ćirić type contractive condition in the setting of complete extended b-metric spaces.

Theorem 9. *Let $(\mathscr{U}, d_\theta)$ be a complete extended b-metric space with θ: $\mathscr{U} \times \mathscr{U} \longrightarrow [1, \infty)$. Let $\mathfrak{R}$: $\mathscr{U} \longrightarrow F(\mathscr{U})$ be a fuzzy mapping and α: $\mathscr{U} \longrightarrow (0,1]$ such that, for each $\mu \in \mathscr{U}$, $[\mathfrak{R}\mu]_\alpha$ is a nonempty closed subset of $\mathscr{U}$ and $\varphi \in \Psi_\theta$ such that*

$$H_\theta([\mathfrak{R}\mu]_\alpha, [\mathfrak{R}\nu]_\alpha) \le \varphi(M(\mu, \nu)), \tag{49}$$

where

$$M(\mu, \nu) = \max\left\{d_\theta(\mu, \nu), d_\theta(\mu, [\mathfrak{R}\mu]_\alpha), d_\theta(\nu, [\mathfrak{R}\nu]_\alpha), \frac{d_\theta(\mu, [\mathfrak{R}\nu]_\alpha) + d_\theta(\nu, [\mathfrak{R}\mu]_\alpha)}{2\theta(\mu, [\mathfrak{R}\nu]_\alpha)}\right\}, \tag{50}$$

for all $\mu, \nu \in \mathscr{U}$, where

$$\theta(\mu, [\mathfrak{R}\nu]_\alpha) = \inf_{\nu \in [\mathfrak{R}\nu]_\alpha} \theta(\mu, \nu). \tag{51}$$

Then, $\mathfrak{R}$ has an α-fuzzy fixed point.

Proof. Let us take an arbitrary point $\mu_0 \in \mathscr{U}$. Suppose there exists $\mu_1 \in [\mathfrak{R}\mu_0]_\alpha$. Recall that $[\mathfrak{R}\mu_0]_\alpha$ is a nonempty closed subset of $\mathscr{U}$. Thus, clearly if $\mu_0 = \mu_1$ and $\mu_1 \in [\mathfrak{R}\mu_1]_\alpha$, we get μ_1 as an α-fuzzy fixed point of T and the proof is complete. Hence, throughout our proof, we will assume $\mu_0 \ne \mu_1$ and

$\mu_1 \notin [\mathfrak{R}\mu_1]_\alpha$. Thus, $d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha) > 0$. From the definition of Hausdorff metric, equation (49), and $\varphi \in \Psi_\theta$, we have

$$\begin{aligned}
0 &< d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha) \le H_\theta([\mathfrak{R}\mu_0]_\alpha, [\mathfrak{R}\mu_1]_\alpha) \\
&\le \varphi(M(d_\theta(\mu_0, \mu_1))), \\
&= \varphi\left(\max\left\{d_\theta(\mu_0, \mu_1), d_\theta(\mu_0, [\mathfrak{R}\mu_0]_\alpha), d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha), \frac{d_\theta(\mu_0, [\mathfrak{R}\mu_1]_\alpha) + d_\theta(\mu_1, [\mathfrak{R}\mu_0]_\alpha)}{2\theta(\mu_0, [\mathfrak{R}\mu_1]_\alpha)}\right\}\right) \\
&\le \varphi(\max\{d_\theta(\mu_0, \mu_1), d_\theta(\mu_0, [\mathfrak{R}\mu_0]_\alpha), d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha), \\
&\quad \left.\frac{\theta(\mu_0, [\mathfrak{R}\mu_1]_\alpha)(d_\theta(\mu_0, \mu_1) + d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha)) + d_\theta(\mu_1, [\mathfrak{R}\mu_0]_\alpha)\}}{2\theta(\mu_0, [\mathfrak{R}\mu_1]_\alpha)}\right) \\
&\le \varphi\left(\max\left\{d_\theta(\mu_0, \mu_1), d_\theta(\mu_0, \mu_1), d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha), \frac{d_\theta(\mu_0, \mu_1) + d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha) + d_\theta(\mu_1, \mu_1)}{2}\right\}\right), \\
&= \varphi\left(\max\left\{d_\theta(\mu_0, \mu_1), d_\theta(\mu_0, \mu_1), d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha), \frac{d_\theta(\mu_0, \mu_1) + d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha)}{2}\right\}\right), \\
&= \varphi(\max\{d_\theta(\mu_0, \mu_1), d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha)\}).
\end{aligned} \tag{52}$$

This implies that

$$d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha) \le \varphi(\max\{d_\theta(\mu_0, \mu_1), d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha)\}). \tag{53}$$

Now, we will take the following two cases:

Case 1: if $\max\{d_\theta(\mu_0, \mu_1), d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha)\} = d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha)$, then we have

$$0 < d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha) \le \varphi(d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha)) < d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha), \tag{54}$$

which is a contradiction.

Case 2: if $\max\{d_\theta(\mu_0, \mu_1), d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha)\} = d_\theta(\mu_0, \mu_1)$, then we have

$$0 < d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha) \le \varphi(d_\theta(\mu_0, \mu_1)) < \varphi(r d_\theta(\mu_0, \mu_1)), \tag{55}$$

where $r > 1$ is a real number.

This ensures that there exists $\mu_2 \in [\mathfrak{R}\mu_1]_\alpha$ with $\mu_1 \ne \mu_2$ such that

$$0 < d_\theta(\mu_1, \mu_2) \le \varphi(d_\theta(\mu_0, \mu_1)) < \varphi(r d_\theta(\mu_0, \mu_1)). \tag{56}$$

As $[\mathfrak{R}\mu_2]_\alpha$ is a nonempty closed subset of $\mathcal{U}$, therefore, we assume $\mu_2 \notin [\mathfrak{R}\mu_2]_\alpha$, so $d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha) > 0$. From the definition of Hausdorff metric, equation (49), and $\varphi \in \Psi_\theta$, we have

$$
\begin{aligned}
0 &< d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha) \le H_\theta([\mathfrak{R}\mu_1]_\alpha, [\mathfrak{R}\mu_2]_\alpha) \\
&\le \varphi(M(d_\theta(\mu_1,\mu_2))) \\
&= \varphi\Bigg(\max\Bigg\{d_\theta(\mu_1,\mu_2), d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha), d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha), \\
&\quad \frac{d_\theta(\mu_1, [\mathfrak{R}\mu_2]_\alpha) + d_\theta(\mu_2, [\mathfrak{R}\mu_1]_\alpha)}{2\theta(\mu_1, [\mathfrak{R}\mu_2]_\alpha)}\Bigg\}\Bigg) \\
&\le \varphi\Bigg(\max\Bigg\{d_\theta(\mu_1,\mu_2), d_\theta(\mu_1, [\mathfrak{R}\mu_1]_\alpha), d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha), \\
&\quad \frac{\theta(\mu_1, [\mathfrak{R}\mu_2]_\alpha)(d_\theta(\mu_1,\mu_2) + d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha)) + d_\theta(\mu_2, [\mathfrak{R}\mu_1]_\alpha)}{2\theta(\mu_1, [\mathfrak{R}_2]_\alpha)}\Bigg\}\Bigg) \\
&\le \varphi\left(\max\left\{d_\theta(\mu_1,\mu_2), d_\theta(\mu_1,\mu_2), d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha), \frac{d_\theta(\mu_1,\mu_2) + d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha) + d_\theta(\mu_2,\mu_2)}{2}\right\}\right), \\
&= \varphi\left(\max\left\{d_\theta(\mu_1,\mu_2), d_\theta(\mu_1,\mu_2), d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha), \frac{d_\theta(\mu_1,\mu_2) + d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha)}{2}\right\}\right) \\
&= \varphi(\max\{d_\theta(\mu_1,\mu_2), d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha)\}).
\end{aligned}
\tag{57}
$$

If $\max\{d_\theta(\mu_1,\mu_2), d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha)\} = d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha)$, then we have

$$
0 < d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha) \le \varphi(d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha)) < d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha), \tag{58}
$$

which is a contradiction. Thus, $\max\{d_\theta(\mu_1,\mu_2), d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha)\} = d_\theta(\mu_1,\mu_2)$. As φ is increasing, we have

$$
0 < d_\theta(\mu_2, [\mathfrak{R}\mu_2]_\alpha) \le \varphi(d_\theta(\mu_1,\mu_2)) < \varphi(r d_\theta(\mu_1,\mu_2)), \tag{59}
$$

where $r > 1$ is a real number. By induction, we can construct a sequence $\{\mu_n\}_{n=0}^{\infty}$ in μ such that $\mu_n \notin [\mathfrak{R}\mu_n]_\alpha$, $\mu_{n+1} \in [\mathfrak{R}\mu_n]_\alpha$, and

$$
\begin{aligned}
0 &< d_\theta(\mu_n, [\mathfrak{R}\mu_n]_\alpha) \le H_\theta([\mathfrak{R}\mu_{n-1}]_\alpha, [\mathfrak{R}\mu_n]_\alpha) \\
&\le \varphi(d_\theta(\mu_{n-1},\mu_n)) \\
&< \varphi^n(r d_\theta(\mu_0,\mu_1)),
\end{aligned}
\tag{60}
$$

for all $n \in \mathbb{N}$. By applying the same procedure as Theorem 8, we prove that μ is an α-fuzzy fixed point of $\mathfrak{R}$. □

Remark 5. If we put $M(\mu,\nu) = d_\theta(\mu,\nu)$ in Theorem 9, we get Theorem 8. Hence, Theorem 9 is an extension of Theorem 8.

Example 3. Let $\mathscr{U} = \{0,1,2\}$. Define d_θ: $\mathscr{U} \times \mathscr{U} \longrightarrow [0,\infty)$ by

$$
d_\theta(\mu,\nu) = \begin{cases} 0, & \text{if } \mu = \nu; \\ \dfrac{1}{6}, & \text{if } \mu \ne \nu, \mu,\nu \in \{0,1\}; \\ \dfrac{1}{4}, & \text{if } \mu \ne \nu, \mu,\nu \in \{0,2\}; \\ 1, & \text{if } \mu \ne \nu, \mu,\nu \in \{1,2\}. \end{cases} \tag{61}
$$

Hence, $(\mathscr{U}, d_\theta)$ is a complete extended b-metric space, where θ: $\mathscr{U} \times \mathscr{U} \longrightarrow [1,\infty)$ is defined by $\theta(\mu,\nu) = \mu + \nu + 1$. Define $\mathfrak{R}$: $\mathscr{U} \longrightarrow F(\mathscr{U})$ by

$$
\begin{aligned}
(\mathfrak{R}0)(t) = (\mathfrak{R}1)(t) &= \begin{cases} \dfrac{1}{4}, & \text{if } t = 0; \\ 0, & \text{if } t = 1, 2, \end{cases} \\
(\mathfrak{R}2)(t) &= \begin{cases} 0, & \text{if } t = 0, 2; \\ \dfrac{1}{4}, & \text{if } t = 1. \end{cases}
\end{aligned}
\tag{62}
$$

Define α: $\mathscr{U} \longrightarrow (0,1]$ by $\alpha(\mu) = (1/2)$, for all $\mu \in \mathscr{U}$. Clearly, we can see that, for all $\mu \in \mathscr{U}$,

$$
[\mathfrak{R}\mu]_\alpha = \begin{cases} \{0\}, & \text{if } \mu = 0, 1; \\ \{1\}, & \text{if } x = 2. \end{cases} \tag{63}
$$

Now, for all $\mu, \nu \in \mathcal{U}$, we get $H_\theta([\mathfrak{R}0]_{(1/2)}, [\mathfrak{R}1]_{(1/2)}) = H_\theta(0,0) = 0$ and $H_\theta([\mathfrak{R}0]_{(1/2)}, [\mathfrak{R}2]_{(1/2)}) = H_\theta(0,1) = (1/6) = H_\theta([\mathfrak{R}1]_{(1/2)}, [\mathfrak{R}2]_{(1/2)})$.

Also since for all $\mu, \nu \in \mathcal{U}$

$$M(\mu,\nu) = \max\left\{d_\theta(\mu,\nu), d_\theta(\mu,[\mathfrak{R}\mu]_\alpha), d_\theta(\nu,[\mathfrak{R}\nu]_\alpha), \frac{d_\theta(\mu,[\mathfrak{R}\nu]_\alpha) + d_\theta(\nu,[\mathfrak{R}\mu]_\alpha)}{2\theta(\mu,[\mathfrak{R}\nu]_\alpha)}\right\}. \tag{64}$$

For $\mu = 0$ any $\nu = 1$, we have

$$\begin{aligned} M(0,1) &= \max\left\{d_\theta(0,1), d_\theta\left(0,[\mathfrak{R}0]_{(1/2)}\right), d_\theta\left(1,[\mathfrak{R}1]_{(1/2)}\right), \frac{d_\theta\left(0,[\mathfrak{R}1]_{(1/2)}\right) + d_\theta\left(1,[\mathfrak{R}0]_{(1/2)}\right)}{2\theta\left(0,[\mathfrak{R}1]_{(1/2)}\right)}\right\},\\ M(0,1) &= \max\left\{\frac{1}{6}, 0, \frac{1}{6}, \frac{0+(1/6)}{2}\right\}. \end{aligned} \tag{65}$$

This implies that $M(0,1) = (1/6)$. Similarly, $M(0,2) = M(1,2) = 1$. Define $\varphi\colon [0,\infty) \longrightarrow [0,\infty)$ by $\varphi(t) = (1/2)t$ for all $t > 0$. Hence, for all $\mu, \nu \in \mathcal{U}$, we have

$$\begin{aligned} H_\theta\left([\mathfrak{R}0]_{(1/2)}, [\mathfrak{R}1]_{(1/2)}\right) &= 0 < \frac{1}{2}(M(0,1)),\\ H_\theta\left([\mathfrak{R}0]_{(1/2)}, [\mathfrak{R}2]_{(1/2)}\right) &= H_\theta(0,1) = \frac{1}{6} < \frac{1}{2}(M(0,2)),\\ H_\theta\left([\mathfrak{R}1]_{(1/2)}, [\mathfrak{R}2]_{(1/2)}\right) &= H_\theta(0,1) = \frac{1}{6} < \frac{1}{2}(M(1,2)). \end{aligned} \tag{66}$$

Hence, all the conditions of Theorem 9 hold, and therefore, there exists $0 \in \mathcal{U}$ such that $0 \in [\mathfrak{R}0]_{(1/2)}$ is an α-fuzzy fixed point of $\mathfrak{R}$.

Next, we will show that, by utilizing Corollary 3, we can prove fixed point results for multivalued mappings.

Corollary 4. *Let $(\mathcal{U}, d_\theta)$ be a complete extended b-metric space with $\theta\colon \mathcal{U}\times\mathcal{U} \longrightarrow [1,\infty)$. Let $\mathfrak{I}\colon \mathcal{U} \longrightarrow CLD(\mathcal{U})$ be a multivalued mapping such that*

$$H_\theta(\mathfrak{I}\mu, \mathfrak{I}\nu) \leq k(d_\theta(\mu,\nu)), \tag{67}$$

for all $\mu, \nu \in \mathcal{U}$, where $k \in (0,1)$. Assume $k < (1/r)$, where $r \geq 1$ is a real constant. Then, $\mathfrak{I}$ has a fixed point.

Proof. Let $\alpha\colon \mathcal{U} \longrightarrow (0,1]$ be any arbitrary mapping and define $\mathfrak{R}\colon \mathcal{U} \longrightarrow F(\mathcal{U})$ by

$$(\mathfrak{R}\mu)(t) = \begin{cases} \alpha, & \text{if } t \in \mathfrak{I}\mu;\\ 0, & \text{if } t \neq \mathfrak{I}\mu. \end{cases} \tag{68}$$

Hence, we obtain

$$[\mathfrak{R}\mu]_\alpha = \{t\colon (\mathfrak{R}\mu)t \geq \alpha\} = \mathfrak{I}\mu. \tag{69}$$

Thus, condition (37) becomes (49). Therefore, Corollary 3 can be applied to get an α-fuzzy fixed point $\mu \in [\mathfrak{R}\mu]_\alpha = \mathfrak{I}\mu$, where $\mu \in \mathcal{U}$. Hence, the multivalued mapping $\mathfrak{I}$ has a fixed point. ☐

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article.

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Retraction

Retracted: Pythagorean *m*-Polar Fuzzy Weighted Aggregation Operators and Algorithm for the Investment Strategic Decision Making

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Pythagorean m-Polar Fuzzy Weighted Aggregation Operators and Algorithm for the Investment Strategic Decision Making

Muhammad Riaz,[1] **Khalid Naeem**,[2] **Ronnason Chinram**,[3] **and Aiyared Iampan**[4]

[1]*Department of Mathematics, University of the Punjab, Lahore, Pakistan*
[2]*Department of Mathematics & Statistics, The University of Lahore, Lahore, Pakistan*
[3]*Algebra and Applications Research Unit, Division of Computational Science, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand*
[4]*Department of Mathematics, School of Science, University of Phayao Mae Ka, Phayao 56000, Thailand*

Correspondence should be addressed to Ronnason Chinram; ronnason.c@psu.ac.th

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The role of multipolar uncertain statistics cannot be unheeded while confronting daily life problems on well-founded basis. Fusion (aggregation) of a number of input values in multipolar form into a sole multipolar output value is an essential tool not merely of physics or mathematics but also of widely held problems of economics, commerce and trade, engineering, social sciences, decision-making problems, life sciences, and many more. The problem of aggregation is very wide-ranging and fascinating, in general. We use, in this article, Pythagorean fuzzy numbers (PFNs) in multipolar form to contrive imprecise information. We introduce Pythagorean m-polar fuzzy weighted averaging (PmFWA), Pythagorean m-polar fuzzy weighted geometric (PmFWG), symmetric Pythagorean m-polar fuzzy weighted averaging (SPmFWA), and symmetric Pythagorean m-polar fuzzy weighted geometric (SPmFWG) operators for aggregating uncertain data. Finally, we present a practical example to illustrate the application of the proposed operators and to demonstrate its practicality and effectiveness towards investment strategic decision making.

1. Introduction and Literature Review

The process of MCGDM focuses upon assisting the choice makers in evaluating the most appropriate choice amongst a finite number of options according to some criteria in such a manner that inclination of any member from the group towards a particular choice is diffused. Such knotty problems occur frequently in daily life situations. Due to the presence of uncertain, imprecise, and ever changing information, the decision makers face problems in reaching some unanimous decision. To address the issue of uncertainty, Zadeh [1] founded fuzzy set (FS) theory by annexing membership map to each element of the traditional set. The so-called membership function yields information about level of association of some particular element with the underlying set.

Soft set (SS), initiated by Molodtsov [2], is yet another model to handle imprecisions available in data. Zhang [3] suggested bipolar fuzzy sets as a generality of FSs. Lee [4] proposed bipolar-valued fuzzy sets. Ensuing the realization of Zhang and Lee, Chen et al. [5] inaugurated m-polar fuzzy sets as an extension of bipolar fuzzy sets.

After the actuation of FSs, the researchers around the globe initiated working on its further expansions in different directions. Atanassov [6] supplemented FSs by including anti-membership map and denominated the resulting family as intuitionistic fuzzy set (IFS). According to Atanassov, the mappings used in an IFS drag members of underlying universe to [0, 1] with the additional restriction that their aggregate should also fall in the same interval.

Yager [7] initiated the concept of ordered weighted averaging aggregation operators and information aggregation. Yager [8, 9] adjusted the curtailment imposed on parameters in IFSs so that the sum total of their squared values should lie in [0, 1] and acknowledged the evolved structure as Pythagorean fuzzy set (PFS). Yager [10] further acquainted the notion of q-ROFS as an enlargement of PFS. A short time ago, Pythagorean m-polar fuzzy sets with their practical implementations have been unveiled by Naeem et al. [11, 12]. Well along, Riaz et al. [13] extended the notion of soft sets towards Pythagorean m-polar fuzzy soft sets and prooffered some fascinating utilizations of this model. Riaz et al. [14] unveiled Pythagorean fuzzy multisets with their applications.

Peng and Yang [15, 16] proposed some properties of PFSs and interval-valued Pythagorean fuzzy aggregation operators. Peng and Yuan [17] studied fundamental properties of PF aggregation operators. Selvachandran and Peng [18] presented a new approach for the supplier selection problem based on the modified TOPSIS method under vague parameterized vague soft information. Peng and Selvachandran [19] proposed state of the art and future directions for Pythagorean fuzzy set. Peng [20] introduced a new similarity measure and distance measure for Pythagorean fuzzy set. Feng et al. [21] discussed generalized intuitionistic fuzzy soft sets with their practical usage. Feng et al. [22] proposed Minkowski weighted score functions of intuitionistic fuzzy values and developed an algorithm for solving decision-making problems.

Aggregation operators are used to fuse a given information as a single resultant from the same structure. Diverse sorts of operators employed on different expansions of FSs along with their practical usage are studied by different researchers. Jose and Kuriaskose [23] studied aggregation operators, score function, and accuracy function for multicriteria decision making in intuitionistic fuzzy context. Kaur and Garg [24] studied cubic intuitionistic fuzzy aggregation operators. Garg and Arora [25] presented t-norm-based generalized intuitionistic fuzzy soft power aggregation operator accompanied by its practical implementation. Garg and Arora [26] proposed scaled prioritized intuitionistic fuzzy soft interaction averaging operator. Garg [27] suggested neutrality operations-based Pythagorean fuzzy aggregation operators. Garg and Kaur [28] introduced a robust correlation coefficient for probabilistic dual hesitant fuzzy sets and its applications. Karaaslan and Hunu [29] introduced type-2 single-valued neutrosophic sets and their applications in multicriteria group decision making based on the TOPSIS method.

Liu and Wang [30] discussed some q-rung orthopair fuzzy aggregation operator. Liu et al. [31] extended prioritized weighted aggregation operators. Liu et al. [32] explored the ranking range-based approach to MADM under incomplete context. Li et al. [33] established decision making based on interval-valued complex single-valued neutrosophic hesitant fuzzy generalized hybrid weighted averaging operators. Liu et al. [34] proposed group decision making using complex q-rung orthopair fuzzy Bonferroni mean. Liu and Wang [35] introduced the multiattribute group decision-making method based on intuitionistic fuzzy Einstein interactive operations. Liu et al. [36] introduced the concept of hesitant intuitionistic fuzzy linguistic aggregation operators and their applications to multiattribute decision making.

Akram et al. [37] studied Pythagorean Dombi fuzzy aggregation operators. Akram et al. [38, 39] introduced decision-making analysis based on q-rung picture fuzzy graph structures and complex picture fuzzy Hamacher aggregation operators.

Lu et al. [40] coined hesitant Pythagorean fuzzy Hamacher aggregation operators. Ma and Xu [41] launched symmetric Pythagorean fuzzy weighted geometric/averaging operators. Akram and Shahzadi [42] established q-rung orthopair fuzzy Yager aggregation operators.

Zararsiz and Sengönül [43] introduced certain concepts on the gravity of center of sequence of fuzzy numbers. Zararsiz [44] proposed new similarity measures of sequence of fuzzy numbers and fuzzy risk analysis. Riaz and Hashmi [45] introduced a novel approach to censuses process by using Pythagorean m-polar fuzzy Dombi's aggregation operators. Riaz et al. [46] introduced a robust q-rung orthopair fuzzy Einstein prioritized aggregation operators with application towards MCGDM. Riaz and Tehrim [47, 48] introduced the concept of cubic bipolar fuzzy set with application to multicriteria group decision making using geometric aggregation operators. They proposed a robust extension of the VIKOR method for bipolar fuzzy sets using connection numbers of SPA theory-based metric spaces.

Wei and Lu [49] unveiled PF power aggregation operators. Wei [50] coined PF interaction aggregation operators. Faizi et al. [51] developed Einstein aggregation operational laws for intuitionistic 2-tuple linguistic set and further developed weighted averaging and weighted geometric operators. Xu [52] studied intuitionistic fuzzy aggregation operators. Xu and Cai [53] explored IF information aggregation.

The motive behind this article is to study (symmetric) Pythagorean fuzzy weighted averaging and geometric aggregation operators encompassing multipolar information and their characteristics. Contribution of multipolar data cannot be overlooked in coping with daily life problems. Pythagorean m-polar fuzzy sets have a range of applications in diverse real-life circumstances, and these models boost the management of uncertainty and vagueness by using multipolarity in the membership and nonmembership grades in a broader way. The practical characteristic of PmFSs is that the decision makers (DMs) can be asked to assign multipolar ordered pairs of membership and nonmembership grades with the condition that their sum of squares may not exceed unity. Before reaching a solid decision, we think time and again about the pros and cons of the problem which is indeed a process of manipulating multipolar information.

The leftover part of this article is organized as follows. Section 2 gives access to preliminary notions mainly including operational laws of Pythagorean m-polar fuzzy numbers. The next segment presents Pythagorean m-polar fuzzy weighted averaging operator in company with its desirable qualities, whereas Section 4 deals with the corresponding geometric operator. Section 5 deals with symmetric Pythagorean m-polar fuzzy weighted averaging operator as well as its worthwhile characteristics, whereas Section 6 is dedicated to deal with the corresponding geometric operator. The four suggested operators are applied on MCGDM problem of capital investment analysis accompanied by an algorithm in Section 7. Comparative analysis and superiority of the proposed work is also rendered in the same segment. We conclude the paper in Section 8 with some further future directions.

2. Preliminaries

We recall some fundamentals of Pythagorean m-polar fuzzy sets and their operational laws accompanied by operational laws of corresponding numbers in this segment.

Definition 1 (see [11]). A Pythagorean m-polar fuzzy set (PmFS) O is characterized by two sets of mappings $\Upsilon_O^{(i)}$ (denoting affiliation degrees) and $o_O^{(i)}$ (meant for dissociation grades) dropping members of X to $[0,1]$ constrained to obey $0 \leq (\Upsilon_O^{(i)}(g))^2 + (o_O^{(i)}(g))^2 \leq 1$, for all i. The quantity $\varepsilon_O^{(i)}(g) = \sqrt{1 - (o_O^{(i)}(g))^2 - (\Upsilon_O^{(i)}(g))^2}$ is known as hesitation margin or indeterminacy degree of $g \in X$ to O. $\varepsilon_O^{(i)}\colon X \mapsto [0,1]$ are mappings expressing lack of knowledge regarding $g \in O$ or $g \notin O$. The pair $(\Upsilon^{(i)}, \underset{-}{o}^{(i)})$ is commonly acknowledged as Pythagorean fuzzy number (PFN).

A PmFS is usually expressed as

$$\mathscr{B} = \left\{ \frac{g}{\left(\Upsilon_O^{(i)}(g), \underline{o}_O^{(i)}(g)\right)} \right\}_{i=1}^{m}. \tag{1}$$

If $|X| = r$, then tabulatory array of O is as in Table 1. The corresponding matrix format is

$$O = \begin{pmatrix} \left(\Upsilon_O^{(1)}(g_1), o_O^{(1)}(g_1)\right) & \left(\Upsilon_O^{(2)}(g_1), o_O^{(2)}(g_1)\right) & \cdots & \left(\Upsilon_O^{(m)}(g_1), o_O^{(m)}(g_1)\right) \\ \left(\Upsilon_O^{(1)}(g_2), o_O^{(1)}(g_2)\right) & \left(\Upsilon_O^{(2)}(g_2), o_O^{(2)}(g_2)\right) & \cdots & \left(\Upsilon_O^{(m)}(g_2), o_O^{(m)}(g_2)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\Upsilon_O^{(1)}(g_r), o_O^{(1)}(g_r)\right) & \left(\Upsilon_O^{(2)}(g_r), o_O^{(2)}(g_r)\right) & \cdots & \left(\Upsilon_O^{(m)}(g_r), o_O^{(m)}(g_r)\right) \end{pmatrix}. \tag{2}$$

This matrix of size $r \times m$ is titled as PmF matrix.

2.1. Operational Laws of Pythagorean *m*-Polar Fuzzy Sets (P*m*FSs)

Definition 2 (see [11]). Let $O_1 = \langle \Upsilon_{O_1}^{(i)}(\gamma), o_{O_1}^{(i)}(\gamma)\rangle$ and $O_2 = \langle \Upsilon_{O_2}^{(i)}(\gamma), o_{O_2}^{(i)}(\gamma)\rangle$ be PmFSs on $\mathbb{X}$ and λ be a fuzzy number. Then,

(1) $(O_1)^c = \langle o_{O_1}^{(i)}(\gamma), \Upsilon_{O_1}^{(i)}(\gamma)\rangle$.

(2) $O_1 \sqsubseteq O_2$ on condition that $\Upsilon_{O_1}^{(i)}(\wp) \leq \Upsilon_{O_2}^{(i)}(\wp)$ and $o_{O_2}^{(i)}(\wp) \leq o_{O_1}^{(i)}(\wp)$.

(3) $O_1 \sqcup O_2 = \langle \max\{\Upsilon_{O_1}^{(i)}(\wp), \Upsilon_{O_2}^{(i)}(\wp)\}, \min\{o_{O_1}^{(i)}(\wp), o_{O_2}^{(i)}(\wp)\}\rangle$.

(4) $O_1 \sqcap O_2 = \langle \min\{\Upsilon_{O_1}^{(i)}(\wp), \Upsilon_{O_2}^{(i)}(\wp)\}, \max\{o_{O_1}^{(i)}(\wp), o_{O_2}^{(i)}(\wp)\}\rangle$.

(5) $O_1 \oplus O_2 = \langle \sqrt{(\Upsilon_{O_1}^{(i)}(\wp))^2 + (\Upsilon_{O_2}^{(i)}(\wp))^2 - (\Upsilon_{O_1}^{(i)}(\wp))^2 (\Upsilon_{O_2}^{(i)}(\wp))^2}, o_{O_1}^{(i)}(\wp)\, o_{O_2}^{(i)}(\wp)\rangle$.

(6) $O_1 \otimes O_2 = \langle \Upsilon_{O_1}^{(i)}(\wp)\Upsilon_{O_2}^{(i)}(\wp), \sqrt{(o_{O_1}^{(i)}(\wp))^2 + (o_{O_2}^{(i)}(\wp))^2 - (o_{O_1}^{(i)}(\wp))^2 (o_{O_2}^{(i)}(\wp))^2}\rangle$.

(7) $\lambda O_1 = \langle \sqrt{1 - (1 - (\Upsilon_{O_1}^{(i)}(\wp))^2)^\lambda}, (o_{O_1}^{(i)}(\wp))^\lambda\rangle$.

(8) $O_1^\lambda = \langle (\Upsilon_{O_1}^{(i)}(\wp))^\lambda, \sqrt{1 - (1 - (o_{O_1}^{(i)}(\wp))^2)^\lambda}\rangle$.

2.2. *Operational Laws of Pythagorean* $O_1^\lambda = \langle (\Upsilon_{O_1}^{(i)}(\wp))^\lambda, \sqrt{1 - (1 - (o_{O_1}^{(i)}(\wp)^2)^\lambda}\rangle$*-Polar Fuzzy Numbers (PmFNs)*

Definition 3. Suppose $O_1 = \left\{\langle \Upsilon_1^{(i)}, o_1^{(i)}\rangle\right\}_{i=1}^m$ and $O_2 = \left\{\langle \Upsilon_2^{(i)}, o_2^{(i)}\rangle\right\}_{i=1}^m$ are two PmFNs and λ is a fuzzy number. Then,

(1) $(O_1)^c = \left\{\langle o_1^{(i)}, \Upsilon_1^{(i)}\rangle\right\}_{i=1}^m$.

(2) $O_1 \sqcup O_2 = \left\{\langle \max\{\Upsilon_1^{(i)}, \Upsilon_2^{(i)}\}, \min\{o_1^{(i)}, o_2^{(i)}\}\rangle\right\}_{i=1}^m$.

(3) $O_1 \sqcap O_2 = \left\{\langle \min\{\Upsilon_1^{(i)}, \Upsilon_2^{(i)}\}, \max\{o_1^{(i)}, o_2^{(i)}\}\rangle\right\}_{i=1}^m$.

(4) $O_1 \oplus O_2 = \left\{\langle \sqrt{(\Upsilon_1^{(i)})^2 + (\Upsilon_2^{(i)})^2 - (\Upsilon_1^{(i)})^2 (\Upsilon_2^{(i)})^2}, o_1^{(i)} o_2^{(i)}\rangle\right\}_{i=1}^m$.

(5) $O_1 \otimes O_2 = \left\{\langle \Upsilon_1^{(i)}\Upsilon_2^{(i)}, \sqrt{(o_1^{(i)})^2 + (o_2^{(i)})^2 - (o_1^{(i)})^2 (o_2^{(i)})^2}\rangle\right\}_{i=1}^m$.

(6) $O_1 \boxplus O_2 = \left\{\langle (\Upsilon_1^{(i)}\Upsilon_2^{(i)} / \sqrt{[1 - (\Upsilon_1^{(i)})^2][1 - (\Upsilon_2^{(i)})^2] + [\Upsilon_1^{(i)}\Upsilon_2^{(i)}]^2}), (o_1^{(i)} o_2^{(i)} / \sqrt{[1 - (o_1^{(i)})^2]\,[1 - (o_2^{(i)})^2] + [o_1^{(i)} o_2^{(i)}]^2})\rangle\right\}_{i=1}^m$.

(7) $O_1 \dotplus O_2 = \left\{\langle \sqrt{(1 - [1 - (\Upsilon_1^{(i)})^2][1 - (\Upsilon_2^{(i)})^2]/2 - [1 - (\Upsilon_1^{(i)})^2][1 - (\Upsilon_2^{(i)})^2] - [\Upsilon_1^{(i)}\Upsilon_2^{(i)}]^2)}, \sqrt{(1 - [1 - (o_1^{(i)})^2][1 - (o_2^{(i)})^2]/2 - [1 - (o_1^{(i)})^2][1 - (o_2^{(i)})^2] - [o_1^{(i)} o_2^{(i)}]^2)}\rangle\right\}_{i=1}^m$.

(8) $\lambda O_1 = \left\{\langle \sqrt{1 - (1 - (\Upsilon_1^{(i)})^2)^\lambda}, (o_1^{(i)})^\lambda\rangle\right\}_{i=1}^m$.

(9) $\lambda \boxdot O_1 = \left\{\langle ((\Upsilon_1^{(i)})^\lambda / \sqrt{[1 - (\Upsilon_1^{(i)})^2]^\lambda + [\Upsilon_1^{(i)}]^{2\lambda}}), ((o_1^{(i)})^\lambda / \sqrt{[1 - (o_1^{(i)})^2]^\lambda + [o_1^{(i)}]^{2\lambda}})\rangle\right\}_{i=1}^m$.

Table 1: Tabulatory array of O.

O				
$\mathfrak{g}_1$	$(\Upsilon^{(1)}_O(\mathfrak{g}_1), \underline{o}^{(1)}_O(\mathfrak{g}_1))$	$(\Upsilon^{(2)}_O(\mathfrak{g}_1), \underline{o}^{(2)}_O(\mathfrak{g}_1))$	$\cdots$	$(\Upsilon^{(m)}_O(\mathfrak{g}_1), \underline{o}^{(m)}_O(\mathfrak{g}_1))$
$\mathfrak{g}_2$	$(\Upsilon^{(1)}_O(\mathfrak{g}_2), \underline{o}^{(1)}_O(\mathfrak{g}_2))$	$(\Upsilon^{(2)}_O(\mathfrak{g}_2), \underline{o}^{(2)}_O(\mathfrak{g}_2))$	$\cdots$	$(\Upsilon^{(m)}_O(\mathfrak{g}_2), \underline{o}^{(m)}_O(\mathfrak{g}_2))$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\mathfrak{g}_r$	$(\Upsilon^{(1)}_O(\mathfrak{g}_r), \underline{o}^{(1)}_O(\mathfrak{g}_r))$	$(\Upsilon^{(2)}_O(\mathfrak{g}_r), \underline{o}^{(2)}_O(\mathfrak{g}_r))$	$\cdots$	$(\Upsilon^{(m)}_O(\mathfrak{g}_r), \underline{o}^{(m)}_O(\mathfrak{g}_r))$

(10) $\lambda \cdot O_1 = \{\langle \sqrt{(1 - [1 - (\Upsilon_1^{(i)})^2]^\lambda/2 - [1 - (\Upsilon_1^{(i)})^2]^\lambda - [\Upsilon_1^{(i)}]^{2\lambda})}, \sqrt{(1 - [1 - (o_1^{(i)})^2]^\lambda/2 - [1 - (o_1^{(i)})^2]^\lambda - [o_1^{(i)}]^{2\lambda})} \rangle\}_{i=1}^m$.

(11) $O_1^\lambda = \left\{\langle (\Upsilon_1^{(i)})^\lambda, \sqrt{1 - (1 - (o_1^{(i)})^2)^\lambda} \rangle\right\}_{i=1}^m$.

Definition 4 (see [12]). The score function of a PmFN $O = \left\{\langle \Upsilon^{(i)}, \underset{-}{\overset{(i)}{o}} \rangle\right\}_{i=1}^m$ is specified by

$$s(O) = \frac{1}{m}\sum_{i=1}^m \left\{\left(\Upsilon_O^{(i)}\right)^2 - \left(o_O^{(i)}\right)^2\right\}. \tag{3}$$

The value of this score function always falls in $[-1, 1]$.

Definition 5 (see [12]). The accuracy function of a PmFN $O = \left\{\langle \Upsilon^{(i)}, \underset{-}{\overset{(i)}{o}} \rangle\right\}_{i=1}^m$ is determined by

$$a(O) = \frac{1}{m}\sum_{i=1}^m \left\{\left(\Upsilon_O^{(i)}\right)^2 + \left(o_O^{(i)}\right)^2\right\}. \tag{4}$$

The value of this accuracy function always falls in $[0, 1]$.

We get advantage of score and accuracy functions of two PmFNs O_1 and O_2 in deciding ordering of O_1 and O_2 as described in Definition 6.

Definition 6 (see [12]). Let $O_1 = \left\{\langle \Upsilon_1^{(i)}, o_1^{(i)} \rangle\right\}_{i=1}^m$ and $O_2 = \left\{\langle \Upsilon_2^{(i)}, o_2^{(i)} \rangle\right\}_{i=1}^m$ be two PmFNs.

(1) If $s(O_1) < s(O_2)$, then $O_1 \prec O_2$.

(2) If, however, $s(O_1) = s(O_2)$ and

(i) $a(O_1) < a(O_2)$, then $O_1 \prec O_2$.

(ii) $a(O_1) = a(O_2)$, then $O_1 \sim O_2$.

Note that $O_1 \prec O_2$ means O_1 precedes O_2, and $O_1 \sim O_2$ means O_1 and O_2 are identical (same).

3. Pythagorean m-Polar Fuzzy Weighted Averaging Operator

We dedicate this segment for inauguration of the notion of Pythagorean m-polar fuzzy weighted averaging operator for Pythagorean m-polar fuzzy numbers along with some of its prime characteristics.

Definition 7. Let $O_k = \left\{\langle \Upsilon_k^{(i)}, o_k^{(i)} \rangle\right\}_{i=1}^m$ $(k = 1, 2, \ldots, n)$ be an assemblage of PmFNs. Define PmFWA: $T^n \longrightarrow T$ given by

$$\begin{aligned} \text{PmFWA}(O_1, O_2, \ldots, O_n) &= \oplus_{k=1}^n \hbar_k O_k \\ &= \hbar_1 O_1 \oplus \hbar_2 O_2 \oplus \cdots \oplus \hbar_n O_n, \end{aligned} \tag{5}$$

where T^n is the collection of all PmFNs and $\hbar_k$'s are fuzzy weights of $(O_1, O_2, \ldots, O_n)$, such that addition of all $\hbar_k$'s results in unity. Then, PmFWA is called the Pythagorean m-polar fuzzy weighted averaging operator.

If weight vector $W = ((1/n), (1/n), \ldots, (1/n))^t$, then P$m$FWA operator reduces to Pythagorean m-polar fuzzy averaging (PmFA) operator of dimension n and is given as

$$\begin{aligned} \text{PmFA}(O_1, O_2, \ldots, O_n) &= \frac{1}{n}\oplus_{k=1}^n O_k \\ &= \frac{1}{n}(O_1 \oplus O_2 \oplus \cdots \oplus O_n). \end{aligned} \tag{6}$$

As maintained by operational laws of PmFNs given in Definition 3, the following theorem assists in computing PmFWA for any PmFNs.

Theorem 1. *Let* $O_k = \left\{\langle \Upsilon_k^{(i)}, o_k^{(i)} \rangle\right\}_{i=1}^m$ $(k = 1, 2, \ldots, n)$ *be an assemblage of PmFNs; then,*

$$\text{PmFWA}(O_1, O_2, \ldots, O_n) = \left\{\left\langle \sqrt{1 - \prod_{k=1}^n \left(1 - \left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}}, \prod_{k=1}^n \left(o_k^{(i)}\right)^{\hbar_k} \right\rangle\right\}_{i=1}^m. \tag{7}$$

Proof. We establish the result by means of induction.

By definition,

$$\begin{aligned}\hbar_1 O_1 &= \left\{\left\langle\sqrt{1-\left(1-\left(\Upsilon_1^{(i)}\right)^2\right)^{\hbar_1}}, \left(o_1^{(i)}\right)^{\hbar_1}\right\rangle\right\}_{i=1}^{m},\\ \hbar_2 O_2 &= \left\{\left\langle\sqrt{1-\left(1-\left(\Upsilon_2^{(i)}\right)^2\right)^{\hbar_2}}, \left(o_2^{(i)}\right)^{\hbar_2}\right\rangle\right\}_{i=1}^{m},\end{aligned} \tag{8}$$

so that

$$\operatorname{PmFWA}\left(O_1, O_2\right) \,\&\, = \hbar_1 O_1 \oplus \hbar_2 O_2 \,\&\, = \left\{\left\langle\sqrt{1-\left(1-\left(\Upsilon_1^{(i)}\right)^2\right)^{\hbar_1}}, \left(o_1^{(i)}\right)^{\hbar_1}\right\rangle\right\}_{i=1}^{m} \oplus \left\{\left\langle\sqrt{1-\left(1-\left(\Upsilon_2^{(i)}\right)^2\right)^{\hbar_2}}, \left(o_2^{(i)}\right)^{\hbar_2}\right\rangle\right\}_{i=1}^{m}. \tag{9}$$

The x-component of the resultant is

$$\begin{aligned}&\sqrt{\left(\sqrt{1-\left(1-\left(\Upsilon_1^{(i)}\right)^2\right)^{\hbar_1}}\right)^2+\left(\sqrt{1-\left(1-\left(\Upsilon_2^{(i)}\right)^2\right)^{\hbar_2}}\right)^2-\left(\sqrt{1-\left(1-\left(\Upsilon_1^{(i)}\right)^2\right)^{\hbar_1}}\sqrt{1-\left(1-\left(\Upsilon_2^{(i)}\right)^2\right)^{\hbar_2}}\right)^2}\\ &= \sqrt{1-\left(1-\left(\Upsilon_1^{(i)}\right)^2\right)^{\hbar_1}+1-\left(1-\left(\Upsilon_2^{(i)}\right)^2\right)^{\hbar_2}-\left(1-\left(1-\left(\Upsilon_1^{(i)}\right)^2\right)^{\hbar_1}\right)\left(1-\left(1-\left(\Upsilon_2^{(i)}\right)^2\right)^{\hbar_2}\right)}\\ &= \left\{1-\left(1-\left(\Upsilon_1^{(i)}\right)^2\right)^{\hbar_1}+1-\left(1-\left(\Upsilon_2^{(i)}\right)^2\right)^{\hbar_2}-1+\left(1-\left(\Upsilon_1^{(i)}\right)^2\right)^{\hbar_1}\right)+\left(1-\left(\Upsilon_2^{(i)}\right)^2\right)^{\hbar_2}\right)-\left(1-\left(\Upsilon_1^{(i)}\right)^2\right)^{\hbar_1}\left(1-\left(\Upsilon_2^{(i)}\right)^2\right)^{\hbar_2}\right\}^{(1/2)}\\ &= \sqrt{1-\left(1-\left(\Upsilon_1^{(i)}\right)^2\right)^{\hbar_1}\left(1-\left(\Upsilon_2^{(i)}\right)^2\right)^{\hbar_2}}\\ &= \sqrt{1-\prod_{k=1}^{2}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}},\end{aligned} \tag{10}$$

and the y-component is $(o_1^{(i)})^{\hbar_1}(o_2^{(i)})^{\hbar_2}$. Thus,

$$\operatorname{PmFWA}\left(O_1, O_2\right) = \left\{\left\langle\sqrt{1-\prod_{k=1}^{2}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}}, \prod_{k=1}^{2}\left(o_k^{(i)}\right)^{\hbar_k}\right\rangle\right\}_{i=1}^{m}. \tag{11}$$

Now assuming that the result is valid for n PmFNs, we exhibit its validity for $n+1$ PmFNs. By definition,

$$\begin{aligned}\mathrm{PmFWA}\left(O_1,O_2,\ldots,O_{n+1}\right) &= \oplus_{k=1}^{n}\hbar_k O_k \oplus \hbar_{n+1}O_{n+1}\\ &= \left\{\langle\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}},\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\rangle\right\}_{i=1}^{m}\\ &\oplus\left\{\langle\sqrt{1-\left(1-\left(\Upsilon_{n+1}^{(i)}\right)^2\right)^{\hbar_{n+1}}},\left(o_{n+1}^{(i)}\right)^{\hbar_{n+1}}\rangle\right\}_{i=1}^{m}.\end{aligned}\tag{12}$$

The x-component of the resultant would be

$$\begin{aligned}&\left\{\left(\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}}\right)^2+\left(\sqrt{1-\left(1-\left(\Upsilon_{n+1}^{(i)}\right)^2\right)^{\hbar_{n+1}}}\right)^2-\left(\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}}\right)\left(\sqrt{1-\left(1-\left(\Upsilon_{n+1}^{(i)}\right)^2\right)^{\hbar_{n+1}}}\right)^2\right\}^{1/2}\\ &=\left\{\left(1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}\right)+\left(1-\left(1-\left(\Upsilon_{n+1}^{(i)}\right)^2\right)^{\hbar_{n+1}}\right)-\left(1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}\right)\left(1-\left(1-\left(\Upsilon_{n+1}^{(i)}\right)^2\right)^{\hbar_{n+1}}\right)\right\}^{1/2}\\ &=\left\{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}+1-\left(1-\left(\Upsilon_{n+1}^{(i)}\right)^2\right)^{\hbar_{n+1}}-1+\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}+\left(1-\left(\Upsilon_{n+1}^{(i)}\right)^2\right)^{\hbar_{n+1}}-\prod_{k=1}^{n+1}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}\right\}^{1/2}\\ &=\sqrt{1-\prod_{k=1}^{n+1}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}},\end{aligned}\tag{13}$$

and the y-component is

$$\prod_{k=1}^{n}\left(\overset{(i)}{\underset{-}{o}}\right)^{\hbar_k}\times\left(\overset{(i)}{\underset{-}{o}}\right)^{\hbar_{n+1}}=\prod_{k=1}^{n+1}\left(\overset{(i)}{\underset{-}{o}}\right)^{\hbar_k}.\tag{14}$$

This concludes the proof. □

Theorem 2. *If* $O_k=\left\{\langle\Upsilon_k^{(i)},o_k^{(i)}\rangle\right\}_{i=1}^{m}$ $(k=1,2,\ldots,n)$ *is an aggregate of PmFNs, then*

$$\begin{aligned}&\mathrm{PmFWA}\left(O_1,O_2,\ldots,O_n\right)\\ &=\left\{\langle\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}},\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\rangle\right\}_{i=1}^{m}\end{aligned}\tag{15}$$

is also a PmFN.

Proof. Since $\Upsilon_k^{(i)},o_k^{(i)}\in[0,1]$, for each k and i,

$$\begin{aligned}&0\le\left(\Upsilon_k^{(i)}\right)^2\le 1\\ &\Longrightarrow 0\le 1-\left(\Upsilon_k^{(i)}\right)^2\le 1\\ &\Longrightarrow 0\le\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}\le 1\\ &\Longrightarrow 0\le\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}\le 1\\ &\Longrightarrow 0\le 1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}\le 1\\ &\Longrightarrow 0\le\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}}\le 1,\\ &0\le\left(o_k^{(i)}\right)^{\hbar_k}\le 1\\ &\Longrightarrow 0\le\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\le 1.\end{aligned}\tag{16}$$

Thus, $\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}},\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\in[0,1]$. Now,

$$
\begin{aligned}
\left(\Upsilon_k^{(i)}\right)^2+\left(o_k^{(i)}\right)^2 \le 1 \\
&\Rightarrow \left(o_k^{(i)}\right)^2 \le 1-\left(\Upsilon_k^{(i)}\right)^2 \\
&\Rightarrow \left(\left(o_k^{(i)}\right)^2\right)^{\hbar_k} \le \left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k} \\
&\Rightarrow \prod_{k=1}^{n}\left(\left(o_k^{(i)}\right)^2\right)^{\hbar_k} \le \prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k} \\
&\Rightarrow \prod_{k=1}^{n}\left(\left(o_k^{(i)}\right)^{\hbar_k}\right)^2 \le \prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k},
\end{aligned} \tag{17}
$$

so that

$$
\begin{aligned}
0 &\le \left(\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}}\right)^2 + \left(\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\right)^2 \\
&= 1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k} + \left(\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\right)^2 \\
&\le 1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k} + \prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k} \\
&= 1,
\end{aligned} \tag{18}
$$

i.e., $0 \le \left(\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}}\right)^2 + \left(\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\right)^2 \le 1.$ □

Example 1. Let

$$
\begin{aligned}
O_1 &= \{\langle 0.62, 0.39\rangle, \langle 0.55, 0.68\rangle, \langle 0.37, 0.26\rangle, \langle 0.46, 0.61\rangle\}, \\
O_2 &= \{\langle 0.41, 0.37\rangle, \langle 0.19, 0.73\rangle, \langle 0.10, 0.05\rangle, \langle 0.37, 0.46\rangle\}, \\
O_3 &= \{\langle 0.47, 0.68\rangle, \langle 0.39, 0.40\rangle, \langle 0.84, 0.35\rangle, \langle 0.15, 0.92\rangle\},
\end{aligned} \tag{19}
$$

be three P4FNs with corresponding weights $\hbar_1 = 0.4$ and $\hbar_2 = \hbar_3 = 0.3$. We aggregate the three PmFNs utilizing the result rendered in Theorem 1 as below:

$$
\begin{aligned}
\text{P4FWA}(O_1, O_2, O_3) &= \left\{\left\langle \sqrt{1-\prod_{k=1}^{3}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}}, \prod_{k=1}^{3}\left(o_k^{(i)}\right)^{\hbar_k}\right\rangle\right\}_{i=1}^{4} \\
&= \{\langle 0.526, 0.454\rangle, \langle 0.430, 0.592\rangle, \langle 0.590, 0.173\rangle, \langle 0.369, 0.634\rangle\}.
\end{aligned} \tag{20}
$$

Theorem 3. *Assume that* $O_k = \langle \Upsilon_k^{(i)}, o_k^{(i)}\rangle_{i=1}^{m}$ $(k = 1, \ldots, n)$ *is an assembly of PmFNs. Then,*

(1) (Idempotency) if $O_k = O = \langle \Upsilon^{(i)}, \underline{o}^{(i)}\rangle_{i=1}^{m}$ $(k = 1, 2, \ldots, n)$ *for all k, then*

$$\text{PmFWA}(O_1, O_2, \ldots, O_n) = O. \tag{21}$$

(2) (Boundedness) if $O^- = (\min(\Upsilon_k^{(i)}), \max(o_k^{(i)}))$ *and* $O^+ = (\max(\Upsilon_k^{(i)}), \min(o_k^{(i)}))$, *then*

$$O^- \le \text{PmFWA}(O_1, O_2, \ldots, O_n) \le O^+. \tag{22}$$

(3) (Monotonicity) let $O_k = \langle \Upsilon_k^{(i)}, o_k^{(i)}\rangle_{i=1}^{m}$ *and* $O_k^* = \langle (\Upsilon_k^{(i)})^*, (o_k^{(i)})^*\rangle_{i=1}^{m}$ $(k = 1, \ldots, n)$ *be two sets of PmFNs such that* $\Upsilon_k^{(i)} \ge (\Upsilon_k^{(i)})^*$ *and* $o_k^{(i)} \le (o_k^{(i)})^*$ *for all k and all permissible value of i; then,*

$$\text{PmFWA}(O_1, O_2, \ldots, O_n) \ge \text{PmFWA}(O_1^*, O_2^*, \ldots, O_n^*). \tag{23}$$

Proof. For idempotency, consider

$$
\begin{aligned}
&\text{PmFWA}(O_1, O_2, \ldots, O_n) \\
&= \left\{\left\langle \sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon^{(i)}\right)^2\right)^{\hbar_k}}, \prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\right\rangle\right\}_{i=1}^{m} \\
&= \left\{\left\langle \sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon^{(i)}\right)^2\right)^{\hbar_k}}, \prod_{k=1}^{n}\left(\underline{o}^{(i)}\right)^{\hbar_k}\right\rangle\right\}_{i=1}^{m} \\
&= \left\{\left\langle \sqrt{1-\left(1-\left(\Upsilon^{(i)}\right)^2\right)^{\sum_{k=1}^{n}\hbar_k}}, \left(\underline{o}^{(i)}\right)^{\sum_{k=1}^{n}\hbar_k}\right\rangle\right\}_{i=1}^{m} \\
&= \left\{\left\langle \sqrt{1-\left(1-\left(\Upsilon^{(i)}\right)^2\right)}, \underline{o}^{(i)}\right\rangle\right\}_{i=1}^{m} \\
&= \left\{\left\langle \Upsilon^{(i)}, \underline{o}^{(i)}\right\rangle\right\}_{i=1}^{m} \\
&= O.
\end{aligned} \tag{24}
$$

Now, we establish boundedness. For membership grades of PmFWA$(O_1, O_2, \ldots, O_n)$, we have

$$
\begin{aligned}
&\sqrt{1-\prod_{k=1}^{n}\left(1-\min\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}} \leq \sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}} \leq \sqrt{1-\prod_{k=1}^{n}\left(1-\max\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}}\\
&\Rightarrow \sqrt{1-\left(1-\min\left(\Upsilon_k^{(i)}\right)^2\right)^{\sum_{k=1}^{n}\hbar_k}} \leq \sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}} \leq \sqrt{1-\left(1-\max\left(\Upsilon_k^{(i)}\right)^2\right)^{\sum_{k=1}^{n}\hbar_k}}\\
&\Rightarrow \sqrt{1-\left(1-\min\left(\Upsilon_k^{(i)}\right)^2\right)} \leq \sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}} \leq \sqrt{1-\left(1-\max\left(\Upsilon_k^{(i)}\right)^2\right)}\\
&\Rightarrow \min\left(\Upsilon_k^{(i)}\right) \leq \sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^2\right)^{\hbar_k}} \leq \max\left(\Upsilon_k^{(i)}\right),
\end{aligned}
\tag{25}
$$

and for the nonmembership grades, we have

$$
\begin{aligned}
&\prod_{k=1}^{n}\min\left(o_k^{(i)}\right)^{\hbar_k} \leq \prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k} \leq \prod_{k=1}^{n}\max\left(o_k^{(i)}\right)^{\hbar_k}\\
&\Rightarrow \min\left(o_k^{(i)}\right)^{\sum_{k=1}^{n}\hbar_k} \leq \prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k} \leq \max\left(o_k^{(i)}\right)^{\sum_{k=1}^{n}\hbar_k}\\
&\Rightarrow \min\left(o_k^{(i)}\right) \leq \prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k} \leq \max\left(o_k^{(i)}\right).
\end{aligned}
\tag{26}
$$

Now, we prove the monotonicity. Since $\Upsilon_k^{(i)} \geq (\Upsilon_k^{(i)})^*$ and $\overline{o}_k^{(i)} \leq (\overline{o}_k^{(i)})^*$ for all k and all permissible value of i,

$$
\begin{aligned}
&1-\Upsilon_k^{(i)} \leq 1-\left(\Upsilon_k^{(i)}\right)^*\\
&\Rightarrow \left(1-\Upsilon_k^{(i)}\right)^{\hbar_k} \leq \left(1-\left(\Upsilon_k^{(i)}\right)^*\right)^{\hbar_k}\\
&\Rightarrow \prod_{k=1}^{n}\left(1-\Upsilon_k^{(i)}\right)^{\hbar_k} \leq \prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^*\right)^{\hbar_k}\\
&\Rightarrow 1-\prod_{k=1}^{n}\left(1-\Upsilon_k^{(i)}\right)^{\hbar_k} \geq 1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^*\right)^{\hbar_k}\\
&\Rightarrow \sqrt{1-\prod_{k=1}^{n}\left(1-\Upsilon_k^{(i)}\right)^{\hbar_k}} \geq \sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^*\right)^{\hbar_k}},\\
&\left(o_k^{(i)}\right)^{\hbar_k} \leq \left(\left(o_k^{(i)}\right)^*\right)^{\hbar_k}\\
&\Rightarrow \prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\prod_{k=1}^{n}\left(\left(o_k^{(i)}\right)^*\right)^{\hbar_k}.
\end{aligned}
\tag{27}
$$

Therefore,

$$
\begin{aligned}
&\left(\sqrt{1-\prod_{k=1}^{n}\left(1-\Upsilon_k^{(i)}\right)^{\hbar_k}}\right)^2-\left(\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\right)^2\\
&\geq \left(\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^*\right)^{\hbar_k}}\right)^2-\left(\prod_{k=1}^{n}\left(\left(o_k^{(i)}\right)^*\right)^{\hbar_k}\right)^2.
\end{aligned}
\tag{28}
$$

Assume that $O = \text{PmFWA}(O_1, O_2, \ldots, O_n)$ and $O^* = \text{PmFWA}(O_1^*, O_2^*, \ldots, O_n^*)$; then, $s(O) \geq s(O^*)$.

(i) If $s(O) > s(O^*)$, then $O > O^*$, i.e., $\text{PmFWA}(O_1, O_2, \ldots, O_n) > \text{PmFWA}(O_1^*, O_2^*, \ldots, O_n^*)$.

(ii) If $s(O) = s(O^*)$, then

$$
\begin{aligned}
&\left(\sqrt{1-\prod_{k=1}^{n}\left(1-\Upsilon_k^{(i)}\right)^{\hbar_k}}\right)^2-\left(\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\right)^2\\
&= \left(\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^*\right)^{\hbar_k}}\right)^2-\left(\prod_{k=1}^{n}\left(\left(o_k^{(i)}\right)^*\right)^{\hbar_k}\right)^2,
\end{aligned}
\tag{29}
$$

along with the given conditions $\Upsilon_k^{(i)} \geq (\Upsilon_k^{(i)})^*$ and $o_k^{(i)} \leq (o_k^{(i)})^*$ which yield

$$
\begin{aligned}
&\left(\sqrt{1-\prod_{k=1}^{n}\left(1-\Upsilon_k^{(i)}\right)^{\hbar_k}}\right)^2 = \left(\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^*\right)^{\hbar_k}}\right)^2\\
&\left(\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\right)^2 = \left(\prod_{k=1}^{n}\left(\left(o_k^{(i)}\right)^*\right)^{\hbar_k}\right)^2,
\end{aligned}
\tag{30}
$$

so that

$$
\begin{aligned}
a(O) &= \frac{1}{m}\left\{\left(\sqrt{1-\prod_{k=1}^{n}\left(1-\Upsilon_k^{(i)}\right)^{\hbar_k}}\right)^2 + \left(\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}\right)^2\right\}\\
&= \frac{1}{m}\left\{\left(\sqrt{1-\prod_{k=1}^{n}\left(1-\left(\Upsilon_k^{(i)}\right)^{*}\right)^{\hbar_k}}\right)^2 + \left(\prod_{k=1}^{n}\left(\left(o_k^{(i)}\right)^{*}\right)^{\hbar_k}\right)^2\right\}\\
&= a\left(O^{*}\right).
\end{aligned}
\tag{31}
$$

Thus, $O = O^*$, i.e., $\mathrm{PmFWA}(O_1, O_2, \ldots, O_n) = \mathrm{PmFWA}(O_1^*, O_2^*, \ldots, O_n^*)$, and hence

$$
\mathrm{PmFWA}(O_1, O_2, \ldots, O_n) \geq \mathrm{PmFWA}(O_1^*, O_2^*, \ldots, O_n^*). \tag{32}
$$

□

4. Pythagorean m-Polar Fuzzy Weighted Geometric Operator

In this segment, we present the notion of Pythagorean m-polar fuzzy weighted geometric operator for Pythagorean m-polar fuzzy numbers accompanied by some of its prime characteristics.

Definition 8. Let $O_k = \left\{\langle \Upsilon_k^{(i)}, o_k^{(i)}\rangle\right\}_{i=1}^{m}$ $(k = 1, 2, \ldots, n)$ be an assemblage of PmFNs. Define PmFWG: $T^n \longrightarrow T$ given by

$$
\begin{aligned}
\mathrm{PmFWG}(O_1, O_2, \ldots, O_n) &= \otimes_{k=1}^{n} O_k^{\hbar_k}\\
&= O_1^{\hbar_1} \otimes O_2^{\hbar_2} \otimes \cdots \otimes O_n^{\hbar_n},
\end{aligned}
\tag{33}
$$

where T^n is the collection of all PmFNs and $\hbar_k$'s are fuzzy weights of $(O_1, O_2, \ldots, O_n)$, such that their sum equals unity. PmFWG is called Pythagorean m-polar fuzzy weighted geometric operator.

If each $\hbar_k$ equals $(1/n)$, then PmFWG operator turns down to n-dimensional Pythagorean m-polar fuzzy geometric (PmFG) operator and is given as

$$
\begin{aligned}
\mathrm{PmFG}(O_1, O_2, \ldots, O_n) &= \left(\otimes_{k=1}^{n} O_k\right)^{(1/n)}\\
&= \left(O_1 \otimes O_2 \otimes \cdots \otimes O_n\right)^{(1/n)}.
\end{aligned}
\tag{34}
$$

In conformity with operational laws of PmFNs given in Definition 3, the following theorem accommodates in aggregating any finite number of PmFNs.

Theorem 4. *Let $O_k = \left\{\langle \Upsilon_k^{(i)}, o_k^{(i)}\rangle\right\}_{i=1}^{m}$ $(k = 1, 2, \ldots, n)$ be an assemblage of PmFNs; then,*

$$
\begin{aligned}
&\mathrm{PmFWG}(O_1, O_2, \ldots, O_n)\\
&\quad = \left\{\langle \prod_{k=1}^{n}\left(\Upsilon_k^{(i)}\right)^{\hbar_k}, \sqrt{1-\prod_{k=1}^{n}\left(1-\left(o_k^{(i)}\right)^2\right)^{\hbar_k}}\rangle\right\}_{i=1}^{m}.
\end{aligned}
\tag{35}
$$

Proof. The proof may be furnished on the parallel track as proof of Theorem 1. □

Theorem 5. *If $O_k = \left\{\langle \Upsilon_k^{(i)}, o_k^{(i)}\rangle\right\}_{i=1}^{m}$ $(k = 1, 2, \ldots, n)$ is an aggregate of PmFNs, then*

$$
\begin{aligned}
&\mathrm{PmFWG}(O_1, O_2, \ldots, O_n)\\
&\quad = \left\{\langle \prod_{k=1}^{n}\left(\Upsilon_k^{(i)}\right)^{\hbar_k}, \sqrt{1-\prod_{k=1}^{n}\left(1-\left(o_k^{(i)}\right)^2\right)^{\hbar_k}}\rangle\right\}_{i=1}^{m}
\end{aligned}
\tag{36}
$$

is also a PmFN.

Proof. The proof may be established in the same manner as the proof of Theorem 2. □

Example 2. We utilize the input of Example 1. The P4FWG, using Theorem 4, is

$$
\begin{aligned}
\mathrm{P4FWG}(O_1, O_2, O_3) &= \left\{\langle \prod_{k=1}^{3}\left(\Upsilon_k^{(i)}\right)^{\hbar_k}, \sqrt{1-\prod_{k=1}^{3}\left(1-\left(o_k^{(i)}\right)^2\right)^{\hbar_k}}\rangle\right\}_{i=1}^{4}\\
&= \left\{\langle 0.504, 0.506\rangle, \langle 0.361, 0.641\rangle, \langle 0.320, 0.256\rangle, \langle 0.308, 0.748\rangle\right\}.
\end{aligned}
\tag{37}
$$

Theorem 6. *Assume that* $O_k = \langle \Upsilon_k^{(i)}, o_k^{(i)} \rangle_{i=1}^m \, (k = 1, \ldots, n)$ *is an assemblage of PmFNs. Then,*

(1) *(Idempotency) if* $O_k = O = \langle \Upsilon^{(i)}, \underline{o}^{(i)} \rangle_{i=1}^m \, (k = 1, 2, \ldots, n)$ *for all k, then*

$$\mathrm{PmFWG}(O_1, O_2, \ldots, O_n) = O. \tag{38}$$

(2) *(Boundedness) if* $O^- = (\min(\Upsilon_k^{(i)}), \max(o_k^{(i)}))$ *and* $O^+ = (\max(\Upsilon_k^{(i)}), \min(o_k^{(i)}))$*, then*

$$O^- \leq \mathrm{PmFWG}(O_1, O_2, \ldots, O_n) \leq O^+. \tag{39}$$

(3) *(Monotonicity) let* $O_k = \langle \Upsilon_k^{(i)}, o_k^{(i)} \rangle_{i=1}^m$ *and* $O_k^* = \langle (\Upsilon_k^{(i)})^*, (o_k^{(i)})^* \rangle_{i=1}^m \, (k = 1, \ldots, n)$ *be two sets of PmFNs such that* $\Upsilon_k^{(i)} \geq (\Upsilon_k^{(i)})^*$ *and* $o_k^{(i)} \leq (o_k^{(i)})^*$ *for all k and all permissible value of i; then,*

$$\mathrm{PmFWG}(O_1, O_2, \ldots, O_n) \geq \mathrm{PmFWG}(O_1^*, O_2^*, \ldots, O_n^*). \tag{40}$$

Proof. The proof may be established in the same fashion as the proof of Theorem 3. □

5. Symmetric Pythagorean m-Polar Fuzzy Weighted Averaging Operator

In this portion, we render the notion of symmetric Pythagorean m-polar fuzzy weighted averaging operator for Pythagorean m-polar fuzzy numbers accompanied by some of its prime characteristics.

Definition 9. Let $O_k = \{\langle \Upsilon_k^{(i)}, o_k^{(i)} \rangle\}_{i=1}^m \, (k = 1, 2, \ldots, n)$ be an assemblage of PmFNs. Define SPmFWA: $T^n \longrightarrow T$ given by

$$\begin{aligned}\mathrm{SPmFWA}(O_1, O_2, \ldots, O_n) &= \dotplus_{k=1}^n (\hbar_k \cdot O_k) \\ &= (\hbar_1 \cdot O_1) \dotplus (\hbar_2 \cdot O_2) \dotplus \cdots \\ &\quad \dotplus (\hbar_n \cdot O_n),\end{aligned} \tag{41}$$

where T^n is the collection of all PmFNs and $\hbar_k$'s are fuzzy weights of $(O_1, O_2, \ldots, O_n)$, such that their addition yields unity. SPmFWA is called symmetric Pythagorean m-polar fuzzy weighted averaging operator.

If $W = (w, w, \ldots, w)^t = ((1/n), (1/n), \ldots, (1/n))^t$, then SP$m$FWA operator shrinks to symmetric Pythagorean m-polar fuzzy averaging (SPmFA) operator of dimension n and is given as

$$\begin{aligned}\mathrm{SPmFA}(O_1, O_2, \ldots, O_n) &= \frac{1}{n} \cdot \left(\dotplus_{k=1}^n O_k\right) \\ &= \frac{1}{n} \cdot (O_1 \dotplus O_2 \dotplus \cdots \dotplus O_n).\end{aligned} \tag{42}$$

In keeping with operational laws of PmFNs given in Definition 3, the forthcoming theorem benefits in aggregating PmFNs.

Theorem 7. *Let* $O_k = \{\langle \Upsilon_k^{(i)}, o_k^{(i)} \rangle\}_{i=1}^m \, (k = 1, \ldots, n)$ *be a family of PmFNs; then,*

$$\begin{aligned}&\mathrm{SPmFWA}(O_1, O_2, \ldots, O_n) \\ &= \left\{ \left\langle \sqrt{\frac{1 - \prod_{k=1}^n \left[1 - \left(\Upsilon_k^{(i)}\right)^2\right]^{\hbar_k}}{2 - \prod_{k=1}^n \left[1 - \left(\Upsilon_k^{(i)}\right)^2\right]^{\hbar_k} - \prod_{k=1}^n \left[\Upsilon_k^{(i)}\right]^{2\hbar_k}}}, \right.\right. \\ &\qquad \left.\left. \sqrt{\frac{1 - \prod_{k=1}^n \left[1 - \left(o_k^{(i)}\right)^2\right]^{\hbar_k}}{2 - \prod_{k=1}^n \left[1 - \left(o_k^{(i)}\right)^2\right]^{\hbar_k} - \prod_{k=1}^n \left[o_k^{(i)}\right]^{2\hbar_k}}} \right\rangle \right\}_{i=1}^m .\end{aligned} \tag{43}$$

Proof. The proof may be furnished by means of induction. □

Theorem 8. *If* $O_k = \{\langle \Upsilon_k^{(i)}, o_k^{(i)} \rangle\}_{i=1}^m \, (k = 1, 2, \ldots, n)$ *is an aggregate of PmFNs, then*

$$\begin{aligned}&\mathrm{SPmFWA}(O_1, O_2, \ldots, O_n) \\ &= \left\{ \left\langle \sqrt{\frac{1 - \prod_{k=1}^n \left[1 - \left(\Upsilon_k^{(i)}\right)^2\right]^{\hbar_k}}{2 - \prod_{k=1}^n \left[1 - \left(\Upsilon_k^{(i)}\right)^2\right]^{\hbar_k} - \prod_{k=1}^n \left[\Upsilon_k^{(i)}\right]^{2\hbar_k}}}, \right.\right. \\ &\qquad \left.\left. \sqrt{\frac{1 - \prod_{k=1}^n \left[1 - \left(o_k^{(i)}\right)^2\right]^{\hbar_k}}{2 - \prod_{k=1}^n \left[1 - \left(o_k^{(i)}\right)^2\right]^{\hbar_k} - \prod_{k=1}^n \left[o_k^{(i)}\right]^{2\hbar_k}}} \right\rangle \right\}_{i=1}^m\end{aligned} \tag{44}$$

is also a PmFN.

Proof. Straight forward. □

Example 3. Consider the data of Example 1. The SP4FWA, using Theorem 7, is

$$
\begin{aligned}
&\mathrm{SP}m\mathrm{FWA}(O_1,O_2,O_3)\\
&=\left\{\left\langle\sqrt{\frac{1-\prod_{k=1}^{3}\left[1-\left(\Upsilon_k^{(i)}\right)^2\right]^{\hbar_k}}{2-\prod_{k=1}^{3}\left[1-\left(\Upsilon_k^{(i)}\right)^3\right]^{\hbar_k}-\prod_{k=1}^{3}\left[\Upsilon_k^{(i)}\right]^{2\hbar_k}}},\right.\right.\\
&\left.\left.\sqrt{\frac{1-\prod_{k=1}^{3}\left[1-\left(o_k^{(i)}\right)^2\right]^{\hbar_k}}{2-\prod_{k=1}^{3}\left[1-\left(o_k^{(i)}\right)^2\right]^{\hbar_k}-\prod_{k=1}^{3}\left[o_k^{(i)}\right]^{2\hbar_k}}}\right\rangle\right\}_{i=1}^{4}\\
&=\{\langle 0.520,0.494\rangle,\langle 0.419,0.623\rangle,\langle 0.529,0.252\rangle,\\
&\langle 0.361,0.695\rangle\}.
\end{aligned}
\tag{45}
$$

Theorem 9. *Assume that* $O_k=\langle\Upsilon_k^{(i)},o_k^{(i)}\rangle_{i=1}^{m}\,(k=1,\ldots,n)$ *is a setting of PmFNs. Then,*

(1) *(Idempotency) if* $O_k=O=\langle\Upsilon^{(i)},\underset{-}{o}^{(i)}\rangle_{i=1}^{m}\,(k=1,2,\ldots,n)$ *for all k, then*

$$\mathrm{SP}m\mathrm{FWA}(O_1,O_2,\ldots,O_n)=O. \tag{46}$$

(2) *(Boundedness) if* $O^-=(\min(\Upsilon_k^{(i)}),\max(o_k^{(i)}))$ *and* $O^+=(\max(\Upsilon_k^{(i)}),\min(o_k^{(i)}))$*, then*

$$O^-\leq\mathrm{SP}m\mathrm{FWA}(O_1,O_2,\ldots,O_n)\leq O^+. \tag{47}$$

(3) *(Monotonicity) let* $O_k=\langle\Upsilon_k^{(i)},o_k^{(i)}\rangle_{i=1}^{m}$ *and* $O_k^*=\langle(\Upsilon_k^{(i)})^*,(o_k^{(i)})^*\rangle_{i=1}^{m}\,(k=1,\ldots,n)$ *be two collections of PmFNs such that* $\Upsilon_k^{(i)}\geq(\Upsilon_k^{(i)})^*$ *and* $o_k^{(i)}\leq(o_k^{(i)})^*$ *for all k and all permissible value of i; then,*

$$\mathrm{SP}m\mathrm{FWA}(O_1,O_2,\ldots,O_n)\geq\mathrm{SP}m\mathrm{FWA}(O_1^*,O_2^*,\ldots,O_n^*). \tag{48}$$

Proof. Straight forward. □

6. Symmetric Pythagorean m-Polar Fuzzy Weighted Geometric Operator

We assign this unit to render the notion of symmetric Pythagorean m-polar fuzzy weighted geometric operator for Pythagorean m-polar fuzzy numbers in company with some of its prime characteristics.

Definition 10. Let $O_k=\left\{\langle\Upsilon_k^{(i)},o_k^{(i)}\rangle\right\}_{i=1}^{m}\,(k=1,2,\ldots,n)$ be an assemblage of PmFNs. Define SPmFWG: $T^n\longrightarrow T$ given by

$$
\begin{aligned}
\mathrm{SP}m\mathrm{FWG}(O_1,O_2,\ldots,O_n)&=\boxplus_{k=1}^{n}(\hbar_k\boxdot O_k)\\
&=(\hbar_1\boxdot O_1)\boxplus(\hbar_2\boxdot O_2)\\
&\boxplus\cdots\boxdot(\hbar_n\boxdot O_n),
\end{aligned}
\tag{49}
$$

where T^n is the collection of all PmFNs and $\hbar_k$'s are fuzzy weights of $(O_1,O_2,\ldots,O_n)$, bearing the constraint that they add up to unity. SPmFWG is called symmetric Pythagorean m-polar fuzzy weighted geometric operator.

If $W=(w,w,\ldots,w)^t=((1/n),(1/n),\ldots,(1/n))^t$, then SP$m$FWG operator diminishes to symmetric Pythagorean m-polar fuzzy geometric (SPmFG) operator of dimension n and is specified as

$$
\begin{aligned}
\mathrm{SP}m\mathrm{FG}(O_1,O_2,\ldots,O_n)&=\frac{1}{n}\boxdot(\boxplus_{k=1}^{n}O_k)\\
&=\frac{1}{n}\boxdot(O_1\boxplus O_2\boxplus\cdots\boxplus O_n).
\end{aligned}
\tag{50}
$$

Relying upon the operational laws of PmFNs given in Definition 3, the approaching theorem suggests mathematical formulation of SPmFWG operator meant for aggregating PmFNs.

Theorem 10. *Let* $O_k=\left\{\langle\Upsilon_k^{(i)},o_k^{(i)}\rangle\right\}_{i=1}^{m}\,(k=1,2,\ldots,n)$ *be an assemblage of PmFNs; then,*

$$
\begin{aligned}
&\mathrm{SP}m\mathrm{FWG}(O_1,O_2,\ldots,O_n)\\
&=\left\{\left\langle\frac{\prod_{k=1}^{n}\left(\Upsilon_k^{(i)}\right)^{\hbar_k}}{\sqrt{\prod_{k=1}^{n}\left[1-\left(\Upsilon_k^{(i)}\right)^2\right]^{\hbar_k}+\prod_{k=1}^{n}\left[\Upsilon_k^{(i)}\right]^{2\hbar_k}}},\frac{\prod_{k=1}^{n}\left(o_k^{(i)}\right)^{\hbar_k}}{\sqrt{\prod_{k=1}^{n}\left[1-\left(o_k^{(i)}\right)^2\right]^{\hbar_k}+\prod_{k=1}^{n}\left[o_k^{(i)}\right]^{2\hbar_k}}}\right\rangle\right\}_{i=1}^{m}.
\end{aligned}
\tag{51}
$$

Proof. The proof may be furnished by means of induction. □

Theorem 11. *If* $O_k=\left\{\langle\Upsilon_k^{(i)},o_k^{(i)}\rangle\right\}_{i=1}^{m}\,(k=1,2,\ldots,n)$ *is an aggregate of PmFNs, then*

$$\text{SP}m\text{FWG}(O_1, O_2, \ldots, O_n)$$
$$= \left\{ \left\langle \frac{\prod_{k=1}^{n} \left(\Upsilon_k^{(i)}\right)^{\hbar_k}}{\sqrt{\prod_{k=1}^{n} \left[1 - \left(\Upsilon_k^{(i)}\right)^2\right]^{\hbar_k} + \prod_{k=1}^{n} \left[\Upsilon_k^{(i)}\right]^{2\hbar_k}}}, \frac{\prod_{k=1}^{n} \left(o_k^{(i)}\right)^{\hbar_k}}{\sqrt{\prod_{k=1}^{n} \left[1 - \left(o_k^{(i)}\right)^2\right]^{\hbar_k} + \prod_{k=1}^{n} \left[o_k^{(i)}\right]^{2\hbar_k}}} \right\rangle \right\}_{i=1}^{m} \tag{52}$$

is also a PmFN.

Proof. Straight forward. □

Example 4. Consider the data of Example 1. The SP4FWA, using Theorem 10, is

$$\text{SP}m\text{FWG}(O_1, O_2, O_3)$$
$$= \left\{ \left\langle \frac{\prod_{k=1}^{3} \left(\Upsilon_k^{(i)}\right)^{\hbar_k}}{\sqrt{\prod_{k=1}^{3} \left[1 - \left(\Upsilon_k^{(i)}\right)^2\right]^{\hbar_k} + \prod_{k=1}^{3} \left[\Upsilon_k^{(i)}\right]^{2\hbar_k}}}, \frac{\prod_{k=1}^{3} \left(o_k^{(i)}\right)^{\hbar_k}}{\sqrt{\prod_{k=1}^{3} \left[1 - \left(o_k^{(i)}\right)^2\right]^{\hbar_k} + \prod_{k=1}^{3} \left[o_k^{(i)}\right]^{2\hbar_k}}} \right\rangle \right\}_{i=1}^{4} \tag{53}$$
$$= \{\langle 0.510, 0.466\rangle, \langle 0.134, 0.362\rangle, \langle 0.118, 0.177\rangle, \langle 0.314, 0.691\rangle\}.$$

Theorem 12. *Assume that* $O_k = \langle \Upsilon_k^{(i)}, o_k^{(i)} \rangle_{i=1}^{m}$ $(k = 1, \ldots, n)$ *is a family of PmFNs. Then,*

(1) *(Idempotency) if* $O_k = O = \langle \Upsilon^{(i)}, \underset{-}{\overset{(i)}{o}} \rangle_{i=1}^{m}$ $(k = 1, 2, \ldots, n)$ *for all k, then*

$$\text{SP}m\text{FWG}(O_1, O_2, \ldots, O_n) = O. \tag{54}$$

(2) *(Boundedness) if* $O^- = (\min(\Upsilon_k^{(i)}), \max(o_k^{(i)}))$ *and* $O^+ = (\max(\Upsilon_k^{(i)}), \min(o_k^{(i)}))$, *then*

$$O^- \leq \text{SP}m\text{FWG}(O_1, O_2, \ldots, O_n) \leq O^+. \tag{55}$$

(3) *(Monotonicity) let* $O_k = \langle \Upsilon_k^{(i)}, o_k^{(i)} \rangle_{i=1}^{m}$ *and* $O_k^* = \langle (\Upsilon_k^{(i)})^*, (o_k^{(i)})^* \rangle_{i=1}^{m}$ $(k = 1, \ldots, n)$ *be two sets of PmFNs such that* $\Upsilon_k^{(i)} \geq (\Upsilon_k^{(i)})^*$ *and* $o_k^{(i)} \leq (o_k^{(i)})^*$ *for all k and all permissible value of i; then,*

$$\text{SP}m\text{FWG}(O_1, O_2, \ldots, O_n) \geq \text{SP}m\text{FWG}(O_1^*, O_2^*, \ldots, O_n^*). \tag{56}$$

Proof. Straight forward. □

7. Robust Decision Making through Pythagorean *m*-Polar Fuzzy Weighted Aggregation Operators

In the wake of investment, a venture capitalist usually faces manifold challenges in deciding about pros and cons of the trade and commerce industry. Companies entice the investor by cutting down the prices of their commodities, despite the fact that they have evaluated that consumer consummation is one of the most significant and fundamental features to stay alive and subsist in the market. A view of capital market is shown in Figure 1.

To take a clearer, rewarding, and intelligent decision, a financier will definitely want to have awareness about which market is suitable for investment and then consult a team of experts to get benefitted from their experience to have better safeguard for his investment. So, subsequent upon their prefatory scrutinization, a commission has been instituted to act as aide in investing the finances in the paramount markets where there is least chance of loss, according to the following major criteria:

Safeguard of principal: protection of funds financed is one of the indispensable components of any worthy investment program. Security of principal indicates fortification against any probable forfeiture under fluctuating environments. Protection of principal may be accomplished over and done with a watchful analysis of fiscal and industrial inclinations afore deciding on nature of investment. Obviously, no one can predict the yet to come commercial conditions with ultimate exactitude. To defend against definite slips which may sneak in while taking a decision on investment, far-reaching diversification is recommended.

Liquidity and collateral value: an investment that may be transformed into cash instantly without having any financial loss is known as liquid investment. Liquid investments comfort financiers to meet crises and disasters. Stocks are with no trouble merchantable only when they make available satisfactory profit through dividends and funds appreciation. Assortment of liquid investments empowers the financiers to raise funds through sale of liquid securities or borrowing by proposing them as collateral security. The venture capitalist finances in top ranked and readily profit-making investments for ensuring their liquidity and collateral value.

Figure 1: Capital market (source: cushmanwakefield.com).

Tax implications: associated tax implications should be earnestly and well thought out before scheduling an investment plan. Singularly, the amount of revenue that investment offers and the liability of income tax levied on that revenue must be pondered well. Financiers in small revenue brackets go on to make best use of cash earnings on their monies and hence are diffident to take extreme jeopardies. On the contrary, venture capitalists who are not specific about cash returns do not cogitate tax implications earnestly.

Steady revenue: financiers endow their treasuries in such assets that offer steady revenue. Monotony of revenue is consistent with a good investment program. Investors are attracted towards those programs that generate revenue not only stably but also adequately.

Permanency of buying power: investment is utilization of money with the aim of receiving capital appreciation or profits. Stated differently, current assets are surrendered with the object of getting loftier volumes of future funds. Thus, the financier must deliberate the buying power of future funds. For maintaining the constancy of buying power, the financier must scrutinize the projected price level inflation and the likelihoods of additions and sufferings in the investment accessible to them.

Capital growth: capital appreciation is one of the essential main beliefs of investment. The firmness of an industry warranties its allied companies to flourish and progress. So, by identifying the association flanked by industry evolution and assets appreciation, the financiers should capitalize in growth stocks. In brief, right matter in the suitable business must be taken on board at the appropriate stage.

Lawfulness: the financier must capitalize only in those assets which are legitimate and sanctioned by law. Illegitimate securities land the financier in misfortune. In addition to being mollified with the rightfulness of investment, the financier ought to be at liberty from administration of securities.

We develop an algorithm (Algorithm 1) first to intelligibly decipher a decision-making problem.

The flowchart of the algorithm is portrayed in Figure 2.

Example 5. Consider the decision-making problem of capital investment comprising three experts. Assume that there are five choices l_1, l_2, l_3, l_4, l_5 that are to be assessed by the financial experts keeping in view four attributes c_1, c_2, c_3, and c_4, where

$$\begin{aligned} c_1 &= \text{analysis of permanency of buying power}, \\ c_2 &= \text{analysis of liquidity and collateral value}, \\ c_3 &= \text{principal safeguard analysis}, \\ c_4 &= \text{analysis of capital growth}. \end{aligned} \tag{57}$$

The weights of three experts, in order, are assigned as $\hbar_1 = 0.40$ and $\hbar_2 = \hbar_3 = 0.30$ according to their expertise and importance of their opinion. The three experts provide the information in the form of PFNs which are transformed in the form of PF matrices in which rows represent choices and the columns are meant for criteria.

$$\begin{aligned} \mathbb{M}_1 &= \begin{pmatrix} (0.37, 0.49) & (0.76, 0.36) & (0.72, 0.41) & (0.61, 0.08) \\ (0.77, 0.48) & (0.81, 0.39) & (0.32, 0.89) & (0.21, 0.56) \\ (0.42, 0.71) & (0.56, 0.54) & (0.37, 0.80) & (0.11, 0.39) \\ (0.56, 0.21) & (0.42, 0.06) & (0.58, 0.60) & (0.45, 0.82) \\ (0.54, 0.21) & (0.31, 0.73) & (0.50, 0.59) & (0.62, 0.15) \end{pmatrix}, \\ \mathbb{M}_2 &= \begin{pmatrix} (0.76, 0.29) & (0.54, 0.09) & (0.11, 0.23) & (0.37, 0.52) \\ (0.48, 0.61) & (0.58, 0.63) & (0.67, 0.36) & (0.91, 0.40) \\ (0.52, 0.53) & (0.48, 0.21) & (0.34, 0.79) & (0.54, 0.21) \\ (0.28, 0.09) & (0.48, 0.19) & (0.21, 0.86) & (0.40, 0.90) \\ (0.33, 0.76) & (0.79, 0.21) & (0.67, 0.71) & (0.49, 0.36) \end{pmatrix}, \\ \mathbb{M}_3 &= \begin{pmatrix} (0.54, 0.11) & (0.28, 0.56) & (0.38, 0.21) & (0.06, 0.82) \\ (0.47, 0.54) & (0.39, 0.46) & (0.41, 0.43) & (0.35, 0.11) \\ (0.37, 0.24) & (0.54, 0.11) & (0.48, 0.42) & (0.47, 0.18) \\ (0.36, 0.29) & (0.58, 0.16) & (0.39, 0.22) & (0.41, 0.32) \\ (0.41, 0.49) & (0.54, 0.41) & (0.37, 0.18) & (0.46, 0.33) \end{pmatrix}. \end{aligned} \tag{58}$$

We present these PF matrices in lamellar formation in Table 2.

The values of PmFWA(O_1, O_2, O_3) for each choice are given in Table 3.

The values of score function against each choice are given in Table 4.

Hence, the rank of choices is

$$l_1 \succ l_2 \succ l_5 \succ l_4 \succ l_3. \tag{59}$$

Let us experience what happens if we proceed with PmFWG operator instead of PmFWA operator. The values of PmFWA(O_1, O_2, O_3) for each choice are given in Table 5.

The score function values against each choice are given in Table 6.

Hence, the rank of choices is

Input:
(1) Analyze the problem: Let $X = \{l_1, l_2, \ldots, l_p\}$ be the collection of choices and the set of criteria be $E = \{c_1, c_2, \ldots, c_q\}$. Suppose that $\hbar_1, \hbar_2, \ldots, \hbar_n$ are respective weights of n decision experts.
Computations:
(2) Present the information given by experts in the form of PF matrices as $\mathbb{M}_1, \mathbb{M}_2, \ldots, \mathbb{M}_n$.
(3) Present the matrices $\mathbb{M}_1, \mathbb{M}_2, \ldots, \mathbb{M}_n$ in Pythagorean m-polar fuzzy array.
(4) Use PmFWA, PmFWG, SPmFWA, or SPmFWG operator to aggregate the PFNs for each choice.
(5) Compute value of score function s for each choice.
Output:
(6) The alternative with highest value of s is the desired alternative.

Algorithm 1

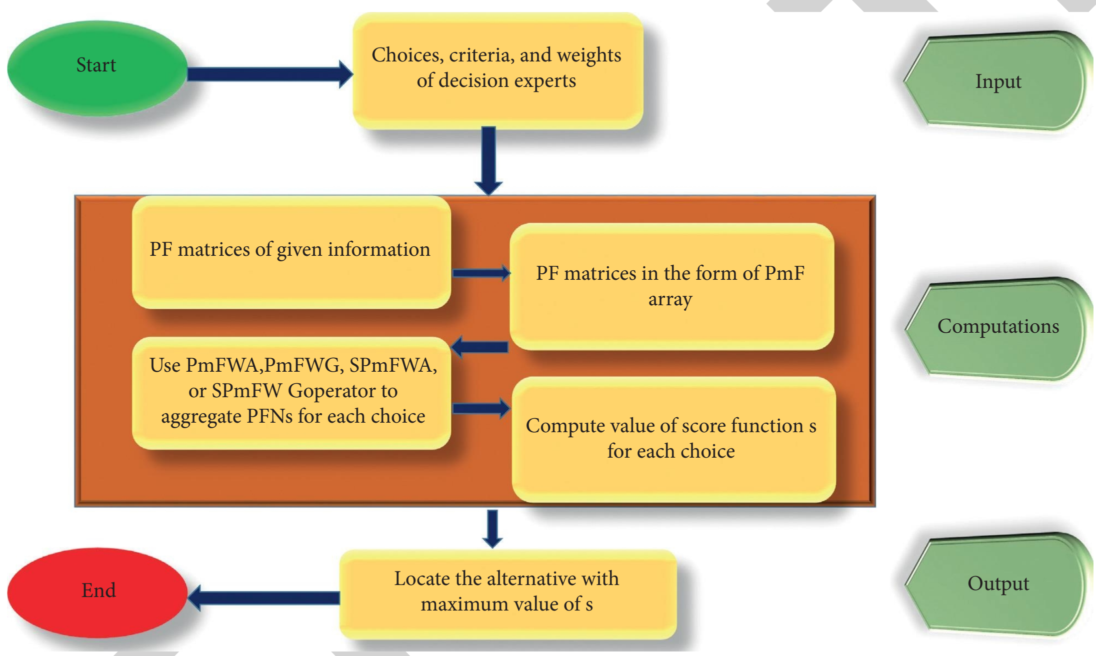

Figure 2: Flowchart of the algorithm.

Table 2: PmFNs for each choice.

Choice	PFNs
l_1	$O_1 = \{\langle 0.37, 0.49\rangle, \langle 0.76, 0.36\rangle, \langle 0.72, 0.41\rangle, \langle 0.61, 0.08\rangle\}$ $O_2 = \{\langle 0.76, 0.29\rangle, \langle 0.54, 0.09\rangle, \langle 0.11, 0.23\rangle, \langle 0.37, 0.52\rangle\}$ $O_3 = \{\langle 0.54, 0.11\rangle, \langle 0.28, 0.56\rangle, \langle 0.38, 0.21\rangle, \langle 0.06, 0.82\rangle\}$
l_2	$O_1 = \{\langle 0.77, 0.48\rangle, \langle 0.81, 0.39\rangle, \langle 0.32, 0.89\rangle, \langle 0.21, 0.56\rangle\}$ $O_2 = \{\langle 0.48, 0.61\rangle, \langle 0.58, 0.63\rangle, \langle 0.67, 0.36\rangle, \langle 0.91, 0.40\rangle\}$ $O_3 = \{\langle 0.47, 0.54\rangle, \langle 0.39, 0.46\rangle, \langle 0.41, 0.43\rangle, \langle 0.35, 0.11\rangle\}$
l_3	$O_1 = \{\langle 0.42, 0.71\rangle, \langle 0.56, 0.54\rangle, \langle 0.37, 0.80\rangle, \langle 0.11, 0.39\rangle\}$ $O_2 = \{\langle 0.52, 0.53\rangle, \langle 0.48, 0.21\rangle, \langle 0.34, 0.79\rangle, \langle 0.54, 0.21\rangle\}$ $O_3 = \{\langle 0.37, 0.24\rangle, \langle 0.54, 0.11\rangle, \langle 0.48, 0.42\rangle, \langle 0.47, 0.18\rangle\}$
l_4	$O_1 = \{\langle 0.56, 0.21\rangle, \langle 0.42, 0.06\rangle, \langle 0.58, 0.60\rangle, \langle 0.45, 0.82\rangle\}$ $O_2 = \{\langle 0.28, 0.09\rangle, \langle 0.48, 0.19\rangle, \langle 0.21, 0.86\rangle, \langle 0.40, 0.90\rangle\}$ $O_3 = \{\langle 0.36, 0.29\rangle, \langle 0.58, 0.16\rangle, \langle 0.39, 0.22\rangle, \langle 0.41, 0.32\rangle\}$
l_5	$O_1 = \{\langle 0.54, 0.21\rangle, \langle 0.31, 0.73\rangle, \langle 0.50, 0.59\rangle, \langle 0.62, 0.15\rangle\}$ $O_2 = \{\langle 0.33, 0.76\rangle, \langle 0.79, 0.21\rangle, \langle 0.67, 0.71\rangle, \langle 0.49, 0.36\rangle\}$ $O_3 = \{\langle 0.41, 0.49\rangle, \langle 0.54, 0.41\rangle, \langle 0.37, 0.18\rangle, \langle 0.46, 0.33\rangle\}$

Table 3: Values of PmFWA(O_1, O_2, O_3) for each choice.

Choice	PmFWA(O_1, O_2, O_3)
l_1	$\{\langle 0.586, 0.267\rangle, \langle 0.614, 0.271\rangle, \langle 0.539, 0.282\rangle, \langle 0.455, 0.282\rangle\}$
l_2	$\{\langle 0.633, 0.534\rangle, \langle 0.671, 0.473\rangle, \langle 0.492, 0.545\rangle, \langle 0.666, 0.311\rangle\}$
l_3	$\{\langle 0.441, 0.470\rangle, \langle 0.532, 0.252\rangle, \langle 0.400, 0.657\rangle, \langle 0.409, 0.257\rangle\}$
l_4	$\{\langle 0.441, 0.179\rangle, \langle 0.494, 0.114\rangle, \langle 0.451, 0.495\rangle, \langle 0.424, 0.636\rangle\}$
l_5	$\{\langle 0.451, 0.398\rangle, \langle 0.595, 0.423\rangle, \langle 0.536, 0.437\rangle, \langle 0.542, 0.247\rangle\}$

Table 4: Values of score function against each choice.

Choice	s
l_1	0.229
l_2	0.158
l_3	0.006
l_4	0.032
l_5	0.137

Table 5: Values of PmFWA(O_1, O_2, O_3) for each choice.

Choice	PmFWA(O_1, O_2, O_3)
l_1	$\{\langle 0.514, 0.361\rangle, \langle 0.508, 0.396\rangle, \langle 0.338, 0.313\rangle, \langle 0.262, 0.592\rangle\}$
l_2	$\{\langle 0.576, 0.542\rangle, \langle 0.588, 0.501\rangle, \langle 0.430, 0.720\rangle, \langle 0.380, 0.432\rangle\}$
l_3	$\{\langle 0.431, 0.573\rangle, \langle 0.529, 0.379\rangle, \langle 0.390, 0.730\rangle, \langle 0.274, 0.292\rangle\}$
l_4	$\{\langle 0.398, 0.214\rangle, \langle 0.482, 0.142\rangle, \langle 0.380, 0.671\rangle, \langle 0.422, 0.790\rangle\}$
l_5	$\{\langle 0.429, 0.549\rangle, \langle 0.485, 0.558\rangle, \langle 0.499, 0.569\rangle, \langle 0.528, 0.286\rangle\}$

Table 6: Score function values against each choice.

Choice	s
l_1	−0.008
l_2	−0.061
l_3	−0.099
l_4	−0.107
l_5	−0.018

$$l_1 \succ l_5 \succ l_2 \succ l_3 \succ l_4. \tag{60}$$

Let us employ SPmFWA operator to experience if there is any change in the optimal choice. The values of SPmFWA(O_1, O_2, O_3) for each choice are given in Table 7.

The values of score function against each choice are given in Table 8.

Hence, the rank of choices is

$$l_1 \succ l_2 \succ l_5 \succ l_4 \succ l_3 \tag{61}$$

Finally, we wield SPmFWG operator to discuss whether this operator brings any revision in the choice of optimal option. The values of SPmFWA(O_1, O_2, O_3) for each choice are given in Table 9.

The values of score function against each choice are given in Table 10.

Hence, the rank of choices is

$$l_1 \succ l_5 \succ l_2 \succ l_4 \succ l_3. \tag{62}$$

From these four ranking indices, we observe that the optimal choice, which is l_1, stays unaltered. We exhibit the four ranking catalogues through the medium of horizontal bar chart cited in Figure 3.

7.1. Comparison Analysis and Superiority of the Proposed Work. We have observed that the optimal solution remains the same by use of either of the four proposed operators in this article. Further, the optimal choice attained through our suggested techniques does not alter by use of other methods. No computationally easy to use aggregation operator for PmFSs has yet been introduced so far, according to our best knowledge. Our suggested technique is simple to apply and yields definitive outputs. It can handle the data given at repeated spans of times or by different decision experts efficiently. The comparison of presented aggregation operators with some existing operators is given in Table 11.

Table 7: Values of SPmFWA(O_1, O_2, O_3) for each choice.

Choice	SPmFWA(O_1, O_2, O_3)
l_1	$\{\langle 0.564, 0.351\rangle, \langle 0.580, 0.381\rangle, \langle 0.497, 0.311\rangle, \langle 0.426, 0.525\rangle\}$
l_2	$\{\langle 0.613, 0.540\rangle, \langle 0.639, 0.494\rangle, \langle 0.479, 0.652\rangle, \langle 0.584, 0.414\rangle\}$
l_3	$\{\langle 0.439, 0.544\rangle, \langle 0.531, 0.365\rangle, \langle 0.398, 0.695\rangle, \langle 0.391, 0.290\rangle\}$
l_4	$\{\langle 0.434, 0.212\rangle, \langle 0.491, 0.141\rangle, \langle 0.438, 0.611\rangle, \langle 0.424, 0.715\rangle\}$
l_5	$\{\langle 0.447, 0.514\rangle, \langle 0.563, 0.524\rangle, \langle 0.526, 0.535\rangle, \langle 0.538, 0.283\rangle\}$

Table 8: Values of score function against each choice.

Choice	s
l_1	0.111
l_2	0.056
l_3	−0.053
l_4	−0.037
l_5	0.044

Table 9: Values of SPmFWA(O_1, O_2, O_3) for each choice.

Choice	SPmFWA(O_1, O_2, O_3)
l_1	$\{\langle 0.536, 0.276\rangle, \langle 0.541, 0.283\rangle, \langle 0.373, 0.285\rangle, \langle 0.282, 0.093\rangle\}$
l_2	$\{\langle 0.344, 0.537\rangle, \langle 0.622, 0.480\rangle, \langle 0.443, 0.618\rangle, \langle 0.454, 0.326\rangle\}$
l_3	$\{\langle 0.433, 0.497\rangle, \langle 0.530, 0.263\rangle, \langle 0.392, 0.693\rangle, \langle 0.288, 0.259\rangle\}$
l_4	$\{\langle 0.406, 0.181\rangle, \langle 0.484, 0.114\rangle, \langle 0.391, 0.555\rangle, \langle 0.423, 0.720\rangle\}$
l_5	$\{\langle 0.433, 0.430\rangle, \langle 0.517, 0.454\rangle, \langle 0.509, 0.469\rangle, \langle 0.532, 0.247\rangle\}$

Table 10: Values of score function against each choice.

Choice	s
l_1	0.138
l_2	−0.025
l_3	−0.040
l_4	−0.035
l_5	0.081

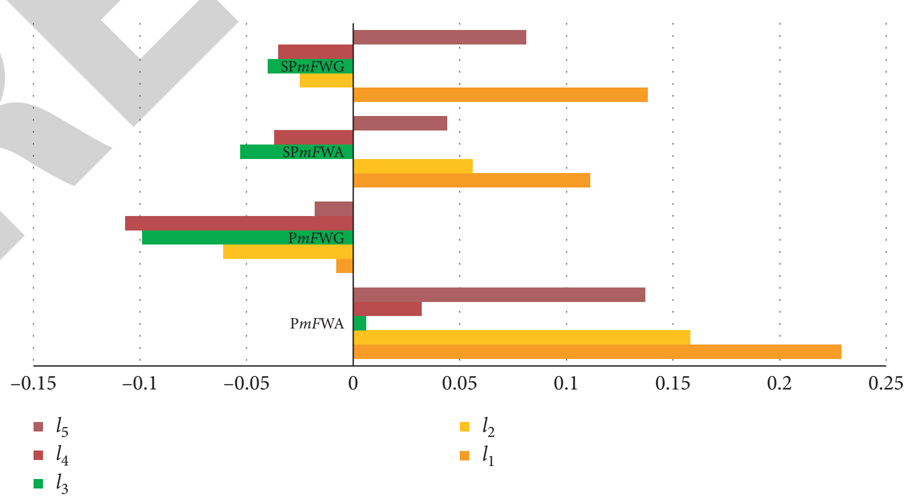

Figure 3: Horizontal bar chart of the two rankings.

Table 11: Comparison of proposed operators with some existing operators.

Aggregation operators	Optimal choice
Ordered weighted averaging aggregation operators (Yager [7])	l_1
Pythagorean fuzzy aggregation operators (Peng and Yuan [17])	l_1
Generalized intuitionistic fuzzy soft power aggregation operator (Garg and Arora [25])	l_1
q-Rung orthopair fuzzy aggregation operator (Liu and Wang [30])	l_1
Prioritized weighted aggregation operators (Liu et al. [31])	l_1
Pythagorean m-polar fuzzy Dombi's aggregation operators (Hashmi and Riaz [45])	l_1
Intuitionistic fuzzy aggregation operators (Xu [52])	l_1
Pythagorean m-polar fuzzy weighted averaging operator (proposed)	l_1
Pythagorean m-polar fuzzy weighted geometric operator (proposed)	l_1
Symmetric Pythagorean m-polar fuzzy weighted averaging operator (proposed)	l_1
Symmetric Pythagorean m-polar fuzzy weighted geometric operator (proposed)	l_1

8. Conclusion

Pythagorean m-polar fuzzy set is a mighty model for examining the information given in multipolar form. We suggested four operators, namely, Pythagorean m-polar fuzzy weighted averaging operator, Pythagorean m-polar fuzzy weighted geometric operator, symmetric Pythagorean m-polar fuzzy weighted averaging operator, and symmetric Pythagorean m-polar fuzzy weighted geometric operator for the sake of aggregating the statistics given in multipolar form. The aggregated resultant falling in the same structure has also been manifested. We established the desirable qualities of idempotency, monotonicity, and boundedness for the proposed operators.

The results presented in this article are also valid for intuitionistic m-polar fuzzy sets and have potential to be generalized to q-rung orthopair m-polar fuzzy sets and many other models. We rendered an algorithm for capital investment analysis problem as practical usage of the suggested operators in daily life situations and found that our computed results are compatible with the existing techniques. The suggested algorithm may be used in human resource management problems, life sciences, economics analysis, business and trade analysis, pattern recognition, water management problems, agribusiness, and many other areas. We anticipate that this article will attract the attention of vibrant researchers working in this field.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this study and read and approved the final manuscript.

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Retraction

Retracted: Ordered-Theoretic Fixed Point Results in Fuzzy *b*-Metric Spaces with an Application

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] K. Javed, F. Uddin, H. Aydi, A. Mukheimer, and M. Arshad, "Ordered-Theoretic Fixed Point Results in Fuzzy *b*-Metric Spaces with an Application," *Journal of Mathematics*, vol. 2021, Article ID 6663707, 7 pages, 2021.

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Research Article

Ordered-Theoretic Fixed Point Results in Fuzzy *b*-Metric Spaces with an Application

Khalil Javed,[1] Fahim Uddin,[1] Hassen Aydi ,[2,3,4] Aiman Mukheimer,[5] and Muhammad Arshad[1]

[1]*Department of Mathematics and Statistics, International Islamic University, Islamabad, Pakistan*
[2]*Université de Sousse, Institut Supérieur dínformatique et des Techniques de Communication, H. Sousse 4000, Tunisia*
[3]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University, Ga-Rankuwa, South Africa*
[4]*China Medical University Hospital, China Medical University, Taichung 40402, Taiwan*
[5]*Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia*

Correspondence should be addressed to Hassen Aydi; hassen.aydi@isima.rnu.tn

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The aim of this manuscript is to initiate the study of the Banach contraction in R-fuzzy b-metric spaces and discuss some related fixed point results to ensure the existence and uniqueness of a fixed point. A nontrivial example is imparted to illustrate the feasibility of the proposed methods. Finally, to validate the superiority of the provided results, an application is presented to solve the first kind of a Fredholm-type integral equation.

1. Introduction and Preliminaries

Since the axiomatic interpretation of metric spaces and the inception of the Banach contraction principle, many authors have studied fixed point theory vividly. A number of results have been introduced, and metric fixed point has been generalized in different directions. In this connectedness, Bakhtin [1] and Czerwik [2] gave a generalization of a metric space and named it as a b-metric space. Zadeh [3] introduced the concept of fuzzy sets and generalized the concept of metric spaces and fuzzy sets and named them as fuzzy metric spaces, which became a point of interest for many authors [2, 4]. Nădăban [5] extended the concept of a fuzzy metric and introduced the notion of fuzzy b-metric spaces. For related works in this setting, refer to [6–9].

Recently, Baghani and Ramezani [10] tossed the concept of orthogonal sets and gave an extension of the Banach contraction principle. For more details, refer to [10–24].

In this article, we further aim to establish fixed point results in the setting of R-complete fuzzy b-metric spaces. We provide an example dealing with an R-fuzzy b-metric space, but it is not a fuzzy b-metric space. The presented results improve and generalize many results in the literature.

First, we recall some basic definitions and notions, which are essential for this work.

Definition 1 (see [11]). A binary operation $*: [0,1]\times[0,1] \longrightarrow [0,1]$ is referred to as a continuous t-norm if the following assumptions hold:

(1) $e * f = f * e, \forall e, f \in [0,1]$
(2) $e * 1 = e, \forall e \in [0,1]$
(3) $(e * f) * s = e * (f * s), \forall e, f, s \in [0,1]$
(4) If $e \le s$ and $f \le u$, with $e, f, s, u \in [0,1]$, then $e * f \le s * u$

Some fundamental examples of a t-norm are $e * f = e \cdot f, e * f = \min\{e, f\}$, and $e * f = \max\{e + f - 1, 0\}$.

Definition 2 (see [12, 13]). A 3-tuple $(H, M, *)$ is said to be a fuzzy metric space if $H \neq M$ is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $H \times H \times (0, \infty)$ meeting the following conditions for all $\sigma, H, z \in M, \tau, \mathbf{s} > 0$:

(B1) $M(\sigma, M, \tau) > 0$
(B2) $M(\sigma, M, \tau) = 1$ iff $\sigma = M$
(B3) $M(\sigma, M, \tau) = M(M, \sigma, \tau)$
(B4) $M(\sigma, z, \tau + \mathbf{s}) \geq M(\sigma, M, \tau) * M(M, z, \mathbf{s})$
(B5) $M(\sigma, M, M)$: $(0, \infty) \longrightarrow [0, 1]$ is continuous

Example 1 (see [12]). Let (H, d) be a metric space with a continuous t-norm $\mathsf{a} * M = \mathsf{a} \cdot M$, and let M be a fuzzy set defined on $H \times H \times (0, \infty)$ by

$$M(\sigma, M, \tau) = \frac{\tau}{\tau + d(\sigma, M)}. \tag{1}$$

Then, $(H, M, *)$ is called a standard fuzzy metric space.

Definition 3 (see [6]). A 4-tuple $(H, M, *, u)$ is said to be a fuzzy b-metric space if $H \neq M$ is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set on $H \times H \times (0, \infty)$ meeting the following conditions for all $\sigma, M, z \in H, \tau, \mathbf{s} > 0$ and for a given real number $u \geq 1$:

(B1) $M(\sigma, M, \tau) > 0$
(B2) $M(\sigma, M, \tau) = 1$ iff $\sigma = M$
(B3) $M(\sigma, M, \tau) = M(M, \sigma, \tau)$
(B4) $M(\sigma, z, \tau + \mathbf{s}) \geq M(\sigma, M, \tau/u) * M(M, z, \mathbf{s}/u)$
(B5) $M(\sigma, M, M)$: $(0, \infty) \longrightarrow [0, 1]$ is continuous

Example 2 (see [7]). Let $M(\sigma, M, \tau) = e^{-|\sigma - M|^p/\tau}$, where $p > 1$ represents a real number. It is then simple to prove that M is a fuzzy b-metric with $u = 2^{p-1}$. It should be noted that, for $p = 2$, $(H, M, *)$ is not a fuzzy metric space.

Definition 4. Assume $H \neq M$ and $R \in H \times H$ is a binary relation. Suppose there exists $\sigma_0 \in M$ such that $\sigma_0 R \sigma$ or $\sigma R \sigma_0$ for all $\sigma \in H$. Then, we say that H is an R-set.

Example 3

(i) Let $H = [0, \infty)$ and define $\sigma R M$ if $\sigma M = \min\{\sigma, M\}$; then, by putting $\sigma_0 = 1$, (H, R) is an R-set.
(ii) Suppose M is a set of scalar matrices of order 2×2 with entries from natural numbers (i.e., $M = \begin{bmatrix} \mathsf{a} & 0 \\ 0 & \mathsf{a} \end{bmatrix}$, for all $\mathsf{a} \in N$). Define the relation R by

$$ARB \text{ if } \quad \det(A) \leq \det(B). \tag{2}$$

Then, by taking $A = I$, (M, R) is an R-set.

Definition 5 (see [10]). Suppose that (H, R) is an R-set. A sequence $\{\sigma_n\}$ for all $n \in \mathbb{N}$ is said to be an R-sequence if $(\forall n;\ \sigma_n R \sigma_{n+1})$ or $(\forall n;\ \sigma_{n+1} R \sigma_n)$.

Definition 6 (see [14])

(a) A metric space (H, d) is an R-metric space if (H, R) is an R-set.
(b) A mapping F: $H \longrightarrow H$ is R-continuous at $\sigma \in H$ if for each R-sequence $\{\sigma_n\}$ for all $n \in \mathbb{N}$ in H if $\lim_{n \longrightarrow \infty} d(\sigma_n, \sigma) = 0$, then $\lim_{n \longrightarrow \infty} d(\mathsf{F}\sigma_n, \mathsf{F}\sigma) = 0$. Furthermore, F is R-continuous if F is R-continuous at each $\sigma \in H$.
(c) A mapping F: $H \longrightarrow H$ is called R-preserving if $\sigma R \mathsf{F}$, then $\mathsf{F}\sigma R \mathsf{F} M$ for all $\sigma, M \in H$.
(d) An R-sequence $\{\sigma_n\}$ in H is said to be an R-Cauchy sequence if for every $\varepsilon > 0$, there exists an integer N such that $\mathrm{d}(\sigma_n, \sigma_m) < \varepsilon$ for all $n \geq \mathbb{N}$ and $m \geq \mathbb{N}$. It is clear that $\sigma_n R \sigma_m$ or $\sigma_m R \sigma_n$.
(e) H is R-complete if every R-Cauchy sequence is convergent.

2. Main Results

We start this section with the introduction of R-fuzzy b-metric spaces.

Definition 7. Let $H \neq M$ and R be a reflexive binary relation on H. Let $*$ be a continuous t-norm and H be a fuzzy set on $H \times H \times (0, \infty)$. Suppose that, for all $\tau, \mathbf{s} > 0$ and for all $\sigma, M, z \in H$, with either $(\sigma R z$ or $z R \sigma)$, either $(\sigma R M$ or $M R \sigma)$, and either $(M R z$ or $z R M)$, the following conditions hold:

(1) $M(\sigma, M, \tau) > 0$
(2) $M(\sigma, M, \tau) = 1$ if and only if $\sigma = M$
(3) $M(\sigma, M, \tau) = M(M, \sigma, \tau)$
(4) $M(\sigma, z, \tau + \mathbf{s}) \geq M(\sigma, M, \tau/u) * M(M, z, \mathbf{s}/u)$, where $u \geq 1$
(5) $M(\sigma, M, M)$: $(0, \infty) \longrightarrow [0, 1]$ is continuous

Then, $(H, M, *, u, R)$ is called an R-fuzzy b-metric space with the coefficient $u \geq 1$.

Remark 1. In the above definition, the set H is endowed with a reflexive binary relation R, and M is a fuzzy set on $H \times H \times (0, \infty)$ satisfying (1)–(5) for those comparable elements with respect to the reflexive binary relation R. An R-fuzzy b-metric may not be a fuzzy b-metric.

The following simplest example shows that the R-fuzzy b-metric with $u = 4$ does not need to be a fuzzy b-metric with $u = 4$.

Example 4. Let $H = [-1, 1]$ and $M(\sigma, M, \tau) = e^{-(\sigma - M)^3/\tau}$. Define a binary relation such that $\sigma R M$ iff $|\sigma| \geq |M|$. It is clear that $M(\sigma, M, \tau)$ is an R-fuzzy b-metric on H with $u = 4$.

Note that for $\sigma = 0.1$, $M = 0.5$, and $z = 0.8$, the following condition does not hold:

$$M(\sigma, z, \tau + \mathbf{s}) \geq M\left(\sigma, M, \frac{\tau}{u}\right) * M\left(M, z, \frac{\mathbf{s}}{u}\right). \tag{3}$$

So, $M(\sigma, M, \tau)$ is not a fuzzy b-metric.

Definition 8. Let $(H, M, *, u, R)$ represent an R-fuzzy b-metric space.

(a) A sequence $\{\sigma_n\}$ for all $n \in \mathbb{N}$ is said to be an R-sequence if $(\forall n;\ \sigma_n R \sigma_{n+1})$ or $(\forall n;\ \sigma_{n+1} R \sigma_n)$.

(b) A Cauchy sequence $\{\sigma_n\}$ is said to be an R-Cauchy sequence if $(\forall n;\ \sigma_n R \sigma_{n+1})$ or $(\forall n;\ \sigma_{n+1} R \sigma_n)$.

(c) A mapping $\digamma\colon H \longrightarrow H$ is R-continuous at $\sigma \in M$ if for each R-sequence $\{\sigma_n\}$ for all $n \in \mathbb{N}$ in M with $\lim_{n\longrightarrow\infty} M(\sigma_n, \sigma, \tau) = 1$ for all $\tau > 0$, then $\lim_{n\longrightarrow\infty} M(\digamma\sigma_n, \digamma\sigma, \tau) = 1$ for all $\tau > 0$. Furthermore, $\digamma$ is R-continuous if $\digamma$ is R-continuous at each $\sigma \in H$.

(d) A mapping $\digamma\colon H \longrightarrow H$ is called R-preserving if $\sigma R M$, then $\digamma\sigma R \digamma M$ for all $\sigma, M \in H$.

(e) If each R-Cauchy sequence is convergent, then M is R-complete.

Motivated by the work of Baghani and Ramezani [10] and Hezarjaribi et al. [14], we introduce the concept of Banach contraction principle in the setting of R-fuzzy b-metric spaces.

Definition 9. Let $(H, M, *, u, R)$ be an R-fuzzy b-metric space. A map $\digamma\colon H \longrightarrow H$ is an R-contraction if there exists $q \in (0, 1)$ such that, for every $\tau > 0$ and $\sigma, M \in H$ with $\sigma R M$, we have

$$M(\digamma\sigma, \digamma M, q\tau) \geq M(\sigma, M, \tau). \tag{4}$$

Theorem 1. *Assume that $(H, M, *, u, R)$ is an R-complete fuzzy b-metric space. Let $\digamma\colon H \longrightarrow H$ be an R-continuous, R-contraction, and R-preserving mapping. Thus, $\digamma$ has a unique fixed point $\sigma_* \in H$. Furthermore,*

$$\lim_{n\longrightarrow\infty} M(\digamma^n\sigma, \sigma_*, \tau) = 1, \text{ for all } \sigma \in H \text{ and } \tau > 0. \tag{5}$$

Proof. Since $(H, M, *, u, R)$ is an R-complete fuzzy b-metric space, there exists $\sigma_0 \in H$ such that

$$\sigma_0 R M, \text{ for all } M \in H. \tag{6}$$

This yields that $\sigma_0 R \digamma\sigma_0$. Assume that

$$\sigma_1 = \digamma\sigma_0, \sigma_2 = \digamma^2\sigma_0 = \digamma\sigma_1, \ldots, \sigma_n = \digamma^n\sigma_0 = \digamma\sigma_{n-1}, \text{ for all } n \in N. \tag{7}$$

Since $\digamma$ is R-preserving, $\{\sigma_n\}$ is an R-sequence and $\digamma$ is an R-contraction. Thus,

$$M(\sigma_{n+1}, \sigma_n, q\tau) = M(\digamma\sigma_n, \digamma\sigma_{n-1}, q\tau) \geq M(\sigma_n, \sigma_{n-1}, \tau), \tag{8}$$

for all $n \in N$ and $\tau > 0$. Therefore, by applying the above expression, we deduce

$$\begin{aligned} M(\sigma_{n+1}, \sigma_n, \tau) &\geq M(\sigma_{n+1}, \sigma_n, q\tau) = M(\digamma\sigma_n, \digamma\sigma_{n-1}, q\tau) \geq M(\sigma_n, \sigma_{n-1}, \tau) \\ &= M(\digamma\sigma_{n-1}, \digamma\sigma_{n-2}, \tau) \geq M\left(\sigma_{n-1}, \sigma_{n-2}, \frac{\tau}{q}\right) \geq \ldots \geq M\left(\sigma_1, \sigma_0, \frac{\tau}{q^n}\right), \end{aligned} \tag{9}$$

for all $n \in N$ and $\tau > 0$. Thus, from (9) and (B4), we have

$$\begin{aligned} M\left(\sigma_n, \sigma_{n+p}, \tau\right) &\geq M\left(\sigma_n, \sigma_{n+1}, \frac{\tau}{u}\right) * M\left(\sigma_{n+1}, \sigma_{n+p}, \frac{\tau}{u}\right) \\ &\geq M\left(\sigma_n, \sigma_{n+1}, \frac{\tau}{u}\right) * M\left(\sigma_{n+1}, \sigma_{n+2}, \frac{\tau}{u^2}\right) * M\left(\sigma_{n+2}, \sigma_{n+3}, \frac{\tau}{u^3}\right) * \ldots * M\left(\sigma_{n+p-1}, \sigma_{n+p}, \frac{\tau}{u^{n+p}}\right) \\ &\geq M\left(\sigma_1, \sigma_0, \frac{\tau}{uq^n}\right) * M\left(\sigma_1, \sigma_0, \frac{\tau}{u^2q^n}\right) * \ldots * M\left(\sigma_1, \sigma_0, \frac{\tau}{u^{n+p}q^n}\right). \end{aligned} \tag{10}$$

Here, u is an arbitrary positive integer. We know that $\lim_{n\longrightarrow\infty} M(\sigma, M, \tau) = 1$ for all $\sigma, M \in H$ and $\tau > 0$. From (10), we get

$$\lim_{n\longrightarrow\infty} M\left(\sigma_n, \sigma_{n+p}, \tau\right) \geq 1 * 1 * \ldots * 1 = 1. \tag{11}$$

Then, $\{\sigma_n\}$ is an R-Cauchy sequence. The hypothesis of R-completeness of the fuzzy b-metric space $(H, M, *, u, R)$ ensures that there exists $\sigma_* \in H$ such that $M(\sigma_n, \sigma_*, \tau) \longrightarrow 1$ as $n \longrightarrow +\infty$ for all $\tau > 0$. Since $\digamma$ is an R-continuous mapping, one writes $M(\sigma_{n+1}, \digamma\sigma_*, \tau) = M(\digamma\sigma_n, \digamma\sigma_*, \tau) \longrightarrow 1$ as $n \longrightarrow +\infty$. Hence,

$$M(\sigma_*, \digamma\sigma_*, \tau) \geq M\left(\sigma_*, \sigma_{n+1}, \frac{\tau}{2u}\right) * M\left(\sigma_{n+1}, \digamma\sigma_*, \frac{\tau}{2u}\right). \tag{12}$$

As $n \longrightarrow +\infty$, we get $\digamma(\sigma_*, \digamma\sigma_*, \tau) = 1 * 1 = 1$; hence, $\digamma\sigma_* = \sigma_*$.

To show the uniqueness of the fixed point for the mapping $\digamma$, assume that σ_* and M_* are two fixed points of $\digamma$ such that $\sigma_* \neq M_*$. We have

$$\sigma_0 R \sigma_* \text{ and } \sigma_0 R M_*. \tag{13}$$

Since M is R-preserving, we can write

$$\digamma^n \sigma_0 R \digamma^n \sigma_* \text{ and } \digamma^n \sigma_0 R \digamma^n \digamma v_*, \tag{14}$$

for all $n \in N$. Using (4), we have

$$\begin{aligned} M(\digamma^n\sigma_0, \digamma^n\sigma_*, \tau) &\geq M(\digamma^n\sigma_0, \digamma^n\sigma_*, q\tau) \geq M\left(\sigma_0, \sigma_*, \frac{\tau}{q^n}\right), \\ M(\digamma^n\sigma_0, \digamma^n v_*, \tau) &\geq M(\digamma^n\sigma_0, \digamma^n v_*, q\tau) \geq M\left(\sigma_0, v_*, \frac{\tau}{q^n}\right). \end{aligned} \tag{15}$$

Hence,

$$\begin{aligned} M(\sigma_*, v_*, \tau) = M(\digamma^n\sigma_*, \digamma^n v_*, \tau) &\geq M\left(\digamma^n\sigma_0, \digamma^n\sigma_*, \frac{\tau}{2u}\right) * M\left(\digamma^n\sigma_0, \digamma^n v_*, \frac{\tau}{2u}\right) \\ &\geq M\left(\sigma_0, \sigma_*, \frac{\tau}{2uq^n}\right) * M\left(\sigma_0, v_*, \frac{\tau}{2uq^n}\right) \longrightarrow 1 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{16}$$

So, $\sigma_* = M_*$; hence, σ_* is the unique fixed point.

Corollary 1. *Let $(H, M, *, u, R)$ be an R-complete fuzzy b-metric space. Let H: $H \longrightarrow H$ be an R-contraction and R-preserving. Also, if $\{\sigma_n\}$ is an R-sequence with $\sigma_n \longrightarrow \sigma \in \digamma$, then $\sigma R \sigma_n$ for all $n \in \mathbb{N}$. Therefore, M has a unique fixed point $\sigma_* \in H$. Furthermore, $\lim_{n\longrightarrow\infty} M(\digamma^n\sigma, \sigma_{\∗}, \tau) = 1$, for all $\sigma \in H$ and $\tau > 0$.*

Proof. The proof of this result moves along the same lines as in Theorem 1, that is, $\{\sigma_n\}$ is an R-Cauchy sequence and converges to $\sigma_* \in H$. Hence, $\sigma_* R \sigma_n$ for all $n \in \mathbb{N}$. From (4), we have

$$\begin{aligned} M(\digamma\sigma_*, \sigma_{n+1}, \tau) = M(\digamma\sigma_*, \digamma\sigma_n, \tau) &\geq M(\digamma\sigma_*, \digamma\sigma_n, \tau q) \\ &\geq M(\sigma_*, \sigma_n, \tau). \end{aligned} \tag{17}$$

Also,

$$\lim_{n\longrightarrow\infty} M(\digamma\sigma_*, \sigma_{n+1}, \tau) = 1. \tag{18}$$

Hence,

$$M(\sigma_*, \digamma\sigma_*, \tau) \geq M\left(\sigma_*, \sigma_{n+1}, \frac{\tau}{2u}\right) * M\left(\sigma_{n+1}, \digamma\sigma_*, \frac{\tau}{2u}\right). \tag{19}$$

As $n \longrightarrow +\infty$, we get $M(\sigma_*, \digamma\sigma_*, \tau) = 1 * 1 = 1$, and so, $\digamma\sigma_* = \sigma_*$. The rest of the proof is the same as in Theorem 1.

Corollary 2. *Let $(\digamma, M, *, u, R)$ be an R-complete fuzzy b-metric space and $\digamma$: $H \longrightarrow H$ be an R-continuous and R-preserving mapping. Suppose that there exist $q \in (0, 1/2)$ and $\tau > 0$ such that*

$$M(\digamma\sigma, \digamma v, q\tau) \geq M\left(\digamma\sigma, \sigma, \frac{\tau}{2}\right) + M\left(\digamma v, v, \frac{\tau}{2}\right). \tag{20}$$

Then, M has a unique fixed point.

Corollary 3. *Let $(H, M, *, u, R)$ be an R-complete fuzzy b-metric space and $\digamma$: $H \longrightarrow H$ be an R-continuous and R-preserving mapping. Assume that there exist $q \in (0, 1/u)$ and $\tau > 0$ such that*

$$M(\digamma\sigma, \digamma v, q\tau) \geq \min\{M(\digamma\sigma, \sigma, \tau), M(\digamma v, v, \tau)\}. \tag{21}$$

Then, $\digamma$ has a unique fixed point.

Proof. The proof is a part of the next corollary.

Corollary 4. *Let $(\digamma, M, *, u, R)$ be an R-complete fuzzy b-metric space and $\digamma$: $H \longrightarrow H$ be an R-continuous and R-preserving mapping. Assume that there exist $q \in (0, 1/u)$ and $\tau > 0$ such that*

$$M(\digamma\sigma, \digamma v, q\tau) \geq \min\{M(\digamma\sigma, \sigma, \tau), M(\digamma v, v, \tau), M(\sigma, v, \tau)\}. \tag{22}$$

Then, $\digamma$ *has a unique fixed point.*

Proof. This corollary is a generalization of Theorem 2.5 in [8]. It is easy to prove this result by the help of Theorem 1 of this article and Theorem 2.5 of [8].

Example 5. Let $H = [-1, 1]$. The relation on H is defined as $\sigma R M M|\sigma| \geq |M|$. Define the R-fuzzy b-metric given as in Example 4:

$$M(\sigma, M, \tau) = \begin{cases} e^{-(\sigma-M)^3/\tau}, & \text{if } \tau > 0, \\ 0, & \text{if } \tau \leq 0, \end{cases} \tag{23}$$

with the t-norm $\alpha * M = \alpha M M$. Let $\{\sigma_n\}$ be an R-sequence in H such that $\sigma_n = 1$. Hence, $\{\sigma_n\}$ converges to 1. Therefore, $(H, M, *, u, R)$ is an R-complete fuzzy b-metric space with $u = 4$.

Define $\digamma: H \longrightarrow H$ by

$$\digamma(\sigma) = \begin{cases} \dfrac{\sigma}{4} & \text{if } \sigma \in [0, 1], \\ 0 & \text{if } \sigma \in [-1, 0). \end{cases} \tag{24}$$

Note the following:

(1) If $\sigma \in [0, 1]$ and $v \in [0, 1]$, then $\digamma(\sigma) = \sigma/4$ and $\digamma(v) = v/4$

(2) If $\sigma \in [0, 1]$ and $v \in [-1, 0)$, then $\digamma(\sigma) = \sigma/4$ and $\digamma(v) = 0$

(3) If $\sigma \in [-1, 0)$ and $\sigma \in [-1, 0)$, then $\digamma(\sigma) = 0$ and $\digamma(v) = 0$

In all cases, we have $|\digamma(\sigma)| \geq |\digamma(M)|$. Thus, $\digamma$ is an R-preserving map.

Let $\{\sigma_n\}$ be an arbitrary R-sequence in H so that $\{\sigma_n\}$ converges to $\sigma \in H$. Now,

$$\lim_{n\longrightarrow\infty} M(\sigma_n, \sigma, \tau) = \lim_{n\longrightarrow\infty} e^{-(\sigma_n-\sigma)^3/\tau}. \tag{25}$$

As $\{\sigma_n\}$ converges to $\sigma \in H$, we have $e^{-(0)^3/\tau} = e^0 = 1$.

Now, we need to show that $\lim_{n\longrightarrow\infty} M(\digamma\sigma_n, \digamma\sigma, q\tau) = 1$. For this purpose, there are some cases.

(1) Take $\sigma_n, \sigma \in [-1, 0)$; then,

$$\lim_{n\longrightarrow\infty} M(\digamma\sigma_n, \digamma\sigma, q\tau) = \lim_{n\longrightarrow\infty} M(0, 0, q\tau) = \lim_{n\longrightarrow\infty} e^0 = 1. \tag{26}$$

(2) Take $\sigma_n, \sigma \in [0, 1]$; then,

$$\begin{aligned} \lim_{n\longrightarrow\infty} M(\digamma\sigma_n, \digamma\sigma, q\tau) &= \lim_{n\longrightarrow\infty} M\left(\frac{\sigma_n}{4}, \frac{\sigma}{4}, q\tau\right) \\ &= \lim_{n\longrightarrow\infty} e^{-(\sigma_n-\sigma)^3/64q\tau}. \end{aligned} \tag{27}$$

As $\{\sigma_n\}$ converges to $\sigma \in H$, we have $e^{-(0)^3/64\tau} = e^0 = 1$.

(3) Now, take $\sigma_n \in [0, 1]$ and $\sigma \in [-1, 0)$; then,

$$\begin{aligned} \lim_{n\longrightarrow\infty} M(\digamma\sigma_n, \digamma\sigma, q\tau) &= \lim_{n\longrightarrow\infty} M\left(\frac{\sigma_n}{4}, 0, q\tau\right) \\ &= \lim_{n\longrightarrow\infty} e^{-(\sigma_n)^3/64q\tau}. \end{aligned} \tag{28}$$

As $n \longrightarrow \infty$, we can easily see $\lim_{n\longrightarrow\infty} e^{-(\sigma_n)^3/64q\tau} = e^0 = 1$.

Hence, $\digamma$ is R-continuous.

For each $\sigma, M \in H$ with $\sigma R M$, we have the following.

Case (a) For $\sigma, H \in [0, 1]$, we have

$$\begin{aligned} M(\digamma\sigma, \digamma v, q\tau) &= M\left(\frac{\sigma}{4}, \frac{v}{4}, q\tau\right) = e^{-(\sigma-M)^3/64q\tau} \\ &\geq e^{-(\sigma-M)^3/\tau} = M(\sigma, v, \tau). \end{aligned} \tag{29}$$

Case (b) For $\sigma, v \in [-1, 0)$, we have

$$\begin{aligned} M(\digamma\sigma, \digamma v, q\tau) &= M(0, 0, q\tau) = e^0 \\ &\geq e^{-(\sigma-M)^3/\tau} = M(\sigma, v, \tau). \end{aligned} \tag{30}$$

Hence, $\digamma$ is an R-contraction. Hence, by Theorem 1, $\digamma$ has a unique fixed point.

3. An Application to an Integral Equation

Within this part, we apply Theorem 1.

Let $\hbar = C([\alpha, M], \mathbb{R})$ be the set of all continuous real-valued functions defined on $[\alpha, M]$.

Now, we consider the following Fredholm-type integral equation of first kind:

$$\sigma(l) = \int_{\mathfrak{a}}^{M} F(l, \tau)\sigma(l)d\tau, \quad \text{for } l, \tau \in [\mathfrak{a}, M], \tag{31}$$

where $F \in H$. Define M as in Example 4, that is,

$$M(\sigma(l), v(l), \tau) = \sup_{l\in[\mathfrak{a},M]}\left(e^{-(\sigma(l)-M(l))^3/\tau}\right), \quad \text{for all } \sigma, v \in \text{ƕ} \text{ and } \tau > 0. \tag{32}$$

Then, $(\text{ƕ}, M, *, u, R)$ is an R-complete fuzzy b-metric space.

Theorem 2. *Assume that* $(F(l, \tau)\sigma(l) - F(l, \tau)M(l)) \leq q(\sigma(l) - M(l))$ *for* $\sigma, M \in H$, $q \in (0, 1)$, *and* $\forall l, \tau \in [\mathfrak{a}, M]$. *Also, consider* $\int_{\mathfrak{a}}^{M} d\tau = M - \mathfrak{a} = 1$. *Let* $\text{Ƒ}\colon H \longrightarrow H$ *be*

(i) R-preserving

(ii) R-contraction

(iii) R-continuous

Then, the Fredholm-type integral equation of first kind in equation (31) has a unique solution.

Proof. Define $\text{Ƒ}\colon \text{ƕ} \longrightarrow \text{ƕ}$ by

$$\text{Ƒ}\sigma(l) = \int_{\mathfrak{a}}^{M} F(l, \tau)\sigma(l)d\tau, \quad \text{for all } l, \tau \in [\mathfrak{a}, M]. \tag{33}$$

(i) Define R as $\sigma(l)\, R\, M(l) M|\sigma(l)| \geq |M(l)|$. We have $\sigma(l)\, R\, M(l)$, and we want to show that $\text{Ƒ}\sigma(l)\, R\, \text{Ƒ}M(l)$. We see that $\sigma(l)$ and $\text{Ƒ}\sigma(l)$ belong to H because Ƒ is a self-map. So, observe that if $\sigma(l)\, R\, M(l)$, then it must be $\text{Ƒ}\sigma(l)\, R\, \text{Ƒ}M(l)$. Hence, Ƒ is R-preserving.

(ii) Observe that the existence of a fixed point of Ƒ is equivalent to the existence of a solution of Fredholm-type integral equation (31). Now, for all $\sigma, M \in M$, we have

$$\begin{aligned}
M(M\sigma(l), \text{Ƒ}M(l), q\tau) &= e^{-(\text{Ƒ}\sigma(l) - \text{Ƒ}M(l))^3/q\tau} \\
&= e^{-\left(\int_{\mathfrak{a}}^{M} F(l, \tau)\sigma(l)\,d\tau\right) - \int_{\mathfrak{a}}^{M} \mathrm{F}(l, \tau)\mathrm{M}(l)\mathrm{d}\tau^3/q\tau} \\
&= e^{-\int_{\mathfrak{a}}^{M}(F(l,\tau)\sigma(l) - \mathrm{F}(l, \tau)\mathrm{M}(l))\mathrm{d}\tau/q\tau} \\
&\geq e^{-\int_{\mathfrak{a}}^{M} q(\sigma(l) - \mathrm{M}(l))\mathrm{d}\tau/q\tau} \geq \sup_{l\in[\mathfrak{a},M]}\left(e^{-q(\sigma(l)-M(l))^3\int_{\mathfrak{a}}^{M} d\tau/q\tau}\right) \\
&= \sup_{l\in[\mathfrak{a},M]}\left(e^{-(\sigma(l)-M(l))^3/\tau}\right) \\
&= M(\sigma(l), M(l), \tau).
\end{aligned} \tag{34}$$

Hence, Ƒ is an R-contraction.

(iii) Suppose $\{p_n\}$ is an R-sequence in H such that $\{p_n\}$ converges to $p \in H$. Because Ƒ is R-preserving, $\{\text{Ƒ}p_n\}$ is an R-sequence for each $n \in \mathbb{N}$. From (ii), we have

$$M(\text{Ƒ}p_n(l), \text{Ƒ}p(l), q\tau) \geq \text{Ƒ}(p_n(l), p(l), \tau). \tag{35}$$

As $\lim_{n\longrightarrow\infty} M(p_n(l), p(l), \tau) = 1$, for all $\tau > 0$, it is clear that

$$\lim_{n\longrightarrow\infty} \text{Ƒ}(Mp_n(l), \text{Ƒ}p(l), q\tau) = 1. \tag{36}$$

Hence, Ƒ is R-continuous.

Now, assume that σ and M are two fixed points of $\mathscr{F}$; then, we have

$$M(\sigma(l), M(l), \tau) \geq M\left(\sigma(l), M(l), \frac{\tau}{q}\right). \tag{37}$$

Thus, for all $n \in \mathbb{N}$,

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Retraction

Retracted: Process Monitoring for Gamma Distributed Product under Neutrosophic Statistics Using Resampling Scheme

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] A. M. Almarashi and M. Aslam, "Process Monitoring for Gamma Distributed Product under Neutrosophic Statistics Using Resampling Scheme," *Journal of Mathematics*, vol. 2021, Article ID 6635846, 12 pages, 2021.

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Research Article

Process Monitoring for Gamma Distributed Product under Neutrosophic Statistics Using Resampling Scheme

Abdullah M. Almarashi and Muhammad Aslam

Department of Statistics, Faculty of Science, King Abdulaziz University, Jeddah 21551, Saudi Arabia

Correspondence should be addressed to Muhammad Aslam; aslam_ravian@hotmail.com

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In this article, a repetitive sampling control chart for the gamma distribution under the indeterminate environment has been presented. The control chart coefficients, probability of in-control, probability of out-of-control, and average run lengths have been determined under the assumption of the symmetrical property of the normal distribution using the neutrosophic interval method. The performance of the designed chart has been evaluated using the average run length measurements under different process settings for an indeterminate environment. In-control and out-of-control nature of the proposed chart under different levels of shifts have been described. The comparison of the proposed chart has been made with the existing chart. A real-world example from the healthcare department has been included for the practical application of the proposed chart. It has been observed from the simulation study and real example that the proposed control chart is efficient in quick monitoring of the out-of-control process. It can be concluded that the proposed control chart can be applied effectively in uncertainty.

1. Introduction

The control chart is considered as the most efficient, fabulous, and powerful tool of statistical process control. The control charts have been widely used in various fields. Suman and Prajapati [1] discussed the application in the healthcare department. Zaman et al. [2] applied a control chart in the wind turbine field. Hossain et al. [3] discussed the application of a control chart for monitoring the glass fiber process. The effectiveness and efficiency of the control chart are judged by its reaction behavior against changes in its designed parameters. There are two types of changes observed in the control chart literature, i.e., common changes and special changes. Common changes also known as common causes are natural and have no threatening effect on the interested quality characteristic as compared to the special changes or special causes [4]. The early and quick detection of the special cause of variation is the prime property of any control chart which not only detects the out-of-control process quickly but also timely stops the process from producing a bulk of defective items which ultimately cause a bad impression for the producer and results in heavy losses [5]. The idea of the control chart was floated by Shewhart during the 1920s [6], and researchers are endeavoring to propose a robust control chart since its inception but remained unsuccessful. The proposed chart is an efficient struggle for the quick monitoring of the manufacturing process. The variable control chart is used when the data obtained from the measurement process and attribute control charts are applied when the data is obtained from the counting process. Abbas et al. [7] proposed the control chart for monitoring healthcare. Aslam et al. [8] designed the control chart for the process capability index. Nazir et al. [9] proposed the improved control chart for the industrial processes. Saghir et al. [10] proposed the improved control chart for modified gamma data. Saghir et al. [11] incorporated auxiliary information and repetitive sampling for the monitoring of the process.

Repetitive sampling scheme (RSS) is an efficient sampling scheme for the statistical process control techniques that attracted the attention of many researchers during the last two decades. The RSS was basically introduced by

Sherman [12] in the attribute acceptance sampling plans. The acceptance sampling plans for the normal distribution and the log-normal distribution using the variable RSS were proposed by Balamurali et al. [13]. Later on, the RSS for the variable acceptance sampling plan was developed by Balamurali and Jun [14]. The efficiency of the RSS for the average sample number is intermediate between the single sampling scheme and the probability to ratio sampling scheme Balamurali et al. [13]. Ahmad et al. [15] developed the Shewhart X-bar control chart for the RSS for monitoring the mean value of the process capability index C_p. Ahmad et al. [15] applied the RSS for the efficient monitoring of the coal quality. Azam et al. [16] developed plans for the exponentially weighted moving average regression estimators. Repetitive sampling plans based on one-sided specifications limits were presented by Yen et al. [17] Recently, Saghir et al. [10] developed a repetitive control chart for exponentially weighted moving average (EWMA) statistic using auxiliary information for monitoring process means. During the last few years, repetitive sampling has been explored by many authors including Adeoti and Olaomi [18], Aslam et al. [19], Aslam et al. [19], Aslam et al. [20], Balamurali and Jun [14], Balamurali et al. [13], Jun et al. [21], Liu and Wu [22], and Radhakrishnan and Sivakumaran [23].

In probability theory, the gamma distribution is considered as the family of two-parameter continuous probability distributions and is extremely useful in quality control literature when used under appropriate conditions. The normal probability distribution which is also very common in quality control literature but may lead to erroneous results when the shape of the underlying observations or the variable of quality of interest is unknown [24] or does not follow the normal distribution [25]. Another reason in which the normal distribution is inappropriate is the size of the collected data, particularly the single size data. However, these situations are handled by using the gamma distribution as an excellent substitute for the normal distribution in the study carried out by Khan et al. [26] and Saghir et al. [11]. In general, the gamma distribution is very common in modeling the waiting time of the events or modeling the failure time of the systems or the processes of Aksoy [27] and Saghir et al. [10]. Many other distributions such as chi-square distribution, Erlang distribution, and exponential distribution are the special cases of the gamma distribution. For larger values of the shape parameter, the gamma distribution approaches to the normal probability distribution [28]. The gamma distribution is considered as a better approximation of the interested quality characteristic when its distribution is skewed [29, 28]. Many control charts have been developed for monitoring the skewed statistic and proved to be effective and useful, for example, Jearkpaporn et al. [30] developed a monitoring scheme to detect a shift in the shape parameter, Zhang et al. [31] developed the gamma chart based on the random shift model for monitoring the out-of-control process, Chen and Yeh [32] developed an X-bar chart for nonnormal distribution using the gamma distribution, and Gonzalez and Viles [33] presented the method to monitor the variable quality characteristic using the r-chart under the gamma distribution.

Several control chart schemes have been developed for the processes having clear, certain, determined, and crisp observations of the interested quality characteristic. There are many situations when the observations are unclear, uncertain, vague, indeterminate, incomplete, and fuzzy. Bradshaw [34] developed a control chart for monitoring the observations from the fuzzy set theory. Williams and Zigli [35] proposed charts for fuzzy logic for the service industry. Taleb and Limam [36] constructed procedures for monitoring of linguistic data based on probability and fuzzy theory. Gülbay et al. [37] developed a fuzzy control chart for linguistic data. Hsieh et al. [38] explained a Poisson-based control chart for monitoring wafer defects for fuzzy theory. Sorooshian [39] investigated the fuzzy theory for monitoring attribute quality characteristics.

The neutrosophic logic which is the extension of the fuzzy logic was proposed by Smarandache [40]. The neutrosophic provides information about the measure of indeterminacy which fuzzy logic is unable to provide. Smarandache [41] discussed the generalization of intuitionistic fuzzy logic. Smarandache [42] introduced neutrosophic theory using the generalization form of the fuzzy set theory. Abu Qamar and Hassan [43] and Abu Qamar and Hassan [44] discussed Q-neutrosophic with appellations in decision-making. More information on the applications of neutrosophic logic can be found in the study carried out by Alhabib et al. [45], Abdel-Baset et al. [46], and Jana and Pal [47].

Smarandache [48] introduced the generalized class of the traditional statistics under the neutrosophic logic and called it the neutrosophic statistics. The neutrosophic statistics tend to transform to the classical statistics if all the observations are clear, certain, complete, or determined. Chen et al. [49] analyzed the scale effect and anisotropy for neutrosophic numbers of rock joint roughness coefficient based on neutrosophic statistics. Aslam [50] introduced a new sampling plan for the indeterminate environment under the process loss consideration. Aslam et al. [51] studied the indeterminate environment for testing of grouped product using the Weibull distribution. Aslam and Raza [8] developed a novel neutrosophic sampling plan for the multiple manufacturing lines using an exponentially weighted moving average and classical process capability index under the neutrosophic optimization solution method. Recently, Aslam et al. [52] designed the control chart for the gamma distribution using the indeterminate environment. More information regarding the control charts can be found in the study carried out by Intaramo and Pongpullponsak [53], Charongrattanasakul and Pongpullponsak [54], Panthong and Pongpullponsak [55], Aslam et al. [29], Aslam et al. [56], Fernández [57], Khan et al. [26], Aslam et al. [58], and Mashuri and Ahsan [59].

Average run length (ARL) is used very commonly in control chart literature as the evaluation tool of any proposed chart. ARL is defined as the average number of samples falling inside the control limits before the process shows an out-of-control condition Montgomery [4]. In a statistically controlled process, the values of neutrosophic ARL (NARL) must be larger, but for the shifted process, the smaller NARL values are preferred under the indeterminate environment for quick indication of out-of-control process and thus resulting in a smaller amount of defective items. More information about ARL can be found in the study carried out by Woodall [60], Molnau et al. [61], Kim [62], Knoth [63], Li et al. [64], Chananet et al. [65], and Phanyaem et al. [66].

In this article, a control chart scheme has been developed for a repetitive sampling scheme using the gamma distribution for the indeterminate environment with the objective that it will be an efficient monitoring scheme. To the best of the author's knowledge, no work has been done on a repetitive sampling control chart for gamma distribution using the indeterminate environment. The rest of the paper is organized as follows. The Neutrosophic gamma distribution is introduced in Section 2. The design of the proposed neutrosophic gamma distribution chart has been given in Section 3. In Section 3, the control chart for $a_N \in [3,5]$ and $b_N \in [1.9, 2.1]$ and $a_N \in [5,10]$ and $b_N \in [1.45, 1.55]$ has been discussed. In addition, tables of NARLs have been generated and the simulation study of the neutrosophic statistics has been explained. In Section 4, a comparison of the proposed chart with an existing chart has been given. In Section 5, a real example has been explained for the practical application of the proposed chart. Conclusion and the direction for future research have been given in the Section 6.

2. Neutrosophic Gamma Distribution

Let the neutrosophic failure time be $T_N \in [T_L, T_U]$, where T_L and T_U represent the indeterminacy interval of lower and upper failures of an item that follows the neutrosophic gamma distribution with neutrosophic scale parameter $b_N \in [b_L, b_U]$ and neutrosophic shape parameter $a_N \in [a_L, a_U]$. Then, the neutrosophic probability density function (npdf) of the neutrosophic gamma distribution is given as

$$f(t_N) = \frac{b_N^{a_N}}{\Gamma(a_N)} t_N^{a_N-1} e^{-b_N t_N}; \quad t_N, a_N, b_N > 0;\ a_N \in [a_L, a_U],\ b_N \in [b_L, b_U], \tag{1}$$

where $\Gamma(x)$ describes the neutrosophic gamma function; for more details, readers may refer to [20].

The resultant neutrosophic cumulative distribution (ncd) of the neutrosophic Gamma distribution (NGD) is

$$P(T_N \le t_N) = 1 - \sum_{j=1}^{a_N-1} \frac{e^{-(t_N/b_N)} (t_N/b_N)^j}{j!}; \quad T_N \in [T_L, T_U],\ a_N \in [a_L, a_U],\ b_N \in [b_L, b_U]. \tag{2}$$

It is to be noted that the NGD under the classic statistics is the generalization of the traditional gamma distribution. The mean and variance of the neutrosophic statistics can be written as

$$\mu_N = \frac{a_N}{b_N}; \quad a_N \in [a_L, a_U],\ b_N \in [b_L, b_U],$$
$$\sigma_N^2 = \frac{a_N}{b_N^2}; \quad a_N \in [a_L, a_U],\ b_N \in [b_L, b_U]. \tag{3}$$

To construct control chart, we need the neutrosophic normal distribution which is developed using the approximation developed by [67] as $T_N^* = T_N^{1/3}$ and $T_N \in [T_L, T_U]$. More information regarding neutrosophic distribution can be found in the study carried out by Smarandache [48], Peng and Dai [68], Peng and Dai [69], Aslam et al. [51], Aslam et al. [51], Aslam and Raza [8], and Aslam [50]. Then, the mean and variance of the transformed neutrosophic distribution $T_N^* \in [T_L^*, T_U^*]$ can be written as

$$\mu_{T_N^*} = \frac{b_N^{1/3}\Gamma(a_N + 1/3)}{\Gamma(a_N)}, \quad a_N \in [a_L, a_U],\ b_N \in [b_L, b_U],$$
$$\sigma_{T_N^*} = \frac{b_N^{2/3}\Gamma(a_N + 2/3)}{\Gamma(a_N)} - \left(\frac{b_N^{1/3}\Gamma(a_N + 1/3)}{\Gamma(a_N)}\right)^2, \quad a_N \in [a_L, a_U],\ b_N \in [b_L, b_U]. \tag{4}$$

3. Design of the Proposed Control Chart

In this section, we described the designing of the proposed neutrosophic control chart for the transformed variable $T_N^* = T_N^{1/3}, T_N^* \in [T_L, T_U]$. According to Wilson and Hilferty [67], the random variable $T_N^* = T_N^{1/3}$, $T_N^* \in [T_L, T_U]$, has the symmetry property of the normal probability distribution. We developed the neutrosophic control chart using the neutrosophic statistical interval method under the condition that the interested quality characteristic follows the NGD.

As mentioned by Wilson and Hilferty [67], the transformed variable $T_N^* = T_N^{1/3}, T_N^* \in [T_L, T_U]$, has the symmetry property of the neutrosophic normal distribution. We propose the following control chart under the NISM when the quality of interest follows the NGD. The following two steps have been adopted to develop the neutrosophic control chart:

(1) Determine $T_N^* = T_N^{1/3}$, where T_N^* is the transformed random variable based on the randomly selected items from the manufacturing process.
(2) Using control limits, plot T_N^*; then, declare the process as out-of-control when $T_N^* \ge \mathrm{UCL}_{1N}$ or $T_N^* \le \mathrm{LCL}_{1N}$, where $\mathrm{LCL}_{1N} \in [\mathrm{LCL}_{1L}, \mathrm{LCL}_{1U}]$ and $\mathrm{UCL}_{1N} \in [\mathrm{UCL}_{1L}, \mathrm{UCL}_{1U}]$ are neutrosophic lower and upper control limits, respectively. Note here that the decision about the process is out-of-control and is taken if T_N^* is beyond the outer of neutrosophic control limits.

The proposed neutrosophic control chart under the neutrosophic statistical interval method is the extension of the Sheu and Lin [70] control chart under the classical statistics. The proposed chart converts to Sheu and Lin [70] control chart when developed under the crisp, complete, or certain observations. Let the process lie in-control state under the neutrosophic scale parameter $b_{0N} \in [b_{0L}, b_{0U}]$. Then, the control limits of the proposed neutrosophic control chart can be developed as

$$\begin{aligned}
\mathrm{LCL}_{1N} &= \mu_{T_N^*} - k_{1N}\sigma_{T_N^*} = \frac{b_{0N}^{1/3}\Gamma(a_N + (1/3))}{\Gamma(a_N)} - k_{1N}\sqrt{\frac{b_{0N}^{2/3}\Gamma(a_N + 2/3)}{\Gamma(a_N)} - \mu_{T_N^*}^2},\\
\mathrm{LCL}_{2N} &= \mu_{T_N^*} - k_{2N}\sigma_{T_N^*} = \frac{b_{0N}^{1/3}\Gamma(a_N + (1/3))}{\Gamma(a_N)} - k_{2N}\sqrt{\frac{b_{0N}^{2/3}\Gamma(a_N + 2/3)}{\Gamma(a_N)} - \mu_{T_N^*}^2},\\
\mathrm{UCL}_{1N} &= \mu_{T_N^*} + k_{1N}\sigma_{T_N^*} = \frac{b_{0N}^{1/3}\Gamma(a_N + 1/3)}{\Gamma(a_N)} + k_{1N}\sqrt{\frac{b_{0N}^{2/3}\Gamma(a_N + 2/3)}{\Gamma(a_N)} - \mu_{T_N^*}^2},\\
\mathrm{UCL}_{2N} &= \mu_{T_N^*} + k_{2N}\sigma_{T_N^*} = \frac{b_{0N}^{1/3}\Gamma(a_N + 1/3)}{\Gamma(a_N)} + k_{2N}\sqrt{\frac{b_{0N}^{2/3}\Gamma(a_N + 2/3)}{\Gamma(a_N)} - \mu_{T_N^*}^2},
\end{aligned} \tag{5}$$

where $k_{1N} \in [k_{1L}, k_{1U}]$ and $k_{2N} \in [k_{2L}, k_{2U}]$ are the neutrosophic control limit coefficients.

Furthermore, we define

$$\begin{aligned}
\mathrm{LL}_{1N} &= \left[\frac{\Gamma(a_N + 1/3)}{\Gamma(a_N)} - k_{1N}\sqrt{\frac{\Gamma(a_N + 2/3)}{\Gamma(a_N)} - \left(\frac{\Gamma(a_N + 1/3)}{\Gamma(a_N)}\right)^2}\right],\\
\mathrm{LL}_{2N} &= \left[\frac{\Gamma(a_N + 1/3)}{\Gamma(a_N)} - k_{2N}\sqrt{\frac{\Gamma(a_N + 2/3)}{\Gamma(a_N)} - \left(\frac{\Gamma(a_N + 1/3)}{\Gamma(a_N)}\right)^2}\right],\\
\mathrm{UL}_{1N} &= \left[\frac{\Gamma(a_N + 1/3)}{\Gamma(a_N)} + k_{1N}\sqrt{\frac{\Gamma(a_N + 2/3)}{\Gamma(a_N)} - \left(\frac{\Gamma(a_N + 1/3)}{\Gamma(a_N)}\right)^2}\right],\\
\mathrm{UL}_{2N} &= \left[\frac{\Gamma(a_N + 1/3)}{\Gamma(a_N)} + k_{2N}\sqrt{\frac{\Gamma(a_N + 2/3)}{\Gamma(a_N)} - \left(\frac{\Gamma(a_N + 1/3)}{\Gamma(a_N)}\right)^2}\right].
\end{aligned} \tag{6}$$

Therefore, the neutrosophic control limits can be written as follows:

$$\begin{aligned}
\mathrm{LCL}_{1N} &= b_{0N}^{1/3}\mathrm{LL}_{1N},\\
\mathrm{LCL}_{2N} &= b_{0N}^{1/3}\mathrm{LL}_{2N},\\
\mathrm{UCL}_{1N} &= b_{0N}^{1/3}\mathrm{UL}_{1N},\\
\mathrm{UCL}_{2N} &= b_{0N}^{1/3}\mathrm{UL}_{2N}.
\end{aligned} \tag{7}$$

For a shifted process, note that a shift occurs in the neutrosophic scale parameter, whereas the shape parameter remains constant. Then, the probability under the neutrosophic statistical interval method of the in-control process when the process shows the state of in-control can be calculated as

$$P_{\mathrm{out},N}^0 = P\left(T_N^* < \mathrm{LCL}_{1N} | b_N = b_{0N}\right) + P\left(T_N^* > \mathrm{UCL}_{1N} | b_N = b_{0N}\right), \tag{8}$$

$$\text{or } P_{\mathrm{out},N}^0 = 1 - \sum_{j=1}^{a_N-1} \frac{e^{-\mathrm{LL}_{1N}^3}\left(\mathrm{LL}_{1N}^3\right)^j}{j!} + \sum_{j=1}^{a_N-1} \frac{e^{-\mathrm{UL}_{1N}^3}\left(\mathrm{UL}_{1N}^3\right)^j}{j!},$$

$$\begin{aligned}
P_{\mathrm{rep},N}^0 &= P\left(\mathrm{LCL}_{1N} < T_N^* < \mathrm{LCL}_{2N} | b_N = b_{0N}\right) + P\left(\mathrm{UCL}_{1N} < T_N^* < \mathrm{UCL}_{2N} | b_N = b_{0N}\right),\\
P_{\mathrm{rep},N}^0 &= \sum_{j=1}^{a_N-1} \frac{e^{-\mathrm{UL}_{2N}^3}\left(\mathrm{UL}_{2N}^3\right)^j}{j!} - \sum_{j=1}^{a_N-1} \frac{e^{-\mathrm{UL}_{1N}^3}\left(\mathrm{UL}_{1N}^3\right)^j}{j!} + \sum_{j=1}^{a_N-1} \frac{e^{-\mathrm{LL}_{1N}^3}\left(\mathrm{LL}_{1N}^3\right)^j}{j!} - \sum_{j=1}^{a_N-1} \frac{e^{-\mathrm{LL}_{2N}^3}\left(\mathrm{LL}_{2N}^3\right)^j}{j!}.
\end{aligned} \tag{9}$$

The probability of out-of-control under neutrosophic statistics is given by

$$P_{\mathrm{out}}^0 = \frac{P_{\mathrm{out},N}^0}{1 - P_{\mathrm{rep},N}^0}. \tag{10}$$

As mentioned earlier the ARL is used to evaluate the developed scheme for its efficiency to declare the shifted process as out-of-control quickly. So, the neutrosophic ARL (NARL) for the in-control process ARL_{0N} can be defined as

$$\mathrm{ARL}_{0N} = \frac{1}{P^0_{\mathrm{out}}}; \quad \mathrm{ARL}_{0N} \in [\mathrm{ARL}_{0L}, \mathrm{ARL}_{0U}]. \tag{11}$$

We will measure the efficiency of the proposed control chart under the neutrosophic average run length (NARL) which shows on the average when the process is out-of-control and is defined by

$$\mathrm{ARL}_{0N} = \frac{1}{P^0_{\mathrm{out}}}; \quad \mathrm{ARL}_{0N} \in [\mathrm{ARL}_{0L}, \mathrm{ARL}_{0U}]. \tag{12}$$

Let a shift occur in the process; then, the process is shifted from the targeted $b_{0N} \in [b_{0L}, b_{0U}]$ to $b_{1N} = cb_{0N}, b_{1N} \in [b_{1L}, b_{1U}]$, where the constant c shows the shift in the process. Then, the probability of the out-of-process under the neutrosophic statistical interval method can be developed as

$$P^1_{\mathrm{out},N} = P(T^*_N < \mathrm{LCL}_{1N} | b_N = cb_{0N}) + P(T^*_N > \mathrm{UCL}_{1N} | b_N = cb_{0N}), \tag{13}$$

$$\text{or } P^1_{\mathrm{out},N} = 1 - \sum_{j=1}^{a_N-1} \frac{e^{(-\mathrm{LL}^3_{1N}/c)}(\mathrm{LL}^3_{1N}/c)^j}{j!} + \sum_{j=1}^{a-1} \frac{e^{(-\mathrm{UL}^3_{1N}/c)}(\mathrm{UL}^3_{1N}/c)^j}{j!},$$

$$P^1_{\mathrm{rep},N} = P(\mathrm{LCL}_{1N} < T^*_N < \mathrm{LCL}_{2N} | b_N = cb_{0N}) + P(\mathrm{UCL}_{1N} < T^*_N < \mathrm{UCL}_{2N} | b_N = cb_{0N}),$$

$$P^1_{\mathrm{rep},N} = \sum_{j=1}^{a_N-1} \frac{e^{(-\mathrm{UL}^3_{2N}/c)}(\mathrm{UL}^3_{2N}/c)^j}{j!} - \sum_{j=1}^{a_N-1} \frac{e^{(-\mathrm{UL}^3_{1N}/c)}(\mathrm{UL}^3_{1N}/c)^j}{j!} + \sum_{j=1}^{a_N-1} \frac{e^{(-\mathrm{LL}^3_{1N}/c)}(\mathrm{LL}^3_{1N}/c)^j}{j!} - \sum_{j=1}^{a_N-1} \frac{e^{(-\mathrm{LL}^3_{2N}/c)}(\mathrm{LL}^3_{2N}/c)^j}{j!}. \tag{14}$$

The probability of out-of-control under neutrosophic statistics for the shifted process is given by

$$P^1_{\mathrm{out}} = \frac{P^1_{\mathrm{out},N}}{1 - P^1_{\mathrm{rep},N}}. \tag{15}$$

Thus, the NARL for the shifted process ARL_{1N} is defined as

$$\mathrm{ARL}_{1N} = \frac{1}{P^1_{\mathrm{out},N}}; \quad \mathrm{ARL}_{1N} \in [\mathrm{ARL}_{1L}, \mathrm{ARL}_{1U}]. \tag{16}$$

Using the abovementioned equations, the R-language code program was written to estimate the neutrosophic parameters of the proposed chart for different process settings. Tables 1 and 2 have been generated for $a_N \in [3, 5]$ and $b_N \in [1.9, 2.1]$ and $a_N \in [5, 10]$ and $b_N \in [1.45, 1.55]$ with NARL values for different shifts from 1.0 to 4.0.

Table 1 provides NARL values for the in-control $\mathrm{NARL}_0 = 200$, 300, and 370 with $ka_N = [4.594878, 5.233344]$, [5.282686, 5.430229], and [5.000939, 5.409798] and $kr_N = [1.527915, 2.881848]$, [0.3242994, 2.66222], and [0.9223276, 4.060355]. Figure 1 has been given for the plotting of $a_N \in [3, 5]$ and $b_N \in [1.9, 2.1]$.

From Tables 1 and 2, we made the following trends in NARL:

(1) As the values of the shift c increase from 1.0 to 4.0, the indeterminacy intervals $\mathrm{ARL}_{1N} \in [\mathrm{ARL}_{1L}, \mathrm{ARL}_{1U}]$ decrease

(2) As the values of $a_N \in [a_L, a_U]$ and $b_N \in [b_L, b_U]$ increase from $a_N \in [3, 5]$ and $b_N \in [1.9, 2.1]$ to $a_N \in [5, 10]$ and $b_N \in [1.45, 1.55]$, the indeterminacy intervals decrease

4. Comparison of the Proposed Chart with the Existing Chart

In this section, the comparative advantages and efficiency of the proposed chart over the existing chart of the traditional chart for gamma distribution under the indeterminacy environment have been discussed with the help of the simulated data. For the purpose of fair comparison, we fixed the same values of the process parameters. Table 3 shows the in-control NARL_0 and out-of-control NARL_1 values for different shifts from 1.0 to 4.0.

A simple comparison shows that the proposed chart has smaller NARL_1 values as compared to the existing chart [52]. From example, when $c = 1.1$, the indeterminacy intervals of NARL for the existing chart is $\mathrm{ARL}_{1N} \in [89.86, 101.98]$ and for the proposed chart is $\mathrm{ARL}_{1N} \in [80.02, 86.99]$. From this comparison, it can be concluded that the proposed control chart will indicate the shift in the process between 80th to 86th samples. On the contrary, the chart proposed by Aslam et al. [8] will indicate the shift in the process between 89th and 101st samples. Therefore, the proposed control chart has the ability to detect a shift in the process earlier than the existing control chart.

We will now discuss the efficiency of the proposed control chart over the existing control chart using the simulated data. According to the proposed chart, the process is said to out-of-control if $T^*_N \geq \mathrm{UCL}_{1N}$ or $T^*_N \leq \mathrm{LCL}_{1N}$. The first 20 observations are generated from the neutrosophic gamma distribution when the process is an in-control state.

Table 1: Neutrosophic average run length of the proposed chart for $a_N \in [3, 5]$ and $b_N \in [1.9, 2.1]$.

ka_N	[4.594878, 5.233344]	[5.282686, 5.430229]	[5.000939, 5.409798]
kr_N	[1.527915, 2.881848]	[0.3242994, 2.66222]	[0.9223276, 4.060355]
a_N	[3, 5]	[3, 5]	[3, 5]
b_N	[1.9, 2.1]	[1.9, 2.1]	[1.9, 2.1]
c		ARL_N	
1.0	[200, 200.01]	[300.01, 300]	[370, 370]
1.1	[80.02, 86.99]	[101.62, 111.28]	[149.4, 138.84]
1.2	[37.51, 43.40]	[41.19, 48.86]	[71.14, 61.11]
1.3	[19.92, 24.14]	[19.33, 24.49]	[38.39, 30.54]
1.4	[11.71, 14.68]	[10.28, 13.69]	[22.84, 16.95]
1.5	[7.50, 9.62]	[6.11, 8.40]	[14.7, 10.29]
1.6	[5.18, 6.72]	[4.01, 5.58]	[10.1, 6.75]
1.7	[3.81, 4.96]	[2.88, 3.99]	[7.32, 4.75]
1.8	[2.97, 3.85]	[2.23, 3.03]	[5.56, 3.55]
1.9	[2.42, 3.11]	[1.84, 2.43]	[4.39, 2.80]
2.0	[2.05, 2.60]	[1.59, 2.03]	[3.58, 2.31]
2.3	[1.55, 1.88]	[1.28, 1.51]	[2.44, 1.66]
2.5	[1.32, 1.54]	[1.15, 1.29]	[1.88, 1.37]
2.8	[1.18, 1.33]	[1.08, 1.16]	[1.53, 1.21]
3.0	[1.13, 1.25]	[1.06, 1.11]	[1.39, 1.15]
4.0	[1.03, 1.08]	[1.01, 1.03]	[1.12, 1.04]

Table 2: Neutrosophic average run length of the proposed chart for $a_N \in [5, 10]$ and $b_N \in [1.45, 1.55]$.

ka_N	[4.006202, 4.571112]	[3.939843, 4.788404]	[4.14799, 4.867394]
kr_N	[1.086602, 2.200099]	[2.939107, 1.818469]	[1.414571, 1.799174]
a_N	[5, 10]	[5, 10]	[5, 10]
b_N	[1.45, 1.55]	[1.45, 1.55]	[1.45, 1.55]
c		ARL_N	
1.0	[200.01, 200]	[300.01, 300.01]	[370, 370.02]
1.1	[58.89, 72.91]	[77.12, 125.12]	[91.79, 130.98]
1.2	[21.72, 31.51]	[25.26, 61.28]	[29.13, 55.15]
1.3	[9.65, 15.64]	[10.15, 33.91]	[11.36, 26.64]
1.4	[5.06, 8.74]	[4.93, 20.64]	[5.36, 14.42]
1.5	[3.09, 5.42]	[2.87, 13.56]	[3.04, 8.60]
1.6	[2.15, 3.68]	[1.97, 9.48]	[2.04, 5.60]
1.7	[1.68, 2.72]	[1.53, 6.98]	[1.57, 3.93]
1.8	[1.42, 2.15]	[1.31, 5.37]	[1.33, 2.95]
1.9	[1.27, 1.79]	[1.19, 4.29]	[1.20, 2.35]
2.0	[1.18, 1.57]	[1.12, 3.54]	[1.12, 1.96]
2.3	[1.07, 1.27]	[1.04, 2.45]	[1.04, 1.46]
2.5	[1.03, 1.15]	[1.02, 1.90]	[1.02, 1.25]
2.8	[1.01, 1.08]	[1.01, 1.56]	[1.01, 1.13]
3.0	[1.01, 1.05]	[1, 1.42]	[1, 1.09]
4.0	[1, 1.01]	[1, 1.13]	[1, 1.02]

The next 20 observations are from the out-of-control process when $c = 1.4$. The proposed control chart for simulated data is shown in Figure 1. The existing control chart for the simulated data is shown in Figure 2. From Table 1, it is expected that the shift should be detected between 16^{th} sample to 22^{nd} sample. From Figure 1, it can be seen that the proposed control chart detects a shift in the process according to expectation. The determinate part (lower value) of the statistic T_N^* is beyond UCL_{1N} between 16^{th} samples to 22^{nd} sample. We also note that several observations are within indeterminacy interval and resampling areas. On the contrary, the existing control chart does not show any shift in the process. From this simulation study, it is concluded that the proposed chart has the ability to detect a shift in the process as compared to the existing control chart.

5. Application of the Proposed Chart

In this section, we will discuss the application of the proposed control chart in the healthcare department. A large hospital management is interested to track the urinary tract infections (UTIs) patients. According to Santiago and Smith [71], "data were provided from a large hospital system concerned with a very high rate of hospital-acquired UTIs. Specifically, the hospital would like to track the frequency of patients being discharged who had acquired a UTI while in the hospital as a way to quickly identify an increase in infection rate or, conversely, monitor whether the forthcoming process or material changes result in fewer infections because the root cause often differs based on gender, male and female patients." The UTIs' data of male

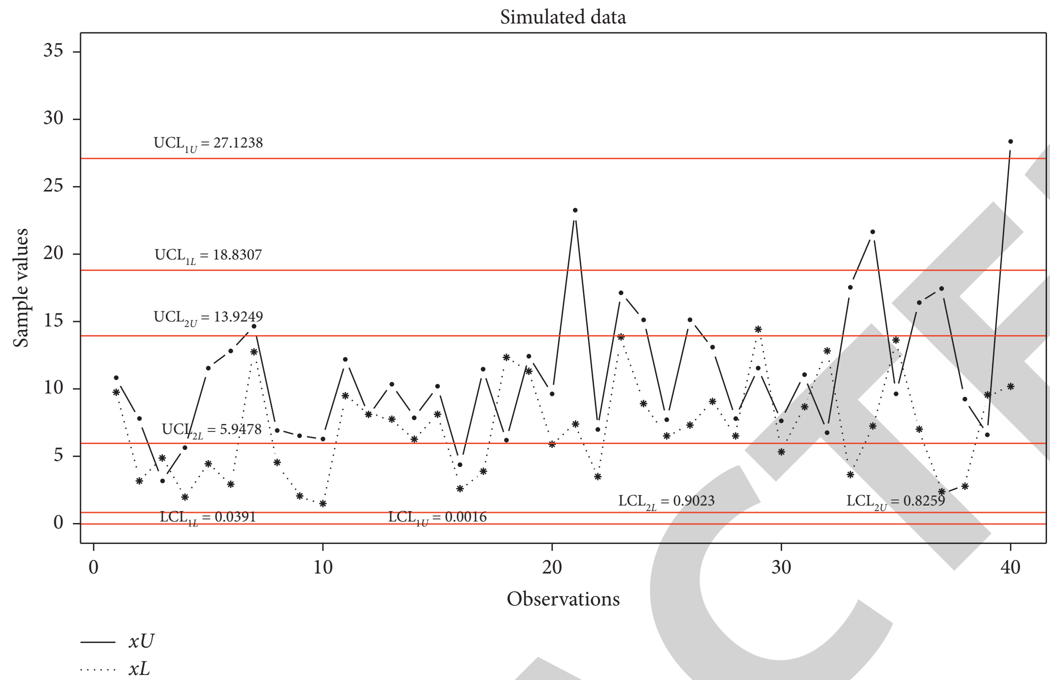

Figure 1: The proposed control for simulated data when $a_N \in [3, 5], b_N \in [1.9, 2.1], n_N \in [20, 20], k_{1N} \in [4.5948, 5.2333]$, and $k_{2N} \in [1.5279, 2.8818]$.

Table 3: Comparison of proposed control chart with neutrosophic Shewhart control chart.

Existing	Proposed	Existing	Proposed	Existing	Proposed
[200, 200]	[200, 200.01]	[300, 300]	[300.01, 300]	[370.01, 370.01]	[370, 370]
[89.86, 101.98]	[80.02, 86.99]	[128.26, 146.33]	[101.62, 111.28]	[154.21, 176.4]	[138.84, 149.4]
[47.25, 58.86]	[37.51, 43.40]	[64.73, 81.41]	[41.19, 48.86]	[76.20, 96.30]	[61.11, 71.14]
[27.98, 37.33]	[19.92, 24.14]	[37.02, 50.05]	[19.33, 24.49]	[42.82, 58.27]	[30.54, 38.39]
[18.15, 25.47]	[11.71, 14.68]	[23.32, 33.26]	[10.28, 13.69]	[26.57, 38.20]	[16.95, 22.84]
[12.64, 18.42]	[7.50, 9.62]	[15.84, 23.51]	[6.11, 8.40]	[17.82, 26.68]	[10.29, 14.70]
[9.32, 13.95]	[5.18, 6.72]	[11.43, 17.45]	[4.01, 5.58]	[12.72, 19.61]	[6.75, 10.1]
[7.2, 10.97]	[3.81, 4.96]	[8.66, 13.49]	[2.88, 3.99]	[9.54, 15.02]	[4.75, 7.32]
[5.77, 8.90]	[2.97, 3.85]	[6.83, 10.77]	[2.23, 3.03]	[7.46, 11.90]	[3.55, 5.56]
[4.77, 7.41]	[2.42, 3.11]	[5.56, 8.85]	[1.84, 2.43]	[6.03, 9.70]	[2.80, 4.39]
[4.04, 6.30]	[2.05, 2.60]	[4.65, 7.43]	[1.59, 2.03]	[5.01, 8.10]	[2.31, 3.58]
[2.92, 4.53]	[1.55, 1.88]	[3.27, 5.21]	[1.28, 1.51]	[3.47, 5.61]	[1.66, 2.44]
[2.30, 3.52]	[1.32, 1.54]	[2.53, 3.97]	[1.15, 1.29]	[2.66, 4.23]	[1.37, 1.88]
[1.88, 2.80]	[1.18, 1.33]	[2.02, 3.10]	[1.08, 1.16]	[2.10, 3.27]	[1.21, 1.53]
[1.69, 2.48]	[1.13, 1.25]	[1.81, 2.72]	[1.06, 1.11]	[1.87, 2.85]	[1.15, 1.39]
[1.27, 1.69]	[1.03, 1.08]	[1.31, 1.79]	[1.01, 1.03]	[1.33, 1.85]	[1.04, 1.12]

patients are selected from [8] and shown in Table 4. From the UTIs' data, it is clear that the data is presented in the interval. Therefore, the existing control chart proposed by [71] cannot apply for the monitoring of UTIs patients. The hospital management can apply the proposed control chart for tracking UTIs patients. Suppose that $ARL_{0N} \in [370, 370]$, $a_N \in [7.6666, 7.7777], b_N \in [1.0959, 1.1559]$, and $n_N \in [50, 50]$. The control limit coefficients are $k_{1N} \in [3.3590, 3.7703]$ and $k_{2N} \in [0.1637, 2.0479]$. Figure 3 shows the proposed control chart for UTIs patients. From Figure 3, it can be seen that two points are outside the upper control limits. Aslam et al. [8] presented a control chart for UTIs data. The neutrosophic control chart proposed by Aslam et al. [8] shows that all points are within the control limits. In addition, it can be noted from the proposed chart that several points are within the indeterminacy interval and between repetitive areas. It means that the hospital management can be indeterminate about the several observations in the UTIs data and need to repeat the process from those observations in the repetitive areas. By comparing the proposed UTIs chart with the UTIs chart proposed by Aslam et al. [8], it can be concluded that the proposed control chart clearly indicates some issues in tracking the UTIs' patient, and therefore, the hospital management should take action to bring back the process to in-control state. The proposed control chart can be applied to any other data in the same way.

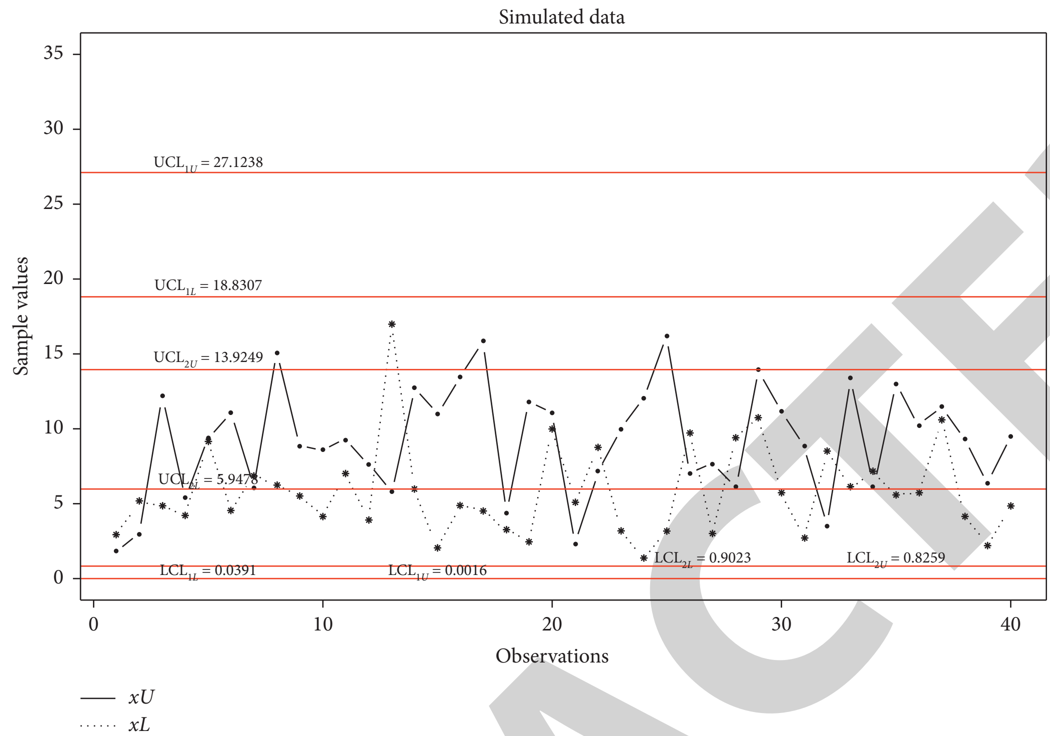

Figure 2: The existing control chart for simulated data when $a_N \in [3, 5], b_N \in [1.9, 2.1], n_N \in [20, 20], k_{1N} \in [4.5948, 5.2333]$, and $k_{2N} \in [1.5279, 2.8818]$.

Table 4: The neutrosophic UTIs' data.

Sr#	T_N	T_N^*
1	[13.13, 13.56]	[2.35, 2.38]
2	[3.57, 15.55]	[1.52, 2.49]
3	[4.31, 16.50]	[1.62, 2.54]
4	[2.76, 25.53]	[1.40, 2.94]
5	[7.75, 15.38]	[1.97, 2.48]
6	[11.45, 13.18]	[2.25, 2.36]
7	[9.20, 15.18]	[2.09, 2.47]
8	[5.51, 9.77]	[1.76, 2.13]
9	[8.18, 13.07]	[2.01, 2.35]
10	[7.07, 19.91]	[1.91, 2.71]
11	[7.35, 14.89]	[1.94, 2.46]
12	[5.62, 11.09]	[1.77, 2.23]
13	[8.38, 16.72]	[2.03, 2.55]
14	[9.49, 10.06]	[2.11, 2.15]
15	[4.90, 23.67]	[1.69, 2.87]
16	[4.45, 14.68]	[1.64, 2.44]
17	[7.11, 16.44]	[1.92, 2.54]
18	[9.37, 15.95]	[2.10, 2.51]
19	[12.00, 16.38]	[2.28, 2.53]
20	[7.41, 16.62]	[1.95, 2.55]
21	[10.64, 15.15]	[2.19, 2.47]
22	[6.63, 11.21]	[1.87, 2.23]
23	[2.87, 14.27]	[1.42, 2.42]
24	[6.87, 10.37]	[1.90, 2.18]
25	[6.16, 18.85]	[1.83, 2.66]
26	[6.53, 12.47]	[1.87, 2.31]
27	[6.85, 12.13]	[1.89, 2.29]
28	[8.08, 22.69]	[2.00, 2.83]
29	[11.61, 17.14]	[2.26, 2.57]

Table 4: Continued.

Sr#	T_N	T_N^*
30	[3.98, 17.16]	[1.58, 2.57]
31	[6.81, 17.25]	[1.89, 2.58]
32	[4.42, 12.53]	[1.64, 2.32]
33	[6.53, 13.96]	[1.86, 2.40]
34	[8.73, 9.30]	[2.05, 2.10]
35	[5.37, 9.43]	[1.75, 2.11]
36	[8.44, 6.35]	[2.03, 1.85]
37	[11.79, 17.01]	[2.27, 2.57]
38	[5.33, 14.90]	[1.74, 2.46]
39	[4.20, 21.20]	[1.61, 2.76]
40	[5.74, 11.95]	[1.79, 2.28]
41	[5.24, 11.09]	[1.73, 2.23]
42	[5.10, 10.10]	[1.72, 2.16]
43	[9.11, 24.54]	[2.08, 2.90]
44	[8.39, 10.21]	[2.03, 2.16]
45	[5.33, 18.03]	[1.74, 2.62]
46	[7.90, 11.43]	[1.99, 2.25]
47	[3.62, 13.00]	[1.53, 2.35]
48	[5.01, 13.62]	[1.71, 2.38]
49	[4.09, 12.88]	[1.60, 2.34]
50	[9.38, 17.45]	[2.10, 2.59]

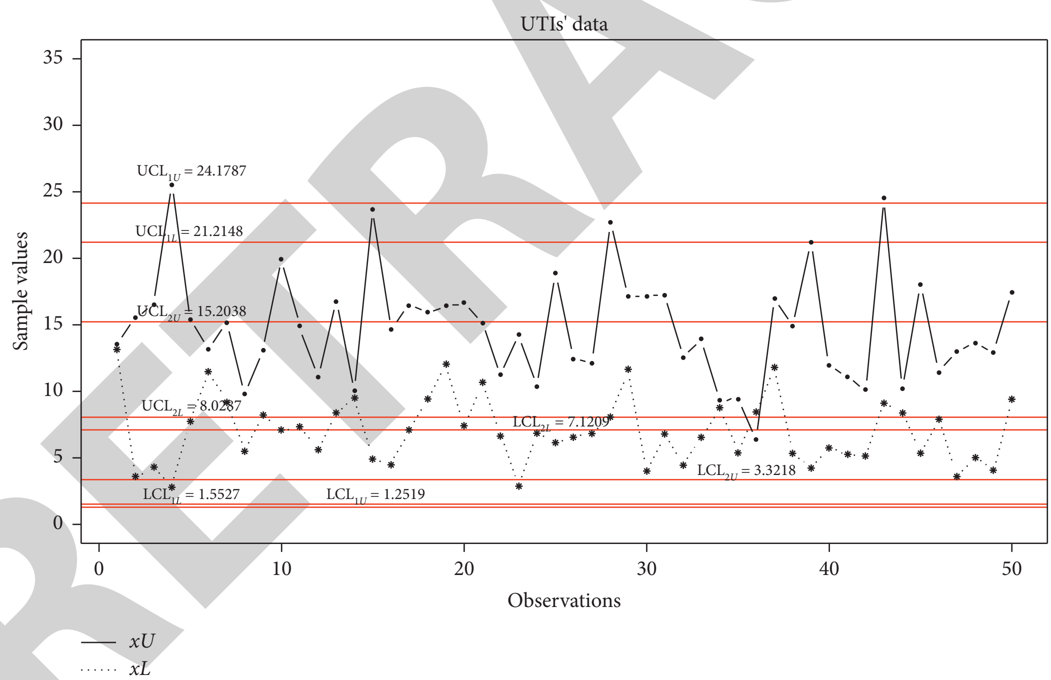

Figure 3: The proposed control chart for UTIs' patients.

6. Concluding Remarks

In this article, we presented the control chart using repetitive sampling under neutrosophic statistics when the data follow the gamma distribution. We presented some necessary measures to evaluate the proposed control chart. A simulation study and real example from the healthcare were included to show the efficiency of the proposed control chart over the existing control chart. From the study, it is observed that the proposed chart is an efficient addition in the tool kit of the quality control personnel. The proposed scheme can be extended for the multivariate case as future research. The proposed control using some other transformation for nonnormal distribution and different datasets can be considered as future research. The proposed chart using the cost model can be studied as future research. The proposed control chart for monitoring imbalanced data can be considered as future research.

Data Availability

The data use to support the findings of the study are included within the article.

Conflicts of Interest

The authors declare no conflicts of interest regarding this paper.

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Retraction

Retracted: Some Novel Cosine Similarity Measures Based on Complex Hesitant Fuzzy Sets and Their Applications

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] R. Chinram, T. Mahmood, U. Ur Rehman, Z. Ali, and A. Iampan, "Some Novel Cosine Similarity Measures Based on Complex Hesitant Fuzzy Sets and Their Applications," *Journal of Mathematics*, vol. 2021, Article ID 6690728, 20 pages, 2021.

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Research Article

Some Novel Cosine Similarity Measures Based on Complex Hesitant Fuzzy Sets and Their Applications

Ronnason Chinram [1] **Tahir Mahmood** [2] **Ubaid Ur Rehman** [2] **Zeeshan Ali** [2] **and Aiyared Iampan** [3]

[1] *Algebra and Applications Research Unit, Faculty of Science, Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand*
[2] *Department of Mathematics and Statistics, International Islamic University Islamabad, Islamabad, Pakistan*
[3] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Phayao 56000, Thailand*

Correspondence should be addressed to Tahir Mahmood; tahirbakhat@iiu.edu.pk

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The theory of complex hesitant fuzzy set (CHFS) is a modification technique of the complex fuzzy set (CFS) to cope with awkward and unreliable information's in daily life issues. CHFS contains the grade of truth in the form of complex number, whose real and imaginary parts are in the form of the finite subset of the unit interval. CHFS is the mixture of hesitant fuzzy set (HFS) and CFS, which handles the complex and uncertain information in real-world issues which is compared with fuzzy sets and complex fuzzy sets. The positive membership in CHFS is in the form a polar coordinate belonging to unit disc in the complex plane. The aims of this manuscript are to explore some similarity measures (SMs), weighted SMs (WSMs) such as cosine SMs, weighted cosine SMs, SMs based on cosine function, WSMs based on cosine function, SMs based on tangent function, and WSMs based on tangent function of CHFS. Some special cases of the presented measures are discussed in detail. Moreover, we use our described SMs and weighted SMs of CHFS in the environment of medical diagnosis and pattern recognition to assess the practicality and competence of the described SMs. Finally, to find the validity and proficiency of the investigated measures based on CHFSs, the comparison between explored measures with some already defined measures and their graphical representations are also discussed in detail.

1. Introduction

The fuzzy set (FS) is the modification of crisp set which was given by Zadeh [1] to manage the vagueness and uncertainty in the information in real-life decisions. In the theory of FS, the positive grade belongs to closed interval [0, 1], where greatest value designated greatest positive grade. FS has numerous applications in various fields [2–4]. Bustince et al. [5] operated on FSs and their models, extensions portrayal, and aggregation. SMs between FSs play an essential role in the theory of FS, which attracted a lot of attention from the authors. SMs have a lot of applications in real-world problems and are extremely useful in numerous fields [6, 7]. Chen [8] interpreted the similarity function to find the similarity degree among FSs. Pedrycz [9] presented fuzzy control and fuzzy systems. FSs in pattern recognition, methodology, and methods are also presented by Pedrycz [10]. Rangel-Valdez et al. [11] described parallel designs for metaheuristics that solve portfolio selection problems using fuzzy outranking relations. Mahmood [12] described a novel approach towards bipolar soft set and their applications.

Numerous authors mentioned the issue, what will be the impact when we alter the range of FS into a unit circle of a complex plane. To deal with such sorts of circumstance, Ramot et al. [13] described the notion of CFS as a modification of FS to handle the complex and tricky data in real-world. The idea of CFS is represented by complex-valued positive grade, which carries two-dimensional data in a particular set. Moreover, Tamir et al. [14] presented the Cartesian form of CFS and the Cartesian complex fuzzy positive grade where both real and imaginary parts carry the fuzzy data. In polar portrayal, the fuzzy data carries the phase value and absolute value of complex positive grade. The complex fuzzy number is not the same as the CFS. The δ-equalities and operation properties of CFS were introduced by Zhang et al. [15].

HFS are the significant expansions of the theory of FS. Torra [16, 17] described the notion of HFS. An HFS is represented by positive grade which is in the shape of a finite subset of closed interval $[0, 1]$. Torra and Narukawa [17] characterized some fundamental operations on HFS. Rodriguez et al. [18] built up the idea of hesitant fuzzy linguistic term sets. Farhadinia [19] interpreted the idea of similarity and distance measures for higher order HFS. The notion of hesitant fuzzy data aggregation in decision-making (DM) was described by Xia and Xu [20]. Wei et al. [21] interpreted the idea of hesitant fuzzy Choquet aggregation operators and their applications to multiple attribute DM (MADM). Zhang [22] characterized hesitant fuzzy aggregation operators and their application to MADM. Xu and Xia [23] explored the idea of separation and correlation measures of HFS. Zhu et al. [24] gave the idea of hesitant fuzzy geometric Bonferroni means. Herrera et al. [25] described HFSs, an emerging tool in decision-making. A review of HFSs, quantitative and qualitative extensions, was explored by Rodriguez et al. [26]. Li et al. [27] described the consistency of hesitant fuzzy linguistic preference relations. Muhiuddin et al. [28] interpreted the generalized hesitant fuzzy ideals in semigroups.

The idea of similarity is a fundamental idea in human cognizance. Similarity has a key role in recognition, taxonomy, and several different fields. There are numerous aspects of the notion of the similarity that have escaped formalization. As per (HFS) detailing of a substantial, broadly useful definition of similarity is a difficult issue. There does not exist a legitimate, universally useful definition of similarity. There exist numerous specific definitions that have been utilized with accomplishment in diagnostics, classification, cluster analysis, and recognition. There are a few comparability measures that are interpreted and utilized for different purposes [29]. The SMs are categorized into 3 classifications: (1) measures based on implicators. (2) Measure based on metric. (3) Measure based on set-theoretic. While managing SMs based on distance, examples have been developed for perceptual similarity where each distance adage is obviously damaged by dissimilarity measures, especially the triangle inequality [17], and thusly the relating SM ignores transitivity. This model hypothesizes that the perceptual distance fulfils the metric adages, the observational legitimacy of which has been tentatively tested by a few authors, especially the triangle inequality (for subtleties see [16] and [17, 30, 31]). Thus, in the event of set-theoretic SMs, it is seen that crisp transitivity is a lot more grounded condition to be put upon SM. Set-theoretic SMs are additionally partitioned in three gatherings: (i) measures dependent on crisp logic; (ii) measures dependent on fuzzy logic; (iii) measures dependent on HFSs.

In this paper, we present complex HFSs. The inspiration is that when characterizing the positive grade of the element, the struggle of establishing the positive grade is not as we have margin of error (as in complex intuitionistic FS [32]), or some chance circulations (as in type 2 CFSs) on the probable values. In the existing theories, numerous scholars have faced several troubles. When a decision-maker provides such types of information for the grade of truth in the form of $0.22e^{(0.3)}$ and $0.5e^{(0.31)}$, this circumstance can emerge in a multicriteria DM. Basically, the theory of complex hesitant fuzzy set contains the grade of truth in the form of complex number, whose real and imaginary parts are in the form of the finite subset of the unit interval. In this unique situation, rather than considering only an aggregation operator [12], it is helpful to manage all the possible values. This circumstance, as we will talk about later, can be demonstrated utilizing multisets. Therefore, the existing theories are not able to cope with such types of troubles. The investigated ideas are more able to cope with it effectivelyg.

Due to this and preserving the advantages of the SMs, in this manuscript, the notion of CHFS is explored, which is the fusion of HFS and CFS to manage the uncertainty and complicated data in real world. The positive membership in CHFS is in the form of a finite subset of unit disc in the complex plane. Moreover, in this manuscript, we interpreted some similarity measures (SMs) and weighted SMs (WSMs). Additionally, we use our explored SMs and weighted SMs of CHFS in the environment of medical diagnosis and pattern recognition to assess the practicality and competence of the described SMs. The comparison between explored measures with some already defined measures and their graphical representations are also discussed in detail.

The structure of this manuscript is given as follows: in Section 2 of this manuscript, we present preliminaries. In Section 3, the notion of the CHFS and its fundamental properties are explored. In Section 4 of this manuscript, we explore some similarity measures (SMs) and weighted SMs (WSMs) of CHFS. In Section 5, we use proposed SMs and weighted SMs in the environment of medical diagnosis and pattern recognition. The comparison between explored measure with some already defined measures and their graphical representations are also discussed in detail in Section 6. In Section 7, we discuss the conclusion of the article.

2. Preliminaries

In this section, we revise fundamental definitions such as FS, CFS, and HFS. Throughout this paper, x denotes the fix set.

Definition 1 (see [1]). An FS E is of the shape,

$$E = \{(x, \mu_{\mathrm{E}}(x)) | x \in \chi\}, \tag{1}$$

with a condition $0 \le \mu_{\mathrm{E}}(x) \le 1$, where $\mu_E(x)$ stands for the grade of membership. Throughout this paper, the family of all FSs on X are designated by $\mathrm{FS}(X)$. The pair $E = (x, \mu_E(x))$ is said to be fuzzy number (FN).

Definition 2 (see [13]). A CFS E is of the shape,

$$E = \{(x, \mu_E(x)) | x \in \chi\}, \tag{2}$$

where $\mu_E(x) = \gamma_E(x).e^{i2\pi(\omega\gamma_E^{(x)})}$ stands for the complex-valued membership grade in the shape of polar coordinate, where

$\gamma_E(x), \omega_{\gamma_E}(x) \in [0,1]$. Moreover, the pair $E = (x, \gamma_E(x).e^{i2\pi(\omega\gamma_E^{(x)})})$ is said to be complex fuzzy number (CFN).

Definition 3 (see [16, 17]). An HFS E is of the shape,

$$E = \{(x, \mu_{\mathrm{E}}(x)) | x \in \chi\}, \tag{3}$$

where $\mu_E(x)$ is a finite subset of $[0,1]$ standing for the grade of membership for every element $x \in \chi$. Moreover, the pair $E = (x, \mu_E(x))$ is said to be hesitant fuzzy number (HFN).

Definition 4 (see [29]). For any two HFSs E and F, the SM $\mathbb{D}(E,F)$ fulfils the following axioms:

(1) $0 \le \mathbb{S}(E,F) \le 1$;
(2) $\mathbb{S}(E,F) = 1 \Longleftrightarrow E = F$;
(3) $\mathbb{S}(E,F) = \mathbb{S}(F,E)$.

Definition 5 (see [29]). For any two HFSs E and F, the distance measure $\mathbb{d}(E,F)$ fulfils the following properties:

(1) $0 \le \mathbb{D}(E,F) \le 1$;
(2) $\mathbb{D}(E,F) = 1 \Longleftrightarrow E = F$;
(3) $\mathbb{D}(E,F) = \mathbb{D}(F,E)$.

From the discussion we did above, we get that the $\mathbb{S}(E,F) = 1 = \mathbb{D}(E,F)$.

3. Complex Hesitant Fuzzy Sets

In this section, we explored the notion of complex hesitant fuzzy sets (CHFSs) and some of its properties.

Definition 6. A CHFS E is of the shape,

$$E = \{(x, \mu_E(x)) | x \in X\}, \tag{4}$$

where

$$\begin{aligned}\mu_{\mathrm{E}}(x) &= \left\{\gamma_{E_j}(x).e^{i2\pi\left(\omega\gamma_{E_j}^{(x)}\right)}, \quad j = 1,2,3,\ldots,n\right\}\\ &= \left\{\gamma_{E_1}(x).e^{i2\pi\left(\omega_{\gamma_{E1}}(x)\right)}, \gamma E_2(x).e^{i2\pi\left(\omega_{\gamma_{E2}}(x)\right)}, \ldots \ \ldots, \gamma E_n(x).e^{i2\pi\left(\omega_{\gamma_{En}}(x)\right)}\right\},\end{aligned} \tag{5}$$

expressed the complex-valued grade of membership which is the subset of unit disc in complex plane with acondition $\gamma_{E_j}(x), \omega_{\gamma E_j}(x) \in [0,1]$. Further, $E = (x, \gamma_{E_j}(x).e^{i2\pi(\omega_{\gamma_{E_j}}^x)})$ is known as the complex hesitant fuzzy number (CHFN).

Definition 7. Let $E = (x, \gamma_{E_j}(x).e^{i2\pi(\omega_{\gamma_{E_j}}^x)})$ and $F = (x, \gamma_{E_j}(x).e^{i2\pi(\omega_{\gamma_{E_j}}^x)})$ be two CHFNs. Then,

(1) $c(\gamma_E(x)) = \left\{(x, \left\{1 - \gamma_{E_j}(x)\right\}.e^{i2\pi(\left\{1-\omega_{\gamma_{E_j}}^{(x)}\right\})}\right\}$;
(2) $E \cup F = \left\{(x, \max(\gamma_{E_j}(x), \gamma_{F_j}(x)).e^{i2\pi(\max(\omega_{\gamma E_j}(x), \omega_{\gamma E_j}))})\right\}$;
(3) $E \cap F = \left\{(x, \min(\gamma_{E_j}(x), \gamma_{F_j}(x)).e^{i2\pi(\min(\omega_{\gamma E_j}(x), \omega_{\gamma E_j}))})\right\}$.

The theory of CHFS is a powerful tool to deal with unsure and complicated data in real-world issues. The CHFS holds the grade of membership in the shape of a finite subset of the unit disc in the complex plane, whose entities are in the shape of polar coordinates. Essentially, the CHFS holds two-dimensional data in a particular set. The explored CHFS is more general than the existing notions such as FS, CFS, and HFS.

Example 1. Let $E = \left\{\begin{matrix}(x_1, \{0.8e^{i2\pi(0.9)}, 0.6e^{i2\pi(0.5)}\}), (x_2, \{0.5e^{i2\pi(0.7)}, 0.2e^{i2\pi(1)}, 1e^{i2\pi(0.5)}\}), (x_3, \{0.1e^{i2\pi(0.2)}\})\\ (x_4, \{0.4e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.4)}\}), (x_5, \{0.1e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.5)}\})\end{matrix}\right\}$ and $F = \left\{\begin{matrix}(x_1, \{0.5e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.4)}, 1e^{i2\pi(0.5)}\}), (x_2, \{0.4e^{i2\pi(0.6)}\}), (x_3, \{0.8e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.8)}\})\\ (x_4, \{0.9e^{i2\pi(0.6)}, 0.4e^{i2\pi(0.3)}\}), (x_5, \{0.5e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.5)}\}),\end{matrix}\right\}$ be two CHFSs. Then,

(1) $E^c = \left\{\begin{matrix}\{0.2e^{i2\pi(0.1)}, 0.4e^{i2\pi(0.5)}\}, \{0.5e^{i2\pi(0.3)}, 0.8e^{i2\pi(0.0)}, 0.0e^{i2\pi(0.5)}\}, \{0.9e^{i2\pi(0.8)}\}\\ \{0.6e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.4)}, 0.7e^{i2\pi(0.6)}\}, \{0.9e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.5)}\}\end{matrix}\right\}$;
(2) $E \cup F = \left\{\begin{matrix}\{0.8e^{i2\pi(0.9)}, 0.7e^{i2\pi(0.5)}\}, \{0.5e^{i2\pi(0.7)}, 0.2e^{i2\pi(1)}, 1e^{i2\pi(0.5)}\}, \{0.8e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.8)}\}\\ \{0.9e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.4)}\}, \{0.5e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.5)}\}\end{matrix}\right\}$;
(3) $E \cap F = \left\{\begin{matrix}\{0.5e^{i2\pi(0.8)}, 0.6e^{i2\pi(0.4)}\}, \{0.4e^{i2\pi(0.6)}\}, \{0.1e^{i2\pi(0.2)}\}\\ \{0.4e^{i2\pi(0.5)}, 0.4e^{i2\pi(0.3)}\}, \{0.1e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.5)}\}\end{matrix}\right\}$;

4. Similarity Measures Based on the Cosine Function for CHFSs

In this section, we interpreted some SMs such as cosine SMs for CHFSs, SMs of CHFSs based on cosine function, and SMs of CHFSs based on cotangent function.

Definition 8. Let E and F be two CHFSs on set X. Then, SM between E and F is represented by $\mathbb{S}_c(E,F)$, which fulfils the following postulate:

(1) $0 \le \mathbb{S}_c(E,F) \le 1$;
(2) $\mathbb{S}_c(E,F) = 1$ if and only if $E = F$;
(3) $\mathbb{S}_c(E,F) = \mathbb{S}_c(F,E)$.

4.1. Cosine Similarity Measures for CHFS

4.1. Cosine Similarity Measures for CHFS. Let E be a CHFS on a set X. Then, the elements contained in CHFS can be presented as the function of membership degree $\mu_E(x)$, which is a subset of a unit disc in a complex plane. Consequently, a cosine SM and weighted cosine SM with CHF data are expressed similarly to the cosine SM based on Bhattacharya's distance [33].

Definition 9. Let E and F be two CHFSs on a set X. Then, the cosine SM between E and F can be presented as

$$\mathbb{S}_c^1(E,F)=\frac{1}{n}\sum_{\kappa=1}^{n}\left(\frac{(1/\mathbb{L})\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_\kappa)\cdot\gamma_{F_j}(x_\kappa)+(1/\mathbb{L})\sum_{j}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_\kappa)\cdot\omega_{\gamma_{F_j}}(x_\kappa)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa)+(1/\mathbb{L})\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}\sqrt{(1/\mathbb{L})\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa)+(1/\mathbb{L})\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}}\right). \tag{6}$$

In Definition 9, if we assume the imaginary parts zero, then the interpreted SM transforms for HFS. Likewise, if we assume the CHFS as a singleton set, then the interpreted SM transforms for CFS. Moreover, if we assume the CHFS as a singleton set and the imaginary part zero, then the interpreted SM transforms for FS. Its structure makes it important and expert to deal with unknown and undependable data in real decision theory.

Theorem 1. *The SM $\mathbb{S}_c^1(E,F)$ fulfils the following postulates:*

(1) $0\le\mathbb{S}_c^1(E,F)\le 1$;

(2) $\mathbb{S}_c^1(E,F)=1$ *if* $E=F$;

(3) $\mathbb{S}_c^1(E,F)=\mathbb{S}_c^1(F,E)$.

Proof

(1) Since $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_\kappa)\cdot\gamma_{F_j}(x_\kappa)\in[0,1]$, $1/\mathbb{L}\sum_{j}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_\kappa)\cdot\omega_{\gamma_{F_j}}(x_\kappa)\in[0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa)\in[0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)\in[0,1]$, and $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa)\in[0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)\in[0,1]$ and denominator will always remain greater than the nominator. So, for $\kappa=1$, we have

$$\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_1)\cdot\gamma_{F_j}(x_1)+1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_1)\cdot\omega_{\gamma_{F_j}}(x_1)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_1)+1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_1)}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_1)+1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_1)}}\in[0,1]. \tag{7}$$

For $\kappa=2$, we have

$$\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_2)\cdot\gamma_{F_j}(x_2)+1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_2)\cdot\omega_{\gamma_{F_j}}(x_2)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_2)+1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_2)}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_2)+1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_2)}}\in[0,1]. \tag{8}$$

By continuing this procedure, we obtain

$$\sum_{\kappa=1}^{n}\left(\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_\kappa)\cdot\gamma_{F_j}(x_\kappa)+1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_\kappa)\cdot\omega_{\gamma_{F_j}}(x_\kappa)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa)+1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa)+1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}}\right)\in n[0,1]. \tag{9}$$

This implies that

$$
\begin{aligned}
&0 \leq \sum_{\kappa=1}^{n}\left(\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_\kappa)\cdot\gamma_{F_j}(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_\kappa)\cdot\omega_{\gamma_{F_j}}(x_\kappa)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}}\right) \leq n,\\
&0 \leq \frac{1}{2}\sum_{\kappa=1}^{n}\left(\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_\kappa)\cdot\gamma_{F_j}(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_\kappa)\cdot\omega_{\gamma_{F_j}}(x_\kappa)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}}\right) \leq 1,
\end{aligned}
\tag{10}
$$

which implies that

$$
0 \leq \mathbb{S}_c^1(x_\kappa) \leq 1. \tag{11}
$$

(2) We have

$$
\mathbb{S}_c^1(E,F) = \frac{1}{n}\sum_{\kappa=1}^{n}\left(\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_\kappa)\cdot\gamma_{F_j}(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_\kappa)\cdot\omega_{\gamma_{F_j}}(x_\kappa)}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}}\right). \tag{12}
$$

Now, as $E = F \Longrightarrow \mu_E(x_\kappa) = \mu_F(x_\kappa)$, for $\kappa = 1, 2, \ldots, n \Longrightarrow \gamma_{E_j}(x_\kappa)e^{i2\pi(\omega_{E_j}(x_\kappa))} = \gamma_{F_j}(x_\kappa)e^{i2\pi(\omega_{F_j}(x_\kappa))}$ for $\kappa = 1, 2, \ldots, n \Longrightarrow \gamma_{E_j}(x_\kappa) = \gamma_{F_j}(x_\kappa)$ and $e^{i2\pi(\omega_{E_j}(x_\kappa))} = e^{i2\pi(\omega_{F_j}(x_\kappa))}$ for $\kappa = 1, 2, \ldots, n$. Then,

$$
\begin{aligned}
&\mathbb{S}_c^1(E,F) = \frac{1}{n}\sum_{\kappa=1}^{n}\left(\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}{\left(\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_\kappa) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_\kappa)}\right)^2}\right).\\
&\mathbb{S}_c^1(E,F) = \frac{1}{n}\sum_{\kappa=1}^{n}\left(\frac{1/\mathbb{L}\left(\gamma_{E_1}^2(x_\kappa) + \gamma_{E_2}^2(x_\kappa) + \cdots + \gamma_{E_\mathbb{L}}^2(x_\kappa)\right) + 1/\mathbb{L}\left(\omega_{\gamma_{E_1}}^2(x_\kappa) + \omega_{\gamma_{E_2}}^2(x_\kappa) + \cdots + \omega_{\gamma_{E_\mathbb{L}}}^2(x_\kappa)\right)}{1/\mathbb{L}\left(\gamma_{E_1}^2(x_\kappa) + \gamma_{E_2}^2(x_\kappa) + \cdots + \gamma_{E_\mathbb{L}}^2(x_\kappa)\right) + 1/\mathbb{L}\left(\omega_{\gamma_{E_1}}^2(x_\kappa) + \omega_{\gamma_{E_2}}^2(x_\kappa) + \cdots + \omega_{\gamma_{E_\mathbb{L}}}^2(x_\kappa)\right)}\right).
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\mathbb{S}_c^1(E,F) = \frac{1}{n}\Bigg[&\frac{1/\mathbb{L}\left(\gamma_{E_1}^2(x_1) + \gamma_{E_2}^2(x_1) + \cdots + \gamma_{E_\mathbb{L}}^2(x_1)\right) + 1/\mathbb{L}\left(\omega_{\gamma_{E_1}}^2(x_1) + \omega_{\gamma_{E_2}}^2(x_1) + \cdots + \omega_{\gamma_{E_\mathbb{L}}}^2(x_1)\right)}{1/\mathbb{L}\left(\gamma_{E_1}^2(x_1) + \gamma_{E_2}^2(x_1) + \cdots + \gamma_{E_\mathbb{L}}^2(x_1)\right) + 1/\mathbb{L}\left(\omega_{\gamma_{E_1}}^2(x_1) + \omega_{\gamma_{E_2}}^2(x_1) + \cdots + \omega_{\gamma_{E_\mathbb{L}}}^2(x_1)\right)}\\
&+ \frac{1/\mathbb{L}\left(\gamma_{E_1}^2(x_2) + \gamma_{E_2}^2(x_2) + \cdots + \gamma_{E_\mathbb{L}}^2(x_2)\right) + 1/\mathbb{L}\left(\omega_{\gamma_{E_1}}^2(x_2) + \omega_{\gamma_{E_2}}^2(x_2) + \cdots + \omega_{\gamma_{E_\mathbb{L}}}^2(x_2)\right)}{1/\mathbb{L}\left(\gamma_{E_1}^2(x_2) + \gamma_{E_2}^2(x_2) + \cdots + \gamma_{E_\mathbb{L}}^2(x_2)\right) + 1/\mathbb{L}\left(\omega_{\gamma_{E_1}}^2(x_2) + \omega_{\gamma_{E_2}}^2(x_2) + \cdots + \omega_{\gamma_{E_\mathbb{L}}}^2(x_2)\right)} + \cdots\\
&+ \frac{1/\mathbb{L}\left(\gamma_{E_1}^2(x_n) + \gamma_{E_2}^2(x_n) + \cdots + \gamma_{E_\mathbb{L}}^2(x_n)\right) + 1/\mathbb{L}\left(\omega_{\gamma_{E_1}}^2(x_n) + \omega_{\gamma_{E_2}}^2(x_n) + \cdots + \omega_{\gamma_{E_\mathbb{L}}}^2(x_n)\right)}{1/\mathbb{L}\left(\gamma_{E_1}^2(x_n) + \gamma_{E_2}^2(x_n) + \cdots + \gamma_{E_\mathbb{L}}^2(x_n)\right) + 1/\mathbb{L}\left(\omega_{\gamma_{E_1}}^2(x_n) + \omega_{\gamma_{E_2}}^2(x_n) + \cdots + \omega_{\gamma_{E_\mathbb{L}}}^2(x_n)\right)}\Bigg],\\
S_c^1(E,F) = 1.&
\end{aligned}
\tag{14}
$$

(3) We have

$$\begin{aligned}\mathbb{S}_c^1(E,F) &= \frac{1}{n}\sum_{\mathcal{K}=1}^{n}\left(\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_{\mathcal{K}}).\gamma_{F_j}(x_{\mathcal{K}}) + 1/\mathbb{L}\sum_{j}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_{\mathcal{K}}).\omega_{\gamma_{F_j}}(x_{\mathcal{K}})}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_{\mathcal{K}})}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{F_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{F_j}}^2(x_{\mathcal{K}})}}\right)\\ &= \frac{1}{n}\sum_{\mathcal{K}=1}^{n}\left(\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{F_j}(x_{\mathcal{K}}).\gamma_{E_j}(x_{\mathcal{K}}) + 1/\mathbb{L}\sum_{j}^{\mathbb{L}}\omega_{\gamma_{F_j}}(x_{\mathcal{K}}).\omega_{\gamma_{E_j}}(x_{\mathcal{K}})}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{F_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{F_j}}^2(x_{\mathcal{K}})}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_{\mathcal{K}})}}\right),\\ &= \mathbb{S}_c^1(F,E).\end{aligned} \tag{15}$$

We defined distance measure of the angle as $d(E,F) = \arccos(\mathbb{S}_c^1(E,F))$. It holds the following axioms:

(1) $d(E,F) \geq 0$ if $0 \leq \mathbb{S}_c^1(F,E) \leq 1$;

(2) $d(E,F) = \arccos(1) = 0$ if $\mathbb{S}_c^1(E,F) = 1$;

(3) $d(E,F) = d(F,E)$ if $\mathbb{S}_c(E,F) = \mathbb{S}_c(F,E)$. □

Definition 10. Let E and F be two CHFSs on a set X. Then, the weighted cosine SM between E and F can be presented as

$$\mathbb{S}_{cw}^1(E,F) = \sum_{\mathcal{K}=1}^{n} w_{\mathcal{K}}\frac{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}(x_{\mathcal{K}}).\gamma_{F_j}(x_{\mathcal{K}}) + 1/\mathbb{L}\sum_{j}^{\mathbb{L}}\omega_{\gamma_{E_j}}(x_{\mathcal{K}}).\omega_{\gamma_{F_j}}(x_{\mathcal{K}})}{\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{E_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{E_j}}^2(x_{\mathcal{K}})}\sqrt{1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\gamma_{F_j}^2(x_{\mathcal{K}}) + 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}\omega_{\gamma_{F_j}}^2(x_{\mathcal{K}})}}, \tag{16}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathcal{T}}$ represents the weight vector of every element $x_{\mathcal{K}}\,(\mathcal{K} = 1, 2, \ldots, .n)$ included in CHFS and the weight vector satisfies $w_{\mathcal{K}} \in [0,1]$ for every $\mathcal{K} = 1, 2, \ldots, n$, $\sum_{\mathcal{K}=1}^{n} w_{\mathcal{K}} = 1$. When we suppose the weight vector to be $w = (1/n, 1/n, \ldots, 1/n)^{\mathcal{T}}$, the weighted cosine SM will transform into cosine SM. Otherwise speaking, when $w_{\mathcal{K}} = 1/n$, $\mathcal{K} = 1, 2, 3, \ldots, n$, the $\mathbb{S}_{cw}^1(E,F) = \mathbb{S}_c^1(E,F)$.

4.2. Similarity Measures of CHFSs Based on Cosine Function. In this part of the paper, we interpreted SMs of CHFSs based on cosine function and studied their properties.

Definition 11. Let E and F be two CHFSs on a set X. Then, the SMs based on the cosine function between E and F can be presented as

$$\mathbb{S}_c^2(E,F) = \frac{1}{n}\sum_{\mathcal{K}=1}^{n}\operatorname{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right], \tag{17}$$

where $\mathbb{S}_c^2(E,F)$ means the SM based on the cosine function between E and F, which considers the maximum distance based on the amplitude and phase terms.

$$\mathbb{S}_c^3(E,F) = \frac{1}{n}\sum_{\mathcal{K}=1}^{n}\operatorname{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}}) - \gamma_{F_j}(x_{\mathcal{K}})\right| + \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right], \tag{18}$$

where $\mathbb{S}_c^3(E,F)$ means the SM based on the cosine function between E and F, which considers the sum of the distance based on the amplitude and phase terms.

Theorem 2. *The SM $\mathbb{S}_c^2(E,F)$ fulfils the following postulates:*

(1) $0 \leq \mathbb{S}_c^2(E,F) \leq 1$;

(2) $\mathbb{S}_c^2(E,F) = 1$ *if* $E = F$;
(3) $\mathbb{S}_c^2(E,F) = \mathbb{S}_c^2(F,E)$.

Proof

1. Since $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})| \in [0,1]$, $1/\mathbb{L}\sum_{j}^{\mathbb{L}}|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})| \in [0,1]$, this implies that $\max(1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})|, 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})|) \in [0,1]$. So, for $\mathscr{K} = 1$, we have

$$\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_1) - \gamma_{F_j}(x_1)\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_1) - \omega_{\gamma_{F_j}}(x_1)\right|\right)\right)\right] \in [0,1]. \tag{19}$$

For $\mathscr{K} = 2$, we have

$$\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_2) - \gamma_{F_j}(x_2)\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_2) - \omega_{\gamma_{F_j}}(x_2)\right|\right)\right)\right] \in [0,1]. \tag{20}$$

By continuing this procedure, we obtain

$$\sum_{\mathscr{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right] \in n[0,1]. \tag{21}$$

This implies that

$$\begin{aligned}
&0 \leq \sum_{\mathscr{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right] \leq n,\\
&0 \leq \frac{1}{n}\sum_{\mathscr{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right] \leq 1,
\end{aligned} \tag{22}$$

which implies that

$$0 \leq \mathbb{S}_c^2(x_{\mathscr{K}}) \leq 1. \tag{23}$$

(2) We have

$$\mathbb{S}_c^2(E,F) = \frac{1}{n}\sum_{\mathscr{K}=1}^{n}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right]. \tag{24}$$

Now, as $E = F \Longrightarrow \mu_E(x_{\mathscr{K}}) = \mu_F(x_{\mathscr{K}})$, for $\mathscr{K} = 1, 2, \ldots, n \Longrightarrow \gamma_{E_j}(x_{\mathscr{K}})e^{i2\pi(\omega_{E_j}(x_{\mathscr{K}}))} = \gamma_{F_j}(x_{\mathscr{K}})e^{i2\pi(\omega_{F_j}(x_{\mathscr{K}}))}$ for $\mathscr{K} = 1, 2, \ldots, n \Longrightarrow \gamma_{E_j}(x_{\mathscr{K}}) = \gamma_{F_j}(x_{\mathscr{K}})$ and $e^{i2\pi(\omega_{E_j}(x_{\mathscr{K}}))} = e^{i2\pi(\omega_{F_j}(x_{\mathscr{K}}))}$ for $\mathscr{K} = 1, 2, \ldots, n$. Then, $|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})| = 0$ and $|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})| = 0$ for $\mathscr{K} = 1, 2, \ldots, n$. This implies that

$$\mathbb{S}_c^2(E,F)=1. \tag{25}$$

(3) We have

$$\begin{aligned}\mathbb{S}_c^2(E,F)&=\frac{1}{n}\sum_{\mathscr{K}=1}^{n}\operatorname{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_\mathscr{K})-\gamma_{F_j}(x_\mathscr{K})\right|,\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_\mathscr{K})-\omega_{\gamma_{F_j}}(x_\mathscr{K})\right|\right)\right)\right],\\&=\frac{1}{n}\sum_{\mathscr{K}=1}^{n}\operatorname{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{F_j}(x_\mathscr{K})-\gamma_{E_j}(x_\mathscr{K})\right|,\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{F_j}}(x_\mathscr{K})-\omega_{\gamma_{E}}(x_\mathscr{K})\right|\right)\right)\right],\\&=\mathbb{S}_c^2(F,E).\end{aligned} \tag{26}$$

□

Theorem 3. *The SM $\mathbb{S}_c^3(E,F)$ fulfils the following postulates:*

(4) $0\le\mathbb{S}_c^3(E,F)\le 1$;

(5) $\mathbb{S}_c^3(E,F)=1$ if $E=F$;

(6) $\mathbb{S}_c^3(E,F)=\mathbb{S}_c^3(F,E)$.

Proof

(1) Since $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\gamma_{E_j}(x_\mathscr{K})-\gamma_{F_j}(x_\mathscr{K})|\in[0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\omega_{E_j}(x_\mathscr{K})-\omega_{F_j}(x_\mathscr{K})|\in[0,1]$, this implies that, $1/2\max(1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\gamma_{E_j}(x_\mathscr{K})-\gamma_{F_j}(x_\mathscr{K})|\in[0,1], 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\omega_{E_j}(x_\mathscr{K})-\omega_{F_j}(x_\mathscr{K})|\in[0,1])$. So, for $\mathscr{K}=1$, we have

$$\operatorname{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_1)-\gamma_{F_j}(x_1)\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_1)-\omega_{\gamma_{F_j}}(x_1)\right|\right)\right]\in[0,1]. \tag{27}$$

For $\mathscr{K}=2$, we have

$$\operatorname{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_2)-\gamma_{F_j}(x_2)\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_2)-\omega_{\gamma_{F_j}}(x_2)\right|\right)\right]\in[0,1]. \tag{28}$$

By continuing this procedure, we obtain

$$\sum_{\mathscr{K}=1}^{n}\operatorname{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_\mathscr{K})-\gamma_{F_j}(x_\mathscr{K})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_\mathscr{K})-\omega_{\gamma_{F_j}}(x_\mathscr{K})\right|\right)\right]\in n[0,1]. \tag{29}$$

This implies that

$$\begin{aligned}&0\le\sum_{\mathscr{K}=1}^{n}\operatorname{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_\mathscr{K})-\gamma_{F_j}(x_\mathscr{K})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_\mathscr{K})-\omega_{\gamma_{F_j}}(x_\mathscr{K})\right|\right)\right]\le n,\\&0\le\frac{1}{n}\sum_{\mathscr{K}=1}^{n}\operatorname{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_\mathscr{K})-\gamma_{F_j}(x_\mathscr{K})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_\mathscr{K})-\omega_{\gamma_{F_j}}(x_\mathscr{K})\right|\right)\right]\le 1,\end{aligned} \tag{30}$$

which implies that

$$0 \leq \mathbb{S}_c^3(x_{\mathscr{K}}) \leq 1. \tag{31}$$

(2) We have

$$\mathbb{S}_c^3(E,F) = \frac{1}{n}\sum_{\mathscr{K}=1}^{n} \mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right| + \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right]. \tag{32}$$

Now, as $E = F \Longrightarrow \mu_E(x_{\mathscr{K}}) = \mu_F(x_{\mathscr{K}})$. for $\mathscr{K} = 1, 2, \ldots, n \Longrightarrow \gamma_{E_j}(x_{\mathscr{K}})e^{i2\pi(\omega_{E_j}(x_{\mathscr{K}}))} = \gamma_{F_j}(x_{\mathscr{K}})e^{i2\pi}$ $(\omega_{F_j}(x_{\mathscr{K}}))$ for $\mathscr{K} = 1, 2, \ldots, n \Longrightarrow \gamma_{E_j}(x_{\mathscr{K}}) = \gamma_{F_j}(x_{\mathscr{K}})$ and $e^{i2\pi(\omega_{E_j}(x_{\mathscr{K}}))} = e^{i2\pi(\omega_{F_j}(x_{\mathscr{K}}))}$ for $\mathscr{K} = 1, 2, \ldots, n$.

Then, $|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})| = 0$ and $|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})| = 0$ for $\mathscr{K} = 1, 2, \ldots, n$. This implies that

$$\mathbb{S}_c^3(E,F) = 1. \tag{33}$$

(3) We have

$$\begin{aligned}
\mathbb{S}_c^3(E,F) &= \frac{1}{n}\sum_{\mathscr{K}=1}^{n} \mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right| + \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right],\\
&= \frac{1}{n}\sum_{\mathscr{K}=1}^{n} \mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\left|\gamma_{F_j}(x_{\mathscr{K}}) - \gamma_{E_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{F_j}}(x_{\mathscr{K}}) - \omega_{\gamma_E}(x_{\mathscr{K}})\right|\right)\right],\\
&= \mathbb{S}_c^3(F,E).
\end{aligned} \tag{34}$$

□

Definition 12. Let E and F be two CHFSs on a set X. Then, the weighted SMs based on the cosine function between E and F can be presented as

$$\begin{aligned}
\mathbb{S}_{\mathbf{cw}}^2(E,F) &= \sum_{\mathscr{K}=1}^{n} \mathrm{w}_{\mathscr{K}}\mathrm{Cos}\left[\frac{\pi}{2}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right],\\
\mathbb{S}_{\mathbf{cw}}^3(E,F) &= \sum_{\mathscr{K}=1}^{n} \mathrm{w}_{\mathscr{K}}\mathrm{Cos}\left[\frac{\pi}{4}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right| + \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right],
\end{aligned} \tag{35}$$

where $w = (w_1, w_2, \ldots, w_n)^{\mathscr{T}}$ represents the weight vector of every element $x_{\mathscr{K}}(\mathscr{K} = 1, 2, , , .n)$ carried in CHFS and the weight vector satisfies $w_{\mathscr{K}} \in [0,1]$ for every $\mathscr{K} = 1, 2, 3, , , , . n$, $\sum_{\mathscr{K}=1}^{n} w_{\mathscr{K}} = 1$. When we assume the weight vector to be $w = (1/n, 1/n, \ldots, 1/n)^{\mathscr{T}}$, the weighted SMs based on the cosine function will transform into SMs based on the cosine function. Otherwise speaking, when $w_{\mathscr{K}} = 1/n$, $\mathscr{K} = 1, 2, 3, \ldots, n$, the $\mathbb{S}_{cw}^m(E,F) = \mathbb{S}_c^m(E,F)\, m = 2, 3$

4.3. Similarity Measures of CHFSs Based on Cotangent Function. In this section, according to the cotangent function, we interpreted some cotangent SMs between CHFSs and studied their properties.

Definition 13. Let E and F be two CHFSs on a set X. Then, the cotangent SMs between E and F can be presented as

$$\mathbb{S}_{\mathbf{c}}^4(E,F) = \frac{1}{n}\sum_{\mathscr{K}=1}^{n} \mathrm{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right], \tag{36}$$

where $\mathbb{S}_{\mathbf{c}}^4(E, F)$ means the cotangent SM between E and F, which considers the maximum distance based on the amplitude and phase terms.

$$\mathbb{S}_{\mathbf{c}}^5(E,F)=\frac{1}{n}\sum_{\mathcal{K}=1}^{n}\operatorname{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|,\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right], \tag{37}$$

where $\mathbb{S}_c^5(E,F)$ means the cotangent SM between E and F, which considers the sum of distance based on the amplitude and phase terms.

Theorem 4. *The SM $\mathbb{S}_c^4(E,F)$ fulfils the following postulates:*

(7) $0\leq\mathbb{S}_c^4(E,F)\leq 1$;

(8) $\mathbb{S}_c^4(E,F)=1$ *if* $E=F$;

(9) $\mathbb{S}_c^4(E,F)=\mathbb{S}_c^2(F,E)$.

Proof

1. Since $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})|\in[0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\omega_{E_j}(x_{\mathcal{K}})-\omega_{F_j}(x_{\mathcal{K}})|\in[0,1]$, this implies that $\max(1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})|, 1/\mathbb{L}\sum_{j=1}^{\mathbb{L}}|\omega_{E_j}(x_{\mathcal{K}})-\omega_{F_j}(x_{\mathcal{K}})|)\in[0,1]$. So, for $\mathcal{K}=1$, we have

$$\operatorname{Cot}\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_1)-\gamma_{F_j}(x_1)\right|,\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_1)-\omega_{\gamma_{F_j}}(x_1)\right|\right)\right)\right]\in[0,1]. \tag{38}$$

For $\mathcal{K}=2$, we have

$$\operatorname{Cot}\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_2)-\gamma_{F_j}(x_2)\right|,\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_2)-\omega_{\gamma_{F_j}}(x_2)\right|\right)\right)\right]\in[0,1]. \tag{39}$$

By continuing this procedure, we obtain

$$\sum_{\mathcal{K}=1}^{n}\operatorname{Cot}\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|,\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right]\in n[0,1]. \tag{40}$$

This implies that

$$\begin{aligned}
&0\leq\sum_{\mathcal{K}=1}^{n}Cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|,\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right]\leq n,\\
&0\leq\frac{1}{n}\sum_{\mathcal{K}=1}^{n}Cot\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|,\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right)\right]\leq 1,
\end{aligned} \tag{41}$$

which implies that

$$0 \leq \mathbb{S}_c^4(x_{\mathscr{K}}) \leq 1. \tag{42}$$

(2) We have

$$\mathbb{S}_c^4(E,F) = \frac{1}{n}\sum_{\mathscr{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right]. \tag{43}$$

Now, as $E = F \Longrightarrow \mu_E(x_{\mathscr{K}}) = \mu_F(x_{\mathscr{K}})$, for $\mathscr{K} = 1, 2, \ldots, n \Longrightarrow \gamma_{E_j}(x_{\mathscr{K}}) e^{i2\pi(\omega_{E_j}(x_{\mathscr{K}}))} = \gamma_{F_j}(x_{\mathscr{K}}) e^{i2\pi(\omega_{F_j}(x_{\mathscr{K}}))}$ for $\mathscr{K} = 1, 2, \ldots, n \Longrightarrow \gamma_{E_j}(x_{\mathscr{K}}) = \gamma_{F_j}(x_{\mathscr{K}})$ and $e^{i2\pi(\omega_{E_j}(x_{\mathscr{K}}))} = e^{i2\pi(\omega_{F_j}(x_{\mathscr{K}}))}$ for $\mathscr{K} = 1, 2, \ldots, n$. Then, $|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})| = 0$ and $|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})| = 0$ for $\mathscr{K} = 1, 2, \ldots, n$. This implies that

$$\mathbb{S}_c^4(E,F) = 1. \tag{44}$$

(3) We have

$$\begin{aligned}
\mathbb{S}_c^4(E,F) &= \frac{1}{n}\sum_{\mathscr{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right] \\
&= \frac{1}{n}\sum_{\mathscr{K}=1}^{n} \operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})\right|, \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathscr{K}}) - \omega_{\gamma_{F_j}}(x_{\mathscr{K}})\right|\right)\right)\right] \\
&= \mathbb{S}_c^4(F,E).
\end{aligned} \tag{45}$$

□

Theorem 5. *The SM $\mathbb{S}_c^5(E,F)$ fulfils the following postulates:*

(10) $0 \leq \mathbb{S}_c^5(E,F) \leq 1$;

(11) $\mathbb{S}_c^5(E,F) = 1$ *if* $E = F$;

(12) $\mathbb{S}_c^5(E,F) = \mathbb{S}_c^5(F,E)$.

Proof

(1) Since $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}} |\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})| \in [0,1]$, $1/\mathbb{L}\sum_{j=1}^{\mathbb{L}} |\omega_{E_j}(x_{\mathscr{K}}) - \omega_{F_j}(x_{\mathscr{K}})| \in [0,1]$, this implies that, $1/2\max(1/\mathbb{L}\sum_{j=1}^{\mathbb{L}} |\gamma_{E_j}(x_{\mathscr{K}}) - \gamma_{F_j}(x_{\mathscr{K}})|, \sum_{j=1}^{\mathbb{L}} |\omega_{E_j}(x_{\mathscr{K}}) - \omega_{F_j}(x_{\mathscr{K}})|) \in [0,1]$. So, for $\mathscr{K} = 1$, we have

$$\operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_1) - \gamma_{F_j}(x_1)\right| + \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_1) - \omega_{\gamma_{F_j}}(x_1)\right|\right)\right] \in [0,1]. \tag{46}$$

For $\mathscr{K} = 2$, we have

$$\operatorname{Cot}\left[\frac{\pi}{4} + \frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_2) - \gamma_{F_j}(x_2)\right| + \frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_2) - \omega_{\gamma_{F_j}}(x_2)\right|\right)\right] \in [0,1]. \tag{47}$$

By continuing this procedure, we obtain

$$\sum_{\mathcal{K}=1}^{n} \mathrm{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right] \in n[0,1]. \tag{48}$$

This implies that

$$\begin{aligned} 0 &\leq \sum_{\mathcal{K}=1}^{n} \mathrm{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right] \leq n, \\ 0 &\leq \frac{1}{n}\sum_{\mathcal{K}=1}^{n} \mathrm{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right] \leq 1, \end{aligned} \tag{49}$$

which implies that

$$0 \leq \mathbb{S}_c^5(x_{\mathcal{K}}) \leq 1. \tag{50}$$

(2) We have

$$\mathbb{S}_c^5(E,F) = \frac{1}{n}\sum_{\mathcal{K}=1}^{n} \mathrm{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right]. \tag{51}$$

Now, as $E = F \Longrightarrow \mu_E(x_{\mathcal{K}}) = \mu_F(x_{\mathcal{K}})$, for $\mathcal{K} = 1,2,\ldots,n \Longrightarrow \gamma_{E_j}(x_{\mathcal{K}})e^{i2\pi(\omega_{E_j}(x_{\mathcal{K}}))} = \gamma_{F_j}(x_{\mathcal{K}})e^{i2\pi(\omega_{F_j}(x_{\mathcal{K}}))}$ for $\mathcal{K} = 1,2,\ldots,n \Longrightarrow \gamma_{E_j}(x_{\mathcal{K}}) = \gamma_{F_j}(x_{\mathcal{K}})$ and $e^{i2\pi(\omega_{E_j}(x_{\mathcal{K}}))} = e^{i2\pi(\omega_{F_j}(x_{\mathcal{K}}))}$ for $\mathcal{K} = 1,2,\ldots,n$. Then, $|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})| = 0$ and $|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})| = 0$ for $\mathcal{K} = 1,2,\ldots,n$. This implies that

$$\mathbb{S}_c^5(E,F) = 1. \tag{52}$$

(3) We have

$$\begin{aligned} \mathbb{S}_c^5(E,F) &= \frac{1}{n}\sum_{\mathcal{K}=1}^{n} \mathrm{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right], \\ &= \frac{1}{n}\sum_{\mathcal{K}=1}^{n} \mathrm{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\gamma_{E_j}(x_{\mathcal{K}})-\gamma_{F_j}(x_{\mathcal{K}})\right|+\frac{1}{\mathbb{L}}\sum_{j=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_j}}(x_{\mathcal{K}})-\omega_{\gamma_{F_j}}(x_{\mathcal{K}})\right|\right)\right], \\ &= \mathbb{S}_c^5(F,E). \end{aligned} \tag{53}$$

□

Definition 14. Let E and F be two CHFSs on a set X. Then, the weighted cotangent SMs between E and F can be presented as

$$\begin{aligned}
\mathbb{S}^4_{\mathbf{cw}}(E,F) &= \sum_{\mathscr{K}=1}^{n} w_{\mathscr{K}} \mathrm{Cot}\left[\frac{\pi}{4}+\frac{\pi}{4}\left(\max\left(\frac{1}{\mathbb{L}}\sum_{\mathrm{j}=1}^{\mathbb{L}}\left|\gamma_{E_{\mathrm{j}}}(x_{\mathscr{K}})-\gamma_{F_{\mathrm{j}}}(x_{\mathscr{K}})\right|,\frac{1}{\mathbb{L}}\sum_{\mathrm{j}=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_{\mathrm{j}}}}(x_{\mathscr{K}})-\omega_{\gamma_{F_{\mathrm{j}}}}(x_{\mathscr{K}})\right|\right)\right)\right],\\
\mathbb{S}^5_{\mathbf{cw}}(E,F) &= \sum_{\mathscr{K}=1}^{n} w_{\mathscr{K}} \mathrm{Cot}\left[\frac{\pi}{4}+\frac{\pi}{8}\left(\frac{1}{\mathbb{L}}\sum_{\mathrm{j}=1}^{\mathbb{L}}\left|\gamma_{E_{\mathrm{j}}}(x_{\mathscr{K}})-\gamma_{F_{\mathrm{j}}}(x_{\mathscr{K}})\right|,\frac{1}{\mathbb{L}}\sum_{\mathrm{j}=1}^{\mathbb{L}}\left|\omega_{\gamma_{E_{\mathrm{j}}}}(x_{\mathscr{K}})-\omega_{\gamma_{F_{\mathrm{j}}}}(x_{\mathscr{K}})\right|\right)\right],
\end{aligned}\tag{54}$$

where $w=(w_1,w_2,\ \ldots,w_n)^{\mathscr{T}}$ represents the weight vector of every element $x_{\mathscr{K}}\ (\mathscr{K}=1,2,,,.n)$ carried in CHFS and the weight vector satisfies $w_{\mathscr{K}}\in[0,1]$ for every $\mathscr{K}=1,2,3,,,,.n$, $\sum_{\mathscr{K}=1}^{n} w_{\mathscr{K}}=1$. When we assume the weight vector to be $w=(1/n,1/n,\ldots,1/n)^{\mathscr{T}}$, the weighted cotangent SMs will transform into cotangent SMs. Otherwise speaking, when $w_{\mathscr{K}}=1/n$, $\mathscr{K}=1,2,\ldots,n$, the $\mathbb{S}^m_{cw}(E,F)=\mathbb{S}^m_{c}(E,F)\ m=4,5$.

5. Applications

In this section, we gave two applications about cosine SM, SMs based on cosine function, and cotangent SM under CHF environment. The interpreted SMs are applied to pattern recognition and medical diagnosis to express the usefulness of these SMs.

5.1. Pattern Recognition

Example 2. Without any hesitancy, the quantity of construction usually relies on the standard of building materials. Accordingly, building material scrutiny is the assumption of good engineering standards. The selection of material must be strictly controlled. Scrutiny authorizes the builders to correctly recognize qualified materials and upgrade the standard of the project. To resolve the abovementioned issues, we choose the building meterials E_j(j = 1, 2, 3, 4, 5), which are discussed as follows:

$$\begin{aligned}
E_1 &= \left\{\begin{array}{c}\left(x_1,\left\{0.6e^{i2\pi(1)},0.5e^{i2\pi(0.5)}\right\}\right),\left(x_2,\left\{0.7e^{i2\pi(0.4)}\right\}\right),\left(x_3,\left\{0.6e^{i2\pi(0.8)},0.4e^{i2\pi(0.7)}\right\}\right)\\ \left(x_4,\left\{0.8e^{i2\pi(0.9)},0.2e^{i2\pi(0.7)}\right\}\right),\left(x_5,\left\{0.2e^{i2\pi(0.3)},0.6e^{i2\pi(0.5)},0.4e^{i2\pi(0.6)}\right\}\right)\end{array}\right\},\\
E_2 &= \left\{\begin{array}{c}\left(x_1,\left\{0.1e^{i2\pi(0.4)}\right\}\right),\left(x_2,\left\{0.5e^{i2\pi(0.1)},0.1e^{i2\pi(0.6)}\right\}\right),\left(x_3,\left\{0.2e^{i2\pi(0.6)},0.7e^{i2\pi(0.4)}\right\}\right)\\ \left(x_4,\left\{0.1e^{i2\pi(0.4)},0.3e^{i2\pi(0.1)}\right\}\right),\left(x_5,\left\{0.5e^{i2\pi(0.6)}\right\}\right)\end{array}\right\},\\
E_3 &= \left\{\begin{array}{c}\left(x_1,\left\{1e^{i2\pi(0.8)},0.6e^{i2\pi(0.8)},0.5e^{i2\pi(0.9)}\right\}\right),\left(x_2,\left\{0.5e^{i2\pi(0.7)}\right\}\right),\left(x_3,\left\{0.8e^{i2\pi(1)},0.7e^{i2\pi(0.9)}\right\}\right)\\ \left(x_4,\left\{0.9e^{i2\pi(0.8)},0.7e^{i2\pi(0.6)},\right\}\right),\left(x_5,\left\{0.7e^{i2\pi(0.5)},0.2e^{i2\pi(0.4)},0.3e^{i2\pi(0.7)}\right\}\right)\end{array}\right\},\\
E_4 &= \left\{\begin{array}{c}\left(x_1,\left\{0.3e^{i2\pi(0.9)},1\,e^{i2\pi(1)},\right\}\right),\left(x_2,\left\{0.4.e^{i2\pi(0.2)},0.2e^{i2\pi(0.5)}\right\}\right),\left(x_3,\left\{0.2e^{i2\pi(1)}\right\}\right)\\ \left(x_4,\left\{0.8e^{i2\pi(0.6)}\right\}\right),\left(x_5,\left\{0.5e^{i2\pi(0.1)},0.6e^{i2\pi(0.3)},0.8e^{i2\pi(0.5)}\right\}\right)\end{array}\right\},\\
E_5 &= \left\{\begin{array}{c}\left(x_1,\left\{0.4e^{i2\pi(0.2)},0.2e^{i2\pi(0.5)}\right\}\right),\left(x_2,\left\{0.4.e^{i2\pi(0.2)},0.4e^{i2\pi(0.1)}\right\}\right),\left(x_3,\left\{0.2e^{i2\pi(0.4)},0.3e^{i2\pi(0.1)}\right\}\right)\\ \left(x_4,\left\{0.6e^{i2\pi(0.7)},0.3e^{i2\pi(0.5)}0.6e^{i2\pi(0.1)}\right\}\right),\left(x_5,\left\{0.1e^{i2\pi(0.3)}\right\}\right)\end{array}\right\}.
\end{aligned}\tag{55}$$

To resolve the abovementioned issue, we choose the complex hesitant fuzzy set in the form of unknown materials.

$$E=\left\{\begin{array}{c}\left(x_1,\left\{0.9e^{i2\pi(0.8)},0.7e^{i2\pi(0.4)},1e^{i2\pi(0.8)}\right\}\right),\left(x_2,\left\{0.4e^{i2\pi(0.6)}\right\}\right),\left(x_3,\left\{0.8e^{i2\pi(0.6)},0.5e^{i2\pi(0.8)}\right\}\right)\\ \left(x_4,\left\{0.9e^{i2\pi(0.6)},0.4e^{i2\pi(0.3)}\right\}\right)\left(x_5,\left\{0.5e^{i2\pi(0.7)},0.3e^{i2\pi(0.9)},0.2e^{i2\pi(0.5)}\right\}\right),\end{array}\right\}.\tag{56}$$

The aim of this issue is to categorize the unspecified building material E in one of the categories $E_j\,(j = 1, 2, 3, 4, 5)$. For it, the cosine SM, SMs based on cosine function, and cotangent SMs which are explored in this paper have been used to determine the similarity from E to $E_j\,(j = 1, \ldots, 5)$ and calculations are introduced in Tables 1 and 2.

As stated by the above-computed calculations given in Table 1, we simply note that the degree of similarity between E and E_3 is the greatest one as an extract by all five SMs. This specifies that all five SMs assign the unspecified building material E to the specified building material E_3 based on the principle of the maximum degree of similarity. Ranking of the explored cosine and cotangent SMs between E and $E_j\,(j = 1, \ldots, 5)$ is also introduced in Table 1. The graphical representation of the interpreted SMs between E and $E_j\,(j = 1, \ldots, 5)$ is indicated in Figure 1.

The weight of elements has great significance to suppose in real decision-making problems. If we suppose the weight of elements $x_{\mathscr{K}}\,(\mathscr{K} = 1, 2, 3, 4, 5)$ to be $w_{\mathscr{K}} = (0.15, 0.1, 0.25, 0.2, 0.3)$, respectively, then the interpreted WSMs (weighted cosine SMs and weighted cotangent SMs) have been used to determine the similarity from E to $E_j\,(j = 1, \ldots, 5)$ and calculations are introduced in Table 2.

As stated by the above-computed calculations given in Table 2, we simply note that the degree of similarity between E and E_3 is the greatest one as an extract by all five WSMs. This specifies that all five WSMs assign the unspecified building material E to the specified building material E_3 based on the principle of the maximum degree of similarity. Ranking of the explored weighted cosine SMs, weighted SMs based on cosine function, and weighted cotangent SMs between E and $E_j\,(j = 1, \ldots, 5)$ is also introduced in Table 2. The graphical representation of the interpreted WSMs between E and $E_j\,(j = 1, \ldots, 5)$ is indicated in Figure 2.

5.2. Medical Diagnosis. Symptoms of every diseases are almost different. To examine that the victim is suffering from what type of diseases, the medical diagnosis relies on the victim's symptoms. The victim's symptoms are a set of symptoms and unspecified diseases will be a set of diagnostic diseases. The interpreted SMs are illustrated by a following numerical example of medical diagnosis.

Example 3. Let a set of diagnosis $D = \{D_1\,(\text{Typhoid}), D_2\,(\text{Flu}), D_3\,(\text{Heart problem}), D_4\,(\text{Pneumonia}), D_5\,(\text{Coronavirus})\}$ and set of symptoms $\mathscr{X} = \{x_1\,(\text{fever}), x_2\,(\text{cough}), x_3\,(\text{heart pain}), x_4\,(\text{loss of ppetite}), x_5\,(\text{short of breath})\}$. The victim's symptoms are represented in the form of CHFSs as follows:

$$P = \left\{ \begin{array}{c} \left(x_1, \left\{0.8e^{i2\pi(0.9)}, 0.6e^{i2\pi(0.5)}\right\}\right), \left(x_2, \left\{0.5e^{i2\pi(0.7)} 0.9e^{i2\pi(1)}, 1e^{i2\pi(0.5)}\right\}\right), \left(x_3, \left\{0.1e^{i2\pi(0.2)}\right\}\right) \\ \left(x_4, \left\{0.4e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.6)}, 0.7e^{i2\pi(0.4)}\right\}\right), \left(x_5, \left\{0.1e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.5)}\right\}\right) \end{array} \right\}. \tag{57}$$

The indications of each disease $D_j\,(j = 1, 2, 3, 4, 5)$ are represented in the form CHFSs as follows:

$$\begin{aligned} D_1 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.7e^{i2\pi(1)}, 0.9e^{i2\pi(0.8)}\right\}\right), \left(x_2, \left\{1e^{i2\pi(0.8)}, 0.5e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.9)}\right\}\right), \left(x_3, \left\{0.4e^{i2\pi(0.6)},\right\}\right) \\ \left(x_4, \left\{0.9e^{i2\pi(0.8)}, 0.7e^{i2\pi(0.6)}, 0.5e^{i2\pi(0.7)}\right\}\right), \left(x_5, \left\{0.2e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.4)}\right\}\right) \end{array} \right\}, \\ D_2 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.5e^{i2\pi(0.6)}, 0.9e^{i2\pi(0.8)}\right\}\right), \left(x_2, \left\{0.8e^{i2\pi(1)}, 0.7e^{i2\pi(0.8)}\right\}\right), \left(x_3, \left\{0.1e^{i2\pi(0.05)}\right\}\right) \\ \left(x_4, \left\{0.2e^{i2\pi(0.1)}, 0.5e^{i2\pi(0.2)}\right\}\right), \left(x_5, \left\{0.5e^{i2\pi(0.3)}\right\}\right) \end{array} \right\}, \\ D_3 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.6e^{i2\pi(0.5)}\right\}\right), \left(x_2, \left\{0.2e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.1)}\right\}\right), \left(x_3, \left\{0.8e^{i2\pi(1)}, 1e^{i2\pi(1)}, 0.7e^{i2\pi(0.9)}\right\}\right) \\ \left(x_4, \left\{0.5e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.4)},\right\}\right), \left(x_5, \left\{0.7e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.7)}\right\}\right) \end{array} \right\}, \\ D_4 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.6e^{i2\pi(0.9)}, 0.7e^{i2\pi(0.8)}, 0.4e^{i2\pi(0.7)}\right\}\right), \left(x_2, \left\{0.5e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.3)}, 0.1e^{i2\pi(0.6)}\right\}\right), \\ \left(x_3, \left\{0.1e^{i2\pi(0.4)}\right\}\right), \left(x_4, \left\{0.6e^{i2\pi(0.8)}\right\}\right), \left(x_5, \left\{0.4e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.4)}\right\}\right) \end{array} \right\}, \\ D_5 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.8e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.7)}\right\}\right), \left(x_2, \left\{0.6e^{i2\pi(0.7)}, 0.7e^{i2\pi(0.9)}\right\}\right), \left(x_3, \left\{0.1e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.2)}\right\}\right) \\ \left(x_4, \left\{0.3e^{i2\pi(0.4)}\right\}\right), \left(x_5, \left\{0.8e^{i2\pi(0.7)}, 0.9e^{i2\pi(1)}, 1e^{i2\pi(0.7)}\right\}\right) \end{array} \right\}. \end{aligned} \tag{58}$$

The aim of this issue is to find the disease of the victim P in one of the diseases $D_j\,(j = 1, 2, 3, 4, 5)$. For it, the cosine SM, SMs based on cosine function, and cotangent SMs which are explored in this paper have been utilized to determine the similarity from P to $D_j\,(j = 1, \ldots, 5)$ and calculations are introduced in Tables 3 and 4.

Table 1: The explored SMs between E and E_j $(j = 1, 2, 3, 4, 5)$.

SMs	$(\mathbf{E}, \mathbf{E}_1)$	$(\mathbf{E}, \mathbf{E}_2)$	$(\mathbf{E}, \mathbf{E}_3)$	$(\mathbf{E}, \mathbf{E}_4)$	$(\mathbf{E}, \mathbf{E}_5)$	Ranking
$\mathbb{S}_c^1(\mathbf{E}, \mathbf{E}_j)$	0.4733	0.2829	0.517	0.3429	0.2844	$E_3 \geq E_1 \geq E_4 \geq E_5 \geq E_2$
$\mathbb{S}_c^2(\mathbf{E}, \mathbf{E}_j)$	0.8674	0.6372	0.9321	0.6047	0.7278	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
$\mathbb{S}_c^3(\mathbf{E}, \mathbf{E}_j)$	0.9022	0.796	0.959	0.7405	0.8054	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
$\mathbb{S}_c^4(\mathbf{E}, \mathbf{E}_j)$	0.6056	0.3777	0.6988	0.3762	0.4605	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
$\mathbb{S}_c^5(\mathbf{E}, \mathbf{E}_j)$	0.6528	0.5264	0.7563	0.4746	0.7479	$E_3 \geq E_5 \geq E_1 \geq E_2 \geq E_4$

Table 2: The explored WSMs between E and E_j $(j = 1, 2, 3, 4, 5)$.

SMs	$(\mathbf{E}, \mathbf{E}_1)$	$(\mathbf{E}, \mathbf{E}_2)$	$(\mathbf{E}, \mathbf{E}_3)$	$(\mathbf{E}, \mathbf{E}_4)$	$(\mathbf{E}, \mathbf{E}_5)$	Ranking
$\mathbb{S}_{cw}^1(\mathbf{E}, \mathbf{E}_j)$	0.4247	0.2933	0.4557	0.3128	0.2826	$E_3 \geq E_1 \geq E_4 \geq E_2 \geq E_5$
$\mathbb{S}_{cw}^2(\mathbf{E}, \mathbf{E}_j)$	0.8829	0.6705	0.9219	0.672	0.7072	$E_3 \geq E_1 \geq E_5 \geq E_4 \geq E_2$
$\mathbb{S}_{cw}^3(\mathbf{E}, \mathbf{E}_j)$	0.911	0.816	0.9551	0.7603	0.7964	$E_3 \geq E_1 \geq E_2 \geq E_5 \geq E_4$
$\mathbb{S}_{cw}^4(\mathbf{E}, \mathbf{E}_j)$	0.628	0.4011	0.6737	0.4207	0.4359	$E_3 \geq E_1 \geq E_5 \geq E_4 \geq E_2$
$\mathbb{S}_{cw}^5(\mathbf{E}, \mathbf{E}_j)$	0.6664	0.5434	0.7431	0.4933	0.7386	$E_3 \geq E_5 \geq E_1 \geq E_2 \geq E_4$

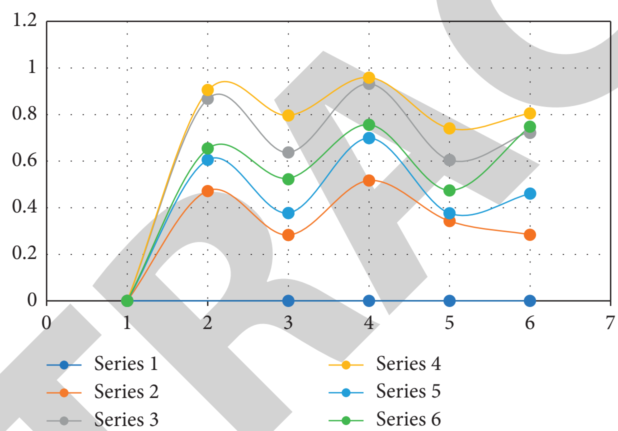

Figure 1: The graphical representation of interpreted SMs.

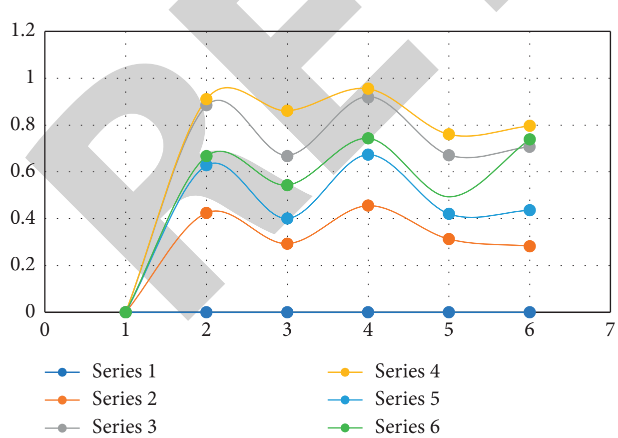

Figure 2: The graphical representation of interpreted WSMs.

As stated by the above-computed calculations described in Table 3, we simply note that the degree of similarity between P and D_1 is the greatest one as an extract by five SMs. This specifies that all five SMs express that the victim P has typhoid based on the principle of the maximum similarity degree. Ranking of the explored cosine and cotangent SMs between P and D_j $(j = 1, \ldots, 5)$ is also introduced in Table 3. Next, the graphical representation of the interpreted SMs between P and D_j $(j = 1, \ldots, 5)$ is indicated in Figure 3.

The weight of elements has great significance to suppose in real decision-making problems. If we suppose the weight of elements $x_{\mathscr{K}}$ $(\mathscr{K} = 1, 2, 3, 4, 5)$ to be $w_{\mathscr{K}} = (0.15, 0.1, 0.25, 0.2, 0.3)$, respectively, then the interpreted WSMs (weighted cosine SMs and weighted cotangent SMs) have been used to determine the similarity from P to D_j $(j = 1, \ldots, 5)$ and calculations are introduced in Table 4.

As stated by the above-computed calculations described in Table 3, we simply note that the degree of similarity between P and D_1 is the greatest one as an extract by WSMs, except $\mathbb{S}_{cw}^5$. This specifies that WSMs $\mathbb{S}_{cw}^1$, $\mathbb{S}_{cw}^2$, $\mathbb{S}_{cw}^3$, and $\mathbb{S}_{cw}^4$ show that the victim P has typhoid based on the principle of the maximum similarity degree. We also note that degree of similarity between P and D_5 is the highest one as an extract by WSM $\mathbb{S}_{cw}^5$. This specifies that the WSM $\mathbb{S}_{cw}^5$ shows that the victim P has coronavirus. Ranking of the explored weighted cosine SMs, weighted SMs based on cosine function, and

Table 3: The explored SMs between P and $D_j\,(j = 1, 2, 3, 4, 5)$.

Similarity measures	$(\mathbf{P}, \mathbf{D}_1)$	$(\mathbf{P}, \mathbf{D}_2)$	$(\mathbf{P}, \mathbf{D}_3)$	$(\mathbf{P}, \mathbf{D}_4)$	$(\mathbf{P}, \mathbf{D}_5)$	Ranking
$\mathbb{S}^1_C(\mathbf{P}, \mathbf{D}_j)$	0.5132	0.4279	0.3019	0.4216	0.3039	$D_1 \geq D_2 \geq D_4 \geq D_5 \geq D_3$
$\mathbb{S}^2_C(\mathbf{P}, \mathbf{D}_j)$	0.8833	0.8465	0.6363	0.8701	0.7434	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}^3_C(\mathbf{P}, \mathbf{D}_j)$	0.9119	0.8843	0.6736	0.8985	0.8249	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}^4_C(\mathbf{P}, \mathbf{D}_j)$	0.6367	0.5777	0.3953	0.6044	0.4858	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}^5_C(\mathbf{P}, \mathbf{D}_j)$	0.7288	0.6797	0.6361	04314	0.6621	$D_1 \geq D_2 \geq D_5 \geq D_3 \geq D_4$

Table 4: The explored WSMs between P and $D_j\,(j = 1, 2, 3, 4, 5)$

Similarity measures	$(\mathbf{P}, \mathbf{D}_1)$	$(\mathbf{P}, \mathbf{D}_2)$	$(\mathbf{P}, \mathbf{D}_3)$	$(\mathbf{P}, \mathbf{D}_4)$	$(\mathbf{P}, \mathbf{D}_5)$	Ranking
$\mathbb{S}^1_{cw}(\mathbf{P}, \mathbf{D}_j)$	0.5575	0.393	0.3089	0.4598	0.2974	$D_1 \geq D_4 \geq D_2 \geq D_3 \geq D_5$
$\mathbb{S}^2_{cw}(\mathbf{P}, \mathbf{D}_j)$	0.8972	0.8652	0.6369	0.8856	0.7065	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}^3_{cw}(\mathbf{P}, \mathbf{D}_j)$	0.9224	0.8992	0.6785	0.9111	0.7898	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}^4_{cw}(\mathbf{P}, \mathbf{D}_j)$	0.6635	0.601	0.4026	0.6302	0.465	$D_1 \geq D_4 \geq D_2 \geq D_5 \geq D_3$
$\mathbb{S}^5_{cw}(\mathbf{P}, \mathbf{D}_j)$	0.7058	0.6618	0.4433	0.6893	0.7526	$D_5 \geq D_1 \geq D_4 \geq D_2 \geq D_3$

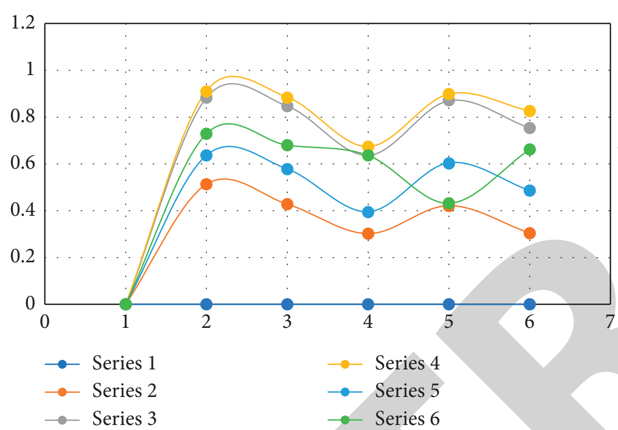

Figure 3: The graphical representation of interpreted SMs.

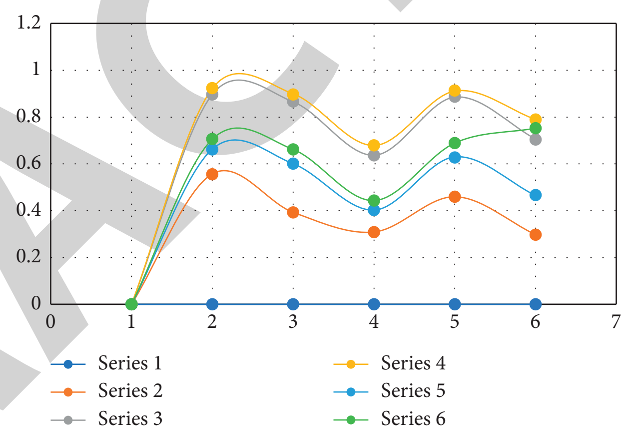

Figure 4: The graphical representation of interpreted WSMs.

weighted cotangent SMs between P and $D_j\,(j = 1, \ldots, 5)$ is also introduced in Table 4. Next, we have the graphical representation of the interpreted WSMs between P and $D_j\,(j = 1, \ldots, 5)$ in Figure 4.

6. Comparison

In this section of the paper, we expressed the effectiveness and advantages of the interpreted SMs by comparing with some already defined SMs.

Example 4. Without any hesitancy, the quantity of construction usually relies on the standard of building materials. Accordingly, building material scrutiny is the assumption of good engineering standards. The selection of material must be strictly controlled. Scrutiny authorizes the builders to correctly recognize qualified materials and upgrade the standard of the project. Suppose pattern recognition problem about the categorization of building materials, Let five specified building materials $E_j\,(j = 1, 2, 3, 4, 5)$ which are represented in the form of HFSs as follows:

$$\begin{aligned}
E_1 &= \left\{ \begin{matrix} (x_1, \{0.6, 0.5\}), (x_2, \{0.7\}), (x_3, \{0.6, 0.4\}) \\ (x_4, \{0.8, 0.2\}), (x_5, \{0.2, 0.6, 0.4\}) \end{matrix} \right\}, \\
E_2 &= \left\{ \begin{matrix} (x_1, \{0.1\}), (x_2, \{0.5, 0.1\}), (x_3, \{0.2, 0.7\}) \\ (x_4, \{0.1, 0.3\}), (x_5, \{0.5\}) \end{matrix} \right\}, \\
E_3 &= \left\{ \begin{matrix} (x_1, \{1, 0.6, 0.5\}), (x_2, \{0.5\}), (x_3, \{0.8, 0.7\}) \\ (x_4, \{0.9, 0.7,\}), (x_5, \{0.7, 0.2, 0.3\}) \end{matrix} \right\}, \\
E_4 &= \left\{ \begin{matrix} (x_1, \{0.3, 1\}), (x_2, \{0.4, 0.2\}), (x_3, \{0.2\}) \\ (x_4, \{0.8\}), (x_5, \{0.5, 0.6, 0.8\}) \end{matrix} \right\}, \\
E_5 &= \left\{ \begin{matrix} (x_1, \{0.4, 0.2\}), (x_2, \{0.4, 0.4\}), (x_3, \{0.2, 0.3\}) \\ (x_4, \{0.6, 0.3, 0.6\}), (x_5, \{0.1\}) \end{matrix} \right\}.
\end{aligned} \tag{59}$$

Next, let an unspecified building material E in the form of CHFS which needs to be recognized be

$$E = \left\{ \begin{matrix} (x_1, \{0.9, 0.7, 1\}), (x_2, \{0.4\}), (x_3, \{0.8, 0.5\}) \\ (x_4, \{0.9, 0.4\})(x_5, \{0.5, 0.3, 0.2\}), \end{matrix} \right\}. \tag{60}$$

Table 5: The comparison between interpreted and some already defined SMs of Example 4.

Method	Score value	Ranking
Xu and Xia [29]	$\mathbb{S}(E, E_1) = 0.7667$, $\mathbb{S}(E, E_2) = 0.5967$ $\mathbb{S}(E, E_3) = 0.7733$, $\mathbb{S}(E, E_4) = 0.3467$ $\mathbb{S}(E, E_5) = 0.7067$	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
Zeng et al. [30]	$\mathbb{S}_h(E, E_1) = 0.8367$, $\mathbb{S}_h(E, E_2) = 0.5283$, $\mathbb{S}_h(E, E_3) = 0.845$, $\mathbb{S}_h(E, E_4) = 0.415$, $\mathbb{S}_h(E, E_4) = 0.6872$	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
Jun [31]	$\mathrm{Cos}_{\mathrm{HFS}}(E, E_1) = 0.963$, $\mathrm{Cos}_{\mathrm{HFS}}(E, E_2) = 0.841$, $\mathrm{Cos}_{\mathrm{HFS}}(E, E_3) = 0.911$, $\mathrm{Cos}_{\mathrm{HFS}}(E, E_4) = 0.883$, $\mathrm{Cos}_{\mathrm{HFS}}(E, E_5) = 0.925$	$E_1 \geq E_5 \geq E_3 \geq E_4 \geq E_2$
Proposed SM	$\mathbb{S}_c^1(E, E_1) = 0.5006$, $\mathbb{S}_c^1(E, E_2) = 0.3334$, $\mathbb{S}_c^1(E, E_3) = 0.5234$ $\mathbb{S}_c^1(E, E_4) = 0.453$, $\mathbb{S}_c^1(E, E_5) = 0.3147$	$E_3 \geq E_1 \geq E_4 \geq E_2 \geq E_5$
Proposed SM	$\mathbb{S}_c^2(E, E_1) = 0.8914$, $\mathbb{S}_c^2(E, E_2) = 0.7565$, $\mathbb{S}_c^2(E, E_3) = 0.9719$, $\mathbb{S}_c^2(E, E_4) = 0.7922$, $\mathbb{S}_c^2(E, E_5) = 0.8036$	$E_3 \geq E_1 \geq E_5 \geq E_4 \geq E_2$
Proposed SM	$\mathbb{S}_c^3(E, E_1) = 0.9721$, $\mathbb{S}_c^3(E, E_2) = 0.9342$, $\mathbb{S}_c^3(E, E_3) = 0.9929$, $\mathbb{S}_c^3(E, E_4) = 0.9456$, $\mathbb{S}_c^3(E, E_5) = 0.9486$	$E_3 \geq E_1 \geq E_5 \geq E_4 \geq E_2$
Proposed SM	$\mathbb{S}_c^4(E, E_1) = 0.6513$, $\mathbb{S}_c^4(E, E_2) = 0.5454$, $\mathbb{S}_c^4(E, E_3) = 0.7994$, $\mathbb{S}_c^4(E, E_4) = 0.5352$, $\mathbb{S}_c^4(E, E_5) = 0.5395$	$E_3 \geq E_1 \geq E_2 \geq E_5 \geq E_4$
Proposed SM	$\mathbb{S}_c^5(E, E_1) = 0.8105$, $\mathbb{S}_c^5(E, E_2) = 0.7457$, $\mathbb{S}_c^5(E, E_3) = 0.8953$, $\mathbb{S}_c^5(E, E_4) = 0.7418$, $\mathbb{S}_c^5(E, E_5) = 0.8671$	$E_3 \geq E_5 \geq E_1 \geq E_2 \geq E_4$

We convert the HFSs in the CHFSs by taking $1 = e^0$ as follows:

$$
\begin{aligned}
E_1 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.6e^{i2\pi(0.0)}, 0.5e^{i2\pi(0.0)}\right\}\right), \left(x_2, \left\{0.7e^{i2\pi(0.0)}\right\}\right), \left(x_3, \left\{0.6e^{i2\pi(0.0)}, 0.4e^{i2\pi(0.0)}\right\}\right) \\ \left(x_4, \left\{0.8e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}\right\}\right), \left(x_5, \left\{0.2e^{i2\pi(0.0)}, 0.6e^{i2\pi(0.0)}, 0.4e^{i2\pi(0.0)}\right\}\right) \end{array} \right\}, \\
E_2 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.1e^{i2\pi(0.0)}\right\}\right), \left(x_2, \left\{0.5e^{i2\pi(0.0)}, 0.1e^{i2\pi(0.0)}\right\}\right), \left(x_3, \left\{0.2e^{i2\pi(0.0)}, 0.7e^{i2\pi(0.0)}\right\}\right) \\ \left(x_4, \left\{0.1e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}\right\}\right), \left(x_5, \left\{0.5e^{i2\pi(0.0)}\right\}\right) \end{array} \right\}, \\
E_3 &= \left\{ \begin{array}{c} \left(x_1, \left\{1e^{i2\pi(0.0)}, 0.6e^{i2\pi(0.0)}, 0.5e^{i2\pi(0.0)}\right\}\right), \left(x_2, \left\{0.5e^{i2\pi(0.0)}\right\}\right), \left(x_3, \left\{0.8e^{i2\pi(0.0)}, 0.7e^{i2\pi(0.0)}\right\}\right) \\ \left(x_4, \left\{0.9e^{i2\pi(0.0)}, 0.7e^{i2\pi(0.0)}\right\} \left(x_5, \left\{0.7e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}, 0.3e^{i2\pi(0.0)}\right\}\right)\right) \end{array} \right\}, \\
E_4 &= \left\{ \begin{array}{c} \left(x_1, \left\{0.3e^{i2\pi(0.0)}, 1\, e^{i2\pi(0.0)}\right\}\right), \left(x_2, \left\{0.4e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}\right\}\right), \left(x_3, \left\{0.2e^{i\pi(0.0)}\right\}\right) \\ \left(x_4, \left\{0.8e^{i2\pi(0.0)}\right\}\right)\left(x_5, \left\{0.5e^{i\pi(0.0)}, 0.6e^{i2\pi(0.0)}, 0.8e^{i2\pi(0.0)}\right\}\right) \end{array} \right\},
\end{aligned}
\tag{61}
$$

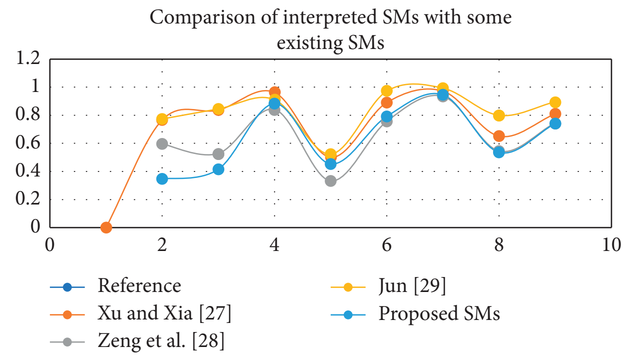

FIGURE 5: The graphical representation of interpreted SMs with some existing SMs for Example 4.

TABLE 6: The comparison between interpreted and some already defined SMs of Example 2.

Method	Score value	Ranking
Xu and Xia [29]	Unsuccessful	Unsuccessful
Zeng et al. [30]	Unsuccessful	Unsuccessful
Jun [31]	Unsuccessful	Unsuccessful
Proposed SM	$\mathbb{S}_c^1(E, E_1) = 0.4733$, $\mathbb{S}_c^1(E, E_2) = 0.2829$ $\mathbb{S}_c^1(E, E_3) = 0.517$ $\mathbb{S}_c^1(E, E_4) = 0.3429$, $\mathbb{S}_c^1(E, E_5) = 0.2844$	$E_3 \geq E_1 \geq E_4 \geq E_5 \geq E_2$
Proposed SM	$\mathbb{S}_c^2(E, E_1) = 0.8674$, $\mathbb{S}_c^2(E, E_2) = 0.6372$, $\mathbb{S}_c^2(E, E_3) = 0.9321$, $\mathbb{S}_c^2(E, E_4) = 0.6047$, $\mathbb{S}_c^2(E, E_5) = 0.7278$	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
Proposed SM	$\mathbb{S}_c^3(E, E_1) = 0.9022$, $\mathbb{S}_c^3(E, E_2) = 0.796$, $\mathbb{S}_c^3(E, E_3) = 0.959$, $\mathbb{S}_c^3(E, E_4) = 0.7405$, $\mathbb{S}_c^3(E, E_5) = 0.8054$	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
Proposed SM	$\mathbb{S}_c^4(E, E_1) = 0.6056$, $\mathbb{S}_c^4(E, E_2) = 0.3777$, $\mathbb{S}_c^4(E, E_3) = 0.6988$, $\mathbb{S}_c^4(E, E_4) = 0.3762$, $\mathbb{S}_c^4(E, E_5) = 0.4605$	$E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$
Proposed SM	$\mathbb{S}_c^5(E, E_1) = 0.6528$, $\mathbb{S}_c^5(E, E_2) = 0.5264$, $\mathbb{S}_c^5(E, E_3) = 0.7563$, $\mathbb{S}_c^5(E, E_4) = 0.4746$, $\mathbb{S}_c^5(E, E_5) = 0.7479$	$E_3 \geq E_5 \geq E_1 \geq E_2 \geq E_4$

and

$$E = \left\{ \begin{array}{c} \left(x_1, \left\{0.9e^{i2\pi(0.0)}, 0.7e^{i2\pi(0.0)}, 1e^{i2\pi(0.0)}\right\}\right), \left(x_2, \left\{0.4e^{i2\pi(0.0)}\right\}\right), \left(x_3, \left\{0.8e^{i2\pi(0.0)}, 0.5e^{i2\pi(0.0)}\right\}\right) \\ \left(x_4, \left\{0.9e^{i2\pi(0.0)}, 0.4e^{i2\pi(0.0)}\right\}\right)\left(x_5, \left\{0.5e^{i2\pi(0.0)}, 0.3e^{i2\pi(0.0)}, 0.2e^{i2\pi(0.0)}\right\}\right) \end{array} \right\}. \quad (62)$$

For Example 4, we need to find that the unknown building material E belongs to which of the specified building material $E_j\,(j = 1, 2, 3, 4, 5)$. In Example 4, the data are in the shape of HFSs. We found similarity between E and $E_j\,(j = 1, \ldots, 5)$ through some already defined SMs for HFSs, as shown in Table 5. As $1 = e^0$, then the data given in Example 4 are transformed into CHFSs. Then, through interpreted SMs, we found the similarity between E and $E_j\,(j = 1, 2, 3, 4, 5)$ which is also given in Table 5. Our interpreted SMs showed that unspecified building material E belongs to the specified building material E_3 because the similarity between E and E_3 is the greatest one. Ranking of the interpreted and already defined SMs is also described in Table 5. Next, we have the graphical representation of the

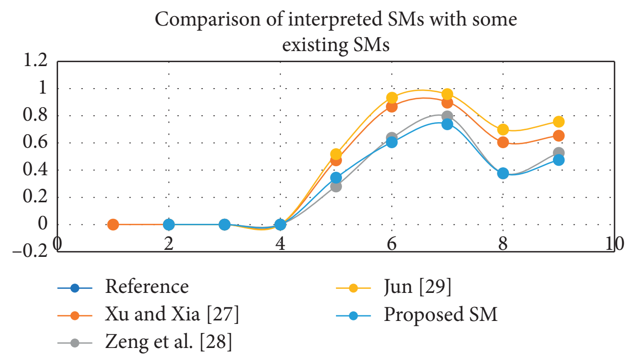

Figure 6: The graphical representation of interpreted SMs with some existing SMs for Example 2.

comparison of the proposed and already defined SMs which is represented in Figure 5.

Now, we discuss the comparison between interpreted and already defined SMs for Example 2. In Example 2, the data are in the shape of CHFSs. We know that no SM exists in the literature to solve this kind of data. The existing SMs are ineffective to find the similarity between E and $E_j\,(j = 1, \ldots, 5)$ as demonstrated in Table 6. From Table 6, we observe that the data given in Example 2 are solvable by the interpreted SMs. The interpreted SMs get the similarity between E and $E_j\,(j = 1, \ldots, 5)$, as demonstrated in Table 6. Our interpreted SMs showed that unspecified building material E belongs to the specified building material E_3 because the similarity between E and E_3 is the greatest one. Ranking of the explored SMs is also introduced in Table 6. Next, we have the graphical representation of the comparison of proposed and already defined SMs which is represented in Figure 6.

From the above discussion, our explored SMs can represent extra fuzzy information and put it broadly in circumstances in real-life problems. Based on CHFS, we explored the SMs; our SMs are more satisfactory for real-life problems, and the existing SMs and our SMs are more general than the existing SMs.

7. Conclusion

The CHFS is one of the enlargements of the CFS in which the possibility of the enrollment work is stretched out from the subset of the genuine number to the unit disc which is interpreted. In this article, we explored another type of similarity measure (SM) which relies on the cosine and cotangent functions. At that stage, we use our introduced SMs and weighted SMs (based on the cosine and cotangent functions) between CHFSs to manage pattern recognition and medical diagnosis problems including design acknowledgment and plan choice. Finally, two numerical models are given to represent the logic and effectiveness of the likeness measures for design acknowledgment and conspire choice. The comparison between explored measure with some existing measures and their graphical representations are also discussed in detail.

Consequently, the measures defined in this manuscript can be utilized in a larger range of applications. In future research, we will extend this work to suppose the two facts: (1) similarity measures and aggregation operators [34–42]; (2) methods [43].

Data Availability

The data used in this article are artificial and hypothetical, and anyone can use these data before prior permission by just citing this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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Retraction

Retracted: Bipolar Fuzzy Implicative Ideals of BCK-Algebras

Journal of Mathematics

Received 10 October 2023; Accepted 10 October 2023; Published 11 October 2023

This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

Bipolar Fuzzy Implicative Ideals of BCK-Algebras

G. Muhiuddin[1] and D. Al-Kadi[2]

[1]*Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia*
[2]*Department of Mathematics and Statistic, College of Science, Taif University, P. O. Box 11099, Taif 21944, Saudi Arabia*

Correspondence should be addressed to G. Muhiuddin; chishtygm@gmail.com

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The notion of bipolar fuzzy implicative ideals of a BCK-algebra is introduced, and several properties are investigated. The relation between a bipolar fuzzy ideal and a bipolar fuzzy implicative ideal is studied. Characterizations of a bipolar fuzzy implicative ideal are given. Conditions for a bipolar fuzzy set to be a bipolar fuzzy implicative ideal are provided. Extension property for a bipolar fuzzy implicative ideal is stated.

1. Introduction

Fuzzy sets are characterized by a membership function which associates elements with real numbers in the interval [0, 1] that represents its membership degree to the fuzzy set. Several kinds of fuzzy set extensions have been introduced such as interval-valued, intuitionistic, and bipolar-valued fuzzy sets. The bipolar-valued fuzzy set notion [1] was introduced to treat imprecision as in traditional fuzzy sets, where the degree of membership belongs to the interval [0, 1], and we cannot tell apart unrelated elements from the opposite elements. The extension here enlarges the range of the membership degree from the interval [0, 1] to the interval [−1, 1] to solve such a problem (we refer the reader to [2–4]). The membership degrees which lie in the interval [−1, 1] represent the satisfaction degree to the corresponding property in a fuzzy set and its counter property as follows: having a membership degree in the interval [−1, 0) means that the elements are satisfying implicit counter property, having (0, 1] means that the elements are satisfying the property, and having 0 means that the elements are unrelated to the corresponding property.

The bipolar-valued fuzzification has been used to study different notions in BCK/BCI-algebras such as subalgebras and ideals of BCK/BCI-algebras [5], a-ideals of BCI-algebras [6], and more, see the references [7–10]. Other researches also added their contribution to the study in this field on different branches of algebra in various aspects (see, e.g., [11–26]). Also, some more general concepts on bipolar fuzzy have been studied in [27–31].

Recently, the bipolar fuzzy BCI-implicative ideals of BCI-algebras were studied in [32]. Moreover, new types of bipolar fuzzy ideals of BCK-algebras have been investigated in [33], typically bipolar fuzzy (closed, positive implicative, and implicative) ideals. Moreover, some related concepts on fuzzy sets and their useful generalizations were applied in various algebraic structures (see, e.g., [33–53]).

In this paper, we apply the notion of a bipolar-valued fuzzy set to implicative ideals of BCK-algebras and obtain further results in this manner. Furthermore, we consider the relation of a bipolar fuzzy ideal with a bipolar fuzzy implicative ideal. We provide characterizations of a bipolar fuzzy implicative ideal. Moreover, we display conditions for a bipolar fuzzy set to be a bipolar fuzzy implicative ideal. Finally, we discuss extension property for a bipolar fuzzy implicative ideal.

2. Preliminaries

The basic results on BCK-algebras are given in this section.

By a *BCK-algebra*, we mean an algebra $(\text{Ł};^{*}, 0)$ of type (2, 0) satisfying the axioms:

(a1) $(\forall \varkappa, \ell, \upsilon \in \text{Ł})(((\varkappa * \ell) * (\varkappa * \upsilon)) * (\upsilon * \ell) = 0)$
(a2) $(\forall \varkappa, \ell \in \text{Ł})((\varkappa * (\varkappa * \ell)) * \ell = 0)$

(a3) $(\forall \varkappa \in Ł)(\varkappa * \varkappa = 0, 0 * \varkappa = 0)$

(a4) $(\forall \varkappa, \ell \in Ł)(\varkappa * \ell = 0, \ell * \varkappa = 0 \Longrightarrow \varkappa = \ell)$

We can define a partial ordering $\leq$ by $\varkappa \leq \ell$ if and only if $\varkappa * \ell = 0$.

In any BCK-algebra Ł, the following hold:

(b1) $(\forall \varkappa \in Ł)(\varkappa * 0 = \varkappa)$

(b2) $(\forall \varkappa, \ell, \upsilon \in Ł)((\varkappa * \ell) * \upsilon = (\varkappa * \upsilon) * \ell)$

(b3) $(\forall \varkappa, \ell, \upsilon \in Ł)((\varkappa * \upsilon) * (\ell * \upsilon) \leq \varkappa * \ell)$

(b4) $(\forall \varkappa, \ell, \upsilon \in Ł)(\varkappa \leq \ell \Rightarrow \varkappa * \upsilon \leq \ell * \upsilon, \upsilon * \ell \leq \upsilon * \varkappa)$

Let us consider a subset $(\varnothing \neq I)$ of a BCK-algebra Ł. We say I is an *ideal* if

(c1) $0 \in I$, (c2) $(\forall \varkappa \in Ł)(\forall \ell \in I)(\varkappa * \ell \in I \Rightarrow \varkappa \in I)$

A nonempty subset I of a BCK-algebra Ł is called an *implicative ideal* of Ł if it satisfies (c1) and

(c3) $(\forall \varkappa, \ell, \upsilon \in Ł)((\varkappa * \ell) * \upsilon \in I, \ell * \upsilon \in I \Rightarrow \varkappa * \upsilon \in I)$

3. Bipolar Fuzzy Ideals

In the following sections, Ł denotes a BCK-algebra.

For any family $\{\delta_i | i \in \Delta\}$ of real numbers, we define

$$\begin{aligned} \vee\{\delta_i | i \in \Delta\} &:= \begin{cases} \max\{\delta_i | i \in \Delta\}, & \text{if } \Delta \text{ is finite,} \\ \sup\{\delta_i | i \in \Delta\}, & \text{otherwise,} \end{cases} \\ \wedge\{\delta_i | i \in \Delta\} &:= \begin{cases} \min\{\delta_i | i \in \Delta\}, & \text{if } \Delta \text{ is finite,} \\ \inf\{\delta_i | i \in \Delta\} & \text{otherwise.} \end{cases} \end{aligned} \tag{1}$$

Moreover, if $\Delta = \{1, 2, \ldots, n\}$, then $\vee\{\delta_i | it \in n\Delta\}$ and $\wedge\{\delta_i | it \in n\Delta\}$ are denoted by $\delta_1 \vee \delta_2 \vee \cdots \vee \delta_n$ and $\delta_1 \wedge \delta_2 \wedge \cdots \wedge \delta_n$, respectively.

For a bipolar fuzzy set $q = (Ł; q_n, q_p)$, we define *negative* α-cut of $q = (Ł; q_n, q_p)$ and the *positive* β-cut of $q = (Ł; q_n, q_p)$, respectively, as follows:

$$\begin{aligned} N(q; \alpha) &:= \{\varkappa \in Ł | q_n(\varkappa) \leq \alpha\}, \\ P(q; \beta) &:= \{\varkappa \in Ł | q_p(\varkappa) \geq \beta\}, \end{aligned} \tag{2}$$

where $(\alpha, \beta) \in [-1, 0) \times (0, 1]$. The set

$$C(q; (\alpha, \beta)) := N(q; \alpha) \cap P(q; \beta) \tag{3}$$

is called the (α, β)-cut of $q = (Ł; q_n, q_p)$. For every $k \in (0, 1)$, if $(\alpha, \beta) = (-k, k)$, then the set

$$C(q; k) := N(q; -k) \cap P(q; k) \tag{4}$$

is called the k-cut of $q = (Ł; q_n, q_p)$.

Definition 1 (see [5]). A bipolar fuzzy set $q = (Ł; q_n, q_p)$ in a BCK-algebra Ł is called a *bipolar fuzzy ideal* of Ł if it satisfies the following assertions:

(i) $(\forall \varkappa \in Ł)(q_n(0) \leq q_n(\varkappa),\ q_p(0) \geq q_p(\varkappa))$

(ii) $(\forall \varkappa, \ell \in Ł)\begin{pmatrix} q_n(\varkappa) \leq q_n(\varkappa * \ell) \vee q_n(\ell), \\ q_p(\varkappa) \geq q_p(\varkappa * \ell) \wedge q_p(\ell). \end{pmatrix}$

For any $w \in Ł$ and any bipolar fuzzy set $q = (Ł; q_n, q_p)$ in Ł, we let

$$I(w) = \{\varkappa \in Ł | q_n(\varkappa) \leq q_n(w), q_p(\varkappa) \geq q_p(w)\}. \tag{5}$$

Obviously, $w \in I(w)$. If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł, then $0 \in I(w)$. The following is our question: *For a bipolar fuzzy set $q = (Ł; q_n, q_p)$ in* Ł *satisfying* Definition 1 (*i*), *is $I(w)$an ideal of* Ł? The following example provides a negative answer; that is, there exists an element $w \in Ł$ such that $I(w)$ is not an ideal of Ł.

Example 1. Let $Ł = \{\theta, \ell, \upsilon, \omega, \delta\}$ be a set with a Cayley table which is given in Table 1.

Then, $(Ł; *, \theta)$ is a BCK-algebra. Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł defined by

	θ	l	υ	w	δ
q_n	−0.7	−0.5	−0.3	−0.1	−0.4
q_p	0.8	0.7	0.4	0.2	0.5

Then, $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i), and it is not a bipolar fuzzy ideal of Ł because

$$q_n(\upsilon) = -0.3 > -0.4 = q_n(\upsilon * \delta) \vee q_n(\delta), \tag{6}$$

and/or

$$q_p(\upsilon) = 0.4 < 0.5 = q_p(\upsilon * \delta) \wedge q_p(\delta). \tag{7}$$

Then, $I(\delta) = \{\theta, \ell, \delta\}$ is not an ideal of Ł since $\upsilon * \delta = \theta \in I(\delta)$ and $\delta \in I(\delta)$, while $\upsilon \notin I(\delta)$. Note that $I(\upsilon) = \{\theta, \ell, \upsilon, \delta\}$ is an ideal of Ł.

We give conditions for the set $I(w)$ to be an ideal.

Theorem 1. *Let $w \in Ł$. If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł, then $I(w)$ is an ideal of Ł.*

Proof. We recall that $0 \in I(w)$. Let $\varkappa, \ell \in Ł$ such that $\varkappa * \ell \in I(w)$ and $\ell \in I(w)$. Then, $q_n(w) \geq q_n(\varkappa * \ell)$, $q_p(w) \leq q_p(\varkappa * \ell)$, $q_n(w) \geq q_n(\ell)$ and $q_p(w) \leq q_p(\ell)$. Since $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł, we have from Definition 1 (ii) that

$$\begin{aligned} q_n(\varkappa) &\leq q_n(\varkappa * \ell) \vee q_n(\ell) \leq q_n(w), \\ q_p(\varkappa) &\geq q_p(\varkappa * \ell) \wedge q_p(\ell) \geq q_p(w), \end{aligned} \tag{8}$$

and so, $\varkappa \in I(w)$. Therefore, $I(w)$ is an ideal of Ł. □

Theorem 2. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł and $w \in Ł$.*

(1) If $I(w)$ is an ideal of Ł, then $q = (Ł; q_n, q_p)$ satisfies the following implications for all $\varkappa, \ell, \upsilon \in Ł$:

$$\begin{aligned} q_n(\varkappa) \geq q_n(\ell * \upsilon) \vee q_n(\upsilon) &\Rightarrow q_n(\varkappa) \geq q_n(\ell), \\ q_p(\varkappa) \leq q_p(\ell * \upsilon) \wedge q_p(\upsilon) &\Rightarrow q_p(\varkappa) \leq q_p(\ell). \end{aligned} \tag{9}$$

(2) If $q = (Ł; q_n, q_p)$ satisfies Definition 1(i) and (9), then $I(w)$ is an ideal of Ł.

Table 1: Cayley table.

$*$	θ	ℓ	υ	ω	δ
θ	θ	θ	θ	θ	θ
ℓ	ℓ	θ	ℓ	θ	θ
υ	υ	υ	θ	υ	θ
ω	ω	ω	ω	θ	ω
δ	δ	δ	δ	δ	θ

Proof

(1) We assume that $I(w)$ is an ideal of Ł for each $w \in$ Ł. We suppose that $q_n(\varkappa) \geq q_n(\ell * \upsilon) \vee q_n(\upsilon)$ and $q_p(\varkappa) \leq q_p(\ell * \upsilon) \wedge q_p(\upsilon)$ for all $\varkappa, \ell, \upsilon \in$ Ł. Then, $\ell * \upsilon \in I(\varkappa)$ and $\upsilon \in I(\varkappa)$. Since $I(\varkappa)$ is an ideal of Ł, it follows that $\ell \in I(\varkappa)$, that is, $q_n(\varkappa) \geq q_n(\ell)$ and $q_p(\varkappa) \leq q_p(\ell)$.

(2) We suppose that $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i) and (9). For each $w \in$ Ł, let $\varkappa, \ell \in$ Ł such that $\varkappa * \ell \in I(w)$ and $\ell \in I(w)$. Then, $q_n(\varkappa * \ell) \leq q_n(w)$, $q_p(\varkappa * \ell) \geq q_p(w)$, $q_n(\ell) \leq q_n(w)$, and $q_p(\ell) \geq q_p(w)$, which imply that $q_n(w) \geq q_n(\varkappa * \ell) \vee q_n(\ell)$ and $q_p(w) \leq q_p(\varkappa * \ell) \wedge q_p(\ell)$. Using (9), we have $q_n(w) \geq q_n(\varkappa)$ and $q_p(w) \leq q_p(\varkappa)$, and so, $\varkappa \in I(w)$. Since $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i), it follows that $0 \in I(w)$. Therefore, $I(w)$ is an ideal of Ł. □

Lemma 1 (see [5]). *Every bipolar fuzzy ideal $q = (Ł; q_n, q_p)$ of Ł satisfies the following implication:*

$$(\forall \varkappa, \ell \in Ł)\left(\varkappa \leq \ell \Rightarrow q_n(\varkappa) \leq q_n(\ell), q_p(\varkappa) \geq q_p(\ell)\right). \tag{10}$$

Proposition 1. *For any bipolar fuzzy ideal $q = (Ł; q_n, q_p)$ of Ł, the following are equivalent:*

(1) $(\forall \varkappa, \ell \in Ł)\begin{pmatrix} q_n(\varkappa * \ell) \leq q_n((\varkappa * \ell) * \ell), \\ q_p(\varkappa * \ell) \geq q_p((\varkappa * \ell) * \ell). \end{pmatrix}$

(2) $(\forall \varkappa, \ell, \upsilon \in Ł)\begin{pmatrix} q_n((\varkappa * \upsilon) * (\ell * \upsilon)) \leq q_n((\varkappa * \ell) * \upsilon), \\ q_p((\varkappa * \upsilon) * (\ell * \upsilon)) \geq q_p((\varkappa * \ell) * \upsilon). \end{pmatrix}$

Proof We assume that condition (2) is valid. Note that

$$((\varkappa * (\ell * \upsilon)) * \upsilon) * \upsilon = ((\varkappa * \upsilon) * (\ell * \upsilon)) * \upsilon \leq (\varkappa * \ell) * \upsilon, \tag{11}$$

for all $\varkappa, \ell, \upsilon \in$ Ł by using (b2), (b3), and (b4). It follows from Lemma 1 that

$$\begin{aligned} q_n((\varkappa * \ell) * \upsilon) &\geq q_n(((\varkappa * (\ell * \upsilon)) * \upsilon) * \upsilon), \\ q_p((\varkappa * \ell) * \upsilon) &\leq q_p(((\varkappa * (\ell * \upsilon)) * \upsilon) * \upsilon). \end{aligned} \tag{12}$$

So, from (b2) and (2), it follows that

$$\begin{aligned} q_n((\varkappa * \upsilon) * (\ell * \upsilon)) &= q_n((\varkappa * (\ell * \upsilon)) * \upsilon) \\ &\leq q_n(((\varkappa * (\ell * \upsilon)) * \upsilon) * \upsilon) \\ &\leq q_n((\varkappa * \ell) * \upsilon), \\ q_p((\varkappa * \upsilon) * (\ell * \upsilon)) &= q_p((\varkappa * (\ell * \upsilon)) * \upsilon) \\ &\geq q_p(((\varkappa * (\ell * \upsilon)) * \upsilon) * \upsilon) \\ &\geq q_p((\varkappa * \ell) * \upsilon). \end{aligned} \tag{13}$$

Thus, (9) holds. Now, we suppose that (9) is valid. Using (b1), (a3), and (9) with replacing υ by ℓ, we have

$$\begin{aligned} q_n(\varkappa * \ell) &= q_n((\varkappa * \ell) * 0) = q_n((\varkappa * \ell) * (\ell * \ell)) \\ &\leq q_n((\varkappa * \ell) * \ell), \\ q_p(\varkappa * \ell) &= q_p((\varkappa * \ell) * 0) = q_p((\varkappa * \ell) * (\ell * \ell)) \\ &\geq q_p((\varkappa * \ell) * \ell), \end{aligned} \tag{14}$$

which proves (2). □

Proposition 2 (see [5]). *A bipolar fuzzy set $q = (Ł; q_n, q_p)$ in Ł is a bipolar fuzzy ideal of Ł if and only if for all $\varkappa, \ell, \upsilon \in$ Ł, $(\varkappa * \ell) * \upsilon = 0$ implies $q_n(\varkappa) \leq q_n(\ell) \vee q_n(\upsilon)$ and $q_p(\varkappa) \geq q_p(\ell) \wedge q_p(\upsilon)$.*

As a generalization of Proposition 2, we have the following results.

Theorem 3. *If a bipolar fuzzy set $q = (Ł; q_n, q_p)$ in Ł is a bipolar fuzzy ideal of Ł, then for all $\varkappa, w_1, w_2, \ldots, w_n \in$ Ł,*

$$\prod_{i=1}^{n} \varkappa * w_i = 0 \Rightarrow \begin{pmatrix} q_n(\varkappa) \leq q_n(w_1) \vee q_n(w_2) \vee \cdots \vee q_n(w_n), \\ q_p(\varkappa) \geq q_p(w_1) \wedge q_p(w_2) \wedge \cdots \wedge q_p(w_n), \end{pmatrix} \tag{15}$$

where $\prod_{i=1}^{n} \varkappa * w_i = (\cdots((\varkappa * w_1) * w_2) * \cdots) * w_n$.

Proof. The proof is by induction on n. Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy ideal of Ł. Lemma 1 and Proposition 2 show that condition (15) is valid for $n = 1, 2$. We assume that $q = (Ł; q_n, q_p)$ satisfies condition (15) for $n = k$, that is, for all $\varkappa, w_1, w_2, \ldots, w_k \in$ Ł, $\prod_{i=1}^{k} \varkappa * w_i = 0$ implies

$$\begin{aligned} q_n(\varkappa) &\leq q_n(w_1) \vee q_n(w_2) \vee \cdots \vee q_n(w_k), \\ q_p(\varkappa) &\geq q_p(w_1) \wedge q_p(w_2) \wedge \cdots \wedge q_p(w_k). \end{aligned} \tag{16}$$

Let $\varkappa, w_1, w_2, \ldots, w_k, w_{k+1} \in$ Ł such that $\prod_{i=1}^{k+1} \varkappa * w_i = 0$. Then,

$$\begin{aligned} q_n(\varkappa * w_1) &\leq q_n(w_2) \vee q_n(w_3) \vee \cdots \vee q_n(w_{k+1}), \\ q_p(\varkappa * w_1) &\geq q_p(w_2) \wedge q_p(w_3) \wedge \cdots \wedge q_p(w_{k+1}). \end{aligned} \tag{17}$$

Since $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł, it follows from Definition 1 (ii) that

$$
\begin{aligned}
q_n(\varkappa) &\leq q_n(\varkappa * w_1) \vee q_n(w_1) \\
&\leq q_n(w_1) \vee q_n(w_2) \vee \cdots \vee q_n(w_{k+1}), \\
q_p(\varkappa) &\geq q_p(\varkappa * w_1) \wedge q_p(w_1) \\
&\geq q_p(w_1) \wedge q_p(w_2) \wedge \cdots \wedge q_p(w_{k+1}).
\end{aligned}
\tag{18}
$$

This completes the proof. □

Now, we consider the converse of Theorem 3.

Theorem 4. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł satisfying condition (15). Then, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł.*

Proof. Note that $(\cdots((0 * \underbrace{\varkappa) * \varkappa) * \cdots) * \varkappa}_{n \text{ times}} = 0$ for all $\varkappa \in Ł$. It follows from (15) that $q_n(0) \leq q_n(\varkappa)$ and $q_p(0) \geq q_p(\varkappa)$ for all $\varkappa \in Ł$. Let $\varkappa, \ell, \upsilon \in Ł$ such that $\varkappa * \ell \leq \upsilon$. Then,

$$
0 = (\varkappa * \ell) * \upsilon = (\cdots(((\varkappa * \ell) * \upsilon) * \underbrace{0) * \cdots) * 0}_{n-2 \text{ times}},
\tag{19}
$$

and so,

$$
\begin{aligned}
q_n(\varkappa) &\leq q_n(\ell) \vee q_n(\upsilon) \vee q_n(0) = q_n(\ell) \vee q_n(\upsilon), \\
q_p(\varkappa) &\geq q_p(\ell) \wedge q_p(\upsilon) \wedge q_p(0) = q_p(\ell) \wedge q_p(\upsilon).
\end{aligned}
\tag{20}
$$

Hence, by Proposition 2, we conclude that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł. □

4. Bipolar Fuzzy Implicative Ideals

Definition 2. A bipolar fuzzy set $q = (Ł; q_n, q_p)$ in Ł is called a *bipolar fuzzy implicative ideal* of Ł if both the nonempty negative α-cut and the nonempty positive β-cut of $q = (Ł; q_n, q_p)$ are implicative ideals of Ł for all $(\alpha, \beta) \in [-1, 0] \times [0, 1]$.

Example 2. Let $Ł = \{\theta, \ell, \upsilon\}$ be a set in which the operation $*$ is defined by Table 2.

Then, $(Ł; *, \theta)$ is a BCK-algebra. Let $(t_0, s_0), (t_1, s_1) \in [-1, 0] \times [0, 1]$ satisfy $(t_0, s_0) > (t_1, s_1)$, that is, $t_0 < t_1$ and $s_0 > s_1$. Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł given by

	θ	l	υ
q_n	t_0	t_0	t_1
q_p	s_0	s_0	s_1

By routine calculations, we know that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł.

Theorem 5. *A bipolar fuzzy set $q = (Ł; q_n, q_p)$ in Ł is a bipolar fuzzy implicative ideal of Ł if and only if it satisfies Definition 1 (i) and the following assertions:*

$$
(\forall \varkappa, \ell, \upsilon \in Ł) \begin{pmatrix} q_n(\varkappa * \upsilon) \leq q_n((\varkappa * \ell) * \upsilon) \vee q_n(\ell * \upsilon), \\ q_p(\varkappa * \upsilon) \geq q_p((\varkappa * \ell) * \upsilon) \wedge q_p(\ell * \upsilon). \end{pmatrix}
\tag{21}
$$

Table 2: Cayley table.

$*$	θ	ℓ	υ
θ	θ	θ	θ
ℓ	ℓ	θ	θ
υ	υ	υ	θ

Proof. We suppose that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. If $q_n(0) > q_n(b)$ or $q_p(0) < q_p(d)$ for some $b, d \in Ł$, then $0 \notin N(q; q_n(b))$ or $0 \notin P(q; q_p(d))$, which contradicts the fact. Hence, $q_n(0) \leq q_n(\varkappa)$ and $q_p(0) \geq q_p(\varkappa)\ \forall \varkappa \in Ł$. For some $b, d, c \in Ł$, we assume that we have the following relation:

$$
q_n(b * c) > q_n((b * d) * c) \vee q_n(d * c) = s.
\tag{22}
$$

Then, $(b * d) * c \in N(q; s)$ and $d * c \in N(q; s)$, but $b * c \notin N(q; s)$. This is not possible; therefore, we have

$$
q_n(\varkappa * \upsilon) \leq q_n((\varkappa * \ell) * \upsilon) \vee q_n(\ell * \upsilon),
\tag{23}
$$

for all $\varkappa, \ell, \upsilon \in Ł$. If $q_p(b * c) < q_p((b * d) * c) \wedge q_p(d * c) = t$ for some $b, d, c \in Ł$, then $(b * d) * c \in P(q; t)$ and $d * c \in P(q; t)$, but $b * c \notin P(q; t)$. We reach a contradiction because $P(q; t)$ is an implicative ideal of Ł. Henceforth,

$$
q_p(\varkappa * \upsilon) \geq q_p((\varkappa * \ell) * \upsilon) \wedge q_p(\ell * \upsilon),
\tag{24}
$$

for all $\varkappa, \ell, \upsilon \in Ł$. Consequently, a bipolar fuzzy implicative ideal $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i) and (21).

Conversely, we suppose that $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i) and (21) and let $(\alpha, \beta) \in [-1, 0] \times [0, 1]$ s.th. $N(q; \alpha) \neq \varnothing$ and $P(q; \beta) \neq \varnothing$. It is clear that $0 \in N(q; \alpha) \cap P(q; \beta)$. Let $\varkappa, \ell, \upsilon \in Ł$ be such that $(\varkappa * \ell) * \upsilon \in N(q; \alpha)$ and $\ell * \upsilon \in N(q; \alpha)$. Then, $q_n((\varkappa * \ell) * \upsilon) \leq \alpha$ and $q_n(\ell * \upsilon) \leq \alpha$. It follows from (21) that

$$
q_n(\varkappa * \upsilon) \leq q_n((\varkappa * \ell) * \upsilon) \vee q_n(\ell * \upsilon) \leq \alpha,
\tag{25}
$$

and so, $\varkappa * \upsilon \in N(q; \alpha)$. Hence, $N(q; \alpha)$ is an implicative ideal of Ł. Similarly, we can show that

$$
q_p(\varkappa * \upsilon) \geq q_p((\varkappa * \ell) * \upsilon) \wedge q_p(\ell * \upsilon) \geq \beta,
\tag{26}
$$

for all $\varkappa, \ell, \upsilon \in Ł$, and so, $\varkappa * \upsilon \in P(q; \beta)$. Therefore, $P(q; \beta)$ is an implicative ideal of Ł. Consequently, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. □

Next, we have the following theorems.

Theorem 6. *A bipolar fuzzy ideal $q = (Ł; q_n, q_p)$ of Ł is a bipolar fuzzy implicative ideal of Ł if and only if it satisfies Proposition 1 (1).*

Proof. Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy implicative ideal of Ł. If υ is replaced by ℓ in (21), then

$$
\begin{aligned}
q_n(\varkappa * \ell) &\leq q_n((\varkappa * \ell) * \ell) \vee q_n(\ell * \ell), \\
&= q_n((\varkappa * \ell) * \ell) \vee q_n(0), \\
&= q_n((\varkappa * \ell) * \ell),
\end{aligned}
$$

$$\begin{aligned} q_p(\varkappa * \ell) &\geq q_p((\varkappa * \ell) * \ell) \wedge q_p(\ell * \ell), \\ &= q_p((\varkappa * \ell) * \ell) \wedge q_p(0), \\ &= q_p((\varkappa * \ell) * \ell), \end{aligned} \tag{27}$$

which is Proposition 1 (1). Conversely, let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy ideal of Ł satisfying Proposition 1 (1). Note that

$$((\varkappa * \upsilon) * \upsilon) * (\ell * \upsilon) \leq (\varkappa * \upsilon) * \ell = (\varkappa * \ell) * \upsilon, \tag{28}$$

for all $\varkappa, \ell, \upsilon \in Ł$. Using Lemma 1, we have

$$\begin{aligned} q_n((\varkappa * \ell) * \upsilon) &\geq q_n(((\varkappa * \upsilon) * \upsilon) * (\ell * \upsilon)), \\ q_p((\varkappa * \ell) * \upsilon) &\leq q_p(((\varkappa * \upsilon) * \upsilon) * (\ell * \upsilon)). \end{aligned} \tag{29}$$

It follows from Definition 1 (ii) and Proposition 1 (1) that

$$\begin{aligned} q_n(\varkappa * \upsilon) &\leq q_n((\varkappa * \upsilon) * \upsilon) \\ &\leq q_n(((\varkappa * \upsilon) * \upsilon) * (\ell * \upsilon)) \vee q_n(\ell * \upsilon) \\ &\leq q_n((\varkappa * \ell) * \upsilon) \vee q_n(\ell * \upsilon), \\ q_p(\varkappa * \upsilon) &\geq q_p((\varkappa * \upsilon) * \upsilon) \\ &\geq q_p(((\varkappa * \upsilon) * \upsilon) * (\ell * \upsilon)) \wedge q_p(\ell * \upsilon) \\ &\geq q_p((\varkappa * \ell) * \upsilon) \wedge q_p(\ell * \upsilon). \end{aligned} \tag{30}$$

Thus, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. □

Combining Proposition 1 and Theorem 6, we have the following characterization of a bipolar fuzzy implicative ideal.

Theorem 7. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy ideal of Ł. Then, it is a bipolar fuzzy implicative ideal of Ł if and only if it satisfies Proposition 1 (2).*

Theorem 8 (see [33]). *Every bipolar fuzzy implicative ideal is a bipolar fuzzy ideal.*

Theorem 9. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł. Then, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł if and only if it satisfies Definition 1 (i) and*

$$(\forall \varkappa, \ell, \upsilon \in Ł) \begin{pmatrix} q_n(\varkappa * \ell) \leq q_n(((\varkappa * \ell) * \ell) * \upsilon) \vee q_n(\upsilon), \\ q_p(\varkappa * \ell) \geq q_p(((\varkappa * \ell) * \ell) * \upsilon) \wedge q_p(\upsilon). \end{pmatrix} \tag{31}$$

Proof. We suppose that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. Then, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł by Theorem 8, and so, Definition 1 (i) is true. From Theorem 7, it follows that $q = (Ł; q_n, q_p)$ satisfies Proposition 1 (2). Thus,

$$\begin{aligned} q_n(\varkappa * \ell) &\leq q_n((\varkappa * \ell) * \upsilon) \vee q_n(\upsilon), \\ &= q_n(((\varkappa * \upsilon) * \ell) * (\ell * \ell)) \vee q_n(\upsilon) \\ &\leq q_n(((\varkappa * \upsilon) * \ell) * \ell) \vee q_n(\upsilon), \\ &= q_n(((\varkappa * \ell) * \ell) * \upsilon) \vee q_n(\upsilon) \\ q_p(\varkappa * \ell) &\geq q_p((\varkappa * \ell) * \upsilon) \wedge q_p(\upsilon), \\ &= q_p(((\varkappa * \upsilon) * \ell) * (\ell * \ell)) \wedge q_p(\upsilon) \\ &\geq q_p(((\varkappa * \upsilon) * \ell) * \ell) \wedge q_p(\upsilon), \\ &= q_p(((\varkappa * \ell) * \ell) * \upsilon) \wedge q_p(\upsilon), \end{aligned} \tag{32}$$

which proves (31). Conversely, let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł satisfying Definition 1 (i) and (31). Then,

$$\begin{aligned} q_n(\varkappa) &= q_n(\varkappa * 0) \leq q_n(((\varkappa * 0) * 0) * \upsilon) \vee q_n(\upsilon) \\ &= q_n(\varkappa * \upsilon) \vee q_n(\upsilon), \\ q_p(\varkappa) &= q_p(\varkappa * 0) \geq q_p(((\varkappa * 0) * 0) * \upsilon) \wedge q_p(\upsilon) \\ &= q_p(\varkappa * \upsilon) \wedge q_p(\upsilon). \end{aligned} \tag{33}$$

Thus, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł. Now, we take $\upsilon = 0$ in (31) and use (b1) and Definition 1 (i) to get

$$\begin{aligned} q_n(\varkappa * \ell) &\leq q_n(((\varkappa * \ell) * \ell) * 0) \vee q_n(0), \\ &= q_n((\varkappa * \ell) * \ell) \vee q_n(0), \\ &= q_n((\varkappa * \ell) * \ell), \\ q_p(\varkappa * \ell) &\geq q_p(((\varkappa * \ell) * \ell) * 0) \wedge q_p(0), \\ &= q_p((\varkappa * \ell) * \ell) \wedge q_p(0), \\ &= q_p((\varkappa * \ell) * \ell). \end{aligned} \tag{34}$$

It follows from Theorem 6 that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. □

Summarizing the abovementioned results, we have a characterization of a bipolar fuzzy implicative ideal of Ł.

Theorem 10. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł. Then, the following assertions are equivalent:*

(1) $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł

(2) $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i) and (21)

(3) $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł satisfying Proposition 1 (1)

(4) $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł satisfying Proposition 1 (2)

(5) $q = (Ł; q_n, q_p)$ satisfies Definition 1 (i) and (31)

Theorem 11. *Let $w \in Ł$. If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, then $I(w)$ is an implicative ideal of Ł.*

Proof. We recall that $0 \in I(w)$. Let $\varkappa, \ell, \upsilon \in Ł$ such that $(\varkappa * \ell) * \upsilon \in I(w)$ and $\ell * \upsilon \in I(w)$. Then, $q_n(w) \geq q_n((\varkappa * \ell) * \upsilon)$, $q_p(w) \leq q_p((\varkappa * \ell) * \upsilon)$, $q_n(w) \geq q_n(\ell * \upsilon)$, and $q_p(w) \leq q_p(\ell * \upsilon)$. Since $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, it follows from (21) that

$$\begin{aligned} q_n(\varkappa * v) &\le q_n((\varkappa * \ell) * v) \vee q_n(\ell * v) \le q_n(w), \\ q_p(\varkappa * v) &\ge q_p((\varkappa * \ell) * v) \wedge q_p(\ell * v) \ge q_p(w), \end{aligned} \tag{35}$$

so that $\varkappa * v \in I(w)$. Therefore, $I(w)$ is an implicative ideal of Ł. □

Theorem 12. *If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, then for all $\varkappa, \ell, v, a, b \in Ł$,*

(1) $((\varkappa * \ell) * \ell) * a \le b \Rightarrow \begin{pmatrix} q_n(\varkappa * \ell) \le q_n(a) \vee q_n(b), \\ q_p(\varkappa * \ell) \ge q_p(a) \wedge q_p(b). \end{pmatrix}$

(2) $((\varkappa * \ell) * v) * a \le b \Rightarrow \begin{pmatrix} q_n((\varkappa * v) * (\ell * v)) \le q_n(a) \vee q_n(b), \\ q_p((\varkappa * v) * (\ell * v)) \ge q_p(a) \wedge q_p(b). \end{pmatrix}$

Proof. Let $\varkappa, \ell, a, b \in Ł$ such that $((\varkappa * \ell) * \ell) * a \le b$. Using Proposition 2, we have $q_n((\varkappa * \ell) * \ell) \le q_n(a) \vee q_n(b)$ and $q_p((\varkappa * \ell) * \ell) \ge q_p(a) \wedge q_p(b)$. It follows that

$$\begin{aligned} q_n(\varkappa * \ell) &\le q_n((\varkappa * \ell) * \ell) \vee q_n(\ell * \ell), \\ &= q_n((\varkappa * \ell) * \ell) \vee q_n(0), \\ &= q_n((\varkappa * \ell) * \ell) \\ &\le q_n(a) \vee q_n(b), \\ q_p(\varkappa * \ell) &\ge q_p((\varkappa * \ell) * \ell) \wedge q_p(\ell * \ell), \\ &= q_p((\varkappa * \ell) * \ell) \wedge q_p(0), \\ &= q_p((\varkappa * \ell) * \ell) \\ &\ge q_p(a) \wedge q_p(b). \end{aligned} \tag{36}$$

Now, let $\varkappa, \ell, v, a, b \in Ł$ such that $((\varkappa * \ell) * v) * a \le b$, that is,

$$(((\varkappa * \ell) * v) * a) * b = 0. \tag{37}$$

Since $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, it follows from Theorem 7 and Proposition 2 that

$$\begin{aligned} q_n((\varkappa * v) * (\ell * v)) &\le q_n((\varkappa * \ell) * v) \le q_n(a) \vee q_n(b), \\ q_p((\varkappa * v) * (\ell * v)) &\ge q_p((\varkappa * \ell) * v) \ge q_p(a) \wedge q_p(b). \end{aligned} \tag{38}$$

This completes the proof. □

Theorem 13. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł satisfying Theorem 12 (1). Then, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł.*

Proof. We first prove that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł. Let $\varkappa, \ell, v \in Ł$ such that $\varkappa * \ell \le v$. Then,

$$\begin{aligned} (((\varkappa * 0) * 0) * \ell) * v = (\varkappa * \ell) * v = 0, \\ \text{that is, } ((\varkappa * 0) * 0) * \ell \le v, \end{aligned} \tag{39}$$

which implies from (b1) and Theorem 12 (1) that $q_n(\varkappa) = q_n(\varkappa * 0) \le q_n(\ell) \vee q_n(v)$ and $q_p(\varkappa) = q_p(\varkappa * 0) \ge q_p(\ell) \wedge q_p(v)$. Therefore, by Proposition 2, we know that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy ideal of Ł. Note that $(((\varkappa * \ell) * \ell) * ((\varkappa * \ell) * \ell)) * 0 = 0$ for all $\varkappa, \ell \in Ł$. Using Theorem 12 (1) and Definition 1 (i), we have

$$\begin{aligned} q_n(\varkappa * \ell) &\le q_n((\varkappa * \ell) * \ell) \vee q_n(0) = q_n((\varkappa * \ell) * \ell), \\ q_p(\varkappa * \ell) &\ge q_p((\varkappa * \ell) * \ell) \wedge q_p(0) = q_p((\varkappa * \ell) * \ell), \end{aligned} \tag{40}$$

and so, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł by Theorem 6. □

Theorem 14. *Let $q = (Ł; q_n, q_p)$ be a bipolar fuzzy set in Ł satisfying Theorem 12 (2). Then, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł.*

Proof. Let $\varkappa, \ell, a, b \in Ł$ such that $((\varkappa * \ell) * \ell) * a \le b$, that is,

$$(((\varkappa * \ell) * \ell) * a) * b = 0. \tag{41}$$

Then,

$$\begin{aligned} q_n(\varkappa * \ell) &= q_n((\varkappa * \ell) * 0) = q_n((\varkappa * \ell) * (\ell * \ell)) \\ &\le q_n(a) \vee q_n(b), \\ q_p(\varkappa * \ell) &= q_p((\varkappa * \ell) * 0) = q_p((\varkappa * \ell) * (\ell * \ell)) \\ &\ge q_p(a) \wedge q_p(b), \end{aligned} \tag{42}$$

and so, $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł by Theorem 13. □

Corollary 1. *If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, then*

$$\begin{aligned} q_n((\varkappa * v) * (\ell * v)) &\le \vee\{q_n(w_i) | i = 1, 2, \cdots, n\}, \\ q_p((\varkappa * v) * (\ell * v)) &\ge \wedge\{q_p(w_i) | i = 1, 2, \cdots, n\}, \end{aligned} \tag{43}$$

whenever $\prod_{i=1}^{n} ((\varkappa * \ell) * v) * w_i = 0$ for all $\varkappa, \ell, v, w_1, \ldots, w_n \in Ł$.

Proof. Let $\varkappa, \ell, v, w_1, \ldots, w_n \in Ł$ such that $\prod_{i=1}^{n} ((\varkappa * \ell) * v) * w_i = 0$. Then,

$$\begin{aligned} q_n((\varkappa * v) * (\ell * v)) &\le q_n((\varkappa * \ell) * v) \\ &\le \vee\{q_n(w_i) | i = 1, 2, \cdots, n\}, \\ q_p((\varkappa * v) * (\ell * v)) &\ge q_p((\varkappa * \ell) * v) \\ &\ge \wedge\{q_p(w_i) | i = 1, 2, \cdots, n\}. \end{aligned} \tag{44}$$

This completes the proof. □

Theorem 15 (Extension Property). *Let $q = (Ł; q_n, q_p)$ and $g = (Ł; g_n, g_p)$ be bipolar fuzzy ideals of Ł such that $q_n(0) = g_n(0)$ and $q_p(0) = g_p(0)$ and $q_n(\varkappa) \ge g_n(\varkappa)$ and $q_p(\varkappa) \le g_p(\varkappa)$ for all $\varkappa \in Ł$. If $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł, then so is $g = (Ł; g_n, g_p)$.*

Proof. We assume that $q = (Ł; q_n, q_p)$ is a bipolar fuzzy implicative ideal of Ł. For any $\varkappa, \ell, v \in Ł$, we have

$$
\begin{aligned}
& g_n(((\varkappa * v) * (\ell * v)) * ((\varkappa * \ell) * v)), \\
& \quad = g_n(((\varkappa * v) * ((\varkappa * \ell) * v)) * (\ell * v)), \\
& \quad = g_n(((\varkappa * ((\varkappa * \ell) * v)) * v) * (\ell * v)) \\
& \quad \leq q_n(((\varkappa * ((\varkappa * \ell) * v)) * v) * (\ell * v)) \\
& \quad \leq q_n(((\varkappa * ((\varkappa * \ell) * v)) * \ell) * v), \\
& \quad = q_n(((\varkappa * \ell) * ((\varkappa * \ell) * v)) * v), \\
& \quad = q_n(((\varkappa * \ell) * v) * ((\varkappa * \ell) * v)), \\
& \quad = q_n(0) = g_n(0), \\
& g_p(((\varkappa * v) * (\ell * v)) * ((\varkappa * \ell) * v)), \\
& \quad = g_p(((\varkappa * v) * ((\varkappa * \ell) * v)) * (\ell * v)), \\
& \quad = g_p(((\varkappa * ((\varkappa * \ell) * v)) * v) * (\ell * v)) \\
& \quad \geq q_p(((\varkappa * ((\varkappa * \ell) * v)) * v) * (\ell * v)) \\
& \quad \geq q_p(((\varkappa * ((\varkappa * \ell) * v)) * \ell) * v), \\
& \quad = q_p(((\varkappa * \ell) * ((\varkappa * \ell) * v)) * v), \\
& \quad = q_p(((\varkappa * \ell) * v) * ((\varkappa * \ell) * v)), \\
& \quad = q_p(0) = g_p(0).
\end{aligned} \tag{45}
$$

It follows from Definition 1 (i) and (ii) that

$$
\begin{aligned}
& g_n((\varkappa * v) * (\ell * v)) \\
& \quad \leq g_n(((\varkappa * v) * (\ell * v)) * ((\varkappa * \ell) * v)) \vee g_n((\varkappa * \ell) * v) \\
& \quad \leq g_n(0) \vee g_n((\varkappa * \ell) * v), \\
& \quad = g_n((\varkappa * \ell) * v), \\
& g_p((\varkappa * v) * (\ell * v)) \\
& \quad \geq g_p(((\varkappa * v) * (\ell * v)) * ((\varkappa * \ell) * v)) \wedge g_p((\varkappa * \ell) * v) \\
& \quad \geq g_p(0) \wedge g_p((\varkappa * \ell) * v), \\
& \quad = g_p((\varkappa * \ell) * v),
\end{aligned} \tag{46}
$$

for all $\varkappa, \ell, v \in$ Ł. Hence, by Theorem 7, $g = (\text{Ł}; g_n, g_p)$ is a bipolar fuzzy implicative ideal of Ł. □

5. Conclusions

In the present paper, we apply the notion of a bipolar-valued fuzzy set to implicative ideals of BCK-algebras and obtain more related results. We considered the relation of a bipolar fuzzy ideal with a bipolar fuzzy implicative ideal and provided characterizations of a bipolar fuzzy implicative ideal. Also, we studied conditions for a bipolar fuzzy set to be a bipolar fuzzy implicative ideal. Furthermore, an extension property for a bipolar fuzzy implicative ideal is discussed.

We hope that this work will give a deep impact on the upcoming research in this field and other fuzzy algebraic study to open up new horizons of interest and innovations. One may apply this concept to study some application fields such as decision making, knowledge base system, and data analysis. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as subtraction algebras, B-algebras, MV-algebras, d-algebras, and Q-algebras.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: Interval Information Content of Fuzzy Relation and the Application in the Fuzzy Implication Operators

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] Y. Shi, "Interval Information Content of Fuzzy Relation and the Application in the Fuzzy Implication Operators," *Journal of Mathematics*, vol. 2021, Article ID 6611367, 11 pages, 2021.

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Research Article

Interval Information Content of Fuzzy Relation and the Application in the Fuzzy Implication Operators

Yiying Shi

School of Science, Shenyang Ligong University, Shenyang 110159, China

Correspondence should be addressed to Yiying Shi; shiyiying98@163.com

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In rule optimization, some rule characteristics were extracted to describe the uncertainty correlations of fuzzy relations, but the concrete numbers cannot express correlations with uncertainty, such as "at least 0.1 and up to 0.5." To solve this problem, a novel definition concerning interval information content of fuzzy relation has been proposed in this manuscript to realize the fuzziness measurement of the fuzzy relation. Also, its definition and expressions have also been constructed. Meanwhile based on the interval information content, the issues of fuzzy implication ranking and clustering were analyzed. Finally, utilizing the combination of possibility's interval comparison equations and interval value's similarity measure, the classifications of implication operators were proved to be realizable. The achievements in the presented work will provide a reasonable index to measure the fuzzy implication operators and lay a solid foundation for further research.

1. Introduction

Nowadays, we are in the midst of an information revolution, which is driving the development and deployment of new kinds of science and technology with ever-increasing depth and breadth. Information is related to data and knowledge, as data represents the values attributed to parameters, and knowledge signifies the understanding of real things or abstract concepts [1]. With the development of computer science, the amount of information generated by people has grown from a trickle to a torrent. In 1948, the definition of information theory was first proposed by Shannon, in which the statistics method was used to measure the information content quantitatively.

The rapid progress of information theory makes people realize its significance [2], and its conception has been applied in many regions such as communication, decision making, and pattern recognition [3–5]. But unfortunately, the application research studies of information theory in semantic and pragmatic information science have not been conducted widely until now. As the era of big data has been opened, useful information must be mined from more and more data. In this process, the information needs to be expressed by various rules. From a practical standpoint, it is difficult to describe decision makers' experience with precise mathematical models. So, how to select and evaluate rules is the key issue to realize the control of fuzzy system, which can be summarized as rule optimization [6, 7].

To solve this issue, many researches have been conducted to develop several methods, which can be divided into two categories: (1) by means of extracting some rule characteristics [8–13], such as uncertainties of operators by Yu et al. [8], information entropy by Sendi and Ayoubi [7], and fuzzy reliance by Hu et al. [12], the optimizations of fuzzy systems have been realized. (2) First, the structure of fuzzy rules was established; then, some algorithms [14–19], such as the gradient descent method [14] and neural networks [16], have been used to optimize the variable parameters in fuzzy systems. In the classical compositional rule of inference methods, the fuzzy rules were often converted into implication operators. So, many fuzzy implication operators can be constructed [20–24], and for them, the research on how to realize better control of fuzzy systems is still lacking. To address these issues, a novel method has been proposed in the paper, utilizing which the interval information contents of fuzzy relations have been extracted to realize the ranking, clustering, and classification.

Information content is used to describe the correlations between the sets in fuzzy relation. But, owing to the complexities of the things and the uncertainties of human cognition, the concrete numbers cannot be used to express the correlations between two sets. For example, when the correlation is "at least 0.1 and up to 0.5," how to measure it is still an unsolved problem. In order to solve this problem, a new definition of uncertainty measurement is constructed, which is named as the interval information content of the fuzzy relation. First, the fuzzy relation of the interval information content was proposed, and five different expressions were developed, with which the ranking, clustering, and classification of fuzzy implication operators have been realized.

2. Preliminaries

In this section, some definitions and theories involved in this paper are introduced.

Definition 1 (see [8]). Let X and Y be two sets, a fuzzy relation R from X to Y be a fuzzy subset of $X \times Y$, and $R(x, y)$ be the membership degree of x and y to fuzzy relation R, and the class of all fuzzy relations from X to Y can be denoted by $\mathcal{F}(X \times Y)$.

Let $X = \{x_1, x_2, \ldots, x_m\}, Y = \{y_1, y_2, \ldots, y_n\}$ be the finite sets and $r_{ij} = R(x_i, y_i)$; then, the fuzzy relation R can be denoted by fuzzy relation matrix $R = (r_{ij})_{m\times n}$.

Remark 1

(1) For fuzzy relation matrix $R = (r_{ij})_{m\times n}, S = (S_{ij})_{m\times n}$, the operations of the fuzzy relation matrix are defined as follows:

$$\begin{aligned} R \cap S &= \left(r_{ij} \wedge S_{ij}\right)_{m\times n},\\ R \cup S &= \left(r_{ij} \vee S_{ij}\right)_{m\times n},\\ R^c &= \left(1 - r_{ij}\right)_{m\times n}, \end{aligned} \tag{1}$$

where $r_{ij} \wedge s_{ij} \triangleq \min(r_{ij}, s_{ij})$ and $r_{ij} \vee s_{ij} \triangleq \max(r_{ij}, s_{ij})$.

(2) $R_\lambda = \{(x, y) | R(x, y) \geq \lambda\}$ is defined as the λ-cut relations of R. Furthermore, the $((r_{ij})_\lambda)_{m\times n}$ is defined as λ-cut matrix of R with the expression as follows:

$$\left(r_{ij}\right)_\lambda = \begin{cases} 1, & r_{ij} \geq \lambda,\\ 0, & r_{ij} < \lambda. \end{cases} \tag{2}$$

Definition 2 (see [10]). Let $X = \{x_1, x_2, \ldots, x_m\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and R be a fuzzy relation from X to Y, and the information content of R is measured as follows:

$$\mathrm{IC}(R) = \frac{m}{m+n}\mathrm{IC}(R|X) + \frac{n}{m+n}\mathrm{IC}(R|Y), \tag{3}$$

where $\mathrm{IC}(R|X), \mathrm{IC}(R|Y)$ are the information contents of R restricted on X and Y, respectively with the expression as follows:

$$\begin{aligned} \mathrm{IC}(R \mid X) &= -\sum_{i=1}^{m} \frac{\sum_{j=1}^{n} R\left(x_i, y_j\right)}{\sum_{i=1}^{m}\sum_{j=1}^{n} R\left(x_i, y_j\right)} \log_2 \frac{\sum_{j=1}^{n} R\left(x_i, y_j\right)}{n},\\ \mathrm{IC}(R|Y) &= -\sum_{j=1}^{n} \frac{\sum_{i=1}^{m} R^{-1}\left(y_j, x_i\right)}{\sum_{j=1}^{n}\sum_{i=1}^{m} R^{-1}\left(y_j, x_i\right)} \log_2 \frac{\sum_{i=1}^{m} R^{-1}\left(y_j, x_i\right)}{m}. \end{aligned} \tag{4}$$

The U-uncertainty of A is also used to measure the information content of fuzzy sets.

Definition 3 (see [25]). A is a fuzzy set defined on $X = \{x_1, x_2, \ldots, x_m\}$, and all $A(x_i)$ ($i = 1, 2, \ldots, m$) can be designed to an ordered possibility distribution $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. It is always the case that $\lambda_{i+1} \leq \lambda_i$; then,

$$\begin{aligned} U(A) &= -\sum_{i=1}^{m} (\lambda_i - \lambda_{i+1}) \log_2 \left|A_{\lambda_i}\right|\\ &= -\sum_{i=1}^{m} \lambda_i \left(\log_2 \left|A_{\lambda_i}\right| - \log_2 \left|A_{\lambda_i - 1}\right|\right) \end{aligned} \tag{5}$$

is defined as the U-uncertainty of A, $|\cdot|$ is the cardinality of a set, and

$$A_{\lambda_i} = \{x \in X | A(x) \geq \lambda_i\}. \tag{6}$$

Definition 4 (see [20]). A fuzzy implication operator is any mapping I: $[0,1] \times [0,1] \longrightarrow [0,1]$ satisfying the border conditions:

(P1) $\exists\ a \in [0,1]$, $b \in [0,1]$, $I(a, b) = 1$

(P2) $\exists\ c \in [0,1]$, $d \in [0,1]$, $I(c, d) = 0$

Furthermore,

(P3) If $I(1,0) = 0$, $I(0,1) = I(1,1) = I(0,0) = 1$, then I is a normal implication operator. Otherwise, it is called an abnormal implication operator.

For instance,

(1) Zadeh operator: $I_1(a,b) = (1-a) \vee (a \wedge b)$
(2) Kleene–Dienes operator: $I_2(a,b) = (1-a) \vee b$
(3) Lukasiewicz operator: $I_3(a,b) = (1-a+b) \wedge 1$
(4) Reichenbach operator: $I_4(a,b) = 1 - a + ab$
(5) Mamdani operator: $I_5(a,b) = a \wedge b$
(6) Probability product operator: $I_6(a,b) = ab$
(7) R_0 operator:

$$I_7(a,b) = \begin{cases} 1, & a \leq b,\\ (1-a) \vee b, & a > b \end{cases} \tag{7}$$

(8) Goguen operator:

$$I_8(a,b) = \begin{cases} 1, & a = 0,\\ \left(\dfrac{b}{a}\right) \wedge 1, & a > 0 \end{cases} \tag{8}$$

(9) Gaines–Reseher operator:

$$I_9(a,b)=\begin{cases}1, & a\le b,\\ 0, & a>b\end{cases} \tag{9}$$

(10) Yager operator:

$$I_{10}(a,b)=b^a \tag{10}$$

(11) Bounded product operator: $I_{11}(a,b)=(a+b-1)\vee 0$

(12) Gödel operator:

$$I_{12}(a,b)=\begin{cases}1, & a\le b,\\ b, & a>b\end{cases} \tag{11}$$

(13) $I_{13}(a,b)=\begin{cases}1, & a\le b,\\ 1-a, & a>b.\end{cases}$

To indicate the degree of similarity of two fuzzy sets, the concept of similarity measure is proposed as follows.

Definition 5 (see [26, 27]). A real function S_I: $D\times D\longrightarrow [0,1]$ is called similarity measure, where $D=\{[a^-,a^+]|0\le a^-\le a^+\le 1\}$, if S_I satisfies the following properties:

(S_I1) $S_I(A,A^c)=0$ if A is a crisp set

(S_I2) $S_I(A,B)=1\Longleftrightarrow A=B$

(S_I 3) $S_I(A,B)=S_I(B,A)$

(S_I 4) $\forall A,B,C\in D$, if $A\subseteq B\subseteq C$, then $S_I(A,C)\le S_I(A,B)$, $S_I(A,C)\le S_I(B,C)$

For instance, let $A=[a^-,a^+], B=[b^-,b^+]\in D$, and

$$S_I(A,B)=S_I([a^-,a^+],[b^-,b^+])=\frac{1}{2}\left(\frac{a^-\wedge b^-}{a^-\vee b^-}+\frac{a^+\wedge b^+}{a^+\vee b^+}\right). \tag{12}$$

3. The Construction of Interval Information Content of the Fuzzy Relation

In fact, IC(R) can be used to measure information content transferred by two fuzzy sets by means of an exact value. But, with uncertainty, the value of the information content between two fuzzy sets cannot be measured precisely. For instance, when it is measured as a maximum of 0.7 and a minimum of 0.1, how about it? It is necessary to extend the value from the exact number to interval value, and then, the definition of interval information content is proposed as follows:

Definition 6. Let $X=\{x_1,x_2,\ldots,x_m\}$, $Y=\{y_1,y_2,\ldots,y_n\}$, the interval information content of fuzzy relation R be the mapping IIC(R): $X\times Y\longrightarrow D$, and

$$\mathrm{IIC}_1(R)=[\mathrm{IC}(R|X)\wedge \mathrm{IC}(R|Y),\mathrm{IC}(R|X)\vee \mathrm{IC}(R|Y)]. \tag{13}$$

Based on U-uncertainty, interval information content of the fuzzy relation can also be expressed as follows.

Definition 7. Let $X=\{x_1,x_2,\ldots,x_m\}$, $Y=\{y_1,y_2,\ldots,y_n\}$, R be the fuzzy relation from X to Y, and $\{R(x_1,y_1),R(x_1,y_2),\ldots,R(x_i,y_j)\}$ be ranked in descending order $\{\lambda_1,\lambda_2,\ldots,\lambda_{mn}\}$, where $\lambda_{i+1}\le\lambda_i,\lambda_{mn+1}=0$, and then,

$$\mathrm{IIC}_2(R)=\left[\sum_{i=1}^{mn}(\lambda_i-\lambda_{i+1})\log_2\frac{mn}{|R_{\lambda_{i+1}}|},\sum_{i=1}^{mn}(\lambda_i-\lambda_{i+1})\log_2\frac{mn}{|R_{\lambda_i}|}\right] \tag{14}$$

is the interval information content of R from X to Y.

Similarly, by Definition 3, the interval information content of R can also be constructed as

$$\begin{aligned}
\mathrm{IIC}_3(R)&=\left[\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\log_2\left(\frac{mn}{|R_{\lambda_{i+1}}|}\right),\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\log_2\left(\frac{mn}{|R_{\lambda_i}|}\right)\right],\\
\mathrm{IIC}_4(R)&=\left[\sum_{i=1}^{mn}(\lambda_i-\lambda_{i+1})\left(\mathrm{IC}(R_{\lambda_i}|X)\wedge\mathrm{IC}(R_{\lambda_i}|Y)\right),\sum_{i=1}^{mn}(\lambda_i-\lambda_{i+1})\left(\mathrm{IC}(R_{\lambda_i}|X)\vee\mathrm{IC}(R_{\lambda_i}|Y)\right)\right],\\
\mathrm{IIC}_5(R)&=\left[\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\left(\mathrm{IC}(R_{\lambda_i}|X)\wedge\mathrm{IC}(R_{\lambda_i}|Y)\right),\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\left(\mathrm{IC}(R_{\lambda_i}|X)\vee\mathrm{IC}(R_{\lambda_i}|Y)\right)\right].
\end{aligned} \tag{15}$$

Then, we have $\mathrm{IIC}_i(R) \in D, \quad i = 1, 2, \ldots, 5.$

Example 1. Let $X = \{x_1, x_2, \ldots, x_9\}$, $Y = \{y_1, y_2, \ldots, y_9\}$, and R be the fuzzy relation from X to Y; the results of $R(x_i, y_j)$ are listed in Table 1.

Taking $\mathrm{IIC}_1(R)$ for example, we have

$$\begin{aligned}
\mathrm{IC}(R|X) &= -\sum_{i=1}^{9} \frac{\sum_{j=1}^{9} R(x_i, y_j)}{\sum_{i=1}^{9}\sum_{j=1}^{9} R(x_i, y_j)} \log_2 \frac{\sum_{j=1}^{9} R(x_i, y_j)}{9},\\
&= \frac{2\times 3.1}{63.4}\log_2\frac{9}{3.1} + \frac{5.2}{63.4}\log_2\frac{9}{5.2} + \frac{7}{63.4}\log_2\frac{9}{7} + \frac{5\times 9}{63.4}\log_2\frac{9}{9},\\
&= 0.2553,\\
\mathrm{IC}(R|Y) &= -\sum_{j=1}^{9} \frac{\sum_{i=1}^{9} R^{-1}(y_j, x_i)}{\sum_{j=1}^{9}\sum_{i=1}^{9} R^{-1}(y_j, x_i)} \log_2 \frac{\sum_{i=1}^{9} R^{-1}(y_j, x_i)}{9},\\
&= \frac{5\times 5.9}{63.4}\log_2\frac{9}{5.9} + \frac{7.5}{63.4}\log_2\frac{9}{7.5} + \frac{8.4}{63.4}\log_2\frac{9}{8.4} + \frac{2\times 9}{63.4}\log_2\frac{9}{9},\\
&= 0.3278,
\end{aligned} \tag{16}$$

and then, $\mathrm{IIC}_1(R) = [0.\ 2553,\ 0.\ 3278]$.

$\{R(x_i, y_j)\}$ is ranked in descending order $\{1, 0.7, 0.6, 0.4, 0.3, 0\}$; then, $\lambda_1 = 1$, $\lambda_2 = 0.7$, $\lambda_3 = 0.6$, $\lambda_4 = 0.4$, $\lambda_5 = 0.3$, and $\lambda_i = 0$ $(i = 6, 7, \ldots, 81)$. Taking the case of $\lambda_1 = 1$, the values of $R_1(x_i, y_j)$ are listed in Table 2.

So,

$$\begin{aligned}
\mathrm{IC}(R_1|X) &= \frac{2\times 2}{56}\log_2\frac{9}{2} + \frac{3}{56}\log_2\frac{9}{3} + \frac{4}{56}\log_2\frac{9}{4} + \frac{5\times 9}{56}\log_2\frac{9}{9} = 0.3235,\\
\mathrm{IC}(R_1|Y) &= \frac{5\times 5}{56}\log_2\frac{9}{5} + \frac{6}{56}\log_2\frac{9}{6} + \frac{7}{56}\log_2\frac{9}{7} + \frac{2\times 9}{56}\log_2\frac{9}{9} = 0.4866.
\end{aligned} \tag{17}$$

Then,

$$\mathrm{IIC}_2(R) = \left[\sum_{i=1}^{mn}(\lambda_i - \lambda_{i+1})\log_2\frac{mn}{\left|R_{\lambda_{i+1}}\right|}, \sum_{i=1}^{mn}(\lambda_i - \lambda_{i+1})\log_2\frac{mn}{\left|R_{\lambda_i}\right|}\right] = [0.2492, 0.36]. \tag{18}$$

Similarly, we have

$$\begin{aligned}
\mathrm{IIC}_3(R) &= \left[\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\log_2\left(\frac{mn}{\left|R_{\lambda_{i+1}}\right|}\right), \sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\log_2\left(\frac{mn}{\left|R_{\lambda_i}\right|}\right)\right] = [0.3161, 0.4106],\\
\mathrm{IIC}_4(R) &= \left[\sum_{i=1}^{mn}(\lambda_i - \lambda_{i+1})\left(\mathrm{IC}\left(R_{\lambda_i}|X\right)\wedge\mathrm{IC}\left(R_{\lambda_i}|Y\right)\right), \sum_{i=1}^{mn}(\lambda_i - \lambda_{i+1})\left(\mathrm{IC}\left(R_{\lambda_i}|X\right)\vee\mathrm{IC}\left(R_{\lambda_i}|Y\right)\right)\right] = [0.2342, 0.3292],\\
\mathrm{IIC}_5(R) &= \left[\sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\left(\mathrm{IC}\left(R_{\lambda_i}|X\right)\wedge\mathrm{IC}\left(R_{\lambda_i}|Y\right)\right), \sum_{i=1}^{mn}\frac{\lambda_i}{\sum_{i=1}^{mn}\lambda_i}\left(\mathrm{IC}\left(R_{\lambda_i}|X\right)\vee\mathrm{IC}\left(R_{\lambda_i}|Y\right)\right)\right] = [0.2683, 0.3722].
\end{aligned} \tag{19}$$

4. The Ranking for Fuzzy Implication Operators Based on Interval Information Content

In data mining, it is necessary to extract rules from large databases, which means that a large number of rules will be generated during the process. So, how to evaluate these rules and get valid and useful information by determining the ranking of rules has become a new hotspot of data mining area. Here, the ranking of fuzzy implication operators can be realized by the interval information content of fuzzy relation. Let $I = \{I_1, I_2, \ldots, I_n\}$be the set of fuzzy implication operators, and the ranking method is defined as follows:

Step 1: to calculate the interval information content $\mathrm{IIC}(I_i)$ of fuzzy implication operator I_i $(i = 1, 2, \ldots, n)$.

Step 2: to calculate the possibility-based comparison value of interval information content p_{ij}, where $p_{ij} = P(\mathrm{IIC}(I_i) > \mathrm{IIC}(I_j))$ [28].

Step 3: to construct interval information content possibility-based comparison matrix P, where $P = (p_{ij})$.

Step 4: let $P_i = \sum_{j=1}^{n} p_{ij}$, and the ranking of implication operator is determined by the value of P_i. That is to say, if $P_i > P_j$, then $I_i > I_j$.

For implication operators, the ranking can be confirmed by extracting the interval information content of the corresponding fuzzy relation, but as the fuzzy relation matrix is only aimed for the discrete domain, it is necessary discretize the interval [0,1] by dividing into n parts, that is to say, let $X = \{m_0 = 0, m_1, m_2, \ldots, m_{n-1}, m_n = 1\}$, and the implication operators can be expressed as

$$I\colon X\times X \longrightarrow [0,1]\ \left(m_i, m_j\right) \longmapsto I\left(m_i, m_j\right), \quad i, j \in \{0, 1, 2, \ldots, n\}. \tag{20}$$

Here, four insertions can be adopted for the discretization: the average insertion of 9 points (a scale of zero

Table 1: The value of $R(x_i, y_j)$ in Example 1.

R	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	$\sum_{j=1}^{9} R(x_i, y_j)$
x_1	0	0	0	0	0	0.4	0.7	1	1	3.1
x_2	0	0	0	0	0	0.4	0.7	1	1	3.1
x_3	0.3	0.3	0.3	0.3	0.3	0.7	1	1	1	5.2
x_4	0.6	0.6	0.6	0.6	0.6	1	1	1	1	7
x_5	1	1	1	1	1	1	1	1	1	9
x_6	1	1	1	1	1	1	1	1	1	9
x_7	1	1	1	1	1	1	1	1	1	9
x_8	1	1	1	1	1	1	1	1	1	9
x_9	1	1	1	1	1	1	1	1	1	9
$\sum_{i=1}^{9} R(x_i, y_j)$	5.9	5.9	5.9	5.9	5.9	7.5	8.4	9	9	63.4

Table 2: The value of $R_1(x_i, y_j)$.

R_1	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	$\sum_{j=1}^{9} R_1(x_i, y_j)$
x_1	0	0	0	0	0	0	0	1	1	2
x_2	0	0	0	0	0	0	0	1	1	2
x_3	0	0	0	0	0	0	1	1	1	3
x_4	0	0	0	0	0	1	1	1	1	4
x_5	1	1	1	1	1	1	1	1	1	9
x_6	1	1	1	1	1	1	1	1	1	9
x_7	1	1	1	1	1	1	1	1	1	9
x_8	1	1	1	1	1	1	1	1	1	9
x_9	1	1	1	1	1	1	1	1	1	9
$\sum_{i=1}^{9} R_1(x_i, y_j)$	5	5	5	5	5	6	7	9	9	56

to ten), the average insertion of 19 points, the average insertion of 99 points, and the random insertion of 9 points. By equation (13), the interval information content of implication operators $I_1, \ldots, I_{13}$ is listed in Table 3.

Taking the average insertion of 9 points among the interval [0,1] as an example, the interval information content comparison matrix can be constructed as follows:

$$P = \left(p_{ij}\right) = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0.8199 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0.6501 & 0 & 0 & 0 & 0.0061 & 0.0061 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0.3591 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.3499 & 0 & 0 & 0.6409 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0.1801 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0.9939 & 0 & 0 & 1 & 0.8067 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.9939 & 0 & 0 & 1 & 0.8067 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

Table 3: Interval information content of I_i.

I_i	The average insertion of 9 points	The average insertion of 19 points	The average insertion of 99 points	The random insertion of 9 points
I_1	[0.5995, 0.6340]	[0.6155, 0.6477]	[0.6292, 0.6595]	[0.4978, 0.5060]
I_2	[0.5126, 0.5126]	[0.5304, 0.5304]	[0.5460, 0.5460]	[0.3506, 0.3944]
I_3	[0.2592, 0.2592]	[0.2495, 0.2495]	[0.2410, 0.2410]	[0.2400, 0.2679]
I_4	[0.3825, 0.3825]	[0.3853, 0.3853]	[0.3875, 0.3875]	[0.2972, 0.3305]
I_5	[1.0813, 1.0813]	[1.2214, 1.2214]	[1.3760, 1.3760]	[0.7492, 0.7905]
I_6	[1.2058, 1.2058]	[1.4142, 1.4142]	[1.6487, 1.6487]	[0.8869, 0.9217]
I_7	[0.3509, 0.3509]	[0.3603, 0.3603]	[0.3692, 0.3692]	[0.3301, 0.3301]
I_8	[0.3119, 0.4205]	[0.3077, 0.4075]	[0.3067, 0.3925]	[0.2545, 0.3775]
I_9	[0.6514, 0.6514]	[0.6854, 0.6854]	[0.7141, 0.7141]	[0.5656, 0.6152]
I_{10}	[0.4320, 0.5303]	[0.4394, 0.5192]	[0.4531, 0.5043]	[0.3051, 0.4310]
I_{11}	[1.2174, 1.2174]	[1.4865, 1.4865]	[1.8393, 1.8393]	[0.9064, 0.9215]
I_{12}	[0.3817, 0.5126]	[0.3916, 0.5304]	[0.4016, 0.5460]	[0.2917, 0.4293]
I_{13}	[0.3817, 0.5126]	[0.3916, 0.5304]	[0.4016, 0.5460]	[0.4228, 0.5118]

Then,

$$P_1 = \sum_{j=1}^{n} p_{1j} = 0+1+1+1+0+0+1+1+0+1+0+1+1 = 8. \tag{22}$$

Similarly, we have $P_2 = 6.8199$, $P_3 = 0$, $P_4 = 2.6562$, $P_5 = 10$, $P_6 = 11$, $P_7 = 1.3591$, $P_8 = 1.9908$, $P_9 = 9$, $P_{10} = 4.1801$, $P_{11} = 12$, $P_{12} = 3.8006$, and $P_{13} = 3.8006$; then,

$$I_{11} \succ I_6 \succ I_5 \succ I_9 \succ I_1 \succ I_2 \succ I_{10} \succ \{I_{12}, I_{13}\} \succ I_4 \succ I_8 \succ I_7 \succ I_3. \tag{23}$$

The ranking results indicated that I_{11}, I_6, and I_5 transfer large amounts of information content, whereas I_3 (Luckasiewz operators) transmits the least amount of information content. Also, when the average insertion is concerned, the ranking results of I_{12} and I_{13} cannot be sure, but could be improved with the help of the random ways. All results with different insertions are listed in Table 4. From Table 4, it can be concluded that even though the insertion is different, the ranking results are different, but I_{11}, I_6, and I_5 always transfer large amounts of information content, whereas I_3 (Luckasiewz operators) transmits the least. In fact, the former three operators are used in the construction of fuzzy systems with higher frequency. However, there is almost no research on the advantages of these operators in the construction of fuzzy control systems, and the ranking results based on interval information content provide theoretical basis for the study of the abovementioned problems.

5. The Clustering and Classification of Fuzzy Implication Operators

The clustering analysis is focused on cluster the things with similar attributes into a category by means of extracting the things' attribute. Also, whether the classification is reasonable is a question worth considering. In this section, clustering analysis is carried out for 13 fuzzy operators according to the attributes of interval information content, which are commonly used to construct fuzzy control systems. After confirming the best classification, the similarity measure is used to classify the category of the implication operator.

5.1. Clustering of Fuzzy Implication Operators Based on Interval Information Content. Based on similarity measure of the interval value, fuzzy implication operators can be clustered utilizing interval information content. Let $I = \{I_1, I_2, \ldots, I_n\}$be the set containing finite implication operators; the cluster analysis can be undertaken as follows:

Step 1: to complete the interval information content $\mathrm{IIC}(I_i)$ $(i = 1, 2, \ldots, n)$ of I_i

Step 2: to complete the similarity measure $s_{ij} = S_I(\mathrm{IIC}(I_i), \mathrm{IIC}(I_j))$ by equation (12)

Step 3: to construct similarity matrix $S = (s_{ij})$ based on interval information content

Step 4: to compute transitive closure matrix $t(S)$

Step 5: to cluster the implication operators according to the value of λ

In the same way, four methods are used to disperse the interval [0,1]: the average insertion of 9 points, the average insertion of 19 points, the average insertion of 99 points, and the random insertion of 9 points. By equation (12), the interval information content of $I_1, \ldots, I_{13}$ is listed in Table 3. Next, taking the average insertion of 9 points for example, the 13×13 similarity matrix S_{10} based on interval information content is expressed as

Table 4: Ranking results of implication operators.

Interval segmentation	Ranking results
The average insertion of 9 points	$I_{11} \succ I_6 \succ I_5 \succ I_9 \succ I_1 \succ I_2 \succ I_{10} \succ \{I_{12}, I_{13}\} \succ I_4 \succ I_8 \succ I_7 \succ I_3$
The average insertion of 19 points	$I_{11} \succ I_6 \succ I_5 \succ I_9 \succ I_1 \succ I_2 \succ I_{10} \succ \{I_{12}, I_{13}\} \succ I_4 \succ I_8 \succ I_7 \succ I_3$
The average insertion of 99 points	$I_{11} \succ I_6 \succ I_5 \succ I_9 \succ I_1 \succ I_2 \succ I_{10} \succ \{I_{12}, I_{13}\} \succ I_4 \succ I_8 \succ I_7 \succ I_3$
The random insertion of 9 points	$I_{11} \succ I_6 \succ I_5 \succ I_9 \succ I_1 \succ I_{13} \succ I_2 \succ I_7 \succ I_{10} \succ I_{12} \succ I_4 \succ I_8 \succ I_3$

$$S_{10} = \begin{pmatrix} 1 & 0.8318 & 0.4206 & 0.6207 & 0.5704 & 0.5115 & 0.5694 & 0.5918 & \cdots & 0.7226 \\ 0.8318 & 1 & 0.5057 & 0.7462 & 0.4741 & 0.4251 & 0.6845 & 0.7144 & \cdots & 0.8723 \\ 0.4206 & 0.5057 & 1 & 0.6776 & 0.2397 & 0.2150 & 0.7387 & 0.7237 & \cdots & 0.5924 \\ 0.6207 & 0.7462 & 0.6776 & 1 & 0.3537 & 0.3172 & 0.9174 & 0.8625 & \cdots & 0.8721 \\ 0.5704 & 0.4741 & 0.2397 & 0.3537 & 1 & 0.8967 & 0.3245 & 0.3387 & \cdots & 0.4135 \\ 0.5115 & 0.4251 & 0.2150 & 0.3172 & 0.8967 & 1 & 0.2910 & 0.3037 & \cdots & 0.3708 \\ 0.5694 & 0.6845 & 0.7387 & 0.9174 & 0.3245 & 0.2910 & 1 & 0.8617 & \cdots & 0.8019 \\ 0.5918 & 0.7144 & 0.7237 & 0.8625 & 0.3387 & 0.3037 & 0.8617 & 1 & \cdots & 0.8187 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.7226 & 0.8723 & 0.5924 & 0.8721 & 0.4135 & 0.3708 & 0.8019 & 0.8187 & \cdots & 1 \end{pmatrix}. \quad (24)$$

Furthermore, the transitive closure matrix is constructed as follows:

$$t_{10}(S) = \begin{pmatrix} 1 & 0.8318 & 0.7387 & 0.8318 & 0.6024 & 0.6024 & 0.8318 & 0.8318 & \cdots & 0.8318 \\ 0.8318 & 1 & 0.7387 & 0.8721 & 0.6024 & 0.6024 & 0.8721 & 0.8625 & \cdots & 0.9047 \\ 0.7387 & 0.7387 & 1 & 0.7387 & 0.6024 & 0.6024 & 0.7387 & 0.7387 & \cdots & 0.7381 \\ 0.8318 & 0.8721 & 0.7387 & 1 & 0.6024 & 0.6024 & 0.9174 & 0.8625 & \cdots & 0.8721 \\ 0.6024 & 0.6024 & 0.6024 & 0.6024 & 1 & 0.8967 & 0.6024 & 0.6024 & \cdots & 0.6024 \\ 0.6024 & 0.6024 & 0.6024 & 0.6024 & 0.8967 & 1 & 0.6024 & 0.6024 & \cdots & 0.6024 \\ 0.8318 & 0.8721 & 0.7387 & 0.9174 & 0.6024 & 0.6024 & 1 & 0.8625 & \cdots & 0.8721 \\ 0.8318 & 0.8625 & 0.7387 & 0.8625 & 0.6024 & 0.6024 & 0.8625 & 1 & \cdots & 0.8625 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0.8318 & 0.9047 & 0.7387 & 0.8721 & 0.6024 & 0.6024 & 0.8721 & 0.8625 & \cdots & 1 \end{pmatrix}. \quad (25)$$

The elements in the abovementioned matrix are arranged in ascending order {0.6024, 0.7387, 0.8318, 0.8625, 0.8721, 0.8967, 0.9047, 0.9174, 0.9251, 0.9468, 0.9905, 1}, then the cluster can be conducted by the abovementioned value, and all cluster results are listed in Figure 1.

Similarly, cluster analyses of average insertion of 19 and 99 points, as well as random insertion of 9 points, are listed in Figures 2–4.

Judging from the abovementioned four figures, the uniformity clustering results are divided into two categories of 13 fuzzy implication operators: $\{I_5, I_6, I_{11}\}$ and $\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$. Therefore, it can be granted as the optimum category. According to Definition 4, all fuzzy implication operators can be strictly divided into two categories. Evenly, I_5, I_6, and I_{11} are abnormal implications, and others are normal implications. That is to say, the optimum cluster of the fuzzy implication operators based on interval information content is divided into two categories: normal and abnormal. Therefore, the classification method is reasonable.

5.2. Classification of Implication operators. In the problem of pattern recognition, as soon as the best classifications are selected, it is necessary to determine which category of classification features is the closest to the sample. For any fuzzy implication operator, after determining the best classification by extracting the interval information content characteristics, the final categories are confirmed by the similarity measure between the sample implication operators and the clustering centers of each category. Concretely,

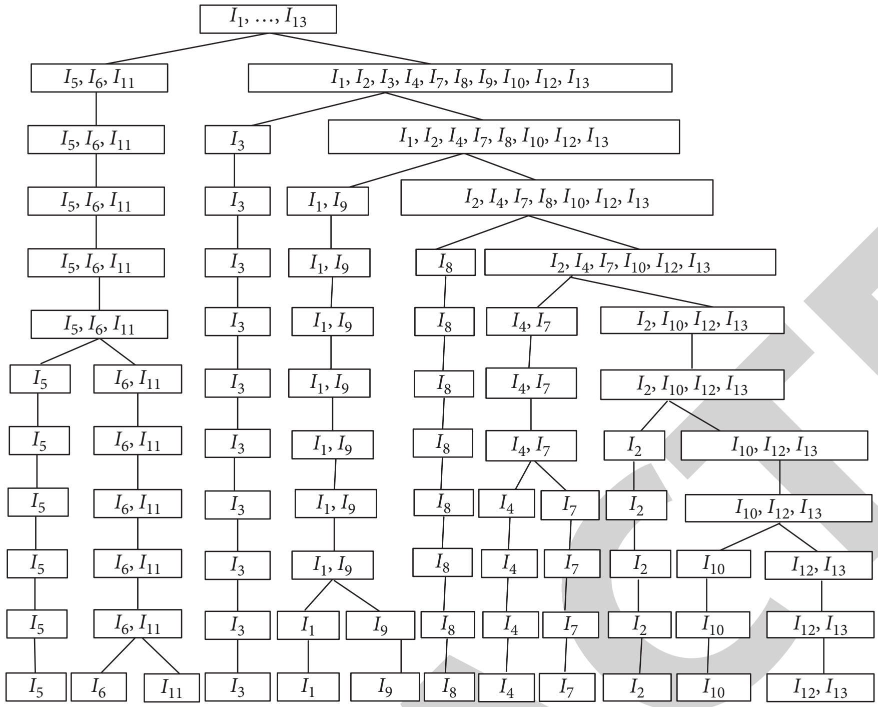

Figure 1: Cluster analysis of average insertion of 9 points.

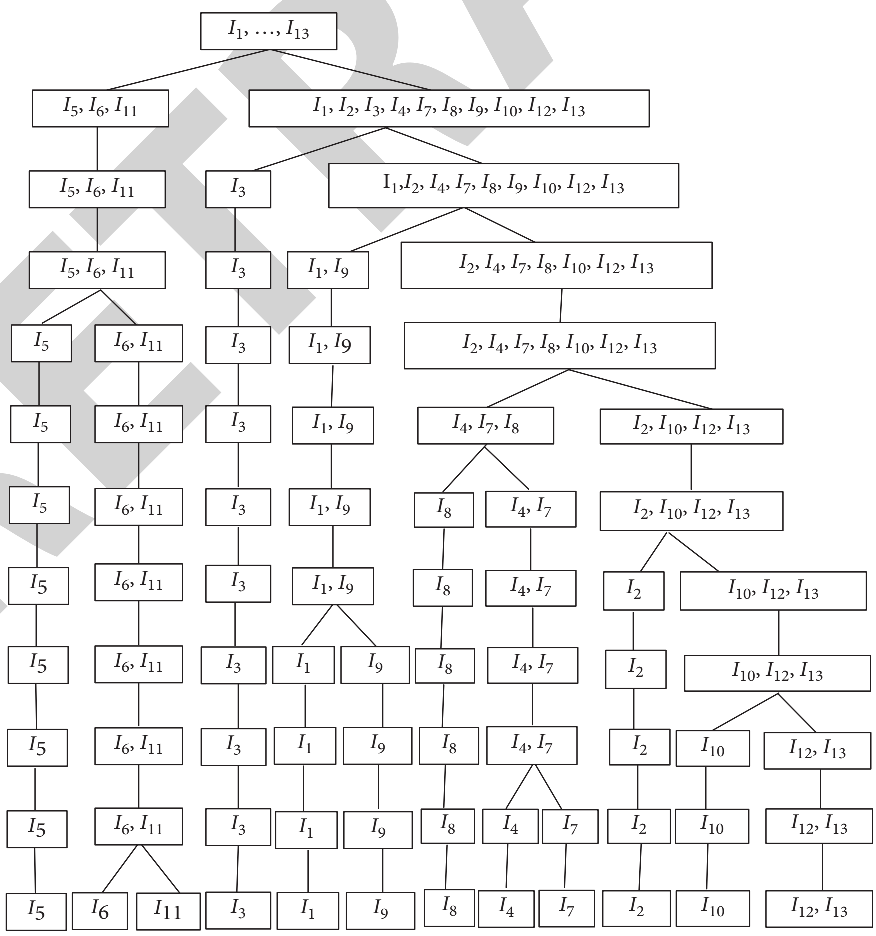

Figure 2: Cluster analysis of average insertion of 19 points.

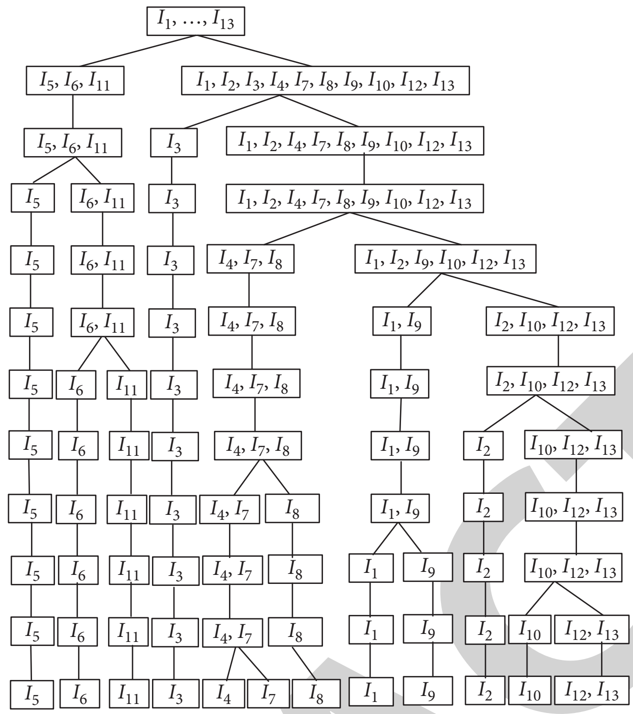

Figure 3: Cluster analysis of average insertion of 99 points.

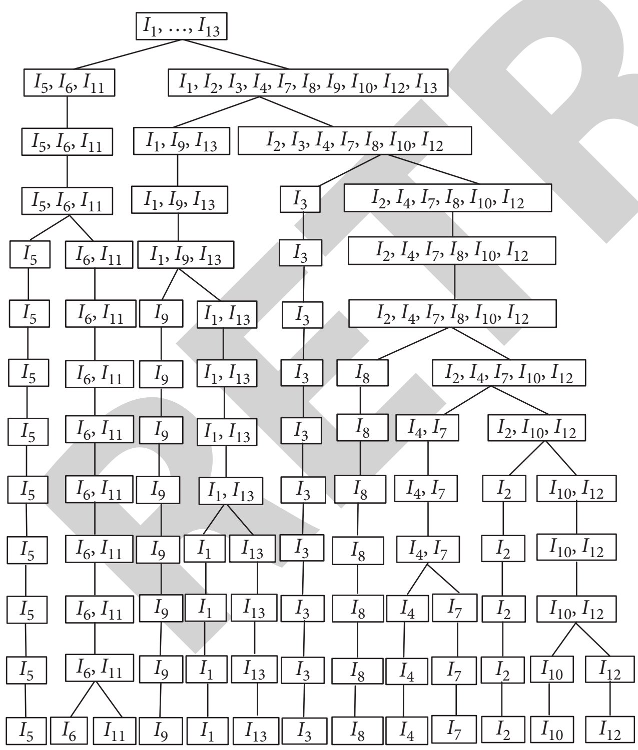

Figure 4: Cluster analysis of random insertion of 9 points.

Step 1: to compute IIC(I) of the sample operator I

Step 2: by equation (12), to compute the similarity measure between the sample operators and the clustering centers of each category, where the center of the i^{th} category $\overline{x}^i = [(\overline{x}^i)^-, (\overline{x}^i)^+]$ and

$$\left(\overline{x}^i\right)^- = \frac{1}{n_i}\sum_{j=1}^{n_i}\left(x_j^i\right)^-, \left(\overline{x}^i\right)^+ = \frac{1}{n_i}\sum_{j=1}^{n_i}\left(x_j^i\right)^+, \tag{26}$$

n_i is the sample number of the i^{th} category

Step 3: to determine the categories according to the maximum similarity principle

For instance, I_{14} and I_{15} are selected as the sample operators for classification:

$$\begin{aligned} I_{14}(a,b) &= \begin{cases} (1-a)\vee b, & (1-a)\wedge b = 0, \\ 1, & \text{else}, \end{cases} \\ I_{15}(a,b) &= \begin{cases} a\wedge b, & a\vee b = 1, \\ 0, & a\vee b < 1. \end{cases} \end{aligned} \tag{27}$$

I_{14} and I_{15} are the normal and abnormal fuzzy implication operators, respectively. Under the abovementioned optimum category, can these two implication operators be classified into correct categories? Firstly, to compute the interval information content with different insertions by equation (13), the results are listed in Table 5.

Secondly, to compute the similarity measure between them and cluster centers, the results are shown in Table 6.

From Table 6, it can be seen that I_{14} is always divided into the category of normal implication operators $\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$ and I_{15} is always divided into the contrary, which is consistent with the nature of I_{14} and I_{15} as normal and abnormal implication.

Table 5: Interval information content of I_{14} and I_{15}.

I_i	The average insertion of 9 points	The average of 19 points	The average of 99 points	The random of 9 points
I_{14}	[0.1063, 0.1063]	[0.0570, 0.0570]	[0.0120, 0.0120]	[0.0916, 0.0916]
I_{15}	[1.1719, 1.1719]	[1.2999, 1.2999]	[2.0703, 2.0703]	[1.1093, 1.1093]

Table 6: Similarity measure between the implication operator and cluster center.

Interval segmentation	Cluster Result	Cluster center	Similarity measure between I_{14} and clustering centers	Similarity measure between I_{15} and clustering centers
The average insertion of 9 points	$\{I_5, I_6, I_{11}\}$	[1.1682, 1.1682]	0.0910	**1.0032**
	$\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$	[0.4263, 0.4263]	**0.2494**	0.3628
The average insertion of 19 points	$\{I_5, I_6, I_{11}\}$	[1.3790, 1.3790]	0.0413	**0.9426**
	$\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$	[0.4357, 0.4357]	**0.1308**	0.3352
The average insertion of 99 points	$\{I_5, I_6, I_{11}\}$	[1.6213, 1.6213]	0.0074	**0.7831**
	$\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$	[0.4450, 0.4450]	**0.027**	0.2149
The random insertion of 9 points	$\{I_5, I_6, I_{11}\}$	[0.8475, 0.8779]	0.1156	**0.7495**
	$\{I_1, I_2, I_3, I_4, I_7, I_8, I_9, I_{10}, I_{12}, I_{13}\}$	[0.3555, 0.4194]	**0.2576**	0.3358

6. Conclusions

Facing the era of big data, it is essential to process a large amount of data. So, it is a key issue to extract the attribute of data. The novel attribute in the presented work can be used to realize the rules' ranking and clustering effectively. Utilizing the interval information content, the Mamdani, probability product, and Yager operators show better ranking results than others, which provides a solid theoretical base for the operator selection in constructing fuzzy system. For clustering issues, by means of extracting the interval information content, the operators can be divided into two categories: normal and abnormal. Then, the correct clustering result of the operator with known attribute proves valid.

In the future, the following works will be carried out:

(1) If the axiomatic representation of interval information quantity of fuzzy relation can be established, the research of information quantity will be of great theoretical significance

(2) For the defined interval information content, if it can be applied to data mining to optimize and ranking the inference rules, it will be of practical significance to improve the accuracy of the fuzzy system

Data Availability

All the data have been included in the manuscript, and no additional information can be provided.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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Retraction

Retracted: General Complex-Valued Overlap Functions

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] Y. Chen, L. Bi, B. Hu, and S. Dai, "General Complex-Valued Overlap Functions," *Journal of Mathematics*, vol. 2021, Article ID 6613730, 6 pages, 2021.

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Research Article

General Complex-Valued Overlap Functions

Ying Chen,[1] Lvqing Bi,[2] Bo Hu,[3] and Songsong Dai[1]

[1]*School of Electronics and Information Engineering, Taizhou University, Taizhou 318000, China*
[2]*School of Electronics and Communication Engineering,*
Guangxi Colleges and Universities Key Laboratory of Complex System Optimization and Big Data Processing,
Yulin Normal University, Yulin 537000, China
[3]*School of Mechanical and Electrical Engineering, Guizhou Normal University, Guiyang 550025, China*

Correspondence should be addressed to Songsong Dai; ssdai@stu.xmu.edu.cn

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Overlap function is a special type of aggregation function which measures the degree of overlapping between different classes. Recently, complex fuzzy sets have been successfully applied in many applications. In this paper, we extend the concept of overlap functions to the complex-valued setting. We introduce the notions of complex-valued overlap, complex-valued 0-overlap, complex-valued 1-overlap, and general complex-valued overlap functions, which can be regarded as the generalizations of the concepts of overlap, 0-overlap, 1-overlap, and general overlap functions, respectively. We study some properties of these complex-valued overlap functions and their construction methods.

1. Introduction

Bustince et al. [1] introduced the concept of overlap function in order to express the overlapping degree between two different classes. Overlap functions are a special type of aggregation functions [2] that are used in many applications such as image processing [1, 3], classification [4, 5], and decision making [6, 7]. It has gained a rapid development with various forms. The concepts of Archimedean, general, 0-overlap, 1-overlap, n-dimensional, interval-valued overlap functions have been proposed [8–11]. Various properties including migrativity, distributivity, idempotency, and homogeneity of overlap functions have been investigated [7, 8, 12–17]. The additive generators [11, 18] and multiplicative generators [19] of overlap functions have been given. Implications derived from overlap functions have been studied [20, 21].

Ramot et al. [22, 23] introduced the concept of complex fuzzy sets. It is an effective tool to handle uncertainty and periodicity simultaneously. It has been successfully applied in signal processing [23–25], image processing [26], time series forecasting [27–30], and decision making [31, 32]. Different measures including distance, similarity, and entropy of complex fuzzy sets have been proposed [33–38]. Various properties including δ-equality, parallelity, orthogonality, and rotational invariance of complex fuzzy sets have been investigated [39–42].

Complex fuzzy sets have been successfully applied in many different fields. In some cases, overlapping degree is needed for complex-valued information of two or more objects. In this paper, we extend traditional real-valued overlap functions to complex-valued overlap function. The starting point is that complex-valued overlap differs from other real-valued overlap functions. For example, $e^{j\cdot x}$ ($j = \sqrt{-1}$) is a periodic function and negative operation (−) is closed in the range of complex unit circle. These features may lead to special properties and construction methods of complex-valued overlap functions and provide a good issue for generation of overlap functions. As far as we know, nowadays, there are no corresponding discussions to propose the complex-valued overlap functions. Therefore, in this paper, from the theoretical point of view, we propose the definitions and construction methods of complex-valued overlap functions.

This paper is organized as follows. In Section 2, we recall the concepts of overlap functions. In Section 3, we introduce complex-valued overlap functions and their properties. In Section 4, we propose some construction methods of complex-valued overlap functions. Conclusions are given in Section 5.

2. Preliminaries

In this section, we recall some concepts of bivariate overlap functions and n-dimensional overlap functions, which are largely studied [1, 10, 11].

2.1. Overlap Functions

Definition 1 (see [1]). A mapping O: $[0,1]^2 \longrightarrow [0,1]$ is an overlap function if, for all $a, b \in [0,1]$, it is commutative, nondecreasing, and continuous and satisfies the following conditions:

(O1) $O(a,b) = 0$ if and only if $ab = 0$;
(O2) $O(a,b) = 1$ if and only if $ab = 1$.

As introduced in [11], a mapping O: $[0,1]^2 \longrightarrow [0,1]$ is a 0-overlap function if we loose the condition (O1) to (O1') $ab = 0 \Rightarrow O(a,b) = 0$ without changing any other condition.

Similarly, a mapping O: $[0,1]^2 \longrightarrow [0,1]$ is a 0-overlap function if we loose the condition (O2) to (O2') $ab = 1 \Rightarrow O(a,b) = 1$ without changing any other condition.

Definition 2 (see [10]). A mapping O_n: $[0,1]^n \longrightarrow [0,1]$ is a n-dimensional overlap function if, for all $a_1, \ldots, a_n \in [0,1]$, it is commutative, nondecreasing, and continuous and satisfies the following conditions:

(O_n1) $O_n(a_1, \ldots, a_n) = 0$ if and only if $\prod_{i=1}^n a_i = 0$;
(O_n2) $O_n(a_1, \ldots, a_n) = 1$ if and only if $\prod_{i=1}^n a_i = 1$.

As introduced in [11], a mapping O_n: $[0,1]^n \longrightarrow [0,1]$ is an n-dimensional 0-overlap function if we loose condition (O_n1) to (O_n1') $\prod_{i=1}^n a_i = 0 \Rightarrow O_n(a_1, \ldots, a_n) = 0$ without changing any other condition.

Analogously, a mapping O_n: $[0,1]^n \longrightarrow [0,1]$ is an n-dimensional 1-overlap function if we loose condition (O_n2) to (O_n2') $\prod_{i=1}^n a_i = 1 \Rightarrow O_n(a_1, \ldots, a_n) = 1$without changing any other condition.

Based on the concepts of n-dimensional 0-overlap and 1-overlap functions, the general overlap functions are defined as follows:

Definition 3 (see [10]). A mapping O_n: $[0,1]^n \longrightarrow [0,1]$ is an n-dimensional general overlap function if, for all $a_1, \ldots, a_n \in [0,1]$, it is commutative, nondecreasing, and continuous and satisfies the following conditions:

(GO$_n$1) if $\prod_{i=1}^n a_i = 0$, then $O_n(a_1, \ldots, a_n) = 0$;
(GO$_n$2) if $\prod_{i=1}^n a_i = 1$, then $O_n(a_1, \ldots, a_n) = 1$.

3. N-Dimensional Complex-Valued Overlap Functions

Let $\mathbf{D} = \{\alpha \in \mathbb{C} \| \alpha | t \leq n1\}$, then we define n-dimensional complex-valued overlap functions.

Definition 4. A mapping CO$_n$: $\mathbf{D}^n \longrightarrow \mathbf{D}$ is an n-dimensional complex-valued overlap function if, for all $a_1, \ldots, a_n \in \mathbf{D}$, it is commutative and continuous and satisfies the following conditions:

(CO$_n$1) $CO_n(a_1, \ldots, a_n) = 0$ if and only if $\prod_{i=1}^n a_i = 0$;
(CO$_n$2) $CO_n(a_1, \ldots, a_n) = 1$ if and only if $\prod_{i=1}^n a_i = 1$;
(CO$_n$3) CO_n is amplitude monotonic in the first component: $|\mathrm{CO}_n(a, a_2, \ldots, a_n) \leq |\mathrm{CO}_n(b, a_2, \ldots, a_n)|$ when $|a| \leq |b|$.

Since CO$_n$ is commutative, n-dimensional complex-valued overlap functions also are amplitude monotonic in any other component based on (GCO$_n$3). Obviously, these conditions are analogous to those of Definition 1. When the domain is limited to [0,1], n-dimensional complex-valued overlap function reduces to n-dimensional real-valued overlap function of Definition 1.

Example 1. Nevertheless, there are mappings that are overlap functions in the domain [0,1] but are not complex-valued overlap functions. The function f: $\mathbf{D}^2 \longrightarrow \mathbf{D}$ given by

$$f(a,b) = ab\frac{a+b}{2} \tag{1}$$

is an overlap function but not a complex-valued overlap function.

There are many types of real-valued overlap functions. Similarly, we extend the concept of 0-overlap and 1-overlap functions to n-dimensional complex-valued overlap functions.

A mapping CO$_n$: $\mathbf{D}^n \longrightarrow \mathbf{D}$ is an n-dimensional complex-valued 0-overlap function if we loose condition (CO$_n$1) to (CO$_n$1') $\prod_{i=1}^n a_i = 0 \Rightarrow \mathrm{CO}_n(a_1, \ldots, a_n) = 0$ without changing any other condition.

A mapping CO$_n$: $\mathbf{D}^n \longrightarrow \mathbf{D}$ is an n-dimensional 1-overlap complex-valued function if we loose condition (CO$_n$2) to (CO$_n$2') $\prod_{i=1}^n a_i = 1 \Rightarrow \mathrm{CO}_n(a_1, \ldots, a_n) = 1$ without changing any other condition.

Based on the concepts of n-dimensional complex-valued 0-overlap and 1-overlap functions, we define the concept of n-dimensional general complex-valued overlap functions

Definition 5. A mapping CO$_n$: $\mathbf{D}^n \longrightarrow \mathbf{D}$ is an n-dimensional general complex-valued overlap function if, for all $a_1, \ldots, a_n \in \mathbf{D}$, it is commutative and continuous and satisfies the following conditions:

(GCO$_n$1) if $\prod_{i=1}^n a_i = 0$, then $\mathrm{CO}_n(a_1, \ldots, a_n) = 0$;
(GCO$_n$2) if $\prod_{i=1}^n a_i = 1$, then $\mathrm{CO}_n(a_1, \ldots, a_n) = 1$;

GCO$_n$ CO$_n$ is amplitude monotonic in the first component: $|CO_n(a, a_2, \ldots, a_n) \leq |CO_n(b, a_2, \ldots, a_n)|$ when $|a| \leq |b|$.

The relations between n-dimensional complex-valued overlap functions, complex-valued 0-overlap functions, complex-valued 1-overlap functions, and general complex-valued overlap functions are shown in Figure 1. Asmus et al. [17] gave the relations between n-dimensional interval-valued overlap functions, interval-valued 0-overlap functions, interval-valued 1-overlap functions, and general interval-valued overlap functions, which are similar to that of Figure 1.

Clearly, we have the following.

Proposition 1. *If a mapping* $g: \mathbf{D}^n \longrightarrow \mathbf{D}$ *is either an n-dimensional complex-valued overlap, complex-valued 0-overlap, or complex-valued 1-overlap function, then g is also a general complex-valued overlap function.*

We give some examples of these complex-valued overlap functions to demonstrate their relations.

Example 2. The function $\pi: \mathbf{D}^2 \longrightarrow \mathbf{D}$ given by

$$\pi(a, b) = a \cdot b \tag{2}$$

is a complex-valued overlap function.

Example 3. The function $\pi_2: \mathbf{D}^2 \longrightarrow \mathbf{D}$ given by

$$\pi_2(a, b) = a^2 \cdot b^2 \tag{3}$$

is a general complex-valued overlap function. Moreover, it is a complex-valued 1-overlap function but not a complex-valued 0-overlap function.

Example 4. The function $g: \mathbf{D}^2 \longrightarrow \mathbf{D}$ given by

$$g(a, b) = \min(|a|, |b|) \cdot a \cdot b \tag{4}$$

is a general complex-valued overlap function. Moreover, it is a complex-valued 1-overlap function but not a complex-valued 0-overlap function.

Example 5. The function $h: \mathbf{D}^2 \longrightarrow \mathbf{D}$ given by

$$h(a, b) = \max(|a| + |b| - 1, 0) \cdot a \cdot b \tag{5}$$

is a general complex-valued overlap function. Moreover, it is a complex-valued 0-overlap function but not a complex-valued 1-overlap function.

Note that the class of overlap functions is convex. But the convex sun is not amplitude monotonic [?], then the class of complex-valued overlap functions is not convex.

Negative operation (−) is closed in the range of complex unit circle, but is not closed in [0,1]. Then, we have the following properties only for complex-valued overlap functions.

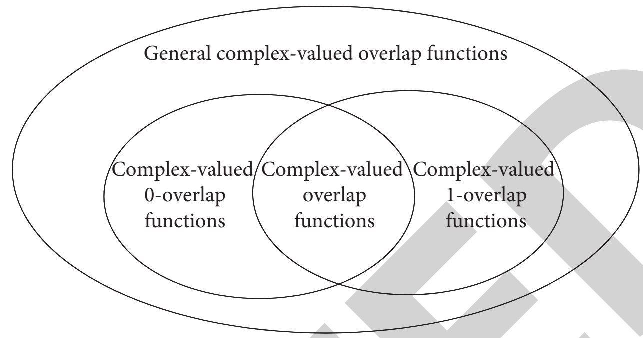

Figure 1: Relations between complex-valued overlap functions, complex-valued 0-overlap functions, complex-valued 1-overlap functions, and general complex-valued overlap functions.

Definition 6. We say the complex-valued overlap function CO: $\mathbf{D}^n \longrightarrow \mathbf{D}$ satisfies the self-duality property, if it satisfies

$$CO(a_1, \ldots, a_n) = -CO(-a_1, -a_2, \ldots, -a_n), \tag{6}$$

for any $a_1, \ldots, a_n \in \mathbf{D}$.

Definition 7. We say the complex-valued overlap function CO: $\mathbf{D}^n \longrightarrow \mathbf{D}$ is symmetric with respect to the point 0, if it satisfies

$$CO(a_1, \ldots, a_n) = CO(-a_1, -a_2, \ldots, -a_n), \tag{7}$$

for any $a_1, \ldots, a_n \in \mathbf{D}$.

There are complex-valued overlap functions satisfying the abovementioned properties.

Example 8. The function $g: \mathbf{D}^n \longrightarrow \mathbf{D}$ given by

$$g(a_1, \ldots, a_n) = \max\left(\sum_{i=1}^{n} |a_i| - n + 1, 0\right) \cdot \prod_{i=1}^{n} a_i \tag{8}$$

is a complex-valued overlap function. Interestingly, it satisfies the self-duality property when n is an odd number. It is symmetric with respect to the point 0 when n is an even number.

4. Construction of Complex-Valued Overlap Functions

We assume that the complex numbers are used in the form of exponent, i.e., $a \in \mathbf{D}$ is of the form $r_a e^{j\theta_a}$, where $j = \sqrt{-1}$, the amplitude term $r_a \in \mathbb{R}$, and the phase term $\theta_a \in [0, 2\pi)$. In order to let the phase term within valid range, we compute the least positive residue modulo 2π of the phase term when it is out of range. For simplicity, we omit the symbol $(\mathrm{mod}\, 2\pi)$.

Proposition 2. *If a mapping* $f: \mathbf{D}^n \longrightarrow \mathbf{D}$ *is n-dimensional complex-valued overlap (complex-valued 0-overlap, complex-valued 1-overlap, or general complex-valued overlap) function is expressed as*

$$f(a_1,\ldots,a_n) = g\left(r_{a_1}, r_{a_2},\ldots,r_{a_n}\right)e^{jh\left(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n}\right)}, \tag{9}$$

then the function g is an n-dimensional overlap (0-overlap, 1-overlap, or general overlap) function.

Theorem 1. *If the function $g\colon [0,1]^n \longrightarrow [0,1]$ is an n-dimensional overlap function, the function $h\colon [0,2\pi)^n \longrightarrow [0,2\pi)$ satisfies the following properties:*

(i) h is commutative;

(ii) $\sum_{i=1}^n \theta_{a_i} = 0$ if and only if $h(\theta_{a_1},\ldots,\theta_{a_n}) = 0$;

(iii) h is continuous.

then, the function $f\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ defined by equation (9) is an n-dimensional complex-valued overlap (complex-valued 0-overlap, complex-valued 1-overlap, or general complex-valued overlap) function.

Proof. It is immediate that f is commutative, amplitude monotonic, and continuous, since g is nondecreasing, and g and h are both commutative and continuous.

(CO$_n$1): ($\Rightarrow$) if $f(a_1,\ldots,a_n) = g(r_{a_1}, r_{a_2},\ldots r_{a_n})e^{jh}(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n}) = 0$, then $g(r_{a_1}, r_{a_2},\ldots r_{a_n}) = 0$. Then, $\prod_{i=1}^n r_{a_i} = 0$ since g is an overlap function. Then, $\prod_{i=1}^n a_i = \prod_{i=1}^n r_{a_i}\cdot e^{j(\sum_{i=1}^n \theta_{a_i})} = 0\cdot e^{j(\sum_{i=1}^n \theta_{a_i})} = 0$.
(CO$_n$1): ($\Leftarrow$) if $\prod_{i=1}^n a_i = \prod_{i=1}^n r_{a_i}\cdot e^{j(\sum_{i=1}^n \theta_{a_i})} = 0$, this means $\prod_{i=1}^n r_{a_i} = 0$, then $g(r_{a_1}, r_{a_2},\ldots r_{a_n}) = 0$ since g is an overlap function. Then, $f(a_1,\ldots,a_n) = g(r_{a_1}, r_{a_2},\ldots r_{a_n})e^{jh(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n})} = 0\cdot e^{jh(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n})} = 0$.
(CO$_n$2): ($\Rightarrow$) if $f(a_1,\ldots,a_n) = g(r_{a_1}, r_{a_2},\ldots r_{a_n})e^{jh}(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n}) = 1$, then $g(r_{a_1}, r_{a_2},\ldots r_{a_n}) = 1$ and $h(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n}) = 0$. Then, $\prod_{i=1}^n r_{a_i} = 1$ since g is an overlap function, and $\sum_{i=1}^n \theta_{a_i} = 0$ since h satisfies (ii). Then, $\prod_{i=1}^n a_i = \prod_{i=1}^n r_{a_i}\cdot e^{j(\sum_{i=1}^n \theta_{a_i})} = 1\cdot e^{j0} = 1$.
(CO$_n$2): ($\Leftarrow$) if $\prod_{i=1}^n a_i = \prod_{i=1}^n r_{a_i}\cdot e^{j(\sum_{i=1}^n \theta_{a_i})} = 1$, this means $\prod_{i=1}^n r_{a_i} = 1$ and $\sum_{i=1}^n \theta_{a_i} = 0$, then $g(r_{a_1}, r_{a_2},\ldots r_{a_n}) = 1$ since g is an overlap function, and $h(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n}) = 0$ since h satisfies (ii). Then, $f(a_1,\ldots,a_n) = g(r_{a_1}, r_{a_2},\ldots r_{a_n})e^{jh(\theta_{a_1},\theta_{a_2},\ldots\theta_{a_n})} = 1\cdot e^{j0} = 1$. □ □

Theorem 2. *If the function $g\colon [0,1]^n \longrightarrow [0,1]$ is an n-dimensional 0-overlap function, the function $h\colon [0,2\pi)^n \longrightarrow [0,2\pi)$ satisfies the following properties:*

(i) h is commutative;

(ii) $\sum_{i=1}^n \theta_{a_i} = 0$ if and only if $h(\theta_{a_1},\ldots,\theta_{a_n}) = 0$;

(iii) h is continuous.

Then, the function $f\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ defined by equation (9) is an n-dimensional complex-valued 0-overlap function.

Proof. Analogous to the proof of Theorem 1. □

Theorem 3. *If the function $g\colon [0,1]^n \longrightarrow [0,1]$ is an n-dimensional 1-overlap (or general overlap) function, the function $h\colon [0,2\pi)^n \longrightarrow [0,2\pi)$ satisfies the following properties:*

(i) h is commutative;

(ii) If $\sum_{i=1}^n \theta_{a_i} = 0$, then $h(\theta_{a_1},\ldots,\theta_{a_n}) = 0$;

(iii) h is continuous.

Then, the function $f\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ defined by equation (9) is an n-dimensional complex-valued 1-overlap (or general complex-valued overlap) function.

Proof. Analogous to the proof of Theorem 1. □

There are several construction methods of (general) overlap functions. Here, we consider the construction of (general) complex-valued overlap functions. If the n-dimensional complex-valued function $f\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ is defined by equation (9), then we can easily see that it is a key step to construct the function $h\colon [0,2\pi)^n \longrightarrow [0,2\pi)$, which satisfies the condition (ii) of Theorem 2 (or 3).

Now, we give some examples of bivariate functions $h\colon [0,2\pi)^2 \longrightarrow [0,2\pi)$ satisfying condition (ii) of Theorem 2 (or 3).

Example 6. The function $h_1\colon [0,2\pi)^2 \longrightarrow [0,2\pi)$ given by

$$h_1(a,b) = a+b \tag{10}$$

satisfies condition (ii) of Theorem 2. The function $h_2\colon [0,2\pi)^2 \longrightarrow [0,2\pi)$ given by

$$h_2(a,b) = -a-b \tag{11}$$

satisfies condition (ii) of Theorem 2. The function $h_3\colon [0,2\pi)^2 \longrightarrow [0,2\pi)$ given by

$$h_3(a,b) = 2(a+b) \tag{12}$$

satisfies condition (ii) of Theorem 3. But it does not satisfy condition (ii) of Theorem 2.

Note that we omit the operation of $\bmod 2\pi$. If $a = b = (\pi/2)$, then $h_3(a,b) = 2(a+b) = 2\pi = 0$, but $a+b = \pi \neq 0$. So, h_3 does not satisfy condition (ii) of Theorem 2. In general, we have the following results.

Example 7. The function $h_4\colon [0,2\pi)^2 \longrightarrow [0,2\pi)$ given by

$$h_4(a,b) = k(a+b),\quad k = \pm 2, \pm 3,\ldots, \tag{13}$$

satisfies condition (ii) of Theorem 3. But it does not satisfy condition (ii) of Theorem 2.

Based on results of complex-valued overlap functions, we give the following examples.

Example 8. The function $h_{n,p,k}\colon \mathbf{D}^n \longrightarrow \mathbf{D}$ given by

$$h_{n,p,k} = \left(r_{a_1}\cdot r_{a_2},\ldots,r_{a_n}\right)^p \cdot e^{jk\left(\theta_{a_1}+\theta_{a_2}+\cdots+\theta_{a_n}\right)} \tag{14}$$

is a complex-valued overlap function when $p > 0$ and $k = \pm 1$.

The function $h_{n,p,k}$: $\mathbf{D}^n \longrightarrow \mathbf{D}$ given by

$$h_{n,L,k} = \max\left(\sum_{i=1}^{n} r_{a_i} - n + 1, 0\right) \cdot e^{jk\left(\theta_{a_1}+\theta_{a_2}+\cdots+\theta_{a_n}\right)} \tag{15}$$

is a complex-valued 0-overlap function when $k = \pm 1$.

The function $h_{n,p,k}$: $\mathbf{D}^n \longrightarrow \mathbf{D}$ given by

$$h_{n,\wedge,k} = \left(\min_{i=1}^{n} r_{a_i}\right) \cdot e^{jk\left(\theta_{a_1}+\theta_{a_2}+\cdots+\theta_{a_n}\right)} \tag{16}$$

is a complex-valued 1-overlap function when $k = \pm 2, \pm 3, \ldots$.

5. Conclusions

In this paper, we introduced the concepts of complex-valued overlap, complex-valued 0-overlap, complex-valued 1-overlap, and general complex-valued overlap functions. We gave the relationship between them and studied their properties. Different from the traditional real-valued overlap functions, we added the following properties for complex-valued overlap functions since the domain of each variable is the unit disk of complex plane:

$$\begin{aligned} f(a_1, \ldots, a_n) = -f(-a_1, -a_2 \ldots, -a_n), f(a_1, \ldots, a_n) \\ = f(-a_1, -a_2 \ldots, -a_n). \end{aligned} \tag{17}$$

Then, we presented some construction methods for complex-valued overlap functions. Because of the periodicity of exponential function e^{jx}, our method includes the construction of a continuous, commutative function h: $[0, 2\pi)^n \longrightarrow [0, 2\pi)$ satisfying the following property:

$$\sum_{i=1}^{n} \theta_{a_i} = 0 \quad \text{if and only if } h\left(\theta_{a_1}, \ldots, \theta_{a_n}\right) = 0. \tag{18}$$

Of course, we should note that complex-valued overlap functions have many differences with the traditional real-valued overlap functions. Some interesting properties are useful for complex-valued overlap functions but they do not appear in traditional real-valued overlap functions. As further works, we intend to investigate these special properties of complex-valued overlap functions.

Complex-valued overlap functions can be viewed as a special class of complex fuzzy aggregation functions which have been widely used in many application fields. How to apply the complex-valued overlap functions is another problem of interest.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: On Three Types of Soft Rough Covering-Based Fuzzy Sets

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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Research Article

On Three Types of Soft Rough Covering-Based Fuzzy Sets

Mohammed Atef,[1] Shokry Nada,[1] Abdu Gumaei,[2,3] and Ashraf S. Nawar[1]

[1]*Department of Mathematics and Computer Science, Faculty of Science, Menoufia University, Menoufia, Egypt*
[2]*Research Chair of Pervasive and Mobile Computing, Department of Information Systems, College of Computer and Information Sciences, King Saud University, Riyadh 11543, Saudi Arabia*
[3]*Computer Science Department, Faculty of Applied Science, Taiz University, Taiz, Yemen*

Correspondence should be addressed to Abdu Gumaei; abdugumaei@gmail.com

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Recently, the concept of a soft rough fuzzy covering (briefly, SRFC) by means of soft neighborhoods was defined and their properties were studied by Zhan's model. As a generalization of Zhan's method and in order to increase the lower approximation and decrease the upper approximation, the present work aims to define the complementary soft neighborhood and hence three types of soft rough fuzzy covering models (briefly, 1-SRFC, 2-SRFC, and 3-SRFC) are proposed. We discuss their axiomatic properties. According to these results, we investigate three types of fuzzy soft measure degrees (briefly, 1-SMD, 2-SMD, and 3-SMD). Also, three kinds of ψ-soft rough fuzzy coverings (briefly, 1-ψ-SRFC, 2-ψ-SRFC, and 3-ψ-SRFC) and three kinds of $\mathscr{D}$-soft rough fuzzy coverings (briefly, 1-$\mathscr{D}$-SRFC, 2-$\mathscr{D}$-SRFC, and 3-$\mathscr{D}$-SRFC) are discussed and some of their properties are studied. Finally, the relationships among these three models and Zhan's model are presented.

1. Introduction

Pawlak [1, 2] developed the rough set theory for addressing the vagueness and granularity of information systems and data analysis. His theory and its generalizations since then have produced applications in different areas [3–15]. As mentioned above, a large variety of generalized rough set models have been investigated. These extensions include variable precision rough sets, covering-based rough sets (CRSs), fuzzy rough sets and rough fuzzy sets, covering-based multigranulation fuzzy rough sets, decision-theoretic rough sets, soft fuzzy rough sets, and probabilistic rough sets [16–19].

Covering-based rough sets are arguably one of the most studied generalizations of rough sets. Pomykala [20, 21] produced two pairs of operators with dual approximation. The definitions of neighborhood and granularity gave further insights of these approximation operators I (cf., Yao [22, 23]). Under the assumption of incomplete knowledge, Couso and Dubois [24] studied both pairs as well. Bonikowski et al. [25] proposed a model of CRS that depends on the concept of minimal description. There are other CRS models and relationships between them in [26–29]. Some CRS models were proposed by Tsang et al. [30] and Xu and Zhang [31]. Liu and Sai [32] compared CRS models defined by Zhu [26] and Xu and Zhang [31]. Ma [33] developed some neighborhood-related forms of covering rough sets using the neighborhood and complementary neighborhood concepts in 2012.

The fuzzy covering from a fuzzy relation is introduced by Deng et al. [34] in 2007. In 2016, Ma [35] introduced the concept of a fuzzy β-neighborhood to generate two types of fuzzy rough coverings. In 2017, Yang and Hu [36] defined the fuzzy β-complementary neighborhood to establish some types of the fuzzy covering-based rough sets. Also, Yang and Hu [37] in 2019 introduced the concept of fuzzy β-minimal description and fuzzy β-maximal description to propose four types of fuzzy neighborhood operators and studied their properties. D'eer et al. [38] discussed the fuzzy neighborhoods according to fuzzy coverings.

Dubois and Prade [39] presented the concepts of rough fuzzy set and fuzzy rough set in 1990. Lately, some scholars worked on covering-based rough fuzzy sets and fuzzy rough sets, for more information see [40–45].

Molodtsov [46] conceived the soft set theory as another valuable mathematical method for tackling the uncertainty problem. The soft set theory has a unique benefit compared to conventional mathematical methods, namely, parameterization by attributes. Maji et al. [47] introduced the concept of fuzzy soft sets (briefly, FSSs) in 2002. Recently, many researchers have studied the soft set theory, see [48–62]. Recently, the notion of a soft rough fuzzy covering by using soft neighborhoods was defined and their properties were studied by Zhan and Sun [63].

The aim of the paper is to increase the lower approximation and decrease the upper approximation of Zhan's model; this paper's contribution is to introduce three new kinds of soft rough fuzzy covering based on soft neighborhoods and complementary soft neighborhoods. Also, some of the related properties are studied. Further, the relationships among these models are discussed. The outline of this paper is as follows. Section 2 gives technical preliminaries. Sections 3 and 4 describe the three types of SRFC by using the notions of soft neighborhoods and complementary soft neighborhoods. In Section 5, we establish relationships among our model and Zhan's model. We conclude in Section 6.

2. Preliminaries

In this section, we review some concepts and results related to RST, CRS, SST, and SRFC.

Definition 1 (see [64]). Let Ω be a nonempty finite universe. A fuzzy subset on the universe Ω is defined by the mapping $\mathscr{A}(\bullet)\colon \Omega \longrightarrow [0,1]$, where the $\mathscr{A}(x)$ denotes the membership grade of the element $x\,(x \in \Omega)$ in the fuzzy set $\mathscr{A}.\mathscr{F}(\Omega)$ for the set of all fuzzy subsets of the Ω.

Definition 2 (see [65]). Let Ω be a universe of discourse, $\mathscr{A}, \mathscr{B} \in \mathscr{F}(\Omega)$. Then, we have the following statements:

(1) $\mathscr{A} \subseteq \mathscr{B} \Leftrightarrow \mathscr{A}(x) \leq \mathscr{B}(x)$,

(2) $\mathscr{A} = \mathscr{B} \Leftrightarrow \mathscr{A} \subseteq \mathscr{B}$ and $\mathscr{B} \subseteq \mathscr{A}$,

(3) $(\mathscr{A} \cap \mathscr{B})(x) = \mathscr{A}(x) \wedge \mathscr{B}(x)$ and $(\mathscr{A} \cup \mathscr{B})(x) = \mathscr{A}(x) \vee \mathscr{B}(x)$,

(4) $\mathscr{A}^c(x) = 1 - \mathscr{A}(x)$.

Definition 3 (see [26]). Let Ω be a universe and $\mathbf{C}$ be a family of subsets of Ω. If the empty set does not belong to $\mathbf{C}$ and $\Omega = \cup_{C \in \mathbf{C}} C$, then $\mathbf{C}$ is called a covering of Ω, and the ordered pair $(\Omega, \mathbf{C})$ is called a covering approximation space.

Definition 4 (see [26]). Let $(\Omega, \mathbf{C})$ be a covering approximation space. Then, for each $x \in \Omega$, define the neighborhood of x as follows:

$$N_{\mathbf{C}}(x) = \cap\{C \in \mathbf{C}\colon x \in C\}. \tag{1}$$

As already mentioned, the notion of soft sets was introduced in [46]. The beauty of soft sets lies in their quality of hybridization with other theories such as fuzzy sets and rough sets.

Definition 5 (see [46]). Let Ω be a universe of discourse, and let $\mathscr{E}$ be a finite set of relevant parameters regarding Ω. The pair $\mathscr{S} = (\widetilde{\mathscr{F}}, tA)$ is a soft set over Ω, when $\mathscr{A} \subseteq \mathscr{E}$ and $\widetilde{\mathscr{F}}\colon \mathscr{A} \longrightarrow \mathscr{P}(\Omega)$ (i.e., $\widetilde{\mathscr{F}}$ is a set-valued mapping from the subset of attributes $\mathscr{A}$ to Ω and $\mathscr{P}(\Omega)$ denotes the set of all subsets of Ω).

Definition 6 (see [52, 54]). The soft set $\mathscr{S} = (\widetilde{\mathscr{F}}, tA)$ is called a full soft set if $\cup_{a \in \mathscr{A}} \widetilde{\mathscr{F}}(a) = \Omega$ and a full soft set $\mathscr{S} = (\widetilde{\mathscr{F}}, tA)$ is called a soft covering (briefly, SC) over Ω if for each $a \in \mathscr{A}$, then $\widetilde{\mathscr{F}}(a) \neq \varnothing$. In addition, $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ is called a soft covering approximation space (briefly, SCAS).

Zhan et al. [63] introduced the concept of soft rough fuzzy covering (briefly, SRFC). So, in the following, some basic concepts related to SRFC are given.

Definition 7 (see [63]). Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS. For each $x \in \Omega$, then we define a soft neighborhood of x as follows:

$$N_S(x) = \cap\left\{\widetilde{\mathscr{F}}(a)t\colon naq \in hA\text{,}xx7 \in C\widetilde{\mathscr{F}};(a)\right\}. \tag{2}$$

Definition 8 (see [63]). Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be a SCAS of Ω. For each $\mathscr{A} \in \mathscr{F}(\Omega)$, the set $\mathscr{S}^{-0}(\mathscr{A})$ (resp. $\mathscr{S}^{+0}(\mathscr{A})$) is called the soft covering lower approximation (resp. the soft covering upper approximation), briefly 0-SCLA (resp. 0-SCUA), where

$$\begin{aligned} \mathscr{S}^{-0}(\mathscr{A})(x) &= \wedge\{\mathscr{A}(y)\colon y \in N_S(x)\}, \\ \mathscr{S}^{+0}(\mathscr{A})(x) &= \vee\{\mathscr{A}(y)\colon y \in N_S(x)\}, \quad \forall x \in \Omega. \end{aligned} \tag{3}$$

If $\mathscr{S}^{-0}(\mathscr{A}) \neq \mathscr{S}^{+0}(\mathscr{A})$, then $\mathscr{A}$ is called a soft rough covering-based fuzzy set (briefly, 0-SRFC); otherwise, it is definable.

3. The First Kind of Soft Rough Covering-Based Fuzzy Sets

This section deals with the 1-SRFC, 1-SMD, 1-ψ-SRFC, and 1-$\mathscr{D}$-SRFC as complementary soft neighborhoods and studies some of their properties.

Definition 9 Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS. Then, for each $x \in \Omega$, define the complementary soft neighborhood of x as follows:

$$M_S(x) = \{y \in \Omega, x \in N_S(y)\}. \tag{4}$$

Example 1. Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS and $(\widetilde{\mathscr{F}}, tA)$ be a soft set given as Table 1.

Compute the soft neighborhoods and complementary soft neighborhoods as the following:

$$
\begin{aligned}
N_S(x_1) &= \{x_1, x_2\},\\
N_S(x_2) &= \{x_1, x_2\},\\
N_S(x_3) &= \{x_3\},\\
N_S(x_4) &= \{x_4, x_5\},\\
N_S(x_5) &= \{x_5\},\\
N_S(x_6) &= \{x_3, x_5, x_6\},\\
M_S(x_1) &= \{x_1, x_2\},\\
M_S(x_2) &= \{x_1, x_2\},\\
M_S(x_3) &= \{x_3, x_6\},\\
M_S(x_4) &= \{x_4\},\\
M_S(x_5) &= \{x_4, x_5, x_6\},\\
M_S(x_6) &= \{x_6\}.
\end{aligned}
\tag{5}
$$

Definition 10. Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω. For each $\mathscr{A} \in \mathscr{F}(\Omega)$, the set $\mathscr{S}^{-1}(\mathscr{A})$ (resp. $\mathscr{S}^{+1}(\mathscr{A})$) is called the first type of a soft covering lower approximation (resp. the first type of a soft covering upper approximation), briefly 1-SCLA (resp. 1-SCUA), where

$$
\begin{aligned}
\mathscr{S}^{-1}(\mathscr{A})(x) &= \wedge\{\mathscr{A}(y)\colon y \in M_S(x)\},\\
\mathscr{S}^{+1}(\mathscr{A})(x) &= \vee\{\mathscr{A}(y)\colon y \in M_S(x)\}, \quad \forall x \in \Omega.
\end{aligned}
\tag{6}
$$

If $\mathscr{S}^{-1}(\mathscr{A}) \neq \mathscr{S}^{+1}(\mathscr{A})$, then $\mathscr{A}$ is called a soft rough covering-based fuzzy set (briefly, 1-SRFC); otherwise, it is definable.

Example 2 (continued from Example 1). If we take fuzzy set $\mathscr{A} = (0.1/x_1) + (0.3/x_2) + (0.8/x_3) + (0.2/x_4) + (0.5/x_5) + (0.7/x_6)$, then we have the following results:

$$
\begin{aligned}
\mathscr{S}^{-1}(\mathscr{A}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.2}{x_4} + \frac{0.2}{x_5} + \frac{0.7}{x_6},\\
\mathscr{S}^{+1}(\mathscr{A}) &= \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.7}{x_5} + \frac{0.7}{x_6}.
\end{aligned}
\tag{7}
$$

Therefore, $\mathscr{A}$ is a 1-SRFC. In addition, we can obtain

$$
\begin{aligned}
\mathscr{S}^{-0}(\mathscr{A}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.5}{x_5} + \frac{0.5}{x_6},\\
\mathscr{S}^{+0}(\mathscr{A}) &= \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.8}{x_3} + \frac{0.5}{x_4} + \frac{0.5}{x_5} + \frac{0.8}{x_6}.
\end{aligned}
\tag{8}
$$

Thus, $\mathscr{A}$ is a 0-SRFC.

Remark 1. From Example 2, we can see that

Table 1: Table for $(\widetilde{\mathscr{F}}, tA)$.

Ω	ν_1	ν_2	ν_3	ν_4	ν_5
x_1	1	1	1	0	0
x_2	1	1	1	0	0
x_3	0	1	0	1	1
x_4	0	0	1	1	0
x_5	0	0	1	1	1
x_6	0	0	0	1	1

(1) $\mathscr{S}^{-1}(\mathscr{A}) \nsubseteq \mathscr{S}^{-0}(\mathscr{A})$ and $\mathscr{S}^{-0}(\mathscr{A}) \nsubseteq \mathscr{S}^{-1}(\mathscr{A})$,
(2) $\mathscr{S}^{+1}(\mathscr{A}) \nsubseteq \mathscr{S}^{+0}(\mathscr{A})$ and $\mathscr{S}^{+0}(\mathscr{A}) \nsubseteq \mathscr{S}^{+1}(\mathscr{A})$.

Therefore, it is clear that 0-SRFC model and 1-SRFC model cannot contain each other.

Theorem 1. *Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω and $\mathscr{A}, \mathscr{B} \in \mathscr{F}(\Omega)$. Then, we have the following properties:*

(1) (L1) $\mathscr{S}^{-1}(\mathscr{A}^c) = (\mathscr{S}^{+1}(\mathscr{A}))^c$.
(H1) $\mathscr{S}^{+1}(\mathscr{A}^c) = (\mathscr{S}^{-1}(\mathscr{A}))^c$.
(2) If $\mathscr{A} \subseteq \mathscr{B}$, then
(L2) $\mathscr{S}^{-1}(\mathscr{A}) \subseteq \mathscr{S}^{-1}(\mathscr{B})$.
(H2) $\mathscr{S}^{+1}(\mathscr{A}) \subseteq \mathscr{S}^{+1}(\mathscr{B})$.
(3) (L3) $\mathscr{S}^{-1}(\mathscr{A} \cap \mathscr{B}) = \mathscr{S}^{-1}(\mathscr{A}) \cap \mathscr{S}^{-1}(\mathscr{B})$.
(H3) $\mathscr{S}^{+1}(\mathscr{A} \cap \mathscr{B}) \subseteq \mathscr{S}^{+1}(\mathscr{A}) \cap \mathscr{S}^{+1}(\mathscr{B})$.
(4) (L4) $\mathscr{S}^{-1}(\mathscr{A} \cup \mathscr{B}) \supseteq \mathscr{S}^{-1}(\mathscr{A}) \cup \mathscr{S}^{-1}(\mathscr{B})$.
(H4) $\mathscr{S}^{+1}(\mathscr{A} \cup \mathscr{B}) = \mathscr{S}^{+1}(\mathscr{A}) \cup \mathscr{S}^{+1}(\mathscr{B})$.
(5) (L5) $\mathscr{S}^{-1}(\mathscr{A}) = \mathscr{S}^{-1}(\mathscr{S}^{-1}(\mathscr{A}))$.
(H5) $\mathscr{S}^{+1}(\mathscr{A}) = \mathscr{S}^{+1}(\mathscr{S}^{+1}(\mathscr{A}))$.
(6) (LH) $\mathscr{S}^{-1}(\mathscr{A}) \subseteq \mathscr{A} \subseteq \mathscr{S}^{+1}(\mathscr{A})$.

Proof. We shall only prove (L1), (L2), (L3), (L5), and (LH), since (L1) (resp. (L2), (L4), and (L5)) is equivalent to (H1) (resp. (H2), (H4), and (H5)) and (L3), (L4), (H3), and (H4) are all equivalent to each other.

(1) (L1):

$$
\begin{aligned}
\mathscr{S}^{-1}(\mathscr{A}^c) &= \wedge\{\mathscr{A}^c(y)\colon y \in M_S(x)\}\\
&= \wedge\{1 - \mathscr{A}(y)\colon y \in M_S(x)\}\\
&= 1 - \vee\{\mathscr{A}(y)\colon y \in M_S(x)\} = \left(\mathscr{S}^{+1}(\mathscr{A})\right)^c.
\end{aligned}
\tag{9}
$$

(2) (L2): let $\mathscr{A}, \mathscr{B} \in \mathscr{F}(\Omega)$ such that $\mathscr{A} \subseteq \mathscr{B}$ and $x \in \Omega$. Then, we get the following result:

$$
\begin{aligned}
\mathscr{S}^{-1}(\mathscr{A})(x) &= \wedge\{\mathscr{A}(y)\colon y \in M_S(x)\}\\
&\leq \wedge\{\mathscr{B}(y)\colon y \in M_S(x)\} = \mathscr{S}^{-1}(\mathscr{B})(x).
\end{aligned}
\tag{10}
$$

(3) (L3): if $x \in \Omega$, then we have

$$\begin{aligned}\mathcal{S}^{-1}(\mathcal{A}\cap\mathcal{B})(x) &= \wedge\{(\mathcal{A}\cap\mathcal{B})(y)\colon\ y\in M_S(x)\}\\ &= \wedge\{\mathcal{A}(y)\colon\ y\in M_S(x)\}\cap\wedge\{\mathcal{B}(y)\colon\ y\in M_S(x)\} = \mathcal{S}^{-1}(\mathcal{A})(x)\cap\mathcal{S}^{-1}(\mathcal{B})(x).\end{aligned} \tag{11}$$

(4) (L5):

$$\begin{aligned}\mathcal{S}^{-1}\left(\mathcal{S}^{-1}(\mathcal{A})\right)(x) &= \wedge\left\{\mathcal{S}^{-1}(\mathcal{A})(y)\colon\ y\in M_S(x)\right\} = \wedge\{\wedge\{\mathcal{A}(w)\colon\ w\in M_S(y)\}\colon\ y\in M_S(x)\}\\ &= \wedge\{\mathcal{A}(w)\colon\ w\in M_S(y)\wedge y\in M_S(x)\}\\ &= \wedge\{\mathcal{A}(w)\colon\ w\in M_S(y)\subseteq M_S(x)\} = \wedge\{\mathcal{A}(w)\colon\ w\in M_S(x)\} = \mathcal{S}^{-1}(\mathcal{A})(x).\end{aligned} \tag{12}$$

(5) (LH): it is clear from Definition 10. □

Let us define the first type of a soft measure degree (briefly, 1-SMD) as follows.

Definition 11. Let $(\Omega, \widetilde{\mathcal{F}}, \mathcal{A})$ be an SCAS of Ω and $x, y\in\Omega$. The first kind of a soft measure degree between x and y (briefly, 1-SMD), denoted by $\mathcal{D}_S^1(x,y)$, is defined as follows:

$$\mathcal{D}_S^1(x,y) = \frac{|M_S(x)\cap M_S(y)|}{|M_S(x)\cup M_S(y)|}. \tag{13}$$

Obviously, $\mathcal{D}_S^1(x,x) = 1$ and $\mathcal{D}_S^1(x,y) = \mathcal{D}_S^1(y,x)$. Also, $0\le\mathcal{D}_S^1(x,y)\le 1$.

Example 3 (continued from Example 1). We have the following results as shown in Table 2.

From the concept of 1-SMD, we define a new kind called a first type of a soft rough covering-based ψ-fuzzy set (briefly, 1-ψ-SRFC) as follows.

Definition 12. Let $(\Omega, \widetilde{\mathcal{F}}, \mathcal{A})$ be an SCAS of Ω and $\mathcal{D}_S^1(x,y)$ be a 1-SMD of Ω. For each $\mathcal{A}\in\mathcal{F}(\Omega)$, the set $\mathcal{S}_\psi^{-1}(\mathcal{A})$ (resp. $\mathcal{S}_\psi^{+1}(\mathcal{A})$) is called the first type of a soft covering ψ-lower approximation (resp. the first type of a soft covering ψ-upper approximation), briefly 1-ψ-SCLA (resp. 1-ψ-SCUA), where

$$\begin{aligned}&\mathcal{S}_\psi^{-1}(\mathcal{A})(x) = \wedge\left\{\mathcal{A}(y)\colon\ \mathcal{D}_S^1(x,y)>\psi\right\},\\ &\mathcal{S}_\psi^{+1}(\mathcal{A})(x) = \vee\left\{\mathcal{A}(y)\colon\ \mathcal{D}_S^1(x,y)>\psi\right\},\quad \forall x\in\Omega.\end{aligned} \tag{14}$$

If $\mathcal{S}_\psi^{-1}(\mathcal{A})\neq\mathcal{S}_\psi^{+1}(\mathcal{A})$, then $\mathcal{A}$ is called 1-ψ-SRFC; otherwise, it is definable.

Example 4 (continued from Example 3). If $\psi = 0.2$ and $\mathcal{A} = (0.1/x_1) + (0.3/x_2) + (0.8/x_3) + (0.2/x_4) + (0.5/x_5) + (0.7/x_6)$, then we have the following results:

$$\begin{aligned}&\mathcal{S}_\psi^{-1}(\mathcal{A}) = \frac{0.1}{x_1}+\frac{0.1}{x_2}+\frac{0.5}{x_3}+\frac{0.2}{x_4}+\frac{0.2}{x_5}+\frac{0.5}{x_6},\\ &\mathcal{S}_\psi^{+1}(\mathcal{A}) = \frac{0.3}{x_1}+\frac{0.3}{x_2}+\frac{0.8}{x_3}+\frac{0.5}{x_4}+\frac{0.8}{x_5}+\frac{0.8}{x_6}.\end{aligned} \tag{15}$$

The proof of the following theorem is similar to Theorem 1, so we omit it.

Theorem 2. *Let $(\Omega, \widetilde{\mathcal{F}}, \mathcal{A})$ be an SCAS of Ω and $\mathcal{A}, \mathcal{B}\in\mathcal{F}(\Omega)$. Then, we have the following properties:*

(1) (L1) $\mathcal{S}_\psi^{-1}(\mathcal{A}^c) = (\mathcal{S}_\psi^{+1}(\mathcal{A}))^c$.

(H1) $\mathcal{S}_\psi^{+1}(\mathcal{A}^c) = (\mathcal{S}_\psi^{-1}(\mathcal{A}))^c$.

(2) If $\mathcal{A}\subseteq\mathcal{B}$, then

(L2) $\mathcal{S}_\psi^{-1}(\mathcal{A})\subseteq\mathcal{S}_\psi^{-1}(\mathcal{B})$.

(H2) $\mathcal{S}_\psi^{+1}(\mathcal{A})\subseteq\mathcal{S}_\psi^{+1}(\mathcal{B})$.

(3) (L3) $\mathcal{S}_\psi^{-1}(\mathcal{A}\cap\mathcal{B}) = \mathcal{S}_\psi^{-1}(\mathcal{A})\cap\mathcal{S}_\psi^{-1}(\mathcal{B})$.

(H3) $\mathcal{S}_\psi^{+1}(\mathcal{A}\cap\mathcal{B})\subseteq\mathcal{S}_\psi^{+1}(\mathcal{A})\cap\mathcal{S}_\psi^{+1}(\mathcal{B})$.

(4) (L4) $\mathcal{S}_\psi^{-1}(\mathcal{A}\cup\mathcal{B})\supseteq\mathcal{S}_\psi^{-1}(\mathcal{A})\cup\mathcal{S}_\psi^{-1}(\mathcal{B})$.

(H4) $\mathcal{S}_\psi^{+1}(\mathcal{A}\cup\mathcal{B}) = \mathcal{S}_\psi^{+1}(\mathcal{A})\cup\mathcal{S}_\psi^{+1}(\mathcal{B})$.

(5) If $\alpha\le\beta$, then

(L5) $\mathcal{S}_\alpha^{-1}(\mathcal{A})\subseteq\mathcal{S}_\beta^{-1}(\mathcal{A})$.

(H5) $\mathcal{S}_\alpha^{+1}(\mathcal{A})\subseteq\mathcal{S}_\beta^{+1}(\mathcal{A})$.

(6) (LH) $\mathcal{S}_\psi^{-1}(\mathcal{A})\subseteq\mathcal{A}\subseteq\mathcal{S}_\psi^{+1}(\mathcal{A})$.

Next, we define other SRFC models induced by 1-SMD as follows.

Definition 13. Let $(\Omega, \widetilde{\mathcal{F}}, \mathcal{A})$ be an SCAS of Ω and $\mathcal{D}_S^1(x,y)$ be a 1-SMD of Ω. For each $\mathcal{A}\in\mathcal{F}(\Omega)$, the set $\mathcal{S}_\mathcal{D}^{-1}(\mathcal{A})$ (resp. $\mathcal{S}_\mathcal{D}^{+1}(\mathcal{A})$) is called the first type of soft covering $\mathcal{D}$-lower approximation (resp. the first type of soft covering $\mathcal{D}$-upper approximation), briefly 1-$\mathcal{D}$-SCLA (resp. 1-$\mathcal{D}$-SCUA), where

$$\begin{aligned}&\mathcal{S}_\mathcal{D}^{-1}(\mathcal{A})(x) = \underset{y\in\Omega}{\wedge}\left\{\left(1-\mathcal{D}_S^1\right)(x,y)\vee\mathcal{A}(y)\right\},\\ &\mathcal{S}_\mathcal{D}^{+1}(\mathcal{A})(x) = \underset{y\in\Omega}{\vee}\left\{\mathcal{D}_S^1(x,y)\wedge\mathcal{A}(y)\right\},\quad\forall x\in\Omega.\end{aligned} \tag{16}$$

If $\mathcal{S}_\mathcal{D}^{-1}(\mathcal{A})\neq\mathcal{S}_\mathcal{D}^{+1}(\mathcal{A})$, then $\mathcal{A}$ is called 1-$\mathcal{D}$-SRFC; otherwise, it is definable.

Example 5 (continued from Example 3). If we take the fuzzy set $\mathcal{A} = (0.1/x_1) + (0.3/x_2) + (0.8/x_3) + (0.2/x_4) + (0.5/x_5) + (0.7/x_6)$, then we have the following results:

TABLE 2: Table for $\mathscr{D}_S^1(x_i, x_j)\,\forall i,\ j \in \{1, 2, \ldots, 6\}$.

Ω	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1	1	0	0	0	0
x_2	1	1	0	0	0	0
x_3	0	0	1	0	(1/4)	(1/2)
x_4	0	0	0	1	(1/3)	0
x_5	0	0	(1/4)	(1/3)	1	(1/3)
x_6	0	0	(1/2)	0	(1/3)	1

$$\begin{aligned}
\mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{A}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.2}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6},\\
\mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{A}) &= \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.8}{x_3} + \frac{0.3}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}.
\end{aligned} \tag{17}$$

Theorem 3. *Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω and $\mathscr{A}, \mathscr{B} \in \mathscr{F}(\Omega)$. Then, we have the following properties:*

(1) (L1) $\mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{A}^c) = (\mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{A}))^c$.

(H1) $\mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{A}^c) = (\mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{A}))^c$.

(2) If $\mathscr{A} \subseteq \mathscr{B}$, then

(L2) $\mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{A}) \subseteq \mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{B})$.

(H2) $\mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{A}) \subseteq \mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{B})$.

(3) (L3) $\mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{A} \cap \mathscr{B}) = \mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{A}) \cap \mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{B})$.

(H3) $\mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{A} \cap \mathscr{B}) \subseteq \mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{A}) \cap \mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{B})$.

(4) (L4) $\mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{A} \cup \mathscr{B}) \supseteq \mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{A}) \cup \mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{B})$.

(H4) $\mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{A} \cup \mathscr{B}) = \mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{A}) \cup \mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{B})$.

(5) (LH) $\mathcal{S}_{\mathscr{D}}^{-1}(\mathscr{A}) \subseteq \mathscr{A} \subseteq \mathcal{S}_{\mathscr{D}}^{+1}(\mathscr{A})$.

Proof. It is similar to Theorem 1. □

4. The Other Two SRFC Models

The implementation of the other two types of SRFC models (i.e., 2-SRFC and 3-SRFC) will be the subject of this section by merging soft neighborhoods and complementary soft neighborhoods. We list only the baseline concepts and omit the properties.

4.1. Type 2-SRFC

Definition 14. Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω. For each $\mathscr{A} \in \mathscr{F}(\Omega)$, the set $\mathcal{S}^{-2}(\mathscr{A})$ (resp. $\mathcal{S}^{+2}(\mathscr{A})$) is called the second type of a soft covering lower approximation (resp. the second type of a soft covering upper approximation), briefly 2-SCLA (resp. 2-SCUA), where

$$\begin{aligned}
\mathcal{S}^{-2}(\mathscr{A})(x) &= \wedge\{\mathscr{A}(y)\colon\ y \in (N_S \cap M_S)(x)\},\\
\mathcal{S}^{+2}(\mathscr{A})(x) &= \vee\{\mathscr{A}(y)\colon\ y \in (N_S \cap M_S)(x)\}, \quad \forall x \in \Omega.
\end{aligned} \tag{18}$$

If $\mathcal{S}^{-2}(\mathscr{A}) \neq \mathcal{S}^{+2}(\mathscr{A})$, then $\mathscr{A}$ is called a soft rough covering-based fuzzy set (briefly, 2-SRFC); otherwise, it is definable.

Example 6. Let us consider Examples 1 and 2. Then, for all $x \in \Omega$, we have

$$\begin{aligned}
(N_S \cap M_S)(x_1) &= \{x_1, x_2\},\\
(N_S \cap M_S)(x_2) &= \{x_1, x_2\},\\
(N_S \cap M_S)(x_3) &= \{x_3\},\\
(N_S \cap M_S)(x_4) &= \{x_4\},\\
(N_S \cap M_S)(x_5) &= \{x_5\},\\
(N_S \cap M_S)(x_6) &= \{x_6\}.
\end{aligned} \tag{19}$$

Also, we get $\mathcal{S}^{-2}(\mathscr{A})$ and $\mathcal{S}^{+2}(\mathscr{A})$ as the following:

$$\begin{aligned}
\mathcal{S}^{-2}(\mathscr{A}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6},\\
\mathcal{S}^{+2}(\mathscr{A}) &= \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}.
\end{aligned} \tag{20}$$

We define the second type of a soft measure degree (briefly, 2-SMD) as follows.

Definition 15. Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω and $x, y \in \Omega$. The second type of a soft measure degree between x and y (briefly, 2-SMD), denoted by $\mathscr{D}_S^2(x, y)$, is defined as follows:

$$\mathscr{D}_S^2(x, y) = \frac{\left|(N_S \cap M_S)(x) \cap (N_S \cap M_S)(y)\right|}{\left|(N_S \cap M_S)(x) \cup (N_S \cap M_S)(y)\right|}. \tag{21}$$

Obviously, $\mathscr{D}_S^2(x, x) = 1$ and $\mathscr{D}_S^2(x, y) = \mathscr{D}_S^2(y, x)$. Also, $0 \leq \mathscr{D}_S^2(x, y) \leq 1$.

Example 7 (continued from Example 6). We have the following results as set in Table 3.

From the concept of 2-SMD, we define a second type of a soft rough covering-based ψ-fuzzy set (briefly, 2-ψ-SRFC) as follows.

Definition 16. Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω and $\mathscr{D}_S^2(x, y)$ be a 2-SMD of Ω. For each $\mathscr{A} \in \mathscr{F}(\Omega)$, the set $\mathcal{S}_{\psi}^{-2}(\mathscr{A})$ (resp. $\mathcal{S}_{\psi}^{+2}(\mathscr{A})$) is called the second type of a soft covering ψ-lower approximation (resp. the second type of a soft covering ψ-upper approximation), briefly 2-ψ-SCLA (resp. 2-ψ-SCUA), where

$$\begin{aligned}
\mathcal{S}_{\psi}^{-2}(\mathscr{A})(x) &= \wedge\left\{\mathscr{A}(y)\colon\ \mathscr{D}_S^2(x, y) > \psi\right\},\\
\mathcal{S}_{\psi}^{+2}(\mathscr{A})(x) &= \vee\left\{\mathscr{A}(y)\colon\ \mathscr{D}_S^2(x, y) > \psi\right\}, \quad \forall x \in \Omega.
\end{aligned} \tag{22}$$

If $\mathcal{S}_{\psi}^{-2}(\mathscr{A}) \neq \mathcal{S}_{\psi}^{+2}(\mathscr{A})$, then $\mathscr{A}$ is called 2-ψ-SRFC; otherwise, it is definable.

Example 8. Let us consider Example 7. If we take $\psi = 0.2$ and $\mathscr{A} = (0.1/x_1) + (0.3/x_2) + (0.8/x_3) + (0.2/x_4) + (0.5/x_5) + (0.7/x_6)$, then 2-$\psi$-SCLA and 2-$\psi$-SCUA are obtained as follows:

Table 3: Table for $\mathscr{D}_S^2(x_i, x_j)\,\forall i,\ j \in \{1, 2, \ldots, 6\}$.

Ω	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1	1	0	0	0	0
x_2	1	1	0	0	0	0
x_3	0	0	1	0	0	0
x_4	0	0	0	1	0	0
x_5	0	0	0	0	1	0
x_6	0	0	0	0	0	1

$$\begin{aligned}\mathcal{S}_{\psi}^{-2}(\mathscr{A}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6},\\ \mathcal{S}_{\psi}^{+2}(\mathscr{A}) &= \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}.\end{aligned} \tag{23}$$

Now, we define other SRFC models induced by 2-SMD as follows.

Definition 17. Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω and $\mathscr{D}_S^2(x, y)$ be a 2-SMD of Ω. For each $\mathscr{A} \in \mathscr{F}(\Omega)$, the set $\mathcal{S}_{\mathscr{D}}^{-2}(\mathscr{A})$ (resp. $\mathcal{S}_{\mathscr{D}}^{+2}(\mathscr{A})$) is called the second type of soft covering $\mathscr{D}$-lower approximation (resp. the second type of soft covering $\mathscr{D}$-upper approximation), briefly 2-$\mathscr{D}$-SCLA (resp. 2-$\mathscr{D}$-SCUA), if

$$\begin{aligned}\mathcal{S}_{\mathscr{D}}^{-2}(\mathscr{A})(x) &= \bigwedge_{y\in\Omega}\left\{\left(1 - \mathscr{D}_S^2\right)(x, y) \vee \mathscr{A}(y)\right\},\\ \mathcal{S}_{\mathscr{D}}^{+2}(\mathscr{A})(x) &= \bigvee_{y\in\Omega}\left\{\mathscr{D}_S^2(x, y) \wedge \mathscr{A}(y)\right\}, \quad \forall x \in \Omega.\end{aligned} \tag{24}$$

If $\mathcal{S}_{\mathscr{D}}^{-2}(\mathscr{A}) \neq \mathcal{S}_{\mathscr{D}}^{+2}(\mathscr{A})$, then $\mathscr{A}$ is called 2-$\mathscr{D}$-SRFC; otherwise, it is definable.

Example 9. Let us consider Example 7 and fuzzy set $\mathscr{A} = (0.1/x_1) + (0.3/x_2) + (0.8/x_3) + (0.2/x_4) + (0.5/x_5) + (0.7/x_6)$ obtains 2-$\mathscr{D}$-SCLA and 2-$\mathscr{D}$-SCUA as follows:

$$\begin{aligned}\mathcal{S}_{\mathscr{D}}^{-2}(\mathscr{A}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6},\\ \mathcal{S}_{\mathscr{D}}^{+2}(\mathscr{A}) &= \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}.\end{aligned} \tag{25}$$

4.2. Type 3-SRFC

Definition 18. Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω. For each $\mathscr{A} \in \mathscr{F}(\Omega)$, the set $\mathcal{S}^{-3}(\mathscr{A})$ (resp. $\mathcal{S}^{+3}(\mathscr{A})$) is called the third type of a soft covering lower approximation (resp. the third type of a soft covering upper approximation), briefly 3-SCLA (resp. 3-SCUA), if

$$\begin{aligned}\mathcal{S}^{-3}(\mathscr{A})(x) &= \wedge\{\mathscr{A}(y)\colon\ y \in (N_S \cup M_S)(x)\},\\ \mathcal{S}^{+3}(\mathscr{A})(x) &= \vee\{\mathscr{A}(y)\colon\ y \in (N_S \cup M_S)(x)\}, \quad \forall x \in \Omega.\end{aligned} \tag{26}$$

If $\mathcal{S}^{-3}(\mathscr{A}) \neq \mathcal{S}^{+3}(\mathscr{A})$, then $\mathscr{A}$ is called a soft rough covering-based fuzzy set (briefly, 3-SRFC); otherwise, it is definable.

Example 10. Let us consider Examples 1 and 2. Then, for all $x \in \Omega$, we have

$$\begin{aligned}(N_S \cup M_S)(x_1) &= \{x_1, x_2\},\\ (N_S \cup M_S)(x_2) &= \{x_1, x_2\},\\ (N_S \cup M_S)(x_3) &= \{x_3, x_6\},\\ (N_S \cup M_S)(x_4) &= \{x_4, x_5\},\\ (N_S \cup M_S)(x_5) &= \{x_4, x_5, x_6\},\\ (N_S \cup M_S)(x_6) &= \{x_3, x_5, x_6\}.\end{aligned} \tag{27}$$

Also, $\mathcal{S}^{-3}(\mathscr{A})$ and $\mathcal{S}^{+3}(\mathscr{A})$ are obtained as follows:

$$\begin{aligned}\mathcal{S}^{-3}(\mathscr{A}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.7}{x_3} + \frac{0.2}{x_4} + \frac{0.2}{x_5} + \frac{0.5}{x_6},\\ \mathcal{S}^{+3}(\mathscr{A}) &= \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.8}{x_3} + \frac{0.5}{x_4} + \frac{0.7}{x_5} + \frac{0.8}{x_6}.\end{aligned} \tag{28}$$

In the following definition, third type of a soft measure degree (briefly, 3-SMD) is given.

Definition 19. Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω and $x, y \in \Omega$. The third kind of a soft measure degree (briefly, 3-SMD) between x and y, denoted by $\mathscr{D}_S^3(x, y)$, is defined by

$$\mathscr{D}_S^3(x, y) = \frac{\left|(N_S \cup M_S)(x) \cap (N_S \cup M_S)(y)\right|}{\left|(N_S \cup M_S)(x) \cup (N_S \cup M_S)(y)\right|}. \tag{29}$$

Obviously, $\mathscr{D}_S^3(x, x) = 1$ and $\mathscr{D}_S^3(x, y) = \mathscr{D}_S^3(y, x)$. Also, $0 \leq \mathscr{D}_S^3(x, y) \leq 1$.

Example 11 (continued from Example 10). We have the following results as summarized in Table 4.

From the concept of 3-SMD, we define a third type of a soft rough covering-based ψ-fuzzy set (briefly, 3-ψ-SRFC) as follows.

Definition 20. Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω and $\mathscr{D}_S^3(x, y)$ be a 3-SMD of Ω. For each $\mathscr{A} \in \mathscr{F}(\Omega)$, the set $\mathcal{S}_{\psi}^{-3}(\mathscr{A})$ (resp. $\mathcal{S}_{\psi}^{+3}(\mathscr{A})$) is called the third type of a soft covering ψ-lower approximation (resp. the third type of a soft covering ψ-upper approximation), briefly 3-ψ-SCLA (resp. 3-ψ-SCUA), if

$$\begin{aligned}\mathcal{S}_{\psi}^{-3}(\mathscr{A})(x) &= \wedge\left\{\mathscr{A}(y)\colon\ \mathscr{D}_S^3(x, y) > \psi\right\},\\ \mathcal{S}_{\psi}^{+3}(\mathscr{A})(x) &= \vee\left\{\mathscr{A}(y)\colon\ \mathscr{D}_S^3(x, y) > \psi\right\}, \quad \forall x \in \Omega.\end{aligned} \tag{30}$$

If $\mathcal{S}_{\psi}^{-3}(\mathscr{A}) \neq \mathcal{S}_{\psi}^{+3}(\mathscr{A})$, then $\mathscr{A}$ is called 3-ψ-SRFC; otherwise, it is definable.

Table 4: Table for $\mathscr{D}_S^3(x_i, x_j)\,\forall i,\ j \in \{1, 2, \ldots, 6\}$.

Ω	x_1	x_2	x_3	x_4	x_5	x_6
x_1	1	1	0	0	0	0
x_2	1	1	0	0	0	0
x_3	0	0	1	0	0	0
x_4	0	0	0	1	(2/3)	(1/4)
x_5	0	0	0	(2/3)	1	(1/2)
x_6	0	0	0	(1/4)	(1/2)	1

Example 12. Let us consider Example 11. If we take $\psi = 0.2$ and $\mathscr{A} = (0.1/x_1) + (0.3/x_2) + (0.8/x_3) + (0.2/x_4) + (0.5/x_5) + (0.7/x_6)$, then 3-$\psi$-SCLA and 3-$\psi$-SCUA of fuzzy sets $\mathscr{A}$ are obtained as follows:

$$\begin{aligned} \mathscr{S}_\psi^{-3}(\mathscr{A}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.2}{x_5} + \frac{0.2}{x_6}, \\ \mathscr{S}_\psi^{+3}(\mathscr{A}) &= \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.8}{x_3} + \frac{0.7}{x_4} + \frac{0.7}{x_5} + \frac{0.7}{x_6}. \end{aligned} \tag{31}$$

We define other SRFC models induced by 3-SMD as follows.

Definition 21. Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω and $\mathscr{D}_S^3(x, y)$ be a 3-SMD of Ω. For each $\mathscr{A} \in \mathscr{F}(\Omega)$, the set $\mathscr{S}_\mathscr{D}^{-3}(\mathscr{A})$ (resp. $\mathscr{S}_\mathscr{D}^{+3}(\mathscr{A})$) is called the third type of soft covering $\mathscr{D}$-lower approximation (resp. the third type of soft covering $\mathscr{D}$-upper approximation), briefly 3-$\mathscr{D}$-SCLA (resp. 3-$\mathscr{D}$-SCUA), where

$$\begin{aligned} \mathscr{S}_\mathscr{D}^{-3}(\mathscr{A})(x) &= \bigwedge_{y\in\Omega} \left\{\left(1 - \mathscr{D}_S^3\right)(x, y) \vee \mathscr{A}(y)\right\}, \\ \mathscr{S}_\mathscr{D}^{+3}(\mathscr{A})(x) &= \bigvee_{y\in\Omega} \left\{\mathscr{D}_S^3(x, y) \wedge \mathscr{A}(y)\right\}, \quad \forall x \in \Omega. \end{aligned} \tag{32}$$

If $\mathscr{S}_\mathscr{D}^{-3}(\mathscr{A}) \neq \mathscr{S}_\mathscr{D}^{+3}(\mathscr{A})$, then $\mathscr{A}$ is called 3-$\mathscr{D}$-SRFC; otherwise, it is definable.

Example 13. Consider Example 11 and fuzzy set $\mathscr{A} = (0.1/x_1) + (0.3/x_2) + (0.8/x_3) + (0.2/x_4) + (0.5/x_5) + (0.7/x_6)$, then 3-$\mathscr{D}$-SCLA and 3-$\mathscr{D}$-SCUA of fuzzy set $\mathscr{A}$ are obtained as follows:

$$\begin{aligned} \mathscr{S}_\mathscr{D}^{-3}(\mathscr{A}) &= \frac{0.1}{x_1} + \frac{0.1}{x_2} + \frac{0.8}{x_3} + \frac{0.2}{x_4} + \frac{0.3}{x_5} + \frac{0.5}{x_6}, \\ \mathscr{S}_\mathscr{D}^{+3}(\mathscr{A}) &= \frac{0.3}{x_1} + \frac{0.3}{x_2} + \frac{0.8}{x_3} + \frac{0.5}{x_4} + \frac{0.5}{x_5} + \frac{0.7}{x_6}. \end{aligned} \tag{33}$$

5. The Relationships between Zhan's Model and Our's

Now, we proceed to explain some relationships among the models presented in previous sections.

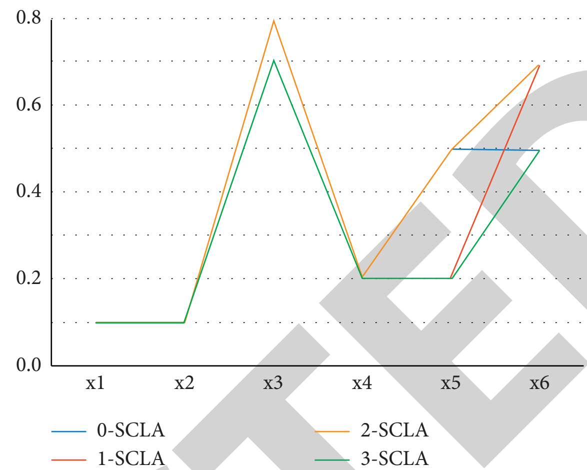

Figure 1: The representations of the four types of SCLA models.

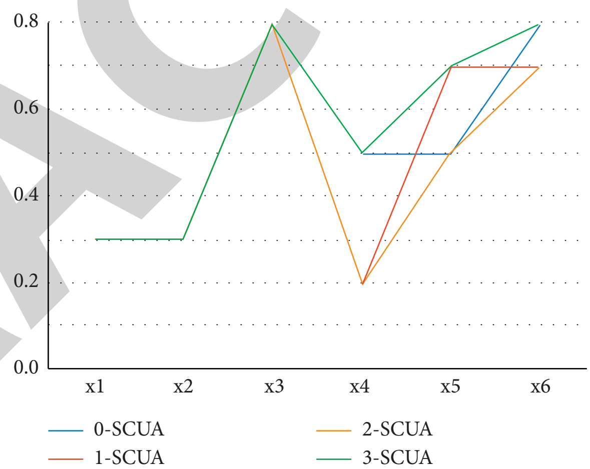

Figure 2: The representations of the four types of SCUA models.

Proposition 1. *Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω and $\mathscr{A} \in \mathscr{F}(\Omega)$. Then, we have the following properties.*

(1) $\mathscr{S}^{-3}(\mathscr{A}) \subseteq \mathscr{S}^{-1}(\mathscr{A}) \subseteq \mathscr{S}^{-2}(\mathscr{A})$.
(2) $\mathscr{S}^{-3}(\mathscr{A}) \subseteq \mathscr{S}^{-0}(\mathscr{A}) \subseteq \mathscr{S}^{-2}(\mathscr{A})$.
(3) $\mathscr{S}^{+2}(\mathscr{A}) \subseteq \mathscr{S}^{+1}(\mathscr{A}) \subseteq \mathscr{S}^{+3}(\mathscr{A})$.
(4) $\mathscr{S}^{+2}(\mathscr{A}) \subseteq \mathscr{S}^{+0}(\mathscr{A}) \subseteq \mathscr{S}^{+3}(\mathscr{A})$.

Proof. The proof is clear from Definitions 8, 10, 14, and 18. □

Proposition 2. *Let $(\Omega, \widetilde{\mathscr{F}}, \mathscr{A})$ be an SCAS of Ω and $\mathscr{A} \in \mathscr{F}(\Omega)$. Then, we have the following properties:*

(1) $\mathscr{S}^{-2}(\mathscr{A}) = \mathscr{S}^{-0}(\mathscr{A}) \cup \mathscr{S}^{-1}(\mathscr{A})$.
(2) $\mathscr{S}^{+2}(\mathscr{A}) = \mathscr{S}^{+0}(\mathscr{A}) \cap \mathscr{S}^{+1}(\mathscr{A})$.
(3) $\mathscr{S}^{-3}(\mathscr{A}) = \mathscr{S}^{-0}(\mathscr{A}) \cap \mathscr{S}^{-1}(\mathscr{A})$.
(4) $\mathscr{S}^{+3}(\mathscr{A}) = \mathscr{S}^{+0}(\mathscr{A}) \cup \mathscr{S}^{+1}(\mathscr{A})$.

Proof. Straightforward. □

The comparison of the results is given in Figures 1 and 2. Clearly, it is easy to see that the 2-SRFC model is better than 0-SRFC, 1-SRFC, and 2-SRFC model. Thus, this study indicates that our models are reasonable and effective.

6. Conclusion

In this paper, three new types of SRFC models are constructed as a generalization of definitions given in [63] by Zhan and Sun and their related properties are studied. The relationships between our model and Zhan's model are established. From Figures 1 and 2, it is obvious to see that the 2-SRFC is the best model (i.e., the increasing of the lower approximation and the decreasing of the upper approximation against Zhan's method) among the other models which are presented.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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